Chapter 4

Fermi-liquid theory *(Last version: 6 September 2019)*

Contents

One of the main goals of this book is to introduce the theoretical methods necessary to study (strongly) correlated quantum systems. This chapter is devoted to a class of fermion systems – known as Landau Fermi liquids or merely Fermi liquids – that can be understood without resorting to sophisticated many-body techniques. The Fermi-liquid paradigm describes not only the low-energy behavior of electrons in metals but also Fermi systems such as He^3, nuclear matter, or ultracold fermion gases.

In a Fermi liquid, the elementary excitations (quasi-particles and quasi-holes) are in direct correspondence with the (particle or hole) excitations of the ideal Fermi gas;

they carry the same quantum numbers and satisfy the Fermi-Dirac statistics.[1,2] This correspondence can be made explicit by means of an adiabatic switching-on of the interactions. The quasi-particles and quasi-holes determine both the low-temperature thermodynamics and the response of the system to macroscopic perturbations. In a very elegant phenomenological theory [1–3], Landau has shown that the low-energy behavior of the system can be expressed in terms of a few unknown parameters (the Landau parameters) that depend on the interactions between quasi-particles.

In the first part of the chapter (Secs. 4.1, 4.2 and 4.3), we review the main aspects of Landau Fermi-liquid theory starting from the quasi-particle concept. We mainly consider neutral Fermi liquids.[3] In the second part (Sec. 4.4), we discuss the microscopic underpinning of Fermi-liquid theory. We conclude the chapter by a discussion of Fermi-liquid theory in the framework of the renormalization group (Sec. 4.5).

4.1 The quasi-particle concept

Landau Fermi-liquid theory relies on the assumption that the low-lying eigenstates of the ideal Fermi gas continuously evolve into eigenstates of the real system as the interaction is adiabatically switched on. The quasi-particle concept, which is the starting point of Fermi-liquid theory, is a direct consequence of this assumption.

Before discussing this concept in detail, it should be noted that the adiabatic continuity assumption is quite restrictive and there are a number of cases where it is obviously violated. For instance, in a superconductor – and more generally whenever an instability of the Fermi surface leads to a broken symmetry state – the ground state is not related in any direct way to any one state of the free Fermi gas but rather to a coherent superposition of a large number of states. Fermi-liquid theory can also break down without the occurrence of a broken symmetry state as in the 1D interacting fermion gas (chapter 15).

The ideal Fermi gas

Let us start with the ideal Fermi gas and for simplicity consider a three-dimensional isotropic system. The eigenstates are antisymmetric combinations of plane waves, and a state of the system is fully determined by the momentum distribution function $n_{\mathbf{k}\sigma}$ giving the number of particles with momentum $\mathbf{k}$ and spin σ. The ground state corresponds to the distribution function $n^0_{\mathbf{k}} = \Theta(k_F - |\mathbf{k}|)$ where the Fermi momentum k_F is related to the mean particle density

$$n = \frac{1}{V}\sum_{\mathbf{k},\sigma} \Theta(k_F - |\mathbf{k}|) = 2\int \frac{d^3k}{(2\pi)^3}\Theta(k_F - |\mathbf{k}|) = \frac{k_F^3}{3\pi^2} \tag{4.1}$$

[1] Generally, the term "quasi-particles" refers to the elementary excitations whatever their relation to the bare particles. In the Fermi-liquid theory context, it has a narrower sense as explained in section 4.1.

[2] The Fermi-liquid theory elucidates the success of the ideal Fermi gas model in explaining some physical properties of electrons in metals despite the importance of the Coulomb interaction at metallic densities (this latter point is discussed in detail in chapter 5).

[3] The electromagnetic response of charged systems was the subject of section 3.4. The electron liquid will be studied in chapter 5 within the framework of the random-phase approximation.

(the sum over σ gives a factor of 2). The ground state energy is given by

$$E_0 = \sum_{\substack{\mathbf{k},\sigma \\ |\mathbf{k}|\leq k_F}} \epsilon^0_{\mathbf{k}} = \frac{3}{5} n \epsilon^0_F V, \tag{4.2}$$

where $\epsilon^0_{\mathbf{k}} = \mathbf{k}^2/2m$ is the free fermion dispersion and $\epsilon^0_F = k_F^2/2m = \mu(T=0)$ the Fermi energy.

Low-lying excited states are defined by their distribution function

$$n_{\mathbf{k}\sigma} = n^0_{\mathbf{k}} + \delta n_{\mathbf{k}\sigma}. \tag{4.3}$$

The change in the total energy corresponding to $\delta n_{\mathbf{k}\sigma}$ is

$$\delta E[\delta n] = E[n] - E_0 = \sum_{\mathbf{k},\sigma} \epsilon^0_{\mathbf{k}} \delta n_{\mathbf{k}\sigma}. \tag{4.4}$$

We denote by $n \equiv \{n_{\mathbf{k}\sigma}\}$ the momentum distribution function (not to be confused with the mean particle density). The particle energy can be defined as the functional derivative of the total energy with respect to the momentum distribution function $n_{\mathbf{k}\sigma}$,

$$\epsilon^0_{\mathbf{k}} = \frac{\delta E[n]}{\delta n_{\mathbf{k}\sigma}}. \tag{4.5}$$

The particle group velocity and the Fermi velocity are obtained from

$$\mathbf{v}_{\mathbf{k}} = \boldsymbol{\nabla}_{\mathbf{k}} \epsilon^0_{\mathbf{k}}, \qquad v_F = |\mathbf{v}_{\mathbf{k}}|_{k_F} = \left.\frac{\partial \epsilon^0_{\mathbf{k}}}{\partial |\mathbf{k}|}\right|_{k_F} = \frac{k_F}{m}, \tag{4.6}$$

respectively. Near the Fermi surface $|\mathbf{k}| = k_F$, one can therefore write the dispersion law as

$$\epsilon^0_{\mathbf{k}} = \epsilon^0_F + v_F(|\mathbf{k}| - k_F) + \mathcal{O}\left((|\mathbf{k}| - k_F)^2\right). \tag{4.7}$$

An elementary excitation corresponds to a particle added to or removed from the ground state. Any excited eigenstate of the system can be constructed by creating a certain number of these elementary particle or hole excitations. Since the latter are non-interacting, the total energy δE of the excited state is simply the sum of the particle and hole excitation energies [Eq. (4.4)].

The interacting Fermi liquid

According to the central hypothesis of Landau Fermi-liquid theory, any state of the ideal Fermi gas, characterized by a momentum distribution function $n_{\mathbf{k}\sigma} = n^0_{\mathbf{k}} + \delta n_{\mathbf{k}\sigma}$, generates an eigenstate of the interacting system as the interactions are switched on. This eigenstate can therefore be labeled by the distribution function $n_{\mathbf{k}\sigma}$. For reasons given below, this distribution function is referred to as the quasi-particle distribution function of the interacting Fermi liquid. As we shall see in section 4.4.1, it differs from – and should not be confused with – the momentum distribution $\langle \hat\psi^\dagger_\sigma(\mathbf{k}) \hat\psi_\sigma(\mathbf{k}) \rangle$ of the interacting system.

For reasons of symmetry, the Fermi surface of the interacting (isotropic) system is spherical.[4] Basic to Fermi-liquid theory is the fact that the volume of the Fermi surface is not changed by interactions (Luttinger theorem, derived in section 4.4.6) so that the Fermi momentum k_F of the interacting system is the same as that of the ideal gas. The ground state of the interacting system is then generated adiabatically from that of the ideal gas.[5]

Let us now add a particle with momentum $\mathbf{k}$ ($|\mathbf{k}| > k_F$) and spin σ to the ground state of the ideal gas. According to the adiabatic continuity assumption, as the interaction is slowly turned on we generate an (excited) eigenstate of the interacting system. However, because of the interactions the state under study is damped and acquires a finite life-time. Central to Fermi-liquid theory is the fact that the life-time becomes larger and larger at low energy ($|\mathbf{k}| \to k_F$). This property is a consequence of the Pauli principle which makes interactions ineffective near the Fermi surface (Sec. 4.4.1).[6] Thus, the state obtained by adding a low-lying ($|\mathbf{k}| \gtrsim k_F$) particle to the non-interacting Fermi sea evolves into a quasi-eigenstate of the interacting system, which is referred to as a quasi-particle. Similarly, one can define a quasi-hole by removing a particle with momentum $|\mathbf{k}| \lesssim k_F$ from the non-interacting Fermi sea. Since the total momentum and spin are conserved, quasi-particles and quasi-holes can be labeled by the same quantum numbers as in the non-interacting case, namely the momentum $\mathbf{k}$ and the spin projection σ along a given axis. As the bare particles, they carry charge $e < 0$ (quasi-particles) and $-e > 0$ (quasi-holes).

Near the Fermi surface, the quasi-particle dispersion can be expanded as

$$\epsilon_{\mathbf{k}} = \epsilon_{k_F} + v_F^*(|\mathbf{k}| - k_F) + \mathcal{O}\left((|\mathbf{k}| - k_F)^2\right), \qquad v_F^* = \frac{k_F}{m^*}, \tag{4.8}$$

which defines the (renormalized) Fermi velocity v_F^* and the effective mass m^*. $\epsilon_F = \mu(T=0)$ is the Fermi energy. The quasi-particle group velocity is defined by

$$\mathbf{v}_{\mathbf{k}}^* = \boldsymbol{\nabla}_{\mathbf{k}} \epsilon_{\mathbf{k}} \to v_F^* \hat{\mathbf{k}} \quad \text{for} \quad |\mathbf{k}| \to k_F. \tag{4.9}$$

At the Fermi level, the density of quasi-particle states (per spin) $N^*(\xi) = V^{-1} \sum_{\mathbf{k}} \delta(\xi - \xi_{\mathbf{k}})$ takes the value

$$N^*(0) = \frac{m^* k_F}{2\pi^2}. \tag{4.10}$$

The only difference with the case of the ideal Fermi gas is that the mass is replaced by the effective mass.

A generic low-lying excited state of the ideal Fermi gas, defined by its momentum distribution function $n_{\mathbf{k}\sigma} \neq n^0_{\mathbf{k}}$, can be constructed by combining particle and hole

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[4] Note that at this stage we have not rigorously defined the Fermi surface of an interacting system; this will be done in section 4.4.1.

[5] This is not true in anisotropic systems, where in general interactions deform the Fermi surface. The ground state then follows adiabatically from an excited state of the non-interacting system.

[6] This result can be obtained from a simple phase space argument. Consider the process where a particle above the Fermi sea ($|\mathbf{k}| > k_F$) is scattered into the state $\mathbf{k}+\mathbf{q}$ ($|\mathbf{k}+\mathbf{q}| > k_F$) by creating a particle-hole pair ($\mathbf{k}', \mathbf{k}' - \mathbf{q}$) ($|\mathbf{k}'| < k_F$ and $|\mathbf{k}' - \mathbf{q}| > k_F$). Because of energy conservation, $\omega = \epsilon_{\mathbf{k}+\mathbf{q}} - \epsilon_{\mathbf{k}} = -\epsilon_{\mathbf{k}'-\mathbf{q}} + \epsilon_{\mathbf{k}'} < 0$, the phase phase available for this scattering process is proportional to $(|\mathbf{k}| - k_F)^2$. This result can be seen by evaluating the integral $\int d^3k' d^3q \delta(\epsilon_{\mathbf{k}+\mathbf{q}} + \epsilon_{\mathbf{k}'-\mathbf{q}} - \epsilon_{\mathbf{k}} + \epsilon_{\mathbf{k}'})$ where the momenta satisfied the above mentioned constraints. Higher-order processes, involving multi-pair excitations, are more strongly suppressed as the corresponding phase space is smaller.

excitations. As the interaction is switched on, it evolves into a quasi-eigenstate of the interacting system characterized by the *quasi-particle* distribution function $n_{\mathbf{k}\sigma}$.[7,8] Because of the one-to-one correspondence between particle (or hole) excitations in the ideal Fermi gas and quasi-particle excitations in the Fermi liquid, quasi-particles follow the Fermi-Dirac statistics. Since the concept of quasi-particles refers only to low-lying excited states, $\delta n_{\mathbf{k}\sigma} = n_{\mathbf{k}\sigma} - n^0_{\mathbf{k}}$ should be appreciable only in the vicinity of the Fermi surface.

More precisely, for the notion of quasi-particles (or quasi-holes) to make sense, their life-time $\tau_{\mathbf{k}}$ and excitation energy $\xi_{\mathbf{k}} = \epsilon_{\mathbf{k}} - \mu$ should satisfy

$$\frac{1}{\tau_{\mathbf{k}}} \ll |\xi_{\mathbf{k}}|, \tag{4.11}$$

since $1/|\xi_{\mathbf{k}}|$ is the minimum time required to observe (or create with an external field) the quasi-particle. We shall see in section 4.4.1 that in a three-dimensional Fermi liquid[6]

$$\frac{1}{\tau_{\mathbf{k}}} = \mathcal{O}\left((|\mathbf{k}| - k_F)^2\right) \tag{4.12}$$

at zero temperature, so that the condition (4.11) is satisfied in the vicinity of the Fermi surface (recall that $\xi_{\mathbf{k}} = \mathcal{O}(|\mathbf{k}| - k_F)$). Suppose that the interaction is switched on within a characteristic time η^{-1}: $\hat{H}_{\text{int}}(t) = \hat{H}_{\text{int}}(t=0)e^{\eta t}$. Quasi-particles can be observed if their life-time is larger than η^{-1} and $1/|\xi_{\mathbf{k}}|$ smaller than η^{-1}, i.e.

$$\frac{1}{\tau_{\mathbf{k}}} \ll \eta \ll |\xi_{\mathbf{k}}|. \tag{4.13}$$

When condition (4.11) is fulfilled, it is possible to satisfy the inequality (4.13).

It should be emphasized that quasi-particle and quasi-hole excitations are not necessarily the only elementary excitations in the interacting system. The adiabatic continuity hypothesis does not exclude the possibility of other elementary excitations of the real system which disappear when the interaction is reduced to zero. These states correspond to collective excitations and emerge naturally in Landau Fermi-liquid theory (Sec. 4.3).

4.1.1 Landau energy functional $E[n]$

In the interacting Fermi liquid, the change in energy due a change $\delta n = n - n^0$ in the distribution function reads

$$\delta E[\delta n] = \sum_{\mathbf{k},\sigma} \epsilon_{\mathbf{k}} \delta n_{\mathbf{k}\sigma} \tag{4.14}$$

to first order in δn. Here $\epsilon_{\mathbf{k}}$ is the energy of a single quasi-particle added to the ground state of the system (as defined in the preceding section by (4.8)). According

[7]Note that the term "quasi-particle" often refers both to quasi-particles and/or quasi-holes. We shall explicitely distinguish between quasi-particles ($|\mathbf{k}| > k_F$) and quasi-holes ($|\mathbf{k}| < k_F$) only when necessary.

[8]If the system is anisotropic in spin space and the spin projection not a good quantum number, the quasi-particle distribution function $n_{\mathbf{k}\sigma\sigma'}$ becomes a matrix in spin space.

to (4.14), there is no interaction between quasi-particles or quasi-holes, since the total energy is simply additive. This suggests to push (4.14) one step further,

$$\delta E[\delta n] = \sum_{\mathbf{k},\sigma} \epsilon_{\mathbf{k}} \delta n_{\mathbf{k}\sigma} + \frac{1}{2V} \sum_{\mathbf{k},\mathbf{k}',\sigma,\sigma'} f_{\sigma\sigma'}(\mathbf{k},\mathbf{k}') \delta n_{\mathbf{k}\sigma} \delta n_{\mathbf{k}'\sigma'}. \tag{4.15}$$

The quadratic term in (4.15) is due to the interactions between quasi-particles. The Landau function f is defined as the second-order functional derivative of the total energy,

$$\frac{1}{V} f_{\sigma\sigma'}(\mathbf{k},\mathbf{k}') = \left. \frac{\delta^{(2)} E[n]}{\delta n_{\mathbf{k}\sigma} \delta n_{\mathbf{k}'\sigma'}} \right|_{n=n^0}, \tag{4.16}$$

and is therefore symmetric under the exchange $(\mathbf{k},\sigma) \leftrightarrow (\mathbf{k}',\sigma')$.

In the grand-canonical ensemble at $T = 0$, the relevant thermodynamic potential is $\Omega(T=0) = E - \mu N$, where $N = \sum_{\mathbf{k},\sigma} n_{\mathbf{k}\sigma}$ is the total number of quasi-particles (see below). Its variation is given by

$$\delta E[\delta n] - \mu \delta N[\delta n] = \sum_{\mathbf{k},\sigma} (\epsilon_{\mathbf{k}} - \mu) \delta n_{\mathbf{k}\sigma} + \frac{1}{2V} \sum_{\mathbf{k},\mathbf{k}',\sigma,\sigma'} f_{\sigma\sigma'}(\mathbf{k},\mathbf{k}') \delta n_{\mathbf{k}\sigma} \delta n_{\mathbf{k}'\sigma'}. \tag{4.17}$$

Suppose that $\delta n_{\mathbf{k}\sigma}$ is appreciable only for $||\mathbf{k}| - k_F| \lesssim \delta$. Then both terms in (4.17) are of order δ^2, which shows the need to push the expansion in δn to second order.

The quasi-particle energy $\tilde{\epsilon}_{\mathbf{k}}$ is defined as the variation of the total energy of the system due to the introduction of this quasi-particle. Mathematically, this means that $\tilde{\epsilon}_{\mathbf{k}}$ is given by the functional derivative of $E[n]$ with respect to the distribution function $n_{\mathbf{k}\sigma}$,

$$\tilde{\epsilon}_{\mathbf{k}} = \frac{\delta E[n]}{\delta n_{\mathbf{k}\sigma}} = \epsilon_{\mathbf{k}} + \frac{1}{V} \sum_{\mathbf{k}',\sigma'} f_{\sigma\sigma'}(\mathbf{k},\mathbf{k}') \delta n_{\mathbf{k}'\sigma'}. \tag{4.18}$$

Thus the quasi-particle energy $\tilde{\epsilon}_{\mathbf{k}} \equiv \tilde{\epsilon}_{\mathbf{k}}[\delta n]$ depends on the distribution $\delta n_{\mathbf{k}\sigma} = n_{\mathbf{k}\sigma} - n^0_{\mathbf{k}}$ of quasi-particles present in the system; it coincides with $\epsilon_{\mathbf{k}}$ only when $\delta n = 0$. In an isotropic liquid, spin rotation invariance ensures that $\tilde{\epsilon}_{\mathbf{k}}$ is independent of σ. Equation (4.18) gives the quasi-particle energy change coming from the average field due to the other quasi-particles. This mean-field-like description is characteristic of Landau Fermi-liquid theory. It also shows up in the random-phase-approximation form of the response functions (Sec. 4.3.4).

Landau parameters

The Landau function f plays a crucial role in Fermi-liquid theory. Spin rotation invariance implies that it can be written in terms of a spin symmetric and a spin anti-symmetric part,

$$f_{\sigma\sigma'}(\mathbf{k},\mathbf{k}') = f^s(\mathbf{k},\mathbf{k}') + \sigma\sigma' f^a(\mathbf{k},\mathbf{k}'). \tag{4.19}$$

Furthermore, for states near the Fermi surface one can set $|\mathbf{k}| = |\mathbf{k}'| = k_F$ so that $f_{\sigma\sigma'}(\mathbf{k},\mathbf{k}')$ depends only on the angle θ between $\mathbf{k}_F = k_F \hat{\mathbf{k}}$ and $\mathbf{k}'_F = k_F \hat{\mathbf{k}}'$,

$$f_{\sigma\sigma'}(\mathbf{k},\mathbf{k}') = f_{\sigma\sigma'}(\mathbf{k}_F,\mathbf{k}'_F) = f^s(\theta) + \sigma\sigma' f^a(\theta), \tag{4.20}$$

where the functions $f^s(\theta)$ and $f^a(\theta)$ can be expanded in Legendre polynomials,

$$f^{s,a}(\theta) = \sum_{l=0}^{\infty} f_l^{s,a} P_l(\cos\theta), \qquad f_l^{s,a} = (2l+1)\int_0^{\pi} \frac{d\Omega}{4\pi} f^{s,a}(\theta) P_l(\cos\theta) \tag{4.21}$$

($d\Omega = d\varphi d\theta \sin\theta$ denotes the elementary solid angle in the direction (φ, θ)). It is convenient to introduce dimensionless parameters – the Landau parameters – by multiplying $f_l^{s,a}$ by the density of states at the Fermi level,

$$F_l^{s,a} = 2N^*(0) f_l^{s,a} \tag{4.22}$$

(recall that $N^*(\xi)$ is the quasi-particle density of states per spin and $2N^*(\xi)$ the total density of states).

Entropy and thermodynamic potential

Since quasi-particles obey the Fermi-Dirac statistics, their entropy takes the form

$$S[n] = -\sum_{\mathbf{k},\sigma} [n_{\mathbf{k}\sigma} \ln n_{\mathbf{k}\sigma} + (1-n_{\mathbf{k}\sigma})\ln(1-n_{\mathbf{k}\sigma})]\,. \tag{4.23}$$

The thermodynamic potential is given by

$$\Omega[n] = E[n] - \mu N[n] - TS[n], \tag{4.24}$$

where $N[n] = \sum_{\mathbf{k},\sigma} n_{\mathbf{k}\sigma}$ is the total quasi-particle number. The equilibrium distribution function $\bar n \equiv \{\bar n_{\mathbf{k}\sigma}\}$ is obtained from the stationarity condition $\delta\Omega[n]/\delta n_{\mathbf{k}\sigma} = 0$,

$$\bar n_{\mathbf{k}\sigma} = n_F(\tilde{\xi_{\mathbf{k}}}), \tag{4.25}$$

where

$$\tilde\epsilon_{\mathbf{k}} = \frac{\delta E[n]}{\delta n_{\mathbf{k}\sigma}}\bigg|_{\bar n} = \epsilon_{\mathbf{k}} + \frac{1}{V}\sum_{\mathbf{k}',\sigma'} f_{\sigma\sigma'}(\mathbf{k},\mathbf{k}')\left(\bar n_{\mathbf{k}'\sigma'} - n^0_{\mathbf{k}'\sigma'}\right) \tag{4.26}$$

is the quasi-particle energy corresponding to the equilibrium distribution $\bar n$. If we expand $\Omega[n]$ about its equilibrium value, we obtain[9]

$$\Omega[\bar n + \delta n] - \Omega[\bar n] = \frac{1}{2}\sum_{\mathbf{k},\mathbf{k}',\sigma,\sigma'}\left[-\frac{\delta_{\sigma,\sigma'}\delta_{\mathbf{k},\mathbf{k}'}}{n'_F(\tilde{\xi_{\mathbf{k}}})} + \frac{1}{V} f_{\sigma\sigma'}(\mathbf{k},\mathbf{k}')\right]\delta n_{\mathbf{k}\sigma}\delta n_{\mathbf{k}'\sigma'} \tag{4.27}$$

to lowest order in δn. There is no linear term since $\Omega[n]$ is stationary for $n = \bar n$. Equation (4.27) shows that the f function can also be defined from the thermodynamic potential,

$$\frac{1}{V} f_{\sigma\sigma'}(\mathbf{k},\mathbf{k}') = \frac{\delta_{\sigma,\sigma'}\delta_{\mathbf{k},\mathbf{k}'}}{n'_F(\tilde{\xi_{\mathbf{k}}})} + \frac{\delta^{(2)}\Omega[n]}{\delta n_{\mathbf{k}\sigma}\delta n_{\mathbf{k}'\sigma'}}\bigg|_{\bar n}. \tag{4.28}$$

This relation will be used in section 4.4.2 to obtain a microscopic definition of the Landau function.

[9] The first term in the rhs of (4.27) comes from $\frac{\delta^{(2)} S[n]}{\delta n_{\mathbf{k}\sigma}\delta n_{\mathbf{k}'\sigma'}}\Big|_{\bar n} = -\frac{\delta_{\mathbf{k},\mathbf{k}'}\delta_{\sigma,\sigma'}}{\bar n_{\mathbf{k}\sigma}(1-\bar n_{\mathbf{k}\sigma})} = \delta_{\mathbf{k},\mathbf{k}'}\delta_{\sigma,\sigma'}\beta/n'_F(\tilde{\xi_{\mathbf{k}}})$.

4.1.2 Stability of the ground state

Because of the thermal factor $1/n'_F(\tilde{\xi_{\mathbf{k}}})$ in (4.27), small variations of the thermodynamic potential are due to quasi-particle excitations lying in the thermal broadening of the Fermi surface ($|\tilde{\xi_{\mathbf{k}}}| \lesssim T$). When $T \to 0$, these excitations have vanishing energies and can be viewed as resulting from a displacement (that can depend on spin) of the Fermi surface. Suppose that in the direction $\hat{\mathbf{k}}$, the Fermi momentum k_F varies by an infinitesimal amount $u_\sigma(\hat{\mathbf{k}})$ for spin-σ particles. This induces a change

$$\begin{aligned} \delta n_{\mathbf{k}\sigma} &= \lim_{u_\sigma(\hat{\mathbf{k}})\to 0} \left\{ n_F[\tilde{\xi_{\mathbf{k}}} - v_F^* u_\sigma(\hat{\mathbf{k}})] - n_F(\tilde{\xi_{\mathbf{k}}}) \right\} \\ &= -v_F^* n'_F(\tilde{\xi_{\mathbf{k}}}) u_\sigma(\hat{\mathbf{k}}) = v_F^* \delta(\xi_{\mathbf{k}}) u_\sigma(\hat{\mathbf{k}}) \qquad (T \to 0) \end{aligned} \tag{4.29}$$

in the distribution function. We have used $\tilde{\epsilon}_{\mathbf{k}} \to \epsilon_{\mathbf{k}}$ and $n'_F(x) \to -\delta(x)$ when $T \to 0$. The corresponding variation of the thermodynamic potential reads

$$\begin{aligned} \delta\Omega[u] = V \frac{{v_F^*}^2 N^*(0)}{2} \sum_{\sigma,\sigma'} \bigg\{ & \delta_{\sigma,\sigma'} \int \frac{d\Omega_{\hat{\mathbf{k}}}}{4\pi} u_\sigma^2(\hat{\mathbf{k}}) \\ & + \frac{1}{2} \int \frac{d\Omega_{\hat{\mathbf{k}}}}{4\pi} \frac{d\Omega_{\hat{\mathbf{k}}'}}{4\pi} F_{\sigma\sigma'}(\mathbf{k}_F, \mathbf{k}'_F) u_\sigma(\hat{\mathbf{k}}) u_{\sigma'}(\hat{\mathbf{k}}') \bigg\} \end{aligned} \tag{4.30}$$

in the limit $T \to 0$. $d\Omega_{\hat{\mathbf{k}}}$ denotes the elementary solid angle in the direction of $\hat{\mathbf{k}}$. To proceed further, we expand $u_\sigma(\hat{\mathbf{k}})$ in spherical harmonics,

$$u_\sigma(\hat{\mathbf{k}}) = u^s(\hat{\mathbf{k}}) + \sigma u^a(\hat{\mathbf{k}}) = \sum_{l=0}^\infty \sum_{m=-l}^{l} (u_{lm}^s + \sigma u_{lm}^a)\, Y_l^m(\hat{\mathbf{k}}), \tag{4.31}$$

where $u_{l,-m}^{s,a} = (-1)^m u_{lm}^{s,a*}$ since $u_\sigma(\hat{\mathbf{k}})$ is real. Using the addition theorem and other standard properties of spherical harmonics, we obtain

$$\delta\Omega[u] = V \frac{{v_F^*}^2 N^*(0)}{4\pi} \sum_{l=0}^\infty \sum_{m=-l}^{l} \left[|u_{lm}^s|^2 \left(1 + \frac{F_l^s}{2l+1} \right) + |u_{lm}^a|^2 \left(1 + \frac{F_l^a}{2l+1} \right) \right]. \tag{4.32}$$

The stability of the spherical Fermi surface requires $\delta\Omega[u]$ to be positive for any deformation $u_\sigma(\hat{\mathbf{k}})$, i.e.

$$F_l^s > -2l - 1, \qquad F_l^a > -2l - 1. \tag{4.33}$$

The instabilities occurring when the conditions (4.33) are violated are known as Pomeranchuk instabilities.

4.1.3 Effective mass

The current carried by a quasi-particle, as well as its effective mass, can be obtained by considering the system from a reference frame moving at a velocity $\mathbf{v} = \mathbf{q}/m$ with respect to the laboratory frame. In the moving frame, the Hamiltonian is

$$\hat{H}' = \hat{H} - \mathbf{P} \cdot \mathbf{v} + \mathcal{O}(\mathbf{v}^2), \tag{4.34}$$

where M is the total mass, and $\mathbf{P}$ the total momentum measured with respect to the laboratory frame (Sec. 2.2.5). Since the momentum coincides with the current in a translation invariant system,

$$\mathbf{J} = \frac{\mathbf{P}}{m} = -\frac{1}{m}\frac{\partial E}{\partial \mathbf{v}}\bigg|_{\mathbf{v}=0} = -\frac{\partial E}{\partial \mathbf{q}}\bigg|_{\mathbf{q}=0}, \tag{4.35}$$

where E is the energy in the moving frame.

Let us consider the state corresponding to a quasi-particle of momentum $\mathbf{k}$ (in the laboratory frame) added to the ground state. The current in that state is simply

$$\mathbf{j}_\mathbf{k} = \frac{\mathbf{k}}{m}. \tag{4.36}$$

The current can also be calculated by considering the same physical state in the moving frame, where the quasi-particle has momentum $\mathbf{k}-\mathbf{q}$ and the ground state is a shifted Fermi sea: $n_\mathbf{k} = n^0_{\mathbf{k}+\mathbf{q}}$. Since equation (4.15) is valid in any Galilean reference frame, the quasi-particle energy in the moving frame reads[10]

$$\tilde\epsilon_{\mathbf{k}-\mathbf{q}} = \epsilon_{\mathbf{k}-\mathbf{q}} + \frac{1}{V}\sum_{\mathbf{k}',\sigma'} f_{\sigma\sigma'}(\mathbf{k},\mathbf{k}')[n^0_{\mathbf{k}'+\mathbf{q}} - n^0_{\mathbf{k}'}], \tag{4.37}$$

so that the current $\mathbf{j}_\mathbf{k}$ – measured in the laboratory frame – carried by a quasi-particle of momentum $\mathbf{k}$ is

$$\begin{aligned} \mathbf{j}_\mathbf{k} &= -\frac{\partial \tilde\epsilon_{\mathbf{k}-\mathbf{q}}}{\partial \mathbf{q}}\bigg|_{\mathbf{q}=0} = \boldsymbol{\nabla}_\mathbf{k}\epsilon_\mathbf{k} - \frac{1}{V}\sum_{\mathbf{k}',\sigma'} f_{\sigma\sigma'}(\mathbf{k},\mathbf{k}')\boldsymbol{\nabla}_{\mathbf{k}'} n^0_{\mathbf{k}'} \\ &= \mathbf{v}^*_\mathbf{k} + \frac{1}{V}\sum_{\mathbf{k}',\sigma'} f_{\sigma\sigma'}(\mathbf{k},\mathbf{k}')\mathbf{v}^*_{\mathbf{k}'}\delta(\xi_{\mathbf{k}'}). \end{aligned} \tag{4.38}$$

The first term in (4.38) can be seen as the contribution of a localized wave packet containing one extra particle and moving with the group velocity $\mathbf{v}^*_\mathbf{k}$ (in a picture where we see the quasi-particle as a localized excitation). The second term is a drag current that comes from the interaction of the moving wave packet with the surrounding fluid.

Near the Fermi surface, where $\xi_\mathbf{k} \simeq v^*_F(|\mathbf{k}| - k_F)$ and $\mathbf{v}^*_\mathbf{k} \simeq v^*_F\hat{\mathbf{k}}$, equation (4.38) gives

$$\mathbf{j}_\mathbf{k} = v^*_F\hat{\mathbf{k}}\Big[1 + 2N^*(0)\int\frac{d\Omega_{\mathbf{k}'}}{4\pi} f^s(\theta)\cos\theta\Big] = v^*_F\hat{\mathbf{k}}\left(1 + \frac{F^s_1}{3}\right). \tag{4.39}$$

By comparing (4.36) and (4.39), we obtain

$$\frac{m^*}{m} = 1 + \frac{F^s_1}{3}. \tag{4.40}$$

Depending on the sign of F^s_1, the effective mass can be larger or smaller than the bare mass. When $F^s_1 < -3$, the effective mass is negative and the system unstable since quasi-particle excitations across the Fermi surface are energetically favorable. This is a special case of Pomeranchuk instabilities [Eqs. (4.33)].

[10] Note that $\mathbf{k}$ should not be shifted in the Landau function since the interaction between particles is velocity independent.

4.2 Thermodynamics

In this section, we show how to obtain the thermodynamics quantities from the Landau energy functional $E[n]$.

4.2.1 Specific heat

The specific heat is defined by

$$C_V = \frac{\partial E}{\partial T}\bigg|_{V,N}, \tag{4.41}$$

where

$$E = E(T=0) + \sum_{\mathbf{k},\sigma} \epsilon_{\mathbf{k}} \delta n_{\mathbf{k}\sigma} + \frac{1}{2V} \sum_{\substack{\mathbf{k},\mathbf{k}' \\ \sigma,\sigma'}} f_{\sigma\sigma'}(\mathbf{k},\mathbf{k}') \delta n_{\mathbf{k}\sigma} \delta n_{\mathbf{k}'\sigma'} \tag{4.42}$$

and $\delta n_{\mathbf{k}\sigma} = n_F(\tilde\epsilon_{\mathbf{k}}) - n^0_{\mathbf{k}}$. If we neglect the interactions between quasi-particles, then

$$C_V = V \frac{2\pi^2}{3} N^*(0) T = V \frac{m^* k_F}{3} T \tag{4.43}$$

($T \to 0$) is simply the specific heat of non-interacting fermions with mass m^*. The interaction term in (4.42) is $\mathcal{O}(T^4)$, since $\int d|\mathbf{k}|\mathbf{k}^2 \delta n_{\mathbf{k}\sigma} = \mathcal{O}(T^2)$ in the grand-canonical ensemble, and can be neglected. Clearly, this conclusion will not change if we consider the $\mathcal{O}(T^2)$ shift of the chemical potential necessary to keep the total number of particles constant.[11]

4.2.2 Compressibility

The isothermal compressibility is defined as

$$\kappa = -\frac{1}{V}\frac{\partial V}{\partial P}\bigg|_{T,N} = \frac{1}{n^2}\frac{\partial n}{\partial \mu}\bigg|_T \tag{4.44}$$

(Sec. 3.3), so that what we need to calculate is $\partial n/\partial\mu$. A variation in the density $n = k_F^3/3\pi^2$ is equivalent to a variation of the Fermi momentum k_F: $\partial k_F/\partial n = \pi^2/k_F^2$. When k_F varies, the quasi-particle distribution also varies, so that the change in the chemical potential $\mu = \epsilon_{k_F}$ is given by

$$\frac{\partial\mu}{\partial n} = \frac{\partial \epsilon_{k_F}}{\partial k_F}\frac{\partial k_F}{\partial n} + \frac{1}{V}\sum_{\mathbf{k}',\sigma'} f_{\sigma\sigma'}(\mathbf{k}_F,\mathbf{k}') \frac{\partial n_{\mathbf{k}'\sigma'}}{\partial k_F}\frac{\partial k_F}{\partial n}. \tag{4.45}$$

Using $\partial\epsilon_{k_F}/\partial k_F = v_F^* = k_F/m^*$ and $\partial n_{\mathbf{k}\sigma}/\partial k_F = \delta(k_F - |\mathbf{k}|)$ ($T=0$), we obtain

$$\begin{aligned} \frac{\partial\mu}{\partial n} &= \frac{\pi^2}{k_F^2}\Big[v_F^* + \sum_{\sigma'} \int \frac{d^3k'}{(2\pi)^3} f_{\sigma\sigma'}(\mathbf{k}_F,\mathbf{k}') \delta(k_F - |\mathbf{k}'|) \Big] \\ &= \frac{\pi^2}{k_F^2}\left[v_F^* + 2N^*(0) v_F^* f_0^s \right] = \frac{1+F_0^s}{2N^*(0)}, \end{aligned} \tag{4.46}$$

[11] If we approximate $f_{\sigma\sigma'}(\mathbf{k},\mathbf{k}')$ by $f_{\sigma\sigma'}(\mathbf{k}_F,\mathbf{k}'_F)$, then the interacting term in (4.42) vanishes in the canonical ensemble (N fixed) where $\frac{1}{V}\sum_{\mathbf{k}} \delta n_{\mathbf{k}\sigma} = (2\pi^2)^{-1}\int d|\mathbf{k}|\mathbf{k}^2 \delta n_{\mathbf{k}\sigma} = 0$.

and in turn

$$n^2\kappa = \frac{2N^*(0)}{1+F_0^s}. \tag{4.47}$$

The interactions between quasi-particles lead to a renormalization by a factor $1/(1+F_0^s)$ of the naive result $2N^*(0)$ obtained from the compressibility of the ideal Fermi gas by the mere replacement $m \to m^*$. Again, we note that the stability of the system requires $F_0^s > -1$ in agreement with (4.33).

Equation (4.47) yields the macroscopic sound velocity

$$c_s = \frac{1}{\sqrt{\kappa n m}} = \frac{v_F}{\sqrt{3}}\left(\frac{1+F_0^s}{1+F_1^s/3}\right)^{1/2} = \frac{v_F^*}{\sqrt{3}}\left[(1+F_0^s)\left(1+\frac{F_1^s}{3}\right)\right]^{1/2} \tag{4.48}$$

(see Eq. (3.108)). In the absence of interaction, we recover the sound velocity $v_F/\sqrt{3}$ of the ideal Fermi gas.

4.2.3 Spin susceptibility

In the presence of a magnetic field $\mathbf{B} = B\hat{\mathbf{z}}$, the energy $\epsilon_{\mathbf{k}\sigma}$ is shifted by $\frac{\sigma}{2}g\mu_B B$, where μ_B is the Bohr magneton. A spin-σ quasi-particle being an eigenstate of $\hat{S}_z$ with eigenvalue $\sigma/2$, the Landé factor $g=2$ coincides with that of the bare particle. Since the field displaces the Fermi surface and changes the quasi-particle distribution, the quasi-particle energy becomes

$$\tilde\epsilon_{\mathbf{k}\sigma} = \epsilon_{\mathbf{k}} + \frac{\sigma}{2}g\mu_B B + \frac{1}{V}\sum_{\mathbf{k}',\sigma'} f_{\sigma\sigma'}(\mathbf{k},\mathbf{k}')\delta n_{\mathbf{k}'\sigma'}. \tag{4.49}$$

The new (spin-dependent) Fermi surface is defined by $\tilde\epsilon_{k_{F\uparrow},\uparrow} = \tilde\epsilon_{k_{F\downarrow},\downarrow} = \mu$,[12] i.e.

$$\begin{aligned}\epsilon_{k_{F\uparrow}} &+ \frac{g\mu_B B}{2} + \frac{1}{V}\sum_{\mathbf{k}',\sigma'} f_{\uparrow\sigma'}(\mathbf{k}_F,\mathbf{k}')\delta n_{\mathbf{k}'\sigma'} \\ &= \epsilon_{k_{F\downarrow}} - \frac{g\mu_B B}{2} + \frac{1}{V}\sum_{\mathbf{k}',\sigma'} f_{\downarrow\sigma'}(\mathbf{k}_F,\mathbf{k}')\delta n_{\mathbf{k}'\sigma'},\end{aligned} \tag{4.50}$$

where

$$\delta n_{\mathbf{k}\sigma} = \Theta(k_F+\delta k_{F\sigma} - |\mathbf{k}|) - \Theta(k_F - |\mathbf{k}|) = \delta k_{F\sigma} v_F^* \delta(\xi_{\mathbf{k}}) \tag{4.51}$$

($\delta k_{F\sigma} = k_{F\sigma} - k_F$) to leading order in B. Using $\epsilon_{k_{F\sigma}} = \epsilon_{k_F} + v_F^*\delta k_{F\sigma}$, equation (4.50) gives

$$v_F^*(\delta k_{F\uparrow} - \delta k_{F\downarrow}) = -\frac{g\mu_B B}{1+F_0^a}. \tag{4.52}$$

The magnetization per unit volume is given by

$$\begin{aligned}M &= -\frac{1}{2V}g\mu_B\sum_{\mathbf{k}}(\delta n_{\mathbf{k}\uparrow} - \delta n_{\mathbf{k}\downarrow}) = -\frac{1}{2}g\mu_B N^*(0)v_F^*(\delta k_{F\uparrow} - \delta k_{F\downarrow}) \\ &= \left(\frac{g\mu_B}{2}\right)^2\frac{2N^*(0)}{1+F_0^a}B.\end{aligned} \tag{4.53}$$

[12] Since $\delta\mu$ cannot depend on the direction of $\mathbf{B}$, it is at least of order B^2 and the chemical potential is constant to leading order in B.

This yields the spin susceptibility

$$\chi = \frac{\partial M}{\partial B} = \left(\frac{g\mu_B}{2}\right)^2 \frac{2N^*(0)}{1+F_0^a}. \tag{4.54}$$

Stability against ferromagnetism requires $F_0^a > -1$.

4.3 Non-equilibrium properties

The Landau energy function $\delta E[\delta n]$ enables to compute the thermodynamic properties of the Fermi liquid but does not contain any information about the quasi-particle dynamics. To study the latter, one has to extend the definition of δE to non-equilibrium states. When physical properties vary only on macroscopic scales ($\gg k_F^{-1}$), one can adopt a semiclassical description and define a local quasi-particle distribution function $n_{\mathbf{k}\sigma}(\mathbf{r},t)$ giving the density of quasi-particles with momentum $\mathbf{k}$ and spin σ in the vicinity of point $\mathbf{r}$ at time t. By analogy with (4.15), we define the time-dependent functional

$$\begin{aligned}\delta E[\delta n,t] &= \sum_{\mathbf{k},\sigma}\int d^3r\,\epsilon_{\mathbf{k}}\delta n_{\mathbf{k}\sigma}(\mathbf{r},t) \\ &\quad + \frac{1}{2V}\sum_{\mathbf{k},\mathbf{k}',\sigma,\sigma'}\int d^3r d^3r' f_{\sigma\sigma'}(\mathbf{k},\mathbf{k}';\mathbf{r}-\mathbf{r}')\delta n_{\mathbf{k}\sigma}(\mathbf{r},t)\delta n_{\mathbf{k}'\sigma'}(\mathbf{r}',t).\end{aligned} \tag{4.55}$$

For a homogeneous system, $\epsilon_{\mathbf{k}}$ is independent of the spin σ and position $\mathbf{r}$ of the quasi-particle. The interaction is assumed to be instantaneous in time and short-range in space. $f_{\sigma\sigma'}(\mathbf{k},\mathbf{k}';\mathbf{r}-\mathbf{r}')$ then decreases rapidly in space and we can approximate $\delta n_{\mathbf{k}'\sigma'}(\mathbf{r}',t)$ by $\delta n_{\mathbf{k}'\sigma'}(\mathbf{r},t)$ in (4.55). This leads to

$$\begin{aligned}\delta E[\delta n,t] &= \sum_{\mathbf{k},\sigma}\int d^3r\,\epsilon_{\mathbf{k}}\delta n_{\mathbf{k}\sigma}(x) \\ &\quad + \frac{1}{2V}\sum_{\mathbf{k},\mathbf{k}',\sigma,\sigma'}\int d^3r\, f_{\sigma\sigma'}(\mathbf{k},\mathbf{k}')\delta n_{\mathbf{k}\sigma}(x)\delta n_{\mathbf{k}'\sigma'}(x),\end{aligned} \tag{4.56}$$

where $f_{\sigma\sigma'}(\mathbf{k},\mathbf{k}') = \int d^3r' f_{\sigma\sigma'}(\mathbf{k},\mathbf{k}';\mathbf{r}-\mathbf{r}')$ and $x=(\mathbf{r},t)$.

4.3.1 Kinetic equation

To obtain the equation governing the quasi-particle dynamics, we consider the time-dependent quasi-particle energy

$$\tilde\epsilon_{\mathbf{k}\sigma}(x) = \frac{\delta E[n,t]}{\delta n_{\mathbf{k}\sigma}(x)} = \epsilon_{\mathbf{k}} + \frac{1}{V}\sum_{\mathbf{k}',\sigma'} f_{\sigma\sigma'}(\mathbf{k},\mathbf{k}')\delta n_{\mathbf{k}'\sigma'}(x) \tag{4.57}$$

as a quasi-classical Hamiltonian. This assumption leads to the equations of motion

$$\partial_t\mathbf{r} = \boldsymbol{\nabla}_{\mathbf{k}}\tilde\epsilon_{\mathbf{k}\sigma}(x), \qquad \partial_t\mathbf{k} = -\boldsymbol{\nabla}_{\mathbf{r}}\tilde\epsilon_{\mathbf{k}\sigma}(x). \tag{4.58}$$

The time evolution of the quasi-particle distribution is then governed by the usual Boltzmann equation[13]

$$\frac{dn_{\mathbf{k}\sigma}(x)}{dt} = \partial_t n_{\mathbf{k}\sigma}(x) - \boldsymbol{\nabla}_{\mathbf{k}} n_{\mathbf{k}\sigma}(x) \cdot \boldsymbol{\nabla}_{\mathbf{r}} \tilde{\epsilon}_{\mathbf{k}\sigma}(x) \\ + \boldsymbol{\nabla}_{\mathbf{r}} n_{\mathbf{k}\sigma}(x) \cdot \boldsymbol{\nabla}_{\mathbf{k}} \tilde{\epsilon}_{\mathbf{k}\sigma}(x) = I[n_{\mathbf{k}\sigma}(x)], \tag{4.59}$$

where the "collision integral" $I[n_{\mathbf{k}\sigma}(x)] = \partial_t n_{\mathbf{k}\sigma}(x)|_{\rm coll}$ takes into account the collisions between particles. To first order in $\delta n_{\mathbf{k}\sigma}(x) = n_{\mathbf{k}\sigma}(x) - n^0_{\mathbf{k}}$,

$$\partial_t \delta n_{\mathbf{k}\sigma}(x) - \boldsymbol{\nabla}_{\mathbf{k}} n^0_{\mathbf{k}} \cdot \boldsymbol{\nabla}_{\mathbf{r}} \tilde{\epsilon}_{\mathbf{k}\sigma}(x) + \boldsymbol{\nabla}_{\mathbf{r}} \delta n_{\mathbf{k}\sigma}(x) \cdot \boldsymbol{\nabla}_{\mathbf{k}} \epsilon_{\mathbf{k}} = I[n_{\mathbf{k}\sigma}(x)], \tag{4.60}$$

so that we finally obtain

$$\partial_t \delta n_{\mathbf{k}\sigma}(x) + \mathbf{v}^*_{\mathbf{k}} \cdot \boldsymbol{\nabla}_{\mathbf{r}} \delta n_{\mathbf{k}\sigma}(x) \\ + \frac{1}{V} \sum_{\mathbf{k}',\sigma'} f_{\sigma\sigma'}(\mathbf{k},\mathbf{k}') \boldsymbol{\nabla}_{\mathbf{r}} \delta n_{\mathbf{k}'\sigma'}(x) \cdot \mathbf{v}^*_{\mathbf{k}} \delta(\xi_{\mathbf{k}}) = I[n_{\mathbf{k}\sigma}(x)]. \tag{4.61}$$

Note that this equation involves only states near the Fermi surface where the quasi-particle concept is valid.

4.3.2 Conservation laws

Particle number conservation

Since the collisions conserve the total number of particles,

$$\sum_{\mathbf{k},\sigma} \frac{dn_{\mathbf{k}\sigma}(x)}{dt} = \sum_{\mathbf{k},\sigma} I[n_{\mathbf{k}\sigma}(x)] = 0. \tag{4.62}$$

Making use of (4.59), this equation can be written as the continuity equation $\partial_t n(x) + \boldsymbol{\nabla} \cdot \mathbf{j}(x) = 0$, where

$$n(x) = \frac{1}{V} \sum_{\mathbf{k},\sigma} n_{\mathbf{k}\sigma}(x) \tag{4.63}$$

is the particle density at point $\mathbf{r}$ and time t and

$$\mathbf{j}(x) = \frac{1}{V} \sum_{\mathbf{k},\sigma} n_{\mathbf{k}\sigma}(x) \boldsymbol{\nabla}_{\mathbf{k}} \tilde{\epsilon}_{\mathbf{k}\sigma}(x) \tag{4.64}$$

the current density. To linear order in δn, we obtain

$$\mathbf{j}(x) = \frac{1}{V} \sum_{\mathbf{k},\sigma} \delta n_{\mathbf{k}\sigma}(x) \Big[\mathbf{v}^*_{\mathbf{k}} + \frac{1}{V} \sum_{\mathbf{k}',\sigma'} f_{\sigma\sigma'}(\mathbf{k},\mathbf{k}') \mathbf{v}^*_{\mathbf{k}'} \delta(\xi_{\mathbf{k}'}) \Big] \\ = \frac{1}{V} \sum_{\mathbf{k},\sigma} \delta n_{\mathbf{k}\sigma}(x) \mathbf{j}_{\mathbf{k}}, \tag{4.65}$$

where $\mathbf{j}_{\mathbf{k}}$ is the current carried by a quasi-particle of momentum $\mathbf{k}$ [Eqs. (4.36,4.38)].

[13] See, for instance, N. W. Ashcroft and N. D. Mermin, *Solid State Physics*, chapter 16 (Saunders College Publishing, 1976).

Momentum conservation

Similarly, by multiplying (4.59) by k_i and summing over $\mathbf{k}$ and σ, we obtain

$$\partial_t g_i(x) + \frac{1}{V}\sum_{\mathbf{k},\sigma,j} k_i \left[\frac{\partial}{\partial r_j}\left(n_{\mathbf{k}\sigma}(x) \frac{\partial \tilde\epsilon_{\mathbf{k}\sigma}(x)}{\partial k_j} \right) - \frac{\partial}{\partial k_j}\left(n_{\mathbf{k}\sigma}(x) \frac{\partial \tilde\epsilon_{\mathbf{k}\sigma}(x)}{\partial r_j} \right) \right] = 0, \quad (4.66)$$

where

$$\mathbf{g}(x) = \frac{1}{V}\sum_{\mathbf{k},\sigma} \mathbf{k} n_{\mathbf{k}\sigma}(x) \quad (4.67)$$

is the momentum density. Again, from momentum conservation in collisions, the collision term does not appear in (4.66). By integrating by part, we rewrite the last term of (4.66) as

$$\begin{aligned} \frac{1}{V}\sum_{\mathbf{k},\sigma} n_{\mathbf{k}\sigma}(x) \frac{\partial \tilde\epsilon_{\mathbf{k}\sigma}(x)}{\partial r_i} &= \frac{1}{V}\sum_{\mathbf{k},\sigma} \left[\frac{\partial}{\partial r_i}\big(n_{\mathbf{k}\sigma}(x)\tilde\epsilon_{\mathbf{k}\sigma}(x) \big) - \frac{\partial n_{\mathbf{k}\sigma}(x)}{\partial r_i} \tilde\epsilon_{\mathbf{k}\sigma}(x) \right] \\ &= \frac{1}{V}\sum_{\mathbf{k},\sigma} \frac{\partial}{\partial r_i} [n_{\mathbf{k}\sigma}(x)\tilde\epsilon_{\mathbf{k}\sigma}(x)] - \frac{1}{V}\frac{\partial E}{\partial r_i}, \end{aligned} \quad (4.68)$$

where we have used (4.18). From (4.66,4.68), we deduce the equation

$$\partial_t g_i(x) + \sum_j \nabla_{r_j} \Pi_{ij}(x) = 0, \quad (4.69)$$

where

$$\Pi_{ij}(x) = \frac{1}{V}\sum_{\mathbf{k},\sigma} k_i n_{\mathbf{k}\sigma}(x) \frac{\partial \tilde\epsilon_{\mathbf{k}\sigma}(x)}{\partial k_j} + \delta_{i,j}\left[\frac{1}{V}\sum_{\mathbf{k},\sigma} n_{\mathbf{k}\sigma}(x)\tilde\epsilon_{\mathbf{k}\sigma}(x) - \frac{E}{V} \right] \quad (4.70)$$

is the momentum-current tensor. To linear order in δn,

$$\Pi_{ij}(x) = \frac{1}{V}\sum_{\mathbf{k},\sigma} k_i \Big[\delta n_{\mathbf{k}\sigma}(x) + \delta(\xi_{\mathbf{k}}) \frac{1}{V}\sum_{\mathbf{k}',\sigma'} f_{\sigma\sigma'}(\mathbf{k},\mathbf{k}')\delta n_{\mathbf{k}'\sigma'}(x) \Big] v^*_{\mathbf{k}j}. \quad (4.71)$$

Energy conservation

Last, we obtain the expression of the energy current by multiplying (4.59) by $\tilde\epsilon_{\mathbf{k}\sigma}$ and summing over $\mathbf{k}$ and σ,

$$\partial_t E + \frac{1}{V}\sum_{\mathbf{k},\sigma,i} \tilde\epsilon_{\mathbf{k}\sigma}(x) \left[\frac{\partial}{\partial r_i}\left(n_{\mathbf{k}\sigma}(x) \frac{\partial \tilde\epsilon_{\mathbf{k}\sigma}(x)}{\partial k_i} \right) - \frac{\partial}{\partial k_i}\left(n_{\mathbf{k}\sigma}(x) \frac{\partial \tilde\epsilon_{\mathbf{k}\sigma}(x)}{\partial r_i} \right) \right] = 0, \quad (4.72)$$

where

$$\partial_t E = \frac{1}{V}\sum_{\mathbf{k},\sigma} \frac{\delta E}{\delta n_{\mathbf{k}\sigma}(x)} \partial_t n_{\mathbf{k}\sigma}(x) = \frac{1}{V}\sum_{\mathbf{k},\sigma} \tilde\epsilon_{\mathbf{k}\sigma}(x) \partial_t n_{\mathbf{k}\sigma}(x). \quad (4.73)$$

is the time derivative of the energy. Integrating by part the last term in (4.72), we obtain

$$\partial_t E + \boldsymbol{\nabla}\cdot \mathbf{j}_E(x) = 0, \quad (4.74)$$

where the energy current is given by

$$\mathbf{j}_E(x) = \frac{1}{V}\sum_{\mathbf{k},\sigma} \tilde\epsilon_{\mathbf{k}\sigma}(x) n_{\mathbf{k}\sigma}(x) \boldsymbol{\nabla}_{\mathbf{k}} \tilde\epsilon_{\mathbf{k}\sigma}(x). \tag{4.75}$$

To linear order in δn,

$$\mathbf{j}_E(x) = \frac{1}{V}\sum_{\mathbf{k},\sigma} \epsilon_{\mathbf{k}} \Big[\delta n_{\mathbf{k}\sigma}(x) + \delta(\xi_{\mathbf{k}}) \frac{1}{V}\sum_{\mathbf{k}',\sigma'} f_{\sigma\sigma'}(\mathbf{k},\mathbf{k}')\delta n_{\mathbf{k}'\sigma'}(x)\Big] \mathbf{v}^*_{\mathbf{k}}. \tag{4.76}$$

4.3.3 Collective modes

A collective mode with momentum $\mathbf{q}$ and frequency ω is a coherent superposition of quasi-particle–quasi-hole pair excitations. When $\mathbf{q} \to 0$, the quasi-particles (holes) excitations have vanishing energy and the collective mode can be seen as a time-dependent displacement $u_\sigma(\hat{\mathbf{k}})e^{i(\mathbf{q}\cdot\mathbf{r}-\omega t)}$ + c.c. of the Fermi surface. Generalizing (4.29), we therefore consider

$$\delta n_{\mathbf{k}\sigma}(x) = v_F^* \delta(\xi_{\mathbf{k}}) u_\sigma(\hat{\mathbf{k}}) e^{i(\mathbf{q}\cdot\mathbf{r}-\omega t)} + \text{c.c.} \tag{4.77}$$

(Note that $u_\sigma(\hat{\mathbf{k}})$ is now complex.) As before we expand $u_\sigma(\hat{\mathbf{k}})$ in spherical harmonics [Eq. (4.31)] and choose $\mathbf{q}$ as the polar axis. Since $\delta n_{\mathbf{k}\sigma}(x)$ satisfies the kinetic equation (4.61), we have

$$(\cos\theta - s) u^\nu(\hat{\mathbf{k}}) + \cos\theta \int \frac{d\Omega_{\hat{\mathbf{k}}'}}{4\pi} F^\nu(\mathbf{k},\mathbf{k}') u^\nu(\hat{\mathbf{k}}') = I[u] \tag{4.78}$$

($\nu = s, a$), where $s = \omega/v_F^*|\mathbf{q}|$ and θ is the angle between $\mathbf{k}$ and $\mathbf{q}$. Using standard properties of the spherical harmonics, we obtain

$$(\cos\theta - s)\sum_{l=0}^\infty \sum_{m=-l}^{l} u^\nu_{lm} Y_l^m(\hat{\mathbf{k}}) + \cos\theta \sum_{l=0}^\infty \frac{F_l^\nu}{2l+1} \sum_{m=-l}^{l} u^\nu_{lm} Y_l^m(\hat{\mathbf{k}}) = I[u]. \tag{4.79}$$

A set of equations for the u^ν_{lm}'s can be obtained multiplying (4.79) by $\int d\Omega_{\hat{\mathbf{k}}} Y_l^m(\hat{\mathbf{k}})^*$. One readily sees that m is a good quantum number (but l is not) if one ignores the collision term.

The "longitudinal" mode $m = 0$ is particular as it is the only one to involve density fluctuations. Indeed, we have

$$\begin{aligned} \delta n(x) &= 2N^*(0) v_F^* \int \frac{d\Omega_{\hat{\mathbf{k}}}}{4\pi} u^s(\hat{\mathbf{k}}) e^{i(\mathbf{q}\cdot\mathbf{r}-\omega t)} + \text{c.c.} \\ &= N^*(0) v_F^* \frac{u^s_{00}}{\sqrt{\pi}} e^{i(\mathbf{q}\cdot\mathbf{r}-\omega t)} + \text{c.c.} \\ &= \frac{k_F^2}{2\pi^{5/2}} u^s_{00} e^{i(\mathbf{q}\cdot\mathbf{r}-\omega t)} + \text{c.c.} \end{aligned} \tag{4.80}$$

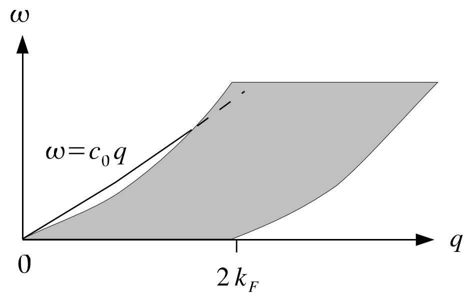

Figure 4.1: Dispersion $\omega = c_0|\mathbf{q}|$ of an undamped zero-sound mode ($c_0 > v_F^*$). The shaded area shows the continuum of particle-hole excitations.

After straightforward manipulations, one finds that in the mode $m = 0$ the current is longitudinal and takes the form

$$\begin{aligned}\mathbf{j}(x) &= \hat{\mathbf{q}} N^*(0) v_F^{*\,2} \frac{u_{10}^s}{\sqrt{3\pi}} \left(1 + \frac{F_1^s}{3}\right) e^{i(\mathbf{q}\cdot\mathbf{r}-\omega t)} + \text{c.c.} \\ &= \hat{\mathbf{q}} \frac{k_F^3}{2\sqrt{3}\pi^{5/2} m} u_{10}^s e^{i(\mathbf{q}\cdot\mathbf{r}-\omega t)} + \text{c.c.}\end{aligned} \tag{4.81}$$

Zero sound

Let us consider the longitudinal mode $m = 0$ in the frequency range $\omega\tau \gg 1$ where the collision term $I[n_{\mathbf{k}\sigma}] \sim -\delta n_{\mathbf{k}\sigma}/\tau$ can be neglected with respect to $\partial_t \delta n_{\mathbf{k}\sigma}(x)$. τ is a characteristic quasi-particle collision time. We shall see later that $\tau \sim 1/T^2$ at low temperatures (Sec. 4.4.1). We further assume that $f_{\sigma\sigma'}(\mathbf{k},\mathbf{k}') = f_0^s$. For a solution where both spin states oscillate in phase (sound mode: $u^a = 0$), equation (4.78) gives

$$(\cos\theta - s) u^s(\hat{\mathbf{k}}) + F_0^s \cos\theta \int \frac{d\Omega_{\hat{\mathbf{k}}'}}{4\pi} u^s(\hat{\mathbf{k}}') = 0, \tag{4.82}$$

the solution of which is

$$\begin{aligned} u^s(\hat{\mathbf{k}}) &= \text{const} \times \frac{\cos\theta}{s + i\eta - \cos\theta}, \\ \frac{1}{F_0^s} &= \int \frac{d\Omega}{4\pi} \frac{\cos\theta}{s + i\eta - \cos\theta} = -1 + \frac{s}{2} \ln\left(\frac{s + i\eta + 1}{s + i\eta - 1}\right). \end{aligned} \tag{4.83}$$

We have added to the real frequency s an infinitesimal imaginary part $i\eta = i0^+$, which amounts to switching the collective fluctuations adiabatically. This makes the logarithm in (4.83) well defined even when $|s| \leq 1$. As when considering a retarded response function (Sec. 3.2.3), one can allow s to take complex values and interpret the imaginary part of ω as the inverse life-time of the collective mode.

For a repulsive interaction $F_0^s > 0$, s is real and larger then unity. The limiting cases are

$$\begin{aligned} s &\to 1 + 2e^{-2/F_0^s - 2} \quad \text{for} \quad F_0^s \to 0, \\ s &\to (F_0^s/3)^{1/2} \quad \text{for} \quad F_0^s \to \infty. \end{aligned} \tag{4.84}$$

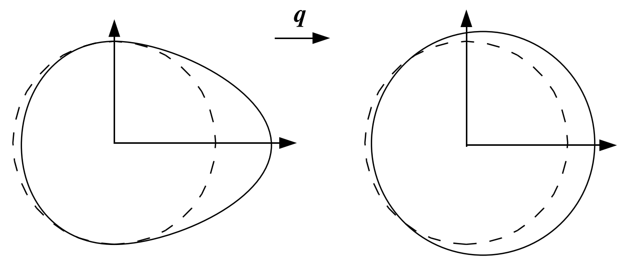

Figure 4.2: Fermi surface deformations in the zero-sound and first-sound modes for a constant interaction $f_{\sigma\sigma'}(\mathbf{k},\mathbf{k}') = f_0^s$. The dashed lines show the equilibrium Fermi surface. In the first-sound mode, the Fermi surface keeps its spherical shape.

The solution corresponds to an undamped mode – known as the zero-sound mode – propagating at the velocity $c_0 = \omega/|\mathbf{q}| = sv_F^*$ larger than v_F^* (Fig. 4.1). The corresponding Fermi surface deformation is shown in figure 4.2. In practice, the collisions between particles will give a finite life-time to the zero-sound mode. In the limit $\omega\tau \gg 1$, this effect is however negligible and the main source of damping comes from multi-pair excitations (Sec. 4.3.5).

For moderate attractive interactions, $-1 < F_0^s < 0$, one can numerically verify that s is complex and satisfies $|\Re(s)| < 1$ and $\Im(s) < 0$, corresponding to a damped zero-sound mode. From (4.83), it is clear that the imaginary part of s is due to the interaction of the collective mode with quasi-hole–quasi-particle pair excitations. When $\omega = \epsilon_{\mathbf{k}+\mathbf{q}} - \epsilon_{\mathbf{k}} \simeq v_F^*|\mathbf{q}|\cos\theta$, the interaction is resonant and gives rise to a damping of the collective mode (known as Landau damping). The collective mode has then a short life-time and does not represent a well-defined excitation of the system.

Last, for $F_0^s < -1$, there are two purely imaginary solutions. Substituting $s = i\alpha$ into (4.83), one finds

$$\frac{1}{F_0^s} = -1 + \frac{i}{2}\alpha \ln\left(\frac{1+i\alpha}{i\alpha - 1}\right) = -1 - \alpha\left(\gamma - \frac{\pi}{2}\right), \tag{4.85}$$

where $\gamma \in]-\pi,\pi]$ is defined by $1 + i\alpha = \sqrt{1+\alpha^2}e^{i\gamma}$ and $-1 + i\alpha = \sqrt{1+\alpha^2}e^{i(\pi-\gamma)}$. Since $\tan\gamma = \alpha$, we eventually obtain

$$\frac{1}{F_0^s} = -1 - \alpha\left(\arctan\alpha - \frac{\pi}{2}\right) = -1 + \alpha \arctan\left(\frac{1}{\alpha}\right). \tag{4.86}$$

For $F_0^s < -1$, this equation possesses two real solutions of opposite signs. One of these ($\alpha > 0$, i.e. $\Im(\omega) > 0$) corresponds to an unstable collective mode. This instability, characterized by divergent density fluctuations, also shows up in the negative compressibility [Eq. (4.47)].

It should be noted that the zero-sound mode exists only in neutral Fermi liquids. In a charged system, it is replaced by a plasmon mode as discussed in section 3.4.1. We shall come back to this point in section 4.3.4.

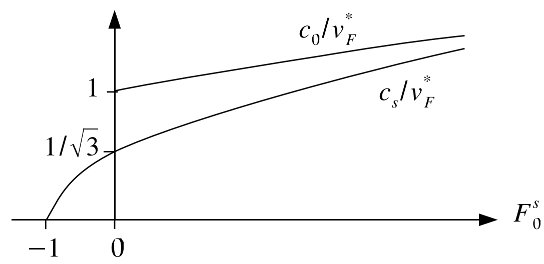

Figure 4.3: Velocities of the zero- and first-sound modes when $f_{\sigma\sigma'}(\mathbf{k},\mathbf{k'}) = f_0^s$. For $F_0^s \to \infty$, $c_s \simeq c_0 \simeq v_F^*(F_0^s/3)^{1/2}$.

First sound

Since the Fermi surface relaxes towards its equilibrium position within a characteristic time τ, in the hydrodynamic regime $\omega\tau \ll 1$ the displacement $u_\sigma(\hat{\mathbf{k}})$ of the Fermi surface is expected to be extremely small. In fact, this is true of all components u^s_{lm} except u^s_{00} and u^s_{10} whose fluctuations (and return to equilibrium) are constrained by the conservation of particle number [Eqs. (4.80,4.81)]. Thus, the sound propagation in the hydrodynamic regime can be studied by retaining only the hydrodynamic components u^s_{00} and u^s_{10} (which are not affected by the collisions).

With $Y_0^0(\hat{\mathbf{k}}) = 1/2\sqrt{\pi}$ and $Y_1^0(\hat{\mathbf{k}}) = \cos\theta\sqrt{3}Y_0^0(\hat{\mathbf{k}})$, the kinetic equation then becomes

$$(\cos\theta - s)\left(u^s_{00} + u^s_{10}\sqrt{3}\cos\theta\right) + \cos\theta\left(F_0^s u^s_{00} + \frac{F_1^s}{\sqrt{3}}u^s_{10}\cos\theta\right) = 0. \tag{4.87}$$

Multiplying this equation by $\int d\Omega_{\hat{\mathbf{k}}} Y_0^0(\hat{\mathbf{k}})$ and $\int d\Omega_{\hat{\mathbf{k}}} Y_1^0(\hat{\mathbf{k}})$, we deduce

$$\begin{aligned} s u^s_{00} - \frac{u^s_{10}}{\sqrt{3}}\left(1 + \frac{F_1^s}{3}\right) &= 0, \\ u^s_{00}\left(1 + F_0^s\right) - s\sqrt{3}u^s_{10} &= 0. \end{aligned} \tag{4.88}$$

The first of these equations is nothing else but the continuity equation $\partial_t n(x) + \boldsymbol{\nabla}\cdot\mathbf{j}(x) = 0$ in the longitudinal mode $m = 0$ [Eqs. (4.80,4.81)]. The second one can be identified with (4.69).[14] They admit a solution if

$$s^2 = \frac{\omega^2}{(v_F^*\mathbf{q})^2} = \frac{1}{3}(1 + F_0^s)\left(1 + \frac{F_1^s}{3}\right)^{1/2}, \tag{4.89}$$

which agrees with the macroscopic sound velocity c_s obtained from the compressibility by the usual hydrodynamic arguments [Eq. (4.48)]. The solution reads

$$u^s(\hat{\mathbf{k}}) = \frac{u^s_{00}}{2\sqrt{\pi}}\left[1 + \sqrt{3}\left(\frac{1+F_0^s}{1+F_1^s/3}\right)^{1/2}\cos\theta\right]. \tag{4.90}$$

[14] In the hydrodynamic mode $m = 0$ (where only u^s_{00} and u^s_{10} are considered), the momentum current tensor $\Pi_{ij} = \delta_{i,j}\Pi$ is diagonal with $\Pi(x) = \frac{1}{3\sqrt{\pi}}N^*(0)k_F {v_F^*}^2 u^s_{00}(1+F_0^s)e^{i(\mathbf{q}\cdot\mathbf{r}-\omega t)}$ + c.c. The second of equations (4.88) can be rewritten as $m\partial_t\mathbf{j}(x) + \boldsymbol{\nabla}\Pi(x) = 0$.

The Fermi surface keeps its spherical shape but its center oscillates about the origin in momentum space (hence the $\cos\theta$ term in (4.90)).

It is instructive to compare the zero- and first-sound modes within the simple model where $f_{\sigma\sigma'}(\mathbf{k},\mathbf{k}') = f_0^s$. In the limit $F_0^s \to \infty$, the sound velocity of the two modes tend to the same value $v_F^*(F_0^s/3)^{1/2}$ (Fig. 4.3), while the Fermi surface deformation takes the simple form $u^s(\hat{\mathbf{k}}) \propto \cos\theta$.

An example of transverse mode

Let us consider the transverse mode $m = 1$ where

$$u_\sigma(\hat{\mathbf{k}}) = \sum_{l=1}^{\infty} u^s_{l,m=1} Y_l^1(\hat{\mathbf{k}}) \equiv u^s(\theta)e^{i\varphi}. \tag{4.91}$$

There are no density fluctuations in this mode. Making use of (4.65), one easily finds that the current is transverse (recall that $\mathbf{q} \parallel \hat{\mathbf{z}}$ defines the polar axis) and circularly polarized,

$$\mathbf{j}(x) = j_0[\hat{\mathbf{x}}\cos(\mathbf{q}\cdot\mathbf{r} - \omega t) - \hat{\mathbf{y}}\sin(\mathbf{q}\cdot\mathbf{r} - \omega t)], \tag{4.92}$$

with j_0 a constant depending on $u^s(\theta)$ and $F^s(\theta)$.

In the collisionless regime, the kinetic equation (4.78) then gives

$$(\cos\theta - s)u^s(\theta)e^{i\varphi} + \cos\theta \int \frac{d\Omega'}{4\pi} F^s(\Omega,\Omega')u^s(\theta')e^{i\varphi'} = 0. \tag{4.93}$$

In the following, we assume that only F_0^s and F_1^s are non-zero, i.e. $F^s(\alpha) = F_0^s + F_1^s\cos\alpha$ (α is the angle between Ω and Ω'). Using $\cos\alpha = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\varphi - \varphi')$, we obtain

$$(\cos\theta - s)u^s(\theta) + \frac{F_1^s}{4}\cos\theta\sin\theta \int_0^\pi d\theta' \sin^2(\theta')u^s(\theta') = 0, \tag{4.94}$$

so that

$$u^s(\theta,\varphi) = \text{const} \times \frac{\cos\theta\sin\theta}{s + i\eta - \cos\theta} e^{i\varphi}, \tag{4.95}$$

$$\frac{4}{F_1^s} = \int_0^\pi d\theta \frac{\sin^3\theta\cos\theta}{s + i\eta - \cos\theta} = -\frac{4}{3} + 2s^2 + s(1 - s^2)\ln\left(\frac{s + i\eta + 1}{s + i\eta - 1}\right). \tag{4.96}$$

As previously, we have added an infinitesimal imaginary part to the frequency ω. A real solution must satisfy $|s| > 1$. Since the rhs of (4.96) is maximum at $|s| = 1$ where it takes the value $2/3$, a real solution is possible only if $F_1^s > 6$. The interaction between quasi-particles must be quite strong for the mode $m = 1$ to propagate. For $F_1^s \le 6$, the collective mode mixes with quasi-particle–quasi-hole pair excitations and is not a well-defined excitation of the system any more.

Spin-wave modes

Until now we have only considered spin symmetric solutions ($u^a = 0$) corresponding to density oscillations (sound modes). We could repeat the same discussion for spin antisymmetric solutions ($u^a \neq 0$) corresponding to spin density oscillations. The spin collective modes are similar to their charge counterparts but involve the spin antisymmetric Landau parameters F^a instead of F^s. For instance, for $F_0^a > 0$, one finds a "spin zero-sound" mode analogous to the (charge) zero-sound mode.

4.3.4 Response functions

In this section, we compute the density-density and current-current response functions in the long-wavelength low-energy limit. To this end we consider the quasi-classical Hamiltonian

$$\tilde{\epsilon}_{\mathbf{k}\sigma}(x) + \phi(x) - e\frac{\mathbf{k}}{m} \cdot \mathbf{A}(x). \tag{4.97}$$

For a neutral Fermi liquid, ϕ and $e\mathbf{A}$ should be considered as fictitious external fields introduced in order to derive the response functions; in a charged system they correspond to the usual scalar and vector potential. The linearized kinetic equation now reads

$$\begin{aligned} \partial_t \delta n_{\mathbf{k}\sigma}(x) + \mathbf{v}^*_{\mathbf{k}} \cdot \boldsymbol{\nabla}_{\mathbf{r}} \delta n_{\mathbf{k}\sigma}(x) + \frac{1}{V} \sum_{\mathbf{k}',\sigma'} f_{\sigma\sigma'}(\mathbf{k},\mathbf{k}') \boldsymbol{\nabla}_{\mathbf{r}} \delta n_{\mathbf{k}'\sigma'}(x) \cdot \mathbf{v}^*_{\mathbf{k}} \delta(\xi_{\mathbf{k}}) \\ + \delta(\xi_{\mathbf{k}}) \mathbf{v}^*_{\mathbf{k}} \cdot \boldsymbol{\nabla} \left[\phi(x) - \frac{e}{m} \mathbf{k} \cdot \mathbf{A}(x) \right] = I[n_{\mathbf{k}\sigma}(x)]. \end{aligned} \tag{4.98}$$

Following the analysis of section 4.3.2, we can verify that the conservation of particle number implies the continuity equation $\partial_t n(x) + \boldsymbol{\nabla} \cdot \mathbf{J}(x) = 0$, where the current

$$\mathbf{J}(x) = \mathbf{j}(x) - \frac{ne}{m} \mathbf{A}(x), \qquad \mathbf{j}(x) = \frac{1}{V} \sum_{\mathbf{k},\sigma} \frac{\mathbf{k}}{m} \delta n_{\mathbf{k}\sigma}(x) \tag{4.99}$$

includes the usual diamagnetic part.

Density-density response function in the collisionless regime

We first consider the response to a scalar potential potential

$$\phi(x) = \phi(\mathbf{q},\omega) e^{i[\mathbf{q}\cdot\mathbf{r} - (\omega + i\eta)t]} + \text{c.c.} \tag{4.100}$$

in the regime $\omega\tau \gg 1$ where collisions can be neglected. We use (4.77) – valid for $\mathbf{q} \to 0$ – and write the induced density as

$$\begin{aligned} \delta n(x) &= \frac{1}{V} \sum_{\mathbf{k},\sigma} \delta n_{\mathbf{k}\sigma}(x) = \delta n(\mathbf{q},\omega) e^{i[\mathbf{q}\cdot\mathbf{r} - (\omega+i\eta)t]} + \text{c.c.}, \\ \delta n(\mathbf{q},\omega) &= \frac{1}{V} \sum_{\mathbf{k},\sigma} v^*_F \delta(\xi_{\mathbf{k}}) u_\sigma(\hat{\mathbf{k}}) = 2N^*(0) v^*_F \int \frac{d\Omega_{\hat{\mathbf{k}}}}{4\pi} u^s(\hat{\mathbf{k}}). \end{aligned} \tag{4.101}$$

As shown in section 3.3, the linear response to the external field is determined by the density-density response function,

$$\delta n(\mathbf{q},\omega) = -\chi^R_{nn}(\mathbf{q},\omega) \phi(\mathbf{q},\omega) \tag{4.102}$$

(note the minus sign).

Without the collision terms, the kinetic equation gives

$$(\omega+i\eta-\mathbf{v}^*_{\mathbf{k}}\cdot\mathbf{q})u_\sigma(\hat{\mathbf{k}})-\mathbf{v}^*_{\mathbf{k}}\cdot\mathbf{q}\frac{1}{V}\sum_{\mathbf{k}',\sigma'}f_{\sigma\sigma'}(\mathbf{k},\mathbf{k}')\delta(\xi_{\mathbf{k}'})u_{\sigma'}(\hat{\mathbf{k}}')=\mathbf{v}^*_{\mathbf{k}}\cdot\mathbf{q}\frac{\phi(\mathbf{q},\omega)}{v^*_F}. \quad (4.103)$$

In the absence of interactions, equation (4.103) gives $\delta n(\mathbf{q},\omega)=-\chi^{0R}_{nn}(\mathbf{q},\omega)\phi(\mathbf{q},\omega)$, where

$$\begin{aligned}\chi^{0R}_{nn}(\mathbf{q},\omega)&=-\frac{2}{V}\sum_{\mathbf{k}}\frac{\mathbf{v}^*_{\mathbf{k}}\cdot\mathbf{q}}{\omega+i\eta-\mathbf{v}^*_{\mathbf{k}}\cdot\mathbf{q}}\delta(\xi_{\mathbf{k}})\\&=2N^*(0)\left[1-\frac{s}{2}\ln\left(\frac{s+i\eta+1}{s+i\eta-1}\right)\right]\end{aligned} \quad (4.104)$$

($s=\omega/v^*_F|\mathbf{q}|$) is the non-interacting density-density response function. Equation (4.104) agrees with the direct evaluation of the density-density correlation function of the ideal Fermi gas in the limit $\mathbf{q}\to 0$ [Eq. (3.73)].

It is difficult to solve (4.103) in the general case. Assuming that $f_{\sigma\sigma'}(\mathbf{k},\mathbf{k}')=f^s_0$, we find

$$\chi^R_{nn}(\mathbf{q},\omega)=\frac{\chi^{0R}_{nn}(\mathbf{q},\omega)}{1+f^s_0\chi^{0R}_{nn}(\mathbf{q},\omega)}. \quad (4.105)$$

The density-density response function possesses a pole for $1+f^s_0\chi^{0R}_{nn}(\mathbf{q},\omega)=0$, which is precisely the equation determining the zero-sound frequency obtained previously [Eq. (4.83)]. Such a pole (for ω real) exists only for a repulsive interaction ($f^s_0>0$). More generally, the density excitation spectrum can be obtained from the imaginary part of the response function or, equivalently, the structure factor $S_{nn}(\mathbf{q},\omega;T=0)=2\Theta(\omega)\Im[\chi^R_{nn}(\mathbf{q},\omega)]=2\Theta(\omega)\chi''_{nn}(\mathbf{q},\omega)$ (Sec. 3.2.5). In the non-interacting case

$$S^0_{nn}(\mathbf{q},\omega)=\begin{cases}2\pi N(0)\dfrac{\omega}{v_F|\mathbf{q}|}=\dfrac{m^2}{\pi}\dfrac{\omega}{|\mathbf{q}|} & \text{if}\quad 0\le\omega\le v_F|\mathbf{q}|,\\ 0\quad\text{otherwise.}\end{cases} \quad (4.106)$$

In the interacting case, there are two contributions to the structure factor

$$S_{nn}(\mathbf{q},\omega)=2\Theta(\omega)\frac{\chi^{0\,\prime\prime}_{nn}(\mathbf{q},\omega)}{\left(1+f^s_0\Re[\chi^{0R}_{nn}(\mathbf{q},\omega)]\right)^2+\left(f^s_0\chi^{0\,\prime\prime}_{nn}(\mathbf{q},\omega)\right)^2}. \quad (4.107)$$

The first one is due to a non-vanishing $\chi^{0\,\prime\prime}_{nn}$ and comes from the quasi-particle–quasi-hole excitations. These are quite similar to the particle-hole excitations of the non-interacting system and give a continuous excitation spectrum for $|\omega|\le v^*_F|\mathbf{q}|$.[15] When $f^s_0>0$, there is a second contribution due to the pole of $\chi^R_{nn}(\mathbf{q},\omega)$ at the (real) zero-sound frequency $\omega=c_0|\mathbf{q}|$. For ω near $c_0|\mathbf{q}|$,

$$\begin{aligned}\chi^R_{nn}(\mathbf{q},\omega)&\simeq\frac{1}{\omega+i\eta-c_0|\mathbf{q}|}\frac{1}{f^{s2}_0\partial_\omega\chi^{0R}_{nn}(\mathbf{q},\omega)|_{\omega=c_0|\mathbf{q}|}},\\ \chi''_{nn}(\mathbf{q},\omega)&=-\frac{\pi}{f^{s2}_0\partial_\omega\chi^{0R}_{nn}(\mathbf{q},\omega)|_{\omega=c_0|\mathbf{q}|}}\delta(\omega-c_0|\mathbf{q}|).\end{aligned} \quad (4.108)$$

[15] Note that $m^*=m$ and $v^*_F=v_F$ in the simple model ($f_{\sigma\sigma'}(\mathbf{k},\mathbf{k}')=f^s_0$) we are considering here.

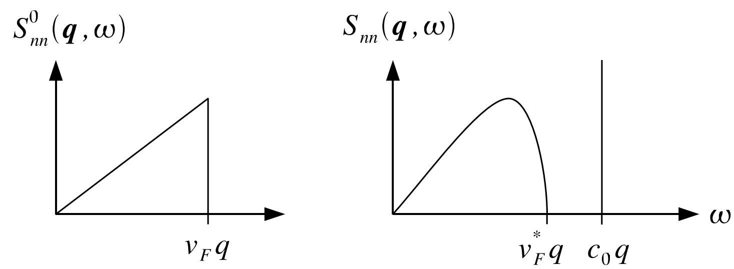

Figure 4.4: Structure factors $S^0_{nn}(\mathbf{q},\omega)$ and $S_{nn}(\mathbf{q},\omega)$ obtained within the Landau Fermi-liquid theory for $f^s(\mathbf{k},\mathbf{k}') = f^s_0$ and $|\mathbf{q}| \ll k_F$.

Thus the zero-sound mode gives a delta peak contribution to the structure factor $S_{nn}(\mathbf{q},\omega)$ (Fig. 4.4). When $-1 < F^s_0 < 0$, the zero-sound mode strongly couples to the quasi-particle–quasi-hole pair excitations (Landau damping) and is not a well defined excitation of the interacting Fermi liquid; it appears as a pole of $\chi^R_{nn}(\mathbf{q},\omega)$ at a complex frequency. In the structure factor $S_{nn}(\mathbf{q},\omega)$, it manifests itself as a broad resonance, characteristic of a damped collective mode, located at the frequency ω defined by $1 + f^s_0 \Re[\chi^{0R}_{nn}(\mathbf{q},\omega)] = 0$ ($\omega < v^*_F|\mathbf{q}|$).

We have already pointed out that the zero-sound mode exists only in a neutral Fermi liquid. In the presence of long-range Coulomb interactions, it is convenient to write the density-density response function as in (3.120). Its poles then appear as zeros of the longitudinal dielectric function

$$\epsilon^R_\parallel(\mathbf{q},\omega) = 1 + \frac{e^2}{\epsilon_0\mathbf{q}^2}\Pi^R_{nn}(\mathbf{q},\omega), \tag{4.109}$$

where Π^R_{nn} is the irreducible part of the density-density response function (Sec. 3.4.1). The simplest approximation amounts to approximating Π_{nn} by the density-density response function of the neutral (i.e. without the long-range Coulomb interaction) system. Making use of

$$\chi^R_{nn}(\mathbf{q},\omega) \simeq \chi^{0R}_{nn}(\mathbf{q},\omega) \simeq -\frac{2N^*(0){v^*_F}^2\mathbf{q}^2}{3\omega^2} = -\frac{n\mathbf{q}^2}{m\omega^2} \quad \text{for} \quad \omega \gg v^*_F|\mathbf{q}|, \tag{4.110}$$

where χ_{nn} denotes the density-density response function of the neutral system [Eq. (4.105)], we obtain

$$\epsilon^R_\parallel(\mathbf{q},\omega) \simeq 1 - \frac{\omega_p^2}{\omega^2} \quad \text{for} \quad \omega \gg v^*_F|\mathbf{q}|, \tag{4.111}$$

where $\omega_p = (ne^2/\epsilon_0 m)^{1/2}$ is the plasma frequency (Sec. 3.4). $\epsilon^R_\parallel(\mathbf{q},\omega)$ possesses a pole at $\omega = \omega_p$. We conclude that the zero-sound mode of the neutral Fermi liquid has been replaced by the plasmon mode of the charged Fermi liquid.

Density-density response function in the hydrodynamic regime

We have shown earlier that in the regime $\omega\tau \ll 1$, the effect of collisions is to suppress all components u^s_{lm} except the "hydrodynamic" variables u^s_{00} and u^s_{10}. The kinetic equation (4.103) then reduces to

$$\begin{aligned}(\omega + i\eta - \mathbf{v}^*_{\mathbf{k}} \cdot \mathbf{q}) \left[u^s_{00} Y^0_0(\hat{\mathbf{k}}) + u^s_{10} Y^0_1(\hat{\mathbf{k}}) \right] \\ - \mathbf{v}^*_{\mathbf{k}} \cdot \mathbf{q} \left[F^s_0 u^s_{00} Y^0_0(\hat{\mathbf{k}}) + \frac{F^s_1}{3} Y^0_1(\hat{\mathbf{k}}) \right] = \frac{\mathbf{v}^*_{\mathbf{k}} \cdot \mathbf{q}}{v^*_F} \phi(\mathbf{q}, \omega). \end{aligned} \tag{4.112}$$

Multiplying this equation by $\int d\Omega_{\hat{\mathbf{k}}} Y^0_0(\hat{\mathbf{k}})$ and $\int d\Omega_{\hat{\mathbf{k}}} Y^0_1(\hat{\mathbf{k}})$, we obtain

$$\begin{aligned} &(\omega + i\eta) u^s_{00} - v^*_F |\mathbf{q}| \frac{u^s_{10}}{\sqrt{3}} \left(1 + \frac{F^s_1}{3} \right) = 0, \\ &v^*_F |\mathbf{q}| u^s_{00} (1 + F^s_0) - (\omega + i\eta) \sqrt{3} u^s_{10} = -2\sqrt{\pi} |\mathbf{q}| \phi(\mathbf{q}, \omega). \end{aligned} \tag{4.113}$$

From (4.101), $\delta n(\mathbf{q}, \omega) = \pi^{-1/2} N^*(0) v^*_F u^s_{00}$, we deduce

$$\chi^R_{nn}(\mathbf{q}, \omega) = - \frac{n\mathbf{q}^2/m}{(\omega + i\eta)^2 - c_s^2 \mathbf{q}^2} \tag{4.114}$$

and

$$S_{nn}(\mathbf{q}, \omega) = \Theta(\omega) \frac{\pi n}{m c_s} |\mathbf{q}| \delta(\omega - c_s |\mathbf{q}|). \tag{4.115}$$

The structure factor exhibits a single delta peak at the first-sound frequency $c_s|\mathbf{q}|$. The quasi-particle–quasi-hole pair excitations have been washed out by collisions. Remarkably, the structure factor (4.115) satisfies the f-sum rule (3.103) and the compressibility sum rule (3.108), namely

$$\begin{aligned} \int_0^\infty \frac{d\omega}{\pi} \omega S_{nn}(\mathbf{q}, \omega) &= \frac{n\mathbf{q}^2}{m}, \\ \int_0^\infty \frac{d\omega}{\pi} \frac{S_{nn}(\mathbf{q}, \omega)}{\omega} &= \frac{n}{m c_s^2}. \end{aligned} \tag{4.116}$$

This shows that the Landau theory, in spite of being a low-energy theory, describes all excitations of the system in the hydrodynamic regime.

At finite temperature, one should in principle include the coupling between mass and energy density fluctuations. The latter give rise to a thermal diffusive mode $\omega = -iD_T \mathbf{q}^2$ (D_T is the thermal diffusion coefficient), which contributes a term proportional to

$$\Im \left[\frac{D_T \mathbf{q}^2}{-i\omega + D_T \mathbf{q}^2} \right] = \frac{\omega D_T \mathbf{q}^2}{\omega^2 + D_T^2 \mathbf{q}^4} \tag{4.117}$$

to the structure factor. Since its spectral weight extends to infinity, the diffusive mode broadens the first-sound delta peak at $\omega = c_s|\mathbf{q}|$. It can be shown that the weight of the diffusive part is of order $1 - C_p/C_v$ and is therefore negligible at very low temperatures [4].

Current-current response function

In this section, we compute the transverse current-current response function and the conductivity in the collisionless regime. We take

$$\mathbf{A}(x) = \mathbf{A}(\mathbf{q}, \omega) e^{i[\mathbf{q}\cdot\mathbf{r} - (\omega + i\eta)t]} + \text{c.c.} \tag{4.118}$$

and $\mathbf{q} \parallel \hat{\mathbf{z}}$. We write the induced paramagnetic current as

$$\begin{aligned} \mathbf{j}(x) &= \frac{1}{V}\sum_{\mathbf{k},\sigma} \frac{\mathbf{k}}{m}\delta n_{\mathbf{k}\sigma}(x) = \mathbf{j}(\mathbf{q},\omega)e^{i[\mathbf{q}\cdot\mathbf{r}-(\omega+i\eta)t]} + \text{c.c.} \\ \mathbf{j}(\mathbf{q},\omega) &= \frac{1}{V}\sum_{\mathbf{k},\sigma} \frac{\mathbf{k}}{m} v_F^*\delta(\xi_\mathbf{k})u_\sigma(\hat{\mathbf{k}}) = \frac{3n}{m}\int \frac{d\Omega_{\hat{\mathbf{k}}}}{4\pi}\hat{\mathbf{k}}u^s(\hat{\mathbf{k}}). \end{aligned} \tag{4.119}$$

The linear response to the potential is given by the current-current response function,

$$J_\mu(\mathbf{q},\omega) = \sum_{\mu'}\left[\chi^R_{j_\mu j_{\mu'}}(\mathbf{q},\omega) + \frac{n}{m}\delta_{\mu,\mu'}\right] eA_{\mu'}(\mathbf{q},\omega) \tag{4.120}$$

(see Sec. 3.4.4).

Without the collision term, the kinetic equation gives

$$\begin{aligned} (\omega + i\eta - \mathbf{v}^*_\mathbf{k}\cdot\mathbf{q})u_\sigma(\hat{\mathbf{k}}) - \mathbf{v}^*_\mathbf{k}\cdot\mathbf{q}\frac{1}{V}\sum_{\mathbf{k}',\sigma'} f_{\sigma\sigma'}(\mathbf{k},\mathbf{k}')\delta(\xi_{\mathbf{k}'})u_{\sigma'}(\hat{\mathbf{k}}') \\ = -\frac{e}{mv_F^*}(\mathbf{v}^*_\mathbf{k}\cdot\mathbf{q})\mathbf{k}\cdot\mathbf{A}(\mathbf{q},\omega). \end{aligned} \tag{4.121}$$

In the absence of interaction, equation (4.121) gives $j_\mu(\mathbf{q},\omega) = \sum_{\mu'}\chi^{0R}_{j_\mu j_{\mu'}}(\mathbf{q},\omega)A_{\mu'}(\mathbf{q},\omega)$ where

$$\chi^{0R}_{j_\mu j_{\mu'}}(\mathbf{q},\omega) = -\frac{2}{V}\sum_\mathbf{k}\delta(\xi_\mathbf{k})\frac{\mathbf{v}^*_\mathbf{k}\cdot\mathbf{q}}{\omega+i\eta-\mathbf{v}^*_\mathbf{k}\cdot\mathbf{q}}\frac{k_\mu k_{\mu'}}{m^2} \tag{4.122}$$

is the non-interacting (paramagnetic) current-current response function. To evaluate the transverse response, we take $\mu = \mu' = x$,

$$\chi^{0R}_\perp(\mathbf{q},\omega) = -\frac{2}{V}\sum_\mathbf{k}\delta(\xi_\mathbf{k})\frac{\mathbf{v}^*_\mathbf{k}\cdot\mathbf{q}}{\omega+i\eta-\mathbf{v}^*_\mathbf{k}\cdot\mathbf{q}}\frac{k_x^2}{m^2} = -\frac{3}{4}\frac{nm^*}{m^2}I(s+i\eta), \tag{4.123}$$

where $s = \omega/v_F^*|\mathbf{q}|$ and

$$I(x) = \int_0^\pi d\theta\frac{\sin^3\theta\cos\theta}{x-\cos\theta} = -\frac{4}{3} + 2x^2 + x(1-x^2)\ln\left(\frac{x+1}{x-1}\right). \tag{4.124}$$

In order to solve (4.121) in the interacting case, we assume that $f_{\sigma\sigma}(\mathbf{k},\mathbf{k}') = f_0^s + f_1^s\cos(\alpha)$ (α is the angle between $\mathbf{k}$ and $\mathbf{k}'$). After some algebra, one finds

$$\chi^R_\perp(\mathbf{q},\omega) = \frac{\chi^{0R}_\perp(\mathbf{q},\omega)}{1+\frac{F_1^s}{3+F_1^s}\frac{m}{n}\chi^{0R}_\perp(\mathbf{q},\omega)} = \frac{\chi^{0R}_\perp(\mathbf{q},\omega)}{1-\frac{F_1^s}{4}I(s+i\eta)}. \tag{4.125}$$

The transverse current-current correlation function possesses a pole for $1 - \frac{F_1^s}{4}I(s+i\eta) = 0$. We recover the equation (4.96) determining the frequency of the transverse mode $m = 1$.

The preceding results enable us to calculate the response to the (local) electromagnetic field in a charged Fermi liquid.[16] The transverse conductivity is defined by

$$\sigma_\perp(\mathbf{q},\omega) = \frac{e^2}{i(\omega+i\eta)}\left[\chi^R_\perp(\mathbf{q},\omega) - \frac{n}{m}\right] \tag{4.126}$$

(see Sec. 3.4.4). Using $\chi^{0R}_\perp(0,\omega)=0$, we obtain

$$\sigma_\perp(0,\omega) \equiv \sigma(0,\omega) = \frac{i}{\omega+i\eta}\frac{ne^2}{m}, \tag{4.127}$$

which is the expected result for a translation invariant system (Sec. 3.4.4). Making use of

$$\chi^{0R}_\perp(\mathbf{q},\omega) = \frac{n}{m^*}\left(1+i\frac{3}{4}\pi s\right) + \mathcal{O}(s^2), \tag{4.128}$$

we obtain the static transverse conductivity

$$\sigma_\perp(\mathbf{q},0) = \frac{3\pi}{4}\frac{ne^2}{k_F|\mathbf{q}|}, \tag{4.129}$$

a result that does not depend on the mass of the particles.

The transverse mode $m=1$ which appears as a pole of $\chi^R_\perp(\mathbf{q},\omega)$ when $F^s_1 > 6$ [Eq. (4.96)] is modified by the coupling to the electromagnetic field. The dispersion of the transverse modes of the electron system coupled to the electromagnetic field is obtained from (3.159). Solving this equation together with (4.125), one finds that the transverse excitations of the neutral system – that appear as a pole of $\chi^R_\perp$ – are little affected by the coupling to the electromagnetic field for $|\mathbf{q}| \gg \omega_p/c_l$ (c_l is the velocity of light). But at low frequency, when $|\mathbf{q}| \lesssim \omega_p/c_l$, the transverse mode disappears in the continuum of particle-hole excitations (see figure 3.9 and the discussion in section 3.4.2).

4.3.5 Multi-pair excitations

The preceeding study of the response functions $\chi^R_{nn}(\mathbf{q},\omega)$ and $\chi^R_\perp(\mathbf{q},\omega)$ shows that the Landau theory describes single-pair excitations and collective modes but does not take into account multi-pair excitations. In an interacting Fermi liquid, a single quasi-particle–quasi-hole pair excitation can decay into multiple excited pairs. In other words – focusing on the density-density response function – the density operator $\hat n(\mathbf{q})$ couples the ground state to excited states with an arbitrary number of quasi-particle–quasi-hole pairs. In this section, we briefly discuss to what extent the multi-pair excitations are expected to affect the structure factor $S_{nn}(\mathbf{q},\omega)$. For a thorough analysis, we refer to Refs. [4,5].

When $|\mathbf{q}| \ll k_F$, the excitation energy $\omega = \epsilon_{\mathbf{k}+\mathbf{q}} - \epsilon_{\mathbf{k}}$ of a single pair is necessarily small since the Pauli principle requires $|\mathbf{k}| < k_F$ and $|\mathbf{k}+\mathbf{q}| > k_F$ (Fig. 4.5). Energy conservation imposes $\epsilon_{\mathbf{k}}$ and $\epsilon_{\mathbf{k}+\mathbf{q}}$ to be within ω of the Fermi surface. The density

[16] The Landau theory gives a transverse current-current response function $\chi^R(\mathbf{q},0) \equiv \lim_{\mathbf{q}\to 0}\chi^R_\perp(\mathbf{q},0)$ which is independent of $\mathbf{q}$; $\chi^R_\perp(\mathbf{q},0)$ is obtained from (4.123) and (4.125) with $s=0$. As a result, the Landau theory does not describe the orbital diamagnetism of the charged Fermi liquid (Sec. 3.4.5).

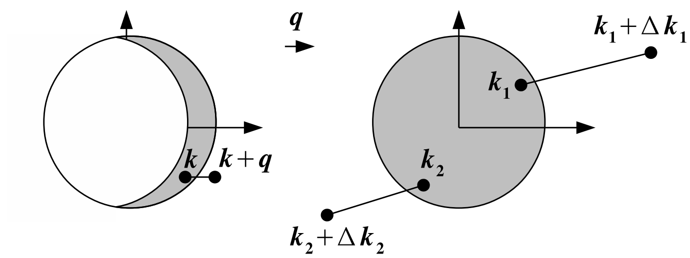

Figure 4.5: Single-pair *vs* 2-pair excitations ($\Delta\mathbf{k}_1 + \Delta\mathbf{k}_2 = \mathbf{q}$). The shaded areas show the allowed regions for the hole wave vectors $\mathbf{k}_i$.

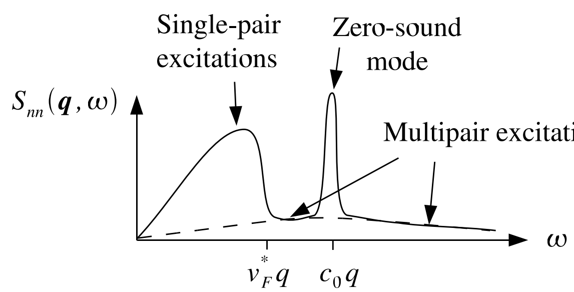

Figure 4.6: Structure factor $S_{nn}(\mathbf{q}, \omega)$ of a Fermi liquid when multi-pair excitations are taken into account. The zero-sound mode is broadened due to the coupling to multi-pair excitations. (After Ref. [4].)

per unit energy $\rho^{(1)}(\omega)$ of the single-pair excitations is proportional to ω at small energy. In the ideal gas, $S^0_{nn}(\mathbf{q}, \omega)$ is directly given by $\rho^{(1)}(\omega)$ since the matrix element $\langle 0|\hat{n}(\mathbf{q})|m\rangle$ is equal to unity when the transition is allowed by momentum conservation.[17]

Matters are otherwise for the multi-pair excitations. Since only the total momentum $\mathbf{q}$ is fixed, the excitation energy can extend up to infinity. An example of a two-pair excitation is given in figure 4.5. Multi-pair excitations are therefore expected to contribute a broad spectrum to the structure factor $S_{nn}(\mathbf{q}, \omega)$. For a n-pair excitation, the excitation energy is determined by $\omega = \sum_{i=1}^n (\epsilon_{\mathbf{k}_i + \Delta\mathbf{k}_i} - \epsilon_{\mathbf{k}_i})$ with $|\mathbf{k}_i| < k_F$, $|\mathbf{k}_i + \Delta\mathbf{k}_i| > k_F$ and $\mathbf{q} = \sum_{i=1}^n \Delta\mathbf{k}_i$. Energy conservation requires that the $2n$ quasi-particles and quasi-holes lie within ω of the Fermi surface. The density per unit energy $\rho^{(n)}(\omega)$ of the n-pair excitations is therefore of order ω^{2n-1} for $\omega \to 0$. At low energies, multi-pair excitations are therefore negligible with respect to single-pair excitations. Their main effect is to produce a continuous excitation spectrum extended up to very high energies and leading to a small damping of the zero-sound mode (Fig. 4.6).

It can also be shown that multi-pair excitations are negligible in the limit $\mathbf{q} \to 0$

[17] Recall that the $T = 0$ structure factor reads $S_{nn}(\mathbf{q}, \omega) = 2\pi \sum_{m\neq 0} |\langle 0|\hat{n}(\mathbf{q})|m\rangle|^2 \delta(\omega + \epsilon_0 - \epsilon_m)$ [see Eq. (3.37)].

regardless of the value of the energy ω. The reason is that the matrix element $\langle m|\hat{n}(\mathbf{q})|0\rangle$ is of order $\mathbf{q}^2$ (in a translation invariant system) for a multi-pair excited state $|m\rangle$. ($|0\rangle$ denotes the ground state.) By contrast, $\langle m|\hat{n}(\mathbf{q})|0\rangle$ is $\mathcal{O}(1)$ for single-pair excitations (as in the ideal Fermi gas) and $\mathcal{O}(\sqrt{|\mathbf{q}|})$ for the zero-sound mode. The suppression of multi-pair excitations in the long-wavelength limit is a direct consequence of translation invariance. The latter implies $[\hat{H}, \hat{\mathbf{j}}(\mathbf{q}=0)] = 0$, so that we expect $[\hat{H}, \hat{\mathbf{j}}(\mathbf{q})] = \mathcal{O}(\mathbf{q})$. Let us now consider the continuity equation

$$\partial_t \hat{n}(\mathbf{q}) + i\mathbf{q}\cdot\hat{\mathbf{j}}(\mathbf{q}) = i[\hat{H}, \hat{n}(\mathbf{q})] + i\mathbf{q}\cdot\hat{\mathbf{j}}(\mathbf{q}) = 0. \tag{4.130}$$

It implies

$$\begin{aligned} 0 &= \langle m|[\hat{H}, \hat{n}(\mathbf{q})]|0\rangle + \mathbf{q}\cdot\langle m|\hat{\mathbf{j}}(\mathbf{q})|0\rangle \\ &= (\epsilon_m - \epsilon_0)\langle m|\hat{n}(\mathbf{q})|0\rangle + \mathbf{q}\cdot\langle m|\hat{\mathbf{j}}(\mathbf{q})|0\rangle. \end{aligned} \tag{4.131}$$

Since the excitation energy $\epsilon_m - \epsilon_0$ remains finite as $\mathbf{q}\to 0$, $\langle m|\hat{n}(\mathbf{q})|0\rangle = \mathcal{O}(\mathbf{q}^2)$.

4.4 Microscopic basis of Fermi-liquid theory

The main goal of a microscopic approach to Fermi-liquid theory is to show how the quasi-particle concept emerges from the single-particle Green function. But it should also give a microscopic interpretation of the Landau functional $E[n]$ and the Landau parameters, reproduce the collective modes and response to macroscopic perturbations as obtained in Landau theory, and prove Luttinger theorem. Before discussing these points in more detail, let us summarize the characteristic features of a Fermi liquid that emerge from a microscopic theory:

- The central (and often taken as the defining) property of a Fermi liquid is that the self-energy satisfies

$$\Im[\Sigma^R(\mathbf{k},\omega)] \propto -(\omega^2 + \pi^2 T^2) \tag{4.132}$$

at low energies and temperatures. Equation (4.132) implies that the $T = 0$ scattering rate $1/\tau_{\mathbf{k}} \sim -\Im[\Sigma^R(\mathbf{k},\xi_{\mathbf{k}})] \propto (|\mathbf{k}| - k_F)^2$ vanishes faster than $\xi_{\mathbf{k}}$ as one approaches the Fermi surface – a necessary condition for the existence of quasi-particles.

- Another (and related) fundamental property is that

$$\frac{\partial}{\partial\omega}\Re[\Sigma^R(\mathbf{k},\omega)]\Big|_{\omega=\xi_{\mathbf{k}}} \leq 0. \tag{4.133}$$

This equation implies that the "quasi-particle weight"

$$z_{\mathbf{k}} = \frac{1}{1 - \partial_\omega \Re[\Sigma^R(\mathbf{k},\omega)]}\Bigg|_{\omega=\xi_{\mathbf{k}}}, \tag{4.134}$$

which measures the overlap between a particle excitation $\hat{\psi}^\dagger_\sigma(\mathbf{k})|0\rangle$ and a quasi-particle state is finite ($0 < z_{\mathbf{k}} \leq 1$). $z_{\mathbf{k}}$ also determines the discontinuity at k_F in the momentum distribution $\langle\hat{\psi}^\dagger_\sigma(\mathbf{k})\hat{\psi}_\sigma(\mathbf{k})\rangle$. The existence of such a discontinuity is a characteristic property of a Fermi liquid.

- The Landau energy functional $E[n]$ – or, equivalently, the thermodynamic potential $\Omega[n]$ – can be obtained as a suitably defined Legendre transform of the grand potential $-T\ln Z$. The Landau f function is related in a simple way to the particle-hole vertex $\Gamma_{\rm ph}$.
- The volume of the Fermi surface is independent of the interactions (Luttinger theorem).

4.4.1 Quasi-particles

Spectral function of the ideal Fermi gas

We start by considering an ideal Fermi gas with (grand-canonical) Hamiltonian

$$\hat{H}_0 = \sum_{\mathbf{k},\sigma} \xi^0_{\mathbf{k}} \hat{\psi}^\dagger_\sigma(\mathbf{k})\hat{\psi}_\sigma(\mathbf{k}), \qquad \xi^0_{\mathbf{k}} = \epsilon^0_{\mathbf{k}} - \mu, \tag{4.135}$$

where μ is the chemical potential ($\mu(T=0)=\epsilon^0_F$). The ground state reads

$$|0\rangle = \prod_{\substack{\mathbf{k},\sigma \\ |\mathbf{k}|\leq k_F}} \hat{\psi}^\dagger_\sigma(\mathbf{k})|\text{vac}\rangle, \tag{4.136}$$

and its (grand-canonical) energy is $E_0 - \mu N$ (E_0 is given by (4.2)).

The state $\hat{\psi}^\dagger_\sigma(\mathbf{k})|0\rangle$ with an additional particle of momentum $\mathbf{k}$ ($|\mathbf{k}|>k_F$) and spin σ has an energy $E_0-\mu N+\xi^0_{\mathbf{k}}$ and evolves in time according to

$$\exp\bigl(-i\hat{H}_0 t\bigr)\hat{\psi}^\dagger_\sigma(\mathbf{k})|0\rangle = \exp[-i(E_0-\mu N+\xi^0_{\mathbf{k}})t]\hat{\psi}^\dagger_\sigma(\mathbf{k})|0\rangle. \tag{4.137}$$

The purely oscillating time dependence is due to the fact that $\hat{\psi}^\dagger_\sigma(\mathbf{k})|0\rangle$ is an exact eigenstate. This property also shows up in the retarded Green function (which will turn out to be the quantity of interest in the interacting case)

$$G^R_0(\mathbf{k},t) = \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{e^{-i\omega t}}{\omega+i\eta-\xi^0_{\mathbf{k}}} = -i\Theta(t)\exp(-i\xi^0_{\mathbf{k}}t) \tag{4.138}$$

and the spectral function

$$A_0(\mathbf{k},\omega) = -\frac{1}{\pi}\Im[G^R_0(\mathbf{k},\omega)] = \delta(\omega-\xi^0_{\mathbf{k}}), \tag{4.139}$$

where the exact excited state appears as a Dirac peak. A similar reasoning can be made for a hole excitation [$\hat{\psi}_\sigma(\mathbf{k})|0\rangle$ with $|\mathbf{k}|<k_F$].

Spectral function of the interacting Fermi liquid

In the interacting system, the retarded Green function and the spectral function,

$$\begin{aligned} G^R(\mathbf{k},\omega) &= \frac{1}{\omega+i\eta-\xi^0_{\mathbf{k}}-\Sigma^R(\mathbf{k},\omega)}, \\ A(\mathbf{k},\omega) &= -\frac{1}{\pi}\frac{\Im[\Sigma^R(\mathbf{k},\omega)]}{\bigl(\omega-\xi^0_{\mathbf{k}}-\Re[\Sigma^R(\mathbf{k},\omega)]\bigr)^2+\bigl(\Im[\Sigma^R(\mathbf{k},\omega)]\bigr)^2} \end{aligned} \tag{4.140}$$

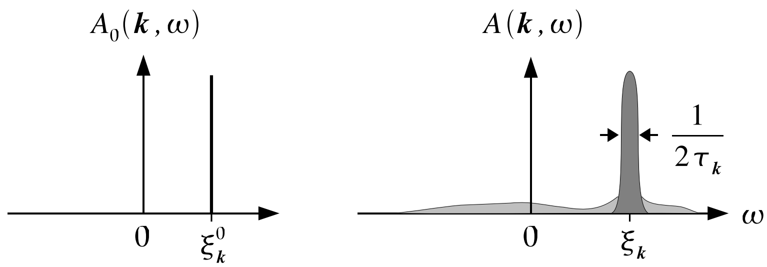

Figure 4.7: Spectral function in an ideal Fermi gas and in a Fermi liquid. The dark shaded area shows the quasi-particle peak (spectral weight $z_{\mathbf{k}}$) and the light shaded one the incoherent part of the spectrum (spectral weight $1 - z_{\mathbf{k}}$). The width $1/\tau_{\mathbf{k}}$ of the quasi-particle peak goes to zero faster than $|\xi_{\mathbf{k}}|$ when $|\mathbf{k}| \to k_F$.

can be expressed in terms of the retarded self-energy $\Sigma^R(\mathbf{k},\omega)$. Equation (4.140) holds for $\Im[\Sigma^R(\mathbf{k},\omega)] \neq 0$; when $\Im[\Sigma^R(\mathbf{k},\omega)] = 0$, $A(\mathbf{k},\omega) = \delta(\omega - \xi^0_{\mathbf{k}} - \Sigma^R(\mathbf{k},\omega))$. A Fermi liquid is defined by a spectral function $A(\mathbf{k},\omega)$ which, for $\mathbf{k}$ in the vicinity of the Fermi surface, exhibits a sharp peak at an energy $\xi_{\mathbf{k}}$ with a width $1/2\tau_{\mathbf{k}}$ which goes to zero faster than $\xi_{\mathbf{k}}$ when $|\mathbf{k}| \to k_F$ (Fig. 4.7). As we shall see, such a peak is the signature of a quasi-particle ($|\mathbf{k}| > k_F$) or quasi-hole ($|\mathbf{k}| < k_F$) excitation.

If $\Im[\Sigma^R(\mathbf{k},\omega)]$ varies weakly for $\omega \approx \xi_{\mathbf{k}}$, then the position of the maximum is determined by

$$\xi_{\mathbf{k}} - \xi^0_{\mathbf{k}} - \Re[\Sigma^R(\mathbf{k},\xi_{\mathbf{k}})] = 0 \tag{4.141}$$

(using $\partial_\omega \Im[\Sigma^R(\mathbf{k},\omega)] = 0$ for $\omega \simeq \xi_{\mathbf{k}}$). This equation determines the energy of a quasi-particle with momentum $\mathbf{k}$. In particular, the Fermi momentum k_F is obtained from $\xi_{k_F} = 0$, i.e.

$$\xi^0_{k_F} + \Sigma^R(k_F, 0) = 0. \tag{4.142}$$

Here we have used the fact that $\Sigma^R(\mathbf{k}_F, 0)$ is real at zero temperature (see Eq. (4.171) below). Note that in general μ differs from $\epsilon^0_F = k_F^2/2m$ in the interacting system so that $\xi^0_{k_F} \neq 0$. We shall see in section 4.4.6 that k_F (for a given density n) is not changed by the interactions (Luttinger theorem) and therefore given by (4.1).[18] For ω near $\xi_{\mathbf{k}}$, we have

$$\begin{aligned} \omega - \xi^0_{\mathbf{k}} - \Re[\Sigma^R(\mathbf{k},\omega)] &\simeq \omega - \xi^0_{\mathbf{k}} - \Re[\Sigma^R(\mathbf{k},\xi_{\mathbf{k}})] - (\omega - \xi_{\mathbf{k}})\partial_\omega \Re[\Sigma^R(\mathbf{k},\omega)]\big|_{\omega=\xi_{\mathbf{k}}} \\ &= \frac{\omega - \xi_{\mathbf{k}}}{z_{\mathbf{k}}}, \end{aligned} \tag{4.143}$$

where $z_{\mathbf{k}}$, defined in (4.134), is referred to as the quasi-particle weight (for reasons that will be explained below). We deduce that the spectral function can be written as

$$A(\mathbf{k},\omega) = \frac{z_{\mathbf{k}}}{\pi} \frac{1/2\tau_{\mathbf{k}}}{(\omega - \xi_{\mathbf{k}})^2 + (1/2\tau_{\mathbf{k}})^2} + A_{\text{inc}}(\mathbf{k},\omega), \tag{4.144}$$

[18] More generally, in anisotropic systems Luttinger theorem states that the volume of the Fermi surface is not affected by the interactions, but of course its shape is likely to depend on the interactions.

where

$$\frac{1}{\tau_{\mathbf{k}}} = -2z_{\mathbf{k}} \Im[\Sigma^R(\mathbf{k}, \xi_{\mathbf{k}})] \tag{4.145}$$

is, as we shall see later, the inverse quasi-particle life-time. The first term in the rhs of (4.144) – the "quasi-particle peak" – follows from (4.143) and determines $A(\mathbf{k}, \omega)$ near the maximum at $\omega = \xi_{\mathbf{k}}$. It corresponds to a Lorentzian peak of width $\sim 1/\tau_{\mathbf{k}}$ and spectral weight (defined as the area under the peak) $z_{\mathbf{k}}$. Since $A(\mathbf{k}, \omega) \geq 0$ and $\int_{-\infty}^{\infty} d\omega A(\mathbf{k}, \omega) = 1$, one has $0 \leq z_{\mathbf{k}} \leq 1$ and in turn (4.133). $A_{\text{inc}}(\mathbf{k}, \omega)$ denotes the "incoherent" part of the spectral function. It typically corresponds to a broad (featureless) excitation spectrum extending up to very high energies. In order for the total spectral weight to be unity, its weight should be equal to $1 - z_{\mathbf{k}}$. From (4.144) and the spectral representation ???, we deduce that the retarded Green function takes the form

$$G^R(\mathbf{k}, \omega) = \int_{-\infty}^{\infty} d\omega' \frac{A(\mathbf{k}, \omega')}{\omega + i\eta - \omega'} = \frac{z_{\mathbf{k}}}{\omega - \xi_{\mathbf{k}} + \frac{i}{2\tau_{\mathbf{k}}}} + G^R_{\text{inc}}(\mathbf{k}, \omega). \tag{4.146}$$

The quasi-particle peak gives rise to a pole at the complex energy $\xi_{\mathbf{k}} - i/2\tau_{\mathbf{k}}$ with a residue determined by the quasi-particle weight $z_{\mathbf{k}}$. In time space, equation (4.146) gives

$$G^R(\mathbf{k}, t) = -iz_{\mathbf{k}} \Theta(t) \exp\left(-i\xi_{\mathbf{k}} t - \frac{t}{2\tau_{\mathbf{k}}}\right) + G^R_{\text{inc}}(\mathbf{k}, t). \tag{4.147}$$

Because $A_{\text{inc}}(\mathbf{k}, \omega)$ has no sharp structure in the variable ω, the incoherent part $G^R_{\text{inc}}(\mathbf{k}, t)$ of the Green function decays quickly in time, and the long-time behavior of $G^R(\mathbf{k}, t)$ is dominated by the quasi-particle pole. Thus, for $1/|\xi_{\mathbf{k}}| \ll t \ll \tau_{\mathbf{k}}$ – which requires $1/\tau_{\mathbf{k}} \ll \xi_{\mathbf{k}}$ – one observes the oscillating behavior characteristic of an eigenstate of the Hamiltonian. Equation (4.147) confirms the interpretation of $\tau_{\mathbf{k}}$ as the life-time of the quasi-particle. The oscillating part of the Green function (4.147) is reduced by a factor $z_{\mathbf{k}}$ in the interacting system. Thus, $z_{\mathbf{k}}$ is a measure of the overlap between the state $\hat\psi^\dagger_\sigma(\mathbf{k})|0\rangle$ and the quasi-particle state with momentum $\mathbf{k}$ and spin σ; it can be seen as the fraction of bare particle contained in the quasi-particle.[19]

Momentum distribution

The existence of quasi-particles with reduced spectral weight has an important consequence for the momentum distribution function of the bare particles,

$$\begin{aligned}
\langle \hat\psi^\dagger_\sigma(\mathbf{k}) \hat\psi_\sigma(\mathbf{k}) \rangle = G(\mathbf{k}, \tau = 0^-) &= \frac{1}{\beta} \sum_{\omega_n} e^{i\omega_n \eta} G(\mathbf{k}, i\omega_n) \\
&= \frac{1}{\beta} \sum_{\omega_n} e^{i\omega_n \eta} \int_{-\infty}^{\infty} d\omega \frac{A(\mathbf{k}, \omega)}{i\omega_n - \omega} = \int_{-\infty}^{\infty} d\omega n_F(\omega) A(\mathbf{k}, \omega) \\
&= \int_{-\infty}^{0} d\omega A(\mathbf{k}, \omega),
\end{aligned} \tag{4.148}$$

[19]Recall that the $G^R(\mathbf{k}, t)$ essentially measures the probability amplitude for a bare particle (or hole) with momentum $\mathbf{k}$ created at $t = 0$ to be in the same quantum state at time t.

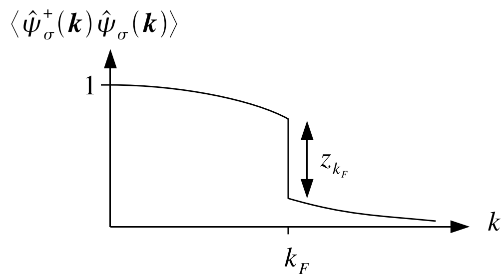

Figure 4.8: Momentum distribution $\langle\hat{\psi}^\dagger_\sigma(\mathbf{k})\hat{\psi}_\sigma(\mathbf{k})\rangle$ in a Fermi liquid.

where the last result holds at zero temperature. Here $G(\mathbf{k},\tau)$ is the imaginary time Green function. When $1/\tau_{\mathbf{k}} \ll \xi_{\mathbf{k}}$, the quasi-particle peak in $A(\mathbf{k},\omega)$ becomes sharper and sharper as we approach the Fermi surface and tends to $z_{k_F}\delta(\omega-\xi_{\mathbf{k}})$ for $\xi_{\mathbf{k}} \to 0$. Since the incoherent part of the spectral function varies smoothly with $\mathbf{k}$, it is continuous across the Fermi level $\xi_{\mathbf{k}} = 0$. We then deduce

$$\left[\lim_{|\mathbf{k}|\to k_F^+} - \lim_{|\mathbf{k}|\to k_F^-}\right]\langle\hat{\psi}^\dagger_\sigma(\mathbf{k})\hat{\psi}_\sigma(\mathbf{k})\rangle = \left[\lim_{|\mathbf{k}|\to k_F^+} - \lim_{|\mathbf{k}|\to k_F^-}\right]\int_{-\infty}^0 d\omega z_{k_F}\delta(\omega-\xi_{\mathbf{k}}), \quad (4.149)$$

where $k_F^\pm = k_F \pm 0^+$. Since $\xi_{k_F^+} > 0$ whereas $\xi_{k_F^-} < 0$, we conclude that the momentum distribution function exhibits a jump

$$\left[\lim_{|\mathbf{k}|\to k_F^+} - \lim_{|\mathbf{k}|\to k_F^-}\right]\langle\hat{\psi}^\dagger_\sigma(\mathbf{k})\hat{\psi}_\sigma(\mathbf{k})\rangle = -z_{k_F} \quad (4.150)$$

across the Fermi level (Fig. 4.8). The existence of quasi-particles requiring $z_{k_F} > 0$, the discontinuity in the momentum distribution function $\langle\hat{\psi}^\dagger_\sigma(\mathbf{k})\hat{\psi}_\sigma(\mathbf{k})\rangle$ of the bare particles is an important characteristics of a Fermi liquid. This momentum distribution should not be confused with the quasi-particle momentum distribution $n^0_{\mathbf{k}} = \Theta(k_F - |\mathbf{k}|)$ introduced in section 4.1.

Effective mass

The quasi-particle group velocity is defined by

$$\mathbf{v}^*_{\mathbf{k}} = \boldsymbol{\nabla}\xi_{\mathbf{k}}. \quad (4.151)$$

Using (4.141), we obtain

$$\begin{aligned} \mathbf{v}^*_{\mathbf{k}} &= \boldsymbol{\nabla}\{\xi^0_{\mathbf{k}} + \Re[\Sigma^R(\mathbf{k},\xi_{\mathbf{k}})]\} \\ &= \mathbf{v}_{\mathbf{k}} + \left(\frac{\partial}{\partial\mathbf{k}}\Re[\Sigma^R(\mathbf{k},\omega)] + \frac{\partial}{\partial\omega}\Re[\Sigma^R(\mathbf{k},\omega)]\boldsymbol{\nabla}\xi_{\mathbf{k}}\right)_{\omega=\xi_{\mathbf{k}}} \\ &= z_{\mathbf{k}}\left(\mathbf{v}_{\mathbf{k}} + \frac{\partial}{\partial\mathbf{k}}\Re[\Sigma^R(\mathbf{k},\omega)]\Big|_{\omega=\xi_{\mathbf{k}}}\right). \end{aligned} \quad (4.152)$$

For symmetry reasons, $\mathbf{v}^*_{\mathbf{k}} = v^*_{\mathbf{k}}\hat{\mathbf{k}}$ and $\Sigma^R(\mathbf{k},\omega)$ is a function of $|\mathbf{k}|$. From (4.152), we then obtain

$$v^*_{\mathbf{k}} = z_{\mathbf{k}} \left(\frac{|\mathbf{k}|}{m} + \frac{\partial}{\partial|\mathbf{k}|} \Re[\Sigma^R(\mathbf{k},\omega)]\Big|_{\omega=\xi_{\mathbf{k}}} \right). \tag{4.153}$$

From the definition (4.8) of the effective mass, we finally deduce

$$\frac{m}{m^*} = z_{k_F} \left(1 + \frac{m}{k_F} \frac{\partial}{\partial|\mathbf{k}|} \Re[\Sigma^R(\mathbf{k},\omega)]\Big|_{|\mathbf{k}|=k_F,\omega=0} \right). \tag{4.154}$$

When the self-energy is momentum independent, the effective mass is simply determined by $m^* = m/z_{k_F}$ and is larger than the bare mass. More generally however, $\partial_{|\mathbf{k}|}\Re[\Sigma^R(\mathbf{k},\omega)]|_{|\mathbf{k}|=k_F,\omega=0}$ can have either sign and the effective mass can be larger or smaller than the bare mass.[20] The same conclusion was reached in the phenomenological approach (Sec. 4.1.3). In section 4.4.4 we shall show, using the Ward identities and the microscopic definition of the Landau f function, that equation (4.154) agrees with (4.40).

Quasi-particle operators

We can formally define quasi-particle operators (or fields) as follows [8]. In a first step, one eliminates the incoherent part of the spectral function $A(\mathbf{k},\omega)$ by filtering out $A_{\rm inc}(\mathbf{k},\omega)$. The retarded Green function then exhibits a purely propagating behavior $-iz_{\mathbf{k}}\Theta(t)e^{-i\xi_{\mathbf{k}}t}$ for $t \ll \tau_{\mathbf{k}}$. Let us suppose that this step can be seen as a change $\hat\psi_\sigma(\mathbf{k}) \to \hat\psi'_\sigma(\mathbf{k})$ of the fermion operator. In a second step, one introduces rescaled operators $\hat{\bar\psi}_\sigma(\mathbf{k}) = z_{\mathbf{k}}^{-1/2}\hat\psi'_\sigma(\mathbf{k})$ in order to recover a spectral function normalized to unity (Fig. 4.9),

$$\bar A(\mathbf{k},\omega) = \frac{1}{\pi}\frac{1/2\tau_{\mathbf{k}}}{(\omega-\xi_{\mathbf{k}})^2 + (1/2\tau_{\mathbf{k}})^2}. \tag{4.155}$$

The quasi-particle retarded Green function then reads

$$\bar G^R(\mathbf{k},\omega) = \frac{1}{\omega - \xi_{\mathbf{k}} + \frac{i}{2\tau_{\mathbf{k}}}} \tag{4.156}$$

and is related to the fermion Green function by

$$G^R(\mathbf{k},\omega) = z_{\mathbf{k}}\bar G^R(\mathbf{k},\omega) + G_{\rm inc}(\mathbf{k},\omega). \tag{4.157}$$

The corresponding distribution function $\langle \hat{\bar\psi}^\dagger_\sigma(\mathbf{k})\hat{\bar\psi}_\sigma(\mathbf{k})\rangle = \bar G^R(\mathbf{k},\tau=0^-)$ takes the quasi-particle form $n^0_{\mathbf{k}} = \Theta(k_F - |\mathbf{k}|)$. In section 4.5 we shall see that the quasi-particle operators $\hat{\bar\psi}$ arise very naturally in the renormalization group framework.

Quasi-particle life-time

The divergence of the quasi-particle life-time near the Fermi surface is a consequence of the reduced phase space available for a decay process of an incoming particle induced by the excitation of a single or many quasi-particle–quasi-hole pairs (see footnote 6

[20] For instance, the effective mass in a metal at high density is expected to be smaller than the bare mass (chapter 5).

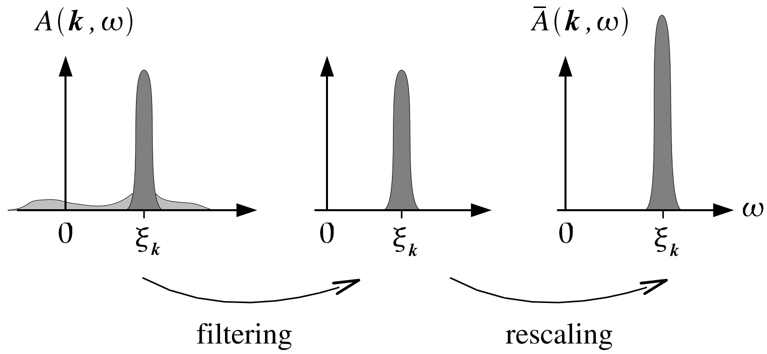

Figure 4.9: Quasi-particle spectral function $\bar{A}(\mathbf{k},\omega)$ obtained by filtering out the incoherent part of $A(\mathbf{k},\omega)$ and rescaling in order to have a total spectral weight normalized to unity.

page 288). Thus this essential property of the Fermi liquid does not depend on the strength of the interactions.

In this section, we want to substantiate the phase space argument by computing explicitly the second-order self-energy

$$\Sigma(k) = \frac{1}{\beta V}\sum_q v_{\mathbf{q}}^2 \chi_0(q) G_0(k+q) \tag{4.158}$$

shown in figure 4.10, where

$$\chi_0(q) = -\frac{2}{\beta V}\sum_k G_0(k)G_0(k+q) \tag{4.159}$$

is the bare particle-hole response function. To calculate the sum over ω_ν in (4.158), we consider the integral

$$\oint_{(\mathcal{C})} \frac{dz}{2i\pi} n_B(z)\chi_0(\mathbf{q},z)G_0(\mathbf{k}+\mathbf{q},i\omega_n+z) \tag{4.160}$$

where $(\mathcal{C})$ is the contour shown in figure 4.11. Using the residue theorem and noting that the part of the contour at infinity does not contribute, we obtain

$$\begin{aligned} \frac{1}{\beta}\sum_{\omega_\nu\neq 0} \chi_0(q)G_0(k+q) + n_B(-i\omega_n+\xi^0_{\mathbf{k}})\chi_0(\mathbf{q},\xi^0_{\mathbf{k}}-i\omega_n) \\ = \mathcal{P}\int_{-\infty}^{\infty} \frac{d\omega}{2i\pi} n_B(\omega) \frac{\chi_0(\mathbf{q},\omega+i\eta)-\chi_0(\mathbf{q},\omega-i\eta)}{i\omega_n+\omega-\xi^0_{\mathbf{k}}} \\ - \frac{1}{\beta}\chi_0(\mathbf{q},0)G_0(\mathbf{k}+\mathbf{q},i\omega_n), \end{aligned} \tag{4.161}$$

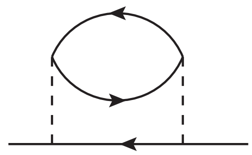

Figure 4.10: Second-order contribution to the self-energy.

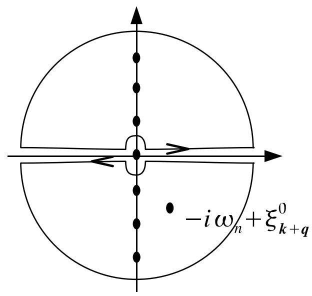

Figure 4.11: Contour ($\mathcal{C}$) used in equation (4.158). The black dots indicate the position of the bosonic Matsubara frequencies $i\omega_\nu$ as well as $-i\omega_n + \xi^0_{\mathbf{k}}$.

where the last term comes from the part of the contour near the origin. Thus we have

$$\begin{aligned}\frac{1}{\beta}\sum_{\omega_\nu}\chi_0(q)G_0(k+q) &= n_F(\xi^0_{\mathbf{k}})\chi_0(\mathbf{q},\xi^0_{\mathbf{k}}-i\omega_n) + \mathcal{P}\int_{-\infty}^{\infty}\frac{d\omega}{\pi}n_B(\omega)\frac{\chi''_0(\mathbf{q},\omega)}{i\omega_n+\omega-\xi^0_{\mathbf{k}}} \\ &= \mathcal{P}\int_{-\infty}^{\infty}\frac{d\omega}{\pi}\frac{\chi''_0(\mathbf{q},\omega)}{i\omega_n+\omega-\xi^0_{\mathbf{k}}}\left[n_B(\omega)+n_F(\xi^0_{\mathbf{k}})\right] \end{aligned} \tag{4.162}$$

($\chi''_0(\mathbf{q},\omega) = \Im[\chi_0(\mathbf{q},\omega+i\eta)]$) making use of the spectral representation (3.33) of $\chi_0(\mathbf{q},\xi^0_{\mathbf{k}}-i\omega_n)$. This gives

$$\Sigma(k) = \frac{1}{V}\sum_{\mathbf{q}} v^2_{\mathbf{q}}\,\mathcal{P}\int_{-\infty}^{\infty}\frac{d\omega}{\pi}\frac{\chi''_0(\mathbf{q},\omega)}{i\omega_n+\omega-\xi^0_{\mathbf{k+q}}}\left[n_B(\omega)+n_F(\xi^0_{\mathbf{k+q}})\right]. \tag{4.163}$$

To obtain the quasi-particle life-time, we need to calculate the imaginary part of the retarded self-energy,

$$\begin{aligned}\Im[\Sigma^R(\mathbf{k},\omega)] &= -\frac{1}{V}\sum_{\mathbf{q}} v^2_{\mathbf{q}}\int_{-\infty}^{\infty}d\omega'\chi''_0(\mathbf{q},\omega')\delta(\omega+\omega'-\xi^0_{\mathbf{k+q}}) \\ &\qquad\times\left[n_B(\omega')+n_F(\xi^0_{\mathbf{k+q}})\right] \\ &= -\int_0^{\infty}\frac{d|\mathbf{q}|}{2\pi^2}\mathbf{q}^2 v^2_{\mathbf{q}}\int\frac{d\Omega_{\hat{\mathbf{q}}}}{4\pi}\int_{-\infty}^{\infty}d\omega'\chi''_0(\mathbf{q},\omega')\delta(\omega+\omega'-\xi^0_{\mathbf{k+q}}) \\ &\qquad\times\left[n_B(\omega')+n_F(\omega+\omega')\right]. \end{aligned} \tag{4.164}$$

The last line holds for a three-dimensional system. Since $\chi''_0(\mathbf{q},\omega) = \chi''_0(|\mathbf{q}|,\omega)$, we can carry out the angular integration,

$$\begin{aligned}\int \frac{d\Omega_{\hat{\mathbf{q}}}}{4\pi}\delta\left(\omega+\omega'-\xi^0_{\mathbf{k}}-\frac{\mathbf{q}^2}{2m}-\frac{\mathbf{q}\cdot\mathbf{k}}{m}\right)&\\ &=\frac{1}{2}\int_0^\pi d\theta\sin\theta\delta\left(\omega+\omega'-\xi^0_{\mathbf{k}}-\frac{\mathbf{q}^2}{2m}-\frac{|\mathbf{q}||\mathbf{k}|\cos\theta}{m}\right)\\ &=\frac{m}{2|\mathbf{k}||\mathbf{q}|}\Theta\left(1-\frac{|\omega+\omega'-\xi^0_{\mathbf{k}}-\mathbf{q}^2/2m|}{|\mathbf{k}||\mathbf{q}|/m}\right).\end{aligned} \tag{4.165}$$

From (4.164,4.165), we deduce

$$\begin{aligned}\Im[\Sigma^R(\mathbf{k},\omega)]=-\frac{m}{2|\mathbf{k}|}\int_0^\infty\frac{d|\mathbf{q}|}{2\pi^2}|\mathbf{q}|\int_{-\infty}^\infty d\omega'\chi_0''(\mathbf{q},\omega')v^2_{\mathbf{q}}\left[n_B(\omega')+n_F(\omega+\omega')\right]&\\ \times\Theta\left(1-\frac{|\omega+\omega'-\xi^0_{\mathbf{k}}-\mathbf{q}^2/2m|}{|\mathbf{k}||\mathbf{q}|/m}\right).&\end{aligned} \tag{4.166}$$

Because of the Bose and Fermi functions, the relevant part of the integration over ω' corresponds to $|\omega'|\lesssim\max(|\omega|,T)$. Since we are interested in the low-energy behavior of the self-energy where $|\omega|,|\xi^0_{\mathbf{k}}|,T\ll\epsilon^0_F$, we have $|\omega'|\ll\epsilon^0_F$,

$$\Theta\left(1-\frac{|\omega+\omega'-\xi^0_{\mathbf{k}}-\mathbf{q}^2/2m|}{|\mathbf{k}||\mathbf{q}|/m}\right)\simeq\Theta\left(1-\frac{|\mathbf{q}|}{2k_F}\right), \tag{4.167}$$

and

$$\Im[\Sigma^R(\mathbf{k},\omega)]=-\frac{m}{4\pi^2k_F}\int_0^{2k_F}v^2_{\mathbf{q}}|\mathbf{q}|d|\mathbf{q}|\int_{-\infty}^\infty d\omega'\chi_0''(\mathbf{q},\omega')\left[n_B(\omega')+n_F(\omega'+\omega)\right]. \tag{4.168}$$

For $|\mathbf{q}|\le 2k_F$, $\chi_0''(\mathbf{q},\omega)$ is given by

$$\chi_0''(\mathbf{q},\omega)=\pi N(0)\frac{\omega}{v_F|\mathbf{q}|}=\frac{m^2\omega}{2\pi|\mathbf{q}|}\quad\text{for}\quad|\omega|\le\omega_-=v_F|\mathbf{q}|-\frac{\mathbf{q}^2}{2m} \tag{4.169}$$

(the function $\chi_0''(\mathbf{q},\omega)$ is studied in detail in section 5.3.1). Since $|\omega'|\ll\epsilon^0_F$ in (4.168), we can use the low-energy expression (4.169), which gives[21]

$$\begin{aligned}\Im[\Sigma^R(\mathbf{k},\omega)]&=-\frac{m^3}{8\pi^3k_F}\int_0^{2k_F}d|\mathbf{q}|v^2_{\mathbf{q}}\int_{-\infty}^\infty d\omega'\omega'\left[n_B(\omega')+n_F(\omega'+\omega)\right]\\ &=-\frac{m^3}{16\pi^3k_F}\int_0^{2k_F}d|\mathbf{q}|v^2_{\mathbf{q}}\left(\omega^2+\pi^2T^2\right).\end{aligned} \tag{4.170}$$

From (4.170), we finally obtain the quasi-particle life-time

$$\frac{1}{\tau_{\mathbf{k}}}=-2z_{\mathbf{k}}\Im[\Sigma^R(\mathbf{k},\xi_{\mathbf{k}})]=z_{\mathbf{k}}\frac{m^3}{8\pi^3k_F}\left(\xi^2_{\mathbf{k}}+\pi^2T^2\right)\int_0^{2k_F}d|\mathbf{q}|v^2_{\mathbf{q}}. \tag{4.171}$$

At zero temperature, $1/\tau_{\mathbf{k}}\sim(|\mathbf{k}|-k_F)^2$ when approaching the Fermi surface.

[21] The frequency integral is done using $\int_{-\infty}^\infty dy\frac{y-x}{(1-e^{y-x})(1+e^{-y})}=\frac{1}{2}\frac{x^2+\pi^2}{1+e^{-x}}$.

Similarly, one also could calculate higher-order self-energy diagrams. One would find that they give weaker contributions (i.e. $\sim \omega^n, T^n$ with $n > 2$) than the second-order one. The reason is that the phase space available for multi-pair excitations (i.e. the density per unit energy of the multi-pair excitations) is very small at low energies (Sec. 4.3.5).

Consistency of the Fermi-liquid picture

The result obtained above,

$$\Im[\Sigma^R(k_F,\omega)] = -\gamma\omega^2 + \mathcal{O}(\omega^4) \tag{4.172}$$

($T = 0$) with γ a positive constant, can also be derived from the assumption that $\Sigma^R(\mathbf{k},\omega)$ is an analytic function of ω near $\omega = 0$. Since $\Im[\Sigma^R(k_F,\omega)] \leq 0$ (as required by the analyticity of $G^R(\mathbf{k},\omega)$ in the upper half complex plane), the additional condition $\Im[\Sigma^R(k_F,0)] = 0$ is sufficient to obtain (4.172). The Kramers-Kronig relations then give

$$\begin{aligned}
\lim_{\omega\to 0} & \Re[\Sigma^R(k_F,\omega)] - \Re[\Sigma^R(k_F,\infty)] \\
&= \lim_{\omega\to 0} \mathcal{P}\int_{-\infty}^{\infty} \frac{d\omega'}{\pi} \frac{\Im[\Sigma^R(k_F,\omega')]}{\omega'-\omega} \\
&= \mathcal{P}\int_{-\infty}^{\infty} \frac{d\omega'}{\pi} \frac{\Im[\Sigma^R(\mathbf{k},\omega')]}{\omega'} + \omega\mathcal{P}\int_{-\infty}^{\infty} \frac{d\omega'}{\pi} \frac{\Im[\Sigma^R(\mathbf{k},\omega')]}{\omega'^2} + \mathcal{O}(\omega^2)
\end{aligned} \tag{4.173}$$

(see Sec. 3.2.4), i.e.

$$\frac{\partial}{\partial\omega}\Re[\Sigma^R(k_F,\omega)]\Big|_{\omega=0} = \mathcal{P}\int_{-\infty}^{\infty} \frac{d\omega'}{\pi} \frac{\Im[\Sigma^R(k_F,\omega')]}{\omega'^2}. \tag{4.174}$$

Using (4.172), one sees that the integral in (4.174) exists (assuming that the high-energy part does not cause any trouble) and is negative, in agreement with the requirement $0 < z_{\mathbf{k}} \leq 1$ [Eq. (4.133)]. Thus, the assumption (4.172) appears to be consistent with the existence of quasi-particles.

The quasi-particle picture in the Matsubara formalism

In the Matsubara formalism, the quasi-particle propagator naturally emerges if one assumes that $\Sigma(\mathbf{k},z)$ is analytic near $z = 0$,

$$\Sigma(\mathbf{k},i\omega_n) = \Sigma(\mathbf{k},0) + i\omega_n \frac{\partial\Sigma(\mathbf{k},z)}{\partial z}\Big|_{z=0} + \mathcal{O}(\omega_n^2). \tag{4.175}$$

Making use of (4.175) and keeping track of the incoherent part of the Green function, we obtain

$$G(\mathbf{k},i\omega_n) = \frac{z_{\mathbf{k}}}{i\omega_n - \xi_{\mathbf{k}}} + G_{\rm inc}(\mathbf{k},i\omega_n) \equiv z_{\mathbf{k}}\bar G(\mathbf{k},i\omega_n) + G_{\rm inc}(\mathbf{k},i\omega_n), \tag{4.176}$$

where

$$z_{\mathbf{k}} = \left(1 - \frac{\partial\Sigma(\mathbf{k},z)}{\partial z}\Big|_{z=0}\right)^{-1}, \qquad \xi_{\mathbf{k}} = z_{\mathbf{k}}[\xi^0_{\mathbf{k}} + \Sigma(\mathbf{k},0)]. \tag{4.177}$$

These two equations can also be deduced from (4.141) and (4.134) by expanding $\Sigma^R(\mathbf{k},\xi_\mathbf{k})$ about $\Sigma^R(\mathbf{k},0)=\Sigma(\mathbf{k},z=0)$. The finite quasi-particle life-time is not taken into account in the simple expansion (4.175). For many purposes, however, this expression is sufficient at low energies. In the following we will often approximate the quasi-particle weight $z_\mathbf{k}$ by its value at the Fermi level $z_{k_F}\equiv z$.

4.4.2 Thermodynamic potential $\Omega[n]$

In this section, we derive the thermodynamic potential $\Omega[n]$ introduced in Landau Fermi-liquid theory (Sec. 4.1.1) and obtain a microscopic definition of the Landau f function. We then relate f to the particle-hole vertex $\Gamma_{\rm ph}$.

Microscopic definitions of $\Omega[n]$ and the Landau f function

We consider the partition function

$$Z[h]=\int\mathcal{D}[\psi^*,\psi]\exp\Bigl\{-S[\psi^*,\psi]+\sum_{\mathbf{k},\sigma}h_{\mathbf{k}\sigma}\int_0^\beta d\tau\hat n_{\mathbf{k}\sigma}(\tau)\Bigr\} \tag{4.178}$$

in the presence of a static external field that couples to the quasi-particle number operator

$$\hat n_{\mathbf{k}\sigma}(\tau)=\bar\psi^*_\sigma(\mathbf{k},\tau)\bar\psi_\sigma(\mathbf{k},\tau). \tag{4.179}$$

Note that $\hat n_{\mathbf{k}\sigma}$ is defined as a function of the quasi-particle field $\bar\psi$ which differs from the (bare) fermion field ψ. The quasi-particle occupation number is then obtained from a functional derivative of the partition function,

$$n_{\mathbf{k}\sigma}=\langle\hat n_{\mathbf{k}\sigma}(\tau)\rangle=\frac{1}{\beta}\frac{\delta\ln Z[h]}{\delta h_{\mathbf{k}\sigma}}. \tag{4.180}$$

In order to write the grand potential $\Omega[n]$ as a function of a the quasi-particle distribution function $n\equiv\{n_{\mathbf{k}\sigma}\}$, we consider the Legendre transform

$$\Omega[n]=-\frac{1}{\beta}\ln Z[h]+\sum_{\mathbf{k},\sigma}h_{\mathbf{k}\sigma}n_{\mathbf{k}\sigma}, \tag{4.181}$$

where $h_{\mathbf{k}\sigma}\equiv h_{\mathbf{k}\sigma}[n]$ is obtained by inverting (4.180). $\Omega[n]$ satisfies the equation of state

$$\frac{\delta\Omega[n]}{\delta n_{\mathbf{k}\sigma}}=h_{\mathbf{k}\sigma}. \tag{4.182}$$

At equilibrium ($h=0$), it is stationary with respect to variations of the quasi-particle distribution.

For a non-interaction system, the calculation of $\Omega[n]$ is straightforward. In that case, the quasi-particles coincide with the bare fermions ($\bar\psi=\psi$) so that

$$\begin{aligned}Z[h]&=\int\mathcal{D}[\psi^*,\psi]\exp\Bigl\{\sum_{\mathbf{k},\sigma,\omega_n}\psi^*_\sigma(\mathbf{k},i\omega_n)(i\omega_n-\xi_\mathbf{k}+h_{\mathbf{k}\sigma})\psi_\sigma(\mathbf{k},i\omega_n)\Bigr\}\\&=\prod_{\mathbf{k},\sigma}\Bigl[1+e^{-\beta(\xi_\mathbf{k}-h_{\mathbf{k}\sigma})}\Bigr].\end{aligned} \tag{4.183}$$

This gives $n_{\mathbf{k}\sigma} = n_F(\xi_{\mathbf{k}} - h_{\mathbf{k}\sigma})$ and

$$\Omega[n] = \sum_{\mathbf{k},\sigma} \xi_{\mathbf{k}} n_{\mathbf{k}\sigma} + \frac{1}{\beta} \sum_{\mathbf{k},\sigma} \left[n_{\mathbf{k}\sigma} \ln n_{\mathbf{k}\sigma} + (1 - n_{\mathbf{k}\sigma}) \ln(1 - n_{\mathbf{k}\sigma}) \right], \tag{4.184}$$

which is the expected result for non-interacting fermions.

For interacting fermions, it is not possible to calculate exactly the grand potential. However, we do need require the whole knowledge of $\Omega[n]$, but only its variation $\delta\Omega$ when the quasi-particle distribution n varies from its equilibrium value $\bar n = n|_{h=0}$ by δn. For $T \to 0$, the case we are interested in, $\bar n_{\mathbf{k}\sigma} = \Theta(k_F - |\mathbf{k}|)$. Expanding $\delta\Omega[\delta n] = \Omega[n + \delta n] - \Omega[n]$ to second-order in δn, we obtain

$$\delta\Omega[\delta n] = \frac{1}{2} \sum_{\mathbf{k},\mathbf{k}',\sigma,\sigma'} \frac{\delta^{(2)}\Omega[n]}{\delta n_{\mathbf{k}\sigma} \delta n_{\mathbf{k}'\sigma'}} \bigg|_{n=\bar n} \delta n_{\mathbf{k}\sigma} \delta n_{\mathbf{k}'\sigma'}. \tag{4.185}$$

There is no linear term since we expand about the stationary (equilibrium) state. Taking the functional derivative of the equation of state (4.182), one easily obtains

$$\frac{1}{\beta} \sum_{\mathbf{k}_3,\sigma_3} \frac{\delta^{(2)}\Omega[n]}{\delta n_{\mathbf{k}_1\sigma_1} \delta n_{\mathbf{k}_3\sigma_3}} \frac{\delta^{(2)} \ln Z[h]}{\delta h_{\mathbf{k}_3\sigma_3} \delta h_{\mathbf{k}_2\sigma_2}} = \delta_{\mathbf{k}_1,\mathbf{k}_2} \delta_{\sigma_1,\sigma_2}. \tag{4.186}$$

This allows us to rewrite $\delta\Omega$ as

$$\delta\Omega[\delta n] = \frac{1}{2} \sum_{\mathbf{k},\mathbf{k}',\sigma,\sigma'} \bar\chi^{-1}_{\sigma\sigma'}(\mathbf{k},\mathbf{k}') \delta n_{\mathbf{k}\sigma} \delta n_{\mathbf{k}'\sigma'}, \tag{4.187}$$

where $\bar\chi^{-1}$ is the inverse (in a matrix sense) of the correlation function

$$\bar\chi_{\sigma\sigma'}(\mathbf{k},\mathbf{k}') = \frac{1}{\beta} \frac{\delta^{(2)} \ln Z[h]}{\delta h_{\mathbf{k}\sigma} \delta h_{\mathbf{k}'\sigma'}} \bigg|_{h=0} = \frac{1}{\beta} \int_0^\beta d\tau d\tau' \langle \hat n_{\mathbf{k}\sigma}(\tau) \hat n_{\mathbf{k}'\sigma'}(\tau') \rangle. \tag{4.188}$$

Note that $\bar\chi$ is nothing but the linear response function to the external field h. Comparing (4.187) and (4.28), we obtain the following microscopic definition of the Landau f function,

$$\frac{1}{V} f_{\sigma\sigma'}(\mathbf{k},\mathbf{k}') = \frac{\delta_{\sigma,\sigma'} \delta_{\mathbf{k},\mathbf{k}'}}{n'_F(\xi_{\mathbf{k}})} + \bar\chi^{-1}_{\sigma\sigma'}(\mathbf{k},\mathbf{k}'). \tag{4.189}$$

Since we are interested in the limit $T \to 0$, we have taken $\tilde\xi_{\mathbf{k}} = \xi_{\mathbf{k}}$ in (4.28).

Thus the calculation of the Landau f function reduces to that of the response function $\bar\chi$. In the following, we show that f can be identified to the irreducible (2PI) quasi-particle–quasi-hole vertex. This will enable us to related f to the particle-hole vertex.

Relation between f and the particle-hole vertex $\Gamma_{\rm ph}$

Let us introduce the correlation functions

$$\begin{aligned} \bar\chi_{\sigma\sigma'}(k,k';q) &= \langle \bar\psi^*_\sigma(k) \bar\psi_\sigma(k+q) \bar\psi^*_{\sigma'}(k'+q) \bar\psi_{\sigma'}(k') \rangle \\ &\quad - \langle \bar\psi^*_\sigma(k) \bar\psi_\sigma(k+q) \rangle \langle \bar\psi^*_{\sigma'}(k'+q) \bar\psi_{\sigma'}(k') \rangle, \\ \bar\chi_{\sigma\sigma'}(\mathbf{k},\mathbf{k}';q) &= \frac{1}{\beta} \sum_{\omega_n,\omega_{n'}} \bar\chi_{\sigma\sigma'}(k,k';q). \end{aligned} \tag{4.190}$$

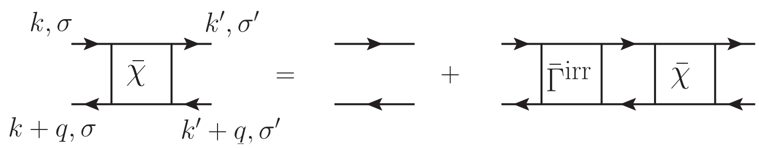

Figure 4.12: Diagrammatic representation of the Bethe-Salpeter equation satisfied by $\bar\chi_{\sigma\sigma'}(k,k';q)$.

The function $\bar\chi_{\sigma\sigma'}(\mathbf{k},\mathbf{k}')$, which is related to the Landau f function by (4.189), corresponds to the $q\to 0$ limit of $\bar\chi_{\sigma\sigma'}(\mathbf{k},\mathbf{k}';q)$. We shall see below that this limit is ill defined, since the limits $\mathbf{q}\to 0$ and $i\omega_\nu\to 0$ do not commute. At zero temperature, it is not possible to create a quasi-particle–quasi-hole pair with a finite total energy and a vanishing total momentum because of the Pauli principle.[22] Thus for the external field $h_{\mathbf{k}\sigma}$ to modify the quasi-particle distribution function, it should be understood as

$$\begin{aligned} h_{\mathbf{k}\sigma}\int_0^\beta d\tau \bar\psi^*_\sigma(\mathbf{k},\tau)\bar\psi_\sigma(\mathbf{k},\tau) &= h_{\mathbf{k}\sigma}\sum_{\omega_n}\bar\psi^*_\sigma(k)\bar\psi_\sigma(k) \\ &\equiv h_{\mathbf{k}\sigma}\lim_{\mathbf{q}\to 0}\biggl[\lim_{\omega_\nu\to 0}\sum_{\omega_n}\bar\psi^*_\sigma(k)\bar\psi_\sigma(k+q)\biggr]. \end{aligned} \tag{4.191}$$

This leads us to define $\bar\chi_{\sigma\sigma'}(\mathbf{k},\mathbf{k}')$ as

$$\bar\chi_{\sigma\sigma'}(\mathbf{k},\mathbf{k}') = \lim_{\mathbf{q}\to 0}\Bigl[\lim_{\omega_\nu\to 0}\bar\chi_{\sigma\sigma'}(\mathbf{k},\mathbf{k}';q)\Bigr]. \tag{4.192}$$

It is customary to refer to the limits $(i\omega_\nu\to 0,\mathbf{q}=0)$ and $(i\omega_\nu=0,\mathbf{q}\to 0)$ as the ω- and $\mathbf{q}$-limits, respectively.

The two-particle Green function $\bar\chi_{\sigma\sigma'}(k,k';q)$ satisfies the Bethe-Salpeter equation (Fig. 4.12)

$$\begin{aligned} \bar\chi_{\sigma\sigma'}(k,k';q) &= \bar\Pi_{\sigma\sigma'}(k,k';q) \\ &\quad -\frac{1}{\beta V}\sum_{k_1,k_2,\sigma_1,\sigma_2}\bar\Pi_{\sigma\sigma_1}(k,k_1;q)\bar\Gamma^{\rm irr}_{\sigma_1\sigma_2}(k_1,k_2;q)\bar\chi_{\sigma_2\sigma'}(k_2,k';q) \end{aligned} \tag{4.193}$$

where

$$\bar\Gamma^{\rm irr}_{\sigma\sigma'}(k,k';q) = \bar\Gamma^{\rm irr}_{{\rm ph},\sigma\sigma\sigma'\sigma'}(k+q,k;k',k'+q) \tag{4.194}$$

is the irreducible (2PI) vertex in the particle-hole channel, and

$$\bar\Pi_{\sigma\sigma'}(k,k';q) = -\delta_{\sigma,\sigma'}\delta_{k,k'}\bar G(k)\bar G(k+q) \tag{4.195}$$

the quasi-particle–quasi-hole pair propagator. The quasi-particle propagator $\bar G$ is defined by (4.176). In the following we approximate $z_{\mathbf{k}}$ by its value $z_{k_F}\equiv z$ at the Fermi level.

[22]Technically, this means that the equation $\omega=\epsilon_{\mathbf{k}+\mathbf{q}}-\epsilon_{\mathbf{k}}$, together with the constraints $|\mathbf{k}|<k_F$ and $|\mathbf{k}+\mathbf{q}|>k_F$, has no solution for ω finite and $\mathbf{q}\to 0$.

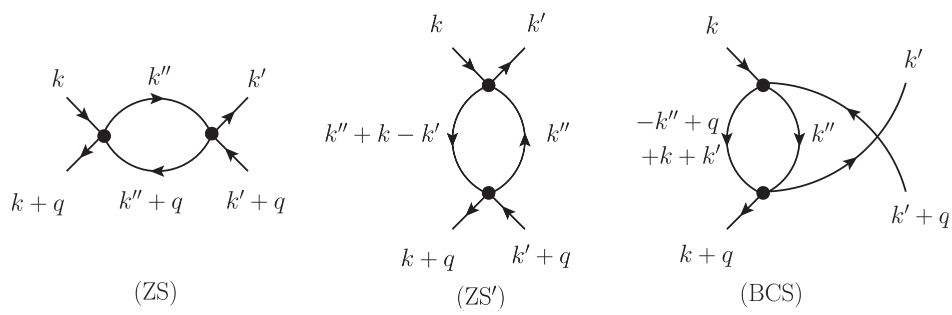

Figure 4.13: First-order (one-loop) corrections to the response function $\bar\chi$ (or, without the external legs, the particle-hole vertex $\bar\Gamma$). The line denotes the quasi-particle propagator $\bar G$ and the dot the interaction.

The quasi-particle–quasi-hole propagator $\bar\Pi$ is singular in the limit $q \to 0$. To see this, we consider

$$\begin{aligned}\bar\Pi_{\sigma\sigma'}(\mathbf{k},\mathbf{k}';q) &= \frac{1}{\beta}\sum_{\omega_n,\omega_{n'}} \bar\Pi_{\sigma\sigma'}(k,k';q) = -\delta_{\sigma,\sigma'}\delta_{k,k'}\frac{1}{\beta}\sum_{\omega_n}\bar G(k)\bar G(k+q) \\ &= \delta_{\sigma,\sigma'}\delta_{k,k'}\frac{n_F(\xi_{\mathbf{k+q}})-n_F(\xi_{\mathbf{k}})}{i\omega_\nu-\xi_{\mathbf{k+q}}+\xi_{\mathbf{k}}} \\ &\simeq -\delta_{\sigma,\sigma'}\delta_{k,k'}\frac{\mathbf{v}^*_{\mathbf{k}}\cdot\mathbf{q}}{i\omega_\nu-\mathbf{v}^*_{\mathbf{k}}\cdot\mathbf{q}}\delta(\xi_{\mathbf{k}}) \end{aligned} \tag{4.196}$$

in the limit $\mathbf{q}\to 0$ and $T\to 0$. Thus for $q\to 0$, we obtain 0 if $|\mathbf{q}|/\omega_\nu \to 0$, and $\delta_{\sigma,\sigma'}\delta_{k,k'}\delta(\xi_{\mathbf{k}})$ if $\omega_\nu/|\mathbf{q}|\to 0$. In Fermi-liquid theory, one assumes that only the quasi-particle–quasi-hole pair propagator $\bar\Pi$ leads to singularities when $q\to 0$. This can be checked in perturbation theory, at least to lowest order. The one-loop corrections to $\bar\chi$ (or $\bar\Gamma$) are shown in figure 4.13. The Feynman diagrams are labeled according to the type of fluctuations they describe. The zero-sound (ZS) channel corresponds to propagation of particle-hole pairs with a small total frequency-momentum. The crossed particle-hole channel is referred to as (ZS'), while the BCS channel involves particle-particle pair propagation. One readily sees that only the zero-sound channel leads to a singularity in the $q\to 0$ limit; the ZS' and BCS loops (after summation over the internal frequency-momentum) are not singular in that limit.[23]

Since the irreducible vertex $\bar\Gamma^{\rm irr}$ does not contain the ZS loop (the latter being two-particle reducible), it has a well-defined limit when $q\to 0$. We can therefore set $q=0$ in $\bar\Gamma^{\rm irr}$. Furthermore, since the singularity of $\bar\Pi$ is due to states near the Fermi surface [Eq. (4.196)], we can ignore the $|\mathbf{k}|$ and ω_n dependence of $\bar\Gamma^{\rm irr}$ which then becomes a function $\bar\Gamma^{\rm irr}_{\sigma\sigma'}(\mathbf{k}_F,\mathbf{k}'_F)$ of $\mathbf{k}_F = k_F\hat{\mathbf{k}}$ and $\mathbf{k}'_F = k_F\hat{\mathbf{k}}'$. This enables us to carry out the frequency sum in the equation (4.193) satisfied by $\bar\chi_{\sigma\sigma'}(k,k';q)$, so that

[23]For $\mathbf{k}\to\mathbf{k}'$, the ZS' channel also becomes singular since the total frequency-momentum $\mathbf{k}-\mathbf{k}'$ in the ZS' loop vanishes (Fig. 4.13). For $\mathbf{k}\to\mathbf{k}'$, both the ZS and ZS' channels should therefore be taken into account. In particular, this is necessary for a correct description of the Pauli principle. But for most physical properties, the ZS' channel can be safely discarded. This issue is discussed in detail in Ref. [13].

we eventually obtain

$$\bar{\chi}_{\sigma\sigma'}(\mathbf{k},\mathbf{k}';q) = \bar{\Pi}_{\sigma\sigma'}(\mathbf{k},\mathbf{k}';q) - \frac{1}{V}\sum_{\mathbf{k}_1,\mathbf{k}_2,\sigma_1,\sigma_2} \bar{\Pi}_{\sigma\sigma_1}(\mathbf{k},\mathbf{k}_1;q)\bar{\Gamma}^{\rm irr}_{\sigma_1\sigma_2}(\mathbf{k}_{F1},\mathbf{k}_{F2})\bar{\chi}_{\sigma_2\sigma'}(\mathbf{k}_2,\mathbf{k}';q). \tag{4.197}$$

This equation can be rewritten in a matrix form as

$$\bar{\chi} = \bar{\Pi} - \frac{1}{V}\bar{\Pi}\bar{\Gamma}^{\rm irr}\bar{\chi}, \quad \text{i.e.} \quad \bar{\chi}^{-1} = \bar{\Pi}^{-1} + \frac{1}{V}\bar{\Gamma}^{\rm irr}. \tag{4.198}$$

In the limit $\mathbf{q}$-limit, this gives

$$\bar{\chi}^{-1}_{\sigma\sigma'}(\mathbf{k},\mathbf{k}') = -\frac{\delta_{\sigma,\sigma'}\delta_{\mathbf{k},\mathbf{k}'}}{n'_F(\xi_{\mathbf{k}})} + \frac{1}{V}\bar{\Gamma}^{\rm irr}_{\sigma\sigma'}(\mathbf{k}_F,\mathbf{k}'_F). \tag{4.199}$$

Comparing this result with (4.189), we deduce

$$f_{\sigma\sigma'}(\mathbf{k},\mathbf{k}') = \bar{\Gamma}^{\rm irr}_{\sigma\sigma'}(\mathbf{k}_F,\mathbf{k}'_F). \tag{4.200}$$

One can proceed one step further by relating the irreducible (2PI) vertex $\bar{\Gamma}^{\rm irr}$ to the full (1PI) vertex $\bar{\Gamma} \equiv \bar{\Gamma}_{\rm ph}$. The latter satisfies the Bethe-Salpeter equation

$$\bar{\Gamma}_{\sigma\sigma'}(k,k';q) = \bar{\Gamma}^{\rm irr}_{\sigma\sigma'}(k,k') - \frac{1}{\beta V}\sum_{k_1,k_2,\sigma_1,\sigma_2} \bar{\Gamma}^{\rm irr}_{\sigma\sigma_1}(k,k_1)\bar{\Pi}_{\sigma_1\sigma_2}(k_1,k_2;q)\bar{\Gamma}_{\sigma_2\sigma'}(k_2,k';q), \tag{4.201}$$

where again we set $q=0$ in $\bar{\Gamma}^{\rm irr}$. Since the frequency dependence of $\bar{\Gamma}$, as that of $\bar{\Gamma}^{\rm irr}$, can be neglected, we can carry out the frequency sum,

$$\bar{\Gamma}_{\sigma\sigma'}(\mathbf{k},\mathbf{k}';q) = \bar{\Gamma}^{\rm irr}_{\sigma\sigma'}(\mathbf{k},\mathbf{k}') - \frac{1}{V}\sum_{\mathbf{k}_1,\mathbf{k}_2,\sigma_1,\sigma_2} \bar{\Gamma}^{\rm irr}_{\sigma\sigma_1}(\mathbf{k},\mathbf{k}_1)\bar{\Pi}_{\sigma_1\sigma_2}(\mathbf{k}_1,\mathbf{k}_2;q)\bar{\Gamma}_{\sigma_2\sigma'}(\mathbf{k}_2,\mathbf{k}';q). \tag{4.202}$$

$\bar{\Pi}$ vanishing in the ω-limit,

$$\bar{\Gamma}^{\rm irr}_{\sigma\sigma'}(\mathbf{k}_F,\mathbf{k}'_F) = \lim_{\omega_\nu\to 0}\left[\lim_{\mathbf{q}\to 0}\bar{\Gamma}_{\sigma\sigma'}(\mathbf{k}_F,\mathbf{k}'_F;q)\right] \equiv \bar{\Gamma}^{\omega}_{\sigma\sigma'}(\mathbf{k}_F,\mathbf{k}'_F). \tag{4.203}$$

Equations (4.200) and (4.203) relate the Landau f function to $\bar{\Gamma}^\omega$.

The final step is to relate the quasi-particle–quasi-hole vertex $\bar{\Gamma}^\omega$ to the particle-hole vertex Γ. Let us introduce the correlation function

$$\chi_{\sigma\sigma'}(k,k';q) = \langle \psi^*_\sigma(k)\psi_\sigma(k+q)\psi^*_{\sigma'}(k'+q)\psi_{\sigma'}(k')\rangle - \langle \psi^*_\sigma(k)\psi_\sigma(k+q)\rangle\langle \psi^*_{\sigma'}(k'+q)\psi_{\sigma'}(k')\rangle. \tag{4.204}$$

χ satisfies the equation

$$\chi_{\sigma\sigma'}(k,k';q) = \Pi_{\sigma\sigma'}(k,k';q) - \frac{1}{\beta V}\sum_{k_1,k_2,\sigma_1,\sigma_2} \Pi_{\sigma\sigma_1}(k,k_1;q)\Gamma_{\sigma_1\sigma_2}(k_1,k_2;q)\Pi_{\sigma_2\sigma'}(k_2,k';q), \tag{4.205}$$

where

$$\begin{aligned}\Pi_{\sigma\sigma'}(k,k';q) &= -\delta_{\sigma,\sigma'}\delta_{k,k'}G(k)G(k+q)\\ &= -\delta_{\sigma,\sigma'}\delta_{k,k'}\left[z^2\bar G(k)\bar G(k+q)+\varphi(k)\right]\end{aligned}\tag{4.206}$$

is the particle-hole pair propagator. We have separated the coherent part $z^2\bar G\bar G = z^2\bar\Pi$ from the incoherent one (φ) using (4.176). Since φ is non-singular in the $q\to 0$ limit, it is evaluated at $q=0$. Retaining only the quasi-particle (coherent) part in (4.205), we obtain

$$\chi\big|_{\rm coh} = z^2\bar\Pi - z^2\bar\Pi\Gamma z^2\bar\Pi \tag{4.207}$$

(with matrix notations). Since $\bar\chi = \bar\Pi - \bar\Pi\bar\Gamma\bar\Pi$, we conclude that $\chi\big|_{\rm coh} = z^2\bar\chi$ with $\bar\Gamma = z^2\Gamma$. This yields our final expression for the Landau f function,

$$f_{\sigma\sigma'}(\mathbf{k},\mathbf{k}') = z^2\Gamma^\omega_{\sigma\sigma'}(\mathbf{k}_F,\mathbf{k}'_F). \tag{4.208}$$

4.4.3 Quantum Boltzmann equation

The preceding analysis can easily be extended to non-equilibrium cases. Instead of the quasi-particle distribution function we should consider the quasi-particle Wigner distribution function

$$n_{\mathbf{k}\sigma}(\mathbf{r},\tau) = \int d^3r' e^{-i\mathbf{k}\cdot\mathbf{r}'}\left\langle \bar\psi^*_\sigma\Big(\mathbf{r}-\frac{\mathbf{r}'}{2},\tau\Big)\bar\psi_\sigma\Big(\mathbf{r}+\frac{\mathbf{r}'}{2},\tau\Big)\right\rangle \tag{4.209}$$

and its Fourier transform

$$n_{\mathbf{k}\sigma}(q) = \frac{1}{\beta V}\int_0^\beta d\tau\int d^3r\, e^{-i(\mathbf{q}\cdot\mathbf{r}-\omega_\nu\tau)}n_{\mathbf{k}\sigma}(\mathbf{r},\tau) = \frac{1}{\beta}\sum_{\omega_n}\langle\bar\psi^*_\sigma(k)\bar\psi_\sigma(k+q)\rangle. \tag{4.210}$$

The Wigner distribution function is the quantum analog of the semiclassical distribution function considered in the phenomenological Fermi-liquid theory (Sec. 4.3). Although it is not positive definite and therefore not a true distribution function, as far as its moments are concerned it generally behaves similarly to a distribution function [18, 19].

We consider the system in the presence of a source field $h_{\mathbf{k}\sigma}(q) = h^*_{\mathbf{k}\sigma}(-q)$ that couples to the quasi-particle operator

$$\hat n_{\mathbf{k}\sigma}(q) = \frac{1}{\beta}\sum_{\omega_n}\bar\psi^*_\sigma(k)\bar\psi_\sigma(k+q). \tag{4.211}$$

The source term in the action reads

$$S_h = -\beta\sum_{\mathbf{k},\sigma,q}h_{\mathbf{k}\sigma}(-q)\hat n_{\mathbf{k}\sigma}(q) \tag{4.212}$$

and the Wigner distribution function is given by

$$n_{\mathbf{k}\sigma}(q) = \langle\hat n_{\mathbf{k}\sigma}(q)\rangle = \frac{1}{\beta}\frac{\delta\ln Z[h]}{\delta h_{\mathbf{k}\sigma}(-q)}. \tag{4.213}$$

We are now in a position to introduce a functional of the Wigner distribution function – analog to the grand potential $\Omega[n]$ in the equilibrium case – by means of a Legendre transform,

$$\Omega[n] = -\frac{1}{\beta}\ln Z[h] + \sum_{\mathbf{k},\sigma,q} h_{\mathbf{k}\sigma}(-q) n_{\mathbf{k}\sigma}(q). \tag{4.214}$$

The "equation of state" reads

$$\frac{\delta\Omega[n]}{\delta n_{\mathbf{k}\sigma}(q)} = h_{\mathbf{k}\sigma}(-q). \tag{4.215}$$

Even for non-interacting fermions, $\Omega[n]$ cannot be calculated exactly. We shall therefore consider only small fluctuations about the equilibrium state,

$$n_{\mathbf{k}\sigma}(q) = \delta_{q,0}\bar{n}_{\mathbf{k}} + \delta n_{\mathbf{k}\sigma}(q). \tag{4.216}$$

To lowest order in δn,

$$\delta\Omega[\delta n] = \frac{1}{2}\sum_{\mathbf{k},\mathbf{k}',\sigma,\sigma',q,q'} \frac{\delta^{(2)}\Omega[n]}{\delta n_{\mathbf{k}\sigma}(-q)\delta n_{\mathbf{k}'\sigma'}(q')}\bigg|_{n=\bar{n}} \delta n_{\mathbf{k}\sigma}(-q)\delta n_{\mathbf{k}'\sigma'}(q'). \tag{4.217}$$

Equation (4.186) can easily be generalized into

$$\frac{\delta^{(2)}\Omega[n]}{\delta n_{\mathbf{k}\sigma}(-q)\delta n_{\mathbf{k}'\sigma'}(q')}\bigg|_{n=\bar{n}} = \delta_{q,q'}\bar{\chi}^{-1}_{\sigma\sigma'}(\mathbf{k},\mathbf{k}';q), \tag{4.218}$$

where $\bar{\chi}_{\sigma\sigma'}(\mathbf{k},\mathbf{k}';q)$ is the correlation function defined in (4.190). The Kronecker symbol in (4.218) results from translation invariance. Making use of (4.196,4.198,4.200) we find

$$\begin{aligned}
\delta\Omega[\delta n] &= \frac{1}{2}\sum_{\mathbf{k},\mathbf{k}',\sigma,\sigma',q} \bar{\chi}^{-1}_{\sigma\sigma'}(\mathbf{k},\mathbf{k}';q)\delta n_{\mathbf{k}\sigma}(-q)\delta n_{\mathbf{k}'\sigma'}(q) \\
&= \frac{1}{2}\sum_{\mathbf{k},\mathbf{k}',\sigma,\sigma',q} \bigg\{ \frac{\delta_{\sigma,\sigma'}\delta_{\mathbf{k},\mathbf{k}'}}{n'_F(\xi_{\mathbf{k}})} \frac{i\omega_\nu - \mathbf{v}^*_{\mathbf{k}}\cdot\mathbf{q}}{\mathbf{v}^*_{\mathbf{k}}\cdot\mathbf{q}} \\
&\qquad + \frac{1}{V} f_{\sigma\sigma'}(\mathbf{k},\mathbf{k}')\bigg\} \delta n_{\mathbf{k}\sigma}(-q)\delta n_{\mathbf{k}'\sigma'}(q).
\end{aligned} \tag{4.219}$$

In the absence of a source field, the stationary condition (4.215) gives the quantum Boltzmann equation

$$(i\omega_\nu - \mathbf{v}^*_{\mathbf{k}}\cdot\mathbf{q})\delta n_{\mathbf{k}\sigma}(q) - \delta(\xi_{\mathbf{k}})\mathbf{v}^*_{\mathbf{k}}\cdot\mathbf{q}\frac{1}{V}\sum_{\mathbf{k}',\sigma'} f_{\sigma\sigma'}(\mathbf{k},\mathbf{k}')\delta n_{\mathbf{k}'\sigma'}(q) = 0 \tag{4.220}$$

satisfied by the Wigner distribution function $n_{\mathbf{k}\sigma}(q)$. (As usual, the real-time formalism can be recovered from the analytic continuation $i\omega_\nu \to \omega + i\eta$.) This equation is identical to the semiclassical kinetic equation obtained in section 4.3. Its solution can be written as

$$\delta n_{\mathbf{k}\sigma}(q) = v^*_F\delta(\xi_{\mathbf{k}})u_\sigma(\hat{\mathbf{k}},q) \tag{4.221}$$

where $u_\sigma(\hat{\mathbf{k}}, q)$ is the displacement of the Fermi surface in the direction $\hat{\mathbf{k}}$. The functional $\delta\Omega$ can be expressed in terms of u,

$$\begin{aligned}\delta\Omega[u] = \frac{V}{2} N^*(0) {v_F^*}^2 \sum_{q,\sigma,\sigma'} \Biggl\{ \delta_{\sigma,\sigma'} \int \frac{d\Omega_{\hat{\mathbf{k}}}}{4\pi} \frac{v_F^* \hat{\mathbf{k}} \cdot \mathbf{q}}{v_F^* \mathbf{k} \cdot \mathbf{q} - i\omega_\nu} u_\sigma(\hat{\mathbf{k}}, -q) u_\sigma(\hat{\mathbf{k}}, q) \\ + \frac{1}{2} \int \frac{d\Omega_{\hat{\mathbf{k}}}}{4\pi} \frac{d\Omega_{\hat{\mathbf{k}}'}}{4\pi} F_{\sigma\sigma'}(\mathbf{k}_F, \mathbf{k}'_F) u_\sigma(\hat{\mathbf{k}}, -q) u_{\sigma'}(\hat{\mathbf{k}}', q) \Biggr\}. \quad (4.222)\end{aligned}$$

This equation generalizes (4.30) – to which it reduces in the $\mathbf{q}$-limit – to dynamic fluctuations of the Fermi surface. It can be used as the starting point for the calculation of both the static (thermodynamics) and dynamic (collective modes, response functions) properties of the Fermi liquid.

4.4.4 Ward identities for the Fermi liquid

In section 4.4.1, we have seen that the quasi-particle properties – velocity, effective mass, life-time – are related to the self-energy of the single-particle Green function. In section 4.4.2, we have obtained a relation between the Landau f function and the particle-hole vertex Γ. To complete the microscopic description of the Fermi liquid, we should also consider the relations between the self-energy and the particle-hole vertex that follow from the symmetries of the physical system.

Symmetries and their consequences were discussed in chapter 2. Each symmetry of the action implies a set of relations (known as Ward identities) between vertices. Gauge and Galilean invariances imply

$$\begin{aligned}\Sigma(k) - \Sigma(k+q) &= \frac{1}{\beta V} \sum_{k',\sigma'} [G_0^{-1}(k'+q) - G_0^{-1}(k')] \\ &\qquad \times G(k') G(k'+q) \Gamma_{\sigma\sigma'}(k, k'; q), \\ (\mathbf{k}+\mathbf{q})\Sigma(k) - \mathbf{k}\Sigma(k+q) &= \frac{1}{\beta V} \sum_{k',\sigma'} [\mathbf{k}' G_0^{-1}(k'+q) - (\mathbf{k}'+\mathbf{q}) G_0^{-1}(k')] \\ &\qquad \times G(k') G(k'+q) \Gamma_{\sigma\sigma'}(k, k'; q) \quad (4.223)\end{aligned}$$

(see Eqs. (2.170,2.179)). Considering the first identity both in the ω- and $\mathbf{q}$-limits and the second one in the ω-limit, we obtain

$$\frac{\partial \Sigma(k)}{\partial i\omega} = -\frac{1}{\beta V} \sum_{k',\sigma'} \left\{ G(k')^2 \right\}_\omega \Gamma^\omega_{\sigma\sigma'}(k, k'), \quad (4.224)$$

$$\boldsymbol{\nabla}_{\mathbf{k}} \Sigma(k) = \frac{1}{\beta V} \sum_{k',\sigma'} \boldsymbol{\nabla}_{\mathbf{k}'} \xi^0_{\mathbf{k}'} \left\{ G(k')^2 \right\}_{\mathbf{q}} \Gamma^{\mathbf{q}}_{\sigma\sigma'}(k, k') \quad (4.225)$$

$$\mathbf{k} \frac{\partial \Sigma(i\omega)}{\partial i\omega} = -\frac{1}{\beta V} \sum_{k',\sigma'} \mathbf{k}' \left\{ G(k')^2 \right\}_\omega \Gamma^\omega_{\sigma\sigma'}(k, k'), \quad (4.226)$$

where

$$\begin{aligned}\left\{G(k)^2\right\}_\omega &= \lim_{\omega_\nu\to 0}\left[\lim_{\mathbf{q}\to 0} G(k)G(k+q)\right],\\ \left\{G(k)^2\right\}_{\mathbf{q}} &= \lim_{\mathbf{q}\to 0}\left[\lim_{\omega_\nu\to 0} G(k)G(k+q)\right].\end{aligned} \tag{4.227}$$

We consider the $T\to 0$ limit where the Matsubara frequency $i\omega_n \equiv i\omega$ becomes a continuous variable and $\frac{1}{\beta}\sum_{\omega_n} \equiv \int \frac{d\omega}{2\pi}$.

These three Ward identities should be supplemented by the relation between Γ^ω and $\Gamma^{\mathbf{q}}$. The particle-hole vertex satisfies the Bethe-Salpeter equation (4.201) with $\bar\Gamma^{\rm irr}$ and $\bar\Pi$ replaced by $\Gamma^{\rm irr}$ and Π, i.e.

$$\Gamma = \Gamma^{\rm irr} - \Gamma^{\rm irr}\Pi\Gamma \tag{4.228}$$

in matrix form. Writing the particle-hole pair propagator as in (4.206),

$$\Pi = z^2\bar\Pi + \varphi, \tag{4.229}$$

we have

$$\Gamma = \Gamma^{\rm irr} - \Gamma^{\rm irr}(z^2\bar\Pi + \varphi)\Gamma. \tag{4.230}$$

Since the coherent part $z^2\bar\Pi$ of the pair propagator does not contribute in the ω limit,

$$\Gamma^\omega = \Gamma^{\rm irr} - \Gamma^{\rm irr}\varphi\Gamma^\omega, \quad \text{i.e.} \quad \Gamma^\omega = (1+\Gamma^{\rm irr}\varphi)^{-1}\Gamma^{\rm irr}. \tag{4.231}$$

From (4.228,4.231), we deduce $\Gamma = \Gamma^\omega - z^2\Gamma^\omega\bar\Pi\Gamma$, i.e.

$$\begin{aligned}\Gamma_{\sigma\sigma'}(k,k';q) &= \Gamma^\omega_{\sigma\sigma'}(k,k') \\ &\quad - \frac{1}{\beta V}\sum_{k_1,k_2,\sigma_1,\sigma_2} z^2\Gamma^\omega_{\sigma\sigma_1}(k,k_1)\bar\Pi_{\sigma_1\sigma_2}(k_1,k_2;q)\Gamma_{\sigma_2\sigma'}(k_2,k';q)\end{aligned} \tag{4.232}$$

and, taking the $\mathbf{q}$-limit,

$$\begin{aligned}\Gamma^{\mathbf{q}}_{\sigma\sigma'}(k,k') &= \Gamma^\omega_{\sigma\sigma'}(k,k') + \frac{1}{\beta V}\sum_{k'',\sigma''} z^2\Gamma^\omega_{\sigma\sigma''}(k,k'')\left\{\bar G(k'')^2\right\}_{\mathbf{q}}\Gamma^{\mathbf{q}}_{\sigma''\sigma'}(k'',k') \\ &= \Gamma^\omega_{\sigma\sigma'}(k,k') - z^2 N^*(0)\sum_{\sigma''}\int\frac{d\Omega_{\hat{\mathbf{k}}''}}{4\pi}\Gamma^\omega_{\sigma\sigma''}(k,\mathbf{k}''_F)\Gamma^{\mathbf{q}}_{\sigma''\sigma'}(\mathbf{k}''_F,k').\end{aligned} \tag{4.233}$$

To obtain the last line, we have carried out the Matsubara sum and used

$$\frac{1}{\beta}\sum_{\omega_n}\left\{\bar G(k)^2\right\}_{\mathbf{q}} = -\delta(\xi_{\mathbf{k}}), \tag{4.234}$$

and the fact that $\Gamma^{\omega,\mathbf{q}}(k,k')$ depends only on $\mathbf{k}_F$ and $\mathbf{k}'_F$ for $\mathbf{k}$ and $\mathbf{k}'$ near the Fermi surface.

Quasi-particle current and effective mass

We are now in a position to compute the current carried by a quasi-particle and the quasi-particle effective mass. From (4.225) and (4.233), we obtain

$$\begin{aligned} \boldsymbol{\nabla}_{\mathbf{k}}\Sigma(k) &= \frac{1}{\beta V}\sum_{k',\sigma'} \mathbf{v}_{\mathbf{k}'} \left\{G(k')^2\right\}_{\mathbf{q}} \Gamma^{\omega}_{\sigma\sigma'}(k,k') \\ &- \frac{1}{\beta V}\sum_{k',\sigma'} z^2 N^*(0)\mathbf{v}_{\mathbf{k}'} \left\{G(k')^2\right\}_{\mathbf{q}} \sum_{\sigma''}\int \frac{d\Omega_{\hat{\mathbf{k}}''}}{4\pi} \Gamma^{\omega}_{\sigma\sigma''}(k,\mathbf{k}''_F)\Gamma^{\mathbf{q}}_{\sigma''\sigma'}(\mathbf{k}''_F,k'), \end{aligned} \tag{4.235}$$

where $\mathbf{v}_{\mathbf{k}} = \boldsymbol{\nabla}_{\mathbf{k}}\xi^0_{\mathbf{k}} = \mathbf{k}/m$. The second line can be simplified using again (4.225) and (4.229),

$$\begin{aligned} \boldsymbol{\nabla}_{\mathbf{k}}\Sigma(k) = \frac{1}{\beta V}\sum_{k',\sigma'} \mathbf{v}_{\mathbf{k}'} \left[\left\{G(k')^2\right\}_{\omega} + z^2\left\{\bar{G}(k')^2\right\}_{\mathbf{q}}\right] \Gamma^{\omega}_{\sigma\sigma'}(k,k') \\ - z^2 N^*(0)\sum_{\sigma'}\int \frac{d\Omega_{\hat{\mathbf{k}}'}}{4\pi}\Gamma^{\omega}_{\sigma\sigma'}(k,\mathbf{k}'_F)\nabla_{\mathbf{k}'}\Sigma(k')|_{|\mathbf{k}'|=k_F}. \end{aligned} \tag{4.236}$$

The first term in the rhs simplifies using the Ward identity (4.226), which leads to

$$\begin{aligned} \boldsymbol{\nabla}_{\mathbf{k}}\Sigma(k) = &-\frac{\mathbf{k}}{m}\frac{\partial\Sigma(k)}{\partial i\omega} \\ &- z^2 N^*(0)\sum_{\sigma'}\int\frac{d\Omega_{\hat{\mathbf{k}}'}}{4\pi}\left[\mathbf{v}_{\mathbf{k}'_F} + \boldsymbol{\nabla}_{\mathbf{k}'}\Sigma(k')|_{|\mathbf{k}'|=\mathbf{k}'_F}\right]\Gamma^{\omega}_{\sigma\sigma'}(k,\mathbf{k}'_F). \end{aligned} \tag{4.237}$$

For $i\omega\to 0$ and $|\mathbf{k}|\to k_F$, using

$$\frac{\partial\Sigma(k)}{\partial i\omega}\to 1-\frac{1}{z}, \qquad \boldsymbol{\nabla}_{\mathbf{k}}\Sigma(k)\to\frac{\mathbf{v}^*_{\mathbf{k}}}{z}-\mathbf{v}_{\mathbf{k}}, \tag{4.238}$$

we finally obtain

$$\begin{aligned} \mathbf{v}_{\mathbf{k}} &= \mathbf{v}^*_{\mathbf{k}} + z^2 N^*(0)\sum_{\sigma'}\int\frac{d\Omega_{\hat{\mathbf{k}}'}}{4\pi}\mathbf{v}^*_{\mathbf{k}'_F}\Gamma^{\omega}_{\sigma\sigma'}(\mathbf{k}_F,\mathbf{k}'_F) \\ &= \mathbf{v}^*_{\mathbf{k}} + N^*(0)\sum_{\sigma'}\int\frac{d\Omega_{\hat{\mathbf{k}}'}}{4\pi}\mathbf{v}^*_{\mathbf{k}'_F}f_{\sigma\sigma'}(\mathbf{k}_F,\mathbf{k}'_F) \\ &= v^*_F\hat{\mathbf{k}}\left(1+\frac{F^s_1}{3}\right). \end{aligned} \tag{4.239}$$

This reproduces the expression for the quasi-particle current $\mathbf{j}_{\mathbf{k}} = \mathbf{v}_k$ obtained within Landau Fermi-liquid theory [Eq. (4.38)]. Equation (4.239) implies that the quasi-particle effective mass is determined by (4.40)

4.4.5 Response to external fields

In Landau Fermi-liquid theory, one calculates the quasi-particle response to an external field assuming that the latter couples to the quasi-particles in the standard way.

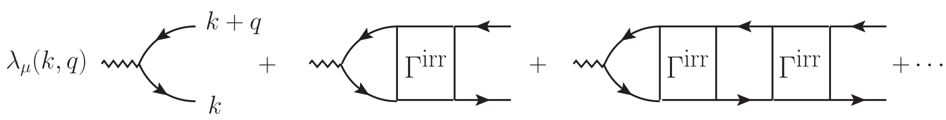

Figure 4.14: Diagrammatic representation of the renormalization of the coupling between the fermions and the external field.

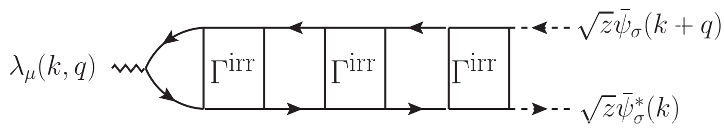

Figure 4.15: Diagrammatic representation of the coupling of the external field to the quasi-particles. The solid lines represent the incoherent part of the particle-hole propagator (i.e. φ in (4.229)), and the dashed ones the coherent part $z\bar G$ of the single-particle Green function G.

That this assumption is correct is by no means obvious. In this section, we show that the absence of renormalization of the coupling between quasi-particles and the external field is a consequence of the Ward identities discussed in the previous section.

An external field contributes to the action a term

$$\begin{aligned} S_{\rm ext}[\psi^*,\psi] &= \int_0^\beta d\tau \int d^3r\, [\phi(\mathbf{r},\tau)n(\mathbf{r},\tau) - \mathbf{j}(\mathbf{r},\tau)\cdot\mathbf{A}(\mathbf{r},\tau)] \\ &= -\sum_{\mu=0,x,y,z}\sum_{k,q,\sigma} A_\mu(-q)\lambda_\mu(k,q)\psi^*_\sigma(k)\psi_\sigma(k+q), \end{aligned} \tag{4.240}$$

where ϕ and $\mathbf{A}$ are the scalar and vector potentials, respectively. In the second line of (4.240), we have introduced a compact notation with $A_0 = -\phi$ and

$$\lambda_0(k,q) = 1, \qquad \lambda_{\mu\neq 0}(k,q) = \frac{1}{m}\left(k_\mu + \frac{q_\mu}{2}\right). \tag{4.241}$$

The interaction renormalizes the coupling of the field to the particles as shown in figure 4.14. The renormalized vertex satisfies the equation

$$\Lambda_\mu(k,q) = \lambda_\mu(k,q) + \frac{1}{\beta V}\sum_{k',\sigma'} \Lambda_\mu(k',q)G(k')G(k'+q)\Gamma^{\rm irr}_{\sigma'\sigma}(k',k). \tag{4.242}$$

Let us now try to formulate the perturbation theory in terms of the quasi-particles only. The effective coupling between the external field and the quasi-particles is then represented by diagrams of the type shown in figure 4.15 where only the incoherent part of the particle-hole pair propagator is involved. It can be obtained from $\Lambda_\mu(k,q)$ by considering the ω-limit where no quasi-particle propagation is possible

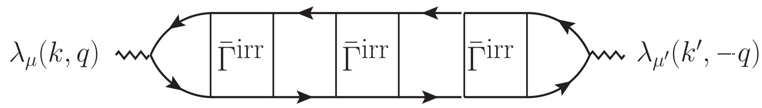

Figure 4.16: A typical diagram contributing to the quasi-particle response function $\bar\chi_{\mu\mu'}(q)$. The solid lines denote the quasi-particle propagator $\bar G$.

(since $z^2\bar G(k)\bar G(k+q)$ does not contribute in that limit),

$$\begin{aligned}\Lambda^\omega_\mu(k,q) &= \lambda_\mu(k,q) + \frac{1}{\beta V}\sum_{k',\sigma'} \Lambda^\omega_\mu(k',q)\left\{G(k')^2\right\}_\omega \Gamma^{\rm irr}_{\sigma'\sigma}(k',k) \\ &= \lambda_\mu(k,q) + \frac{1}{\beta V}\sum_{k',\sigma'} \lambda_\mu(k',q)\left\{G(k')^2\right\}_\omega \Gamma^{\omega}_{\sigma'\sigma}(k',k). \end{aligned} \tag{4.243}$$

Using the Ward identities (4.224) and (4.226), we obtain the very simple result

$$\Lambda^\omega_\mu(k,q) = \frac{\lambda_\mu(k,q)}{z}. \tag{4.244}$$

The effective coupling between the external field and the quasi-particles is therefore described by the action

$$\begin{aligned} S_{\rm ext,eff}[\bar\psi^*,\bar\psi] &= -\sum_{\mu=0,x,y,z}\sum_{k,q,\sigma} A_\mu(-q)\Lambda^\omega_\mu(k,q)\sqrt{z}\bar\psi^*_\sigma(k)\sqrt{z}\bar\psi_\sigma(k+q) \\ &= -\sum_{\mu=0,x,y,z}\sum_{k,q,\sigma} A_\mu(-q)\lambda_\mu(k,q)\bar\psi^*_\sigma(k)\bar\psi_\sigma(k+q). \end{aligned} \tag{4.245}$$

The renormalization of the coupling cancels the rescaling introduced in the definition of the quasi-particle field $\bar\psi$. This explains why the response functions do not depend on the quasi-particle weight z. In order for Fermi liquid theory to be valid we need z to be finite, but its precise value does not influence the physical properties of the system that can be measured experimentally.[24]

The response functions $\chi_{\mu\mu'}(q)$ are defined by

$$\langle j_\mu(q)\rangle = \sum_{\mu'}\chi_{\mu\mu'}(q)A_{\mu'}(q) = \sum_{\mu'}\left[\chi^{\rm inc}_{\mu\mu'}(q) + \bar\chi_{\mu\mu'}(q)\right]A_{\mu'}(q), \tag{4.246}$$

where $\chi^{\rm inc}$ denotes the purely incoherent response function and $\bar\chi$ the one due to the quasi-particles. $\bar\chi$ is computed using

$$\bar\chi_{\mu\mu'}(q) = \frac{1}{V}\sum_{\mathbf{k},\mathbf{k}',\sigma,\sigma'} \lambda_\mu(\mathbf{k},q)\bar\chi_{\sigma\sigma'}(\mathbf{k},\mathbf{k}';q)\lambda_{\mu'}(\mathbf{k}',-q), \tag{4.247}$$

where $\bar\chi_{\sigma\sigma'}(\mathbf{k},\mathbf{k}';q)$ satisfies the Bethe-Salpeter equation (4.197). A typical diagram contributing to $\bar\chi_{\mu\mu'}(q)$ is shown in figure 4.16. We leave it to the reader to show that (4.197) and (4.247) reproduce the results obtained in section 4.3.4.

[24] From a diagrammatic point of view, this is a very clear exemple of the importance of vertex corrections when calculating response functions. A mere renormalization of the one-particle propagator ($G_0 \to zG$) in the perturbation expansion would violate the Ward identities and give response functions that depend on z.

4.4.6 Luttinger theorem

At zero temperature, the density $n = N/V$ is given by

$$\begin{aligned} n &= 2\int \frac{d^dk}{(2\pi)^d}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{i\omega\eta}G(\mathbf{k},i\omega) \\ &= 2\int \frac{d^dk}{(2\pi)^d}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}e^{i\omega\eta}\left\{\frac{\partial}{\partial i\omega}\ln[-i\omega_n+\xi_{\mathbf{k}}+\Sigma(\mathbf{k},i\omega)] + G(\mathbf{k},i\omega)\frac{\partial\Sigma(\mathbf{k},i\omega)}{\partial i\omega}\right\}. \end{aligned} \tag{4.248}$$

In this section, we do not make any assumption on the dispersion $\xi_{\mathbf{k}}$ of the non-interacting fermions and consider a d-dimensional system.

Let us first show that the last term in the rhs of (4.248) vanishes. Integrating by part, we find

$$\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}G(\mathbf{k},i\omega)\frac{\partial\Sigma(\mathbf{k},i\omega)}{\partial i\omega} = -\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\Sigma(\mathbf{k},i\omega)\frac{\partial G(\mathbf{k},i\omega)}{\partial i\omega}. \tag{4.249}$$

Here we have used $\lim_{|\omega|\to\infty} G(\mathbf{k},i\omega) = 1/i\omega$ and $\lim_{|\omega|\to\infty}\Sigma(\mathbf{k},i\omega) = \text{const}$. The self-energy $\Sigma(\mathbf{k},i\omega)$ can be expressed as the functional derivative of the Luttinger-Ward functional $\Phi[G]$ with respect to the Green function $G(\mathbf{k},i\omega)$ (chapter 1). This functional is given by the sum of skeleton diagrams; it is clearly invariant if we shift all Matsubara frequencies $i\omega$ in the propagators of these diagrams by an infinitesimal amount $i\epsilon$,

$$\begin{aligned} 0 &= \int \frac{d^dk}{(2\pi)^d}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{\delta\Phi[G]}{\delta G(\mathbf{k},i\omega)}\frac{\partial G(\mathbf{k},i\omega)}{\partial i\omega} \\ &= \int \frac{d^dk}{(2\pi)^d}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\Sigma(\mathbf{k},i\omega)\frac{\partial G(\mathbf{k},i\omega)}{\partial i\omega}. \end{aligned} \tag{4.250}$$

Equations (4.249) and (4.250) prove our assertion.

We therefore have

$$n = -2\int \frac{d^dk}{(2\pi)^d}\int_{-i\infty}^{i\infty}\frac{dz}{2i\pi}e^{z\eta}\frac{\partial}{\partial z}\ln[-G(\mathbf{k},z)]. \tag{4.251}$$

$G(\mathbf{k},z)$ is an analytic function of the complex variable z except possibly on the real axis. For $\Im(z)>0$ (<0) it coincides with $G^R(\mathbf{k},z)$ ($G^A(\mathbf{k},z)$). Furthermore, $\Sigma(\mathbf{k},z)$ being analytic for $\Im(z)\neq 0$ (Sec. 3.5), $G(\mathbf{k},z)$ has no zero for $\Im(z)\neq 0$. The convergence factor $e^{z\eta}$ then enables us to change the contour of integration as shown in figure 4.17,

$$\begin{aligned} n &= -2\int \frac{d^dk}{(2\pi)^d}\left\{\int_{-\infty}^{0}\frac{d\omega}{2i\pi}e^{\omega\eta}\frac{\partial}{\partial\omega}\ln[-G(\mathbf{k},\omega-i\eta)] \right. \\ &\qquad \left. + \int_{0}^{-\infty}\frac{d\omega}{2i\pi}e^{\omega\eta}\frac{\partial}{\partial\omega}\ln[-G(\mathbf{k},\omega+i\eta)]\right\} \\ &= -\frac{i}{\pi}\int \frac{d^dk}{(2\pi)^d}\int_{-\infty}^{0}d\omega\frac{\partial}{\partial\omega}\ln\frac{G^R(\mathbf{k},\omega)}{G^A(\mathbf{k},\omega)} \end{aligned} \tag{4.252}$$

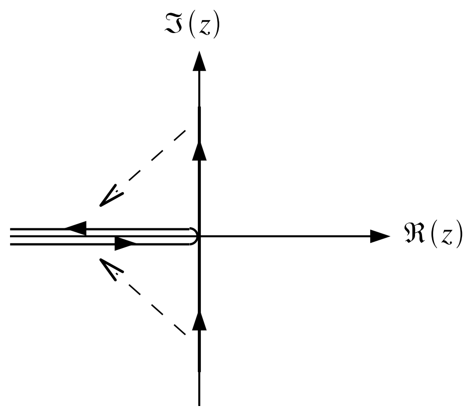

Figure 4.17: Contour of integration used in equation (4.252).

(we have dropped the convergence factor which is not necessary anymore). Denoting the phase of $G^R(\mathbf{k},\omega) = [\omega + i\eta - \xi_\mathbf{k} - \Sigma^R(\mathbf{k},\omega)]^{-1}$ by $\varphi(\mathbf{k},\omega)$, we obtain

$$n = \frac{2}{\pi} \int \frac{d^dk}{(2\pi)^d} \int_{-\infty}^{0} d\omega \frac{\partial}{\partial\omega} \varphi(\mathbf{k},\omega). \tag{4.253}$$

$G^R(\mathbf{k},-\infty)$ being real and negative (with an infinitesimal negative imaginary part), $\varphi(\mathbf{k},-\infty) = -\pi$. Since $\Im[\Sigma^R(\mathbf{k},\omega)]$ is always negative (Sec. 3.5), $G^R(\mathbf{k},\omega)$ remains in the lower complex plane and its phase $\varphi(\mathbf{k},\omega)$ can only vary between $-\pi$ and 0. There is therefore no winding of the phase $\varphi(\mathbf{k},\omega)$ as ω varies between $-\infty$ and 0, and

$$n = \frac{2}{\pi} \int \frac{d^dk}{(2\pi)^d} [\varphi(\mathbf{k},0) - \varphi(\mathbf{k},-\infty)] = \frac{2}{\pi} \int \frac{d^dk}{(2\pi)^d} [\varphi(\mathbf{k},0) + \pi]. \tag{4.254}$$

Given that $\Im[\Sigma(\mathbf{k},0)] = 0$ in a Fermi liquid (Sec. 4.4.1), the phase $\varphi(\mathbf{k},0)$ is either 0 or $-\pi$ depending on the sign of $G^R(\mathbf{k},0)$. We finally obtain

$$n = 2 \int \frac{d^dk}{(2\pi)^d} \Theta[G^R(\mathbf{k},0)]. \tag{4.255}$$

In a Fermi liquid, the region $G^R(\mathbf{k},0) > 0$, i.e. $\xi_\mathbf{k} + \Sigma^R(\mathbf{k},0) < 0$, is bounded by the Fermi surface defined by $\xi_\mathbf{k} + \Sigma^R(\mathbf{k},0) = 0$. Equation (4.255) then states that for a given density the volume of the Fermi surface in $\mathbf{k}$ space is the same as that of the non-interacting Fermi system.

4.5 Fermi-liquid theory and renormalization group

Guide to the bibliography

- Besides Landau's original papers [1–3], there are excellent textbooks on Landau Fermi-liquid theory [4–7]. sections 4.1, 4.2 and 4.3 rely heavily on Refs. [4,5].
- The microscopic foundations of Fermi-liquid theory are discussed in Refs. [3,6–8].
- The derivation of the thermodynamic potential $\Omega[n]$ and the quantum Boltzmann equation follows Ref. [9]. Similar ideas were discussed in the context of the so-called statistical Fermi-liquid theory [10–12]. A direct derivation of $E[n]$ can be found in Ref. [7].
- Ward identities for the Fermi liquid are discussed in Refs. [6,8,14].
- The Luttinger theorem [15] is discussed in Refs. [6,7]. For a discussion of the validity of possible extensions of this theorem, see Refs. [16,17].

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