Anderson localisation

"Very few believed it at the time, and even fewer saw its importance; among those who failed to fully understand it at first was certainly its author"

-P.W. Anderson

What was new?

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Absence of Diffusion in Certain Random Lattices

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This paper presents a simple model for such processes as spin diffusion or conduction in the "impurity band." These processes involve transport in a lattice which is in some sense random, and in them diffusion is expected to take place via quantum jumps between localized sites. In this simple model the essential randomness is introduced by requiring the energy to vary randomly from site to site. It is shown that at low enough densities no diffusion at all can take place, and the criteria for transport to occur are given.

- Anderson suggested localisation which was against the common thing(diffusion) everyone believed at that time.
- The derivation of diffusion was quite general and it had at it's heart an intuitive assumption that the process was Markovian(not remembering history).
- Localisation meant particle was remembering its history and was destructively interfering to cancel out the probability density.

Introduction

- **Strong disorder:** In Anderson's seminal paper he proposed that if the disorder is strong enough the localisation of states will happen no matter what the dimension of system d is. However delocalisation and critical behaviour are not well studied in this article.
- **Lower dimensions:** For d=1,2; motivated by Anderson's work people tried to solve the problem when disorder is finite for d=1 it was proved that the localisation must happen however small the disorder is and for d=2 there is no exact solution however the renormalisation group analysis shows localisation must happen in a two dimensional system no matter how small the disorder is.
- **Higher dimensions:** d=3; For higher dimensions there is delocalisation and there is a critical disorder strength.

http://web.mit.edu/8.334/www/grades/projects/projects19/GuanChenguang.pdf

Localisation in one dimension

Transfer matrix method

- We consider the Anderson Hamiltonian, a tight binding model with nearest neighbour hopping and random on-site energy:
- $\hat{H} = W \sum_n \epsilon_n \hat{c}_n^{\dagger} \hat{c}_n + t \sum_{<n,m>} (\hat{c}_n^{\dagger} \hat{c}_m + hc).$

  where the on-site energy $\epsilon_n \in [-1/2, 1/2]$ is drawn from uniform random distribution and W is the disorder parameter.
- Writing the wave function as $|\Psi\rangle = \sum_n \psi(n) \hat{c}_n^{\dagger} |0\rangle$ one then gets the energy relation as $E\psi(n) = \epsilon_n \psi(n) + \psi(n+1) + \psi(n-1)$.

https://pdfs.semanticscholar.org/51cb/3249779f035066619c1339d194a91eb087cc.pdf

- Rewriting the recursion relation in terms of transfer matrices

$$\begin{pmatrix} \psi(n+1) \\ \psi(n) \end{pmatrix} = \prod_{i=1}^{L} \hat{T}_i \begin{pmatrix} \psi(1) \\ \psi(0) \end{pmatrix}, \quad \hat{T}_i = \begin{pmatrix} E - \epsilon_i & -1 \\ 1 & 0 \end{pmatrix}.$$

- By proving Fuerstenberg's theorem for products of random matrices, it can be rigorously shown that solutions $\psi(n)$ of the recursion relation decay exponentially for $n \to \infty$, so that the overall shape of the wavefunction is $\psi(n) \sim \exp(-n/\xi)$. Here, ξ is the localization length in units of lattice constant a.

- As a consequence, eigenstates are not extended over real-space but are exponentially confined to certain areas of the lattice. We will refer to these states as localized.

Dynamical transport

- This was the method used by Anderson in his 1958 paper.
- The main idea of dynamical method is that we write a wave function and evolve with the time and see that after long time whether the wave function decays or not.
- A non-decaying return probability would indicate localization.
- **Note:** In his 1958 paper Anderson has only showed that there will always be localisation for large disorder

Anderson's method

- We consider the Anderson Hamiltonian, a tight binding model with hopping matrix V and random on-site energy:
- $\hat{H} = \sum_n E_n \hat{c}_n^{\dagger} \hat{c}_n + \sum_{j,k(j\neq k)} V_{jk} \hat{c}_j^{\dagger} \hat{c}_k$. where the on-site energy $E_n \in [-W/2, W/2]$ is drawn from uniform random distribution and W is the disorder parameter.
- Writing the wave function as $|\Psi\rangle = \sum_n a_n(t) \hat{c}_n^{\dagger} |0\rangle$ one then gets the relation

$$i\hbar \dot{a}_j = E_j a_j + \sum_{k\neq j} V_{jk} a_k$$

- Anderson asked the simplest question to assume $a_0(0) = 1$, and asked the question how a_j varies with time.

- Doing the Laplace transform $f_j(s) = \int_0^\infty e^{-st} a_j(t) dt$ transforms the equation to

$$i[sf_j(s) - a_j(0)] = E_j f_j + \sum_{k \neq j} V_{jk} f_k$$

- Also noting that small s implies large t behaviour as can be seen by $\lim_{s \to 0^+} sf_j(s) = < a_j(\infty) >_{Av.}$. So all we need to do now is to see how f_j behaves for small s.

- So the solution comes out as $f_j(s) = i\frac{\delta_{0j}}{is - E_j} + \sum_{k \neq j} \frac{1}{is - E_j} V_{jk} f_k(s)$

- The solution can be found by iteration as

$$f_j(s) = \frac{1}{is - E_j} V_{j0} f_0(s) + \sum_k \frac{1}{is - E_j} V_{jk} \frac{1}{is - E_k} V_{k0} f_0(s) + \dots$$

- Substituting this we get the zeroth order equation as

$$f_0(s) = \frac{i}{is - E_0} + \sum_k \frac{1}{is - E_0} V_{0k} \left(\frac{1}{is - E_k} V_{k0} + \sum_l \frac{1}{is - E_k} V_{kl} \frac{1}{is - E_l} V_{l0} + \dots \right) f_0(s)$$

- Define, $V_c(j) = \sum_k V_{jk} \frac{1}{is - E_k} V_{kj} + \sum_{k,l} V_{jk} \frac{1}{is - E_k} V_{kl} \frac{1}{is - E_l} V_{lj} + \dots$

  then $V_c(0) = \sum_k V_{0k} \frac{1}{is - E_k} V_{k0} + \sum_{k,l} V_{0k} \frac{1}{is - E_k} V_{kl} \frac{1}{is - E_l} V_{l0} + \dots$

- Then equation simplifies to $f_0(s) = \frac{i}{is - E_0} + \frac{1}{is - E_0} V_c(0) f_0(s)$
- This then gives $f_0(s) = \frac{i}{is - E_0 - V_c(0)}$
- Or $f_0(s) = \frac{1}{s - Im(V_c(0) + i(E_0 + Re(V_c(0))}$

In many cases the first term suffices. Studying this first term, we see that it can be written

$$V_c(0)=\sum_k (V_{0k})^2\left(\frac{-E_k}{s^2+E_k^2}-\frac{is}{s^2+E_k^2}\right). \tag{8}$$

In the limit as $s\to 0$, the first part of this is obviously just the second-order perturbation $-\Delta E^{(2)}$ of the energy. The second part may be written

$$\lim_{s\to 0}(V_c)=-i\sum_k (V_{0k})^2\delta(E_k)-is\sum_{k,E_k\neq 0}\frac{(V_{0k})^2}{E_k^2}$$

$$\equiv -\frac{i}{\tau}-isK. \tag{9}$$

Of course, if the first of these terms, the usual transition probability, is finite the second is indeterminate. We include both in order to see what happens if the first term does vanish.

The solution for f_0 is

$$f_0=\frac{i}{is(1+K)+(i/\tau)-(E_0-\Delta E^{(2)})}. \tag{10}$$

If τ is finite, one gets the usual result of perturbation theory:

$$f_0(s)=\frac{1}{s+(1/\tau)+i(E_0-\Delta E^{(2)})}, \tag{10A}$$

which represents a state of perturbed energy $E_0-\Delta E^{(2)}$ decaying at a rate $e^{-t/\tau}$. If, on the other hand, τ is infinite [$\mathrm{Im}(V_c)\to 0$ as $s\to 0$], then the constant K enters and the amplitude is

$$f_0(s)=\frac{1}{s(1+K)+i(E_0-\Delta E^{(2)})}. \tag{10B}$$

Laplace transform

time domain	s domain
$e^{at}f(t)$	$F(s-a)$
Unit step function, u(t)	$\frac{1}{s}$
$e^{-\alpha t}u(t)$	$\frac{1}{s+\alpha}$

- To first order $\mathrm{Im}(V_c(0)) = -s\,X(s)$, where $X(s) = \sum_k \frac{|V_{0k}|^2}{s^2 + E_k^2}$
- One can also see that $X(s) = \sum_{j\neq 0} \frac{|f_j(s)|^2}{|f_0(s)|^2}$ to first order approximation
- Both ways show that in the limit $s \to 0^+$ **finite X(s)** means **no transport**.

PROBABILITY DISTRIBUTION OF THE FIRST TERM OF Vc

- **Idea:** Directly taking expectation value gives the wrong result; correct way is to find the distribution first. To find the distribution we have some terms in getting the distribution, the finiteness of those terms give us what limit will be there for convergence.

The integration here depends on the more stringent condition $V \sim 1/r^{3+\epsilon}$ for large r. The probability distribution, which will be valid for large X at least, is familiar from line-broadening theory[12]:

$$P(X) = \frac{n\langle V\rangle}{WX^{\frac{3}{2}}} \exp\left[-\left(2n\Gamma(\tfrac{1}{2})\frac{\langle V\rangle}{W}\right)^2 \cdot \frac{1}{X}\right]. \quad (33)$$

For large X, this falls off as $X^{-\frac{3}{2}}$, as stated in Sec. II: while the mean of X is divergent, the probability that X is larger than some value X_0 decreases as $X_0^{-\frac{1}{2}}$. Thus, for any given starting atom n, the renormalization constant K may be large, but the probability that τ is finite is exactly zero.

Stochastic problems in physics and astronomy by S. Chandrashekhar https://journals.aps.org/rmp/pdf/10.1103/RevModPhys.15.1

The probability distribution of such a sum is best calculated by the Markoff method,[11] as modified by Holtsmark.[6] In this method we find the Fourier transform of the probability distribution $P(X)$:

$$P(X) = \int_{-\infty}^{\infty} e^{ixX}\varphi(x)dx, \quad (29)$$

where

$$\varphi(x) = \exp\left\{-n\left\langle \int \left[1 - \exp\left(ix\frac{V^2(r)}{E^2+s^2}\right)\right]d\tau\right\rangle\right\}. \quad (30)$$

The average is to be taken over the probability distribution of E, $P(E)$, and n is the density of sites.

Let us write out the important integral in the exponent of (30):

$$I = \int_{-\infty}^{\infty} P(E)dE 4\pi \int_0^{\infty} r^2 dr\left[1 - \exp\left(\frac{ixV^2(r)}{E^2+s^2}\right)\right]. \quad (31)$$

The behavior of $P(X)$ for large X depends on the behavior of I for sufficiently small x. Let us first consider the case $s=0$. Now for small enough x, and a finite E (say of order W), the exponential $\exp ixV^2/E^2$ can be expanded in a power series in x, and the integration over r done (so long as V is finite and falls off faster than $r^{-\frac{3}{2}}$) to obtain terms which go as x^1 or higher powers for small x. Thus only the behavior for small E is important, and we can neglect the variation of $P(E)$, replacing it by a constant, $1/W$. Then

$$I \cong \frac{4\pi}{W}\int_0^{\infty} r^2 dr \int_{-\infty}^{\infty} dE\left\{1 - \exp\left[\frac{ixV^2(r)}{E^2}\right]\right\}$$

$$= \frac{4\pi}{W}(x)^{\frac{1}{2}}\int_0^{\infty} r^2 dr V(r)\int_{-\infty}^{\infty} du[1 - \exp(i/u^2)] \quad (32)$$

$$= 2\left(\frac{x}{i}\right)^{\frac{1}{2}}\Gamma(\tfrac{1}{2})\frac{\langle V(r)\rangle}{W}.$$

Watson technique

$$V_c(0) = \sum_k V_{0k} \frac{1}{is - E_k} V_{k0} + \sum_{k,l} V_{0k} \frac{1}{is - E_k} V_{kl} \frac{1}{is - E_l} V_{l0} + \dots$$

- The main idea is to write the series without repetition in a self consistent way.
- **Note:** To apply Watson technique we have to write the terms in the terminology of multiple scattering theory which can be done by some elementary matrix operations.

The technique of Watson[7,9] shows us that we may eliminate all repeated indices in a self-consistent way by including in the energy denominator for atom k the perturbed energy $V_c(k)$ calculated from just such a series of terms as (13). A complicating factor is that $V_c(k)$ must be calculated from a series of diagrams which do not include *any* indices which have previously appeared before in the particular term of $V_c(0)$ we are calculating. That is, if we want the term

$$V_{03} \underset{e_3}{1} V_{32} \underset{e_2}{1} V_{21} \underset{e_1}{1} V_{10},$$

where for brevity we introduce the usual "propagator" notation

$$is - E_j - V_c(j) = e_j, \tag{14}$$

then the propagator e_2, for instance, is given by

$$e_2 = is - E_2 - V_c^{0,1}(2)$$

$$= is - E_2 - \sum_{j,k,l \neq 0,1} V_{2j} \underset{e_j}{1} V_{jl} \underset{e_l}{1} \cdots \underset{e_k}{1} V_{k2} \tag{15}$$

and again, each of the propagators in this series must be appropriately modified not to include either 0, 1, or any of the previous indices in the $V_c^{0,1}(2)$ series.

Thus we may now write

$$V_c(0) = \sum_{\substack{i \neq 0 \\ j \neq i,0 \\ k \neq i,j,0 \\ \cdots \\ l \neq \cdots i,j,k,0}} V_{0l} \cdots \frac{1}{is - E_k - V_c^{0,i,j}(k)}$$

$$\times V_{kj} \frac{1}{is - E_j - V_c^{0,i}(j)} V_{ji} \frac{1}{is - E_i - V_c^{0}(i)} V_{i0}. \tag{16}$$