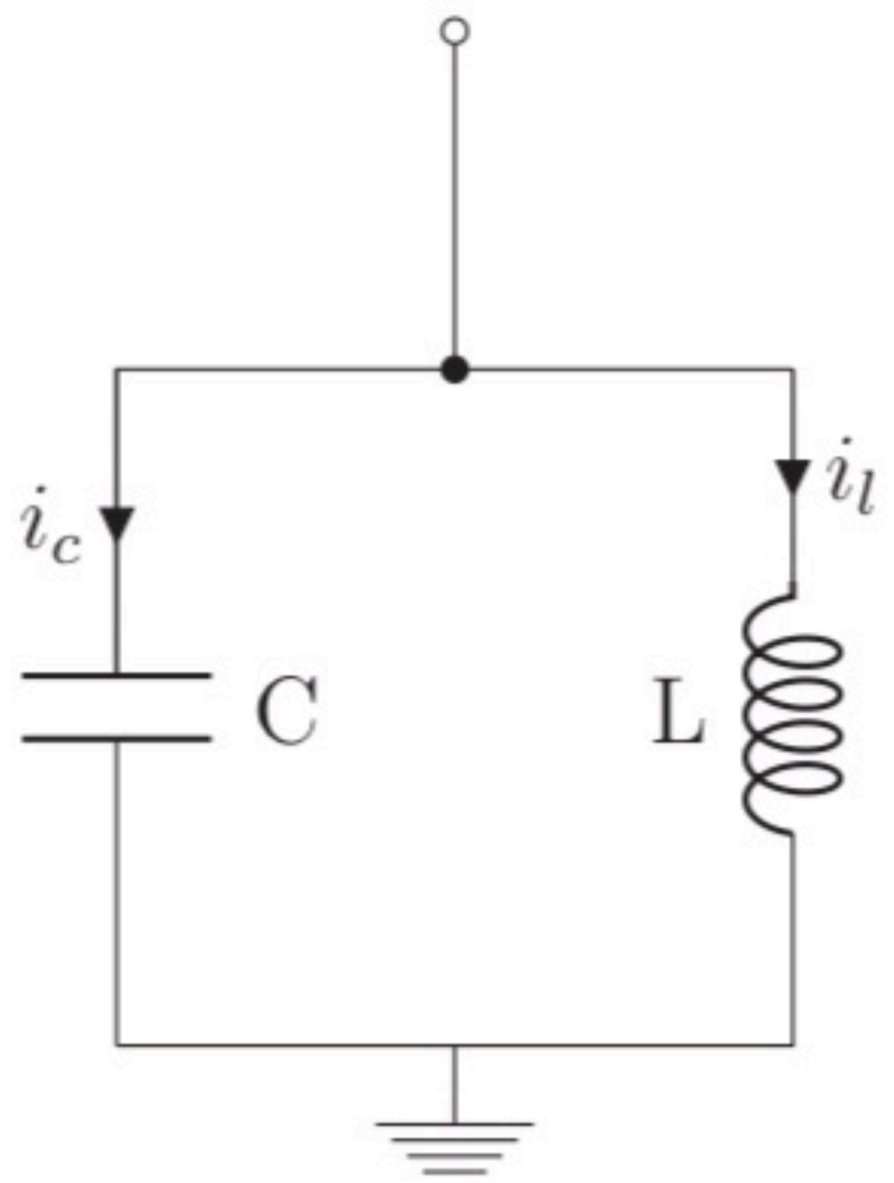
CQED
&
Master Equation

CQED

- $i_C + i_l = 0$
- $C\dot{V}_c + \dfrac{\phi}{L} = 0$
- $C\ddot{\phi} + \dfrac{\phi}{L} = 0$ (SHO)
- $\mathscr{L} = \dfrac{1}{2}C\dot{\phi}^2 - \dfrac{\phi^2}{2L}$, this Lagrangian gives us the correct equation of motion
- The corresponding Hamiltonian where $Q = C\dot{\phi}$, is $H = \dfrac{Q^2}{2C} + \dfrac{\phi^2}{2L}$



- We can write the Hamiltonian in terms of dimensionless creation and annihilation operators, where $Z_r = \sqrt{L/C}$

$$\hat{a} = \sqrt{\frac{1}{2\hbar Z_r}}\left(\hat{\Phi} + iZ_r\hat{Q}\right)$$
$$\hat{a}^\dagger = \sqrt{\frac{1}{2\hbar Z_r}}\left(\hat{\Phi} - iZ_r\hat{Q}\right)$$

- We can quantise the system by putting the commutator relation, $[a, a^\dagger] = 1$

- This gives the Hamiltonian as, $H = \hbar\omega_r\left(a^\dagger a + \frac{1}{2}\right)$

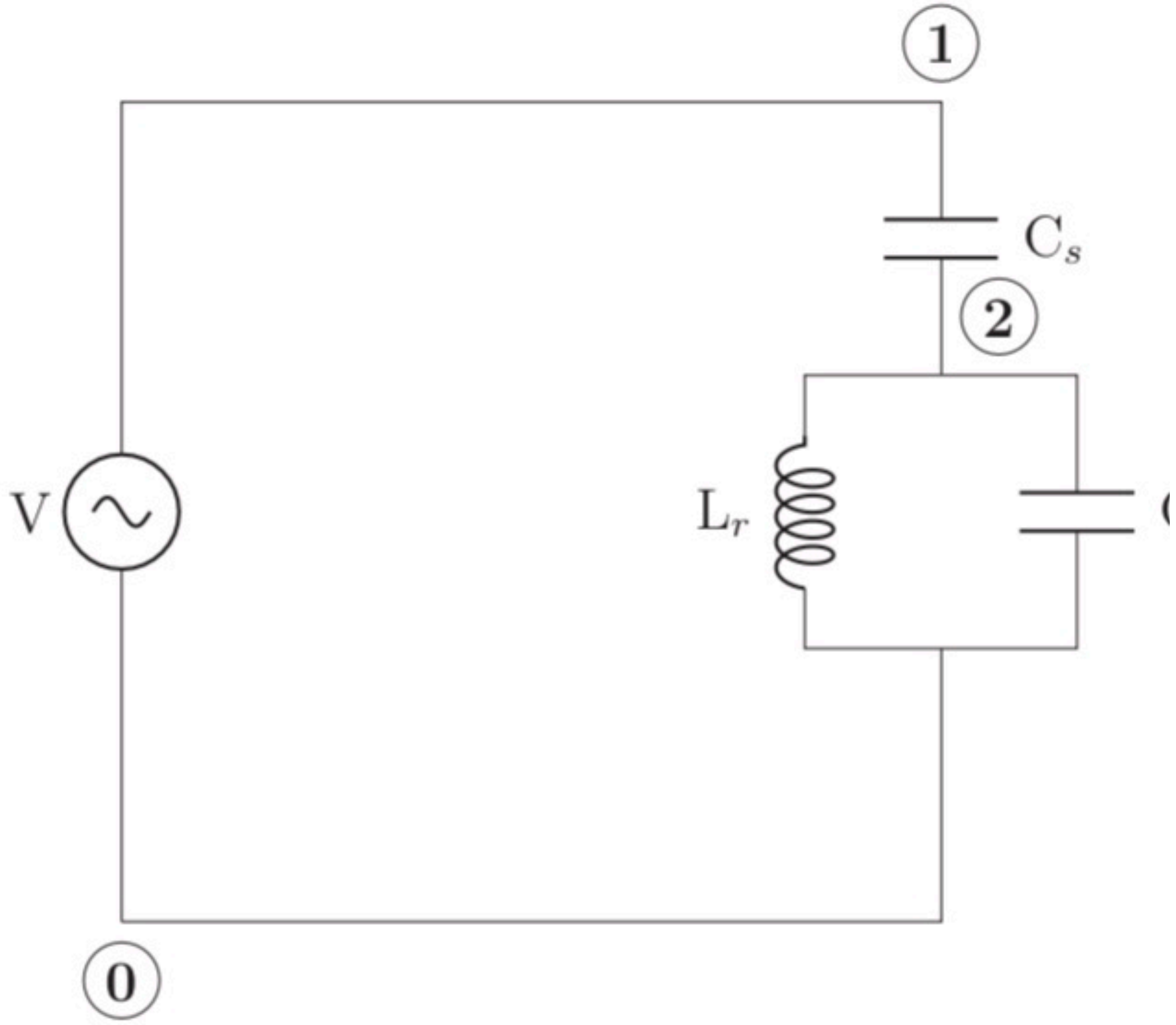
Different components of CQED

Resonator

- The Hamiltonian for a driven resonator

$$\mathcal{H} = \frac{Q_r^2}{2(C_r + C_s)} + \frac{\phi_r^2}{2L_r} + \frac{C_s}{C_r + C_s} Q_r V.$$

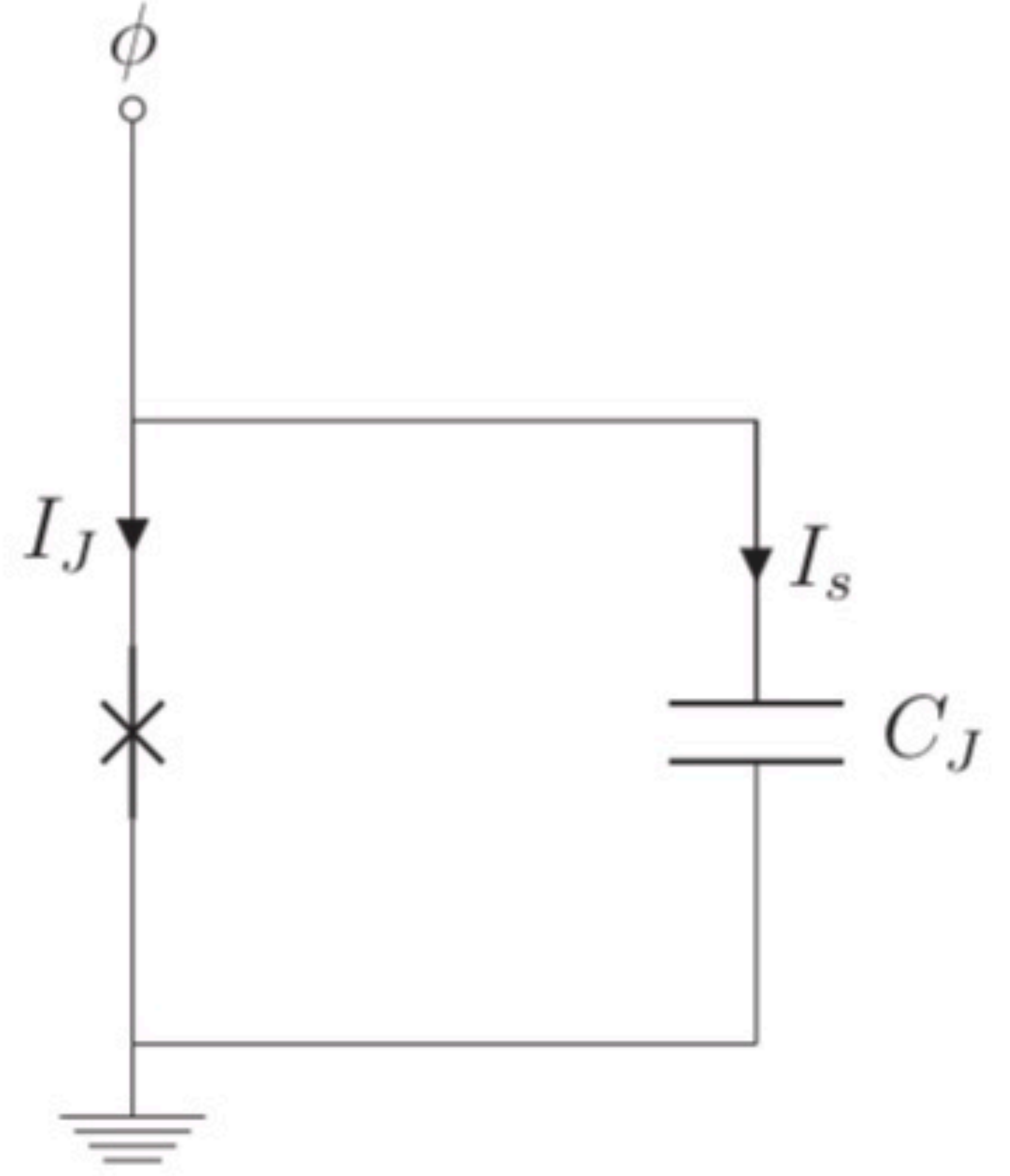
$$H = \hbar\omega_r(a^\dagger a + \frac{1}{2}) - \hbar\Omega_r \, (a + a^\dagger) \cos(\omega t)$$



Transmon(Qubit)

$$I_J + I_s = 0$$

$$C_J \ddot{\phi} + I_c \sin\left(2\pi \frac{\phi}{\Phi_0}\right) = 0$$



- The above equations of motion can be derived from the Lagrangian:

$$\mathcal{L} = \frac{C_J \dot{\phi}^2}{2} + E_J \cos\left(2\pi \frac{\phi}{\Phi_0}\right).$$

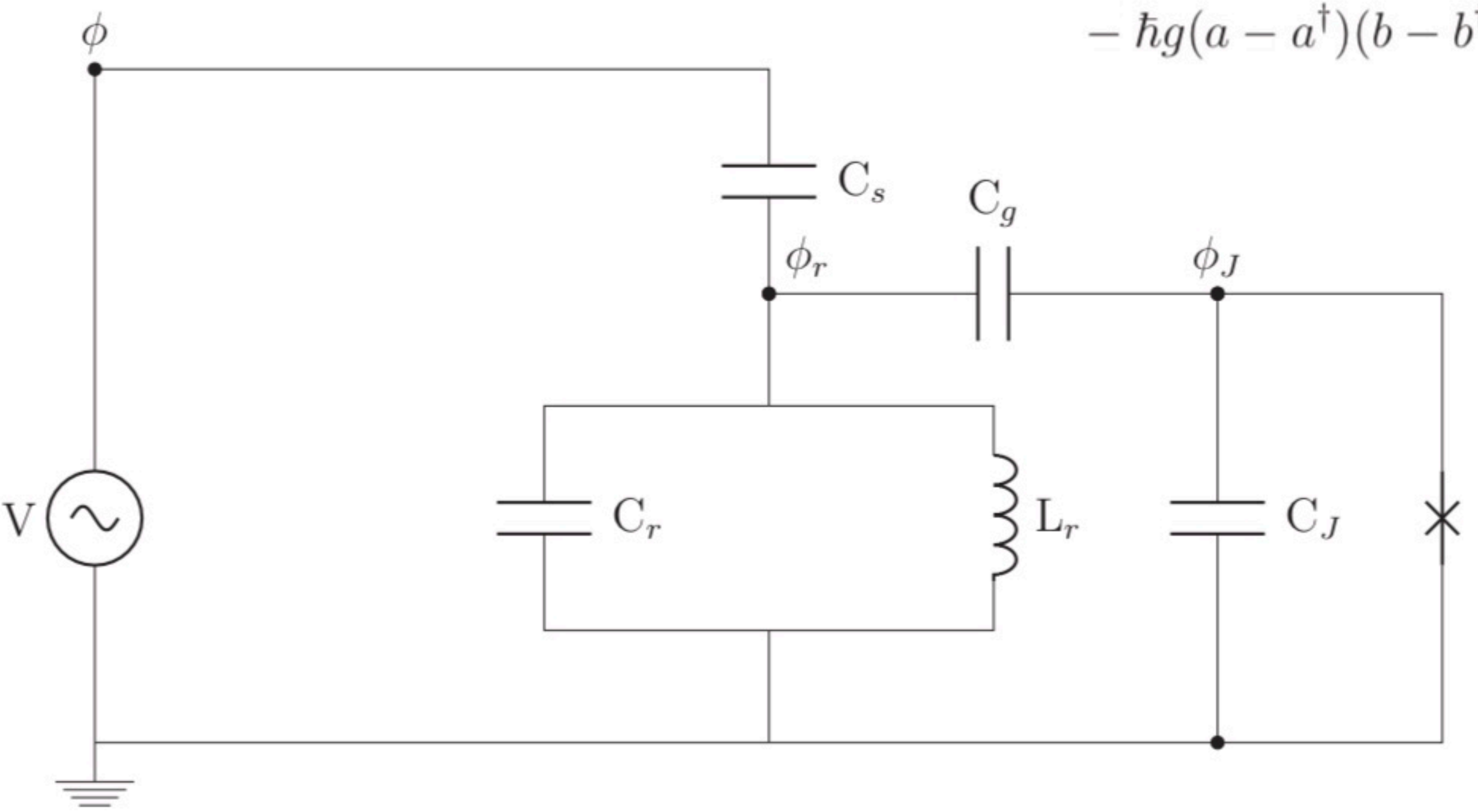
- Expanding the cosine terms we get a harmonic oscillator with a quartic anharmonicity.

- In certain energy range we can approximate this system by a TLS

$$\mathcal{H} = \underbrace{\frac{Q_r^2}{2C_{res}} + \frac{\Phi_r^2}{2L_r}}_{\text{Resonator}} + \underbrace{\frac{Q_J^2}{2C_\Sigma} - E_J \cos\left(2\pi\frac{\phi_J}{\Phi_0}\right)}_{\text{Transmon}} + \underbrace{\beta_{rJ} Q_r Q_J}_{\text{coupling}} + \underbrace{\beta_r V Q_r + \beta_J V Q_J}_{\text{drive terms}}$$

$$H = \hbar\omega_r\left(a^\dagger a + \frac{1}{2}\right) + \hbar\omega_J\left(b^\dagger b + \frac{1}{2}\right) - \frac{E_c}{12}\left(6\,(b^\dagger b)^2 + 6\,b^\dagger b + 3\right)$$

$$-\,\hbar g(a - a^\dagger)(b - b^\dagger) - i\hbar\Omega_r(a - a^\dagger)\cos\omega t - i\hbar\Omega_J(b - b^\dagger)\cos\omega t$$



Transmission line

- A transmission line is a waveguide that supports propagation of EM waves.
- A coaxial transmission line can be modelled like this(ignoring the resistance one can write a Hamiltonian for this system).
- $$L = \sum_{i=0}^{N} \frac{1}{2} C \dot{\phi}_i^2 - \frac{(\phi_{i+1} - \phi_i)^2}{2L}$$

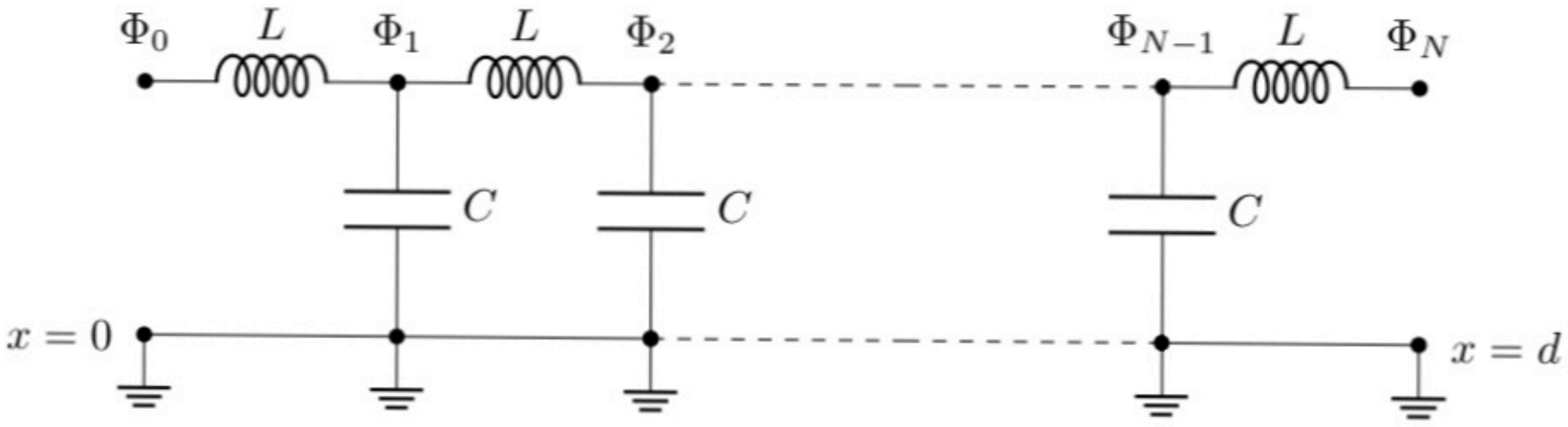


Figure 13.4: Lumped element circuit model for a transmission line

- For a continuum version of the above Lagrangian we get the solution as the wave equation.

- We can decouple the equations to get an uncoupled Lagrangian, which then gives us our decoupled Hamiltonian as

- $$H = \sum_{n=0}^{\infty} \left[\frac{Q_n^2}{2C_n} + \frac{\phi_n^2}{2L_n} \right]$$ where $C_n \equiv cd/2$ & $L_n \equiv 2ld/(n^2\pi^2)$

- $$H = \sum_k \hbar\omega_k \left(a_k^\dagger a_k + \frac{1}{2} \right)$$

- So we can physically think of **transmission line** as a **collection of infinite harmonic oscillator.**

How does dissipation enter into quantum mechanics?

When the number of bath degrees of freedom is infinite, the time-scale for returning information from the bath to the system becomes arbitrarily long, consistent with dissipation and a loss of information from the system.

Master Equation

Describes the time evolution of a system in terms of probability of states at a given time

Is it the only way to deal with dissipation?

No. There are other ways like introducing the environment action, coupling it with the system and then finding the effective action of the system (Caldeira-Leggett model).
A formally exact quantum master equation is the Nakajima–Zwanzig equation, which is in general as difficult to solve as the full quantum problem.
The Redfield equation and **Lindblad equation** are examples of approximate Markovian quantum master equations. These equations are very easy to solve, but are not generally accurate.
Some modern approximations based on quantum master equations, which show better agreement with exact numerical calculations in some cases, include the polaron transformed quantum master equation and the VPQME (variational polaron transformed quantum master equation).

Assumptions/Approximations for deriving the "Master equation"

- Weak coupling between system & environment which allows us to take density matrix as direct product of system and environment density matrix
- Born approximation(taking only till 2nd order terms in Dyson series)
- Markov approximation(bath operators are taken to be Markovian or "memory-less")

- We need to use two formulas for density matrix:
- $< A > = Tr\{A\rho\}$
- $i\hbar\frac{\partial\rho}{\partial t} = [H, \rho]$ in interaction picture also the equation remains the same

$$i\hbar\frac{\partial\rho}{\partial t} = [H_I, \rho]$$

$$\rho(t) = \rho(0) - \frac{i}{\hbar}\int_0^t d\tau\ [H_I(\tau), \rho(\tau)]$$

the $\rho(\tau)$ in the RHS can be substituted using the above equation recursively,giving us

$$\rho(t) = \rho(0) + \left(\frac{-i}{\hbar}\right)\int_0^t dt_1\ [H_I(t_1), \rho(0)] + \left(\frac{-i}{\hbar}\right)^2\int_0^t\int_0^{t_1} dt_1 dt_2\ [H_I(t_1), [H_I(t_2), \rho(t_2)]]$$

This process can be repeated *ad infinitum* resulting in the infinite series solution

$$\rho(t) = \left[\sum_{n=0}^{\infty}\left(\frac{-i}{\hbar}\right)^n\int_0^t\int_0^{t_1}\cdots\int_0^{t_{n-1}} dt_1 dt_2\cdots dt_n \times [H_I(t_1), [H_I(t_2), \cdots [H_I(t_n), \rho(0)]]]\right]$$

where, $t \geq t_1 \geq t_2 \geq \cdots \geq t_{n-1} \geq t_n \cdots \geq 0$

This infinite series solution for the density matrix is called the Dyson Series.

- Hamiltonian for a resonator coupled to the transmission line(environment).

$$H = \underbrace{\hbar\omega_r a^\dagger a}_{\text{resonator}} + \underbrace{\sum_k \hbar\omega_k b_k^\dagger b_k}_{\text{transmission line}} + \underbrace{\hbar\sum_k g_k \left(b_k a^\dagger + b_k^\dagger a\right)}_{\text{interaction}}$$

$$= H_0 + H_1$$

- The density matrix is taken as $\rho(0) = \rho_o(0) \otimes \rho_e(0)$

- The trace of first term vanishes in Dyson series and the we only expand till second term which gives us

$$\boxed{\frac{\partial \rho_o(t)}{\partial t} = \frac{-1}{\hbar^2}\int_0^t d\tau \, \mathrm{Tr}_e \left[H_I(t), \left[H_I(\tau), \rho_e(0) \otimes \rho_o(t)\right]\right]}$$

- The density matrix of environment(transmission line) is given as:

$$\rho_e = \Pi_k \left[\sum_{n_k=0}^{\infty} \frac{1}{\bar{n}_k + 1} \left(\frac{\bar{n}_k}{\bar{n}_k + 1} \right)^{n_k} | n_k > < n_k | \right] \text{ where } \bar{n}_k = \frac{1}{e^{\beta\hbar\omega_k} - 1}$$

- And the bath operators are Markovian, $< b_k^\dagger(t) b_l(t') > = \delta_{kl} \bar{n}_k \delta(t - t')$

- We also go from the sum of states in TL to integration with appropriate density of states.

- We also introduce a constant $\kappa = 2\pi D(\omega_r) g_{\omega_r}^2$, also known as decay rate.

$$\dot{\rho} = -\frac{i}{\hbar}[H,\rho] + \sum \left(\kappa_i \mathscr{D}[a_i]\rho + e^{-\beta\hbar\omega}\kappa_i \mathscr{D}[a_i^\dagger]\rho\right)$$

where $\mathscr{D}[A_i]\rho \equiv A_i \rho A_i^\dagger - \frac{1}{2}(A_i^\dagger A_i \rho + \rho A_i^\dagger A_i)$ is known as the Lindblad super operator.

In the TL resonator coupling case we don't have the i index

$$\frac{\mathrm{d}\langle a\rangle}{\mathrm{d}t} = \frac{\mathrm{d}\,\mathrm{Tr}\{a\rho_o^S\}}{\mathrm{d}t}$$

Using the Master equation derived above, we get

$$\frac{\mathrm{d}\langle a\rangle}{\mathrm{d}t} = -i\omega_o\langle a\rangle - \frac{\kappa}{2}\langle a\rangle + if_o e^{-i\omega_d t}$$

This is exactly the same as the classical oscillator equation once we identify the quantity $\langle a\rangle \leftrightarrow$

$$\dot{\alpha} = -i\omega_o\alpha - \frac{\kappa}{2}\alpha + if_o e^{-i\omega_d t}$$

Similarly, one can also check that:

$$\langle a(t)\rangle = \langle a(0)\rangle\, e^{-i\omega_r t - \frac{\kappa t}{2}}$$
$$|\langle a(t)\rangle| = |\langle a(0)\rangle| e^{-\frac{\kappa t}{2}}$$
$$\frac{\mathrm{d}\langle a^\dagger a\rangle}{\mathrm{d}t} = if_o\left[\langle a^\dagger\rangle e^{-i\omega_d t} - \langle a\rangle e^{i\omega_d t}\right] - \kappa\langle a^\dagger a\rangle + \kappa\bar{n}(\omega_r)$$

For an undriven oscillator ($f_o = 0$) at $T = 0$K ($N = 0$), we have

$$\frac{\mathrm{d}\langle a^\dagger a\rangle}{\mathrm{d}t} = -\kappa\langle a^\dagger a\rangle$$
$$\langle n(t)\rangle = \langle n\rangle_0\, e^{-\kappa t}$$

Two practical examples of the Master Equation

- Dissipation in a resonator: brings friction from a microscopic theory.
- Steady state of a damped driven SHO: Coherent state.