Calculating entanglement entropy

1 S_{EE} for Kitaev chain

1.1 Characteristic function for thermal state

$$\chi_N(\boldsymbol{\xi}) = Z^{-1}\int \mathcal{D}(\bar{c},c)\exp\Bigg[-\int_0^\beta d\tau\Bigg\{\sum_i \bar{c}_i(\tau)\partial_\tau c_i(\tau)+H(\bar{c}_i(\tau),c_i(\tau))-\sum_{i\in A'}\bar{c}_i(\tau)\delta(\tau^+)\xi_i+\sum_{i\in A'}\bar{\xi}_i\delta(\tau)\bar{c}_i(\tau)\Bigg\}\Bigg], \tag{1}$$

The Kitaev chain hamiltonian is

$$H=\sum_j\Bigg[-w(a_j^\dagger a_{j+1}+a_{j+1}^\dagger a_j)-\mu\Big(a_j^\dagger a_j-\frac{1}{2}\Big)+\Delta a_j a_{j+1}+\Delta^* a_{j+1}^\dagger a_j^\dagger\Bigg]. \tag{2}$$

In terms of Majorana operators the Hamiltonian becomes

$$H=i\sum_j\Bigg[-\mu d_{2j-1}d_{2j}+(w+|\Delta|)d_{2j}d_{2j+1}+(-w+|\Delta|)d_{2j-1}d_{2j+2}\Bigg], \tag{3}$$

where $d_{2j-1}=(e^{i\theta/2}a_j+e^{-i\theta/2}a_j^\dagger)/\sqrt{2}$ and $d_{2j}=(e^{i\theta/2}a_j-e^{-i\theta/2}a_j^\dagger)/\sqrt{2}i$. Any arbitrary quadratic Hamiltonian can be written in the form

$$H=\frac{i}{2}\sum_{l,m}A_{lm}d_l d_m, \qquad A_{lm}^*=A_{lm}=-A_{ml}. \tag{4}$$

Thus the Hamiltonian can be reduced to a canonical form

$$H_{canonical}=\frac{i}{2}\sum_{m=1}^N \epsilon_m\gamma_i^A\gamma_i^B, \tag{5}$$

where

$$\boldsymbol{\gamma}\equiv\begin{pmatrix}\gamma_1^A\\ \gamma_1^B\\ .\\ .\\ .\\ \gamma_N^A\\ \gamma_N^B\end{pmatrix}=W\begin{pmatrix}d_1\\ d_2\\ .\\ .\\ .\\ d_{2N-1}\\ d_{2N}\end{pmatrix}; \qquad P\equiv WAW^T=\begin{pmatrix}0 & \epsilon_1 & & & & & \\ -\epsilon_1 & 0 & & & & & \\ & & . & & & & \\ & & & . & & & \\ & & & & . & & \\ & & & & & 0 & \epsilon_N\\ & & & & & -\epsilon_N & 0\end{pmatrix}. \tag{6}$$

The characteristic function can now be written in terms of Majorana variables as

$$\chi_N(\boldsymbol{\xi})=Z^{-1}\cdot(i)^N\det(\mathrm{W})^{-1}\cdot\int\mathcal{D}(\boldsymbol{\gamma}^A,\boldsymbol{\gamma}^B)\exp\Bigg[-\int_0^\beta d\tau\Bigg\{\frac{1}{2}\sum_i(\gamma_i^A\partial_\tau\gamma_i^A+\gamma_i^B\partial_\tau\gamma_i^B)+\frac{i}{2}\boldsymbol{\gamma}^TP\boldsymbol{\gamma}-\sum_{i\in A'}(\gamma_i^A\eta_i^A+\gamma_i^B\eta_i^B)\Bigg\}\Bigg] \tag{7}$$

where $\boldsymbol{\gamma}^{A/B}=\{\gamma_i^{A/B}\}$, $\mathcal{D}(\boldsymbol{\gamma}^A,\boldsymbol{\gamma}^B)=\lim_{N\to\infty}\prod_{n=1}^N d(\boldsymbol{\gamma}^{A,n},\boldsymbol{\gamma}^{B,n})$, $\eta_i^A=(\delta(\tau^+)\xi_i+\delta(\tau)\bar{\xi}_i)/\sqrt{2}$ and $\eta_i^B=i(\delta(\tau^+)\xi_i-\delta(\tau)\bar{\xi}_i)/\sqrt{2}$. Also because W is an orthogonal matrix, $\det(W)=1$.

Going to Matsubara space by setting: $\gamma_i^{A/B}=\sum_{\omega_n}\gamma_i^{A/B}(\omega_n)e^{-i\omega_n\tau}$

$$S=\frac{\beta}{2}\sum_{\omega_n}\sum_i(\gamma_i^A(\omega_n)(i\omega_n)\gamma_i^A(-\omega_n)+\gamma_i^B(\omega_n)(i\omega_n)\gamma_i^B(-\omega_n)+2i\epsilon_i\gamma_i^A(\omega_n)\gamma_i^B(-\omega_n))-\sum_{\omega_n}\sum_{i\in A'}(\gamma_i^A(\omega_n)\zeta_i^A+\gamma_i^B(\omega_n)\zeta_i^B) \tag{8}$$

where $\zeta_i^A=(\xi_i+\bar{\xi}_i)/\sqrt{2}$ and $\zeta_i^B=(\xi_i-\bar{\xi}_i)/\sqrt{2}i$. Note that for $i\notin A'$, $\zeta_i^A=\zeta_i^B=\xi_i=\bar{\xi}_i=0$.

$$S=\frac{\beta}{2}\sum_{\omega_n}\begin{pmatrix}\gamma_1^A(\omega_n)\\ \gamma_1^B(\omega_n)\\ .\\ .\\ .\\ \gamma_N^A(\omega_n)\\ \gamma_N^B(\omega_n)\end{pmatrix}^T\begin{pmatrix}i\omega_n & i\epsilon_1 & & & & & \\ -i\epsilon_1 & i\omega_n & & & & & \\ & & . & & & & \\ & & & . & & & \\ & & & & . & & \\ & & & & & i\omega_n & i\epsilon_N\\ & & & & & -i\epsilon_N & i\omega_n\end{pmatrix}\begin{pmatrix}\gamma_1^A(-\omega_n)\\ \gamma_1^B(-\omega_n)\\ .\\ .\\ .\\ \gamma_N^A(-\omega_n)\\ \gamma_N^B(-\omega_n)\end{pmatrix}-\sum_{\omega_n}\begin{pmatrix}\gamma_1^A(\omega_n)\\ \gamma_1^B(\omega_n)\\ .\\ .\\ .\\ \gamma_N^A(\omega_n)\\ \gamma_N^B(\omega_n)\end{pmatrix}^T\begin{pmatrix}\zeta_1^A\\ \zeta_1^B\\ .\\ .\\ .\\ \zeta_N^A\\ \zeta_N^B\end{pmatrix} \tag{9}$$

$$\text{OR,}\quad S=\frac{\beta}{2}\sum_{\omega_n}\boldsymbol{\gamma}(\omega_n)^TB(\omega_n)\boldsymbol{\gamma}(-\omega_n)-\sum_{\omega_n}\boldsymbol{\gamma}(\omega_n)^T\boldsymbol{\zeta} \tag{10}$$

$$\implies S = \frac{\beta}{2}\sum_{\omega_n>0}[\boldsymbol{\gamma}(\omega_n)^T B(\omega_n)\boldsymbol{\gamma}(-\omega_n) + \boldsymbol{\gamma}(-\omega_n)^T B(-\omega_n)\boldsymbol{\gamma}(\omega_n)] - \sum_{\omega_n>0}[\boldsymbol{\gamma}(\omega_n)^T\boldsymbol{\zeta} + \boldsymbol{\gamma}(-\omega_n)^T\boldsymbol{\zeta}] \tag{11}$$

$$\implies S = \frac{\beta}{2}\sum_{\omega_n>0}\boldsymbol{\gamma}(\omega_n)^T[B(\omega_n) - B(-\omega_n)^T]\boldsymbol{\gamma}(-\omega_n) - \sum_{\omega_n>0}[\boldsymbol{\gamma}(\omega_n)^T\boldsymbol{\zeta} - \boldsymbol{\zeta}^T\boldsymbol{\gamma}(-\omega_n)] \tag{12}$$

$$\implies S = \beta\sum_{\omega_n>0}\boldsymbol{\gamma}(\omega_n)^T B(\omega_n)\boldsymbol{\gamma}(-\omega_n) + \sum_{\omega_n>0}[\boldsymbol{\zeta}^T\boldsymbol{\gamma}(-\omega_n) - \boldsymbol{\gamma}(\omega_n)^T\boldsymbol{\zeta}] \tag{13}$$

Now we use the standard independent Gaussian Grassmann integration result:

$$\int d(\bar{\phi},\phi)\ e^{-\bar{\phi}^T\boldsymbol{A}\phi + \bar{\eta}^T\cdot\phi + \bar{\phi}^T\cdot\eta} = \det\boldsymbol{A}\ e^{-\bar{\eta}^T\boldsymbol{A}^{-1}\eta}. \tag{14}$$

$$\chi_N(\boldsymbol{\xi}) = Z^{-1}\cdot(i)^N\cdot\int\mathcal{D}(\boldsymbol{\gamma}^A,\boldsymbol{\gamma}^B)\ e^{-S} \tag{15}$$

where now $\mathcal{D}(\boldsymbol{\gamma}^A,\boldsymbol{\gamma}^B) = \prod_{\omega_n>0} d(\boldsymbol{\gamma}(-\omega_n),\boldsymbol{\gamma}(\omega_n))$. Hence we get that

$$\chi_N(\boldsymbol{\xi}) = Z^{-1}\cdot(i)^N\cdot\left[\prod_{\omega_n>0}\det(\beta B(\omega_n))\right]\cdot\ \exp\left[\frac{1}{\beta}\sum_{\omega_n>0}\bar{\boldsymbol{\zeta}}^T B(\omega_n)^{-1}\boldsymbol{\zeta}\right] \tag{16}$$

The exponential part simplifies to

$$\exp\left[\frac{1}{\beta}\sum_{\omega_n>0}\sum_{ij}\bar{\xi}_i X_{ij}\xi_j\right],\ \text{ where }\ X_{ij} = -\frac{2\epsilon_i\delta_{ij}}{\omega_n^2+\epsilon_i^2} \tag{17}$$

Now the frequency sum becomes

$$-T\sum_{\omega_n>0}\frac{2\epsilon_i}{\omega_n^2+\epsilon_i^2} = T\sum_{\omega_n}\frac{1}{i\omega_n-\epsilon_i} = n_F(\epsilon_i) = \frac{1}{\exp(\epsilon_i-\mu)+1} = \langle c_i^\dagger c_i\rangle \tag{18}$$

This implies

$$\chi_N(\boldsymbol{\xi}) = Z^{-1}\cdot(i)^N\cdot\left[\prod_{\omega_n>0}\det(\beta B(\omega_n))\right]\cdot\ \exp\left[\sum_{ij}\bar{\xi}_i C_{ij}^T\xi_j\right],\ \text{ where }\ C_{ij}^T = n_F(\epsilon_i)\delta_{ij} = \langle c_i^\dagger c_i\rangle\delta_{ij}. \tag{19}$$

If we put all the source terms to zero($\boldsymbol{\xi}=0$), then $\chi_N(\boldsymbol{\xi}) = \mathrm{Tr}[\rho D_N(\boldsymbol{\xi}\in A')] = 1$. So this implies the constant factor infront of exponential term should be 1. Hence,

$$\chi_N(\boldsymbol{\xi}) = \exp\left[\sum_{ij\in A'}\bar{\xi}_i C_{ij}^T\xi_j\right]. \tag{20}$$

1.2 Calculating entanglement entropy for thermal state

If characteristic function of the form of eq.(20), then the n^{th} Renyi-entropy is given(in the master note):

$$S_A^{(n)} = \frac{1}{1-n}\mathrm{Tr}[\ln[(\mathbf{1}-\boldsymbol{C})^n + \boldsymbol{C}^n]]. \tag{21}$$

Only thing left is what are the ϵ_i's in terms of the Kitaev parameters?