Green's function

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1 Introduction

The Hamiltonian is given as $H = H_0 + V$ and it follows $H|\psi\rangle = i\hbar\frac{\partial\psi}{\partial t}$

1. **Schrodinger:** This is the normal Schrodinger equation, and here, wavefunction evolves in time, and operators are not time-dependent.

$$\psi_S(t) = e^{-iHt}\psi(0) \tag{1}$$

2. **Heisenberg:** Here, the wavefunction is time-independent, but the operators vary with time as below:

$$\psi_H(t) = e^{iHt}\psi_S(t) = \psi_S(0) \tag{2}$$

$$A_H(t) = e^{iHt}A_S\, e^{-iHt} \tag{3}$$

$$\implies i\frac{dA_H(t)}{dt} = [A_H(t), H] \tag{4}$$

3. **Interaction:** Here, both the wavefunction and operators vary with time as follows:

$$\psi_I(t) = e^{iH_0t}\psi_S(t) = e^{iH_0t}e^{-iHt}\psi_S(0) \tag{5}$$

$$A_I(t) = e^{iH_0t}A_S\, e^{-iH_0t} \tag{6}$$

$$\implies i\frac{dA_I(t)}{dt} = [A_I(t), H_0] \tag{7}$$

Unitary evolution as, $\psi_I(t) = U(t)\psi_I(0)$.

$$\frac{\partial U(t)}{\partial t} = -iV(t)U(t) \tag{8}$$

where the $V(t) \equiv V_I(t)$, is in the interaction picture.

$$
\begin{aligned}
\implies U(t) &= 1 - i\int_0^t dt_1 V(t_1) + (-i)^2 \int_0^t dt_1 \int_0^{t_1} dt_2 V(t_1)V(t_2) + ... \\
&= \sum_{n=0}^{\infty} (-i)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \; ... \int_0^{t_n} dt_n V(t_1)V(t_2)...V(t_n) \\
&= 1 - i\int_0^t dt_1 V(t_1) + \frac{(-i)^2}{2!} \int_0^t dt_1 \int_0^t dt_2 \; T[V(t_1)V(t_2)] + ... \qquad (9) \\
&= 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int_0^t dt_1 \int_0^t dt_2 \; ... \int_0^t dt_n \; T[V(t_1)V(t_2)...V(t_n)] \\
&\equiv T\exp\left[-i\int_0^t dt_1 \; V(t_1)\right]
\end{aligned}
$$

Note: In the above equations, V(t) is in the interaction picture operator ($V_I(t)$). S-matrix operator is defined as, $\psi_I(t) = S(t,t')\psi_I(t')$. Then the S-matrix is given as

$$S(t,t') = T\exp\left[-i\int_{t'}^t dt_1 \; V(t_1)\right] \qquad (10)$$

1.1 Example

Q. Consider a particle in ground state (at time t') of a harmonic oscillator, now add a time dependent force on it and then see what is the probability for it to be in ground state (at time t)? (Coleman MBP 2015)

$$H_0(t) = \hbar\omega\left(b^\dagger(t)b(t) + \frac{1}{2}\right), \quad V_I(t) = b^\dagger(t)\bar{z}(t) + b(t)z(t) \qquad (11)$$

where the time dependent operators in Heisenberg notation are $b(t) = be^{-i\omega t}$ and $b^\dagger(t) = b^\dagger e^{i\omega t}$. Then the quantity of interest that we need to calculate is

$$\langle 0|S(t,t')|0\rangle = \langle 0|T\exp\left[-i\int_{t'}^t dt_1 \; V(t_1)\right]|0\rangle \qquad (12)$$

$$S_N = e^{-iV(t_N)\Delta t_N} e^{-iV(t_{N-1})\Delta t_{N-1}} \cdots e^{-iV(t_1)\Delta t_1} = \prod_{j=1}^{N} e^{-iV(t_j)\Delta t_j} \qquad (13)$$

Define the quantities

$$A_r = b(t_r)z(t_r), \quad A_r^\dagger = b^\dagger(t_r)\bar{z}(t_r). \qquad (14)$$

We bring all annihilation operator to the right and creation ones to the left, with the use of below commutators

$$[A_i, A_j] = [A_i^\dagger, A_j^\dagger] = 0, \quad [A_i, A_j^\dagger] = z(t_i)\bar{z}(t_j)e^{-i\omega(t_i - t_j)}. \qquad (15)$$

Now, $e^{-iV(t_j)\Delta t_j} = e^{-iA_j\Delta t_j - iA_j^\dagger \Delta t_j}$ and we use the BCH identities

$$e^{\hat{\alpha}+\hat{\beta}} = e^{\hat{\alpha}} e^{\hat{\beta}} e^{-\frac{1}{2}[\hat{\alpha},\hat{\beta}]}, \quad e^{\hat{\alpha}} e^{\hat{\beta}} = e^{\hat{\beta}} e^{\hat{\alpha}} e^{[\hat{\alpha},\hat{\beta}]}. \tag{16}$$

when the commutator of $\hat{\alpha}$ and $\hat{\beta}$ is a c-number.

$$\implies e^{-iA_j\Delta t_j - iA_j^\dagger \Delta t_j} = e^{-iA_j^\dagger \Delta t_j} e^{-iA_j \Delta t_j} e^{-\frac{1}{2}\Delta t_j^2 z(t_j)\bar{z}(t_j)} \tag{17}$$

$$\implies e^{-iA_i\Delta t_i} e^{-iA_j^\dagger \Delta t_j} = e^{-iA_j^\dagger \Delta t_j} e^{-iA_i \Delta t_i} e^{-[A_i, A_j^\dagger]\Delta t_i \Delta t_j} \tag{18}$$

Hence, we can write our S_N as

$$S_N = e^{-i\sum_r A_r^\dagger \Delta t_r} e^{-i\sum_r A_r \Delta t_r} \exp\Big[-\sum_{i\geq j} [A_i, A_j^\dagger](1 - \frac{1}{2}\delta_{ij})\Delta t_i \Delta t_j \Big] \tag{19}$$

Taking all time intervals to be equal ($\Delta t_i = \Delta\tau$) and in the limit of $\Delta\tau \to 0$,

$$\langle 0|S(t,t')|0\rangle = \lim_{\Delta\tau\to 0} S_N = \lim_{\Delta\tau\to 0} \langle 0| e^{-i\sum_r A_r^\dagger \Delta\tau} e^{-i\sum_r A_r \Delta\tau} |0\rangle \exp\Big[-\sum_{i\geq j} [A_i, A_j^\dagger](1 - \frac{1}{2}\delta_{ij})\Delta\tau^2 \Big] \tag{20}$$

Note that the first part of the product becomes one, as all annihilation operators are on right and creation on left.

$$\begin{aligned}
\implies \langle 0|S(t,t')|0\rangle &= \lim_{\Delta\tau\to 0} \exp\Big[-\sum_{i\geq j} z(t_i)\bar{z}(t_j) e^{-i\omega(t_i - t_j)} (1 - \frac{1}{2}\delta_{ij})\Delta\tau^2 \Big] \\
&= \exp\Big[-\int_{t'}^{t} d\tau' d\tau \; \bar{z}(\tau)\theta(\tau' - \tau) e^{-i\omega(\tau' - \tau)} z(\tau') \Big] \qquad (21) \\
&= \exp\Big[-i \int_{t'}^{t} d\tau' d\tau \; \bar{z}(\tau) G(\tau - \tau') z(\tau') \Big]
\end{aligned}$$

Note, that we ignored the term δ_{ij} as it didn't contribute to the double integration significantly. Thus, our one-particle Green's function is given as

$$G(\tau - \tau') = -i\theta(\tau' - \tau) e^{-i\omega(\tau' - \tau)}. \tag{22}$$