Green's function

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September 2023

1 Introduction

The Hamiltonian is given as $H = H_0 + V$ and it follows $H|\psi\rangle = i\hbar\frac{\partial\psi}{\partial t}$

1. **Schrodinger:** This is the normal Schrodinger equation, and here, wavefunction evolves in time, and operators are not time-dependent.

$$\psi_S(t) = e^{-iHt}\psi(0) \tag{1}$$

2. **Heisenberg:** Here, the wavefunction is time-independent, but the operators vary with time as below:

$$\psi_H(t) = e^{iHt}\psi_S(t) = \psi_S(0) \tag{2}$$

$$A_H(t) = e^{iHt}A_S\, e^{-iHt} \tag{3}$$

$$\implies i\frac{dA_H(t)}{dt} = [A_H(t), H] \tag{4}$$

3. **Interaction:** Here, both the wavefunction and operators vary with time as follows:

$$\psi_I(t) = e^{iH_0t}\psi_S(t) = e^{iH_0t}e^{-iHt}\psi_S(0) \tag{5}$$

$$A_I(t) = e^{iH_0t}A_S\, e^{-iH_0t} \tag{6}$$

$$\implies i\frac{dA_I(t)}{dt} = [A_I(t), H_0] \tag{7}$$

Unitary evolution as, $\psi_I(t) = U(t)\psi_I(0)$.

$$\frac{\partial U(t)}{\partial t} = -iV(t)U(t) \tag{8}$$

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\begin{aligned}
\implies U(t) &= 1 - i\int_0^t dt_1 V(t_1) + (-i)^2 \int_0^t dt_1 \int_0^{t_1} dt_2 V(t_1)V(t_2) + ... \\
&= \sum_{n=0}^{\infty} (-i)^n \int_0^t dt_1 \int_0^{t_1} dt_2 \; ... \int_0^{t_n} dt_n V(t_1)V(t_2)...V(t_n) \\
&= 1 - i\int_0^t dt_1 V(t_1) + \frac{(-i)^2}{2!} \int_0^t dt_1 \int_0^t dt_2 \; T[V(t_1)V(t_2)] + ... \qquad (9) \\
&= 1 + \sum_{n=1}^{\infty} \frac{(-i)^n}{n!} \int_0^t dt_1 \int_0^t dt_2 \; ... \int_0^t dt_n \; T[V(t_1)V(t_2)...V(t_n)] \\
&\equiv T \exp\left[- i \int_0^t dt_1 \; V(t_1) \right]
\end{aligned}
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Note: In the above equations, V(t) is in the interaction picture operator ($V_I(t)$). S-matrix operator is defined as, $\psi_I(t) = S(t, t')\psi_I(t')$.