Problem Set 2 — ECE367

Problem 2.8: Quadratic Approximations

Part A: Gradients and Hessians

f_1:

```
Gradient ∇f1:
```

Out[81]:

$$\begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

```
∇2f1:
```

Out[83]:

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

f_2:

Gradient ∇f2:

Out[55]:
$$\begin{bmatrix} 2x - y \\ -x + 2y \end{bmatrix}$$

∇2f2:

Out[57]:
$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$$

f_3:

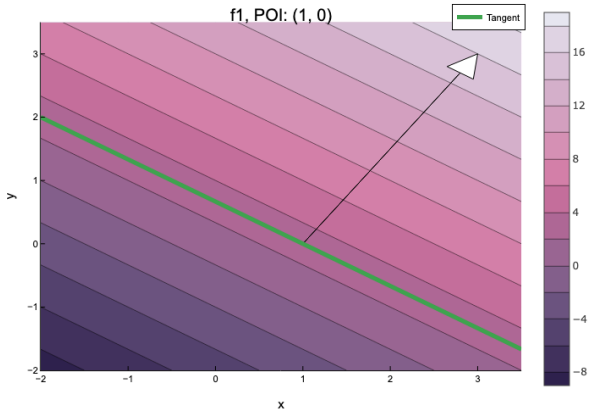
Gradient ∇f3:

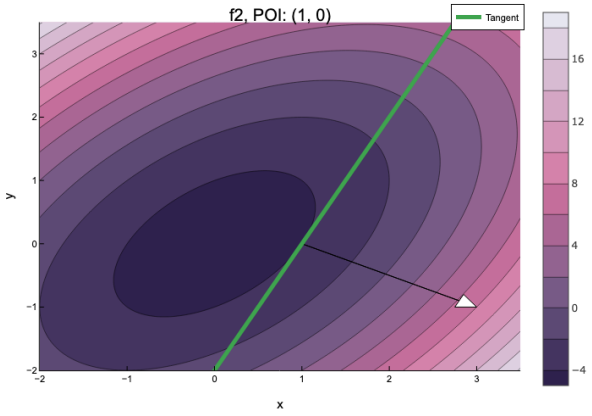
Out[56]:
$$\begin{bmatrix} (5 - y)\cos(x - 5) + \cos(y - 5) \\ -(x - 5)\sin(y - 5) - \sin(x - 5) \end{bmatrix}$$

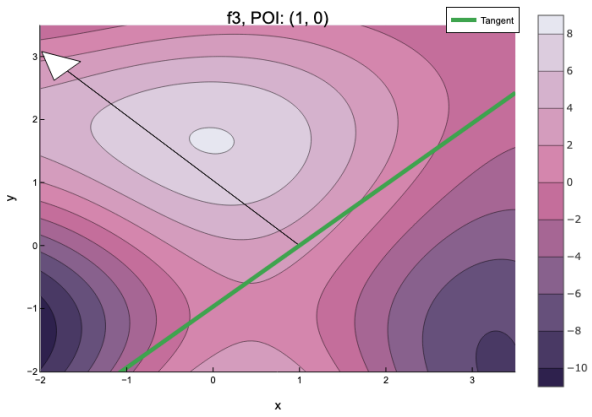
∇2f3:

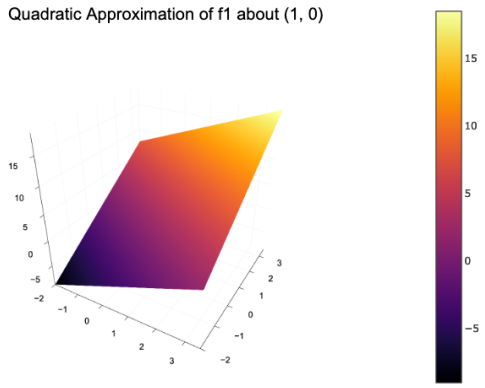
Out[58]:
$$\begin{bmatrix} (y - 5)\sin(x - 5) & -(\sin(y - 5) + \cos(x - 5)) \\ -(\sin(y - 5) + \cos(x - 5)) & -(x - 5)\cos(y - 5) \end{bmatrix}$$

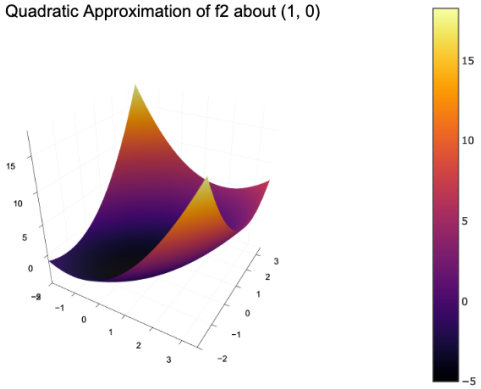
Part B: Plots Centered at $(1, 0)$

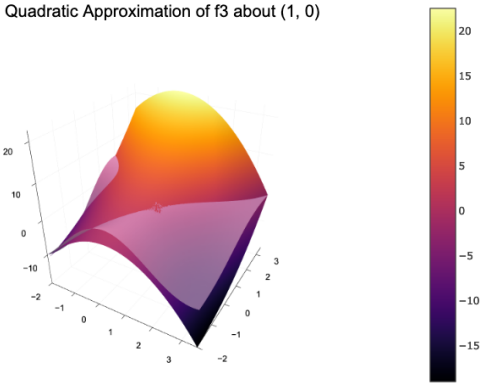




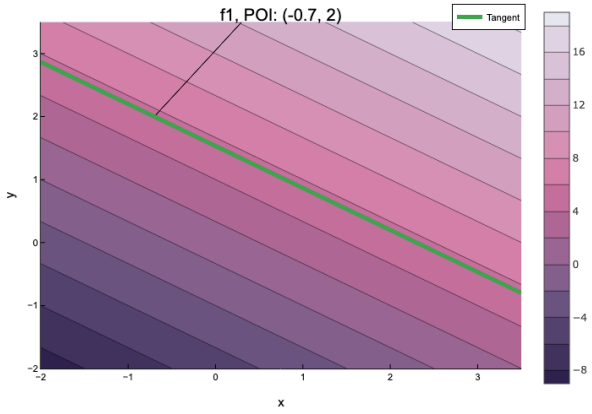


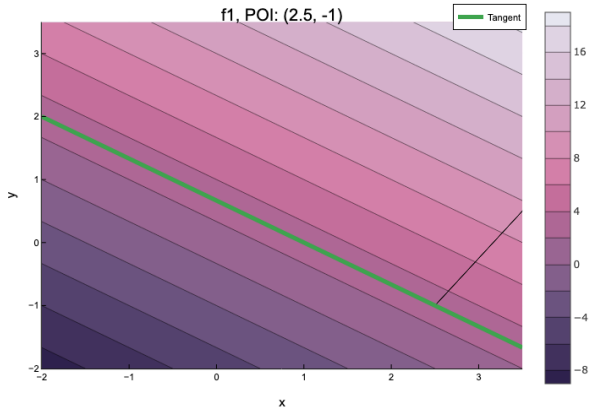


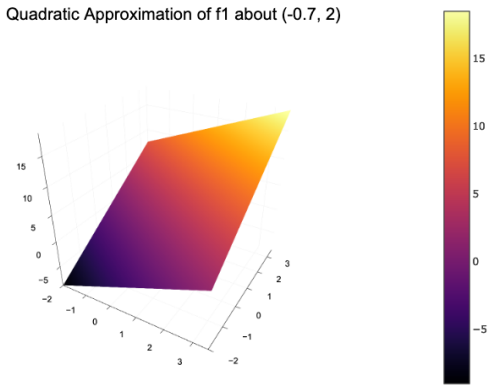


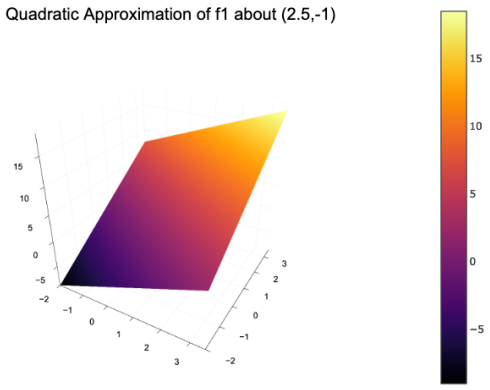


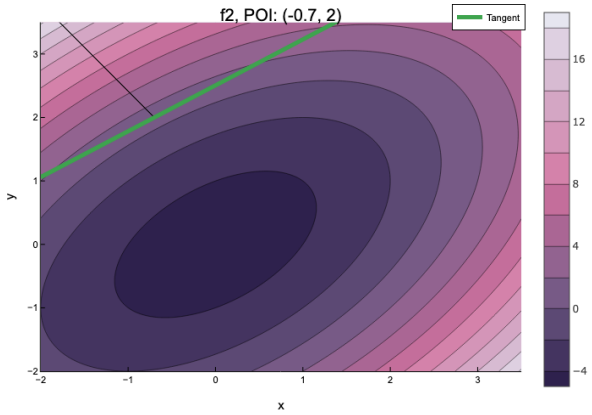
Part C: Plots Centered at $(-.7, 2)$ and $(2.5, -1)$

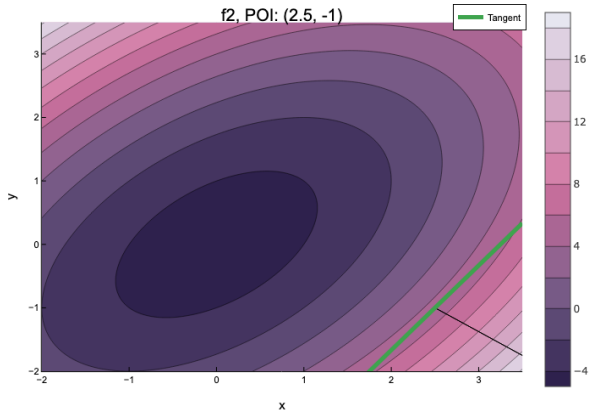


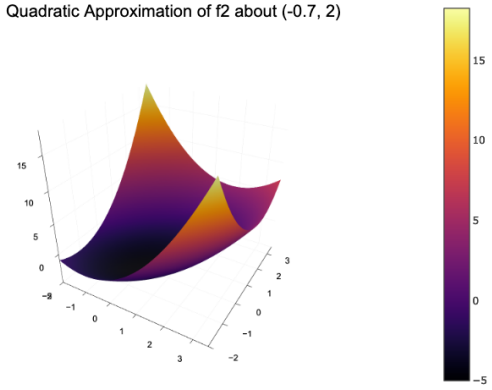


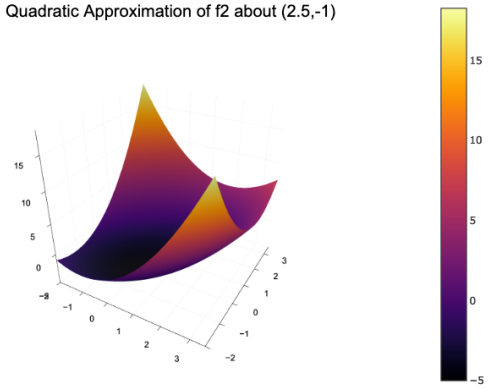

Quadratic Approximation of f1 about (-0.7, 2)

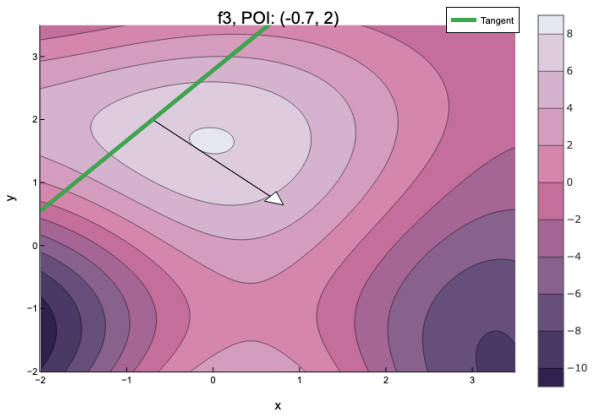

Quadratic Approximation of f1 about (2.5,-1)

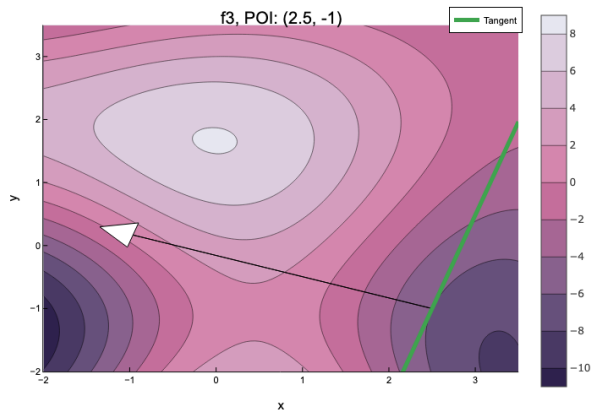

f2, POI: (-0.7, 2)
Tangent
x
y

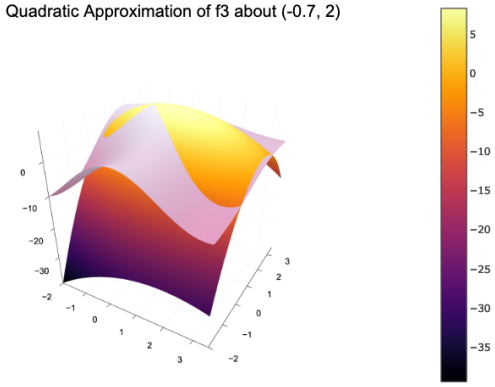

f2, POI: (2.5, -1)
Tangent
x
y

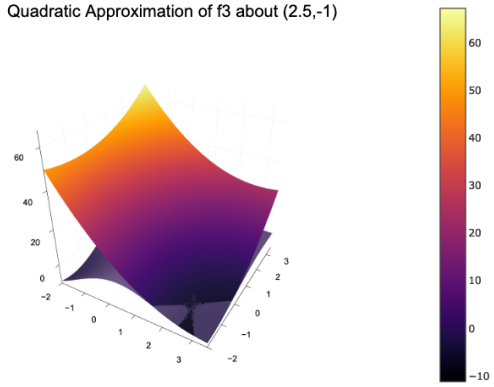

Quadratic Approximation of f2 about (-0.7, 2)


Quadratic Approximation of f2 about (2.5,-1)









Part D: Comments

Where are the approximations accurate? Where are they not? Why?

- The approximations are perfectly accurate for f_1 and f_2.
- The approximations are somewhat less accurate for f_3.
  - However, it still does a decent job of approximating f_3 very close to $\bar{x}$.
- This is because f_1 and f_2 are planar and quadratic surfaces respectively.
  - One can infer all the parameters of their governing equations from the value $f(\bar{x})$, the gradient, and the hessian.
  - Therefore, the approximations perfectly overlap with the original functions.

- Meanwhile, f_3 has a sinusoidal component.
  - One cannot infer all the parameters of their governing equations from the value $f(\bar{x})$, the gradient, and the hessian.
  - Furthermore, the quadratic approximation is not of the same form as the initial equation. One would need an infinite Taylor expansion about $\bar{x}$ to perfectly approximate the function.
  - That said, the function is still two-times differentiable and relatively smooth, so the second-order approximation has relatively low error in the neighbourhood of $\bar{x}$.

Problem 2.9: PageRank

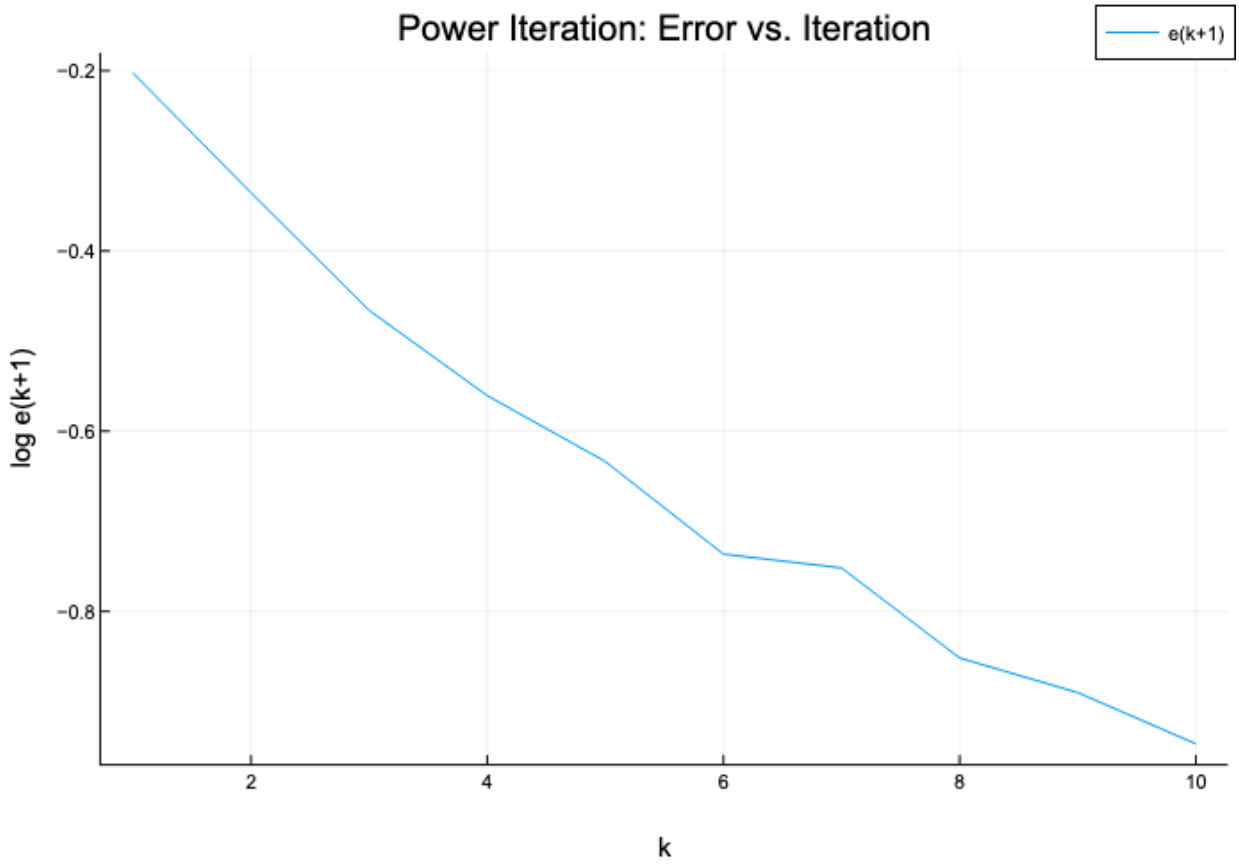
Part A

Why is the sum of each column being 1 important for the algorithm?

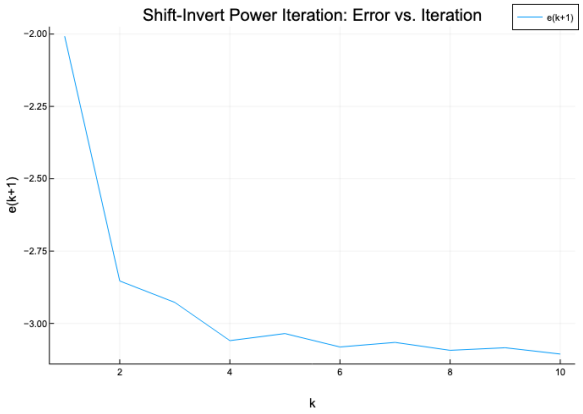
- The product of Ax where $x = [0, 0, ..., 1, ..., 0]^T$ describes the probability mass function (PMF) for x_{t+1}.
- Each column of A, therefore, must sum to 1 to meet the requirements for a PMF (i.e. the probability of the sample space S must be 1).
- If one multiplies a probability distribution x by A, they arrive at the probability distribution for x_{t+1}, necessitating the sum of each column of A to be 1.

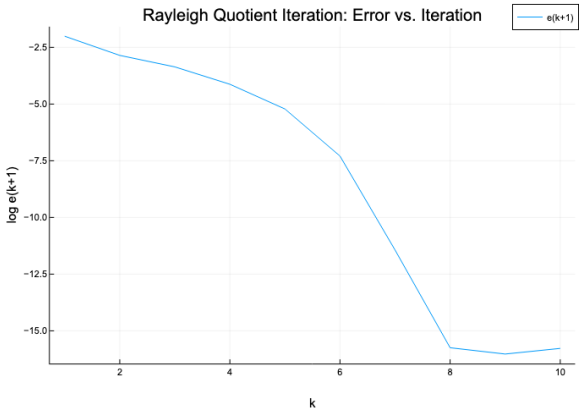
Part B: Power Iteration

See code in `PS02_2.9.ipynb`.



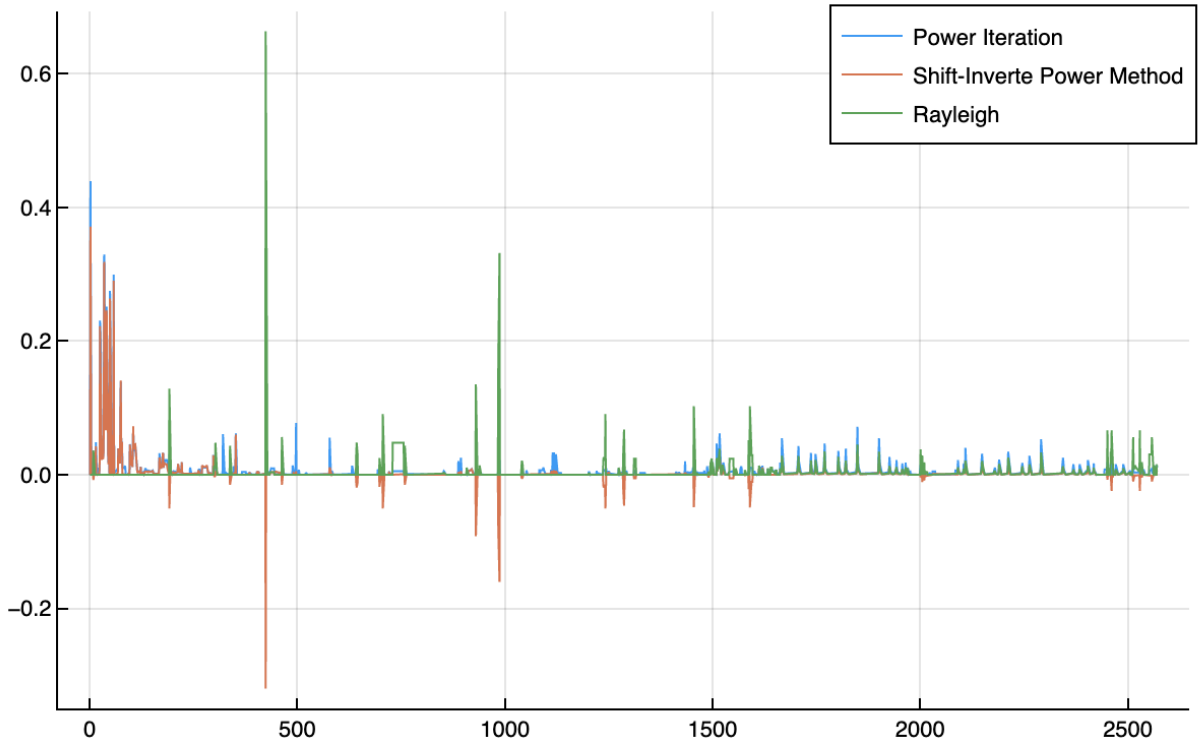
Part C: Shift-Invert Power Iteration & Rayleigh Quotient Iteration

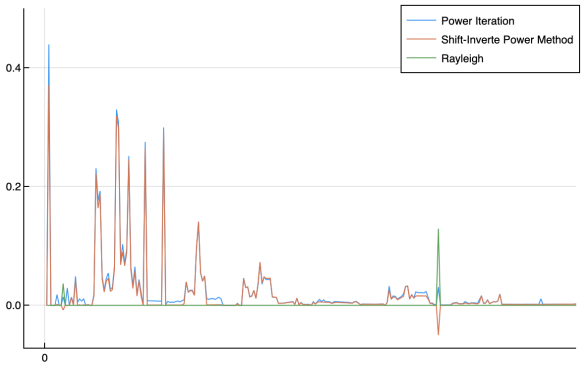


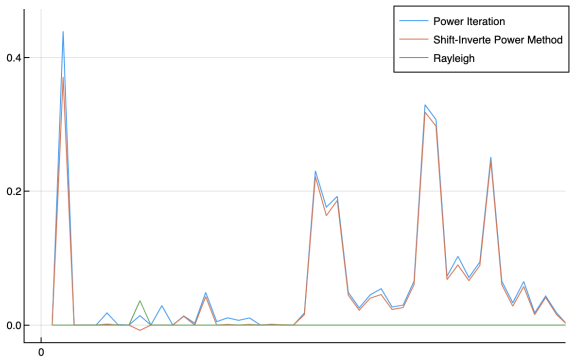


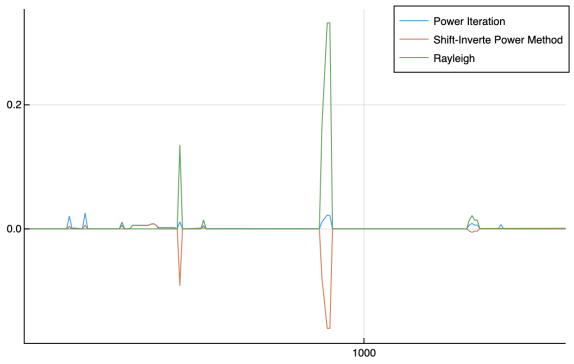
- The results for the loss are similar to the initial power iteration method, but they are much lower errors. In particular, the Rayleigh Quotient Iteration method appears to minimize to an extremely low error at the 8th iteration.
- However, as we will see in Part D, our results for the top and bottom 5 pages differ by each method.
- Furthermore, the Shift-Invert Power Iteration method at times yields negative values in the eigenvector.
- The below line plots overlaying the values of the final eigenvectors calculated through each method shows the differences futher:

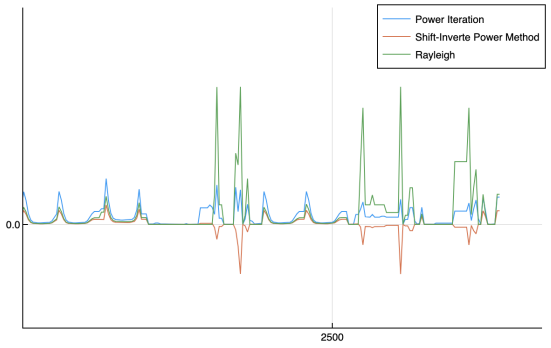
Diagnostic line plots comparing the values of the eigenvector indexes.











Part D: Top and Bottom 5 Pages

Power Iteration Scores

```
For Power Iteration
Top 1 Page: http://www.hollins.edu/ at index 2 with score 0.4393669395950616
Top 2 Page: http://www.hollins.edu/admissions/visit/visit.htm at index 35 with score 0.3295553218972332
Top 3 Page: http://www.hollins.edu/about/about_tour.htm at index 36 with score 0.3072679100390794
Top 4 Page: http://www.hollins.edu/htdig/index.html at index 58 with score 0.29959997614868467
Top 5 Page: http://www.hollins.edu/admissions/info-request/info-request.cfm at index 49 with score 0.2750916071185774
4
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Bottom 1 Page: http://www1.hollins.edu/ at index 1 with score 0.0
Bottom 2 Page: http://www1.hollins.edu/Docs/Forms/GetForms.htm at index 3 with score 0.0
Bottom 3 Page: http://www1.hollins.edu/Docs/misc/travel.htm at index 4 with score 0.0
Bottom 4 Page: http://www1.hollins.edu/Docs/GVCalendar/gvmain.htm at index 5 with score 0.0
Bottom 5 Page: http://www1.hollins.edu/Docs/comptech/comptech.htm at index 10 with score 0.0
```

Shift-Invert Power Iteration

```
For Shift-Invert Power Iteration (Not Taking Absolute Values)
Top 1 Page: http://www.hollins.edu/ at index 2 with score 0.371076800177363
Top 2 Page: http://www.hollins.edu/admissions/visit/visit.htm at index 35 with score 0.3184300571822739
Top 3 Page: http://www.hollins.edu/about/about_tour.htm at index 36 with score 0.2974898527981775
Top 4 Page: http://www.hollins.edu/htdig/index.html at index 58 with score 0.2903683424743939
Top 5 Page: http://www.hollins.edu/admissions/info-request/info-request.cfm at index 49 with score 0.2637480447079890
7
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Bottom 1 Page: http://www1.hollins.edu/homepages/hammerpw/qrhomepage.htm at index 424 with score -0.3196296934984905
Bottom 2 Page: http://www1.hollins.edu/homepages/hammerpw/qrcourses2.htm at index 987 with score -0.16021771066678325
Bottom 3 Page: http://www1.hollins.edu/homepages/hammerpw/qrcourses.htm at index 986 with score -0.16019371004811028
Bottom 4 Page: http://www1.hollins.edu/homepages/hammerpw/qractivities.htm at index 985 with score -0.120018672955278
38
Bottom 5 Page: http://www1.hollins.edu/homepages/godardrd/homepage.htm at index 930 with score -0.09168793383643452


For Shift-Invert Power Iteration (Absolute Values)
Top 1 Page: http://www.hollins.edu/ at index 2 with score 0.371076800177363
Top 2 Page: http://www1.hollins.edu/homepages/hammerpw/qrhomepage.htm at index 424 with score -0.3196296934984905
Top 3 Page: http://www.hollins.edu/admissions/visit/visit.htm at index 35 with score 0.3184300571822739
Top 4 Page: http://www.hollins.edu/about/about_tour.htm at index 36 with score 0.2974898527981775
Top 5 Page: http://www.hollins.edu/htdig/index.html at index 58 with score 0.2903683424743939
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Bottom 1 Page: http://www1.hollins.edu/homepages/hammerpw/qrhomepage.htm at index 424 with score -0.3196296934984905
Bottom 2 Page: http://www1.hollins.edu/homepages/hammerpw/qrcourses2.htm at index 987 with score -0.16021771066678325
Bottom 3 Page: http://www1.hollins.edu/homepages/hammerpw/qrcourses.htm at index 986 with score -0.16019371004811028
Bottom 4 Page: http://www1.hollins.edu/homepages/hammerpw/qractivities.htm at index 985 with score -0.120018672955278
38
Bottom 5 Page: http://www1.hollins.edu/homepages/godardrd/homepage.htm at index 930 with score -0.09168793383643452
```

Rayleigh Quotient Iteration

```
For Rayleigh Quotient Iteration
Top 1 Page: http://www1.hollins.edu/homepages/hammerpw/qrhomepage.htm at index 424 with score 0.663327738197711
Top 2 Page: http://www1.hollins.edu/homepages/hammerpw/qrcourses.htm at index 986 with score 0.33166386909885553
Top 3 Page: http://www1.hollins.edu/homepages/hammerpw/qrcourses2.htm at index 987 with score 0.3316638690988555
Top 4 Page: http://www1.hollins.edu/homepages/hammerpw/qractivities.htm at index 985 with score 0.24874790182414167
Top 5 Page: http://www1.hollins.edu/homepages/hammerpw/qrgrantsummary.htm at index 984 with score 0.16583193454942774
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Bottom 1 Page: http://www1.hollins.edu/ at index 1 with score 3.713492277236943e-74
Bottom 2 Page: http://www.hollins.edu/academics/library/libtoc.htm at index 48 with score 3.713492277236943e-74
Bottom 3 Page: http://www1.hollins.edu/Docs/GVCalendar/gvmain.htm at index 5 with score 5.404751202091903e-74
Bottom 4 Page: http://www1.hollins.edu/Docs/Forms/GetForms.htm at index 3 with score 5.4047512020919126e-74
Bottom 5 Page: http://www1.hollins.edu/Docs/misc/travel.htm at index 4 with score 5.4047512020919126e-74
```

Discussion

- This discussion pertains primarily to the Rayleigh Quotient Iteration method results as this yielded the lowest error.
    - The ordering seems to make sense. The highest ranked page is a homepage, and it makes sense that many pages would link back to it. Similar arguments can be made for the course pages, the activities, and the grant summary.
    - For the least popular pages, the most unpopular one is the homepage with `www1` prefix. That makes sense because it seems likely that people would tend to link back to the `www` prefix version. The library subpage, the gvmain, forms, and travel pages are similarly uninteresting.

- That said, it is concerning that low errors were achieved for the three approaches but the top 5 and bottom 5 pages differ so drastically.
    - In fact, the bottom 5 pages for the Shift-Invert Power Iteration method include pages that are listed in the top 5 pages for the Rayleigh Quotient Iteration method.
    - This suggest that there *may* be local optima for eigenvalues that are calculated through this method that are far from the globally optimal eigenvalue(s).