job if she receives a strong recommendation, a 40 percent chance if she receives a moderately good recommendation, and a 10 percent chance if she receives a weak recommendation. She further estimates that the probabilities that the recommendation will be strong, moderate, and weak are .7, .2, and .1, respectively.

(a) How certain is she that she will receive the new job offer?

(b) Given that she does receive the offer, how likely should she feel that she received a strong recommendation? a moderate recommendation? a weak recommendation?

(c) Given that she does not receive the job offer, how likely should she feel that she received a strong recommendation? a moderate recommendation? a weak recommendation?

3.52. A high school student is anxiously waiting to receive mail telling her whether she has been accepted to a certain college. She estimates that the conditional probabilities of receiving notification on each day of next week, given that she is accepted and that she is rejected, are as follows:

Day	P(mail|accepted)	P(mail|rejected)
Monday	.15	.05
Tuesday	.20	.10
Wednesday	.25	.10
Thursday	.15	.15
Friday	.10	.20

She estimates that her probability of being accepted is .6.

(a) What is the probability that she receives mail on Monday?

(b) What is the conditional probability that she received mail on Tuesday given that she does not receive mail on Monday?

(c) If there is no mail through Wednesday, what is the conditional probability that she will be accepted?

(d) What is the conditional probability that she will be accepted if mail comes on Thursday?

(e) What is the conditional probability that she will be accepted if no mail arrives that week?

3.53. A parallel system functions whenever at least one of its components works. Consider a parallel system of n components, and suppose that each component works independently with probability $\frac{1}{2}$. Find the conditional probability that component 1 works given that the system is functioning.

3.54. If you had to construct a mathematical model for events E and F, as described in parts (a) through (e), would you assume that they were independent events? Explain your reasoning.

(a) E is the event that a businesswoman has blue eyes, and F is the event that her secretary has blue eyes.

(b) E is the event that a professor owns a car, and F is the event that he is listed in the telephone book.

(c) E is the event that a man is under 6 feet tall, and F is the event that he weighs over 200 pounds.

(d) E is the event that a woman lives in the United States, and F is the event that she lives in the Western Hemisphere.

(e) E is the event that it will rain tomorrow, and F is the event that it will rain the day after tomorrow.

3.55. In a class, there are 4 freshman boys, 6 freshman girls, and 6 sophomore boys. How many sophomore girls must be present if sex and class are to be independent when a student is selected at random?

3.56. Suppose that you continually collect coupons and that there are m different types. Suppose also that each time a new coupon is obtained, it is a type i coupon with probability $p_i, i = 1, \ldots, m$. Suppose that you have just collected your nth coupon. What is the probability that it is a new type?
Hint: Condition on the type of this coupon.

3.57. A simplified model for the movement of the price of a stock supposes that on each day the stock's price either moves up 1 unit with probability p or moves down 1 unit with probability $1 - p$. The changes on different days are assumed to be independent.

(a) What is the probability that after 2 days the stock will be at its original price?

(b) What is the probability that after 3 days the stock's price will have increased by 1 unit?

(c) Given that after 3 days the stock's price has increased by 1 unit, what is the probability that it went up on the first day?

3.58. Suppose that we want to generate the outcome of the flip of a fair coin, but that all we have at our disposal is a biased coin which lands on heads with some unknown probability p that need not be equal to $\frac{1}{2}$. Consider the following procedure for accomplishing our task:

1. Flip the coin.
2. Flip the coin again.
3. If both flips land on heads or both land on tails, return to step 1.
4. Let the result of the last flip be the result of the experiment.

(a) Show that the result is equally likely to be either heads or tails.
(b) Could we use a simpler procedure that continues to flip the coin until the last two flips are different and then lets the result be the outcome of the final flip?

3.59. Independent flips of a coin that lands on heads with probability p are made. What is the probability that the first four outcomes are
(a) H, H, H, H?
(b) T, H, H, H?
(c) What is the probability that the pattern T, H, H, H occurs before the pattern H, H, H, H?
Hint for part (c): How can the pattern H, H, H, H occur first?

3.60. The color of a person's eyes is determined by a single pair of genes. If they are both blue-eyed genes, then the person will have blue eyes; if they are both brown-eyed genes, then the person will have brown eyes; and if one of them is a blue-eyed gene and the other a brown-eyed gene, then the person will have brown eyes. (Because of the latter fact, we say that the brown-eyed gene is *dominant* over the blue-eyed one.) A newborn child independently receives one eye gene from each of its parents, and the gene it receives from a parent is equally likely to be either of the two eye genes of that parent. Suppose that Smith and both of his parents have brown eyes, but Smith's sister has blue eyes.
(a) What is the probability that Smith possesses a blue-eyed gene?
(b) Suppose that Smith's wife has blue eyes. What is the probability that their first child will have blue eyes?
(c) If their first child has brown eyes, what is the probability that their next child will also have brown eyes?

3.61. Genes relating to albinism are denoted by A and a. Only those people who receive the a gene from both parents will be albino. Persons having the gene pair A, a are normal in appearance and, because they can pass on the trait to their offspring, are called carriers. Suppose that a normal couple has two children, exactly one of whom is an albino. Suppose that the nonalbino child mates with a person who is known to be a carrier for albinism.
(a) What is the probability that their first offspring is an albino?
(b) What is the conditional probability that their second offspring is an albino given that their firstborn is not?

3.62. Barbara and Dianne go target shooting. Suppose that each of Barbara's shots hits a wooden duck target with probability p_1, while each shot of Dianne's hits it with probability p_2. Suppose that they shoot simultaneously at the same target. If the wooden duck is knocked over (indicating that it was hit), what is the probability that
(a) both shots hit the duck?
(b) Barbara's shot hit the duck?
What independence assumptions have you made?

3.63. A and B are involved in a duel. The rules of the duel are that they are to pick up their guns and shoot at each other simultaneously. If one or both are hit, then the duel is over. If both shots miss, then they repeat the process. Suppose that the results of the shots are independent and that each shot of A will hit B with probability p_A, and each shot of B will hit A with probability p_B. What is
(a) the probability that A is not hit?
(b) the probability that both duelists are hit?
(c) the probability that the duel ends after the nth round of shots?
(d) the conditional probability that the duel ends after the nth round of shots given that A is not hit?
(e) the conditional probability that the duel ends after the nth round of shots given that both duelists are hit?

3.64. A true–false question is to be posed to a husband-and-wife team on a quiz show. Both the husband and the wife will independently give the correct answer with probability p. Which of the following is a better strategy for the couple?
(a) Choose one of them and let that person answer the question.
(b) Have them both consider the question, and then either give the common answer if they agree or, if they disagree, flip a coin to determine which answer to give.

3.65. In Problem 3.5, if $p = .6$ and the couple uses the strategy in part (b), what is the conditional probability that the couple gives the correct answer given that the husband and wife (a) agree? (b) disagree?

3.66. The probability of the closing of the ith relay in the circuits shown in Figure 3.4 is given by $p_i, i = 1, 2, 3, 4, 5$. If all relays function independently, what is the probability that a current flows between A and B for the respective circuits?
Hint for (b): Condition on whether relay 3 closes.

3.67. An engineering system consisting of n components is said to be a k-out-of-n system ($k \leq n$) if the system functions if and only if at least k of the n components function. Suppose that all components function independently of each other.
(a) If the ith component functions with probability $P_i, i = 1, 2, 3, 4$, compute the probability that a 2-out-of-4 system functions.

over, and so on. Let P_1 be the probability of the coin's landing on heads when A flips and P_2 when B flips. The winner of the game is the first one to get

(a) 2 heads in a row;
(b) a total of 2 heads;
(c) 3 heads in a row;
(d) a total of 3 heads.

In each case, find the probability that A wins.

3.83. Die A has 4 red and 2 white faces, whereas die B has 2 red and 4 white faces. A fair coin is flipped once. If it lands on heads, the game continues with die A; if it lands on tails, then die B is to be used.

(a) Show that the probability of red at any throw is $\frac{1}{2}$.
(b) If the first two throws result in red, what is the probability of red at the third throw?
(c) If red turns up at the first two throws, what is the probability that it is die A that is being used?

3.84. An urn contains 12 balls, of which 4 are white. Three players—A, B, and C—successively draw from the urn, A first, then B, then C, then A, and so on. The winner is the first one to draw a white ball. Find the probability of winning for each player if

(a) each ball is replaced after it is drawn;
(b) the balls that are withdrawn are not replaced.

3.85. Repeat Problem 3.84 when each of the 3 players selects from his own urn. That is, suppose that there are 3 different urns of 12 balls with 4 white balls in each urn.

3.86. Let $S = \{1, 2, \ldots, n\}$ and suppose that A and B are, independently, equally likely to be any of the 2^n subsets (including the null set and S itself) of S.

(a) Show that

$$P\{A \subset B\} = \left(\frac{3}{4}\right)^n$$

Hint: Let $N(B)$ denote the number of elements in B. Use

$$P\{A \subset B\} = \sum_{i=0}^{n} P\{A \subset B | N(B) = i\} P\{N(B) = i\}$$

Show that $P\{AB = \emptyset\} = \left(\frac{3}{4}\right)^n$.

3.87. In Example 5e, what is the conditional probability that the ith coin was selected given that the first n trials all result in heads?

3.88. In Laplace's rule of succession (Example 5e), are the outcomes of the successive flips independent? Explain.

3.89. A person tried by a 3-judge panel is declared guilty if at least 2 judges cast votes of guilty. Suppose that when the defendant is in fact guilty, each judge will independently vote guilty with probability .7, whereas when the defendant is in fact innocent, this probability drops to .2. If 70 percent of defendants are guilty, compute the conditional probability that judge number 3 votes guilty given that

(a) judges 1 and 2 vote guilty;
(b) judges 1 and 2 cast 1 guilty and 1 not guilty vote;
(c) judges 1 and 2 both cast not guilty votes.

Let $E_i, i = 1, 2, 3$ denote the event that judge i casts a guilty vote. Are these events independent. Are they conditionally independent? Explain.

3.90. Suppose that n independent trials, each of which results in any of the outcomes 0, 1, or 2, with respective probabilities p_0, p_1, and p_2, $\sum_{i=0}^{2} p_i = 1$, are performed. Find the probability that outcomes 1 and 2 both occur at least once.

THEORETICAL EXERCISES

3.1. Show that if $P(A) > 0$, then

$$P(AB|A) \geq P(AB|A \cup B)$$

3.2. Let $A \subset B$. Express the following probabilities as simply as possible:

$$P(A|B), \quad P(A|B^c), \quad P(B|A), \quad P(B|A^c)$$

3.3. Consider a school community of m families, with n_i of them having i children, $i = 1, \ldots, k$, $\sum_{i=1}^{k} n_i = m$. Consider the following two methods for choosing a child:

1. Choose one of the m families at random and then randomly choose a child from that family.
2. Choose one of the $\sum_{i=1}^{k} in_i$ children at random.

Show that method 1 is more likely than method 2 to result in the choice of a firstborn child.

Hint: In solving this problem, you will need to show that

$$\sum_{i=1}^{k} in_i \sum_{j=1}^{k} \frac{n_j}{j} \geq \sum_{i=1}^{k} n_i \sum_{j=1}^{k} n_j$$

if the top-ranked person is female.) Find P{X = i}, $i = 1, 2, 3, \ldots, 8, 9, 10$.

4.5. Let X represent the difference between the number of heads and the number of tails obtained when a coin is tossed n times. What are the possible values of X?

4.6. In Problem 5, for $n = 3$, if the coin is assumed fair, what are the probabilities associated with the values that X can take on?

4.7. Suppose that a die is rolled twice. What are the possible values that the following random variables can take on:
- **(a)** the maximum value to appear in the two rolls;
- **(b)** the minimum value to appear in the two rolls;
- **(c)** the sum of the two rolls;
- **(d)** the value of the first roll minus the value of the second roll?

4.8. If the die in Problem 7 is assumed fair, calculate the probabilities associated with the random variables in parts (a) through (d).

4.9. Repeat Example 1b when the balls are selected with replacement.

4.10. In Example 1d, compute the conditional probability that we win i dollars, given that we win something; compute it for $i = 1, 2, 3$.

4.11. **(a)** An integer N is to be selected at random from $\{1, 2, \ldots, (10)^3\}$ in the sense that each integer has the same probability of being selected. What is the probability that N will be divisible by 3? by 5? by 7? by 15? by 105? How would your answer change if $(10)^3$ is replaced by $(10)^k$ as k became larger and larger?

(b) An important function in number theory—one whose properties can be shown to be related to what is probably the most important unsolved problem of mathematics, the Riemann hypothesis—is the Möbius function $\mu(n)$, defined for all positive integral values n as follows: Factor n into its prime factors. If there is a repeated prime factor, as in $12 = 2 \cdot 2 \cdot 3$ or $49 = 7 \cdot 7$, then $\mu(n)$ is defined to equal 0. Now let N be chosen at random from $\{1, 2, \ldots (10)^k\}$, where k is large. Determine $P\{\mu(N) = 0\}$ as $k \to \infty$.

Hint: To compute $P\{\mu(N) \neq 0\}$, use the identity

$$\prod_{i=1}^{\infty} \frac{P_i^2 - 1}{P_i^2} = \left(\frac{3}{4}\right)\left(\frac{8}{9}\right)\left(\frac{24}{25}\right)\left(\frac{48}{49}\right)\cdots = \frac{6}{\pi^2}$$

where P_i is the ith-smallest prime. (The number 1 is not a prime.)

4.12. In the game of Two-Finger Morra, 2 players show 1 or 2 fingers and simultaneously guess the number of fingers their opponent will show. If only one of the players guesses correctly, he wins an amount (in dollars) equal to the sum of the fingers shown by him and his opponent. If both players guess correctly or if neither guesses correctly, then no money is exchanged. Consider a specified player, and denote by X the amount of money he wins in a single game of Two-Finger Morra.
- **(a)** If each player acts independently of the other, and if each player makes his choice of the number of fingers he will hold up and the number he will guess that his opponent will hold up in such a way that each of the 4 possibilities is equally likely, what are the possible values of X and what are their associated probabilities?
- **(b)** Suppose that each player acts independently of the other. If each player decides to hold up the same number of fingers that he guesses his opponent will hold up, and if each player is equally likely to hold up 1 or 2 fingers, what are the possible values of X and their associated probabilities?

4.13. A salesman has scheduled two appointments to sell encyclopedias. His first appointment will lead to a sale with probability .3, and his second will lead independently to a sale with probability .6. Any sale made is equally likely to be either for the deluxe model, which costs \$1000, or the standard model, which costs \$500. Determine the probability mass function of X, the total dollar value of all sales.

4.14. Five distinct numbers are randomly distributed to players numbered 1 through 5. Whenever two players compare their numbers, the one with the higher one is declared the winner. Initially, players 1 and 2 compare their numbers; the winner then compares her number with that of player 3, and so on. Let X denote the number of times player 1 is a winner. Find $P\{X = i\}, i = 0, 1, 2, 3, 4$.

4.15. The National Basketball Association (NBA) draft lottery involves the 11 teams that had the worst won–lost records during the year. A total of 66 balls are placed in an urn. Each of these balls is inscribed with the name of a team: Eleven have the name of the team with the worst record, 10 have the name of the team with the second-worst record, 9 have the name of the team with the third-worst record, and so on (with 1 ball having the name of the team with the 11th-worst record). A ball is then chosen at random, and the team whose name is on the ball is given the first pick in the draft of players about to enter the league. Another ball is then chosen, and if it "belongs" to a team different from the one that received the first draft pick, then the team to which it belongs receives the second draft pick. (If the ball belongs

N_1 matches and the right-hand box contained N_2 matches.

4.77. In the Banach matchbox problem, find the probability that, at the moment when the first box is emptied (as opposed to being found empty), the other box contains exactly k matches.

4.78. An urn contains 4 white and 4 black balls. We randomly choose 4 balls. If 2 of them are white and 2 are black, we stop. If not, we replace the balls in the urn and again randomly select 4 balls. This continues until exactly 2 of the 4 chosen are white. What is the probability that we shall make exactly n selections?

4.79. Suppose that a batch of 100 items contains 6 that are defective and 94 that are not defective. If X is the number of defective items in a randomly drawn sample of 10 items from the batch, find (a) $P\{X = 0\}$ and (b) $P\{X > 2\}$.

4.80. A game popular in Nevada gambling casinos is Keno, which is played as follows: Twenty numbers are selected at random by the casino from the set of numbers 1 through 80. A player can select from 1 to 15 numbers; a win occurs if some fraction of the player's chosen subset matches any of the 20 numbers drawn by the house. The payoff is a function of the number of elements in the player's selection and the number of matches. For instance, if the player selects only 1 number, then he or she wins if this number is among the set of 20, and the payoff is \$2.2 won for every dollar bet. (As the player's probability of winning in this case is $\frac{1}{4}$, it is clear that the "fair" payoff should be \$3 won for every \$1 bet.) When the player selects 2 numbers, a payoff (of odds) of \$12 won for every \$1 bet is made when both numbers are among the 20,

(a) What would be the fair payoff in this case?
Let $P_{n,\,k}$ denote the probability that exactly k of the n numbers chosen by the player are among the 20 selected by the house.

(b) Compute $P_{n,\,k}$

(c) The most typical wager at Keno consists of selecting 10 numbers. For such a bet the casino pays off as shown in the following table. Compute the expected payoff:

Keno Payoffs in 10 Number Bets

Number of matches	Dollars won for each \$1 bet
0–4	-1
5	1
6	17
7	179
8	1,299
9	2,599
10	24,999

4.81. In Example 8i, what percentage of i defective lots does the purchaser reject? Find it for $i = 1, 4$. Given that a lot is rejected, what is the conditional probability that it contained 4 defective components?

4.82. A purchaser of transistors buys them in lots of 20. It is his policy to randomly inspect 4 components from a lot and to accept the lot only if all 4 are nondefective. If each component in a lot is, independently, defective with probability .1, what proportion of lots is rejected?

4.83. There are three highways in the county. The number of daily accidents that occur on these highways are Poisson random variables with respective parameters .3, .5, and .7. Find the expected number of accidents that will happen on any of these highways today.

4.84. Suppose that 10 balls are put into 5 boxes, with each ball independently being put in box i with probability p_i, $\sum_{i=1}^{5} p_i = 1$.

(a) Find the expected number of boxes that do not have any balls.

(b) Find the expected number of boxes that have exactly 1 ball.

4.85. There are k types of coupons. Independently of the types of previously collected coupons, each new coupon collected is of type i with probability p_i, $\sum_{i=1}^{k} p_i = 1$. If n coupons are collected, find the expected number of distinct types that appear in this set. (That is, find the expected number of types of coupons that appear at least once in the set of n coupons.)

THEORETICAL EXERCISES

4.1. There are N distinct types of coupons, and each time one is obtained it will, independently of past choices, be of type i with probability $P_i, i = 1, \ldots, N$. Let T denote the number one need select to obtain at least one of each type. Compute $P\{T = n\}$.
Hint: Use an argument similar to the one used in Example 1e.

5.7. The density function of X is given by

$$f(x) = \begin{cases} a + bx^2 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

If $E[X] = \frac{3}{5}$, find a and b.

5.8. The lifetime in hours of an electronic tube is a random variable having a probability density function given by

$$f(x) = xe^{-x} \qquad x \ge 0$$

Compute the expected lifetime of such a tube.

5.9. Consider Example 4b of Chapter 4, but now suppose that the seasonal demand is a continuous random variable having probability density function f. Show that the optimal amount to stock is the value s^* that satisfies

$$F(s^*) = \frac{b}{b + \ell}$$

where b is net profit per unit sale, ℓ is the net loss per unit unsold, and F is the cumulative distribution function of the seasonal demand.

5.10. Trains headed for destination A arrive at the train station at 15-minute intervals starting at 7 A.M., whereas trains headed for destination B arrive at 15-minute intervals starting at 7:05 A.M.

(a) If a certain passenger arrives at the station at a time uniformly distributed between 7 and 8 A.M. and then gets on the first train that arrives, what proportion of time does he or she go to destination A?

(b) What if the passenger arrives at a time uniformly distributed between 7:10 and 8:10 A.M.?

5.11. A point is chosen at random on a line segment of length L. Interpret this statement, and find the probability that the ratio of the shorter to the longer segment is less than $\frac{1}{4}$.

5.12. A bus travels between the two cities A and B, which are 100 miles apart. If the bus has a breakdown, the distance from the breakdown to city A has a uniform distribution over (0, 100). There is a bus service station in city A, in B, and in the center of the route between A and B. It is suggested that it would be more efficient to have the three stations located 25, 50, and 75 miles, respectively, from A. Do you agree? Why?

5.13. You arrive at a bus stop at 10 o'clock, knowing that the bus will arrive at some time uniformly distributed between 10 and 10:30.

(a) What is the probability that you will have to wait longer than 10 minutes?

(b) If, at 10:15, the bus has not yet arrived, what is the probability that you will have to wait at least an additional 10 minutes?

5.14. Let X be a uniform (0, 1) random variable. Compute $E[X^n]$ by using Proposition 2.1, and then check the result by using the definition of expectation.

5.15. If X is a normal random variable with parameters $\mu = 10$ and $\sigma^2 = 36$, compute

(a) $P\{X > 5\}$;

(b) $P\{4 < X < 16\}$;

(c) $P\{X < 8\}$;

(d) $P\{X < 20\}$;

(e) $P\{X > 16\}$.

5.16. The annual rainfall (in inches) in a certain region is normally distributed with $\mu = 40$ and $\sigma = 4$. What is the probability that, starting with this year, it will take over 10 years before a year occurs having a rainfall of over 50 inches? What assumptions are you making?

5.17. A man aiming at a target receives 10 points if his shot is within 1 inch of the target, 5 points if it is between 1 and 3 inches of the target, and 3 points if it is between 3 and 5 inches of the target. Find the expected number of points scored if the distance from the shot to the target is uniformly distributed between 0 and 10.

5.18. Suppose that X is a normal random variable with mean 5. If $P\{X > 9\} = .2$, approximately what is $\text{Var}(X)$?

5.19. Let X be a normal random variable with mean 12 and variance 4. Find the value of c such that $P\{X > c\} = .10$.

5.20. If 65 percent of the population of a large community is in favor of a proposed rise in school taxes, approximate the probability that a random sample of 100 people will contain

(a) at least 50 who are in favor of the proposition;

(b) between 60 and 70 inclusive who are in favor;

(c) fewer than 75 in favor.

5.21. Suppose that the height, in inches, of a 25-year-old man is a normal random variable with parameters $\mu = 71$ and $\sigma^2 = 6.25$. What percentage of 25-year-old men are over 6 feet, 2 inches tall? What percentage of men in the 6-footer club are over 6 feet, 5 inches?

5.22. The width of a slot of a duralumin forging is (in inches) normally distributed with $\mu = .9000$ and $\sigma = .0030$. The specification limits were given as $.9000 \pm .0050$.

(a) What percentage of forgings will be defective?