E0-226 Linear Algebra Module

Assignment 1

1. Find the eigenvalues and eigenvectors of

$$\begin{pmatrix} 2 & 1 & 5 \\ 4 & 2 & 10 \\ 2 & 1 & 5 \end{pmatrix}$$

2. Find the rank and all the eigenvalues of the following matrix:

$$\boldsymbol{A} = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$$

   Which eigenvectors correspond to non-zero eigenvalues? Find the eigenvalues and determinant of $\boldsymbol{A} - 7I$.

3. Find the determinants of $\boldsymbol{A}$ and $\boldsymbol{A}^{-1}$ if

$$\boldsymbol{A} = \boldsymbol{S} \begin{pmatrix} \lambda_1 & 2 \\ 0 & \lambda_2 \end{pmatrix} \boldsymbol{S}^{-1}$$

4. If $\boldsymbol{A}$ has eigenvalues 0 and 1 corresponding to the eigenvectors $(1, 2)^T$ and $(2, -1)^T$, is $\boldsymbol{A}$ symmetric? What are its trace and determinant? What is $\boldsymbol{A}$? What will be the eigenvalues and eigenvectors of $\boldsymbol{A}^2$?

5. If $\boldsymbol{Ax} = \lambda_1 \boldsymbol{x}$ and $\boldsymbol{A}^T\boldsymbol{y} = \lambda_2 \boldsymbol{y}$ (all real), show that $\boldsymbol{x}^T\boldsymbol{y} = 0$.

6. Consider the projection matrix, $\boldsymbol{P} = \frac{\boldsymbol{x}\boldsymbol{x}^T}{\boldsymbol{x}^T\boldsymbol{x}}$, which projects onto a line. What is the trace of this matrix? Is $\boldsymbol{P}$ invertible? Why or why not?

7. Let $\boldsymbol{A} \in \mathbb{R}^{3\times 3}$ be a matrix whose columns are $\boldsymbol{u}, \boldsymbol{v}$ and $\boldsymbol{w}$ and are linearly independent. Let $\boldsymbol{Q} = (\boldsymbol{q}_1|\boldsymbol{q}_2|\boldsymbol{q}_3)$ be the matrix whose columns are obtained by using Gram-Schmidt orthogonalization process on the set $\{\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}\}$. Find the matrix $\boldsymbol{R}$ such that $\boldsymbol{A} = \boldsymbol{QR}$.

8. If $\boldsymbol{A}$ and $\boldsymbol{B}$ are invertible matrices, do the matrices $\boldsymbol{AB}$ and $\boldsymbol{BA}$ have same eigenvalues? Prove this claim if is true. Otherwise, give a counterexample.

9. For any $\boldsymbol{A}$ and $\boldsymbol{b}$, prove that one and only one of the following systems has a solution:

   (a) $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$

   (b) $\boldsymbol{A}^T\boldsymbol{y} = \boldsymbol{0}, \boldsymbol{y}^T\boldsymbol{b} \neq 0$

10. Give an example of a 3 by 3 matrix to show that the eigenvalues of the matrix can be changed when a multiple of one row is subtracted from another.

11. Show that the quadratic $f(x_1, x_2) = x_1^2 + 4x_1x_2 + 2x_2^2$ has a saddle point at the origin.

12. Find the minimum of the function $f(x_1, x_2, x_3) = 2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3$.

13. Construct 2×2 real symmetric matrices $\boldsymbol{A}$ and $\boldsymbol{B}$ to verify that $\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{B}\boldsymbol{x}$ might not have real eigenvalues.

14. For the ellipse $x_1^2 + x_1x_2 + x_2^2 = 1$, find the half lengths of its axes from the eigenvalues of the corresponding matrix $\boldsymbol{A}$.

15. Find the minimum value of

$$R(x_1, x_2) = \frac{x_1^2 - x_1x_2 + x_2^2}{x_1^2 + x_2^2}$$