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Probability Assignment - 3

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(a) Let us consider the ~~event~~ experiment of tossing the coin twice. The sample space is,

$$S = \{HH, HT, TH, TT\}$$

Let us now compute the probabilities of each outcome. We know that the coin is biased and,

$$P(H) = p$$
$$\text{So, } P(T) = 1 - p = q$$

→ Therefore,

$$P(HH) = p^2$$
$$P(HT) = p \cdot q$$
$$P(TH) = q \cdot p$$
$$P(TT) = q^2$$

The outcome of tosses is independent for every toss. So, we simply multiply the probabilities of the independent events to get the result.

→ In every iteration of our procedure, we toss the coin twice. We terminate only if we get different outcomes for the two tosses. So, we define;

C → Denotes the event that both the outcomes of the two tosses are identical, and we need to continue tossing.

$$P(C) = P(\{HH, TT\}) = p^2 + q^2$$

→ We now view our experiment as a series of trials, where each trial means tossing of coin twice. It is guaranteed that the total no. of tosses in the entire experiment is always even, since we do two tosses exactly per iteration.

→ Our experiment terminates when the outcome of our trial is:

i) HT → In this case, we conclude the outcome of entire experiment as 'T'.

or

ii) TH → In this case, we conclude the outcome of entire experiment as 'H'.

→ We define H' and T' as,

H' → The event that the outcome of last two tosses is (T,H) in that order.

T' → The event that the outcome of last two tosses is (H,T) in that order.

→ Now, the result of entire experiment is 'H' if the outcome of our trials follows the pattern,

$P(H') = P(TH) = qp$

H' → (TH)

or, CH' → (HH/TT)(TH)

or, CCH' → (HH/TT)(HH/TT)(TH)

or, CCCH' → (HH/TT)(HH/TT)(HH/TT)(TH)

⋮

and so on.

→ Therefore, the probability of getting the result of entire experiment as 'Head' =

$P[(H') \cup (CH') \cup (CCH') \cup \cdots]$

$= P(H') + P(CH') + P(CCH') + \cdots$ (Since, each is disjoint)

$= qp + (p^2+q^2)qp + (p^2+q^2)(p^2+q^2)qp + \cdots$

$= qp + (p^2+q^2)qp + (p^2+q^2)^2 qp + \cdots$

$= qp[1 + (p^2+q^2) + (p^2+q^2)^2 + \cdots]$

$= qp\left\{\frac{1}{1-(p^2+q^2)}\right\}$

$= \frac{qp}{1-p^2-q^2} = \frac{(1-p)p}{(1-p^2)-(1-p)^2} = \frac{p}{(1+p)-(1-p)} = \frac{p}{2p} = \frac{1}{2}$ —(1)

→ Now, the result of entire experiment is 'T' if the outcome of our trials follows the pattern,

$P(T') = P(HT) = pq$

T' → (HT)

or, CT' → (HH/TT)(HT)

or, CCT' → (HH/TT)(HH/TT)(HT)

or, CCCT' → (HH/TT)(HH/TT)(HH/TT)(HT)

⋮

and so on.

→ Therefore, the probability of getting the result of entire experiment as 'Tail' =

$$P[(T') \cup (CT') \cup (CCT') \cup \ldots]$$
$$= P(T') + P(CT') + P(CCT') + \ldots$$
$$= pq + (p^2+q^2)pq + (p^2+q^2)(p^2+q^2)pq + \ldots$$
$$= pq\left[1 + (p^2+q^2) + (p^2+q^2)^2 + \ldots\right]$$
$$= pq\left\{\frac{1}{1-(p^2+q^2)}\right\}$$

$$= \frac{pq}{1-p^2-q^2} = \frac{p(1-p)}{(1-p^2)-(1-p)^2} = \frac{p}{1+p-1+p} = \frac{p}{2p} = \frac{1}{2} \quad \text{—— ②}$$

From ① & ②, we conclude that the probability of getting 'H' or 'T' is equal. We could thus accomplish our task of simulating the outcome of a fair coin, using our biased coin in above manner.

(b) In the second procedure, we terminate our experiment, when the outcome of last two tosses is different. ~~So,~~ Also, the outcome of last flip is considered as the result of entire experiment.

→ The outcome of entire experiment is 'H' if our outcomes are of the following pattern:

TH, TTH, TTTH, ……

Hence, the probability of getting 'head' as the result of entire experiment is:

$$P[(TH) \cup (TTH) \cup (TTTH) \ldots]$$
$$= P(TH) + P(TTH) + P(TTTH) \ldots \qquad (\text{disjoint outcomes})$$
$$= qp + q^2p + q^3p + \ldots$$
$$= p[q + q^2 + q^3 + \ldots] = p\left(\frac{q}{1-q}\right) = p\left(\frac{q}{p}\right) = q$$

→ The outcome of entire experiment is 'T' if our outcomes are of the following pattern:

HT, HHT, HHHT, ……

Hence, the probability of getting 'tail' as the result of entire experiment is :

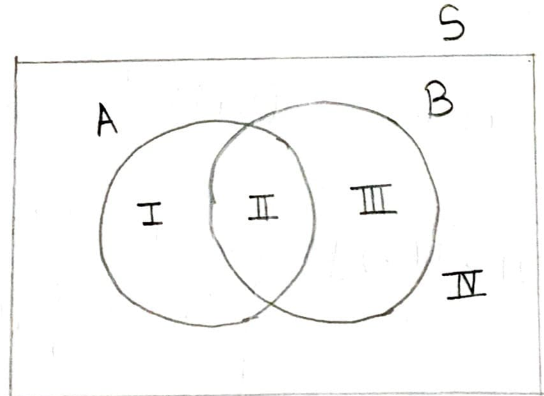
$$P[(HT) \cup (HHT) \cup (HHHT) \cdots]$$
$$= P(HT) + P(HHT) + P(HHHT) + \cdots$$
$$= pq + p^2 q + p^3 q + \cdots$$
$$= q[p + p^2 + p^3 + \cdots] = q\left(\frac{p}{1-p}\right) = q\left(\frac{p}{q}\right) = p$$

→ Probability of getting 'H' as result of entire exp. $= q$
________ " ________ 'T' ________ " ________ $= p$

Therefore, we conclude that this simpler procedure does not ensure that the result is equally likely to 'H' or 'T'. It does not help us simulate a fair coin.

② (a) $S = \{1, 2, \ldots, n\}$

Now, A and B are equally likely to be any of the 2^n subsets. Every choice of A and B can be represented by a venn diagram. For e.g.,



We now divide the venn diagram into four regions.

I → $A - B$
II → $A \cap B$
III → $B - A$
IV → $(A \cup B)^c$

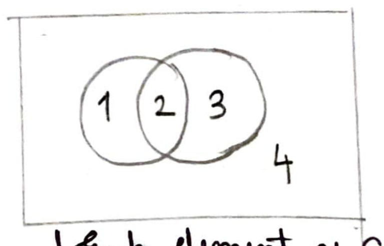
→ These four regions are non-overlapping. We claim that any choice of A and B can be represented by a unique mapping of the elements of S to one of the four regions. For e.g.,

$n = 4$

$S = \{1, 2, 3, 4\}$

If $A = \{1, 2\}$
$B = \{2, 3\}$

1 → I
2 → II
3 → III
4 → IV



Then we have the following mapping.

Each element of S is mapped to one region exactly.

5 → Our claim is true because for any choice of A and B, we can draw the venn diagram or just compute the sets $(A-B)$, $(B-A)$, $(A \cap B)$ and $(A \cup B)^c$ and assign the mapping. The same can be done in reverse order. That is, given a mapping of elements of S to the regions, we can figure out A and B.

→ For $A \subseteq B$, we require that $A - B = \emptyset$. That is, the elements of S can be either in $(A \cap B)$, or $(B-A)$ or $(A \cup B)^c$ but not in $\underline{A-B}$.

→ Now the probability that we select A and B such that $A \subseteq B$, is the product of probabilities of independent events E_i for $i = 1, 2, \ldots, n$.

E_i → Denotes the event that $i \in S$, is mapped to any region except $(A-B)$.

$P(E_i) = 3/4$, this is because out of 4 possible regions, we have eliminated one (I) region.

So,

$$P(A \subseteq B) = P(E_1 \cap E_2 \cap \ldots \cap E_n)$$
$$= \underbrace{P(E_1) \cdot P(E_2) \ldots P(E_n)}$$

$\boxed{\forall i,\ P(E_i) = 3/4}$

These are mutually independent events as mapping of one element does not influence mapping for another.

$$P(A \subseteq B) = \left(\frac{3}{4}\right)\left(\frac{3}{4}\right) \cdots \left(\frac{3}{4}\right) = \left(\frac{3}{4}\right)^n$$

(b) — For $A \cap B = \emptyset$, we require that all elements of S get mapped to either $(A-B)$ or $(B-A)$ or $(A \cup B)^c$ but not $A \cap B$. Just like above, we now eliminate region II from our choice.

F_i → Denotes the event that $i \in S$, is mapped to any region except $(A \cap B)$. Hence $P(F_i) = 3/4$ just like above. We have 3 choices for all $i \in S$.

So,

$$P(A\cap B=\phi) = P(F_1 \cap F_2 \cap \ldots \cap F_n)$$

$$= \underbrace{P(F_1).\,P(F_2)\ldots P(F_n)}_{\text{mutually independent events}}$$

$\boxed{\forall i,\ P(F_i) = 3/4}$

$$= \left(\tfrac{3}{4}\right)\left(\tfrac{3}{4}\right)\ldots\left(\tfrac{3}{4}\right)$$

$$P(A\cap B=\phi) = \left(\tfrac{3}{4}\right)^n$$

③ (a) $P\{X=0\}$

This means player 1 looses in first round. Let N_i denote the value / number assigned to player i.

So, for $P\{X=0\}$,

$$N_1 < N_2$$

The numbers assigned to other players do not follow any restriction. N_3, N_4 and N_5 could be any number as long as $N_1 < N_2$.

Let every assignment follow the bellow pattern.

$$\begin{pmatrix} N_4 & N_1 & N_3 & N_2 & N_5 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix} \leftarrow \text{An example permutation}$$

Thus, any unique assignment of numbers can be represented as a one-to-one mapping between a permutation of N_1, N_2, N_3, N_4 and N_5 with the sequence $(1, 2, 3, 4, 5)$. In the above case,

Player 1 $(N_1) \rightarrow 2$
Player 2 $(N_2) \rightarrow 4$
Player 3 $(N_3) \rightarrow 3$
Player 4 $(N_4) \rightarrow 1$
Player 5 $(N_5) \rightarrow 5$

Thus, N_i ~~is~~ can also be thought of as the place holder for value assigned.

For above problem let us construct all permutations such that $N_1 < N_2$. We know N_1 lies ahead of N_2 in the permutation. $(\_\ N_1\ \_\ N_2\ \_)$

N_3 has 3 choices

Using same argument, after choosing a place for N_3, we have 4 possible choices for N_4, and after that 5 choices for N_5. Thus, the total number of ways, we can construct permutation subject to restriction

$N_1 < N_2 \quad = \quad 3 \times 4 \times 5 = 60$

Thus, we have 60 permutations of $(N_1, N_2, \ldots N_5)$ where N_1 lies before N_2 in the sequence, and hence after assignment to $(1, 2, 3, 4, 5)$ $N_1 < N_2$.

Now, total possible permutations of $(N_1, N_2, N_3, N_4, N_5)$ $= 5! = 120$

Hence, $P(\{X=0\}) = \dfrac{60}{120} = \dfrac{1}{2}$

(b) $P\{X=1\}$

This happens when, $N_1 > N_2$
but, $N_1 < N_3$

That is, $N_2 < N_1 < N_3$.

There are no restriction for N_4 and N_5

$\_\, N_2 \,\_\, N_1 \,\_\, N_3 \,\_$

Possible choices for placing $N_4 = 4$
$\quad\quad$ " $\quad\quad$ $N_5 = 5$

Total permutation subject to $N_2 < N_1 < N_3$ $= 4 \times 5 = 20$

Hence, $P(\{X=1\}) = \dfrac{20}{120} = \dfrac{1}{6}$

(c) $P\{X=2\}$

This happens when, $N_1 > N_2$
$N_1 > N_3$
$N_1 < N_4$

Now we have two cases:

i) $N_2 < N_3 < N_1 < N_4$
ii) $N_3 < N_2 < N_1 < N_4$

For case (i),

$$\_N_2\_N_3\_N_1\_N_4\_$$

N_5 can be placed in 5 ways.

Similarly, for case (ii), N_5 can be placed in 5 ways. In total, we have 10 permutations as both cases are disjoint.

Total permutation subject to $N_2 < N_3 < N_1 < N_4$ or, & $N_3 < N_2 < N_1 < N_4$ $= 10$

$$P(\{X=2\}) = 10/120 = 1/12$$

(d) $P\{X=3\}$

This happens when,
$$N_1 > N_2$$
$$N_1 > N_3$$
$$N_1 > N_4$$
$$N_1 < N_5$$

That is,

$$(N_2, N_3, N_4) < N_1 < N_5$$

The resulting permutations are of the type,

$$\underbrace{\_\ \_\ \_}_{(N_2, N_3, N_4)}\ N_1\, N_5$$

To arrange N_2, N_3, N_4 we have $3! = 6$ possibilities.

Therefore, $P(\{X=3\}) = \dfrac{6}{120} = \dfrac{1}{20}$

(e) $P\{X=4\}$

This happens when
$$N_1 > N_2$$
$$N_1 > N_3$$
$$N_1 > N_4$$
$$N_1 > N_5$$

That is, $N_1 \Rightarrow 5$.

The resulting permutations are of the type,

$$\underbrace{\_\ \_\ \_\ \_}_{(N_2, N_3, N_4, N_5)}\ N_1$$

The N_2, N_3, N_4, N_5 can be arranged in $4! = 24$ ways.

Therefore,

$$P\{X=4\} = \frac{24}{120} = \frac{1}{5}$$

4 Our experiment can be viewed as a series of trials, where each trial involves drawing 4 balls randomly. The urn contains 4 white and 4 black balls. We can draw 4 balls out of 8 in ${}^{8}C_4$ ways.

→ Let us define a trial to be 'successful' when we draw 4 balls such that 2 of them are black, 2 are white. Otherwise, the trial to be a 'failure'.

→ Now, the trials are independent because the balls are replaced if the trial is a failure. Also if the trial is a 'success', we halt.

S → 'Successful' Trial

F → 'Failure' Trial

$$P(S) = \frac{{}^{4}C_2 \times {}^{4}C_2}{{}^{8}C_4} = \frac{36}{70} = \frac{18}{35}$$

$$P(F) = 1 - P(S) = \frac{17}{35}$$

→ The experiment will contain exactly 'n' trials if we 'fail' exactly (n-1) times, and then have a 'success'.

$$\underbrace{F\ F\ F \ldots\ldots F}_{n-1}\ S$$

→ The probability that we have exactly 'n' trials is:

$$\underbrace{P(F).\,P(F).\,P(F) \ldots P(F).\,P(S)}_{\text{All trials are mutually independent, hence we take product of probabilities.}}$$

$$= \left(\frac{17}{35}\right)^{n-1}\left(\frac{18}{35}\right)$$

5. Let X be a continuous random ~~r~~ variable representing the seasonal demand. Also,

$s \rightarrow$ units stocked
$b \rightarrow$ profit per unit
$l \rightarrow$ loss per unit

$f(x) \rightarrow$ probability density function of X

Hence, we can write profit as,

$$P(s) = \begin{cases} bX - l(s-X), & X \leq s \\ bs & X > s \end{cases}$$

$$= \begin{cases} (b+l)X - ls, & X \leq s \\ bs, & X > s \end{cases}$$

The expected profit equals,

$$E[P(s)] = \int_{-\infty}^{s} [(b+l)x - ls]\, f(x).dx + \int_{s}^{\infty} [bs] f(x).dx$$

Now,

$$\Rightarrow \int_{-\infty}^{s} f(x).dx + \int_{s}^{\infty} f(x).dx = 1$$

$$\Rightarrow \int_{s}^{\infty} f(x).dx = 1 - \int_{-\infty}^{s} f(x).dx \quad \text{——(1)}$$

Using (1),

$$E[P(s)] = (b+l)\int_{-\infty}^{s} x f(x).dx - ls\int_{-\infty}^{s} f(x).dx + bs\left(1 - \int_{-\infty}^{s} f(x).dx\right)$$

$$E[P(s)] = (b+l)\int_{-\infty}^{s} x f(x).dx - s(b+l)\int_{-\infty}^{s} f(x).dx + bs$$

Now, we want to maximize our expected profit. For that we compute $\frac{d}{ds}(E[P(s)]) = 0$. The value satisfying this eq^n, will be the optimal amount of stock s^*, giving us maximum expected profit.

Before proceeding let us compute the following,

$$\frac{d}{ds}\int_{-\infty}^{s} x\, f(x).\,dx$$

Here, we apply Lebinitz rule i.e.,

$$\frac{d}{ds}\int_{u(s)}^{v(s)} g(x).\,dx = g(v(s))\,v'(s) - g(u(s))\,u'(s)$$

In our case,

$$u(s) = -\infty \quad , u'(s) = 0$$
$$v(s) = s \quad , v'(s) = 1$$

So,

$$\frac{d}{ds}\int_{-\infty}^{s} x\, f(x).\,dx = s f(s) \quad \text{——— (2)}$$

Also,

$$\frac{d}{ds}\int_{-\infty}^{s} f(x).\,dx = f(s) \quad \text{——— (3)}$$

Now,

$$E[P(s)] = (b+l)\int_{-\infty}^{s} x\, f(x).\,dx - s(b+l)\int_{-\infty}^{s} f(x).\,dx + bs$$

$$\frac{d}{ds}[E(P(s))] = (b+l)[s f(s)] - \left[s(b+l) f(s) + (b+l)\int_{-\infty}^{s} f(x).\,dx \right] + b$$

Equating $\frac{d}{ds}[E[P(s)]] = 0$,

$$0 = b - (b+l)\int_{-\infty}^{s} f(x).\,dx$$

Here, $\int_{-\infty}^{s} f(x).\,dx = F(s) \longrightarrow$ Cumulative distribution function.

$$\boxed{F(s) = \frac{b}{b+l}}$$

Thus, expected profit is maximal if we choose $s = s^*$, where $F(s^*) = b/_{b+l}$.

But, we need to also verify the point s^*, by computing $\frac{d^2[E[P(s)]]}{ds^2}$.

Differentiating ④,

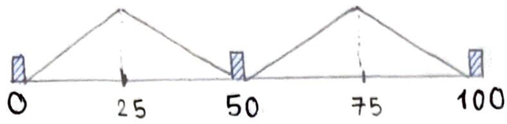
$$\frac{d^2[E[P(s)]]}{ds^2} = -(b+l)\,f(s) \quad \text{——} ⑤$$

Since, $f(s)$ is the probability that demand is equal to s, it is non-negative. The probability density function always assumes non-negative values for all values of x. Also, 'b' and 'l' are positive constants. Therefore, $-[(b+l)f(s)]$ is negative.

→ Thus, expected profit is maximum if we choose $s = s^*$ where,

$$\boxed{F(s^*) = \frac{b}{b+l}}$$

(6) In the first case,



The figure shows refill stations at 0, 50 and 100. It also shows a representative figure of the nearest service station.
Let X be the uniform random variable representing location of breakdown.

$$f_X(x) = \begin{cases} \frac{1}{100}, & 0 \le x \le 100 \\ 0, & \text{otherwise} \end{cases}$$

Let Y be the function of random variable X representing distance to nearest service station.

$$Y(x) = \cancel{f_Y(x)} = \begin{cases} x, & x \le 25 \\ 50-x, & 25 < x \le 50 \\ x-50, & 50 < x \le 75 \\ 100-x, & 75 < x \le 100 \end{cases}$$

Thus expected distance to nearest service station,

$$E[Y] = \int_{-\infty}^{\infty} Y(x) \cdot f_X(x) \cdot dx$$

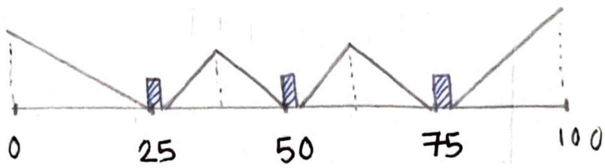
$$= \left[\int_0^{25} x \cdot \frac{1}{100} \cdot dx\right] + \left[\int_{25}^{50} (50-x)\frac{1}{100} \cdot dx\right] + \left[\int_{50}^{75} (x-50) \cdot \frac{1}{100} \cdot dx\right] + \left[\int_{75}^{100} (100-x) \cdot \frac{1}{100} \cdot dx\right]$$

$$= \frac{1}{100}\left\{\left[\frac{x^2}{2}\right]_0^{25} + \left[50x - \frac{x^2}{2}\right]_{25}^{50} + \left[\frac{x^2}{2} - 50x\right]_{50}^{75} + \left[100x - \frac{x^2}{2}\right]_{75}^{100}\right\}$$

$$= \frac{1}{100}(312.5 + 312.5 + 312.5 + 312.5)$$

$$= \frac{1}{100}(1250) = 12.50 \text{ miles}$$

In the second case,



Let Z be the function of random variable X representing distance to nearest service station.

$$Z(x) = \begin{cases} 25 - x, & x \leq 25 \\ x - 25, & 25 < x \leq 37.5 \\ 50 - x, & 37.5 < x \leq 50 \\ x - 50, & 50 < x \leq 62.5 \\ 75 - x, & 62.5 < x \leq 75 \\ x - 75, & 75 < x \leq 100 \end{cases}$$

The expected distance to nearest service station,

$$E[Z(x)] = \int_{-\infty}^{\infty} Z(x) . f_X(x) . dx$$

$$= \int_0^{25} (25 - x) \frac{1}{100} . dx + \int_{25}^{37.5} (x - 25) \frac{1}{100} . dx + \int_{37.5}^{50} (50 - x) \frac{1}{100} . dx + \int_{50}^{62.5} (x - 50) \frac{1}{100} . dx + \int_{62.5}^{75} (75 - x) \frac{1}{100} . dx + \int_{75}^{100} (x - 75) \frac{1}{100} . dx$$

$$= \frac{1}{100} \left\{ \left(25x - \frac{x^2}{2}\right)_0^{25} + \left(\frac{x^2}{2} - 25x\right)_{25}^{37.5} + \left(50x - \frac{x^2}{2}\right)_{37.5}^{50} + \left(\frac{x^2}{2} - 50x\right)_{50}^{62.5} + \left(75x - \frac{x^2}{2}\right)_{62.5}^{75} + \left(\frac{x^2}{2} - 75x\right)_{75}^{100} \right.$$

$$= \frac{1}{100} (312.5 + 78.125 + 78.125 + 78.125 + 78.125 + 312.5)$$

$$= \frac{1}{100} (937.5) = 9.375 \text{ miles}$$ (< 12.50 miles of case 1

The second arrangement is more efficient as it reduces the expected distance to nearest service station.