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Problem 1

Let, $A_K = \{B_K \cup B_{K+1} \cup B_{K+2} \cup \ldots\}$ be an event.

Therefore,

$$P\left[\bigcap_{N=0}^{\infty} \bigcup_{n=N}^{\infty} B_n\right]$$

$$= P\left[\bigcap_{N=0}^{\infty} A_N\right]$$

Now,

$A_0 \supseteq A_1 \supseteq A_2 \supseteq A_3 \supseteq \ldots$

because

$A_0 = B_0 \cup A_1$

$A_1 = B_1 \cup A_2$

$A_2 = B_2 \cup A_3$

and so on ....

Therefore, we can use continuity of ~~pro~~probability,

$$P\left[\bigcap_{N=0}^{\infty} A_N\right] = \lim_{N \to \infty} P[A_N]$$

It is given that,

$$\sum_{i=0}^{\infty} P(B_i) < \infty$$

This means that this sum converges to a finite value, lets say S.

$$\lim_{K \to \infty} \sum_{i=0}^{K} P(B_i) = S$$

Now, $\lim_{N \to \infty} P[A_N]$

$$= \lim_{N \to \infty} P(B_N \cup B_{N+1} \cup B_{N+2} \ldots)$$

$$\leq \lim_{N \to \infty} P(B_N) + P(B_{N+1}) + P(B_{N+2}) + \ldots \quad \text{(Union Bound Property)}$$

$$\leq \lim_{N \to \infty} \sum_{i=N}^{\infty} P(B_i)$$

$$\leq \lim_{N \to \infty} S - \sum_{i=0}^{N-1} P(B_i)$$

$$\leq S - \lim_{N \to \infty} \sum_{i=0}^{N-1} P(B_i) \quad \{\text{Let, } N' = N-1\}$$

$$\leq S - \lim_{N' \to \infty} \sum_{i=0}^{N'} P(B_i)$$

$$\leq S - S$$

$$\leq 0$$

Now, $$\lim_{N \to \infty} P[A_N] = P\left(\bigcap_{N=0}^{\infty} A_N\right) \leq 0$$

Since, probability of an event can only be between 0 and 1, we can conclude,

$$P\left(\bigcap_{N=0}^{\infty} A_N\right) = 0$$

i.e., $$\boxed{P\left(\bigcap_{N=0}^{\infty} \bigcup_{n=N}^{\infty} B_n\right) = 0}$$

Problem 2

Case 1: X is discrete

$$P(X \geq a) = \sum_{x=a}^{\infty} p(x)$$

$$\Rightarrow a P(X \geq a) = \sum_{x=a}^{\infty} a p(x) = a p(a) + a p(a+1) + a p(a+2) + \ldots$$

$$\Rightarrow a P(X \geq a) = a p(a) + a p(a+1) + a p(a+2) + \ldots$$

$$\Rightarrow a P(X \geq a) \leq a p(a) + (a+1) p(a+1) + (a+2) p(a+2) + \ldots$$

↳ Inequality holds true as pmf has non-negative values, and a is positive

$$\Rightarrow a P(X \geq a) \leq \sum_{x=a}^{\infty} x p(x)$$

$$\Rightarrow a P(X \geq a) \leq \sum_{x=-\infty}^{a-1} x p(x) + \sum_{a}^{\infty} x p(x)$$

Since, X is non negative,

$$P(X<0) = 0$$

$$\Rightarrow a P(X \geq a) \leq \sum_{x=0}^{\infty} x p(x)$$

$$\Rightarrow a P(X \geq a) \leq E(X)$$

$$\Rightarrow P(X \geq a) \leq \frac{1}{a} E(X) \text{ —— (Proved)}$$

Now,

$$e^{-ta} M_X(t) = e^{-ta} E(e^{tX})$$

$$= E(e^{t(X-a)})$$

$$= E(e^{tZ}) \quad \{ \text{Let, } Z = X - a \}$$

$$= \sum_{z=-\infty}^{\infty} e^{tz} p(z)$$

$$= \sum_{z=-\infty}^{-1} e^{tz} p(z) + \sum_{z=0}^{\infty} e^{tz} p(z) \text{ —— (1)}$$

Since, p(z) is a probability mass function, and exponential function is always positive, both terms on R.H.S are non-negative.

$$(1) \geq \sum_{z=0}^{\infty} e^{tz} p(z)$$

$$\geq \sum_{z=0}^{\infty} p(z)$$

$$\geq P(Z \geq 0)$$

$$\geq P(X - a \geq 0)$$

$$\geq P(X \geq a)$$

That is,

$$P(X \geq a) \leq (1)$$

$$\Rightarrow P(X \geq a) \leq e^{-ta} M_X(t) \text{ —— (Proved)}$$

Case 2: X is continuous

$$P(X \geq a) = \int_a^{\infty} f(x).dx$$

$$\Rightarrow a P(X \geq a) = \int_a^{\infty} a f(x).dx$$

$$\leq \int_a^{\infty} x f(x).dx \longrightarrow$$

Inequality holds true as pdf has only non-negative values

$$\leq \int_{-\infty}^{\infty} x f(x).dx$$

$$\leq \int_{0}^{\infty} x f(x).dx \longrightarrow$$

$\int_{-\infty}^{0} x f(x).dx = 0$, as X is non-negative random variable.

$$\Rightarrow a P(X \geq a) \leq E(X)$$

$$\Rightarrow P(X \geq a) \leq E(X)/a \text{ —— (Proved)}$$

Now,

$$e^{-ta} M_X(t) = e^{-ta} E(e^{tX})$$

$$= E(e^{t(X-a)})$$

$$= E(e^{tY}) \quad \text{Let, } Y = t - a$$

$$= \int_{-\infty}^{\infty} e^{tY} f(y).dy$$

$$= \int_{-\infty}^{0} e^{tY} f(y).dy + \int_{0}^{\infty} e^{tY} f(y).dy$$

$$\geq \int_{0}^{\infty} e^{tY} f(y).dy \longleftarrow$$

because $e^{tY}.f(y)$ is non-negative quantity, (as explained above also)

$$\geq \int_{0}^{\infty} f(y).dy$$

$$\geq P(Y \geq 0)$$

$$\geq P(X - a \geq 0)$$

$$\geq P(X \geq a)$$

$$e^{-ta} M_X(t)$$

$$\Rightarrow P(X \geq a) \leq e^{-ta} M_X(t) \text{ —— (Proved)}$$

Problem 6.6

The testing of 5 transistors can be done in $5! = 120$ ways.

Let the first defective transistor be detected on ith test, while the second be detected after 'j' further tests. Basically,

$$N_1 = i$$
$$N_2 = j$$

Any such permutation is of the following form:

transistors before D_1: $(i-1)$ | D_1 | # transistors b/w D_1 & D_2: $(j-1)$ | D_2 | # transistors after D_2: $(5-i-j)$

D_1 & D_2 represents the first and second defective transistors. Note i & j must follow these constraints:

a) $i \leq 4$

b) $j \leq 4$

c) $i + j \leq 5$

Condition (a) says $i = 5$ is not possible, because it is given that 2 out of 5 transistors are known to be faulty, so after testing every one of them, it cannot be the case, that we come across only one.

Condition (c) says that the no. of tests cannot exceed the no. of transistors.

Condition (b) basically follows from (a) & (c).

Now, we define joint probability mass function of N_1 & N_2 as,

$$P(N_1 = i,\ N_2 = j)$$
$$= P\left[(N_1 = i) \cap (N_2 = j)\right]$$

From conditional probability,

$$= P\left[\frac{(N_2 = j)}{(N_1 = i)}\right] \cdot P(N_1 = i)$$

$P(N_1 = i) \rightarrow$ Probability of the event that out of 'i' transistors tested, first $(i-1)$ were not defective, while ith one was faulty.

$$P(N_1 = i) = \frac{\left({}^3P_{i-1} \times {}^2P_1\right)}{\left({}^5P_i\right)} \qquad \text{——— (1)}$$

Now, after testing 'i' pieces, $(5-i)$ remain. We calculate $P\left[(N_2 = j)/(N_1 = i)\right]$. This requires that out of the next 'j' pieces tested, first $(j-1)$ were defective, while jth piece was not. Hence,

$$P\left[\frac{(N_2 = j)}{(N_1 = i)}\right] = \frac{\left({}^{3-(i-1)}P_{j-1} \times {}^1P_1\right)}{\left({}^{5-i}P_j\right)} \qquad \text{——— (2)}$$

Therefore, from (1) & (2),

$$P\left[(N_1 = i) \cap (N_2 = j)\right]$$
$$= \frac{{}^3P_{i-1} \times {}^2P_1}{{}^5P_i} \times \frac{{}^{3-(i-1)}P_{j-1} \times {}^1P_1}{{}^{5-i}P_j} \times {}^{5-i-j}P_{5-i-j}$$
$$= \frac{3!}{(3-i+1)!} \times \frac{2!}{1!} \times \frac{(5-i)!}{5!} \times \frac{(3-i+1)!}{(5-i-j)!} \times \frac{1!}{0!} \times \frac{(5-i-j)}{(5-i)!}$$
$$= \frac{3! \times 2!}{5!} = \frac{12}{120} = \frac{1}{10}$$

Therefore, we can define $P(N_1 = i,\ N_2 = j)$ as,

N_1 \ N_2	1	2	3	4	5
1	1/10	1/10	1/10	1/10	0
2	1/10	1/10	1/10	0	0
3	1/10	1/10	0	0	0
4	1/10	0	0	0	0
5	0	0	0	0	0

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Problem 6.9

$$f(x,y) = \frac{6}{7}\left(x^2 + \frac{xy}{2}\right), \quad 0 < x < 1, \; 0 < y < 2$$

When, $0 < x < 1$, $0 < y < 2$

$$\frac{6}{7}\left(x^2 + \frac{xy}{2}\right) \geq 0$$

i.e. $f(x,y) \geq 0$ as must be the case for any pmf.

(a) To verify that this is indeed a joint density function, we need to show that,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y).\,dx.\,dy = 1$$

Now,

$$L.H.S = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{6}{7}\left(x^2 + \frac{xy}{2}\right).\,dx\,dy$$

$$= \int_0^2\int_0^1 \frac{6}{7}\left(x^2 + \frac{xy}{2}\right).\,dx.\,dy$$

$$= \int_0^2 \frac{6}{7}\left[\frac{x^3}{3} + \frac{x^2y}{4}\right]_0^1.\,dx.\,dy$$

$$= \frac{6}{7}\int_0^2 \left[\left(\frac{1}{3} + \frac{y}{4}\right) - 0\right].\,dy$$

$$= \frac{6}{7}\int_0^2 \frac{3y+4}{12}.\,dy$$

$$= \frac{1}{14}\left[\frac{3y^2}{2} + 4y\right]_0^2$$

$$= \frac{1}{14}\left[\left(\frac{12}{2} + 8\right) - 0\right]$$

$$= \frac{14}{14} = 1 = R.H.S \quad \text{(Hence, proved)}$$

(b) To compute the density function of X, we use,

$$\{X \in A\} = \int_A \underbrace{\left[\int_{-\infty}^{\infty} f(x,y).\,dy\right]}_{f_X(x) \rightarrow \text{ density function of } X} dx$$

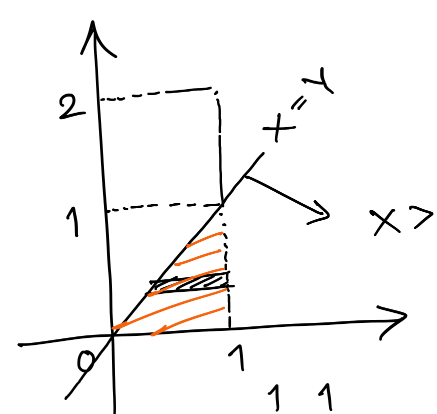
$$f_X(x) = \int_{-\infty}^{\infty} \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) dy$$

$$= \frac{6}{7}\int_0^2 \left(x^2 + \frac{xy}{2}\right).\,dy$$

$$= \frac{6}{7}\left(x^2y + \frac{xy^2}{4}\right)_0^2$$

$$f_X(x) = \frac{6}{7}\left(2x^2 + x\right)$$

(c) $P[X > Y]$



$$P[X > Y] = \int_0^1\int_y^1 \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) dx\,dy$$

$$= \int_0^1 \frac{6}{7}\left(\frac{x^3}{3} + \frac{x^2y}{4}\right)_y^1 dy$$

$$= \int_0^1 \frac{6}{7}\left[\left(\frac{1}{3} + \frac{y}{4}\right) - \left(\frac{y^3}{3} + \frac{y^3}{4}\right)\right] dy$$

$$= \frac{6}{7}\int_0^1 \left(-\frac{7}{12}y^3 + \frac{y}{4} + \frac{1}{3}\right).\,dy$$

$$= \int_0^1 \left(-\frac{y^3}{2} + \frac{3y}{14} + \frac{2}{7}\right) dy$$

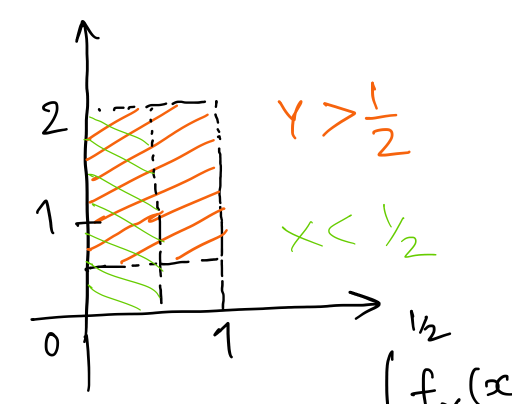
$$= \left(-\frac{y^4}{8} + \frac{3y^2}{28} + \frac{2y}{7}\right)_0^1$$

$$= \frac{-1}{8} + \frac{3}{28} + \frac{2}{7}$$

$$= \frac{-7+6+16}{56}$$

$$P[X > Y] = \frac{15}{56}$$

(d) $P\left\{Y > \frac{1}{2} / X < \frac{1}{2}\right\}$



$$P\left(X < \tfrac{1}{2}\right) = \int_{-\infty}^{1/2} f_X(x).\,dx$$

$$= \int_0^{1/2} \frac{6}{7}(2x^2 + x).\,dx$$

$$= \frac{6}{7}\left(\frac{2x^3}{3} + \frac{x^2}{2}\right)_0^{1/2}.\,dx$$

$$= \frac{6}{7}\left(\frac{1}{12} + \frac{1}{8}\right)$$

$$= \frac{6}{7}\left(\frac{20}{96}\right) = \frac{20}{112} \quad \text{—— (1)}$$

$$P\left\{\left(X < \tfrac{1}{2}\right) \cap \left(Y > \tfrac{1}{2}\right)\right\}$$

$$= \int_{1/2}^2\int_0^{1/2} \frac{6}{7}\left(x^2 + \frac{xy}{2}\right).\,dx.\,dy$$

$$= \int_{1/2}^2 \frac{6}{7}\left(\frac{x^3}{3} + \frac{x^2y}{4}\right)_0^{1/2} dy$$

$$= \int_{1/2}^2 \frac{6}{7}\left(\frac{1}{24} + \frac{y}{16}\right) dy$$

$$= \int_{1/2}^2 \left(\frac{1}{28} + \frac{3y}{56}\right) dy$$

$$= \frac{1}{56}\int_{1/2}^2 (3y + 2)\,dy$$

$$= \frac{1}{56}\left(\frac{3y^2}{2} + 2y\right)_{1/2}^2$$

$$= \frac{1}{56}\left[(6+4) - \left(3/8 + 1\right)\right]$$

$$= \frac{1}{56}\left(10 - \frac{11}{8}\right) = \frac{69}{448} \quad \text{—— (2)}$$

Using (1) & (2),

$$P\left(Y > \tfrac{1}{2} / X < \tfrac{1}{2}\right) = \frac{P\left\{\left(X < \frac{1}{2}\right) \cap \left(Y > \frac{1}{2}\right)\right\}}{P\left(X < \frac{1}{2}\right)}$$

$$= \frac{69/448}{20/112}$$

$$= 0.8625$$

(e)

$$E[X] = \int_{-\infty}^{\infty} x\,f_X(x)$$

$$= \int_0^1 \frac{6}{7}(2x^2 + x)\,x.\,dx$$

$$= \frac{6}{7}\int_0^1 2x^3 + x^2.\,dx$$

$$= \frac{6}{7}\left[\frac{x^4}{2} + \frac{x^3}{3}\right]_0^1 dx$$

$$= \frac{6}{7}\left(\left(\frac{1}{2} + \frac{1}{3}\right) - 0\right)$$

$$= \frac{5}{7}$$

(f) To find $E[Y]$, we first compute $f_Y(y)$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y).\,dx$$

$$= \int_0^1 \frac{6}{7}\left(x^2 + \frac{xy}{2}\right).\,dx$$

$$= \frac{6}{7}\left[\frac{x^3}{3} + \frac{x^2y}{4}\right]_0^1$$

$$= \frac{6}{7}\left(\frac{1}{3} + \frac{y}{4}\right)$$

$$= \frac{3y+4}{14}$$

Now,

$$E[Y] = \int_{-\infty}^{\infty} y\left(\frac{3y+4}{14}\right).\,dy$$

$$= \frac{1}{14}\int_0^2 (3y^2 + 4y)\,dy$$

$$= \frac{1}{14}\left(y^3 + 2y^2\right)_0^2$$

$$= \frac{1}{14}(8+8)$$

$$= 8/7$$