compute the probability that no row or file contains more than one rook.

18. Two cards are randomly selected from an ordinary playing deck. What is the probability that they form a blackjack? That is, what is the probability that one of the cards is an ace and the other one is either a ten, a jack, a queen, or a king?

19. Two symmetric dice have both had two of their sides painted red, two painted black, one painted yellow, and the other painted white. When this pair of dice is rolled, what is the probability that both dice land with the same color face up?

20. Suppose that you are playing blackjack against a dealer. In a freshly shuffled deck, what is the probability that neither you nor the dealer is dealt a blackjack?

21. A small community organization consists of 20 families, of which 4 have one child, 8 have two children, 5 have three children, 2 have four children, and 1 has five children.

(a) If one of these families is chosen at random, what is the probability it has i children, $i = 1, 2, 3, 4, 5$?

(b) If one of the children is randomly chosen, what is the probability that child comes from a family having i children, $i = 1, 2, 3, 4, 5$?

22. Consider the following technique for shuffling a deck of n cards: For any initial ordering of the cards, go through the deck one card at a time and at each card, flip a fair coin. If the coin comes up heads, then leave the card where it is; if the coin comes up tails, then move that card to the end of the deck. After the coin has been flipped n times, say that one round has been completed. For instance, if $n = 4$ and the initial ordering is 1, 2, 3, 4, then if the successive flips result in the outcome h, t, t, h, then the ordering at the end of the round is 1, 4, 2, 3. Assuming that all possible outcomes of the sequence of n coin flips are equally likely, what is the probability that the ordering after one round is the same as the initial ordering?

23. A pair of fair dice is rolled. What is the probability that the second die lands on a higher value than does the first?

24. If two dice are rolled, what is the probability that the sum of the upturned faces equals i? Find it for $i = 2, 3, \ldots, 11, 12$.

25. A pair of dice is rolled until a sum of either 5 or 7 appears. Find the probability that a 5 occurs first.
Hint: Let E_n denote the event that a 5 occurs on the nth roll and no 5 or 7 occurs on the first $n - 1$ rolls. Compute $P(E_n)$ and argue that $\sum_{n=1}^{\infty} P(E_n)$ is the desired probability.

26. The game of craps is played as follows: A player rolls two dice. If the sum of the dice is either a 2, 3, or 12, the player loses; if the sum is either a 7 or an 11, the player wins. If the outcome is anything else, the player continues to roll the dice until she rolls either the initial outcome or a 7. If the 7 comes first, the player loses, whereas if the initial outcome reoccurs before the 7 appears, the player wins. Compute the probability of a player winning at craps.
Hint: Let E_i denote the event that the initial outcome is i and the player wins. The desired probability is $\sum_{i=2}^{12} P(E_i)$. To compute $P(E_i)$, define the events $E_{i,n}$ to be the event that the initial sum is i and the player wins on the nth roll. Argue that $P(E_i) = \sum_{n=1}^{\infty} P(E_{i,n})$.

27. An urn contains 3 red and 7 black balls. Players A and B withdraw balls from the urn consecutively until a red ball is selected. Find the probability that A selects the red ball. (A draws the first ball, then B, and so on. There is no replacement of the balls drawn.)

28. An urn contains 5 red, 6 blue, and 8 green balls. If a set of 3 balls is randomly selected, what is the probability that each of the balls will be (a) of the same color? (b) of different colors? Repeat under the assumption that whenever a ball is selected, its color is noted and it is then replaced in the urn before the next selection. This is known as *sampling with replacement*.

29. An urn contains n white and m black balls, where n and m are positive numbers.

(a) If two balls are randomly withdrawn, what is the probability that they are the same color?

(b) If a ball is randomly withdrawn and then replaced before the second one is drawn, what is the probability that the withdrawn balls are the same color?

(c) Show that the probability in part (b) is always larger than the one in part (a).

30. The chess clubs of two schools consist of, respectively, 8 and 9 players. Four members from each club are randomly chosen to participate in a contest between the two schools. The chosen players from one team are then randomly paired with those from the other team, and each pairing plays a game of chess. Suppose that Rebecca and her sister Elise are on the chess clubs at different schools. What is the probability that

(a) Rebecca and Elise will be paired?

(b) Rebecca and Elise will be chosen to represent their schools but will not play each other?

(c) either Rebecca or Elise will be chosen to represent her school?

31. A 3-person basketball team consists of a guard, a forward, and a center.
 (a) If a person is chosen at random from each of three different such teams, what is the probability of selecting a complete team?
 (b) What is the probability that all 3 players selected play the same position?

32. A group of individuals containing b boys and g girls is lined up in random order; that is, each of the $(b + g)!$ permutations is assumed to be equally likely. What is the probability that the person in the ith position, $1 \le i \le b + g$, is a girl?

33. A forest contains 20 elk, of which 5 are captured, tagged, and then released. A certain time later, 4 of the 20 elk are captured. What is the probability that 2 of these 4 have been tagged? What assumptions are you making?

34. The second Earl of Yarborough is reported to have bet at odds of 1000 to 1 that a bridge hand of 13 cards would contain at least one card that is ten or higher. (By *ten or higher* we mean that a card is either a ten, a jack, a queen, a king, or an ace.) Nowadays, we call a hand that has no cards higher than 9 *a Yarborough*. What is the probability that a randomly selected bridge hand is a Yarborough?

35. Seven balls are randomly withdrawn from an urn that contains 12 red, 16 blue, and 18 green balls. Find the probability that
 (a) 3 red, 2 blue, and 2 green balls are withdrawn;
 (b) at least 2 red balls are withdrawn;
 (c) all withdrawn balls are the same color;
 (d) either exactly 3 red balls or exactly 3 blue balls are withdrawn.

36. Two cards are chosen at random from a deck of 52 playing cards. What is the probability that they
 (a) are both aces?
 (b) have the same value?

37. An instructor gives her class a set of 10 problems with the information that the final exam will consist of a random selection of 5 of them. If a student has figured out how to do 7 of the problems, what is the probability that he or she will answer correctly
 (a) all 5 problems?
 (b) at least 4 of the problems?

38. There are n socks, 3 of which are red, in a drawer. What is the value of n if, when 2 of the socks are chosen randomly, the probability that they are both red is $\frac{1}{2}$?

39. There are 5 hotels in a certain town. If 3 people check into hotels in a day, what is the probability that they each check into a different hotel? What assumptions are you making?

40. A town contains 4 people who repair televisions. If 4 sets break down, what is the probability that exactly i of the repairers are called? Solve the problem for $i = 1, 2, 3, 4$. What assumptions are you making?

41. If a die is rolled 4 times, what is the probability that 6 comes up at least once?

42. Two dice are thrown n times in succession. Compute the probability that double 6 appears at least once. How large need n be to make this probability at least $\frac{1}{2}$?

43. **(a)** If N people, including A and B, are randomly arranged in a line, what is the probability that A and B are next to each other?
 (b) What would the probability be if the people were randomly arranged in a circle?

44. Five people, designated as A, B, C, D, E, are arranged in linear order. Assuming that each possible order is equally likely, what is the probability that
 (a) there is exactly one person between A and B?
 (b) there are exactly two people between A and B?
 (c) there are three people between A and B?

45. A woman has n keys, of which one will open her door.
 (a) If she tries the keys at random, discarding those that do not work, what is the probability that she will open the door on her kth try?
 (b) What if she does not discard previously tried keys?

46. How many people have to be in a room in order that the probability that at least two of them celebrate their birthday in the same month is at least $\frac{1}{2}$? Assume that all possible monthly outcomes are equally likely.

47. If there are 12 strangers in a room, what is the probability that no two of them celebrate their birthday in the same month?

48. Given 20 people, what is the probability that, among the 12 months in the year, there are 4 months containing exactly 2 birthdays and 4 containing exactly 3 birthdays?

49. A group of 6 men and 6 women is randomly divided into 2 groups of size 6 each. What is the probability that both groups will have the same number of men?

50. In a hand of bridge, find the probability that you have 5 spades and your partner has the remaining 8.

51. Suppose that n balls are randomly distributed into N compartments. Find the probability that m balls will fall into the first compartment. Assume that all N^n arrangements are equally likely.

5. For any sequence of events $E_1, E_2, \ldots$, define a new sequence $F_1, F_2, \ldots$ of disjoint events (that is, events such that $F_iF_j = \emptyset$ whenever $i \neq j$) such that for all $n \geq 1$,

$$\bigcup_1^n F_i = \bigcup_1^n E_i$$

6. Let E, F, and G be three events. Find expressions for the events so that, of E, F, and G,
(a) only E occurs;
(b) both E and G, but not F, occur;
(c) at least one of the events occurs;
(d) at least two of the events occur;
(e) all three events occur;
(f) none of the events occurs;
(g) at most one of the events occurs;
(h) at most two of the events occur;
(i) exactly two of the events occur;
(j) at most three of the events occur.

7. Find the simplest expression for the following events:
(a) $(E \cup F)(E \cup F^c)$;
(b) $(E \cup F)(E^c \cup F)(E \cup F^c)$;
(c) $(E \cup F)(F \cup G)$.

8. Let S be a given set. If, for some $k > 0$, $S_1, S_2, \ldots, S_k$ are mutually exclusive nonempty subsets of S such that $\bigcup_{i=1}^{k} S_i = S$, then we call the set $\{S_1, S_2, \ldots, S_k\}$ a *partition* of S. Let T_n denote the number of different partitions of $\{1, 2, \ldots, n\}$. Thus, $T_1 = 1$ (the only partition being $S_1 = \{1\}$) and $T_2 = 2$ (the two partitions being $\{\{1, 2,\}\}, \{\{1\}, \{2\}\}$).
(a) Show, by computing all partitions, that $T_3 = 5, T_4 = 15$.
(b) Show that

$$T_{n+1} = 1 + \sum_{k=1}^{n} \binom{n}{k} T_k$$

and use this equation to compute T_{10}.
Hint: One way of choosing a partition of $n + 1$ items is to call one of the items *special*. Then we obtain different partitions by first choosing $k, k = 0, 1, \ldots, n$, then a subset of size $n - k$ of the nonspecial items, and then any of the T_k partitions of the remaining k nonspecial items. By adding the special item to the subset of size $n - k$, we obtain a partition of all $n + 1$ items.

9. Suppose that an experiment is performed n times. For any event E of the sample space, let $n(E)$ denote the number of times that event E occurs and define $f(E) = n(E)/n$. Show that $f(\cdot)$ satisfies Axioms 1, 2, and 3.

10. Prove that $P(E \cup F \cup G) = P(E) + P(F) + P(G) - P(E^cFG) - P(EF^cG) - P(EFG^c) - 2P(EFG)$.

11. If $P(E) = .9$ and $P(F) = .8$, show that $P(EF) \geq .7$. In general, prove Bonferroni's inequality, namely,

$$P(EF) \geq P(E) + P(F) - 1$$

12. Show that the probability that exactly one of the events E or F occurs equals $P(E) + P(F) - 2P(EF)$.

13. Prove that $P(EF^c) = P(E) - P(EF)$.

14. Prove Proposition 4.4 by mathematical induction.

15. An urn contains M white and N black balls. If a random sample of size r is chosen, what is the probability that it contains exactly k white balls?

16. Use induction to generalize Bonferroni's inequality to n events. That is, show that

$$P(E_1E_2 \cdots E_n) \geq P(E_1) + \cdots + P(E_n) - (n - 1)$$

17. Consider the matching problem, Example 5m, and define A_N to be the number of ways in which the N men can select their hats so that no man selects his own. Argue that

$$A_N = (N - 1)(A_{N-1} + A_{N-2})$$

This formula, along with the boundary conditions $A_1 = 0, A_2 = 1$, can then be solved for A_N, and the desired probability of no matches would be $A_N/N!$.
Hint: After the first man selects a hat that is not his own, there remain $N - 1$ men to select among a set of $N - 1$ hats that does not contain the hat of one of these men. Thus, there is one extra man and one extra hat. Argue that we can get no matches either with the extra man selecting the extra hat or with the extra man not selecting the extra hat.

18. Let f_n denote the number of ways of tossing a coin n times such that successive heads never appear. Argue that

$$f_n = f_{n-1} + f_{n-2} \qquad n \geq 2, \text{where } f_0 \equiv 1, f_1 \equiv 2$$

Hint: How many outcomes are there that start with a head, and how many start with a tail? If P_n denotes the probability that successive heads never appear when a coin is tossed n times, find P_n (in terms of f_n) when all possible outcomes of the n tosses are assumed equally likely. Compute P_{10}.

19. An urn contains n red and m blue balls. They are withdrawn one at a time until a total of $r, r \leq n$, To do so, multiply the sums and show that, for all pairs i, j, the coefficient of the term $n_i n_j$ is greater in the expression on the left than in the one on the right.

3.4. A ball is in any one of n boxes and is in the ith box with probability P_i. If the ball is in box i, a search of that box will uncover it with probability α_i. Show that the conditional probability that the ball is in box j, given that a search of box i did not uncover it, is

$$\frac{P_j}{1 - \alpha_i P_i} \quad \text{if } j \neq i$$
$$\frac{(1 - \alpha_i)P_i}{1 - \alpha_i P_i} \quad \text{if } j = i$$

3.5. An event F is said to carry negative information about an event E, and we write $F \searrow E$, if

$$P(E|F) \leq P(E)$$

Prove or give counterexamples to the following assertions:

(a) If $F \searrow E$, then $E \searrow F$.
(b) If $F \searrow E$ and $E \searrow G$, then $F \searrow G$.
(c) If $F \searrow E$ and $G \searrow E$, then $FG \searrow E$.

Repeat parts (a), (b), and (c) when $\searrow$ is replaced by $\nearrow$, where we say that F carries positive information about E, written $F \nearrow E$, when $P(E|F) \geq P(E)$.

3.6. Prove that if $E_1, E_2, \ldots, E_n$ are independent events, then

$$P(E_1 \cup E_2 \cup \cdots \cup E_n) = 1 - \prod_{i=1}^{n}[1 - P(E_i)]$$

3.7. **(a)** An urn contains n white and m black balls. The balls are withdrawn one at a time until only those of the same color are left. Show that, with probability $n/(n + m)$, they are all white.
Hint: Imagine that the experiment continues until all the balls are removed, and consider the last ball withdrawn.

(b) A pond contains 3 distinct species of fish, which we will call the Red, Blue, and Green fish. There are r Red, b Blue, and g Green fish. Suppose that the fish are removed from the pond in a random order. (That is, each selection is equally likely to be any of the remaining fish.) What is the probability that the Red fish are the first species to become extinct in the pond?
Hint: Write $P\{R\} = P\{RBG\} + P\{RGB\}$, and compute the probabilities on the right by first conditioning on the last species to be removed.

3.8. Let A, B, and C be events relating to the experiment of rolling a pair of dice.

(a) If

$$P(A|C) > P(B|C) \quad \text{and} \quad P(A|C^c) > P(B|C^c)$$

either prove that $P(A) > P(B)$ or give a counterexample by defining events A, B, and C for which that relationship is not true.

(b) If

$$P(A|C) > P(A|C^c) \quad \text{and} \quad P(B|C) > P(B|C^c)$$

either prove that $P(AB|C) > P(AB|C^c)$ or give a counterexample by defining events A, B, and C for which that relationship is not true.

Hint: Let C be the event that the sum of a pair of dice is 10; let A be the event that the first die lands on 6; let B be the event that the second die lands on 6.

3.9. Consider two independent tosses of a fair coin. Let A be the event that the first toss results in heads, let B be the event that the second toss results in heads, and let C be the event that in both tosses the coin lands on the same side. Show that the events A, B, and C are pairwise independent—that is, A and B are independent, A and C are independent, and B and C are independent—but not independent.

3.10. Consider a collection of n individuals. Assume that each person's birthday is equally likely to be any of the 365 days of the year and also that the birthdays are independent. Let $A_{i,j}$, $i \neq j$, denote the event that persons i and j have the same birthday. Show that these events are pairwise independent, but not independent. That is, show that $A_{i,j}$ and $A_{r,s}$ are independent, but the $\binom{n}{2}$ events $A_{i,j}, i \neq j$ are not independent.

3.11. In each of n independent tosses of a coin, the coin lands on heads with probability p. How large need n be so that the probability of obtaining at least one head is at least $\frac{1}{2}$?

3.12. Show that $0 \leq a_i \leq 1, i = 1, 2, \ldots$, then

$$\sum_{i=1}^{\infty}\left[a_i \prod_{j=1}^{i-1}(1 - a_j)\right] + \prod_{i=1}^{\infty}(1 - a_i) = 1$$

Hint: Suppose that an infinite number of coins are to be flipped. Let a_i be the probability that the ith coin lands on heads, and consider when the first head occurs.

3.13. The probability of getting a head on a single toss of a coin is p. Suppose that A starts and continues to flip the coin until a tail shows up, at which point