PROBLEMS

1. **(a)** How many different 7-place license plates are possible if the first 2 places are for letters and the other 5 for numbers?
(b) Repeat part (a) under the assumption that no letter or number can be repeated in a single license plate.

2. How many outcome sequences are possible when a die is rolled four times, where we say, for instance, that the outcome is 3, 4, 3, 1 if the first roll landed on 3, the second on 4, the third on 3, and the fourth on 1?

3. Twenty workers are to be assigned to 20 different jobs, one to each job. How many different assignments are possible?

4. John, Jim, Jay, and Jack have formed a band consisting of 4 instruments. If each of the boys can play all 4 instruments, how many different arrangements are possible? What if John and Jim can play all 4 instruments, but Jay and Jack can each play only piano and drums?

5. For years, telephone area codes in the United States and Canada consisted of a sequence of three digits. The first digit was an integer between 2 and 9, the second digit was either 0 or 1, and the third digit was any integer from 1 to 9. How many area codes were possible? How many area codes starting with a 4 were possible?

6. A well-known nursery rhyme starts as follows:

"As I was going to St. Ives

I met a man with 7 wives.

Each wife had 7 sacks.

Each sack had 7 cats.

Each cat had 7 kittens..."

How many kittens did the traveler meet?

7. **(a)** In how many ways can 3 boys and 3 girls sit in a row?
(b) In how many ways can 3 boys and 3 girls sit in a row if the boys and the girls are each to sit together?
(c) In how many ways if only the boys must sit together?
(d) In how many ways if no two people of the same sex are allowed to sit together?

8. How many different letter arrangements can be made from the letters
(a) Fluke?
(b) Propose?
(c) Mississippi?
(d) Arrange?

9. A child has 12 blocks, of which 6 are black, 4 are red, 1 is white, and 1 is blue. If the child puts the blocks in a line, how many arrangements are possible?

10. In how many ways can 8 people be seated in a row if
(a) there are no restrictions on the seating arrangement?
(b) persons A and B must sit next to each other?
(c) there are 4 men and 4 women and no 2 men or 2 women can sit next to each other?
(d) there are 5 men and they must sit next to each other?
(e) there are 4 married couples and each couple must sit together?

11. In how many ways can 3 novels, 2 mathematics books, and 1 chemistry book be arranged on a bookshelf if
(a) the books can be arranged in any order?
(b) the mathematics books must be together and the novels must be together?
(c) the novels must be together, but the other books can be arranged in any order?

12. Five separate awards (best scholarship, best leadership qualities, and so on) are to be presented to selected students from a class of 30. How many different outcomes are possible if
(a) a student can receive any number of awards?
(b) each student can receive at most 1 award?

13. Consider a group of 20 people. If everyone shakes hands with everyone else, how many handshakes take place?

14. How many 5-card poker hands are there?

15. A dance class consists of 22 students, of which 10 are women and 12 are men. If 5 men and 5 women are to be chosen and then paired off, how many results are possible?

16. A student has to sell 2 books from a collection of 6 math, 7 science, and 4 economics books. How many choices are possible if
(a) both books are to be on the same subject?
(b) the books are to be on different subjects?

17. Seven different gifts are to be distributed among 10 children. How many distinct results are possible if no child is to receive more than one gift?

18. A committee of 7, consisting of 2 Republicans, 2 Democrats, and 3 Independents, is to be chosen from a group of 5 Republicans, 6 Democrats, and 4 Independents. How many committees are possible?

19. From a group of 8 women and 6 men, a committee consisting of 3 men and 3 women is to be formed. How many different committees are possible if
(a) 2 of the men refuse to serve together?
(b) 2 of the women refuse to serve together?
(c) 1 man and 1 woman refuse to serve together?

12. A committee of 6 people is to be chosen from a group consisting of 7 men and 8 women. If the committee must consist of at least 3 women and at least 2 men, how many different committees are possible?

***13.** An art collection on auction consisted of 4 Dalis, 5 van Goghs, and 6 Picassos. At the auction were 5 art collectors. If a reporter noted only the number of Dalis, van Goghs, and Picassos acquired by each collector, how many different results could have been recorded if all of the works were sold?

***14.** Determine the number of vectors $(x_1, \ldots, x_n)$ such that each x_i is a positive integer and

$$\sum_{i=1}^{n} x_i \leq k$$

where $k \geq n$.

15. A total of n students are enrolled in a review course for the actuarial examination in probability. The posted results of the examination will list the names of those who passed, in decreasing order of their scores. For instance, the posted result will be "Brown, Cho" if Brown and Cho are the only ones to pass, with Brown receiving the higher score. Assuming that all scores are distinct (no ties), how many posted results are possible?

16. How many subsets of size 4 of the set $S = \{1, 2, \ldots, 20\}$ contain at least one of the elements $1, 2, 3, 4, 5$?

17. Give an analytic verification of

$$\binom{n}{2} = \binom{k}{2} + k(n-k) + \binom{n-k}{2}, \quad 1 \leq k \leq n$$

Now, give a combinatorial argument for this identity.

18. In a certain community, there are 3 families consisting of a single parent and 1 child, 3 families consisting of a single parent and 2 children, 5 families consisting of 2 parents and a single child, 7 families consisting of 2 parents and 2 children, and 6 families consisting of 2 parents and 3 children. If a parent and child from the same family are to be chosen, how many possible choices are there?

19. If there are no restrictions on where the digits and letters are placed, how many 8-place license plates consisting of 5 letters and 3 digits are possible if no repetitions of letters or digits are allowed. What if the 3 digits must be consecutive?

which can be generalized to give

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) - \sum\sum_{i<j} P(A_iA_j) + \sum\sum\sum_{i<j<k} P(A_iA_jA_k) + \cdots + (-1)^{n+1}P(A_1 \cdots A_n)$$

If S is finite and each one point set is assumed to have equal probability, then

$$P(A) = \frac{|A|}{|S|}$$

where $|E|$ denotes the number of outcomes in the event E.

$P(A)$ can be interpreted either as a long-run relative frequency or as a measure of one's degree of belief.

PROBLEMS

1. A box contains 3 marbles: 1 red, 1 green, and 1 blue. Consider an experiment that consists of taking 1 marble from the box and then replacing it in the box and drawing a second marble from the box. Describe the sample space. Repeat when the second marble is drawn without replacing the first marble.

2. In an experiment, die is rolled continually until a 6 appears, at which point the experiment stops. What is the sample space of this experiment? Let E_n denote the event that n rolls are necessary to complete the experiment. What points of the sample space are contained in E_n? What is $\left(\bigcup_{1}^{\infty} E_n\right)^c$?

3. Two dice are thrown. Let E be the event that the sum of the dice is odd, let F be the event that at least one of the dice lands on 1, and let G be the event that the sum is 5. Describe the events $EF, E \cup F, FG, EF^c$, and EFG.

4. A, B, and C take turns flipping a coin. The first one to get a head wins. The sample space of this experiment can be defined by

$$S = \begin{cases} 1, 01, 001, 0001, \ldots, \\ 0000\cdots \end{cases}$$

(a) Interpret the sample space.

(b) Define the following events in terms of S:

(i) A wins $= A$.

(ii) B wins $= B$.

(iii) $(A \cup B)^c$.

Assume that A flips first, then B, then C, then A, and so on.

5. A system is comprised of 5 components, each of which is either working or failed. Consider an experiment that consists of observing the status of each component, and let the outcome of the experiment be given by the vector $(x_1, x_2, x_3, x_4, x_5)$, where x_i is equal to 1 if component i is working and is equal to 0 if component i is failed.

(a) How many outcomes are in the sample space of this experiment?

(b) Suppose that the system will work if components 1 and 2 are both working, or if components 3 and 4 are both working, or if components 1, 3, and 5 are all working. Let W be the event that the system will work. Specify all the outcomes in W.

(c) Let A be the event that components 4 and 5 are both failed. How many outcomes are contained in the event A?

(d) Write out all the outcomes in the event AW.

6. A hospital administrator codes incoming patients suffering gunshot wounds according to whether they have insurance (coding 1 if they do and 0 if they do not) and according to their condition, which is rated as good (g), fair (f), or serious (s). Consider an experiment that consists of the coding of such a patient.

(a) Give the sample space of this experiment.

(b) Let A be the event that the patient is in serious condition. Specify the outcomes in A.

(c) Let B be the event that the patient is uninsured. Specify the outcomes in B.

(d) Give all the outcomes in the event $B^c \cup A$.

7. Consider an experiment that consists of determining the type of job—either blue-collar or white-collar—and the political affiliation—Republican, Democratic, or Independent—of the 15 members of an adult soccer team. How many outcomes are

(a) in the sample space?

(b) in the event that at least one of the team members is a blue-collar worker?

(c) in the event that none of the team members considers himself or herself an Independent?

8. Suppose that A and B are mutually exclusive events for which $P(A) = .3$ and $P(B) = .5$. What is the probability that
(a) either A or B occurs?
(b) A occurs but B does not?
(c) both A and B occur?

9. A retail establishment accepts either the American Express or the VISA credit card. A total of 24 percent of its customers carry an American Express card, 61 percent carry a VISA card, and 11 percent carry both cards. What percentage of its customers carry a credit card that the establishment will accept?

10. Sixty percent of the students at a certain school wear neither a ring nor a necklace. Twenty percent wear a ring and 30 percent wear a necklace. If one of the students is chosen randomly, what is the probability that this student is wearing
(a) a ring or a necklace?
(b) a ring and a necklace?

11. A total of 28 percent of American males smoke cigarettes, 7 percent smoke cigars, and 5 percent smoke both cigars and cigarettes.
(a) What percentage of males smokes neither cigars nor cigarettes?
(b) What percentage smokes cigars but not cigarettes?

12. An elementary school is offering 3 language classes: one in Spanish, one in French, and one in German. The classes are open to any of the 100 students in the school. There are 28 students in the Spanish class, 26 in the French class, and 16 in the German class. There are 12 students that are in both Spanish and French, 4 that are in both Spanish and German, and 6 that are in both French and German. In addition, there are 2 students taking all 3 classes.
(a) If a student is chosen randomly, what is the probability that he or she is not in any of the language classes?
(b) If a student is chosen randomly, what is the probability that he or she is taking exactly one language class?
(c) If 2 students are chosen randomly, what is the probability that at least 1 is taking a language class?

13. A certain town with a population of 100,000 has 3 newspapers: I, II, and III. The proportions of townspeople who read these papers are as follows:

I: 10 percent	I and II: 8 percent	I and II and III: 1 percent
II: 30 percent	I and III: 2 percent	
III: 5 percent	II and III: 4 percent	

(The list tells us, for instance, that 8000 people read newspapers I and II.)
(a) Find the number of people who read only one newspaper.
(b) How many people read at least two newspapers?
(c) If I and III are morning papers and II is an evening paper, how many people read at least one morning paper plus an evening paper?
(d) How many people do not read any newspapers?
(e) How many people read only one morning paper and one evening paper?

14. The following data were given in a study of a group of 1000 subscribers to a certain magazine: In reference to job, marital status, and education, there were 312 professionals, 470 married persons, 525 college graduates, 42 professional college graduates, 147 married college graduates, 86 married professionals, and 25 married professional college graduates. Show that the numbers reported in the study must be incorrect.
Hint: Let M, W, and G denote, respectively, the set of professionals, married persons, and college graduates. Assume that one of the 1000 persons is chosen at random, and use Proposition 4.4 to show that if the given numbers are correct, then $P(M \cup W \cup G) > 1$.

15. If it is assumed that all $\binom{52}{5}$ poker hands are equally likely, what is the probability of being dealt
(a) a flush? (A hand is said to be a flush if all 5 cards are of the same suit.)
(b) one pair? (This occurs when the cards have denominations a, a, b, c, d, where a, b, c, and d are all distinct.)
(c) two pairs? (This occurs when the cards have denominations a, a, b, b, c, where a, b, and c are all distinct.)
(d) three of a kind? (This occurs when the cards have denominations a, a, a, b, c, where a, b, and c are all distinct.)
(e) four of a kind? (This occurs when the cards have denominations a, a, a, a, b.)

16. Poker dice is played by simultaneously rolling 5 dice. Show that
(a) $P\{\text{no two alike}\} = .0926$;
(b) $P\{\text{one pair}\} = .4630$;
(c) $P\{\text{two pair}\} = .2315$;
(d) $P\{\text{three alike}\} = .1543$;
(e) $P\{\text{full house}\} = .0386$;
(f) $P\{\text{four alike}\} = .0193$;
(g) $P\{\text{five alike}\} = .0008$.

17. If 8 rooks (castles) are randomly placed on a chessboard, compute the probability that none of the rooks can capture any of the others. That is,

52. A closet contains 10 pairs of shoes. If 8 shoes are randomly selected, what is the probability that there will be
(a) no complete pair?
(b) exactly 1 complete pair?

53. If 4 married couples are arranged in a row, find the probability that no husband sits next to his wife.

54. Compute the probability that a bridge hand is void in at least one suit. Note that the answer is not

$$\frac{\binom{4}{1}\binom{39}{13}}{\binom{52}{13}}$$

(Why not?)
Hint: Use Proposition 4.4.

55. Compute the probability that a hand of 13 cards contains
(a) the ace and king of at least one suit;
(b) all 4 of at least 1 of the 13 denominations.

56. Two players play the following game: Player A chooses one of the three spinners pictured in Figure 2.6, and then player B chooses one of the remaining two spinners. Both players then spin their spinner, and the one that lands on the higher number is declared the winner. Assuming that each spinner is equally likely to land in any of its 3 regions, would you rather be player A or player B? Explain your answer!

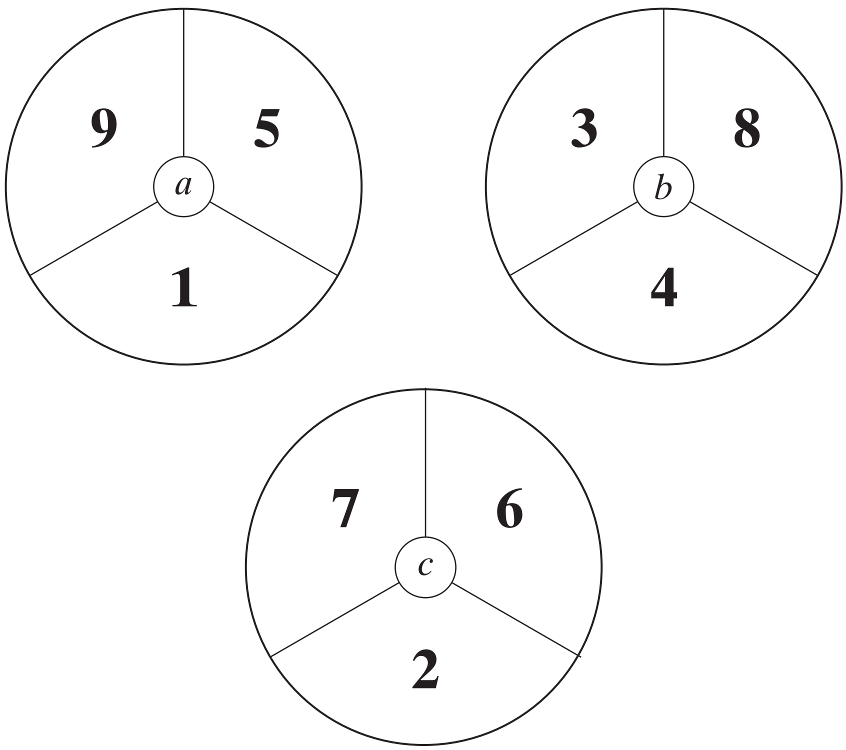

FIGURE 2.6: Spinners

THEORETICAL EXERCISES

Prove the following relations:

1. $EF \subset E \subset E \cup F$.

2. If $E \subset F$, then $F^c \subset E^c$.

3. $F = FE \cup FE^c$ and $E \cup F = E \cup E^cF$.

4. $\left(\bigcup_1^\infty E_i\right) F = \bigcup_1^\infty E_iF$ and

$$\left(\bigcap_1^\infty E_i\right) \cup F = \bigcap_1^\infty (E_i \cup F).$$