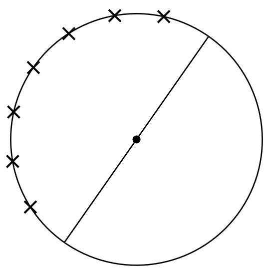
(a) Show that $1/c =$ area of region R.
Suppose that (X, Y) is uniformly distributed over the square centered at $(0, 0)$ and with sides of length 2.
(b) Show that X and Y are independent, with each being distributed uniformly over $(-1, 1)$.
(c) What is the probability that (X, Y) lies in the circle of radius 1 centered at the origin? That is, find $P\{X^2 + Y^2 \le 1\}$.

6.16. Suppose that n points are independently chosen at random on the circumference of a circle, and we want the probability that they all lie in some semicircle. That is, we want the probability that there is a line passing through the center of the circle such that all the points are on one side of that line, as shown in the following diagram:



Let $P_1, \ldots, P_n$ denote the n points. Let A denote the event that all the points are contained in some semicircle, and let A_i be the event that all the points lie in the semicircle beginning at the point P_i and going clockwise for 180°, $i = 1, \ldots, n$.
(a) Express A in terms of the A_i.
(b) Are the A_i mutually exclusive?
(c) Find $P(A)$.

6.17. Three points X_1, X_2, X_3 are selected at random on a line L. What is the probability that X_2 lies between X_1 and X_3?

6.18. Two points are selected randomly on a line of length L so as to be on opposite sides of the midpoint of the line. [In other words, the two points X and Y are independent random variables such that X is uniformly distributed over $(0, L/2)$ and Y is uniformly distributed over $(L/2, L)$.] Find the probability that the distance between the two points is greater than $L/3$.

6.19. Show that $f(x, y) = 1/x$, $0 < y < x < 1$, is a joint density function. Assuming that f is the joint density function of X, Y, find
(a) the marginal density of Y;
(b) the marginal density of X;
(c) $E[X]$;
(c) $E[Y]$.

6.20. The joint density of X and Y is given by

$$f(x, y) = \begin{cases} xe^{-(x+y)} & x > 0,\ y > 0 \\ 0 & \text{otherwise} \end{cases}$$

Are X and Y independent? If, instead, $f(x, y)$ were given by

$$f(x, y) = \begin{cases} 2 & 0 < x < y,\ 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

would X and Y be independent?

6.21. Let

$$f(x, y) = 24xy \quad 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le x + y \le 1$$

and let it equal 0 otherwise.
(a) Show that $f(x, y)$ is a joint probability density function.
(b) Find $E[X]$.
(c) Find $E[Y]$.

6.22. The joint density function of X and Y is

$$f(x, y) = \begin{cases} x + y & 0 < x < 1,\ 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Are X and Y independent?
(b) Find the density function of X.
(c) Find $P\{X + Y < 1\}$.

6.23. The random variables X and Y have joint density function

$$f(x, y) = 12xy(1 - x) \quad 0 < x < 1,\ 0 < y < 1$$

and equal to 0 otherwise.
(a) Are X and Y independent?
(b) Find $E[X]$.
(c) Find $E[Y]$.
(d) Find $\text{Var}(X)$.
(e) Find $\text{Var}(Y)$.

6.24. Consider independent trials, each of which results in outcome i, $i = 0, 1, \ldots, k$, with probability p_i, $\sum_{i=0}^{k} p_i = 1$. Let N denote the number of trials needed to obtain an outcome that is not equal to 0, and let X be that outcome.
(a) Find $P\{N = n\}$, $n \ge 1$.
(b) Find $P\{X = j\}$, $j = 1, \ldots, k$.
(c) Show that $P\{N = n, X = j\} = P\{N = n\}P\{X = j\}$.
(d) Is it intuitive to you that N is independent of X?
(e) Is it intuitive to you that X is independent of N?

6.25. Suppose that 10^6 people arrive at a service station at times that are independent random variables,

(c) Compute $P\{XY \le 3\}, P\{X + Y > 2\}$, $P\{X/Y > 1\}$.

6.41. The joint density function of X and Y is given by

$$f(x,y) = xe^{-x(y+1)} \quad x > 0,\ y > 0$$

(a) Find the conditional density of X, given $Y = y$, and that of Y, given $X = x$.
(b) Find the density function of $Z = XY$.

6.42. The joint density of X and Y is

$$f(x,y) = c(x^2 - y^2)e^{-x} \quad 0 \le x < \infty,\ -x \le y \le x$$

Find the conditional distribution of Y, given $X = x$.

6.43. An insurance company supposes that each person has an accident parameter and that the yearly number of accidents of someone whose accident parameter is λ is Poisson distributed with mean λ. They also suppose that the parameter value of a newly insured person can be assumed to be the value of a gamma random variable with parameters s and α. If a newly insured person has n accidents in her first year, find the conditional density of her accident parameter. Also, determine the expected number of accidents that she will have in the following year.

6.44. If X_1, X_2, X_3 are independent random variables that are uniformly distributed over (0, 1), compute the probability that the largest of the three is greater than the sum of the other two.

6.45. A complex machine is able to operate effectively as long as at least 3 of its 5 motors are functioning. If each motor independently functions for a random amount of time with density function $f(x) = xe^{-x}, x > 0$, compute the density function of the length of time that the machine functions.

6.46. If 3 trucks break down at points randomly distributed on a road of length L, find the probability that no 2 of the trucks are within a distance d of each other when $d \le L/2$.

6.47. Consider a sample of size 5 from a uniform distribution over (0, 1). Compute the probability that the median is in the interval $\left(\frac{1}{4}, \frac{3}{4}\right)$.

6.48. If X_1, X_2, X_3, X_4, X_5 are independent and identically distributed exponential random variables with the parameter λ, compute
(a) $P\{\min(X_1, \ldots, X_5) \le a\}$;
(b) $P\{\max(X_1, \ldots, X_5) \le a\}$.

6.49. Let $X_{(1)}, X_{(2)}, \ldots, X_{(n)}$ be the order statistics of a set of n independent uniform (0,1) random variables. Find the conditional distribution of $X_{(n)}$ given that $X_{(1)} = s_1, X_{(2)} = s_2, \ldots, X_{(n-1)} = s_{n-1}$.

6.50. Let Z_1 and Z_2 be independent standard normal random variables. Show that X, Y has a bivariate normal distribution when $X = Z_1,\ Y = Z_1 + Z_2$.

6.51. Derive the distribution of the range of a sample of size 2 from a distribution having density function $f(x) = 2x,\ 0 < x < 1$.

6.52. Let X and Y denote the coordinates of a point uniformly chosen in the circle of radius 1 centered at the origin. That is, their joint density is

$$f(x,y) = \frac{1}{\pi} \quad x^2 + y^2 \le 1$$

Find the joint density function of the polar coordinates $R = (X^2 + Y^2)^{1/2}$ and $\Theta = \tan^{-1} Y/X$.

6.53. If X and Y are independent random variables both uniformly distributed over (0, 1), find the joint density function of $R = \sqrt{X^2 + Y^2}, \Theta = \tan^{-1} Y/X$.

6.54. If U is uniform on $(0, 2\pi)$ and Z, independent of U, is exponential with rate 1, show directly (without using the results of Example 7b) that X and Y defined by

$$X = \sqrt{2Z}\cos U$$
$$Y = \sqrt{2Z}\sin U$$

are independent standard normal random variables.

6.55. X and Y have joint density function

$$f(x,y) = \frac{1}{x^2y^2} \quad x \ge 1,\ y \ge 1$$

(a) Compute the joint density function of $U = XY, V = X/Y$.
(b) What are the marginal densities?

6.56. If X and Y are independent and identically distributed uniform random variables on (0, 1), compute the joint density of
(a) $U = X + Y, V = X/Y$;
(b) $U = X, V = X/Y$;
(c) $U = X + Y, V = X/(X + Y)$.

6.57. Repeat Problem 6.56 when X and Y are independent exponential random variables, each with parameter $\lambda = 1$.

6.58. If X_1 and X_2 are independent exponential random variables, each having parameter λ, find the joint density function of $Y_1 = X_1 + X_2$ and $Y_2 = e^{X_1}$.

6.59. If X, Y, and Z are independent random variables having identical density functions $f(x) = e^{-x}, 0 < x < \infty$, derive the joint distribution of $U = X + Y,\ V = X + Z, W = Y + Z$.

8.3. Use the central limit theorem to solve part (c) of Problem 2.

8.4. Let $X_1, \ldots, X_{20}$ be independent Poisson random variables with mean 1.

(a) Use the Markov inequality to obtain a bound on

$$P\left\{\sum_1^{20} X_i > 15\right\}$$

(b) Use the central limit theorem to approximate

$$P\left\{\sum_1^{20} X_i > 15\right\}.$$

8.5. Fifty numbers are rounded off to the nearest integer and then summed. If the individual round-off errors are uniformly distributed over $(-.5, .5)$, approximate the probability that the resultant sum differs from the exact sum by more than 3.

8.6. A die is continually rolled until the total sum of all rolls exceeds 300. Approximate the probability that at least 80 rolls are necessary.

8.7. A person has 100 light bulbs whose lifetimes are independent exponentials with mean 5 hours. If the bulbs are used one at a time, with a failed bulb being replaced immediately by a new one, approximate the probability that there is still a working bulb after 525 hours.

8.8. In Problem 7, suppose that it takes a random time, uniformly distributed over (0, .5), to replace a failed bulb. Approximate the probability that all bulbs have failed by time 550.

8.9. If X is a gamma random variable with parameters $(n, 1)$, approximately how large need n be so that

$$P\left\{\left|\frac{X}{n} - 1\right| > .01\right\} < .01?$$

8.10. Civil engineers believe that W, the amount of weight (in units of 1000 pounds) that a certain span of a bridge can withstand without structural damage resulting, is normally distributed with mean 400 and standard deviation 40. Suppose that the weight (again, in units of 1000 pounds) of a car is a random variable with mean 3 and standard deviation .3. Approximately how many cars would have to be on the bridge span for the probability of structural damage to exceed .1?

8.11. Many people believe that the daily change of price of a company's stock on the stock market is a random variable with mean 0 and variance σ^2. That is, if Y_n represents the price of the stock on the nth day, then

$$Y_n = Y_{n-1} + X_n \quad n \geq 1$$

where $X_1, X_2, \ldots$ are independent and identically distributed random variables with mean 0 and variance σ^2. Suppose that the stock's price today is 100. If $\sigma^2 = 1$, what can you say about the probability that the stock's price will exceed 105 after 10 days?

8.12. We have 100 components that we will put in use in a sequential fashion. That is, component 1 is initially put in use, and upon failure, it is replaced by component 2, which is itself replaced upon failure by component 3, and so on. If the lifetime of component i is exponentially distributed with mean $10 + i/10, i = 1, \ldots, 100$, estimate the probability that the total life of all components will exceed 1200. Now repeat when the life distribution of component i is uniformly distributed over $(0, 20 + i/5), i = 1, \ldots, 100$.

8.13. Student scores on exams given by a certain instructor have mean 74 and standard deviation 14. This instructor is about to give two exams, one to a class of size 25 and the other to a class of size 64.

(a) Approximate the probability that the average test score in the class of size 25 exceeds 80.

(b) Repeat part (a) for the class of size 64.

(c) Approximate the probability that the average test score in the larger class exceeds that of the other class by over 2.2 points.

(d) Approximate the probability that the average test score in the smaller class exceeds that of the other class by over 2.2 points.

8.14. A certain component is critical to the operation of an electrical system and must be replaced immediately upon failure. If the mean lifetime of this type of component is 100 hours and its standard deviation is 30 hours, how many of these components must be in stock so that the probability that the system is in continual operation for the next 2000 hours is at least .95?

8.15. An insurance company has 10,000 automobile policyholders. The expected yearly claim per policyholder is \$240, with a standard deviation of \$800. Approximate the probability that the total yearly claim exceeds \$2.7 million.

8.16. A.J. has 20 jobs that she must do in sequence, with the times required to do each of these jobs being independent random variables with mean 50 minutes and standard deviation 10 minutes. M.J. has 20 jobs that he must do in sequence, with the times required to do each of these jobs being independent random variables with mean 52 minutes and standard deviation 15 minutes.

(a) Find the probability that A.J. finishes in less than 900 minutes.

(b) Find the probability that M.J. finishes in less than 900 minutes.

(c) Find the probability that A.J. finishes before M.J.

8.17. Redo Example 5b under the assumption that the number of man–woman pairs is (approximately) normally distributed. Does this seem like a reasonable supposition?

8.18. Repeat part (a) of Problem 2 when it is known that the variance of a student's test score is equal to 25.

8.19. A lake contains 4 distinct types of fish. Suppose that each fish caught is equally likely to be any one of these types. Let Y denote the number of fish that need be caught to obtain at least one of each type.

(a) Give an interval (a, b) such that $P\{a \le Y \le b\} \ge .90$.

(b) Using the one-sided Chebyshev inequality, how many fish need we plan on catching so as to be at least 90 percent certain of obtaining at least one of each type.

8.20. If X is a nonnegative random variable with mean 25, what can be said about

(a) $E[X^3]$?

(b) $E[\sqrt{X}]$?

(c) $E[\log X]$?

(d) $E[e^{-X}]$?

8.21. Let X be a nonnegative random variable. Prove that

$$E[X] \le (E[X^2])^{1/2} \le (E[X^3])^{1/3} \le \cdots$$

8.22. Would the results of Example 5f change if the investor were allowed to divide her money and invest the fraction $\alpha, 0 < \alpha < 1$, in the risky proposition and invest the remainder in the risk-free venture? Her return for such a split investment would be $R = \alpha X + (1 - \alpha)m$.

8.23. Let X be a Poisson random variable with mean 20.

(a) Use the Markov inequality to obtain an upper bound on

$$p = P\{X \ge 26\}$$

(b) Use the one-sided Chebyshev inequality to obtain an upper bound on p.

(c) Use the Chernoff bound to obtain an upper bound on p.

(d) Approximate p by making use of the central limit theorem.

(e) Determine p by running an appropriate program.

THEORETICAL EXERCISES

8.1. If X has variance σ^2, then σ, the positive square root of the variance, is called the *standard deviation*. If X has mean μ and standard deviation σ, show that

$$P\{|X - \mu| \ge k\sigma\} \le \frac{1}{k^2}$$

8.2. If X has mean μ and standard deviation σ, the ratio $r \equiv |\mu|/\sigma$ is called the *measurement signal-to-noise ratio* of X. The idea is that X can be expressed as $X = \mu + (X - \mu)$, with μ representing the signal and $X - \mu$ the noise. If we define $|(X - \mu)/\mu| \equiv D$ as the relative deviation of X from its signal (or mean) μ, show that, for $\alpha > 0$,

$$P\{D \le \alpha\} \ge 1 - \frac{1}{r^2\alpha^2}$$

8.3. Compute the measurement signal-to-noise ratio—that is, $|\mu|/\sigma$, where $\mu = E[X]$ and $\sigma^2 = \text{Var}(X)$—of the following random variables:

(a) Poisson with mean λ;

(b) binomial with parameters n and p;

(c) geometric with mean $1/p$;

(d) uniform over (a, b);

(e) exponential with mean $1/\lambda$;

(f) normal with parameters μ, σ^2.

8.4. Let $Z_n, n \ge 1$, be a sequence of random variables and c a constant such that, for each $\varepsilon > 0$, $P\{|Z_n - c| > \varepsilon\} \to 0$ as $n \to \infty$. Show that, for any bounded continuous function g,

$$E[g(Z_n)] \to g(c) \quad \text{as} \quad n \to \infty$$

8.5. Let $f(x)$ be a continuous function defined for $0 \le x \le 1$. Consider the functions

$$B_n(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k} x^k (1 - x)^{n-k}$$

(called *Bernstein polynomials*) and prove that

$$\lim_{n \to \infty} B_n(x) = f(x)$$

Hint: Let $X_1, X_2, \ldots$ be independent Bernoulli random variables with mean x. Show that

$$B_n(x) = E\left[f\left(\frac{X_1 + \cdots + X_n}{n}\right)\right]$$

and then use Theoretical Exercise 4.

Since it can be shown that the convergence of $B_n(x)$ to $f(x)$ is uniform in x, the preceding reasoning provides a probabilistic proof of the famous Weierstrass theorem of analysis, which states that