DESIGN AND ANALYSIS OF ALGORITHMS

Homework 8

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1 Problem 1

1.1 Primal LP (P)

- **Decision Variables:** x_1, x_2, x_3
- **Linear Constraints:**

  $x_1 - x_2 \leq 1$
  $2x_2 - x_3 \leq 1$
  $x_1, x_2, x_3 \geq 0$

  It can be written as $Ax \leq b$, where $A_{2X3} = \{\{1, -1, 0\}, \{0, 2, -1\}\}$ and $b_{2X1} = \{1, 1\}$.

- **Objective Function:** max $(x_1 - 2x_3)$

  It can be written as $c^T x$, where $c_{3X1} = \{1, 0, -2\}$.

- Let us consider the point $x = (1.5, 0.5, 0)$. It is in fact a feasible solution for P as it satisfies all its constraints:
  $1.5 - 0.5 = 1 \leq 1$
  $2(0.5) - 0 = 1 \leq 1$
  $1.5, 0.5, 0 \geq 0$

- The value of the objective function for P at x is, $c^T x = 1.5 - 2(0) = 1.5$

1.2 Dual LP (D)

Primal	Dual
variables $x_1, \ldots, x_n$	n constraints
m constraints	variables $y_1, \ldots, y_m$
objective function $\mathbf{c}$	right-hand side $\mathbf{c}$
right-hand side $\mathbf{b}$	objective function $\mathbf{b}$
$\max \mathbf{c}^T\mathbf{x}$	$\min \mathbf{b}^T\mathbf{y}$
constraint matrix $\mathbf{A}$	constraint matrix $\mathbf{A}^T$
ith constraint is "$\leq$"	$y_i \geq 0$
ith constraint is "$\geq$"	$y_i \leq 0$
ith constraint is "$=$"	$y_i \in \mathbb{R}$
$x_j \geq 0$	jth constraint is "$\geq$"
$x_j \leq 0$	jth constraint is "$\leq$"
$x_j \in \mathbb{R}$	jth constraint is "$=$"

- We use the above recipe to compute corresponding dual LP, D.
- **Decision Variables:** y_1, y_2
  Since, there are two constraints for P, we correspondingly have two decision variables for D.
- **Linear Constraints:**

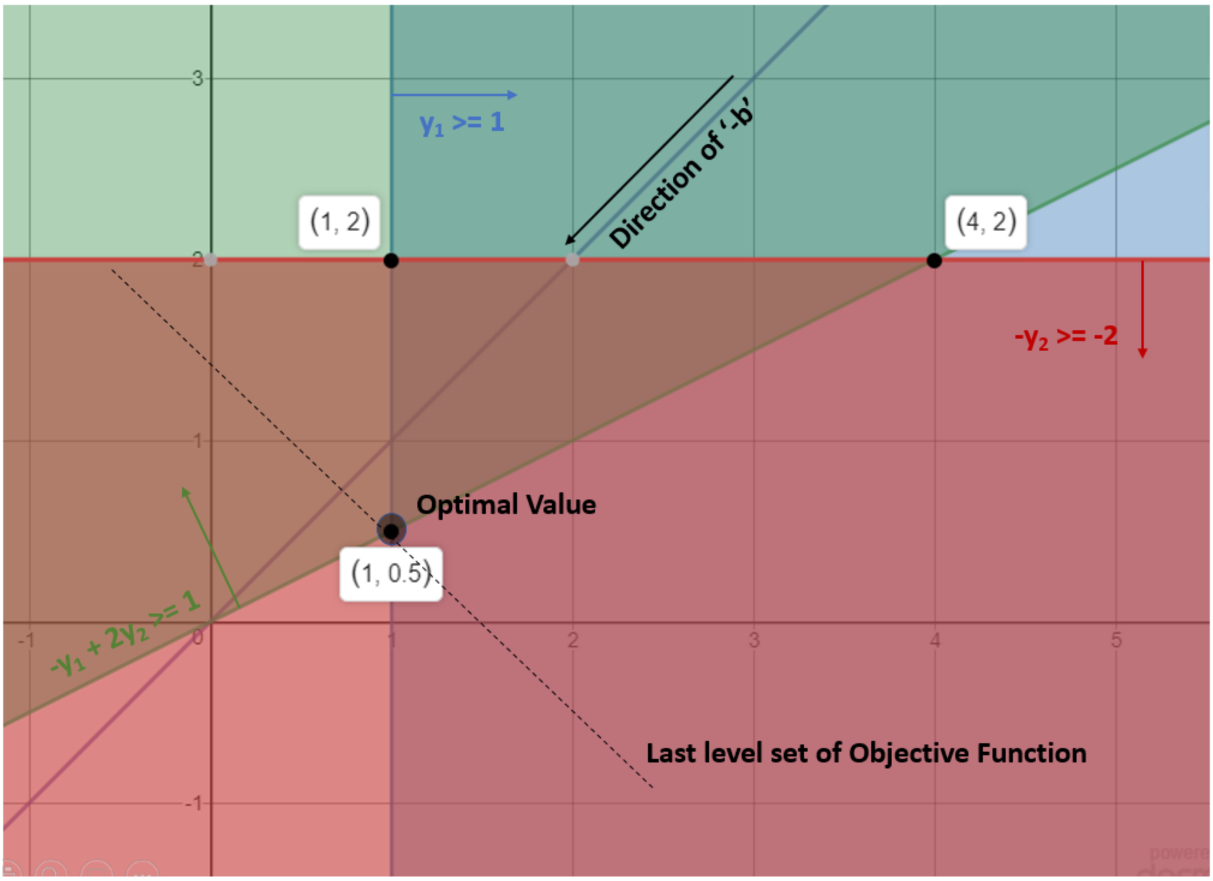
  $y_1 \geq 1$
  $-y_1 + 2y_2 \geq 1$
  $-y_2 \geq -2$
  $y_1, y_2 \geq 0$

  Since, there are three decision variables for P, we correspondingly have three constraints for D. These constraints are of the form $A^T y \geq c$. The $\geq$ inequality is attributed to the non-negativity of the decision variables of P. Finally the non-negativity of the decision variables of D follows from $\leq$ inequality in the linear constraint $Ax \leq b$ for P.

- **Objective Function:** min $(y_1 + y_2)$

  The objective function of D has the form min $(b^T y)$ where b is derived from the right-hand side b of P.

- From the figure below, it is clear that the feasible region of solutions is represented by the triangle formed by coordinates: $(1, 2)$, $(1, 0.5)$ and $(4, 2)$.
- We can also conclude that the optimal solution for D is $y = (1, 0.5)$, courtesy to the fact that this point is the last point of intersection (while traveling in the direction -b) of a level set of the objective function and the feasible region.
- The value of the objective function for D at y is, $b^T y = 1 + 0.5 = 1.5$



Now, $x = (1.5, 0.5, 0)$ and $y = (1, 0.5)$ are feasible for (P), (D) respectively and $c^T x = b^T y = 1.5$. Hence, using **Corollary 5.2 (c) for weak duality** (mentioned in lecture notes), we conclude that x and y are both optimal. The problem only requires us to prove that $x = (1.5, 0.5, 0)$ is optimal for P, which is now implied.

2 Problem 2

- We will construct a linear program L to solve this problem.
- First, we define the **permissible region** where our circle can lie as,

$$\mathcal{F} = \{\ (x_1, x_2)\ \in\ \mathbb{R}^2 \mid a_i x_1 + b_i y_1 + c_i\ \le\ 0\ \text{ for all }\ 1 \le i \le m\ \} \subset \mathbb{R}^2$$

2.1 Linear Program

1. **Decision Variables**

   We have three decision variables: x, y, r. Here, $(x, y) \in \mathbb{R}^2$ together represent the coordinates of the **center** of circle, while $r \in \mathbb{R}$ denotes the **radius** of circle.

2. **Linear Constraints**

   (i) Firstly, we want the centre of circle to lie in $\mathcal{F}$. Correspondingly, we have m constraints as follows:

   $$a_i x + b_i y + c_i \le 0 \text{ for all } 1 \le i \le m \qquad ...(1)$$

   (ii) Next, we want the circle to fit completely inside $\mathcal{F}$. This will only happen if, for every line, the distance between the center and the line is at least as much as the radius of circle. Thus, another m constraints follow:

   $$\frac{|a_i x + b_i y + c_i|}{\sqrt{a_i^2 + b_i^2}} \ge r \text{ for all } 1 \le i \le m$$

   From (1), we can eliminate the modulus in above constraints,

   $$\frac{-a_i x - b_i y - c_i}{\sqrt{a_i^2 + b_i^2}} - r \ge 0 \text{ for all } 1 \le i \le m \qquad ...(2)$$

   (iii) The final constraint says that the radius of circle must be non-negative.

   $$r \ge 0 \qquad ...(3)$$

3. **Linear Objective Function**

   Our primary goal is to find the largest (area wise) circle that can fit inside $\mathcal{F}$. The area of a circle whose radius is r is $A = \pi r^2$. If we choose our objective function to be max (πr^2), it ceases to be linear. However, we note that area is solely a function of r, hence we can simply maximize r to optimize the area. Thus our objective function is,

   $$\max\ (r)$$

2.2 Proof of Correctness

Claim 1: The solution of L is a valid circle inscribed completely in $\mathcal{F}$.

- The solution of L is the set of decision variables $D = \{x, y, r\}$. First of all, we note that $x, y, r \in \mathbb{R}$. Next, constraint (3) guarantees that D represents a valid circle. Thereafter, constraint (1) restricts the circle to be centered at a point which lies in $\mathcal{F}$. Finally, constraint (2) ensures that for each line, the distance between the centre (x, y) and the line is at least r. Hence, we can conclude that the circle is safely inscribed inside $\mathcal{F}$, and there are no cases of the circle overshooting its prescribed boundaries. Consequently, our claim is true.

Claim 2: The feasible region of solutions of L includes every possible valid circle in $\mathcal{F}$.

- Consider any arbitrary valid circle C' lying in $\mathcal{F}$. Let C' be centered at (x', y'), with its radius being r'. This circle can be encoded using our decision variables as $D' = \{x', y', r'\}$. Now, D' is a point in the $\mathbb{R}^3$ space. We need to show that D' lies in the feasible region of L.
- Since, C' is a valid circle, it trivially satisfies constraint (3) of L.
- We next observe that the circle we have chosen lies entirely in $\mathcal{F}$. Therefore, its centre must also reside in $\mathcal{F}$. Thus, D' also satisfies constraint (1).
- Since, C' lies completely inside $\mathcal{F}$, it must never be the case that the circle exceeds the boundaries defined by the lines. As a result, r' will always be upper bounded by the distance between the centre and any of the line. Consequently, constraint (2) is also fulfilled.

- We observe that D' satisfies each of the required constraints of L. Hence, we conclude that it lies in the feasible region of solutions of L.
- Since, we have shown any arbitrary circle C' can be encoded into a set of decision variables $D' = \{x', y', r'\}$, such that D' lies in the feasible region, Claim 2 is hence proved.

Claim 3: The solution returned by L, is a valid circle in $\mathcal{F}$ and of maximum-possible area.

- Claim 1 tells us that the solution computed by L is first of all, a valid circle lying entirely in $\mathcal{F}$. Then, using Claim 2, we established that L was optimizing over all possible circles in $\mathcal{F}$, i.e. its optimization domain was exhaustive. Finally, since the aim of our objective function was to maximize the **area**, and L computed the result over all possible feasible points in the solution space, it thus follows that L indeed returns a circle with the maximum-possible area.

Claim 4: The algorithm is robust to corner cases.

- There might be cases when the permissible region $\mathcal{F}$ is unbounded. This would mean that we can inscribe circles of increasingly larger radii in $\mathcal{F}$. In other words, the **optimal objective function value is also unbounded**. But, we know that linear programming algorithms correctly detect when such cases occur. Hence, our algorithm covers this corner case.
- A case might also arise, when the permissible region $\mathcal{F} = \phi$ (as there does not exist any $(x_1, x_2) \in \mathbb{R}^2$ which is common to every region defined by each of the m lines). In such cases, the feasible set of solutions of L is also empty. In this scenario no solution exists, and the same would be reported by the LP solver.

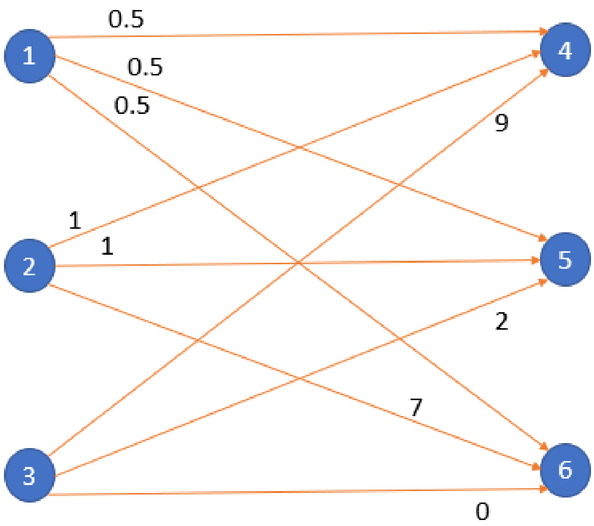
2.3 Time Complexity

- The total number of constraints is $O(m)$, which is linear. Also, the number of terms per constraint is a constant. Therefore the time complexity to encode this problem into a linear program is of linear order.
- Finally, we assume access to an LP solver (which are known to solve linear programs in polynomial time). Therefore, the original problem can also be solved in polynomial time.

Problem 3

Initialization

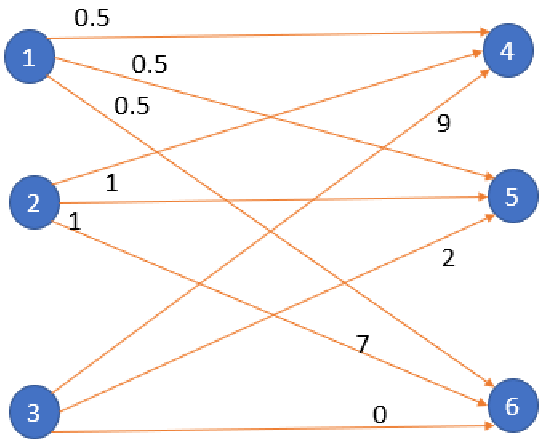
- **Construct G with $d_1d_2d_3d_4$ = 7920**
- **Initialize dual variables**:
  - $y_1 = y_2 = y_3 = 0$
  - $y_4 = \min\{0.5, 1, 9\} = 0.5$
  - $y_5 = \min\{0.5, 1, 2\} = 0.5$
  - $y_6 = \min\{0.5, 7, 0\} = 0$
- **Status of dual variables:**
  $\mathbf{y_1} = 0$, $\mathbf{y_2} = 0$, $\mathbf{y_3} = 0$, $\mathbf{y_4} = 0.5$, $\mathbf{y_5} = 0.5$, $\mathbf{y_6} = 0$



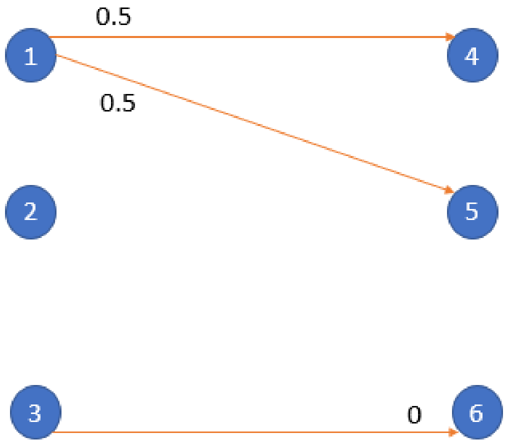
$d_1d_2d_3d_4$ = 7920

Iteration 1

- **Step 1:** Compute Tight set, **T** = { (1,4), (1,5), (3,6) }
- **Status of dual variables:** $\mathbf{y_1} = 0$, $\mathbf{y_2} = 0$, $\mathbf{y_3} = 0$, $\mathbf{y_4} = 0.5$, $\mathbf{y_5} = 0.5$, $\mathbf{y_6} = 0$



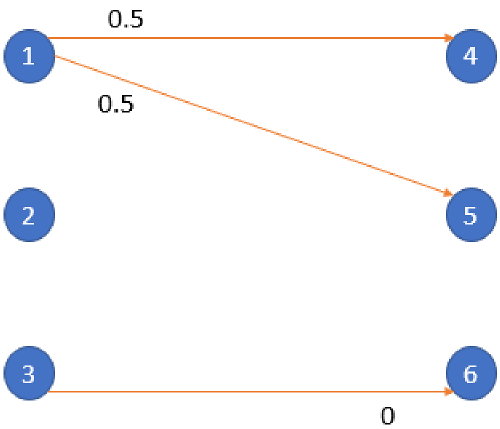
G



T

Iteration 1

- **Step 2:** Run max cardinality matching to obtain **M** (shown in black edges).
- **Status of dual variables:** $\mathbf{y_1} = 0$, $\mathbf{y_2} = 0$, $\mathbf{y_3} = 0$, $\mathbf{y_4} = 0.5$, $\mathbf{y_5} = 0.5$, $\mathbf{y_6} = 0$



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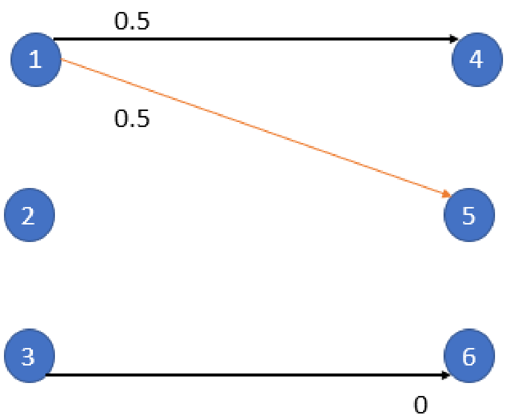


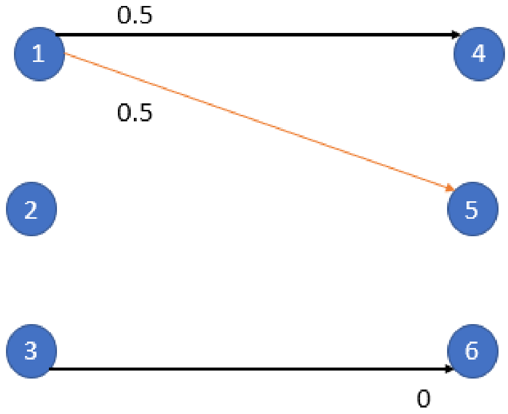
M

Iteration 1

- **Step 3: M** is not a perfect matching, hence we need to update dual variables.

- We compute the min-cut **(S, S^c)** and the following sets:
  1) **L \ S** = {1,3}
  2) **L ∩ S** = {2}
  3) **R \ S** = {4,5,6}
  4) **R ∩ S** = {}

S = {s,2}
S^c = {1,3,4,5,6,t}



M

- Compute **δ** over **(L ∩ S) X (R \ S)** = { (2,4), (2,5), (2,6) }:
- $\delta = \min \{ w(2,4) - y_2 - y_4 ,\ w(2,5) - y_2 - y_5 ,\ w(2,6) - y_2 - y_6 \}$
  $\delta = \min \{ 1 - 0 - 0.5,\ 1 - 0 - 0.5,\ 7 - 0 - 0 \}$
  $\delta = \min \{ 0.5,\ 0.5,\ 7 \} = \mathbf{0.5}$

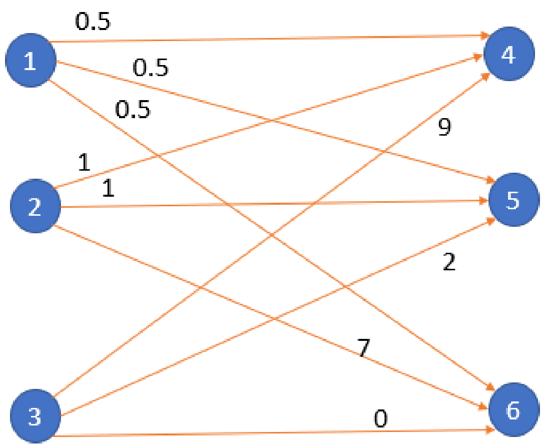
- Update dual variables for **L ∩ S:**
  $y_2 = y_2 + \delta = 0.5$

- No update for dual variables in **R ∩ S:**

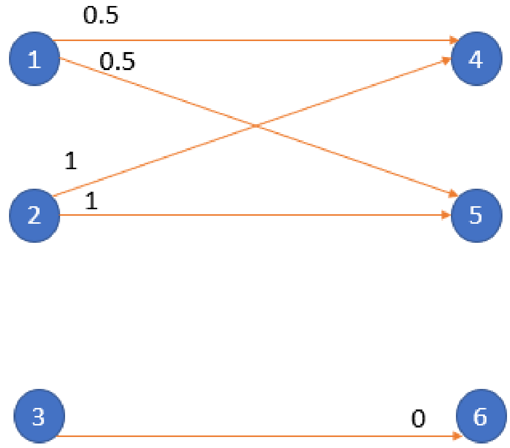
- **Status of dual variables:**
  $y_1 = 0$, $y_2 = 0.5$, $y_3 = 0$, $y_4 = 0.5$, $y_5 = 0.5$, $y_6 = 0$

Iteration 2

- **Step 1:** Compute Tight set, **T** = { (1,4), (1,5), (3,6), (2,4), (2,5) }
- **Status of dual variables:** $y_1 = 0$, $y_2 = 0.5$, $y_3 = 0$, $y_4 = 0.5$, $y_5 = 0.5$, $y_6 = 0$



G



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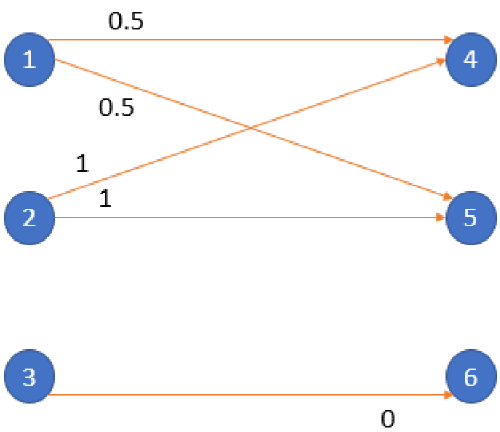
Iteration 2

- **Step 2:** Run max cardinality matching to obtain **M** (shown in black edges).
- **Step 3: M** is a perfect matching, hence we terminate.
- The minimum weight perfect matching for **G** has total weight **1.5** (which is also the sum of all dual variables).



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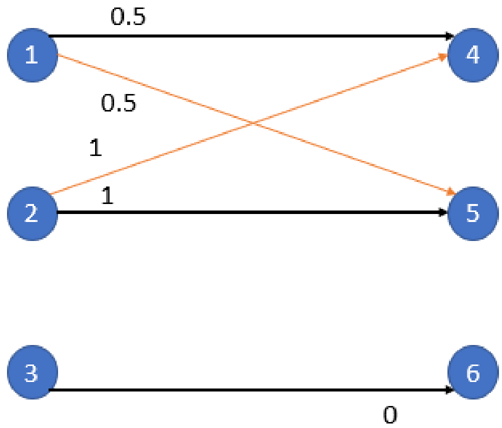

M

Status of dual variables: $y_1 = 0$, $y_2 = 0.5$, $y_3 = 0$, $y_4 = 0.5$, $y_5 = 0.5$, $y_6 = 0$