Probability
Assignment - 1

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Ch-1 (10) In how many ways can 8 people be seated in a row if :-

(a) there are no restrictions on the seating arrangement?

Ans- There are 8 people to be seated in 8 places.
The first person has → 8 choices.
After 1st person, the 2nd person has → 7 choices..
In this way, the no. of seating arrangement =
$8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 8! = \underline{40320}$

(b) persons A and B must sit next to each other?

Ans - Let us assume the pair (A, B) to be one entity. The pair and the rest 6 people together constitute 7 entities, which can be arranged in 7! ways.
Now, the pair can itself be seated in 2! ways.
The total ways $= 7! \times 2! = \underline{10080}$

(c) there are 4 men and 4 women and no 2 men or 2 women can sit next to each other?

Ans- The above arrangement can be done as follows,

Case 1   M W M W M W M W

Case 2   W M W M W M W M

For case 1, 4 men are to be seated in 4 places, which can be done in 4! ways. The 4 women are to be seated in 4 fixed places which can again be done in 4! ways. Hence the total ways for case 1 is $4! \times 4!$ ways.
Similarly, case 2 can be done in $4! \times 4!$ ways.
Hence, the total number of ways $= 4! \times 4! \times 2$
$= \underline{1152}$

(d) there are 5 men and they must sit next to each other?

Ans- Let us consider the 5 men as one entity. Together with 3 women, we have total of 4 entities. 4 entities can be arranged in $4!$ ways.
The group of 5 men can be arranged among themselves in $5!$ ways.
Hence, the total no. of ways $= 4! \times 5! = \underline{2880}$

(e) there are 4 married couples and each couple must sit together?

Ans- Let us consider each couple as one entity. Hence, the 4 entities can be arranged in $4!$ ways. Again, each couple can be seated among themselves in $2!$ ways. Hence, total number of ways $=$
$4! \times 2! \times 2! \times 2! \times 2! = 4! \times 2^4 = \underline{384}$

Ch-1 (14) Determine the number of vectors $(x_1, x_2, \ldots, x_n)$ such that each x_i is a positive integer and

$$\sum_{i=1}^{n} x_i \leq K \quad \text{where } K \geq n.$$

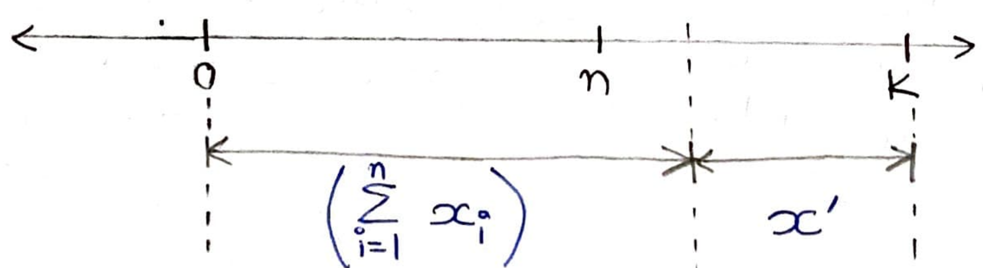
Ans - Since each x_i is a positive integer, $\forall i,\ x_i > 0$

Hence, $n \leq \sum_{i=1}^{n} x_i \leq K$

$$\underline{x_1 + x_2 + \ldots + x_n \leq K} \qquad \text{——— (1)}$$

Let us convert this inequality into an equality by introducing a new variable x', such that

$$\underline{(x_1 + x_2 + \ldots + x_n) + x' = K} \qquad \text{——— (2)}$$



→ x' is a dummy variable. After computing solⁿ to eqⁿ (2), we simply discard it and retain values of x_i.

The value of $\left(\sum_{i=1}^{n} x_i\right)$ lies between n and k. The variable (x') holds the amount by which $\left(\sum_{i=1}^{n} x_i\right)$ falls shy of k. The value of (x') depends on the value of $\left(\sum_{i=1}^{n} x_i\right)$.

$$0 \leq x' \leq k-n$$

More specifically,

$x' = \cancel{\neq}\ 0$, the case when $\sum_{i=1}^{n} x_i = k$

$x' = k-n$, ——— " ——— $\sum_{i=1}^{n} x_i = n$

Now, we have to find sol^n to,

$$(x_1 + x_2 + \dots + x_n) + x' = k \qquad \text{——— (2)}$$

where, $\left\{\begin{matrix} x_i \geq 1 \\ x' \geq 0 \end{matrix}\right\}$

We define,

$$\forall i, \quad x_i = y_i + 1$$

Substituting into (2),

$$(y_1+1) + (y_2+1) + \dots + (y_n+1) + x' = k$$

$$\Rightarrow (y_1 + y_2 + \dots + y_n) + x' = k-n \qquad \text{——— (3)}$$

Since, $\forall i, \ x_i \geq 1$
$\Rightarrow x_i - 1 \geq 0$
$\Rightarrow y_i \geq 0$

So, for eq (3)
$\left\{\begin{matrix} y_i \geq 0 \\ x' \geq 0 \end{matrix}\right\}$

→ For every sol^n of eq^n (3),

$$(y_1, y_2, \dots, y_n, x')$$

There exists a sol^n for eq^n ($\cancel{4}$2),

$$(y_1+1, y_2+1, \dots, y_n+1, x')$$

→ There exists a bijection between sol^n of eq^n (3) and eq^n ($\cancel{4}$)(2).

→ The no. of non-negative sol^n to eq^n (3) is equivalent to no. of ways in which we can partition $m(=k-n)$ 1's into $r(=n+1)$ groups, where the groups could be empty.

→ We can do this as follows,

$$\underbrace{\_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_\ \_}_{m+(r-1)\text{ places.}}$$

Let us assume we have m objects and $(r-1)$ lines. $(r-1)$ lines help us to form r groups by placing these $(r-1)$ lines between m objects. For e.g.,

$m = 4, \quad r = 3$

$m+r-1 = 4+3-1 = 6$ interpreted as,

O O | O | O → (2, 1, 1)

O O O O | | → (4, 0, 0)

| O | O O O → (0, 1, 3)

→ Hence, we can accomplish our task by choosing $(r-1)$ lines from $(m+r-1)$ places. So, no. of ways to form r groups (possibly empty) from m objects are,

$$\boxed{{}^{m+r-1}C_{r-1}}$$

→ We can use the above result to find no. of sol^n to eq^n (3),

$$(y_1+y_2+\dots+y_n)+x' = k-n \quad \text{—— (3)}$$

Here, $m = k-n$, $r = n+1$

So, ${}^{m+r-1}C_{r-1}$

$= {}^{(k-n)+(n+1)-1}C_{(n+1)-1}$

$= \underline{{}^{k}C_{n}}$

→ For every solⁿ of eqⁿ (3),

$(y_1, y_2, \ldots y_n, x')$

→ We obtain solⁿ of eqⁿ (2),

$(y_1+1, y_2+1, \ldots, y_n+1, x')$

$(x_1, x_2, \ldots, x_n, x')$, where $\left(\sum_{i=1}^{n} x_i\right) + x' = K$

→ Here, we discard x',

$(x_1, x_2, \ldots, x_n)$, where $\sum_{i=1}^{n} x_i \leq K$

Thus we obtain solⁿ to our original problem. The total no. of such vectors is also the same as the no. of solⁿ to eqⁿ (2) & (3) i.e. $\underline{{}^{K}C_n}$.

Ch-2 (5) A system is comprised of 5 components, each of which is either working or failed. Consider an experiment that consists of observing the status of each component, and let the outcome of the experiment be given by the vector $(x_1, x_2, x_3, x_4, x_5)$ where x_i is equal to 1 if component i is working and is equal to 0 if component i is failed.

(a) How many outcomes are there in the sample space of this experiment?

Ans - Every component has only 2 possible states, either 1 or 0. Hence, the sample space has all the possible states of system.

$$(\underset{2}{x_1} \times \underset{2}{x_2} \times \underset{2}{x_3} \times \underset{2}{x_4} \times \underset{2}{x_5}) = 32$$

$$|S| = 32$$

(b) Suppose that the system will work if components 1 and 2 are both working, or if components 3 and 4 are both working, or if components 1, 3 and 5 are all working. Let W be the event that the system will work. Specify all the outcomes in W.

Ans → Let X, Y, Z be events such that.

X → Component 1 and 2 are working.

Y → Component 3 and 4 are working.

Z → Component 1, 3 and 5 are working.

X = {(11000), (11001), (11010), (11011), (11100), (11101), (11110), (11111)}

Y = {(00110), (00111), (01110), (01111), (10110), (10111), (11110), (11111)}

Z = {(10101), (10111), (11101), (11111)}

Let,

W = X ∪ Y ∪ Z

↳ Event that system will work.

$$|W| = |X| + |Y| + |Z| - |X \cap Y| - |Y \cap Z| - |Z \cap X| + |X \cap Y \cap Z|$$

$$= 8 + 8 + 4 - 2 - 2 - 2 + 1$$

$$= 15$$

W = {(11000), (11001), (11010), (11011), (11100), (11101), (00110), (00111), (01110), (01111), (10110), (11110), (10101), (10111), (11111)}

(c) Let A be the event that components 4 and 5 are both failed. How many outcomes are contained in the event A?

Ans →

$$\underset{1}{\_}\ \underset{2}{\_}\ \underset{3}{\_}\ \underset{4}{0}\ \underset{5}{0}$$

Since 4 & 5 are failed, we have only 1 possibilities for 4 & 5. However, positions 1, 2 and 3 each still have 2 possibilities. So, the no. of outcomes in A

$$= 2 \times 2 \times 2 \times 1 \times 1 = 8$$

(d) Write out the outcomes in the event AW.

Ans → A → Components 4 and 5 are failed.

W → System will work.

Now, the system works if either

(1, 2) are working. → Only possible choice.

or, (3, 4) are working.

or, (1, 3, 5) are working.

Since 4 & 5 are failed, these cases will not occur in $A \cap W$.

So, (1, 2) are working and (4, 5) are failed.

$\underset{1}{1}\ \underset{2}{1}\ \underset{3}{\_}\ \underset{4}{0}\ \underset{5}{0}$

Component 1, 2, 4, 5 have only 1 choice each, except component 3 which has 2 choices.

$A \cap W$ → The system will work, even when component 4 and 5 are failed.

$$A \cap W = \left\{ \begin{matrix} (1\ 1\ 0\ 0\ 0), \\ (1\ 1\ 1\ 0\ 0) \end{matrix} \right\}$$

Ch 2 (8) Suppose that A and B are mutually exclusive events for which $P(A) = 0.3$ and $P(B) = 0.5$. What is the probability that,

(a) either A or B occurs?

$A \cap B = \phi$

$P(A \cap B) = 0$

Ans → $P(A \cup B) = P(A) + P(B) + \underbrace{P(A \cap B)}_{0}$

$= 0.3 + 0.5 + 0$

$= 0.8$

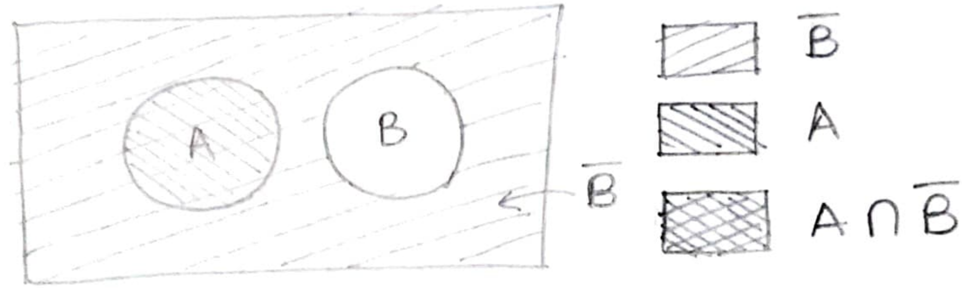
$P(A \cap B) = 0$ because A and B are mutually exclusive events. At any time either event A occurs, or B, or none of them occurs, but they never occur simultaneously.

(b) A occurs but B does not?

Ans → It is the same as asking the P(A) because when A occurs, the B will not occur by condition of mutual exclusiveness of A and B.

$P(A \cap \bar{B}) = P(A) = 0.3$

A venn diagram also shows the above fact,



(c) both A and B occur?

Ans → Again, the above event is not possible as A and B are mutually exclusive. $A \cap B = \phi$

$$P(A \cap B) = 0$$

Ch-2

TE (4)

(a) $$\left(\bigcup_{1}^{\infty} E_i\right) \cap F = \bigcup_{1}^{\infty} (E_i \cap F)$$

Ans → Two sets are equal if and only if they have the same elements. More formally, for any sets A and B, $A = B$ iff,

$$\forall x \, [x \in A \leftrightarrow x \in B]$$

Part 1

We will show ~~if~~ for any aribtrary element x,

$$x \in A \longrightarrow x \in B$$

Let us assume,

$$x \in \left(\bigcup_{1}^{\infty} E_i\right) \cap F$$

This means,

i) $x \in \left(\bigcup_{1}^{\infty} E_i\right)$ ——— (1)

ii) $x \in F$ ——— (2)

$x \in \left(\bigcup_{1}^{\infty} E_i\right)$ means that there exists at least one $i = k$, such that,

$$x \in E_k \quad \text{(at least one such } E_i\text{)} \quad \text{—— (3)}$$

Now, from (2) & (3),

$$x \in F$$

and, $x \in E_k$

This implies, $x \in (E_k \cap F)$.

Hence, we can say $x \in \bigcup_{1}^{\infty} (E_i \cap F)$.

Part 2

We will show for any arbitrary element x,

$$\cancel{x \Rightarrow} \; x \in B \rightarrow x \in A$$

Let us assume,

$$x \in \bigcup_{1}^{\infty} (E_i \cap F)$$

This means, there exists at least one $i = k$, such that,

$$x \in (E_k \cap F) \quad \text{—— (4)}$$

From (4), we can conclude,

$$x \in E_k \quad \text{—— (5)}$$

and $x \in F$ —— (6)

From (5), we can conclude that,

$$x \in \left(\bigcup_{1}^{\infty} E_i\right) \quad \text{—— (7)}$$

From (6) and (7), we thus conclude that,

$$x \in \left(\bigcup_{1}^{\infty} E_i\right) \cap F$$

From Part 1 and Part 2, we can conclude that, $\left(\bigcup_{1}^{\infty} E_i\right) \cap F = \bigcup_{1}^{\infty} (E_i \cap F)$

(b) $\left(\bigcap_{1}^{\infty} E_i\right) \cup F = \bigcap_{1}^{\infty} (E_i \cup F)$

Ans → <u>Part 1</u>

We will show for any arbitrary element x,

$$x \in A \rightarrow x \in B$$

Let us assume that,

$$x \in \left(\bigcap_{1}^{\infty} E_i\right) \cup F$$

We can conclude that,

$$x \in F$$

or, $x \in \left(\bigcap_{1}^{\infty} E_i\right)$

or, both.

→ ~~If,~~ If $x \in F$,

~~We can say,~~ $x \in (E_K \cup F)$, ~~for any random set~~ ~~E_K.~~

We can say that $x \in (E_i \cup F)$ for $\forall i$.

Hence, we can conclude that,

$$x \in \bigcap_{1}^{\infty} (E_i \cup F)$$

→ If $x \in \left(\bigcap_{1}^{\infty} E_i\right)$,

We can say that $x \in E_i$ for $\forall i$.

———— // ———— $x \in (E_i \cup F)$, for $\forall i$

Hence, we can conclude that,

$$x \in \bigcap_{1}^{\infty} (E_i \cup F)$$

<u>Part 2</u>

We will show for any arbitrary element x,

$$x \in B \rightarrow x \in A$$

Let us assume that,

$$x \in \bigcap_{1}^{\infty} (E_i \cup F)$$

We can say that, $x \in (E_i \cup F)$, for $\forall i$.

We can conclude that, also

$x \in F$ (x may belong to E_i, for some i's)

or, $x \notin F$ and $\forall i, x \in E_i$

~~or,~~

→ If $x \in F$,
we can directly say, $x \in \left(\bigcap_1^\infty E_i\right) \cup F$.

→ If $x \notin F$, then $\forall i, x \in E_i$.
So, we can conclude $x \in \left(\bigcap_1^\infty E_i\right)$

Hence, —— " —— $x \in \left(\bigcap_1^\infty E_i\right) \cup F$.

From Part 1 and Part 2, we can say,

$$\left(\bigcap_1^\infty E_i\right) \cup F = \bigcap_1^\infty (E_i \cup F)$$