# Assignment 04

1. Let $\{B_n\}$ be a sequence of any events. If $\sum_n \Pr\{B_n\} < \infty$, then

$$\Pr\left\{\bigcap_{N=0}^{\infty}\bigcup_{n=N}^{\infty} B_n\right\} = 0.$$

   [Hint: Use continuity of probability].

2. For a non-negative random variable $X$, show that

$$\Pr\{X \geq a\} \leq \mathbb{E}[X]/a.$$

   Please deal with the two cases of $X$ being discrete and continuous, individually. Using the above result or otherwise, show that for any random variable $X$, whose moment generating function $M_X(t)$ is well defined for $t \geq 0$, we have

$$\Pr\{X \geq a\} \leq e^{-ta} M_X(t).$$

## PROBLEMS

**6.1.** Two fair dice are rolled. Find the joint probability mass function of $X$ and $Y$ when
**(a)** $X$ is the largest value obtained on any die and $Y$ is the sum of the values;
**(b)** $X$ is the value on the first die and $Y$ is the larger of the two values;
**(c)** $X$ is the smallest and $Y$ is the largest value obtained on the dice.

**6.2.** Suppose that 3 balls are chosen without replacement from an urn consisting of 5 white and 8 red balls. Let $X_i$ equal 1 if the $i$th ball selected is white, and let it equal 0 otherwise. Give the joint probability mass function of
**(a)** $X_1, X_2$;
**(b)** $X_1, X_2, X_3$.

**6.3.** In Problem 2, suppose that the white balls are numbered, and let $Y_i$ equal 1 if the $i$th white ball is selected and 0 otherwise. Find the joint probability mass function of
**(a)** $Y_1, Y_2$;
**(b)** $Y_1, Y_2, Y_3$.

**6.4.** Repeat Problem 2 when the ball selected is replaced in the urn before the next selection.

**6.5.** Repeat Problem 3a when the ball selected is replaced in the urn before the next selection.

**6.6.** A bin of 5 transistors is known to contain 2 that are defective. The transistors are to be tested, one at a time, until the defective ones are identified. Denote by $N_1$ the number of tests made until the first defective is identified and by $N_2$ the number of additional tests until the second defective is identified. Find the joint probability mass function of $N_1$ and $N_2$.

**6.7.** Consider a sequence of independent Bernoulli trials, each of which is a success with probability $p$. Let $X_1$ be the number of failures preceding the first success, and let $X_2$ be the number of failures between the first two successes. Find the joint mass function of $X_1$ and $X_2$.

**6.8.** The joint probability density function of $X$ and $Y$ is given by

$$f(x, y) = c(y^2 - x^2)e^{-y} \qquad -y \le x \le y,\ 0 < y < \infty$$

**(a)** Find $c$.
**(b)** Find the marginal densities of $X$ and $Y$.
**(c)** Find $E[X]$.

**6.9.** The joint probability density function of $X$ and $Y$ is given by

$$f(x, y) = \frac{6}{7}\left(x^2 + \frac{xy}{2}\right) \qquad 0 < x < 1,\ 0 < y < 2$$

**(a)** Verify that this is indeed a joint density function.
**(b)** Compute the density function of $X$.
**(c)** Find $P\{X > Y\}$.
**(d)** Find $P\{Y > \frac{1}{2} | X < \frac{1}{2}\}$.
**(e)** Find $E[X]$.
**(f)** Find $E[Y]$.

**6.10.** The joint probability density function of $X$ and $Y$ is given by

$$f(x, y) = e^{-(x+y)} \qquad 0 \le x < \infty, 0 \le y < \infty$$

Find (a) $P\{X < Y\}$ and (b) $P\{X < a\}$.

**6.11.** A television store owner figures that 45 percent of the customers entering his store will purchase an ordinary television set, 15 percent will purchase a plasma television set, and 40 percent will just be browsing. If 5 customers enter his store on a given day, what is the probability that he will sell exactly 2 ordinary sets and 1 plasma set on that day?

**6.12.** The number of people that enter a drugstore in a given hour is a Poisson random variable with parameter $\lambda = 10$. Compute the conditional probability that at most 3 men entered the drugstore, given that 10 women entered in that hour. What assumptions have you made?

**6.13.** A man and a woman agree to meet at a certain location about 12:30 P.M. If the man arrives at a time uniformly distributed between 12:15 and 12:45, and if the woman independently arrives at a time uniformly distributed between 12:00 and 1 P.M., find the probability that the first to arrive waits no longer than 5 minutes. What is the probability that the man arrives first?

**6.14.** An ambulance travels back and forth at a constant speed along a road of length $L$. At a certain moment of time, an accident occurs at a point uniformly distributed on the road. [That is, the distance of the point from one of the fixed ends of the road is uniformly distributed over $(0, L)$.] Assuming that the ambulance's location at the moment of the accident is also uniformly distributed, and assuming independence of the variables, compute the distribution of the distance of the ambulance from the accident.

**6.15.** The random vector $(X, Y)$ is said to be uniformly distributed over a region $R$ in the plane if, for some constant $c$, its joint density is

$$f(x, y) = \begin{cases} c & \text{if}(x, y) \in R \\ 0 & \text{otherwise} \end{cases}$$

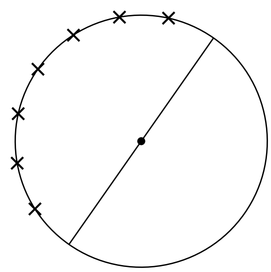
**(a)** Show that $1/c$ = area of region $R$.

Suppose that $(X, Y)$ is uniformly distributed over the square centered at (0, 0) and with sides of length 2.

**(b)** Show that $X$ and $Y$ are independent, with each being distributed uniformly over $(-1, 1)$.

**(c)** What is the probability that $(X, Y)$ lies in the circle of radius 1 centered at the origin? That is, find $P\{X^2 + Y^2 \le 1\}$.

**6.16.** Suppose that $n$ points are independently chosen at random on the circumference of a circle, and we want the probability that they all lie in some semicircle. That is, we want the probability that there is a line passing through the center of the circle such that all the points are on one side of that line, as shown in the following diagram:



Let $P_1, \ldots, P_n$ denote the $n$ points. Let $A$ denote the event that all the points are contained in some semicircle, and let $A_i$ be the event that all the points lie in the semicircle beginning at the point $P_i$ and going clockwise for $180°, i = 1, \ldots, n$.

**(a)** Express $A$ in terms of the $A_i$.

**(b)** Are the $A_i$ mutually exclusive?

**(c)** Find $P(A)$.

**6.17.** Three points $X_1, X_2, X_3$ are selected at random on a line $L$. What is the probability that $X_2$ lies between $X_1$ and $X_3$?

**6.18.** Two points are selected randomly on a line of length $L$ so as to be on opposite sides of the midpoint of the line. [In other words, the two points $X$ and $Y$ are independent random variables such that $X$ is uniformly distributed over $(0, L/2)$ and $Y$ is uniformly distributed over $(L/2, L)$.] Find the probability that the distance between the two points is greater than $L/3$.

**6.19.** Show that $f(x, y) = 1/x, 0 < y < x < 1$, is a joint density function. Assuming that $f$ is the joint density function of $X, Y$, find

**(a)** the marginal density of $Y$;

**(b)** the marginal density of $X$;

**(c)** $E[X]$;

**(c)** $E[Y]$.

**6.20.** The joint density of $X$ and $Y$ is given by

$$f(x, y) = \begin{cases} xe^{-(x+y)} & x > 0,\ y > 0 \\ 0 & \text{otherwise} \end{cases}$$

Are $X$ and $Y$ independent? If, instead, $f(x, y)$ were given by

$$f(x, y) = \begin{cases} 2 & 0 < x < y,\ 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

would $X$ and $Y$ be independent?

**6.21.** Let

$$f(x, y) = 24xy \quad 0 \le x \le 1,\ 0 \le y \le 1,\ 0 \le x + y \le 1$$

and let it equal 0 otherwise.

**(a)** Show that $f(x, y)$ is a joint probability density function.

**(b)** Find $E[X]$.

**(c)** Find $E[Y]$.

**6.22.** The joint density function of $X$ and $Y$ is

$$f(x, y) = \begin{cases} x + y & 0 < x < 1,\ 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

**(a)** Are $X$ and $Y$ independent?

**(b)** Find the density function of $X$.

**(c)** Find $P\{X + Y < 1\}$.

**6.23.** The random variables $X$ and $Y$ have joint density function

$$f(x, y) = 12xy(1 - x) \quad 0 < x < 1,\ 0 < y < 1$$

and equal to 0 otherwise.

**(a)** Are $X$ and $Y$ independent?

**(b)** Find $E[X]$.

**(c)** Find $E[Y]$.

**(d)** Find $\text{Var}(X)$.

**(e)** Find $\text{Var}(Y)$.

**6.24.** Consider independent trials, each of which results in outcome $i, i = 0, 1, \ldots, k$, with probability $p_i, \sum_{i=0}^{k} p_i = 1$. Let $N$ denote the number of trials needed to obtain an outcome that is not equal to 0, and let $X$ be that outcome.

**(a)** Find $P\{N = n\}, n \ge 1$.

**(b)** Find $P\{X = j\}, j = 1, \ldots, k$.

**(c)** Show that $P\{N = n, X = j\} = P\{N = n\}P\{X = j\}$.

**(d)** Is it intuitive to you that $N$ is independent of $X$?

**(e)** Is it intuitive to you that $X$ is independent of $N$?

**6.25.** Suppose that $10^6$ people arrive at a service station at times that are independent random variables,