1

$$A = \begin{bmatrix} 2 & 1 & 5 \\ 4 & 2 & 10 \\ 2 & 1 & 5 \end{bmatrix}$$

Now,

$$|A - \lambda I| = 0$$

$$\Rightarrow \begin{vmatrix} 2-\lambda & 1 & 5 \\ 4 & 2-\lambda & 10 \\ 2 & 1 & 5-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (2-\lambda)\{(2-\lambda)(5-\lambda) - 10\} - 1\{4(5-\lambda) - 20\} + 5\{4 - 2(2-\lambda)\} = 0$$

$$\Rightarrow (2-\lambda)\{\cancel{10} - 7\lambda + \lambda^2 - \cancel{10}\} - \{\cancel{20} - 4\lambda - \cancel{20}\} + 5\{\cancel{4} - \cancel{4} + 2\lambda\} = 0$$

$$\Rightarrow (2-\lambda)(\lambda^2 - 7\lambda) + 4\lambda + 10\lambda = 0$$

$$\Rightarrow \lambda\{(2-\lambda)(\lambda - 7) + 14\} = 0$$

$$\Rightarrow \lambda\{2\lambda - \cancel{14} - \lambda^2 + 7\lambda + \cancel{14}\} = 0$$

$$\Rightarrow \lambda(9\lambda - \lambda^2) = 0$$

$$\Rightarrow \lambda^2(9 - \lambda) = 0$$

$$\Rightarrow \lambda = 0, 0, 9$$

Therefore, the eigen values are 0, 0, 9.

Let us calculate eigen vectors.

i) $\lambda_1 = 0$

$$(A - \lambda I)x = 0$$

$$\begin{bmatrix} 2 & 1 & 5 \\ 4 & 2 & 10 \\ 2 & 1 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Using elimination method, we reduce the matrix to echelon form.

$R_2 \leftarrow R_2 - R_1$

$R_3 \leftarrow R_3 - R_1$

$$\Rightarrow \left[\begin{array}{ccc|c} 2 & 1 & 5 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow 2x_1 + x_2 + 5x_3 = 0$$

$$\Rightarrow x_1 = \frac{-x_2 - 5x_3}{2}$$

General solⁿ,

$$x = \begin{bmatrix} -\frac{1}{2}x_2 - \frac{5}{2}x_3 \\ x_2 \\ x_3 \end{bmatrix}$$

Let, $x_2 = 0, x_3 = 1$

$$v_1 = \begin{bmatrix} -5/2 \\ 0 \\ 1 \end{bmatrix}$$

Let, $x_2 = 1, x_3 = 0$

$$v_2 = \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix}$$

ii) $\lambda_2 = 9$

$$(A - \lambda I)x = 0$$

$$\Rightarrow \begin{bmatrix} -7 & 1 & 5 \\ 4 & -7 & 10 \\ 2 & 1 & -4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$R_1 \leftarrow \frac{-1}{7}R_1$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & \frac{-1}{7} & \frac{-5}{7} & 0 \\ 4 & -7 & 10 & 0 \\ 2 & 1 & -4 & 0 \end{array}\right]$$

$R_2 \leftarrow R_2 - 4R_1$

$R_3 \leftarrow R_3 - 2R_1$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & \frac{-1}{7} & \frac{-5}{7} & 0 \\ 0 & -7 + \frac{4}{7} & 10 + \frac{20}{7} & 0 \\ 2 & 1 + \frac{2}{7} & -4 + \frac{10}{7} & 0 \end{array}\right]$$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & -1/7 & \frac{-5}{7} & 0 \\ 0 & -\frac{45}{7} & \frac{90}{7} & 0 \\ 0 & \frac{9}{7} & \frac{-18}{7} & 0 \end{array}\right]$$

$R_2 \leftarrow -\frac{7}{45}R_2$, $R_3 \leftarrow \frac{7}{9}R_3$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & -1/7 & -5/7 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 1 & -2 & 0 \end{array}\right]$$

$R_3 \leftarrow R_3 - R_2$

$$\Rightarrow \left[\begin{array}{ccc|c} 1 & -1/7 & -5/7 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$$\Rightarrow x_2 - 2x_3 = 0 \quad \text{—— } (1)$$

$$x_1 - \frac{x_2}{7} - \frac{5x_3}{7} = 0 \quad \text{—— } (2)$$

$$\Rightarrow x_2 = 2x_3$$

$$x_1 = \frac{2}{7}x_3 + \frac{5x_3}{7} = x_3$$

General solⁿ,

$$x = \begin{bmatrix} x_3 \\ 2x_3 \\ x_3 \end{bmatrix}$$

Let, $x_3 = 1$

$$V_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

Hence, the eigen vectors of A are,

$$v_1 = \begin{bmatrix} -5/2 \\ 0 \\ 1 \end{bmatrix}, \quad V_2 = \begin{bmatrix} -1/2 \\ 1 \\ 0 \end{bmatrix}, \quad V_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$$

(4)

Thursday, 12 November 2020 11:37 PM

Now,
the eigen vectors of A are,

$\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \end{bmatrix}$

$\lambda_1 = 0$
$\lambda_2 = 1$

Let, $X = \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}$

$|X| = -1-4 = -5$

Since, $|X| \neq 0$, rank$(X) = 2$.
The eigen vectors of A are linearly independent.
Hence A is diagonalizable.

$X^{-1} = \frac{-1}{5}\begin{bmatrix} -1 & -2 \\ -2 & 1 \end{bmatrix}$

$A = X \Lambda X^{-1}$

$= \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & -2 \\ -2 & 1 \end{bmatrix} \cdot \left(-\frac{1}{5}\right)$

$= \begin{bmatrix} 1 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ -2 & 1 \end{bmatrix} \cdot \left(\frac{-1}{5}\right)$

$= \begin{bmatrix} -4 & 2 \\ 2 & -1 \end{bmatrix} \cdot \left(\frac{-1}{5}\right)$

$A = \begin{bmatrix} \frac{4}{5} & \frac{-2}{5} \\ \frac{-2}{5} & \frac{1}{5} \end{bmatrix}$

Hence, A is symetric.

→ Trace $(A) = \frac{4}{5} + \frac{1}{5} = 1$ (Sum of diagonal values / eigen values)

→ $|A| = 0.1 = 0$ (Product of eigen values)

Now,

$Ax = \lambda x$

$\Rightarrow A^2 x = \lambda A x$

$\Rightarrow A^2 x = \lambda^2 x$

→ So, eigen values of A^2 are $0^2, 1^2$ i.e. 0 and 1.

→ From (1), it is clear that the eigen vectors of A^2 are same as A, $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

5

Thursday, 26 November 2020 6:53 PM

$$Ax = \lambda_1 x$$

$$\Rightarrow (Ax)^T = (\lambda_1 x)^T$$

$$\Rightarrow x^T A^T = \lambda_1 x^T$$

$$\Rightarrow x^T A^T y = \lambda_1 x^T y \qquad (A^T y = \lambda_2 y)$$

$$\Rightarrow x^T \lambda_2 y = \lambda_1 x^T y$$

$$\Rightarrow x^T y (\lambda_1 - \lambda_2) = 0$$

When $\lambda_1 \neq \lambda_2$,

$$x^T y = 0$$

9

Thursday, 26 November 2020 10:57 AM

(a) $Ax = b$

(b) $A^T y = 0$, $y^T b \neq 0$

$$\begin{array}{cc} & n \\ m & \begin{bmatrix} \quad A \quad \end{bmatrix} \end{array}$$

I. If $Ax = b$ has a solution,

Then, $x = A^{-1} b$ exists.

This means A^{-1} exists and rank $(A) = m = n$.

→ We know that (column space of A) and (null space of A^T) are orthogonal complements and the sum of their dimensions add up to m.

→ Now, $\dim(col(A)) = rank(A) = m$

$$\dim(col(A)) + \dim(null(A^T)) = m$$

$$\Rightarrow \dim(null(A^T)) = 0 \quad \text{——} (1)$$

From (1), it is clear that,

$$A^T y = 0$$

has no non-trivial sol^n. Also it is given that $y^T b \neq 0$. This means that y cannot be zero/trivial.

Hence, if $Ax = b$ has a sol^n, then $A^T y = 0$ has no solutions ——— (2)

II. Let, $A^T y = 0$ have a non-trivial sol^n.

This means, $\dim(null(A^T)) = K \neq 0$

We know,

$$\dim(col(A)) + \dim(null(A^T)) = m$$

$$\Rightarrow \dim(col(A)) = m - K < m$$

$$\Rightarrow rank(A) < m. \quad \text{——} (3)$$

From (3), we conclude that A is non-invertible.

and $Ax = b \Rightarrow x = A^{-1} b$ does not exist.

Therefore, when $A^T y = 0$ has a non-trivial sol^n, $Ax = b$, does not have a sol^n. ——— (4)

Also since $y^T b \neq 0$, b is non zero. Hence, $Ax = b$ cannot have any trivial sol^n.

From (2) & (4), we can say, that only one of system (a) or (b) can have a sol^n.

(15)

Thursday, 26 November 2020 11:04 PM

$$R(x,y) = \frac{x^2 - xy + y^2}{x^2 + y^2}$$

The above eqn is Rayleigh quotient for symetric matrix,

$$A = \begin{bmatrix} 1 & -1/2 \\ -1/2 & 1 \end{bmatrix}$$

$$R(x,y) = \frac{v^T A v}{v^T v}, \quad v = \begin{bmatrix} x \\ y \end{bmatrix}$$

Also, the Rayleigh quotient is lower bounded by the least eigen value of A.

Now, we calculate eigen values for A,

$$\begin{vmatrix} 1-\lambda & -1/2 \\ -1/2 & 1-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)^2 - 1/4 = 0$$

$$\Rightarrow \lambda^2 - 2\lambda + 1 - \frac{1}{4} = 0$$

$$\Rightarrow \lambda^2 - 2\lambda + \frac{3}{4} = 0$$

$$\Rightarrow \lambda = \frac{2 \pm \sqrt{4-3}}{2}$$

$$\Rightarrow \lambda = 1 \pm \frac{1}{2}$$

$$\Rightarrow \lambda = \frac{1}{2}, \frac{3}{2}$$

So, minimum value of $R(x,y)$ is $\frac{1}{2}$.