Probability Assignment-2

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(1) Consider the following technique for shuffling a deck of n cards : For any initial ordering of the cards, go through the deck one card at a time and at each card, flip a fair coin. If the coin comes up heads, then leave the card where it is; if the coin comes up tails, then move the card to the end of the deck. After the coin has been flipped n times, say that one round has been completed. Assuming that all possible outcomes of the sequence of n coin flips are equally likely, what is the probability that the ordering after one round is the same as the initial ordering?

Sol^n : We claim that for the deck to retain its initial ordering, the outcome of 'n' coin tosses should be of the form :

$$\underbrace{H\,H\,H\cdots H}_{k}\ \underbrace{T\,T\,T\cdots T}_{n-k}$$

It must be k consecutive heads followed by (n-k) consecutive tails. Here k = 0 to k = n. For e.g. for n = 3, there are n+1 possibilities :—

k=0	T T T
k=1	H T T
k=2	H H T
k=3	H H H

In general for a deck of n cards, (n+1) possible outcomes help to ensure that the initial ordering is preserved. We next show why our claim is true.

(2) Let us assume that $n = 5$, and our cards are as follows :—

$$(C_1 C_2 C_3 C_4 C_5)$$

Consider two cases of outcomes,

<u>Case 1</u>: All 'H' appear before 'T' (No <u>interleaving</u> of H and T).

$$(C_1 \; C_2 \; C_3 \; C_4 \; C_5)$$
$$(H \; H \; H \; T \; T)$$

$$H' \rightarrow \underbrace{C_1 \; C_2 \; C_3} \; \vdots \; \underbrace{C_4 \; C_5} \leftarrow T'$$

End

$$\Rightarrow (C_1 \; C_2 \; C_3 \, \vdots \, C_4 \; C_5)$$

We claim that any outcome of 'n' tosses separates the original deck sequence into two sequences :

$H' \rightarrow$ Seq. of elements with outcome H

$T' \rightarrow$ Seq. of elements with outcome T

Within each seq., the relative ordering of initial deck sequence is preserved.

<u>Case 2</u>: 'H' and 'T' are <u>interleaved</u>.

$$(C_1 \; C_2 \; C_3 \; C_4 \; C_5)$$
$$(H \; T \; H \; T \; H)$$

$$H' \rightarrow \underbrace{C_1 \; \_\_ \; C_3 \; \_\_ \; C_5} \; \vdots \; \underbrace{C_2 \; C_4} \leftarrow T'$$

gaps

End

$$\Rightarrow (C_1 \; C_3 \; C_5 \, \vdots \, C_2 \; C_4)$$

An 'element with outcome 'T' has got a <u>passport</u> to go into sequence T'. Here, we see that if there is any interleaving of H and T, the elements ~~of~~ with outcome 'T' (for e.g. C_2 & C_4) go into the

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other sequence T', thus leaving gaps in the sequence H', and hence destroy the continuity. Therefore original sequence is destroyed.

Now, we calculate the total probability of our event,

$$S = \{(HHH\ldots H), \\ \vdots \\ (TTT\ldots T)\}$$

→ The size of sample space is $\underbrace{(2 \times 2 \times \cdots \times 2)}_{n \text{ times}}$

$= \underline{2^n}$, as every position has only 2 possibilities, H or T.

$$E = \{(TTT\ldots T), (HTT\ldots T), (HHT\ldots T), \ldots, (HHH\ldots H)\}$$

→ We already established that the no. of favourable outcomes is $= \underline{(n+1)}$

→ We now show that the probability of any of the 2^n possible outcome is equal. That is for any given outcome,

$P(H) = P(T) = \frac{1}{2}$

$$\underbrace{\frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2}\ \frac{1}{2} \ldots \frac{1}{2}}_{n \text{ places}}$$

Probability of singleton event $= \left(\frac{1}{2}\right)^n$

Probability of E $= P(TTT\ldots T) + P(HTT\ldots T) + P(HHT\ldots T) + \ldots + P(HHH\ldots H)$

All are mutually exclusive, as they are singleton sets.

$$P(E) = (n+1)\left(\frac{1}{2}\right)^n$$

$$\boxed{P(E) = \frac{n+1}{2^n}}$$

4 (2) An urn contains n white and m black balls, where n and m are positive numbers.

(a) If two balls are randomly withdrawn, what is the probability that they are the same color?

Sol^n: Size of sample space, $|S| = {}^{n+m}C_2$

E_1 = No. of ways in which, we can draw 2 balls of same color $= \underbrace{{}^{n}C_2}_{\text{2 white balls}} + \underbrace{{}^{m}C_2}_{\text{2 black balls}}$

(2 white balls) OR (2 black balls)

Hence,
$$P(E_1) = \frac{{}^{n}C_2 + {}^{m}C_2}{{}^{n+m}C_2}$$

$$P(E_1) = \frac{\frac{n(n-1)}{2} + \frac{m(m-1)}{2}}{\frac{(n+m)(n+m-1)}{2}}$$

$$P(E_1) = \frac{n^2 - n + m^2 - m}{n^2 + nm - n + mn + m^2 - m}$$

$$P(E_1) = \frac{(n^2 + m^2) - (n+m)}{(n+m)^2 - (n+m)} \quad \text{——} (1)$$

(b) If a ball is randomly withdrawn and then replaced before the second one is withdrawn, what is the probability that the withdrawn balls are the same color?

Sol^n: Size of sample space, $|S| = (n+m) \times (n+m)$
$= (n+m)^2$

$E_2 \rightarrow$ Withdraw 2 balls of same color when balls are replaced.

(5)

No. of ways in which, we can draw 2 balls of same color when replacement is allowed =

$$(n \times n) + (m \times m)$$

(2 white balls) OR (2 black balls)

Hence,

$$P(E_2) = \frac{(n^2+m^2)}{(n+m)^2} \quad \text{---} (2)$$

(c) Show that probability in part (b) is always larger than the one in part (a)

Sol^n: Divide (1) by (2),

$$\frac{P(E_1)}{P(E_2)} = \frac{(n^2+m^2)-(n+m)}{(n+m)^2-(n+m)} \times \frac{(n+m)^2}{(n^2+m^2)}$$

$$= \frac{(n^2+m^2)-(n+m)}{\cancel{(n+m)}[(n+m)-1]} \times \frac{(n+m)^{\cancel{2}}}{(n^2+m^2)}$$

$$= \frac{(n^2+m^2-n-m)(n+m)}{(n+m-1)(n^2+m^2)}$$

$$= \frac{(n^3+m^3+mn^2+m^2n-n^2-m^2)-2mn}{(n^3+m^3+mn^2+m^2n-n^2-m^2)}$$

$$= 1-\left(\frac{2mn}{n^3+m^3+mn^2+m^2n-n^2-m^2}\right) \quad \text{---} (3)$$

Since, $n>0,\ m>0$

$$\frac{2mn}{n^3+m^3+mn^2+m^2n-n^2-m^2} > 0 \quad \text{---} (4)$$

In denominator,

$$n^3 + m^3 \geq n^2 + m^2$$

$$n^3 + m^3 - n^2 - m^2 \geq 0$$

$$(n^3 + m^3 - n^2 - m^2) + mn^2 + m^2n > 0 \quad \text{—— (5)}$$

In numerator,

$$2mn > 0 \quad \text{—— (6)}$$

From (5) and (6), (4) is justified.

Hence,

$$\left[1 - \left(\frac{2mn}{n^3 + m^3 + mn^2 + m^2n - n^2 - m^2}\right)\right] < 1$$

$$\frac{P(E_1)}{P(E_2)} < 1$$

$$\Rightarrow \boxed{P(E_2) > P(E_1)}$$

(3) An instructor gives her class a set of 10 problems with the information that the final exam will consist of a random selection of 5 of them. If a student has figured out how to do 7 of the problems, what is the probability that he / she will answer correctly the

(a) all 5 problems?

Sol^n: Now, size of sample space, $|S| = {}^{10}C_5$

The student will answer all 5 problems correctly, if the problems given in the exam are from the 7 problems he learnt. So, the total no. of favourable outcomes $= {}^{7}C_5$

$$P(\text{5 problems correct}) = \frac{{}^{7}C_5}{{}^{10}C_5} = \underline{0.083}$$

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(b) at least 4 of the problems ?

The students answers at least 4 problems if :—

i) 4 out of 5 problems are from the 7 problems

ii) All 5 out of 5 ———— " ————

The above cases are mutually exclusive.

$$P(\text{Case i}) = ({}^{7}C_4 \times {}^{3}C_1) / {}^{10}C_5$$

4 known problems from 7 known problems

1 problem from 3 problems he left out

$$P(\text{Case ii}) = \frac{({}^{7}C_5)}{{}^{10}C_5}$$

All 5 problems from 7 problems he studied.

$$P(\text{at least 4 correct}) = P(\text{Case i}) + P(\text{Case ii})$$

$$= \frac{({}^{7}C_4 \times {}^{3}C_1) + ({}^{7}C_5)}{{}^{10}C_5}$$

$$= \underline{0.5}$$

4. Use induction to generalize Bonferroni's inequality to n events. That is show that,

$$P(E_1 \cap E_2 \cap \dots E_n) \geq P(E_1) + \dots + P(E_n) - (n-1)$$

Sol^n: Let us check for base cases,

Case : $n=1$

$$P(E_1) \geq P(E_1) - (1-1)$$

$$\Rightarrow P(E_1) \geq P(E_1) \quad \text{———— (Satisfied)}$$

Case : $n=2$

$$P(E_1 \cap E_2) \geq P(E_1) + P(E_2) - (2-1)$$

$$\Rightarrow P(E_1 \cap E_2) \geq P(E_1) + P(E_2) - 1$$

$$\Rightarrow P(E_1) + P(E_2) - P(E_1 \cap E_2) \leq 1$$

$$\Rightarrow P(E_1 \cup E_2) \leq 1 \quad \text{———— (Satisfied)}$$

(Axiom 1 of Probability) $P(E) \in [0,1]$

Case : $n > 2$

Let us assume that the Bonferroni's equality holds for all $m < n$. That is,

$$P(E_1 \cap E_2 \cap \dots \cap E_{n-1}) \geq \sum_{i=1}^{n-1} P(E_i) - [(n-1)-1]$$

Let,

$$B_k = \bigcap_{i=1}^{k} E_i$$

So,

$$P(B_{n-1}) \geq \sum_{i=1}^{n-1} P(E_i) - n + 2 \quad \text{——— (1)}$$

Now,

$$B_n = B_{n-1} \cap E_n$$

$$P(B_n) = P(B_{n-1} \cap E_n)$$

$$= P(B_{n-1}) + \underbrace{P(E_n) - P(B_{n-1} \cup E_n)}_{\text{Let this be } x} = [P(B_{n-1}) + x]$$

Add x to both sides of (1),

$$P(B_{n-1}) + x \geq \sum_{i=1}^{n-1} P(E_i) - n + 2 + x$$

$$\Rightarrow P(B_{n-1}) + P(E_n) - P(B_{n-1} \cup E_n)$$

$$\sum_{i=1}^{n-1} P(E_i) - n + 2 + P(E_n) - P(B_{n-1} \cup E_n)$$

$$\Rightarrow P(B_n) \geq \sum_{i=1}^{n} P(E_i) - n + [2 - P(B_{n-1} \cup E_n)] \quad \text{——— (2)}$$

Now, from Axiom (1),

$$P(B_{n-1} \cup E_n) \leq 1$$

$$\Rightarrow -P(B_{n-1} \cup E_n) \geq -1$$

$$\Rightarrow [2 - P(B_{n-1} \cup E_n)] \geq 1 \quad \text{——— (3)}$$

From ② and ③,

$$P(B_n) \geq \sum_{i=1}^{n} P(E_i) - n + 1$$

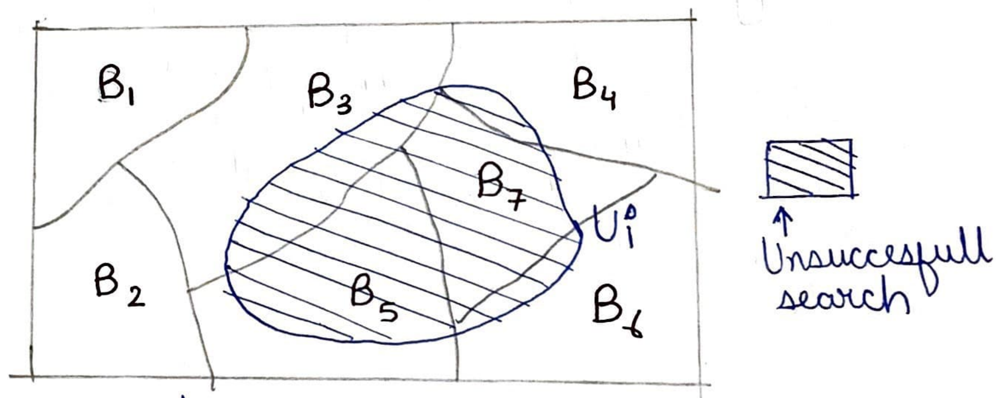
$$\Rightarrow \quad P(B_n) \geq \sum_{i=1}^{n} P(E_i) - (n-1)$$

$$\Rightarrow \quad P(E_1 \cap E_2 \cap \dots \cap E_n) \geq P_1(E_1) + P(E_2) + \dots + P(E_n) - (n-1)$$

Hence, Bonferroni's equality holds for all natural numbers

⑤ A ball is in any one of n boxes and is in the ith box with probability P_i. If the ball is in box i, a search of that box will uncover it with probability α_i. Show that the conditional probability that the ball is in box j, given that a search of box i did not uncover it, is

(a) $\dfrac{P_j}{1-\alpha_i P_i}$, if $j \neq i$ (b) $\dfrac{P_i(1-\alpha_i)}{1-\alpha_i P_i}$, if $j = i$



Sol^n: Let us define events,

$B_K \rightarrow$ The ball belongs to Box 'K'.

$U_i \rightarrow$ The search for ball was unsuccessfull. in Box i

The events B_K for $K = \{1, 2, \dots, n\}$ are mutually exclusive and exhaustive. The ball will belong to ~~only~~ exactly one Box. So, $B_i \cap B_j = \phi$, $(i \neq j)$

Now, let us compute the probability of an unsuccessfull search of Box i.

$$U_i = U_i \cap S$$
$$= U_i \cap (B_1 \cup B_2 \cup \dots \cup B_n)$$
$$= (U_i \cap B_1) \cup (U_i \cap B_2) \cup \dots \cup (U_i \cap B_n) \quad \text{—— (1)}$$

$(U_i \cap B_K) \rightarrow$ denotes the event that the ball belongs to (Box K) and the search for ball in the (Box i) was unsuccessfull.

The ~~tex~~ events on R.H.S. of (1) are again mutually exclusive as the ball can belong to exactly one box. From Axiom 3,

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i)$$

Setting $A_i = \phi$, for $i > n$

$$P\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) \quad \text{since } P(\phi) = 0 \quad \text{—— (2)}$$

Using (2),

$$P(U_i) = P(U_i \cap B_1) + P(U_i \cap B_2) + \dots + P(U_i \cap B_n)$$
$$= \sum_{K=1}^{n} P(U_i \cap B_K) \quad \text{—— (3)}$$

Now,

$$P(U_i \cap B_K) = P(B_K) \cdot P(U_i / B_K)$$ (From rule of conditional probability)

Now, $P(B_K) = P_K$ (given)

$$P(U_i / B_K) = \begin{cases} (1 - \alpha_i) & , K = i \quad \text{(given)} \\ 1 & , K \neq i \end{cases}$$

The event that the search for ball in (Box i) is unsuccessfull given that ball belongs to Box K

(if the ball does not belong to Box i, we can be sure that on search of Box i, we are bound to fail)

So,

$$P(U_i) = \sum_{K=1}^{n} P(B_K) \cdot P(U_i / B_K) \quad \text{—— (4)}$$

$$P(U_i) = P_1(1) + P_2(1) + \dots + P_i(1-\alpha_i) + \dots + P_n(1)$$

$$= P_i(1-\alpha_i) + (P_1 + P_2 + \dots + P_{i-1} + P_{i+1} + \dots + P_n)$$

$$= P_i(1-\alpha_i) + \underbrace{\sum_{\substack{j=1 \\ j\neq i}}^{n} P_j}$$

This term denotes the probability that ball does not belong to Box i.

$$= P_i(1-\alpha_i) + (1-P_i)$$

$$= \cancel{P_i} - \alpha_i P_i + 1 - \cancel{P_i}$$

$$P(U_i) = 1 - \alpha_i P_i \quad \text{———} \quad (5)$$

Now,

a)

$$P\left(B_j / U_i\right) = \frac{P(U_i \cap B_j)}{P(U_i)} \qquad [j \neq i]$$

$$= \frac{P(B_j) \cdot P\left(U_i / B_j\right)}{P(U_i)}$$

$$= \frac{P_j(1)}{1-\alpha_i P_i}$$

$$\boxed{P\left(B_j / U_i\right) = \frac{P_j}{1-\alpha_i P_i}} \qquad [j \neq i]$$

b) Now,

$$P\left(B_i / U_i\right) = \frac{P(U_i \cap B_i)}{P(U_i)}$$

$$= \frac{P(B_i) \cdot P\left(U_i / B_i\right)}{P(U_i)}$$

$$= \frac{P_i(1-\alpha_i)}{1-\alpha_i P_i}$$

$$\boxed{P\left(B_i / U_i\right) = \frac{P_i(1-\alpha_i)}{1-\alpha_i P_i}} \qquad [j = i]$$