6.19

Saturday, 16 January 2021 11:06 AM

$f(x,y) = \frac{1}{x} \quad , 0 < y < x < 1$

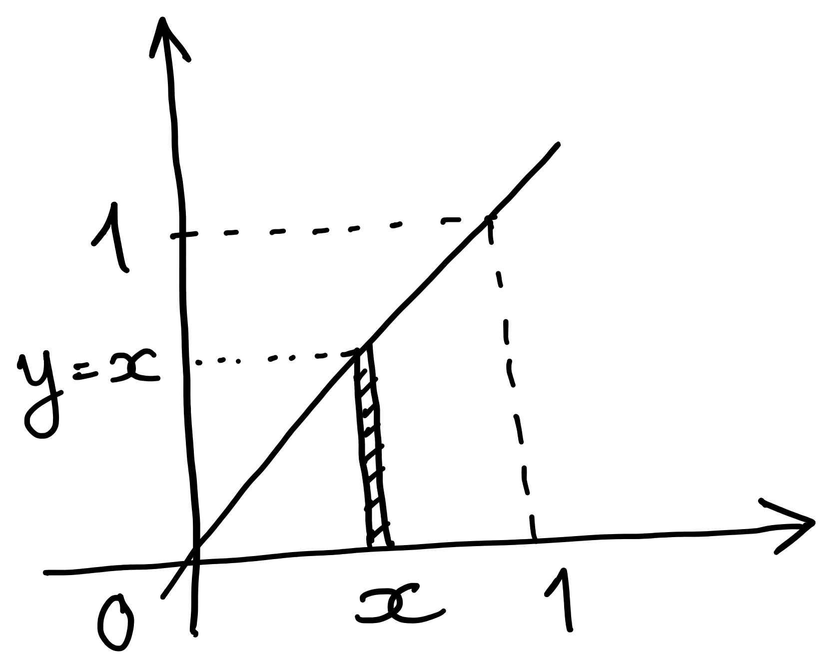
Now,

for $0 < y < x < 1$

$\boxed{f(x,y) = \frac{1}{x} \geq 0}$ ——— ①

We just need to show that,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, dx\, dy = 1$$



$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, dx\, dy$$

$$= \int_0^1\int_0^x \frac{1}{x}\,.\, dy\,.\, dx$$

$$= \int_0^1 \frac{1}{x}\left[y\right]_0^x dx$$

$$= \int_0^1 \frac{x}{x}\,.\, dx = \int_0^1 dx = \left[x\right]_0^1 = 1 \quad \text{——— ②}$$

From ① & ②, we conclude that $f(x,y)$ is a joint density function.

(a) $f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,.\,dx$

$$= \int_y^1 \frac{1}{x}\,.\,dx \quad [0 < y < x < 1]$$

$$= \left[\log x\right]_y^1$$

$$= 0 - \log y$$

$$\boxed{f_Y(y) = -\log y}$$

(b) $f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,.\,dy$

$$= \int_0^x \frac{1}{x}\,.\,dy$$

$$= \frac{1}{x}\left[y\right]_0^x$$

$$= \frac{x}{x} = 1$$

$$\boxed{f_X(x) = 1}$$

(c) $E[X] = \int_{-\infty}^{\infty} x\,.\,f_X(x)\,.\,dx$

$$= \int_0^1 x\,.\,1\,.\,dx$$

$$= \left[\frac{x^2}{2}\right]_0^1 = \frac{1}{2}$$

$$\boxed{E[X] = \frac{1}{2}}$$

(d) $E[Y] = \int_{-\infty}^{\infty} y\, f_Y(y)\,.\,dy$

$$= -\int_0^1 y \log y\,.\, dy$$

Integrating by parts,

$$= -\left[\log y\,.\,\frac{y^2}{2}\right]_0^1 - \int_0^1 \frac{1}{y}\,.\,\frac{y^2}{2}\,.\,dy$$

$$= -(0-0) - \frac{1}{2}\int_0^1 y\,.\,dy$$

$$= +\frac{1}{2}\left[\frac{y^2}{2}\right]_0^1$$

$$= \frac{1}{4}$$

$$\boxed{E[Y] = \frac{1}{4}}$$

Q 6.56

Friday, 22 January 2021 12:11 AM

a) Now,
$$f_X(x) = \begin{cases} \frac{1}{1-0}, & x \in (0,1) \\ 0, & \text{otherwise} \end{cases}$$

$$f_Y(x) = \begin{cases} \frac{1}{1-0}, & x \in (0,1) \\ 0, & \text{otherwise} \end{cases}$$

$f_X(x)$ & $f_Y(x)$ are independent.

Hence, $f_{X,Y}(x,y) = f_Y(x) \cdot f_X(x)$

$$\Rightarrow f_{X,Y}(x,y) = \begin{cases} 1, & x \in (0,1),\ y \in (0,1) \\ 0, & \text{otherwise} \end{cases}$$

i) $U = X+Y, \quad V = X/Y$

Step 1: Solve for x and y in terms of u and v.

$\Rightarrow x = vy$

$\Rightarrow u = vy + y$

$\Rightarrow y = \frac{u}{v+1}$

$\Rightarrow x = v\left(\frac{u}{v+1}\right)$

$\Rightarrow x = \frac{uv}{v+1}, \quad y = \frac{u}{v+1}$

Step 2: Compute Jacobian

$$J(x,y) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ \frac{1}{y} & -\frac{x}{y^2} \end{vmatrix}$$

$$= -\frac{x}{y^2} - \frac{1}{y} = -\left(\frac{x+y}{y^2}\right)$$

This is $\neq 0$ when, $x, y \in (0,1)$

We can write Joint distribution of (U,V) as,

$$f_{U,V}(u,v) = f_{X,Y}(x,y) \cdot \frac{1}{|J(x,y)|}$$

Substituting, $x = \frac{uv}{v+1}, \quad y = \frac{u}{v+1}$

$$\frac{1}{|J(x,y)|} = \frac{y^2}{x+y} = \frac{\left(\frac{u}{v+1}\right)^2}{u} = \frac{u}{(v+1)^2}$$

$$\Rightarrow f_{U,V}(u,v) = f_{X,Y}\left(\frac{uv}{v+1}, \frac{u}{v+1}\right) \frac{u}{(v+1)^2}$$

$$\Rightarrow f_{U,V}(u,v) = \begin{cases} \frac{u}{(v+1)^2}, & x = \frac{uv}{v+1} \in (0,1),\ y = \frac{u}{v+1} \in (0,1) \\ 0, & \text{otherwise} \end{cases}$$

ii) $U = X, \quad V = \frac{X}{Y}$

Step 1: Solve for x and y in terms of u and v.

$\Rightarrow x = u, \quad y = \frac{u}{v}$

Step 2: Compute Jacobian

$$J(x,y) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ \frac{1}{y} & -\frac{x}{y^2} \end{vmatrix} = -\frac{x}{y^2}$$

This is $\neq 0$ when, $x, y \in (0,1)$

We can write the joint distribution of (U,V) as,

$$f_{U,V}(u,v) = f_{X,Y}(x,y) \frac{1}{|J(x,y)|}$$

Substituting, $x = u, \quad y = u/v$

$$\frac{1}{|J(x,y)|} = \frac{y^2}{x} = \frac{u^2}{v^2 \cdot u} = \frac{u}{v^2}$$

$$f_{U,V}(u,v) = f_{X,Y}\left(u, \frac{u}{v}\right) \frac{u}{v^2}$$

$$= \begin{cases} \frac{u}{v^2}, & u \in (0,1),\ u/v \in (0,1) \\ 0, & \text{otherwise} \end{cases}$$

(iii) $U = X+Y, \quad V = \frac{X}{X+Y}$

Step 1: Solve for x and y in terms of u & v.

$x = v(x+y)$

$\Rightarrow x = vu$

$\Rightarrow y = u - uv$

$x = uv, \quad y = u(1-v)$

Step 2: Compute Jacobian

$$J(x,y) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ \frac{(x+y)-x}{(x+y)^2} & \frac{-x}{(x+y)^2} \end{vmatrix}$$

$$= \frac{-x}{(x+y)^2} - \frac{y}{(x+y)^2} = \frac{-1}{x+y}$$

This is $\neq 0$ when, $x, y \in (0,1)$

We can write joint distribution of (U,V) as,

$$f_{U,V}(u,v) = f_{X,Y}(x,y) \frac{1}{|J(x,y)|}$$

Substituting, $x = uv, \quad y = u(1-v)$

$$\frac{1}{|J(x,y)|} = (x+y) = uv + u(1-v) = u$$

$$\Rightarrow f_{U,V}(u,v) = f_{X,Y}(uv, u(1-v)) \cdot u$$

$$= \begin{cases} u, & x = uv \in (0,1),\ y = u(1-v) \in (0,1) \\ 0, & \text{otherwise} \end{cases}$$

8.11

Saturday, 16 January 2021 11:50 AM

Let today's stock price be denoted by Y_0.

So, $Y_0 = 100$

Now, $Y_n = Y_{n-1} + X_n$

So, $Y_1 = Y_0 + X_1$ (After 1 day)

$Y_2 = Y_1 + X_2$
$= Y_0 + X_1 + X_2$ (After 2 days)

⋮

$Y_{10} = Y_0 + \sum_{i=1}^{10} X_i$ (After 10 days)

We need to find the probability that the stock's price will exceed 105 after 10 days, i.e.

$P(Y_{10} > 105)$

Now, $Y_{10} = 100 + \sum_{i=1}^{10} X_i > 105$

$\Rightarrow \sum_{i=1}^{10} X_i > 105 - 100$

$\Rightarrow X = \sum_{i=1}^{10} X_i > 5$

$\Rightarrow P(X > 5)$

Here, X_i's are independent and identically distributed random variables with,

$\mu_i = 0$

$\sigma_i^2 = 1$

Central limit theorem says that, the distribution,

$$\frac{S_n - n\mu}{\sigma\sqrt{n}}$$

tends to that of a standard normal distribution as $n \to \infty$. So, we can get an approximate distribution Z using CLT, i.e.

$$Z = \frac{X - 10(0)}{1(\sqrt{10})} = \frac{X}{\sqrt{10}}$$

$$P(X > 5) = P\left(Z > \frac{5}{\sqrt{10}}\right)$$
$$= P(Z > 1.58)$$
$$= 1 - P(Z \leq 1.58)$$
$$= 1 - \Phi(1.58)$$
$$= 1 - 0.9429$$
$$= 0.0571$$

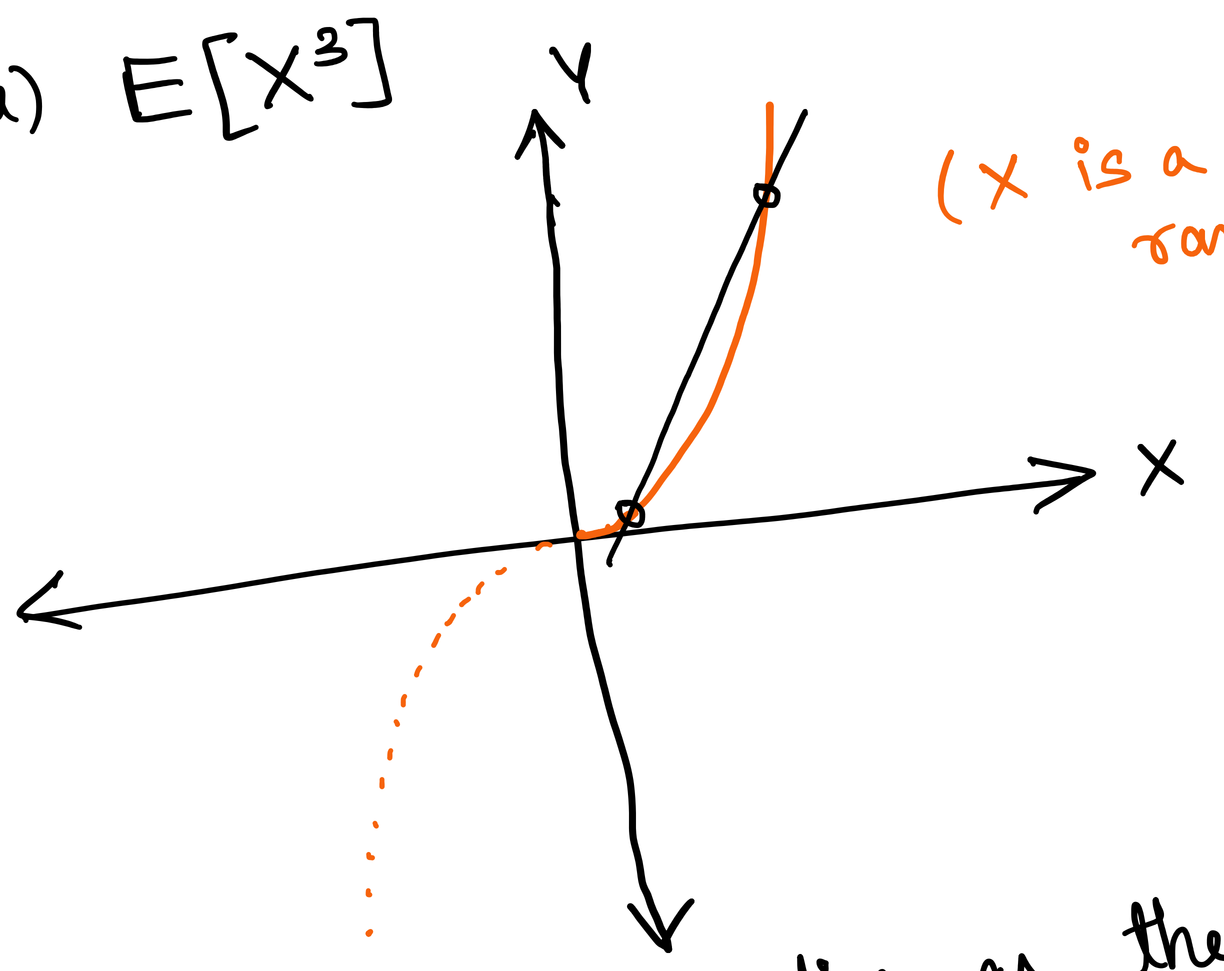
Hence, $\boxed{P(Y_{10} > 105) = 0.0571}$

8.20

Saturday, 16 January 2021 11:27 AM

Now,
$E[X] = 25$ —— (1)

(a) $E[X^3]$



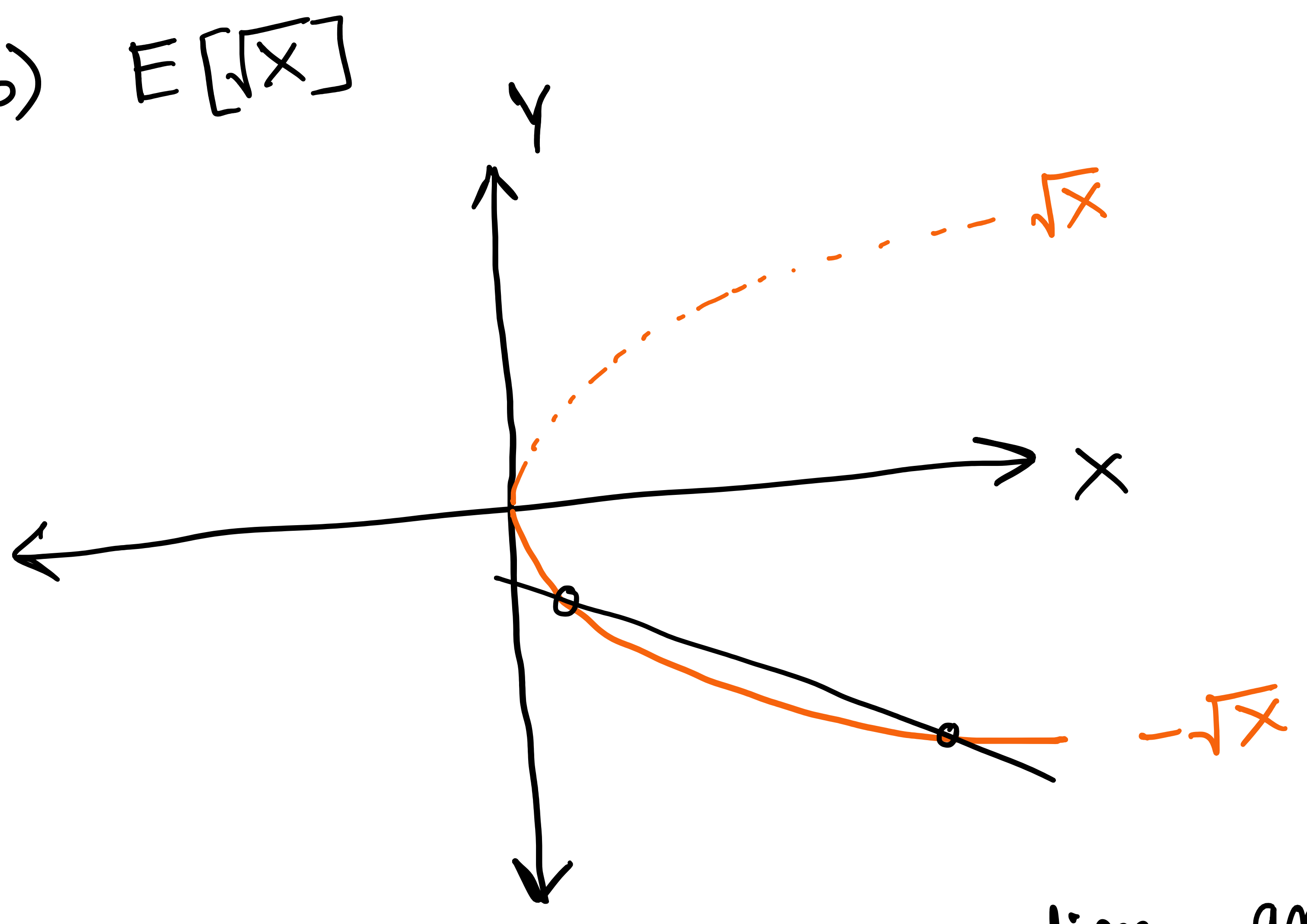
(X is a non-negative random variable)

X^3 is a convex function, as the line drawn between any two points lies above the curve. Here, $E[X] = 25$ is also finite. Hence, using Jensen's inequality, we conclude that,

$$E[X^3] \geqslant \left(E[X]\right)^3$$

$$\Rightarrow \boxed{E[X^3] \geqslant 15625}$$

(b) $E[\sqrt{X}]$



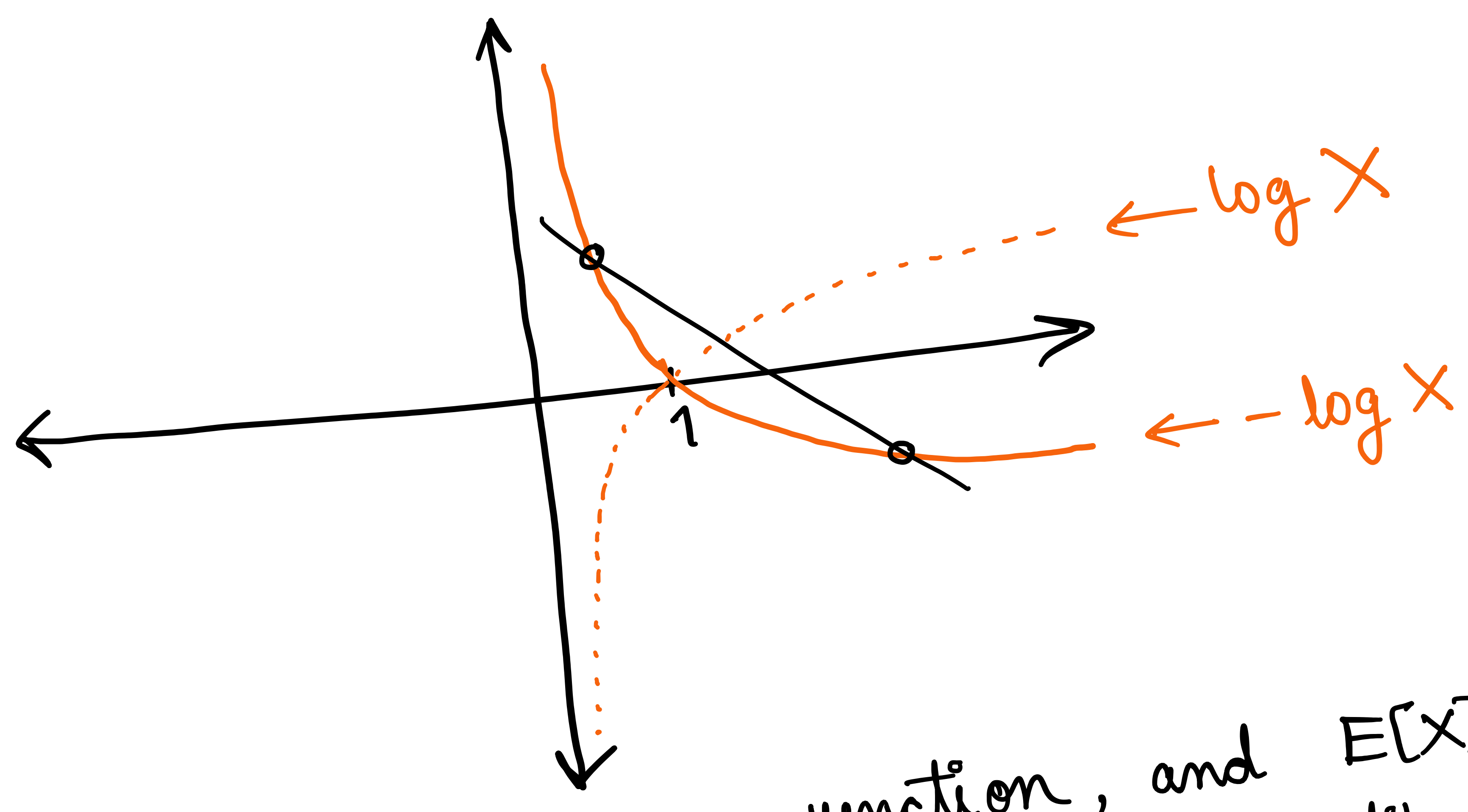
Here, $-\sqrt{X}$ is a convex function, and $E[X] = 25$ is finite. Hence, from Jensen's inequality,

$$E[-\sqrt{X}] \geqslant -\sqrt{E[X]}$$

$$\Rightarrow -E[\sqrt{X}] \geqslant -\sqrt{25}$$

$$\Rightarrow E[\sqrt{X}] \leqslant 5$$

(c) $E[\log X]$



Here, $-\log X$ is a convex function, and $E[X]$ is finite. Hence, from Jensen's inequality,

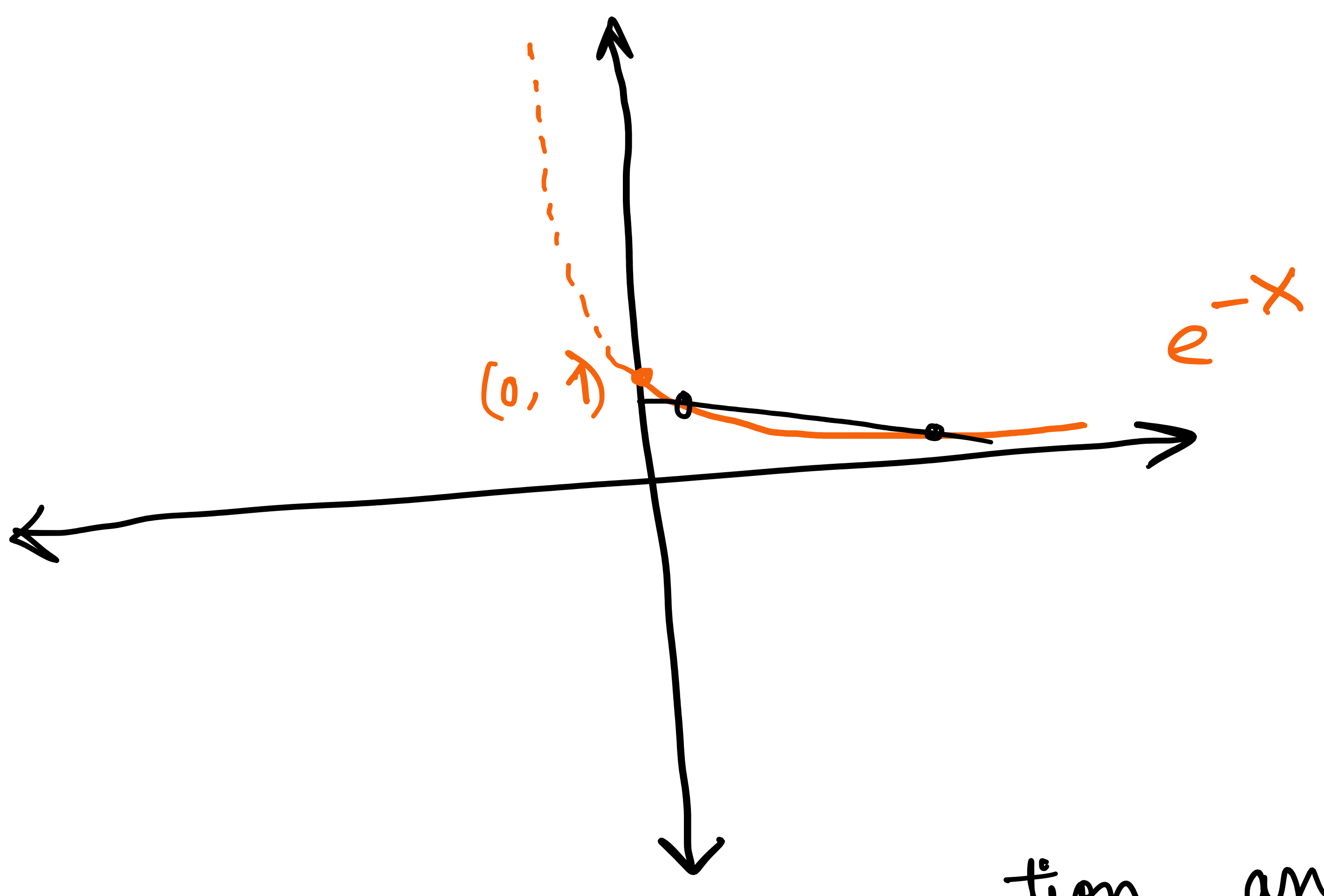
$$E[-\log X] \geqslant -\log(E[X])$$

$$\Rightarrow -E[\log X] \geqslant -\log(25)$$

$$\Rightarrow E[\log X] \leqslant \log(25)$$

$$\Rightarrow \boxed{E[\log X] \leqslant 1.398}$$

(d) $E[e^{-X}]$



Here, e^{-X} is a convex function, and $E[X]$ is finite. Hence, from Jensen's inequality,

$$E[e^{-X}] \geqslant e^{-E[X]}$$

$$\Rightarrow \boxed{E[e^{-X}] \geqslant e^{-25}}$$