Subsequences and the Bolzano-Weierstrass Theorem

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I found Terrence Tao's treatment of subsequences in his *Analysis I* to be fascinating, so I decided to typeset and publish a small set of notes on the section.

Definition. Given a sequence $(a_n)_{n=0}^{\infty}$, say $(b_n)_{n=0}^{\infty}$ is a subsequence iff there exists a strictly increasing function $f : \mathbb{N} \to \mathbb{N}$ such that

$$b_n = a_{f(n)} \quad \text{for all } n \in \mathbb{N}.$$

For example, let $(a_n)_{n=0}^{\infty}$ be the sequence

$$0, 1, 2, 3, ...$$

Then the sequence containing only evens

$$0, 2, 4, 6, ...$$

is a subsequence, as is the sequence containing only primes

$$2, 3, 5, 7, ...$$

Unpacking the definition a bit, I like to think of f as sort of a "selection" function, which picks elements from $(a_n)_{n=0}^{\infty}$ to put into a new sequence. In the above example of the sequence containing only the evens, the function $f : \mathbb{N} \to \mathbb{N}$ would be defined by $f(n) = 2n$.

Proposition 1. *Let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers, and let $L \in \mathbb{R}$. Then the following statements are equivalent:*

(a) *The sequence $(a_n)_{n=0}^{\infty}$ converges to L.*

(b) *Every subsequence of $(a_n)_{n=0}^{\infty}$ converges to L.*

Proof. First, prove that (b) implies (a). Assume every subsequence of $(a_n)_{n=0}^{\infty}$ converges to L. Notice that $(a_n)_{n=0}^{\infty}$ is a subsequence of itself, given by the function $f(n) = n$, and so it converges to L by assumption.

Next, prove that (a) implies (b). Take some subsequence $(b_n)_{n=0}^{\infty}$ given by $b_n = a_{f(n)}$ for some strictly increasing function f. To show that $(b_n)_{n=0}^{\infty}$ converges to L, let $\epsilon > 0$ and find some integer N such that for all $n \geq N$ we have $|b_n - L| < \epsilon$. First, note that since f is strictly increasing, we have $f(n) \geq n$ for all n. This is easily shown by induction: $f(0)$ is a natural number, so we must have $f(0) \geq 0$. And assuming $f(n) \geq n$, f strictly increasing gives us $f(n+1) > f(n) \geq n$ so $f(n+1) \geq n+1$.

With this, use convergence of $(a_n)_{n=0}^{\infty}$ to get an integer N such that for all $n \geq N$, we have $|a_n - L| < \epsilon$. Since $f(n) \geq f(N) \geq N$ for all $n \geq N$, we have in particular that $|a_{f(n)} - L| < \epsilon$. But $a_{f(n)} = b_n$, so

$$|b_n - L| < \epsilon,$$

completing the proof. $\square$

The first direction was relatively trivial, but I think the second highlights the importance of defining our "selection" function f to be strictly increasing. If it was only restricted to be nondecreasing, for instance, we could let $f(n) = 0$ for all n. Then

$$1, 1, 1, ...,$$

which clearly converges to 1, would be a subsequence of

$$1, \frac{1}{2}, \frac{1}{3}, ...,$$

which converges to 0. It would also be a valid subsequence of

$$1, 2, 3, ...,$$

which doesn't converge at all! Thus the strict increasing requirement corresponds to the idea of "filtering out" elements from the original sequence to create a new sequence, without adding or duplicating anything.

Proposition 2. *Again let $(a_n)_{n=0}^{\infty}$ be a sequence of real numbers, and let $L \in \mathbb{R}$. Then the following statements are equivalent:*

(a) *L is a limit point of $(a_n)_{n=0}^{\infty}$.*

(b) *There exists a subsequence of $(a_n)_{n=0}^{\infty}$ which converges to L.*

Proof. First show that (b) implies (a). Let $(b_n)_{n=0}^{\infty}$ be a subsequence that converges to L, given by $b_n = a_{f(n)}$. Take some $\epsilon > 0$ and some natural number N. To show that L is a limit point of $(a_n)_{n=0}^{\infty}$, we want some $n \geq N$ such that $|a_n - L| < \epsilon$. Since $(b_n)_{n=0}^{\infty}$ converges to L, we can take some M such that for all $m \geq M$,

$$|b_m - L| < \epsilon$$

Rewriting, we have that $|a_{f(m)} - L| < \epsilon$ for all $m \geq M$. Now let $n := \max(f(N), f(M))$. Since f is strictly increasing,

$$n \geq f(N) \geq N \quad \text{and} \quad n \geq f(M) \geq M$$

Thus we have some $n \geq N$ such that $|a_n - L| < \epsilon$, and we're done.

Next show that (a) implies (b). We want to define a sequence of natural numbers $m_0, m_1, ...$ such that the sequence given by $b_n := a_{m_n}$ converges to L. Define our first term by

$$m_0 := \min\{k \in \mathbb{N} : |a_k - L| < 1\}$$

The above set is clearly bounded below by 0. To show that it is nonempty, take $\epsilon = 1$ and $N = 0$ in the definition of a limit point to get some $k \in \mathbb{N}$ with $|a_k - L| < 1$. Thus the minimum exists and m_0 is well defined. Next, recursively define the rest of the sequence as follows:

$$m_{n+1} := \min\left\{k > m_n : |a_k - L| < \frac{1}{n+1}\right\}$$

Again, the set is clearly bounded below by m_n. Similarly to the argument above, the set is nonempty since we can take $\epsilon = \frac{1}{n+1}$ and $N = m_n$ in the definition of a limit point to find some $k \geq m_n$ with $|a_k - L| < \frac{1}{n+1}$. Thus the minimum m_{n+1} exists and is well-defined.

Note that $(m_n)_{n=0}^{\infty}$ is strictly increasing by construction, so the sequence $(b_n)_{n=0}^{\infty}$ given by $b_n = a_{m_n}$ is a subsequence. Now show that $(b_n)_{n=0}^{\infty}$ converges to L. Take some $\epsilon > 0$, and let N be some natural number such that $N > \frac{1}{\epsilon}$. Using our definition of $(b_n)_{n=0}^{\infty}$, we see that for all $n \geq N$ we have

$$|b_n - L| = |a_{m_n} - L| < \frac{1}{n+1} < \frac{1}{N} < \epsilon,$$

and thus $(b_n)_{n=0}^{\infty}$ converges to L.

□

With these facts established, we now turn to the Bolzano-Weierstrass theorem, which states that every bounded sequence has a convergent subsequence. Before we prove the theorem, though, it will be useful to establish the following lemmas.

Lemma 1. *Let* $(a_n)_{n=0}^{\infty}$ *be a sequence with* $\limsup_{n\to\infty} a_n = L$ *for some* $L \in \mathbb{R}$. *Then* L *is a limit point of* $(a_n)_{n=0}^{\infty}$.

Proof. Take $\epsilon > 0$ and some natural number N. The limit superior is defined to be

$$L := \inf(a_n^+)_{n=0}^{\infty},$$

where $a_k^+ := \sup(a_n)_{n=k}^{\infty}$. Firstly, notice that the sequence $(a_n^+)_{n=0}^{\infty}$ is decreasing. If it were not, there would exist some n_1, n_2 with $n_1 \leq n_2$ and

$$\sup(a_n)_{n=n_1}^{\infty} < \sup(a_n)_{n=n_2}^{\infty}$$

Then the difference $\delta := \sup(a_n)_{n=n_2}^{\infty} - \sup(a_n)_{n=n_1}^{\infty}$ would be positive, so by the properties of the supremum, there exists some $m \geq n_2$ such that

$$a_m > \sup(a_n)_{n=n_2}^{\infty} - \frac{\delta}{2} > \sup(a_n)_{n=n_1}^{\infty}$$

But this is a contradiction, since $m \geq n_2 \geq n_1$ means we should have $a_m \leq \sup(a_n)_{n=n_1}^{\infty}$. Thus we can conclude that the sequence $(a_n^+)_{n=0}^{\infty}$ is decreasing.

Next, use the properties of the infimum to take some M such that

$$L \leq a_M^+ < L - \frac{\epsilon}{2}$$

Let $m := \max(N, M)$. Since $n \geq M$ and $(a_m^+)_{n=0}^{\infty}$ is decreasing, we have that

$$L \leq a_m^+ \leq a_M^+ < L - \frac{\epsilon}{2}$$

Rewriting, we see that $|a_m^+ - L| < \frac{\epsilon}{2}$. Next, apply the definition of a_m^+ to get some $n \geq m$ such that

$$a_m^+ - \frac{\epsilon}{2} < a_n \leq a_m^+$$

Rewriting once again, we now have some $n \geq m \geq N$ such that $|a_n - a_m^+| < \frac{\epsilon}{2}$. Combining this with the inequality from above, we find

$$|a_n - L| \leq |a_n - a_m^+| + |a_m^+ - L| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Since we've found some $n \geq N$ with $|a_n - L| < \epsilon$, L is a limit point of $(a_n)_{n=0}^{\infty}$. □

The above lemma is perhaps unsurprising, but I wanted to include it for the sake of completeness. Next, another unsurprising but useful lemma.

Lemma 2. *For any sequences $(a_n)_{n=m}^\infty, (b_n)_{n=m}^\infty$ satisfying $a_n \le b_n$ for all n, we have*

$$\sup(a_n)_{n=m}^\infty \le \sup(b_n)_{n=m}^\infty \tag{1}$$

$$\inf(a_n)_{n=m}^\infty \le \inf(b_n)_{n=m}^\infty \tag{2}$$

$$\limsup_{n\to\infty} a_n \le \limsup_{n\to\infty} b_n \tag{3}$$

Proof. To prove inequality (1), write $S_b := \sup(b_n)_{n=m}^\infty$ and $S_a := \sup(a_n)_{n=m}^\infty$. Assume for contradiction that $S_a > S_b$, and let $\epsilon = S_a - S_b > 0$. By the definition of the supremum, there exists some n such that $a_n > S_a - \epsilon = S_b$. But since S_b is the supremum of $(b_n)_{n=0}^\infty$, we have that in particular $S_b \ge b_n$. Combining these inequalities, we see that

$$a_n > S_a - \epsilon = S_b \ge b_n,$$

which is a contradiction since $a_n \le b_n$ for all n by assumption.

To prove inequality (2), write the infima as I_a and I_b, and assume that $\epsilon := I_a - I_b > 0$. Similar to above, take some n such that $b_n < I_b + \epsilon = I_a$. But $I_a \ge a_k$ for all k, which in particular means

$$b_n < I_b + \epsilon = I_a \le a_n,$$

which is again a contradiction.

Finally, to prove inequality (3), write the limits as L_a and L_b and apply the previous two inequalities to show $L_a \le L_b$. Since $a_n \le b_n$, inequality (1) gives us

$$a_k^+ = \sup(a_n)_{n=k}^\infty \le \sup(b_n)_{n=k}^\infty = b_k^+$$

for all $k \ge 0$. Since the sequences $(a_n^+)_{n=0}^\infty$ and $(b_n^+)_{n=0}^\infty$ satisfy $a_k^+ \le b_k^+$, we can apply inequality (2) to conclude:

$$L_a = \inf(a_n^+)_{n=0}^\infty \le \inf(b_n^+)_{n=0}^\infty = L_b$$

□

Now we have all the necessary machinery to prove the Bolzano-Weierstrass theorem. In fact, all the hard work is behind us!

Theorem (Bolzano-Weierstrass)**.** *Any bounded sequence $(a_n)_{n=0}^\infty$ has a convergent subsequence.*

Proof. Since $(a_n)_{n=0}^\infty$ is bounded, we can take some M such that $-M < a_n < M$ for all n. Applying Lemma 2 to the above inequalities, we find

$$-M = \limsup_{n\to\infty}(-M)_{n=0}^\infty \le \limsup_{n\to\infty}(a_n)_{n=0}^\infty \le \limsup_{n\to\infty}(M)_{n=0}^\infty = M$$

Since $L := \limsup_{n\to\infty}(a_n)_{n=0}^\infty$ satisfies $-M \le L \le M$, it must be a real number (i.e. not $+\infty$ or $-\infty$). So, we can apply Lemma 1 to conclude that L is a limit point of $(a_n)_{n=0}^\infty$. Because L is a limit point of $(a_n)_{n=0}^\infty$, Proposition 2 allows us to conclude that there exists some subsequence of $(a_n)_{n=0}^\infty$ that converges to L. □