MATH 4061 - Homework 1

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Problem 1

1. The following is the truth table for $(\neg A \implies \neg B)$.

A	B	$\neg A$	$\neg B$	$\neg A \implies \neg B$
T	T	F	F	T
F	T	T	F	F
T	F	F	T	T
F	F	T	T	T

2. The following is the truth table for $(\neg B \implies \neg A)$.

A	B	$\neg A$	$\neg B$	$\neg B \implies \neg A$
T	T	F	F	T
F	T	T	F	T
T	F	F	T	F
F	F	T	T	T

Comparing the two truth tables above with the following truth table for $A \implies B$, I conclude that $(\neg B \implies \neg A)$ and $A \implies B$ are equivalent.

A	B	$A \implies B$
T	T	T
F	T	T
T	F	F
F	F	T

Problem 2

Given $A, B, C \subset M$.

1. We want to show $(A \subset C) \wedge (B \subset C) \iff ((A \cup B) \subset C)$.

   Proof. First show the forward direction. Assume $(A \subset C) \wedge (B \subset C)$ and take some element $x \in A \cup B$. Thus we have $x \in A \vee x \in B$. If $x \in A$, we have $x \in C$ since $A \subset C$. Similarly, $x \in B$ gives us $x \in C$. So $x \in A \cup B \implies x \in C$, and thus $(A \cup B) \subset C$.

   Next show the reverse direction. Assume $(A \cup B) \subset C$. For all $a \in A$, we have $a \in A \cup B$ and thus $a \in C$. So $A \subset C$. The same argument yields $B \subset C$, and thus $(A \subset C) \wedge (B \subset C)$. $\square$

2. $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Proof. Take some $x \in A \cap (B \cup C)$.

$$\begin{aligned}
&\iff (x \in A) \wedge (x \in B \cup C) \\
&\iff (x \in A) \wedge ((x \in B) \vee (x \in C)) \\
&\iff ((x \in A) \wedge (x \in B)) \vee ((x \in A) \wedge (x \in C)) \\
&\iff (x \in A \cap B) \vee (x \in A \cap C) \\
&\iff x \in (A \cap B) \cup (A \cap C)
\end{aligned}$$

Since $x \in A \cap (B \cup C)$ if and only if $x \in (A \cap B) \cup (A \cap C)$, the two sets are equal. $\square$

3. $M \setminus (A \cup B) = (M \setminus A) \cap (M \setminus B)$.

Proof. Take some $x \in M \setminus (A \cup B)$.

$$\begin{aligned}
&\iff (x \in M) \wedge \neg(x \in A \cup B) \\
&\iff (x \in M) \wedge \neg((x \in A) \vee (x \in B)) \\
&\iff (x \in M) \wedge (\neg(x \in A) \wedge \neg(x \in B)) \\
&\iff (x \in M) \wedge ((x \notin A) \wedge (x \notin B)) \\
&\iff ((x \in M) \wedge (x \notin A)) \wedge ((x \in M) \wedge (x \notin B)) \\
&\iff (x \in M \setminus A) \wedge (x \in M \setminus B) \\
&\iff x \in ((M \setminus A) \cap (M \setminus B))
\end{aligned}$$

Thus the two sets are equal. $\square$

Problem 3

1. The cartesian product $I \times J$ of the line segments $I = J = [0, 1]$ is the unit square.
2. The cartesian product of a line $\mathbb{R}$ and a circle $\mathbb{S}^1$ is an infinite-length cylinder.
3. The cartesian product of two circles is a torus.

The diagonal of $I \times J$ is the line $y = x$, restricted to $x \in [0, 1]$. The diagonal of the torus is essentially a circle along the surface of the torus passing through the outermost point on one side and the innermost point on the other side.

Problem 4

Given a field F and some $x \in F \setminus \{0\}$.

1. $xy = xz \implies y = z$.

Proof. Since $x \neq 0$, x has some inverse x^{-1}. Multiplying both sides of the equation $xy = xz$ by this inverse yields

$$y = 1y = x^{-1}xy = x^{-1}xz = 1z = z,$$

and we're done. $\square$

2. $xy = x \implies y = 1$.

Proof. Notice that $x = x(1)$, and so the initial statement becomes $xy = x(1)$. Apply the previous proof to conclude $y = 1$. $\square$

3. $xy = 1 \implies y = x^{-1}$.

Proof. Multiply both sides by x^{-1}, yielding

$$y = 1y = x^{-1}xy = x^{-1}(1) = x^{-1}$$

$\square$

4. $(x^{-1})^{-1} = x$.

Proof. To show that x is the inverse of x^{-1}, it suffices to show that $xx^{-1} = x^{-1}x = 1$. But this follows from the properties of the inverse, and we're done. $\square$