Quantum Physics and Relativity
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Contents

1 Special Relativity

1.1 Inertial Frames of Reference

Newtonian Physics: Absolute space exists and inertial frames of reference are reference frames which are at rest or moving at a constant speed relative to it. Newton's laws of physics are valid in these.
Modern Physics: No experiment can tell us whether the experiment is moving or not. There is no absolute space, no special inertial frame.

> **Example:**
> The compass of an aircraft shows it is travelling north and its airspeed is $240\frac{km}{h}$. If there is a wind of $100\frac{km}{h}$ from west to east, what is the velocity of the aircraft relative to the ground?
> Simple geometry shows that the aircraft has a velocity of $\approx 260\frac{km}{h}$.
> Let the frame of reference of the ground be S and the one of the air be S'. Now x, y, z are used to label the position of something relative to the ground. x', y', z' label position of something relative to air. Notice that we assume time to be universal.

1.2 Events and Observers

An event takes place at a particular location $\vec{x}$ and stops a clock at that position. This is the time t of the event.

1.3 Galilean Transformation

In special relativity, we will be interested in transforming the coordinates of events from one inertial frame of reference to another.

> **Example**:
> Consider two inertial frames of reference S and S'. Suppose an event occurs in S at (x, y, z, t). What are the coordinates of that event in S', which moves at velocity v relative to S parallel to the x-Axis?
> Obviously $z = z'$ and $y = y'$ and we assume here that $t = t'$. It can also be easily seen that $x = x' + vt$. This can be inverted to $x' = x - vt$.
> So for an observer in S' the event occurs at $(x', y', z', t') = (x - vt, y, z, t)$

Let's prove the seemingly trivial fact that two observers will agree on the length of things:

Example:
A rigid rod of length $1m$ is at rest and lying along the x-Axis in frame S. Show that under Galilean Transformations, the rod is also 1m long as determined by an observer at rest in S' which moves at speed v relative to S in x-direction.
The observer at rest in S measures the coordinates of one end of the rod. This is event 1. This happened at a particular time t. Event 2 is the measurement of coordinates of other end of the rod (at the same time (!)).

$$E_1 = (x_1, 0, 0, t) \tag{1}$$

$$E_2 = (x_2, 0, 0, t) \tag{2}$$

For the second observer (in S'):

$$E_3 = (x'_1, 0, 0, t') \tag{3}$$

$$E_4 = (x'_2, 0, 0, t') \tag{4}$$

Transforming the coordinates of events from S to S' yields

$$x'_1 = x_1 - vt' \tag{5}$$

and

$$x'_2 = x_2 - vt' \tag{6}$$

Still

$$x'_2 - x'_1 = x_2 - x_1 = 1m \tag{7}$$

So for Galilean Transformations all observers agree on lengths.

Similarly one can show that things travelling in straight lines in one inertial frame of reference do so in all. This is proof that if Newtons 1st law holds in one inertial frame of reference it holds in all.
One could define inertial frames as those in which Newtons 1st law holds true.
As we shall see, it turns out that the laws of physics are not invariant under Galilean transformations and they (the transformations) are therefore wrong.

1.4 Einstein's Postulates

Identical isolated experiments in different inertial frames deliver identical results.

The speed of light in a vacuum is the same in all inertial frames.

Einstein was influenced by Maxwell's equations:

$$\boldsymbol{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t} \tag{8}$$

$$\boldsymbol{\nabla} \cdot \vec{E} = \frac{\rho}{\epsilon_0} \tag{9}$$

$$\boldsymbol{\nabla} \cdot \vec{B} = 0 \tag{10}$$

$$\boldsymbol{\nabla} \times \vec{B} = \mu_0 \vec{j} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \tag{11}$$

They imply that there is such a thing as electromagnetic waves, that move at a speed of

$$c = \frac{1}{\sqrt{\mu_0 \epsilon_0}} \tag{12}$$

If ϵ_0 and μ_0 are the same in all inertial frames, this says that c is also the same in all inertial frames.

1.5 Time Dilation

We are going to show that time ticks at different rates for different observers, given Einstein's Postulates.

A good way to define time is to consider a light clock. Imagine two mirrors, some distance d apart and light bouncing off of the two. In the inertial frame of reference of the clock it takes

$$\Delta t_0 = \frac{2d}{c} \tag{13}$$

amount of time for one tick.

For a moving observer the light has to travel a longer distance, because while its on its way to the other mirror it is moving away. Simple geometry can show that for an observer moving at v the light has to travel a distance of $2\sqrt{(\frac{1}{2}v\Delta t)^2 + d^2}$ assuming that the speed of light is the same for both observers. That means that for the moving observer the clock needs

$$\Delta t = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}\Delta t_0 = \gamma \Delta t_0 \tag{14}$$

So moving clocks run more slowly.

Example: Muons
Muons are created in cosmic ray collisions in the upper atmosphere, typically at an altitude of $20km$. They have a very short lifetime of $2.2\mu s$. They shouldn't be able to reach the earth even when travelling at almost the speed of light. However they do.
In the muons rest frame its lifetime is $2.2\mu s$. From the point of view of an observer on Earth that's

$$\Delta t_e = \gamma \Delta t_0 \tag{15}$$

For the particles to reach the earth

$$\gamma \Delta t_0 > \frac{20km}{v} \tag{16}$$

Solving for v one finds that the μ-particles would reach the ground if $v > 0.999c$.

1.6 Perceived time

Perceived lengths of time for an observer are not necessarily the same as the lengths of time that you would get for the observers frame of reference (using the formula for time dilation). The perceived time depends on the position of the observer relative to the event at the beginning and end of it.

1.7 Doppler effect for light

If f_0 is the frequency of light in S', which is moving **towards** S with $\beta = v/c$ then the frequency of light as *seen* from S is:

$$f = \sqrt{\frac{1-\beta}{1+\beta}} f_0 \tag{17}$$

or expressed in terms of wavelength:

$$\lambda = \sqrt{\frac{1+\beta}{1-\beta}}\lambda' \tag{18}$$

since $c = f\lambda$.

Example:
How fast must you be approaching a red traffic light ($\lambda = 675nm$) in order for it to appear amber ($\lambda = 575nm$)?

$$575nm = \sqrt{\frac{1-\beta}{1+\beta}} \times 675nm \tag{19}$$

Notice the minus sign (the traffic light is moving towards us). This gives

$$\beta = 0.16 \tag{20}$$

1.8 Lorentz Transformations

When speeds close to the speed of light are involved the Galilean Transformations are no longer applicable. So what are the correct formulae relating the coordinates of an event in S to the coordinates of the same event in S', moving away at v? Let's have O and O' coincident at $t = t' = 0$. Also let's for now choose the x direction to be in the same direction as v. That way

$$y = y' \text{ and } z = z' \tag{21}$$

Let's start with the linear formulae

$$x' = ax + bt \tag{22}$$

$$t' = dx + et \tag{23}$$

(They need to be linear to satisfy Einstein's postulates.) We know that the origin of S' is moving away from the origin of S at v, so using the equation 22:

$$0 = avt + bt \tag{24}$$

so $-\frac{b}{a} = v$. Similarly the movement of O tells us that $-\frac{b}{e} = v$. So $a = e$ and $b = -av$. So

$$x' = ax - avt \tag{25}$$

$$t' = dx + at \tag{26}$$

Consider a photon emitted at O at $t = t' = 0$. Its position is

$$x = ct \quad \text{and} \quad x' = ct' \tag{27}$$

This gives together with 25 and 26

$$d = \frac{-av}{c^2} \tag{28}$$

So

$$x' = a(x - vt) \qquad t' = a(t - \frac{vx}{c^2}) \tag{29}$$

Now we only need to work out what a is. This is easy if we use that the transformation law should be symmetric. So

$$x = a(x' + vt') \quad \text{and} \quad t = a(t' + \frac{vx'}{c^2}) \tag{30}$$

must be just as true. Now it's very easy to determine that

$$a = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \equiv \gamma \tag{31}$$

So to summarize, the proper way to transform between different inertial frames of reference is to use the **Lorentz Transformations**:

$$x' = \gamma(x - vt) \tag{32}$$

$$t' = \gamma(t - \frac{vx}{c^2}) \tag{33}$$

and

$$x = \gamma(x' + vt') \tag{34}$$

$$t = \gamma(t' + \frac{vx'}{c^2}) \tag{35}$$

For $v \ll c$ these are the Galilean Transformations.

1.9 Velocity addition

If S' is moving away from S with u and in S' there is an object moving at v' then for an observer in S this object has a velocity vector with

$$v_x = \frac{v'_x + u}{1 + \frac{v'_x u}{c^2}} \tag{36}$$

$$v_y = \frac{v'_y}{\gamma(u)(1 + \frac{v'_x u}{c^2})} \tag{37}$$

1.10 Spacetime

From now on we will refer to locations in spacetime with four dimensional vectors

$$a = (t, \vec{x}) \tag{38}$$

The distance between events in spacetime

$$(\Delta s)^2 = (c\Delta t)^2 - (\Delta \vec{x})^2 \tag{39}$$

is Lorentz invariant ($\Delta s = \Delta s'$). So is the inner product

$$a \cdot b = c^2 t_a t_b - \vec{x}_a \cdot \vec{x}_b \tag{40}$$

1.11 Energy-momentum four vector

Energy and momentum are interesting because they are conserved, but $\vec{p} = m\vec{v}$ and $E = \frac{1}{2}mv^2$ will no longer work in Einstein's theory. The correct formulae are

$$\vec{p} = \gamma m \vec{v} \tag{41}$$

and

$$E = \gamma mc^2 = \sqrt{m^2c^4 + \vec{p}^2c^2} \tag{42}$$

We call

$$E_0 = mc^2 \tag{43}$$

is the rest mass energy. And

$$E_{kin} = E - E_0 = (\gamma - 1)mc^2 \tag{44}$$

is the kinetic energy. It becomes $\frac{1}{2}mv^2$ if $v \ll c$.
We can create a four vector

$$k = (\frac{E}{c}, \vec{p}) = (\gamma mc, \gamma m\vec{v}) \tag{45}$$

This is the four momentum. It has the interesting property that

$$k^2 = m^2 = \text{ invariant} \tag{46}$$

This is very useful for describing and calculating scattering processes.

Example:
A ^{236}U nucleus can decay into $^{95}Kr +^{141} Ba$ with a release of $181MeV$ in energy. The mass of the Krypton is $94.94u$ and that of Barium is 140.91. What is the mass of the Uranium?
Using conservation of energy

$$m_U c^2 = m_{Kr}c^2 + m_{Ba}c^2 + 181MeV \tag{47}$$

This is also (not relevant here but true):

$$[= \gamma(v_{Kr})m_{Kr}c^2 + \gamma(v_{Ba})m_{Ba}c^2] \tag{48}$$

This gives $236.04u$ for the mass of the Uranium.

1.12 Summary

Let's write down a quick overview of the new formulae, we have introduced so far:

Einstein	pre-Einstein
$\vec{p} = \gamma(v)m\vec{v}$	$\vec{p} = m\vec{v}$
$T = (\gamma - 1)mc^2$	$T = \frac{1}{2}mv^2$
$E_{tot} = \gamma mc^2 = \sqrt{m^2c^4 + \vec{p}^2c^2}$	-

Conservation of momentum and conservation of energy still hold. Notice however that mass is no longer necessarily conserved.
For a bunch of particles it is also true that

$$E_{tot} = \sqrt{c^2\vec{p}_{tot}^2 + M_{inv}^2c^4} \tag{49}$$

where $M_{inv} = \sqrt{k_{tot}^2}$ is the so called invariant mass. This is not the same as the sum of all masses of the particles combined. It ts that portion of the total mass of a system that is independent of the velocity of the system. That makes it Lorentz invariant.

1.13 Photons

Photons have no mass. Their total energy formula is reduced to

$$E = cp \tag{50}$$

This only is consistent with $E = \gamma mc^2$ if they travel at the speed of light, since that's the only point where this formula becomes useless.

Example: Compton Scattering
Consider a photon scattering on an electron through an angle θ. Find the final energy of the photon as a function of θ.
Let's use $'$s to refer to the time after the scattering process. The four-vectors of the system before the scattering are

$$k_\gamma = \begin{pmatrix} E_\gamma/c \\ p \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} E_\gamma/c \\ E_\gamma/c \\ 0 \\ 0 \end{pmatrix} \quad \text{and} \quad k_e = \begin{pmatrix} m_e c^2 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{51}$$

After the scattering:

$$k'_\gamma = \begin{pmatrix} E'_\gamma/c \\ p' \cos\theta \\ p' \sin\theta \\ 0 \end{pmatrix} = \begin{pmatrix} E'_\gamma/c \\ \cos\theta E'_\gamma/c \\ \sin\theta E'_\gamma/c \\ 0 \end{pmatrix} \quad \text{and} \quad k'_e = \text{(complicated)} \tag{52}$$

Four momentum is conserved

$$k_\gamma + k_e = k'_\gamma + k'_e \tag{53}$$

We want to work our way around the final four momentum. To do that we will exploit that

$$k^2 = m^2 \tag{54}$$

always. We can rewrite 53

$$(k_\gamma + k_e - k'_\gamma)^2 = (k'_e)^2 \tag{55}$$

$$k_\gamma^2 + k_e^2 + k_\gamma'^2 + 2k_e(k_\gamma - k'_\gamma) - 2k_\gamma k'_\gamma = k_e'^2 \tag{56}$$

Plucking in the numbers and simplifying gives:

$$\frac{1}{E'_\gamma} - \frac{1}{E_\gamma} = \frac{1}{mc^2}(1 - \cos\theta) \tag{57}$$

Let's look at another example:

Example: $\gamma + p \to \pi^0 + p$
Given the masses $m_p = 938MeV/c^2$ and $m_\pi = 135MeV/c^2$ what is the minimum photon energy in the rest frame of the proton?
Energy conservation tells us

$$E_\gamma + m_p c^2 = E_\pi + E'_p \tag{58}$$

where we know that $E_\pi > m_\pi c^2$ and $E'_p > m_p c^2$. So

$$E_\gamma > m_\pi c^2 + m_p c^2 - m_p c^2 = m_\pi c^2 = 135MeV \tag{59}$$

The energy must be more than this to conserve momentum.
Let's think about this in the zero momentum frame (centre of mass frame). Before the collision the photon and the proton are moving towards each other with the same momentum. After the collision the proton and the pion will be at rest, if they are produced at the energy threshold. So in the centre of mass frame

$$E_{tot} = m_\pi c^2 + m_p c^2 \quad \text{and} \quad p_{tot} = 0 \tag{60}$$

The invariant mass is

$$M_{inv}^2 = (E_{tot}^2 + p_{tot}^2)/c^2 = (m_\pi + m_p)^2 \tag{61}$$

This must be the same in all frames. In the lab:

$$M_{inv}^2 = \frac{1}{c^2}((E_\gamma + m_p c^2)^2/c^2 - (E_\gamma/c)^2) = 2E_\gamma m_p/c^2 + m_p^2 \tag{62}$$

Equating the invariant mass of both frames gives

$$E_\gamma = m_\pi c^2 + \frac{m_\pi^2 c^2}{2m_p} = 145MeV \tag{63}$$

These kinds of calculations are easiest in a unit system where $c = 1$.

2 Quantum Physics

2.1 Double slit experiment

$$I(\theta) = I_0 \cos^2(\frac{\pi d}{\lambda} \sin\theta) \tag{64}$$

i.e. maxima at $d\sin\theta = n\lambda$.

2.2 The de Broglie Equation

Inspired by the formulae for photons we can assign particles a wavelength

$$\lambda = h/p \tag{65}$$

In interference experiments it turns out that this seems to actually be true.[1]

> **Example**:
> Calculate the de Broglie wavelength of a proton with kinetic energy $7.0GeV$.
> We can calculate the momentum via
>
> $$p^2 = \frac{1}{c^2}(E^2 - m^2c^4) \tag{66}$$
>
> So
>
> $$\lambda = \frac{h}{p} = \frac{hc^2}{E^2 - m^2c^4} \approx 1.6 \times 10^{-16} m \tag{67}$$
>
> This is tiny which is why we usually do not notice the wave nature of particles.

2.3 Probability distributions

<u>Discrete distributions</u>:
If x can have values x_i, each with a probability of P_i, then the mean value is given by

$$< x >= \sum_i P_i x_i \tag{68}$$

The normalisation condition states that $\sum_i P_i = 1$.

<u>Continuous distributions</u>:
If x can take on any value in an interval I, then the probabilities of all possible

[1]This result can be derived using the path integral formulation of QFT.

values are described by a function $\rho(x)$. The mean value can then be calculated via

$$< x >= \int_I x\rho(x)dx \tag{69}$$

The normalisation condition now is $\int_I \rho(x) = 1$

2.4 The wavefunction (1D)

A particle in one dimension is described by a complex wave function $\Psi(x,t)$ that encodes the probability to find the particle in a particular place. The probability to find a particle between x and $x + dx$ is given by:

$$P = |\Psi(x,t)|^2 dx = \Psi(x,t)\Psi^\star(x,t)dx \tag{70}$$

That's why $|\Psi(x,t)|^2$ is also known as probability density (Born). It must be normalised

$$\int_{-\infty}^{+\infty} |\Psi(x,t)|^2 dx = 1 \tag{71}$$

The expectation value for the particle's position is therefore

$$< x >= \int_{-\infty}^{+\infty} x|\Psi(x,t)|^2 dx \tag{72}$$

The uncertainty is given by

$$\Delta x = \sqrt{< x^2 > - < x >^2} \tag{73}$$

2.5 Particle in an infinite box

Consider a particle in the potential

$$V(x) = \begin{cases} \infty & \text{if } x \leq 0 \text{ or } x \geq L \\ 0 & \text{if } 0 < x < L \end{cases} \tag{74}$$

The Schrödinger equation for this system is solved by

$$\Psi = \sqrt{\frac{2}{L}} \sin \frac{2\pi n x}{L} \tag{75}$$

Obviously the expectation value for the position will be in the middle of the well, i.e. at $\frac{L}{2}$. For $n = 1$

$$< x^2 >= \frac{1}{6} L^2 (2 - \frac{3}{\pi^2}) \tag{76}$$

So

$$\Delta x = \sqrt{< x^2 > - < x >^2} = \sqrt{\frac{L^2}{12}(1 - \frac{6}{\pi^2})} \tag{77}$$

2.6 Atoms

One unified atomic mass unit is

$$1u = \frac{1}{12} \text{ mass of a } {}^{12}_{6}C \text{ atom} = 1.6605 \times 10^{-27} kg \tag{78}$$

This is roughly the mass of the proton which is roughly the mass of the neutron. One mole is the amount of substance containing as many elementary particles as there are in $12g$ of ${}^{12}C$

$$1mol = 6.022045 \times 10^{23} \text{ particles} \tag{79}$$

2.7 The momentum operator

The wave function contains everything there is to know about the system, not just position expectation values. We have already seen how to calculate the expectation value of position from $\Psi(x,t)$

$$< x >= \int_{-\infty}^{\infty} \Psi^{\star} \hat{x} \Psi dx \tag{80}$$

where we have introduced the hat notation for operators. We can think of $\hat{x} = x$ as the operator for position that if used like in the equation above tells us about the position expectation value. For a general observable

$$< \hat{A} >= \int \Psi^{\star} \hat{A} \Psi dx \tag{81}$$

The momentum operator in one dimension is

$$\hat{p} = -i\hbar \frac{d}{dx} \tag{82}$$

Let's apply this to one of the wave functions we have already met:

> **Example**:
> Consider the ground state wave function for a particle trapped in an infinite

square well of width L

$$\Psi(x) = \sqrt{\frac{2}{L}} \sin \frac{\pi x}{L} \tag{83}$$

The expectation value for the momentum (in x direction) is

$$< p >= \int_0^L \Psi^\star(x) \hat{p} \Psi(x) \tag{84}$$

$$= \int_0^L (\sqrt{\frac{2}{L}} \sin \frac{\pi x}{L})(-i\hbar \frac{d}{dx})(\sqrt{\frac{2}{L}} \sin \frac{\pi x}{L}) = 0 \tag{85}$$

This does not mean that the particle stands still most of the time. It just means that it is just as likely to move to the left as it is to move to the right. To make this more clear we can write the wave function as

$$\Psi(x) = \sqrt{\frac{2}{L}} \frac{1}{2i} (e^{i\frac{\pi x}{L}} - e^{-i\frac{\pi x}{L}}) \tag{86}$$

The first term represents a particle with momentum $\frac{h}{2L}$ moving in th $+x$ direction

$$(-i\hbar \frac{d}{dx}) e^{i\frac{\pi x}{L}} = \frac{h}{2L} e^{i\frac{\pi x}{L}} \tag{87}$$

, whereas the second one represents a particle with the same momentum in $-x$ direction. So we can think of the wave function as describing a superposition fo a left and a right moving particle.
We can also calculate

$$< p^2 >= -\hbar^2 \int \Psi^\star \frac{d^2}{dx^2} \Psi dx \tag{88}$$

$$= \hbar^2 \frac{2}{L} \frac{\pi^2}{L} \int \sin^2 \frac{\pi x}{L} dx = \frac{h^2}{4L^2} \tag{89}$$

which is not surprising if we again think of the system as a superposition of two particles with momentum $\pm\frac{h}{2L}$.
The uncertainty is therefore

$$(\Delta p)^2 =< p^2 > - < p >^2 = \frac{h}{2L} \tag{90}$$

Recall that we found

$$< x >= \frac{L}{2} \quad \text{and} \quad < x^2 >= \frac{2}{L} (\frac{L^3}{6} - \frac{L^3}{4\pi^2}) \tag{91}$$

$$\Delta x = \sqrt{\frac{L^2}{12}(1 - \frac{6}{\pi^2})} \tag{92}$$

So we can calculate

$$\Delta x \Delta p \approx 1.136\frac{\hbar}{2} \tag{93}$$

2.8 Heisenberg's Uncertainty Principle

Heisenberg's Uncertainty Principle states that

$$\Delta x \Delta p \geq \frac{\hbar}{2} \tag{94}$$

A particle definitely located at one time is equally likely to be anywhere else in the universe some time later.

Example: Single slit diffraction
We now want to use this to estimate the angular spread of photons of wavelength λ after they pass through a single slit of width d.
In the direction that the photons are travelling the photons are not constrained so $\Delta x \approx \infty$ and $\Delta p_x \approx 0$. De Broglie tells us that $p_x = \frac{h}{\lambda}$.
Just after they pass through the slit $\Delta y = d$. So $\Delta p_y \approx \frac{\hbar}{d}$. So even though they only had a momentum in x direction before the slit they now have a momentum in y direction as well. For the angular spread

$$\tan\theta \approx \frac{\Delta p_y}{p_x} \approx \frac{h}{d}\frac{\lambda}{h} \approx \theta \text{ for small } \theta \tag{95}$$

So

$$\theta \approx \frac{\lambda}{d} \tag{96}$$

There is a similar relationship between energy and time. It can be approximately written as

$$\Delta E \Delta t \approx h \tag{97}$$

Notice that because time is not an observable in quantum mechanics(it doesn't have an operator), we cannot write $\Delta t = \sqrt{< t^2 > - < t >^2}$. Rather we should think of Δt as an the timescale.

Example: Z boson decay

The Z boson has a mass of $91.18GeV$ with a width of $\approx 2.5GeV$. Estimate its lifetime.

Using $\Delta E \Delta t \approx h \approx \hbar$

$$\Delta t \approx \frac{\hbar}{\Delta E} \approx 3 \times 10^{-25} s \tag{98}$$

This relationship can also be useful for seemingly classical problems.

Example:

Determine the uncertainty in determing the frequency of a middle C played for $0.1s$

$$\Delta f \Delta t \approx \frac{1}{2\pi} \tag{99}$$

where we used $E = hf$. So

$$\Delta f \approx \frac{1}{2\pi f} \approx 1.7Hz \tag{100}$$

Musical notes of finite duration necessarily involve a range of frequency.

2.9 The time independent Schrödinger Equation

We now want to introduce the operator for energy. The so called Hamiltonian:

$$\hat{H}\Psi = E\Psi \tag{101}$$

Classically $E_{kin} = \frac{p^2}{2m}$ and $E = E_{kin} + V$. In Quantum Mechanics we now that

$$\hat{p} = -i\hbar\frac{d}{dx} \quad \rightarrow \quad \hat{p}^2 = -\hbar^2\frac{d^2}{dx^2} \tag{102}$$

so we could (and can) write

$$\hat{H} = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V \tag{103}$$

Recall that for a particle of definite momentum

$$\psi(x,0) = Ae^{ipx/\hbar} \quad \text{so} \quad -i\hbar\frac{d}{dx}\psi = p\psi \tag{104}$$

Similarly for a particle of definite total energy

$$(-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + V)\psi = E\psi \tag{105}$$

2.10 Revisiting the infinite square well

Recall that we claimed that the wave function for a particle in an infinite square well is

$$\psi = \sqrt{\frac{2}{L}} \sin\frac{n\pi x}{L} \tag{106}$$

We can use the energy operator on it (in the square well $V = 0$):

$$\hat{H}\psi = -\frac{\hbar^2}{2m}\frac{d^2}{dx^2}\sqrt{\frac{2}{L}} \sin\frac{n\pi x}{L} = \frac{n^2 h^2}{8mL^2}\psi \tag{107}$$

$$\rightarrow E_n = \frac{h^2}{8mL^2}n^2 = \frac{\hbar^2\pi^2}{2mL^2}n^2 \tag{108}$$

Recall that for $n = 1$

$$< p >^2 = \frac{h^2}{4L^2} \tag{109}$$

which is consistent with $E = \frac{p^2}{2m}$.

2.11 Hydrogen Atom

The Schrödinger equation for a hydrogen atom[2] is solved by a wavefunction with energy eigenvalues

$$E_n = -13.6eV\frac{Z^2}{n^2} \tag{110}$$

$E = 0$ corresponds to an unbound electron. If $E > 0$ the electron will have additional kinetic energy. And if $E < 0$ then it is bound.

> **Example**:
> A hydrogen atom is in the state with $n = 2$. What wavelength does the emitted photon have if the electron returns to the ground state?

$$E_\gamma = E_2 - E_1 = 13.6eV(\frac{1}{2^2} - \frac{1}{1^2}) \tag{111}$$

$$\lambda = \frac{hc}{E_\gamma} = 122nm \tag{112}$$

[2]The Hamiltonian is now $-\frac{\hbar^2}{2m}\nabla - \frac{Ze^2}{4\pi r}$

What energy of γ can ionise a hydrogen atom in the $n = 2$ state?

$$E_\gamma = E_\infty - E_2 = E_2 = 3.4eV \tag{113}$$

3 Useful identities

$$(1-\frac{v^2}{c^2})^{1/2} \approx 1+\frac{v^2}{2c^2} \text{ if } v \ll c \tag{114}$$

$$\frac{cp}{E}=\frac{v}{c} \tag{115}$$

$$\sin^2(a)=\frac{1}{2}(1-cos(2a)) \tag{116}$$

$$\int \sin(x)\cos(x)dx=\cos^2 x \tag{117}$$