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Contents

1 The Universe and its Physics

- **Units to use in Astrophysics**:
  - $1AU = 1.5 \times 10^8 km$
  - $1pc = 3.1 \times 10^{13} km$
- **Conserved properties**:
  - Energy E
  - Momentum $\vec{p} = m\vec{v}$ ($p = E/c$ for photons)
  - Angular Momentum $L = \vec{r} \times \vec{p}$
- Astrophysics deals with **extreme temperatures and densities**.
- **Newton's laws of motion**:
  - If there is no force on a body, its momentum is constant.
  - $\vec{F} = \frac{d}{dt}\vec{p}$
  - If a body exerts a force on another body, the other body exerts the same force on the first body but in opposite direction.
- **Forces in Astrophysics**:
  Astrophysics really only deals with Gravity and maybe Electromagnetism in stars. The Weak and the Strong Force are not important on the scales of interest.
  Law of Gravity:
  $$F_G = -G\frac{mM}{r^2} \tag{1}$$
  with $G = 6.67 \times 10^{-11} \frac{m^3}{kgs^2}$.
  Newton's shell theorems:
  - The gravitational force from a spherically symmetric mass distribution can be calculated as if the entire mass is located at the centre of the sphere.
  - The gravitational force of a spherically symmetric shell on any object inside (regardless of the object's location within the shell) is zero.
- **Circular Motion**:
  A planet m on a circular orbit around a star M maintains its orbit if
  $$-G\frac{mM}{r^2} = F_g = -F_c = \frac{mv^2}{r} \tag{2}$$

So for $M \gg m$

$$v = \sqrt{\frac{GM}{r}} \tag{3}$$

The period of this orbit is

$$P = \frac{2\pi r}{v} = \sqrt{\frac{4\pi^2}{GM}} r^{1.5} \tag{4}$$

aka Kepler's third law

$$P^2 = \frac{4\pi^2}{GM} r^3 \tag{5}$$

- **Pressure**:

$$P = \frac{F}{A} \tag{6}$$

For randomly moving and colliding gas particles (for ideal gas):

$$P = nkT \tag{7}$$

where n is the number of particles per m^3.
The **thermal energy density** of the moving gas particles is

$$\epsilon_{th} = \frac{E_{th}}{V} = \frac{3}{2} nkT \tag{8}$$

- **Composition of the Universe**:
  - 90% Hydrogen
  - 10% Helium
  - 0.01% Rest

For molecular gas: $n \approx \frac{\rho}{m_{H_2}}$
For atomic gas: $n \approx \frac{\rho}{m_H}$
For ionised gas: $n \approx 2\frac{\rho}{m_H}$

- **Particle flux**:
Flux is the number of particles passing through surface area per time:

$$f_p = nv \tag{9}$$

Each particle carries momentum $p = mv$. The transfered momentum is

$$\frac{dp}{dt} = A f_p m v \tag{10}$$

Pressure is force per area, so

$$P = \frac{F}{A} = \frac{1}{A}\frac{dp}{dt} = \frac{1}{A}\frac{d}{dt}f_p mv = \frac{1}{3}n\langle vp\rangle \tag{11}$$

- **Thermodynamics**:

  $$\Delta E_{th} = Q - W \tag{12}$$

  where W is the work done by the gas.

  $$W = \int F ds = \int P dV \tag{13}$$

  This means that expanding gas cools down.

- **Electromagnetic radiation**:

  $$c = \lambda f \tag{14}$$

  The energy of a photon:

  $$E = hf = \hbar\omega = \frac{hc}{\lambda} \tag{15}$$

  Its momentum:

  $$p = \frac{E}{c} \tag{16}$$

- **Hydrogen**:
  Bohr radius:

  $$a_0 = \frac{4\pi\epsilon_0\hbar^4}{m_e e^2} \tag{17}$$

  Energy levels:

  $$E_n = \frac{1}{n^2}\frac{m_e c^2 e^4}{2(\hbar c)^2} = -\frac{13.6eV}{n^2} \tag{18}$$

  Hydrogen lines:

  $$\frac{1}{\lambda} = R(\frac{1}{n_1^2} - \frac{1}{n_2^2}) \tag{19}$$

  with $R = 1.097 \times 10^7 \frac{1}{m}$

- **Blackbody radiation**:
  Stefan Boltzman law:

  $$F = \frac{P}{A} = \sigma T^4 \tag{20}$$

  For a sphere, the total emitted power is

  $$L = AF = 4\pi R^2 \sigma T^4 \tag{21}$$

The emitted energy over time t is

$$E = Lt \tag{22}$$

Planck's law:

$$B_\lambda(T) = \frac{2hc^2}{\lambda^5}\frac{1}{exp(\frac{hc}{kT\lambda}) - 1} \tag{23}$$

(flux per unit of wavelength) Wien's displacement law:

$$\lambda_{peak} = 2.897 \times 10^{-3}\frac{1}{T}m \tag{24}$$

- **Flux**:
  Flux is power received per surface. At a distance d away from a source of luminosity L:
  $$F = \frac{L}{4\pi d^2} \tag{25}$$

- **Magnitude system**:
  The apparent magnitude m is based on actual flux received and is defined as
  $$m = -2.5log_{10}F + m_0 \tag{26}$$
  where m_0 is a constant defined such that the star Vega has $m = 0$. This constant drops out when calculating the difference between to stars.
  The absolute magnitude M is what the magnitude of a star would be at $10parsec$.
  The distance modulus $\mu = m - M$ is the difference between the apparent magnitude m and the absolute magnitude M. It is related to the distance d in parsecs by
  $$d = 10^{1+\mu/5} \tag{27}$$

- **Colour index**:
  In astronomy, the colour index is a simple numerical expression that determines the colour of an object, which in case of a star gives its temperature. The smaller the colour index, the more blue (or hotter) the object is. This is a consequence of the logarithmic magnitude scale, in which brighter objects have smaller (more negative) magnitudes than dimmer ones.
  To measure the index, one observes the magnitude of an object successively through two different filters, such as U and B, or B and V, where U is sensitive to ultraviolet rays, B is sensitive to blue light and V to visible light. The difference in magnitudes found like that is called $U - B$ or $B - V$ colour index

respectively.

$$B - V = -2.5 \log \frac{F_B}{F_V} + 2.5 \log \frac{F_{B,Vega}}{F_{V,Vega}} \tag{28}$$

This means that

$$B - V = 0 \text{ for Vega} \tag{29}$$

$$B - V < 0 \text{ for a cooler star} \tag{30}$$

$$B - V > 0 \text{ for a hotter star} \tag{31}$$

- **Snellius' law**:

$$\frac{n_2}{n_1} = \frac{\sin \theta_1}{\sin \theta_2} = \frac{v_1}{v_2} \tag{32}$$

- **Grating**
Constructive interference of a double slit at

$$d \sin \theta = m\lambda \tag{33}$$

- **Sensitivity**:
Sensitivity is the ability to detect faint signals.
The Flux F at Earth from a source emitting power P at distance r from Earth

$$F = \frac{P}{4\pi r^2} \tag{34}$$

The Energy absorbed by the telescope over a time t is:

$$E = F \frac{\pi D^2}{4} t \tag{35}$$

where D is the diameter of the telescope.

- **Resolution**:
The ability to separate closely spaced sources

$$\theta = 1.2 \frac{\lambda}{D} \tag{36}$$

is called the diffraction limit of the telescope.
The angular resolution θ of an interferometer array can usually be approximated by

$$\theta = \frac{\lambda}{B} \tag{37}$$

where B is the length of the maximum physical separation of the telescopes in the array called the baseline.

- **Units of angles**:
  Measure Angles in radian and solid angles in steradian (sr):

$$\Omega = \frac{A}{r^2} \tag{38}$$

  The total solid angle of a sphere is therefore $4\pi sr$. One full circle is 2π. 1 degree is 60 arcminutes and 1 arcminute is 60 arcseconds.

- **Lenses**:
  - Focal length f
    positive for convex lenses and negative for concave lenses.
  - Lens formula (for thin lenses)

$$\frac{1}{f} = (n-1)(\frac{1}{R_1} + \frac{1}{R_2}) \tag{39}$$

    where n is the refractive index of the material (depends on λ).
    $\frac{1}{f}$ is the optical power of a lens. An incoming parallel beam will create an image at f(or a virtual image at $-f$ for concave lenses).
  -

$$\frac{1}{f} = \frac{1}{u} + \frac{1}{v} \tag{40}$$

    where u is the distance between the object and the lens and v is the distance between the lens and the image.
  - Magnification

$$M = \frac{V}{U} = -\frac{v}{u} \tag{41}$$

    where U is the height of the object and V is the height of the image.
  - Chromatic aberration:
    The focal length of blue light is shorter than for red light.
  - Image scale: (sometimes plate scale)
    Consider a region of the sky whose angular size is θ. The observed things are far away so rays from the same point arrive parallel. Then

$$\theta = \frac{V}{f} \tag{42}$$

    Therefore the scale of the image in radians per metre (θ/V) is equal to

$1/f$.

$$\frac{\theta}{V} = \frac{1}{f} \tag{43}$$

- **Distance measurement**:

  - **Parallax**:
    Distance of one parsec is defined as a parallax of 1arcsec

    $$D = \frac{1}{p} \tag{44}$$

    Where D is measured in parsec and p is measured in arcsec.

  - **Proper motion**:
    Proper motion is the astronomical measure of the observed changes in the apparent places of stars in the sky, as seen from the centre of mass of the Solar System, compared to the abstract background of the more distant stars.
  - **Cepheid variables**:
    are stars with a regular periodic change in magnitude. The period has a known relation to its absolute magnitude so these objects can be used for distance measurement.
  - **Supernova**:
    Supernovae type I have uniform peak brightness $M = -19.3$ and can therefore also used for distance measurement. They can be detected at very large distances.

2 Stars

Stars come in a variety of colours, temperatures, luminosities, masses and radius.

$$3000 - 200000K \quad 10^{-4} - 10^{6} L_{\odot} \quad 0.08 - 120 M_{\odot} \quad 0.01 - 500 R_{\odot} \tag{45}$$

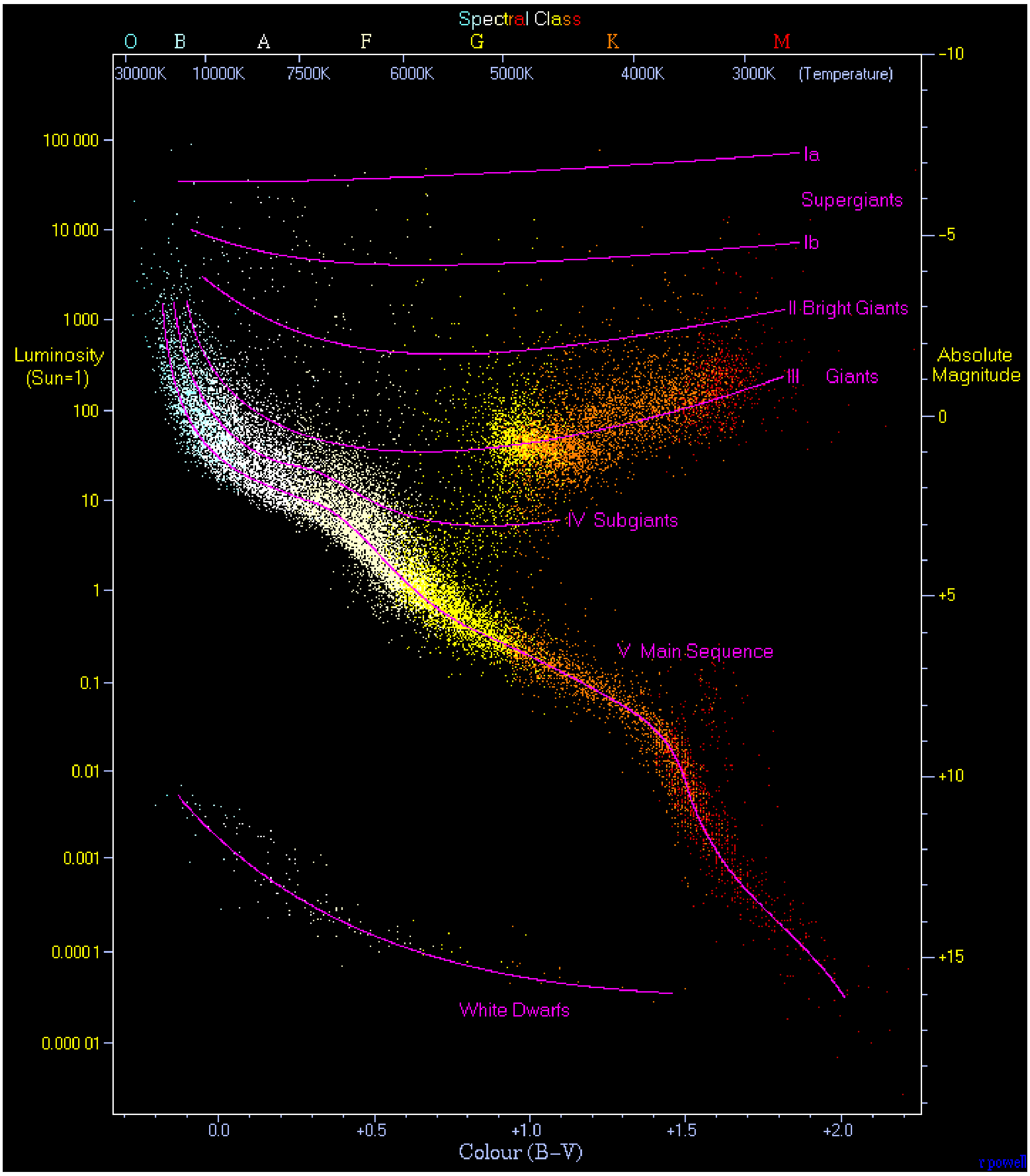

Figure 1: Herzsprung Russel diagram

The above diagram shows a scatter plot of starts that shows colour versus absolute magnitude. Stars only occupy certain regions.

90% of stars are on the main sequence. There are however many luminous, cool stars - Those are red giants and supergiants. Hot faint stars are called white dwarfs.

Stars on the main sequence live longer when they are cooler and less luminous.

2.1 Stellar Spectral Classification

OBAFGKM(LT)
This is a sequence of decreasing temperature. They are subdivided in classes 0-9. O3 are the hottest stars and M9 are the coolest. L, T are brown dwarfs (sub-stellar masses). Our sun is a G2 star.

2.2 Timescales I

We can define the dynamical time as

$$t_{dyn} = \frac{R}{v_e sc} \approx \sqrt{\frac{1}{G\bar{\rho}}} \tag{46}$$

Where $v_{esc} = \sqrt{\frac{2GM}{R}}$ is the escape velocity. This can be used as an estimate for the life time of a thing that was collapsing under gravity, with nothing to hold it. This is also known as the free fall time.
The average density of the sun is $1400kg/m^3$. This would give the sun a lifetime of about $15min$. So the processes involved in keeping it stable must be enormous.

2.3 Equation of Hydrostatic equilibrium

So what keeps the star from collapsing?
Let's look at a star of radius R and consider a shell of thickness dr. A piece of that shell dA experiences the gravitational force an pressure on both sides of the shell. If the star is not collapsing these forces are in equilibrium. So

$$-F_g + P(r)dA - P(r + dr)dA = 0 \tag{47}$$

But $P(r + dr) = P(r) + \frac{dP}{dr}dr$ and $F_g = \frac{Gm(r)}{r^2}$, where $m(r) = \int_0^r \rho(r)4\pi r^2 dr$ is the mass enclosed by the shell, so

$$\frac{Gm(r)\rho(r)}{r^2}drdA + \frac{dP}{dr}dAdr = 0 \tag{48}$$

And we arrive at the Equation of hydrostatic equilibrium:

$$\frac{dP}{dr} = -\frac{Gm(r)\rho(r)}{r^2} \tag{49}$$

Example: Pressure at the centre of a star

$$\frac{dP}{dm} = \frac{dP}{dr}\frac{dr}{dm} = -\frac{Gm(r)}{4\pi r^4} \tag{50}$$

Integrating over m gives

$$\int \frac{dP}{dm} dm = -\int \frac{Gm(r)}{4\pi r^4} dm \tag{51}$$

$$\int_{P(0)}^{P(R_\star)} dP = -\int_0^{m_\star} \frac{Gm(r)}{4\pi r^4} dm \tag{52}$$

where $m_\star$ is the mass of the star. We know that $P(R_\star) = 0$. This is the surface. And we can define $P(0) = P_c$ at the centre. So

$$P_c = \int_0^{m_\star} \frac{Gm(r)}{4\pi r^4} dm \tag{53}$$

We know that $\frac{1}{R_\star} < \frac{1}{r}$. So we can find a lower limit

$$P_c > \int_0^{m_\star} \frac{Gm(r)}{4\pi R_\star^4} dm = \frac{Gm_\star^2}{8\pi R_\star^4} \tag{54}$$

This is a limit for the pressure at the centre of the star.
For the sun this is

$$P_c > 4.4 \times 10^{13} N/m^2 \tag{55}$$

$1atm \approx 10^5 N/m^2$.

2.4 The Mass Equation

The mass of a whole spherical shell is $dm = \rho(r)Adr$. Using $A = 4\pi r^2$ we find

$$\frac{dm}{dr} = 4\pi r^2 \rho(r) dr \tag{56}$$

2.5 The Virial Theorem

We want to use the equation for hydrostatic equilibrium to derive a relationship between potential and kinetic energy.

$$\frac{dP}{dm} = -\frac{Gm}{4\pi r^4} \tag{57}$$

Multiply by $V = \frac{4}{3}\pi r^3$

$$V(r)\frac{dP}{dm} = -\frac{Gm}{3r} \tag{58}$$

Integrate over dm for the whole star

$$\int_{P(0)}^{P(R_\star)} V(r)dP = -\frac{1}{3}\int_0^{m_\star} \frac{Gm(r)}{r}dm \tag{59}$$

Notice that the RHS is just (a third of) the gravitational energy of the star $E_p = -\int \frac{Gm(r)}{r}dm$. The LHS:

$$\int_{P(0)}^{P(R_\star)} V(r)dP = [VP]_{r=0}^{r=R_\star} - \int_0^{V(R_\star)} PdV = -\int_0^{V(R_\star)} PdV \tag{60}$$

So using $dm = \rho dV$:

$$\int_0^{m^\star} \frac{P}{\rho}dm = -\frac{1}{3}E_{pot} \tag{61}$$

For a main sequence star the pressure is due to thermal gas pressure. The equation of state is the ideal gas law

$$P = nkT = \frac{\rho}{m_g}kT \tag{62}$$

where n is the number density. We can rewrite that as

$$\frac{P}{\rho} = \frac{kT}{m_g} = \frac{2E_k}{3m_g} \tag{63}$$

$E_k = \frac{3}{2}kT$ is the kinetic energy of the particles. Let's define the energy per unit mass

$$u = \frac{3}{2}\frac{kT}{m_g} = \frac{3}{2}\frac{P}{\rho} \tag{64}$$

$$\rightarrow -3\int_0^{m^\star} \frac{P}{\rho}dm = -\int_0^{m^\star} 2udm \tag{65}$$

Using that the total internal energy is $U = \int_0^{m^\star} udm$ we get the virial theorem

$$U = -\frac{1}{2}E_p \tag{66}$$

On compression a star becomes hotter so stars have negative heat capacity. This is true for any gravitationally bound system.

2.6 Timescales II

How long does it take before a star radiates away all its gravitational energy E_p?

$$t = \frac{E_p}{L} \tag{67}$$

where L is the Luminosity - the rate at which the sun radiates. A good approximation is given by

$$t_{KH} \approx \frac{GM^2}{RL} \quad \text{Kelvin Helmholtz timescale} \tag{68}$$

The sun - by that formula- has a lifetime of $t_{KH}(\text{Sun}) \approx 10^{15}s \approx 30 \times 10^6 y$

2.7 Timescales III

How long can a star shine?

$$t = \frac{fM}{L} \tag{69}$$

where $f \approx 6.2 \times 10^{13} J/kg$ is the efficiency factor.
Massive stars burn bright, live fast and die young.

2.8 Energy generation

The temperature and the density in the sun are just high enough for nuclear fusion

$2\times$	$p^+ + p^+$	$\rightarrow$	$d^+ + e^+ + v_e$
$2\times$	$p^+ + d^+$	$\rightarrow$	${}^3He^{+2} + \gamma$
	${}^3He^{+2} + {}^3He^{+2}$	$\rightarrow$	${}^4He + 2p^+$
	$4p$	$\rightarrow$	${}^4He^{+2} + 2e^+ + 2v_e$

where the second reaction happens much faster ($6s$) than the third one ($10^6 yr$) which again is much more likely than the first one ($10^{10} yr$). This in total releases $26.7 MeV$.
Also important are the so called pp-II and pp-III cycles.
Nuclear reaction rates are extremely sensitive to temperature (and density):

$$q_{pp} \propto \rho T^4 \tag{70}$$

The CNO cycle is proportional to ρT^{16} and $q_{3\alpha} \propto \rho^2 T^{40}$.
Stars are only stable because they have negative specific heat capacity.

Eventually all the Hydrogen is used up. Now gravity starts to take over and the inner regions contract and heat up. Around the core of inert helium hydrogen fusion restarts and the star swells up. This is a **red giant**. The helium core grows and heats up more. When the temperature reaches $10^8 K$ the Helium ignites and the triple-alpha process starts

$$^4He +^4 He \rightarrow^8 Be \tag{71}$$

$$^8Be +^4 He \rightarrow^{12} C \tag{72}$$

Once the helium core is exhausted it extinguishes, contracts and heats up. The helium burning continues in the shell. The star is now on the Asymptotic Giant Branch.

- Stars on the main sequence have a burning core of hydrogen
- Stars on the Red Giant Branch (RGB) have a burning shell of hydrogen
- Stars on the Horizontal Branch (HB) have a burning helium core
- Stars on the Asymptotic Giant Branch (AGB) have a burning shell of helium

2.9 Pressure equilibrium

In a burning star gravity is balances by the ideal gas pressure $P = nkT$. So what balances gravity in an inert core? Without energy production the temperature drops. The star will collapse unless a different source of pressure can be found.
The Pauli Exclusion Principle dictates that two fermions cannot occupy the same quantum state. This causes electron degeneracy pressure.

$$P_{deg} = \frac{(3\pi^2)^{2/3}\hbar^2}{5m_e}\rho_N^{5/3} \tag{73}$$

where ρ_N is the free electron density. Interestingly this pressure does not explicitly depend on temperature.
The pressure inside a star is the sum of the different kinds of pressure

$$P = P_{gas} + P_{rad} + P_{deg} \tag{74}$$

each with different equations of state. The degeneracy pressure becomes important at low temperatures and very high densities.

2.10 End States

Low mass stars ($< 5M_\odot$):
The electron pressure stabilises this star before the temperature rises enough for carbon burning. This forms a **white dwarf**. They are supported by electron degeneracy pressure, are tiny ($\approx 0.01R_\odot$) and have very high density ($\approx 10^8 kg/m^3$). They are not completely stable. Their maximum mass is the Chandrasekhar mass $1.46M_\odot$ (observed mass peaks are $\approx 0.6M_\odot$). Above that their theoretical radius would be zero.

Higher mass stars:
Here the temperature rises enough for carbon fusion. After the core runs out of fuel. The fusion moves to shell burning. This causes the core to contract and heat up and the next element ignites. This process repeats and you end up with a star made up of layers of elements. An onion star.
The heavier the fuel the shorter the burning time. For a 25 solar mass star the hydrogen burns for 7 million years, the helium for 700.000 years, the carbon for 600 years, the neon for 1 year, the oxygen for 6 months and silicon burns for only a day. Iron is the end product of nuclear fusion.
When the iron core collapses, densities reach $10^{13} kg/m^3$. Now three processes remove energy

- Neutrino losses
- Electron capture by nuclei
- Photo disintegration $Fe \rightarrow He$

The (now earth sized) core collapses and reaches a $30km$ neutron rich sphere with densities of $10^17 kg/m^3$ (density of the atomic nucleus). The collapse is reversed by strong nuclear force repulsion and outer layers of the star blow off $\rightarrow$ **Supernova**. The end product is a **neutron star**. They are supported by neutron degeneracy pressure. They have a radius of only about $10km$ and a high density of about $10^{17} kg/m^3$. A star ends like this if the mass does not exceed a certain limit of 1.8 to $2.9M_\odot$. After this the neutron degeneracy pressure can't win over gravity and you end up with a **black hole**. The escape velocity of a body is given by

$$v_{esc} = \sqrt{\frac{2GM}{R}} \tag{75}$$

If this exceeds the speed of light nothing can escape[1]. For a mass M this would happen for

$$R = \frac{2GM}{c^2} = R_{Schwartzschild} \tag{76}$$

3 Orbiting objects

Consider two masses $m_1 > m_2$ at points $\vec{r}_1$ and $\vec{r}_2$. The centre of mass will be at

$$\vec{r}_{com} = \frac{m_1\vec{r}_1 + m_2\vec{r}_2}{m_1 + m_2} \tag{77}$$

Its momentum is

$$\vec{p}_{com} = m_1\vec{v}_1 + m_2\vec{v}_2 = \vec{p}_1 + \vec{p}_2 \tag{78}$$

The acceleration is

$$\vec{a}_{com} = \frac{m_1\vec{a}_1 + m_2\vec{a}_2}{m_1 + m_2} = 0 \tag{79}$$

If there are no external forces $m_1\vec{a}_1 = -m_2\vec{a}_2$ then $\vec{a}_{com} = 0$. Note that these equations work for N masses not just 2:

$$\vec{x}_{com} = \frac{\sum m_i\vec{x}_i}{\sum m_i} \quad \vec{p}_{com} = \sum \vec{p}_i \quad \vec{a}_{com} = 0 \tag{80}$$

3.1 Planets

Stars are or used to be powered by nuclear fusion. Objects $< 0.08M_\odot$ don't get hot enough. **Brown dwarfs** will fuse some deuterium at the beginning of their life but don't count as stars or planets. The IAU defines an **exoplanet** as a celestial body which has a mass below the limit for fusion of deuterium and orbits a star or a stellar remnant.

3.2 Finding Exoplanets

Planets are very faint compared to stars. Indirect detection measures are necessary. One way is to look for wobbles in the movement of stars. These indicate that there is mass outside the star pulling out the centre of mass of the system.

3.3 Kepler's Laws

1. Planets orbit the sun in ellipses with the sun at one focus.

[1]see Hawking radiation for a more precise statement.

2. Planets sweep out equal areas in equal times
   We could show from this that the force holding the planet has to be radial.

3. The period P and semi-major axis a of the orbit are related through

$$P^2 = \frac{4\pi^2 a^3}{G(M_1 + M_2)} \tag{81}$$

   In the solar system this simplifies to:

$$\frac{P^2}{yr} = \frac{a^3}{AU} \tag{82}$$

   From this we could follow that $F_G \propto \frac{1}{r^2}$.

3.4 Circular Motion

$$\vec{r} = r(\cos\theta\hat{\vec{x}} + \sin\theta\hat{\vec{y}}) \tag{83}$$

$$\vec{v} = r\frac{d\theta}{dt}\hat{\vec{\theta}} \tag{84}$$

$$\vec{a} = -(\frac{d\theta}{dt})^2\vec{r} \tag{85}$$

3.5 Binary Stars

Consider two stars in circular orbits around their centre of mass.

$$\frac{d^2\vec{r}}{dt^2} = -(\frac{d\theta}{dt})^2\vec{r} = -(\frac{2\pi}{P})^2 r\hat{\vec{r}} \tag{86}$$

This must be true for both stars

$$\ddot{r}_1 = -\frac{4\pi^2}{P^2}r_1 \quad \text{and} \quad \ddot{r}_2 = -\frac{4\pi^2}{P^2}r_2 \tag{87}$$

The gravitational acceleration on M_1 is

$$\ddot{r}_1 = -\frac{GM_2}{(r_1 + r_2)^2} = -\frac{4\pi^2}{P^2}r_1 \tag{88}$$

Similarly for M_2

$$\ddot{r}_2 = -\frac{GM_1}{(r_1 + r_2)^2} = -\frac{4\pi^2}{P^2}r_2 \tag{89}$$

Adding these equations and using $r_1 + r_2 = r$ leaves us with

$$P^2 = \frac{4\pi^2}{G}(\frac{r^3}{M_1 + M_2}) \tag{90}$$

For an elliptical orbit

$$P^2 = \frac{4\pi^2}{G}(\frac{a^3}{M_1 + M_2}) \tag{91}$$

where a is the semi-major axis (= half the maximum separation of the stars).
Two stars in a binary system have the same period

$$P_1 = \frac{2\pi r_1}{v_1} = \frac{2\pi r_2}{v_2} = P_2 \tag{92}$$

So

$$\frac{v_1}{v_2} = \frac{r_1}{r_2} = \frac{M_2}{M_1} \tag{93}$$

Using Kepler's law

$$M_1 + M_2 = \frac{P}{2\pi G}(v_1 + v_2)^3 \tag{94}$$

3.6 Doppler Shift

for light:

$$\frac{v}{c} = \frac{\nu_0 - \nu}{\nu_0} = \frac{\lambda - \lambda_0}{\lambda_0} \tag{95}$$

where v is the line of sight velocity.

3.7 Planet in orbit around a sun

For $M_s \gg M_p$ and $r_p \gg r_s$

$$v_p = \sqrt{\frac{GM_s}{r_p}} \tag{96}$$

and

$$v_s = \frac{M_p}{M_s}\sqrt{\frac{GM_s}{r_p}} \tag{97}$$

A planet with an absorption coefficient of f, reflects a fraction of $1 - f$.
Assume the planet is at a distance r away from the star and has a radius R. The star has luminosity L. The flux of the star light at the location of the planet is

$$F_S = \frac{L}{4\pi r^2} \tag{98}$$

It reflects (effectively the luminosity of the planet)

$$L_P = \pi R^2 F_S(1 - f) \tag{99}$$

Now assume we observe this star and its planet from earth a distance d away from the star. Since $d \gg r$ the star and planet can be assumed to be at the same distance from us. The difference in magnitude of the star and the planet is

$$m_S - m_P = -2.5 \log F_S + 2.5 \log F_P \tag{100}$$

$$= -2.5 \log\left(\frac{L}{4\pi d^2}\right) + 2.5 \log\left(\frac{L_P}{4\pi d^2}\right) \tag{101}$$

$$= -2.5 \log L + 2.5 \log \frac{L\pi R^2(1-f)}{4\pi r^2} \tag{102}$$

$$= +2.5 \log \frac{R^2(1-f)}{4r^2} \tag{103}$$

Notice that this only depends on the radius of the star and its distance to its sun. Using sensible values we find that the star will be 20 million times brighter than the planet. So it is almost impossible to see exoplanets directly.
The transit method exploits that the apparent magnitude of the star drops when a planet moves in front of it.
So you can either measure the radial velocity (RV) of the star to find wobbles in its movement or you can wait for transits. The second method is particularly sensitive to the relative orientation of the solar system. The radial velocity method typically finds rather massive stars.

3.8 Habitable Zone

The planet absorbs

$$L_{abs} = \frac{\pi R^2}{4\pi r^2} f L \tag{104}$$

A perfect black body would emit

$$L_{BB} = 4\pi R^2 \sigma T^4 \tag{105}$$

These should be equal so

$$T^4 = \frac{fL}{16\sigma\pi r^2} \tag{106}$$

We can substitute $L = 4\pi R_\star^2 \sigma T_\star^4$

$$T = (\frac{fR_\star^2}{4r^2})^{1/4} T_\star \tag{107}$$

We can use these calculations to calculate where liquid water could exist. We call these areas habitable zones.

4 Galaxies

Galaxies come in many shapes and sizes.

- **Elliptical Galaxies** have an ellipsoidal shape and no/little internal structure. They have red colours because they contain low mass stars. They do not rotate.
  Size: $0.3kpc$ to several $10^2 kpc$
  Mass: 10^7 to $10^{13} M_\odot$
- **Spiral Galaxies** have a central bulge and are basically a rotating disk. They are host to a full range of types of stars.
  Size: $5kpc$ to $10^2 kpc$ in diameter
  Mass: 10^9 t0 $10^{12} M_\odot$
  The Milky way is one of those. It is $30kpc$ in diameter and the disk is about $0.3kpc$ thick It contains about 10^{11} stars and its mass is about $10^{12} M_0$.
- **Irregular Galaxies** have very diverse structures. They contain both young and old stars.

4.1 Star Formation Conditions

Let's apply the Virial Theorem to a ball of gas

$$E_p + 2U = 0 \tag{108}$$

$$E_p = -\frac{3}{5}\frac{M^2}{R} \qquad U = \frac{3}{2}NkT \tag{109}$$

Also $N = \frac{M}{m}$ where m is the mass of a particle.

$$\frac{3}{5}\frac{GM^2}{R} = 3kT\frac{M}{m} \quad \rightarrow \quad R = (\frac{3M}{4\pi\rho_0})^{1/3} \tag{110}$$

where we used $M = \frac{4\pi}{3}R^3\rho_0$. We can find a critical mass for which gravitational pull wins over:

$$M > (\frac{5kT}{Gm})^{3/2}(\frac{3}{4\pi\rho_0})^{1/2} \tag{111}$$

the system is then no longer stable and will form a mass. This limit is known as the **Jeans mass**.
The **IMF** (initial mass function) $\psi(m)$ describes the distribution of stellar masses when the stars are formed. $\psi(m)$ is the number of stars with mass between m and dm. This is usually something like $\psi = \psi_0 m^{-2.5}$.

The total mass is therefore

$$M_T = \int_{m_1}^{m_2} m\psi(m)dm \tag{112}$$

where we need to be careful about our integration boundaries.

4.2 Stellar Populations in Spiral Galaxies

The spiral arms of galaxies are host to young massive stars along with gas and dust. This is where stars are formed. Low mass stars then move towards the thin disk. Old stars produce a thicker disk. The bulge in the middle is basically a mini-elliptical galaxy full of old stars.

4.3 Spectrum of a Galaxy

When a galaxy ages it becomes redder, less luminous and its spectrum will contain more spectral lines of heavier elements. The mass to luminosity ratio will increase with age.

4.4 The mean free path

The mean free path λ is the average distance travelled before a collision.

> Consider a sphere of diameter d moving through a volume containing other identical spheres. We assume that the volume contains n spheres per unit volume. A moving sphere will collide with an other sphere if their centres are a distance of d or less apart. So the effective cross section for a collision is
>
> $$\sigma = \pi d^2 \tag{113}$$
>
> The moving sphere will in a time t collide with all spheres with their centres within a cylinder of base $A = \sigma$ and length vdt. The volume of this cylinder is $V = \pi d^2 vt$. So the number of collisions is $Vn = n\pi d^2 vt$. The mean distance between collisions is
>
> $$\lambda = \frac{vt}{n\pi d^2 vt} = \frac{1}{\sigma n} \tag{114}$$

So

$$\lambda = \frac{1}{\sigma n} \tag{115}$$

We can use this result to calculate the mean free path of a star in a galaxy. We would find an estimate of $\approx 10^{14} pc$. And this is assuming random motion. So stars never collide. We can do the same for galaxies themselves and we would find something like $\lambda \approx 360 MPc$. Galaxy collisions are much, much more common than stellar collisions.

4.5 Galaxy collisions

The strength of the effects of collisions depend on the mass ratio of the colliding galaxies. The gravitational potential accelerates stars and heats them up. Gas will settle into the disk, clouds collide and compress the gas. This causes star formation. Some gas will be driven into the centre of the galaxy. That gives rise to two phenomena: **Nuclear starburst** (massive star formation) and feeding of the central black hole.

4.6 Central Black holes

The black hole in the milky way has a mass of $4.6 \pm 0.7 \times 10^6 M_\odot$. It has so far only been observed indirectly by measuring the velocity of stars close to the galactic centre.

4.7 Galactic Rotation

Rotational velocity around some mass M is

$$v = \sqrt{\frac{GM(R)}{R}} \tag{116}$$

For a point mass $M(R) = M_0$ this means that velocity goes like $v \propto R^{-1/2}$. If there is a constant density in the system, then M will depend on the radius $M(R) = \frac{4}{3}\pi R^3 \rho_0$. Now $v \propto R$, so the velocity will *increase* as the radius increases.

What we observe however is that the velocity increases with the radius at small radiae (constant density) but eventually becomes constant. We can take the above argument in reverse

$$\frac{mv^2}{R} = \frac{M(R)mG}{R^2} \quad \rightarrow \quad M(R) = \frac{v^2(R)R}{G} \tag{117}$$

If $v(r) \propto 1$ then $M(R) \propto R$ and $\rho \propto R^{-2}$. This is how the density in our and in most galaxies works.
It turns out that all the observed stuff is not enough to explain the observed rotation speeds in galaxies. This can be corrected by introducing what is called a **dark matter halo**.

5 Cosmology

5.1 Why is the sky dark at night?

Consider a spherical shell centred on us with radius r and thickness dr. We assume that there are n stars per unit volume and that each star has a radius $R_\star$. The total surface area (facing us) of stars in this shell is $\pi R_\star^2 n \times 4\pi r^2 dr$. The fraction of the surface area of this shell covered by stars is

$$dF = \frac{\pi R\star^2\, n 4\pi r^2 dr}{4\pi r^2} = n\pi R_\star^2 dr \tag{118}$$

If we integrate this over dr to include all shells

$$F = \int^\infty dF = \int^\infty n\pi R_\star^2 dr \to \infty \tag{119}$$

So there are stars in every direction (since this is ≥ 1). The sky should not be dark at night. This is known as Olbers' Paradox.
Possible solutions to this:

- Dust would not work. It would be heated to the temperature of the stars and radiate itself.
- If the universe is not infinite the integral would have a cut off.
- We assumed that the universe is infinitely old. There could just not have been enough time for photons from far away stars to reach us.

So the conclusion is that the universe is not infinitely old and/or not infinite in size.

5.2 The cosmological principle

The universe on a large scale is **homogenous** (the same in all places) and **isotropic** (the same in all directions).

5.3 Hubble flow

We observe that galaxies that are further away from us are moving away from us and they do so faster if they are further away. We can measure the velocities of galaxies from the redshift z

$$\frac{\Delta\lambda}{\lambda} = \frac{v}{c} = z \tag{120}$$

The velocity increases linearly with distance r

$$v = H_0 r \qquad H_0 = 70.0(+12.0 - 8.0)\frac{km/s}{Mpc} \tag{121}$$

In general the expansion factor is a function of time[2]. Gravity is slowing the expansion down. If

$$E_{pot} = \frac{GM}{r} = G\frac{4\pi\rho r^3}{3r} > \frac{1}{2}v^2 = \frac{1}{2}H_0^2 r^2 \tag{122}$$

If they are equal the galaxy will just expand. This happens at

$$\rho = \frac{3H_0^2}{8\pi G} = \rho_{critical} \tag{123}$$

Depending on the average density of a universe its shape will differ

$\Omega_0 = \frac{\rho_0}{\rho_{critical}}$	
< 1	open (unbound) universe
$= 1$	critical universe
> 1	closed universe

Observations suggest that our universe has a third of the critical density (including dark matter).
The Hubble flow does not affect gravitationally bound systems. So distances within galaxies or to nearby galaxies do not expand.

5.4 Scale factor

Distances change with time as the universe expands. We can define a (time dependent) scale factor

$$a = \frac{r(t)}{r(t_0)} \tag{124}$$

$r(t_0)$ is the current distance. We can express the Hubble constant in terms of it. Using

$$\frac{dr}{dt} = Hr \qquad r(t) = ar(t_0) \quad \rightarrow \quad \frac{da}{dt}r(t_0) = Har(t_0) \tag{125}$$

we find

$$H(t) = \frac{1}{a}\frac{da}{dt} \tag{126}$$

[2] If we assume the Hubble constant to be constant we can calculate the age of the universe $t = \frac{1}{H_0} \approx 4.6 \times 10^{17}s \approx 1.5 \times 10^{10}yr$.

5.5 The evolution of the universe

$$\frac{1}{a^2}(\frac{da}{dt})^2 - \frac{8\pi}{3}G\rho(t) = -\frac{kc^2}{a^2} \tag{127}$$

This is the **Friedmann Equation**. It describes the evolution of the universe. The constant term on the right describes the curvature of space.

$k = 0$	flat Euclidean space	The universe has just enough mass to expand forever but at a decelerating rate with the expansion rate asymptotically approaching zero (Triangle sum of 180°)
$k > 0$	closed, spherical space	The universe expands to a certain point and then shrinks again (Triangle sum $> 180°$)
$k < 0$	open, hyperbolic space	The universe expands with a constant velocity forever (Triangle sum of $< 180°$)

5.6 Cosmological redshift

$$z = \frac{\lambda_o - \lambda_e}{\lambda_e} \tag{128}$$

where λ_o is the observed wavelength and λ_e is the emitted wavelength. Using $\lambda_o = \frac{\lambda_e}{a}$

$$a = \frac{1}{z+1} \tag{129}$$

5.7 The hot early universe

In the past the universe was denser and hotter. A long time ago it was just a mixture of nucleons, electrons, photons and other things interacting. But when the temperature drops below about $3000K$ atoms form and the universe is no longer ionised. At this point photons and matter decouple and the universe is now filled with photons with a blackbody temperature of $3000K$. This is the cosmic microwave background. It appears to have $3K$ tough today. This is due to the expansion of the universe.

The energy density in a blackbody radiation field is

$$u_\gamma = \frac{4\sigma}{c}T^4 \tag{130}$$

$\frac{4\sigma}{c} \approx 7.565 \times 10^{-16} \frac{J}{m^3K^4}$. The total energy content of the universe in photons is $Nh\nu$ where N is the number of photons in the universe. The photon energy density is therefore (r is the size of the universe)

$$u_\gamma \propto \frac{Nhc}{\lambda r^3} \quad \text{so} \quad T^4 \propto \frac{Nhc}{\lambda r^3} \tag{131}$$

At recombination (decoupling) we had T_{rc}, λ_{rc} and r_{rc}

$$T_{rc}^4 \propto \frac{N}{\lambda_{rc} r_{rc}^3} \quad \text{and today} \quad T_0^4 \propto \frac{N}{\lambda_0 r_0^3} \tag{132}$$

$r_{rc} = ar_0$ and $z = \frac{\lambda_0 - \lambda_e}{\lambda_e} = \frac{1}{a} - 1$. N is fixed.

$$\frac{T_{rc}^4}{T_0^4} = \frac{1}{a}\frac{1}{a^3} = \frac{1}{a^4} \quad \rightarrow \quad T_0 = aT_{rc} \tag{133}$$

$$T_0 = \frac{T_{rc}}{1+z} \tag{134}$$

We can use this to calculate that the cosmic microwave density formed at $z \approx 10^3$.

$u_\gamma = \frac{E}{V}$ is an energy density. So using $E = mc^2$ we can write $u = \rho c^2$ and because $u \propto T^4 \propto a^{-4}$ we find $\rho_\gamma \propto a^{-4}$ for a universe where energy is stored in radiation. We can do the same calculation for the density evolution if the energy is stored in matter instead of radiation and we would find $\rho_m \propto a^{-3}$

5.8 Cosmic microwave background

The cosmic microwave background

- is isotropic
- was created 380 000 years after the Big Bang
- was discovered by Penzias and Wilson in 1965

5.9 Cosmological Nucleosynthesis

About 100s after its origin, the universe became cool enough for protons and neutrons to start to form nuclei. This is called **Big Bang Nucleosynthesis (BBN)**. It is not like stellar nucleosynthesis:

Stellar	BNN
Equilibrium	Non equilibrium
high density ($\approx 10^5 kg/m^3$)	low density ($\approx 10^{-2} kg/m^3$)
low temperature ($< 10^7 K$)	high temperature ($\approx 10^9 K$)

The key reason is

$$n + p \to D + \gamma \tag{135}$$

5.10 Evidence for the Big Bang

- Expanding universe
- Cosmic microwave background
- Abundance of light elements a few minutes after the Big Bang

5.11 Evolution since the Big Bang

There was a peak in star formation about $10Gyr$ ago ($z \approx 2$). It declined by more than a factor of 10 since then.
Observations in the 1990s suggest that our universe is expanding at an accelerating rate. This cannot be explained with our observations of matter distribution across the universe. Cosmologist therefore hypothesized an unknown form of energy which permeates all of space. This is known as **Dark Energy**.

5.12 Composition of the universe

We find a composition of

- 68% Dark Energy
- 27% Dark Matter
- 5% Ordinary Matter

6 Appendix

6.1 Classical Derivation of the Friedman Equation

Consider the force on a galaxy of mass m

$$F = mA = m\frac{dv}{dt} = -\frac{Gm_r m}{r^2} \tag{136}$$

where we used A to refer to the acceleration and m_r for the mass enclosed within radius r. So $m_r = \frac{4}{3}\pi r^3 \rho$ for a universe that has a uniform density $\rho = \rho(t)$.

$$v = Hr = \frac{1}{a}\frac{da}{dt}r = r(t_0)\frac{da}{dt} \tag{137}$$

where we used $r = ar(t_0)$. So

$$r(t_0)\frac{d^2a}{dt^2} = -G\frac{4}{3}\frac{\pi r^3 \rho}{r^2} = -\frac{4}{3}\pi Gar(t_0) \tag{138}$$

$$\frac{1}{a}\frac{d^2a}{dt^2} = -\frac{4}{3}\pi G\rho(t) \tag{139}$$

Consider the mass in an initial sphere of radius $r(t_0)$

$$m = \frac{4}{3}\pi r(t_0)^3 \rho(t_0) = \frac{4}{3}\pi r(t_0)^3 \rho_0 \tag{140}$$

After some expansion we have the same mass in the sphere of radius $r(t)$

$$m = \frac{4}{3}\pi r(t)^3 \rho(t) = \frac{4}{3}\pi a^3 r(t_0)^3 \rho(t) \tag{141}$$

The mass stays the same so we can equate the equations for both times. So

$$\rho(t) = \frac{\rho_0}{a^3} \tag{142}$$

Using our previous result

$$\frac{1}{a}\frac{d^2a}{dt^2} = -\frac{4}{3}\pi\frac{G\rho_0}{a^3} \tag{143}$$

Multiplying by $2a\frac{da}{dt}$ gives

$$2\frac{d^2a}{dt^2}\frac{da}{dt} = -\frac{8}{3}\pi\frac{G\rho_0}{a^2}\frac{da}{dt} \tag{144}$$

Integrating both sides over dt finally gives

$$(\frac{da}{dt})^2 = \frac{8}{3}\pi\rho_0\frac{G}{a} - Kc^2 \tag{145}$$

where Kc^2 is the constant of integration. We can use $\rho_0 = a^3\rho$ to finally find

$$\frac{1}{a^2}(\frac{da}{dt})^2 - \frac{8\pi}{3}G\rho(t) = -\frac{Kc^2}{a^2} \tag{146}$$

This is the **Friedmann Equation**.