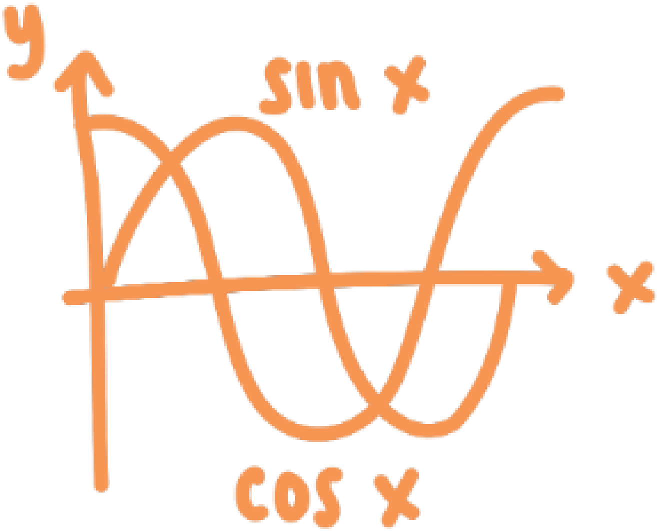
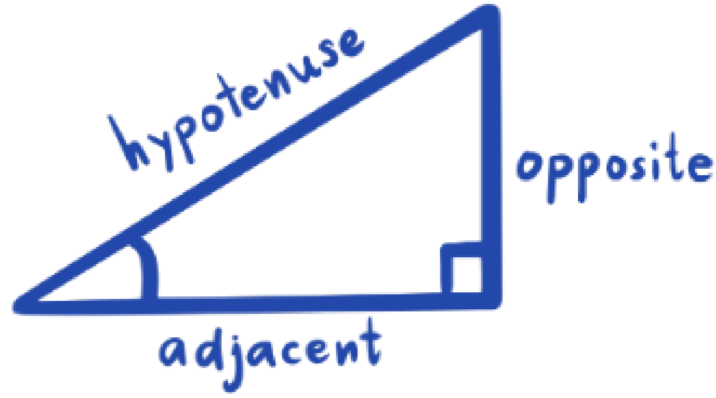
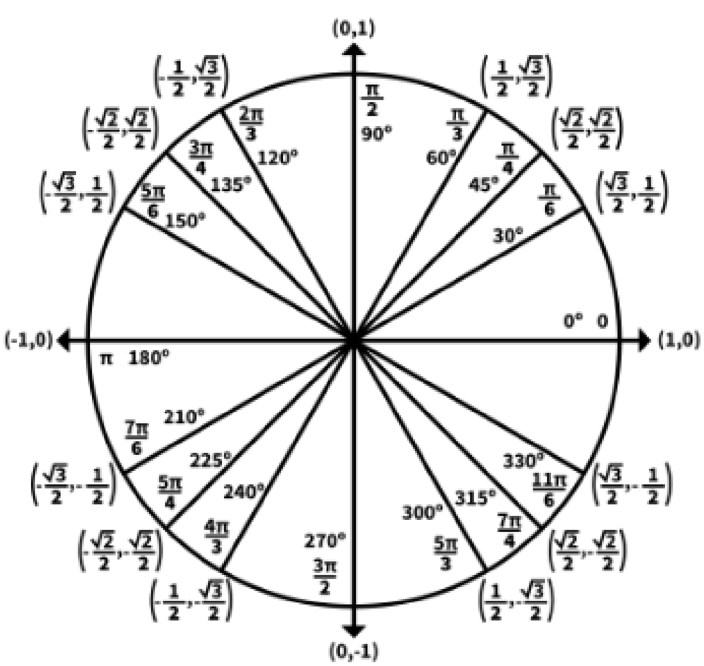
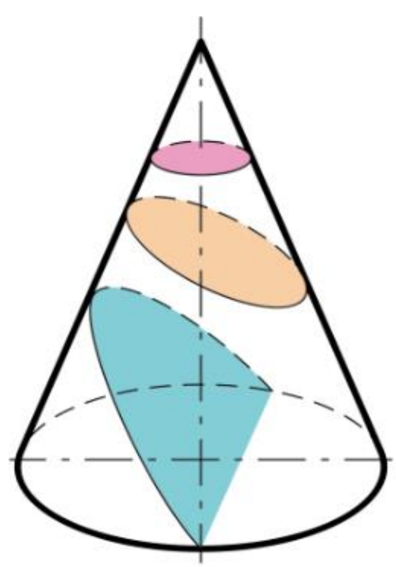




PreCalculus Math Guide





Created by:
Amanda Russell

Contents

Exponential Function

An exponential function is any function where the variable (usually x) sits in the exponent of a constant base b. The most common form looks like this:

$$f(x) = b^x$$

Where

- $b > 0$ (The base must be positive)
- $b \neq 1$

Examples:

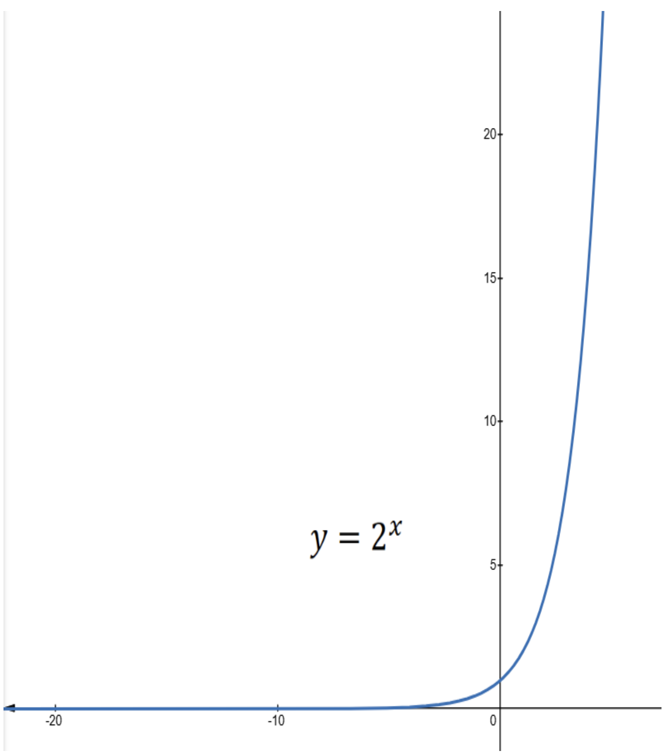
- $f(x) = 2^x$
- $g(x) = 10^x$
- $h(x) = 3^{x+1}$

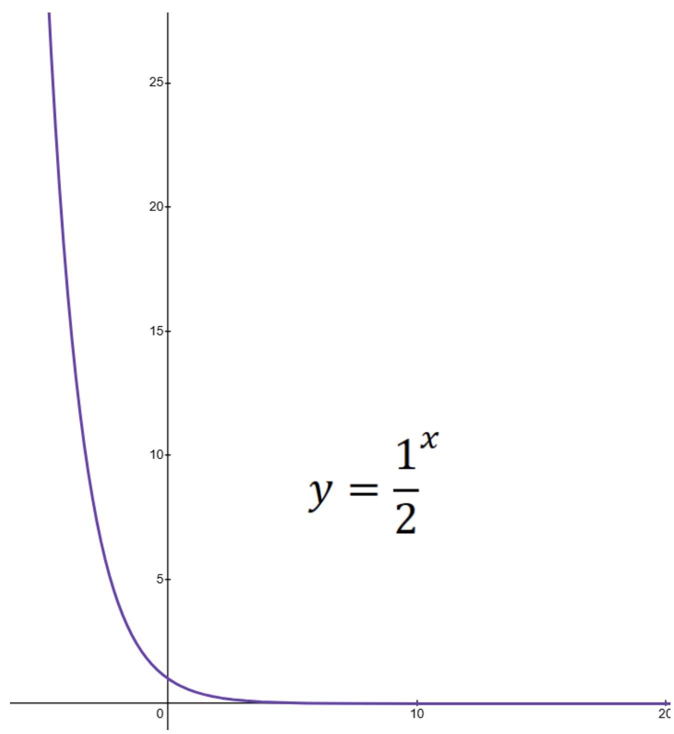
The Rules of Exponential Function

1. **Positive Base (but not 1).**
   The base b must be a positive number, and it can't be 1. (Why not 1? Because $\mathbf{1^x}$ is always just 1—no real "exponential" behavior there!)
2. **The Exponent Is the Variable.**
   In an exponential function, the unknown (or variable) is what sits in the exponent. That's what makes it "exponential" rather than "polynomial."
3. **Constant Base.**
   We keep the base b fixed. That means b doesn't change with x.
4. **The Output Is Always Positive.**
   No matter what x is (positive, negative, or zero), $\boldsymbol{b^x}$ will always come out to be a positive number (greater than 0).

Key Features of the Graph $y = b^x$

1. **Domain:** all real numbers ($-\infty$ to ∞).
2. **Range:** all positive real numbers $(\mathbf{0},\ \infty)$.
3. **Intercept:** It always crosses the y-axis at $\boldsymbol{y = 1}$ (Because $\boldsymbol{b^0 = 1}$)
4. **No x-intercept** (the curve never touches or crosses the x-axis since $\boldsymbol{b^x}$ never reaches 0).
5. **Horizontal Asymptote:** $\boldsymbol{y = 0}$. The graph gets super close to the x-axis but never actually touches it.
6. **Growing or shrinking?**
    - **If $\boldsymbol{b > 1}$: the function *increases* (rises) as x increases.**
    - **If $\boldsymbol{0 < b < 1}$: the function *decreases* (falls) as x increases.**





There is a special irrational number $e \approx 2.718281827$. When we use e as the base, the function is called the **natural exponential function**:

$$f(x) = e^x$$

It pops up in many real-life applications (continuous growth, population models, continuously compounded interest, and so on).

Transformations: Shifting, Stretching, & Reflecting

Just like other functions, exponentials can be **shifted**, **stretched**, or **reflected**. The transformations usually appear inside or outside the exponent:

1. Vertical shifts:
- $g(x) = b^x + c$ shifts the graph up by c
- $g(x) = b^x - c$ shifts the graph down by c
2. Horizontal shifts:
- $g(x) = b^{x+c}$ shifts the graph left by c
- $g(x) = b^{x-c}$ shifts the graph right by c
3. Reflections:
- $g(x) = -b^x$ flips the graph upside down (reflect across the x- axis)
- $g(x) = b^{-x}$ flips it left to right (reflect across the y-axis)
4. Stretching or Shrinking:
- $g(x) = c * b^x$ (Vertical stretch if $c > 1$, shrink if $0 < c < 1$)
- $g(x) = b^{cx}$ (Horizontal shrink if $c > 1$, stretch if $0 < c < 1$)

Exponential Growth vs. Decay

- **Growth:** If $b > 1$ the function b^x grows (gets larger) as x increases.
- **Decay:** If $0 < b < 1$, the function b^x decays (gets smaller) as x increases.

Real-World Example: Compound Interest

One of the most common applications of exponentials is **compound interest** in finance. If you invest P dollars at an annual interest rate r (in decimal form) for t years, compounded n times per year, the balance A is:

$$A = P\left(1 + \frac{r}{n}\right)^{nt}$$

Continuous Compounding uses the special base e:

$$A = Pe^{rt}$$

Tips & Tricks

1. **Get comfortable with $\boldsymbol{b^0 = 1}$.** This always gives the y-intercept in exponential functions.
2. **Remember the horizontal asymptote:** y=0. Exponential graphs approach the x-axis but never touch it.
3. **Check the base first:**
   - If $b > 1$, the graph is *increasing*.
   - If $0 < b > 1$, the graph is *decreasing*.
4. **Be mindful of transformations:** A small shift inside the exponent or outside can move the graph around or flip it.
5. **Calculator tip:** Use the ∧, or specifically the $\boldsymbol{ln}$ and $\boldsymbol{e^x}$ buttons when dealing with e.
6. **Negative Exponents:** $\boldsymbol{b^{-x}}$ is just $\boldsymbol{\frac{1}{b^x}}$. It flips the graph left to right if you rewrite it, but crucually it stays above the x-axis

That's it! Exponential functions are all about having your variable in the exponent and watching how quickly (or slowly) the function zooms up or slides down. Just follow the "positive base" rule, look for that (0,1) anchor, and remember the horizontal asymptote along $y = 0$

Logarithmic Function

A **logarithm** is just another way to write an **exponent**.

Basic Idea:

A logarithm **answers the question**:

"What exponent do I put on this base to get that number?"

Example

$$\log_2 8 = 3$$

This means:

"What power do I raise 2 to, in order to get 8?"

Since $2^3 = 8$, the answer is **3**.

How Logs and Exponents Are Related

A logarithm is the **inverse (opposite)** of an exponent.

Two Ways to Write the Same Thing

1. Exponential Form → $\boldsymbol{b^y = x}$
2. Logarithmic Form → $\boldsymbol{\log_b x = y}$

Example

- $10^2 = 100$ (Exponential Form)
- 100=$\log_{10} 100 = 2$ (Logarithmic Form)

Special Types of Logarithms

There are two very common types of logarithms:

1. Common Logarithm (Base 10)
   - **Written as $\boldsymbol{\log x}$** (no base written = base 10)
   - **Example:** $\log 1000 = 3$ (since $10^3 = 1000$)
2. Natural Logarithm (Base e)
   - **Written as $\boldsymbol{\ln x}$** (log base e, where $e \approx 2.718$)
   - **Example:** $\ln e^4 = 4$

Important Log Properties (Must-Know Rules!)

Logs follow some important rules that make solving easier.
Basic Properties

Property	**Explanation**
$\log_b b = 1$	Any base raised to 1 is itself. Example: $\log_3 3 = 1$
$\log_b 1 = 0$	Any base raised to 0 equals 1. Example: $\log_5 1 = 0$
$\log_b b^x = x$	Log and exponent cancel. Example: $\log_2 2^5 = 5$
$b^{\log_b x} = x$	Base raised to a log its canceled. Example: $3^{\log_3 7} = 7$

How to Solve Log Equations

Example 1: Converting to Exponential Form

Solve for x:

$$\log_5 x = 3$$

Step-by-Step:

1. Convert to exponential form:

$$5^3 = \mathrm{x}$$

2. Simplify:

$$\mathrm{x} = 125$$

Final Answer: $x = 125$

Log Transformations (Shifting, Stretching, and Flipping)

Just like any function, logarithmic functions can be **moved**, **stretched**, and **flipped**.

Transformation	Equation	Effect
Vertical Shift	$f(x) = \log_b x + c$ $f(x) = \log_b x - c$	• Moves the graph up • Moves the graph down
Horizontal Shift	$f(x) = \log_b(x + c)$ $f(x) = \log_b(x - c)$	• Moves the graph right Vertical Asymptotes: $x = c$ • Moves the graph left Vertical Asymptote: $x = -c$
Reflection	$f(x) = -\log_b x$ $f(x) = \log_b(-x)$	• Reflect about the x-axis • Reflect about the y-axis
Vertical Stretch Vertical Shrink	$f(x) = c \log_b x$	• Stretch when $c > 1$ • Shrink when $0 < c < 1$
Horizontal Stretch Horizontal Shrink	$f(x) = \log_b(cx)$	• Stretch when $0 < c < 1$ • Shrink when $c > 1$

Tip:

The graph of $\log_b x$ always has a **vertical asymptote** at $x = 0$, meaning it **never crosses the y-axis**.

Finding the Domain of a Logarithm
REMEMBER:

A logarithm is **undefined for negative numbers or zero**.
You can **only** take the log of **positive numbers**!

Tip:
To find the domain of $\log_b(x - c)$:

1. Set the inside **greater than 0**:
$$x - c > 0$$
2. Solve for x.

Example
Find the domain of $\mathrm{h}(x) = \log_4(x - 8)$.
Step-by-Step:

1. Set inside > 0: $x - 8 > 0$
2. Solve: $x > 8$

Final Answer (in Interval Notation): $(8, \infty)$

Solving Log Equations
Sometimes, you'll need to **get rid of a log** to solve for x. Here's how:
Convert to Exponentials
Example: Solve $\log_3(x - 2) = 4$
Step-by-Step:

1. Convert to exponential for $3^4 = x - 2$
2. Simplify: $81 = x - 2$
3. Solve for x: $x = 83$

Common Logarithm Mistakes
Avoid these mistakes!

- **You can't take the log of zero or a negative number!**
(Example: $\log_2(-5)$ is **undefined**.)
- **Logs do NOT distribute over addition or subtraction!**

$$\log(a + b) \neq \log a + \log b$$

Properties of Logarithms

Logarithms follow **rules** that make solving and simplifying log expressions much easier. These rules come from exponent properties. Let's go step by step.

The Product Rule:

$$\log_b(MN) = \log_b M + \log_b N$$

Explanation:

- When **multiplying** numbers inside a logarithm, you can **split** them into a **sum of two logs**.
- This comes from the exponent rule: $b^m * b^n = b^{m+n}$

Example:

$$\log_2(xy) = \log_2 x + \log_2 y$$

Tip: Use this when **expanding** a logarithm.

The Quotient Rule:

$$\log_b\left(\frac{M}{N}\right) = \log_b M - \log_b N$$

Explanation:

- When **dividing** numbers inside a logarithm, you can **split** them into a **difference of two logs**.
- This comes from the exponent rule: $\frac{b^m}{b^n} = b^{m-n}$

Example:

$$\log_3\left(\frac{b}{a}\right) = \log_3 a - \log_3 b$$

Tip: Use this when **expanding** a logarithm.

The Power Rule

$$\log_b(M^p) = p log_b M$$

Explanation:

- When **raising a number inside a log to a power**, you can **pull the exponent in front**.
- This comes from the exponent rule: $(b^m)^n = b^{m*n}$

Example:

$$\log_5(x^2) = 2\log_5 x$$

Tip:

Use this when **expanding** a logarithm.

Expanding Logarithmic Expressions

When we **break down** a logarithm into separate parts using these rules, it's called **expanding**.

Example: Expand $\ln(\frac{x^3(y+1)^5}{z^2})$

Step-by-Step:

1. **Use Quotient Rule**

$$\ln\left(\frac{x^3(y+1)^5}{z^2}\right) = \ln\left(\frac{x^3\ (y+1)^5}{z^2}\right) = \ln(x^3(y+1)^5) - \ln(z^2))$$

2. **Use Product Rule**

$$\ln(x^3) + \ln(y+1)^5 - \ln(z^2)$$

3. **Use Power Rule**

$$3\ln(x) + 5\ln(y+1) - 2\ln(z)$$

Final Answer:

$$3\ln(x) + 5\ln(y+1) - 2\ln(z)$$

Condensing Logarithmic Expressions

Condensing means **combining multiple logs** into a **single log**.

Example: Condense $\frac{1}{5}[2\ln(x+8) - \ln x - \ln(x^2-16)]$

Step-by-Step:

1. **Use Power Rule** $\frac{1}{5}[\ln(\mathrm{x}+8)^2 - \ln(\mathrm{x}) - \ln(\mathrm{x}^2-16)$
2. **Use Quotient Rule**: $\frac{1}{5}\ln(\frac{(x+8)^2}{x(x^2-16)})$
3. **Use Power Rule**: $\sqrt[5]{\ln(\frac{(\mathrm{x}+8)^2}{\mathrm{x}(\mathrm{x}^2-16)})}$

Final Answer:

$$\sqrt[5]{\ln\left(\frac{(x+8)^2}{x(x^2-16)}\right)}$$

The Change-of-Base Formula

$$\log_b M = \frac{\log M}{\log b}$$

Explanation:

- This allows you to **convert a log from one base to another**.
- Most calculators only have **base 10** (log) and **base e** (ln), so we use this to evaluate logs in other bases.

Example: Convert $\log_5 43$ to common logarithm (base 10).

$$\log_5 43 = \frac{\log 43}{\log 5}$$

Tip: The **original base b goes on the bottom** of the fraction, just like the **base of a pyramid** is at the bottom.

Exponential/Logarithmic Functions

- **Exponential functions** grow (or shrink) really fast.
  Example: $f(x) = 2^x$ (keeps doubling!)
- **Logarithmic functions** are the opposite of exponentials.
  Example: $g(x) = \log_2 x$ (answers: "What exponent gives me x?")

They are **inverse functions** of each other:

$$\log_b b^x = x$$

$$b^{\log_b x} = x$$

Exponential Functions

An exponential function has the form:

$$f(x) = a \cdot b^x$$

where:

- a = starting value
- b = growth (if $b > 1$) or decay (if $0 < b < 1$)
- x = exponent (causes rapid changes!)

Examples:

1. **Growth**: $f(x) = 3(2^x)$ (doubles every step!)
2. **Decay**: $g(x) = 5(0.5^x)$ (shrinks by half every step!)

Graph Facts:

- Always **above the x-axis**.
- **Increases** if $b > 1$, **decreases** if $0 < b < 1$.
- **Horizontal asymptote** at $y = 0$

Logarithmic Functions

A logarithm **undoes** an exponent. It has the form:

$$f(x) = \log_b x$$

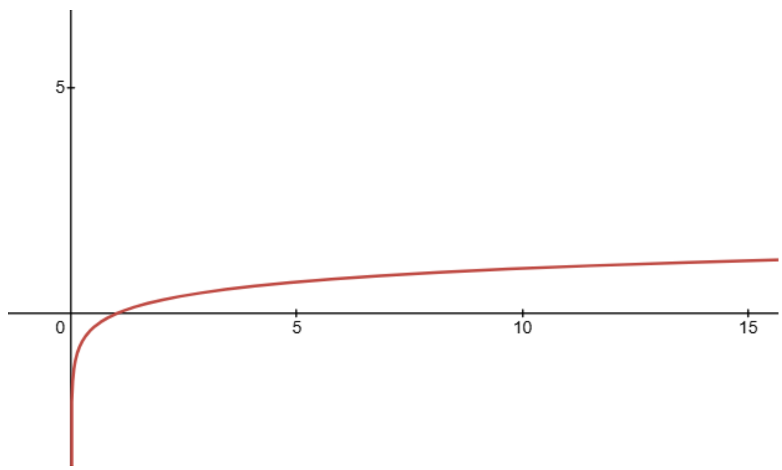
which means: $b^y = x$(Rewriting in exponential form)

Examples

1. $\log_2 8 = 3$ (Because $2^3 = 8$)
2. $\ln(e^5) = 5$ (Because $e^5 = e^5$)

Graph Facts:

- Always **to the right of the y-axis**.
- **Increases slowly** (opposite of exponentials).
- **Vertical asymptote** at $x = 0$



Solving Exponential Equations

Method 1: Same Base Rule

If both sides have the same base:

$$b^M = b^N \Rightarrow M = N$$

Example: Solve $2^{x-1} = 2^4$

Step by step:

1. Since bases are the same, set exponents equal: $x - 1 = 4$
2. Solve for x: $x = 5$

Method 2: Using Logarithms

If bases are different, use **logs**:

$$b^x = M \Rightarrow x = \log_b M$$

Example: Solve $3^x = 20$

Step by step:

1. Take **log** on both sides:

$$\log 3^x = \log 20$$

2. Use **power rule** ($\log_b M^p = plog_b M$):

$$x \cdot \log 3 = \log 20$$

3. Solve for x:

$$x = \frac{\log 20}{\log 3}$$

4. Use a calculator:

$$x \approx 2.73$$

Tip: Use **ln** if base is **e**.

Solving Logarithmic Equations

Method 1: Convert to Exponential Form

If you have $\log_b M = c$, rewrite as:

$$M = b^c$$

Example: Solve $\log_5 x = 3$

Step by step:

1. Convert to exponent form:

$$x = 5^3$$

2. Solve:

$$x = 125$$

Method 2: One-to-One Property

If both sides have logs with the same base:

$$\log_b M = \log_b N \Rightarrow M = N$$

Example: Solve $\log_3(x+1) = \log_3 5$

Step by step:

1. Remove logs: $x + 1 = 5$
2. Solve for x: $x = 4$

Tip: Always check if $x > 0$ (logs can't have negatives!).

The Change-of-Base Formula

Used for logs when the base isn't **10** or **e**.

$$\log_b M = \frac{\log M}{\log b}$$

Example: Find $\log_4 23$

Step by step:

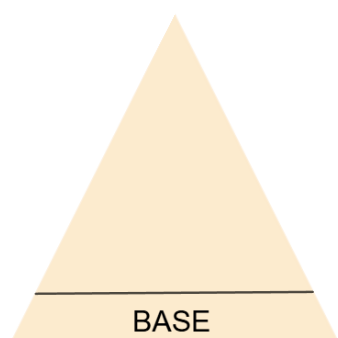
1. Use the formula:

$$\log_4 23 = \frac{\log 23}{\log 4}$$

2. Use a calculator:

$$\approx 2.54$$

Tip: Think of the base going **to the bottom** like a pyramid.



Applications of Exponential & Log Functions

Exponential Growth & Decay

$$A(t) = A_0 e^{rt}$$

where:

- A_0= starting amount
- r = growth/decay rate
- t = time

Example: A bacteria culture grows at $r = 0.2$. If there were 100 bacteria, how many after 5 hours?

Step by step:

$$A(5) = 100e^{(0.2)(5)}$$
$$A(5) \approx 271.83$$

Tip: Use **ln** to solve for time!

Logarithmic Models

Used for things that **slow down** over time, like:

- Sound intensity
- Earthquake magnitude
- Memory recall

Example: A student remembers 90% of a lecture but forgets overtime:

$$A(x) = 90 - 25\log_2 x$$

After how many days do they remember 80%?

Step by step:

1. Set A(x)=80: $$80 = 90 - 25\log_2 x$$

2. Solve for x:

$$-10 = -25\log_2 x$$

$$\frac{-10}{-25} = \log_2 x \Rightarrow \frac{2}{5} = \log_2 x$$

3. Convert to exponent form:

$$x = 2^{\frac{2}{5}} \approx 1.3$$

Final Answer: **1.3 days**.

Quick Recap & Tips

- Exponentials grow FAST; logs grow SLOW.
- Exponentials & logs are inverses—undo each other.
- Convert exponentials to logs for solving.
- Check if your final answer is valid! (logs can't be negative).
- Use change-of-base formula for weird bases.

Exponential Growth and Decay

Exponential functions are used to model **growth** (things increasing rapidly) and **decay** (things decreasing over time). These models appear in finance, population studies, radioactive decay, and more.

The General Formula

The formula for **exponential growth and decay** is:

$$A = A_0 e^{kt}$$

where:

- A_0 = the **starting amount** (initial value)
- A = the **amount at time t**
- e = Euler's number (≈ 2.718)
- k = the **growth** ($k > 0$) or **decay** ($k < 0$) rate
- t = time

Exponential Growth

Occurs when something **increases rapidly** over time.

Formula:

$$A = A_0 e^{kt} \text{ where } k > 0$$

Example (Population Growth):

A city has 100,000 people and grows at a rate of 2% per year. Find the population after 5 years.

Solution:

1. Identify values:
$$A_0 = 100{,}000 \qquad k = 0.02 \qquad t = 5$$
2. Plug into the formula:
$$A = 100000e^{(0.02)(5)}$$
3. Solve using a calculator:
$$A \approx 110{,}517$$

Final Answer: 110,517 people after 5 years.

Exponential Decay
Occurs when something **decreases over time** (losing mass, energy, or quantity).

$$A = A_0 e^{kt}, \text{ where } k < 0$$

Example (Radioactive Decay):
A fossil contains 10 grams of Carbon-14. Carbon-14 has a **half-life** of 5730 years, meaning it decays by half every 5730 years. Find the decay rate.

Solution:

1. Use the **half-life formula**:
$$\frac{1}{2}A = A_0 e^{(k)(5730)}$$
2. Divide both sides by A:
$$\frac{1}{2} = e^{(5730)(k)}$$
3. Take the **natural log (ln)** on both sides:
$$\ln\left(\frac{1}{2}\right) = 5730k$$
4. Solve for k:
$$k = \frac{\ln(\frac{1}{2})}{5730} \approx -0.000121$$

Decay Rate: $k \approx -0.000121$ **per year.**

Logistic Growth Formula

The general formula for logistic growth is:

$$P(t) = \frac{C}{1+ae^{-bt}}$$

where:

- P(t)= population (or amount) at time t
- C = **carrying capacity** (maximum possible population)

- a = constant related to the **initial population size**
- b = growth rate
- t = time

Key Fact: As $t \to \infty$, $P(t) \to C$ (it **approaches** the carrying capacity but never exceeds it).

Example

Virus Spread Model

A logistic model describes the spread of the flu in a community. The function:

$$P(t) = \frac{105000}{1+4200e^{-t}}$$

models the number of infected people, where:

- **C=105,000** (max number of infected people)
- **t** = weeks after the outbreak

Example Questions

(a) How many people were infected at the beginning?

$$P(0) = \frac{105000}{1 + 4200e^{-0}} \approx 25\, people$$

(b) How many people were infected after 4 weeks?

$$P(4) = \frac{105000}{1 + 4200e^{-4}} \approx 1347$$

(c) What is the maximum number of people that will be infected?

The carrying capacity is C=105,000

Quick Recap & Tips

- Use $A = A_0e^{kt}$ for continuous growth/decay.
- If the problem says, “limited growth,” use the logistic model.
- To solve for time, take the natural logarithm (ln) of both sides.
- Check your decay rate k (it should be negative).

Trigonometric Functions

Angles and Radians

- An **angle** is formed by two rays that share a common endpoint (called the **vertex**).
- One ray is called the **initial side**, and the other is the **terminal side**.

Standard Position of an Angle

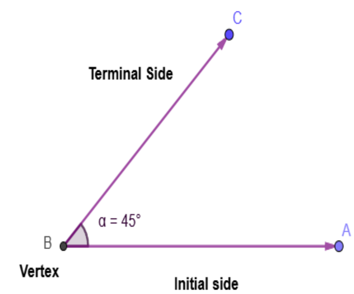
An angle is in **standard position** if:

- Its **vertex is at the origin** of the coordinate plane.
- Its **initial side lies along the positive x-axis**.

Tip: If an angle isn't in a standard position, you can **rotate it or shift it to start at the x-axis** to make calculations easier.

Key Parts of an Angle:

- **Initial Side**: Where the angle starts.
- **Terminal Side**: Where the angle ends after rotation.
- **Vertex**: The common point where the two rays meet.



Tip: Think of a clock! If the hand moves from **12 to 2**, the ray at **12 is the initial side**, and the ray at **2 is the terminal side**.

Measuring Angles

Angles can be measured in **degrees (°)** or **radians**.

Degrees

- A full circle is **360°**.
- A right angle is **90°**.
- A straight angle is **180°**.

Tip:

A **right angle (90°)** is the one that looks like an **L**, and an **acute angle** (less than 90°) is like a "cute" little angle.

Measuring Angles Using Radians

- A radian measures an angle based on the length of the arc it creates.
- 1 radian = the angle where the arc length equals the radius.

Formula:

$$\theta = \frac{s}{r}$$

where:

- **S** = arc length
- **r** = radius
- **θ** = angle in radians

Key Fact: A full circle is **2π radians**, which is the same as **360°**.

Relationship Between Degrees and Radians

Since a full circle is **360°** or **2π radians**, we get the conversion:

$$180° = \pi\, radians$$

Conversion Formulas

To convert **degrees to radians**:

$$\theta \times \frac{\pi}{180}$$

To convert **radians to degrees**:

$$\theta\, rad \times \frac{180}{\pi}$$

Examples:

- Convert 90° to radians:

$$90° \times \frac{\pi}{180} = \frac{\pi}{2}$$

- Convert $\frac{3\pi}{4}$ radians to degrees:

$$\frac{3\pi}{4} \times \frac{180}{\pi} = 135°$$

Tip to Remember: π radians = 180°, so $\frac{\pi}{2}$ is 90°, $\frac{\pi}{4}$ is 45°, and so on.

Coterminal Angles

Coterminal angles are angles that **end at the same position but have different values**.

Formula:

$$\theta \pm 360°k \ \ \text{(for degrees)}$$
$$\theta \pm 2\pi k \ \ \text{(for radians)}$$

where k is any integer.

Example:

- **Find a coterminal angle for -45°**:
$$-45° + 360° = 315°$$
- **Find a coterminal angle for $\frac{\pi}{3}$ radians**:
$$\frac{\pi}{3} + 2\pi = \frac{7\pi}{3}$$

Tip:

- If an angle is **too big (> 360° or > 2π radians)**, **subtract** 360° or 2π to get a smaller equivalent angle.
- If an angle is **negative**, **add** 360° or 2π to make it positive.

Arc Length Formula

Arc length (s) is the distance along a circle’s edge.

Formula:

$$s = r\theta$$

where:

- **s** = arc length
- **r** = radius
- **θ** = angle in radians

Example:

A circle has a **radius of 10 inches**. How far does the tip of the minute hand move from **12 to 1 o’clock**?

A clock is divided into **12 sections**, so each section is $\frac{\pi}{6}$ **radians**.

$$s = 10 \times \frac{\pi}{6} = \frac{10\pi}{6} = \frac{5\pi}{3}$$

If we approximate:

$$\frac{5\pi}{3} \approx 5.24$$

Tip:

- **Always use radians** in this formula! Convert degrees to radians first if needed.

Linear and Angular Speed

When something moves in a **circular** path, it has:

- **Linear speed**: How fast it moves **along** the circle.
- **Angular speed**: How fast it **rotates**.

Formulas:

Linear Speed:

$$v = \frac{s}{t}$$

Angular Speed:

$$\omega = \frac{\theta}{t}$$

Linear Speed (In terms of Angular Speed)

$$v = r\omega \quad \left[v = \frac{s}{t} = \frac{r\theta}{t} = r\left(\frac{\theta}{t}\right) = r\omega\right] \; \boldsymbol{Remember\ s\ arc\ length\ r\theta}$$

Example:

If a wheel completes **1 full turn (2π radians) in 10 seconds**, the angular speed is:

$$\omega = \frac{2\pi}{10} = \frac{\pi}{5}\frac{rad}{sec}$$

Example:
A water wheel rotates **25 times per minute**, with a **radius of 25 feet**. The linear speed is what?

Well first we must convert revolutions per minute (rpm) into angular speed

$$\omega = \text{revolutions per minute} \times \text{radians per minute}$$
$$\omega = 25 \times 2\pi = 50\pi \text{ radians per minute}$$

Linear speed is:

$$v = r \times \omega$$
$$v = 25 \times 50\pi \approx 3927 \text{ feet per minute}$$

Quick Recap & Tip:

- More rotations = Higher speed
- Bigger radius = Higher linear speed (since it travels a longer distance per rotation).
- Linear speed is about distance traveled.
- Angular speed is about how fast the angle changes.

Degrees vs. Radians:

- Use degrees for common angles (90°, 180°, etc.).
- Use radians for circle-based calculations (arc length, speed).

Conversion Cheat Sheet:

- $180° = \pi$ radians
- Multiply by $\frac{\pi}{180}$ to go from **degrees → radians**.
- Multiply by $\frac{180}{\pi}$ to go from **radians → degrees**.

Coterminal Angles:

- Add/subtract **360°** (or **2π** radians) to find them.

Arc Length:

- **Use** $s = r\theta$, but **theta must be in radians!**

Speed:

- Linear speed: $v = \frac{s}{t}$
- Angular speed: $\omega = \frac{\theta}{t}$

Right Triangle Trigonometry

Trigonometry focuses on the **relationships between the angles and sides** of triangles. In **right triangle trigonometry**, we use **trigonometric ratios** to describe these relationships.

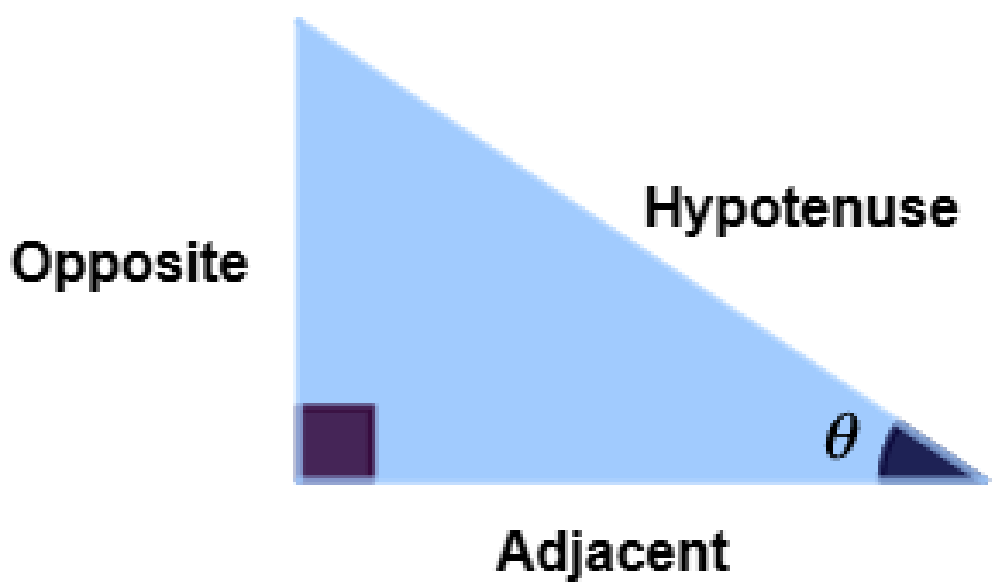
Key Parts of a Right Triangle

Every right triangle has:

1. **Hypotenuse** – The longest side, **always opposite the right angle**.
2. **Opposite Side** – The side **across** from the given angle **θ (theta)**.
3. **Adjacent Side** – The side **next to** the given angle **θ**, but NOT the hypotenuse.

Tip to Remember:

- **Hypotenuse = the longest side**
- **Opposite = across from the angle**
- **Adjacent = next to the angle (but not the hypotenuse)**



The Six Trigonometric Functions

Trigonometry uses **ratios of the sides** to define six functions:

Function	Abbreviation	Formula
Sine	$\sin(\theta)$	$\frac{\text{Opposite}}{\text{Hypotenuse}}$
Cosine	$\cos(\theta)$	$\frac{\text{Adjacent}}{\text{Hypotenuse}}$
Tangent	$\tan(\theta)$	$\frac{\text{Opposite}}{\text{Adjacent}}$
Cosecant	$\csc(\theta)$	$\frac{\text{Hypotenuse}}{\text{Opposite}}$
Secant	$\sec(\theta)$	$\frac{\text{Hypotenuse}}{\text{Adjacent}}$
Cotangent	$\cot(\theta)$	$\frac{\text{Adjacent}}{\text{Opposite}}$

SOH-CAH-TOA Mnemonic

A great way to remember **sine, cosine, and tangent** is:

- **S**ine = **O**pposite / **H**ypotenuse (**SOH**)
- **C**osine = **A**djacent / **H**ypotenuse (**CAH**)
- **T**angent = **O**pposite / **A**djacent (**TOA**)

Tip:

Think of it as a **fun phrase**:
"Some Old Horses Can't Always Hide Their Old Age"

Reciprocal Trigonometric Functions

Each trig function has a **reciprocal** function:

Function	Reciprocal Function
$\sin(\theta)$	$\csc(\theta) = \dfrac{1}{\sin(\theta)}$
$\cos(\theta)$	$\sec(\theta) = \dfrac{1}{\cos(\theta)}$
$\tan(\theta)$	$\cot(\theta) = \dfrac{1}{\tan(\theta)}$

Tip to Remember:

- Cosecant (csc) = 1/sin
- Secant (sec) = 1/cos
- Cotangent (cot) = 1/tan

  These are **just flipped versions** of sine, cosine, and tangent!

Pythagorean Theorem

To find a missing side of a right triangle, we use:

$$a^2 + b^2 = c^2$$

Where:

- **a** and **b** are the legs,
- **c** is the hypotenuse.

Example:

Find the missing side when:

- One leg = **6**
- Hypotenuse = **10**

$$6^2 + b^2 = 10^2$$
$$36 + b^2 = 100$$
$$b^2 = 64$$
$$b = 8$$

Answer: The missing side is 8.

Pythagorean Identities

These are special formulas based on the **Pythagorean Theorem**, used to relate **sine, cosine, and tangent**:

1. Main Identity:

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

    - This equation is always true for any angle.

2. **Other Forms:**
    - Divide by $\cos^2(\theta)$:

$$1 + \tan^2(\theta) = \sec^2(\theta)$$

    - Divide by $\sin^2(\theta)$:

$$1 + \cot^2(\theta) = \csc^2(\theta)$$

Tip to Remember:

- The first equation ($\sin^2 + \cos^2 = 1$) is the most important!
- To get the other two, divide by $\sin^2$ or $\cos^2$.

Quotient Identities

These identities express **tangent** and **cotangent** using sine and cosine:

1. Tangent in terms of sine and cosine:
   $\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$
2. Cotangent in terms of sine and cosine:
   $\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$

Tip to Remember:

- tan = sin/cos
- cot = cos/sin
  Think of tangent as **rise/run** (sine over cosine)!

Cofunction Identities

Cofunctions show how **one trig function is related to another when angles are complementary** (adding to **90°** or **π/2 radians**).

Function	**Cofunction**
$\sin(\theta)$	$\cos(90° - \theta)$
$\cos(\theta)$	$\sin(90° - \theta)$
$\tan(\theta)$	$\cot(90° - \theta)$
$\cot(\theta)$	$\tan(90° - \theta)$
$\sec(\theta)$	$\csc(90° - \theta)$
$\csc(\theta)$	$\sec(90° - \theta)$

Tip to Remember:

- Sine and Cosine are cofunctions
- Tangent and Cotangent are cofunctions
- Secant and Cosecant are cofunctions
  Think of cofunctions as "partner functions" that swap at 90°.

Example:
Find a cofunction identity for **sin (18°)**.
Since $\sin(\theta) = \cos(90° - \theta)$

$$\sin(18°) = \cos(90 - 18)$$
$$\sin(18°) = \cos(72°)$$

Angles of Elevation and Depression
These are **real-world applications** of right triangle trig.

- **Angle of Elevation** – The angle **from the ground up** to an object.
- **Angle of Depression** – The angle **from above an object down**.

Example:
A **ladder** leans against a wall, reaching **10 feet** up. The ladder forms a **60° angle** with the ground. How long is the ladder?

Using cosine:

$$\cos(60) = \frac{10}{\mathrm{h}}$$

$$\mathrm{h} = \frac{10}{\cos(60)} = 20$$

The ladder is 20 feet long

Trigonometric Functions of Any Angle

In the previous sections, we focused on right triangles where acute angles are (less than 90°). But what about angles that go beyond 90° or even full revolutions? That's where **trigonometric functions of any angle** come in.

Instead of restricting trigonometry to right triangles, we extend the six trigonometric functions to **any angle** by placing them in the **coordinate plane**.

Key Idea: Trigonometry in the Coordinate Plane

Each point on the terminal side of an angle can be represented as **P(x,y)** where:

- x is the **horizontal distance** (adjacent side in a right triangle),
- y is the **vertical distance** (opposite side in a right triangle),
- r is the **hypotenuse**, calculated as $r = \sqrt{x^2 + y^2}$

Using this setup, we redefine the trigonometric functions:

Function	Definition in a Right Triangle	Extended Definition in the Coordinate Plane
$\sin(\theta)$	$\frac{opp}{\text{hyp}}$	$\sin(\theta) = \frac{y}{r}$
$\cos(\theta)$	$\frac{adj}{\text{hyp}}$	$\cos(\theta) = \frac{x}{r}$
$\tan(\theta)$	$\frac{opp}{adj}$	$\tan(\theta) = \frac{y}{x}$ (undefined if x=0)
$\csc(\theta)$	$\frac{\text{hyp}}{opp}$	$\csc(\theta) = \frac{r}{y}$ (undefined if y=0)
$\sec(\theta)$	$\frac{\text{hyp}}{adj}$	$\sec(\theta) = \frac{r}{x}$ (undefined if x=0)
$\cot(\theta)$	$\frac{adj}{opp}$	$\cot(\theta) = \frac{x}{y}$ (undefined if y=0)

Tip to Remember:

- **Think of trigonometric functions as ratios of coordinates and distances.**
- r (the hypotenuse) is **always positive**, but x and y can be **positive or negative**, depending on the quadrant.

Signs of Trigonometric Functions in Different Quadrants

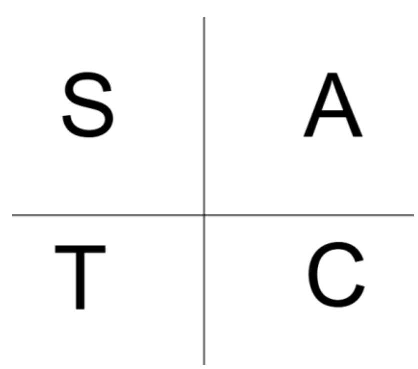
The **sign of a trigonometric function** depends on the quadrant in which the angle lies.

Quadrant	Positive Functions
$I\ (0° \ to\ 90°)$	All functions are positive
$II\ (90° to\ 180°)$	Sin and csc are positive
$III\ (180° \ to\ 270°)$	Tan and cot are positive
$IV\ (270° \ to\ 360°)$	Cos and sec are positive

Mnemonic to Remember:

"All Students Take Calculus"

- **A** – **All** functions positive in Quadrant I
- **S** – **Sine** and its reciprocal csc are positive in Quadrant II
- **T** – **Tangent** and its reciprocal cot are positive in Quadrant III
- **C** – **Cosine** and its reciprocal sec are positive in Quadrant IV



Reference Angles: **Making Any Angle Manageable**

A **reference angle** is the **smallest positive acute angle** (θ′) formed between the terminal side of an angle and the x-axis.

Quadrant	Formula for Reference Angle θ'
$I\ (0° \ to\ 90°)$	$\theta' = \theta$
$II\ (90° to\ 180°)$	$\theta' = 180° - \theta$
$III\ (180°\ to\ 270°)$	$\theta' = \theta - 180°$
$IV\ (270°\ to\ 360°)$	$\theta = 360° - \theta$

Why Use Reference Angles?

- The **trigonometric function values of any angle** are the **same** as those of their reference angle, except for the **sign**.
- It helps simplify calculations without memorizing a lot of values.

Example: Find the Reference Angle of 210°

- 210° is in **Quadrant III**.
- Use the formula:

$$\theta' = 210° - 180° = 30°$$

- So, the reference angle is **30°**.

Coterminal Angles: Same Angle, Different Rotations

Two angles are **coterminal** if they **end at the same position**. You find coterminal angles by **adding or subtracting 360° (or 2π radians)**.

Formula for Coterminal Angles:

$$\theta_{coterminal} = \theta \pm 360° \quad or\ \theta \pm 2\pi$$

Example: Find a Coterminal Angle of 75°

- Adding 360°:

$$75° + 360° = 435$$

- Subtracting 360°:

$$75° - 360° = -285$$

- So, **75°, 435°, and -285°** are all coterminal.

Evaluating Trigonometric Functions Using Reference Angles

1. Find the reference angle θ′
2. Find the trig function value for θ′
3. Use the quadrant to determine the correct sign.

Example: Evaluate sin 225°

- 225° is in **Quadrant III**.
- Reference angle: $255° - 180° = 45°$
- $\sin(45) = \frac{\sqrt{2}}{2}$, but **sine is negative in Quadrant III**.
- So,

$$\sin(225°) = -\frac{\sqrt{2}}{2}$$

Final Tips for Success

- Always **find the reference angle** first, it simplifies everything!
- **Memorize key values**: 30°, 45°, and 60° angles are **commonly used**.
- Use **ASTC** (All Students Take Calculus) to **remember the signs** of trig functions.
- Know how to **convert between degrees and radians**.
- **Coterminal angles** help when simplifying large angles.

Trigonometric Functions of Real Numbers

In the previous topics, we explored trigonometric functions for angles measured in degrees and radians. Now, we extend these functions to apply to **real numbers** using the **unit circle**. This allows us to study functions that **repeat periodically**, which is useful in modeling natural phenomena like sound waves, tides, and planetary motion.

The Unit Circle and Trigonometric Functions

A **unit circle** is a circle with a **radius of 1** centered at the **origin (0,0)** in the coordinate plane. The equation for this circle is:

$$x^2 + y^2 = 1$$

Every point P(x,y) on the unit circle corresponds to an **angle t (in radians)**, where:

- The **x-coordinate** represents **cosine**: $x = \cos(t)$
- The **y-coordinate** represents **sine**: $y = \sin(t)$

This means that the six trigonometric functions can be defined as:

$$\sin(t) = y, \cos(t) = x, \tan(\mathrm{t}) = \frac{\mathrm{y}}{\mathrm{x}}, \csc(\mathrm{t}) = \frac{1}{\mathrm{y}}, \sec(\mathrm{t}) = \frac{1}{\mathrm{x}}, \cot(\mathrm{t}) = \frac{\mathrm{x}}{\mathrm{y}}$$

NOTE: The tangent and secant functions are undefined when x = 0 , and the cotangent and cosecant functions are undefined when y=0)

Even and Odd Trigonometric Functions

Functions can be classified as **even** or **odd**, which tells us how they behave when we replace t with -t.

Function Type	Property	Examples
Even Function	$f(-t) = f(t)$	$\cos(-t) = \cos(t)\,, \sec(-t) = sect(t)$
Odd Function	$f(-t) = -f(t)$	$\sin(-t) = -\sin(t),$ $\tan(-t) = -tan(t),$ $\csc(-t) = -\csc(t),$ $\cot(-t) = -\cot(t)$

Why Does This Matter?
Even functions are **symmetrical** across the **y-axis**.
Odd functions have **rotational symmetry** around the **origin**.

Periodic Functions

A function is **periodic** if it **repeats itself** after a fixed interval, called the **period**. The trig functions **sin**, **cos**, **tan**, etc., are all periodic.

Sine and Cosine Periodicity: **(Period = 2π)**

$$\sin(t + 2\pi) = \sin(t), \qquad \cos(t + 2\pi) = \cos(t)$$

Tangent and Cotangent Periodicity: **(Period = π)**

$$\tan(t + \pi) = \tan(t), \qquad \cot(t + \pi) = \cot(t)$$

Real-Life Applications of Periodic Functions

Tides: Water levels rise and fall periodically.
Sound Waves: Frequencies in music follow periodic functions.
Signals & Circuits: Electrical signals in alternating current (AC) follow sine waves.

Repetitive Behavior of Trigonometric Functions

Since trig functions repeat, you can **add or subtract** multiples of 2π (or 360°) to get an equivalent angle:

$$\sin(t \pm 2\pi n) = \sin(t)$$
$$\cos(t \pm 2\pi n) = \cos(t)$$
$$\tan(t \pm \pi n) = \tan(t)$$

This means if you don't recognize an angle, **subtract or add multiples of 2π** until you get a familiar value.

How to Use the Unit Circle to Find Trigonometric Values

1. Locate the angle t on the **unit circle**.
2. Find the **coordinates** of the point where the terminal side meets the circle.
3. Use x and y to determine $\sin(t)$, $\cos(t)$, $\tan(t)$, etc.

Example

Find the trigonometric functions for $t = \frac{5\pi}{6}$:

- Find the reference angle: $\pi - \frac{5\pi}{6} = \frac{\pi}{6}$
- The coordinates for $\frac{5\pi}{6}$ are $\left(-\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$
- So:
    - $\cos\left(\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2}$
    - $\sin\left(\frac{5\pi}{6}\right) = \frac{1}{2}$
    - $\tan\left(\frac{5\pi}{6}\right) = -\frac{1}{\sqrt{3}}$

Tips & Tricks

- **Remember SOH-CAH-TOA** → It still applies when working with the unit circle!
- **Use the symmetry of the unit circle** → If you know the first quadrant values, you can figure out the rest!
- **For negative angles**, move **clockwise** instead of counterclockwise.
- **Use periodic properties** to simplify large angles by subtracting multiples of 2π.

Graphs of Sine and Cosine Functions

The sine and cosine functions are periodic, meaning they repeat their values in a predictable pattern. These functions are important in modeling waves, oscillations, and other real-world phenomena.

Graph of $y = \sin(x)$

The function $y = \sin(x)$ produces a wave-like curve known as a **sine wave**. It follows these key characteristics:

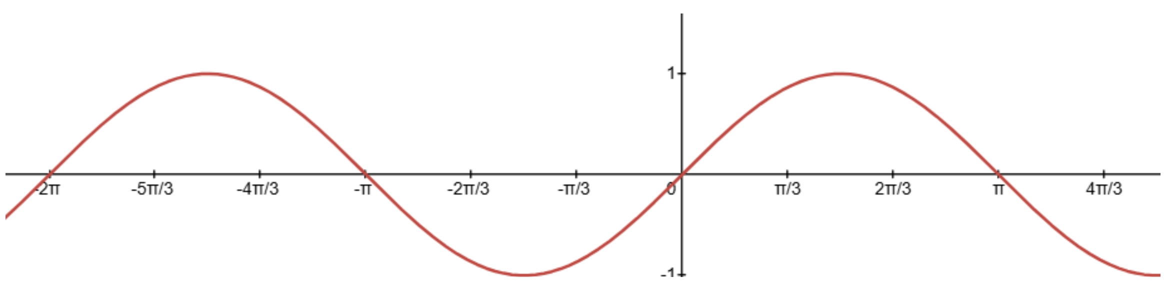
Key Properties of $y = \sin(x)$:

- **Domain**: $(-\infty, \infty)$ (all real numbers)
- **Range**: $[-1,1]$ (values never exceed 1 or go below -1)
- **Period**: 2π (the function repeats every 2π units)
- **X-intercepts**: Occur at $x = 0, \pi, 2\pi, \ldots$
- **Maximum points**: Occur at $x = \frac{\pi}{2}, \frac{5\pi}{2}, \ldots$
- **Minimum points**: Occur at $x = \frac{3\pi}{2}, \frac{7\pi}{2}, \ldots$
- **Odd Function**: Since $\sin(-x) = -\sin(x)$, the graph is symmetric about the origin.

The sine function starts at 0, reaches a maximum of 1 at $\frac{\pi}{2}$, returns to 0 at π, reaches a minimum of -1 at $\frac{3\pi}{2}$, and returns to 0 at 2π.

Table of Key Values for $y = \sin(x)$

X	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
Y	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	0	-1	0



The function $y = \cos(x)$ also produces a wave-like curve, like, $y = \sin(x)$ but shifted horizontally.

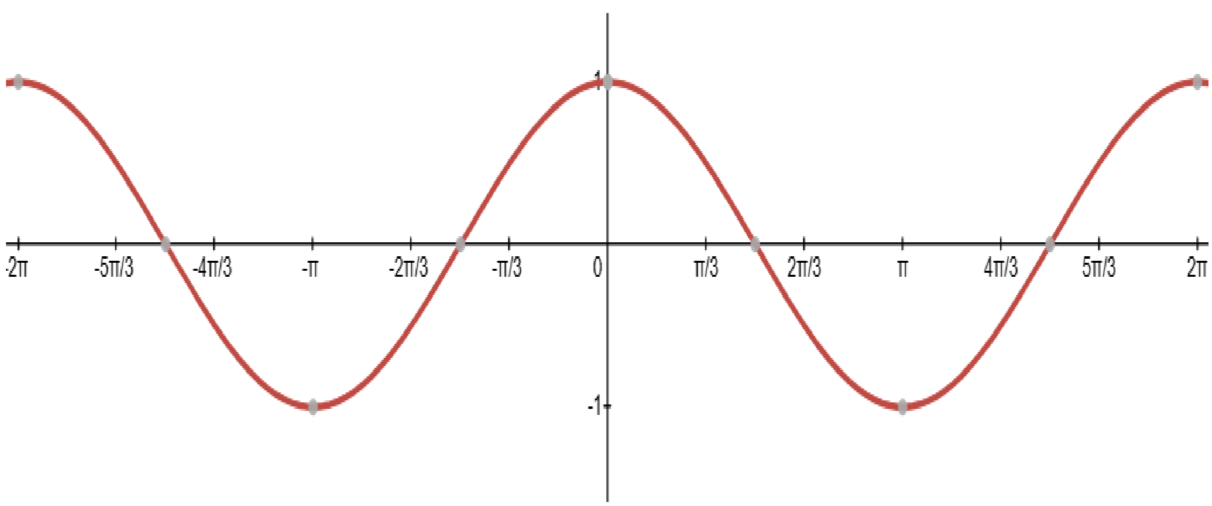
Key Properties of $y = \cos(x)$:

- **Domain**: $(-\infty, \infty)$(all real numbers)
- **Range**: $[-1,1]$ (values never exceed 1 or go below -1)
- **Period**: 2π (the function repeats every 2π units)
- **X-intercepts**: Occur at $x = \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$
- **Maximum points**: Occur a $x = 0, 2\pi, 4\pi, \ldots$
- **Minimum points**: Occur at $x = \pi, 3\pi, 5\pi, \ldots$
- **Even Function**: Since $\cos(-x) = \cos(x)$ the graph is symmetric about the y-axis.

The cosine function starts at 1, decreases to 0 at $\frac{\pi}{2}$, reaches a minimum of −1 at π, increases back to 0 at $\frac{3\pi}{2}$, and returns to 1 at 2π.

Table of Key Values for $y = \cos(x)$

X	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	π	$\frac{3\pi}{2}$	2π
Y	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	-1	0	1



Amplitude, Period, and Phase Shift

More general forms of sine and cosine functions include transformations:

Amplitude

The **amplitude** measures the height of the wave. It is given by the coefficient **A** in the function:

$$y = Asin(x) \quad or\, y = Acos(x)$$

- The amplitude is |A|.
- If $A > 1$, the wave stretches vertically.
- If $0 < A < 1$, the wave shrinks.
- If $A < 0$, the wave reflects over the x-axis.

Period

The **period** tells us how long it takes for the function to repeat. It is determined by **B** in the function:

$$y = \sin(Bx) \;\; or\, y = \cos(Bx)$$

- The period is found using the formula:

$$\text{Period} = \frac{2\pi}{B}$$

- If $B > 1$, the graph is compressed.
- If $0 < B < 1$, the graph stretches.

3. Phase Shift

The **phase shift** moves the graph left or right. It is determined by **C** in the function:

$$y = \sin(Bx - C)\, or\, y = \cos(\mathrm{Bx} - \mathrm{C})$$

- The phase shift is found using:

$$\text{Phase Shift} = \frac{C}{B}$$

- If $C > 0$, the graph shifts **right**.
- If $C < 0$, the graph shifts **left**.

Horizontal Shift

The **horizontal shift** moves the graph up or down. It is determined by **D** in the function:

$$y = \sin(Bx - C) + D \; or \; \cos(Bx - C) + D$$

- If $D > 0$, then graph shifts **up**
- If $D < 0$, then graph shifts **down**

Step-by-Step Example: Graphing $y = 2\sin(3x - \pi) + 1$

We will graph this step by step:

Step 1: Identify the Key Features

- **Amplitude**: $|A| = 2$
- **Period**: $\frac{2\pi}{B} = \frac{2\pi}{3}$
- **Phase Shift**: $\frac{C}{B} = \frac{\pi}{3}$ (Shift **right** by $\frac{\pi}{3}$)
- **Vertical Shift**: (Move the graph **up by 1 unit**)

Step 2: Find the Key x-Values

Since a full period is $\frac{2\pi}{3}$, we divide it into **four equal parts**:

$$\frac{Period}{4} = \frac{2\pi}{3} \times \frac{1}{4} = \frac{\pi}{6}$$

Now, starting at the **phase shift** ($x = \frac{\pi}{3}$), we add $\frac{\pi}{6}$ successively:

1. **Start**: $x = \frac{\pi}{3}$
2. **First Quarter**: $x = \frac{\pi}{3} + \frac{\pi}{6} = \frac{\pi}{2}$
3. **Midpoint**: $x = \frac{\pi}{3} + 2\left(\frac{\pi}{6}\right) = \frac{2\pi}{3}$
4. **Third Quarter**: $x = \frac{\pi}{3} + 3\left(\frac{\pi}{6}\right) = \frac{5\pi}{6}$
5. **End of Period**: $x = \frac{\pi}{3} + 4\left(\frac{\pi}{6}\right) = \pi$

We divide by 4 because it helps to break one full period into **four key points**:

1. **Starting point** (beginning of the cycle)
2. **First quarter point** (where the function reaches its peak if positive, or lowest point if negative)
3. **Midpoint** (returns to the centerline)
4. **Third quarter point** (opposite of the first quarter, lowest point if positive, or highest if negative)
5. **End point** (completing one full cycle)

Step 3: Find the Corresponding y-Values

Since we are graphing **sine**, we use the basic sine wave pattern:

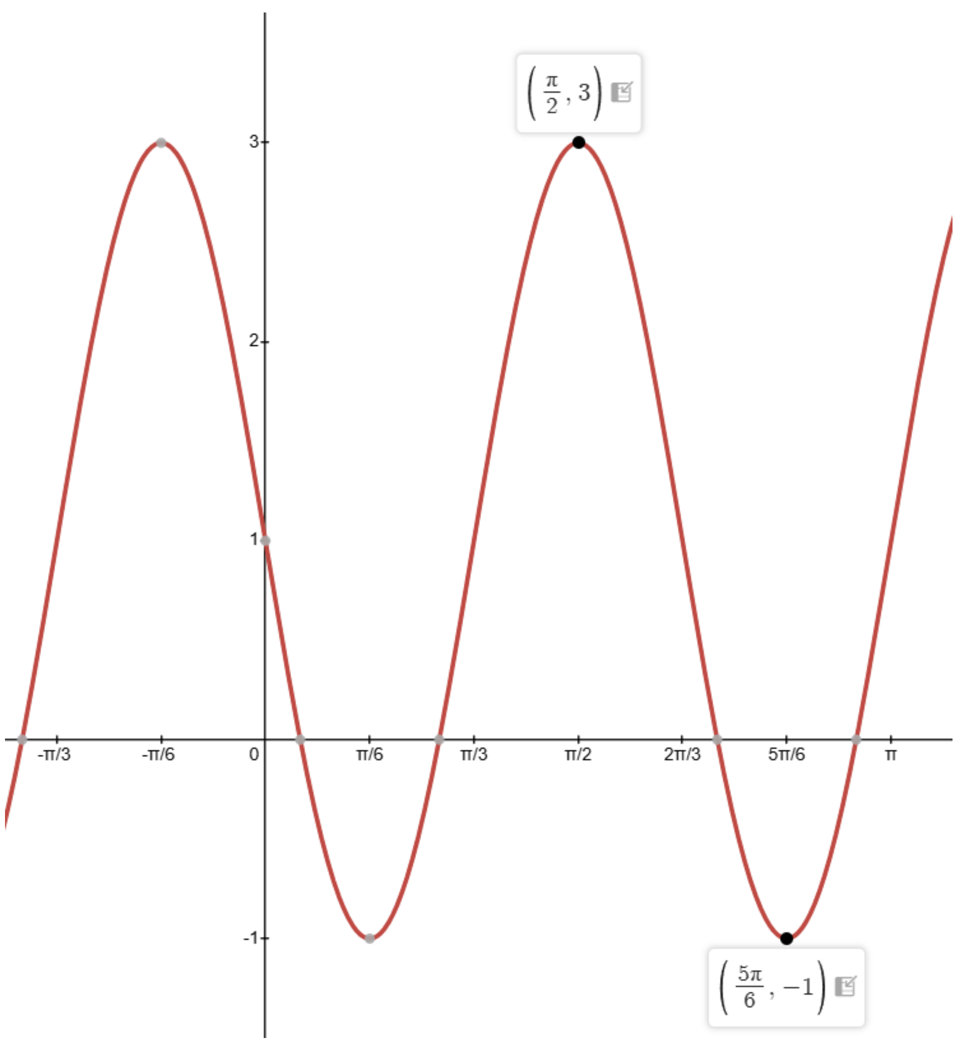
$$0 \to 1 \to 0 \to -1 \to 0$$

Now, we apply transformations:

- **Multiply by Amplitude (2)**: $0, 2, 0, -2, 0$
- **Shift Up by 1**: $1, 3, 1, -1, 1$

Step 4: Plot and Connect the Points

- The graph starts at $\left(\frac{\pi}{3}, 1\right)$
- Peaks at $(\frac{\pi}{2}, 3)$
- Crosses midline at $(\frac{2\pi}{3}, 1)$
- Bottoms at $(\frac{5\pi}{6}, -1)$
- Ends at $(\pi, 1)$



Graphs of Other Trigonometric Functions
(Tangent, Cotangent, Secant, Cosecant)

So far, we've learned about **sine** and **cosine** graphs. But what about the other four trig functions? Let's break them down one by one!

Tangent Function (y = tan x)

Step 1: Understanding the Shape

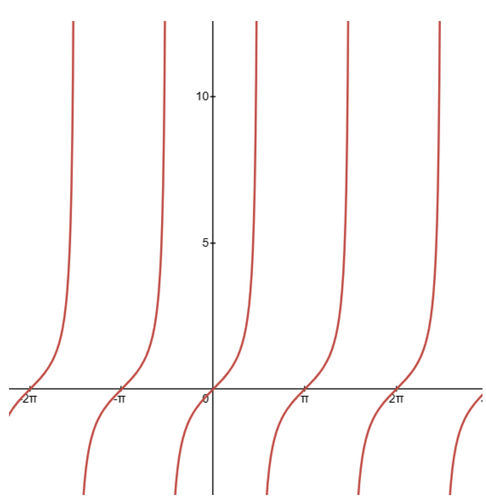
- **Unlike sine and cosine**, the tangent function has **asymptotes**, which are invisible walls where the graph **shoots up or down to infinity**.
- The graph **repeats every π (pi)**, meaning its **period is π**.
- It **crosses through the origin (0,0)** and looks like a repeating curve that starts low, rises, and then jumps up.

Step 2: Key Features

- **Period** = π → One cycle happens every π units.
- **Asymptotes** at odd multiples of $\frac{\pi}{2}$: These are where the function is **undefined** (because you can't divide by zero!).
- **Midpoint at (0,0)** and repeats every π.

Step 3: Plot Key Points

1. **Start at (0,0)**
2. **Move Right:** At $\frac{\pi}{4}$, $\tan\left(\frac{\pi}{4}\right) = 1$ At $\frac{\pi}{2}$, the function is undefined (asymptote).
3. **Move Left:** At $-\frac{\pi}{4}$, $\tan\left(\frac{\pi}{4}\right) = -1$
4. **Keep Repeating**: The pattern repeats every **π**.



Cotangent Function (y = cot x)

Cotangent is just $\frac{\mathbf{1}}{\mathbf{tangent}}$, so it's similar but flipped!

Step 1: Understanding the Shape

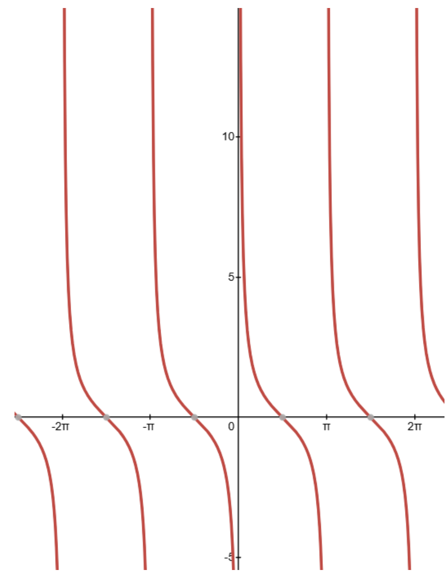
- **Still has asymptotes**, but they appear at multiples of **π**.
- **Period is also π**, just like tangent.
- **Flipped behavior**: Instead of rising from left to right, it **falls from left to right**.

Step 2: Key Features

- **Period** = π (just like tangent)
- **Asymptotes** at **multiples of π** (0, π, 2π, etc.).
- **Intercepts at halfway points** ($\frac{\pi}{2}, \frac{3\pi}{2}, ect$).

Step 3: Plot Key Points

1. Mark asymptotes at $x = 0, \pi, 2\pi$ ect
2. Midpoints at $x = \frac{\pi}{2}, \frac{3\pi}{2}$, ect., where y = 0.
3. At $x = \frac{\pi}{4}$, $y = 1$. At $x = \frac{3\pi}{4}$, $y = -1$
4. Graph falls from left to right in each section.



3. Secant Function (y = sec x)

Secant is $\frac{\mathbf{1}}{\mathbf{cosine}}$, **mean**ing **it follows the shape of cosine but flips up and down**.

Step 1: Understanding the Shape

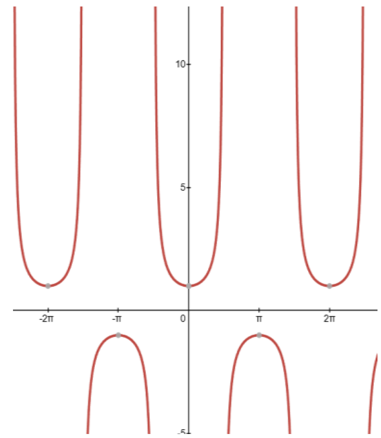
- **Uses cosine graph as a guide** (so always graph cosine lightly first!).
- **Where cosine = 0, secant has asymptotes** (because dividing by zero is undefined).
- **Forms "U" and upside-down "U" shapes**.

Step 2: Key Features

- **Period** = 2π (same as cosine).
- **Asymptotes** at odd multiples of π/2 (π/2, 3π/2, etc.).
- **Where cosine is at its peak, secant touches the peak.**

Step 3: Plot Key Points

1. Graph y = cos x lightly.
2. Mark asymptotes where cos x = 0 (π/2, 3π/2, etc.).
3. At max and min of cosine, secant touches these points.
4. Flip the "U" shapes upwards and downwards.



Cosecant Function (y = csc x)

Cosecant is $\frac{\mathbf{1}}{\mathbf{sine}}$, so it's like secant but follows the sine curve.

Step 1: Understanding the Shape

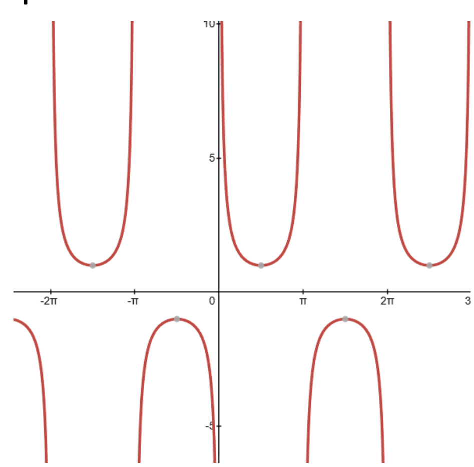
- Uses sine graph as a guide.
- Asymptotes where sine = 0.
- Forms "U" and upside-down "U" shapes.

Step 2: Key Features

- Period = 2π (same as sine).
- Asymptotes at multiples of π (0, π, 2π, etc.).
- Where sine is at its peaks, cosecant touches these points.

Step 3: Plot Key Points

1. Graph y = sin x lightly.
2. Mark asymptotes where sin x = 0 (0, π, 2π, etc.).
3. At max and min of sine, cosecant touches these points.
4. Flip the "U" shapes upwards and downwards.



Final Comparison Chart

Function	Period	Asymptotes	Key Tricks
Tangent (tan x)	π	Odd multiples of $\frac{\pi}{2}$	Rises left to right
Cotangent (cot x)	π	Multiples of π	Falls left to right
Secant (sec x)	2π	Odd multiples of $\frac{\pi}{2}$	Flips from cosine
Cosecant (csc x)	2π	Multiples of π	Flips from sine

Quick Tricks for Graphing These Functions

- **Always mark asymptotes FIRST** – they tell you where the function is undefined.
- **For secant & cosecant, lightly sketch cosine/sine first** – then draw U-shapes.
- **Tangent rises left to right; cotangent falls left to right** – don't mix them up!

Now that we understand the **basic shapes**, let's learn what happens when **shifts and transformations** are added!

Understanding Transformations

When graphing **y = A trig(Bx - C) + D**, each **letter** does something important:

Letter	What it Does	Effect
A	Amplitude	**Makes the graph taller or shorter**
B	Frequency	**Changes the period (how often the Waves repeats)**
C	Phase Shift	**Moves the graph left or right**
D	Vertical Shift	**Moves the graph up or down**

Quick Formula Cheat Sheet

- **Amplitude**: **|A|** → This affects **sine, cosine, secant, and cosecant** (but not tangent or cotangent).
- **Period**:
  - **Sine, Cosine, Secant, Cosecant**: $\frac{2\pi}{B}$
  - **Tangent, Cotangent**: $\frac{\pi}{B}$
- **Phase Shift**: $\frac{C}{B}$
  - If **C > 0**, shift **right**.
  - If **C < 0**, shift **left**.
- **Vertical Shift**: D moves everything up or down.

Transformations of the Tangent Function (y = A tan (Bx - C) + D)

1. **Period** = $\boldsymbol{\frac{\pi}{B}}$
2. **Asymptotes** at $\boldsymbol{x = \frac{C}{D} \pm \frac{\pi}{2B}}$ (mark these first!)
3. **Phase Shift** = $\boldsymbol{\frac{C}{B}}$ (moves left or right)
4. **Vertical Shift** = **D** (moves everything up or down)

Example: Graph $y = \tan(2x - \pi) + 1$

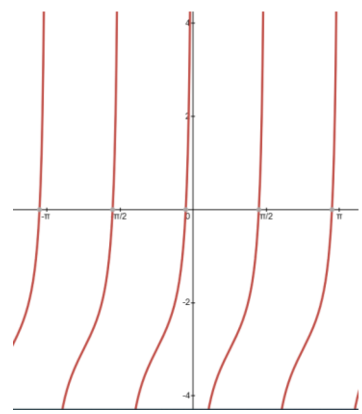
1. **Period** $= \frac{\pi}{B} = \frac{\pi}{2}$
2. **Asymptotes** at $x = \frac{\pi}{2}$ and $x = -\frac{\pi}{2}$
3. **Phase Shift** $= \frac{\pi}{2}$
4. **Vertical Shift** = **+1** (Move everything **up 1 unit**)

Tip: First mark asymptotes, then sketch the basic curve, then apply shifts!

Transformations of the Cotangent Function (y = A cot (Bx - C) + D)

Same steps as tangent, but **cotangent falls from left to right** instead of rising.

Example: Graph $y = -\cot(2x + \pi) - 3$



1. Period $= \frac{\pi}{B} = \frac{\pi}{2}$
2. Asymptotes at $x = 0$ and $x = \frac{\pi}{2}$
3. Phase Shift = $-\frac{\pi}{2}$ (Shift left)
4. Vertical Shift = -3 (Move everything down 3 units)
5. Flipped graph (because of negative A)

Tip: Cotangent is just a flipped tangent!

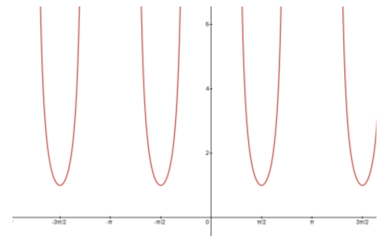
Transformations of the Secant Function (y = A sec(Bx - C) + D)

Step 1: Identify Transformations

1. Graph y = cos (Bx - C) first! (Secant is based on cosine.)
2. Period $\frac{2\pi}{B}$
3. Asymptotes where cosine is zero
4. Phase Shift = $\frac{C}{D}$
5. Vertical Shift = D

Example: Graph $y = 3\sec(2x - \pi) - 2$

1. Graph y = 3 cos (2x - π) - 2 lightly
2. Period = $\frac{2\pi}{2} = \pi$
3. Asymptotes where cos = 0 (odd multiples of $\frac{\pi}{2}$)
4. Shift right by $\frac{\pi}{2}$
5. Stretch by 3 and move down 2

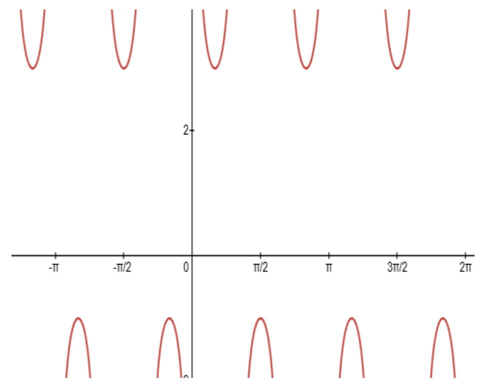


Tip: Draw cosine first, then flip secant over it!

Transformations of the Cosecant Function (y = A csc (Bx - C) + D)

Example: Graph $\boldsymbol{y = -2csc(3x - \pi) + 1}$

1. Graph y = -2 sin (3x - π) + 1 first.
2. Period = $\frac{2\pi}{3}$
3. Asymptotes where sin = 0 (multiples of π)
4. Phase Shift = $\frac{\pi}{3}$
5. Flip downward because of negative A.



Tip: Cosecant follows sine's pattern but flips over it.

Tricks for Graphing Transformed Functions

Step 1: Find the period first → $\frac{2\pi}{B}$ or $\frac{\pi}{B}$

Step 2: Mark asymptotes → Helps structure the graph.

Step 3: Find the phase shift → Shift left/right using $\frac{C}{B}$

Step 4: Apply vertical shift → Move graph up/down by **D**.

Step 5: Sketch lightly first!

Inverse Trigonometric Functions

Inverse trigonometric functions help us find the angle when we already know the sine, cosine, or tangent value. For example, if we know that, we use to find the angle .

The six inverse trigonometric functions are:

- **Inverse Sine (Arcsin)**: $y = \sin^{-1}(x)$
- **Inverse Cosine (Arccos)**: $y = \cos^{-1}(x)$
- **Inverse Tangent (Arctan)**: $y = \tan^{-1}(x)$
- **Inverse Cotangent (Arccot)**: $y = \cot^{-1}(x)$
- **Inverse Secant (Arcsec)**: $y = \sec^{-1}(x)$
- **Inverse Cosecant (Arccsc)**: $y = \csc^{-1}(x)$

Each inverse function finds an angle such that the original trigonometric function equals.

Understanding Domain and Range

Why Are These Restricted?

Trigonometric functions repeat infinitely, so without restrictions, inverse functions would give multiple answers. To keep things simple, we define specific **principal values** (ranges) for each inverse function:

Function	Domain (X-Values)	Range (Y-Values)
$\sin^{-1}(x)$	$-1 \leq x \leq 1$	$-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$
$\cos^{-1}(x)$	$-1 \leq x \leq 1$	$0 \leq y \leq \pi$
$\tan^{-1}(x)$	$-\infty < x < \infty$	$-\frac{\pi}{2} < y < \frac{\pi}{2}$
$\cot^{-1}(x)$	$-\infty < x < \infty$	$0 < y < \pi$
$\sec^{-1}(x)$	$-\infty < x \leq 1 \cup 1 \leq x < \infty$	$0 \leq x < \frac{\pi}{2} \cup \frac{\pi}{2} < x \leq \pi$
$\csc^{-1}(x)$	$-\infty < x \leq -1 \cup 1 \leq x < \infty$	$-\frac{\pi}{2} \leq x < 0 \cup 0 < x \leq \frac{\pi}{2}$

Tip:

- Inverse sine and inverse tangent functions give angles in **Quadrants I & IV**
- Inverse cosine gives angles in **Quadrants I & II**
- Secant, cosecant, and cotangent inverses follow the same quadrant rules as their related primary functions.

How to Find Inverse Trigonometric Values

Step-by-Step Process

1. **Recognize the Given Value**: Identify the trigonometric function and the given value (e.g., $\sin^{-1}(\frac{1}{2})$).
2. **Find the Angle**: Determine the angle that has that value in the restricted range.
3. **Verify the Range**: Ensure the angle falls within the allowed range from the table above.
4. **Write the Answer**: Express the answer in radians or degrees.

Step-by-Step Process to Find Inverse Trigonometric Values

Example 1: Finding $y = \sin^{-1}(\frac{1}{2})$

1. **Recognize the Given Value**: Identify the trigonometric function and value.
   - Here, $y = \sin^{-1}(\frac{1}{2})$
2. **Find the Angle**: Determine which angle gives this value.
   - $\sin\left(\frac{\pi}{6}\right) = \frac{1}{2}, \, so \; y = \frac{\pi}{6}$
3. **Verify the Range**:
   - The range of $\sin^{-1}(x)$ is $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$, which includes $\frac{\pi}{6}$
4. **Write the Answer**:
   - $\sin^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{6}$

Reciprocal Inverse Trigonometric Functions

For **secant, cosecant, and cotangent**, we use their definitions:

Example 2: Finding $y = \sec^{-1}(2)$

1. **Recognize the Given Value**:
    - $y = \sec^{-1}(2)$, meaning $\sec(y) = 2$
2. **Find the Angle**:
    - Since $\sec(y) = \frac{1}{\cos(y)}$, we solve:

$$\frac{1}{\cos(y)} = 2 \Rightarrow \cos(y) = \frac{1}{2}$$

  - From the unit circle, $\cos\left(\frac{\pi}{3}\right) = \frac{1}{2}$
3. **Verify the Range**:
    - The range of $\sec^{-1}(x)$ is $0 \leq y \leq \pi$ (excluding $y = \frac{\pi}{2}$), so $\frac{\pi}{3}$ valid.
4. **Write the Answer**:
    - $\sec^{-1}(2) = \frac{\pi}{3}$

Graphing Inverse Trigonometric Functions

Inverse trig functions **reflect their original functions** across the line y=x

- $\sin^{-1}(x)$: Increasing from $-\frac{\pi}{2}\ to \frac{\pi}{2}$
- $\cos^{-1}(x)$: Decreasing from $0\ to\ \pi$
- $\tan^{-1}(x)$: Increasing from $-\frac{\pi}{2}\ to \frac{\pi}{2}$
- $\csc^{-1}(x)$: Restricted to **Quadrants I and IV**.
- $\sec^{-1}(x)$: Restricted to **Quadrants I and II**.
- $\cot^{-1}(x)$: Restricted to **Quadrants I and II**.

Tip: The inverse function graphs **don't oscillate** like sine or cosine but instead increase or decrease steadily.

Using a Right Triangle to Evaluate Inverse Trigonometric Expressions
We'll solve problems like:

$$\cos(\sin^{-1}(x))$$
$$\tan(\cos^{-1}(x))$$
$$\sec(\tan^{-1}(x))$$

using a **right triangle** to help visualize.

Example 1: $\cos(\sin^{-1}(\frac{3}{5}))$

Step 1: Recognize the Given Value

- We are given $\sin^{-1}(\frac{3}{5})$, which means we need to find an angle θ such that:

$$\sin(\theta) = \frac{3}{5}$$

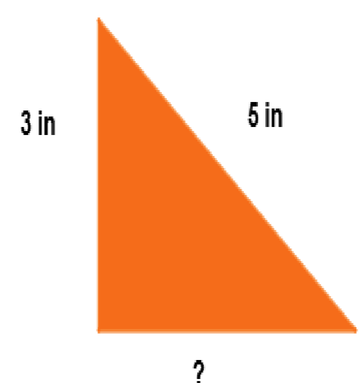
  - **Quadrant Check**: Since the range of $\sin^{-1}(x)$ is $-\frac{\pi}{2} \leq \boldsymbol{\theta} \leq \frac{\pi}{2}$, the angle must be in **Quadrant I or IV**.
  - Since $\sin\theta = \frac{3}{5}$ is positive, the angle must be in **Quadrant I**.

Step 2: Draw a Right Triangle

- Label a right triangle with:
  - **Opposite = 3 in**
  - **Hypotenuse = 5 in**
  - Use the Pythagorean Theorem to find the **adjacent** side:

$a^2 + 3^2 = 5^2$

$$a^2 + 9 = 25$$
$$a^2 = 16 \Rightarrow a = 4$$



- The triangle now has:
  - $\sin\theta = \frac{3}{5}$
  - $\cos\theta = \frac{4}{5}$

Step 3: Solve for $\cos(\sin^{-1}(\frac{3}{5}))$

$$\cos(\sin^{-1}\frac{3}{5}) = \frac{4}{5}$$

Example 2: $\tan(\cos^{-1}\frac{5}{13})$

Step 1: Recognize the Given Value

- We are given $\cos^{-1}(\frac{5}{13})$, meaning we need to find an angle θ such that:

$$\cos(\theta) = \frac{5}{13}$$

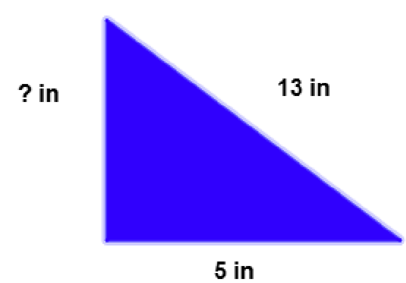
  - **Quadrant Check**: Since the range of $\cos^{-1}(x)$ is $\mathbf{0 \leq \theta \leq \pi}$, the angle must be in **Quadrant I or II**.
  - Since $\cos(\theta) = \frac{5}{13}$ is positive, the angle must be in **Quadrant I**.

Step 2: Draw a Right Triangle

- Label a right triangle with:
  - **Adjacent = 5**
  - **Hypotenuse = 13**
  - Use the Pythagorean Theorem to find the **opposite** side:

$$o^2 + 5^2 = 13^2$$
$$o^2 + 25 = 169$$
$$o^2 = 144 \Rightarrow o = 12$$

- The triangle now has:
  - $\cos(\theta) = \frac{5}{13}$
  - $\tan(\theta) = \frac{12}{5}$



Step 3: Solve for $\tan(\cos^{-1}\frac{5}{13})$

$$\tan(\cos^{-1}\frac{5}{13}) = \frac{12}{5}$$

Quadrant Restrictions Recap

To **correctly evaluate inverse trig functions**, you **must** know their range restrictions:

Inverse Function	Range (Quadrant Allowed)
$\sin^{-1}(x)$	Quadrants I and IV
$\cos^{-1}(x)$	Quadrants I and II
$\tan^{-1}(x)$	Quadrants I and IV
$\cot^{-1}(x)$	Quadrants I and II
$\sec^{-1}(x)$	Quadrants I and II
$\csc^{-1}(x)$	Quadrants I and IV

- **Use Quadrant I for positive values.**
- **Use Quadrant IV for negative values when dealing with** $\sin^{-1}(x), \tan^{-1}(x), \csc^{-1}(x)$
- **Use Quadrant II for negative values when dealing with** $\cos^{-1}(x), \cot^{-1}(x), \sec^{-1}(x)$

Application of trigonometry

Solving Right Triangles (Finding Missing Sides and Angles)

A **right triangle** is a triangle with **one 90-degree angle**. We label its sides as:

- **Opposite**: The side **opposite** the angle we are working with.
- **Adjacent**: The side **next to** the angle (but not the hypotenuse).
- **Hypotenuse**: The **longest side** (across from the right angle).

How to Solve a Right Triangle?

To find missing sides or angles, we use **Soh Cah Toa**:

- **Sine**: $\sin(\theta) = \frac{\text{opposite}}{\text{hypotenuse}}$
- **Cosine**: $\cos(\theta) = \frac{\text{adjacent}}{\text{hypotenuse}}$
- **Tangent**: $\tan(\theta) = \frac{\text{opposite}}{\text{adjacent}}$

Example: Finding a Missing Side

Suppose we have a **right triangle** where:

- The angle is **30°**
- The hypotenuse is **10 units**
- We want to find the **opposite side**.

Using **Soh Cah Toa**, we use **sine** because it involves the opposite and hypotenuse:

$$\sin(30) = \frac{\text{opposite}}{10}$$

$$\text{opposite} = 10 \times \sin(30) = 10 * \frac{1}{2} = 5$$

Answer: The opposite side is **5 units**.

TIP: If you are missing an **angle**, use the inverse trig functions:

- $\theta = \sin^{-1}(\frac{opp}{\text{h}yp})$
- $\theta = \cos^{-1}(\frac{adj}{\text{h}yp})$
- $\theta = \tan^{-1}\left(\frac{opp}{adj}\right)$

Trigonometry and Bearings (Navigation & Directions)

What is a Bearing?

A **bearing** is an angle that tells us **which direction something is moving**. It is measured in **degrees** from **North (0°) in a clockwise direction**.

Think of a compass:

- North (N) = 0°
- East (E) = 90°
- South (S) = 180°
- West (W) = 270°

Two Types of Bearings

Standard Bearings (Measured from North Clockwise)

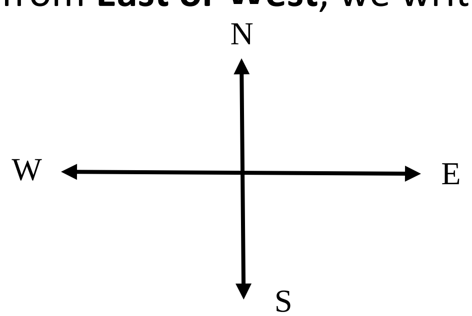
- Always measured **clockwise from North (0°).**
- Example: A boat moving **directly east** has a bearing of **090°**.
- Example: A plane moving **southwest** might have a bearing of **225°**.

Compass Bearings (Uses North/South as a Reference)

- Use **North (N) or South (S) first**, then **how far East (E) or West (W)**.
- Example: **N 40° E** (40° east of north)
- Example: **S 65° W** (65° west of south)

Key Rule:

- If the angle is **measured from North or South**, we write **N or S first**.
- If measured from **East or West**, we write **E or W last**.



How to Find Bearings?

Example: A plane is flying north-east at a bearing of N 40° E.

- This means the plane is **40° east of North**.
- If a second plane is flying at **S 50° W**, it is **50° west of South**.

Example: A ship sails 100 miles on a bearing of 60°. How far has it traveled north and east?

We break it into **right triangle parts**:

- **Northward Distance** = 100 × cos (60°) = 100 × 0.5 = 50 miles
- **Eastward Distance** = 100 × sin (60°) = 100 × 0.866 = 86.6 miles

Answer: The ship has traveled **50 miles north** and **86.6 miles east**.

Identifying the Right Triangle Angle

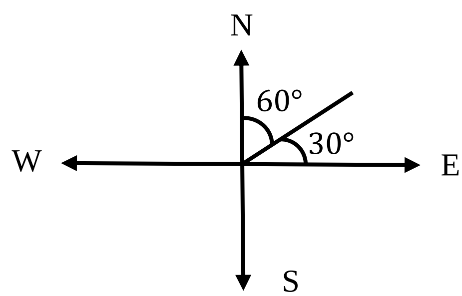
When working with trigonometry, we usually work with **right triangles**, and the angle we use is often inside the triangle.

- Bearings are measured from the **North-South line**.
- But in a right triangle, angles are typically measured from the **horizontal** (East-West line).
- The two angles in the right triangle must add up to **90°** (because it's a right triangle).

So, if you are given a bearing of N 60° E, the angle inside the triangle (from the East-West line) is:

$$90° - 60° = 30°$$

That's why you use **30° instead of 60°** when solving with trigonometry.



Example 1: A Ship Traveling on a Bearing of N 60° E

Imagine a ship sailing **100 miles** on a **bearing of N 60° E**. We want to find how far it has traveled **north** and **east**.

1. **Step 1: Draw the Right Triangle**
   - The ship's path is the hypotenuse (100 miles).
   - The North-South line is the vertical side.
   - The East-West line is the horizontal side.
2. **Step 2: Identify the Right Triangle Angle**
   - The bearing N 60° E is measured from the North line.
   - The angle inside the triangle (from the East axis) is:
     $90° - 60° = 30°$
   - So, the triangle's reference angle is 30°.
3. **Step 3: Use SOH-CAH-TOA**
   - Northward distance (adjacent side) = $100 \times \cos(30°)$
   - Eastward distance (opposite side) = $100 \times \sin(30°)$

Bearing Summary

- Bearings are **measured from North/South**, but trigonometry usually uses angles from the **horizontal (East/West)**.
- To find the **triangle's reference angle**, subtract the bearing from **90°**.
- Then, use **SOH-CAH-TOA** to break it into vertical and horizontal distances.

Simple Harmonic Motion (Springs & Waves)

It describes objects that **move back and forth in a predictable, repeating pattern**, like:

- A **swing**.
- A **mass on a spring**.
- **Sound waves**.

Equation for Simple Harmonic Motion

$d = Acos(\omega t)$ or $d = Asin(\omega t)$

Where:

- **A** = Amplitude (maximum displacement)
- **ω** = Angular speed
- **t** = Time

Example: A spring is pulled 8 inches down and let go. It takes 4 seconds for one full cycle. Find its equation.

1. **Amplitude** = 8 (since the spring stretches 8 inches)
2. **Period** = 4 seconds
3. **Find ω using:** $\boldsymbol{\omega = \frac{2\pi}{period}}$

$$\omega = \frac{2\pi}{4} = \frac{\pi}{2}$$

4. **Equation:** Since the object starts at its lowest position, we use **cosine**:

$$d = 8\cos\left(\frac{\pi}{2}t\right)$$

TIP: If the object starts at **the highest or lowest position**, use **cosine**. If it starts **at zero and moves up**, use **sine**.

Key Takeaways

- Use SOH-CAH-TOA to solve right triangles.
- Bearings measure directions clockwise from North (0°).
- Simple Harmonic Motion follows a cosine or sine wave pattern.
- Divide problems into right triangles when working with navigation.

Analytic Trigonometry

Verifying Trigonometric Identities

Trigonometric identities are like math puzzles where we prove that two sides of an equation are actually the same thing, even if they look different. Think of it like taking different routes to the same destination.

Key Tricks for Verifying Trigonometric Identities

1. **Work on the more complicated side first** – It's easier to simplify a messy expression than to complicate a simple one.
2. **Use trigonometric identities** – These are like cheat codes that help transform an equation.
3. **Change everything into sine and cosine** – Since all trig functions can be rewritten in terms of sine and cosine, this trick often makes things simpler.
4. **Look for common factors** – Sometimes, factoring out terms can reveal a hidden identity.
5. **Try different strategies** – Some problems need you to add fractions, multiply by a conjugate, or simplify using algebra.

Verifying Trigonometric Identities

Step 1: Know Your Trigonometric Identities

Here are the fundamental identities you'll need:

Reciprocal Identities

$$\sin x = \frac{1}{\csc x}, \cos x = \frac{1}{\csc x}, \tan x = \frac{1}{\cot x}$$

Quotient Identities

$$\tan x = \frac{\sin x}{\cos x}, \quad \cot x = \frac{\cos x}{\sin x}$$

Pythagorean Identities

$$\sin^2 x + \cos^2 x = 1$$
$$1 + \tan^2 x = \sec^2 x$$
$$1 + \cot^2 x = \csc^2 x$$

Even-Odd Identities

$$\sin(-x) = -\sin(x), \cos(-x) = \cos(x), \tan(-x) = -\tan(x)$$

Step 2: Start with the More Complicated Side

When verifying an identity, start with the side that looks messier. Your goal is to transform it into the simpler side.

For example, verify:

$$\frac{1+\sin(x)}{\cos(x)} = \sec x + \tan x$$

Solution:

1. Start with the left side: $\frac{1+\sin x}{\cos x}$
2. Split into two fractions: $\frac{1}{\cos x} + \frac{\sin x}{\cos x}$
3. Use identities: $\frac{1}{\cos \mathrm{x}} = \sec x$ and $\frac{\sin x}{\cos x} = \tan x$
4. Now we have: $\sec x + \tan x$

Boom! We proved both sides are the same.

Step 3: Try to Rewrite in Terms of Sine and Cosine

If the identity isn't obvious, rewrite everything in terms of $\sin x$ and $\cos x$

For example, verify:

$$\frac{\cot^2 x}{\csc^2 x} = \cos^2 x$$

Solution:

1. Convert $\cot x$ and $\csc x$:

$$\cot^2 x = \frac{\cos^2 x}{\sin^2 x}, \csc^2 x = \frac{1}{\sin^2 x}$$

2. Rewrite the left side:

$$\frac{\frac{\cos^2 x}{\sin^2 x}}{\frac{1}{\sin^2 x}}$$

3. Simplify:

$$\cos^2 x$$

Step 4: Factor When Needed

If you see squares of trig functions, look for a way to factor.

For example, verify:

$$\cos(x) - \cos(x)\sin^2(x) = \cos^3 x$$

Solution:

1. Factor out $\cos x$ on the LHS: $\cos x(1 - \sin^2 x)$
2. Use the identity $1 - \sin^2 x = \cos^2 x$: $\cos x \cdot \cos^2 x = \cos^3 x$

Boom! We proved it.

Step 5: Look for Pythagorean Identities

If you see $\sin^2 x + \cos^2 x$, try substituting 1.

For example, verify:

$$\frac{1 - \cos^2 x}{\sin^2 x} = 1$$

Solution:

1. Recognize that $1 - \cos^2 x = \sin^2 x$, so rewrite the left side:

$$\frac{\sin^2 x}{\sin^2 x}$$

2. Simplify:

$$1$$

Nice and easy!

Step 6: Don't Be Afraid to Restart

If your method isn't working, don't panic! Go back, try a different approach, and see if rewriting in another way helps.

Final Tips

Break things into smaller pieces – Don't try to do everything in one step.

Use algebra – Many trig verifications involve factoring, distributing, or simplifying fractions.

Know when to stop – The goal is to make one side look exactly like the other

Sum and Difference Formulas

Sum and difference formulas help us **find the sine, cosine, and tangent of the sum or difference of two angles** without needing a calculator. Instead of estimating values, we can **break them down** into values we already know

The Formulas (Must-Know)

1. **Cosine Formulas**
    - **Cosine of a sum:**

$$\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$

- **Cosine of a difference:**

$$\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta$$

- **Tip to Remember: Cosine = Change the Sign** (The formula uses the opposite sign of what's inside the parentheses)

2. **Sine Formulas**
    - **Sine of a sum:**

$$\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta$$

- **Sine of a difference:**

$$\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$$

- **Tip to Remember: Sine = Keep the Sign** (The formula keeps the same sign as in parentheses)

3. T**angent Formulas**

- **Tangent of sum**:

$$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

- **Tangent of a difference:**

$$\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$$

- **Tip to Remember:** Think of **tan as a fraction** (it has division unlike sine and cosine)

Step-by-Step Example: Find Exact Value of Cos 15°

We don't have 15° on the unit circle, but we can break it down using **60° and 45°** because we know their exact values.

Step 1: Recognize the formula

Since **15° = 60° - 45°**, we use the **cosine difference formula**:

$$\cos(60-45)=\cos 60\cos 45+\sin 60\sin 45$$

Step 2: Substitute Known Values

We get these from the unit circle:

- $\cos 60=\frac{1}{2}$
- $\cos 45=\frac{\sqrt{2}}{2}$
- $\sin 60=\frac{\sqrt{3}}{2}$
- $\sin 45=\frac{\sqrt{2}}{2}$

Now, plug them in:

$$\cos(15)=\left(\frac{1}{2}\times\frac{\sqrt{2}}{2}\right)+(\frac{\sqrt{3}}{2}\times\frac{\sqrt{2}}{2})$$

Step 3: Multiply and Simplify

$$\cos(15)=\frac{\sqrt{2}}{4}+\frac{\sqrt{6}}{4}$$

$$\cos(15)=\frac{\sqrt{2}+\sqrt{6}}{4}$$

Final Answer:

$$\cos(15)=\frac{\sqrt{6}+\sqrt{2}}{4}$$

Tricks to Remember Sum and Difference Formulas

1. For cosine: "Change the sign"
   - If it's + inside, use − in the formula.
   - If it's − inside, use + in the formula.

2. For sine: “Keep the sign”
    - If it’s + inside, use + in the formula.
    - If it’s − inside, use − in the formula.
3. For tangent: Think of a fraction:

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \pm \tan\alpha\tan\beta}$$

The numerator keeps the sign. The denominator flips it.

Example: Verify Identity
Prove:

$$\cos(\alpha - \beta) = \frac{\cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)}$$

Step 1: Start with the given expression

$$\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$$

Step 2: Compare with the right-hand side

$$\frac{\cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)}{\cos(\alpha)\cos(\beta) - \sin(\alpha)\sin(\beta)}$$

We notice that the **right side resembles** a sum/difference formula, so we verify by using the known identity.
Since the **cosine of a sum/difference formula is already known**, we conclude that:

$$\cos(\alpha - \beta) = \cos(\alpha)\cos(\beta) + \sin(\alpha)\sin(\beta)$$

So, the identity is verified.

Final Thoughts

- These formulas are super useful when dealing with non-standard angles.
- You don’t need to memorize the unit circle fully—just break angles into known values (30°, 45°, 60°).
- Use sum and difference when working with angles that are not on the unit circle.
- Cosine flips the sign inside, sine keeps the sign.
- Tangent has a fraction, so watch the denominator

Double-Angle, Power-Reducing, and Half-Angle Formulas

Double-angle formulas allow you to **rewrite** trigonometric functions of **double angles** 2θ in terms of single angles θ.

Double-Angle Formulas:

- **Sine**: $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$
  - Trick: Think of it as **double** the product of sin and cos.
- **Cosine** (3 Forms):

$$\cos(2\theta) = \cos^2(\theta) - \sin^2(\theta)$$
$$\cos(2\theta) = 2\cos^2(\theta) - 1$$
$$\cos(2\theta) = 1 - 2\sin^2(\theta)$$

- Trick: Use the Pythagorean identity $\sin^2(\theta) + \cos^2(\theta) = 1$ to change it into different forms.
- **Tangent**: $\tan(2\theta) = \frac{2\tan(\theta)}{1-\tan^2(\theta)}$
- Trick: It's similar to the sum formula for tangent.

Power-Reducing Formulas

These can help by **rewriting squared trigonometric functions** $\sin^2 x$ and $\cos^2 x$ into terms without squares.

Power-Reducing Formulas:

- **Sine**:

$$\sin^2(\theta) = \frac{1 - \cos(2\theta)}{1}$$

- **Cosine**:

$$\cos^2(\theta) = \frac{1 + \cos(2\theta)}{2}$$

- **Tangent**:

$$\tan^2(\theta) = \frac{1 - \cos(2\theta)}{1 + \cos(2\theta)}$$

Half-Angle Formulas

These help to find trigonometric values for **half** an angle (like $\frac{\theta}{2}$).

Half-Angle Formulas:

- **Sine**:

$$\sin\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{2}}$$

- **Cosine**:

$$\cos\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1+\cos(\theta)}{2}}$$

- **Tangent**:

$$\tan\left(\frac{\theta}{2}\right) = \pm\sqrt{\frac{1-\cos(\theta)}{1+\cos(\theta)}}$$

Tips & Tricks

1. **Double-Angle & Power-Reducing are Related**
   - Double-Angle Formulas help derive Power-Reducing Formulas.
   - If you solve for $\sin^2 x$ or $\cos^2(x)$ using the double-angle cosine formula, you get power-reducing formulas!
2. **Use Power-Reducing Formulas When Dealing with Squared Trig Functions**
   - If you see $\sin^2 x$ or $\cos^2(x)$, try using power-reducing formulas to simplify.
3. **Use Half-Angle Formulas for Exact Values**
   - If an angle is not standard (e.g., 22.5° or 67.5°), half-angle formulas are useful.
4. **Memorization Hack:**
   - Power-Reducing is Just the Cosine Double-Angle Formula Rearranged! **Example:** $\boldsymbol{\sin^2\theta = \frac{1-\cos(2\theta)}{2}}$

→ Just **rearrange** $\cos(2\theta) = 1 - 2\sin^2(\theta)$

Example Problems

1. Find the exact value of $\sin(2\theta)$ **if** $\sin(\theta) = \frac{3}{5}$

- We know: $\sin(2\theta) = 2\sin(\theta)\cos(\theta)$
- Find $\cos(\theta)$using **Pythagorean theorem**:

$$\cos(\theta) = \sqrt{1 - \sin^2\theta} = \sqrt{1 - \left(\frac{3}{5}\right)^2} = \sqrt{1 - \frac{9}{25}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$$

- Now plug in values:

$$\sin(2\theta) = 2\left(\frac{3}{5}\right)\left(\frac{4}{5}\right) = 2\left(\frac{12}{25}\right) = \frac{24}{25}$$

Answer: $\frac{24}{25}$

Use Power-Reducing Formula to Rewrite $\sin^4(x)$

- We first use $\sin^2(x) = \frac{1-\cos(2x)}{2}$
- Then, square it to reduce the power further:

$$\sin^4(x) = \left(\frac{1 - \cos(2x)}{2}\right)^2$$

$$= \frac{(1 - 2\cos(2x) + \cos^2(2x))}{4}$$

- Now use $\cos^2(2x) = \frac{1+\cos(4x)}{2}$:

$$= \frac{1 - 2\cos(2x) + \frac{1 + \cos(4x)}{2}}{4}$$

$$= \frac{2 - 4\cos(2x) + 1 + \cos(4x)}{8}$$

$$= \frac{3 - 4\cos(2x) + \cos(4x)}{8}$$

Final Answer:

$$\sin^4 x = \frac{3}{8} - \frac{1}{2}\cos(2x) + \frac{1}{8}\cos(4x)$$

Find tan (15) Using Half-Angle Formula

- We use:

$$\tan\left(\frac{\theta}{2}\right) = \frac{1-\cos(\theta)}{\sin^2(\theta)}$$

- Let $\theta = 30$ so that $\frac{\theta}{2} = 15$
- Use known values:

$$\cos(30) = \frac{\sqrt{3}}{2}, \sin 30 = \frac{1}{2}$$

$$\tan 15 = \frac{1-\frac{\sqrt{3}}{2}}{\frac{1}{2}} = (1-\frac{\sqrt{3}}{2})(\frac{2}{1})$$

$$= \frac{2-\sqrt{3}}{1} = 2-\sqrt{3}$$

Final Answer:

$$\tan 15 = 2-\sqrt{3}$$

Final Thoughts

- **Double-Angle**: Expands trigonometric functions of **double angles**.
- **Power-Reducing**: Rewrites squared trig functions in terms of cosine of double angles.
- **Half-Angle**: Breaks trigonometric functions into half-angle expressions.

Best Trick for Memorization

Think of **Double-Angle** as growing, **Power-Reducing** as shrinking, and **Half-Angle** as splitting!

Product-to-Sum and Sum-to-Product

These formulas express the **product** of sine and/or cosine functions as a **sum or difference**. They help in simplifying trigonometric expressions, especially in integrals and other advanced problems.

Product-to-Sum Identities

$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta)-\cos(\alpha+\beta)]$$
$$\cos\alpha\cos\beta = \frac{1}{2}[\cos(\alpha-\beta)+\cos(\alpha+\beta)]$$
$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta)+\sin(\alpha-\beta)]$$
$$\sin\alpha\cos\beta = \frac{1}{2}[\sin(\alpha+\beta)-\sin(\alpha-\beta)]$$

Key Idea: These formulas allow us to rewrite the **product of sine and cosine** functions as sums or differences, making complex expressions easier to work with.

Sum-to-Product Formulas

These formulas **reverse** the Product-to-Sum formulas, expressing sums or differences of sines and cosines as **products**. They are useful in solving equations and simplifying expressions.

Sum-to-Product Identities

$$\sin\alpha+\sin\beta = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos(\frac{\alpha-\beta}{2})$$
$$\sin\alpha-\sin\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\sin(\frac{\alpha-\beta}{2})$$
$$\cos\alpha+\cos\beta = 2\cos\left(\frac{\alpha+\beta}{2}\right)\cos(\frac{\alpha-\beta}{2})$$
$$\cos\alpha+\cos\beta = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin(\frac{\alpha-\beta}{2})$$

Key Idea: These formulas are useful when simplifying trigonometric expressions that involve **sums or differences** of sine and cosine functions.

Example: Using the Product-to-Sum Formula

Express $\sin(6x)\sin(4x)$ **as a sum**

Using the formula:

$$\sin\alpha\sin\beta = \frac{1}{2}[\cos(\alpha-\beta) - \cos(\alpha+\beta)]$$

Set $\alpha = 6x$ and $\beta = 4x$ then:

$$\sin(6x)\sin(4x) = \frac{1}{2}[\cos(2x) - \cos(10x)]$$

Example: Using the Sum-to-Product Formula

Express $\sin(\ 6x) + \sin(8x)$ **as a product**

Using the formula:

$$\sin(\alpha) + \sin(\beta) = 2\sin\left(\frac{\alpha+\beta}{2}\right)\cos\left(\frac{\alpha-\beta}{2}\right)$$

Set $\alpha = 6x$ and $\beta = 8x$, then:

$$\sin(6x) + \sin(8x) = 2\sin(7x)\cos(x)$$

Example: Verifying an Identity

Verify the identity:

$$\frac{\sin(3x) - \sin(7x)}{\cos(3x) - \cos(7x)} = -\cot(5x)$$

Step 1: Use Sum-to-Product Formulas For the numerator:

$$\sin(3x) - \sin(7x) = 2\cos\left(\frac{3x + 7x}{2}\right)\sin\left(\frac{3x - 7x}{2}\right)$$

$$= 2\cos(5x)\sin(-2x)$$

$\text{Since } \sin(-x) = -\sin(x) \ \text{[sine is odd function]}, \ \text{this simplifies to:}$

$$-2\cos(5x)\sin(2x)$$

$\text{For the denominator:}$

$$\cos(3x) - \cos(7x) = -2\sin\left(\frac{3x + 7x}{2}\right)\sin\left(\frac{3x - 7x}{2}\right)$$
$$= -2\sin(5x)\sin(-2x)$$

$\text{Since } \sin(-x) = -\sin(x) \ \text{[sine is odd function]}, \ \text{this simplifies to:}$

$$= 2\sin(5x)\sin(2x)$$

Step 2: Simplify

$$\frac{-2\cos(5x)\sin(2x)}{2\sin(5x)\sin(2x)}$$

Cancel $-2\sin(2x)$ and the 2 from both numerator and denominator:

$$= \frac{-\cos(5x)}{\sin(5x)}$$
$$= -\cot(5x)$$

Trigonometric Equations

Trigonometric equations are equations that include **sin, cos, tan**, and sometimes **exponential or logarithmic functions**. The goal is to **solve for x (or θ)**, which means finding the angles that make the equation true.

Recognizing the Type of Equation

There are **four** common types of trig equations:

1. Basic equations → Ex: $\sin(x) = \frac{1}{2}$
2. Equations with multiple angles → Ex: $\sin(2x) = \frac{1}{2}$
3. Quadratic form → Ex: $2\cos^2 x - \cos x - 1 = 0$
4. Equations with exponential or logarithmic terms →

   Ex: $e^{\cos(x)} = 1$

Solving Basic Trigonometric Equations

Example 1: Solve $\sin x = \frac{1}{2}$

1. **Find the reference angle**
   - $\sin x = \frac{1}{2}$ happens at $x = \frac{\pi}{6}$
2. **Determine where sine is positive**
   - **Sine is positive in Quadrants I and II**.
   - So, the two solutions in one cycle $0 \leq x < 2\pi$ are:

$$x = \frac{\pi}{6},\ x = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$$

3. **If no interval is given, write the general solution**
   - Since sine has a **period of 2π**, add $2\pi k$, where k is any integer:

$$x = \frac{\pi}{6} + 2\pi k,\ x = \frac{5\pi}{6} + 2\pi k,\ k \in \mathbb{Z}$$

Solving Multiple Angle Equations

Example 2: Solve $\cos(2x) = -\frac{1}{2}$ **for** $0 \leq x < 2\pi$

1. **Find reference angle**
   - $\cos x = \frac{1}{2}\ at\ x = \frac{\pi}{3}$
   - Cosine is negative in Quadrants II and III → solutions for 2x:

$$2x = \pi - \frac{\pi}{3} = \frac{2\pi}{3},\ 2x = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$$

2. **Find general solutions for 2x**

$$2x = \frac{2\pi}{3} + 2\pi k,\ 2x = \frac{4\pi}{3} + 2\pi k,\ k \in \mathbb{Z}$$

3. **Solve for x by dividing everything by 2**

$$x = \frac{\pi}{3} + \pi k,\ x = \frac{2\pi}{3} + \pi k,\ k \in \mathbb{Z}$$

4. **Find solutions in $\boldsymbol{0 \leq x < 2\pi}$**
   - If $k = 0$: $x = \frac{\pi}{3}, \frac{2\pi}{3}$
   - If $k = 1$: $x = \frac{\pi}{3} + \pi = \frac{4\pi}{3}, \frac{2\pi}{3} + \pi = \frac{5\pi}{3}$

Final Answer:

$$x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}$$

Solving Quadratic Trigonometric Equations

Example 3: Solve $2\cos^2 x - \cos x - 1 = 0$

1. **Factor it: (**$2\cos x + 1)(\cos x - 1) = 0$
2. **Set each factor equal to 0**
   - $2\cos x + 1 \rightarrow \cos x = \frac{1}{2}$
   - $\cos x - 1 = 0 \rightarrow \cos x = 1$
3. **Solve each equation**
   - $\cos x = -\frac{1}{2} \rightarrow x = \frac{2\pi}{3}, \frac{4\pi}{3}$ (Quadrants II & III)
   - $\cos x = 1 \rightarrow x = 0$

Final Answer: $x = 0, \frac{2\pi}{3} + 2\pi n, \frac{4\pi}{3} + 2\pi n$

Solving Logarithmic/Exponential Trig Equations

Example 4: Solve $e^{\sin x} = 2$

1. **Take the natural logarithm (ln) of both sides**

$$\ln e^{\sin x} = \ln 2$$
$$\sin(x) = \ln 2$$

2. **Check if ln 2 is within the range of sine**
   - Since $\ln 2 \approx 0.693$ and $-1 \leq \sin x \leq 1$, we proceed.
3. **Find x**
   - Reference angle: $\sin^{-1}(\ln 2)$
   - Since sine is positive, the solutions are in Quadrants I and II:

$$x = \sin^{-1}(\ln 2),\ x = \pi - \sin^{-1}(\ln 2)$$

4. **Final Answer (if a general solution is needed)**

$$x = \sin^{-1}(\ln 2) + 2\pi k,\ x = \pi - \sin^{-1}(\ln 2) + 2\pi k$$

Summary of Key Steps

- Isolate the trig function
- Find the reference angle
- Determine the quadrants
- Write solutions (general solution or within given interval)
- For multiple angles (2x,3x), solve for the full cycle first, then divide
- For quadratic equations, factor or use the quadratic formula
- For exponentials/logarithms, use ln or rewrite in exponential

Applications and Extensions of Trigonometry

Law of Sines

The **Law of Sines** is a formula that helps us **find missing angles or sides** in a triangle that **doesn't have a right angle** (called an **oblique triangle**). It works when we know at least **one angle and its opposite side**.

Formula:

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

where:

- A, B, C are **angles** of the triangle.
- a, b, c are the **sides opposite** those angles.

When to Use the Law of Sines

Case 1: When you know **two angles and one side (AAS or ASA).**

Case 2: When you know **two sides and a non-included angle (SSA)** → *But watch out! This can give 0, 1, or 2 triangles.*

Case 1: Solving a Triangle (AAS or ASA)

Step-by-Step Example

Solve the triangle if:

- $A = 40°$, $B = 65°$, and $a = 20$. Find side b

Step 1: Find the missing angle

Since the sum of angles in a triangle is always **180°**, we can find C:

$$C = 180° - A - B$$

$$C = 180° - 40° - 65° = 75°$$

Step 2: Use Law of Sines to Find Side b

We set up the Law of Sines:

$$\frac{a}{\sin A} = \frac{b}{\sin B} \rightarrow \frac{20}{\sin 40} = \frac{b}{\sin 65}$$

Step 3: Solve for b

Multiply both sides by **sin 65**:

$$b = \frac{20 \times \sin 65}{\sin 40}$$

Using a calculator (make your in-degree mode):

$$b \approx \frac{20 \times 0.9063}{0.6428}$$

$$b \approx 28.2$$

Case 2: SSA (The Ambiguous Case)

WARNING: This case can give **0, 1, or 2 triangles**!

Example: Solve the triangle if:

- $A = 50°,\ a = 10,\ \text{and}\ b = 12$

Step 1: Set Up Law of Sines

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{10}{\sin 50} = \frac{12}{\sin B}$$

Step 2: Solve for B

Multiply both sides by sin B:

$$\sin B = \frac{12 \times \sin 50}{10}$$

Using a calculator (make sure you're in the correct mode):

$$\sin B = \frac{12 \times 0.7660}{10}$$

$$\sin B \approx 0.9192$$

Now use $\sin^{-1}(x)$ inverse sine to find B:

$$B = \sin^{-1}(0.9192)$$

$$B \approx 66.8°$$

Step 3: Find Possible Angles for C
Since angles in a triangle add up to **180°**, we get:

$$C = 180 - A - B$$
$$C = 180 - 50 - 66.8$$
$$C \approx 63.2$$

Now, check if a second triangle is possible:
$$A + B_2 < 180°$$
$$50 + 113.2 = 163.2$$

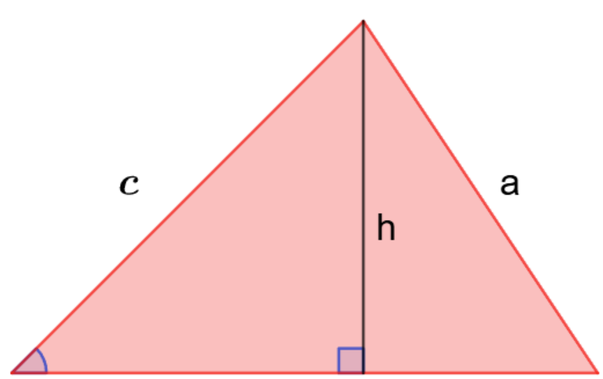
Since $163.2 < 180$, there IS another triangle!

For the second triangle:
$$C_2 = 180 - A - B_2$$
$$C_2 = 180 - 50 - 113.2$$
$$C_2 = 16.8$$

So, **two triangles exist!**

The Ambiguous Case (SSA)
If you are given **two sides** (a and c) and an **angle opposite one of them** (A), the number of possible triangles depends on "h," which is the length of the altitude.
The altitude is calculated as: $\mathrm{h} = c \sin A$

Number of Triangles	**Condition**
One Triangle	**If** $a > \mathrm{h}$, **and** $a > c$ (a is longer than the altitude and the other given side)
One Right Triangle	**If** $a = \mathrm{h}$ (a perfectly matches the altitude, forming a right triangle)
No Triangle	**If** $a < \mathrm{h}$ (a is too short to reach the base, so no triangle is possible)
Two Triangles	**If** $a > \mathrm{h}$, **and** $a < c$ (a is long enough to reach but not longer than the other given side, creating two possible triangles)



$$h = csinA$$

Application Problem: Finding the Area of an Oblique Triangle

If you're asked to find the **area of a triangle using two sides and an included angle**, use:

$$Area = \frac{1}{2}bc\sin A$$

Example: Find the area of a triangle with:

- $b = 8,\ c = 12$ and A=40

$$Area = \frac{1}{2}(8)(12)\sin 40$$

$$Area \approx 30.9$$

Final Answer: The area is about 30.9 square units.

Law of Cosines

The **Law of Cosines** helps you solve triangles when you **don't have a right angle** and cannot use the Pythagorean Theorem. It is especially useful when:

- You know two sides and the included angle (SAS)
- You know all three sides (SSS) and need to find angles

Formula

For a triangle with sides **a, b, c** and opposite angles **A, B, C**, the **Law of Cosines** states:

$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$b^2 = a^2 + c^2 - 2ac \cos B$$
$$c^2 = a^2 + b^2 - 2ab \cos C$$

Think of it as the **Pythagorean Theorem with an adjustment for angles**.

When to Use the Law of Cosines

1. **SAS (Side-Angle-Side)**
   - If you are given two sides and an angle between them, you use the Law of Cosines to find the missing side**.**
2. **SSS (Side-Side-Side)**
   - If you are given all three sides, you use the Law of Cosines to find an angle.

Step-by-Step Examples

Finding a Missing Side (SAS)

- **Given:**
- $b = 7,\ c = 9\ and\ A = 120°$

Find: a

Steps

1. **Use the formula:**
$$a^2 = b^2 + c^2 - 2bc \cos A$$
2. **Plug in the values:**
$$a^2 = 7^2 + 9^2 - 2(7)(9) \cos 120$$

3. **Calculate:**

$$a^2 = 49 + 81 - 126\cos 120$$
$$a^2 = 49 + 81 + 63$$
$$a^2 = 193$$

4. **Take the square root:**

$$a = \sqrt{193} \approx 13.89$$

Final Answer: $a \approx 13.89$

Example 2: Finding a Missing Angle (SSS)

- **Given:**

$$a = 6, b = 9, c = 4$$

- **Find:** B

Steps

1. **Use the rearranged formula for cosine:**

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

2. **Plug in the values:**

$$\cos B = \frac{6^2 + 4^2 - 9^2}{2(6)(4)}$$

$$\cos B = \frac{36 + 16 - 81}{48}$$

$$\cos B = -\frac{29}{48}$$

3. **Take the inverse cosine:**

$$B = \cos^{-1}\left(-\frac{29}{48}\right)$$
$$B \approx 127°$$

- **Final Answer**: $B \approx 127°$

Heron's Formula

If you are given **all three sides** of a triangle and need to find its **area**, use **Heron's Formula**.

Formula

1. **Find s, the semi-perimeter:**

$$s = \frac{a + b + c}{2}$$

2. **Use the formula:**

$$Area = \sqrt{s(s - a)(s - b)(s - c)}$$

Example 3: Finding the Area Using Heron's Formula

- **Given:**

$$a = 12, b = 16, c = 24$$

- **Find:** Area

Steps

1. **Find s:**

$$s = \frac{12 + 16 + 24}{2} = 26$$

2. **Use Heron's Formula:**

$$Area = \sqrt{26(26 - 12)(26 - 16)(26 - 24)}$$
$$Area = \sqrt{26(14)(10)(2)}$$
$$Area = \sqrt{7280} \approx 85$$

Final Answer: **85 square units**

Real-World Application Problems

Example 4: Finding Distance Between Two Ships

Two ships leave a harbor at the same time:

- Ship A travels **S10°W** at **11 mph**.
- Ship B travels **N75°E** at **9 mph**.
- Find the distance between them after **3 hours**.

Steps

1. **Find the distances each ship traveled:**

- Ship A: $d_A = 11 \times 3 = 33$
- Ship B: $d_B = 9 \times 3 = 27$

2. **Find the angle between them:**

$$\theta = 10 + 75 = 85$$

3. **Use the Law of Cosines:**

$$d^2 = 33^2 + 27^2 - 2(33)(27)\cos 85$$

4. **Calculate:**

$$d^2 = 1089 + 729 - 2(33)(27)\cos 85$$
$$d^2 = 1089 + 729 - 155.3$$
$$d^2 = 1662.7$$
$$d = \sqrt{1662.7} \approx 40.8$$

- **Final Answer**: **40.8 miles apart**

Example 5: Finding the Cost of Land

A triangular piece of commercial land has:

- $a = 260\ ft,\ b = 360\ ft,\ 420\ ft$
- The price is **$3.25 per square foot**.
- Find the total cost.

Steps

Use **Heron's Formula** to find the area:

- $s = \frac{260+360+420}{2} = 520$

$$Area = \sqrt{520(520-260)(520-360)(520-420)}$$

$$Area = \sqrt{520(260)(160)(100)}$$

$$Area \approx 46510$$

Multiply by the cost per square foot:

$$Cost = 46510 \times 3.25 = 15157.5,$$

Final Answer: **$15,157.5**

Tips and Tricks

- Use Law of Cosines first if you have SAS or SSS.
- Switch to Law of Sines after finding an angle, it's easier!
- For obtuse angles, check if cos(θ) is negative.
- Use Heron's Formula when given three sides and need the area.

Conclusion

- Law of Cosines: Solves SAS and SSS triangles.
- Heron's Formula: Finds the area when given all three sides.
- Application Problems: Useful in navigation, land surveying, and real-world scenarios.

Polar Coordinates

Polar coordinates are another way to locate points on a plane. Instead of using (x, y) like in rectangular (Cartesian) coordinates, we use (r, θ) where:

r = distance from the origin (how far the point is from the center)

θ = angle measured from the positive x-axis (in degrees or radians)

Plotting Points in Polar Coordinates

Steps to Plot a Point (r, θ)

1. **Find the angle θ**:
    - If θ is positive, move counterclockwise from the positive x-axis
    - If θ is negative, move clockwise
2. **Move out by r**:
    - If r is positive, move in the direction of θ
    - If r is negative, move in the opposite direction of θ

Example 1: Plot (3, 45°)

- Start from the positive x-axis and rotate **45° counterclockwise**
- Move **3 units** along this direction
- That's your point!

Example 2: Plot (-3, 135°)

- Start from the positive x-axis and rotate **135° counterclockwise**
- Since **r is negative**, move **3 units in the opposite direction** (toward -45°)
- That's your point!

Converting Between Polar and Rectangular Coordinates

You can switch between coordinate systems using these formulas.

Convert Polar (r, θ) → Rectangular (x, y)

Use:

$$x = r\cos\theta$$
$$y = r\sin\theta$$

Example: Convert (4, 30°) to rectangular

$$x = 4\cos 30 = 4 \times \frac{\sqrt{3}}{2} = 2\sqrt{3} \approx 3.46$$
$$y = 4\sin 30 = 4 \times \frac{1}{2} = 2$$

So, the rectangular coordinates are **($2\sqrt{3}$, 2)**

Convert Rectangular (x, y) → Polar (r, θ)

Use:

$$r = \sqrt{x^2 + y^2}$$
$$\theta = \tan^{-1}\left(\frac{y}{x}\right)$$

Example: Convert (3, 3) to polar

$$r = \sqrt{3^2 + 3^2} = \sqrt{18} = 3\sqrt{2}$$
$$\theta = \tan^{-1}\left(\frac{3}{3}\right) = \tan^{-1}(1) = 45°$$

So, the polar coordinates are **($3\sqrt{2}$, 45°)**

Multiple Polar Representations

A single point in polar coordinates has **many** representations

Rules:

1. **Adding or subtracting 360° (or 2π in radians) does not change the point**
   - Example: (3, 45°) is the same as (3, 405°) or (3, -315°)
2. **Using a negative r reverses the direction by 180°:**
   - Example: (3, 45°) is the same as (-3, 225°)

Converting Equations Between Polar and Rectangular

You can rewrite equations from one system to another.

Polar to Rectangular

Use:

- $x = r\cos\theta$
- $y = r\sin\theta$
- $x^2 + y^2 = r^2$

Example: Convert $r = 5\sin\theta$ to rectangular.

$$r = 5\sin\theta$$

Multiply both sides by r:

$$r^2 = 5r\sin\theta$$

Using $r^2 = x^2 + y^2$ and $rsin\theta = y$,

$$x^2 + y^2 = 5y$$

$$x^2 + y^2 - 5y = 0$$

$$x^2 + \left(y^2 - 5y - \frac{25}{4}\right) = \frac{25}{4} \text{ [Complete the square]}$$

$$x^2 + \left(y - \frac{5}{2}\right)^2 = \frac{25}{4}$$

which is a circle.

Rectangular to Polar

Use:

- $x^2 + y^2 = r^2$
- $\tan\theta = \frac{y}{x}$
- $r\cos\theta = x,\ r\sin\theta = y$

Example: Convert $x^2 + y^2 = 9$ to polar. Since $x^2 + y^2 = r^2$, we get:

$$(r\cos\theta)^2 + (r\sin\theta)^2 = 9$$

$$r^2\cos^2\theta + r^2\sin^2\theta = 9$$

$$r^2(\cos^2\theta + \sin^2\theta) = 9$$

$$r^2(1) = 9$$

$$r^2 = 9$$

$$r = \pm 3$$

So, the equation in polar form is $r = \pm 3$.

Tricks and Tips

- When converting points, always check the quadrant to get the correct angle!
- Remember that angles can have multiple equivalent representations.
- In polar coordinates, equations often describe spirals, roses, and limacons.

Graphs of Polar Equations

A **polar graph** is a way of plotting points using the **polar coordinate system** instead of the usual rectangular (Cartesian) system. In polar graphs:

- **Each point is given as** (r, θ), where:
  - r is the distance from the **pole** (origin).
  - θ is the angle (measured from the positive x-axis).
- Equations in polar form involve r and θ, such as $r = 2\cos\theta$ or $r = 3 + 2\sin\theta$.

Steps for Graphing Polar Equations

Method 1: Point-Plotting Method

1. **Make a table of values**:
   - Pick angle values (θ) in radians or degrees (e.g. $0°,\ 30°, 45°, 60°, 90°$).
   - Compute r using the given equation.
   - Write down ordered pairs (r, θ).
2. **Plot the points**:
   - Use a polar grid (which looks like concentric circles with angle lines).
   - Move out r units in the direction of θ.
3. **Connect the points smoothly**:
   - Use a smooth curve to connect the plotted points.
   - If needed, continue plotting more points for accuracy.

Example: Graph $r = 4\cos\theta$

- Compute values for angles (e.g., $\theta = 0°, 30°, 60°$, etc.).
- Plot points like $(4{,}0°), (3.46,\ 30°), (2{,}60°)$.
- Connect them smoothly—it forms a circle!

Method 2: Using Symmetry to Graph Quickly

Graphs of polar equations often have **symmetry**, which means you don't need to plot every point. There are **three main types of symmetry**:

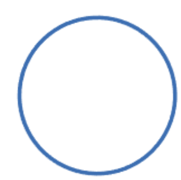
Symmetry Type	How to Check	Meaning
Polar Axis (x-axis)	Replace θ with $-\theta$	Reflect across the x-axis
Line $\theta = \frac{\pi}{2}$ (y-axis)	Replace (r, θ) with $(-r, -\theta)$	Reflect across the y-axis
Pole (Origin)	Replace r with $-r$	Reflect through the origin

Shortcut: If an equation has only sine terms $\sin(\theta)$, it's symmetric with respect to the **y-axis**. If it has only cosine terms $\cos\theta$, it's symmetric with respect to the **x-axis**.

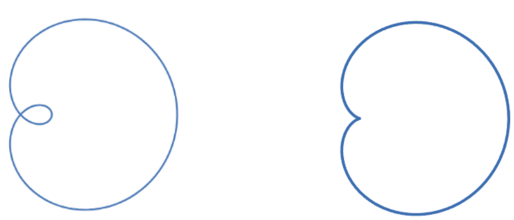
Special Polar Graphs (Common Shapes)

Different polar equations create different graphs. Here are some common ones:

1. **Circles**: $\boldsymbol{r = a\cos\theta}$ or $\boldsymbol{r = a\sin\theta}$
    - Example: $r = 3\cos\theta$→ Circle centered at (1.5, 0).

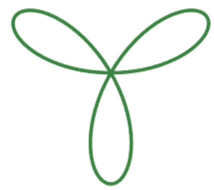


2. **Limaçons (Snail Shapes)**: $\boldsymbol{r = a + b\cos\theta \ \ or\ r = a + b\sin\theta}$
    - When $a < b$ → Inner loop appears.
    - When $a = b$ → Heart-shaped graph (cardioid).

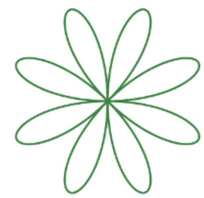


3. **Rose Curves**: $\boldsymbol{r = a\sin(n\theta)}$ **or** $\boldsymbol{r = a\cos(n\theta)}$
   - Even n → 2n petals.
   - Odd n → n petals.
   - Example: $r = 3\cos 4\theta$ → 8-petal flower.

$r = 2\sin(3\theta)$

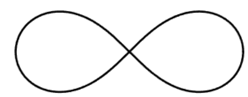


$r = 3\sin(3\theta)$



$r = 3\sin(4\theta)$

4. **Lemniscates (Infinity Shapes)**: $\boldsymbol{r^2 = a^2\cos(2\theta)}$ **or** $\boldsymbol{r^2 = a^2\sin 2\theta}$
   - Graph looks like a sideways figure-eight.



Tips and Tricks

- **Use symmetry**: If you detect symmetry, you only need to plot half or a quarter of the graph.
- **Use a polar grid**: This helps visualize where to plot points.
- **Convert to rectangular if needed**: Sometimes converting polar to rectangular form makes understanding easier.
- **For sine-based graphs**, expect reflections across the **y-axis**.
- **For cosine-based graphs**, expect reflections across the **x-axis**.
- **Use your calculator**: Make sure it's in **radian mode** for accurate calculations.

Final Summary

- **Polar graphs use** (r, θ) instead of (x, y).
- **Plotting points** helps visualize graphs.
- **Symmetry tests** help graph faster.
- **Common graphs**: Circles, limaçons, rose curves, and lemniscates.
- **Use a polar grid** for accuracy.

Complex Numbers in Polar Form; DeMoivre's Theorem

Complex Numbers in Polar Form

A complex number is usually written in the form:

$$z = a + bi$$

where:

- a is the **real part**.
- b is the **imaginary part**.
- i is the imaginary unit, where $i^2 = -1$.

Converting a Complex Number to Polar Form

Instead of writing a complex number as $a + bi$, we can express it using its **magnitude (r)** and **angle (θ)**.

Find the Magnitude (r)

The magnitude (also called modulus) is the distance of the complex number from the origin in the complex plane:

$$r = \sqrt{a^2 + b^2}$$

- This is just the **Pythagorean theorem**.

Find the Angle (θ) (Argument)

The angle is found using the tangent function:

$$\boldsymbol{\theta = \tan^{-1}\left(\frac{b}{a}\right)}$$

- If the complex number is in **Quadrant II or III**, add 180 or π radians to the angle.

Write in Polar Form

$$z = r(\cos\theta + i\sin\theta)$$

- This is also called **trigonometric form**.

Example

Convert $z = 3 + 4i$ to polar form.

- **Step 1: Find r**

$$r = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$$

- **Step 2: Find θ**

$$\theta = \tan^{-1}\left(\frac{4}{3}\right) \approx 53.1°$$

- **Step 3: Write in Polar**

$$z = 5\cos 53.1 + i\sin 53.1$$

Shortcut: You can write this as:

$$5cis(53.1)$$

(where **cis** means $\cos x + i\ \sin x$).

DeMoivre's Theorem

This theorem is super useful for raising complex numbers to powers easily.

$$[z = r(\cos\theta + i\sin\theta]^n = r^n(\cos(n\theta) + i\sin(n\theta))$$

Steps:

1. Raise the modulus r to the power n: r^n
2. Multiply the angle θ by n.
3. Write the result in polar form.

Example

Find $(1 + i)^6$ using DeMoivre's Theorem.

Step 1: Convert to Polar Form

$$r = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\theta = \tan^{-1}\left(\frac{1}{1}\right) = 45°\ \left(or\,\frac{\pi}{4}\right)$$

So,$1 + i = \sqrt{2}\ cis\ 45$

Step 2: Apply DeMoivre's Theorem

$$\left(\sqrt{2}\,cis\ 45\right)^6 = \left(\sqrt{2}\right)^6 cis\ (6 \times 45)$$

$$= 8\ cis\ 270$$

Step 3: Convert Back to Rectangular Form

- $\cos(270) = 0, \sin(270) = -1$
- So:

$$8(\cos 270 + i \sin 270) = 8(0 - i) = -8i$$

Thus,

$$(1 + i)^6 = -8i$$

Finding Roots of Complex Numbers (Using DeMoivre's Theorem)

If you need to find the **nth roots** of a complex number:

$$z_k = \sqrt[n]{r}(\cos\left(\frac{\theta + 360k}{n}\right) + i \sin\left(\frac{\theta + 360k}{n}\right)$$

For $k = 0,1,2, \dots, (n - 1)$

Example: Find the **cube roots** of $8\ cis\ 240$

1. **Find $r^{\frac{1}{3}}$:**

$$\sqrt[3]{8} = 2$$

2. **Find the angles:**

$$\theta_k = \frac{240 + 360k}{3}$$

- For $k = 0: \theta_0 = \frac{240}{3} = 80°$
- For $k = 1: \theta_1 = \frac{240+360}{3} = 200$
- For $k = 2: \theta_2 = \frac{240+720}{3} = 320$

3. **Write the Roots:** $2\ cis\ 80,\ 2\ cis\ 200,\ 2\ cis\ 320$

   These are the three cube roots of 8 cis 240

Key Takeaways & Tricks

- **Magnitude (r)** = Distance from the origin. Use $r = \sqrt{a^2 + b^2}$
- **Angle (θ)** = Use $\theta = \tan^{-1}(\frac{b}{a})$, but adjust for quadrants.
- **Polar Form**: $z = r(\cos\theta + i\sin\theta)$, also written as $r\ cis\theta$
- **DeMoivre's Theorem**: $z^n = r^n(\cos(n\theta) + i\sin(n\theta))$
- **Roots**: Use the formula with k=0,1,2,...(n−1)k = 0, 1, 2, ... (n-1) to find **n** roots.

Shortcut for Multiplication/Division:

- Multiply: Multiply r's and **add** angles.
- Divide: Divide r's and **subtract** angles.

Vectors

A **vector** is a quantity that has **both direction and magnitude (size).** It's different from a regular number (called a **scalar**) because it tells you **how much** and **which way.**

Think of it like this:

- A **scalar** is like saying, "I walked 5 miles."
- A **vector** is like saying, "I walked 5 miles north."

A vector always has two parts:

1. **Magnitude** – How big it is (the length).
2. **Direction** – The way it points.

How Do We Represent a Vector?

A vector is often written in the form **v = (a, b)** or **v = ai + bj**, where:

- **a** = horizontal component (how far it moves left/right).
- **b** = vertical component (how far it moves up/down).
- **i** and **j** are unit vectors (think of them as labels for the x and y directions).

Example:

If **v = (3, 4)**, this means the vector moves **3 units right** and **4 units up**.

Magnitude of a Vector

The **magnitude** (or length) of a vector **v = (a, b)** is found using the **Pythagorean theorem**:

$$|v| = \sqrt{a^2 + b^2}$$

Example:

If **v = (3, 4)**, then:

$$|v| = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$$

Trick to Remember:

It's just the distance formula! Think of the vector as forming a right triangle.

Adding & Subtracting Vectors

You **add** or **subtract** vectors by combining their components.

Vector Addition

If **u = (a_1, b_1)** and **v = (a_2, b_2)**, then:

$$u + v = (a_1 + a_2,\ b_1 + b_2)$$

Example:

If **u = (2, 3)** and **v = (4, 1):**

$$u + v = (2 + 4,3 + 1) = (6,4)$$

Vector Subtraction

If **u = (a_1, b_1)** and **v = (a_2, b_2)**, then:

$$u - v = (a_1 - a_2, b_1 - b_2)$$

Example:

If **u = (5, 7)** and **v = (3, 2):**

$$u - v = (5 - 3,7 - 2) = (2,5)$$

Trick to Remember:

- Adding moves **in the same direction**.
- Subtracting moves **in the opposite direction**.

Scalar Multiplication

Multiplying a vector by a number (**scalar**) changes its length but not its direction.

$$k \cdot v = (ka, kb)$$

Example:

If **v = (2, 3)** and **k = 3**, then:

$$3v = (3 \cdot 2,3 \cdot 3) = (6,9)$$

Trick to Remember:

- If **k > 1**, the vector gets **longer**.
- If **0 < k < 1**, the vector gets **shorter**.
- If **k is negative**, the vector **flips direction**.

Unit Vectors

A **unit vector** is a vector with a length (magnitude) of **1**.
You find a unit vector by **dividing a vector by its magnitude**.

$$u = \frac{v}{|v|}$$

Example:
If **v = (3, 4)**:

1. Find **|v|**: $|v| = \sqrt{3^2 + 4^2} = \sqrt{9 + 16} = \sqrt{25} = 5$
2. Divide each component by 5:

$$u = (\frac{3}{5}, \frac{4}{5})$$

This new vector **still points in the same direction**, but its length is **exactly 1**.

Dot Product of Vectors

The **dot product** helps find the angle between two vectors.
Formula:

$$u \cdot v = a_1 a_2 + b_1 b_2$$

Example:
If **u = (2, 3)** and **v = (4, 5)**:

$$u \cdot v = (2 \cdot 4) + (3 \cdot 5) = 8 + 15 = 23$$

If the dot product = **0**, the vectors are **perpendicular (at 90°)**.

Direction Angle of a Vector

If you want to find **the angle θ a vector makes with the x-axis**, use:

$$\tan\theta = \frac{b}{a}$$

Example: If **v = (3, 4):**

$$\tan\theta = \frac{4}{3}$$

Find **θ** using the inverse tangent function:

$$\theta = \tan^{-1}\left(\frac{4}{3}\right) \approx 53.13°$$

Writing a Vector Using Magnitude & Direction

Sometimes, instead of **(a, b)**, we write vectors using **magnitude and angle**:

$$v = |v|(\cos\theta i + \sin\theta j)$$

Example:

If **|v| = 5** and **θ = 53.13°**:

$$v = 5(\cos(53.13)i + \sin(53.13)j)$$

Use a calculator to find:

$$\cos 53.13 \approx 0.6, \sin 53.13 \approx 0.8$$

So:

$$v \approx (5 \times 0.6)i + (5 \times 0.8)j$$

$$v = (3,4)$$

Trick to Remember:

This is useful when converting between rectangular (x, y) and polar (r, θ) forms of a vector.

Summary of Key Formulas

Concept	Formula
Magnitude	$\sqrt{a^2 + b^2}$
Addition	$u + v = (a_1 + a_2,\ b_1 + b_2)$
Subtraction	$u - v = (a_1 - a_2,\ b_1 - b_2)$
Scalar Multiplication	$kv = (ka, kb)$
Unit Vector	$u = \frac{v}{\lvert v\rvert}$
Dot Product	$u \cdot v = a_1 a_2 + b_1 b_2$
Angle Between Vectors	$\cos\theta = \frac{u \cdot v}{\lvert u\rvert \cdot \lvert v\rvert}$
Direction Angle	$\tan\theta = \frac{b}{a}$
Vector in Magnitude & Direction Form	$v = \lvert v\rvert(\cos\theta i + \sin\theta j)$

Final Tips

- Remember the Pythagorean theorem for finding magnitude.
- Addition and subtraction are just combining x and y parts.
- Multiplying by a negative number flips the direction.
- The dot product = 0 means vectors are perpendicular.
- Use inverse tangent to find the direction angle.

Conics

Ellipse

An **ellipse** is like a stretched-out circle. Instead of having one center like a circle, an ellipse has **two special points** called **foci** (plural of focus).

Definition: An ellipse is the set of all points where the sum of the distances from **two fixed points (the foci)** is always the same.

The Standard Equation of an Ellipse

Ellipses come in **two orientations**:

1. **Horizontal Ellipse (stretched left-right)**

$$\frac{(x-\mathrm{h})^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

- The **major axis** (longest part) is **horizontal**.
- The **foci, vertices, and co-vertices** lie along the **x-axis**.

2. **Vertical Ellipse (stretched up-down)**

$$\frac{(x-\mathrm{h})^2}{b^2}+\frac{(y-k)^2}{a^2}=1$$

- The **major axis** is **vertical**.
- The **foci, vertices, and co-vertices** lie along the **y-axis**.

Key Parts of an Ellipse

Step 1: Find the Center

- The **center** is always at (h, k) in the equation.
- If the equation is not written nicely, **complete the square** to rewrite it in standard form!

Step 2: Identify a and b

- $\boldsymbol{a^2}$ **is always the BIGGER denominator!**
- $\boldsymbol{b^2}$ **is the smaller denominator.**
- **A** is the distance from the **center to the vertices** (longest stretch).
- **b** is the distance from the **center to the co-vertices** (shorter stretch).

Step 3: Find the Major Axis Direction

- If the bigger number is under x^2, the major axis is horizontal.
- If the bigger number is under y^2, the major axis is vertical.

Step 4: Find the Vertices and Co-vertices

For Horizontal Ellipses:

- Vertices: $(\mathrm{h} \pm a, k)$
- Co-vertices: $(\mathrm{h}, k \pm b)$

For Vertical Ellipses:

- Vertices: $(\mathrm{h}, k \pm a)$
- Co-vertices: $(\mathrm{h} \pm b, k)$

Step 5: Find the Foci

The **foci** are always inside the ellipse along the major axis and are found using:

$$c^2 = a^2 - b^2$$

- **For horizontal ellipses:** foci are at $(\mathrm{h} \pm c, k)$
- **For vertical ellipses:** foci are at $(\mathrm{h}, k \pm c)$

Step-by-Step Example

Example 1: Horizontal Ellipse

Given equation:

$$\frac{(x-2)^2}{25} + \frac{(y+3)^2}{9} = 1$$

Step 1: Identify Center

- $\mathrm{h} = 2, k = -3$ → **Center is (2, -3)**.

Step 2: Identify a and b

- $a^2 = 25 \rightarrow a = 5$
- $b^2 = 9 \rightarrow b = 3$

Step 3: Determine Major Axis

- The **bigger denominator (25) is under** x^2, so the major axis is **horizontal**.

Step 4: Find the Vertices

- Since the major axis is **horizontal**, vertices are at $(\mathrm{h} \pm a, k)$:

$$(2 \pm 5, -3) = (-3, -3) \; and \; (7,3)$$

Step 5: Find the Co-vertices

- Since the **minor axis** is vertical, co-vertices are at $(\mathrm{h}, k \pm b)$:

$$(2, -3 \pm 3) = (2,0) \; and \; (2, -6)$$

Step 6: Find the Foci

$$c^2 = a^2 - b^2 = 25 - 9 = 16$$

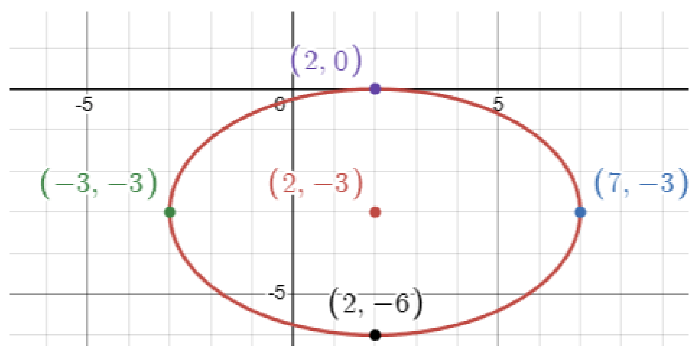
$$c = \sqrt{16} = 4$$

- Foci are at $(\mathrm{h} \pm c, k)$

$$(2 \pm 4, -3) = (-2, -3) \; and \; (6, -3)$$

Final Answer:

- **Center:** $(2, -3)$
- **Vertices:** $(-3, -3), (7, -3)$
- **Co-vertices:** $(2,0),\ (2, -6)$
- **Foci:** $(-2, -3), (6, -3)$
- **Major Axis: Horizontal**



Example 2: Vertical Ellipse

Given equation:

$$\frac{(x+4)^2}{4} + \frac{(y-1)^2}{16} = 1$$

Step 1: Identify Center

- $\mathrm{h} = -4, k = 1 \rightarrow$ **Center is (-4, 1)**.

Step 2: Identify a and b

- $a^2 = 16 \rightarrow a = 4$
- $b^2 = 4 \rightarrow b = 2$

Step 3: Determine Major Axis

- The **bigger denominator (16) is under** y^2, so the major axis is **vertical**.

Step 4: Find the Vertices

- Since the major axis is **vertical**, vertices are at $(\mathrm{h}, k \pm a)$:

$$(-4,1 \pm 4) = (-4,5)\ and\ (-4,-3)$$

Step 5: Find the Co-vertices

- Since the **minor axis** is horizontal, co-vertices are at $(\mathrm{h} \pm b, k)$:

$$(-4 \pm 2,1) = (-6,1)\ and\ (-2,1)$$

Step 6: Find the Foci

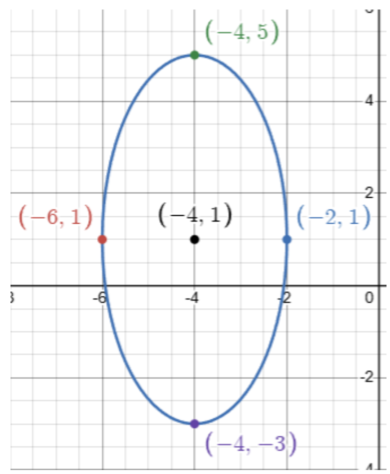
$$c^2 = a^2 - b^2 = 16 - 4 = 12$$
$$c = \sqrt{12} = 2\sqrt{3} \approx 3.46$$

- Foci are at $(\mathrm{h}, k \pm c)$:

$$\left(-4,1 \pm 2\sqrt{3}\right) = \left(-4,1 + 2\sqrt{3}\right)\ and\ \left(-4,\ 1 - 2\sqrt{3}\right)$$

Final Answer:

- **Center:** $(-4,1)$
- **Vertices:** $(-4,5),\ (-4,-3)$
- **Co-vertices:** $(-6,1), (-2,1)$
- **Foci:** $\left(-4,1 + 2\sqrt{3}\right), \left(-4,1 - 2\sqrt{3}\right)$
- **Major Axis: Vertical**



Example 3: Standard Form

Given equation:

$$36x^2 + 100y^2 - 144x + 400y - 3056 = 0$$

Step 1: Identify Center (Complete the Square $c = \left(\frac{b}{2}\right)^2$)

$$(36x^2 - 144x + \quad) + (100y^2 + 400y + \quad) = 3056$$

$$36\left(x^2 - 4x + \left(\frac{4}{2}\right)^2\right) + 100\left(y^2 + 4y + \left(\frac{4}{2}\right)^2\right) = 3056 + 144 + 400$$

$$36(x^2 - 4x + 4) + 100(y^2 + 4y + 4) = 3600$$

$$36(x - 2)^2 + 100(y + 2)^2 = 3600$$

$$\frac{36(x-2)^2}{3600} + \frac{100(y+2)^2}{3600} = 1 \rightarrow \frac{(x-2)^2}{100} + \frac{(y+2)^2}{36} = 1$$

$\mathrm{h} = 2,\ k = -2$ → **Center is (2, -2)**.

Step 2: Identify a and b

- $a^2 = 100 \rightarrow a = 10$
- $b^2 = 36 \rightarrow b = 6$

Step 3: Determine Major Axis

- The **bigger denominator (100) is under x^2**, so the major axis is **horizontal**.

Step 4: Find the Vertices

- Since the major axis is **horizontal**, vertices are at $(\mathrm{h} \pm a, k)$:

$$(2 \pm 10, -2) = (12, -2)\ and\ (-8, -2)$$

Step 5: Find the Co-vertices

- Since the **minor axis** is horizontal, co-vertices are at $(\mathrm{h}, k \pm b)$:

$$(2, -2 \pm 6) = (2,4)\ and\ (2, -8)$$

Step 6: Find the Foci

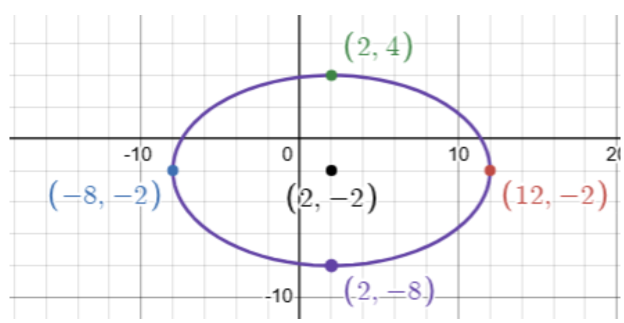
$$c^2 = a^2 - b^2 = 100 - 36 = 64$$
$$c = \sqrt{64} = 8$$

- Foci are at $(\mathrm{h} \pm c, k)$:

$$(2 \pm 8, -2) = (10, -2)\ and\ (-6,\ -2)$$

Final Answer:

- **Center:** $(2, -2)$
- **Vertices:** $(12, -2),\ (-8, -2)$
- **Co-vertices:** $(2,4), (2, -8)$
- **Foci:** $(10, -2), (-6, -2)$
- **Major Axis: Horizontal**



Tricks & Tips

- Always check which denominator is larger! That tells you the direction of the major axis.
- The foci always lie on the major axis. Find c using $c^2 = a^2 - b^2$
- Vertices are always at a distance of a from the center along the major axis.
- If the ellipse equation is not in standard form, complete the square to rewrite it!

Hyperbola

A **hyperbola** is a U-shaped curve that looks like two opposite arcs. Think of it as two parabolas facing away from each other! You'll see them in **satellite dishes**, **bridge structures**, and even **planet orbits**.
A hyperbola is a shape where **the difference in distances from any point on the hyperbola to two fixed points (foci) is always constant**.

Key Parts of a Hyperbola:

- **Foci (plural of focus)** – Two fixed points that define the hyperbola.
- **Center** – The midpoint between the foci.
- **Vertices** – Points closest to the center where the hyperbola starts curving.
- **Asymptotes** – Diagonal lines that the hyperbola approaches but never touches.
- **Transverse Axis** – The line segment that passes through both vertices.
- **Conjugate Axis** – The perpendicular line to the transverse axis.

2. Standard Equations of a Hyperbola

Hyperbolas come in **two types**, depending on the direction they open.

1. **Horizontal Hyperbola (Left-Right Opening)**

$$\frac{(x-\mathrm{h})^2}{a^2}-\frac{(y-k)^2}{b^2}=1$$

- **Opens Left & Right**
- **Vertices:** $(\mathrm{h}\pm a, k)$
- **Foci:** $(\mathrm{h}\pm c, k)$
- **Asymptotes:** $y-k=\pm\frac{b}{a}(x-\mathrm{h})$

2. **Vertical Hyperbola (Up-Down Opening)**

$$\frac{(y-\mathrm{h})^2}{a^2} - \frac{(x-k)^2}{b^2} = 1$$

- **Opens Up & Down**
- **Vertices:** $(\mathrm{h},\ k \pm a)$
- **Foci:** $(\mathrm{h},\ k \pm c)$
- **Asymptotes:** $y - k = \pm\frac{a}{b}(x - \mathrm{h})$

3. How to Identify If a Hyperbola is Horizontal or Vertical
Check which variable comes first!

- If **x^2 is first → It's horizontal** (opens left & right).
- If **y^2 is first → It's vertical** (opens up & down).

4. Finding Key Features of a Hyperbola
Once you have the equation in standard form, follow these steps:

Step 1: Find the Center
The center is (h ,k), taken from the equation's (x−h) and (y−k) terms.

Step 2: Find a and b

- **a** (distance from the center to the vertices) = **√ (denominator under the first term)**.
- **b** (used for asymptotes) = **√ (denominator under the second term)**.

Step 3: Find the Foci Use the formula: $c^2 = a^2 + b^2$
Find **c**, then:

- **Horizontal:** Foci are **(h ± c, k)**
- **Vertical:** Foci are **(h, k ± c)**

Step 4: Find the Asymptotes

- **Horizontal:** $y - k = \pm\frac{b}{a}(x - \mathrm{h})$
- **Vertical:** $y - k = \pm\frac{a}{b}(x - \mathrm{h})$

Step 5: Draw the Hyperbola

1. Plot the **center**.
2. Plot the **vertices**.
3. Draw the **asymptotes** as dashed lines.
4. Sketch the **hyperbola** around the asymptotes.

5. Example Horizontal Hyperbola

Find the key features of:

$$\frac{(x+2)^2}{16} - \frac{(y-3)^2}{9} = 1$$

Step 1: Find the Center

$$(\mathrm{h}, k) = (-2,3)$$

Step 2: Determine the Type

- **x^2 is first**, so it's **horizontal** (opens left & right).

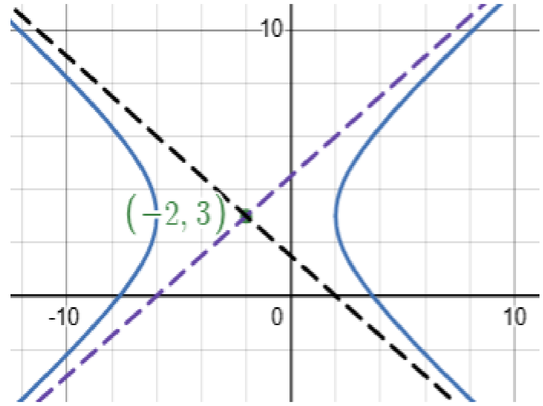
Step 3: Find a and b

- $a^2 = 16$ → **a = 4**
- $b^2 = 9$ → **b = 3**

Step 4: Find the Foci Use:

$$c^2 = a^2 + b^2$$
$$c^2 = 16 + 9 = 25$$
$$c = 5$$

Foci: $(-2 \pm 5,3)$→ $(-7,3)$ **and** $(3,3)$



Step 5: Find the Asymptotes Slope = $\frac{b}{a} = \frac{3}{4}$

Equations:

$$y - 3 = \pm\frac{3}{4}(x + 2)$$

Example Vertical Hyperbola

Find the key features of:

$$\frac{(y+3)^2}{4} - \frac{(x-1)^2}{9} = 1$$

Step 1: Find the Center

$$(\mathrm{h}, k) = (1, -3)$$

Step 2: Determine the Type

- **y² is first**, so it's **vertical** (opens up & down).

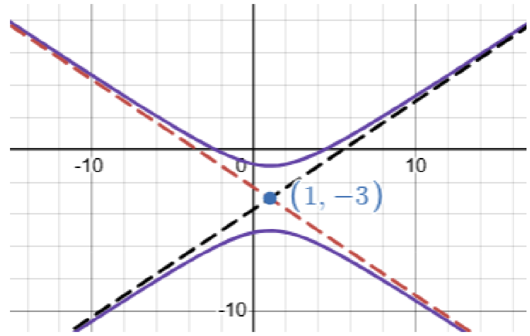
Step 3: Find a and b

- $a^2 = 4 \rightarrow$ **a = 2**
- $b^2 = 9 \rightarrow$ **b = 3**

Step 4: Find the Foci Use:

$$c^2 = a^2 + b^2$$
$$c^2 = 4 + 9 = 13$$
$$c = \sqrt{13}$$

Foci: $(1, -3 \pm \sqrt{13}) \rightarrow (1, -3 + \sqrt{13})$ **and** $(1, -3 - \sqrt{13})$



Step 5: Find the Asymptotes Slope = $\frac{a}{b} = \frac{2}{3}$

Equations:

$$y + 3 = \pm\frac{2}{3}(x - 1)$$

Example Hyperbola (Complete the Square)

Find the key features of:

$$x^2 - y^2 - 4x - 12y - 33 = 0$$

Step 1: Find the Center

$$x^2 - 4x + \left(-\frac{4}{2}\right)^2 - \left(y^2 + 12y + \left(\frac{12}{2}\right)^2\right) = 33 + 4 - 36$$

$$(x-2)^2 - (y+6)^2 = 1$$

$$(\mathrm{h}, k) = (2, -6)$$

Step 2: Determine the Type

- **x² is first**, so it's **horizontal** (opens left & right).

Step 3: Find a and b

- $a^2 = 1$ → **a = 1**
- $b^2 = 1$ → **b = 1**

Step 4: Find the Foci Use:

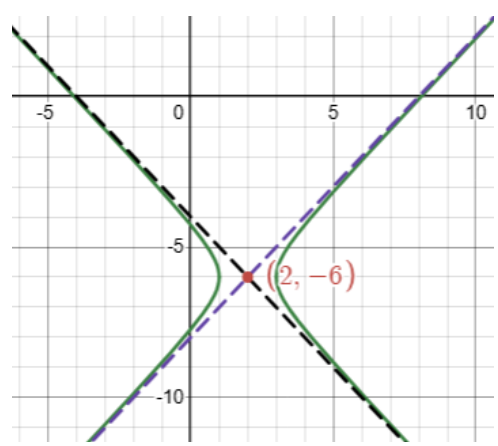
$$c^2 = a^2 + b^2$$
$$c^2 = 1 + 1 = 2$$
$$c = \sqrt{2}$$

Foci: $(2 \pm \sqrt{2}, -6)$→ $(2 + \sqrt{2}, -6)$ **and** $(2 - \sqrt{2}, -6)$

Step 5: Find the Asymptotes Slope = $\frac{b}{a} = \frac{1}{1} = 1$

Equations:

$$y + 6 = \pm 1(x - 2)$$



Quick Tips & Tricks

- Identify if **x^2 or y^2 comes first** to know if it opens **sideways or up/down**.
- Find **a, b, and c** from the denominators.
- Use **a** to find **vertices** and **c** to find **foci**.
- Use **b/a** to find **asymptotes**.
- **Graph the center, vertices, asymptotes, and then sketch the hyperbola.**

Parabola

A **parabola** is a U-shaped curve that appears in many real-world applications, such as satellite dishes, car headlights, and the paths of objects thrown.

Key Parts of a Parabola

- **Vertex (h, k)**: The turning point of the parabola.
- **Axis of Symmetry**: A vertical or horizontal line that splits the parabola into two mirror images.
- **Focus (h, k + p)**: A special point inside the parabola that directs how it curves.
- **Directrix**: A line outside the parabola that helps define its shape.
- **Latus Rectum**: A line segment passing through the focus and perpendicular to the axis of symmetry. Its length is **|4p|**.

Standard Forms of a Parabola

1. Parabola Opening Up or Down

Equation:

$$(y-k)^2 = 4p(x-\mathrm{h})$$

- **Opens right if** $p > 0$, left if $p < 0$.
- **Focus:** $(\mathrm{h}+p, k)$
- **Directrix:** $x = \mathrm{h} - p$
- **Axis of Symmetry:** Horizontal, $y = k$

2. Parabola Opening Left or Right

Equation:

$$(x-k)^2 = 4p(y-\mathrm{h})$$

- **Opens up if** $p > 0$, down if $p < 0$.
- **Focus:** $(\mathrm{h}, k+p)$
- **Directrix:** $y = k - p$
- **Axis of Symmetry:** Vertical, $x = \mathrm{h}$

Tip: p tells you how far the focus and directrix are from the vertex.

Vertex Form of a Parabola

$$y = a(x - \mathrm{h})^2 + k$$

- **If** $a > 0$, parabola opens **up**.
- **If** $a < 0$, parabola opens **down**.
- **Vertex:** (h, k).
- **Axis of Symmetry:** $x = \mathrm{h}$

Tip: This is useful for graphing because the vertex is easily identified.

Finding the Vertex

For a parabola in **standard quadratic form**:

$$y = ax^2 + bx + c$$

The vertex is found using:

$$x = -\frac{b}{2a}$$

Substituting this x value into the equation gives the **y-coordinate** of the vertex.

Step-by-Step Example: Find the Focus and Directrix

Given Equation: $y^2 = 12x$

Step 1: Identify the standard form

Compare the given equation with the standard form of a **horizontally oriented** parabola:

$$y^2 = 4px$$

From $y^2 = 12x$, we see that:

$$4p = 12$$

Solve for p:

$$p = \frac{12}{4} = 3$$

Step 2: Identify the vertex, focus, and directrix

- **Vertex**: $(0,0)$ (since the equation is not shifted)
- **Focus**: $(p, 0) = (3,0)$
- **Directrix**: $x = -p = -3$

Step 3: Graph the Parabola

1. Plot the vertex at (0,0).
2. Plot the focus at (3,0). The parabola **opens right** because $\boldsymbol{p > 0}$.
3. Draw the directrix as the vertical line $\boldsymbol{x = -3}$.
4. Sketch the curve by ensuring the points are equidistant from the focus and directrix.

Answers:

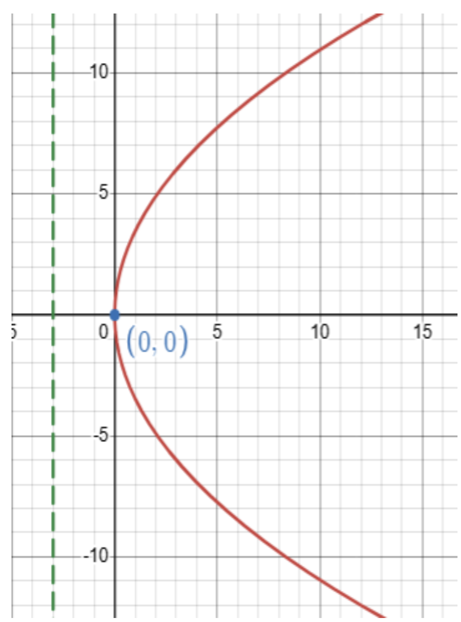
Vertex: (0,0)

Focus: (3,0)

Directrix: $x = -3$

Axis of Symmetry: $y = 0$ (horizontal)

Opening Direction: Right ($p > 0$)



Identifying a Conic Section Without Completing the Square

If you are given an equation in the form:

$$Ax^2 + Cy^2 + Dx + Ey + F = 0$$

You can **quickly** determine what type of conic section it is by checking the values of **A and C**:

Condition	Type of Conic
$A = C$	Circle
$AC = 0$	Parabola
$A \neq C$ and $AC > 0$	Ellipse
$AC < 0$	Hyperbola

Example:
Equation: $8x^2 - 6y^2 - 8x - 36y - 69 = 0$
Here, $A = 8$ and $C = -6$, so **AC < 0** → The equation represents a **hyperbola**.

Graphing a Parabola

1. Identify the **vertex** from the equation.
2. Find the **focus** and **directrix** using p.
3. Determine if it **opens up/down/left/right**.
4. Draw the **axis of symmetry**.
5. Plot a few **points** on each side.
6. Sketch a smooth curve through the points.

Trick: The distance from the **vertex to the focus** = the distance from the **vertex to the directrix**.

Quick Reference Table

Feature	Vertical Parabola	Horizontal Parabola
Standard Form	$(x-\mathrm{h})^2 = 4p(y - k)$	$(y-k)^2 = 4p(x - \mathrm{h})$
Opens	$p > 0$ (Up) $p < 0\ (Down)$	$p > 0$ (Right) $P < 0$ (Left)
Focus	$(\mathrm{h}, k+p)$	$(\mathrm{h}+p, k)$
Directrix	$y = k - p$	$x = \mathrm{h} - p$
Axis of Symmetry	$x = \mathrm{h}$	$y = k$

Parametric Equations

Most of the time, we describe graphs using **x and y** equations (like y = 2x + 3). But sometimes, it's easier to **introduce a third variable** called a **parameter** (usually "t") to describe both x and y.

Instead of writing:

$$y = f(x)$$

We write:

$$x = f(t),\ y = g(t)$$

Where:

- x and y depend on t.
- t is the parameter.

Why Do We Use Parametric Equations?

- They describe motion (like the path of a moving object).
- They are useful for curves that are not functions (like circles).
- They make it easier to describe curves that can't be written as y = f(x).

Basic Example

Let's say a ball is moving in the air. Instead of just using x and y, we can describe the movement using **time (t)**:

$$x = 2t + 3,\ y = -t^2 + 4$$

This means:

- **x changes over time** following $x = 2t + 3$
- **y changes over time** following $y = -t^2 + 4$

If we want to know **where the ball is at t=1:**

$$x = 2(1) + 3 = 5$$
$$y = -(1)^2 + 4 = 3$$

So, the ball is at **(5,3) at t = 1**.

How to Graph Parametric Equations

Steps:

1. Pick values of t (like -2, -1, 0, 1, 2).
2. Find x and y for each t.
3. Plot the points (x, y) on a graph.
4. Connect the points smoothly.
5. Use arrows to show the direction of increasing t.

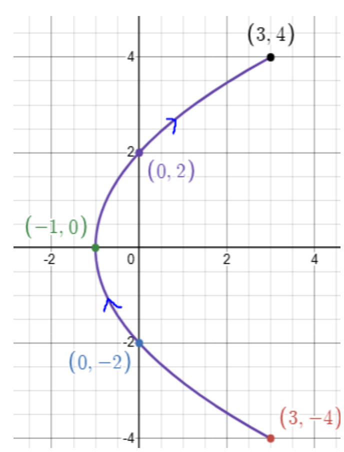
Example:

Given:

$$x = t^2 - 1,\ y = 2t,\ \ -2 \leq t \leq 2$$

t	$x = t^2 - 1$	$y = 2t$	(x, y)
-2	$(-2)^2 - 1 = 3$	$2(-2) = -4$	$(3, -4)$
-1	$(-1)^2 - 1 = 0$	$2(-1) = -2$	$(0, -2)$
0	$(0)^2 - 1 = -1$	$2(0) = 0$	$(-1,0)$
1	$(1)^2 - 1 = 0$	$2(1) = 2$	$(0,2)$
2	$(2)^2 - 1 = 3$	$2(2) = 4$	$(3,4)$

- **Plot the points** (3, -4), (0, -2), (-1,0), (0,2), and (3,4).
- **Connect them smoothly**.
- **Add arrows** to show the direction as t increases.



Eliminating the Parameter (Converting to Rectangular Form)

Sometimes, we want to **get rid of t** and write an equation with just x and y.

How?

1. Solve **one** of the equations for t.
2. Substitute it into the other equation.

Example:

Given:

$$x = t^2,\ y = 2t$$

1. Solve for **t** in terms of **x**: $t = \sqrt{x}$
2. Substitute into **y = 2t**:

$$y = 2\sqrt{x}$$

This is now a normal equation in terms of **x and y**.

- If **t had no restrictions**, we would graph $y = 2\sqrt{x}$ for all x.
- If **t had a range**, we only graph the part that fits.

Graphing Special Parametric Equations

Circles

$$x = r\cos \mathrm{t},\ y = r\sin t,\ 0 \le t \le 2\pi$$

This describes a **circle** with radius r centered at the origin.

Example:

$$x = 3\cos t,\ y = 3\sin t$$

This is a **circle with radius 3**.

Horizontal/Vertical Motion

$$x = at + b,\ y = ct + d$$

- **If a is 0**, movement is vertical.
- **If c is 0**, movement is horizontal.

Final Tricks and Tips

- Always plot points to see the shape.
- Use arrows to show direction.
- If needed, eliminate t to get an x-y equation.
- Circles use $x = r\cos t,\ y = r\sin t$
- Parabolas can be written in parametric form.
- Projectiles follow curved paths (gravity pulls them down).

Conic Sections in Polar Coordinates

What are Conic Sections?
Conic sections are curves formed by slicing a cone at different angles. The four types of conics are:

1. **Circle**
2. **Ellipse**
3. **Parabola**
4. **Hyperbola**

Defining Conics in Polar Form
Instead of the usual x-y coordinate system, we can describe conics using **polar coordinates**, where:

- r = distance from the pole (origin)
- θ = angle from the positive x-axis
- e = **eccentricity** (tells us the shape of the conic)
- p = distance from the focus (pole) to the directrix

The **general equation** of a conic in polar form is:

$$r = \frac{ep}{1 \pm e\cos\theta} \; or \; r = \frac{ep}{1 \pm e\sin\theta}$$

Where:

- **If $e = 1$, the conic is a parabola.**
- **If $e < 1$, the conic is an ellipse.**
- **If $e > 1$, the conic is a hyperbola.**

Identifying the Conic Type Using Eccentricity

- **Parabola**: $e = 1$ (Focus is at the center, one curved branch)
- **Ellipse**: $e < 1$ (Closed, oval shape)
- **Hyperbola**: $e > 1$ (Two separate branches)

Trick to Remember:

- **Ellipses are "squished" circles** because they have eccentricity less than 1.
- **Parabolas are balanced curves**, touching a directrix at one point.
- **Hyperbolas are two curves that never touch**, forming a mirrored shape.

Direction of the Conic

The direction of the conic depends on whether **cosine** or **sine** is in the denominator:

- If the equation has $\cos\theta$ → The directrix is **vertical** (x = p)
- If the equation has $\sin\theta$ → The directrix is **horizontal** (y = p)

Standard Forms of Polar Conics

Conic Type	Equation Form	Directrix
Horizontal Directrix (Parallel to x-axis)	$r = \dfrac{ep}{1 \pm e\sin\theta}$	$y = p$
Vertical Directrix (Parallel to y-axis)	$r = \dfrac{ep}{1 \pm e\cos\theta}$	$x = p$

Positive sign (+) → The directrix is **above** (for sine) or **right** (for cosine).
Negative sign (−) → The directrix is **below** (for sine) or **left** (for cosine).

Step-by-Step: Identifying and Graphing a Conic in Polar Form

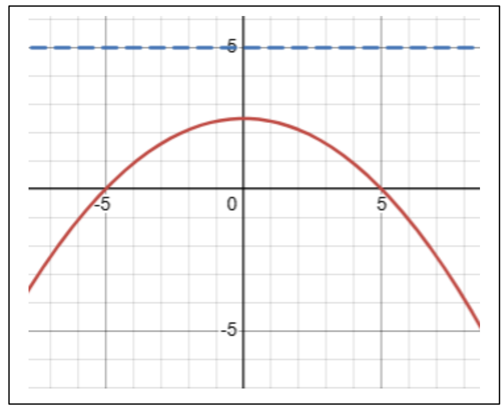
Example 1: Identify and Graph $r = \frac{5}{1+\sin\theta}$

1. **Compare to Standard Form**
   - $\boldsymbol{r = \frac{ep}{1+e\sin\theta}}$
   - Here, $e = 1$, so it's a parabola.
   - The directrix is horizontal, at $y = p$.
   - Since the sign is +, the directrix is above the pole.

2. **Find the Directrix**
   - **Given $ep = 5$ and $e = 1$, solve for p:**

$$p = \frac{5}{1} = 5$$

   - **So, the directrix is at $y = 5$.**



3. **Graphing the Parabola**

- Focus is at the pole.
- Draw a dashed line at $y = 5$.
- Sketch the parabola opening **downward** since the directrix is above.

Example 2: Identify and Graph $r = \frac{12}{4+3\cos\theta}$

1. **Rewrite in Standard Form**
   - Divide numerator and denominator by 4:

$$r = \frac{3}{1 + \frac{3}{4}\cos\theta}$$

- Here, $e = \frac{3}{4}$, so it's an **ellipse**.

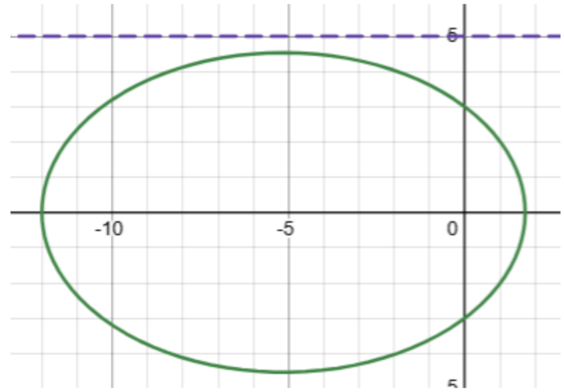
2. **Find the Directrix**
   - Given $ep = 3$, solve for p:

$$p = \frac{3}{\frac{3}{4}} = 4$$

   - Since cosine is present, the directrix is vertical, at $x = 4$.

3. Graphing the Ellipse

- Focus is at the pole.
- Draw a dashed vertical line at $x = 4$.
- The ellipse is horizontally stretched.



Tips for Graphing Polar Conics

- **Find e first** to classify the conic.
- **Look at cosine or sine** to determine vertical or horizontal orientation.
- **Find p** using $ep =$ numerator value.
- **Graph the directrix** first, then draw the conic **opposite the directrix**.
- **Use test points** to plot accurate shapes.

Final Summary

1. Determine e:
   - $e = 1$ → Parabola
   - $e < 1$ → Ellipse
   - $e > 1$ → Hyperbola
2. Check if cos or sin is in the denominator:
   - cos → Vertical directrix
   - sin → Horizontal directrix
3. Find p using $ep =$ numerator.
4. Graph the conic:
   - Focus at the pole.
   - Directrix as a dashed line.
   - Sketch the conic curve in the correct direction.