

# Proofs for Theory Results in The impact of network homophily on bias of multi-hop referrals

## A NOTATIONS

For clarity, we list all notations that are used in our theory presentation in Table 3.

## B PROOF OF THEOREM 4.1

THEOREM 4.1. *For a sequence of networks  $\{\mathcal{G}(N_t, t, r, \rho)\}$  generated by the BPA model with  $r < 0.5$ , for any  $k > 1$ ,*

$$\lim_{t \rightarrow \infty} \frac{\mathbb{E}[AG_{S_L}^{(k)}(R)]}{\mathbb{E}[AG_{S_L}^{(k)}(B)]} = \frac{\beta_2}{\beta_3}, \quad (11)$$

where

$$\beta_2 = \frac{r\rho}{2(1 - (1 - \alpha^*)(1 - \rho))}, \beta_3 = \frac{1 - r}{2(1 - \alpha^*(1 - \rho))}, \quad (12)$$

and  $\alpha^*$  is the unique solution in  $[0, 1]$  of the equation

$$2\alpha^* = 1 - (1 - r) \frac{(1 - \alpha^*)}{1 - \alpha^*(1 - \rho)} + r \frac{\alpha^*}{1 - (1 - \alpha^*)(1 - \rho)}. \quad (13)$$

PROOF. For sequences  $\{a_t, t \geq 1\}$ ,  $\{b_t, t \geq 1\}$  and  $\{c_t, t \geq 1\}$ , we denote by  $c_t = o_t(1)$  if  $\lim_{t \rightarrow \infty} c_t = 0$ ; we denote by  $a_t = O(b_t)$  if there exists some  $C \neq 0$  such that  $a_t/b_t - C = o_t(1)$ .

For the growth model, at each time  $t$ , there are 4 possibilities:

- case 1: a new red node connects a red node;
- case 2: a new red node connects a blue node;
- case 3: a new blue node connects a blue node;
- case 4: a new blue node connects a red node.

In BPA model, the probability for above cases are:

$$\mathbb{P}(\text{case 1 at time } t) = \frac{r\alpha(t)}{2(1 - (1 - \alpha(t))(1 - \rho))}, \quad (14)$$

$$\mathbb{P}(\text{case 2 at time } t) = \frac{r(1 - \alpha(t))\rho}{2(1 - (1 - \alpha(t))(1 - \rho))}, \quad (15)$$

$$\mathbb{P}(\text{case 3 at time } t) = \frac{(1 - r)(1 - \alpha(t))}{2(1 - \alpha(t)(1 - \rho))}, \quad (16)$$

$$\mathbb{P}(\text{case 4 at time } t) = \frac{(1 - r)\alpha(t)\rho}{2(1 - \alpha(t)(1 - \rho))}. \quad (17)$$

$V_C(t)$	the set of nodes in color $C$ at time $t$ ;
$d_i^{(k)}(t)$	the $k$ -hop degree of node $i$ at time $t$ ;
$d_{i,C}^{(k)}(t)$	the number of nodes in color $C$ among the $k$ -hop neighbors of node $i$ at time $t$ ;
$d_{i,C*}^{(k)}(t)$	among the $k$ -hop neighbors of node $i$ at time $t$ , the number of nodes from which the second node on the unique path to $i$ is of color $C$ ;
$D_C^{(k)}(t)$	sum of $k$ -hop degrees over nodes of color $C$ at time $t$ ; that is, $\sum_{i \in V_C(t)} d_i^{(k)}(t)$ ;
$\alpha(t)$	fraction of the sum of 1-hop degree for red nodes over that for all nodes; that is, $D_R^{(1)}(t)/(2t)$ ;
$M_{C_1,**}^{(k)}(t)$	$\sum_{i \in V_{C_1}(t)} d_i^{(1)}(t) d_i^{(k)}(t)$ ;
$M_{C_1,*}^{(k)}(t)$	$\sum_{i \in V_{C_1}(t)} d_i^{(1)}(t) d_i^{(k)}(t)$ ;
$M_{C_1,*C_2}^{(k)}(t)$	$\sum_{i \in V_{C_1}(t)} d_i^{(1)}(t) d_{i,C_2}^{(k)}(t)$ ;
$M_{C_1,C_2*}^{(k)}(t)$	$\sum_{i \in V_{C_1}(t)} d_i^{(1)}(t) d_{i,C_2*}^{(k)}(t)$ with $M_{C_1,C_2*}^{(1)}(t) := M_{C_1,**}^{(1)}(t)$ .

Table 3. Notation

We can write a recursion for  $D_C^{(k)}(t)$  by considering those 4 cases. Denote by  $\{\mathcal{F}_t, t \geq 1\}$  the filtration such that  $\mathcal{F}_t$  contains the information of the graph at time  $t$ . We have

$$\begin{aligned} \mathbb{E}[D_R^{(k)}(t+1) \mid \mathcal{F}_t] &= D_R^{(k)}(t) + \beta_1(t) \frac{M_{R,**}^{(k-1)}(t) + M_{R,*R}^{(k-1)}(t)}{t} \\ &+ \beta_2(t) \frac{M_{B,**}^{(k-1)}(t) + M_{B,*R}^{(k-1)}(t)}{t} + \beta_3(t) \frac{M_{B,*R}^{(k-1)}(t)}{t} + \beta_4(t) \frac{M_{R,*R}^{(k-1)}(t)}{t}, \end{aligned} \quad (18)$$

where

$$\beta_1(t) = \frac{r}{2(1 - (1 - \alpha(t))(1 - \rho))}, \quad (19)$$

$$\beta_2(t) = \frac{r\rho}{2(1 - (1 - \alpha(t))(1 - \rho))}, \quad (20)$$

$$\beta_3(t) = \frac{1-r}{2(1 - \alpha(t)(1 - \rho))}, \quad (21)$$

$$\beta_4(t) = \frac{(1-r)\rho}{2(1 - \alpha(t)(1 - \rho))}. \quad (22)$$

From [2], we see that  $\alpha(t) \rightarrow \alpha^*$  almost surely, where  $\alpha^*$  satisfies the following equation.

$$2\alpha^* = \frac{2r\alpha^* + r(1 - \alpha^*)\rho}{1 - (1 - \alpha^*)(1 - \rho)} + \frac{(1-r)\alpha^*\rho}{1 - \alpha^*(1 - \rho)}. \quad (23)$$

Hence we also have that  $\beta_i(t) \rightarrow \beta_i$  almost surely for  $i = 1, 2, 3, 4$ , where

$$\beta_1 = \frac{r}{2(1 - (1 - \alpha^*)(1 - \rho))}, \beta_2 = \frac{r\rho}{2(1 - (1 - \alpha^*)(1 - \rho))}, \quad (24)$$

$$\beta_3 = \frac{1-r}{2(1 - \alpha^*(1 - \rho))}, \beta_4 = \frac{(1-r)\rho}{2(1 - \alpha^*(1 - \rho))}. \quad (25)$$

Taking expectations on both sides of (18), we get

$$\begin{aligned} \mathbb{E}[D_R^{(k)}(t+1)] &= \mathbb{E}[D_R^{(k)}(t)] + \mathbb{E}\left[\beta_1(t) \frac{M_{R,**}^{(k-1)}(t) + M_{R,*R}^{(k-1)}(t)}{t}\right] \\ &+ \mathbb{E}\left[\beta_2(t) \frac{M_{B,**}^{(k-1)}(t) + M_{B,*R}^{(k-1)}(t)}{t}\right] + \mathbb{E}\left[\beta_3(t) \frac{M_{B,*R}^{(k-1)}(t)}{t}\right] \\ &+ \mathbb{E}\left[\beta_4(t) \frac{M_{R,*R}^{(k-1)}(t)}{t}\right]. \end{aligned} \quad (26)$$

Similarly,

$$\begin{aligned} \mathbb{E}[D_B^{(k)}(t+1)] &= \mathbb{E}[D_B^{(k)}(t)] + \mathbb{E}\left[\beta_3(t) \frac{M_{B,**}^{(k-1)}(t) + M_{B,*B}^{(k-1)}(t)}{t}\right] \\ &+ \mathbb{E}\left[\beta_4(t) \frac{M_{R,**}^{(k-1)}(t) + M_{R,*B}^{(k-1)}(t)}{t}\right] + \mathbb{E}\left[\beta_1(t) \frac{M_{R,*B}^{(k-1)}(t)}{t}\right] \\ &+ \mathbb{E}\left[\beta_2(t) \frac{M_{B,*B}^{(k-1)}(t)}{t}\right]. \end{aligned} \quad (27)$$

We have the following lemma.

LEMMA B.1. *For any color  $C \in \{R, B\}$ , any  $i \in \{1, 2, 3, 4\}$  and any  $k$ , we have that*

$$\frac{\mathbb{E}\left[\beta_i(t) M_{C,**}^{(k-1)}(t)\right]}{\beta_i \mathbb{E}\left[M_{C,**}^{(k-1)}(t)\right]} = 1 + o_t(t^{-1/4}). \quad (28)$$

Using the above lemma, we get

$$\begin{aligned} \mathbb{E}[D_R^{(k)}(t+1)] &= \mathbb{E}[D_R^{(k)}(t)] \\ &+ \beta_1(1 + o_t(1)) \frac{\mathbb{E}\left[M_{R,**}^{(k-1)}(t)\right] + \mathbb{E}\left[M_{R,*R}^{(k-1)}(t)\right]}{t} \\ &+ \beta_2(1 + o_t(1)) \frac{\mathbb{E}\left[M_{B,**}^{(k-1)}(t)\right] + \mathbb{E}\left[M_{B,*R}^{(k-1)}(t)\right]}{t} \\ &+ \beta_3(1 + o_t(1)) \frac{\mathbb{E}\left[M_{B,*R}^{(k-1)}(t)\right]}{t} + \beta_4(1 + o_t(1)) \frac{\mathbb{E}\left[M_{R,*R}^{(k-1)}(t)\right]}{t}. \end{aligned} \quad (29)$$

$$\begin{aligned} \mathbb{E}[D_B^{(k)}(t+1)] &= \mathbb{E}[D_B^{(k)}(t)] \\ &+ \beta_3(1 + o_t(1)) \frac{\mathbb{E}\left[M_{B,**}^{(k-1)}(t)\right] + \mathbb{E}\left[M_{B,*B}^{(k-1)}(t)\right]}{t} \\ &+ \beta_4(1 + o_t(1)) \frac{\mathbb{E}\left[M_{R,**}^{(k-1)}(t)\right] + \mathbb{E}\left[M_{R,*B}^{(k-1)}(t)\right]}{t} \end{aligned}$$

$$+ \beta_1(1 + o_t(1)) \frac{\mathbb{E}[M_{R,*B}^{(k-1)}(t)]}{t} + \beta_2(1 + o_t(1)) \frac{\mathbb{E}[M_{B,*B}^{(k-1)}(t)]}{t}. \quad (30)$$

Assume that  $r < 1/2$ . We need the following lemmas.

LEMMA B.2. *We have that, for  $k \geq 1$ ,*

$$\mathbb{E}[M_{B,**}^{(k)}(t)] = O((\log t)^{k-1} t^{2\beta_2+2\beta_3}), \quad (31)$$

$$\mathbb{E}[M_{B,B*}^{(k)}(t)] = \mathbb{E}[M_{B,**}^{(k)}(t)](1 + o_t(1)), \quad (32)$$

$$\mathbb{E}[M_{B,*B}^{(k)}(t)] = \frac{\beta_3}{\beta_2 + \beta_3} \mathbb{E}[M_{B,**}^{(k)}(t)(1 + o_t(1))], \quad (33)$$

$$\mathbb{E}[M_{R,**}^{(k)}(t)] = o_t(\mathbb{E}[M_{B,**}^{(k)}(t)]). \quad (34)$$

LEMMA B.3. *For a sequence  $\{a_t\}$  satisfying*

$$a_{t+1} = a_t + c_1 d_t \frac{a_t}{t} + c_2 b_t, \quad (35)$$

where  $d_t = (1 + o_t(t^{-\epsilon}))$  for some  $\epsilon > 0$ ,  $b_t = (\log t)^m t^{c_3-1}(1 + o_t(1))$ . We then have that

$$a_t = \begin{cases} O(t^{c_1}), & \text{if } c_1 > c_3 \\ O((\log t)^{m+1} t^{c_3}), & \text{if } c_1 = c_3 \\ \frac{c_2}{c_3 - c_1} (\log t)^m t^{c_3} (1 + o_t(1)), & \text{if } c_1 < c_3. \end{cases} \quad (36)$$

By Lemma B.2, we see that

$$\mathbb{E}[D_R^{(k)}(t+1)] = \mathbb{E}[D_R^{(k)}(t)] + 2\beta_2 \frac{\mathbb{E}[M_{B,**}^{(k-1)}(t)]}{t} (1 + o_t(1)). \quad (37)$$

Similarly, we have

$$\mathbb{E}[D_B^{(k)}(t+1)] = \mathbb{E}[D_B^{(k)}(t)] + 2\beta_3 \frac{\mathbb{E}[M_{B,**}^{(k-1)}(t)]}{t} (1 + o_t(1)). \quad (38)$$

By Lemma B.3 we have that

$$\frac{\mathbb{E}[D_R^{(k)}(t)]}{\mathbb{E}[D_B^{(k)}(t)]} = \frac{\beta_2}{\beta_3} (1 + o_t(1)). \quad (39)$$

□

## C PROOF OF THEOREM 4.2

THEOREM 4.2. *For a sequence of networks  $\{\mathcal{G}(N_t, t, r, \rho)\}$  generated by the BPA model with  $r < 0.5$ , the linear referral strategy exhibits  $k$ -hop referral inequality against red nodes for any  $k \geq 1$ . That is,*

$$\lim_{t \rightarrow \infty} \frac{\mathbb{E}[AG_{S_L}^{(k)}(R)]}{\mathbb{E}[AG_{S_L}^{(k)}(B)]} \leq \frac{r}{1-r}, \quad (40)$$

PROOF. By the definition of  $\beta_2$  and  $\beta_3$ , we know

$$\frac{\beta_2}{\beta_2 + \beta_3} \leq r \Leftrightarrow \frac{\rho}{1 - (1 - \alpha^*)(1 - \rho)} \leq \frac{1}{1 - \alpha^*(1 - \rho)} \quad (41)$$

$$\Leftrightarrow \rho - \alpha^* \rho (1 - \rho) \leq 1 - (1 - \alpha^*)(1 - \rho) \quad (42)$$

$$\Leftrightarrow 1 - \rho - (1 - \rho)(1 - \alpha^* - \alpha^* \rho) \geq 0 \quad (43)$$

$$\Leftrightarrow \alpha^*(1 + \rho)(1 - \rho) \geq 0, \quad (44)$$

which is clearly true by the definition of  $\alpha^*$  and  $\rho$ .  $\square$

#### D PROOF OF COROLLARY 4.4

COROLLARY 4.4. For a sequence of networks  $\{\mathcal{G}(N_t, t, r, \rho)\}$  generated by the BPA model with  $r < 0.5$ , as  $\rho \rightarrow 0$ ,

$$\lim_{t \rightarrow \infty} \frac{\mathbb{E}[AG_{S_L}^{(1)}(R)]}{\mathbb{E}[AG_{S_L}^{(1)}(B)]} = \frac{r}{1 - r} \text{ and } \lim_{t \rightarrow \infty} \frac{\mathbb{E}[AG_{S_L}^{(k)}(R)]}{\mathbb{E}[AG_{S_L}^{(k)}(B)]} = 0. \quad (45)$$

PROOF. From [2], we know that as  $t \rightarrow \infty$ ,  $\frac{\mathbb{E}[D_R^{(1)}(t)]}{\mathbb{E}[D_B^{(1)}(t)]} = \frac{\alpha^*}{(1 - \alpha^*)}$ , where

$$2\alpha^* = \frac{2r\alpha^* + r(1 - \alpha^*)\rho}{1 - (1 - \alpha^*)(1 - \rho)} + \frac{(1 - r)\alpha^*\rho}{1 - \alpha^*(1 - \rho)}. \quad (46)$$

When  $\rho \rightarrow 0$ ,  $\alpha^* \rightarrow r$ , and therefore  $\frac{\mathbb{E}[D_R^{(1)}(t)]}{\mathbb{E}[D_B^{(1)}(t)]} \rightarrow \frac{r}{1 - r}$ .

For  $k \geq 2$ , we know from Theorem 4.1 that, as  $t \rightarrow \infty$ ,

$$\lim_{\rho \rightarrow 0} \frac{\mathbb{E}[D_R^{(k)}(t)]}{\mathbb{E}[D_B^{(k)}(t)]} = \lim_{\rho \rightarrow 0} \frac{\beta_2}{\beta_3} = \lim_{\rho \rightarrow 0} \frac{r\rho}{1 - r} \cdot \frac{2(1 - \alpha^*(1 - \rho))}{2(1 - (1 - \alpha^*)(1 - \rho))} = 0. \quad (47)$$

$\square$

#### E PROOF OF LEMMA 5.1

LEMMA 5.1. For a sequence of networks  $\{\mathcal{G}(N_t, t, r, \rho)\}$  generated by the BPA model with  $r < 0.5$ , as  $t \rightarrow \infty$ , under the random referral strategy and the popularity-driven referral strategy, the ratios of expected 1-hop active gain for the red nodes are the red node population ratio; under the acceptance-driven referral strategy, the ratio of expected 1-hop active gain for red nodes is no greater than the red node population ratio.

PROOF. Denote the expected referral reach for a red (blue) member of degree  $k$  as  $h(k, r)$  ( $h(k, b)$ ), and the probability of a randomly selected red (blue) node having degree  $k$  on  $\mathcal{G}(N_t, t, r, \rho)$  as  $p_t^r(k)$  ( $p_t^b(k)$ ). Note that all the four referral programs satisfy the following three conditions: (1).  $\lim_{t \rightarrow \infty} h(k, i)p_t^i(k) < \infty$ , (2).  $h(k, b) \geq h(k, r)$ , and (3).  $h(k, i)$  being non-decreasing for  $i \in \{r, b\}$ . Furthermore,

$$\lim_{t \rightarrow \infty} \frac{\sum_{k \geq 1} h(k, r)p_t^r(k)}{\sum_{k \geq 1} h(k, b)p_t^b(k)} - \lim_{t \rightarrow \infty} \frac{\sum_{k \geq 1} p_t^r(k)}{\sum_{k \geq 1} p_t^b(k)} \quad (48)$$

$$= \lim_{t \rightarrow \infty} \frac{\sum_{i > j} (h(i, b) - h(j, r)) (p_t^r(i)p_t^b(j) - p_t^r(j)p_t^b(i))}{\sum_{k \geq 1} h(k, b)p_t^b(k) \sum_{k \geq 1} p_t^b(k)}. \quad (49)$$

By [2][Theorem 4.12], we know for all  $i > j$ ,  $\lim_{t \rightarrow \infty} p_t^r(k) \propto k^{-\beta(R)}$  and  $\lim_{t \rightarrow \infty} p_t^b(k) \propto k^{-\beta(B)}$  with  $\beta(R) > \beta(B)$ . Therefore,

$$\lim_{t \rightarrow \infty} \frac{p_t^r(i)}{p_t^r(j)} \leq \lim_{t \rightarrow \infty} \frac{p_t^b(i)}{p_t^b(j)}, \quad (50)$$

which implies that the limit of (49)  $\leq 0$ . As  $\lim_{t \rightarrow \infty} \frac{\sum_{k \geq 1} p_t^r(k)}{\sum_{k \geq 1} p_t^b(k)} = 1$ , we have

$$\lim_{t \rightarrow \infty} \frac{\sum_{k \geq 1} h(k, r) p_t^r(k)}{\sum_{k \geq 1} h(k, b) p_t^b(k)} \leq 1. \quad (51)$$

The equality holds if and only if  $h(k, b) = h(k, r)$  and  $h(k, i)$  is constant in  $k$ , the conditions the *random referral strategy* and the *popularity-driven referral strategy* satisfy.  $\square$

## F PROOF OF COROLLARY 5.1

**COROLLARY 5.1.** *Let  $\{\mathcal{G}(N_t, t, r, \rho)\}$  be a sequence of graphs generated through the BPA model with  $r < 1/2$ . Under the acceptance-driven referral strategy  $\mathcal{S}_a$ , the ratio of expected referral reach for red nodes in the second hop is no greater than that in the first hop. That is,*

$$\lim_{t \rightarrow \infty} \frac{\mathbb{E}[AG_{\mathcal{S}_a}^{(2)}(R)]}{\mathbb{E}[AG_{\mathcal{S}_a}^{(2)}(B)]} \geq \lim_{t \rightarrow \infty} \frac{\mathbb{E}[AG_{\mathcal{S}_a}^{(1)}(R)]}{\mathbb{E}[AG_{\mathcal{S}_a}^{(1)}(B)]}. \quad (52)$$

**PROOF.** We denote the probability that a referral sent from a red (blue) node being accept at the first hop as  $P_r^1$  ( $P_b^1$ ) as  $t \rightarrow \infty$ . As proved in Lemma 5.1, they satisfy  $P_r^1 \leq P_b^1$ . We have the following expression

$$\lim_{t \rightarrow \infty} \frac{\mathbb{E}[AG_{\mathcal{S}_a}^{(1)}(R)]}{\mathbb{E}[AG_{\mathcal{S}_a}^{(1)}(B)]} = \frac{r \cdot P_r^1}{(1-r) \cdot P_b^1}. \quad (53)$$

For the 2-hop, notice that if the 1-hop friend who receives the referral denies referral and randomly gives the referral to one of its friends, the probability that the referral gets accepted at the 2-hop is actually a constant. We thus have that

$$\lim_{t \rightarrow \infty} \frac{\mathbb{E}[AG_{\mathcal{S}_a}^{(2)}(R)]}{\mathbb{E}[AG_{\mathcal{S}_a}^{(2)}(B)]} = \frac{r \cdot (1 - P_r^1)}{(1-r) \cdot (1 - P_b^1)}, \quad (54)$$

Since  $P_r^1 \leq P_b^1$ , it is easy to check that (54) is greater than or equal to (53), and the proof is finished.  $\square$

## G PROOF OF FRIENDSHIP PARADOX IN BPA

**LEMMA G.1. paradox (Friendship Paradox in BPA)** *Let  $\{\mathcal{G}(N_t, t, r, \rho)\}$  be a sequence of graphs generated through the BPA model with  $r < 1/2$ . For  $c_1, c_2 \in \{R, B\}$ , denote  $F_{c_1 c_2}(t)$  as the degree of the  $c_2$  color node of a randomly chosen edge that connects two nodes with  $c_1, c_2$  colors (if  $c_1 = c_2$ , randomly choose one end of the edge). We then have*

$$\lim_{t \rightarrow \infty} \mathbb{E}[F_{c_1 c_2}(t)] \geq \lim_{t \rightarrow \infty} \mathbb{E}[d_{c_2}(t)], \quad (55)$$

where  $d_{c_2}(t)$  is the degree of a randomly chosen  $c_2$  color node.

**PROOF.** We consider the case that  $c_1 = R$  and  $c_2 = B$  (the other 3 cases are similar). The probability that a certain blue node  $v$  is chosen is

$$\frac{d_{v,R}^{(1)}(t)}{\sum_{u \in B} d_{u,R}^{(1)}(t)}. \quad (56)$$

As  $t \rightarrow \infty$ , denote by  $p_k^{(R,B)}$  as the limiting probability that a blue node with degree  $k$  is chosen,  $p_k^B$  as the limiting probability that a blue node has degree  $k$ , and  $R_k(B)$  as the limiting average number of red neighbors for a blue node of size  $k$ . We then have that

$$p_k^{(R,B)} = \frac{p_k^B \cdot R_k(B)}{\sum_{k=1}^{\infty} p_k^B \cdot R_k(B)}. \quad (57)$$

Note that  $R_k(B)$  is non-decreasing with respect to  $k$ . Therefore,  $p_k^{(R,B)}$  puts larger weights on larger  $k$ 's (compared to  $p_k^B$ ), that is, for any  $n$  we have that  $\sum_{k \geq n} p_k^{(R,B)} \geq \sum_{k \geq n} p_k^B$ . Consequently, we have that

$$\lim_{t \rightarrow \infty} \mathbb{E}[F_{c_1 c_2}(t)] = \sum_{k \geq 1} k \cdot p_k^{(R,B)} = \sum_{n \geq 1} \sum_{k \geq n} p_k^{(R,B)} \geq \sum_{n \geq 1} \sum_{k \geq n} p_k^B = \sum_{k \geq 1} k \cdot p_k^B = \lim_{t \rightarrow \infty} \mathbb{E}[d_{c_2}(t)], \quad (58)$$

and it finishes the proof.  $\square$

## H PROOF OF AUXILIARY LEMMAS

LEMMA B.2. *We have that, for  $k \geq 1$ ,*

$$\mathbb{E}[M_{B,**}^{(k)}(t)] = O((\log t)^{k-1} t^{2\beta_2+2\beta_3}), \quad (59)$$

$$\mathbb{E}[M_{B,B*}^{(k)}(t)] = \mathbb{E}[M_{B,**}^{(k)}(t)](1 + o_t(1)), \quad (60)$$

$$\mathbb{E}[M_{B,*B}^{(k)}(t)] = \frac{\beta_3}{\beta_2 + \beta_3} \mathbb{E}[M_{B,**}^{(k)}(t)(1 + o_t(1))], \quad (61)$$

$$\mathbb{E}[M_{R,**}^{(k)}(t)] = o_t(\mathbb{E}[M_{B,**}^{(k)}(t)]). \quad (62)$$

PROOF. We prove it by induction. Recall that  $M_{B,B*}^{(1)}(t) = M_{B,**}^{(1)}(t)$  by definition, and thus (60) trivially holds. We first show that the claims hold for  $k = 1$ . First note that  $M_{B,**}^{(1)}(t+1)$  arises from  $M_{B,**}^{(1)}(t)$  in the following cases:

- (1) A new red node connects with an existing blue node  $i$ : node  $i$  then contributes  $(d_i^{(1)}(t) + 1)^2 - (d_i^{(1)}(t))^2 = 1 + 2d_i(t)$  increments.
- (2) A new blue node connects with an existing red node  $i$ : the new node contributes 1 increment.
- (3) A new blue node connects with an existing blue node  $i$ : node  $i$  then contributes  $(d_i^{(1)}(t) + 1)^2 - (d_i^{(1)}(t))^2 = 1 + 2d_i(t)$  increments, and the the new node contributes 1 increment.

Therefore,

$$\begin{aligned} \mathbb{E}[M_{B,**}^{(1)}(t+1) \mid \mathcal{F}_t] &= M_{B,**}^{(1)}(t) + \beta_2(t) \frac{D_B^{(1)}(t) + 2M_{B,**}^{(1)}(t)}{t} \\ &\quad + \beta_3(t) \left( 1 + \frac{D_B^{(1)}(t) + 2M_{B,**}^{(1)}(t)}{t} \right) + \beta_4(t) \end{aligned} \quad (63)$$

Taking expectations on both sides gives

$$\mathbb{E}[M_{B,**}^{(1)}(t+1)] = \mathbb{E}[M_{B,**}^{(1)}(t)] + \mathbb{E} \left[ \beta_2(t) \frac{D_B^{(1)}(t) + 2M_{B,**}^{(1)}(t)}{t} \right]$$

$$+ \mathbb{E} \left[ \beta_3(t) \left( 1 + \frac{D_B^{(1)}(t) + 2M_{B,**}^{(1)}(t)}{t} \right) + \beta_4(t) \right] \quad (64)$$

$$= \mathbb{E}[M_{B,**}^{(1)}(t)] + \frac{2\beta_2 + 2\beta_3}{t} \mathbb{E}[M_{B,**}^{(1)}(t)] (1 + o(1)) = O(t^{2\beta_2+2\beta_3}). \quad (65)$$

The last step follows by Lemma B.3.

For (61), we can again write a recursive formula:

$$\begin{aligned} \mathbb{E}[M_{B,*B}^{(1)}(t+1) \mid \mathcal{F}_t] &= M_{B,*B}^{(1)}(t) + \beta_2(t) \frac{M_{B,*B}^{(1)}(t)}{t} \\ &+ \beta_3(t) \left( 1 + \frac{D_B(t) + M_{B,**}^{(1)}(t) + M_{B,*B}^{(1)}(t)}{t} \right) \end{aligned} \quad (66)$$

and take the expectation:

$$\begin{aligned} \mathbb{E}[M_{B,*B}^{(1)}(t+1)] &= \mathbb{E} \left[ M_{B,*B}^{(1)}(t) + \beta_2(t) \frac{M_{B,*B}^{(1)}(t)}{t} \right. \\ &\left. + \beta_3(t) \left( 1 + \frac{D_B(t) + M_{B,**}^{(1)}(t) + M_{B,*B}^{(1)}(t)}{t} \right) \right] \end{aligned} \quad (67)$$

$$= \mathbb{E}[M_{B,*B}^{(1)}(t)] + \left( \frac{\beta_2 + \beta_3}{t} \mathbb{E}[M_{B,*B}^{(1)}(t)] + \frac{\beta_3}{t} \mathbb{E}[M_{B,**}^{(1)}(t)] \right) (1 + o(1)) \quad (68)$$

$$= \frac{\beta_3}{\beta_2 + \beta_3} \mathbb{E}[M_{B,**}^{(1)}(t)(1 + o_t(1))]. \quad (69)$$

The last step again follows by Lemma B.3.

Finally, we repeat the same step as in (59) for (62), which gives

$$\mathbb{E}[M_{R,**}^{(1)}(t+1)] = O(t^{2\beta_1+2\beta_4}). \quad (70)$$

By Lemma H.1, we know  $\mathbb{E}[M_{R,**}^{(k)}(t)] = o_t(\mathbb{E}[M_{B,**}^{(k)}(t)])$ . We therefore complete the proof of the basis case.

Now assume that the claims hold for  $\{2, \dots, k-1\}$ . The idea is to write out the recursive equation for  $M_{B,B*}^{(k)}(t)$ ,  $M_{B,R*}^{(k)}(t)$ ,  $M_{B,*B}^{(k)}(t)$ ,  $M_{B,*R}^{(k)}(t)$ , according to the 4 possible cases at each step.

For  $M_{B,B*}^{(k)}(t)$ , notice that  $M_{B,B*}^{(k)}(t+1)$  can arise from  $M_{B,B*}^{(k)}(t)$  in the following cases:

- (1) A new blue node connects with an existing red node  $i$ : then the new node contributes  $1 \cdot d_{i,B*}^{(k-1)}(t)$  increments.
- (2) A new blue node connects with an existing blue node  $i$ : the new node contributes  $1 \cdot d_{i,B*}^{(k-1)}(t)$  increments, node  $i$  contributes  $1 \cdot d_{i,B*}^{(k)}(t)$  increments, and each blue node  $j$  with  $\text{dist}_{i,j}(t) = k-1$  contributes  $d_j(t)$  increments, which sum up to  $d_{i,B*}^{(k)}(t)$  in total.
- (3) A new red node connects with an existing blue node  $i$ : node  $i$  contributes  $1 \cdot d_{i,B*}^{(k)}(t)$  increments, and blue nodes that are  $(k-1)$  away from  $i$  contribute  $d_j(t)$  increments in total.



Therefore,

$$\begin{aligned} \mathbb{E}[M_{B,B*}^{(k)}(t+1) \mid \mathcal{F}_t] &= M_{B,B*}^{(k)}(t) + \beta_2(t) \frac{2M_{B,B*}^{(k)}(t) + M_{B,*B}^{(k-1)}(t)}{t} \\ &+ \beta_3(t) \frac{2M_{B,B*}^{(k)}(t) + M_{B,*B}^{(k-1)}(t) + M_{B,B*}^{(k-1)}(t)}{t} + \beta_4(t) \frac{M_{R,B*}^{(k-1)}(t)}{t}. \end{aligned} \quad (71)$$

Now, taking expectations on both sides of (71), we get

$$\begin{aligned} \mathbb{E}[M_{B,B*}^{(k)}(t+1)] &= \mathbb{E}[M_{B,B*}^{(k)}(t)] + \mathbb{E}\left[\beta_2(t) \frac{2M_{B,B*}^{(k)}(t) + M_{B,*B}^{(k-1)}(t)}{t}\right] \\ &+ \mathbb{E}\left[\beta_3(t) \frac{2M_{B,B*}^{(k)}(t) + M_{B,*B}^{(k-1)}(t) + M_{B,B*}^{(k-1)}(t)}{t}\right] + \mathbb{E}\left[\beta_4(t) \frac{M_{R,B*}^{(k-1)}(t)}{t}\right] \end{aligned} \quad (72)$$

$$\begin{aligned} &= \mathbb{E}[M_{B,B*}^{(k)}(t)] + (2\beta_2 + 2\beta_3)(1 + o_t(t^{-1/4})) \frac{\mathbb{E}[M_{B,B*}^{(k)}(t)]}{t} \\ &+ 2\beta_3 \mathbb{E}[M_{B,**}^{(k-1)}(t)](1 + o_t(1)), \end{aligned} \quad (73)$$

where in the last step we used Lemma B.1 and the induction assumptions that

$$\mathbb{E}[M_{B,B*}^{(k-1)}(t)] = \mathbb{E}[M_{B,**}^{(k-1)}(t)](1 + o_t(1)), \quad (74)$$

$$\mathbb{E}[M_{B,*B}^{(k-1)}(t)] = \frac{\beta_3}{\beta_2 + \beta_3} \mathbb{E}[M_{B,**}^{(k-1)}(t)](1 + o_t(1)), \quad (75)$$

$$\mathbb{E}[M_{R,**}^{(k-1)}(t)] = o_t(\mathbb{E}[M_{B,**}^{(k-1)}(t)]). \quad (76)$$

Note that by induction we also have

$$\mathbb{E}[M_{B,**}^{(k-1)}(t)] = O((\log t)^{k-1} t^{2\beta_2+2\beta_3}).$$

Applying Lemma B.3 on  $\mathbb{E}[M_{B,B*}^{(k)}(t)]$ , we are in the cast that  $c_1 = c_3 = 2\beta_2 + 2\beta_3$  and  $m = k - 1$ , and thus we see that

$$\mathbb{E}[M_{B,B*}^{(k)}(t)] = O((\log t)^k t^{2\beta_2+2\beta_3}). \quad (77)$$

We can then apply the same method on  $M_{B,R*}^{(k)}(t)$ ,  $M_{B,*B}^{(k)}(t)$ ,  $M_{B,R*}^{(k)}(t)$ , and get the desired result.  $\square$

LEMMA H.1. Let  $\beta_1(t)$ ,  $\beta_2(t)$ ,  $\beta_3(t)$ ,  $\beta_4(t)$  defined as in the previous proof, and  $\beta_i := \lim_{t \rightarrow \infty} \beta_i(t)$ . Then

$$\beta_1 + \beta_4 < \beta_2 + \beta_3. \quad (78)$$

PROOF. We first simplify the expression by defining  $g(x, y) := \frac{x}{1-(1-x)(1-y)}$ , then  $\beta_1 = g(\alpha^*, \rho)$ ,  $\beta_2 = 1 - g(\alpha^*, \rho)$ ,  $\beta_3 = g(1 - \alpha^*, \rho)$ , and  $\beta_r = 1 - g(1 - \alpha^*, \rho)$ . Furthermore, we define  $h(x, y) := \frac{g(x, y)}{x}$ . Then  $\frac{1-g(x, y)}{1-x} = y \cdot h(x, y)$ . Now,

$$\lim_{t \rightarrow \infty} \frac{\beta_1(t) - \beta_2(t)}{\beta_3(t) - \beta_4(t)} = \frac{\frac{r}{2}(1 - \rho)h(\alpha^*, \rho)}{\frac{1-r}{2}(1 - \rho)h(1 - \alpha^*, \rho)} = \frac{rh(\alpha^*, \rho)}{(1 - r)h(1 - \alpha^*, \rho)}. \quad (79)$$

It suffices to show that the above quantity is smaller than 1. Theorem 4.6 in [2] has shown that as  $t \rightarrow \infty$ ,

$$\begin{aligned} 2\alpha^* - 1 &= rg(\alpha^*, \rho) - (1 - r)g(1 - \alpha^*, \rho) \\ \alpha^* - (1 - \alpha^*) &= rh(\alpha^*, \rho)\alpha^* - (1 - r)h(1 - \alpha^*, \rho)(1 - \alpha^*) \end{aligned}$$

$$\alpha^* (1 - rh(\alpha^*, \rho)) = (1 - \alpha^*) (1 - (1 - r)h(1 - \alpha^*, \rho))$$

By Part 5 of Theorem 4.4,  $\alpha^* < 1/2$ . Therefore,

$$1 - rh(\alpha^*, \rho) > 1 - (1 - r)h(1 - \alpha^*, \rho). \quad (80)$$

Thus,  $rh(\alpha^*, \rho) < (1 - r)h(1 - \alpha^*, \rho)$  and  $\beta_1 - \beta_2 < \beta_3 - \beta_4$  as  $t \rightarrow \infty$ .  $\square$

LEMMA B.1. *For any color  $C \in \{R, B\}$ , any  $i \in \{1, 2, 3, 4\}$  and any  $k$ , we have that*

$$\frac{\mathbb{E} \left[ \beta_i(t) M_{C,**}^{(k-1)}(t) \right]}{\beta_i \mathbb{E} \left[ M_{C,**}^{(k-1)}(t) \right]} = 1 + o_t(t^{-1/4}). \quad (81)$$

PROOF. From [2][Corollary 4.11], denoting by

$$\sigma_t = \max(2(\log t)t^{-1/2}, t^{-1/3}), \quad (82)$$

we have that

$$\mathbb{P}(|\alpha(t) - \alpha^*| > \sigma_t) < t^{-4}. \quad (83)$$

Based on the above result, we can check that there exists a constant  $C_1 > 0$ , such that for any  $i \in \{1, 2, 3, 4\}$ ,

$$\mathbb{P}(|\beta_i(t) - \beta_i| > C_1 \sigma_t) < t^{-4}. \quad (84)$$

Note that by definition,  $M_{C,**}^{(k-1)}(t)$  is upper bounded by  $t^3$ ; and there exists a constant  $C_2 > 0$  such that  $\beta_i(t), \beta_i < C_2$  for any  $i, t$ . We have

$$\left| \mathbb{E} \left[ (\beta_i(t) - \beta_i) M_{C,**}^{(k-1)}(t) \right] \right| \leq \mathbb{E} \left[ |\beta_i(t) - \beta_i| M_{C,**}^{(k-1)}(t) \right] \quad (85)$$

$$\begin{aligned} &\leq \mathbb{E} \left[ |\beta_i(t) - \beta_i| M_{C,**}^{(k-1)}(t) \cdot 1_{|\beta_i(t) - \beta_i| > C_1 \sigma_t} \right] \\ &+ \mathbb{E} \left[ |\beta_i(t) - \beta_i| M_{C,**}^{(k-1)}(t) \cdot 1_{|\beta_i(t) - \beta_i| \leq C_1 \sigma_t} \right] \end{aligned} \quad (86)$$

$$\leq C_2 t^3 \cdot t^{-4} + C_1 \sigma_t \mathbb{E} \left[ M_{C,**}^{(k-1)}(t) \right]. \quad (87)$$

The above bound implies that for some  $C_2 > 0$

$$\frac{\left| \mathbb{E} \left[ (\beta_i(t) - \beta_i) M_{C,**}^{(k-1)}(t) \right] \right|}{\beta_i \mathbb{E} \left[ M_{C,**}^{(k-1)}(t) \right]} \leq C_2 \sigma_t = o_t(t^{-1/4}), \quad (88)$$

where  $-1/4$  is just an arbitrary number between  $-1/3$  and  $0$ . The proof is finished.  $\square$

LEMMA B.3. *For a sequence  $\{a_t\}$  satisfying*

$$a_{t+1} = a_t + c_1 d_t \frac{a_t}{t} + c_2 b_t, \quad (89)$$

where  $d_t = (1 + o_t(t^{-\epsilon}))$  for some  $\epsilon > 0$ ,  $b_t = (\log t)^m t^{c_3 - 1} (1 + o_t(1))$ . We then have that

$$a_t = \begin{cases} O(t^{c_1}), & \text{if } c_1 > c_3 \\ O((\log t)^{m+1} t^{c_3}), & \text{if } c_1 = c_3 \\ \frac{c_2}{c_3 - c_1} (\log t)^m t^{c_3} (1 + o_t(1)), & \text{if } c_1 < c_3. \end{cases} \quad (90)$$

PROOF. By the recursive equation for  $a_t$ , we see that

$$a_t = \sum_{s=1}^{t-1} c_2 b_s \prod_{j=s+1}^{t-1} (1 + c_1 d_j / j). \quad (91)$$

We can find a constant  $C_0 > 0$ , such that as  $c_1 d_j / j < 1/2$ ,

$$|\log(1 + c_1 d_j / j) - c_1 d_j / j| < C_0 (c_1 d_j / j)^2,$$

and thus

$$\left| \sum_{j=s+1}^{t-1} \log(1 + c_1 d_j / j) - \sum_{j=s+1}^{t-1} c_1 / j \right| \quad (92)$$

$$\leq \left| \sum_{j=s+1}^{t-1} c_1 (1 - d_j) / j \right| + \left| \sum_{j=s+1}^{t-1} C_0 (c_1 d_j / j)^2 \right|. \quad (93)$$

By the assumption that  $d_t = (1 + o_t(t^{-\epsilon}))$ , it is easy to check that

$$\left| \sum_{j=s+1}^{t-1} c_1 (1 - d_j) / j \right| + \left| \sum_{j=s+1}^{t-1} C_0 (c_1 d_j / j)^2 \right| \quad (94)$$

$$\leq \left| \sum_{j=s+1}^{\infty} c_1 o_j(j^{-1-\epsilon}) \right| + \left| \sum_{j=s+1}^{\infty} C_0 (c_1 / j)^2 (1 + o_j(1)) \right| = o_s(1). \quad (95)$$

Also, by basic mathematical analysis we have that

$$\left| \sum_{j=s+1}^{t-1} c_1 / j - c_1 (\log(t) - \log(s)) \right| = o_s(1). \quad (96)$$

Hence we see that

$$\prod_{j=s+1}^{t-1} (1 + c_1 d_j / j) = \exp \left( \sum_{j=s+1}^{t-1} \log(1 + c_1 d_j / j) \right) \quad (97)$$

$$= \exp \left( \sum_{j=s+1}^{t-1} c_1 / j + o_s(1) \right) = (t/s)^{c_1} (1 + o_s(1)). \quad (98)$$

Combining with the expression of  $b_t$ , we get

$$a_t = \sum_{s=1}^{t-1} c_2 (\log s)^m s^{c_3-1} (t/s)^{c_1} (1 + o_s(1)) \quad (99)$$

$$= t^{c_1} \sum_{s=1}^{t-1} c_2 (\log s)^m s^{c_3-c_1-1} (1 + o_s(1)). \quad (100)$$

We have that

$$\sum_{s=1}^{t-1} c_2 (\log s)^m s^{c_3 - c_1 - 1} (1 + o_s(1)) \quad (101)$$

$$= \begin{cases} O(1), & \text{if } c_1 > c_3 \\ O((\log t)^{m+1}), & \text{if } c_1 = c_3 \\ \frac{c_2}{c_3 - c_1} (\log t)^m t^{c_3 - c_1} (1 + o_t(1)), & \text{if } c_1 < c_3, \end{cases} \quad (102)$$

which finishes the proof.  $\square$