Proofs for Theory Results in The impact of network homophily on bias of multi-hop referrals

A NOTATIONS

For clarity, we list all notations that are used in our theory presentation in Table 3.

B PROOF OF THEOREM 4.1

THEOREM 4.1. *For a sequence of networks $\{\mathcal{G}(N_t, t, r, \rho)\}$ generated by the BPA model with $r < 0.5$, for any $k > 1$,*

$$\lim_{t\to\infty} \frac{\mathbb{E}[AG^{(k)}_{\mathcal{S}_L}(R,t)]}{\mathbb{E}[AG^{(k)}_{\mathcal{S}_L}(B,t)]} = \frac{\beta_2}{\beta_3}, \tag{11}$$

where

$$\beta_2 = \frac{r\rho}{2(1-(1-\alpha^*)(1-\rho))}, \beta_3 = \frac{1-r}{2(1-\alpha^*(1-\rho))}, \tag{12}$$

and α^* is the unique solution in $[0, 1]$ of the equation

$$2\alpha^* = 1 - (1-r)\frac{(1-\alpha^*)}{1-\alpha^*(1-\rho)} + r\frac{\alpha^*}{1-(1-\alpha^*)(1-\rho)}. \tag{13}$$

PROOF. For sequences $\{a_t, t \geq 1\}$, $\{b_t, t \geq 1\}$ and $\{c_t, t \geq 1\}$, we denote by $c_t = o_t(1)$ if $\lim_{t\to\infty} c_t = 0$; we denote by $a_t = O(b_t)$ if there exists some $C \neq 0$ such that $a_t/b_t - C = o_t(1)$.

For the growth model, at each time t, there are 4 possibilities:

- case 1: a new red node connects a red node;
- case 2: a new red node connects a blue node;
- case 3: a new blue node connects a blue node;
- case 4: a new blue node connects a red node.

In BPA model, the probability for above cases are:

$$\mathbb{P}(\text{case 1 at time t}) = \frac{r\alpha(t)}{2(1-(1-\alpha(t))(1-\rho))}, \tag{14}$$

$$\mathbb{P}(\text{case 2 at time t}) = \frac{r(1-\alpha(t))\rho}{2(1-(1-\alpha(t))(1-\rho))}, \tag{15}$$

$$\mathbb{P}(\text{case 3 at time t}) = \frac{(1-r)(1-\alpha(t))}{2(1-\alpha(t)(1-\rho))}, \tag{16}$$

$$\mathbb{P}(\text{case 4 at time t}) = \frac{(1-r)\alpha(t)\rho}{2(1-\alpha(t)(1-\rho))}. \tag{17}$$

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$V_C(t)$	the set of nodes in color C at time t;
$d_i^{(k)}(t)$	the k-hop degree of node i at time t;
$d_{i,C}^{(k)}(t)$	the number of nodes in color C among the k-hop neighbors of node i at time t;
$d_{i,C*}^{(k)}(t)$	among the k-hop neighbors of node i at time t, the number of nodes from which the second node on the unique path to i is of color C;
$D_C^{(k)}(t)$	sum of k-hop degrees over nodes of color C at time t; that is, $\sum_{i\in V_C(t)} d_i^{(k)}(t)$;
$\alpha(t)$	fraction of the sum of 1-hop degree for red nodes over that for all nodes; that is, $D_R^{(1)}(t)/(2t)$;
$M_{C_1,**}^{(k)}(t)$	$\sum_{i\in V_{C_1}(t)} d_i^{(1)}(t)d_i^{(k)}(t)$;
$M_{C_1,**}^{(k)}(t)$	$\sum_{i\in V_{C_1}(t)} d_i^{(1)}(t)d_i^{(k)}(t)$;
$M_{C_1,*C_2}^{(k)}(t)$	$\sum_{i\in V_{C_1}(t)} d_i^{(1)}(t)d_{i,C_2}^{(k)}(t)$;
$M_{C_1,C_2*}^{(k)}(t)$	$\sum_{i\in V_{C_1}(t)} d_i^{(1)}(t)d_{i,C_2*}^{(k)}(t)$ with $M_{C_1,C_2*}^{(1)}(t) := M_{C_1,**}^{(1)}(t)$.

Table 3. Notation

We can write a recursion for $D_C^{(k)}(t)$ by considering those 4 cases. Denote by $\{\mathcal{F}_t, t \geq 1\}$ the filtration such that $\mathcal{F}_t$ contains the information of the graph at time t. We have

$$\begin{aligned}\mathbb{E}[D_R^{(k)}(t+1) \mid \mathcal{F}_t] = D_R^{(k)}(t) + \beta_1(t)\frac{M_{R,**}^{(k-1)}(t) + M_{R,*R}^{(k-1)}(t)}{t}\\ + \beta_2(t)\frac{M_{B,**}^{(k-1)}(t) + M_{B,*R}^{(k-1)}(t)}{t} + \beta_3(t)\frac{M_{B,*R}^{(k-1)}(t)}{t} + \beta_4(t)\frac{M_{R,*R}^{(k-1)}(t)}{t},\end{aligned} \tag{18}$$

where

$$\beta_1(t) = \frac{r}{2(1-(1-\alpha(t))(1-\rho))}, \tag{19}$$

$$\beta_2(t) = \frac{r\rho}{2(1-(1-\alpha(t))(1-\rho))}, \tag{20}$$

$$\beta_3(t) = \frac{1-r}{2(1-\alpha(t)(1-\rho))}, \tag{21}$$

$$\beta_4(t) = \frac{(1-r)\rho}{2(1-\alpha(t)(1-\rho))}. \tag{22}$$

From [2], we see that $\alpha(t) \to \alpha^*$ almost surely, where α^* satisfies the following equation.

$$2\alpha^* = \frac{2r\alpha^* + r(1-\alpha^*)\rho}{1-(1-\alpha^*)(1-\rho)} + \frac{(1-r)\alpha^*\rho}{1-\alpha^*(1-\rho)}. \tag{23}$$

Hence we also have that $\beta_i(t) \to \beta_i$ almost surely for $i = 1, 2, 3, 4$, where

$$\beta_1 = \frac{r}{2(1-(1-\alpha^*)(1-\rho))}, \beta_2 = \frac{r\rho}{2(1-(1-\alpha^*)(1-\rho))}, \tag{24}$$

$$\beta_3 = \frac{1-r}{2(1-\alpha^*(1-\rho))}, \beta_4 = \frac{(1-r)\rho}{2(1-\alpha^*(1-\rho))}. \tag{25}$$

Taking expectations on both sides of (18), we get

$$\mathbb{E}[D_R^{(k)}(t+1)] = \mathbb{E}[D_R^{(k)}(t)] + \mathbb{E}\left[\beta_1(t)\frac{M_{R,**}^{(k-1)}(t) + M_{R,*R}^{(k-1)}(t)}{t}\right] + \mathbb{E}\left[\beta_2(t)\frac{M_{B,**}^{(k-1)}(t) + M_{B,*R}^{(k-1)}(t)}{t}\right] + \mathbb{E}\left[\beta_3(t)\frac{M_{B,*R}^{(k-1)}(t)}{t}\right] + \mathbb{E}\left[\beta_4(t)\frac{M_{R,*R}^{(k-1)}(t)}{t}\right]. \tag{26}$$

Similarly,

$$\mathbb{E}[D_B^{(k)}(t+1)] = \mathbb{E}[D_B^{(k)}(t)] + \mathbb{E}\left[\beta_3(t)\frac{M_{B,**}^{(k-1)}(t) + M_{B,*B}^{(k-1)}(t)}{t}\right] + \mathbb{E}\left[\beta_4(t)\frac{M_{R,**}^{(k-1)}(t) + M_{R,*B}^{(k-1)}(t)}{t}\right] + \mathbb{E}\left[\beta_1(t)\frac{M_{R,*B}^{(k-1)}(t)}{t}\right] + \mathbb{E}\left[\beta_2(t)\frac{M_{B,*B}^{(k-1)}(t)}{t}\right]. \tag{27}$$

We have the following lemma.

Lemma B.1. *For any color $C \in \{R, B\}$, any $i \in \{1, 2, 3, 4\}$ and any k, we have that*

$$\frac{\mathbb{E}\left[\beta_i(t) M_{C,**}^{(k-1)}(t)\right]}{\beta_i \mathbb{E}\left[M_{C,**}^{(k-1)}(t)\right]} = 1 + o_t(t^{-1/4}). \tag{28}$$

Using the above lemma, we get

$$\begin{aligned}\mathbb{E}[D_R^{(k)}(t+1)] &= \mathbb{E}[D_R^{(k)}(t)] \\ &+ \beta_1(1+o_t(1))\frac{\mathbb{E}\left[M_{R,**}^{(k-1)}(t)\right] + \mathbb{E}\left[M_{R,*R}^{(k-1)}(t)\right]}{t} \\ &+ \beta_2(1+o_t(1))\frac{\mathbb{E}\left[M_{B,**}^{(k-1)}(t)\right] + \mathbb{E}\left[M_{B,*R}^{(k-1)}(t)\right]}{t} \\ &+ \beta_3(1+o_t(1))\frac{\mathbb{E}\left[M_{B,*R}^{(k-1)}(t)\right]}{t} + \beta_4(1+o_t(1))\frac{\mathbb{E}\left[M_{R,*R}^{(k-1)}(t)\right]}{t}.\end{aligned} \tag{29}$$

$$\begin{aligned}\mathbb{E}[D_B^{(k)}(t+1)] &= \mathbb{E}[D_B^{(k)}(t)] \\ &+ \beta_3(1+o_t(1))\frac{\mathbb{E}\left[M_{B,**}^{(k-1)}(t)\right] + \mathbb{E}\left[M_{B,*B}^{(k-1)}(t)\right]}{t} \\ &+ \beta_4(1+o_t(1))\frac{\mathbb{E}\left[M_{R,**}^{(k-1)}(t)\right] + \mathbb{E}\left[M_{R,*B}^{(k-1)}(t)\right]}{t}\end{aligned}$$

$$+\beta_1(1+o_t(1))\frac{\mathbb{E}\left[M_{R,*B}^{(k-1)}(t)\right]}{t}+\beta_2(1+o_t(1))\frac{\mathbb{E}\left[M_{B,*B}^{(k-1)}(t)\right]}{t}. \tag{30}$$

Assume that $r < 1/2$. We need the following lemmas.

Lemma B.2. *We have that, for $k \geq 1$,*

$$\mathbb{E}[M_{B,**}^{(k)}(t)] = O((\log t)^{k-1} t^{2\beta_2+2\beta_3}), \tag{31}$$

$$\mathbb{E}[M_{B,B*}^{(k)}(t)] = \mathbb{E}[M_{B,**}^{(k)}(t)](1+o_t(1)), \tag{32}$$

$$\mathbb{E}[M_{B,*B}^{(k)}(t)] = \frac{\beta_3}{\beta_2+\beta_3}\mathbb{E}[M_{B,**}^{(k)}(t)(1+o_t(1))], \tag{33}$$

$$\mathbb{E}[M_{R,**}^{(k)}(t)] = o_t(\mathbb{E}[M_{B,**}^{(k)}(t)]). \tag{34}$$

Lemma B.3. *For a sequence $\{a_t\}$ satisfying*

$$a_{t+1} = a_t + c_1 d_t \frac{a_t}{t} + c_2 b_t, \tag{35}$$

where $d_t = (1+o_t(t^{-\epsilon}))$ for some $\epsilon > 0$, $b_t = (\log t)^m t^{c_3-1}(1+o_t(1))$. We then have that

$$a_t = \begin{cases} O(t^{c_1}), & \text{if } c_1 > c_3 \\ O((\log t)^{m+1} t^{c_3}), & \text{if } c_1 = c_3 \\ \frac{c_2}{c_3-c_1}(\log t)^m t^{c_3}(1+o_t(1)), & \text{if } c_1 < c_3. \end{cases} \tag{36}$$

By Lemma B.2, we see that

$$\mathbb{E}[D_R^{(k)}(t+1)] = \mathbb{E}[D_R^{(k)}(t)] + 2\beta_2\frac{\mathbb{E}[M_{B,**}^{(k-1)}(t)]}{t}(1+o_t(1)). \tag{37}$$

Similarly, we have

$$\mathbb{E}[D_B^{(k)}(t+1)] = \mathbb{E}[D_B^{(k)}(t)] + 2\beta_3\frac{\mathbb{E}[M_{B,**}^{(k-1)}(t)]}{t}(1+o_t(1)). \tag{38}$$

By Lemma B.3 we have that

$$\frac{\mathbb{E}[D_R^{(k)}(t)]}{\mathbb{E}[D_B^{(k)}(t)]} = \frac{\beta_2}{\beta_3}(1+o_t(1)). \tag{39}$$

□

C PROOF OF THEOREM 4.2

Theorem 4.2. *For a sequence of networks $\{\mathcal{G}(N_t, t, r, \rho)\}$ generated by the BPA model with $r < 0.5$, the* linear referral strategy *exhibits k-hop referral inequality against red nodes for any $k \geq 1$. That is,*

$$\lim_{t\to\infty}\frac{\mathbb{E}[AG_{\mathcal{S}_L}^{(k)}(R,t)]}{\mathbb{E}[AG_{\mathcal{S}_L}^{(k)}(B,t)]} \leq \frac{r}{1-r}, \tag{40}$$

Proof. By the definition of β_2 and β_3, we know

$$\frac{\beta_2}{\beta_2+\beta_3} \le r \Leftrightarrow \frac{\rho}{1-(1-\alpha^*)(1-\rho)} \le \frac{1}{1-\alpha^*(1-\rho)} \tag{41}$$

$$\Leftrightarrow \rho - \alpha^*\rho(1-\rho) \le 1-(1-\alpha^*)(1-\rho) \tag{42}$$

$$\Leftrightarrow 1-\rho-(1-\rho)(1-\alpha^*-\alpha^*\rho) \ge 0 \tag{43}$$

$$\Leftrightarrow \alpha^*(1+\rho)(1-\rho) \ge 0, \tag{44}$$

which is clearly true by the definition of α^* and ρ. □

D PROOF OF COROLLARY 4.4

Corollary 4.4. *For a sequence of networks $\{\mathcal{G}(N_t,t,r,\rho)\}$ generated by the BPA model with $r<0.5$, as $\rho \longrightarrow 0$,*

$$\lim_{t\to\infty} \frac{\mathbb{E}[AG^{(1)}_{\mathcal{S}_L}(R,t)]}{\mathbb{E}[AG^{(1)}_{\mathcal{S}_L}(B,t)]} = \frac{r}{1-r} \text{ and } \lim_{t\to\infty} \frac{\mathbb{E}[AG^{(k)}_{\mathcal{S}_L}(R,t)]}{\mathbb{E}[AG^{(k)}_{\mathcal{S}_L}(B,t)]} = 0. \tag{45}$$

Proof. From [2], we know that as $t \longrightarrow \infty$, $\frac{\mathbb{E}[D_R^{(1)}(t)]}{\mathbb{E}[D_B^{(1)}(t)]} = \frac{\alpha^*}{(1-\alpha^*)}$, where

$$2\alpha^* = \frac{2r\alpha^* + r(1-\alpha^*)\rho}{1-(1-\alpha^*)(1-\rho)} + \frac{(1-r)\alpha^*\rho}{1-\alpha^*(1-\rho)}. \tag{46}$$

When $\rho \longrightarrow 0$, $\alpha^* \longrightarrow r$, and therefore $\frac{\mathbb{E}[D_R^{(1)}(t)]}{\mathbb{E}[D_B^{(1)}(t)]} \longrightarrow \frac{r}{1-r}$.

For $k \ge 2$, we know from Theorem 4.1 that, as $t \longrightarrow \infty$,

$$\lim_{\rho\to 0} \frac{\mathbb{E}[D_R^{(k)}(t)]}{\mathbb{E}[D_B^{(k)}(t)]} = \lim_{\rho\to 0} \frac{\beta_2}{\beta_3} = \lim_{\rho\to 0} \frac{r\rho}{1-r} \cdot \frac{2(1-\alpha^*(1-\rho))}{2(1-(1-\alpha^*)(1-\rho))} = 0. \tag{47}$$

□

E PROOF OF LEMMA 5.1

Lemma 5.1. *For a sequence of networks $\{\mathcal{G}(N_t,t,r,\rho)\}$ generated by the BPA model with $r<0.5$, as $t \longrightarrow \infty$, under the* random referral strategy *and the* popularity-driven referral strategy, *the ratios of expected 1-hop active gain for the red nodes are the red node population ratio; under the* acceptance-driven referral strategy, *the ratio of expected 1-hop active gain for red nodes is no greater than the red node population ratio.*

Proof. Denote the expected referral reach for a red (blue) member of degree k as $h(k,r)$ ($h(k,b)$), and the probability of a randomly selected red (blue) node having degree k on $\mathcal{G}(N_t,t,r,\rho)$ as $p_t^r(k)$ ($p_t^b(k)$). Note that all the four referral programs satisfy the following three conditions: (1). $\lim_{t\to\infty} h(k,i)p_t^i(k) < \infty$, (2). $h(k,b) \ge h(k,r)$, and (3). $h(k,i)$ being non-decreasing for $i \in \{r,b\}$. Furthermore,

$$\lim_{t\to\infty} \frac{\sum_{k\ge1} h(k,r)p_t^r(k)}{\sum_{k\ge1} h(k,b)p_t^b(k)} - \lim_{t\to\infty} \frac{\sum_{k\ge1} p_t^r(k)}{\sum_{k\ge1} p_t^b(k)} \tag{48}$$

$$= \lim_{t\to\infty} \frac{\sum_{i>j} \left(h(i,b)-h(j,r)\right)\left(p_t^r(i)p_t^b(j) - p_t^r(j)p_t^b(i)\right)}{\sum_{k\ge1} h(k,b)p_t^b(k) \sum_{k\ge1} p_t^b(k)}. \tag{49}$$

By [2][Theorem 4.12], we know for all $i > j$, $\lim_{t\to} p_t^r(k) \propto k^{-\beta(R)}$ and $\lim_{t\to} p_t^b(k) \propto k^{-\beta(B)}$ with $\beta(R) > \beta(B)$. Therefore,

$$\lim_{t\to\infty} \frac{p_t^r(i)}{p_t^r(j)} \leq \lim_{t\to\infty} \frac{p_t^b(i)}{p_t^b(j)}, \tag{50}$$

which implies that the limit of (49) ≤ 0. As $\lim_{t\to\infty} \frac{\sum_{k\geq 1} p_t^r(k)}{\sum_{k\geq 1} p_t^b(k)} = 1$, we have

$$\lim_{t\to\infty} \frac{\sum_{k\geq 1} h(k,r) p_t^r(k)}{\sum_{k\geq 1} h(k,b) p_t^b(k)} \leq 1. \tag{51}$$

The equality holds if and only if $h(k,b) = h(k,r)$ and $h(k,i)$ is constant in k, the conditions the *random referral strategy* and the *popularity-driven referral strategy* satisfy. □

F PROOF OF COROLLARY 5.1

Corollary 5.1. *Let $\{\mathcal{G}(N_t, t, r, \rho)\}$ be a sequence of graphs generated through the BPA model with $r < 1/2$. Under the* acceptance-driven referral strategy $\mathcal{S}_a$*, the ratio of expected referral reach for red nodes in the second hop is no greater than that in the first hop. That is,*

$$\lim_{t\to\infty} \frac{\mathbb{E}[AG^{(2)}_{\mathcal{S}_A}(R,t)]}{\mathbb{E}[AG^{(2)}_{\mathcal{S}_A}(B,t)]} \geq \lim_{t\to\infty} \frac{\mathbb{E}[AG^{(1)}_{\mathcal{S}_A}(R,t)]}{\mathbb{E}[AG^{(1)}_{\mathcal{S}_A}(B,t)]}. \tag{52}$$

Proof. We denote the probability that a referral sent from a red (blue) node being accept at the first hop as P_r^1 (P_b^1) as $t \to \infty$. As proved in Lemma 5.1, they satisfy $P_r^1 \leq P_b^1$. We have the following expression

$$\lim_{t\to\infty} \frac{\mathbb{E}[AG^{(1)}_{\mathcal{S}_A}(R,t)]}{\mathbb{E}[AG^{(1)}_{\mathcal{S}_A}(B,t)]} = \frac{r \cdot P_r^1}{(1-r)\cdot P_b^1}. \tag{53}$$

For the 2-hop, notice that if the 1-hop friend who receives the referral denies referral and randomly gives the referral to one of its friends, the probability that the referral gets accepted at the 2-hop is actually a constant. We thus have that

$$\lim_{t\to\infty} \frac{\mathbb{E}[AG^{(2)}_{\mathcal{S}_A}(R,t)]}{\mathbb{E}[AG^{(2)}_{\mathcal{S}_A}(B,t)]} = \frac{r \cdot (1-P_r^1)}{(1-r)\cdot (1-P_b^1)}, \tag{54}$$

Since $P_r^1 \leq P_b^1$, it is easy to check that (54) is greater than or equal to (53), and the proof is finished. □

G PROOF OF FRIENDSHIP PARADOX IN BPA

Lemma G.1. *paradox (Friendship Paradox in BPA) Let $\{\mathcal{G}(N_t, t, r, \rho)\}$ be a sequence of graphs generated through the BPA model with $r < 1/2$. For $c_1, c_2 \in \{R, B\}$, denote $F_{c_1c_2}(t)$ as the degree of the c_2 color node of a randomly chosen edge that connects two nodes with c_1, c_2 colors (if $c_1 = c_2$, randomly choose one end of the edge). We then have*

$$\lim_{t\to\infty} \mathbb{E}[F_{c_1c_2}(t)] \geq \lim_{t\to\infty} \mathbb{E}[d_{c_2}(t)], \tag{55}$$

where $d_{c_2}(t)$ is the degree of a randomly chosen c_2 color node.

Proof. We consider the case that $c_1 = R$ and $c_2 = B$ (the other 3 cases are similar). The probability that a certain blue node v is chosen is

$$\frac{d^{(1)}_{v,R}(t)}{\sum_{u\in B} d^{(1)}_{u,R}(t)}. \tag{56}$$

As $t \to \infty$, denote by $p_k^{(R,B)}$ as the limiting probability that a blue node with degree k is chosen, p_k^B as the limiting probability that a blue node has degree k, and $R_k(B)$ as the limiting average number of red neighbors for a blue node of size k. We then have that

$$p_k^{(R,B)} = \frac{p_k^B \cdot R_k(B)}{\sum_{k=1}^{\infty} p_k^B \cdot R_k(B)}. \tag{57}$$

Note that $R_k(B)$ is non-decreasing with respect to k. Therefore, $p_k^{(R,B)}$ puts larger weights on larger k's (compared to p_k^B), that is, for any n we have that $\sum_{k\geq n} p_k^{(R,B)} \geq \sum_{k\geq n} p_k^B$. Consequently, we have that

$$\lim_{t\to\infty} \mathbb{E}[F_{c_1c_2}(t)] = \sum_{k\geq 1} k \cdot p_k^{(R,B)} = \sum_{n\geq 1}\sum_{k\geq n} p_k^{(R,B)} \geq \sum_{n\geq 1}\sum_{k\geq n} p_k^B = \sum_{k\geq 1} k \cdot p_k^B = \lim_{t\to\infty} \mathbb{E}[d_{c_2}(t)], \tag{58}$$

and it finishes the proof. □

H PROOF OF AUXILIARY LEMMAS

Lemma B.2. *We have that, for $k \geq 1$,*

$$\mathbb{E}[M_{B,**}^{(k)}(t)] = O((\log t)^{k-1} t^{2\beta_2+2\beta_3}), \tag{59}$$

$$\mathbb{E}[M_{B,B*}^{(k)}(t)] = \mathbb{E}[M_{B,**}^{(k)}(t)](1 + o_t(1)), \tag{60}$$

$$\mathbb{E}[M_{B,*B}^{(k)}(t)] = \frac{\beta_3}{\beta_2+\beta_3}\mathbb{E}[M_{B,**}^{(k)}(t)(1 + o_t(1))], \tag{61}$$

$$\mathbb{E}[M_{R,**}^{(k)}(t)] = o_t(\mathbb{E}[M_{B,**}^{(k)}(t)]). \tag{62}$$

Proof. We prove it by induction. Recall that $M_{B,B*}^{(1)}(t) = M_{B,**}^{(1)}(t)$ by definition, and thus (60) trivially holds. We first show that the claims hold for $k = 1$. First note that $M_{B,**}^{(1)}(t+1)$ arises from $M_{B,**}^{(1)}(t)$ in the following cases:

(1) A new red node connects with an existing blue node i: node i then contributes $\left(d_i^{(1)}(t)+1\right)^2 - \left(d_i^{(1)}(t)\right)^2 = 1 + 2d_i(t)$ increments.
(2) A new blue node connects with an existing red node i: the new node contributes 1 increment.
(3) A new blue node connects with an existing blue node i: node i then contributes $\left(d_i^{(1)}(t)+1\right)^2 - \left(d_i^{(1)}(t)\right)^2 = 1 + 2d_i(t)$ increments, and the the new node contributes 1 increment.

Therefore,

$$\begin{aligned}\mathbb{E}[M_{B,**}^{(1)}(t+1) \mid \mathcal{F}_t] = M_{B,**}^{(1)}(t) &+ \beta_2(t)\frac{D_B^{(1)}(t) + 2M_{B,**}^{(1)}(t)}{t} \\ &+ \beta_3(t)\left(1 + \frac{D_B^{(1)}(t) + 2M_{B,**}^{(1)}(t)}{t}\right) + \beta_4(t)\end{aligned} \tag{63}$$

Taking expectations on both sides gives

$$\mathbb{E}[M_{B,**}^{(1)}(t+1)] = \mathbb{E}[M_{B,**}^{(1)}(t)] + \mathbb{E}\left[\beta_2(t)\frac{D_B^{(1)}(t) + 2M_{B,**}^{(1)}(t)}{t}\right]$$

$$+\mathbb{E}\left[\beta_3(t)\left(1+\frac{D_B^{(1)}(t)+2M_{B,**}^{(1)}(t)}{t}\right)+\beta_4(t)\right] \tag{64}$$

$$=\mathbb{E}[M_{B,**}^{(1)}(t)]+\frac{2\beta_2+2\beta_3}{t}\mathbb{E}[M_{B,**}^{(1)}(t)]\,(1+o(1))=O\left(t^{2\beta_2+2\beta_3}\right). \tag{65}$$

The last step follows by Lemma B.3.

For (61), we can again write a recursive formula:

$$\mathbb{E}[M_{B,*B}^{(1)}(t+1)\mid \mathcal{F}_t]=M_{B,*B}^{(1)}(t)+\beta_2(t)\frac{M_{B,*B}^{(1)}(t)}{t}$$
$$+\beta_3(t)\left(1+\frac{D_B(t)+M_{B,**}^{(1)}(t)+M_{B,*B}^{(1)}(t)}{t}\right) \tag{66}$$

and take the expectation:

$$\mathbb{E}[M_{B,*B}^{(1)}(t+1)]=\mathbb{E}\left[M_{B,*B}^{(1)}(t)+\beta_2(t)\frac{M_{B,*B}^{(1)}(t)}{t}\right.$$
$$\left.+\beta_3(t)\left(1+\frac{D_B(t)+M_{B,**}^{(1)}(t)+M_{B,*B}^{(1)}(t)}{t}\right)\right] \tag{67}$$

$$=\mathbb{E}[M_{B,*B}^{(1)}(t)]+\left(\frac{\beta_2+\beta_3}{t}\mathbb{E}[M_{B,*B}^{(1)}(t)]+\frac{\beta_3}{t}\mathbb{E}[M_{B,**}^{(1)}(t)]\right)(1+o(1)) \tag{68}$$

$$=\frac{\beta_3}{\beta_2+\beta_3}\mathbb{E}[M_{B,**}^{(1)}(t)(1+o_t(1))]. \tag{69}$$

The last step again follows by Lemma B.3.

Finally, we repeat the same step as in (59) for (62), which gives

$$\mathbb{E}[M_{R,**}^{(1)}(t+1)]=O\left(t^{2\beta_1+2\beta_4}\right). \tag{70}$$

By Lemma H.1, we know $\mathbb{E}[M_{R,**}^{(k)}(t)]=o_t(\mathbb{E}[M_{B,**}^{(k)}(t)])$. We therefore complete the proof of the basis case.

Now assume that the claims hold for $\{2,\ldots,k-1\}$. The idea is to write out the recursive equation for $M_{B,B*}^{(k)}(t)$, $M_{B,R*}^{(k)}(t)$,$M_{B,*B}^{(k)}(t)$, $M_{B,*R}^{(k)}(t)$, according to the 4 possible cases at each step.

For $M_{B,B*}^{(k)}(t)$, notice that $M_{B,B*}^{(k)}(t+1)$ can arise from $M_{B,B*}^{(k)}(t)$ in the following cases:

(1) A new blue node connects with an existing red node i: then the new node contributes $1\cdot d_{i,B^*}^{(k-1)}(t)$ increments.

(2) A new blue node connects with an existing blue node i: the new node contributes $1\cdot d_{i,B^*}^{(k-1)}(t)$ increments, node i contributes $1\cdot d_{i,B^*}^{(k)}(t)$ increments, and each blue node j with $dist_{i,j}(t)=k-1$ contributes $d_j(t)$ increments, which sum up to $d_{i,B^*}^{(k)}(t)$ in total.

(3) A new red node connects with an existing blue node i: node i contributes $1\cdot d_{i,B^*}^{(k)}(t)$ increments, and blue nodes that are $(k-1)$ away from i contribute $d_j(t)$ increments in total.

Therefore,

$$\mathbb{E}[M_{B,B*}^{(k)}(t+1) \mid \mathcal{F}_t] = M_{B,B*}^{(k)}(t) + \beta_2(t)\frac{2M_{B,B*}^{(k)}(t) + M_{B,*B}^{(k-1)}(t)}{t} + \beta_3(t)\frac{2M_{B,B*}^{(k)}(t) + M_{B,*B}^{(k-1)}(t) + M_{B,B*}^{(k-1)}(t)}{t} + \beta_4(t)\frac{M_{R,B*}^{(k-1)}(t)}{t}. \quad (71)$$

Now, taking expectations on both sides of (71), we get

$$\mathbb{E}[M_{B,B*}^{(k)}(t+1)] = \mathbb{E}[M_{B,B*}^{(k)}(t)] + \mathbb{E}\left[\beta_2(t)\frac{2M_{B,B*}^{(k)}(t) + M_{B,*B}^{(k-1)}(t)}{t}\right] + \mathbb{E}\left[\beta_3(t)\frac{2M_{B,B*}^{(k)}(t) + M_{B,*B}^{(k-1)}(t) + M_{B,B*}^{(k-1)}(t)}{t}\right] + \mathbb{E}\left[\beta_4(t)\frac{M_{R,B*}^{(k-1)}(t)}{t}\right] \quad (72)$$

$$= \mathbb{E}[M_{B,B*}^{(k)}(t)] + (2\beta_2 + 2\beta_3)(1 + o_t(t^{-1/4}))\frac{\mathbb{E}[M_{B,B*}^{(k)}(t)]}{t} + 2\beta_3\mathbb{E}[M_{B,**}^{(k-1)}(t)](1 + o_t(1)), \quad (73)$$

where in the last step we used Lemma B.1 and the induction assumptions that

$$\mathbb{E}[M_{B,B*}^{(k-1)}(t)] = \mathbb{E}[M_{B,**}^{(k-1)}](1 + o_t(1)), \quad (74)$$

$$\mathbb{E}[M_{B,*B}^{(k-1)}(t)] = \frac{\beta_3}{\beta_2 + \beta_3}\mathbb{E}[M_{B,**}^{(k-1)}](1 + o_t(1)), \quad (75)$$

$$\mathbb{E}[M_{R,**}^{(k-1)}(t)] = o_t(\mathbb{E}[M_{B,**}^{(k-1)}(t)]). \quad (76)$$

Note that by induction we also have

$$\mathbb{E}[M_{B,**}^{(k-1)}(t)] = O((\log t)^{k-1}t^{2\beta_2+2\beta_3}).$$

Applying Lemma B.3 on $\mathbb{E}[M_{B,B*}^{(k)}(t)]$, we are in the cast that $c_1 = c_3 = 2\beta_2 + 2\beta_3$ and $m = k - 1$, and thus we see that

$$\mathbb{E}[M_{B,B*}^{(k)}(t)] = O((\log t)^k t^{2\beta_2+2\beta_3}). \quad (77)$$

We can then apply the same method on $M_{B,R*}^{(k)}(t)$,$M_{B,*B}^{(k)}(t)$, $M_{B,*R}^{(k)}(t)$, and get the desired result. □

Lemma H.1. *Let $\beta_1(t), \beta_2(t), \beta_3(t), \beta_4(t)$ defined as in the previous proof, and $\beta_i := \lim_{t\to\infty} \beta_i(t)$. Then*

$$\beta_1 + \beta_4 < \beta_2 + \beta_3. \quad (78)$$

Proof. We first simplify the expression by defining $g(x, y) := \frac{x}{1-(1-x)(1-y)}$, then $\beta_1 = g(\alpha^*, \rho)$, $\beta_2 = 1-g(\alpha^*, \rho)$, $\beta_3 = g(1-\alpha^*, \rho)$, and $\beta_r = 1-g(1-\alpha^*, \rho)$. Furthermore, we define $h(x, y) := \frac{g(x,y)}{x}$. Then $\frac{1-(g(x,y)}{1-x} = y \cdot h(x, y)$. Now,

$$\lim_{t\to\infty}\frac{\beta_1(t) - \beta_2(t)}{\beta_3(t) - \beta_4(t)} = \frac{\frac{r}{2}(1-\rho)h(\alpha^*, \rho)}{\frac{1-r}{2}(1-\rho)h(1-\alpha^*, \rho)} = \frac{rh(\alpha^*, \rho)}{(1-r)h(1-\alpha^*, \rho)}. \quad (79)$$

It suffices to show that the above quantity is smaller than 1. Theorem 4.6 in [2] has shown that as $t \to \infty$,

$$2\alpha^* - 1 = rg(\alpha^*, \rho) - (1-r)g(1-\alpha^*, \rho)$$
$$\alpha^* - (1-\alpha^*) = rh(\alpha^*, \rho)\,\alpha^* - (1-r)h(1-\alpha^*, \rho)\,(1-\alpha^*)$$

$$\alpha^* \left(1 - rh(\alpha^*, \rho)\right) = (1 - \alpha^*) \left(1 - (1 - r)h\left(1 - \alpha^*, \rho\right)\right)$$

By Part 5 of Theorem 4.4, $\alpha^* < 1/2$. Therefore,

$$1 - rh\left(\alpha^*, \rho\right) > 1 - (1 - r)h\left(1 - \alpha^*, \rho\right). \tag{80}$$

Thus, $rh\left(\alpha^*, \rho\right) < (1 - r)h\left(1 - \alpha^*, \rho\right)$ and $\beta_1 - \beta_2 < \beta_3 - \beta_4$ as $t \to \infty$. □

Lemma B.1. *For any color $C \in \{R, B\}$, any $i \in \{1, 2, 3, 4\}$ and any k, we have that*

$$\frac{\mathbb{E}\left[\beta_i(t) M_{C,**}^{(k-1)}(t)\right]}{\beta_i \mathbb{E}\left[M_{C,**}^{(k-1)}(t)\right]} = 1 + o_t(t^{-1/4}). \tag{81}$$

Proof. From [2][Corollary 4.11], denoting by

$$\sigma_t = \max(2(\log t)t^{-1/2}, t^{-1/3}), \tag{82}$$

we have that

$$\mathbb{P}\left(|\alpha(t) - \alpha^*| > \sigma_t\right) < t^{-4}. \tag{83}$$

Based on the above result, we can check that there exists a constant $C_1 > 0$, such that for any $i \in \{1, 2, 3, 4\}$,

$$\mathbb{P}\left(|\beta_i(t) - \beta_i| > C_1 \sigma_t\right) < t^{-4}. \tag{84}$$

Note that by definition, $M_{C,**}^{(k-1)}(t)$ is upper bounded by t^3; and there exists a constant $C_2 > 0$ such that $\beta_i(t), \beta_i < C_2$ for any i, t. We have

$$\left|\mathbb{E}\left[(\beta_i(t) - \beta_i) M_{C,**}^{(k-1)}(t)\right]\right| \leq \mathbb{E}\left[|\beta_i(t) - \beta_i| M_{C,**}^{(k-1)}(t)\right] \tag{85}$$

$$\leq \mathbb{E}\left[|\beta_i(t) - \beta_i| M_{C,**}^{(k-1)}(t) \cdot 1_{|\beta_i(t) - \beta_i| > C_1 \sigma_t}\right]$$

$$+ \mathbb{E}\left[|\beta_i(t) - \beta_i| M_{C,**}^{(k-1)}(t) \cdot 1_{|\beta_i(t) - \beta_i| \leq C_1 \sigma_t}\right] \tag{86}$$

$$\leq C_2 t^3 \cdot t^{-4} + C_1 \sigma_t \mathbb{E}\left[M_{C,**}^{(k-1)}(t)\right]. \tag{87}$$

The above bound implies that for some $C_2 > 0$

$$\frac{\left|\mathbb{E}\left[(\beta_i(t) - \beta_i) M_{C,**}^{(k-1)}(t)\right]\right|}{\beta_i \mathbb{E}\left[M_{C,**}^{(k-1)}(t)\right]} \leq C_2 \sigma_t = o_t(t^{-1/4}), \tag{88}$$

where $-1/4$ is just an arbitrary number between $-1/3$ and 0. The proof is finished. □

Lemma B.3. *For a sequence $\{a_t\}$ satisfying*

$$a_{t+1} = a_t + c_1 d_t \frac{a_t}{t} + c_2 b_t, \tag{89}$$

where $d_t = (1 + o_t(t^{-\epsilon}))$ for some $\epsilon > 0$, $b_t = (\log t)^m t^{c_3 - 1}(1 + o_t(1))$. We then have that

$$a_t = \begin{cases} O(t^{c_1}), & if\, c_1 > c_3 \\ O((\log t)^{m+1} t^{c_3}), & if\, c_1 = c_3 \\ \frac{c_2}{c_3 - c_1} (\log t)^m t^{c_3} (1 + o_t(1)), & if\, c_1 < c_3. \end{cases} \tag{90}$$

Proof. By the recursive equation for a_t, we see that

$$a_t = \sum_{s=1}^{t-1} c_2 b_s \prod_{j=s+1}^{t-1} (1 + c_1 d_j/j). \tag{91}$$

We can find a constant $C_0 > 0$, such that as $c_1 d_j/j < 1/2$,

$$|\log(1 + c_1 d_j/j) - c_1 d_j/j| < C_0 (c_1 d_j/j)^2,$$

and thus

$$\left|\sum_{j=s+1}^{t-1} \log(1 + c_1 d_j/j) - \sum_{j=s+1}^{t-1} c_1/j\right| \tag{92}$$

$$\leq \left|\sum_{j=s+1}^{t-1} c_1(1 - d_j)/j\right| + \left|\sum_{j=s+1}^{t-1} C_0 (c_1 d_j/j)^2\right|. \tag{93}$$

By the assumption that $d_t = (1 + o_t(t^{-\epsilon}))$, it is easy to check that

$$\left|\sum_{j=s+1}^{t-1} c_1(1 - d_j)/j\right| + \left|\sum_{j=s+1}^{t-1} C_0 (c_1 d_j/j)^2\right| \tag{94}$$

$$\leq \left|\sum_{j=s+1}^{\infty} c_1 o_j(j^{-1-\epsilon})\right| + \left|\sum_{j=s+1}^{\infty} C_0 (c_1/j)^2 (1 + o_j(1))\right| = o_s(1). \tag{95}$$

Also, by basic mathematical analysis we have that

$$\left|\sum_{j=s+1}^{t-1} c_1/j - c_1(\log(t) - \log(s))\right| = o_s(1). \tag{96}$$

Hence we see that

$$\prod_{j=s+1}^{t-1} (1 + c_1 d_j/j) = \exp\left(\sum_{j=s+1}^{t-1} \log(1 + c_1 d_j/j)\right) \tag{97}$$

$$= \exp\left(\sum_{j=s+1}^{t-1} c_1/j + o_s(1)\right) = (t/s)^{c_1}(1 + o_s(1)). \tag{98}$$

Combining with the expression of b_t, we get

$$a_t = \sum_{s=1}^{t-1} c_2 (\log s)^m s^{c_3 - 1} (t/s)^{c_1} (1 + o_s(1)) \tag{99}$$

$$= t^{c_1} \sum_{s=1}^{t-1} c_2 (\log s)^m s^{c_3 - c_1 - 1} (1 + o_s(1)). \tag{100}$$

We have that

$$\sum_{s=1}^{t-1} c_2(\log s)^m s^{c_3-c_1-1}(1+o_s(1)) \tag{101}$$

$$= \begin{cases} O(1), & \text{if } c_1 > c_3 \\ O((\log t)^{m+1}), & \text{if } c_1 = c_3 \\ \frac{c_2}{c_3-c_1}(\log t)^m t^{c_3-c_1}(1+o_t(1)), & if c_1 < c_3, \end{cases} \tag{102}$$

which finishes the proof. □