# Lecture 30. Optimization of Energy Systems

"When a subsystem's goals dominate at the expense of the total system's goals, the resulting behavior is called suboptimization."

— Donella H. Meadows [4]

## 1 What is optimization?

Optimization is a process for finding the best solution or set of solutions to a defined problem, *as we have defined 'best,' and within some acceptable limits for labor and computational time to perform the search*. We do this by carefully identifying what our options are, expressing the criteria that we want to be maximized or minimized, and restricting our choices for the set of solutions to those which are physically possible and within the limits of cost and performance that are acceptable to the decision maker(s).

## 2 Optimization terminology

We do have a few terms that have specific meanings within the context of optimization.

**decision variables** "describe choices that are under our control"[3]

**objective function** "describes a criterion that we wish to minimize (e.g., cost) . . ."[3]

**constraints** "describe the limitations that restrict our choices for decision variables"[3]

**constrained optimization problem** "how to find the single best arrangement of a set of variables, given particular rules and a scorekeeping measure"[2]

**linear program** what an optimization problem is called "if the objective and constraint functions are linear"[1]

**suboptimization** "behavior resulting from a subsystem's goals dominating at the expense of the total system's goals."[4]

## 3 Steps for setting up an optimization

Here is a general procedure that can be applied many different optimization methods. Note that a clearly defined problem will make many other steps in the process fall into place, and the first five steps are just as important as the actual mathematical and computational implementation.

1. Define the problem.
2. Define your goal (what it is you want to achieve by using optimization).
3. Identify decision variables.
4. Identify constraints.
5. Identify inputs under your control.
6. Use mathematical expressions to represent the objective function and constraints.
7. "Sanity check": Is the model formulated in a way that makes sense for this problem? Do the answers make sense given your domain expertise?

## 4 Understanding optimization problems

An optimization can get mathematically and computationally complex very quickly! I like this simple advice from Christian and Griffiths: "If you can't solve the problem in front of you, solve an easier version of it—and then see if that solution offers you a starting point, or a beacon, in the full-blown problem."[2] It's a legitimate strategy for approaching even intractable problems.

Optimization problems can be constrained by either equalities or inequalities, or some combination of these:

*Minimize* $f(x)$ *subject to* $x_1 = 0$, $10 \leq x_2 \leq 100$, *and* $0 \leq x_3 \leq 1$

When it comes to framing the objective function with mathematical notation, it is conventional (but not necessary) to express the problem as a minimization. A problem where you ultimately want to maximize something (e.g. revenue) would simply be negated so that *f(x)* is a minimization problem.

## 5 Examples of optimization problems

Here is a free book (web version) from Cambridge University Press that is a mathematically rigorous resource for learning about convex optimization (with Matlab code):

Convex Optimization [1]

The two pages that follow are from an open textbook on calculus from the University of Georgia that can be found in its entirety here:

UGA Calculus, Calculus Volume 1 - Custom University of Georgia Version [5]



## References

[1] Stephen Boyd and Lieven Vandenberghe. *Convex Optimization*. Cambridge University Press, 2010. ISBN: 9780521833783. DOI: 10.1080/10556781003625177. URL: http://stanford.edu/~boyd/cvxbook/.

[2] Brian Christian and Tom Griffiths. *Algorithms to Live By: The Computer Science of Human Decisions*. en. Macmillan, 19 4 2016. ISBN: 9781627790369.

[3] Ebrahim Nasrabadi James Orlin. *Optimization Methods in Management Science*. Creative Commons BY-NC-SA. 2013. URL: https://ocw.mit.edu.

[4] Donella H Meadows. *Thinking in Systems: A Primer*. en. Chelsea Green Publishing, 2008. ISBN: 9781603580557.

[5] UGA Calculus. *Calculus Volume 1 - Custom University of Georgia Version*. 2019. URL: http://cnx.org/contents/1c1513d3-de69-41ee-bc4b-8266100f2958@2.1.

# 4.7 | Applied Optimization Problems

## Learning Objectives

**4.7.1** Set up and solve optimization problems in several applied fields.

One common application of calculus is calculating the minimum or maximum value of a function. For example, companies often want to minimize production costs or maximize revenue. In manufacturing, it is often desirable to minimize the amount of material used to package a product with a certain volume. In this section, we show how to set up these types of minimization and maximization problems and solve them by using the tools developed in this chapter.

## Solving Optimization Problems over a Closed, Bounded Interval

The basic idea of the **optimization problems** that follow is the same. We have a particular quantity that we are interested in maximizing or minimizing. However, we also have some auxiliary condition that needs to be satisfied. For example, in Example 4.32, we are interested in maximizing the area of a rectangular garden. Certainly, if we keep making the side lengths of the garden larger, the area will continue to become larger. However, what if we have some restriction on how much fencing we can use for the perimeter? In this case, we cannot make the garden as large as we like. Let's look at how we can maximize the area of a rectangle subject to some constraint on the perimeter.

### Example 4.32

#### Maximizing the Area of a Garden

A rectangular garden is to be constructed using a rock wall as one side of the garden and wire fencing for the other three sides (Figure 4.62). Given $100$ ft of wire fencing, determine the dimensions that would create a garden of maximum area. What is the maximum area?

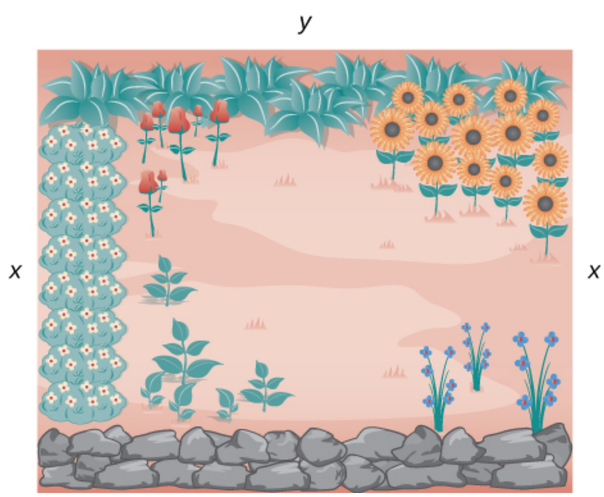


**Figure 4.62** We want to determine the measurements $x$ and $y$ that will create a garden with a maximum area using $100$ ft of fencing.

#### Solution

Let $x$ denote the length of the side of the garden perpendicular to the rock wall and $y$ denote the length of the side parallel to the rock wall. Then the area of the garden is

$$A = x \cdot y.$$

We want to find the maximum possible area subject to the constraint that the total fencing is $100$ ft. From **Figure 4.62**, the total amount of fencing used will be $2x + y$. Therefore, the constraint equation is

$$2x + y = 100.$$

Solving this equation for $y$, we have $y = 100 - 2x$. Thus, we can write the area as

$$A(x) = x \cdot (100 - 2x) = 100x - 2x^2.$$

Before trying to maximize the area function $A(x) = 100x - 2x^2$, we need to determine the domain under consideration. To construct a rectangular garden, we certainly need the lengths of both sides to be positive. Therefore, we need $x > 0$ and $y > 0$. Since $y = 100 - 2x$, if $y > 0$, then $x < 50$. Therefore, we are trying to determine the maximum value of $A(x)$ for $x$ over the open interval $(0, 50)$. We do not know that a function necessarily has a maximum value over an open interval. However, we do know that a continuous function has an absolute maximum (and absolute minimum) over a closed interval. Therefore, let's consider the function $A(x) = 100x - 2x^2$ over the closed interval $[0, 50]$. If the maximum value occurs at an interior point, then we have found the value $x$ in the open interval $(0, 50)$ that maximizes the area of the garden. Therefore, we consider the following problem:

Maximize $A(x) = 100x - 2x^2$ over the interval $[0, 50]$.

As mentioned earlier, since $A$ is a continuous function on a closed, bounded interval, by the extreme value theorem, it has a maximum and a minimum. These extreme values occur either at endpoints or critical points. At the endpoints, $A(x) = 0$. Since the area is positive for all $x$ in the open interval $(0, 50)$, the maximum must occur at a critical point. Differentiating the function $A(x)$, we obtain

$$A'(x) = 100 - 4x.$$

Therefore, the only critical point is $x = 25$ (**Figure 4.63**). We conclude that the maximum area must occur when $x = 25$. Then we have $y = 100 - 2x = 100 - 2(25) = 50$. To maximize the area of the garden, let $x = 25$ ft and $y = 50$ ft. The area of this garden is $1250\text{ ft}^2$.

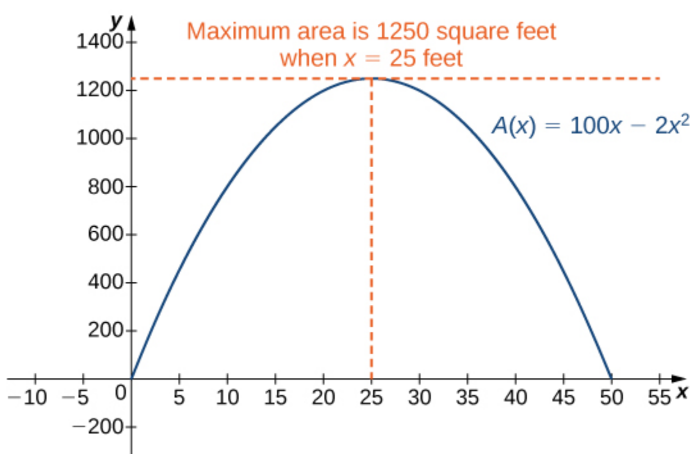


**Figure 4.63** To maximize the area of the garden, we need to find the maximum value of the function $A(x) = 100x - 2x^2$.