MAX PLANCK GESELLSCHAFT

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Understanding formalisms in GR

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The basic variable in GR is the metric $g_{\mu\nu}$, it can be decomposed in its components, a 3+1 decomposition, a tetrad or in some other variable. In this case, we make use of the *gothic metric* or density $\mathfrak{g}$ defined as [?]

$$\mathfrak{g}^{\mu\nu} = \sqrt{-g} g^{\mu\nu} \tag{1}$$

where g is the determinant of the metric. The harmonic gauge or deDonder gauge is

$$\partial_\nu \mathfrak{g}^{\mu\nu} = 0 \tag{2}$$

The Einstein field equations or relaxed equations, in this formalism are

$$\Box \mathfrak{g}^{\mu\nu} = 16\pi(-g)(T^{\mu\nu} + t_{LL}^{\mu\nu}) \tag{3}$$

where $t_{LL}^{\mu\nu}$ is the Landau-Lifshitz pseudotensor and $T^{\mu\nu}$ is the energy-momentum tensor.

The ansatz we took was of the form

$$\mathfrak{g}^{\mu\nu} = \begin{bmatrix} \sqrt{-g}\,a(r) & 0 & 0 & 0 \\ 0 & \multicolumn{3}{c}{\multirow{3}{*}{$\begin{bmatrix} \sqrt{-g}(\delta_{jk}/\sqrt{-g}) - b(r)\Omega_j\Omega_i) & & \\ & & \\ & & \end{bmatrix}_{3x3}$}} \\ 0 & & & \\ 0 & & & \end{bmatrix}$$

Where $\sqrt{-g} = (r + GM/c^2)^2/r^2$.

Spherical expansion

Once we obtain the field equations, we need to calculate the coefficients of the spherical harmonic expansion with the equation

$$a_{l,m} = \int_0^{2\pi} \int_0^{\pi} f(\theta,\phi) Y_l^m(\theta,\phi) sin\theta d\theta d\phi \tag{4}$$

In this way we eliminate the angular dependence and we can solve the differential equations as functions of the radial component. The expansion of the function is then

$$f(r,\theta,\phi) \approx \sum_{l=0}^{N} \sum_{m=-l}^{l} a_{l,m} r^l Y_l^{*m}(\theta,\phi) \tag{5}$$

Results

```
ET[{0,S},{0,S}]//ToValues//ToValues;
%//TraceBasisDummy//ToValues//NoScalar//TraceBasisDummy//ToValues;
G00=%;
(*%//FullSimplify;*)
eq0=%
```

$$\frac{1}{8}\Bigg(-\frac{16\,a[r]\,b[r]^2\,\mathrm{Cos}[\theta]^2\,\mathrm{Cos}[\phi]\,\mathrm{Sin}[\theta]^2\left(-1+b[r]\,\mathrm{Sin}[\theta]^2\right)}{-1+b[r]}+\frac{8\,a[r]\,b[r]^3\,\mathrm{Cos}[\theta]^2\,\mathrm{Cos}[\phi]^2\,\mathrm{Sin}[\theta]^2\left(\mathrm{Cos}[\theta]^2\,\mathrm{Cos}[\phi]-\mathrm{Cos}[\phi]\,\mathrm{Sin}[\theta]^2\right)}{1-b[r]}+\cdots 320\cdots+$$
$$8\,a'[r]\left(b[r]\,\mathrm{Cos}[\theta]\,\mathrm{Sin}[\theta]\,\mathrm{Sin}[\phi]-2\,b[r]\,\mathrm{Cos}[\theta]\,\mathrm{Cos}[\phi]\,\mathrm{Sin}[\theta]\,\mathrm{Sin}[\phi]-\mathrm{Cos}[\phi]^2\,\mathrm{Sin}[\theta]^2\,b'[r]\right)+8\,a''[r]-8\left(1-b[r]\,\mathrm{Cos}[\phi]^2\,\mathrm{Sin}[\theta]^2\right)a''[r]\Bigg)$$

large output | show less | show more | show all | set size limit...

```
% // ExpandAll;
% /. θ[] → θ /. ϕ[] → ϕ
Integrate[Integrate[Conjugate[SphericalHarmonicY[0, 0, θ, ϕ]] % Sin[θ], {θ, 0, Pi}], {ϕ, 0, 2 Pi}]
```

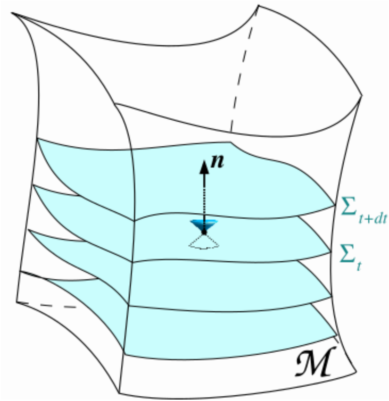
$$\frac{1}{60\,a[r]\,(-1+b[r])^2}\sqrt{\pi}\left(-35\,(-3+b[r])\,(-1+b[r])^2\,a'[r]^2+a[r]^2\left(62\,(-1+b[r])\,b[r]^2+5\,(1-3\,b[r])\,b'[r]^2\right)+10\,a[r]\,(-1+b[r])\,\left((7-5\,b[r])\,a'[r]\,b'[r]+4\,(-1+b[r])\,b[r]\,a''[r]\right)\right)$$

The solution

$$
\begin{aligned}
a(r) &= -\frac{(r+GM/c^2)}{(r-GM/c^2)} \\
b(r) &= (\frac{GM/c^2}{r})^2
\end{aligned} \tag{6}
$$

corresponds to the vacuum solution.

3+1 Decomposition



In this formalism, space-time is said to be foliated by a family of 3D hypersurfaces Σ_t (submanifolds of the manifold of dimension 3). We can orthogonally decompose the tangent manifold T_p

$$T_p = T_p(\Sigma_t) \times Vect(n) \tag{7}$$

Where Vect(n) is the 1-dimensional subspace of the tangent manifold generator by the orthogonal vector n. We can write

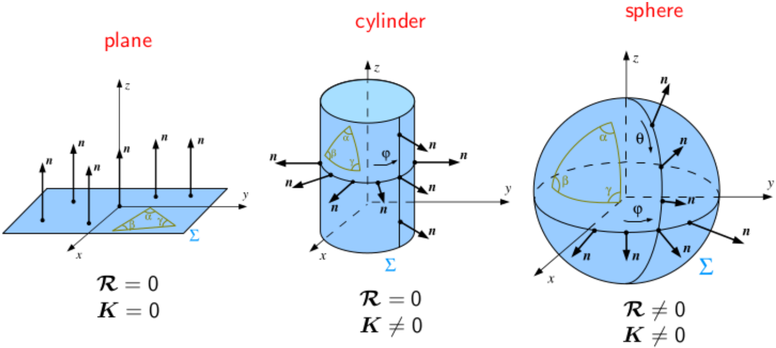
$$\gamma_{\mu\nu} = g_{\mu\nu} + \hat{n_\mu} \hat{n_\nu} \tag{8}$$

γ is said to be the orthogonal projector onto the hypersurface, the 1-form n associated with n is proportional to the gradient of $n_\mu = -\alpha \nabla_\mu t$

The extrinsic curvature K_{ij} (or second fundamental form) of Σ_t is the bilinear form defined by

$$
\begin{aligned}
K : T_p(\Sigma_t) \times T_p(\Sigma_t) &\to \mathbb{R} \\
(u, v) &\to -u \cdot \nabla_v n
\end{aligned}
\tag{9}
$$

It measures the "bending" of Σ_t by evaluating the change of direction of the normal vector n as one moves on the hypersurface.



Field equations

$$
\begin{aligned}
\frac{\partial}{\partial t}\gamma_{ij} &= -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i, \\
\frac{\partial}{\partial t}K_{ij} &= \alpha(R_{ij} + KK_{ij} - 2K_{il}K^l{}_j) - D_iD_j\alpha + (D_j\beta^m)K_{mi} + (D_i\beta^m)K_{mj} \\
&\quad + \beta^m D_m K_{ij} - 8\pi\alpha\left(S_{ij} + \frac{1}{2}\gamma_{ij}(\rho_H - S^l{}_l)\right), \\
\frac{\partial}{\partial t}\gamma &= 2\gamma(-\alpha K + D_i\beta^i), \\
\frac{\partial}{\partial t}K &= \alpha(\mathrm{tr}R + K^2) - D^iD_i\alpha + \beta^j D_j K + 4\pi\alpha(S^l{}_l - 3\rho_H),
\end{aligned}
$$

```
{EvolutionExtrinsicADM, HamiltonianConstraint, MomentumConstraint}
```

$$\left\{\mathcal{L}_n K_{bd} == \frac{1}{6\varphi^4}\left(-6\epsilon\varphi^6 A_b A_d + 6\epsilon K^a{}_a K_{bd} + 6\left(-\epsilon + \varphi^4\right) K_b{}^a K_{da} - 6\varphi^2 \bar{R}_{bd} + 3\varphi^2 h_{bd} S^a{}_a + 3\varphi^2 S_{bd} - 3\epsilon\varphi^6 S_{bd} - 4\varphi^2 h_{bd}\left(S^a{}_a\right) + 2\epsilon\varphi^6 h_{bd}\left(S^a{}_a\right) - 4\epsilon h_{bd}\,\Xi - \varphi^4 h_{bd}\,\Xi + 6\epsilon\varphi^6\left(D_d A_b\right)\right),\ \Xi == \frac{1}{2}\epsilon\left(-\varphi^2\bar{R} + \epsilon\left(K^a{}_a\right)^2 - \epsilon\left(K_{ab}K^{ab}\right)\right),\ J_d == D_c K_d{}^c - D_d K^c{}_c\right\}$$

```
LieD[n[b]]@metrich[-a, -b]
```

$K_{ab} + K_{ba}$

```
% // ToCanonical
```

$2\,K_{ab}$

```
ElectricPart
```

$$E_{bd} == \frac{-6\epsilon K^a{}_a K_{bd} + 6\epsilon K_b{}^a K_{da} + 6\varphi^2\bar{R}_{bd} - 3\varphi^2 h_{bd} S^a{}_a - 3\varphi^2 S_{bd} + 4\varphi^2 h_{bd}\left(S^a{}_a\right) + 4\epsilon h_{bd}\,\Xi}{6\varphi^4}$$

```
MagneticPart
```

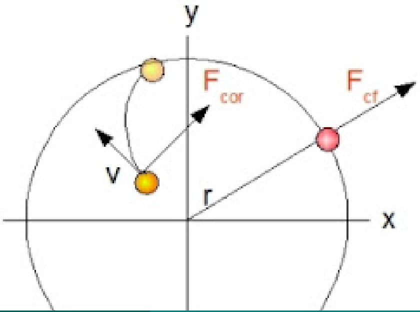
$$B_{fb} == \frac{\epsilon_b{}^c{}_f J_c + \epsilon_b{}^d{}_f J_d - 2\epsilon^{cd}{}_f\left(D_c K_{bd}\right) + 2\epsilon^{cd}{}_f\left(D_d K_{bc}\right)}{4\varphi}$$

Tetrad formalism

- The Coriolis force is an inertial force that acts on objects that are in rotation with respect to a Newtonian inertial frame of reference.
- It disappears when one undoes the rotation, so an inertial acceleration field can have at least locally the value of zero.

The so-called vierbein field components, consists of the sixteen transformation functions $e^k_\alpha(x)$ connecting the displacements dy^k and dx^α by

$$dy^k = e^k_\alpha(x) dx^a \qquad (10)$$



The functions

$$G^k_{\alpha\beta} = e^k_{\alpha,\beta} - e^k_{\beta,\alpha} \tag{11}$$

which vanish for pseudogravitational fields, will be given the name "true gravitational field strenghts."
Employing the metric tensor $g_{\mu\nu}$ to induce the product of the vierbein field and inverse vierbein field, the inner product-signature constraint is

$$g_{\alpha\beta}(x)e^{\alpha}_k(x)e^{\beta}_b(x) = \eta_{kb} \tag{12}$$

The inverse vierbein serves as a transformation matrix that allows one to represent the tetrad basis $\hat{e}_k(x)$ in terms of the coordinate basis $\hat{e}_\alpha$:

$$\hat{e}_k = e^{\alpha}_k \hat{e}_\alpha \tag{13}$$

Using the orthonormality conditions we have

$$g_{\alpha\beta}(x) = h^k_\alpha(x) h^\beta_b = \eta_{kb} \tag{14}$$

So, the vierbein field is the "square root" of the metric.
We include the vierbein field theory as a member of our tribe of "square root" theories. These include:

- The Pythagorean theorem for the distance interval $ds = \sqrt{\eta_{\alpha\beta} dx^\alpha dx^\beta}$
- Complex analysis based on $\sqrt{-1}$) as the imaginary number
- Quantum mechanics (e.g. pathways are assigned amplitudes which are the square root of probabilities)
- Quantum field theory (e.g. Dirac equation as the square root of the Klein Gordon equation)
- Quantum computation based on the universal $\sqrt{SWAP}$ conservative quantum logic gate

xPPN Package

Parametric Post-Newtonian expansion as easy as pie.

Express einstein tensor

$\rightarrow$ *expand into velocity order*

$\rightarrow$ *get each order of the equations*

$\rightarrow$ *define ansatz for the metric*

$\rightarrow$ *get equations*

```
(*Solve for the coefficients*)
sola2 = FullSimplify[First[Solve[# == 0& /@ eqnsc2, aa /@ Range[1, 3]]]]
```

$$\left\{a_1 \to \frac{\kappa^2}{4\pi},\ a_2 \to \frac{\kappa^2}{4\pi},\ a_3 \to 0\right\}$$

```
(*Check solution*)
Simplify[eqnsa2 /. sola2]
```

{0, 0}

```
(*Insert the solution into the ansatz*)
sol2def = ans2def /. sola2
sol2ru = Flatten[mkrg /@ sol2def];
```

$$\left\{\overset{2}{g}_{00} == \frac{\kappa^2 U}{4\pi},\ \overset{2}{g}_{ab} == \frac{\kappa^2 \delta_{ab} U}{4\pi}\right\}$$

```
(*Check that the metric components solve the second order field eqs*)
eqns2 /. sol2ru;
Expand[%];
PotentialToSource /@ %;
ToCanonical /@ %;
SortPDs /@ %;
Simplify[%]
```

{0, 0}

Any tensor field A is expanded as

$$A = \sum_{n=0}^{\infty} A^{n} \tag{15}$$

where each term is of the corresponding velocity order $A^n \approx O(n)$.
The expansion of expressions $F = f(A)$ where f is a scalar, or more general a tensorial function is a Taylor expansion of the function f around the background $\overset{0}{A}$,

$$f(A) = \sum_{k=0}^{\infty} \frac{(A - \overset{0}{A})^{k}}{f^{(k)}(\overset{0}{A})} \tag{16}$$

where $f^{(k)}$ denotes the k'th derivative.

Conclusions

- All of them are different representations of the Einstein tensor.
- Using the same energy-momentum tensor, yields to equivalent field equations between two different formalisms.
- Physics are added by the constraint equations of the variables used.
- It is not deeply explored
- Choosing a variable or formalism depends on what we need of the problem.

Attribute	Landau-Lifshitz formalism	3+1 Decomposition	Tetrad formalism
Variables	Components of the inverse gothic metric	Extrinsic curvature, lapse function, shift vector, and spatial metric	16 components of the vierbein field
Change the basis	No	No	Yes
Metric as 3+1 components	Yes	Yes	Yes
Complexity of field equations	low	medium	low
Usage	Helpful to solve the differential equations, wave-like field equations	Using orthogonal projections of the field equations, the field equations are even simpler	Einstein's action expressed explicitly in terms of the vierbein field reproduces the law of the pure gravitational field in a weak field limit
	The exact field equations can be expressed as a set of ten wave equations in Minkowski spacetime (with complicated and highly nonlinear source terms)	PN with 3+1 formalism allows us to consider this metric in a weak field limit as an **asymptotically flat perturbation** around a flat Minkowski background. The source of this perturbation is assumed to be of compact support and modeled by the energy-momentum tensor	The Vierbein field is taken as the "square root" of the metric tensor field.

Future work

- Different approaches to solve the Landau-Lifshitz field equations
- Find more equivalences between different formalisms
- xAct package to generate the waveform of different theories
- Find and propose different formalisms