Van der Waerden’s Theorem

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What is van der Waerden's Theorem?

Definition: An arithmetic progression (AP) is a list of equally spaced integers.

Question: Suppose we 2-color the set $[100]$. Are we always guaranteed to find a monochromatic 3-AP?

This question leads to us a fundamental result from Ramsey theory.

- Ramsey theory seeks to answer questions concerning the size of an object needed to guarantee a certain property holds.
- The proofs are generally non-constructive.

Theorem. (van der Waerden, 1927):

Given positive integers r and k, there exists an integer $W(r, k)$ such that every r-coloring of the set $[W(r, k)]$ has a monochromatic k-AP.

We know $W(r, k)$ exists. But what is it equal to?

- This is still very much an open problem in math.
- The table to the right shows all values of r and k for which a precise value is know.
- We have upper bounds on $W(r, k)$ for many of the other small values of r and k , but they tend to get very large very quickly.
- The most general bound for any $r \geq 2$ and k is

$$W(r, k) \leq 2^{2^{r^{2^{2^{k+9}}}}}$$

r	k	$W(r, k)$
2	3	9
2	4	35
2	5	178
2	6	1132
3	3	27
3	4	293
4	3	76

The simplest case to bound: $W(2,3)$

Proposition: $W(2,3) \leq 325$

Proof. Consider an arbitrary 2-coloring of $[325]$.

We can partition $[325]$ into 65 "blocks" of length 5.

A given block can have one of $2^5 = 32$ color patterns.

Hence, by the Pigeonhole Principle, within the first 33 blocks, at least 2 blocks are guaranteed to have the same color pattern.

- Call these blocks b_1 and b_2 so that the elements in block b_i are those of the form $5b_i + 1, \dots, 5b_i + 5$ for $b_i \in [0,32]$

Within a given block of length length 5, since we are only working with 2 colors, the Pigeonhole Principle also implies that two of the first three elements in a block are the same color.

- Call the two positions a_1, a_2 so that $5b_1 + a_1,\ 5b_1 + a_2,\ 5b_2 + a_1,\ 5b_2 + a_2$ are all numbers that we know to be the

  same color (blue, for the sake of brevity)
- These elements are contained within the first 33 blocks.

The simplest case to bound: $W(2,3)$, cont.

Case I: $5b_1 + a_3$ is blue.

- Then $5b_1 + a_1, 5b_1 + a_2$, and $5b_3 + a_3$ form a monochromatic 3-AP.

Case II: $5b_1 + a_3$ is red.

- Let $b_3 = 2b_2 - b_1$. Now, since $0 \leq b_1 \leq b_2 \leq 32$, we know $b_3 \leq 64$, further implying that $5b_3 + a_3 \leq 32$.

  Either:

  - $5b_3 + a_3$ is blue.
    - In which case, $5b_1 + a_1, 5b_2 + a_2$, and $5b_3 + a_3$ form a monochromatic 3-AP.

    or
  - $5b_3 + a_3$ is red.
    - Then, we know $5b_1 + a_3$ is red, meaning $5b_2 + a_3$ is also red, and so $5b_1 + a_3, 5b_2 + a_3, 5b_3 + a_3$ form a monochromatic 3-AP.

In all cases, we have show that it is possible to form a monochromatic 3-AP with a maximum of 325 numbers.

Very loose upper bounds

As you may have seen earlier, $W(2,3) = 9 < 325$. Not great.

Unfortunately, as we increase the number of colors or the desired length of an AP, it gets so much worse.

Consider the $W(3,3)$ case. Without spending too much time on it, by a similar argument (but using subblocks to apply Pigeonhole Principle, we have

$$W(3,3) \leq 2 \cdot 3^{7 \cdot (2 \cdot 3^7 + 1)}$$

The actual value of $W(3,3)$ is 27.

We now move on to the Jupyter notebook where these ideas are further explored.