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Hydrodynamic Impact Analysis of a Cylinder

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ABSTRACT

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The determination of the dynamic behavior in waves of semisubmersibles having a large permanent list following structural, systems or operational failure is a matter of extreme concern. The main difficulty lies in the determination of the forces acting on the partially emerged pontoons. Impact forces, in particular, may have a considerable importance. Insight into this problem is gained from the consideration of the simpler situation, yet not fully understood, of the hydrodynamic impact of a rigid horizontal cylinder.

The mathematical formulation of the problem is stated. The method of matched asymptotic expansions is used to solve it. The singularity which appears at the water line is investigated. A new formula for the impact force is obtained, which differs from the classical von Karman's formula by a corrective term, the time derivative of which is infinite at the time of impact. The results obtained are compared with those of experimental observations and numerical calculations. Agreement is excellent for small penetration. The method may be extended to different geometries and non-vertical velocities.

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Introduction

Among the various types of mobile offshore drilling units presently in operation, semisubmersibles were, up until recently, considered to enjoy good safety records. Because of their geometrical spread and moderate motions, they were regarded as inherently safer than ships. As a matter of fact, the capsizing of an intact semisubmersible is a highly unlikely occurence under normal or even severe environmental conditions. The situation, however, may change completely for damaged semisubmersibles following structural, operational or systems failure. Two recent accidents resulting in heavy loss of lives have demonstrated that the apparently large reserve of intact stability in semisubmersibles can be deceptive. A partial failure leading to modifications of the stability characteristics must be recognized as a likely event, and every possible step should be taken to ensure that it does not result in the total loss of the rig and its crew.

The awareness of the current lack of knowledge concerning the various mechanisms affecting the stability of damaged semisubmersibles has motivated worldwide research, both theoretical and experimental. The Norvegian Maritime Directorate initiated the Mobile Platform Stability (or MOPS) project in 1981. A project synthesis and some recommendations are given by Dahle (1985). An other research project was launched in Japan in 1982 to investigate stability of twin pontoons semisubmersibles (e.g. Himeno *et al.*, 1982). If the experimental results clearly demonstrate the importance of non-linearities when the pontoons pierce the free surface (Huang, Naess, and Hoff, 1982), the mathematical modelization of the problem represents a formidable challenge. Even the nature and the importance of the different phenomena involved is not known.

As a part of the MOPS project, Huang and Naess (1983) and Naess and Hoff (1984) developed a numerical simulation technique for heavily listed semisubmersibles. This numerical model was built following the general method given by Paulling (1977). The main difficulty laid in the determination of the forces acting on the partially emerged pontoons. Huang, Naess, and Hoff used a strip theory, the justification of which is not obvious in such a case. They split the forces acting on a section into various components, representing a mix between rigorous expressions derived from hydrodynamic theory (neglecting, however, the frequency dependence of the hydrodynamic coefficients) and semi-empirical formulae of the Morison type. The relatively poor agreement between observed and calculated response could very well be explained by these uncertainties on the loading terms which should be determined with higher accuracy.

A rational approach to the problem of the determination of the wave and current forces acting on the inclined pontoons is therefore needed. As Lin, Newman, and Yue (1984) point out, when viscous forces are neglected, the two main difficulties appearing in such a problem are the memory effect of the free surface (or the presence of outgoing waves) and the mathematical treatment of the flow near the intersection of the body and the free surface. The latter problem is considered here. It must be adressed in order to deal with the non-linearities introduced by the fact that the pontoons are piercing the free surface. Such non-linearities apparently play an essential role.

Insight into this problem is gained from the consideration of the simpler situation, yet not fully understood, of the hydrodynamic impact of a circular cylinder. This situation ideally models the impact forces acting on the partially emerged pontoons, the importance of which may be considerable, but should also give a first idea of the phenomena involved in the determination of the flow around the pontoons. First, a review of the existing results concerning the water impact problem is made. It shows that most of the fundamental concepts were developed before the second world war, but seem to be almost forgotten. In order to define a rational way to address the problem. a brief survey of the use of the method of matched asymptotic expansions for solving similar problems follows. Then the mathematical formulation of the water impact of a circular cylinder is stated. The method of matched asymptotic expansions is used to find an approximate solution for the flow, and therefore for the impact force. This solution agrees to the first order with Wagner's solution (1932), and justifies the fact that the piled-up water has to be considered for a circular cylinder at the beginning of impact. An approximate second order correction is proposed. Its effect is to decrease the impact force very rapidly. The results obtained are compared with those of experimental observations and numerical calculations. Agreement is excellent within the assumptions made.

The Water Impact Problem

The problem of water impact was first addressed in the early thirties. At that time, studies were performed to predict the impact force on a landing seaplane. Von Karman (1929) introduced the main physical concepts of the phenomenon. He studied the vertical entry of a wedge, but his results may also apply to the case of a circle. He equated the impact force on the penetrating object to half the force acting on a flat plate of width the width of the submerged part of the body moving in an unbounded fluid at the impacting velocity (figure 1). This force is given by

$$F = \frac{d}{dt}(m_{vK} V),$$

where m_{vK} is half the added mass of the plate.

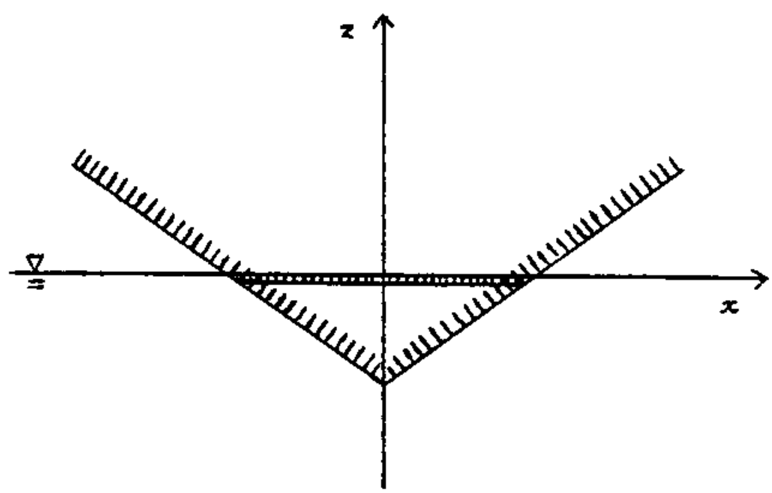

Fig. 1 Von Karman : flat plate fitting

A few years later (1931 and 1932), Wagner refined von Karman's analysis and gave the basic ideas for the mathematical treatment of the problem. His approach will be followed here. He neglected gravity and surface tension and assumed the body to be rigid. Chosing the velocity potential Φ as the unknown, he linearized the problem. This led for a wedge with small deadrise angle to the approximation of the body by a flat plate and by the free surface boundary condition $\Phi=0$. If this free surface boundary condition had been taken on the undisturbed free surface and the linearized pressure integrated on the body, this would have led to von Karman's expression for the impact force. But Wagner approximated the shape of the free surface using the relation $\eta_t=\Phi_z$. [1] He took into account the piled-up water to determine the wetted width of the wedge which gave the width of the flat plate (figure 2). Moreover, he recognized that the linearization was not valid near the intersection of the free surface and the body. There he found a solution with a jet, and gave its thickness (figure 2). Because the thickness of this jet was very small , he neglected the pressure distribution in the jet as well as the quadratic terms in the pressure.[2] Therefore he found for the impact force

$$F=\frac{d}{dt}(m_w V),$$

where m_w is half the added mass of the plate of width the wetted width of the body. This modification to von Karman's formula will be referred later as the ***wetting correction***. Wagner obtained other important results and in particular introduced an 'exact' solution for the wedge using the self-similarity of the flow. A lot of studies of the self-similar flow around a wedge or a cone followed

[1] η is the free surface elevation. The subscripts indicate differentiation.

[2] It is important to notice that the quadratic pressure would have led to an infinite suction force if it were integrated over the whole plate. Only the study of the flow near the singularity which appeared in the linearized problem allowed this approximation.

Wagner's work (e.g. Schiffman and Spencer, 1951; Mackie, 1967). This self-similarity will be used here to find the wetting correction for a circle at the beginning of impact.

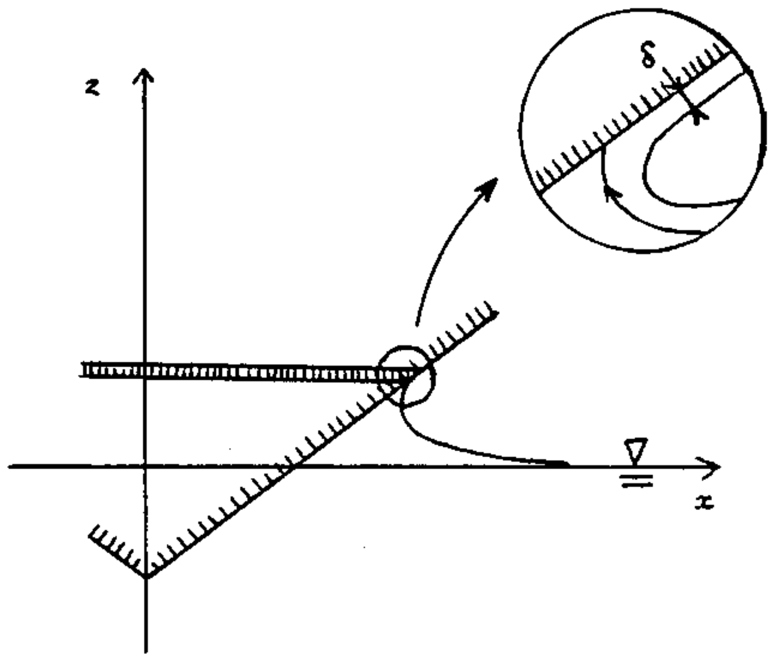

Fig.2 Wagner : flat plate fitting

The second world war served as an impetus for conducting further research in the domain of water impact because of the interest in water entry of projectiles. Von Karman's and Wagner results were used with other fitting shapes than a flat plate. In particular this led to the use of half the added mass of the body together with its image within the free surface. This is also equal to the added mass of the submerged part of the body at infinite frequency, m_∞. This was done for a sphere by Schiffman and Spencer (1945) and for a wedge by Bisplinghoff and Doherty (1952). Later on Schiffman and Spencer (1951) clarified the concept of added mass for the water impact problem. They made a distinction between the kinetic energy and momentum added masses. Only the last one, defined by

$$\frac{d}{dt}(mV)=F,$$

will be considered here. Schiffman and Spencer pointed out that even when the free surface boundary condition was taken as $\Phi=0$ on an horizontal line, the momentum added mass was *not* equal to m_∞. A ***free surface correction*** had to be made which took into account the momentum of the free surface. Fabula (1957) showed that this correction led to an infinite suction force when the flat plate fitting was used. This is due to the singularity which appears in the linearized problem. As mentioned above, the impact force *is* infinite within the linear theory. Therefore, following Schiffman and Spencer (1955), Fabula used an ellipse fitting method to calculate the wetting correction [3] (and in some cases the free surface correction) together with m_∞ for the exact shape. For the wedge, Fabula compared this method with the exact diamond fitting for the wetting correction (figure 3). [4] He got a better agreement with experimental data using the method of ellipse fitting. He explained this better agreement by a "crude spray root approximation" produced by his method. This seems to show that at least a part of the problem lies in the determination of the pressure distribution near the water line. It became clear at this period that von Karman's

[3] When ellipse fitting is used, the linear problem is not singular.

[4] When diamond fitting is used, the linear problem is singular but the quadratic pressure is integrable over the whole body boundary.

formula had to be used with m_∞ for the submerged body, the wetting correction (for flat bodies), and a *drag correction* taking into account the quadratic terms in the pressure and the relative motion of the body and the free surface. Yet the problem remained that the quadratic pressure was unbounded and negative near the water line within the linearization of the free surface boundary condition.

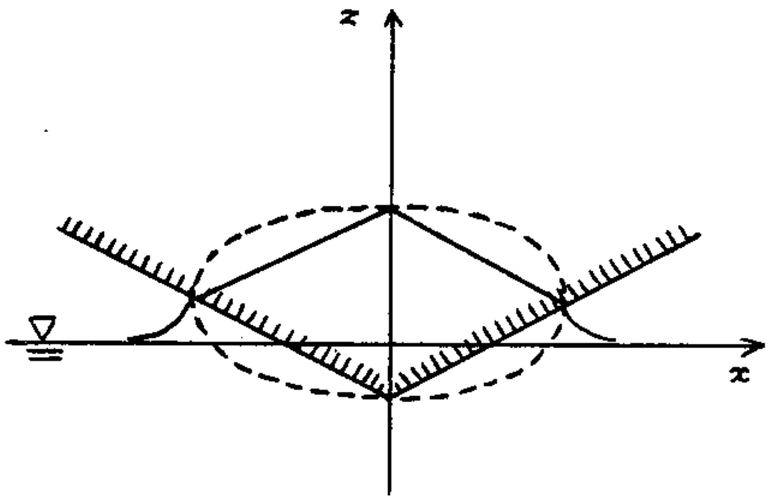

Fig. 3 Fabula : diamond and ellipse fittings

In the early sixties, the connection between ship slamming and the water entry problem was first investigated. This led to extensive reviews on the subject (e.g. Szebehely, 1959; Chu and Abramson, 1961, Szebehely and Ochi, 1966). The capsule re-entry problem also gave a new impulse for the study of water impact, with a special emphasis on the hydroelastic interaction (Wilkinson *et al.*, 1967). Only recently a new application of the theory of water impact was found in offshore technology. Miloh (1981) developed a solution for the impact of a sphere giving an analytical formula for m_∞ and calculating the height of the piled-up water. Both experimental and theoretical studies were published concerning the wave slam on horizontal members of cylindrical cross-sections (Kaplan and Silbert, 1976; Dalton and Nash, 1976; Faltinsen *et al.*, 1977; Arhan and Deleuil, 1978; Sarpkaya, 1978). All the theoretical results were obtained using von Karman's formula with the value of m_∞ for a circular lens which was given by Taylor (1930). It is found surprising that none of these papers used the wetting correction and/or the drag correction while the underestimation of the impact force by von Karman's formula was generaly admitted.

Finally both numerical and experimental results concerning the water impact of a circular cylinder were given in an EPRI report concerned with boiling water reactors. The four numerical models presented in this report are :

1. an explicit Lagrangian method (Gross, 1978),
2. a boundary integral method (Geers, 1978 and 1982),
3. a finite element method (Marcal, 1978),
4. an incompressible Eulerian fluid method (Nichols and Hirt, 1978).

These numerical results, together with experimental results and some of the analytical formulas described above, are shown on figure 4. The non-dimensional impact force (often referred to as the *slamming coefficient*) is plotted as a function of the non-dimensional time. Von Karman's and Wagner results are given using flat plate fitting. Wagner's result is obtained with a *wetting factor* of two. [5] The result using von Karman's formula together with the

[5] The wetting factor is defined by $C_w = \frac{h+\xi}{h}$, where h is the penetration depth and ξ the height of the piled-up water. This value of C_w will be derived later.

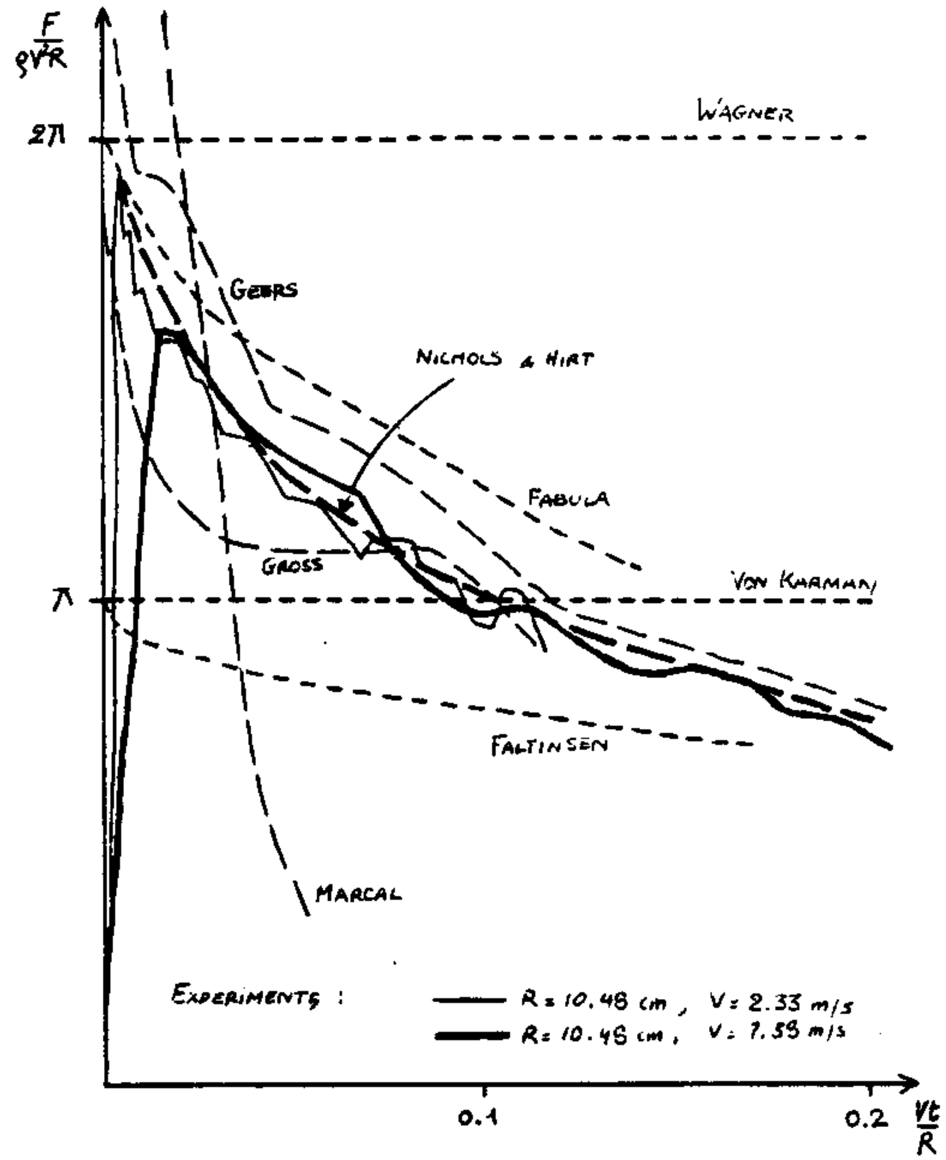

Fig. 4 Experimental, numerical, and theoretical slamming coefficients

added mass of the circular lens is also plotted and attributed to Faltinsen (1977). The main results which may be drawn from this figure are :

1. von Karman's formula underestimates the impact force, while Wagner's formula overestimates it,

2. the use of the added mass of the circular lens in von Karman's formula doesn't improve the accuracy. On the other hand, Fabula's method is very accurate,

3. at least one of the numerical simulations is in excellent agreement with the experimental data. Since these simulations assume that the body is rigid, this hypothesis, which was found unacceptable by Sarpkaya (1978), [6] seems reasonable, at least in this case,

4. in all the analytical results given, the impact force rises instantaneously from zero to a finite value. This is not in agreement with the experimental data. As pointed out by Sarpkaya (1978) several factors, specifically the compressibility of the air between the cylinder and water surface, entrapped gases in the water, and surface irregularities, might account for some finite rise time. Yet this interesting problem will not be addressed here,

5. The compressibility of the water should also be considered at the beginning of impact, when the apparent growth of the circular lens is greater than the sound velocity in the liquid. This gives here (with the wetting correction)

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[6] Sarpkaya explained the underestimation of the impact force by von Karman's formula by dynamic effects associated with the vibration of the cylinder. Because the period of this vibration is only a function of the cylinder in Sarpkaya's model (where hydroelastic interaction is neglected), this is in contradiction with the fact that the non-dimensional curves are superimposed for two experiments with the same cylinder and different velocities.

$\frac{Vt}{R} < \frac{V^2}{V_s^2}$, where V_s is the water sound velocity. For the results shown, this seems negligible compared to the phenomena described in 4. above.

In conclusion, it is believed that an accurate prediction of the impact force can be made using the general assumptions stated by Wagner (incompressible and inviscid fluid, irrotational flow, rigid cylinder, no surface tension, and eventually no gravity). The value predicted by Wagner's theory seems to be a good limit for the impact force when the time t goes to zero (with the restrictions mentioned in 4. and 5. above). This leads to consider a perturbation method where the small parameter is related to the time or the penetration depth. Using this approach, it is expected that the first correction to Wagner's result will involve geometric effects (and in particular the geometry of the body considered by Faltinsen), but also the quadratic terms in the pressure. As that was pointed out, it seems difficult to take into account the latter correction without removing the singularity which appears at the water line. This will be done here using the method of matched asymptotic expansions, which will appear very similar to Wagner's approach of the problem. Before going into the mathematical details, a brief survey of the use of the method of matched asymptotic expansions to similar problems will be made. This should provide a useful guideline for the following developments.

The Method of Matched Asymptotic Expansions

The method of matched asymptotic expansions was successfully applied in some cases for the determination of the singular flow near a body piercing a free surface. This method is in fact very similar to Wagner's approach of the impact problem. Therefore Wagner's work (1931 and 1932) will be briefly rediscussed to point out the major features of the method. This author used linearized equations in a domain which will be referred later as the ***outer domain*** or ***far field***. Yet a singularity appeared in the outer solution. This was in contradiction with the assumptions made to linearize the problem. [1] Therefore Wagner defined an ***inner domain*** or ***near field*** near the water line. In this domain the non-linear equations had to be solved. Yet, because of geometric simplifications, an exact solution was possible using the powerful method of conformal mapping. This solution involved the presence of a jet. Its thickness was a priori unknown and needed to completely determine the problem. It was found by matching the inner and outer solutions, i.e. demanding that the behavior of the inner solution at infinity be the same as the behavior of the outer solution near the singularity.

If Wagner's outer solution was the basis of most of the following studies of the water entry problem, Wagner's inner solution was also at the origin of other studies. Green (1936) studied the steady gliding of a plate on a free surface in two dimensions. As Wagner did, he assumed the formation of a jet. But, since his plate had a finite width, [2] he considered the direction of the jet as an unknown. Yet, he only got one equation between the direction of the jet and its thickness.

[1] In particular, the quadratic pressure was going to $-\infty$ near the water line. This was not consistent with the linearization which disregarded the quadratic terms.

[2] In Wagner's inner domain the flat plate had an infinite width.

Therefore the problem was not fully determined. Moreover, the asymptotic behavior of the free surface at infinity was in $\log x$. Despite the similarity of this solution with Wagner's inner solution [3], it is only recently that the matching with an appropriate outer solution was performed. Rispin (1966) and Wu (1967) studied the two-dimensional problem at large Froude number, $Fr \gg 1$. They indeed showed that in the inner domain the problem was, at first approximation, Green's problem. Therefore the physically unacceptable behavior at infinity of this solution was not necessarily a source of concern : one should only ensure that the solution properly matched with a far field solution. Shen and Ogilvie (1972) used another method. They neglected gravity ($Fr=\infty$) but showed that the indetermination of Green's problem might be removed by studying the three-dimensional case. The constants of the problem [4], direction of the incident flow, direction and thickness of the jet, were to be found in matching the inner and outer solutions. Yet the matching of the zeroth order solutions only gave the direction of the incident flow. Therefore they had to go to the next order to find the missing equation. Shen and Ogilvie took as expansions for the potential and the free surface elevation

$$\Phi=\Phi_0+\varepsilon\Phi_1+o(\varepsilon)$$
$$\eta=\eta_0+\varepsilon\eta_1+o(\varepsilon).$$

After rather intricate calculations, they obtained the missing equation. But Nguyen (1975) and Nguyen and Rojdestvenski (1975) pointed out that this approach was not consistent, because this equation explicitely involved ε. Yet the basic assumption of the perturbation technique is that

[3] Where in particular the behavior of the free surface at infinity is in $\sqrt{x}$

[4] Which are in fact functions of the three-dimensional geometry

$$f(\varepsilon,.) = \sum_{i=0}^{\infty} \delta_i(\varepsilon) f_i(.)\ ,\ \lim_{\varepsilon \to 0} \frac{\delta_{i+1}(\varepsilon)}{\delta_i(\varepsilon)} = 0\ ,$$

where the f_i are not functions of ε. These authors used an expansion of the zeroth order inner solution to guess the adequate expansion for Φ and ε, which are

$$\Phi = \Phi_0 + \varepsilon \Phi_1 + o(\varepsilon)$$
$$\eta = \eta_0 + \varepsilon \log(\varepsilon) \eta_1 + \varepsilon \eta_2 + o(\varepsilon).$$

Using this expansion (and, again, some intricate calculations) they succeeded in completely determining the problem.

The method of matched asymptotic expansions was also used to study the steady motion of a ship at large draft Froude number (Dagan and Tulin, 1972; Fernandez, 1981). In this case, a jet may appear at the bow and a wave at the stern. Without going into the mathematical details of these papers, one of the main difficulties which appeared was the determination of the adequate expansions for Φ and η. The problem is the frequent appearance of terms involving $\log(\varepsilon)$, which was pointed out by van Dyke (1964, p. 200). In that sense, it is important to verify that the successive orders solutions are independent of ε.

This method will be here applied to the impact of an infinite circular cylinder. The major difference with the works mentioned is that this problem is by essence time dependent. Actually, the small parameter ε will be closely related to the time. Yet the formal approach of the problem is quite similar.

Water Impact of a Circular Cylinder

Equations of the Problem

The penetration at constant vertical velocity $-V$ ($V>0$) of an infinite circular cylinder of radius R into a fluid domain initially at rest is considered here. The undisturbed free surface coincides with the x axis of the fixed (x,z) coordinate system. It is assumed that the cylinder is rigid and that the fluid is incompressible and inviscid. Thus the rotation is zero (e.g. Germain, 1982), and the velocity potential is taken as variable. [1] The surface tension of the water is neglected. The free surface is given by

$$F(x,z,t) = 0$$

with, whenever possible,

$$F(x,z,t) = z - \eta(x,t),$$

where η is the free surface elevation. Because of the symmetry of the problem, only the domain $x \geq 0$ is considered. Moreover, only the beginning of impact is studied, and $\frac{Vt}{R} < 1$. [2] The body boundary is given by

$$z = B(x,t),$$

where B satisfies

$$x^2 = 2R(B(x,t)+Vt) - (B(x,t)+Vt)^2\,, \quad B \leq R - Vt.$$

The different boundaries are shown on figure 5.

With these definitions and assumptions, the equations of the problem are easily found. They are :

$$\Delta\Phi = 0 \qquad \text{(FD)}$$

[1] The velocity potential is defined here by $\nabla\Phi = u$, where u is the fluid velocity.

[2] $t = 0$ at the instant of impact.

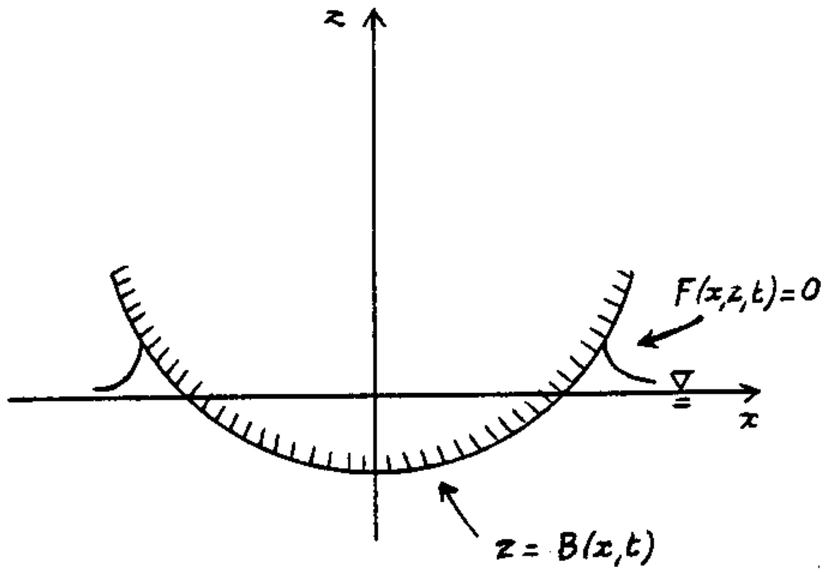

Fig. 5 Water impact of a cylinder : geometric definitions

in the fluid domain,

$$\Phi(x,z,0) = 0\ ;\ F(x,z,0) = z \qquad \text{(IC)}$$

as initial conditions,

$$\left.\begin{aligned} F_t + \nabla F \cdot \nabla\Phi = 0 \\ gz + \Phi_t + \tfrac{1}{2}\nabla\Phi \cdot \nabla\Phi = 0 \end{aligned}\right\} \text{ for } F(x,z,t) = 0 \qquad \text{(FSBC)}$$

as free surface boundary conditions,

$$\Phi_x x - \Phi_z(R - Vt - z) = V(R - Vt - z) \ \text{ for } \ z = B(x,t) \qquad \text{(BBC)}$$

as body boundary condition,

$$\Phi_x = 0 \ \text{ for } \ x = 0\ ,\ z \leq -Vt \qquad \text{(SC)}$$

as symmetry condition,

$$\nabla\Phi \to 0 \ \text{ and } \ \Phi_t \to 0 \ \text{ for } \ (x^2 + z^2) \to \infty \qquad \text{(BI)}$$

as behavior at infinity.

Even with the simplifying assumptions introduced, there remain in the problem two fundamental non-linearities. The first concerns the fact that the free surface boundary conditions are given on a surface which itself is to be found as a part of the problem. The second arises from non-linear terms in the boundary conditions themselves. It is impossible to solve exactly such a problem, and a perturbation method will be used to find an approximate solution. In order to perform the appropriate simplifications, the equations of the problem are first non-dimensionalized.

Non-Dimensional Equations

It is convenient to introduce T, the characteristic time of impact. Taking

$$\varepsilon = \sqrt{\frac{VT}{R}}, \; Fr = \frac{V}{\sqrt{gR}}, \; \text{and } L = \sqrt{VTR}\;,$$

the following non-dimensional variables are defined :

$$\overline{\Phi} = \frac{\Phi}{VR}, \; \overline{x} = \frac{x}{L}, \; \overline{z} = \frac{z}{L}, \; \overline{t} = \frac{t}{T}, \; \overline{F} = \frac{F}{L}, \; \text{and } \overline{B} = \frac{B}{L}$$

L characterizes the width of the intersection of the cylinder and the undisturbed free surface. Since $\varepsilon\Phi$ varies like VL, $\varepsilon\Phi_t$ will vary like $\frac{VL}{T}$ and the first order impact force should vary like ρV^2R; it should therefore be a constant. The quadratic terms in the pressure, $\varepsilon^2\nabla\Phi.\nabla\Phi$, vary like V^2 and the quadratic correction of the preceding term should be in $\rho V^2R\sqrt{\frac{VT}{R}}$. These simple considerations justify the choice of the perturbation parameter.

The non-dimensional equations are : [3]

$$\Delta\overline{\Phi} = 0 \qquad ([illegible])$$

[3] For simplicity, and when no confusion is possible, bars are omitted on the derivatives which are implicitely taken with respect to the current variables.

$$\bar{\Phi}(\bar{x},\bar{z},0) = 0\,;\ \bar{F}(\bar{x},\bar{z},0) = \bar{z} \tag{IC}$$

$$\left.\begin{aligned} \bar{F}_t + \nabla\bar{F}\,.\,\nabla\bar{\Phi} = 0 \\ \frac{\varepsilon^3}{Fr^2}\bar{z} + \bar{\Phi}_t + \tfrac{1}{2}\nabla\bar{\Phi}\,.\,\nabla\bar{\Phi} = 0 \end{aligned}\right\} \text{ for } \bar{F}(\bar{x},\bar{z},\bar{t}) = 0 \tag{FSBC}$$

$$\varepsilon\bar{\Phi}_x\bar{x} - \bar{\Phi}_z(1-\varepsilon\bar{z}-\varepsilon^2\bar{t}) = \varepsilon(1-\varepsilon\bar{z}-\varepsilon^2\bar{t}) \text{ for } \bar{z} = \bar{B}(\bar{x},\bar{t}) \tag{BBC}$$

$$\varepsilon\bar{B}(\bar{x},\bar{t}) = (1-\varepsilon^2\bar{t})-(1-\varepsilon^2\bar{x}^2)^{1/2}$$

$$\bar{\Phi}_x = 0 \text{ for } \bar{x}=0\,,\ \bar{z}\le-\varepsilon\bar{t} \tag{SC}$$

$$\nabla\bar{\Phi} \to 0 \text{ and } \bar{\Phi}_t \to 0 \text{ for } (\bar{x}^2+\bar{z}^2) \to \infty \tag{BI}$$

These equations are the exact equations of the problem. Only now additional assumptions will be made to linearize it in the outer domain. It will be assumed in the following sections that

$$\varepsilon \ll 1 \quad \text{and} \quad \varepsilon \ll Fr\,.$$

Outer Domain Equations

The assumption is made that in an outer domain the free surface is given by

$$\bar{F}(\bar{x},\bar{z},\bar{t}) = \bar{z} - \bar{\eta}(\bar{x},\bar{t}).$$

The intersection of the free surface and the body (or water line) is the point $(\bar{l}(\bar{t}),\bar{\xi}(\bar{t}))$ (figure 6), where

$$\bar{\xi}(\bar{t}) = \bar{\eta}(\bar{l}(\bar{t}),\bar{t}) = \bar{B}(\bar{l}(\bar{t}),\bar{t}).$$

The equations in the outer domain are therefore :

$$\Delta\bar{\Phi} = 0 \tag{FD}$$

$$\bar{\Phi}(\bar{x},\bar{z},0) = 0\,;\ \bar{\eta}(\bar{x},0) = 0 \tag{IC}$$

$$\left.\begin{aligned} \bar{\eta}_t + \bar{\eta}_x\bar{\Phi}_x - \bar{\Phi}_z = 0 \\ \frac{\varepsilon^3}{Fr^2}\bar{\eta} + \bar{\Phi}_t + \tfrac{1}{2}\nabla\bar{\Phi}\,.\,\nabla\bar{\Phi} = 0 \end{aligned}\right\} \text{ for } \bar{z} = \bar{\eta}(\bar{x},\bar{t})\,,\ \bar{x}\ge\bar{l}(\bar{t}) \tag{FSBC}$$

$$\varepsilon\bar{\Phi}_x\bar{x} - \bar{\Phi}_z(1-\varepsilon\bar{z}-\varepsilon^2\bar{t}) = \varepsilon(1-\varepsilon\bar{z}-\varepsilon^2\bar{t}) \text{ for } \bar{z} = \bar{B}(\bar{x},\bar{t})\,,\ \bar{x}\le\bar{l}(\bar{t}) \tag{BBC}$$

$$\bar{B}(\bar{x},\bar{t}) = \varepsilon(\tfrac{1}{2}\bar{x}^2-\bar{t}) + o(\varepsilon^2)$$

$$\bar{\Phi}_x = 0 \text{ for } \bar{x}=0\,,\ \bar{z}\le-\varepsilon\bar{t} \tag{SC}$$

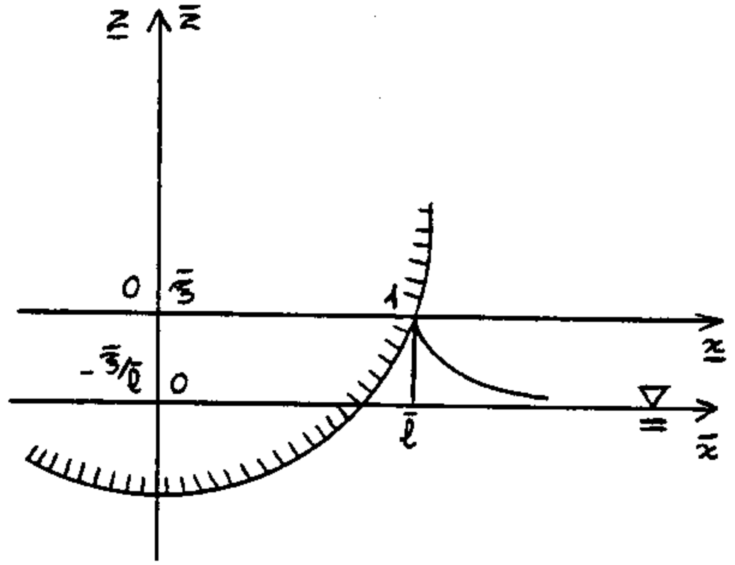

Fig. 6 Water impact of a cylinder : outer domain

$$\nabla\bar{\Phi} \to 0 \text{ and } \bar{\Phi}_t \to 0 \text{ for } (\bar{x}^2+\bar{z}^2) \to \infty \tag{BI}$$

A major difficulty in order to use a perturbation scheme is that the position of the water line is an unknown. Therefore it is *a priori* a function of ε. In order to obtain successive orders problems independent of ε, a Taylor expansion of the boundary conditions near the water line should be made. This is unfortunately not possible because a singularity is expected near this water line. Thus a change of variable is made to avoid this difficulty. Its effect is to fix the position of the water line at the point (1,0). The new variables are defined by (figure 6)

$$\underline{x} = \frac{\bar{x}}{\bar{l}},\ \underline{z} = \frac{\bar{z}-\bar{\xi}}{\bar{l}},\ \underline{\Phi} = \frac{\bar{\Phi}}{\bar{l}},\ \underline{\eta} = \frac{\bar{\eta}-\bar{\xi}}{\bar{l}^2},\ \underline{B} = \frac{\bar{B}-\bar{\xi}}{\bar{l}},\ \text{and } \underline{t} = \bar{t}\,.$$

Since $\bar{l}$ is a function of $\bar{t}$ the partial derivative with respect to $\bar{t}$ is given by

$$\frac{\partial}{\partial\bar{t}}(.) = \frac{\partial}{\partial\underline{t}}(.) - \frac{\partial}{\partial\underline{x}}(.)\,\frac{\underline{x}\bar{v}}{\bar{l}} - \frac{\partial}{\partial\underline{z}}(.)\,(\frac{\underline{z}\bar{v}}{\bar{l}} + \frac{1}{\bar{l}}\,\frac{d\bar{\xi}}{d\bar{t}})\,.$$

where $\bar{v} = \frac{d\bar{l}}{d\bar{t}}$. The equations of the problem in term of the new variables are then easily found. They are :

$$\Delta \underline{\Phi} = 0 \tag{FD}$$

$$\bar{l}(0) = 0 \tag{IC}$$

$$\begin{cases} 2\bar{l}\bar{v}\underline{\eta} + \frac{d}{dt}\bar{\xi} + \bar{l}^2\,\underline{\eta}_t - \bar{l}\bar{v}\underline{x}\,\underline{\eta}_x + \bar{l}\,\underline{\eta}_x\underline{\Phi}_x - \underline{\Phi}_z = 0 \\ \frac{\varepsilon^3}{Fr^2}(\bar{l}^2\underline{\eta} + \bar{\xi}) + \bar{v}\underline{\Phi} + \bar{l}\underline{\Phi}_t - \bar{v}\underline{x}\,\underline{\Phi}_x - (\bar{v}\bar{l}\underline{\eta} + \frac{d\bar{\xi}}{d\bar{t}})\underline{\Phi}_z + \frac{1}{2}\nabla\underline{\Phi}\,.\,\nabla\underline{\Phi} = 0 \end{cases} \tag{FSBC}$$

for $\underline{z} = \bar{l}\underline{\eta}(\underline{x},\underline{t})$, $\underline{x} \geq 1$

$$\varepsilon\bar{l}\underline{\Phi}_x\underline{x} - \underline{\Phi}_z(1-\varepsilon\bar{l}\underline{z}-\varepsilon\bar{\xi}-\varepsilon^2\underline{t}) = \varepsilon(1-\varepsilon\bar{l}\underline{z}-\varepsilon\bar{\xi}-\varepsilon^2\underline{t})\ ,\ \text{for } \underline{z} = \underline{B}\ ,\ \underline{x} \leq 1 \tag{BBC}$$

$$\underline{B}(\underline{x},\underline{t}) = \varepsilon(\tfrac{1}{2}\bar{l}\underline{x}^2 - \frac{\underline{t}}{\bar{l}}) - \frac{\bar{\xi}}{\bar{l}} + o(\varepsilon^2)$$

$$\underline{\Phi}_z = 0 \text{ for } \underline{z} = 0\ ,\ \underline{z} \leq -\varepsilon\frac{\underline{t}}{\bar{l}} - \frac{\bar{\xi}}{\bar{l}} \tag{SC}$$

$$\nabla\underline{\Phi} \to 0 \text{ and } \frac{\partial\underline{\Phi}}{\partial\bar{t}} \to 0 \text{ for } (\underline{x}^2+\underline{z}^2) \to \infty \tag{BI}$$

It is on this system that a perturbation method will be applied.

First Order Outer Domain Problem

It is assumed that

$$\underline{\Phi} = \varepsilon\underline{\Phi}_1 + o(\varepsilon)\ ,\quad \underline{\eta} = \varepsilon\underline{\eta}_1 + o(\varepsilon)\ ,$$
$$\bar{l} = \bar{l}_1 + o(1)\ ,\quad \bar{v} = \bar{v}_1 + o(1)\ ,\quad \bar{\xi} = \varepsilon\bar{\xi}_1 + o(\varepsilon)\ ,$$

where all the functions with subscripts 1 are by hypothesis of order 1 and independent of ε. The equation

$$\underline{B}(1,0) = 0$$

gives at this order

$$\bar{l}_1^2 = 2(\underline{t} + \bar{\xi}_1)\ .$$

The first order system is easily obtained. A Taylor expansion of the boundary conditions is made to remove the ε dependence of the solution. This gives : [4]

$$\Delta \underline{\Phi}_1 = 0 \qquad \text{(FD)}$$

$$\bar{l}_1(0) = 0 \qquad \text{(IC)}$$

$$\left.\begin{aligned} 2\bar{l}_1\bar{v}_1\underline{\eta}_1 + \bar{l}_1\bar{v}_1 - 1 + \bar{l}_1^{\,2}\underline{\eta}_{1t} - \bar{l}_1\bar{v}_1\underline{x}\underline{\eta}_{1x} - \underline{\Phi}_{1z} &= 0 \\ \bar{v}_1\underline{\Phi}_1 + \bar{l}_1\underline{\Phi}_{1t} - \bar{v}_1\underline{x}\underline{\Phi}_{1x} &= 0 \end{aligned}\right\} \text{ for } \underline{z}=0\,,\ \underline{x}\geq 1 \qquad \text{(FSBC)}$$

$$\underline{\Phi}_{1z} = -1 \text{ for } \underline{z} = 0\,,\ \underline{x} \leq 1 \qquad \text{(BBC)}$$

$$\underline{\Phi}_{1x} = 0 \text{ for } \underline{x} = 0\,,\ \underline{z} \leq 0 \qquad \text{(SC)}$$

$$\nabla\underline{\Phi}_1 \to 0 \text{ and } \left(\frac{\partial\underline{\Phi}}{\partial \bar{t}}\right)_1 \to 0 \text{ for } (\underline{x}^2 + \underline{z}^2) \to \infty \qquad \text{(BI)}$$

As in the case of the wedge (Wagner, 1931 and 1932), a self-similar solution is expected at this order. The self-similarity corresponds to a stationary solution in terms of the new variables. Using the symmetry condition and removing the $\underline{t}$-dependent terms, the dynamic free surface boundary condition becomes

$$\underline{\Phi}_1 = 0 \text{ for } z = 0\,,\ x \geq 1\,. \qquad \text{(DFSBC)}$$

This defines $\underline{\Phi}_1$ as the potential of the unbounded flow around a flat plate of width 1. Going back to the bar-variables, $\bar{\Phi}_1$ is the unbounded flow around a flat plate of width $\bar{l}_1$, with

$$\bar{l}_1 = \sqrt{2(\bar{t} + \bar{\xi}_1)}\,.$$

If this were the width of the circular lens defined by the cylinder and the $\bar{x}$ axis, this would correspond to von Karman's solution for the flow (von Karman, 1929). But this is the wetted width of the cylinder and is equivalent to Wagner's solution where the piled-up water is taken into account. It is therefore a justification of the wetting correction. Since the flow around a flat plate is easily obtained using conformal mapping (e.g. Lamb, 1932), $\underline{\Phi}_1$ is known. Yet in order to find $\bar{\Phi}_1$, $\bar{l}_1$ or

[4] The condition $Fr \gg \varepsilon$ is sufficient for the gravity effects not to appear in the first order outer problem. This would still be true with only $Fr \gg \varepsilon^{\frac{1}{2}}$.

equivalently the free surface elevation has to be found. This is done using the kinematic free surface boundary condition, and, again, looking for a stationary solution. This equation gives

$$\bar{l}_1\bar{v}_1\,(2\underline{\eta}_1 - \underline{x}\,\underline{\eta}_{1x} + 1) = \underline{\Phi}_{1x}(\underline{x},0) + 1 = \frac{\underline{x}}{\sqrt{(\underline{x}^2 - 1)}} \text{ for } \underline{x}\geq 1 \qquad \text{(KFSBC)}$$

For the solution to be stationary, $\bar{l}_1\bar{v}_1$ has to be a constant, which will be called C_{w1}. The KFSBC equation is easily integrated and gives

$$\underline{\eta}_1 = -\frac{1}{2} - \frac{\underline{x}\sqrt{\underline{x}^2 - 1}}{C_{w1}} + \frac{\underline{x}^2}{C_{w1}},$$

where the constant of integration was found demanding that $\underline{\eta}_1$ be bounded at infinity. [5] Using the equation $\underline{\eta}_1(1,0) = 0$, this finally gives

$$C_{w1} = 2\ ,\ \ \bar{l}_1 = 2\sqrt{\bar{t}}\ \ ,\ \ \bar{\xi}_1 = \bar{t}\ ,$$
$$\underline{\eta}_1 = -\frac{1}{2} - \frac{\underline{x}\sqrt{\underline{x}^2 - 1}}{2} + \frac{\underline{x}^2}{2}.$$

The constant C_{w1} is the first order wetting factor, which is therefore equal to 2 for a circle at the beginning of impact. For a wedge with small deadrise angle, this wetting factor is $\frac{\pi}{2}$ (Wagner, 1931 and 1932) and for a sphere during the first stage of impact it is $\frac{3}{2}$ (Miloh, 1981). As will be seen, this value of C_w for the cylinder will produce an important wetting correction. The analytical result for the free surface elevation (re-expressed in the dimensional variables) [6] is compared on figure 7 with a numerical calculation using a discretization of the fluid domain by finite differences (Nichols and Hirt, 1978). The agreement is

[5] Then the condition

$$\lim_{\bar{x}\to\infty} \bar{\eta}_1 = 0 \;\; or \;\; \lim_{\underline{x}\to\infty} \underline{\eta}_1 = -\frac{\bar{\xi}_1}{\bar{l}_1^{\,2}}.$$

is automatically satisfied due to (B').

[6] i.e. $\eta(x,t) = Vt\,(-1 + \frac{x^2}{2VRt} - 2(\frac{x}{\sqrt{4VRt}}(\frac{x^2}{4VRt} - 1)^{1/2}))$.

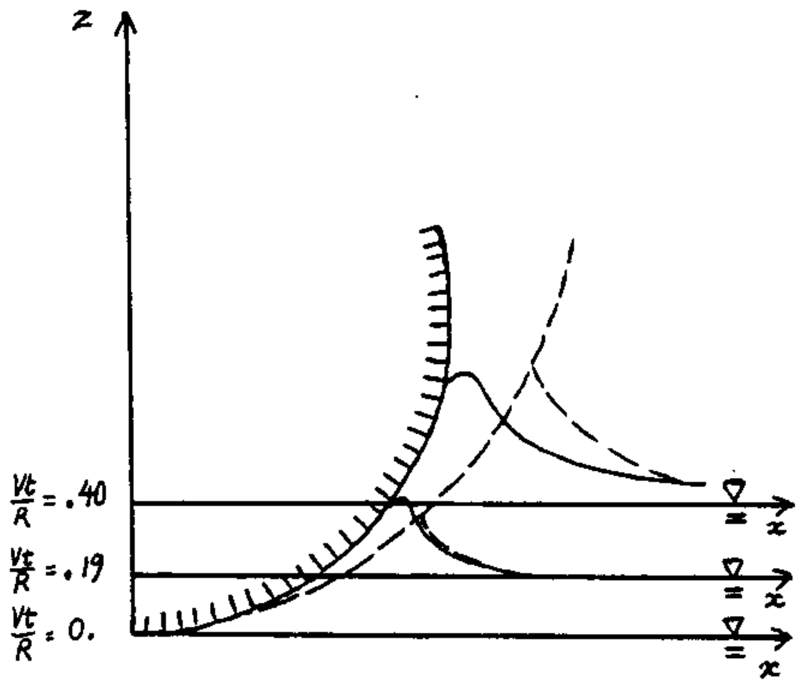

Fig. 7 Free surface elevation :
numerical (—) and first order analytical (- - -) solutions

excellent for $\frac{Vt}{R} = \varepsilon^2 \bar{t}$ small. To improve the accuracy of the calculation for larger times, one should go to the next order approximation because ε^2 is essentially the non-dimensional time. Yet a major problem appears because the first order solution, which is given by the flow around a flat plate, is singular at the edges of the plate. The fact that $\underline{\Phi}_1$ is not bounded near the water line is in contradiction with the assumptions made to linearize the problem. Therefore the outer solution is not valid near the intersection of the body and the free surface. This was first pointed out by Wagner (1931 and 1932). Near the singularity, nonlinearities have to be taken into account. This will be done here using the method of matched asymptotic expansions to define the non-linear inner problem and solve it. Because with the new variables the singularity is fixed, the inner domain will also be fixed and this will greatly facilitate the following computations.

Inner Domain Equations

The inner variables are defined by

$$x^* = \frac{1}{\varepsilon^2}(\underline{x} - 1) \ , \quad z^* = \frac{1}{\varepsilon^2}\underline{z} \ , \quad t^* = \underline{t} \ ,$$
$$\Phi^* = \frac{1}{\varepsilon^2}\underline{\Phi} \ , \quad F^* = \frac{1}{\varepsilon^2}\underline{F} = \frac{1}{\varepsilon^2}\frac{\overline{F}}{\overline{l}} \ .$$

F is used in the inner domain because it is not possible there to parametrize the free surface by η. Only the zeroth order inner problem will be studied here, where

$$\Phi^* = \Phi_0^* + o(1) \ , \quad F^* = F_0^* + o(1) \ .$$

It is given by [7]

$$\Delta\Phi_0^* = 0 \tag{FD}$$

$$\left.\begin{aligned} \nabla\Phi_0^* \cdot \nabla F_0^* - F_{0x}^*\overline{v}_1 &= 0 \\ \tfrac{1}{2}\nabla\Phi_0^* \cdot \nabla\Phi_0^* - \Phi_{0x}^*\overline{v}_1 &= 0 \end{aligned}\right\} \text{ for } F_0^*(x^*, z^*, t^*) = 0 \tag{FSBC}$$

$$\Phi_{0z}^* = 0 \text{ for } z^* = 0 \tag{BBC}$$

The other equations disappear in the inner problem. The resulting indetermination will be removed matching the inner and outer solutions. As expected, this problem, which is exactly the near field problem found by Wagner for a wedge (1931 and 1932), is non-linear. Yet the time appears only as a parameter in it.

[7] The condition $Fr \gg \varepsilon$ is sufficient for the gravity effects not to appear in the zeroth order inner problem. This would still be true with only $Fr \gg \varepsilon^2$.

Therefore it may be solved using conformal mapping. The problem is first simplified by the change of variable

$$\varphi^* = \Phi_0^* - \bar{v}_1 x^* ,$$

which leads to

$$\Delta\varphi^* = 0 \tag{FD}$$

$$\left.\begin{array}{r} \nabla\varphi^* . \nabla F_0^* = 0 \\ \nabla\varphi^* . \nabla\varphi^* = \bar{v}_1^{\,2} \end{array}\right\} \text{ for } F_0^*(x^*, z^*, t^*) = 0 \tag{FSBC}$$

$$\varphi_z^* = 0 \text{ for } z^* = 0 \tag{BBC}$$

This φ^*-problem is a classical jet problem. The normal velocity is zero on the x^*-axis, which defines the body boundary, and the velocity is tangential and of constant magnitude $\bar{v}_1$ on the free surface (figure 8). A solution with a jet is possible, which is found using the Schwarz-Christoffel method and following the work of Gurevitch (1966, p. 179). ζ^* is the complex variable, $x^* + iz^*$; ζ_s^* its value at the stagnation point; w^* the complex potential associated to φ^*; and W_0^* the one associated to Φ_0^*.[8] An intermediate variable u is defined (figure 9). The relations giving ζ^* and w^* as functions of u are

$$\frac{d\zeta^*}{dw^*} = \frac{1}{\bar{v}_1}\frac{1+iu}{1-iu}$$

$$\frac{dw^*}{du} = -\frac{2}{\pi}\bar{v}_1\delta^*\frac{u^2+1}{u}.$$

Because in the u-plane $-i$ corresponds to the stagnation point, this gives[9]

$$w^* = W_0^* + \bar{v}_1 x^* = -\frac{2}{\pi}\bar{v}_1\delta^*\left(\frac{u^2}{2} + \log u + \frac{1}{2} + i\,\frac{\pi}{2}\right)$$

[8] Similar notations will be used for the outer problem.

[9] w^* is taken equal to zero at the stagnation point. This is just a convention and a constant (function of the time which is just a parameter in the inner problem) may have to be added to this solution in order to match it with the outer solution.

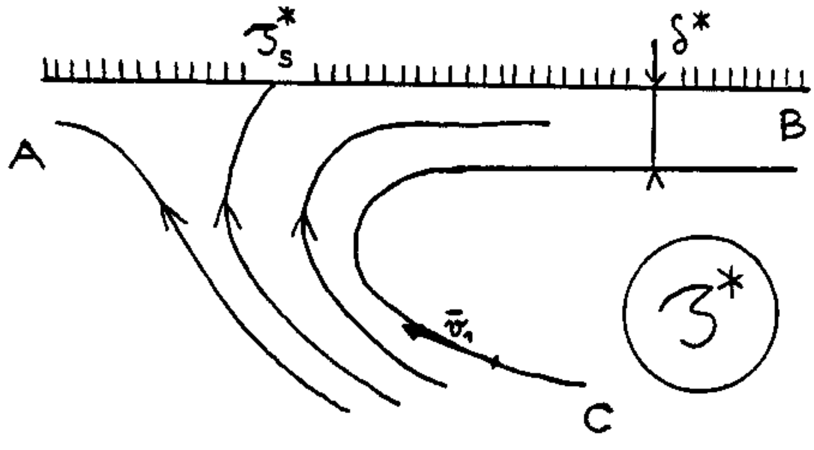

Fig. 8 Inner domain solution : ζ^*-plane

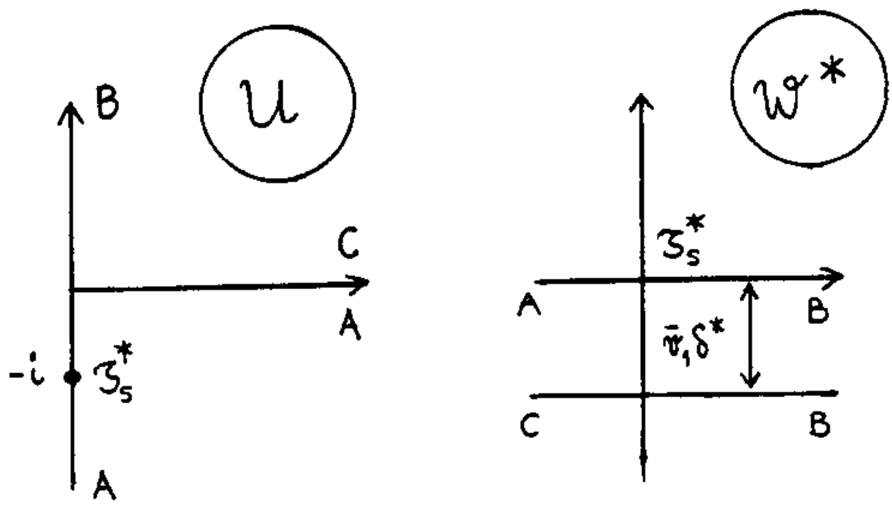

Fig. 9 Inner domain solution : u and w^*-planes

$$\zeta^* - \zeta_s^* = \frac{2}{\pi}\delta^* \left(\frac{u^2}{2} - 2iu - \log u + \frac{5}{2} - i\,\frac{\pi}{2}\right).$$

This totally defines the velocity field in the inner domain, provided that δ^* and ζ_s^* are known. For the zeroth order inner potential, W_0^*, the velocity on the free surface, is obtained as the sum of a tangential and a horizontal component of same magnitude $\bar{v}_1$. In particular, the velocity in the jet is just $2\bar{v}_1$. In order to determine the inner problem, the matching is now performed.

Matching of the Inner and Outer Solutions

The matching is performed demanding that the outer solution near the singularity and the inner solution at infinity have the same behavior. [10] First, the behavior of the inner solution at infinity is found. ζ^* goes to infinity when u goes to infinity. There

$$u = \left(\frac{\pi}{\delta^*}\right)^{1/2} \sqrt{\zeta^*} + 2i + o(1)\ , \quad \text{thus}$$
$$w^* = -\bar{v}_1\zeta^* - 4i\left(\frac{\delta^*}{\pi}\right)^{1/2} \bar{v}_1\sqrt{\zeta^*} + o(\sqrt{\zeta^*})\ .$$

This gives, expressed in the outer variables,

$$\varepsilon^2\, W_0^* = -4i\varepsilon\left(\frac{\delta^*}{\pi}\right)^{1/2} \bar{v}_1(\underline{\zeta} - 1)^{1/2} + o(\varepsilon)\ .$$

But the outer solution is given by

$$\underline{W}_1 = -i(\underline{\zeta}^2 - 1)^{1/2} + i\underline{\zeta} \approx -i\sqrt{2}(\underline{\zeta} - 1)^{1/2} + i \quad \text{for } \underline{\zeta} \to 1\ .$$

Therefore the matching of the inner and outer solutions is possible, provided that

$$\delta^* = \frac{\pi}{8\,\bar{v}_1^2} = \frac{\pi}{8}t^*\ .$$

[10] As stated by van Dyke (1964), the matching principle is : the m-term inner expansion of the (n-term outer expansion) = the n-term outer expansion of the (m-term inner expansion).

Yet the matching of the potentials only determines δ^*. Therefore the matching of the free surfaces will be performed in the hope of determining ζ_s^*. In the inner domain the free surface is given by u real and positive or

$$z^* = -\frac{4\delta^*}{\pi}u - \delta^*$$
$$x^* - x_s^* = \frac{\delta^*}{\pi}u^2 - \frac{2\delta^*}{\pi}logu + \frac{5\delta^*}{\pi}.$$

For large u, this leads to

$$z^* = \bar{l}_1\eta^*(x^*) = -4(\frac{\delta^*}{\pi})^{½}\sqrt{x^*} + o(\sqrt{x^*})$$

or, expressed in the outer variables ,

$$\varepsilon^2\eta^* = -4\ \frac{\varepsilon}{\bar{l}_1}(\frac{\delta^*}{\pi})^{½}\sqrt{\underline{x}-1} + o(\varepsilon)\ .$$

But the outer solution for the free surface elevation is

$$\underline{\eta}_1 \approx -\frac{1}{\sqrt{2}}\sqrt{\underline{x}-1}\ \text{ for }\ \underline{x} \to 1\ .$$

Therefore, the matching of the free surfaces is verified for the value of δ^* found precedently. This is true whatever the value of ζ_s^*, and the inner problem is not fully determined by the matching with the first order outer solution. Apparently, Wagner (1932) assumed for the wedge $\zeta_s^*=0$. He got a similar formula for the thickness of the jet. Re-expressed in the dimensional variables, this thickness is

$$\delta = \frac{\pi R}{4}(\frac{Vt}{R})^{\frac{3}{2}}.$$

In the numerical model given by Nichols and Hirt (1978), no jet appears (see figure 7). This is due to the procedure used to get a smooth transition between the body and the free surface boundary conditions. This essentially consists in applying a pressure on the free surface before the actual contact with the body.

and clearly does not allow the creation of a jet. Greenhow and Hill (1983) made photographic studies of the water impact of a cylinder, and figure 10 shows the free surface elevation they got. A jet appears at each water line, in qualitative agreement with the preceding results. Yet it is difficult to judge of the quantitative agreement with the formula given above for the thickness of the jet.

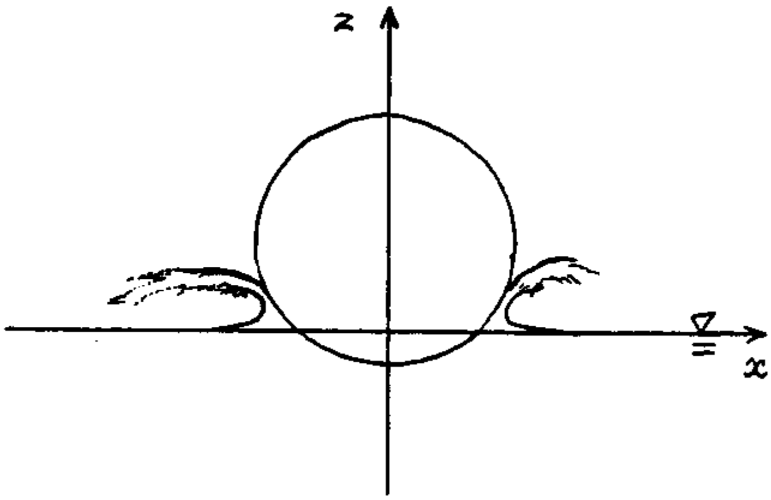

Fig. 10 Water impact of a cylinder : experimental sketch

Nevertheless, because it was possible to remove the singularity appearing in the outer approximation, and despite the indetermination of the inner solution, the first order impact force may now be computed.

First Order Impact Force

In order to find the impact force on the cylinder, the pressure on the body is first computed. This approach is used rather than a global approach (such as the one introducing the added mass for the force on a body moving in an unbounded fluid) because of the singularity appearing in the outer domain. [11] The pressure is obtained using Bernoulli's theorem. The non-dimensional pressure is defined by

$$\bar{p} = \underline{p} = \frac{p}{\rho}\frac{T}{VR}.$$

In the outer domain and at the first order, this leads to the following expression for the pressure on the body

$$\bar{p}(\bar{x},\bar{B},\bar{t}) = -\frac{\varepsilon^3}{Fr^2}\bar{B} - \bar{\Phi}_t - \tfrac{1}{2}\nabla\bar{\Phi}\,.\,\nabla\bar{\Phi} \approx -\varepsilon\bar{\Phi}_{1t} = \frac{2\varepsilon}{\sqrt{\bar{l}_1^2 - \bar{x}^2}}\ , \quad \text{or}$$

$$\underline{p}(\underline{x},\underline{B},\underline{t}) \approx -\varepsilon(\bar{v}_1\underline{\Phi}_1 - \bar{v}_1\underline{x}\underline{\Phi}_{1x}) = \frac{\varepsilon\bar{v}_1}{\sqrt{1-\underline{x}^2}}.$$

In the inner domain, the pressure on the body at the zeroth order is given by

$$p^*(x^*) = \frac{p}{\rho}\frac{T}{VR} \approx \Phi_{0x}^*\bar{v}_1 - \tfrac{1}{2}\nabla\Phi_0^*\,.\,\nabla\Phi_0^* = \tfrac{1}{2}(\bar{v}_1^2 - \varphi_x^{*\,2})\,.$$

Since the body boundary is given in the u-plane by u pure imaginary, this leads to

$$p^*(x^*) = \frac{\bar{v}_1^2}{2}\left(1 - \left(\frac{y-1}{y+1}\right)^2\right)\ , \quad \text{with}$$
$$x^* = b(y) = x_s^* + \frac{2\delta^*}{\pi}\left(-\frac{y^2}{2} - 2y - log\,y + \frac{5}{2}\right).$$

The maximum pressure is $\frac{\bar{v}_1^2}{2}$, reached at the stagnation point. The integral of the pressure in the inner domain over the interval $[x_1^*, x_2^*]$ is

$$F_{x_1x_2}^* = \frac{\bar{v}_1^2}{2}\int_{x_1^*}^{x_2^*} p^*(x^*)\,dx^* = \frac{1}{2}(b^{-1}(x_1^*) - b^{-1}(x_2^*))\,.$$

This shows that the contribution of the inner pressure to the impact force is of

[11] For the unbounded flow around a flat plate, a singularity does appear but has no effect on the total force exerced on the plate. This is due to the fact that the quadratic terms in the pressure have the same (negative) sign on both sides of the plate. Here the pressure is only integrated on one side and therefore the singularity remains even for the total force.

order ε^2, the width of the inner domain. Therefore the first order impact force is only given by the integration of the first order outer pressure. This gives

$$\frac{F_1}{\rho V^2 R} = \frac{\bar{F}_1}{\varepsilon} = \frac{\bar{l}_1 \underline{F}_1}{\varepsilon} = 2\pi \; .$$

This result was not obvious *a priori* because the contribution of the inner domain solution might have been of the first order. It corresponds to Wagner's modification of the original von Karman's formula, and may be rewritten as

$$F_1 = \frac{d}{dt}(m_{\psi} V) \; ,$$

where m_{ψ} is half the added mass of the flat plate of width the wetted width of the cylinder, l_1. The wetting correction multiplies the width of the flat plate by $\sqrt{2}$, but the impact force by 2. As was already seen, this value of the force overestimates the experimental results which lie between Wagner and von Karman's approximations (figure 4). The fact that the agreement between the first order impact force and the experimental data decreases with time is not surprising because the perturbation parameter, ε, is proportional to the square root of the time scale. Since each successive approximation is, by hypothesis, independent of ε, it is logical that the first one be independent of time. To improve the accuracy, and to take into account the time dependence, the next order must be considered. Allowing an expansion with $\varepsilon \log(\varepsilon)$ terms (cf. van Dyke, 1964), one may *a priori* predict that the expression giving the impact force will be [12]

$$\frac{F}{\rho V^2 R} = 2\pi + A\left(\frac{Vt}{R}\right)^{1/2} + B\left(\frac{Vt}{R}\right)^{1/2} \log\left(\frac{Vt}{R}\right) .$$

In order to determine the constants A and B, it is necessary to take into

[12] This type of expansion (except the log term whose physical interpretation is not obvious) was predicted by dimensional analysis.

account :

1. the zeroth order inner domain pressure which is of order 1 on a domain of width ε^2,
2. the first order outer domain quadratic pressure which is of order ε^2 on a domain of order 1,
3. the next order outer domain linear pressure which is of order ε^2 on a domain of order 1.

Therefore an 'exact' expression for the second order impact force demands the determination of ζ_s^* and the solution of the next order outer problem.

Second Order Outer Domain Problem

The behavior of the zeroth order inner solution at infinity is first investigated. Expressed in terms of the outer variables, this gives [13]

$$\varepsilon W_0^* = -4i\bar{v}_1\left(\frac{\delta^*}{\pi}\right)^{1/2}\sqrt{\underline{\zeta}-1} - 2\varepsilon\bar{v}_1\,\frac{\delta^*}{\pi}\log(\underline{\zeta}-1) + o(\varepsilon)$$
$$\varepsilon\bar{l}\eta^* = -4\left(\frac{\delta^*}{\pi}\right)^{1/2}\sqrt{\underline{x}-1} - \varepsilon\delta^* + o(\varepsilon)\ .$$

Therefore the proper choice for the expansions of the unknowns seems to be [14]

$$\underline{\Phi} = \varepsilon\underline{\Phi}_1 + \varepsilon^2\underline{\Phi}_2 + o(\varepsilon^2)\ ,\quad \underline{\eta} = \varepsilon\underline{\eta}_1 + \varepsilon^2\underline{\eta}_2 + o(\varepsilon^2)\ ,$$
$$\bar{l} = \bar{l}_1 + \varepsilon\bar{l}_2 + o(\varepsilon)\ ,\quad \bar{v} = \bar{v}_1 + \varepsilon\bar{v}_2 + o(\varepsilon)\ ,\quad \bar{\xi} = \varepsilon\bar{\xi}_1 + \varepsilon^2\bar{\xi}_2 + o(\varepsilon^2)\ .$$

The equation $\underline{B}(1,0) = 0$ gives at this order $\bar{\xi}_2 = \bar{l}_1\bar{l}_2$. For simplicity, the notation $\overline{lv}_2$ will be employed for $\bar{l}_1\bar{v}_2 + \bar{l}_2\bar{v}_1$. The second order outer problem is found

[13] Constant terms have been omitted in the expression of the potential since they are not involved in the matching procedure.

[14] The preceding equations indicate that no $\log\varepsilon$ terms are introduced by the zeroth order inner problem. Yet the fact that x_s^* does not appear in them shows that the matching at the next order cannot determine it. Moreover, there is a constant term appearing in the second order free surface which cannot exist in the outer problem. Therefore, the matching should also involve the first order inner problem and will not be performed here.

using a Taylor expansion of the boundary conditions and taking advantage of the fact that the first order potential is equal to zero on the free surface. This leads to

$$\Delta\underline{\Phi}_2 = 0 \tag{FD}$$

$$\bar{l}_2(0) = 0 \tag{IC}$$

$$\left\{ \begin{array}{l} 2\bar{l}\bar{v}_2\underline{\eta}_1 + 4\underline{\eta}_2 + \bar{l}\bar{v}_2 + \bar{l}_1^2\underline{\eta}_{2t} - 2\underline{x}\,\underline{\eta}_{2x} - \bar{l}\bar{v}_2\underline{x}\,\underline{\eta}_{1x} = \underline{\Phi}_{2z} \\ \bar{v}_1\underline{\Phi}_2 + \bar{l}_1\underline{\Phi}_{2t} - \bar{v}_1\underline{x}\,\underline{\Phi}_{2x} + \tfrac{1}{2}\underline{\Phi}_{1z}(\underline{\Phi}_{1z} - 2) = 0 \end{array} \right. \tag{FSBC}$$

for $\underline{z} = 0$, $\underline{x} \geq 1$

$$\bar{l}_1\underline{x}\,\underline{\Phi}_{1x} - \underline{\Phi}_{2z} - \underline{\Phi}_{1zz}\,\frac{\bar{l}_1}{2}(\underline{x}^2 - 1) = 0 \;\text{ for }\; \underline{z} = 0,\; \underline{x} \leq 1 \tag{BBC}$$

$$\underline{\Phi}_{2x} = 0 \;\text{ for }\; \underline{x} = 0,\; \underline{z} \leq 0 \tag{SC}$$

$$\nabla\underline{\Phi}_2 \to 0 \;\text{ and }\; \left(\frac{\partial\underline{\Phi}}{\partial\bar{t}}\right)_2 \to 0 \;\text{ for }\; (\underline{x}^2 + \underline{z}^2) \to \infty \tag{BI}$$

Looking for a solution with $\underline{\Phi}_2(\underline{x},\underline{z},\underline{t}) = \sqrt{\underline{t}}\,\underline{\varphi}_2(\underline{x},\underline{z})$, this problem reduces for $\underline{\varphi}_2$ to a Neumann-Dirichlet problem, with $\underline{\varphi}_2$ known for $\underline{z} = 0$, $\underline{x} > 1$, $\underline{\varphi}_{2z}$ known for $\underline{z} = 0$, $\underline{x} < 1$. Yet it was not found possible to solve it analytically.[15] Therefore, and in order to get at least an idea of its importance toward the determination of the constants A and B, a crude simplification will be made to allow its resolution. The dynamic free surface boundary condition will be taken as

$$\underline{\Phi}_2 = 0 \;\text{ for }\; \underline{z} = 0,\; \underline{x} \geq 1 \tag{DFSBC}$$

This means that for $\Psi = \underline{\Phi}_1 + \varepsilon\underline{\Phi}_2$ the dynamic free surface boundary condition is taken as $\Psi = 0$. Since the unbounded flow around a circular lens is well known (Taylor, 1930), Ψ will be found using an expansion of this solution. Then the kinematic free surface boundary condition will be used to find the wetting

[15] Moreover, the fact that a singularity in log appears for $\underline{\Phi}_2$ near the water line seems to imply that $\underline{\eta}_2$ be unbounded near the body. This is a major problem since by hypothesis $\underline{\eta}_2(1,0) = 0$.

correction or, equivalently, $\bar{l}_2$. Yet it must be pointed out that this approach is not consistent, because it only takes into account the corrections due to the geometry of the body. Nevertheless, it provides an easy way to approximate $\underline{\Phi}_2$ and $\bar{l}_2$.

The circular lens of width 2 being defined by the exterior angle of intersection, $\frac{2\pi}{n}$, Taylor's solution is given in term of the new variable τ (figure 11) by

$$\underline{\zeta} = \coth \tau$$
$$\underline{W}_T = -\frac{ni}{\sinh n\tau} + i\underline{\zeta}$$
$$\frac{d\underline{W}_T}{d\underline{\zeta}} = -in^2 \cosh n\tau \left(\frac{\sinh \tau}{\sinh n\tau}\right)^2 + i.$$

The ε-dependence of n is given by

$$\underline{B}(0,\underline{t}) = -\varepsilon\ \frac{\bar{l}}{2} = -\cot\ \frac{\pi}{2n},$$

and therefore, at the first order,

$$n = 1 + \varepsilon\ \frac{\bar{l}_1}{\pi} + o(\varepsilon)\ .$$

Using this expansion of n, Taylor's solution may be expanded in term of ε. The approximation made is then to write

$$\underline{W}_2 \approx \frac{\underline{W}_T - \underline{W}_1}{\varepsilon}.$$

With this approximation, $\bar{l}_2$ is to be found using the kinematic free surface boundary condition. Therefore, the vertical velocity on the free surface is needed. On the $\underline{x}$-axis, and for $\underline{x} \geq 1$, τ is real and positive. Therefore

$$\underline{x} = \coth \tau \ \text{ and } \ \tau = \frac{1}{2} \log \frac{\underline{x} + 1}{\underline{x} - 1}.$$

It is then possible to find an explicit expression giving the vertical velocity which is

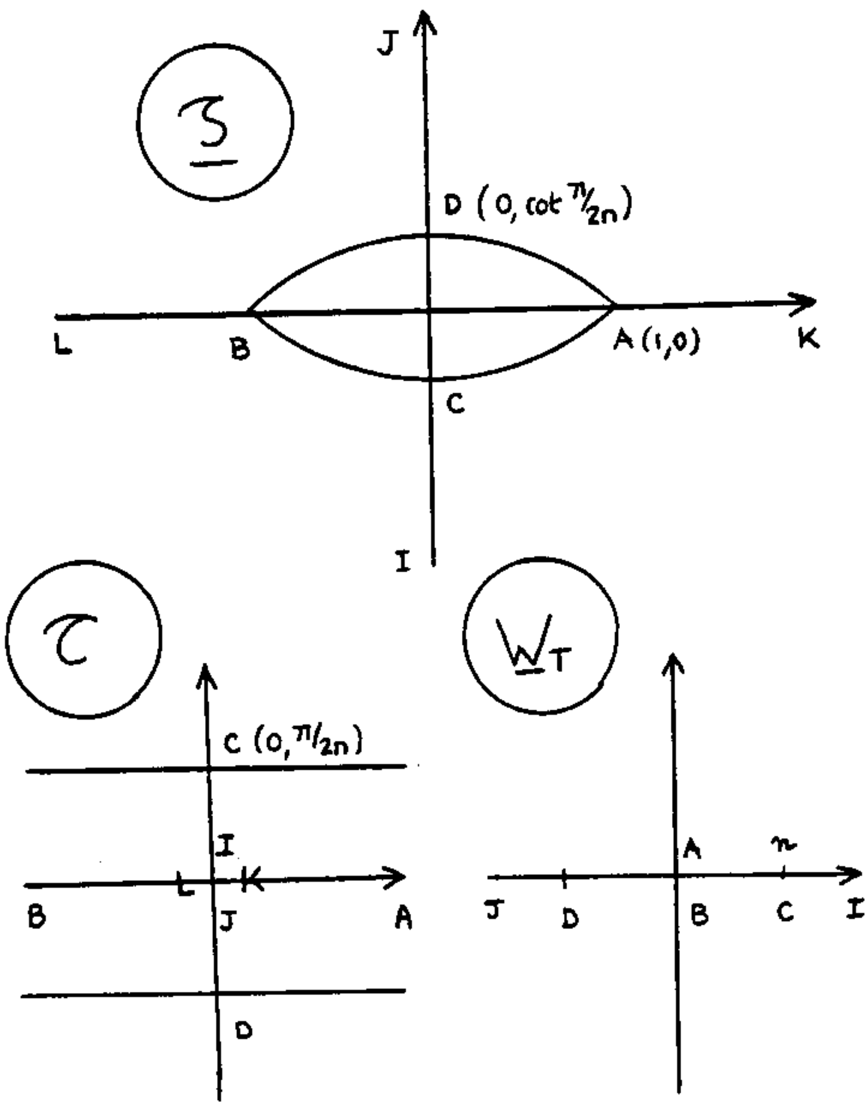

Fig. 11 Flow around a circular lens

$$v_T = -\operatorname{Im}(\frac{dW_T}{d\zeta}) = \frac{n^2}{2}\,\frac{(\underline{x}+1)^n + (\underline{x}-1)^n}{(\underline{x}^2-1)^{\frac{n}{2}}}\,\left(\frac{2(\underline{x}^2-1)^{\frac{n-1}{2}}}{(\underline{x}+1)^n - (\underline{x}-1)^n}\right)^2 - 1\,.$$

The expansion of this expression in ε is straightforward and finally gives

$$\underline{\Phi}_{2x} \approx \frac{\bar{l}_1}{\pi}\,\frac{\underline{x}}{\sqrt{\underline{x}^2-1}}\left(2 + \frac{1}{2}\log\left(\frac{\underline{x}+1}{\underline{x}-1}\right)\frac{1-2\underline{x}^2}{\underline{x}}\right).$$

Then a solution of the kinematic free surface boundary condition may be found with

$$\bar{w}_2 = \lambda\sqrt{\bar{t}}\ ,\quad \bar{l}_2 = \frac{\lambda}{3}\bar{t}\ ,\quad \underline{\eta}_2 = \chi(\underline{x})\sqrt{\underline{t}}\ .$$

This leads to

$$6\chi - 2\underline{x}\chi_x = \left(\frac{4}{\pi} - \frac{\lambda}{2}\right)\frac{\underline{x}}{\sqrt{\underline{x}^2-1}} - \frac{1}{\pi}\,\frac{2\underline{x}^2-1}{\sqrt{\underline{x}^2-1}}\log\left(\frac{\underline{x}+1}{\underline{x}-1}\right).$$

This equation can be integrated, the constant of integration being found by demanding that χ be bounded at infinity. [16] This allows the determination of λ which is obtained by writing $\chi(1) = 0$. This gives

$$\lambda = -\frac{4}{3\pi}.$$

This entirely determines the approximate second order solution. In particular,

$$\bar{l}_2 = -\frac{4}{9\pi}\bar{t}\ ,\quad \bar{\xi}_2 = -\frac{8}{9\pi}\bar{t}^{\frac{3}{2}}\ ,$$

$$\underline{\eta}_2 = -\frac{4}{3\pi}\underline{x}\sqrt{\underline{x}^2-1}\,\sqrt{\underline{t}} \;-\; \frac{1}{6\pi}(4\underline{x}^2-1)\sqrt{\underline{x}^2-1}\,\sqrt{\underline{t}}\,\log\left(\frac{\underline{x}-1}{\underline{x}+1}\right).$$

The wetting factor is therefore

$$C_w \approx 2 - \frac{8}{9\pi}\sqrt{\frac{Vt}{R}}.$$

Within the approximation made, the first correction to the wetting factor is relatively small, but its time derivative is infinite at the time of impact. The second

[16] Then, again, the condition $\lim_{\underline{x}\to\infty} \bar{\eta}_2 = 0$ is automatically satisfied.

order solution being approximated, it is now possible to find the second order impact force. This should give an idea of the respective importance of the phenomena involved.

Second Order Impact Force

This force will be obtained first integrating the pressure distribution on the cylinder. As was already mentioned, different effects have to be considered whose importance will be evaluated. This will be done here showing what corrections, beside the wetting correction, have to be made to von Karman's formula.

The pressure distribution in the inner domain was found before. Yet the pressure in the outer domain has to be calculated using the first order approximation and Taylor's solution. It is given on the body by [17]

$$\begin{aligned} \underline{p} = & -\varepsilon\,(\bar{v}_1\underline{\Phi}_1 - \bar{v}_1\underline{x}\underline{\Phi}_{1x}) \\ & -\varepsilon^2(\bar{v}_1\underline{\Phi}_2 + \bar{l}_1\underline{\Phi}_{2t} - \bar{v}_1\underline{x}\underline{\Phi}_{2x} + \bar{v}_2\underline{\Phi}_1 - \bar{v}_2\underline{x}\underline{\Phi}_{1x} + \tfrac{1}{2}\underline{\Phi}_{1x}^2 + \underline{x}^2 + \frac{1}{2}). \end{aligned}$$

To obtain this pressure, $\underline{\Phi}_2$ has to be found on the body boundary. This boundary is given in Taylor's solution (figure 11) by

$$\tau = \mu + i\,\frac{\pi}{2n}\,, \quad \mu \geq 0\,.$$

Expanding the relations giving $\underline{\zeta}$, $\underline{W}_T$, and $\dfrac{d\underline{W}_T}{d\underline{\zeta}}$ up to the second order, one gets

$$\begin{aligned} \underline{\zeta} &= \underline{x} + i\underline{B}(\underline{x},\underline{t}) \approx \tanh\mu - \varepsilon i\,\frac{\bar{l}_1}{2\cosh^2\mu} \\ \underline{W}_T &\approx -\sqrt{1-\underline{x}^2}\,(1 + \varepsilon\,\frac{\bar{l}_1}{\pi}(1 - \frac{\underline{x}}{2}\log(\frac{1+\underline{x}}{1-\underline{x}})) + i\underline{\zeta} \\ \frac{d\underline{W}_T}{d\underline{\zeta}} &\approx \frac{\underline{x}}{\sqrt{1-\underline{x}^2}}(1 + \varepsilon\,\frac{\bar{l}_1}{\pi}(2 + \frac{1}{2}\,\frac{1-2\underline{x}^2}{\underline{x}}\log(\frac{1+\underline{x}}{1-\underline{x}})) - \varepsilon i\underline{x}\bar{l}_1) + i\,. \end{aligned}$$

[17] Because of the method used to approximate the second order potential, where the geometry of the body is taken into account, the pressure is computed on the exact boundary and no Taylor expansion is needed.

Therefore, on the body,

$$\underline{\Phi}_2 \approx \frac{\bar{l}_1}{\pi}\sqrt{1-\underline{x}^2}\,(-1+\frac{\underline{x}}{2}\log(\frac{1+\underline{x}}{1-\underline{x}})) + \frac{\bar{l}_1}{2}(1-\underline{x}^2)$$

$$\underline{\Phi}_{2t} \approx \frac{\bar{v}_1}{\pi}\sqrt{1-\underline{x}^2}\,(-1+\frac{\underline{x}}{2}\log(\frac{1+\underline{x}}{1-\underline{x}}))$$

$$\underline{\Phi}_{2x} \approx \frac{\bar{l}_1}{\pi}\,\frac{\underline{x}}{\sqrt{1-\underline{x}^2}}(2+\frac{1}{2}\log(\frac{1+\underline{x}}{1-\underline{x}})\,\frac{1-2\underline{x}^2}{\underline{x}}) .$$

This gives the pressure,

$$\underline{p} \approx \varepsilon\,\frac{\bar{v}_1}{\sqrt{1-\underline{x}^2}} - \varepsilon^2(-\frac{32}{9\pi}\,\frac{1}{\sqrt{1-\underline{x}^2}} + \frac{1}{2(1-\underline{x}^2)} + 1 + \frac{\underline{x}}{\pi\sqrt{1-\underline{x}^2}}\log(\frac{1+\underline{x}}{1-\underline{x}})) .$$

Expressed in the inner variables, and at the zeroth order in ε, this pressure is

$$\underline{p} \approx \frac{\bar{v}_1}{\sqrt{-2x^*}} + \frac{1}{4x^*} .$$

The maximum value of this quantity is $\frac{\bar{v}_1^2}{2}$, reached at

$$x^* = -\frac{1}{2\bar{v}_1^2} = -\frac{4}{\pi}\delta^* .$$

This maximum value is the same as the maximum inner pressure, reached at the stagnation point. Therefore, x_s^*, which was not determined by the matching, will be taken as $-\frac{4}{\pi}\delta^*$. This choice locates the stagnation point of the inner domain at the point, fixed in the inner domain, where the outer pressure is maximum. To find the second order impact force, the pressure will be integrated from 0 to $1+\varepsilon^2 x_s^*$ in the outer domain and from x_s^* to $+\infty$ in the inner domain. The contribution of the inner domain is $\varepsilon^2 F^* = \frac{\varepsilon^2}{2}$ for each water line and therefore, for the whole cylinder,

$$\underline{F}_{inner} = \varepsilon^2 .$$

The contribution of the outer domain is

$$\underline{F}_{outer} = 2 \int_0^{1+\varepsilon^2 \underline{x}_s^*} \underline{p} d\underline{x} \, .$$

This gives the total impact force which is

$$\underline{F} \approx \varepsilon \bar{v}_1 \pi - \varepsilon^2 (5 - \frac{32}{9} + \log 2 - \frac{1}{2} \log(\varepsilon^2 \bar{t})) \, .$$

Expressed in the bar-variables, this force is

$$\bar{F} \approx (2\sqrt{\bar{t}} - \varepsilon \frac{4}{9\pi} \bar{t}) \underline{F} \approx \varepsilon \, 2\pi - \varepsilon^2 \sqrt{\bar{t}} \, (\frac{10}{3} + 2 \log 2 - \log(\varepsilon^2 \bar{t})) \, .$$

Going back to the dimensional variables, this finally leads to

$$\frac{F}{\rho V^2 R} = 2\pi - \sqrt{\frac{Vt}{R}} (\frac{10}{3} + 2 \log 2 - \log(\frac{Vt}{R})) \, .$$

In order to assess the importance of the different phenomena involved, it is interesting to find the added mass of the circular lens and to see what are the modifications to von Karman's formula implied by the preceding developments. $\bar{m}_\infty$ could be obtained studying the far field potential but it is found here directly quite easily. By definition, the added mass at infinite frequency of the circular lens is

$$\bar{m}_\infty = -2\varepsilon \int_0^{\bar{l}} \bar{\Phi}(\bar{x}, \bar{B}, \bar{t}) \, d\bar{x} = -2\varepsilon \bar{l}^2 \int_0^1 \underline{\Phi}(\underline{x}, \underline{B}, \underline{t}) \, d\underline{x} \, .$$

At the second order, and using Taylor's solution, the velocity potential on the body boundary is given by

$$\underline{\Phi} \approx \underline{\Phi}_T \approx -\sqrt{1 - \underline{x}^2} \, (1 + \varepsilon \frac{\bar{l}_1}{\pi} (1 - \frac{\underline{x}}{2} \log(\frac{1 + \underline{x}}{1 - \underline{x}})) + \varepsilon \frac{\bar{l}_1}{2} (1 - \underline{x}^2) \, ,$$

and therefore

$$\bar{m}_\infty \approx -2\varepsilon \bar{l}^2 \int_0^1 \underline{\Phi}_T(\underline{x}, \underline{B}, \underline{t}) \, d\underline{x} \approx \varepsilon \, 2\pi \bar{t} - \varepsilon^2 \frac{32}{9} \bar{t}^{\frac{3}{2}} \, .$$

The direct application of von Karman's formula, taking into account the wetting correction up to the second order, would give

$$\overline{F}_{-} = \frac{d}{d\overline{t}}\overline{m}_{-} \approx \varepsilon\, 2\pi - \varepsilon^2\, \frac{16}{3}\sqrt{\overline{t}} \,.$$

Yet, this correction to Wagner's result is not sufficient. Other corrections, of the same order of magnitude, have also to be made. The first one is necessary to take into account the relative motion of the cylinder and the free surface. This comes from the fact that

$$\frac{d}{d\overline{t}}\overline{m}_{-} = \frac{d}{d\overline{t}}\Big(-2\varepsilon \int_0^{\overline{l}} \overline{\Phi}(\overline{x},\overline{B},\overline{t})\, d\overline{x}\Big) = -2\varepsilon \int_0^{\overline{l}} [\overline{\Phi}_t + \overline{\Phi}_x\, \overline{B}_t]\, (\overline{x},\overline{B},\overline{t})\, d\overline{x} \,.$$

Therefore

$$-2\varepsilon \int_0^{\overline{l}} \overline{\Phi}_t(\overline{x},\overline{B},\overline{t})\, d\overline{x} \approx \overline{F}_{-} + 2\varepsilon^2 \overline{l}_1 = \overline{F}_{-} + \overline{F}_B \,,$$

with

$$\overline{F}_B = 4\, \varepsilon^2 \sqrt{\overline{t}} \,.$$

The importance of this correction is tremendous. In particular, if the second order geometry of the body decreases $\overline{F}_{-}$, it *increases* $\overline{F}_{-} + \overline{F}_B$ which represents the integration of the linear pressure (up to the second order, and with the approximations made) over the whole cylinder. Since this integration should only be performed over the outer domain, and the inner pressure integrated over the inner domain, a new correction has to be made,

$$\overline{F}_I = \overline{l}_1 \underline{E}_{inner} - 2\overline{l}_1 \varepsilon \int_{1+\varepsilon^2 \underline{x}_s^*}^{1} \frac{\overline{v}_1}{\sqrt{1-\underline{x}^2}}\, d\underline{x} = -2\, \varepsilon^2 \sqrt{\overline{t}} \,.$$

This correction arises from the fact that the integral of the first order linear pressure over the inner domain (of width $\approx \varepsilon^2$) is of order ε^2, because this pressure is not bounded. The same approach is needed in order to consider the first order quadratic pressure which would lead to an infinite suction force if integrated over the whole cylinder. Both these corrections are of the same order as the geometric correction introduced above and cannot be neglected. This leads to the correction

$$\overline{F}_Q = -\varepsilon^2 \overline{l}_1 \int_0^{1+\varepsilon^2 x_s^*} \frac{1}{1-\underline{x}^2} d\underline{x} = \varepsilon^2 \sqrt{\overline{t}} \, (\log(\varepsilon^2 \overline{t}) - 2\log 2) \, .$$

Therefore the total impact force is

$$\overline{F} = \overline{F}_\infty + \overline{F}_B + \overline{F}_I + \overline{F}_Q = \varepsilon \, 2\pi - \varepsilon^2 \sqrt{\overline{t}} \, (\frac{10}{3} + 2\log 2 - \log(\varepsilon^2 \overline{t})) \, ,$$

which is the result obtained before by direct integration of the pressure.

This formula for the impact force is compared on figure 12 with the experimental results already shown. The agreement is excellent for $\sqrt{\frac{Vt}{R}}$ small, except for the finite rise time not predicted by this theory. The analytical result overestimates the impact force relatively rapidly. This is due to the fact that the perturbation parameter varies like the square root of the time scale, and also to the fact that this result is only an approximation of the second order problem, and that a new correction, at this order, should be made to take into account the geometry of the free surface. In any cases, it seems quite illegitimate to neglect non-linear free surface effects at higher order, i.e. for a penetration depth which is not much smaller than the cylinder radius. Nevertheless the rapid decrease of the non dimensional impact force from its initial value of 2π can be predicted within the potential flow theory neglecting gravity and the non-rigidity of the cylinder, and is clearly related to non-linear effects.

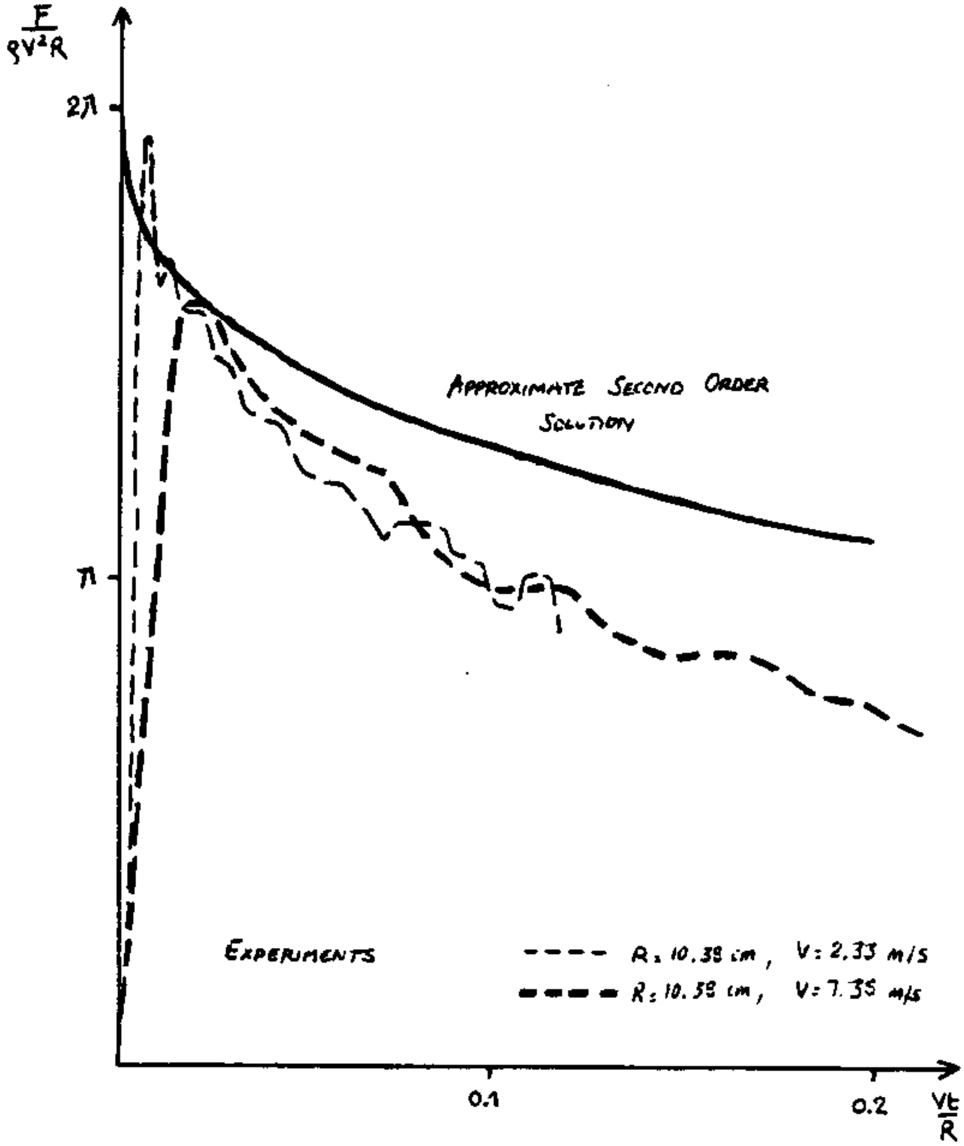

Fig. 12 Second order and experimental slamming coefficients

Conclusion

A rational approach was derived to deal with the water impact of a circular cylinder. This method justifies the wetting correction introduced by Wagner in 1931, and gives its domain of validity in this special case. The rapid decrease of the slamming coefficient from its initial value of 2π predicted by the present theory together with the finite time rise observed experimentally serve to explain the scatter in the experimental observations concerning the maximum impact force on a circular cylinder. This is true without taking into account the flexibility of the cylinder. Another important conclusion is that one must be very careful in using other corrections than the wetting correction. This is due to the fact that, for the circular cylinder, different effects lead to corrections of the same order.

This method seems to be applicable to other cases, such as different geometries, non-vertical and/or non-constant velocities, and eventually moving free surfaces. These cases which have been much less studied than the problem adressed here could lead to interesting results concerning the different parameters involved in the analysis of the impact loads exerced by the waves on a body piercing the free surface.

The method of matched asymptotic expansions, already used to study the flow past blunt ships and hydroplaning seems therefore to be well appropriate to deal with the singularity which appears at the water line for large motions of the partially emerged pontoons of a damaged semisubmersible.

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HYDRODYNAMIC IMPACT ANALYSIS OF A CYLINDER

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ABSTRACT

The problem of the vertical entry of a rigid horizontal cylinder into an incompressible inviscid fluid initially at rest is considered. The method of matched asymptotic expansions is used to solve the resulting boundary-value problem. A new formula for the impact force is obtained, which differs from the classical von Karman's formula by a corrective term. The results obtained are compared with those of experimental observations and numerical calculations. The method may be extended to different geometries and non-vertical velocities to provide an estimate of the impact forces on the partially emerged pontoons of damaged semi-submersibles.

NOMENCLATURE

B	body boundary (Eq. (3))
F	impact force
F_B, F_I, F_Q, F_V	contributions to the impact force
Fr	Froude number (Eq. (5))
i	square root of -1
l	half wetted width
L	length scale (Eq. (5))
m	added mass
o	'much smaller than'
p	pressure
R	cylinder radius
S	parametrization of the free surface (Eq. (2))
t	time
T	time scale
u	intermediate variable (Eqs. (18) and (19))
v	time derivative of l
V	impacting velocity
w	complex potential associated to φ
W	complex potential associated to Φ
x	horizontal coordinate
z	vertical coordinate
δ	jet thickness
Δ	Laplacian (∇^2)
ε	perturbation parameter (Eq. (5))
ζ	complex variable, $x + iz$
η	free surface elevation (Eq. (8))
ξ	height of the piled-up water
ρ	density of water
ϕ	complex potential
∇	gradient

Subscripts

i	i^{th} order approximation ($i = 0, 1, 2$)
s	stagnation point
t	partial derivative with respect to t
T	Taylor's solution [16]
x	partial derivative with respect to x
z	partial derivative with respect to z

Other Symbols

The following definitions apply to any dimensional variable a :

$\bar{a}$	non-dimensional variable (Eq. (6))
$\underline{a}$	outer domain variable (Eq. (9))
a^*	inner domain variable (Eq. (14))

These symbols are not used on subscripts indicating differentiation. Except when otherwise specified, derivatives are taken with respect to the current variables.

INTRODUCTION

The awareness of the current lack of knowledge concerning the various mechanisms affecting the stability of damaged semi-submersibles has motivated worldwide research, both theoretical and experimental. The Norwegian Maritime Directorate initiated the Mobile Platform Stability (or MOPS) project in 1981. A project synthesis and some recommendations are given by Dahle [1]. Another very comprehensive research project was launched in Japan in 1982 to investigate stability of twin pontoons semisubmersibles, with the first results reported in a Shipbuilding Research Association of Japan interim report [2]. If the experimental results clearly demonstrate the importance of non-linearities in the equations of motion when the pontoons pierce the free surface (Huang, Naess, and Hoff, [3]), the mathematical modelization of the problem of the dynamic behavior of a semi-submersible in waves represents a formidable challenge. Even the nature and the importance of the different phenomena involved are not known.

As a part of the MOPS project, Huang and Naess [4] and Naess and Hoff [5] developed a numerical simulation technique for heavily listed semi-submersibles. This numerical model was built following the general method given by Paulling [6]. The main difficulty laid in the determination of the forces acting on the partially emerged pontoons. Huang, Naess, and Hoff used a strip theory, and split the forces acting on a section into various components, representing a mix between rigorous expressions derived from hydrodynamic theory (neglecting the frequency dependence of the hydrodynamic coefficients) and semi-empirical formulas of the Morison type. The discrepancies reported between measured and calculated responses could very well be explained by these uncertainties on the loading terms which should be determined with higher accuracy.

The need for a rational approach to the problem of the determination of the wave and current forces acting on the inclined pontoons is apparent. As Lin, Newman, and Yue [7] point out, when viscous forces are neglected, the two main difficulties appearing in a problem of this kind are the memory effect of the free surface (or the presence of outgoing waves) and the mathematical treatment of the flow near the intersection of the body and the free surface, which must be adressed in order to deal with the non-linearities introduced by the fact that the pontoons are piercing the free surface. Such non-linearities apparently play an essential role.

...ion of the simpler situation, yet not fully understood, of the hydrodynamic impact of a circular cylinder. This situation ideally models the impact forces acting on the partially emerged pontoons, the importance of which may be considerable. It should also help understand the phenomena involved in the determination of the flow around the pontoons.

THE WATER IMPACT PROBLEM

The problem of water impact was first addressed in order to predict the impact force on a landing seaplane. Von Karman [8] introduced the main physical concepts of the phenomenon. He studied the vertical entry of a wedge, but his results may also apply to the case of a circular cylinder. He equated the impact force on the penetrating object to half the force acting on a flat plate moving in an unbounded fluid at the impacting velocity. This force is given by

$$F = \frac{d}{dt}(mV) \tag{1}$$

where m is half the added mass of the plate of width given by the intersection of the body and the undisturbed free surface.

Wagner [9,10] refined von Karman's analysis and offered the first mathematical treatment of the problem. Linearizing it in the case of a wedge with small deadrise angle, he approximated the body by a flat plate and obtained a zero potential free surface boundary condition. He took into account the piled-up water to determine the wetted width, and found von Karman's formula (Equation (1)) for the impact force. Yet he took for m half the added mass of the plate of width the wetted width of the wedge. This modification to von Karman's formula will be referred later as the *wetting correction*.

Fabula [11] used an ellipse-fitting method to calculate the wetting correction but considered the added mass of the exact shape in Equation (1). This method was developed to avoid the singularity which appears in the linearized problem when flat plate-fitting is used. It was then suggested that von Karman's formula had to be used with the added mass of the submerged body, the wetting correction, but also a *drag correction* taking into account the quadratic terms in the pressure and other effects.

Recently both experimental and theoretical studies were published concerning the wave slam on horizontal members of cylindrical cross-sections. In particular, Faltinsen *et al.* [12] used von Karman's formula with the value of the added mass of a circular lens, but did not consider the wetting and drag corrections.

Finally both numerical and experimental results concerning the water impact of a circular cylinder were given in an EPRI report [13] concerned with boiling water reactors. The four numerical models presented in this report are :

1. an explicit Lagrangian method (Gross, [13]).
2. a boundary integral method (Geers, [13]).
3. a finite element method (Marcal, [13]).
4. an incompressible Eulerian fluid method (Nichols and Hirt, [13]).

These numerical results, together with experimental results and some of the analytical formulas described above, are shown on Figure 1. The non-dimensional impact force (often referred to as the *slamming coefficient*) is plotted as a function of the non-dimensional time. The main results which may be drawn from this figure are :

(a) von Karman's formula underestimates the impact force, while Wagner's formula overestimates it.

(b) the use of the added mass of the circular lens in von Karman's formula doesn't improve the accuracy. On the other hand, Fabula's method is very accurate.

(c) at least one of the numerical simulatio is in excellent agreement with the experimental data. Since these simulations assume that the body is rigid this hypothesis seems reasonable, at least in this case.

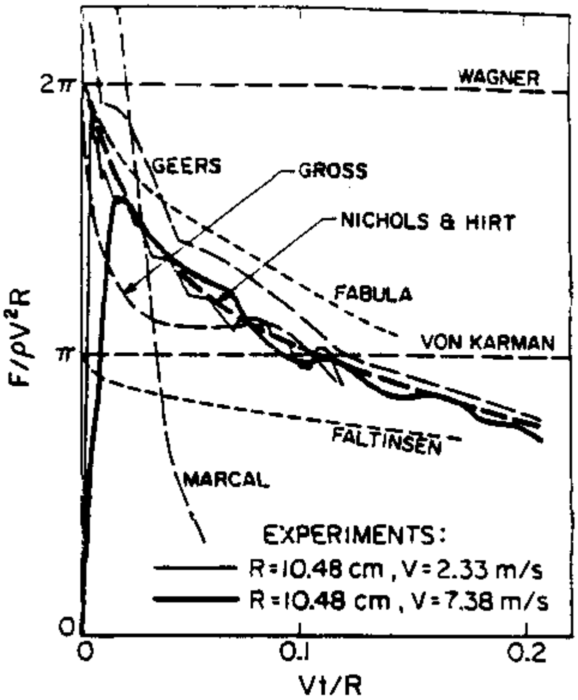

Fig.1 Experimental, numerical, and theoretical slamming coefficients

(d) in all the analytical results given, the impact force rises instantaneously from zero to a finite value. This is not in agreement with the experimental data. As pointed out by Sarpkaya [14] several factors, specifically the compressibility of the air between the cylinder and water surface, entrapped gases in the water, and surface irregularities, might account for some finite rise time. Yet this interesting problem will not be addressed here.

(e) the compressibility of the water should also be considered at the beginning of impact, when the apparent growth velocity of the circular lens is greater than the sound velocity in the liquid. However, for the results shown, this is negligible compared to the phenomena described in (d) above.

In conclusion, it is believed that an accurate prediction of the impact force can be made using the general assumptions stated by Wagner (incompressible and inviscid fluid, irrotational flow, rigid cylinder, no surface tension, and eventually no gravity). The value predicted by Wagner's theory seems to be a good limit for the impact force when the time t goes to zero (with the restrictions mentioned in (d) and (e) above). This leads to consider a perturbation method where the small parameter is related to the time or the penetration depth.

WATER IMPACT OF A CIRCULAR CYLINDER

Equations of the Problem

The penetration at constant vertical velocity $-V$ (V>0) of an infinite circular cylinder of radius R into a fluid domain initially at rest is considered here. The undisturbed free surface coincides with the x axis of the fixed (x,z) coordinate system. It is assumed that the cylinder is rigid and that the fluid is incompressible and inviscid. Thus the velocity potential can be taken as variable. The surface tension of the water is neglected. The equation of the free surface is given in implicit form by

$$S(x,z,t) = 0 \tag{2}$$

If $t = 0$ at the instant of impact, the body boundary is given for $Vt < R$ by

$$z = B(x,t) = R - \sqrt{R^2 - x^2} - Vt. \tag{3}$$

Because of the symmetry of the problem, only the domain $x \geq 0$ is considered. The different boundaries are shown on Figure 2. With these definitions and assumptions, the equations of the problem are easily derived. They are :

$$\Delta\Phi = 0 \tag{4a}$$

in the fluid domain,

$$\Phi(x,z,0) = 0\ ;\ S(x,0,0) = 0 \tag{4b}$$

as initial conditions,

$$\left.\begin{array}{l} S_t + \nabla S \,.\, \nabla\Phi = 0 \\ gz + \Phi_t + \tfrac{1}{2}\nabla\Phi \,.\, \nabla\Phi = 0 \end{array}\right\} \text{ for } S(x,z,t) = 0 \tag{4c}$$

as kinematic and dynamic free surface boundary conditions,

$$\Phi_x x - \Phi_z(R - Vt - z) = V(R - Vt - z) \ \text{ for } \ z = B(x,t) \tag{4d}$$

as body boundary condition,

$$\Phi_x = 0 \ \text{ for } \ x = 0\,,\ z \leq -Vt \tag{4e}$$

as symmetry condition,

$$\nabla\Phi \to 0 \ \text{ and } \ \Phi_t \to 0 \ \text{ for } \ (x^2 + z^2) \to \infty \tag{4f}$$

as behavior at infinity.

Even with the simplifying assumptions introduced, there remain in the problem two fundamental non-linearities. The first concerns the fact that the free surface boundary conditions are given on a surface which itself is to be found as a part of the problem. The second arises from non-linear terms in the boundary conditions themselves. A perturbation method will be used to find an approximate solution to this boundary-value problem. In order to perform the appropriate simplifications, non-dimensional variables are first defined.

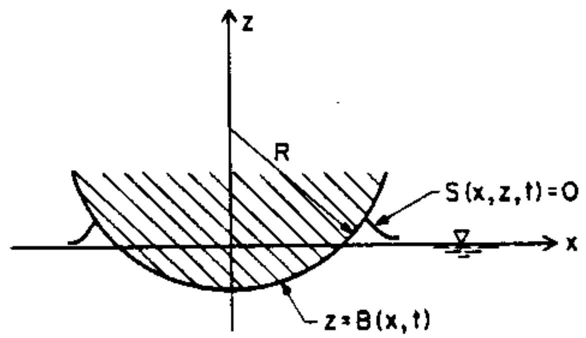

Fig.2 Geometric definitions

Non-Dimensional Variables

It is convenient to introduce T, the characteristic time of impact. Taking

$$\varepsilon = \sqrt{\frac{VT}{R}},\ \ Fr = \frac{V}{\sqrt{gR}},\ \text{ and } L = \sqrt{VTR}\,, \tag{5}$$

the following non-dimensional variables are defined :

$$\bar\Phi = \frac{\Phi}{VR},\ \bar x = \frac{x}{L},\ \bar z = \frac{z}{L},\ \bar t = \frac{t}{T},\ \bar S = \frac{S}{L},\ \bar B = \frac{B}{L}. \tag{6}$$

Since $\varepsilon\bar\Phi$ varies like VL, $\varepsilon\bar\Phi_{\bar t}$ will vary like VLT^{-1} and the first order impact force should vary like $\rho V^2 R$; it should therefore be a constant. The quadratic terms in the pressure, $\varepsilon^2\nabla\bar\Phi.\nabla\bar\Phi$, vary like V^2 and the quadratic correction of the preceding term should be in $\varepsilon\rho V^2 R$. These simple considerations justify the choice of the perturbation parameter, ε. It will therefore be assumed that

$$\varepsilon \ll 1 \quad \text{and} \quad \varepsilon \ll Fr. \tag{7}$$

Outer Domain Equations

It is moreover assumed that in an outer domain far from the water line the free surface is given by

$$\bar S(\bar x,\bar z,\bar t) = \bar z - \bar\eta(\bar x,\bar t). \tag{8}$$

The intersection of the free surface and the body (or water line) being the point $(\bar l(\bar t),\bar\xi(\bar t))$, a change of variable is made to fix its position at the point $(1,0)$. The new variables are defined by

$$\underline{x} = \frac{\bar x}{\bar l},\ \underline{z} = \frac{\bar z - \bar\xi}{\bar l},\ \underline{\phi} = \frac{\bar\Phi}{\bar l},\ \underline{\eta} = \frac{\bar\eta - \bar\xi}{\bar l^2},\ \underline{B} = \frac{\bar B - \bar\xi}{\bar l},\ \underline{t} = \bar t. \tag{9}$$

Taking as expansions of the unknowns

$$\underline{\phi} = \varepsilon\underline{\phi}_1 + o(\varepsilon),\ \underline{\eta} = \varepsilon\underline{\eta}_1 + o(\varepsilon), \tag{10a}$$

$$\bar l = \bar l_1 + o(1),\ \frac{d\bar l}{d\bar t} = \bar v = \bar v_1 + o(1),\ \bar\xi = \varepsilon\bar\xi_1 + o(\varepsilon), \tag{10b}$$

the first order outer domain problem is easily found. The equation $\underline{B}(1,0) = 0$ gives at this order $\bar l_1^2 = 2(\underline{t} + \bar\xi_1)$ and the Equations (4a) to (4f) become

$$\Delta\underline{\phi}_1 = 0 \tag{11a}$$

$$\bar l_1(0) = 0 \tag{11b}$$

$$\left.\begin{array}{l} 2\bar l_1\bar v_1\underline{\eta}_1 + \bar l_1\bar v_1 - 1 + \bar l_1^2\underline{\eta}_{1t} \\ \quad - \bar l_1\bar v_1\underline{x}\underline{\eta}_{1x} - \underline{\phi}_{1z} = 0 \\ \bar v_1\underline{\phi}_1 + \bar l_1\underline{\phi}_{1t} - \bar v_1\underline{x}\underline{\phi}_{1x} = 0 \end{array}\right\} \text{for } \underline{z}=0,\ \underline{x}\geq 1 \tag{11c}$$

$$\underline{\phi}_{1z} = -1 \ \text{ for } \ \underline{z} = 0\,,\ \underline{x} \leq 1 \tag{11d}$$

$$\underline{\phi}_{1x} = 0 \ \text{ for } \ \underline{x} = 0\,,\ \underline{z} \leq 0 \tag{11e}$$

$$\nabla\underline{\phi}_1 \to 0 \ \text{ and } \ (\frac{\partial\underline{\phi}}{\partial\bar t})_1 \to 0 \ \text{ for } \ (\underline{x}^2 + \underline{z}^2) \to \infty \tag{11f}$$

Looking for a stationary solution in terms of the new variables, the dynamic free surface boundary condition reduces to

$$\underline{\phi}_1 = 0 \ \text{ for } \ z = 0\,,\ x \geq 1\,. \tag{12}$$

This defines $\underline{\phi}_1$ as the potential of the unbounded flow around a flat plate of half width 1. Going back to the bar-variables, $\bar\Phi_1$ is the unbounded flow around a flat plate of half width $\bar l_1$. If this were half the width of the circular lens defined by the cylinder and the $\bar x$ axis, this would correspond to von Karman's solution for the flow [8]. But this is half the wetted width of the cylinder and is equivalent to Wagner's solution [9,10] where the piled-up water is taken into account. It is therefore a justification of the wetting correction which appears at the first order. Since the flow around a flat plate is easily obtained using conformal mapping, $\underline{\phi}_1$ is known. Yet in order to find $\bar\Phi_1$, $\bar l_1$ or equivalently the free surface elevation has to be found. This can be done using the kinematic free surface boundary condition. For a stationary solution to exist, $\bar l_1\bar v_1$ has to be a constant. Using the fact that $\underline{\eta}_1(1,\underline{t}) = 0$, and imposing that $\underline{\eta}_1$ be bounded at infinity, we obtain

$$\bar l_1 = 2\sqrt{\bar t}\,,\ \bar\xi_1 = \bar t,\ \underline{\eta}_1 = -\frac{1}{2} - \frac{\underline{x}\sqrt{\underline{x}^2 - 1}}{2} + \frac{\underline{x}^2}{2}. \tag{13}$$

The height of the piled-up water, $\bar\xi_1$, is just equal to the penetration depth. This explains the importance of the wetting correction in this case. The analytical result for the free surface elevation is compared on Figure 3 with a numerical calculation using a discretization of the fluid domain by finite differences (Nichols and Hirt, [13]). The agreement is excellent for Vt much smaller than R. To improve the accuracy of the calculation for larger times, one should go to the next order approximation because ε^2 is essentially the non-dimensional time. Yet a major problem appears because the first order solution, which is given by the flow around a flat plate, is singular at the edges of the plate. The fact that $\underline{\phi}_1$ is not bounded near the water line is in contradiction with the assumptions made to linearize the problem. Therefore the outer solution is not valid near the intersection of the body and the free surface. Near the singularity, non-linearities have to be taken into account. This will be done here using the method of matched asymptotic expansions to define the non-linear inner problem and solve it.

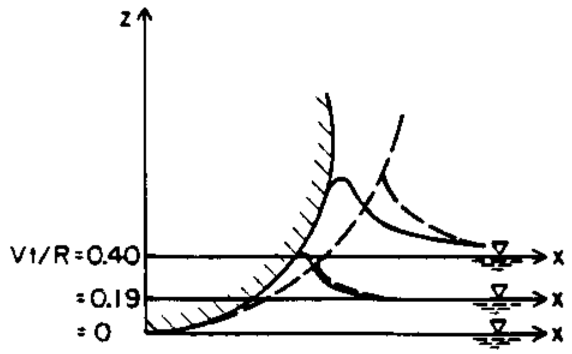

Fig.3 Free surface elevation : numerical (—) and first order analytical (- - -) solutions (the dotted parabola represents the first order approximation of the body boundary)

Inner Domain Equations

The inner variables are defined by

$$x^* = \frac{1}{\varepsilon^2}(\underline{x} - 1)\ ,\quad z^* = \frac{1}{\varepsilon^2}\underline{z}\ ,\quad t^* = \underline{t}\ . \tag{14a}$$

$$\Phi^* = \frac{1}{\varepsilon^2}\underline{\Phi}\ ,\quad S^* = \frac{1}{\varepsilon^2}\underline{S} = \frac{1}{\varepsilon^2}\frac{\overline{S}}{l}\ . \tag{14b}$$

Only the zeroth order inner problem will be studied here, where

$$\Phi^* = \Phi_0^* + o(1)\ ,\quad S^* = S_0^* + o(1)\ . \tag{15}$$

With the change of variable

$$\varphi^* = \Phi_0^* - \bar{v}_1 x^*\ , \tag{16}$$

the zeroth order inner problem is given by

$$\Delta\varphi^* = 0 \tag{17a}$$

$$\left.\begin{array}{l}\nabla\varphi^* \cdot \nabla S_0^* = 0\\ \nabla\varphi^* \cdot \nabla\varphi^* = \bar{v}_1{}^2\end{array}\right\}\ \text{for}\ S_0^*(x^*,z^*,t^*) = 0 \tag{17b}$$

$$\varphi_z^* = 0\ \text{ for }\ z^* = 0 \tag{17c}$$

This φ^*-problem is a classical jet problem. The normal velocity is zero on the x^*-axis, which defines the body boundary, and the velocity is tangential and of constant magnitude $\bar{v}_1$ on the free surface. A solution can be found using the Schwarz-Christoffel method. In terms of an intermediate variable u, this gives for the complex potential, w^*, and the complex variable, ζ^*,

$$w^* = -\frac{2}{\pi}\bar{v}_1\delta^*\left(\frac{u^2}{2} + \log u + \frac{1}{2} + i\,\frac{\pi}{2}\right) \tag{18}$$

$$\zeta^* - \zeta_s^* = \frac{2}{\pi}\delta^*\left(\frac{u^2}{2} - 2iu - \log u + \frac{5}{2} - i\,\frac{\pi}{2}\right). \tag{19}$$

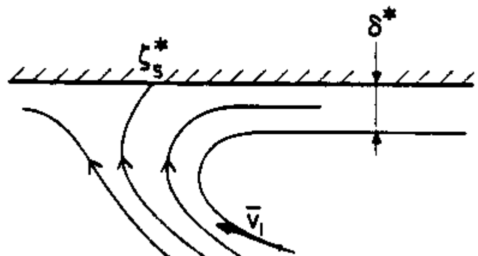

Fig.4 Inner domain solution (φ^*-problem)

This totally defines the velocity field in the inner domain, provided that δ^*, the thickness of the jet, and ζ_s^*, the position of the stagnation point, are known (Figure 4). In order to determine these constants, the inner and outer solutions have to be matched.

Matching of the Inner and Outer Solutions

The matching is performed demanding that the outer solution near the singularity and the inner solution at infinity have the same behavior. It is easy to show that the matching of the potentials and the free surfaces is verified for

$$\delta^* = \frac{\pi}{8\bar{v}_1^2} = \frac{\pi}{8}t^*\ . \tag{20}$$

This is true whatever the value of ζ_s^*, and the inner problem is not fully determined by the matching with the first order outer solution. Apparently, Wagner [10] assumed for the wedge $\zeta_s^* = 0$. He got a similar formula for the thickness of the jet. Re-expressed in the dimensional variables, this thickness is

$$\delta = \frac{\pi R}{4}\left(\frac{Vt}{R}\right)^{\frac{3}{2}}. \tag{21}$$

In the numerical model given by Nichols and Hirt [13], no jet appears (Figure 3). This is due to the procedure used to get a smooth transition between the body and the free surface boundary conditions which essentially consists in applying a pressure on the free surface before the actual contact with the body, and clearly does not allow the creation of a jet. Greenhow and Hill [15] made photographic studies of the water impact of a cylinder where a jet appeared at each water line, in qualitative agreement with the preceding results.

Because it was possible to remove the singularity appearing in the outer approximation, and despite the indetermination of the inner solution, the first order impact force may now be computed.

IMPACT FORCE

First Order Impact Force

In order to find the first order impact force on the cylinder, the pressure on the body is first computed using Bernoulli's theorem. The non-dimensional pressure is defined by

$$\bar{p} = \underline{p} = p^* = \frac{p}{\rho}\,\frac{T}{VR}\ . \tag{22}$$

In the outer domain and at the first order, this leads to the following expression for the pressure on the body :

$$\underline{p}(\underline{x},\underline{B},\underline{t}) \approx -\varepsilon(\bar{v}_1\underline{\Phi}_1 - \bar{v}_1\underline{x}\underline{\Phi}_{1x}) = \frac{\varepsilon\bar{v}_1}{\sqrt{1-\underline{x}^2}}\ . \tag{23}$$

In the inner domain, the pressure on the body at the zeroth order is given by

$$p^*(x^*,0,t^*) = \tfrac{1}{2}(\bar{v}_1^2 - \varphi_x^{*2})\ . \tag{24}$$

An exact expression can be found for this pressure using Equations (18) and (19). The maximum pressure is $\frac{1}{2}\bar{v}_1^2$, reached at the stagnation point. It can be shown that the contribution of the inner pressure to the impact force is of order ε^2, the width of the inner domain. Therefore the first order impact force is only given by the integration of the first order outer pressure. This gives

$$\frac{F}{\rho V^2 R} \approx 2\pi\ . \tag{25}$$

This result was not obvious *a priori* because the contribution of the inner domain solution might have been of the first order. It corresponds to Wagner's modification [9,10] of the original von Karman's formula [8], and may be rewritten as

$$F = \frac{d}{dt}(mV) + o(\rho V^2 R)\ \text{ for }\ \frac{Vt}{R} \to 0\ , \tag{26}$$

where m is half the added mass of the flat plate of width the wetted width of the cylinder, $2l_1$. Therefore, von Karman's formula, Equation (1), when it is used with flat plate-fitting and the wetting correction, corresponds to the first order approximation of the impact force. The wetting correction multiplies the width of the flat plate by $\sqrt{2}$, but the impact force by 2. As was already seen, this value of the force overestimates the experimental results which lie between Wagner and von Karman's approximations (Figure 1). The fact that the agreement between the first order impact force and the experimental data decreases with time is not surprising because the perturbation parameter, ε, is proportional to the square root of the time scale. Since each successive approximation is, by hypothesis, independent of ε, it is logical that the first one be independent of time. To improve the accuracy, and to take into account the time dependence, the next order must be considered. For this purpose, it is necessary to take into account

(a) the zeroth order inner domain pressure which is of order 1 on a domain of width ε^2,

(b) the first order outer domain quadratic pressure which is of order ε^2 on a domain of order 1,

(c) the second order outer domain linear pressure which is of order ε^2 on a domain of order 1.

Therefore an 'exact' expression for the second order impact force demands the determination of ζ_s^* and the solution of the next order outer problem.

Second Order Impact Force

The following expansions of the unknowns are now assumed

$$\underline{\Phi} = \varepsilon\underline{\Phi}_1 + \varepsilon^2\underline{\Phi}_2 + o(\varepsilon^2),\ \underline{\eta} = \varepsilon\underline{\eta}_1 + \varepsilon^2\underline{\eta}_2 + o(\varepsilon^2). \tag{27a}$$

$$\bar{l} = \bar{l}_1 + \varepsilon\bar{l}_2 + o(\varepsilon),\ \bar{v} = \bar{v}_1 + \varepsilon\bar{v}_2 + o(\varepsilon), \tag{27b}$$

$$\bar{\xi} = \varepsilon\bar{\xi}_1 + \varepsilon^2\bar{\xi}_2 + o(\varepsilon^2). \tag{27c}$$

The equation $\underline{B}(1,0) = 0$ gives at this order $\bar{\xi}_2 = \bar{l}_1\bar{l}_2$. The second order outer problem can be found using a Taylor expansion of the boundary conditions. It is possible to remove the time dependence of this problem which reduces to a Neumann-Dirichlet boundary-value problem. Yet it was not found possible to solve it analytically. Therefore an important simplification will be made to allow its resolution. The dynamic free surface boundary condition will be taken as

$$\underline{\Phi}_2 = 0 \ \text{ for } \underline{z} = 0,\ \underline{x} \geq 1 \tag{28}$$

The resulting boundary-value problem will be solved using the solution for the unbounded flow around a circular lens given by Taylor [16]. This solution can be expanded in term of ε, which characterizes the exterior angle of intersection of the two circles. It can be verified that a solution of the approximate second order problem, where the dynamic free surface boundary condition has been taken equal to Equation (28), is given by

$$\underline{\Phi}_2 = \lim_{\varepsilon \to 0} \frac{\underline{\Phi}_T - \underline{\Phi}_1}{\varepsilon}. \tag{29}$$

Using this approximation for the second order potential, the kinematic free surface boundary condition can be used to find the free surface elevation as that was done for the first order outer domain problem. This yields

$$\bar{l}_2 = -\frac{4}{9\pi}\bar{t}\ ,\ \ \bar{\xi}_2 = -\frac{8}{9\pi}\bar{t}^{\frac{3}{2}}, \tag{30a}$$

$$\underline{\eta}_2 = -\frac{4}{3\pi}\underline{x}\sqrt{\underline{x}^2 - 1}\,\sqrt{\underline{t}} \tag{30b}$$

$$-\ \frac{1}{6\pi}(4\underline{x}^2 - 1)\sqrt{\underline{x}^2 - 1}\,\sqrt{\underline{t}}\,\log\left(\frac{\underline{x} - 1}{\underline{x} + 1}\right).$$

The second order solution being approximated, it is now possible to find the second order impact force.

This force could be obtained integrating the pressure distribution on the cylinder. Yet, it will be found here showing what corrections, beside the wetting correction, have to be made to von Karman's formula. The added mass of the circular lens could be obtained studying the far field potential but it is found here directly quite easily. It is given by

$$\bar{m} = -2\varepsilon\int_0^{\bar{l}} \bar{\Phi}(\bar{x},\bar{B},\bar{t})\,d\bar{x} = -2\varepsilon\bar{l}^2\int_0^1 \underline{\Phi}(\underline{x},\underline{B},\underline{t})\,d\underline{x}. \tag{31}$$

At the second order, and using Taylor's solution, this leads to

$$\bar{m} \approx \varepsilon\,2\pi\bar{t} - \varepsilon^2\,\frac{32}{9}\,\bar{t}^{\frac{3}{2}}. \tag{32}$$

The direct application of von Karman's formula, with the added mass of the circular lens and the wetting correction up to the second order, would give

$$\bar{F}_V = \frac{d}{d\bar{t}}\bar{m} \approx \varepsilon\,2\pi - \varepsilon^2\,\frac{16}{3}\sqrt{\bar{t}} \tag{33}$$

Yet, this correction to Wagner's result is not sufficient. Other corrections, of the same order of magnitude, have also to be made. The first one is necessary to take into account the relative motion of the cylinder and the free surface. Differentiating Eq. (31) yields

$$\frac{d}{d\bar{t}}\bar{m} = -2\varepsilon\int_0^{\bar{l}} [\bar{\Phi}_t + \bar{\Phi}_z\,\bar{B}_t]\,(\bar{x},\bar{B},\bar{t})\,d\bar{x}. \tag{34}$$

Therefore the following term has to be added to $\bar{F}_V$:

$$\bar{F}_B = 2\varepsilon\int_0^{\bar{l}} [\bar{\Phi}_z\,\bar{B}_t]\,(\bar{x},\bar{B},\bar{t})\,d\bar{x} \approx 4\,\varepsilon^2\sqrt{\bar{t}}. \tag{35}$$

The importance of this correction should not be underestimated. In particular, $\bar{F}_V + \bar{F}_B$, which represents the integration of the linear pressure (up to the second order, and with the approximations made) over the whole cylinder, is much closer to 2π than $\bar{F}_V$.

Since this integration should only be performed over the outer domain, and the inner pressure integrated over the inner domain, a new correction has to be made.

$$\bar{F}_I = 2\bar{l}_1\varepsilon^2\int_{\text{inner domain}} p_0^*\,dx^* - 2\bar{l}_1\varepsilon\int_{\text{inner domain}} \underline{p}_1\,d\underline{x}. \tag{36}$$

Part of this correction arises from the fact that, at the first order, the integral of the linear pressure over the inner domain (of width $\approx \varepsilon^2$) is of order ε^2, because this pressure is not bounded. In order to compute $\bar{F}_I$, the bounds of the inner domain have to be found. It can be shown that at the lowest order the maximum pressures in the inner and outer domains are the same. This leads to locate the stagnation point of the inner domain at the point, fixed in the inner domain, where the outer pressure is maximum and gives

$$x_s^* = -\frac{4}{\pi}\delta^*. \tag{37}$$

The outer domain now extends from the origin to the stagnation point, and the inner domain from the stagnation point to infinity. This gives for $\bar{F}_I$

$$\bar{F}_I = -2\,\varepsilon^2\sqrt{\bar{t}}. \tag{38}$$

The same approach is needed in order to consider the first order quadratic pressure which would lead to an infinite suction force if integrated over the whole cylinder. This leads to the correction

$$\bar{F}_Q = -\varepsilon^2\bar{l}_1\int_{\text{outer domain}} \frac{1}{1 - \underline{x}^2}\,d\underline{x} \tag{39}$$

$$= \varepsilon^2\sqrt{\bar{t}}\,\left(\log(\varepsilon^2\bar{t}) - 2\log 2\right).$$

The total impact force is now

$$\bar{F} = \bar{F}_V + \bar{F}_B + \bar{F}_I + \bar{F}_Q \tag{40}$$

$$= \varepsilon\,2\pi - \varepsilon^2\sqrt{\bar{t}}\,\left(\frac{10}{3} + 2\log 2 - \log(\varepsilon^2\bar{t})\right),$$

which is the result which could be obtained by direct integration of the pressure. This formula for the impact force is compared on Figure 5 with the experimental results already shown. The agreement is excellent for $\varepsilon\bar{t}$ small, except for the finite rise time not predicted by this theory. The analytical result overestimates the impact force relatively rapidly. This is due to the fact that the perturbation parameter varies like the square root of the time scale, and also to the fact that this result is only an approximation of the second order problem, and that a new correction, at this order, should be made to take into account the geometry of the free surface. Nevertheless the rapid decrease of the non-dimensional impact force from its initial value of 2π can be predicted within the potential flow theory neglecting gravity and the flexibility of the cylinder, and is clearly related to non-linear effects.

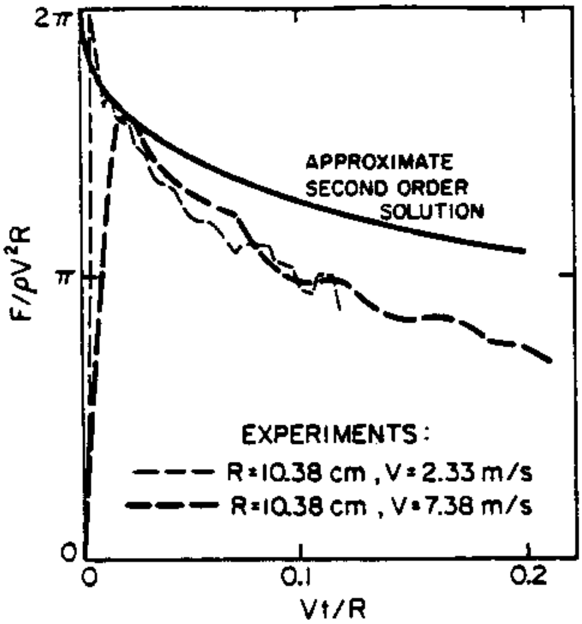

Fig.5 Second order and experimental slamming coefficients

CONCLUSION

A rational approach was derived to deal with the water impact of a circular cylinder. This method justifies the wetting correction introduced by Wagner in 1931, and gives its domain of validity in this special case. The rapid decrease of the slamming coefficient from its initial value of 2π predicted by the present theory together with the finite time rise observed experimentally serve to explain the scatter in the experimental observations concerning the maximum impact force on a circular cylinder. This is true without taking into account the flexibility of the cylinder. Another important conclusion is that one must be very careful in using other corrections than the wetting correction because, for the circular cylinder, different effects lead to corrections of the same order.

This method seems to be applicable to other cases, such as different geometries, non-vertical and/or non-constant velocities, and eventually moving free surfaces. Such cases, which have not been studied as much as the problem adressed here, could lead to interesting results concerning the different parameters involved in the analysis of the impact loads exerted by the waves on a body piercing the free surface.

The method of matched asymptotic expansions, already used to study the flow past blunt ships [17,18] and hydroplaning [19,20], seems therefore to be well appropriate to deal with the singularity which appears at the water line for large motions of the partially emerged pontoons of a damaged semi-submersible.

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