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# A DESIGN APPROACH FOR OPTIMIZING SEGMENTED TRANSDUCERS

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## Abstract

Acoustic system design normally requires a transducer to possess a specific radiation pattern. Yet simplicity of hardware is, as always, an overriding design criterion. One way to minimize hardware is to approximate the required amplitude distribution in the transducer with a small number of constant amplitude segments. The problem then becomes one of optimally choosing the sizes of the segments and their amplitudes.

In this paper we describe a technique, the optimum mean square quantizer, that minimizes the mean square error between a desired amplitude distribution and a design having L constant amplitude levels. It is demonstrated that this design provides the globally minimum error solution. The technique is illustrated by the design of a 4 level segmented transducer that generates a Gaussian beam.


## Introduction

This paper addresses the problem of designing an acoustic transducer with a desired radiation pattern. Control over the radiation pattern is established through control of the amplitude distribution in the transducer. Building transducers with a continuously varying amplitude distribution is difficult to do in a controlled manner, while building a transducer that generates sound at constant amplitude is relatively easy. One way to approximate a desired radiation pattern is to approximate the required continuous amplitude distribution with a number of discrete segments, each driven at a constant amplitude (Figure 1). Since the degree of hardware complexity grows with an increased number of segments, it is desirable to use as few segments as possible to meet design goals. Assuming a fixed number of segments, the problem then becomes one of optimally choosing their sizes and the levels at which they are driven.

The idea of using discrete constant amplitude segments to approximate a continuous amplitude distribution is not new. The National Defense Research Committee examined it during the Second World War [1,2]. However, it appears that their work was limited to experimentation with only 2 and 3 level distributions. No rigorous attempt at optimization appears to have been made at that time. Later, Martin and Hickman [3] considered the problem of optimizing a bilevel amplitude distribution as a function of 2 parameters: the amplitude ratio between the central and outer segments, and their size ratio. More recently, Drost [4] approached the problem of optimizing a bilevel amplitude distribution by coherently adding the predicted far-field radiation patterns of the 2 levels. The sizes of the segments were chosen so that the first positive sidelobe of the wider segment canceled the first negative sidelobe of the narrower one. He then varied the amplitude ratio to minimize all the sidelobes. Most recently Zielinski and Wu [5] have used linear optimization techniques to create arrays of ring radiators with radial radiation patterns that approximate the radiation patterns of $\lambda/2$ spaced linear arrays.

In this paper we describe the optimum mean square quantizer approach as an optimal way to approximate a continuous amplitude distribution using a small number of constant amplitude segments. The analysis applies to a stave transducer producing a 1-dimensional beam.

### Method - the optimum mean square quantizer approach

Quantization is the mapping of a continuously valued variable into a discretely valued variable. Figure 1 shows the quantization of a continuous amplitude distribution wherein all values between decision levels $d_1$ and $d_2$ are mapped into reconstruction level $r_1$, and so forth. We seek the "best" choice of $r_1$ through $r_L$ and $d_1$ through $d_{L+1}$. In the optimum mean square quantizer approach, the best choice is the one that minimize the mean square error between the continuous amplitude distribution and its discrete approximation. The approach takes its name from a similar method in digital communication theory [6]. If the desired amplitude distribution is symmetric and monotonically decreases to zero with distance from the origin as shown in the figure, $d_1$ through $d_{L+1}$ map directly into $X_1$ through $X_{L+1}$, locations that define the sizes of the transducer elements.

The error to be minimized is:

$$\varepsilon = \| f(x) - f_{approx}(x)\|^2 = \int_{-\infty}^{\infty}[f(x) - f_{approx}(x)]^2\, dx \qquad (1)$$

where $f(x)$ = desired amplitude distribution function
$f_{approx}(x)$ = approximate amplitude distribution function

By symmetry, and exploiting the discrete nature of $f_{approx}(x)$:

$$\varepsilon = 2\sum_{i=1}^{L}\int_{X_i}^{X_{i+1}}[f(x) - r_i]^2\, dx \qquad (2)$$

To minimize $\varepsilon$ over the sets of $\{r_k\}$ and $\{X_k\}$ we require that:

$$\frac{\partial\varepsilon}{\partial r_k} = 0 \qquad k = 1, ..., L - 1 \qquad (3)$$

$$\frac{\partial\varepsilon}{\partial X_k} = 0 \qquad k = 2, ..., L \qquad (4)$$

with the following constraints:

$$r_L = 0 \qquad (5)$$

$$X_1 = 0 \qquad (6)$$

$$X_{L+1} = \infty \qquad (7)$$

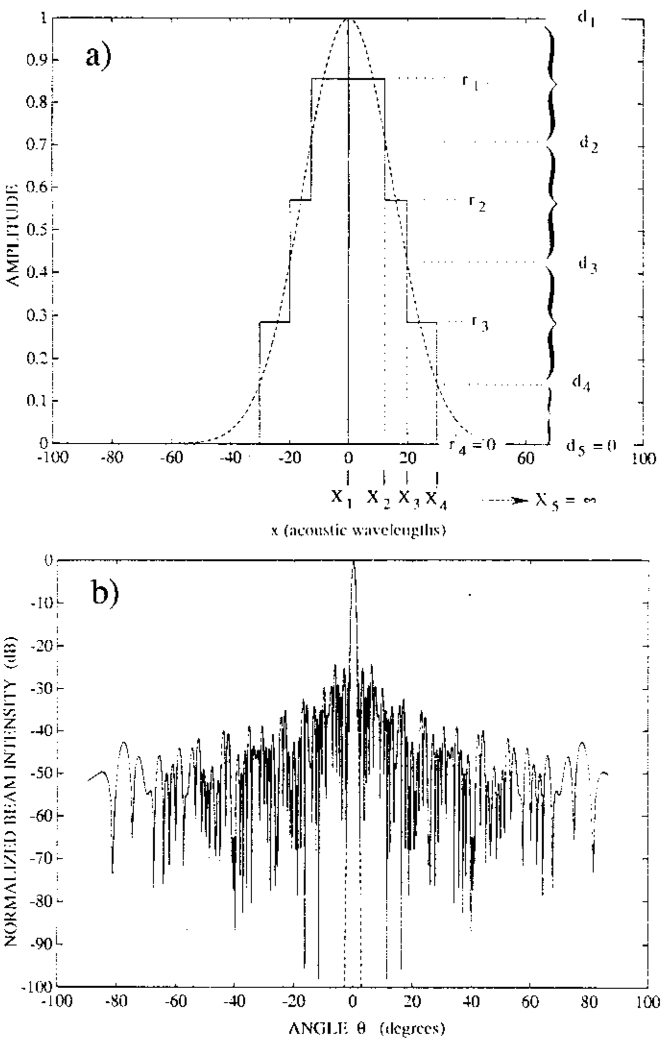


*Figure 1. a) The dashed curve is a continuous Gaussian amplitude distribution. The solid stepped curve is a 4 level quantized approximation. $r_k$ is the kth reconstruction level. All values in the continuous distribution between decisions levels $d_k$ and $d_{k+1}$ are mapped into $r_k$. The kth segment of the transducer is delimited by $X_k$ and $X_{k+1}$.*
*b) The solid curve is the radiation pattern resulting from the 4 level quantized distribution shown in (a). The dashed curve is the ideal Gaussian radiation pattern from the continuous amplitude distribution shown in (a).*

The optimum solution is given by the 3 constraints (5), (6), and (7), and the following simultaneous set of equations. The derivation of these is in Appendix A.

$$d_k = f(X_k) = \frac{r_k + r_{k-1}}{2} \qquad k = 2, \ldots, L \tag{8}$$

$$r_k = \frac{\int_{X_k}^{X_{k+1}} f(x)\,dx}{X_{k+1} - X_k} \qquad k = 1, \ldots, L-1 \tag{9}$$

If the error surface described by Eq. (1) is convex at the point of solution, the solution is guaranteed to be a minimizer [7]. It can be shown (Appendix B) that if the error surface can be described by Eq. (2), it is convex over the entire domain defined by $\{r_k\}$ and $\{X_k\}$. This guarantees that the solution is not only a minimizer, but is a global minimizer over the domain.

Many iterative techniques exist for solving the set of equations (8) and (9). Newton-based methods are among the fastest of these to converge. In this paper we have elected to use a "pure" Newton method which, as shown in Appendix C, is identical to alternately solving equations (8) and (9) until they converge.

## Results

To demonstrate the optimum mean square quantizer approach, a Gaussian beam will be approximated using a 4 level segmented transducer (the 4th level is the 0 level). We will also examine how the number of quantization levels affects the quality of the approximation.

Assuming that a 1° wide Gaussian beam is desired, the radiation pattern is:

$$F(s) = e^{-\pi(38.066s)^2} \tag{10}$$

where $s = \sin(\theta)$; $\theta$ is the angle from broadside.

Goodman [8] has shown that the far-field radiation pattern is proportional to the Fourier transform of the amplitude distribution function. Therefore, inverse Fourier transforming Eq. (10), the desired amplitude distribution is:

$$f(x) = e^{-\pi(.02627x)^2} \tag{11}$$

where x is the distance along the transducer in multiples of $\lambda$, the acoustic wavelength. This distribution is the dashed curve in Figure 1a.

Figure 1a also shows a naively chosen set of quantization levels and segment sizes for a 4 level segmented transducer. The selection is called naive because a "natural" choice has been made to divide the range of amplitudes into 4 equal ranges, then to map each range into its central value (the lowest range is only half as large because it maps into zero).

$$\mathbf{d}_{\text{naive}} = \begin{bmatrix} 1 \\ 5/7 \\ 3/7 \\ 1/7 \\ 0 \end{bmatrix} \Rightarrow \mathbf{r}_{\text{naive}} = \begin{bmatrix} 6/7 \\ 4/7 \\ 2/7 \\ 0 \end{bmatrix}, \quad \mathbf{X}_{\text{naive}} = \begin{bmatrix} 0 \\ 12.4577 \\ 19.7689 \\ 29.9589 \\ \infty \end{bmatrix}$$

Figure 1b shows the expected radiation pattern resulting from the naive quantization. Numerically computing the error using Eq. (1), $\varepsilon_{\text{naive}} = 0.62$.

We will use the naive design as a starting point from which to optimize segment sizes and amplitude levels. Eq. (9) becomes:

$$r_k = \frac{\int_{X_k}^{X_{k+1}} e^{-\pi(.02627x)^2}\,dx}{X_{k+1} - X_k} \qquad k = 1, 2, 3 \tag{12}$$

and Eq. (8) becomes:

$$X_k = f^{-1}\left(\frac{r_k + r_{k-1}}{2}\right)$$

$$= \sqrt{\frac{-\ln\left(\frac{r_k + r_{k-1}}{2}\right)}{\pi * .02627^2}} \quad k = 2, 3, 4 \qquad (13)$$

Also $r_4 = 0$, $X_1 = 0$, and $X_5 = \infty$.

Using $r_{naive}$ as a starting point, iterative solution of the set of equations (12) and (13) gives:

$$\mathbf{r}_{optimum} = \begin{bmatrix} .9099 \\ .5799 \\ .2543 \\ 0 \end{bmatrix}, \qquad \mathbf{X}_{optimum} = \begin{bmatrix} 0 \\ 16.4596 \\ 28.3633 \\ 43.5599 \\ \infty \end{bmatrix}$$

Figure 2a shows the optimized amplitude distribution; Figure 2b shows its radiation pattern. $\varepsilon_{optimum} = 0.56$, approximately 10% better than $\varepsilon_{naive}$. Figure 2c magnifies the main lobe in the radiation pattern to permit comparison of the naive and optimum quantized beams with the ideal beam. Rayleigh's theorem,

$$\int_{-\infty}^{\infty} |f(x) - f_{approx}(x)|^2 dx = \int_{-\infty}^{\infty} |F(s) - F_{approx}(s)|^2 ds$$

indicates that the error between the optimum quantized beam and the ideal beam should also be 10% better that the error between the naive quantized beam and the ideal beam. This improvement is apparent in the figure.

The error between the ideal amplitude distribution and its discrete approximation can also be reduced by increasing the allowed number of quantization levels. Figure 3 shows $\varepsilon$ as a function of L, the number of quantization levels, for quantized approximations to a Gaussian amplitude distribution. The figure indicates that an increase in L results in a big improvement in $\varepsilon$ if L is relatively small. The improvement is less for larger L. Figure 3 is valid for all Gaussian beams.

## Conclusions

The optimum mean square quantizer approach chooses the "best" segment sizes and amplitude levels for a segmented transducer by minimizing the 2-norm error between a desired amplitude distribution and its L level discrete approximation.

The method has several advantages: One is that it cuts the number of independent variables in half, from 2L - 2 to L - 1, by linking the element sizes to the amplitudes at which they are driven. A second advantage is that the error surface is convex over the entire domain. This guarantees the existence of a global minimizer. Another valuable attribute of global convexity is that it permits the use of rapidly converging algorithms,. In this paper we have used a "pure" Newton method, but "modified" Newton methods exist which are guaranteed to converge to a solution more rapidly.

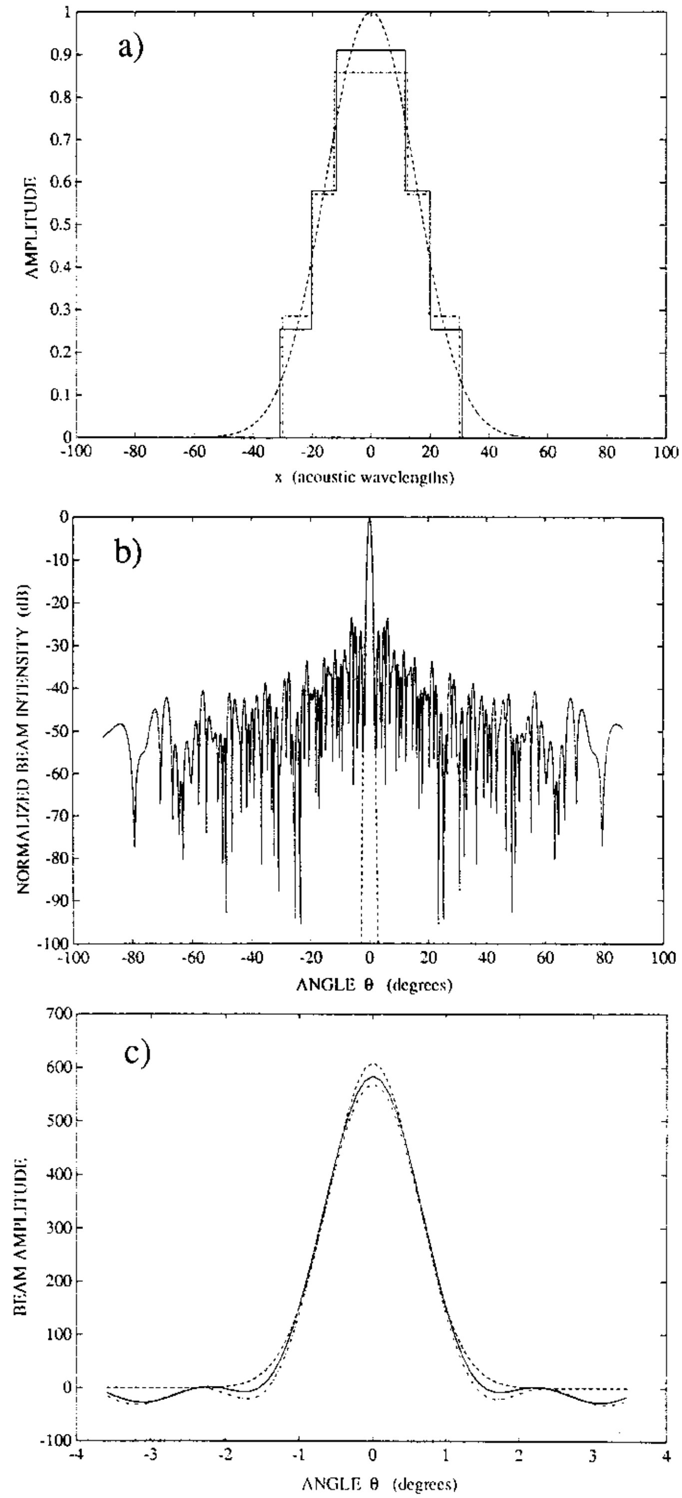


*Figure 2. a) The solid stepped curve is the optimum 4 level quantized amplitude distribution; the dot-dashed stepped curve is the naive choice of a 4 level quantized amplitude distribution; the dashed Gaussian is the continuous amplitude distribution.*
*b) The solid curve is the radiation pattern generated by the optimum 4 level quantized amplitude distribution; the dashed curve is the ideal Gaussian radiation pattern.*
*c) Main lobe of the radiation pattern. The solid curve is the optimum quantized directivity pattern;, the dot-dashed curve is the naive quantized directivity pattern; the dashed curve is the ideal Gaussian radiation pattern.*

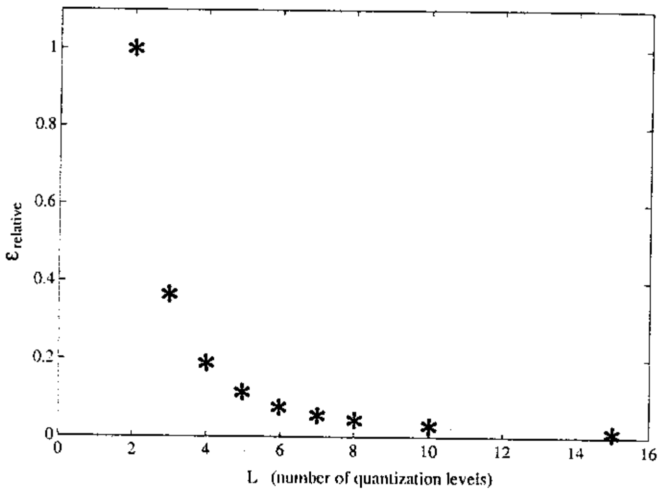


*Figure 3. Relative error $\varepsilon_{relative}$ as a function of the number of quantization levels. $\varepsilon_{relative} = \varepsilon_L/\varepsilon_2$, where $\varepsilon_L$ = error from Eq. (1) for the optimum L level quantized amplitude distribution and $\varepsilon_2$ is the error for the optimum 2 level quantized amplitude distribution.*

There are 2 ways to decrease the error between the desired amplitude distribution and its discrete approximation. The first is to optimize the segment sizes and drive levels, as shown above. The other is to increase L, the allowed number of quantization levels. For small L, the improvements can be substantial. For larger L, the incremental advantage of increasing L is negligible. Using the design methodology developed in this article and a given error criterion, the minimum number of quantization levels needed to achieve this error can be determined.


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### References


[1] Design and Construction of Crystal Transducers, National Defense Research Committee, Division 6, Summary Tech. Rpt. Vol. 12, Washington, D. C., 1946.

[2] The Design and Construction of Magnetostriction Transducers, National Defense Research Committee, Division 6, Summary Tech. Rpt. Vol. 12, Washington, D. C., 1946.

[3] G. E. Martin and J. S. Hickman, "Directional Properties of Continuous Plane Radiators with Bizonal Amplitude Shading," J. Acoust. Soc. Am. 27: 1120-1127 (1955).

[4] C. J. Drost, "Near and farfield of strip-shaped acoustic radiators," J. Acoust. Soc. Am. 65: 565-572 (1979).

[5] A. Zielinski and L. Wu, "A Novel Array of Ring Radiators," IEEE J. Oceanic Eng. 16: 136-141 (1991).

[6] A. K. Jain, Fundamentals of Digital Image Processing, Prentice-Hall, Englewood, NJ, 1989.

[7] P. Gill, W. Murray, and M. Wright, Practical Optimization, Academic Press, New York, NY, 1981.

[8] J. W. Goodman, Introduction to Fourier Optics, McGraw-Hill, New York, NY, 1968.

[9] J. M. Ortega and W. C. Rheinboldt, Iterative Solution of Nonlinear Equations in Several Variables, Academic Press, New York, NY, 1970.


### Appendix A. Mathematical development of the optimum mean square quantizer

The error to be minimized is:

$$\varepsilon = \| f(x) - f_{approx}(x)\|^2 = \int_{-\infty}^{\infty} [f(x) - f_{approx}(x)]^2\, dx \qquad (A1)$$

where f(x) = desired amplitude distribution function
$f_{approx}(x)$ = approximate amplitude distribution function

By symmetry, and exploiting the discrete nature of $f_{approx}(x)$:

$$\varepsilon = 2\sum_{i=1}^{L}\int_{X_i}^{X_{i+1}} [f(x) - r_i]^2\, dx \qquad (A2)$$

To minimize $\varepsilon$ over the sets of $\{r_k\}$ and $\{X_k\}$ we require that:

$$\frac{\partial\varepsilon}{\partial r_k} = 0 \qquad k = 1, \ldots, L-1 \qquad (A3)$$

$$\frac{\partial\varepsilon}{\partial X_k} = 0 \qquad k = 2, \ldots, L \qquad (A4)$$

with the following constraints:

$$r_L = 0 \qquad (A5)$$

$$X_1 = 0 \qquad (A6)$$

$$X_{L+1} = \infty \qquad (A7)$$

Let:

$$\int_{X_i}^{X_{i+1}} [f(x) - r_i]^2 \, dx = G_i(x) \big|_{X_i}^{X_{i+1}} = G_i(X_{i+1}) - G_i(X_i) \qquad \text{(A8)}$$

Then:

$$\varepsilon = 2(G_1(X_2) - G_1(X_1) + \ldots + G_{k-1}(X_k) - G_{k-1}(X_{k-1}) + G_k(X_{k+1}) - G_k(X_k) + \ldots) \qquad \text{(A9)}$$

$$\frac{\partial \varepsilon}{\partial X_k} = 2 \frac{\partial}{\partial X_k} [G_{k-1}(X_k) - G_k(X_k)]$$

$$= 2[(f(X_k) - r_{k-1})^2 - (f(X_k) - r_k)^2] \qquad \text{(A10)}$$

Recall that $d_k = f(X_k)$ and set:

$$\frac{\partial \varepsilon}{\partial X_k} = 0$$

Then:

$$0 = d_k^2 - 2d_k r_{k-1} + r_{k-1}^2 - d_k^2 + 2d_k r_k - r_k^2$$

$$= 2d_k(r_k - r_{k-1}) - (r_k^2 - r_{k-1}^2)$$

$$= (2d_k - (r_k + r_{k-1}))(r_k - r_{k-1}) \qquad \text{(A11)}$$

As long as

$$r_k \neq r_{k-1}$$

then

$$0 = 2d_k - (r_k + r_{k-1})$$

$$d_k = f(X_k) = \frac{r_k + r_{k-1}}{2} \qquad k = 2, \ldots, L \qquad \text{(A12)}$$

Thus, the optimum choice of decision level $d_k$ is midway between reconstruction levels $r_k$ and $r_{k+1}$. Now solve:

$$\frac{\partial \varepsilon}{\partial r_k} = 0$$

Recall from Eq. (A2) that:

$$\varepsilon = 2 \sum_{i=1}^{L} \int_{X_i}^{X_{i+1}} [f(x) - r_i]^2 \, dx$$

$$\frac{\partial \varepsilon}{\partial r_k} = 2 \int_{X_k}^{X_{k+1}} 2(f(x) - r_k)(-1) \, dx = 0 \qquad \text{(A13)}$$

$$\int_{X_k}^{X_{k+1}} f(x) \, dx = \int_{X_k}^{X_{k+1}} r_k \, dx \qquad \text{(A14)}$$

$$r_k = \frac{\int_{X_k}^{X_{k+1}} f(x) \, dx}{X_{k+1} - X_k} \qquad k = 1, \ldots, L-1 \qquad \text{(A15)}$$

In summary, the optimum solution is given by simultaneous solution of Eqs. (A12) and (A15) and by the constraints (A5), (A6), and (A7).

## Appendix B. Proof that the solution is a global minimizer

From Eq. (8), $\{X_k\}$ is completely specified by $\{r_k\}$. Therefore, for a given amplitude distribution $f(x)$, $\varepsilon = \varepsilon(\mathbf{r})$ where:

$$\mathbf{r} = \begin{bmatrix} r_1 \\ r_2 \\ \ldots \\ r_{L-1} \end{bmatrix} \qquad \text{(B1)}$$

The gradient vector $\mathbf{g}$ is defined as:

$$\mathbf{g} = \begin{bmatrix} \partial\varepsilon/\partial r_1 \\ \partial\varepsilon/\partial r_2 \\ \ldots \\ \partial\varepsilon/\partial r_{L-1} \end{bmatrix} \qquad \text{(B2)}$$

A necessary condition for $\varepsilon(\mathbf{r})$ to be a minimum at the point $\mathbf{r}^*$ is that $\mathbf{g} = 0$ and this is what we solve for using Eqs. (8) and (9). However, $\mathbf{g}$ is also 0 at maxima and at saddle points as well. To guarantee that $\mathbf{r}^*$ is a minimizer, we add the condition that $\varepsilon(\mathbf{r})$ be convex at $\mathbf{r}^*$. This can be shown equivalent to the condition that the Hessian matrix H be positive semidefinite [7] where:

$$H = \begin{bmatrix} \partial^2\varepsilon/\partial r_1^2 & \partial^2\varepsilon/\partial r_1 \partial r_2 & \ldots & \partial^2\varepsilon/\partial r_1 \partial r_{L-1} \\ \ldots & & & \\ \partial^2\varepsilon/\partial r_{L-1} \partial r_1 & & \ldots & \partial^2\varepsilon/\partial^2 r_{L-1}^2 \end{bmatrix} \qquad \text{(B3)}$$

From Eq. (A13):

$$\frac{\partial \varepsilon}{\partial r_k} = -4 \int_{X_k}^{X_{k+1}} (f(x) - r_k) \, dx$$

$$\frac{\partial^2 \varepsilon}{\partial r_j \partial r_k} = 0 \qquad \text{for } k \neq j \qquad \text{(B4)}$$

$$\frac{\partial^2 \varepsilon}{\partial^2 r_k} = 4 \int_{X_k}^{X_{k+1}} \frac{\partial r_k}{\partial r_k} \, dx = 4\,[X_{k+1} - X_k] > 0 \qquad \text{(B5)}$$

as long as $X_{k+1} > X_k$.

H is therefore a diagonal matrix with only positive values. The test for positive definiteness is that H be decomposable by Cholesky factorization:

$$H = R^T R$$

where R is upper triangular.

By inspection:

$$R = \begin{bmatrix} 2(X_2 - X_1)^{1/2} & & & \\ & 2(X_3 - X_2)^{1/2} & & \\ & & \cdots & \\ & & & 2(X_{L+1} - X_L)^{1/2} \end{bmatrix} \tag{B6}$$

$\mathbf{r}^*$ is a minimizer because H is positive definite. Indeed, $\mathbf{r}^*$ is a global minimizer because $\varepsilon(\mathbf{r})$ is everywhere convex (H is always positive definite) [9].

## Appendix C. A "pure" Newton method

Newton methods are iterative methods for solving sets of nonlinear equations. With a "pure" Newton method, the iterates of $\mathbf{r}$ are given by [7]:

$$\mathbf{r}_{j+1} = \mathbf{r}_j + \mathbf{p}_j \tag{C1}$$

where the Newton step $\mathbf{p}_j$ satisfies:

$$H_j \mathbf{p}_j = -\mathbf{g}_j \tag{C2}$$

Since H is diagonal (Appendix B), $\mathbf{p}_j$ can be written:

$$\mathbf{p}_j = -\begin{bmatrix} g_{1_j}/H_{11_j} \\ \cdots \\ g_{k_j}/H_{kk_j} \\ \cdots \end{bmatrix} \tag{C3}$$

From Eqs. (B2) and (A13), the kth element of $\mathbf{g}_j$ is given by:

$$g_{k_j} = \left(\frac{\partial \varepsilon}{\partial r_k}\right)_j = -4\int_{X_{k_j}}^{X_{k+1_j}} (f(x) - r_{k_j})\, dx$$

$$g_{k_j} = -4\left[\int_{X_{k_j}}^{X_{k+1_j}} f(x)\, dx - r_{k_j}(X_{k+1_j} - X_{k_j})\right] \tag{C4}$$

From Eq. (B5),

$$H_{kk_j} = 4(X_{k+1_j} - X_{k_j}) \tag{C5}$$

Substituting Eqs. (C4) and (C5) into Eq. (C3),

$$\mathbf{p}_j = \begin{bmatrix} \cdots \\ \dfrac{\int_{X_{k_j}}^{X_{k+1_j}} f(x)\, dx}{X_{k+1_j} - X_{k_j}} \\ \cdots \end{bmatrix} - \mathbf{r}_j \tag{C6}$$

Substituting Eq. (C6) into Eq. (C1),

$$\mathbf{r}_{j+1} = \begin{bmatrix} \cdots \\ \dfrac{\int_{X_{k_j}}^{X_{k+1_j}} f(x)\, dx}{X_{k+1_j} - X_{k_j}} \\ \cdots \end{bmatrix} \tag{C7}$$

Eq. (C7) is identical to Eq. (9). Therefore, solution of the system of equations (8) and (9) by a pure Newton method is identical to alternately solving Eqs. (8) and (9) until they converge.