SI231B: Matrix Computations, 2024 Fall

Homework Set #1

Acknowledgements:

1) Deadline: **2024-10-22 23:59:59**
2) Please submit the PDF file to gradescope. Course entry code: 8KJ345.
3) You have 5 "free days" in total for all late homework submissions.
4) If your homework is handwritten, please make it clear and legible.
5) All your answers are required to be in English.

Problem 1. (Subspace) (20 points)

1) Let $\mathcal{V} = \mathbb{R}^2$. Whether or not each of the following is a subspace of $\mathcal{V}$? Justify your answer.
   a) $\mathcal{S}_1 = \{(x, y) \in \mathbb{R}^2 \mid x + y = 0\}$. (2 points)
   b) $\mathcal{S}_2 = \{(x, y) \in \mathbb{R}^2 \mid xy = 0\}$. (2 points)
2) Let $\mathcal{V} = \mathbb{C}^{n\times n}$ be the set of all $n \times n$ complex matrices. $\mathcal{V}$ is a vector space over $\mathbb{C}$: the addition is defined by standard addition of two complex matrices, and the scalar multiplication is defined by standard multiplication of a complex number and a complex matrix; $\mathcal{V}$ is also a vector space over $\mathbb{R}$, the addition is the same, but the scalar multiplication is defined by standard multiplication of a real number and a complex matrix, i.e., the scalars in this vector space are form $\mathbb{R}$, not $\mathbb{C}$. Let $\mathcal{S} = \{\mathbf{A} \in \mathbb{C}^{n\times n} \mid \mathbf{A}^H = \mathbf{A}\}$ be the set of all $n \times n$ Hermitian matrices.
   a) Whether or not $\mathcal{S}$ is a subspace of $\mathcal{V}$ over $\mathbb{R}$? Justify your answer. (Note: You need to check whether any linear combination of the elements in $\mathcal{S}$ lies in $\mathcal{S}$. In the vector space $\mathcal{V}$ over $\mathbb{R}$, a linear combination is in the form of $\alpha\mathbf{A} + \beta\mathbf{B}$, $\forall\alpha, \beta \in \mathbb{R}$, $\forall\mathbf{A}, \mathbf{B} \in \mathcal{V}$) (4 points)
   b) Whether or not $\mathcal{S}$ is a subspace of $\mathcal{V}$ over $\mathbb{C}$? Justify your answer. (Note: You need to check whether any linear combination of the elements in $\mathcal{S}$ lies in $\mathcal{S}$. In the vector space $\mathcal{V}$ over $\mathbb{C}$, a linear combination is in the form of $\alpha\mathbf{A} + \beta\mathbf{B}$, $\forall\alpha, \beta \in \mathbb{C}$, $\forall\mathbf{A}, \mathbf{B} \in \mathcal{V}$) (4 points)
   c) Prove that each $\mathbf{A} \in \mathcal{V}$ can be written in exactly one way as $\mathbf{A} = H(\mathbf{A}) + iK(\mathbf{A})$, in which $H(\mathbf{A})$ and $K(\mathbf{A})$ are Hermitian. (8 points)

Solution:

1) a) $\mathcal{S}_1$ is a subspace. Reason: for any $(x_1, y_1), (x_2, y_2) \in \mathcal{S}_1$ and $a, b \in \mathbb{R}$, $a(x_1, y_1) + b(x_2, y_2) = (ax_1 + bx_2, ay_1 + by_2) \in \mathcal{S}_1$, since $(ax_1 + bx_2) + (ay_1 + by_2) = a(x_1 + y_1) + b(x_2 + y_2) = 0 + 0 = 0$. (2 points)
   b) $\mathcal{S}_2$ is not a subspace, since $(0, 1), (1, 0) \in \mathcal{S}_2$ but $(0, 1) + (1, 0) = (1, 1) \notin \mathcal{S}_2$. (2 points)
2) a) $\mathcal{S}$ is a subspace of $\mathcal{V}$ over $\mathbb{R}$ (1 points). Reason: for any Hermitian matrices $\mathbf{A} = \mathbf{A}^H, \mathbf{B} = \mathbf{B}^H$, and real number $\alpha, \beta \in \mathbb{R}$, $\alpha\mathbf{A} + \beta\mathbf{B}$ is still Hermitian, since $(\alpha\mathbf{A} + \beta\mathbf{B})_{ij} = \alpha a_{ij} + \beta b_{ij} = \alpha a_{ji}^* + \beta b_{ji}^* = (\alpha a_{ji} + \beta b_{ji})^* = (\alpha\mathbf{A} + \beta\mathbf{B})_{ji}^*$. (3 points)
   b) $\mathcal{S}$ is not a subspace of $\mathcal{V}$ over $\mathbb{C}$ (1 points). Reason: consider the case of $n = 1$, $\mathbf{A} = \mathbf{B} = \begin{bmatrix}1\end{bmatrix} = \mathbf{A}^H = \mathbf{B}^H \in \mathcal{S}$ and $\alpha = 1, \beta = i \in \mathbb{C}$. $\alpha\mathbf{A} + \beta\mathbf{B} = \begin{bmatrix}1 + i\end{bmatrix} \neq \begin{bmatrix}1 - i\end{bmatrix} = (\alpha\mathbf{A} + \beta\mathbf{B})^H$. The linear combination of $\mathbf{A}, \mathbf{B} \in \mathcal{S}$ with scalars in $\mathbb{C}$ does not lie in $\mathcal{S}$, thus $\mathcal{S}$ is not a subspace. (3 points)
   c) Let $H(\mathbf{A}) = \frac{1}{2}(\mathbf{A} + \mathbf{A}^H)$ and $K(A) = \frac{1}{2i}(\mathbf{A} - \mathbf{A}^H)$, it is easy to see that both of them are Hermitian (4 points for construction). To show the uniqueness of the decomposition, let $\mathbf{A} = \mathbf{H}_1 + i\mathbf{K}_1 = \mathbf{H}_2 + i\mathbf{K}_2$, where $\mathbf{H}_1, \mathbf{H}_2, \mathbf{K}_1$ and $\mathbf{K}_2$ are all Hermitian. Then $\mathbf{H}_1 - \mathbf{H}_2 = i(\mathbf{K}_2 - \mathbf{K}_1)$. Note that $\mathbf{H}_1 - \mathbf{H}_2$ is Hermitian while $i(\mathbf{K}_2 - \mathbf{K}_1)$ is skew-Hermitian, hence both of them are zero-matrices. Thus $\mathbf{H}_1 = \mathbf{H}_2$ and $\mathbf{K}_1 = \mathbf{K}_2$ (4 points for uniqueness).
   The representation $\mathbf{A} = H(\mathbf{A}) + iK(\mathbf{A})$ of a complex or real matrix is its *Toeplitz decomposition*.

Problem 2. (Range and Nullspace) (15 points)

1) Consider two matrices $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{B} \in \mathbb{R}^{n\times p}$. What is the relationship between $\mathcal{R}(\mathbf{A})$ and $\mathcal{R}(\mathbf{AB})$? Are they necessarily equal? If yes, prove your statement, otherwise, give a counter example. (3 points)
2) Let $\mathbf{A} \in \mathbb{R}^{n\times n}$. Consider the following chain:

$$\mathcal{R}(\mathbf{A}^0) \supseteq \mathcal{R}(\mathbf{A}^1) \supseteq \mathcal{R}(\mathbf{A}^2) \supseteq \cdots \supseteq \mathcal{R}(\mathbf{A}^k) \supseteq \mathcal{R}(\mathbf{A}^{k+1}) \supseteq \cdots . \qquad (\star)$$

   a) Prove that there is equality at some point of the chain, i.e., there exists $k \in \{0, 1, 2, 3, \cdots\}$ such that $\mathcal{R}(\mathbf{A}^k) = \mathcal{R}(\mathbf{A}^{k+1})$. (2 points)
   b) Prove that once the equality is attained, it is maintained throughout the rest of the chain, i.e., for some positive integer k,

$$\mathcal{R}(\mathbf{A}^0) \supseteq \mathcal{R}(\mathbf{A}^1) \supseteq \mathcal{R}(\mathbf{A}^2) \supseteq \cdots \supseteq \mathcal{R}(\mathbf{A}^k) = \mathcal{R}(\mathbf{A}^{k+1}) = \mathcal{R}(\mathbf{A}^{k+2}) = \cdots .$$

   (3 points)
   c) Prove that for the integer k in b), we have $\mathcal{R}(\mathbf{A}^k) + \mathcal{N}(\mathbf{A}^k) = \mathbb{R}^n$ and $\mathcal{R}(\mathbf{A}^k) \cap \mathcal{N}(\mathbf{A}^k) = \{\mathbf{0}\}$. In other words, $\mathcal{R}(\mathbf{A}^k) \oplus \mathcal{N}(\mathbf{A}^k) = \mathbb{R}^n$ is a *direct sum*. (7 points)
   (Hint: You might use Grassmann's formula: Let $\mathcal{M}, \mathcal{N}$ be subspaces of a finite-dimensional vector space $\mathcal{V}$. Then $\dim \mathcal{M} + \dim \mathcal{N} = \dim(\mathcal{M} + \mathcal{N}) - \dim(\mathcal{M} \cap \mathcal{N})$.)

Solution:

1) $\mathcal{R}(\mathbf{AB}) \subseteq \mathcal{R}(\mathbf{A})$ in sense that for all $\mathbf{y} \in \mathcal{R}(\mathbf{AB})$, there exists some $\mathbf{x}$ such that $y = (\mathbf{AB})\mathbf{x}$. Hence $\mathbf{y}$ is also in $\mathcal{R}(\mathbf{A})$, since $\mathbf{y} = \mathbf{Az}$, in which $\mathbf{z} = \mathbf{Bx}$ (2 points).
   $\mathcal{R}(\mathbf{AB})$ and $\mathcal{R}(\mathbf{A})$ are not necessarily equal. Consider $\mathbf{A} = \mathbf{I} \in \mathbb{R}^{m\times m}$ and $\mathbf{B}$ is a $m \times p$ matrix with all entries equal to 0. Then $\mathcal{R}(\mathbf{A}) = \mathbb{R}^m$ but $\mathcal{R}(\mathbf{AB}) = \{\mathbf{0}\}$, which are not equal for any $m \geq 1$ (1 points).
2) a) If there is strict containment at each link in the chain $(\star)$, then the sequence of inequalities

$$\dim \mathcal{R}(\mathbf{A}^0) > \dim \mathcal{R}(\mathbf{A}^1) \dim \mathcal{R}(\mathbf{A}^2) > \cdots$$

   holds, and this forces $\dim \mathcal{R}(\mathbf{A}^{n+1}) < 0$, which is impossible. (2 points)
   b) Observe that if k is the smallest nonnegative integer such that $\mathcal{R}(\mathbf{A}^k) = \mathcal{R}(\mathbf{A}^{k+1})$, then for all $i \geq 1$

$$\mathcal{R}(\mathbf{A}^{i+k}) = \mathcal{R}(\mathbf{A}^i\mathbf{A}^k) = \mathbf{A}^i\,\mathcal{R}(\mathbf{A}^k) = \mathbf{A}^i\,\mathcal{R}(\mathbf{A}^{k+1}) = \mathcal{R}(\mathbf{A}^{i+k+1}).$$

   (3 points)
   c) If $\mathbf{x} \in \mathcal{R}(\mathbf{A}^k) \cap \mathcal{N}(\mathbf{A}^k)$, then $\mathbf{A}^k\mathbf{y} = \mathbf{x}$ for some $\mathbf{y} \in \mathbb{R}^n$, and $\mathbf{A}^k\mathbf{x} = \mathbf{0}$. Hence $\mathbf{A}^{2k}\mathbf{y} = \mathbf{A}^k\mathbf{x} = \mathbf{0} \Rightarrow \mathbf{y} \in \mathcal{N}(\mathbf{A}^{2k}) = \mathcal{N}(\mathbf{A}^k) \Rightarrow \mathbf{x} = \mathbf{0}$ (3 points).
   On the other hand, by Grassmann's formula, we have

$$\begin{aligned}\dim(\mathcal{R}(\mathbf{A}^k) + \mathcal{N}(\mathbf{A}^k)) &= \dim \mathcal{R}(\mathbf{A}^k) + \dim \mathcal{N}(\mathbf{A}^k) - \dim(\mathcal{R}(\mathbf{A}^k) \cap \mathcal{N}(\mathbf{A}^k)) \\ &= \dim \mathcal{R}(\mathbf{A}^k) + \dim \mathcal{N}(\mathbf{A}^k) = n \quad \text{(by rank plus nullity theorem)} \\ &\Longrightarrow \quad \mathcal{R}(\mathbf{A}^k) + \mathcal{N}(\mathbf{A}^k) = \mathbb{R}^n. \quad \text{(4 points)}\end{aligned}$$

Problem 3. (Flops Counting, Complexity (15 points)

1) Recall that for scalars $a, x, y \in \mathbb{R}$, this is a 2-flop operation.

```
y = y + a*x;
```

Complete the following table of flops required by the common operations. Briefly explain your answer. (6 points)

Operation	Dimension	Flops
$\alpha = \mathbf{x}^T\mathbf{y}$	$\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$	$2n$
$\mathbf{y} = \mathbf{y} + \alpha\mathbf{x}$	$\alpha \in \mathbb{R}$, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$	$2n$
$\mathbf{y} = \mathbf{y} + \mathbf{A}\mathbf{x}$	$\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{y} \in \mathbb{R}^m$	__
$\mathbf{A} = \mathbf{A} + \mathbf{y}\mathbf{x}^T$	$\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{y} \in \mathbb{R}^m$	__
$\mathbf{C} = \mathbf{C} + \mathbf{A}\mathbf{B}$	$\mathbf{A} \in \mathbb{R}^{m\times r}$, $\mathbf{B} \in \mathbb{R}^{r\times n}$, $\mathbf{C} \in \mathbb{R}^{m\times n}$	__

2) Let $\mathbf{H} \in \mathbb{R}^{n\times n}$ be such that each of its entry is given by

$$h_{ij} = \sum_{p=1}^{n}\sum_{q=1}^{n} a_{ip}b_{pq}c_{qj}.$$

Using this formula for each h_{ij}, then it requires $\mathcal{O}(n^4)$ flops to set up $\mathbf{H}$. Design a procedure to compute $\mathbf{H}$ that only needs $\mathcal{O}(n^3)$ operations. (3 points)

3) Use the same methodology as in 2) to develop an $\mathcal{O}(n^3)$ procedure for computing $\mathbf{H} \in \mathbb{R}^{n\times n}$ defined by

$$h_{ij} = \sum_{k_1=1}^{n}\sum_{k_2=1}^{n}\sum_{k_3=1}^{n} a_{k_1 i}b_{k_1 i}c_{k_2 k_1}d_{k_2 k_3}b_{k_2 k_3}e_{k_3 j}.$$

(Hint: Transposes and pointwise products, i.e., $\mathbf{X} \cdot \mathbf{Y} = \mathbf{Z}$ such that $z_{ij} = x_{ij}y_{ij}$, are involved.) (6 points)

Solution:

1) If $\mathbf{A} \in \mathbb{R}^{m\times r}$, $\mathbf{B} \in \mathbb{R}^{r\times n}$, and $\mathbf{C} \in \mathbb{R}^{m\times n}$ are give, then the following algorithm overwrites $\mathbf{C}$ with $\mathbf{C}+\mathbf{A}\mathbf{B}$.

```
for i = 1:m
    for j = 1:n
        for k = 1:r
            C(i,j) = C(i,j) + A(i,k)*B(k,j);
        end
    end
end
```

The algorithm requires $2mnr$ flops. The table can be completed by special cases of this matrix multiplication algorithm, with different settings of m, n, r. (3 points for correct answers, 3 points for explanation)

2) Note that

$$h_{ij} = \sum_{p=1}^{n} a_{ip}\left(\sum_{q=1}^{n} b_{pq}c_{qj}\right) = \sum_{p=1}^{n} x_{ip}m_{pj},$$

Operation	Dimension	Flops
$\alpha = \mathbf{x}^T\mathbf{y}$	$\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$	$2n$
$\mathbf{y} = \mathbf{y} + \alpha\mathbf{x}$	$\alpha \in \mathbb{R}$, $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$	$2n$
$\mathbf{y} = \mathbf{y} + \mathbf{A}\mathbf{x}$	$\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{y} \in \mathbb{R}^m$	$\underline{2mn}$
$\mathbf{A} = \mathbf{A} + \mathbf{y}\mathbf{x}^T$	$\mathbf{A} \in \mathbb{R}^{m\times n}$, $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{y} \in \mathbb{R}^m$	$\underline{2mn}$
$\mathbf{C} = \mathbf{C} + \mathbf{A}\mathbf{B}$	$\mathbf{A} \in \mathbb{R}^{m\times r}$, $\mathbf{B} \in \mathbb{R}^{r\times n}$, $\mathbf{C} \in \mathbb{R}^{m\times n}$	$\underline{2mnr}$

where $\mathbf{M} = \mathbf{B}\mathbf{C}$. Thus $\mathbf{H} = \mathbf{A}\mathbf{M} = \mathbf{A}\mathbf{B}\mathbf{C}$ and only require $\mathcal{O}(n^3)$ operations. (3 points)

3) Note that

$$h_{ij} = \sum_{k_2=1}^{n} \left(\sum_{k_1=1}^{n} c_{k_2 k_1} (a_{k_1 i} b_{k_1 i}) \right) \left(\sum_{k_3=1}^{n} (d_{k_2 k_3} b_{k_2 k_3}) e_{k_3 j} \right) = \sum_{k_2=1}^{n} [\mathbf{X}]_{k_2 i} [\mathbf{Y}]_{k_2 j},$$

where $\mathbf{X} = \mathbf{C}(\mathbf{A} \cdot \mathbf{B})$ and $\mathbf{Y} = (\mathbf{D} \cdot \mathbf{B})\mathbf{E}$. Thus $\mathbf{H} = \mathbf{X}^T\mathbf{Y}$ and each of these computation steps requires no more than $\mathcal{O}(n^3)$ operations. (6 points)

Problem 4. (Norms) (15 points)

For any $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{x} \in \mathbb{R}^n$, prove the following arguments:

1) Prove that: $\frac{1}{\sqrt{n}}||\mathbf{A}||_\infty \le ||\mathbf{A}||_2 \le \sqrt{m}||\mathbf{A}||_\infty$, (7.5 points)
2) Prove that: $\frac{1}{\sqrt{m}}||\mathbf{A}||_1 \le ||\mathbf{A}||_2 \le \sqrt{n}||\mathbf{A}||_1$, (7.5 points)

Solution:

With the theorems of vector norm:

$$||\mathbf{x}||_2 \le ||\mathbf{x}||_1 \le \sqrt{n}||\mathbf{x}||_2, \tag{1}$$

$$||\mathbf{x}||_\infty \le ||\mathbf{x}||_2 \le \sqrt{n}||\mathbf{x}||_\infty, \tag{2}$$

$$||\mathbf{x}||_\infty \le ||\mathbf{x}||_1 \le n||\mathbf{x}||_\infty. \tag{3}$$

$$\tag{4}$$

(Mentioning the theorems above: 3 points; or 1.5 points for 1) and 2) separately)

1) Left: For any $\mathbf{x} \in \mathbb{R}^n$, We have

$$||\mathbf{Ax}||_\infty \le ||\mathbf{Ax}||_2 \le ||\mathbf{A}||_2||\mathbf{x}||_2 \le \sqrt{n}||\mathbf{A}||_2||\mathbf{x}||_\infty. \tag{5}$$

   Hence, $\frac{1}{\sqrt{n}}\frac{||\mathbf{Ax}||_\infty}{||\mathbf{x}||_\infty} \le ||\mathbf{Ax}||_2$.

   Since we can arbitrarily choose $\mathbf{x}$, we can have that $\frac{1}{\sqrt{n}}||\mathbf{A}||_\infty \le ||\mathbf{A}||_2$.

   (3 points)

   Right:

$$||\mathbf{Ax}||_2 \le \sqrt{m}||\mathbf{Ax}||_\infty \le \sqrt{m}||\mathbf{A}||_\infty||\mathbf{x}||_\infty \le \sqrt{m}||\mathbf{A}||_\infty||\mathbf{x}||_2. \tag{6}$$

   In the same way, we can have that $||\mathbf{A}||_2 \le \sqrt{m}||\mathbf{A}||_\infty$.

   (3 points)

2) Left:

$$||\mathbf{Ax}||_1 \le \sqrt{m}||\mathbf{Ax}||_2 \le \sqrt{m}||\mathbf{A}||_2||\mathbf{x}||_2 \le \sqrt{m}||\mathbf{A}||_2||\mathbf{x}||_1. \tag{7}$$

   We can have that $\frac{1}{\sqrt{m}}||\mathbf{A}||_1 \le ||\mathbf{A}||_2$.

   (3 points)

   Right:

$$||\mathbf{Ax}||_2 \le ||\mathbf{Ax}||_1 \le ||\mathbf{A}||_1||\mathbf{x}||_1 \le \sqrt{n}||\mathbf{A}||_1||\mathbf{x}||_2. \tag{8}$$

   We can have that $||\mathbf{A}||_2 \le \sqrt{n}||\mathbf{A}||_1$.

   (3 points)

Problem 5. (LU Decomposition) (15 points)

Consider $\mathbf{A} = \begin{bmatrix} 2 & 2 & 1 & 1 \\ 1 & \xi & 3 & -2 \\ 3 & 9 & \xi+6 & -10 \\ 0 & 10 & -5 & 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 5 \\ 2 \\ 3 \\ 1 \end{bmatrix}$.

1) Use two different methods to determine the condition on the value of ξ such that $\mathbf{A}$ always has an LU decomposition. (6 points)
2) With the range of ξ you find in 1), determine the further restriction on the value of ξ such that the LU decomposition of $\mathbf{A}$ is unique. (3 points)
3) Let $\xi = -2$. Use LU decomposition to solve $\mathbf{Ax} = \mathbf{b}$. (Hint: You can use forward substitution and back substitution learnt from class.) (6 points)

Solution:

1) a) method 1: compute the LU decomposition of $\mathbf{A}$

The existence of the LU decomposition of a square matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ means that there exists Gaussian transformation matrices $\mathbf{M}_1, \mathbf{M}_2, \mathbf{M}_3$ such that $\mathbf{M}_1\mathbf{M}_2\mathbf{M}_3\mathbf{A} = \mathbf{U}$. Suppose the given matrix $\mathbf{A}$ has an LU decomposition, applying a Gaussian transformation to $\mathbf{A}$ obtains

$$\mathbf{M}_1\mathbf{A} = \begin{bmatrix} 2 & 2 & 1 & 1 \\ 0 & \xi-1 & \frac{5}{2} & -\frac{5}{2} \\ 0 & 6 & \frac{2\xi+9}{2} & -\frac{23}{2} \\ 0 & 10 & -5 & 0 \end{bmatrix} \tag{9}$$

, which implies $\xi \neq 1$. (1 points)

With $\xi \neq 1$, applying a Gaussian transformation to $\mathbf{M}_1\mathbf{A}$ obtains

$$\mathbf{M}_1\mathbf{M}_2\mathbf{A} = \begin{bmatrix} 2 & 2 & 1 & 1 \\ 0 & \xi-1 & \frac{5}{2} & -\frac{5}{2} \\ 0 & 0 & \frac{2\xi^2+7\xi-39}{2\xi-2} & \frac{-23\xi+53}{2\xi-2} \\ 0 & 0 & \frac{-5\xi-20}{\xi-1} & \frac{25}{\xi-1} \end{bmatrix}, \tag{10}$$

which implies that $2\xi^2 + 7\xi - 39 = (\xi-3)(2\xi+13) \neq 0$ for upper triangularizing. (1 points)

Consequently, when $\xi \neq 1, 3, -\frac{13}{2}$, there always exists an LU decomposition.

(1 points)

b) method 2: using the theorem in class

The matrix $\mathbf{A}$ has an LU decomposition if every leading principle submatrix $\mathbf{A}(1:k, 1:k)$ satisfies

$$\det(\mathbf{A}(1:k, 1:k)) \neq 0 \tag{11}$$

for $k = 1, 2, 3$.

$$\det(\mathbf{A}(1:1,1:1)) = 2 \neq 0, \tag{12}$$

$$\det(\mathbf{A}(1:2,1:2)) = 2\xi - 2 \neq 0, \longrightarrow \xi \neq 1; (1 \text{ points}) \tag{13}$$

$$\det(\mathbf{A}(1:3,1:3)) = 2\xi^2 + 7\xi - 39 = (\xi - 3)(2\xi + 13) \neq 0. \longrightarrow \xi \neq 3; \xi \neq -\frac{13}{2}. (1 \text{ points}) \tag{14}$$

$$\tag{15}$$

Hence, the range of ξ is $\xi \neq 1, 3, -\frac{13}{2}$. (1 points)

2) The LU decomposition is unique if $\mathbf{A}$ is nonsingular.

$\det(\mathbf{A}) = -65\xi - 85 \neq 0 \Longrightarrow \xi \neq \frac{17}{13}$.

(3 points)

3) Let $\xi = -2$,

$$\mathbf{A} = \begin{bmatrix} 2 & 2 & 1 & 1 \\ 1 & -2 & 3 & -2 \\ 3 & 9 & 4 & -10 \\ 0 & 10 & -5 & 0 \end{bmatrix}. \tag{16}$$

Its LU decomposition is

$$\mathbf{L} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2} & 1 & 0 & 0 \\ \frac{3}{2} & -2 & 1 & 0 \\ 0 & -\frac{10}{3} & \frac{4}{9} & 1 \end{bmatrix}, (1 \text{ points}) \tag{17}$$

$$\mathbf{U} = \begin{bmatrix} 2 & 2 & 1 & 1 \\ 0 & -3 & \frac{5}{2} & -\frac{5}{2} \\ 0 & 0 & \frac{15}{2} & -\frac{33}{2} \\ 0 & 0 & 0 & -1 \end{bmatrix}. (1 \text{ points}) \tag{18}$$

a) solve $\mathbf{Lz} = \mathbf{b}$ for $\mathbf{z}$

$$\mathbf{z}_1 = \frac{\mathbf{b}_1}{\mathbf{L}_{11}} = \frac{5}{1} = 5 \tag{19}$$

$$\mathbf{z}_2 = \frac{\mathbf{b}_2 - \mathbf{L}_{21}\mathbf{z}_1}{\mathbf{L}_{22}} = \frac{2 - \frac{5}{2}}{1} = -\frac{1}{2} \tag{20}$$

$$\mathbf{z}_3 = \frac{\mathbf{b}_3 - \mathbf{L}_{31}\mathbf{z}_1 - \mathbf{L}_{32}\mathbf{z}_2}{\mathbf{L}_{33}} = \frac{3 - \frac{3}{2} \times 5 - (-2) \times (-\frac{1}{2})}{1} = -\frac{11}{2} \tag{21}$$

$$\mathbf{z}_4 = \frac{\mathbf{b}_4 - \mathbf{L}_{41}\mathbf{z}_1 - \mathbf{L}_{42}\mathbf{z}_2 - \mathbf{L}_{43}\mathbf{z}_3}{\mathbf{L}_{44}} = \frac{1 - 0 - (-\frac{10}{3}) \times (-\frac{1}{2}) - \frac{4}{9} \times (-\frac{11}{2})}{1} = \frac{16}{9} \tag{22}$$

(1 points)

$\mathbf{z} = \begin{bmatrix} 5 \\ -\frac{1}{2} \\ -\frac{11}{2} \\ \frac{16}{9} \end{bmatrix}$. (1 points)

b) solve $\mathbf{U}\mathbf{x} = \mathbf{z}$ for $\mathbf{x}$

$$\mathbf{x}_4 = \frac{\mathbf{z}_4}{\mathbf{U}_{44}} = \frac{\frac{16}{9}}{-1} = -\frac{16}{9} \tag{23}$$

$$\mathbf{x}_3 = \frac{\mathbf{z}_3 - \mathbf{U}_{34}\mathbf{x}_4}{\mathbf{U}_{33}} = \frac{-\frac{11}{2} - (-\frac{33}{2}) \times (-\frac{16}{9})}{\frac{15}{2}} = -\frac{209}{45} \tag{24}$$

$$\mathbf{x}_2 = \frac{\mathbf{z}_2 - \mathbf{U}_{23}\mathbf{x}_3 - \mathbf{U}_{24}\mathbf{x}_4}{\mathbf{U}_{22}} = \frac{-\frac{1}{2} - \frac{5}{2} \times (-\frac{209}{45}) - (-\frac{5}{2}) \times (-\frac{16}{9})}{-3} = -\frac{20}{9} \tag{25}$$

$$\mathbf{x}_1 = \frac{\mathbf{z}_1 - \mathbf{U}_{12}\mathbf{x}_2 - \mathbf{U}_{13}\mathbf{x}_3 - \mathbf{U}_{14}\mathbf{x}_4}{\mathbf{U}_{11}} = \frac{5 - 2 \times (-\frac{20}{9}) - (-\frac{209}{45}) - (-\frac{16}{9})}{2} = \frac{119}{15} \tag{26}$$

(1 points) Hence, $\mathbf{x} = \begin{bmatrix} \frac{119}{15} \\ -\frac{20}{9} \\ -\frac{209}{45} \\ -\frac{16}{9} \end{bmatrix}$.

(1 points)

Problem 6. (block Gaussian elimination) (20 points)

In this exercise, we extend the idea of Gaussian elimination with block matrix operations. Let $\mathbf{A}$ be a nonsingular matrix, whose leading principle submatrices are all nonsingular. Partition $\mathbf{A}$ as follows:

$$\mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}, \tag{27}$$

where the size of $\mathbf{A}_{11}$ is $k \times k$. Since $\mathbf{A}_{11}$ is a leading principle submatrix, it is nonsingular.

1) Show that there is exactly one matrix $\mathbf{M}$ such that

$$\begin{bmatrix} \mathbf{I}_k & \mathbf{0} \\ -\mathbf{M} & \mathbf{I}_{n-k} \end{bmatrix} \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{0} & \widetilde{\mathbf{A}}_{22} \end{bmatrix}. \tag{28}$$

   (4 points)

   In this equation we place no restriction on the form of $\widetilde{\mathbf{A}}_{22}$. The point is that we seek a transformation that makes the $(2,1)-$block zero. This is a block Gaussian elimination operation; $\mathbf{M}$ is a block multiplier.

   Show that the unique $\mathbf{M}$ that works is given by $\mathbf{M} = \mathbf{A}_{21}\mathbf{A}_{11}^{-1}$, and this implies that

$$\widetilde{\mathbf{A}}_{22} = \mathbf{A}_{22} - \mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12}. \tag{29}$$

   Hence, the matrix $\widetilde{\mathbf{A}}_{22}$ is called the *Schur complement* of $\mathbf{A}_{11}$ in $\mathbf{A}$.

   (4 points)

2) Show that

$$\begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{I}_k & \mathbf{0} \\ \mathbf{M} & \mathbf{I}_{n-k} \end{bmatrix} \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{0} & \widetilde{\mathbf{A}}_{22} \end{bmatrix}. \tag{30}$$

   This is a block LU decomposition. (4 points)

3) We know that the leading principle submatrices of $\mathbf{A}_{11}$ are all nonsingular. Prove that $\widetilde{\mathbf{A}}_{22}$ is nonsigular. More generally, prove that all of the leading principle submatrices of $\widetilde{\mathbf{A}}_{22}$ are nonsigular.

   (4 points)

4) Prove that the Schur complement $\widetilde{\mathbf{A}}_{22}$ is symmetric if $\mathbf{A}$ is.

   (4 points)

Solution:

1) Assume that there are at least two $\mathbf{M}$ that will satisfy the given expression. What is important of this expression is that the block matrices $\mathbf{A}_{11}$ and $\mathbf{A}_{12}$ don't change while the transformation introduces zeros below the block matrix $\mathbf{A}_{11}$. As specified in the problem, the matrix $\widetilde{\mathbf{A}}_{22}$ in each case can be different. This means we will assume that there exists matrices $\mathbf{M}_1$ and $\mathbf{M}_2$ such that

$$\begin{bmatrix} \mathbf{I}_k & \mathbf{0} \\ -\mathbf{M}_1 & \mathbf{I}_{n-k} \end{bmatrix} \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{0} & \widetilde{\mathbf{A}}_{22}^{(1)} \end{bmatrix}, \text{(2 points)} \tag{31}$$

where we have indicated that $\widetilde{\mathbf{A}}_{22}$ may depend on the "M" matrix by providing it with a subscript. For the matrix $\mathbf{M}_2$ we have a similar expression. Multiplying on the left by the inverse of this block matrix

$$\begin{bmatrix} \mathbf{I}_k & \mathbf{0} \\ -\mathbf{M}_1 & \mathbf{I}_{n-k} \end{bmatrix} \tag{32}$$

which is

$$\begin{bmatrix} \mathbf{I}_k & \mathbf{0} \\ \mathbf{M}_1 & \mathbf{I}_{n-k} \end{bmatrix} \tag{33}$$

, gives that

$$\begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{I}_k & \mathbf{0} \\ \mathbf{M}_1 & \mathbf{I}_{n-k} \end{bmatrix} \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{0} & \widetilde{\mathbf{A}}_{22}^{(1)} \end{bmatrix} = \begin{bmatrix} \mathbf{I}_k & \mathbf{0} \\ \mathbf{M}_2 & \mathbf{I}_{n-k} \end{bmatrix} \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{0} & \widetilde{\mathbf{A}}_{22}^{(2)} \end{bmatrix}. \tag{34}$$

Equating the (2,1) component of the block multiplication above gives $\mathbf{M}_1\mathbf{A}_{11} = \mathbf{M}_2\mathbf{A}_{11}$, which implies that $\mathbf{M}_1 = \mathbf{M}_2$, since $\mathbf{A}_{11}$ is nonsingular. (1 points) This shows the uniqueness of this block Gaussian factorization. (1 points)

Returning to a nonsingular $\mathbf{M}$, by multiplying the given factorization out we have

$$\begin{bmatrix} \mathbf{I}_k & \mathbf{0} \\ -\mathbf{M} & \mathbf{I}_{n-k} \end{bmatrix} \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ -\mathbf{M}\mathbf{A}_{11} + \mathbf{A}_{21} & -\mathbf{M}\mathbf{A}_{12} + \mathbf{A}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{0} & \widetilde{\mathbf{A}}_{22} \end{bmatrix}, \tag{35}$$

so equating the $(2,1)-$block component of the above expression we see that $-\mathbf{M}\mathbf{A}_{11} + \mathbf{A}_{21} = \mathbf{0}$, or $\mathbf{M} = \mathbf{A}_{21}\mathbf{A}_{11}^{-1}$.(2 points) In the same way equating the $(2,2)-$block components of the above gives that

$$\widetilde{\mathbf{A}}_{22} = -\mathbf{M}\mathbf{A}_{12} + \mathbf{A}_{22} = -\mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12} + \mathbf{A}_{22}, \text{(2 points)} \tag{36}$$

which is the Schur complement of $\mathbf{A}_{11}$ in $\mathbf{A}$.

2) By multiplying on the left by the block matrix and its inverse, we have

$$\begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{I}_k & \mathbf{0} \\ \mathbf{M} & \mathbf{I}_{n-k} \end{bmatrix} \begin{bmatrix} \mathbf{I}_k & \mathbf{0} \\ -\mathbf{M} & \mathbf{I}_{n-k} \end{bmatrix} \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} = \begin{bmatrix} \mathbf{I}_k & \mathbf{0} \\ \mathbf{M} & \mathbf{I}_{n-k} \end{bmatrix} \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{0} & \widetilde{\mathbf{A}}_{22} \end{bmatrix}. \tag{37}$$

(4 points)

3) Taking the determinant of the above expression gives that

$$|\mathbf{A}| = \left|\begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{M} & \mathbf{I} \end{bmatrix}\right| \left|\begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{0} & \widetilde{\mathbf{A}}_{22} \end{bmatrix}\right| = 1|\mathbf{A}_{11}||\widetilde{\mathbf{A}}_{22}| \neq 0. \tag{38}$$

(2 points)

So $|\widetilde{\mathbf{A}}_{22}| \neq 0$ and therefore $\widetilde{\mathbf{A}}_{22}$ is nonsingular. (2 points)

4) The Schur complement is given by $\widetilde{\mathbf{A}}_{22} = -\mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12} + \mathbf{A}_{22}$ and involves the submatrices in $\mathbf{A}$. To determine properties of these submatrices consider the transpose of $\mathbf{A}$ given by

$$\mathbf{A}^T = \begin{bmatrix} \mathbf{A}_{11}^T & \mathbf{A}_{21}^T \\ \mathbf{A}_{12}^T & \mathbf{A}_{22}^T \end{bmatrix} = \mathbf{A} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} \tag{39}$$

which gives the following:

$$\mathbf{A}_{11}^T = \mathbf{A}_{11} \tag{40}$$

$$\mathbf{A}_{21}^T = \mathbf{A}_{12} \tag{41}$$

$$\mathbf{A}_{12}^T = \mathbf{A}_{21} \tag{42}$$

$$\mathbf{A}_{22}^T = \mathbf{A}_{22}. \tag{43}$$

(2 points) With these components we can compute the transpose of the Schur complement given by

$$\widetilde{\mathbf{A}}_{22}^T = -\mathbf{A}_{12}^T(\mathbf{A}_{11}^{-1})^T\mathbf{A}_{21}^T + \mathbf{A}_{22}^T = -\mathbf{A}_{21}\mathbf{A}_{11}^{-1}\mathbf{A}_{12} + \mathbf{A}_{22} = \widetilde{\mathbf{A}}_{22}, \tag{44}$$

(2 points) showing that $\widetilde{\mathbf{A}}_{22}$ is symmetric.