SI 140A-02 Probability & Statistics for EECS, Fall 2024
Homework 3

Name:
Student ID:

Due on Oct. 22, 2024, 11:59 UTC+8

Read all the instructions below carefully before you start working on the assignment, and before you make a submission.

- You are required to write down all the major steps towards making your conclusions; otherwise you may obtain limited points of the problem.
- Write your homework in English; otherwise you will get no points of this homework.
- Any form of plagiarism will lead to 0 point of this homework.

Problem 1

A fair die is rolled repeatedly, and a running total is kept (which is, at each time, the total of all the rolls up until that time). Let p_n be the probability that the running total is ever exactly n (assume the die will always be rolled enough times so that the running total will eventually exceed n, but it may or may not ever equal n).

(a) Write down a recursive equation for p_n (relating p_n to earlier terms p_k in a simple way). Your equation should be true for all positive integers n, so give a definition of p_0 and p_k for $k < 0$ so that the recursive equation is true for small values of n.

(b) Find p_7.

(c) Give an intuitive explanation for the fact that $p_n \to 1/3.5 = 2/7$ as $n \to \infty$.

(a) For any n, it can be rolled through $(n-1)+1$ or $(n-2)+2$ or $(n-3)+3$ ...

so $p_n = \frac{1}{6}(p_{n-1}+p_{n-2}+p_{n-3}+\cdots+p_{n-6})$

It's easy to know $p_1 = \frac{1}{6}$ and $p_1 = \frac{1}{6}(p_0+p_{-1}+p_{-2}+p_{-3}+p_{-4}+p_{-5})$

For practice, 0 can always be rolled without rolling.

$$\Rightarrow \begin{cases} p_0 = 1 \\ p_i = 0, \ i = -1, -2, -3, -4, -5 \end{cases}$$

(b) Since $p_7 = \frac{1}{6}(p_1+p_2+\cdots+p_6)$

and $p_1 = \frac{1}{6}p_0 = \frac{1}{6}$

$p_2 = \frac{1}{6}(p_0+p_1) = \frac{7}{36}$

$p_3 = \frac{1}{6}(p_0+p_1+p_3) = \frac{49}{216}$

...

$p_6 = \frac{1}{6}(p_0+p_1+\cdots+p_5) = \frac{16807}{46656}$

So we have

$$p_7 = \frac{70993}{279936}$$

(c) As $n \to +\infty$, the gap of rolling different numbers is negligible, so the expectation for each is given by

$\frac{1}{6}(1+2+3+4+5+6) = \frac{7}{2}$

So the probability is $\frac{1}{\frac{7}{2}} = \frac{2}{7}$

Problem 2

A message is sent over a noisy channel. The message is a sequence $x_1, x_2, \ldots, x_n$ of n bits ($x_i \in \{0, 1\}$). Since the channel is noisy, there is a chance that any bit might be corrupted, resulting in an error (a 0 becomes a 1 or vice versa). Assume that the error events are independent. Let p be the probability that an individual bit has an error ($0 < p < 1/2$). Let $y_1, y_2, \ldots, y_n$ be the received message (so $y_i = x_i$ if there is no error in that bit, but $y_i = 1 - x_i$ if there is an error there).
To help detect errors, the nth bit is reserved for a parity check: x_n is defined to be 0 if $x_1 + x_2 + \cdots + x_{n-1}$ is even, and 1 if $x_1 + x_2 + \cdots + x_{n-1}$ is odd. When the message is received, the recipient checks whether y_n has the same parity as $y_1 + y_2 + \cdots + y_{n-1}$. If the parity is wrong, the recipient knows that at least one error occurred; otherwise, the recipient assumes that there were no errors.

(a) For $n = 5, p = 0.1$, what is the probability that the received message has errors which go undetected?

(b) For general n and p, write down an expression (as a sum) for the probability that the received message has errors which go undetected.

(c) Give a simplified expression, not involving a sum of a large number of terms, for the probability that the received message has errors which go undetected.

(a) The situation should be the number of error bits is even
If errors in n-1 is even, then x_n should be correct. If errors in n-1 is odd, then x_n should be wrong.

$$P(\text{undetected}) = \binom{5}{4} p^4 (1-p) + \binom{5}{2} p^2 (1-p)^3 = 0.07335$$

(b) According to (a), we have

$$P(\text{undetected}) = \sum_{k \text{ is even}}^{n} \binom{n}{k} p^k (1-p)^{n-k}$$

(c) $$P(\text{undetected}) = \frac{(p+(1-p))^n + (-1)^n (p-(1-p))^n}{2} - (1-p)^n$$

$$= \frac{1+(1-2p)^n}{2} - (1-p)^n$$

Problem 3

In Monty Hall problem, now suppose the car is not placed randomly with equal probability behind the three doors. Instead, the car is behind door one with probability p_1, behind door two with probability p_2, and behind door three with probability p_3. Here $p_1 + p_2 + p_3 = 1$ and $p_1 \geq p_2 \geq p_3 > 0$. You are to choose one of the three doors, after which Monty will open a door he knows to conceal a goat. Monty always chooses randomly with equal probability among his options in those cases where your initial choice is correct. What strategy should you follow?

Initial choice	Car is behind	Monty opens	If switch	If stay
Door 1	$\xrightarrow{p_1}$ Door 1	$\xrightarrow{\frac{1}{2}}$ Door 2	✗	✓
		$\xrightarrow{\frac{1}{2}}$ Door 3	✗	✓
	$\xrightarrow{p_2}$ Door 2	$\xrightarrow{1}$ Door 3	✓	✗
	$\xrightarrow{p_3}$ Door 3	$\xrightarrow{1}$ Door 2	✓	✗

Therefore, $P_1^{switch}(win) = p_2 + p_3$

$$P_1^{stay}(win) = p_1$$

$$P_2^{switch}(win) = p_1 + p_3$$

$$P_2^{stay}(win) = p_2$$

$$P_3^{switch}(win) = p_1 + p_2$$

$$P_3^{stay}(win) = p_3$$

As $p_1 \geq p_2 \geq p_3$, we choose $P_3^{switch}(win)$

Which is: choose door 3 first, then switch to the unopened door.

Problem 4

(a) Consider the following 7 -door version of the Monty Hall problem. There are 7 doors, behind one of which there is a car (which you want), and behind the rest of which there are goats (which you don't want). Initially, all possibilities are equally likely for where the car is. You choose a door. Monty Hall then opens 3 goat doors, and offers you the option of switching to any of the remaining 3 doors.
Assume that Monty Hall knows which door has the car, will always open 3 goat doors and offer the option of switching, and that Monty chooses with equal probabilities from all his choices of which goat doors to open. Should you switch? What is your probability of success if you switch to one of the remaining 3 doors?

(b) Generalize the above to a Monty Hall problem where there are $n \geq 3$ doors, of which Monty opens m goat doors, with $1 \leq m \leq n-2$.

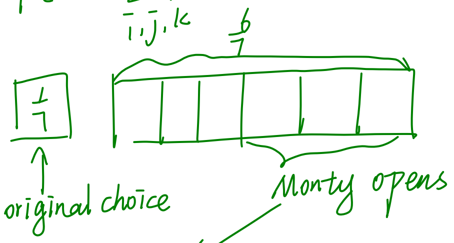
(a) Similar to the last question, if stick to the original strategy.

$P(S) = \frac{1}{7}$

Let $M_{i,j,k}$ be the event that opens Door i,j,k. Then

$P(S) = \sum_{i,j,k} P(S \mid M_{i,j,k})$, $2 \leq i < j < k \leq 7$

So =



Then rest 6 doors share $\frac{6}{7}$

Now 3 doors are open with empty result, then $\frac{6}{7}$ is shared by the left 3 unopened doors, so each door is $\frac{2}{7}$, switch is better.

(b) Similar to (a), after switch a door as the original choice, the rest $n-1$ doors share the left $1-\frac{1}{n}$ winning probability.

If not switch, the winning prob. is still $\frac{1}{n}$.

If switch, after m doors are opened, $(1-\frac{1}{n})$ is shared by the left $n-m-1$ doors which means each unopened door shares $\frac{1-\frac{1}{n}}{n-m-1}$

Obviously, $\frac{1-\frac{1}{n}}{n-m-1} > \frac{1}{n}$, so switch is better.

Problem 5

A/B testing is a form of randomized experiment that is used by many companies to learn about how customers will react to different treatments. For example, a company may want to see how users will respond to a new feature on their website (compared with how users respond to the current version of the website) or compare two different advertisements.
As the name suggests, two different treatments, Treatment A and Treatment B, are being studied. Users arrive one by one, and upon arrival are randomly assigned to one of the two treatments. The trial for each user is classified as "success" (e.g., the user made a purchase) or "failure". The probability that the nth user receives Treatment A is allowed to depend on the outcomes for the previous users. This set-up is known as a *two-armed bandit*.
Many algorithms for how to randomize the treatment assignments have been studied. Here is an especially simple (but fickle) algorithm, called a "stay-with-a-winner" procedure:

(i) Randomly assign the first user to Treatment A or Treatment B, with equal probabilities.

(ii) If the trial for the nth user is a success, stay with the same treatment for the $(n+1)$ st user; otherwise, switch to the other treatment for the $(n+1)$st user.

Let a be the probability of success for Treatment A, and b be the probability of success for Treatment B. Assume that $a \neq b$, but that a and b are unknown (which is why the test is needed). Let p_n be the probability of success on the nth trial and a_n be the probability that Treatment A is assigned on the nth trial (using the above algorithm).

(a) Show that

$$p_n = (a-b)a_n + b, a_{n+1} = (a+b-1)a_n + 1 - b$$

(b) Use the results from (a) to show that p_{n+1} satisfies the following recursive equation:

$$p_{n+1} = (a+b-1)p_n + a + b - 2ab$$

(c) Use the result from (b) to find the long-run probability of success for this algorithm, $\lim_{n\to\infty} p_n$, assuming that this limit exists.

(a) $p_n = P\{n\text{-th trial succeed}\}$

$= P\{n\text{-th} \checkmark \mid A \text{ is } n\text{-th}\}P\{A \text{ is } n\text{-th}\} + P\{n\text{-th} \checkmark \mid B \text{ is } n\text{-th}\}P\{B \text{ is } n\text{-th}\}$

$= a \cdot a_n + b \cdot (1-a_n)$

$= (a-b)a_n + b$

$a_{n+1} = P\{A \text{ is } (n+1)\text{-th}\} = P\{A \text{ is } n\text{-th}\}P\{n\text{-th} \checkmark\} + P\{B \text{ is } n\text{-th}\}P\{n\text{-th} \times\}$

$= a_n \cdot a + (1-a_n)(1-b) = (a+b-1)a_n + 1 - b$

(b) $p_{n+1} = P\{(n+1)\text{-th succeed}\}$

$= P\{(n+1) \checkmark \mid A \text{ is } (n+1)\text{-th}\} \cdot P\{A \text{ is } (n+1)\text{-th}\}$

$+ P\{(n+1)\text{-th succeed} \mid B \text{ is } (n+1)\text{-th}\} \cdot P\{B \text{ is } (n+1)\}$

$= a \cdot a_{n+1} + b(1-a_{n+1})$

(c) set $\lim p_n = p$, we have

$p = (a+b-1)p + a + b - 2ab \Rightarrow p = \dfrac{a+b-2ab}{2-a-b}$

Problem 6

(**Optional Challenging Problem I**) By LOTP for problems with recursive structure, we generate many different equations.

(a) Explain the principle behind the method of characteristic equation.

(b) Solve the following difference equation:

$$p \cdot f_{i+1} - f_i + q \cdot f_{i-1} = -1, 1 \leq i \leq N-1$$

where $0 < p < 1, q = 1-p, N$ is a constant, $f_0 = 0, f_N = 0$.

(c) Solve the following difference equation:

$$f_{i+1} = b \cdot f_i + a \cdot f_{i-1} + h, i \geq 1$$

where h is a constant.

(d) Solve the following difference equation:

$$f_{i+1} = b \cdot f_i + a \cdot f_{i-1} + g(i), i \geq 1$$

where $g(i)$ is a function of i.

Problem 7

Optional Challenging Problem II

(a) An event E_{n+1} is mutually independent of the set of events $E_1, \ldots, E_n$ if for any subset $I \subseteq [1, n]$

$$P\left(E_{n+1} \mid \bigcap_{j \in I} E_j\right) = P\left(E_{n+1}\right)$$

(b) A dependence graph for the set of events $E_1, \ldots, E_n$ is a graph $G = (V, E)$ such that $V = \{1, \ldots, n\}$, and for $i = 1, \ldots, n$, event E_i is mutually independent of the events $\{E_j \mid (i, j) \notin E\}$.

(c) Assume there exist real numbers $x_1, \ldots, x_n \in [0, 1]$ such that, for any $i (1 \leq i \leq n)$,

$$P\left(E_i\right) \leq x_i \prod_{j:(i,j) \in E} \left(1 - x_j\right)$$

Then show the following inequality hold:

$$P\left(\bigcap_{i=1}^{n} E_i^c\right) \geq \prod_{i=1}^{n} \left(1 - x_i\right)$$

(d) Find the possible applications of the above inequality in the field of EECS.