Problem 2. (QR decomposition for ill-conditioned matrices.)

Consider the Vandemonde matrix

$$\mathbf{V} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ z_1 & z_2 & \cdots & z_n \\ z_1^2 & z_2^2 & \cdots & z_n^2 \\ \vdots & \vdots & \ddots & \vdots \\ z_1^{m-1} & z_2^{m-1} & \cdots & z_n^{m-1} \end{bmatrix}$$

which can be ill-conditioned sometimes. When $z_i = 1 + 0.001 * (i - 1)$, its condition number is large, and the increase in the dimension n causes significant numerical errors in solving linear system.

In this problem, you are encouraged to explore the decomposition stability of the Gram-Schmidt (CGS), Modified Gram-Schmidt (MGS), Householder Reflections, and Givens Rotations methods against ill-conditioned matrices in this case. For $m = 20$, $n = 2, 3, 4, \dots, 20$, you are required to conduct the above four methods for obtaining the $\mathbf{QR}$ decomposition of $\mathbf{V}$ and plot two figures:

(1) the relationship between the decomposition accuracy $\|\mathbf{QR} - \mathbf{V}\|_2$ of the four methods and the dimension n;

(2) the relationship between the orthogonality error $\|\mathbf{Q^T Q} - \mathbf{I}\|_2$ of the four methods and the dimension n. (For Gram-Schmidt and Modified Gram-Schmidt methods, Q is an $m \times n$ matrix. The orthogonality error is $\|\mathbf{Q^T Q} - \mathbf{I}_n\|_2$. For Householder Reflections and Givens Rotations methods, Q is an m-dimensional square matrix. The orthogonality error is $\|\mathbf{Q^T Q} - \mathbf{I}_m\|_2$.)