SI252 Reinforcement Learning 2025/02/26

Homework 1

Professor: Ziyu Shao *Due: 2020/03/09 11:59am*

1. Let X, Y be jointly Gaussian random variables. Show the following equality:

$$E[Y|X] = L[Y|X] = E(Y) + \frac{\text{Cov}(X,Y)}{\text{Var}(X)}(X - E(X)).$$

2. We wish to estimate the probability of landing heads, denoted by θ, of a biased coin. We model θ as the value of a random variable Θ with a known prior PDF $f_\Theta \sim \text{Unif}(0,1)$. We consider n independent tosses and let X be the number of heads observed. Find the MMSE $E[\Theta|X]$ and the LLSE $L[\Theta|X]$.

3. Sampling from the discrete distribution.
(a) Given n positive real number $a_1, \ldots, a_n$, where $\sum_{j=1}^{n} a_j = 1$; and n i.i.d. random variables $X_1, \ldots, X_n \sim\sim \text{Gumbel}(0,1)$. show the following equality:

$$P\left(\log a_i + X_i = \max_{j \in \{1,\ldots,n\}} (\log a_j + X_j)\right) = a_i.$$

(b) Illustrate how the above result leads to new sampling methods for the discrete distribution.

4. Adopt the Acceptance-Rejection method to estimate the value of π, then evaluate the performance of Monte Carlo algorithms with finite number of samples.

5. Sampling from probability distributions. Show histograms and compare them to corresponding PDFs.
(a) Sampling from the standard Normal distribution with both the Box-Muller method and the Acceptance-Rejection method. Discuss the pros and cons of both methods.
(b) Sampling from the distribution with the following pdf:

$$f(x) \propto \exp\left(-\frac{1}{2}x^2\right)\left(\sin^2(6x) + 3\cos^2(x)\sin^2(4x) + 1\right).$$

6. Given a random variable $X \sim N(0,1)$, evaluate the tail probability $c = P(X > 8)$ by Monte Carlo methods with & without importance sampling. Discuss the pros and cons of importance sampling.

7. Generate uniform distributions over the following geometric objects:
(a) Elliptic ($a = 2, b = 1$):

$$E_2(a, b) = \left\{ (x, y) \in \mathbb{R}^2 : \left(\frac{x}{a}\right)^2 + \left(\frac{y}{b}\right)^2 \leq 1 \right\}.$$

(b) Sphere ($r = 1$):

$$S_2(r) = \left\{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 = r^2 \right\}.$$

(c) Ball ($r = 1$):

$$B_3(r) = \left\{ (x, y, z) \in \mathbb{R}^3 : x^2 + y^2 + z^2 \leq r^2 \right\}.$$

(d) Torus ($r_0 = 2, r = 1$):

$$T_2(r_0, r) = \left\{ (x, y, z) \in \mathbb{R}^3 : \left(r_0 - \sqrt{x^2 + y^2}\right)^2 + z^2 = r^2 \right\}.$$

8. The Curse and Blessing of Dimensionality. Denote $\boldsymbol{x} = (x_1, \ldots, x_d)$.

(a) The d-dimensional hyperball of radius r is denoted as

$$B_d(r) = \left\{ \boldsymbol{x} \in \mathbb{R}^d : \sum_{i=1}^{d} x_i^2 \leq r^2 \right\}$$

Find the volume of $B_d(r)$ and plot a figure to show how such volume changes with d when $r = 1$.

(b) The d-dimensional hypersphere of radius r is denoted as

$$S_{d-1}(r) = \left\{ \boldsymbol{x} \in \mathbb{R}^d : \sum_{i=1}^{d} x_i^2 = r^2 \right\}$$

When $d \gg 1$, adopt concentration inequalities to show almost all the volume of the high-dimensional hypersphere lies near its equator.

(c) The d-dimensional hypercube of radius r is denoted as

$$C_d(r) = \left\{ \boldsymbol{x} \in \mathbb{R}^d : -r \leq x_i \leq r \right\}$$

When $d \gg 1$, adopt concentration inequalities to show almost all the volume of the high-dimensional cube is located in its corners.