CS243: Introduction to Algorithmic Game Theory

Cost Sharing and Public Goods (Dengji ZHAO)

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Recap: Coalitional/Cooperative Game

- A set of agents N.
- Each subset of agents (coalition) $S \subseteq N$ cooperate together can generate some value $v(S) \in \mathbb{R}$. Assume $v(\emptyset) = 0$. N is called grand coalition. $v : 2^N \to \mathbb{R}$ is called the characteristic function of the game.
- The possible outcomes of the game is defined by $V(S) = \{x \in \mathbb{R}^S : \sum_{i \in S} x_i \leq v(S)\}$.

Recap: Core and Shapley Value

Definition (Core)

The core of the coalitional game (N, v) is a set of vectors $x \in \mathbb{R}^N$ such that x is efficient and $\forall_{S \subseteq N} \sum_{i \in S} x_i \geq v(S)$.

Definition (Shapley Value)

Given a coalitional game (N, v), the Shapley value of each player i is:

$$\phi_i(v) = \sum_{S \subseteq N \setminus \{i\}} \frac{|S|!(n - |S| - 1)!}{n!} (v(S \cup \{i\}) - v(S))$$

Recap: Cost Sharing

Definition

A cost sharing game (N, c) is defined by

- a set of n agents N.
- a cost function $c : 2^N \rightarrow \mathbb{R}$ and assume $c(\emptyset) = 0$.

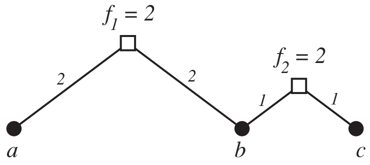

Figure 15.1. An example of the facility location game.

- $c(\{a\}) = 4, c(\{b\}) = 3, c(\{c\}) = 3$
- $c(\{a, b\}) = 6, c(\{b, c\}) = 4, c(\{a, c\}) = 7, c(\{a, b, c\}) = 8$

Public Goods

- A set of agents N.
- These agents have to choose to whether to produce some indivisible, nonexcludable public good. We denote the decision by $g \in \{0, 1\}$.
- There is a commonly known cost c to build the public good.
- Each agent i has a (private) valuation v_i for the public good.

Optimal Decision Making

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- How much will each agent share the cost?
  - Can we apply VCG here?

Application of VCG

Assume that the decision is to build the public good, i.e. $\sum_{i \in N} v_i > c$. What is the payment for i if

- $\sum_{j \neq i} v_j > c$
- $\sum_{j \neq i} v_j \leq c$

Application of VCG

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Assume that the decision is to NOT build the public good, i.e. $\sum_{i \in N} v_i \leq c$. What is the payment for i if

- $\sum_{j \neq i} v_j \leq c$
  - Question: is the payment (1) > 0, (2) $= 0$, (3) < 0?
- $\sum_{j \neq i} v_j > c$
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Question

Can their VCG payments cover the cost of the public good?

IC, IR, Budget Balanced Mechanisms

In the public good setting, a mechanism is budget balanced if

- the total payments of all agents is not less than the cost to build the public good, if the decision is to build the good.
- the total payments of all agents is non-negative, if the decision is to not build the good.

Theorem

When there are only two agents, the only mechanisms that are IC, IR and budget balanced are fixed-price mechanisms. (It does not hold for more than three agents settings) [Tilman Borgers, 2015]

For example: when the public good is built, agent 1 has to pay p_1 and agent 2 has to pay p_2, where $p_1 + p_2 = c$ and p_1 and p_2 are independent of their valuations.

Cost Sharing of Excludable Good Production

- A set of agents N.
- There is a good can be produced with a cost c.
- Each agent has a valuation v_i for sharing the good.
- The good can be shared by a subset of agents.

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Question

How to design an IC, IR and budget balanced mechanism?

One Example [Moulin and Shenker, 2001]

- Find the largest k such that the highest k agents' valuation reports are at lease c/k.
- Charge these k agents c/k and reject all others, i.e. the good is only shared by the k agents.

IC Mechanism for the General Case

- A set of agents N.
- A cost function $c : 2^N \to \mathbb{R}$ and assume $c(\emptyset) = 0$.
- Each agent has a valuation v_i for sharing the good.
- A cost sharing method is a formula α. Given $S \subseteq N, \alpha_i(S)$ means agent i's cost share under coalition S.

The Shapley Value Mechanism

Definition (Cross-monotonic)

Agent i's cost share cannot increase when the set of agents receiving service expands: $S \subseteq T, i \in S \Rightarrow \alpha_i(T) \leq \alpha_i(S)$.

Definition (Shapley Value Cost Sharing Method)

$\alpha_i^*(S) = \sum_{T \subseteq S-i} \frac{|T|!(|S|-|T|-1)!}{|S|!}[c(T \cup i) - c(T)]$ $\forall S$ *and* $i \in S$.

- If c is submodular, then the Shapley Value Method satisfies cross-monotonic. (How to prove?)

The Strategy-proof Mechanism

The mechanism $M(\alpha)$ is denoted by:

- Begin with the largest set S^0 and then iteratively search for a set S^* that can precisely cover the cost.

$$S^0 = N, S^{t+1} = \{i | v_i \geq \alpha_i(S^t)\}$$

- Allocate the cost to the agents in S^*.

Theorem

If α is cross-monotonic then mechanism $M(\alpha)$ is strategy-proof.

Advanced Reading

- Tilman Borgers, *an introduction to the theory of mechanism design*, 2015.
- Moulin H, Shenker S. *Strategyproof sharing of submodular costs: budget balance versus efficiency*. Economic Theory. 2001 Nov 1:511-33.