CS243: Introduction to Algorithmic Game Theory

Facility Location Games (Dengji ZHAO)

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Recap: Social Choice

- A set of n players/voters.
- A set of alternatives A (the candidates).
- Let L be the set of all linear orders on A.
- Each voter i has a preference $\succ_i \in L$, a total order on A (antisymmetric, transitive). $a \succ_i b$ means i prefers a to b.

Definition 9.7 Voter i is a *dictator* in social choice function f if for all $\prec_1, \ldots, \prec_n \in L$, $\forall b \neq a,\ a \succ_i b \Rightarrow f(\prec_1, \ldots, \prec_n) = a$. f is called a *dictatorship* if some i is a dictator in it.

Theorem 9.8 (Gibbard–Satterthwaite) *Let f be an incentive compatible social choice function onto A, where $|A| \geq 3$, then f is a dictatorship.*

Recap: Social Choice

Theorem (Arrow's Theorem)

Every social welfare function over a set of more than 2 candidates ($|A| \geq 3$) that satisfies unanimity and independence of irrelevant alternatives is a dictatorship.

- F satisfies unanimity if for every $\succ \in L$, $F(\succ, \ldots, \succ) = \succ$.
- F satisfies independence of irrelevant alternatives if for every $a, b \in A$, every $\succ_1, \ldots, \succ_n, \succ'_1, \ldots, \succ'_n \in L$, if $\succ = F(\succ_1, \ldots, \succ_n)$ and $\succ' = F(\succ'_1, \ldots, \succ'_n)$, then $a \succ_i b \Leftrightarrow a \succ'_i b$ for all i implies $a \succ b \Leftrightarrow a \succ' b$.
- Voter i is a dictator in F if for all $\succ_1, \ldots, \succ_n \in L$, $F(\succ_1, \ldots, \succ_n) = \succ_i$. F is not a dictatorship if no i is a dictator in F.

Outline

Single-Peaked Preference

The setting:

- There are n players, who decide one point from space $A = [0, 1]$
- Each player i has a single-peaked preference $\succeq_i$ over A, i.e. there exists a point $p_i \in A$ s.t. $\forall x \in A \setminus \{p_i\}$ and $\forall \lambda \in [0, 1)$, $(\lambda x + (1 - \lambda)p_i) \succ_i x$. Let $\mathcal{R}$ denote the class of single-peaked preferences.

Design the Social Choice Rule

A rule $f : \mathcal{R}^n \rightarrow A$ assigns one outcome to any preference profile.

- f is incentive compatible if it is a dominant strategy for each agent to report his preferences truthfully when f is being used to choose a point.

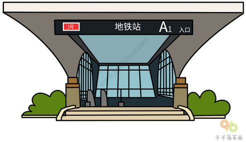
The Median Voter Rule

The median voter rule:

- Consider a fixed set of points $y_1, y_2, \cdots, y_{n-1} \in A$.
- Given any profile of preferences, choose the median of the $2n-1$ points consisting of the n players' peaks and the $n-1$ y points.

Outline

Facility Location Games



- A government wants to locate a facility (e.g. a subway station) on a line to serve people.
- Each agent $i \in N = \{1, \cdots, n\}$ has a private location $x_i \in \mathbb{R}$.
- Each agent has a cost $c_i(y, x_i)$ when the facility is located at y.
- The government asks agents to report their locations and then locates the facility, aiming to optimize some objective of agents' costs.
- Agents may misreport their locations ($x_i \rightarrow x_i'$) to get good outcomes for themselves.

Facility Location Games with Single-peaked preferences

- The cost of agent i with respect to the facility located at y is her distance away from the facility, i.e. $c_i(y, x_i) = dist(y, x_i)$
- A government wants to design a truthful mechanism f which minimizes/approximates social cost (SC), i.e. $SC_f(x') = \sum_i c_i(f(x'), x_i) = \sum_i dist(f(x'), x_i)$, where $x' = (x'_1, \cdots, x'_n)$.
- A government wants to design a truthful mechanism f which minimizes/approximates maximum cost (MC), i.e. $MC_f(x') = max c_i(f(x'), x_i) = max dist(f(x'), x_i)$.

Facility Location Games with Single-peaked preferences

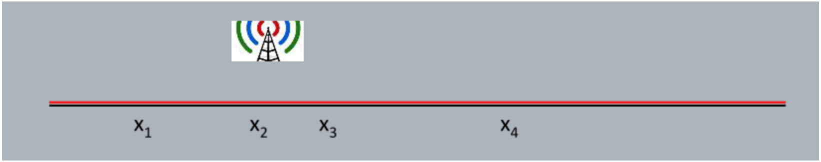
Definition

Truthfulness: A mechanism f is truthful if for every agent $i \in N$, we have $c_i((x_i, x'_{-i}), x_i) \leq c_i((x'_i, x'_{-i}), x_i)$, where $x'_{-i} = (x'_1, \cdots, x'_{i-1}, x'_{i+1}, \cdots, x'_n)$.

Facility Location Games with Single-peaked preferences

Mechanism 1

Locate the facility at the location of the median agent.



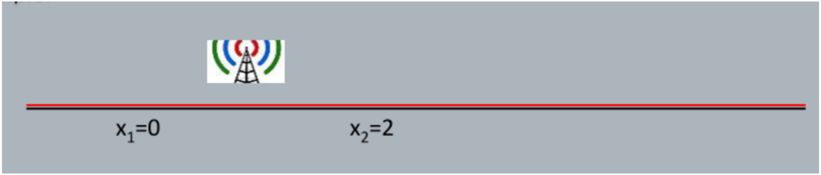
Theorem (Procaccia and Tennenholtz EC'09)

Mechanism 1 is truthful and gives the optimal (minimum) social cost.

Facility Location Games with Single-peaked preferences

Mechanism 2

Locate the facility at the location that minimizes the maximum cost.



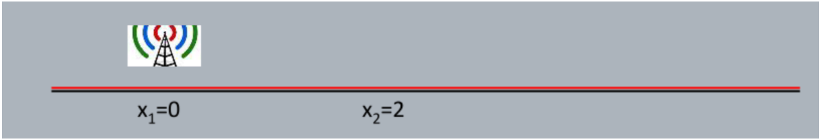
Question

Is the Mechanism 2 truthful?

Facility Location Games with Single-peaked preferences

Mechanism 3

Locate the facility at the location of the first agent.



Question

Is the Mechanism 3 truthful?

Facility Location Games with Single-peaked preferences

Theorem (Procaccia and Tennenholtz EC'09)

Mechanism 3 is truthful and gives 2-approximation for the maximum cost (i.e. the maximum cost generated by Mechanism 3 is at most 2 times of Mechanism 2).

Obnoxious Facility Location Games

Agents want to stay away from the facility, e.g.

- A polluting factory
- A garbage dump site
- A prison

Definition

For any point $x, y \in I = [0, l]$, the distance between them is $dist(x, y) = |x - y|$.

Definition

The utility of agent i is her distance to the facility, i.e. $u(f(x'), x_i) = dist(f(x'), x_i)$.

Obnoxious Facility Location Games

Definition

The obnoxious social welfare of a mechanism f on reported locations x' is defined as the total utilities of n agents: $SW(f, x') = \sum_{i=1}^{n} u(f(x'), x_i)$.

Definition

Let $OPT(x)$ be the optimal social welfare, i.e. $OPT(x) = maxSW(f, x)$. We say a mechanism f has an approximation ratio γ if for all x, $OPT(x) \leq \gamma \cdot SW(f, x)$.

Obnoxious Facility Location Games

Definition

A mechanism f is truthful if it holds that $u(f(x_i, x'_{-i}), x_i) \geq u(f(x'_i, x'_{-i}), x_i)$.

Mechanism 4

Given a reported locations x' on $[0, l]$. Let n_1 be the number of agents located on $[0, \frac{l}{2}]$ and n_2 be the number of agents located on $[\frac{l}{2}, l]$. If $n_1 \geq n_2$, return $f(x') = l$ and otherwise return $f(x') = 0$.

Mechanism 4 is truthful and has an approximation ratio of 3 for the obnoxious facility game.

References

- Single-Peaked Preferences over Policies [AGT Chapter 10.2]
- Facility Location [AGT Chapter 19.4]
- Li Minming. "FACILITY LOCATION GAMES ORIGIN TO RECENT DEVELOPMENT."
- Ariel D. Procaccia and Moshe Tennenholtz. 2009. Approximate mechanism design without money. In Proceedings of the 10th ACM conference on Electronic commerce (EC '09).
- Cheng Y, Yu W, Zhang G. Strategy-proof approximation mechanisms for an obnoxious facility game on networks[J]. Theoretical Computer Science, 2013.