CS243: Introduction to Algorithmic Game Theory

Price of Anarchy (Dengji ZHAO)

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Outline

Definition of PoA

Price of Anarchy for Nash equilibria:

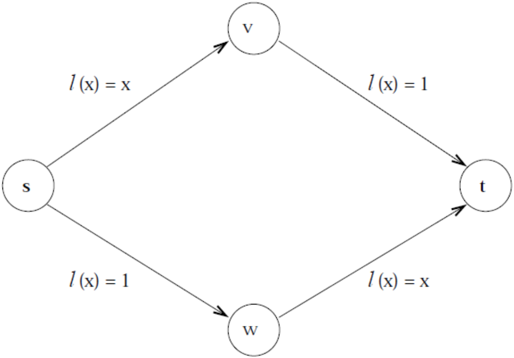
- A *cost-minimization* game Γ with $\mathcal{N} = \{1, \ldots, n\}$ players, we assume the social cost $cost(s)$ is the sum of all players' cost.
- Let Σ be the set of all states of the game Γ. For every state $s \in \Sigma$, let $cost(s) = \sum_{i \in \mathcal{N}} c_i(s)$
- Consider Σ^{PNE} as the set of pure Nash equilibria (PNE) of Γ.
- Price of Anarchy is a ratio:

$$PoA = \frac{\max_{s' \in \Sigma^{PNE}} cost(s')}{\min_{s \in \Sigma} cost(s)}$$

PoA is the worst-case ratio and measures how much the worst PNE costs in comparison to an optimal state of the game.

Example: Braess's Paradox

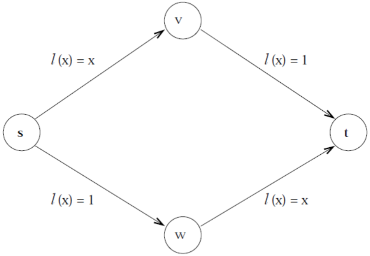
Consider a network with a source s, a destination t, and two hubs v and w. Suppose on each edge there is a function $l(x)$ corresponding the latency (time to travel through) where x is the size of vehicles flow travelling on it ($0 \leq x \leq 1$, we normalize the total flow as 1).



Example: Braess's Paradox

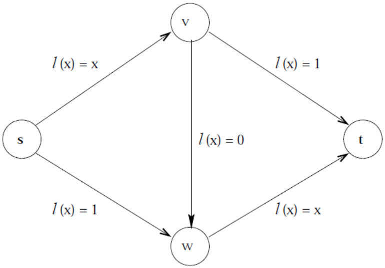
For each vehicle,

- the Nash equilibrium is to choose two path uniformly randomly;
- then the expected latency of the traffic is $1 + \frac{1}{2} = \frac{3}{2}$.



Example: Braess's Paradox

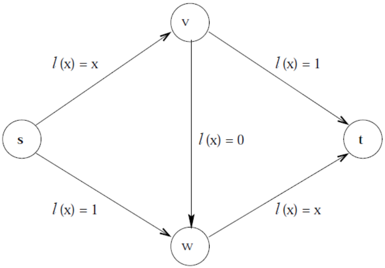
Now we consider to additionally add a new road from v to w with $l(x) = 0$ (a really efficient road).



Example: Braess's Paradox

Now for each vehicle,

- the Nash equilibrium is to choose path $s - v - w - t$;
- then the expected latency of the traffic becomes $2 > \frac{3}{2}$.



Price of Anarchy in Selfish Routing

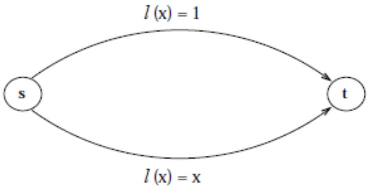
Sometimes it will not achieve the best social welfare or even achieve a worse social welfare when everyone in the game is self-interested. This is called the price of anarchy.

How Bad is Selfish Routing?

Consider the graph on the right. For each vehicle,

- the Nash equilibrium is still to choose the lower path;
- then the expected latency of the traffic is 1.

However, the minimum average-latency is $3/4$ (assign $1/2$ units on lower path).

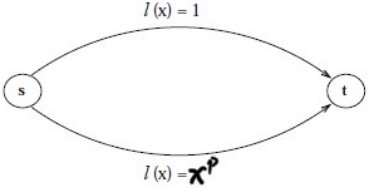


How Bad is Selfish Routing?

It can be much worse! Suppose the latency function on the lower path is now $l(x) = x^p$. Then for every vehicle,

- the Nash equilibrium is still to choose the lower path;
- then the expected latency of the traffic is still 1.

However, the minimum average-latency is now $1 - p(p+1)^{-(p+1)/p}$ (assign $(p+1)^{-1/p}$ units on lower path), which tends to 0 as $p \to \infty$.



How Bad is Selfish Routing?

Theorem

If all the latency functions are linear, then the price of anarchy in worst case is $\frac{4}{3}$. (Tim Roughgarden and Eva Tardos, 2000)

Outline

Definition of Smoothness

A *smooth game* is a cost-minimization game that admits PoA bound of a canonical type.

Definition

A *cost-minimization game* is called (λ, μ)*-smooth* for $\lambda > 0$ and $\mu < 1$ if, for every pair of states $s, s^* \in \Sigma$, we have

$$\sum_{i \in \mathcal{N}} c_i(s_i^*, s_{-i}) \leq \lambda \cdot cost(s^*) + \mu \cdot cost(s)$$

Smoothness directly gives a bound for the PoA.

Theorem

In a (λ, μ)*-smooth game, the PoA for pure Nash equilibria is at most* $\frac{\lambda}{1-\mu}$.

Proof PoA for PNE

Proof.

Let s be the worst PNE and s^* be an optimum solution. Then:

$$\begin{aligned} cost(s) = \sum_{i \in \mathcal{N}} c_i(s) &\leq \sum_{i \in \mathcal{N}} c_i(s_i^*, s_{-i}) \quad (as\ s\ is\ NE) \\ &\leq \lambda \cdot cost(s^*) + \mu \cdot cost(s) \quad (by\ smoothness) \end{aligned}$$

On both sides subtract $\mu \cdot cost(s)$, this gives

$$(1 - \mu) \cdot cost(s) \leq \lambda \cdot cost(s^*)$$

$$\frac{cost(s)}{cost(s^*)} \leq \frac{\lambda}{1 - \mu}.$$

$\square$

Advanced Reading

- *AGT Chapter 18 and 19.3*
- *How Bad is Selfish Routing?*
  by Tim Roughgarden and Eva Tardos (FOCS 2000)
- *Intrinsic Robustness of the Price of Anarchy.*
  by Tim Roughgarden (STOC 2009)