Convex Optimization Problems

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Outline

Optimization Problems in Standard Form I

$$
\begin{array}{lll}
\underset{\boldsymbol{x}}{\text{minimize}} & f_0(\boldsymbol{x}) & \\
\text{subject to} & f_i(\boldsymbol{x}) \leq 0 & i = 1, \cdots, m \\
 & h_i(\boldsymbol{x}) = 0 & i = 1, \cdots, p
\end{array}
$$

- $\boldsymbol{x} = (x_1, \cdots, x_n)$ is the optimization variable
- $f_0 : \mathbb{R}^n \to \mathbb{R}$ is the objective function
- $f_i : \mathbb{R}^n \to \mathbb{R} \quad i = 1, \cdots, m$ are the inequality constraint functions
- $h_i : \mathbb{R}^n \to \mathbb{R} \quad i = 1, \cdots, p$ are the equality constraint functions

Optimization Problems in Standard Form II

Feasibility:

- a point $\boldsymbol{x} \in \mathrm{dom}\, f_0$ is feasible if it satisfies all the constraints and infeasible otherwise
- a problem is feasible if it has at least one feasible point and infeasible otherwise.

Optimal value:

$$p^\star = \inf\{f_0(\boldsymbol{x}) \mid f_i(\boldsymbol{x}) \leq 0,\ i = 1, \cdots, m,\ h_i(\boldsymbol{x}) = 0,\ i = 1, \cdots, p\}$$

- $p^\star = \infty$ if problem is infeasible (no $\boldsymbol{x}$ satisfies the constraints)
- $p^\star = -\infty$ if problem is unbounded below

Optimal solution: $\boldsymbol{x}^\star$ such that $f(\boldsymbol{x}^\star) = p^\star$ (and $\boldsymbol{x}^\star$ feasible).

Global and Local Optimality

- A feasible $\boldsymbol{x}$ is **optimal** if $f_0(\boldsymbol{x}) = p^\star$; X_{opt} is the set of optimal points.
- A feasible $\boldsymbol{x}$ is **locally optimal** if it is optimal within a ball, i.e., there is an $R > 0$ such that $\boldsymbol{x}$ is optimal for

$$
\begin{array}{ll}
\underset{\boldsymbol{z}}{\text{minimize}} & f_0(\boldsymbol{z}) \\
\text{subject to} & f_i(\boldsymbol{z}) \leq 0 \quad i = 1, \cdots, m \\
& h_i(\boldsymbol{z}) = 0 \quad i = 1, \cdots, p \\
& \|\boldsymbol{z} - \boldsymbol{x}\|_2 \leq R
\end{array}
$$

Example:

- $f_0(x) = 1/x$, $\text{dom}\, f_0 = \mathbb{R}_{++}$: $p^\star = 0$, no optimal point
- $f_0(x) = x^3 - 3x$: $p^\star = -\infty$, local optimum at $x = 1$.

Implicit Constraints

- The standard form optimization problem has an explicit constraint:

$$\boldsymbol{x} \in \mathcal{D} = \bigcap_{i=0}^{m} \operatorname{dom} f_i \;\cap\; \bigcap_{i=1}^{p} \operatorname{dom} h_i$$

- $\mathcal{D}$ is the domain of the problem
- The constraints $f_i(\boldsymbol{x}) \leq 0, h_i(\boldsymbol{x}) = 0$ are the explicit constraints
- A problem is unconstrained if it has no explicit constraints
- Example:

$$\underset{\boldsymbol{x}}{\text{minimize}} \quad \log(b - \boldsymbol{a}^T \boldsymbol{x})$$

is an unconstrained problem with implicit constraint $b > \boldsymbol{a}^T \boldsymbol{x}$

Feasibility Problem

- Sometimes, we don't really want to minimize any objective, just to find a feasible point:

$$\begin{array}{ll} \underset{\boldsymbol{x}}{\text{find}} & \boldsymbol{x} \\ \text{subject to} & f_i(\boldsymbol{x}) \leq 0 \quad i = 1, \cdots, m \\ & h_i(\boldsymbol{x}) = 0 \quad i = 1, \cdots, p \end{array}$$

- This feasibility problem can be considered as a special case of a general problem:

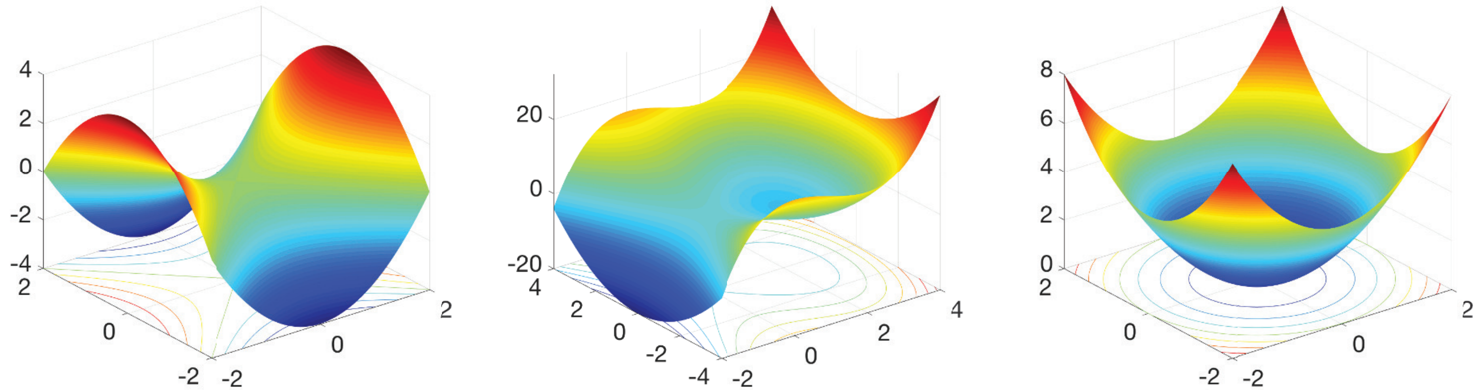
$$\begin{array}{ll} \underset{\boldsymbol{x}}{\text{minimize}} & 0 \\ \text{subject to} & f_i(\boldsymbol{x}) \leq 0 \quad i = 1, \cdots, m \\ & h_i(\boldsymbol{x}) = 0 \quad i = 1, \cdots, p \end{array}$$

where $p^\star = 0$ if constraints are feasible and $p^\star = \infty$ otherwise.

Stationary Points

Given a smooth function $f : \mathbb{R}^n \to \mathbb{R}$, a point $\boldsymbol{x} \in \mathbb{R}^n$ is called

- A **stationary point**, if $\nabla f(\boldsymbol{x}) = \mathbf{0}$;
- A **local minimum**, if $\boldsymbol{x}$ is a stationary point and there exists a neighborhood $\mathcal{B} \subseteq \mathbb{R}^n$ of $\boldsymbol{x}$ such that $f(\boldsymbol{x}) \leq f(\boldsymbol{y})$ for any $\boldsymbol{y} \in \mathcal{B}$;
- A **global minimum**, if $\boldsymbol{x}$ is a stationary point and $f(\boldsymbol{x}) \leq f(\boldsymbol{y})$ for any $\boldsymbol{y} \in \mathbb{R}^n$;
- **Saddle point**, if $\boldsymbol{x}$ is a stationary point and for any neighborhood $\mathcal{B} \subseteq \mathbb{R}^n$ of $\boldsymbol{x}$, there exist $\boldsymbol{y}, \boldsymbol{z} \in \mathcal{B}$ such that $f(\boldsymbol{z}) \leq f(\boldsymbol{x}) \leq f(\boldsymbol{y})$ and $\lambda_{\min}(\nabla^2 f(\boldsymbol{x})) \leq 0$.



(a) strict saddle (b) local minimum (c) global minimum

Outline

Convex Optimization Problem

- Convex optimization problem in standard form:

$$
\begin{array}{lll}
\underset{\boldsymbol{x}}{\text{minimize}} & f_0(\boldsymbol{x}) & \\
\text{subject to} & f_i(\boldsymbol{x}) \leq 0 & i = 1, \cdots, m \\
& \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} &
\end{array}
$$

  where $f_0, f_1, \cdots, f_m$ are convex and equality constraints are affine.

- **Local and global optima:** any locally optimal point of a convex problem is globally optimal

  x^*

- Most problems are not convex when formulated

- Reformulating a problem in convex form is an art, there is no systematic way

Proof: Suppose x^* is local optimum, $\forall y$, there exists a small $\theta > 0$

such that $f(x^*) < f(x^* + \theta(y - x^*))$ ①

since f is convex,

$$f(x^* + \theta(y - x^*)) = f(\theta y + (1-\theta) x^*)$$

$$\leq \theta f(y) + (1-\theta) f(x^*) \quad ②$$

①, ② $\Rightarrow$ $f(x^*) < \theta f(y) + (1-\theta) f(x^*)$

$\Rightarrow$ $\theta f(x^*) < \theta f(y)$, $\forall y$.

$\Rightarrow$ $f(x^*) < f(y)$, $\forall y$. $\square$

Example

- The following problem is nonconvex (why not?):

$$\begin{array}{ll} \underset{\boldsymbol{x}}{\text{minimize}} & x_1^2 + x_2^2 \\ \text{subject to} & x_1/(1+x_2^2) \leq 0 \\ & (x_1 + x_2)^2 = 0 \end{array}$$

- The objective is convex.
- The equality constraint function is not affine; however, we can rewrite it as $x_1 = -x_2$ which is then a linear equality constraint.
- The inequality constraint function is not convex; however, we can rewrite it as $x_1 \leq 0$ which again is linear.
- We can rewrite it as

$$\begin{array}{ll} \underset{\boldsymbol{x}}{\text{minimize}} & x_1^2 + x_2^2 \\ \text{subject to} & x_1 \leq 0 \\ & x_1 = -x_2 \end{array}$$

Global and Local Optimality

Any locally optimal point of a convex problem is globally optimal.
Proof: Suppose $\boldsymbol{x}$ is locally optimal (around a ball of radius R) and $\boldsymbol{y}$ is optimal with $f_0(\boldsymbol{y}) < f_0(\boldsymbol{x})$. We will show this cannot be.

Just take the segment from $\boldsymbol{x}$ to $\boldsymbol{y}$: $\boldsymbol{z} = \theta\boldsymbol{y} + (1-\theta)\boldsymbol{x}$.
Obviously the objective function is strictly decreasing along the segment since $f_0(\boldsymbol{y}) < f_0(\boldsymbol{x})$:

$$\theta f_0(\boldsymbol{y}) + (1-\theta) f_0(\boldsymbol{x}) < f_0(\boldsymbol{x}) \qquad \theta \in (0,1]$$

Using now the convexity of the function, we can write

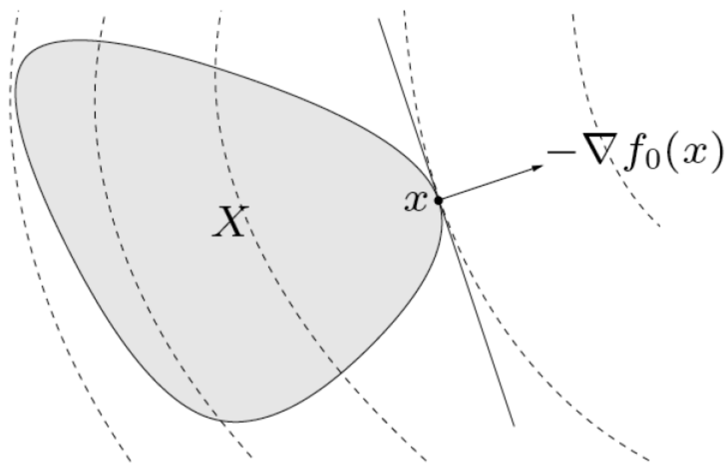
$$f_0(\theta\boldsymbol{y} + (1-\theta)\boldsymbol{x}) < f_0(\boldsymbol{x}) \qquad \theta \in (0,1]$$

Finally, just choose θ sufficiently small such that the point $\boldsymbol{z}$ is in the ball of local optimality of $\boldsymbol{x}$, arriving at a contradiction.

Optimality Criterion for Differentiable f_0 I

Minimum Principle: A feasible point $\boldsymbol{x}$ is optimal if and only if

$$\nabla f_0(\boldsymbol{x})^T(\boldsymbol{y}-\boldsymbol{x}) \geq 0 \qquad \text{for all feasible } \boldsymbol{y}$$



$$\textcircled{1} \quad f_0(x) \le f_0(y), \quad \forall y \in \operatorname{dom} f_0$$

$$\Updownarrow$$

$$\textcircled{2} \quad \nabla f_0^T(x)(y-x) \ge 0, \quad \forall y \in \operatorname{dom} f_0.$$

② ⇒ ①: By convexity: $f_0(y) \ge f_0(x) + \underbrace{\nabla f^T(x_0)(y-x)}_{\ge 0}$

$\ge f_0(x)$ $\square$

① ⇒ ② Consider $z(\theta) = x + \theta(y-x)$. $\theta \in [0,1]$

Suppose $f_0(x) \le f_0(y)$, $\Rightarrow \nabla f_0^T(x)(y-x) < 0$

$$\frac{d}{d\theta} f_0(z(\theta)) = \frac{d}{d\theta} f_0(x + \theta(y-x))\Big|_{\theta=0}$$

$$= \nabla f_0^T(x)(y-x) < 0$$

$\downarrow$

So for small $\theta > 0$, we have $f_0(z(\theta)) < f_0(x)$, conflict!

$$\lim_{\theta \to 0} \frac{f_0(x+\theta(y-x)) - f_0(x)}{\theta(y-x)} = f_0'(x + \theta(y-x))$$

③ $\nabla f_0(x) = 0$

③ $\Rightarrow$ ② Clearly!

② $\Rightarrow$ ③ Let $y = x - \theta \nabla f_0(x)$, $\theta > 0$

for small $\theta > 0$, $y \in \mathrm{dom} f_0$

$\nabla f_0^T(x)(y-x) = -\theta \|\nabla f_0(x)\|^2 \geq 0$

$\Rightarrow \nabla f_0(x) = 0$

Optimality Criterion for Differentiable f_0 II

- **Unconstrained problem:** $\boldsymbol{x}$ is optimal iff

$$\boldsymbol{x} \in \operatorname{dom} f_0, \qquad \nabla f_0(\boldsymbol{x}) = 0$$

- **Equality constrained problem:** $\min_{\boldsymbol{x}} f_0(\boldsymbol{x}) \quad \text{s.t. } \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}$
  $\boldsymbol{x}$ is optimal iff

$L(x, \nu) = f_0(x) + \nu^T(Ax - b)$

$$\boldsymbol{x} \in \operatorname{dom} f_0, \qquad \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b},\ \nabla f_0(\boldsymbol{x}) + \boldsymbol{A}^T\boldsymbol{\nu} = \boldsymbol{0}$$

$\frac{\partial L}{\partial x} = \nabla f(x) + A^T\nu = 0$, $\frac{\partial L}{\partial \nu} = Ax - b = 0$

- **Minimization over nonnegative orthant:** $\min_{\boldsymbol{x}} f_0(\boldsymbol{x}) \quad \text{s.t. } \boldsymbol{x} \succeq \boldsymbol{0}$ $\boldsymbol{x}$ is optimal iff

$$\boldsymbol{x} \in \operatorname{dom} f_0, \qquad \boldsymbol{x} \succeq 0, \begin{cases} \nabla_i f_0(\boldsymbol{x}) \geq 0 & x_i = 0 \\ \nabla_i f_0(\boldsymbol{x}) = 0 & x_i > 0 \end{cases}$$

$\nabla f_0^T(x)(y - x) \geq 0 \quad \forall y \geq 0$

$\nabla f_0^T(x)\, y \geq \nabla f_0^T(x)\, x$

$x_i(\nabla_i f_0(x)) = 0$

$$\min_{x} \quad f_0(x) + \lambda \|Ax - b\|^2$$

$\lambda \to \infty$, $Ax - b = 0$. $f_0(x) \uparrow$

$\lambda > 0$. $f_0(x) \downarrow$ $Ax - b = 0$

Equivalent Reformulations I

❧ **Eliminating/introducing equality constraints:**

$$\begin{array}{ll} \underset{\boldsymbol{x}}{\text{minimize}} & f_0(\boldsymbol{x}) \\ \text{subject to} & f_i(\boldsymbol{x}) \leq 0 \quad i = 1, \cdots, m \\ & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \end{array}$$

is equivalent to

$$\begin{array}{ll} \underset{\boldsymbol{z}}{\text{minimize}} & f_0(\boldsymbol{F}\boldsymbol{z} + \boldsymbol{x}_0) \\ \text{subject to} & f_i(\boldsymbol{F}\boldsymbol{z} + \boldsymbol{x}_0) \leq 0 \quad i = 1, \cdots, m \end{array}$$

where $\boldsymbol{F}$ and $\boldsymbol{x}_0$ are such that $\boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \Longleftrightarrow \boldsymbol{x} = \boldsymbol{F}\boldsymbol{z} + \boldsymbol{x}_0$ for some $\boldsymbol{z}$.

Equivalent Reformulations II

- **Introducing slack variables for linear inequalities:**

$$\begin{array}{ll} \underset{\boldsymbol{x}}{\text{minimize}} & f_0(\boldsymbol{x}) \\ \text{subject to} & \boldsymbol{a}_i^T \boldsymbol{x} \leq b_i \quad i = 1, \cdots, m \end{array}$$

is equivalent to

$$\begin{array}{ll} \underset{\boldsymbol{x}, \boldsymbol{s}}{\text{minimize}} & f_0(\boldsymbol{x}) \\ \text{subject to} & \boldsymbol{a}_i^T \boldsymbol{x} + s_i = b_i \quad i = 1, \cdots, m \\ & s_i \geq 0 \end{array}$$

Equivalent Reformulations III

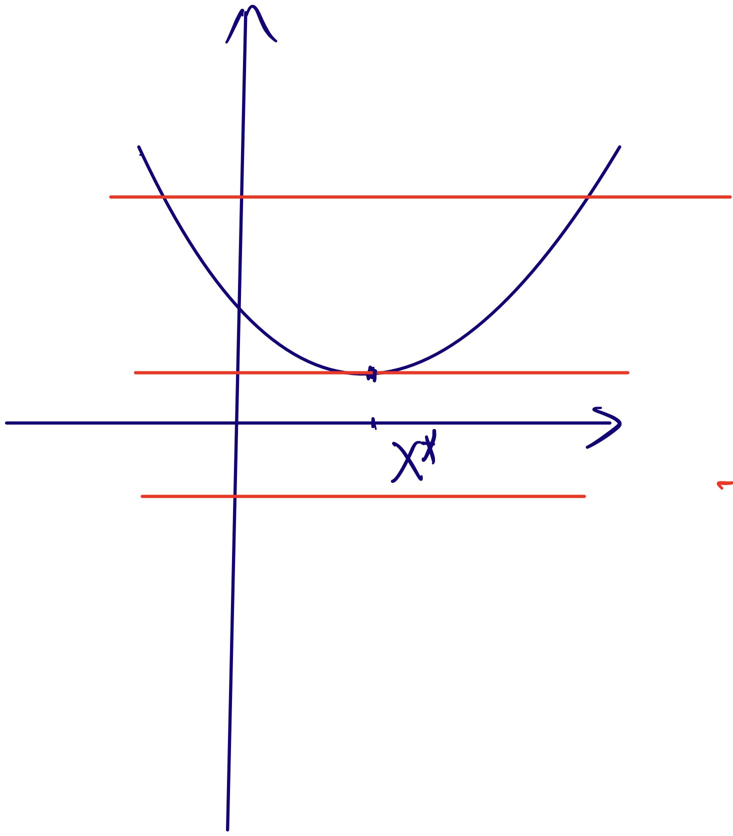
- **Epigraph form:** a standard form convex problem is equivalent to

$$
\begin{array}{ll}
\underset{\boldsymbol{x},t}{\text{minimize}} & t \\
\text{subject to} & f_0(\boldsymbol{x}) - t \le 0 \\
& f_i(\boldsymbol{x}) \le 0 \quad i = 1, \cdots, m \\
& \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}
\end{array}
$$

(x^*, t^*)

$$
\begin{array}{ll}
\min\limits_{x} & f_0(x) \\
\text{s.t.} & f_i(x) \le 0 \quad i = 1 \cdots m \\
& Ax = b
\end{array}
$$

$f_0(x^*) = t^*$.



$f_0(x) < t$

$f_0(x^*) = t$ minimum

$f_0(x) > t$ ✗

Equivalent Reformulations IV

- **Minimizing over some variables:**

$$\begin{array}{ll} \underset{\boldsymbol{x},\boldsymbol{y}}{\text{minimize}} & \underbrace{f_0(\boldsymbol{x},\boldsymbol{y})}_{} \\ \text{subject to} & f_i(\boldsymbol{x}) \leq 0 \quad i = 1, \cdots, m \end{array}$$

jointly convex in (x,y)

is equivalent to

$$\begin{array}{ll} \underset{\boldsymbol{x}}{\text{minimize}} & \tilde{f}_0(\boldsymbol{x}) \\ \text{subject to} & f_i(\boldsymbol{x}) \leq 0 \quad i = 1, \cdots, m \end{array}$$

where $\underbrace{\tilde{f}_0(\boldsymbol{x})}_{\text{convex}} = \inf_{\boldsymbol{y}} f_0(\boldsymbol{x}, \boldsymbol{y})$

convex

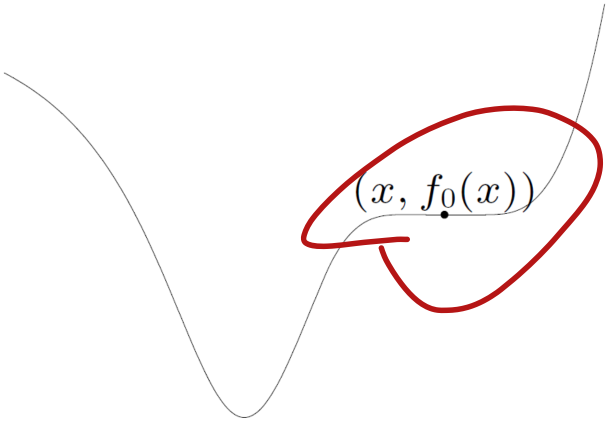
Outline

Quasiconvex Optimization

$$\begin{array}{ll} \underset{\boldsymbol{x}}{\text{minimize}} & f_0(\boldsymbol{x}) \\ \text{subject to} & f_i(\boldsymbol{x}) \leq 0 \quad i = 1, \cdots, m \\ & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \end{array}$$

where $f_0 : \mathbb{R}^n \longrightarrow \mathbb{R}$ is quasiconvex and $f_1, \cdots, f_m$ are convex

- Observe that it can have locally optimal points that are not (globally) optimal:



Quasiconvex Optimization

- **Convex representation** of sublevel sets of a quasiconvex function f_0: there exists a family of convex functions $\phi_t(\boldsymbol{x})$ for fixed t such that

$$f_0(\boldsymbol{x}) \le t \iff \phi_t(\boldsymbol{x}) \le 0$$

$S_t(x) = \{x \mid f_0(x) \le t\}$

convex set

- **Example:**

$$f_0(\boldsymbol{x}) = \frac{p(\boldsymbol{x})}{q(\boldsymbol{x})}$$

with p convex, q concave, and $p(\boldsymbol{x}) \ge 0$, $q(\boldsymbol{x}) > 0$ on dom f_0. We can choose:

$$\phi_t(\boldsymbol{x}) = p(\boldsymbol{x}) - tq(\boldsymbol{x})$$

convex

  - for $t \ge 0$, $\phi_t(\boldsymbol{x})$ is convex in $\boldsymbol{x}$
  - $p(\boldsymbol{x})/q(\boldsymbol{x}) \le t$ if and only if $\phi_t(\boldsymbol{x}) \le 0$

minimize $f_0(x)$ → quasiconvex

subject to $f_i(x) \le 0$

$Ax = b$

$\Downarrow$ fixed t

convex feasibility

minimize $\cancel{x}$ t (fixed)

subject to $f_0(x) \le t$ $\Rightarrow$ $\phi_t(x) \le 0$

$f_i(x) \le 0$

$Ax = b$

l $t_0 = \frac{l+u}{2}$ t_1 u

t_0 feasible, $t_1 = \frac{l+t_0}{2}$

t_0 infeasible

$t_1 = \frac{t_0+u}{2}$

$\#$ steps of bi-section : $\log_2 \left\lceil \frac{u-l}{\epsilon} \right\rceil$

Quasiconvex Optimization

Solving a quasiconvex problem via convex feasibility problems: the idea is to solve the epigraph form of the problem with a sandwich technique in t:

- for fixed t the epigraph form of the original problem reduces to a feasibility convex problem

convex feasibility
$$\begin{cases} \text{find} & x \\ \text{s.t.} & \phi_t(\boldsymbol{x}) \leq 0, \quad f_i(\boldsymbol{x}) \leq 0 \forall i, \quad \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b} \end{cases}$$

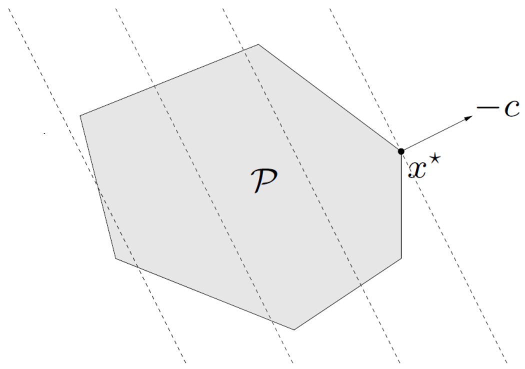
- if t is too small, the feasibility problem will be infeasible
- if t is too large, the feasibility problem will be feasible
- start with upper and lower bounds on t (termed u and l, resp.) and use a sandwich technique (bisection method): at each iteration use $t = (l+u)/2$ and update the bounds according to the feasibility or infeasibility of the problem.

Outline

Linear Programming (LP)

$$\begin{aligned} \underset{\boldsymbol{x}}{\text{minimize}} \quad & \boldsymbol{c}^T \boldsymbol{x} + d \\ \text{subject to} \quad & \boldsymbol{G}\boldsymbol{x} \leq \boldsymbol{h} \\ & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \end{aligned}$$

- Convex problem: affine objective and constraint functions.
- Feasible set is a polyhedron:



ℓ_1- and ℓ_∞- Norm Problems as LPs I

- ℓ_∞-norm minimization:

$$\begin{array}{ll} \underset{\boldsymbol{x}}{\text{minimize}} & \underbrace{\|\boldsymbol{x}\|_\infty \leq t} \\ \text{subject to} & \boldsymbol{G}\boldsymbol{x} \leq \boldsymbol{h} \\ & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \end{array}$$

is equivalent to the LP

$$\max_i |x_i| \leq t$$

$$\begin{array}{ll} \underset{t,\boldsymbol{x}}{\text{minimize}} & t \\ \text{subject to} & -t\mathbf{1} \preceq \boldsymbol{x} \preceq t\mathbf{1} \\ & \boldsymbol{G}\boldsymbol{x} \leq \boldsymbol{h} \\ & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \end{array}$$

$\forall i . |x_i| \leq t$

$\forall i, \ -t \leq x_i \leq t$

ℓ_1- and ℓ_∞- Norm Problems as LPs II

- **ℓ_1-norm minimization:**

$$
\begin{array}{ll}
\underset{\boldsymbol{x}}{\text{minimize}} & \|\boldsymbol{x}\|_1 \\
\text{subject to} & \boldsymbol{G}\boldsymbol{x} \le \boldsymbol{h} \\
& \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}
\end{array}
$$

$\sum_i |x_i|$

$|x_i| \le t_i$

is equivalent to the LP

$$
\begin{array}{ll}
\underset{\boldsymbol{t},\boldsymbol{x}}{\text{minimize}} & \sum_i t_i \\
\text{subject to} & -\boldsymbol{t} \preceq \boldsymbol{x} \preceq \boldsymbol{t} \\
& \boldsymbol{G}\boldsymbol{x} \le \boldsymbol{h} \\
& \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}
\end{array}
$$

$-t_i \le x_i \le t_i$

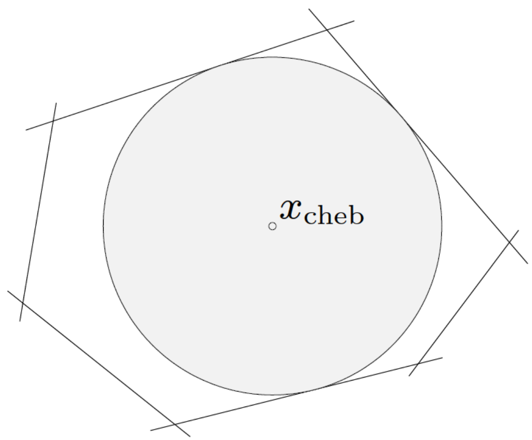
Examples: Chebyshev Center of a Polyhedron I

- Chebyshev center of a polyhedron

$$\mathcal{P} = \{\boldsymbol{x} \mid \boldsymbol{a}_i^T \boldsymbol{x} \le b_i,\ i = 1, \cdots, m\}$$

is center of largest inscribed ball

$$\mathcal{B} = \{\boldsymbol{x}_c + \boldsymbol{u} \mid \|\boldsymbol{u}\| \le r\}$$



- Let's solve the problem

$$\begin{array}{ll} \underset{r, \boldsymbol{x}_c}{\text{maximize}} & r \\ \text{subject to} & \boldsymbol{x} \in \mathcal{P} \quad \text{for all } \boxed{\boldsymbol{x} = \boldsymbol{x}_c + \boldsymbol{u}} \text{ with } \|\boldsymbol{u}\| \le r \end{array}$$

- Observe that $\boldsymbol{a}_i^T \boldsymbol{x} \le b_i$ for all $\boldsymbol{x} \in \mathcal{B}$ if and only if

$$\sup_{\boldsymbol{u}}\{\boldsymbol{a}_i^T(\boldsymbol{x}_c + \boldsymbol{u}) \mid \|\boldsymbol{u}\| \le r\} \le b_i$$

$$a_i^T x_c + a_i^T u \le a_i^T x_c + \|a_i\| \, \|u\| \le a_i^T x_c + r\|a_i\|$$

Examples: Chebyshev Center of a Polyhedron II

- Using Schwartz inequality, the supremum condition can be rewritten as

$$\boldsymbol{a}_i^T \boldsymbol{x}_c + r\|\boldsymbol{a}_i\|_2 \leq b_i$$

- Hence, the Chebyshev center can be obtained by solving:

$$\begin{array}{ll} \underset{r, \boldsymbol{x}_c}{\text{maximize}} & r \\ \text{subject to} & \boldsymbol{a}_i^T \boldsymbol{x}_c + r\|\boldsymbol{a}_i\|_2 \leq b_i, \quad i = 1, \cdots, m \end{array}$$

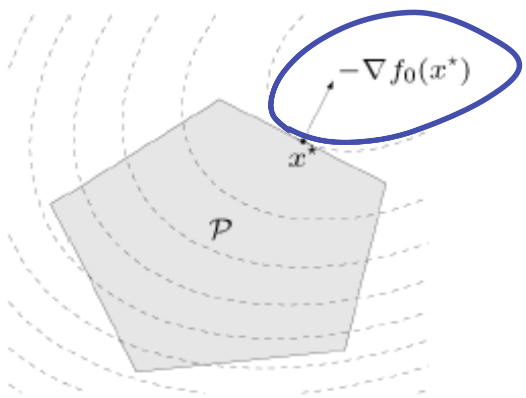
which is an LP.

Quadratic Programming (QP)

$$\begin{array}{ll} \underset{\boldsymbol{x}}{\text{minimize}} & (1/2)\,\boldsymbol{x}^T\boldsymbol{P}\boldsymbol{x} + \boldsymbol{q}^T\boldsymbol{x} + r \\ \text{subject to} & \boldsymbol{G}\boldsymbol{x} \leq \boldsymbol{h} \\ & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \end{array}$$

- Convex problem (assuming $\boldsymbol{P} \in \mathbb{S}_+^n \succeq \mathbf{0}$): convex quadratic objective and affine constraint functions.

- Minimization of a convex quadratic function over a polyhedron:



Quadratically Constrained QP (QCQP)

$$\begin{array}{lll} \underset{\boldsymbol{x}}{\text{minimize}} & (1/2)\, \boldsymbol{x}^T \boldsymbol{P}_0 \boldsymbol{x} + \boldsymbol{q}_0^T \boldsymbol{x} + r_0 & \\ \text{subject to} & (1/2)\, \boldsymbol{x}^T \boldsymbol{P}_i \boldsymbol{x} + \boldsymbol{q}_i^T \boldsymbol{x} + r_i \leq 0 & i = 1, \cdots, m \\ & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} & \end{array}$$

- Convex problem (assuming $\boldsymbol{P}_i \in \mathbb{S}_+^n \succeq \mathbf{0}$): convex quadratic objective and constraint functions.

nonconvex: $X^T P X \leq 0$, $P \in S^n$

$\Downarrow$

$\begin{cases} \text{trace}(PM) \leq 0 & \text{convex in } M \\ \underline{\text{rank}(M) = 1} & \text{non convex} \end{cases}$

Second-Order Cone Programming (SOCP)

second-order cone

$$
\begin{array}{ll}
\underset{x,\, y,\, s}{\text{minimize}} & \boldsymbol{f}^T \boldsymbol{x} \\
\text{subject to} & \underbrace{\|\boldsymbol{A}_i \boldsymbol{x} + \boldsymbol{b}_i\|}_{y_i} \leq \underbrace{\boldsymbol{c}_i^T \boldsymbol{x} + d_i}_{s_i} \quad i = 1, \cdots, m \\
& \boldsymbol{F}\boldsymbol{x} = \boldsymbol{g}
\end{array}
$$

$\Rightarrow \|y\| \leq s$

$A_i x + b_i = y_i \ , \quad c_i^T x + d_i = s_i$

- Convex problem: linear objective and second-order cone constraints
- For $\boldsymbol{A}_i$ row vector, it reduces to an LP
- For $\boldsymbol{c}_i = \boldsymbol{0}$, it reduces to a QCQP
- More general than QCQP and LP

Robust LP as an SOCP

- Sometimes, the parameters of an optimization problem are imperfect
- Consider the robust LP:

$$\begin{aligned} \underset{\boldsymbol{x}}{\text{minimize}} \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{subject to} \quad & \boldsymbol{a}_i^T \boldsymbol{x} \le b_i \quad \forall \boldsymbol{a}_i \in \mathcal{E}_i, \; i = 1, \cdots, m \end{aligned}$$

uncertainty set

where $\mathcal{E}_i = \{\bar{\boldsymbol{a}}_i + \boldsymbol{P}_i \boldsymbol{u} \mid \|\boldsymbol{u}\| \le 1\}$

- It can be rewritten as the SOCP:

$$\begin{aligned} \underset{\boldsymbol{x}}{\text{minimize}} \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{subject to} \quad & \bar{\boldsymbol{a}}_i^T \boldsymbol{x} + \|\boldsymbol{P}_i^T \boldsymbol{x}\|_2 \le b_i \quad i = 1, \cdots, m \end{aligned}$$

$\sup a_i^T x = \sup \{ (\bar{a}_i + P_i u)^T x \mid \|u\| \le 1 \}$

$\bar{a}_i^T x + u^T P_i^T x \le \bar{a}_i^T x + \|u\| \|P_i^T x\|$

Generalized Inequality Constraints

- Convex problem with generalized inequality constraints:

$$\begin{array}{ll} \underset{\boldsymbol{x}}{\text{minimize}} & f_0(\boldsymbol{x}) \\ \text{subject to} & \boldsymbol{f}_i(\boldsymbol{x}) \preceq_{K_i} \mathbf{0} \quad i = 1, \cdots, m \\ & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \end{array}$$

  where f_0 is convex and f_i are K_i-convex w.r.t. proper cone K_i
- It has the same properties as a standard convex problem
- **Conic form problem:** special case with affine objective and constraints:

$$\begin{array}{ll} \underset{\boldsymbol{x}}{\text{minimize}} & \boldsymbol{c}^T\boldsymbol{x} \\ \text{subject to} & \boldsymbol{F}\boldsymbol{x} + \boldsymbol{g} \preceq_K \mathbf{0} \\ & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b} \end{array}$$

conic optimization

P_1 minimize$_{x}$ $c^T x$ subject to $Ax=b$, $x \in K$

conic form

tractable conic optimization

① the nonnegative orthant, $R_+^n \Rightarrow LP$

② the second-order cone, $Q^n = \{(x,t) \in R^{n+1}, \|x\|_2 \le t\}$

$\Rightarrow$ SOCP

③ the semidefinite cone, $S_+^n = \{X \mid X = X^T \succeq 0\}$

$\Rightarrow$ SDP

$$\boxed{P_0 \quad \begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & f_i(x) \le 0,\ Ax=b \end{array}}$$

mathematical form

$P_0 \xrightarrow[\text{lect. 1-3}]{\text{CVX}} P_1$

SDPT3 } interior-point
Sedumi
SCS → ADMM

$\xrightarrow[\text{lect. 4}]{}$ solution

KKT condition

Semidefinite Programming (SDP)

$$
\begin{array}{ll}
\underset{\boldsymbol{x}}{\text{minimize}} & \boldsymbol{c}^T \boldsymbol{x} \\
\text{subject to} & x_1 \boldsymbol{F}_1 + x_2 \boldsymbol{F}_2 + \cdots + x_n \boldsymbol{F}_n \preceq \boldsymbol{G} \\
 & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}
\end{array}
$$

- Inequality constraint is called linear matrix inequality (LMI)
- Convex problem: linear objective and linear matrix inequality (LMI) constraints
- Observe that multiple LMI constraints can always be written as a single one

SDP I

- **LP and equivalent SDP:**

$$\begin{array}{ll} \underset{\boldsymbol{x}}{\text{minimize}} & \boldsymbol{c}^T\boldsymbol{x} \\ \text{subject to} & \boldsymbol{A}\boldsymbol{x} \preceq \boldsymbol{b} \end{array} \qquad \begin{array}{ll} \underset{\boldsymbol{x}}{\text{minimize}} & \boldsymbol{c}^T\boldsymbol{x} \\ \text{subject to} & \operatorname{diag}(\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}) \preceq \boldsymbol{0} \end{array}$$

- **SOCP and equivalent SDP:**

$$\begin{array}{ll} \underset{\boldsymbol{x}}{\text{minimize}} & \boldsymbol{f}^T\boldsymbol{x} \\ \text{subject to} & \|\boldsymbol{A}_i\boldsymbol{x} + \boldsymbol{b}_i\| \leq \boldsymbol{c}_i^T\boldsymbol{x} + d_i, \quad i = 1, \cdots, m \end{array}$$

$$\begin{array}{ll} \underset{\boldsymbol{x}}{\text{minimize}} & \boldsymbol{f}^T\boldsymbol{x} \\ \text{subject to} & \begin{bmatrix} (\boldsymbol{c}_i^T\boldsymbol{x} + d_i)\boldsymbol{I} & \boldsymbol{A}_i\boldsymbol{x} + \boldsymbol{b}_i \\ \boldsymbol{A}_i\boldsymbol{x} + \boldsymbol{b}_i & \boldsymbol{c}_i^T\boldsymbol{x} + d_i \end{bmatrix} \succeq \boldsymbol{0}, \quad i = 1, \cdots, m \end{array}$$

SDP II

- **Eigenvalue minimization:**

$$\underset{\boldsymbol{x}}{\text{minimize}} \quad \lambda_{\max}(\boldsymbol{A}(\boldsymbol{x}))$$

where $\boldsymbol{A}(\boldsymbol{x}) = \boldsymbol{A}_0 + x_1\boldsymbol{A}_1 + \cdots + x_n\boldsymbol{A}_n$, is equivalent to SDP

$$\begin{array}{ll} \underset{\boldsymbol{x},t}{\text{minimize}} & t \\ \text{subject to} & \boldsymbol{A}(\boldsymbol{x}) \preceq t\boldsymbol{I} \end{array}$$

- It follows from

$$\lambda_{\max}(\boldsymbol{A}(\boldsymbol{x})) \leq t \iff \boldsymbol{A}(\boldsymbol{x}) \preceq t\boldsymbol{I}$$

Reference

Chapter 4 of:

- Stephen Boyd and Lieven Vandenberghe, *Convex Optimization*. Cambridge, U.K.: Cambridge University Press, 2004.