



Lecture 12

- Frequency Response

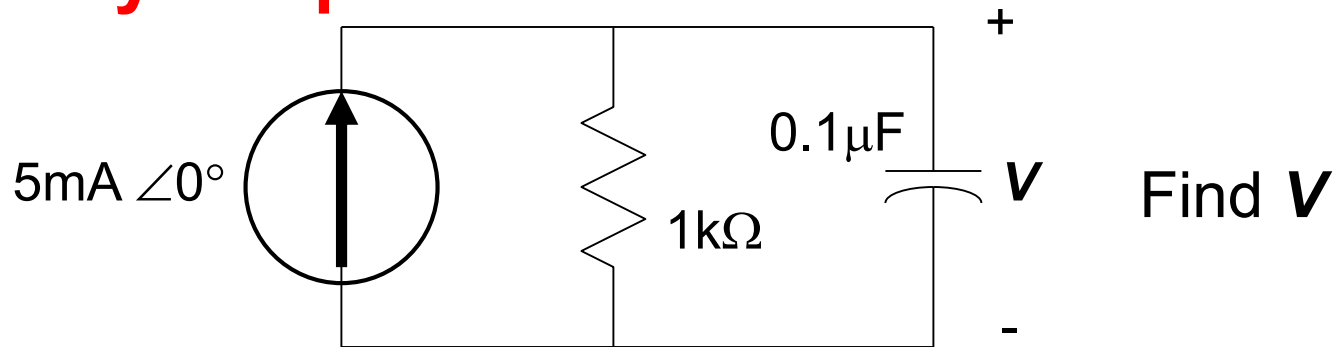


Outline

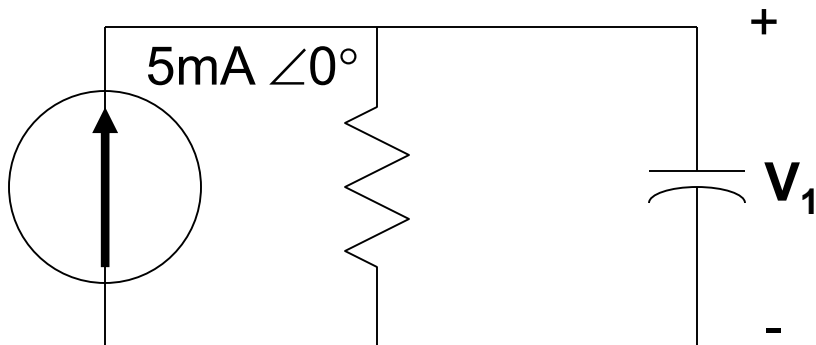
- Frequency response
 - *Transfer function*
 - ~~*Bode plots (or diagram)*~~



Frequency Response



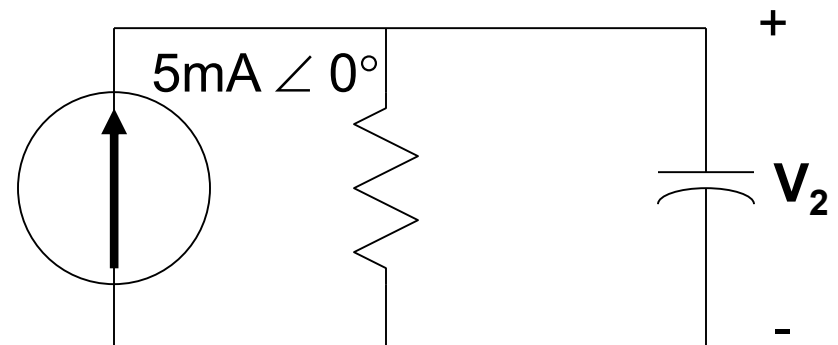
Case 1: $\omega = 2\pi \times 3000$



$$Z_{eq} = 468.2 \angle -62.1^\circ \Omega$$

$$V_1 = 2.34 \angle -62.1^\circ \text{V}$$

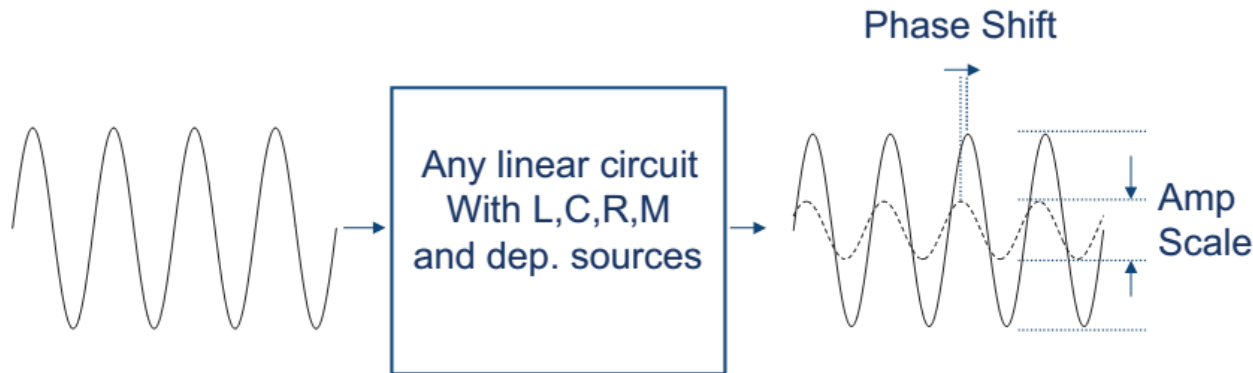
Case 2: $\omega = 2\pi \times 455000$



$$Z_{eq} = 3.5 \angle -89.8^\circ \Omega$$

$$V_2 = 17.5 \angle -89.8^\circ \text{mV}$$

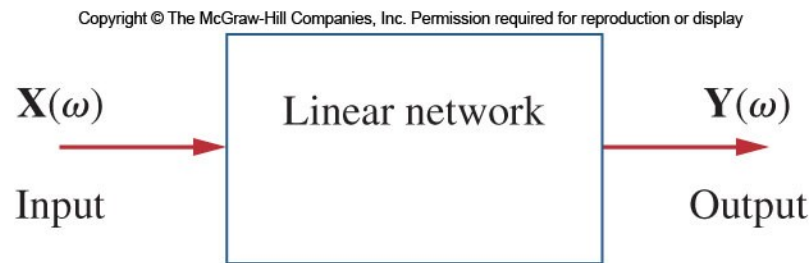
Frequency Response



- The “Frequency Response” is a characterization of the input-output relation for sinusoidal inputs at **all** frequencies.
 - Its output is also a sinusoid at the ***same*** frequency.
 - Only the magnitude and phase of the output differ from the input.
 - Significant for applications, esp. in communications and control systems.

Transfer Function

- The transfer function $H(\omega)$ is the frequency-dependent ratio of a forced function $Y(\omega)$ to the forcing function $X(\omega)$.



$$H(\omega) = \frac{Y(\omega)}{X(\omega)}$$

$$H(\omega) = \text{Voltage gain} = \frac{V_o(\omega)}{V_i(\omega)}$$

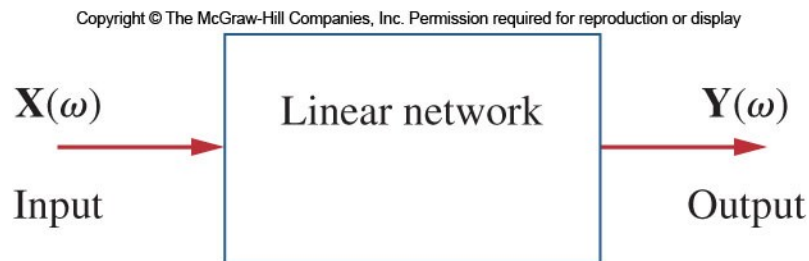
$$H(\omega) = \text{Current gain} = \frac{I_o(\omega)}{I_i(\omega)}$$

$$H(\omega) = \text{Transfer impedance} = \frac{V_o(\omega)}{I_i(\omega)}$$

$$H(\omega) = \text{Transfer admittance} = \frac{I_o(\omega)}{V_i(\omega)}$$

Transfer Function

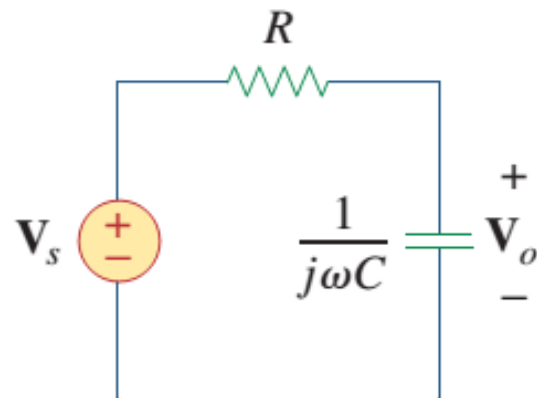
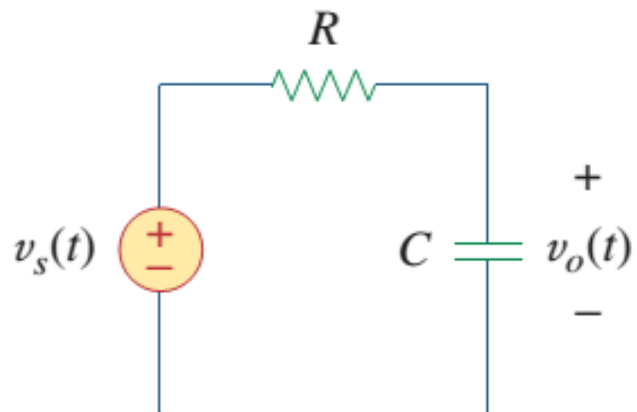
- Complex quantity
- Both *magnitude and phase* of the **output** are functions of frequency



$$\frac{\mathbf{V}_{\text{out}}}{\mathbf{V}_{\text{in}}} = \frac{V_{\text{out}}}{V_{\text{in}}} \angle (\theta_{\text{out}} - \theta_{\text{in}})$$



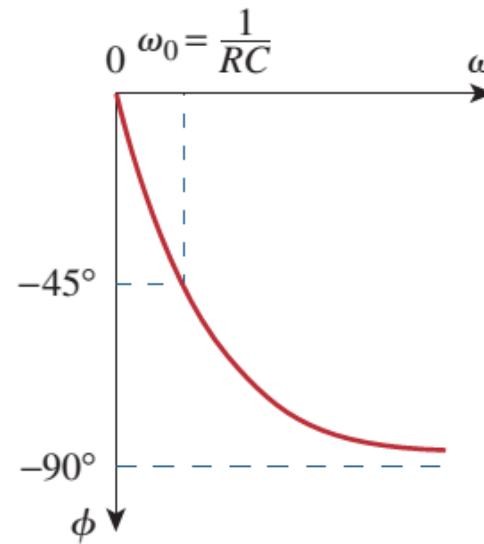
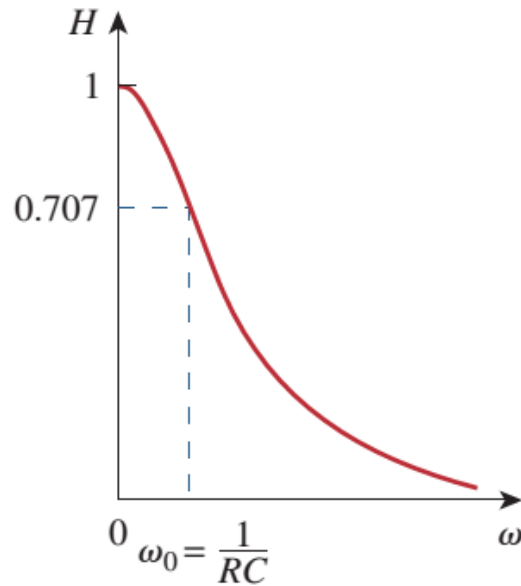
Example





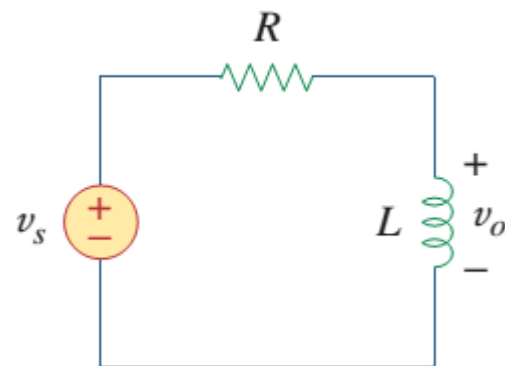
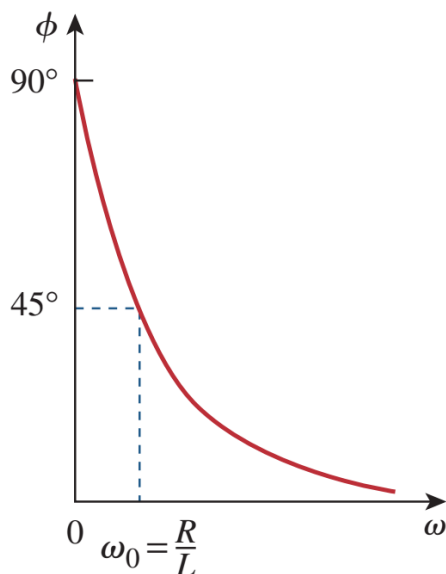
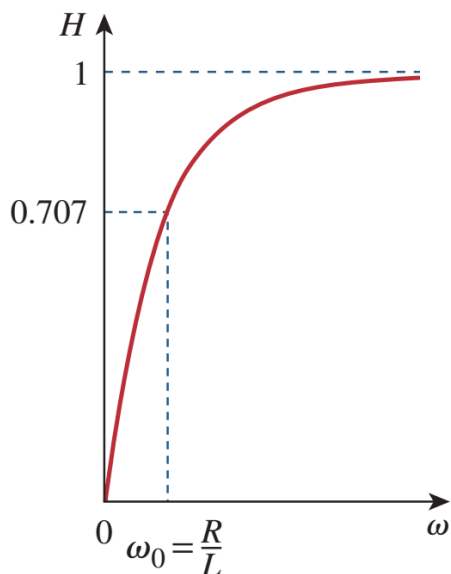
$$H = \frac{1}{\sqrt{1 + (\omega/\omega_0)^2}}, \quad \phi = -\tan^{-1} \frac{\omega}{\omega_0}$$

ω/ω_0	H	ϕ	ω/ω_0	H	ϕ
0	1	0	10	0.1	-84°
1	0.71	-45°	20	0.05	-87°
2	0.45	-63°	100	0.01	-89°
3	0.32	-72°	∞	0	-90°



Exercise

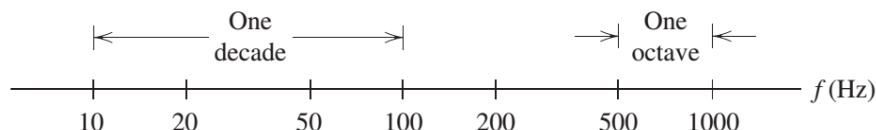
- Obtain the transfer function V_o/V_s of the RL circuit.
Assuming $v_s = V_m \cos \omega t$.



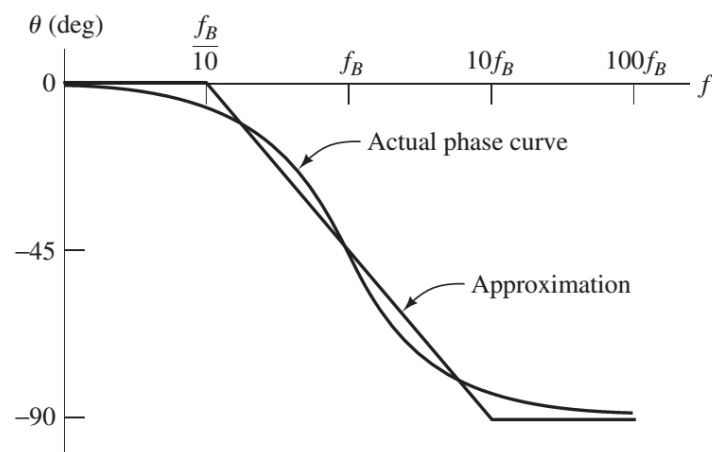
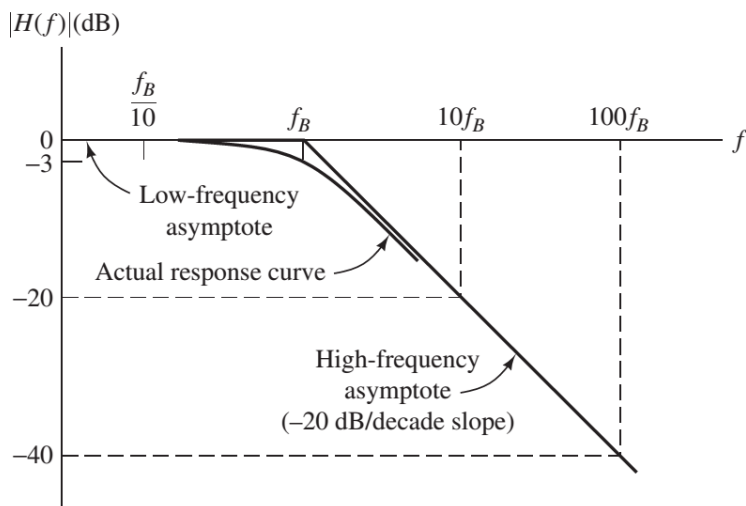
Bode Plots

Plotting the frequency response, magnitude & phase, on plots with

- Frequency X in log scale



- Y scale in dB (for magnitude) & degree (for phase)





Bel and Decibel (dB)

- A **bel** (symbol **B**) is a unit of measure of ratios of power levels, i.e. relative power levels.

- The name was coined in the early 20th century in honor of Alexander Graham Bell, a telecommunication pioneer.
- Definition of bel:

$$\text{Ratio with a unit of B} = \log_{10}(P_1/P_2)$$

where P_1 and P_2 are power levels.

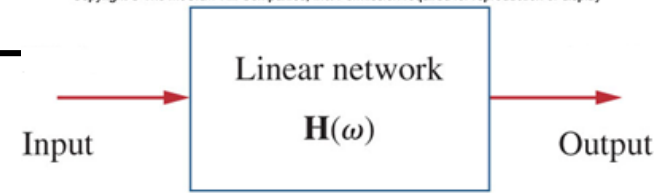
- One bel is too large for everyday use, so the **decibel (dB)**, equal to 0.1B, is more commonly used.

$$\text{Ratio with a unit of dB} = 10 \log_{10}(P_1/P_2)$$

- used to measure electric power, gain or loss of amplifiers, etc.



dB for Voltage or Current



- We can similarly relate the reference voltage or current to the reference power, as

$$P = (V)^2/R \text{ or } P = (I)^2R$$

Hence,

$$\begin{aligned}\text{Voltage, } V \text{ in decibels} &= 20\log_{10}(V_1/V_2) \\ \text{Current, } I, \text{ in decibels} &= 20\log_{10}(I_1/I_2)\end{aligned}$$

Question: How many decibels larger is the voltage of a 9-volt transistor battery than that of a 1.5-volt AA battery?

Question: **The voltage gain** of an amplifier with input = 0.2 mV and output = 0.5 V is ?



Summary

If G is defined as the power gain,

$$G = \frac{P}{P_0},$$

then the corresponding gain in dB is defined as

$$G \text{ [dB]} = 10 \log G = 10 \log \left(\frac{P}{P_0} \right) \quad (\text{dB}).$$

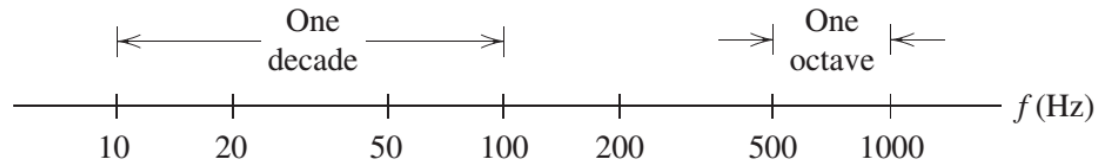
$$G \text{ [dB]} = 10 \log \left(\frac{\frac{1}{2} |\mathbf{V}|^2 / R}{\frac{1}{2} |\mathbf{V}_0|^2 / R} \right) = 20 \log \left(\frac{|\mathbf{V}|}{|\mathbf{V}_0|} \right)$$

$\frac{P}{P_0}$	dB
10^N	$10N$ dB
10^3	30 dB
100	20 dB
10	10 dB
4	$\simeq 6$ dB
2	$\simeq 3$ dB
1	0 dB
0.5	$\simeq -3$ dB
0.25	$\simeq -6$ dB
0.1	-10 dB
10^{-N}	$-10N$ dB

$\left \frac{\mathbf{V}}{\mathbf{V}_0} \right $ or $\left \frac{\mathbf{I}}{\mathbf{I}_0} \right $	dB
10^N	$20N$ dB
10^3	60 dB
100	40 dB
10	20 dB
4	$\simeq 12$ dB
2	$\simeq 6$ dB
1	0 dB
0.5	$\simeq -6$ dB
0.25	$\simeq -12$ dB
0.1	-20 dB
10^{-N}	$-20N$ dB

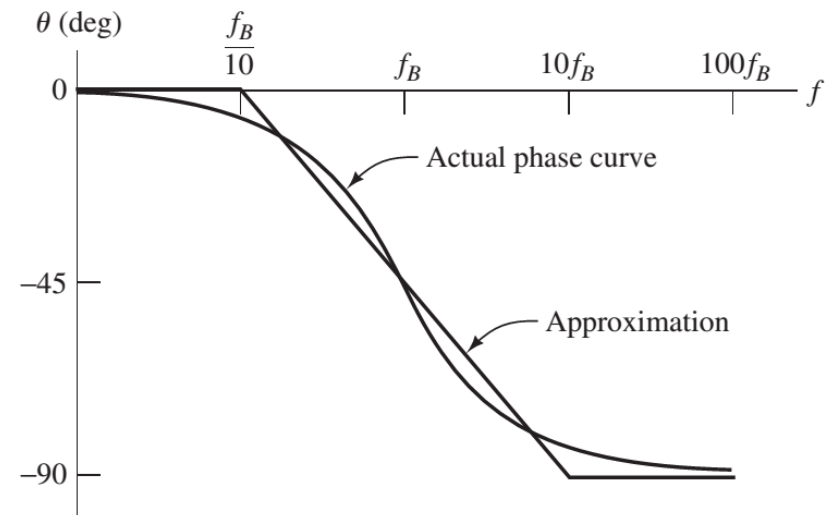
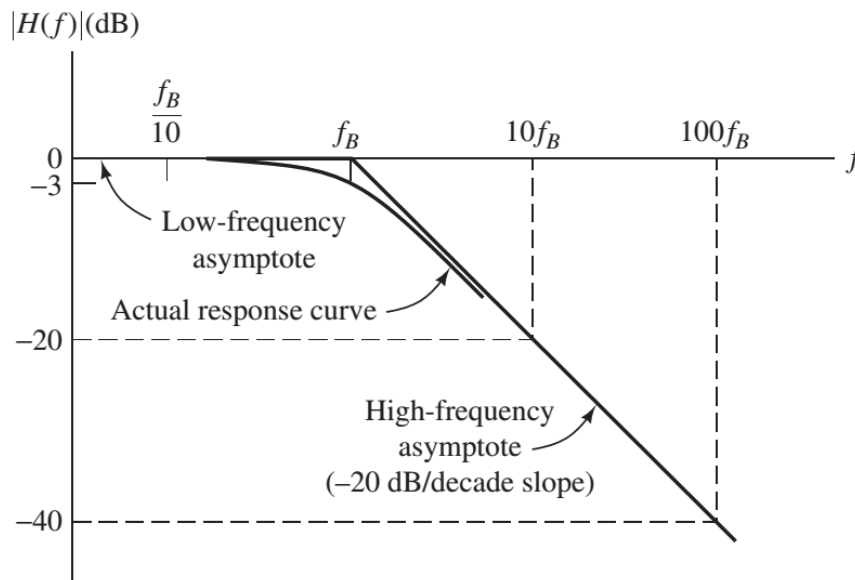


Bode Plots



Plotting the frequency response, magnitude or phase, on plots with

- Frequency X in log scale
- Y scale in dB (for magnitude) or degree (for phase)





Bode Plots

- Bode plot is particularly useful for displaying transfer function-- a general form is displayed as:

$$\mathbf{H}(\omega) = \frac{K(j\omega)^{\pm 1} (1 + j\omega/z_1)[1 + j2\zeta_1\omega/\omega_k + (j\omega/\omega_k)^2]\cdots}{(1 + j\omega/p_1)[1 + j2\zeta_2\omega/\omega_n + (j\omega/\omega_n)^2]\cdots}$$

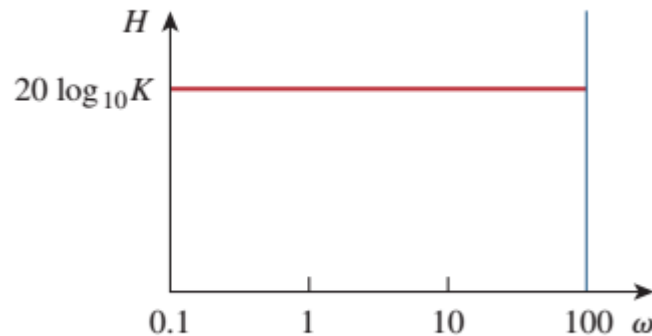
In constructing a Bode plot, we plot each factor separately and then add them graphically. The factors can be considered one at a time and then combined additively because of the logarithms involved. It is this mathematical convenience of the logarithm that makes Bode plots a powerful engineering tool.



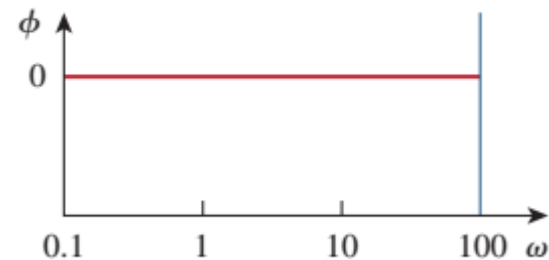
Constant term K

$$\mathbf{H}(\omega) = \frac{K(j\omega)^{\pm 1} (1 + j\omega/z_1)[1 + j2\zeta_1\omega/\omega_k + (j\omega/\omega_k)^2]\cdots}{(1 + j\omega/p_1)[1 + j2\zeta_2\omega/\omega_n + (j\omega/\omega_n)^2]\cdots}$$

$K > 0$



(a)



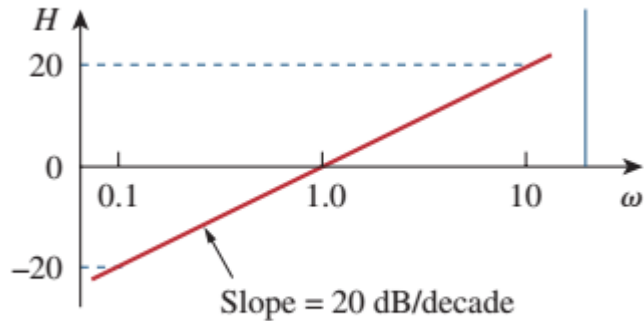
(b)

$K < 0$

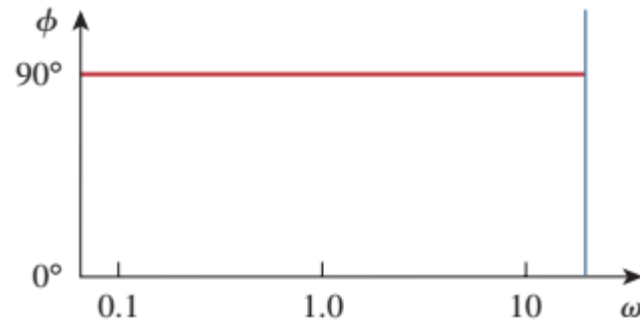


$j\omega$

$$\mathbf{H}(\omega) = \frac{K(j\omega)^{\pm 1} (1 + j\omega/z_1)[1 + j2\zeta_1\omega/\omega_k + (j\omega/\omega_k)^2]\cdots}{(1 + j\omega/p_1)[1 + j2\zeta_2\omega/\omega_n + (j\omega/\omega_n)^2]\cdots}$$



(a)



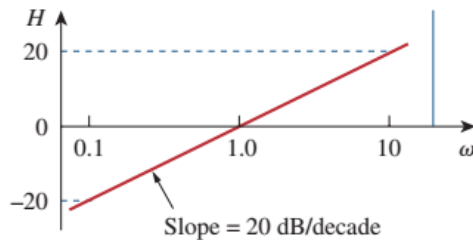
(b)

$(j\omega)^{-1}$

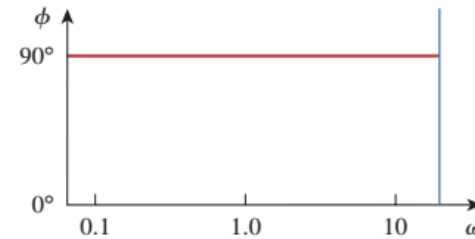


$j\omega$

$$\mathbf{H}(\omega) = \frac{K(j\omega)^{\pm 1} (1 + j\omega/z_1)[1 + j2\zeta_1\omega/\omega_k + (j\omega/\omega_k)^2] \cdots}{(1 + j\omega/p_1)[1 + j2\zeta_2\omega/\omega_n + (j\omega/\omega_n)^2] \cdots}$$

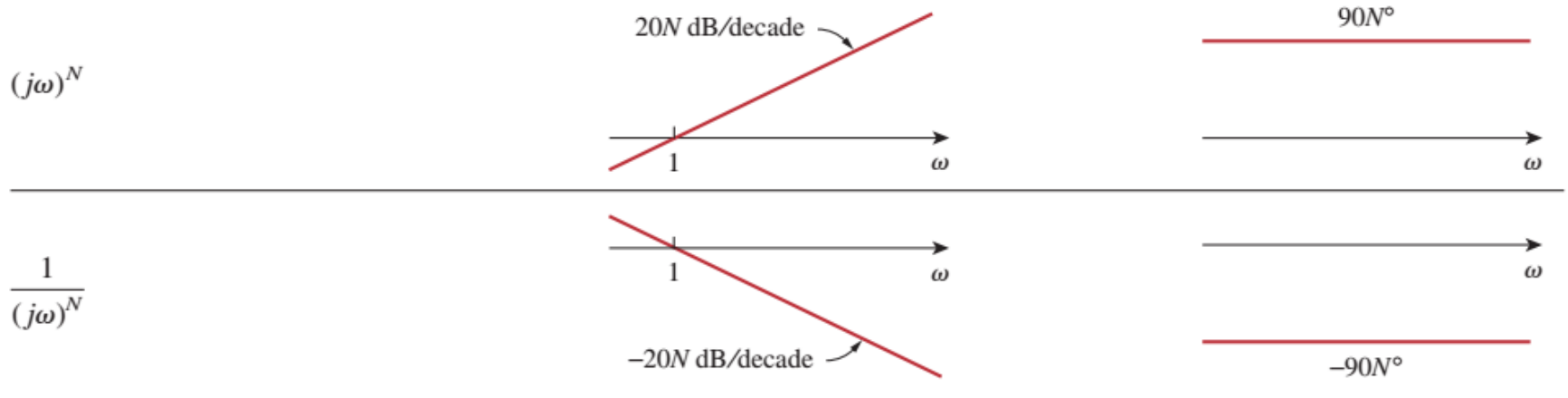


(a)



(b)

- In general:





$$1 + j\omega/z_1$$

$$\mathbf{H}(\omega) = \frac{K(j\omega)^{\pm 1} (1 + j\omega/z_1)[1 + j2\zeta_1\omega/\omega_k + (j\omega/\omega_k)^2]\cdots}{(1 + j\omega/p_1)[1 + j2\zeta_2\omega/\omega_n + (j\omega/\omega_n)^2]\cdots}$$

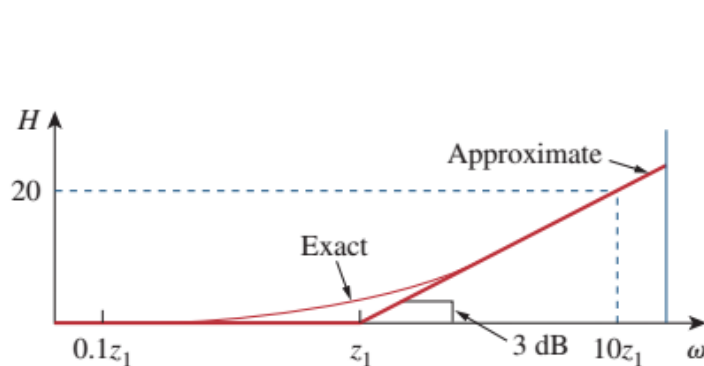
Simple pole/zero: For the simple zero $(1 + j\omega/z_1)$, the magnitude is $20 \log_{10} |1 + j\omega/z_1|$ and the phase is $\tan^{-1} \omega/z_1$. We notice that

$$H_{\text{dB}} = 20 \log_{10} \left| 1 + \frac{j\omega}{z_1} \right| \Rightarrow 20 \log_{10} 1 = 0$$

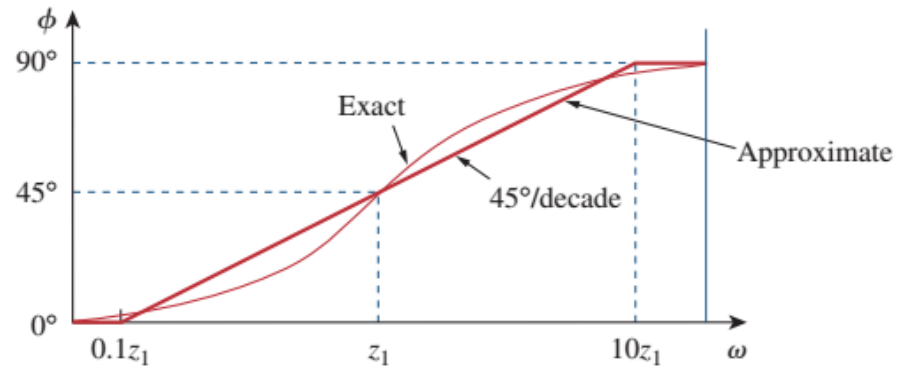
as $\omega \rightarrow 0$

$$H_{\text{dB}} = 20 \log_{10} \left| 1 + \frac{j\omega}{z_1} \right| \Rightarrow 20 \log_{10} \frac{\omega}{z_1}$$

as $\omega \rightarrow \infty$



(a)



(b)



$$1/(1+j\omega/p_1)$$



$$1/[1+2j\zeta_2\omega/\omega_n + (j\omega/\omega_n)^2]$$

$$\mathbf{H}(\omega) = \frac{K(j\omega)^{\pm 1} (1 + j\omega/z_1)[1 + j2\zeta_1\omega/\omega_k + (j\omega/\omega_k)^2]\cdots}{(1 + j\omega/p_1)[1 + j2\zeta_2\omega/\omega_n + (j\omega/\omega_n)^2]\cdots}$$

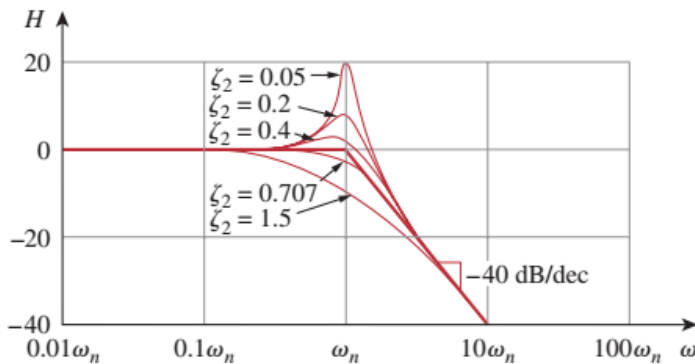
Magnitude:

$$H_{dB} = -20 \log_{10} \left| 1 + \frac{j2\zeta_2\omega}{\omega_n} + \left(\frac{j\omega}{\omega_n}\right)^2 \right| \Rightarrow 0$$

and

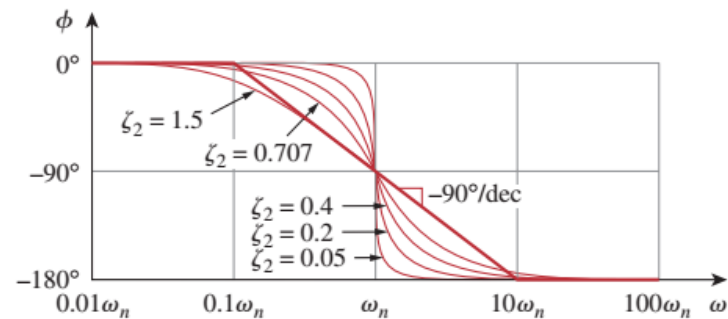
$$H_{dB} = -20 \log_{10} \left| 1 + \frac{j2\zeta_2\omega}{\omega_n} + \left(\frac{j\omega}{\omega_n}\right)^2 \right| \Rightarrow -40 \log_{10} \frac{\omega}{\omega_n}$$

as $\omega \rightarrow \infty$



(a)

the phase is $-\tan^{-1}(2\zeta_2\omega/\omega_n)/(1 - \omega^2/\omega_n^2)$.



(b)



$$1 + 2j\zeta_1\omega/\omega_k + (j\omega/\omega_k)^2$$

$$\mathbf{H}(\omega) = \frac{K(j\omega)^{\pm 1} (1 + j\omega/z_1)[1 + j2\zeta_1\omega/\omega_k + (j\omega/\omega_k)^2]\cdots}{(1 + j\omega/p_1)[1 + j2\zeta_2\omega/\omega_n + (j\omega/\omega_n)^2]\cdots}$$



TABLE 14.3

Summary of Bode straight-line magnitude and phase plots.

Factor	Magnitude	Phase
K	$20 \log_{10} K$ 	0°
$(j\omega)^N$	$20N \text{ dB/decade}$ 	$90N^\circ$
$\frac{1}{(j\omega)^N}$	$-20N \text{ dB/decade}$ 	$-90N^\circ$
$\left(1 + \frac{j\omega}{z}\right)^N$	$20N \text{ dB/decade}$ 	0° to $90N^\circ$
$\frac{1}{(1 + j\omega/p)^N}$	$-20N \text{ dB/decade}$ 	0° to $-90N^\circ$
$\left[1 + \frac{2j\omega\zeta}{\omega_n} + \left(\frac{j\omega}{\omega_n}\right)^2\right]^N$	$40N \text{ dB/decade}$ 	0° to $180N^\circ$
$\frac{1}{[1 + 2j\omega\zeta/\omega_k + (j\omega/\omega_k)^2]^N}$	$-40N \text{ dB/decade}$ 	0° to $-180N^\circ$



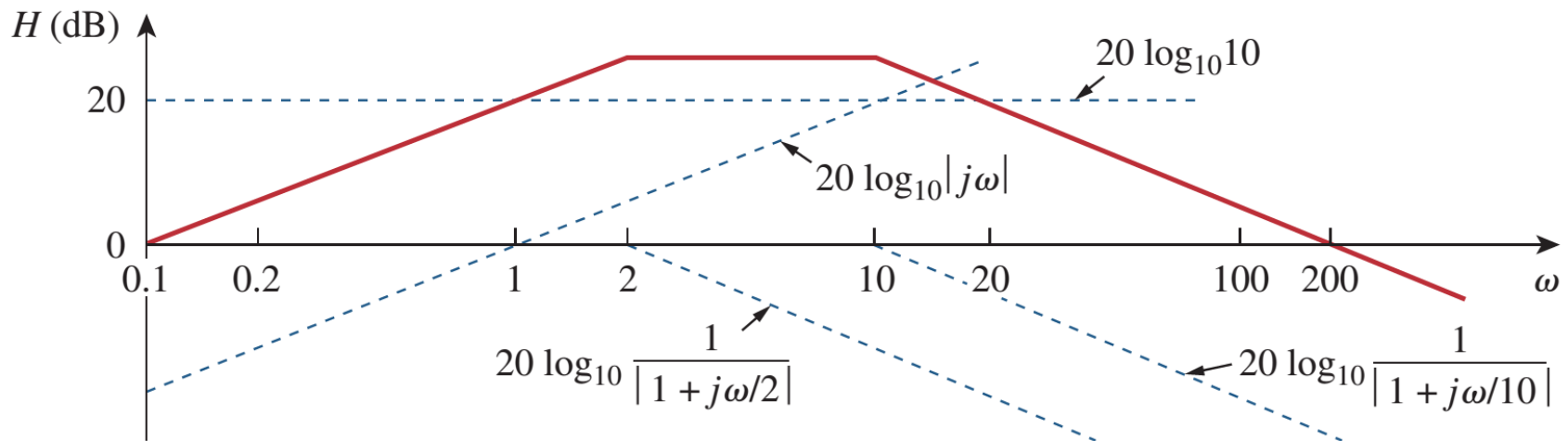
Example--Standard Form

$$\mathbf{H}(\omega) = \frac{200j\omega}{(j\omega + 2)(j\omega + 10)}$$

$$\mathbf{H}(\omega) = \frac{10j\omega}{(1 + j\omega/2)(1 + j\omega/10)}$$



Example - Magnitude

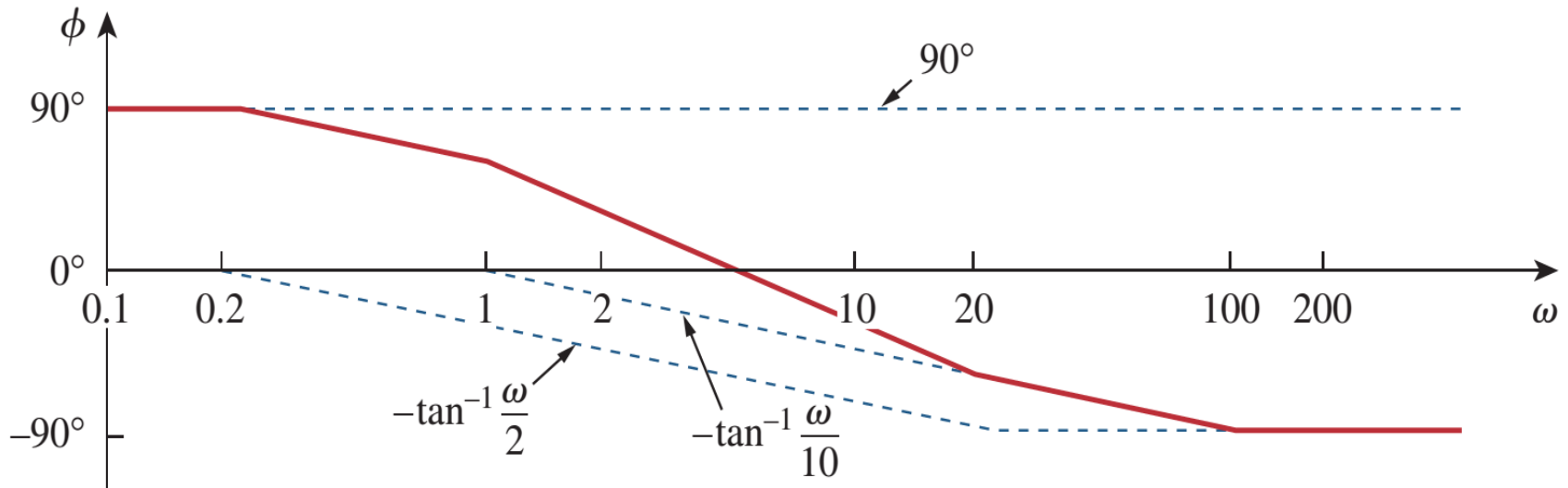


$$H_{\text{dB}} = 20 \log_{10} 10 + 20 \log_{10} |j\omega| - 20 \log_{10} \left| 1 + \frac{j\omega}{2} \right| - 20 \log_{10} \left| 1 + \frac{j\omega}{10} \right|$$



Example - Phase

$$\begin{aligned} \mathbf{H}(\omega) &= \frac{10j\omega}{(1 + j\omega/2)(1 + j\omega/10)} \\ &= \frac{10|j\omega|}{|1 + j\omega/2||1 + j\omega/10|} \angle 90^\circ - \tan^{-1} \omega/2 - \tan^{-1} \omega/10 \\ \phi &= 90^\circ - \tan^{-1} \frac{\omega}{2} - \tan^{-1} \frac{\omega}{10} \end{aligned}$$





Exercises

- $$\mathbf{H}(\omega) = \frac{(20 + j4\omega)^2}{j40\omega(100 + j2\omega)}$$

- $$\mathbf{H}(\omega) = \frac{(j10\omega + 30)^2}{(300 - 3\omega^2 + j90\omega)}$$



Obtain the transfer function

