EE150 -Signals and Systems, Fall 2024

Homework Set #7

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Problem 1 (20 pt)

1. Determine the **unilateral Laplace transform** of each of the following signals, and specify the corresponding regions of convergence:

   (a) $x(t) = e^{-2t}u(t+1) + e^{-4t}u(t)$ (b) $x(t) = \delta(t+2) + \delta(t) + e^{-3(t+2)}u(t+2)$

2. Determine the **Laplace transform** and the associated region of convergence for each of the following functions of time:

   (a) $x(t) = te^{-3|t|}$ (b) $x(t) = \delta(2t) + u(2t) + e^{-5t}sin(5t)u(t)$

Solution:

1. (a) $\chi(s) = \frac{1}{s+2} + \frac{1}{s+4}$ $Re\{s\} > -2$

   (b) $\chi(s) = 1 + \frac{e^{-6}}{s+3}$ $Re\{s\} > -3$

2. (a) $x(t) = te^{-3t}u(t) + te^{3t}u(-t)$

$$e^{-3t}u(t) \overset{LT}{\leftrightarrow} \frac{1}{s+3} \quad Re\{s\} > -3 \qquad e^{3t}u(-t) \overset{LT}{\leftrightarrow} -\frac{1}{s-3} \quad Re\{s\} < 3$$

$$te^{-3t}u(t) \overset{LT}{\leftrightarrow} -\frac{d}{ds}\left(\frac{1}{s+3}\right) = \frac{1}{(s+3)^2} \quad Re\{s\} > -3$$

$$te^{3t}u(-t) \overset{LT}{\leftrightarrow} -\frac{d}{ds}\left(-\frac{1}{s-3}\right) = -\frac{1}{(s-3)^2} \quad Re\{s\} < 3$$

$$X(s) = \frac{1}{(s+3)^2} - \frac{1}{(s-3)^2} = \frac{-12s}{(s^2-9)^2} \quad -3 < Re\{s\} < 3$$

$$(b)\ \delta(2t) = \frac{1}{2}\delta(t) \quad u(2t) = u(t) \qquad sin(5t)\,u(t) \overset{LT}{\leftrightarrow} \frac{5}{s^2+25}$$

$$e^{-5t}\,sin(5t)\,u(t) \overset{LT}{\leftrightarrow} \frac{5}{(s+5)^2+25} \quad Re\{s\} > -5$$

$$X(s) = \frac{1}{2} + \frac{1}{s} + \frac{5}{(s+5)^2+25} \quad Re\{s\} > 0$$

Problem 2 (10 pt)

Consider a signal $y(t)$ obtained by convolving two signals $x_1(t-3)$ and $x_2(-t+2)$

$$y(t) = x_1(t-3) * x_2(-t+2)$$

where $x_1(t) = e^{-2t}u(t)$ *and* $x_2(t) = e^{-3t}u(t)$

use properties of the Laplace transform to determine the Laplace transform $Y(s)$ of $y(t)$.

Solution:

$$x_1(t) = e^{-2t}u(t) \overset{LT}{\leftrightarrow} \frac{1}{s+2} \quad Re\{s\} > -2 \qquad x_2(t) = e^{-3t}u(t) \overset{LT}{\leftrightarrow} \frac{1}{s+3} \quad Re\{s\} > -3$$

$$x_1(t-3) \overset{LT}{\leftrightarrow} \frac{1}{s+2}e^{-3s} \quad Re\{s\} > -2$$

$$x_2(t+2) \overset{LT}{\leftrightarrow} \frac{1}{s+2}e^{2s} \quad Re\{s\} > -3 \qquad x_2(-t+2) \overset{LT}{\leftrightarrow} \frac{1}{-s+2}e^{-2s} \quad Re\{s\} < 3$$

$$y(t) = x_1(t-3) * x_2(-t+2)$$

$$Y(s) = \frac{1}{s+2}e^{-3s} \cdot \frac{1}{-s+3}e^{-2s} = -\frac{e^{-5s}}{(s+2)(s-3)} \qquad -2 < Re\{s\} < 3$$

Problem 3 (20 pt)

Consider a continuous-time LTI system for which the input $x(t)$ and output $y(t)$ are related by the differential equation

$$\frac{d^2y(t)}{dt^2} - 2\frac{dy(t)}{dt} - 3y(t) = x(t)$$

Let $X(s)$ and $Y(s)$ denote Laplace transforms of $x(t)$ and $y(t)$, respectively, and let $H(s)$ denote the Laplace transform of $h(t)$, the system impulse response.

(a) Determine $H(s)$ as a ratio of two polynomials in s. Sketch the pole-zero pattern of $H(s)$.

(b) Determine $h(t)$ for each of the following cases:

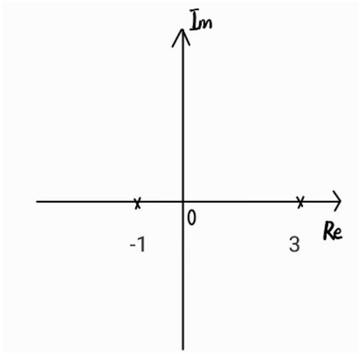
1. The system is stable.
2. The system is causal.
3. The system is neither stable nor causal

Solution:

(a) $H(s) = \dfrac{1}{s^2 - 2s - 3} = \dfrac{1}{(s-3)(s+1)}$

There are no finite zeros, there are two poles :$s = 3$ and $s = -1$

The pole-zero plot is shown in Figure



$$H(s) = \frac{\frac{1}{4}}{(s-3)} + \frac{-\frac{1}{4}}{(s+1)}$$

(i) If the system is stable, then the ROC of $H(s)$ must be $-1 < Re\{s\} < 3$

$$h(t) = -\frac{1}{4}e^{3t}u(-t) - \frac{1}{4}e^{-t}u(t)$$

(ii) If the system is causal, then the ROC of $H(s)$ must be $Re\{s\} > 3$

$$h(t) = \frac{1}{4}e^{3t}u(t) - \frac{1}{4}e^{-t}u(t)$$

(iii) If the system is neither stable nor causal, then the ROC of $H(s)$ must be $Re\{s\} < -1$

$$h(t) = -\frac{1}{4}e^{3t}u(-t) + \frac{1}{4}e^{-t}u(-t)$$

Problem 4 (15 pt)

Suppose we are given the following information about a causal and stable LTI system S with impulse response $h(t)$ and a rational system function $H(s)$:

1.$H(1) = 0.1$.

2. When the input is $u(t)$, the output is absolutely integrable.

3. When the input is $tu(t)$, the output is not absolutely integrable.

4. The signal $\frac{d^2h(t)}{dt^2} + \frac{4dh(t)}{dt} + 5h(t)$is of finite duration.

5.$H(s)$ has exactly one zero at infinity.

Determine $H(s)$ and its region of convergence.

Solution:

According to information (2), when the input is $u(t)$, the Laplace transform of the output signal is$\frac{1}{s}H(s)$, and this output is absolutely integrable, indicating that the ROC of $H(s)$ contains the jw-axis. In other words, $H(s)$ has a zero point at $s = 0$.

According to information (3), when the input is $tu(t)$, the output is not absolutely integrable, indicating that the ROC of $\frac{1}{s^2}H(s)$ does not contain jw axis, so it can be concluded that $s = 0$ is only the first-order zero of $H(s)$.

According to information (4), the Laplace transform of the finite-width signal has the ROC of the full s-plane. Since the Laplace transform of the signal $\frac{d^2h(t)}{dt^2} + \frac{4dh(t)}{dt} + 5h(t)$ is $(s^2 + 4s + 5)H(s)$, it can be deduced that $(s^2 + 4s + 5)H(s)$ has no finite poles. In other words, the denominator of $H(s)$ is $s^2 + 4s + 5$.

From information (5), $H(s)$ has only one zero point at $s = \infty$, which means that the denominator polynomial of $H(s)$ is only one order higher than the numerator polynomial. We know that the numerator of $H(s)$ contains the factor s, so we can basically write the expression of $H(s)$ as

$$H(s) = \frac{As}{s^2 + 4s + 5}$$

Finally, according to information (1), when $s = 1$, $H(s) = 0.1$, and $A = 1$ can be obtained.

Since both complex poles of $H(s)$ that are conjugate to each other have real parts -2 and the system is causal, the ROC of $H(s)$ can be inferred to be $Re\{s\} > -2$

$$H(s) = \frac{s}{s^2 + 4s + 5} \qquad Re\{s\} > -2$$

Problem 5 (15 pt)

The system function of a continuous system is

$$H(s) = \frac{2s + 4}{s^3 + 3s^2 + 5s + 3}$$

Try to draw the direct, cascaded and parallel block diagrams respectively

Solution:

(a) direct-form

$$H(s) = \frac{2s + 4}{s^3 + 3s^2 + 5s + 3} = \frac{2s^{-2} + 4s^{-3}}{1 + 3s^{-1} + 5s^{-2} + 3s^{-3}}$$

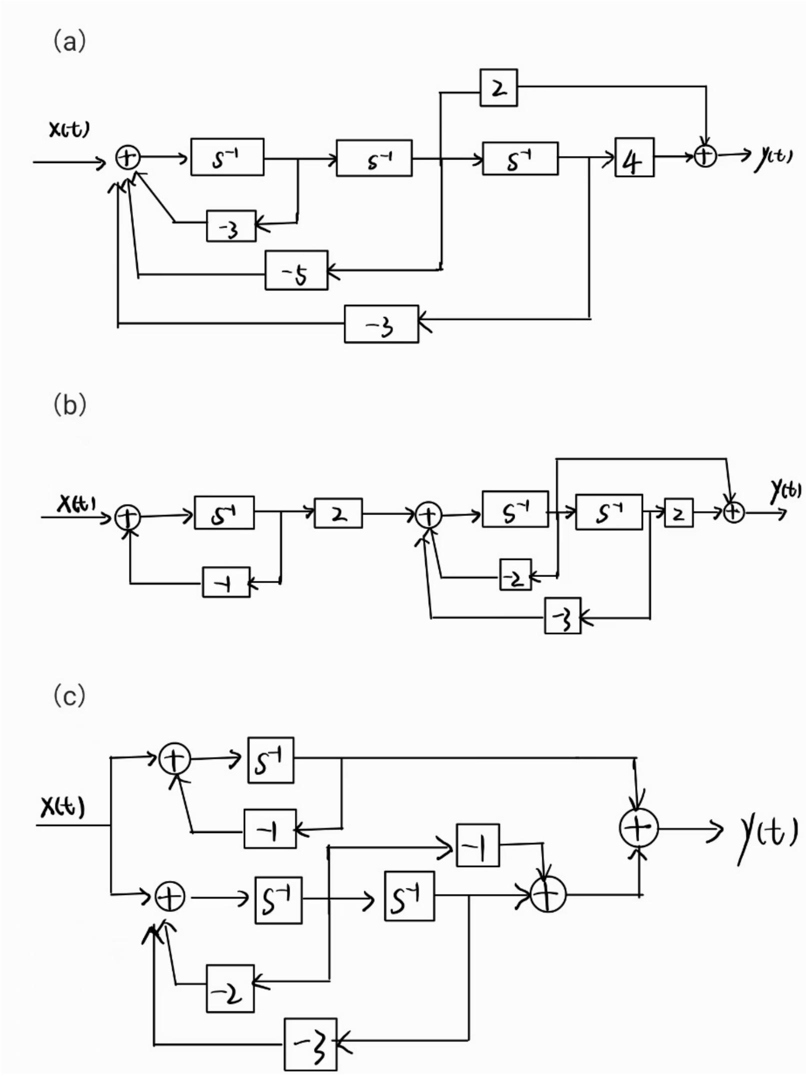
(b) cascade-form

$$H(s)=\frac{2s+4}{s^3+3s^2+5s+3}=\frac{2}{s+1}\cdot\frac{s+2}{s^2+2s+3}=\frac{2s^{-1}}{1+s^{-1}}\cdot\frac{s^{-1}+2s^{-2}}{1+2s^{-1}+3s^{-2}}$$

(c) parallel-form

$$H(s)=\frac{2s+4}{s^3+3s^2+5s+3}=\frac{2}{s+1}*\frac{s+2}{s^2+2s+3}=\frac{1}{s+1}+\frac{-s+1}{s^2+2s+3}$$

$$=\frac{s^{-1}}{1+s^{-1}}+\frac{-s^{-1}+s^{-2}}{1+2s^{-1}+3s^{-2}}$$



Problem 6 (20 pt)

Consider the system S characterized by the differential equation

$$\frac{d^2y(t)}{dt^2}+5\frac{dy(t)}{dt}+6y(t)=x(t)$$

(a) Determine the zero-state response of this system for the input $x(t)=e^{-4t}u(t)$.

(b) Determine the zero-input response of the system for $t > 0$, given that

$$y(0^-) = -2 \quad \frac{dy(t)}{dt}|_{t=0^-} = 1$$

(c) Determine the output of *S* when the input is $x(t) = e^{-4t}u(t)$ and the initial conditions are the same as those specified in part (b).

Solution:

$$s^2\mathcal{Y}(s) - sy(0^-) - y'(0^-) + 5s\mathcal{Y}(s) - 5y(0^-) + 6\mathcal{Y}(s) = \mathcal{X}(s)$$

$$\mathcal{Y}(s) = \frac{(s+5)y(0^-) + y'(0^-)}{s^2 + 5s + 6} + \frac{\mathcal{X}(s)}{s^2 + 5s + 6}$$

(a) $x(t) = e^{-4t}u(t) \quad \mathcal{X}(s) = \frac{1}{s+4} \quad Re\{s\} > -4$

$$\mathcal{Y}_{zs}(s) = \frac{1}{s^2 + 5s + 6} \cdot \frac{1}{s+4} = \frac{1}{(s+2)(s+3)(s+4)}$$

$$= \frac{\frac{1}{2}}{s+2} + \frac{-1}{s+3} + \frac{\frac{1}{2}}{s+4} \quad Re\{s\} > -2$$

$$y_{zs}(t) = \left(\frac{1}{2}e^{-2t} - e^{-3t} + \frac{1}{2}e^{-4t}\right)u(t)$$

(b) $\mathcal{Y}_{zi}(s) = \frac{(s+5)y(0^-)+y'(0^-)}{s^2+5s+6} = \frac{-2s-9}{(s+3)(s+2)} = -\frac{5}{s+2} + \frac{3}{s+3}$

$$y_{zi}(t) = -5e^{-2t}u(t) + 3e^{-3t}u(t)$$

(c) $y(t) = y_{zs}(t) + y_{zi}(t) = \left(-\frac{9}{2}e^{-2t} + 2e^{-3t} + \frac{1}{2}e^{-4t}\right)u(t)$