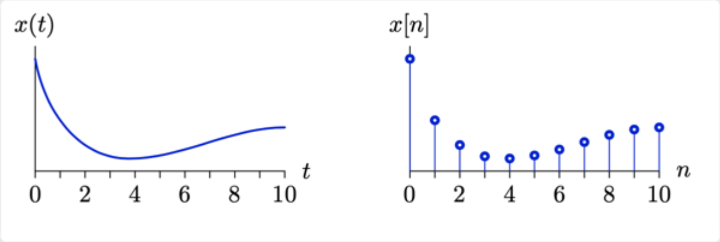
EE150 Signals and Systems
– Part 7: Sampling

May 8, 2024

Sampling

Conversion of a continuous-time signal to a discrete-time signal.



Sampling

Sampling allows the use of modern digital electronics to process, record, transmit, store, and retrieve CT signals.

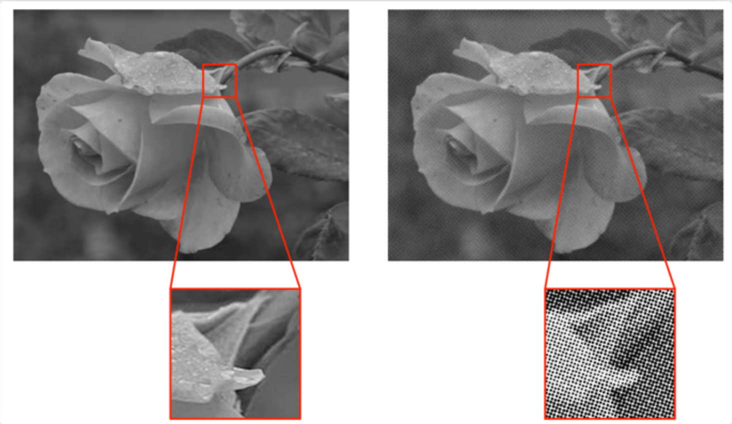
- audio: MP3, CD, cell phone
- pictures: digital camera, printer
- video: DVD
- everything on the web

Sampling

Photographs in newsprint are “half-tone” images. Each point is black or white and the average conveys brightness.

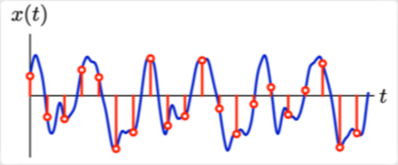
Sampling

Zoom in to see the binary pattern.

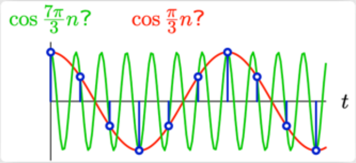


Sampling

We would like to sample in a way that preserves information, which may not seem possible.

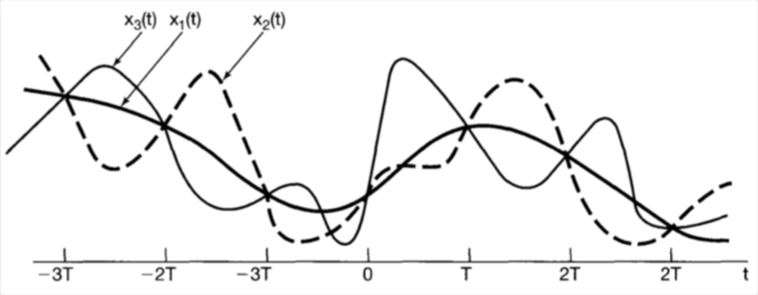


Information between samples is lost. Therefore, the same samples can represent multiple signals.



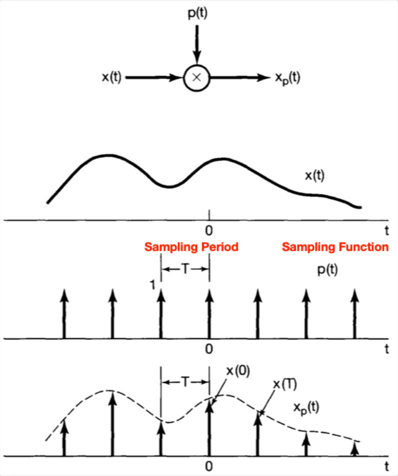
Sampling

Another example: $x_1(kT) = x_2(kT) = x_3(kT)$



Impulse-Train Sampling

We use a periodic impulse train to multiply the continuous-time signal.



$$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$$

$$x_p(t) = x(t)p(t)$$

Impulse-Train Sampling

$$x_p(t) = \sum_{n=-\infty}^{\infty} x(nT)\delta(t - nT)$$

Based on the multiplication property, we have

$$X_p(j\omega) = \frac{1}{2\pi}[X(j\omega) * P(j\omega)]$$

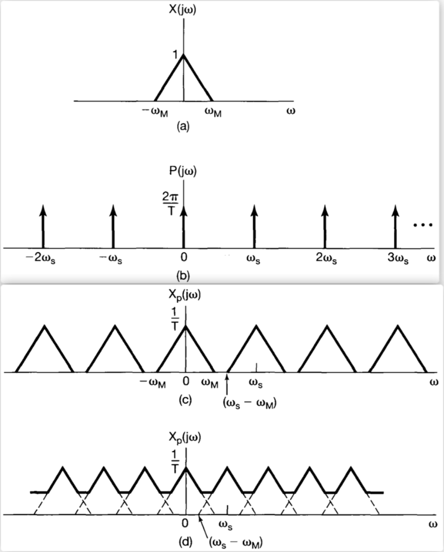
where

$$P(j\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_s)$$

As a result

$$X_p(j\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} X(j(\omega - k\omega_s))$$

Impulse-Train Sampling



Sampling Theorem

Theorem (Sampling Theorem)

Let $x(t)$ be a band-limited signal with $X(j\omega) = 0$ for $|\omega| > \omega_M$. Then $x(t)$ is uniquely determined by its samples $x(nT)$, $n = \cdots, -2, -1, 0, 1, 2, \cdots$, if $\omega_s > 2\omega_M$, where $\omega_s = \frac{2\pi}{T}$.

Given these samples, we can reconstruct $x(t)$ by generating a periodic impulse train in which successive impulses have amplitudes that are successive sample values. This impulse train is then processed through an ideal low-pass filter with gain T and cutoff frequency greater than ω_M and less than $\omega_s - \omega_M$. The resulting output signal will exactly equal $x(t)$.

Sampling Theorem

If signal is band-limited $\rightarrow$ sample without loosing information.

If $x(t)$ is band-limited so that

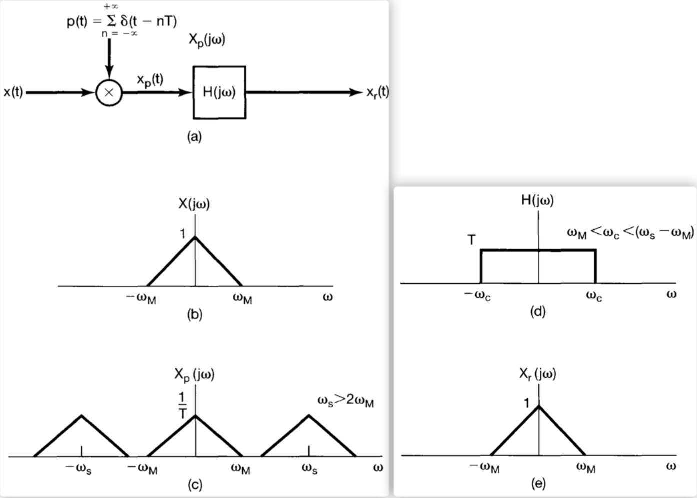
$$X(j\omega) = 0 \quad \text{for} \quad |\omega| > \omega_M$$

Then $x(t)$ is uniquely determined by its samples $x(nT)$ if

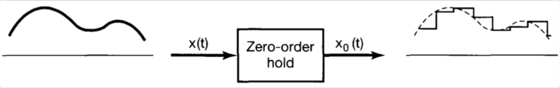
$$\omega_s = \frac{2\pi}{T} > 2\omega_M$$

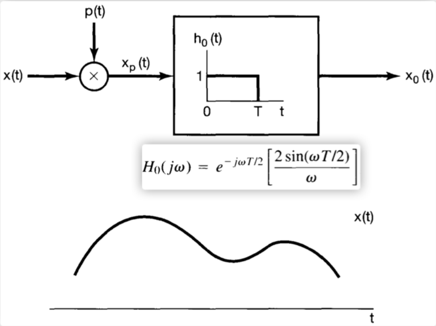
The minimum sampling frequency, $2\omega_m$, is called the "Nyquist rate"

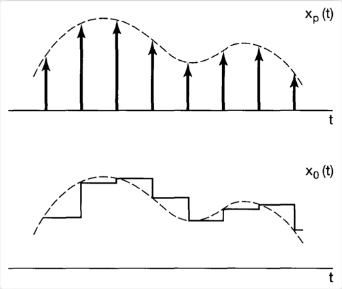
Impulse-Train Sampling



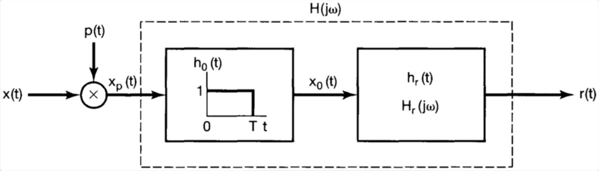
In practice, narrow and large-amplitude pulses are difficult to generate.



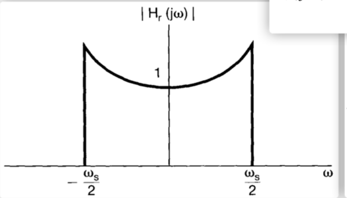


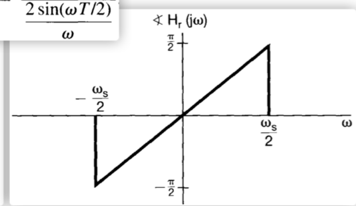


Sampling with a Zero-Order Hold



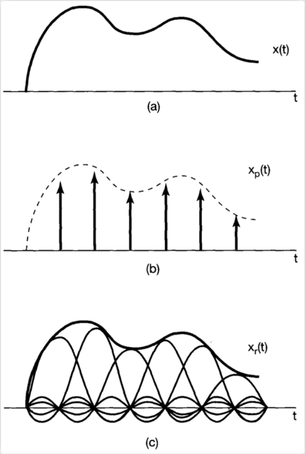
$$H_r(j\omega) = \frac{e^{j\omega T/2} H(j\omega)}{\frac{2\sin(\omega T/2)}{\omega}}$$

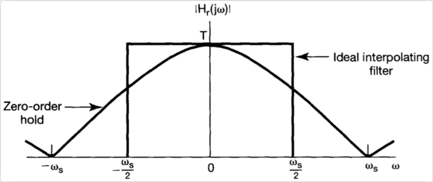




Signal Reconstruction Using Interpolation

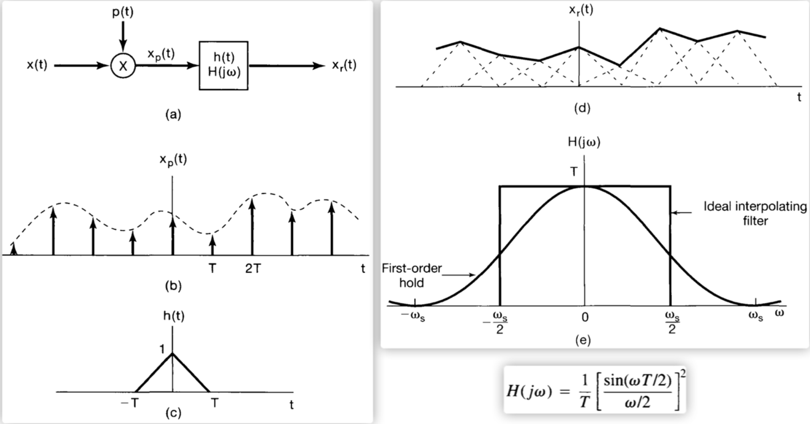
Interpolation, the fitting of a continuous signal to a set of sample values, is a commonly used procedure for reconstructing a function, either approximately or exactly, from samples.





$$
\begin{aligned}
x_r(t) &= x_p(t) * h(t) \\
&= \sum_{n=-\infty}^{\infty} x(nT)h(t-nT) \\
&= \sum_{n=-\infty}^{\infty} x(nT)\frac{\omega_c T}{\pi}\frac{\sin(\omega_c(t-nT))}{\omega_c(t-nT)}
\end{aligned}
$$

Signal Reconstruction Using Interpolation



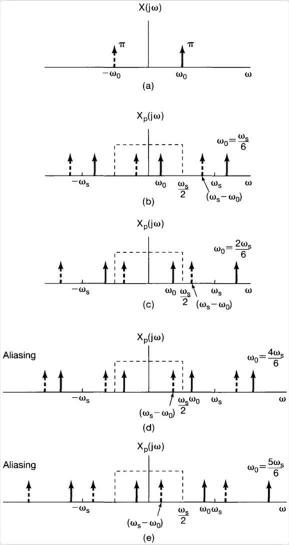
$$H(j\omega) = \frac{1}{T}\left[\frac{\sin(\omega T/2)}{\omega/2}\right]^2$$

Effect of Undersampling: Aliasing

When $\omega_s < 2\omega_M$, the spectrum of $x(t)$, is no longer replicated in $X_p(j\omega)$ and thus is no longer recoverable by low-pass filtering. This effect is referred to as aliasing.

The original signal and the signal $x_r(t)$ that is reconstructed using band-limited interpolation will always be equal at the sampling instants, i.e., $x_r(nT) = x(nT)$.

Effect of Undersampling: Aliasing



$$x(t) = \cos(\omega_0 t)$$

$$\omega_0 = \frac{\omega_s}{6}; x_r(t) = \cos(\omega_0 t) = x(t)$$

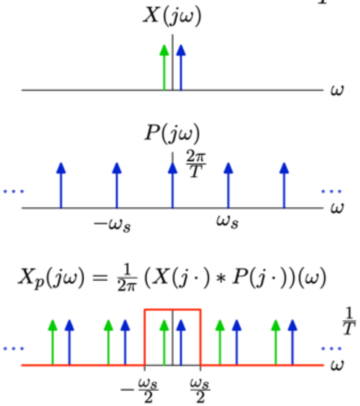
$$\omega_0 = \frac{2\omega_s}{6}; x_r(t) = \cos(\omega_0 t) = x(t)$$

$$\omega_0 = \frac{4\omega_s}{6}; x_r(t) = \cos(\omega_s - \omega_0)t \neq x(t)$$

$$\omega_0 = \frac{5\omega_s}{6}; x_r(t) = \cos(\omega_s - \omega_0)t \neq x(t)$$

Effect of Undersampling: Aliasing

What happens if X contains frequencies $|\omega| > \frac{\pi}{T}$?



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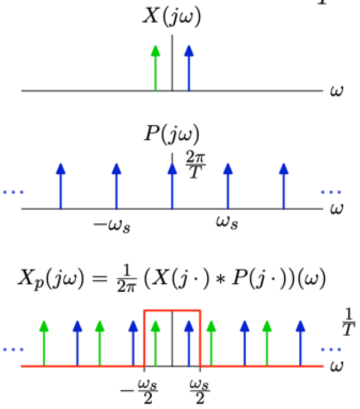

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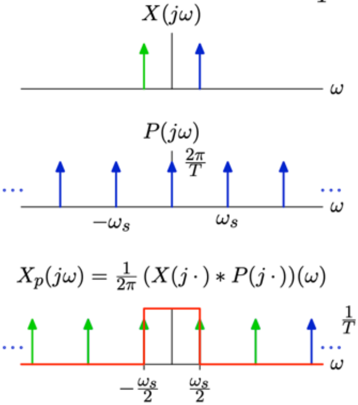

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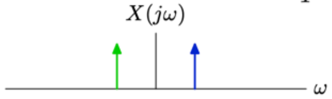
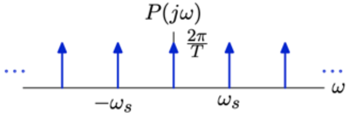
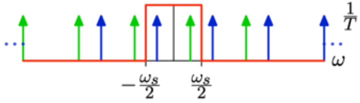


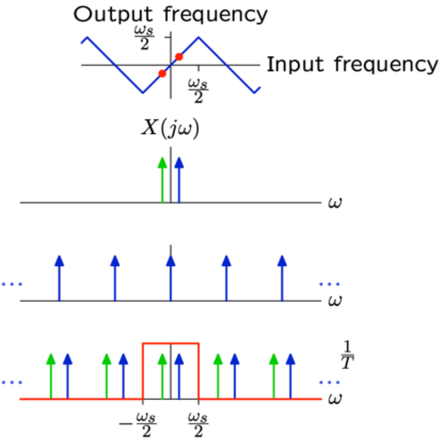


$$X_p(j\omega) = \frac{1}{2\pi}\left(X(j\,\cdot\,) * P(j\,\cdot\,)\right)(\omega)$$



Effect of Undersampling: Aliasing

The effect of aliasing is to wrap frequencies.



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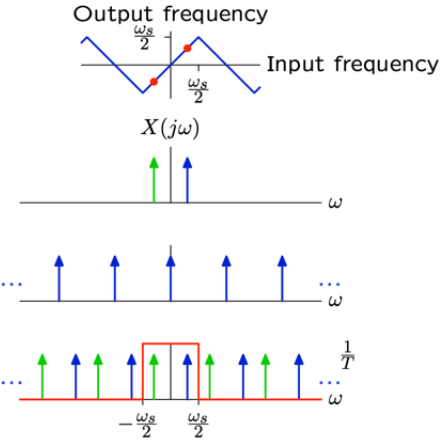

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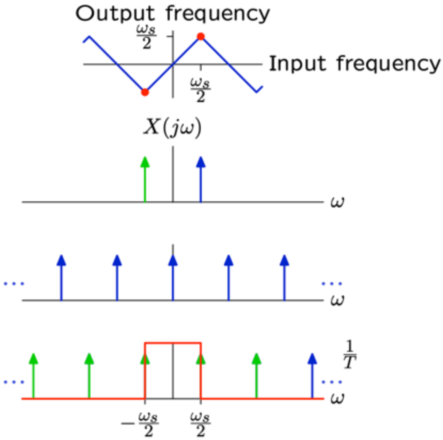

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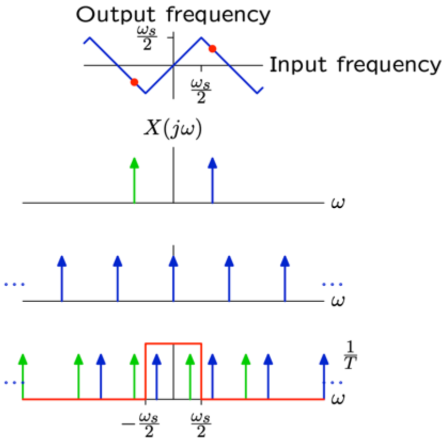
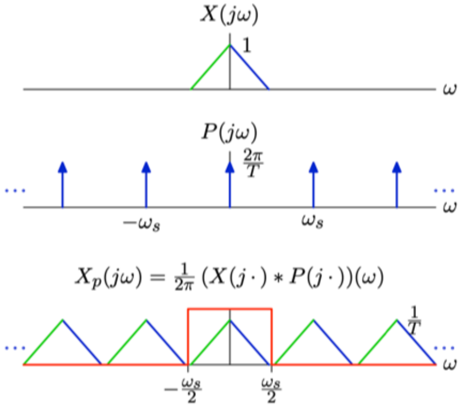


Effect of Undersampling: Aliasing

Aliasing increases as the sampling rate decreases.



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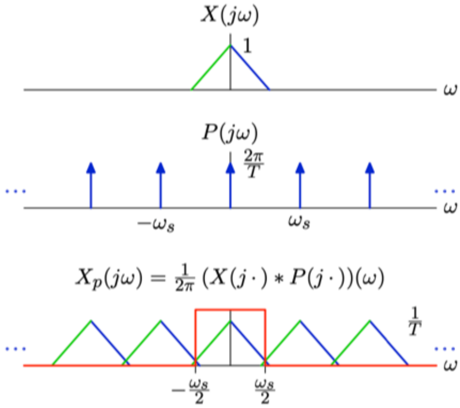

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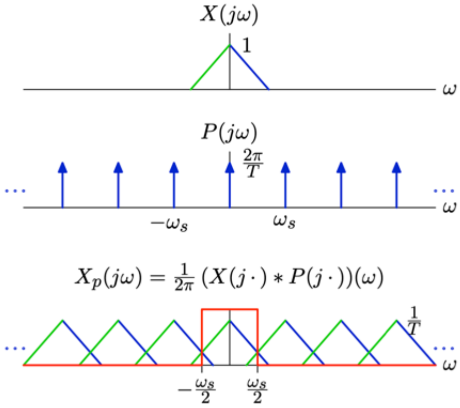

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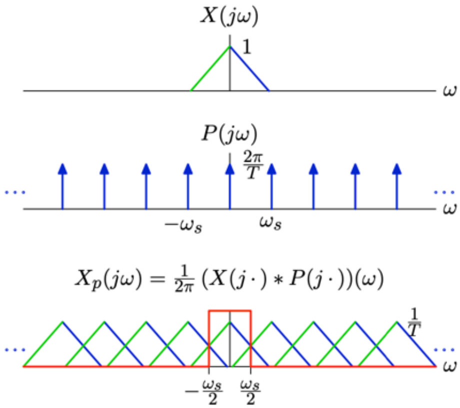

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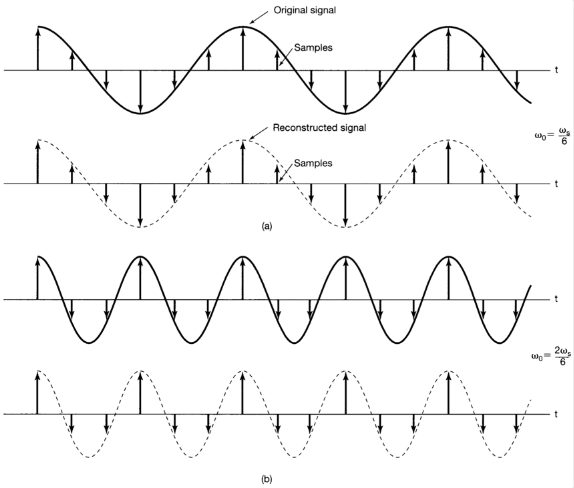

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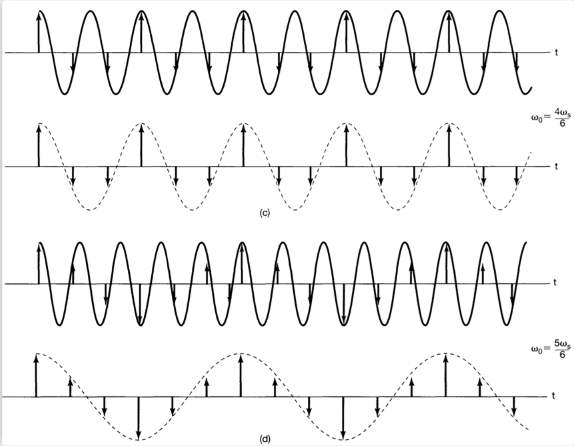
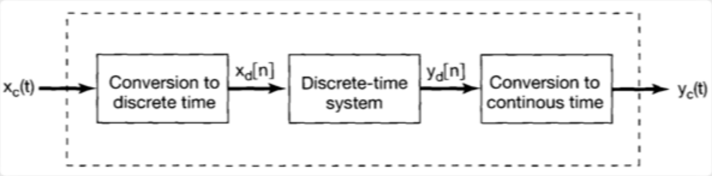
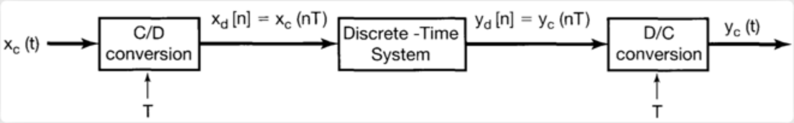


Discrete-Time Processing of Continuous-Time Signals





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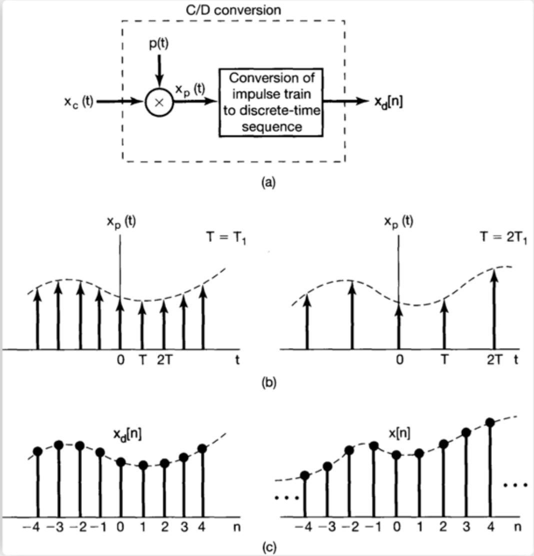


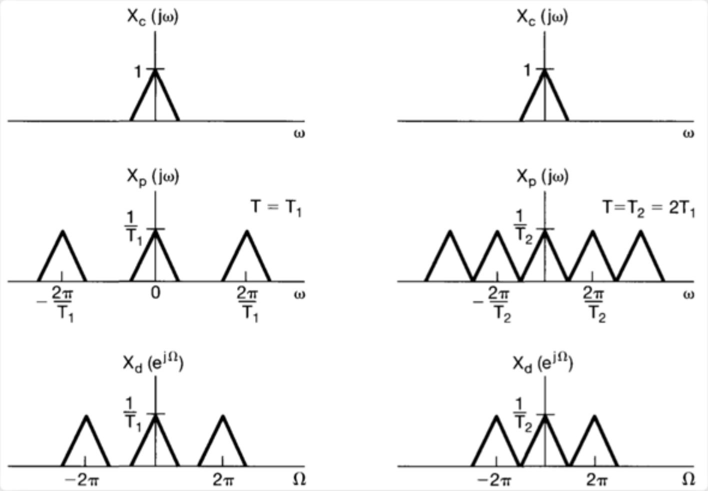
$$x_p(t) = \sum_{n=-\infty}^{\infty} x_c(nT)\delta(t - nT)$$

As $\mathcal{F}\{\delta(t - nT)\} = e^{-j\omega nT}$, we have

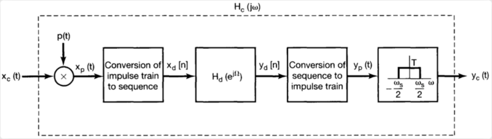
$$X_p(j\omega) = \sum_{n=-\infty}^{\infty} x_c(nT)e^{-j\omega nT}$$

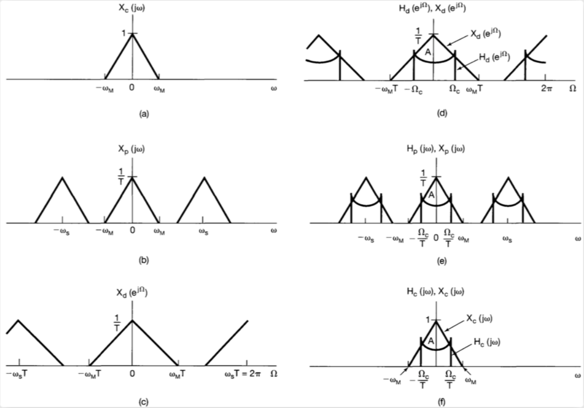
$$\begin{aligned}
X_d(e^{j\Omega}) &= \sum_{n=-\infty}^{\infty} x_d[n]e^{-j\Omega n} = \sum_{n=-\infty}^{\infty} x_c(nT)e^{-j\Omega n} = X_p(j\Omega/T) \\
&= \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c(j(\Omega - 2\pi k)/T)
\end{aligned}$$

Discrete-Time Processing of Continuous-Time Signals

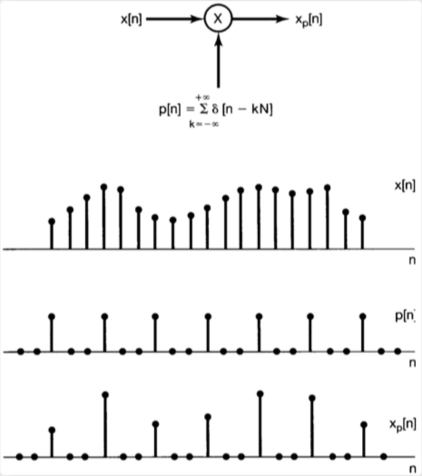


Discrete-Time Processing of Continuous-Time Signals





Sampling of Discrete-Time Signals



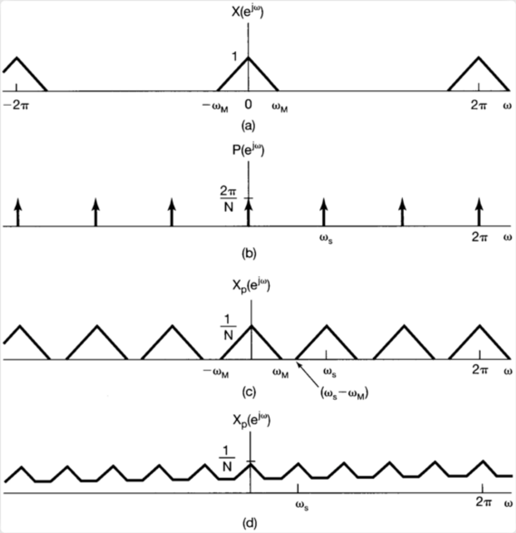
$$x_p[n] = x[n]p[n] = \sum_{n=-\infty}^{\infty} x[kN]\delta[n-kN]$$

$$X_p(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} P(e^{j\theta})X(e^{j(\omega-\theta)})d\theta$$

$$P(e^{j\omega}) = \frac{2\pi}{N}\sum_{k=-\infty}^{\infty}\delta(\omega-k\omega_s)$$

$$X_p(e^{j\omega}) = \frac{1}{N}\sum_{k=0}^{N-1} X(e^{j(\omega-k\omega_s)})$$

Sampling of Discrete-Time Signals



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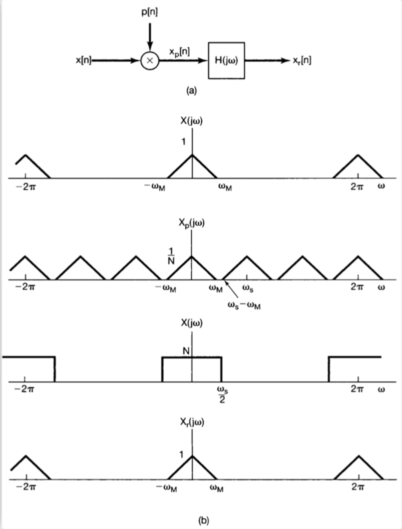

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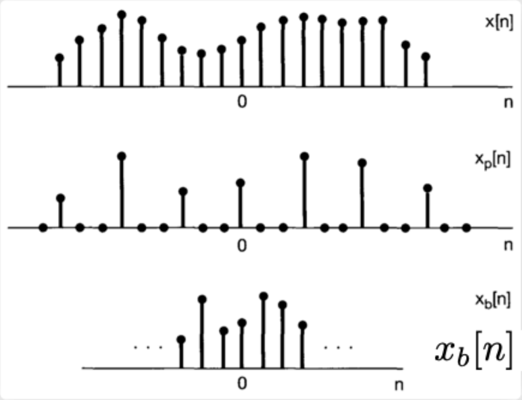
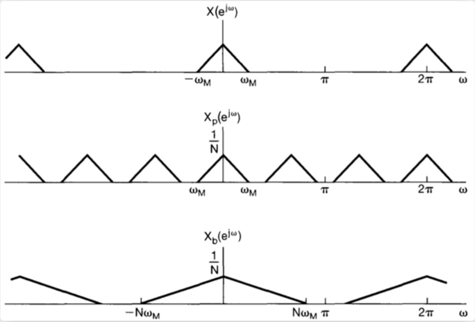


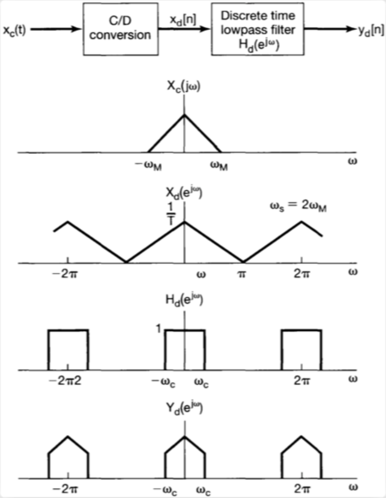
$$x_b[n] = x_p[nN] = x[nN]$$

Sampling of Discrete-Time Signals



$$
\begin{aligned}
X_b(e^{j\omega}) &= \sum_{k=-\infty}^{\infty} x_b[k]e^{-j\omega k} \\
&= \sum_{k=-\infty}^{\infty} x_p[kN]e^{-j\omega k} \\
&= \sum_{n=-\infty}^{\infty} x_p[n]e^{-j\omega n/N} \\
&= X_p(e^{j\omega/N})
\end{aligned}
$$

Sampling of Discrete-Time Signals



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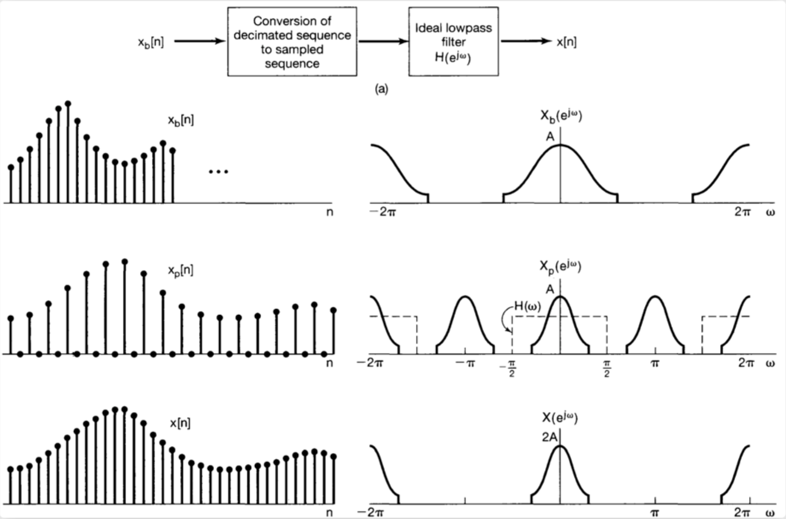

Summary

- Effects of sampling are easy to visualize with Fourier representations
- Signals that are band-limited in frequency (e.g., $-W < \omega < W$) can be sampled without loss of information
- The minimum sampling frequency for sampling without loss of in formation is called the Nyquist rate. The Nyquist rate is twice the highest frequency contained in a band-limited signal
- Sampling at frequencies below the Nyquist rate causes aliasing
- Aliasing can be eliminated by pre-filtering to remove frequency components that would otherwise alias