# Ch.2 *Linear Time-Invariant Systems*

Lecturer: Yijie Mao

# Part III *Properties of LTI Systems*

# Outline

- The Commutative Property
- The Distributive Property
- The Associative Property
- LTI systems with and without memory
- Invertibility for LTI systems
- Causality for LTI systems
- Stability for LTI systems
- The unit step response of LTI systems

# The Commutative Property

- Discrete-time

$$x[n] * h[n] = h[n] * x[n]$$

  - Proof:

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \overset{n-k=m}{=} \sum_{m=-\infty}^{\infty} h[m]x[n-m] = h[n] * x[n]$$

- Continuous-time

$$x(t) * h(t) = h(t) * x(t)$$

  - Proof:

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\, d\tau \overset{t-\tau=\tau'}{=} \int_{-\infty}^{\infty} h(\tau')x(t-\tau')d\tau' = h(t) * x(t)$$

# The Distributive Property

- Discrete-time

$$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$$

- Proof:

$$x[n] * (h_1[n] + h_2[n]) = \sum_{k=-\infty}^{\infty} x[k]\,(h_1[n-k] + h_2[n-k])$$

$$= \sum_{k=-\infty}^{\infty} x[k]\,h_1[n-k] + \sum_{k=-\infty}^{\infty} x[k]\,h_2[n-k]$$

$$= x[n] * h_1[n] + x[n] * h_2[n]$$

# The Distributive Property

- Continuous-time

$$x(t) * \big(h_1(t) + h_2(t)\big) = x(t) * h_1(t) + x(t) * h_2(t)$$

- Proof：

$$x(t) * \big(h_1(t) + h_2(t)\big) = \int_{-\infty}^{\infty} x(\tau)\big(h_1(t-\tau) + h_2(t-\tau)\big)d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)h_1(t-\tau)d\tau + \int_{-\infty}^{\infty} x(\tau)h_2(t-\tau)d\tau$$

$$= x(t) * h_1(t) + x(t) * h_2(t)$$

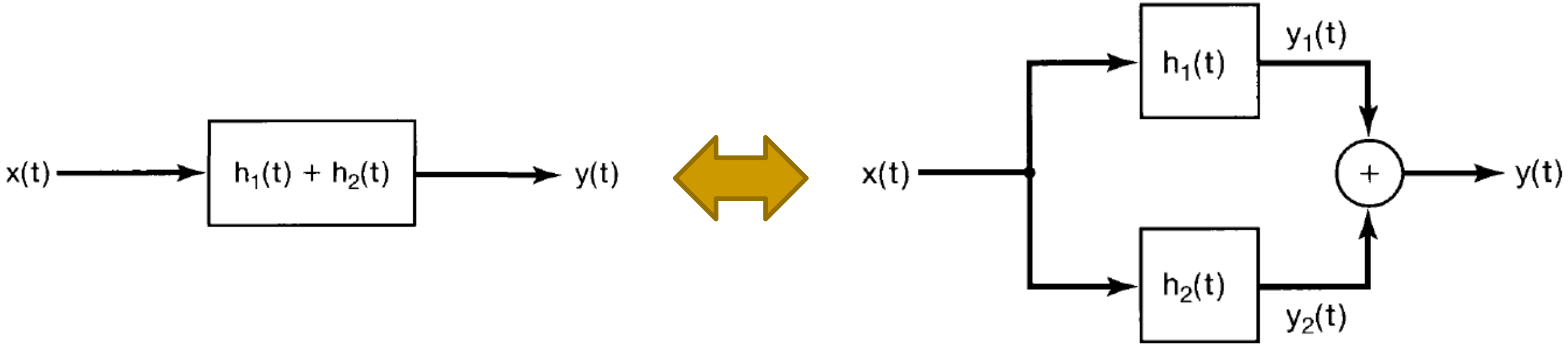
# The Distributive Property

- Discrete-time

$$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$$

- Continuous-time

$$x(t) * \big(h_1(t) + h_2(t)\big) = x(t) * h_1(t) + x(t) * h_2(t)$$



# The Distributive Property

- Discrete-time

$$(x_1[n] + x_2[n]) * h[n] = x_1[n] * h[n] + x_2[n] * h[n]$$

- Continuous-time

$$\big(x_1(t) + x_2(t)\big) * h(t) = x_1(t) * h(t) + x_2(t) * h(t)$$

- The response of an LTI system to the sum of two inputs must equal the sum of the responses of these signals individually.

# The Distributive Property

- Example. Let $y[n]$ denote the convolution of the following two sequences

$$x[n] = 0.5^n u[n] + 2^n u[-n]$$
$$h[n] = u[n]$$

Calculate $y[n]$.

# The Associative Property

- Discrete-time

$$x[n] * (h_1[n] * h_2[n]) = (x[n] * h_1[n]) * h_2[n]$$

- Proof: Let $y[n] = h_1[n] * h_2[n] = \sum_{m=-\infty}^{\infty} h_1[m]h_2[n-m]$

$$x[n] * (h_1[n] * h_2[n]) = x[n] * y[n]$$

$$= \sum_{k=-\infty}^{\infty} x[k]y[n-k] = \sum_{k=-\infty}^{\infty} x[k] \sum_{m=-\infty}^{\infty} h_1[m]h_2[n-k-m]$$

Let $k + m = l$

$$= \sum_{k=-\infty}^{\infty} x[k] \sum_{l=-\infty}^{\infty} h_1[l-k]h_2[n-l]$$

$$= \sum_{l=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x[k]h_1[l-k]\, h_2[n-l]$$

$$= \sum_{l=-\infty}^{\infty} (x[l] * h_1[l])\, h_2[n-l] = (x[n] * h_1[n]) * h_2[n]$$

# The Associative Property

- Continuous-time

$$x(t) * \big(h_1(t) * h_2(t)\big) = \big(x(t) * h_1(t)\big) * h_2\,(t)$$

  - Proof:

$$x(t) * \big(h_1(t) * h_2(t)\big) = x(t) * \int_{-\infty}^{\infty} h_1(\tau)h_2(t-\tau)d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau') \int_{-\infty}^{\infty} h_1(\tau)h_2(t-\tau'-\tau)d\tau\, d\tau'$$

Let $\tau' + \tau = \tau''$

$$= \int_{-\infty}^{\infty} x(\tau') \int_{-\infty}^{\infty} h_1(\tau''-\tau')h_2(t-\tau'')d\tau''\, d\tau'$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau')h_1(\tau''-\tau')d\tau'\; h_2(t-\tau'')d\tau''$$

$$= \int_{-\infty}^{\infty} x(\tau'') * h_1(\tau'')\; h_2(t-\tau'')d\tau''$$

$$= \big(x(t) * h_1(t)\big) * h_2\,(t)$$

# The Associative Property

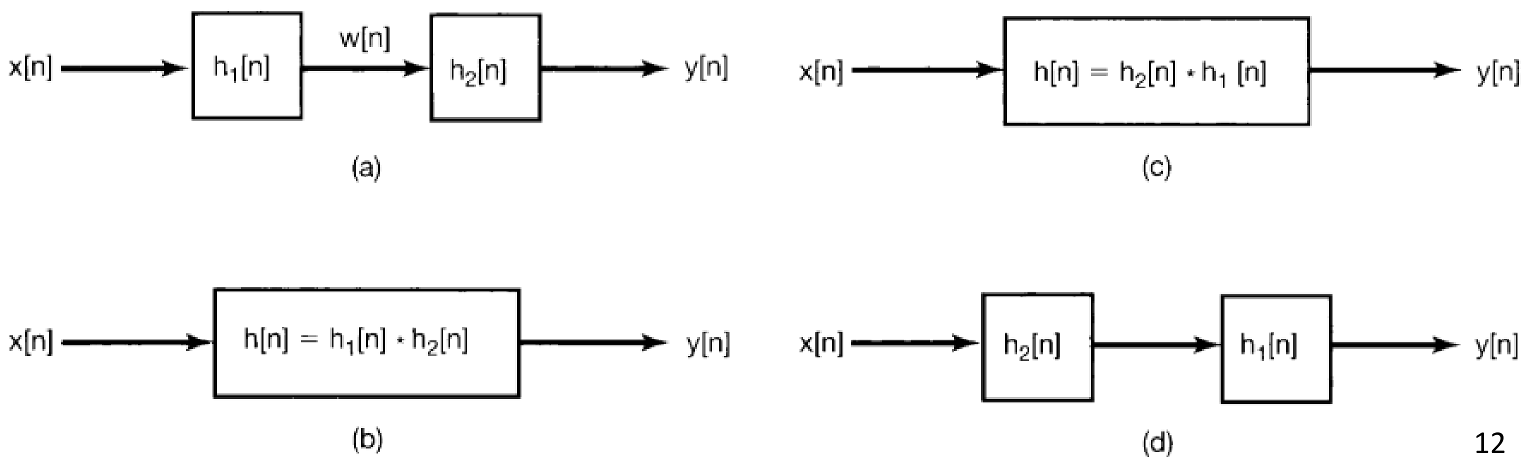
- Discrete-time

$$x[n] * (h_1[n] * h_2[n]) = (x[n] * h_1[n]) * h_2[n]$$

- Continuous-time

$$x(t) * \big(h_1(t) * h_2(t)\big) = \big(x(t) * h_1(t)\big) * h_2\,(t)$$

- The unit impulse response of a cascade of two LTI systems does not depend on the order in which they are cascaded.



# LTI Systems With and Without Memory

- A system is memoryless if its output at any time depends only on the value of the input at that same time.

- Discrete-time system without memory only if

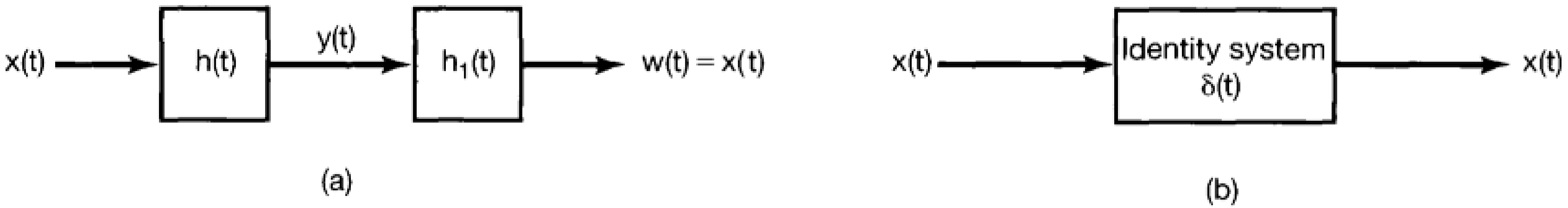
$$h[n] = 0 \text{ for all } n \neq 0$$

- Continuous-time system without memory only if

$$h(t) = 0 \text{ for all } t \neq 0$$

# *Invertibility of LTI Systems*

- A system is invertible only if an inverse system exists.




- In continuous time, the system $h_1(t)$ is the inverse of the system $h_0(t)$ if

$$h_0(t) * h_1(t) = \delta(t)$$

- In discrete time, the system $h_1[n]$ is the inverse system of $h_0[n]$ if

$$h_0[n] * h_1[n] = \delta[n]$$

# *Invertibility of LTI Systems*

- Example: Consider an LTI system with $h_0[n] = u[n]$, determine the inverse system $h_1[n]$.

# *Invertibility of LTI Systems*

- Example: Consider the LTI system consisting of a pure time shift $y(t) = x(t - t_0)$, determine the inverse system.

# Causality for LTI Systems

- For a discrete-time LTI system to be casual, $y[n]$ must not depend on $x[k]$ for $k > n$. Hence, the impulse response

$$h[n] = 0 \text{ for } \mathrm{n} < 0$$

- Thus, convolution sum

$$y[n] = \sum_{k=-\infty}^{n} x[k]h[n-k]$$

- or equivalently

$$y[n] = \sum_{k=0}^{\infty} h[k]x[n-k]$$

# Causality for LTI Systems

- For a continuous-time LTI system to be casual, the impulse response

$$h(t) = 0 \text{ for } t < 0$$

- Thus, convolution integral

$$y(t) = \int_{-\infty}^{t} x(\tau)h(t-\tau)\, d\tau$$

- or equivalently

$$y(t) = \int_{0}^{\infty} h(\tau)x(t-\tau) d\tau$$

# Causality for LTI Systems

- Examples
  - $y[n] = \sum_{l=-\infty}^{n} x[l]$
  - $y[n] = x_u[n] + \frac{1}{2}(x_u[n-1] + x_u[n+1])$

# Stability of LTI Systems

- A system is stable if every bounded input produces a bounded output.

- A discrete LTI system is stable if and only if $h[n]$ is absolutely summable

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty \qquad \text{absolutely summable}$$

- A continuous LTI system is stable if and only if $h(t)$ is absolutely integrable

$$\int_{0}^{\infty} |h(\tau)| d\tau < \infty \qquad \text{absolutely integrable}$$

# *Stability of LTI Systems*

- Proof: discrete-time case

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} h[k]x[n-k] \right| \le \sum_{k=-\infty}^{\infty} |h[k]|\,|x[n-k]|$$

  - If $|x[n-k]| \le B_x$ and $\sum_{k=-\infty}^{\infty} |h[k]| < \infty$, then

$$|y[n]| \le B_x \sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

  - If and only if $\sum_{k=-\infty}^{\infty} |h[k]| < \infty$, we have $|y[n]| < \infty$

# *Stability of LTI Systems*

- Proof: continuous-time case

$$|y(t)| = \left|\int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau\right|$$

$$\leq \int_{-\infty}^{\infty} |h(\tau)| \cdot |x(t-\tau)| d\tau$$

If $|x(t-\tau)| \leq B_x$

$$\leq B_x \int_{-\infty}^{\infty} |h(\tau)| d\tau$$

If and only if $\int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty$, we have $|y(t)| < \infty$

# Stability of LTI Systems

- Example: Consider a causal LTI discrete-time system with an impulse response
$$h[n] = \alpha^n u[n]$$
Is this a stable system?

# The Unit Step Response of LTI Systems

- The unit step response, $s(t)$ or $s[n]$, corresponding to the output with input $x(t) = u(t)$ or $x[n] = u[n]$.

$$u[n] \text{ or } u(t) \longrightarrow \boxed{\text{LTI}} \longrightarrow s[n] \text{ or } s(t)$$

- For discrete time

$$s[n] = u[n] * h[n] = \sum_{k=-\infty}^{\infty} h[k]u[n-k] = \sum_{k=-\infty}^{n} h[k]$$

- $h[n]$ can be recovered from $s[n]$ using the relation

$$h[n] = s[n] - s[n-1]$$

# The Unit Step Response of LTI Systems

- The unit step response, $s(t)$ or $s[n]$, corresponding to the output with input $x(t) = u(t)$ or $x[n] = u[n]$.

- For continuous time

$$s(t) = u(t) * h(t) = \int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau = \int_{-\infty}^{t} h(\tau)d\tau$$

- The unit impulse response is the first derivative of the unit step response

$$h(t) = \frac{ds(t)}{dt} = s'(t)$$

# Summary

- The Commutative Property
- The Distributive Property
- The Associative Property
- LTI systems with and without memory
- Invertibility for LTI systems
- Causality for LTI systems
- Stability for LTI systems
- The unit step response of LTI systems
- Reference in textbook:
  - 2.3