Signals and Systems

Lecturer: Dr. Lin Xu, Dr. Xiran Cai

Email: xulin1@shanghaitech.edu.cn

caixr@shanghaitech.edu.cn

Office: 3-428(Xu), 3-438(Cai) SIST

Tel: 20684449(Xu), 20684431(Cai)

ShanghaiTech University

上海科技大学
ShanghaiTech University

Chapter 3: Fourier Series Representation of Periodic signals

- The response of LTI systems to complex exponentials
- Fourier series representation of continuous periodic signals
- Convergence of the Fourier series
- Properties of continuous-time Fourier series
- Fourier series representation of discrete –time periodic signals
- Properties of discrete FS
- Fourier series and LTI systems

The response of LTI systems to complex exponentials

Recall Chapter 2

- Objective: characterization of a LTI system

$$x(t) \longrightarrow \boxed{\text{LTI}} \longrightarrow y(t)$$

- $x(t)$ is considered as linear combinations of a basis signal $\delta(t)$

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\, d\tau \quad \rightarrow \quad y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\, d\tau$$

- $\delta(t)$ is not the only one. In general, a basic signal should satisfy
  - It can be used to construct a broad and useful class of signals
  - The response of an LTI system to the basic signal is simple

The response of LTI systems to complex exponentials

Continuous-time

$$e^{st} \longrightarrow \boxed{\text{LTI}} \longrightarrow y(t) = ?$$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}d\tau = e^{st}\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau$$

Let $\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau = H(s) \quad \rightarrow \quad y(t) = H(s)e^{st}$

- e^{st} is an eigenfunction of the system
- For a specific value s, $H(s)$ is the corresponding eigenvalue

The response of LTI systems to complex exponentials

Discrete-time

$$z^n \longrightarrow \boxed{\text{LTI}} \longrightarrow y[n] = ?$$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k] z^{n-k} = z^n \sum_{k=-\infty}^{\infty} h[k] z^{-k}$$

Let $$H[z] = \sum_{k=-\infty}^{\infty} h[k] z^{-k} \quad \rightarrow y[n] = H[z] z^n$$

- z^n is an eigenfunction of the system
- For a specific value z, $H[z]$ is the corresponding eigenvalue

The response of LTI systems to complex exponentials

Continuous-time

$$e^{st} \longrightarrow \boxed{\text{LTI}} \longrightarrow \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau\, e^{st} = H(s)e^{st}$$

If $\quad x(t) = a_1 e^{s_1 t} + a_2 e^{s_2 t} + a_3 e^{s_3 t} \qquad y(t) = ?$

$$y(t) = a_1 H(s_1) e^{s_1 t} + a_2 H(s_2) e^{s_2 t} + a_3 H(s_3) e^{s_3 t}$$

Generally, if $\quad x(t) = \sum_k a_k e^{s_k t}$

$$y(t) = \sum_k a_k H(s_k) e^{s_k t}$$

The response of LTI systems to complex exponentials

Discrete-time

$$z^n \longrightarrow \boxed{\text{LTI}} \longrightarrow \sum_{k=-\infty}^{\infty} h[k] z^{-k} z^n = H[z] z^n$$

If $$x[n] = \sum_k a_k Z_k^n$$

$$y[n] = \sum_k a_k H(z_k) Z_k^n$$

The response of LTI systems to complex exponentials

Examples

For a LTI system $y(t) = x(t-3)$, determine $H(s)$

Solution 1:

let $x(t) = e^{st}$, $y(t) = e^{s(t-3)} = e^{-3s}e^{st}$

$$\therefore H(s) = e^{-3s}$$

Solution 2:

$$H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau = \int_{-\infty}^{\infty} \delta(\tau-3)e^{-s\tau}d\tau = e^{-3s}$$

上海科技大学 ShanghaiTech University

The response of LTI systems to complex exponentials

Examples

For a LTI system $y(t) = x(t-3)$

If $x(t) = \cos(4t) + \cos(7t)$, $y(t) = ?$

Solution 1: $y(t) = \cos\big(4(t-3)\big) + \cos\big(7(t-3)\big)$

Solution 2: $x(t) = \frac{1}{2}e^{j4t} + \frac{1}{2}e^{-j4t} + \frac{1}{2}e^{j7t} + \frac{1}{2}e^{-j7t}$

$$y(t) = \frac{1}{2}H(j4)e^{j4t} + \frac{1}{2}H(-j4)e^{-j4t} + \frac{1}{2}H(j7)e^{j7t} + \frac{1}{2}H(-j7)e^{-j7t}$$

$H(s) = e^{-3s}$

$$= \frac{1}{2}e^{-j12}e^{j4t} + \frac{1}{2}e^{j12}e^{-j4t} + \frac{1}{2}e^{-j21}e^{j7t} + \frac{1}{2}e^{j21}e^{-j7t}$$

$$= \frac{1}{2}e^{j4(t-3)} + \frac{1}{2}e^{-j4(t-3)} + \frac{1}{2}e^{j7(t-3)} + \frac{1}{2}e^{-j7(t-3)}$$

上海科技大学 ShanghaiTech University

Chapter 3: Fourier Series Representation of Periodic signals

- The response of LTI systems to complex exponentials
- Fourier series representation of continuous periodic signals
- Convergence of the Fourier series
- Properties of continuous-time Fourier series
- Fourier series representation of discrete –time periodic signals
- Properties of discrete FS
- Fourier series and LTI systems

Fourier series representation of C-T periodic signals

Recall

$$x(t) \longrightarrow \boxed{\text{LTI}} \longrightarrow y(t)$$

$$e^{st} \qquad\qquad\qquad\qquad H(s)e^{st}$$

- Decompose $x(t)$ into linear combinations of basis signals, which should satisfy
  - It can be used to construct a broad and useful class of signals
  - The response of an LTI system to the basic signal is simple

- Complex exponentials are eigenfunctions of a LTI system

- Can we represent $x(t)$ as linear combinations of complex exponentials?

Fourier series representation of C-T periodic signals

Linear combination of harmonically related complex exponentials

- Harmonically related complex exponentials (consider e^{st} with s purely imaginary)

$$\emptyset_k(t) = e^{jk\omega_0 t} = e^{jk(2\pi/T_0)t},\ k = 0, \pm 1, \pm 2, \ldots$$

For any $k \neq 0$, fundamental frequency $|k|\omega_0$; fundamental period $\frac{2\pi}{|k|\omega_0} = \frac{T_0}{|k|}$

- Linear combination of $\emptyset_k(t)$ is also periodic

$$\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T_0)t} = x(t)$$

$x(t)$ is periodic

Fourier series representation of C-T periodic signals

Linear combination of harmonically related complex exponentials

- Can any $x(t)$ (periodic) be decomposed as Linear combination of $\emptyset_k(t)$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad ?$$

Yes for most periodic signals

Because $e^{jk\omega_0 t}$ are orthonormal: $\left\langle e^{jk_1\omega_0 t}, e^{jk_2\omega_0 t} \right\rangle = 0$

$$\left\langle e^{jk_1\omega_0 t}, e^{jk_2\omega_0 t} \right\rangle = \frac{1}{T}\int_0^T e^{jk_1\omega_0 t} e^{-jk_2\omega_0 t} dt = \begin{cases} 1, k_1 = k_2 \\ 0, k_1 \neq k_2 \end{cases}$$

Fourier series representation of C-T periodic signals

Linear combination of harmonically related complex exponentials

- Fourier Series representation

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

- ω_0 is the fundamental frequency

- For $a_k e^{jk\omega_0 t}$
  - $k = 0$: DC component
  - $k = \pm 1$: fundamental (first harmonic) components
  - $k = \pm N$: Nth harmonic components

Fourier series representation of C-T periodic signals

Linear combination of harmonically related complex exponentials

- An example

If $x(t) = \sum_{k=-3}^{3} a_k e^{jk2\pi t}$

And $a_0 = 1$, $a_1 = a_{-1} = 1/4$, $a_2 = a_{-2} = 1/2$, $a_3 = a_{-3} = 1/3$

$$x(t) = 1 + \frac{1}{4}\left(e^{j2\pi t} + e^{-j2\pi t}\right) + \frac{1}{2}\left(e^{j4\pi t} + e^{-j4\pi t}\right) + \frac{1}{3}\left(e^{j6\pi t} + e^{-j6\pi t}\right)$$

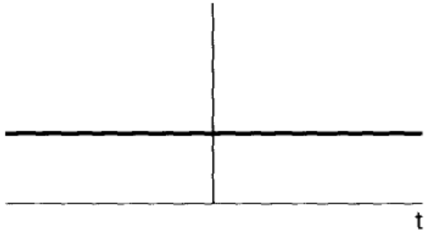
$$= 1 + \frac{1}{2}\cos 2\pi t + \cos 4\pi t + \frac{2}{3}\cos 6\pi t$$

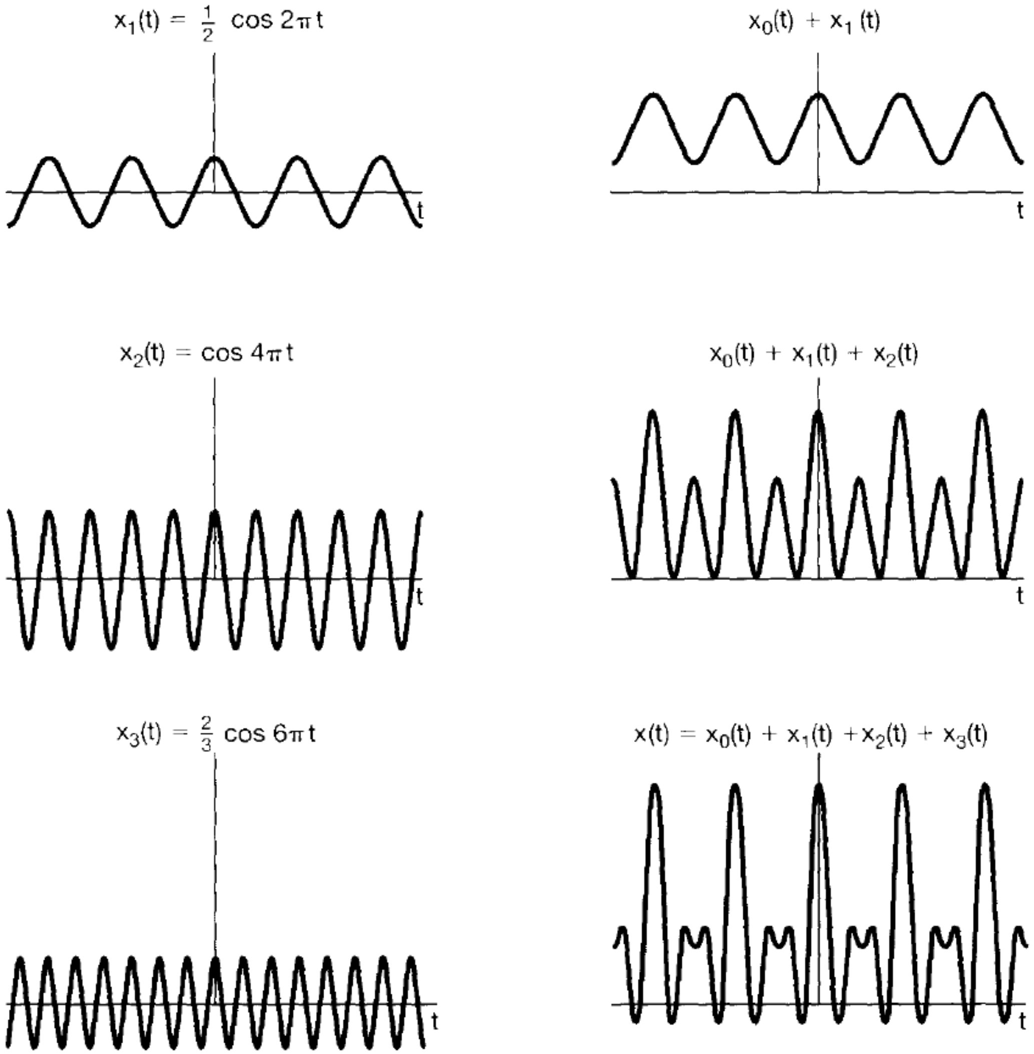
Fourier series representation of C-T periodic signals

Linear combination of harmonically related complex exponentials

- An example

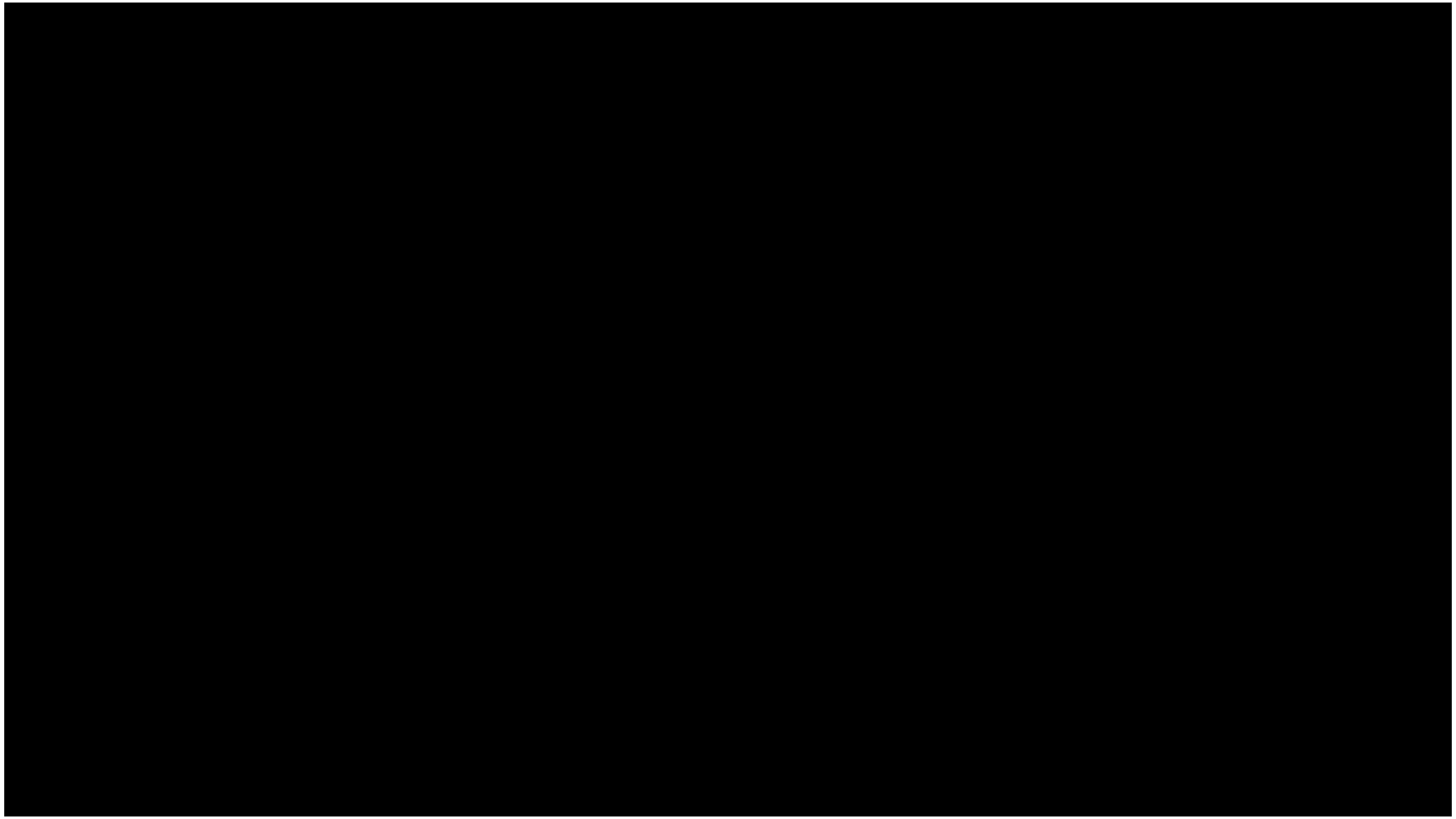
$$1+\frac{1}{2}\cos 2\pi t+\cos 4\pi t+\frac{2}{3}\cos 6\pi t$$





Fourier series representation of C-T periodic signals

https://www.youtube.com/watch?v=cUD1gMAl6W4



Fourier series representation of C-T periodic signals

Linear combination of harmonically related complex exponentials

- Real signal

$$x(t)=\sum_{k=-\infty}^{\infty}a_k e^{jk\omega_0 t} \qquad x^*(t)=\sum_{k=-\infty}^{\infty}a_k^* e^{-jk\omega_0 t}=\sum_{k=-\infty}^{\infty}a_{-k}^* e^{jk\omega_0 t}$$

Real $\Rightarrow x(t)=x^*(t) \quad \Rightarrow a_k=a_{-k}^*$, or $a_k^*=a_{-k}$ (Conjugate symmetry)

- Alternative form of Fourier Series for real signal

$$x(t)=a_0+\sum_{k=1}^{\infty}\left[a_k e^{jk\omega_0 t}+a_{-k}e^{-jk\omega_0 t}\right]$$

$$=a_0+\sum_{k=1}^{\infty}2\mathcal{R}e\left[a_k e^{jk\omega_0 t}\right]=a_0+2\sum_{k=1}^{\infty}A_k\cos(k\omega_0 t+\theta_k)$$

$$a_k=A_k e^{j\theta_k}$$

上海科技大学 ShanghaiTech University

Fourier series representation of C-T periodic signals

Determine the Fourier Series Representation

$$\int_0^T x(t)e^{-jn\omega_0 t}dt = \int_0^T \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}\, e^{-jn\omega_0 t}dt$$

$$= \begin{cases} T, k = n \\ 0, k \neq n \end{cases} = T\delta[k-n]$$

$$= \sum_{k=-\infty}^{\infty} a_k \boxed{\left[\int_0^T e^{j(k-n)\omega_0 t}dt\right]} \;=\; Ta_n$$

$$\therefore a_n = \frac{1}{T}\int_0^T x(t)e^{-jn\omega_0 t}dt$$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$$

Fourier series representation of C-T periodic signals

Fourier Series pair

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \qquad \text{Synthesis equation}$$

$$a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} dt \qquad \text{Analysis equation}$$

- a_k: Fourier Series coefficients or spectral coefficients of $x(t)$

$$a_0 = \frac{1}{T}\int_T x(t) dt$$

Fourier series representation of C-T periodic signals

Determine the Fourier Series Representation

- Examples: determine the FS coefficients of $x(t)$

$$x(t) = \sin \omega_0 t$$

$$\sin \omega_0 t = \frac{1}{2j} e^{j\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t}$$

$$\therefore a_1 = \frac{1}{2j} \qquad a_{-1} = -\frac{1}{2j} \qquad a_k = 0, \text{ for } k \neq \pm 1$$

Fourier series representation of C-T periodic signals

Determine the Fourier Series Representation

- Examples: determine the FS coefficients of $x(t)$

$$x(t) = 1 + \sin\omega_0 t + 2\cos\omega_0 t + \cos\left(2\omega_0 t + \frac{\pi}{4}\right)$$

$$x(t) = 1 + \frac{1}{2j}\left[e^{j\omega_0 t} - e^{-j\omega_0 t}\right] + \left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right] + \frac{1}{2}\left(e^{j(2\omega_0 t+\pi/4)} + e^{-j(2\omega_0 t+\pi/4)}\right)$$

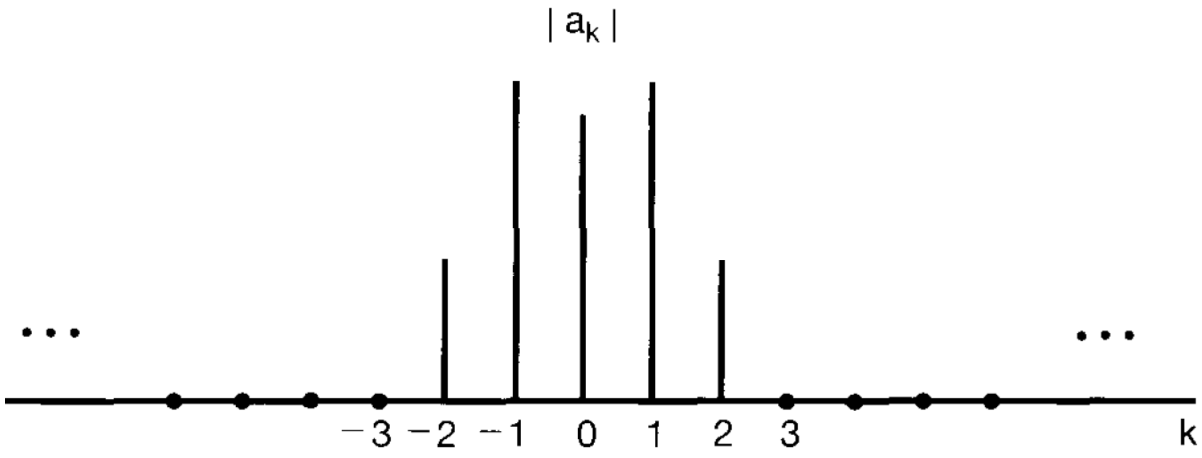
$$\therefore x(t) = \underbrace{\boxed{1}}_{a_0} + \underbrace{\boxed{\left(1+\frac{1}{2j}\right)}}_{a_1} e^{j\omega_0 t} + \underbrace{\boxed{\left(1-\frac{1}{2j}\right)}}_{a_{-1}} e^{-j\omega_0 t} + \underbrace{\boxed{\frac{1}{2}e^{j\pi/4}}}_{a_2} e^{j2\omega_0 t} + \underbrace{\boxed{\frac{1}{2}e^{-j\pi/4}}}_{a_{-2}} e^{-j2\omega_0 t}$$

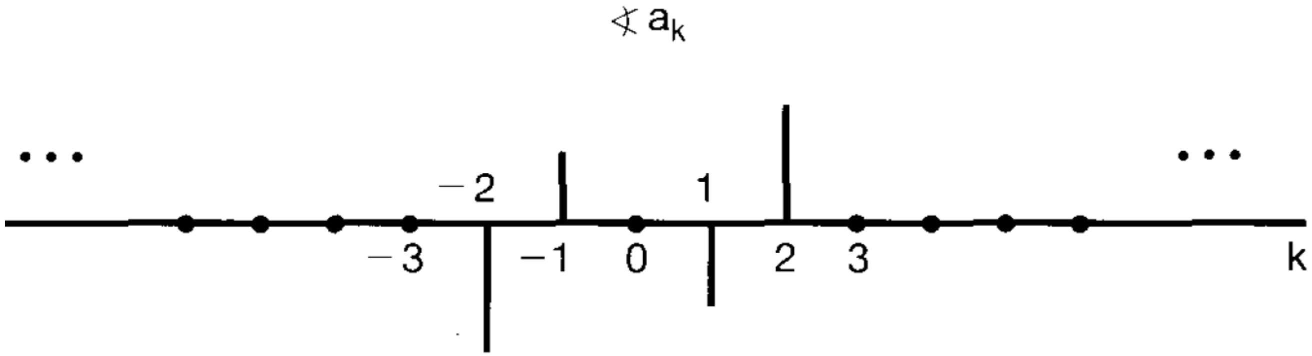
Fourier series representation of C-T periodic signals

Determine the Fourier Series Representation

- Examples: determine the FS coefficients of $x(t)$

$$x(t) = \underbrace{\boxed{1}}_{a_0} + \underbrace{\boxed{\left(1+\frac{1}{2j}\right)}}_{a_1} e^{j\omega_0 t} + \underbrace{\boxed{\left(1-\frac{1}{2j}\right)}}_{a_{-1}} e^{-j\omega_0 t} + \underbrace{\boxed{\frac{1}{2}e^{j\pi/4}}}_{a_2} e^{j2\omega_0 t} + \underbrace{\boxed{\frac{1}{2}e^{-j\pi/4}}}_{a_{-2}} e^{-j2\omega_0 t}$$

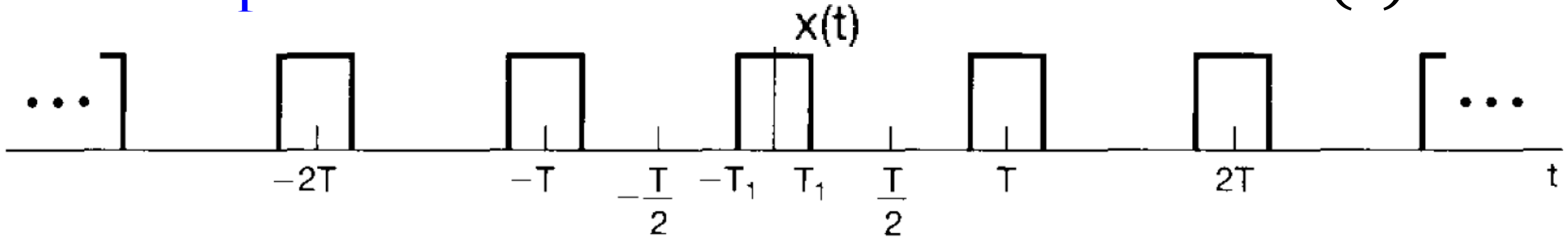




Fourier series representation of C-T periodic signals

Determine the Fourier Series Representation

- Examples: determine the FS coefficients of $x(t)$



$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$$

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} x(t)dt = \frac{1}{T}\int_{-T_1}^{T_1} 1\, dt = \frac{2T_1}{T}$$

$$a_k = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}dt = -\frac{1}{jk\omega_0 T} e^{-jk\omega_0 t}\Big|_{-T_1}^{T_1} = \frac{2}{k\omega_0 T}\left[\frac{e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}}{2j}\right]$$

$$\mathrm{sinc}(x) = \frac{\sin(\pi x)}{\pi x}$$

$$= \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T} = \frac{\sin(k\omega_0 T_1)}{k\pi} = \frac{2T_1}{T}\boxed{\frac{\sin(k\omega_0 T_1)}{k\omega_0 T_1}}, k \neq 0$$

上海科技大学
ShanghaiTech University

Fourier series representation of C-T periodic signals

Determine the Fourier Series Representation

- Examples: determine the FS coefficients of $x(t)$

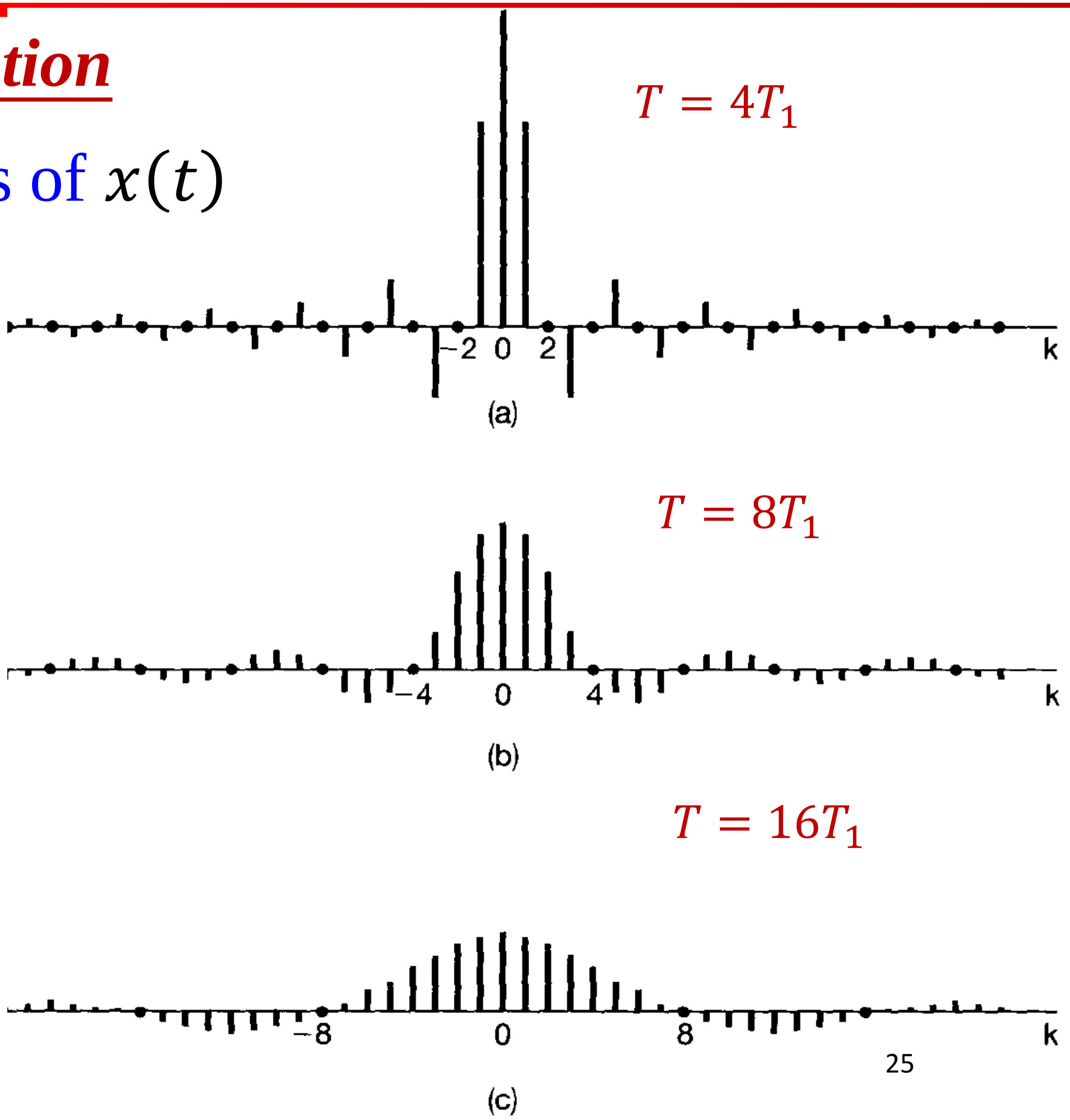
$$a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T} = \frac{\sin(k\omega_0 T_1)}{k\pi}$$

$$= \frac{2T_1}{T}\frac{\sin(k\omega_0 T_1)}{k\omega_0 T_1}, k \neq 0$$

$T = 4T_1$

$T = 8T_1$

$T = 16T_1$



Chapter 3: Fourier Series Representation of Periodic signals

- The response of LTI systems to complex exponentials
- Fourier series representation of continuous periodic signals
- Convergence of the Fourier series
- Properties of continuous-time Fourier series
- Fourier series representation of discrete –time periodic signals
- Properties of discrete FS
- Fourier series and LTI systems

Convergence of the Fourier series

History

- Using "trigonometric sum" to describe periodic signal can be tracked back to Babylonians who predicted astronomical events similarly.
- L. Euler (in 1748) and Bernoulli (in 1753) used the "normal mode" concept to describe the motion of a vibrating string; though JL Lagrange strongly criticized this concept.
- Fourier (in 1807) had found series of harmonically related sinusoids to be useful to describe the temperature distribution through body, and he claimed "any" periodic signal can be represented by such series.
- Dirichlet (in 1829) provide a precise condition under which a periodic signal can be represented by a Fourier series.

Jean Baptiste Joseph Fourier
March 21 1768 - May 16 1830
Born Auxerre, France. Died Paris, France.

Convergence of the Fourier series

Convergence problem

- Approximate periodic signal $x(t)$ by $x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$
- How good the approximation is?

$$e_N(t) = x(t) - x_N(t) = x(t) - \sum_{k=-N}^{N} a_k e^{jk\omega_0 t} \qquad E_N = \int_T |e_N(t)|^2 dt$$

- When $a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} dt$, E_N is minimized;
- If $x_N(t)$ can be expressed as $\sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$, $N \to \infty \Rightarrow E_N \to 0$

- Problem:
  - a_k may be infinite
  - $N \to \infty$, $x_N(t)$ may be infinite

Convergence problem!

Convergence of the Fourier series

Two different classes of conditions

- Condition 1: Finite energy condition

If $\int_T |x(t)|^2 dt < \infty$, $x(t)$ can be represented by a FS

- Guarantees no energy in their difference; FS is not equal to $x(t)$

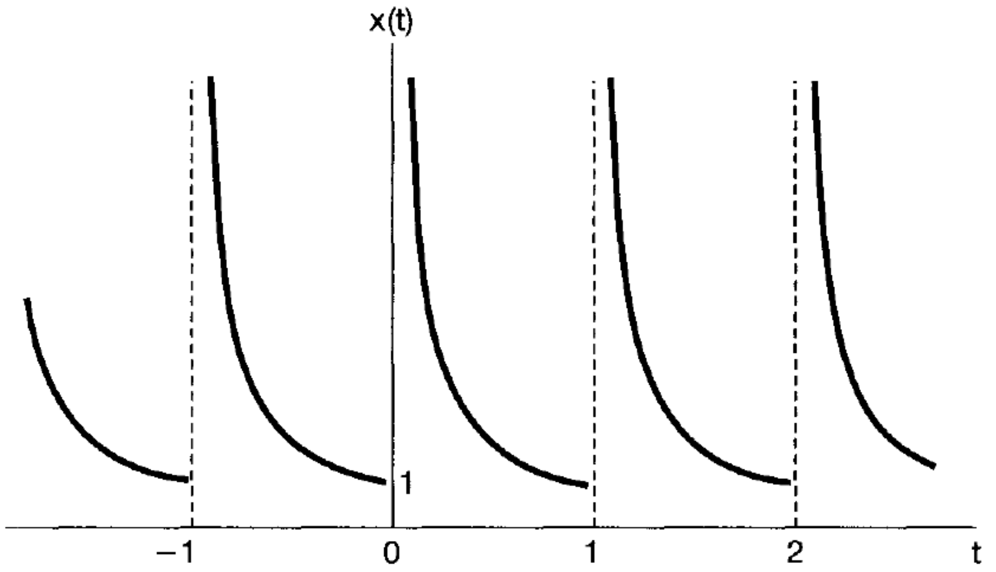
- Condition 2: Dirichlet condition

(1) Absolutely integrable $\int_T |x(t)| dt < \infty$

An example: a periodic signal

$$x(t) = \frac{1}{t}, 0 < t \leq 1$$

is not absolutely integrable.



Convergence of the Fourier series

Two different classes of conditions

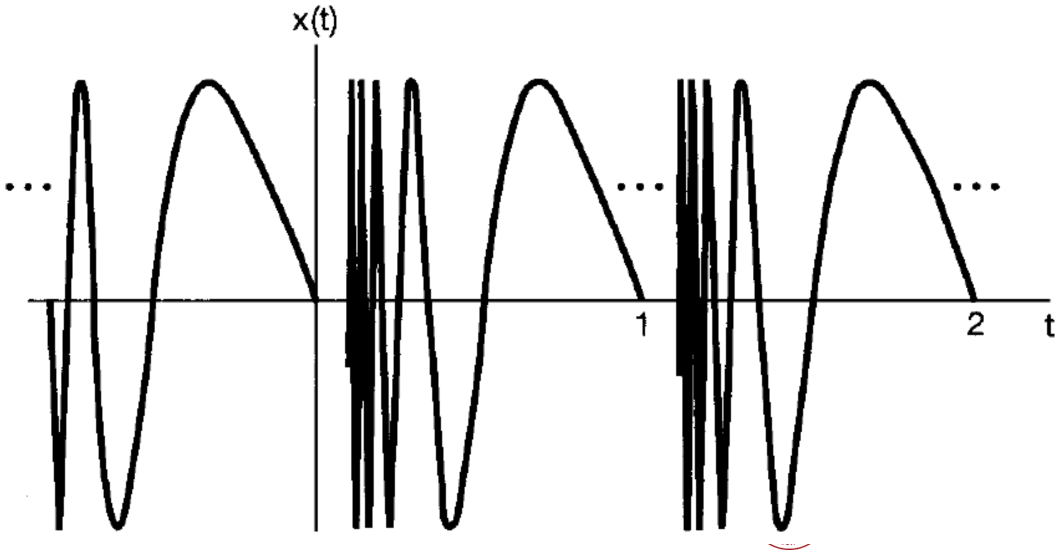
- Condition 2: Dirichlet condition

(2) In any finite interval of time, x(t) is of bounded variation; finite maxima and minima in one period

An example: a periodic signal

$$x(t) = \sin\left(\frac{2\pi}{t}\right), 0 < t \leq 1$$

meets (1) but not (2).



Convergence of the Fourier series

Two different classes of conditions

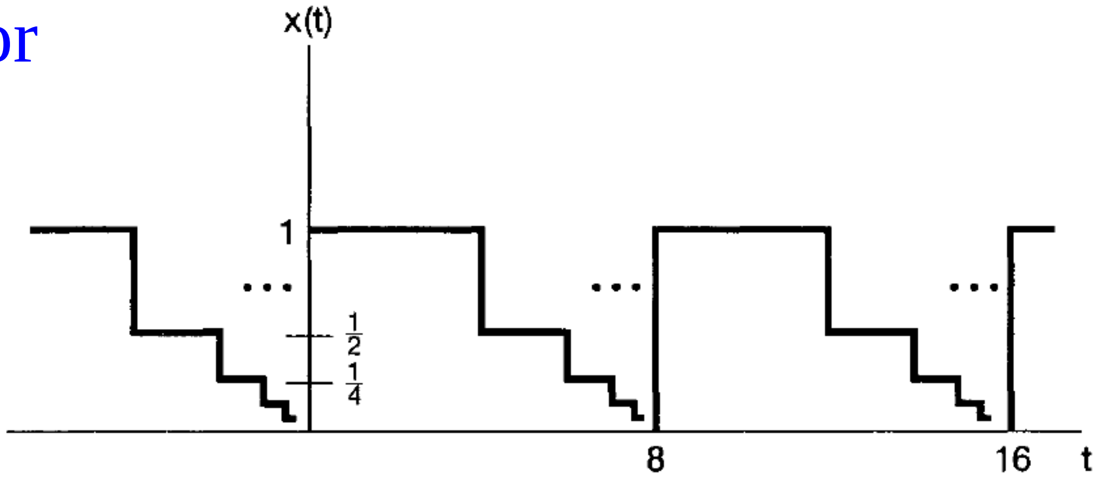
- Condition 2: Dirichlet condition

(3) In any finite interval of time, only a finite number of finite discontinuities

An example: a periodic signal meets (1) and (2) but not (3).

- Dirichlet condition guarantees $x(t)$ equals its Fourier Series representation, except for discontinuous points.

- Three examples are pathological in nature and do not typically arise in practical contexts.



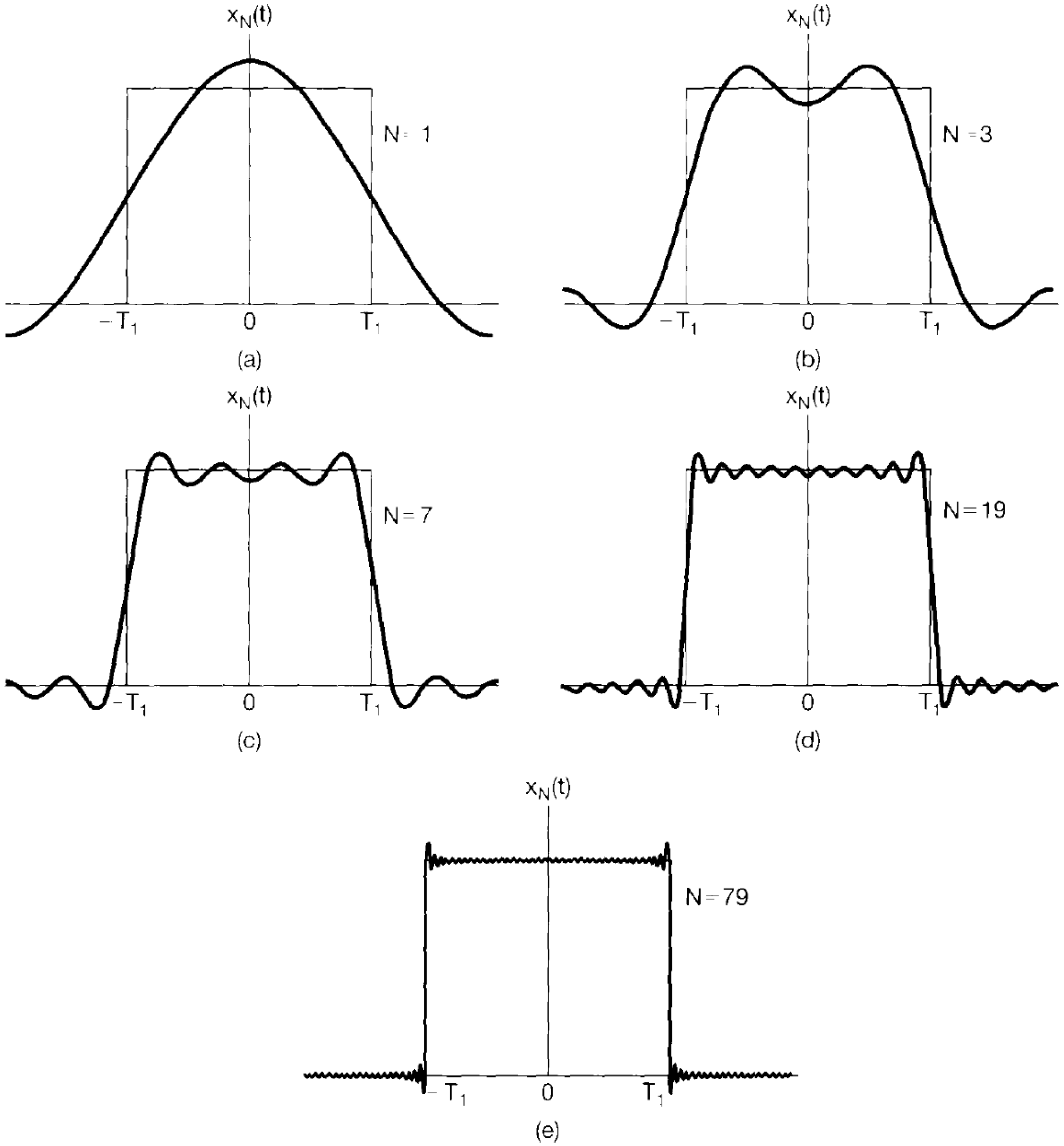
Convergence of the Fourier series

Example

- $x(t)$ is a square wave

$$x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$$

$$\lim_{N\to\infty} x_N(t_1) = x(t_1)$$



Chapter 3: Fourier Series Representation of Periodic signals

- The response of LTI systems to complex exponentials
- Fourier series representation of continuous periodic signals
- Convergence of the Fourier series
- Properties of continuous-time Fourier series
- Fourier series representation of discrete –time periodic signals
- Properties of discrete FS
- Fourier series and LTI systems

Properties of continuous-time FS

- Use the notation

$$x(t) \xleftrightarrow{\mathcal{FS}} a_k$$

to signify the paring of a periodic signal with its FS coefficients.

- Linearity: if $x(t)$ and $y(t)$ are periodic signals with the same period T

$$\begin{aligned} x(t) &\xleftrightarrow{\mathcal{FS}} a_k \\ y(t) &\xleftrightarrow{\mathcal{FS}} b_k \end{aligned} \quad \Rightarrow \quad z(t) = Ax(t) + By(t) \xleftrightarrow{\mathcal{FS}} c_k = Aa_k + Bb_k$$

Properties of continuous-time FS

- Time shifting

$$x(t) \xleftrightarrow{\mathcal{FS}} a_k \quad \Rightarrow \quad x(t-t_0) \xleftrightarrow{\mathcal{FS}} e^{-jk\omega_0 t_0} a_k$$

- Proof

$$t - t_0 = \tau$$

$$\frac{1}{T}\int_T x(t-t_0)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_T x(\tau)e^{-jk\omega_0(\tau+t_0)}d\tau$$

$$= e^{-jk\omega_0 t_0}\frac{1}{T}\int_T x(\tau)e^{-jk\omega_0 \tau}d\tau$$

$$= e^{-jk\omega_0 t_0} a_k$$

Properties of continuous-time FS

- Time reversal

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k \quad \Rightarrow \quad y(t) = x(-t) \overset{\mathcal{FS}}{\longleftrightarrow} b_k = a_{-k}$$

- Proof

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \Rightarrow x(-t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0(-t)} = \sum_{k=-\infty}^{\infty} a_k e^{j(-k)\omega_0 t}$$

$$= \sum_{m=-\infty}^{\infty} a_{-m} e^{jm\omega_0 t}$$

- If $x(t)$ even, $a_{-k} = a_k$, if $x(t)$ odd, $a_{-k} = -a_k$

上海科技大学 ShanghaiTech University

Properties of continuous-time FS

- Time scaling

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k \quad \Rightarrow \quad y(t) = x(\alpha t) \overset{\mathcal{FS}}{\longleftrightarrow} b_k = a_k$$

- Proof

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \Rightarrow x(\alpha t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 \alpha t} = \sum_{k=-\infty}^{\infty} a_k e^{jk(\alpha\omega_0)t}$$

FS coefficients the same, but fundamental frequency changed.

Properties of continuous-time FS

Multiplication

$$\begin{aligned} x(t) &\xleftrightarrow{\mathcal{FS}} a_k \\ y(t) &\xleftrightarrow{\mathcal{FS}} b_k \end{aligned} \Rightarrow z(t) = x(t)y(t) \xleftrightarrow{\mathcal{FS}} h_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}$$

Proof

$k + n = l$

$$x(t)y(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \sum_{n=-\infty}^{\infty} b_n e^{jn\omega_0 t} = \sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} a_k b_n e^{j(k+n)\omega_0 t}$$

$$= \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} a_k b_{l-k} e^{jl\omega_0 t} = \sum_{l=-\infty}^{\infty} \underbrace{\boxed{\sum_{k=-\infty}^{\infty} a_k b_{l-k}}}_{h_l} e^{jl\omega_0 t}$$

$$= \sum_{l=-\infty}^{\infty} h_l e^{jl\omega_0 t}$$

上海科技大学 ShanghaiTech University

Properties of continuous-time FS

- Conjugation and conjugate symmetry

$$x(t) \xleftrightarrow{\mathcal{FS}} a_k \quad \Rightarrow \quad z(t) = x^*(t) \xleftrightarrow{\mathcal{FS}} b_k = a_{-k}^*$$

- Proof

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad \therefore x^*(t) = \sum_{k=-\infty}^{\infty} a_k^* e^{-jk\omega_0 t} \overset{-k=m}{=} \sum_{m=-\infty}^{\infty} a_{-m}^* e^{jm\omega_0 t}$$

- If $x(t)$ real, $a_k^* = a_{-k}$ (conjugate symmetry) $\Rightarrow |a_k| = |a_{-k}|$
  - $x(t)$ real and even ($a_{-k} = a_k$) $\Rightarrow a_k = a_k^* \Rightarrow a_k$ real and even
  - $x(t)$ real and odd ($a_{-k} = -a_k$) $\Rightarrow a_k = -a_k^* \Rightarrow a_k$ pure imaginary and odd
  - $x(t)$ real and odd, $a_0 =$?

Properties of continuous-time FS

- Differentiation and Integration

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k \quad \Rightarrow \begin{cases} dx(t)/dt \overset{\mathcal{FS}}{\longleftrightarrow} jk\omega_0 a_k \\ \int_{-\infty}^{t} x(\tau)d\tau \overset{\mathcal{FS}}{\longleftrightarrow} a_k/(jk\omega_0) \end{cases}$$

- Proof

$$\frac{dx(t)}{dt} = \sum_{k=-\infty}^{\infty} a_k \frac{d(e^{jk\omega_0 t})}{dt} = \sum_{k=-\infty}^{\infty} a_k jk\omega_0 e^{jk\omega_0 t}$$

$$\int_{-\infty}^{t} x(\tau)d\tau = \sum_{k=-\infty}^{\infty} a_k \int_{-\infty}^{t} e^{jk\omega_0 \tau} d\tau = \sum_{k=-\infty}^{\infty} a_k/(jk\omega_0) e^{jk\omega_0 t}$$

Properties of continuous-time FS

- Frequency shifting

$$x(t) \xleftrightarrow{\mathcal{FS}} a_k \quad \Rightarrow \quad e^{jM\omega_0 t}x(t) \xleftrightarrow{\mathcal{FS}} a_{k-M}$$

- Proof

$$e^{jM\omega_0 t}x(t) = e^{jM\omega_0 t}\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{j(k+M)\omega_0 t}$$

$k + M = l$

$$= \sum_{l=-\infty}^{\infty} a_{l-M} e^{jl\omega_0 t}$$

上海科技大学 ShanghaiTech University

Properties of continuous-time FS

- Periodic convolution

$$\begin{array}{l} x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k \\ y(t) \overset{\mathcal{FS}}{\longleftrightarrow} b_k \end{array} \Rightarrow \int_T x(\tau)y(t-\tau)d\tau \overset{\mathcal{FS}}{\longleftrightarrow} Ta_k b_k$$

- Proof

$$\int_T x(\tau)y(t-\tau)d\tau = \int_T \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0\tau} \sum_{n=-\infty}^{\infty} b_n e^{jn\omega_0(t-\tau)}\, d\tau$$

$$= \int_T \sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} a_k e^{jk\omega_0\tau} b_n e^{-jn\omega_0\tau} e^{jn\omega_0 t}\, d\tau$$

$$= \sum_{k=-\infty}^{\infty} a_k \sum_{n=-\infty}^{\infty} e^{jn\omega_0 t} b_n \overbrace{\boxed{\int_T e^{jk\omega_0\tau} e^{-jn\omega_0\tau} d\tau}}^{T\delta[k-n]} = \sum_{k=-\infty}^{\infty} Ta_k b_k e^{jk\omega_0 t}$$

Properties of continuous-time FS

- Parseval's relation

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

- Proof

$$\frac{1}{T}\int_T |x(t)|^2 dt = \frac{1}{T}\int_T x(t)x^*(t)dt = \frac{1}{T}\int_T x(t) \sum_{k=-\infty}^{\infty} a_k^* e^{-jk\omega_0 t}\, dt$$

$$= \sum_{k=-\infty}^{\infty} a_k^* \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \sum_{k=-\infty}^{\infty} a_k^* a_k = \sum_{k=-\infty}^{\infty} |a_k|^2$$

Properties of continuous-time FS

- Parseval's relation

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

$$\frac{1}{T}\int_T |a_k e^{jk\omega_0 t}|^2 dt = \frac{1}{T}\int_T |a_k|^2 dt = |a_k|^2$$

- $|a_k|^2$ is the average power in the kth harmonic component of $x(t)$

- Total average power in $x(t)$ equals the sum of the average powers in all of its harmonic components

Properties

- Summary

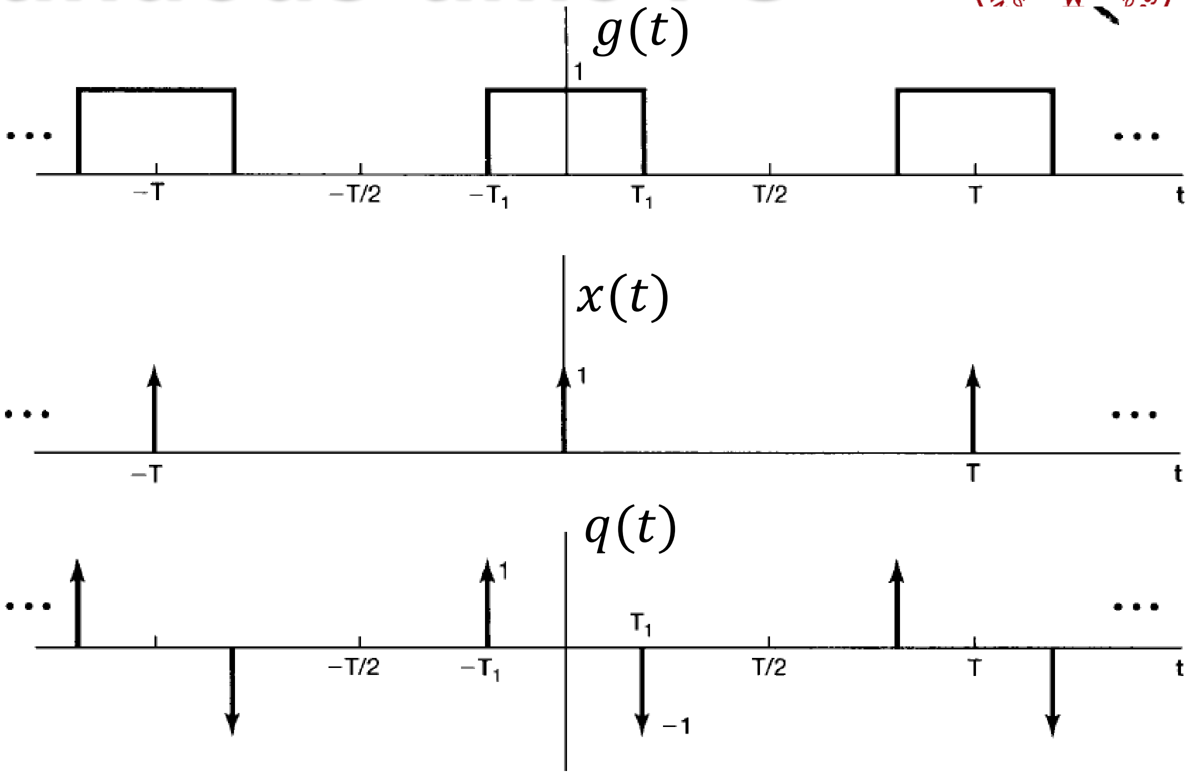
Property	Section	Periodic Signal	Fourier Series Coefficients
		$\left.\begin{matrix} x(t) \\ y(t) \end{matrix}\right\}$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k b_k
Linearity	3.5.1	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	3.5.2	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting		$e^{jM\omega_0 t} = e^{jM(2\pi/T)t} x(t)$	a_{k-M}
Conjugation	3.5.6	$x^*(t)$	a^*_{-k}
Time Reversal	3.5.3	$x(-t)$	a_{-k}
Time Scaling	3.5.4	$x(\alpha t), \alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution		$\int_T x(\tau)y(t - \tau)d\tau$	$T a_k b_k$
Multiplication	3.5.5	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation		$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T} a_k$
Integration		$\int_{-\infty}^{t} x(t)\,dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$
Conjugate Symmetry for Real Signals	3.5.6	$x(t)$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{Re}\{a_k\} = \mathcal{Re}\{a_{-k}\} \\ \mathcal{Im}\{a_k\} = -\mathcal{Im}\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	3.5.6	$x(t)$ real and even	a_k real and even
Real and Odd Signals	3.5.6	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals		$\begin{cases} x_e(t) = \mathcal{Ev}\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = \mathcal{Od}\{x(t)\} & [x(t) \text{ real}] \end{cases}$	$\mathcal{Re}\{a_k\}$ $j\mathcal{Im}\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

Properties of continuous-time FS

- Examples FS coefficients of $g(t)$?



- Solution
  - Let $x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t) e^{-jk\omega_0 t} dt = \frac{1}{T}$$

  - Let $q(t) = x(t + T_1) - x(t - T_1)$

$$b_k = e^{jk\omega_0 T_1} a_k - e^{-jk\omega_0 T_1} a_k = \frac{1}{T}\left(e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}\right) = \frac{2j\sin(k\omega_0 T_1)}{T}$$

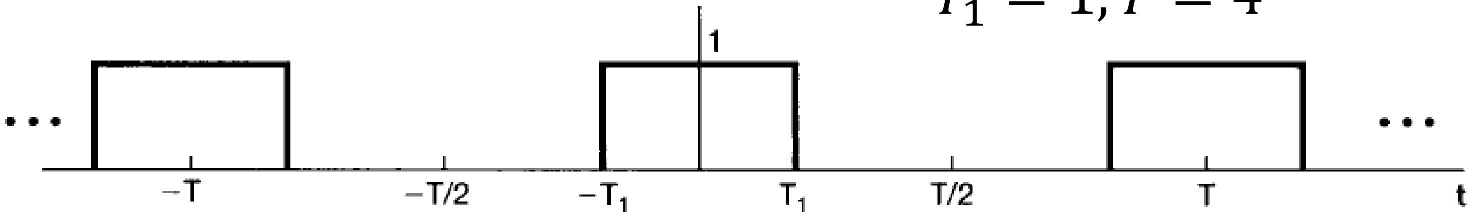
  - $g(t) = \int_{-\infty}^{t} q(\tau) d\tau$

$$\therefore c_k = \frac{b_k}{jk\omega_0} = \frac{2j\sin(k\omega_0 T_1)}{jk\omega_0 T} = \frac{\sin(k\omega_0 T_1)}{k\pi}, k \neq 0$$

Properties of continuous-time FS

- Examples

$x(t)$ $\qquad T_1 = 1, T = 4$



$$a_k = \frac{\sin(k\omega_0 T_1)}{k\pi}, k \neq 0$$
$$= \frac{\sin(k\pi/2)}{k\pi}, k \neq 0$$

$g(t) = x(t-1) - 1/2$ $\qquad$ FS coefficients of $g(t)$?

- Solution

$$x(t-1) \overset{\mathcal{FS}}{\leftrightarrow} e^{-jk\omega_0 t_0} a_k = e^{-jk\pi/2} a_k, k \neq 0$$

$$-1/2 \overset{\mathcal{FS}}{\leftrightarrow} \begin{cases} 0, k \neq 0 \\ -\frac{1}{2}, k = 0 \end{cases} \qquad \therefore x(t-1) - 1/2 \overset{\mathcal{FS}}{\leftrightarrow} \begin{cases} e^{-jk\pi/2} a_k, k \neq 0 \\ a_0 - \frac{1}{2}, k = 0 \end{cases}$$

Properties of continuous-time FS

❑ Examples

Given a signal $x(t)$ with the following facts, determine $x(t)$

1. $x(t)$ is real;
2. $x(t)$ is periodic with T=4 and FS coefficients $a_k = 0$ for $|\mathrm{k}| > 1$;
3. A signal with FS coefficients $b_k = e^{-j\pi k/2} a_{-k}$ is odd;
4. $\frac{1}{4}\int_4 |x(t)|^2 dt = \frac{1}{2}$.

❑ Solution

- From 2, $x(t) = a_0 + a_1 e^{j\pi t/2} + a_{-1} e^{-j\pi t/2}$
- $b_k = e^{-j\pi k/2} a_{-k}$ corresponds to the signal $x(-t+1)$, which is real and odd
- $\frac{1}{4}\int_4 |x(t)|^2 dt = \frac{1}{4}\int_4 |x(-t+1)|^2 dt = \sum_{k=-\infty}^{\infty} |b_k|^2 = |b_0|^2 + |b_1|^2 + |b_{-1}|^2 = \frac{1}{2}$
- $x(-t+1)$ is real and odd $\Rightarrow b_k = -b_{-k} \Rightarrow b_0 = 0, b_1 = -b_{-1} = \frac{\mathrm{j}}{2}$ or $-\frac{\mathrm{j}}{2}$
- $a_0 = 0, a_1 = -\frac{1}{2}$ or $\frac{1}{2}, a_{-1} = -\frac{1}{2}$ or $\frac{1}{2}$

上海科技大学 ShanghaiTech University

Chapter 3: Fourier Series Representation of Periodic signals

- The response of LTI systems to complex exponentials
- Fourier series representation of continuous periodic signals
- Convergence of the Fourier series
- Properties of continuous-time Fourier series
- Fourier series representation of discrete –time periodic signals
- Properties of discrete FS
- Fourier series and LTI systems

Fourier series representation of D-T periodic signals

Linear combination of harmonically related complex exponentials

- Harmonically related complex exponentials

$$\emptyset_k[n] = e^{jk(2\pi/N)n},\ k = 0, \pm 1, \pm 2, \ldots$$

  - Fundamental frequency $|k|(\frac{2\pi}{N})$
  - Only N distinct signals in $\emptyset_k[n]$, since $\emptyset_k[n] = \emptyset_{k+rN}[n]$

- Linear combination of $\emptyset_k[n]$ is also periodic

$$x[n] = \sum_{k=\langle N\rangle} a_k \emptyset_k[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n} = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$$

- $\sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$: Discrete-Time Fourier Series; a_k: Fourier Series coefficients

Fourier series representation of D-T periodic signals

Determine the Fourier Series Representation

$$\sum_{n=\langle N\rangle} x[n]e^{-jr(2\pi/N)n} = \sum_{n=\langle N\rangle}\sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}\, e^{-jr(2\pi/N)n}$$

$$= \sum_{k=\langle N\rangle} a_k \boxed{\sum_{n=\langle N\rangle} e^{j(k-r)(2\pi/N)n}} = N a_r$$

$$\sum_{n=\langle N\rangle} e^{j(k-r)(2\pi/N)n} = \begin{cases} N, k=r \\ 0, k\neq r \end{cases} = N\delta[k-r]$$

$$\therefore a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk(2\pi/N)n}$$

Fourier series representation of D-T periodic signals

Determine the Fourier Series Representation

- Discrete Fourier series pair

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk(2\pi/N)n}$$

Analysis equation; a_k: Fourier Series coefficients or spectral coefficients

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n}$$

Synthesis equation; Fourier Series (Finite)

- a_k is periodic

$$x[n] = \sum_{k=\langle N \rangle} a_k \emptyset_k[n] = a_0 \emptyset_0[n] + a_1 \emptyset_1[n] + \cdots + a_{N-1} \emptyset_{N-1}[n]$$
$$= a_1 \emptyset_1[n] + a_2 \emptyset_2[n] + \cdots + a_N \emptyset_N[n]$$
$$= a_2 \emptyset_2[n] + a_3 \emptyset_3[n] + \cdots + a_{N+1} \emptyset_{N+1}[n]$$

$$\therefore a_k = a_{k+rN}$$

上海科技大学 ShanghaiTech University

Fourier series representation of D-T periodic signals

Determine the Fourier Series Representation

- Examples $\quad x[n] = \sin \omega_0 n$

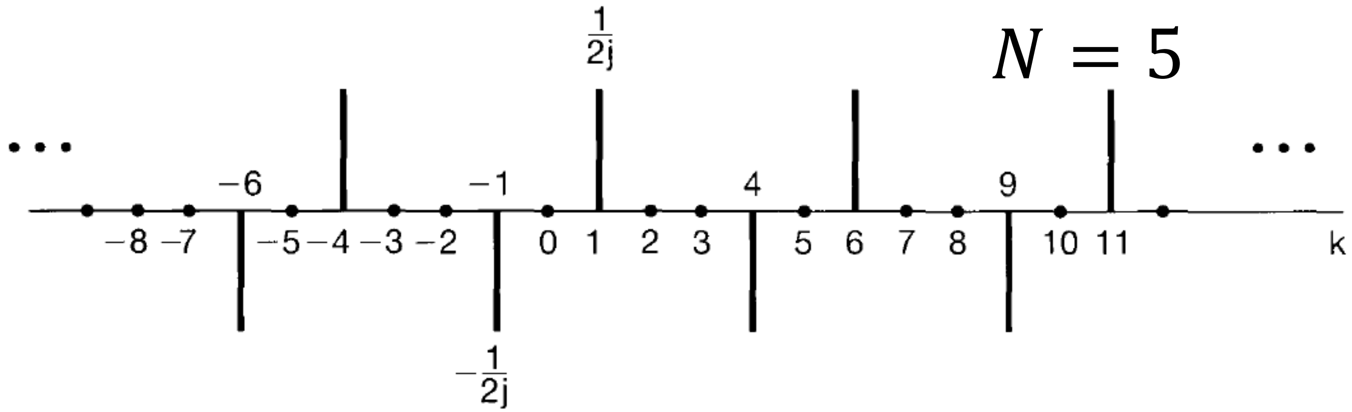
If $\omega_0 = \frac{2\pi}{N}$, $x[n]$ is periodic with fundamental period of N.

$$x[n] = \sin \omega_0 n = \frac{1}{2j} e^{j(2\pi/N)n} - \frac{1}{2j} e^{-j(2\pi/N)n}$$

$$\therefore a_1 = \frac{1}{2j} \qquad a_{-1} = -\frac{1}{2j} \qquad a_k = 0, \text{ for } k \neq \pm 1 \text{ in one period}$$

- a_k is periodic and only one period is utilized in the synthesis equation

$N = 5$



Fourier series representation of D-T periodic signals

Determine the Fourier Series Representation

- Examples $x[n] = 1 + \sin(\frac{2\pi}{N})n + 3\cos(\frac{2\pi}{N})n + \cos\left(\frac{4\pi}{N}n + \frac{\pi}{2}\right)$

$$x[n] = 1 + \frac{1}{2j}\left[e^{j(2\pi/N)n} - e^{-j(2\pi/N)n}\right] + \frac{3}{2}\left[e^{j(2\pi/N)n} + e^{-j(2\pi/N)n}\right]$$

$$+ \frac{1}{2}\left(e^{j(4\pi n/N+\pi/2)} + e^{-j(4\pi n/N+\pi/2)}\right)$$

$$\therefore x[n] = \underbrace{\boxed{1}}_{a_0} + \underbrace{\boxed{\left(\frac{3}{2} + \frac{1}{2j}\right)}}_{a_1} e^{j(2\pi/N)n} + \underbrace{\boxed{\left(\frac{3}{2} - \frac{1}{2j}\right)}}_{a_{-1}} e^{-j(2\pi/N)n}$$

$$+ \underbrace{\boxed{\frac{1}{2}e^{j\pi/2}}}_{a_2} e^{j2(2\pi/N)n} + \underbrace{\boxed{\frac{1}{2}e^{-j\pi/2}}}_{a_{-2}} e^{-j2(2\pi/N)n}$$

Fourier series representation of D-T periodic signals

Linear combination of harmonically related complex exponentials

- Real signal $a_k = a^*_{-k}$, or $a^*_k = a_{-k}$

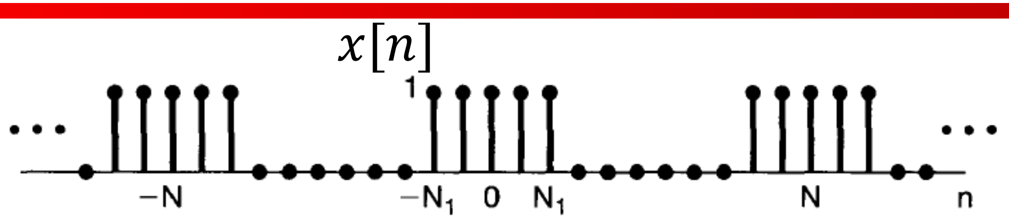
$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n}$$

$$x^*[n] = \sum_{k=\langle N \rangle} a^*_k e^{-jk(2\pi/N)n} = \sum_{k=\langle N \rangle} a^*_{-k} e^{jk(2\pi/N)n}$$

$$x[n] = x^*[n] \qquad \Longrightarrow a_k = a^*_{-k}$$

Fourier series representation of D-T periodic signals

Determine the Fourier Series Representation



- Examples: $x[n]$ discrete square

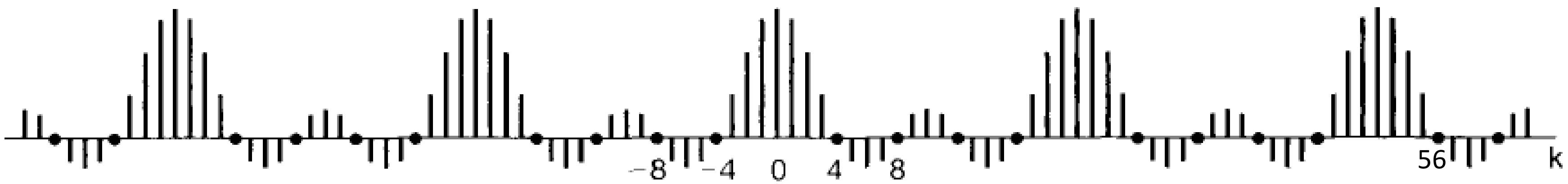
$$a_k = \frac{1}{N}\sum_{n=-N_1}^{N_1} x[n]e^{-jk(2\pi/N)n} = \frac{1}{N}\sum_{n=-N_1}^{N_1} e^{-jk(2\pi/N)n}$$

$m = n + N_1$

$$= \frac{1}{N}\sum_{m=0}^{2N_1} e^{-jk(2\pi/N)(m-N_1)} = \frac{1}{N}e^{jk(2\pi/N)N_1}\sum_{m=0}^{2N_1} e^{-jk(2\pi/N)m}$$

$$= \begin{cases} \dfrac{2N_1+1}{N}, k = 0, \pm N, \pm 2N, \ldots \\ \dfrac{1}{N}\dfrac{\sin[2k\pi(N_1+1/2)/N]}{\sin(k\pi/N)}, k \neq 0, \pm N, \pm 2N, \ldots \end{cases}$$

$a_k \ (2N_1 + 1 = 5, N = 20)$



Fourier series representatio signals

Linear combination of harmonically related cc

- Approximate a discrete square by $\hat{x}[n]$

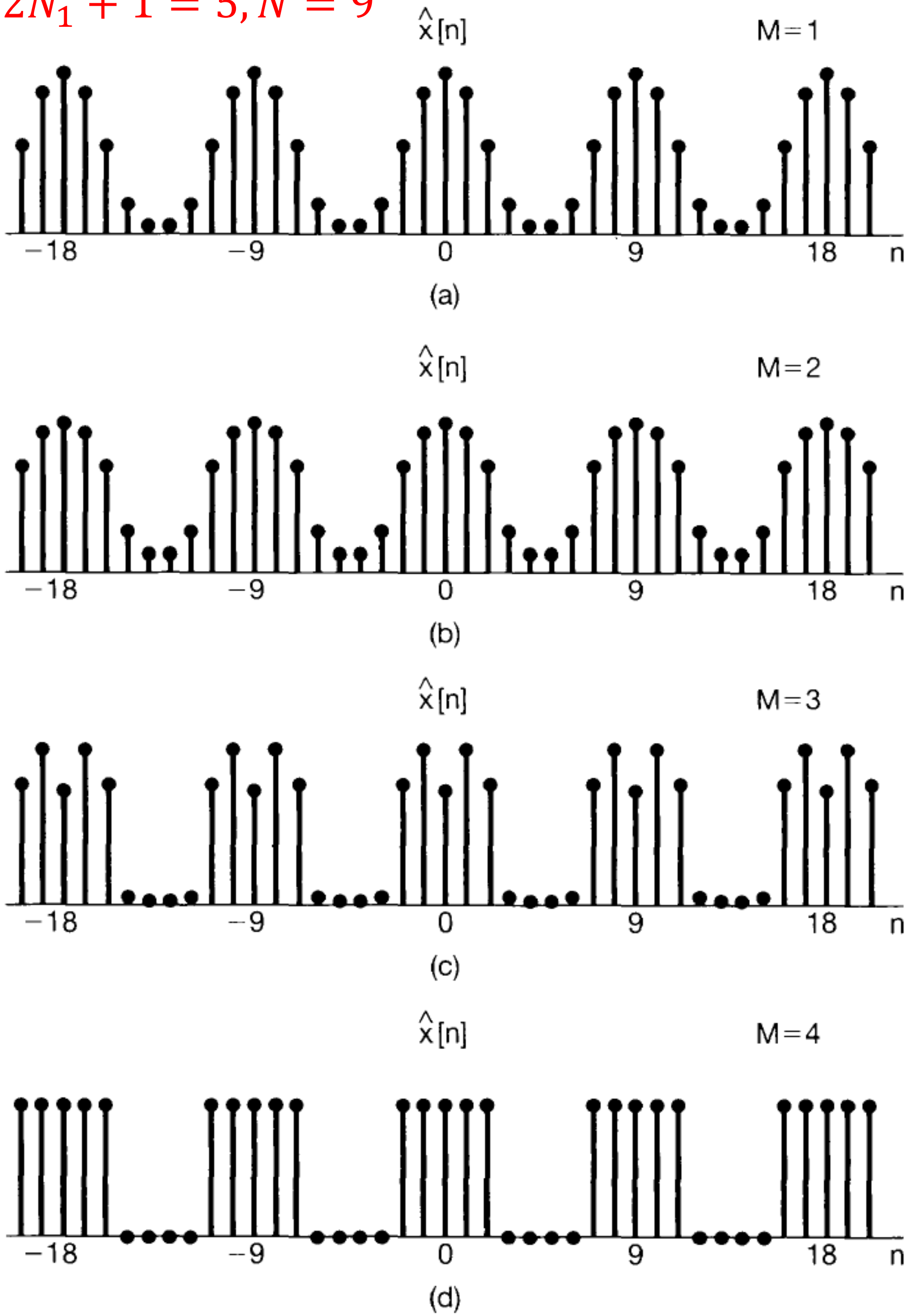
$$\hat{x}[n] = \sum_{k=-M}^{M} a_k e^{jk(2\pi/N)n}$$

$$\text{With } a_k = \begin{cases} \frac{2N_1+1}{N}, k = 0, \pm N, \pm 2N, \ldots \\ \frac{1}{N}\frac{\sin[2k\pi(N_1+1/2)/N]}{\sin(k\pi/N)}, \text{else} \end{cases}$$

- For M=4, $\hat{x}[n] = x[n]$

- No convergence issues for the discrete–time Fourier series!

$2N_1 + 1 = 5, N = 9$



Chapter 3: Fourier Series Representation of Periodic signals

- The response of LTI systems to complex exponentials
- Fourier series representation of continuous periodic signals
- Convergence of the Fourier series
- Properties of continuous-time Fourier series
- Fourier series representation of discrete –time periodic signals
- Properties of discrete FS
- Fourier series and LTI systems

Properties

$$x[n] \overset{\mathcal{FS}}{\longleftrightarrow} a_k \qquad y[n] \overset{\mathcal{FS}}{\longleftrightarrow} b_k$$

- Multiplication

$$x[n]y[n] \overset{\mathcal{FS}}{\longleftrightarrow} \sum_{l=\langle N\rangle} a_l b_{k-l}$$

- First difference

$$x[n]-x[n-1] \overset{\mathcal{FS}}{\longleftrightarrow} \left(1-e^{-jk(2\pi/N)}\right)a_k$$

- Parseval's relation

$$\frac{1}{N}\sum_{n=\langle N\rangle} |x[n]|^2 = \sum_{k=\langle N\rangle} |a_k|^2$$

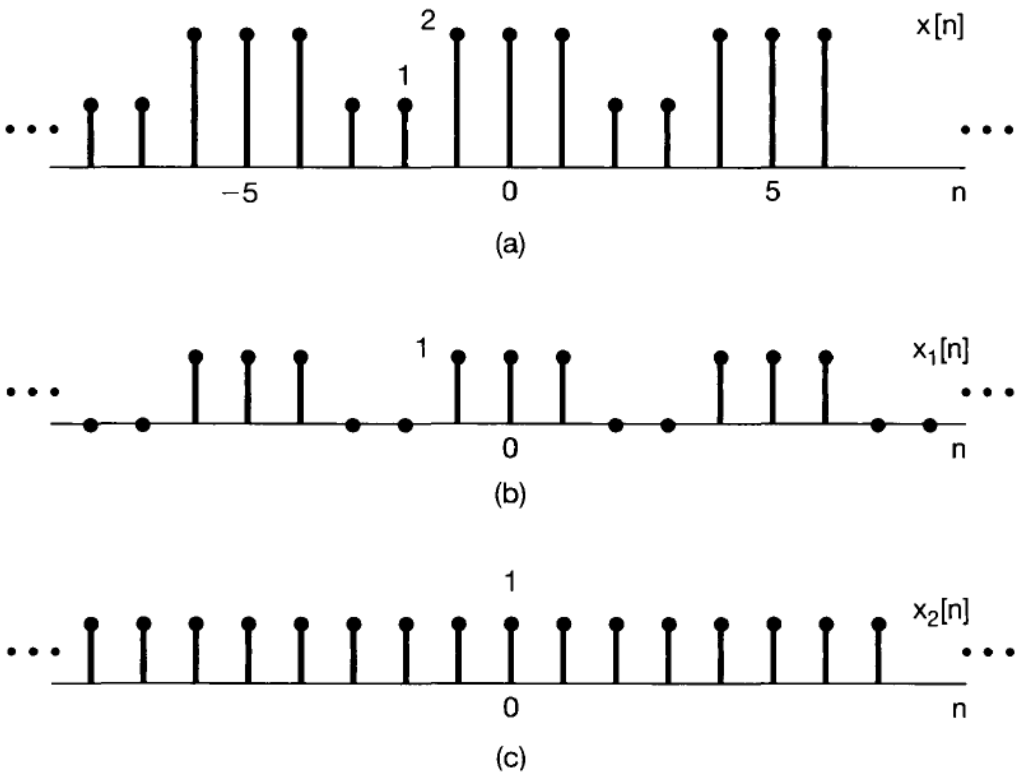
Property	Periodic Signal	Fourier Series Coefficients
	$\left.\begin{matrix} x[n] \\ y[n] \end{matrix}\right\}$ Periodic with period N and fundamental frequency $\omega_0 = 2\pi/N$	$\left.\begin{matrix} a_k \\ b_k \end{matrix}\right\}$ Periodic with period N
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n-n_0]$	$a_k e^{-jk(2\pi/N)n_0}$
Frequency Shifting	$e^{jM(2\pi/N)n}x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}
Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m \end{cases}$ (periodic with period mN)	$\frac{1}{m}a_k \begin{pmatrix}\text{viewed as periodic} \\ \text{with period } mN\end{pmatrix}$
Periodic Convolution	$\sum_{r=\langle N\rangle} x[r]y[n-r]$	Na_kb_k
Multiplication	$x[n]y[n]$	$\sum_{l=\langle N\rangle} a_l b_{k-l}$
First Difference	$x[n] - x[n-1]$	$(1 - e^{-jk(2\pi/N)})a_k$
Running Sum	$\sum_{k=-\infty}^{n} x[k] \begin{pmatrix}\text{finite valued and periodic only} \\ \text{if } a_0 = 0\end{pmatrix}$	$\left(\frac{1}{(1-e^{-jk(2\pi/N)})}\right)a_k$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{R}e\{a_k\} = \mathcal{R}e\{a_{-k}\} \\ \mathcal{I}m\{a_k\} = -\mathcal{I}m\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	$x[n]$ real and even	a_k real and even
Real and Odd Signals	$x[n]$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e[n] = \mathcal{E}v\{x[n]\} & \text{[x[n] real]} \\ x_o[n] = \mathcal{O}d\{x[n]\} & \text{[x[n] real]} \end{cases}$	$\mathcal{R}e\{a_k\}$ $j\mathcal{I}m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{N}\sum_{n=\langle N\rangle} |x[n]|^2 = \sum_{k=\langle N\rangle} |a_k|^2$$

Properties of discrete-time FS

- Examples $\quad x[n] = x_1[n] + x_2[n]$



- $x_1[n]$ is a square wave with $N = 5$ and $N_1 = 1$

$$b_k = \begin{cases} \frac{2N_1+1}{N}, k = \pm N, \pm 2N, \dots \\ \frac{1}{N}\frac{\sin[2k\pi(N_1+1/2)/N]}{\sin(k\pi/N)}, \text{else} \end{cases} = \begin{cases} \frac{3}{5}, k = \pm 5, \pm 10, \dots \\ \frac{1}{5}\frac{\sin(3k\pi/5)}{\sin(k\pi/5)}, \text{else} \end{cases}$$

- For $x_2[n]$

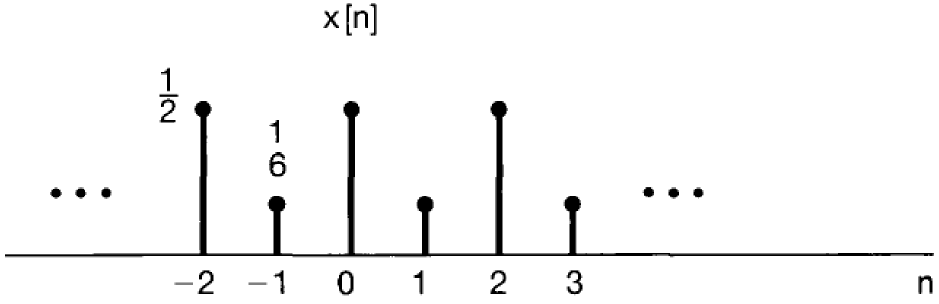
$$c_k = \begin{cases} 1, k = \pm N, \pm 2N, \dots \\ 0, \qquad \text{else} \end{cases}$$

$$\therefore a_k = b_k + c_k = \begin{cases} \frac{8}{5}, \qquad k = \pm 5, \pm 10, \dots \\ \frac{1}{5}\frac{\sin(3k\pi/5)}{\sin(k\pi/5)}, \qquad \text{else} \end{cases}$$

Properties of discrete-time FS

Examples

Suppose we are given the following facts about a sequence $x[n]$:



1. $x[n]$ is periodic with period $N = 6$.
2. $\sum_{n=0}^{5} x[n] = 2$.
3. $\sum_{n=2}^{7} (-1)^n x[n] = 1$.
4. $x[n]$ has the minimum power per period among the set of signals satisfying the preceding three conditions.

Solution

- $\sum_{n=0}^{5} x[n] = 2 \Longrightarrow a_0 = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-j0(2\pi/N)n} = 1/3.$
- $\sum_{n=2}^{7}(-1)^n x[n] = 1 \Longrightarrow \sum_{n=\langle N\rangle} x[n]e^{-j3(2\pi/N)n} = 1 \Longrightarrow a_3 = 1/6$
- from 4, $a_1 = a_2 = a_4 = a_5 = 0$
- $\therefore x[n] = a_0 e^{-j0(2\pi/N)n} + a_3 e^{-j3(2\pi/N)n} = \frac{1}{3} + \frac{1}{6}e^{-j\pi n} = \frac{1}{3} + \frac{1}{6}(-1)^n$

上海科技大学 ShanghaiTech University

Chapter 3: Fourier Series Representation of Periodic signals

- The response of LTI systems to complex exponentials
- Fourier series representation of continuous periodic signals
- Convergence of the Fourier series
- Properties of continuous-time Fourier series
- Fourier series representation of discrete –time periodic signals
- Properties of discrete FS
- Fourier series and LTI systems

Fourier series and LTI systems

- Recall

$$e^{st} \longrightarrow \boxed{\text{LTI}} \longrightarrow H(s)e^{st} \qquad H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau$$

$$z^{n} \longrightarrow \boxed{\text{LTI}} \longrightarrow H[z]z^{n} \qquad H[z] = \sum_{k=-\infty}^{\infty} h[k]z^{-k}$$

- System functions: $H(s)$ and $H[z]$

e^{st}
- s pure imaginary $e^{st} \rightarrow e^{j\omega t}$
  - For periodic signal, CT Fourier Series (Ch3)
  - For aperiodic signal, CT Fourier Transform (Ch4)
- s complex number e^{st} — Laplace Transform (Ch9)

z^{n}
- $|z|=1$, $z^{n} \rightarrow e^{j\omega n}$
  - For periodic signal, DT Fourier Series (Ch3)
  - For aperiodic signal, DT Fourier Transform (Ch5)
- $|z|$ not limited to 1, z^{n} — Z-Transform (Ch10)

Fourier series and LTI systems

- Frequency response for CT system: $H(j\omega)$

$$H(s)=\int_{-\infty}^{\infty}h(\tau)e^{-s\tau}d\tau \quad \xRightarrow{s=j\omega} \quad H(j\omega)=\int_{-\infty}^{\infty}h(\tau)e^{-j\omega\tau}d\tau$$

$$e^{j\omega t} \longrightarrow \boxed{\text{LTI}} \longrightarrow H(j\omega)e^{j\omega t}$$

$$x(t)=\sum_{k=-\infty}^{\infty}a_k e^{jk\omega_0 t} \longrightarrow \boxed{\text{LTI}} \longrightarrow y(t)=\sum_{k=-\infty}^{\infty}a_k H(jk\omega_0)e^{jk\omega_0 t}$$

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k \qquad y(t) \overset{\mathcal{FS}}{\longleftrightarrow} b_k$$

$$b_k = a_k H(jk\omega_0)$$

Fourier series and LTI systems

- Frequency response for CT system: example

$x(t)=\sum_{k=-3}^{3} a_k e^{jk2\pi t}$ $(a_0=1,\ a_1=a_{-1}=\frac{1}{4},\ a_2=a_{-2}=\frac{1}{2},\ a_3=a_{-3}=\frac{1}{3})$ is the input of a LTI system with $h(t)=e^{-t}u(t)$, determine $y(t)$

- Solution

$$y(t)=\sum_{k=-\infty}^{\infty} a_k H(j\omega) e^{jk2\pi t} \qquad H(j\omega)=\int_0^{\infty} e^{-\tau} e^{-j\omega\tau} d\tau=\frac{1}{1+j\omega}$$

$$b_k=a_k H(j\omega)=a_k\frac{1}{1+jk2\pi} \qquad b_0=1\cdot 1=1 \qquad b_1=\frac{1}{4}\frac{1}{1+j2\pi} \qquad b_{-1}=\frac{1}{4}\frac{1}{1-j2\pi}$$

$$b_2=\frac{1}{2}\frac{1}{1+j4\pi} \qquad b_{-2}=\frac{1}{2}\frac{1}{1-j4\pi} \qquad b_3=\frac{1}{3}\frac{1}{1+j6\pi} \qquad b_{-3}=\frac{1}{3}\frac{1}{1-j6\pi}$$

Fourier series and LTI systems

- Frequency response DT system: $H\left(e^{j\omega}\right)$

$$H[z] = \sum_{n=-\infty}^{\infty} h[k] z^{-n} \quad \xRightarrow{z=e^{j\omega}} \quad H\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n}$$

$$e^{j\omega n} \longrightarrow \boxed{\text{LTI}} \longrightarrow H\left(e^{j\omega}\right) e^{j\omega n}$$

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n} \longrightarrow \boxed{\text{LTI}} \longrightarrow y[n] = \sum_{k=\langle N\rangle} a_k H\left(e^{j\omega}\right) e^{jk(2\pi/N)n}$$

$$b_k = a_k H\left(e^{j\omega}\right)$$

上海科技大学 ShanghaiTech University

Fourier series and LTI systems

- Frequency response DT system: example

$$h[n] = \alpha^n u[n], |\alpha| < 1$$

$$x[n] = \cos\frac{2\pi n}{N} \longrightarrow \boxed{\text{LTI}} \longrightarrow y[n]?$$

- Solution

$$x[n] = \frac{1}{2}e^{j(2\pi/N)n} + \frac{1}{2}e^{-j(2\pi/N)n}$$

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} = \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n} = \frac{1}{1-\alpha e^{-j\omega}}$$

$$y[n] = \frac{1}{2}\left(\frac{1}{1-\alpha e^{-j2\pi/N}}\right)e^{j(2\pi/N)n} + \frac{1}{2}\left(\frac{1}{1-\alpha e^{j2\pi/N}}\right)e^{-j(2\pi/N)n}$$