上海科技大学
ShanghaiTech University

CS240 Algorithm Design and Analysis

Lecture 0

Introduction and Overview

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Fall 2024
2024.09.19

Organization

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Organization

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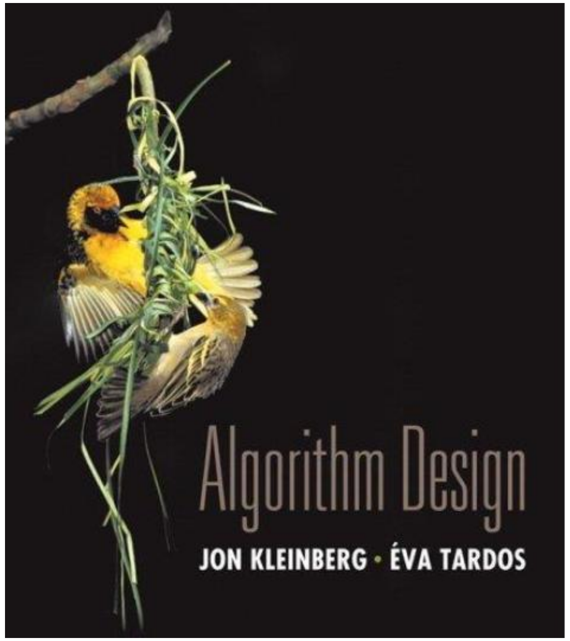
Prerequisites

- **Algorithm and Data Structure (Undergraduate course)**
  - Sorting and searching, divide & conquer, greedy, dynamic programming, graph algorithms
  - Analysis of algorithms
- **Basic discrete mathematics**
  - Recurrences, logic and proofs, basic graph theory
- **Basic probability theory**
  - Probability space, random variables, expectation, variance
- **Computer programming**
  - Doesn't matter which language(s) you know
  - But you should be capable of translating high-level algorithm descriptions into working programs in some programming language

Organization

- **Textbook**
  - [KT] Algorithm Design, by Jon Kleinberg and Eva Tardos.
  - [CLRS] Introduction to Algorithms (3rd edition), by T. Cormen, C. Leiserson, R. Rivest, and C. Stein.
  - [V] Approximation Algorithms, by Vijay V. Vazirani.
  - [MR] Randomized Algorithms, by Rajeev Motwani and Prabhakar Raghavan.
- **Piazza** (https://piazza.com/shanghaitech.edu.cn/fall2024/cs240) **(Please JOIN as students!)**
  - Lecture slides, announcements, homework assignment, QA and discussions, etc.
- **Gradescope**
  - Homework submission and grading
- **Academic Integrity**
  - Unless explicitly noted, work turned in should reflect your own/independent capabilities
  - No cheating (We will check carefully!)
    - Don't share your homework/code!
    - No fake solutions!
    - No plagiarism!
    - Serious consequences!



Organization

- Grading
  - Assignments (20%)
  - Midterm (35%)
  - Final (35%)
  - Course Project (10%)
  - Exams will be open-book with only one A4 cheating cheet

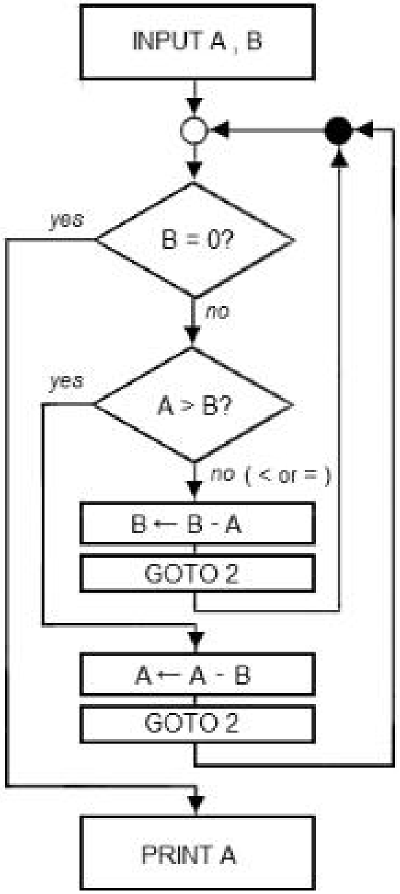
上海科技大学2024-2025学年校历

	八月		九月				十月					十一月				十二月				一月		
星期一	19	26	2	9	16	23	30	7	14	21	28	4	11	18	25	2	9	16	23	30	6	13
星期二	20	27	3	10	17 中秋节	24	1 国庆节	8	15	22	29	5	12	19	26	3	10	17	24	31	7	14
星期三	21	28	4	11	18	25	2	9	16	23	30	6	13	20	27	4	11	18	25	1 元旦	8	15
星期四	22	29	5	12	19	26	3	10	17	24	31	7	14	21	28	5	12	19	26	2	9	16
星期五	23	30	6	13	20	27	4	11	18	25	1	8	15	22	29	6	13	20	27	3	10	17
星期六	24	31	7	14	21	28	5	12	19	26	2	9	16	23	30	7	14	21	28	4	11	18
星期日	25	1	8	15	22	29	6	13	20	27	3	10	17	24	1	8	15	22	29	5	12	19
周数	4	5	6	7	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18
学期	暑假				秋学期																	

Dates for homework assignments, midterm, and course project will be announced in due course~

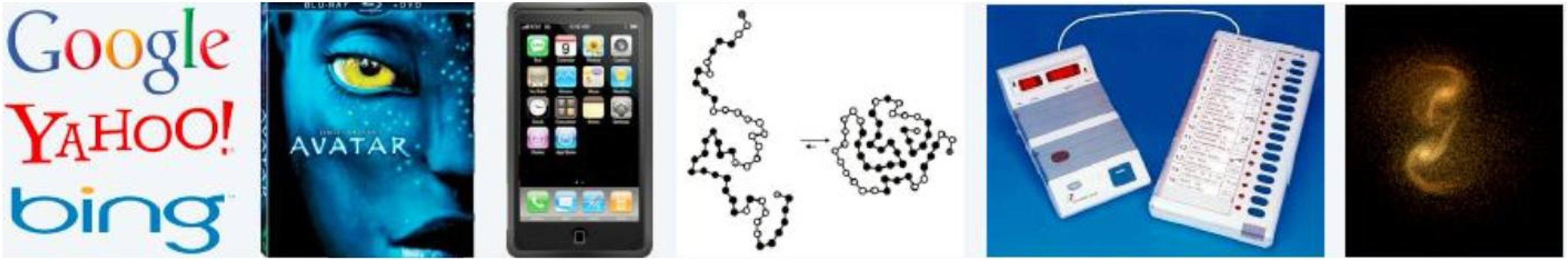
Algorithms

- [Knuth, TAOCP] An algorithm is a finite, definite, effective procedure, with some input and some output
- [Wikipedia] An algorithm is a finite sequence of well-defined, computer-implementable instructions, typically to solve a class of problems or to perform a computation
- Important for all other branches of computer science
- Plays a key role in modern technological innovation
- Provides novel "lens" on processes outside of computer science and technology
  - Internet: Web search, packet routing, distributed file sharing, ...
  - Biology: Human genome project, protein folding, ...
  - Computers: Circuit layout, databases, caching, networking, compilers, ...
  - Computer graphics: Movies, video games, virtual reality, ...
  - Security: Cell phones, e-commerce, voting machines, federated learning, ...
  - Multimedia: MP3, JPG, DivX, HDTV, face recognition, ...
  - Social networks: Recommendations, news feeds, advertisements, ...
  - Physics. N-body simulation, particle collision simulation, ...



Why Study Algorithms?

- Wide range of applications
  - Internet. Web search, packet routing, distributed file sharing, …
  - Biology. Human genome project, protein folding, …
  - Computers. Circuit layout, databases, caching, networking, compilers, …
  - Computer graphics. Movies, video games, virtual reality, …
  - Security. Cell phones, e-commerce, voting machines, …
  - Multimedia. MP3, JPG, DivX, HDTV, face recognition, …
  - Social networks. Recommendations, news feeds, advertisements, …
  - Physics. N-body simulation, particle collision simulation, …



Typical Undergraduate Algorithm Course

Understanding and implementing classic algorithms

- Sorting
- Searching
- String algorithms
- Graph algorithms

Critical thinking, problem-solving, coding

This Course

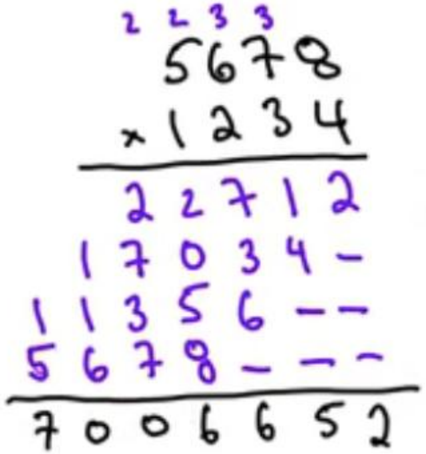
Design and analysis of computer algorithms

- Vocabulary for design and analysis of algorithms
- Greedy algorithms
- Divide-and-conquer
- Dynamic programming
- Network flow
- Intractability (complexity classes)
- Amortized analysis
- Approximation algorithms
- Randomized algorithms
- Local search

Critical thinking, problem-solving, rigorous analysis

Integer Multiplication

- Input: two n-digit numbers x and y
- Output: the product x · y
- “Primitive operation”: add or multiply two single-digit numbers
- The grade-school algorithm: 5678 * 1234 = 7006652
- **2n** operations per row and there are **n** rows
- Upshot: #operations overall <= constant · n^2



- The Algorithm Designers’ Mantra

“*Perhaps the most important principle for the good algorithm designers is to refuse to be content.*” – Aho, Hopcroft, and Ullman, The Design and Analysis of Computer Algorithms, 1974

- **CAN WE DO BETTER?**

Karatsuba Multiplication

- Example: $x = 5678$ $y = 1234$ to compute the product $x \cdot y$
- Assume $56 = a$, $78 = b$, $12 = c$, $34 = d$
- Step 1: Compute $a \cdot c = 672$
- Step 2: Compute $b \cdot d = 2652$
- Step 3: Compute $(a+b) \cdot (c+d) = 134 \cdot 46 = 6164$
- Step 4: Compute Step3 − step2 − step1 = 2840

$$
\begin{array}{r}
6720000 \\
2652 \\
284000 \\
\hline
7006652
\end{array}
$$

A Recursive Algorithm

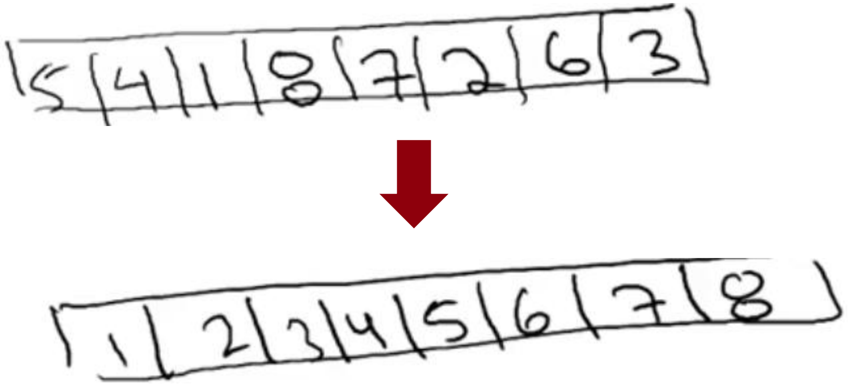
- Write $x = 10^{n/2}a + b$ and $y = 10^{n/2}c + d$ where a, b, c, d are n/2 − digit numbers
- [Example: a = 56, b = 78, c = 12, d = 34]
- Then: $x \cdot y = (10^{n/2}a + b) \cdot (10^{n/2}c + d) = 10^{n}ac + 10^{n/2}(ad + bc) + bd$
- Idea; recursively compute ac, ad, bc, bd, then compute the above equation in the straightforward way
- Simple base case omitted (if input is very small, get the result immediately)

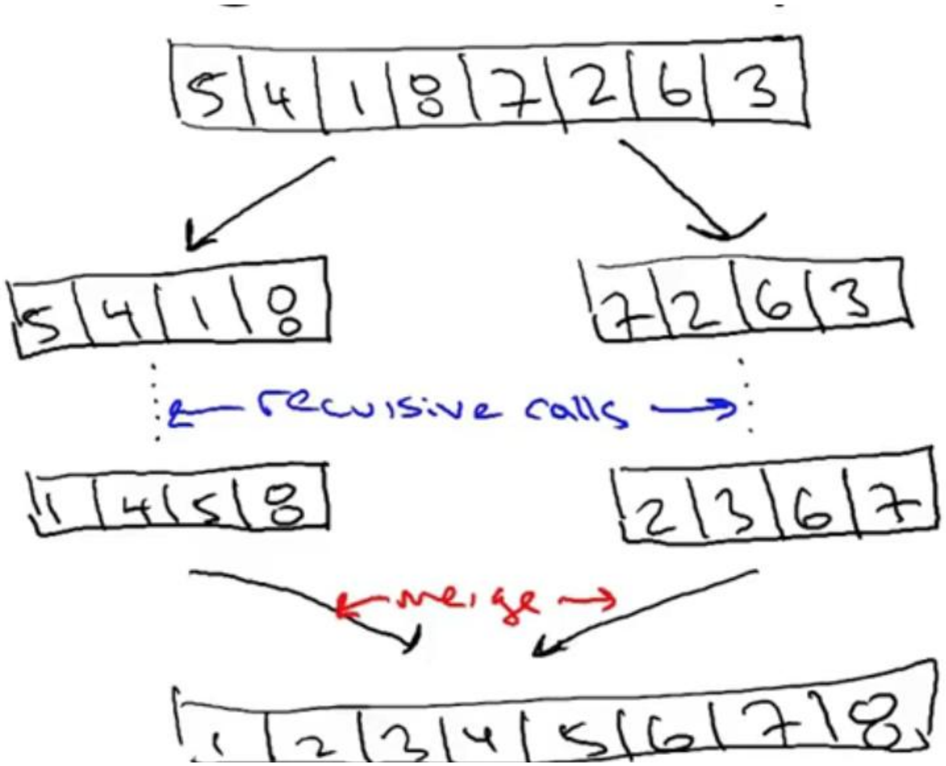
- **Karatsuba Multiplication**
  - Recall: $x \cdot y = 10^{n}ac + 10^{n/2}(ad + bc) + bd$ (seems having 4 recursive multiplications…)
  - Step 1: recursively compute ac
  - Step 2: recursively compute bd
  - Step 3: recursively compute (a+b)(c+d) = ac+ad+bc+bd
  - **Gaoss's trick: step3 − step1 − step2 = ad + bc**
  - Upshot: only need 3 recursive multiplications and some additions

Why study merge sort?

- Good introduction to divide & conquer
  - Improves over selection, insertion, bubble sorts
- Motivates guiding principles for algorithm analysis (worst-case and asymptotic analysis)
- Analysis generalizes to “Master Method”

- The Sorting Problem
- Input: array of n numbers, unsorted
- Output: Same numbers, sorted in increasing order





Merge Sort: Pseudocode

- Recursively sort 1st half of input array
- Recursively sort 2nd half of input array
- Merge two sorted sublists into one

- Pseudocode for Merge:

C = output array [length = n]

A = 1st sorted array [n/2]

B = 2nd sorted array [n/2]

i = 1

j = 1

```
for k  =  1 to n
    if A[i] < B[j]
        C[k] = A[i]
        i++
    else B[j] < A[i]
        C[k] = B[j]
        j++
End
(ignores end cases)
```

Merge Sort Running Time?

- **Key question:** running the MergesSort on array of n numbers?
- Running time ≈ # of lines of code executed

- Pseudocode for Merge:

C = output array [length = n]

A = 1st sorted array [n/2]

B = 2nd sorted array [n/2]

i = 1, j = 1

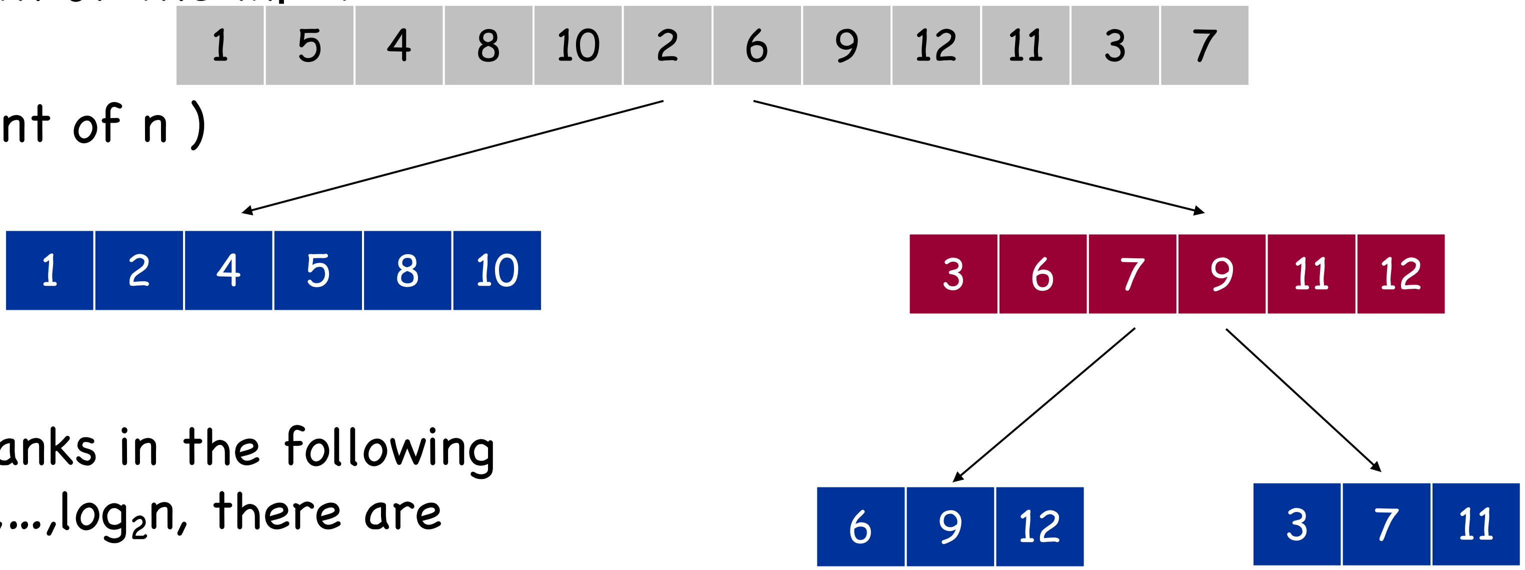
2 operations

```
for k  =  1 to n
    if A[i] < B[j]
        C[k] = A[i]
        i++
    else B[j] < A[i]
        C[k] = B[j]
        j++
End
(ignores end cases)
```

- **Upshot:** running time of merge on array of n numbers is <= 4n + 2 <= 6n (since n >= 1)
- **Claim:** MergeSort requires <= 6nlogn + 6n operations to sort n numbers
- **Recall:** logn = # d times you divide by 2 until you get down to 1

Proof of claim (assuming n = power of 2)

- Will use "recursion tree"
- Q: Roughly how many levels does this recursion tree have (as a function of n, the length of the input array)?
  - A constant number (independent of n)
  - $\sqrt{n}$
  - $\log_2 n$
  - n



- Q: What is the pattern? Fill in blanks in the following statement: at each level $j = 0,1,2,\ldots,\log_2 n$, there are ___ subproblems, each of size ____.
  - 2^j and 2^j, respectively
  - $n/2^j$ and $n/2^j$, respectively
  - 2^j and $n/2^j$, respectively
  - $n/2^j$ and 2^j, respectively

Proof of claim (assuming n = power of 2)

- At each level $j = 0,1,2,\ldots,\log_2 n$, there are 2^j subproblems, each of size $n/2^j$
- Total # of operations at level j:

[each $j=0,1,2,\ldots,\log_2 n$]

$<= 2^j * 6(n/2^j) = 6n$ ← independent of j

- Total $<= 6n(\log_2 n+1)$

- **Claim:** For every input array of n numbers, MergeSort produces a sorted output array and uses at most $6n\log_2 n+6n$ operations

Guiding Principles

- **Guiding Principle 1**
  - "worst-case analysis" : running time bound holds for every input of length n
  - Particularly appropriate for "general-purpose" routines
  - As opposed to "average-case" analysis and benchmarks (requires domain knowledge)
  - Worst case usually easier to analyze
- **Guiding Principle 2**
  - Won't pay much attention to constant factors, lower-order terms
  - Way easier
  - Constants depend on architecture/compiler/programmer anyways
  - Lose very little predictive power
- **Guiding Principle 3**
  - Asymptotic analysis: focus on running time for large input size n
  - E.g., $6n\log_2 n + 6n$ "better than" $\frac{1}{2} n^2$ (e.g., insertion sort)
  - Justification: any big problems are interesting

Fast algorithm: worst-case running time grows slowly with input size

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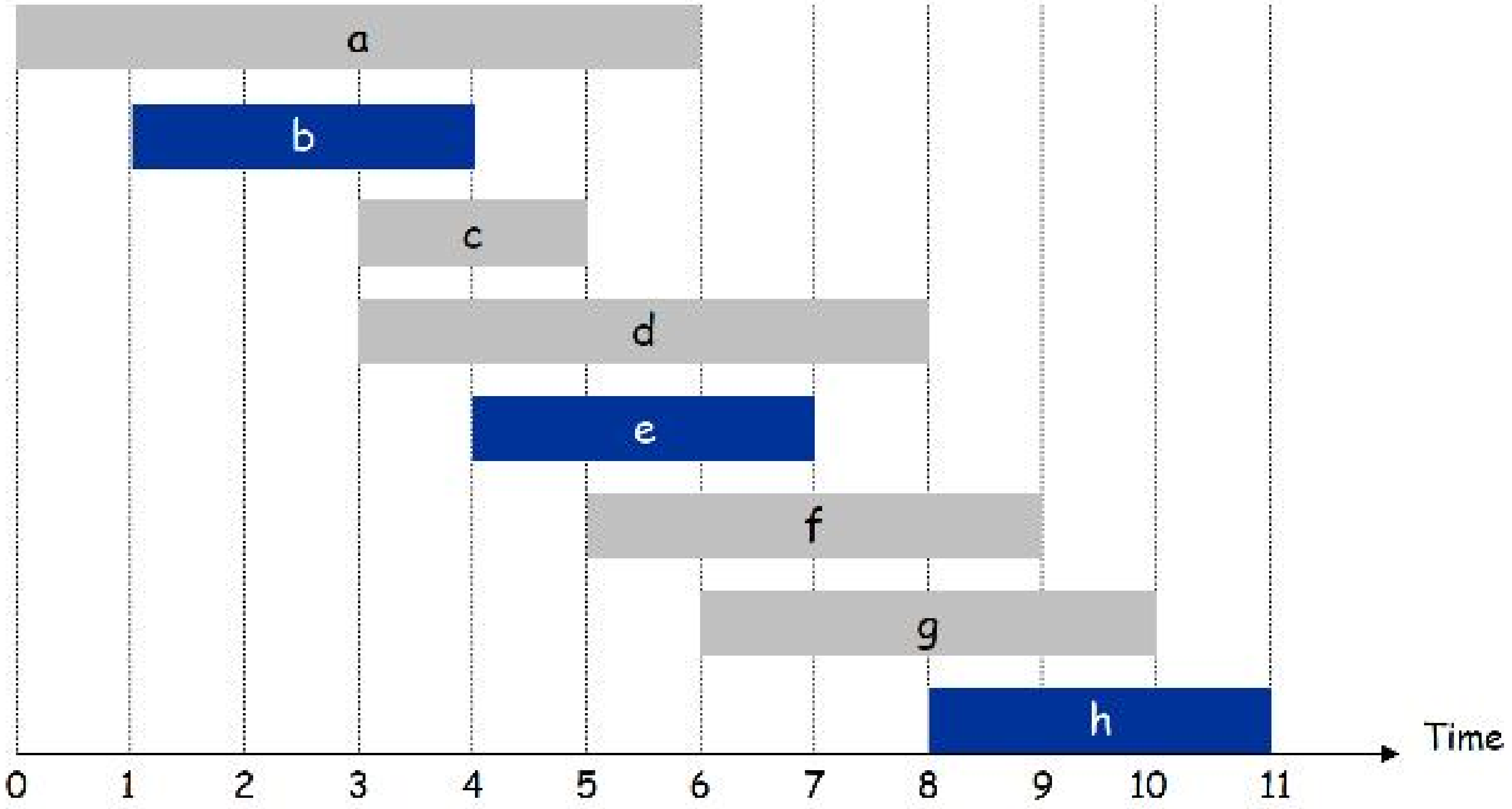
Five Representative Problems

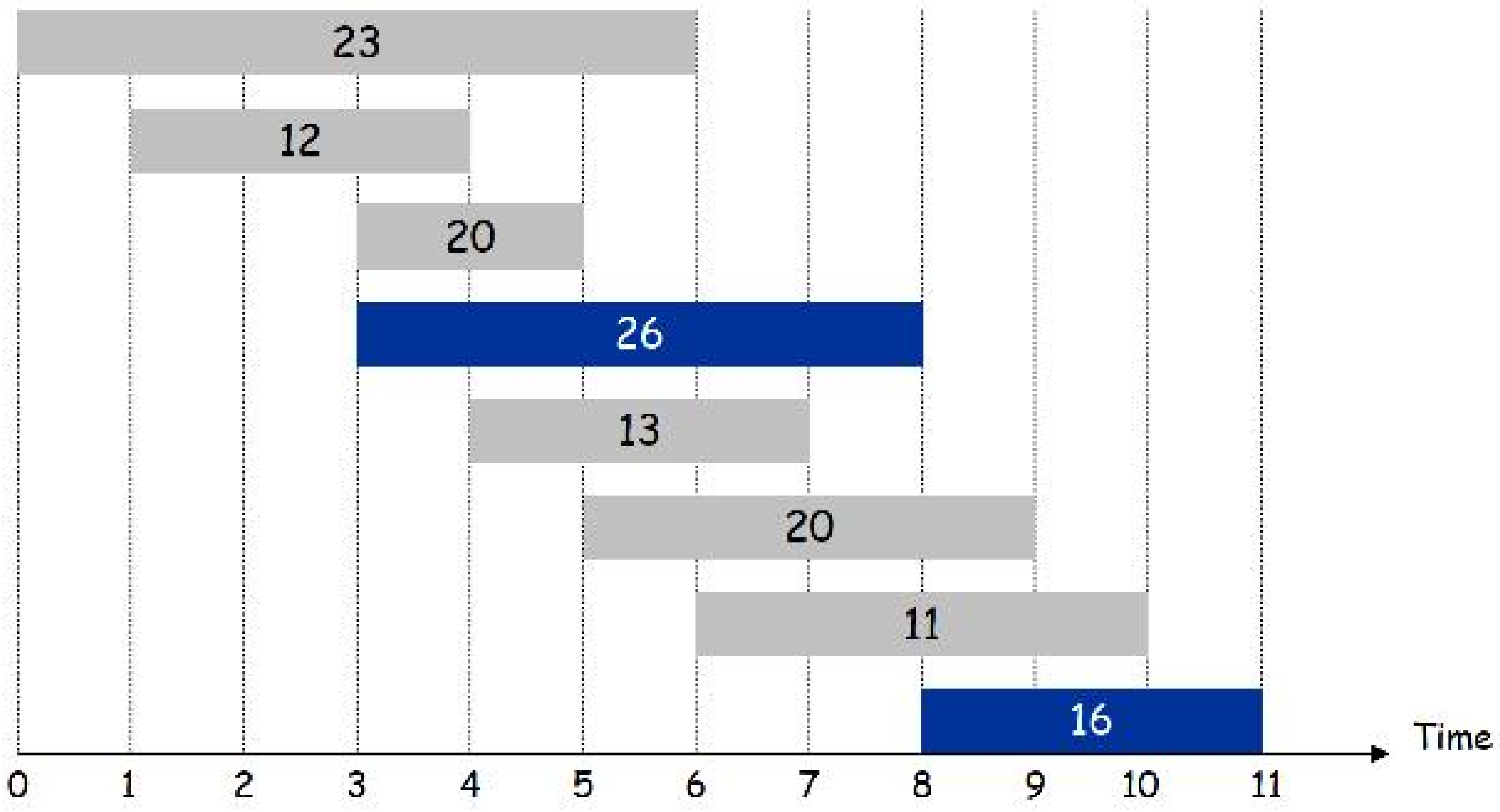
Interval Scheduling

- **Input:** Set of jobs with start times and finish times
- **Goal:** Find **maximum cardinality** subset of mutually compatible (i.e., jobs don't overlap) jobs



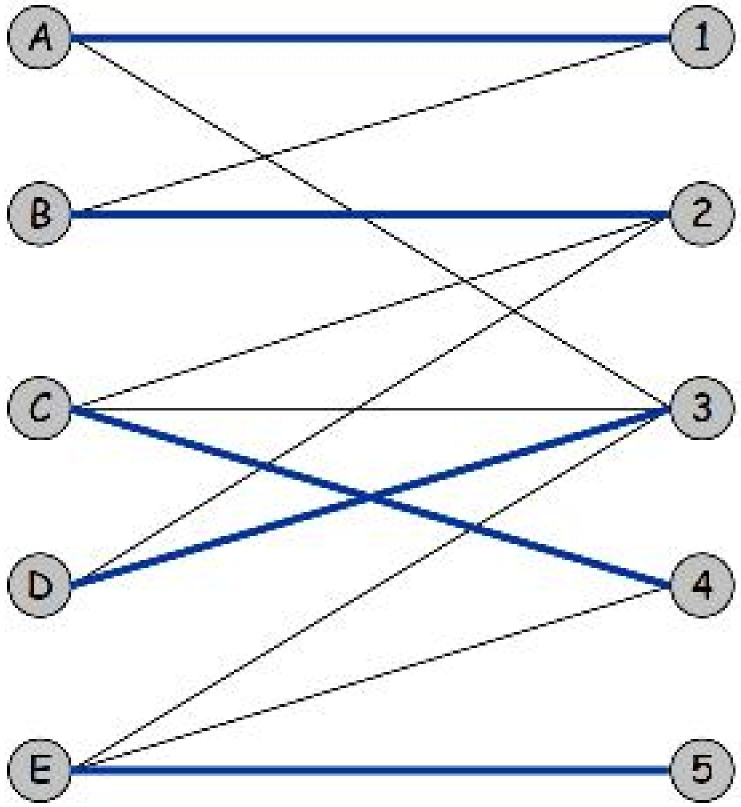
Weighted Interval Scheduling

- **Input:** Set of jobs with start times, finish times, and weights
- **Goal:** Find maximum weight subset of mutually compatible jobs



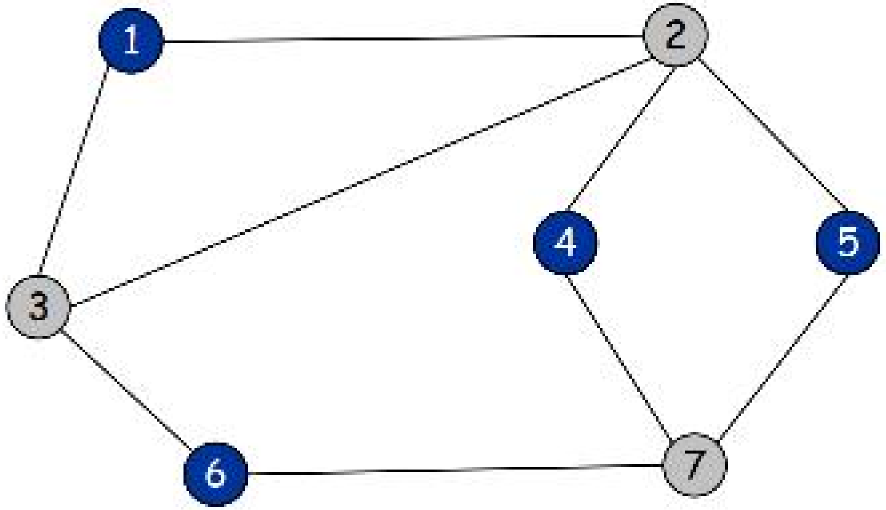
Bipartite Matching

- **Input:** Bipartite graph
- **Goal:** Find **maximum cardinality** matching



Independent Set

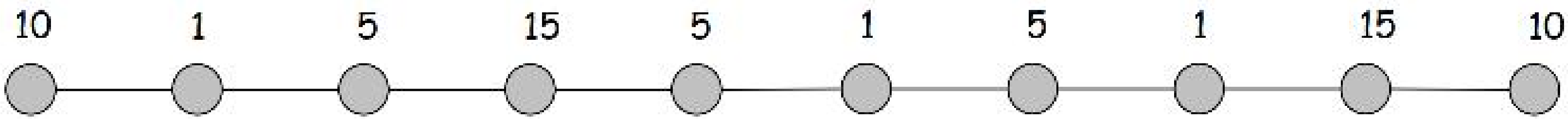
- **Input:** Graph
- **Goal:** Find **maximum cardinality** independent (i.e., subset of nodes such that no two joined by an edge) set



- **Extension:** Weighted independent set

Competitive Facility Location

- **Input:** Graph with weight on each node
- **Game:** Two competing players alternate in selecting nodes. Not allowed to select a node if any of its neighbors have been selected
- **Goal:** Select a **maximum weight** subset of nodes



Second player can guarantee 20, but not 25

Five Representative Problems

- **Variations on a theme:** independent set

- **Interval scheduling:** nlogn greedy algorithm
- **Weighted interval scheduling:** nlogn dynamic programming algorithm
- **Bipartite matching:** n^2 max-flow-based algorithm
- **Independent set:** NP-complete
- **Competitive facility location:** PSPACE-complete

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CS240 Algorithm Design and Analysis

Lecture 1

Computational Tractability

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"For me, great algorithms are the poetry of computation. Just like verse, they can be terse, allusive, dense, and even mysterious. But once unlocked, they cast a brilliant new light on some aspect of computing." – Francis Sullivan, Science, Vol. 287, No. 5454, p.799, February 2000

Polynomial-Time

- **Brute force.** For many non-trivial problems, there is a natural brute force search algorithm that checks every possible solution
  - Typically takes 2^N time or worse for inputs of size N
  - Unacceptable in practice

- **Desirable scaling property.** When the input size doubles, the algorithm should only slow down by some constant factor C

Poly-time: There exists constants $c > 0$ and $d > 0$ such that on every input of size N, its running time is bounded by cN^d steps

- **Thm.** An algorithm is poly-time iff the above scaling property holds (i.e., choose $C = 2^d$)

Polynomial-Time

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- **Def.** An algorithm is **efficient** if its running time is polynomial

- **Exceptions**
  - Some poly-time algorithms do have high constants and/or exponents, and are useless in practice. Ex. $6.02 * 10^{23} * N^{20}$

- **Justification: It really works in practice!**
  - In practice, the poly-time algorithms that people develop almost always have low constants and low exponents
  - Breaking through the exponential barrier of brute force typically exposes some crucial structure of the problem

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Why It Matters

	n	$n \log_2 n$	n^2	n^3	1.5^n	2^n	$n!$
$n = 10$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	4 sec
$n = 30$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	< 1 sec	18 min	10^{25} years
$n = 50$	< 1 sec	< 1 sec	< 1 sec	< 1 sec	11 min	36 years	very long
$n = 100$	< 1 sec	< 1 sec	< 1 sec	1 sec	12,892 years	10^{17} years	very long
$n = 1{,}000$	< 1 sec	< 1 sec	1 sec	18 min	very long	very long	very long
$n = 10{,}000$	< 1 sec	< 1 sec	2 min	12 days	very long	very long	very long
$n = 100{,}000$	< 1 sec	2 sec	3 hours	32 years	very long	very long	very long
$n = 1{,}000{,}000$	1 sec	20 sec	12 days	31,710 years	very long	very long	very long

The running times (rounded up) of different algorithms on inputs of increasing size, for a processor performing a million high-level instructions per second.
In cases where the running time exceeds 10^{25} years, we simply record the algorithm as taking a very long time

Worst-Case Analysis

- **Worst case running time.** Obtain bound on largest possible running time of algorithm on input of a given size N
  - Generally captures efficiency in practice

- **Exceptions**
  - Some exponential-time (or worse) algorithms are widely used because the worst-case instances seem to be rare (e.g., simplex method Unix grep)

Average-Case Analysis

- **Average case running time.** Obtain bound on running time of algorithm on **random** input as a function of input size N
  - Need to choose a distribution over input instances
  - Algorithm tuned for a certain distribution may perform poorly on other inputs
  - Average-case analysis may tell us more about the choice of distributions than about the algorithm itself

Asymptotic Order of Growth

Asymptotic Order of Growth

- **Importance:** vocabulary for the design and analysis of algorithms (e.g., bit-oh notation)
- Sweet spot for high-level reasoning about algorithms
- Coarse enough to suppress architecture/language/compiler-dependent details
- Sharp enough to make useful comparisons between different algorithms, especially on large inputs (e.g., sorting or integer multiplication)
- High-level idea: Suppress constant factors (too system-dependent) and lower-order terms → irrelevant for large inputs
- **Example:** equate $6n\log_2 n + 6n$ with just $n\log n$
- **Terminology:** running time is $O(n\log n)$ where n = input size (e.g., length of input array)

Examples

- Problem: does array A contain the integer t?
- Given A (array of length n) and t (an integer)

```
for i = 1 to n
    if A[i] == t return TRUE
return FALSE
```

- Problem: Given A, B (array of length n) and t (an integer), does A or B contain t?

```
for i = 1 to n
    if A[i] == t return TRUE
for i = 1
    if B[i] == t return TRUE
return FALSE
```

- Problem: Do arrays A and B have a number in common?

```
for i = 1 to n
    for j = 1 to n
        if A[i] == B[j] return
        TRUE
return FALSE
```

- Problem: Do arrays A have duplicated entities?

```
for i = 1 to n
    for j = i+1 to n
        if A[i] == A[j] return
        TRUE
return FALSE
```

Question: What is the running time?

$O(1)$

$O(n)$

$O(\log n)$

$O(n^2)$

Asymptotic Order of Growth

- **Upper bounds.** $T(n)$ is $O(f(n))$ if there exist constants $c > 0$ and $n_0 >= 0$ such that for all $n >= n_0$ we have $T(n) <= c \cdot f(n)$
- Example #1
  - Claim: If $T(n) = a_k n^k + \ldots + a_1 n + a_0$ then $T(n) = O(n^k)$
  - Proof: Choose $n_0 = 1$ and $c = |a_k|+|a_{k-1}|+\ldots+|a_1|+|a_0|$
  - Need to show that exist $n >= 1$, $T(n) <= cn^k$
  - We have for every $n >= 1$, $T(n) <= |a_k|n^k + \ldots + |a_1|n + |a_0| <= |a_k|n^k + \ldots + |a_1|n^k + |a_0|n^k = cn^k$
- Example #2
  - Claim: for every $k >= 1$, n^k is not $O(n^{k-1})$
  - Proof: by contradiction. Suppose $n^k = O(n^{k-1})$
  - Then exist constants $c, n_0 > 0$ such that $n^k <= cn^{k-1}$ for every $n >= n_0$
  - But then cancelling n^{k-1} from both sides → $n <= c$ which is clearly false

Asymptotic Order of Growth

- **Upper bounds.** $T(n)$ is $O(f(n))$ if there exist constants $c > 0$ and $n_0 >= 0$ such that for all $n >= n_0$ we have $T(n) <= c \cdot f(n)$
- **Lower bounds.** $T(n)$ is $\Omega(f(n))$ if there exist constants $c > 0$ and $n_0 >= 0$ such that for all $n >= n_0$ we have $T(n) >= c \cdot f(n)$
- **Tight bounds.** $T(n)$ is $\Theta(f(n))$ if $T(n)$ is both $O(f(n))$ and $\Omega(f(n))$.
  - Exist constants c_1, c_2, n_0, such that $c_1 f(n) <= T(n) <= c_2 f(n)$ for all $n >= n_0$
- Let $T(n) = ½n^2 + 3n$. Which of the following statements are true?
  - $T(n) = O(n)$
  - $T(n) = \Omega(n)$
  - $T(n) = \Theta(n)$
  - $T(n) = O(n^3)$

Ex: $T(n) = 32n^2 + 17n + 32$

$T(n)$ is $O(n^2)$, $O(n^3)$ ← choose $c = 50$, $n_0 = 1$

$T(n)$ is $\Omega(n^2)$, $\Omega(n)$ ← choose $c=32$, $n_0 = 1$

$T(n)$ is $\Theta(n^2)$

$T(n)$ is not $O(n)$, $\Omega(n^3)$, $\Theta(n)$, or $\Theta(n^3)$

More Examples

- **Claim:** $2^{n+10} = O(2^n)$
- **Proof:** need to pick constants c, n_0 such that $2^{n+10} <= c2^n$ for every $n >= n_0$
- **Note:** $2^{n+10} = 2^{10}2^n = (1024)2^n$
- **So:** if we choose c = 1024, n_0 = 1, then $2^{n+10} <= c2^n$ holds

- **Claim:** 2^{10n} is not $O(2^n)$
- **Proof:** By contradiction. If $2^{10}n = O(2^n)$, then exist constants c, $n_0 > 0$, such that
- $2^{10n} <= c2^n$ for every $n >= n_0$
- But then cancelling 2^n:
- $2^{9n} <= c$ for every $n >= n_0$ which is certainly false

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More Examples

- **Claim:** for every pair of (positive) functions f(n), g(n), max{f, g} = $\Theta(f(n) + g(n))$
- **Proof:** max{f, g} = $\Theta(f(n) + g(n))$

For every n, we have max{f(n), g(n)} <= f(n) + g(n)

and

2max{f(n), g(n)} >= f(n) + g(n) → max{f(n), g(n)} >= ½ (f(n) + g(n))

Thus: ½ (f(n) + g(n)) <= max{f(n), g(n} <= f(n) + g(n) for all n >= 1

→ max{f, g} = $\Theta(f(n) + g(n))$, where $n_0 = 1$, $c_1 = ½$, $c_2 = 1$

Notation and Properties

- **Slight abuse of notation.** $T(n) = O(f(n))$
- **Asymmetric**
  - $f(n) = 5n^3$; $g(n) = 3n^2$
  - $f(n) = O(n^3) = g(n)$
  - But $f(n) \neq g(n)$
  - Better notation: $T(n) \in O(f(n))$

$O(f(n))$ is a set of functions, but we often write $T(n) = O(f(n))$ instead of $T(n) \in O(f(n))$

- **Transitivity**
  - If $f=O(g)$ and $g=O(h)$ then $f=O(h)$
  - If $f=\Omega(g)$ and $g=\Omega(h)$ then $f=\Omega(h)$
  - If $f=\Theta(g)$ and $g=\Theta(h)$ then $f=\Theta(h)$
- **Additivity**
  - If $f = O(h)$ and $g = O(h)$ then $f + g = O(h)$
  - If $f = \Omega(h)$ and $g = \Omega(h)$ then $f + g = \Omega(h)$
  - If $f = \Theta(h)$ and $g = \Theta(h)$ then $f + g = \Theta(h)$

Asymptotic Bounds for Some Common Functions

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- **Polynomials.** $a_0 + a_1 n + \ldots + a_d n^d$ is $\Theta(n^d)$ if $a_d > 0$
- **Polynomial time.** Running time is $O(n^d)$ for some constant d independent of the input size n
- **Logarithms.** $O(\log_a n) = O(\log_b n)$ for any constants $a, b > 1$
- **Logarithms.** For every $x > 0$, $\log n = O(n^x)$

log grows slower than every polynomial

- **Exponentials.** For every $r > 1$ and every $d > 0$, $n^d = O(r^n)$

every exponential grows faster than every polynomial

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A Survey of Common Running Times

Linear Time: O(n)

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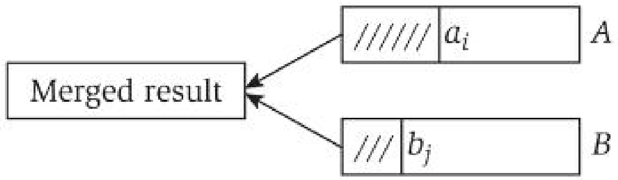
- **Linear time.** Running time is at most a constant factor times the size of the input
- **Computing the maximum.** Compute the maximum of n numbers $a_1, \ldots, a_n$

```
max ← a1
for i = 2 to n {
   if (ai > max)
      max ← ai
}
```

- **Merge.** Combine two sorted lists $A = a_1, a_2, \ldots, a_n$ with $B = b_1, b_2, \ldots, b_n$ into sorted whole




```
i = 1, j = 1
while (both lists are nonempty) {
   if (ai ≤ bj) append ai to output list and increment i
   else         append bj to output list and increment j
}
append remainder of nonempty list to output list
```

- **Claim.** Merging two lists of size n takes O(n) time
- **Pf.** After each comparison, the length of output list increases by 1

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O(nlogn) Time

- **O(nlogn) time.** Arises in divide-and-conquer algorithms
- **Sorting.** Mergesort and heapsort are sorting algorithms that perform O(nlogn) comparisons
- **Largest empty interval.** Given n timestamps $x_1, \ldots, x_n$ on which copies of a file arrive at a server, what is largest interval of time when no copies of the file arrive?
- **O(nlogn) solution.** Sort the timestamps. Scan the sorted list in order, identifying the maximum gap between successive timestamps

Quadratic Time: $O(n^2)$

- **Quadratic time.** Enumerate all pairs of elements
- **Closest pair of points.** Given a list of n points in the plane $(x_1, y_1), \ldots, (x_n, y_n)$, find the pair that is closest
- **$O(n^2)$ solution.** Try all pairs of points

```
min ← (x1 - x2)² + (y1 - y2)²
for i = 1 to n {
   for j = i+1 to n {
      d ← (xi - xj)² + (yi - yj)²
      if (d < min)
         min ← d
   }
}
```

Don't need to take square roots

- **Remark.** $\Omega(n^2)$ seems inevitable, but this is just an illusion

Cubic Time: $O(n^3)$

- **Cubic time.** Enumerate all triples of elements
- **Set disjointness.** Given n sets $S_1, \ldots, S_n$ each of which is a subset of 1, 2, …, n, is there some pair of these which are disjoint?
- **$O(n^3)$ solution.** For each pairs of sets, determine if they are disjoint

```
foreach set Si {
    foreach other set Sj {
        foreach element p of Si {
            determine whether p also belongs to Sj
        }
        if (no element of Si belongs to Sj)
            report that Si and Sj are disjoint
    }
}
```

Polynomial Time: $O(n^k)$ Time

- **Independent set of size k.** Given a graph, are there k nodes such that no two are joined by an edge?
- **$O(n^k)$ solution.** Enumerate all subsets of k nodes

```
foreach subset S of k nodes {
   check whether S in an independent set
   if (S is an independent set)
      report S is an independent set
   }
}
```

- Check whether S is an independent set = $O(k^2)$
- Number of k element subsets = $$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k(k-1)(k-2)\cdots(2)(1)} \leq \frac{n^k}{k!}$$
- $O(k^2 n^k / k!) = O(n^k)$

poly-time for k=17,
but not practical

Exponential Time

- **Independent set.** Given a graph, what is maximum size of an independent set?
- **$O(n^2 2^n)$ solution.** Enumerate all subsets.

```
S* ← ϕ
foreach subset S of nodes {
   check whether S in an independent set
   if (S is largest independent set seen so far)
      update S* ← S
   }
}
```