EE160 Control Theory

Lab 2 Time-Domain Analysis of Linear Systems (1)

Objectives

1. Master the methods of time-domain analysis for linear systems.
2. Be familiar with typical control blocks.

Time-domain analysis refers to the study of a system's stability, transient (dynamic) performance, and steady-state performance by analyzing the time-domain expression of the system's output under a typical input. The output of the system can be obtained from a differential equation, or from a transfer function (when initial conditions are zero, the transfer function is commonly used for analysis).

Analyzing the system's transient (dynamic) response can provide insights into the system's stability, response speed, and damping conditions. Analyzing the system's steady-state response can reveal how well the system's final output follows the input, thus offering information about steady-state error.

To analyze the system's responses, it is necessary to know the system's input signals. However, the input signals of a control system are random and cannot be predetermined. Therefore, it is necessary to select several typical signals as the input signals to study the response process and the performance of linear control systems under typical input signals. In control systems, typical input signals include: step functions, ramp functions, parabolic functions, impulse functions, and sine functions. Although different input signals correspond to different output responses, for linear systems, they represent the same system characteristics (transfer function). To compare the performance of various control systems on a unified basis, in this course, we will use unit step functions as the input signals for explanation. Students are encouraged to choose other signals for research in practical cases.

1 Time-Domain Analysis for First-order System

Typical first-order control systems include: Proportional control (P) , Integral control (I), Proportional-Integral control (PI), Proportional-Derivative control (PD), Inertia control, and Proportional-Integral-Derivative control (PID).

Before we delve into these control elements, let's first understand some basic electrical circuit knowledge.

1.1 Circuit Analysis Basics

1.1.1 S-domain Equivalent Circuit

To simplify circuit analysis, we usually first transform the circuit from time domain to s-domain, which is a complex frequency domain used in control systems and signal processing, before analyzing. This transformation is typically done using Laplace transforms to analyze the system's behavior in terms of its transfer functions, poles, and zeros, which are easier to handle mathematically than time-domain representations. In this context, the s-domain is a powerful tool for analyzing linear- time-invariant (LTI) systems, as it:

- Convert differential equations into algebraic equations, simplifying the analysis of system dynamics.
- Determine system stability by examining the poles of the transfer function.
- Analyze frequency response by evaluating the system's behavior at different frequencies.

- Design control systems by manipulating the transfer function to achieve desired performance characteristics.

This approach is fundamental in control theory and is used extensively in the design and analysis of control systems.

To convert a circuit from time domain to s-domain, we first need to transform the representation of each component from time domain to s-domain. Typical circuit components include resistors (R), capacitors (C), and inductors (L). The transformation processes are as follows:

1. **Resistor (R)**:In s-domain, the impedance of a resistor remains the same as its resistance value in the time domain.

$$\begin{aligned} u(t) &= R \cdot i(t) \\ \mathcal{L}(u(t)) &= \mathcal{L}(R \cdot i(t)) \\ U(s) &= R \cdot I(s) \\ \frac{U(s)}{I(s)} &= R(s) \\ Z_R(s) &= R \end{aligned}$$

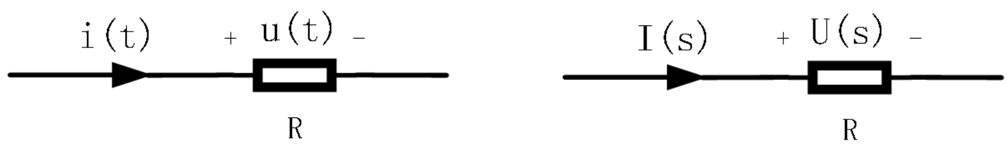


Figure 1: s-domain Equivalent Circuit of Resistor

2. **Capacitors (C)**: The impedance of a capacitor in s-domain is the reciprocal of the product of its capacitance and the complex frequency s.

$$\begin{aligned} i(t) &= C \cdot \frac{du(t)}{dt} \\ \mathcal{L}(i(t)) &= \mathcal{L}(C \cdot \frac{du(t)}{dt}) \\ I(s) &= C \cdot [sU(s) - u(0_-)] \end{aligned}$$

When the initial state is zero, that is $u(0_-) = 0, (t \leq 0)$, the equation simplifies to:

$$\begin{aligned} I(s) &= CsU(s) \\ \frac{U(s)}{I(s)} &= \frac{1}{sC} \\ Z_C(s) &= \frac{1}{sC} \end{aligned}$$

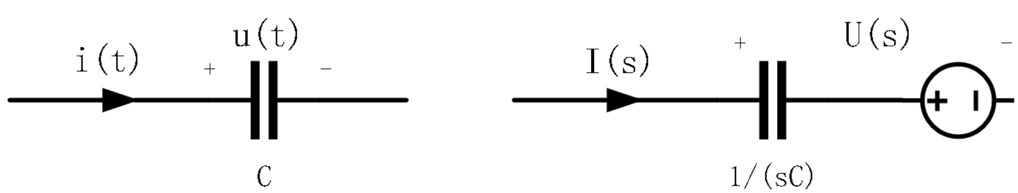


Figure 2: s-domain Equivalent Circuit of Capacitor

3. **Inductors (L)**: The impedance of an inductor in the s-domain is the product of its inductance and the complex frequency s.

$$u(t) = L \cdot \frac{di(t)}{dt}$$
$$\mathcal{L}(u(t)) = \mathcal{L}(L \cdot \frac{di(t)}{dt})$$
$$U(s) = L \cdot [sI(s) - i(0_-)]$$

When the initial state is zero, that is $i(0_-) = 0, (t \leq 0)$, the equation simplifies to:

$$U(s) = LsI(s)$$
$$\frac{U(s)}{I(s)} = Ls$$
$$Z_L(s) = sL$$

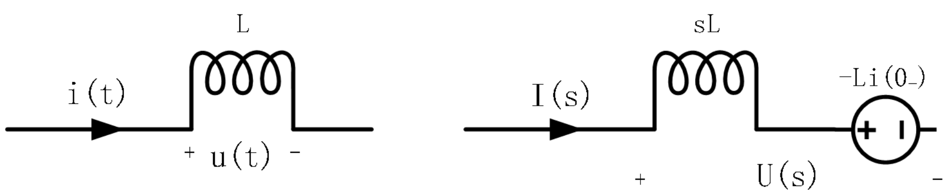


Figure 3: s-domain Equivalent Circuit of Inductor

1.1.2 Operational Amplifier (op-amp)

An operational amplifier (op-amp for short) is an integrated circuit that is designed to perform a wide range of linear and non-linear operations. They are fundamental building blocks in many electronic systems.

As shown in Figure 4:, an op-amp has two input terminals (a non-inverting input (+), an inverting input (-)) and an output terminal. The output is generally the difference between the voltages at the two inputs. An op-amp can work in the linear region or the saturation region. This course will utilize the characteristics of the operational amplifier when it works in the linear region.

When an op-amp works in the linear region, the voltage difference between the inverting and non-inverting inputs is very small, almost zero, which is referred to as a virtual short circuit; While due to the extremely high input impedance of an op-amp, the inverting input effectively acts as an open circuit, which means there is no significant current flows into the inverting input terminal, which is referred to as a virtual open circuit. Taking Figure 4 as an example, since the non-inverting input terminal of the amplifier is grounded, its voltage is 0, therefore the inverting input terminal is also at 0 volts, that is $U_- = 0$. And the current flowing into the inverting input terminal is zero, that is $i = 0$, following the virtual open circuit characteristic.

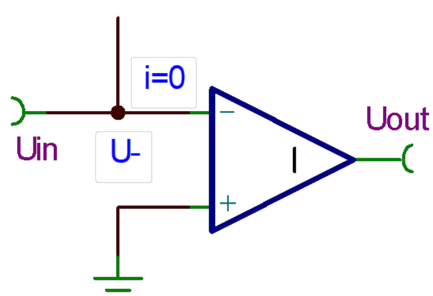


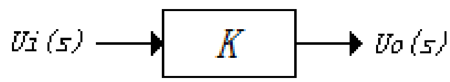
Figure 4: Operational Amplifier

1.2 Basic Control Circuit

1.2.1 Proportional Control Circuit (P)

Transfer Function for Proportional Control Circuit

$$H(s) = \frac{U_o(s)}{U_i(s)} = K$$



Circuit Analysis

A typical circuit for proportional control is shown in Figure 5.

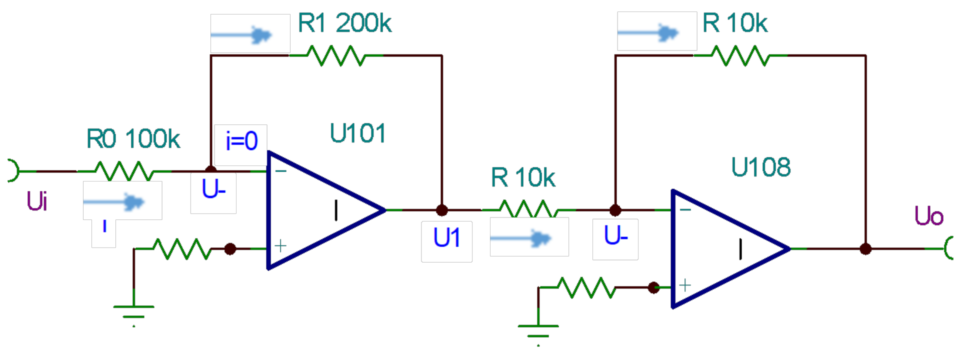


Figure 5: Circuit for Proportional

Taking advantage of the virtual short and virtual open circuit characteristics of the op-amp, we can determine that the current into the op-amp's inverting input is zero; the voltage at the inverting input is the same as that at the non-inverting input, which is also zero, that is $i = 0$ and $U_- = 0$. The arrow in Figure 5 shows the reference direction of the current. Using Kirchhoff's Current Law, the transfer function of U101 can be derived as:

$$\frac{U_i - U_-}{R_0} = \frac{U_- - U_1}{R_1}$$
$$H_1(s) = \frac{U_1}{U_i} = -\frac{R_1}{R_0}$$

Obviously, U101 is an inverting proportional amplifier.

Similarly, the transfer function of U108 can be derived as:

$$\frac{U_1 - U_-}{R} = \frac{U_- - U_o}{R}$$
$$H_2(s) = \frac{U_o}{U_1} = -\frac{R}{R} = -1$$

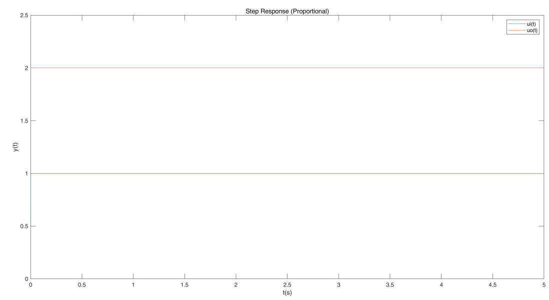
Obviously, U108 is an inverting amplifier.

So the transfer function of the system shown in Figure 5 is

$$H(s) = \frac{U_o}{U_i} = \frac{U_1}{U_i} \times \frac{U_o}{U_1} = H_1(s) \times H_2(s) = (-\frac{R_1}{R_0}) \times (-1) = \frac{R_1}{R_0}$$

MATLAB Simulation

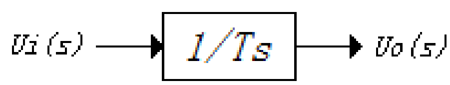
```
syms H(s) ui(t)
R0 = 100e3; R1 = 200e3;
K = R1/R0; H(s) = K;
ui(t) = heaviside(t);
Ui(s) = laplace(ui);
Uo(s) = Ui(s)*H(s);
uo(t) = ilaplace(Uo);
fplot(ui(t)); hold on;
fplot(uo(t));
axis([0 5 0 2.5]);
legend('ui(t)','uo(t)');
xlabel('t(s)'); ylabel('y(t)');
title('Step Response (Proportional)');
```



1.2.2 Integral Control Circuit (I)

Transfer Function for Integral Control Circuit

$$H(s) = \frac{U_o(s)}{U_i(s)} = \frac{1}{Ts}$$



Circuit Analysis

A typical circuit for integral control is shown in Figure 6.

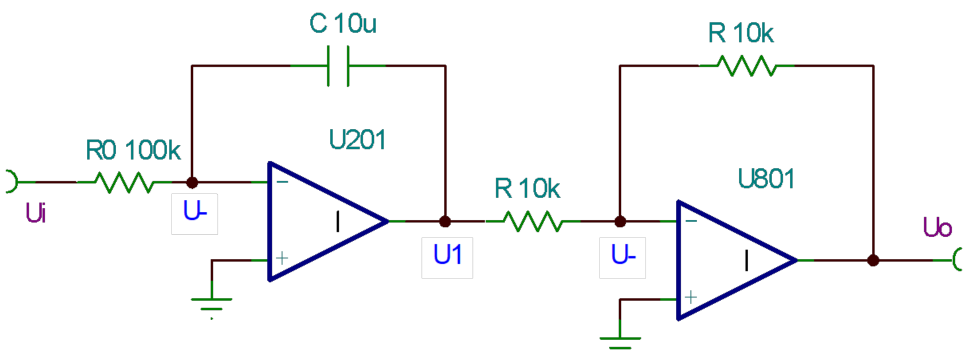


Figure 6: Circuit for Integral

Based on the analysis method of the s-domain equivalent circuit and the linear working principle of op-amp, for

U201 we have:

$$\frac{U_i - U_-}{R_0} = \frac{U_- - U_1}{\frac{1}{Cs}}$$

$$H_1(s) = \frac{U_1}{U_i} = -\frac{1}{R_0 C s} = -\frac{1}{Ts}$$

$$T = R_0 C$$

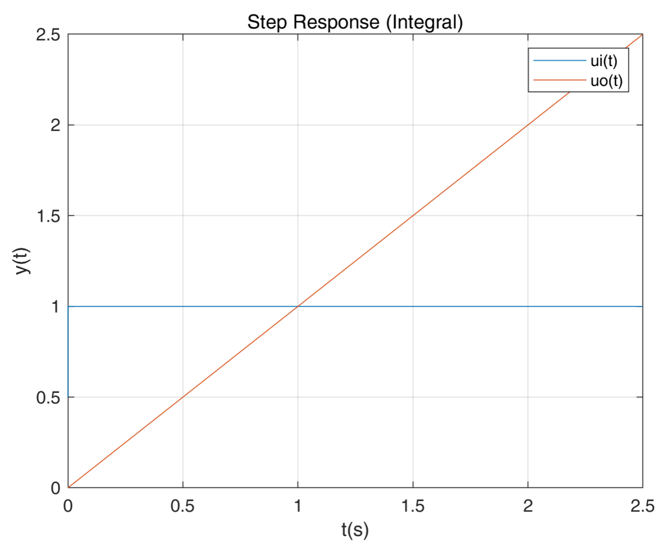
As we mentioned before, U801 is an inverting amplifier. So it has the transfer Function like $H_2(s) = -1$.

So the transfer function of the integral control circuit shown in Figure 6 is:

$$H(s) = \frac{U_o}{U_i} = \frac{U_1}{U_i} \times \frac{U_o}{U_1} = H_1(s) \times H_2(s) = (-\frac{1}{Ts}) \times (-1) = \frac{1}{Ts}$$

MATLAB Simulation

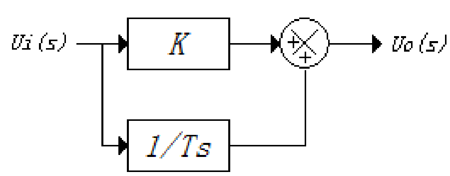
```
syms H(s) ui(t)
R0 = 100e3; C = 10e-6;
T = R0*C; H(s) = 1/(T*s);
ui(t) = heaviside(t);
Ui(s) = laplace(ui);
Uo(s) = Ui(s)*H(s);
uo(t) = ilaplace(Uo);
fplot(ui(t)); hold on;
fplot(uo(t)); grid on;
axis([0 2.5 0 2.5]);
legend('ui(t)','uo(t)')
xlabel('t(s)'); ylabel('y(t)');
title('Step Response (Integral)')
```



1.2.3 Proportional-Integral Control Circuit (PI)

Transfer Function for Proportional-Integral Control Circuit

$$H(s) = \frac{U_o(s)}{U_i(s)} = K + \frac{1}{Ts}$$



Circuit Analysis

A typical circuit for proportional-integral control is shown in Figure 7.

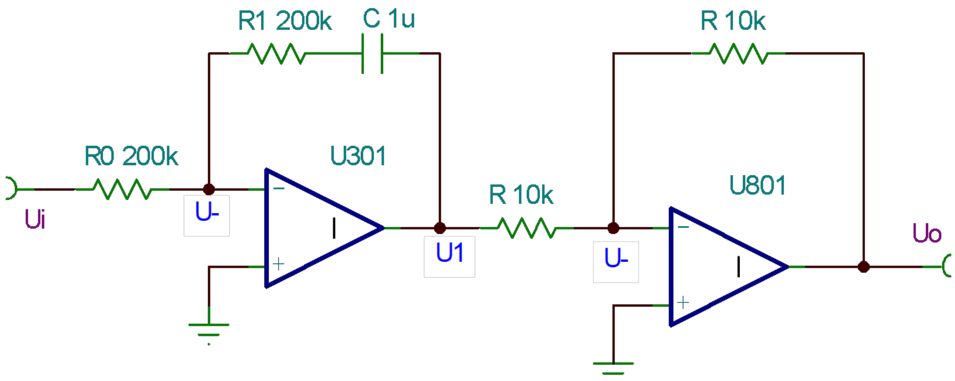


Figure 7: Circuit for Proportional-integral

For U301 we have:

$$\frac{U_i - U_-}{R_0} = \frac{U_- - U_1}{R_1 + \frac{1}{Cs}}$$

$$H_1(s) = \frac{U_1}{U_i} = -(\frac{R_1}{R_0} + \frac{1}{R_0 Cs}) = -(K + \frac{1}{Ts})$$

$$K = \frac{R_1}{R_0}, T = R_0 C$$

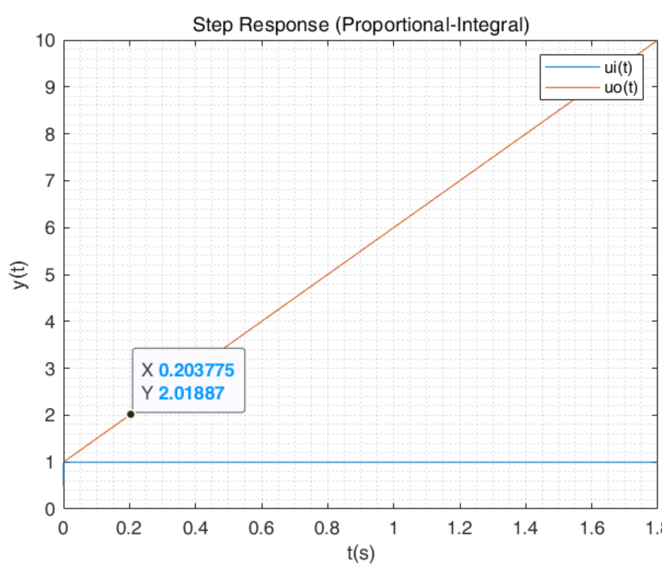
U801 is an inverting amplifier and has the transfer Function like $H_2(s) = -1$.

So the transfer function of the integral control circuit shown in Figure 7 is:

$$H(s) = \frac{U_o}{U_i} = \frac{U_1}{U_i} \times \frac{U_o}{U_1} = H_1(s) \times H_2(s) = -(K + \frac{1}{Ts}) \times (-1) = K + \frac{1}{Ts}$$

MATLAB Simulation

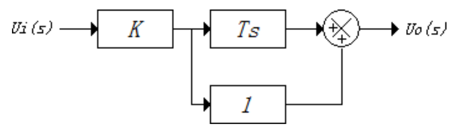
```
syms H(s) ui(t)
R0 = 200e3; R1 = 200e3; C = 1e-6;
K = R1/R0; T = R0*C;
H(s) = K+1/(T*s);
ui(t) = heaviside(t);
Ui(s) = laplace(ui);
Uo(s) = Ui(s)*H(s);
uo(t) = ilaplace(Uo);
fplot(ui); hold on;
fplot(uo); grid on;
axis([0 1.8 0 10])
legend('ui(t)','uo(t)')
xlabel('t(s)'); ylabel('y(t)');
title('Step Response (Proportional-Integral)')
```



1.2.4 Proportional-Derivative Control Circuit (PD)

Transfer Function for Proportional-Derivative Control Circuit

$$H(s) = \frac{U_o(s)}{U_i(s)} = K(1 + Ts)$$



Circuit Analysis

A typical circuit for proportional-derivative control is shown in Figure 8.

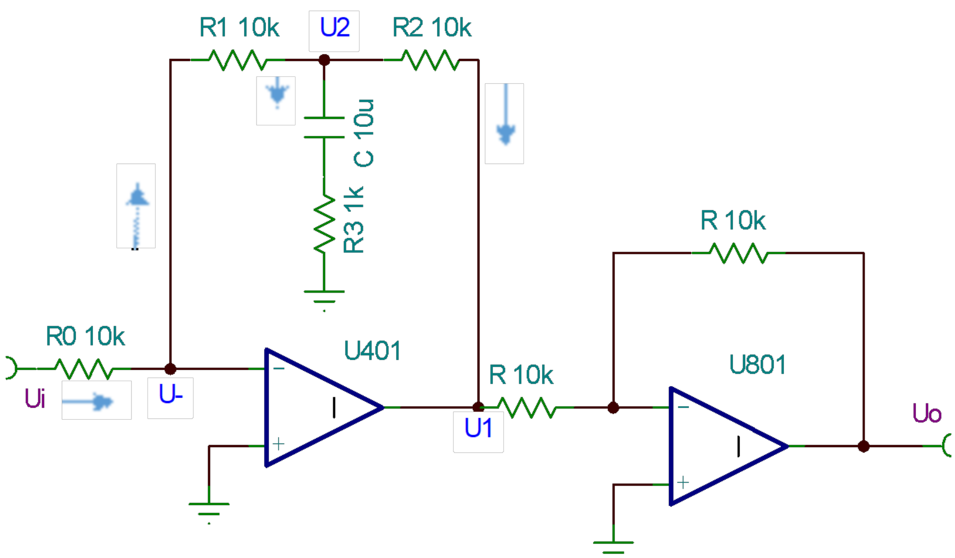


Figure 8: Circuit for Proportional-derivative

For U401 we have:

$$\begin{cases} \frac{U_i}{R_0} = -\frac{U_2}{R_1} \Rightarrow U_2 = -\frac{R_1}{R_0} \cdot U_i & \text{Apply KCL at } U_- \\ -\frac{U_2}{R_1} = \frac{U_2}{R_3+\frac{1}{Cs}} + \frac{U_2-U_1}{R_2} = U_2 \cdot (\frac{Cs}{R_3Cs+1} + \frac{1}{R_2}) - \frac{U_1}{R_2} & \text{Apply KCL at } U_2 \\ U_1 = -U_o & \text{U801 is an inverting amplifier} \end{cases}$$

After rearranging the above equation, we will get:

$$\frac{U_i}{R_0} = -\frac{R_1}{R_0} \cdot U_i \cdot (\frac{Cs}{R_3Cs+1} + \frac{1}{R_2}) + \frac{U_o}{R_2}$$

$$U_i \cdot (\frac{1}{R_0} + \frac{R_1Cs}{R_0R_3Cs+R_0} + \frac{R_1}{R_0R_2}) = \frac{U_o}{R_2}$$

$$H(s) = \frac{U_o}{U_i} = \frac{R_2}{R_0} + \frac{R_1R_2Cs}{R_0R_3Cs+R_0} + \frac{R_1}{R_0} = \frac{(R_1R_2+R_2R_3+R_3R_1)Cs + (R_1+R_2)}{R_0R_3Cs+R_0}$$

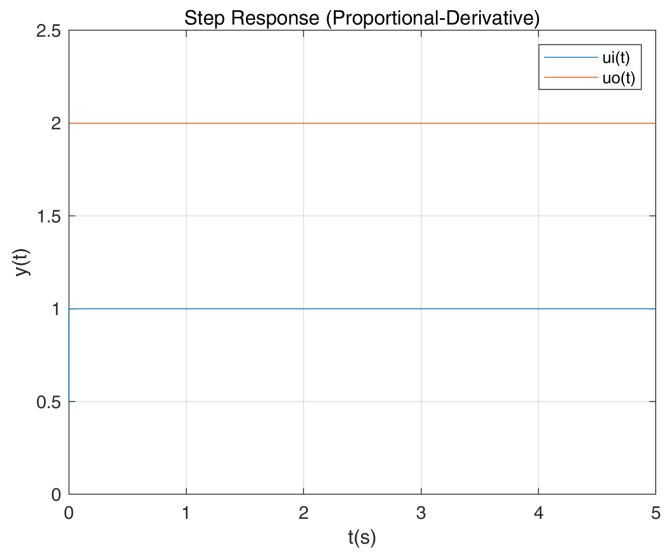
By taking $R_1, R_2 \gg R_3$, that is taking $R_3 = 0$, the transfer function can be simplified to:

$$H(s) = \frac{R_1 R_2 C s + (R_1 + R_2)}{R_0} = \frac{R_1 + R_2}{R_0} \cdot (\frac{R_1 R_2}{R_1 + R_2} \cdot Cs + 1) = K \cdot (Ts + 1)$$

$$K = \frac{R_1 + R_2}{R_0}, T = \frac{R_1 R_2}{R_1 + R_2} \cdot C$$

MATLAB Simulation

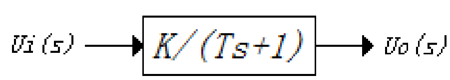
```
syms H(s) ui(t)
R0 = 10e3; R1 = 10e3; R2 = 10e3; R4 = 1e3;
C = 10e-6;
K = (R1+R2)/R0; T = R1*R2*C/(R1+R2);
H(s) = K*(1+T*s);
ui(t) = heaviside(t);
Ui(s) = laplace(ui);
Uo(s) = Ui(s)*H(s);
uo(t) = ilaplace(Uo);
fplot(ui); hold on;
fplot(uo); grid on;
axis([0 5 0 2.5])
legend('ui(t)','uo(t)')
xlabel('t(s)'); ylabel('y(t)');
title('Step Response (Proportional-Derivative)')
```



1.2.5 Inertia Control Circuit

Transfer Function for Inertia Control Circuit

$$H(s) = \frac{U_o(s)}{U_i(s)} = \frac{K}{Ts + 1}$$



Circuit Analysis

A typical circuit for inertia control is shown in Figure 9.

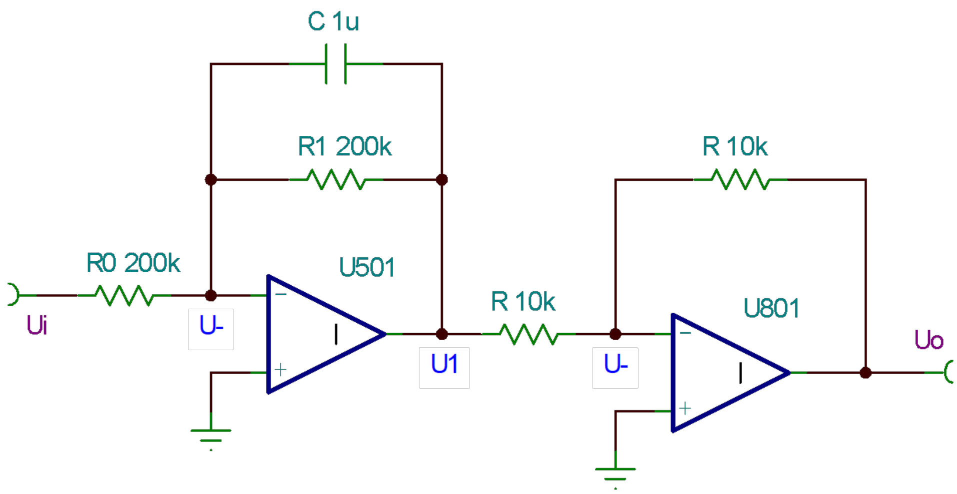


Figure 9: Circuit for Inertia

For U501 we have:

$$\frac{U_i - U_-}{R_0} = \frac{U_- - U_1}{R_1} + \frac{U_- - U_1}{\frac{1}{Cs}}$$

$$H_1(s) = \frac{U_1}{U_i} = -\frac{R_1}{R_0} \cdot \frac{1}{1 + R_1 Cs} = -K \cdot \frac{1}{Ts + 1}$$

$$K = \frac{R_1}{R_0}, T = R_1 C$$

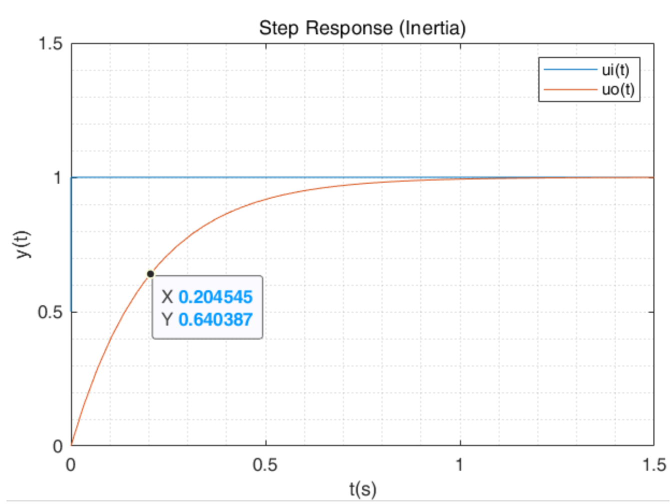
U801 is an inverting amplifier and has the transfer Function like $H_2(s) = -1$.

So the transfer function of the integral control circuit shown in Figure 9 is:

$$H(s) = \frac{U_o}{U_i} = \frac{U_1}{U_i} \times \frac{U_o}{U_1} = H_1(s) \times H_2(s) = -K \cdot \frac{1}{Ts + 1} \times (-1) = \frac{K}{Ts + 1}$$

MATLAB Simulation

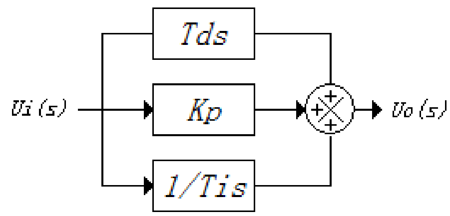
```
syms H(s) ui(t)
R0 = 200e3; R1 = 200e3; C = 1e-6;
K = R1/R0; T = R1*C;
H(s) = K/(1+T*s);
ui(t) = heaviside(t);
Ui(s) = laplace(ui);
Uo(s) = Ui(s)*H(s);
uo(t) = ilaplace(Uo);
fplot(ui); hold on;
fplot(uo); grid minor;
axis([0 1.5 0 1.5]);
legend('ui(t)','uo(t)');
xlabel('t(s)'); ylabel('y(t)');
title('Step Response (Inertia)')
```



1.2.6 Proportional-Integral-Derivative Control Circuit (PID)

Transfer Function for Proportional-Integral-Derivative Control Circuit

$$H(s) = \frac{U_o(s)}{U_i(s)} = K_p + \frac{1}{T_i s} + T_d s$$



Circuit Analysis

A typical circuit for proportional-integral-derivative control is shown in Figure 10.

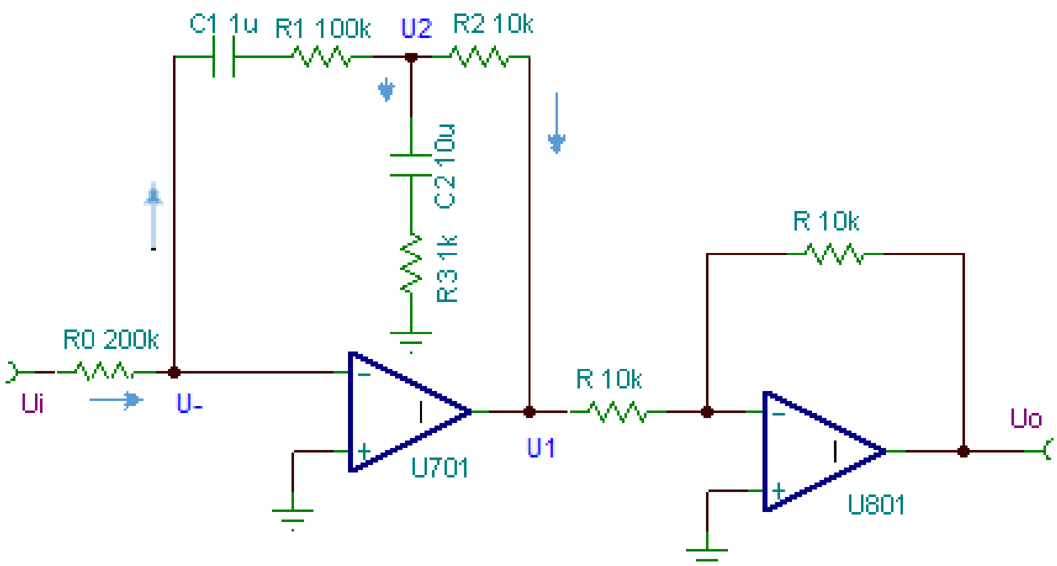


Figure 10: Circuit for Proportional-derivative

For U601 we have:

$$\begin{cases} \frac{U_i}{R_0} = -\frac{U_2}{R_1+\frac{1}{C_1 s}} \Rightarrow U_2 = -\frac{R_1 C_1 s+1}{R_0 C_1 s} \cdot U_i & \text{Apply KCL at } U_- \\ -\frac{U_2}{R_1+\frac{1}{C_1 s}} = \frac{U_2}{R_3+\frac{1}{C_2 s}} + \frac{U_2-U_1}{R_2} = U_2 \cdot (\frac{C_2 s}{R_3 C_2 s+1} + \frac{1}{R_2}) - \frac{U_1}{R_2} & \text{Apply KCL at } U_2 \\ U_1 = -U_o & \text{U801 is an inverting amplifier} \end{cases}$$

After rearranging the above equation, we will get:

$$\frac{U_i}{R_0} = -\frac{R_1 C_1 s + 1}{R_0 C_1 s} \cdot U_i \cdot (\frac{C_2 s}{R_3 C_2 s + 1} + \frac{1}{R_2}) + \frac{U_o}{R_2}$$

$$U_i \cdot [\frac{1}{R_0} + \frac{1 + R_1 C_1 s}{R_0 C_1 s} \cdot (\frac{C_2 s}{R_3 C_2 s} + \frac{1}{R_2})] = \frac{U_o}{R_2}$$

$$H(s) = \frac{U_o}{U_i} = \frac{R_2}{R_0} + \frac{1 + R_1 C_1 s}{R_0 C_1 s} \cdot \left(\frac{R_2 C_2 s}{R_3 C_2 s + 1} + 1\right) = \frac{R_1 + R_2}{R_0} + \frac{1}{R_0 C_1 s} + \frac{R_2 C_2 (1 + R_1 C_1 s)}{R_0 C_1 (1 + R_3 C_2 s)}$$

By taking $R_1 \gg R_2 \gg R_3$, that is taking $R_3 = 0$, the transfer function can be simplified to:

$$H(s) = \frac{R_1 + R_2}{R_0} + \frac{1}{R_0 C_1 s} + \frac{R_2 C_2 (1 + R_1 C_1 s)}{R_0 C_1} \approx \frac{R_1 + R_2}{R_0} + \frac{1}{R_0 C_1 s} + \frac{R_1 R_2 C_1 C_2 s}{R_0 C_1}$$

$$= \frac{R_1 + R_2}{R_0} + \frac{1}{R_0 C_1 s} + \frac{R_1 R_2 C_2}{R_0} s = K_p + \frac{1}{T_i s} + T_d s$$

$$K_p = \frac{R_1 + R_2}{R_0}, T_i = R_0 C_1, T_d = \frac{R_1 R_2}{R_0} \cdot C_2$$

MATLAB Simulation

```matlab
syms H(s) ui(t)
R0=200e3; R1=100e3; R2=10e3; R3=1e3;
C1=1e-6; C2=10e-6;
Kp=(R1+R2)/R0; Ti=R0*C1; Td=R1*R2*C2/R0;
H(s) = Kp+1/(Ti*s)+Td*s;
ui(t) = heaviside(t);
Ui(s) = laplace(ui);
Uo(s) = H(s)*Ui(s);
uo(t) = ilaplace(Uo);
fplot(ui); hold on;
fplot(uo); grid minor;
axis([0 1 0 2.5])
legend('ui(t)','uo(t)')
xlabel('t(s)'); ylabel('y(t)');
title('Step Response (PID)')
```