Homework 5

Problem 1 (15 points)

Compute the Fourier transform of each of the following signals:

$$(a)\ \ x[n] = \left(\frac{2}{3}\right)^{-n} u[-n]$$

$$(b)\ \ x[n] = \sin\left(\frac{\pi}{6}n\right)\cos\left(\frac{\pi}{6}n\right)$$

$$(c)\ \ x[n] = \begin{cases} 1, \ n = 8k-1, \ 8k, \ 8k+1 \ (k\epsilon Z) \\ 0, \ others \end{cases}$$

Solutions:

(a) $Method\ 1:$

$$x[n] = \left(\frac{2}{3}\right)^{-n}(u[-n])$$

$$X(e^{jw}) = \sum_{n=-\infty}^{\infty}\left(\frac{2}{3}\right)^{-n} u[-n]\cdot e^{-jwn} = \sum_{n=-\infty}^{0}\left(\frac{2}{3}\right)^{-n}\cdot e^{-jwn}$$

$$X(e^{jw}) \overset{k=-n}{=} \sum_{k=0}^{\infty}\left(\frac{2}{3}\right)^{k}\cdot e^{jwk} = \frac{1-\lim\limits_{k\to\infty}\left(\frac{2}{3}e^{jw}\right)^{k+1}}{1-\frac{2}{3}e^{jw}} = \frac{1}{1-\frac{2}{3}e^{jw}}$$

$Method\ 2:$

$$let\ x_0[n] = \left(\frac{2}{3}\right)^{n}(u[n]), \ \ then\ x[n] = x_0[-n]\ ,\ a=-1$$

$$X_0(e^{jw}) = \frac{1}{1-\frac{2}{3}e^{-jw}} \Longrightarrow X(e^{jw}) \overset{a\ =\ -1}{=} X_0(e^{j(-w)}) = \frac{1}{1-\frac{2}{3}e^{jw}}$$

(b)

$$Owing\ to\ \sin\left(\frac{\pi}{6}n\right)\cos\left(\frac{\pi}{6}n\right) = \frac{1}{2}\sin\left(\frac{\pi}{3}n\right)$$

$$x[n]=\frac{1}{2}\sin\left(\frac{\pi}{3}n\right) = \frac{1}{2}\frac{e^{j\frac{\pi}{3}n}-e^{-j\frac{\pi}{3}n}}{2j} = \frac{1}{4j}e^{j\frac{\pi}{3}n} - \frac{1}{4j}e^{-j\frac{\pi}{3}n}$$

$$X(e^{jw})=\frac{1}{4j}\sum_{l=-\infty}^{\infty}2\pi\cdot\delta\left(w-\frac{\pi}{3}-2\pi l\right) - \frac{1}{4j}\sum_{l=-\infty}^{\infty}2\pi\cdot\delta\left(w+\frac{\pi}{3}-2\pi l\right)$$

$$=\frac{\pi}{2j}\sum_{l=-\infty}^{\infty}\left(\delta\left(w-\frac{\pi}{3}-2\pi l\right) - \delta\left(w+\frac{\pi}{3}-2\pi l\right)\right)$$

(c) $Method\ 1:$

$$a_k=\frac{1}{N}\sum_{n=\langle N\rangle}x[n]e^{-jk\left(\frac{2\pi}{N}\right)n} = \frac{1}{8}\left(e^{-j\frac{\pi}{4}k}+1+e^{-j\frac{\pi}{4}k}\right) = \frac{1}{8}\left(1+2\cos\left(\frac{\pi}{4}k\right)\right)$$

$$X_0(e^{jw})=\sum_{k=-\infty}^{\infty}2\pi\cdot a_k\cdot\delta\left[w-\frac{2\pi k}{N}\right]$$

$$=\frac{\pi}{4}\sum_{k=-\infty}^{\infty}\left(\left(1+2\cos\left(\frac{\pi}{4}k\right)\right)\delta\left[w-\frac{2\pi k}{N}\right]\right)$$

$Method\ 2: \quad From\ PPT\ 14-17$

$$x_0[n] = \sum_{k=-\infty}^{\infty} \delta[n-kN] \implies a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk\left(\frac{2\pi}{N}\right)n} = \frac{1}{N}$$

$$X_0(e^{jw}) = \sum_{k=-\infty}^{\infty} 2\pi \cdot a_k \cdot \delta\left[w - \frac{2\pi k}{N}\right] = \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \delta\left[w - \frac{2\pi k}{N}\right]$$

$N = 8,\ for\ Problem\ (c)$

$$\begin{aligned} x[n] &= \sum_{k=-\infty}^{\infty} \delta[n-kN] + \delta[n-1-kN] + \delta[n+1-kN] \\ &= \left(\sum_{k=-\infty}^{\infty} \delta[n-kN]\right) * (\delta[n] + \delta[n-1] + \delta[n+1]) \\ &= x_0[n]\,|_{N=8} * (\delta[n] + \delta[n-1] + \delta[n+1]) \end{aligned}$$

$$\begin{aligned} X(e^{jw}) &= X_0(e^{jw})\,|_{N=8} \cdot (1 + e^{-jw} + e^{jw}) \\ &= \frac{2\pi}{N} \sum_{k=-\infty}^{\infty} \left(\delta\left[w - \frac{2\pi k}{N}\right]\Big|_{N=8} \cdot (1 + e^{-jw} + e^{jw})\right) \\ &= \frac{\pi}{4} \sum_{k=-\infty}^{\infty} \left(\delta\left[w - \frac{\pi k}{4}\right] \cdot (1 + e^{-jw} + e^{jw}) \right) \\ &= \frac{\pi}{4} \sum_{k=-\infty}^{\infty} \left(\left(1 + 2\cos\left(\frac{\pi k}{4}\right)\right) \cdot \delta\left[w - \frac{\pi k}{4}\right] \right) \end{aligned}$$

Problem 2 (15 points)

Compute the inverse Discrete-Time Fourier transform of $X(e^{j\omega})$ of each of the following signals:

$$(a)\ X_1(e^{jw}) \ = \ \sum_{k=-\infty}^{\infty}\left\{2\pi\delta(w-2\pi k)-\pi\delta\left(w-\frac{\pi}{3}-2\pi k\right)-\pi\delta\left(w+\frac{\pi}{3}-2\pi k\right)\right\}$$

$$(b)\ X_2(e^{jw}) \ = \ \frac{1}{(1-ae^{-jw})^2}\ ,\quad |a|<1$$

$$(c)\ X_3(e^{jw}) \ = \ \frac{1-\frac{1}{729}e^{-j6w}}{1-\frac{1}{3}e^{-jw}},\quad hint:\ \frac{1}{729}=\frac{1}{3^6}$$

Solutions:

(a)

$$From\ page\ 14\ of\ PPT\ chapter 5$$

$$x[n]=e^{jw_0 n}\longleftrightarrow X(e^{jw})=\sum_{k=-\infty}^{\infty}2\pi\delta(w-w_0-2\pi k)$$

$$\sum_{k=-\infty}^{\infty}2\pi\delta(w-0-2\pi k)\longleftrightarrow e^{j0n}=1$$

$$\left\{\pi\delta\left(w-\frac{\pi}{3}-2\pi k\right)+\pi\delta\left(w+\frac{\pi}{3}-2\pi k\right)\right\}\longleftrightarrow\frac{e^{j\frac{\pi}{3}n}+e^{-j\frac{\pi}{3}n}}{2}=\cos\left(\frac{\pi n}{3}\right)$$

$$Then,\ x_1[n]=1-\cos\left(\frac{\pi n}{3}\right)$$

(b)

$$-j\frac{d\frac{1}{(1-ae^{-jw})}}{dw}=\frac{ae^{-jw}}{(1-ae^{-jw})^2}=\frac{1}{(1-ae^{-jw})}-\frac{1}{(1-ae^{-jw})^2}$$

$$\frac{1}{(1-ae^{-jw})}\leftrightarrow a^n u[n]$$

$$j\frac{d\frac{1}{(1-ae^{-jw})}}{dw}\leftrightarrow na^n u[n]$$

$$\frac{1}{(1-ae^{-jw})^2}\ =\ \frac{1}{(1-ae^{-jw})}\ +\ j\frac{d\frac{1}{(1-ae^{-jw})}}{dw}$$

$$x_2[n]=(n+1)a^n u[n]$$

(c)

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}e^{-jwn_0}\,e^{jwn}dw\ =\ \begin{cases}1,\ n=n_0\\ 0,\ others\end{cases}$$

$$X_3(e^{jw})\ =\ \frac{1-\frac{1}{729}e^{-j6w}}{1-\frac{1}{3}e^{-jw}}=\frac{1-\left(\frac{1}{3}e^{-jw}\right)^6}{1-\frac{1}{3}e^{-jw}}=\sum_{n=0}^{5}\left(\frac{1}{3}\right)^n e^{-jwn}$$

$$X_3(e^{jw})\ =\ \left(\frac{1}{3}\right)^0 e^{-jw0}\ +\ \left(\frac{1}{3}\right)^1 e^{-jw1}\ +\ \left(\frac{1}{3}\right)^2 e^{-jw2}\ +\ \left(\frac{1}{3}\right)^3 e^{-jw3}\ +\ \left(\frac{1}{3}\right)^4 e^{-jw4}\ +\ \left(\frac{1}{3}\right)^5 e^{-jw5}$$

$$1\ \overset{\mathcal{F}^{-1}}{\rightarrow}\ \delta[n]\ ,\quad e^{-jwn_0}\ \overset{\mathcal{F}^{-1}}{\rightarrow}\ \delta[n-n_0]$$

$$x_3[n]\ =\ \delta[n]\ +\ \left(\frac{1}{3}\right)^1\delta[n-1]\ +\ \left(\frac{1}{3}\right)^2\delta[n-2]\ +\ \left(\frac{1}{3}\right)^3\delta[n-3]\ +\ \left(\frac{1}{3}\right)^4\delta[n-4]\ +\ \left(\frac{1}{3}\right)^5\delta[n-5]$$

$$x_3[n]\ =\ \left(\frac{1}{3}\right)^n(u[n]-u[n-6])$$

Problem 3 (20 points)

Let $X(e^{jw})$ denote the Fourier transform of the signal $x[n]$ depicted in Figure below. Perform the following calculations without explicitly evaluating $X(e^{jw})$:

(a) Evaluate $X(e^{i0})$. **(2 points)**

(b) Evaluate $\int_{-\pi}^{\pi} X(e^{jw})\,dw$. **(2 points)**

(c) Find $X(e^{-j\pi})$ **(2 points)**

(d) Determine and sketch the signal whose Fourier transform is $\mathrm{Re}\{X(e^{jw})\}$ **(3 points)**

(e) If a signal whose Fourier transform is $(1-e^{-2jw})X(e^{jw})$, draw its figure please. **(4 points)**

(f) Evaluate:

$$(i)\ \int_{\pi}^{-\pi} |X(e^{jw})|^2 dw$$

$$(ii)\ \int_{\pi}^{-\pi} \left|\frac{dX(e^{jw})}{dw}\right|^2 dw$$

(i) 3 points (ii) 4 points

Note:

In question (f), the upper and lower limits were mistakenly reversed (they should be -π and π). It is supposed to be $(i)\ \int_{-\pi}^{\pi} |X(e^{jw})|^2 dw,\ (ii)\ \int_{-\pi}^{\pi} \left|\frac{dX(e^{jw})}{dw}\right|^2 dw$ .Therefore, if your answers are numerically correct but have the opposite sign, you should still get full marks as well.

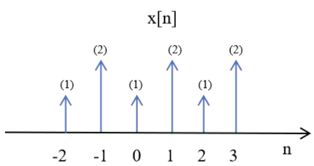


Figure of Problem 3

Solutions:

$(a)\ X(e^{j0}) = \sum_{n=-\infty}^{\infty} x[n] = 1+2+1+2+1+2 = 9$ **(2 points)**

$(b)\quad x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{jw})\, e^{jwn} dw$, $so\ \int_{-\pi}^{\pi} X(e^{jw})\,dw = 2\pi\, x[0] = 2\pi$ **(2 points)**

$(c)\ X(e^{jw}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-jwn},\ so\ X(e^{-j\pi}) = \sum_{n=-\infty}^{\infty} x[n]\, e^{-j(-\pi)n} = \sum_{n=-\infty}^{\infty} x[n] e^{j\pi n}$

$X(e^{-j\pi}) = \sum_{n=-\infty}^{\infty} x[n](-1)^n = 1-2+1-2+1-2 = -3$ **(2 points)**

$(d)\ \mathrm{Re}\{X(e^{jw})\} \longleftrightarrow \frac{x[n]+x[-n]}{2}$ $(only\ this\ formula\ is\ rigth\ ,\ get\ 1.5\ poins)$

$\frac{x[n]+x[-n]}{2} = \delta[n+3] + \delta[n+2] + 2\delta[n+1] + \delta[n] + 2\delta[n-1] + \delta[n-2] + \delta[n-3]$ **(3 points)**

$(e)\ (1-e^{-2jw})X(e^{jw}) \longleftrightarrow x[n]-x[n-2]$ **(1 point)**

The figure worths **3 points**.

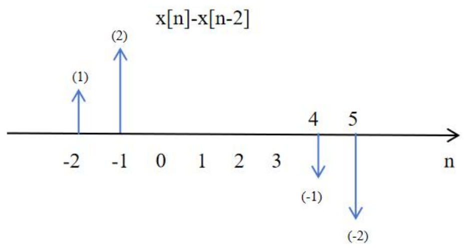


Figure of solution 3(e)

$$(i)\quad \int_{-\pi}^{\pi} |X(e^{jw})|^2 dw = 2\pi \sum_{n=-\infty}^{\infty} |x[n]|^2 = 2\pi\ (1+2^2+1+2^2+1+2^2)=30\pi \qquad (\textbf{3 points})$$

$$(ii)\quad \int_{-\pi}^{\pi} \left|\frac{dX(e^{jw})}{dw}\right|^2 dw = 2\pi \sum_{n=-\infty}^{\infty} |-jnx[n]|^2 = 2\pi \sum_{n=-\infty}^{\infty} |nx[n]|^2$$

$$\begin{aligned}\int_{-\pi}^{\pi} \left|\frac{dX(e^{jw})}{dw}\right|^2 dw &= 2\pi\ [(-2\cdot 1)^2+(-1\cdot 2)^2+0^2+(1\cdot 2)^2+(2\cdot 1)^2+(3\cdot 2)^2] \qquad (\textbf{4 points})\\ &= 2\pi\ (4+4+0+4+4+36)\\ &= 104\ \pi\end{aligned}$$

Problem 4 (15 points)

Simple calculation，it is known that $x[n]=\left(\frac{1}{2}\right)^n u[n-4]$**.**

(a) Determine $X(e^{jw})$. **(5 points)**

(b) If $y[n]=\sum_{k=-\infty}^{n-2} x[k]$, determine $Y(e^{jw})$. Note: use the answer in (a) to find the final expression of $Y(e^{jw})$. **(10 points)**

Solutions:

$$(a)\quad \left(\frac{1}{2}\right)^n u[n] \leftrightarrow \frac{1}{1-\frac{1}{2}e^{-jw}}$$

$$x[n] = \left(\frac{1}{2}\right)^n u[n-4] = \left(\frac{1}{2}\right)^4 \left(\frac{1}{2}\right)^{n-4} u[n-4]$$

$$x[n] = \frac{1}{16}\left[\left(\left(\frac{1}{2}\right)^n u[n]\right) * \delta[n-4]\right]$$

$$X(e^{jw}) = \frac{1}{16}\frac{1}{1-\frac{1}{2}e^{-jw}}e^{-jw4}$$

$$(b)\quad y[n]=\sum_{k=-\infty}^{n-2} x[k] \longleftrightarrow y[n+2]=\sum_{k=-\infty}^{n} x[k]$$

$$\sum_{k=-\infty}^{n} x[k] \longleftrightarrow \frac{1}{1-e^{-jw}}X(e^{jw})+\pi X(e^{j0})\sum_{k=-\infty}^{\infty}\delta(w-2\pi k)$$

$$y[n+2] \longleftrightarrow e^{jw2}Y(e^{jw})$$

$$Y(e^{jw})=e^{-jw2}\left(\frac{1}{1-e^{-jw}}X(e^{jw})+\pi X(e^{j0})\sum_{k=-\infty}^{\infty}\delta(w-2\pi k)\right)$$

$$It's\ known\ that\ X(e^{jw}) = \frac{1}{16}\frac{1}{1-\frac{1}{2}e^{-jw}}e^{-jw4}$$

$$X(e^{j0}) = \frac{1}{16}\frac{1}{1-\frac{1}{2}e^{-j0}}e^{-jw0} = \frac{1}{16}\frac{1}{1-\frac{1}{2}} = \frac{1}{8}$$

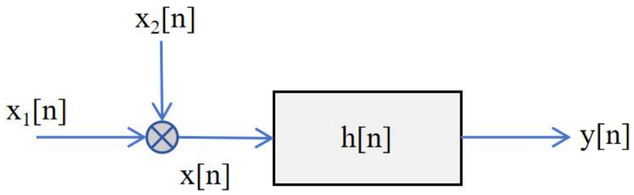
$$so,\quad Y(e^{jw}) = e^{-jw2}\left(\frac{1}{16}\frac{1}{1-e^{-jw}}\frac{1}{1-\frac{1}{2}e^{-jw}}e^{-jw4} + \frac{\pi}{8}\sum_{k=-\infty}^{\infty}\delta(w-2\pi k)\right)$$

Problem 5 (15 points)

Given $x[n] = x_1[n] \cdot x_2[n]$, $x_1[n] = \dfrac{\sin\left(\frac{\pi n}{4}\right)}{\pi n}$, $x_2[n] = \cos\left(\dfrac{3\pi n}{4}\right)$.

(a) Draw the spectrum diagram of $X(e^{jw})$. **(9 points)**

(b) Given a discrete-time LTI system (see the system block diagram below) whose unit impulse response is $h[n] = \dfrac{\sin\left(\frac{3\pi}{4} n\right)}{\pi n}$, draw the spectrum diagram of $Y(e^{jw})$ in one period. **(6 points)**



System block figure of Problem 5

Solutions:

$$\textit{From page 410 of the textbook, example 5.15}$$

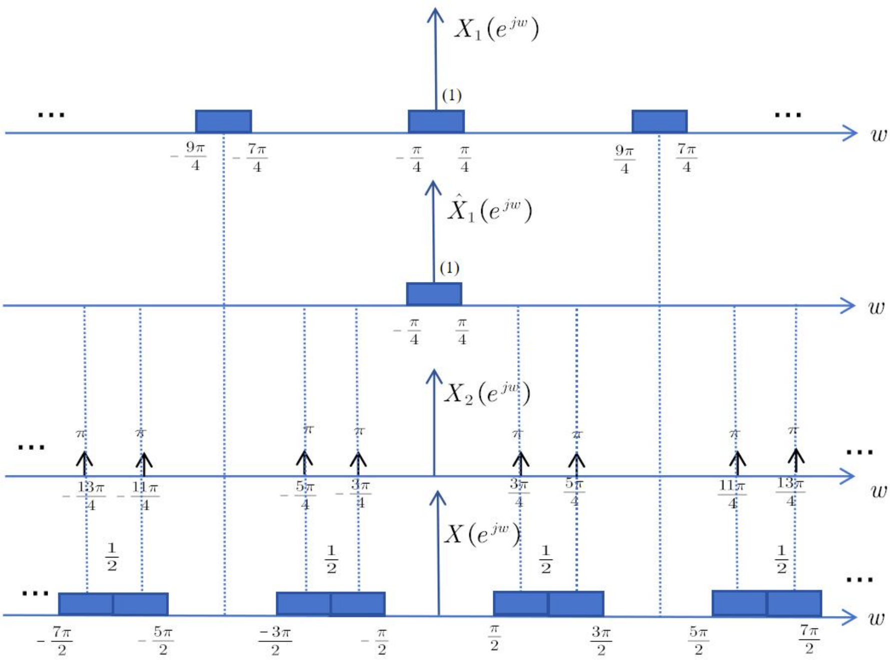
(a)

$$x[n] = x_1[n] \cdot x_2[n] \longleftrightarrow X(e^{jw}) = \frac{1}{2\pi} X_1(e^{jw}) * X_2(e^{jw})$$

$$X(e^{jw}) = \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta}) \cdot X_2(e^{j(w-\theta)})d\theta \ , \quad let\ \hat{X}_1(e^{jw}) = \begin{cases} X_1(e^{jw}), & -\pi < w < \pi \\ 0 \ , & otherwise \end{cases}$$

$$X_2(e^{jw}) = \pi \sum_{l=-\infty}^{\infty}\left[\delta\left(w - \frac{3\pi}{4} - 2\pi l\right) + \delta\left(w + \frac{3\pi}{4} - 2\pi l\right)\right]$$

$$X(e^{jw}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{X}_1(e^{j\theta}) \cdot X_2(e^{j(w-\theta)})d\theta = \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{X}_1(e^{j\theta}) \cdot X_2(e^{j(w-\theta)})d\theta$$



The figure of $X(e^{jw})$

(b)

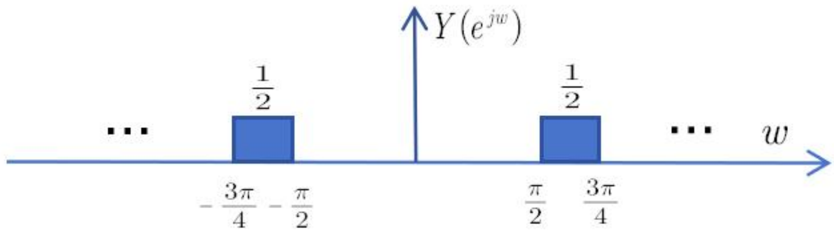
$$h[n]=\frac{\sin\left(\frac{3\pi}{4}n\right)}{\pi n}\longleftrightarrow H(e^{jw})=\begin{cases}1\ ,\ -\frac{3\pi}{4}+2k\pi<w<\frac{3\pi}{4}+2k\pi\ \ (k\in Z)\\ 0,\quad otherwise\end{cases}$$

$$H(e^{jw})\ =\begin{cases}1\ ,\ |w|<\frac{3\pi}{4} & in\ one\ period\\ 0,\ others\end{cases}$$

$$Y(e^{jw})\ =\ X(e^{jw})\cdot H(e^{jw})$$

$$Y(e^{jw})\ =\ \begin{cases}\frac{1}{2}\ ,\ \frac{\pi}{2}<|w-2k\pi|<\frac{3\pi}{4}\ (k\in Z)\\ 0,\ others\end{cases}$$

$$Y(e^{jw})\ =\begin{cases}\frac{1}{2}\ ,\ \frac{\pi}{2}<|w|<\frac{3\pi}{4} & in\ one\ period\\ 0,\ others\end{cases}$$



The figure of $Y(e^{jw})$ in one period

Problem 6 (20 points)

We are given a discrete-time, liner, time-invariant, causal system with input denoted by $x[n]$ and output denoted by $y[n]$. This system is specified by the following difference equations, involving an intermediate signal $w(n)$:

$$y[n]+\frac{1}{4}y[n-1]+w[n]+\frac{1}{2}w[n-1]=\frac{2}{3}x[n]$$

$$y[n]-\frac{5}{4}y[n-1]+2w[n]-2w[n-1]=-\frac{5}{3}x[n]$$

(a) Find a difference equation relating $y[n]$ and $x[n]$, directly (without using $w[n]$) for the system. **(8points)**

(b) Calculate $h[n]$ and $H(jw)$. **(4 points)**

(c) If the input $x[n] = \left(\frac{1}{3}\right)^n u(n)$, find $y[n]$. **(8 points)**

Solutions:

(a):

$$y[n]+\frac{1}{4}y[n-1]+w[n]+\frac{1}{2}w[n-1]=\frac{2}{3}x[n] \qquad (1)$$

$$y[n]-\frac{5}{4}y[n-1]+2w[n]-2w[n-1]=-\frac{5}{3}x[n] \qquad (2)$$

$$(1)*2: \quad 2y[n]+\frac{1}{2}y[n-1]+2w[n]+w[n-1]=\frac{4}{3}x[n]$$

$$(3)-(2): \quad y[n]+\frac{7}{4}y[n-1]+3w[n-1]=3x[n]$$

$$which\ means, \quad w[n-1]=x[n]-\frac{1}{3}y[n]-\frac{7}{12}y[n-1] \qquad (a) \ \textbf{2 points}$$

$$(1)*4: \quad 4y[n]+y[n-1]+4w[n]+2w[n-1]=\frac{8}{3}x[n]$$

$$(4)+(2): \ 5y[n]-\frac{1}{4}y[n-1]+6w[n]=x[n]$$

$$which\ means, \quad w[n]=\frac{1}{6}x[n]-\frac{5}{6}y[n]+\frac{1}{24}y[n-1] \quad (b) \ \textbf{2 points}$$

With (a) and (b), we can get $w(n)$ and $w(n-1)$, $w(n) = w(n-1+1)$, so we can get a signal difference equation relating $x(n)$ and $y(n)$.

$$
\begin{aligned}
x[n+1]-\frac{1}{3}y[n+1]-\frac{7}{12}y[n] &= \frac{1}{6}x[n]-\frac{5}{6}y[n]+\frac{1}{24}y[n-1] \\
-\frac{1}{3}y[n+1]+\frac{1}{4}y[n]-\frac{1}{24}y[n-1] &= -x[n+1]+\frac{1}{6}x[n] \\
\frac{1}{3}y[n+1]-\frac{1}{4}y[n]+\frac{1}{24}y[n-1] &= x[n+1]-\frac{1}{6}x[n] \quad \textbf{4 points}
\end{aligned}
$$

(b):

$$
\begin{aligned}
H(j\omega) &= \frac{Y(j\omega)}{X(j\omega)} = \frac{e^{jw}-\frac{1}{6}}{\frac{1}{3}e^{jw}-\frac{1}{4}+\frac{1}{24}e^{-jw}} = \frac{3\left(1-\frac{1}{6}e^{-jw}\right)}{\left(1-\frac{1}{2}e^{-jw}\right)\left(1-\frac{1}{4}e^{-jw}\right)} \\
H(j\omega) &= \frac{4}{1-\frac{1}{2}e^{-jw}}-\frac{1}{1-\frac{1}{4}e^{-jw}} \quad \textbf{2 points}
\end{aligned}
$$

$$
h(n) = \left[4\left(\frac{1}{2}\right)^n-\left(\frac{1}{4}\right)^n\right]u[n] \quad \textbf{2 points}
$$

(c):

$$
x[n] = \left(\frac{1}{3}\right)^n u(n) \Longleftrightarrow X(jw)=\frac{1}{1-\frac{1}{3}e^{-jw}} \quad \textbf{2 points}
$$

$$
\begin{aligned}
Y(jw) &= X(jw)H(jw) \\
Y(jw) &= \frac{3\left(1-\frac{1}{6}e^{-jw}\right)}{\left(1-\frac{1}{2}e^{-jw}\right)\left(1-\frac{1}{4}e^{-jw}\right)\left(1-\frac{1}{3}e^{-jw}\right)} \\
&= \frac{12}{1-\frac{1}{2}e^{-jw}}+\frac{3}{1-\frac{1}{4}e^{-jw}}+\frac{-12}{1-\frac{1}{3}e^{-jw}} \quad \textbf{3 points} \\
y(n) &= \left[12\left(\frac{1}{2}\right)^n+3\left(\frac{1}{4}\right)^n-12\left(\frac{1}{3}\right)^n\right]u[n] \quad \textbf{3 points}
\end{aligned}
$$