Lecture 8
- Sinusoidal Steady-State Analysis

Outline

- Kirchhoff's laws in phasor domain

- Generalized impedance

- General AC phasor analysis

- Phasor diagram

Kirchhoff's Laws in the Phasor Domain

- Let $v_1, v_2, \cdots v_n$ be the voltages around a closed loop. Then according to KVL

$$v_1 + v_2 + \cdots + v_n = 0$$

$$\mathbf{V}_1 + \mathbf{V}_2 + \cdots + \mathbf{V}_n = 0$$

Similarly, KCL holds for phasors:

$$i_1 + i_2 + \cdots + i_n = 0$$

$$\mathbf{I}_1 + \mathbf{I}_2 + \cdots + \mathbf{I}_n = 0,$$

Proof

If

$$v_1 + v_2 + \cdots + v_n = 0$$

where v_i are sinusoidal voltages <u>of the same frequency</u>, then

$$\mathbf{V}_1 + \mathbf{V}_2 + \cdots + \mathbf{V}_n = 0$$

Proof:

$$v_1 + v_2 + \cdots + v_n = 0$$

$$\Updownarrow$$

$$V_{m1}\cos(\omega t + \theta_1) + V_{m2}\cos(\omega t + \theta_2) + \cdots + V_{mn}\cos(\omega t + \theta_n) = 0$$

$$\Updownarrow$$

$$\text{Re}(V_{m1}e^{j\theta_1} \cdot e^{j\omega t}) + \cdots + \text{Re}(V_{mn}e^{j\theta_n} \cdot e^{j\omega t}) = 0$$

$$\Updownarrow$$

$$\text{Re}\left((\mathbf{V_1} + \cdots + \mathbf{V_n}) \cdot e^{j\omega t}\right) = 0 \quad Where \ \mathbf{V_k} = V_{mk}e^{j\theta_k}$$

$$\Updownarrow$$

$$\mathbf{V}_1 + \mathbf{V}_2 + \cdots + \mathbf{V}_n = 0$$

Outline

- Kirchhoff's laws in phasor domain

- Generalized impedance
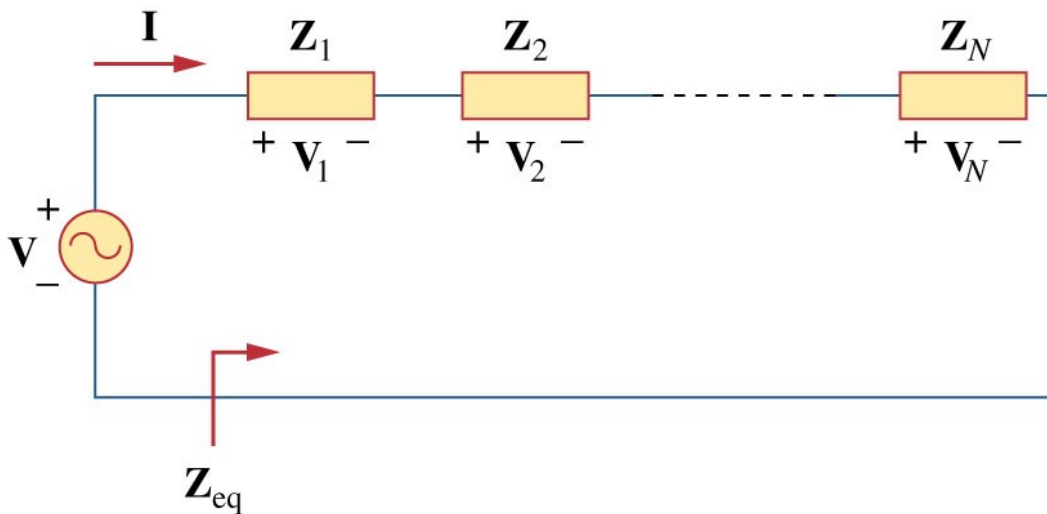
- General AC phasor analysis

- Phasor diagram

Impedance and Admittance

Resistor	$\mathbf{Z} = R$	$\mathbf{Y} = 1/R$
Inductor	$\mathbf{Z} = j\omega L$	$\mathbf{Y} = 1/j\omega L$
Capacitor	$\mathbf{Z} = 1/j\omega C$	$\mathbf{Y} = j\omega C$

Series Impedance

• In phasor domain, combinations of *impedance* will follow the rules for ***resistors:***

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$$Z_{eq} = Z_1 + Z_2 + Z_3 + \cdots$$

$$V_1 = \frac{Z_1}{Z_1 + \cdots + Z_N} V$$

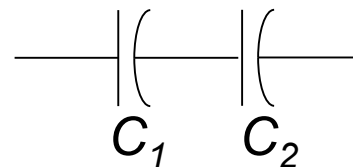$$V_2 = \frac{Z_2}{Z_1 + \cdots + Z_N} V$$

Series Impedance

$$Z_{eq} = Z_1 + Z_2 + Z_3$$

Z_1

Z_3

Z_2

Z_{eq}

For example:

L_1 L_2

$Z_{eq} =$

C_1 C_2
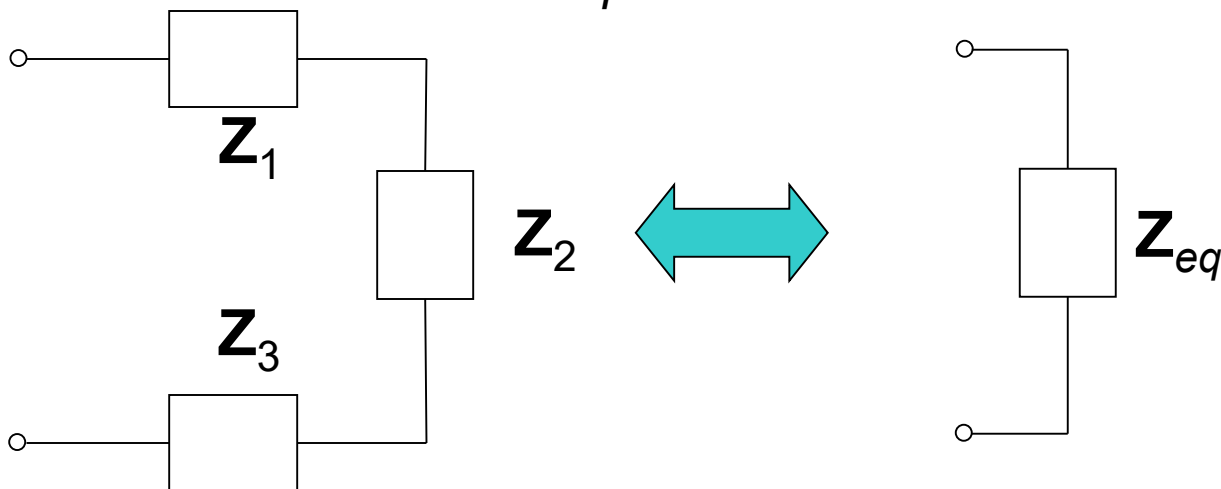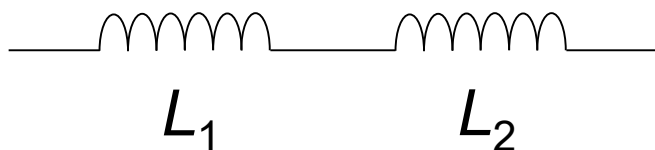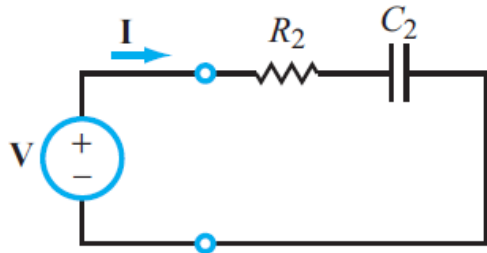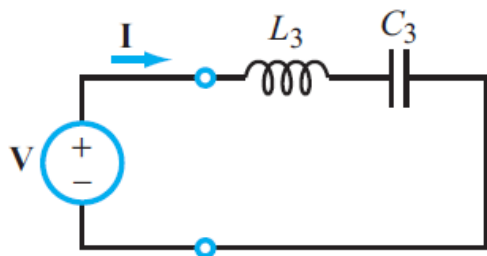
$Z_{eq} =$

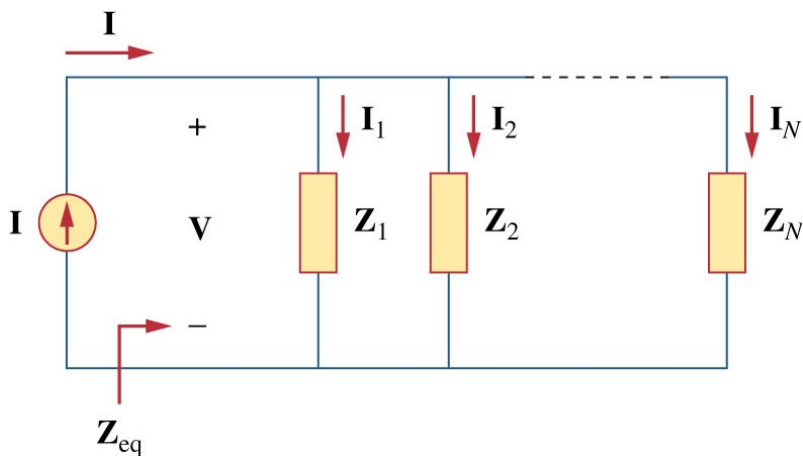# Impedance combination for RLC Circuit
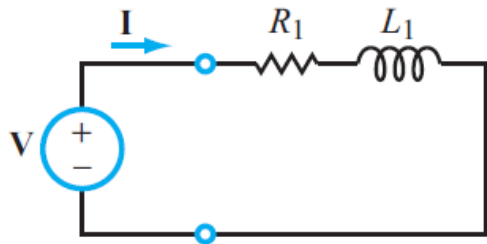
(a) RL

(b) RC

(c) LC

Parallel Combination

• Likewise, elements in parallel will combine in the same fashion as resistors in parallel:

$$\frac{1}{Z_{eq}} = \frac{1}{Z_1} + \frac{1}{Z_2} + \frac{1}{Z_3} + \cdots \frac{1}{Z_N}$$
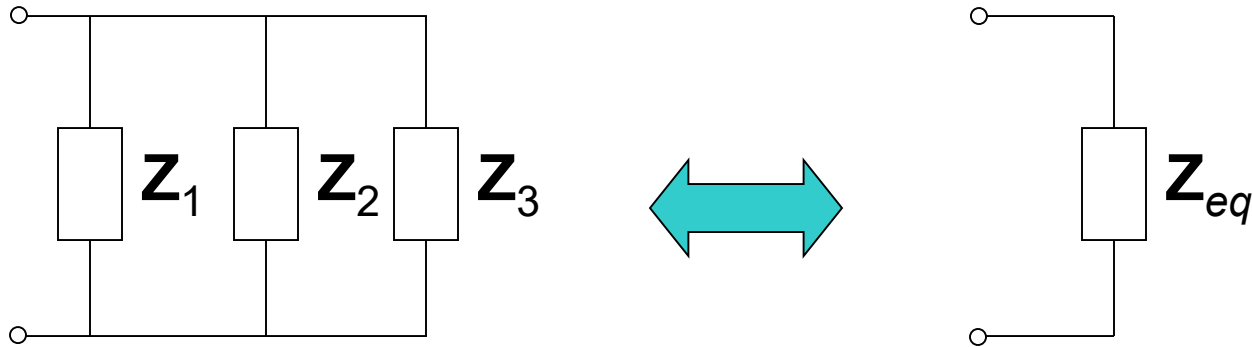
$$Y_{eq} = Y_1 + Y_2 + Y_3 + \cdots$$

$$I_1 = \frac{Y_1}{Y_1 + \cdots + Y_N} I$$
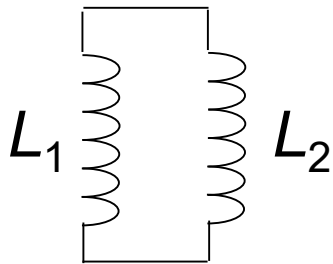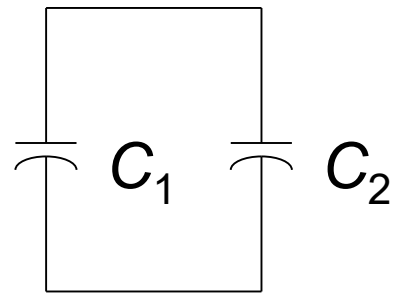
$$I_2 = \frac{Y_2}{Y_1 + \cdots + Y_N} I$$

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Parallel Impedance

$$1/\mathbf{Z}_{eq} = 1/\mathbf{Z}_1 + 1/\mathbf{Z}_2 + 1/\mathbf{Z}_3$$

For example:

$$\mathbf{Z}_{eq} = j\omega \frac{L_1 L_2}{(L_1 + L_2)}$$
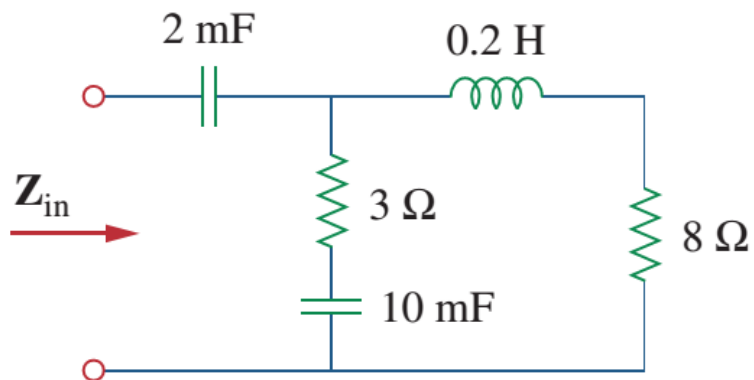
$$\mathbf{Z}_{eq} = \frac{1}{j\omega(C_1 + C_2)}$$

Exercise

- Find the input impedance of the circuit below. $\omega = 50$ rad/s.

Delta-Wye Transformation in Phasor Domain

$$Z_1 = \frac{Z_b Z_c}{Z_a + Z_b + Z_c}$$

$$Z_2 = \frac{Z_c Z_a}{Z_a + Z_b + Z_c}$$

$$Z_3 = \frac{Z_a Z_b}{Z_a + Z_b + Z_c}$$

$$Z_a = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1}{Z_1}$$

$$Z_b = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1}{Z_2}$$

$$Z_c = \frac{Z_1 Z_2 + Z_2 Z_3 + Z_3 Z_1}{Z_3}$$

Impedance and Admittance

Resistor	$\mathbf{Z} = R$	$\mathbf{Y} = 1/R$
Inductor	$\mathbf{Z} = j\omega L$	$\mathbf{Y} = 1/j\omega L$
Capacitor	$\mathbf{Z} = 1/j\omega C$	$\mathbf{Y} = j\omega C$

Impedance is voltage/current

$$\mathbf{Z} = R + jX$$

R = resistance = Re(Z)

X = reactance = Im(Z)

Admittance is current/voltage

$$\mathbf{Y} = \frac{1}{\mathbf{Z}} = G + jB$$

G = conductance = Re(Y)

B = susceptance = Im(Y)

Outline

- Kirchhoff's laws in phasor domain

- Generalized impedance

- General AC phasor analysis
  - Nodal/mesh analysis
  - Superposition
  - Source transformation/Thevenin/Norton

- Phasor diagram

AC Phasor Analysis
General Procedure

Step 1: Adopt cosine reference

$$v_s(t) = 12\sin(\omega t - 45°)$$

$$= 12\cos(\omega t - 45° - 90°) = 12\cos(\omega t - 135°) \text{ V}.$$

$$\mathbf{V}_s = 12e^{-j135°} \text{ V}.$$

Step 2: Transform circuit to phasor domain

Step 3: Cast KCL and/or KVL equations in phasor domain

$$\mathbf{Z_R I} + \mathbf{Z_C I} = \mathbf{V}_s,$$

which is equivalent to

$$\left(R + \frac{1}{j\omega C}\right)\mathbf{I} = 12e^{-j135°}.$$
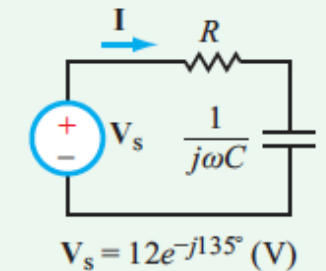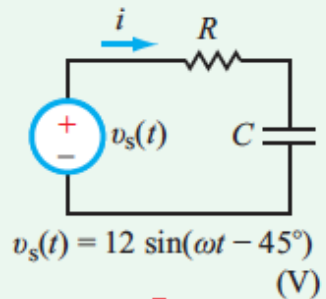
Step 1
Adopt Cosine Reference
(Time Domain)

$v_s(t) = 12\sin(\omega t - 45°)$ (V)

Step 2
Transfer to Phasor Domain
$i \longrightarrow \mathbf{I}$
$v \longrightarrow \mathbf{V}$
$R \longrightarrow \mathbf{Z_R} = R$
$L \longrightarrow \mathbf{Z_L} = j\omega L$
$C \longrightarrow \mathbf{Z_C} = 1/j\omega C$

$\mathbf{V}_s = 12e^{-j135°}$ (V)

Step 3
Cast Equations in
Phasor Form

$$\mathbf{I}\left(R + \frac{1}{j\omega C}\right) = \mathbf{V}_s$$

AC Phasor Analysis
General Procedure

Step 4: Solve for unknown variable

$$\mathbf{I} = \frac{12e^{-j135°}}{R + \frac{1}{j\omega C}} = \frac{j12\omega Ce^{-j135°}}{1 + j\omega RC}.$$

Using the specified values, namely $R = \sqrt{3}$ kΩ, $C = 1$ μF, and $\omega = 10^3$ rad/s,

$$\mathbf{I} = \frac{j12 \times 10^3 \times 10^{-6}e^{-j135°}}{1 + j10^3 \times \sqrt{3} \times 10^3 \times 10^{-6}} = \frac{j12e^{-j135°}}{1 + j\sqrt{3}} \text{ mA.}$$

$$\mathbf{I} = \frac{12e^{-j135°} \cdot e^{j90°}}{2e^{j60°}} = 6e^{j(-135°+90°-60°)} = 6e^{-j105°} \text{ mA.}$$
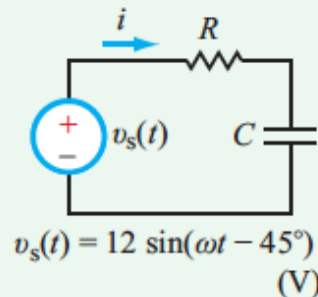
Step 5: Transform solution back to time domain

To return to the time domain, we apply the fundamental relation between a sinusoidal function and its phasor counterpart, namely

$$i(t) = \Re[\mathbf{I}e^{j\omega t}] = \Re[6e^{-j105°}e^{j\omega t}] = 6\cos(\omega t - 105°) \text{ mA.}$$

Step 1
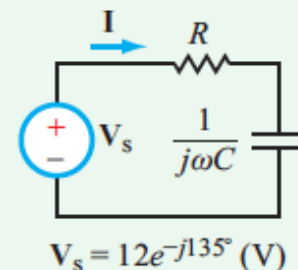Adopt Cosine Reference
(Time Domain)

$v_s(t) = 12 \sin(\omega t - 45°)$
(V)

Step 2
Transfer to Phasor Domain

$i \longrightarrow \mathbf{I}$
$v \longrightarrow \mathbf{V}$
$R \longrightarrow Z_R = R$
$L \longrightarrow Z_L = j\omega L$
$C \longrightarrow Z_C = 1/j\omega C$

$\mathbf{V}_s = 12e^{-j135°}$ (V)

Step 3
Cast Equations in
Phasor Form

$$\mathbf{I}\left(R + \frac{1}{j\omega C}\right) = \mathbf{V}_s$$

Step 4
Solve for Unknown Variable
(Phasor Domain)

$$\mathbf{I} = \frac{\mathbf{V}_s}{R + \frac{1}{j\omega C}}$$

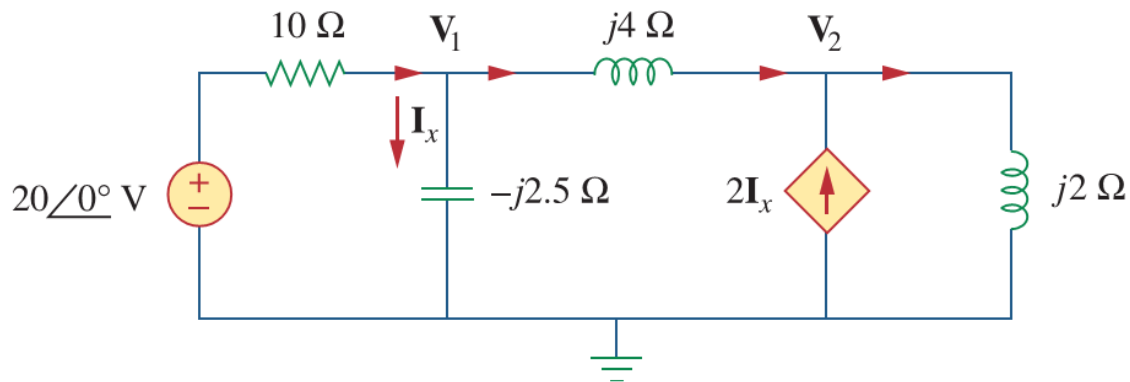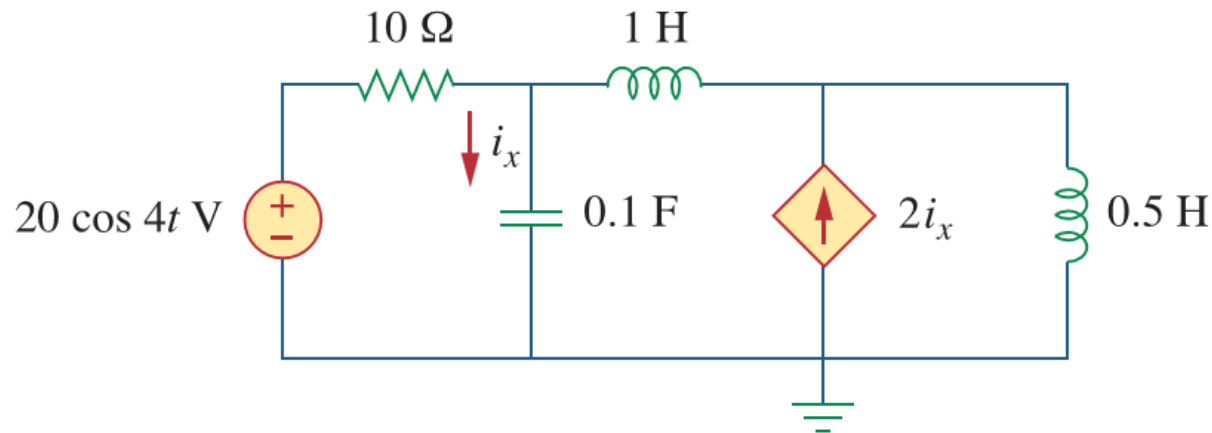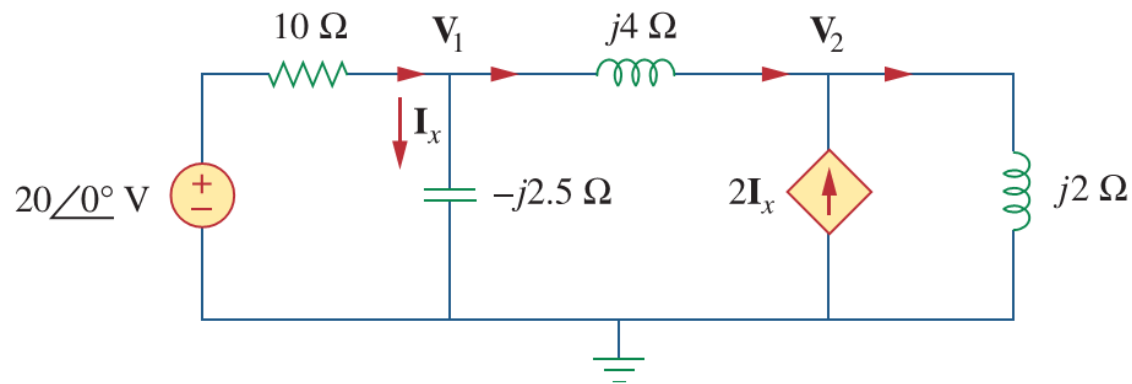**Step 5**
Transform Solution
Back to Time Domain

$i(t) = \Re[\mathbf{I}e^{j\omega t}]$
$= 6\cos(\omega t - 105°)$
(mA)
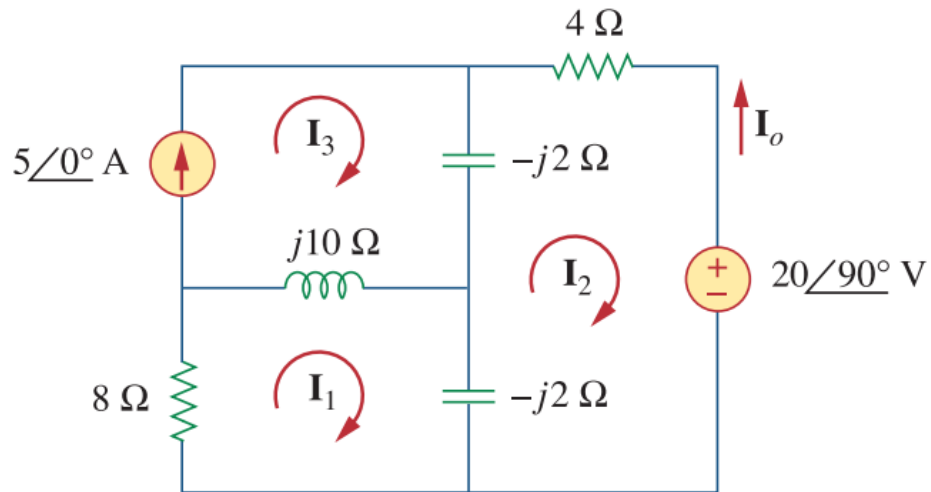
Nodal Analysis

- Example---Find i_x

$$\frac{20 - \mathbf{V}_1}{10} = \frac{\mathbf{V}_1}{-j2.5} + \frac{\mathbf{V}_1 - \mathbf{V}_2}{j4}$$

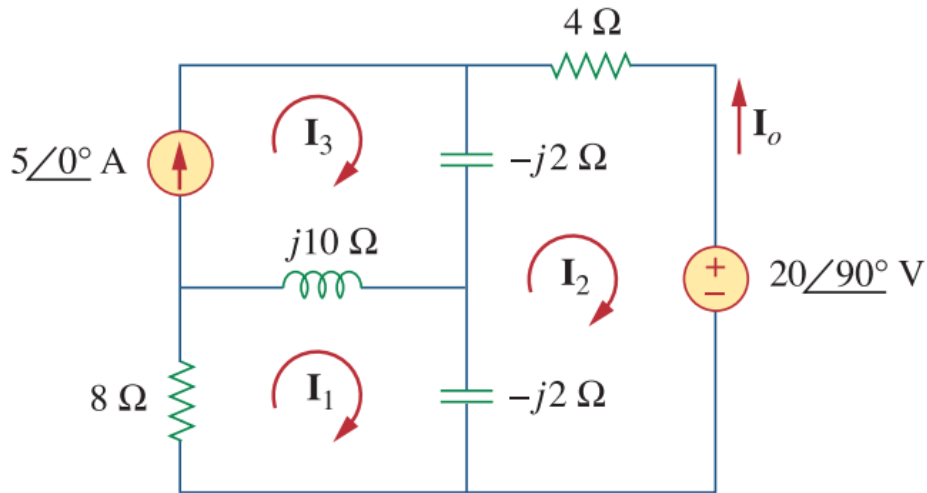$$2\mathbf{I}_x + \frac{\mathbf{V}_1 - \mathbf{V}_2}{j4} = \frac{\mathbf{V}_2}{j2}$$

Mesh Analysis

Mesh Analysis

Applying KVL to mesh 1, we obtain

$$(8 + j10 - j2)\mathbf{I}_1 - (-j2)\mathbf{I}_2 - j10\mathbf{I}_3 = 0 \qquad \textbf{(10.3.1)}$$

For mesh 2,

$$(4 - j2 - j2)\mathbf{I}_2 - (-j2)\mathbf{I}_1 - (-j2)\mathbf{I}_3 + 20\underline{/90°} = 0 \qquad \textbf{(10.3.2)}$$
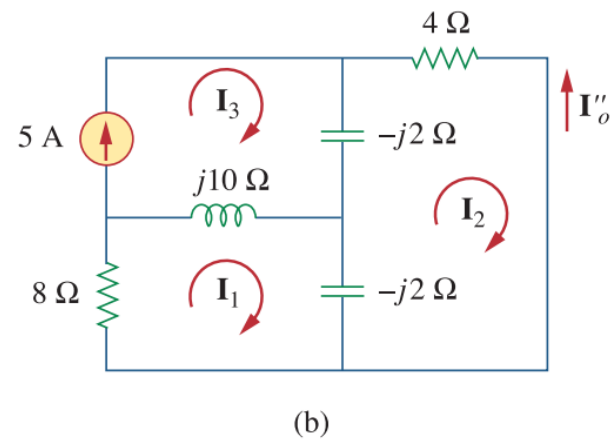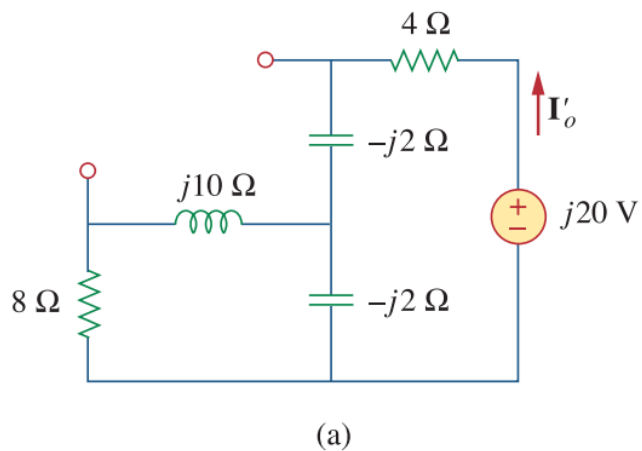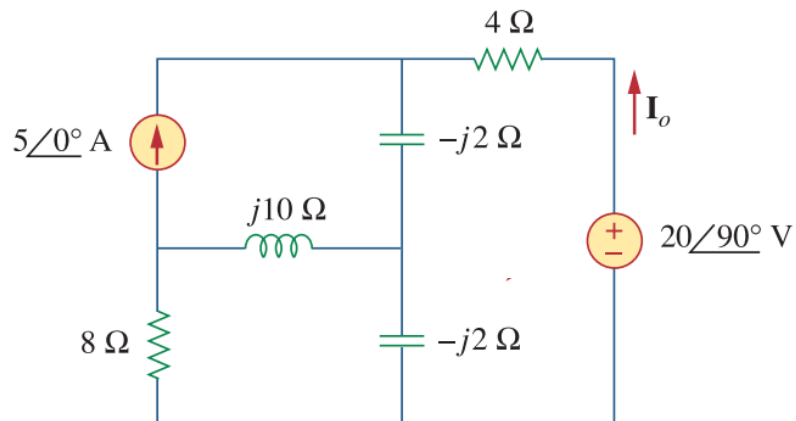
For mesh 3, $\mathbf{I}_3 = 5$. Substituting this in Eqs. (10.3.1) and (10.3.2), we get

$$(8 + j8)\mathbf{I}_1 + j2\mathbf{I}_2 = j50 \qquad \textbf{(10.3.3)}$$

$$j2\mathbf{I}_1 + (4 - j4)\mathbf{I}_2 = -j20 - j10 \qquad \textbf{(10.3.4)}$$

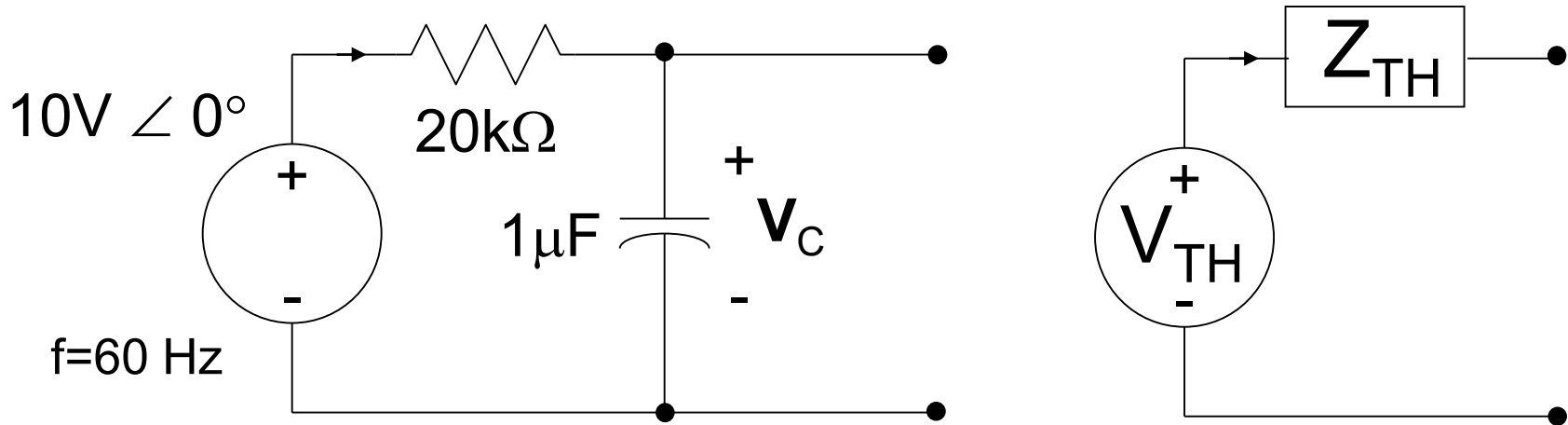# Superposition-Example

Superposition-Example 2

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Thevenin Equivalent

$\mathbf{Z}_R$ = R= 20kΩ = 20kΩ $\angle$ 0°

$\mathbf{Z}_C$ = 1/j (2πf x 1μF) = 2.65kΩ $\angle$ -90°

$$\mathbf{V}_{TH} = \mathbf{V}_{OC} = 10\text{V} \angle 0° \left(\frac{2.65\text{k}\Omega\angle\text{-}90°}{2.65\text{k}\Omega\angle\text{-}90° + 20\text{k}\Omega\angle 0°} \right) = 1.31\angle - 82.4$$

$$\mathbf{Z}_{TH} = \mathbf{Z}_R \parallel \mathbf{Z}_C = \left(\frac{20\text{k}\Omega\angle 0° \cdot 2.65\text{k}\Omega\angle\text{-}90°}{2.65\text{k}\Omega\angle\text{-}90° + 20\text{k}\Omega\angle 0°} \right) = 2.62\angle - 82.4$$

Source transformation/Norton
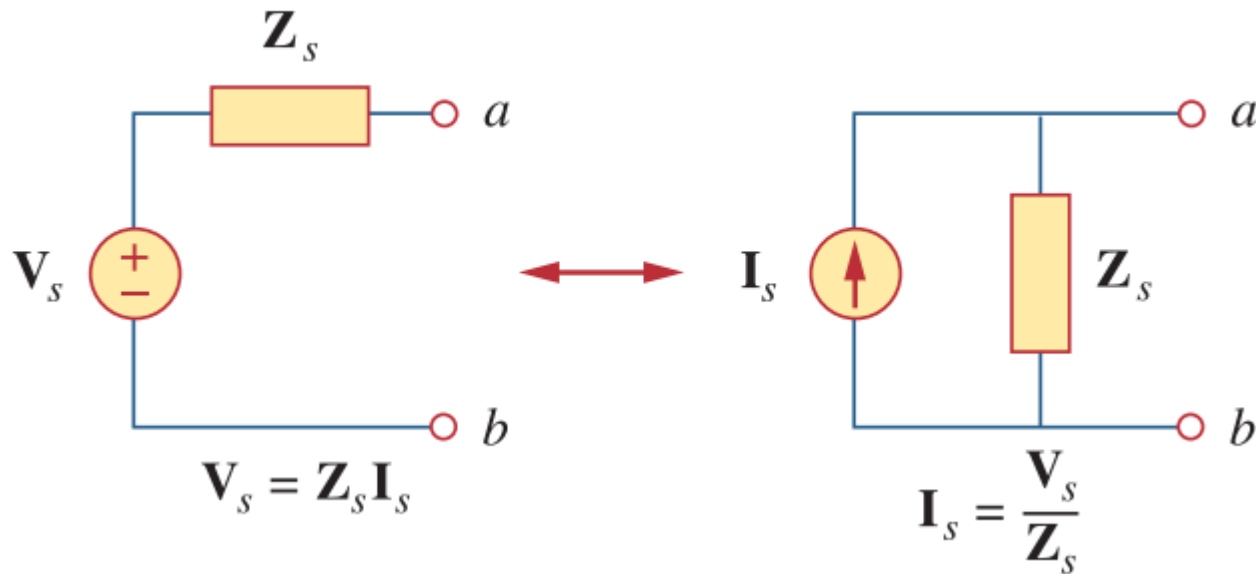
$$V_s = Z_s I_s \quad \Leftrightarrow \quad I_s = \frac{V_s}{Z_s}$$

AC Op Amp Circuits

Question 1: Are op amps used in ac circuits?
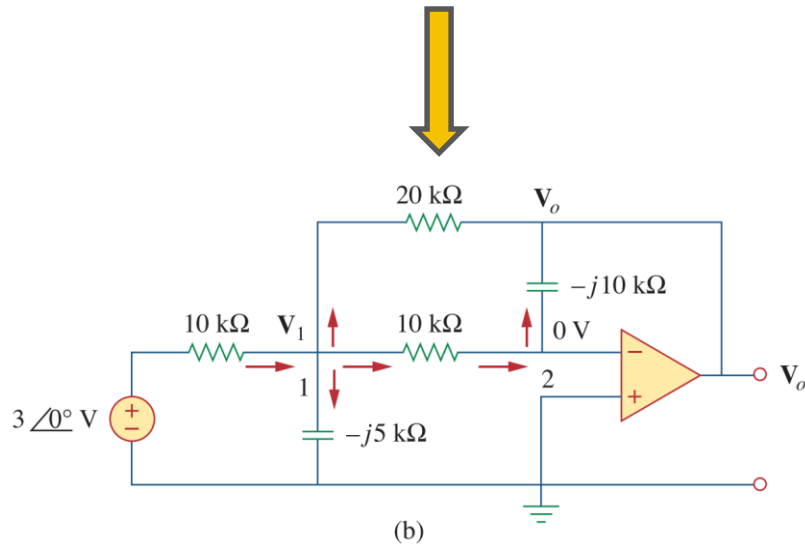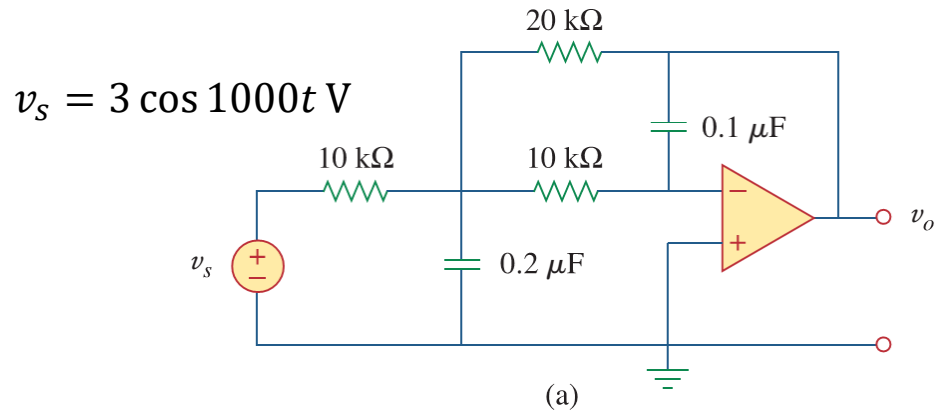
Answer 1: Yes.

Question 2: Is the ideal op-amp model applicable to ac circuits?

Answer 2: The ideal op-amp model is based on the assumption that the open-loop gain A is very large $(> 10^4)$, which is true at dc and low frequencies, but not necessarily so at high frequencies. The range of frequencies over which A is large depends on the specific op-amp design.
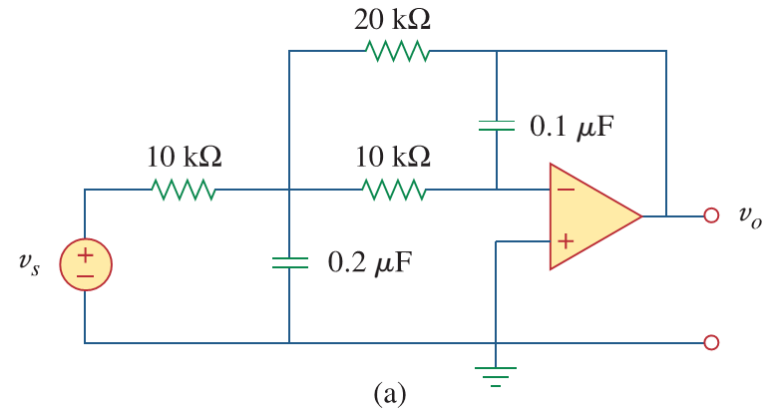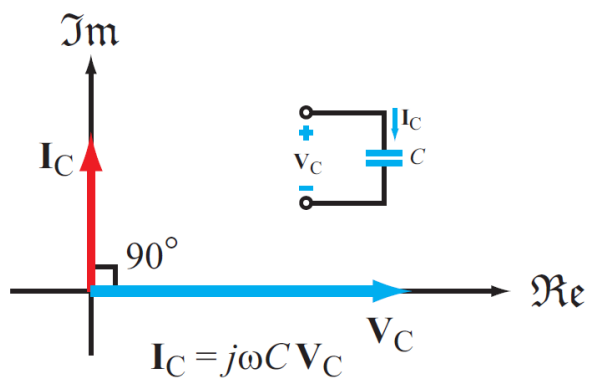

(a)

Example –find v_o

$$v_s = 3\cos 1000t \text{ V}$$

(a)

(b)

Outline

- Kirchhoff's laws in phasor domain

- Generalized impedance

- General AC phasor analysis

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Phasor Diagrams

Capacitor
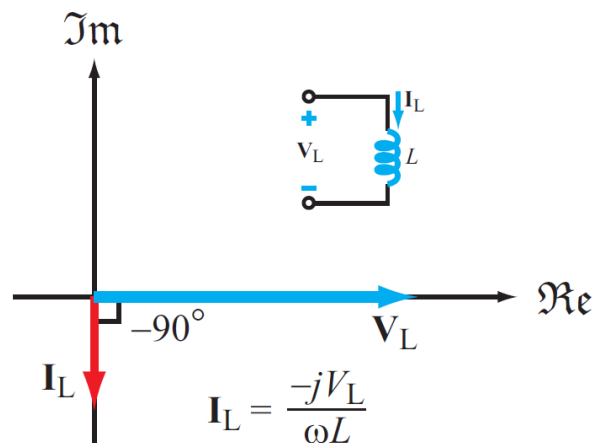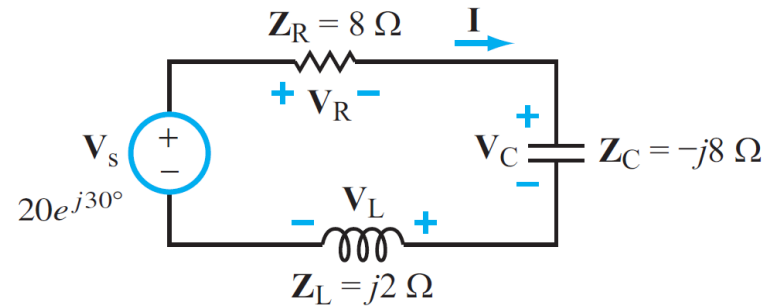
$\mathbf{I}_C = j\omega C \mathbf{V}_C$

Inductor

$\mathbf{I}_L = \dfrac{-jV_L}{\omega L}$

$$\mathbf{I} = \cfrac{\mathbf{V}_s}{R + j\omega L - \cfrac{j}{\omega C}} = \frac{20e^{j30°}}{8 + j2 - j8} = \frac{20e^{j30°}}{8 - j6} = \frac{20e^{j30°}}{10e^{-j36.87°}} = 2e^{j66.87°} \text{ A}$$

(c) Relative phasor diagram to that of **I**.