CS240 Algorithm Design and Analysis

Lecture 8

Network Flow (Cont.)

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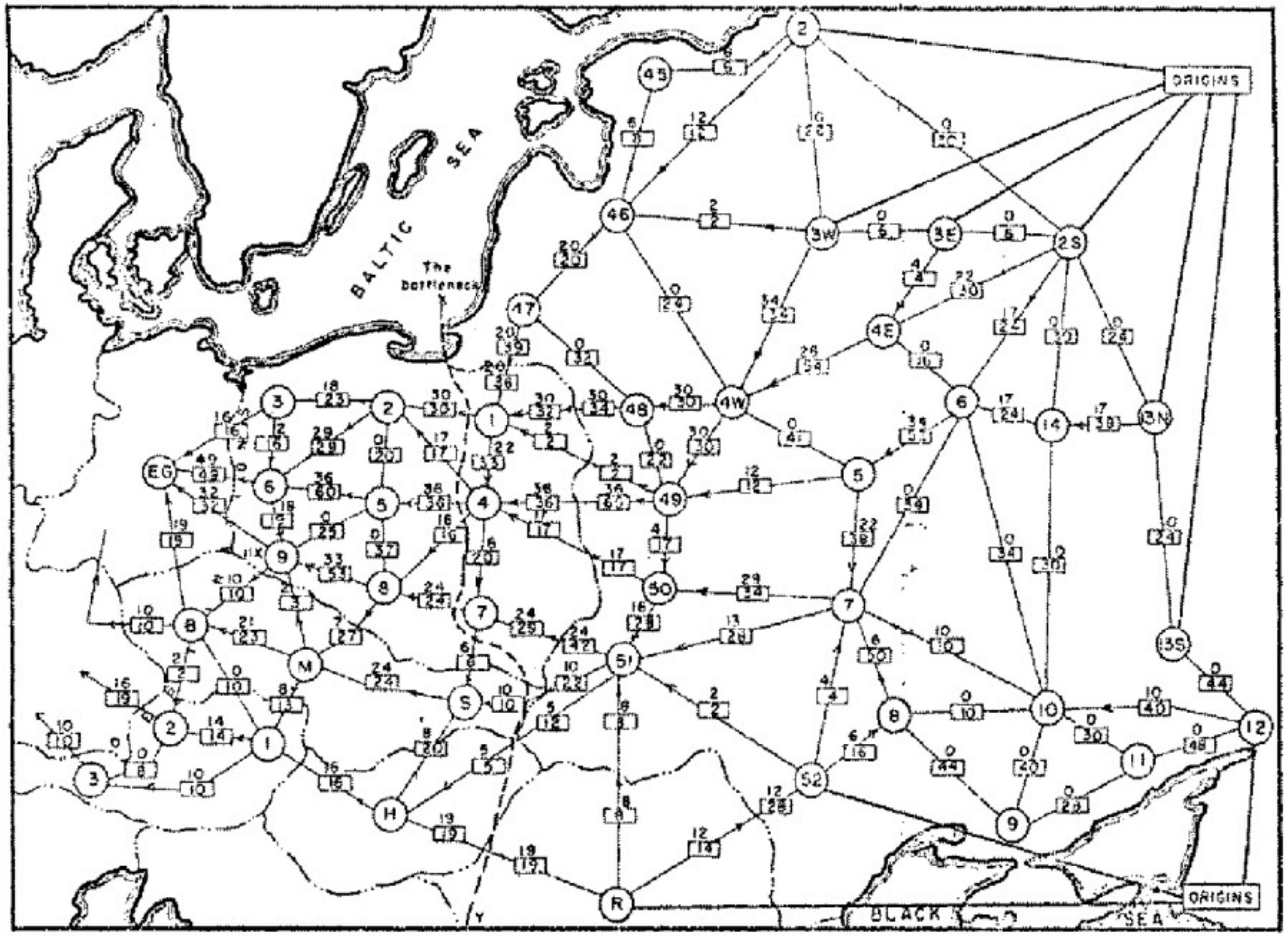
2024.10.24

Applications of max-flow

Soviet Rail Network, 1955



Reference: On the history of the transportation and maximum flow problems. Alexander Schrijver in Math Programming, 91: 3, 2002

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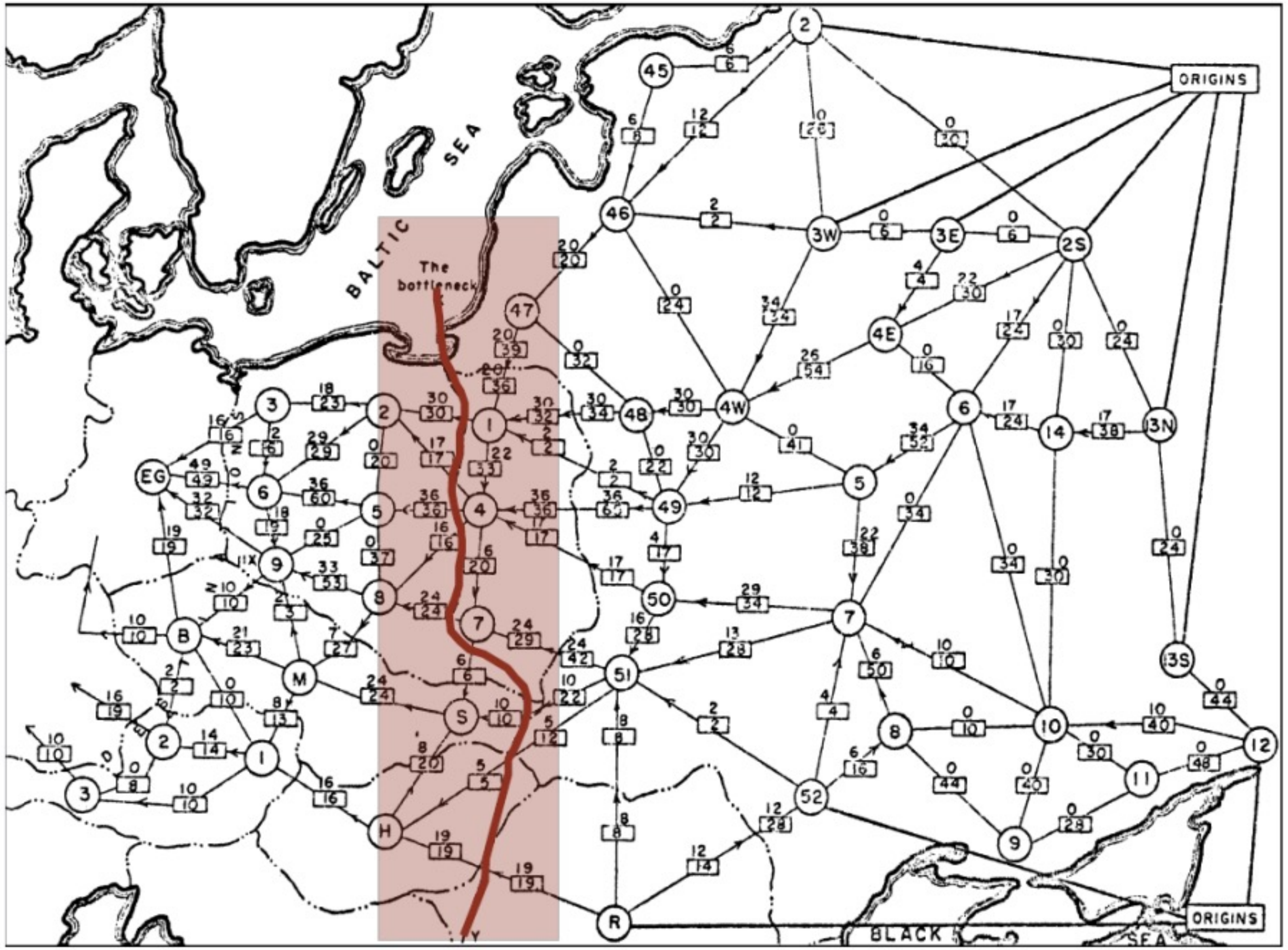

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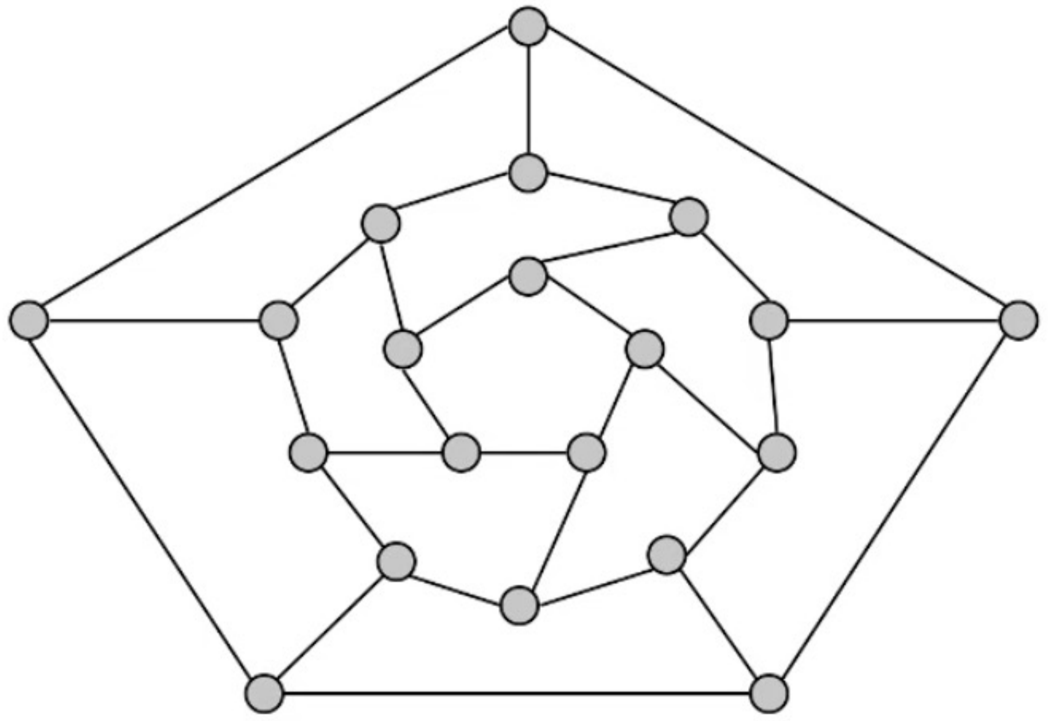
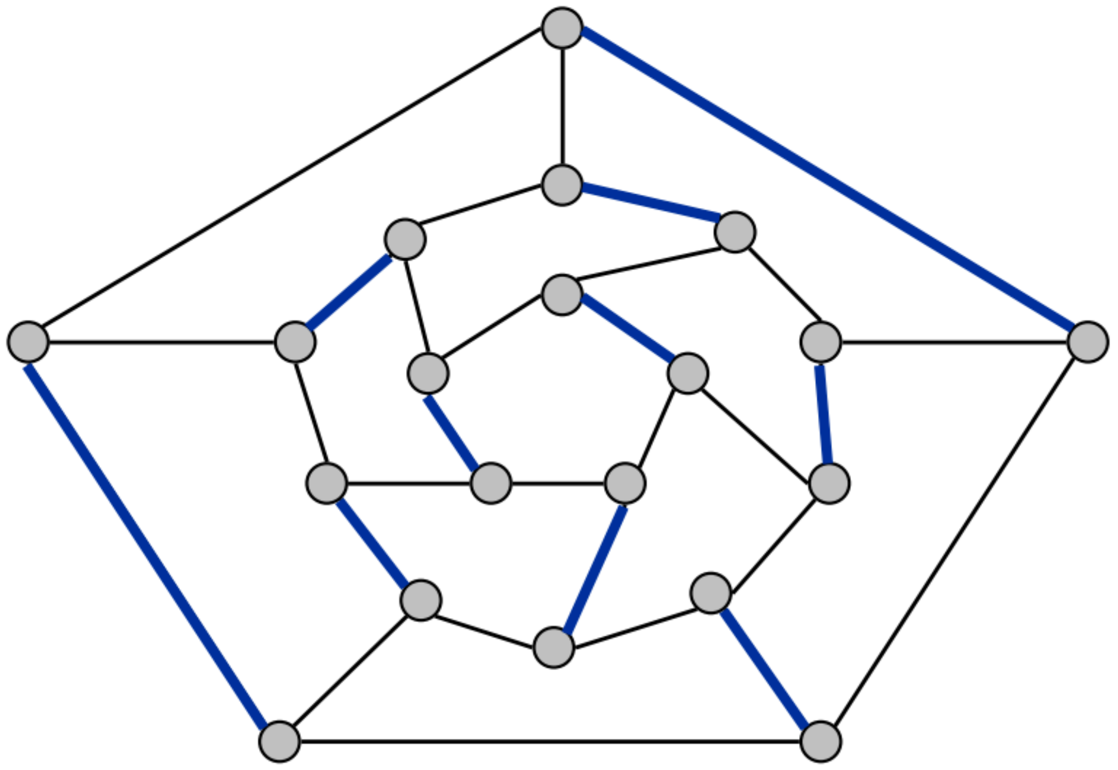
Bipartite Matching

Matching

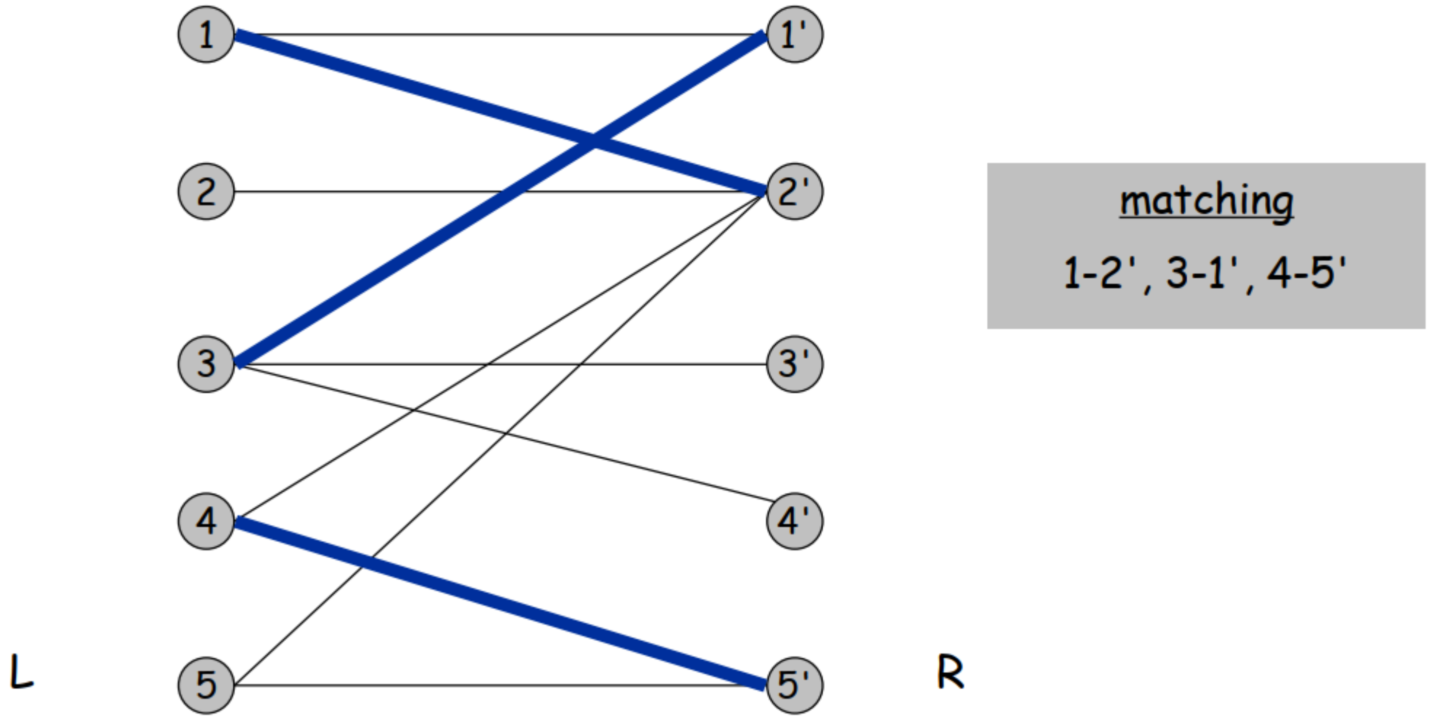
- **Matching**
  - Input: undirected graph G = (V, E)
  - M ⊆ E is a **matching** if each node appears in at most one edge in M
  - Max matching: find a max cardinality matching





Bipartite Matching

- Bipartite matching
  - Input: undirected, **bipartite** graph $G = (L \cup R, E)$
  - $M \subseteq E$ is a **matching** if each node appears in at most one edge in M
  - Max matching: find a max cardinality matching



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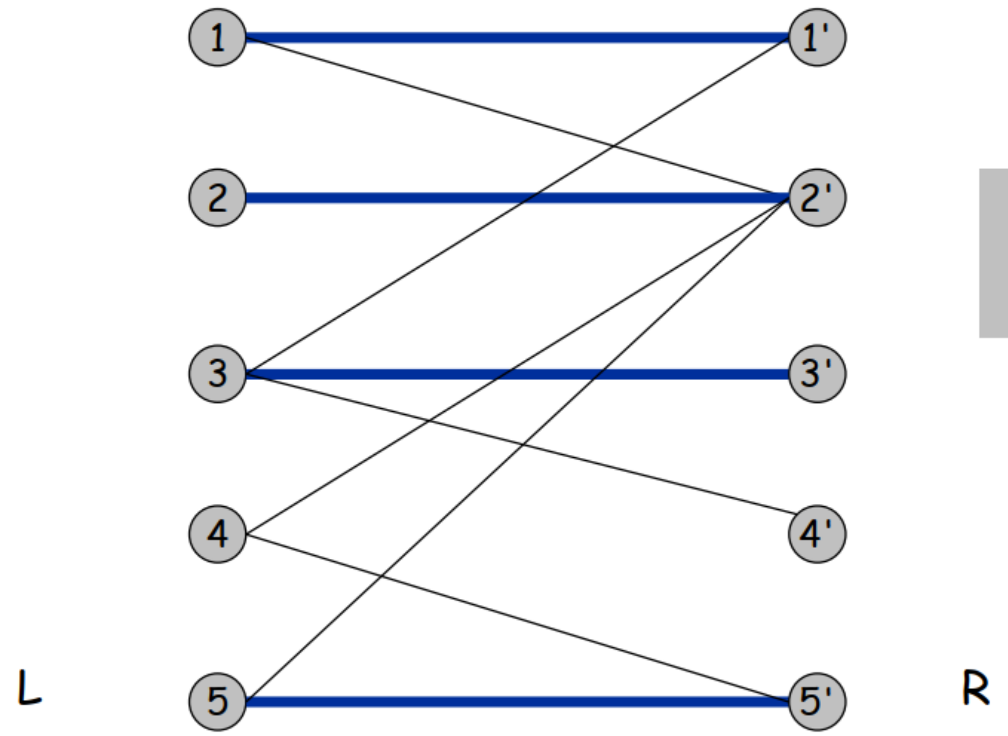
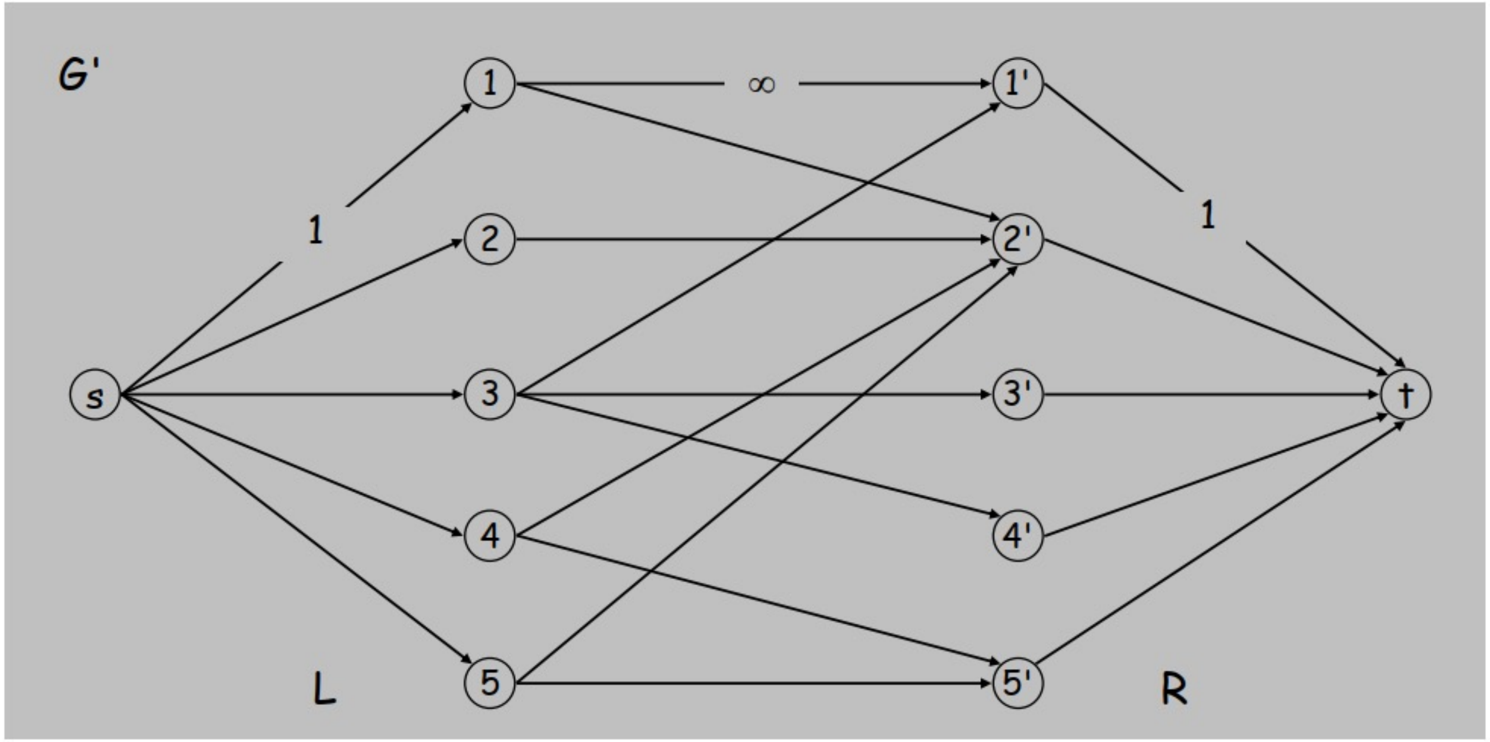


max matching

1-1', 2-2', 3-3' 4-4'

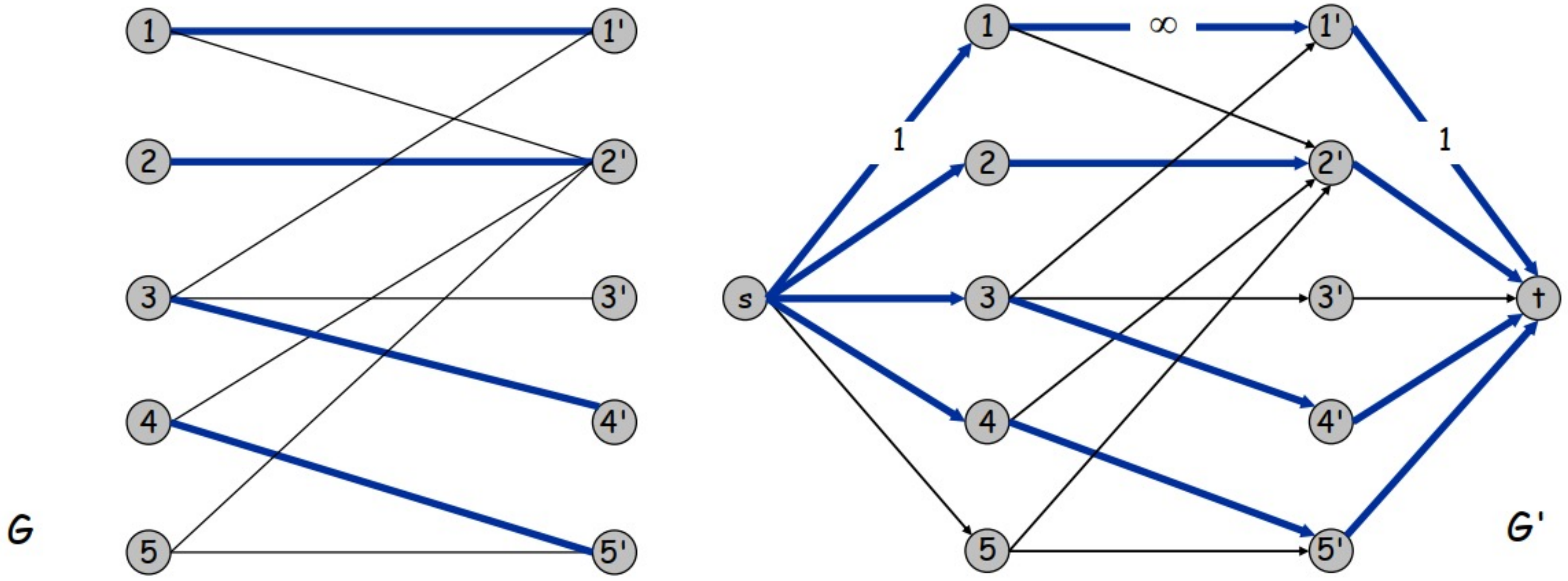
Bipartite Matching

- **Max flow formulation**
  - Create digraph $G' = (L \cup R \cup \{s, t\}, E')$
  - Direct all edges from L to R, and assign infinite (or unit) capacity
  - Add source s, and unit capacity edges from s to each node in L
  - Add sink t, and unit capacity edges from each node in R to t



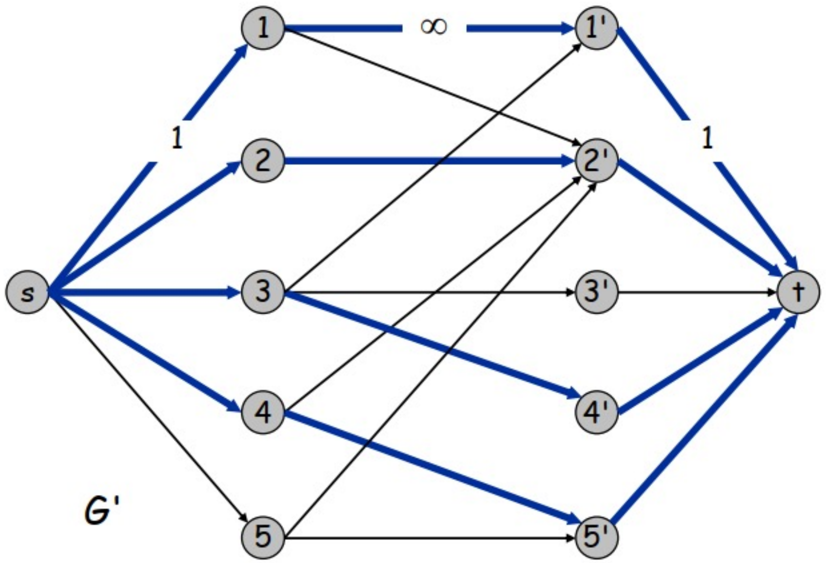
Bipartite Matching: Proof of Correctness

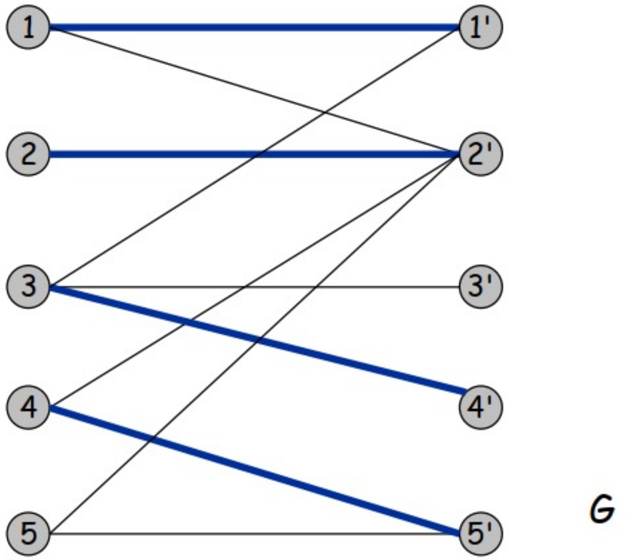
- **Theorem.** Max cardinality matching in G = value of max flow in G′
- **Pf. <=**
  - Given max matching M of cardinality k
  - Consider flow f that sends 1 unit along each of k paths
  - f is a flow, and has cardinality k



Bipartite Matching: Proof of Correctness

- **Theorem.** Max cardinality matching in G = value of max flow in G′
- **Pf. >=**
  - Let f be a max flow in G′ of value k
  - Integrality theorem → k is integral and can assume f is 0-1
  - Consider M = set of edges from L to R with f(e) = 1
    - Each node in L and R participates in at most one edge in M
    - |M| = k: consider cut (L∪s, R∪t)





Perfect Matching

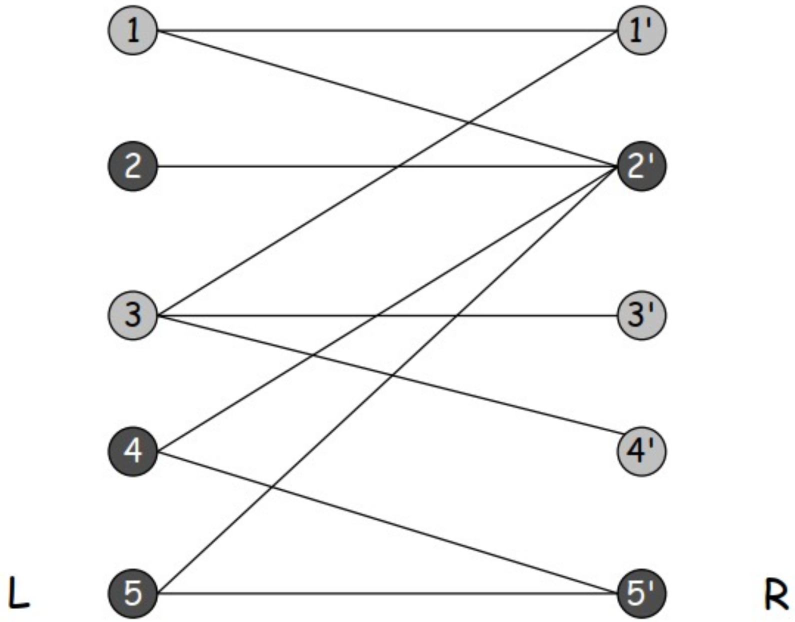
- **Def.** A matching $M \subseteq E$ is **perfect** if each node appears in exactly one edge in M

- **Q.** When does a bipartite graph have a perfect matching?

- **Structure of bipartite graphs with perfect matchings**
  - Clearly we must have $|L| = |R|$
  - What other conditions are necessary?
  - What conditions are sufficient?

Perfect Matching

- **Notation.** Let S be a subset of nodes, and let N(S) be the set of nodes adjacent to nodes in S
- **Observation.** If a bipartite graph G = (L∪R, E), has a perfect matching, then |N(S)| >= |S| for all subsets $S \subseteq L$
- **Pf.** Each node in S has to be matched to a different node in N(S)



<u>No perfect matching:</u>

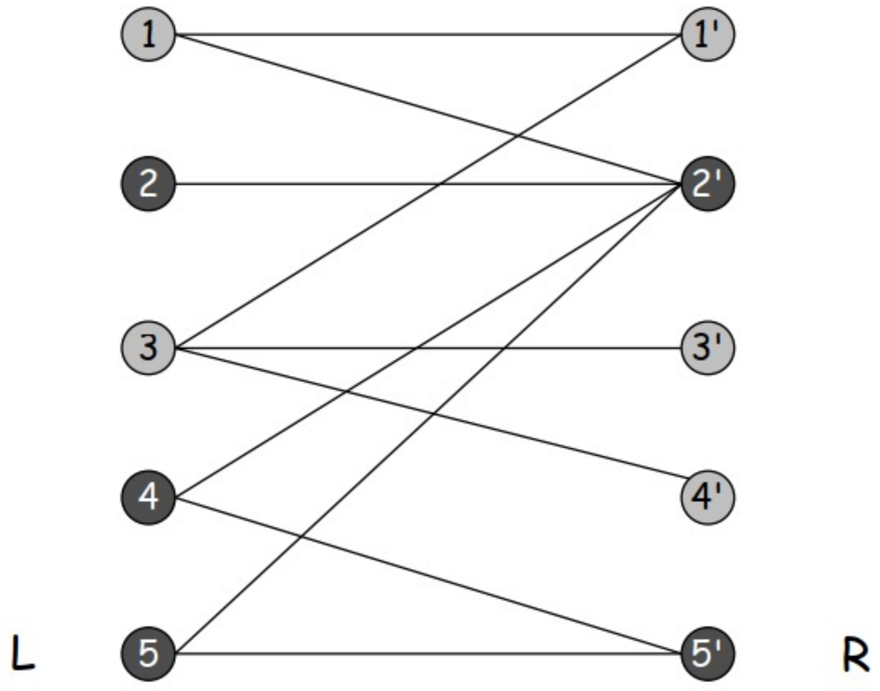
S = { 2, 4, 5 }

N(S) = { 2', 5' }.

Marriage Theorem

- **Marriage Theorem.** [Frobenius 1917, Hall 1935] Let $G = (L \cup R, E)$ be a bipartite graph with $|L| = |R|$. Then, G has a perfect matching iff $|N(S)| >= |S|$ for all subsets $S \subseteq L$

- **Pf.** → This was the previous observation



No perfect matching:

$S = \{ 2, 4, 5 \}$

$N(S) = \{ 2', 5' \}$.

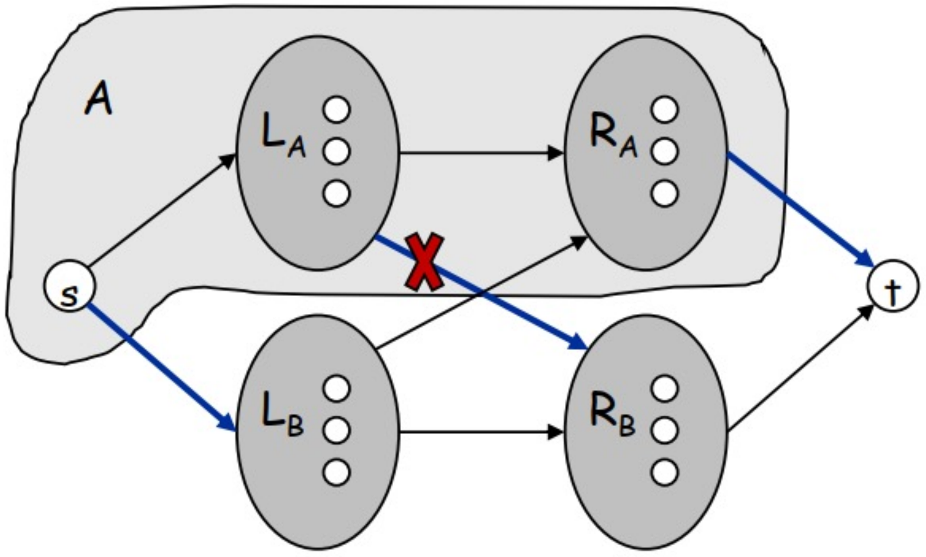
Proof of Marriage Theorem

- **Marriage Theorem.** G has a perfect matching iff $|N(S)| >= |S|$ for all subsets $S \subseteq L$
- **Pf.** ← Suppose G does not have a perfect matching
  - Formulate as a max flow problem and let (A, B) be min cut in G′
  - Define $L_A = L \cap A$, $L_B = L \cap B$, $R_A = R \cap A$, $R_B = R \cap B$
  - $Cap(A, B) = v(f^*) = |M| < |L|$ ("<": because no perfect matching)
  - Since min cut can't use ∞ edges, no edge between L_A and R_B
    - $Cap(A, B) = |L_B| + |R_A|$
    - $N(L_A) \subseteq R_A$
  - $|N(L_A)| <= |R_A|$

    $= cap(A, B) - |L_B|$

    $< |L| - |L_B|$

    $= |L_A|$
  - This contradicts the condition



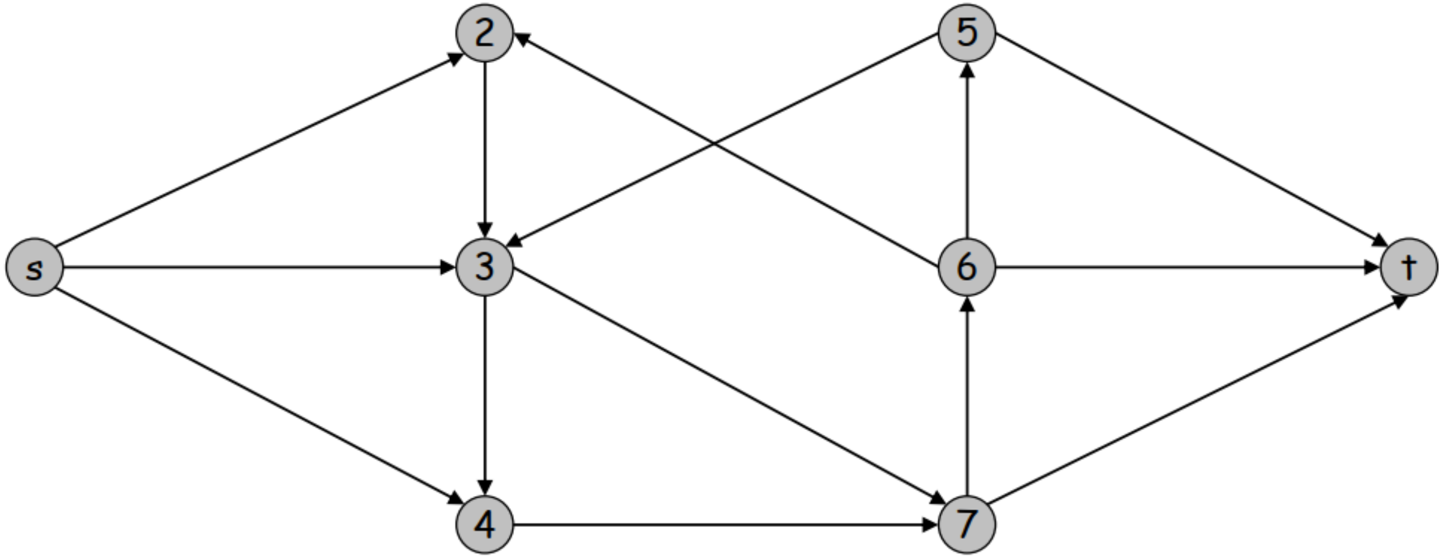
Disjoint Paths

Edge Disjoint Paths

- **Disjoint path problem.** Given a digraph G = (V, E) and two nodes s and t, find the max number of edge-disjoint s-t paths

- **Def.** Two paths are **edge-disjoint** if they have no edge in common

- **Ex:** communication networks



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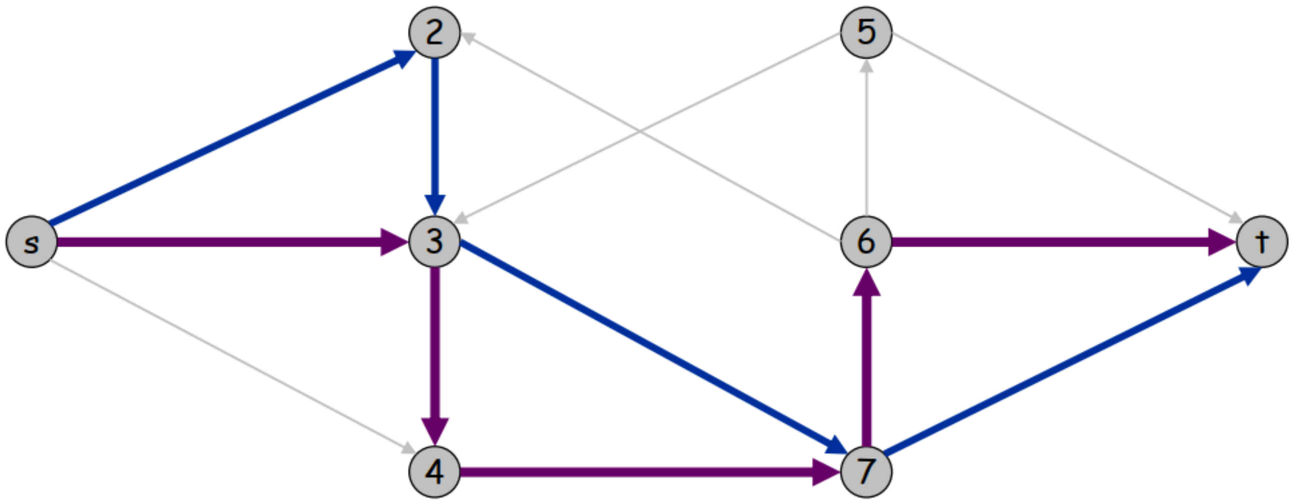
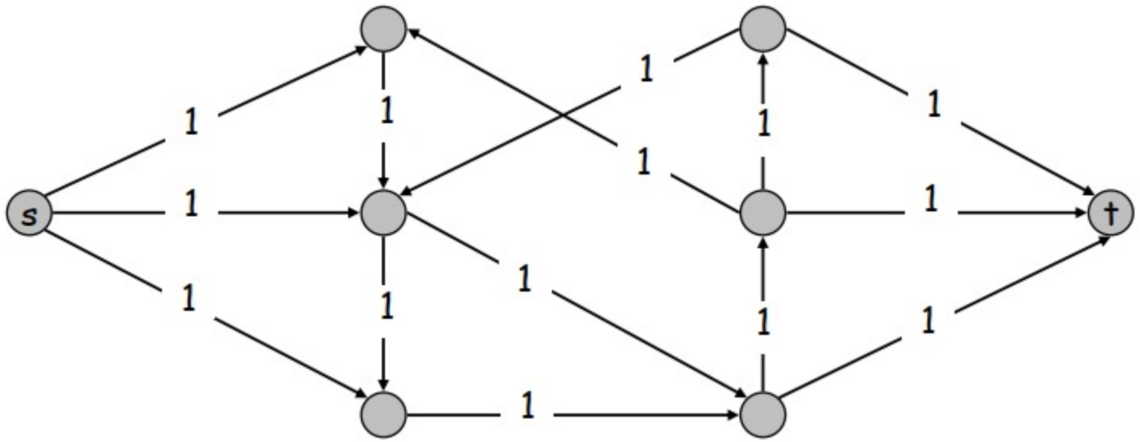


Edge Disjoint Paths

- **Max flow formulation:** assign unit capacity to every edge



- **Theorem.** Max number edge-disjoint s-t paths equals to max flow value

Edge Disjoint Paths

- **Theorem.** Max number edge-disjoint s-t paths equals max flow value
- **Pf. <=**
  - Suppose there are k edge-disjoint paths P_1, …, P_k
  - Set f(e) = 1 if e participates in some path P_i; else set f(e) = 0
  - Since paths are edge-disjoint, f is a flow of value k

Edge Disjoint Paths

- **Theorem.** Max number edge-disjoint s-t paths equals max flow value
- **Pf. >=**
- Suppose max flow value is k
- Integrality theorem → there exists 0-1 flow f of value k
- Consider edge (s, u) with f(s, u) = 1
  - By conservation, there exists an edge (u, v) with f(u, v) = 1
  - Continue until reach t, always choosing a new edge
  - So we get a s-t path
- Reduce the flow to 0 along the path, so we get a flow of value k-1
- Repeat the process for k times, then we get k (not necessarily simple) edge-disjoint paths

Can eliminate cycles to get simple paths if desired

Extensions to Max Flow

Circulation with Demands

- **Circulation with demands**
  - Directed graph G = (V, E)
  - Edge capacities c(e), e ∈ E
  - Node supply and demands d(v), v ∈ V

demand if d(v) > 0; supply if d(v) < 0; transshipment if d(v) = 0

- **Def.** A **circulation** is a function that satisfies:
  - For each e ∈ E: 0 <= f(e) <= c(e) (capacity)
  - For each v ∈ V: $\sum_{e \text{ in to } v} f(e) - \sum_{e \text{ out of } v} f(e) = d(v)$ (conservation)

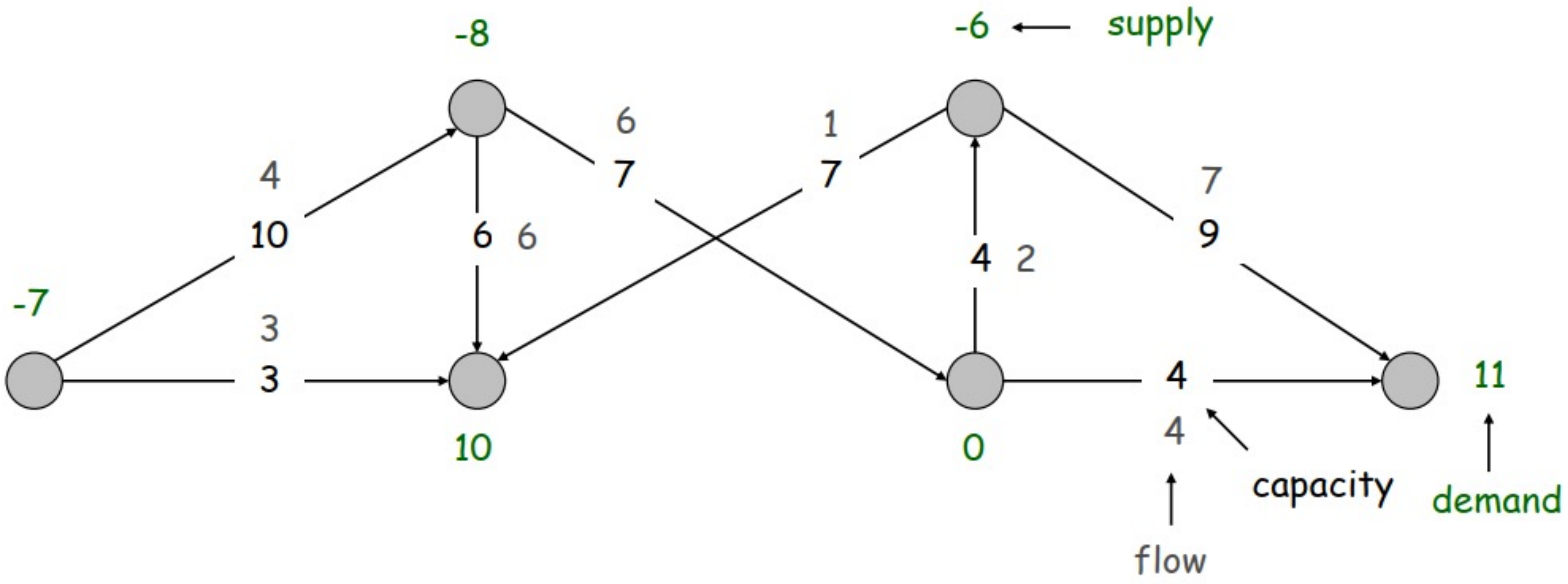
- **Circulation problem:** given (V, E, c, d), does there exist a circulation?

Circulation with Demands

- **Necessary condition:** sum of supplies = sum of demands

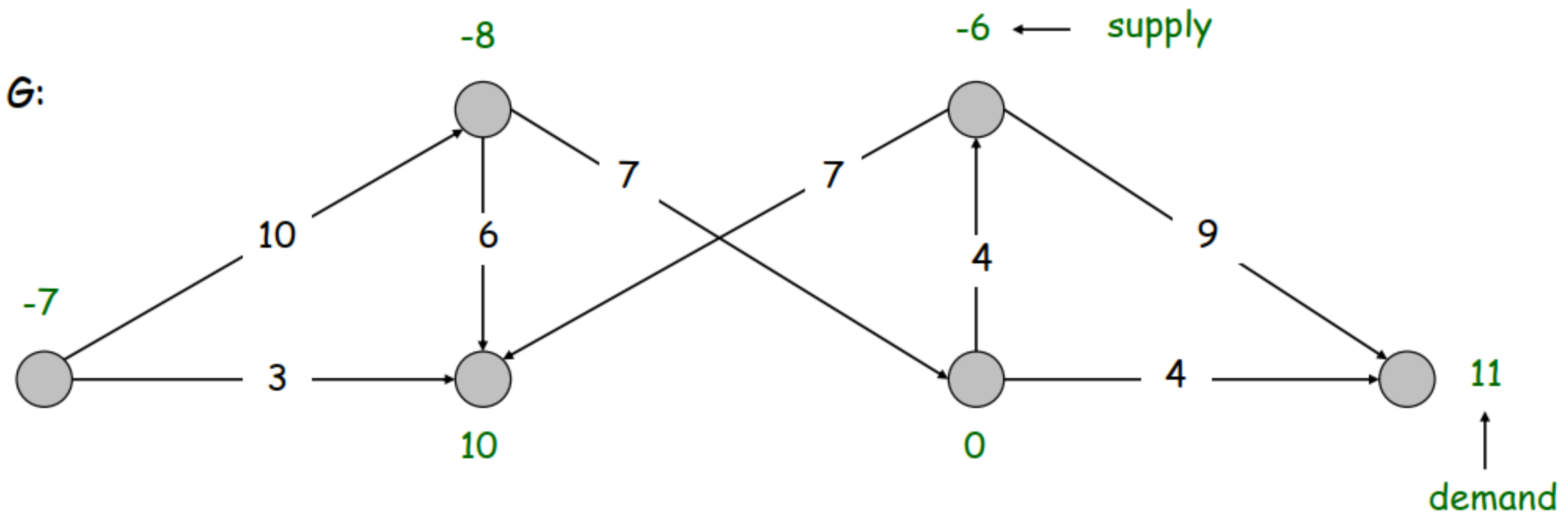
$$\sum_{v : d(v) > 0} d(v) = \sum_{v : d(v) < 0} -d(v) =: D$$

- **Pf.** Sum conservation constraints for every demand node v



Circulation with Demands

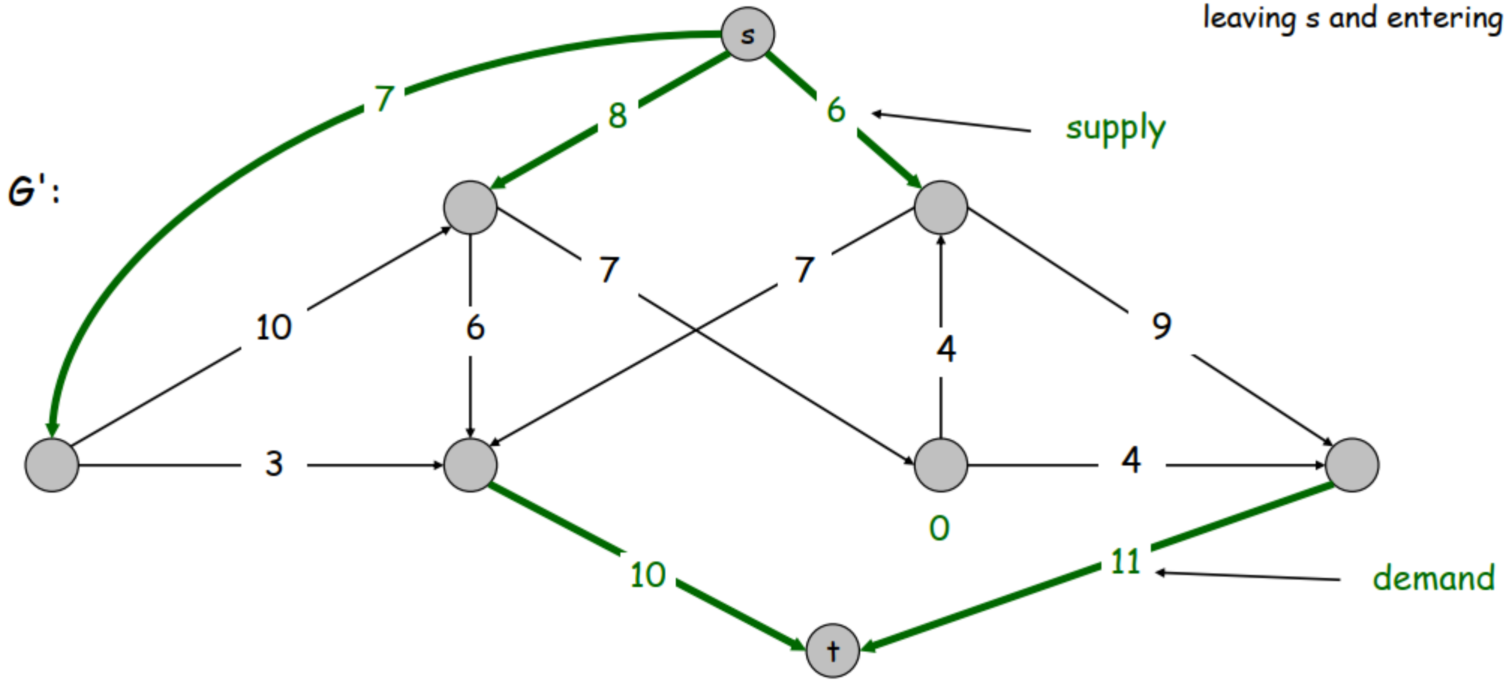
- Max flow formulation



Circulation with Demands

- **Max flow formulation**
  - Add new source s and sink t
  - For each v with d(v) < 0, add edge (s, v) with capacity −d(v)
  - For each v with d(v) > 0, add edge (v, t) with capacity d(v)
- **Claim.** G has circulation iff G′ has max flow of value D

saturates all edges
leaving s and entering t



Circulation with Demands

- **Integrality theorem.** If all capacities and demands are integers, and there exists a circulation, then there exists one that is integer-valued
- **Pf.** Follows from max flow formulation and integrality theorem for max flow

- **Characterization.** Given (V, E, c, d), there does **not** exist a circulation iff there exists a node partition (A, B) such that $\Sigma_{v \in B} d_v > cap(A, B)$

demand by nodes in B exceeds supply of nodes in B plus max capacity of edges going from A to B

- **Pf idea.** Look at max flow and min cut in G`

Circulation with Demands and Lower Bounds

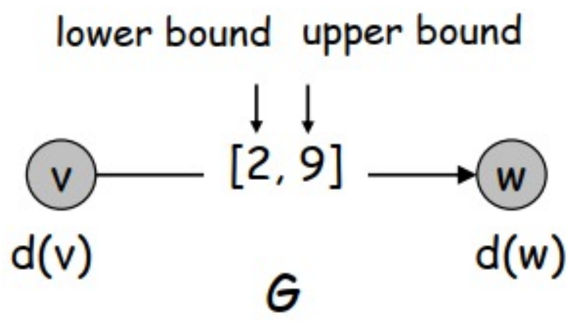
- **Feasible circulation**
  - Directed graph G = (V, E)
  - Edge capacities c(e) and **lower bounds** $\ell(e)$, $e \in E$
  - Node supply and demands d(v), $v \in V$

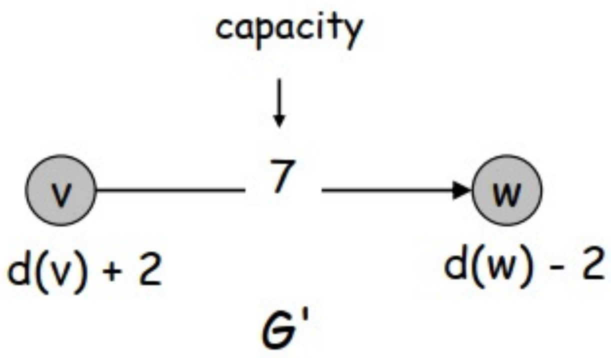
- **Def.** A **circulation** is a function that satisfies:
  - For each $e \in E$: $\ell(e) <= f(e) <= c(e)$ (capacity)
  - For each $v \in V$: $$\sum_{e \text{ in to } v} f(e) - \sum_{e \text{ out of } v} f(e) = d(v)$$ (conservation)

- **Circulation problem with lower bounds.** Given (V, E, ℓ, c, d), does there exist a circulation?

Circulation with Demands and Lower Bounds

- **Idea.** Model lower bounds with demands
  - Send $\ell(e)$ units of flow along edge e
  - Update demands of both endpoints





- **Theorem.** There exists a circulation in G iff there exists a circulation in G′. If all demands, capacities, and lower bounds in G are integers, then there is a circulation in G that is integer-valued

- **Pf sketch.** f(e) is a circulation in G iff $f'(e) = f(e) - \ell(e)$ is a circulation in G′

Survey Design

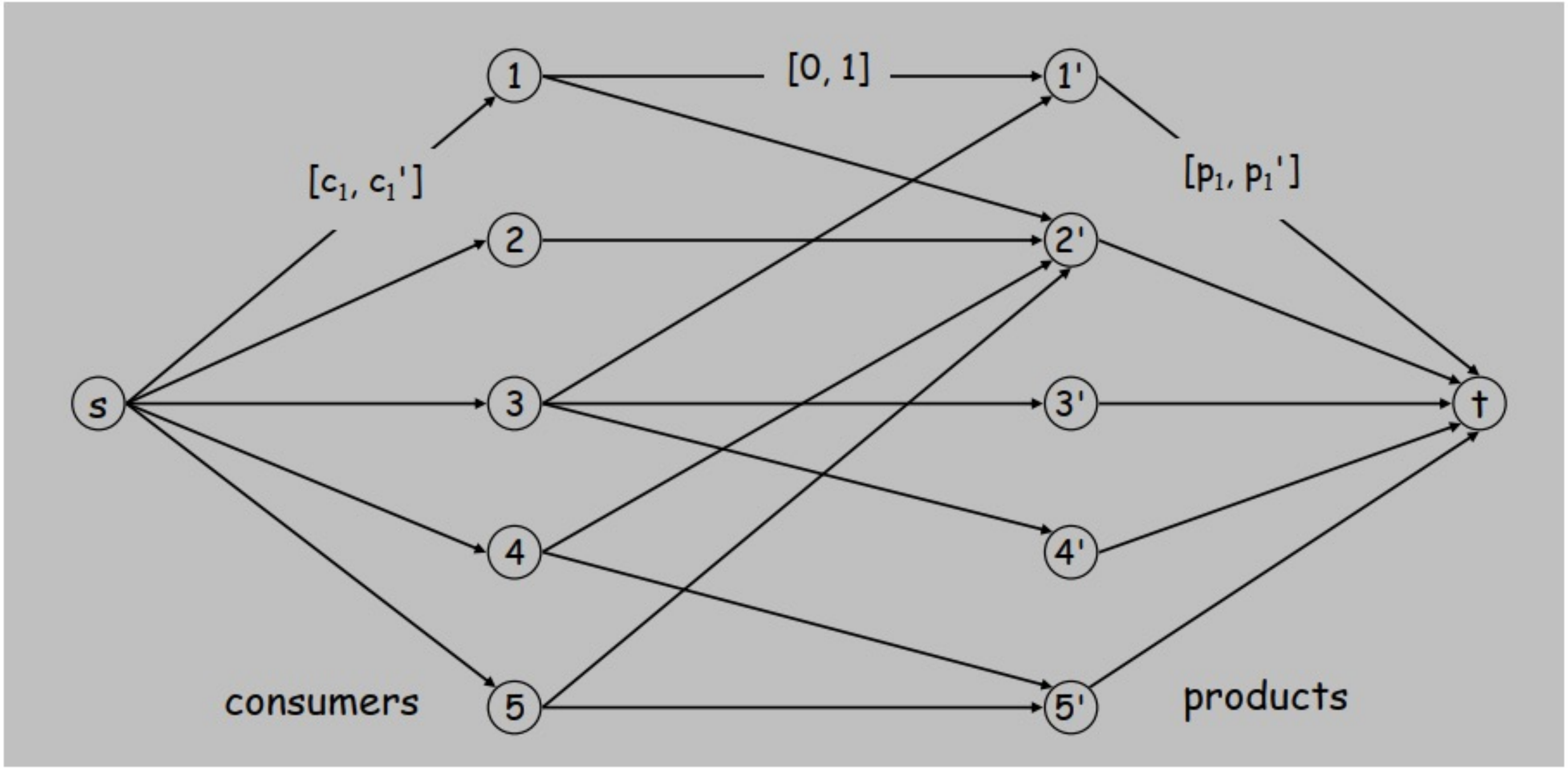
Survey Design

- **Survey design**
  - Design survey asking n_1 consumers about n_2 products
  - Can only survey consumer i about a product j if they own it
  - Ask consumer i between c_i and c'_i questions
  - Ask between p_j and p'_j consumers about product j

- **Goal.** Design a survey that meets these specs, if possible

Survey Design

- **Algorithm.** Formulate as a flow-network?
  - Include an edge (i, j) if customer own product i
  - Goal: find a flow that satisfies edge upper & lower bounds. How?



Survey Design

- **Algorithm.** Formulate as a circulation problem with lower bounds
  - Include an edge (i, j) if customer own product i
  - Integer circulation → feasible survey design

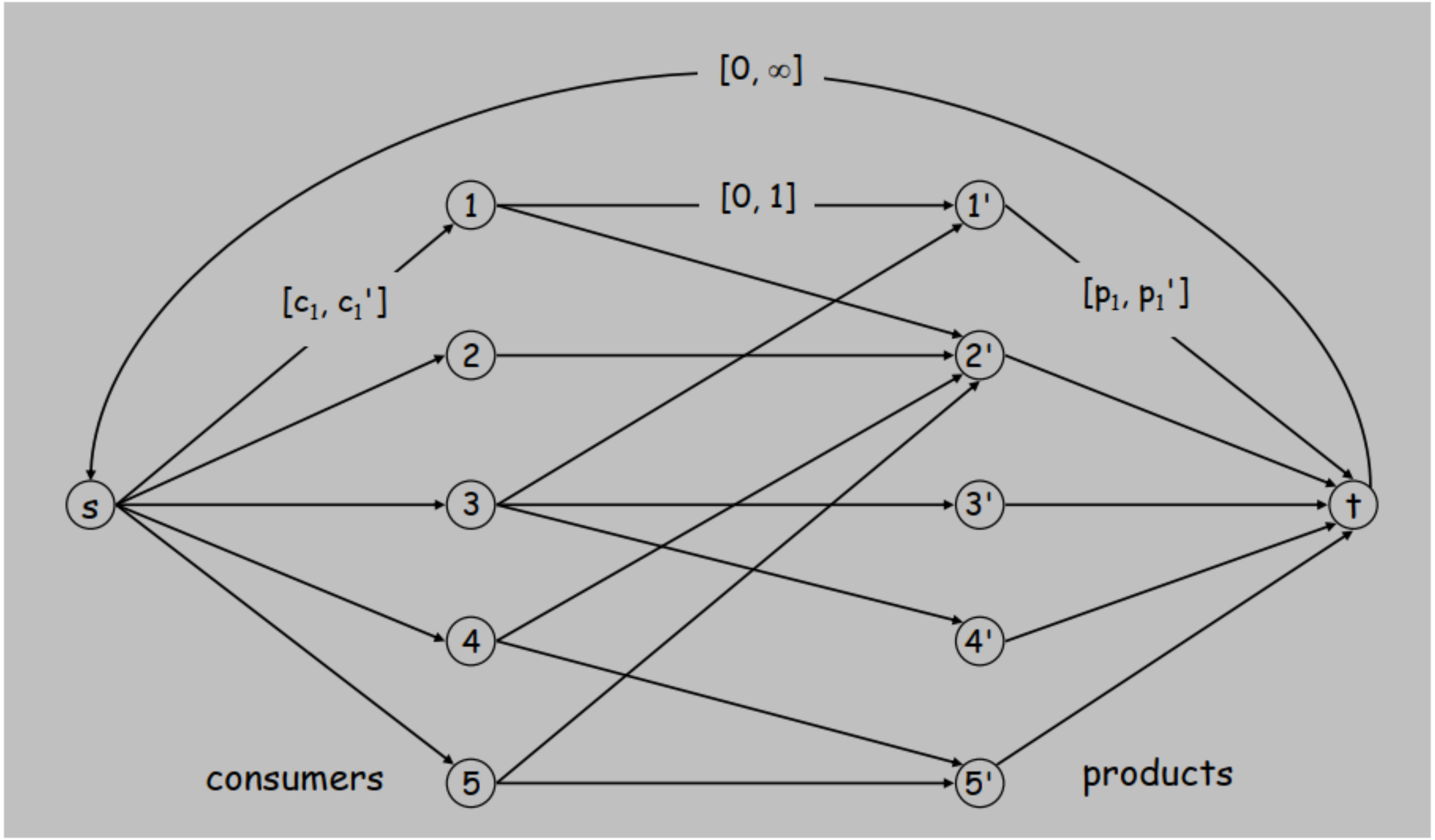


Image Segmentation

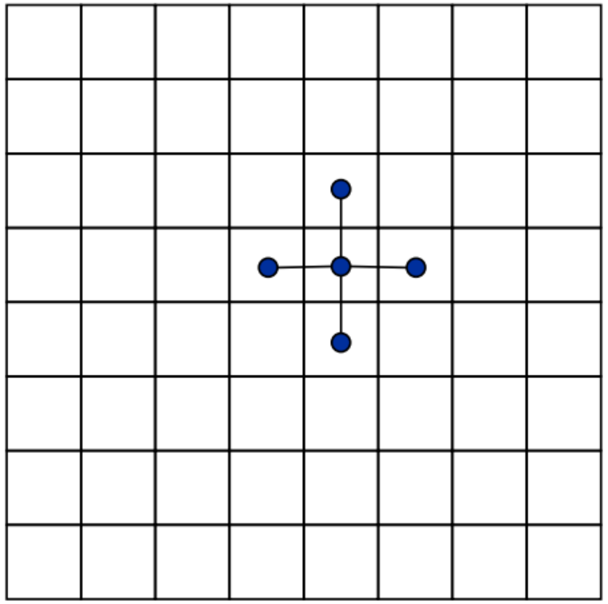
Image Segmentation

- **Image segmentation**
  - Central problem in image processing
  - Divide image into coherent regions

- **Ex:** Two people standing in front of complex background scene. Identify each person as a coherent object

Image Segmentation

- **Foreground / Background segmentation**
  - Label each pixel in picture as belonging to foreground or background
  - V = set of pixels, E = pairs of neighboring pixels
  - $a_i >= 0$ is likelihood pixel i in foreground
  - $b_i >= 0$ is likelihood pixel i in background
  - $p_{ij} >= 0$ is separation penalty for labeling one of i and j as foreground, and the other as background



- **Goals**
  - Accuracy: if $a_i > b_i$ in isolation, prefer to label i in foreground
  - Smoothness: if many neighbors of i are labeled foreground, we should be inclined to label i as foreground
  - Find partition (A, B) that maximizes:

$$\underbrace{\sum_{i \in A} a_i}_{\text{foreground}} + \underbrace{\sum_{j \in B} b_j}_{\text{background}} - \sum_{\substack{(i,j) \in E \\ |A \cap \{i,j\}| = 1}} p_{ij}$$

Image Segmentation

- **Formulate as min cut problem**
  - Maximization
  - No source or sink
  - Undirected graph

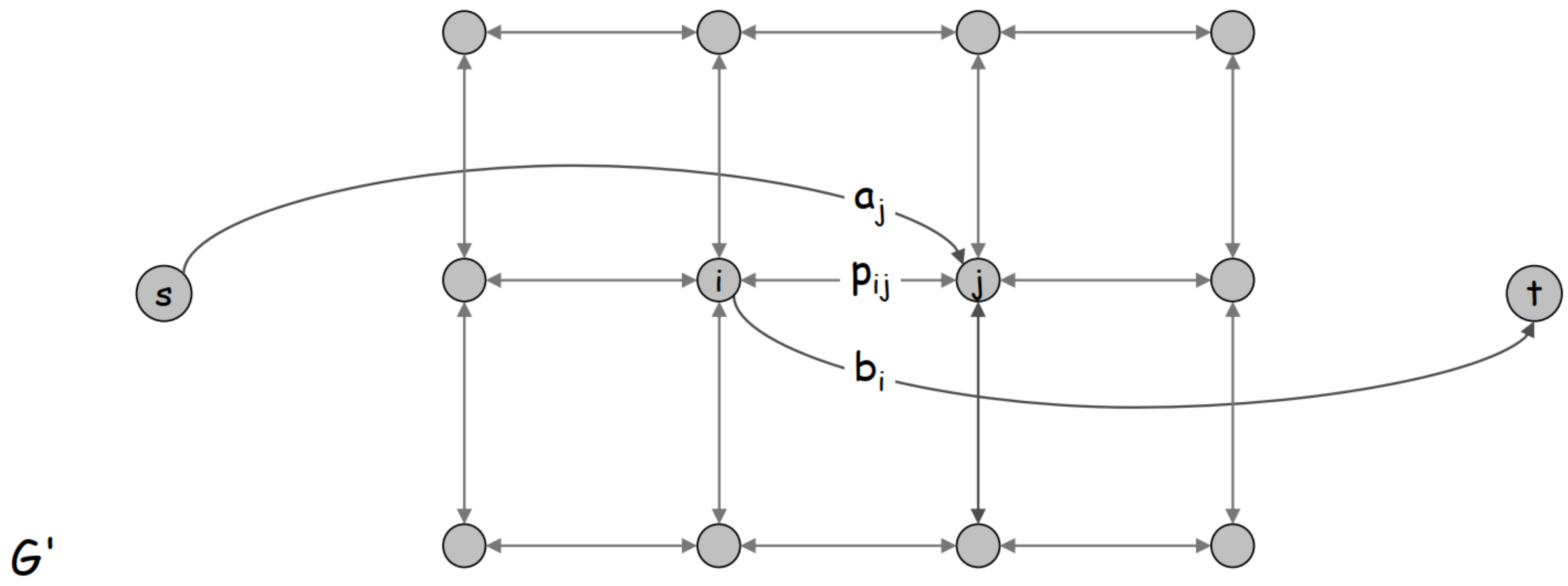
- **Turn into minimization problem**
  - Maximizing $$\sum_{i \in A} a_i + \sum_{j \in B} b_j - \sum_{\substack{(i,j) \in E \\ |A \cap \{i,j\}| = 1}} p_{ij}$$

is equivalent to minimizing $$\underbrace{\left(\sum_{i \in V} a_i + \sum_{j \in V} b_j\right)}_{\text{a constant}} - \sum_{i \in A} a_i - \sum_{j \in B} b_j + \sum_{\substack{(i,j) \in E \\ |A \cap \{i,j\}| = 1}} p_{ij}$$

or alternatively $$\sum_{j \in B} a_j + \sum_{i \in A} b_i + \sum_{\substack{(i,j) \in E \\ |A \cap \{i,j\}| = 1}} p_{ij}$$

Image Segmentation

- **Formulate as min cut problem**
  - G′ = (V′, E′)
  - Add source to correspond to foreground; add sink to correspond to background
  - Use two anti-parallel edges instead of undirected edge



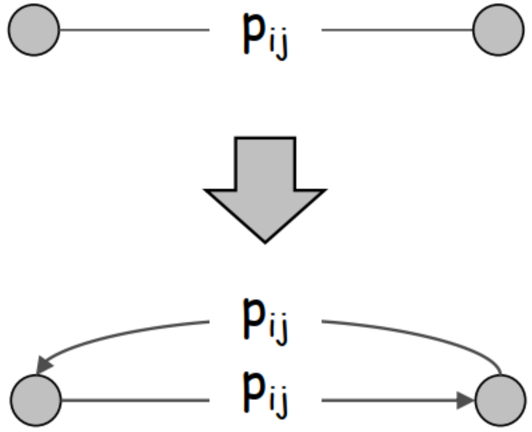


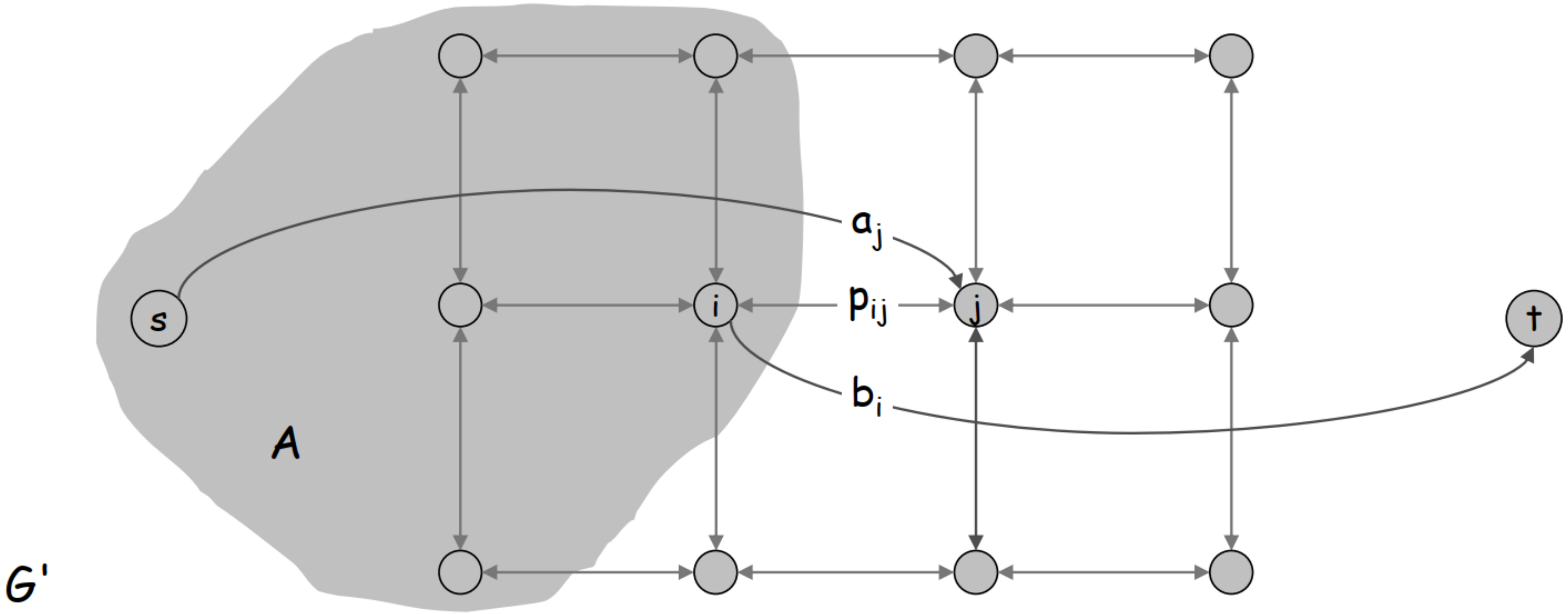
Image Segmentation

- **Consider min cut (A, B) in G'**
- A = foreground

$$cap(A,B) \;=\; \sum_{j\in B} a_j + \sum_{i\in A} b_i \;+\; \sum_{\substack{(i,j)\,\in\,E\\ i\in A,\; j\in B}} p_{ij}$$

if i and j on different sides, p_{ij} counted exactly once

- Precisely the quantity we want to minimize



Next Time:
Network Flow (Cont.)