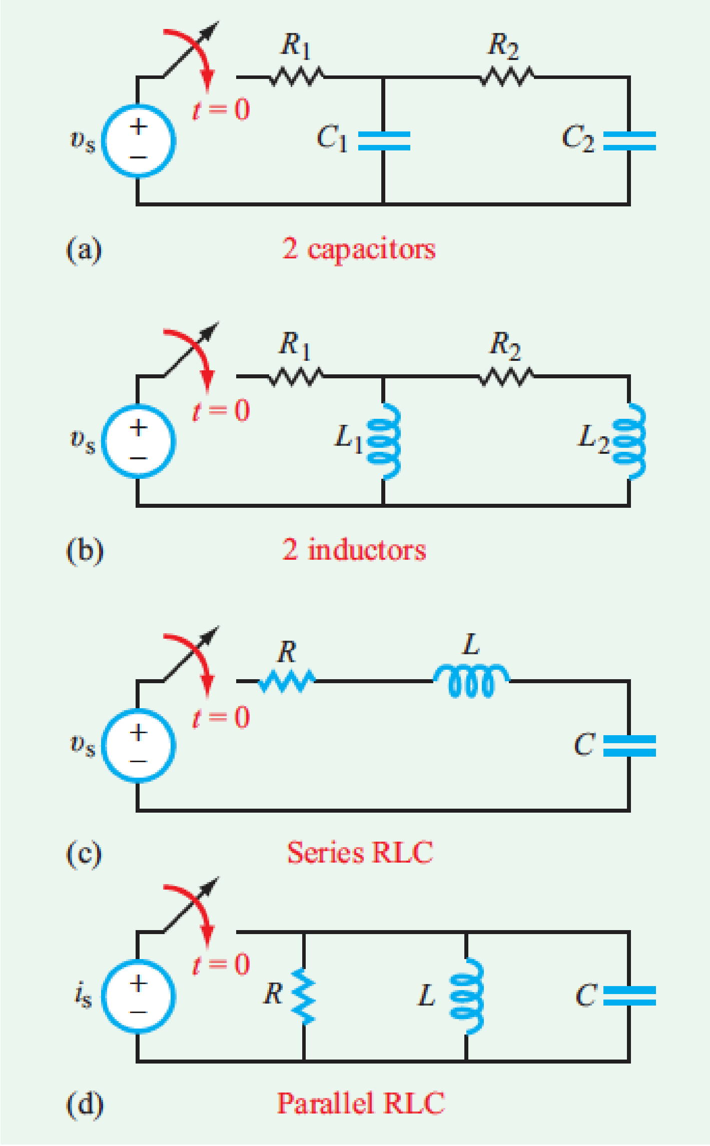
# Lecture 6

## - Second-Order Circuits

# Second-Order Circuits

- **Two** energy storage elements
- **Analysis**: basically determine voltage or current as a function of time
- A second-order circuit is characterized by a second-order differential equation.




[Source: Berkeley]

# Outline

- Natural Response Series/Parallel RLC circuit

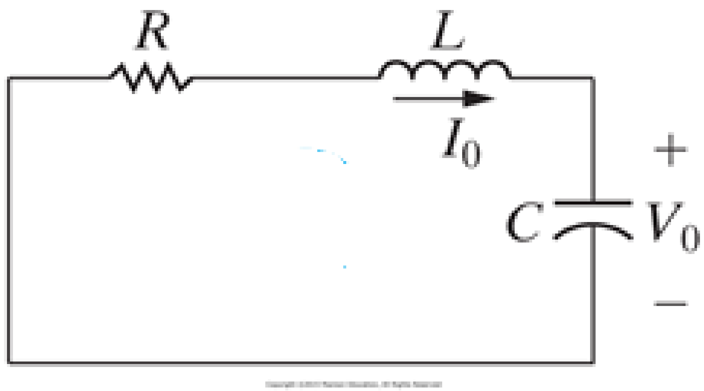
  *Source-free*

- Step Response of a Series/Parallel RLC Circuit

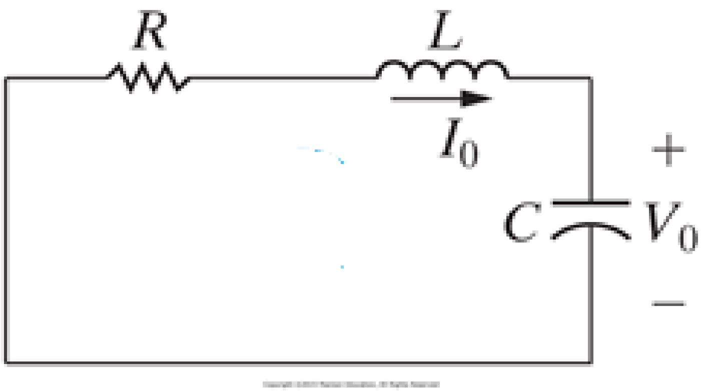
  *With Independent Source*

- General 2nd-order circuits

# Source-Free Series RLC Circuit



# Source-Free Series RLC Circuit




$$\frac{d^2v}{d^2t} + \frac{R}{L}\frac{dv}{dt} + \frac{1}{\mathrm{LC}}v = 0$$

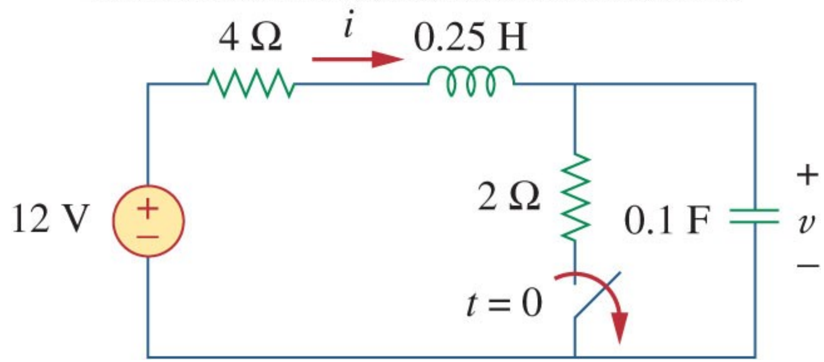
$$\boxed{s^2 + \frac{R}{L}s + \frac{1}{LC} = 0}$$

$$s = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

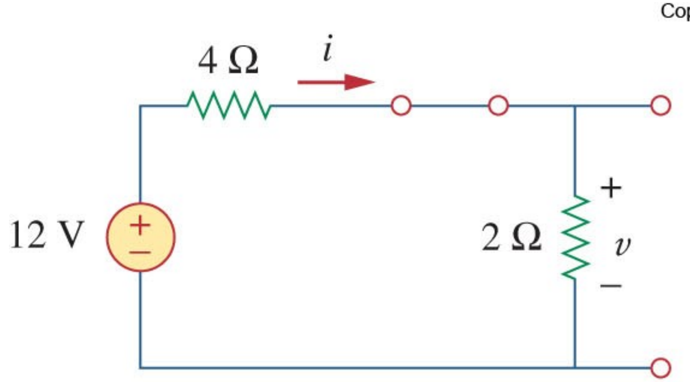
# Example

- The switch has been closed for a long time. It is open at $t = 0$. Find
  - $v(0^+), dv(0^+)/dt$
  - $i(0^+), di(0^+)/dt$

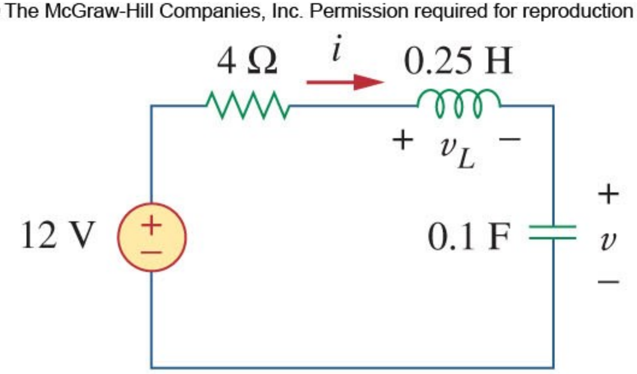







(a)

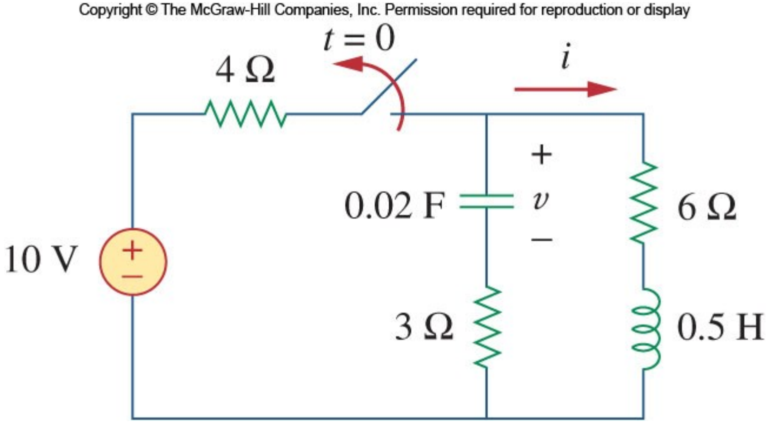



(b)

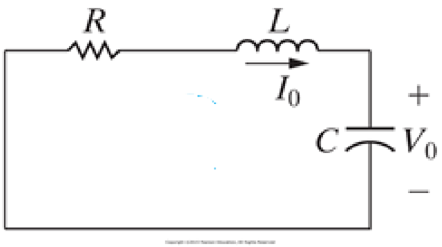
# Exercise

- Assume the circuit has reached steady state at $t = 0^{-}$. Find
  - $v(0^{+}), dv(0^{+})/dt$
  - $i(0^{+}), di(0^{+})/dt$





# Case 1: Overdamped ($\alpha>\omega_0$)




$$\frac{d^2v}{dt^2}+\frac{R}{L}\frac{dv}{dt}+\frac{1}{LC}v=0$$

$$s=-\frac{R}{2L}\pm\sqrt{\left(\frac{R}{2L}\right)^2-\frac{1}{LC}}$$

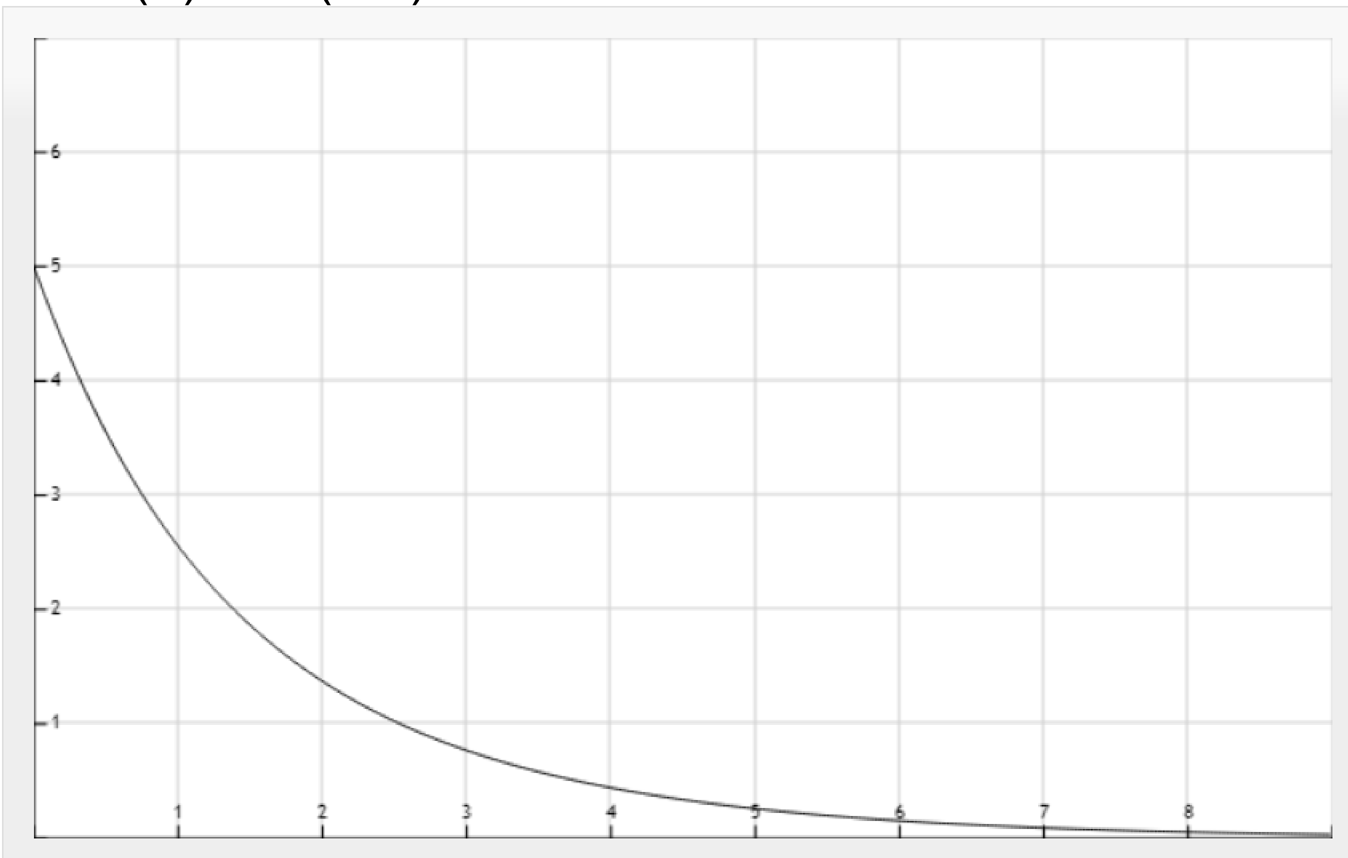
$$\alpha=\frac{R}{2L}\quad \omega_0=\frac{1}{\sqrt{LC}}$$

$$s_1=-\alpha+\sqrt{\alpha^2-\omega_o^2}\qquad s_2=-\alpha-\sqrt{\alpha^2-\omega_o^2}$$

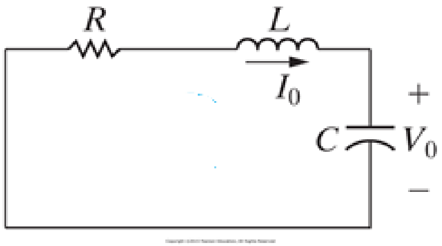
$$v(t)=A_1e^{s_1t}+A_2e^{s_2t}$$

# An example

$V_c =$ 2e^(-t)+3e^(-t/2)



# Case 2: Critically Damped ($\alpha=\omega_0$)




$$\frac{d^2v}{dt^2}+\frac{R}{L}\frac{dv}{dt}+\frac{1}{LC}v=0 \qquad s=-\frac{R}{2L}\pm\sqrt{\left(\frac{R}{2L}\right)^2-\frac{1}{LC}}$$

$$\alpha=\frac{R}{2L} \qquad \omega_0=\frac{1}{\sqrt{LC}}$$

$$s_1=-\alpha+\sqrt{\alpha^2-\omega_o^2} \qquad s_2=-\alpha-\sqrt{\alpha^2-\omega_o^2}$$

$$s_1=s_2=-\alpha=-\frac{R}{2L}$$

# Case 2: Critically Damped ($\alpha=\omega_0$)

$$v(t) = (A_1 t + A_2)e^{-\alpha t}$$

# Case 3: Underdamped ($\alpha<\omega_0$)

$$\frac{d^2v}{dt^2}+\frac{R}{L}\frac{dv}{dt}+\frac{1}{LC}v=0$$

$$\alpha=\frac{R}{2L}\qquad \omega_0=\frac{1}{\sqrt{LC}}$$

$$\boxed{s=-\alpha\pm\sqrt{\alpha^2-\omega_0^2}}$$

$$s_1=-\alpha+\sqrt{\alpha^2-\omega_0^2}=-\alpha+\sqrt{-(\omega_0^2-\alpha^2)}=-\alpha+j\omega_d$$

$$s_2=-\alpha-\sqrt{\alpha^2-\omega_0^2}=-\alpha-\sqrt{-(\omega_0^2-\alpha^2)}=-\alpha-j\omega_d$$

where $j=\sqrt{-1}$ and $\omega_d=\sqrt{\omega_0^2-\alpha^2}$.

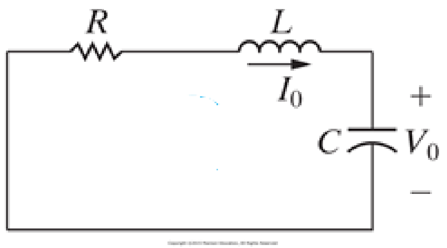
- $\omega_0$ is often called the resonant frequency;
- $\omega_d$ is called the damping frequency.

The natural response

$$v(t)=A_1e^{s_1t}+A_2e^{s_2t}$$

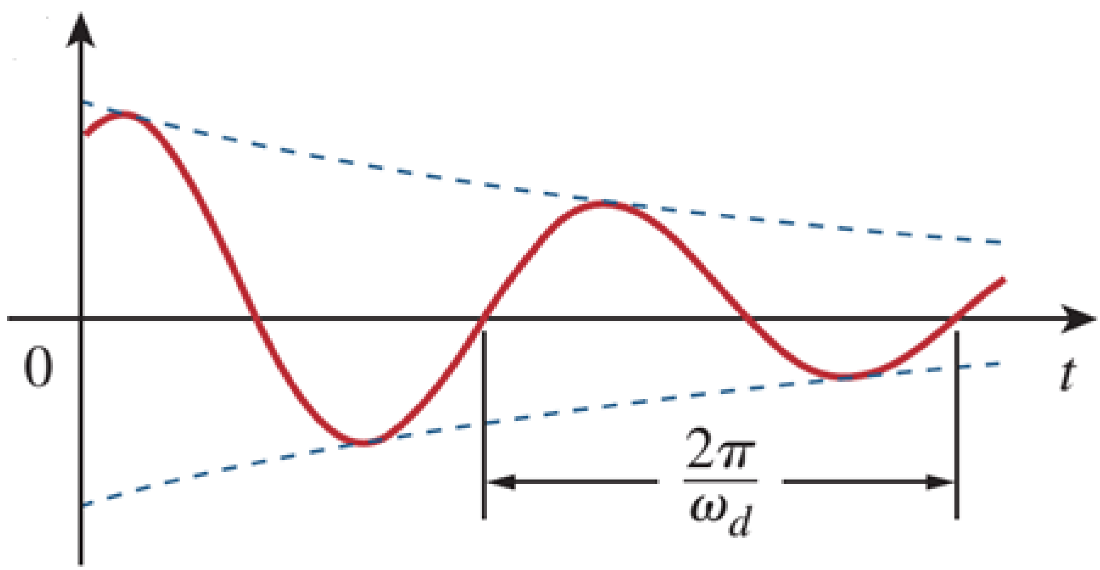
becomes

$$v(t)=e^{-\alpha t}(B_1\cos\omega_d t+B_2\sin\omega_d t)$$

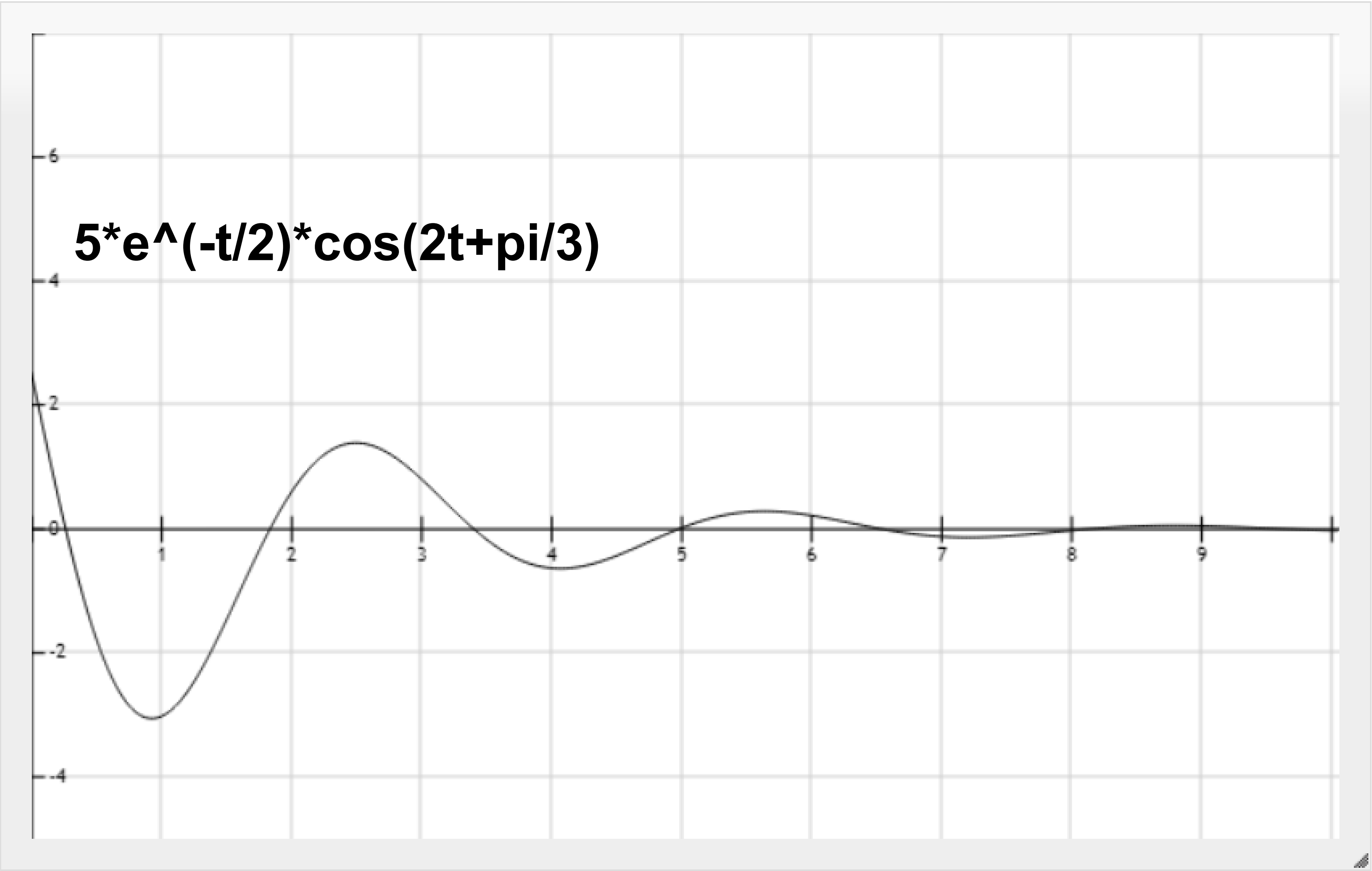


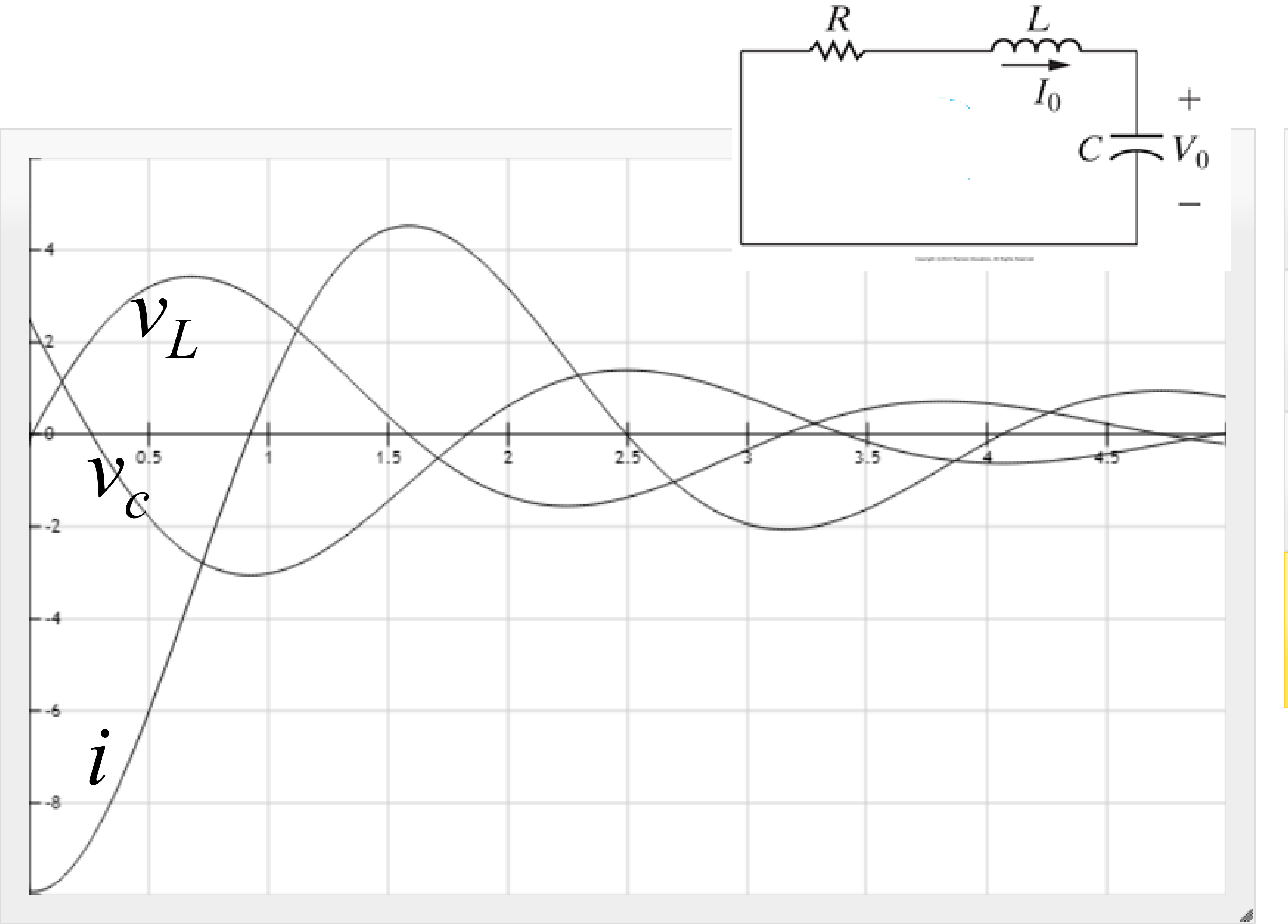

$$v(t) = e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t)$$

- Exponential $e^{-\alpha t}$ * Sine/Cosine term
  - Exponentially damped, time constant = $1/\alpha$
  - Oscillatory, period $T = \frac{2\pi}{\omega_d}$



# Example




R
L
$I_0$
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3
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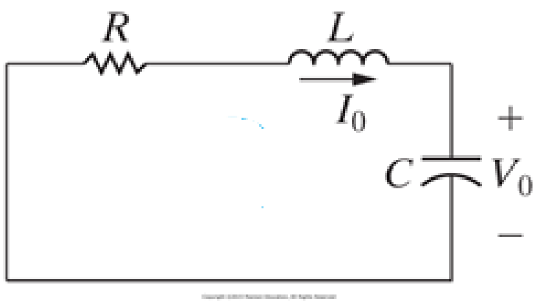
# Properties of Series RLC Network - *v(t)*




- Behavior captured by damping
  - Gradual loss of the initial stored energy
  - $\alpha$ determines the rate of damping

$$\alpha = \frac{R}{2L} \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

- $\alpha > \omega_0$ , overdamped

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

- $\alpha = \omega_0$ , critically damped

$$v(t) = (A_1 t + A_2) e^{-\alpha t}$$

- $\alpha < \omega_0$, underdamped

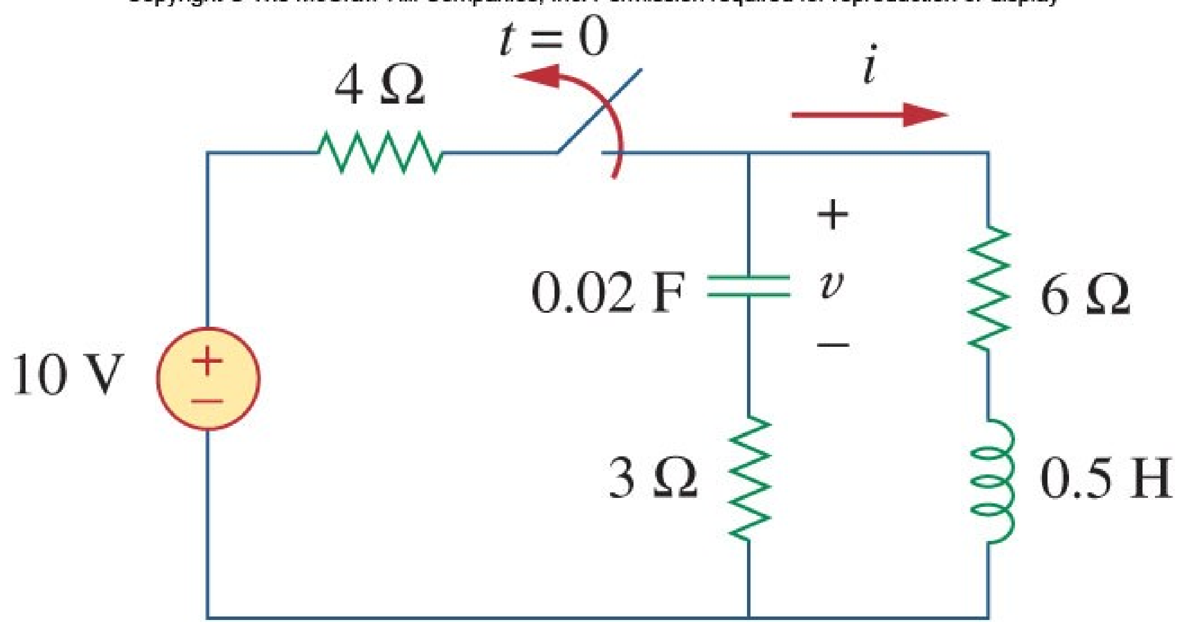
$$v(t) = e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t)$$

[Source: Berkeley]

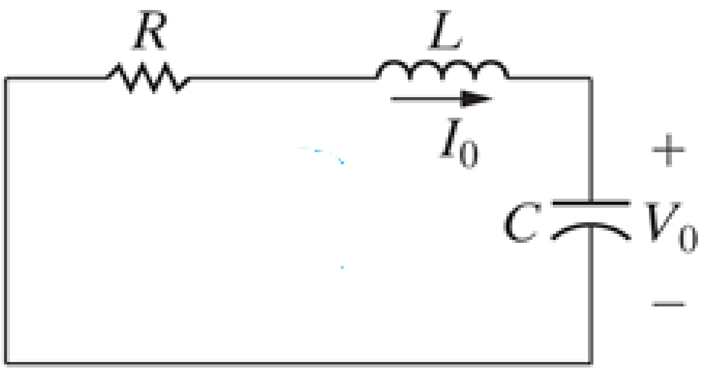
# Example

- Find $v(t)$ in the circuit below. Assume the circuit has reached steady state at $t = 0^{-}$.

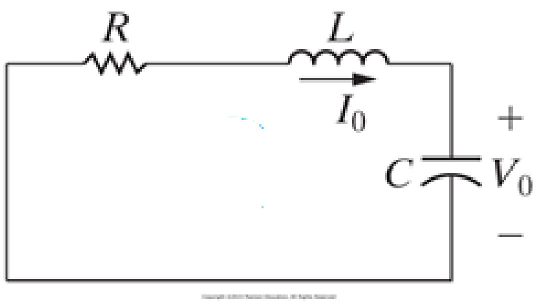




# Source-Free Series RLC Circuit



# Properties of Series RLC Network - *i(t)*

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$$\alpha = \frac{R}{2L} \quad \omega_0 = \frac{1}{\sqrt{LC}}$$

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$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

- $\alpha = \omega_0$ , critically damped

$$i(t) = (A_1 t + A_2) e^{-\alpha t}$$

- $\alpha < \omega_0$, underdamped

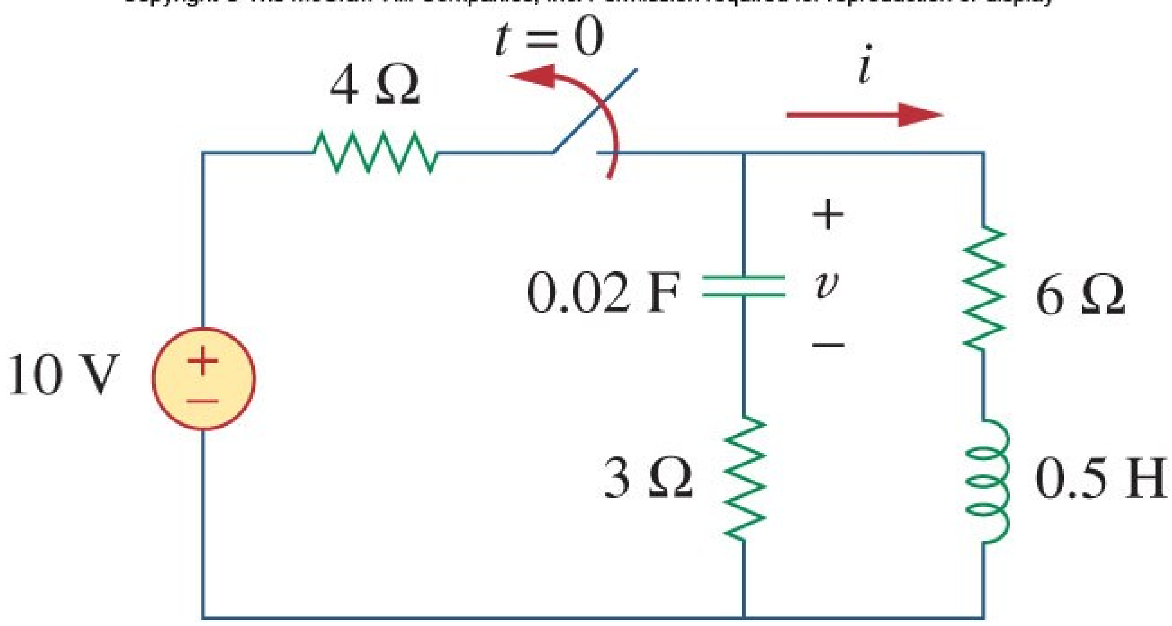
$$i(t) = e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t)$$

[Source: Berkeley]

# Example

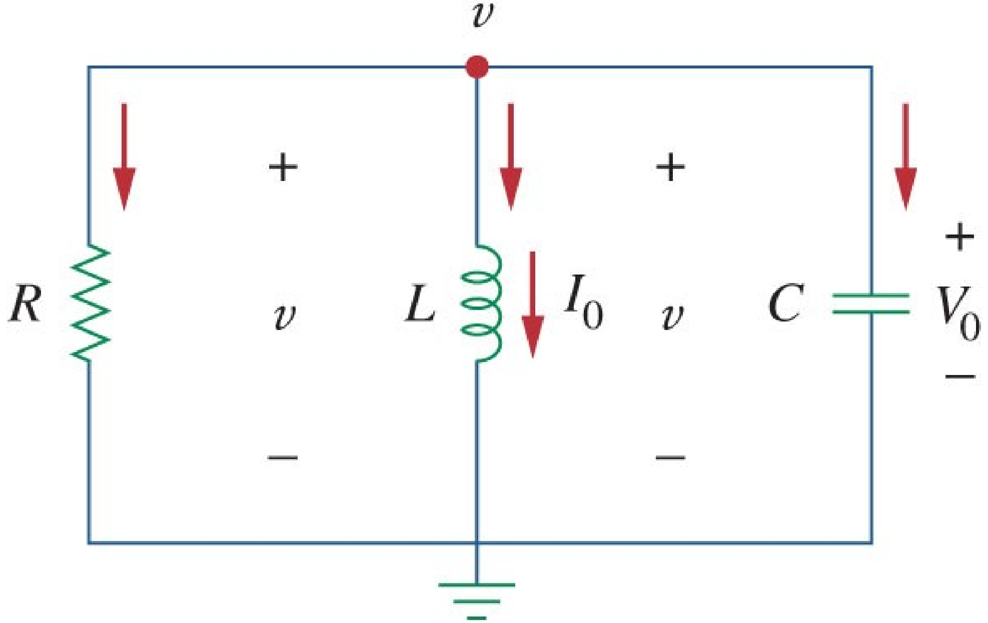
- Find $i(t)$ in the circuit below. Assume the circuit has reached steady state at $t = 0^-$.





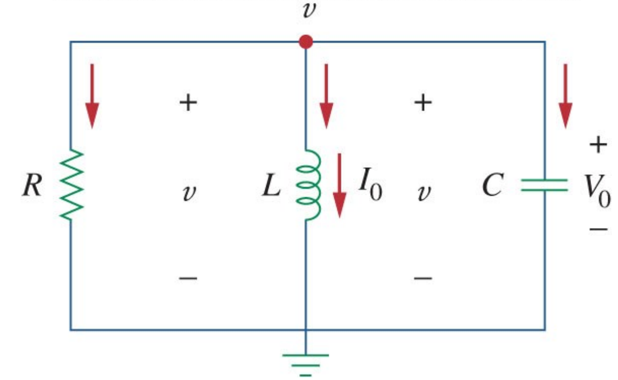
# Source-Free Parallel RLC Network





# Source-Free Parallel RLC Network - *v(t)*

$$\frac{d^2v}{dt^2}+\frac{1}{RC}\frac{dv}{dt}+\frac{1}{LC}v=0$$




- The <u>characteristic equation</u> is:

$$s^2+\frac{1}{RC}s+\frac{1}{LC}=0$$

From this, we can find the roots of the characteristic equation to be:

$$s_{1,2}=-\alpha\pm\sqrt{\alpha^2-\omega_0^2}$$

$$\alpha=\frac{1}{2RC}\qquad \omega_0=\frac{1}{\sqrt{LC}}$$

As in last time, there are three scenarios to consider.

# Three Damping Cases - *v(t)*

$$\frac{d^2v}{dt^2}+\frac{1}{RC}\frac{dv}{dt}+\frac{1}{LC}v=0$$

$$s_{1,2}=-\alpha\pm\sqrt{\alpha^2-\omega_0^2}$$

$$\alpha=\frac{1}{2RC}\qquad \omega_0=\frac{1}{\sqrt{LC}}$$

- For the overdamped case, the roots are real and negative,

$$v(t)=A_1e^{s_1t}+A_2e^{s_2t}$$

- For critically damped, the roots are real and equal

$$v(t)=(A_1t+A_2)e^{-\alpha t}$$

- In the underdamped case, the roots are complex

$$v(t)=e^{-\alpha t}(B_1\cos\omega_d t+B_2\sin\omega_d t)$$

# Three Damping Cases - *i(t)*

$$s_{1,2} = -\alpha \pm \sqrt{\alpha^2 - \omega_0^2}$$

$$\alpha = \frac{1}{2RC} \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

- For the overdamped case, the roots are real and negative,

$$i(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

- For critically damped, the roots are real and equal

$$i(t) = (A_1 t + A_2)e^{-\alpha t}$$

- In the underdamped case, the roots are complex

$$i(t) = e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t)$$

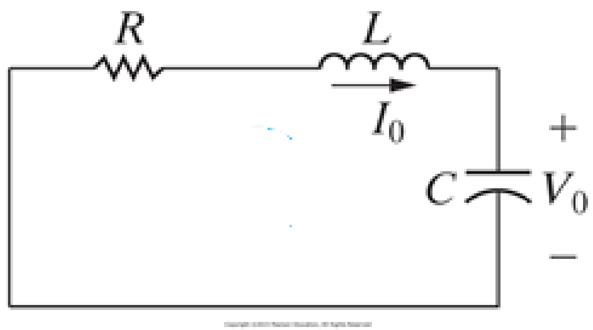
# Series vs. Parallel (Source-Free RLC Circuit)

- Series $\alpha = \frac{R}{2L} \quad \omega_0 = \frac{1}{\sqrt{LC}}$

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

$$v(t) = (A_1 t + A_2) e^{-\alpha t}$$

$$v(t) = e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t)$$




- Parallel $\alpha = \frac{1}{2RC} \quad \omega_0 = \frac{1}{\sqrt{LC}}$

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

$$v(t) = (A_1 t + A_2) e^{-\alpha t}$$

$$v(t) = e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t)$$