CS243: Introduction to Algorithmic Game Theory

VCG-based Mechanism (Dengji ZHAO)

SIST, ShanghaiTech University, China

Recap: VCG Mechanisms

Definition 9.16 A mechanism $(f, p_1, \ldots, p_n)$ is called a Vickrey–Clarke–Groves (VCG) mechanism if

- $f(v_1, \ldots, v_n) \in \operatorname{argmax}_{a \in A} \sum_i v_i(a)$; that is, f maximizes the social welfare, and
- for some functions $h_1, \ldots, h_n$, where $h_i : V_{-i} \to \Re$ (i.e., h_i does not depend on v_i), we have that for all $v_1 \in V_1, \ldots, v_n \in V_n$: $p_i(v_1, \ldots, v_n) = h_i(v_{-i}) - \sum_{j \neq i} v_j(f(v_1, \ldots, v_n))$.

Complexity of VCG Mechanisms

Question

Considering a combinatorial auction with n buyers and m items. What is the time complexity of computing the optimal allocation that maximizes $\sum_i v_i(a)$?

Question

Since the computation of the optimal allocation is infeasible, what if the optimal outcome is replaced by the results of a sub-optimal algorithm?

Complexity of VCG Mechanisms

Question

Considering a combinatorial auction with n buyers and m items. What is the time complexity of computing the optimal allocation that maximizes $\sum_i v_i(a)$?

The lower bound is $O(n \cdot 2^m)$, which is exponential. Actually, it is known to be NP-complete!

Question

Since the computation of the optimal allocation is infeasible, what if the optimal outcome is replaced by the results of a sub-optimal algorithm?

VCG-based Mechanisms

Definition

We call $m = (k, p_1, \ldots, p_n)$ a VCG-based mechanism, where

- k is an algorithm that maps type profile to a feasible allocation.
- for all i, calculate the payment $p_i(v) = \sum_{j \neq i} v_i(k(v)) + h_i(v_{-i})$.

Example: Non-optimal Vickery Auction

Consider the second-price auction where the allocation function is replaced by choosing the second highest agent. The mechanism will now give the object to the agent with the second highest declaration for the price of the third highest agent.

Question

Is the above mechanism truthful?

Truthful VCG-based Mechanism

Question

When will the VCG-based Mechanism be truthful?

Truthful VCG-based Mechanism

Definition

Let $k(v)$ be an algorithm that maps type profile into allocations. Let $V = \prod_{i=1}^{n} V_i$ be the space of all possible types and $V' \subseteq V$ be a subspace of V. Let $\mathcal{O}$ denote the range of k at V', i.e., $\mathcal{O} = \{k(v) \mid v \in V'\}$. We say that k is maximal in its range at V' if for all type $v \in V'$, $k(v)$ maximizes the total welfare over $\mathcal{O}$ (i.e., $\mathcal{SW}(k(v)) \in \arg\max_{o \in \mathcal{O}} \mathcal{SW}(o)$). We say that k is maximal in its range if it is maximal in its range at V.

Truthful VCG-based Mechanism

Theorem

A VCG-based mechanism with a function k that is maximal in its range is truthful.

Theorem

If a VCG-based mechanism for the combinatorial auction problem is truthful, then its output algorithm k is maximal in its range at $\tilde{V}$. ($\tilde{V}$ denotes the space of all type profiles v such that for any two different allocations x and y, $\mathcal{SW}(x) \neq \mathcal{SW}(y)$.)

Polynomial Mechanism

Definition

A mechanism $(k, p_1, \ldots, p_n)$ is called polynomial time computable if both $k(v)$ and $p(v)$ run in polynomial time (in the size of v).

Limitations of Truthful VCG-based Mechanisms

Definition

A mechanism for combinatorial auctions is reasonable if whenever there exists an item j and an agent i such that

- for all set of items S, if $j \notin S$, then $v_i(S \cup \{j\}) > v_i(S)$, and,
- for all agent $i' \neq i$, $v_{i'}(S \cup \{j\}) = v_{i'}(S)$,

then j is allocated to agent i.

Intuitively, in situations where only one agent desires an item, that agent gets it.

Limitations of Truthful VCG-based Mechanisms

Theorem

Unless P=NP, any polynomial time truthful VCG-based mechanism for combinatorial auctions is not reasonable. (Noam Nisan and Amir Ronen, 2007)

That is to say a reasonable mechanism for combinatorial auction is either

- truthful but the computation is intractable, or,
- can be computed in polynomial time but is not truthful.

Examples of VCG-based Mechanisms

Consider two allocation functions in a combinatorial auction, function f_1 allocates all items as a bundle to the bidder who bids the highest value, function f_2 allocates each item to the bidder who gives the highest bid for the specific item and each bidder can only get one item.

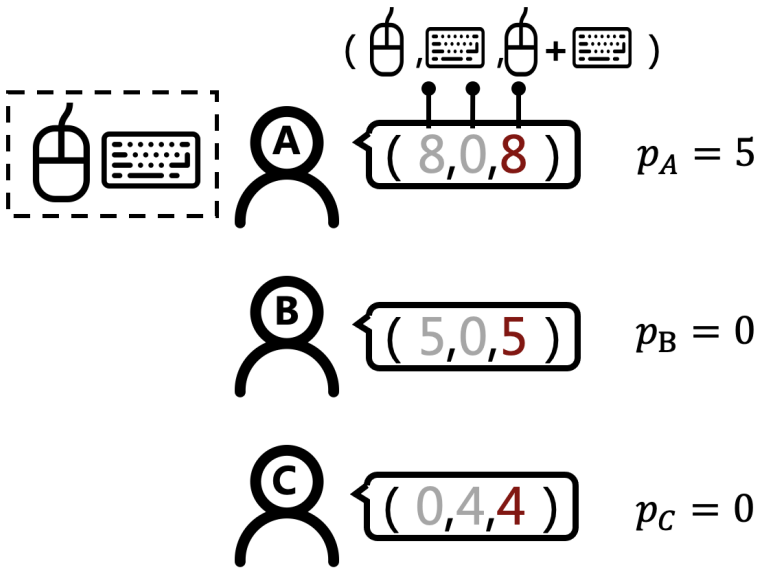
Question 1

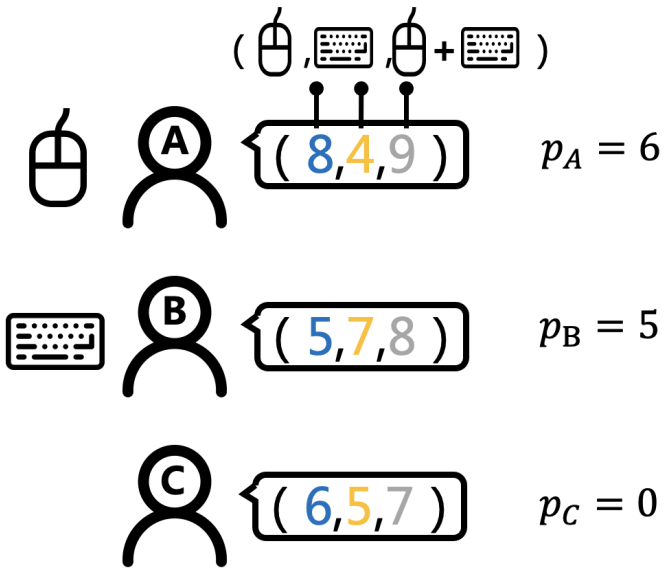
Is the VCG-based mechanism with function f_1 reasonable?

Question 2

Is the VCG-based mechanism with function f_2 polynomial time computable?

Examples of VCG-based Mechanisms





Question 1

Is the VCG-based mechanism with function f_1 reasonable?

Question 2

Is the VCG-based mechanism with function f_2 polynomial time computable?

Advanced Reading

- *AGT Chapter 2*
- *Computationally Feasible VCG Mechanisms* by Noam Nisan and Amir Ronen (JAIR 2007)