Signal and System
Homework 8

1. (15 pts) Given the following discrete time signal

$$x[n] = -(\frac{1}{3})^n \, u[-n-1]$$

$$x[n] = (\frac{1}{2})^n \{u[n] - u[n-5]\}$$

$$x[n] = 11(\frac{1}{3})^n \cos\left[\frac{\pi}{3}n + \frac{\pi}{4}\right] u[n]$$

Use the definition of the Z-transform to find their Z-transform expression and give the ROC (5 points each)

Solution:

(1)

$$X(z) = -\sum_{n=-\infty}^{\infty} \left(\frac{1}{3z}\right)^n \, u[-n-1]$$

$$X(z) = -\sum_{n=-\infty}^{-1} \left(\frac{1}{3z}\right)^n$$

$$X(z) = -\sum_{m=1}^{\infty} (3z)^m$$

$$X(z) = 1 - \sum_{m=0}^{\infty} (3z)^m$$

$$X(z) = 1 - \frac{1}{1-3z} = \frac{z}{z-\frac{1}{3}} \text{ with ROC } |z| < \frac{1}{3}$$

(2)

$$X(z) = \sum_{n=0}^{4} \left(\frac{1}{2z}\right)^n = \frac{1-\left(\frac{1}{2z}\right)^4}{1-\frac{1}{2z}} = \frac{(z)^4 - \left(\frac{1}{2}\right)^4}{(z)^4 - \frac{1}{2}(z)^3}$$

The signal is finite duration, it covers whole z plane except $z = 0$ and $z = \frac{1}{2}$. In this case, the ROC contains $z = \frac{1}{2}$ since one zero cancels one pole at $z = \frac{1}{2}$. Therefore ROC:

$$|z| > 0$$

(3)

$$x[n] = 11(\frac{1}{3})^n(e^{j(\frac{\pi}{3}n+\frac{\pi}{4})} + e^{-j(\frac{\pi}{3}n+\frac{\pi}{4})})\frac{1}{2}u[n]$$

$$X(z) = \frac{11}{2}\sum_{n=0}^{\infty}(\frac{1}{3z})^n[e^{j(\frac{\pi}{3}n+\frac{\pi}{4})} + e^{-j(\frac{\pi}{3}n+\frac{\pi}{4})}]$$

$$= \frac{11}{2}[e^{j\frac{\pi}{4}}\sum_{n=0}^{\infty}\left(\frac{e^{j\frac{\pi}{3}}}{3z}\right)^n + e^{-j\frac{\pi}{4}}\sum_{n=0}^{\infty}\left(\frac{e^{-j\frac{\pi}{3}}}{3z}\right)^n]$$

$$= \frac{11}{2}(\frac{e^{j\frac{\pi}{4}}}{1-\frac{e^{j\frac{\pi}{3}}}{3z}} + \frac{e^{-j\frac{\pi}{4}}}{1-\frac{e^{-j\frac{\pi}{3}}}{3z}})$$

$$= \frac{11z}{2}\frac{\sqrt{2}z - \frac{2}{3}\cos(\frac{\pi}{12})}{(z-\frac{e^{j\frac{\pi}{3}}}{3})(z-\frac{e^{-j\frac{\pi}{3}}}{3})}$$

The ROC is $|z| > \frac{1}{3}$

2. (15 pts) The expression of Z-transform $X(z)$ of the discrete time signal $x[n]$ is shown below, please discuss all possibilities of region of convergence and find the corresponding $x[n]$. What will the region of convergence be like if the Fourier transform of the $x[n]$ converges.

$$X(z) = \frac{8 - 13z^{-1}}{2 - 5z^{-1} + 2z^{-2}}$$

Solution:

$X(z)$ could be rewritten as:

$$X(z) = \frac{3}{1 - \frac{1}{2z}} + \frac{1}{1 - \frac{2}{z}}$$

It is obvious to determine two poles: $z = 2$ (corresponding: $\frac{1}{1-\frac{2}{z}}$) and $z = \frac{1}{2}$ (corresponding: $\frac{3}{1-\frac{1}{2z}}$)

There are three cases for inverse Z-transform, using the linearity, to analyze each sector separately:

1. ROC: $|z| < \frac{1}{2}$

$$x[n] = -3(\frac{1}{2})^n u[-n-1] - (2)^n u[-n-1]$$

2. ROC: $2 > |z| > \frac{1}{2}$

$$x[n] = 3(\frac{1}{2})^n u[n] - (2)^n u[-n-1]$$

3. ROC: $|z| > 2$

$$x[n] = 3(\frac{1}{2})^n u[n] + (2)^n u[n]$$

If we want to make sure the FT of the signal is convergent, unit circle must be included in the region of convergence, the ROC should be in $2 > |z| > \frac{1}{2}$

3. (10 pts) Given the equation below, determine the Z transform or inverse Z transform using the properties of the Z transform

(a) $X(z) = \frac{\frac{2}{3}z^{-1}}{(1-\frac{1}{3}z^{-1})^2}$, $|z| > \frac{1}{3}$

(b) $\mathrm{y[n]} = \{((\frac{1}{2})^n \mathrm{u[n]}) * ((3)^n \mathrm{u[-n]})\}$

Note: Sign "∗" represents the convolution operator.

Solution:

(a) For signal $h[n] = (\frac{1}{3})^n u[n]$, its Z transformation is $H(z) = \frac{1}{1-\frac{1}{3}z^{-1}}$ where ROC: $|z| > \frac{1}{3}$

Using Z transformation properties, Z transformation of the signal $nh[n]$ is identical to the differentiation of $H(z)$ by z multiplies $-z$, which is $-z\frac{d}{dz}[H(z)] = \frac{\frac{1}{3}z^{-1}}{(1-\frac{1}{3}z^{-1})^2}$.

Therefore, for Z domain signal $X(z) = \frac{\frac{2}{3}z^{-1}}{(1-\frac{1}{3}z^{-1})^2}$ with ROC $|z| > \frac{1}{3}$, its inverse Z transform is $x[n] = 2n(\frac{1}{3})^n u[n]$

(b) $\mathrm{y[n]} = \{((\frac{1}{2})^n \mathrm{u[n]}) * ((3)^n \mathrm{u[-n]})\}$

denote $\mathrm{a[n]} = (\frac{1}{2})^n \mathrm{u[n]}$ and $\mathrm{b[n]} = (3)^n \mathrm{u[-n]}$

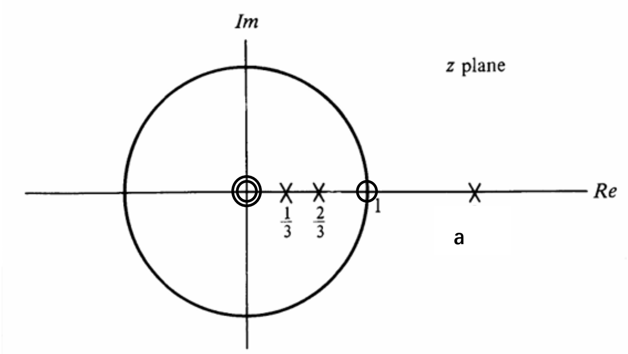
for $a[n]$, its Z transform is $A(z) = \frac{z}{z-\frac{1}{2}}$ with ROC $|z| > \frac{1}{2}$

for $b[n]$, it could be written as $b[n] = (\frac{1}{3})^{-n} \mathrm{u[-n]}$

for signal in time domain $(\frac{1}{3})^n \mathrm{u[n]}$, its Z transform is $\frac{z}{z-\frac{1}{3}}$ with ROC $|z| > \frac{1}{3}$ using time reversal, its Z transform would be $B(z) = \frac{z^{-1}}{z^{-1}-\frac{1}{3}}$ with ROC $|z| < 3$.

For $y[n]$, using convolution property, its Z transform would be $Y(z) = \frac{z}{(z-\frac{1}{2})(1-\frac{1}{3}z)}$ with ROC $\frac{1}{2} < |z| < 3$

4. (20 pts) Consider a LTI system whose zero-pole plot is shown below and the system function $H(z)$ is rational, with the impulse response function defined as *h[n]*:



a. Determine whether the following statements are true or false and explain your judgement.
   a) If the system is causal, then it is stable.
   b) $H(e^{j\omega})$ could not be zero for any ω
   c) $h[n]$ is finite duration
   d) If the system is stable, then $F\{\left(\frac{5}{4}\right)^n h[n]\}$ converges

b. Find out H(z) if we know:
   a) $H(z)$ has exactly three poles and three zeros.
   b) $H(z)$'s region of convergence is $|z| > 2$
   c) $h[0] = 3$

Solution:

a. True and False judgement
   a) False. If the system is causal, then the ROC is $|z| > a > 1$, which does not include unit circle. Therefore, it's not stable.
   b) False. There is a zero lies on the unit circle, therefore, for $H(e^{j\omega})$, it's z non-zero value for any given ω.
   c) False. If the system is causal, then the ROC is $|z| > a > 1$, it does not include the whole plane except zero and infinity. Therefore, it couldn't be finite duration.
   d) True. If the system is stable, then determine from the zero-pole plot, the ROC is $a > |z| > \frac{2}{3}$. For $F\{\left(\frac{5}{4}\right)^n h[n]\}$, it could be seen as Z-transform where $z = \frac{4}{5}e^{j\omega}$ which lies in the ROC, therefore it converges.

b. Since $H(z)$ has three poles, it is the three poles plotted on the figure and a is 2 since the impulse response is right-sided and ROC is $|z| > 2$ Consider there is one zero at 1 and an 2nd order zero at 0. Therefore the $H(z)$ could be written as:

$$H(z) = \frac{Az^2(z-1)}{(z-2)(z-\frac{1}{3})(z-\frac{2}{3})}$$

Where A is a constant multiplies an unknown number of zeros. Consider $h[0] = 3$, using initial value theorem,

$$h[0] = \lim_{z\to\infty} H(z) = 3$$

It's obvious that there are 3 zeros on the z plane and A is merely a constant with value 3. Therefore,

$$H(z) = \frac{3z^2(z-1)}{(z-2)(z-\frac{1}{3})(z-\frac{2}{3})}$$

5. (30 pts) For a causal LTI system characterized by the difference equation:

$$y[n-2] - 7y[n-1] + 10y[n] = 10x[n]$$

a). Find the transfer function of the system.

b). Sketch the zero-pole plot of this system and find its region of convergence. Is this system stable? Why?

c). Find the unit impulse response of the system.

d). Sketch the block diagram of the system in **parallel**, **cascade**, and **direct form.**

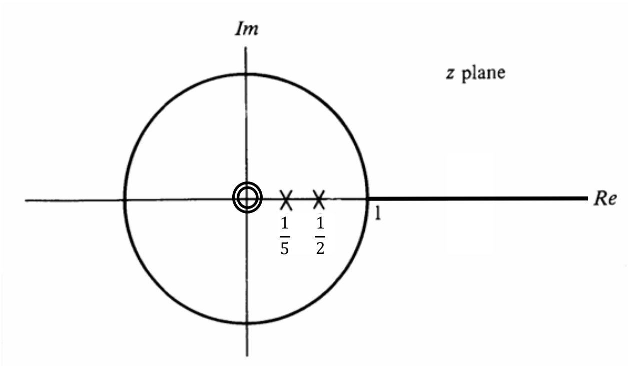
Solution:

a. Conducting Z transform to the system function

$$z^{-2}Y(z) - 7z^{-1}Y(z) + 10Y(z) = 10X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{z^2}{(z-\frac{1}{2})(z-\frac{1}{5})}$$

b. From the transfer function, the system has poles at $z = \frac{1}{2}$ and $z = \frac{1}{5}$ with a second order zero at $z = 0$. Since the system is causal, therefore the ROC is $|z| > \frac{1}{2}$. The system is stable since ROC includes the unit circle.



c. Since

$$H(z) = \frac{z^2}{(z-\frac{1}{2})(z-\frac{1}{5})}$$

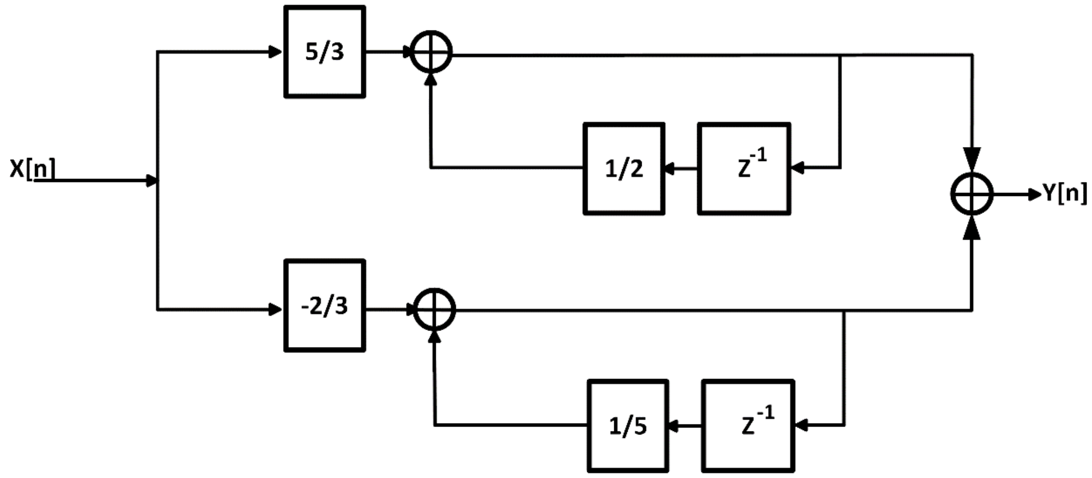
$$H(z) = \frac{5}{3(1-\frac{1}{2}z^{-1})} - \frac{2}{3(1-\frac{1}{5}z^{-1})}$$

Since ROC in determined, therefore,

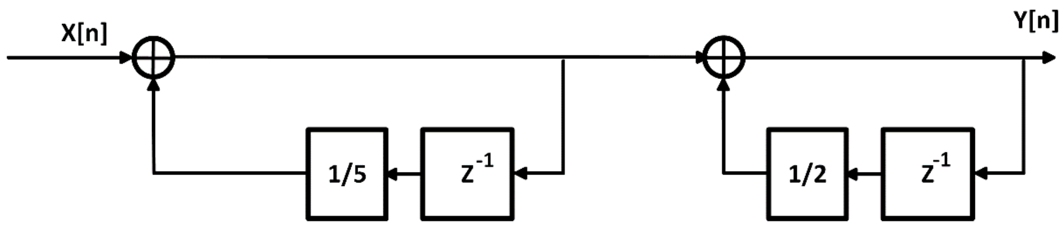
$$h[n] = \frac{5}{3}\left(\frac{1}{2}\right)^n u[n] - \frac{2}{3}\left(\frac{1}{5}\right)^n u[n]$$

d. Parallel, cascade, and direct block diagram

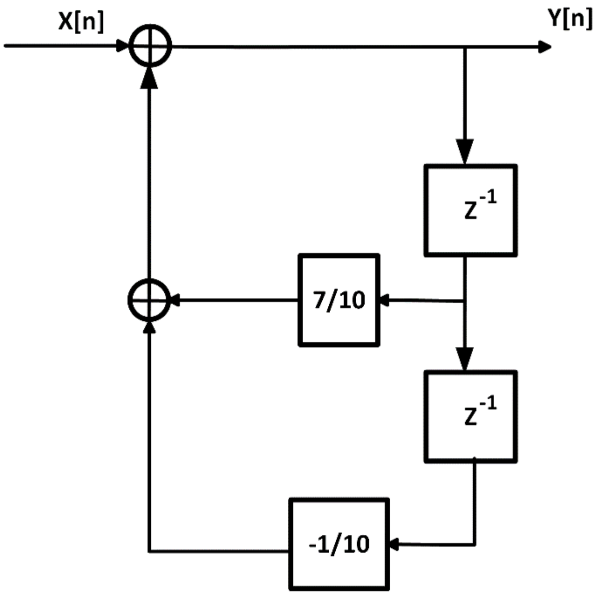
Parallel



Cascade



Direct



6. (10 pts) The system equation is characterized as:

$$y[n]-y[n-1]-2y[n-2]=x[n]+2x[n-2]$$

Given the initial condition of the system $y[-1]=\beta$ and $y[-2]=\gamma$,and the input signal $x[n]=\alpha u[n]$, find the zero-input response and zero-state response of the system.

Solution:

The characteristic equation could be written as:

$$Y(z)-\{z^{-1}Y(z)+y[-1]\}-2\{z^{-2}Y(z)+y[-1]z^{-1}+y[-2]\}=X(z)+2z^{-2}X(z)$$

It could be sorted as:

$$Y(z)=\frac{\{y[-1]+2y[-2]\}z^2+2y[-1]z}{z^2-z-2}+\frac{z^2+2}{z^2-z-2}X(z)$$

Take initial condition into the Y(z)

$$Y(z)=\frac{(\beta+2\gamma)z^2+2\beta z}{z^2-z-2}+\frac{z^2+2}{z^2-z-2}X(z)$$

For zero-input and zero-state response:

$$Y_{zi}(z)=\frac{(\beta+2\gamma)z^2+2\beta z}{z^2-z-2}=\frac{4(\gamma+\beta)z}{3(z-2)}+\frac{(2\gamma-\beta)z}{3(z+1)}$$

$$Y_{zs}(z)=\alpha\frac{z^2+2}{z^2-z-2}\frac{z}{z-1}=\alpha[\frac{2z}{z-2}+\frac{z}{2(z+1)}-\frac{3z}{2(z-1)}]$$

Conducting inverse Z transform:

Zero-input response:

$$y_{zi}[n]=\left\{\frac{\gamma+\beta}{3}2^{n+2}+\frac{2\gamma-\beta}{3}(-1)^n\right\}u[n]$$

Zero-state response:

$$y_{zs}[n]=\alpha\left\{2^{n+1}+\frac{1}{2}(-1)^n-\frac{3}{2}\right\}u[n]$$