Matrix Computations
Chapter 4: Eigenvalues, Eigenvectors, and Eigendecomposition

Section 4.3 Hermitian matrices and the Variational Characterizations of Eigenvalues

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Hermitian Matrices

Recall that

- A square matrix $\mathbf{A} \in \mathbb{C}^{n\times n}$ is said to be Hermitian if $a_{ij} = a_{ji}^*$ for all i, j with $i \neq j$, or equivalently, if $\mathbf{A}^H = \mathbf{A}$
- We denote the set of all $n \times n$ complex Hermitian matrices by $\mathbb{H}^n$
- By definition, a real symmetric matrix is also Hermitian, i.e., $\mathbb{S}^n \subset \mathbb{H}^n$
- When we say that a matrix is Hermitian, we often imply that the matrix is complex–a real Hermitian matrix is simply real symmetric

Eigenvalues and Eigenvectors of Hermitian Matrices

Property

The following properties hold for $\mathbf{A} \in \mathbb{H}^n$*:*

1. *The* n *eigenvalues of* $\mathbf{A}$ *are real*
2. *Suppose* $\{\lambda_1, \ldots, \lambda_k\}$ *is the set of all distinct eigenvalues of* $\mathbf{A}$*, and let* $\mathbf{v}_i$ *be* any *eigenvector associated with* λ_i*. Then* $\mathbf{v}_1, \ldots, \mathbf{v}_k$ *are orthogonal*
3. *There exists an orthonormal basis of* $\mathbb{C}^n$ *consisting of the eigenvectors of* $\mathbf{A}$

Corollary: For any $\mathbf{A} \in \mathbb{S}^n$, there exist n real orthonormal eigenvectors

Proof of the Property

Idea: Use invariant subspace

Given $\mathbf{A} \in \mathbb{C}^{n \times n}$, a subspace $\mathcal{S} \subseteq \mathbb{C}^n$ with

$$\mathbf{x} \in \mathcal{S} \Longrightarrow \mathbf{A}\mathbf{x} \in \mathcal{S}$$

is said to be an invariant subspace for $\mathbf{A}$

E.g., any eigenvector of $\mathbf{A}$ spans a 1-dimensional invariant subspace

E.g., any k eigenvectors of $\mathbf{A}$ spans an invariant subspace for $\mathbf{A}$

Fact: If $\mathcal{Z}$ is a nonzero invariant subspace for $\mathbf{A}$, then $\mathbf{A}$ has an eigenvector in $\mathcal{Z}$

- A Consequence of the Fundamental Theorem of Algebra

Proof of the Property (cont'd)

Proof of the Property (cont'd)

Eigendecomposition for Hermitian Matrices

Theorem

Every $\mathbf{A} \in \mathbb{H}^n$ *admits an eigendecomposition*

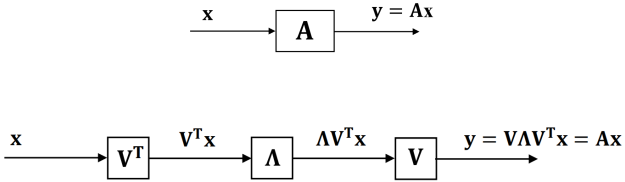
$$\mathbf{A} = \mathbf{V}\Lambda\mathbf{V}^H,$$

where $\mathbf{V} \in \mathbb{C}^{n\times n}$ *is unitary,* $\Lambda = \mathrm{Diag}(\lambda_1, \ldots, \lambda_n)$ *with* $\lambda_i \in \mathbb{R}$ *for all* i*.* *In addition, if* $\mathbf{A} \in \mathbb{S}^n$*,* $\mathbf{V}$ *can be taken as a real orthogonal matrix.*

- A special case of Schur decomposition
- No need of assuming distinct eigenvalues

Corollary: If $\mathbf{A} \in \mathbb{H}^n$, $\mu_i = \gamma_i$ for all i

Interpretation of Eigendecomposition in $\mathbb{S}^n$



1. $\mathbf{V}^T\mathbf{x}$: Let $\mathbf{x}$ resolve into $\mathbf{v}_1, \ldots, \mathbf{v}_n$
2. $\Lambda(\mathbf{V}^T\mathbf{x})$: Scale the ith coordinate of $(\mathbf{V}^T\mathbf{x})$ by λ_i
3. $\mathbf{V}(\Lambda\mathbf{V}^T\mathbf{x})$: Reconstitute $(\Lambda\mathbf{V}^T\mathbf{x})$ with basis $\mathbf{v}_1, \ldots, \mathbf{v}_n$

Courant-Fischer Min-Max Theorem

For $\mathbf{A} \in \mathbb{H}^{n \times n}$, let $\lambda_k(\mathbf{A})$ denote the kth largest eigenvalue of $\mathbf{A}$, i.e.,

$$\lambda_n(\mathbf{A}) \leq \cdots \leq \lambda_1(\mathbf{A})$$

Theorem

For any $\mathbf{A} \in \mathbb{H}^{n \times n}$ and $k = 1, \ldots, n$,

$$\begin{aligned} \lambda_k(\mathbf{A}) &= \max_{\substack{\mathcal{S} \subseteq \mathbb{C}^n: \\ \dim(\mathcal{S}) = k}} \ \min_{\substack{\mathbf{y} \in \mathcal{S}, \\ \mathbf{y} \neq \mathbf{0}}} \frac{\mathbf{y}^H \mathbf{A} \mathbf{y}}{\mathbf{y}^H \mathbf{y}} \\ &= \min_{\substack{\mathcal{S} \subseteq \mathbb{C}^n: \\ \dim(\mathcal{S}) = n-k+1}} \ \max_{\substack{\mathbf{y} \in \mathcal{S}, \\ \mathbf{y} \neq \mathbf{0}}} \frac{\mathbf{y}^H \mathbf{A} \mathbf{y}}{\mathbf{y}^H \mathbf{y}} \end{aligned}$$

$R_{\mathbf{A}}(\mathbf{y}) = \frac{\mathbf{y}^H \mathbf{A} \mathbf{y}}{\mathbf{y}^H \mathbf{y}}$, $\mathbf{y} \neq \mathbf{0}$ is called the Rayleigh–Ritz quotient

Courant-Fischer Min-Max Theorem (Cont'd)

- $R_{\mathbf{A}}(\mathbf{y})$ can be replaced with $\mathbf{y}^H\mathbf{A}\mathbf{y}$, $\|\mathbf{y}\|_2 = 1$
- If $\mathbf{y}$ is an eigenvector of $\mathbf{A}$, $R_{\mathbf{A}}(\mathbf{y})$ is its associated eigenvalue
- **Consequence** of theorem: $\lambda_n(\mathbf{A}) \leq R_{\mathbf{A}}(\mathbf{y}) \leq \lambda_1(\mathbf{A})$

Proof

Poincaré's Inequality: Let S be a subspace of $\mathbb{C}^n$ with $\dim(S) = k$. There exist unit vectors $\mathbf{x}, \mathbf{y} \in S$ s.t. $\mathbf{x}^H \mathbf{A} \mathbf{x} \leq \lambda_k(\mathbf{A})$ and $\mathbf{y}^H \mathbf{A} \mathbf{y} \geq \lambda_{n+1-k}(\mathbf{A})$.

Proof (cont'd)

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