Signals and Systems

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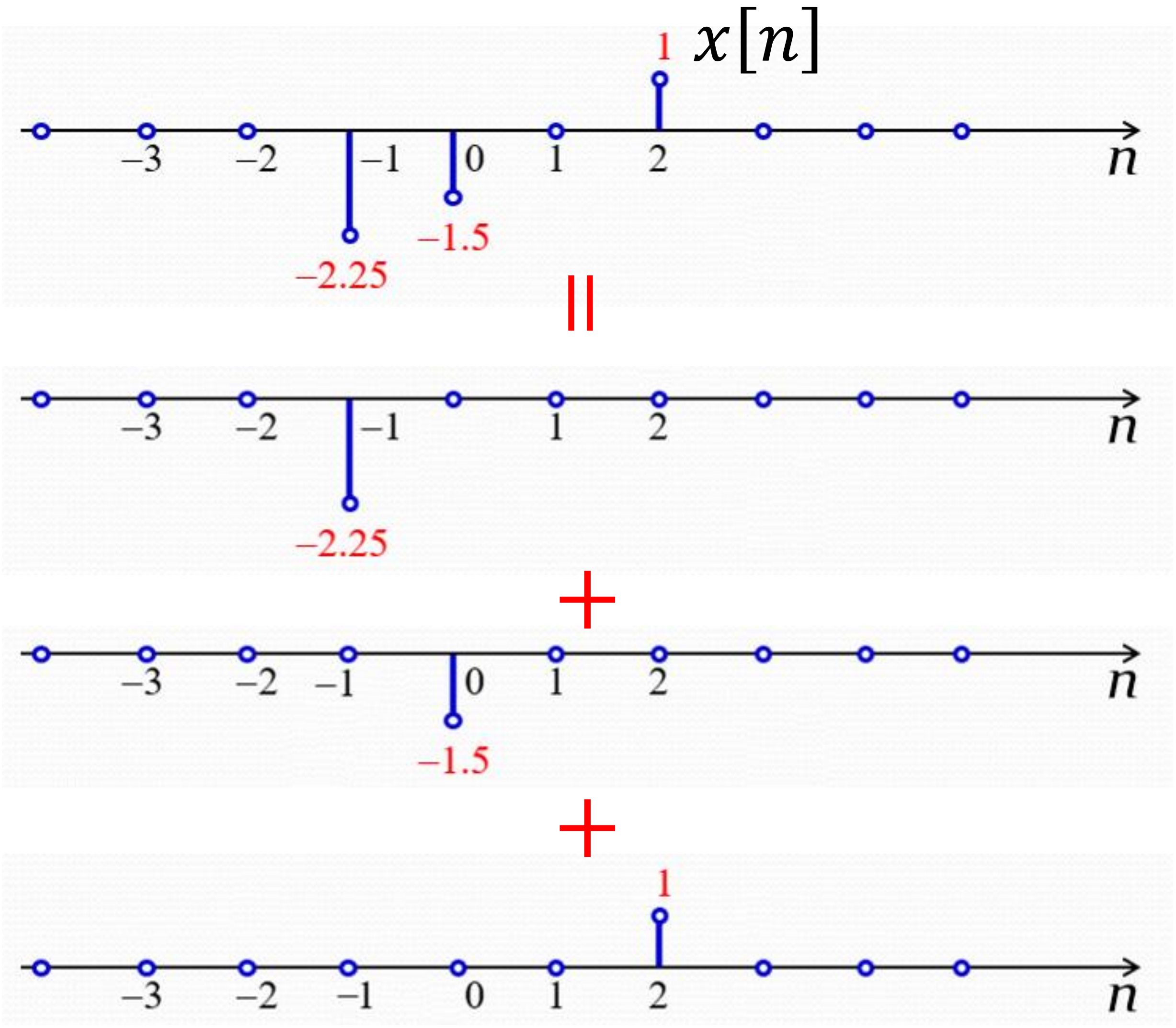
Chapter 2: Linear Time-Invariant Systems

- ☐ **Discrete-Time LTI Systems**
- ☐ **Continuous-Time LTI Systems**
- ☐ **Properties of LTI Systems**
- ☐ **Differential or Difference Equations**

Discrete-Time LTI Systems

Representation of Discrete-Time Signals in Terms of Impulse



$x[n]$

$\|$

$\leftrightarrow x_1[n] = -2.25 \times \delta[n+1]$

$+$

$\leftrightarrow x_2[n] = -1.5 \times \delta[n]$

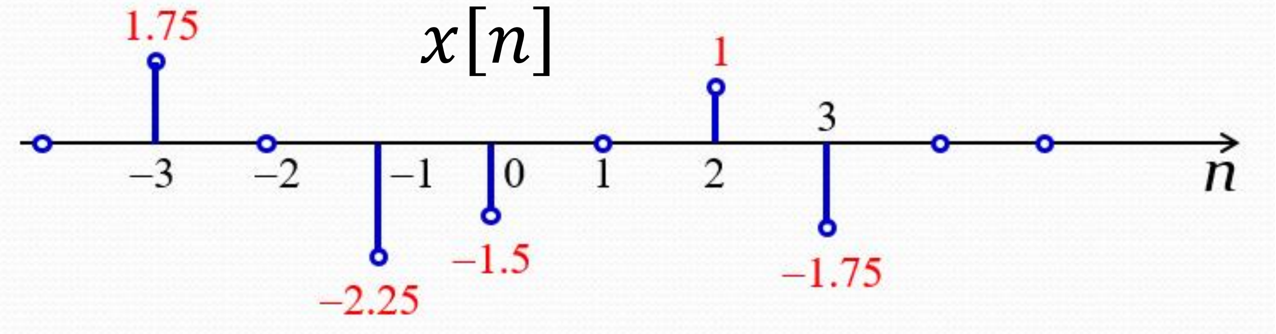
$+$

$\leftrightarrow x_3[n] = 1 \times \delta[n-2]$

Discrete-Time LTI Systems

Representation of Discrete-Time Signals in Terms of Impulse

- ❑ An arbitrary sequence can be represented as the weighted sum of shifted unit impulses



$$x[n]=1.75\delta[n+3]-2.25\delta[n+1]-1.5\delta[n]+\delta[n-2]-1.75\delta\left[n-3\right]$$

- ❑ A general form

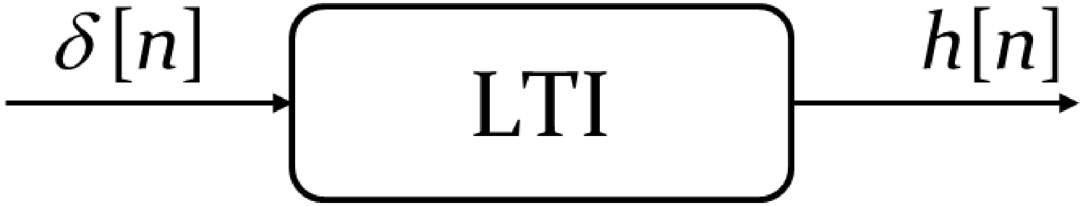
$$x[n]=\sum_{k=-\infty}^{\infty}x[k]\delta[n-k]$$

Sifting property of $\delta[n]$

Discrete-Time LTI Systems

Discrete-Time Unit Impulse Response and the Convolution-Sum

- The response of a system to a unit impulse sequence $\delta[n]$ is called impulse response, denoted by $h[n]$



Discrete-Time LTI Systems

Discrete-Time Unit Impulse Response and the Convolution-Sum

- ☐ How to calculate the impulse response of a system

- ➢ For any system whose input-output relationship is defined by

$$y[n] = f\{x[n]\}$$

the impulse response $h[n]$ is calculated as

$$h[n] = f\{\delta[n]\}$$

replace $x[n]$ by $\delta[n]$

Discrete-Time LTI Systems

Discrete-Time Unit Impulse Response and the Convolution-Sum

- How to calculate the impulse response of a system

- Examples: a system is defined as

$$y[n] = a_1 x[n] + a_2 x[n-1] + a_3 x[n-2] + a_4 x[n-3]$$

its impulse response $h[n]$ is

$$h[n] = a_1 \delta[n] + a_2 \delta[n-1] + a_3 \delta[n-2] + a_4 \delta[n-3]$$

Discrete-Time LTI Systems

Discrete-Time Unit Impulse Response and the Convolution-Sum

- ❑ How to calculate the impulse response of a system

- ➢ Examples: a system is defined as

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

its impulse response $h[n]$ is

$$h[n] = \sum_{k=-\infty}^{n} \delta[k]$$

Discrete-Time LTI Systems

Discrete-Time Unit Impulse Response and the Convolution-Sum

- How to calculate the impulse response of a system

- Examples: a system is defined as

$$y[n] = x[n-1] + \frac{1}{2}(x[n-2] + x[n])$$

its impulse response $h[n]$ is

$$h[n] = \delta[n-1] + \frac{1}{2}(\delta[n-2] + \delta[n])$$

Discrete-Time LTI Systems

Discrete-Time Unit Impulse Response and the Convolution-Sum

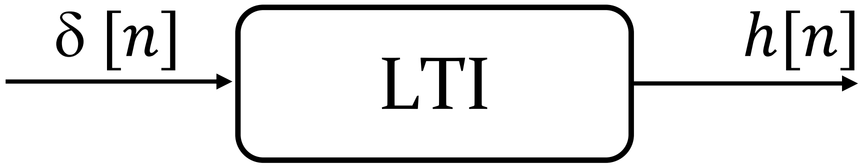
- ❑ An LTI discrete system is completely characterized by its impulse response

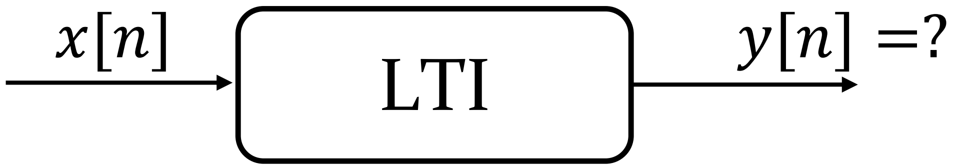
- ❑ In other words, knowing the impulse response one can compute the output of the LTI system for an arbitrary input

Discrete-Time LTI Systems

Discrete-Time Unit Impulse Response and the Convolution-Sum

- The impulse response completely characterizes an LTI system





- Recall, an arbitrary input $x[n]$ can be expressed as a linear combination of shifted unit impulses

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

Discrete-Time LTI Systems

Discrete-Time Unit Impulse Response and the Convolution-Sum

- For any $k = k_0$

$$\delta[n] \longrightarrow \boxed{\text{LTI}} \longrightarrow h[n]$$

$$\delta[n - k_0] \longrightarrow \boxed{\text{LTI}} \longrightarrow h[n - k_0]$$

$$x[k_0]\delta[n - k_0] \longrightarrow \boxed{\text{LTI}} \longrightarrow x[k_0]h[n - k_0]$$

$$\Rightarrow \quad x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n - k] \longrightarrow \boxed{\text{LTI}} \longrightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n - k]$$

Discrete-Time LTI Systems

Discrete-Time Unit Impulse Response and the Convolution-Sum

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \longrightarrow \boxed{\text{LTI}} \longrightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

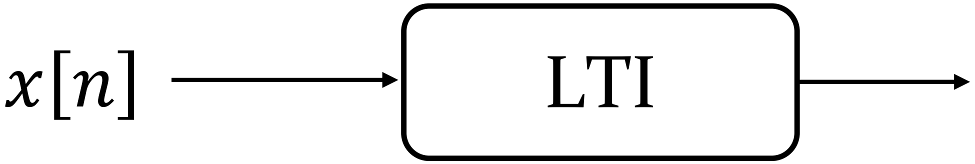
- ❑ $\sum_{k=-\infty}^{\infty} x[k]h[n-k]$ is referred as to the **convolution-sum**

$$x[n] \longrightarrow \boxed{\text{LTI}} \longrightarrow y[n] = x[n] * h[n]$$

Discrete-Time LTI Systems

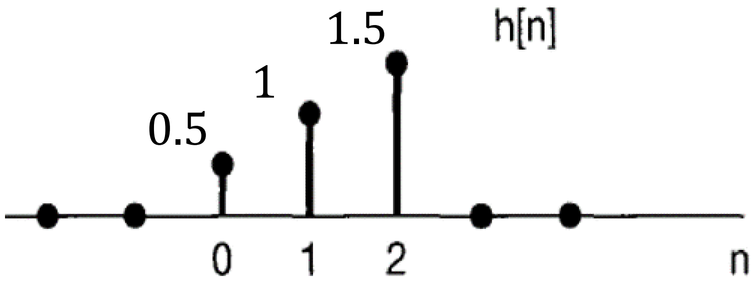
Discrete-Time Unit Impulse Response and the Convolution-Sum

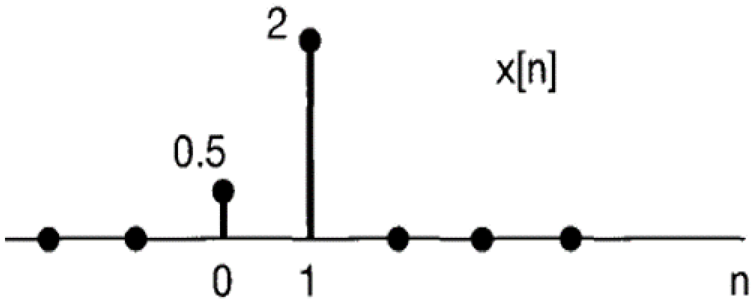
- ❑ Convolution-Sum calculation – Method 1: sum of k shifted and scaled $h[n]$

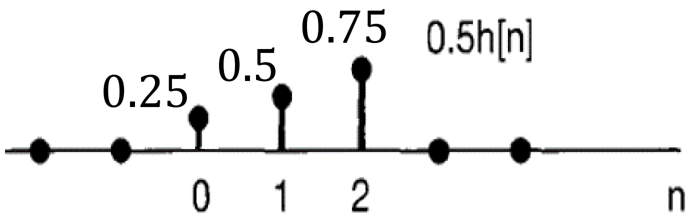


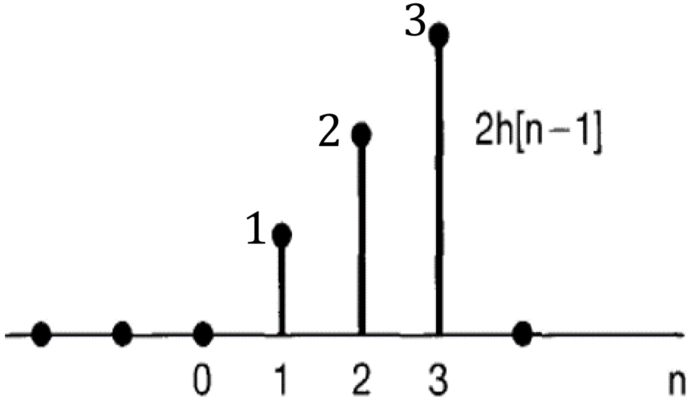
$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n] * h[n]$$

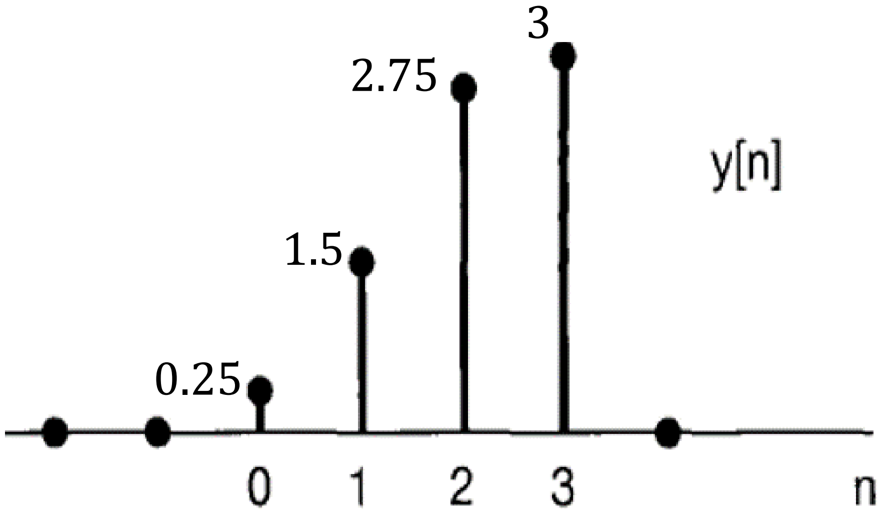
n: variable, k:constant











Discrete-Time LTI Systems

Discrete-Time Unit Impulse Response and the Convolution-Sum

- ❑ Convolution-Sum calculation – Method 2: calculate $y[n]$ for each n

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n] * h[n]$$

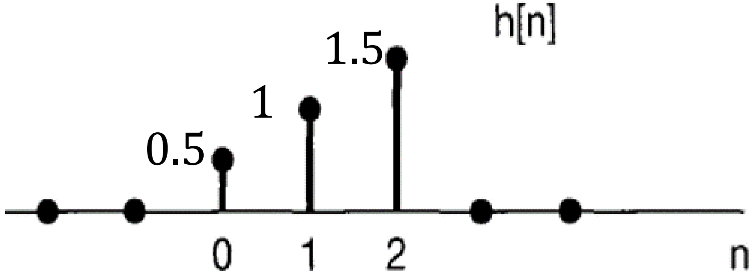
- ➢ Step 1: determine the range of k

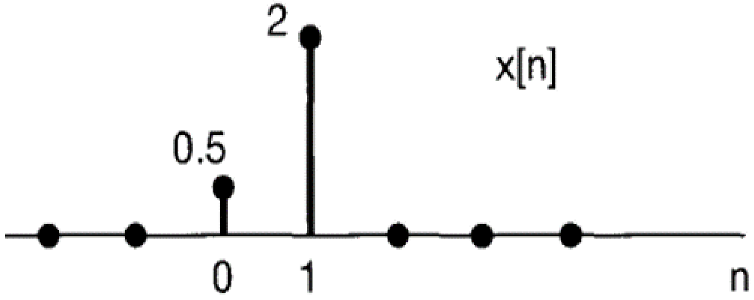
$$k \in \{0,1\}$$

- ➢ Step 2: determine the range of n

$$[n-k] \in \{0,1,2\} \leftrightarrow n \in \{0,1,2,3\},$$

For other n, $y[n]$=0





Discrete-Time LTI Systems

Discrete-Time Unit Impulse Response and the Convolution-Sum

❑ Convolution-Sum calculation – Method 2: calculate $y[n]$ for each n

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n] * h[n]$$

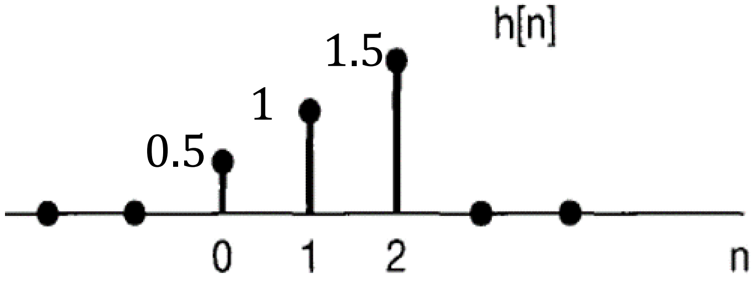
➢ Step 3: calculate $y[n]$ for each n

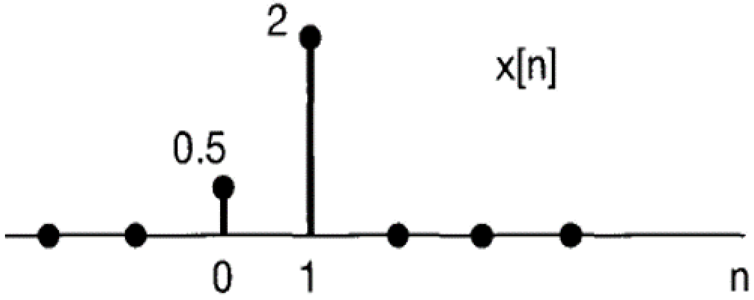
$$y[0] = \sum_{k=0}^{1} x[k]h[0-k] = x[0]h[0] + x[1]h[-1] = 0.25$$

$$y[1] = \sum_{k=0}^{1} x[k]h[1-k] = x[0]h[1] + x[1]h[0] = 1.5$$

$$y[2] = \sum_{k=0}^{1} x[k]h[2-k] = x[0]h[2] + x[1]h[1] = 2.75$$

$$y[3] = \sum_{k=0}^{1} x[k]h[3-k] = x[0]h[3] + x[1]h[2] = 3$$





Discrete-Time LTI Systems

Discrete-Time Unit Impulse Response and the Convolution-Sum

- ☐ Convolution-Sum calculation – Method 3

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

For each n :

- ➢ Step 1: change time variables $x[n] \rightarrow x[k]$, $h[n] \rightarrow h[k]$, and reverse $h[k] \rightarrow h[-k]$
- ➢ Step 2: Shift $h[-k] \rightarrow h[n-k]$, n is considered as a constant
- ➢ Step 3: multiply $x[k] \cdot h[n-k]$
- ➢ Step 4: Summation $\sum_{k=-\infty}^{\infty} x[k] \cdot h[n-k]$

Change n, repeat step 1 to 4, calculate another $y[n]$

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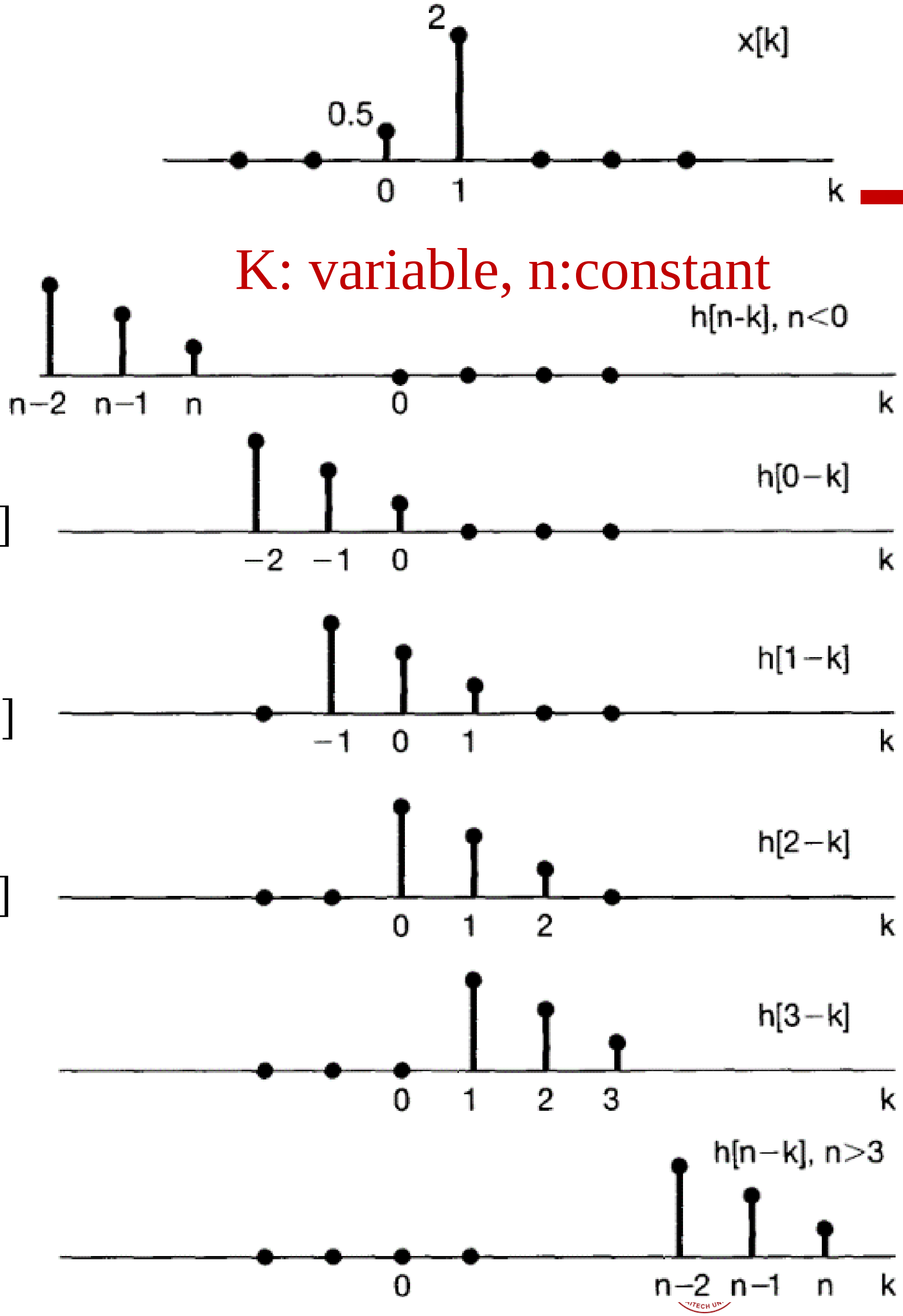
Discrete-Time LTI Systems

The Convolution-Sum

- ❑ Convolution-Sum calculation
  - – Method 3

- If the lengths of the two sequences are *M* and *N*, then the sequence generated by the convolution is of length *M*+*N*–1

K: variable, n:constant



2
0.5
x[k]
0
1
k
h[n-k], n<0
n−2
n−1
n
0
k
h[0−k]
−2
−1
0
k
h[1−k]
−1
0
1
k
h[2−k]
0
1
2
k
h[3−k]
0
1
2
3
k
h[n−k], n>3
0
n−2
n−1
n
k

$y[n] = 0$, for $n<0$

$$y[0] = \sum_{k=0}^{1} x[k]h[0-k]$$

$$y[1] = \sum_{k=0}^{1} x[k]h[1-k]$$

$$y[2] = \sum_{k=0}^{1} x[k]h[2-k]$$

$$y[3] = \sum_{k=0}^{1} x[k]h[3-k]$$

$y[n] = 0$, for $n>3$

Discrete-Time LTI Systems

The Convolution-Sum

- ❑ Examples

$$y[n] = x[n] * \delta[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] = x[n]$$

$$y[n] = x[n] * \delta[n-d] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k-d]$$

Let $k + d = k'$

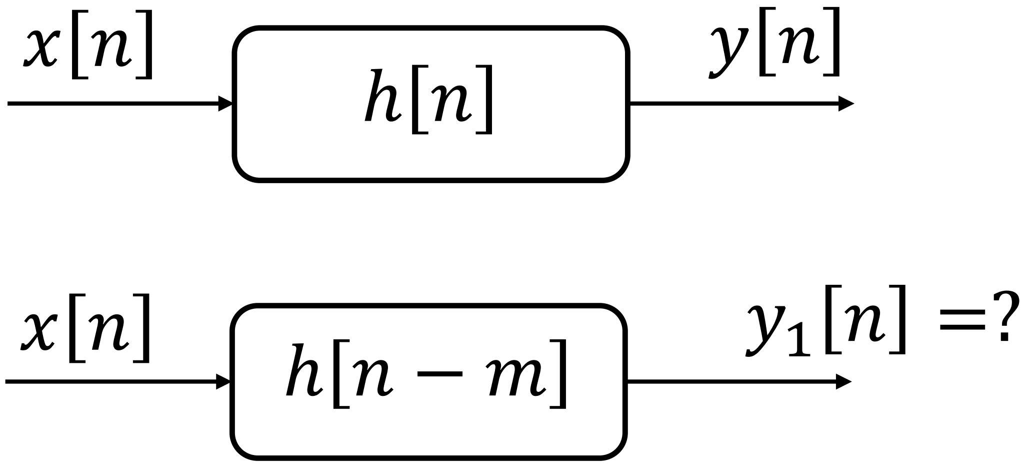
$$= \sum_{k'=-\infty}^{\infty} x[k'-d]\delta[n-k']$$

$$= x[n-d] * \delta[n] = x[n-d]$$

Discrete-Time LTI Systems

The Convolution-Sum

- ❑ Examples



$$y_1[n] = x[n] * h[n-m] = \sum_{k=-\infty}^{\infty} x[k]h[n-k-m]$$

Let $k+m=k'$

$$= \sum_{k'=-\infty}^{\infty} x[k'-m]h[n-k']$$

$$= x[n-m] * h[n] = y[n-m]$$

Chapter 2: Linear Time-Invariant Systems

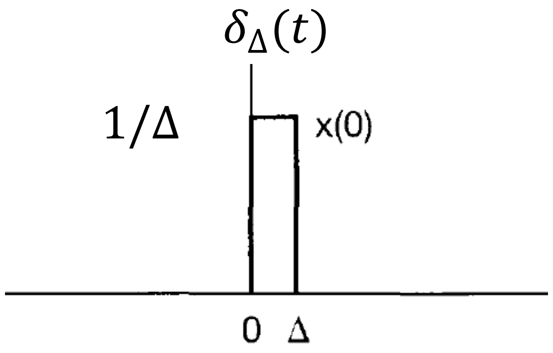
- ☐ Discrete-Time LTI Systems
- ☐ **Continuous-Time LTI Systems**
- ☐ Properties of LTI Systems
- ☐ Differential or Difference Equations

Continuous-Time LTI Systems

Continuous-Time Signals in Terms of Impulse

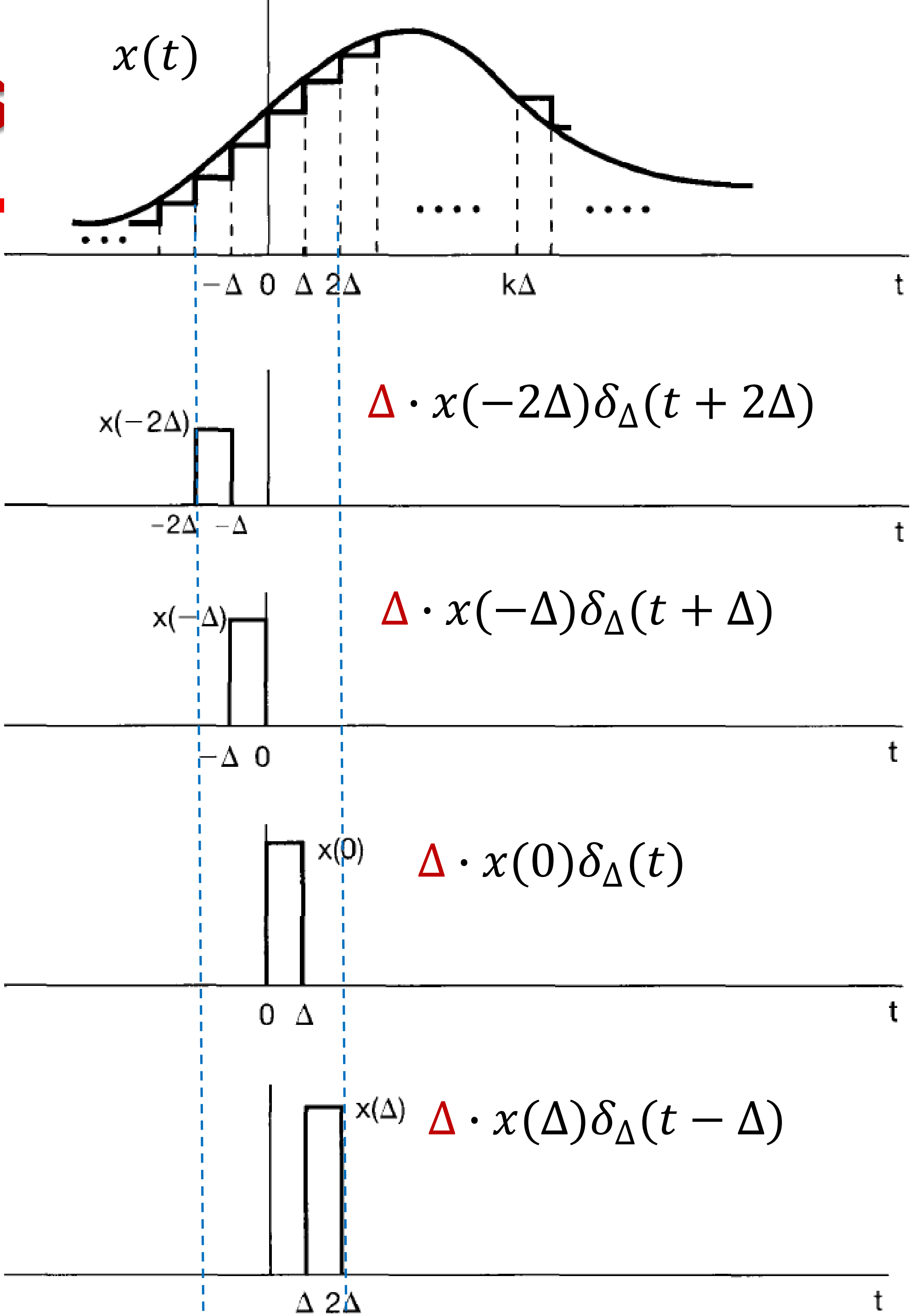
- ❑ "staircase" approximation of $x(t)$

$$\delta_\Delta(t) = \begin{cases} \frac{1}{\Delta}, & 0 \le t \le \Delta \\ 0, & other\ wise \end{cases}$$


$\delta_\Delta(t)$
1/Δ
x(0)
0
Δ

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t-k\Delta)\cdot\Delta$$

$$x(t) = \lim_{\Delta\to 0}\hat{x}(t)$$


$x(t)$
...
....
....
−Δ
0
Δ
2Δ
kΔ
t
x(−2Δ)
−2Δ
−Δ
t
x(−Δ)
−Δ
0
t
x(0)
0
Δ
t
x(Δ)
Δ
2Δ
t

$$\Delta\cdot x(-2\Delta)\delta_\Delta(t+2\Delta)$$

$$\Delta\cdot x(-\Delta)\delta_\Delta(t+\Delta)$$

$$\Delta\cdot x(0)\delta_\Delta(t)$$

$$\Delta\cdot x(\Delta)\delta_\Delta(t-\Delta)$$

Continuous-Time LTI Systems

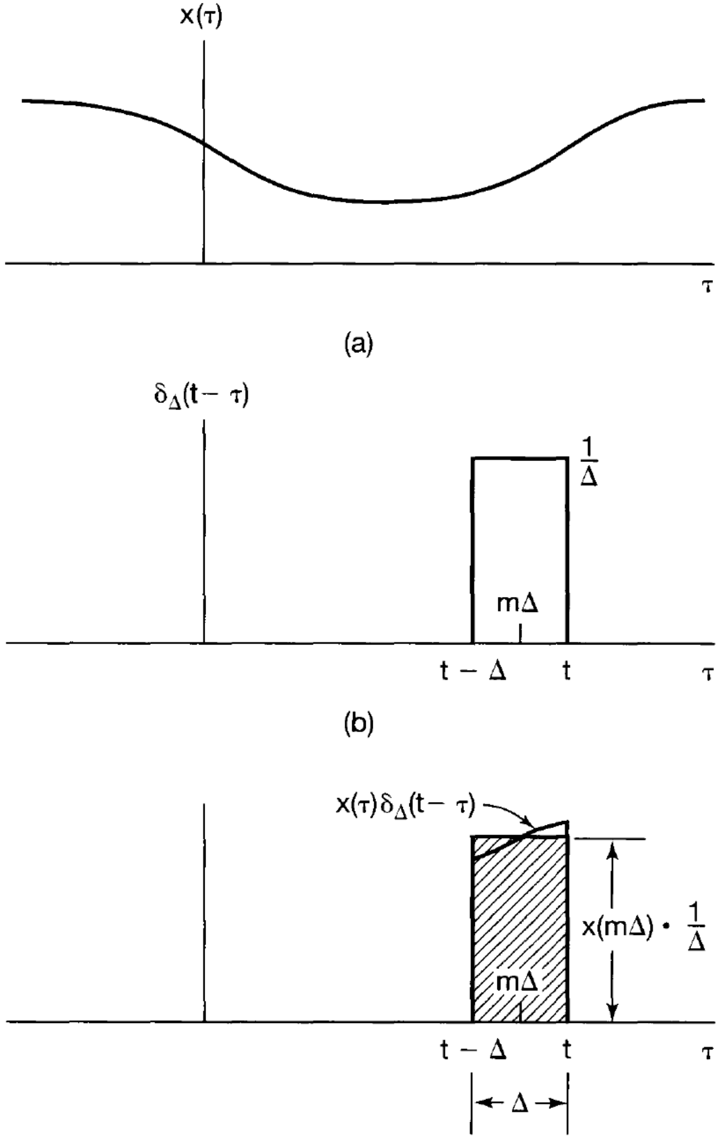
Continuous-Time Signals in Terms of Impulse

- ❑ "staircase" approximation of $x(t)$

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t - k\Delta) \cdot \Delta$$

$$x(t) = \lim_{\Delta \to 0} \hat{x}(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t - \tau)\, d\tau$$

Sifting property of $\delta(t)$


x(τ)
τ
(a)
δ_Δ(t − τ)
1/Δ
mΔ
t − Δ
t
τ
(b)
x(τ)δ_Δ(t − τ)
x(mΔ) · 1/Δ
mΔ
t − Δ
t
τ
Δ

Continuous-Time LTI Systen

Continuous-Time Signals in Terms of Impul

- ☐ Using sampling property of $\delta(t)$

$$\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau = ?$$

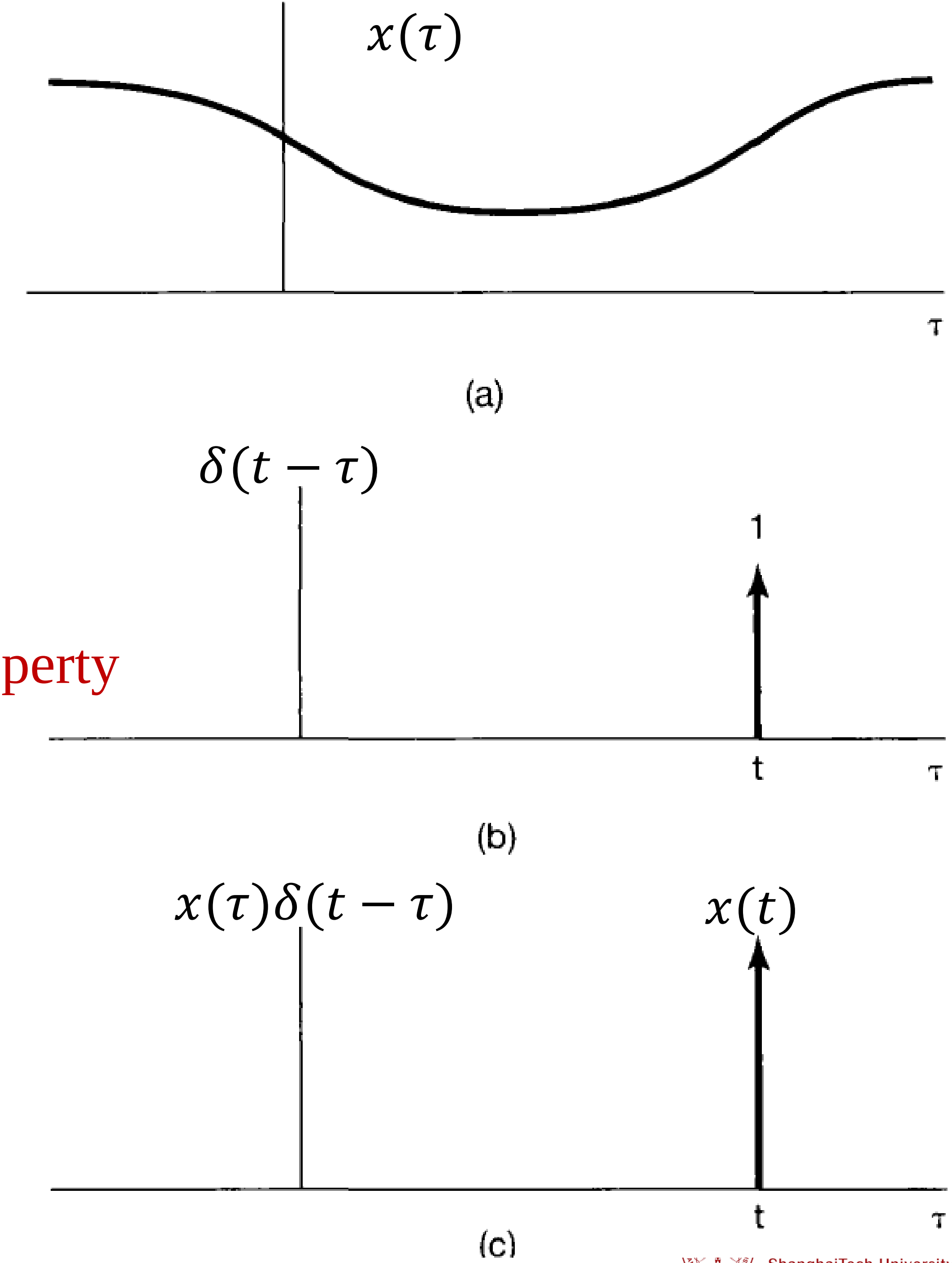
$$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$$ sampling property

$$x(\tau)\delta(t-\tau) = x(t)\delta(t-\tau)$$ t: constant

$$\therefore \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau = \int_{-\infty}^{\infty} x(t)\delta(t-\tau)\,d\tau$$

$$= x(t)\int_{-\infty}^{\infty} \delta(t-\tau)\,d\tau$$

$$= x(t)$$


$x(\tau)$
τ
(a)
$\delta(t-\tau)$
1
t
τ
(b)
$x(\tau)\delta(t-\tau)$
$x(t)$
t
τ
(c)

Continuous-Time LTI Systems

Continuous-Time Signals in Terms of Impulse

- An example

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\, d\tau$$

$$u(t) = \int_{-\infty}^{\infty} u(\tau)\delta(t-\tau)\, d\tau = \int_{0}^{\infty} \delta(t-\tau)\, d\tau$$

Continuous-Time LTI Systems

Continuous-Time Unit Impulse Response and Convolution Integral

- ❑ Continuous-Time Unit Impulse Response

$$\delta(t) \longrightarrow \boxed{\text{LTI}} \longrightarrow h(t)$$

- ❑ What about

$$x(t) \longrightarrow \boxed{\text{LTI}} \longrightarrow y(t) = ? \qquad \delta_\Delta(t) \longrightarrow \boxed{\text{LTI}} \longrightarrow h_\Delta(t)$$

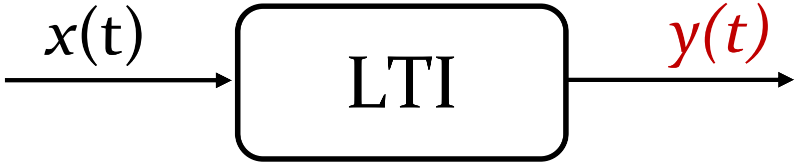
$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_\Delta(t - k\Delta) \cdot \Delta \qquad \hat{y}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)h_\Delta(t - k\Delta) \cdot \Delta$$

$$x(t) = \lim_{\Delta \to 0} \hat{x}(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t - \tau)\, d\tau \qquad y(t) = \lim_{\Delta \to 0} \hat{y}(t) = \int_{-\infty}^{\infty} x(\tau)h(t - \tau)\, d\tau$$

Continuous-Time LTI Systems

Continuous-Time Unit Impulse Response and Convolution Integral

- ❑ Continuous-Time Unit Impulse Response



$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\, d\tau$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\, d\tau$$

Integral of weighted and shift impulses

- ❑ Convolution integral

$$\int_{-\infty}^{\infty} x(\tau)h(t-\tau)\, d\tau = x(t) * h(t)$$

Continuous-Time LTI Systems

Continuous-Time Unit Impulse Response and Convolution Integral

- ☐ Compute convolution integral

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\, d\tau$$

  - ➢ Change time variables $x(t) \rightarrow x(\tau)$, $h(t) \rightarrow h(\tau)$, and reverse $h(\tau) \rightarrow h(-\tau)$
  - ➢ Shift $h(-\tau) \rightarrow h(t-\tau)$
  - ➢ Multiply $x(\tau) \cdot h(t-\tau)$
  - ➢ Integral $\int_{-\infty}^{\infty} x(\tau) \cdot h(t-\tau)\mathrm{d}\tau$

Continuous-Time LTI Systems

Continuous-Time Unit Impulse Response and Convolution Integral

- ❑ Compute convolution integral: examples

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

$$x(t) * \delta(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau = x(t)$$

$$x(t) * \delta(t-t_0) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau-t_0)\,d\tau = \int_{-\infty}^{\infty} x(\tau)\delta(t-(\tau+t_0))\,d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau'-t_0)\delta(t-\tau')\,d\tau' = x(t-t_0) * \delta(t)$$

$$= x(t-t_0)$$

Continuous-Time LTI Systems

Continuous-Time Unit Impulse Response and Convolution Integral

- ☐ Compute convolution integral: examples

$$x(t) = e^{-at}u(t),\ h(t) = u(t),\quad a > 0 \quad x(t) * h(t) = ?$$

$$y(t) = \int_{-\infty}^{\infty} e^{-a\tau} u(\tau) \cdot u(t-\tau) d\tau$$

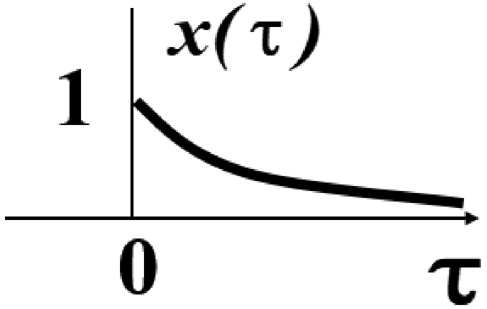
For $t<0$ $\quad x(\tau) \cdot h(t-\tau) = 0 \quad \Rightarrow \quad y(t) = 0$

For $t\geq 0$ $\quad y(t) = \int_{0}^{t} e^{-a\tau} d\tau = \frac{-1}{a} e^{-at} \Big|_{0}^{t} = \frac{1}{a}(1-e^{-at})$

Continuous-Time LTI Systems

Continuous-Time Unit Impulse Response and Convolution Integral

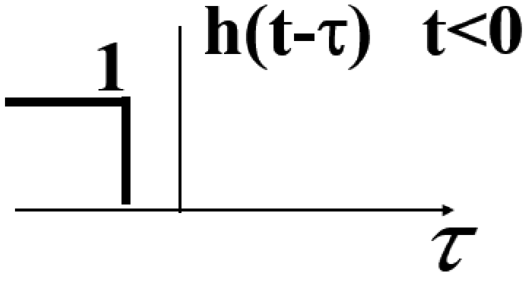
- ☐ Compute convolution integral: Graphical Solution

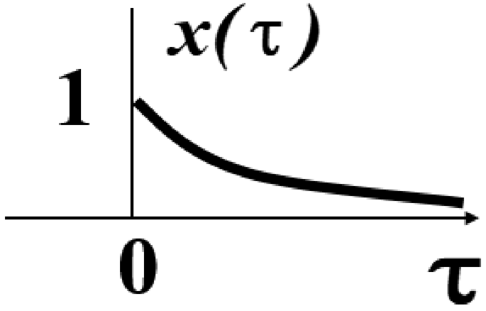


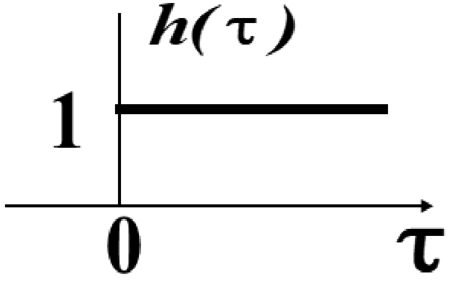
$$x(t) = e^{-at}u(t),\ h(t) = u(t),\quad a > 0$$

$$x(t) * h(t) = ?$$

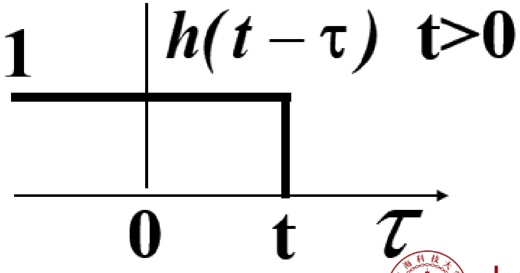
τ : variable, t: constant







$$y(t) = \int_0^t e^{-a\tau}d\tau = \frac{-1}{a}e^{-at}\Big|_0^t = \frac{1}{a}(1 - e^{-at})$$

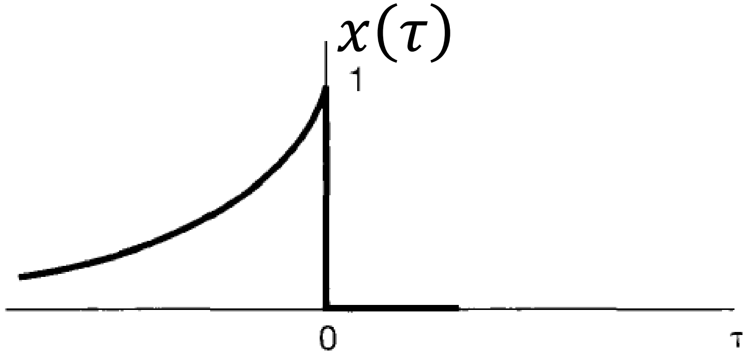


Continuous-Time LTI Systems

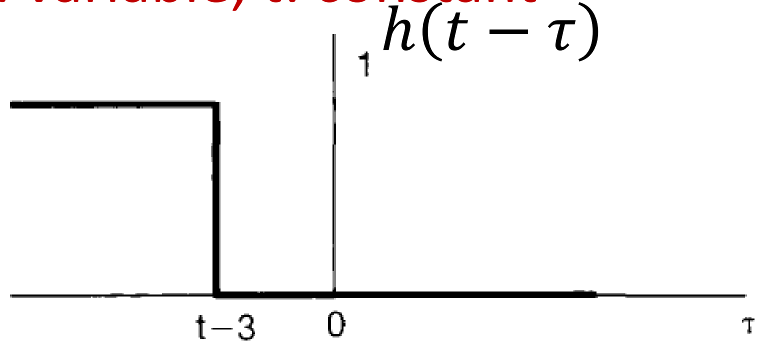
Continuous-Time Unit Impulse Response and Convolution Integral

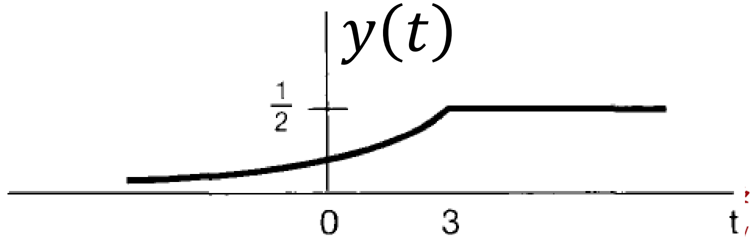
- Computation convolution integral: examples

$x(t) = e^{2t}u(-t) \quad h(t) = u(t-3) \quad x(t) * h(t) = ?$



τ : variable, t: constant



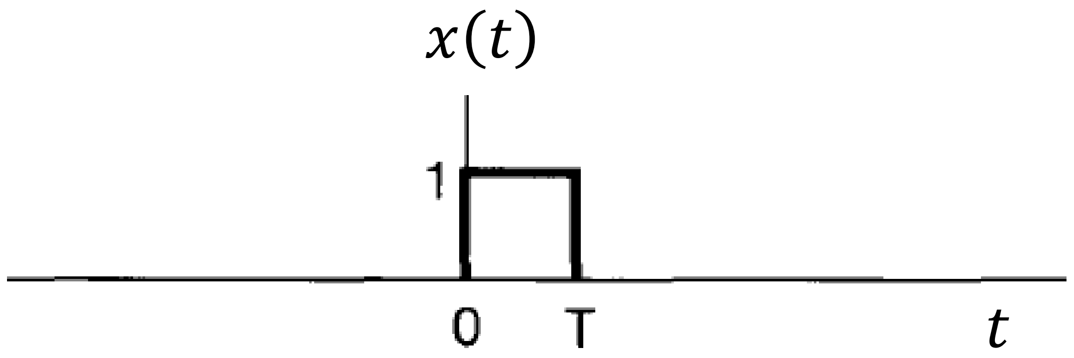


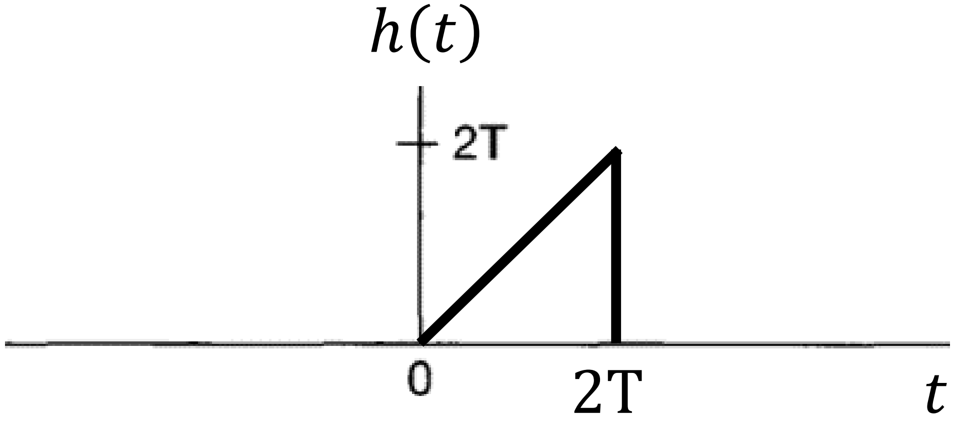
Continuous-Time LTI Systems

Continuous-Time Unit Impulse Response and Convolution Integral

- Compute convolution integral: examples

$$x(t) = \begin{cases} 1, 0 < t < T \\ 0, \text{otherwise} \end{cases} \qquad h(t) = \begin{cases} t, 0 < t < 2T \\ 0, \text{otherwise} \end{cases}$$





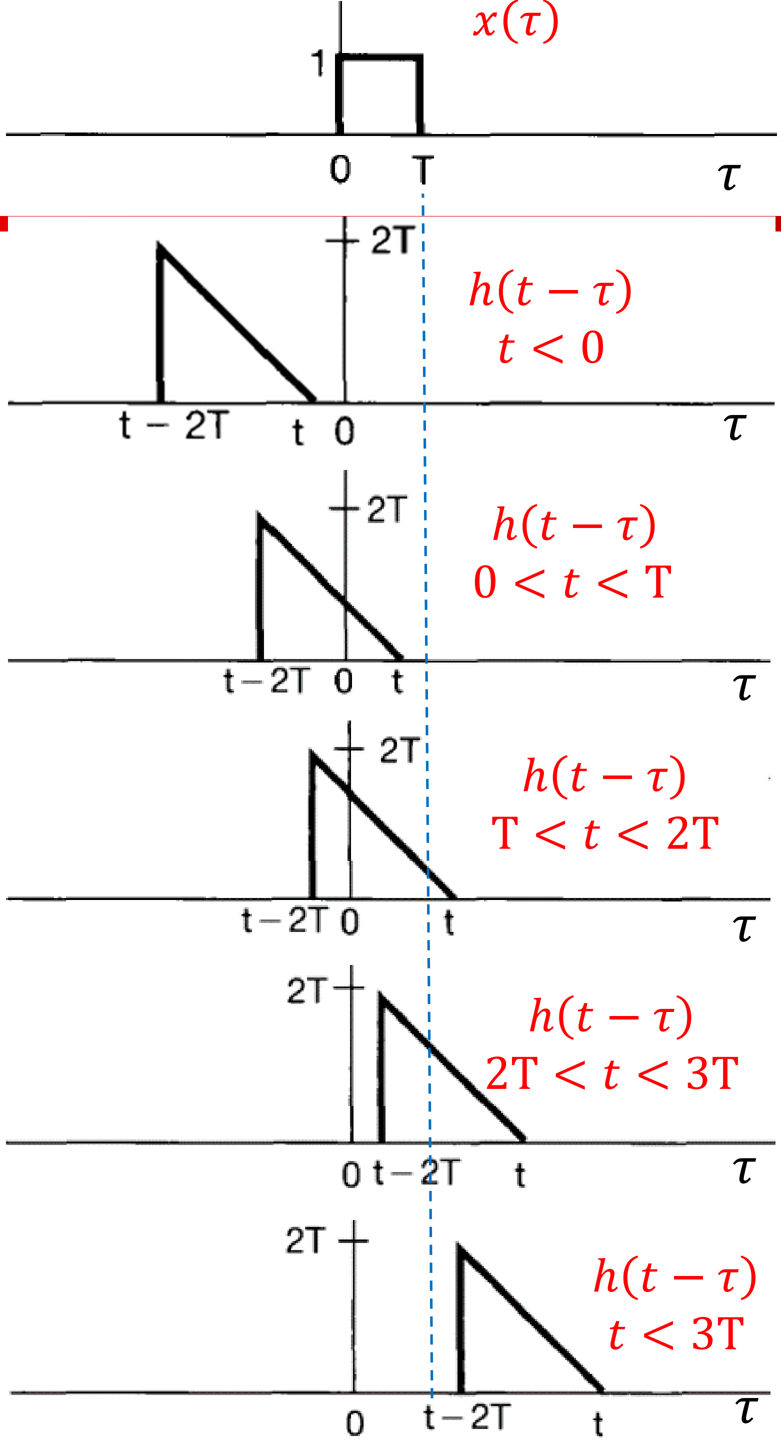
$$x(t) * h(t) = ?$$

Continuous-Time LTI Systems

Convolution Integral

- ❑ Computation: examples

$$y(t) = \begin{cases} 0, t < 0 \\ \int_{0}^{t} (t-\tau)d\tau = \frac{1}{2}t^2, 0 < t < T \\ \int_{0}^{T} (t-\tau)d\tau = Tt - \frac{1}{2}T^2, T < t < 2T \\ \int_{t-2T}^{T} (t-\tau)d\tau = -\frac{1}{2}t^2 + Tt + \frac{3}{2}T^2, 2T < t < 3T \\ 0, t > 3T \end{cases}$$


$x(\tau)$
1
0
T
τ
2T
$h(t-\tau)$
$t < 0$
t − 2T
t
0
τ
2T
$h(t-\tau)$
$0 < t < \mathrm{T}$
t − 2T
0
t
τ
2T
$h(t-\tau)$
$\mathrm{T} < t < 2\mathrm{T}$
t − 2T
0
t
τ
2T
$h(t-\tau)$
$2\mathrm{T} < t < 3\mathrm{T}$
0
t − 2T
t
τ
2T
$h(t-\tau)$
$t < 3\mathrm{T}$
0
t − 2T
t
τ

Chapter 2: Linear Time-Invariant Systems

- ☐ **Discrete-Time LTI Systems**
- ☐ **Continuous-Time LTI Systems**
- ☐ **Properties of LTI Systems**
- ☐ **Differential or Difference Equations**

Properties of LTI Systems

The commutative property

- ❑ Discrete-time $x[n] * h[n] = h[n] * x[n]$

$$x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] \overset{n-k=m}{=} \sum_{m=-\infty}^{\infty} h[m]x[n-m] = h[n] * x[n]$$

- ❑ Continuous-time $x(t) * h(t) = h(t) * x(t)$

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau \overset{t-\tau=\tau'}{=} \int_{-\infty}^{\infty} h(\tau')x(t-\tau')d\tau' = h(t) * x(t)$$

Properties of LTI Systems

The distribute property

- Discrete-time

$$x[n] * (h_1[n] + h_2[n]) = x[n] * h_1[n] + x[n] * h_2[n]$$

- Proof

$$x[n] * (h_1[n] + h_2[n]) = \sum_{k=-\infty}^{\infty} x[k]\,(h_1[n-k] + h_2[n-k])$$

$$= \sum_{k=-\infty}^{\infty} x[k]\,h_1[n-k] + \sum_{k=-\infty}^{\infty} x[k]\,h_2[n-k])$$

$$= x[n] * h_1[n] + x[n] * h_2[n]$$

Properties of LTI Systems

The distribute property

- ☐ Continuous-time

$$x(t) * \big(h_1(t) + h_2(t)\big) = x(t) * h_1(t) + x(t) * h_2(t)$$

- ☐ Proof

$$x(t) * \big(h_1(t) + h_2(t)\big) = \int_{-\infty}^{\infty} x(\tau)\big(h_1(t-\tau) + h_2(t-\tau)\big)d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)h_1(t-\tau)d\tau + \int_{-\infty}^{\infty} x(\tau)h_2(t-\tau)d\tau$$

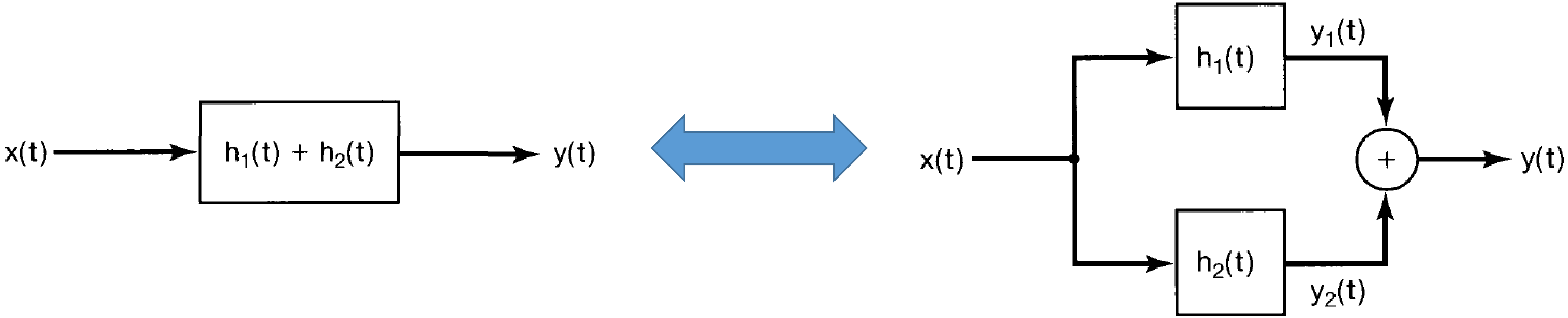
$$= x(t) * h_1(t) + x(t) * h_2(t)$$

Properties of LTI Systems

The distribute property

- Continuous-time

$$x(t) * \left(h_1(t) + h_2(t)\right) = x(t) * h_1(t) + x(t) * h_2(t)$$


$x(t)$
$h_1(t) + h_2(t)$
$y(t)$
$x(t)$
$h_1(t)$
$y_1(t)$
$+$
$y(t)$
$h_2(t)$
$y_2(t)$

Properties of LTI Systems

The associative property

- Discrete-time $$x[n] * \boxed{h_1[n] * h_2[n]} = (x[n] * h_1[n]) * h_2[n]$$

$$x[n] * (h_1[n] * h_2[n]) = x[n] * y[n], \qquad y[n] = \sum_{m=-\infty}^{\infty} h_1[m]h_2[n-m]$$

$$= \sum_{k=-\infty}^{\infty} x[k]y[n-k] = \sum_{k=-\infty}^{\infty} x[k] \sum_{m=-\infty}^{\infty} h_1[m]h_2[n-k-m]$$

Let $k + m = l$

$$= \sum_{k=-\infty}^{\infty} x[k] \sum_{l=-\infty}^{\infty} h_1[l-k]h_2[n-l]$$

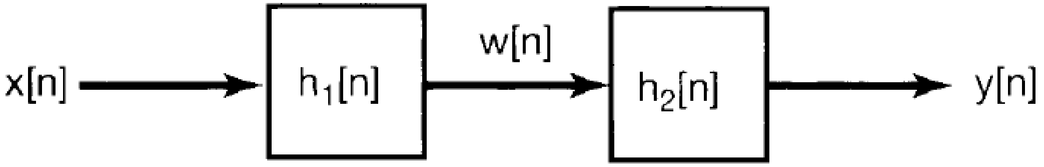
$$= \sum_{l=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x[k]h_1[l-k]\, h_2[n-l]$$

$$= \sum_{l=-\infty}^{\infty} (x[l] * h_1[l])\, h_2[n-l] = (x[n] * h_1[n]) * h_2[n]$$

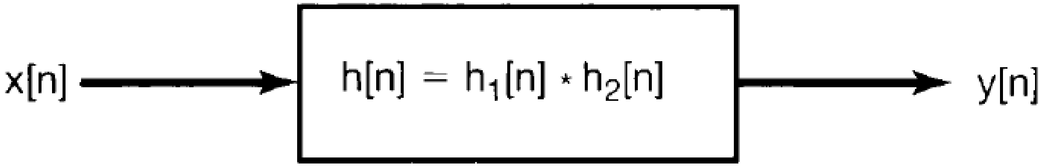
Properties of LTI Systems

The associative property

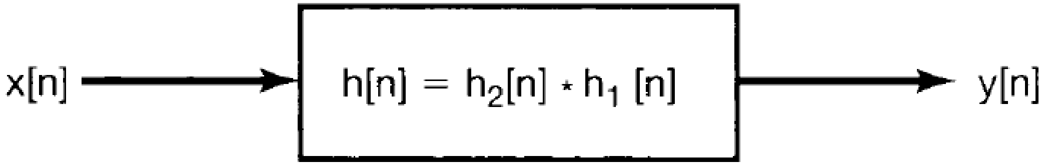
- ❑ Discrete-time


$x[n]$
$h_1[n]$
$w[n]$
$h_2[n]$
$y[n]$

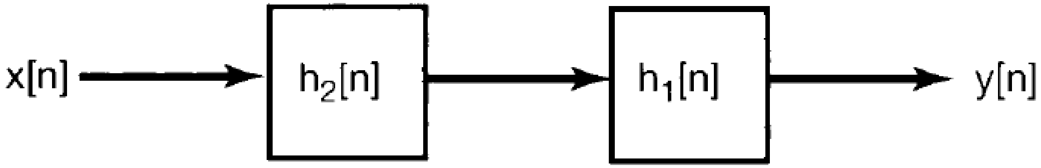
(a)


$x[n]$
$h[n] = h_1[n] * h_2[n]$
$y[n]$

(b)


$x[n]$
$h[n] = h_2[n] * h_1[n]$
$y[n]$

(c)


$x[n]$
$h_2[n]$
$h_1[n]$
$y[n]$

(d)

Properties of LTI Systems

The associative property

- ☐ Continuous-time

$$x(t) * \big(h_1(t) * h_2(t)\big) = \big(x(t) * h_1(t)\big) * h_2\,(t)$$

$$x(t) * \big(h_1(t) * h_2(t)\big) = x(t) * \int_{-\infty}^{\infty} h_1(\tau)h_2(t-\tau)d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau') \int_{-\infty}^{\infty} h_1(\tau)h_2(t-\tau'-\tau)d\tau\, d\tau'$$

Let $\tau' + \tau = \tau''$

$$= \int_{-\infty}^{\infty} x(\tau') \int_{-\infty}^{\infty} h_1(\tau''-\tau')h_2(t-\tau'')d\tau''\, d\tau'$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x(\tau')h_1(\tau''-\tau')d\tau'\; h_2(t-\tau'')d\tau''$$

$$= \int_{-\infty}^{\infty} x(\tau'') * h_1(\tau'')\; h_2(t-\tau'')d\tau'' = \big(x(t) * h_1(t)\big) * h_2\,(t)$$

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Properties of LTI Systems

LTI systems with and without memory

- Discrete-time system without memory only if $h[n] = 0$ for all $n \neq 0$

$$h[n] = h[0]\delta[n] = k\delta[n] \qquad y[n] = kx[n]$$

Why?

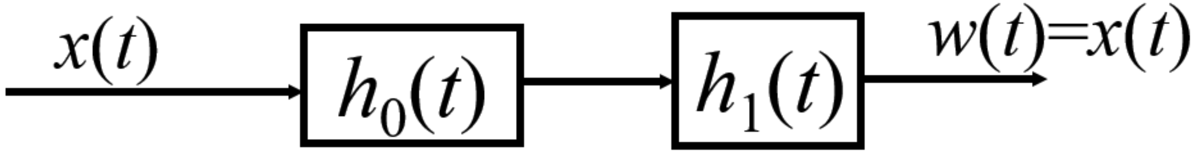
- Continuous-time system without memory only if $h(t) = 0$ for all $t \neq 0$

$$h(t) = h(0)\delta(t) = k\delta(t) \qquad y(t) = kx(t)$$

Properties of LTI Systems

Invertibility for LTI systems

- If $h_0(t) * h_1(t) = \delta(t)$, the system with impulse response $h_1(t)$ is the inverse of the system with impulse response $h_0(t)$



- Similarly, if $h_0[n] * h_1[n] = \delta[n]$, the system with impulse response $h_1[n]$ is the inverse of the system with impulse response $h_0[n]$

Properties of LTI Systems

Invertibility for LTI systems

- Examples

Consider $h_0[n] = u[n]$, determine the inverse system $h_1[n]$

$$\because\ h_0[n] * h_1[n] = u[n] * h_1[n] \overset{\textbf{\textit{hold}}}{=} \delta[n]$$

$$\delta[n] = u[n] - u[n-1] = u[n] * (\delta[n] - \delta[n-1])$$

$$\therefore\ h_1[n] = \delta[n] - \delta[n-1]$$

Properties of LTI Systems

Invertibility for LTI systems

- Examples

Consider the LTI system consisting of a pure time shift

$$y(t) = x(t - t_0),$$

determine the inverse system.

Properties of LTI Systems

Causality for LTI systems

- ☐ If $h[n] = 0$ for n < 0, or $h(t) = 0$ for $t < 0$, the system is causal
- ☐ Equivalent to the condition of initial rest: if $t \leq t_0, x(t) = 0$, then $y(t_0) = 0$

$$y[n] = \sum_{k=-\infty}^{n} x[k]h[n-k] \quad \text{or} \quad y[n] = \sum_{k=0}^{\infty} h[k]x[n-k]$$

$$y(t) = \int_{-\infty}^{t} x(\tau)h(t-\tau)\,d\tau \quad \text{or} \quad y(t) = \int_{0}^{\infty} h(\tau)x(t-\tau)d\tau$$

Properties of LTI Systems

Causality for LTI systems

- ❑ Examples

  - Accumulator: $y[n] = \sum_{l=-\infty}^{n} x[l]$ Causal LTI system

    $h[n] = \sum_{l=-\infty}^{n} \delta[l] = u[n]$ $h[n] = 0$ for n < 0

  - Factor 2 interpolator: $y[n] = x[n] + \frac{1}{2}(x[n-1] + x[n+1])$

    Non-Causal LTI system

    $$h[n] = \delta[n] + \frac{1}{2}(\delta[n-1] + \delta[n+1])$$

    $h[n] \neq 0$ for n $= -1 < 0$

Properties of LTI Systems

Stability for LTI systems

- ☐ A discrete LTI system is stable if $h[n]$ is absolutely summable
- ☐ A continuous LTI system is stable if $h(t)$ is absolutely integrable

$$\sum_{k=-\infty}^{\infty} |h[k]| < \infty$$

absolutely summable

$$\int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty$$

absolutely integrable

Properties of LTI Systems

Stability for LTI systems

- ❑ Proof: "if and only if" (Sufficient and necessary condition)

$$|y[n]| = \left|\sum_{k=-\infty}^{\infty} h[k]x[n-k]\right| \leq \sum_{k=-\infty}^{\infty} |h[k]x[n-k]| \leq \sum_{k=-\infty}^{\infty} |h[k]|\,|x[n-k]|$$

$$\therefore\ |y[n]| \leq \sum_{k=-\infty}^{\infty} |h[k]|\,|x[n-k]|$$

If $|x[n-k]| \leq B_x$ $\qquad |y[n]| \leq B_x \sum_{k=-\infty}^{\infty} |h[k]|$

If and only if $\sum_{k=-\infty}^{\infty} |h[k]| < \infty$ $\qquad |y[n]| < \infty$

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Properties of LTI Systems

Stability for LTI systems

- ❑ Proof: continuous case

If $|x(t-\tau)| \le B_x$

$$|y(t)| = \left| \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau \right| \le \int_{-\infty}^{\infty} |h(\tau)| \cdot |x(t-\tau)| d\tau \le B_x \int_{-\infty}^{\infty} |h(\tau)| d\tau$$

If and only if $\int_{-\infty}^{\infty} |h(\tau)| d\tau < \infty$ $\quad$ $|y(t)| < \infty$

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Properties of LTI Systems

Stability for LTI systems

- Examples

$$y[n] = x[n - n_0]$$

$$h[n] = \delta[n - n_0]$$

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=-\infty}^{\infty} |\delta[n - n_0]| = 1$$

Properties of LTI Systems

Stability for LTI systems

- ☐ Examples $h[n] = \alpha^n u[n] \qquad |\alpha| \leq 1$

$$\sum_{n=-\infty}^{\infty} |h[n]| = \sum_{n=-\infty}^{\infty} |\alpha^n| u[n] = \sum_{n=0}^{\infty} |\alpha^n| = \frac{1}{1-|\alpha|}$$

If $|\alpha| < 1$, the system is stable

If $|\alpha| = 1$, the system is unstable

Properties of LTI Systems

The unit step response of LTI systems

- The unit step response, $s[n]$, corresponding to the output with input $x[n] = u[n]$

$$s[n] = u[n] * h[n] = \sum_{-\infty}^{\infty} h[k]u[n-k] = \sum_{k=-\infty}^{n} h[k]$$

$$u[n] = \sum_{k=-\infty}^{n} \delta[k] \qquad\qquad s[n] = \sum_{k=-\infty}^{n} h[k]$$

$$h[n] = s[n] - s[n-1]$$

Properties of LTI Systems

The unit step response of LTI systems

- ☐ The unit step response, $s(t)$, corresponding to the output with input x(t)=u(t)

$$s(t) = u(t) * h(t) = \int_{-\infty}^{\infty} h(\tau)u(t-\tau)d\tau = \int_{-\infty}^{t} h(\tau)d\tau$$

$$u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau \qquad s(t) = \int_{-\infty}^{t} h(\tau)d\tau$$

$$h(t) = \frac{ds(t)}{dt} = s'(t)$$

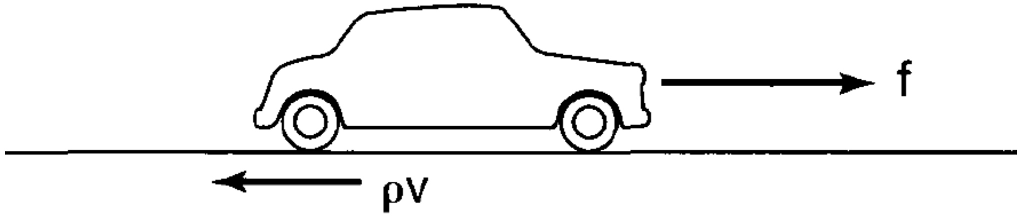
Chapter 2: Linear Time-Invariant Systems

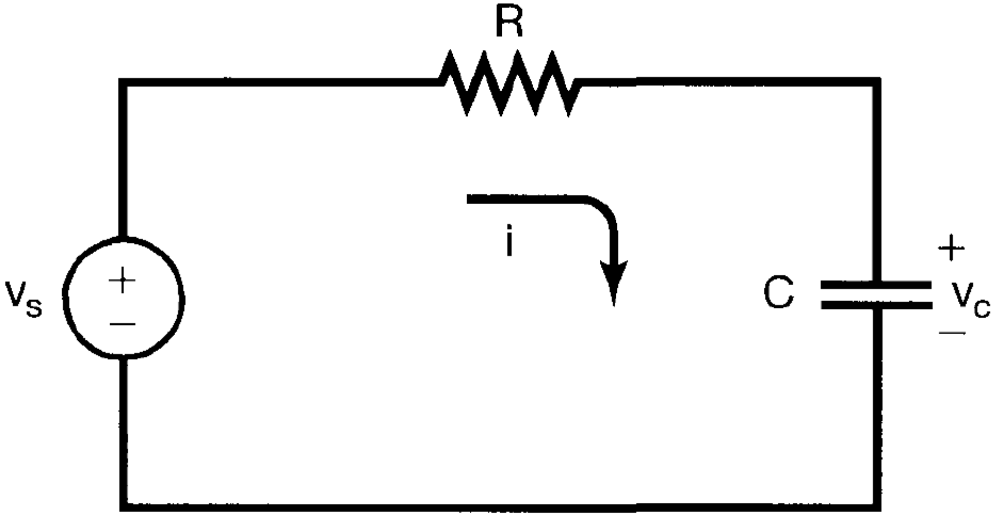
- ❑ Discrete-Time LTI Systems
- ❑ Continuous-Time LTI Systems
- ❑ Properties of LTI Systems
- ❑ Differential or Difference Equations

Differential or Difference Equations

Differential equation

- ❑ First order system





$$\frac{dv(t)}{dt} + \frac{\rho}{m} v(t) = \frac{1}{m} f(t)$$

$$\frac{dv_c(t)}{dt} + \frac{1}{RC} v_c(t) = \frac{1}{RC} v_s(t)$$

- ❑ In general: $\frac{dy(t)}{dt} + ay(t) = bx(t)$

Differential or Difference Equations

Differential equation

- ☐ Linear constant-coefficient DE $$\frac{dy(t)}{dt} + ay(t) = bx(t)$$

- ☐ Describes an implicit relationship between the input and the output

- ☐ Can not completely characterize a LTI system

- ☐ Auxiliary conditions are required to solve the DE: causal (initial rest condition)

Differential or Difference Equations

Differential equation

- ☐ First order system: example

$$\frac{dy(t)}{dt} + 2y(t) = x(t)$$

If $x(t) = Ke^{3t}u(t)$ $\quad y(t) = ?$

- ☐ Solution:

$$y(t) = y_p(t) + y_h(t)$$

$y_p(t)$: particular solution, *forced response (same form as input)*

$y_h(t)$: Homogenous solution $\quad \frac{dy(t)}{dt} + 2y(t) = 0$

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Differential or Difference Equations

Differential equation

☐ First order system: example $\frac{dy(t)}{dt} + 2y(t) = x(t)$

If $x(t) = Ke^{3t}u(t)$ $\quad y(t) = ?$

☐ Particular solution:

$$3Ye^{3t} + 2Ye^{3t} = Ke^{3t} \longrightarrow Y = K/5 \longrightarrow y_p(t) = \frac{K}{5}e^{3t} \text{ for } t>0$$

☐ Homogenous solution: Let $y_h(t) = Ae^{st}$, for t>0

$$Ase^{st} + 2Ae^{st} = 0 \longrightarrow s = -2 \longrightarrow y_h(t) = Ae^{-2t}$$

$$y(t) = Ae^{-2t} + \frac{K}{5}e^{3t}, \text{ for } t > 0$$

Differential or Difference Equations

Differential equation

$$y(t) = Ae^{-2t} + \frac{K}{5}e^{3t}, \text{ for } t > 0$$

☐ Auxiliary condition is required to determine A

☐ Initial rest as auxiliary condition for causal LTI systems: $y(0) = 0$

$$A + \frac{K}{5} = 0 \Rightarrow A = -\frac{K}{5} \Rightarrow y(t) = \frac{K}{5}\left(e^{3t} - e^{-2t}\right), \text{ for } t > 0$$

$$= \frac{K}{5}\left(e^{3t} - e^{-2t}\right)u(t)$$

Differential or Difference Equations

Differential equation

- ❑ General case: *N*th-order linear constant-coefficient differential equation

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

- ❑ Particular solution + Homogenous solution: $y(t) = y_p(t) + y_h(t)$
  - $y_p(t)$: *forced response (same form as input)*
  - $y_h(t)$: Natural response, $\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = 0$
- ❑ Initial rest as auxiliary condition, that is if $x(t) = 0$ for $t \le t_0$,

$$y(t_0) = \frac{dy(t_0)}{dt} = \cdots = \frac{d^{N-1}y(t_0)}{dt^{N-1}} = 0$$

Differential or Difference Equations

Difference equation

- ❑ General case: *N*th-order linear constant-coefficient difference equation

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

- ❑ Particular solution + Homogenous solution: $y[n] = y_p[n] + y_h[n]$
  - $y_p[n]$: *forced response (same form as input)*
  - $y_h[n]$: Natural response, $\sum_{k=0}^{N} a_k y[n-k] = 0$
- ❑ Initial rest as auxiliary condition, that is if $x[n] = 0$ for $n \le n_0$,

$$y[n_0] = y[n_0-1] = \cdots = y[n_0-(N-1)] = 0$$

Differential or Difference Equations

Difference equation

- ☐ Recursive solution:

$$y[n] = \frac{1}{a_0}\left\{\sum_{k=0}^{M} b_k x[n-k] - \sum_{k=1}^{N} a_k y[n-k]\right\}$$

- ➢ Particular case *N*=0

$$y[n] = \frac{1}{a_0}\sum_{k=0}^{M} b_k x[n-k]$$

Non-recursive equation

$$h[n] = \frac{1}{a_0}\sum_{k=0}^{M} b_k \delta[n-k]$$

Finite impulse response (FIR) system

Differential or Difference Equations

Difference equation

- ☐ Recursive solution: example $$y[n] - \frac{1}{2}y[n-1] = x[n]$$

  - Consider $x[n] = K\delta[n]$ and take initial rest: $y[-1] = 0$

$$y[0] = x[0] + \frac{1}{2}y[-1] = K \qquad y[1] = x[1] + \frac{1}{2}y[0] = \frac{1}{2}K$$

$$y[2] = x[2] + \frac{1}{2}y[1] = \left(\frac{1}{2}\right)^2 K \quad \cdots \quad y[n] = x[n] + \frac{1}{2}y[n-1] = \left(\frac{1}{2}\right)^n K$$

$$\therefore h[n] = \left(\frac{1}{2}\right)^n u[n]$$

Infinite impulse response (IIR) system

- ☐ Generally $$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k] \begin{cases} N = 0, \text{FIR system} \\ N > 0, \text{ usually IIR system} \end{cases}$$

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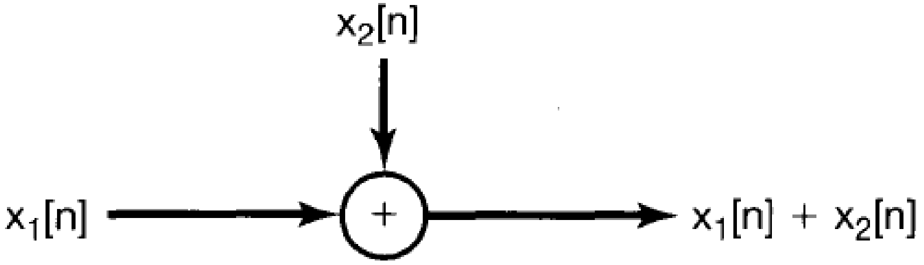
Differential or Difference Equations

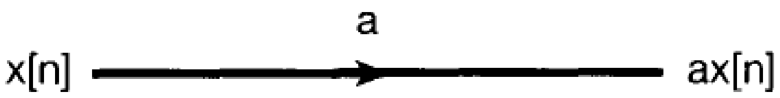
Block Diagram Representations

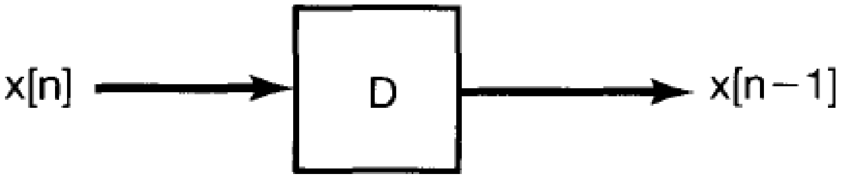
- Basic elements: discrete-time

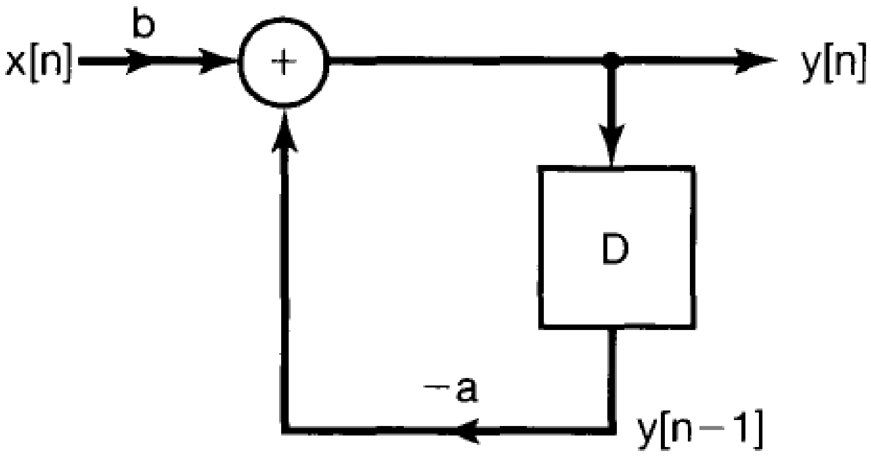
$$y[n] + ay[n-1] = bx[n]$$

$$y[n] = -ay[n-1] + bx[n]$$


$x_2[n]$
$x_1[n]$
+
$x_1[n] + x_2[n]$



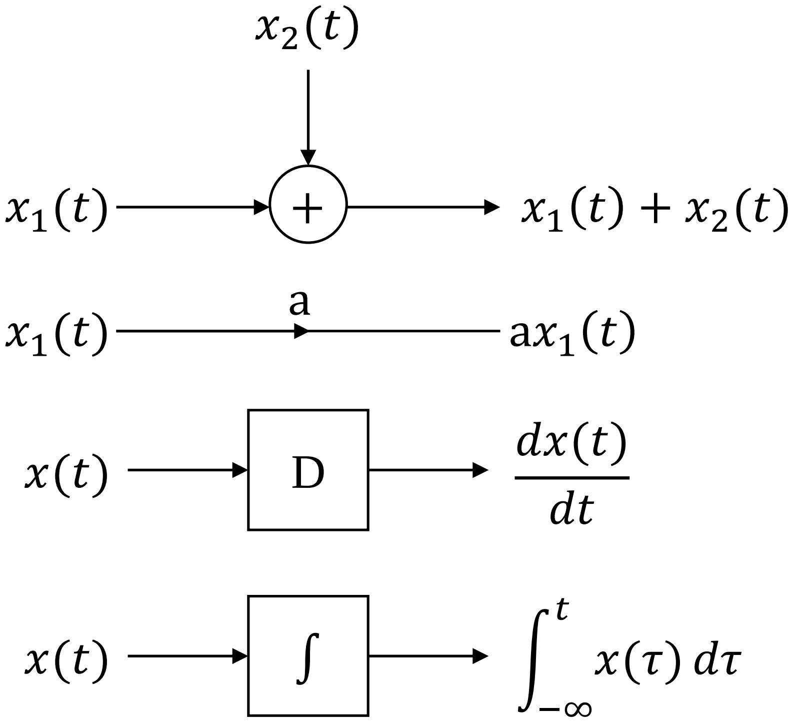



x[n]
b
+
y[n]
D
−a
y[n−1]

Differential or Difference Equations

Block Diagram Representations

- ❑ Basic elements: continuous-time



$x_2(t)$

$x_1(t)$ $+$ $x_1(t)+x_2(t)$

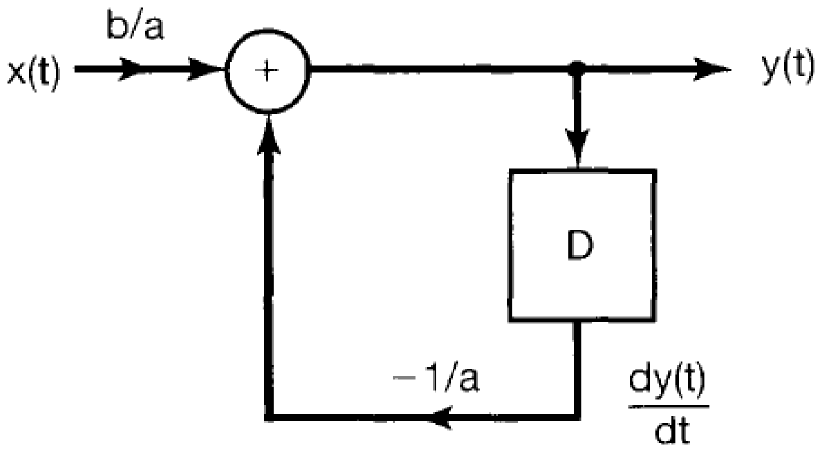
$x_1(t)$ a $ax_1(t)$

$x(t)$ D $\frac{dx(t)}{dt}$

$x(t)$ $\int$ $\int_{-\infty}^{t} x(\tau)\,d\tau$

$$\frac{dy(t)}{dt} + ay(t) = bx(t)$$

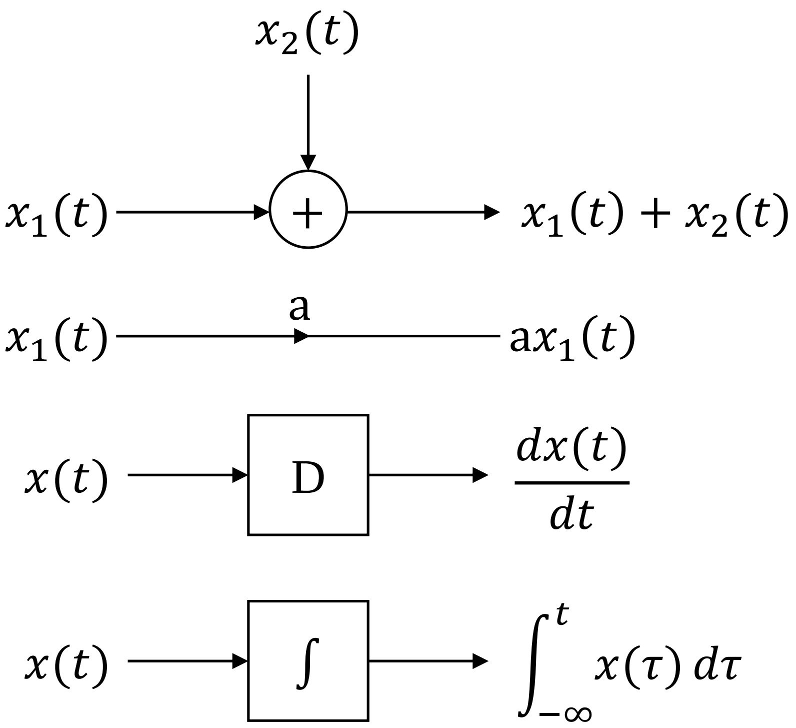
$$y(t) = -\frac{1}{a}\frac{dy(t)}{dt} + \frac{b}{a}x(t)$$



Differential or Difference Equations

Block Diagram Representations

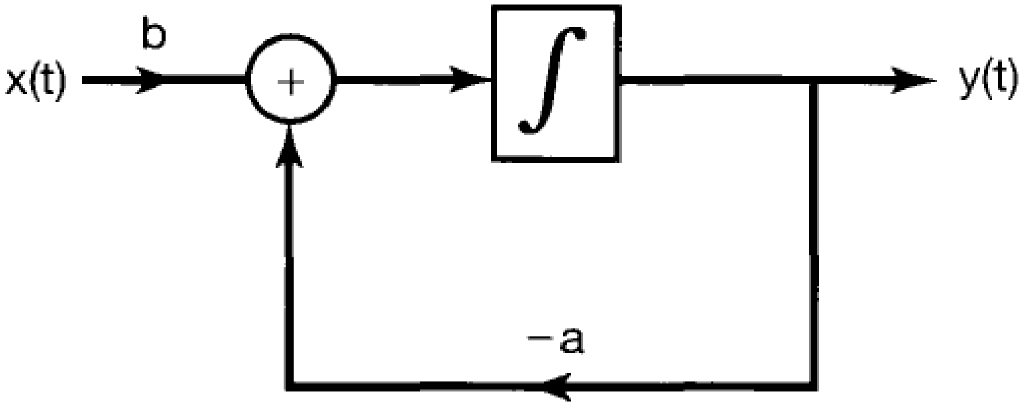
- ❑ Basic elements: continuous-time



$x_2(t)$
$x_1(t)$
+
$x_1(t) + x_2(t)$
$x_1(t)$
a
$\mathrm{a}x_1(t)$
$x(t)$
D
$\frac{dx(t)}{dt}$
$x(t)$
$\int$
$\int_{-\infty}^{t} x(\tau)\, d\tau$

$$\frac{dy(t)}{dt} + ay(t) = bx(t)$$

$$\frac{dy(t)}{dt} = -ay(t) + bx(t)$$

$$y(t) = \int_{-\infty}^{t} [bx(\tau) - ay(\tau)]d\tau$$