Homework 3

Due: 2025-04-20
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1 One-sided Matching (3pt)

Consider a House Allocation problem with 5 buyers $\{b_1, \dots, b_5\}$ and 5 houses$\{A, \dots, E\}$. Each buyer's strict preference is as follows:

$$\begin{aligned}
\mathrm{b}_1 &: A \succ_{b_1} B \succ_{b_1} C \succ_{b_1} D \succ_{b_1} E \\
\mathrm{b}_2 &: B \succ_{b_2} C \succ_{b_2} A \succ_{b_2} E \succ_{b_2} D \\
\mathrm{b}_3 &: A \succ_{b_3} E \succ_{b_3} D \succ_{b_3} C \succ_{b_3} B \\
\mathrm{b}_4 &: D \succ_{b_4} E \succ_{b_4} C \succ_{b_4} A \succ_{b_4} B \\
\mathrm{b}_5 &: D \succ_{b_5} B \succ_{b_5} E \succ_{b_5} C \succ_{b_5} A
\end{aligned}$$

1.1 Serial Dictatorship (1pt)

Suppose the ranking of the buyers are $\{b_1, b_2, b_3, b_4, b_5\}$, buyers will choose their preferred house one by one according to this sequence. What will the final allocation be? Is this allocation Pareto optimal?

The final allocation will be: $\{A : b_1, B : b_2, C : b_5, D : b_4, E : b_3\}$. It is pareto optimal.

1.2 TTC (2pt)

Suppose the initial allocation is $\{A : b_1, B : b_2, C : b_3, D : b_4, E : b_5\}$. Run the Top Trading Cycle algorithm on this allocation, give the step-by-step process and the final allocation.

Step 1: Buyer b_1 points to herself, buyer b_2 points to herself, buyer b_3 points to buyer b_1, b_4 to herself and b_5 to b_4. So buyers b_1, b_2, b_4 trades with themselves and goes out of the cycle.
Step 2: Buyer b_3 and buyer b_5 is left. b_5 now points to herself and b_3 to b_5. So b_5 trades with herself and goes out of the cycle.
Step 3: Buyer b_3 is now the only one left. She has to trade with herself.
So the final allocation is:$\{A : b_1, B : b_2, C : b_3, D : b_4, E : b_5\}$

2 Two-sided Matching (4pt)

Consider a Public School Choice problem with five students $\{s_1, \ldots, s_5\}$ and four schools $\{c_1, \ldots, c_4\}$. Only c_2 has two slots; every other school has only one slot. All schools have the same priority list: $s_1 \succ s_2 \succ s_3 \succ s_4 \succ s_5$ since they prefer students with higher scores. The students' preferences are as follows:

$$
\begin{aligned}
s_1 &: c_1 \succ_{s_1} c_2 \succ_{s_1} c_3 \succ_{s_1} c_4 \\
s_2 &: c_1 \succ_{s_2} c_2 \succ_{s_2} c_4 \succ_{s_2} c_3 \\
s_3 &: c_1 \succ_{s_3} c_3 \succ_{s_3} c_2 \succ_{s_3} c_4 \\
s_4 &: c_2 \succ_{s_4} c_1 \succ_{s_4} c_3 \succ_{s_4} c_4 \\
s_5 &: c_3 \succ_{s_5} c_4 \succ_{s_5} c_2 \succ_{s_5} c_1
\end{aligned}
$$

2.1 Boston Mechanism

The Boston mechanism, which was used in Boston high schools until 2005, is defined as follows:

1. In the first step, students submit proposals to their top preferred school and are matched based on school priority and available capacity.
2. In every following step $(k > 1)$: unmatched students propose to their k-th most preferred school. They are then matched to a school according to the priority of the school based on school priority and available capacity. The process terminates when all students have been matched.

(a) Show the matching results of the Boston mechanism and the steps involved. (1pt).

1. proposal: $(c_1 : s_1, s_2, s_3), (c_2 : s_4), (c_3 : s_5)$. match: $(c_1, s_1), (c_2 : s_4), (c_3 : s_5)$.
2. proposal: $(c_2 : s_2), (c_3 : s_3)$. match: $(c_2 : s_2)$.
3. proposal: $(c_2 : s_3)$. match: $\emptyset$.
4. proposal: $(c_4 : s_3)$. match: $(c_4 : s_3)$.

matching results: $(c_1 : s_1), (c_2 : s_2, s_4), (c_3 : s_5), (c_4 : s_3)$.

(b) Does any student have an incentive to misreport their preference (assuming that other students report truthfully)? Give a proof. (1pt)

Yes. s_3 proposes to c_3 in step 1 and gets matched, which better than c_4.

2.2 Shanghai Mechanism

The Shanghai mechanism was first implemented as a high school admissions mechanism in Shanghai. In 2008, variations of this mechanism were implemented in other provinces as parallel college admissions mechanisms, replacing the previously used sequential mechanisms. We now describe a stylized version of the parallel mechanism, specifically adapted for the school choice context as follows:

1. Students propose to their top preferred school. If a school receives more proposals than its capacity permits, it will retain the students with the highest priority up to the limit of its capacity. Any extra proposals will be rejected (Note that throughout the allocation process, a school cannot hold more proposals than its capacity).

   When students are rejected by schools, their proposal is sent to their next highest-ranked school. When a school receives new proposals, they are evaluated along with the existing retained proposals for that school. The proposals with the highest priority, up to the capacity of the school, are selected and retained from the pool of new and retained proposals.

2. The allocation process is completed at regular intervals of e steps, where $e = 2$ here. Each student is assigned to a school that holds their proposal during that step. Once assigned, these students and their assignments are removed from the system.

3. The allocation process terminates when all students have been matched.

(a) Show the matching results of the Shanghai mechanism and the steps involved. (1pt)

1. proposal: $(c_1 : s_1, s_2, s_3), (c_2 : s_4), (c_3 : s_5)$.
2. proposal: $(c_1 : s_1), (c_2 : s_2, s_4), (c_3 : s_3, s_5)$. match: $(c_1 : s_1), (c_2 : s_2, s_4), (c_3 : s_3)$.
3. proposal: $(c_4 : s_5)$. match: $(c_4 : s_5)$.

matching results: $(c_1 : s_1), (c_2 : s_2, s_4), (c_3 : s_3), (c_4 : s_5)$.

(b) Given the above students' preferences, does any student have an incentive to misreport their preference (assuming that other students report truthfully)? Give a proof. (1pt)

No. s_1 and s_4 already get their most preferred school. s_2 and s_3 get their second preferred school since they compete with s_1 for c_1. s_5 gets his second preferred school since it competes with s_3 for c_3.
Note: the Shanghai mechanism itself is not incentive compatible.

3 Social Choice and Voting (3pt)

The veto rule is the following social-choice rule:

- Every voter names their least favorite alternative.
- The rule then selects the alternative that is named the least number of times.

3.1 Positional Scoring Rule (1pt)

A positional scoring rule (PSR) is defined by a so-called scoring vector

$$s = (s_1, s_2, \ldots, s_m) \in \mathbb{R}^m$$

with $s_1 \geq s_2 \geq \cdots \geq s_m$ and $s_1 > s_m$.
Each voter submits a ranking of the m alternatives. Each alternative receives s_i points for every voter putting it at the ith position.
The alternative(s) with the highest score (sum of points) win(s).
Formalize the veto rule as a positional scoring rule.

scoring vector $(0, 0, \ldots, -1)$.

3.2 Plurality (1pt)

Formalize the plurality rule as a positional scoring rule. The veto rule is also called the anti-plurality rule since the plurality rule selects the alternative that was placed first by the largest number of voters.

scoring vector $(1, 0, \ldots, 0)$.

3.3 Incentive Compatible (1pt)

Consider whether the veto rule is incentive compatible and give a proof.

The veto rule is not incentive compatible.
For lexicographic tie-breaking, suppose A and B are both winners, but A wins by tie-breaking. Voter v prefers $B \succ A \succ \ldots$. v can name A as his least favorite alternative and B becomes the unique winner.
For more than one alternatives can all be winners, suppose A wins over B by one fewer least favourite. Voter v prefers $B \succ A \succ \ldots$. v can name A as his least favorite alternative and B can also be a winner.