Ch.2 *Linear Time-Invariant Systems*

Lecturer: Yijie Mao

Part II *Continuous-Time LTI Systems*

Outline

- Representation of Continuous-Time Signals in Terms of Impulses
- The Continuous-Time Unit Impulse Response
- The Convolution-Integral Representation of LTI Systems

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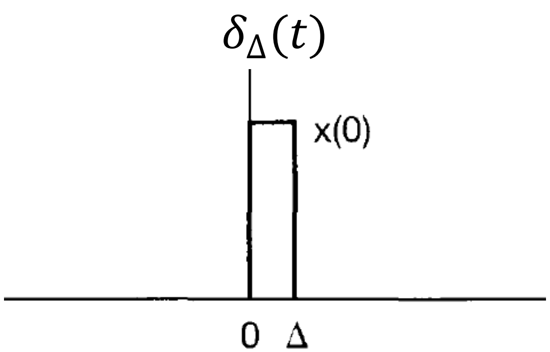
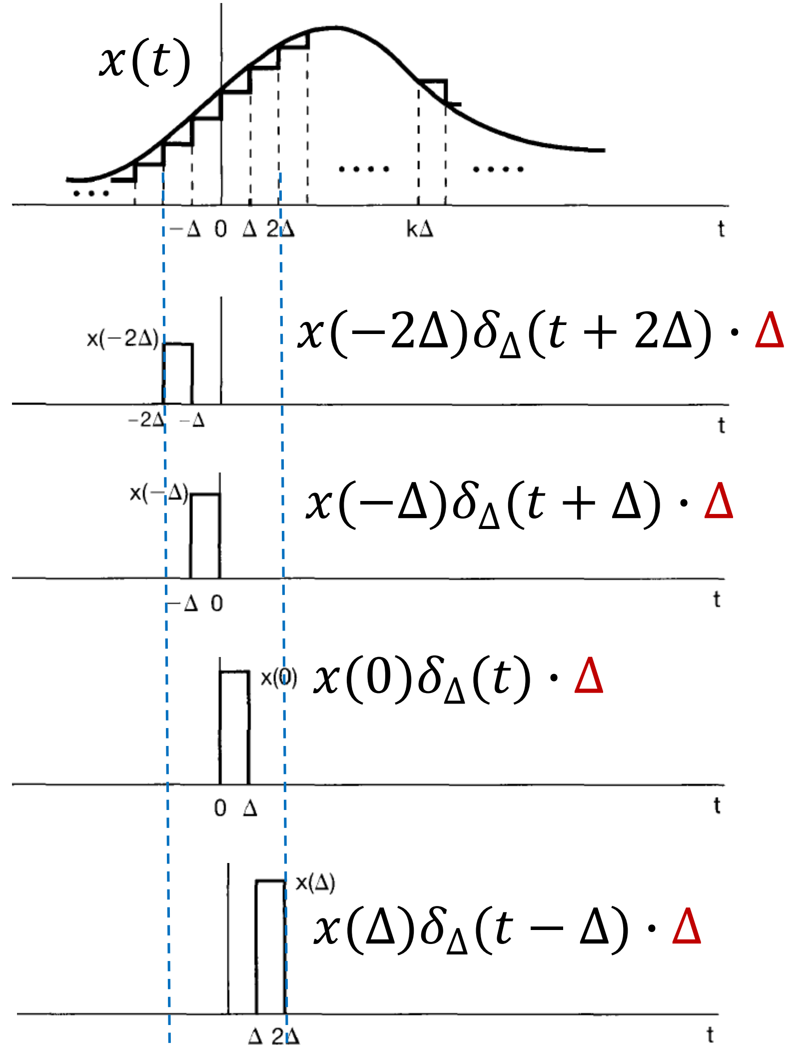
Representation of Continuous-Time Signals in Terms of Impulse

- "staircase" approximation of $x(t)$

$$\delta_\Delta(t) = \begin{cases} \dfrac{1}{\Delta}, & 0 \le t \le \Delta \\ 0, & otherwise \end{cases}$$





Representation of Continuous-Time Signals in Terms of Impulse

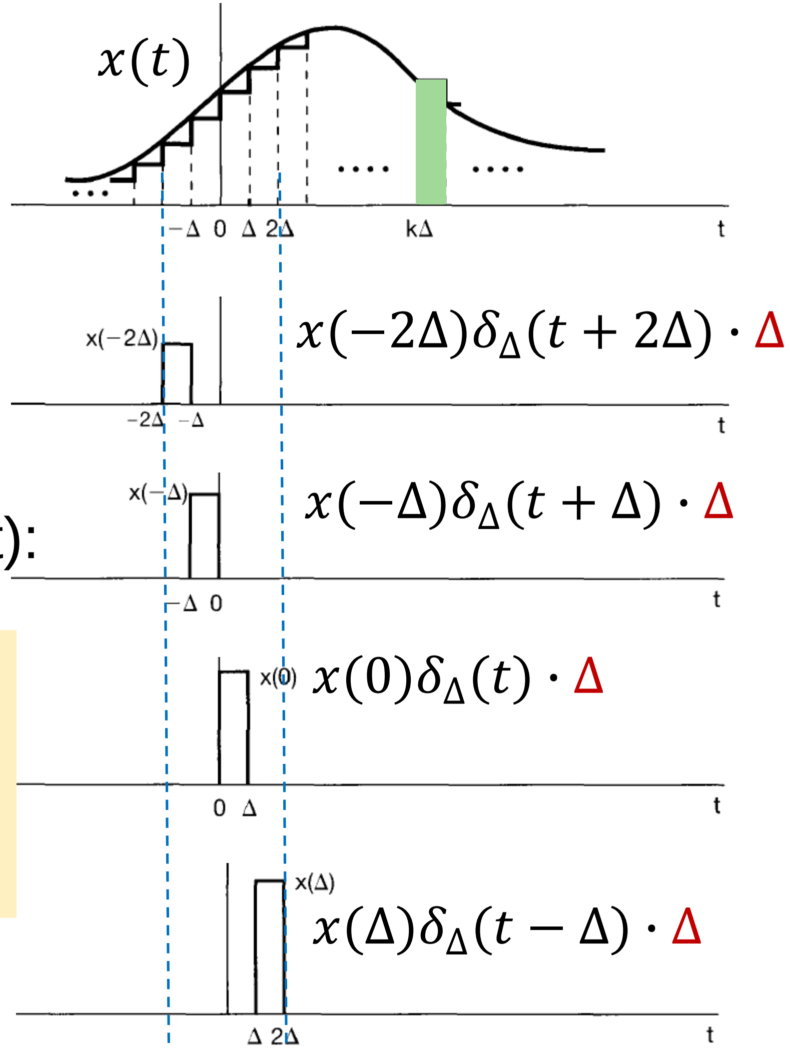
- "staircase" approximation of $x(t)$

$$\hat{x}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\delta_{\Delta}(t-k\Delta)\cdot\Delta$$

- When Δ approaches 0, the approximation $\hat{x}(t)$ becomes better and better, and in the limit equals x(t):

$$x(t) = \lim_{\Delta\to 0}\hat{x}(t)$$
$$= \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau$$

Sifting property of $\delta(t)$



Representation of Continuous-Time Signals in Terms of Impulse

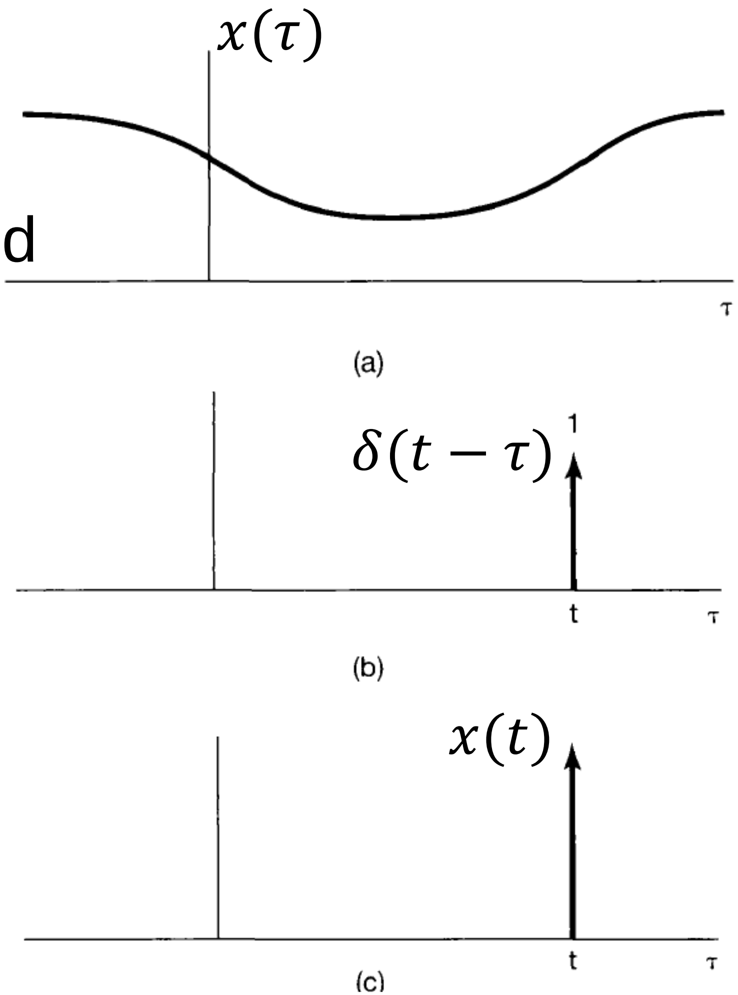
- A continuous-time signal is the superposition of scaled and shifted pulses.

$$\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau$$

$$= \int_{-\infty}^{\infty} x(t)\delta(t-\tau)\,d\tau$$

$$= x(t)\int_{-\infty}^{\infty} \delta(t-\tau)\,d\tau$$

$$= x(t)$$



$$x(\tau)\delta(t-\tau) = x(t)\delta(t-\tau)$$

Representation of Continuous-Time Signals in Terms of Impulse

- Example. Represent $u(t)$ in terms of impulse.

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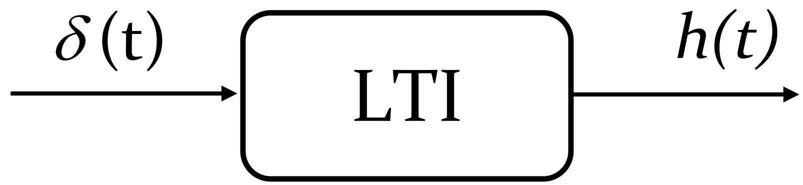
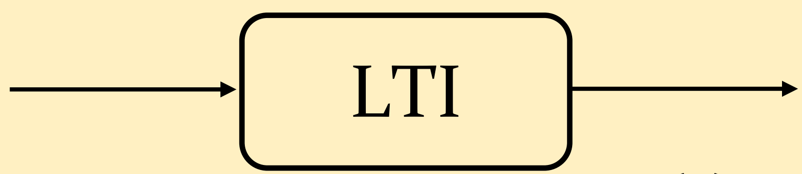
Continuous-Time Unit Impulse Response

- The response of a system to a unit impulse $\delta(t)$ is called the **unit impulse response**, denoted by $h(t)$.



- Unit impulse response completely characterizes an LTI system.



$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\, d\tau$$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\, d\tau$$

Integral of weighted and shift impulses

Integral of weighted and shift impulse response

Continuous-Time Unit Impulse Response

- Response $\hat{y}(t)$ of an LTI system is the superposition of the responses to the scaled and shifted versions of $\delta_\Delta(t)$.

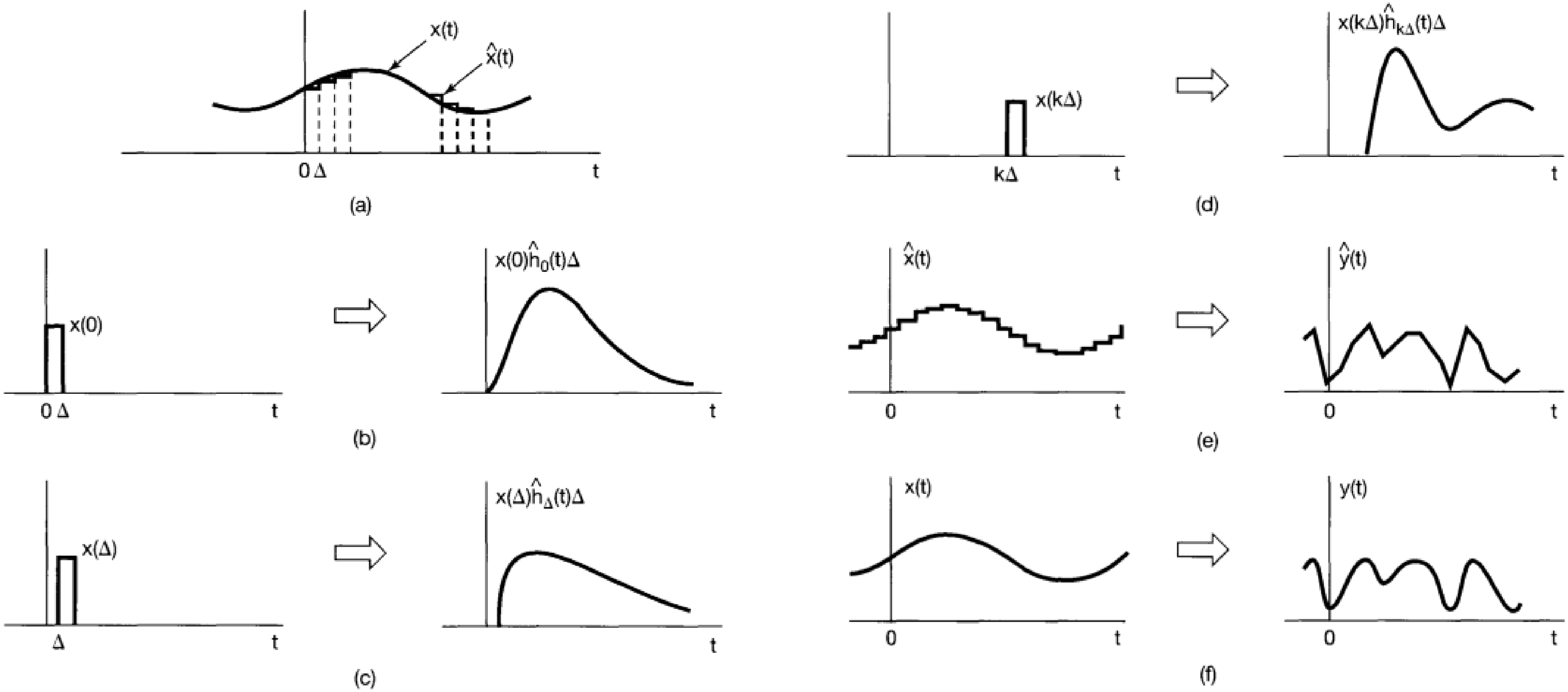
- Denote $\hat{h}_{k\Delta}(t)$ as the response to input $\delta_\Delta(t-k\Delta)$.

$$\hat{y}(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\hat{h}_{k\Delta}(t)\Delta$$

- As $\Delta \rightarrow 0$, convolution integral is

$$y(t) = \int_{-\infty}^{\infty} x(\tau)h_\tau(t)\, d\tau$$
$$= \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\, d\tau$$

Graphical Interperation



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The Convolution Integral

- The convolution integral

$$x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\, d\tau$$

- **Step 1**: Change time variables $x(t) \to x(\tau)$, $h(t) \to h(\tau)$, and reverse $h(\tau) \to h(-\tau)$

- **Step 2**: Shift $h(-\tau) \to h(t-\tau)$

- **Step 3**: Multiply $x(\tau) \cdot h(t-\tau)$

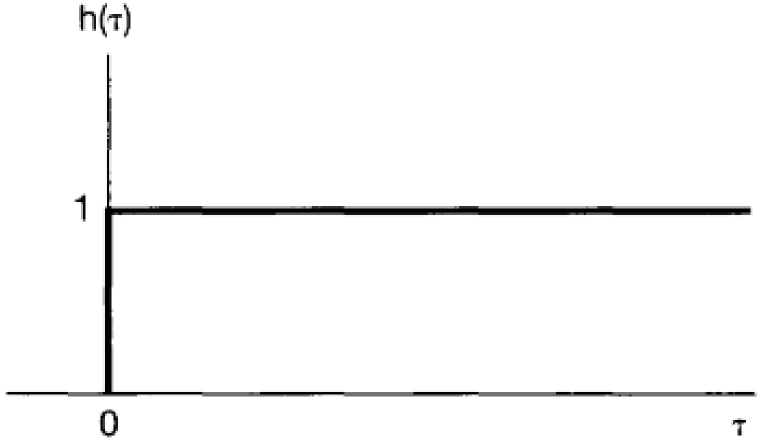
- **Step 4**: Integral $\int_{-\infty}^{\infty} x(\tau) \cdot h(t-\tau)\mathrm{d}\tau$

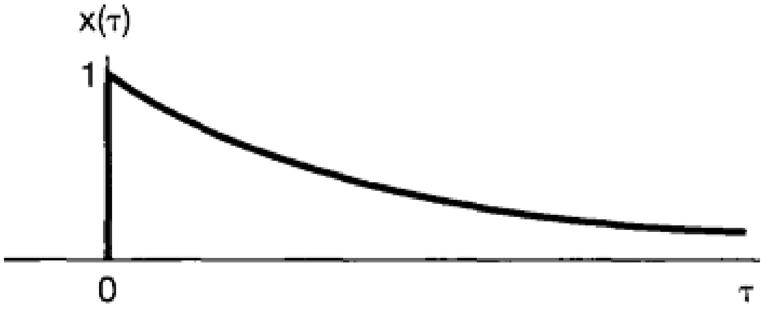
The Convolution Integral

- Example 1: Let x(t) be the input to an LTI system with unit impulse response h(t), where

$$x(t) = e^{-at}u(t),\ h(t) = u(t)$$

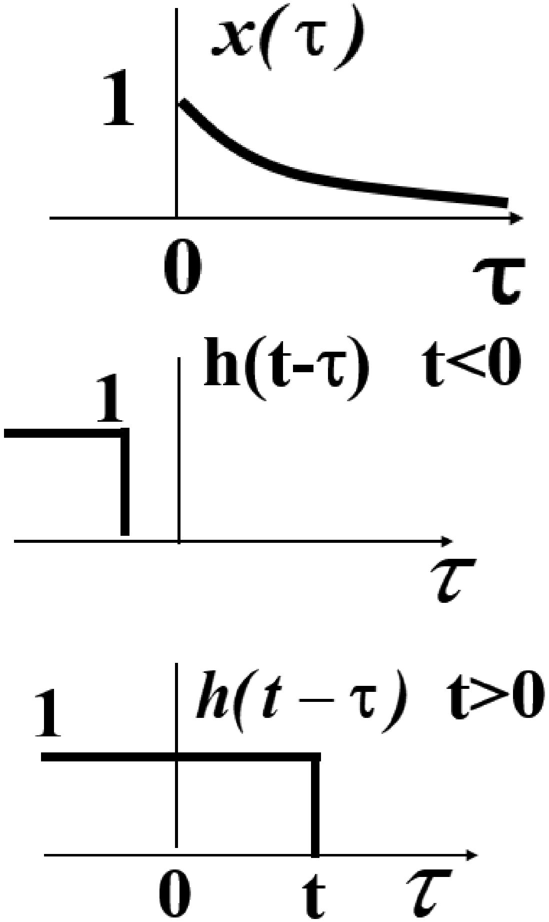
$x(t) * h(t) = ?$





The Convolution Integral

- Solution: τ : variable, t: constant



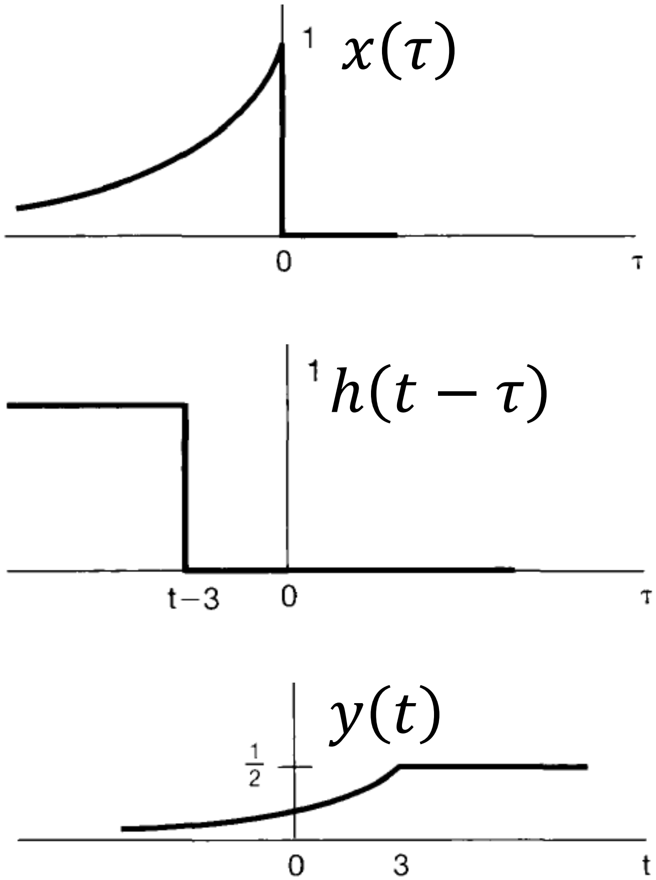
The Convolution Integral

- Example 2: Let x(t) be the input to an LTI system with unit impulse response h(t), where

$$x(t) = e^{2t}u(-t),\ \ h(t) = u(t-3)$$

$$x(t) * h(t) = ?$$

- Solution:

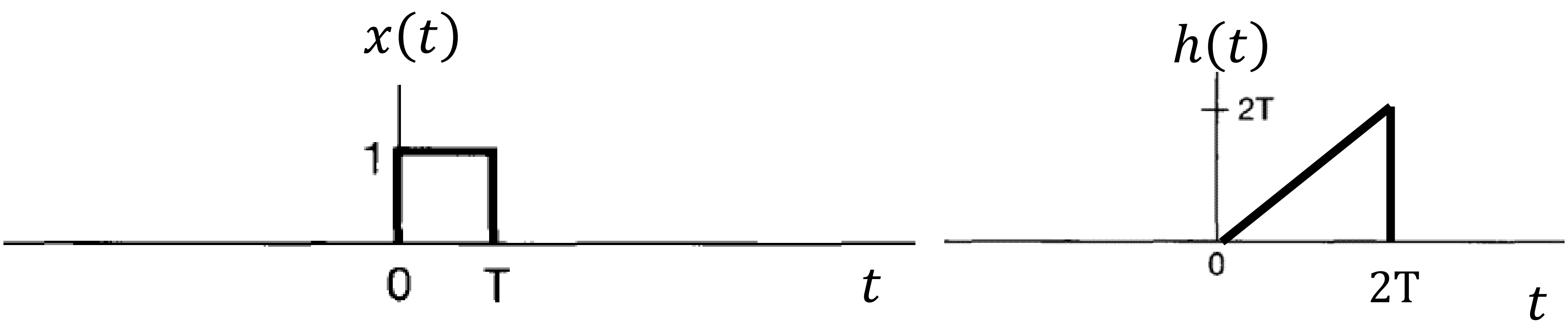


The Convolution Integral

- Example 3: Let x(t) be the input to an LTI system with unit impulse response h(t), where

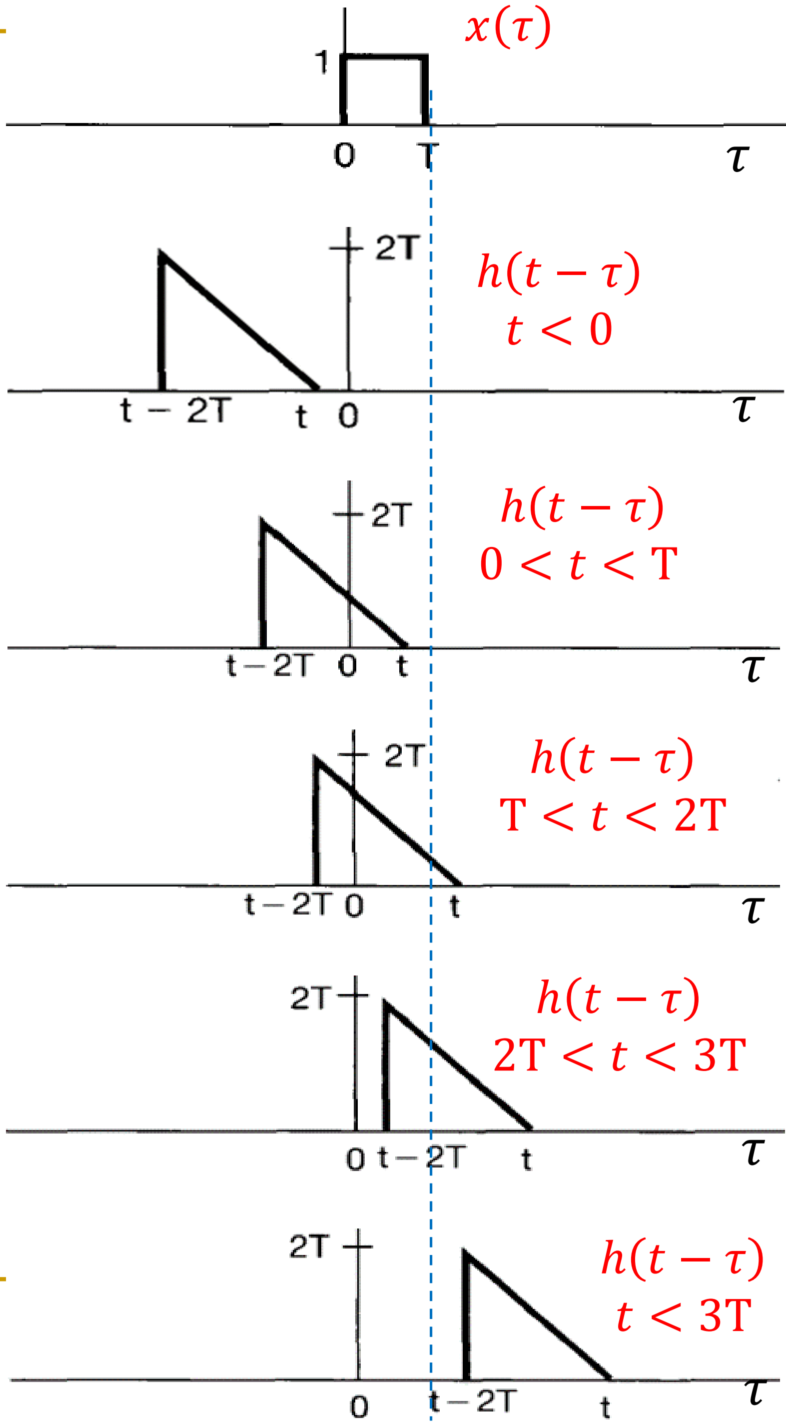
$$x(t) = \begin{cases} 1, 0 < t < T \\ 0, \text{otherwise} \end{cases} \qquad h(t) = \begin{cases} t, 0 < t < 2T \\ 0, \text{otherwise} \end{cases}$$

$$x(t) * h(t) = ?$$



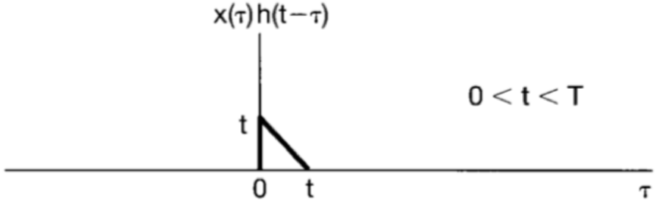
The Convolution Integral

- Solution:

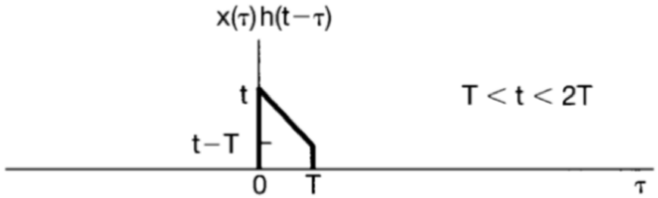


The Convolution Integral

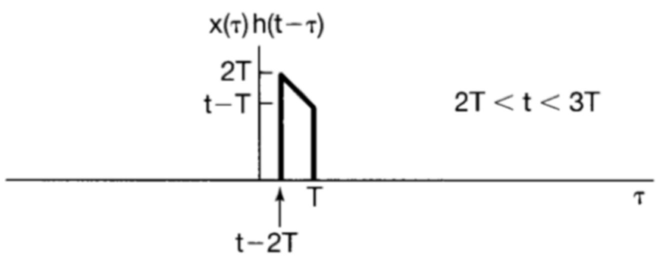
- Solution:



(a)

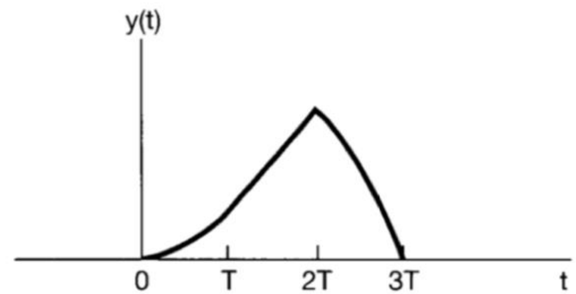


(b)



(c)

$$y(t) = \begin{cases} 0, t < 0 \\ \int_0^t (t-\tau)d\tau = \frac{1}{2}t^2, 0 < t < T \\ \int_0^T (t-\tau)d\tau = Tt - \frac{1}{2}T^2, T < t < 2T \\ \int_{t-2T}^T (t-\tau)d\tau = -\frac{1}{2}t^2 + Tt + \frac{3}{2}T^2, 2T < t < 3T \\ 0, t > 3T \end{cases}$$



The Convolution Operation

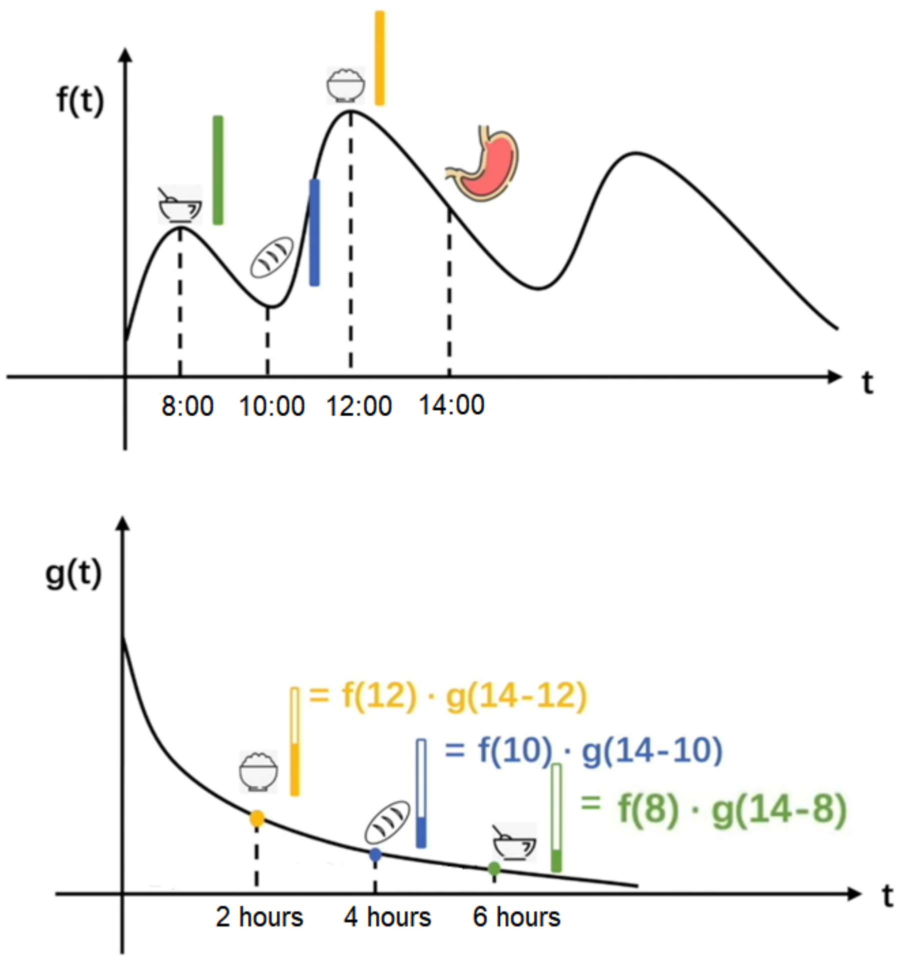
- Convolution of two signals $x(t)$ and $h(t)$, denoted by $x(t) * h(t)$ is defined by

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\, d\tau$$

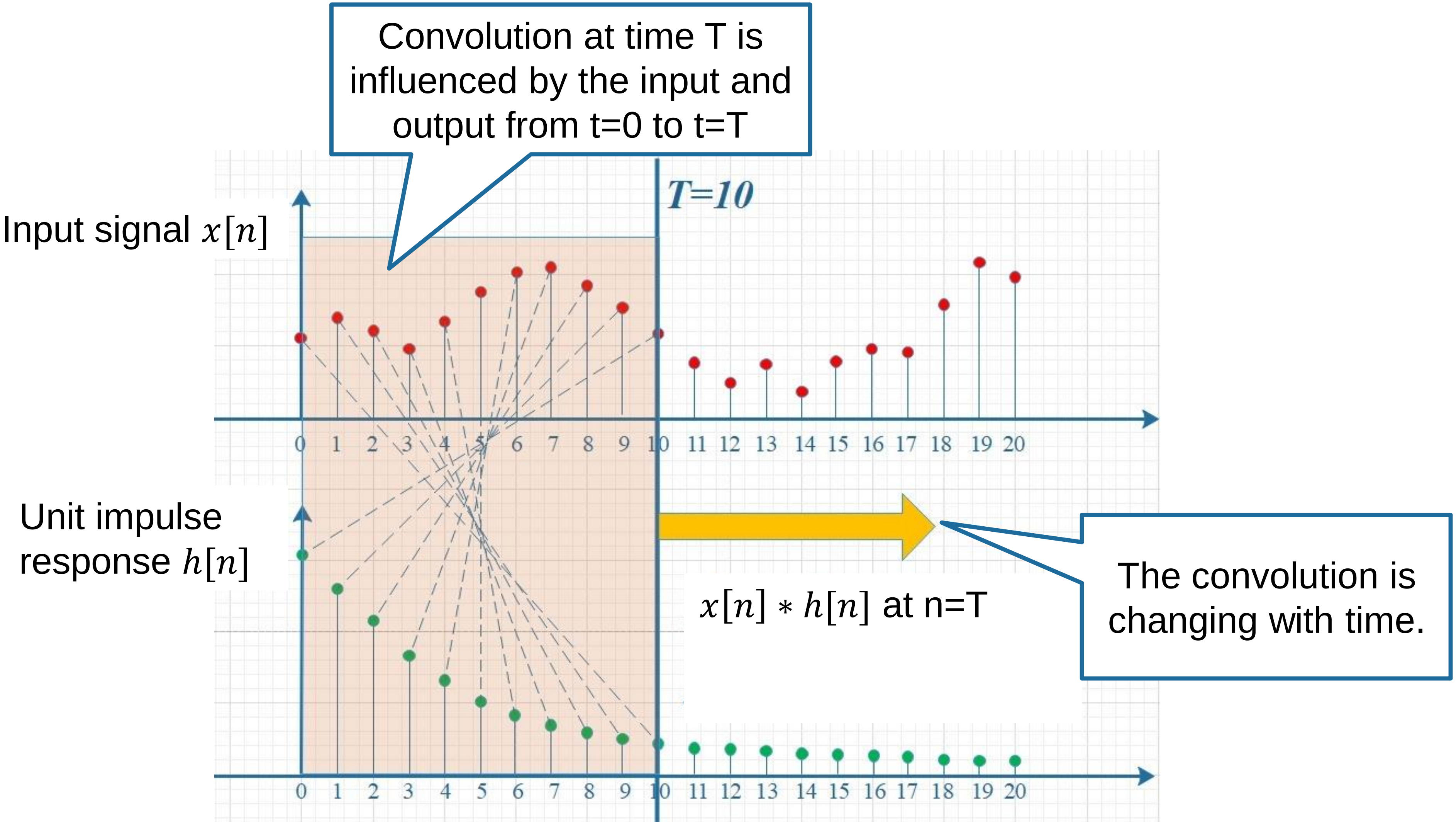
- For discrete-time

$$y[n] = x[n] * h[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

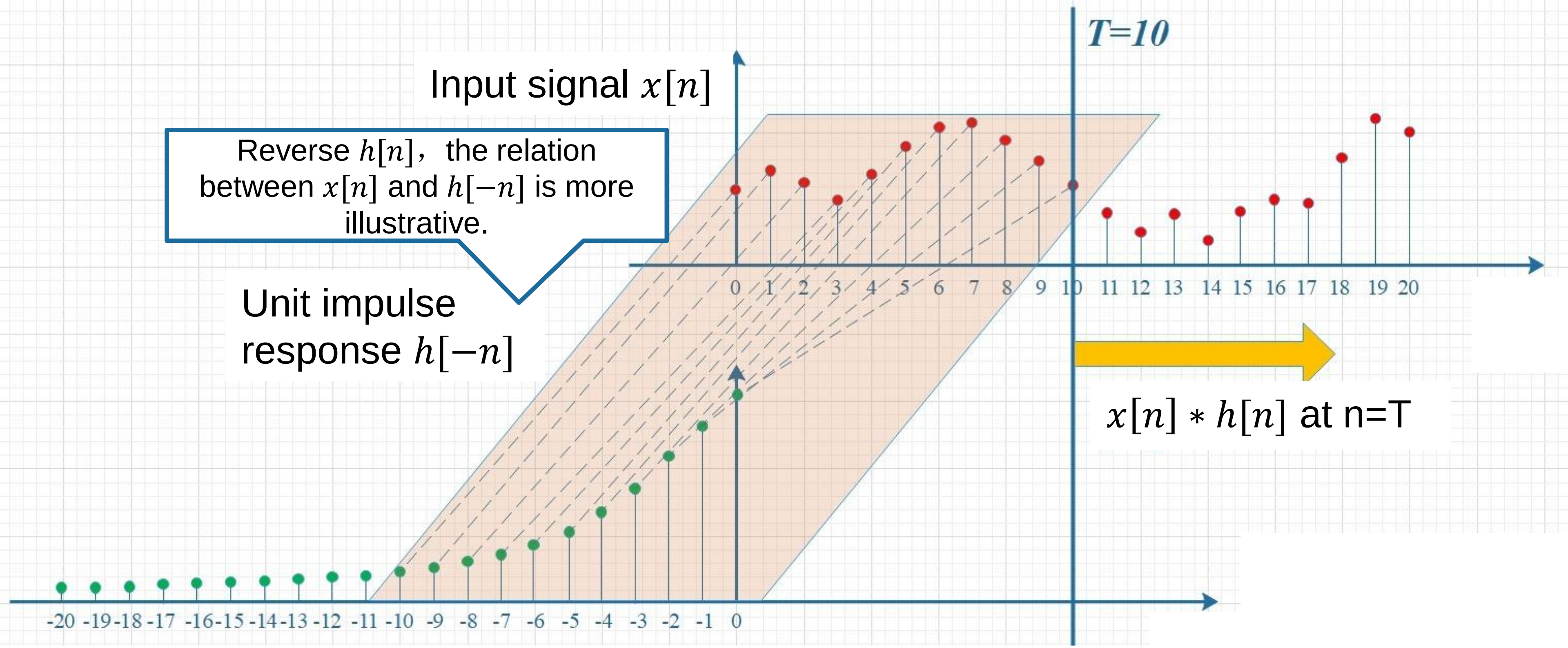
The Convolution Operation



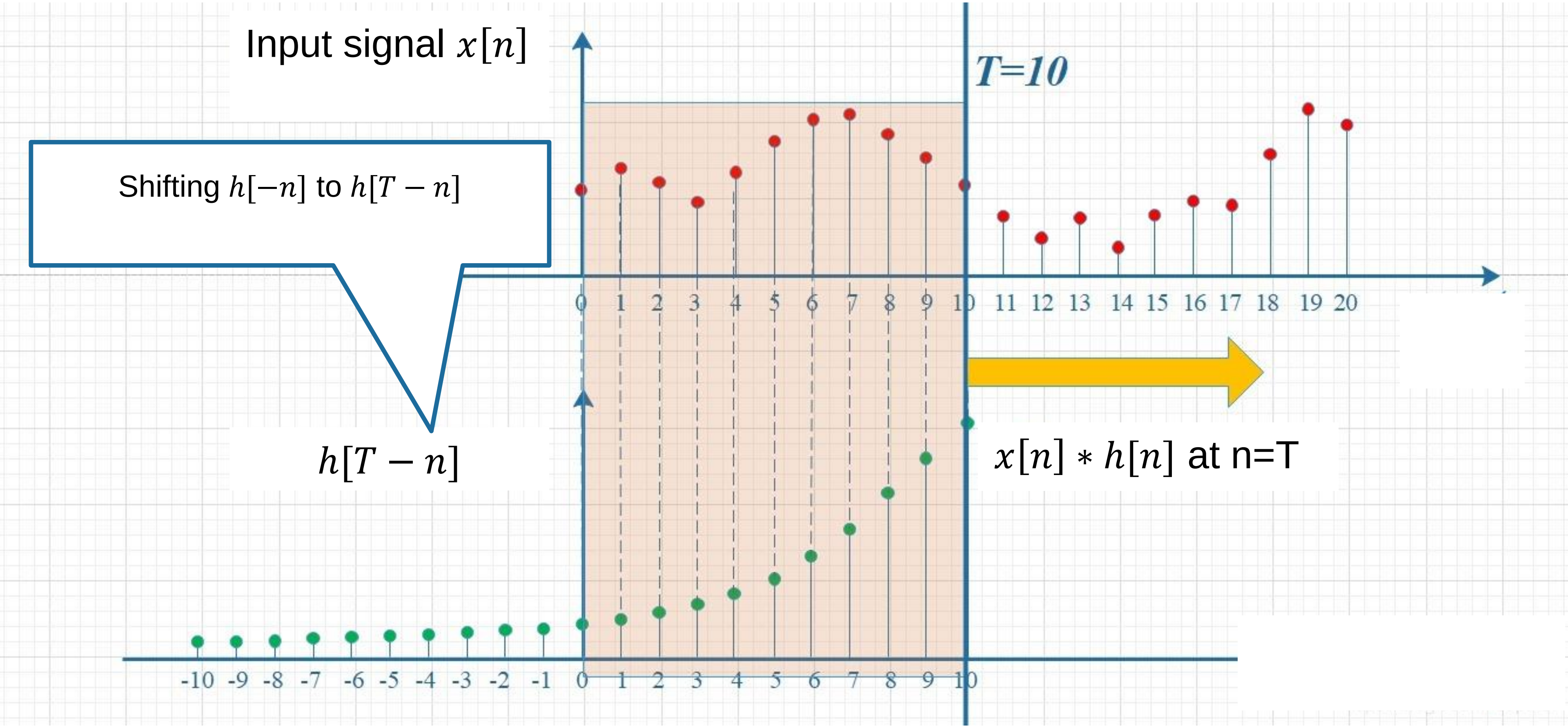
The Convolution Operation



The Convolution Operation



The Convolution Operation



Summary

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- Reference in textbook:
  - 2.2