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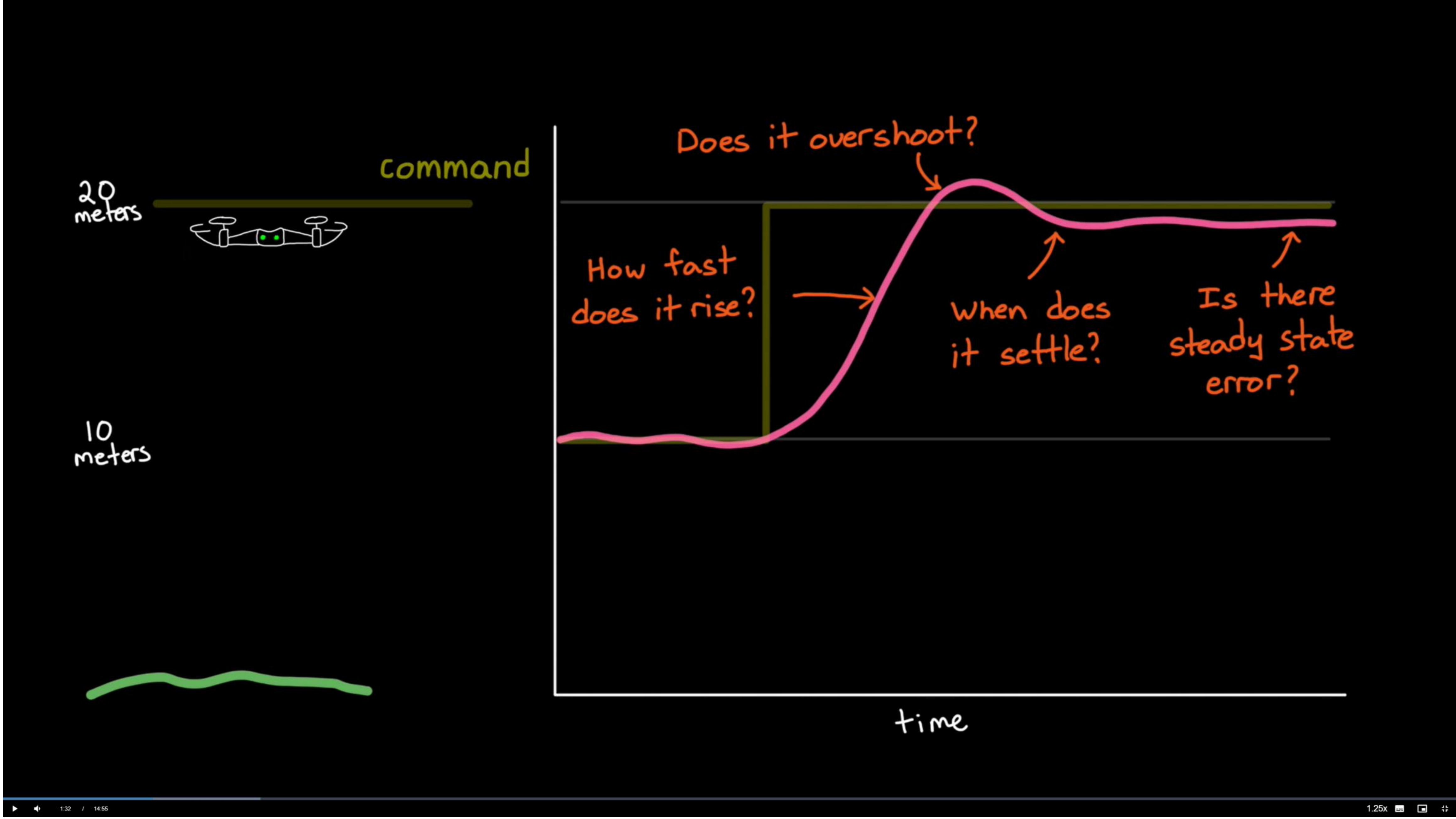
Lecture 4 The performance of Feedback Control System

J. Chen



[The Step Response | Control Systems in Practice Video - MATLAB (mathworks.cn)]

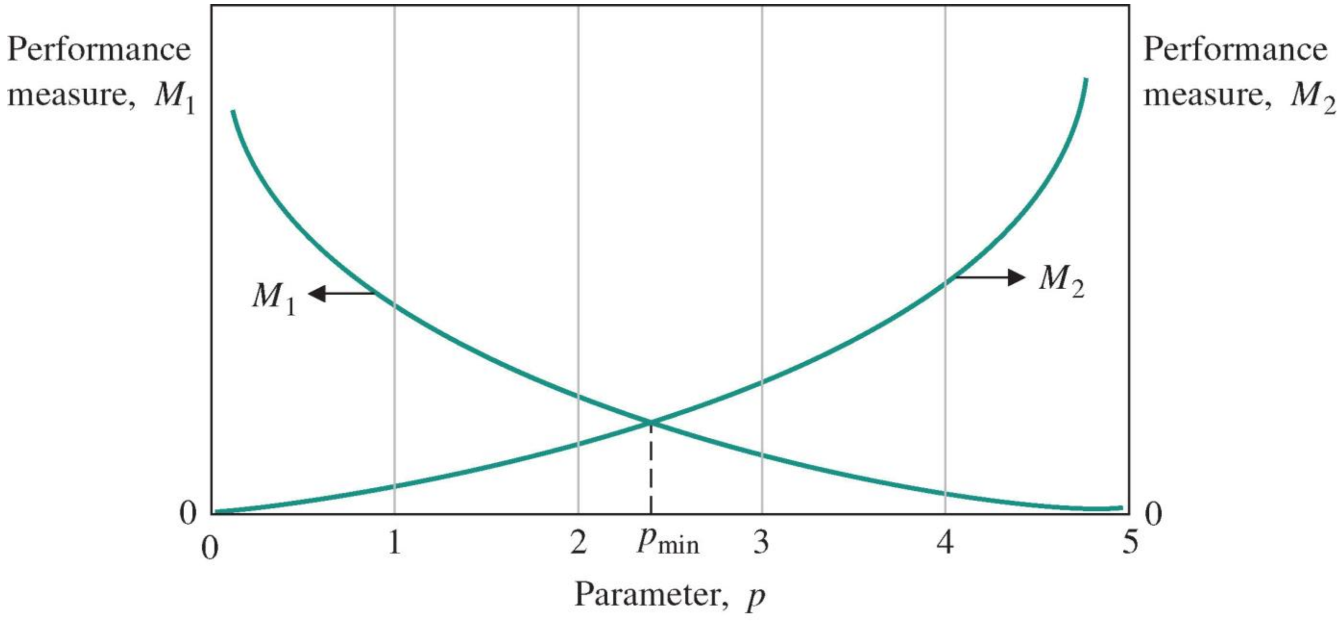
Introduction

Feedback has the ability to adjust the transient and steady-state response of a control system, but How?

I. Define and measure its performance, i.e. specifications in terms of transient response and steady state

II. Figure out the correlation between the system performance and transfer function, to be specific, the poles and zeros. Even, some tunable parameter.

III. Optimize and compromise



Test Input Signals

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The response to a specific input signal will provide several measures of the performance.

Because the actual input signal of the system is usually unknown, **a standard test input signal is normally chosen**.

A rectangular function

$$f_\epsilon(t) = \begin{cases} 1/\epsilon, & -\frac{\epsilon}{2} \le t \le \frac{\epsilon}{2}; \\ 0, & \text{otherwise}, \end{cases}$$

As ϵ approaches zero, $f_\epsilon(t)$ approaches the unit impulse function $\delta(t)$

$$\int_{-\infty}^{\infty} \delta(t)\, dt = 1 \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(t-a) g(t)\, dt = g(a).$$

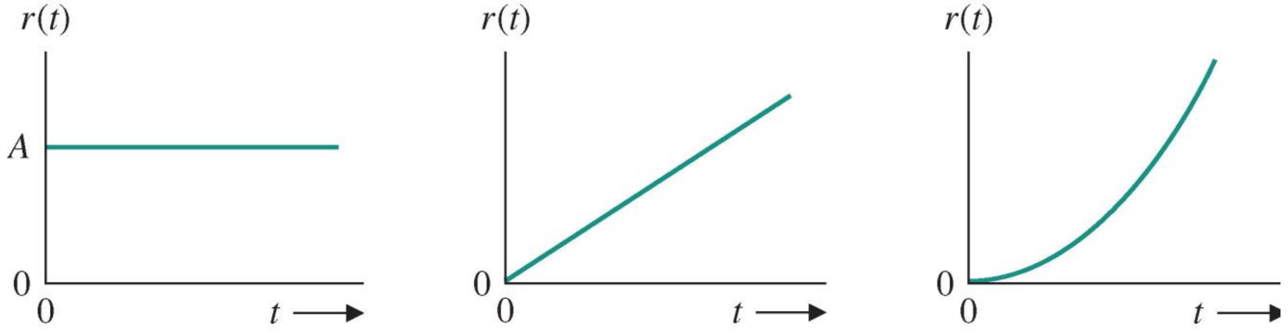
The impulse input is useful when we consider

$$y(t) = \int_{-\infty}^{t} g(t-\tau)\delta(\tau)\, d\tau = g(t),$$

is the impulse response of the system $G(s)$.

Test Input Signals

Other typical test signals



(a) step (b) ramp (c) parabolic.

$$t^n \longleftrightarrow \frac{n!}{s^{n+1}}$$

Table 5.1 Test Signal Inputs

Test Signal	$r(t)$	$R(s)$
Step	$r(t) = A, t > 0$ $= 0, t < 0$	$R(s) = A/s$
Ramp	$r(t) = At, t > 0$ $= 0, t < 0$	$R(s) = A/s^2$
Parabolic	$r(t) = At^2, t > 0$ $= 0, t < 0$	$R(s) = 2A/s^3$

Test Input Signals

Consider a system injected by a unit step input

$$G(s) = \frac{9}{s + 10}. \qquad R(s) = 1/s,$$

Then the output is

$$Y(s) = \frac{9}{s(s + 10)},$$

$$y(t) = 0.9(1 - e^{-10t}),$$

and the steady-state response is

$$y(\infty) = 0.9.$$

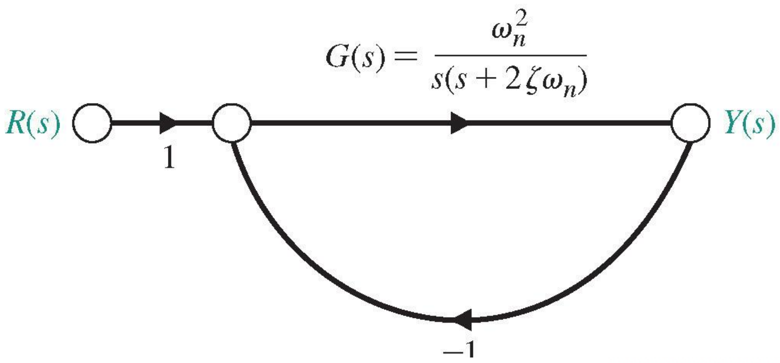
Leads to the steady-state error

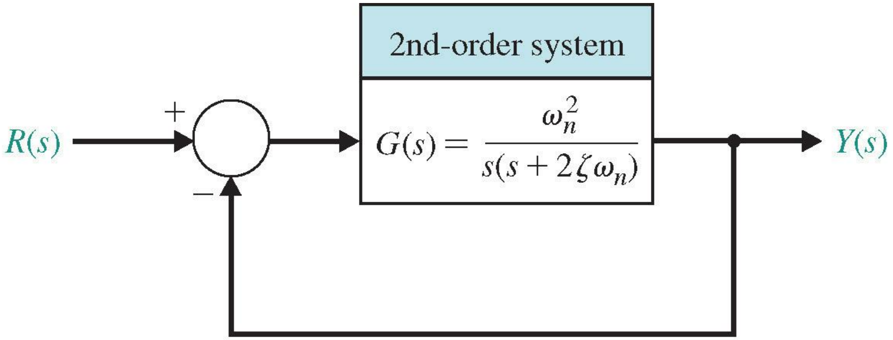
$$e_{ss} = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{s + 1}{s + 10} = 0.1.$$

Q: can we design a good open loop controller $G_c(s)$?

Performance of 2nd-order System

Let us consider a **single-loop second-order system**





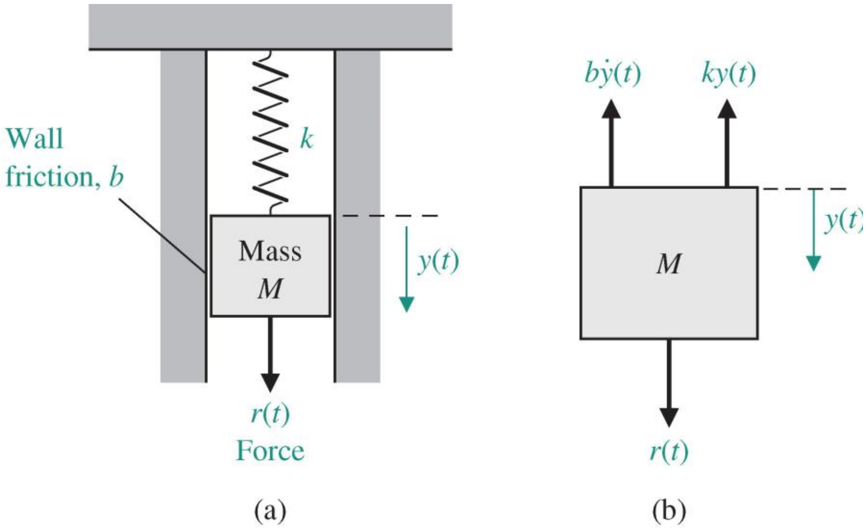
damping ratio $0 < \zeta < 1.$

natural frequency $\omega_n > 0$

$$Y(s) = \frac{G(s)}{1 + G(s)} R(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} R(s).$$

Recall the spring-mass-damper system

$$G(s) = \frac{Y(s)}{R(s)} = \frac{1}{Ms^2 + bs + k}$$



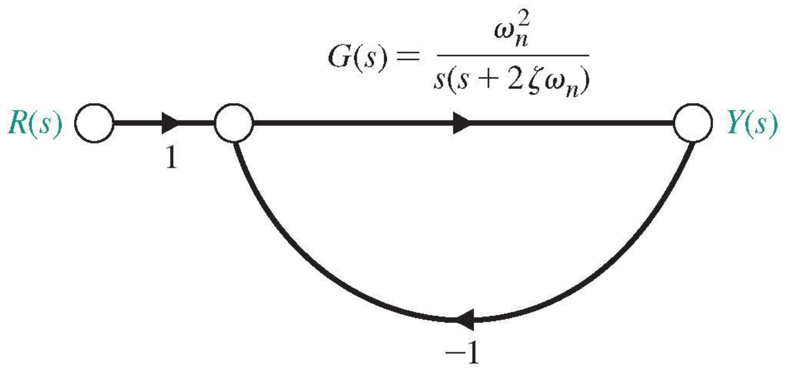
Recall useful Laplace Transform Pairs

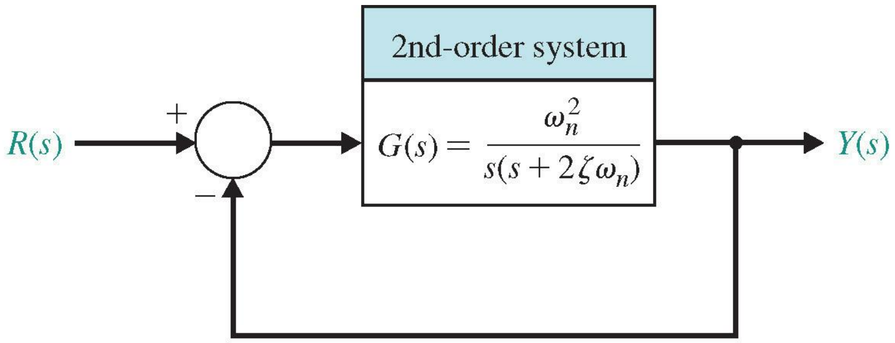
Appendix D

16.	$\dfrac{\omega}{(s+a)^2+\omega^2}$	$e^{-at}\sin\omega t$
17.	$\dfrac{(s+\alpha)}{(s+a)^2+\omega^2}$	$e^{-at}\cos\omega t$
18.	$\dfrac{s+\alpha}{(s+a)^2+\omega^2}$	$\dfrac{1}{\omega}[(\alpha-a)^2+\omega^2]^{1/2}e^{-at}\sin(\omega t+\phi)$, $\phi=\tan^{-1}\dfrac{\omega}{\alpha-a}$
19.	$\dfrac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$	$\dfrac{\omega_n}{\sqrt{1-\zeta^2}}e^{-\zeta\omega^n t}\sin\omega_n\sqrt{1-\zeta^2}t, \zeta<1$
20.	$\dfrac{1}{s[(s+a)^2+\omega^2]}$	$\dfrac{1}{a^2+\omega^2}+\dfrac{1}{\omega\sqrt{a^2+\omega^2}}e^{-at}\sin(\omega t-\phi)$, $\phi=\tan^{-1}\dfrac{\omega}{-a}$
21.	$\dfrac{\omega_n^2}{s(s^2+2\xi\omega_n s+\omega_n^2)}$	$1-\dfrac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega^n t}\sin(\omega_n\sqrt{1-\zeta^2}t+\phi)$, $\phi=\cos^{-1}\zeta, \zeta<1$

Performance of 2nd-order System

Let us consider a **single-loop second-order system**





damping ratio $0 < \zeta < 1.$

natural frequency $\omega_n > 0$

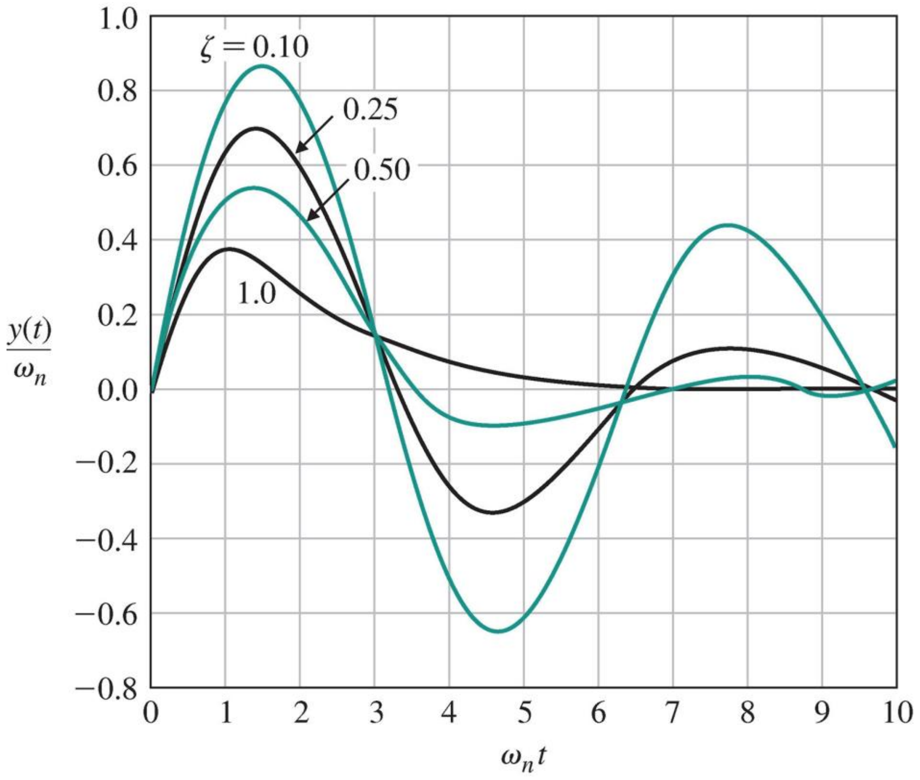
$$Y(s) = \frac{G(s)}{1 + G(s)} R(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} R(s).$$

For the unit impulse function

$$Y(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

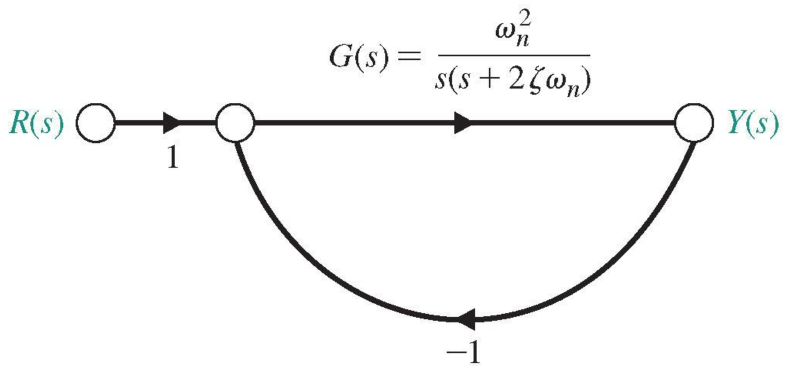
$$y(t) = \frac{\omega_n}{\beta} e^{-\zeta\omega_n t} \sin(\omega_n \beta t),$$

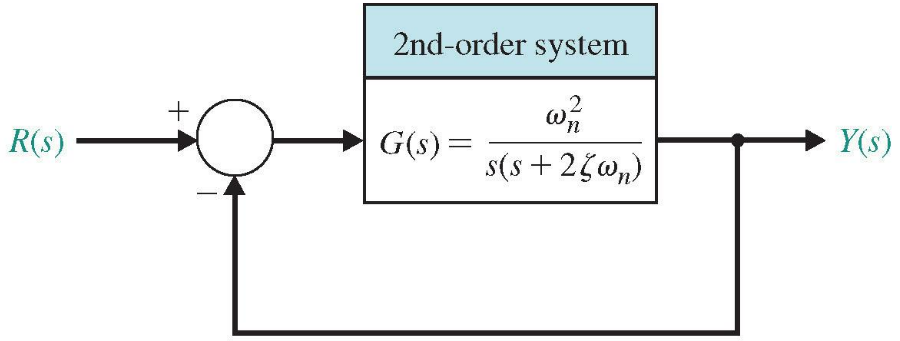
where $\beta = \sqrt{1 - \zeta^2}, \theta = \cos^{-1} \zeta,$



Performance of 2nd-order System

Let us consider a **single-loop second-order system**





damping ratio $0 < \zeta < 1.$

natural frequency $\omega_n > 0$

$$Y(s) = \frac{G(s)}{1 + G(s)} R(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} R(s).$$

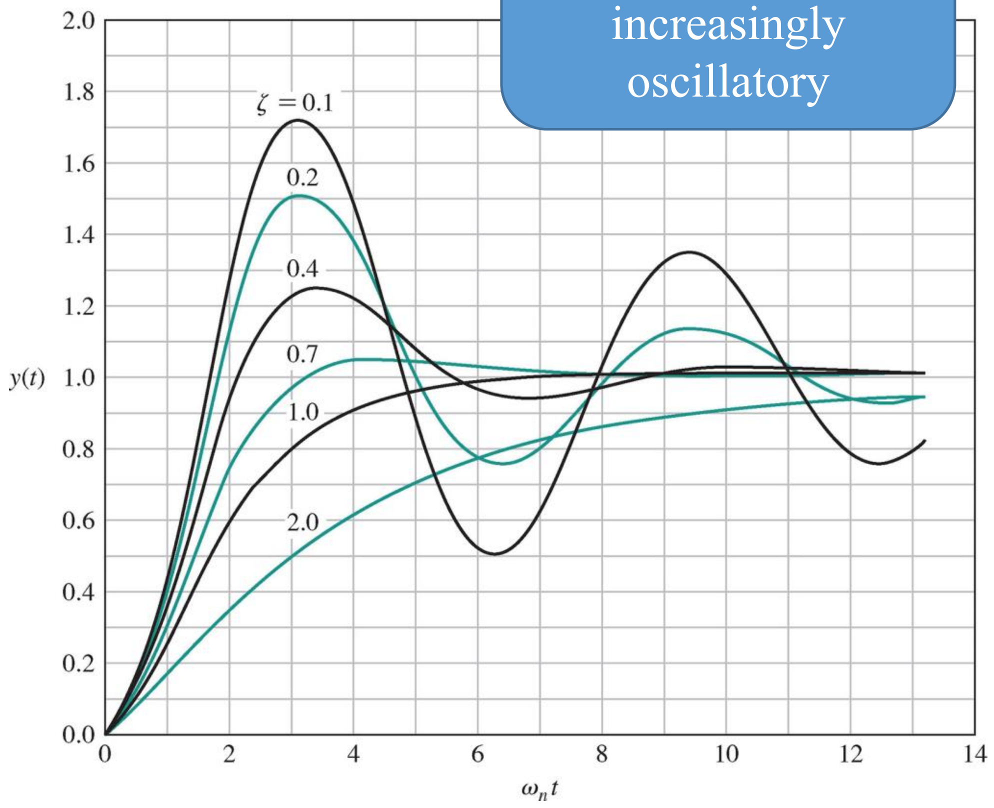
For the unit step

$$Y(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)},$$

$$y(t) = 1 - \frac{1}{\beta} e^{-\zeta\omega_n t} \sin(\omega_n \beta t + \theta),$$

where $\beta = \sqrt{1 - \zeta^2}, \theta = \cos^{-1} \zeta,$

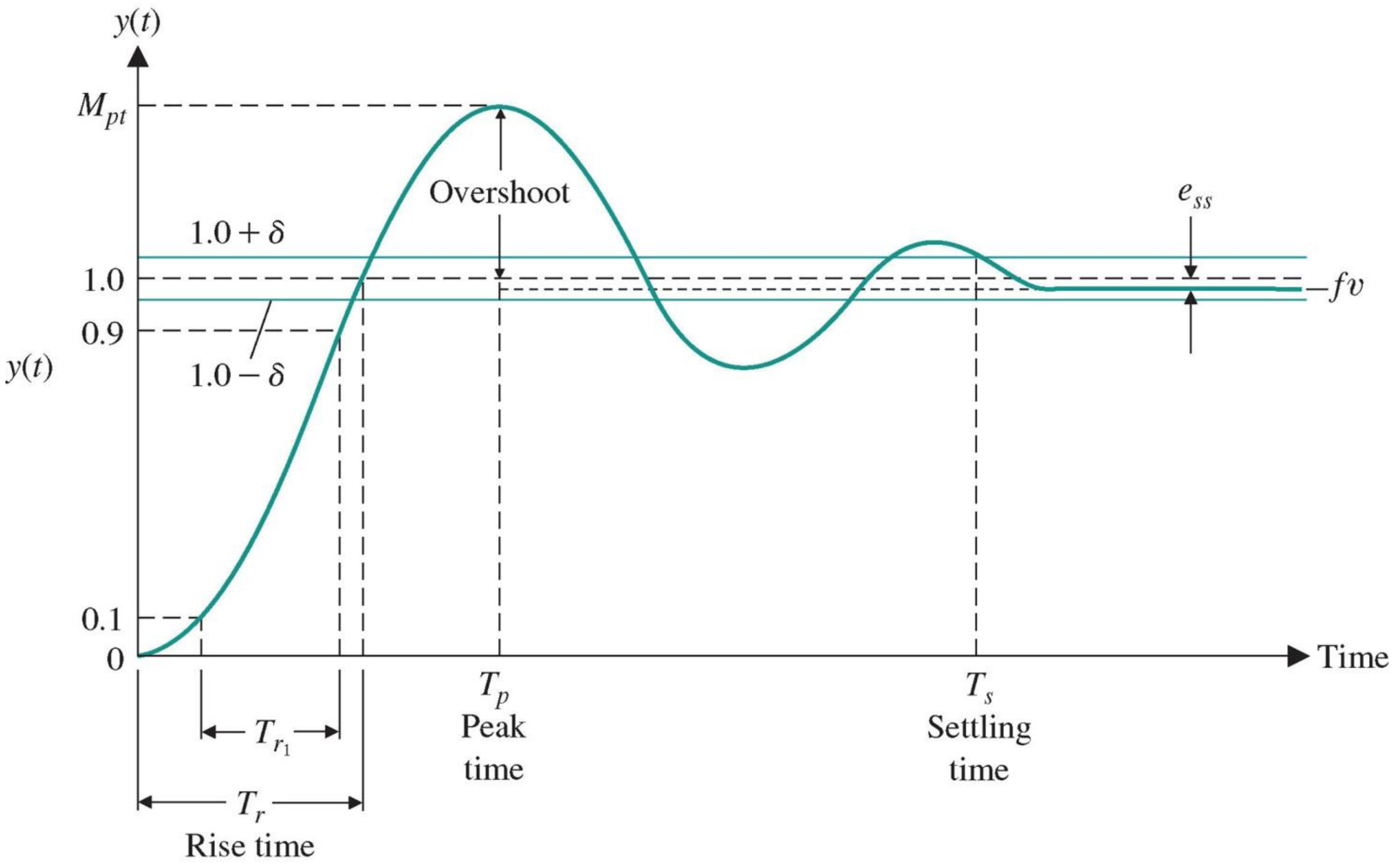
As ζ decreases, the response become increasingly oscillatory



Performance of 2nd-order System

For the unit step
$$Y(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)},$$

$$y(t) = 1 - \frac{1}{\beta} e^{-\zeta\omega_n t} \sin(\omega_n \beta t + \theta), \quad \text{where } \beta = \sqrt{1-\zeta^2}, \theta = \cos^{-1}\zeta, \qquad 0 < \zeta < 1.$$



a) Peak time T_p
b) Rise time T_r

For underdamped systems with an overshoot, the 0–100% rise time is a useful index. If the system is overdamped, then the peak time is not defined, and the 10–90% rise time is normally used.

c) The percent overshoot

$$P.O. = \frac{M_{Pt} - fv}{fv} \times 100\%$$

d) The settling time

is defined as the time required for the system to settle within a certain percentage δ of the input amplitude

Performance of 2nd-order System

For the typical 2nd-order system $$Y(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)},$$

$$y(t) = 1 - \frac{1}{\beta} e^{-\zeta\omega_n t} \sin(\omega_n \beta t + \theta), \quad \text{where } \beta = \sqrt{1-\zeta^2}, \theta = \cos^{-1}\zeta, \qquad 0 < \zeta < 1.$$

We now study how the parameters of plant (ξ and ω_n) influence the performance of system?

- peak time

$$\dot{y}(t) = \frac{\omega_n}{\beta} e^{-\zeta\omega_n t} \sin(\omega_n \beta t),$$

Happen to be same as the time response of unit impulse!

which is equal to zero when $\omega_n \beta t = n\pi$, where $n = 0, 1, 2, \ldots.$

For n=1, we have peak time relationship

$$T_p = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}},$$

- the peak response and percent overshoot

Irrelevant with ω_n

$$M_{pt} = 1 + e^{-\zeta\pi/\sqrt{1-\zeta^2}}$$

$$P.O. = 100 e^{-\zeta\pi/\sqrt{1-\zeta^2}}$$

Performance of 2nd-order System

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For the typical 2nd-order system
$$Y(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)},$$

$$y(t) = 1 - \frac{1}{\beta} e^{-\zeta\omega_n t} \sin(\omega_n \beta t + \theta), \quad \text{where } \beta = \sqrt{1-\zeta^2}, \theta = \cos^{-1}\zeta, \qquad 0 < \zeta < 1.$$

We now study how the parameters of plant (ξ and ω_n) influence the performance of system?

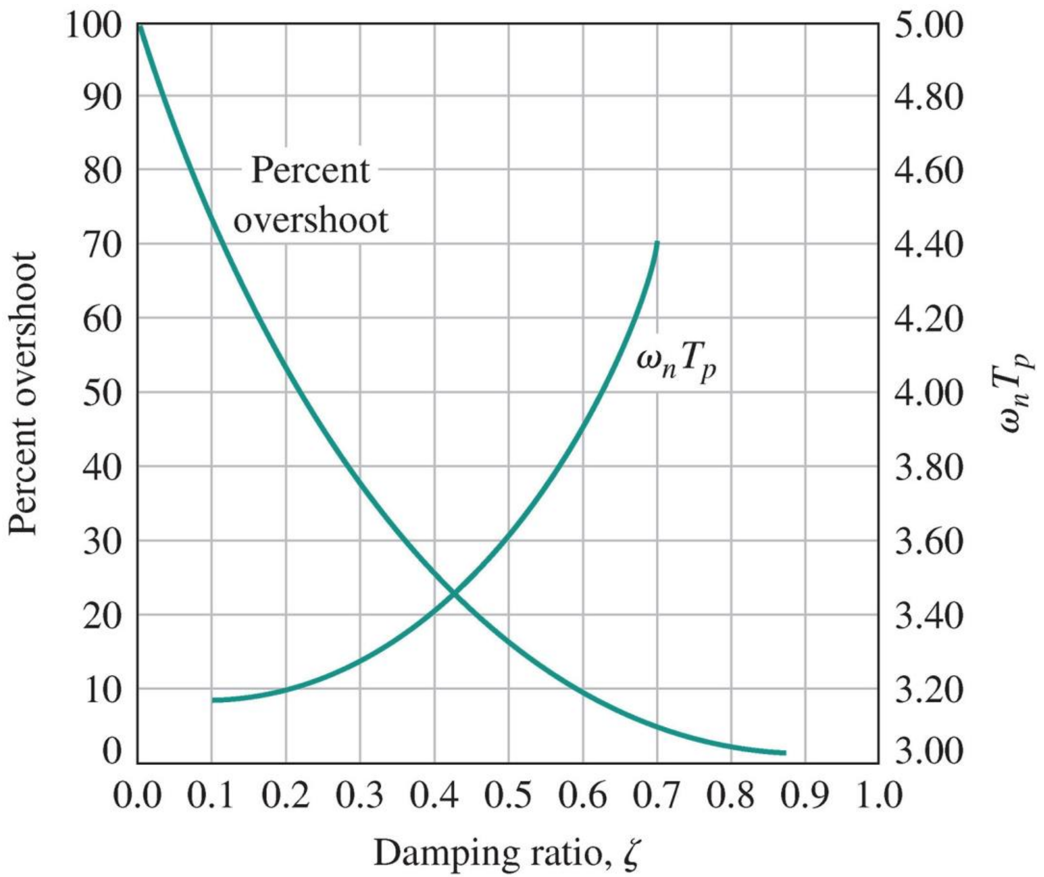
$$T_p = \frac{\pi}{\omega_n \sqrt{1-\zeta^2}},$$

$$P.O. = 100e^{-\zeta\pi/\sqrt{1-\zeta^2}}$$

We should

$\zeta \uparrow \quad \Rightarrow \quad P.O.\downarrow \quad T_p \uparrow$

$\omega_n \uparrow \quad \Rightarrow \quad T_p \downarrow$



Performance of 2nd-order System

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For the typical 2nd-order system
$$Y(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)},$$

$$y(t) = 1 - \frac{1}{\beta} e^{-\zeta\omega_n t} \sin(\omega_n \beta t + \theta), \quad \text{where } \beta = \sqrt{1 - \zeta^2}, \theta = \cos^{-1}\zeta,$$

We now study how the parameters of plant (ξ and ω_n) influence the performance of system?

- The settling time : to determine the time T_s for $\delta \approx 2\%$ of the final value.

$$e^{-\zeta\omega_n T_s} < 0.02,$$

$$\zeta\omega_n T_s \cong 4.$$

Therefore, we have

$$T_s = 4\tau = \frac{4}{\zeta\omega_n}.$$

where $\tau = 1/\zeta\omega_n$ is a time constants of the dominant roots of the characteristic equation.

Notice

$$\left.\begin{array}{lcl} \zeta \uparrow & \Longrightarrow & P.O. \downarrow \\ & & T_p \uparrow \\ \omega_n \uparrow & \Longrightarrow & T_p \downarrow \end{array}\right\} \Longrightarrow T_s \downarrow$$

Performance of 2nd-order System

For the typical 2nd-order system

$$Y(s) = \frac{\omega_n^2}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)},$$

$$y(t) = 1 - \frac{1}{\beta} e^{-\zeta\omega_n t} \sin(\omega_n \beta t + \theta), \quad \text{where } \beta = \sqrt{1-\zeta^2}, \theta = \cos^{-1}\zeta,$$

We now study how the parameters of plant (ξ and ω_n) influence the performance of system?

What does this mean for transfer function?

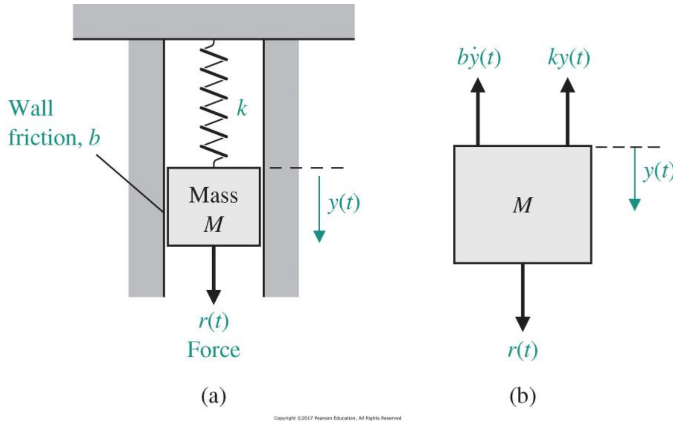
$$\left.\begin{array}{l} \zeta\uparrow \;\Longrightarrow\; P.O.\downarrow \\ \qquad\qquad\quad T_p\uparrow \\ \omega_n\uparrow \;\Longrightarrow\; T_p\downarrow \end{array}\right\} \Longrightarrow T_s\downarrow$$

$$\Downarrow$$

k fixed $\quad M\downarrow \quad b\uparrow$

Change the poles of the transfer function significantly influence the performance of the system

$$G(s) = \frac{Y(s)}{R(s)} = \frac{1}{Ms^2 + bs + k}$$

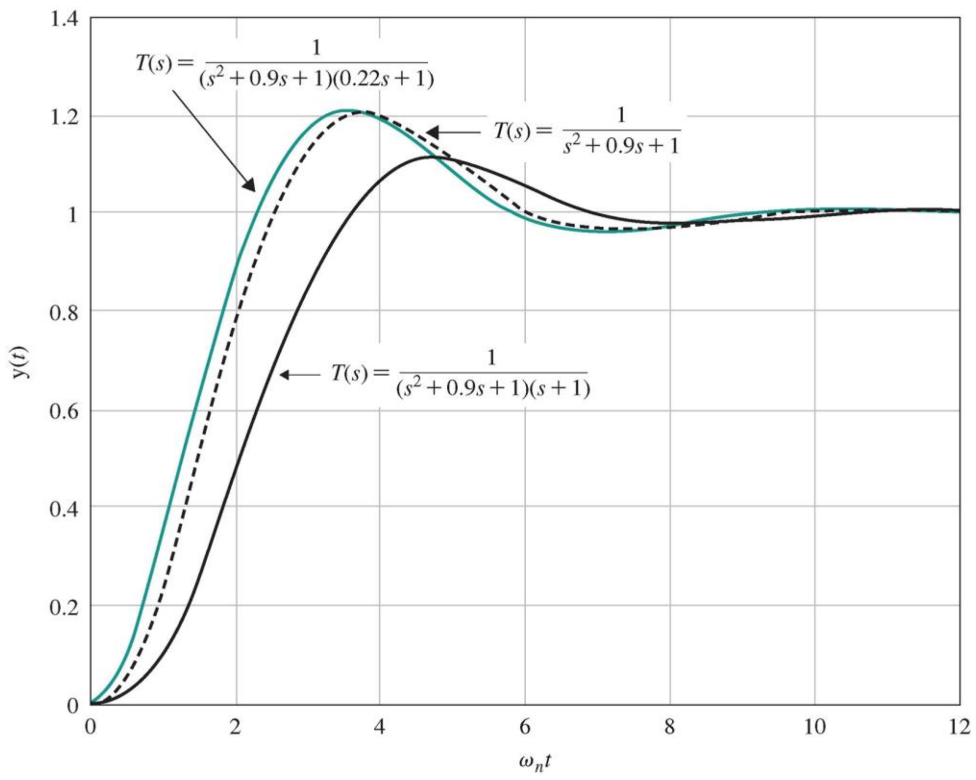


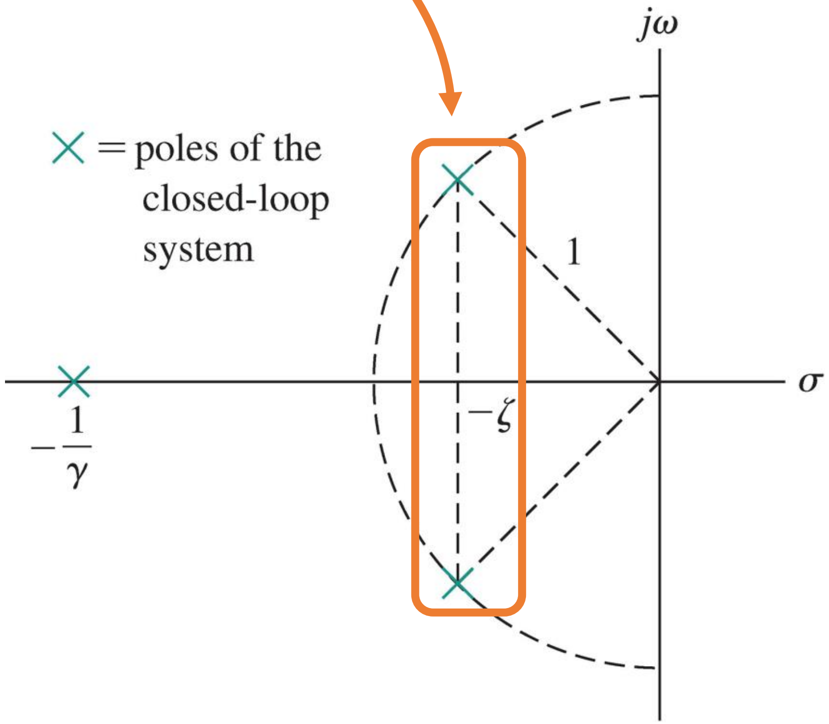
The effect of the third pole or zero

The curves presented previous are exact only for the second-order system.However, they provide important information because many systems **possess a dominant pair of roots**.

Consider the third-order system

$$T(s) = \frac{1}{(s^2 + 2\zeta\omega_n s + 1)(\gamma s + 1)}$$





The effect of the third pole or zero

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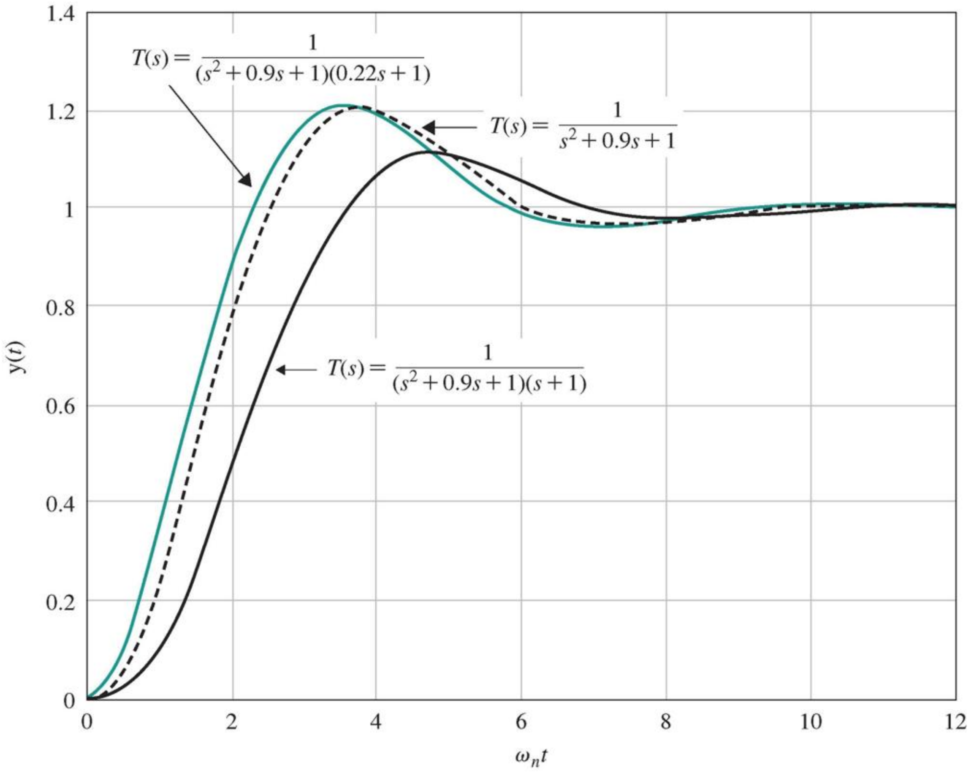
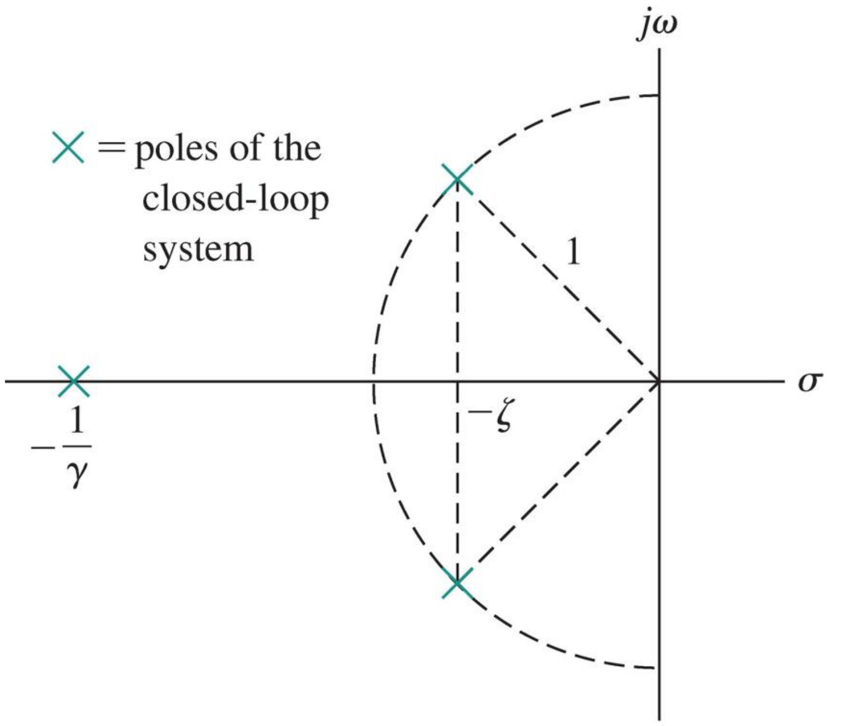
- the response of a third-order system can be approximated by the dominant roots of the second-order system

$$T_3(s) = \frac{\omega_n^2}{(s^2+2\xi\omega_n s+\omega_n^2)(\gamma s+1)} \approx \quad T_2(s) = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

as long as the real part of the dominant roots is less than **one tenth** of the real part of the third root:

$$|1/\gamma| \geq 10|\zeta\omega_n|$$





The effect of the third pole or zero

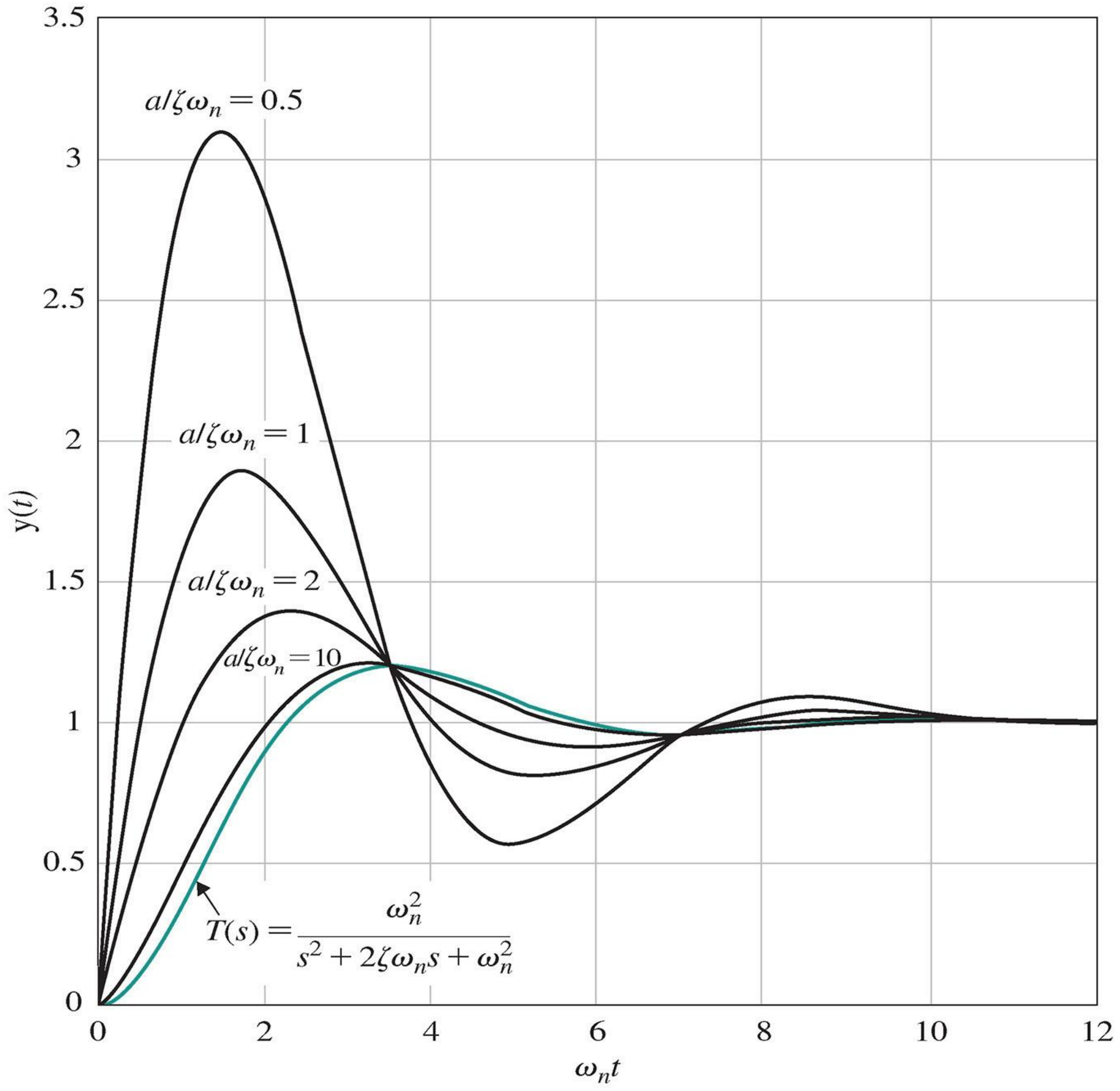
BUT only for transfer functions without finite zeros. zeros will materially affect the transient response of the system.

Consider

$$T(s) = \frac{(\omega_n^2/a)(s + a)}{s^2 + 2\zeta\omega_n s + \omega_n^2}.$$

- Suppose $\xi = 0.45$, as $a/\xi\omega_n$ increases, the finite zero moves farther into the left half-plane and away from the poles, and the step response approaches the second-order system response.

- If the transfer function of a system possesses a finite zero and it is located relatively near the dominant complex poles, then the zero will materially affect the transient response of the system



the s-Plane root location and the transient response

The output of a system (with DC gain = 1) **without repeated roots** and a unit step input can be formulated as a partial fraction expansion as

$$Y(s) = \frac{1}{s} + \sum_{i=1}^{M} \frac{A_i}{s + \sigma_i} + \sum_{k=1}^{N} \frac{B_k s + C_k}{s^2 + 2\alpha_k s + (\alpha_k^2 + \omega_k^2)},$$

The roots of the system must be either

$$s = -\sigma_i$$

or complex conjugate pairs such as

$$s = -\alpha_k \pm j\omega_k$$

Then the inverse transform results in the transient response as the sum of terms

$$y(t) = 1 + \sum_{i=1}^{M} A_i e^{-\sigma_i t} + \sum_{k=1}^{N} D_k e^{-\alpha_k t} \sin(\omega_k t + \theta_k).$$

steady-state output

exponential terms

damped sinusoidal

the s-Plane root location and the transient response

$$y(t) = 1 + \sum_{i=1}^{M} A_i e^{-\sigma_i t} + \sum_{k=1}^{N} D_k e^{-\alpha_k t} \sin(\omega_k t + \theta_k)$$

- For the response **to be stable**—that is, bounded for a step input—the real part of the poles must be in the left-hand portion of the s-plane.

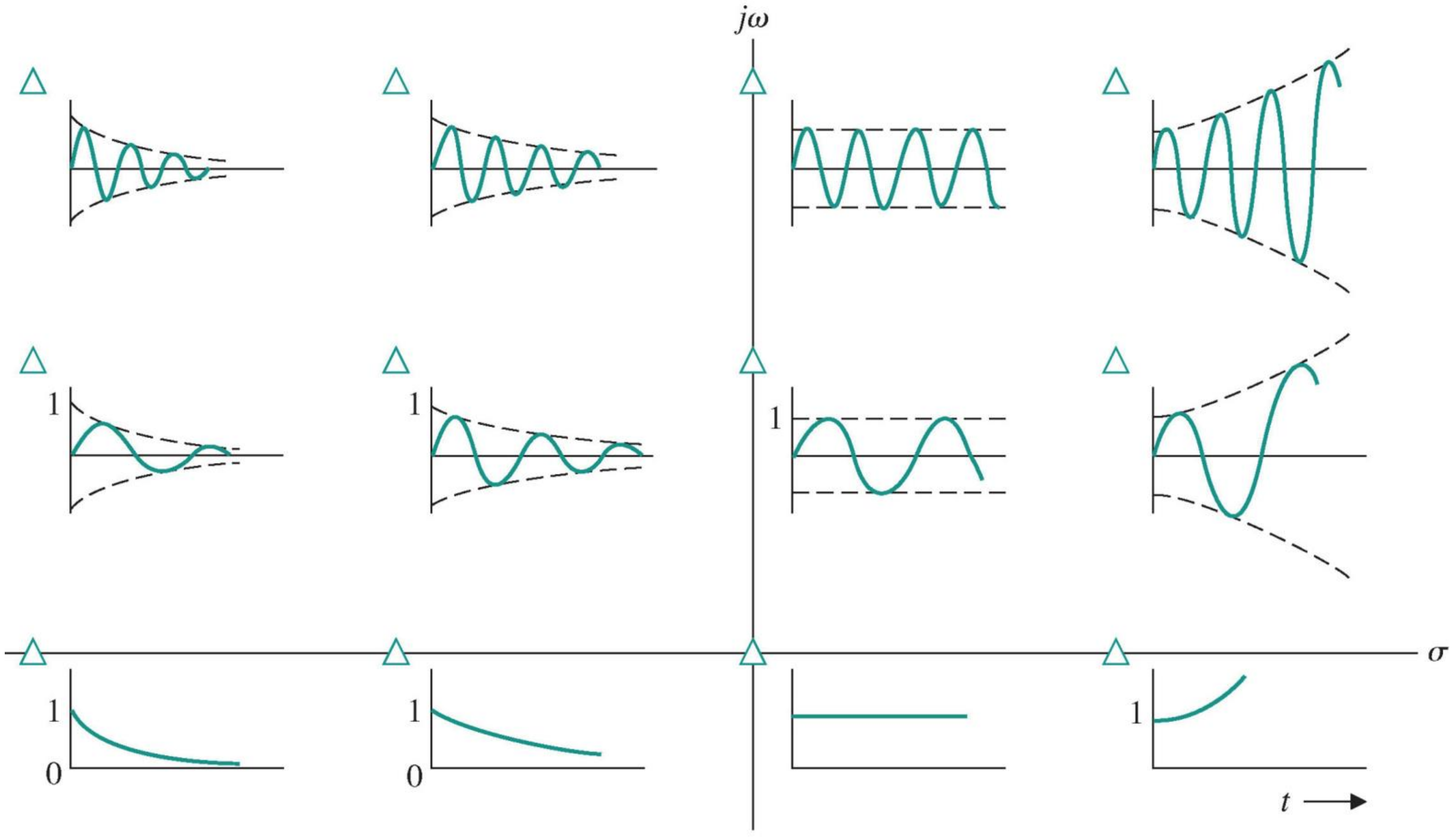


FIGURE Impulse response for various root locations in the *s*-plane. (The conjugate root is not shown.)

the s-Plane root location and the transient response

It is important for the control system designer to understand

time-domain response
to step and other inputs
(Objective)

the poles and zeros
location
(design)

Tunable parameters of the
physical application
(Reality)

- The poles of $T(s)$ determine the particular response modes that will be present
- The zeros of $T(s)$ establish the relative weightings of the individual mode functions
- Control engineers can envision the effects on the step and impulse responses of adding, deleting, or moving poles and zeros of $T(s)$ in the s-plane

The steady-state error

Recall :
for the unity feedback system, in the absence of $T_d\ (s)$and $N(s)$, the tracking error of a unity feedback system is

$$E(s) = \frac{1}{1 + G_c(s)G(s)} R(s).$$

Using the final value theorem and computing the steady-state tracking error yields

$$\lim_{t\to\infty} e(t) = e_{ss} = \lim_{s\to 0} s \frac{1}{1 + G_c(s)G(s)} R(s)$$

The loop gain $L(s) = G_c(s)G(s)$ determines the steady state error.

High frequency gain

zeros

poles

$$G_c(s)G(s) = \frac{K\prod_{i=1}^{M}(s + z_i)}{s^N \prod_{k=1}^{Q}(s + p_k)}$$

The number of integrations is often indicated by labeling a system with a type number that is equal to N

The steady-state error

$$G_c(s)G(s) = \frac{K\prod_{i=1}^{M}(s+z_i)}{s^N\prod_{k=1}^{Q}(s+p_k)},$$

$$\lim_{t\to\infty} e(t) = e_{ss} = \lim_{s\to 0} s\frac{1}{1+G_c(s)G(s)}R(s)$$

Table 5.2 Summary of Steady-State Errors

Number of Integrations in $G_c(s)G(s)$, Type Number	Input		
	Step, $r(t) = A$, $R(s) = A/s$	Ramp, $r(t) = At$, $R(s) = A/s^2$	Parabola, $r(t) = At^2/2$, $R(s) = A/s^3$
0	$e_{ss} = \frac{A}{1+K_p}$	∞	∞
1	$e_{ss} = 0$	$\frac{A}{K_v}$	∞
2	$e_{ss} = 0$	0	$\frac{A}{K_a}$

The steady-state error

$$G_c(s)G(s) = \frac{K\prod_{i=1}^{M}(s+z_i)}{s^N\prod_{k=1}^{Q}(s+p_k)}$$

$$\lim_{t\to\infty} e(t) = e_{ss} = \lim_{s\to 0} s\frac{1}{1+G_c(s)G(s)}R(s)$$

Control systems are often described in terms of their type number and the error constants

Table 5.2 Summary of Steady-State Errors

Number of Integrations in $G_c(s)G(s)$, Type Number	Step, $r(t) = A$, $R(s) = A/s$	… $r(t)$ … $s) = A/$ …	Parabola, $r(t) = At^2/2$, … s^3
0	$e_{ss} = \frac{A}{1+K_p}$	∞	
1	$e_{ss} = 0$	$\frac{A}{K_v}$	∞
2	$e_{ss} = 0$	0	$\frac{A}{K_a}$

position error constant
$K_p = \lim_{s\to 0} G_c(s)G(s)$

velocity error constant
$K_v = \lim_{s\to 0} sG_c(s)G(s)$

acceleration error constant
$K_a = \lim_{s\to 0} s^2G_c(s)G(s)$

Other performance indices

A performance index is a quantitative measure of the performance of a system and is chosen so that emphasis is given to the important system specifications.

It is convenient to choose $T = T_s$

- integral of the square of the error

$$\mathrm{ISE} = \int_0^T e^2(t)\, dt.$$

- integral of the absolute magnitude of the error

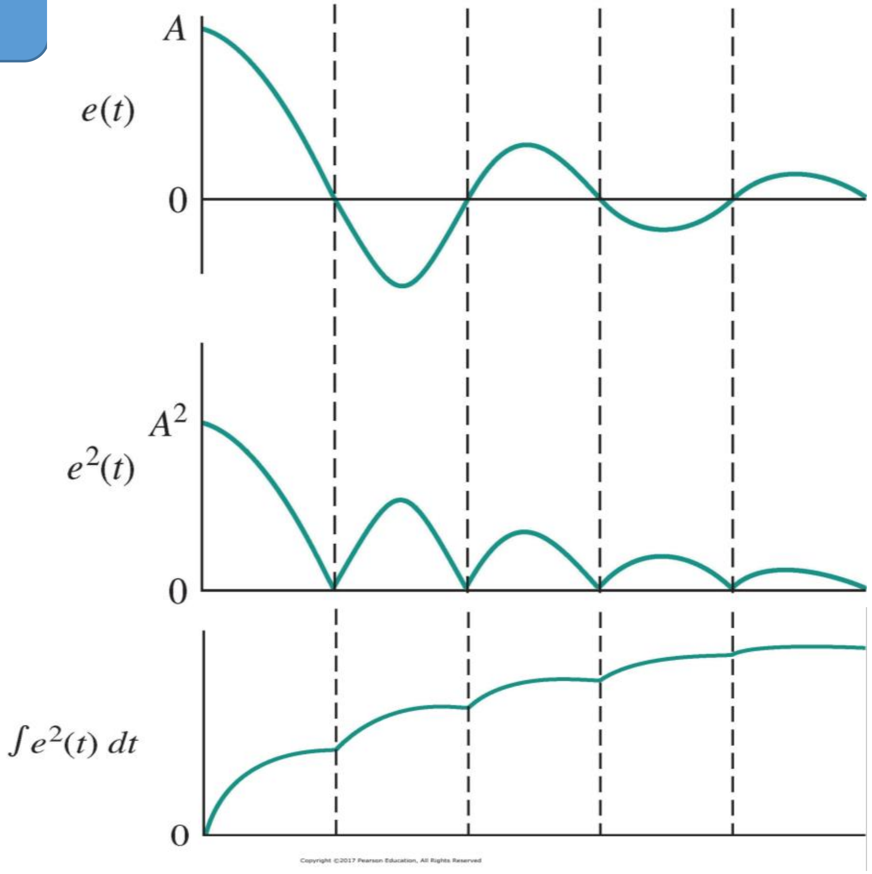
$$\mathrm{IAE} = \int_0^T |e(t)|\, dt,$$

- integral of time multiplied absolute error

$$\mathrm{ITAE} = \int_0^T t|e(t)|\, dt,$$

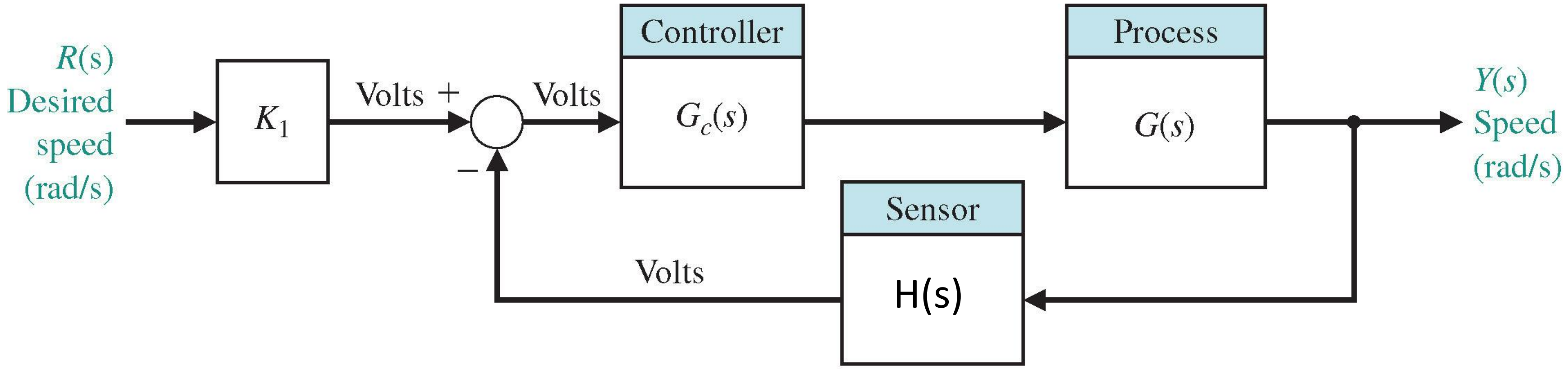
- integral of time multiplied squared error

$$\mathrm{ITSE} = \int_0^T t e^2(t) dt.$$



Exercise

Consider a speed control system



$$G_c(s) = 40,\ G(s) = \frac{1}{s+5},\ \text{and}\ \ H(s) = \frac{2}{0.1s+1}.$$

1. Selecting K_1 such that the steady state error is less than 6% of the magnitude of the step input

2. Analyze the transient behavior of step response in terms of rising time, overshoot, settling time, etc. with K_1=2

Summary

Feedback has the ability to adjust the transient and steady-state response of a control system, but How?

I. Define and measure its performance, i.e. specifications in terms of transient response and steady state

II. Figure out the correlation between the system performance and transfer function, to be specific, the poles and zeros. Even, some tunable parameter.

III. Optimize and compromise

Key words list:
Impulse, step, ramp, parabolic functions
Damping ratio & natural frequency
2nd order system
Transient Vs steady state
Peak time, rising time, P.O., settling time
Poles and zeros
Dominate roots
Position, velocity, acceleration error constants

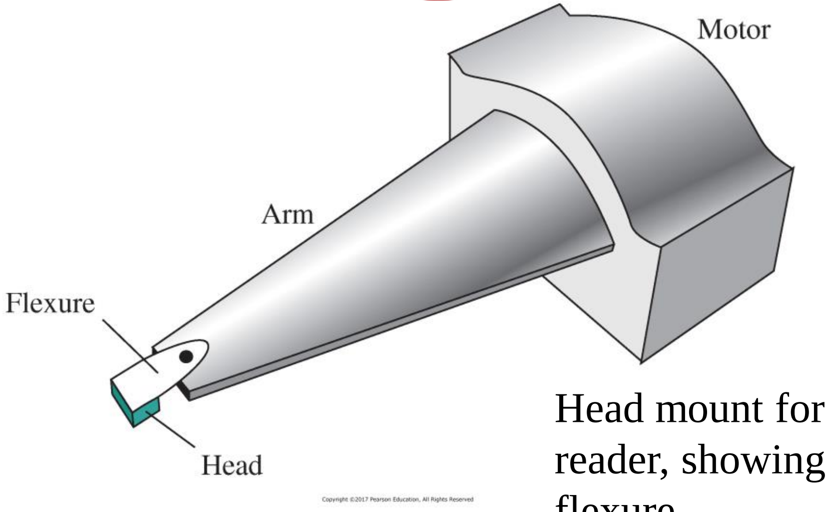
Disk Drive Read System

Recall: Our goal is to position the reader head accurately at the desired track and to move from one track to another.

We attempt to adjust the amplifier gain K_a in order to obtain the best performance possible

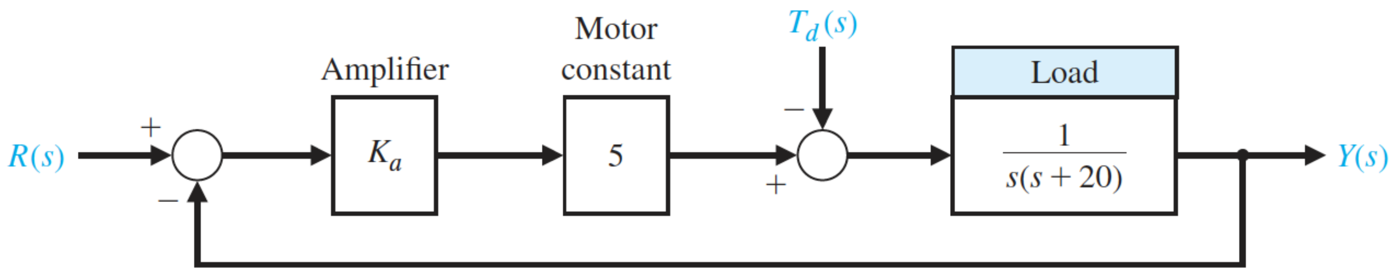
Table 5.5 Specifications for the Transient Response

Performance Measure	Desired Value
Percent overshoot	Less than 5%
Settling time	Less than 250 ms
Maximum value of response to a unit step disturbance	Less than 5×10^{-3}



Head mount for reader, showing flexure.

Let us consider the second-order model of the motor and arm,



$$Y(s) = \frac{5K_a}{s(s+20)+5K_a}R(s)$$

$$= \frac{5K_a}{s^2+20s+5K_a}R(s)$$

$$= \frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}R(s).$$

Therefore, $\omega_n^2 = 5K_a$, and $2\zeta\omega_n = 20$.

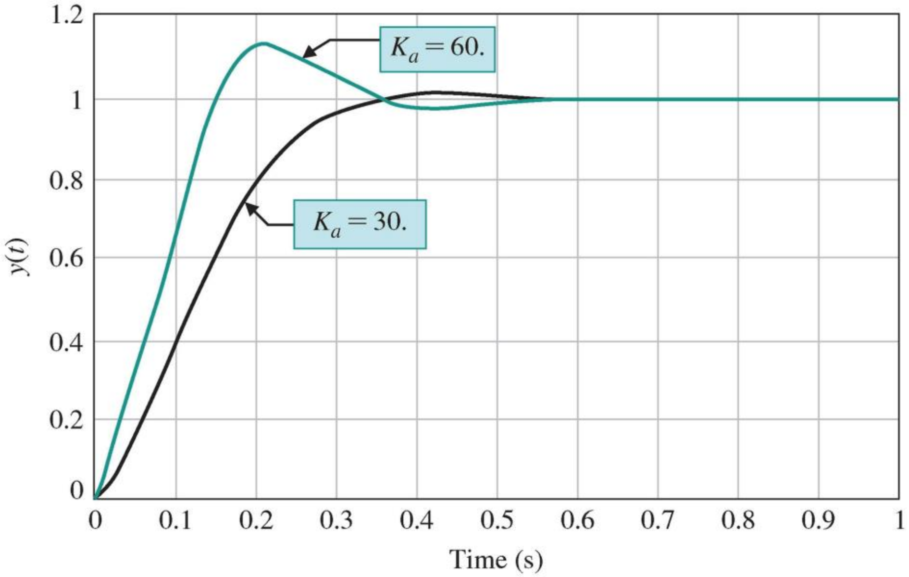
Disk Drive Read System

Response of the system to a unit step input

```
Ka=30;
t=[0:0.01:1];
nc=[Ka*5];dc=[1]; sysc=tf(nc,dc);
ng=[1];dg=[1 20 0]; sysg=tf(ng,dg);
sys1=series(sysc,sysg);
sys=feedback(sys1, [1]);
y=step(sys,t);
plot(t,y), grid
xlabel('Time (s)')
ylabel('y(t)')
```

Select K_a.

Compute the closed-loop transfer function.

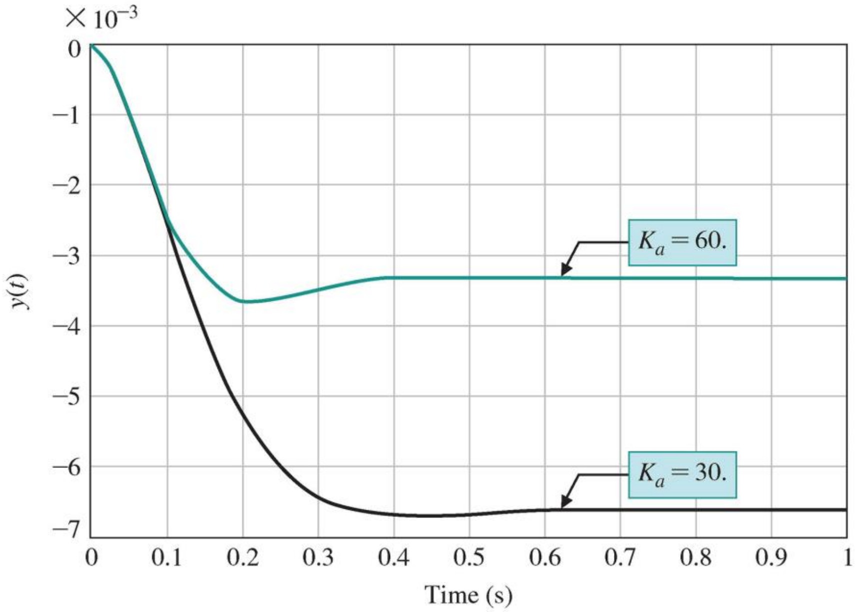


Response of the system to a unit step disturbance

```
Ka=30;
t=[0:0.01:1];
nc=[Ka*5];dc=[1]; sysc=tf(nc,dc);
ng=[1];dg=[1 20 0]; sysg=tf(ng,dg);
sys=feedback(sysg,sysc);
sys=-sys;
y=step(sys,t); plot(t,y)
xlabel('Time (s)'), ylabel('y(t)'), grid
```

Select K_a.

Disturbance enters summer with a negative sign.



Disk Drive Read System

Table 5.6 Response for the Second-Order Model for a Step Input

K_a	20	30	40	60	80
Percent overshoot	0	1.2%	4.3%	10.8%	16.3%
Settling time (s)	0.55	0.40	0.40	0.40	0.40
Damping ratio	1	0.82	0.707	0.58	0.50
Maximum value of the response $y(t)$ to a unit disturbance	-10×10^{-3}	-6.6×10^{-3}	-5.2×10^{-3}	-3.7×10^{-3}	-2.9×10^{-3}

- When Ka is increased to 60, the effect of a disturbance is reduced by a factor of 2.
- Clearly, if we wish to meet our goals with this system, we need to select a compromise gain. In this case, we select Ka = 40 as the best compromise.
- However, this compromise does not meet all the specifications.

Thanks!

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