SI231B: Matrix Computations, 2024 Fall

Homework Set #3

Acknowledgements:

1) Deadline: **2024-12-02 23:59:59**
2) Please submit the PDF file to gradescope. Course entry code: 8KJ345.
3) You have 5 “free days” in total for all late homework submissions.
4) If your homework is handwritten, please make it clear and legible.
5) All your answers are required to be in English.
6) Please include the main steps in your answer; otherwise, you may not get the points.

Problem 1. (Diagonalization) (15 points)

1) Determine whether each of the following matrices is diagonalizable. (10 points)

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 4 & -2 \\ -3 & -3 & 5 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 & 0 & 1 \\ 3 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix}. \tag{1}$$

2) If it is diagnolizable, diagonalize the matrix using a similarity transformation. (5 points)

Solution:

1) (10 points)

1 The characteristic equation of $\mathbf{A}$ is

$$p(\lambda) = |\lambda\mathbf{I} - \mathbf{A}| = \begin{vmatrix} \lambda-1 & 1 & -1 \\ -2 & \lambda-4 & 2 \\ 3 & 3 & \lambda-5 \end{vmatrix} = (\lambda-6)(\lambda-2)^2 = 0. \text{ (1 point)} \tag{2}$$

$\lambda_1(\mathbf{A}) = 6$. Its algebraic multiplicity and geometric multiplicity: $\mu_1(\mathbf{A}) = \gamma_1(\mathbf{A}) = 1$. (1 point)

$\lambda_2(\mathbf{A}) = 2$. Its algebraic multiplicity $\mu_2(\mathbf{A}) = 2$.

$$\lambda_2(\mathbf{A})\mathbf{I} - \mathbf{A} = \begin{vmatrix} 1 & 1 & -1 \\ -2 & -2 & 2 \\ 3 & 3 & -3 \end{vmatrix}, \tag{3}$$

$$\gamma_2(\mathbf{A}) = \dim \mathcal{N}(\lambda_2(\mathbf{A})\mathbf{I} - \mathbf{A}) = n - \text{rank}(\lambda_2(\mathbf{A})\mathbf{I} - \mathbf{A}) = 3 - 1 = 2 = \mu_2(\mathbf{A}). \text{ (2 points)} \tag{4}$$

Hence, $\mathbf{A}$ is diagonalizable. (1 point)

2 The characteristic equation of $\mathbf{B}$ is

$$p(\lambda) = |\lambda\mathbf{I} - \mathbf{B}| = \begin{vmatrix} \lambda & 0 & -1 \\ -3 & \lambda-1 & -2 \\ -1 & 0 & \lambda \end{vmatrix} = (\lambda+1)(\lambda-1)^2 = 0. \text{ (1 point)} \tag{5}$$

$\lambda_1(\mathbf{B}) = -1$. Its algebraic multiplicity and geometric multiplicity: $\mu_1(\mathbf{B}) = \gamma_1(\mathbf{B}) = 1$. (1 point)

$\lambda_2(\mathbf{B}) = 1$. Its algebraic multiplicity $\mu_2(\mathbf{B}) = 2$.

$$\lambda_2(\mathbf{B})\mathbf{I} - \mathbf{B} = \begin{vmatrix} 1 & 0 & -1 \\ -3 & 0 & -2 \\ -1 & 0 & 1 \end{vmatrix}, \tag{6}$$

$$\gamma_2(\mathbf{B}) = \dim \mathcal{N}(\lambda_2(\mathbf{B})\mathbf{I} - \mathbf{B}) = n - \text{rank}(\lambda_2(\mathbf{B})\mathbf{I} - \mathbf{B}) = 3 - 2 = 1 < \mu_2(\mathbf{B}). \text{ (2 points)} \tag{7}$$

Hence, $\mathbf{B}$ is not diagonalizable. (1 point)

2) (5 points)

From problem 1), $\lambda_1(\mathbf{A}) = 6$, $\mu_1(\mathbf{A}) = \gamma_1(\mathbf{A}) = 1$. $\lambda_2(\mathbf{A}) = 2$, $\mu_2(\mathbf{A}) = \gamma_2(\mathbf{A}) = 2$.

$$\lambda_1(\mathbf{A})\mathbf{I} - \mathbf{A} = \begin{bmatrix} 5 & 1 & -1 \\ -2 & 2 & 2 \\ 3 & 3 & 1 \end{bmatrix}, \mathcal{N}(\lambda_1(\mathbf{A})\mathbf{I} - \mathbf{A}) = \text{span}\{\begin{bmatrix} \frac{1}{3} \\ -\frac{2}{3} \\ 1 \end{bmatrix}\}. \text{ (1 point)}$$

$$\lambda_2(\mathbf{A})\mathbf{I} - \mathbf{A} = \begin{bmatrix} 1 & 1 & -1 \\ -2 & -2 & 2 \\ 3 & 3 & -3 \end{bmatrix}, \mathcal{N}(\lambda_2(\mathbf{A})\mathbf{I} - \mathbf{A}) = \text{span}\{\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\}. \text{ (2 points)}$$

It's easy to check that when combined these three eigenvectors constitute a linearly independent set.

To explicitly exhibit the similarity transformation that diagonalizes $\mathbf{A}$,

set $\mathbf{P} = \begin{bmatrix} \frac{1}{3} & -1 & 1 \\ -\frac{2}{3} & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, and verify $\mathbf{P}^{-1}A\mathbf{P} = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = \mathbf{D}$. (2 points)

Hence, we obtain $\mathbf{A} = \mathbf{PDP}^{-1}$.

Problem 2. (Eigenvector, eigenvalue) (15 points)

Suppose $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{2\times 2}$ are unitary and the dimension of $\mathcal{N}(\mathbf{A}-\mathbf{B})$ is 1. Prove the following statements:

1) $\mathbf{A^{-1}B}$ and $\mathbf{B^{-1}A}$ have the same eigenvalues. (5 points)
2) $\mathbf{A^{-1}B}$ and $\mathbf{B^{-1}A}$ have a shared eigenvector. (5 points)
3) Suppose that $\mathbf{A^{-1}B}$ is diagonalizable. Show that $\mathbf{A}$ and $\mathbf{B}$ are not similar. (5 points)

Solution:

1) Since $\mathbf{A}$ and $\mathbf{B}$ are unitary,

$$(\mathbf{A}^{-1}\mathbf{B})^T = \mathbf{B}^T(\mathbf{A}^{-1})^T = \mathbf{B}^T(\mathbf{A}^T)^{-1} = \mathbf{B}^{-1}(\mathbf{A}^{-1})^{-1} = \mathbf{B}^{-1}\mathbf{A} = (\mathbf{A}^{-1}\mathbf{B})^{-1}. \tag{8}$$

   $\mathbf{A}^{-1}\mathbf{B}$ is unitary. (2 points)

   $(\mathbf{B}^{-1}\mathbf{A})(\mathbf{A}^{-1}\mathbf{B}) = (\mathbf{A}^{-1}\mathbf{B})(\mathbf{B}^{-1}\mathbf{A}) = \mathbf{I}$.

   $\mathbf{A}^{-1}\mathbf{B} = (\mathbf{B}^{-1}\mathbf{A})^T$. (2 points)

   They have the same eigenvalues. (1 point)

2) For any $\mathbf{x} \in \mathbb{R}^2$,

$$(\mathbf{A}-\mathbf{B})\mathbf{x} = \mathbf{0} \longrightarrow \mathbf{A}\mathbf{x} = \mathbf{B}\mathbf{x} \longrightarrow \begin{cases} \mathbf{A}^{-1}\mathbf{B}\mathbf{x} = \mathbf{x} \\ \mathbf{B}^{-1}\mathbf{A}\mathbf{x} = \mathbf{x} \end{cases} \quad (\text{ 2 points}) \tag{9}$$

   If α is a solution to $(\mathbf{A}-\mathbf{B})\mathbf{x} = \mathbf{0}$, α is the eigenvector of $\mathbf{A}^{-1}\mathbf{B}$ and $\mathbf{B}^{-1}\mathbf{A}$ associated with eigenvalue $\lambda = 1$. (2 points)

   Hence, $\mathbf{A^{-1}B}$ and $\mathbf{B^{-1}A}$ have a shared eigenvector.(1 point)

3) Since $\mathbf{A^{-1}B}$ is diagonalizable and has only one eigenvector associated with $\lambda = 1$, its algebraic multiplicity and geometric multiplicity are 1.

   In 1), we know that $\mathbf{A^{-1}B}$ is unitary. Its eigenvalues can only be 1 or -1.

   $\det(\mathbf{A}^{-1}\mathbf{B}) = \det(\mathbf{A}^{-1})\det(\mathbf{B}) = -1 \longrightarrow \det(\mathbf{A}) = -\det(\mathbf{B})$. (2 points)

   Hence, $\mathbf{A}$ and $\mathbf{B}$ have different determinants. (2 points) They are not similar. (1 point)

Problem 3. (Eigenvector) (20 points)

Given $\mathbf{A} \in \mathbb{R}^{3\times3}$, $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are eigenvectors associated with three different eigenvalues $\lambda_1, \lambda_2, \lambda_3$. Let $\mathbf{b} = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$.

1) Prove that $\mathbf{b}, \mathbf{A}\mathbf{b}, \mathbf{A}^2\mathbf{b}$ are linearly independent. (10 points)
2) If $\mathbf{A}^3\mathbf{b} = \mathbf{A}\mathbf{b}$, what is the rank of $\mathbf{A} - \mathbf{I}$? Compute $\det(\mathbf{A} + \mathbf{I})$. (10 points)

Solution:

1) Assume that there exist constants k_1, k_2, k_3, such that

$$k_1\mathbf{b} + k_2\mathbf{A}\mathbf{b} + k_3\mathbf{A}^2\mathbf{b} = 0, \tag{10}$$

Since $\mathbf{A}\mathbf{v_i} = \lambda_i\mathbf{v_i}$, $(i = 1, 2, 3)$,

$$\mathbf{A}\mathbf{b} = \mathbf{A}\mathbf{v_1} + \mathbf{A}\mathbf{v_2} + \mathbf{A}\mathbf{v_3} = \lambda_1\mathbf{v_1} + \lambda_2\mathbf{v_2} + \lambda_3\mathbf{v_3}, (2 \text{ points})$$
$$\mathbf{A}^2\mathbf{b} = \lambda_1^2\mathbf{v_1} + \lambda_2^2\mathbf{v_2} + \lambda_3^2\mathbf{v_3}, (2 \text{ points})$$

$$k_1\mathbf{b} + k_2\mathbf{A}\mathbf{b} + k_3\mathbf{A}^2\mathbf{b} = (k_1 + k_2\lambda_1 + k_3\lambda_1^2)\mathbf{v_1} + (k_1 + k_2\lambda_2 + k_3\lambda_2^2)\mathbf{v_2} + (k_1 + k_2\lambda_3 + k_3\lambda_3^2)\mathbf{v_3} = 0, \tag{11}$$

$\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ associate with different eigenvalues, they are linearly independent.

$$\begin{cases} k_1 + k_2\lambda_1 + k_3\lambda_1^2 = 0, \\ k_1 + k_2\lambda_2 + k_3\lambda_2^2 = 0, \quad (2 \text{ points}) \\ k_1 + k_2\lambda_3 + k_3\lambda_3^2 = 0, \end{cases} \tag{12}$$

It is a linear equation with parameters k_1, k_2, k_3. $\det(\begin{bmatrix} 1 & \lambda_1 & \lambda_1^2 \\ 1 & \lambda_2 & \lambda_2^2 \\ 1 & \lambda_3 & \lambda_3^2 \end{bmatrix}) \neq 0$. (2 points)

Hence, $k_1 = k_2 = k_3 = 0$. (2 points) $\mathbf{b}, \mathbf{A}\mathbf{b}, \mathbf{A}^2\mathbf{b}$ are linearly independent.

2) Since $\mathbf{A}^3\mathbf{b} = \mathbf{A}\mathbf{b}$,

$$\mathbf{A}[\mathbf{b}, \mathbf{A}\mathbf{b}, \mathbf{A}^2\mathbf{b}] = [\mathbf{A}\mathbf{b}, \mathbf{A}^2\mathbf{b}, \mathbf{A}^3\mathbf{b}] = [\mathbf{A}\mathbf{b}, \mathbf{A}^2\mathbf{b}, \mathbf{A}\mathbf{b}] = [\mathbf{b}, \mathbf{A}\mathbf{b}, \mathbf{A}^2\mathbf{b}]\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, (2 \text{ points}) \tag{13}$$

Let $\mathbf{P} = [\mathbf{b}, \mathbf{A}\mathbf{b}, \mathbf{A}^2\mathbf{b}]$, $\mathbf{P}$ is invertible. (2 points)

$\mathbf{P}^{-1}\mathbf{A}\mathbf{P} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \mathbf{B}$. (2 points)

Hence, $\text{rank}(\mathbf{A} - \mathbf{I}) = \text{rank}(\mathbf{B} - \mathbf{I}) = \text{rank}(\begin{bmatrix} -1 & 0 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & -1 \end{bmatrix}) = 2$. (2 points)

$$\det(\mathbf{A}+\mathbf{I}) = \det(\mathbf{B}+\mathbf{I}) = \begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{vmatrix} = 0.$$ (2 points)

Problem 4. (Schur decomposition) (20 points)

1) Let $\mathbf{A} = \begin{bmatrix} 5 & 7 \\ -2 & -4 \end{bmatrix}$. Compute the Schur decomposition of $\mathbf{A} = \mathbf{U}\mathbf{T}\mathbf{U}^T$, where $\mathbf{U}$ is (real) orthogonal. (The answer is not unique.) (10 points)
2) Let $\mathbf{B} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$.Compute the Schur decomposition of $\mathbf{B} = \mathbf{U}\mathbf{T}\mathbf{U}^H$, where $\mathbf{U}$ is unitary. (The answer is not unique.) (10 points)

Solution:

1) First compute the eigenpairs of $\det(\lambda\mathbf{I} - \mathbf{A}) = 0 \implies (\lambda_1, \mathbf{v}_1) = (-2, [1, -1]^T), (\lambda_2, \mathbf{v}_2) = (3, [7, -2]^T)$ (3 points).

   Take $\mathbf{U}^{(1)} = [\mathbf{v}_1/\left\|\mathbf{v}_1\right\|, \mathbf{u}_1] = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ to be an orthogonal matrix (4 points).

   Then $(\mathbf{U}^{(1)})^T \mathbf{A} \mathbf{U}^{(1)} = \begin{bmatrix} -2 & 9 \\ 0 & 3 \end{bmatrix} = \mathbf{T}$ (3 points).
2) First compute an eigenpair of $\det(\lambda\mathbf{I} - \mathbf{B}) = 0 \implies (\lambda_1, \mathbf{v}_1) = (i, [-1-i, 2]^T)$ (3 points).

   Take $\mathbf{U}^{(1)} = [\mathbf{v}_1/\left\|\mathbf{v}_1\right\|, \mathbf{u}_1] = \frac{1}{\sqrt{6}} \begin{bmatrix} -1-i & 2 \\ 2 & 1-i \end{bmatrix}$ to be an orthogonal matrix (4 points).

   Then $(\mathbf{U}^{(1)})^H \mathbf{A} \mathbf{U}^{(1)} = \begin{bmatrix} i & -2+i \\ 0 & -i \end{bmatrix} = \mathbf{T}$ (3 points).

Problem 5. (Variational Characterizations) (20 points)

For $\mathbf{M} \in \mathbb{S}^{n\times n}$, let $\lambda_k(\mathbf{M})$ denote the kth largest eigenvalue of $\mathbf{M}$, i.e., $\lambda_n(\mathbf{M}) \le \cdots \le \lambda_1(\mathbf{M})$. We make the convention that $\lambda_i(\mathbf{M}) = -\infty, \forall i \ge n+1$ and $\lambda_i(\mathbf{M}) = +\infty, \forall i \le 0$.

1) Let $\mathbf{A}, \mathbf{B} \in \mathbb{S}^{n\times n}$. Suppose that $\mathbf{B}$ has exactly π positive eigenvalues and exactly ν negative eignevalues. Prove that

$$\lambda_i(\mathbf{A}+\mathbf{B}) \le \lambda_{i-\pi}(\mathbf{A}), \quad i = \pi+1, \cdots, n,$$
$$\lambda_{i+\nu}(\mathbf{A}) \le \lambda_i(\mathbf{A}+\mathbf{B}), \quad i = 1, \cdots, n-\nu.$$

(Hint: you might use Weyl's inequality and the interlacing theorem in section 4.6 of the lectures.) (15 points)

2) Let $\mathbf{A}, \mathbf{B} \in \mathbb{S}^{n\times n}$. Suppose that $\mathbf{B}$ is singular and has rank r. Prove that

$$\lambda_i(\mathbf{A}+\mathbf{B}) \le \lambda_{i-r}(\mathbf{A}), \quad i = r+1, \cdots, n,$$
$$\lambda_{i+r}(\mathbf{A}) \le \lambda_i(\mathbf{A}+\mathbf{B}), \quad i = 1, \cdots, n-r.$$

(Hint: use the result of 1).) (5 points)

Solution:

1) We first prove another version of Weyl's inequality:

$$\lambda_{i+j-1}(\mathbf{A}+\mathbf{B}) \overset{\text{(a)}}{\le} \lambda_i(\mathbf{A}) + \lambda_j(\mathbf{B}) \overset{\text{(b)}}{\le} \lambda_{i+j-n}(\mathbf{A}+\mathbf{B}), \quad \forall i, j \in \{1, 2, \cdots, n\}.$$

Proof. Let $\mathbf{u}_1, \cdots, \mathbf{u}_n$ and $\mathbf{v}_1 \cdots, \mathbf{v}_n$ be orthonormal lists of eigenvectors of $\mathbf{A}$ and $\mathbf{B}$ respectively, such that $\mathbf{u}_i$ is associated with $\lambda_i(\mathbf{A})$ and $\mathbf{v}_i$ is associated with $\lambda_i(\mathbf{B})$. Denote by $\mathbf{A}^{(i-1)} = \sum_{\alpha=1}^{i-1} \lambda_\alpha(\mathbf{A})\mathbf{u}_\alpha\mathbf{u}_\alpha^T$, $\mathbf{B}^{(j-1)} = \sum_{\alpha=1}^{j-1} \lambda_\alpha(\mathbf{B})\mathbf{v}_\alpha\mathbf{v}_\alpha^T$.

First assume that $\mathbf{A}, \mathbf{B} \succeq 0$. We have

$$\begin{aligned}
\lambda_{i+j-1}(\mathbf{A}+\mathbf{B}) &\overset{\text{Interlacing}}{\le} \lambda_1(\mathbf{A}-\mathbf{A}^{(i-1)}+\mathbf{B}-\mathbf{B}^{(j-1)}) \\
&\overset{\text{Weyl}}{\le} \lambda_1(\mathbf{A}-\mathbf{A}^{(i-1)}) + \lambda_1(\mathbf{B}-\mathbf{B}^{(j-1)}) \\
&\overset{\mathbf{A},\mathbf{B}\succeq 0}{=} \lambda_i(\mathbf{A}) + \lambda_j(\mathbf{B}).
\end{aligned}$$

(5 points)

Now let $\mathbf{A}, \mathbf{B}$ be general symmetric matrices. Then apply what we just proved for $\mathbf{A}+\delta\mathbf{I}_n, \mathbf{B}+\delta\mathbf{I}_n$ for a sufficiently large number $\delta > 0$ and cancel the δ's on both sides. Inequality (a) is proved (3 points).

Now we prove inequality (b). Applying the first inequality to $-\mathbf{A}$ and $-\mathbf{B}$ yields

$$\begin{aligned}
&-\lambda_{i'}(-\mathbf{A}) - \lambda_{j'}(\mathbf{B}) \le -\lambda_{i'+j'-1}(-\mathbf{A}-\mathbf{B}) \\
\iff &\lambda_{n-i'+1}(\mathbf{A}) + \lambda_{n-j'+1}(\mathbf{B}) \le \lambda_{n-i'-j'+2}(\mathbf{A}+\mathbf{B})
\end{aligned}$$

Inequality (b) is obtained by letting $i' = n-i+1$ and $j' = n-j+1$ (4 points). $\square$

Using inequality (a), we have $\lambda_i(\mathbf{A}+\mathbf{B}) \leq \lambda_{i-\pi}(\mathbf{A}) + \lambda_{\pi+1}(\mathbf{B}) = \lambda_{i-\pi}(\mathbf{A})$. Similarly, using inequality (b), we have $\lambda_i(\mathbf{A}+\mathbf{B}) \geq \lambda_{i+\nu}(\mathbf{A}) + \lambda_{n-\nu}(\mathbf{B}) = \lambda_{i+\nu}(\mathbf{A})$ (3 points).

2) Note that $\lambda_{r+1}(\mathbf{B}) \leq 0$ and $\lambda_{n-r}(\mathbf{B}) \geq 0$. Combining this with the proof of 1) gives the result (5 points).

Problem 6. (Power Iteration) (10 points)

Let $\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$. Starting with $\mathbf{v}^{(0)} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ such that $\left\|\mathbf{v}^{(0)}\right\| = 1$, $\{\mathbf{v}^{(0)}, \mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \cdots\}$ is generated by the power iteration.

1) Does the sequence $\{\mathbf{v}^{(k)}\}_{k\geq 0}$ converge? (There are three choices: it converges; it may converge, and may not; it does not converge.) Briefly explains why. (4 points)
2) Let $a \neq b$. Describe the sequence $\{\mathbf{v}^{(k)}\}_{k\geq 0}$ in terms of the eigenvectors of $\mathbf{A}$. Specifically, if we decompose $\mathbf{v}^{(k)}$ as a linear combination of the eigenvectors of $\mathbf{A}$ to get $\mathbf{v}^{(k)} = \alpha^{(k)}\mathbf{u}_1 + \beta^{(k)}\mathbf{u}_2 + \gamma^{(k)}\mathbf{u}_3$, and consider the scalar sequences $\{\alpha^{(k)}\}_{k\geq 0}$, $\{\beta^{(k)}\}_{k\geq 0}$ and $\{\gamma^{(k)}\}_{k\geq 0}$, whether their magnitudes increase or decrease? Do they keep changing signs? Finally, does $\{\mathbf{v}^{(k)}\}_{k\geq 0}$ converge? (6 points)

   (Hint: $\mathbf{A} = \begin{bmatrix} \mathbf{B} & 0 \\ 0 & 1/2 \end{bmatrix}$ is block diagonal and $\mathbf{B}$ is orthogonal.)

Solution:

1) Diagonalize $\mathbf{A}$ to get

$$\mathbf{A} = \mathbf{U} \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} \mathbf{U}^{-1},$$

   where $\mathbf{U} = [\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3]$, $\mathbf{u}_1 = [1, 1, 0]^T$, $\mathbf{u}_2 = [-1, 1, 0]^T$, $\mathbf{u}_3 = [0, 0, 1]^T$, $\lambda_1 = 1$, $\lambda_2 = -1$ and $\lambda_3 = \frac{1}{2}$ (2 points). We can see that the assumption of power iteration that "the greatest magnitude of eignevalue is strictly larger than the others" is violated, hence generally speaking, we shall not expect the algorithm to converge (2 points). But in some special cases, it may converge, e.g., when $a = b$.

2) Decompose $\mathbf{v}^{(0)}$ as a linear combination of $\mathbf{u}_1, \mathbf{u}_2$ and $\mathbf{u}_3$ to get $\mathbf{v}^{(0)} = \alpha^{(0)}\mathbf{u}_1 + \beta^{(0)}\mathbf{u}_2 + \gamma^{(0)}\mathbf{u}_3$, where $\beta \neq 0$ since $a \neq b$ (1 point). Note that

$$\mathbf{A}\mathbf{v}^{(0)} = \mathbf{B}(\alpha^{(0)}\mathbf{u}_1 + \beta^{(0)}\mathbf{u}_2) + \frac{1}{2}\gamma^{(0)}\mathbf{u}_3 = \alpha^{(0)}\mathbf{u}_1 - \beta^{(0)}\mathbf{u}_2 + \frac{1}{2}\gamma^{(0)}\mathbf{u}_3.$$

   Moreover, by the fact that $\mathbf{B}$ is orthogonal and $\left\|\mathbf{v}^{(0)}\right\| = 1$, we have

$$\left\|\mathbf{A}\mathbf{v}^{(0)}\right\|^2 = \left\|\mathbf{B}(\alpha^{(0)}\mathbf{u}_1 + \beta^{(0)}\mathbf{u}_2)\right\|^2 + \left\|\frac{1}{2}\gamma^{(0)}\mathbf{u}_3\right\|^2 = \left\|\alpha^{(0)}\mathbf{u}_1 + \beta^{(0)}\mathbf{u}_2\right\|^2 + \left\|\frac{1}{2}\gamma^{(0)}\mathbf{u}_3\right\|^2,$$

   hence $1/2 < \left\|\mathbf{A}\mathbf{v}^{(0)}\right\| \leq 1$ (2 points). Therefore, when normalize $\mathbf{A}\mathbf{v}^{(0)}$ to get $\mathbf{v}^{(1)} = \alpha^{(1)}\mathbf{u}_1 + \beta^{(1)}\mathbf{u}_2 + \gamma^{(1)}\mathbf{u}_3$, we have

$$\alpha^{(1)} = \frac{\alpha^{(0)}}{\left\|\mathbf{A}\mathbf{v}^{(0)}\right\|}, \quad \beta^{(1)} = -\frac{\beta^{(0)}}{\left\|\mathbf{A}\mathbf{v}^{(0)}\right\|}, \quad \gamma^{(1)} = \frac{\gamma^{(0)}}{2\left\|\mathbf{A}\mathbf{v}^{(0)}\right\|}.$$

   We can see that $\left|\alpha^{(1)}\right| \geq \left|\alpha^{(0)}\right|$, $\left|\beta^{(1)}\right| \geq \left|\beta^{(0)}\right|$, $\left|\gamma^{(1)}\right| \leq \left|\gamma^{(0)}\right|$, and only β changes its sign (2 points). Repeat the above procedure, we further have $\left|\alpha^{(k+1)}\right| \geq \left|\alpha^{(k)}\right|$, $\left|\beta^{(k+1)}\right| \geq \left|\beta^{(k)}\right|$, $\left|\gamma^{(k+1)}\right| \leq \left|\gamma^{(k)}\right|$ and $\beta^{(k+1)}\beta^{(k)} < 0$, for all $k \geq 1$ (1 point). The magnitude of $\beta^{(k)}$ is nondecreasing as k increases, and its sign is alternating, thus the $\{\mathbf{v}^{(k)}\}_{k\geq 0}$ does not converge.