SI231B: Matrix Computations, 2024 Fall

Homework Set #3

Acknowledgements:

1) Deadline: **2024-12-02 23:59:59**
2) Please submit the PDF file to gradescope. Course entry code: 8KJ345.
3) You have 5 “free days” in total for all late homework submissions.
4) If your homework is handwritten, please make it clear and legible.
5) All your answers are required to be in English.
6) Please include the main steps in your answer; otherwise, you may not get the points.

Problem 1. (Diagonalization) (15 points)

1) Determine whether each of the following matrices is diagonalizable. (10 points)

$$\mathbf{A} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 4 & -2 \\ -3 & -3 & 5 \end{bmatrix}, \mathbf{B} = \begin{bmatrix} 0 & 0 & 1 \\ 3 & 1 & 2 \\ 1 & 0 & 0 \end{bmatrix}. \tag{1}$$

2) If it is diagnolizable, diagonalize the matrix using a similarity transformation. (5 points)

Problem 2. (Eigenvector, eigenvalue) (15 points)

Suppose $\mathbf{A}$, $\mathbf{B} \in \mathbb{R}^{2\times 2}$ are unitary and the dimension of $\mathcal{N}(\mathbf{A}-\mathbf{B})$ is 1. Prove the following statements:

1) $\mathbf{A^{-1}B}$ and $\mathbf{B^{-1}A}$ have the same eigenvalues. (5 points)
2) $\mathbf{A^{-1}B}$ and $\mathbf{B^{-1}A}$ have a shared eigenvector. (5 points)
3) Suppose that $\mathbf{A^{-1}B}$ is diagonalizable. Show that $\mathbf{A}$ and $\mathbf{B}$ are not similar. (5 points)

Problem 3. (Eigenvector) (20 points)

Given $\mathbf{A} \in \mathbb{R}^{3\times 3}$, $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are eigenvectors associated with three different eigenvalues $\lambda_1, \lambda_2, \lambda_3$. Let $\mathbf{b} = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3$.

1) Prove that $\mathbf{b}, \mathbf{A}\mathbf{b}, \mathbf{A}^2\mathbf{b}$ are linearly independent. (10 points)
2) If $\mathbf{A}^3\mathbf{b} = \mathbf{A}\mathbf{b}$, what is the rank of $\mathbf{A} - \mathbf{I}$? Compute $\det(\mathbf{A} + \mathbf{I})$. (10 points)

Problem 4. (Schur decomposition) (20 points)

1) Let $\mathbf{A} = \begin{bmatrix} 5 & 7 \\ -2 & -4 \end{bmatrix}$. Compute the Schur decomposition of $\mathbf{A} = \mathbf{U}\mathbf{T}\mathbf{U}^T$, where $\mathbf{U}$ is (real) orthogonal. (The answer is not unique.) (10 points)
2) Let $\mathbf{B} = \begin{bmatrix} 1 & 1 \\ -2 & -1 \end{bmatrix}$.Compute the Schur decomposition of $\mathbf{B} = \mathbf{U}\mathbf{T}\mathbf{U}^H$, where $\mathbf{U}$ is unitary. (The answer is not unique.) (10 points)

Problem 5. (Variational Characterizations) (20 points)

For $\mathbf{M} \in \mathbb{S}^{n\times n}$, let $\lambda_k(\mathbf{M})$ denote the kth largest eigenvalue of $\mathbf{M}$, i.e., $\lambda_n(\mathbf{M}) \leq \cdots \leq \lambda_1(\mathbf{M})$. We make the convention that $\lambda_i(\mathbf{M}) = -\infty, \forall i \geq n+1$ and $\lambda_i(\mathbf{M}) = +\infty, \forall i \leq 0$.

1) Let $\mathbf{A}, \mathbf{B} \in \mathbb{S}^{n\times n}$. Suppose that $\mathbf{B}$ has exactly π positive eigenvalues and exactly ν negative eignevalues. Prove that

$$\lambda_i(\mathbf{A}+\mathbf{B}) \leq \lambda_{i-\pi}(\mathbf{A}), \quad i = \pi+1, \cdots, n,$$
$$\lambda_{i+\nu}(\mathbf{A}) \leq \lambda_i(\mathbf{A}+\mathbf{B}), \quad i = 1, \cdots, n-\nu.$$

(Hint: you might use Weyl's inequality and the interlacing theorem in section 4.6 of the lectures.) (15 points)

2) Let $\mathbf{A}, \mathbf{B} \in \mathbb{S}^{n\times n}$. Suppose that $\mathbf{B}$ is singular and has rank r. Prove that

$$\lambda_i(\mathbf{A}+\mathbf{B}) \leq \lambda_{i-r}(\mathbf{A}), \quad i = r+1, \cdots, n,$$
$$\lambda_{i+r}(\mathbf{A}) \leq \lambda_i(\mathbf{A}+\mathbf{B}), \quad i = 1, \cdots, n-r.$$

(Hint: use the result of 1).) (5 points)

Problem 6. (Power Iteration) (10 points)

Let $\mathbf{A} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1/2 \end{bmatrix}$. Starting with $\mathbf{v}^{(0)} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ such that $\left\|\mathbf{v}^{(0)}\right\| = 1$, $\{\mathbf{v}^{(0)}, \mathbf{v}^{(1)}, \mathbf{v}^{(2)}, \cdots\}$ is generated by the power iteration.

1) Does the sequence $\{\mathbf{v}^{(k)}\}_{k\geq 0}$ converge? (There are three choices: it converges; it may converge, and may not; it does not converge.) Briefly explains why. (4 points)
2) Let $a \neq b$. Describe the sequence $\{\mathbf{v}^{(k)}\}_{k\geq 0}$ in terms of the eigenvectors of $\mathbf{A}$. Specifically, if we decompose $\mathbf{v}^{(k)}$ as a linear combination of the eigenvectors of $\mathbf{A}$ to get $\mathbf{v}^{(k)} = \alpha^{(k)}\mathbf{u}_1 + \beta^{(k)}\mathbf{u}_2 + \gamma^{(k)}\mathbf{u}_3$, and consider the scalar sequences $\{\alpha^{(k)}\}_{k\geq 0}$, $\{\beta^{(k)}\}_{k\geq 0}$ and $\{\gamma^{(k)}\}_{k\geq 0}$, whether their magnitudes increase or decrease? Do they keep changing signs? Finally, does $\{\mathbf{v}^{(k)}\}_{k\geq 0}$ converge? (6 points)
(Hint: $\mathbf{A} = \begin{bmatrix} \mathbf{B} & 0 \\ 0 & 1/2 \end{bmatrix}$ is block diagonal and $\mathbf{B}$ is orthogonal.)