Convex Sets

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Outline

Definition of Affine Set

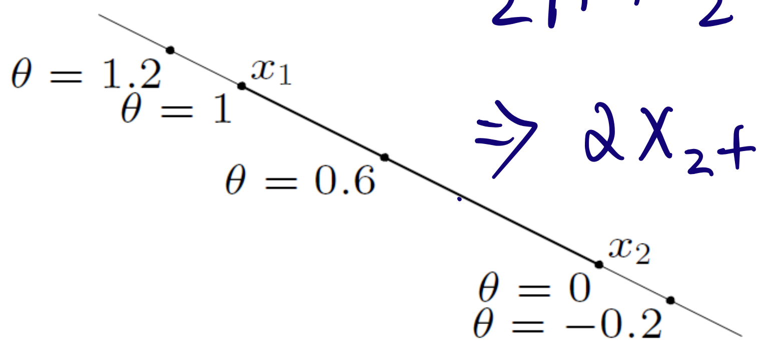
$P_1: \ 2\alpha x_2 + (1-2\alpha) x_1 \in C$

$P_2: \ 2\beta x_3 + (1-2\beta) x_1 \in C$

- **Line:** through $\boldsymbol{x}_1, \boldsymbol{x}_2$: all points

$$\boldsymbol{x} = \theta \boldsymbol{x}_1 + (1-\theta)\boldsymbol{x}_2 \quad (\theta \in \mathbb{R})$$

$\frac{1}{2} P_1 + \frac{1}{2} P_2 \in C$



$\Rightarrow \alpha x_2 + \beta x_3 + (1-\alpha-\beta) x_1 \in C$

- **Affine set:** contains the line through any two distinct points in the set
- **Example:** solution set of linear equations $\{\boldsymbol{x} | \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}\}$ (conversely, every affine set can be expressed as solution set of system of linear equations)

generalize $\Rightarrow \sum_{i=1}^{k} \theta_i x_i \in C, \ \sum_{i=1}^{k} \theta_i = 1$

$$A x_1 = b$$
$$A x_2 = b$$

$$\begin{aligned} A(\theta x_1 + (1-\theta) x_2) &= \theta A x_1 + (1-\theta) A x_2 \\ &= \theta b + (1-\theta) b \\ &= b \end{aligned}$$

$$\begin{aligned} a_1^T x &= b_1 \\ a_2^T x &= b_2 \\ &\vdots \\ a_m^T x &= b_m \end{aligned} \qquad A = \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_m^T \end{bmatrix}$$

Assume $C \neq \phi$, we are to show that there exist A, b such that $Ax = b \iff x \in C$, C is an affine set

Take any $x_0 \in C$, then $C_0 = C - x_0 = \{ x - x_0 \mid x_0 \in C \}$ is a subspace.

Since every subspace is complementable, so there exists a subspace $C_0^\perp$ such that $C_0 \perp C_0^\perp$, i.e. $z \in C_0 \iff z \perp a, \forall a \in C_0^\perp$

Let $\{a_1, \cdots a_m\}$ be a basis of $C_0^{\perp}$. then

$z \perp a, \forall a \in C_0^{\perp} \Leftrightarrow z \perp a_i, i=1, \cdots, m \Leftrightarrow a_i^T z = 0, \forall i=1, \cdots, m$

Define a matrix collect a_i^T as the row, then $Az=0$.

Since $z \in C_0 \Leftrightarrow z = x - x_0, x \in C$, then $A(x-x_0)=0$

$\Leftrightarrow x \in x_0 + C_0 := C, \quad Ax_0 := b$

$\Rightarrow Ax = \underbrace{b}_{Ax_0} \Leftrightarrow x \in C$

To verify C_0 is a subspace:

$\alpha z_1 + \beta z_2 \in C_0, \forall z_1 \in C_0, z_2 \in C_0. \forall \alpha, \beta \in \mathbb{R}$

$z_1 + x_0 \in C, \quad z_2 + x_0 \in C$

$\alpha(z_1 + x_0) + \beta(z_2 + x_0) + (1-\alpha-\beta)x_0 \in C$

$\Rightarrow \alpha z_1 + \beta z_2 - x_0 \in C$

$\Rightarrow \alpha z_1 + \beta z_2 \in C - x_0 := C_0$

Definition of Convex Set

- **Line segment:** between $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$: all points

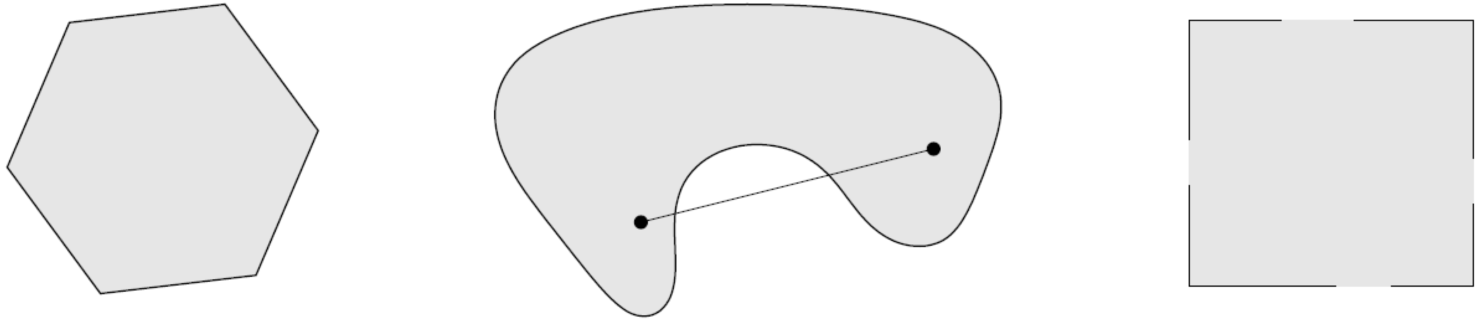
$$\boldsymbol{x} = \theta \boldsymbol{x}_1 + (1-\theta)\boldsymbol{x}_2$$

  with $0 \leq \theta \leq 1$

- **Convex set:** contains line segment between any two points in the set C

$$\boldsymbol{x}_1, \boldsymbol{x}_2 \in C, 0 \leq \theta \leq 1 \quad \Longrightarrow \quad \theta \boldsymbol{x}_1 + (1-\theta)\boldsymbol{x}_2 \in C$$

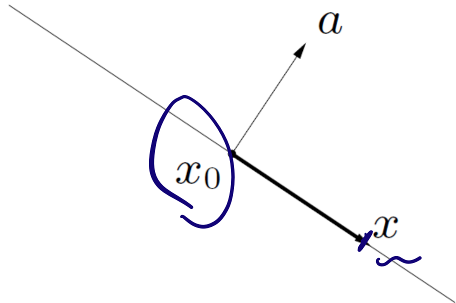
- **Examples** (one convex, two nonconvex sets)



Outline

Examples: Hyperplanes and Halfspaces

- **Hyperplane:** set of the form $\{\boldsymbol{x} \mid \boldsymbol{a}^T\boldsymbol{x} = b\}(\boldsymbol{a} \neq \boldsymbol{0})$

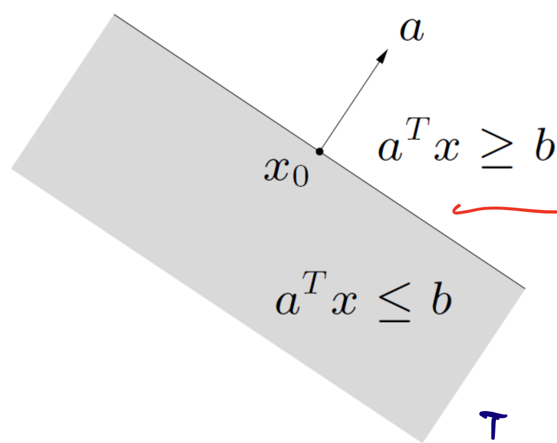


$a^T(x - x_0) = 0$

$\Rightarrow a^T x = a^T x_0 \triangleq b$

- **Halfspace:** set of the form $\{\boldsymbol{x} \mid \boldsymbol{a}^T\boldsymbol{x} \leq b\}(\boldsymbol{a} \neq \boldsymbol{0})$

$a^T x_1 = b, \quad a^T x_2 = b$



$a^T(\theta x_1 + (1-\theta) x_2) = b$

$a^T x_1 \leq b, \quad a^T x_2 \leq b$

$a^T(\theta x_1 + (1-\theta) x_2) \leq b \quad \theta \in [0,1]$

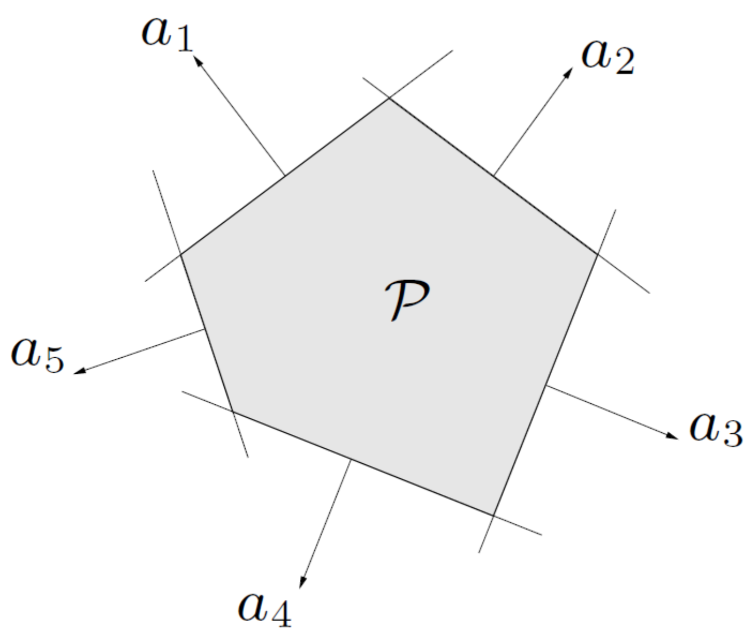
  - $\boldsymbol{a}$ is the normal vector
  - hyperplanes are affine and convex; halfspaces are convex

Example: Polyhedra

Solution set of finitely many linear inequalities and equalities

$$\boldsymbol{A}\boldsymbol{x} \preceq \boldsymbol{b}, \qquad \boldsymbol{C}\boldsymbol{x} = \boldsymbol{d}$$

($\boldsymbol{A} \in \mathbb{R}^{m\times n}$, $\boldsymbol{C} \in \mathbb{R}^{p\times n}$, $\preceq$ is componentwise inequality)



polyhedron is intersection of finite number of halfspaces and hyperplanes

Examples: Euclidean Balls and Ellipsoids

$x - x_c := r u \Rightarrow \|r u\|_2 \le r \Rightarrow \|u\|_2 \le 1$

- **(Euclidean) Ball** with center $\boldsymbol{x}_c$ and radius r:

$$B(\boldsymbol{x}_c, r) = \{\boldsymbol{x} \mid \|\boldsymbol{x} - \boldsymbol{x}_c\|_2 \le r\} = \{\boldsymbol{x}_c + r\boldsymbol{u} \mid \|\boldsymbol{u}\|_2 \le 1\}$$

- **Ellipsoid:** set of the form $\|x - x_c\|_2^2 \le r^2 \Leftrightarrow (x - x_c)^T \cdot (x - x_c) \le r^2$

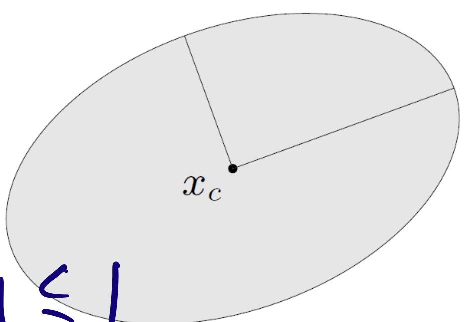
$$\begin{aligned} E(\boldsymbol{x}_c, \boldsymbol{P}) &= \{\boldsymbol{x} \mid (\boldsymbol{x} - \boldsymbol{x}_c)^T \boldsymbol{P}^{-1}(\boldsymbol{x} - \boldsymbol{x}_c) \le 1\} \\ &= \{\boldsymbol{x}_c + \boldsymbol{A}\boldsymbol{u} \mid \|\boldsymbol{u}\|_2 \le 1\} \end{aligned}$$

$\frac{1}{r^2}(x - x_c)^T \cdot (x - x_c) \le 1$

with $\boldsymbol{P} \in \mathbb{S}^n_{++}$ (i.e., $\boldsymbol{P}$ symmetric positive definite), $\boldsymbol{A}$ square and nonsingular

$\Rightarrow (x - x_c)^T \cdot \begin{bmatrix} \frac{1}{r^2} & & \\ & \ddots & \\ & & \frac{1}{r^2} \end{bmatrix} \cdot (x - x_c) \le 1$

Let $x - x_c = Au$. $A = P^{\frac{1}{2}}$



$u^T A^T P^{-1} A u = u^T P^{\frac{1}{2}} \cdot P^{-1} \cdot P^{\frac{1}{2}} u \le 1$

$\Rightarrow \|u\|_2 \le 1$

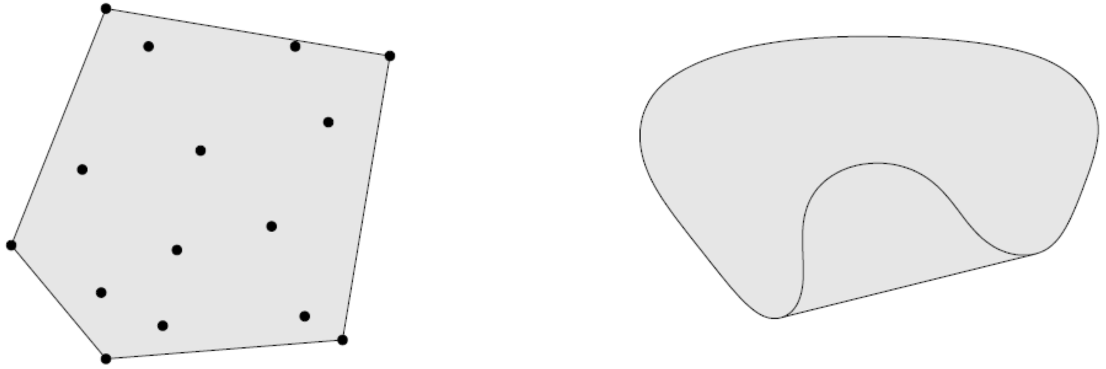
Convex Combination and Convex Hull

- **Convex combination** of $\boldsymbol{x}_1, \cdots, \boldsymbol{x}_k$: any point $\boldsymbol{x}$ of the form

$$\boldsymbol{x} = \theta_1 \boldsymbol{x}_1 + \theta_2 \boldsymbol{x}_2 + \cdots + \theta_k \boldsymbol{x}_k$$

  with $\theta_1 + \cdots + \theta_k = 1, \theta_i \geq 0$

- **Convex hull** $\mathsf{conv}\, S$: set of all convex combinations of points in S

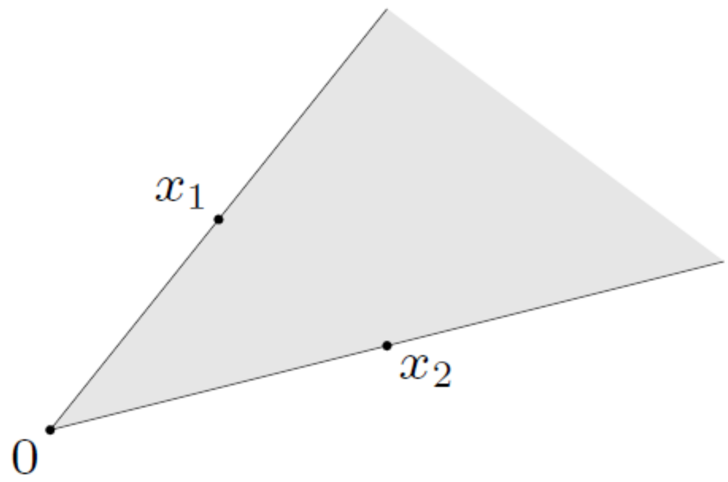


Conic Combination and Convex Cone

- **Conic (nonnegative) combination** of $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$: any point of the form

$$\boldsymbol{x} = \theta_1 \boldsymbol{x}_1 + \theta_2 \boldsymbol{x}_2$$

with $\theta_1 \geq 0, \theta_2 \geq 0$



- **Convex cone:** set that contains all conic combinations of points in the set

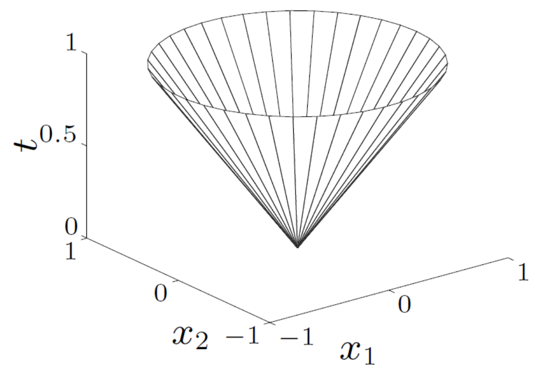
Convex Cones: Norm Balls and Norm Cones

- **Norm:** a function $\|\cdot\|$ that satisfies
  - $\|\boldsymbol{x}\| \geq 0$; $\|\boldsymbol{x}\| = 0$ if and only if $\boldsymbol{x} = \mathbf{0}$
  - $\|t\boldsymbol{x}\| = |t|\|\boldsymbol{x}\|$ for $t \in \mathbb{R}$
  - $\|\boldsymbol{x} + \boldsymbol{y}\| \leq \|\boldsymbol{x}\| + \|\boldsymbol{y}\|$

  notation: $\|\cdot\|$ general (unspecified) norm; $\|\cdot\|_{\mathrm{symb}}$ a particular norm

- **Norm ball** with center $\boldsymbol{x}_c$ and radius r: $\{\boldsymbol{x} | \|\boldsymbol{x} - \boldsymbol{x}_c\| \leq r\}$

- **Norm cone:** $\{(\boldsymbol{x}, t) \in \mathbb{R}^{n+1} | \|\boldsymbol{x}\| \leq t\}$



  Euclidean norm cone or second-order cone (aka ice-cream cone)

Positive Semidefinite Cone

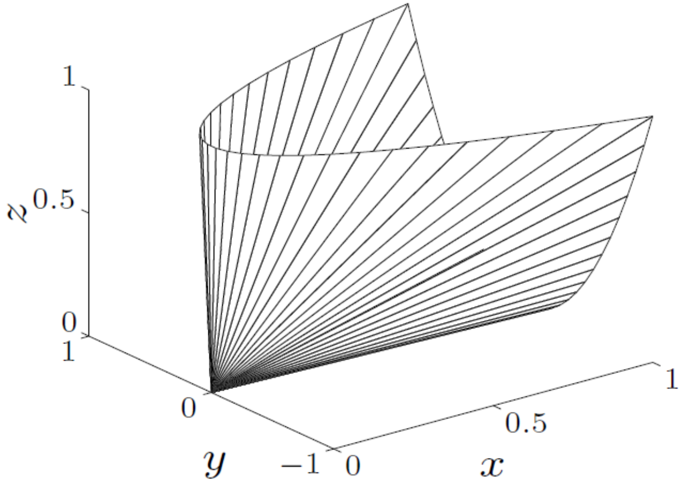
Notation

- $\mathbb{S}^n$ is set of symmetric $n \times n$ matrices
- $\mathbb{S}^n_+ = \{\boldsymbol{X} \in \mathbb{S}^n | \boldsymbol{X} \succeq 0\}$: positive semidefinite $n \times n$ matrices

$$\boldsymbol{X} \in \mathbb{S}^n_+ \quad \Longleftrightarrow \quad \boldsymbol{z}^\top \boldsymbol{X} \boldsymbol{z} \geq 0 \text{ for all } \boldsymbol{z}$$

  $\mathbb{S}^n_+$ is a convex cone
- $\mathbb{S}^n_{++} = \{\boldsymbol{X} \in \mathbb{S}^n | \boldsymbol{X} \succ 0\}$: positive definite $n \times n$ matrices

Example: $\begin{bmatrix} x & y \\ y & z \end{bmatrix} \in \mathbb{S}^2_+$



Outline

Operations that Preserve Convexity

How to establish the convexity of a given set C

- Apply the definition (can be cumbersome)

$$\boldsymbol{x}_1, \boldsymbol{x}_2 \in C, 0 \leq \theta \leq 1 \quad \Longrightarrow \quad \theta \boldsymbol{x}_1 + (1-\theta)\boldsymbol{x}_2 \in C$$

- Show that C is obtained from simple convex sets(hyperplanes, halfspaces, norm balls,$\cdots$) by operations that preserve convexity
  - intersection
  - affine functions
  - perspective function
  - linear-fractional functions

Intersection

- **Intersection:** if $S_1, S_2, \ldots, S_k$ are convex, then $S_1 \cap S_2 \cap \cdots \cap S_k$ is convex (k can be any positive integer)

- Example 1: a polyhedron is the intersection of halfspaces and hyperplanes

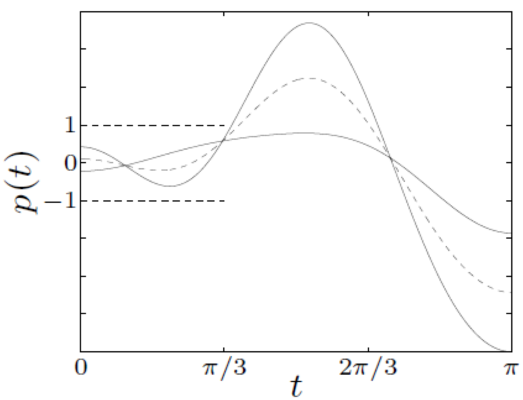
$-1 \le \underbrace{g(t)}\cdot\vec{X} \le 1, \quad |t| \le \frac{\pi}{3}$

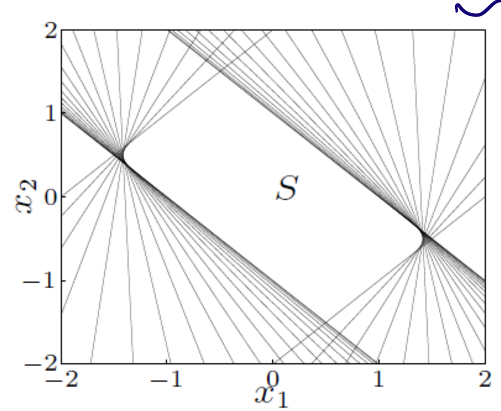
- Example 2:

$$S = \{\boldsymbol{x} \in \mathbb{R}^m \,|\, |\underbrace{p(t)}| \le 1 \text{ for } |t| \le \pi/3\}$$

where $p(t) = x_1 \cos t + x_2 \cos 2t + \cdots + x_m \cos mt$ $= \underbrace{(\cos t, \cos 2t, \cdots, \cos mt)} \cdot \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix}$

$\vec{X}$





for $m = 2$

Affine Function

suppose $f : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ is affine ($f(\boldsymbol{x}) = \boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}$ with $\boldsymbol{A} \in \mathbb{R}^{m \times n}$, $\boldsymbol{b} \in \mathbb{R}^m$)

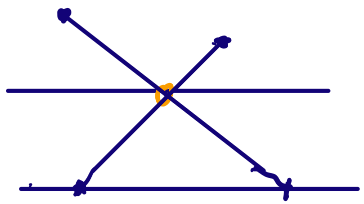
- the image of a convex set under f is convex
$$S \subseteq \mathbb{R}^n \text{ convex} \quad \Longrightarrow \quad f(S) = \{f(\boldsymbol{x}) | \boldsymbol{x} \in S\} \text{ convex}$$
- the inverse image $f^{-1}(C)$ a convex set under f is convex
$$C \subseteq \mathbb{R}^m \text{ convex} \quad \Longrightarrow \quad f^{-1}(C) = \{\boldsymbol{x} \in \mathbb{R}^n | f(\boldsymbol{x}) \in C\} \text{ convex}$$

Examples

- scaling, translation, projection
- solution set of linear matrix inequality $\{\boldsymbol{x} | x_1 \boldsymbol{A}_1 + \cdots + x_m \boldsymbol{A}_m \preceq \boldsymbol{B}\}$ (with $\boldsymbol{A}_i, \boldsymbol{B} \in \mathbb{S}^p$)
- $\{(\boldsymbol{x}, t) \in \mathbb{R}^{n+1} | \|\boldsymbol{x}\| \leq t\}$ is convex, so is
$$\{\boldsymbol{x} \in \mathbb{R}^n | \|\boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}\| \leq \boldsymbol{c}^T \boldsymbol{x} + d\}$$

$$f(x,t) = \begin{pmatrix} Ax+b \\ c^T x + d \end{pmatrix} = \begin{pmatrix} A \\ c^T \end{pmatrix} x + \begin{pmatrix} b \\ d \end{pmatrix}$$

Perspective and Linear-fractional Function I



- **Perspective function** $P : \mathbb{R}^{n+1} \to \mathbb{R}^n$

$$P(\boldsymbol{x}, t) = \boldsymbol{x}/t, \quad \mathrm{dom}P = \{(\boldsymbol{x}, t) | t > 0\}$$

images and inverse images of convex sets under perspective are convex

- **Linear-fractional function** $f : \mathbb{R}^n \to \mathbb{R}^m$

$$f(\boldsymbol{x}) = \frac{\boldsymbol{A}\boldsymbol{x} + \boldsymbol{b}}{\boldsymbol{c}^T\boldsymbol{x} + d}, \quad \mathrm{dom}f = \{\boldsymbol{x} | \boldsymbol{c}^T\boldsymbol{x} + d > 0\}$$

images and inverse images of convex sets under linear-fractional functions are convex

The inverse image of a convex set under perspective function is convex.

$\Leftrightarrow$ If $C \subseteq \mathbb{R}^n$ is convex, then $P^{-1}(C) = \{(x,t) \in \mathbb{R}^{n+1} \mid x/t \in C, t>0\}$ is convex

suppose $(x,t) \in P^{-1}(C)$, $(y,s) \in P^{-1}(C)$. $0 \le \theta \le 1$

we need to show $\theta(x,t) + (1-\theta)(y,s) \in P^{-1}(C)$

$$(\theta x + (1-\theta)y,\ \theta t + (1-\theta)s) \ ?$$

i.e. $\dfrac{\theta x + (1-\theta)y}{\theta t + (1-\theta)s} \overset{?}{\in} C$ ✓

This follows from $\dfrac{\theta x + (1-\theta)y}{\theta t + (1-\theta)s} = u \cdot \dfrac{x}{t} + (1-u)\dfrac{y}{s} \in C$

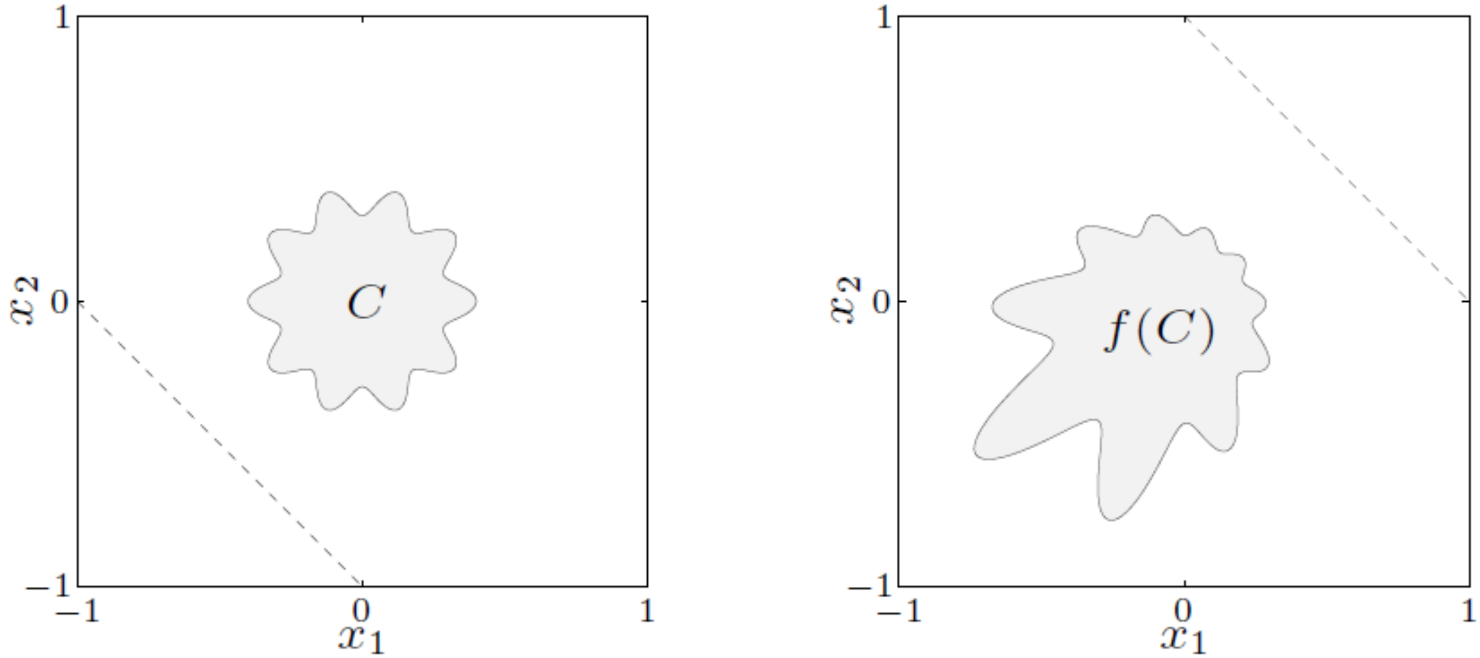
$$u = \frac{\theta t}{\theta t + (1-\theta)s} \in [0,1]$$

$\frac{x}{t} \in C$, $\frac{y}{s} \in C$

Perspective and Linear-fractional Function II

- **Examples** of a linear-fractional function

$$f(\boldsymbol{x}) = \frac{1}{x_1 + x_2 + 1}\boldsymbol{x}$$



Outline

Generalized Inequalities I

- A convex cone $K \subseteq \mathbb{R}^n$ is a **proper cone** if
  - K is closed (contains its boundary)
  - K is solid (has nonempty interior)
  - K is pointed (contains no line)

- **Examples**
  - nonnegative orthant

$$K = \mathbb{R}^n_+ = \{\boldsymbol{x} \in \mathbb{R}^n | x_i \geq 0, i = 1, \cdots, n\}$$

  - positive semidefinite cone

$$K = \mathbb{S}^n_+ = \{\boldsymbol{X} \in \mathbb{R}^{n \times n} | \boldsymbol{X} = \boldsymbol{X}^T \succeq \boldsymbol{0}\}$$

  - nonnegative polynomials on $[0, 1]$:

$$K = \{\boldsymbol{x} \in \mathbb{R}^n | x_1 + x_2 t + x_3 t^2 + \cdots + x_n t^{n-1} \geq 0 \text{ for } t \in [0, 1]\}$$

Generalized Inequalities II

Generalized inequality defined by a proper cone K:

$$\boldsymbol{y} \succeq_K \boldsymbol{x} \quad \Longleftrightarrow \quad \boldsymbol{y} - \boldsymbol{x} \succeq_K \mathbf{0} \text{ or } \boldsymbol{y} - \boldsymbol{x} \in K$$

Examples

- Componentwise inequality ($K = \mathbb{R}^n_+$)

$$\boldsymbol{y} \succeq_{\mathbb{R}^n_+} \boldsymbol{x} \quad \Longleftrightarrow \quad y_i \geq x_i, \quad i = 1, \cdots, n$$

- Matrix inequality ($K = \mathbb{S}^n_+$)

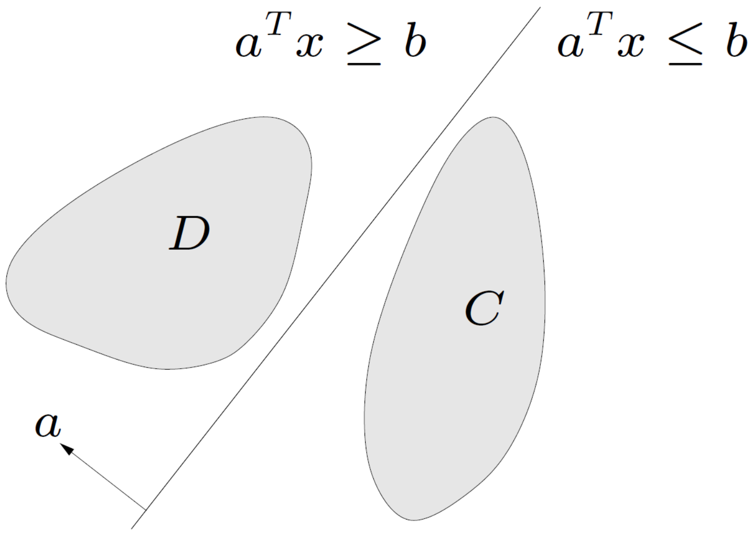
$$\boldsymbol{Y} \succeq_{\mathbb{S}^n_+} \boldsymbol{X} \quad \Longleftrightarrow \quad \boldsymbol{Y} - \boldsymbol{X} \text{ positive semidefinite}$$

Outline

Separating Hyperplane Theorem

If C and D are nonempty disjoint convex sets, there exist $\boldsymbol{a} \neq \mathbf{0}$ and b, such that

$$\boldsymbol{a}^T\boldsymbol{x} \leq b \text{ for } \boldsymbol{x} \in C, \quad \boldsymbol{a}^T\boldsymbol{x} \geq b \text{ for } \boldsymbol{x} \in D$$



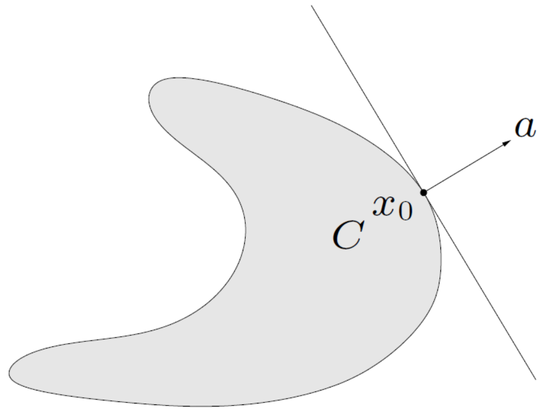
the hyperplane $\{\boldsymbol{x} | \boldsymbol{a}^T\boldsymbol{x} = b\}$ separates C and D

Supporting Hyperplane Theorem

Supporting hyperplane to set C at boundary point $\boldsymbol{x}_0$:

$$\{\boldsymbol{x} | \boldsymbol{a}^T\boldsymbol{x} = \boldsymbol{a}^T\boldsymbol{x}_0\}$$

where $\boldsymbol{a} \neq \boldsymbol{0}$ and $\boldsymbol{a}^T\boldsymbol{x} \leq \boldsymbol{a}^T\boldsymbol{x}_0$ for all $\boldsymbol{x} \in C$



Supporting hyperplane theorem: if C is convex, then there exists a supporting hyperplane at every boundary point of C

Dual Cones and Generalized Inequalities

- **Dual cone** of a cone K:

$$K^* = \{\boldsymbol{y} | \boldsymbol{y}^T \boldsymbol{x} \geq 0 \text{ for all } \boldsymbol{x} \in K\}$$

- Examples
  - $K = \mathbb{R}^n_+$: $K^* = \mathbb{R}^n_+$
  - $K = \mathbb{S}^n_+$: $K^* = \mathbb{S}^n_+$
  - $K = \{(\boldsymbol{x}, t) | \|\boldsymbol{x}\|_2 \leq t\}$: $K^* = \{(\boldsymbol{x}, t) | \|\boldsymbol{x}\|_2 \leq t\}$
  - $K = \{(\boldsymbol{x}, t) | \|\boldsymbol{x}\|_1 \leq t\}$: $K^* = \{(\boldsymbol{x}, t) | \|\boldsymbol{x}\|_\infty \leq t\}$

  First three examples are **self-dual** cones

- Dual cones of proper cones are proper, hence define generalized inequalities:

$$\boldsymbol{y} \succeq_{K^*} \mathbf{0} \quad \Longleftrightarrow \quad \boldsymbol{y}^T \boldsymbol{x} \geq 0 \text{ for all } \boldsymbol{x} \succeq_K \mathbf{0}$$

Reference

Chapter 2 of:

- Stephen Boyd and Lieven Vandenberghe, *Convex Optimization*. Cambridge, U.K.: Cambridge University Press, 2004.