TA Lecture 09

Dec. 10-11

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Outline

HW 9

Problem 1

Use the methods of inverse transform sampling (or called the method of inverse CDF) to obtain samples from each of the following continuous distributions:

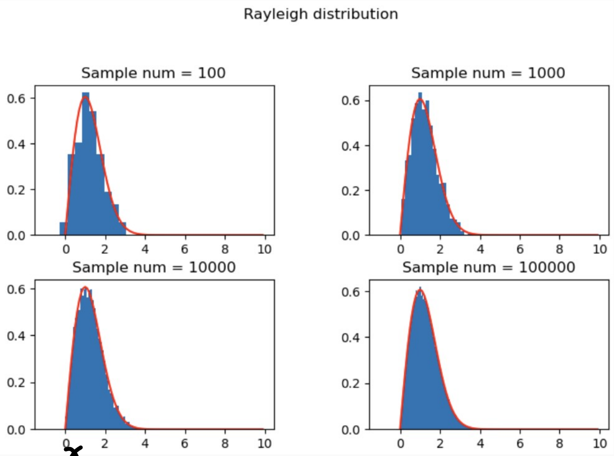
(a) Logistic distribution with CDF $F(x) = e^x/(1+e^x), x \in R$.

(b) Rayleigh distribution with CDF $F(x) = 1 - e^{-x^2/2}, x > 0$.

(c) Exponential distribution with CDF $F(x) = 1 - e^{-x}, x > 0$.

After obtaining enough samples, please plot the corresponding histogram and corresponding theoretical PDF.

Problem 1(a) Solution



$f(x) = \frac{e^x}{(1+e^x)^2}$

Figure 1: $F(x) = e^x/(1+e^x), x \in R.$

Problem 1(b) Solution

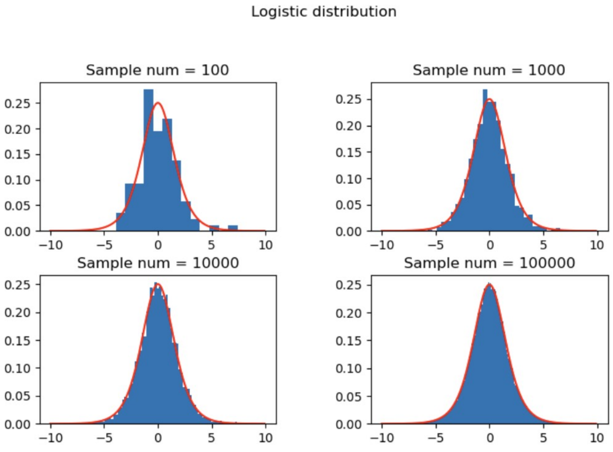
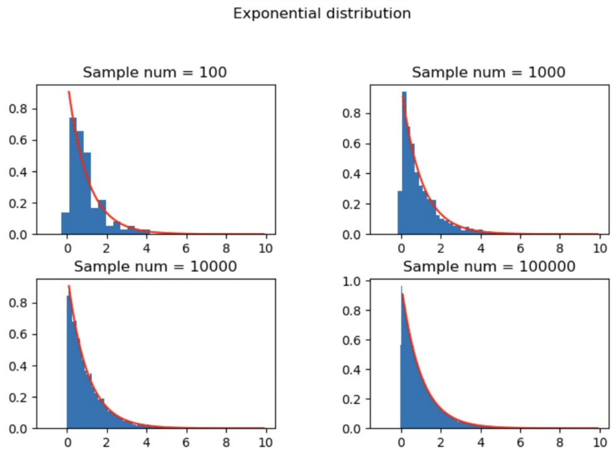


Figure 2: $F(x) = 1 - e^{-x^2/2}, x > 0.$

$f(x) = xe^{-\frac{x^2}{2}}$

Problem 1(c) Solution



$f(x) = e^{-x}$ **Figure 3:** $F(x) = 1 - e^{-x}, x > 0.$

Problem 2

Acceptance-Rejection Method

(a) Use the Acceptance-Rejection Method to obtain samples from Beta distribution $\text{Beta}(4, 6)$.

(b) Use both the Box-Muller method and the Acceptance-Rejection Method to obtain samples from the standard Normal distribution $\mathcal{N}(0, 1)$, then discuss the pros and cons of each method.

After obtaining enough samples, please plot the corresponding histogram and corresponding theoretical PDF.

Problem 2(a) Solution

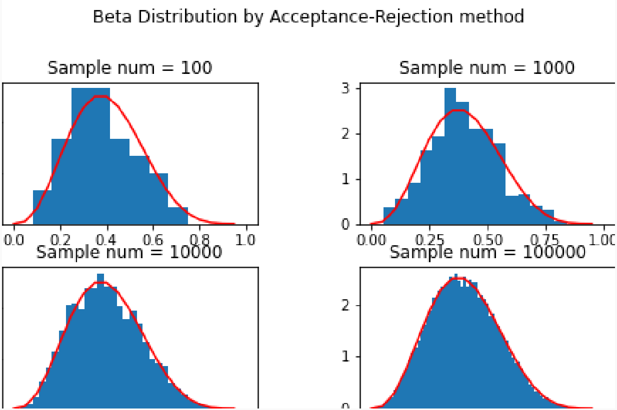
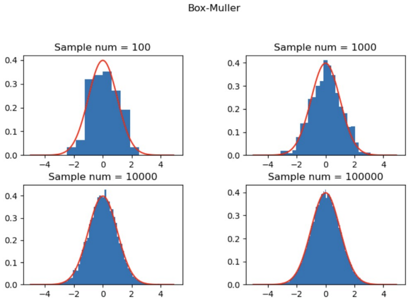


Figure 4: Acceptance-Rejection Method

Problem 2(b) Solution



Let $U \sim \mathrm{Unif}(0, 2\pi)$, and let $T \sim \mathrm{Expo}(1)$ be independent of U. Define $X = \sqrt{2T}\cos U$ and $Y = \sqrt{2T}\sin U$. Then X and Y are independent, and their marginal distributions are standard normal distribution.

Algorithm Normal Random Variable Generation: Box-Muller Approach

output: Independent standard normal random variables X and Y.

1: Generate two independent random variables, U_1 and U_2, from $Unif(0, 1)$.

2: $X \leftarrow (-2\ln U_1)^{1/2}\cos(2\pi U_2)$

3: $Y \leftarrow (-2\ln U_1)^{1/2}\sin(2\pi U_2)$

4: **return** X, Y

Problem 2(b) Solution

In terms of sampling Normal distribution, their variance are similar, while the sample efficiency of Box-Muller is higher with also higher running speed.

Box-Muller:

- Pros: It is easy to implement, and the method only uses Unif(0, 1) as the basis data sample, which is simple to sample.
- Cons: Only the standard normal distribution can be sampled by this method.

Problem 2(b) Solution

Acceptance-Rejection:

- Pros: It can sample many kinds of probability distribution including many distributions that is difficult to sample directly
- Cons: The domain of function $g(x)$ must cover the domain of function $f(x)$. If c is closed to 1, the basis distribution g is still difficult to sample; while if c is closed to 0, the probability of acceptance success will be small, which will cause low efficiency

Problem 3

Monte Carlo Integration

(a) Evaluate the integration

$$\int_0^1 \frac{4}{1+x^2}\,dx.$$

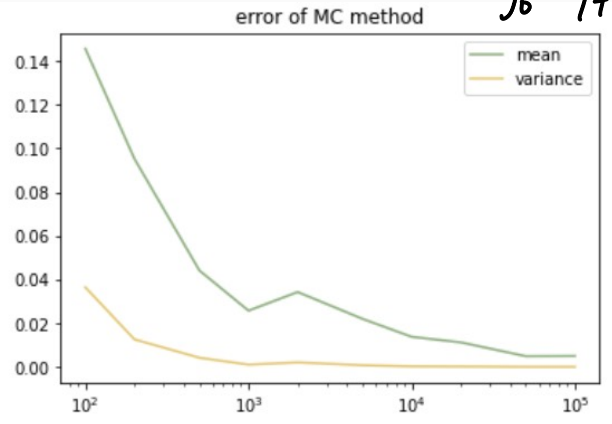
(b) Evaluate the integration

$$\int_0^4 \sqrt{x+\sqrt{x+\sqrt{x+\sqrt{x}}}}\,dx.$$

(c) Evaluate the probability of rare event $c = \mathbb{P}(Y > 8)$, where $Y \sim \mathcal{N}(0,1)$.

Problem 3(a) Solution

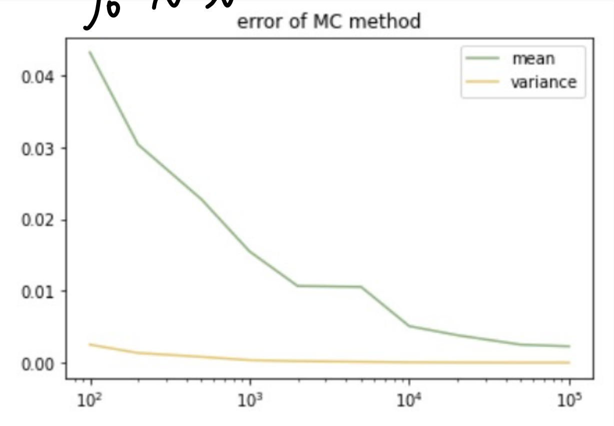
$$\int_0^1 \frac{4}{1+x^2}\,dx$$



3.14160782958

Problem 3(b) Solution

$$\int_0^4 \sqrt{x\sqrt{x+\sqrt{x+\sqrt{x}}}}\,dx$$



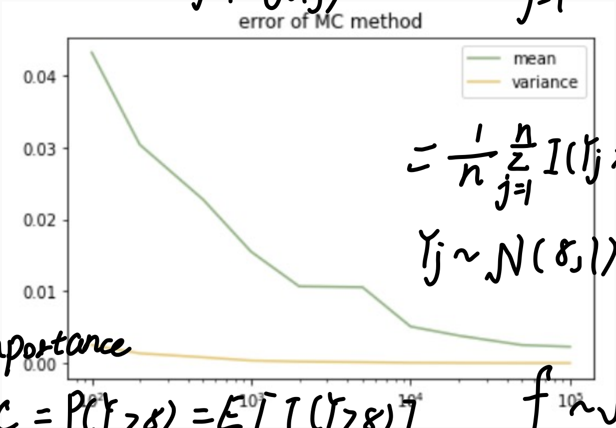
7.6765401290

Problem 3(c) Solution

importance: $c \approx \frac{1}{n}\sum_{j=1}^{n} \frac{h(Y_i)}{g(Y_j)} \cdot f(Y_j) = \frac{1}{n}\sum_{j=1}^{n} I(Y_j > 8) \cdot \frac{\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}Y_j^2}}{\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}(Y_j-8)^2}}$

$= \frac{1}{n}\sum_{j=1}^{n} I(Y_j > 8) e^{-8Y_j+32}$

$Y_j \sim N(8,1)$



without importance

$c \approx 0$

$c = P(Y > 8) = E[I(Y>8)]$

$= \int_{-\infty}^{+\infty} I(y>8) f(y)\, dy$

$f \sim N(0,1)$

$c \approx 6.228 \times 10^{-6}$

Problem 4

Use your own words to describe the geometric perspective of Jacobian Matrix and Jacobian Determinant.

Problem 4 Solution

Geometrically, the Jacobian Matrix at a point encapsulates the best linear approximation of a nonlinear transformation near that point. Each entry in the Jacobian represents the partial derivative of one transformed coordinate with respect to an original coordinate, effectively describing how each direction stretches, compresses, or rotates under the transformation.

The Jacobian Determinant provides a scalar measure of this transformation's local scaling effect on volume. Specifically, when a region in the original space is transformed, the absolute value of the Jacobian Determinant indicates how the volume of that region changes. For example:

Problem 4 Solution

- If the determinant is greater than 1, the transformation locally expands the volume.
- If it is between 0 and 1, the transformation contracts the volume.
- A negative determinant also indicates a reflection in addition to scaling.

Problem 5 i

The PDF of Gamma distribution Gamma(a, λ) is:

$$f(x) = \frac{1}{\Gamma(a)}(\lambda x)^a e^{-\lambda x}\frac{1}{x}, x > 0,$$

where $a > 0, \lambda > 0$, and $\Gamma(a)$ is the Gamma Function:

$$\Gamma(a) = \int_0^\infty z^{a-1} e^{-z} dz.$$

(a) If $X \sim \text{Gamma}(a, \lambda)$, find $E(X)$ and $\text{Var}(X)$.

(b) If $Y \sim \mathcal{N}(0, 1)$, show that $Y^2 \sim \text{Gamma}\left(\frac{1}{2}, \frac{1}{2}\right)$.

(c) If $V = Y_1^2 + \ldots + Y_n^2$, where $Y_i, i = 1, \ldots, n$ are i.i.d. random variables and $Y_i \sim \mathcal{N}(0, 1)$, then V satisfies chi-square distribution, i.e. $V \sim \chi_n^2$. Show that $V \sim \text{Gamma}\left(\frac{n}{2}, \frac{1}{2}\right)$ and find the PDF of V.

(d) If Y and V are independent, define random variable Z as follows

$$Z = \frac{Y}{\sqrt{\frac{V}{n}}}.$$

then Z satisfies Student's t-distribution, i.e. $Z \sim t_n$. Please adopt the change of variable method to find the PDF of Z.

(e) Given two independent random variables V_1 and V_2, where $V_1 \sim \chi^2_m$ and $V_2 \sim \chi^2_n$. Define random variable W as follows

$$W = \frac{\frac{V_1}{m}}{\frac{V_2}{n}}.$$

then W satisfies F -distribution, i.e. $W \sim F(m, n)$. Please adopt the change of variable method to find the PDF of W.

Problem 5 Solution

(a) $X \sim \text{Gamma}(a, \lambda)$

$E(X)$, $Var(X)$

$$E(X) = \int_0^\infty \frac{1}{\Gamma(a)} (\lambda x)^a e^{-\lambda x} \cdot \frac{1}{x} \cdot x \, dx$$

$$= \frac{\lambda^a}{\Gamma(a)\lambda^{a+1}} \int_0^\infty (\lambda x)^a e^{-\lambda x} \, d(\lambda x)$$

$$= \frac{\Gamma(a)}{\lambda} = \frac{a}{\lambda}$$

$$Var(X) = E(X^2) - E(X)^2$$

$$= \int_0^\infty x^2 \frac{1}{\Gamma(a)} (\lambda x)^a e^{-\lambda x} \frac{1}{x} dx - \frac{a^2}{\lambda^2}$$

$$= \frac{\Gamma(a+2)}{\Gamma(a)\lambda^2} - \frac{a^2}{\lambda^2}$$

$$= \frac{a}{\lambda^2}$$

(b) $Y \sim N(0,1)$

$$f(Y=y) = \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}}$$

$$Z = Y^2 \qquad \frac{dy}{dz} = \pm \frac{1}{2\sqrt{z}}$$

$$Y = \pm\sqrt{Z}$$

$$f_Z(z) = f_Y(\sqrt{z}) \cdot \left|\frac{dy}{dz}\right| + f_Y(-\sqrt{z}) \cdot \left|\frac{dy}{dz}\right)$$

$$= \frac{-e^{-\frac{z}{2}}}{\sqrt{2\pi z}} \quad \sim Ga \quad \begin{matrix} a = \frac{1}{2} \\ \lambda = \frac{1}{2} \end{matrix}$$

$$Y^2 \sim Gamma\left(\frac{1}{2}, \frac{1}{2}\right)$$

(c) $V = Y_1^2 + \cdots + Y_n^2$

$Y_i \sim N(0,1)$

$V \sim Gamma(\frac{n}{2}, \frac{1}{2})$

$X_1 \sim Gamma(\alpha_1, \beta)$

$X_2 \sim Gamma(\alpha_2, \beta)$

$X = X_1 + X_2 \sim Gamma(\alpha_1 + \alpha_2, \beta)$

$$f_X = \int_0^x f_{X_1}(t) f_{X_2}(x-t)\, dt$$

$$= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)} \cdot \frac{1}{\beta^{\alpha_1+\alpha_2}} \cdot e^{-\frac{x}{\beta}} \int_0^x t^{\alpha_1 - 1}(x-t)^{\alpha_2 - 1}\, dt$$

$$= \frac{1}{\Gamma(\alpha_1)\Gamma(\alpha_2)\beta^{\alpha_1+\alpha_2}} \cdot e^{-\frac{x}{\beta}} \cdot x^{\alpha_1+\alpha_2-1}$$

$Beta(\alpha_1, \alpha_2)$

$$Beat(\alpha_1, \alpha_2) = \frac{\Gamma(\alpha_1)\Gamma(\alpha_2)}{\Gamma(\alpha_1+\alpha_2)}$$

$$f_X = \frac{x^{\alpha_1+\alpha_2-1} e^{-\frac{x}{\beta}}}{\Gamma(\alpha_1+\alpha_2) \cdot \beta^{\alpha_1+\alpha_2}}$$

Problem 5 Solution

(d) $Z = \frac{Y}{\sqrt{\frac{V}{n}}}$ ✓ $\quad \left|\frac{\partial(Y,V)}{\partial(Z,V)}\right| = \sqrt{\frac{V}{n}} = \frac{\partial V}{\partial Z}$

$$f_{Y,V}(y,v) = f_Y(y) \cdot f_V(v)$$

$$= \frac{e^{-\frac{y^2}{2}}}{\sqrt{2\pi}} \cdot \frac{v^{\frac{n}{2}-1} e^{-\frac{y}{2}}}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})}$$

$$f_{Z,V}(z,v) = f_{Y,V}\left(z\sqrt{\frac{v}{n}},\, v\right)\sqrt{\frac{v}{n}}$$

$$= \frac{1}{\sqrt{2\pi n}} \cdot \frac{1}{2^{\frac{n}{2}}\Gamma(\frac{n}{2})}\, v^{\frac{n-1}{2}} \cdot e^{-\frac{v}{2}(1+\frac{z^2}{n})}$$

$$\int_0^\infty v^{\alpha-1} e^{-\beta v}\,dv = \frac{\Gamma(\alpha)}{\beta^{\alpha}} \qquad \alpha = \frac{n}{2}+\frac{n}{2} \qquad \beta = \frac{n+z^2}{2n}$$

$$\int_0^\infty v^{\frac{n}{2}-\frac{1}{2}} e^{-\frac{n+z^2}{2n}v}\,dv = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\left(\frac{n+z^2}{2n}\right)^{\frac{n+1}{2}}}$$

PDF: $$f_Z(z) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)\sqrt{\pi n}}\left(1+\frac{z^2}{n}\right)^{-\frac{n+1}{2}}$$

Problem 5 Solution

$$W = \frac{\frac{V_1}{m}}{\frac{V_2}{n}}$$

$$f_{V_1,V_2}(v_1, v_2) = \frac{v_1^{\frac{m}{2}-1} e^{-\frac{v_1}{2}}}{2^{\frac{m}{2}} \Gamma(\frac{m}{2})} \cdot \frac{v_2^{\frac{n}{2}-1} e^{-\frac{v_2}{2}}}{2^{\frac{n}{2}} \Gamma(\frac{n}{2})}$$

$$= \frac{v_1^{\frac{m}{2}-1} v_2^{\frac{m}{2}-1} e^{-\frac{v_1+v_2}{2}}}{2^{\frac{m+n}{2}} \Gamma(\frac{m}{2}) \Gamma(\frac{n}{2})}$$

$$V_1 = \frac{W V_2 m}{n} \qquad \left| \frac{\partial(V_1, V_2)}{\partial(W, V_2)} \right| = \frac{mW}{n}$$

Problem 5 Solution

$$f_{W,V_2}(w, v_2) = f_{V_1,V_2}\left(\frac{mwv_2}{n}, v_2\right) \cdot \frac{mw}{n}$$

$$= \frac{\left(\frac{1}{2}\right)^{m+n}}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)} \left(\frac{mw}{n}\right)^{\frac{m}{2}-1} v_2^{\frac{m+n}{2}-1} \cdot e^{-\frac{v_2}{2}\left(1+\frac{mw}{n}\right)}$$

$$f_W(w) = \frac{\left(\frac{1}{2}\right)^{m+n}}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)} \left(\frac{mw}{n}\right)^{\frac{m}{2}-1} \int_0^\infty v_2^{\frac{m+n}{2}-1} e^{-\frac{v_2}{2}\left(1+\frac{mw}{n}\right)} dv_2$$

$$= \frac{\left(\frac{1}{2}\right)^{m+n}}{\Gamma\left(\frac{m}{2}\right)\Gamma\left(\frac{n}{2}\right)} \frac{w^{\frac{m}{2}-1}\left(\frac{m}{n}\right)^{\frac{m}{2}}}{\left(1+\frac{mw}{n}\right)^{\frac{m+n}{2}}}, \quad w > 0$$

v

Problem 5 Solution