1 Game Playing

Alice and Bob are playing a game. They both have a coin. In this game, they each will chose a side of the coin and show it to each other simultaneously. The result is decided with the following rules:

1. If they show different sides, Bob wins \$3 from Alice.
2. If they both show head, Alice wins \$1 from Bob. If they both show tails, Alice wins \$5 from Bob.

1.1 Mixed Strategy(1.5pt)

Suppose Alice shows head with probability p, Bob shows head with probability q. Calculate Alice's supposed payoff in terms of p and q.

The expected payoff of Alice is

$$pq - 3p(1-q) - 3(1-p)q + 5(1-p)(1-q) = 12pq - 8(p+q) + 5$$

1.2 Nash Equilibrium(1.5pt)

Calculate the Nash equilibrium of this game. What are Alice and Bob's expected payoffs in the equilibrium? Do you think this game is fair?

Suppose the Nash equilibrium is (p, q), then we have:

$$\begin{cases} q - 3(1-q) = -3q + 5(1-q) \\ -p + 3(1-p) = 3p - 5(1-p) \end{cases}$$

From the above we derive that $p = q = \frac{2}{3}$, the equilibrium is $(\frac{2}{3}, \frac{2}{3})$. The expected payoff of Alice and Bob is $-\frac{1}{3}, \frac{1}{3}$. According to this result, we can see that the game is actually unfair because if Bob chooses $q = \frac{2}{3}$, Alice always has an negative expectation.

2 True or False

Consider whether the following statements are true or false. If true, give the proof, otherwise give a counter-example

2.1 (1pt)

Given a two player game where the action space of both players is $\{A, B\}$. Suppose (A, A) is the unique pure strategy Nash equilibrium, then action A is a dominant strategy for at least one of the players.

The statement is True.
Since (B, B) is not a Nash equilibrium, then one of the following equations must be satisfied:

$$u_1(A, B) \geq u_1(B, B) \tag{1}$$
$$u_2(B, A) \geq u_2(B, B) \tag{2}$$

Suppose equation (1) is satisfied w.l.o.g.. Since (A, A) is the Nash equilibrium, then

$$u_1(A, A) \geq u_1(B, A) \tag{3}$$

Together with equation (1) and (3), A is the dominant strategy for player 1.

2.2 (1pt)

Given a two player game where the action space of both players is $\{A, B\}$. Suppose (A, A) is the unique Nash equilibrium, then action A is a dominant strategy for both players.

The statement is False.
The counterexample is not unique. A counterexample should satisfy the following requirements:

1. (A, A) is the *unique* Nash equilibrium;
2. A is not the dominant strategy for one of the players.

For example, A is not the dominant strategy for the column player, player 2.

1/2	A	B
A	5,5	3,4
B	1,3	2,6

3 Auctions

3.1 Truthful Auctions (1pt)

Explain whether the flowing auctions are truthful:

1. First price auctions.
   First price auctions are not truthful. Let v_i represent the i^{th} highest valuation. The bidder with the highest valuation will misreport as $v_2 + \epsilon$, where $\epsilon > 0$.

2. Fixed price auctions.
   First price auctions are truthful. Bidders will not have any incentive to misreport their valuation.

3.2 Second Price Auction with Budget (2pt)

Consider a second price auction for a single indivisible item. Suppose each bidder i has a value $v_i > 0$ and a budget $c_i > 0$. If a bidder wins the object and has to pay higher than the budget, the bidder will simply drop out from the auction but is charged with a small penalty $\epsilon > 0$. Compute a bid in the auction for each player i which will be a weakly dominant strategy for the player.

If the bidder i bids a value larger than v_i or c_i, she may get a negative utility; if the bidder i bids a value smaller than v_i and c_i, she may lose the auction. Therefore, the weakly dominant strategy for the player i is $\min\{v_i, c_i\}$.

4 Weighted Vickrey-Clarke-Groves Mechanism (2pt)

A mechanism $(f, p_1, \ldots, p_n)$ is called a weighted VCG mechanism if

1. $f(v_1, \ldots, v_n) \in \arg\max_{a \in A} \left(c_a + \sum_i w_i v_i(a)\right)$, where $c_a, w_1, \ldots, w_n \in \mathbb{R}^+$;
2. for some functions $h_1, \ldots, h_n$, where $h_i : V_{-i} \mapsto \mathbb{R}$, we have that for all $v_1 \in V_1, \ldots, v_n \in V_n$: $p_i(v_1, \ldots, v_n) = h_i(v_{-i}) - c_a/w_i - \sum_{j \neq i}(w_j/w_i)v_j(a)$

Prove that the weighted VCG mechanism is incentive compatible.

First, we can assume w.l.o.g. that $h_i = 0$. The utility of player i if alternative a is chosen is $u_i(a) = v_i(a) - p_i(a) = v_i(a) + c_a/w_i + \sum_{j \neq i}(w_j/w_i)v_j(a)$. By multiplying by $w_i > 0$, $u_i(a) = c_a + \sum_j w_j v_j(a)$. $u_i(a)$ is maximized when $c_a + \sum_j w_j v_j(a)$ is maximized if i reports v_i truthfully.