Matrix Computations, Fall 2024

Midterm Exam Solution

Note: All answers should be in English. Definitions and notation follow the lectures.

1. [30 points] Consider $\mathbf{A} = \begin{bmatrix} 4 & -2 & 3 \\ -2 & 6 & 3 \\ 3 & 3 & 18 \end{bmatrix}$.

(a) Show that $\mathbf{A}$ is a positive definite matrix. (7 points)
(b) Find the LU decomposition of $\mathbf{A}$. (8 points)
(c) Find the Cholesky decomposition of $\mathbf{A}$. (8 points)
(d) Given any $\mathbf{d} \in \mathbb{R}^3$, prove that $\mathbf{A}^{-1} \succeq \mathbf{d}\mathbf{d}^T$ if and only if $\mathbf{d}^T\mathbf{A}\mathbf{d} \leq 1$. (7 points)

Solution:

(a) Use Schur complement. Since $18 > 0$, consider the matrix

$$\mathbf{Z} = \begin{bmatrix} 4 & -2 \\ -2 & 6 \end{bmatrix} - \begin{bmatrix} 3 \\ 3 \end{bmatrix} \cdot \frac{1}{18} \cdot \begin{bmatrix} 3 & 3 \end{bmatrix} = \begin{bmatrix} \frac{7}{2} & -\frac{5}{2} \\ -\frac{5}{2} & \frac{11}{2} \end{bmatrix}$$

Given any $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$,

$$\mathbf{x}^T\mathbf{Z}\mathbf{x} = \frac{7}{2}x_1^2 + \frac{11}{2}x_2^2 - 5x_1x_2 \geq \frac{7}{2}x_1^2 + \frac{11}{2}x_2^2 - \frac{5}{2}(x_1^2 + x_2^2) = x_1^2 + 3x_2^2 \geq 0,$$

where the equality holds if and only if $x_1 = x_2 = 0$. Therefore, $\mathbf{Z}$ is positive definite, so that $\mathbf{A}$ is positive definite.

Similarly, you may consider the matrix $\begin{bmatrix} 6 & 3 \\ 3 & 18 \end{bmatrix} - \begin{bmatrix} -2 \\ 3 \end{bmatrix} \cdot \frac{1}{4} \cdot \begin{bmatrix} -2 & 3 \end{bmatrix} = \begin{bmatrix} 5 & \frac{9}{2} \\ \frac{9}{2} & \frac{63}{4} \end{bmatrix}$ and show it is positive definite.

OR: The eigenvalues of $\mathbf{A}$ are 1.7064, 7.1714, 19.1222, so $\mathbf{A}$ is positive definite.

OR: Compute the leading principal minors: $4 > 0$, $\det(\begin{bmatrix} 4 & -2 \\ -2 & 6 \end{bmatrix}) = 20$, $\det(\begin{bmatrix} 4 & -2 & 3 \\ -2 & 6 & 3 \\ 3 & 3 & 18 \end{bmatrix}) = 234$, which are all positive, so $\mathbf{A}$ is positive definite.

(b)

$$\mathbf{M}_1 = \mathbf{I} - \boldsymbol{\tau}^{(1)}\mathbf{e}_1^T, \quad \boldsymbol{\tau}^{(1)} = \begin{bmatrix} 0 & -\frac{1}{2} & \frac{3}{4} \end{bmatrix}^T$$

$$\mathbf{A}^{(1)} = \mathbf{M}_1\mathbf{A} = \begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{2} & 1 & 0 \\ -\frac{3}{4} & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & -2 & 3 \\ -2 & 6 & 3 \\ 3 & 3 & 18 \end{bmatrix} = \begin{bmatrix} 4 & -2 & 3 \\ 0 & 5 & \frac{9}{2} \\ 0 & \frac{9}{2} & \frac{63}{4} \end{bmatrix}$$

$$\mathbf{M}_2 = \mathbf{I} - \boldsymbol{\tau}^{(2)}\mathbf{e}_2^T, \quad \boldsymbol{\tau}^{(2)} = \begin{bmatrix} 0 & 0 & \frac{9}{10} \end{bmatrix}^T$$

$$\mathbf{A}^{(2)} = \mathbf{M}_2\mathbf{A}^{(1)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -\frac{9}{10} & 1 \end{bmatrix} \begin{bmatrix} 4 & -2 & 3 \\ 0 & 5 & \frac{9}{2} \\ 0 & \frac{9}{2} & \frac{63}{4} \end{bmatrix} = \begin{bmatrix} 4 & -2 & 3 \\ 0 & 5 & \frac{9}{2} \\ 0 & 0 & \frac{117}{10} \end{bmatrix} = \mathbf{U}$$

$$\mathbf{L} = \mathbf{M}_1^{-1}\mathbf{M}_2^{-1} = \mathbf{I} + \boldsymbol{\tau}^{(1)}\mathbf{e}_1^T + \boldsymbol{\tau}^{(2)}\mathbf{e}_2^T = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & 1 & 0 \\ \frac{3}{4} & \frac{9}{10} & 1 \end{bmatrix}.$$

(c) We first find the LDL decomposition of $\mathbf{A}$ based on the LU decomposition in (b). Since $\mathbf{L}$ is unit lower triangular, so is $\mathbf{L}^{-1}$. Thus, we obtain $\mathbf{L}^{-T} = \begin{bmatrix} 1 & \frac{1}{2} & -\frac{6}{5} \\ 0 & 1 & -\frac{9}{10} \\ 0 & 0 & 1 \end{bmatrix}$. Let $\mathbf{D} = \mathbf{U}\mathbf{L}^{-T} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & \frac{117}{10} \end{bmatrix}$. Then, $\mathbf{A} = \mathbf{L}\mathbf{D}\mathbf{L}^T$. Finally, let $\mathbf{G} = \mathbf{L}\mathbf{D}^{\frac{1}{2}} = \begin{bmatrix} 2 & 0 & 0 \\ -1 & \sqrt{5} & 0 \\ \frac{3}{2} & \frac{9\sqrt{5}}{10} & \frac{3\sqrt{130}}{10} \end{bmatrix}$ and we have $\mathbf{A} = \mathbf{G}\mathbf{G}^T$.

(d) Again, use Schur complement. Consider the matrix $\mathbf{X} = \begin{bmatrix} 1 & \mathbf{d}^T \\ \mathbf{d} & \mathbf{A}^{-1} \end{bmatrix}$. Since $\mathbf{A} \succeq \mathbf{0}$, $\mathbf{A}^{-1} \succeq \mathbf{0}$. Thus, $\mathbf{X} \succeq \mathbf{0} \iff 1 - \mathbf{d}^T\mathbf{A}\mathbf{d} \geq 0 \iff \mathbf{A}^{-1} - \mathbf{d}\mathbf{d}^T \succeq \mathbf{0}$.

2. [20 points]

(a) Let $\mathbf{P} \in \mathbb{R}^{n\times n}$ be a permutation matrix (i.e., with exactly one entry of 1 in each row/column, and 0 elsewhere). Show that $\mathbf{P}^{-1} = \mathbf{P}^T$. (6 points)

(b) Suppose $\mathbf{A} \in \mathbb{R}^{4\times 4}$ has the following decomposition

$$\mathbf{A} = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & 0 & \alpha \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

where $\alpha, \beta \in \mathbb{R}$. Find the sufficient and necessary condition on α, β such that $\mathbf{A}$ is nonsingular. Also explain why. (6 points)

(c) For the matrix $\mathbf{A}$ in (b), now let $\alpha = 0$. Find the sufficient and necessary condition on β such that $\mathbf{A}$ admits an eigendecomposition. Also explain why. (8 points)

Solution:

(a) Since $\mathbf{P}$ can be expressed as $\mathbf{P} = \mathbf{\Pi}_1 \cdots \mathbf{\Pi}_k$ for some $k = 1, \ldots, n$. Here, each $\mathbf{\Pi}_i$ is obtained by swapping two rows in the $n \times n$ identity matrix, so that it is symmetric and satisfies $\mathbf{\Pi}_i \mathbf{\Pi}_i = \mathbf{I}$, i.e., $\mathbf{\Pi}_i^{-1} = \mathbf{\Pi}_i$. Hence, $\mathbf{P}^{-1} = \mathbf{\Pi}_k^{-1} \cdots \mathbf{\Pi}_1^{-1} = \mathbf{\Pi}_k \cdots \mathbf{\Pi}_1 = \mathbf{\Pi}_k^T \cdots \mathbf{\Pi}_1^T = (\mathbf{\Pi}_1 \cdots \mathbf{\Pi}_k)^T = \mathbf{P}^T$.

(b) Note that $\begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$ is a permutation matrix, so that its inverse is $\begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}$. Therefore, the equation is indeed a similarity transformation, which indicates that $\mathbf{A}$ is similar to the upper triangular matrix $\tilde{\mathbf{A}} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & \alpha & \beta \\ 0 & 0 & 0 & \alpha \end{bmatrix}$. Consequently, $\mathbf{A}$ is nonsingular if and only if $\tilde{\mathbf{A}}$ is nonsingular, i.e., $\det(\tilde{\mathbf{A}}) \neq 0$, which is equivalent to $\alpha \neq 0$.

(c) $\mathbf{A}$ admits an eigendecomposition if and only if for each eigenvalue of $\mathbf{A}$, the algebraic multiplicity is the same as the geometric multiplicity. Since $\mathbf{A}$ and $\tilde{\mathbf{A}}$ share the same eigenvalues as well as the algebraic and geometric multiplicities, all we need is to guarantee that for each eigenvalue of $\tilde{\mathbf{A}}$, the algebraic multiplicity is the same as the geometric multiplicity. Note that the eigenvalues of $\tilde{\mathbf{A}}$ are $-1, -2, 0$. For the first two eigenvalues, their algebraic multiplicities are both 1, so that the geometric multiplies have to be 1. For the eigenvalue 0, its algebraic multiplicity is 2. Its geometric multiplicity is $\dim \mathcal{N}(\tilde{\mathbf{A}}) = 4 - \text{rank}(\tilde{\mathbf{A}})$. Therefore, we need $\text{rank}(\tilde{\mathbf{A}}) = 2$, which is equivalent to $\beta = 0$.

3. [25 points] Consider $\mathbf{A} = \begin{bmatrix} 2 & 3 & 5 \\ -1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

(a) Find a thin $\mathbf{QR}$ decomposition for $\mathbf{A}$ where $\mathbf{R} \in \mathbb{R}^{3\times 3}$. (10 points)

(b) Does the linear system $\mathbf{Ax} = \mathbf{y}$ has a solution? If your answer is YES, provide all the solutions to the linear system. If your answer is NO, provide all the least-squares solutions to the linear system. (8 points)

(c) For any matrix $\mathbf{B} \in \mathbb{R}^{n\times n}$, is it true that the range of $\mathbf{B}$ is the same as the range of $\mathbf{BB}^T$? If your answer is YES, prove this statement. If your answer is NO, provide a counterexample. (7 points)

Solution:

(a)First, let

$$\mathbf{a}_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{a}_2 = \begin{bmatrix} 3 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{a}_3 = \begin{bmatrix} 5 \\ 0 \\ 1 \\ 2 \end{bmatrix}.$$

Apply the Gram-Schmidt procedure:

$$\tilde{\mathbf{q}}_1 = \mathbf{a}_1, \quad \mathbf{q}_1 = \frac{\tilde{\mathbf{q}}_1}{\|\tilde{\mathbf{q}}_1\|_2} = \frac{1}{\sqrt{6}} \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix},$$

$$\tilde{\mathbf{q}}_2 = \mathbf{a}_2 - (\mathbf{q}_1^T \mathbf{a}_2)\mathbf{q}_1 = \begin{bmatrix} 3 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \sqrt{6}\frac{1}{\sqrt{6}} \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix},$$

$$\mathbf{q}_2 = \frac{\tilde{\mathbf{q}}_2}{\|\tilde{\mathbf{q}}_2\|_2} = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix},$$

$$\tilde{\mathbf{q}}_3 = \mathbf{a}_3 - (\mathbf{q}_1^T \mathbf{a}_3)\mathbf{q}_1 - (\mathbf{q}_2^T \mathbf{a}_3)\mathbf{q}_2 = \begin{bmatrix} 5 \\ 0 \\ 1 \\ 2 \end{bmatrix} - 2\sqrt{6}\frac{1}{\sqrt{6}} \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \end{bmatrix} - \sqrt{6}\frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Then, we have the QR decomposition as $\mathbf{A} = \mathbf{QR}$, where

$$\mathbf{Q} = \frac{1}{\sqrt{6}} \begin{bmatrix} 2 & 1 & 1 \\ -1 & 2 & 0 \\ 0 & 1 & -1 \\ 1 & 0 & -2 \end{bmatrix}, \quad \mathbf{R} = \sqrt{6} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

(Note: the third column of the matrix $\mathbf{Q}$ is not necessarily $\begin{bmatrix} 1 & 0 & -1 & -2 \end{bmatrix}^T$ and can be replaced with any unit vector that is orthogonal to the first and second columns.)

(b) No, because $\mathbf{y} \notin \mathcal{R}(\mathbf{A})$. Let $\hat{\mathbf{Q}} = \begin{bmatrix} \mathbf{Q}_1 & \mathbf{Q}_2 \end{bmatrix}$ be an orthogonal matrix such that $\mathbf{A} = \begin{bmatrix} \mathbf{Q}_1 & \mathbf{Q}_2 \end{bmatrix} \begin{bmatrix} \mathbf{R}_1 \\ \mathbf{0} \end{bmatrix}$, where $\mathbf{Q}_1 = \frac{1}{\sqrt{6}} \begin{bmatrix} 2 & 1 \\ -1 & 2 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\mathbf{R}_1 = \sqrt{6} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix}$. From $\|\mathbf{Ax} - \mathbf{y}\|_2^2 = \|\mathbf{R}_1\mathbf{x} - \mathbf{Q}_1^T\mathbf{y}\|_2^2 + \|\mathbf{Q}_2^T\mathbf{y}\|_2^2$, the least-squares solutions to the linear systems $\mathbf{Ax} = \mathbf{y}$ are equivalent to the solutions of the linear system $\mathbf{R}_1\mathbf{x} = \mathbf{Q}_1^T\mathbf{y}$.

By solving the linear system

$$\sqrt{6} \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & 1 \end{bmatrix} \mathbf{x} = \frac{1}{\sqrt{6}} \begin{bmatrix} 2 & -1 & 0 & 1 \\ 1 & 2 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{6}} \begin{bmatrix} 2 \\ 1 \end{bmatrix},$$

we have $\mathbf{x} = \begin{bmatrix} 0 \\ 0 \\ \frac{1}{6} \end{bmatrix} + k \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$ for all $k \in \mathbb{R}$.

(c) True. For any $\mathbf{x} \in \mathcal{R}(\mathbf{BB}^T)$, there exists $\mathbf{z}$ such that $\mathbf{x} = \mathbf{BB}^T\mathbf{z} = \mathbf{By}$ with $\mathbf{y} = \mathbf{B}^T\mathbf{z}$. Therefore, $\mathbf{x} \in \mathcal{R}(\mathbf{B})$. This implies that $\mathcal{R}(\mathbf{BB}^T) \subseteq \mathcal{R}(\mathbf{B})$. From SVD, we know that the number of zero singular values of $\mathbf{B}$ is the same as the number of zero eigenvalues of $\mathbf{BB}^T$, denoted by γ. Also, since $\mathbf{BB}^T$ is symmetric, it always admits an eigendecomposition, and the algebraic multiplicity of the zero eigenvalue, i.e., γ, is the same as its geometric multiplicity. Thus, the rank of $\mathbf{BB}^T$ is $n - \gamma$. Note that the rank of $\mathbf{B}$ is the number of nonzero singular values of $\mathbf{B}$, which is also $n - \gamma$. Therefore, $\mathbf{B}$ and $\mathbf{BB}^T$ have the same rank, so that the dimensions of the subspaces $\mathcal{R}(\mathbf{B})$ and $\mathcal{R}(\mathbf{BB}^T)$ are the same. This, along with $\mathcal{R}(\mathbf{BB}^T) \subseteq \mathcal{R}(\mathbf{B})$, implies that $\mathcal{R}(\mathbf{BB}^T) = \mathcal{R}(\mathbf{B})$.

4. [25 points] Let $\mathbf{A} \in \mathbb{R}^{5\times 3}$ and suppose $\mathbf{A}$ has the following decomposition:

$$\mathbf{A} = \underbrace{\begin{bmatrix} \mathbf{y}_1 & \mathbf{y}_2 & \mathbf{y}_3 & \mathbf{y}_4 & \mathbf{y}_5 \end{bmatrix}}_{\mathbf{Y}} \begin{bmatrix} \alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \underbrace{\begin{bmatrix} \mathbf{z}_1^T \\ \mathbf{z}_2^T \\ \mathbf{z}_3^T \end{bmatrix}}_{\mathbf{Z}^T},$$

where $\alpha < 0 < \beta$, and $\mathbf{Y} \in \mathbb{R}^{5\times 5}$ and $\mathbf{Z} \in \mathbb{R}^{3\times 3}$ are orthogonal matrices.

(a) Find a singular value decomposition (SVD) for $\mathbf{A}$. (6 points)

(b) Given $\mathbf{b} \in \mathbb{R}^5$, find the projection of $\mathbf{b}$ onto the range of $\mathbf{A}$. (7 points)

(c) Find all the eigenvalues of $\mathbf{y}_1\mathbf{y}_1^T$. (6 points)

(d) Let $\{i_1, i_2, i_3\} \subset \{1, \dots, 5\}$. Find all the eigenvalues of $\mathbf{y}_{i_1}\mathbf{y}_{i_1}^T + \mathbf{y}_{i_2}\mathbf{y}_{i_2}^T + \mathbf{y}_{i_3}\mathbf{y}_{i_3}^T$. (6 points)

Note: In (a), all the singular values should be nonnegative. In (c)(d), if there are repeated eigenvalues, you need to identify their algebraic multiplicities.

Solution:

(a)

$$\mathbf{A} = \underbrace{\begin{bmatrix} -\mathbf{y}_1 & \mathbf{y}_2 & \mathbf{y}_3 & \mathbf{y}_4 & \mathbf{y}_5 \end{bmatrix}}_{\mathbf{U}} \underbrace{\begin{bmatrix} -\alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\mathbf{\Sigma}} \underbrace{\begin{bmatrix} \mathbf{z}_1^T \\ \mathbf{z}_2^T \\ \mathbf{z}_3^T \end{bmatrix}}_{\mathbf{V}^T} \quad \text{or}$$

$$\mathbf{A} = \underbrace{\begin{bmatrix} \mathbf{y}_1 & \mathbf{y}_2 & \mathbf{y}_3 & \mathbf{y}_4 & \mathbf{y}_5 \end{bmatrix}}_{\mathbf{U}} \underbrace{\begin{bmatrix} -\alpha & 0 & 0 \\ 0 & \beta & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\mathbf{\Sigma}} \underbrace{\begin{bmatrix} -\mathbf{z}_1^T \\ \mathbf{z}_2^T \\ \mathbf{z}_3^T \end{bmatrix}}_{\mathbf{V}^T}$$

(b) The projection of $\mathbf{b}$ onto the range of $\mathbf{A}$ is equal to $\mathbf{A}\mathbf{x}_{LS}$ where $\mathbf{x}_{LS}$ is a least-squares solution to $\mathbf{A}\mathbf{x} = \mathbf{b}$. Let $\mathbf{U}_1 = \begin{bmatrix} -\mathbf{y}_1 & \mathbf{y}_2 \end{bmatrix}$, $\tilde{\mathbf{\Sigma}} = \begin{bmatrix} -\alpha & 0 \\ 0 & \beta \end{bmatrix}$, and $\mathbf{V}_1 = \begin{bmatrix} \mathbf{z}_1 & \mathbf{z}_2 \end{bmatrix}$, so that $\mathbf{A} = \mathbf{U}_1\tilde{\mathbf{\Sigma}}\mathbf{V}_1^T$. Thus, $\mathbf{A}\mathbf{x}_{LS} = \mathbf{U}_1\mathbf{U}_1^T\mathbf{b} = (\mathbf{y}_1\mathbf{y}_1^T + \mathbf{y}_2\mathbf{y}_2^T)\mathbf{b}$.

(c) Note that $\operatorname{rank}(\mathbf{y}_1\mathbf{y}_1^T) = 1$, so that $\dim\mathcal{N}(\mathbf{y}_1\mathbf{y}_1^T) = 5 - 1 = 4$. Therefore, $\mathbf{y}_1\mathbf{y}_1^T$ has an eigenvalue at 0 with geometric multiplicity 4, so that its algebraic multiplicity is either 4 or 5. Also, since $\mathbf{y}_1^T\mathbf{y}_1 = 1$, we have $\mathbf{y}_1\mathbf{y}_1^T\mathbf{y}_1 = \mathbf{y}_1$, so that $\mathbf{y}_1\mathbf{y}_1^T$ has an eigenvalue at 1. As a result, the eigenvalues of $\mathbf{y}_1\mathbf{y}_1^T$ are $0, 0, 0, 0, 1$.

(d) Note that $\mathbf{y}_{i_1}\mathbf{y}_{i_1}^T + \mathbf{y}_{i_2}\mathbf{y}_{i_2}^T + \mathbf{y}_{i_3}\mathbf{y}_{i_3}^T = \begin{bmatrix} \mathbf{y}_{i_1} & \mathbf{y}_{i_2} & \mathbf{y}_{i_3} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \mathbf{y}_{i_1}^T \\ \mathbf{y}_{i_2}^T \\ \mathbf{y}_{i_3}^T \end{bmatrix}$. From this SVD, we know that the eigenvalues of $(\mathbf{y}_{i_1}\mathbf{y}_{i_1}^T + \mathbf{y}_{i_2}\mathbf{y}_{i_2}^T + \mathbf{y}_{i_3}\mathbf{y}_{i_3}^T)^T(\mathbf{y}_{i_1}\mathbf{y}_{i_1}^T + \mathbf{y}_{i_2}\mathbf{y}_{i_2}^T + \mathbf{y}_{i_3}\mathbf{y}_{i_3}^T)$ are $1, 1, 1, 0, 0$. Due to the symmetry, $(\mathbf{y}_{i_1}\mathbf{y}_{i_1}^T + \mathbf{y}_{i_2}\mathbf{y}_{i_2}^T + \mathbf{y}_{i_3}\mathbf{y}_{i_3}^T)^T(\mathbf{y}_{i_1}\mathbf{y}_{i_1}^T + \mathbf{y}_{i_2}\mathbf{y}_{i_2}^T + \mathbf{y}_{i_3}\mathbf{y}_{i_3}^T) = (\mathbf{y}_{i_1}\mathbf{y}_{i_1}^T + \mathbf{y}_{i_2}\mathbf{y}_{i_2}^T + \mathbf{y}_{i_3}\mathbf{y}_{i_3}^T)^2$. For any symmetric matrix $\mathbf{B}$, the eigenvalues of $\mathbf{B}^2$ are equal to the square of the eigenvalues of $\mathbf{B}$. Therefore, the eigenvalues of the positive definite matrix $\mathbf{y}_{i_1}\mathbf{y}_{i_1}^T + \mathbf{y}_{i_2}\mathbf{y}_{i_2}^T + \mathbf{y}_{i_3}\mathbf{y}_{i_3}^T$ are $1, 1, 1, 0, 0$.