Numerical Optimization, Fall 2024
Homework 3

Due 23:59 (CST), Oct. 31, 2024

Problem 1

Prove the dual of the dual of the primal problem is itself. [20pts]

Solution:
Consider the following primal question (standard form)

$$\begin{aligned} \min \quad & \boldsymbol{c}^T\boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \\ & \boldsymbol{x} \geq \boldsymbol{0}, \end{aligned}$$

its Lagrangian is given by

$$L(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\nu}) = \boldsymbol{c}^T\boldsymbol{x} - \boldsymbol{\lambda}^T(\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}) - \boldsymbol{\nu}^T\boldsymbol{x}.$$

The dual objective is then

$$g(\boldsymbol{\lambda}, \boldsymbol{\nu}) = \min_{\boldsymbol{x}} L(\boldsymbol{x}, \boldsymbol{\lambda}, \boldsymbol{\nu}) = \min_{\boldsymbol{x}} (\boldsymbol{c} - \boldsymbol{A}^T\boldsymbol{\lambda} - \boldsymbol{\nu})^T\boldsymbol{x} + \boldsymbol{b}^T\boldsymbol{\lambda}.$$

Maximize $g(\boldsymbol{\lambda}, \boldsymbol{\nu})$, only care about $g(\boldsymbol{\lambda}, \boldsymbol{\nu}) > -\infty$, meaning $\boldsymbol{c} - \boldsymbol{A}^T\boldsymbol{\lambda} - \boldsymbol{\nu} = \boldsymbol{0}$. So the dual problem is

$$\begin{aligned} \max \quad & \boldsymbol{b}^T\boldsymbol{\lambda} \\ \text{s.t.} \quad & \boldsymbol{A}^T\boldsymbol{\lambda} \leq \boldsymbol{c}. \end{aligned}$$

The Lagrangian of dual problem is

$$\hat{L}(\boldsymbol{\lambda}, \boldsymbol{x}) = \boldsymbol{b}^T\boldsymbol{\lambda} - \boldsymbol{x}^T(\boldsymbol{A}^T\boldsymbol{\lambda} - \boldsymbol{c}), \quad \boldsymbol{x} \geq \boldsymbol{0}.$$

The dual objective of dual is then

$$f(\boldsymbol{x}) = \max_{\boldsymbol{\lambda}} \hat{L}(\boldsymbol{\lambda}, \boldsymbol{x}) = \max_{\lambda} (\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x})^T\boldsymbol{\lambda} + \boldsymbol{c}^T\boldsymbol{x}, \quad \boldsymbol{x} \geq \boldsymbol{0}.$$

Minimize $f(\boldsymbol{x})$, only care about $f(\boldsymbol{x}) < \infty$, meaning $\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x} = \boldsymbol{0}$. So the dual of dual problem is

$$\begin{aligned} \min \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \\ & \boldsymbol{x} \geq \boldsymbol{0}, \end{aligned}$$

which is the same as the primal problem.

Problem 2

Write the optimality conditions for the following linear programming problem. [15pts]

$$\begin{aligned} \min \quad & x_1 + 2x_2 \\ \text{s.t.} \quad & x_1 + x_2 \geq 1, \\ & 2x_1 + x_2 \geq 2, \\ & x_1, x_2 \geq 0. \end{aligned}$$

Solution(1):
The dual problem is

$$\begin{aligned} \max \quad & y_1 + 2y_2 \\ \text{s.t.} \quad & y_1 + 2y_2 \leq 1, \\ & y_1 + y_2 \leq 2, \\ & y_1, y_2 \geq 0. \end{aligned}$$

optimality conditions:
1. primal feasibility: $x_1, x_2 \geq 0$, $x_1 + x_2 \geq 1$, $2x_1 + x_2 \geq 2$;
2. dual feasibility: $y_1, y_2 \geq 0$, $y_1 + 2y_2 \leq 1$, $y_1 + y_2 \leq 2$;
3. dual gap: $x_1 + 2x_2 = y_1 + 2y_2$.

Solution(2):
The Lagrangian is given by

$$L(x_1, x_2, y_1, y_2) = x_1 + 2x_2 - y_1(x_1 + x_2 - 1) - y_2(2x_1 + x_2 - 2).$$

KKT conditions:
1. primal feasibility: $x_1, x_2 \geq 0$, $x_1 + x_2 \geq 1$, $2x_1 + x_2 \geq 2$;
2. dual feasibility: $y_1, y_2 \geq 0$, $y_1 + 2y_2 \leq 1$, $y_1 + y_2 \leq 2$;
3. complementary: $y_1(x_1 + x_2 - 1) = 0$, $y_2(2x_1 + x_2 - 2) = 0$;
4. gradient vanishing: $\frac{\partial L}{\partial x_1} = 1 - y_1 - 2y_2 = 0$, $\frac{\partial L}{\partial x_2} = 2 - y_1 - y_2 = 0$.

Problem 3

Write the dual problem for the following linear programming problem. [15pts]

$$
\begin{aligned}
\min \quad & 10x_1 + 15x_2 \\
\text{s.t.} \quad & 2x_1 + x_2 \geq 3, \\
& x_1 + 3x_2 \geq 5, \\
& x_1, x_2 \geq 0.
\end{aligned}
$$

Solution:

$$
\begin{aligned}
\max \quad & 3y_1 + 5y_2 \\
\text{s.t.} \quad & 2y_1 + y_2 \leq 10, \\
& y_1 + 3y_2 \leq 15, \\
& y_1, y_2 \geq 0.
\end{aligned}
$$

Problem 4

Give an example where neither the primal problem nor the dual problem is feasible. [20pts]

Solution:
The primal problem is

$$
\begin{aligned}
\min \quad & x_1 + 2x_2 \\
\text{s.t.} \quad & x_1 + x_2 \geq 5, \\
& x_1 + x_2 \leq 2, \\
& x_1 \geq 0, x_2 \leq 0.
\end{aligned}
$$

The dual problem is

$$
\begin{aligned}
\max \quad & 5y_1 + 2y_2 \\
\text{s.t.} \quad & y_1 + y_2 \leq 1, \\
& y_1 + y_2 \geq 2, \\
& y_1 \geq 0, y_2 \leq 0.
\end{aligned}
$$

Problem 5

(1) Prove that one and only one of $(\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{0}, \boldsymbol{c}^{\boldsymbol{T}}\boldsymbol{x} > 0)$ or $(\boldsymbol{A}^T\boldsymbol{y} = \boldsymbol{c}, \boldsymbol{y} \geq \boldsymbol{0})$ is solvable, where $\boldsymbol{A} \in \mathbb{R}^{m\times n}, \boldsymbol{c} \in \mathbb{R}^n$. [15pts]

Solution:

Consider the primal problem

$$\begin{aligned} \min \quad & \mathbf{0}^T \boldsymbol{y} \\ \text{s.t.} \quad & \boldsymbol{A}^T \boldsymbol{y} = \boldsymbol{c}, \\ & \boldsymbol{y} \geq \mathbf{0}, \end{aligned}$$

its dual problem is

$$\begin{aligned} \max \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} \leq \mathbf{0}. \end{aligned}$$

Denote the optimal solution of the primal and dual problems as $f(\boldsymbol{y}^*), g(\boldsymbol{x}^*)$. By weak duality we have $f(\boldsymbol{y}^*) \geq g(\boldsymbol{x}^*)$.

If $\{\boldsymbol{A}\boldsymbol{x} \leq \mathbf{0}, \boldsymbol{c}^{\boldsymbol{T}}\boldsymbol{x} > 0\} \neq \varnothing$, suppose that $\{\boldsymbol{A}^T\boldsymbol{y} = \boldsymbol{c}, \boldsymbol{y} \geq \mathbf{0}\} \neq \varnothing$. Then $f(\boldsymbol{y}^*) \geq g(\boldsymbol{x}^*) > 0$, conflict! So $\{\boldsymbol{A}^T\boldsymbol{y} = \boldsymbol{c}, \boldsymbol{y} \geq \mathbf{0}\} = \varnothing$.

If $\{\boldsymbol{A}\boldsymbol{x} \leq \mathbf{0}, \boldsymbol{c}^{\boldsymbol{T}}\boldsymbol{x} > 0\} = \varnothing$, $g(\boldsymbol{x}^*) \leq 0$. Then $f(\boldsymbol{y}^*) = 0$ and (1) is feasible. So $\{\boldsymbol{A}^T\boldsymbol{y} = \boldsymbol{c}, \boldsymbol{y} \geq \mathbf{0}\} \neq \varnothing$.

(2) Prove that one and only one of $(\boldsymbol{B}\boldsymbol{y} + \boldsymbol{C}\boldsymbol{w} = \boldsymbol{g}, \boldsymbol{y} > \mathbf{0})$ or $(\boldsymbol{g}^T\boldsymbol{d} < 0, \boldsymbol{B}^T\boldsymbol{d} \geq \mathbf{0}, \boldsymbol{C}^T\boldsymbol{d} = \mathbf{0})$ is solvable, where $\boldsymbol{B} \in \mathbb{R}^{n\times m}, \boldsymbol{C} \in \mathbb{R}^{n\times p}, \boldsymbol{g} \in \mathbb{R}^n$. [15pts]

Solution:
Consider the primal problem

$$\begin{aligned} \min \quad & \mathbf{0}^T \boldsymbol{y} + \mathbf{0}^T \boldsymbol{w} \\ \text{s.t.} \quad & \boldsymbol{B}\boldsymbol{y} + \boldsymbol{C}\boldsymbol{w} = \boldsymbol{g}, \\ & \boldsymbol{y} > \mathbf{0}, \end{aligned}$$

its dual problem is

$$\begin{aligned} \max \quad & -\boldsymbol{g}^T \boldsymbol{d} \\ \text{s.t.} \quad & \boldsymbol{B}^T \boldsymbol{d} \geq \mathbf{0}, \\ & \boldsymbol{C}^T \boldsymbol{d} = \mathbf{0}. \end{aligned}$$

Denote the optimal solution of the primal and dual problems as $f(\boldsymbol{y}^*, \boldsymbol{w}^*), g(\boldsymbol{d}^*)$. By weak duality we have $f(\boldsymbol{y}^*, \boldsymbol{w}^*) \geq g(\boldsymbol{d}^*)$.

If $\{\boldsymbol{g}^T\boldsymbol{d} < 0, \boldsymbol{B}^T\boldsymbol{d} \geq \mathbf{0}, \boldsymbol{C}^T\boldsymbol{d} = \mathbf{0}\} \neq \varnothing$, suppose that $\{\boldsymbol{B}\boldsymbol{y} + \boldsymbol{C}\boldsymbol{w} = \boldsymbol{g}, \boldsymbol{y} > \mathbf{0}\} \neq \varnothing$. Then $f(\boldsymbol{y}^*, \boldsymbol{w}^*) \geq g(\boldsymbol{d}^*) > 0$, conflict! So $\{\boldsymbol{B}\boldsymbol{y} + \boldsymbol{C}\boldsymbol{w} = \boldsymbol{g}, \boldsymbol{y} > \mathbf{0}\} = \varnothing$.

If $\{\boldsymbol{g}^T\boldsymbol{d} < 0, \boldsymbol{B}^T\boldsymbol{d} \geq \mathbf{0}, \boldsymbol{C}^T\boldsymbol{d} = \mathbf{0}\} = \varnothing$, $g(\boldsymbol{d}^*) \leq 0$. Then $f(\boldsymbol{y}^*, \boldsymbol{w}^*) = 0$ and (1) is feasible. So $\{\boldsymbol{B}\boldsymbol{y} + \boldsymbol{C}\boldsymbol{w} = \boldsymbol{g}, \boldsymbol{y} > \mathbf{0}\} \neq \varnothing$.