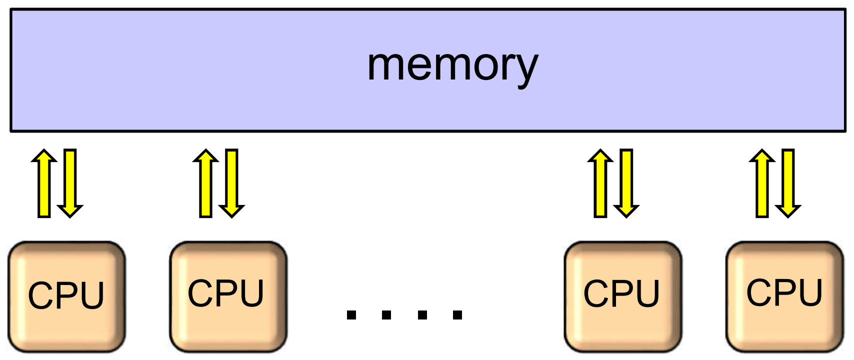
PRAM 1
Model and basic algorithms

CS121 Parallel Computing

Fall 2024

PRAM



- Parallel Random Access Machine, generalizes von Neumann model for sequential computing.
  - Given input of size n, we have f(n) processors accessing a shared memory.
    - f(n) can be very large, even larger than n.
  - All processors execute in synchronized steps.
  - In each step, each processor reads a memory location, computes, then writes a memory location.

PRAM

- Theoretically interesting model, but not practical.
  - Assumes unrealistically large number of processors.
  - Also assumes all processors can communicate every time step; ignores memory latency and bandwidth.
- PRAM's main use is as a simple, clean model to develop parallel algorithms.
  - First maximize parallelism inherent in problem using PRAM.
  - Then simulate the algorithm with real hardware, i.e. map it onto hardware with limited processors / communication.
  - Ex Some GPU algorithms are adaptations of PRAM algorithms.

Memory conflicts

- What if processors read / write to the same memory location in same time step?
- EREW Exclusive read exclusive write.
  - Most restrictive model. Algorithm returns error if processors read/write same location simultaneously.
- CREW Concurrent read exclusive write.
  - Several processors can read same location simultaneously, but error if they write.
- ERCW Exclusive read concurrent write.
  - Uncommon.
- CRCW Concurrent read concurrent write.
  - If multiple writes to same location, can either
    - Let an arbitrary write succeed.
    - Choose a write according to some priority to succeed.

Work and depth

- Depth is the number of (parallel) steps till a PRAM algorithm terminates.
  - Polylogarithmic depth means the algorithm terminates in $O(\log(n)^k)$ steps, where n is input size and k is constant.
  - Goal for PRAM algorithms is often polylog depth using $O(n^k)$ number of processors.
- Work is total number of steps taken by the algorithm.
  - Work of parallel algorithm $\geq$ O(work of best sequential algorithm).
  - If the work is equal, the parallel algorithm is work-efficient.
- In practice, minimizing work of PRAM algorithm is more important than minimizing depth.

Parallel carry lookahead addition

```
a            1  0  1  1  0  0  1
b        +   1  0  1  0  0  1  1
         -----------------------
carry       1  0  1  0  0  1  1
sum      1   0  1  0  1  1  0  0
```

- Suppose we want to add two n-digit binary numbers, but we can only add a single digit at a time and compute its carry.
  - This is what’s provided by full adders in a CPU.
- If we add digit by digit using the grade school method, it takes O(n) time.
  - For n=32 or n=64, this is much too slow.
- Each digit in the sum depends on the digit from the summands, but also a carry bit from the previous digit.
  - The summand digits can be added in parallel, but it seems the carry bits must be computed sequentially.

a_i	b_i	c_i	s_i	c_{i+1}
0	0	0	0	0
0	0	1	1	0
0	1	0	1	0
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1

Parallel carry lookahead addition

- We'll show how to compute all the carry bits in parallel in O(log n) time using n processors.
- After this, all the sum bits can be computed in O(1) parallel time, since $s_i = a_i \oplus b_i \oplus c_i$.
- Denote bitwise AND and OR by $\cdot$ and $+$.
- Define $g_i = a_i b_i$ as i'th "carry generate" bit.
  - If $a_i=b_i=1$, $c_{i+1}=1$ no matter what c_i is.
- Define $p_i = a_i \oplus b_i$ as i'th "carry propagate" bit.
  - If $p_i=1$, then $c_{i+1}=c_i$.

Parallel carry lookahead addition

- We have $c_{i+1} = g_i+c_ip_i$.
- Carry the i+1'st bit if
  - i'th bit of a and b generate a carry, OR
  - We carried the i'th bit, and this was propagated by a and b's i'th bit.
- We can also verify $c_{i+1} = g_i+c_ip_i$ directly.

a_i	b_i	c_i	g_i	p_i	c_{i+1}
0	0	0	0	0	0
0	0	1	0	0	0
0	1	0	0	1	0
0	1	1	0	1	1
1	0	0	0	1	0
1	0	1	0	1	1
1	1	0	1	0	1
1	1	1	1	0	1

Parallel carry lookahead addition

- Observation Can write $\begin{bmatrix} p_i & g_i \\ F & T \end{bmatrix} \begin{bmatrix} c_i \\ T \end{bmatrix} = \begin{bmatrix} c_i p_i + g_i \\ T \end{bmatrix} = \begin{bmatrix} c_{i+1} \\ T \end{bmatrix}$
  - Recall · and + represent AND and OR.
  - Boolean matrix multiplication done same way as for reals.
- Applying this repeatedly, we get

$$\begin{aligned} \begin{bmatrix} c_{i+1} \\ T \end{bmatrix} &= \begin{bmatrix} p_i & g_i \\ F & T \end{bmatrix} \begin{bmatrix} c_i \\ T \end{bmatrix} \\ &= \begin{bmatrix} p_i & g_i \\ F & T \end{bmatrix} \begin{bmatrix} p_{i-1} & g_{i-1} \\ F & T \end{bmatrix} \begin{bmatrix} c_{i-1} \\ T \end{bmatrix} = \cdots \\ &= \begin{bmatrix} p_i & g_i \\ F & T \end{bmatrix} \cdots \begin{bmatrix} p_1 & g_1 \\ F & T \end{bmatrix} \begin{bmatrix} c_0 \\ T \end{bmatrix} \end{aligned}$$

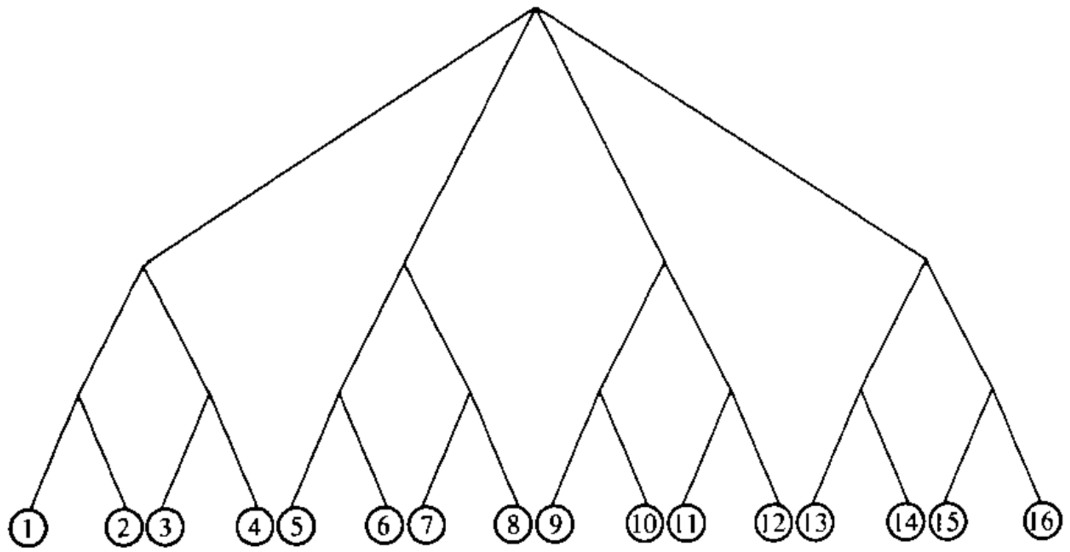
- Since all the p_i and g_i values are known, the final product can be computed using prefix sum in O(log n) time with n processors.
- This algorithm or variants are implemented in most real CPUs.

Constant time max finding

- Using a balanced binary tree we can find the max of n numbers in $O(\log n)$ time and $O(n)$ work.
- We show how to find max in $O(\log \log n)$ time using $O(n)$ work on a min priority CRCW PRAM.
  - I.e. when multiple Boolean values are written to same location, the min value wins.
- First, we can find the max of p numbers $x_1, \ldots, x_p$ in $O(1)$ time and $O(p^2)$ work on the CRCW PRAM.
  - For $1 \le i, j \le p$, in parallel set $B(i,j) = 1$ if $x_i \ge x_j$, and $B(i,j) = 0$ otherwise.
    - Uses p^2 processors.
  - For $1 \le i \le p$, in parallel set $M_i = B(i,1) \land B(i,2) \land \cdots \land B(i,p)$.
    - $M_i = 1$ iff x_i is the max value.
    - This requires that when 0's and 1's are written to the same M_i, the minimum value (i.e. 0) gets written.

Doubly logarithmic tree

- Create a tree with the x_i's at the leaves.
- For each internal node u, let n_u be the number of leaves in the subtree rooted at u. Make the degree of u be $\lceil\sqrt{n_u}\,\rceil$, and all subtrees be the same size.
  - For simplicity, assume $n = 2^{2^k}$. Then the tree has $k = \log\log n$ levels.
  - The root of the tree has degree $2^{2^{k-1}} = \sqrt{n}$.
  - Each child of the root has degree $2^{2^{k-2}}$.
  - In general, at level $0 \le i \le k-1$, each node has degree $2^{2^{k-i-1}}$, and there are $2^{2^k - 2^{k-i}}$ nodes total at the level.



Source: Introduction to Parallel Algorithms, Jaja

Superfast max finding

- Suppose each node computes the max of all its children.
  - Then each node has the max of all the leaf nodes in its subtree, and the root has the overall max value.
  - To compute the max of p children takes $O(p^2)$ work.
- Total time for algorithm is $O(\log\log n)$.
- Total work per level is $O\left(\left(2^{2^{k-i-1}}\right)^2 \cdot 2^{2^k - 2^{k-i}}\right) = O\left(2^{2^k}\right) = O(n).$
  - Total overall work is $O(n \log\log n)$. So the algorithm isn't work efficient.

Superfast max finding

- To make the previous algorithm work efficient, we use a technique called accelerated cascading.
  - Start with a work optimal algorithm until problem size is sufficiently small.
  - Then switch to fast but non-work optimal algorithm.
- First, partition the n values into $n' = n/\log\log n$ blocks of size $\log\log n$ each.
  - Use $n/\log\log n$ processors. Each processor sequentially finds the max of one block of values.
  - This takes $O(\log\log n)$ time and does $O(n)$ work.
  - Then use the doubly logarithmic tree on the n' values.
    - This runs for $O(\log\log n') = O(\log\log n)$ time.
    - It does $O(n' \log\log n') = O(n)$ work.

List ranking

- Given a linked list, compute the distance of each node to the end.
  - Linked list is represented by an array next, where `next[i]` initially points to node following node i.
  - `next[i]=NULL` for the last node.
- Let `d[i]` be i's estimate of its distance to the end.
  - Initially `d[i]`=0 for the last node, and `d[i]`=1 for all other nodes.

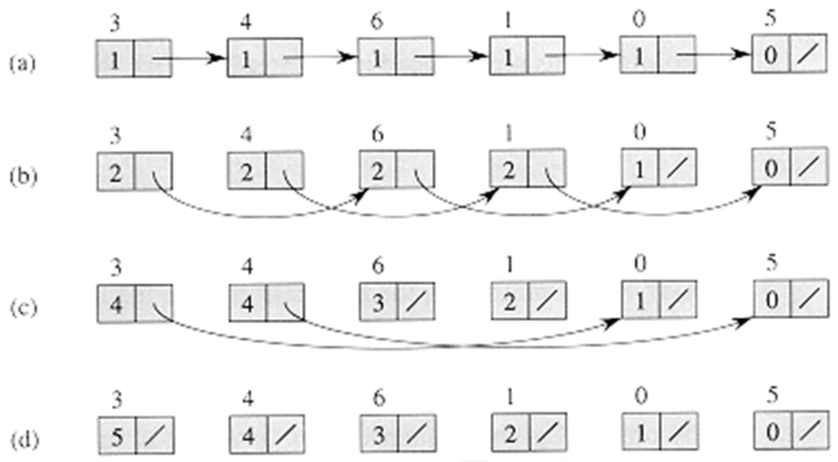
i	0	1	2	3	4	5
next[i]	2	4	⊥	0	5	3

$$\begin{tikzcd}
\text{i} & 1 \arrow[r] & 4 \arrow[r] & 5 \arrow[r] & 3 \arrow[r] & 0 \arrow[r] & 2 \\
\text{d[i]} & 5 & 4 & 3 & 2 & 1 & 0
\end{tikzcd}$$

List ranking

- Repeatedly apply pointer jumping.
  - If currently $i \to j$ and $j \to k$, set $i \to k$. Also increase d[i] by d[j].
- Let k be a node that's distance m away from the end, for some m.
  - After pointer jumping t times, $d[k]=\min(m,2^t)$, and next[k] points $\min(m,2^t)$ distance away.
- Since $d[*] \le n$, algorithm terminates in O(log n) steps.
- Work is O(n log n).
  - Not efficient, since sequential list ranking takes O(n) work.
- List ranking has many applications, including Euler tour technique, connected components, expression tree evaluation, ear decomposition, etc.

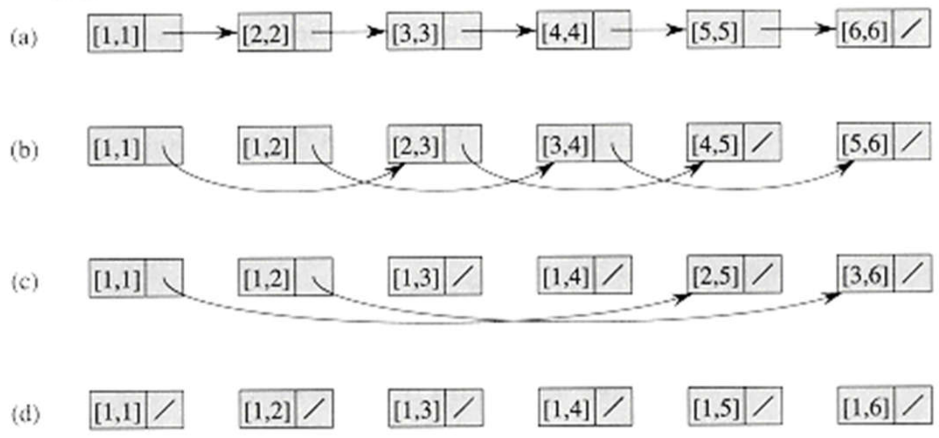
```
do parallel for all i
  d[i]=1
while next[i]≠NULL for some i
   do parallel for all i
      if next[i]≠NULL
         d[i]=d[i]+d[next[i]]
         next[i]=next[next[i]]
```



Prefix sum on linked list

- We've seen how to do prefix sum on an array.
- Using pointer jumping, can also do prefix sum on a linked list.
  - Initially each node i has a value x[i].
  - The output, i.e. prefix sum of node i is stored in d[i].
  - Only difference with list ranking is update `d[next[i]]` instead of `d[i]`.
- After t steps, first 2^t nodes have correct prefix sum, and other nodes have the sum of the preceding 2^t values.
- Takes O(log n) time, does O(n log n) work.

```
do parallel for all i
  d[i]=x[i]
while next[i]≠NULL for some i
  do parallel for all i
    if next[i]≠NULL
      d[next[i]]=d[i]+d[next[i]]
      next[i]=next[next[i]]
```

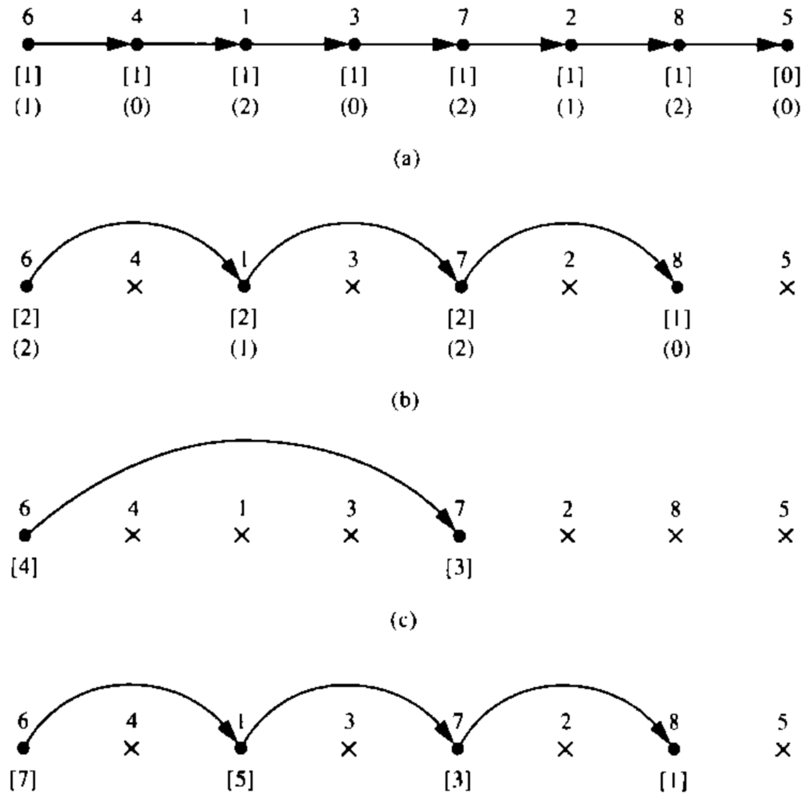


Work efficient list ranking

- List ranking using pointer jumping does $O(n \log n)$ work.
- To make list ranking efficient, we can
  - Shrink the list until only $O(n/\log n)$ nodes remain.
  - Apply pointer jumping to remaining nodes.
  - Restore the removed nodes and determine their ranks.
- Assume first and third steps take $O(n)$ work.
- Then second step takes $O(\frac{n}{\log n} \log(\frac{n}{\log n})) = O(n)$ work, so total work is $O(n)$.

Work efficient list ranking

- To shrink the list, we repeatedly remove an independent set of nodes.
  - A set of nodes I is independent if $\forall i \in I: (prev(i) \notin I) \wedge (next(i) \notin I)$.
  - Suppose we have a set of n nodes. We show next lecture how find $\Omega(n)$ independent nodes in $O(\log n)$ time and $O(n)$ work.
- Given an independent set I, for each $i \in I$ set $dist[prev[i]] = dist[prev[i]] + dist[i]$.
- To compute distance of a removed node i, set $dist[i] = dist[i] + dist[next[i]]$.



- values in parentheses are used to find independent set.
- dist values are shown in brackets.

Work efficient list ranking

- Since each round we remove $\Omega(n)$ number of remaining nodes, it takes $O(\log\log n)$ rounds to shrink the list to size $O(n/\log n)$.
  - After this the pointer jumping takes $O(\log n)$ time.
- Each round takes $O(\log n)$ time to find the independent set.
- So total time is $O(\log n \log\log n)$.
  - Time can be reduced to $O(\log n)$ using more efficient algorithm.
- In round k, number of remaining nodes is $O(c^k n)$ for some $c < 1$.
- So total work to find independent sets in all rounds is $\sum_{k=0}^{\log\log n} O(c^k n) = O(n)$.
- Pointer jumping does $O(n)$ work, so total work is $O(n)$.