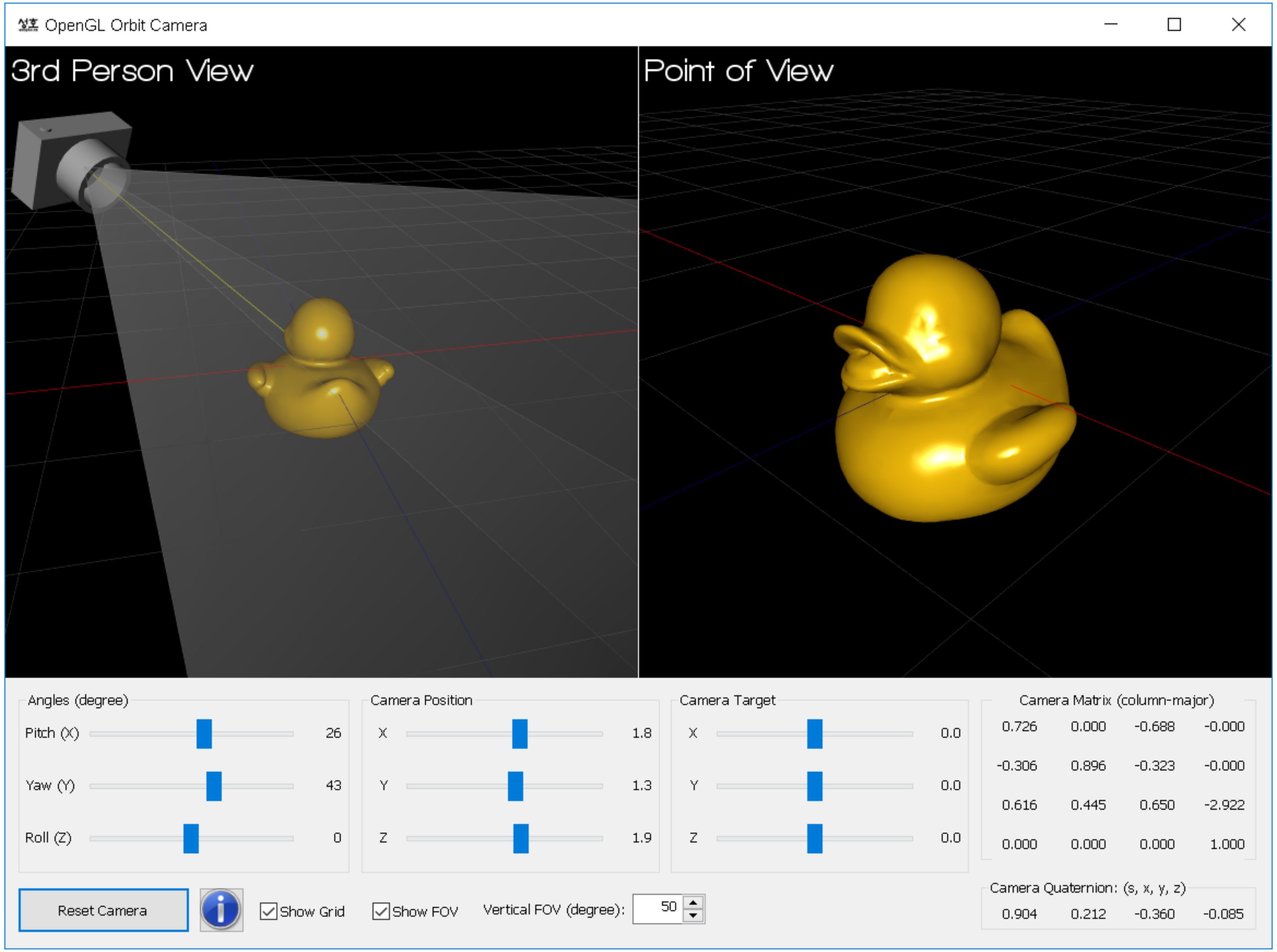
Camera Calibration

Jiayuan Gu

gujy1@shanghaitech.edu.cn

How does a camera take images?



https://www.songho.ca/opengl/gl_camera.html

Three Components

- In the context of rendering geometries (computer graphics)

- We know
  - Camera
  - 3D points (model)

- We render
  - 2D pixels (image)

Review: Pinhole Camera

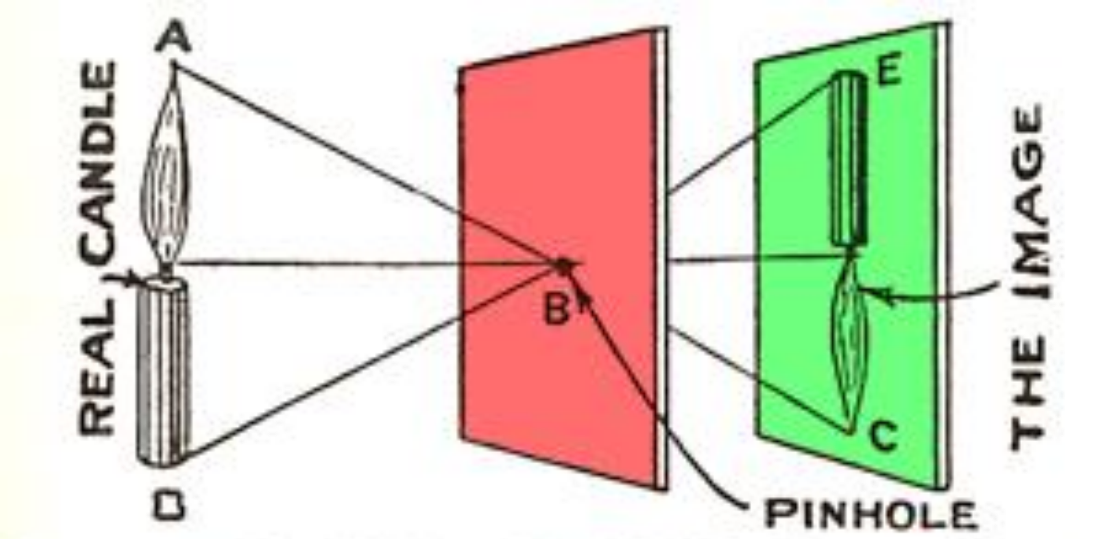
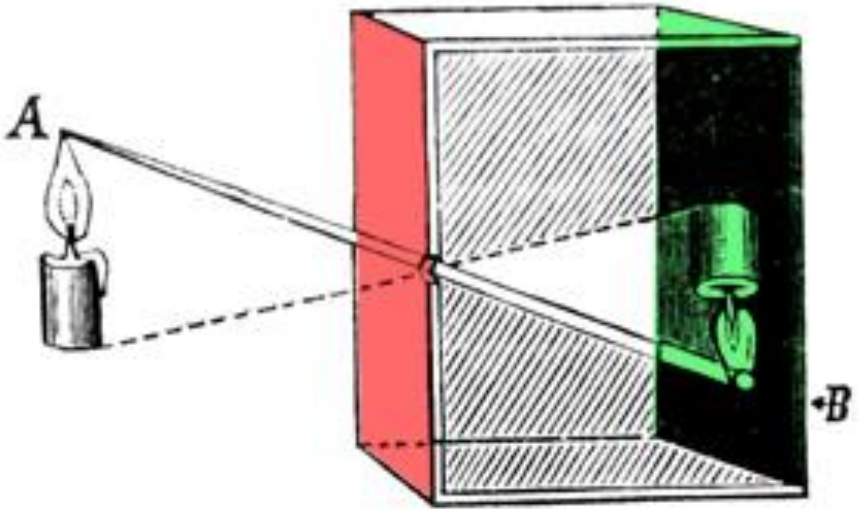


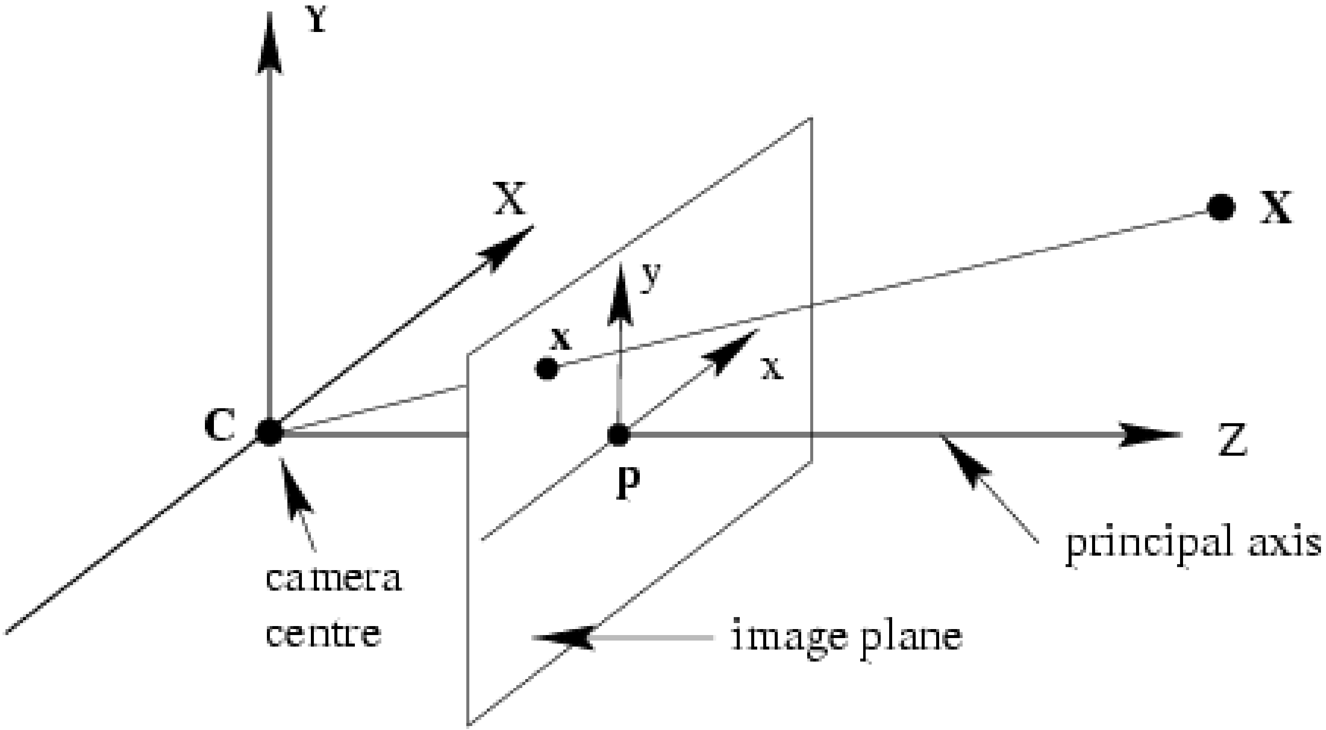
FIG. 131.—How Light and a Pinhole Form an Image.



Illustrated in 1925, in *The Boy Scientist*

Figure from tutorial

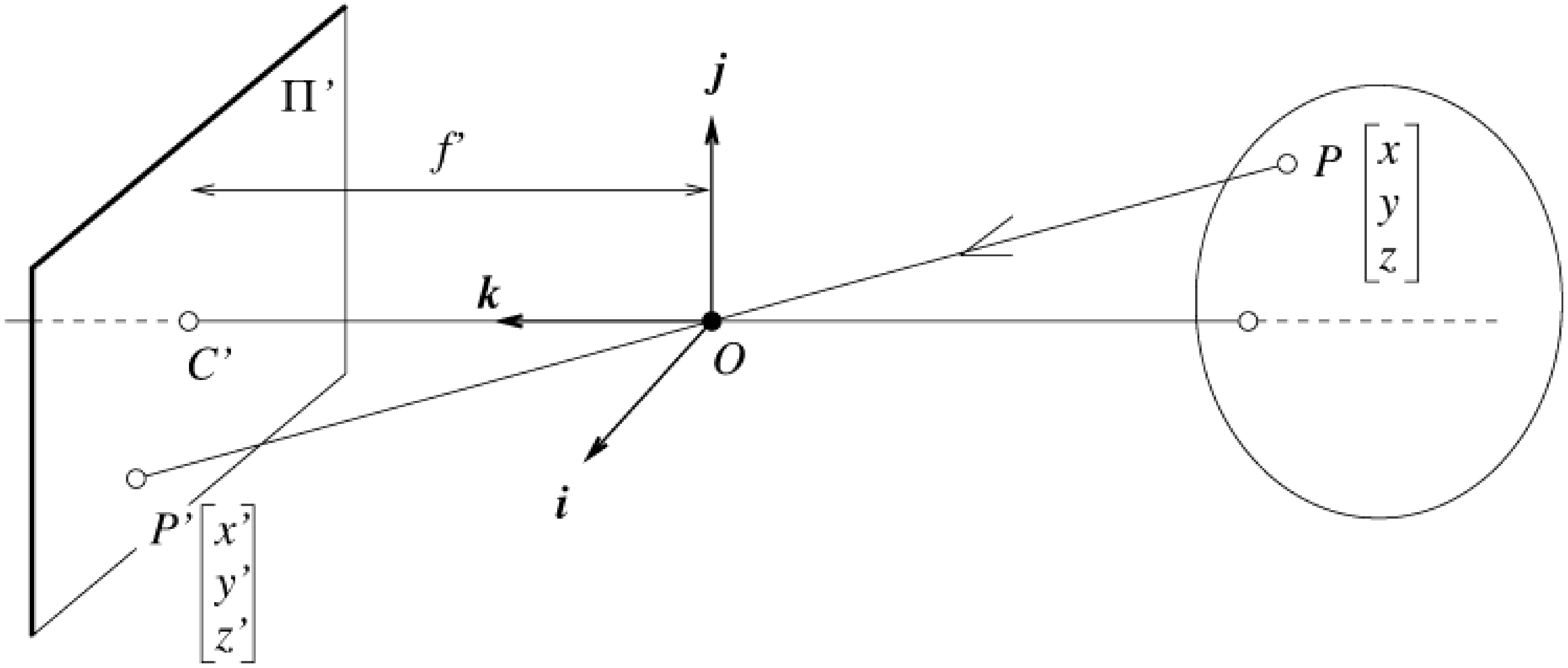
Review: Pinhole Camera



Normalized (camera) coordinate system: camera center is at the origin, the **principal axis** is the z-axis, x and y axes of the image plane are parallel to x and y axes of the camera

Figure from "[Multiple View Geometry in Computer Vision]"

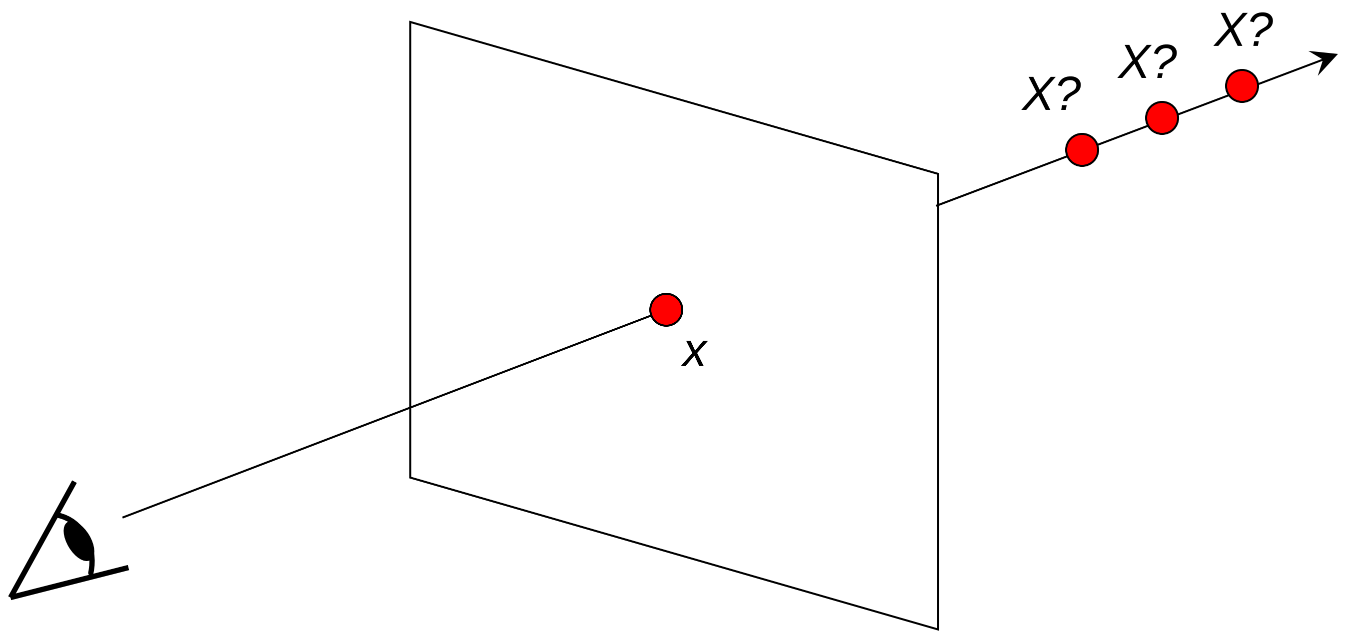
Perspective Projection



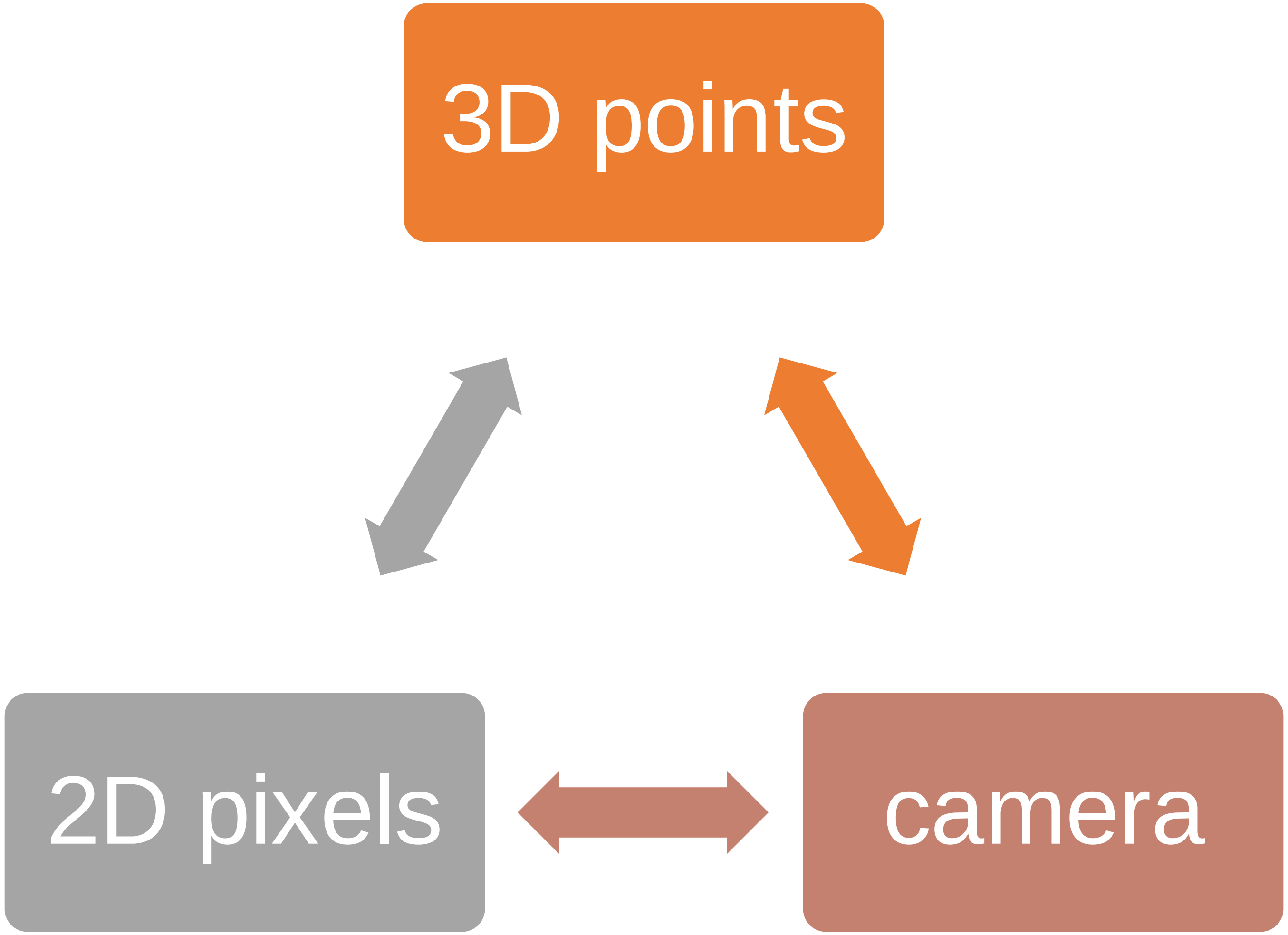
$$\frac{x'}{x} = \frac{y'}{y} = \frac{z'}{z}$$

Source: Bahadir K. Gunturk

Question: Can we recover 3D from images?



Core: Pixels, Points, Camera

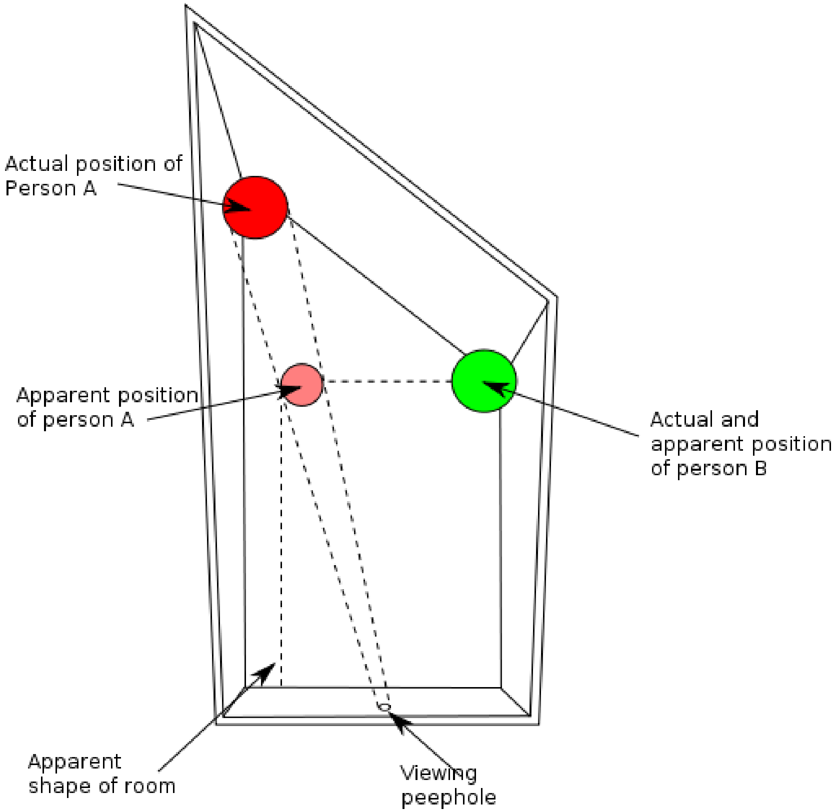


Single-View Ambiguity

Single-View Ambiguity

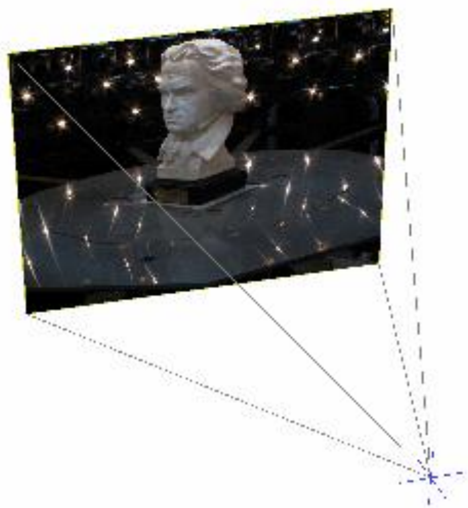
Rashad Alakbarov shadow sculptures

Single-View Ambiguity



[Ames room]

Multi-view Geometry

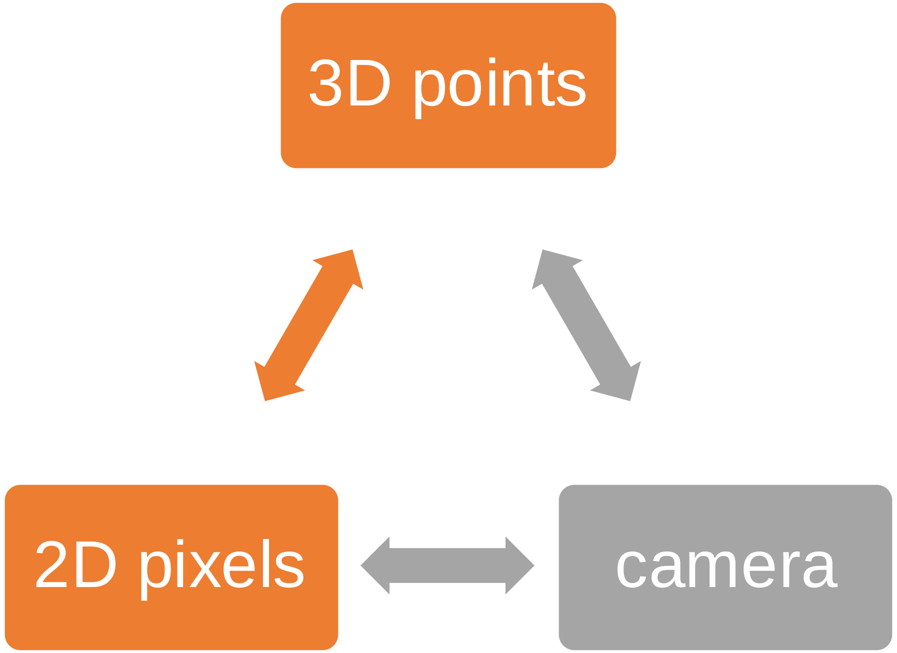


Animation from [TUM]

Camera Calibration

- Known: 3D points and their 2D pixels

- To solve: camera parameters



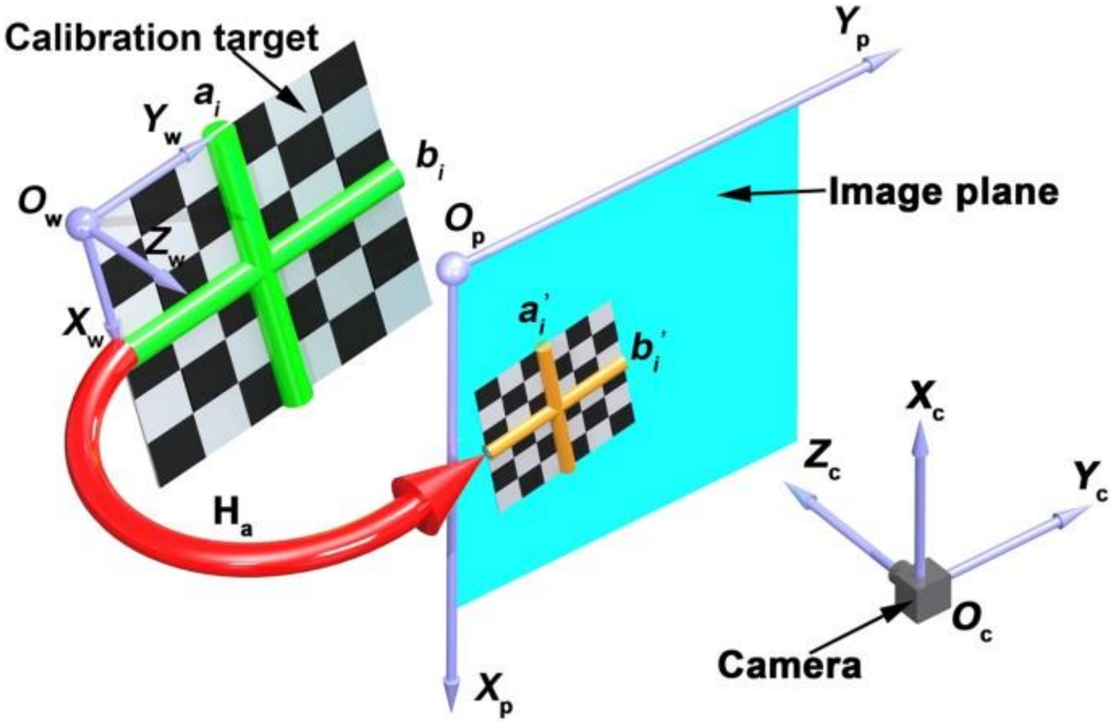
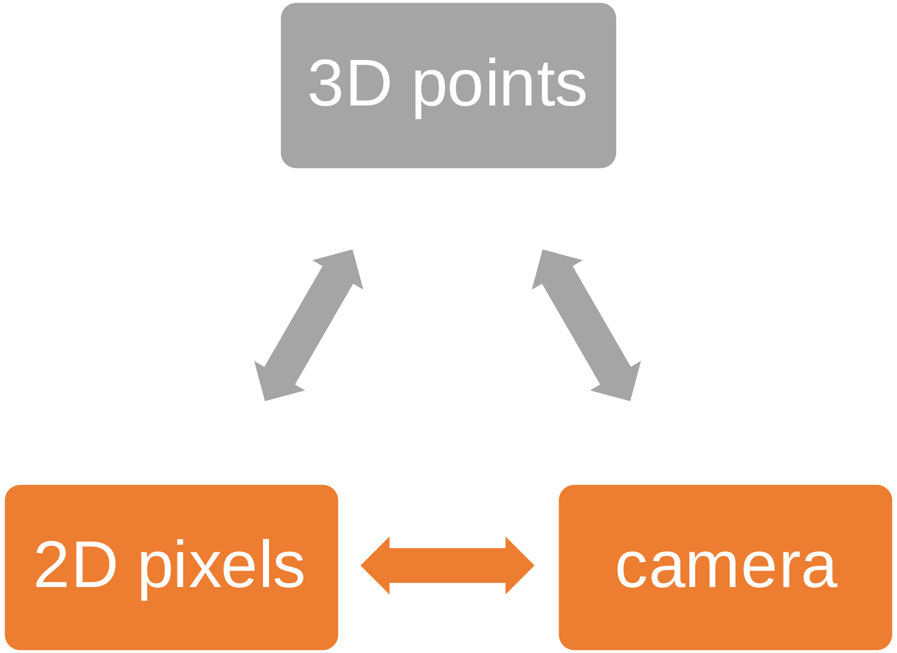
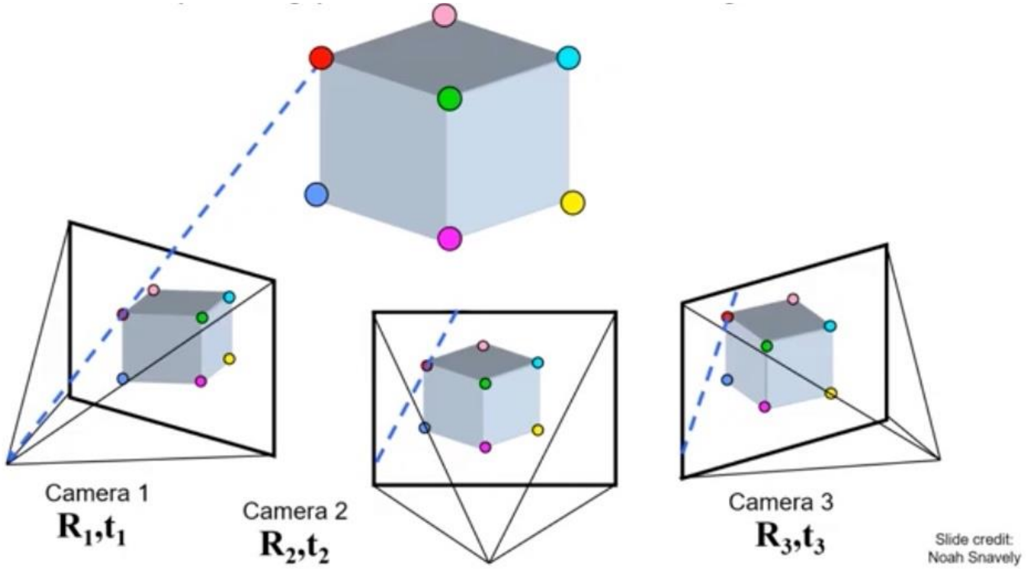


Figure from [Nature]

Multi-View Stereo (MVS)

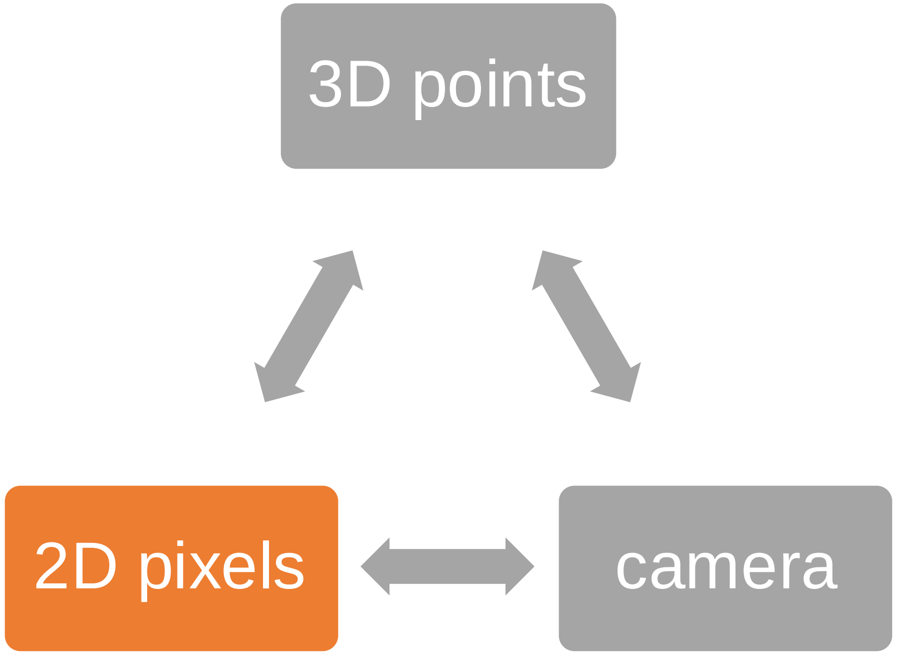
- Known:
  - multiple images (corresponding pixels in each view)
  - camera parameters (including relative poses between views)
- To solve: 3D points (corresponding to pixels in each view)

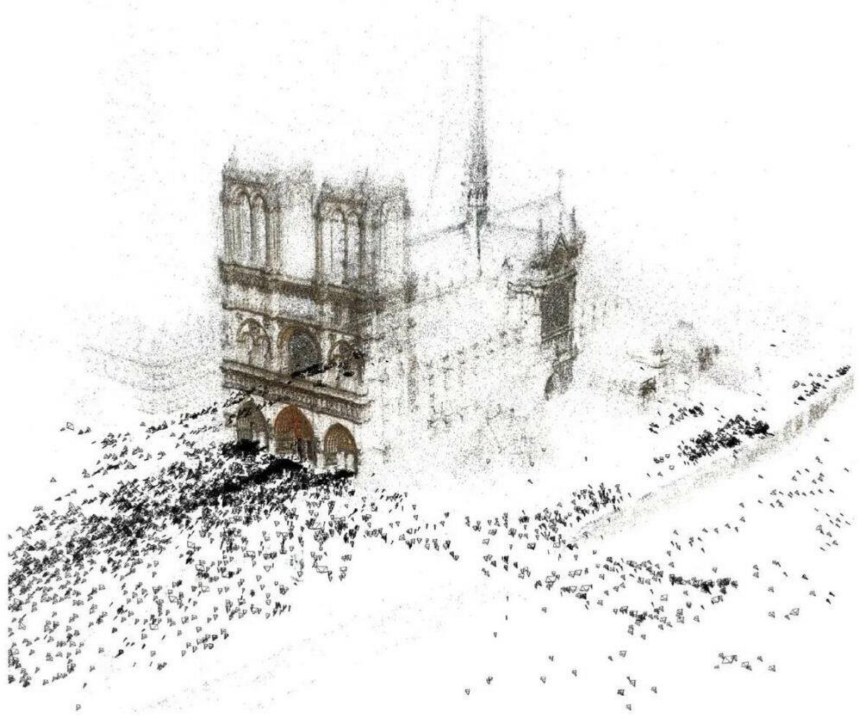




Structure from Motion (SFM)

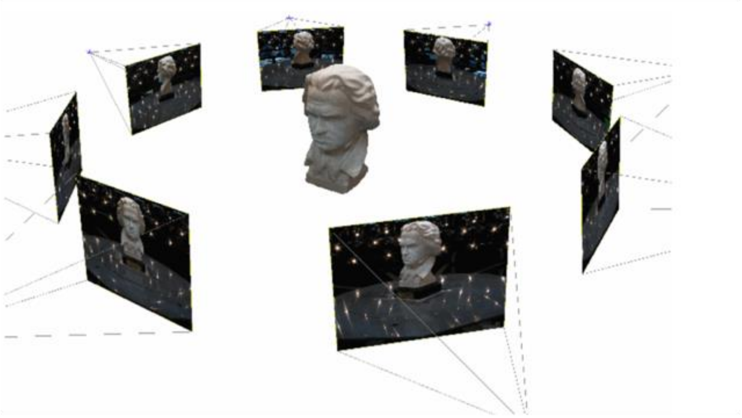
- Known:
  - multiple images (corresponding pixels in each view)
- To solve:
  - 3D points (corresponding to pixels in each view)
  - camera parameters (including relative poses between views)





MVS vs. SFM

Multi-View Stereo

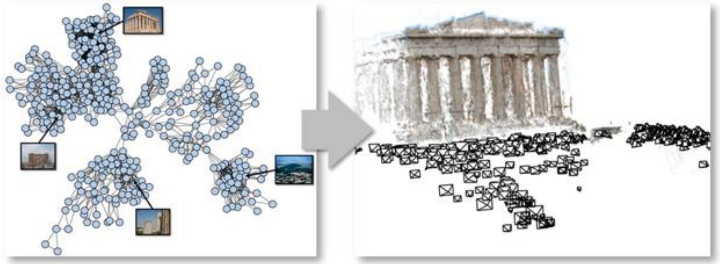


✅ Camera parameters known

✅ Camera position known

📐 Based on Epipolar Geometry

Structure From Motion

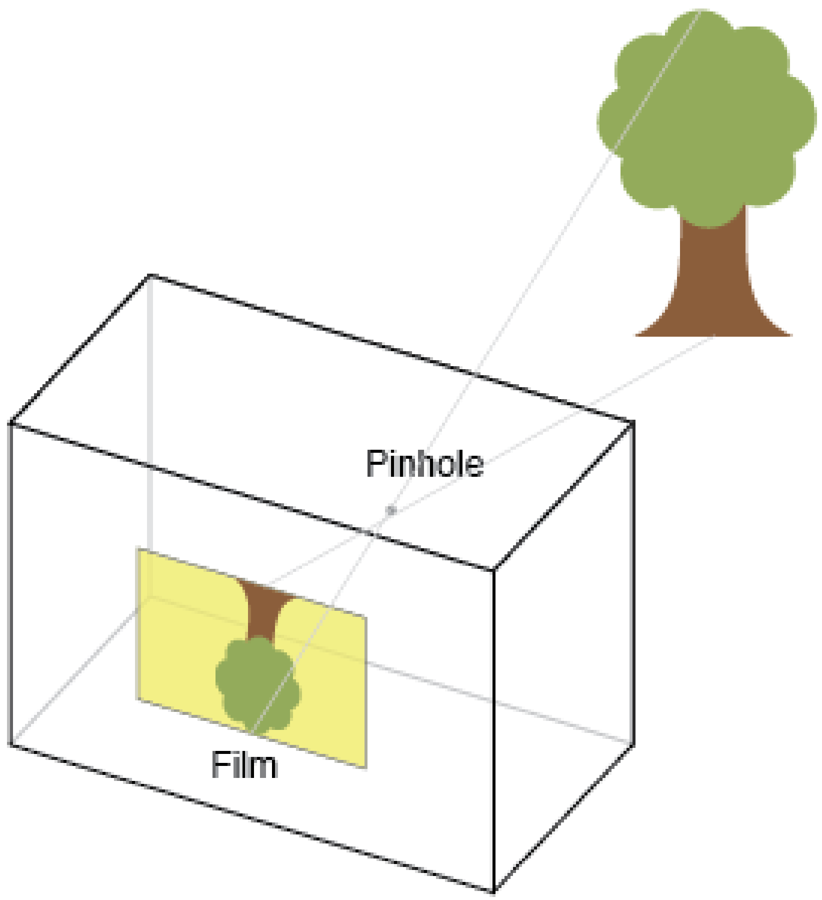


❌ Camera parameters unknown

❌ Camera positions unknown

🔗 Based on Feature Matching

Today’s Topic: Camera Calibration

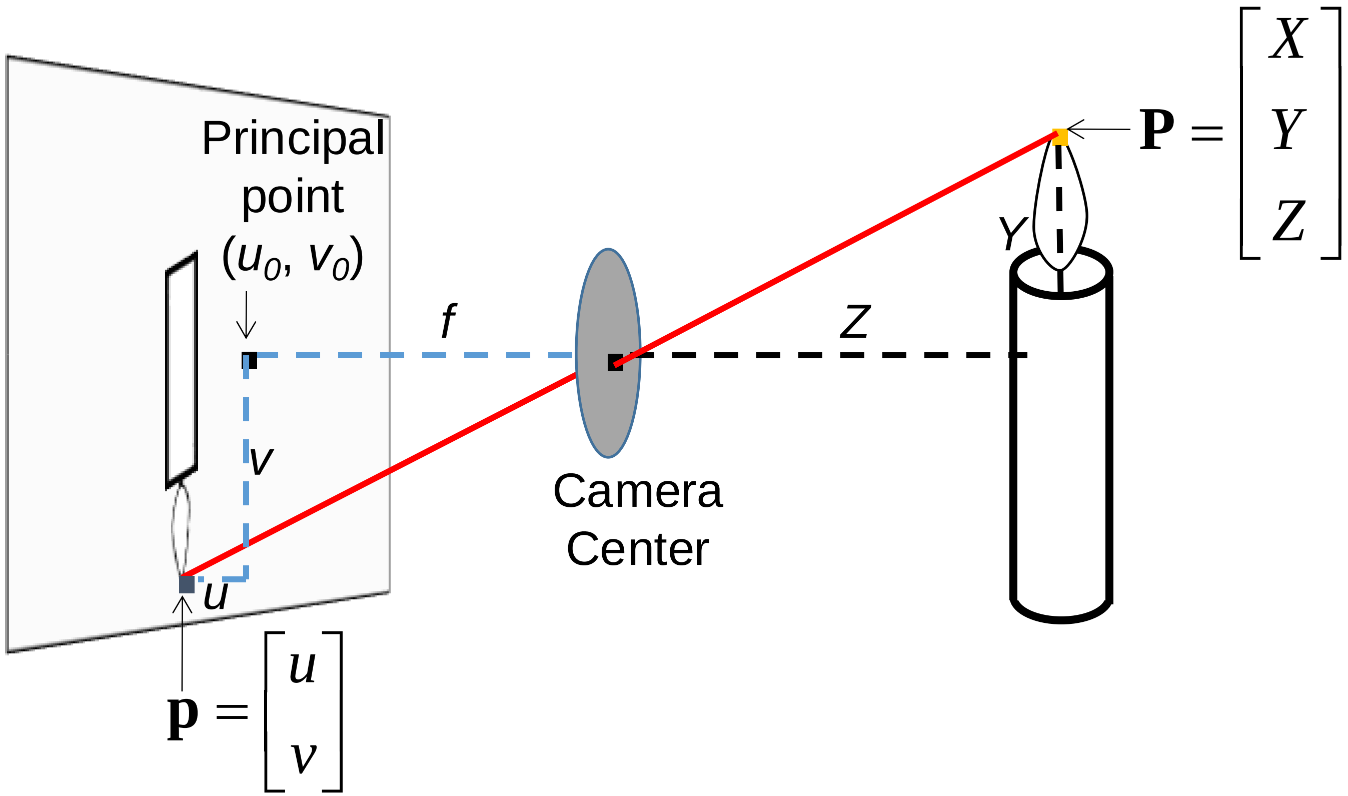


How to parameterize a camera?

How to solve camera parameters?

Figure from [blog]

From Camera Space to Image Space



$$u = f\frac{x}{z}$$

$$v = f\frac{y}{z}$$

Need a new math tool to conveniently describe the conversion

Source: Derek Hoiem

Homogeneous Coordinates

Converting **to** homogeneous coordinates

$$(x, y) \Rightarrow \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}$$

homogeneous image coordinates

$$(x, y, z) \Rightarrow \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

homogeneous scene coordinates

Converting **from** homogeneous coordinates

$$\begin{bmatrix} x \\ y \\ w \end{bmatrix} \Rightarrow (x/w, y/w)$$

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \Rightarrow (x/w, y/w, z/w)$$

Homogeneous Coordinates

$$u = f\frac{x}{z}$$

$$v = f\frac{y}{z}$$

$$\begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} fx \\ fy \\ z \end{bmatrix}$$

Non-linear formula with inhomogeneous coordinates (Cartesian)

Linear formula with homogeneous coordinates

Homogeneous Coordinates

$$k\begin{bmatrix} x \\ y \\ w \end{bmatrix} = \begin{bmatrix} kx \\ ky \\ kw \end{bmatrix} \Rightarrow \begin{bmatrix} \frac{kx}{kw} \\ \frac{ky}{kw} \end{bmatrix} = \begin{bmatrix} \frac{x}{w} \\ \frac{y}{w} \end{bmatrix}$$

Homogeneous Coordinates   Cartesian Coordinates

A point in Cartesian space is a ray in projective space

Homogeneous: invariant to scaling

Geometry in Homogeneous Coordinates

2D line

$$ax + by + c = 0$$

$$[a, b, c]\begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$$

2D line passing two points

$$ax_1 + by_1 + c = 0$$
$$ax_2 + by_2 + c = 0$$

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} x_1 \\ y_1 \\ 1 \end{bmatrix} \times \begin{bmatrix} x_2 \\ y_2 \\ 1 \end{bmatrix}$$

Cartesian Coordinates

Homogeneous Coordinates

Geometry in Homogeneous Coordinates

Intersection of two lines

$$a_1 x + b_1 y + c_1 = 0$$
$$a_2 x + b_2 y + c_2 = 0$$

Cartesian Coordinates

$$[a_1, b_1, c_1] \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$$

$$[a_2, b_2, c_2] \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = 0$$

$$\begin{bmatrix} x \\ y \\ w \end{bmatrix} = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix} \times \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}$$

Homogeneous Coordinates

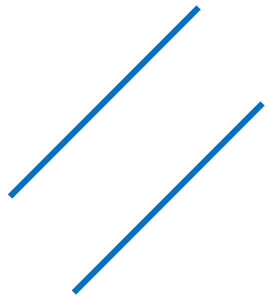
Infinity in Homogeneous Coordinates

A very interesting property of homogeneous coordinates is to represent points at infinity

A point at infinity is $[x, y, 0]^T$

A line at infinity is $[0,0, c]^T$

Two parallel lines will intersect at an infinity point



E.g., Lines $[1,1,1]^T$ and $[1,1,-1]^T$ will intersect at [-2, 2, 0]

Parallel Lines Intersect in Projective Space

https://www.songho.ca/math/homogeneous/homogeneous.html

Projection Matrix (Intrinsic)

$$w \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} f & 0 & 0 & 0 \\ 0 & f & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} fx \\ fy \\ z \end{bmatrix}$$

$x \in \mathcal{R}^{3\times 1}$: 2D point (homogeneous)

$$x = K[I\ 0]X$$

$X \in \mathcal{R}^{4\times 1}$: 3D point (homogeneous)

$K \in \mathcal{R}^{3\times 3}$: Intrinsic matrix

Note that the 3D point is represented in the camera space

More General Intrinsic Matrix

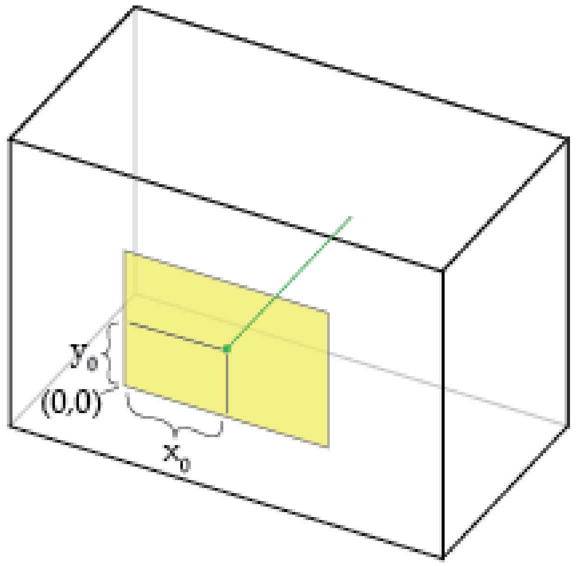
$$K = \begin{pmatrix} f_x & s & x_0 \\ 0 & f_y & y_0 \\ 0 & 0 & 1 \end{pmatrix}$$

A common practice is to measure the intrinsic matrix in **pixels** rather than physical units

Previous assumption:
- Unit aspect ratio
- No skew
- Principal point at (0, 0)

Principal Point Offset

$$w \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} f & 0 & u_0 & 0 \\ 0 & f & v_0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

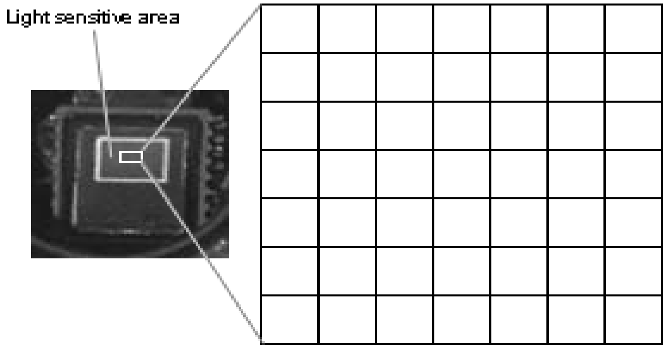


Commonly, the offset is (W/2, H/2)
so that a 3D point on the principal axis will be at the image center

Figure from blog

Focal Length and Aspect Ratio

$$w \begin{bmatrix} u \\ v \\ 1 \end{bmatrix} = \begin{bmatrix} f_x & 0 & u_0 & 0 \\ 0 & f_y & v_0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$



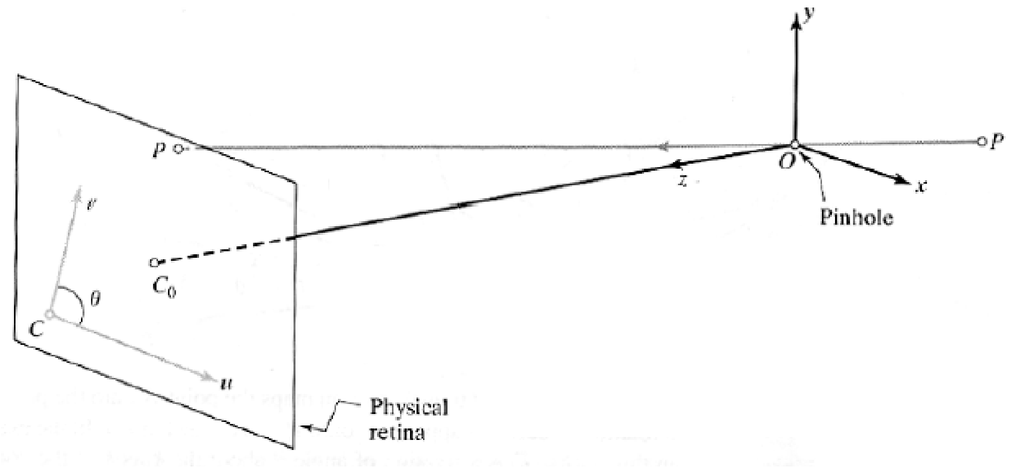
The focal lengths differ for a number of reason:
- Flaws in the digital camera sensor.
- The image has been non-uniformly scaled in post-processing.
- The camera's lens introduces unintentional distortion.
- The camera uses an anamorphic format, where the lens compresses a widescreen scene into a standard-sized sensor.
- Errors in camera calibration.

Source: Bahadir K. Gunturk

Axis Skew

$$w\begin{bmatrix}u\\v\\1\end{bmatrix} = \begin{bmatrix}f_x & -f_x\cot\theta & u_0 & 0\\ 0 & f_y/\sin\theta & v_0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}x\\y\\z\\1\end{bmatrix}$$

$$w\begin{bmatrix}u\\v\\1\end{bmatrix} = \begin{bmatrix}f_x & s & u_0 & 0\\ 0 & f_y & v_0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix}\begin{bmatrix}x\\y\\z\\1\end{bmatrix}$$



Due to manufacturing error

Source: Bahadir K. Gunturk

Summary: Intrinsic Matrix

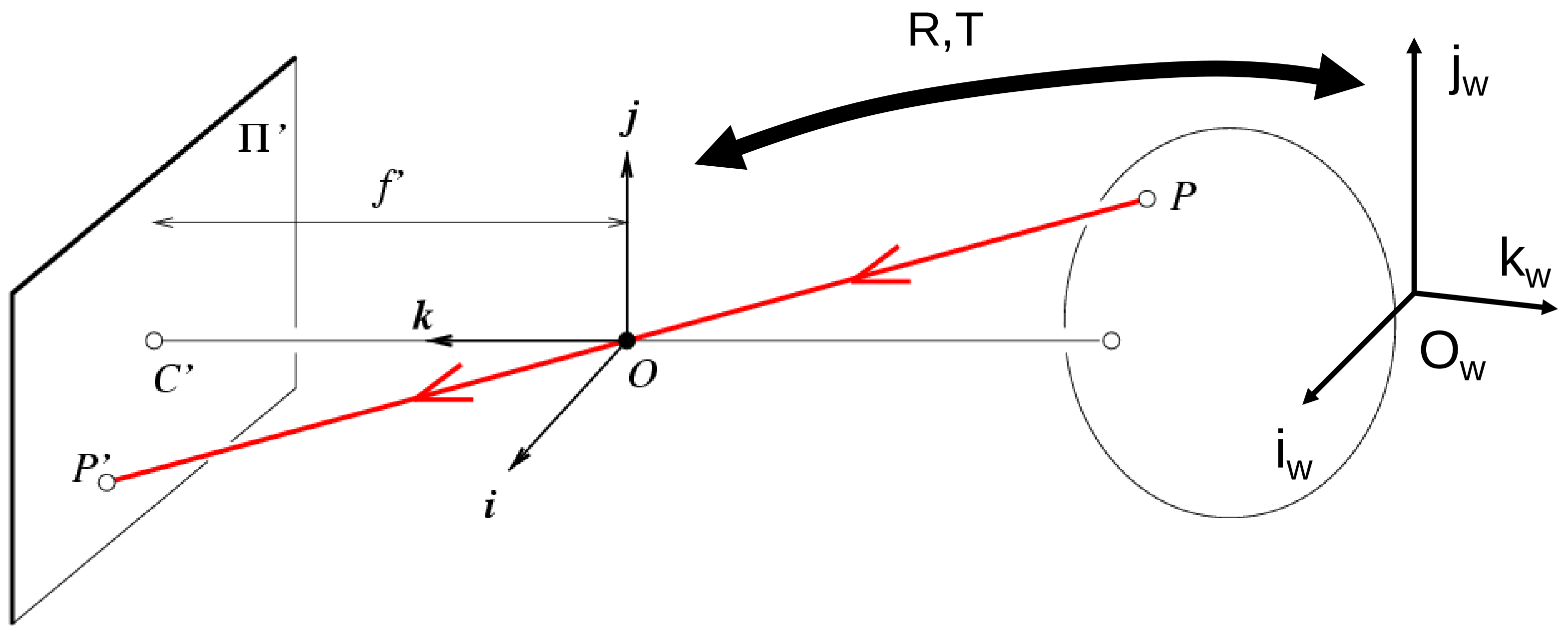
$$K = \begin{pmatrix} f_x & s & x_0 \\ 0 & f_y & y_0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \underbrace{\begin{pmatrix} 1 & 0 & x_0 \\ 0 & 1 & y_0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{2D Translation}} \times \underbrace{\begin{pmatrix} f_x & 0 & 0 \\ 0 & f_y & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{2D Scaling}} \times \underbrace{\begin{pmatrix} 1 & s/f_x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{2D Shear}}$$

5 parameters:
- Focal length: fx, fy
- Skew: s
- Principal point offset: (x0, y0)

There are other ways to parameterize, like FOV (field of view)

Camera Extrinsic

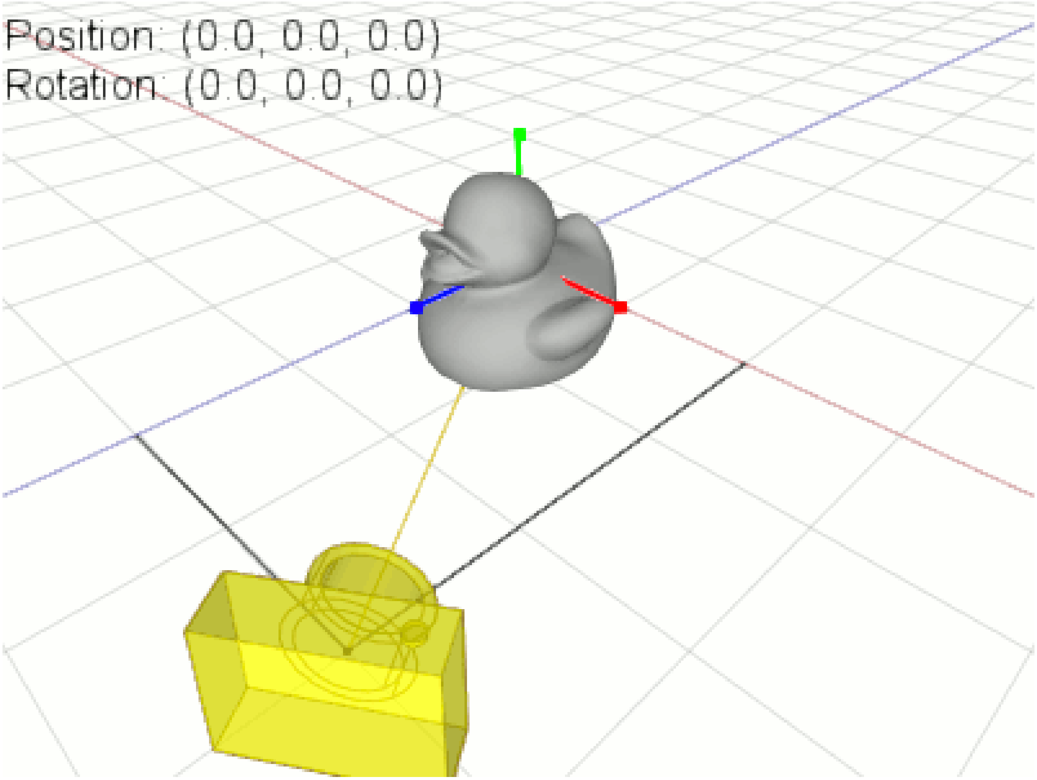


Previously, coordinates are in camera space

Now, we want to record coordinates in **world space** (convenient for multiple cameras)

Source: Derek Hoiem

Camera Extrinsic



Transform the world coordinate system to the camera coordinate system

Animation from blog

Camera Extrinsic

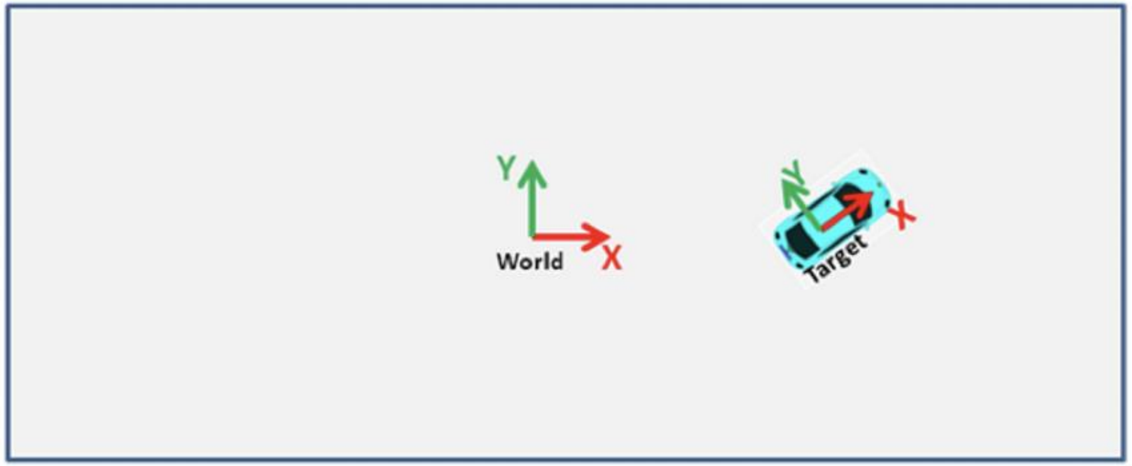
$$x = K[I\ 0]X^C \quad \Longrightarrow \quad x = K[I\ 0]T^{W\to C}X^W$$

X^C is in the camera space

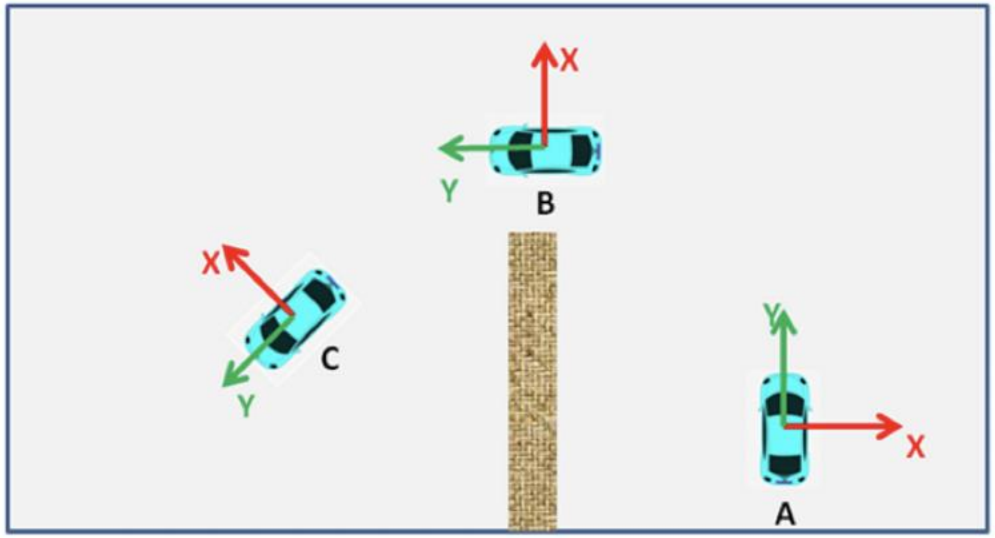
X^W is in the world space

$T^{W\to C}$ is the coordinate transformation from world to camera

Pose: Transformation between Frames



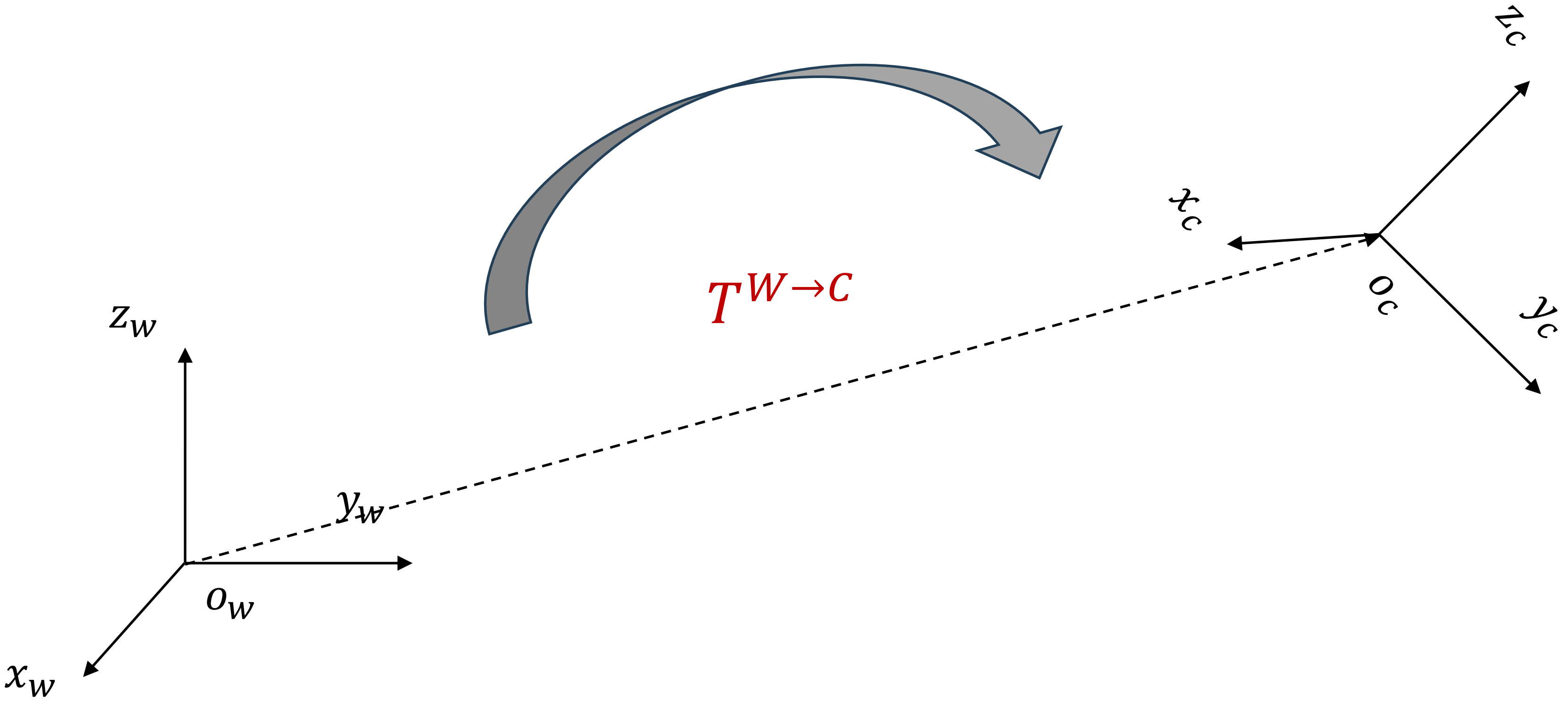
Where is the car in the world?



Where is the car B observed by the driver of the car A?

https://www.rosroboticslearning.com/rigid-body-transformations

Coordinate Transformation

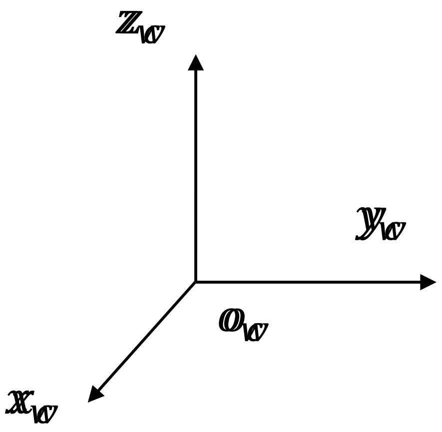


$$x^c = T^{W \to C} x^W$$

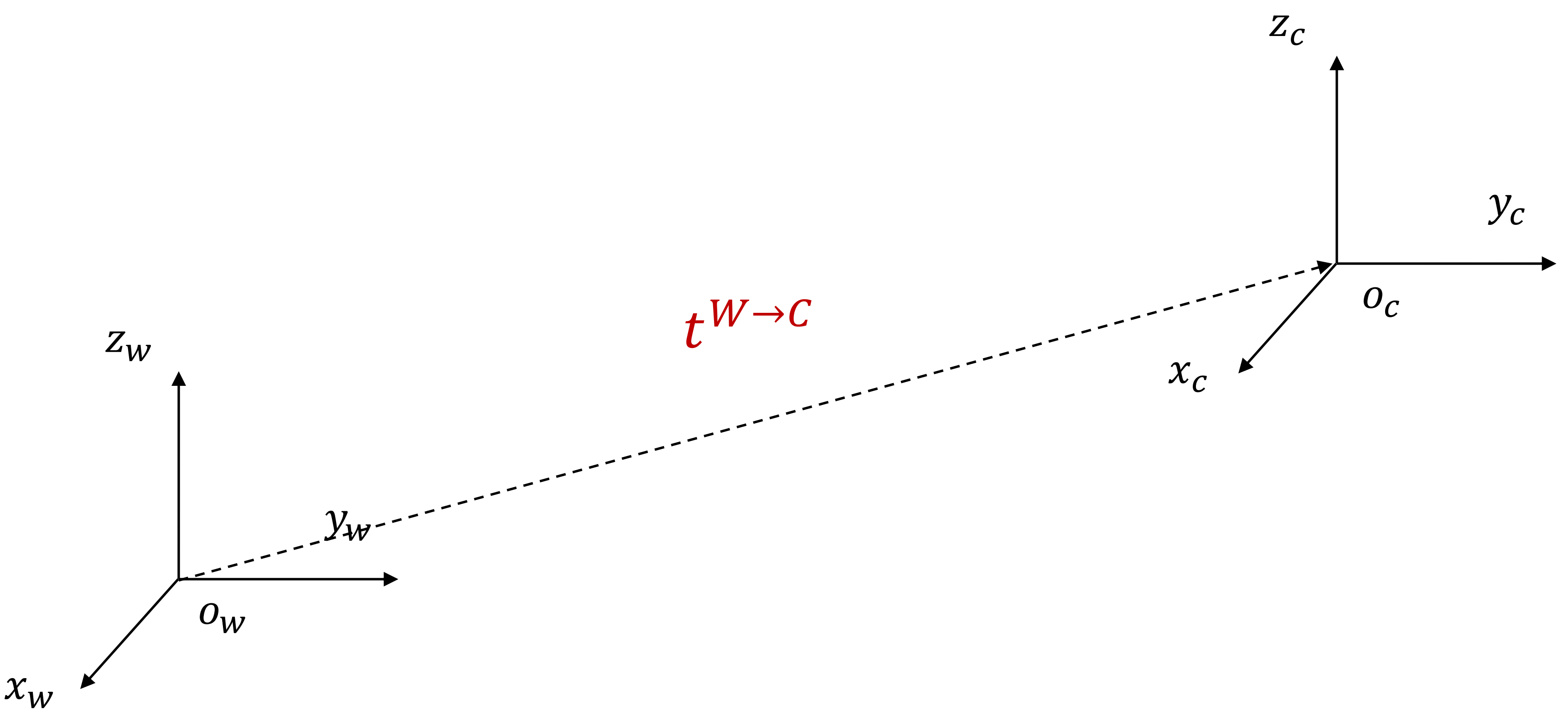
Point in camera space

Point in world space

Translation

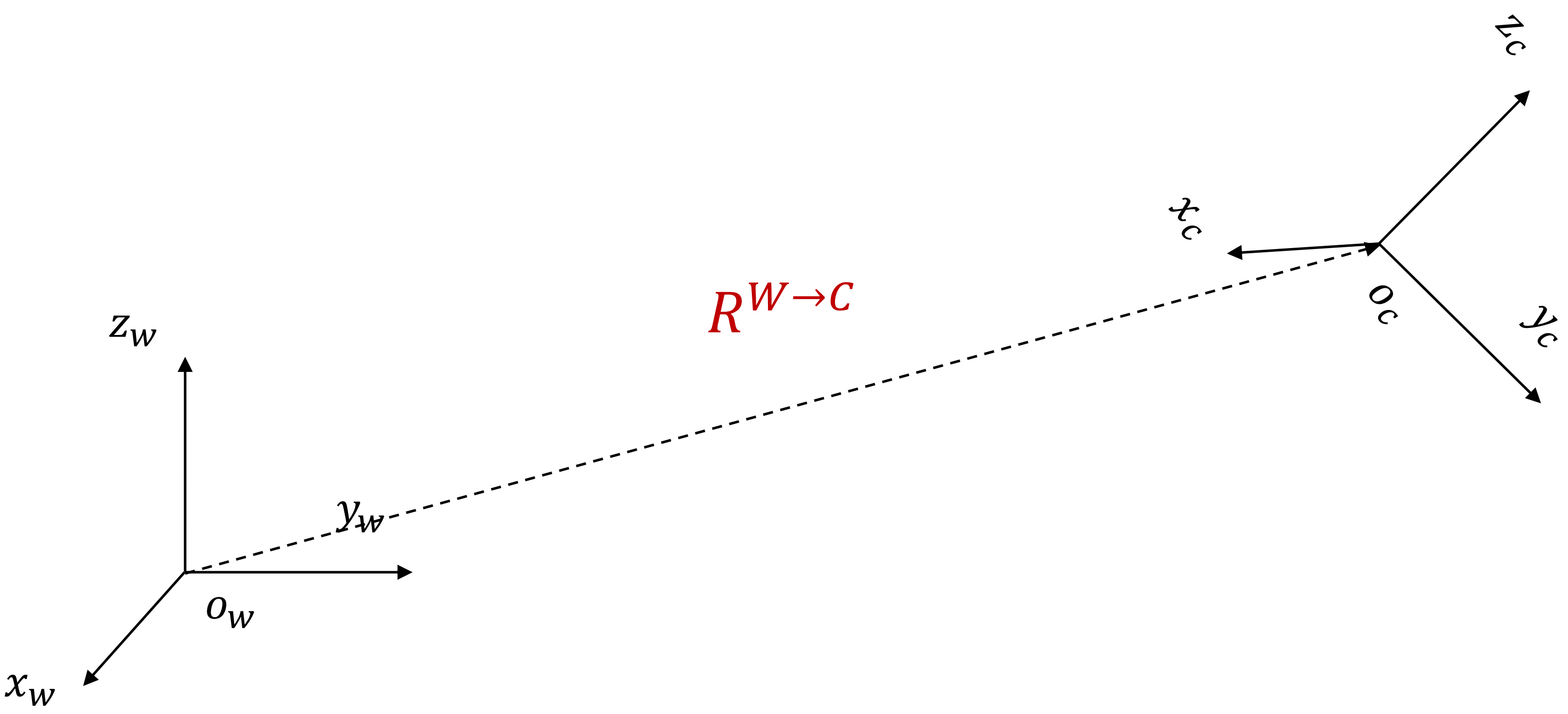


Translation



$$x^c = x^W + t^{W\to C}$$

Rotation

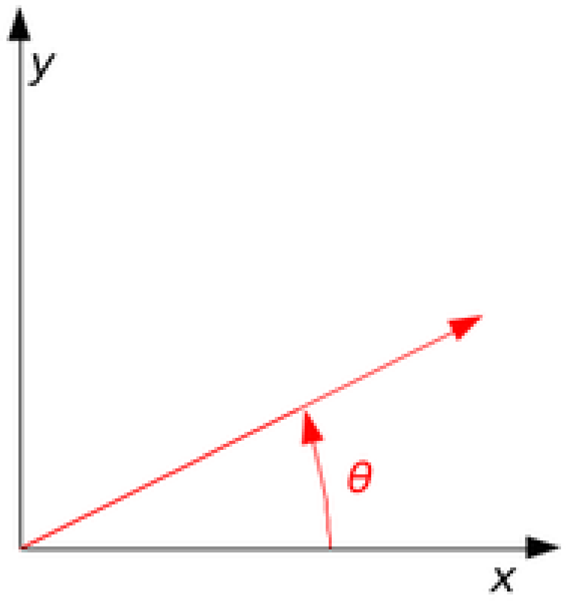


$$x^c = R^{W \to C} x^W + t^{W \to C}$$

Rotation Matrix (2D)

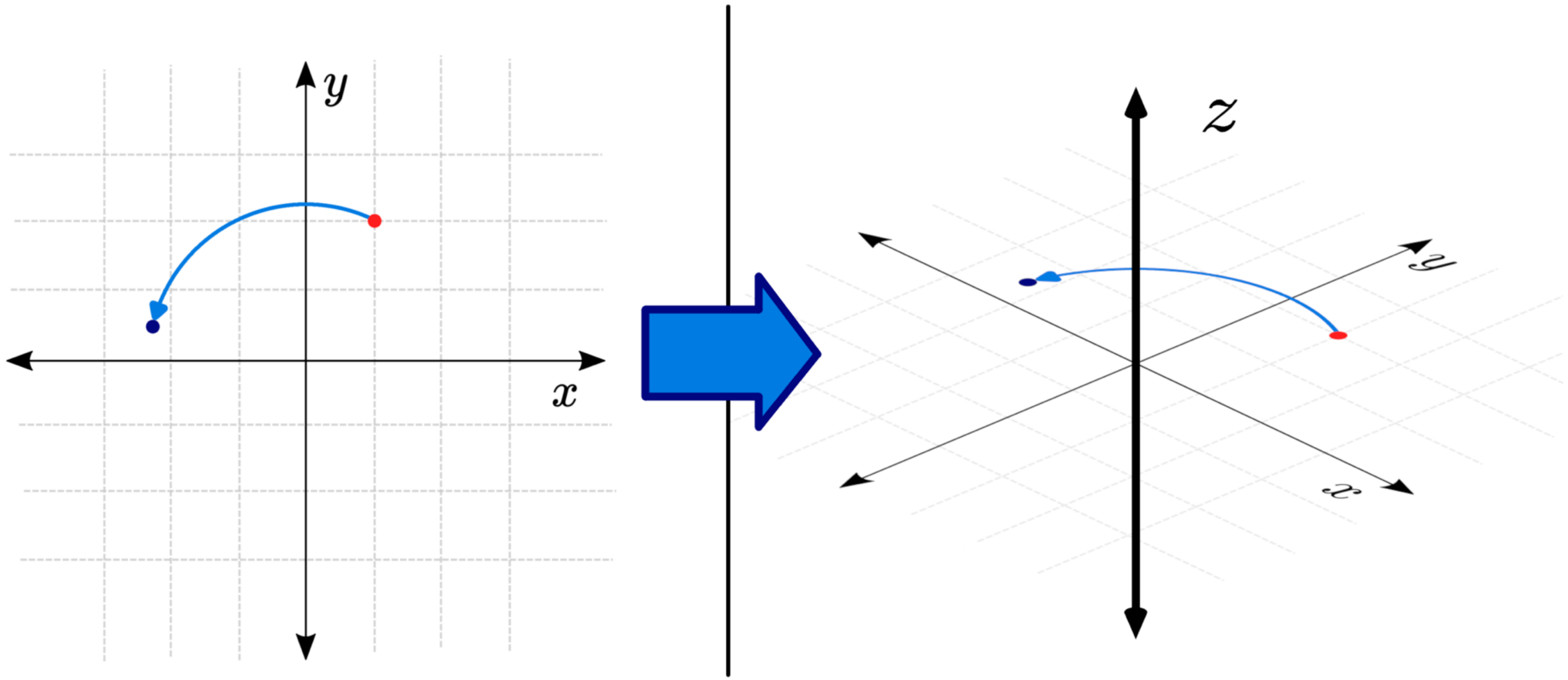
$$R(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$



From wiki

Rotation Matrix (3D)

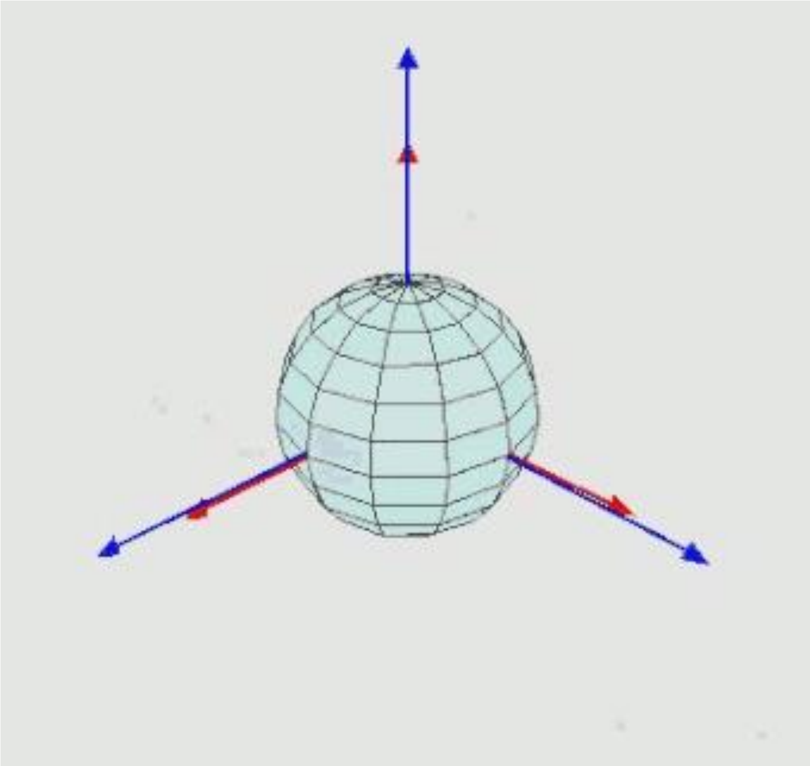


$$\begin{bmatrix} x_{new} \\ y_{new} \\ z_{new} \end{bmatrix} = \begin{bmatrix} \cos(\theta) & -\sin(\theta) & 0 \\ \sin(\theta) & \cos(\theta) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Example: 3D rotation along z-axis

From tutorial

Rotation Representation: Euler Angles



$$R_x(\alpha) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}$$

$$R_y(\beta) = \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix}$$

$$R_z(\gamma) = \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R = R_x(\alpha)R_y(\beta)R_z(\gamma)$$

From Euler Angles Wiki

Rotation Matrix is in SO(3)

- SO(3): Special Orthogonal Group
  - "Group": roughly, closed under matrix multiplication
  - "Orthogonal": $R^T R = I$
  - "Special": $\det(R) = 1$

$$\mathbb{SO}(n) = \{R \in \mathbb{R}^{n\times n} : \det(R) = 1, RR^T = I\}$$

Camera Extrinsic

$$x = K[I\ 0]X^C$$

$$x = K[I\ 0]T^{W\rightarrow C}X^W$$

X^C is in the camera space

X^W is in the world space

$T^{W\rightarrow C}$ is the coordinate transformation from world to camera

$$T^{W\rightarrow C} = \begin{bmatrix} R^{W\rightarrow C} & t^{W\rightarrow C} \\ 0 & 1 \end{bmatrix} \in \mathcal{R}^{4\times 4}$$

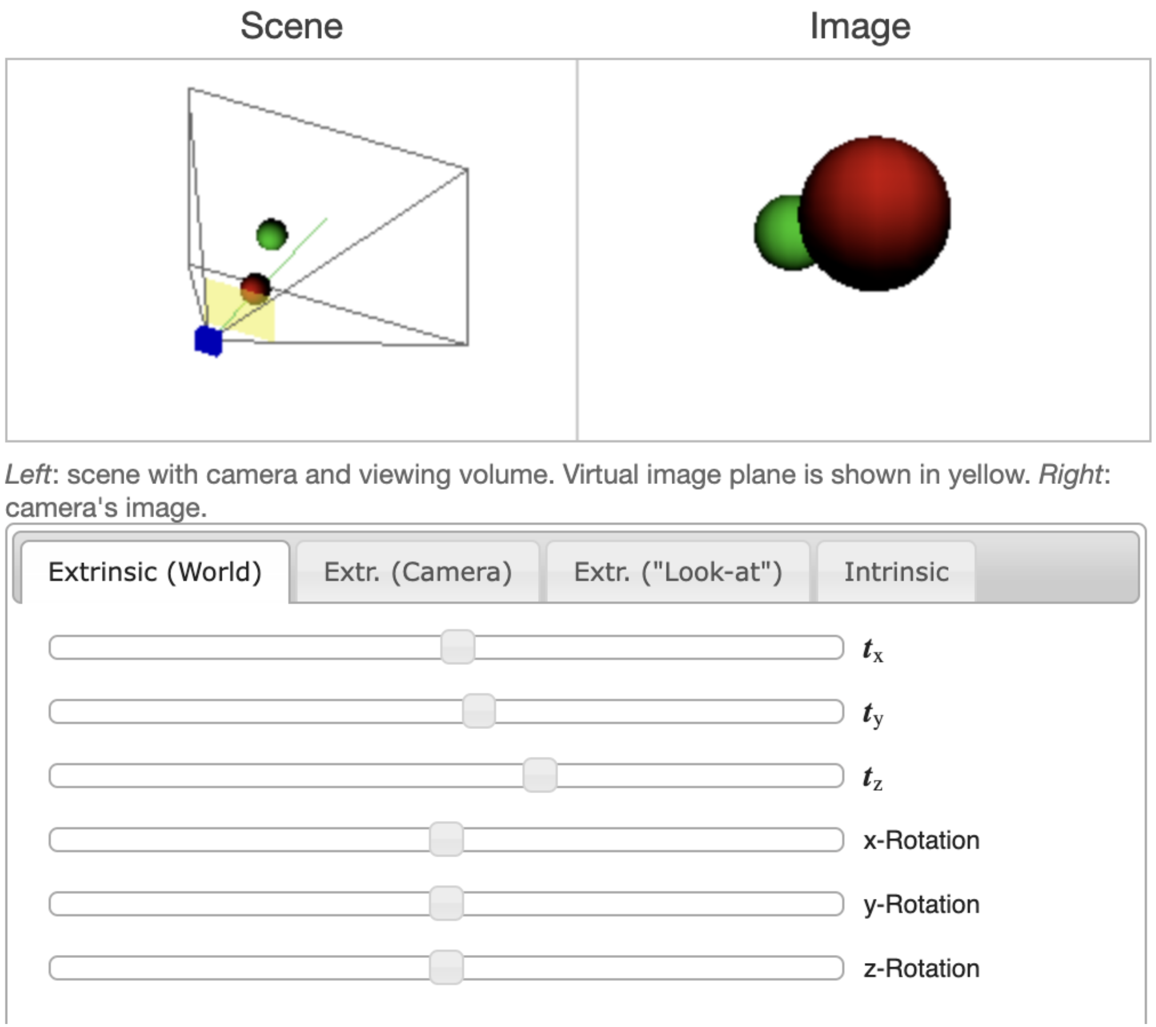
Decomposition of Projection Matrix

$$P = \overbrace{K}^{\text{Intrinsic Matrix}} \times \overbrace{[R \mid \mathbf{t}]}^{\text{Extrinsic Matrix}}$$

$$= \overbrace{\underbrace{\begin{pmatrix} 1 & 0 & x_0 \\ 0 & 1 & y_0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{2D Translation}} \times \underbrace{\begin{pmatrix} f_x & 0 & 0 \\ 0 & f_y & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{2D Scaling}} \times \underbrace{\begin{pmatrix} 1 & s/f_x & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}_{\text{2D Shear}}}^{\text{Intrinsic Matrix}} \times \overbrace{\underbrace{\left(I \mid \mathbf{t} \right)}_{\text{3D Translation}} \times \underbrace{\left(\begin{array}{c|c} R & 0 \\ \hline 0 & 1 \end{array} \right)}_{\text{3D Rotation}}}^{\text{Extrinsic Matrix}}$$

Figure from blog

Demo: WebGL



https://ksimek.github.io/2012/08/22/extrinsic/

Extrinsic Matrix from Camera Pose

Consider the fact:
The origin of camera coordinate system is the camera center in world coordinate system
The axis of camera coordinate system is the forward direction in world coordinate system

Thus, the camera pose is equivalent to the coordinate transformation from camera to world

$$R_C \triangleq R^{C \to W}, C \triangleq t^{C \to W}$$

We can invert $T^{C \to W}$ to compute $T^{W \to C}$

Inverse Rigid Transformation

$$
\begin{aligned}
\left[\begin{array}{c|c} \mathrm{R} & \boldsymbol{t} \\ \hline \mathbf{0} & 1 \end{array}\right] &= \left[\begin{array}{c|c} \mathrm{R_c} & \mathrm{C} \\ \hline \mathbf{0} & 1 \end{array}\right]^{-1} \\
&= \left[\left[\begin{array}{c|c} \mathrm{I} & \mathrm{C} \\ \hline \mathbf{0} & 1 \end{array}\right] \left[\begin{array}{c|c} \mathrm{R_c} & 0 \\ \hline \mathbf{0} & 1 \end{array}\right]\right]^{-1} && \text{(decomposing rigid transform)} \\
&= \left[\begin{array}{c|c} \mathrm{R_c} & 0 \\ \hline \mathbf{0} & 1 \end{array}\right]^{-1} \left[\begin{array}{c|c} \mathrm{I} & \mathrm{C} \\ \hline \mathbf{0} & 1 \end{array}\right]^{-1} && \text{(distributing the inverse)} \\
&= \left[\begin{array}{c|c} \mathrm{R_c^T} & 0 \\ \hline \mathbf{0} & 1 \end{array}\right] \left[\begin{array}{c|c} \mathrm{I} & -\mathrm{C} \\ \hline \mathbf{0} & 1 \end{array}\right] && \text{(applying the inverse)} \\
&= \left[\begin{array}{c|c} \mathrm{R_c^T} & -\mathrm{R_c^T C} \\ \hline \mathbf{0} & 1 \end{array}\right] && \text{(matrix multiplication)}
\end{aligned}
$$

Animation from blog

Camera Calibration

$$x = PX = K[R\ t]X$$

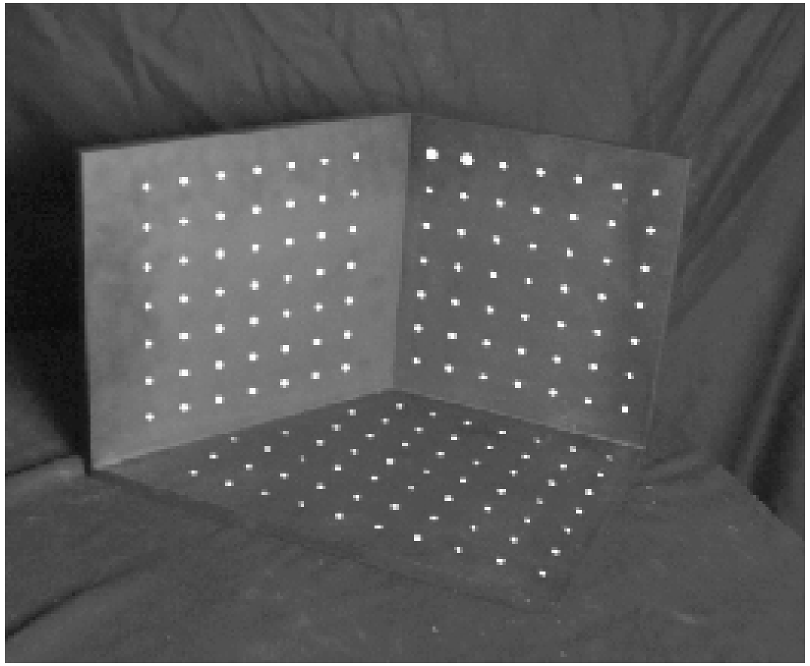
$$\begin{bmatrix} wu \\ wv \\ w \end{bmatrix} = \begin{bmatrix} * & * & * & * \\ * & * & * & * \\ * & * & * & * \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

How many degrees of freedom (DoF)?

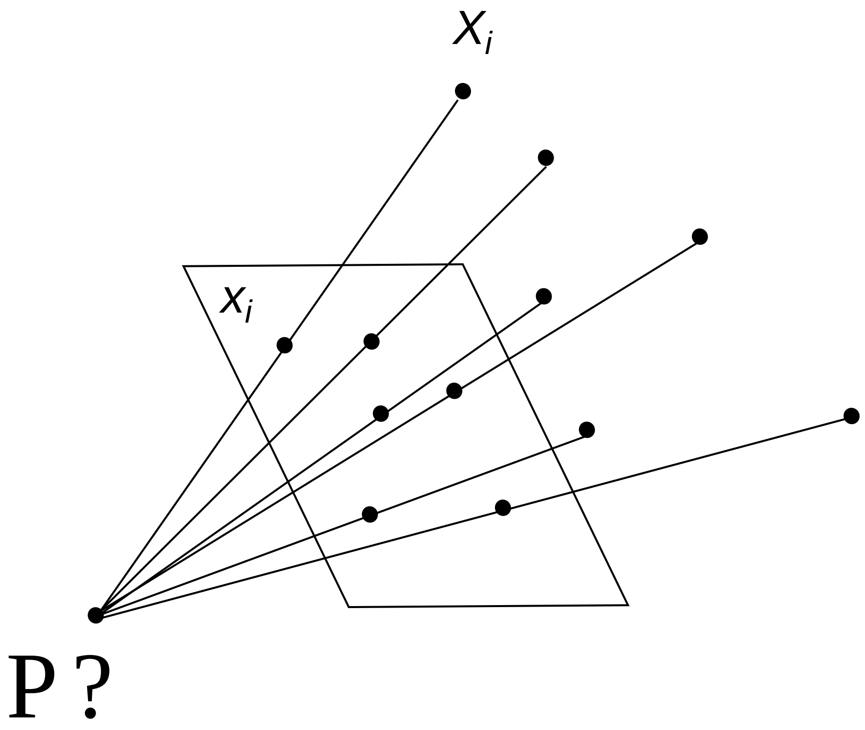
Source: Derek Hoiem

Camera Calibration

- Given n points with known 3D coordinates X_i and known image projections x_i, estimate the camera parameters



A calibration grid with known geometry



Camera Calibration

$$\begin{bmatrix} wu \\ wv \\ w \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \\ 1 \end{bmatrix}$$

Known Unknown Known

One pair of corresponding 3D point and 2D pixel provides an equation (2 constraints)

Direct Linear Transformation (DLT)

$$\begin{bmatrix} X_1 & Y_1 & Z_1 & 1 & 0 & 0 & 0 & 0 & -u_1X_1 & -u_1Y_1 & -u_1Z_1 & -u_1 \\ 0 & 0 & 0 & 0 & X_1 & Y_1 & Z_1 & 1 & -v_1X_1 & -v_1Y_1 & -v_1Z_1 & -v_1 \\ & & & & & & \vdots & & & & & \\ X_n & Y_n & Z_n & 1 & 0 & 0 & 0 & 0 & -u_nX_n & -u_nY_n & -u_nZ_n & -u_n \\ 0 & 0 & 0 & 0 & X_n & Y_n & Z_n & 1 & -v_nX_n & -v_nY_n & -v_nZ_n & -v_n \end{bmatrix} \begin{bmatrix} m_{11} \\ m_{12} \\ m_{13} \\ m_{14} \\ m_{21} \\ m_{22} \\ m_{23} \\ m_{24} \\ m_{31} \\ m_{32} \\ m_{33} \\ m_{34} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}$$

2n x 12

12 x1

Reorganize the equation with the format Ax=0

Least Square Solution

$$\min_{x} x^T A^T A x \text{ , s.t. } \left||\mathrm{x}|\right| = 1$$

Least square objective

$$L(x, \lambda) = x^T A^T A x - \lambda(x^T x - 1)$$

Lagrange multiplier

$$\frac{\partial L}{\partial x} = 2A^T A x - 2\lambda x = 0$$

$$A^T A x = \lambda x$$

x is the eigenvector with the smallest eigenvalue of $A^T A$

Pros and Cons of DLT

- Pros:
  - Easy to formulate and solve
  - No need for any initialization, which non-linear methods need

- Cons
  - Projection matrix is got, but decomposition is not
  - Can not impose constraints (like known focal length)
  - Does not minimize projection error

Decomposing Projection Matrix

$$P = K[R\ t] = [KR\ Kt]$$

$M \triangleq KR$, K is upper triangle and R is SO(3)

We can use RQ decomposition to find K, R

QR Decomposition

Any real square matrix A may be decomposed as

$$A = QR,$$

where Q is an orthogonal matrix (its columns are orthogonal unit vectors meaning $Q^\mathsf{T} = Q^{-1}$) and R is an upper triangular matrix (also called right triangular matrix). If A is invertible, then the factorization is unique if we require the diagonal elements of R to be positive.

$$A = QR$$

where:

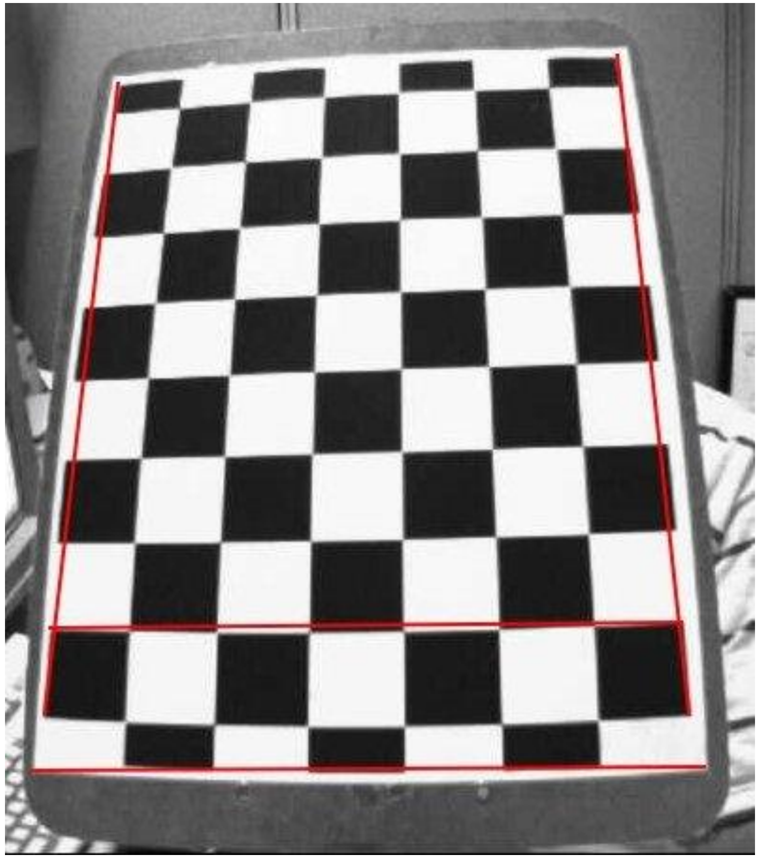
$$Q = \begin{bmatrix} \mathbf{e}_1 & \cdots & \mathbf{e}_n \end{bmatrix}$$

and

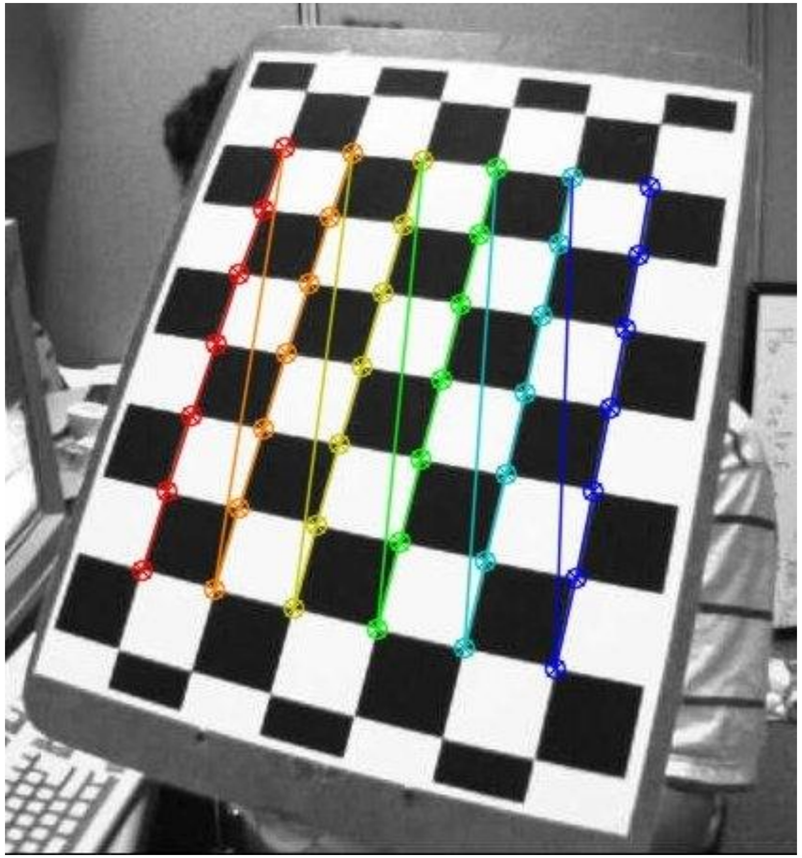
$$R = \begin{bmatrix} \langle \mathbf{e}_1, \mathbf{a}_1 \rangle & \langle \mathbf{e}_1, \mathbf{a}_2 \rangle & \langle \mathbf{e}_1, \mathbf{a}_3 \rangle & \cdots & \langle \mathbf{e}_1, \mathbf{a}_n \rangle \\ 0 & \langle \mathbf{e}_2, \mathbf{a}_2 \rangle & \langle \mathbf{e}_2, \mathbf{a}_3 \rangle & \cdots & \langle \mathbf{e}_2, \mathbf{a}_n \rangle \\ 0 & 0 & \langle \mathbf{e}_3, \mathbf{a}_3 \rangle & \cdots & \langle \mathbf{e}_3, \mathbf{a}_n \rangle \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \langle \mathbf{e}_n, \mathbf{a}_n \rangle \end{bmatrix}$$

Using the Gram–Schmidt process

Real-World Camera Calibration



Use checkboard as the known geometry



Find the corners

From [OpenCV tutorial]

Beyond Pinhole: Radial Distortion

- Common in wide-angle lenses
- Create non-linear terms in projection
- Usually handled by solving for non-linear terms and then correcting image

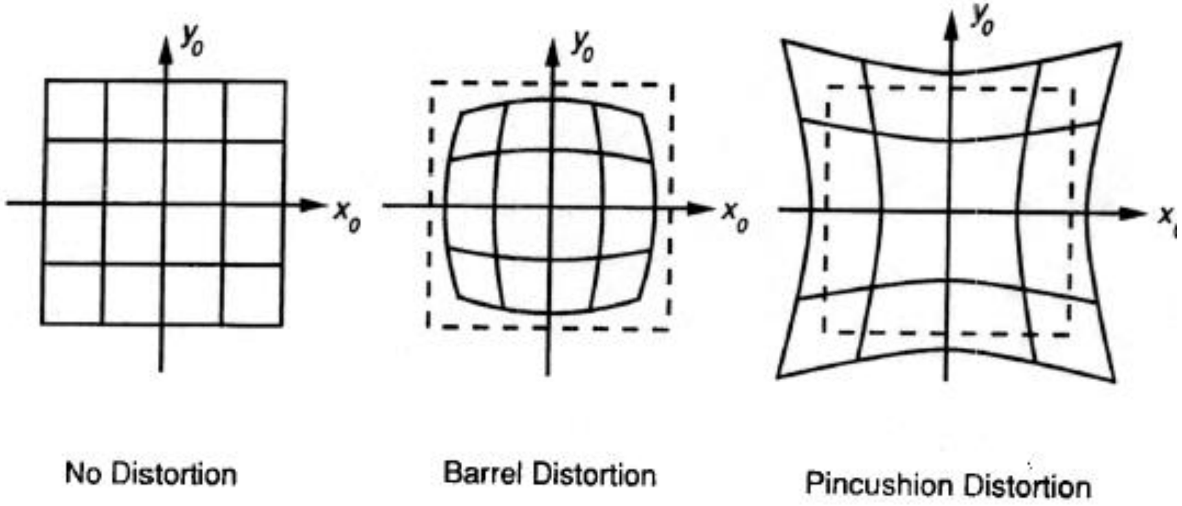


Image from Martin Habbecke

Real-World Camera Calibration

Undistort the image

From [OpenCV tutorial]