Matrix Computations

Chapter 2 Linear systems and LU decomposition

Section 2.3 Special Linear Systems and Other Decompositions

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LDM Decomposition

Given $\mathbf{A} \in \mathbb{R}^{n \times n}$, find matrices $\mathbf{L}, \mathbf{D}, \mathbf{M} \in \mathbb{R}^{n \times n}$ such that

$$\mathbf{A} = \mathbf{L}\mathbf{D}\mathbf{M}^T \qquad \text{(LDM decomposition)}$$

where

$\mathbf{L}$ is **unit** lower triangular

$\mathbf{D} = \mathrm{Diag}(d_1, \ldots, d_n)$

$\mathbf{M}$ is **unit** lower triangular

If $\mathbf{A} = \mathbf{L}\mathbf{U}$ is an LU decomposition, then the LDM decomposition uses the same $\mathbf{L}$ and sets

$$\mathbf{D} = \mathrm{Diag}(u_{11}, \ldots, u_{nn}), \qquad \mathbf{M} = \mathbf{U}^T\mathbf{D}^{-1}$$

The existence of LDM decomposition follows that of LU decomposition

Solving LDM Decomposition

Examine $\mathbf{A} = \mathbf{LDM}^T$ column by column. For each $j = 1, \ldots, n$,

$$\mathbf{A}(:, j) = \mathbf{Ae}_j = \mathbf{Lv}$$
$$\mathbf{v} = \mathbf{DM}^T \mathbf{e}_j$$

Example: Let n=4 and find $\mathbf{v}$ with $j = 3$

Solving LDM Decomposition (cont'd)

Observations: For $i, j = 1, \ldots, n$,

$$v_i = d_i m_{ji}$$

- For $i \geq j+1$, $v_i = 0$ because $m_{ji} = 0$
- For $i = j$, $v_j = d_j$ because $m_{jj} = 1$

Therefore, $\mathbf{A}(:, j) = \mathbf{Lv}$ can be partitioned as

$$\begin{bmatrix} \mathbf{A}(1:j, j) \\ \mathbf{A}(j+1:n, j) \end{bmatrix} = \begin{bmatrix} \mathbf{L}(1:j, 1:j) & \mathbf{0} \\ \mathbf{L}(j+1:n, 1:j) & \mathbf{L}(j+1:n, j+1:n) \end{bmatrix} \begin{bmatrix} \mathbf{v}(1:j) \\ \mathbf{0} \end{bmatrix}$$
$$= \begin{bmatrix} \mathbf{L}(1:j, 1:j)\mathbf{v}(1:j) \\ \mathbf{L}(j+1:n, 1:j)\mathbf{v}(1:j) \end{bmatrix}$$

Solving LDM Decomposition (cont'd)

It follows from the above equation that

$$\mathbf{A}(1:j,j) = \mathbf{L}(1:j,1:j)\mathbf{v}(1:j)$$
$$\mathbf{A}(j+1:n,j) = \mathbf{L}(j+1:n,1:j)\mathbf{v}(1:j)$$

Idea: Recursively find each column of **L**, each row of **M**, and each diagonal entry of **D**

For $j = 1 : n$

Step 1. Form $\mathbf{L}(1:j,1:j)$ using the columns $1,\ldots,j-1$ of **L** and $\mathbf{L}(j,j) = 1$

Step 2. Solve the linear system $\mathbf{A}(1:j,j) = \mathbf{L}(1:j,1:j)\mathbf{v}(1:j)$ for $\mathbf{v}(1:j)$

Step 3. Compute $\mathbf{L}(j+1:n,j)$ according to (not needed for $j = n$)

$$\mathbf{L}(j+1:n,j) = (\mathbf{A}(j+1:n,j) - \mathbf{L}(j+1:n,1:j-1)\mathbf{v}(1:j-1))/\mathbf{v}(j)$$

Step 4. Set $d_j = v_j$, $m_{ji} = v_i/d_i$ for all $i = 1,\ldots,j-1$

% Recall that $\mathbf{L}(1:j,j) = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}^T$ and $\mathbf{M}(j,j:n) = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}$

LDM Code

```
function [L,D,M]= LDMdecomposition(A)
n= size(A,1);
L= eye(n); d= zeros(n,1); M= eye(n);
v= zeros(n,1);
for j=1:n
     v(1:j)= ForwardSubstitution(L(1:j,1:j),A(1:j,j));
     % solve A(1 : j,j) = L(1 : j,1 : j)v(1 : j) using forward
substitution
     d(j)= v(j);
     for i=1:j-1,
         M(j,i)= v(i)'/d(i);
     end;
     L(j+1:n,j)= (A(j+1:n,j)-L(j+1:n,1:j-1)*v(1:j-1))/v(j);
end;
D= diag(d);
```

- Complexity: $\mathcal{O}(2n^3/3)$ (same as the previous LU code)

LDL Decomposition for Symmetric Matrices

For any real symmetric matrix $\mathbf{A}$, i.e., $\mathbf{A} \in \mathbb{S}^{n \times n}$,

$$\mathbf{A} = \mathbf{L}\mathbf{D}\mathbf{L}^T \qquad \text{(LDL decomposition)}$$

where $\mathbf{L}$ is **unit** lower triangular and $\mathbf{D} = \mathrm{Diag}(d_1, \ldots, d_n)$

Theorem

If $\mathbf{A} \in \mathbb{S}^{n \times n}$ is nonsingular, then its LDL decomposition is unique. In addition, if $\mathbf{A} = \mathbf{L}\mathbf{D}\mathbf{M}^T$ is the LDM decomposition, then $\mathbf{L} = \mathbf{M}$.

Solving LDL Decomposition

- In solving LDM decomposition, the key is to solve $\mathbf{A}(1:j,j) = \mathbf{L}(1:j,1:j)\mathbf{v}(1:j)$ for

$$\mathbf{v} = \mathbf{D}\mathbf{M}^T\mathbf{e}_j \;\Rightarrow\; v_i = d_i m_{ji}$$

  via forward substitution

- Now for LDL decomposition, we have $\mathbf{M} = \mathbf{L}$

$$v_i = d_i \ell_{ji}$$

- Finding $\mathbf{v}$ is much easier and no need for forward substitution
  - With the knowledge of the columns $1, \ldots, j-1$ of $\mathbf{L}$, we can easily find $v_i = d_i \ell_{ji}$, $i = 1, \ldots, j-1$
  - Then, find v_j by $v_j = \mathbf{A}(j,j) - \mathbf{L}(j, 1:j-1) * \mathbf{v}(1:j-1)$

LDL Code

```
function [L,D]= LDLdecomposition(A)
n= size(A,1);
L= eye(n); d= zeros(n,1); M= eye(n);
v= zeros(n,1);
for j=1:n
     v(1:j)= ForwardSubstitution(L(1:j,1:j),A(1:j,j));
     v(1:j-1)= L(j,1:j-1)'.*d(1:j-1);
     v(j)= A(j,j)- L(j,1:j-1)*v(1:j-1);
     d(j)= v(j);
     for i=1:j-1,
         M(j,i)= v(i)'/d(i);
     end;
     L(j+1:n,j)= (A(j+1:n,j)-L(j+1:n,1:j-1)*v(1:j-1))/v(j);
end;
D= diag(d);
```

- Complexity: $\mathcal{O}(n^3/3)$, half of LU or LDM

Diagonal Dominance

A matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ is said to be row diagonally dominant if

$$|a_{ii}| \geq \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ij}|, \quad \forall i = 1, \ldots, n$$

It is said to be column diagonally dominant if

$$|a_{ii}| \geq \sum_{\substack{j=1 \\ j\neq i}}^{n} |a_{ji}|, \quad \forall i = 1, \ldots, n$$

It is strictly row/column diagonally dominant if the above inequalities are strict

Diagonally dominant matrices may be singular (e.g., $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$)

LU for Diagonally Dominant Matrices

Theorem

If $\mathbf{A}$ *is nonsingular and column diagonally dominant, then it has an LU decomposition* $\mathbf{A} = \mathbf{LU}$ *and* $|\ell_{ij}| \leq 1$ *for all* i, j*.*

LU for Diagonally Dominant Matrices (cont'd)

LU for Diagonally Dominant Matrices (cont'd)

Positive Definite Matrices

A Matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ is said to be

- positive definite if $\mathbf{x}^T \mathbf{A} \mathbf{x} > 0$ for all **nonzero** $\mathbf{x} \in \mathbb{R}^n$
- positive semi-definite if $\mathbf{x}^T \mathbf{A} \mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$
- negative definite if $\mathbf{x}^T \mathbf{A} \mathbf{x} < 0$ for all **nonzero** $\mathbf{x} \in \mathbb{R}^n$
- negative semi-definite if $\mathbf{x}^T \mathbf{A} \mathbf{x} \leq 0$ for all $\mathbf{x} \in \mathbb{R}^n$
- indefinite if there exist $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ s.t. $(\mathbf{x}^T \mathbf{A} \mathbf{x})(\mathbf{y}^T \mathbf{A} \mathbf{y}) < 0$

Properties: For any positive definite $\mathbf{A} \in \mathbb{R}^{n \times n}$,

- $\mathbf{A}$ is nonsingular
- If $\mathbf{X} \in \mathbb{R}^{n \times q}$ has full column rank, then $\mathbf{X}^T \mathbf{A} \mathbf{X} \in \mathbb{R}^{q \times q}$ is positive definite
- All the principal submatrices are positive definite
- All the diagonal entries of $\mathbf{A}$ are positive
- $\mathbf{A}$ has an LU decomposition $\mathbf{A} = \mathbf{L}\mathbf{U}$ s.t. the diagonal entries of $\mathbf{U}$ are positive

Positive Definite Matrices (cont'd)

Given $\mathbf{A} \in \mathbb{R}^{n \times n}$, define its symmetric part as

$$\mathbf{T} = \frac{\mathbf{A} + \mathbf{A}^T}{2}$$

and its skew-symmetric part as

$$\mathbf{S} = \frac{\mathbf{A} - \mathbf{A}^T}{2}$$

Clearly,

$$\mathbf{A} = \frac{\mathbf{T} + \mathbf{S}}{2}$$

$\mathbf{A}$ is positive definite if and only if $\mathbf{T}$ is positive definite (That's why one mostly considers symmetric positive definite matrices)

Cholesky Decomposition for Positive Definite Matrices

Given a positive definite $\mathbf{A} \in \mathbb{S}^n$, there exists a unique lower triangular matrix $\mathbf{G} \in \mathbb{R}^{n \times n}$ with positive diagonal entries such that

$$\mathbf{A} = \mathbf{G}\mathbf{G}^T \qquad \text{(Cholesky decomposition)}$$

- Can be computed in $\mathcal{O}(n^3/3)$ (similar to LDL), no pivoting, numerically very stable

Banded Systems

A matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ has upper bandwidth q if $a_{ij} = 0 \ \forall j > i + q$ and lower bandwidth p if $a_{ij} = 0 \ \forall i > j + p$

Example: $\mathbf{A} \in \mathbb{R}^{5 \times 5}$ has upper bandwidth $q = 1$ and lower bandwidth $p = 2$

$$\mathbf{A} = \begin{bmatrix} \times & \times & 0 & 0 & 0 \\ \times & \times & \times & 0 & 0 \\ \times & \times & \times & \times & 0 \\ 0 & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times \end{bmatrix}$$

The triangular factors in $\mathbf{LU}$, $\mathbf{GG}^T$, and $\mathbf{LDL}^T$ are also banded $\Longrightarrow$ save a lot of computations

Banded LU Decomposition

Theorem

Suppose $\mathbf{A} \in \mathbb{R}^{n \times n}$ *has an LU decomposition* $\mathbf{A} = \mathbf{LU}$ *and has upper bandwidth* q *and lower bandwidth* p*. Then,* $\mathbf{U}$ *has upper bandwidth* q *and* $\mathbf{L}$ *has lower bandwidth* p*.*

Band LU Decomposition (cont'd)

Suppose $\mathbf{A} = \mathbf{LU}$ exists and $\mathbf{A}$ has upper bandwidth q and lower bandwidth p

```
for k=1:n-1
     for i=k+1:min(k+p,n)
        A(i,k)=A(i,k)/A(k,k)
    end
    for j=k+1:min(k+q,n)
         for i=k+1:min(k+p,n)
            A(i,j)=A(i,j)-A(i,k)*A(k,j)
        end
    end
end
```

Complexity: $\mathcal{O}(2npq)$ flops, much smaller than $\mathcal{O}(2n^3/3)$ when $n \gg p, q$

Solving Band Triangular Systems

Solving $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A} = \mathbf{LU}$ exists and

$\mathbf{L}$ is unit lower triangular with lower bandwidth p

$\mathbf{U}$ is **nonsingular** upper triangular with upper bandwidth q

1. Solve $\mathbf{Lz} = \mathbf{b}$ for $\mathbf{z}$ using band forward substitution

```
for j=1:n
    for i=j+1:min(j+p,n)      % L(i,j)=0 for i>j+p
        b(i)=b(i)-L(i,j)*b(j);
    end
end     % Overwrite b with z
```

Complexity: $\mathcal{O}(2np) \ll \mathcal{O}(n^2)$ if $p \ll n$

Solving Band Triangular Systems (cont'd)

2. Solve $\mathbf{Ux} = \mathbf{z}$ for $\mathbf{x}$ using band backward substitution

```
for j=n:-1:1      % b is z after applying band
forward substitution
    b(j)=b(j)/U(j,j);
    for i=max(1,j-q):j-1      % U(i,j)=0 for j>i+q
        b(i)=b(i)-U(i,j)*b(j);
    end
end      % Overwrite b with x
```

Complexity: $\mathcal{O}(2nq) \ll \mathcal{O}(n^2)$ if $q \ll n$

Read Chapter 4 of textbook for more on special linear systems and their decompositions