Matrix Computations
Chapter 4: Eigenvalues, Eigenvectors, and Eigendecomposition

Section 4.2 Schur Decomposition

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Schur Decomposition

Theorem

Let $\mathbf{A} \in \mathbb{C}^{n \times n}$ with eigenvalues $\lambda_1, \ldots, \lambda_n \in \mathbb{C}$. The matrix $\mathbf{A}$ admits a decomposition

$$\mathbf{A} = \mathbf{U}\mathbf{T}\mathbf{U}^H$$

for some unitary $\mathbf{U} \in \mathbb{C}^{n \times n}$ and some upper triangular $\mathbf{T} \in \mathbb{C}^{n \times n}$ with $t_{ii} = \lambda_i$ for all i. If $\mathbf{A}$ is real and $\lambda_1, \ldots, \lambda_n$ are all real, $\mathbf{U}$ and $\mathbf{T}$ can be taken as real.

- The above decomposition is called the Schur decomposition
- Suppose $\mathbf{A} = \mathbf{U}\mathbf{T}\mathbf{U}^H$ for some unitary $\mathbf{U}$ and upper triangular $\mathbf{T}$, but it's unknown whether $t_{ii} = \lambda_i$. Indeed, $t_{ii} = \lambda_i$ has to be true:

$$\det(\lambda \mathbf{I} - \mathbf{A}) = \det(\lambda \mathbf{I} - \mathbf{T}) = \prod_{i=1}^{n} (\lambda - t_{ii})$$

- Any square matrix is similar to an upper triangular matrix whose diagonal entries are its eigenvalues and the "triangularizer" is unitary

Proof of Schur Decomposition

Lemma

Let $\mathbf{X} \in \mathbb{C}^{n \times n}$ *be block upper triangular in the form of*

$$\mathbf{X} = \begin{bmatrix} \mathbf{X}_{11} & \mathbf{X}_{12} \\ \mathbf{0} & \mathbf{X}_{22} \end{bmatrix}$$

with $\mathbf{X}_{11} \in \mathbb{C}^{k \times k}$, $\mathbf{X}_{22} \in \mathbb{C}^{(n-k)\times(n-k)}$, $0 \leq k < n$. *There exists a unitary* $\mathbf{U} \in \mathbb{C}^{n \times n}$ *s.t.*

$$\mathbf{U}^H \mathbf{X} \mathbf{U} = \begin{bmatrix} \mathbf{X}_{11} & \mathbf{Y}_{12} \\ \mathbf{0} & \mathbf{Y}_{22} \end{bmatrix}, \quad \mathbf{Y}_{22} = \begin{bmatrix} \bar{\lambda} & \times \\ \mathbf{0} & \times \end{bmatrix} \in \mathbb{C}^{(n-k)\times(n-k)}, \ \bar{\lambda} \in \mathbb{C}$$

Proof of lemma:

Proof of Schur Decomposition (cont'd)

Proof of Schur Decomposition (cont'd)

Computations of Schur Decomposition

- The proof of Schur Decomposition indicates how to compute the Schur factors $\mathbf{U}$ and $\mathbf{T}$
- From the lemma in the proof, we need two sub-algorithms to construct $\mathbf{U}$ and $\mathbf{T}$
  - An algorithm for computing an eigenvector of a given matrix (the power method, will be studied later)
  - An algorithm that finds a unitary matrix $\mathbf{Q}$ s.t. its first column is given (QR decomposition)
- There are other computationally more efficient methods for computing the Schur factors (key: QR decomposition)

Discussion

- The Schur decomposition is a powerful tool
- For example, we can use it to show that for *any* square $\mathbf{A}$ (with or without eigendecomposition), $\det(\mathbf{A}) = \prod_{i=1}^{n} \lambda_i$, $\mathrm{tr}(\mathbf{A}) = \sum_{i=1}^{n} \lambda_i$
- We can also use it to prove the convergence of the power method (later) when eigendecomposition does not exist
- An enhancement of the Schur decomposition: Every square matrix $\mathbf{A}$ is also similar to a block diagonal (indeed upper triangular and tri-diagonal) matrix $\mathbf{J}$ called Jordan canonical form

$$\mathbf{A} = \mathbf{S}\mathbf{J}\mathbf{S}^{-1}, \quad \mathbf{S} \text{ is nonsingular}$$

  - We can apply the Schur decomposition to the proof of Jordan canonical form by showing that the Schur factor $\mathbf{T}$ is similar to $\mathbf{J}$ (non-trivial)

A Consequence of Schur Decomposition

Proposition

Let $\mathbf{A} \in \mathbb{C}^{n\times n}$*. For any* $\varepsilon > 0$*, there exists a matrix* $\tilde{\mathbf{A}} \in \mathbb{C}^{n\times n}$ *s.t. the* n *eigenvalues of* $\tilde{\mathbf{A}}$ *are distinct and*

$$\|\mathbf{A} - \tilde{\mathbf{A}}\|_F^2 \leq \varepsilon.$$

Implication: For any square $\mathbf{A}$, we can always find $\tilde{\mathbf{A}}$ that is arbitrarily close to $\mathbf{A}$ and admits an eigendecomposition

Proof (construction of $\tilde{\mathbf{A}}$):

- Let $\mathbf{A} = \mathbf{U}\mathbf{T}\mathbf{U}^H$ be the Schur decomposition of $\mathbf{A}$. Let $\mathbf{D} = \mathrm{Diag}(d_1, \ldots, d_n)$ where $d_1, \ldots, d_n$ are chosen such that (1) $|d_i| \leq \left(\frac{\varepsilon}{n}\right)^{1/2}$ for all i and (2) $t_{11} + d_1, \ldots, t_{nn} + d_n$ are distinct
- Let $\tilde{\mathbf{A}} = \mathbf{U}(\mathbf{T} + \mathbf{D})\mathbf{U}^H$
- We have $\|\mathbf{A} - \tilde{\mathbf{A}}\|_F^2 = \|\mathbf{U}\mathbf{D}\mathbf{U}^H\|_F^2 = \|\mathbf{D}\|_F^2 \leq \varepsilon$