Ch.4 *The Continuous-Time Fourier Transform (CTFT)*

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Part II *Fourier Transform for Periodic Signals*

Fourier Transform for Periodic Signal

- A period signal can be represented by a FS, but also a FT:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \qquad x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}\,d\omega$$

- Consider $x(t)$ and its FT, $X(j\omega)$. Assume $X(j\omega) = 2\pi\delta(\omega - \omega_0)$, find $x(t)$.

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi \cdot \delta(\omega - \omega_0)e^{j\omega t}\,d\omega = e^{j\omega_0 t}$$

Fourier Transform for Periodic Signal

- Now for more general case

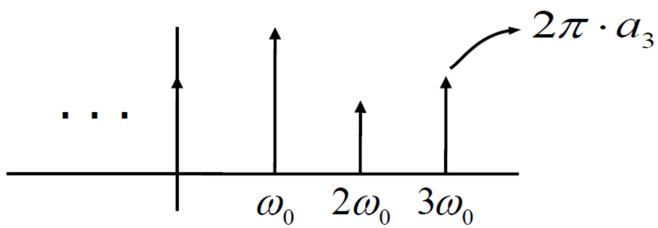
$$X(j\omega) = \sum_{k=-\infty}^{\infty} a_k 2\pi\delta(\omega - k\omega_0) \rightarrow x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

  exactly Fourier Series representation of a periodic signal.

- We can find the FT for a periodic signal by

$$x(t) \xleftrightarrow{\mathcal{FS}} a_k \quad \Rightarrow \quad X(j\omega) = \sum_{k=-\infty}^{\infty} a_k 2\pi\delta(\omega - k\omega_0)$$

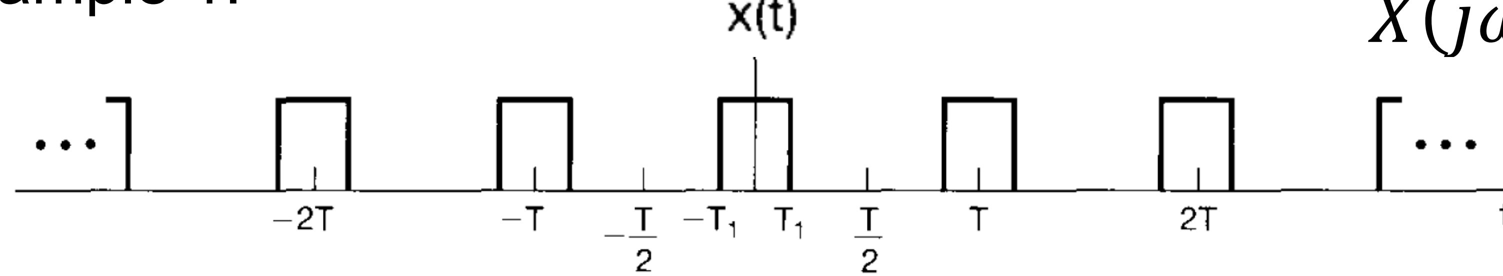
- Note: If $x(t)$ is periodic with period T $\rightarrow X(j\omega)$ is discrete, with frequency spacing $\omega_0 = \frac{2\pi}{T}$



Fourier Transform for Periodic Signal

- Example 1:

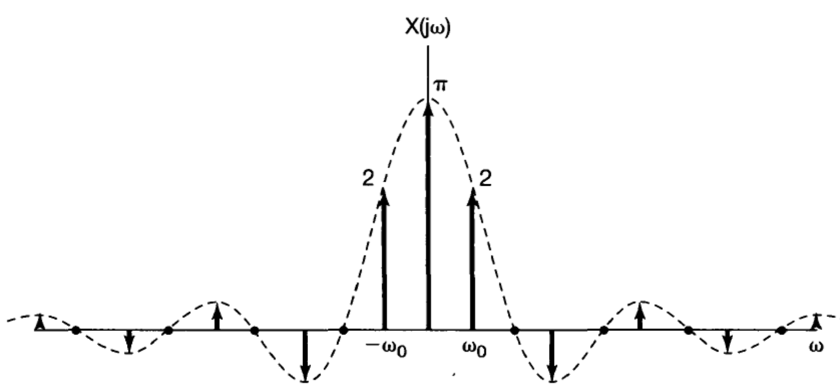
$X(j\omega)$?



- Solution:

$$a_k = \frac{\sin(k\omega_0 T_1)}{\pi k}$$

$$X(j\omega) = \sum_{k=-\infty}^{\infty} a_k 2\pi\delta(\omega - k\omega_0) = \sum_{k=-\infty}^{\infty} \frac{2\sin(k\omega_0 T_1)}{k}\delta(\omega - k\omega_0)$$



Fourier Transform for Periodic Signal

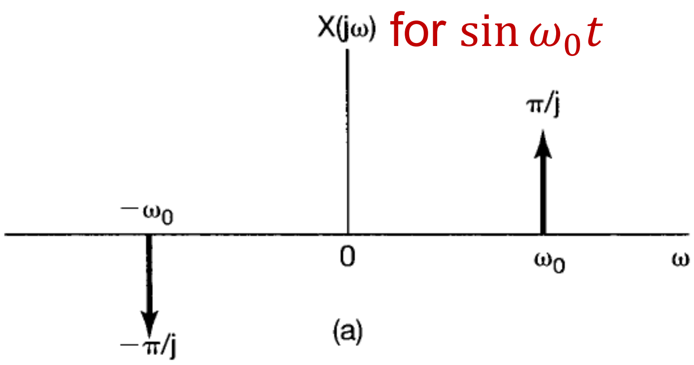
- Example 2: $x_1(t) = \sin\omega_0 t$ and $x_2(t) = \cos\omega_0 t$
  Please find $X_1(j\omega)$ and $X_2(j\omega)$.

- Solution:

$$x_1(t) = \sin\omega_0 t = \frac{1}{2j}e^{j\omega_0 t} - \frac{1}{2j}e^{-j\omega_0 t}$$

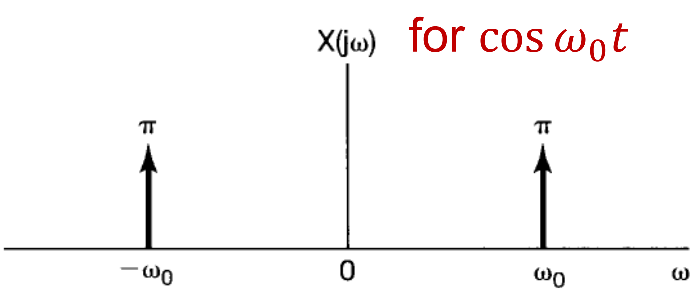
$$X_1(j\omega) = \sum_{k=-\infty}^{\infty} a_k 2\pi\delta(\omega - k\omega_0)$$

$$= \frac{\pi}{j}\delta(\omega - \omega_0) - \frac{\pi}{j}\delta(\omega + \omega_0)$$



$$x_2(t) = \cos\omega_0 t = \frac{1}{2}e^{j\omega_0 t} + \frac{1}{2}e^{-j\omega_0 t}$$

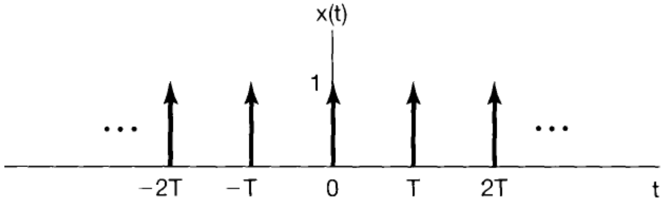
$$X_2(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$



Fourier Transform for Periodic Signal

- Example 3: $x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$. Please find $X(j\omega)$.

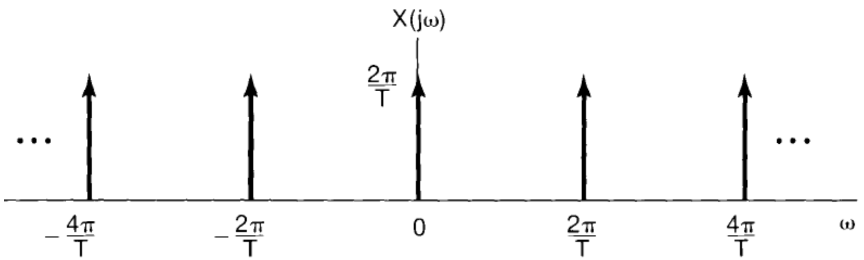
- Solution:



$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t) e^{-jk\omega_0 t} dt = \frac{1}{T}$$

$$X(j\omega) = \sum_{k=-\infty}^{\infty} a_k 2\pi\delta(\omega - k\omega_0)$$

$$= \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta(\omega - \frac{2k\pi}{T})$$



Summary

- Fourier Transform for Periodic Signals

- Reference in textbook:
  - 4.2