EE160 Control Theory

Lab 4 Frequency-Domain Analysis of Linear Systems

Objectives

1. Master the method of frequency domain analysis using Bode diagram (simulation and measurement).
2. Understand the relationship between the transfer function of the system and the Bode plot.

Time-domain analysis find out the system response by solving the system's differential equations, and then analyze the system based on the system response. While for high-order or more complex systems, it is difficult to solve the differential equations. Frequency-domain analysis analyze and design control systems by studying the behavior of the system in frequency domain. It can predict the system's performance without solving differential equations, while also provide methods to achieve the desired performance specifications by adjusting the system parameters.

When doing frequency-domain analysis, sinusoidal signals of varying frequencies are applied at the input end of the system, and the output signals are detected at the output end. For a Linear Time-Invariant (LTI) system, the output signals share the same frequency with the input signals, but differ in amplitude and phase. By measuring the amplitude ratio and the phase difference between the output and input signals, we can determine the system's frequency characteristics.

In frequency domain, the amplitudes and phases are functions of the frequency ω. So the frequency characteristics $G(j\omega)$ of the system can be represented as:

$$G(j\omega) = A(\omega)e^{j\varphi(\omega)}$$

$$A(\omega) = |G(j\omega)|$$

$$\varphi(\omega) = \angle G(j\omega)$$

In addition, there is a very simple conversion relationship between the frequency characteristics $G(j\omega)$ and the transfer function $G(s)$ of the system, for general LTI systems it is:

$$G(s)|_{s=j\omega} = G(j\omega)$$

The frequency characteristics of a system are often represented by Nyquist diagrams (polar coordinate diagrams), Bode plots (logarithmic coordinate diagrams), or Nichols charts (logarithmic amplitude-phase diagrams) etc. In this course, we will focus on Bode plots.

1 Bode Plots Introduction

The Bode plot is a widely used graphical representation in control system analysis, primarily used to describe the frequency response of linear time-invariant (LTI) systems. It consists of two parts: the logarithmic amplitude-frequency plot and the phase-frequency plot. Both horizontal axes are $\omega(rad/s)$, described in terms of tenfold frequency. The vertical axis of the logarithmic amplitude-frequency plot is described in $20logA(\omega)$, denoted as $L(\omega)$, with units in decibels (dB). The vertical axis of the phase-frequency plot is described in $\varphi(\omega)$, with units in degrees °, and the axes are uniformly distributed. Refer to Figure 1 for details.

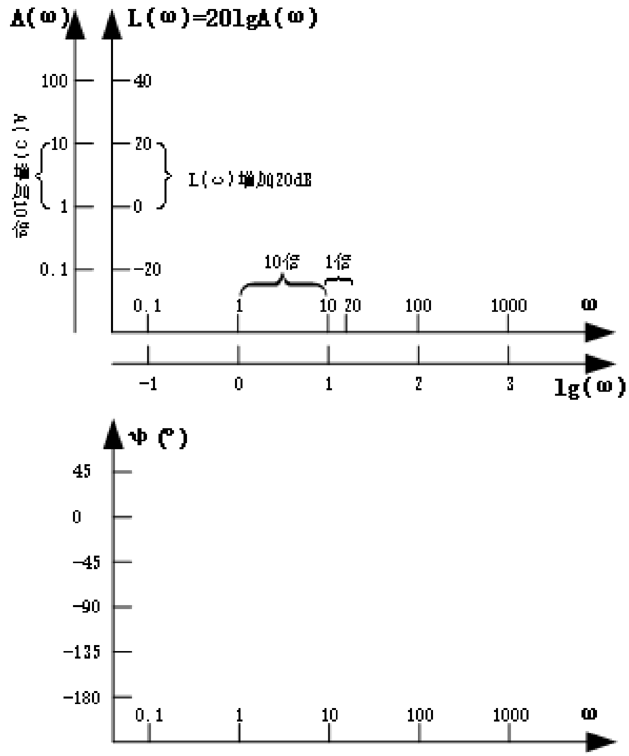

Figure 1: Bode Plot Coordinate

In the Bode plot coordinate system, it can be observed that the frequency ω is not uniformly distributed, while the corresponding $lg\omega$ is uniformly distributed. The frequency axis is transformed from 1ω to 10ω, which is referred as a "decade". The term "dec" is used to denote a tenfold increase in frequency. Points with the same frequency ratio are evenly spaced on the horizontal axis, such as 0.1, 1, 10, 100, 1000, etc. For the logarithmic amplitude-frequency plot, the vertical axis is labeled with the logarithmic values, while the horizontal axis is labeled with the natural values of the frequency, not the logarithmic values. Therefore, the logarithmic frequency characteristic plot is also known as a semi-logarithmic coordinate plot, or a Bode plot.

2 Bode Plot of Basic Control Unit

The transfer functions of basic control units have been provided in Lab2, and this chapter will not derive them again. The frequency characteristics $G(j\omega)$ can be simply represented as:

$$G(j\omega) = G(s)|_{s=j\omega}$$

2.1 Proportional Control

The frequency characteristics of proportional control is:

$$G(j\omega) = K$$

so we have:

$$\begin{cases} A(\omega) = |G(j\omega)| = K \\ \varphi(\omega) = \angle G(j\omega) = 0^\circ \end{cases} \Rightarrow \begin{cases} L(\omega) = 20lgA(\omega) = 20lgK \\ \varphi(\omega) = 0^\circ \end{cases} \tag{1}$$

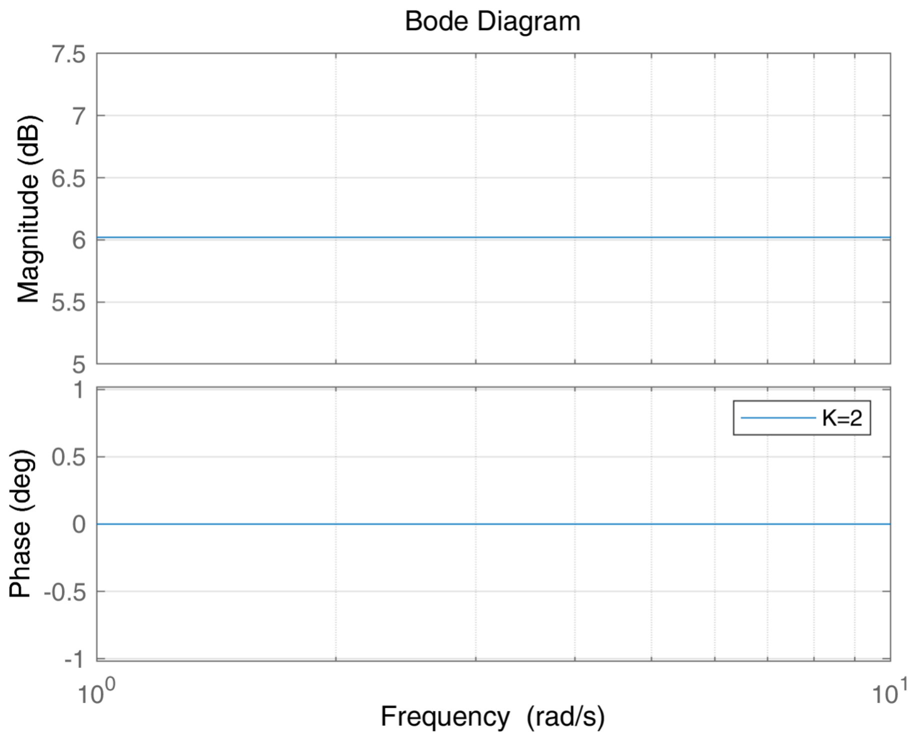

Figure 2: Proportional Control

The Bode show in Figure 2.

2.2 Integral Control

The frequency characteristics of integral control is:

$$G(j\omega) = \frac{1}{j\omega T} = \frac{1}{\omega T} \cdot e^{j(-\frac{\pi}{2})}$$

so we have:

$$\begin{cases} A(\omega) = |G(j\omega)| = \frac{1}{\omega T} \\ \varphi(\omega) = \angle G(j\omega) = -90^\circ \end{cases} \Rightarrow \begin{cases} L(\omega) = 20lgA(\omega) = 20lg(\frac{1}{\omega T}) = -20lg(T) - 20lg(\omega) \\ \varphi(\omega) = -90^\circ \end{cases} \tag{2}$$

The Bode show in Figure 3.

2.3 Derivative Control

The frequency characteristics of derivative control is:

$$G(j\omega) = j\omega T = \omega T \cdot e^{j(\frac{\pi}{2})}$$

so we have:

$$\begin{cases} A(\omega) = |G(j\omega)| = \omega T \\ \varphi(\omega) = \angle G(j\omega) = 90^\circ \end{cases} \Rightarrow \begin{cases} L(\omega) = 20lgA(\omega) = \omega T = 20lg(T) + 20lg(\omega) \\ \varphi(\omega) = 90^\circ \end{cases} \tag{3}$$

The Bode show in Figure 4.

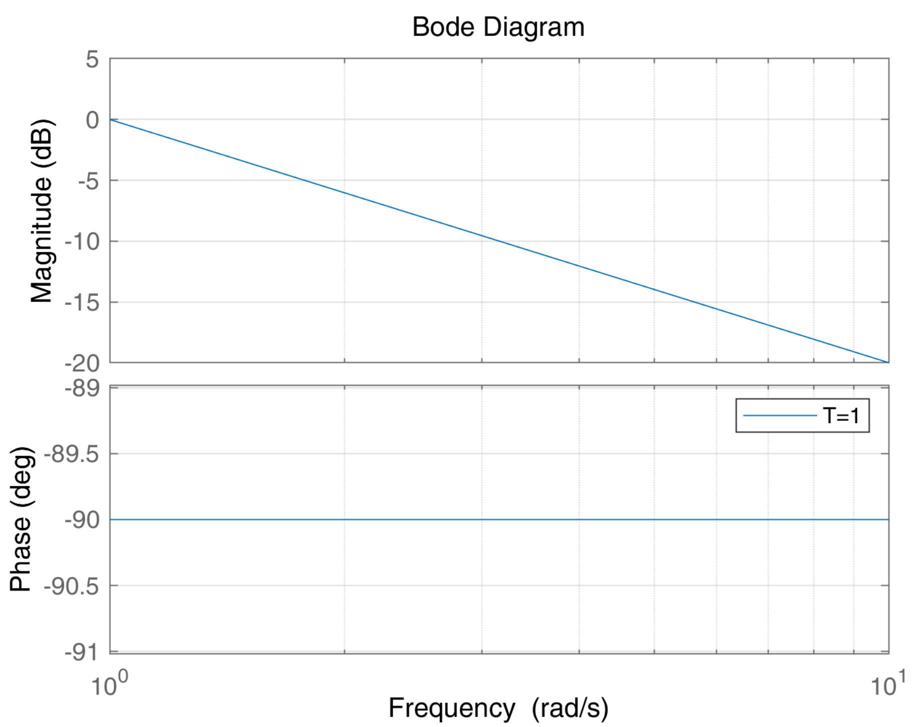

Figure 3: Integral Control

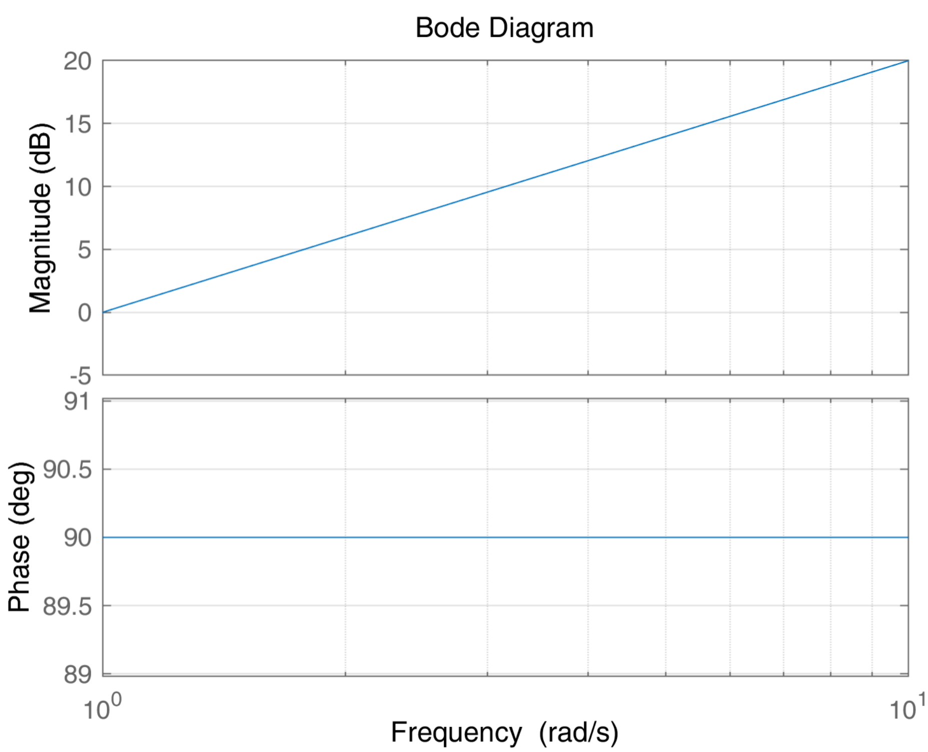

Figure 4: Derivative Control

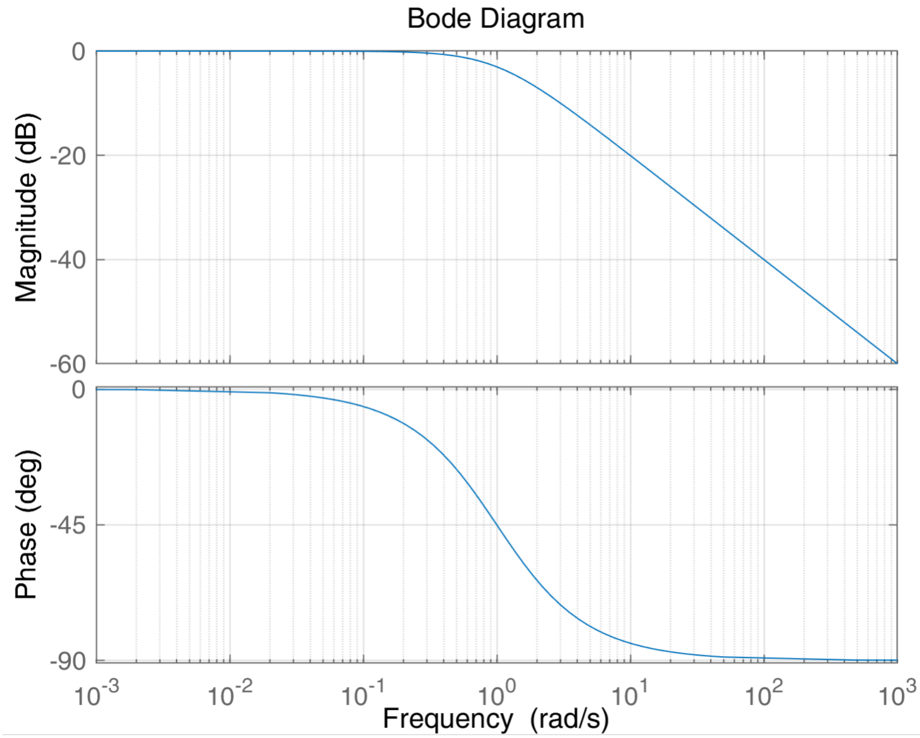

Figure 5: Inertia Control

2.4 Inertia Control

The frequency characteristics of inertia control is:

$$G(j\omega) = \frac{1}{j\omega T + 1} = \frac{1 - j\omega T}{1 + (\omega T)^2} = \frac{1}{\sqrt{1 + (\omega T)^2}} \cdot e^{j(-arctan(\omega T))}$$

so we have:

$$\begin{cases} A(\omega) = |G(j\omega)| = \frac{1}{\sqrt{1+(\omega T)^2}} \\ \varphi(\omega) = \angle G(j\omega) = -arctan(\omega T) \end{cases} \Rightarrow \begin{cases} L(\omega) = 20lgA(\omega) = -20lg(\sqrt{1 + (\omega T)^2}) \\ \varphi(\omega) = -arctan(\omega T) \end{cases} \tag{4}$$

Based on the different values of ω, $L(\omega)$ can be discussed in three segments.

$$\begin{cases} L(\omega) = -20lg(1) = 0, & \omega \to 0 \\ L(\omega) = -20lg(\sqrt{2}) = -3, & \omega = \frac{1}{T} \\ L(\omega) = -20lg(\omega T), & \omega \to \infty \end{cases}$$

The Bode show in Figure 5.

2.5 Proportional-Derivative Control

The frequency characteristics of proportional-derivative control is:

$$G(j\omega) = j\omega T + 1 = \sqrt{1 + (\omega T)^2} \cdot e^{j(arctan(\omega T))}$$

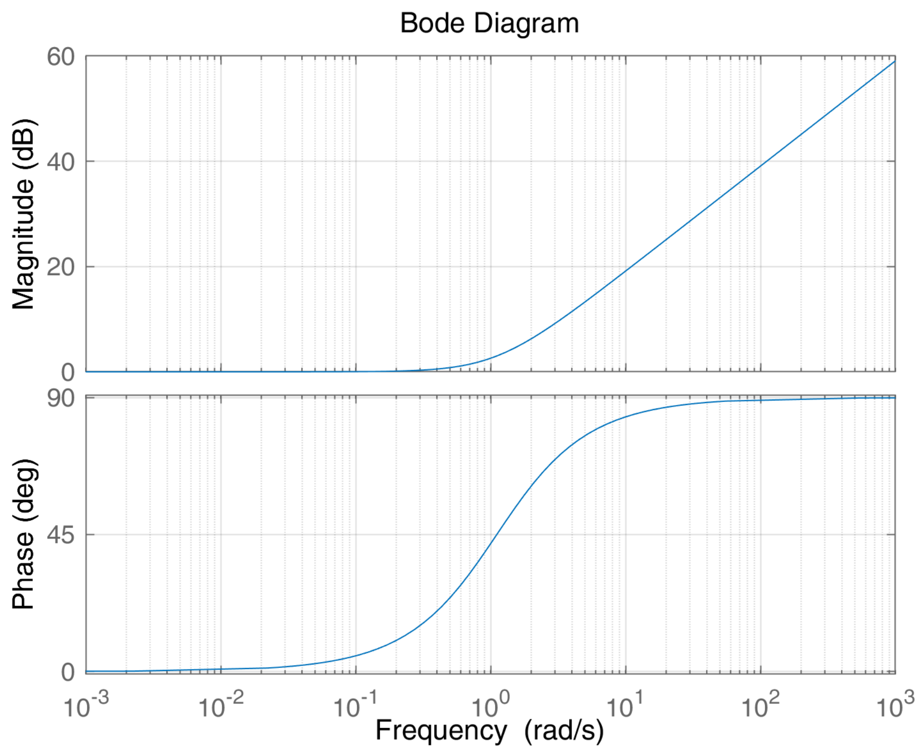

Figure 6: Proportional-Derivative Control

so we have:

$$\begin{cases} A(\omega) = |G(j\omega)| = \sqrt{1+(\omega T)^2} \\ \varphi(\omega) = \angle G(j\omega) = arctan(\omega T) \end{cases} \Rightarrow \begin{cases} L(\omega) = 20lgA(\omega) = 20lg(\sqrt{1+(\omega T)^2}) \\ \varphi(\omega) = arctan(\omega T) \end{cases} \tag{5}$$

Based on the different values of ω, $L(\omega)$ can be discussed in three segments.

$$\begin{cases} L(\omega) = 20lg(1) = 0, & \omega \to 0 \\ L(\omega) = 20lg(\sqrt{2}) = 3, & \omega = \frac{1}{T} \\ L(\omega) = 20lg(\omega T), & \omega \to \infty \end{cases}$$

The Bode show in Figure 6.

3 Bode Plot Analysis

The open-loop transfer function of a system can typically be represented as several subsystems connected in series, that is:

$$G(s) = G_1(s)G_2(s)\cdots G_n(s)$$

the frequency of which is like:

$$\begin{aligned} G(j\omega) &= G_1(j\omega)G_2(j\omega)\cdots G_n(j\omega) \\ &= A_1(\omega)e^{j\varphi_1(\omega)} \cdot A_2(\omega)e^{j\varphi_2(\omega)} \cdots A_n(\omega)e^{j\varphi_n(\omega)} \\ &= \prod_{i=1}^{n} A_i(\omega)e^{j\sum_{i=1}^{n}\varphi_i(\omega)} \\ &= A(\omega)e^{j\varphi(\omega)} \end{aligned}$$

so we get:

$$L(\omega) = 20lgA(\omega) = 20lgA_1(\omega) \cdot 20lgA_2(\omega) \cdots 20lgA_n(\omega)$$
$$\varphi(\omega) = \varphi_1(\omega) + \varphi_2(\omega) \cdots \varphi_n(\omega)$$

Therefore, when drawing the open-loop logarithmic frequency characteristic curve of a system, we can first transform the open-loop transfer function into the product of typical links. Then, draw the logarithmic amplitude and phase frequency curves for each typical link. Finally, sum the logarithmic amplitude and phase frequency curves of the links.

For example, here is a transfer function, try to draw its Bode plot.

$$G(s) = \frac{10 \cdot (0.1s + 1)}{s \cdot (2s + 1)} = 10 \cdot \frac{1}{s} \cdot (0.1s + 1) \cdot \frac{1}{2s + 1} = G_1(s) \cdot G_2(s) \cdot G_3(s) \cdot G_4(s)$$

$$G_1(j\omega) = 10 \Rightarrow \begin{cases} L1(\omega) = 10lg10 = 20 \\ \varphi_1 = 0 \end{cases}$$

$$G_2(j\omega) = \frac{1}{j\omega} \Rightarrow \begin{cases} L2(\omega) = -20log\omega \\ \varphi_2 = -90^\circ \end{cases}$$

$$G_3(j\omega) = (j0.1\omega + 1) \Rightarrow \begin{cases} L3(\omega) = 20lg\sqrt{1 + (0.1\omega)^2} \\ \varphi_3 = arctan(0.1\omega) \end{cases}$$

$$G_4(j\omega) = \frac{1}{j2\omega + 1} \Rightarrow \begin{cases} L4(\omega) = -20lg\sqrt{1 + (2\omega)^2} \\ \varphi_4 = -arctg(2\omega) \end{cases}$$

The Bode plot is shown in Figure 7.

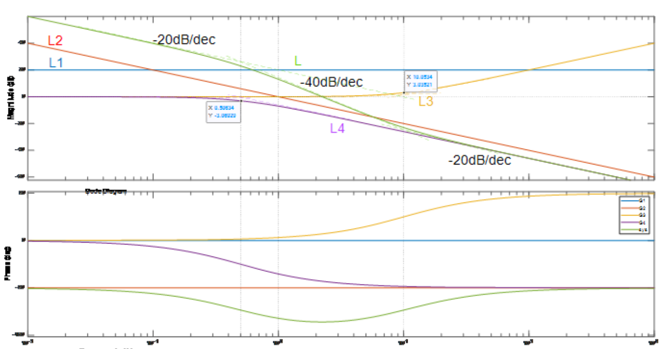

Figure 7: Bode Plot for $G(s)$

Similarly, by analyzing the Bode plot, one can determine the components of the system and thereby deduce the system's transfer function.