Ch.3 *Fourier Series Representation of Periodic Signals*

Lecturer: Yijie Mao

Outline

- The Response of LTI Systems to Complex Exponentials
- Fourier Series Representation of Continuous-Time Periodic Signals
- Fourier Series Representation of Discrete-Time Periodic Signals
- Fourier Series and LTI Systems

Part I *The Response of LTI Systems to Complex Exponentials*

Outline

- The Response of Continuous-Time LTI systems to Complex Exponentials

- The Response of Discrete-Time LTI systems to Complex Exponentials

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- The Response of Continuous-Time LTI systems to Complex Exponentials
- The Response of Discrete-Time LTI systems to Complex Exponentials

Recall Chapter 2

- Objective: characterization of a LTI system

$$x(t)/x[n] \longrightarrow \boxed{\text{LTI}} \longrightarrow y(t)/y[n]$$

- Previously, we use the weighted sum (integral) of shifted impulses to represent an input and then derive the convolution sum (integral).

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau \quad \rightarrow \quad y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau$$

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \quad \rightarrow \quad y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

Objective of Chapter 3

- $\delta(t)$ is not the only one. In general, a basic signal should satisfy
  - It can be used to construct a broad and useful class of signals
  - The response of an LTI system to the basic signal is simple

- In this chapter:
  - We use different basic signal, the complex exponential, to represent the input.

- Why we use complex exponential?

Eigenfunction of LTI System

- A signal for which the system output is just a constant (possibly complex) times the input is referred to as an eigenfunction of the system.

$$f(t) \longrightarrow \boxed{H} \longrightarrow C\cdot f(t)$$

C:constant $\rightarrow$ the eigenvalue

- Objective:

  The output to an input $x(t)$ can be found easily if $x(t)$ can be expressed as weighted sum of the eigenfunctions.

Eigenfunction of CT LTI Systems

- Consider an input $x(t) = e^{st}$ and a CT LTI system:

$$e^{st} \longrightarrow \boxed{\text{LTI}} \longrightarrow y(t) = ?$$

$$y(t) = \int_{-\infty}^{\infty} h(\tau)e^{s(t-\tau)}d\tau = e^{st}\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau$$

- Let $\int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau = H(s)$, we have $y(t) = H(s)e^{st}$.

$$e^{st} \rightarrow H(s)e^{st}$$

  - e^{st} is an eigenfunction of the system
  - For a specific value s, $H(s)$ is the corresponding eigenvalue

The Response of Continuous-Time LTI systems to Complex Exponentials

$$e^{st} \longrightarrow \boxed{\text{LTI}} \longrightarrow \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau\, e^{st} = H(s)e^{st}$$

- If $x(t) = a_1 e^{s_1 t} + a_2 e^{s_2 t} + a_3 e^{s_3 t}$, based on eigenfunction property and superposition property, the the response is
$$y(t) = a_1 H(s_1) e^{s_1 t} + a_2 H(s_2) e^{s_2 t} + a_1 H(s_3) e^{s_3 t}$$

- Generally, for a CT LTI system, if the input is a linear combination of complex exponentials, then

$$x(t) = \sum_k a_k e^{s_k t} \longrightarrow \boxed{\text{LTI}} \longrightarrow y(t) = \sum_k a_k H(s_k) e^{s_k t}$$

The Response of LTI Systems to Complex Exponentials

- Example. For a LTI system $y(t) = x(t-3)$, determine $H(s)$.

The Response of LTI Systems to Complex Exponentials

- Example. For a LTI system $y(t) = x(t-3)$,
  if $x(t) = \cos(4t) + \cos(7t)$, $y(t) =$?

Outline

- The Response of Continuous-Time LTI systems to Complex Exponentials
- The Response of Discrete-Time LTI systems to Complex Exponentials

Eigenfunction of DT LTI Systems

- Consider an input $x[n] = z^n$ and a DT LTI system:

$$z^n \longrightarrow \boxed{\text{LTI}} \longrightarrow y[n] = ?$$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k] z^{n-k} = \sum_{k=-\infty}^{\infty} h[k] z^{-k} \, z^n$$

- Let $\sum_{k=-\infty}^{\infty} h[k] z^{-k} = H(z)$, we have $y[n] = H(z) z^n$.

$$z^n \rightarrow H(z) z^n$$

  - z^n is an eigenfunction of the system
  - For a specific value z, $H(z)$ is the corresponding eigenvalue

The Response of Discrete-Time LTI systems to Complex Exponentials

$$z^n \longrightarrow \boxed{\text{LTI}} \longrightarrow \sum_{k=-\infty}^{\infty} h[k] z^{-k} \, z^n = H(z) z^n$$

$$x[n] = \sum_k a_k z_k^n \longrightarrow \boxed{\text{LTI}} \longrightarrow y[n] = \sum_k a_k H(z_k) z_k^n$$

Fourier Analysis

- For Fourier analysis, we consider
  - **Continuous-time**: purely imaginary $s = j\omega$: $e^{j\omega t}$

$$e^{j\omega t} \longrightarrow \boxed{\text{LTI}} \longrightarrow H(j\omega)e^{j\omega t}$$

sinusoid

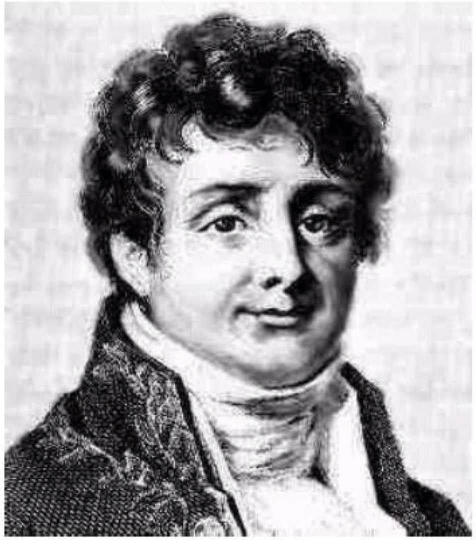
Sinusoid with the same frequency

  - **Discrete-time**: unit magnitude $z = e^{j\omega}$: $e^{j\omega n}$

$$e^{j\omega n} \longrightarrow \boxed{\text{LTI}} \longrightarrow H(e^{j\omega})e^{j\omega n}$$

Periodic signals & Fourier Series Expansion

- Using "trigonometric sum" to describe periodic signal can be tracked back to Babylonians who predicted astronomical events similarly.
- L. Euler (in 1748) and Bernoulli (in 1753) used the "normal mode" concept to describe the motion of a vibrating string; though JL Lagrange strongly criticized this concept.
- Fourier (in 1807) had found series of harmonically related sinusoids to be useful to describe the temperature distribution through body, and he claimed "any" periodic signal can be represented by such series.
- Dirichlet (in 1829) provide a precise condition under which a periodic signal can be represented by a Fourier series.



Jean Baptiste Joseph Fourier
March 21 1768 - May 16 1830
Born Auxerre, France. Died Paris, France.

Summary

- The Response of Continuous-Time LTI systems to Complex Exponentials

- The Response of Discrete-Time LTI systems to Complex Exponentials

- Reference in textbook:
  - 3.1, 3.2