SI152: Numerical Optimization

Lecture 5: Duality

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Outline

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Constrained to unconstrained

Constrained optimization

$$\min\ x^2 + y^2$$
$$\text{s.t. } x + y = 1.$$

- Formulate the unconstrained problem

$$L(x, y, p) = x^2 + y^2 + \lambda(1 - x - y)$$

- The minimizer is

$$x = y = \frac{\lambda}{2}$$

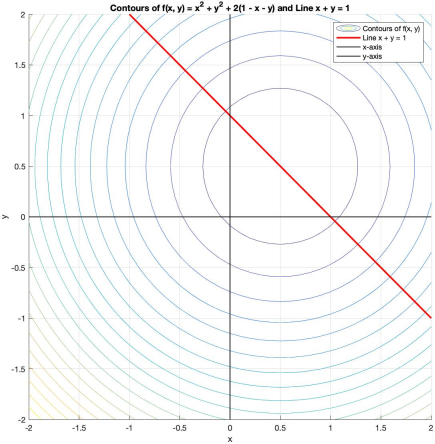


Figure: $p = 1$

$$\min_x \; c^T x \quad \text{s.t. } a_i^T x \leq b_i, i = 1, ..., m.$$

Lagrange function (Lagrangian):

$$L(x, \lambda) = c^T x + \sum_{i=1}^{m} \lambda_i (a_i^T x - b_i)$$

Fixed $\lambda \geq 0$, unconstrained problem :

$$\min_x L(x, \lambda) = c^T x + \sum_{i=1}^{m} \lambda_i (a_i^T x - b_i)$$

- Allow the violation of constraints: $a_i^T x \leq b_i \implies a_i^T x > b_i$
- This violation $a_i^T x > b_i$ yields a cost: $\lambda_i (a_i^T x - b_i)$
- λ_i is the price per unit of violation

$$L(x,\lambda) = c^T x + \sum_{i=1}^{m} \lambda_i (a_i^T x - b_i)$$

Given x, the cost:

$$f(x) := \max_{\lambda \geq 0} L(x,\lambda) = c^T x + \sum_{i=1}^{m} \lambda_i (a_i^T x - b_i)$$

- Given x, the dual player maximize his revenue by manipulating λ, i.e., find the best price λ_i for violation to maximize the revenue: $L(x,\lambda)$.
- Now x is prescribed parameter. Optimal λ depends on x: $\lambda = \lambda(x)$

$$\min_x f(x) = \min_x \max_{\lambda \geq 0} L(x,\lambda) = c^T x + \sum_{i=1}^{m} \lambda_i (a_i^T x - b_i)$$

- The primal player want to minimize his penalty when seeing the price λ.

$$L(x,\lambda) = c^T x + \sum_{i=1}^{m} \lambda_i (a_i^T x - b_i)$$

Given $\lambda \geq 0$, the cost:

$$g(\lambda) := \min_x L(x,\lambda) = c^T x + \sum_{i=1}^{m} \lambda_i (a_i^T x - b_i)$$

- Given $\lambda \geq 0$, the primal player minimize his cost by manipulating x, possibly violating the constraints, to minimize $L(x,\lambda)$.
- Now $\lambda \geq 0$ is the prescribed parameter. Optimal x depends on λ: $x = x(\lambda)$

$$\max_{\lambda \geq 0} g(\lambda) = \max_{\lambda \geq 0} \min_x L(x,\lambda) = c^T x + \sum_{i=1}^{m} \lambda_i (a_i^T x - b_i)$$

- The dual player want to maximize his revenue when seeing the action x.

$$\min\max \text{ v.s. } \max\min$$

- Primal problem (min-max):

$$\min_x f(x) = \min_x \max_{\lambda \geq 0} L(x, \lambda)$$

- Primal objective: $f(x) = \max_{\lambda \geq 0} L(x, \lambda)$
- Primal variable: x
- Dual problem:

$$\max_{\lambda \geq 0} g(\lambda) = \max_{\lambda \geq 0} \min_x L(x, \lambda)$$

- Dual objective: $g(\lambda) = \min_x L(x, \lambda)$
- Dual variable: λ
- Dual feasibility: $\lambda \geq 0$

Primal problem:

$$\min_x \max_{\lambda \geq 0} L(x, \lambda) = c^T x + \sum_{i=1}^{m} \lambda_i (a_i^T x - b_i)$$

Let's take a closer look:

$$f(x) \begin{cases} = +\infty & \text{if } \exists i, a_i^T x > b_i \implies \text{We don't care about this case.} \\ < +\infty & \text{if } \forall i, a_i^T x \leq b_i \implies \text{This implies primal feasibility.} \end{cases}$$

This is equivalent to the original LP:

$$\min \; c^T x \quad \text{s.t. } a_i^T x \leq b_i, \; i = 1, ..., m$$

Notice that:

$$\text{If } \exists i, a_i^T < b_i \implies \text{This implies } \lambda_i = 0.$$

Dual problem:

$$\max_{\lambda \geq 0} \min_{x} L(x, \lambda) = c^T x + \sum_{i=1}^{m} \lambda_i (a_i^T x - b_i) = (c + \sum_{i=1}^{m} \lambda_i a_i)^T x - b^T \lambda$$

Let's take a closer look:

$$g(\lambda) \begin{cases} = -\infty & \text{if } c + \sum_{i=1}^{m} \lambda_i a_i \neq 0 \implies \text{We don't care about this case.} \\ > -\infty & \text{if } c + \sum_{i=1}^{m} \lambda_i a_i = 0 \implies \text{This implies dual feasibility.} \end{cases}$$

This is equivalent to:

$$\max \; -b^T \lambda \quad \text{s.t. } c + A^T \lambda = 0, \lambda \geq 0.$$

Split the equality: $Ax = b \implies Ax \geq b, Ax \leq b$
The Lagrangian ($u \geq 0, w \geq 0, v \geq 0$) is:

$$\begin{aligned} L(x,u,w,v) &= c^T x + u^T(-Ax + b) + w^T(Ax - b) - v^T x, \\ &= c^T x - (u - w)^T (Ax - b) - v^T x \\ &= c^T x - \underbrace{\lambda^T}_{\text{It is free!}} (Ax - b) - v^T x \end{aligned}$$

The dual objective is then:

$$g(\lambda, v) = \min_x L(x, \lambda, v) = \min_x (c - A^T\lambda - v)^T x + b^T\lambda$$

Maximize $g(\lambda, v)$, only care about $g(\lambda, v) > -\infty$, meaning $c - A^T\lambda - v = 0$
Dual problem is:

$$\max b^T\lambda, \quad \text{s.t. } A^T\lambda \leq c.$$

Example

$$\begin{array}{ll} \text{minimize} & -x_1 - 4x_2 - 3x_3 \\ \text{subject to} & 2x_1 + 2x_2 + x_3 = 4 \\ & x_1 + 2x_2 + 2x_3 \le 6 \\ & x_1 \ge 0, x_2 \ge 0, x_3 \ge 0 \end{array}$$

$$\begin{aligned} L(x,\lambda) &= -x_1 - 4x_2 - 3x_3 + \lambda_1(2x_1 + 2x_2 + x_3 - 4) \\ &\quad + \lambda_2(x_1 + 2x_2 + 2x_3 - 6) - \mu_1 x_1 - \mu_2 x_2 - \mu_3 x_3 \\ &= \boxed{(-1 + 2\lambda_1 + \lambda_2 - \mu_1)}x_1 + \boxed{(-4 + 2\lambda_1 + 2\lambda_2 - \mu_2)}x_2 \\ &\quad + \boxed{(-3 + \lambda_1 + 2\lambda_2 - \mu_3)}x_3 \boxed{- 4\lambda_1 - 6\lambda_2} \\ &\quad \lambda_1 \text{ is free }, \lambda_2 \ge 0,\ \mu_1 \ge 0,\ \mu_2 \ge 0,\ \mu_3 \ge 0 \end{aligned}$$

Given $A \in \mathbb{R}^{m\times n}, b \in \mathbb{R}^m, c \in \mathbb{R}^n$

- The primal problem:

$$\min\ c^T x \quad \text{s.t. } Ax = b, x \geq 0$$

- Variables: $x \in \mathbb{R}^n$

- The dual problem:

$$\max\ b^T \lambda \quad \text{s.t. } A^T \lambda \leq c$$

- Variables: $\lambda \in \mathbb{R}^m$

- Show the dual problem of the dual problem is the primal problem.
- Careful about the dual variables, dual feasibility, dual objective, dual problem, duality, and duality gap.

Explanation

Primal：

$$\begin{array}{ll} \text{minimize} & c_1x_1 + c_2x_2 + \cdots + c_nx_n \\ \text{subject to} & a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \geq b_1 \\ & a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \geq b_2 \\ & \vdots \\ & a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \geq b_m \\ & x_1 \geq 0, x_2 \geq 0, \cdots, x_n \geq 0 \end{array}$$

Dual：

$$\begin{array}{ll} \text{maximize} & \lambda_1 b_1 + \lambda_2 b_2 + \cdots + \lambda_m b_m \\ \text{subject to} & \lambda_1 a_{11} + \cdots + \lambda_m a_{m1} \leq c_1 \\ & \lambda_1 a_{12} + \cdots + \lambda_m a_{m2} \leq c_2 \\ & \vdots \\ & \lambda_1 a_{1n} + \cdots + \lambda_m a_{mn} \leq c_n \\ & \lambda_i \geq 0, i = 1, \cdots, m. \end{array}$$

$$\min c^T x$$
$$Ax \begin{Bmatrix} \geq \\ = \\ \leq \end{Bmatrix} b$$
$$x \begin{Bmatrix} \geq \\ = \\ \leq \end{Bmatrix} 0$$

$$\max b^T y$$
$$A^T y \begin{Bmatrix} \geq \\ = \\ \leq \end{Bmatrix} c$$
$$y \begin{Bmatrix} \geq \\ = \\ \leq \end{Bmatrix} 0$$

Primal problem	**Dual problem**
minimize	maximize
Constraints	**Variables**
$\sum_{j=1}^{n} a_{ij}x_j \geq b_i$	$y_i \geq 0$
$\sum_{j=1}^{n} a_{ij}x_j = b_i$	y_i is free
$\sum_{j=1}^{n} a_{ij}x_j \leq b_i$	$y_i \leq 0$
Variables	**Constraints**
x_j is free	$a_j^T y = c_j$
$x_j \geq 0$	$a_j^T y \leq c_j$
$x_j \leq 0$	$a_j^T y \geq c_j$
$x_j = 0$	no constraint

$$\begin{array}{ll} \text{minimize} & x_1 + 2x_2 + 3x_3 \\ \text{subject to} & -x_1 + 3x_2 = 5 \\ & 2x_1 - x_2 + 3x_3 \geq 6 \\ & x_3 \leq 4 \\ & x_1 \geq 0 \\ & x_2 \leq 0 \\ & x_3 \text{ is free} \end{array}$$

$$\begin{array}{ll} \text{maximize} & 5\lambda_1 + 6\lambda_2 + 4\lambda_3 \\ \text{subject to} & \lambda_1 \text{ is free} \\ & \lambda_2 \geq 0 \\ & \lambda_3 \leq 0 \\ & -\lambda_1 + 2\lambda_2 \leq 1 \\ & 3\lambda_1 - \lambda_2 \geq 2 \\ & 3\lambda_2 + \lambda_3 = 3 \end{array}$$

$$\min\ c^T x \quad \text{s.t. } Ax = b, x \geq 0 \qquad\qquad \max\ b^T\lambda \quad \text{s.t. } A^T\lambda \leq c$$

Theorem 1 (Weak duality)

$c^T x \geq \lambda^T b$ *for any primal-dual feasible* (x, λ).

For any primal-dual feasible (x, λ):

$$f(x) = \max_{\lambda \geq 0} L(x, \lambda) \geq L(x, \lambda) \geq \min_{x} L(x, \lambda) = g(\lambda).$$

Corollary 2

If $c^T x = b^T \lambda$ *holds at primal-dual feasible* (x, λ)*, then* x *is primal optimal and* λ *is dual optimal.*

Corollary 3

If the primal (dual) problem is unbounded, then the dual (primal) problem is infeasible.

$$\min \; c^T x \quad \text{s.t. } Ax = b, x \geq 0 \qquad\qquad \max \; b^T \lambda \quad \text{s.t. } A^T \lambda \leq c$$

Theorem 4

Let x^* be the optimal solution of the standard form LP and B be the optimal basis. Then

$$\lambda^* = (c_B^T B^{-1})^T$$

is the optimal dual variable.

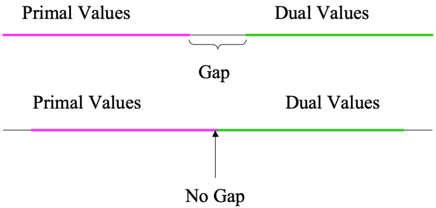
Hint: $r = c - A^T (c_B^T B^{-1})^T = c - A^T B^{-T} c_B = c - A^T \lambda$

Theorem 5

If the primal (dual) problem is feasible, then the dual (primal) is also feasible and the optimal values are equal (duality gap = 0).

Dual Prob \ Primal Prob	Infeasible	Unbounded below	Optimal
Infeasible	√	√	×
Unbounded above	√	×	×
Optimal	×	×	√

Question: write the dual problem of the Phase I problem. Is it feasible? Does it have optimal solution?



Optimal $\lambda^* = (c_B^T B^{-1})^T$

Where is the reduced cost?

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array}$$

λ is whose multipliers?

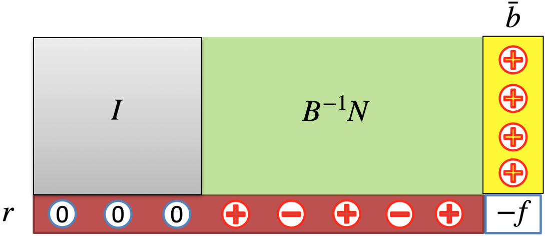
$$\begin{array}{ll} \text{maximize} & b^T \lambda \\ \text{subject to} & \lambda^T A \leq c^T \end{array}$$

λ, r are whose multipliers?

$$\begin{aligned} r^T &= c^T - c_B^T B^{-1} A \\ &= c^T - \lambda^T A \end{aligned}$$

$r^T \geq 0 \implies$ dual feasibility

$$\begin{array}{ll} \text{maximize} & b^T \lambda \\ \text{subject to} & \lambda^T A + r^T = c^T \\ & r \geq 0 \end{array}$$



$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \\ & x \geq 0 \end{array} \qquad \begin{array}{ll} \text{maximize} & b^T \lambda \\ \text{subject to} & \lambda^T A + r^T = c^T \\ & r \geq 0 \end{array}$$

$$\min\ c^T x \quad \text{s.t. } Ax = b, x \geq 0 \qquad\qquad \max\ b^T \lambda \quad \text{s.t. } A^T \lambda \leq c$$

We need to solve

$$\begin{array}{ccc} Ax = b & & A^T\lambda + r = c \\ x \geq 0 & & r \geq 0 \\ & \boxed{b^T\lambda = c^T x} & \end{array}$$

The last condition $b^T \lambda = c^T x$ can be written as

$$0 = c^T x - b^T \lambda = c^T x - (Ax)^T \lambda = (c - A^T \lambda)^T x = r^T x$$

Becase $r \geq 0$ and $x \geq 0$, the condition $r^T x = 0$ implies

$$\boxed{r_i x_i = 0, \quad \forall i}$$

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b \longleftarrow \lambda \\ & x \geq 0 \longleftarrow r \end{array} \qquad \textbf{complementary: } r^T x = 0$$

- It means the inactive constraint at x^* has a 0 multiplier:

$$x_i^* > 0 \implies r_i^* = 0$$

- For general LP

$$\min\ c^T x \quad \text{s.t. } a_i^T x \leq b_i,\ i = 1, ..., m$$

Recall from writing the dual, $a_i^T x^* - b_i < 0 \implies \lambda_i^* = 0$, i.e.,

$$(a_i^T x^* - b_i)\lambda_i^* = 0$$

- The strictly satisfied constraint, the penalty should be 0

$$L(x, \lambda) = c^T x + \sum_{i=1}^{m} \lambda_i (a_i^T x - b_i)$$

Complementary condition

$$\left.\begin{array}{l}\text{Primal feasible}\\ \text{Dual feasible}\\ \text{Duality Gap} = \text{Primal}-\text{Dual} = 0\end{array}\right\} \Longleftrightarrow \text{Primal-dual optimal}$$

$$\left.\begin{array}{l}\text{Primal feasible}\\ \text{Dual feasible}\\ \text{Complementarity}\end{array}\right\} \Longleftrightarrow \text{Primal-dual optimal}$$

$$\begin{array}{lll} \text{Primal} & \text{minimize} & -x_1 - 4x_2 - 3x_3 \\ & \text{subject to} & 2x_1 + 2x_2 + x_3 \leq 4 \\ & & x_1 + 2x_2 + 2x_3 \leq 6 \\ & & x_1 \geq 0, x_2 \geq 0, x_3 \geq 0 \end{array}$$

$$\begin{array}{lll} \text{Dual} & \text{maximize} & 4\lambda_1 + 6\lambda_2 \\ & \text{subject to} & 2\lambda_1 + \lambda_2 \leq -1 \\ & & 2\lambda_1 + 2\lambda_2 \leq -4 \\ & & \lambda_1 + 2\lambda_2 \leq -3 \\ & & \lambda_1 \leq 0, \lambda_2 \leq 0 \end{array}$$

	x_1	x_2	x_3	x_4	x_5	$B^{-1}b$
	2	$\boxed{2}$	1	1	0	4
	1	2	2	0	1	6
r^{T}	-1	-4	-3	0	0	0

	x_1	x_2	x_3	x_4	x_5	$B^{-1}b$
	1	1	$\frac{1}{2}$	$\frac{1}{2}$	0	2
	-1	0	$\boxed{1}$	-1	1	2
r^{T}	3	0	-1	2	0	8

	x_1	x_2	x_3	x_4	x_5	$B^{-1}b$
	$\frac{3}{2}$	1	0	1	$-\frac{1}{2}$	1
	-1	0	1	-1	1	2
r^{T}	2	0	0	1	1	10

why?

Optimal primal

$$x_1^* = 0$$
$$x_2^* = 1$$
$$x_3^* = 2$$

Optimal multipliers

$$\lambda_1^* = -1$$
$$\lambda_2^* = -1$$

$$\max\ b^T\lambda$$
$$\text{s.t. } \lambda^T A \le c^T$$

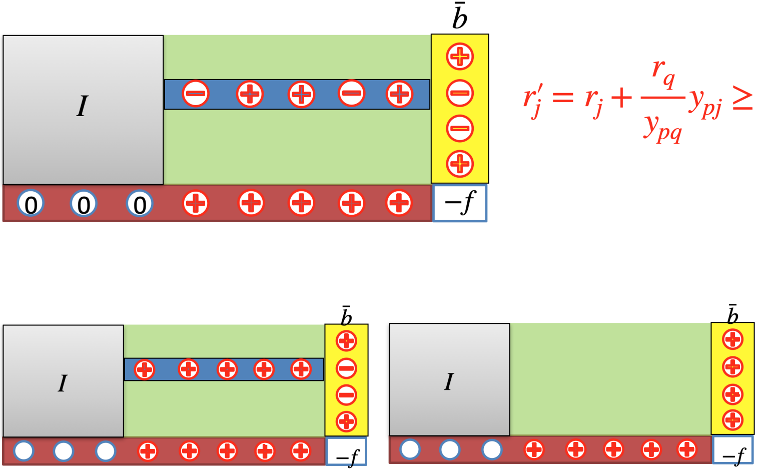
It would be perfect to know the active constraints

Definition 6

Suppose $x_B = B^{-1}b$ is a basic solution (possibly infeasible) for $Ax = b$. If $\lambda^T = c_B^T B^{-1}$ is feasible for the dual problem, i.e., $(r^T =)c^T - \lambda^T A \ge 0$, then we call x is the dual feasible solution for the standard form.

- How about an algorithm:
  Dual BS $\Longrightarrow$ Dual BS $\Longrightarrow$ ... $\Longrightarrow$ Dual BS & Primal feasible
- In the tableau, complementarity is always satisfied!
  Dual feasible + Primal feasible + Complementary = Optimal

This is called the Duality Simplex Method:



$$r'_j = r_j + \frac{r_q}{y_{pq}} y_{pj} \geq 0$$

$$\begin{array}{ll}
\text{minimize} & 3x_1 + 4x_2 + 5x_3 \\
\text{subject to} & x_1 + 2x_2 + 3x_3 \geq 5 \\
& 2x_1 + 2x_2 + x_3 \geq 6 \\
& x_1 \geq 0, x_2 \geq 0, x_3 \geq 0
\end{array}$$

The first tableau:

	x_1	x_2	x_3	x_4	x_5	$B^{-1}b$
	-1	-2	-3	1	0	-5
	$\boxed{-2}$	-2	-1	0	1	-6
r^T	3	4	5	0	0	0

Example

	x_1	x_2	x_3	x_4	x_5	$B^{-1}b$
	0	$\boxed{-1}$	$-\frac{5}{2}$	1	$-\frac{1}{2}$	-2
	1	1	$\frac{1}{2}$	0	$-\frac{1}{2}$	3
r^T	0	1	$\frac{7}{2}$	0	$\frac{3}{2}$	-9

	x_1	x_2	x_3	x_4	x_5	$B^{-1}b$
	0	1	$\frac{5}{2}$	-1	$\frac{1}{2}$	2
	1	0	-2	1	-1	1
r^T	0	0	1	1	1	-11

Optimal primal solution: $x^* = (1, 2, 0)^T$

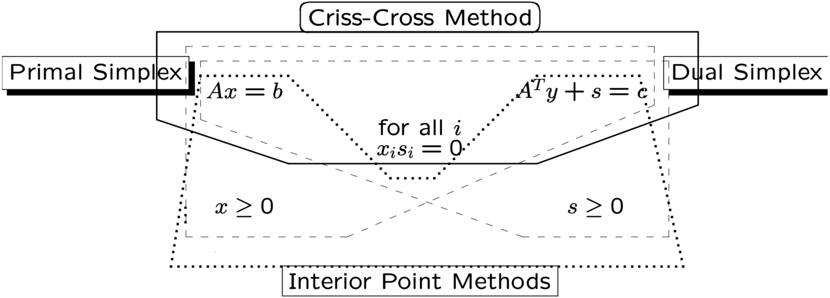
Dual iterates: $(0, 0) \to (0, 3/2) \to (1, 1)$

Summary

- Initialization: see textbook. If every dual feasible BS has positive reduced cost, then the dual simplex method terminates finitely.

- The primal simplex has two cases: optimal solution exists, or unbounded
- The dual simplex has two cases: optimal solution exists, or infeasible

- Two Phase method: can handle every case.
  - Infeasibility is detected in Phase I.
  - Unboundedness is detected in Phase II.



All algorithms for LP keep a part of the optimality criteria valid while working towards the others.