Ch.3 *Fourier Series Representation of Periodic signals*

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Part IV *Fourier Series and LTI Systems*

Outline

- Fourier Series and LTI Systems
- Filtering

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Fourier Series and LTI Systems

- Recall: eigenfunction

$$e^{st} \longrightarrow \boxed{\text{LTI}} \longrightarrow H(s)e^{st} \qquad H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau$$

$$z^n \longrightarrow \boxed{\text{LTI}} \longrightarrow H[z]z^n \qquad H[z] = \sum_{k=-\infty}^{\infty} h[k]z^{-k}$$

- $H(s)$ and $H[z]$ are referred to as "system function" (transfer functions).

- In this chapter, we consider:
  - Continuous-time: $Re\{s\} = 0 \rightarrow s = j\omega$
  - Discrete-time: $|z| = 1 \rightarrow z = e^{j\omega}$

Fourier Series and LTI Systems

- Frequency response for CT system: $H(j\omega)$

$$H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau \xRightarrow{s=j\omega} H(j\omega) = \int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau$$

$$e^{j\omega t} \longrightarrow \boxed{\text{LTI}} \longrightarrow H(j\omega)e^{j\omega t}$$

$$x(t) = \sum_{k-\infty}^{\infty} a_k e^{jk\omega_0 t} \longrightarrow \boxed{\text{LTI}} \longrightarrow y(t) = \sum_{k-\infty}^{\infty} a_k H(jk\omega_0) e^{jk\omega_0 t}$$

$$\mathcal{FS} \updownarrow \qquad\qquad \mathcal{FS} \updownarrow$$

$$a_k \qquad b_k = a_k H(jk\omega_0) \qquad b_k$$

Fourier Series and LTI Systems

- Frequency response for DT system: $H(e^{j\omega})$

$$H[z] = \sum_{n=-\infty}^{\infty} h[k] z^{-n} \quad \xRightarrow{z=e^{j\omega}} \quad H\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n}$$

$$e^{j\omega n} \longrightarrow \boxed{\text{LTI}} \longrightarrow H\left(e^{j\omega}\right) e^{j\omega n}$$

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n} \longrightarrow \boxed{\text{LTI}} \longrightarrow y[n] = \sum_{k=\langle N\rangle} a_k H\left(e^{jk2\pi/N}\right) e^{jk(2\pi/N)n}$$

$$b_k = a_k H\left(e^{jk2\pi/N}\right)$$

Fourier Series and LTI Systems

e^{st}

- s pure imagery
  $e^{st} \rightarrow e^{j\omega t}$
  - For periodic signal, CT Fourier Series (Ch3)
  - For aperiodic signal, CT Fourier Transform (Ch4)
- s complex number
  e^{st}
  - Laplase Transform (Ch9)

z^n

- z pure imagery
  $z^n \rightarrow e^{j\omega n}$
  - For periodic signal, DT Fourier Series (Ch3)
  - For aperiodic signal, DT Fourier Transform (Ch5)
- z complex number
  z^n
  - Z-Transform (Ch10)

Fourier Series and LTI Systems

- Example: $x(t) = \sum_{k=-3}^{3} a_k e^{jk2\pi t}$ ($a_0 = 1$, $a_1 = a_{-1} = \frac{1}{4}$, $a_2 = a_{-2} = \frac{1}{2}$, $a_3 = a_{-3} = \frac{1}{3}$) is the input of a LTI system with $h(t) = e^{-t}u(t)$, determine $y(t)$.

Fourier Series and LTI Systems

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- Solution: $b_k = a_k H(j\omega) = a_k \dfrac{1}{1 + jk2\pi}$

$$b_0 = 1 \cdot 1 = 1 \qquad b_1 = \frac{1}{4}\frac{1}{1 + j2\pi} \qquad b_{-1} = \frac{1}{4}\frac{1}{1 - j2\pi}$$

$$b_2 = \frac{1}{2}\frac{1}{1 + j4\pi} \qquad b_{-2} = \frac{1}{2}\frac{1}{1 - j4\pi} \qquad b_3 = \frac{1}{3}\frac{1}{1 + j6\pi} \qquad b_{-3} = \frac{1}{3}\frac{1}{1 - j6\pi}$$

Fourier Series and LTI Systems

- Example:

$$h[n] = \alpha^n u[n], |\alpha| < 1$$

$$x[n] = \cos\frac{2\pi n}{N} \longrightarrow \boxed{\text{LTI}} \longrightarrow y[n]?$$

- Solution: $x[n] = \frac{1}{2} e^{j(2\pi/N)n} + \frac{1}{2} e^{-j(2\pi/N)n}$

$$H\left(e^{j\omega}\right) = \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n} = \frac{1}{1-\alpha e^{-j\omega}}, \text{ where } \omega = k\left(\frac{2\pi}{N}\right), k = \pm 1.$$

$$y[n] = \sum_{k=\langle N \rangle} a_k H\left(e^{jk\left(\frac{2\pi}{N}\right)}\right) e^{jk\left(\frac{2\pi}{N}\right)n}$$

$$= \frac{1}{2}\left(\frac{1}{1-\alpha e^{-j2\pi/N}}\right) e^{j(2\pi/N)n} + \frac{1}{2}\left(\frac{1}{1-\alpha e^{j2\pi/N}}\right) e^{-j(2\pi/N)n}$$

Outline

- Fourier Series and LTI Systems
- Filtering

Filtering

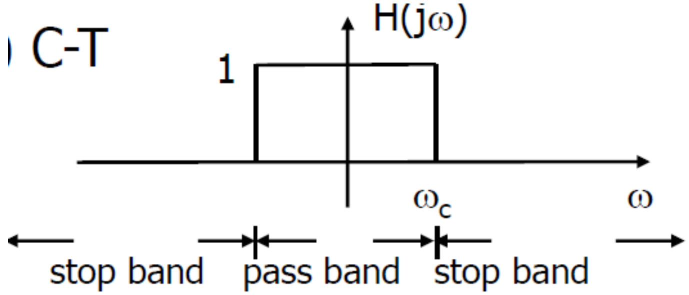
- Filtering: A process that changes the relative amplitude (or phase) of some frequency components.

- e.g.,
  - frequency-shaping filter
    (like equalizer in a Hi-Fi system)

  - frequency-selective filter
    (like low-pass, band-pass, high-pass filters)

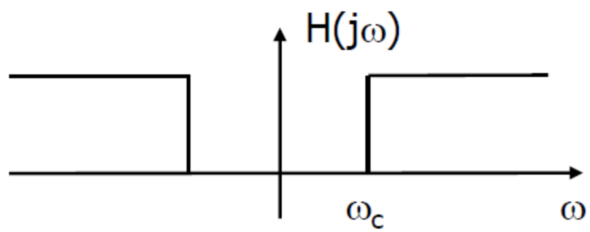
Frequency-Selective Filter

- Select some bands of frequencies and reject others.

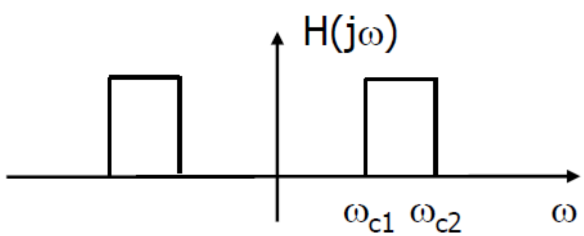


Ideal low-pass filter (LPF)

$$H(j\omega) = \begin{cases} 1 \text{ in pass band} \\ 0 \text{ in stop band} \end{cases}$$



Ideal high-pass filter (HPF)

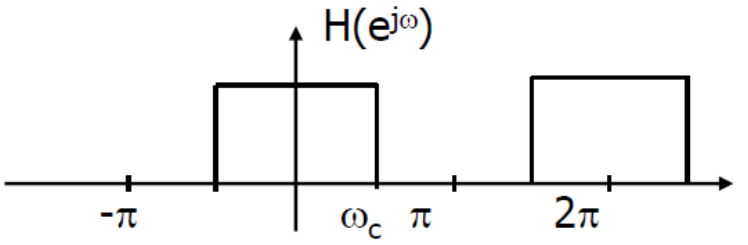


Ideal band-pass filter (BPF)

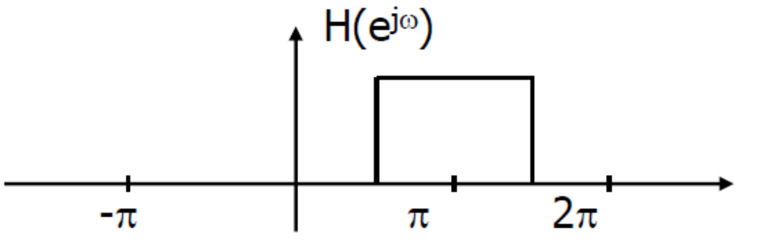
Frequency-Selective Filter

- Discrete-time

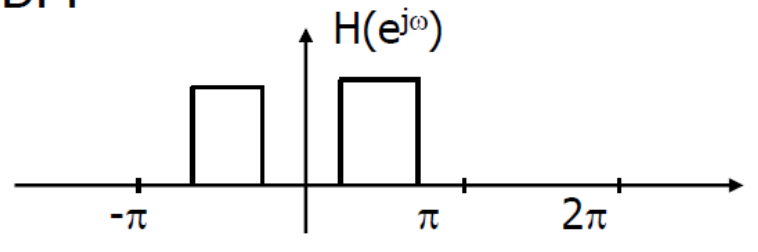
LPF



HPF



BPF



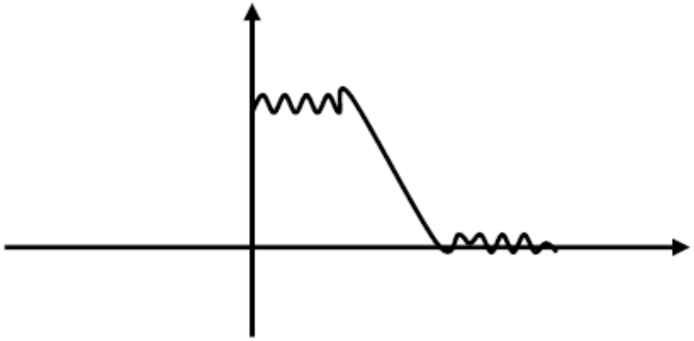
Frequency-Selective Filter

- For D-T: $H(e^{j\omega})$ is periodic with period 2π
  - Low frequencies: at around $\omega = 0, \pm 2\pi, \pm 4\pi, \ldots$
  - High frequencies: at around $\omega = \pm \pi, \pm 3\pi, \ldots$

- Note: ideal filters are not realizable

- Practical filters have transition band, and may have ripple in stopband and passband

e.g. LPF



Summary

- Fourier Series and LTI Systems
- Filtering
- Reference in textbook:
  - 3.8, 3.9