

Signals and Systems

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Chapter 9: The Laplace Transform

- ❑ The Laplace transform
- ❑ The region of convergence for Laplace transforms
- ❑ The inverse Laplace transform
- ❑ Geometric evaluation of the Fourier transform from the pole-zero plot
- ❑ Properties of the Laplace transform
- ❑ Some Laplace transform pairs
- ❑ Analysis and characterization of LTI systems using the Laplace transform
- ❑ System function algebra and block diagram representations
- ❑ The unilateral Laplace transform



The Laplace transform

Recall the response of LTI systems to complex exponentials

$$y(t) = H(s)e^{st}$$

$$H(s) = \int_{-\infty}^{+\infty} h(t)e^{-st} dt$$

$$\begin{aligned} y(t) &= \int_{-\infty}^{+\infty} h(\tau)e^{s(t-\tau)} d\tau \\ &= e^{st} \int_{-\infty}^{+\infty} h(\tau)e^{-s\tau} d\tau \end{aligned}$$

Definition

$$X(s) \triangleq \int_{-\infty}^{+\infty} x(t)e^{-st} dt$$

$$x(t) \xleftrightarrow{\mathcal{L}} X(s)$$



The Laplace transform

Laplace transform vs Fourier transform

$$x(t) \xleftrightarrow{\mathcal{L}} X(s)$$
$$X(s) \triangleq \int_{-\infty}^{+\infty} x(t)e^{-st} dt$$

$$s = j\omega \quad \Downarrow$$

$$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t} dt$$

$$X(s) \Big|_{s=j\omega} = \mathcal{F}\{x(t)\}$$

$$\Downarrow s = \sigma + j\omega$$

$$X(\sigma + j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-(\sigma + j\omega)t} dt$$

$$X(\sigma + j\omega) = \int_{-\infty}^{+\infty} [x(t)e^{-\sigma t}]e^{-j\omega t} dt$$

$$X(s) \Big|_{s=\sigma + j\omega} = \mathcal{F}\{x(t)e^{-\sigma t}\}$$



The Laplace transform

Examples

$$x(t) = e^{-at}u(t) \quad X(s) = ?$$

Solution

$$X(j\omega) = \int_{-\infty}^{+\infty} e^{-at}u(t)e^{-j\omega t}dt = \int_0^{\infty} e^{-(j\omega+a)t}dt = \frac{1}{a+j\omega}, \quad a > 0$$

$$X(\sigma + j\omega) = \int_0^{\infty} e^{-(\sigma+a)t}e^{-j\omega t}dt = \frac{1}{(\sigma+a)+j\omega}, \quad \sigma + a > 0$$

$$X(s) = \int_0^{\infty} e^{-(s+a)t}dt = \frac{1}{s+a}, \quad \operatorname{Re}\{s\} > -a$$

$$e^{-at}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+a} \quad \operatorname{Re}\{s\} > -a$$



The Laplace transform

Examples

$$x(t) = -e^{-at}u(-t) \quad X(s) = ?$$

Solution

$$X(s) = -\int_{-\infty}^{+\infty} e^{-at}u(-t)e^{-st}dt = -\int_{-\infty}^0 e^{-(s+a)t}dt = \frac{1}{s+a}, \quad \operatorname{Re}\{s\} < -a$$

$$-e^{-at}u(-t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+a} \quad \operatorname{Re}\{s\} < -a$$

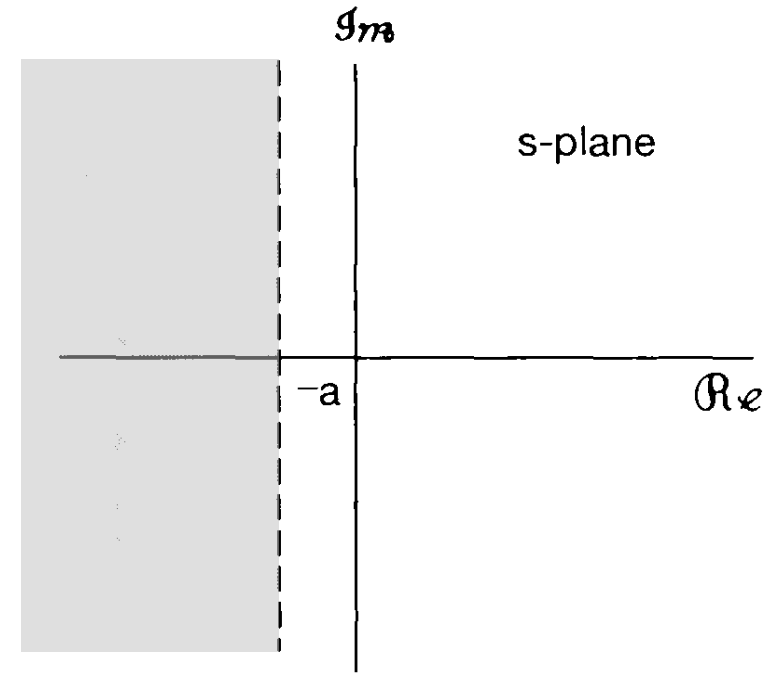
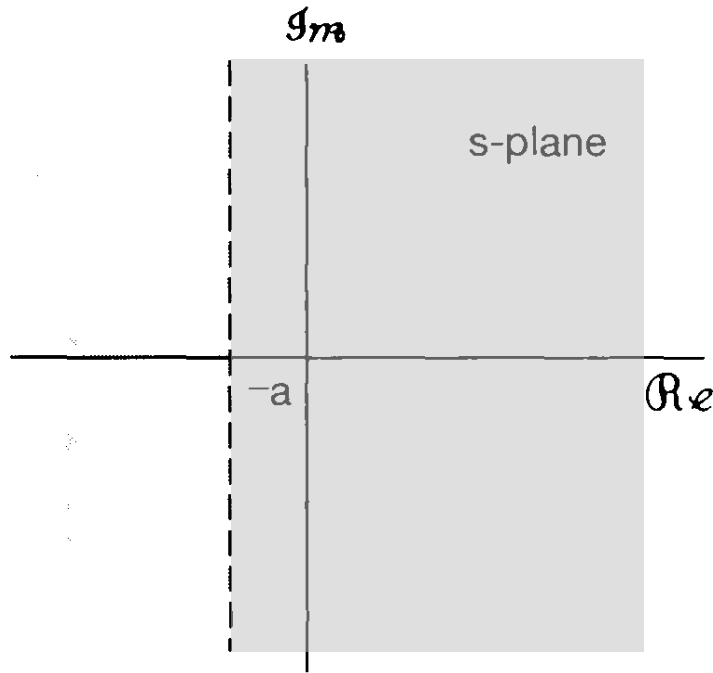


The Laplace transform

Region of convergence (ROC)

$$e^{-at}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+a} \quad \text{Re}\{s\} > -a$$

$$-e^{-at}u(-t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+a} \quad \text{Re}\{s\} < -a$$



The Laplace transform

Examples

$$x(t) = 3e^{-2t}u(t) - 2e^{-t}u(t) \quad X(s) = ?$$

Solution

$$\begin{aligned} X(s) &= \int_{-\infty}^{+\infty} [3e^{-2t}u(t) - 2e^{-t}u(t)]e^{-st}dt \\ &= 3 \int_{-\infty}^{+\infty} e^{-2t}e^{-st}u(t)dt - 2 \int_{-\infty}^{+\infty} e^{-t}e^{-st}u(t)dt = \frac{3}{s+2} - \frac{2}{s+1} = \frac{s-1}{s^2+3s+2} \end{aligned}$$

$$e^{-t}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+1} \quad \mathcal{Re}\{s\} > -1$$

$$e^{-2t}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+2} \quad \mathcal{Re}\{s\} > -2$$

$$3e^{-2t}u(t) - 2e^{-t}u(t) \xleftrightarrow{\mathcal{L}} \frac{s-1}{s^2+3s+2} \quad \mathcal{Re}\{s\} > -1$$



The Laplace transform

Examples $x(t) = e^{-2t}u(t) + e^{-t}(\cos 3t)u(t)$ $X(s) = ?$

Solution

$$x(t) = \left[e^{-2t} + \frac{1}{2}e^{-(1-3j)t} + \frac{1}{2}e^{-(1+3j)t} \right] u(t)$$

$$X(s) = \int_{-\infty}^{+\infty} e^{-2t}u(t)e^{-st}dt + \frac{1}{2} \int_{-\infty}^{+\infty} e^{-(1-3j)t}u(t)e^{-st}dt + \frac{1}{2} \int_{-\infty}^{+\infty} e^{-(1+3j)t}u(t)e^{-st}dt$$

$$e^{-2t}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+2} \quad \operatorname{Re}\{s\} > -2$$

$$e^{-(1-3j)t}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+(1-3j)} \quad \operatorname{Re}\{s\} > -1$$

$$e^{-(1+3j)t}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+(1+3j)} \quad \operatorname{Re}\{s\} > -1$$

$$X(s) = \frac{1}{s+2} + \frac{1}{2} \left(\frac{1}{s+(1-3j)} \right) + \frac{1}{2} \left(\frac{1}{s+(1+3j)} \right) = \frac{2s^2 + 5s + 12}{(s^2 + 2s + 10)(s+2)}, \operatorname{Re}\{s\} > -1$$

The Laplace transform

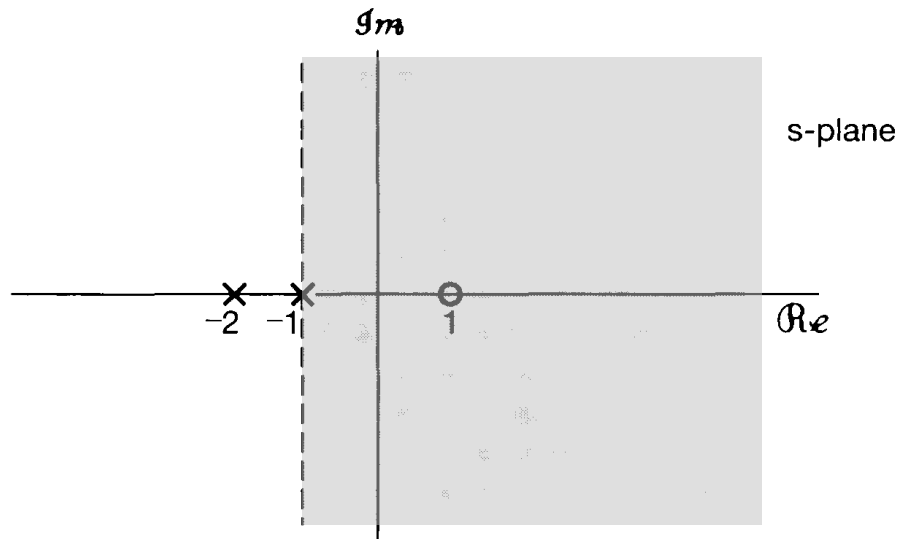
Pole-zero plot of $X(s)$

$$X(s) = \frac{N(s)}{D(s)}$$

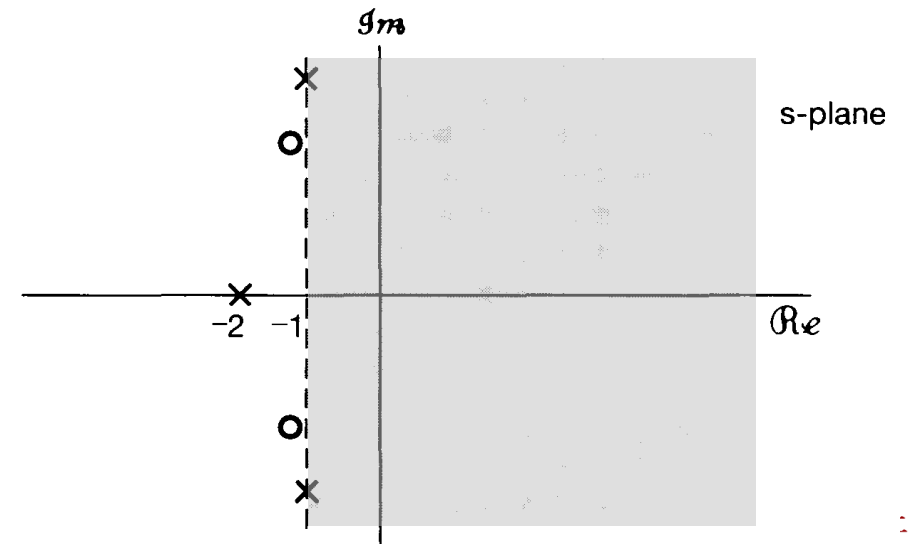
“x”: the location of the root of the numerator polynomial
“o”: the location of the root of the denominator polynomial

Examples

$$X(s) = \frac{s - 1}{s^2 + 3s + 2}, \operatorname{Re}\{s\} > -1$$



$$X(s) = \frac{2s^2 + 5s + 12}{(s^2 + 2s + 10)(s + 2)}, \operatorname{Re}\{s\} > -1$$



The Laplace transform

Examples

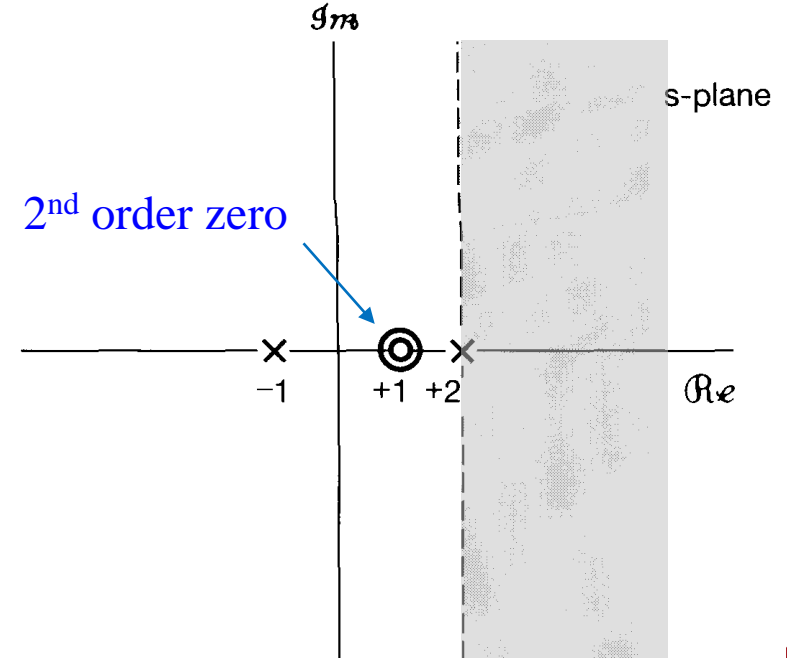
$$x(t) = \delta(t) - \frac{4}{3}e^{-t}u(t) + \frac{1}{3}e^{2t}u(t) \quad X(s) = ?$$

Solution

$$\mathcal{L}\{\delta(t)\} = \int_{-\infty}^{+\infty} \delta(t)e^{-st}dt = 1 \quad \text{valid for any value of } s$$

$$X(s) = 1 - \frac{4}{3} \frac{1}{s+1} + \frac{1}{3} \frac{1}{s-2} \quad \operatorname{Re}\{s\} > 2$$

$$= \frac{(s-1)^2}{(s+1)(s-2)} \quad \operatorname{Re}\{s\} > 2$$



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The region of convergence for Laplace transforms

Properties

1. The ROC of $X(s)$ consists of strips parallel to the $j\omega$ -axis in the s -plane

ROC of $X(s)$: Fourier transform of $x(t)e^{-\sigma t}$ converges (absolutely integrable)

$$\int_{-\infty}^{+\infty} |x(t)| e^{-\sigma t} dt < \infty \quad \text{depends only on } \sigma, \text{ the real part of } s$$

2. For rational Laplace transforms, the ROC does not contain any poles.

$X(s)$ is infinite at a pole



The region of convergence for Laplace transforms

Properties

3. If $x(t)$ is of finite duration and is absolutely integrable, then the ROC is the entire s -plane.

For convergence, require

$$\int_{T_1}^{T_2} |x(t)| e^{-\sigma t} dt < \infty$$

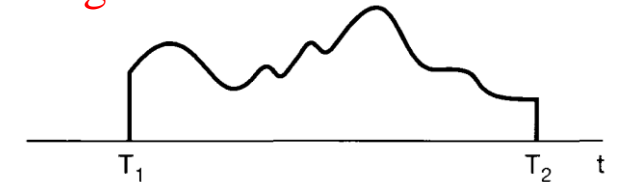
If $\sigma > 0$,

$$\int_{T_1}^{T_2} |x(t)| e^{-\sigma t} dt \leq e^{-\sigma T_1} \int_{T_1}^{T_2} |x(t)| dt$$

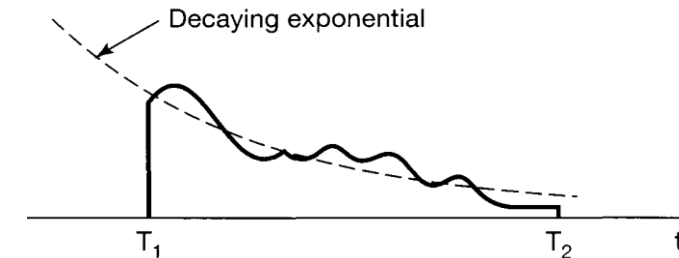
If $\sigma < 0$,

$$\int_{T_1}^{T_2} |x(t)| e^{-\sigma t} dt \leq e^{-\sigma T_2} \int_{T_1}^{T_2} |x(t)| dt$$

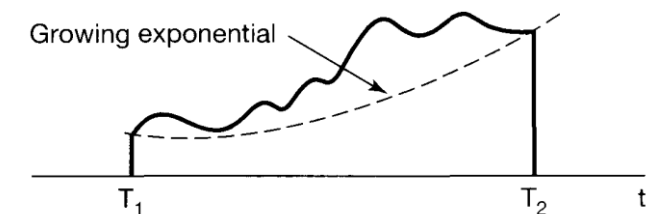
Finite-duration signal



Multiplied by a decaying exponential



Multiplied by a growing exponential



The region of convergence for Laplace transforms

Examples

$$x(t) = \begin{cases} e^{-at} & 0 < t < T \\ 0 & \text{otherwise} \end{cases} \quad X(s) = ?$$

Solution

$$X(s) = \int_0^T e^{-at} e^{-st} dt = \frac{1}{s+a} [1 - e^{-(s+a)T}]$$

$$\lim_{s \rightarrow -a} X(s) = \lim_{s \rightarrow -a} \left[\frac{\frac{d}{ds} (1 - e^{-(s+a)T})}{\frac{d}{ds} (s+a)} \right] = \lim_{s \rightarrow -a} T e^{-aT} e^{-sT}$$

$$X(-a) = T$$

ROC = the entire s-plane



The region of convergence for Laplace transforms

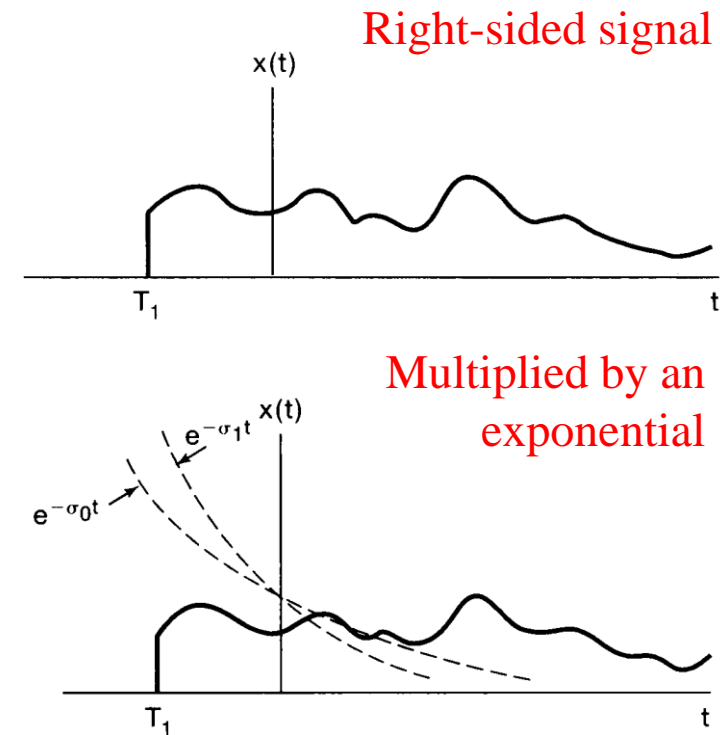
Properties

4. If $x(t)$ is right-sided, and **if** the line $\mathcal{Re}\{s\} = \sigma_0$ is in the ROC, then all values of s for which $\mathcal{Re}\{s\} > \sigma_0$ will also be in the ROC.

For convergence, require $\int_{T_1}^{+\infty} |x(t)| e^{-\sigma_0 t} dt < \infty$

For $\sigma_1 > \sigma_0$,

$$\begin{aligned} \int_{T_1}^{+\infty} |x(t)| e^{-\sigma_1 t} dt &= \int_{T_1}^{+\infty} |x(t)| e^{-\sigma_0 t} e^{-(\sigma_1 - \sigma_0)t} dt \\ &\leq e^{-(\sigma_1 - \sigma_0)T_1} \int_{T_1}^{+\infty} |x(t)| e^{-\sigma_0 t} dt \end{aligned}$$



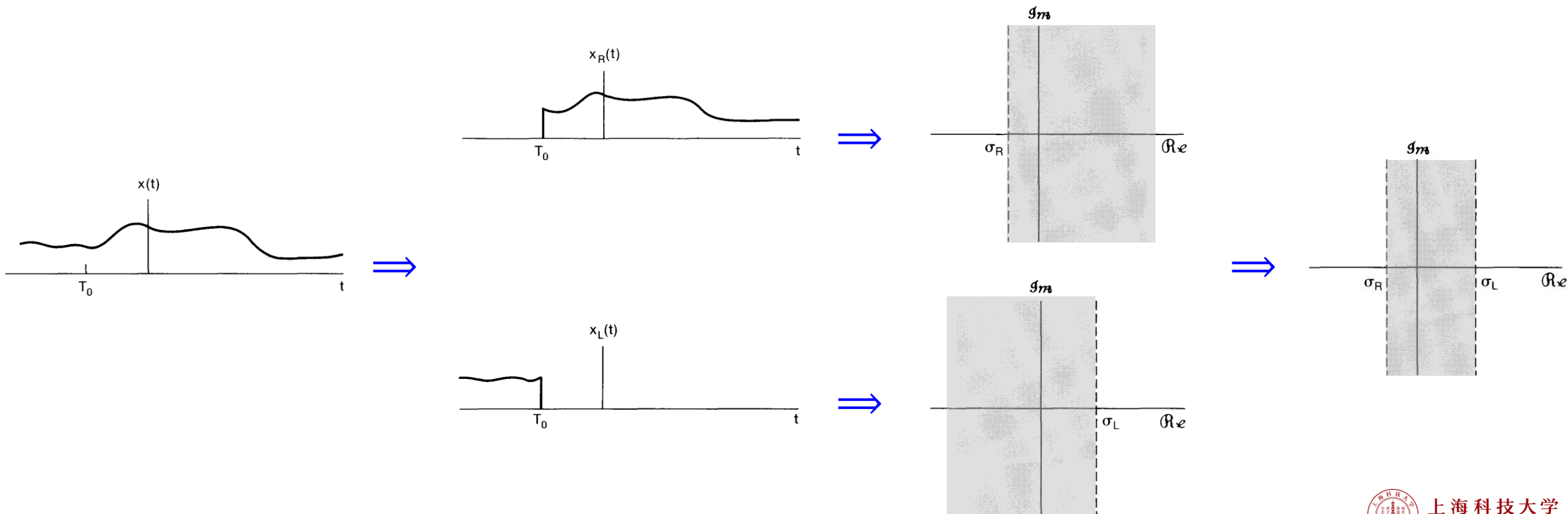
5. If $x(t)$ is left-sided, and if the line $\mathcal{Re}\{s\} = \sigma_0$ is in the ROC, then all values of s for which $\mathcal{Re}\{s\} < \sigma_0$ will also be in the ROC.



The region of convergence for Laplace transforms

Properties

6. If $x(t)$ is two-sided, and if the line $\mathcal{R}e\{s\} = \sigma_0$ is in the ROC, then the ROC will consist of a strip in the s -plane that includes the line $\mathcal{R}e\{s\} = \sigma_0$.



The region of convergence for Laplace transforms

Examples

$$x(t) = e^{-b|t|} \quad X(s) = ?$$

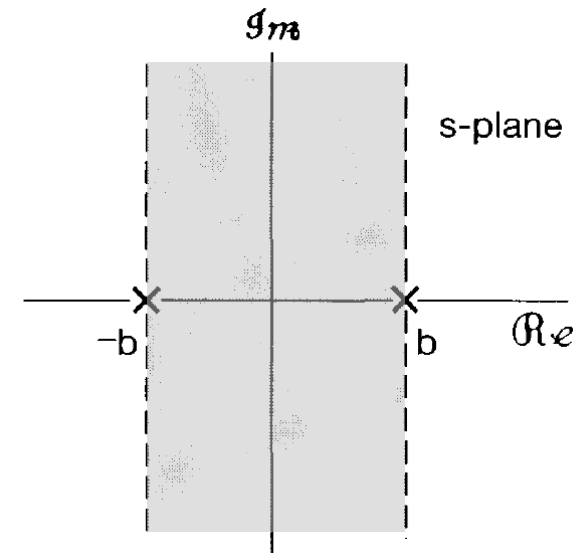
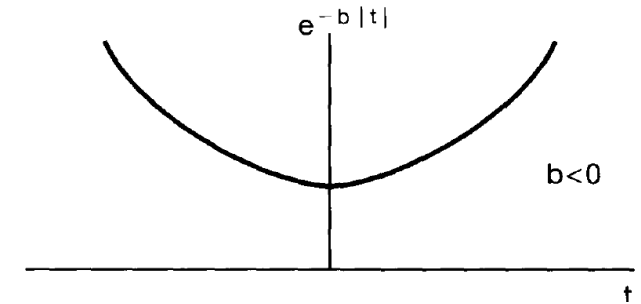
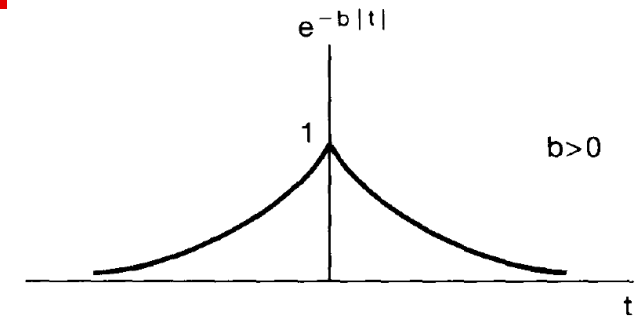
Solution

$$x(t) = e^{-bt}u(t) + e^{bt}u(-t)$$

$$e^{-bt}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+b} \quad \operatorname{Re}\{s\} > -b$$

$$e^{bt}u(-t) \xleftrightarrow{\mathcal{L}} \frac{-1}{s-b} \quad \operatorname{Re}\{s\} < b$$


$$e^{-b|t|} \xleftrightarrow{\mathcal{L}} \frac{1}{s+b} - \frac{1}{s-b} = -\frac{2b}{s^2 - b^2} \quad -b < \operatorname{Re}\{s\} < b, \quad b > 0$$



The region of convergence for Laplace transforms

Properties

7. If the Laplace transform $X(s)$ of $x(t)$ is rational, then its ROC is bounded by poles or extends to infinity. No poles are contained in the ROC.

- 
- ❑ If $x(t)$ is left-sided, and if the line $\mathcal{Re}\{s\} = \sigma_0$ is in the ROC, then all values of s for which $\mathcal{Re}\{s\} < \sigma_0$ will also be in the ROC.
 - ❑ If $x(t)$ is right-sided, and if the line $\mathcal{Re}\{s\} = \sigma_0$ is in the ROC, then all values of s for which $\mathcal{Re}\{s\} > \sigma_0$ will also be in the ROC.

8. If the Laplace transform $X(s)$ of $x(t)$ is rational, then if $x(t)$ is right-sided, the ROC is the region in the s -plane to the right of the right-most pole. The same applies to the left.



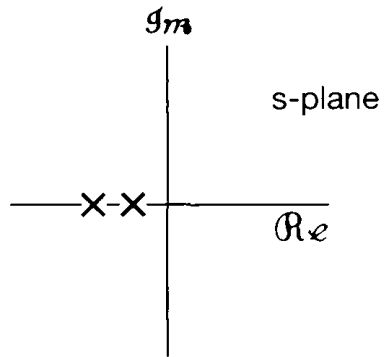
The region of convergence for Laplace transforms

Examples

$$X(s) = \frac{1}{(s+1)(s+2)}$$

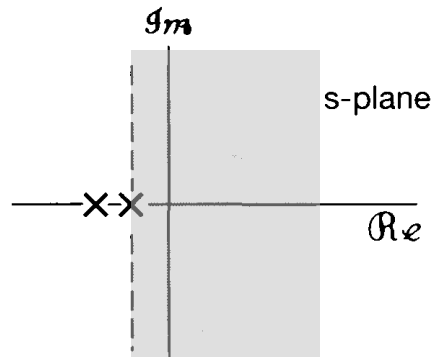
ROCs and convergence of FT?

Solution



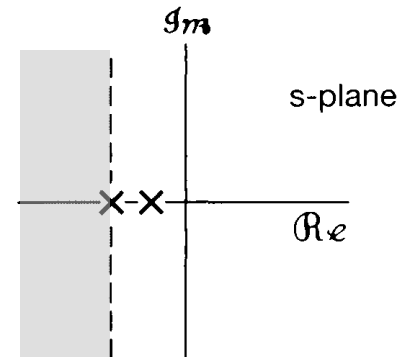
Right-sided

FT converges



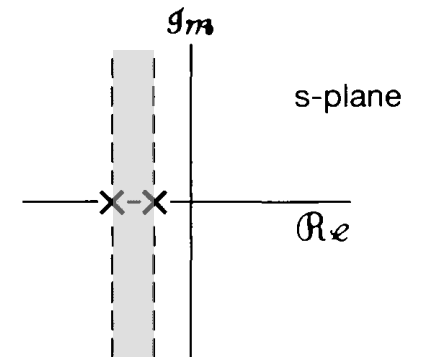
Left-sided

Has no FT



Two-sided

Has no FT



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The inverse Laplace transform

$$X(\sigma + j\omega) = \mathcal{F}\{x(t)e^{-\sigma t}\} = \int_{-\infty}^{+\infty} [x(t)e^{-\sigma t}]e^{-j\omega t} dt$$

$$x(t)e^{-\sigma t} = \mathcal{F}^{-1}\{X(\sigma + j\omega)\} = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\sigma + j\omega)e^{j\omega t} d\omega$$

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} X(\sigma + j\omega)e^{(\sigma + j\omega)t} d\omega$$

$$x(t) = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} X(s)e^{st} ds$$


$$\begin{aligned}s &= \sigma + j\omega \\ ds &= j d\omega\end{aligned}$$



The inverse Laplace transform

Examples

$$X(s) = \frac{1}{(s+1)(s+2)}, \quad \operatorname{Re}\{s\} > -1 \quad x(t) = ?$$

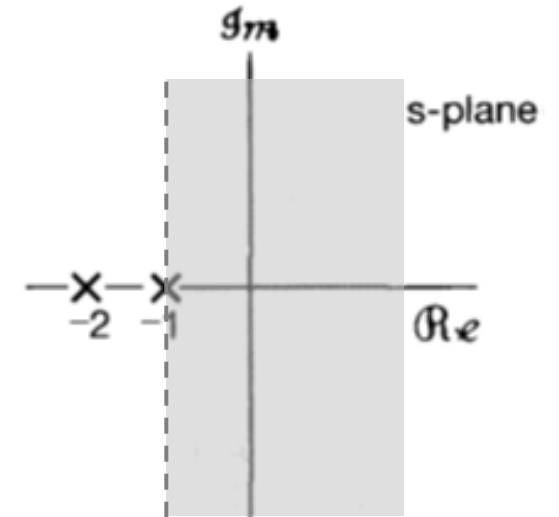
Solution

$$X(s) = \frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}$$

$$e^{-t}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+1} \quad \operatorname{Re}\{s\} > -1$$

$$e^{-2t}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+2} \quad \operatorname{Re}\{s\} > -2$$

$$x(t) = (e^{-t} - e^{-2t})u(t)$$



The inverse Laplace transform

Examples

$$X(s) = \frac{1}{(s+1)(s+2)}, \quad \operatorname{Re}\{s\} < -2 \quad x(t) = ?$$

Solution

$$X(s) = \frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}$$

$$-e^{-t}u(-t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+1} \quad \operatorname{Re}\{s\} < -1$$

$$-e^{-2t}u(-t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+2} \quad \operatorname{Re}\{s\} < -2$$

$$x(t) = (-e^{-t} + e^{-2t})u(-t)$$



The inverse Laplace transform

Examples

$$X(s) = \frac{1}{(s+1)(s+2)}, \quad -2 < \operatorname{Re}\{s\} < -1 \quad x(t) = ?$$

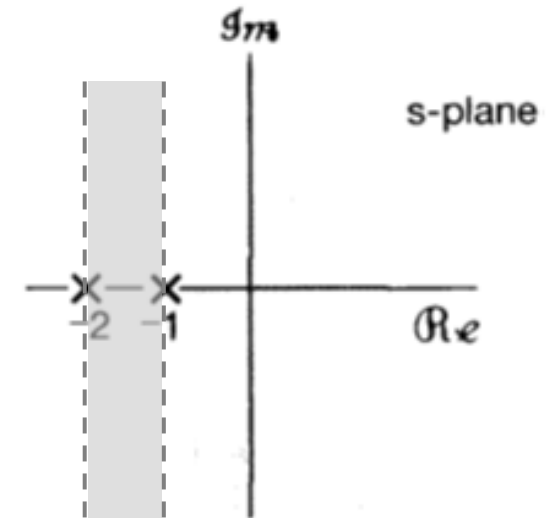
Solution

$$X(s) = \frac{1}{(s+1)(s+2)} = \frac{1}{s+1} - \frac{1}{s+2}$$

$$-e^{-t}u(-t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+1} \quad \operatorname{Re}\{s\} < -1$$

$$e^{-2t}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+2} \quad \operatorname{Re}\{s\} > -2$$

$$x(t) = -e^{-t}u(-t) - e^{-2t}u(t)$$



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Geometry evaluation of the Fourier transform from the pole-zero plot

□ Consider $X(s) = s - a$

$$|X(s_1)| = |\overrightarrow{s_1 - a}|$$

$$\angle X(s_1) = \angle \overrightarrow{s_1 - a}$$

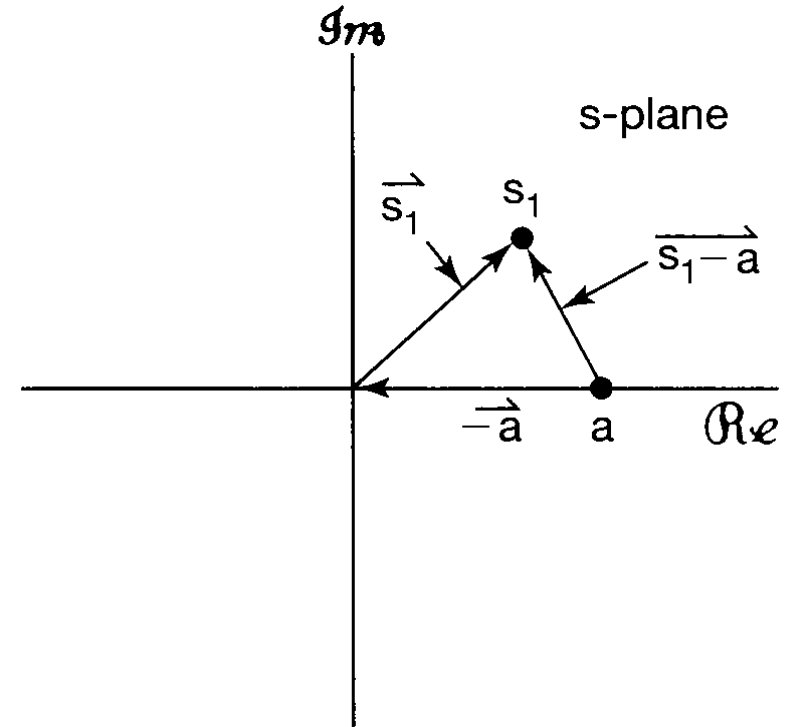
□ Consider $X(s) = 1/(s - a)$

$$|X(s_1)| = \frac{1}{|\overrightarrow{s_1 - a}|}$$

$$\angle X(s_1) = -\angle \overrightarrow{s_1 - a}$$

□ Consider $X(s) = M \frac{\prod_{i=1}^R (s - \beta_i)}{\prod_{j=1}^P (s - \alpha_j)}$

$$|X(s_1)| = |M| \frac{\prod_{i=1}^R |s_1 - \beta_i|}{\prod_{j=1}^P |s_1 - \alpha_j|} \quad \angle X(s_1) = \angle M + \sum_{i=1}^R \angle \overrightarrow{s_1 - \beta_i} - \sum_{j=1}^P \angle \overrightarrow{s_1 - \alpha_j}$$



Geometry evaluation of the Fourier transform from the pole-zero plot

Examples

$$X(s) = \frac{1}{s + 1/2}, \quad \operatorname{Re}\{s\} > -\frac{1}{2}$$

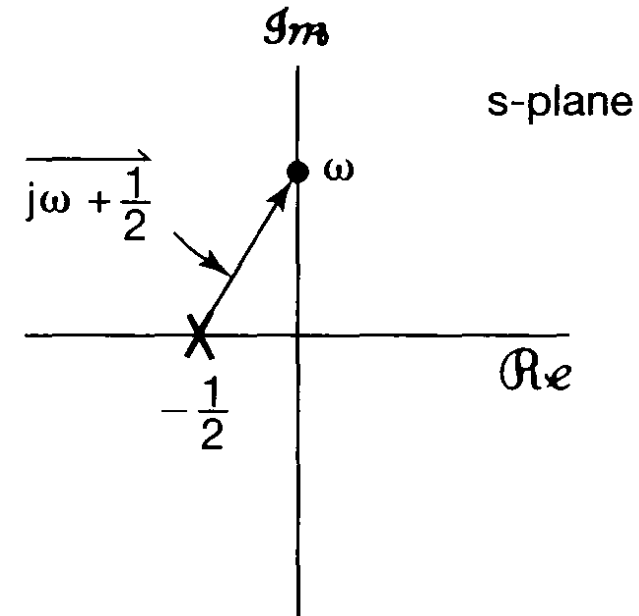
Magnitude and angle at $s = j\omega$?

Solution

$$X(j\omega) = \frac{1}{j\omega + 1/2}$$

$$|X(j\omega)|^2 = \frac{1}{\omega^2 + (1/2)^2}$$

$$\angle X(j\omega) = -\tan^{-1} 2\omega$$



Behavior of the Fourier transform can be obtained from the pole-zero plot



Geometry evaluation of the Fourier transform from the pole-zero plot

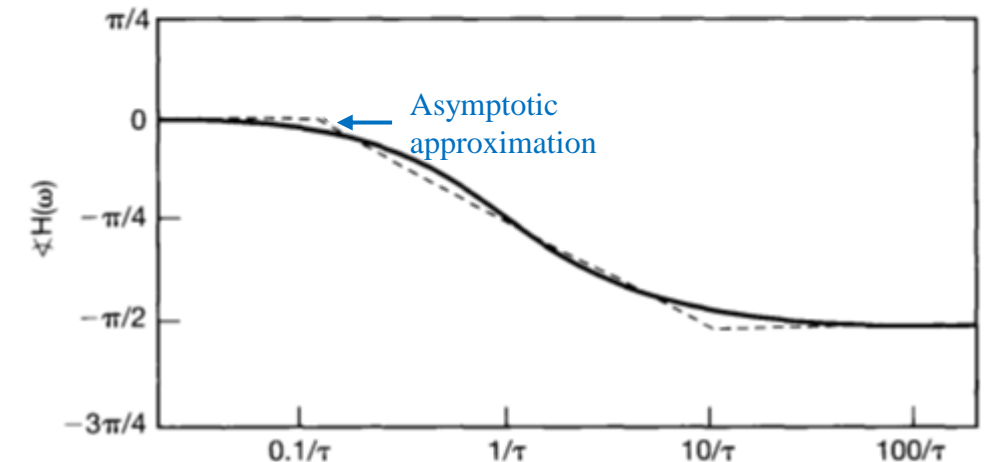
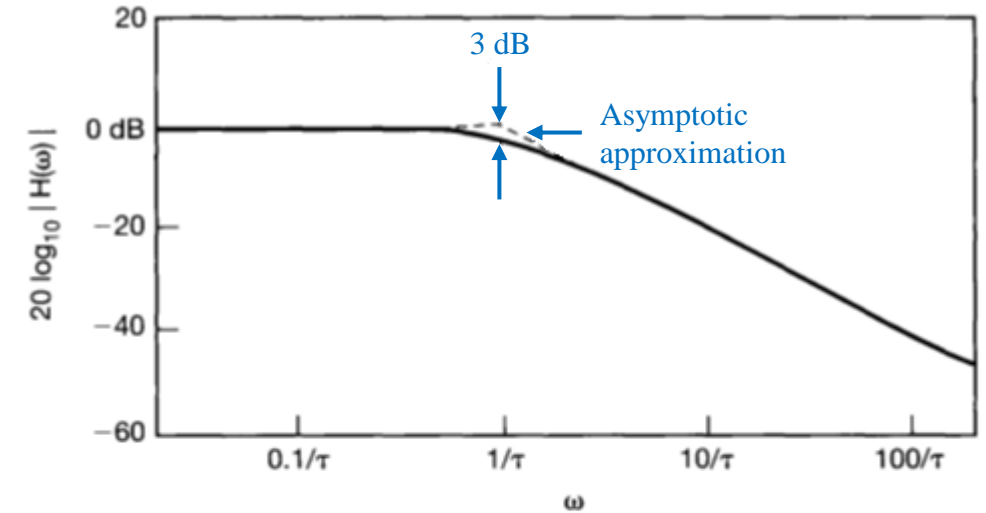
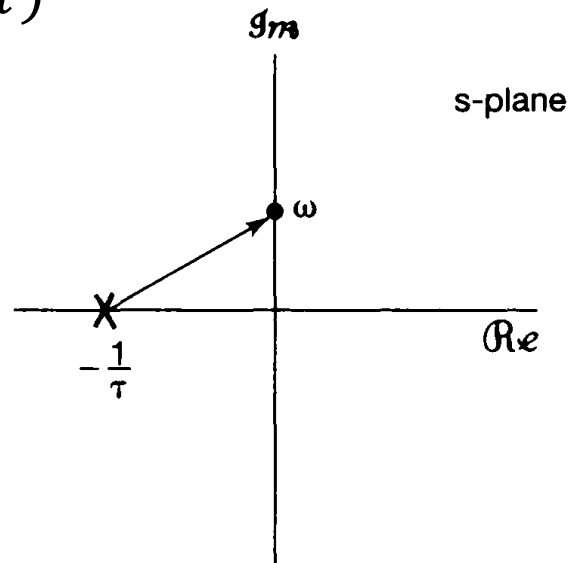
First-order systems

Consider $h(t) = \frac{1}{\tau} e^{-\frac{t}{\tau}} u(t)$

$$H(s) = \frac{1}{s\tau + 1}, \quad \operatorname{Re}\{s\} > -\frac{1}{\tau}$$

$$|H(j\omega)|^2 = \frac{1}{\tau^2} \cdot \frac{1}{\omega^2 + (1/\tau)^2}$$

$$\angle H(j\omega) = -\tan^{-1} \tau\omega$$



Geometry evaluation of the Fourier transform from the pole-zero plot

Second-order systems

$$h(t) = M[e^{c_1 t} - e^{c_2 t}]u(t)$$

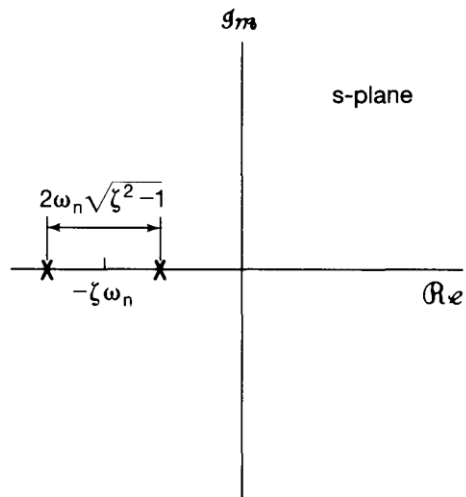
$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s - c_1)(s - c_2)}$$

$$c_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}$$

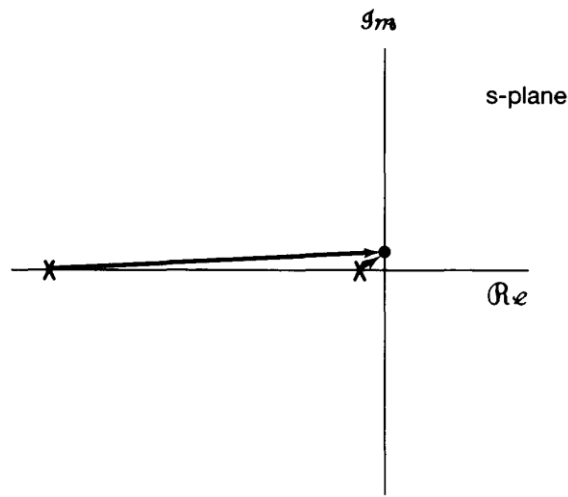
$$c_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$$

$$M = \frac{\omega_n}{2\sqrt{\zeta^2 - 1}}$$

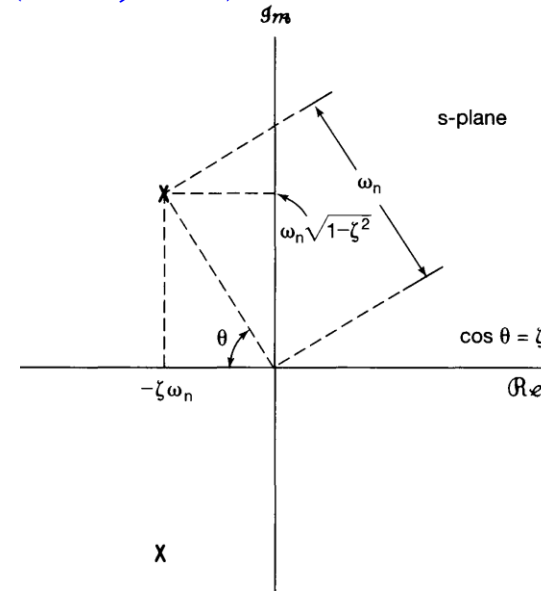
Pole-zero plot
($\zeta > 1$)



Pole vectors
($\zeta \gg 1$)



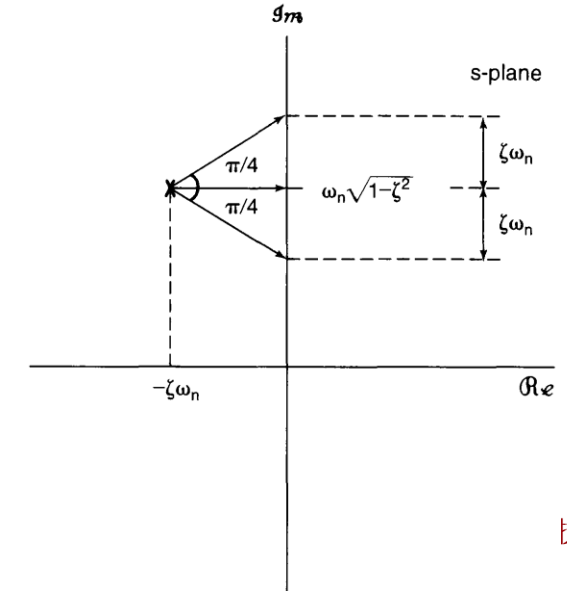
Pole-zero plot
($0 < \zeta < 1$)



Pole vectors ($0 < \zeta < 1$)

$$\omega = \omega_n\sqrt{\zeta^2 - 1}$$

$$\text{or } \omega = \omega_n\sqrt{\zeta^2 - 1} \pm \zeta\omega_n$$



Geometry evaluation of the Fourier transform from the pole-zero plot

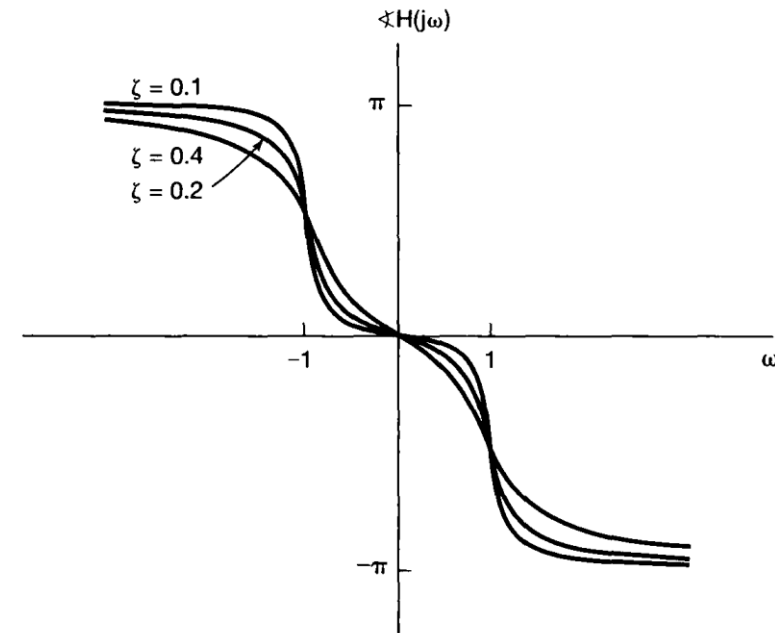
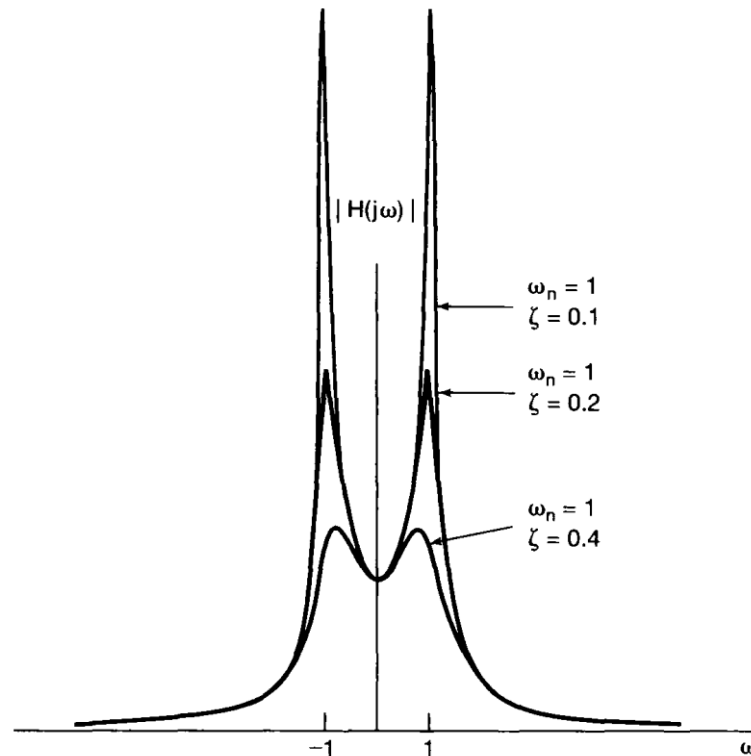
Second-order systems

$$h(t) = M[e^{c_1 t} - e^{c_2 t}]u(t)$$

$$H(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n^2}{(s - c_1)(s - c_2)}$$

$$0 < \zeta < 1$$

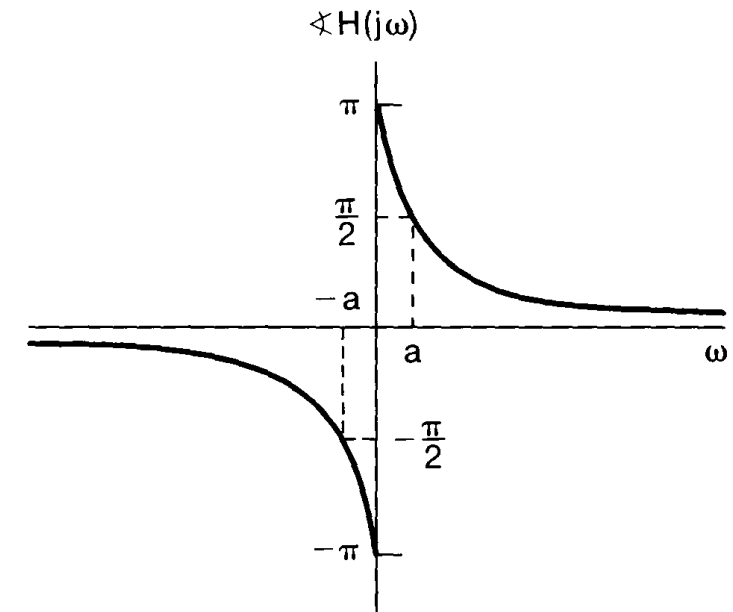
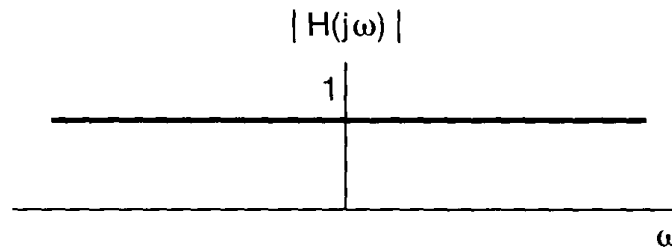
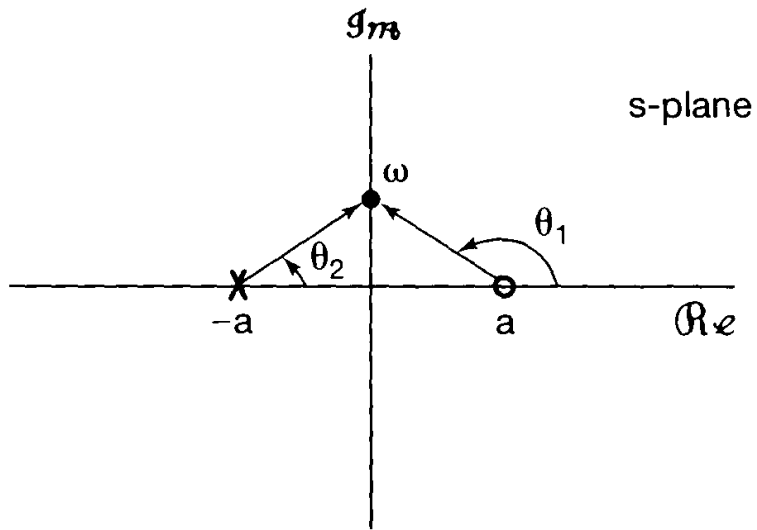
$$\begin{aligned} c_1 &= -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1} \\ c_2 &= -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1} \\ M &= \frac{\omega_n}{2\sqrt{\zeta^2 - 1}} \end{aligned}$$



Geometry evaluation of the Fourier transform from the pole-zero plot

All-pass systems

$$\angle H(j\omega) = \theta_1 - \theta_2 = \pi - 2\theta_2 = \pi - 2 \tan^{-1} \left(\frac{\omega}{a} \right)$$



Chapter 9: The Laplace Transform

- ❑ The Laplace transform
- ❑ The region of convergence for Laplace transforms
- ❑ The inverse Laplace transform
- ❑ Geometric evaluation of the Fourier transform from the pole-zero plot
- ❑ Properties of the Laplace transform
- ❑ Some Laplace transform pairs
- ❑ Analysis and characterization of LTI systems using the Laplace transform
- ❑ System function algebra and block diagram representations
- ❑ The unilateral Laplace transform



Properties of the Laplace transform

Linearity

$$x_1(t) \xleftrightarrow{\mathcal{L}} X_1(s) \quad \text{ROC} = R_1$$

$$\Rightarrow x(t) = ax_1(t) + bx_2(t) \xleftrightarrow{\mathcal{L}} aX_1(s) + bX_2(s)$$

$$x_2(t) \xleftrightarrow{\mathcal{L}} X_2(s) \quad \text{ROC} = R_2$$

ROC contains $R_1 \cap R_2$

$R_1 \cap R_2$ can be empty: $x(t)$ has no Laplace transform

ROC of $X(s)$ can also be larger than $R_1 \cap R_2$



Properties of the Laplace transform

Example

Consider $x(t) = x_1(t) - x_2(t)$

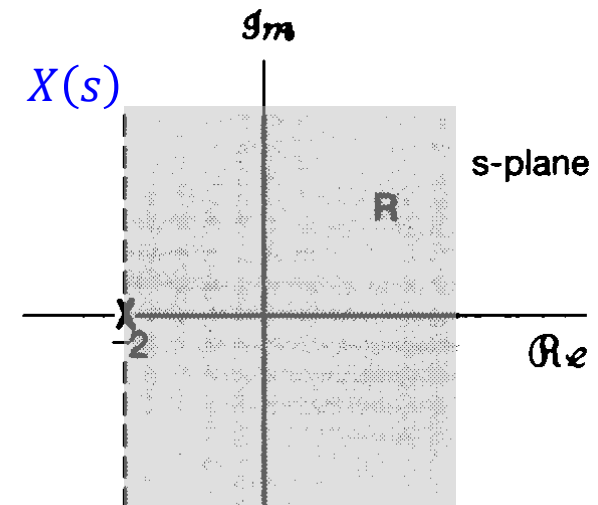
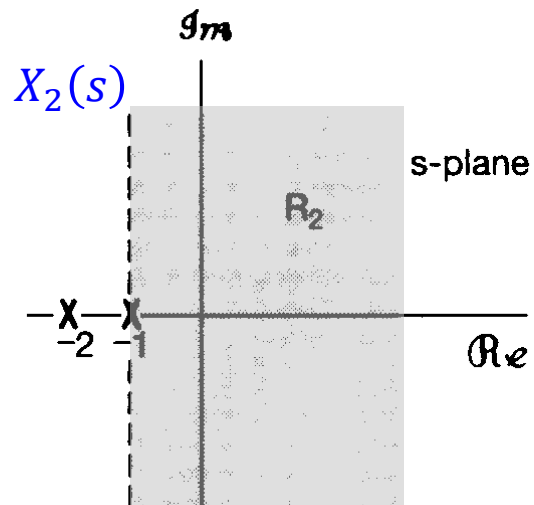
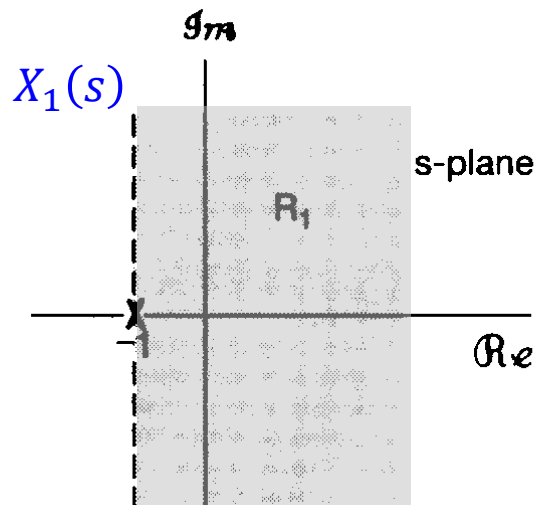
$$X_1(s) = \frac{1}{s+1}, \operatorname{Re}\{s\} > -1$$

$$X_2(s) = \frac{1}{(s+1)(s+2)}, \operatorname{Re}\{s\} > -1$$

$$X(s) = ?$$

Solution

$$X(s) = \frac{1}{s+1} - \frac{1}{(s+1)(s+2)} = \frac{s+1}{(s+1)(s+2)} = \frac{1}{s+2}, \quad \operatorname{Re}\{s\} > -2$$



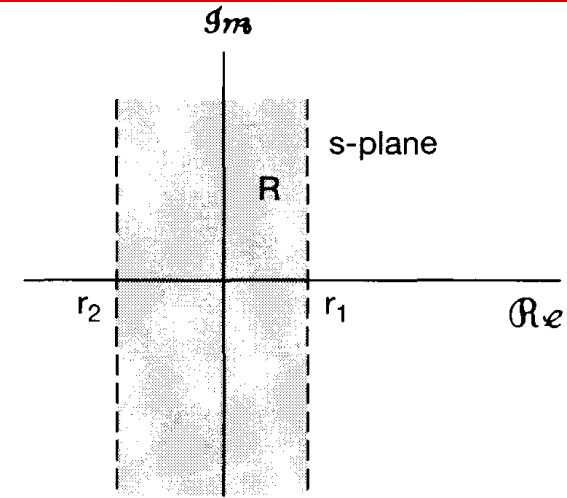
Properties of the Laplace transform

Time shifting

$$x(t) \xleftrightarrow{\mathcal{L}} X(s) \quad \text{ROC} = R$$

\Downarrow

$$x(t - t_0) \xleftrightarrow{\mathcal{L}} e^{-st_0} X(s) \quad \text{ROC} = R$$



\Downarrow

Shifting in the s-domain

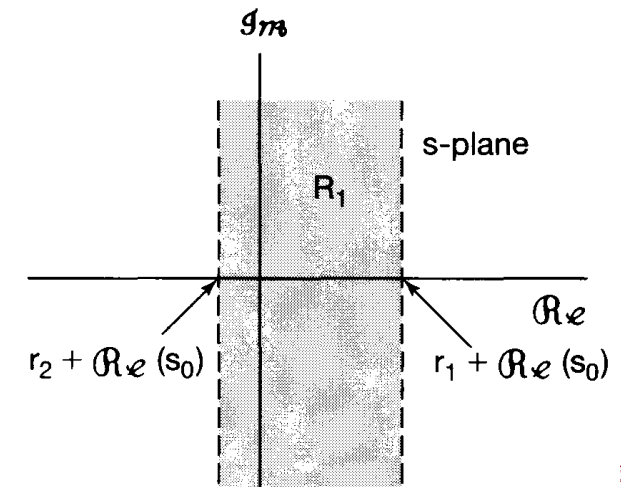
$$x(t) \xleftrightarrow{\mathcal{L}} X(s) \quad \text{ROC} = R$$

\Downarrow

$$e^{s_0 t} x(t) \xleftrightarrow{\mathcal{L}} X(s - s_0) \quad \text{ROC} = R + \mathcal{R}\{s_0\}$$

$\Downarrow \quad s_0 = j\omega_0$

$$e^{j\omega_0 t} x(t) \xleftrightarrow{\mathcal{L}} X(s - j\omega_0) \quad \text{ROC} = R$$



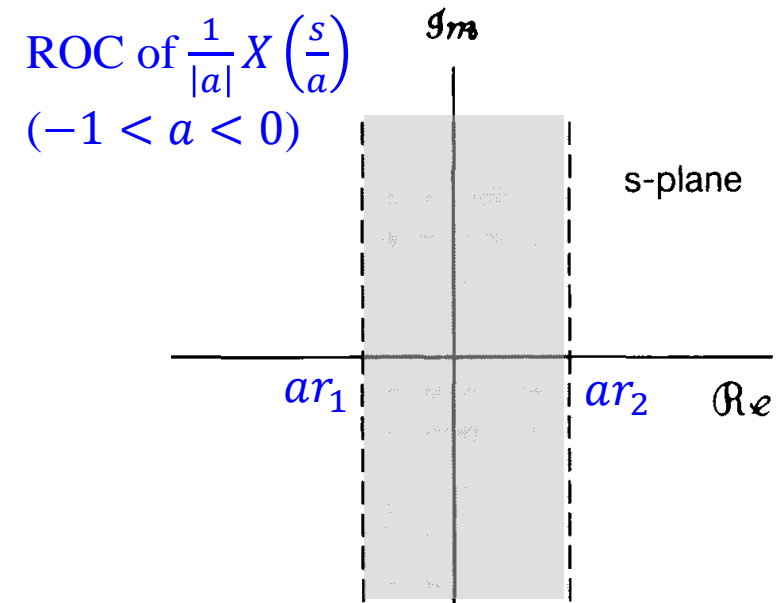
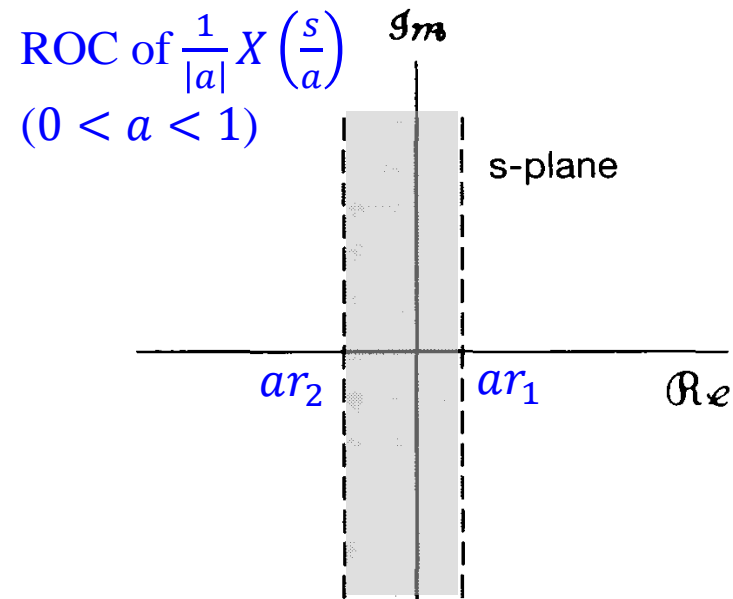
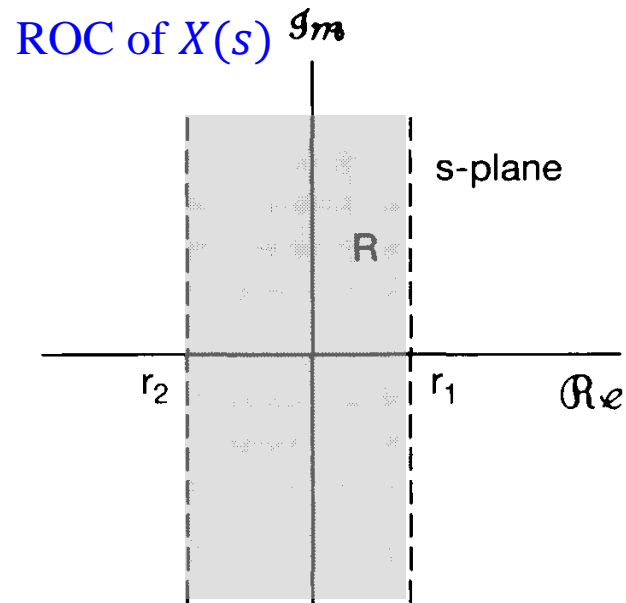
Properties of the Laplace transform

Time scaling

$$x(t) \xleftrightarrow{\mathcal{L}} X(s) \quad \text{ROC} = R$$

\Downarrow

$$x(at) \xleftrightarrow{\mathcal{L}} \frac{1}{|a|} X\left(\frac{s}{a}\right) \quad \text{ROC} = aR \quad \begin{matrix} a = -1 \\ \Rightarrow \end{matrix} \quad x(-t) \xleftrightarrow{\mathcal{L}} X(-s) \quad \text{ROC} = -R$$



Properties of the Laplace transform

Conjugation

$$x(t) \xleftrightarrow{\mathcal{L}} X(s) \quad \text{ROC} = R$$



$$x^*(t) \xleftrightarrow{\mathcal{L}} X^*(s^*) \quad \text{ROC} = R$$

$$X(s) = X^*(s^*) \text{ if } x(t) \text{ is real}$$

Convolution property

$$x_1(t) \xleftrightarrow{\mathcal{L}} X_1(s) \quad \text{ROC} = R_1$$

$$x_2(t) \xleftrightarrow{\mathcal{L}} X_2(s) \quad \text{ROC} = R_2$$

$$\Rightarrow x_1(t) * x_2(t) \xleftrightarrow{\mathcal{L}} X_1(s)X_2(s)$$

$$\text{ROC contains } R_1 \cap R_2$$



Properties of the Laplace transform

Differentiation in the time domain

$$x(t) \xleftrightarrow{\mathcal{L}} X(s) \quad \text{ROC} = R$$



$$\frac{dx(t)}{dt} \xleftrightarrow{\mathcal{L}} sX(s) \quad \text{ROC contains } R$$

$$x(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} X(s)e^{st} ds$$

$$\frac{dx(t)}{dt} = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} sX(s)e^{st} ds$$

Differentiation in the s-domain

$$x(t) \xleftrightarrow{\mathcal{L}} X(s) \quad \text{ROC} = R$$



$$-tx(t) \xleftrightarrow{\mathcal{L}} \frac{dX(s)}{ds} \quad \text{ROC} = R$$

$$X(s) = \int_{-\infty}^{+\infty} x(t)e^{-st} dt$$

$$\frac{dX(s)}{ds} = \int_{-\infty}^{+\infty} (-t)x(t)e^{-st} dt$$



Properties of the Laplace transform

Examples

$$\frac{t^{n-1}}{(n-1)!} e^{-at} u(t) \xleftrightarrow{\mathcal{L}} ?$$

Solution

Consider $x(t) = te^{-at}u(t)$

$$e^{-at}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+a} \quad \mathcal{Re}\{s\} > -a$$

$$te^{-at}u(t) \xleftrightarrow{\mathcal{L}} -\frac{d}{ds} \left[\frac{1}{s+a} \right] = \frac{1}{(s+a)^2} \quad \mathcal{Re}\{s\} > -a$$

$$\frac{t^2}{2} e^{-at}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{(s+a)^3} \quad \mathcal{Re}\{s\} > -a$$

$$\frac{t^{n-1}}{(n-1)!} e^{-at}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{(s+a)^n} \quad \mathcal{Re}\{s\} > -a$$



Properties of the Laplace transform

Examples

$$X(s) = \frac{2s^2 + 5s + 5}{(s + 1)^2(s + 2)}, \quad \operatorname{Re}\{s\} > -1 \quad x(t) = ?$$

Solution

$$X(s) = \frac{2}{(s + 1)^2} - \frac{1}{s + 1} + \frac{3}{s + 2}, \quad \operatorname{Re}\{s\} > -1$$

$$x(t) = [2te^{-t} - e^{-t} + 3e^{-2t}]u(t)$$



Properties of the Laplace transform

Integration in the time domain

$$x(t) \xleftrightarrow{\mathcal{L}} X(s) \quad \text{ROC} = R$$



$$\int_{-\infty}^t x(\tau) d\tau \xleftrightarrow{\mathcal{L}} \frac{1}{s} X(s) \quad \text{ROC contains } R \cap \{\mathcal{R}e\{s\} > 0\}$$

Proof

$$\int_{-\infty}^t x(\tau) d\tau = u(t) * x(t)$$

$$u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s} \quad \mathcal{R}e\{s\} > 0$$

$$u(t) * x(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s} X(s) \quad \text{ROC contains } R \cap \{\mathcal{R}e\{s\} > 0\}$$



Properties of the Laplace transform

The initial- and final-theorems

□ Initial-value theorem

If

$$x(t) = 0 \text{ for } t < 0,$$

$x(t)$ contains no impulses or higher order singularities at the origin,

Then,

$$x(0^+) = \lim_{s \rightarrow \infty} sX(s)$$

□ Final-value theorem

If

$$x(t) = 0 \text{ for } t < 0,$$

$x(t)$ has a finite limit as $t \rightarrow \infty$,

Then,

$$\lim_{t \rightarrow \infty} x(t) = \lim_{s \rightarrow 0} sX(s)$$



Properties of the Laplace transform

Summary

Section	Property	Signal	Laplace Transform	ROC
		$x(t)$ $x_1(t)$ $x_2(t)$	$X(s)$ $X_1(s)$ $X_2(s)$	R R_1 R_2
9.5.1	Linearity	$ax_1(t) + bx_2(t)$	$aX_1(s) + bX_2(s)$	At least $R_1 \cap R_2$
9.5.2	Time shifting	$x(t - t_0)$	$e^{-st_0} X(s)$	R
9.5.3	Shifting in the s -Domain	$e^{s_0 t} x(t)$	$X(s - s_0)$	Shifted version of R (i.e., s is in the ROC if $s - s_0$ is in R)
9.5.4	Time scaling	$x(at)$	$\frac{1}{ a } X\left(\frac{s}{a}\right)$	Scaled ROC (i.e., s is in the ROC if s/a is in R)
9.5.5	Conjugation	$x^*(t)$	$X^*(s^*)$	R
9.5.6	Convolution	$x_1(t) * x_2(t)$	$X_1(s)X_2(s)$	At least $R_1 \cap R_2$
9.5.7	Differentiation in the Time Domain	$\frac{d}{dt} x(t)$	$sX(s)$	At least R
9.5.8	Differentiation in the s -Domain	$-tx(t)$	$\frac{d}{ds} X(s)$	R
9.5.9	Integration in the Time Domain	$\int_{-\infty}^t x(\tau) d(\tau)$	$\frac{1}{s} X(s)$	At least $R \cap \{\Re\{s\} > 0\}$
Initial- and Final-Value Theorems				
9.5.10	If $x(t) = 0$ for $t < 0$ and $x(t)$ contains no impulses or higher-order singularities at $t = 0$, then			
	$x(0^+) = \lim_{s \rightarrow \infty} sX(s)$			
	If $x(t) = 0$ for $t < 0$ and $x(t)$ has a finite limit as $t \rightarrow \infty$, then			
	$\lim_{t \rightarrow \infty} x(t) = \lim_{s \rightarrow 0} sX(s)$			



Chapter 9: The Laplace Transform

- ❑ The Laplace transform
- ❑ The region of convergence for Laplace transforms
- ❑ The inverse Laplace transform
- ❑ Geometric evaluation of the Fourier transform from the pole-zero plot
- ❑ Properties of the Laplace transform
- ❑ **Some Laplace transform pairs**
- ❑ Analysis and characterization of LTI systems using the Laplace transform
- ❑ System function algebra and block diagram representations
- ❑ The unilateral Laplace transform



Some Laplace transform pairs

Transform pair	Signal	Transform	ROC
1	$\delta(t)$	1	All s
2	$u(t)$	$\frac{1}{s}$	$\Re\{s\} > 0$
3	$-u(-t)$	$\frac{1}{s}$	$\Re\{s\} < 0$
4	$\frac{t^{n-1}}{(n-1)!}u(t)$	$\frac{1}{s^n}$	$\Re\{s\} > 0$
5	$-\frac{t^{n-1}}{(n-1)!}u(-t)$	$\frac{1}{s^n}$	$\Re\{s\} < 0$
6	$e^{-\alpha t}u(t)$	$\frac{1}{s + \alpha}$	$\Re\{s\} > -\alpha$
7	$-e^{-\alpha t}u(-t)$	$\frac{1}{s + \alpha}$	$\Re\{s\} < -\alpha$
8	$\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(t)$	$\frac{1}{(s + \alpha)^n}$	$\Re\{s\} > -\alpha$

9	$-\frac{t^{n-1}}{(n-1)!}e^{-\alpha t}u(-t)$	$\frac{1}{(s + \alpha)^n}$	$\Re\{s\} < -\alpha$
10	$\delta(t - T)$	e^{-sT}	All s
11	$[\cos \omega_0 t]u(t)$	$\frac{s}{s^2 + \omega_0^2}$	$\Re\{s\} > 0$
12	$[\sin \omega_0 t]u(t)$	$\frac{\omega_0}{s^2 + \omega_0^2}$	$\Re\{s\} > 0$
13	$[e^{-\alpha t} \cos \omega_0 t]u(t)$	$\frac{s + \alpha}{(s + \alpha)^2 + \omega_0^2}$	$\Re\{s\} > -\alpha$
14	$[e^{-\alpha t} \sin \omega_0 t]u(t)$	$\frac{\omega_0}{(s + \alpha)^2 + \omega_0^2}$	$\Re\{s\} > -\alpha$
15	$u_n(t) = \frac{d^n \delta(t)}{dt^n}$	s^n	All s
16	$u_{-n}(t) = \underbrace{u(t) * \cdots * u(t)}_{n \text{ times}}$	$\frac{1}{s^n}$	$\Re\{s\} > 0$



Chapter 9: The Laplace Transform

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Analysis and characterization of LTI systems using the Laplace transform

e^{st} \longrightarrow LTI \longrightarrow $y(t) = H(s)e^{st}$

$$H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau$$

$x(t)$ \longrightarrow LTI \longrightarrow $y(t) = x(t) * h(t)$

$$Y(s) = X(s) H(s)$$

$H(s)$: system function or transfer function



Analysis and characterization of LTI systems using the Laplace transform

Causality

Causal \Rightarrow ROC of $H(s)$ is a right-half plane **Converse is not necessarily true**

A system with rational $H(s)$ is causal \Leftrightarrow ROC of $H(s)$ is the right-half plane to the right of the right-most pole

Examples $h(t) = e^{-t}u(t)$ Causal?

Solution 1

$$h(t) = 0 \text{ for } t < 0$$

\Rightarrow Causal

Solution 2

$$H(s) = \frac{1}{s+1} \quad \text{Re}\{s\} > -1$$

\Rightarrow Causal

Examples $h(t) = e^{-|t|}$ Causal?

Solution 1

$$h(t) \neq 0 \text{ for } t < 0$$

\Rightarrow Non-causal

Solution 2

$$H(s) = \frac{-2}{s^2 - 1} \quad -1 < \text{Re}\{s\} < 1$$

\Rightarrow Non-causal



Analysis and characterization of LTI systems using the Laplace transform

Examples

$$H(s) = \frac{e^s}{s+1}, \quad \operatorname{Re}\{s\} > -1 \quad \text{Causal?}$$

Solution

$$e^{-t}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+1} \quad \operatorname{Re}\{s\} > -1$$

$$e^{-(t+1)}u(t+1) \xleftrightarrow{\mathcal{L}} \frac{e^s}{s+1} \quad \operatorname{Re}\{s\} > -1$$

Time-shifting

$$h(t) = e^{-(t+1)}u(t+1)$$

⇒ Non-causal

$$x(t) \xleftrightarrow{\mathcal{L}} X(s) \quad \text{ROC} = R$$

$$\Downarrow$$
$$x(t-t_0) \xleftrightarrow{\mathcal{L}} e^{-st_0}X(s) \quad \text{ROC} = R$$



Analysis and characterization of LTI systems using the Laplace transform

Anti-causality

Anti-causal \Rightarrow ROC of $H(s)$ is a left-half plane Converse is not necessarily true

A system with rational $H(s)$ is anti-causal \Leftrightarrow ROC of $H(s)$ is the left-half plane to the left of the left-most pole



Analysis and characterization of LTI systems using the Laplace transform

Stability

Stable \Leftrightarrow The impulse response of $H(s)$ is absolutely integrable



Stable \Leftrightarrow The ROC of $H(s)$ includes the entire $j\omega$ -axis



Analysis and characterization of LTI systems using the Laplace transform

Examples

$$H(s) = \frac{s - 1}{(s + 1)(s - 2)}$$

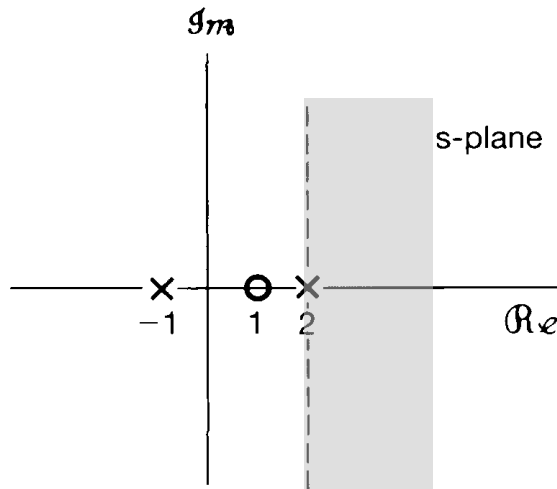
Causal? Stable?

Solution

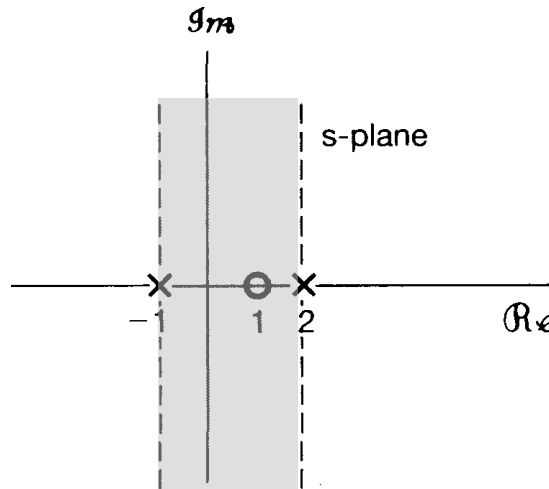
$$h(t) = \left(\frac{2}{3}e^{-t} + \frac{1}{3}e^{2t} \right) u(t)$$

$$h(t) = \frac{2}{3}e^{-t}u(t) - \frac{1}{3}e^{2t}u(-t)$$

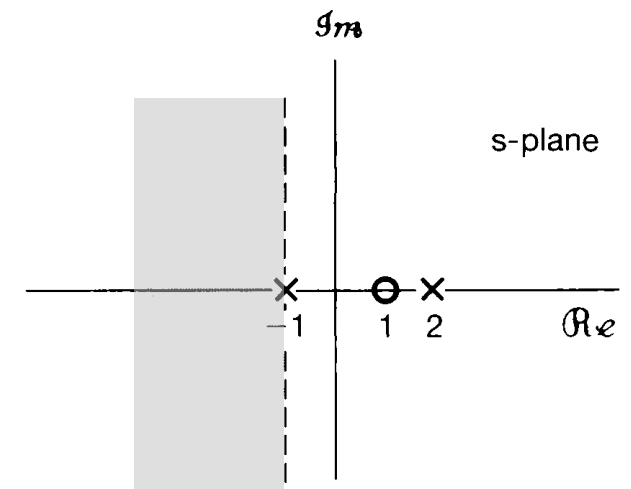
$$h(t) = -\left(\frac{2}{3}e^{-t} + \frac{1}{3}e^{2t} \right) u(-t)$$



Causal
Unstable system



Non-causal
Stable system



Anti-causal
Unstable system



Analysis and characterization of LTI systems using the Laplace transform

Stability

For a causal system, with rational system function $H(s)$,

Stable \Leftrightarrow All the poles of $H(s)$ lie in the left-half of the s -plane

OR

Stable \Leftrightarrow All the poles have negative real parts

Examples

$$H(s) = \frac{1}{(s + 1)}$$

Pole: $s = -1$

Causal \Rightarrow Stable

$$H(s) = \frac{1}{(s - 2)}$$

Pole: $s = 2$

Causal \Rightarrow Unstable



Analysis and characterization of LTI systems using the Laplace transform

Examples

Consider the class of causal second-order systems

$$h(t) = M[e^{c_1 t} - e^{c_2 t}]u(t)$$

$$H(s) = \frac{\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{\omega_n}{(s - c_1)(s - c_2)}$$

$$c_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}$$

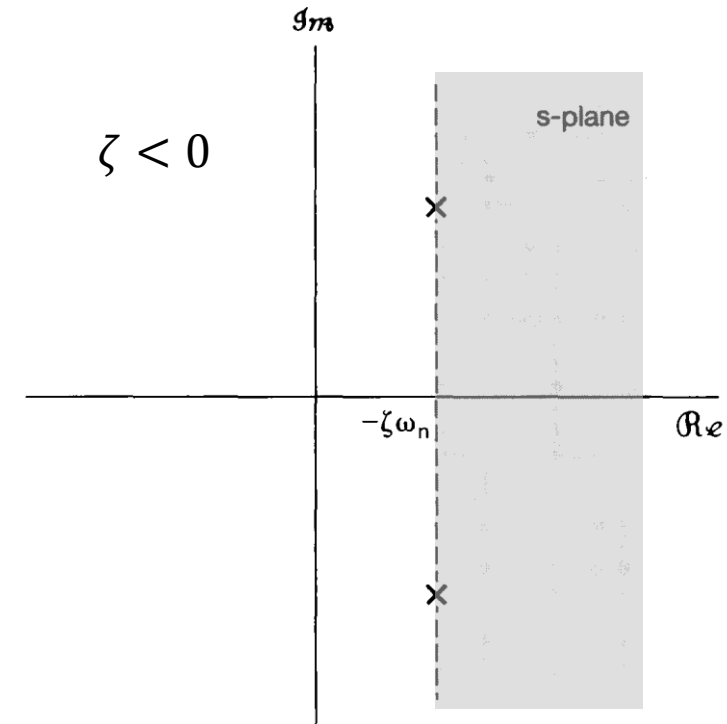
$$c_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$$

$$M = \frac{\omega_n}{2\sqrt{\zeta^2 - 1}}$$

Is the system stable when $\zeta < 0$?

Solution

Unstable



Analysis and characterization of LTI systems using the Laplace transform

LTI systems characterized by linear constant-coefficient differential equations

□ Examples

$$\frac{dy(t)}{dt} + 3y(t) = x(t)$$

$$sY(s) + 3Y(s) = X(s)$$

$$H(s) = \frac{1}{s + 3}$$

Differential equation: not a complete specification of the LTI system!

Pre-knowledge: if causal $h(t) = e^{-3t}u(t)$

Anti-causal $h(t) = -e^{-3t}u(-t)$



Analysis and characterization of LTI systems using the Laplace transform

LTI systems characterized by linear constant-coefficient differential equations

□ Generally

$$\sum_{k=0}^N a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^M b_k \frac{d^k x(t)}{dt^k}$$

$$\left(\sum_{k=0}^N a_k s^k \right) Y(s) = \left(\sum_{k=0}^M b_k s^k \right) X(s)$$

$$H(s) = \frac{\sum_{k=0}^M b_k s^k}{\sum_{k=0}^N a_k s^k}$$

\Rightarrow

Poles at the solution of
Zeros at the solution of

$$\sum_{k=0}^N a_k s^k = 0$$

$$\sum_{k=0}^M b_k s^k = 0$$



Analysis and characterization of LTI systems using the Laplace transform

Examples

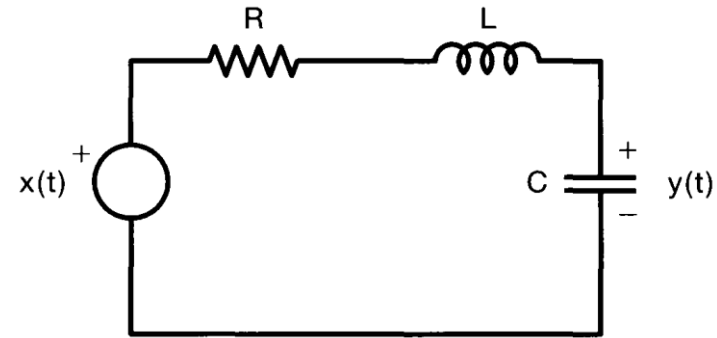
$$RC \frac{dy(t)}{dt} + LC \frac{d^2y(t)}{dt^2} + y(t) = x(t)$$

Solution

$$H(s) = \frac{1/LC}{s^2 + (R/L)s + (1/LC)}$$

Poles have negative real parts when $R > 0$, $L > 0$, and $C > 0$

⇒ Stable



Analysis and characterization of LTI systems using the Laplace transform

Examples relating system behavior to the system function

If the input to an LTI system is $x(t) = e^{-3t}u(t)$

Then the output is $y(t) = [e^{-t} - e^{-2t}]u(t)$

System function?

Solution

$$X(s) = \frac{1}{s+3}, \quad \operatorname{Re}\{s\} > -3$$

$$Y(s) = \frac{1}{(s+1)(s+2)}, \quad \operatorname{Re}\{s\} > -1$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{s+3}{(s+1)(s+2)} = \frac{s+3}{s^2+3s+2}$$

Causal and stable

$$\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = \frac{dx(t)}{dt} + 3x(t)$$



Analysis and characterization of LTI systems using the Laplace transform

Examples relating system behavior to the system function

Given the following information about an LTI system, determine $H(s)$.

1. The system is causal;
2. $H(s)$ is rational and has only two poles at $s = -2$ and $s = 4$;
3. If $x(t) = 1$, then $y(t) = 0$;
4. $h(0^+) = 4$

Solution

$$H(s) = \frac{p(s)}{(s+2)(s-4)} = \frac{p(s)}{s^2 - 2s - 8} \quad p(s) \text{ is a polynomial in } s$$

$$p(0) = 0 \Rightarrow p(s) = sq(s) \quad q(s) \text{ is a polynomial in } s$$

$$\lim_{s \rightarrow \infty} sH(s) = \lim_{s \rightarrow \infty} \frac{s^2 q(s)}{s^2 - 2s - 8} = \lim_{s \rightarrow \infty} \frac{Ks^2}{s^2 - 2s - 8} = 4 \quad q(s) = K \text{ is a constant}$$

$$K = 4 \Rightarrow H(s) = \frac{4s}{(s+2)(s-4)}, \quad \operatorname{Re}\{s\} > 4$$



Analysis and characterization of LTI systems using the Laplace transform

Examples relating system behavior to the system function

A stable and causal system with impulse response $h(t)$ and system function $H(s)$, which is rational and contains a pole at $s = -2$, and does not have a zero at the origin.

- ☐ $\mathcal{F}\{h(t)e^{3t}\}$ converges. False
- ☐ $\int_{-\infty}^{+\infty} h(t)dt = 0$ False
- ☐ $th(t)$ is the impulse response of a causal and stable system. True
- ☐ $dh(t)/dt$ contains at least one pole in its Laplace transform. True
- ☐ $h(t)$ has finite duration. False
- ☐ $H(s) = H(-s)$. False
- ☐ $\lim_{s \rightarrow \infty} H(s) = 2$. Insufficient information



Chapter 9: The Laplace Transform

- ❑ The Laplace transform
- ❑ The region of convergence for Laplace transforms
- ❑ The inverse Laplace transform
- ❑ Geometric evaluation of the Fourier transform from the pole-zero plot
- ❑ Properties of the Laplace transform
- ❑ Some Laplace transform pairs
- ❑ Analysis and characterization of LTI systems using the Laplace transform
- ❑ System function algebra and block diagram representations
- ❑ The unilateral Laplace transform



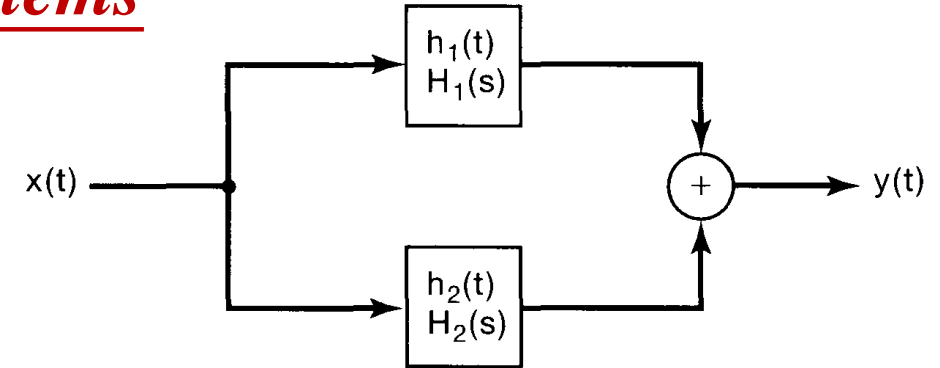
System function algebra and block diagram representations

System functions for interconnections of LTI systems

□ Parallel interconnection

$$h(t) = h_1(t) + h_2(t)$$

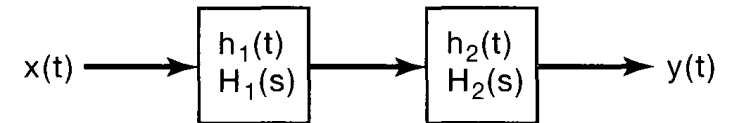
$$H(s) = H_1(s) + H_2(s)$$



□ Series interconnection

$$h(t) = h_1(t) * h_2(t)$$

$$H(s) = H_1(s)H_2(s)$$



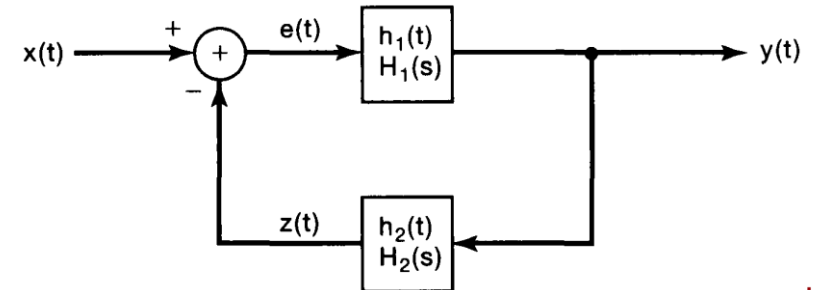
□ Feedback interconnection

$$Y(s) = H_1(s)E(s)$$

$$E(s) = X(s) - Z(s)$$

$$Z(s) = H_2(s)Y(s)$$

$$H(s) = \frac{H_1(s)}{1 + H_1(s)H_2(s)}$$



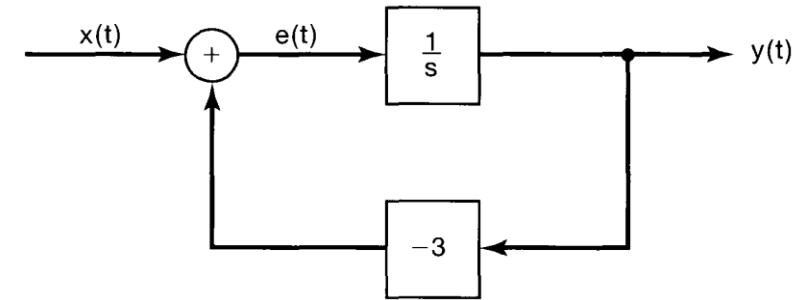
System function algebra and block diagram representations

Block diagram representations for causal LTI systems

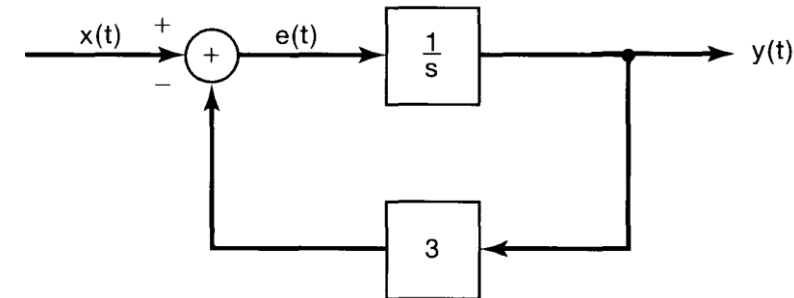
$$H(s) = \frac{1}{s + 3}$$

$$\frac{dy(t)}{dt} + 3y(t) = x(t)$$

$$H(s) = \frac{1/s}{1 + 3/s}$$



Or equivalently



Using basic operations: addition,
multiplication, and integration



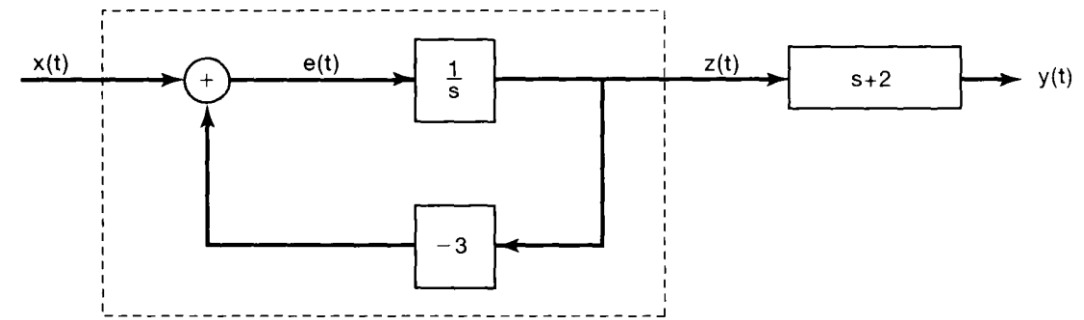
System function algebra and block diagram representations

Examples: block diagram representations for causal LTI systems

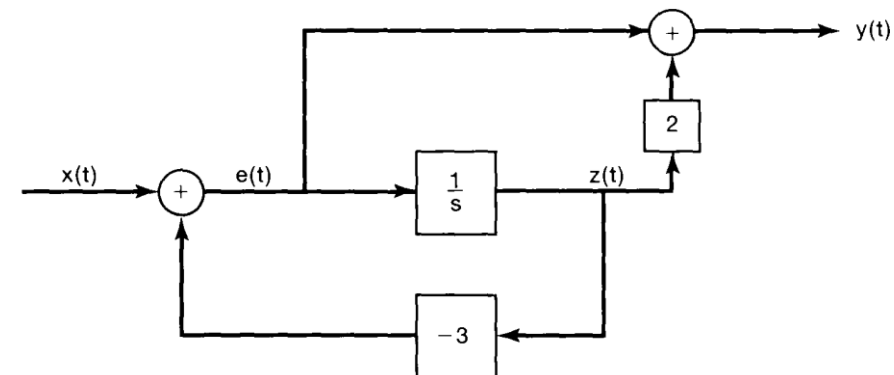
$$H(s) = \frac{s+2}{s+3} = \left(\frac{1}{s+3} \right) (s+2)$$

$$y(t) = \frac{dz(t)}{dt} + 2z(t)$$

$$y(t) = e(t) + 2z(t)$$



Or equivalently



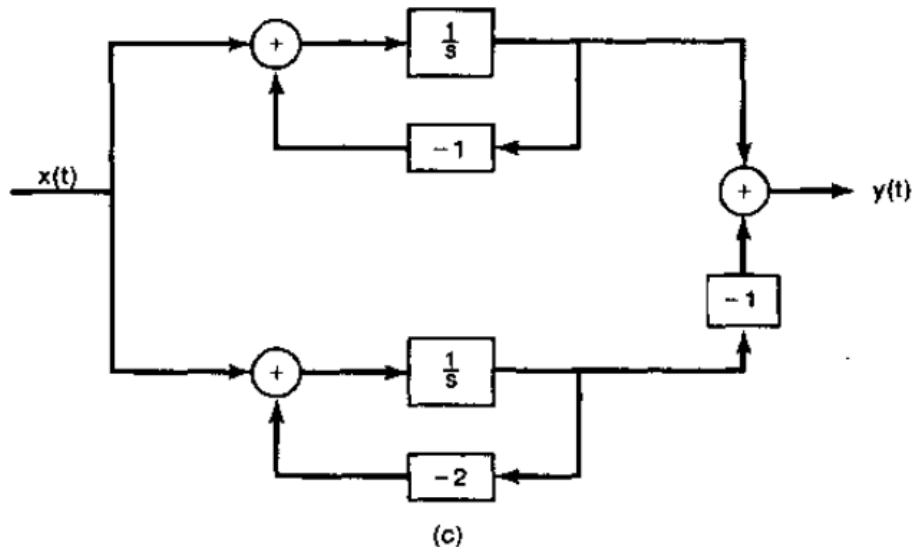
System function algebra and block diagram representations

Examples: block diagram representations for causal LTI systems

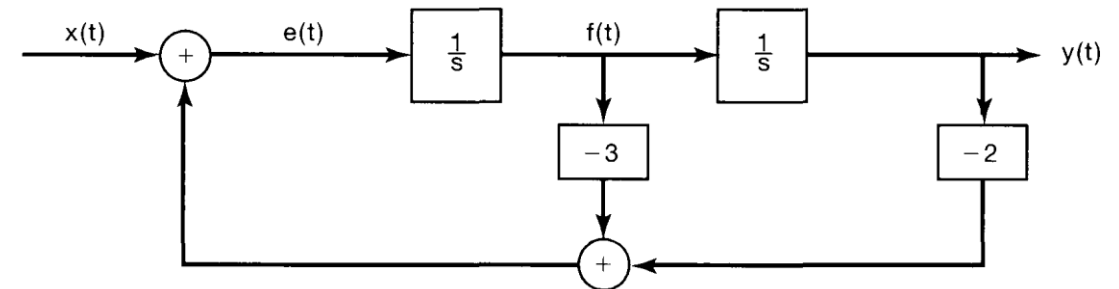
$$H(s) = \frac{1}{s^2 + 3s + 2} = \frac{1}{(s + 1)} \cdot \frac{1}{(s + 2)} = \frac{1}{(s + 1)} - \frac{1}{(s + 2)}$$

$$\frac{d^2 y(t)}{dt^2} + 3 \frac{dy(t)}{dt} + 2y(t) = x(t)$$

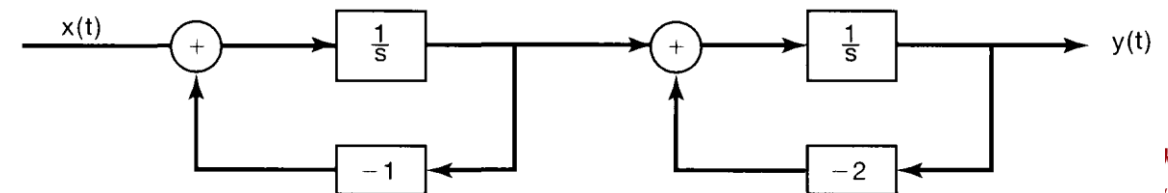
Parallel form



Direct form $e(t) = \frac{d^2 y(t)}{dt^2}$



Cascade form

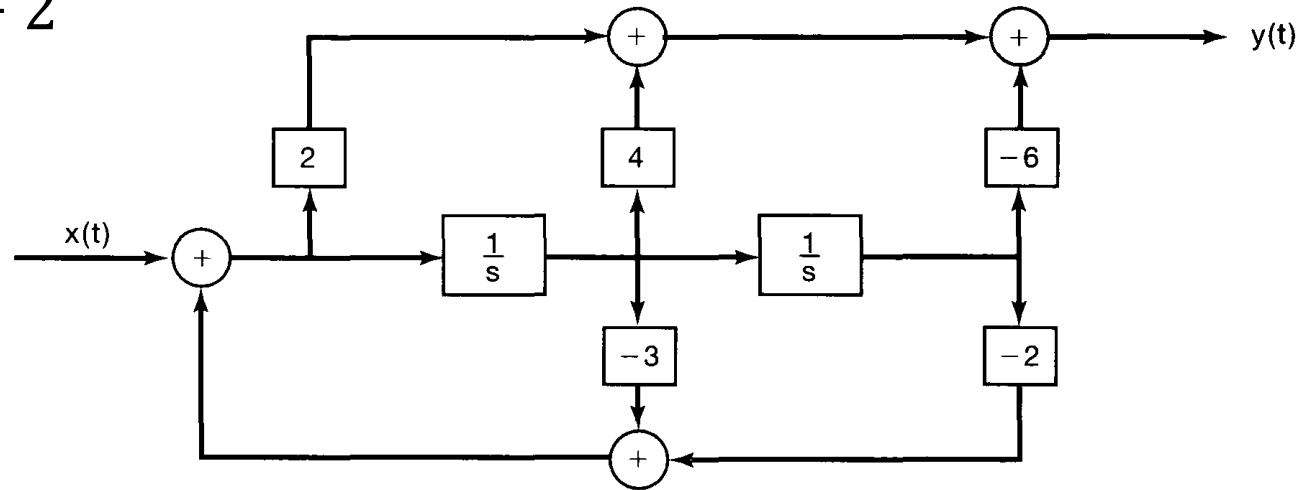


System function algebra and block diagram representations

Examples: block diagram representations for causal LTI systems

$$H(s) = \frac{2s^2 + 4s - 6}{s^2 + 3s + 2}$$

Direct form



Parallel form

$$H(s) = 2 + \frac{6}{s+2} - \frac{8}{s+1}$$

Cascade form

$$H(s) = \left(\frac{2(s-1)}{s+2} \right) \left(\frac{s+3}{s+1} \right)$$

$$H(s) = \left(\frac{s+3}{s+2} \right) \left(\frac{2(s-1)}{s+2} \right)$$



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The unilateral Laplace transform


$$x(t) \xleftrightarrow{\mathcal{U}\mathcal{L}} \mathcal{X}(s) = \mathcal{U}\mathcal{L}\{x(t)\}$$

$$\mathcal{X}(s) \triangleq \int_{0^-}^{\infty} x(t)e^{-st}dt$$

Examples

$$x(t) = \frac{t^{n-1}}{(n-1)!} e^{-at} u(t)$$

$$\mathcal{X}(s) \triangleq \frac{1}{(s+a)^n}, \quad \operatorname{Re}\{s\} > -a$$

 $x(t) = 0, \text{ for } t < 0$

- $x(t) = 0, \text{ for } t < 0$, the unilateral and bilateral transforms are identical



The unilateral Laplace transform

Examples

$$x(t) = e^{-a(t+1)}u(t+1)$$

$$X(s) = \frac{e^s}{s+a}, \quad \operatorname{Re}\{s\} > -a$$

$$\begin{aligned}\mathcal{X}(s) &= \int_{0^-}^{\infty} e^{-a(t+1)}u(t+1)e^{-st}dt \\ &= \int_{0^-}^{\infty} e^{-a}e^{-t(s+a)}dt \\ &= \frac{e^{-a}}{s+a}, \quad \operatorname{Re}\{s\} > -a\end{aligned}$$

- $x(t) \neq 0$, for $-1 < t < 0$, the unilateral and bilateral transforms are different



The unilateral Laplace transform

Examples

$$x(t) = \delta(t) + 2u_1(t) + e^t u(t)$$

$x(t) = 0, \text{ for } t < 0$

$$\mathcal{X}(s) = X(s)$$

$$\begin{aligned} &= 1 + 2s + \frac{1}{s-1} \\ &= \frac{s(2s-1)}{s-1}, \quad \operatorname{Re}\{s\} > 1 \end{aligned}$$

$$u_n(t) = \frac{d^n \delta(t)}{dt^n} \xleftrightarrow{\mathcal{L}} s^n$$

Examples

$$\mathcal{X}(s) = \frac{1}{(s+1)(s+2)}, \quad \operatorname{Re}\{s\} > -1$$

$$x(t) = [e^{-t} - e^{-2t}]u(t) \quad \text{for } t > 0^-$$



The unilateral Laplace transform

Examples

$$\begin{aligned} \mathcal{X}(s) &= \frac{s^2 - 3}{s + 2} \\ &= -2 + s + \frac{1}{s + 2}, \quad \operatorname{Re}\{s\} > -2 \end{aligned}$$

$$x(t) = -2\delta(t) + u_1(t) + e^{-2t}u(t) \quad \text{for } t > 0^-$$

Note: $u_n(t) = \frac{d^n \delta(t)}{dt^n}$



The unilateral Laplace transform

Properties of the unilateral Laplace transform

Property	Signal	Unilateral Laplace Transform
	$x(t)$ $x_1(t)$ $x_2(t)$	$\mathfrak{X}(s)$ $\mathfrak{X}_1(s)$ $\mathfrak{X}_2(s)$
Linearity	$ax_1(t) + bx_2(t)$	$a\mathfrak{X}_1(s) + b\mathfrak{X}_2(s)$
Shifting in the s -domain	$e^{s_0 t} x(t)$	$\mathfrak{X}(s - s_0)$
Time scaling	$x(at), \quad a > 0$	$\frac{1}{a} \mathfrak{X}\left(\frac{s}{a}\right)$
Conjugation	$x^*(t)$	$x^*(s)$
Convolution (assuming that $x_1(t)$ and $x_2(t)$ are identically zero for $t < 0$)	$x_1(t) * x_2(t)$	$\mathfrak{X}_1(s)\mathfrak{X}_2(s)$

Property	Signal	Unilateral Laplace Transform
Differentiation in the time domain	$\frac{d}{dt} x(t)$	$s\mathfrak{X}(s) - x(0^-)$
Differentiation in the s -domain	$-tx(t)$	$\frac{d}{ds} \mathfrak{X}(s)$
Integration in the time domain	$\int_{0^-}^t x(\tau) d\tau$	$\frac{1}{s} \mathfrak{X}(s)$

Initial- and Final-Value Theorems

If $x(t)$ contains no impulses or higher-order singularities at $t = 0$, then

$$x(0^+) = \lim_{s \rightarrow \infty} s\mathfrak{X}(s)$$

$$\lim_{t \rightarrow \infty} x(t) = \lim_{s \rightarrow 0} s\mathfrak{X}(s)$$

Note: no ROC is specified cause it is always the right-half plane

The unilateral Laplace transform

Convolution property

$$x_1(t) \xleftrightarrow{\mathcal{U}\mathcal{L}} X_1(s)$$

$$x_2(t) \xleftrightarrow{\mathcal{U}\mathcal{L}} X_2(s)$$

$$x_1(t) * x_2(t) \xleftrightarrow{\mathcal{U}\mathcal{L}} X_1(s)X_2(s)$$

Only if $x_1(t)$ and $x_2(t)$ are zero for $t < 0$

□ Example A **causal** LTI system:

$$\frac{d^2 y(t)}{dt^2} + 3 \frac{dy(t)}{dt} + 2y(t) = x(t)$$

If: $x(t) = \alpha u(t)$, $y(t) = ?$

□ Solution

$$\text{Causal} \Rightarrow \mathcal{H}(s) = H(s) = \frac{1}{s^2 + 3s + 2}$$

$$Y(s) = \mathcal{H}(s)X(s) = \frac{\alpha}{s(s+1)(s+2)} = \frac{\alpha/2}{s} - \frac{\alpha}{s+1} + \frac{\alpha/2}{s+2}$$

convolution property for unilateral Laplace transforms

$$\therefore y(t) = \alpha \left[\frac{1}{2} - e^{-t} + \frac{1}{2} e^{-2t} \right] u(t)$$

Note: this can also be done by bilateral Laplace transforms



The unilateral Laplace transform

Differentiation property

$$x(t) \xleftrightarrow{\mathcal{U}\mathcal{L}} \mathcal{X}(s) \quad \frac{dx(t)}{dt} \xleftrightarrow{\mathcal{U}\mathcal{L}} s\mathcal{X}(s) - x(0^-)$$

$$\int_{0^-}^{\infty} \frac{dx(t)}{dt} e^{-st} dt = x(t)e^{-st} \Big|_{0^-}^{\infty} + s \int_{0^-}^{\infty} x(t)e^{-st} dt = s\mathcal{X}(s) - x(0^-)$$

Similarly

$$\frac{d^2x(t)}{dt^2} \xleftrightarrow{\mathcal{U}\mathcal{L}} s^2\mathcal{X}(s) - sx(0^-) - x'(0^-)$$



The unilateral Laplace transform

Solving differential equations using the unilateral Laplace transform

□ Why unilateral Laplace transform? Non-zero initial condition

□ Example: A LTI system: $\frac{d^2 y(t)}{dt^2} + 3 \frac{dy(t)}{dt} + 2y(t) = x(t)$ $y(0^-) = \beta$ $y'(0^-) = \gamma$
If $x(t) = \alpha u(t)$, $y(t) = ?$

□ Solution

$$s^2 Y(s) - \beta s - \gamma + 3sY(s) - 3\beta + 2Y(s) = \frac{\alpha}{s}$$

$$Y(s) = \underbrace{\frac{\beta(s+3)}{(s+1)(s+2)} + \frac{\gamma}{(s+1)(s+2)}}_{\text{Zero-input response}} + \underbrace{\frac{\alpha}{s(s+1)(s+2)}}_{\text{Zero-state response}}$$

