EE160: Introduction to Control

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Lecture 11 The Design of State Variable Feedback Systems

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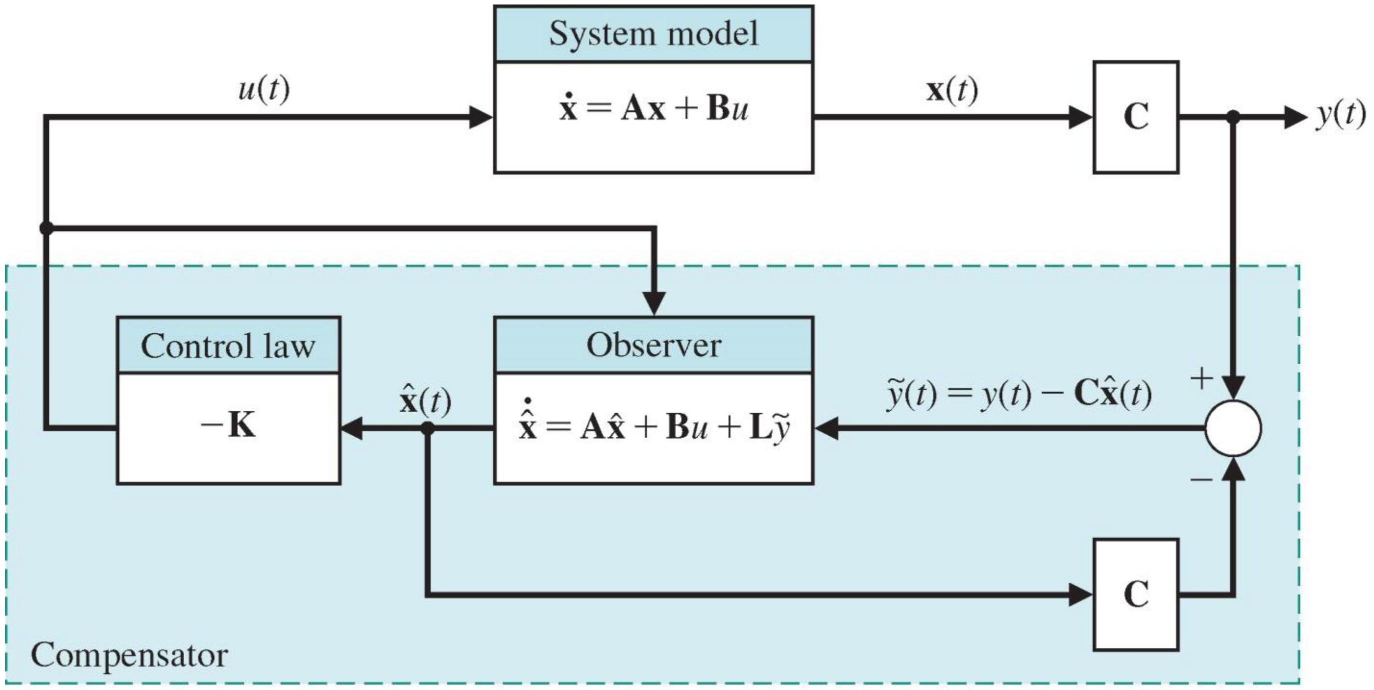
Introduction

The design of controllers utilizing state feedback is the subject of this chapter.

State feedback design typically comprises three steps:



1 we assume that all the state variables are measurable and utilize them in a full-state feedback control law.

2 to construct an observer to estimate the states that are not directly measured and available as outputs.

3 is to appropriately connect the observer to the full-state feedback control law

Upon completion of Lecture 11, we should:

- *Be familiar with the concepts of controllability and observability.*
- *Be able to design full-state feedback controllers and observers.*
- *Appreciate pole-placement methods.*
- *Understand the separation principle and how to construct state variable compensators.*
- *Have a working knowledge of optimal control, and internal model design.*

Controllability and Observability

A key question that arises in the design of state variable compensators is whether or not all the poles of the closed-loop system can be arbitrarily placed in the complex plane.

if the system is controllable and observable, then we can.

The concepts of controllability and observability were introduced by Kalman in the 1960s:

A system is completely controllable if there exists an unconstrained control $\mathbf{u}(t)$ that can transfer any initial state $\mathbf{x}(t_0)$ to any other desired location $\mathbf{x}(t)$ in a finite time, $t_0 \le t \le T$.

For the SISO LTI system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t),$$

we can determine whether the system is controllable by examining the algebraic condition

$$\operatorname{rank}[\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A}^2\mathbf{B} \ldots \mathbf{A}^{n-1}\mathbf{B}] = n.$$

dimension of the system.

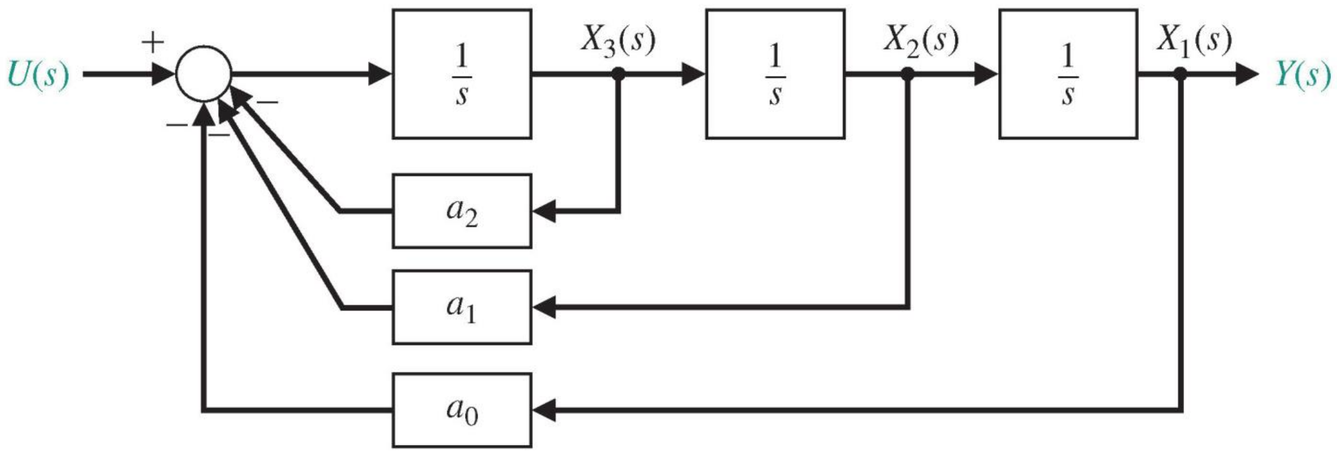
The controllability matrix Pc is described in terms of A and B as

$$\mathbf{P}_c = [\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A}^2\mathbf{B} \ldots \mathbf{A}^{n-1}\mathbf{B}],$$

Therefore, if the determinant of Pc is nonzero, the system is controllable.

Controllability and Observability

Example: *Let us consider the system*



Check whether the system is controllable or not?

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \mathbf{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u(t).$$

$$y(t) = [1 \quad 0 \quad 0]\mathbf{x}(t) + [0]u(t)$$

$$\mathbf{P}_c = [\mathbf{B} \quad \mathbf{AB} \quad \mathbf{A}^2\mathbf{B}] = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & -a_2 \\ 1 & -a_2 & a_2^2 - a_1 \end{bmatrix}.$$

The determinant of Pc = -1, hence this system is controllable.

Controllability and Observability

Observability refers to the ability to estimate a state variable.

A system is completely observable if and only if there exists a finite time T such that the initial state $\mathbf{x}(0)$ can be determined from the observation history $y(t)$ given the control $\mathbf{u}(t)$, $0 \leq t \leq T$.

For the SISO LTI system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t),$$

This system is completely observable when the determinant of the observability matrix Po is nonzero,

$$\mathbf{P}_o = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \\ \vdots \\ \mathbf{CA}^{n-1} \end{bmatrix},$$

Supplementary:

1. Advanced state variable design techniques can handle situations wherein the system is not completely controllable, but where the states (or linear combinations thereof) that cannot be controlled are inherently stable. These systems are classified as stabilizable.

2. These same techniques can handle cases wherein the system is not completely observable, but where the states (or linear combinations thereof) that cannot be observed are inherently stable. These systems are classified as detectable.

Controllability and Observability

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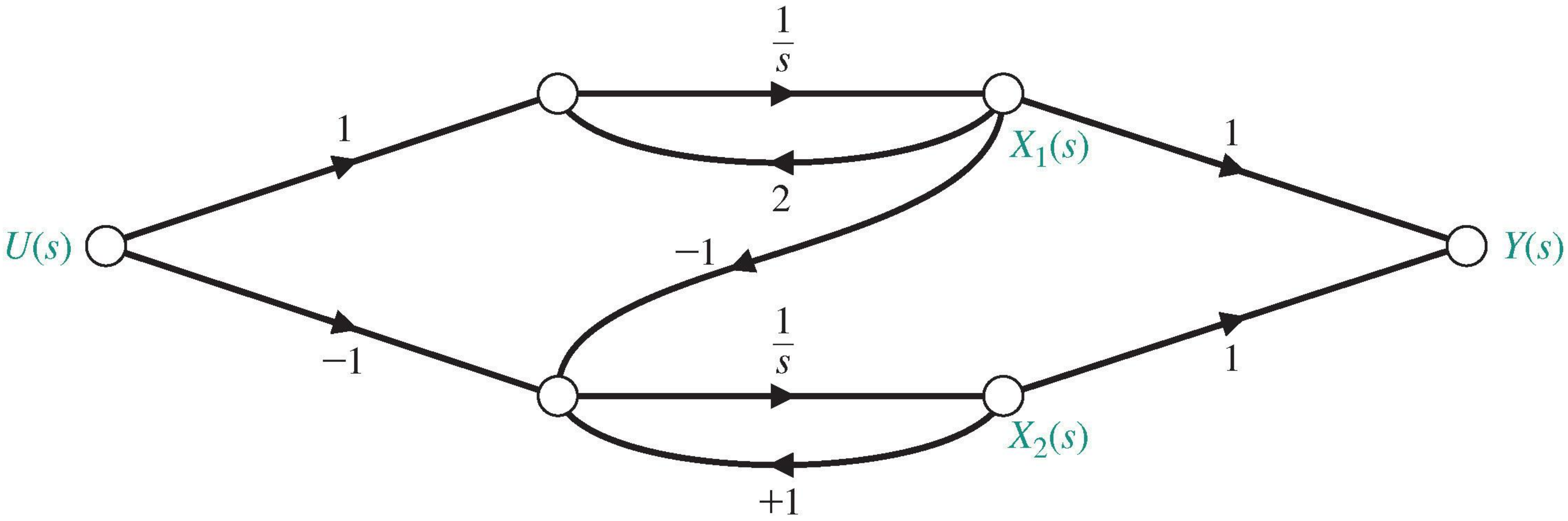
Exercise: *Check the controllability and observability of the following system*



Full-state Feedback Control Design

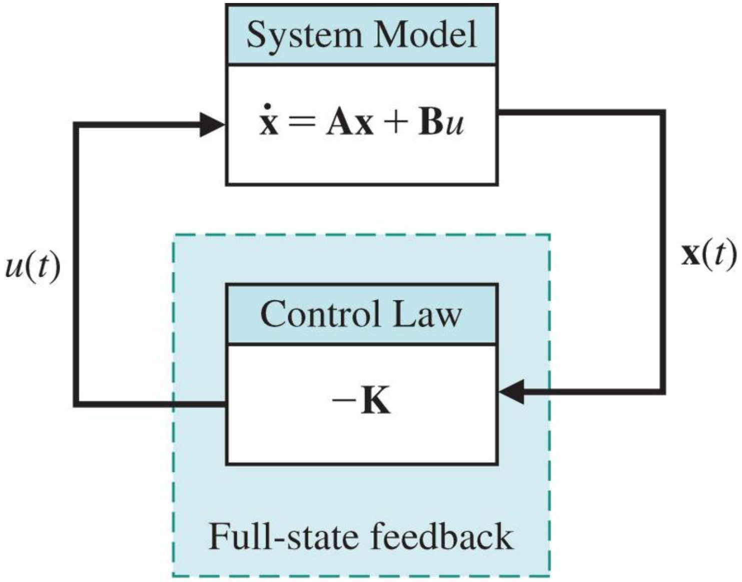
Assume that all the states are available for feedback, we have

dimension of K?

$$u(t) = -\mathbf{K}\mathbf{x}(t).$$

Determining the gain matrix K is the objective of the full-state feedback design procedure.

achieve the desired pole locations of the closed-loop system.



Theorem : Given the pair (A, B), we can always determine K to place all the system closed-loop poles in the left half-plane if and only if the system is completely controllable—that is, if and only if the controllability matrix Pc is full rank.

Full-state Feedback Control Design

Assume that all the states are available for feedback, we have

$$u(t) = -\mathbf{K}\mathbf{x}(t).$$

Determining the gain matrix K is the objective of the full-state feedback design procedure.

achieve the desired pole locations of the closed-loop system.

we find the closed-loop system to be

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) = \mathbf{A}\mathbf{x}(t) - \mathbf{B}\mathbf{K}\mathbf{x}(t) = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}(t).$$

The characteristic equation

$$\det(\lambda\mathbf{I} - (\mathbf{A} - \mathbf{B}\mathbf{K})) = 0.$$

If all the roots of the characteristic equation lie in the left half-plane, then the closed-loop system is stable.

In other words, for any initial condition, it follows that

$$\mathbf{x}(t) = e^{(\mathbf{A}-\mathbf{B}\mathbf{K})t}\mathbf{x}(t_0) \rightarrow 0 \qquad \text{as } t \rightarrow \infty.$$

*This is known as the **regulator problem**. That is, we want to compute K so that all initial conditions are driven to zero in a specified fashion.*

Full-state Feedback Control Design

Assume that all the states are available for feedback, we have

$$u(t) = -\mathbf{K}\mathbf{x}(t).$$

Determining the gain matrix K is the objective of the full-state feedback design procedure.

For tracking purpose, addition of a reference input can be written as

$$u(t) = -\mathbf{K}\mathbf{x}(t) + Nr(t),$$

where r(t) is the reference input.

Example: *Let us consider the third-order system*

$$\frac{d^3y(t)}{dt^3} + 5\frac{d^2y(t)}{dt^2} + 3\frac{dy(t)}{dt} + 2y(t) = u(t).$$

We can select the state variables as

$$x_1(t) = y(t),$$

$$x_2(t) = dy(t)/dt,$$

$$x_3(t) = d^2y(t)/dt^2,$$

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2 & -3 & -5 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}u(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$$

$$y(t) = [1 \quad 0 \quad 0]\mathbf{x}(t).$$

Full-state Feedback Control Design

Example: *We design a state feedback controller as*

$$u(t) = -\mathbf{K}\mathbf{x}(t), \qquad \mathbf{K} = [k_1 \quad k_2 \quad k_3]$$

then the closed-loop system is

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) - \mathbf{B}\mathbf{K}\mathbf{x}(t) = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}(t).$$

The state feedback matrix is

$$\mathbf{A} - \mathbf{B}\mathbf{K} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -2-k_1 & -3-k_2 & -5-k_3 \end{bmatrix},$$

and the characteristic equation is

$$\Delta(\lambda) = \det(\lambda\mathbf{I} - (\mathbf{A} - \mathbf{B}\mathbf{K})) = \lambda^3 + (5 + k_3)\lambda^2 + (3 + k_2)\lambda + (2 + k_1) = 0.$$

If we seek a rapid response with a low overshoot, we choose a desired characteristic equation such as

$$\Delta(\lambda) = (\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2)(\lambda + \zeta\omega_n).$$

Full-state Feedback Control Design

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Example:

$$\Delta(\lambda) = (\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2)(\lambda + \zeta\omega_n).$$

We choose $\xi = 0.8$ *for minimal overshoot and* ω_n *to meet the settling time requirement*

$$T_s = \frac{4}{\zeta\omega_n} = \frac{4}{(0.8)\omega_n} \approx 1. \longrightarrow \omega_n = 6.$$

the desired characteristic equation is

$$(\lambda^2 + 9.6\lambda + 36)(\lambda + 4.8) = \lambda^3 + 14.4\lambda^2 + 82.1\lambda + 172.8.$$

Recall the characteristic equation to be designed

$$\Delta(\lambda) = \det(\lambda\mathbf{I} - (\mathbf{A} - \mathbf{BK})) = \lambda^3 + (5 + k_3)\lambda^2 + (3 + k_2)\lambda + (2 + k_1) = 0.$$

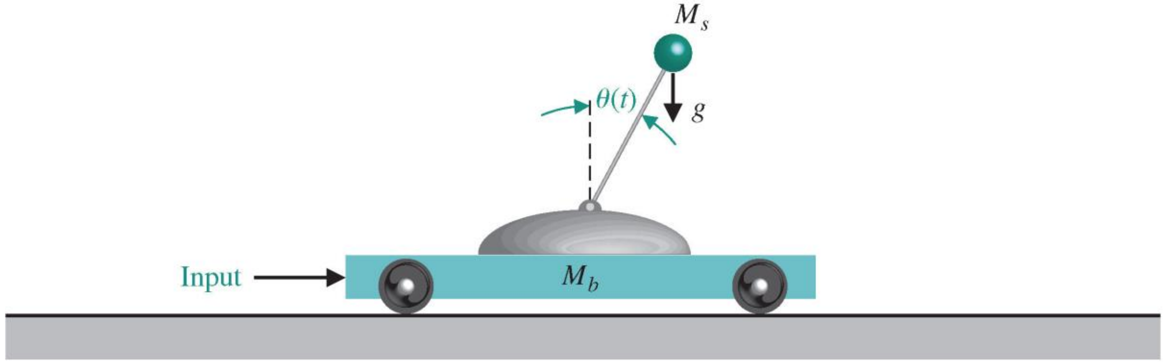
yields the three equations

$$\begin{aligned} 5 + k_3 &= 14.4 \\ 3 + k_2 &= 82.1 \\ 2 + k_1 &= 172.8. \end{aligned} \longrightarrow K = [170.8, 79.1, 9.4]$$

Full-state Feedback Control Design

Example: *Inverted pendulum control*

Consider the control of an unstable inverted pendulum balanced on a moving cart. Now we tend to design a suitable state variable feedback control system to keep the pendulum staying its unstable position.



To simplify, we assume that the control input, $u(t)$, is an acceleration signal, we can focus on the unstable dynamics of the pendulum.

$$\ddot{\theta}(t) = \frac{g}{l}\theta(t) - \frac{1}{l}u(t).$$

Let the state vector be $(x_1(t), x_2(t)) = (\theta(t), \dot{\theta}(t))$.

The A matrix has the characteristic equation $\lambda^2 - \frac{g}{l} = 0$ with one root in the right-hand s-plane.

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ g/l & 0 \end{bmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{bmatrix} 0 \\ -1/l \end{bmatrix}u(t).$$

Full-state Feedback Control Design

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Example: *Inverted pendulum control*

To stabilize the system, we generate a control signal

$$u(t) = -\mathbf{K}\mathbf{x}(t) = -[k_1 \quad k_2]\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = -k_1x_1(t) - k_2x_2(t).$$

Substituting this control signal relationship into the system

$$\begin{pmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ (g + k_1)/l & k_2/l \end{bmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}.$$

Obtaining the characteristic equation, we have

$$\begin{bmatrix} \lambda & -1 \\ -(g + k_1)/l & \lambda - k_2/l \end{bmatrix} = \lambda\left(\lambda - \frac{k_2}{l}\right) - \frac{g + k_1}{l} = \lambda^2 - \left(\frac{k_2}{l}\right)\lambda + \frac{g + k_1}{l}.$$

Thus, for the system to be stable, we require that

$$k_2/l < 0 \text{ and } k_1 > -g.$$

If we wish to achieve a rapid response with modest overshoot, we select

$$\omega_n = 10 \text{ and } \zeta = 0.8.$$

Then we require

$$\frac{k_2}{l} = -16 \quad \text{and} \quad \frac{k_1 + g}{l} = 100.$$

Full-state Feedback Control Design

Tips for making our life easier:

1. *For a SISO LTI, Given the desired characteristic equation*

$$q(\lambda) = \lambda^n + \alpha_{n-1}\lambda^{n-1} + \cdots + \alpha_o,$$

Then Ackermann's formula is useful for determining the state variable feedback mat

Invertible Matrix

$$\mathbf{K} = [0 \quad 0 \ldots 0 \quad 1]\mathbf{P}_c^{-1} q(\mathbf{A}),$$

where Pc is the controllability matrix and

$$q(\mathbf{A}) = \mathbf{A}^n + \alpha_{n-1}\mathbf{A}^{n-1} + \cdots \alpha_1\mathbf{A} + \alpha_0\mathbf{I},$$

2. *Matlab Code:*

Feedback gain matrix $\mathbf{K}$.

$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$

Vector containing desired closed-loop poles.

K=acker(A,B,P)
K=place(A,B,p)

Controllability matrix $\quad \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$, $y = \mathbf{C}\mathbf{x} + \mathbf{D}u$

Pc=ctrb(A,B)

Po=obsv(A,C)

Observability matrix $\quad \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$, $y = \mathbf{C}\mathbf{x} + \mathbf{D}u$

Observer Design

$$u(t) = -\mathbf{K}\mathbf{x}(t).$$

Next big question,

- *what if the state x(t) is not available?*
- *Especially when x(t) does not have a good physical meaning.*
- *Or when you can afford an expensive sensor?*

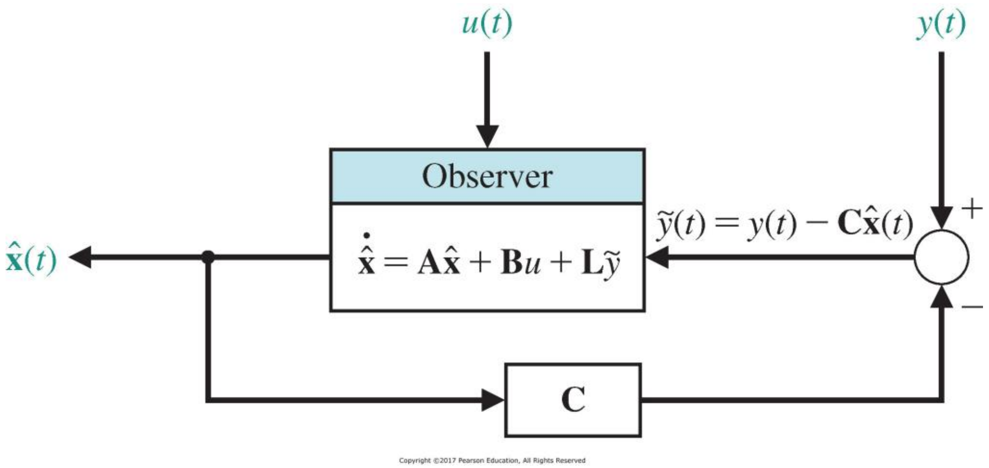
> *Fortunately, **if the system is completely observable with a given set of outputs**, then it is possible to determine (or to estimate) the states that are not directly measured (or observed).*

According to Luenberger , the full-state observer for the system

$$\begin{aligned}\dot{\mathbf{x}}(t) &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)\\ y(t) &= \mathbf{C}\mathbf{x}(t)\end{aligned}$$

is given by

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}u(t) + \mathbf{L}(y(t) - \mathbf{C}\hat{\mathbf{x}}(t))$$



where

- $\hat{x}(t)$ *is the so-called estimates of the state* $x(t)$
- *matrix L is the observer gain matrix and is to be determined as part of the observer design procedure.*

Observer Design

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$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}u(t) + \mathbf{L}(y(t) - \mathbf{C}\hat{\mathbf{x}}(t)) \qquad \hat{x}(t_0) = \hat{x}_0$$

in general, not equal x_0

The goal of the observer is to provide an estimate $\hat{x}(t) \to x(t)$ *as* $t \to \infty$.

Define the observer estimation error as

$$\mathbf{e}(t) = \mathbf{x}(t) - \hat{\mathbf{x}}(t).$$

The observer design should produce an observer with the property that

$$e(t) \to 0 \text{ as } t \to \infty$$

Taking the time-derivative of the estimation error

$$\begin{aligned}\dot{\mathbf{e}}(t) &= \dot{\mathbf{x}}(t) - \dot{\hat{\mathbf{x}}}(t)\\ &= \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) - \mathbf{A}\hat{\mathbf{x}}(t) - \mathbf{B}u(t) - \mathbf{L}(y(t) - \mathbf{C}\hat{\mathbf{x}}(t))\\ &= (\mathbf{A} - \mathbf{L}\mathbf{C})\mathbf{e}(t).\end{aligned}$$

Conclusion: We can achieve our goal if the characteristic equation

$$\det(\lambda\mathbf{I} - (\mathbf{A} - \mathbf{L}\mathbf{C})) = 0$$

has all its roots in the left half-plane.

Observer Design

Example: *Consider the second-order system*

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u(t)$$

$$y(t) = [1 \quad 0]\mathbf{x}(t).$$

In this example, we can only directly observe the state $y(t) = x_1(t)$.

checking the system observability

$$\mathbf{P}_o = \begin{bmatrix} \mathbf{C} \\ \mathbf{CA} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix}.$$

The full-state observer for the system

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}u(t) + \mathbf{L}(y(t) - \mathbf{C}\hat{\mathbf{x}}(t))$$

where

$$\mathbf{L} = [L_1 \quad L_2]^T.$$

Then the characteristic equation of the estimation error yields

$$\det(\lambda\mathbf{I} - (\mathbf{A} - \mathbf{LC})) = \lambda^2 + (L_1 - 6)\lambda - 4(L_1 - 2) + 3(L_2 + 1),$$

Observer Design

what does this "settling" mean?

Suppose that the desired characteristic equation is given by

$$\Delta_d(\lambda) = \lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2.$$

We can select $\xi = 0.8$ *and* $\omega_n = 10$*, resulting in an expected settling time of less than 0.5 second.*

Equating the coefficients

$$L_1 - 6 = 16$$

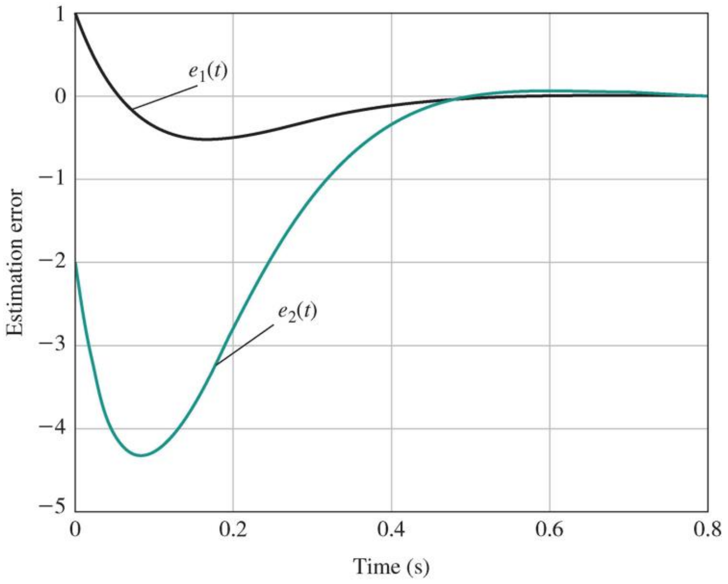
$$-4(L_1 - 2) + 3(L_2 + 1) = 100$$

which, when solved, produces

$$\mathbf{L} = \begin{bmatrix} L_1 \\ L_2 \end{bmatrix} = \begin{bmatrix} 22 \\ 59 \end{bmatrix}.$$

The observer is thus given by

$$\dot{\hat{\mathbf{x}}}(t) = \begin{bmatrix} 2 & 3 \\ -1 & 4 \end{bmatrix}\hat{\mathbf{x}}(t) + \begin{bmatrix} 0 \\ 1 \end{bmatrix}u(t) + \begin{bmatrix} 22 \\ 59 \end{bmatrix}(y(t) - \hat{x}_1(t)).$$



Observer Design

Note:

1. *Given the desired observer characteristic equation*

$$p(\lambda) = \lambda^n + \beta_{n-1}\lambda^{n-1} + \cdots + \beta_1\lambda + \beta_0.$$

Ackermann's formula can also be employed to place the roots of the observer

$$\mathbf{L} = p(\mathbf{A})\mathbf{P}_o^{-1}[0 \cdots 0 \quad 1]^T,$$

where Po is the observability matrix

$$p(\mathbf{A}) = \mathbf{A}^n + \beta_{n-1}\mathbf{A}^{n-1} + \cdots + \beta_1\mathbf{A} + \beta_0\mathbf{I}.$$

2. *Up to now, the effectiveness of the observer has NOTHING to do with the control input and it will NOT alter the behaviour of the system*

Full-state observer-based Feedback controller

Recall

$$u(t) = -\mathbf{K}\mathbf{x}(t).$$

It seems reasonable that we can employ the state estimate in the feedback control law in place of $x(t)$.

$$u(t) = -\mathbf{K}\hat{\mathbf{x}}(t).$$

We need to verify that, using the estimate still retain the stability of the closed-loop system.

Proof: *Substituting the observer-based feedback law into the system model yields*

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) = \mathbf{A}\mathbf{x}(t) - \mathbf{B}\mathbf{K}\hat{\mathbf{x}}(t),$$

and with $\hat{\mathbf{x}}(t) = \mathbf{x}(t) - \mathbf{e}(t)$ *we obtain*

$$\dot{\mathbf{x}}(t) = (\mathbf{A} - \mathbf{B}\mathbf{K})\mathbf{x}(t) + \mathbf{B}\mathbf{K}\mathbf{e}(t).$$

Recall the the estimation error

$$\dot{\mathbf{e}}(t) = \dot{\mathbf{x}}(t) - \dot{\hat{\mathbf{x}}}(t) = (\mathbf{A} - \mathbf{L}\mathbf{C})\mathbf{e}(t).$$

$$\Rightarrow \begin{pmatrix} \dot{\mathbf{x}}(t) \\ \dot{\mathbf{e}}(t) \end{pmatrix} = \begin{bmatrix} \mathbf{A} - \mathbf{B}\mathbf{K} & \mathbf{B}\mathbf{K} \\ \mathbf{0} & \mathbf{A} - \mathbf{L}\mathbf{C} \end{bmatrix} \begin{pmatrix} \mathbf{x}(t) \\ \mathbf{e}(t) \end{pmatrix}.$$

So if A-BK and A-LC are both Hurwitz, then the overall system is stable.

Full-state observer-based Feedback controller

*The fact that the full-state feedback law and the observer can be designed independently is an illustration of **the separation principle.***

The design procedure is summarized as follows:

1. Determine $\mathbf{K}$ such that $\det(\lambda \mathbf{I} - (\mathbf{A} - \mathbf{B}\mathbf{K})) = 0$ has roots in the left half-plane and place the poles appropriately to meet the control system design specifications. The ability to place the poles arbitrarily in the complex plane is guaranteed if the system is completely controllable.

2. Determine $\mathbf{L}$ such that $\det(\lambda \mathbf{I} - (\mathbf{A} - \mathbf{L}\mathbf{C})) = 0$ has roots in the left half-plane and place the poles to achieve acceptable observer performance. The ability to place the observer poles arbitrarily in the complex plane is guaranteed if the system is completely observable.

3. Connect the observer to the full-state feedback law using

$$u(t) = -\mathbf{K}\hat{\mathbf{x}}(t).$$

Compensator Transfer Function.

$$U(s) = [-\mathbf{K}(s\mathbf{I} - (\mathbf{A} - \mathbf{B}\mathbf{K} - \mathbf{L}\mathbf{C}))^{-1}\mathbf{L}]Y(s).$$

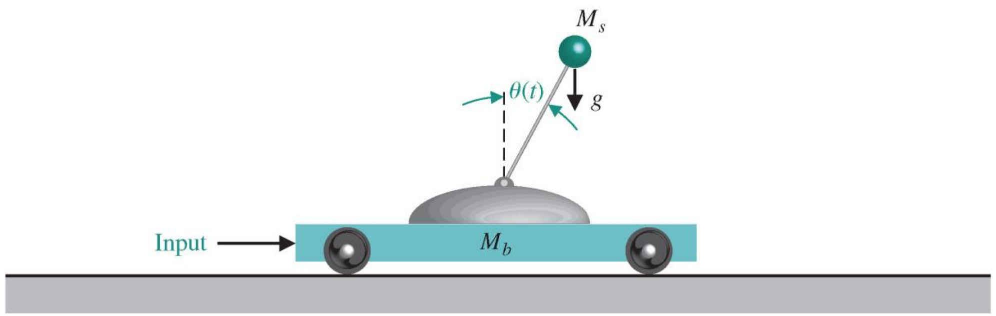
Note that, even though A - BK is stable and A - LC is stable, it does not necessarily follow that A - BK - LC is stable.

*This controller is also commonly referred to as a **stabilizing controller.***

Full-state observer-based Feedback controller

Example: *Compensator design for the inverted pendulum*

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & \dfrac{-mg}{M} & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & \dfrac{g}{l} & 0 \end{bmatrix}\mathbf{x}(t) + \begin{bmatrix} 0 \\ \dfrac{1}{M} \\ 0 \\ \dfrac{-1}{Ml} \end{bmatrix}u(t),$$



$$y(t) = [1 \quad 0 \quad 0 \quad 0]\mathbf{x}(t).$$

Let the system parameters be

$$l = 0.098\text{ m},\ g = 9.8\text{ m/s}^2,\ m = 0.825\text{ kg} \quad M = 8.085\text{ kg}.$$

Checking controllability yields the controllability matrix

$$\mathbf{P}_c = \begin{bmatrix} 0 & 0.1237 & 0 & 1.2621 \\ 0.1237 & 0 & 1.2621 & 0 \\ 0 & -1.2621 & 0 & -126.21 \\ -1.2621 & 0 & -126.21 & 0 \end{bmatrix}.$$

$$\mathbf{P}_o = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}$$

Full-state observer-based Feedback controller

Step 1: Design the Full-State Feedback Control Law

The open-loop system poles are located at

$$\lambda = 0, 0, -10, \text{ and } 10,$$

hence the open-loop system is unstable (there is a pole in the right half-plane).

$$q(\lambda) = (\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2)(\lambda^2 + a\lambda + b),$$

To obtain a settling time less than 10 seconds with low overshoot, we can select

$$(\zeta, \omega_n) = (0.8, 0.5).$$

Then, we choose a separation factor of 20 between the dominant poles and the remaining poles, from which it follows that

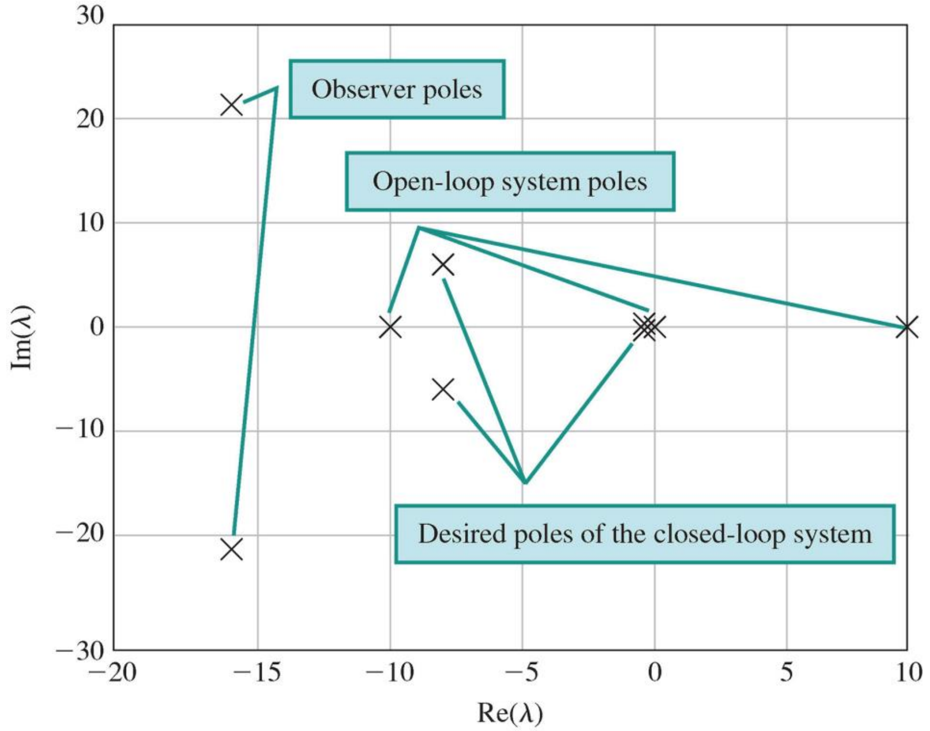
$$(a, b) = (16, 100)$$

The desired roots are

$$\det(\lambda \mathbf{I} - (\mathbf{A} - \mathbf{BK})) = (\lambda + 8 \pm j6)(\lambda + 0.4 \pm j0.3).$$

Using Ackermann's formula yields the feedback gain matrix

$$\mathbf{K} = [-2.2509 \quad -7.5631 \quad -169.0265 \quad -14.0523].$$



Full-state observer-based Feedback controller

Step 2: Observer Design

The desired observer characteristic equation is selected to be of the form

$$p(\lambda) = (\lambda^2 + c_1\lambda + c_2)^2$$

where

$$c_1 = 32 \text{ and } c_2 = 711.11.$$

These values should produce a response to an initial state estimation error that settles in less than 0.5 second with minimal percent overshoot.

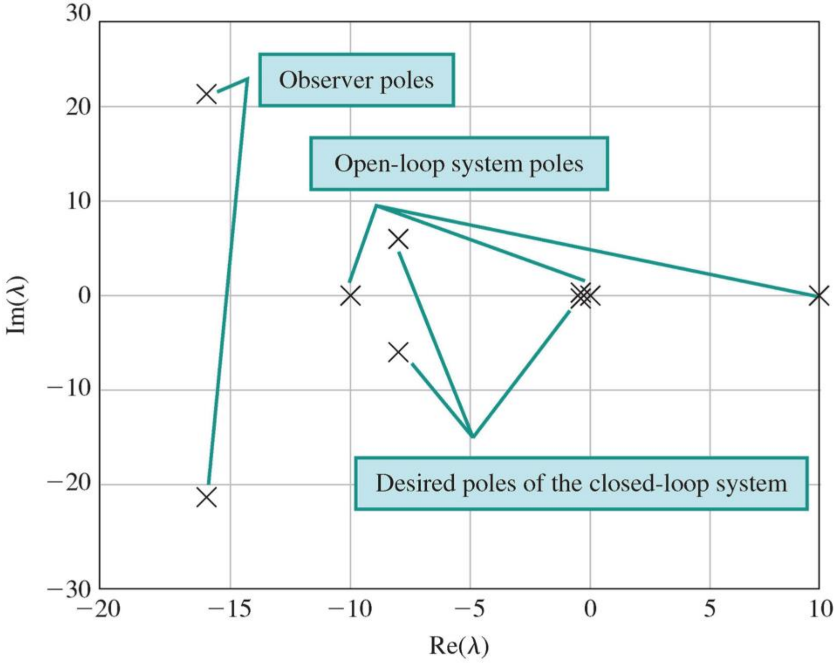
The goal is to achieve an accurate estimate as fast as possible without resulting in too large a gain matrix L. How large is too large depends on the problem under consideration. The trade-off between the time required to obtain accurate observer performance and the amount of noise amplification is a primary design issue

Using Ackermann's formula, we have

$$\mathbf{L} = \begin{bmatrix} 64.0 \\ 2546.22 \\ -5.1911\text{E}04 \\ -7.6030\text{E}05 \end{bmatrix}.$$

achieves the desired observer pole locations

$$\det(\lambda\mathbf{I} - (\mathbf{A} - \mathbf{LC})) = ((\lambda + 16 + j21.3)(\lambda + 16 - j21.3))^2$$



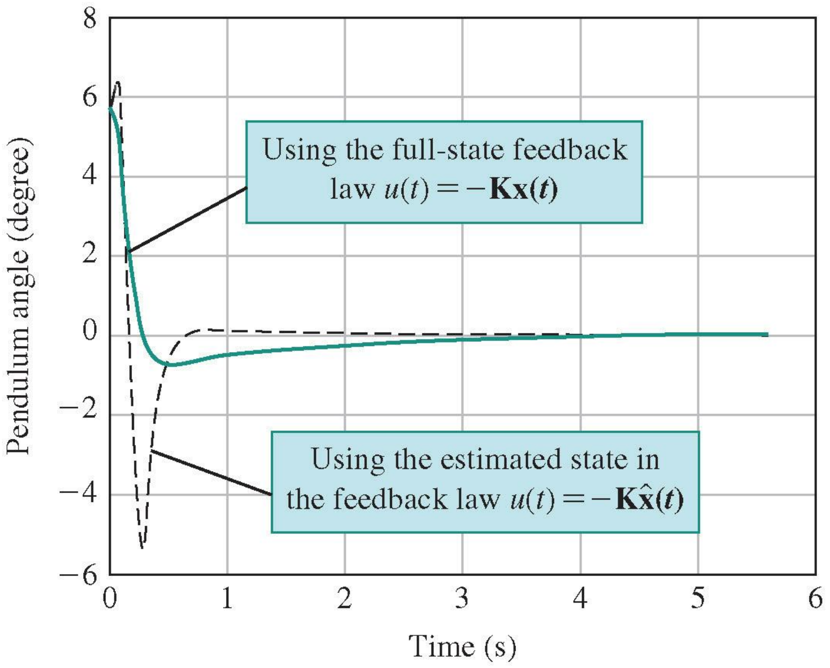
Full-state observer-based Feedback controller

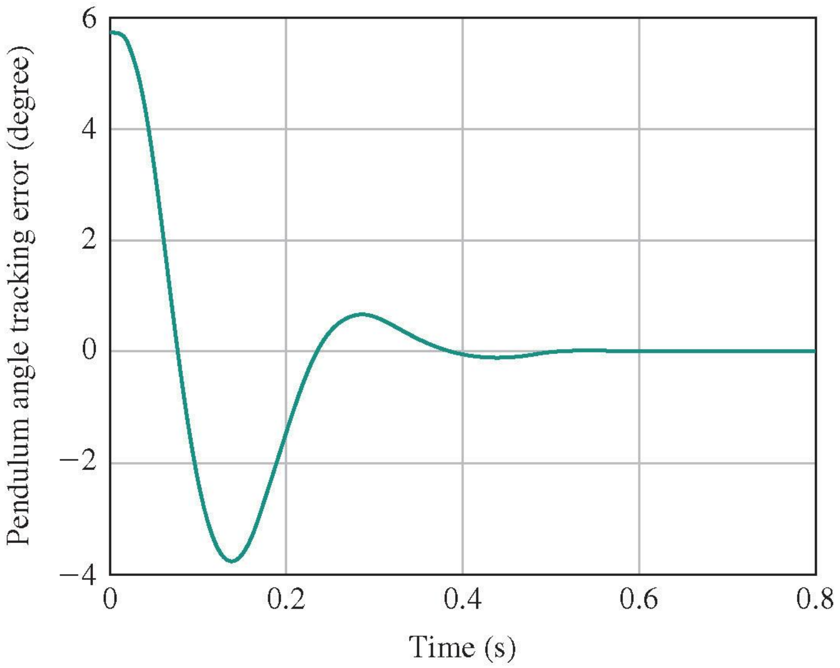
Step 3: Compensator Design

The final step in the design is to connect the observer to the full-state feedback control law via

$$u(t) = -\mathbf{K}\hat{\mathbf{x}}(t)$$

This stabilize the closed-loop system, however, we should not expect the closed-loop performance to be as good when using the state estimate from the observer.





Reference Inputs

We referred to the previous design of state variable feedback stabilizing compensators without reference inputs (i.e., $r(t) = 0$) as ***regulators,*** *however,* ***command following (trajectory tracking)*** *is also an important aspect of feedback design*

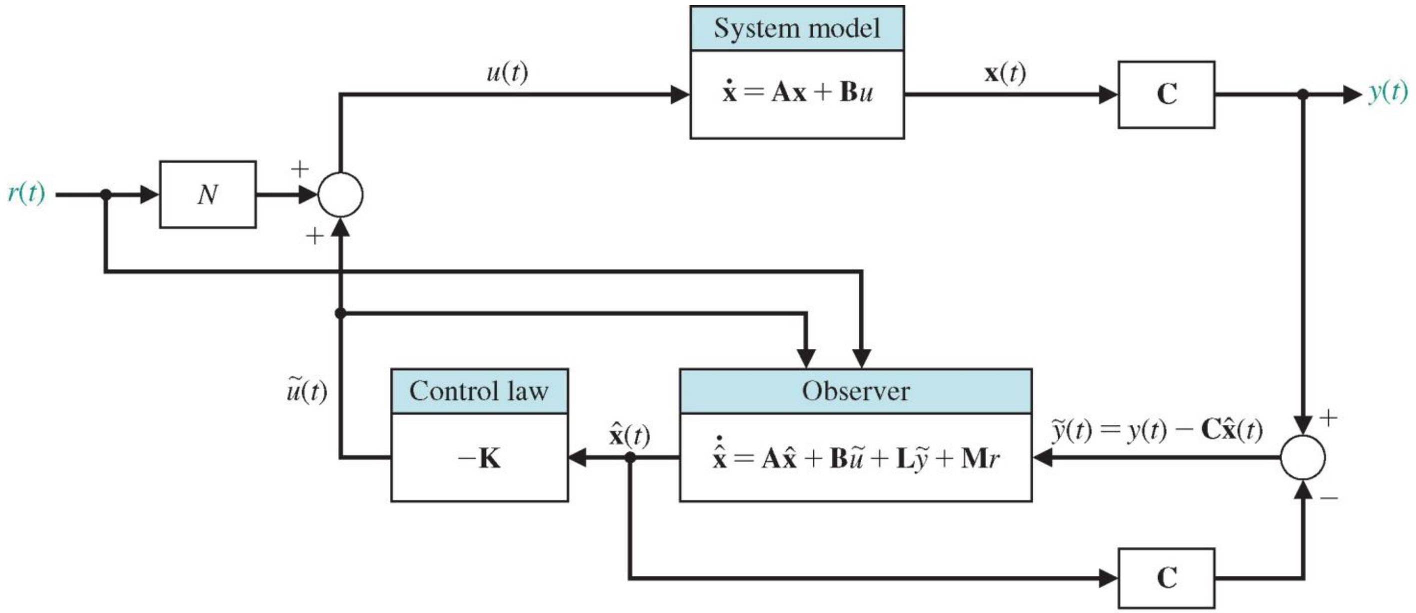
Let's consider ***how we can introduce a reference signal*** *into the state variable feedback compensator.*

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}\tilde{u}(t) + \mathbf{L}\tilde{y}(t) + \mathbf{M}r(t)$$

$$u(t) = \tilde{u}(t) + Nr(t) = -\mathbf{K}\hat{\mathbf{x}}(t) + Nr(t),$$

where

$$\tilde{y}(t) = y(t) - \mathbf{C}\hat{\mathbf{x}}(t)$$



The compensator key design parameters required to implement the command tracking of the reference input are M and N.

Reference Inputs

Case 1, *we select M and N so that the estimation error e(t) is independent of the reference input r(t)*

the estimation error

$$\begin{aligned}\dot{\mathbf{e}}(t) &= \dot{\mathbf{x}}(t) - \dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t) - \mathbf{A}\hat{\mathbf{x}}(t) - \mathbf{B}\tilde{u}(t) - \mathbf{L}\tilde{y}(t) - \mathbf{M}r(t),\\ &= (\mathbf{A} - \mathbf{LC})\mathbf{e}(t) + (\mathbf{B}N - \mathbf{M})r(t).\end{aligned}$$

Suppose that we select

$$\mathbf{M} = \mathbf{B}N.$$

Then the corresponding estimation error is given by

$$\dot{\mathbf{e}}(t) = (\mathbf{A} - \mathbf{LC})\mathbf{e}(t).$$

In this case, the estimation error is independent of the reference input, and the remaining task is to determine a suitable value of N.

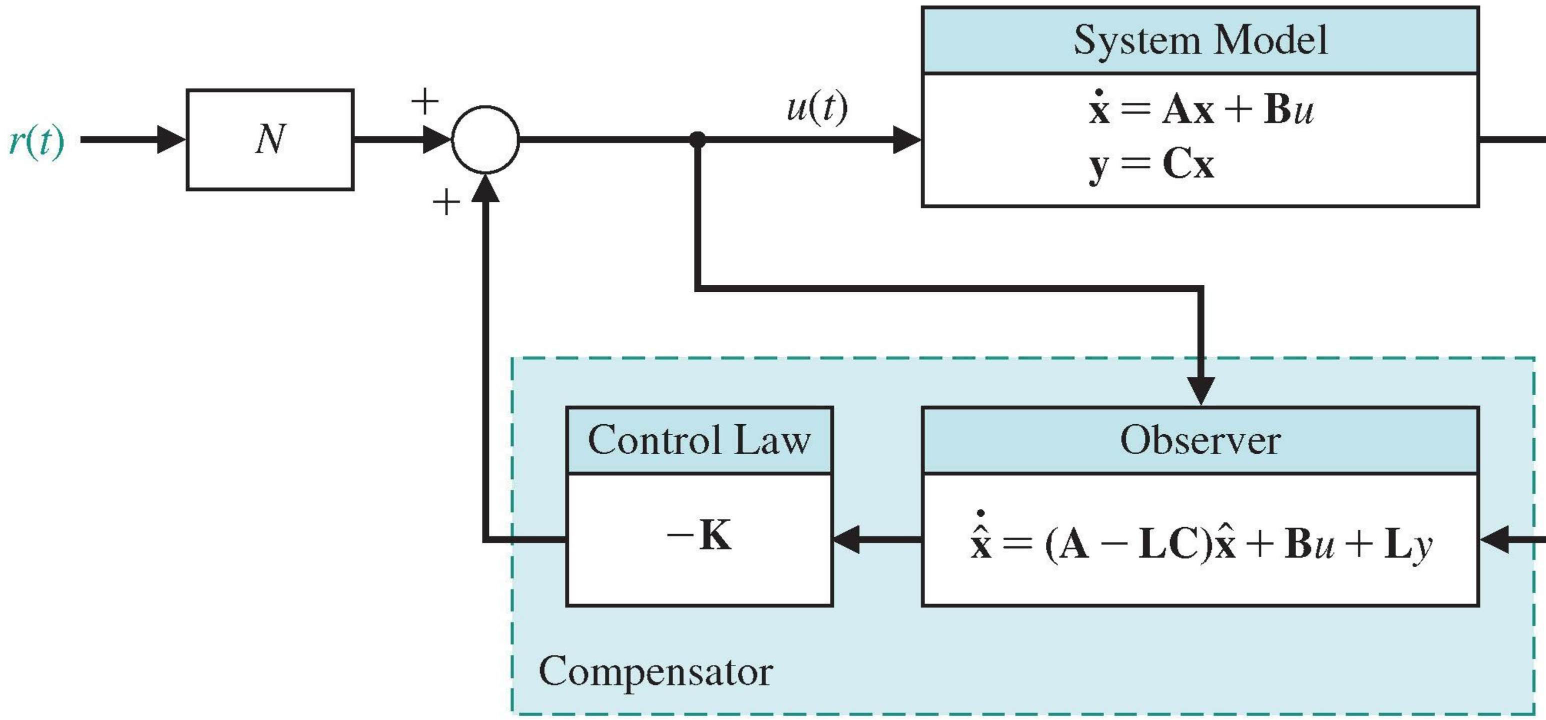
With M = BN, we find that the compensator is given by

$$\begin{aligned}\dot{\hat{\mathbf{x}}}(t) &= \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}u(t) + \mathbf{L}\tilde{y}(t)\\ u(t) &= -\mathbf{K}\hat{\mathbf{x}}(t) + Nr(t).\end{aligned}$$

Reference Inputs

Case 1, we select M and N so that the estimation error $e(t)$ is independent of the reference input $r(t)$

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}u(t) + \mathbf{L}\tilde{y}(t)$$
$$u(t) = -\mathbf{K}\hat{\mathbf{x}}(t) + Nr(t).$$



the compensator is in the feedback loop

Reference Inputs

Case 2, we select M and N so that the tracking error $y(t) - r(t)$ is used as an input to the compensator.

Recall,

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}\tilde{u}(t) + \mathbf{L}\tilde{y}(t) + \mathbf{M}r(t)$$

$$u(t) = \tilde{u}(t) + Nr(t) = -\mathbf{K}\hat{\mathbf{x}}(t) + Nr(t),$$

suppose that we select

$$N = 0 \text{ and } \mathbf{M} = -\mathbf{L}$$

Then, the compensator is given by

$$\dot{\hat{\mathbf{x}}}(t) = \mathbf{A}\hat{\mathbf{x}}(t) + \mathbf{B}u(t) + \mathbf{L}\tilde{y}(t) - \mathbf{L}r(t)$$

$$u(t) = -\mathbf{K}\hat{\mathbf{x}}(t),$$

leads to

$$\dot{e} = \dot{x} - \dot{\hat{x}} = (A - LC)e + Lr$$

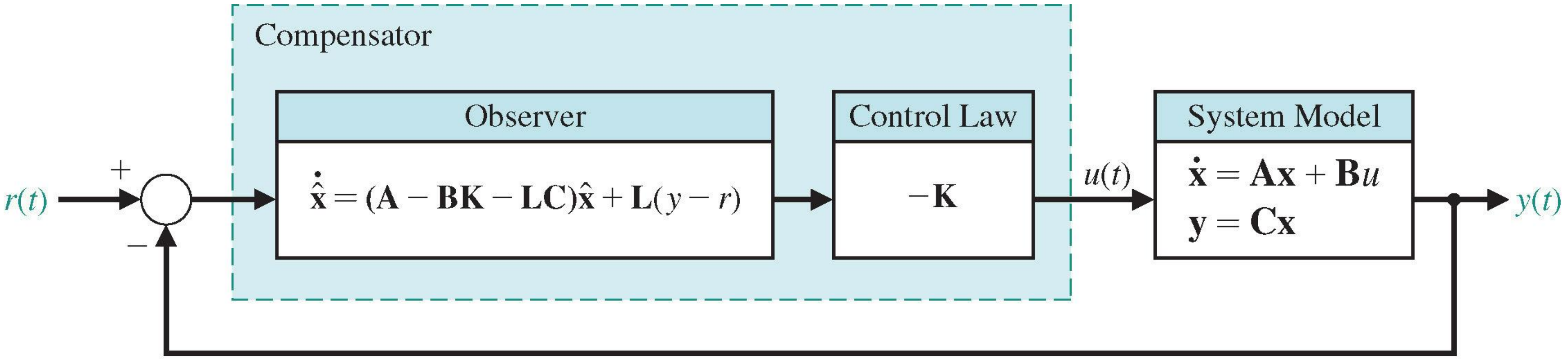
$$\dot{x} = (A - BK)x + BKe$$

Reference Inputs

Case 2, *we select M and N so that the tracking error* $y(t) - r(t)$ *is used as an input to the compensator.*

rewrite the compensator as

$$\dot{\hat{\mathbf{x}}}(t) = (\mathbf{A} - \mathbf{BK} - \mathbf{LC})\hat{\mathbf{x}}(t) + \mathbf{L}(y(t) - r(t))$$
$$u(t) = -\mathbf{K}\hat{\mathbf{x}}(t).$$



the compensator is in the forward path.

Depending on the choice of N and M, other implementations are possible, for instance, the internal model design.

Internal Model Design

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*Now, we consider the problem of designing a compensator that provides asymptotic tracking of **a reference input** with zero steady-state error.*

include steps, ramps, and other persistent signals, such as sinusoids

achieved by type-one system, type-two system, and ??

*This idea is formalized here by **introducing an internal model of the reference input in the compensator***

Consider

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t), \qquad y(t) = \mathbf{C}\mathbf{x}(t).$$

We consider a reference input to be generated by a linear system of the form

$$\dot{\mathbf{x}}_r(t) = \mathbf{A}_r\mathbf{x}_r(t), \qquad r(t) = \mathbf{d}_r\mathbf{x}_r(t),$$

with unknown initial conditions.

For instance, for a step reference input

$$\dot{x}_r(t) = 0, \qquad r(t) = x_r(t),$$

Internal Model Design

Then, the tracking error $e(t)$ *is defined as*

$$e(t) = y(t) - r(t).$$

Taking the time derivative yields

$$\dot{e}(t) = \dot{y}(t) = \mathbf{C}\dot{\mathbf{x}}(t).$$

If we define the two intermediate variables

$$\mathbf{z}(t) = \dot{\mathbf{x}}(t) \quad \text{and} \quad w(t) = \dot{u}(t),$$

we have

$$\begin{pmatrix} \dot{e}(t) \\ \dot{\mathbf{z}}(t) \end{pmatrix} = \begin{bmatrix} 0 & \mathbf{C} \\ 0 & \mathbf{A} \end{bmatrix} \begin{pmatrix} e(t) \\ \mathbf{z}(t) \end{pmatrix} + \begin{bmatrix} 0 \\ \mathbf{B} \end{bmatrix} w(t).$$

If the system is controllable, we can find a feedback of the form

$$w(t) = -K_1 e(t) - \mathbf{K}_2 \mathbf{z}(t)$$

such that the system is stable.

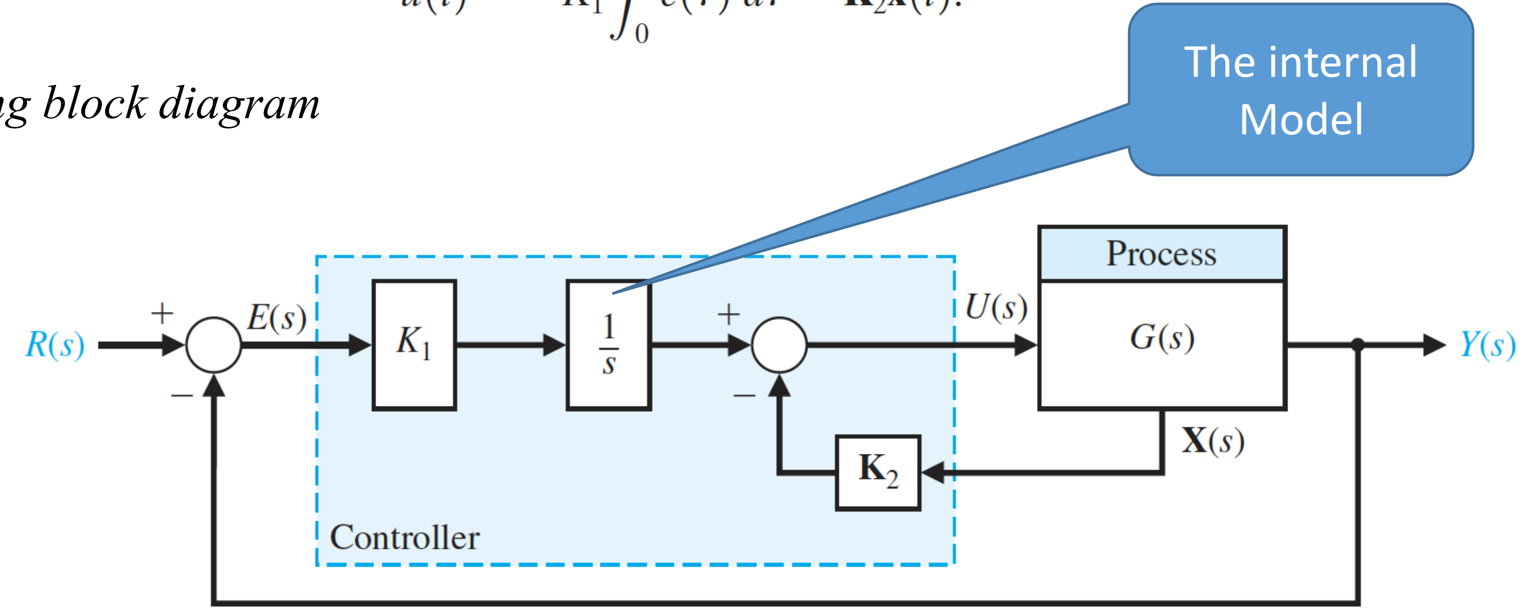
This implies we will have achieved the objective of asymptotic tracking with zero steady state error.

Internal Model Design

The control input, found by

$$u(t) = -K_1 \int_0^t e(\tau)\, d\tau - \mathbf{K}_2\mathbf{x}(t).$$

The corresponding block diagram



The internal model principle states that if $G(s)Gc(s)$ contains $R(s)$, then $y(t)$ will track $r(t)$ asymptotically.

achieved by type-one system, type-two system, and controller contains $\frac{\omega}{s^2+\omega^2}$

Optimal Control Design

Definition: *The design of a systems that are adjusted to provide a minimum performance index such as*

$$J = \int_0^\infty g(\mathbf{x}, \mathbf{u}, t)\, dt,$$

are called optimal control systems

Consider the LTI SISO system

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t).$$

select a feedback controller as

$$u(t) = -\mathbf{K}\mathbf{x}(t),$$

yields

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) - \mathbf{B}\mathbf{K}\mathbf{x}(t) = \mathbf{H}\mathbf{x}(t),$$

where H is the n x n matrix.

Optimal Control Design

Consider an error-squared performance index

$$J = \int_0^\infty \mathbf{x}^T(t)\mathbf{x}(t)\,dt.$$

where

$$\mathbf{x}^T(t)\mathbf{x}(t) = (x_1(t), x_2(t), x_3(t), \ldots, x_n(t))\begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

$$= x_1{}^2(t) + x_2{}^2(t) + x_3{}^2(t) + \cdots + x_n{}^2(t),$$

To obtain the minimum value of J, we postulate the existence of an exact differential so that

$$\frac{d}{dt}(\mathbf{x}^T(t)\mathbf{P}\mathbf{x}(t)) = -\mathbf{x}^T(t)\mathbf{x}(t),$$

where P is to be determined. A symmetric P matrix will be used to simplify the algebra without any loss of generality.

Optimal Control Design

Completing the differentiation indicated on the left-hand side

$$\frac{d}{dt}(\mathbf{x}^T(t)\mathbf{P}\mathbf{x}(t)) = \dot{\mathbf{x}}^T(t)\mathbf{P}\mathbf{x}(t) + \mathbf{x}^T(t)\mathbf{P}\dot{\mathbf{x}}(t).$$

$$= \mathbf{x}^T(t)(\mathbf{H}^T\mathbf{P} + \mathbf{P}\mathbf{H})\mathbf{x}(t).$$

If we let

$$\mathbf{H}^T\mathbf{P} + \mathbf{P}\mathbf{H} = -\mathbf{I},$$

If H is Hurwitz, the existence of an symmetric and positive definite matrix P is guaranteed, this equation is aka the **Lyapunov equation**

then

$$\frac{d}{dt}(\mathbf{x}^T(t)\mathbf{P}\mathbf{x}(t)) = -\mathbf{x}^T(t)\mathbf{x}(t),$$

which indicates to

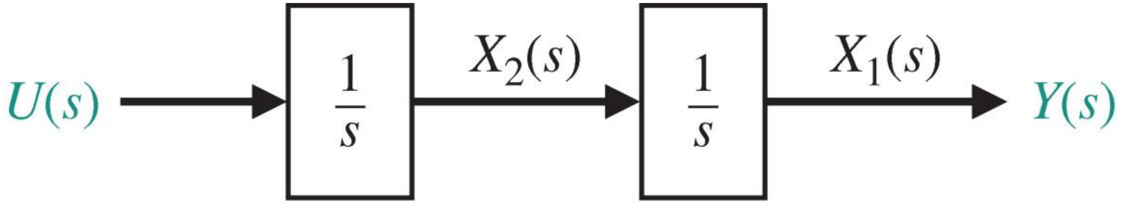
$$J = \int_0^\infty -\frac{d}{dt}(\mathbf{x}^T(t)\mathbf{P}\mathbf{x}(t))\,dt = -\mathbf{x}^T(t)\mathbf{P}\mathbf{x}(t)\Big|_0^\infty = \mathbf{x}^T(0)\mathbf{P}\mathbf{x}(0).$$

The design steps are then as follows

1. Determine the matrix P that satisfies above Lyapunov equation, where H is known.
2. Minimize J by determining the minimum of $x^T(0)Px(0)$ by adjusting one or more unspecified system parameters.

Optimal Control Design

Example:



The vector differential equation of this system is

$$\frac{d}{dt}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t).$$

We choose a feedback control system so that

$$u(t) = -k_1 x_1(t) - k_2 x_2(t),$$

Then the system becomes

$$\begin{aligned} \dot{x}_1(t) &= x_2(t), \\ \dot{x}_2(t) &= -k_1 x_1(t) - k_2 x_2(t). \end{aligned}$$

Optimal Control Design

Example:

In matrix form, we have

$$\dot{\mathbf{x}}(t) = \mathbf{H}\mathbf{x}(t) = \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix}\mathbf{x}(t).$$

Let $k_1 = 1$ *and determine a suitable value for* k_2 *so that the performance index is minimized.*

From the Lyapunov equation, it follows that

$$\begin{bmatrix} 0 & -1 \\ 1 & -k_2 \end{bmatrix}\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} + \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -1 & -k_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Completing the matrix multiplication and addition yields

$$\begin{aligned} -p_{12} - p_{12} &= -1, \\ p_{11} - k_2 p_{12} - p_{22} &= 0, \\ p_{12} - k_2 p_{22} + p_{12} - k_2 p_{22} &= -1. \end{aligned}$$

$$\Rightarrow \quad p_{12} = \frac{1}{2}, \qquad p_{22} = \frac{1}{k_2}, \qquad p_{11} = \frac{{k_2}^2 + 2}{2k_2}.$$

Optimal Control Design

Example:

Consider the integral performance index is then

$$J = \mathbf{x}^T(0)\mathbf{P}\mathbf{x}(0),$$

where

$$\mathbf{x}^T(0) = (1, 1).$$

Therefore J becomes

$$J = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = p_{11} + 2p_{12} + p_{22}.$$

Substituting the values of the elements of P, we have

$$J = \frac{k_2^2 + 2}{2k_2} + 1 + \frac{1}{k_2} = \frac{k_2^2 + 2k_2 + 4}{2k_2}.$$

To minimize as a function of k2,

$$\frac{dJ}{dk_2} = \frac{2k_2(2k_2 + 2) - 2(k_2^2 + 2k_2 + 4)}{(2k_2)^2} = 0.$$

Optimal Control Design

Example:

Therefore

$$k_2 = 2$$

when J is a minimum. The minimum value of J is

$$J_{\min} = 3.$$

The system matrix H, obtained for the compensated system, is then

$$\mathbf{H} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}.$$

The characteristic equation of the compensated system is therefore

$$\det[\lambda\mathbf{I} - \mathbf{H}] = \det\begin{bmatrix} \lambda & -1 \\ 1 & \lambda + 2 \end{bmatrix} = \lambda^2 + 2\lambda + 1.$$

therefore the damping ratio of the compensated system is $\xi = 1$

we recognize that this system is optimal only for the specific set of initial conditions that were assumed.

Optimal Control Design

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Example continue:

let us consider again

$$\dot{\mathbf{x}}(t) = \mathbf{H}\mathbf{x}(t) = \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 \end{bmatrix}\mathbf{x}(t).$$

with $k_1 = k_2 = k.$ *Then system becomes*

$$\dot{\mathbf{x}}(t) = \mathbf{H}\mathbf{x}(t) = \begin{bmatrix} 0 & 1 \\ -k & -k \end{bmatrix}\mathbf{x}(t).$$

To determine the P matrix, we use the Lyapunov equation, yielding

$$p_{12} = \frac{1}{2k}, \qquad p_{22} = \frac{k+1}{2k^2}, \quad \text{and} \quad p_{11} = \frac{1+2k}{2k}.$$

Let us consider the case

$$\mathbf{x}^T(0) = (1 \quad 0)$$

Example continue:

Then the performance index becomes

$$J = \int_0^\infty \mathbf{x}^T(t)\mathbf{x}(t)\,dt = \mathbf{x}^T(0)\mathbf{P}\mathbf{x}(0) = p_{11} = \frac{1 + 2k}{2k} = 1 + \frac{1}{2k}.$$

Then the minimum value of J is obtained when k approaches infinity, which is 1.

Now, we recognize that, in providing a very large gain k, we can cause the feedback signal

$$u(t) = -k(x_1(t) + x_2(t))$$

to be very large, which is unrealistic, cause in many cases, we have physical limits on the control magnitude.

We can limit the control effort by including it within the expression for the performance index

$$J = \int_0^\infty (\mathbf{x}^T(t)\mathbf{I}\mathbf{x}(t) + \lambda\mathbf{u}^T(t)\mathbf{u}(t))\,dt,$$

The weighting factor λ will be chosen so that the relative importance of the state variable performance is contrasted with the importance of the control energy.

Optimal Control Design

Example continue:

Now, let us consider again when λ is other than zero and account for the expenditure of control signal energy.

we still use a state variable feedback

$$u(t) = -\mathbf{K}\mathbf{x}(t) = [-k \quad -k]\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}.$$

The performance function becomes

$$J = \int_0^\infty \mathbf{x}^T(t)(\mathbf{I} + \lambda\mathbf{K}^T\mathbf{K})\mathbf{x}(t)\, dt = \int_0^\infty \mathbf{x}^T(t)\mathbf{Q}\mathbf{x}(t)\, dt,$$

$$\mathbf{Q} = \mathbf{I} + \lambda\mathbf{K}^T\mathbf{K} = \begin{bmatrix} 1 + \lambda k^2 & \lambda k^2 \\ \lambda k^2 & 1 + \lambda k^2 \end{bmatrix}.$$

let $\mathbf{x}^T(0) = (1, \quad 0)$ *yielding*

$$J = p_{11} = (1 + \lambda k^2)\left(1 + \frac{1}{2k}\right) - \lambda k^2. \Rightarrow \frac{dJ}{dk} = \frac{1}{2}\left(\lambda - \frac{1}{k^2}\right) = 0.$$

Therefore, the minimum of the performance index occurs when

$$k = k_{\min} = 1/\sqrt{\lambda},$$

Optimal Control Design

Linear Quadratic Regulator (LQR)

Previous design procedure can be carried out for a more general LTI SISO systems

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}u(t)$$

with feedback

$$u(t) = -\mathbf{K}\mathbf{x}(t) = -[k_1 \quad k_2 \ldots k_n]\mathbf{x}(t).$$

We can consider the performance index

$$J = \int_0^\infty (\mathbf{x}^T(t)\mathbf{Q}\mathbf{x}(t) + Ru^2(t))\, dt,$$

where R>0 is a scalar weighting factor. This index is minimized when

$$\mathbf{K} = R^{-1}\mathbf{B}^T\mathbf{P}.$$

The n×n matrix P is determined from the solution of the equation

$$\mathbf{A}^T\mathbf{P} + \mathbf{P}\mathbf{A} - \mathbf{P}\mathbf{B}R^{-1}\mathbf{B}^T\mathbf{P} + \mathbf{Q} = \mathbf{0}.$$

which is often called the algebraic Riccati equation.

Summary

Regulator problem

Controllability
Full-state feedback control law
Pole placement
Separation principle
Observability
Observer
Estimation error

Command following
Internal model design

Linear quadratic regulator

Output Regulation

Robust control

Kalman state-space decomposition

Kalman filter
Reduced-order observer design

Adaptive control

Optimal control
MPC

Thanks!

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