EE150 - Homework 6

Problem 1

(20 points)

(a) (10 points) Given a band-limited input $x(t)$, and the frequency response of two ideal low-pass filters $H_1(j\omega)$ (zero-phase) and $H_2(j\omega)$ with cut-off frequency ω_c, plot the output signal $y_1(t)$ and $y_2(t)$ filtered by $H_1(j\omega)$ and $H_2(j\omega)$, respectively. Note that the maximum frequency of $x(t)$ is lower than ω_c.

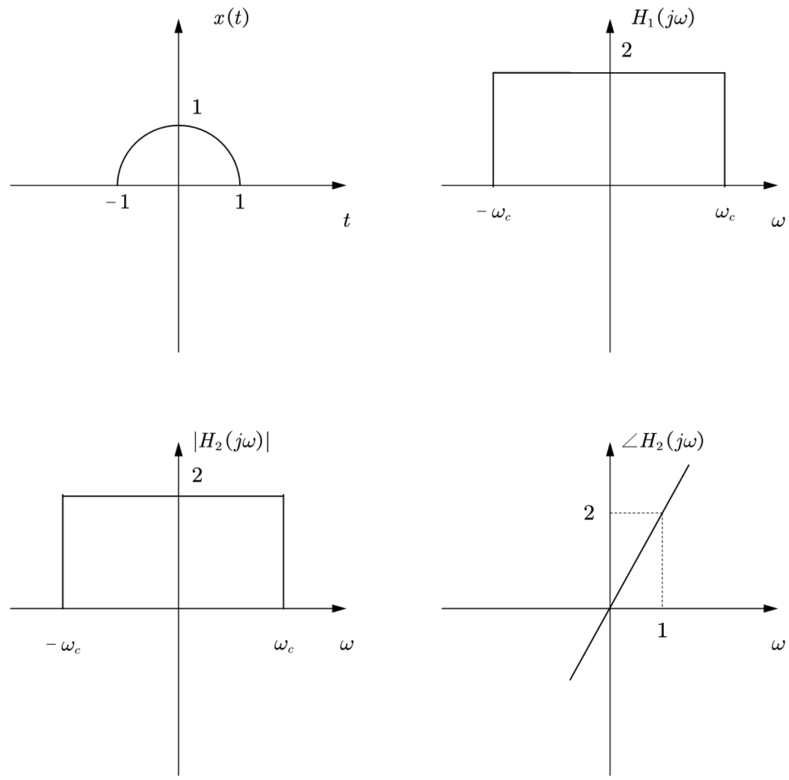


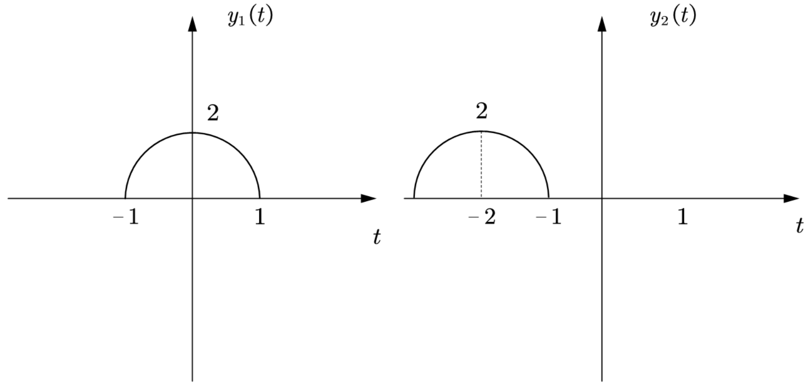
Figure 1: x(t), $H_1(jw)$ and $H_2(jw)$

(b) (10 points) For the continuous ideal low-pass filter, with the following frequency response, calculate the impulse response h(t). When the ω_c increases, is the main lobe of the impulse response more narrow or wider? When $\omega_c \to \infty$, what function will h(t) be approximating to ?

$$H(j\omega) = \begin{cases} 1 & |\omega| \le \omega_c \\ 0 & \omega_c < |\omega| < \pi \end{cases}$$

Solution

(a) (5*2 points)



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$y_1(t)$ **and** $y_2(t)$

$\angle H_2(j\omega) = 2\omega$ and assume that $|H_2(j\omega)| = 2$, then $H_2(j\omega) = 2e^{j\omega 2}$ whose output $y_1(t) = 2x(t)$, $y_2(t) = 2x(t+2)$

(b)

$$h(t) = \frac{\text{sin}\omega_c t}{\pi t}$$

(4 points)

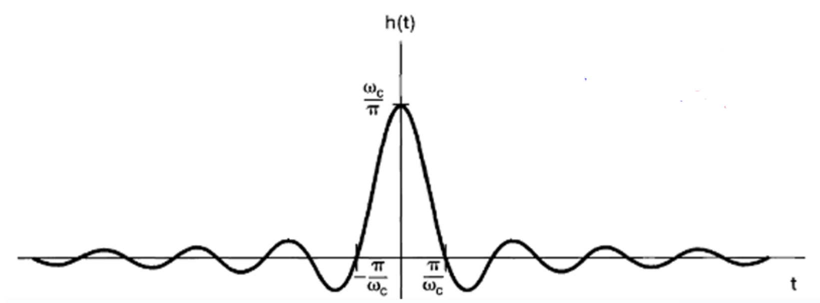


Figure 1: h[n] of ideal low-pass filter

ω_c increases, $\frac{\pi}{\omega_c}$ decreases which is more closer to the $(0,0)$, main lobe of the impulse response will be more narrow. (3 points) When $\omega_c \to \infty$, $h(t) \to \delta(t)$. (3 points)

Problem 2

(20 points) Figure 2 shows the frequency response $H(j\omega)$ of a continuous-time filter referred to as a lowpass differentiator. For the input signal $x(t) = \cos(\pi t + \phi) + \sin(2\pi t + \phi) + \sin(4\pi t + \phi)$, determine the filtered output signal $y(t)$.

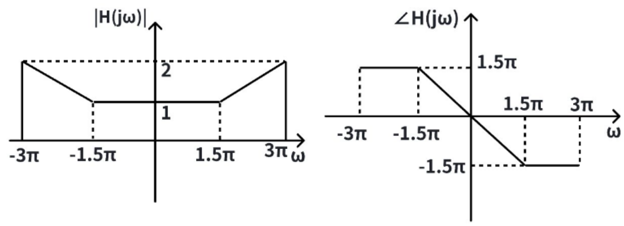


Figure 2: The magnitude and phase spectrum of H(jw)

Solution

We know that (3 points)

$$H(j\omega) = \begin{cases} \frac{2j\omega}{3\pi}, & 1.5\pi \le |\omega| \le 3\pi, \\ e^{-j\omega}, & 0 < |\omega| < 1.5\pi, \\ 0, & \text{otherwise} \end{cases}$$

The input signal is:

$$x(t) = \cos(\pi t + \phi) + \sin(2\pi t + \phi) + \sin(4\pi t + \phi)$$

1. **First Component**: $\cos(\pi t + \phi)$ (5 points)

$$Y_1(j\omega) = H(j\omega)X_1(j\omega) = e^{-j\omega} \cdot X_1(j\omega)$$

Substituting $\omega = \pm\pi$:

$$Y_1(j\omega) = e^{-j\omega} \cdot [\pi e^{i\phi}\delta(\pi - \omega) + \pi e^{-i\phi}\delta(\pi + \omega)]$$

$$= \pi e^{-j\pi} e^{i\phi}\delta(\pi - \omega) + \pi e^{j\pi} e^{-i\phi}\delta(\pi + \omega)$$

$$= -[\pi e^{i\phi}\delta(\pi - \omega) + \pi e^{-i\phi}\delta(\pi + \omega)] = \mathcal{F}\{-\cos(\pi t + \phi)\}$$

Inverse Fourier Transform:

$$y_1(t) = -\cos(\pi t + \phi)$$

2. **Second Component**: $\sin(2\pi t + \phi)$ (5 points)
Substituting $\omega = \pm 2\pi$

$$Y_2(j\omega) = H(j\omega)X_2(j\omega) = \frac{2j\omega}{3\pi} \cdot X_2(j\omega)$$

So using the property $j\omega X(j\omega) = \frac{dx(t)}{dt}$

$$y_2(t) = \frac{2}{3\pi} \frac{dx_2(t)}{dt}$$

Inverse Fourier Transform:

$$y_2(t) = \frac{4}{3} \cos(2\pi t + \phi)$$

3. **Third Component**: $\sin(4\pi t + \phi)$ (5 points)
Since $H(j\omega) = 0$ for $|\omega| > 3\pi$, it follows that:

$$Y_3(j\omega) = H(j\omega)X_3(j\omega) = 0$$

Thus:

$$y_3(t) = 0.$$

The total output signal is:

$$y(t) = y_1(t) + y_2(t) + y_3(t)$$

Substituting the components: (2 points)

$$y(t) = -\cos(\pi t + \phi) + \frac{4}{3} \cos(2\pi t + \phi)$$

Problem 3

(20 points) Given the following properties of a causal LTI system of first-order:

A.If the input signal is $x(t) = a[e^{-(\omega_0-1)t}u(t) - e^{-(\omega_0+1)t}u(t)]$, the output will be $y(t) = be^{-(\omega_0-1)t}u(t) - e^{-(\omega_0)t)}u(t) + be^{-(\omega_0+1)t}u(t)$, where $a, b, \omega_0 \neq 0$, and they are real numbers.

B.The group delay imposed by the system to the input signal is $\tau(\omega) = \frac{5}{25+\omega^2}$. (Hint: $\frac{\mathbf{d}\arctan x}{\mathbf{d}x} = \frac{1}{1+x^2}$)

C.The total energy of the input signal $x(t)$ specified in property(A) is $E_x = \frac{1}{120}$.

please answer the following questions:

1.Find the value of a, b, ω_0.

2.Write out the differential equation of the system in terms of $y(t), x(t)$. And find the frequency response $H(j\omega)$, sketch its Bode plot (use the asymptotic approximation). Note: the lateral axis of the Bode plot should be $\log_{10}(\omega)$.

Solution

1. Firstly, we have the Fourier Transform of $y(t), x(t)$ are

$$Y(j\omega) = \frac{b}{\omega_0 - 1 + j\omega} - \frac{1}{\omega_0 + j\omega} + \frac{b}{\omega_0 + 1 + j\omega}$$

$$= \frac{b(\omega_0 + j\omega)(\omega_0 + 1 + j\omega) - (\omega_0 - 1 + j\omega)(\omega_0 + 1 + j\omega) + b(\omega_0 - 1 + j\omega)(\omega_0 + j\omega)}{(\omega_0 - 1 + j\omega)(\omega_0 + j\omega)(\omega_0 + 1 + j\omega)}$$

$$X(j\omega) = a(\frac{1}{\omega_0 - 1 + j\omega} - \frac{1}{\omega_0 + 1 + j\omega})$$

$$= a \cdot \frac{2}{(\omega_0 - 1 + j\omega)(\omega_0 + 1 + j\omega)}$$

Because this system is LTI, so

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}$$

$$= \frac{b(\omega_0 + j\omega)(\omega_0 + 1 + j\omega) - (\omega_0 - 1 + j\omega)(\omega_0 + 1 + j\omega) + b(\omega_0 - 1 + j\omega)(\omega_0 + j\omega)}{2a(\omega_0 + j\omega)}$$

And we know this LTI system is first-order, so we can get:

$$2b - 1 = 0 \rightarrow b = \frac{1}{2}$$

So we get

$$H(j\omega) = \frac{1}{2a(\omega_0 + j\omega)}$$

Because the group delay is $\tau(\omega) = -\frac{d\angle H(j\omega)}{d\omega} = -\frac{\mathbf{d}[-arctan(\frac{\omega}{\omega_0})]}{\mathbf{d}\omega} = \frac{5}{25+\omega^2}$, so the

$$\omega_0 = 5$$

Now for a, because of the condition(C)

$$E_x = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

$$E_x = \int_{0}^{\infty} |a(e^{-(\omega_0-1)t} - e^{-(\omega_0+1)t})|^2 \, dt$$

$$E_x = a^2 \int_{0}^{\infty} (e^{-2(\omega_0-1)t} - 2e^{-2\omega_0 t} + e^{-2(\omega_0+1)t}) \, dt$$

$$E_x = a^2 \left(\frac{1}{2(\omega_0-1)} - \frac{1}{\omega_0} + \frac{1}{2(\omega_0+1)} \right) = \frac{1}{120}$$

So $a = 1$.In summary, $a = 1, \omega_0 = 5, b = 0.5$.

2.

$$H(j\omega) = \frac{1}{2a(\omega_0 + j\omega)} = \frac{1}{2(5 + j\omega)}$$

And the differential equation is

$$y'(t) + 5y(t) = \frac{1}{2}x(t)$$

So we can get

$$|H(j\omega)| = 20\log_{10} \frac{1}{\sqrt{100(1 + (\frac{\omega}{5})^2)}}$$

$$|H(j\omega)| = -20 - 10\log_{10}\left(1 + \left(\frac{\omega}{5}\right)^2\right)$$

So we have

$$|H(j\omega)| = \begin{cases} -20, & \omega < 5, \\ -20 - 20\log_{10}\omega + 20\log_{10}5, & \omega > 5, \\ 0, & \text{otherwise} \end{cases}$$

and

$$\angle H(j\omega) = -\tan^{-1}\left(\frac{\omega}{5}\right)$$

$$\angle H(j\omega) = \begin{cases} 0, & \omega < 0.5, \\ -\frac{\pi}{4}(\log_{10}\frac{\omega}{5} + 1), & 0.5 < \omega < 50, \\ -\frac{\pi}{2}, & 50 < \omega \end{cases}$$

The Bode plot is as shown in the Figure below.

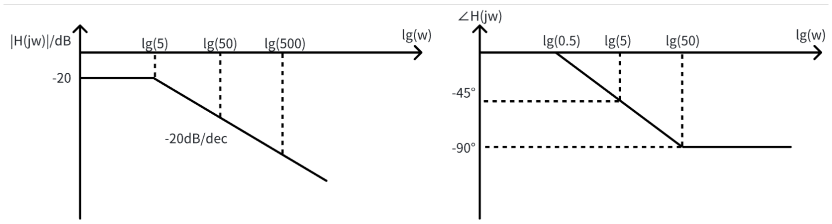


Figure 3: The magnitude and phase spectrum of H(jw)

Problem 4

(20 points) In many practical situations a signal is recorded in the presence of an echo, which we would like to remove by appropriate processing. For example, in Figure 4, we illustrate a system in which a receiver simultaneously receives a signal $x(t)$ and an echo represented by an attenuated delayed replication of $x(t)$. Thus, the receiver output is $s(t) = x(t) + \alpha x(t - T_0)$, where $|\alpha| < 1$. This output is to be processed to recover $x(t)$ by first converting to a sequence and then downsampled by N, and then using an appropriate digital filter $h[n]$, as indicated in Figure 4(b).

Assume that $x(t)$ is band limited [i.e., $X(j\omega) = 0$ for $|\omega| > \omega_M$] and that $|\alpha| < 1$.

(a) If $T_0 < \frac{\pi}{\omega_M}$, and the sampling period is taken to be equal to T_0 (i.e., $NT = T_0$), determine the difference equation for the digital filter $h[n]$ such that $y_c(t)$ equals $x(t)$. (note: for the filter h[n], $s_2[n]$ is the input, y[n] is the output)

(b) Now suppose that $\frac{\pi}{\omega_M} < T_0 < \frac{2\pi}{\omega_M}$. Determine a choice for the sampling period T, the lowpass filter gain A, and the frequency response for the digital filter $h[n]$ such that $y_c(t)$ equals $x(t)$.

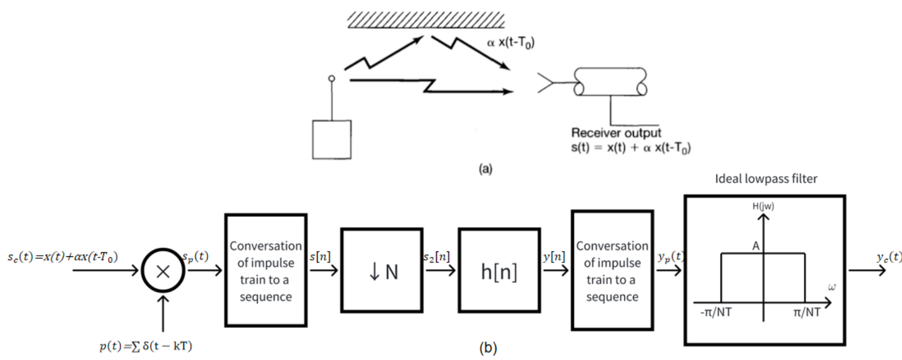


Figure 4.Problem 4

Solution

(a) By sampling $s_c(t)$, we get

$$s[n] = s_c(nT) = x(nT) + \alpha x(nT - T_0)$$

since $s_2[m] = s[mN]$, so

$$s_2[m] = s[mN] = x(mNT) + \alpha x(mNT - T_0)$$

since $NT = T_0$. Let $x[m] = x(mNT)$. Then

$$s_2[m] = x[m] + \alpha x[m - 1]$$

and set n instead of m

$$s_2[n] = x[n] + \alpha x[n-1]$$

Therefore

$$x[n] = -\alpha x[n-1] + s_2[n]$$

This is a first-order difference equation, so given $s_2[n]$, we can find $x[n]$. Since $x(t)$ is appropriately bandlimited, and $y_c(t) = x(t)$ can be reconstructed from $y[n]$, so we can set

$$y[n] = x[n] = x[nNT]$$

Then we can get

$$y[n] = -\alpha y[n-1] + s_2[n]$$

which will make

$$y_c(t) = \frac{A}{NT} x(t)$$

We see that $A = NT$ will make $y_c(t) = x(t)$.

(b) Since we do not want aliasing, we still need $NT < \frac{\pi}{\omega_M}$. Now

$$s_2(t) = x(t) + \alpha x(t - T_0)$$

Taking the continuous Fourier transform, we see that

$$S_2(j\omega) = X(j\omega) + \alpha e^{-j\omega T_0} X(j\omega)$$

Thus, the continuous-time inverse system has frequency response

$$H_c(j\omega) = \frac{1}{1 + \alpha e^{-j\omega T_0}}$$

We want to implement this in discrete time. Therefore, using the relation, we obtain

$$H(j\Omega) = H_c\left(j\frac{\Omega}{NT}\right) = \frac{1}{1 + \alpha e^{-j\Omega\left(\frac{T_0}{NT}\right)}}, \quad \frac{-\pi}{\omega_M} < \Omega < \frac{\pi}{\omega_M}$$

Again, the filter should be $A = NT$.

Problem 5

(20 points)

In Figure 5, we have an input signal $x_c(t) = \frac{\sin(0.5t)}{\pi t} + \sin(0.75t)$, sampling function $p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT)$, given that $T = \frac{\pi}{2}$. After converting $x_p(t)$ to discrete time,, we obtain the sequence $x_d[n]$. We then down sampling the sequence $x_d[n]$ with $M = 2$ to obtain $x_{d2}[n]$. Please plot the spectrum $X_c(j\omega)$, $X_p(j\omega)$, $X_d(e^{j\Omega})$, $X_{d2}(e^{j\Omega})$, $Y_d(e^{j\Omega})$ and $Y_p(j\omega)$. Then write the expression of $y_c(t)$. $H(e^{j\Omega})$ in one period is given below:

$$H(e^{j\Omega}) = \begin{cases} 1, & -\frac{5\pi}{8} \leq \Omega \leq \frac{5\pi}{8} \\ 0, & \text{otherwise} \end{cases}$$

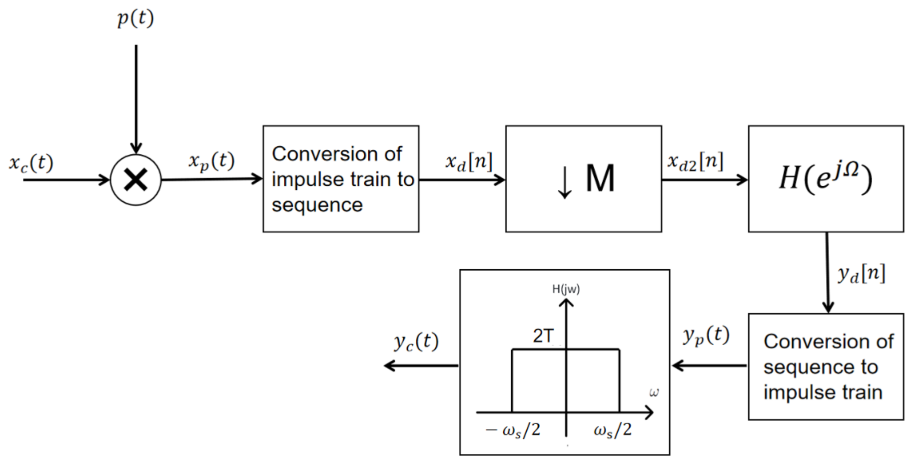


Figure 5. Problem 5

Solution

(a) From Fourier transform,

$$X_c(j\omega) = \begin{cases} j\pi, & \omega = -\frac{3}{4}, \\ 1, & -\frac{1}{2} \leq \omega \leq \frac{1}{2}, \\ -j\pi, & \omega = \frac{3}{4}, \\ 0, & \text{otherwise} \end{cases}$$

After sampling, $x_p(t) = x_c(t)p(t)$, so $X_p(j\omega) = \frac{1}{2\pi}X_c(j\omega) * P(j\omega)$

$$\begin{aligned}
X_p(j\omega) &= \frac{1}{2\pi}X_c(j\omega)\frac{2\pi}{T}\sum_{k=-\infty}^{\infty}\delta(\omega - \frac{2\pi k}{T}) \\
&= \frac{1}{T}X_c(j\omega)\sum_{k=-\infty}^{\infty}\delta(\omega - \frac{2\pi k}{T}) \\
&= \frac{1}{T}\sum_{k=-\infty}^{\infty}X_c(j(\omega - \frac{2\pi k}{T}))
\end{aligned}$$

The spectrum of $x_d[n]$ can be obtained from by $X_p(j\omega)$ replacing ω with $\frac{\Omega}{T}$.

$$X_d(e^{j\Omega}) = \frac{1}{T}\sum_{k=-\infty}^{\infty}X_c(j(\frac{\Omega}{T} - \frac{2\pi k}{T}))$$

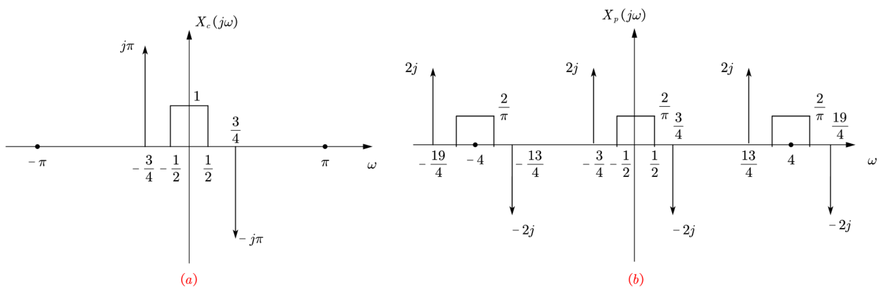
For down sampling, $T \to MT$

$$X_d(e^{j\Omega}) = \frac{1}{MT}\sum_{r=-\infty}^{\infty}X_c(j\frac{\Omega}{MT} - jr\frac{2\pi}{MT})$$

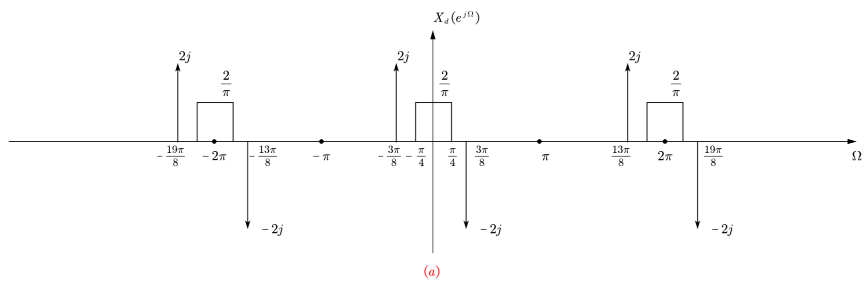
Let $r = i + kM$, $0 \le i \le M-1$, $-\infty \le k \le \infty$,

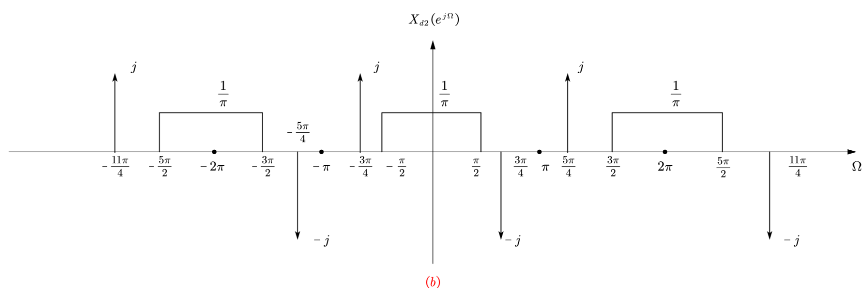
$$\begin{aligned}
X_{dM}(e^{j\Omega}) &= \frac{1}{M}\sum_{i=0}^{M-1}\{\frac{1}{T}\sum_{k=-\infty}^{\infty}X_c(j\frac{\Omega}{MT} - j(i+kM)\frac{2\pi}{MT})\} \\
&= \frac{1}{M}\sum_{i=0}^{M-1}\{\frac{1}{T}\sum_{k=-\infty}^{\infty}X_c(j\frac{\Omega - 2\pi i}{M}\frac{1}{T} - jk\frac{2\pi}{T})\} \\
&= \frac{1}{M}\sum_{i=0}^{M-1}X_d(e^{j\Omega'}) \quad (\Omega' = \frac{\Omega - 2\pi i}{M}) \\
&= \frac{1}{M}\sum_{i=0}^{M-1}X_d(e^{j(\frac{\Omega}{M} - \frac{2\pi i}{M})})
\end{aligned}$$

For $M = 2$, $X_{d2}(e^{j\Omega}) = \frac{1}{2}X_d(e^{j\frac{\Omega}{2}}) + \frac{1}{2}X_d(e^{j(\frac{\Omega}{2} - \pi)})$



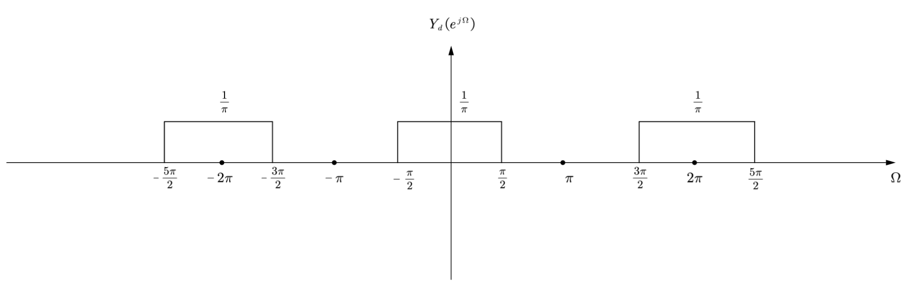
(a) $X_c(j\omega)$ **and (b)** $X_p(j\omega)$





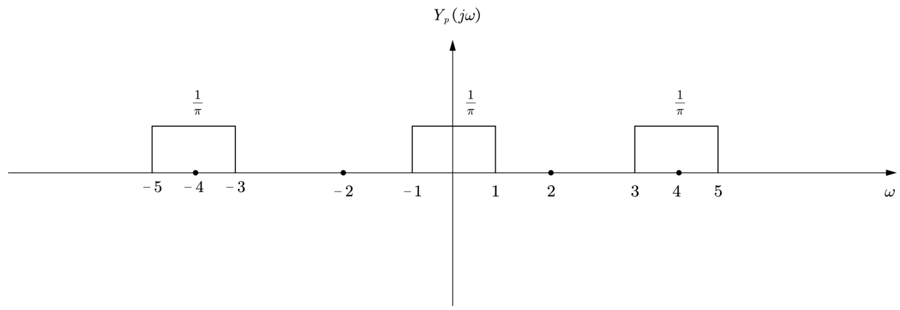
(a) $X_d(e^{j\Omega})$ **and (b)** $X_{d2}(e^{j\Omega})$

$Y_d(e^{j\Omega}) = X_{d2}(e^{j\Omega})H(e^{j\Omega})$



$Y_d(e^{j\Omega})$

$Y_p(j\omega)$ can be obtained from $Y_p(e^{j\Omega})$ by replacing Ω with ωT.



$Y_p(j\omega)$

So we get $y_c(t) = \frac{\sin(t)}{\pi t}$.