Ch.1 *Overview*

Part II *Basic Time Signals*

Lecturer: Yijie Mao

Outline

- Elementary Signals
  - Exponential and Sinusoid Signals
  - The Unit Impulse and Unit Step Functions

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- Elementary Signals
  - Exponential and Sinusoid Signals
  - The Unit Impulse and Unit Step Functions

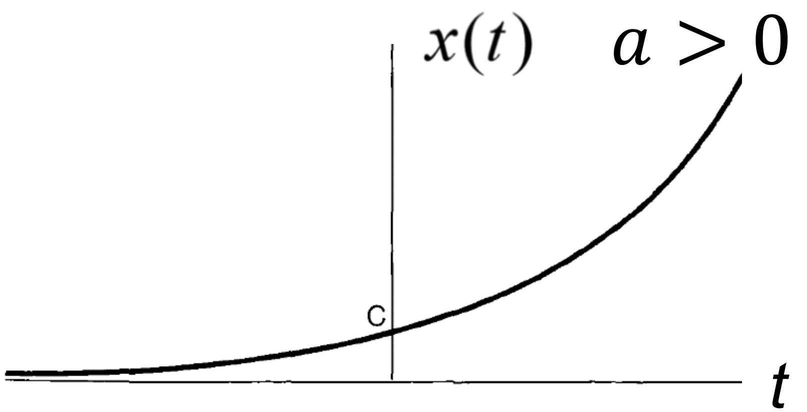
Continuous-time Complex Exponential Signal

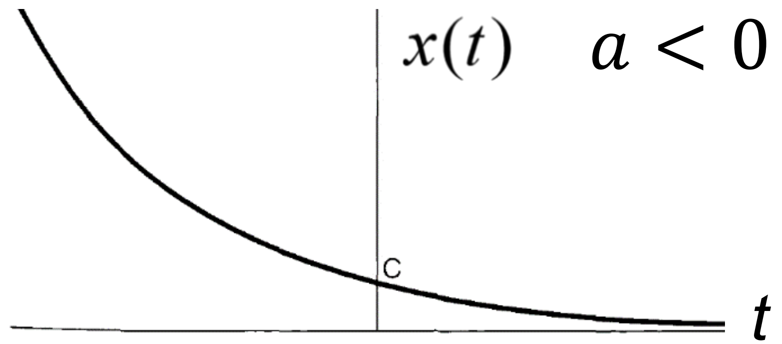
- **General case**

$$x(t) = ce^{at}$$

where c and a are complex numbers

- **Real exponential signal**
  - *When* c and a are real
  - $a > 0$, as $t\uparrow$, $|x(t)| \uparrow$
  - $a < 0$, as $t\uparrow$, $|x(t)| \downarrow$
  - $a = 0$, $|x(t)|$ is constant





$c > 0$

Continuous-time Complex Exponential Signal

- **Periodic exponential signals**

$$x(t) = e^{j\omega_0 t}$$

where c is real, specifically 1 *and* a is purely imaginary

- Fundamental period T_0 of $x(t)$?

$$x(t) = e^{j\omega_0 t} = e^{j\omega_0 (t+T)} = e^{j\omega_0 t} e^{j\omega_0 T} \Rightarrow e^{j\omega_0 T} = 1$$

$$\Rightarrow \omega_0 T = 2k\pi, k = \pm 1, \pm 2, \dots \Rightarrow T = \frac{2k\pi}{\omega_0} \Rightarrow \boxed{T_o = \frac{2\pi}{|\omega_0|}}$$

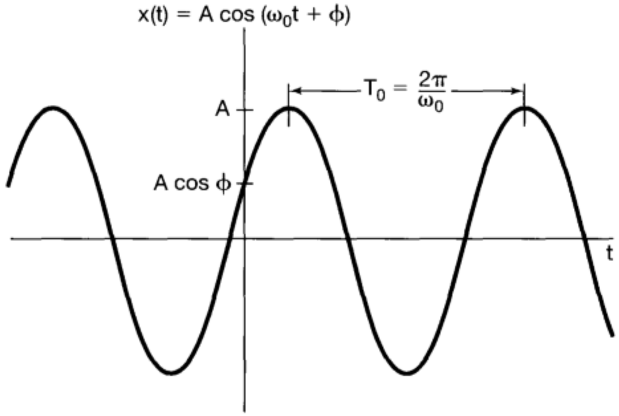
- T_0 is undefined for $\omega_0 = 0$

Continuous-time Sinusoidal Signal

Sinusoidal Signals

$$x(t) = A\cos(\omega_0 t + \emptyset)$$

- A : amplitude
- ω_0 : angular frequency in rad/s
- $\emptyset$: phase angle in radians



- Sinusoidal signal can be written in terms of periodic complex exponentials with the same fundamental frequency

$$e^{j(\omega_0 t+\emptyset)} = \cos(\omega_0 t + \emptyset) + j\sin(\omega_0 t + \emptyset)$$

$$A\cos(\omega_0 t + \emptyset) = A \cdot Re\{e^{j(\omega_0 t+\emptyset)}\} \qquad A\sin(\omega_0 t + \emptyset) = A \cdot Im\{e^{j(\omega_0 t+\emptyset)}\}$$

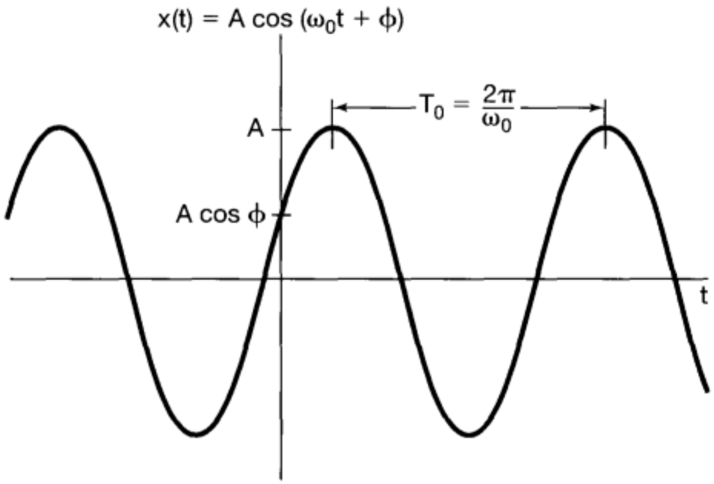
- Fundamental frequency ω_0

Continuous-time Sinusoidal Signal

- **Sinusoidal Signals**

$$x(t) = A\cos(\omega_0 t + \emptyset)$$

  - Fundamental frequency ω_0



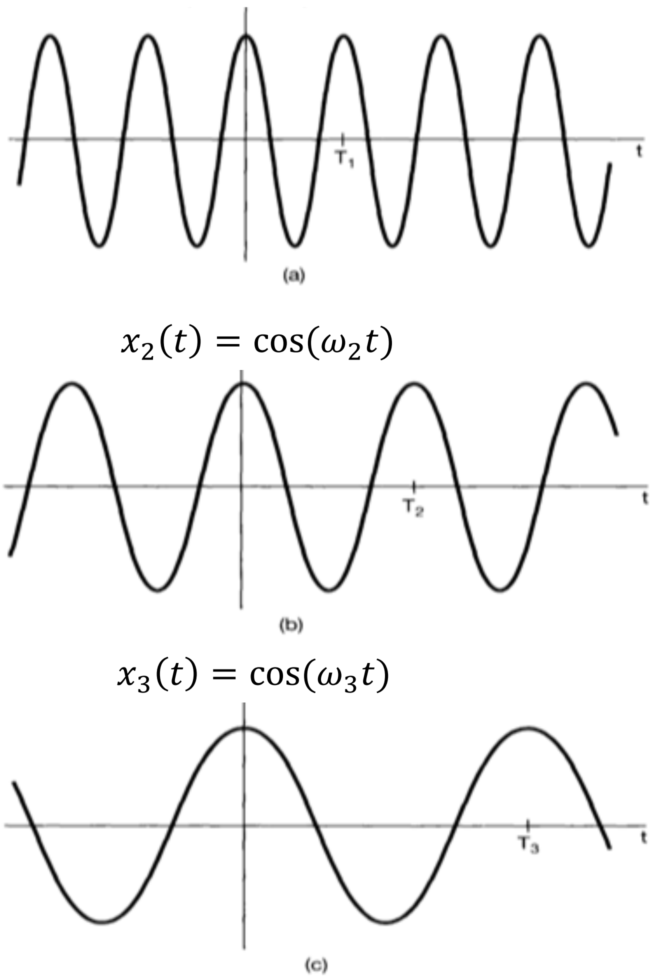
$\omega_3 < \omega_2 < \omega_1$

$T_3 > T_2 > T_1$

$x_1(t) = \cos(\omega_1 t)$

$x_2(t) = \cos(\omega_2 t)$

$x_3(t) = \cos(\omega_3 t)$



Energy and Power of Exponential and Sinusoidal Signals

- $e^{j\omega_0 t}$ and $A\cos(\omega_0 t + \emptyset)$ examples of signals with infinite total energy but finite average power.
- Total Energy over a period

$$E_{period} = \int_0^{T_0} \left| e^{j\omega_0 t} \right|^2 dt = \int_0^{T_0} 1 dt = T_0$$

- Average power over a period

$$p_{period} = \frac{1}{T_0} E_{period} = 1$$

- Complex periodic exponential signal has finite average power

  - Total energy: infinite
  - Average power: finite

$$P_\infty \triangleq \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} \left| e^{j\omega_0 t} \right|^2 dt = 1$$

Harmonically Related Complex Exponentials

- A set of periodic exponentials (with different frequencies), all of which are **periodic** with a **common period** T_0 is known as *a harmonically related set*:

$$\emptyset_k(t) = e^{jk\omega_0 t}, k = 0, \pm 1, \pm 2, \ldots$$

where $\omega_0 = \frac{2\pi}{T_0}$.

- For any $k \neq 0$, the fundamental frequency of $\emptyset_k(t)$ is $|k|\omega_0$; and the fundamental period is

$$\frac{2\pi}{|k|\omega_0} = \frac{T_0}{|k|}$$

Exponential and Sinusoidal Signals

- **Examples – Periodic or not?**

$$(1)\ x_1(t) = je^{j10t}$$

$$(2)\ x_2(t) = e^{(-1+j)t}$$

$$(3)\ x_3(t) = 2\cos(3t + \frac{\pi}{4})$$

$$(4)\ x(t) = 2\cos(3t + \frac{\pi}{4}) + 3\cos(2t - \frac{\pi}{6})$$

Continuous-time Exponential and Sinusoidal Signals

- **Continuous-Time General Complex Exponential**

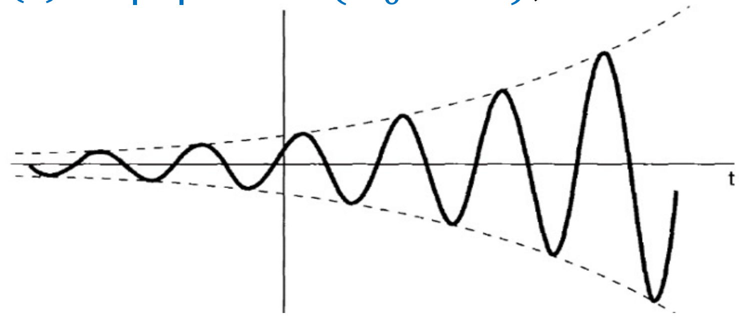
$$x(t) = Ce^{at}$$

C and a are complex numbers

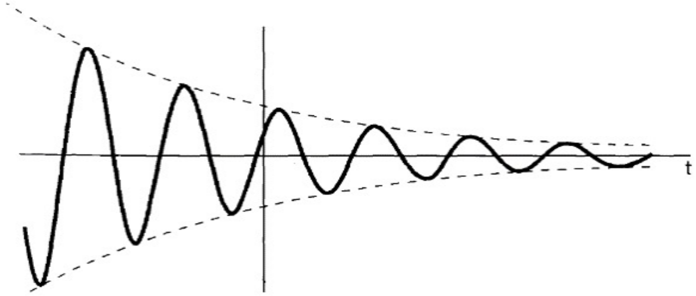
$$C = |C|e^{j\theta},\ a = r + j\omega_0$$

$$Ce^{at} = |C|e^{j\theta}e^{(r+j\omega_0)t}$$
$$= |C|e^{rt}e^{j(\omega_0 t+\theta)}$$

$x(t) = |C|e^{rt}\cos(\omega_0 t + \theta),\ r > 0$



$x(t) = |C|e^{rt}\cos(\omega_0 t + \theta),\ r < 0$



$$Ce^{at} = |C|e^{rt}\cos(\omega_0 t + \theta) + j|C|e^{rt}\sin(\omega_0 t + \theta)$$

Discrete-time Exponential Signals

- **General case**

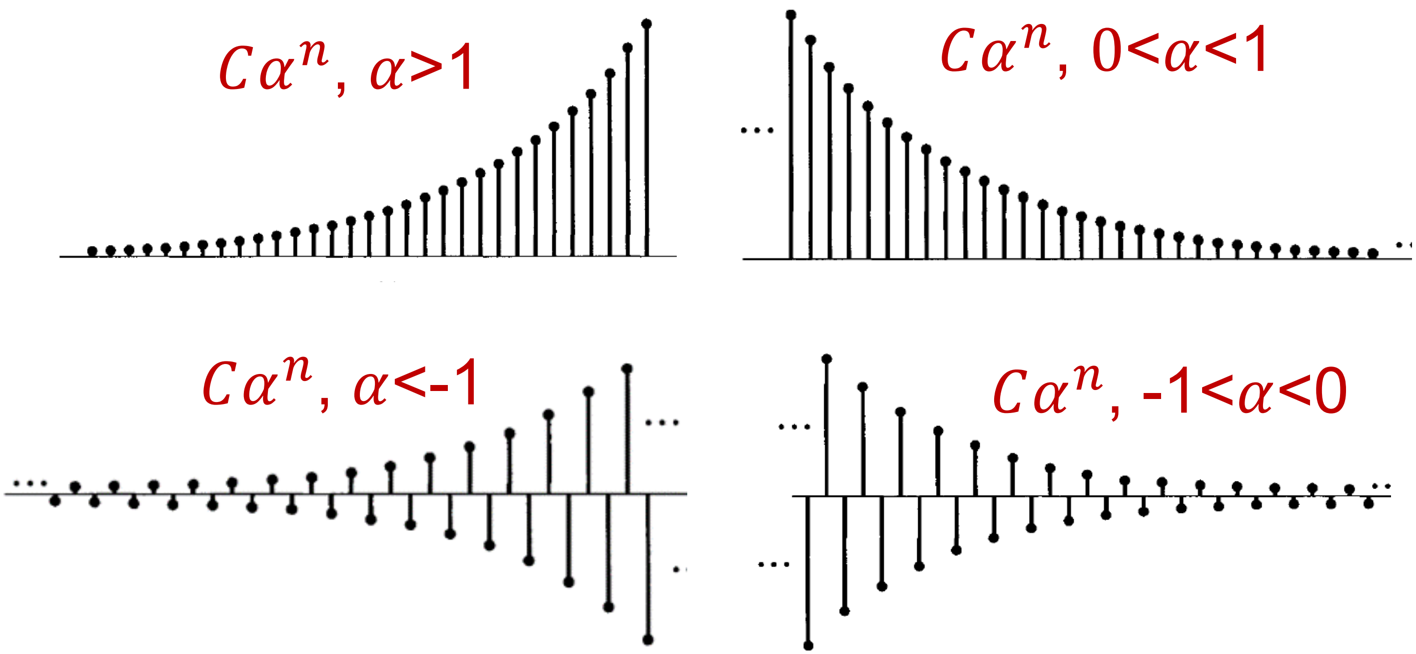
$$x[n] = C\alpha^n$$

C and α are complex numbers

$$x[n] = Ce^{\beta n} \qquad \alpha = e^{\beta}$$

- **Real Exponential Signals**

  C and α are real numbers



Discrete-time Sinusoidal Signals

- **Sinusoidal signals**
  - C is real, specifically 1; β is purely imaginary

$$x[n] = A\cos(\omega_0 n + \emptyset)$$ Closely related $$x[n] = e^{j\omega_0 n}$$

$$e^{j(\omega_0 n+\emptyset)} = \cos(\omega_0 n + \emptyset) + j\sin(\omega_0 n + \emptyset)$$

$$A\cos(\omega_0 n + \emptyset) = A/2 \cdot e^{j\emptyset}e^{j\omega_0 n} + A/2 \cdot e^{-j\emptyset}e^{-j\omega_0 n}$$

$$A\cos(\omega_0 n + \emptyset) = A \cdot Re\{e^{j(\omega_0 n+\emptyset)}\}$$

$$A\sin(\omega_0 n + \emptyset) = A \cdot Im\{e^{j(\omega_0 n+\emptyset)}\}$$

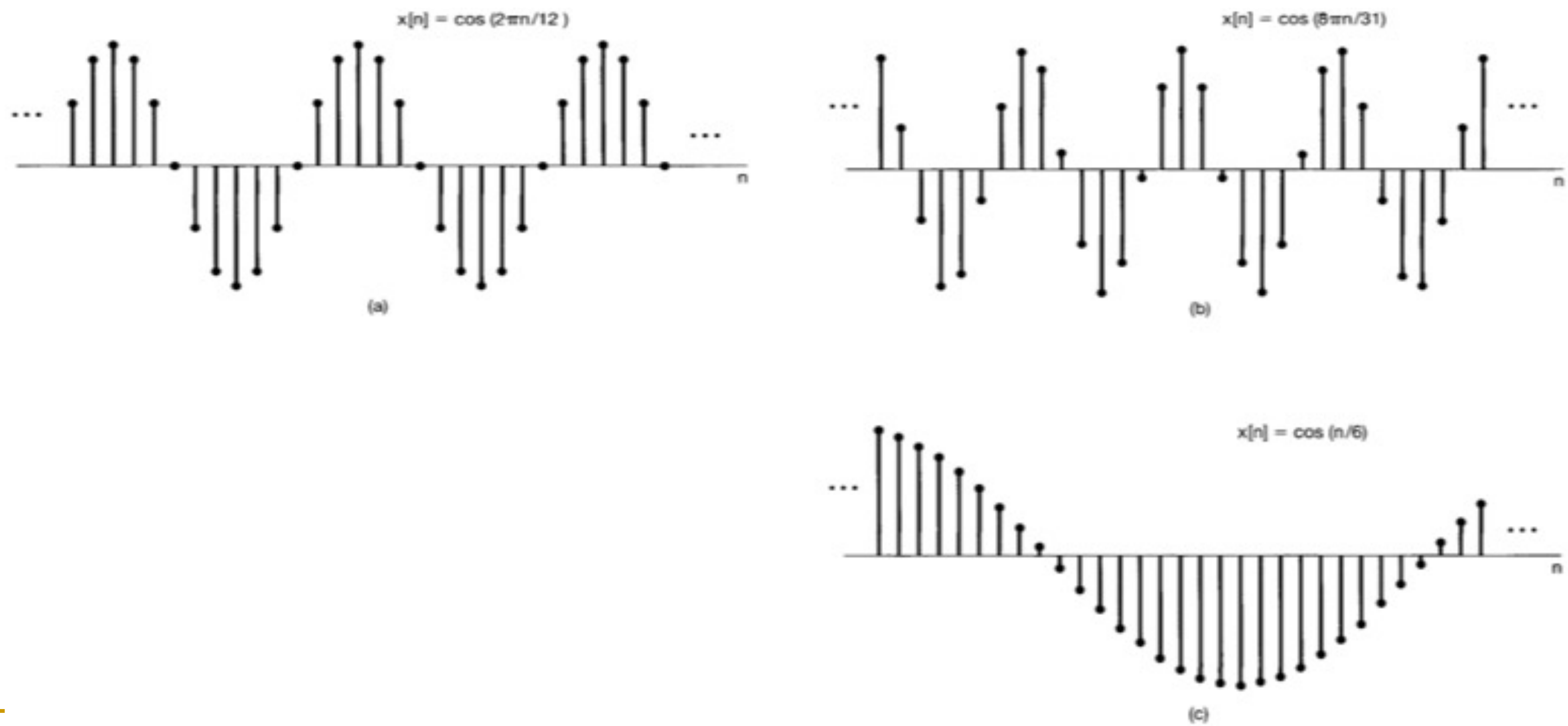
  - Infinite total energy but finite average power

$$|e^{j\omega_0 n}|^2 = 1$$

Discrete-time Sinusoidal Signals

- **Sinusoidal signals**
  - c is real, specifically 1; β is purely imaginary

$$x[n] = A\cos(\omega_0 n + \emptyset)$$



Discrete-time Exponential and Sinusoidal Signals

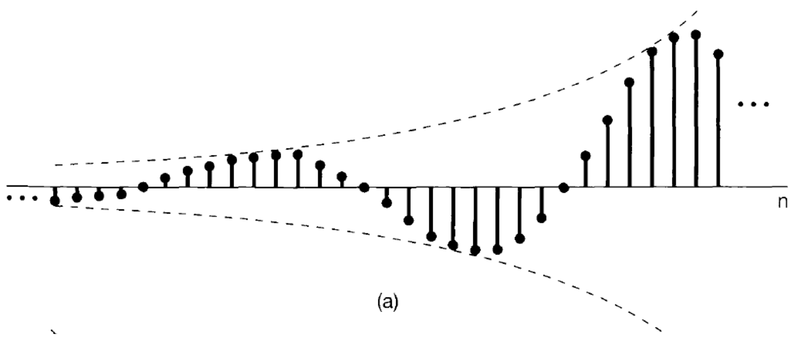
- **Discrete-Time General Complex Exponential**

$$x[n] = C\alpha^n$$

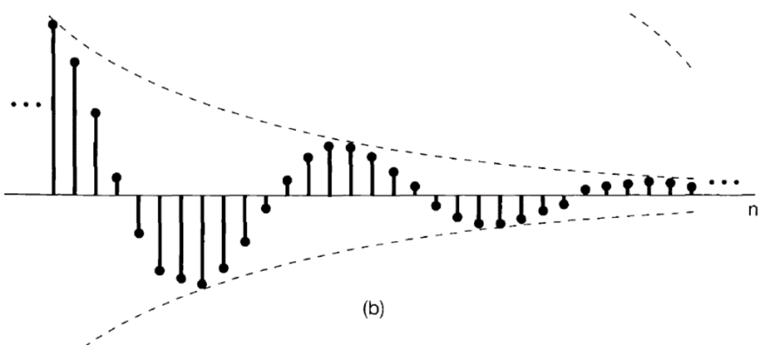
$$C = |C|e^{j\theta}, \qquad \alpha = |\alpha|e^{j\omega_0}$$

$$x[n] = |C||\alpha|^n \cos(\omega_0 n + \theta) + j\,|C||\alpha|^n \sin(\omega_0 n + \theta)$$

(a) $|C||\alpha|^n \cos(\omega_0 n + \theta), |\alpha| > 1$ (b) $|C||\alpha|^n \cos(\omega_0 n + \theta), |\alpha| < 1$



(a)



(b)

Discrete-time Exponential and Sinusoidal Signals

- **Periodic Properties**

$$x[n] = e^{j\omega_0 n}$$

- For continuous-time complex exponential $x(t) = e^{j\omega_0 t}$
  - the larger the ω_0, the higher the rate of oscillation
  - $e^{j\omega_0 t}$ is periodic for any value of ω_0

- Are the above two statements still valid for the discrete case $x[n] = e^{j\omega_0 n}$?

Discrete-time Exponential and Sinusoidal Signals

- **Periodic Properties**

$$x[n] = e^{j\omega_0 n}$$

- ω_0, same value at ω_0 and $\omega_0 + 2k\pi$

$$e^{j(\omega_0+2k\pi)n} = e^{j2k\pi n}e^{j\omega_0 n} = e^{j\omega_0 n}$$

- Only consider interval $0 \leq \omega_0 \leq 2\pi$ or $-\pi \leq \omega_0 \leq \pi$
  - From 0 to π: ω_0 ↑, oscillation rate of $e^{j\omega_0 n}$ ↑
  - From π to 2π: ω_0 ↑, oscillation rate of $e^{j\omega_0 n}$ ↓
  - Maximum oscillation rate at $\omega_0 = \pi$

$$e^{j\pi n} = (e^{j\pi})^n = (-1)^n$$

Discrete-time Exponential and Sinusoidal Signals

- **Periodic Properties**
  - Q: Which one is a higher frequency signal?

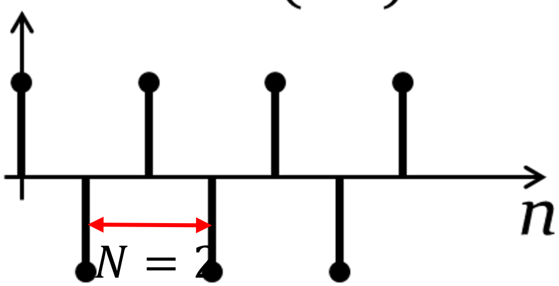
$\omega_0 = \pi$ $\qquad\qquad\qquad\qquad$ $\omega_0 = 3\pi/2$

Discrete-time Exponential and Sinusoidal Signals

- **Periodic Properties**
  - Q: Which one is a higher frequency signal?

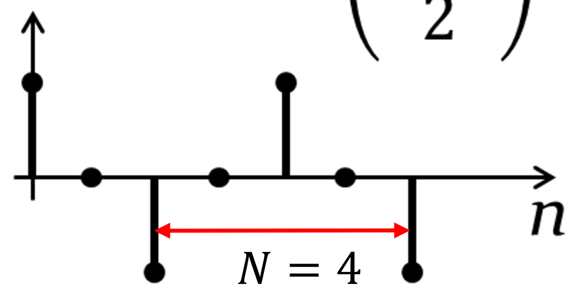
$\omega_0 = \pi$

$\cos(\pi n)$



$\omega_0 = 3\pi/2$

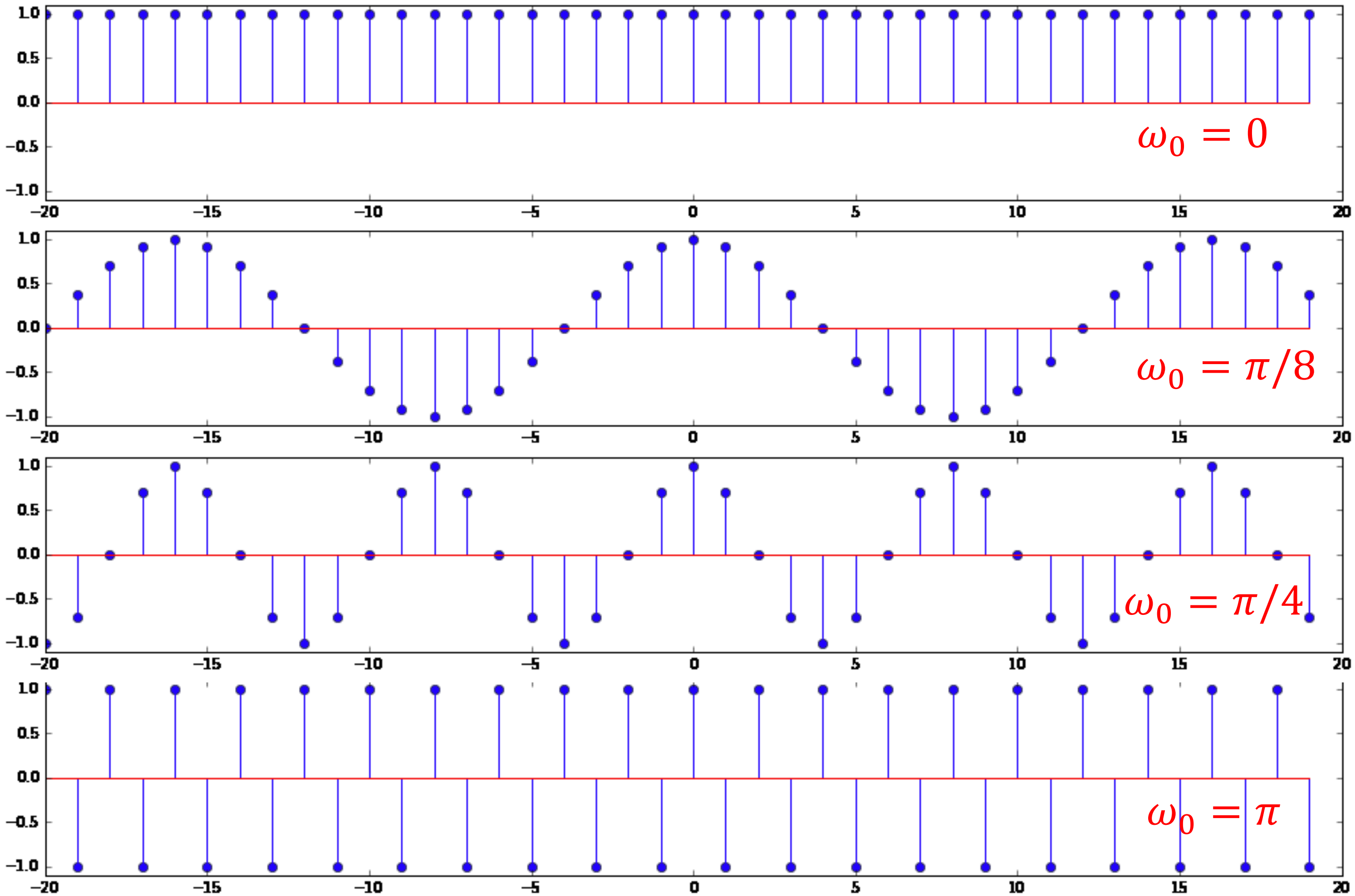
$$\cos\left(\frac{3\pi n}{2}\right) = \cos\left(\frac{\pi n}{2}\right)$$



Discrete-time Exponential and Sinusoidal Signals

- **Periodic Properties**

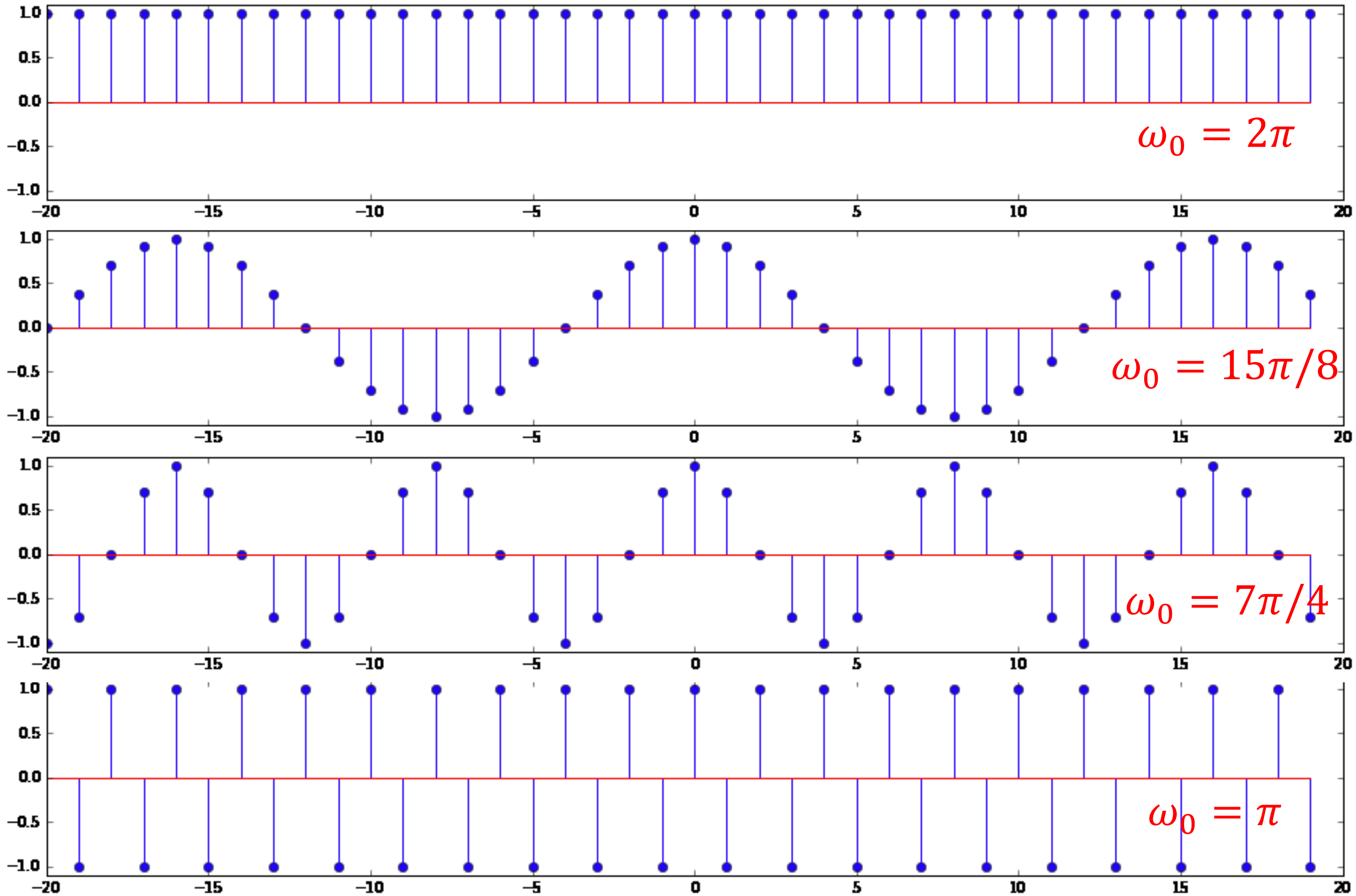
$\cos(\omega_0 n)$



Discrete-time Exponential and Sinusoidal Signals

- **Periodic Properties**

$\cos(\omega_0 n)$



Discrete-time Exponential and Sinusoidal Signals

- **Periodic Properties** $x[n] = e^{j\omega_0 n}$
  - $e^{j\omega_0 n}$ might be non-periodic.
  - In order for $e^{j\omega_0 n}$ to be periodic with *N*>0, must

$$e^{j\omega_0(n+N)} = e^{j\omega_0 N} e^{j\omega_0 n} = e^{j\omega_0 n}$$

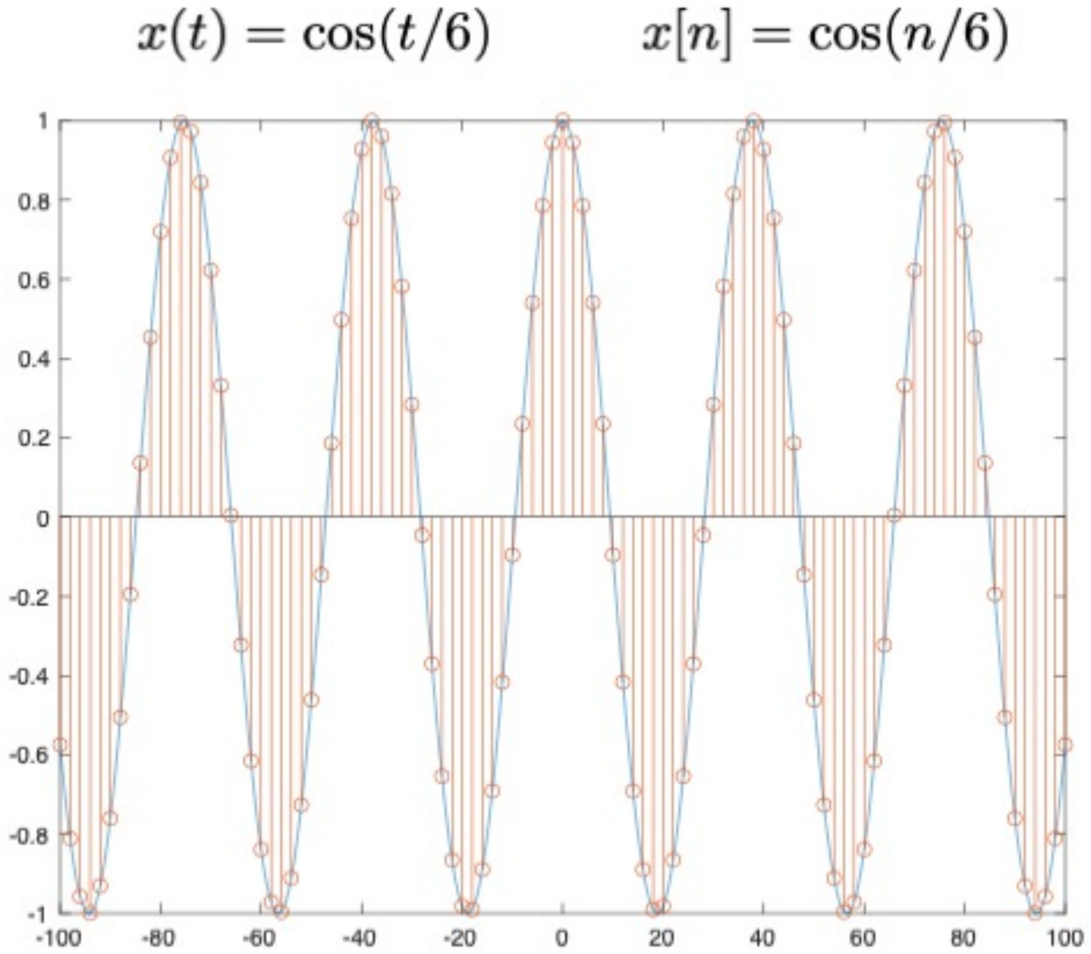
$\omega_0 N = 2\pi m$, m is an integer number

$\dfrac{\omega_0}{2\pi} = \dfrac{m}{N}$ should be a rational number

  - $\omega_0/2\pi$: rational number
  - Fundamental frequency: $2\pi/N = \omega_0/m$
  - Fundamental period: $N = m(2\pi/\omega_0)$

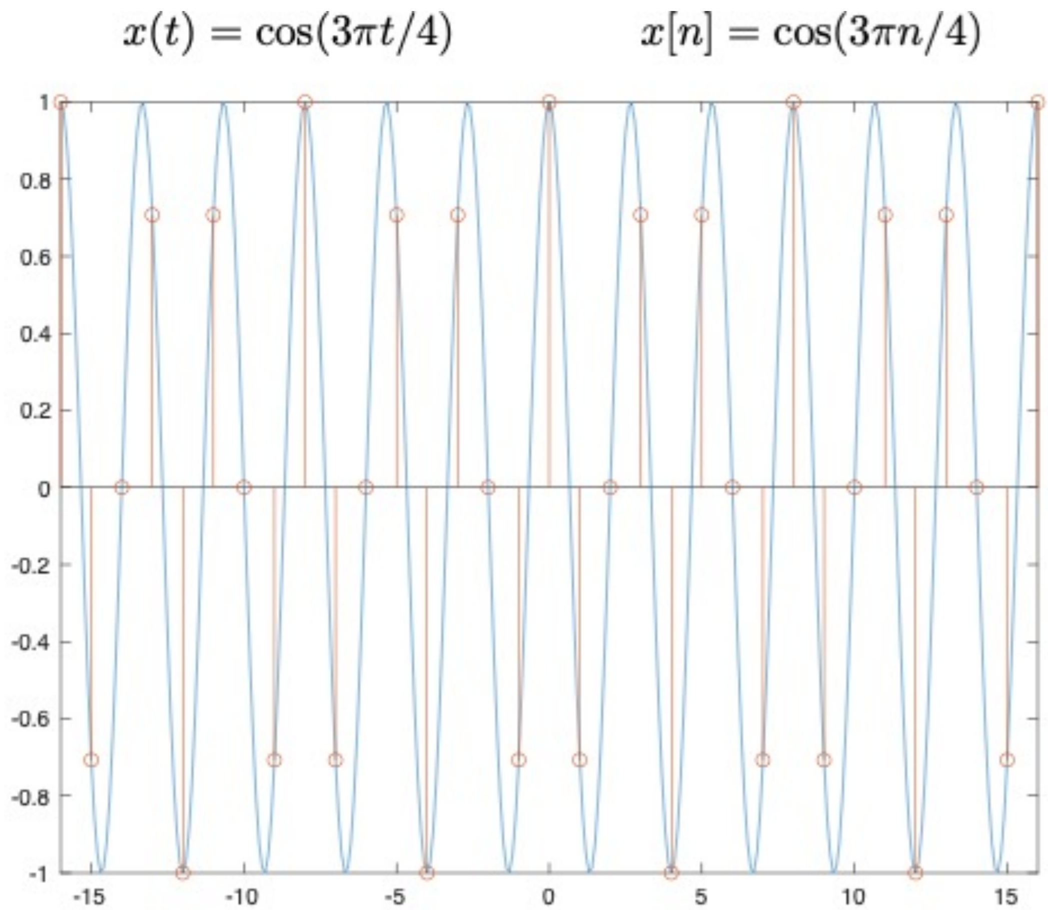
Discrete-time Exponential and Sinusoidal Signals

- **Periodic Properties**



Discrete-time Exponential and Sinusoidal Signals

- **Periodic Properties**



Discrete-time Exponential and Sinusoidal Signals

- **Periodic Properties**

What is the fundamental period of the following discrete-time signals?

$$x[n] = \cos(2\pi n/12)$$

$$x[n] = \cos(8\pi n/31)$$

$$x[n] = \cos(n/6)$$

$$x[n] = e^{j\left(\frac{2\pi n}{3}\right)} + e^{j\left(\frac{3\pi n}{4}\right)}$$

Exponential and Sinusoidal Signals

- **Periodic Properties： continuous-time vs. discrete-time**

$e^{j\omega_0 t}$	$e^{j\omega_0 n}$
Distinct signals for distinct ω_0	**Identical signals for values of ω_0 separated by multiples of 2π**
Periodic for any ω_0	**Only if $\omega_0=2\pi m/N$ for some integers N>0 and m**
Fundamental frequency ω_0	**ω_0/m**
Fundamental period $2\pi/\omega_0$	**$N=m(2\pi/\omega_0)$**

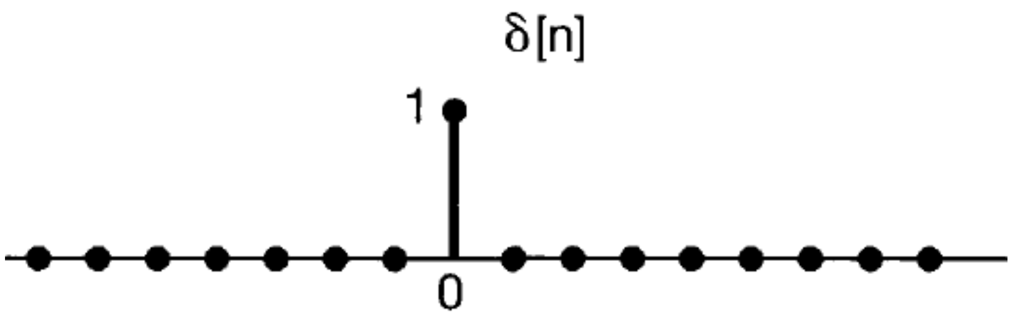
Outline

- Elementary Signals
  - Exponential and Sinusoid Signals
  - The Unit Impulse and Unit Step Functions

Discrete-time Unit Impulse and Unit Step Sequences

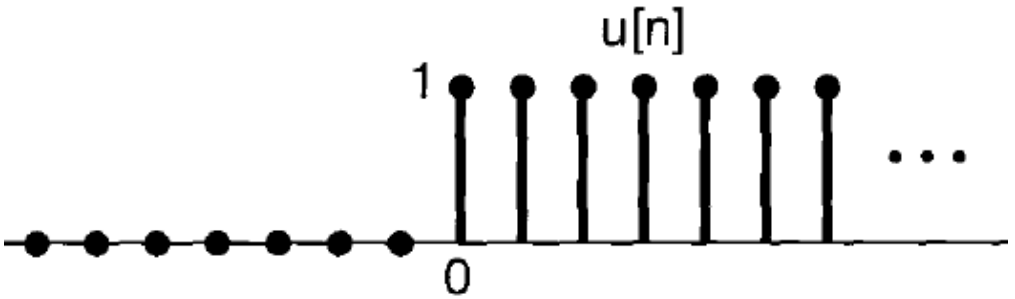
- **Unit impulse (unit sample) is defined as**

$$\delta[n] = \begin{cases} 0, & n \neq 0 \\ 1, & n = 0 \end{cases}$$



- **Unit step is defined as**

$$u[n] = \begin{cases} 0, & n < 0 \\ 1, & n \geq 0 \end{cases}$$



- **Note: u[n] at n = 0 is defined.**

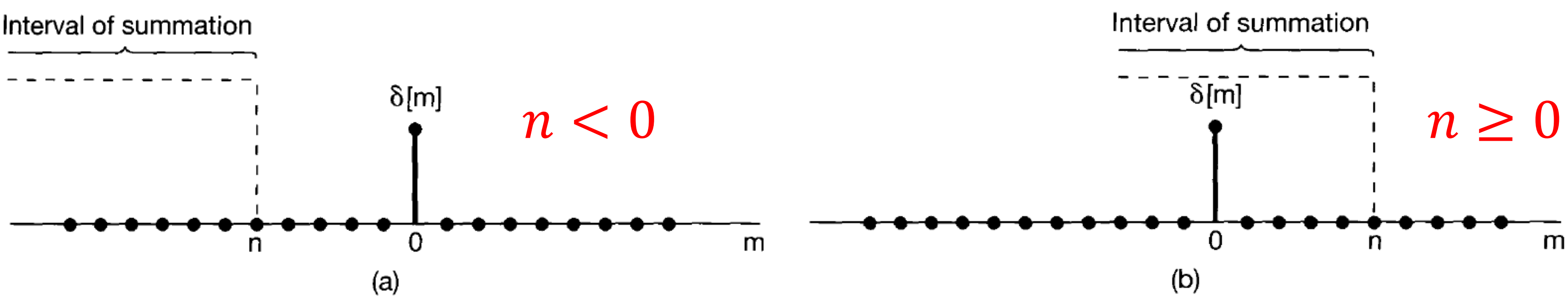
Discrete-time Unit Impulse and Unit Step Sequences

- The impulse is the first difference of the step

$$\delta[n] = u[n] - u[n-1]$$

- Conversely, the step is the running sum of unit sample

$$u[n] = \sum_{m=-\infty}^{n} \delta[m]$$

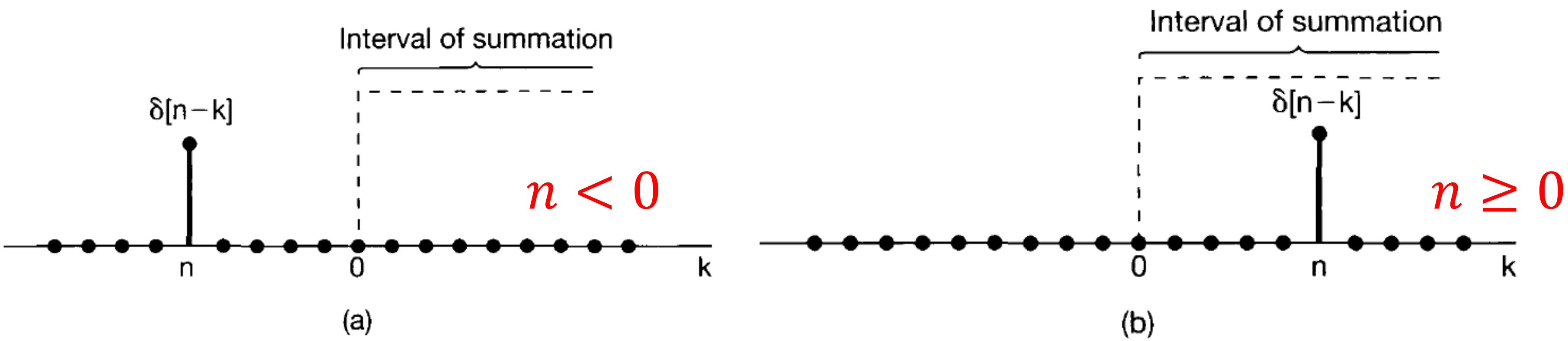


Discrete-time Unit Impulse and Unit Step Sequences

- Let $m = n - k$,

$$u[n] = \sum_{m=-\infty}^{n} \delta[m] = \sum_{k=0}^{+\infty} \delta[n-k]$$

- Running sum of unit sample: superposition of delayed impulses



Discrete-time Unit Impulse and Unit Step Sequences

- Sampling property

$$x[n]\delta[n] = x[0]\delta[n]$$

- More generally

$$x[n]\delta[n - n_0] = x[n_0]\delta[n - n_0]$$

Continuous-time Unit Impulse and Unit Step Functions

- Unit step function

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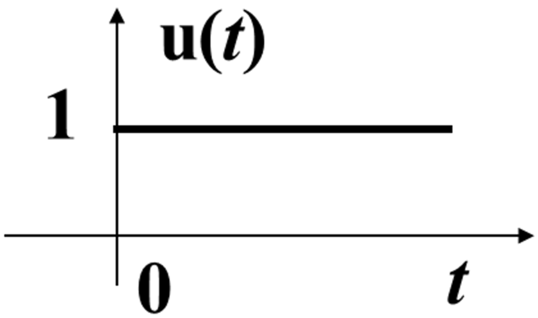
$$u(t) = \begin{cases} 0, & t < o \\ 1, & t > 0 \end{cases}$$



- The continuous unit step $u(t)$ is the running integral of unit impulse $\delta(t)$

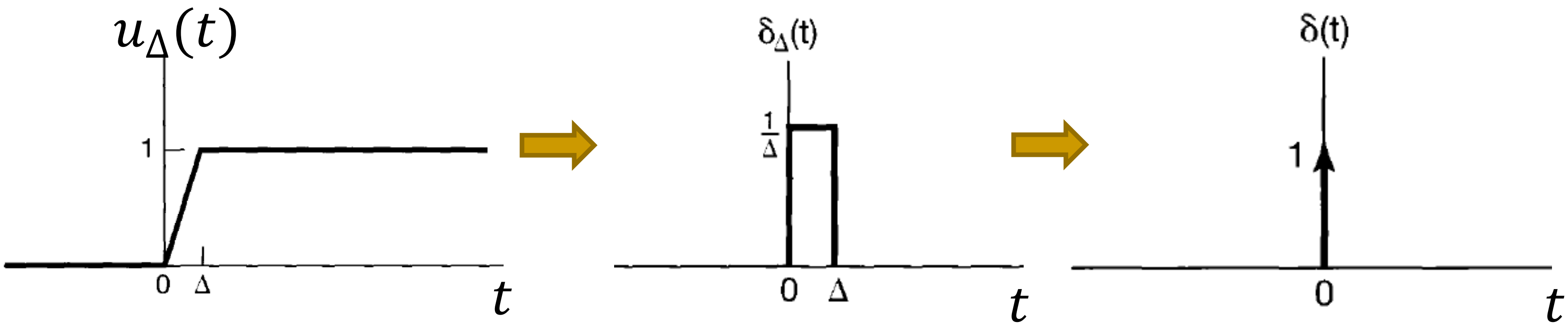
$$u(t) = \int_{-\infty}^{t} \delta(\tau)\mathrm{d}\tau$$

- $\delta(t)$ the first derivative of $u(t)$

$$\delta(t) = \frac{du(t)}{dt}$$

Continuous-time Unit Impulse and Unit Step Functions

- $u(t)$ is discontinuous at $t = 0$, How we get $\delta(t)$?
  - Consider $u_\Delta(t)$

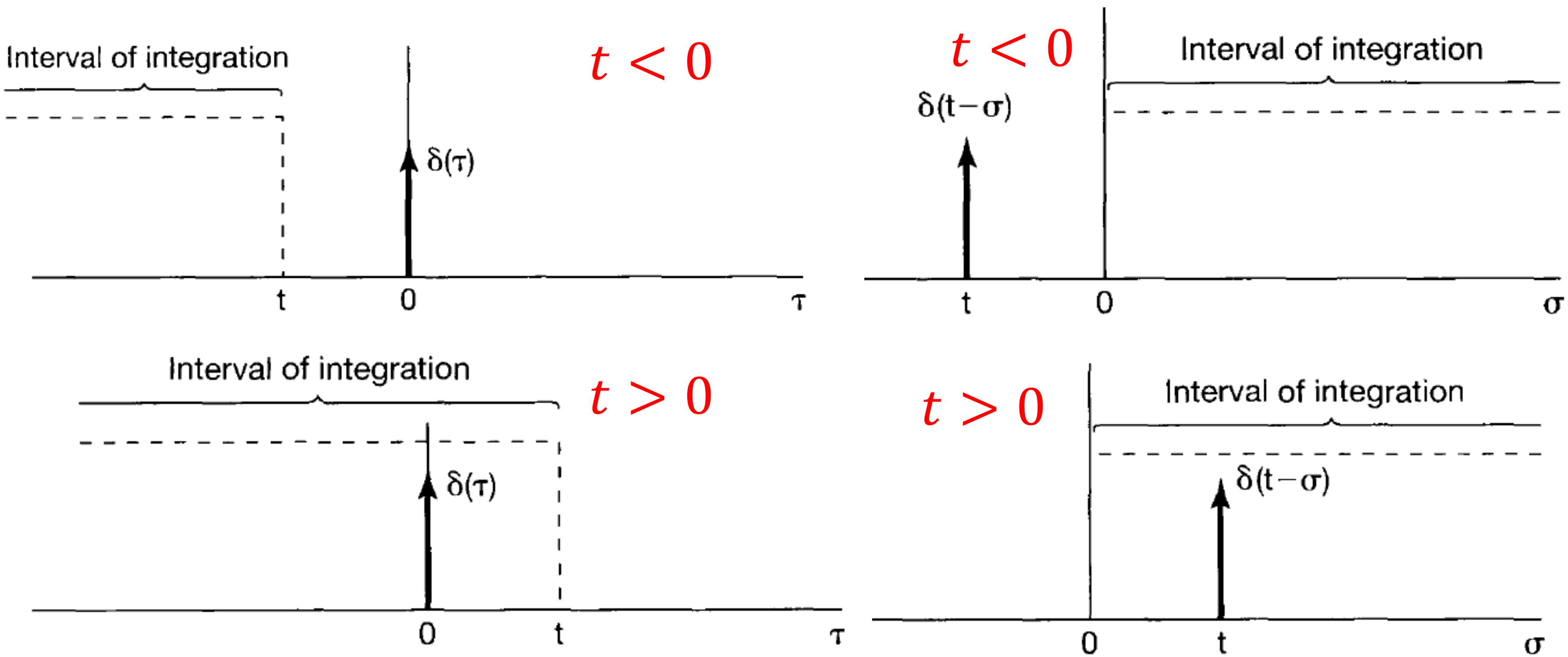


$$u(t) = \lim_{\Delta \to 0} u_\Delta(t) \qquad \delta_\Delta(t) = \frac{d\, u_\Delta(t)}{dt} \qquad \delta(t) = \lim_{\Delta \to 0} \delta_\Delta(t)$$

  - arrow at $t = 0$: area of the pulse is concentrated at $t = 0$
  - arrow height and "1“: *area* of the impulse

Continuous-time Unit Impulse and Unit Step Functions

$$u(t) = \int_{-\infty}^{t} \delta(\tau)d\tau \qquad \text{Let } \sigma = t - \tau \qquad u(t) = \int_{0}^{\infty} \delta(t - \sigma)d\sigma$$



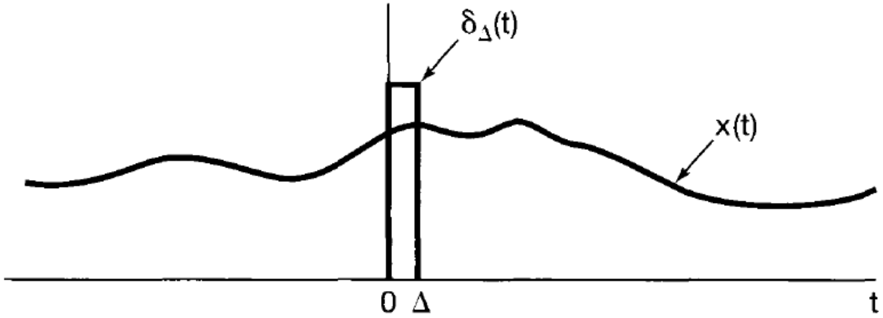
Continuous-time Unit Impulse and Unit Step Functions

- Sampling property

$$x_1(t) = x(t)\delta_\Delta(t)$$

$$x(t)\delta_\Delta(t) \approx x(0)\delta_\Delta(t)$$

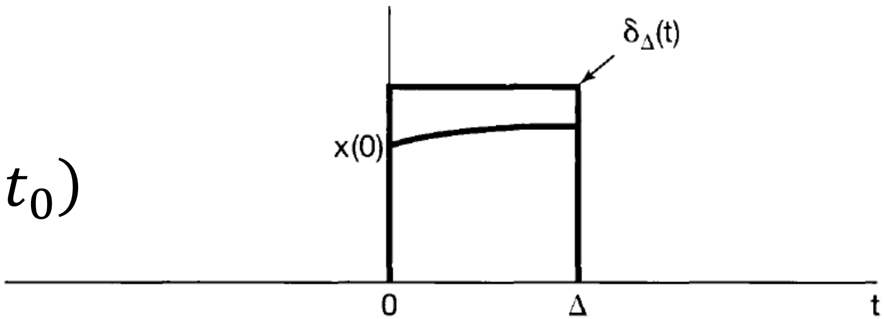
$$x(t)\delta(t) = \lim_{\Delta\to 0} x(t)\delta_\Delta(t) = x(0)\delta(t)$$



(a)

- More generally

$$x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$$

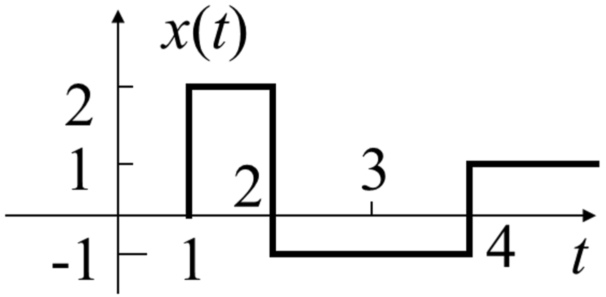


(b)

- Example:

  (1) Calculate and sketch the $x'(t)$;

  (2) Recover $x(t)$ from $x'(t)$.

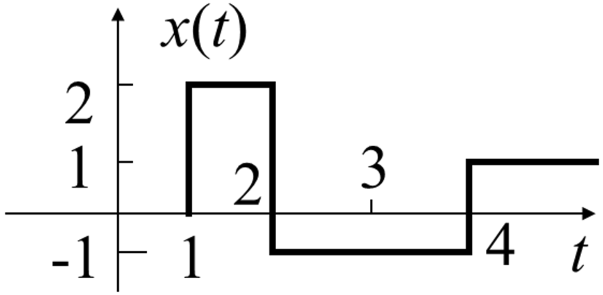


Continuous-time Unit Impulse and Unit Step Functions

- Example:

  (1) Calculate and sketch the $x'(t)$;

  (2) Recover $x(t)$ from $x'(t)$.

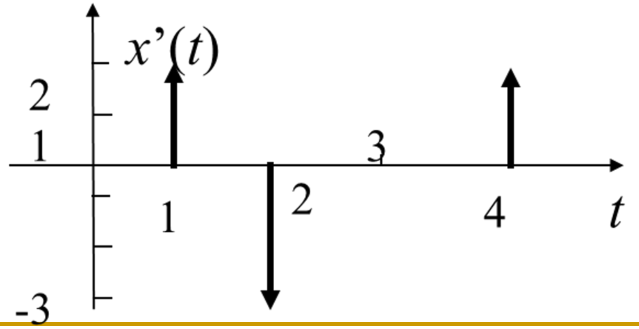


- Solutions:

$$(1)\quad x(t) = 2u(t-1) - 3u(t-2) + 2u(t-4)$$

$$\therefore \quad x'(t) = 2\delta(t-1) - 3\delta(t-2) + 2\delta(t-4)$$

$$(2)\quad x(t) = \int_0^{\infty} x'(t)dt$$



Summary

- Elementary Signals
  - Exponential and Sinusoid Signals
  - The Unit Impulse and Unit Step Functions

- Reference in textbook: 1.3, 1.4