Ch.2 *Linear Time-Invariant Systems*

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Outline

- **Discrete-Time LTI Systems**
- **Continuous-Time LTI Systems**
- **Properties of LTI Systems**
- **Differential or Difference Equations**

Part I *Discrete-Time LTI Systems*

Outline

- The Representation of Discrete-Time Signals in Terms of Impulses
- The Discrete-Time Unit Impulse Response
- The Convolution-Sum Representation of LTI Systems

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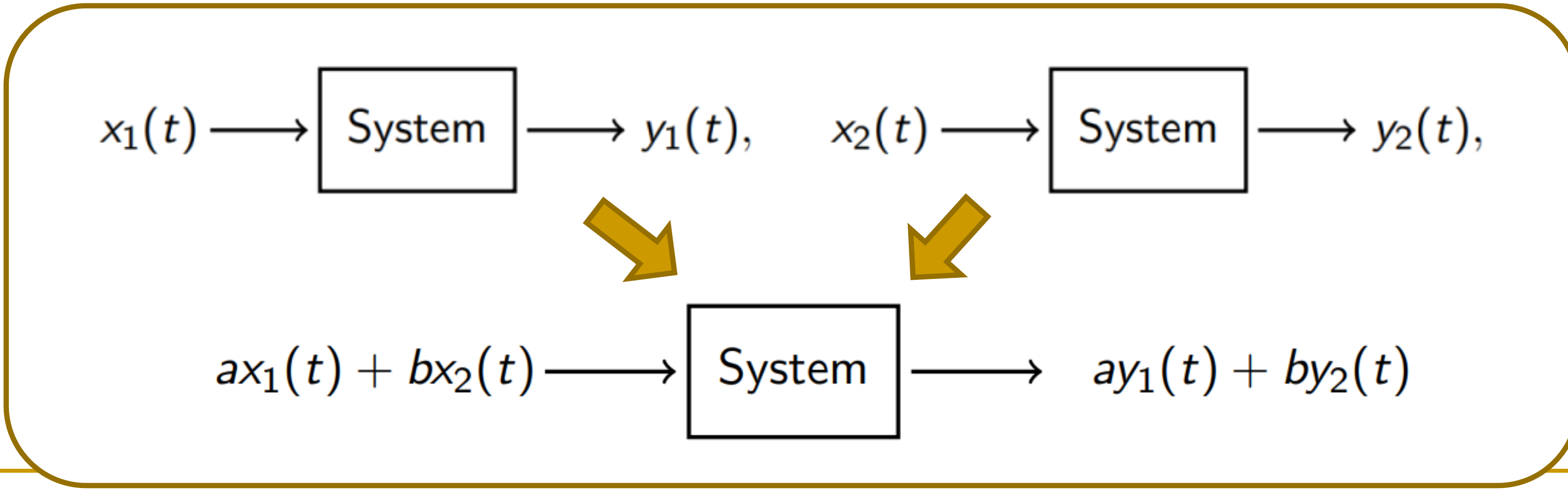
Linear Systems

- A system is linear if the following condition holds for any two inputs $x_1(t)$ and $x_2(t)$:
  - **Additivity**

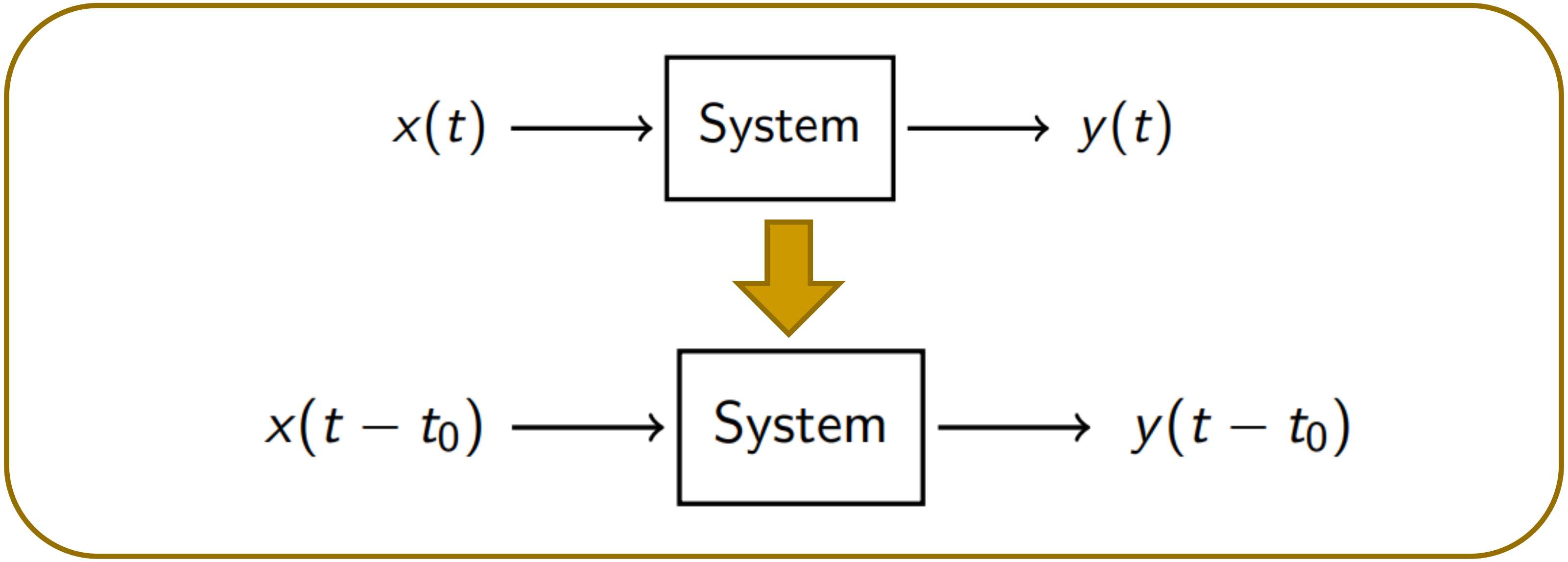
    The response to $x_1(t) + x_2(t)$ is $y_1(t) + y_2(t)$
  - **Scaling/Homogeneity**

    The response to $a \cdot x_1(t)$ is $a \cdot y_1(t)$



Time Invariance

- A system is time-invariant if a time-shift (advance or delay) at the input causes an identical shift at the output.



- The system has no internal way to keep time

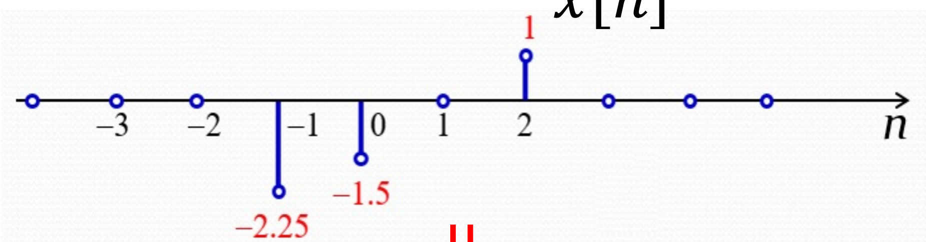
Linear Time-Invariant Systems (LTI)

- LTI system: A system satisfying both the **linearity** and the **time-invariance** properties.
- Many physical processes possess the properties of linearity and time invariance.
- Highly useful signal processing algorithms have been developed utilizing this class of systems.
- LTI systems are mathematically easy to analyze and characterize, and consequently, easy to design. They possess **superposition** theorem.

  If we represent the input to an LTI system in terms of linear combination of a set of basic signals, we can then use superposition to compute the output of the system in terms of responses to these basic signals.

Representation of Discrete-Time Signals in Terms of Impulse

$x[n]$



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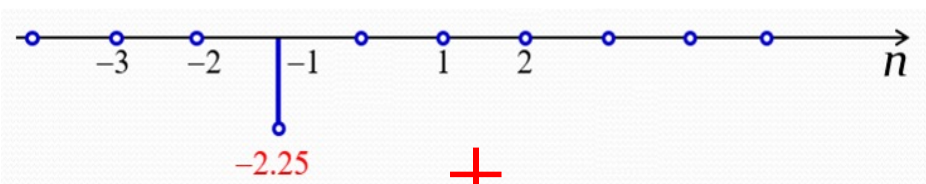

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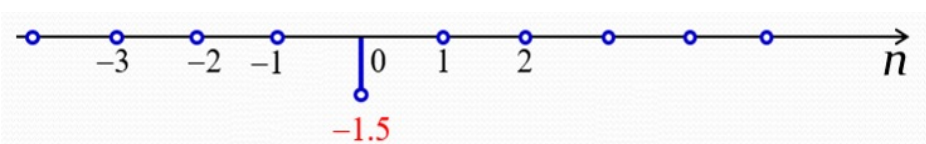

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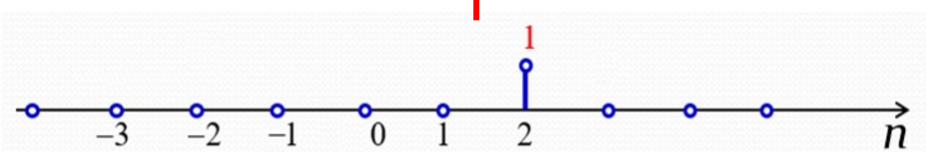


$x[n]$

=

$x_1[n] = -2.25 \times \delta[n+1]$

+

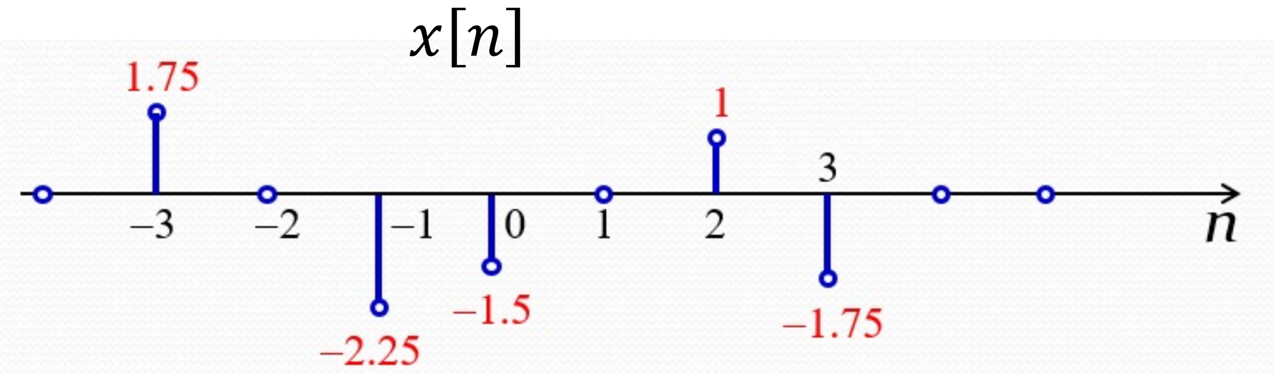
$x_2[n] = -1.5 \times \delta[n]$

+

$x_3[n] = 1 \times \delta[n-2]$

Representation of Discrete-Time Signals in Terms of Impulse

- **An arbitrary sequence can be represented as the weighted sum of shifted unit impulses**



$$x[n] = 1.75\delta[n+3] - 2.25\delta[n+1] - 1.5\delta[n] + \delta[n-2] - 1.75\delta[n-3]$$

Representation of Discrete-Time Signals in Terms of Impulse

- Disctete-time unit impulse can be used to construct any discrete-time signal, because

$$x[k]\delta[n-k] = \begin{cases} x[k], & n = k \\ 0, & n \neq k \end{cases}$$

- **A general form**

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

Sifting property of $\delta[n]$

- Note: for any value of n, only one of the terms on the right-hand side (RHS) is nonzero.

Outline

- The Representation of Discrete-Time Signals in Terms of Impulses
- The Discrete-Time Unit Impulse Response
- The Convolution-Sum Representation of LTI Systems

Discrete-Time Unit Impulse Response

- The response of a system to a unit impulse sequence $\delta[n]$ is called the **unit impulse response**, denoted by $h[n]$.

Discrete-Time Unit Impulse Response

- **How to calculate the impulse response of a system?**

  - Examples: a system is defined as

$$y[n] = a_1 x[n] + a_2 x[n-1] + a_3 x[n-2] + a_4 x[n-3]$$

its impulse response $h[n]$ is

$$h[n] = a_1 \delta[n] + a_2 \delta[n-1] + a_3 \delta[n-2] + a_4 \delta[n-3]$$

Discrete-Time Unit Impulse Response

- **How to calculate the impulse response of a system?**
  - Examples: a system is defined as

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$

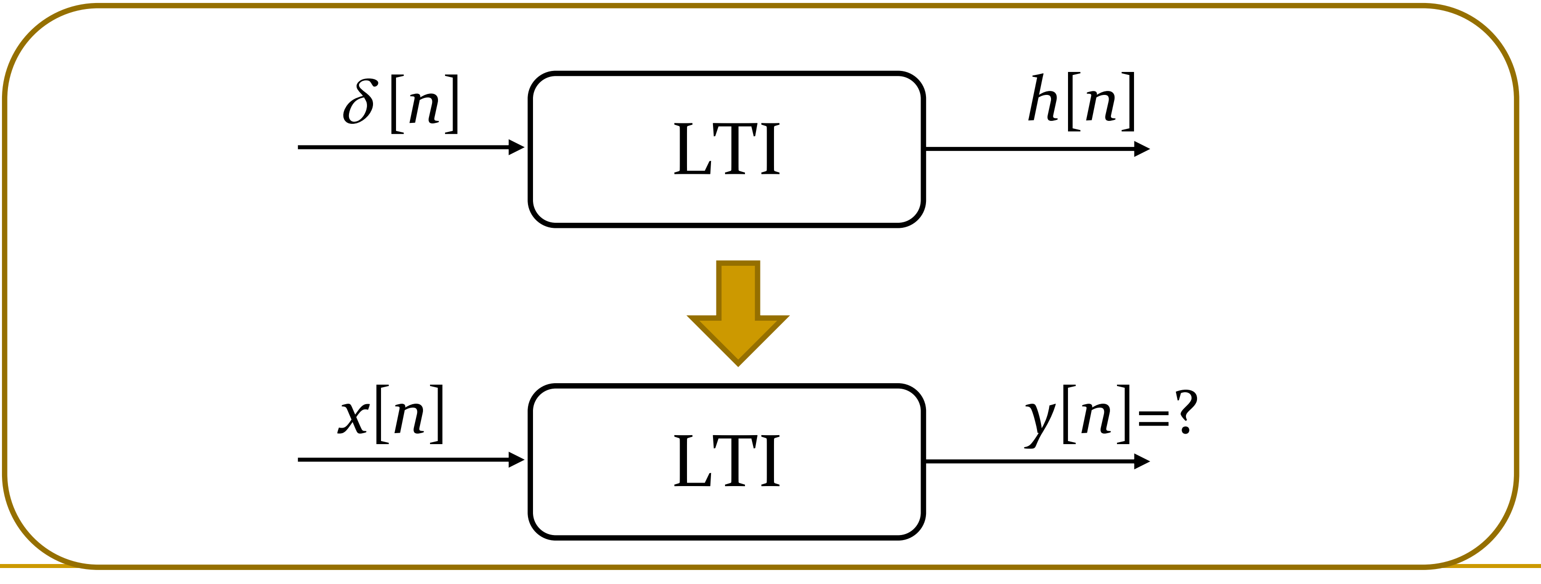
its impulse response $h[n]$ is

$$h[n] = \sum_{k=-\infty}^{n} \delta[k]$$

Discrete-Time Unit Impulse Response

- **An LTI discrete system is completely characterized by its impulse response.**

- **In other words, knowing the impulse response one can compute the output of the LTI system for an arbitrary input.**



Discrete-Time Unit Impulse Response

- Recall, an arbitrary input $x[n]$ can be expressed as a linear combination of shifted unit impulses

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k]$$

Sifting property of $\delta[n]$

- For any $k = k_0$

$$\delta[n] \longrightarrow \boxed{\text{LTI}} \longrightarrow h[n]$$

Time-invariance $$\delta[n-k_0] \longrightarrow \boxed{\text{LTI}} \longrightarrow h[n-k_0]$$

Scaling $$x[k_0]\delta[n-k_0] \longrightarrow \boxed{\text{LTI}} \longrightarrow x[k_0]h[n-k_0]$$

$$\Rightarrow \quad x[n] \longrightarrow \boxed{\text{LTI}} \longrightarrow y[n]$$

Additivity

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \qquad y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

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- The Representation of Discrete-Time Signals in Terms of Impulses
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- The Convolution-Sum Representation of LTI Systems

Discrete-Time Unit Impulse Response and the Convolution-Sum

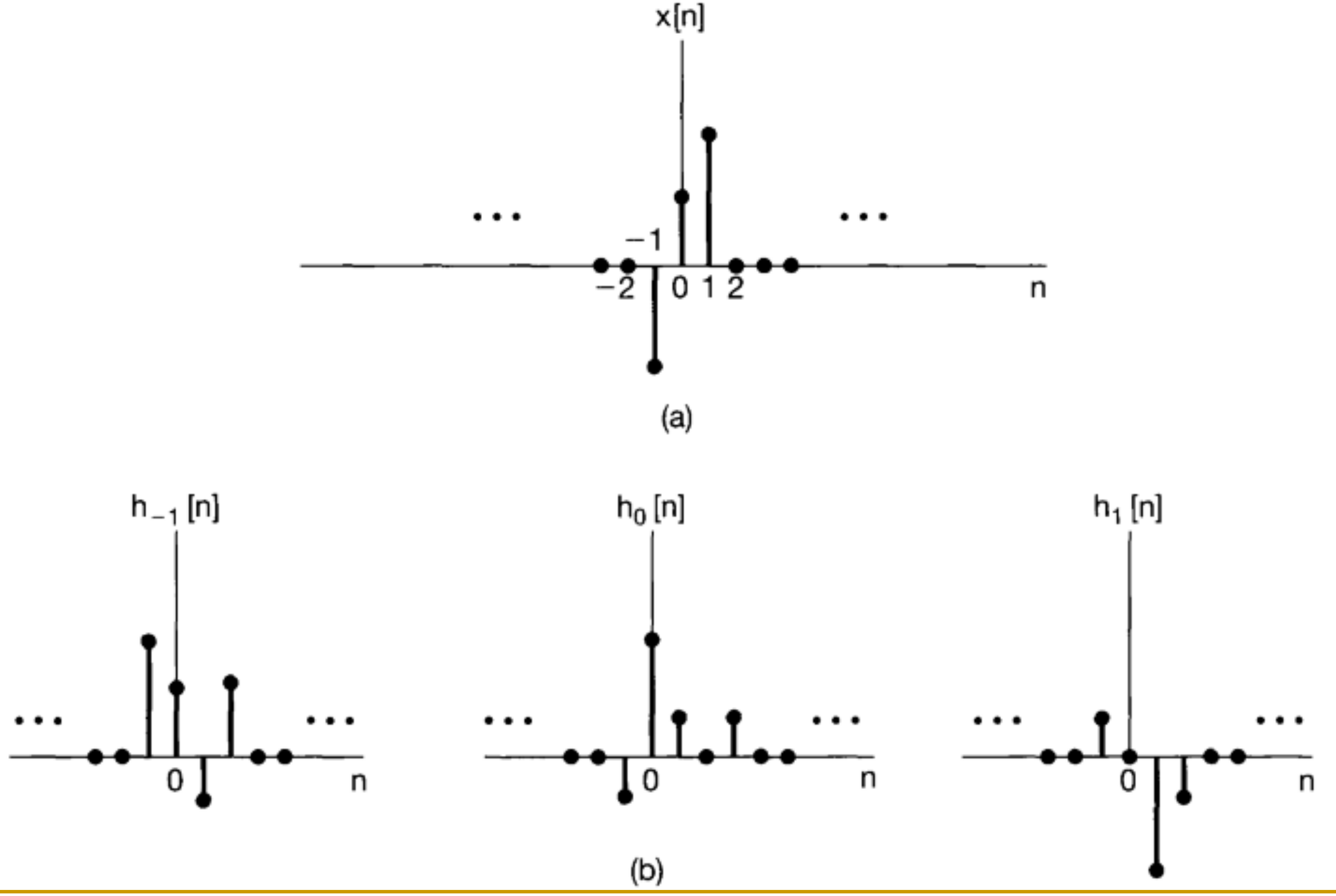
- Unit impulse response completely characterizes an LTI system.

$$x[n] = \sum_{k=-\infty}^{\infty} x[k]\delta[n-k] \longrightarrow \boxed{\text{LTI}} \longrightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$

- $y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k]$ is refereed to as the convolution-sum, denoted by $y[n] = x[n] * h[n]$

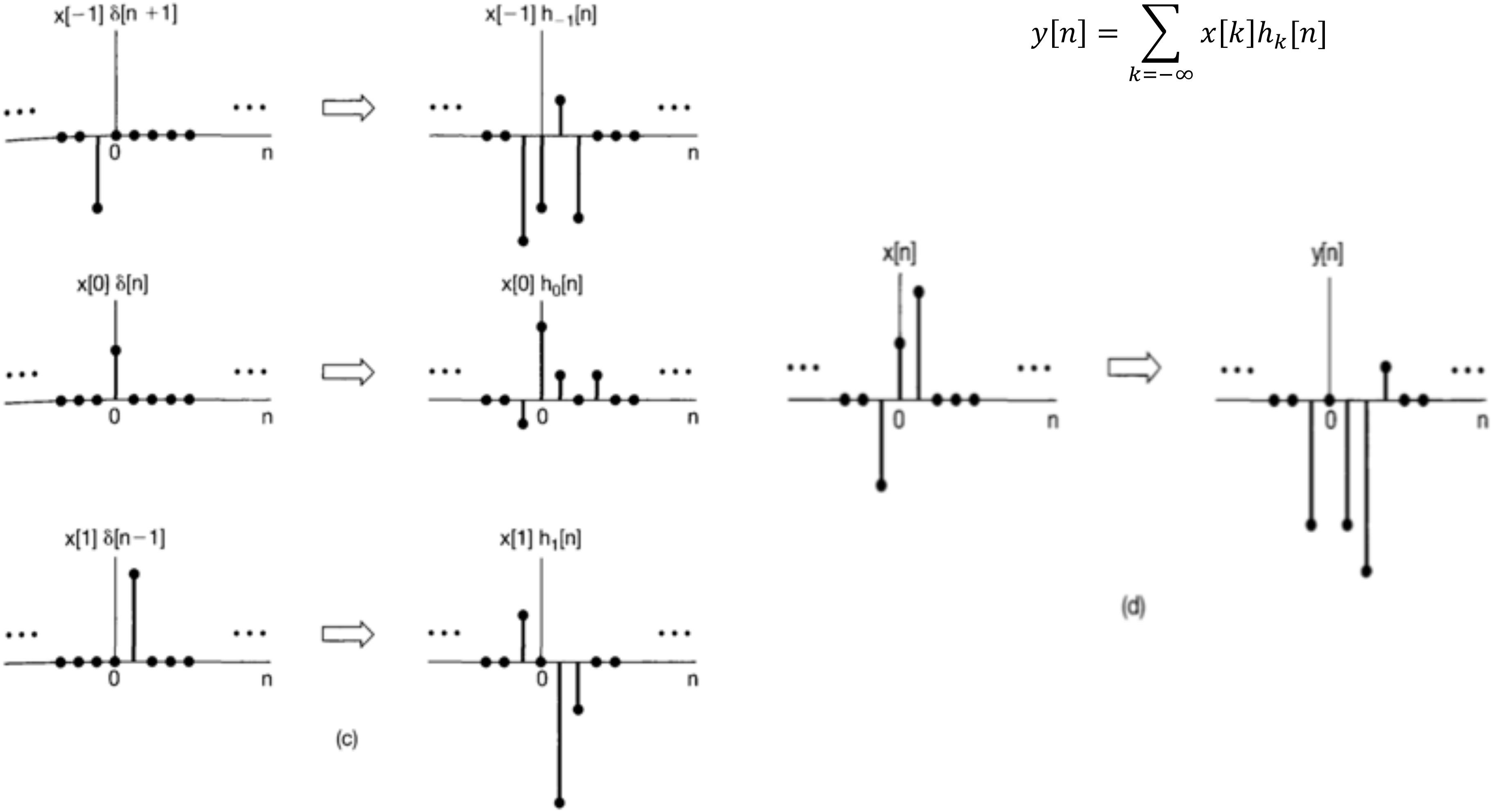
$$x[n] \longrightarrow \boxed{\text{LTI}} \longrightarrow y[n] = x[n] * h[n]$$

Graphical Interpretation

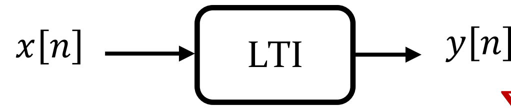


Graphical Interpretation

$$y[n] = \sum_{k=-\infty}^{\infty} x[k]h_k[n]$$



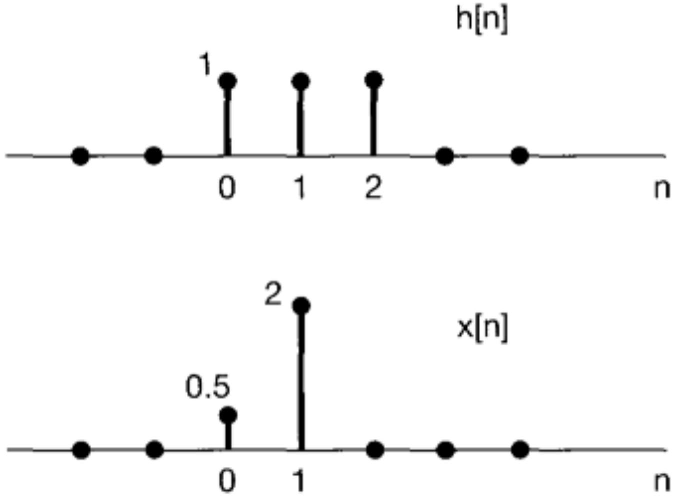
The Convolution-Sum



$$= \sum_{k=-\infty}^{\infty} x[k]h[n-k]$$
$$= x[n] * h[n]$$

How to calculate the convolution sum?

- Example 1: Consider an LTI system with impulse response $h[n]$ and input $x[n]$ given below, calculate $y[n]$.



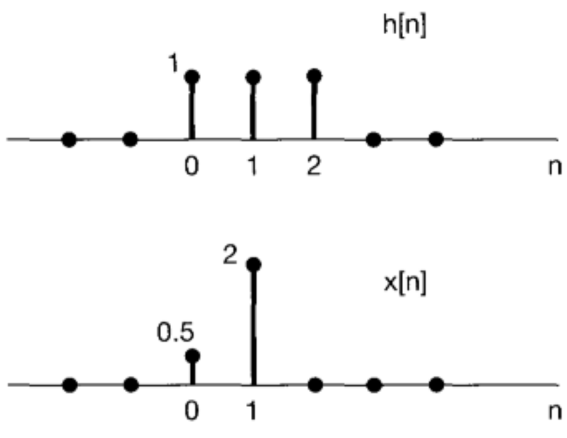
(a)

The Convolution-Sum

$$x[n] \longrightarrow \boxed{\text{LTI}} \longrightarrow y[n] = \sum_{k=-\infty}^{\infty} x[k]h[n-k] = x[n] * h[n]$$

How to calculate the convolution sum?

➢ Method 1: View as functions of n (n: variable, k: constant)



(a)

- Step 1: determine the range of k

$$k \in \{0,1\}$$

- Step 2: determine the range of n

$$[n-k] \in \{0,1,2\} \leftrightarrow n \in \{0,1,2,3\},$$

For other n, $y[n]$=0

- Step 3: calculate y[n] for each n

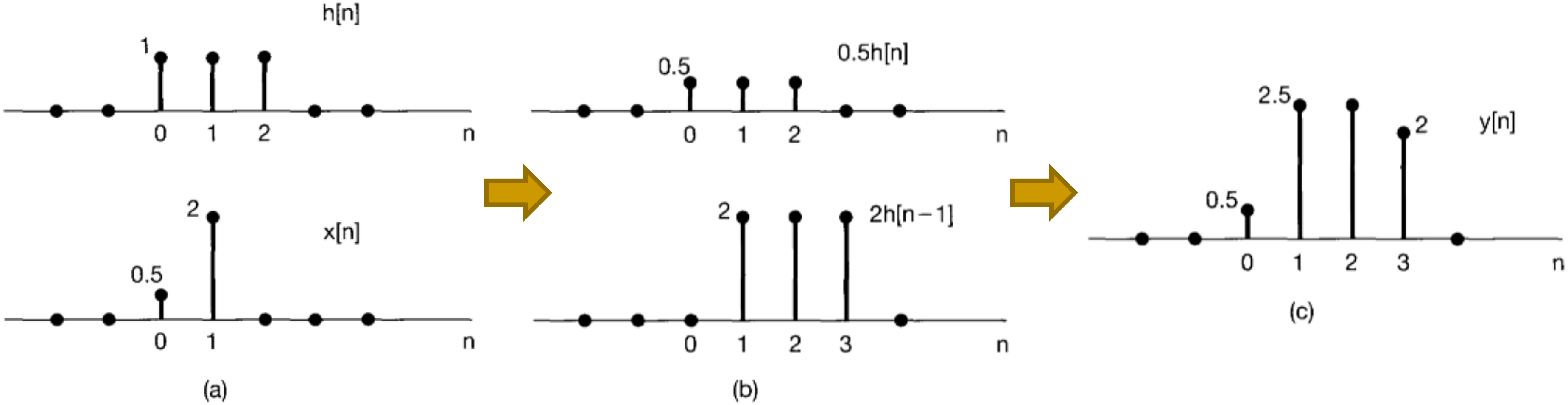
$$\begin{aligned} y[n] &= x[0]h[n-0] + x[1]h[n-1] \\ &= 0.5h[n] + 2h[n-1], n \in \{0,1,2,3\} \end{aligned}$$

The Convolution-Sum

How to calculate the convolution sum?

- Method 1: View as functions of n (n: variable, k: constant)

$$y[n] = x[0]h[n-0] + x[1]h[n-1] = 0.5h[n] + 2h[n-1],$$
$$n \in \{0,1,2,3\}$$



The Convolution-Sum

How to calculate the convolution sum?

- Method 2: View as functions of k (n: constant, k: variable):

For each n :

- Step 1: change time variables

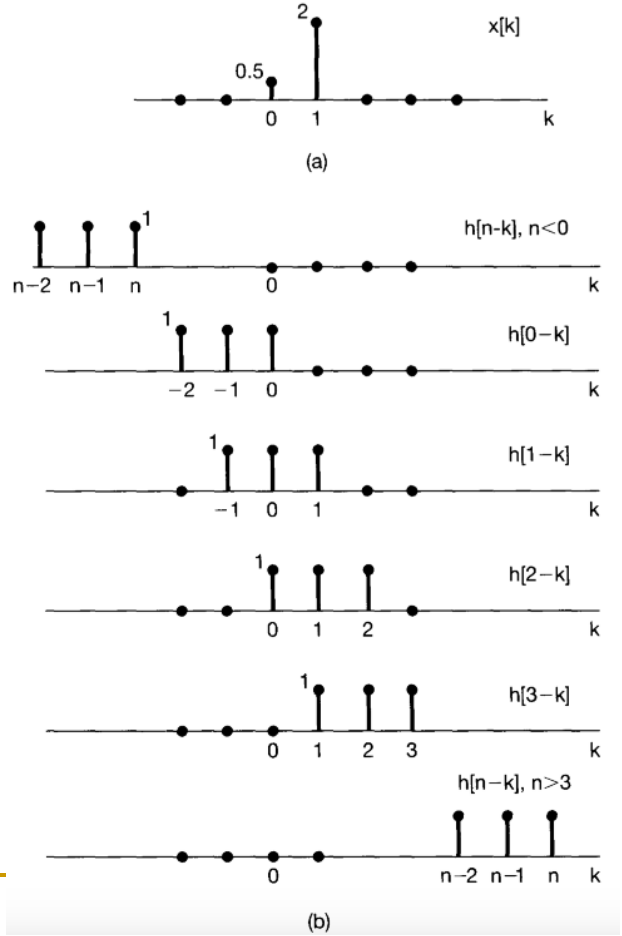
  $x[n] \rightarrow x[k]$, $h[n] \rightarrow h[k]$,

  and reverse $h[k] \rightarrow h[-k]$
- Step 2: Shift $h[-k] \rightarrow h[n-k]$,

  n is considered as a constant
- Step 3: multiply $x[k] \cdot h[n-k]$
- Step 4: Summation

  $$\sum_{k=-\infty}^{\infty} x[k] \cdot h[n-k]$$

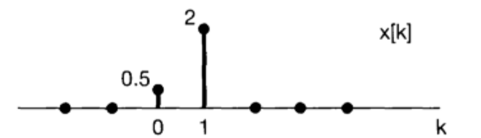
Change n, repeat step 1 to 4, calculate another $y[n]$



The Convolution-Sum

How to calculate the convolution sum?

- Method 2: View as functions of k (n: constant, k: variable):



(a)

$y[n] = 0$, for n<0

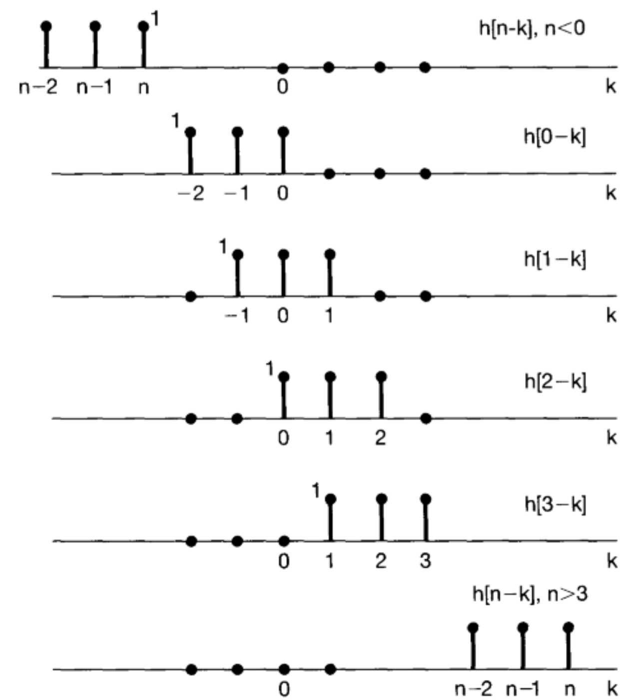
$$y[0] = \sum_{k=0}^{1} x[k]h[0-k]$$

$$y[1] = \sum_{k=0}^{1} x[k]h[1-k]$$

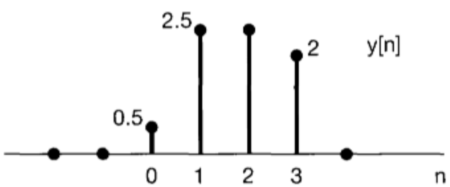
$$y[2] = \sum_{k=0}^{1} x[k]h[2-k]$$

$$y[3] = \sum_{k=0}^{1} x[k]h[3-k]$$

$y[n] = 0$, for n>3



(b)

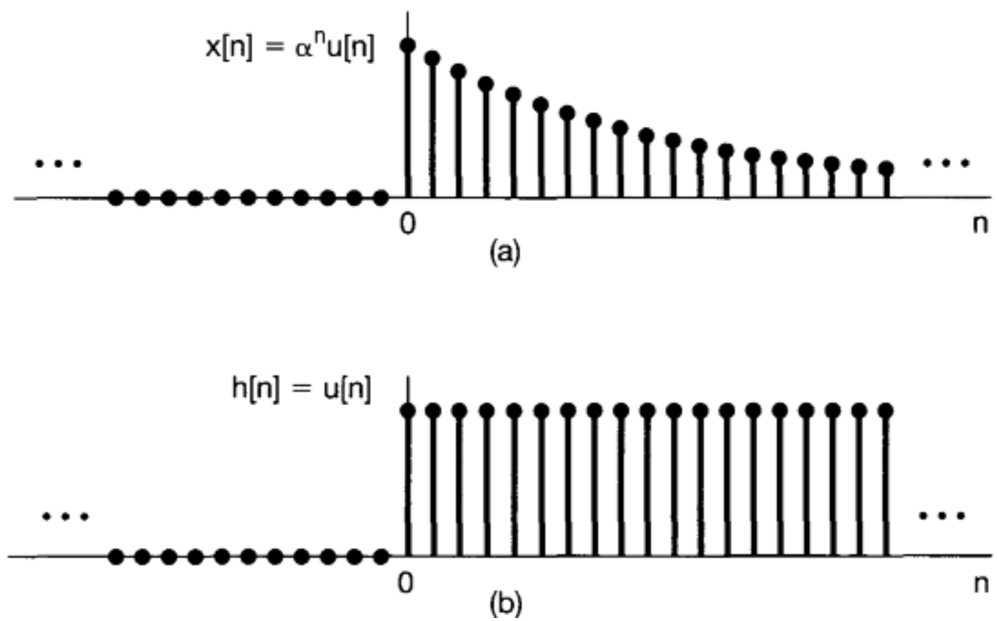


(c)

If the lengths of the two sequences are *M* and *N*, then the sequence generated by the convolution is of length *M*+*N*–1

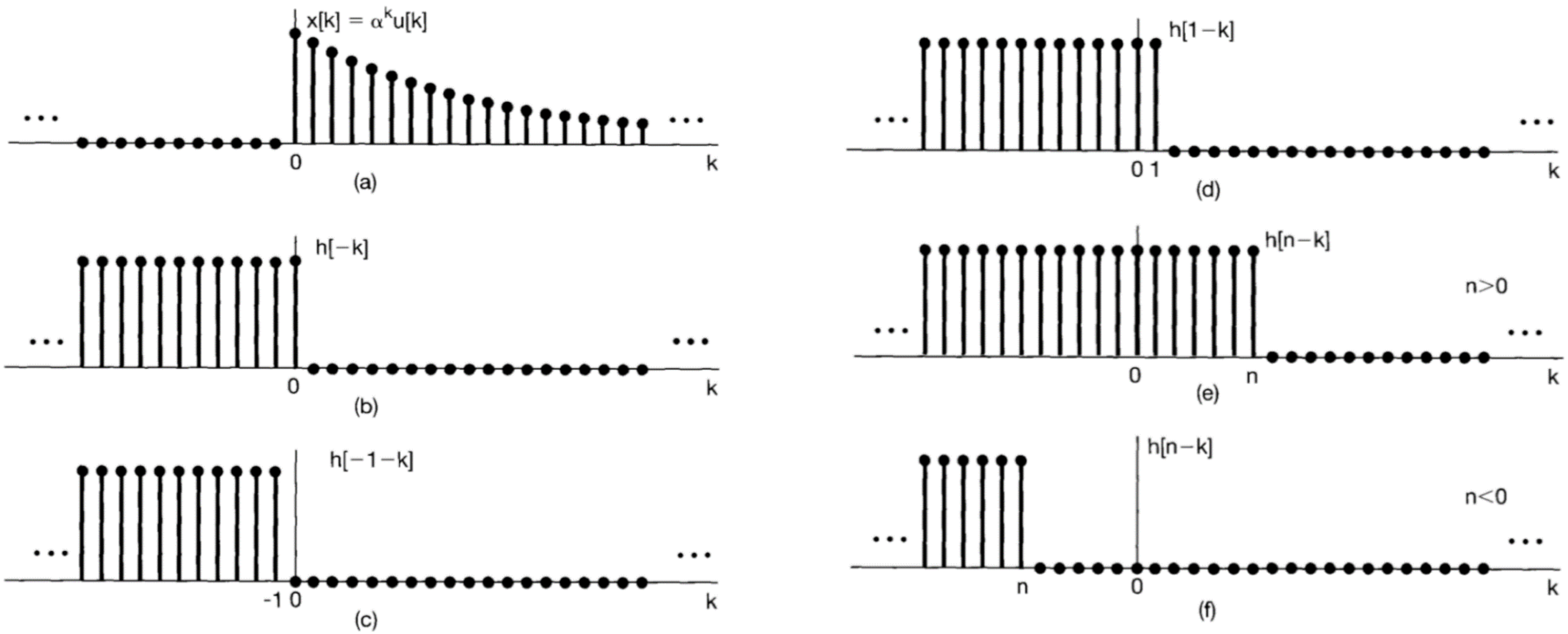
The Convolution-Sum

- Example 2: Consider an input $x[n]$ and a unit impulse response $h[n]$ given by $x[n] = \alpha^n u[n]$ and $h[n] = u[n]$ with $0 < \alpha < 1$.



The Convolution-Sum

- Example 2: Consider an input $x[n]$ and a unit impulse response $h[n]$ given by $x[n] = \alpha^n u[n]$ and $h[n] = u[n]$ with $0 < \alpha < 1$.



For $n < 0$,

we have $x[k]h[n-k] = 0, \forall k$

For $n \geq 0$,

we have $x[k]h[n-k] = \begin{cases} \alpha^k, 0 \leq k \leq n \\ 0, \quad \text{otherwise} \end{cases}$

The Convolution-Sum

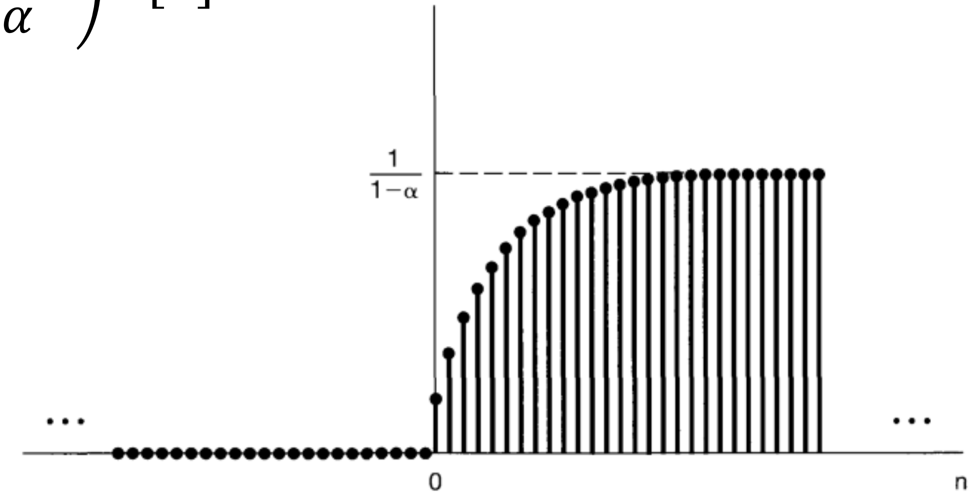
- Example 2: Consider an input $x[n]$ and a unit impulse response $h[n]$ given by $x[n] = \alpha^n u[n]$ and $h[n] = u[n]$ with $0 < \alpha < 1$.

For $n < 0$,
We have $y[n] = 0, n < 0$

For $n \geq 0$,
we have $y[n] = \sum_{k=0}^{n} \alpha^k = \frac{1-\alpha^{n+1}}{1-\alpha}, n \geq 0$

Hence, we have

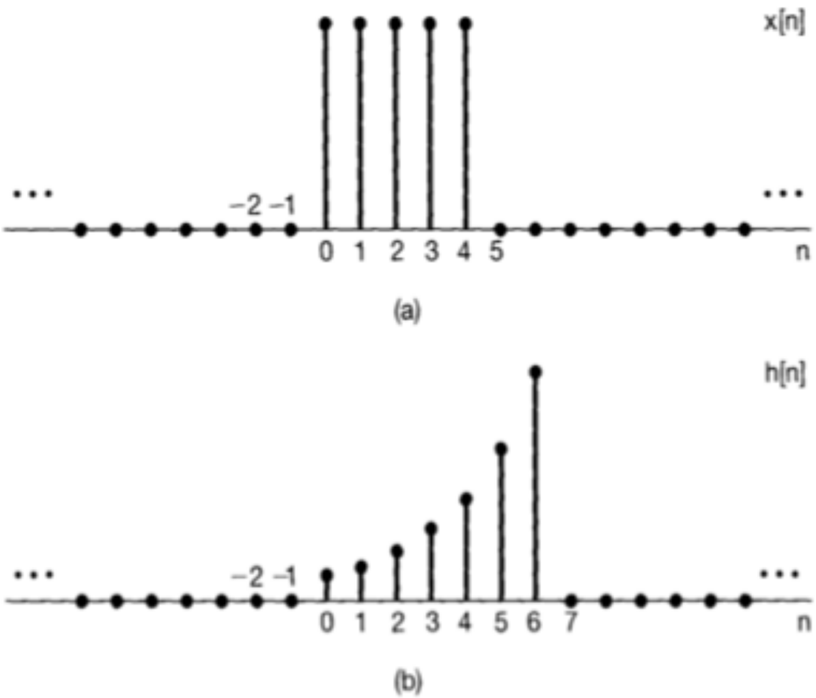
$$y[n] = \left(\frac{1-\alpha^{n+1}}{1-\alpha}\right) u[n]$$

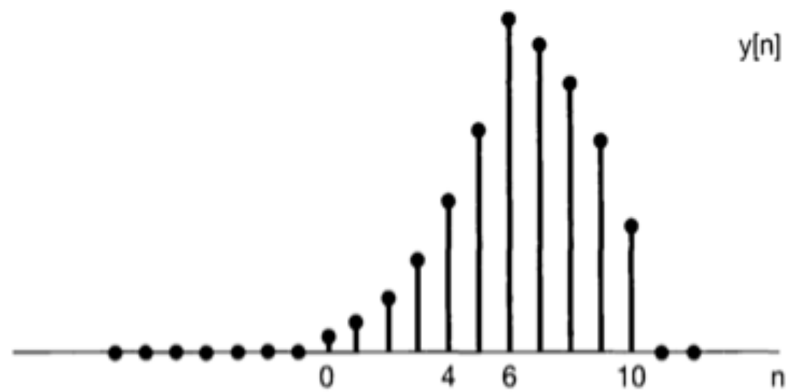


The Convolution-Sum

- Example 3: Consider two sequences

$$x[n] = \begin{cases} 1, & 0 \le n \le 4 \\ 0, & \text{otherwise} \end{cases} \quad \text{and} \quad h[n] = \begin{cases} \alpha^n, & 0 \le n \le 6 \\ 0, & \text{otherwise} \end{cases}$$





The Convolution-Sum

- Example 3: Consider two sequences

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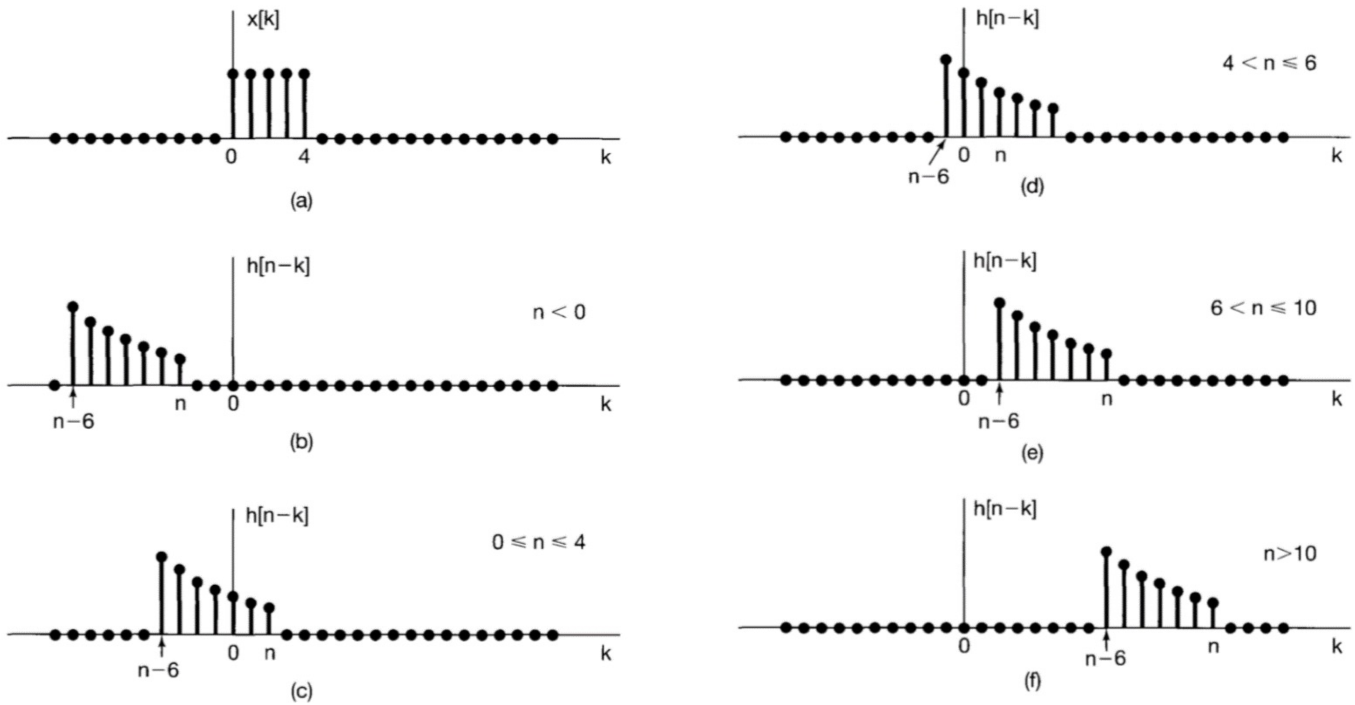

The Convolution-Sum

- Example 4: Consider an LTI system with input $x[n]$ and unit impulse response $h[n]$ specified as follows

$$x[n] = 2^n u[-n] \text{ and } h[n] = u[n]$$

Summary

- The Representation of Discrete-Time Signals in Terms of Impulses
- The Discrete-Time Unit Impulse Response
- The Convolution-Sum Representation of LTI Systems
- Reference in textbook:
  - 2.1