Ch.4 *The Continuous-Time Fourier Transform (CTFT)*

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Part IV *Systems Characterized by Differential Equations* → LTI

Differential Equations

$y(t) = x(t) * \boxed{h(t)}$

- Use differential equations to represent the input and output of continuous-time LTI systems:

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

Also consider the LTI system is characterized by

$$Y(j\omega) = H(j\omega)X(j\omega) \Longrightarrow \boxed{H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}}$$

$h(t) \updownarrow F$

1. Frequency response
2. $h(t) \to FT$
3. System function
4. Transfer function

- Apply Fourier transform to both sides of the equation:

$$\mathcal{F}\left\{\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k}\right\} = \mathcal{F}\left\{\sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}\right\}$$

- Use the linearity property:

$$\sum_{k=0}^{N} a_k \boxed{\mathcal{F}\left\{\frac{d^k y(t)}{dt^k}\right\}} = \sum_{k=0}^{M} b_k \boxed{\mathcal{F}\left\{\frac{d^k x(t)}{dt^k}\right\}}$$

$$x(t) \xleftrightarrow{F} X(j\omega) \qquad \underbrace{\frac{dx(t)}{dt}}_{x_1(t)} \xleftrightarrow{F} \underbrace{j\omega X(j\omega)}_{X_1(j\omega)}$$

$$\frac{dx_1(t)}{dt} = \frac{d^2x(t)}{dt^2} \xleftrightarrow{F} j\omega X_1(j\omega)$$

$$= (j\omega)^2 X(j\omega)$$

$$\frac{d^k x(t)}{dt^k} \xleftrightarrow{F} (j\omega)^k X(j\omega)$$

Differential Equations

- Use the differentiation property

$$\sum_{k=0}^{N} a_k\,(j\omega)^k Y(j\omega) = \sum_{k=0}^{N} b_k\,(j\omega)^k X(j\omega)$$

- Or equivalently:

$$Y(j\omega)\left[\sum_{k=0}^{N} a_k\,(j\omega)^k\right] = X(j\omega)\left[\sum_{k=0}^{N} b_k\,(j\omega)^k\right]$$

$H(j\omega) = \dfrac{\quad}{X(j\omega)}$

- Therefore

$h(t) = F^{-1}(H(j\omega))$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{N} b_k\,(j\omega)^k}{\sum_{k=0}^{N} a_k\,(j\omega)^k}$$

Differential Equations

- Example 1. Consider a stable LTI system characterized by the differential equation

$$\mathcal{F}\left(\frac{dy(t)}{dt} + ay(t)\right) = \mathcal{F}\left(x(t)\right),\ a > 0$$

please find the impulse response of this system.

$$j\omega Y(j\omega) + aY(j\omega) = X(j\omega)$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{1}{a+j\omega} \Rightarrow h(t) = \mathcal{F}^{-1}(H(j\omega))$$

$$= e^{-at}u(t)$$

Differential Equations

- Example 1. Consider a stable LTI system characterized by the differential equation

$$\frac{dy(t)}{dt} + ay(t) = x(t), a > 0$$

please find the impulse response of this system.

- Solution:

$$\mathcal{F}\left\{\frac{dy(t)}{dt} + ay(t)\right\} = \mathcal{F}\{x(t)\}$$

$$j\omega Y(j\omega) + aY(j\omega) = X(j\omega)$$

$$H(j\omega) = \frac{1}{j\omega + a} \Longrightarrow h(t) = e^{-at}u(t)$$

Differential Equations

- Example 2. Consider a stable LTI system characterized by the differential equation

$$F\left(\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t)\right) = F\left(\frac{dx(t)}{dt} + 2x(t)\right)$$

please find the impulse response of this system.

$$(j\omega)^2 Y(j\omega) + 4j\omega Y(j\omega) + 3Y(j\omega) = j\omega X(j\omega) + 2X(j\omega)$$

$$H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{2+j\omega}{(j\omega)^2+4j\omega+3} = \frac{2+j\omega}{(j\omega+1)(j\omega+3)}$$

$$= \frac{a}{j\omega+1} + \frac{b}{j\omega+3}$$

$a = b = \frac{1}{2}$

Differential Equations

- Example 2. Consider a stable LTI system characterized by the differential equation

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + 2x(t)$$

please find the impulse response of this system.

- Solution:

$$H(j\omega) = \frac{(j\omega)+2}{(j\omega)^2+4(j\omega)+3} = \frac{\frac{1}{2}}{j\omega+1} + \frac{\frac{1}{2}}{j\omega+3}$$

$$h(t) = \frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{-3t}u(t)$$

Summary

- Systems Characterized by Differential Equations

- Reference in textbook:
  - 4.7