Signals and Systems

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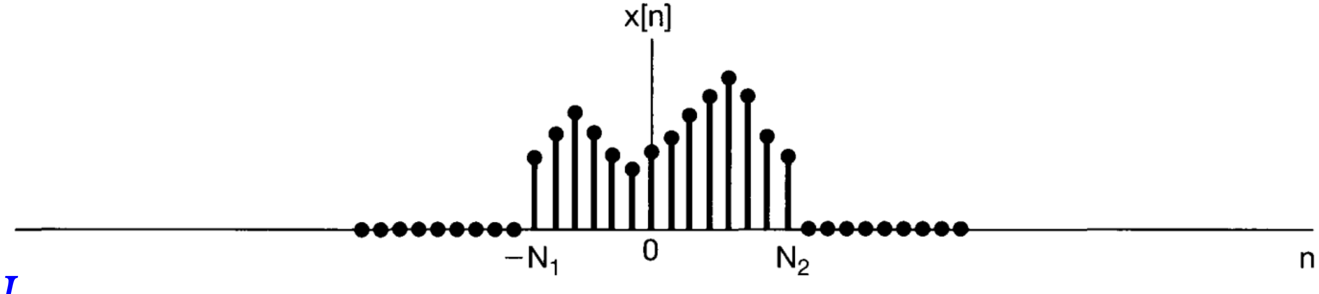
Chapter 5: The Discrete-Time Fourier Transform

- Representation of aperiodic signals - Discrete Fourier transform
- Fourier transform for periodic signals
- Properties of discrete-time Fourier transform
- The convolution property
- The multiplication property
- Duality
- Systems characterized by difference equations

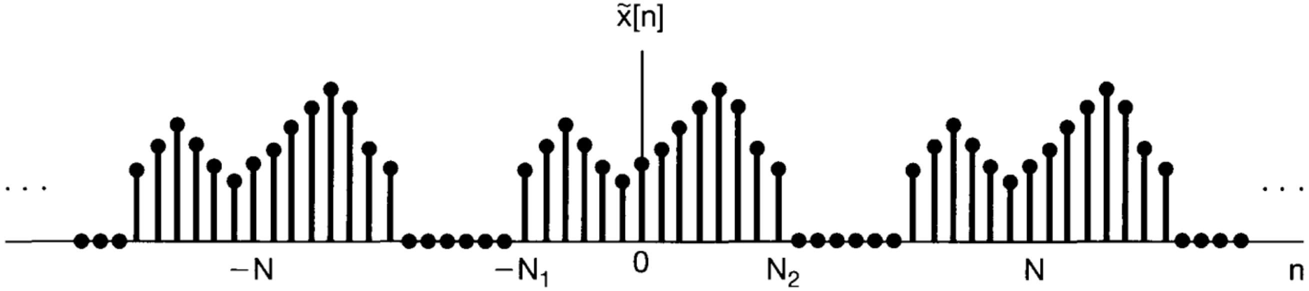
Discrete Fourier Transform

Development of FT

- Consider a general sequence of finite duration: $x[n] = 0$ if $n < N_1$ or $n > N_2$



- Periodic extension of $x[n]$ with N



- FS representation of $\tilde{x}[n]$

$$\tilde{x}[n] = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n} \qquad a_k = \frac{1}{N}\sum_{n=\langle N\rangle} \tilde{x}[n]e^{-jk(2\pi/N)n}$$

Discrete Fourier Transform

Development of FT

- FS coefficients of $\tilde{x}[n]$

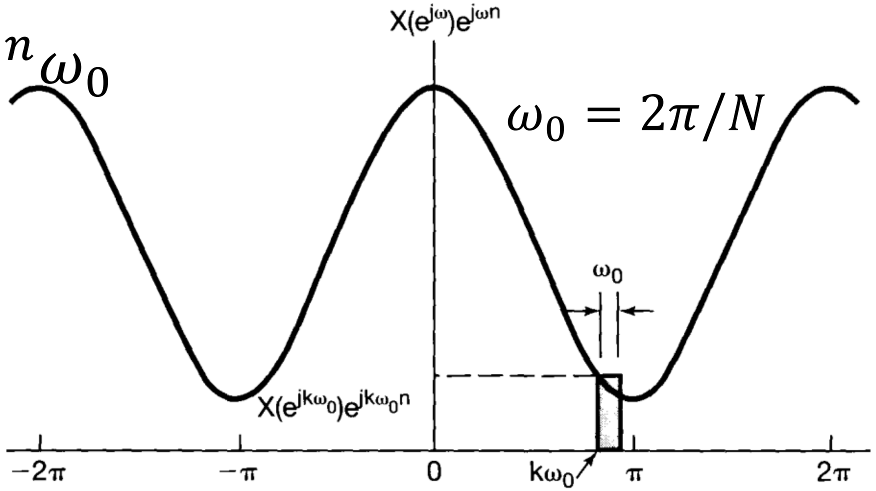
$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} \tilde{x}[n]e^{-jk(2\pi/N)n}$$

Define $X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$

$$= \frac{1}{N}\sum_{n=-N_1}^{N_2} x[n]e^{-jk(2\pi/N)n} = \frac{1}{N}\boxed{\sum_{n=-\infty}^{+\infty} x[n]e^{-jk(2\pi/N)n}} = \frac{1}{N}X(e^{jk\omega_0})$$

- FS of $\tilde{x}[n]$

$$\tilde{x}[n] = \sum_{k=\langle N\rangle} \frac{1}{N}X(e^{jk\omega_0})e^{jk\omega_0 n} = \frac{1}{2\pi}\sum_{k=\langle N\rangle} X(e^{jk\omega_0})e^{jk\omega_0 n}\omega_0$$



$\omega_0 = 2\pi/N$

- $N \to \infty$, $\tilde{x}[n] \to x[n]$

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})\, e^{j\omega n} d\omega$$

Discrete Fourier Transform

FT pairs

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$$

Fourier transform (FT)

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})\, e^{j\omega n} d\omega$$

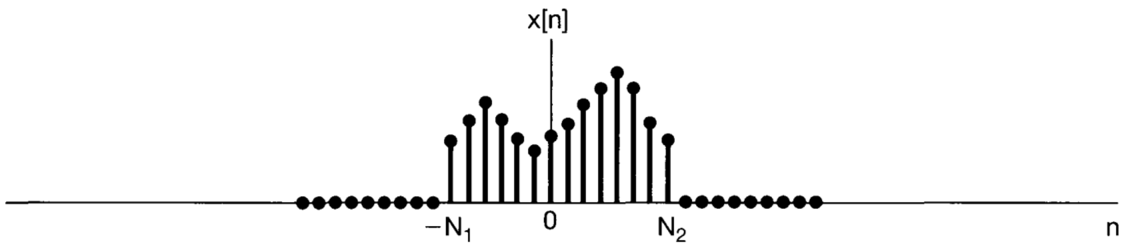
Inverse Fourier transform

- $x[n]$ is a linear combination (specifically, an integral) of complex exponentials at different frequencies
- $X(e^{j\omega})(d\omega/2\pi)$ is the weight for different frequencies
- $X(e^{j\omega})$ is called the spectrum

Discrete Fourier Transform

FT vs. FS

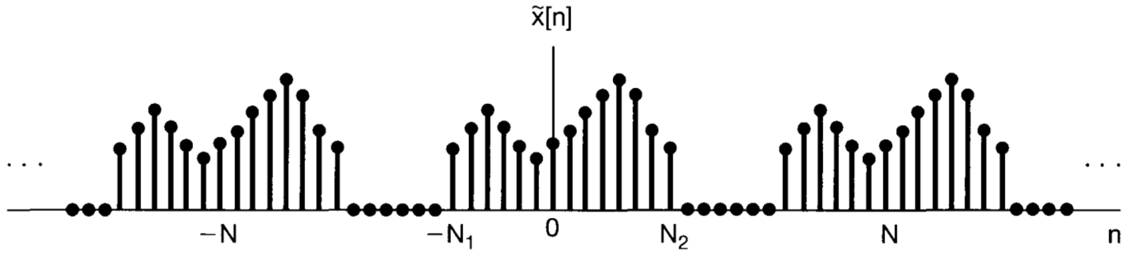
Fourier transform (FT)



$$x[n] = \frac{1}{2\pi} \int_{2\pi} X\left(e^{j\omega}\right) e^{j\omega n} d\omega$$

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$$

Fourier series (FS)



$$\tilde{x}[n] = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$$

$$a_k = \frac{1}{N} \sum_{n=\langle N\rangle} \tilde{x}[n] e^{-jk(2\pi/N)n}$$

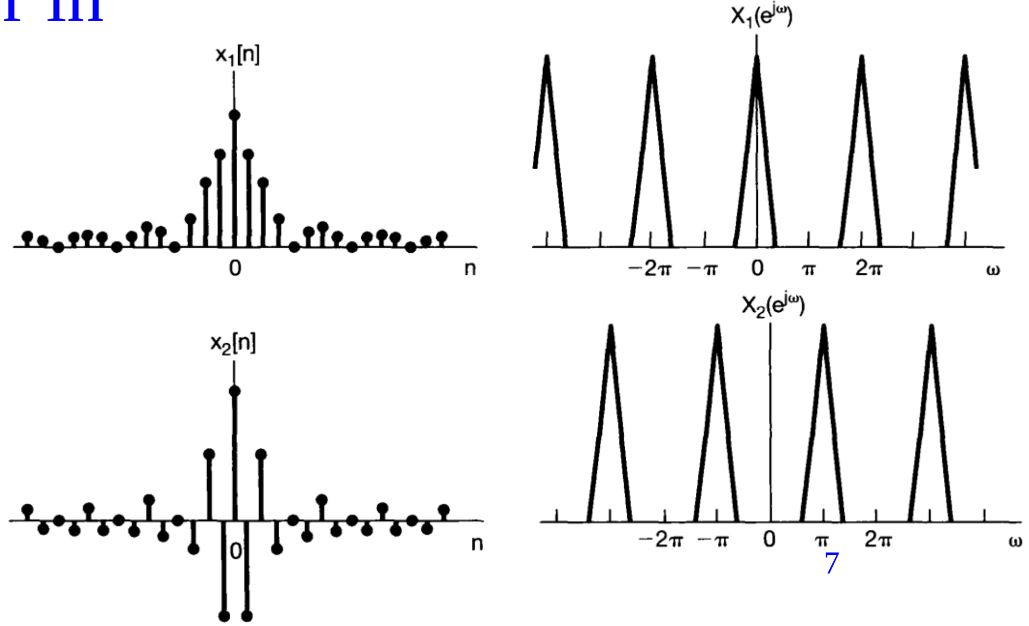
$$a_k = \frac{1}{N} X(e^{j\omega}) \text{ with } \omega = k(2\pi/N)$$

Discrete Fourier Transform

Discrete FT vs. continuous FT

Discrete FT	Continuous FT
$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})\, e^{j\omega n} d\omega$	$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega) e^{j\omega t} d\omega$
$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$	$X(j\omega) = \int_{-\infty}^{+\infty} x(t) e^{-j\omega t} dt$

- Discrete-time complex exponentials that differ in frequency by a multiple of 2π are identical
  - $X(e^{j\omega})$ is periodic
  - Finite interval of integration in the synthesis equation for $x[n]$
  - $\omega = 0, 2\pi, 4\pi, \cdots \Rightarrow$ low-frequency
  - $\omega = \pi, 3\pi, 5\pi, \cdots \Rightarrow$ high-frequency



Discrete Fourier Transform

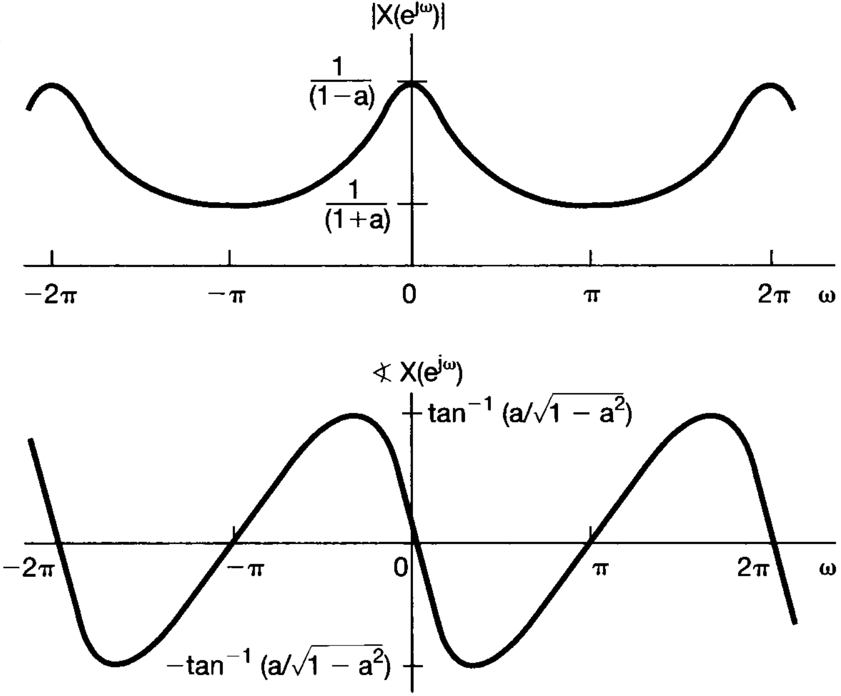
Examples

$$x[n] = a^n u[n], |a| < 1 \qquad X(e^{j\omega}) = ?$$

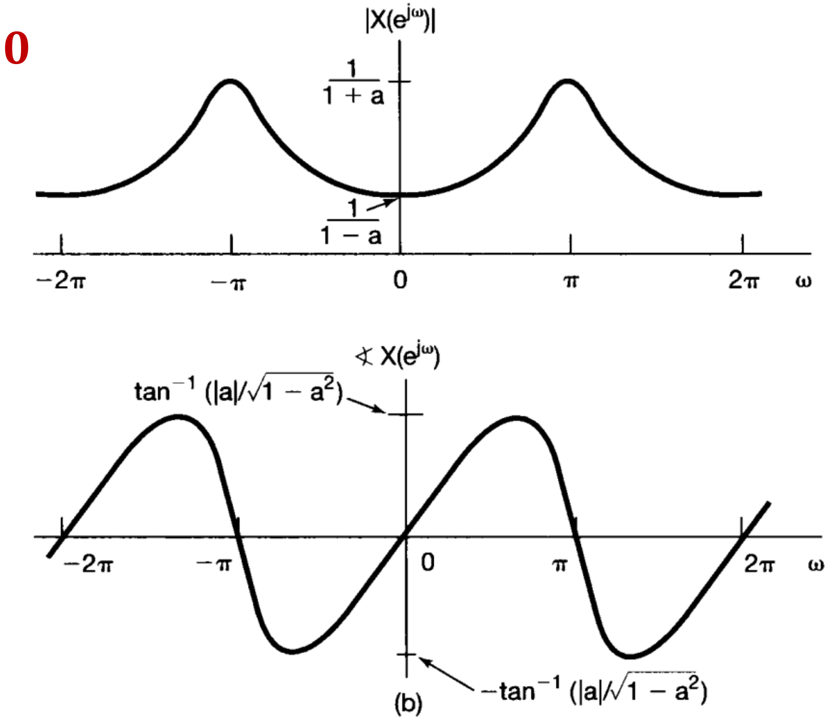
Solution

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} a^n u[n] e^{-j\omega n} = \sum_{n=0}^{\infty} (ae^{-j\omega})^n = \frac{1}{1 - ae^{-j\omega}}$$

***a* > 0**



***a* < 0**



Discrete Fourier Transform

Examples

$$x[n] = a^{|n|}, |a| < 1 \qquad X\left(e^{j\omega}\right) = ?$$

Solution

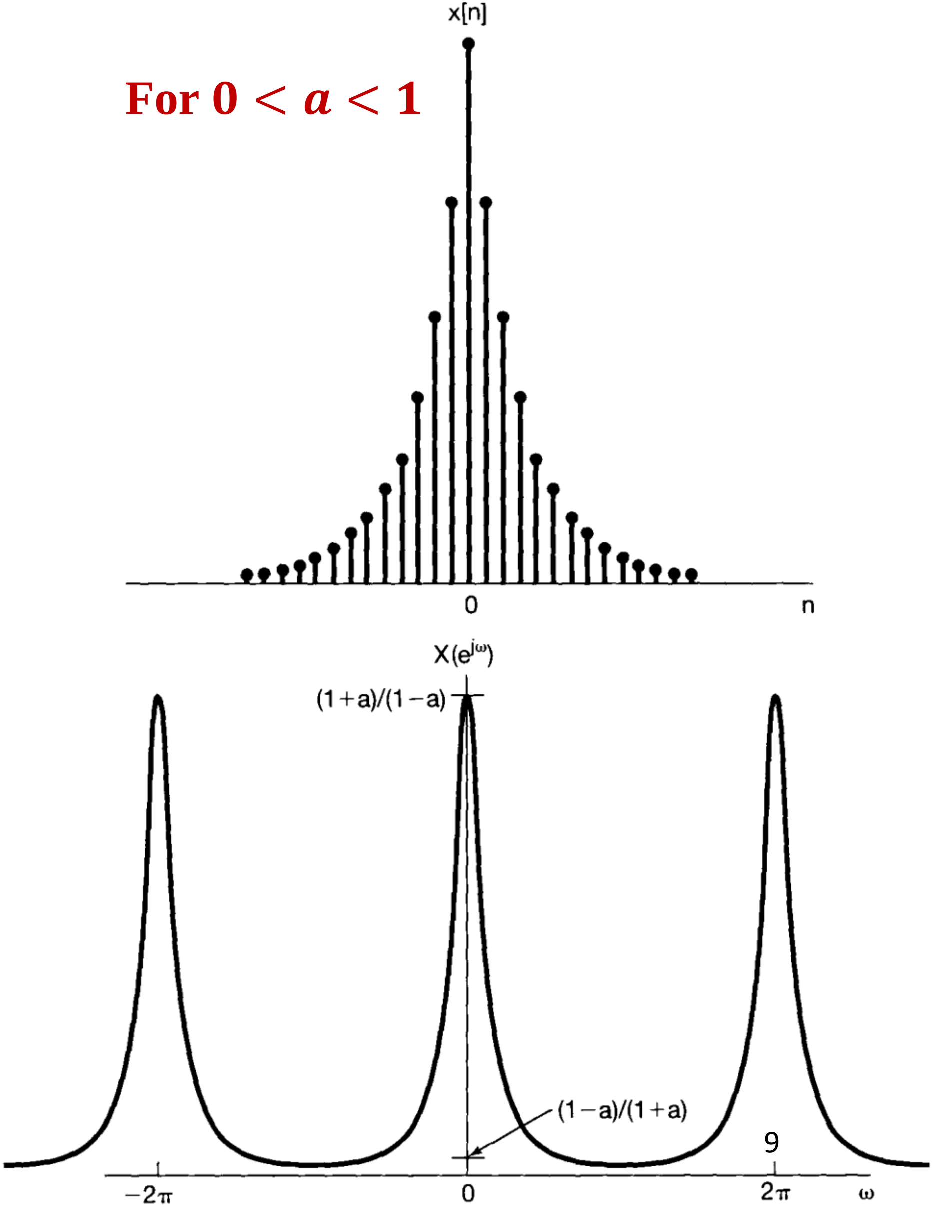
$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{+\infty} a^{|n|} e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} a^{n} e^{-j\omega n} + \sum_{n=-\infty}^{-1} a^{-n} e^{-j\omega n}$$

$$= \sum_{n=0}^{\infty} \left(a e^{-j\omega}\right)^{n} + \sum_{m=1}^{\infty} \left(a e^{j\omega}\right)^{m}$$

$$= \frac{1}{1 - a e^{-j\omega}} + \frac{a e^{j\omega}}{1 - a e^{j\omega}} = \frac{1 - a^2}{1 - 2a\cos\omega + a^2}$$

For $0 < a < 1$



Discrete Fourier Transform

Examples

$$x[n] = \begin{cases} 1, & |n| \le N_1 \\ 0, & |n| > N_1 \end{cases} \qquad X\left(e^{j\omega}\right) = ?$$

Solution

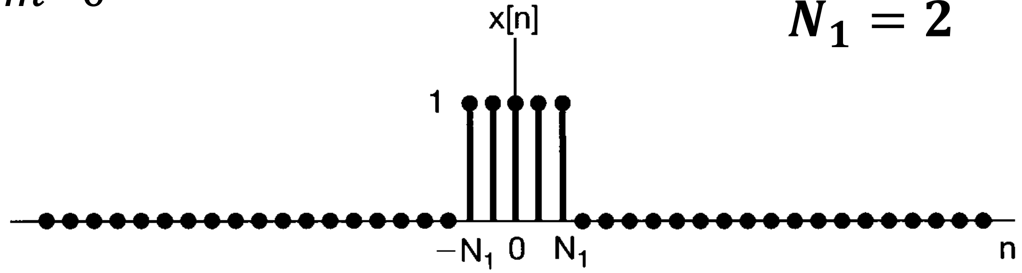
$$X\left(e^{j\omega}\right) = \sum_{n=-N_1}^{N_1} e^{-j\omega n} = \sum_{m=0}^{2N_1} e^{-j\omega(m-N_1)} = e^{j\omega N_1} \sum_{m=0}^{2N_1} e^{-j\omega m}$$

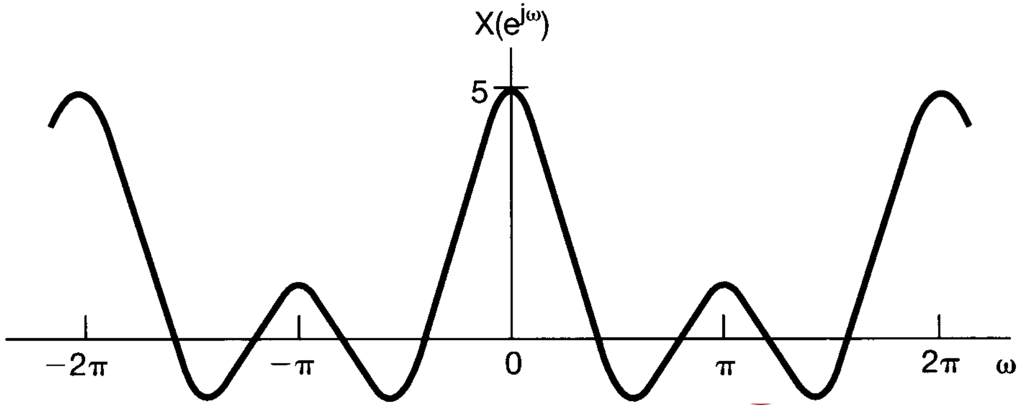
$$= e^{j\omega N_1}\left(\frac{1-e^{-j\omega(2N_1+1)}}{1-e^{-j\omega}}\right)$$

$$= \frac{e^{-j\omega/2}\left(e^{j\omega(N_1+1/2)} - e^{-j\omega(N_1+1/2)}\right)}{e^{-j\omega/2}\left(e^{j\omega/2} - e^{-j\omega/2}\right)}$$

$$= \frac{\sin[\omega(N_1 + 1/2)]}{\sin(\omega/2)}$$

$\boldsymbol{N_1 = 2}$





Discrete Fourier Transform

Convergence of FT

- ❑ For the analysis equation $$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$$
  - Finite energy condition $$\sum_{n=-\infty}^{+\infty} |x[n]|^2 < \infty$$
  - Absolutely summable $$\sum_{n=-\infty}^{+\infty} |x[n]| < \infty$$
- ❑ For the synthesis equation $$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$
  - No convergence issues (finite interval of integration)

Discrete Fourier Transform

Examples

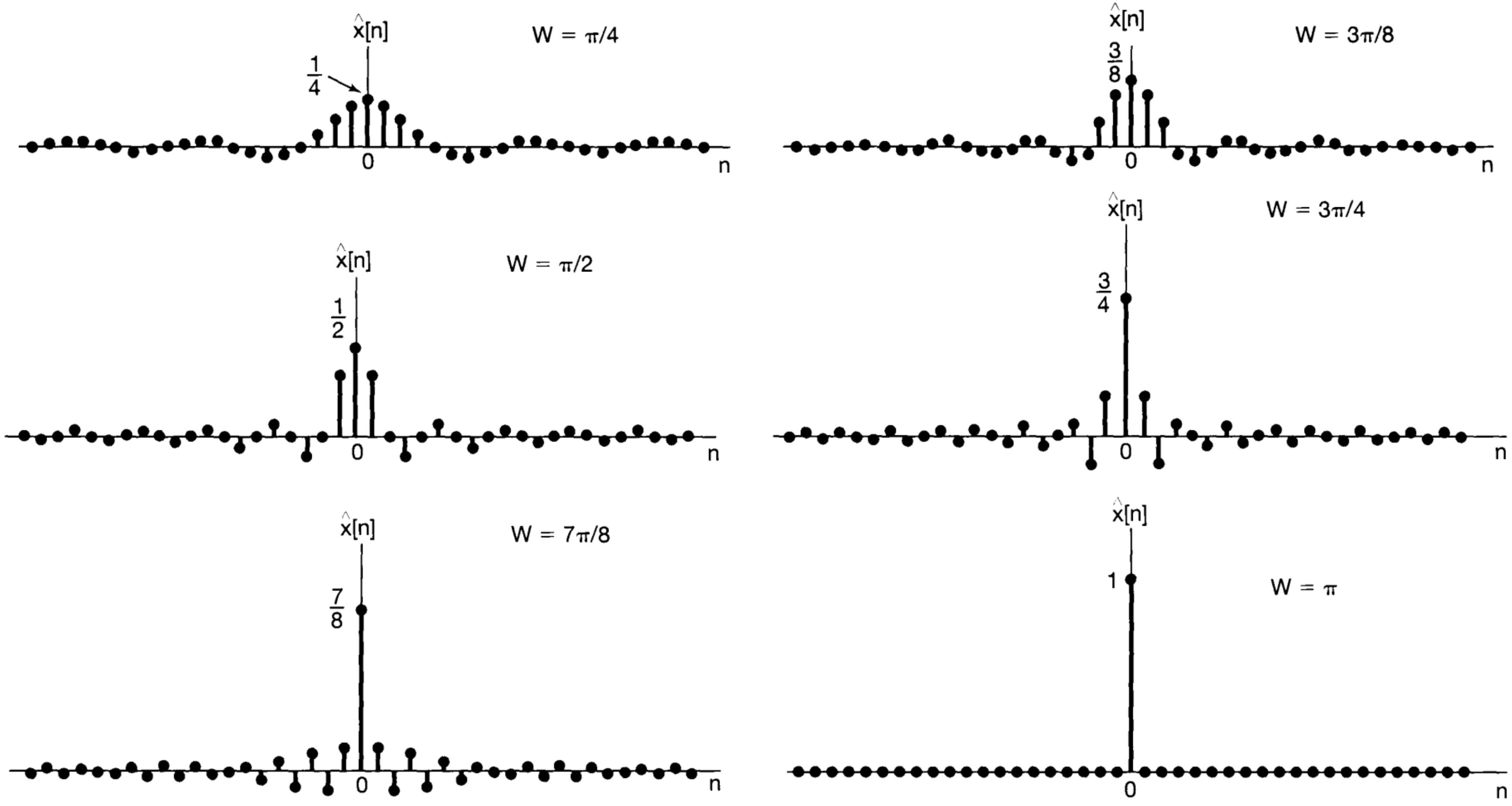
$$x[n] = \delta[n]$$

$$X(e^{j\omega}) = 1$$

$$\hat{x}[n] = \frac{1}{2\pi}\int_{-W}^{W} e^{j\omega n} d\omega$$

$$= \frac{\sin Wn}{\pi n}$$

$$\lim_{W\to\pi} \hat{x}[n] = \delta[n]$$



Chapter 5: The Discrete-Time Fourier Transform

- ❑ Representation of aperiodic signals - Discrete Fourier transform
- ❑ Fourier transform for periodic signals
- ❑ Properties of discrete-time Fourier transform
- ❑ The convolution property
- ❑ The multiplication property
- ❑ Duality
- ❑ Systems characterized by difference equations

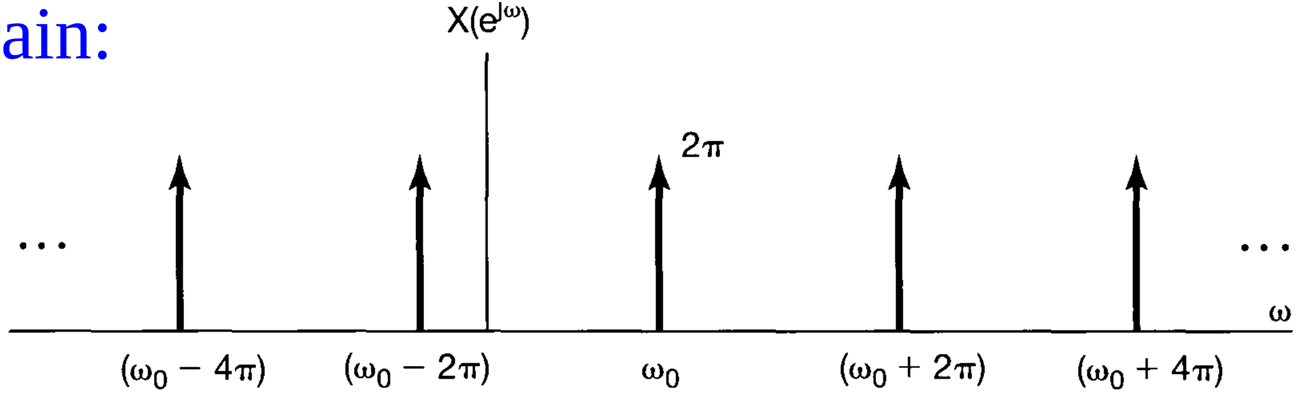
Fourier transform for periodic signals

- ☐ Consider the sinusoidal signal

$$x[n] = e^{j\omega_0 n}$$

- ☐ The FT should be a periodic pulse train:

$$X(e^{j\omega}) = \sum_{l=-\infty}^{+\infty} 2\pi\delta(\omega - \omega_0 - 2\pi l)$$



- ☐ Check validity: evaluate the inverse transform

$$\frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega \quad = \frac{1}{2\pi}\int_{2\pi} \sum_{l=-\infty}^{+\infty} 2\pi\delta(\omega - \omega_0 - 2\pi l)\, e^{j\omega n}d\omega$$

$$= e^{j(\omega_0 + 2\pi r)n}$$ Fixed in one period $l = r$

$$= e^{j\omega_0 n}$$

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Fourier transform for periodic signals

- Consider a periodic sequence

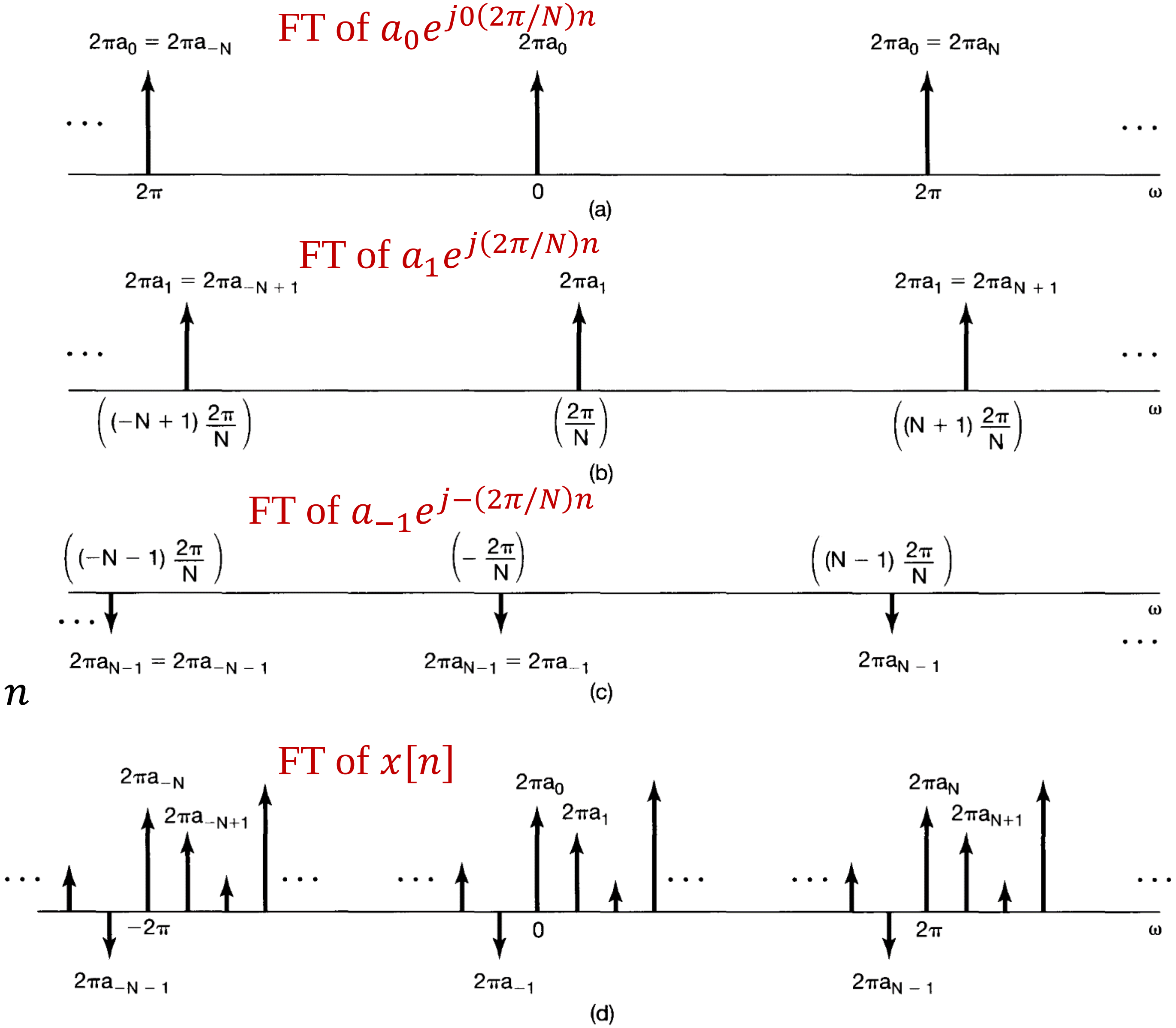
$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$$

- FT

$$X(e^{j\omega}) = \sum_{k=-\infty}^{+\infty} 2\pi a_k \delta(\omega - 2\pi k/N)$$

- Verify

$$x[n] = a_0 + a_1 e^{j(2\pi/N)n} + a_2 e^{j2(2\pi/N)n} + \cdots + a_{N-1} e^{j(N-1)(2\pi/N)n}$$


FT of $a_0 e^{j0(2\pi/N)n}$
$2\pi a_0 = 2\pi a_{-N}$
$2\pi a_0$
$2\pi a_0 = 2\pi a_N$
2π
0
2π
ω
(a)
FT of $a_1 e^{j(2\pi/N)n}$
$2\pi a_1 = 2\pi a_{-N+1}$
$2\pi a_1$
$2\pi a_1 = 2\pi a_{N+1}$
$\left((-N+1)\frac{2\pi}{N}\right)$
$\left(\frac{2\pi}{N}\right)$
$\left((N+1)\frac{2\pi}{N}\right)$
ω
(b)
FT of $a_{-1} e^{j-(2\pi/N)n}$
$\left((-N-1)\frac{2\pi}{N}\right)$
$\left(-\frac{2\pi}{N}\right)$
$\left((N-1)\frac{2\pi}{N}\right)$
ω
$2\pi a_{N-1} = 2\pi a_{-N-1}$
$2\pi a_{N-1} = 2\pi a_{-1}$
$2\pi a_{N-1}$
(c)
FT of $x[n]$
$2\pi a_{-N}$
$2\pi a_{-N+1}$
$2\pi a_0$
$2\pi a_1$
$2\pi a_N$
$2\pi a_{N+1}$
−2π
0
2π
ω
$2\pi a_{-N-1}$
$2\pi a_{-1}$
$2\pi a_{N-1}$
(d)

Fourier transform for periodic signals

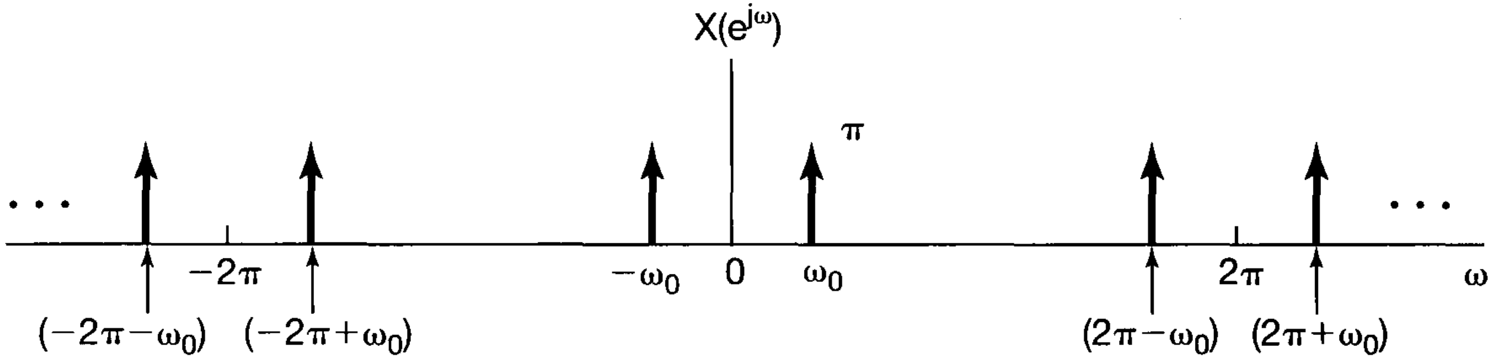
Examples

$$x[n] = \cos\omega_0 n = \frac{1}{2}e^{j\omega_0 n} + \frac{1}{2}e^{-j\omega_0 n}, \omega_0 = \frac{2\pi}{5} \qquad X\left(e^{j\omega}\right) = ?$$

Solution

$$X\left(e^{j\omega}\right) = \sum_{l=-\infty}^{+\infty} \pi\delta\left(\omega - \frac{2\pi}{5} - 2\pi l\right) + \sum_{l=-\infty}^{+\infty} \pi\delta\left(\omega + \frac{2\pi}{5} - 2\pi l\right)$$

$$= \pi\delta\left(\omega - \frac{2\pi}{5}\right) + \pi\delta\left(\omega + \frac{2\pi}{5}\right), -\pi < \omega < \pi$$

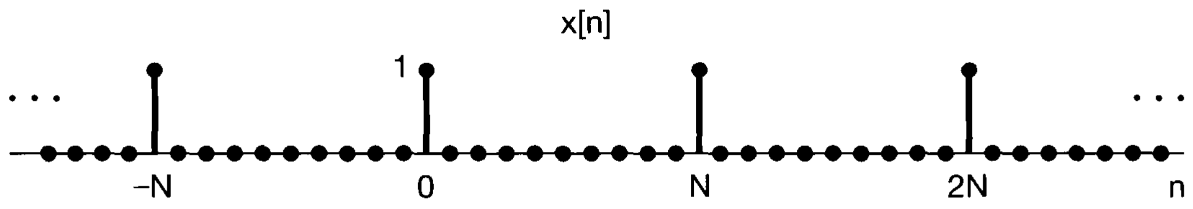


Fourier transform for periodic signals

Examples

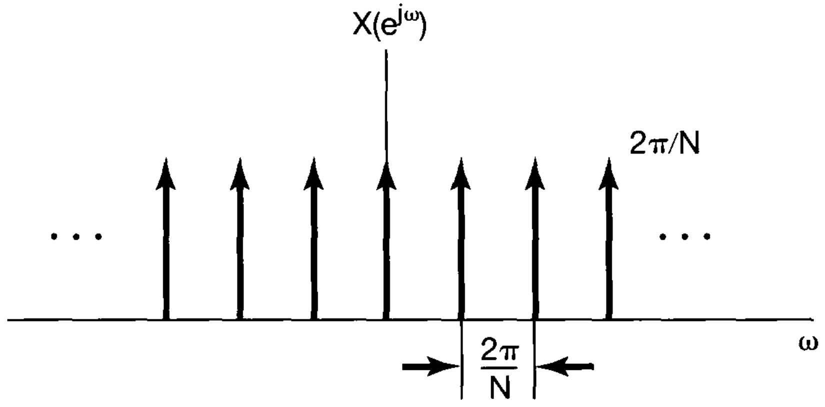
$$x[n] = \sum_{k=-\infty}^{+\infty} \delta[n - kN] \qquad X(e^{j\omega}) = ?$$

Solution



$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk(2\pi/N)n} = \frac{1}{N}$$

$$X(e^{j\omega}) = \frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$$



Chapter 5: The Discrete-Time Fourier Transform

- **Representation of aperiodic signals - Discrete Fourier transform**
- **Fourier transform for periodic signals**
- **Properties of discrete-time Fourier transform**
- **The convolution property**
- **The multiplication property**
- **Duality**
- **Systems characterized by difference equations**

Properties of discrete-time Fourier Transform

Short notation for FT pairs

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X\left(e^{j\omega}\right) e^{j\omega n} d\omega \qquad X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{+\infty} x[n] e^{-j\omega n}$$

$$x[n] \xleftrightarrow{\mathcal{F}} X\left(e^{j\omega}\right)$$

$$X\left(e^{j\omega}\right) = \mathcal{F}\{x[n]\}$$

$$x[n] = \mathcal{F}^{-1}\left\{X\left(e^{j\omega}\right)\right\}$$

Properties of discrete-time Fourier Transform

Periodicity In contrast to continuous FT

$$X\left(e^{j(\omega+2\pi)}\right) = X\left(e^{j\omega}\right)$$

Linearity

$$x_1[n] \xleftrightarrow{\mathcal{F}} X_1\left(e^{j\omega}\right)$$

$$x_2[n] \xleftrightarrow{\mathcal{F}} X_2\left(e^{j\omega}\right)$$

$$\Longrightarrow a x_1[n] + b x_2[n] \xleftrightarrow{\mathcal{F}} a X_1\left(e^{j\omega}\right) + b X_2\left(e^{j\omega}\right)$$

Properties of discrete-time Fourier Transform

Time shifting and frequency shifting

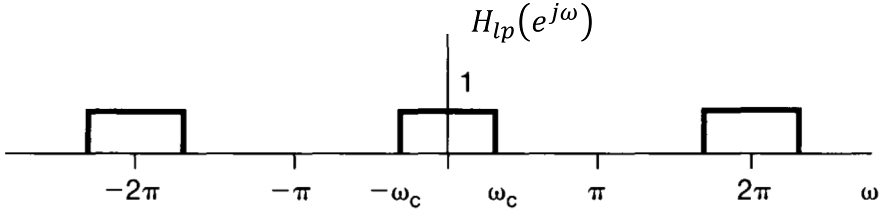
$$x[n] \xleftrightarrow{\mathcal{F}} X\left(e^{j\omega}\right) \quad \Longrightarrow \quad \begin{cases} x[n-n_0] \xleftrightarrow{\mathcal{F}} e^{-j\omega n_0} X\left(e^{j\omega}\right) \\ e^{j\omega_0 n} x[n] \xleftrightarrow{\mathcal{F}} X\left(e^{j(\omega-\omega_0)}\right) \end{cases}$$

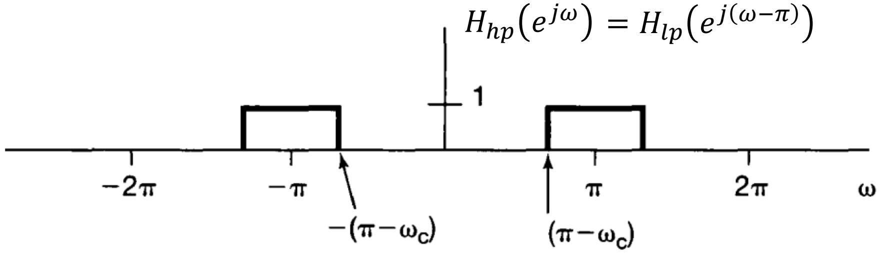
Examples

Low-pass filter $\Longrightarrow$ High-pass filter

$$\begin{tikzcd} H_{lp}\left(e^{j\omega}\right) \arrow[d, leftrightarrow, "\mathcal{F}"] \arrow[r, phantom, "\text{shift by } \pi"] & H_{hp}\left(e^{j\omega}\right) = H_{lp}\left(e^{j(\omega-\pi)}\right) \arrow[d, leftrightarrow, "\mathcal{F}"] \\ h_{lp}[n] \arrow[r, Rightarrow] & h_{hp} = e^{j\pi n} h_{lp}[n] \end{tikzcd}$$

$$= (-1)^n h_{lp}[n]$$





Properties of discrete-time Fourier Transform

Conjugation and Conjugate Symmetry

- ☐ Conjugation property

$$x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega}) \quad \Longrightarrow \quad x^*[n] \overset{\mathcal{F}}{\longleftrightarrow} X^*(e^{-j\omega})$$

- ☐ Conjugation Symmetry

$$X(e^{j\omega}) = X^*(e^{-j\omega}) \quad [x[n] \text{ real}]$$

Magnitude $|X(e^{j\omega})|$ is even

Phase function $\sphericalangle X(e^{j\omega})$ is odd

Properties of discrete-time Fourier Transform

Time reversal

- ☐ $x[n] \xleftrightarrow{\mathcal{F}} X\left(e^{j\omega}\right) \quad \Longrightarrow \quad x[-n] \xleftrightarrow{\mathcal{F}} X\left(e^{-j\omega}\right)$

- ☐ $x[n]$ even, $X\left(e^{j\omega}\right)$ even; $x[n]$ odd, $X\left(e^{j\omega}\right)$ odd

  Recall: $x[n]$ real: $X\left(e^{j\omega}\right) = X^*\left(e^{-j\omega}\right)$

  $\Downarrow$

- ☐ $x[n]$ real and even $\Longrightarrow$ $X\left(e^{j\omega}\right)$ real and even

  $x[n]$ real and odd $\Longrightarrow$ $X\left(e^{j\omega}\right)$ purely imaginary and odd

$$\left.\begin{array}{l} \text{real and even} \\ \text{real and odd} \end{array}\right\} \Longrightarrow \left\{\begin{array}{l} \mathcal{E}v\{x[n]\} \xleftrightarrow{\mathcal{F}} \mathcal{R}e\left\{X\left(e^{j\omega}\right)\right\} \\ \mathcal{O}d\{x[n]\} \xleftrightarrow{\mathcal{F}} j\mathcal{I}m\left\{X\left(e^{j\omega}\right)\right\} \end{array}\right.$$

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Properties of discrete-time Fourier Transform

Differencing and accumulation

- ☐ If $x[n] \xleftrightarrow{\mathcal{F}} X(e^{j\omega})$
- ☐ Then

$$x[n] - x[n-1] \xleftrightarrow{\mathcal{F}} (1 - e^{-j\omega})X(e^{j\omega})$$

$$\sum_{m=-\infty}^{n} x[m] \xleftrightarrow{\mathcal{F}} \frac{1}{1 - e^{-j\omega}} X(e^{j\omega}) + \underbrace{\pi X(e^{j0}) \sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k)}_{\text{DC components}}$$

Properties of discrete-time Fourier Transform

Differencing and accumulation

❑ Examples Determine FT of unit sept $x[n] = u[n]$

Solution

$$g[n] = \delta[n] \xleftrightarrow{\mathcal{F}} G(e^{j\omega}) = 1 \qquad x[n] = \sum_{m=-\infty}^{n} g[m]$$

$$X(e^{j\omega}) = \frac{1}{1-e^{-j\omega}} G(e^{j\omega}) + \pi G(e^{j0}) \sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k)$$

$$= \frac{1}{1-e^{-j\omega}} + \pi \sum_{k=-\infty}^{+\infty} \delta(\omega - 2\pi k)$$

Properties of discrete-time Fourier Transform

Time expansion

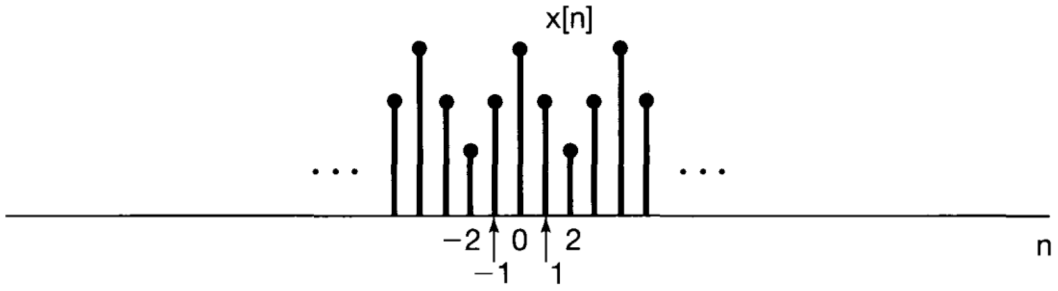
- Recall the continuous time property

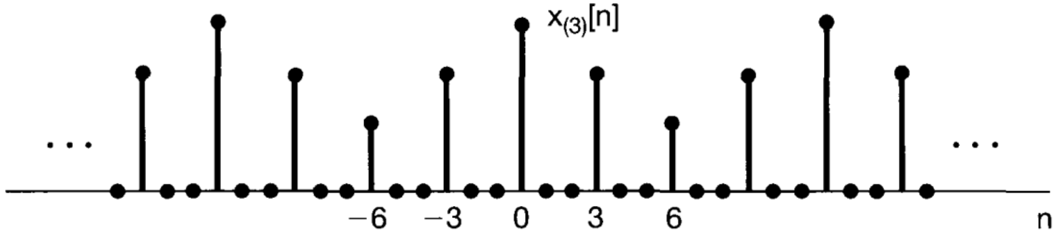
$$x(at) \xleftrightarrow{\mathcal{F}} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$

- Try to define $x[an]$
  - a should be an integer and $a > 1$
  - not merely speed up, but also resample $x[n]$

- Define instead

$$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n \text{ is a multiple of } k \\ 0, & \text{if } n \text{ is not a multiple of } k \end{cases}$$





Properties of discrete-time Fourier Transform

Time expansion

$$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n \text{ is a multiple of } k \\ 0, & \text{if } n \text{ is not a multiple of } k \end{cases}$$

$$x[n] \xleftrightarrow{\mathcal{F}} X\left(e^{j\omega}\right)$$

$$x_{(k)}[n] \xleftrightarrow{\mathcal{F}} X_{(k)}\left(e^{j\omega}\right) = X\left(e^{jk\omega}\right)$$

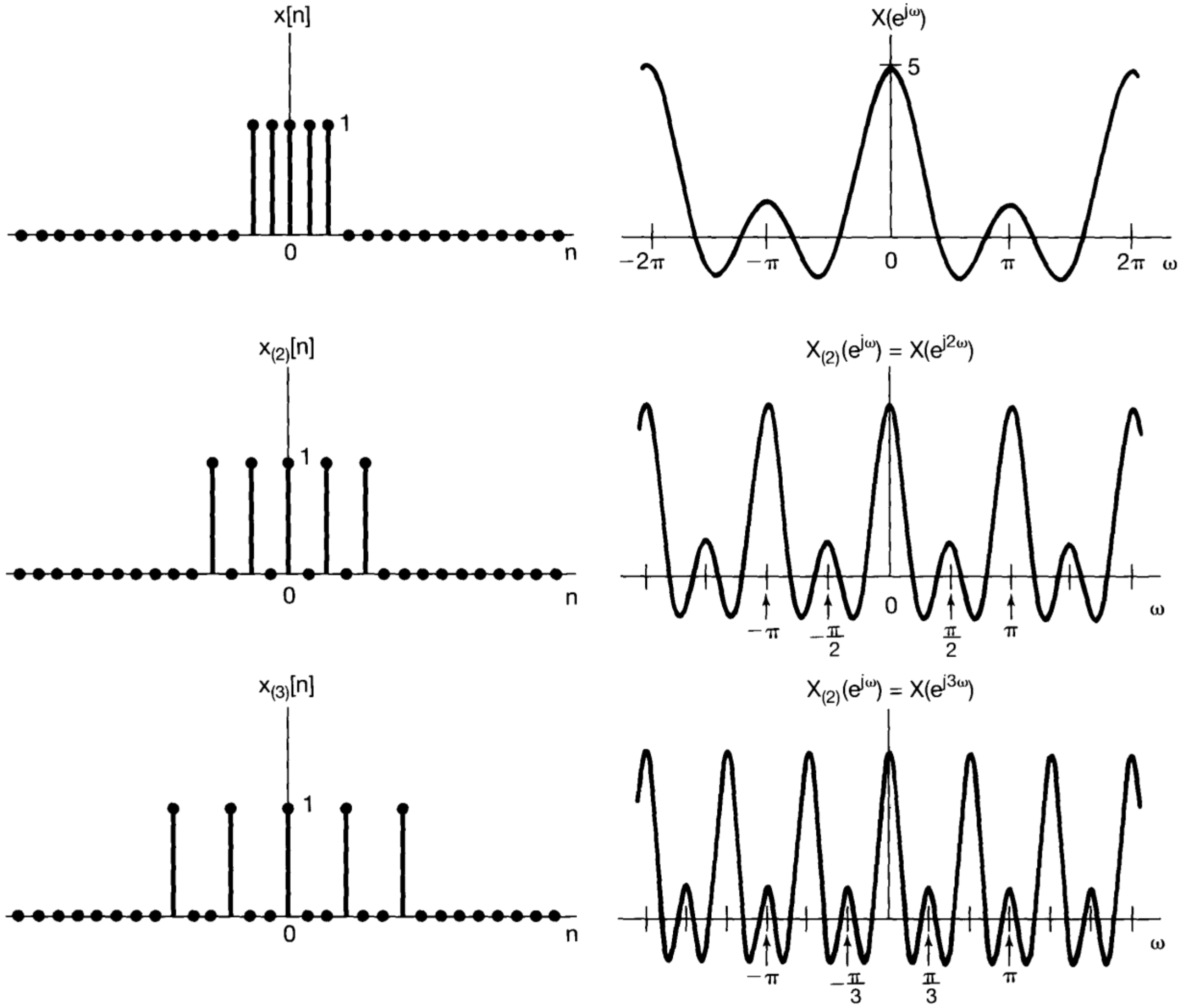
$$X_{(k)}\left(e^{j\omega}\right) = \sum_{n=-\infty}^{+\infty} x_{(k)}[n]e^{-j\omega n}$$

$$n = rk$$

$$= \sum_{r=-\infty}^{+\infty} x_{(k)}[rk]e^{-j\omega rk}$$

$$x_{(k)}[rk] = x[r]$$

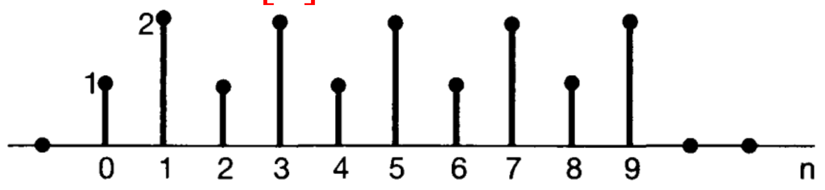
$$X_{(k)}\left(e^{j\omega}\right) = \sum_{r=-\infty}^{+\infty} x[r]e^{-j(k\omega)r} = X\left(e^{jk\omega}\right)$$



Properties of discrete-time Fourier Transform

Examples

$$X\left(e^{j\omega}\right) = ?$$



$x[n]$

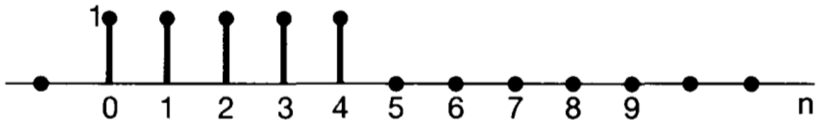
Solution

$$x[n] = y_{(2)}[n] + 2y_{(2)}[n-1]$$

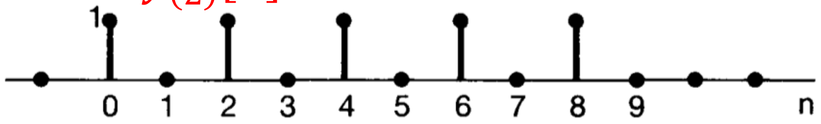
where $y[n] = \begin{cases} 1, 0 \le n \le 4 \\ 0, \quad\quad \text{else} \end{cases}$

$$y_{(2)}[n] = \begin{cases} y[n/2], n \text{ is even} \\ 0, \quad\quad n \text{ is odd} \end{cases}$$

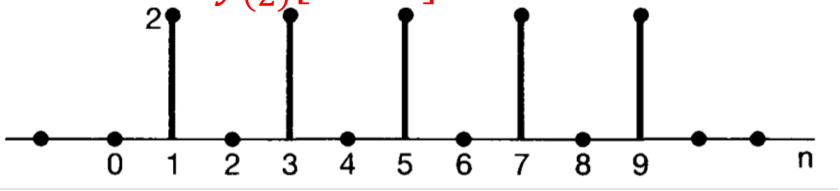
$y[n]$



$y_{(2)}[n]$



$2y_{(2)}[n-1]$



Properties of discrete-time Fourier Transform

Examples

$$X\left(e^{j\omega}\right) = ?$$

Solution

$$Y\left(e^{j\omega}\right) = e^{-j2\omega}\frac{\sin(5\omega/2)}{\sin(\omega/2)}$$

- ☐ Using the time expansion property

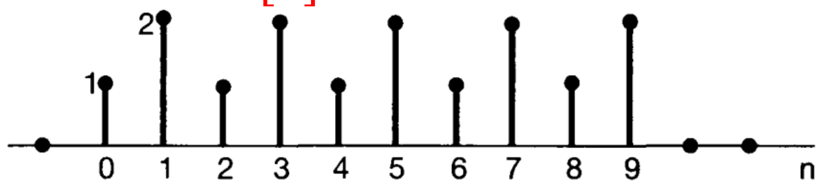
$$y_{(2)}[n] \xleftrightarrow{\mathcal{F}} Y_2\left(e^{j\omega}\right) = Y\left(e^{j2\omega}\right) = e^{-j4\omega}\frac{\sin(5\omega)}{\sin(\omega)}$$

- ☐ Using the linearity and time-shifting properties

$$2y_{(2)}[n-1] \xleftrightarrow{\mathcal{F}} 2e^{-j5\omega}\frac{\sin(5\omega)}{\sin(\omega)}$$

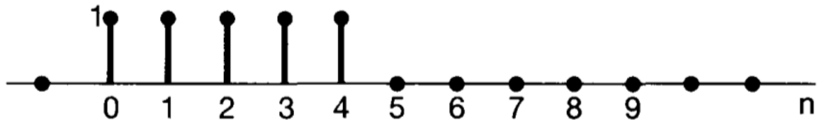
$$X\left(e^{j\omega}\right) = e^{-j4\omega}\left(1+2e^{-j\omega}\right)\left(\frac{\sin(5\omega)}{\sin(\omega)}\right)$$

$x[n]$

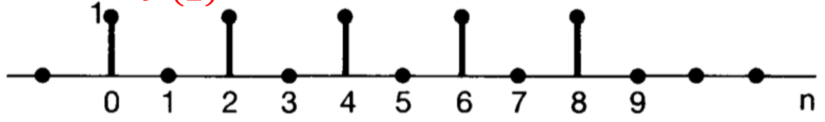


$$x[n] = y_{(2)}[n] + 2y_{(2)}[n-1]$$

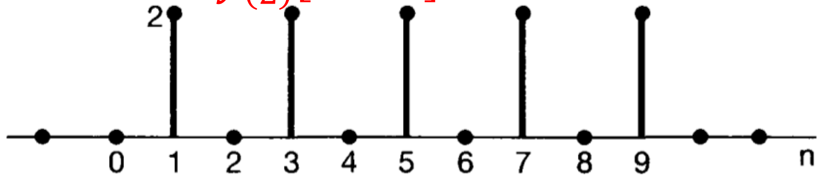
$y[n]$



$y_{(2)}[n]$



$2y_{(2)}[n-1]$



Properties of discrete-time Fourier Transform

Differentiation in frequency

$$nx[n] \xleftrightarrow{\mathcal{F}} j\frac{dX(e^{j\omega})}{d\omega}$$

- Consider

$$x[n] \xleftrightarrow{\mathcal{F}} X(e^{j\omega})$$

$$\frac{dX(e^{j\omega})}{d\omega} = \sum_{n=-\infty}^{+\infty} \boxed{-jnx[n]} e^{-j\omega n}$$

$$\Longrightarrow \quad -jnx[n] \xleftrightarrow{\mathcal{F}} \frac{dX(e^{j\omega})}{d\omega}$$

$$\Longrightarrow \quad nx[n] \xleftrightarrow{\mathcal{F}} j\frac{dX(e^{j\omega})}{d\omega}$$

Parseval's relation

$$\sum_{n=-\infty}^{+\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} \left|X(e^{j\omega})\right|^2 d\omega$$

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Properties of discrete-time Fourier Transform

Examples

$$x[n] \overset{\mathcal{F}}{\longleftrightarrow} X(e^{j\omega})$$

❑ $x[n]$ is

- Periodic?

  No

- Real?

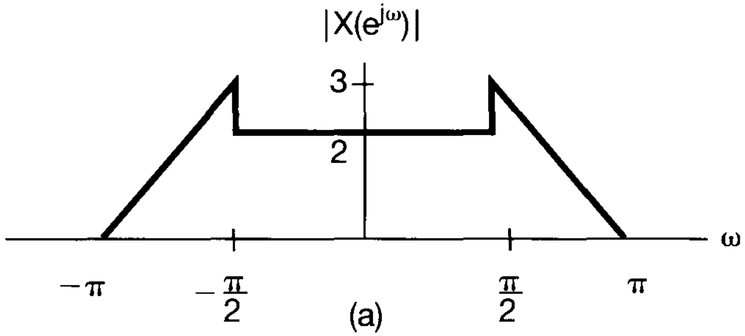
  Yes

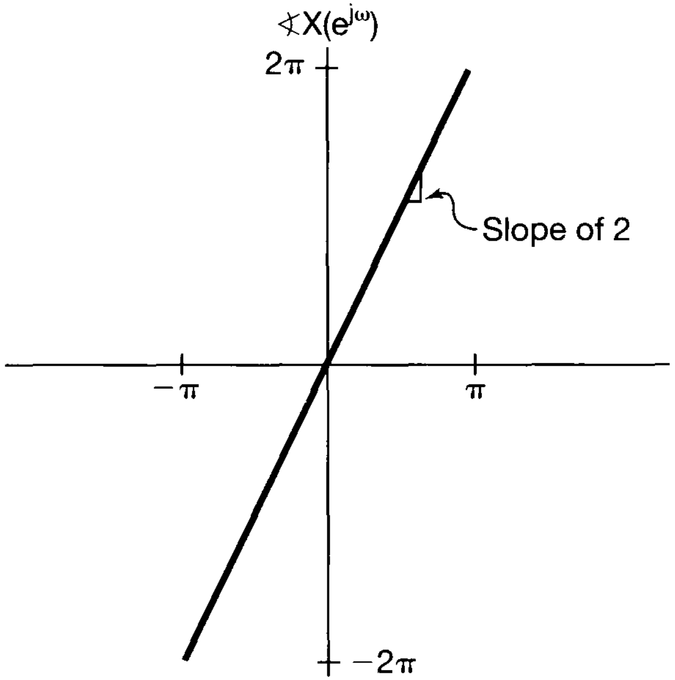
- Even?

  No

- Of finite energy?

  Yes





Chapter 5: The Discrete-Time Fourier Transform

- ❑ **Representation of aperiodic signals - Discrete Fourier transform**
- ❑ **Fourier transform for periodic signals**
- ❑ **Properties of discrete-time Fourier transform**
- ❑ **The convolution property**
- ❑ **The multiplication property**
- ❑ **Duality**
- ❑ **Systems characterized by difference equations**

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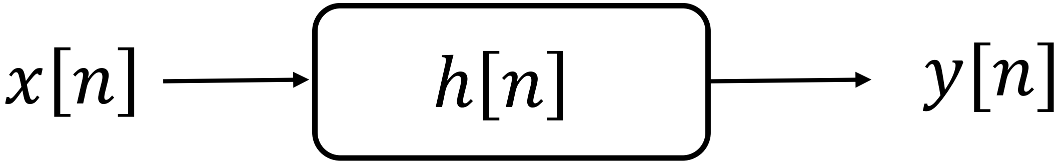
The convolution property



$$y[n] = x[n] * h[n] \xleftrightarrow{\mathcal{F}} Y\left(e^{j\omega}\right) = X\left(e^{j\omega}\right)H\left(e^{j\omega}\right)$$

- $H\left(e^{j\omega}\right)$: Frequency response; FT of the impulse response of the system

The convolution property

Examples

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

$h[n] = \delta[n - n_0]$ and $X(e^{j\omega}) = \mathcal{F}\{x[n]\}$ $Y(e^{j\omega}) = ?$

Solution

$$h[n] = \delta[n - n_0]$$

⇙ ⇘

$$H(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} \delta[n - n_0] e^{-j\omega n}$$

$$= e^{-j\omega n_0}$$

$$y[n] = x[n - n_0]$$

⇘ ⇙

$$Y(e^{j\omega}) = e^{-j\omega n_0} X(e^{j\omega})$$

The convolution property

Examples

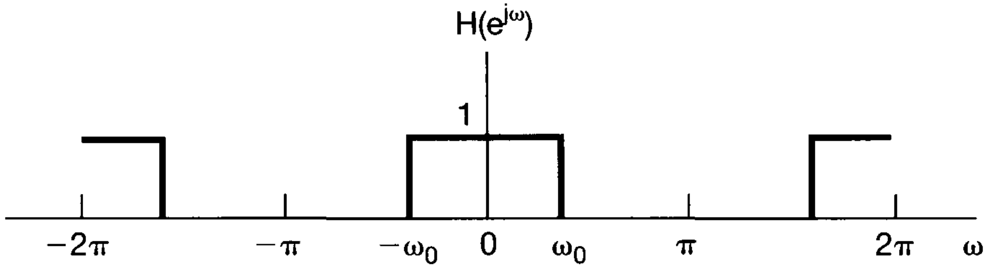
Determine the impulse response of an ideal low-pass filter

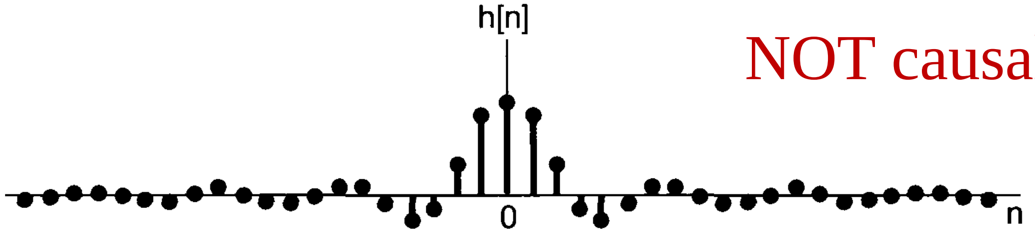
Solution

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H\left(e^{j\omega}\right)e^{j\omega n}d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} e^{j\omega n}d\omega$$

$$= \frac{\sin \omega_0 n}{\pi n}$$





NOT causal

The convolution property

Examples

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

$$h[n] = \alpha^n u[n], (|\alpha| < 1) \quad x[n] = \beta^n u[n], (|\beta| < 1) \quad y[n] = ?$$

Solution

$$H(e^{j\omega}) = \frac{1}{1 - \alpha e^{-j\omega}} \qquad X(e^{j\omega}) = \frac{1}{1 - \beta e^{-j\omega}} \qquad Y(e^{j\omega}) = \frac{1}{(1 - \alpha e^{-j\omega})(1 - \beta e^{-j\omega})}$$

When $\alpha \neq \beta$

$$Y(e^{j\omega}) = \frac{A}{1 - \alpha e^{-j\omega}} - \frac{B}{1 - \beta e^{-j\omega}} \qquad A = \frac{\alpha}{\alpha - \beta} \qquad B = \frac{\beta}{\alpha - \beta}$$

$$y[n] = \frac{\alpha}{\alpha - \beta}\alpha^n u[n] - \frac{\beta}{\alpha - \beta}\beta^n u[n] = \frac{1}{\alpha - \beta}\left(\alpha^{n+1} u[n] - \beta^{n+1} u[n]\right)$$

The convolution property

Examples

$$x[n] \longrightarrow \boxed{h[n]} \longrightarrow y[n]$$

$$h[n] = \alpha^n u[n], (|\alpha| < 1) \quad x[n] = \beta^n u[n], (|\beta| < 1) \quad y[n] = ?$$

Solution

$$H(e^{j\omega}) = \frac{1}{1 - \alpha e^{-j\omega}} \qquad X(e^{j\omega}) = \frac{1}{1 - \beta e^{-j\omega}} \qquad Y(e^{j\omega}) = \frac{1}{(1 - \alpha e^{-j\omega})(1 - \beta e^{-j\omega})}$$

When $\alpha = \beta$

$$Y(e^{j\omega}) = \frac{1}{(1 - \alpha e^{-j\omega})^2} = \frac{j}{\alpha} e^{j\omega} \frac{d}{d\omega}\left(\frac{1}{1 - \alpha e^{-j\omega}}\right)$$

$$y[n] = (n+1)\alpha^n u[n+1] = (n+1)\alpha^n u[n]$$

The convolution property

Examples

Consider the ideal band-stop filter, $Y(e^{j\omega}) = ?$

$$w_1[n] = e^{j\pi n}x[n]$$

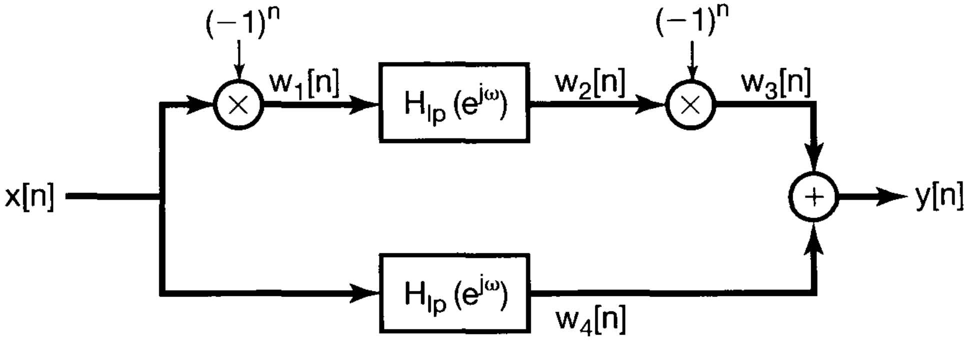
$$W_1(e^{j\omega}) = X(e^{j(\omega-\pi)})$$

$$W_2(e^{j\omega}) = H_{lp}(e^{j\omega})X(e^{j(\omega-\pi)})$$

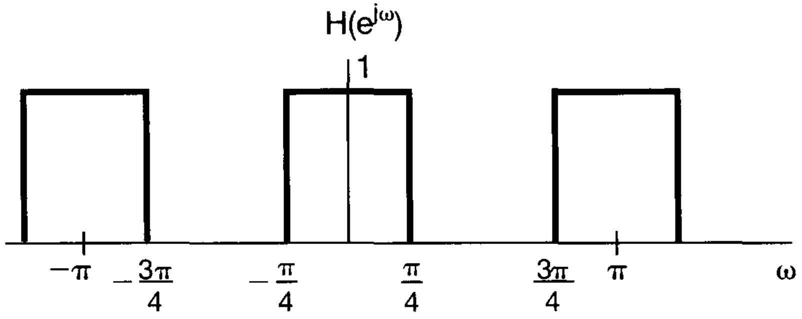
$$\begin{aligned} W_3(e^{j\omega}) &= W_2(e^{j(\omega-\pi)}) \\ &= H_{lp}(e^{j(\omega-\pi)})X(e^{j(\omega-2\pi)}) \\ &= H_{lp}(e^{j(\omega-\pi)})X(e^{j\omega}) \end{aligned}$$

$$W_4(e^{j\omega}) = H_{lp}(e^{j\omega})X(e^{j\omega})$$

$$Y(e^{j\omega}) = W_3(e^{j\omega}) + W_4(e^{j\omega}) = [H_{lp}(e^{j(\omega-\pi)}) + H_{lp}(e^{j\omega})]X(e^{j\omega})$$



$(-1)^n$
$(-1)^n$
$w_1[n]$
$H_{lp}(e^{j\omega})$
$w_2[n]$
$w_3[n]$
$x[n]$
$y[n]$
$H_{lp}(e^{j\omega})$
$w_4[n]$



Chapter 5: The Discrete-Time Fourier Transform

- ☐ Representation of aperiodic signals - Discrete Fourier transform
- ☐ Fourier transform for periodic signals
- ☐ Properties of discrete-time Fourier transform
- ☐ The convolution property
- ☐ The multiplication property
- ☐ Duality
- ☐ Systems characterized by difference equations

The multiplication property

$$y[n] = x_1[n]x_2[n] \xleftrightarrow{\mathcal{F}} Y(e^{j\omega}) = \underbrace{\frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta}_{\text{Periodic convolution}}$$

Proof

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} y[n]e^{-j\omega n} = \sum_{n=-\infty}^{+\infty} x_1[n]x_2[n]e^{-j\omega n}$$

$$x_1[n] = \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})e^{j\theta n}d\theta$$

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x_2[n]\left\{\frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})e^{j\theta n}d\theta\right\}e^{-j\omega n}$$

$$Y(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})\left[\sum_{n=-\infty}^{+\infty} x_2[n]e^{-j(\omega-\theta)n}\right]d\theta$$

$$Y(e^{j\omega}) = \frac{1}{2\pi}\int_{2\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta$$

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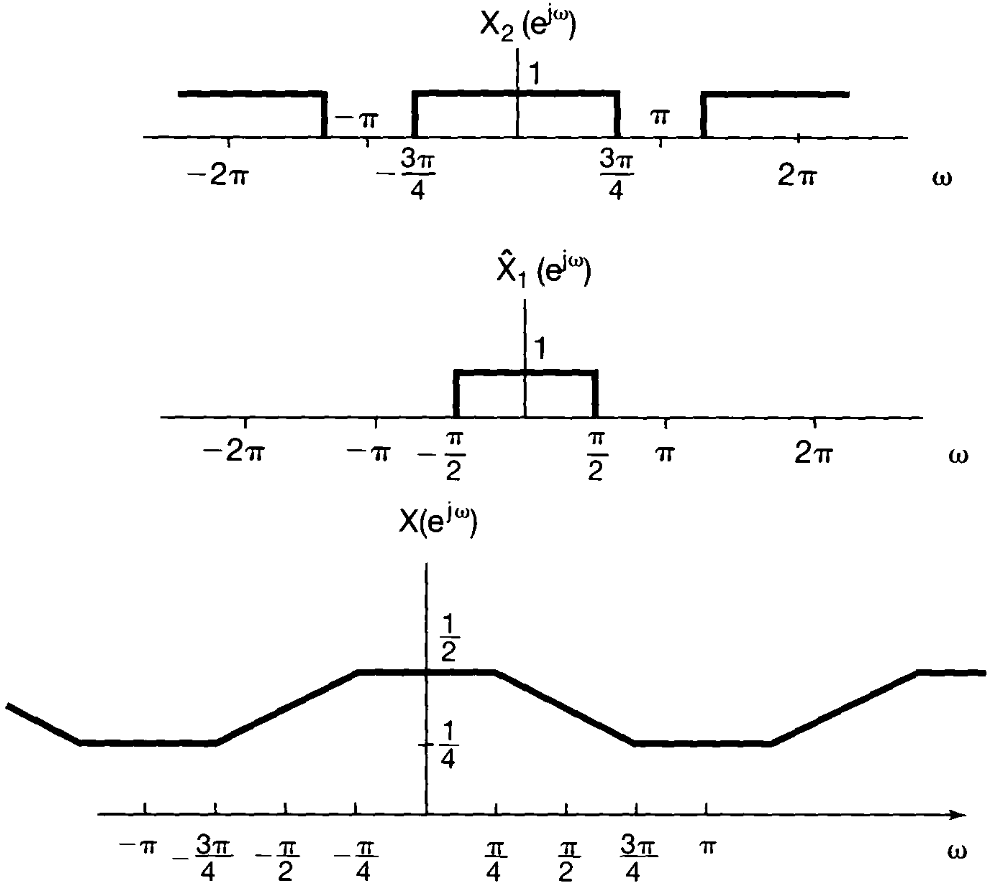
The multiplication property

Examples

$$x_1[n]=\frac{\sin(\pi n/2)}{\pi n} \qquad x_2[n]=\frac{\sin(3\pi n/4)}{\pi n}$$

$$x[n]=x_1[n]x_2[n] \qquad X(e^{j\omega})=?$$



Solution

$$X(e^{j\omega})=\frac{1}{2\pi}\int_{-\pi}^{\pi}X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta$$

- Convert to ordinary convolution

Define $\hat{X}_1(e^{j\omega})=\begin{cases}X_1(e^{j\omega}), & -\pi<\omega<\pi\\ 0, & \text{otherwise}\end{cases}$

$$X(e^{j\omega})=\frac{1}{2\pi}\int_{-\pi}^{\pi}\hat{X}_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{X}_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta$$

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Duality

Duality in the discrete FS

Consider $$f[m] = \frac{1}{N}\sum_{r=\langle N\rangle} g[r]e^{-jr(2\pi/N)m}$$

$f[m]$ and $g[r]$ are periodic

$m = k, r = n$ $\Downarrow$

$$f[k] = \frac{1}{N}\sum_{n=\langle N\rangle} g[n]e^{-jk(2\pi/N)n}$$

$$\Downarrow$$

$$g[n] \xleftrightarrow{\mathcal{FS}} f[k]$$

$m = n, r = -k$ $\Downarrow$

$$f[n] = \frac{1}{N}\sum_{k=\langle N\rangle} g[-k]e^{jk(2\pi/N)n}$$

$$\Downarrow$$

$$f[n] \xleftrightarrow{\mathcal{FS}} \frac{1}{N}g[-k]$$

Every property of the discrete FS has a dual.

- ❑ Examples

$$\begin{cases} x[n-n_0] \xleftrightarrow{\mathcal{FS}} a_k e^{-jk(2\pi/N)n_0} \\ e^{jm(2\pi/N)n}x[n] \xleftrightarrow{\mathcal{FS}} a_{k-m} \end{cases}$$

$$\begin{cases} \sum_{r=\langle N\rangle} x[r]y[n-r] \xleftrightarrow{\mathcal{FS}} Na_k b_k \\ x[n]y[n] \xleftrightarrow{\mathcal{FS}} \sum_{l=\langle N\rangle} a_l b_{k-l} \end{cases}$$

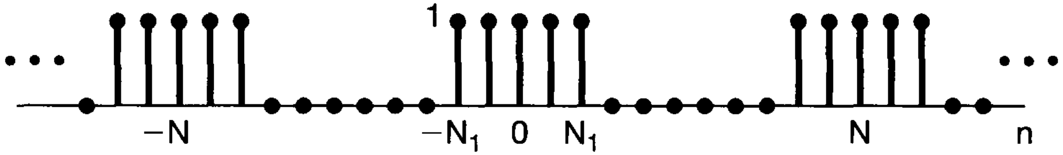
Duality

- ☐ **Examples**

$$x[n]=\begin{cases}\frac{1}{9}\frac{\sin(5\pi n/9)}{\sin(\pi n/9)}, & n\neq \text{multiple of } 9\\ 5/9, & n=\text{multiple of } 9\end{cases}\qquad a_k=?$$

Solution

- Recall the Fourier coefficients of a periodic square wave



$$\Big\updownarrow \mathcal{FS}$$

$$\begin{cases}\frac{1}{N}\frac{\sin[2\pi k(N_1+1/2)/N]}{\sin(\pi k/N)}, & k\neq 0,\pm N,\pm 2N,\cdots\\ (2N_1+1)/N, & k=0,\pm N,\pm 2N,\cdots\end{cases}$$

Duality

Examples

$$x[n] = \begin{cases} \frac{1}{9}\frac{\sin(5\pi n/9)}{\sin(\pi n/9)}, & n \neq \text{multiple of } 9 \\ 5/9, & n = \text{multiple of } 9 \end{cases} \qquad a_k = ?$$

Solution

$$g[n] = \begin{cases} 1, & |n| \leq 2 \\ 0, & 2 < |n| \leq 4 \end{cases} \quad \overset{\mathcal{FS}}{\longleftrightarrow} \quad b_k = \begin{cases} \frac{1}{9}\frac{\sin(5\pi k/9)}{\sin(\pi k/9)}, & k \neq \text{multiple of } 9 \\ 5/9, & k = \text{multiple of } 9 \end{cases}$$

g[n] is periodic (N = 9)

$\Downarrow$ Duality

$$x[n] = \begin{cases} \frac{1}{9}\frac{\sin(5\pi n/9)}{\sin(\pi n/9)}, & n \neq \text{multiple of } 9 \\ 5/9, & n = \text{multiple of } 9 \end{cases} \quad \overset{\mathcal{FS}}{\longleftrightarrow} \quad g[k] = \begin{cases} 1/9, & |k| \leq 2 \\ 0, & 2 < |k| \leq 4 \end{cases}$$

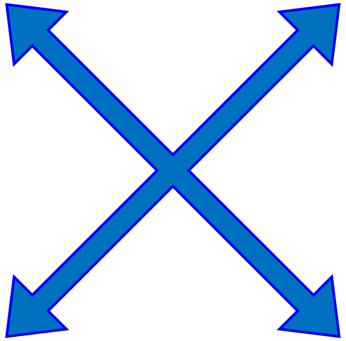
Duality

Duality between discrete FT and continuous FS

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

Discrete FT

$$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$$



$$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$$

Continuous FS

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$$

Duality

Examples

$$x[n] = \frac{\sin(\pi n/2)}{\pi n} \qquad X(e^{j\omega}) = ?$$

Solution

$$g(t) = \begin{cases} 1, & |t| \le T_1 \\ 0, & T_1 < |t| \le \pi \end{cases} \xleftrightarrow{\mathcal{FS}} a_k = \frac{\sin(kT_1)}{k\pi}$$

$T_1 = \pi/2$

$$\frac{\sin(\pi k/2)}{\pi k} = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)e^{-jkt}dt = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}(1)e^{-jkt}dt$$

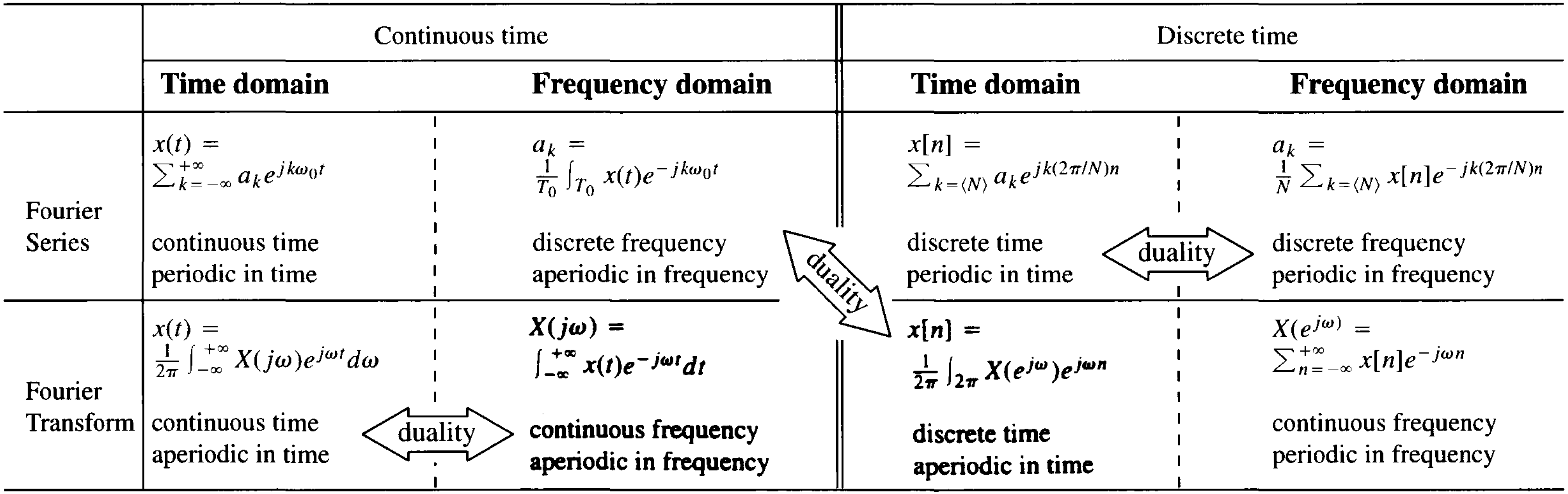
$$\frac{\sin(\pi n/2)}{\pi n} = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}(1)e^{-jn\omega}d\omega$$

$$\frac{\sin(\pi n/2)}{\pi n} = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2}(1)e^{jn\omega}d\omega$$

$$\therefore \quad X(e^{j\omega}) = \begin{cases} 1, & |\omega| \le \pi/2 \\ 0, & \pi/2 < |\omega| \le \pi \end{cases}$$

Duality

Summary FS and FT expressions


duality
duality
duality

	Continuous time		Discrete time	
	Time domain	**Frequency domain**	**Time domain**	**Frequency domain**
Fourier Series	$x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$ continuous time periodic in time	$a_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}$ discrete frequency aperiodic in frequency	$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$ discrete time periodic in time	$a_k = \frac{1}{N}\sum_{k=\langle N\rangle} x[n]e^{-jk(2\pi/N)n}$ discrete frequency periodic in frequency
Fourier Transform	$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega$ continuous time aperiodic in time	$X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$ continuous frequency aperiodic in frequency	$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}$ discrete time aperiodic in time	$X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$ continuous frequency periodic in frequency

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Systems characterized by difference equations

$$\sum_{k=0}^{N} a_k y[n-k] = \sum_{k=0}^{M} b_k x[n-k]$$

$$\sum_{k=0}^{N} a_k e^{-jk\omega} Y\left(e^{j\omega}\right) = \sum_{k=0}^{M} b_k e^{-jk\omega} X\left(e^{j\omega}\right)$$

$$H\left(e^{j\omega}\right) = \frac{Y\left(e^{j\omega}\right)}{X\left(e^{j\omega}\right)} = \frac{\sum_{k=0}^{M} b_k e^{-jk\omega}}{\sum_{k=0}^{N} a_k e^{-jk\omega}}$$

Systems characterized by difference equations

Examples

$y[n] - ay[n-1] = x[n], |a| < 1 \qquad h[n] = ?$

Solution

$$H\left(e^{j\omega}\right) = \frac{1}{1 - ae^{-j\omega}}$$

$$h[n] = a^n u[n]$$

Systems characterized by difference equations

Examples

$$y[n]-\frac{3}{4}y[n-1]+\frac{1}{8}y[n-2]=2x[n] \qquad h[n]=?$$

Solution

$$H\left(e^{j\omega}\right)=\frac{2}{1-\frac{3}{4}e^{-j\omega}+\frac{1}{8}e^{-j2\omega}}=\frac{2}{\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1-\frac{1}{4}e^{-j\omega}\right)}$$

$$=\frac{4}{\left(1-\frac{1}{2}e^{-j\omega}\right)}-\frac{2}{\left(1-\frac{1}{4}e^{-j\omega}\right)}$$

$$h[n]=4\left(\frac{1}{2}\right)^{n}u[n]-2\left(\frac{1}{4}\right)^{n}u[n]$$

Systems characterized by difference equations

Examples

$$y[n]-\frac{3}{4}y[n-1]+\frac{1}{8}y[n-2]=2x[n] \qquad x[n]=\left(\frac{1}{4}\right)^{n}u[n] \qquad y[n]=?$$

Solution

$$Y\left(e^{j\omega}\right)=H\left(e^{j\omega}\right)X\left(e^{j\omega}\right)=\left[\frac{2}{\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1-\frac{1}{4}e^{-j\omega}\right)}\right]\left[\frac{1}{1-\frac{1}{4}e^{-j\omega}}\right]$$

$$=\frac{2}{\left(1-\frac{1}{2}e^{-j\omega}\right)\left(1-\frac{1}{4}e^{-j\omega}\right)^{2}}=-\frac{4}{1-\frac{1}{4}e^{-j\omega}}-\frac{2}{\left(1-\frac{1}{4}e^{-j\omega}\right)^{2}}+\frac{8}{\left(1-\frac{1}{2}e^{-j\omega}\right)}$$

$$y[n]=\left\{-4\left(\frac{1}{4}\right)^{n}-2(n+1)\left(\frac{1}{4}\right)^{n}+8\left(\frac{1}{2}\right)^{\mathrm{n}}\right\}u[n]$$