SI 140A-02 Probability & Statistics for EECS, Fall 2024
Homework 7

Name:
Student ID:

Due on Dec. 3, 2024, 11:59 UTC+8

Read all the instructions below carefully before you start working on the assignment, and before you make a submission.

- You are required to write down all the major steps towards making your conclusions; otherwise you may obtain limited points of the problem.
- Write your homework in English; otherwise you will get no points of this homework.
- Any form of plagiarism will lead to 0 point of this homework.

Problem 1

Let X and Y be two continuous random variables with joint PDF

$$f_{X,Y}(x,y) = \begin{cases} cx^2y & \text{if } 0 \le y \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Find the value of constant c.

(b) Find the conditional probability $P\left(Y \le \frac{X}{4} \,\middle|\, Y \le \frac{X}{2}\right)$.

(a) $\cancel{1} = \iint cx^2y\,dx\,dy$

$= \int_0^1 \int_0^x cx^2y\,dy\,dx = \frac{c}{10} \Rightarrow c = 10$

(b) $P(\cdot) = \dfrac{P(Y \le \frac{X}{4}, Y \le \frac{X}{2})}{P(Y \le \frac{X}{2})} = \dfrac{P(Y \le \frac{X}{4})}{P(Y \le \frac{X}{2})} = \dfrac{1}{4}$

Problem 2

Let X and Y be two integer random variables with joint PMF

$$P_{X,Y}(x,y) = \begin{cases} \frac{1}{6\cdot 2^{\min(x,y)}} & \text{if } x, y \geq 0, |x-y| \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the marginal distributions of X and Y.

(b) Are X and Y independent?

(c) Find $P(X = Y)$. ∞

(a) $P(X) = \sum_{y=0} P(XY)$

$x=0$, $y = 0, 1$

$P(X=0) = P(X=0, Y=0) + P(X=0, Y=1) = \frac{1}{3}$

$x > 0$, $y = x-1, x, x+1$

$P(X=x) = P(X=x, y=x-1) + P(\cdots, Y=x) + P(\cdots, Y=x+1)$

$= \frac{1}{6\cdot 2^{x-2}}$

$P_X(x) = \begin{cases} \frac{1}{3}, & x=0 \\ \frac{1}{6\cdot 2^{x-2}}, & x>0 \\ 0, & \text{else} \end{cases}$ $P_Y(y) = \begin{cases} \frac{1}{3}, & y=0 \\ \frac{1}{6\cdot 2^{y-2}}, & y>0 \\ 0, & \text{else} \end{cases}$

(b) $x=Y=0$ $P(x,y) = \frac{1}{6\cdot 2^0} = \frac{1}{6}$

$P(X=0) = P(Y=0) = \frac{1}{3} \neq P(x,y)$, not independent

(c) We have $P(X=Y) = P(X=Y+1) = P(X=Y-1)$ and $P(X=Y) + P(X=Y+1) + P(X=Y-1) = 1$

$\therefore P(X=Y) = \frac{1}{3}$

Problem 3

Let X and Y be i.i.d. $\mathcal{N}(0,1)$, and let S be a random sign 1 or -1 , with equal probabilities independent of (X, Y).

(a) Determine whether or not $(X, Y, X+Y)$ is Multivariate Normal.

(b) Determine whether or not $(X, Y, SX+SY)$ is Multivariate Normal.

(c) Determine whether or not (SX, SY) is Multivariate Normal.

(a) $aX+bY+c(X+Y)=(a+c)\cdot X+(b+c)Y$

(b) $Z = X+Y+SX+SY$

$Z=0 \Rightarrow S=-1$

$P(Z=0)=P(S=-1)=\frac{1}{2}$, Z is not

(c) $P(SX+SY\le k)=P(SX+SY\le k, S=1)P(S=1)$

$+P(SX+SY\le k, S=-1)P(S=-1)$

$=P(X+Y\le k)\frac{1}{2}+P(-X-Y\le k)\cdot\frac{1}{2}$

$\Rightarrow$ $P(X+Y\le k)\cdot\frac{1}{2}$

$=P(X+Y\le k)$

$\therefore (SX, SY)$ is equally distributed as (X,Y)

Thus, it is.

Problem 4

Let Z_1, Z_2 be two i.i.d. random variables satisfying standard normal distributions, i.e., $Z_1, Z_2 \sim \mathcal{N}(0,1)$. Define

$$\begin{cases} X = \sigma_X Z_1 + \mu_X \\ Y = \sigma_Y \left(\rho Z_1 + \sqrt{1-\rho^2} Z_2\right) + \mu_Y \end{cases}$$

where $\sigma_X > 0, \sigma_Y > 0, -1 < \rho < 1.$

(a) Show that X and Y are bivariate normal.

(b) Find the correlation coefficient between X and Y, i.e., $\mathrm{Corr}(X,Y)$.

(c) Find the joint PDF of X and Y.

(a) $aX+bY = (a\sigma_x + b\sigma_y\rho)Z_1 + b\sqrt{1-\rho^2}\,\sigma_y Z_2 + b\mu_x + b\mu_y$

Linear Combination,

(b) $\mathrm{Corr}(X,Y) = \dfrac{\mathrm{Cov}(X,Y)}{\sqrt{\mathrm{Var}(X)\mathrm{Var}(Y)}} = \dfrac{\sigma_x\sigma_y\rho}{\sqrt{\sigma_x^2}\cdot\sqrt{\sigma_y^2}} = \rho$

$\mathrm{Cov}(X,Y) = \mathrm{Cov}(\sigma_x Z_1 + \mu_x, \sigma_Y(\rho Z_1 + \sqrt{1-\rho^2}Z_2) + \mu_Y)$

$= \sigma_x\sigma_y \cdot \rho$

$X \sim N(\mu_x, \sigma_x^2) \quad Y \sim N(\mu_y, \sigma_y^2)$

(c) $Z_1 = \dfrac{X-\mu_x}{\sigma_x}$

$Z_2 = \dfrac{Y-\mu_y}{\sqrt{1-\rho^2}\cdot\sigma_y} - \rho\dfrac{X-\mu_x}{\sqrt{1-\rho^2}\cdot\sigma_x}$

$f_{Z_1,Z_2} = \dfrac{1}{2\pi} e^{-\frac{Z_1^2+Z_2^2}{2}}$

$f_{XY} = \left|\dfrac{\partial(Z_1,Z_2)}{\partial(X,Y)}\right| f_{Z_1Z_2}$

$= \dfrac{1}{2\pi\sigma_x\sigma_y\sqrt{1-\rho^2}} e^{-\frac{\left(\frac{x-\mu_x}{\sigma_x}\right)^2 - \frac{2\rho(x-\sigma_x)(y-\sigma_y)}{\sigma_x\sigma_y} + \left(\frac{y-\mu_y}{\sigma_y}\right)^2}{2(1-\rho^2)}}$

Problem 5

(a) Let X and Y be i.i.d. $\mathcal{N}(0,1)$, and $Z = \frac{X}{Y}$. Find the PDF of Z.

(b) Let X and Y be i.i.d. $\text{Unif}(0,1)$, $W = X \cdot Y$, and $Z = \frac{X}{Y}$. Find the joint PDF of (W, Z).

(c) A point (X, Y) is picked at random uniformly in the unit circle. Find the joint PDF of R and X, where $R = \sqrt{X^2 + Y^2}$.

(d) A point (X, Y, Z) is picked uniformly at random inside the unit ball of $\mathbb{R}^3$. Find the joint PDF of Z and R, where $R = \sqrt{X^2 + Y^2 + Z^2}$.

(a) Let $X = R\cos\theta$, $Y = R\sin\theta \Rightarrow R = \sqrt{X^2+Y^2}$, $\theta = \tan^{-1}(\frac{Y}{X})$

$f_{X,Y}(x,y) = \frac{1}{2\pi} e^{-\frac{x^2+y^2}{2}}$, $J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$

$f(r,\theta) = |J| f_{X,Y}(x,y) = \frac{r}{2\pi} e^{-\frac{r^2}{2}} \Rightarrow f_\theta(\theta) = \int_0^{+\infty} \frac{r}{2\pi} e^{-\frac{r^2}{2}} dr = \frac{1}{2\pi}$

Then, let $z = \frac{Y}{X}$, $\frac{d\theta}{dz} = \frac{d\tan^{-1}(z)}{dz} = \frac{1}{1+z^2}$

$f_z(z) = f_\theta(\theta) \left|\frac{d\theta}{dz}\right| = \frac{1}{\pi(1+z^2)}$

(b) $f_X(x) = f_Y(y) = 1$, $x, y \in [0,1]$

Define $W = X \cdot Y$, $Z = \frac{X}{Y} \Rightarrow X = \sqrt{WZ}$, $Y = \sqrt{\frac{W}{Z}}$

$J = \begin{vmatrix} \frac{\partial X}{\partial W} & \frac{\partial X}{\partial Z} \\ \frac{\partial Y}{\partial W} & \frac{\partial Y}{\partial Z} \end{vmatrix} = \begin{vmatrix} \frac{1}{2} Z^{-\frac{1}{2}} W^{-\frac{1}{2}} & \frac{1}{2} W^{\frac{1}{2}} Z^{-\frac{3}{2}} \\ \frac{1}{2} Z^{-\frac{1}{2}} W^{-\frac{1}{2}} & -\frac{1}{2} W^{\frac{1}{2}} Z^{-\frac{3}{2}} \end{vmatrix} = -\frac{1}{2Z}$

$f_{WZ}(w,z) = f_{X,Y}(x,y)|J| = \frac{1}{2z}$

3. Set $\begin{cases} R = \sqrt{X^2+Y^2} \\ S = X \end{cases} \Rightarrow \begin{cases} X = S \\ Y = \pm\sqrt{R^2-S^2} \end{cases} \Rightarrow J = \begin{vmatrix} \frac{\partial X}{\partial S} & \frac{\partial X}{\partial R} \\ \frac{\partial Y}{\partial S} & \frac{\partial Y}{\partial R} \end{vmatrix} = \pm \frac{r}{\sqrt{r^2-s^2}}$

$f_{R,S}(r,s) = f_{X,Y}(x,y)|J| = \frac{2r}{\pi\sqrt{r^2-s^2}}$, $|s| < r < 1$, $-1 < s < 1$

4. Set $\begin{cases} X = R\sin\varphi\cos\theta \\ Y = R\sin\varphi\sin\theta \\ Z = R\cos\varphi \end{cases}$, $J = \begin{vmatrix} \frac{\partial X}{\partial R} & \frac{\partial X}{\partial \theta} & \frac{\partial X}{\partial \varphi} \\ \frac{\partial Y}{\partial R} & \frac{\partial Y}{\partial \theta} & \frac{\partial Y}{\partial \varphi} \\ \frac{\partial Z}{\partial R} & \frac{\partial Z}{\partial \theta} & \frac{\partial Z}{\partial \varphi} \end{vmatrix} = R^2\sin\varphi \Rightarrow f_{R,\varphi,\theta}(r,\varphi,\theta) = \frac{3}{4\pi} r^2 \sin\varphi$

$f_{R,\varphi}(r,\varphi) = \int_0^{2\pi} r^2\sin\varphi \, d\theta = \frac{3}{2} r^2 \sin\varphi$

$\Rightarrow f_{R,Z}(r,z) = f_{R,\varphi}(r,\varphi) \left| \det \begin{bmatrix} \frac{\partial r}{\partial r} & \frac{\partial r}{\partial z} \\ \frac{\partial \varphi}{\partial r} & \frac{\partial \varphi}{\partial z} \end{bmatrix} \right| = \frac{3}{2} r$, $|z| \le r$, $r \le 1$

Problem 6

$\frac{X}{Y} > 1 \Rightarrow X > Y$

(**Optional Challenging Problem**)
Let X and Y be i.i.d. Unif$(0,1)$, and $Z = \frac{X}{Y}$. Find the probability that the integer close to Z is even.

$\underbrace{n}_{=2k} - \frac{1}{2} \le Z \le n + \frac{1}{2}$

$F_Z(z) = P\left(\frac{X}{Y} \le z\right) = \int_0^1 P(X \le zy)\,dy$

$z > 1 \quad 0 < y < \frac{1}{z}$

$F(z) = \int_0^{\frac{1}{z}} zy\,dy + \int_{\frac{1}{z}}^1 1 \quad \frac{1}{2z} + 1 - \frac{1}{z} = 1 - \frac{1}{2z}$

$0 \le z \le 1 \quad F(z) = \int_0^1 zy\,dy = \frac{z}{2}$

$f(z) = \begin{cases} \frac{1}{2}, & 0 < z < 1 \\ \frac{1}{2z^2}, & z > 1 \end{cases}$

$P(z \in (0, 0.5)) = \int_0^{0.5} \frac{1}{2}\,dz = \frac{1}{4}$

$\arctan x = \frac{1}{1+x^2}$

$P(z \in (2k - \frac{1}{2}, 2k + \frac{1}{2})) = \int_{2k-\frac{1}{2}}^{2k+\frac{1}{2}} \frac{1}{2z^2}\,dz$

$= \frac{1}{4k-1} - \frac{1}{4k+1}$

$1 - \underbrace{\int_0^1 \frac{1}{1+x^2}\,dx}_{\arctan x}$

$\arctan x \Big|_0^1$

$\sum_{k=1}^{\infty} \left(\frac{1}{4k-1} - \frac{1}{4k+1}\right)$

$\frac{1}{4k-1} = \int_0^1 x^{4k-2}\,dx$

$\frac{1}{4k+1} = \int_0^1 x^{4k}\,dx$

$= \sum_{k=1}^{\infty} \left(\int_0^1 x^{4k-2}\,dx - \int_0^1 x^{4k}\,dx\right)$

$1 - \frac{\pi}{4}$

$= \int_0^1 \left[\sum_{k=1}^{\infty} (x^{4k-2} - x^{4k})\right] dx = \int_0^1 (x^{-2} - 1) \sum_{k=1}^{\infty} x^{4k}\,dx$

$\frac{x^4}{1-x^4}$

$= \int_0^1 (x^{-2} - 1) \frac{x^4}{1-x^4}$

$\frac{1}{4} + 1 - \frac{\pi}{4} = \frac{5 - \pi}{4}$