EE150 - Signals and Systems, Fall 2024

Homework Set #2

Prof. Lin Xu and Prof. Xiran Cai

Problem 1. (20 points)

(a) (5 points)

Given a discrete sequence $x[n]$, represent it as the weighted sum of shifted unit impulses.

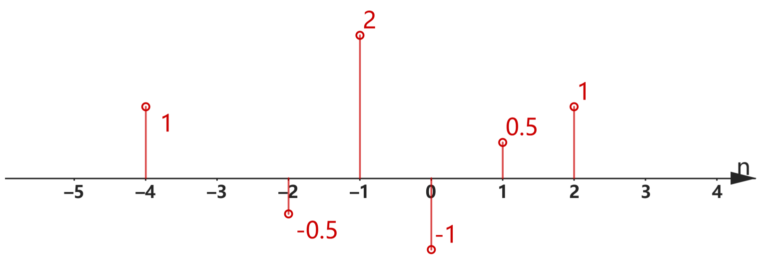

Figure 1

(b) (5 points)

Given $y[n] = x[n+4] - \frac{1}{3}x[n+3] + 4x[n+1] - 2x[n] + \frac{1}{2}x[n-1] - 3x[n-2]$, calculate the impulse response $h[n]$.

(c) (10 points)

Prove that the output of a discrete LTI system $y[n]$ is equal to the convolution of the input $x[n]$ and the impulse response $h[n]$.

Solution.

(a)

$y[n] = \delta[n+4] - \frac{1}{2}\delta[n+2] + 2\delta[n+1] - \delta[n] + \frac{1}{2}\delta[n-1] + \delta[n-2]$

(b)

$h[n] = \delta[n+4] - \frac{1}{3}\delta[n+3] + 4\delta[n+1] - 2\delta[n] + \frac{1}{2}\delta[n-1] - 3\delta[n-2]$

(c)

$$
\begin{aligned}
y[n] &= \mathrm{LTI}\{x[n]\} \\
&= \mathrm{LTI}\{\sum_{k=-\infty}^{\infty} x[k]\delta[n-k]\} \\
&= \sum_{k=-\infty}^{\infty} x[k]\mathrm{LTI}\{\delta[n-k]\} \quad \text{(Linear property of LTI system)} \\
&= \sum_{k=-\infty}^{\infty} x[k]h[n-k] \quad \text{(Defination of impulse response)} \\
&= x[n] * h[n]
\end{aligned}
$$

Problem 2. (20 points)

Compute the following convolutions:

1) $x(t) = \mathrm{e}^{-2t}u(t)$, and $h(t) = u(t) - u(t-1)$,
2) $x(t) = \cos(\omega t)$, and $h(t) = \delta(t+1) - \delta(t-1)$
3) $x[n] = \{1, 2, 0, 2, 1\}, -3 \leq n \leq 1$ with itself.
4) $x[n] = (\frac{1}{3})^n u[n]$, and $h[n] = u[n+2]$

Solution.

(1)

$$
y(t) = \begin{cases} 0 & t < 0 \\ \frac{1}{2}(1 - e^{-2t}) & 0 \leq t < 1 \\ \frac{e^2-1}{2} e^{-2t} & t \geq 1 \end{cases}
$$

(2)

$$y(t) = -2\sin(\omega t)\sin(\omega)$$

(3)

$$y[n] = \{1, 4, 4, 4, 10, 4, 4, 4, 1\}, -6 \leq n \leq 2.$$

(4)

$$y[n] = \frac{3}{2}(1 - (\frac{1}{3})^{n+3})u[n+2]$$

Problem 3. (10 points) Determine the impulse response h[n] of the overall system.

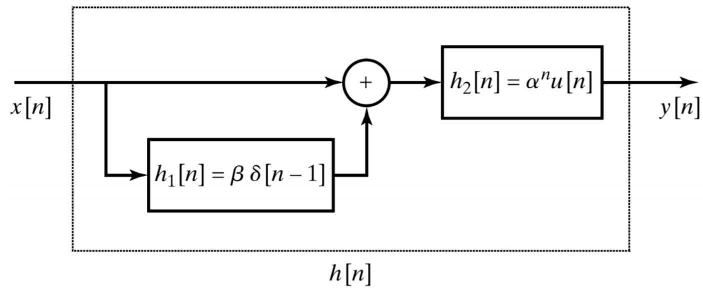

Figure 2

Solution.

From the figure we know that

$$\begin{aligned} y[n] &= (x[n] + x[n] * h_1[n]) * h_2[n] \\ &= (x[n] * (\delta[n] + h_1[n])) * h_2[n] \\ &= x[n] * ((\delta[n] + h_1[n])) * h_2[n]) \end{aligned}$$

For other hand, $y[n] = x[n] * h[n]$. Comparing with two equation, we know that:

$$\begin{aligned} h[n] &= (\delta[n] + h_1[n]) * h_2[n] \\ &= \alpha^n u[n] + \beta\alpha^{n-1} u[n-1] \end{aligned}$$

Problem 4. (20 points)

Given the impulse response of LTI systems. Determine whether each system is causal, whether each system is stable:

1) $h[n] = (\frac{1}{3})^n u[n]$,
2) $h[n] = n(\frac{1}{4})^n u[n-1]$
3) $h(t) = e^{-2t} u(t-1)$.
4) $h(t) = e^{-2|t|}$

Solution.

(1)

causal

$\sum_{n=-\infty}^{\infty} |h[n]| = \frac{3}{2} < \infty$. stable

(2)

causal

$\sum_{n=-\infty}^{\infty} |h[n]| = \frac{4}{9} < \infty$. stable

(3)

causal

$\int_{t=1}^{\infty} |h(t)|dt = \frac{1}{2}e^{-2} < \infty$. stable

(4)

not causal

$\int_{n=-\infty}^{\infty} |h(t)|dt = 1 < \infty$. stable

Problem 5. (30 points)

Draw block diagram representations for causal LTI systems described by the following differential equations, and determine the system output y[n] or y(t).

(a) $y(t) = -\frac{1}{3}\frac{dy(t)}{dt} + 2x(t)$, and $x(t) = 3e^{3t}u(t)$

(b) $y[n] - \frac{1}{3}y[n-1] = x[n]$, and $x[n] = (\frac{1}{2})^n u[n]$

Solution.

(a)

Let particular solution $y_p(t) = Ye^{3t}$, for $t > 0$, we have

$$Ye^{3t} = -Ye^{3t} + 6e^{3t}$$

then $Y = 3$, $y_p(t) = 3e^{3t}$.

Let homogeneous solution $y_h(t) = Ae^{st}$, which should satisfy the equation:

$$\frac{1}{3}\frac{dy(t)}{dt} + y(t) = 0$$

Substituting $y_h(t)$ into equation:

$$\frac{1}{3}Ase^{st} + Ae^{st} = 0$$

then we have $s = -3$, so $y(t) = Ae^{-3t} + 3e^{3t}$. Finally use the initial-rest condition: $y(0) = 0$, so $A = -3$.The answer is:

$$y(t) = (-3e^{-3t} + 3e^{3t})u(t)$$

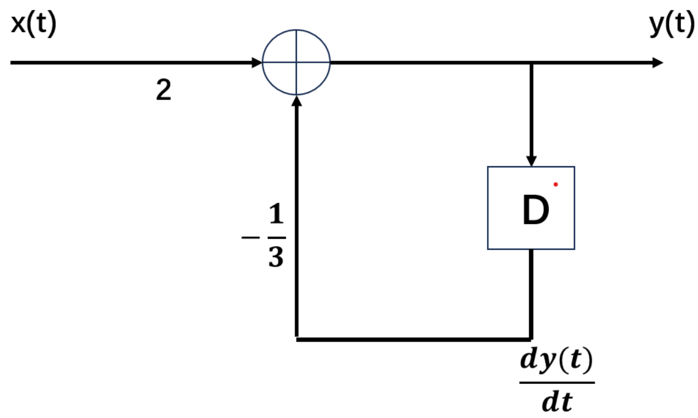

Figure 3: block diagram of (a)

(b)

Let particular solution $y_p[n] = Y(\frac{1}{2})^n$, for $n \geq 0$, we have

$$Y(\frac{1}{2})^n - \frac{1}{3}Y(\frac{1}{2})^{n-1} = (\frac{1}{2})^n$$

then $Y = 3$, $y_p[n] = 3(\frac{1}{2})^n$.

Let homogeneous solution $y_h[n] = A(s)^n$, which should satisfy the equation:

$$y[n] - \frac{1}{3}y[n-1] = 0$$

so $s = \frac{1}{3}$, and $y[n] = A(\frac{1}{3})^n + 3(\frac{1}{2})^n$. Finally use the initial-rest condition: $y[-1] = 0$, then $y[0] = x[0] = 1$, so $A = -2$.The answer is:

$$y[n] = (-2(\frac{1}{3})^n + 3(\frac{1}{2})^n)u[n]$$

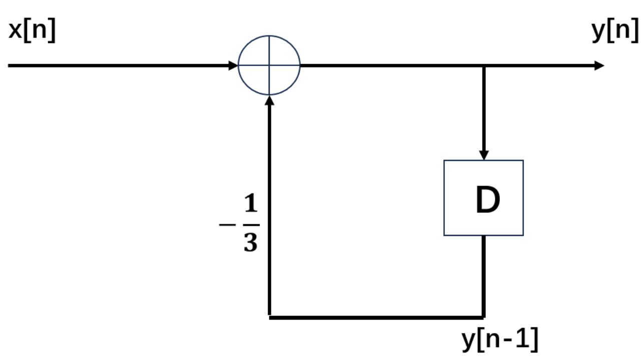

Figure 4: block diagram of (b)