Ch.3 *Fourier Series Representation of Periodic Signals*

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Part II *Fourier Series Representation of Continuous-Time Periodic Signals*

Outline

- Fourier Series Representation of Continuous-Time Periodic Signals
- Convergence of the Fourier Series
- Properties of Continuous-Time Fourier Series

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Fourier Series Representation of Continuous-Time Periodic Signals

- Recall

$$x(t) \longrightarrow \boxed{\text{LTI}} \longrightarrow y(t)$$

$$e^{st} \qquad\qquad\qquad H(s)e^{st}$$

- Complex exponentials are eigenfunctions of a LTI system
- Can we represent $x(t)$ as linear combinations of complex exponentials?

$$x(t) = \sum_k a_k e^{s_k t} \longrightarrow \boxed{\text{LTI}} \longrightarrow y(t) = \sum_k a_k H(s_k) e^{s_k t}$$

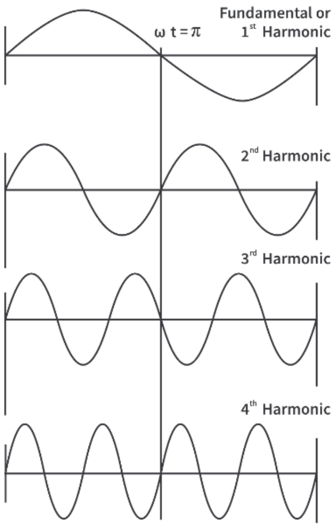
Harmonically Related Complex Exponentials

- Harmonically related complex exponentials (consider e^{st} with s purely imaginary)

$$\emptyset_k(t) = e^{jk\omega_0 t} = e^{jk(2\pi/T_0)t}, k = 0, \pm 1, \pm 2, \ldots$$

- For any $k \neq 0$, the fundamental frequency of $\emptyset_k(t)$ is $|k|\omega_0$; and the fundamental period is

$$\frac{2\pi}{|k|\omega_0} = \frac{T_0}{|k|}$$



Aside: An Orthonormal Set

- Consider the set, $\mathcal{S}$, of $x(t)$ satisfying $x(t) = x(t + T_0)$
- Dot-product (inner-product) defined as

$$< x_1(t), x_2(t) > = \frac{\omega_0}{2\pi} \int_{-\frac{\pi}{\omega_0}}^{\frac{\pi}{\omega_0}} x_1(t) x_2^*(t)\, dt$$

- Consider the set, $\mathcal{B}$, of functions in $\mathcal{S}$

$$\emptyset_k(t) = e^{jk\omega_0 t}; \omega_0 = \frac{2\pi}{T_0}, k \in \mathbb{Z}$$

- Observe that they are orthonormal

$$\frac{\omega_0}{2\pi} \int_{-\frac{\pi}{\omega_0}}^{\frac{\pi}{\omega_0}} e^{jk_1\omega_0 t} e^{-jk_2\omega_0 t} dt = \begin{cases} 0 & k_1 \neq k_2 \\ 1 & k_1 = k_2 \end{cases}$$

Linear Combination of Harmonically Related Complex Exponentials

- The span of the orthonormal functions, $\mathcal{B}$, covers most of $\mathcal{S}$, i.e., span($\mathcal{B}$)

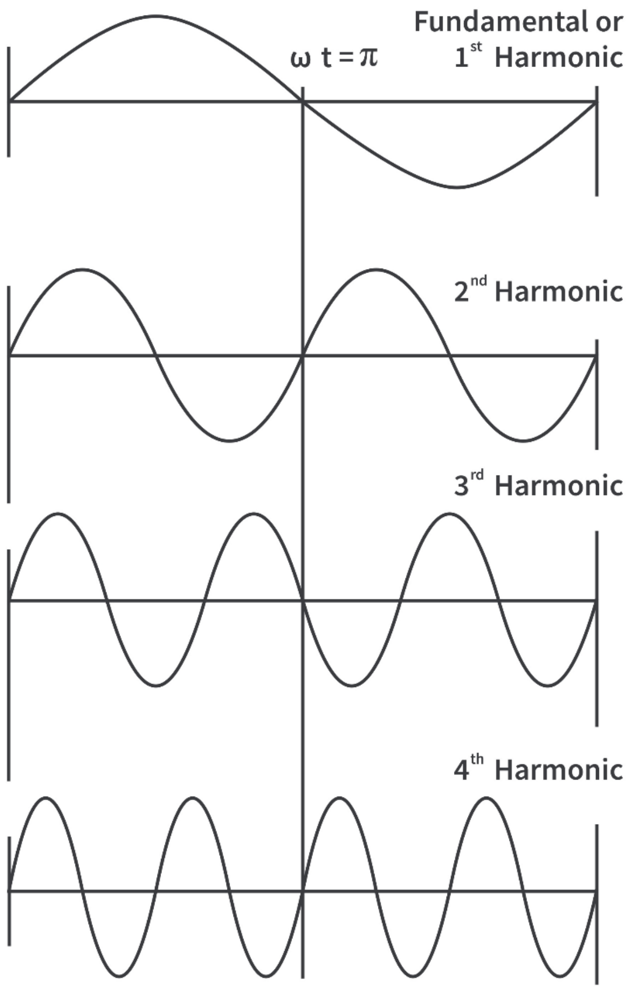
- More precisely, under mild assumptions, $x(t) \in \mathcal{S}$ is a linear combination of $\emptyset_k(t)$, which is also periodic, i.e.,

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T_0)t}$$

- Representation of a periodic signal by linear combination of $\emptyset_k(t)$ is referred to as Fourier Series Representation, ω_0 is the fundamental frequency.

Linear Combination of Harmonically Related Complex Exponentials

- For $a_k e^{jk\omega_0 t}$,
  - $k = 0$: DC component
  - $k = \pm 1$: fundamental (first harmonic) components
  - $k = \pm N$: Nth harmonic components



Linear Combination of Harmonically Related Complex Exponentials

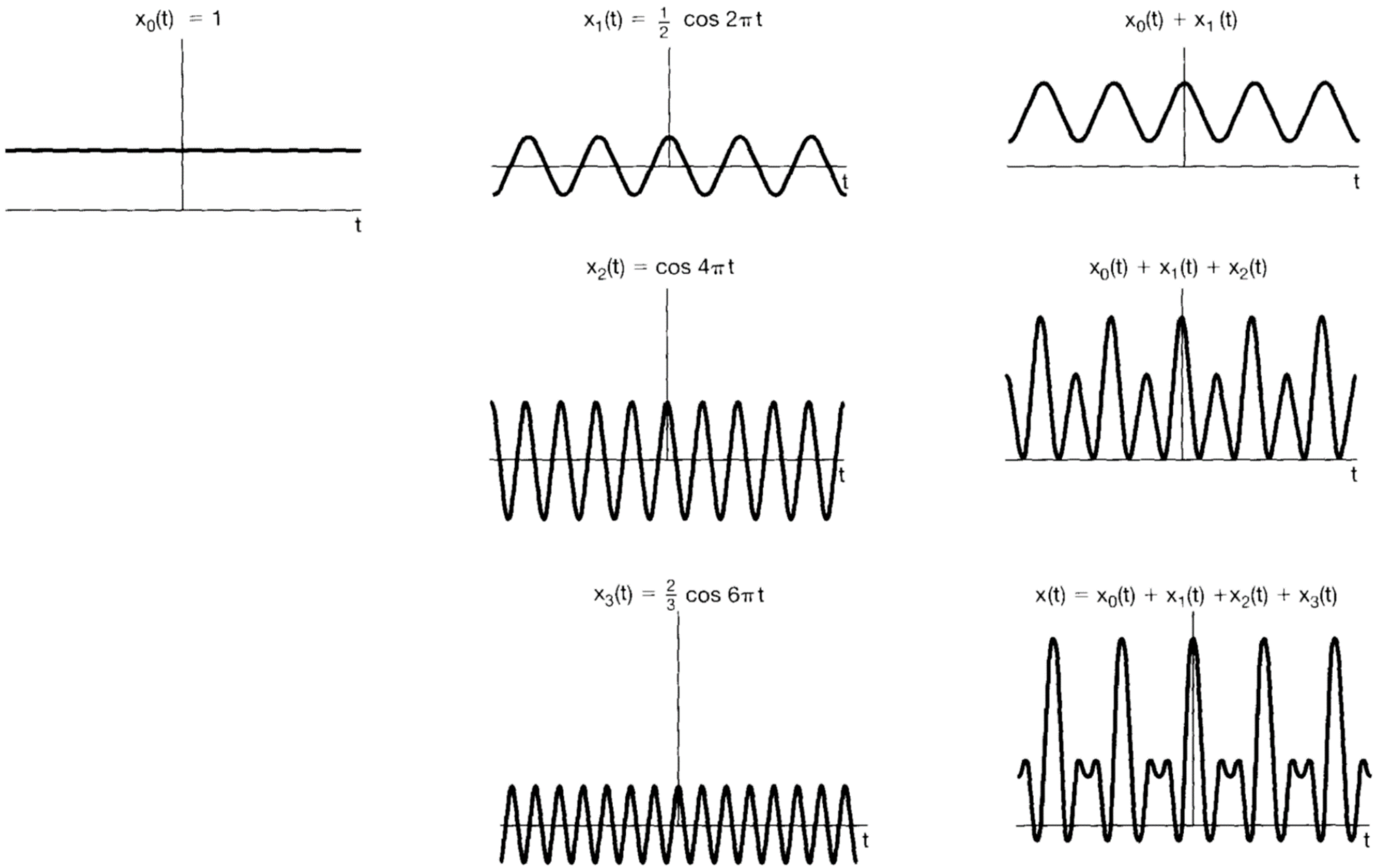
- Example. Consider a periodic signal $x(t) = \sum_{k=-3}^{3} a_k e^{jk2\pi t}$, where $a_0 = 1$, $a_1 = a_{-1} = 1/4$, $a_2 = a_{-2} = 1/2$, $a_3 = a_{-3} = 1/3$.

Collecting each of the harmonic components having the same fundamental frequency:

$$
\begin{aligned}
x(t) &= 1 + \frac{1}{4}\left(e^{j2\pi t} + e^{-j2\pi t}\right) + \frac{1}{2}\left(e^{j4\pi t} + e^{-j4\pi t}\right) + \frac{1}{3}\left(e^{j6\pi t} + e^{-j6\pi t}\right) \\
&= 1 + \frac{1}{2}\cos 2\pi t + \cos 4\pi t + \frac{2}{3}\cos 6\pi t
\end{aligned}
$$

Linear Combination of Harmonically Related Complex Exponentials

- Construction of $x(t) = 1 + \frac{1}{2}\cos 2\pi t + \cos 4\pi t + \frac{2}{3}\cos 6\pi t$:



Linear Combination of Harmonically Related Complex Exponentials

- Real signal

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$

$$x^*(t) = \sum_{k=-\infty}^{\infty} a_k^* e^{-jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_{-k}^* e^{jk\omega_0 t}$$

$x(t)$ is real $\Rightarrow$ $x(t) = x^*(t)$ $\Rightarrow$ $a_k = a_{-k}^*$, or $a_k^* = a_{-k}$ (Conjugate symmetry)

- Alternative form of Fourier Series for real signal

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left[a_k e^{jk\omega_0 t} + a_{-k} e^{-jk\omega_0 t}\right]$$

$$= a_0 + \sum_{k=1}^{\infty} 2\mathcal{R}e\left[a_k e^{jk\omega_0 t}\right]$$

$$a_k = A_k e^{j\theta_k}$$

$$= a_0 + 2\sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k)$$

Determine the Fourier Series Representation

- If $x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$, how to find a_k?

$$x(t)e^{-jn\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}\, e^{-jn\omega_0 t}$$

$$\int_0^T x(t)e^{-jn\omega_0 t}dt = \int_0^T \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}\, e^{-jn\omega_0 t}dt$$

$$= \sum_{k=-\infty}^{\infty} a_k \boxed{\left[\int_0^T e^{j(k-n)\omega_0 t}dt\right]} = \; Ta_n$$

$$\boxed{= \begin{cases} T, k = n \\ 0, k \neq n \end{cases}}$$

$$\therefore a_n = \frac{1}{T}\int_0^T x(t)e^{-jn\omega_0 t}dt \rightarrow a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$$

Fourier Series Pair

- Theorem (for reasonable functions):
  $x(t)$ may be expressed as a Fourier series:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$$ Synthesis equation

- a_k is the Fourier Series Coefficient or spectral coefficient of $x(t)$, which can be obtained by

$$a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} dt$$ Analysis equation

- Note: $e^{-jk\omega_0 t}$, for $k = -\infty$ to ∞, are orthonormal function. (Normal basic signal)

Fourier Series Representation

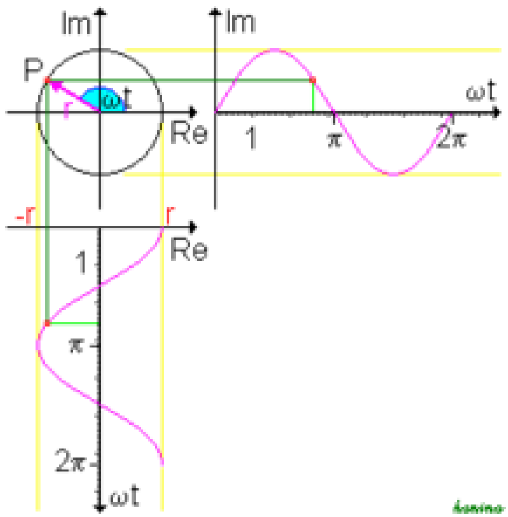
- Example 1: Determine the Fourier Series of $x(t)$

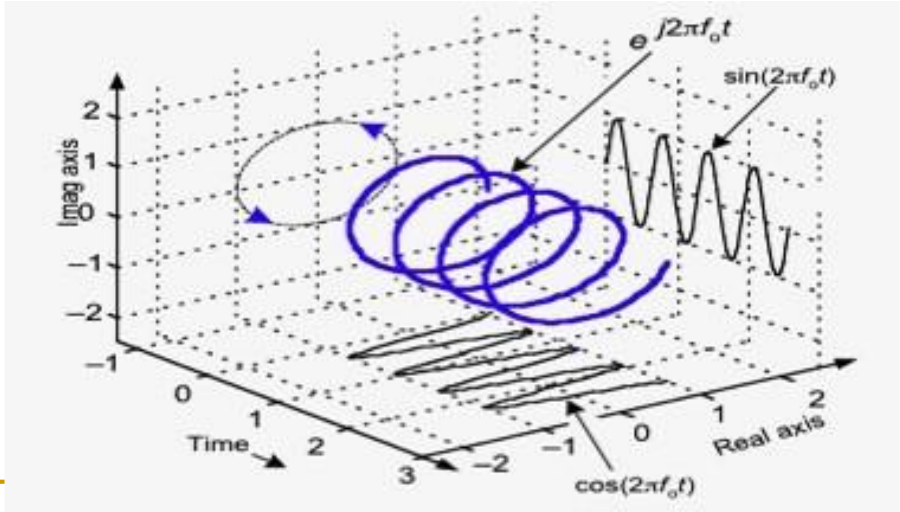
$$x(t) = \cos \omega_0 t$$

Solution:

$$x(t) = \cos \omega_0 t = \frac{1}{2} e^{j\omega_0 t} + \frac{1}{2} e^{-j\omega_0 t}$$

$$\therefore a_1 = a_{-1} = \frac{1}{2}, \quad a_k = 0, \text{ for } k \neq \pm 1$$





Fourier Series Representation

- Example 2: Determine the Fourier Series of $x(t)$

$$x(t) = 1 + \sin\omega_0 t + 2\cos\omega_0 t + \cos\left(2\omega_0 t + \frac{\pi}{4}\right)$$

Solution：

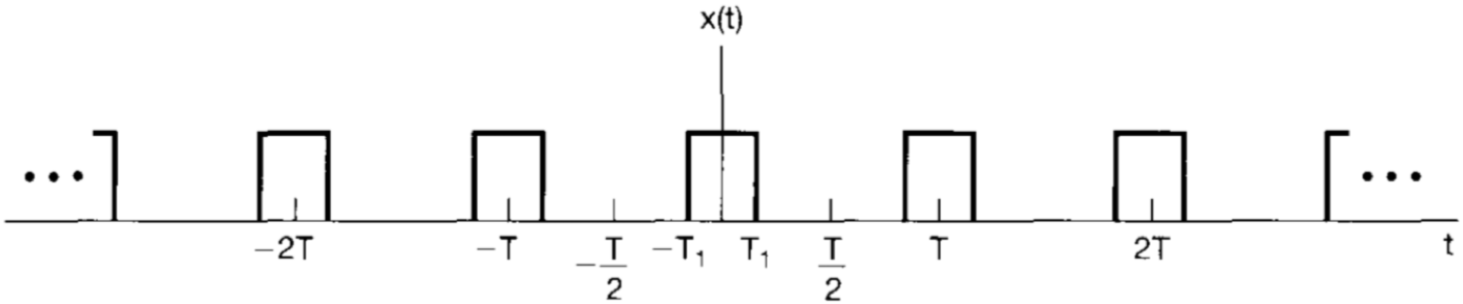
$$x(t) = 1 + \frac{1}{2j}\left[e^{j\omega_0 t} - e^{-j\omega_0 t}\right] + \left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right] + \frac{1}{2}\left(e^{j(2\omega_0 t+\pi/4)} + e^{-j(2\omega_0 t+\pi/4)}\right)$$

$$\therefore x(t) = \underbrace{\boxed{1}}_{a_0} + \underbrace{\boxed{\left(1+\frac{1}{2j}\right)}}_{a_1} e^{j\omega_0 t} + \underbrace{\boxed{\left(1-\frac{1}{2j}\right)}}_{a_{-1}} e^{-j\omega_0 t} + \underbrace{\boxed{\frac{1}{2}e^{j\pi/4}}}_{a_2} e^{j2\omega_0 t} + \underbrace{\boxed{\frac{1}{2}e^{-j\pi/4}}}_{a_{-2}} e^{-j2\omega_0 t}$$

Fourier Series Representation

- Example 3: Determine the Fourier Series of a periodic square wave, the definition over one period is $x(t)$

$$x(t) = \begin{cases} 1, & |t| < T_1, \\ 0, & T_1 < |t| < \dfrac{T}{2} \end{cases}$$



Fourier Series Representation

- Solution of Example 3:
  - For $k = 0$,

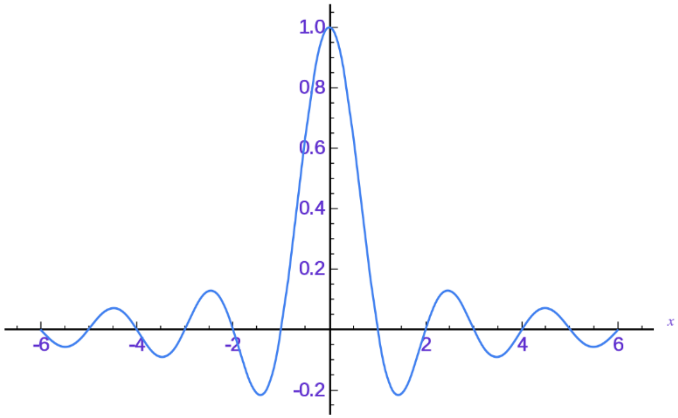
$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} x(t)dt = \frac{1}{T}\int_{-T_1}^{T_1} 1\, dt = \frac{2T_1}{T}$$

  - For $k \neq 0$,

$$sinc(x) = \frac{sin(\pi x)}{\pi x}$$

$$a_k = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t} dt$$

$$= \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T} = \frac{\sin(k\omega_0 T_1)}{k\pi}$$

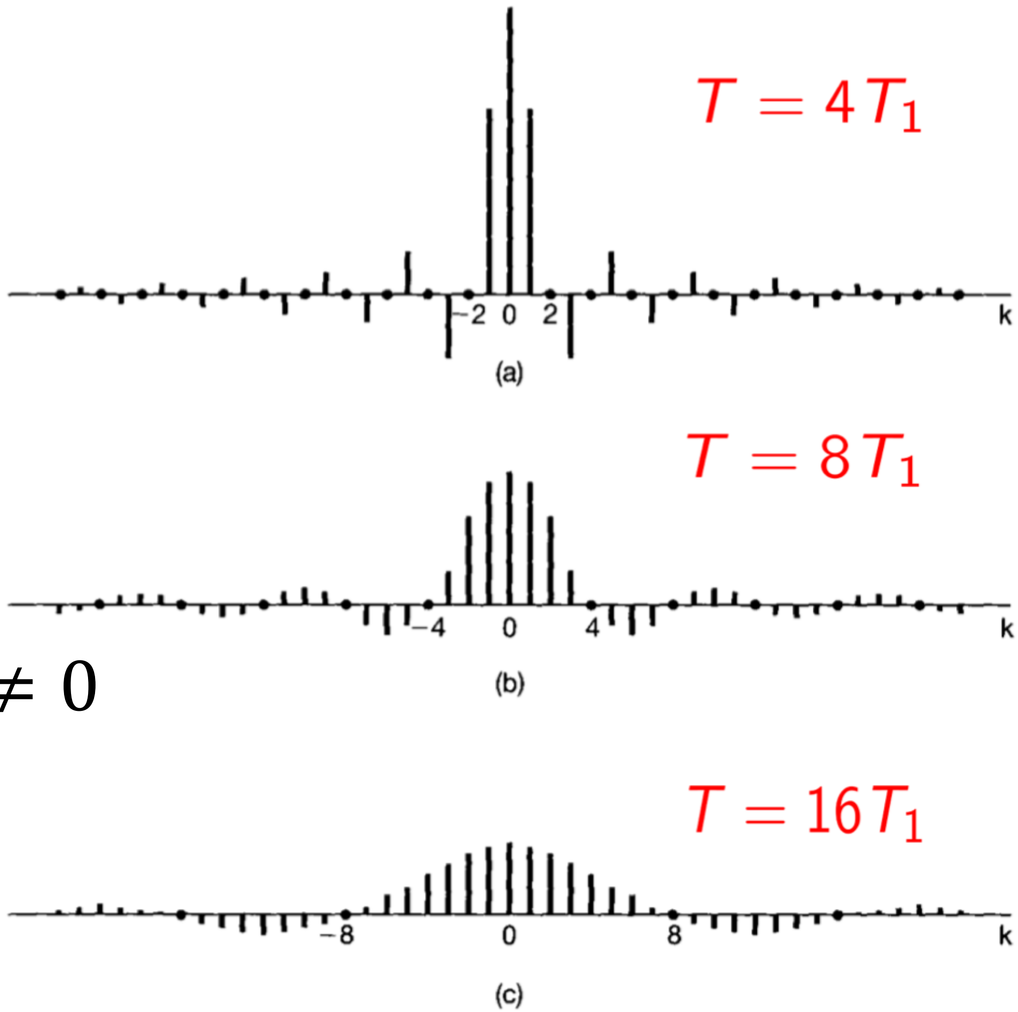


Fourier Series Representation

- Solution of Example 3:

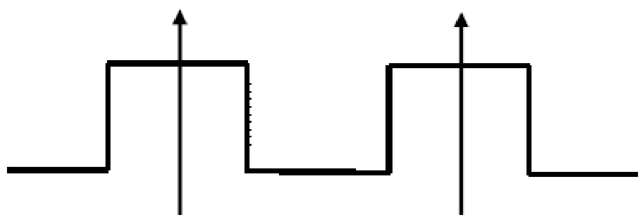
$$a_k = \frac{\sin(k\omega_0 T_1)}{k\pi}$$

$$= \frac{2T_1}{T}\frac{\sin(k\omega_0 T_1)}{k\omega_0 T_1}, k \neq 0$$



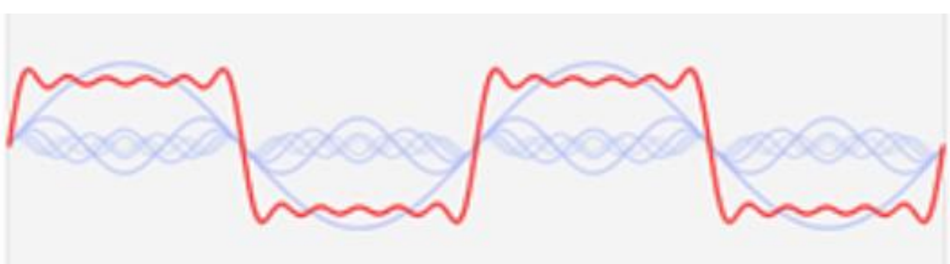
Frequency Domain

- From time-domain to frequency-domain
  - A time-domain graph shows how a signal changes over time, whereas a frequency-domain graph shows how much of the signal lies within each given frequency band over a range of frequencies.



$T = 4T_1$

$$\equiv \frac{1}{2} + \frac{2}{\pi}\cos\omega_0 t - \frac{2}{3\pi}\cos 3\omega_0 t + \cdots$$



Frequency Domain

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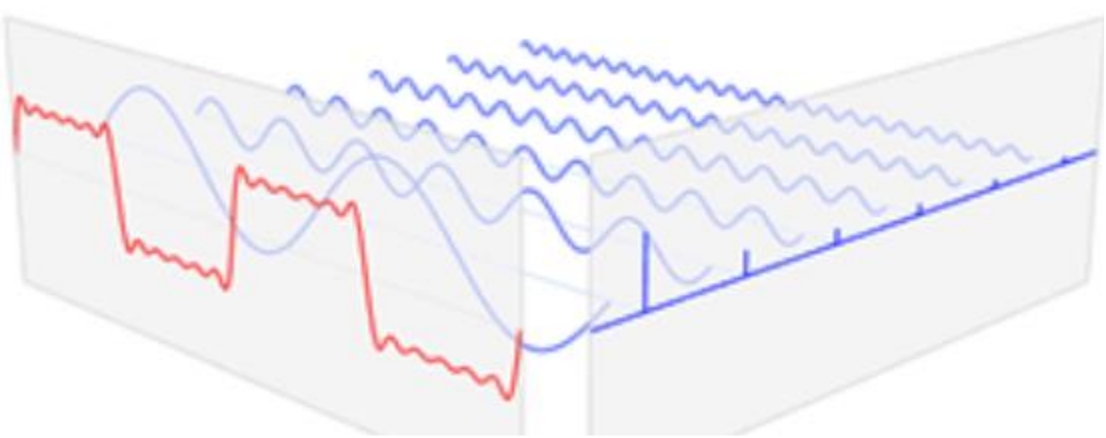
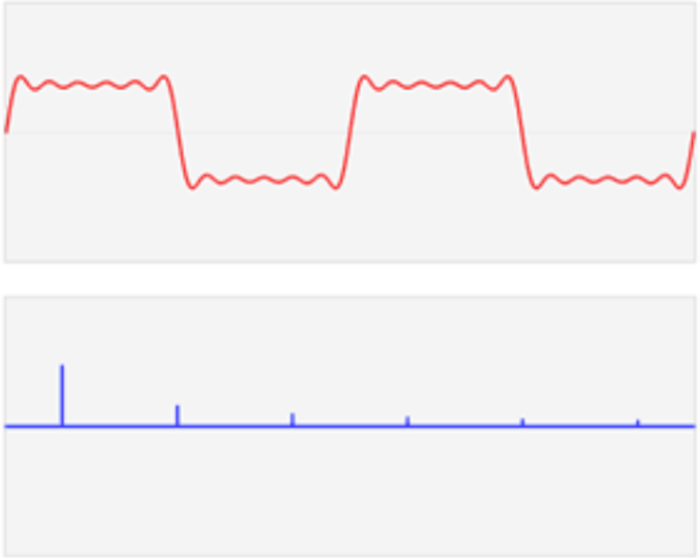




https://zhuanlan.zhihu.com/p/19759362

Frequency Domain

- Advantages of frequency domain

  One of the main reasons for using a frequency-domain representation of a problem is to simplify the mathematical analysis.

  - The output of an LTI system requires a convolution in the time domain, but a simple multiplication in the frequency domain.

  - A frequency domain converts the differential equations to algebraic equations, which are much easier to solve.

  - Frequency-domain analysis can give a better understanding than time domain.

Outline

- Fourier Series Representation of Continuous-Time Periodic Signals
- Convergence of the Fourier Series
- Properties of Continuous-Time Fourier Series

Convergence Problem

- Approximate periodic signal $x(t)$ by

$$x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$$

- How good the approximation is?

$$e_N(t) = x(t) - x_N(t) = x(t) - \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$$

- When $a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} dt$, $E_N = \int_T |e_N(t)|^2 dt$ is minimized.

- Problem:
  - a_k may be infinite
  - $N \to \infty$, $x_N(t)$ may be infinite

Convergence problem!

Two Different Classes of Conditions for Convergence

- **Class 1: Finite energy over a single period**

  If $\int_T |x(t)|^2 dt < \infty$, $x(t)$ can be represented by a FS.

  - Guarantees no energy in their difference; FS is not equal to $x(t)$

Two Different Classes of Conditions for Convergence

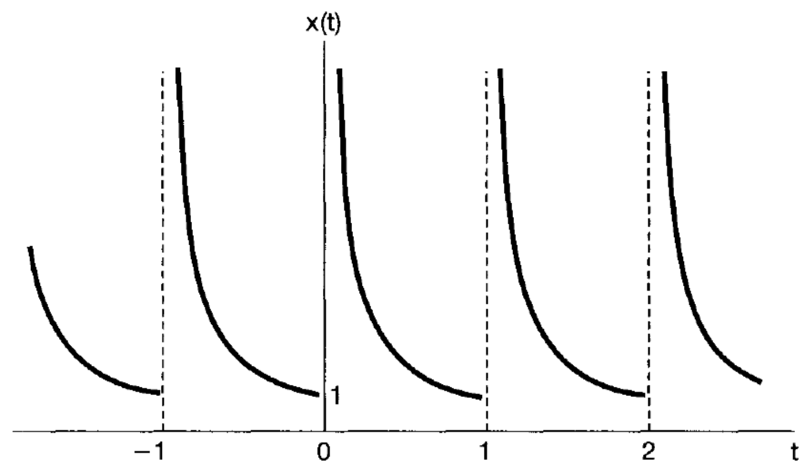
- **Class 2: Dirichlet condition**
  - Condition 1: Over any period, $x(t)$ must be absolutely integrable

$$\int_T |x(t)|dt < \infty$$

An example: a periodic signal

$$x(t) = \frac{1}{t}, 0 < t \leq 1$$

is not absolutely integrable.



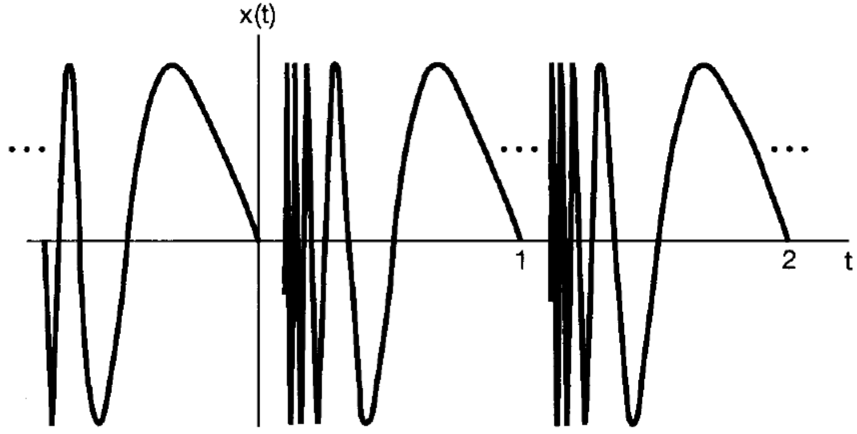
Two Different Classes of Conditions for Convergence

- **Class 2: Dirichlet condition**
  - Condition 2: In any finite interval of time, $x(t)$ is of bounded variation; finite number of maxima and minima in one period.

An example: a periodic signal

$$x(t) = \sin\left(\frac{2\pi}{t}\right), 0 < t \leq 1$$

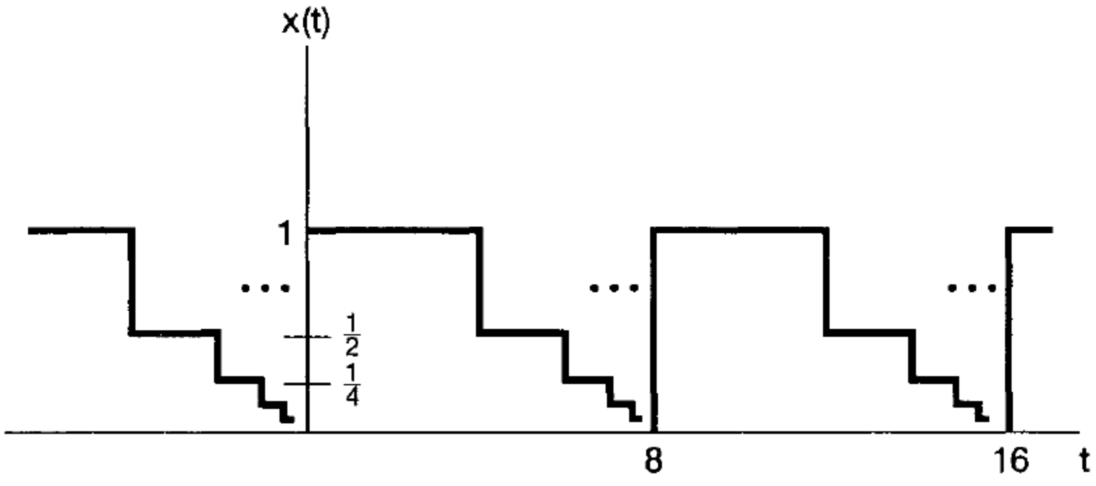
meets Condition 1 but not 2.



Two Different Classes of Conditions for Convergence

- **Class 2: Dirichlet condition**
  - Condition 3: In any finite interval of time, only a finite number of finite discontinuities. Each of these discontinuities is finite.

An example: a periodic signal meets (1) and (2) but not (3).



Two Different Classes of Conditions for Convergence

- **Class 2: Dirichlet condition**

  - Dirichlet condition guarantees $x(t)$ equals its Fourier Series representation, except for discontinuous points.

  - Three examples are pathological in nature and do not typically arise in practical contexts.

Gibbs Phenomenon

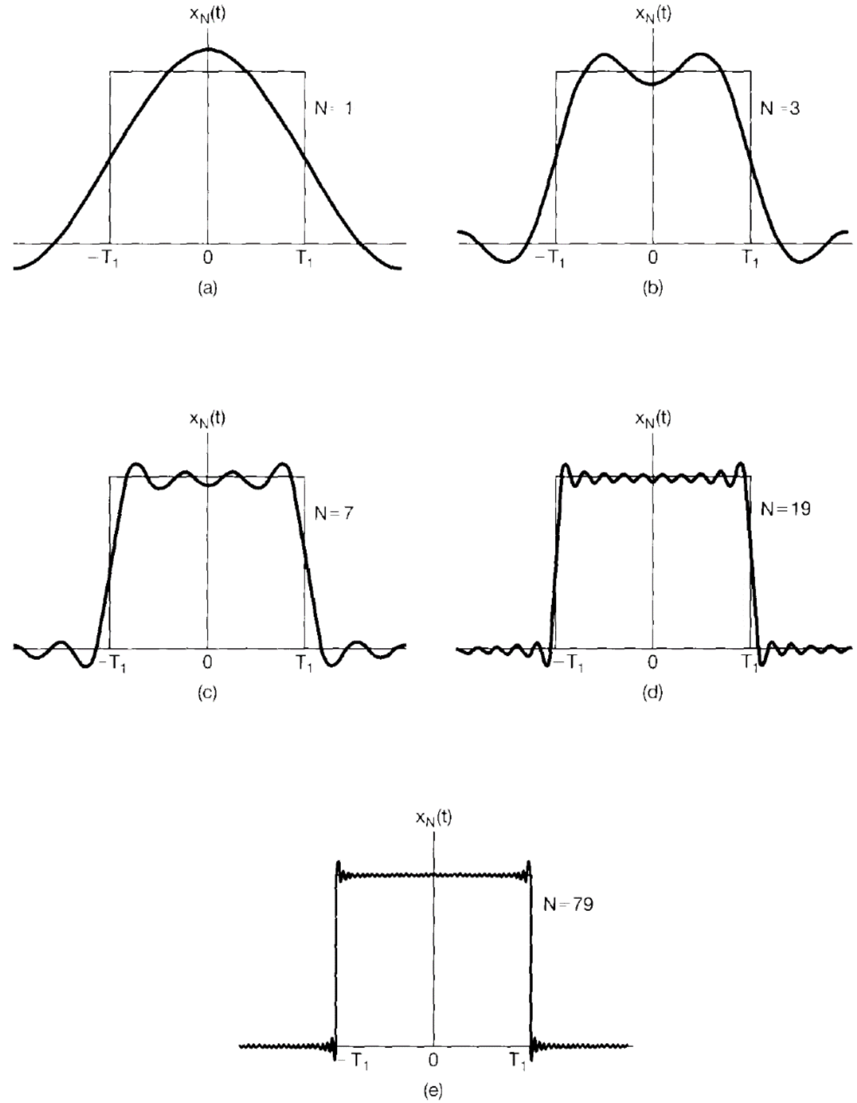
- $x(t)$ is a square wave

$$x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0}$$

$$\lim_{N\to\infty} x_N(t_1) = x(t_1)$$

- Gibbs Phenomenon:

  As N increases, the ripples in the partial sums become compressed toward the discontinuity, but for any finite value of N, the peak amplitude of the ripples remain constant.



Outline

- Fourier Series Representation of Continuous-Time Periodic Signals
- Convergence of the Fourier Series
- Properties of Continuous-Time Fourier Series

Properties of Continuous-Time Fourier Series

- Assume $x(t)$ is periodic with period T and fundamental frequency $\omega_0 = \frac{2\pi}{T}$.

- $x(t)$ and its Fourier-series coefficients a_k are denoted by

$$x(t) \xleftrightarrow{\mathcal{FS}} a_k$$

which signify the paring of a periodic signal with its FS coefficients.

Properties of Continuous-Time Fourier Series

- **Linearity:** if $x(t)$ and $y(t)$ are periodic signals with the same period *T*,

$$x(t) \xleftrightarrow{\mathcal{FS}} a_k \qquad y(t) \xleftrightarrow{\mathcal{FS}} b_k$$

$$z(t) = Ax(t) + By(t) \xleftrightarrow{\mathcal{FS}} c_k = Aa_k + Bb_k$$

  - Proof:

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt, \quad b_k = \frac{1}{T}\int_T y(t)e^{-jk\omega_0 t}dt$$

$$c_k = \frac{1}{T}\int_T z(t)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_T (Ax(t) + By(t))e^{-jk\omega_0 t}dt$$
$$= \frac{A}{T}\int_T x(t)e^{-jk\omega_0 t}dt + \frac{B}{T}\int_T y(t)e^{-jk\omega_0 t}dt = Aa_k + Bb_k$$

Properties of Continuous-Time Fourier Series

- **Time shifting:** if $x(t)$ is a periodic signal with the same period *T*,

$$x(t) \xleftrightarrow{\mathcal{FS}} a_k$$

$$\Downarrow$$

$$x(t-t_0) \xleftrightarrow{\mathcal{FS}} e^{-jk\omega_0 t_0} a_k$$

- Proof:

$$t - t_0 = \tau$$

$$\frac{1}{T}\int_T x(t-t_0)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_T x(\tau)e^{-jk\omega_0(\tau+t_0)}d\tau$$

$$= e^{-jk\omega_0 t_0}\frac{1}{T}\int_T x(\tau)e^{-jk\omega_0 \tau}d\tau$$

$$= e^{-jk\omega_0 t_0}a_k$$

Properties of Continuous-Time Fourier Series

- **Time reversal:** if $x(t)$ is a periodic signal with the same period *T*,

$$x(t) \xleftrightarrow{\mathcal{FS}} a_k$$

$$\Downarrow$$

$$y(t) = x(-t) \xleftrightarrow{\mathcal{FS}} b_k = a_{-k}$$

- Proof:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \Rightarrow x(-t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0(-t)} = \sum_{k=-\infty}^{\infty} a_k e^{j(-k)\omega_0 t}$$

$$= \sum_{m=-\infty}^{\infty} a_{-m} e^{jm\omega_0 t}$$

If $x(t)$ even, $a_{-k} = a_k$, if $x(t)$ odd, $a_{-k} = -a_k$

Properties of Continuous-Time Fourier Series

- **Time scaling:** if $x(t)$ is a periodic signal with the same period *T*,

$$x(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} a_k$$

$$\Downarrow$$

$$y(t) = x(\alpha t) \stackrel{\mathcal{FS}}{\longleftrightarrow} b_k = a_k$$

➢ Proof:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \Rightarrow x(\alpha t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 \alpha t} = \sum_{k=-\infty}^{\infty} a_k e^{jk(\alpha\omega_0)t}$$

FS coefficients the same, but fundamental frequency changed.

Properties of Continuous-Time Fourier Series

- **Multiplication:** if $x(t)$ and $y(t)$ are periodic signals with the same period *T*,

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k \qquad\qquad y(t) \overset{\mathcal{FS}}{\longleftrightarrow} b_k$$

$$z(t) = x(t)y(t) \overset{\mathcal{FS}}{\longleftrightarrow} h_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}$$

> Proof:

$k + n = l$

$$x(t)y(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \sum_{n=-\infty}^{\infty} b_n e^{jn\omega_0 t} = \sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} a_k b_n e^{j(k+n)\omega_0 t}$$

$$= \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} a_k b_{l-k} e^{jl\omega_0 t} = \sum_{l=-\infty}^{\infty} \underbrace{\boxed{\sum_{k=-\infty}^{\infty} a_k b_{l-k}}}_{h_l} e^{jl\omega_0 t} = \sum_{l=-\infty}^{\infty} h_l e^{jl\omega_0 t}$$

Properties of Continuous-Time Fourier Series

- **Conjugation and conjugate symmetry:** if $x(t)$ is a periodic signal with the same period *T*,

$$x(t) \xleftrightarrow{\mathcal{FS}} a_k$$

$$\Downarrow$$

$$z(t) = x^*(t) \xleftrightarrow{\mathcal{FS}} b_k = a^*_{-k}$$

- Proof

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad \therefore x^*(t) = \sum_{k=-\infty}^{\infty} a_k^* e^{-jk\omega_0 t} \overset{-k=m}{=} \sum_{m=-\infty}^{\infty} a^*_{-m} e^{jm\omega_0 t}$$

Properties of Continuous-Time Fourier Series

- **Conjugation and conjugate symmetry:** if $x(t)$ is a periodic signal with the same period *T*,

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k$$

$$\Downarrow$$

$$z(t) = x^*(t) \overset{\mathcal{FS}}{\longleftrightarrow} b_k = a^*_{-k}$$

- If $x(t)$ is real, $a_k^* = a_{-k}$ (conjugate symmetry) $\Rightarrow |a_k| = |a_{-k}|$
- If $x(t)$ is real and even, $(a_{-k} = a_k) \Rightarrow a_k = a_k^* \Rightarrow a_k$ real and even
- If $x(t)$ is real and odd, $(a_{-k} = -a_k) \Rightarrow a_k = -a_k^* \Rightarrow a_k$ pure imagery and odd

Properties of Continuous-Time Fourier Series

- **Differentiation and Integration:** if $x(t)$ is a periodic signal with the same period *T*,

$$x(t) \xleftrightarrow{\mathcal{FS}} a_k$$

$$\Downarrow$$

$$dx(t)/dt \xleftrightarrow{\mathcal{FS}} jk\omega_0 a_k \qquad \int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{\mathcal{FS}} \frac{a_k}{jk\omega_0}$$

- Proof:

$$\frac{dx(t)}{dt} = \sum_{k=-\infty}^{\infty} a_k \frac{d(e^{jk\omega_0 t})}{dt} = \sum_{k=-\infty}^{\infty} a_k jk\omega_0 e^{jk\omega_0 t}$$

$$\int_{-\infty}^{t} x(\tau)d\tau = \sum_{k=-\infty}^{\infty} a_k \int_{-\infty}^{t} e^{jk\omega_0 \tau} d\tau = \sum_{k=-\infty}^{\infty} \frac{a_k}{jk\omega_0} e^{jk\omega_0 t}$$

Properties of Continuous-Time Fourier Series

- **Frequency shifting:** if $x(t)$ is a periodic signal with the same period *T*,

$$x(t) \xleftrightarrow{\mathcal{FS}} a_k$$

$$\Downarrow$$

$$e^{jM\omega_0 t}x(t) \xleftrightarrow{\mathcal{FS}} a_{k-M}$$

➤ Proof:

$$e^{jM\omega_0 t}x(t) = e^{jM\omega_0 t}\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{j(k+M)\omega_0 t}$$

$$\overset{k+M=l}{=} \sum_{l=-\infty}^{\infty} a_{l-M} e^{jl\omega_0 t}$$

Properties of Continuous-Time Fourier Series

- **Periodic convolution:** if $x(t)$ and $y(t)$ are periodic signals with the same period *T*,

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k \qquad y(t) \overset{\mathcal{FS}}{\longleftrightarrow} b_k$$

$$\int_T x(\tau)y(t-\tau)d\tau \overset{\mathcal{FS}}{\longleftrightarrow} Ta_k b_k$$

- Proof:

$$\int_T x(\tau)y(t-\tau)d\tau = \int_T \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0\tau} \sum_{n=-\infty}^{\infty} b_n e^{jn\omega_0(t-\tau)}\, d\tau$$

$$= \int_T \sum_{k=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} a_k e^{jk\omega_0\tau} b_n e^{-jn\omega_0\tau} e^{jn\omega_0 t}\, d\tau$$

$$= \sum_{k=-\infty}^{\infty} a_k \sum_{n=-\infty}^{\infty} e^{jn\omega_0 t} b_n \overset{T\delta[k-n]}{\boxed{\int_T e^{jk\omega_0\tau} e^{-jn\omega_0\tau} d\tau}} = \sum_{k=-\infty}^{\infty} Ta_k b_k e^{jk\omega_0 t}$$

Properties of Continuous-Time Fourier Series

- **Parseval's relation**

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

$$\frac{1}{T}\int_T |a_k e^{jk\omega_0 t}|^2 dt = \frac{1}{T}\int_T |a_k|^2 dt = |a_k|^2$$

- $|a_k|^2$ is the average power in the kth harmonic component of $x(t)$
- Total average power in $x(t)$ equals the sum of the average powers in all of its harmonic components

Properties of Continuous-Time Fourier Series

- **Parseval's relation**

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

- Proof:

$$\frac{1}{T}\int_T |x(t)|^2 dt = \frac{1}{T}\int_T x(t)x^*(t)dt = \frac{1}{T}\int_T x(t)\sum_{k=-\infty}^{\infty} a_k^* e^{-jk\omega_0 t}\, dt$$

$$= \sum_{k=-\infty}^{\infty} a_k^* \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \sum_{k=-\infty}^{\infty} a_k^* a_k = \sum_{k=-\infty}^{\infty} |a_k|^2$$

Properties of Continuous-Time Fourier Series

- **Summary**

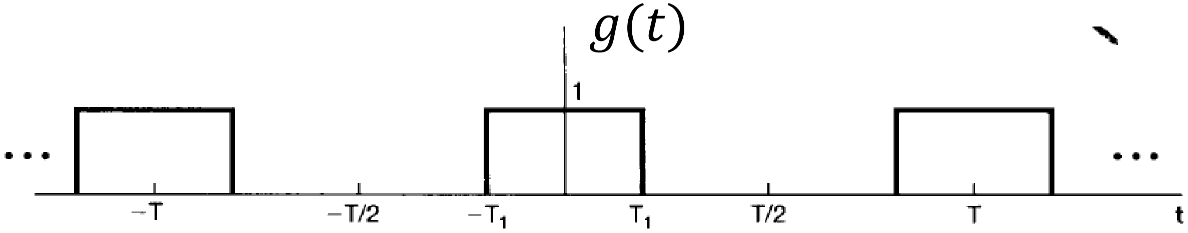
Property	Section	Periodic Signal	Fourier Series Coefficients
		$\left.\begin{array}{l} x(t) \\ y(t) \end{array}\right\}$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k b_k
Linearity	3.5.1	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	3.5.2	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting		$e^{jM\omega_0 t} = e^{jM(2\pi/T)t} x(t)$	a_{k-M}
Conjugation	3.5.6	$x^*(t)$	a^*_{-k}
Time Reversal	3.5.3	$x(-t)$	a_{-k}
Time Scaling	3.5.4	$x(\alpha t), \alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution		$\int_T x(\tau) y(t-\tau) d\tau$	$T a_k b_k$
Multiplication	3.5.5	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation		$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T} a_k$
Integration		$\int_{-\infty}^{t} x(t)\,dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right) a_k = \left(\frac{1}{jk(2\pi/T)}\right) a_k$
Conjugate Symmetry for Real Signals	3.5.6	$x(t)$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{R}e\{a_k\} = \mathcal{R}e\{a_{-k}\} \\ \mathcal{I}m\{a_k\} = -\mathcal{I}m\{a_{-k}\} \\ \lvert a_k \rvert = \lvert a_{-k} \rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	3.5.6	$x(t)$ real and even	a_k real and even
Real and Odd Signals	3.5.6	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals		$\begin{cases} x_e(t) = \mathcal{E}v\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = \mathcal{O}d\{x(t)\} & [x(t) \text{ real}] \end{cases}$	$\mathcal{R}e\{a_k\}$ $j\mathcal{I}m\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

Properties of Continuous-Time Fourier Series

- Example 1: Determine the Fourier Series of $g(t)$

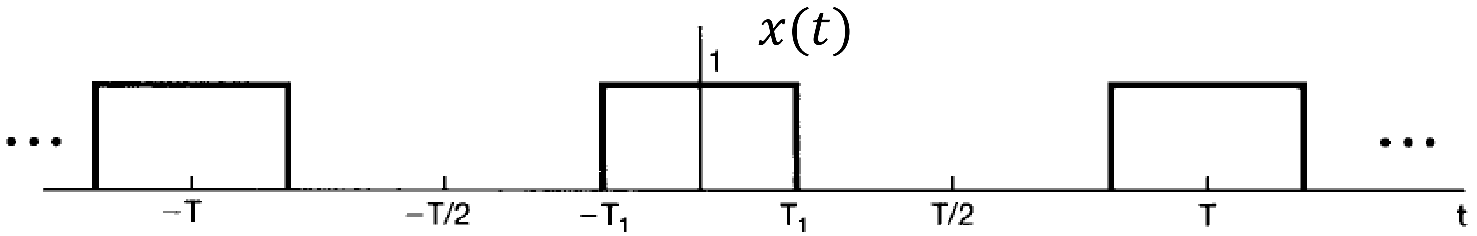


Properties of Continuous-Time Fourier Series

Example 2: Determine the Fourier Series of

$$g(t) = x(t-1) - 1/2$$

when $T_1 = 1, T = 4$.



Properties of Continuous-Time Fourier Series

- Example 3: Given a signal $x(t)$ with the following facts, determine $x(t)$
  1. $x(t)$ is real;
  2. $x(t)$ is periodic with T=4 and it has Fourier series coefficient a_k
  3. FS coefficients $a_k = 0$ for $|\mathrm{k}| > 1$;
  4. A signal with FS coefficients $b_k = e^{-j\pi k/2} a_{-k}$ is odd;
  5. $\frac{1}{4}\int_4 |x(t)|^2 dt = \frac{1}{2}$.

Summary

- Fourier Series Representation of Continuous-Time Periodic Signals
- Convergence of the Fourier Series
- Properties of Continuous-Time Fourier Series
- Reference in textbook:
  - 3.3, 3.4, 3.5