Fourier Series Representation of Periodic signals (ch.3)

- ❑ The response of LTI systems to complex exponentials
- ❑ Fourier series representation of continuous periodic signals
- ❑ Convergence of the Fourier series
- ❑ Properties of continuous-time Fourier series
- ❑ Fourier series representation of discrete –time periodic signals
- ❑ Properties of discrete FS
- ❑ Fourier series and LTI systems

The response of LTI systems to complex exponentials

<u>Recall Chapter 2</u>

- Objective: characterization of a LTI system

$$x(t) \longrightarrow \boxed{\text{LTI}} \longrightarrow y(t)$$

- $x(t)$ is considered as linear combinations of a basis signal $\delta(t)$

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\, d\tau \quad \rightarrow \quad y(t) = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)\, d\tau$$

- $\delta(t)$ is not the only one. In general, a basic signal should satisfy
  - It can be used to construct a broad and useful class of signals
  - The response of an LTI system to the basic signal is simple

The response of LTI systems to complex exponentials

Continuous-time

$$e^{st} \longrightarrow \boxed{\text{LTI}} \longrightarrow y(t) = ?$$

$$y(t) = \int_{-\infty}^{\infty} h(\tau) e^{s(t-\tau)} d\tau = e^{st} \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau$$

Let $\int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau = H(s) \quad \rightarrow \quad y(t) = H(s) e^{st}$

- e^{st} is an eigenfunction of the system
- For a specific value s, $H(s)$ is the corresponding eigenvalue

The response of LTI systems to complex exponentials

Continuous-time

$$e^{st} \longrightarrow \boxed{\text{LTI}} \longrightarrow \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau \, e^{st} = H(s)e^{st}$$

If $x(t) = a_1e^{s_1t} + a_2e^{s_2t} + a_3e^{s_3t}$ $\qquad y(t) = ?$

$$y(t) = a_1H(s_1)e^{s_1t} + a_2H(s_2)e^{s_2t} + a_1H(s_3)e^{s_3t}$$

Generally, if $x(t) = \sum_k a_k e^{s_kt}$

$$y(t) = \sum_k a_k H(s_k)e^{s_kt}$$

The response of LTI systems to complex exponentials

Discrete-time

$$z^n \longrightarrow \boxed{\text{LTI}} \longrightarrow y[n] = ?$$

$$y[n] = \sum_{k=-\infty}^{\infty} h[k] z^{n-k} = z^n \sum_{k=-\infty}^{\infty} h[k] z^{-k}$$

Let $$H[z] = \sum_{k=-\infty}^{\infty} h[k] z^{-k} \quad \rightarrow y[n] = H[z] z^n$$

- z^n is an eigenfunction of the system
- For a specific value z, $H[z]$ is the corresponding eigenvalue

The response of LTI systems to complex exponentials

Discrete-time

$$z^n \longrightarrow \boxed{\text{LTI}} \longrightarrow \sum_{k=-\infty}^{\infty} h[k] z^{-k} \, z^n = H[z] z^n$$

If $\quad x[n] = \sum_k a_k Z_k^n$

$$y[n] = \sum_k a_k H(z_k) Z_k^n$$

The response of LTI systems to complex exponentials

Examples

For a LTI system $y(t) = x(t-3)$, determine $H(s)$

Solution 1:

let $x(t) = e^{st}$, $y(t) = e^{s(t-3)} = e^{-3s}e^{st}$

$\therefore H(s) = e^{-3s}$

Solution 2:

$$H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau = \int_{-\infty}^{\infty} \delta(\tau-3)e^{-s\tau}d\tau = e^{-3s}$$

The response of LTI systems to complex exponentials

Examples

For a LTI system $y(t) = x(t-3)$

If $x(t) = \cos(4t) + \cos(7t)$, $y(t) = ?$

Solution 1: $y(t) = \cos\big(4(t-3)\big) + \cos\big(7(t-3)\big)$

Solution 2: $x(t) = \frac{1}{2}e^{j4t} + \frac{1}{2}e^{-j4t} + \frac{1}{2}e^{j7t} + \frac{1}{2}e^{-j7t}$

$$y(t) = \frac{1}{2}H(j4)e^{j4t} + \frac{1}{2}H(-j4)e^{-j4t} + \frac{1}{2}H(j7)e^{j7t} + \frac{1}{2}H(-j7)e^{-j7t}$$

$H(s) = e^{-3s}$

$$= \frac{1}{2}e^{-j12}e^{j4t} + \frac{1}{2}e^{j12}e^{-j4t} + \frac{1}{2}e^{-j21}e^{j7t} + \frac{1}{2}e^{j21}e^{-j7t}$$

$$= \frac{1}{2}e^{j4(t-3)} + \frac{1}{2}e^{-j4(t-3)} + \frac{1}{2}e^{j7(t-3)} + \frac{1}{2}e^{-j7(t-3)}$$

Fourier Series Representation of Periodic signals (ch.3)

- ❑ The response of LTI systems to complex exponentials
- ❑ Fourier series representation of continuous periodic signals
- ❑ Convergence of the Fourier series
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- ❑ Properties of discrete
- ❑ FS Fourier series and LTI systems

Fourier series representation of C-T periodic signals

Recall

$$x(t) \longrightarrow \boxed{\text{LTI}} \longrightarrow y(t)$$

$$e^{st} \qquad\qquad\qquad\qquad H(s)e^{st}$$

- Decompose $x(t)$ into linear combinations of basis signals, which should satisfy
  - It can be used to construct a broad and useful class of signals
  - The response of an LTI system to the basic signal is simple

- Complex exponentials are eigenfunctions of a LTI system

- Can we represent $x(t)$ as linear combinations of complex exponentials?

Fourier series representation of C-T periodic signals

Linear combination of harmonically related complex exponentials

- Harmonically related complex exponentials (consider e^{st} with s purely imaginary)

$$\emptyset_k(t) = e^{jk\omega_0 t} = e^{jk(2\pi/T)t},\ k = 0, \pm 1, \pm 2, \ldots$$

For any $k \neq 0$, fundamental frequency $|k|\omega_0$; fundamental period $\frac{2\pi}{|k|\omega_0} = \frac{T}{|k|}$

- Linear combination of $\emptyset_k(t)$ is also periodic

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{jk(2\pi/T)t}$$

- Representation of a periodic signal by Linear combination of $\emptyset_k(t)$ is referred to as Fourier Series representation, ω_0 is the fundamental frequency

- For $a_k e^{jk\omega_0 t}$, $k = 0$: DC component; $k = \pm 1$: fundamental (first harmonic) components; $k = \pm N$: Nth harmonic components

Fourier series representation of C-T periodic signals

Linear combination of harmonically related complex exponentials

- An example

If $x(t) = \sum_{k=-3}^{3} a_k e^{jk2\pi t}$

And $a_0 = 1$, $a_1 = a_{-1} = 1/4$, $a_2 = a_{-2} = 1/2$, $a_3 = a_{-3} = 1/3$

$$x(t) = 1 + \frac{1}{4}\left(e^{j2\pi t} + e^{-j2\pi t}\right) + \frac{1}{2}\left(e^{j4\pi t} + e^{-j4\pi t}\right) + \frac{1}{3}\left(e^{j6\pi t} + e^{-j6\pi t}\right)$$

$$= 1 + \frac{1}{2}\cos 2\pi t + \cos 4\pi t + \frac{2}{3}\cos 6\pi t$$

Fourier series representation of C-T periodic signals

Linear combination of harmonically related complex exponentials

- Real signal

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \qquad x^*(t) = \sum_{k=-\infty}^{\infty} a_k^* e^{-jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_{-k}^* e^{jk\omega_0 t}$$

Real $\Rightarrow$ $x(t) = x^*(t)$ $\Rightarrow$ $a_k = a_{-k}^*$, or $a_k^* = a_{-k}$ （Conjugate symmetry）

- Alternative form of Fourier Series for real signal

$$x(t) = a_0 + \sum_{k=1}^{\infty} \left[a_k e^{jk\omega_0 t} + a_{-k} e^{-jk\omega_0 t}\right]$$

$$= a_0 + \sum_{k=1}^{\infty} 2\mathcal{R}e\left[a_k e^{jk\omega_0 t}\right] = a_0 + 2\sum_{k=1}^{\infty} A_k \cos(k\omega_0 t + \theta_k)$$

$$a_k = A_k e^{j\theta_k}$$

Fourier series representation of C-T periodic signals

Determine the Fourier Series Representation

$$\int_0^T x(t)e^{-jn\omega_0 t}dt = \int_0^T \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}\, e^{-jn\omega_0 t}dt$$

$$= \sum_{k=-\infty}^{\infty} a_k \boxed{\left[\int_0^T e^{j(k-n)\omega_0 t}dt\right]} = Ta_n$$

$$= \begin{cases} T, k = n \\ 0, k \neq n \end{cases} = T\delta[k-n]$$

$$\therefore a_n = \frac{1}{T}\int_0^T x(t)e^{-jn\omega_0 t}dt$$

$$a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$$

Fourier series representation of C-T periodic signals

Fourier Series pair

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \qquad \text{Synthesis equation}$$

$$a_k = \frac{1}{T}\int_T x(t) e^{-jk\omega_0 t} dt \qquad \text{Analysis equation}$$

- a_k: Fourier Series coefficients or spectral coefficients of $x(t)$

$$a_0 = \frac{1}{T}\int_T x(t) dt$$

Fourier series representation of C-T periodic signals

Determine the Fourier Series Representation

- Examples: determine the FS coefficients of $x(t)$

$$x(t) = \sin \omega_0 t$$

$$\sin \omega_0 t = \frac{1}{2j} e^{j\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t}$$

$$\therefore a_1 = \frac{1}{2j} \qquad a_{-1} = -\frac{1}{2j} \qquad a_k = 0, \text{ for } k \neq \pm 1$$

Fourier series representation of C-T periodic signals

Determine the Fourier Series Representation

- Examples: determine the FS coefficients of $x(t)$

$$x(t) = 1 + \sin \omega_0 t + 2\cos \omega_0 t + \cos\left(2\omega_0 t + \frac{\pi}{4}\right)$$

$$x(t) = 1 + \frac{1}{2j}\left[e^{j\omega_0 t} - e^{-j\omega_0 t}\right] + \left[e^{j\omega_0 t} + e^{-j\omega_0 t}\right] + \frac{1}{2}\left(e^{j(2\omega_0 t+\pi/4)} + e^{-j(2\omega_0 t+\pi/4)}\right)$$

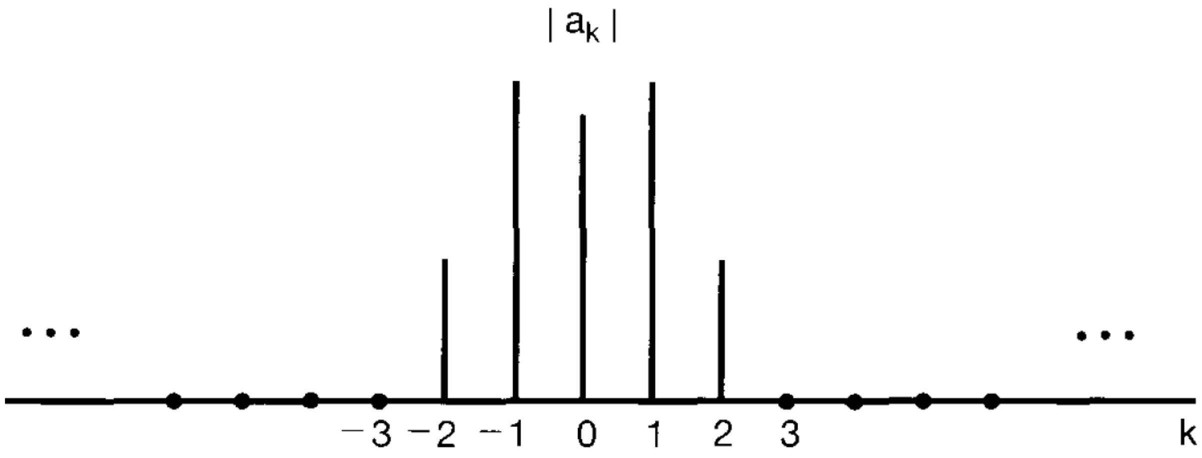
$$\therefore x(t) = \underbrace{\boxed{1}}_{a_0} + \underbrace{\boxed{\left(1+\frac{1}{2j}\right)}}_{a_1} e^{j\omega_0 t} + \underbrace{\boxed{\left(1-\frac{1}{2j}\right)}}_{a_{-1}} e^{-j\omega_0 t} + \underbrace{\boxed{\frac{1}{2}e^{j\pi/4}}}_{a_2} e^{j2\omega_0 t} + \underbrace{\boxed{\frac{1}{2}e^{-j\pi/4}}}_{a_{-2}} e^{-j2\omega_0 t}$$

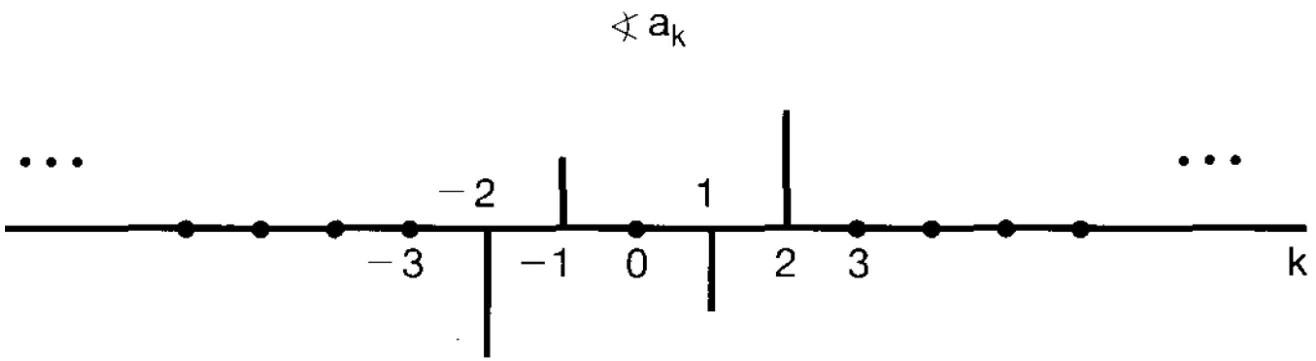
Fourier series representation of C-T periodic signals

Determine the Fourier Series Representation

- Examples: determine the FS coefficients of $x(t)$

$$x(t) = \underbrace{\boxed{1}}_{a_0} + \underbrace{\boxed{\left(1+\frac{1}{2j}\right)}}_{a_1} e^{j\omega_0 t} + \underbrace{\boxed{\left(1-\frac{1}{2j}\right)}}_{a_{-1}} e^{-j\omega_0 t} + \underbrace{\boxed{\frac{1}{2}e^{j\pi/4}}}_{a_2} e^{j2\omega_0 t} + \underbrace{\boxed{\frac{1}{2}e^{-j\pi/4}}}_{a_{-2}} e^{-j2\omega_0 t}$$

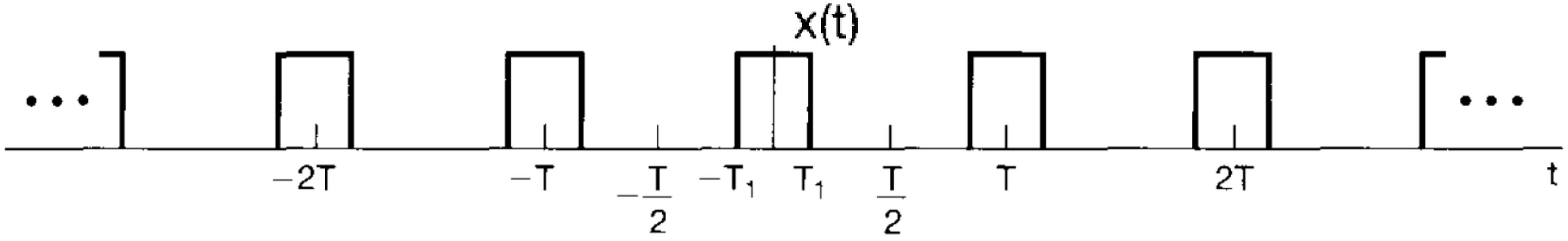




Fourier series representation of C-T periodic signals

Determine the Fourier Series Representation

- Examples: determine the FS coefficients of $x(t)$



$$x(t) = \begin{cases} 1, & |t| < T_1 \\ 0, & T_1 < |t| < T/2 \end{cases}$$

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2} x(t)dt = \frac{1}{T}\int_{-T_1}^{T_1} 1\,dt = \frac{2T_1}{T}$$

$$a_k = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jk\omega_0 t}dt = -\frac{1}{jk\omega_0 T}e^{-jk\omega_0 t}\Big|_{-T_1}^{T_1} = \frac{2}{k\omega_0 T}\left[\frac{e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}}{2j}\right]$$

$$= \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T} = \frac{\sin(k\omega_0 T_1)}{k\pi} = \frac{2T_1}{T}\boxed{\frac{\sin(k\omega_0 T_1)}{k\omega_0 T_1}}, k \neq 0$$

$$\mathrm{sinc}(x) = \frac{\sin(x)}{x}$$

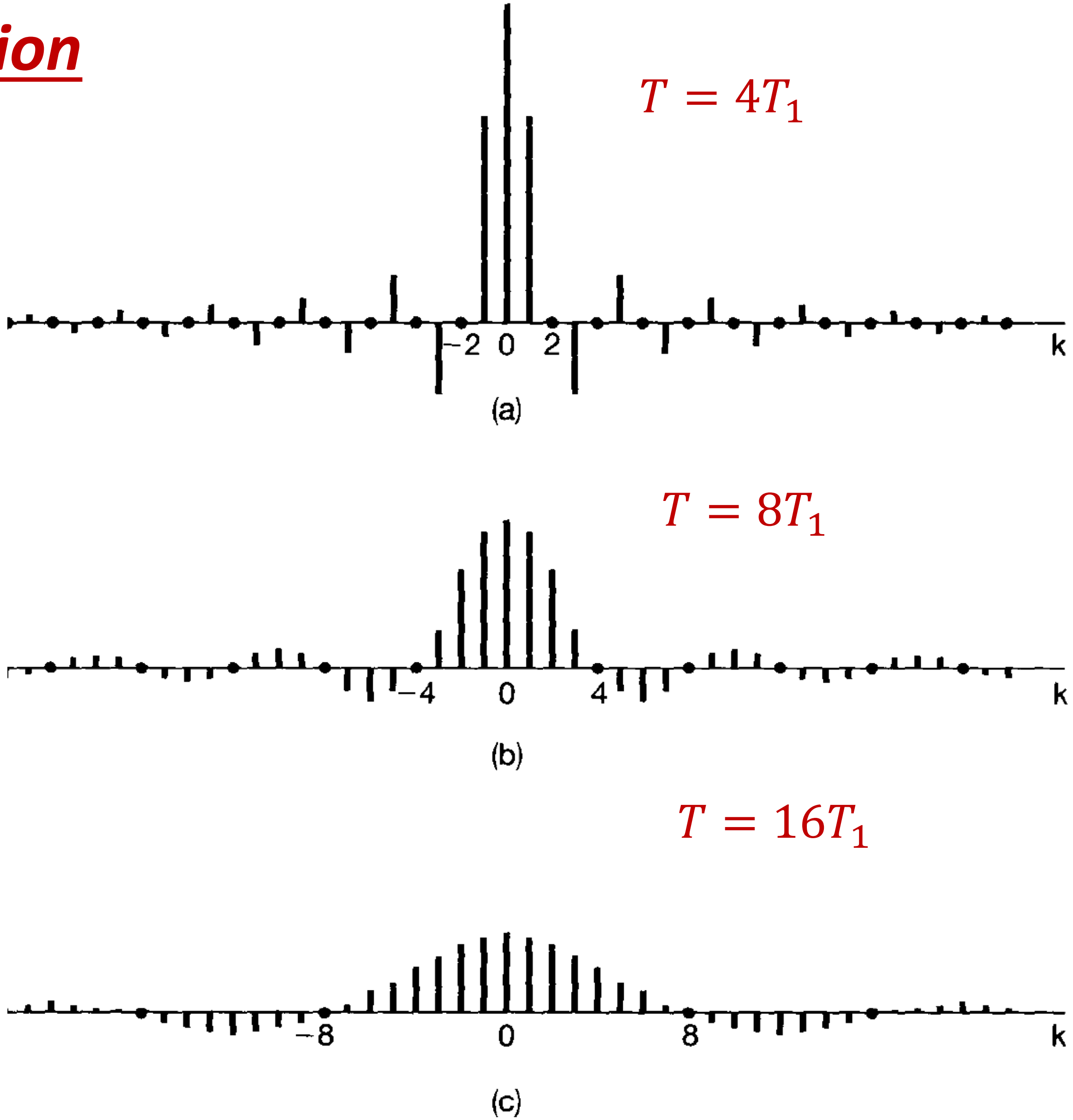
Fourier series representation of C-T periodic signals

Determine the Fourier Series Representation

- Examples: determine the FS coefficients of $x(t)$

$$a_k = \frac{2\sin(k\omega_0 T_1)}{k\omega_0 T} = \frac{\sin(k\omega_0 T_1)}{k\pi}$$

$$= \frac{2T_1}{T}\frac{\sin(k\omega_0 T_1)}{k\omega_0 T_1}, k \neq 0$$



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Convergence of the Fourier series

History

- ❑ Using "trigonometric sum" to describe periodic signal can be tracked back to Babylonians who predicted astronomical events similarly.
- ❑ L. Euler (in 1748) and Bernoulli (in 1753) used the "normal mode" concept to describe the motion of a vibrating string; though JL Lagrange strongly criticized this concept.
- ❑ Fourier (in 1807) had found series of harmonically related sinusoids to be useful to describe the temperature distribution through body, and he claimed "any" periodic signal can be represented by such series.
- ❑ Dirichlet (in 1829) provide a precise condition under which a periodic signal can be represented by a Fourier series.

Jean Baptiste Joseph Fourier
March 21 1768 - May 16 1830
Born Auxerre, France. Died Paris, France.

Convergence of the Fourier series

Convergence problem

- Approximate periodic signal $x(t)$ by $x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$
- How good the approximation is?

$$e_N(t) = x(t) - x_N(t) = x(t) - \sum_{k=-N}^{N} a_k e^{jk\omega_0 t} \qquad E_N = \int_T |e_N(t)|^2 dt$$

  - When $a_k = \frac{1}{T}\int_T x(t)e^{-jk\omega_0 t}dt$, E_N is minimized
  - If $x(t)$ has a FS, then $N \to \infty \Rightarrow E_N \to 0$
- Problem:
  - a_k may be infinite
  - Even a_k is finite, when $N \to \infty$, $x_N(t)$ may not converge to $x(t)$

Convergence problem!

Convergence of the Fourier series

Two different classes of conditions

- Condition 1: Finite energy condition

  If $\int_T |x(t)|^2 dt < \infty$, $x(t)$ can be represented by a FS

  - Guarantees no energy in their difference; FS is not equal to $x(t)$

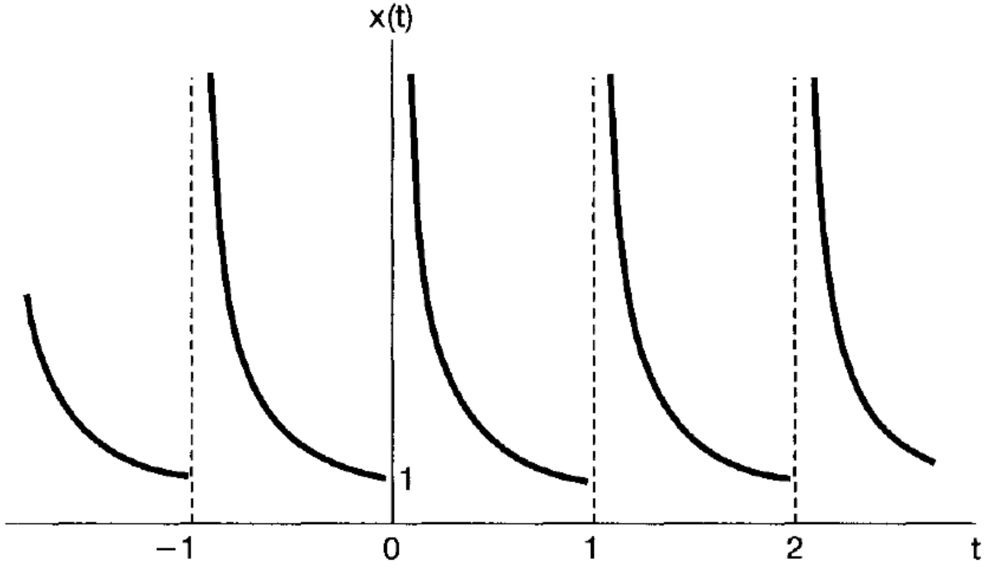
- Condition 2: Dirichlet condition

  (1) Absolutely integrable $\int_T |x(t)| dt < \infty$

  An example: a periodic signal

  $$x(t) = \frac{1}{t}, 0 < t \leq 1$$

  is not absolutely integrable.



Convergence of the Fourier series

Two different classes of conditions

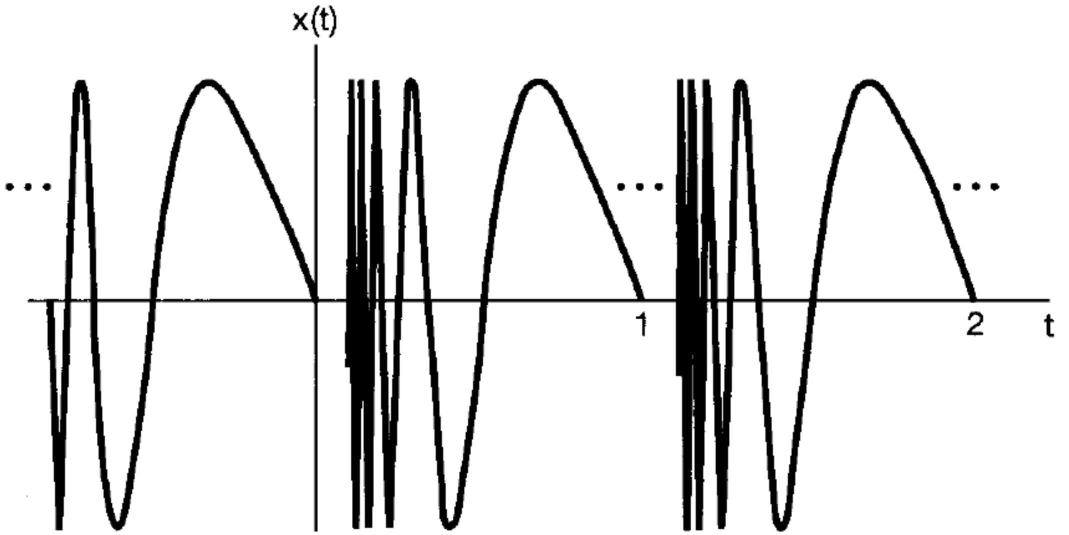
- Condition 2: Dirichlet condition

(2) In any finite interval of time, x(t) is of bounded variation; finite maxima and minima in one period

An example: a periodic signal

$$x(t) = \sin\left(\frac{2\pi}{t}\right), 0 < t \leq 1$$

meets (1) but not (2).



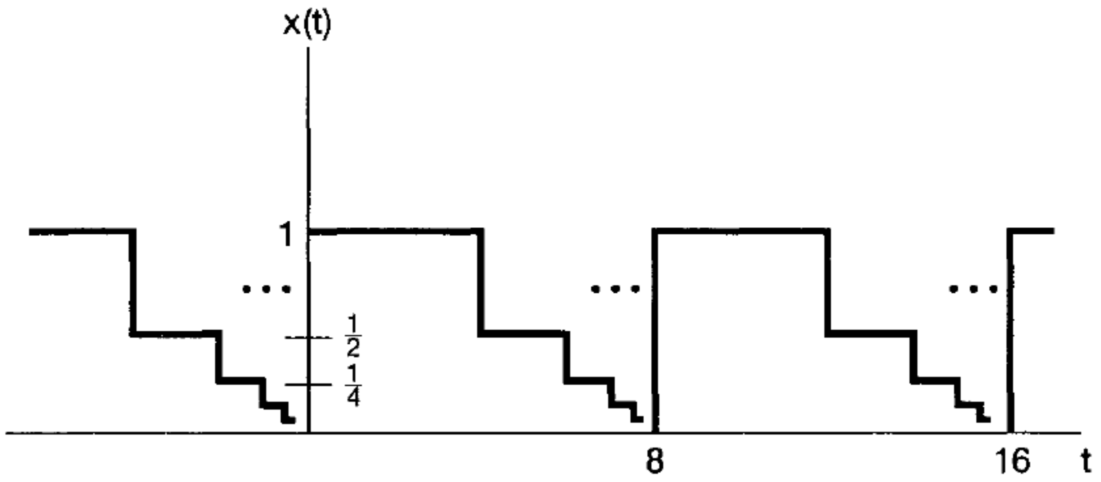
Convergence of the Fourier series

Two different classes of conditions

- Condition 2: Dirichlet condition

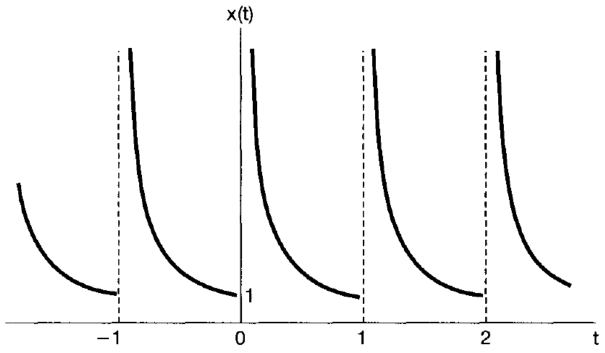
(3) In any finite interval of time, only a finite number of finite discontinuities

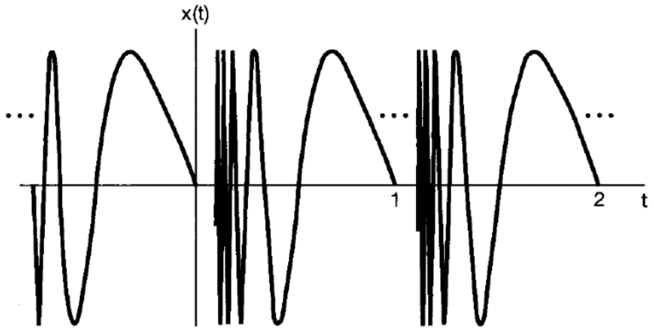
An example: a periodic signal meets (1) and (2) but not (3).

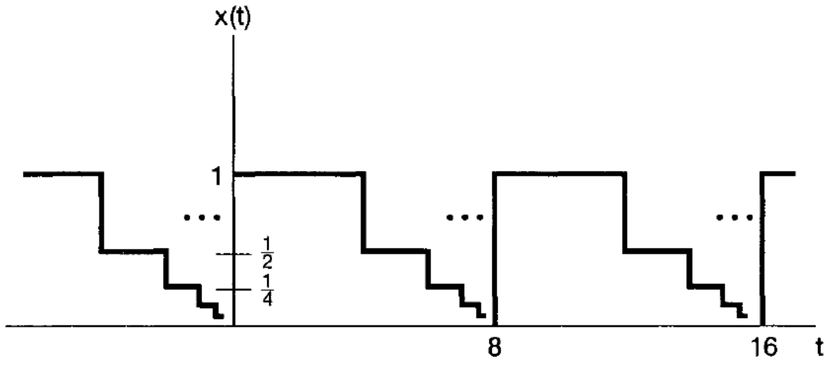


Two different classes of conditions

- Condition 2: Dirichlet condition
  - Three examples are pathological in nature and do not typically arise in practical contexts.







  - For a periodic signal with no discontinuities, FS converge and equals $x(t)$; For a periodic signal with finite discontinuities in one period, FS equals $x(t)$ except for the discontinuities.

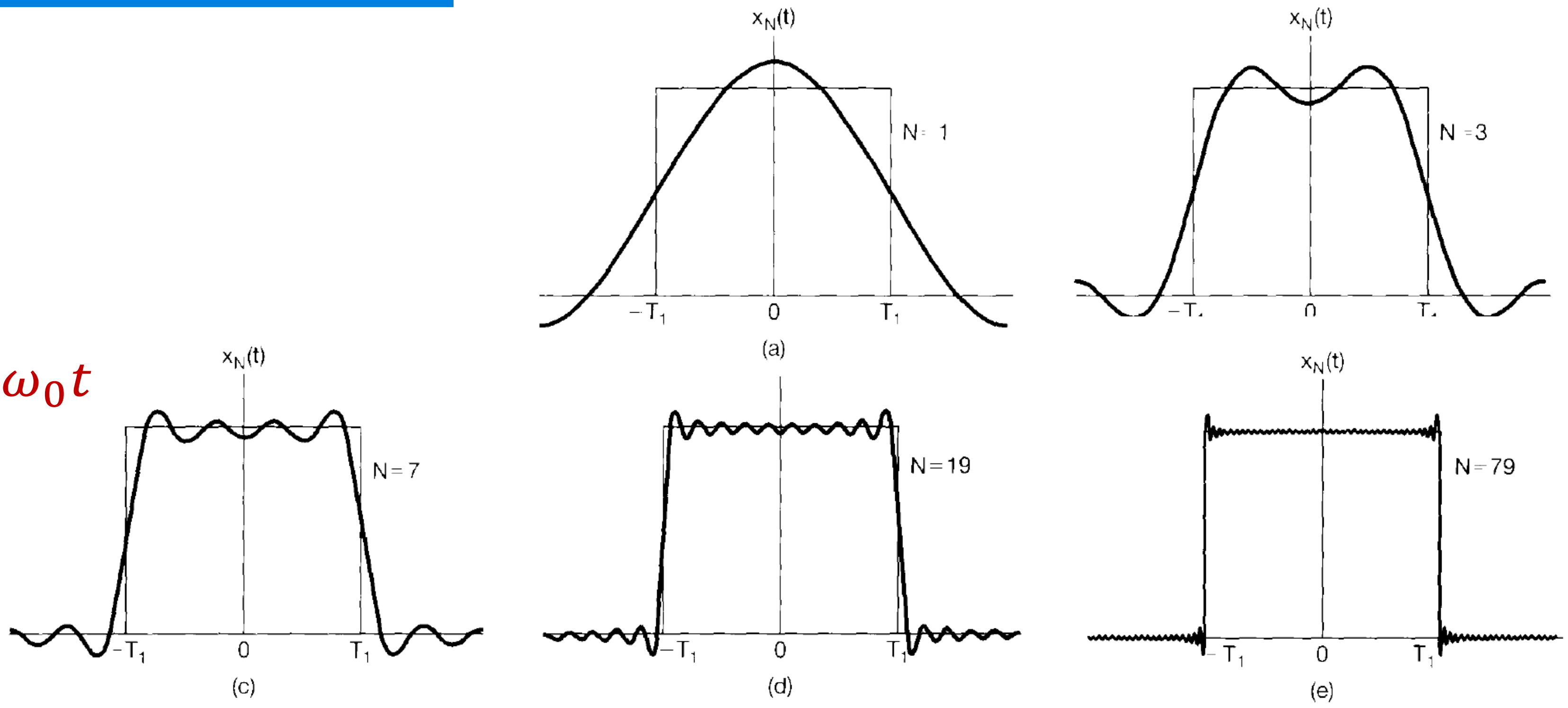
Convergence of the Fourier series

Example for discontinuity

- ❑ $x(t)$ is a square wave

$$x_N(t) = \sum_{k=-N}^{N} a_k e^{jk\omega_0 t}$$

$$\lim_{N\to\infty} x_N(t_1) = x(t_1)$$



- At the discontinuity, FS equals to the average of either side of $x(t)$.
- Ripples at the point (t_1) near the discontinuity (t).
- N increase, amplitude of ripples not change, but t_1 more close to t.

Gibbs phenomenon

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Properties of continuous-time FS

- Use the notation

$$x(t) \xleftrightarrow{\mathcal{FS}} a_k$$

to signify the paring of a periodic signal with its FS coefficients.

- Linearity: if $x(t)$ and $y(t)$ are periodic signals with the same period T

$$\begin{aligned} x(t) &\xleftrightarrow{\mathcal{FS}} a_k \\ y(t) &\xleftrightarrow{\mathcal{FS}} b_k \end{aligned} \quad \Rightarrow \quad z(t) = Ax(t) + By(t) \xleftrightarrow{\mathcal{FS}} c_k = Aa_k + Bb_k$$

Properties of continuous-time FS

- Time shifting

$$x(t) \xleftrightarrow{\mathcal{FS}} a_k \quad \Rightarrow \quad x(t-t_0) \xleftrightarrow{\mathcal{FS}} e^{-jk\omega_0 t_0} a_k$$

- Proof

$$t - t_0 = \tau$$

$$\frac{1}{T}\int_T x(t-t_0)e^{-jk\omega_0 t}dt = \frac{1}{T}\int_T x(\tau)e^{-jk\omega_0(\tau+t_0)}d\tau$$

$$= e^{-jk\omega_0 t_0}\frac{1}{T}\int_T x(\tau)e^{-jk\omega_0 \tau}d\tau$$

$$= e^{-jk\omega_0 t_0} a_k$$

Properties of continuous-time FS

- Time reversal

$$x(t) \xleftrightarrow{\mathcal{FS}} a_k \quad \Rightarrow \quad y(t) = x(-t) \xleftrightarrow{\mathcal{FS}} b_k = a_{-k}$$

- Proof

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \Rightarrow x(-t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0(-t)} = \sum_{k=-\infty}^{\infty} a_k e^{j(-k)\omega_0 t}$$

$$= \sum_{m=-\infty}^{\infty} a_{-m} e^{jm\omega_0 t}$$

- If $x(t)$ even, $a_{-k} = a_k$, if $x(t)$ odd, $a_{-k} = -a_k$

Properties of continuous-time FS

- Time scaling

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k \Rightarrow \quad y(t) = x(\alpha t) \overset{\mathcal{FS}}{\longleftrightarrow} b_k = a_k$$

- Proof

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \Rightarrow x(\alpha t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 \alpha t} = \sum_{k=-\infty}^{\infty} a_k e^{jk(\alpha\omega_0)t}$$

FS coefficients the same, but fundamental frequency changed.

Properties of continuous-time FS

- Multiplication

$$\left.\begin{array}{l} x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k \\ y(t) \overset{\mathcal{FS}}{\longleftrightarrow} b_k \end{array}\right. \Rightarrow z(t) = x(t)y(t) \overset{\mathcal{FS}}{\longleftrightarrow} h_k = \sum_{l=-\infty}^{\infty} a_l b_{k-l}$$

- Proof

$l + m = k$

$$x(t)y(t) = \sum_{l=-\infty}^{\infty} a_l e^{jl\omega_0 t} \sum_{m=-\infty}^{\infty} b_m e^{jm\omega_0 t} = \sum_{l=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} a_l b_m e^{j(l+m)\omega_0 t}$$

$$= \sum_{l=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} a_l b_{k-l} e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} \underbrace{\boxed{\sum_{l=-\infty}^{\infty} a_l b_{k-l}}}_{h_k} e^{jk\omega_0 t}$$

$$= \sum_{k=-\infty}^{\infty} h_k e^{jk\omega_0 t}$$

Properties of continuous-time FS

- Conjugation and conjugate symmetry

$$x(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} a_k \quad \Rightarrow \quad z(t) = x^*(t) \stackrel{\mathcal{FS}}{\longleftrightarrow} b_k = a^*_{-k}$$

- Proof

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \quad \therefore x^*(t) = \sum_{k=-\infty}^{\infty} a_k^* e^{-jk\omega_0 t} \overset{-k=m}{=} \sum_{m=-\infty}^{\infty} a^*_{-m} e^{jm\omega_0 t}$$

- If $x(t)$ real, $a_k^* = a_{-k}$ (conjugate symmetry) $\Rightarrow |a_k| = |a_{-k}|$
  - $x(t)$ real and even $(a_{-k} = a_k) \Rightarrow a_k = a_k^* \Rightarrow a_k$ real and even
  - $x(t)$ real and odd $(a_{-k} = -a_k) \Rightarrow a_k = -a_k^* \Rightarrow a_k$ pure imagery and odd
  - $a_0 =?$

Properties of continuous-time FS

- Differentiation and Integration

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k \quad \Rightarrow \begin{cases} dx(t)/dt \overset{\mathcal{FS}}{\longleftrightarrow} jk\omega_0 a_k \\ \int_{-\infty}^{t} x(\tau)d\tau \overset{\mathcal{FS}}{\longleftrightarrow} a_k/(jk\omega_0) \end{cases}$$

- Proof

$$\frac{dx(t)}{dt} = \sum_{k=-\infty}^{\infty} a_k \frac{d(e^{jk\omega_0 t})}{dt} = \sum_{k=-\infty}^{\infty} a_k jk\omega_0 e^{jk\omega_0 t}$$

$$\int_{-\infty}^{t} x(\tau)d\tau = \sum_{k=-\infty}^{\infty} a_k \int_{-\infty}^{t} e^{jk\omega_0 \tau} d\tau = \sum_{k=-\infty}^{\infty} a_k/(jk\omega_0) e^{jk\omega_0 t}$$

Properties of continuous-time FS

- Frequency shifting

$$x(t) \xleftrightarrow{\mathcal{FS}} a_k \quad \Rightarrow \quad e^{jM\omega_0 t}x(t) \xleftrightarrow{\mathcal{FS}} a_{k-M}$$

- Proof

$$e^{jM\omega_0 t}x(t) = e^{jM\omega_0 t}\sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} = \sum_{k=-\infty}^{\infty} a_k e^{j(k+M)\omega_0 t}$$

$$k + M = l \qquad = \sum_{l=-\infty}^{\infty} a_{l-M} e^{jl\omega_0 t}$$

Properties of continuous-time FS

- Periodic convolution

$$\begin{aligned} x(t) &\overset{\mathcal{FS}}{\longleftrightarrow} a_k \\ y(t) &\overset{\mathcal{FS}}{\longleftrightarrow} b_k \end{aligned} \Rightarrow \int_T x(\tau)y(t-\tau)d\tau \overset{\mathcal{FS}}{\longleftrightarrow} Ta_k b_k$$

- Proof

$$\int_T x(\tau)y(t-\tau)d\tau = \int_T \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0\tau} \sum_{m=-\infty}^{\infty} b_m e^{jm\omega_0(t-\tau)}\, d\tau$$

$$= \int_T \sum_{k=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} a_k e^{jk\omega_0\tau} b_m e^{-jm\omega_0\tau} e^{jm\omega_0 t}\, d\tau$$

$$= \sum_{k=-\infty}^{\infty} a_k \sum_{m=-\infty}^{\infty} e^{jm\omega_0 t} b_m \overbrace{\boxed{\int_T e^{jk\omega_0\tau} e^{-jm\omega_0\tau} d\tau}}^{T\delta[k-m]} = \sum_{k=-\infty}^{\infty} Ta_k b_k e^{jk\omega_0 t}$$

Properties of continuous-time FS

- Parseval's relation

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

- Proof

$$\frac{1}{T}\int_T |x(t)|^2 dt = \frac{1}{T}\int_T x(t)x^*(t)dt = \frac{1}{T}\int_T x(t) \sum_{k=-\infty}^{\infty} a_k^* e^{-jk\omega_0 t}\, dt$$

$$= \sum_{k=-\infty}^{\infty} a_k^* \frac{1}{T}\int_T x(t)\, e^{-jk\omega_0 t}\, dt$$

$$= \sum_{k=-\infty}^{\infty} a_k^* a_k = \sum_{k=-\infty}^{\infty} |a_k|^2$$

Properties of continuous-time FS

- Parseval's relation

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2$$

$$\frac{1}{T}\int_T |a_k e^{jk\omega_0 t}|^2 dt = \frac{1}{T}\int_T |a_k|^2 dt = |a_k|^2$$

- $|a_k|^2$ is the average power in the k-th harmonic component of $x(t)$

- Total average power in $x(t)$ equals the sum of the average powers in all of its harmonic components

Properties of cor

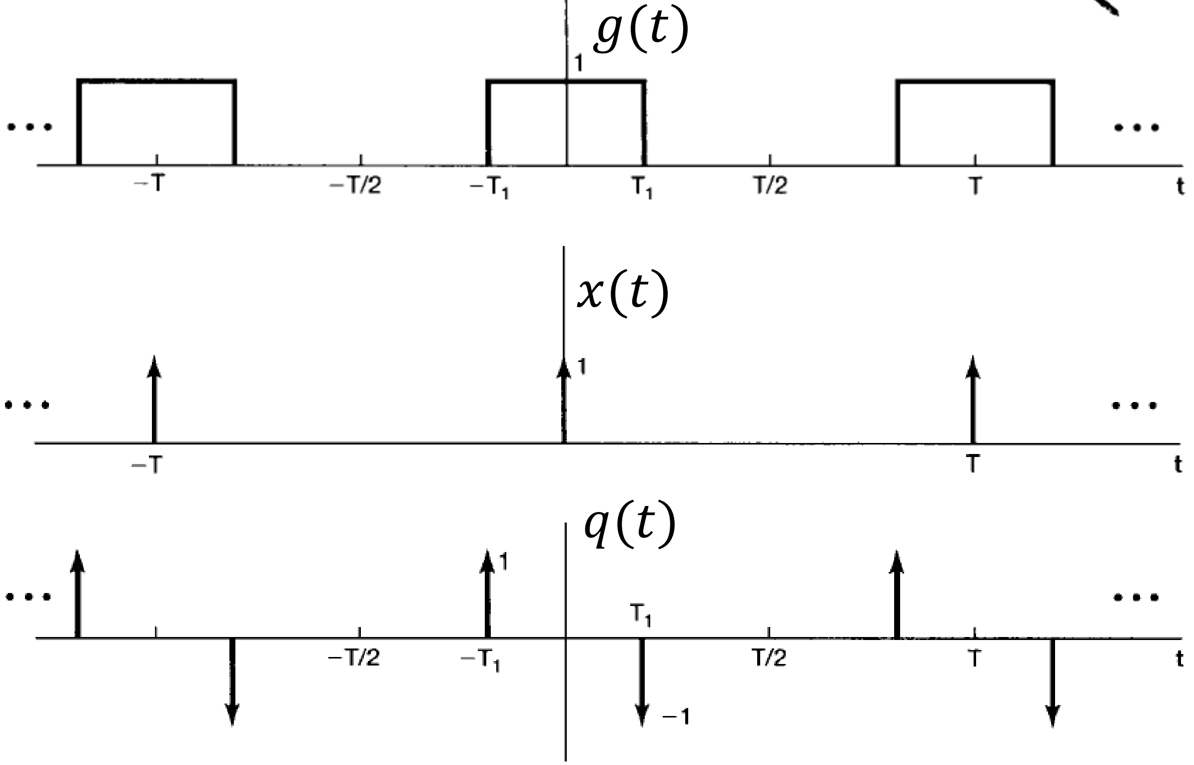
❑ Summary

Property	Section	Periodic Signal	Fourier Series Coefficients
		$\left.\begin{array}{l} x(t) \\ y(t) \end{array}\right\}$ Periodic with period T and fundamental frequency $\omega_0 = 2\pi/T$	a_k b_k
Linearity	3.5.1	$Ax(t) + By(t)$	$Aa_k + Bb_k$
Time Shifting	3.5.2	$x(t - t_0)$	$a_k e^{-jk\omega_0 t_0} = a_k e^{-jk(2\pi/T)t_0}$
Frequency Shifting		$e^{jM\omega_0 t} = e^{jM(2\pi/T)t} x(t)$	a_{k-M}
Conjugation	3.5.6	$x^*(t)$	a^*_{-k}
Time Reversal	3.5.3	$x(-t)$	a_{-k}
Time Scaling	3.5.4	$x(\alpha t), \alpha > 0$ (periodic with period T/α)	a_k
Periodic Convolution		$\int_T x(\tau)y(t-\tau)d\tau$	$Ta_k b_k$
Multiplication	3.5.5	$x(t)y(t)$	$\sum_{l=-\infty}^{+\infty} a_l b_{k-l}$
Differentiation		$\frac{dx(t)}{dt}$	$jk\omega_0 a_k = jk\frac{2\pi}{T}a_k$
Integration		$\int_{-\infty}^{t} x(t)\,dt$ (finite valued and periodic only if $a_0 = 0$)	$\left(\frac{1}{jk\omega_0}\right)a_k = \left(\frac{1}{jk(2\pi/T)}\right)a_k$
Conjugate Symmetry for Real Signals	3.5.6	$x(t)$ real	$\begin{cases} a_k = a^*_{-k} \\ \mathcal{Re}\{a_k\} = \mathcal{Re}\{a_{-k}\} \\ \mathcal{Im}\{a_k\} = -\mathcal{Im}\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k} \end{cases}$
Real and Even Signals	3.5.6	$x(t)$ real and even	a_k real and even
Real and Odd Signals	3.5.6	$x(t)$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals		$\begin{cases} x_e(t) = \mathcal{Ev}\{x(t)\} & [x(t) \text{ real}] \\ x_o(t) = \mathcal{Od}\{x(t)\} & [x(t) \text{ real}] \end{cases}$	$\mathcal{Re}\{a_k\}$ $j\mathcal{Im}\{a_k\}$

Parseval's Relation for Periodic Signals

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{+\infty} |a_k|^2$$

Properties of continuous-



❑ Examples FS coefficients (c_k) of $g(t)$?

❑ Solution

- Let $x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t) e^{-jk\omega_0 t} dt = \frac{1}{T}$$

- Let $q(t) = x(t + T_1) - x(t - T_1)$

$$b_k = e^{jk\omega_0 T_1} a_k - e^{-jk\omega_0 T_1} a_k = \frac{1}{T}\left(e^{jk\omega_0 T_1} - e^{-jk\omega_0 T_1}\right) = \frac{2j\sin(k\omega_0 T_1)}{T}$$

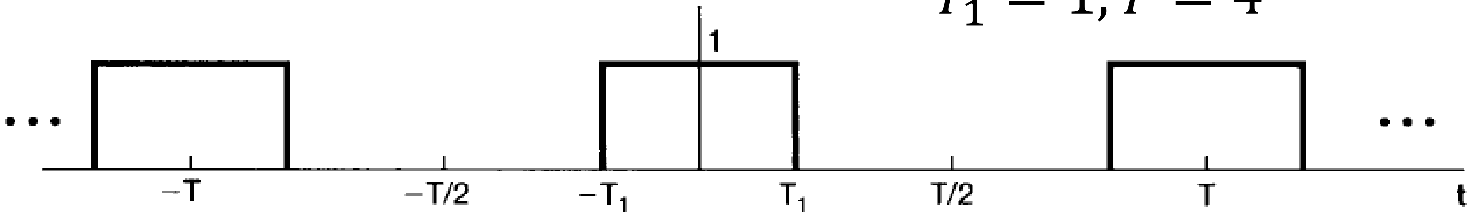
- $q(t) = dg(t)/dt \quad \therefore b_k = jk\omega_0 c_k$

$$\therefore c_k = \frac{b_k}{jk\omega_0} = \frac{2j\sin(k\omega_0 T_1)}{jk\omega_0 T} = \frac{\sin(k\omega_0 T_1)}{k\pi}, k \neq 0 \qquad c_0 = \frac{2T_1}{T}$$

Properties of continuous-time FS

- Examples

$x(t)$ $\quad T_1 = 1, T = 4$



$$a_k = \frac{\sin(k\omega_0 T_1)}{k\pi}, k \neq 0$$
$$= \frac{\sin(k\pi/2)}{k\pi}, k \neq 0$$

$$g(t) = x(t-1) - 1/2$$

FS coefficients of $g(t)$?

- Solution

$$x(t-1) \overset{\mathcal{FS}}{\leftrightarrow} e^{-jk\omega_0 t_0} a_k = e^{-jk\pi/2} a_k, k \neq 0$$

$$-1/2 \overset{\mathcal{FS}}{\leftrightarrow} \begin{cases} 0, k \neq 0 \\ -\frac{1}{2}, k = 0 \end{cases} \qquad \therefore x(t-1) - 1/2 \overset{\mathcal{FS}}{\leftrightarrow} \begin{cases} e^{-jk\pi/2} a_k, k \neq 0 \\ a_0 - \frac{1}{2}, k = 0 \end{cases}$$

Properties of continuous-time FS

Examples

Given a signal $x(t)$ with the following facts, determine $x(t)$

1. $x(t)$ is real;
2. $x(t)$ is periodic with T=4 and FS coefficients $a_k = 0$ for $|\mathrm{k}| > 1$;
3. A signal with FS coefficients $b_k = e^{-j\pi k/2}a_{-k}$ is odd;
4. $\frac{1}{4}\int_4 |x(t)|^2 dt = \frac{1}{2}$.

Solution

- From 2, $x(t) = a_0 + a_1 e^{j\left(\frac{\pi}{2}\right)t} + a_{-1}e^{-j\left(\frac{\pi}{2}\right)t}$
- $b_k = e^{-j\pi k/2}a_{-k}$ corresponds to the signal $x(-t+1)$, which is real and odd
- $\frac{1}{4}\int_4 |x(t)|^2 dt = \frac{1}{4}\int_4 |x(-t+1)|^2 dt = \sum_{k=-\infty}^{\infty}|b_k|^2 = |b_0|^2 + |b_1|^2 + |b_{-1}|^2 = \frac{1}{2}$
- $x(-t+1)$ is real and odd $\Rightarrow b_k = -b_{-k} \Rightarrow b_0 = 0, b_1 = -b_{-1} = \frac{\mathrm{j}}{2}$ or -$\frac{\mathrm{j}}{2}$
- $a_0 = 0, a_1 = -1/2, a_{-1} = 1/2$

Fourier Series Representation of Periodic signals (ch.3)

- ❑ The response of LTI systems to complex exponentials
- ❑ Fourier series representation of continuous periodic signals
- ❑ Convergence of the Fourier series
- ❑ Properties of continuous-time Fourier series
- ❑ Fourier series representation of discrete –time periodic signals
- ❑ Properties of discrete FS
- ❑ Fourier series and LTI systems

Fourier series representation of D-T periodic signals

Linear combination of harmonically related complex exponentials

- Harmonically related complex exponentials

$$\emptyset_k[n] = e^{jk(2\pi/N)n},\ k = 0, \pm 1, \pm 2, \ldots$$

  - Fundamental frequency $|k|(\frac{2\pi}{N})$
  - Only N distinct signals in $\emptyset_k[n]$, since $\emptyset_k[n] = \emptyset_{k+rN}[n]$

- Linear combination of $\emptyset_k[n]$ is also periodic

$$x[n] = \sum_{k=\langle N\rangle} a_k \emptyset_k[n] = \sum_{k=\langle N\rangle} a_k e^{jk\omega_0 n} = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$$

- $\sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$: Discrete-Time Fourier Series; a_k: Fourier Series coefficients

Fourier series representation of D-T periodic signals

Determine the Fourier Series Representation

$$\sum_{n=\langle N\rangle} x[n]e^{-jr(2\pi/N)n} = \sum_{n=\langle N\rangle}\sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}\, e^{-jr(2\pi/N)n}$$

$$= \sum_{k=\langle N\rangle} a_k \boxed{\sum_{n=\langle N\rangle} e^{j(k-r)(2\pi/N)n}} = N a_r$$

$$\sum_{n=\langle N\rangle} e^{j(k-r)(2\pi/N)n} = \begin{cases} N, k=r \\ 0, k \neq n \end{cases} = N\delta[k-r]$$

$$\therefore a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x[n]e^{-jk(2\pi/N)n}$$

Fourier series representation of D-T periodic signals

Determine the Fourier Series Representation

- Discrete Fourier series pair

$$a_k = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-jk(2\pi/N)n}$$

Analysis equation; a_k: Fourier Series coefficients

$$x[n] = \sum_{k=\langle N \rangle} a_k e^{jk(2\pi/N)n}$$

Synthesis equation; Fourier Series (Finite)

- a_k is periodic

$$x[n] = \sum_{k=\langle N \rangle} a_k \emptyset_k[n] = a_0 \emptyset_0[n] + a_1 \emptyset_1[n] + \cdots + a_{N-1} \emptyset_{N-1}[n]$$

$$= a_1 \emptyset_1[n] + a_2 \emptyset_2[n] + \cdots + a_N \emptyset_N[n]$$

$$= a_2 \emptyset_2[n] + a_3 \emptyset_3[n] + \cdots + a_{N+1} \emptyset_{N+1}[n]$$

$$\therefore a_k = a_{k+rN}$$

Fourier series representation of D-T periodic signals

Determine the Fourier Series Representation

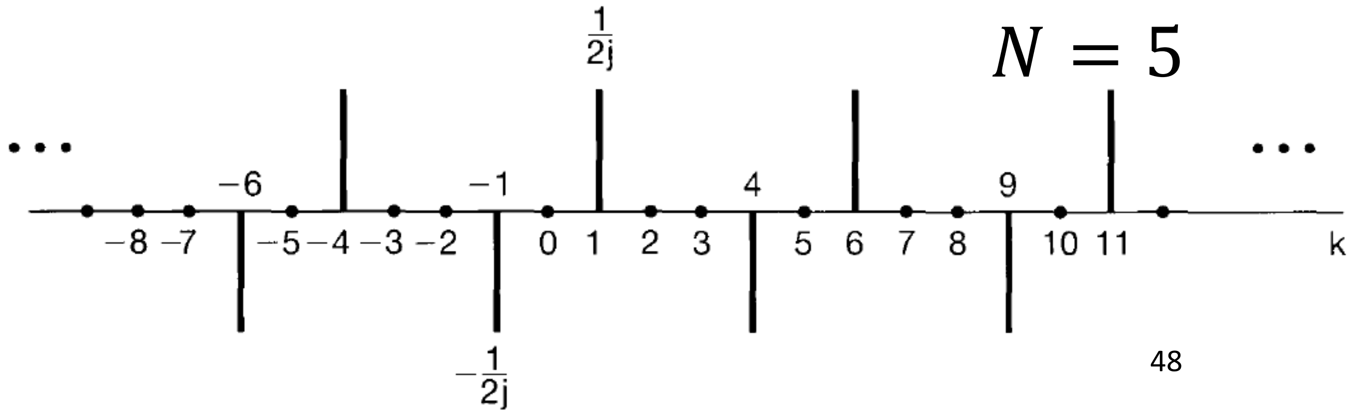
- Examples $$x[n] = \sin \omega_0 n$$

If $\omega_0 = \frac{2\pi}{N}$, $x[n]$ is periodic with fundamental period of N.

$$x[n] = \sin \omega_0 n = \frac{1}{2j} e^{j(2\pi/N)n} - \frac{1}{2j} e^{j(2\pi/N)n}$$

$$\therefore a_1 = \frac{1}{2j} \qquad a_{-1} = -\frac{1}{2j} \qquad a_k = 0, \text{ for } k \neq \pm 1 \text{ in one period}$$

- a_k is periodic and only one period is utilized in the synthesis equation



Fourier series representation of C-T periodic signals

Determine the Fourier Series Representation

❑ Examples: $x[n] = 1 + \sin(\frac{2\pi}{N})n + 3\cos(\frac{2\pi}{N})n + \cos\left(\frac{4\pi}{N}n + \frac{\pi}{2}\right)$

$$x[n] = 1 + \frac{1}{2j}\left[e^{j(2\pi/N)n} - e^{-j(2\pi/N)n}\right] + \frac{3}{2}\left[e^{j(2\pi/N)n} + e^{-j(2\pi/N)n}\right]$$

$$+ \frac{1}{2}\left(e^{j(4\pi n/N+\pi/2)} + e^{-j(4\pi n/N+\pi/2)}\right)$$

$$\therefore x[n] = \overset{a_0}{\boxed{1}} + \overset{a_1}{\boxed{\left(\frac{3}{2} + \frac{1}{2j}\right)}} e^{j(2\pi/N)n} + \overset{a_{-1}}{\boxed{\left(\frac{3}{2} - \frac{1}{2j}\right)}} e^{-j(2\pi/N)n}$$

$$+ \overset{a_2}{\boxed{\frac{1}{2}e^{j\pi/2}}} e^{j2(2\pi/N)n} + \overset{a_{-2}}{\boxed{\frac{1}{2}e^{-j\pi/2}}} e^{-j2(2\pi/N)n}$$

Fourier series representation of D-T periodic signals

Linear combination of harmonically related complex exponentials

- Real signal $a_k = a^*_{-k}$, or $a^*_k = a_{-k}$

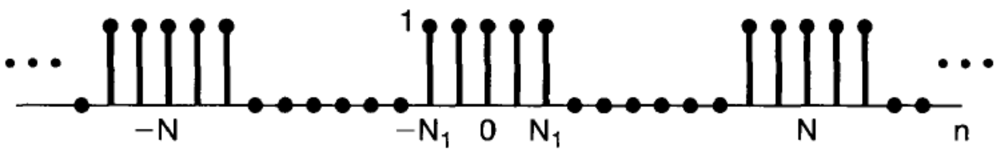
$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$$

$$x^*[n] = \sum_{k=\langle N\rangle} a^*_k e^{-jk(2\pi/N)n} = \sum_{k=\langle N\rangle} a^*_{-k} e^{jk(2\pi/N)n}$$

$$x[n] = x^*[n] \quad \Longrightarrow a_k = a^*_{-k}$$

Fourier series representation of D-T periodic signals

Determine the Fourier Series Representation



- Examples: $x[n]$ discrete square

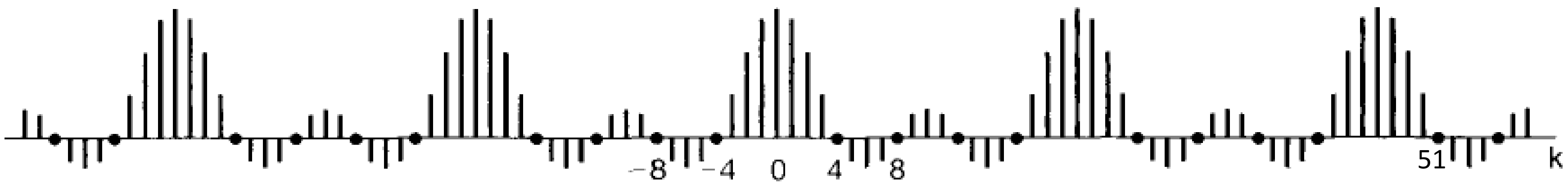
$$a_k = \frac{1}{N}\sum_{n=-N_1}^{N_1} x[n]e^{-jk(2\pi/N)n} = \frac{1}{N}\sum_{n=-N_1}^{N_1} e^{-jk(2\pi/N)n}$$

$m = n + N_1$

$$= \frac{1}{N}\sum_{m=0}^{2N_1} e^{-jk(2\pi/N)(m-N_1)} = \frac{1}{N}e^{jk(2\pi/N)N_1}\sum_{m=0}^{2N_1} e^{-jk(2\pi/N)m}$$

$$= \begin{cases} \dfrac{2N_1+1}{N}, k = 0, \pm N, \pm 2N, \dots \\ \dfrac{1}{N}\dfrac{\sin[2k\pi(N_1+1/2)/N]}{\sin(k\pi/N)}, k \neq 0, \pm N, \pm 2N, \dots \end{cases}$$

$a_k \ (2N_1 + 1 = 5, N = 20)$



Fourier series representation of D-T p

Linear combination of harmonically related co

$2N_1 + 1 = 5, N = 9$

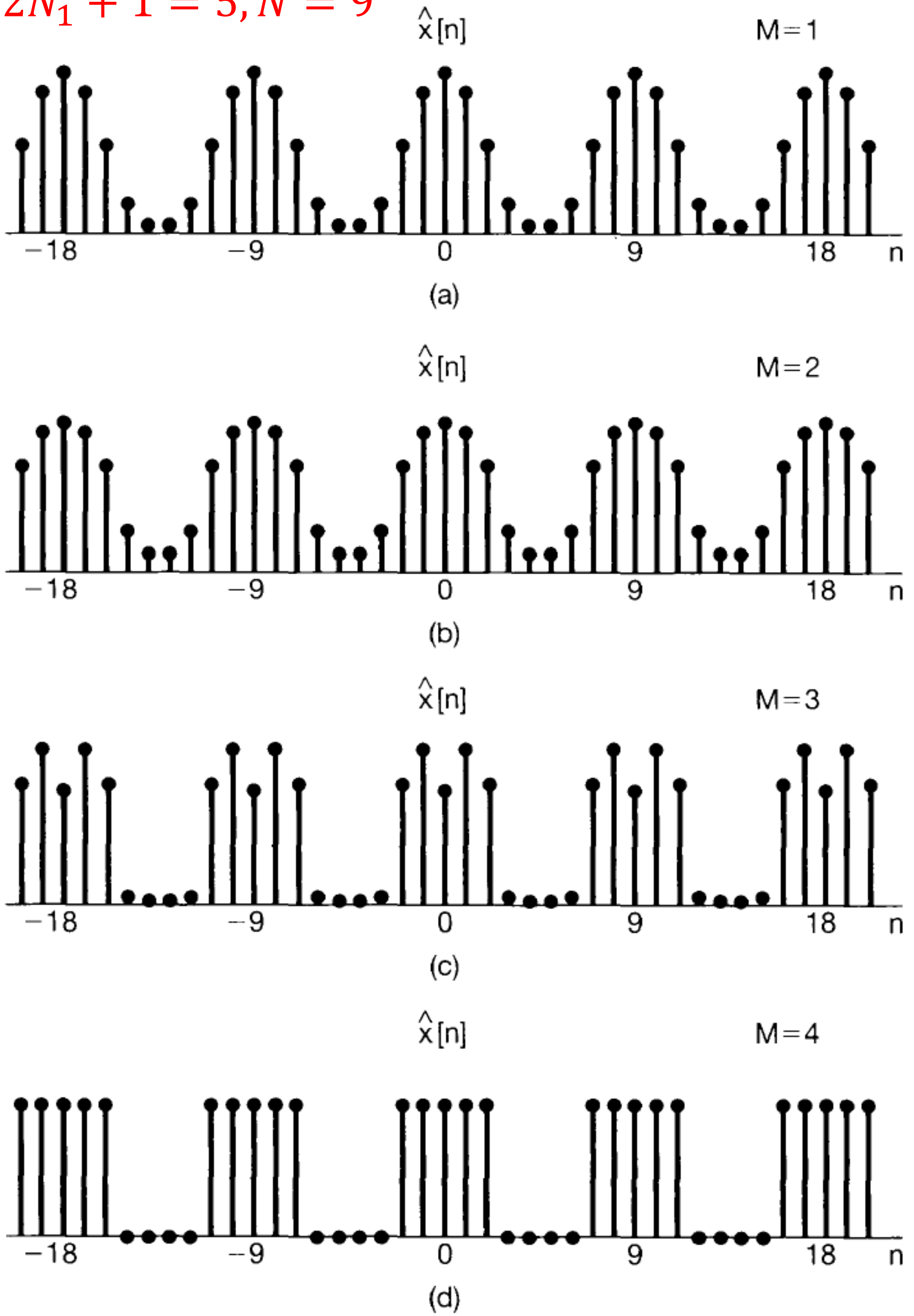
- Approximate a discrete square by $\hat{x}[n]$

$$\hat{x}[n] = \sum_{k=-M}^{M} a_k e^{jk(2\pi/N)n}$$

With $a_k = \begin{cases} \frac{2N_1+1}{N}, k = 0, \pm N, \pm 2N, \ldots \\ \frac{1}{N}\frac{\sin[2k\pi(N_1+1/2)/N]}{\sin(k\pi/N)}, \text{else} \end{cases}$

- For M=4, $\hat{x}[n] = x[n]$

- No convergence issues for the discrete–time Fourier series!



Fourier Series Representation of Periodic signals (ch.3)

- ❑ The response of LTI systems to complex exponentials
- ❑ Fourier series representation of continuous periodic signals
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- ❑ Properties of discrete FS
- ❑ Fourier series and LTI systems

Properties of discre

$$x[n] \overset{\mathcal{FS}}{\longleftrightarrow} a_k \qquad y[n] \overset{\mathcal{FS}}{\longleftrightarrow} b_k$$

- Multiplication

$$x[n]y[n] \overset{\mathcal{FS}}{\longleftrightarrow} \sum_{l=\langle N\rangle} a_l b_{k-l}$$

- First difference

$$x[n]-x[n-1] \overset{\mathcal{FS}}{\longleftrightarrow} \left(1-e^{-jk(2\pi/N)}\right)a_k$$

- Parseval's relation

$$\frac{1}{N}\sum_{l=\langle N\rangle} |x[n]|^2 = \sum_{l=\langle N\rangle} |a_k|^2$$

Property	Periodic Signal	Fourier Series Coefficients
	$\left.\begin{array}{l} x[n] \\ y[n]\end{array}\right\}$ Periodic with period N and fundamental frequency $\omega_0 = 2\pi/N$	$\left.\begin{array}{l} a_k \\ b_k\end{array}\right\}$ Periodic with period N
Linearity	$Ax[n] + By[n]$	$Aa_k + Bb_k$
Time Shifting	$x[n-n_0]$	$a_k e^{-jk(2\pi/N)n_0}$
Frequency Shifting	$e^{jM(2\pi/N)n}x[n]$	a_{k-M}
Conjugation	$x^*[n]$	a^*_{-k}
Time Reversal	$x[-n]$	a_{-k}
Time Scaling	$x_{(m)}[n] = \begin{cases} x[n/m], & \text{if } n \text{ is a multiple of } m \\ 0, & \text{if } n \text{ is not a multiple of } m\end{cases}$ (periodic with period mN)	$\frac{1}{m}a_k$ $\left(\begin{array}{l}\text{viewed as periodic} \\ \text{with period } mN\end{array}\right)$
Periodic Convolution	$\sum_{r=\langle N\rangle} x[r]y[n-r]$	$Na_k b_k$
Multiplication	$x[n]y[n]$	$\sum_{l=\langle N\rangle} a_l b_{k-l}$
First Difference	$x[n]-x[n-1]$	$(1-e^{-jk(2\pi/N)})a_k$
Running Sum	$\sum_{k=-\infty}^{n} x[k]$ $\left(\begin{array}{l}\text{finite valued and periodic only} \\ \text{if } a_0 = 0\end{array}\right)$	$\left(\frac{1}{(1-e^{-jk(2\pi/N)})}\right)a_k$
Conjugate Symmetry for Real Signals	$x[n]$ real	$\begin{cases} a_k = a^*_{-k} \\ \Re e\{a_k\} = \Re e\{a_{-k}\} \\ \Im m\{a_k\} = -\Im m\{a_{-k}\} \\ \lvert a_k\rvert = \lvert a_{-k}\rvert \\ \sphericalangle a_k = -\sphericalangle a_{-k}\end{cases}$
Real and Even Signals	$x[n]$ real and even	a_k real and even
Real and Odd Signals	$x[n]$ real and odd	a_k purely imaginary and odd
Even-Odd Decomposition of Real Signals	$\begin{cases} x_e[n] = \mathcal{E}v\{x[n]\} & [x[n] \text{ real}] \\ x_o[n] = \mathcal{O}d\{x[n]\} & [x[n] \text{ real}]\end{cases}$	$\Re e\{a_k\}$ $j\Im m\{a_k\}$

Parseval's Relation for Periodic Signals

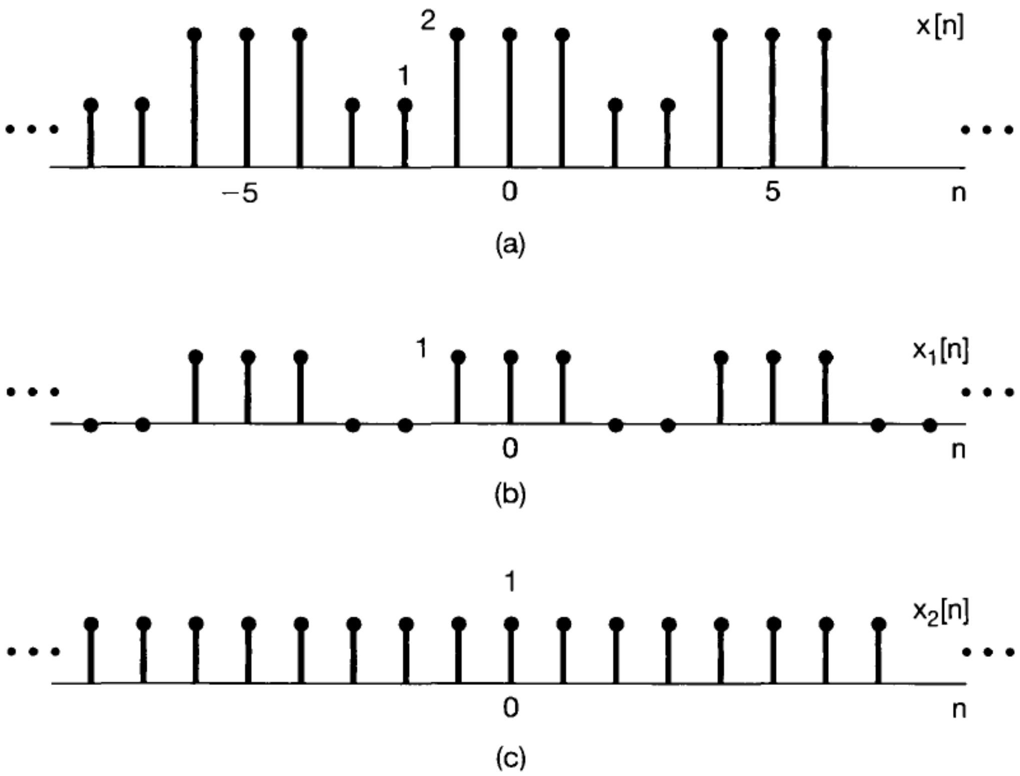
$$\frac{1}{N}\sum_{n=\langle N\rangle} |x[n]|^2 = \sum_{k=\langle N\rangle} |a_k|^2$$

Properties of discrete-time FS

- Examples $\quad x[n] = x_1[n] + x_2[n]$

- $x_1[n]$ is a square wave with $N = 5$ and $N_1 = 1$

$$b_k = \begin{cases} \dfrac{2N_1+1}{N}, k = \pm N, \pm 2N, \ldots \\ \dfrac{1}{N}\dfrac{\sin[2k\pi(N_1+1/2)/N]}{\sin(k\pi/N)}, \text{else} \end{cases} = \begin{cases} \dfrac{3}{5}, k = \pm 5, \pm 10, \ldots \\ \dfrac{1}{5}\dfrac{\sin(3k\pi/5)}{\sin(k\pi/5)}, \text{else} \end{cases}$$



- For $x_2[n]$

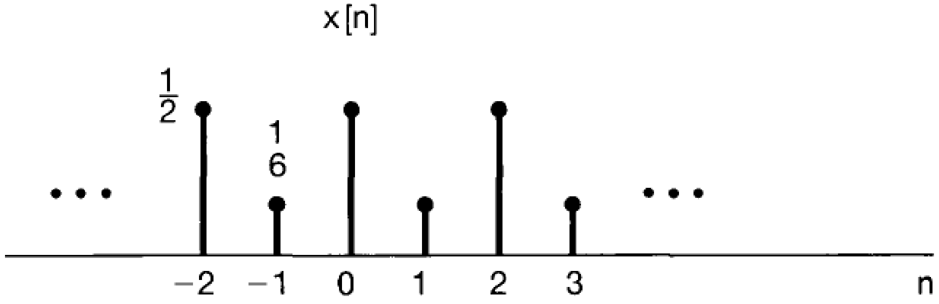
$$c_k = \begin{cases} 1, k = \pm N, \pm 2N, \ldots \\ 0, \quad \text{else} \end{cases}$$

$$\therefore a_k = b_k + c_k = \begin{cases} \dfrac{8}{5}, \quad k = \pm 5, \pm 10, \ldots \\ \dfrac{1}{5}\dfrac{\sin(3k\pi/5)}{\sin(k\pi/5)}, \quad \text{else} \end{cases}$$

Properties of discrete-time FS

Examples

Suppose we are given the following facts about a sequence $x[n]$:



1. $x[n]$ is periodic with period $N = 6$.

2. $\sum_{n=0}^{5} x[n] = 2$.

3. $\sum_{n=2}^{7} (-1)^n x[n] = 1$.

4. $x[n]$ has the minimum power per period among the set of signals satisfying the preceding three conditions.

Solution

- $\sum_{n=0}^{5} x[n] = 2 \Longrightarrow a_0 = \frac{1}{N} \sum_{n=\langle N \rangle} x[n] e^{-j0(2\pi/N)n} = 1/3.$
- $\sum_{n=2}^{7} (-1)^n x[n] = 1 \Longrightarrow \sum_{n=\langle N \rangle} x[n] e^{-j3(2\pi/N)n} = 1 \Longrightarrow a_3 = 1/6$
- from 4, $a_1 = a_2 = a_4 = a_5 = 0$
- $\therefore x[n] = a_0 e^{-j0(2\pi/N)n} + a_3 e^{-j3(2\pi/N)n} = \frac{1}{3} + \frac{1}{6} e^{-j\pi n} = \frac{1}{3} + \frac{1}{6} (-1)^n$

Fourier Series Representation of Periodic signals (ch.3)

- ❑ The response of LTI systems to complex exponentials
- ❑ Fourier series representation of continuous periodic signals
- ❑ Convergence of the Fourier series
- ❑ Properties of continuous-time Fourier series
- ❑ Fourier series representation of discrete –time periodic signals
- ❑ Properties of discrete FS
- ❑ Fourier series and LTI systems

Fourier series and LTI systems

❑ Recall

$$e^{st} \longrightarrow \boxed{\text{LTI}} \longrightarrow H(s)e^{st} \qquad H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau$$

$$z^{n} \longrightarrow \boxed{\text{LTI}} \longrightarrow H[z]z^{n} \qquad H[z] = \sum_{k=-\infty}^{\infty} h[k]z^{-k}$$

❑ System functions: $H(s)$ and $H[z]$

e^{st}
- s pure imagery $e^{st} \rightarrow e^{j\omega t}$
  - For periodic signal, CT Fourier Series (Ch3)
  - For aperiodic signal, CT Fourier Transform (Ch4)
- s complex number e^{st} — Laplase Transform (Ch9)

z^{n}
- z pure imagery $z^{n} \rightarrow e^{j\omega n}$
  - For periodic signal, DT Fourier Series (Ch3)
  - For aperiodic signal, DT Fourier Transform (Ch5)
- z complex number z^{n} — Z-Transform (Ch10)

Fourier series and LTI systems

- Frequency response for CT system: $H(j\omega)$

$$H(s) = \int_{-\infty}^{\infty} h(\tau)e^{-s\tau}d\tau \quad \xRightarrow{s=j\omega} \quad H(j\omega) = \int_{-\infty}^{\infty} h(\tau)e^{-j\omega\tau}d\tau$$

$$e^{j\omega t} \longrightarrow \boxed{\text{LTI}} \longrightarrow H(j\omega)e^{j\omega t}$$

$$x(t) = \sum_{k-\infty}^{\infty} a_k e^{jk\omega_0 t} \longrightarrow \boxed{\text{LTI}} \longrightarrow y(t) = \sum_{k-\infty}^{\infty} a_k H(jk\omega_0)e^{jk\omega_0 t}$$

$$x(t) \overset{\mathcal{FS}}{\longleftrightarrow} a_k \qquad y(t) \overset{\mathcal{FS}}{\longleftrightarrow} b_k$$

$$b_k = a_k H(jk\omega_0)$$

Fourier series and LTI systems

Frequency response for CT system: example

$x(t) = \sum_{k=-3}^{3} a_k e^{jk2\pi t}$ ($a_0 = 1$, $a_1 = a_{-1} = \frac{1}{4}$, $a_2 = a_{-2} = \frac{1}{2}$, $a_3 = a_{-3} = \frac{1}{3}$) is the input of a LTI system with $h(t) = e^{-t}u(t)$, determine $y(t)$

Solution

$$y(t) = \sum_{k=-\infty}^{\infty} a_k H(jk\omega_0) e^{jk2\pi t}$$

$$H(j\omega) = \int_0^{\infty} e^{-\tau} e^{-j\omega\tau} d\tau = \frac{1}{1+j\omega}$$

$$b_k = a_k H(jk\omega_0) = a_k \frac{1}{1+jk2\pi} \quad b_0 = 1 \cdot 1 = 1 \quad b_1 = \frac{1}{4}\frac{1}{1+j2\pi} \quad b_{-1} = \frac{1}{4}\frac{1}{1-j2\pi}$$

$$b_2 = \frac{1}{2}\frac{1}{1+j4\pi} \quad b_{-2} = \frac{1}{2}\frac{1}{1-j4\pi} \quad b_3 = \frac{1}{3}\frac{1}{1+j6\pi} \quad b_{-3} = \frac{1}{3}\frac{1}{1-j6\pi}$$

Fourier series and LTI systems

- Frequency response DT system: $H\left(e^{j\omega}\right)$

$$H[z] = \sum_{n=-\infty}^{\infty} h[k] z^{-n} \quad \xrightarrow{z=e^{j\omega}} \quad H\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} h[n] e^{-j\omega n}$$

$$e^{j\omega n} \longrightarrow \boxed{\text{LTI}} \longrightarrow H\left(e^{j\omega}\right) e^{j\omega n}$$

$$x[n] = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n} \longrightarrow \boxed{\text{LTI}} \longrightarrow y[n] = \sum_{k=\langle N\rangle} a_k H\left(e^{jk(2\pi/N)}\right) e^{jk(2\pi/N)n}$$

$$b_k = a_k H\left(e^{jk(2\pi/N)}\right)$$

Fourier series and LTI systems

❑ Frequency response DT system: example

$$h[n] = \alpha^n u[n], |\alpha| < 1$$

$$x[n] = \cos\frac{2\pi n}{N} \longrightarrow \boxed{\text{LTI}} \longrightarrow y[n]?$$

❑ Solution

$$x[n] = \frac{1}{2}e^{j(2\pi/N)n} + \frac{1}{2}e^{-j(2\pi/N)n}$$

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h[n]e^{-j\omega n} = \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n} = \frac{1}{1-\alpha e^{-j\omega}}$$

$$x[n] = \frac{1}{2}\left(\frac{1}{1-\alpha e^{-j2\pi/N}}\right)e^{j(2\pi/N)n} + \frac{1}{2}\left(\frac{1}{1-\alpha e^{j2\pi/N}}\right)e^{-j(2\pi/N)n}$$