Diagonal Dominance

A matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ is said to be row diagonally dominant if

$$|a_{ii}| \geq \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ij}|, \quad \forall i = 1, \ldots, n$$

It is said to be column diagonally dominant if

$$|a_{ii}| \geq \sum_{\substack{j=1 \\ j \neq i}}^{n} |a_{ji}|, \quad \forall i = 1, \ldots, n$$

It is strictly row/column diagonally dominant if the above inequalities are strict

Diagonally dominant matrices may be singular (e.g., $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$)

LU for Diagonally Dominant Matrices

Theorem c.d.d.

If **A** $\in \mathbb{R}^{n\times n}$ *is nonsingular and column diagonally dominant, then it has an LU decomposition* $\mathbf{A} = \mathbf{LU}$ *and* $|\ell_{ij}| \le 1$ *for all* i, j.

Proof. By induction on n. The statement holds for $n=1$ clearly. Suppose ~~the~~ statement holds for any $(n-1)\times(n-1)$ nonsingular c.d.d. matrix. Partition A as

$$A = \begin{bmatrix} \boxed{\alpha} & w^T \\ v & C \end{bmatrix}$$

$\alpha \in \mathbb{R}$, $v, w \in \mathbb{R}^{n-1}$. $C \in \mathbb{R}^{(n-1)\times(n-1)}$

Note that C is c.d.d.

Also note that $\alpha \neq 0$. To see this, suppose $\alpha = 0$. Then, because A is c.d.d., $v = 0 \Rightarrow A$ singular. Contradiction!

LU for Diagonally Dominant Matrices (cont'd)

block diagonal

Let

$$A = \begin{bmatrix} \alpha & w^T \\ v & C \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{v}{\alpha} & I \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & B \end{bmatrix} \begin{bmatrix} \alpha & w^T \\ 0 & I \end{bmatrix}$$

where $B = C - \frac{1}{\alpha} v w^T$

$$\det(A) = 1 \cdot \det(B) \cdot \alpha \Rightarrow B \text{ is nonsingular}$$

$\det(A) \neq 0$, $\alpha \neq 0$

For each $j = 1, \dots, n-1$,

$$\sum_{\substack{i=1 \\ i \neq j}}^{n-1} |b_{ij}| = \sum_{\substack{i=1 \\ i \neq j}}^{n-1} \left| c_{ij} - \frac{v_i w_j}{\alpha} \right|$$

$$\leq \sum_{\substack{i=1 \\ i \neq j}}^{n-1} |c_{ij}| + \frac{|w_j|}{|\alpha|} \sum_{\substack{i=1 \\ i \neq j}}^{n-1} |v_i| \leq |c_{jj}| - |w_j| + \frac{|w_j|}{|\alpha|} (|\alpha| - |v_j|)$$

A c.d.d

LU for Diagonally Dominant Matrices (cont'd)

$$\leq |c_{jj}| - \left|\frac{w_j v_j}{\alpha}\right| \leq \left| c_{jj} - \frac{w_j v_j}{\alpha} \right| = |b_{jj}|$$

$\Rightarrow B$ is $(n-1)\times(n-1)$ nonsingular, c.d.d.

Using the hypothesis, B has an LU decomposition

$B = L_1 U_1$ where all entries in L_1 have absolute value ≤ 1.

Then, factorize A as

$$A = \underbrace{\begin{bmatrix} 1 & 0 \\ \frac{v}{\alpha} & L_1 \end{bmatrix}}_{\substack{L \\ \text{unit lower triangular}}} \underbrace{\begin{bmatrix} \alpha & w^T \\ 0 & U_1 \end{bmatrix}}_{\substack{U \\ \text{upper triangular}}}$$

Since A is c.d.d, all the entries in $\frac{v}{\alpha}$ have absolute value ≤ 1.

Positive Definite Matrices

$x^T A x$ quadractic

A Matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ is said to be

- positive definite if $\mathbf{x}^T\mathbf{A}\mathbf{x} > 0$ (scalar) for all **nonzero** $\mathbf{x} \in \mathbb{R}^n$
- positive semi-definite if $\mathbf{x}^T\mathbf{A}\mathbf{x} \geq 0$ for all $\mathbf{x} \in \mathbb{R}^n$
- negative definite if $\mathbf{x}^T\mathbf{A}\mathbf{x} < 0$ for all **nonzero** $\mathbf{x} \in \mathbb{R}^n$
- negative semi-definite if $\mathbf{x}^T\mathbf{A}\mathbf{x} \leq 0$ for all $\mathbf{x} \in \mathbb{R}^n$
- indefinite if there exist $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ s.t. $(\mathbf{x}^T\mathbf{A}\mathbf{x})(\mathbf{y}^T\mathbf{A}\mathbf{y}) < 0$

Properties: For any positive definite (p.d.) $\mathbf{A} \in \mathbb{R}^{n\times n}$,

- $\mathbf{A}$ is nonsingular
- ☆ If $\mathbf{X} \in \mathbb{R}^{n\times q}$ has full column rank, then $\mathbf{X}^T\mathbf{A}\mathbf{X} \in \mathbb{R}^{q\times q}$ is positive definite
- All the principal submatrices are positive definite
- All the diagonal entries of $\mathbf{A}$ are positive
- $\mathbf{A}$ has an LU decomposition $\mathbf{A} = \mathbf{L}\mathbf{U}$ s.t. the diagonal entries of $\mathbf{U}$ are positive

☆ Suppose $z \in \mathbb{R}^q$ is s.t.

$z^T(X^TAX)z \leq 0$

$\Updownarrow$

$(Xz)^T A (Xz) \leq 0$

A is p.d. $\Rightarrow Xz = 0$

X full colum rank

$\Downarrow$

$z = 0$

Positive Definite Matrices (cont'd)

Given $\mathbf{A} \in \mathbb{R}^{n\times n}$, define its symmetric part as

$$\mathbf{T} = \frac{\mathbf{A} + \mathbf{A}^T}{2}$$

symmetric

and its skew-symmetric part as

$$\mathbf{S} = \frac{\mathbf{A} - \mathbf{A}^T}{2}$$

Clearly,

$$\mathbf{A} = \frac{\mathbf{T} + \mathbf{S}}{2}$$

A is positive definite if and only if **T** is positive definite (That's why one mostly considers symmetric positive definite matrices)

$$x^T A x \overset{\text{scalar}}{=} (x^T A x)^T = x^T A^T x$$

$$x^T A x = \frac{1}{2} x^T A x + \frac{1}{2} x^T A^T x = x^T \frac{A + A^T}{2} x = x^T T x$$

Cholesky Decomposition for Positive Definite Matrices

symmetric

Given a positive definite $\mathbf{A} \in \mathbb{S}^n$, there exists a unique lower triangular matrix $\mathbf{G} \in \mathbb{R}^{n\times n}$ with positive diagonal entries such that

$$\mathbf{A} = \mathbf{G}\mathbf{G}^T \qquad \text{(Cholesky decomposition)}$$

A p.d. $\Rightarrow$ $\left.\begin{array}{l} A \text{ nonsingular} \\ A \text{ symmetric} \end{array}\right\} \Rightarrow \exists$ unique LDL decomposition $A = LDL^T$

From property ✡, $D = L^{-1} \overset{\text{p.d.}}{A} L^{-T} = \mathrm{Diag}(d_1, \cdots, d_n)$ is p.d.

full rank

$\Rightarrow d_i > 0 \quad \forall i = 1, \cdots, n.$

Let $G = L\,\mathrm{Diag}(\sqrt{d_1}, \cdots, \sqrt{d_n})$ lower triangular positive diagonal

$GG^T = LDL^T = A$

- Can be computed in $\mathcal{O}(n^3/3)$ (similar to LDL), no pivoting, numerically very stable

Banded Systems

$j \geq i+q+1$

A matrix $\mathbf{A} \in \mathbb{R}^{n \times n}$ has upper bandwidth q if $a_{ij} = 0 \; \forall j > i + q$ and lower bandwidth p if $a_{ij} = 0 \; \forall i > j + p$

Example: $\mathbf{A} \in \mathbb{R}^{5 \times 5}$ has upper bandwidth $q = 1$ and lower bandwidth $p = 2$

$$\mathbf{A} = \begin{bmatrix} \times & \times & 0 & 0 & 0 \\ \times & \times & \times & 0 & 0 \\ \times & \times & \times & \times & 0 \\ 0 & \times & \times & \times & \times \\ 0 & 0 & \times & \times & \times \end{bmatrix}$$

The triangular factors in $\mathbf{LU}$, $\mathbf{GG}^T$, and $\mathbf{LDL}^T$ are also banded $\Longrightarrow$ save a lot of computations

Banded LU Decomposition

Theorem

Suppose $\mathbf{A} \in \mathbb{R}^{n\times n}$ has an LU decomposition $\mathbf{A} = \mathbf{LU}$ and has upper bandwidth q and lower bandwidth p. Then, $\mathbf{U}$ has upper bandwidth q and $\mathbf{L}$ has lower bandwidth p.

Proof. By induction on n. Recall that

$$A = \begin{bmatrix} \alpha & w^T \\ v & C \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ \frac{v}{\alpha} & I_{n-1} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & C - \frac{vw^T}{\alpha} \end{bmatrix} \begin{bmatrix} \alpha & w^T \\ 0 & I_{n-1} \end{bmatrix}$$

Since A is banded, only $w(1:q)$ in w and $v(1:p)$ in v are nonzero.

Thus, $C - \frac{vw^T}{\alpha}$ has upper bandwidth q and lower bandwidth p.

Let $C - \frac{vw^T}{\alpha} = L_1 U_1$ be its LU decomposition.

From the hypothesis on $n-1$ and the sparsity of w, v,

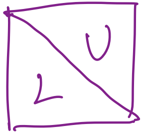
$$L = \begin{bmatrix} 1 & 0 \\ \frac{v}{\alpha} & L_1 \end{bmatrix} \qquad U = \begin{bmatrix} \alpha & w^T \\ 0 & U_1 \end{bmatrix}$$

is the LU decomp of A where L .... U ...

Band LU Decomposition (cont'd)

Suppose $\mathbf{A} = \mathbf{LU}$ exists and $\mathbf{A}$ has upper bandwidth q and lower bandwidth p

A




```
for k=1:n-1
     for i=k+1:min(k+p,n)
        A(i,k)=A(i,k)/A(k,k)
    end
    for j=k+1:min(k+q,n)
         for i=k+1:min(k+p,n)
            A(i,j)=A(i,j)-A(i,k)*A(k,j)
        end
    end
end
```

Complexity: $\mathcal{O}(2npq)$ flops, much smaller than $\mathcal{O}(2n^3/3)$ when $n \gg p, q$

Solving Band Triangular Systems

Solving $\mathbf{Ax} = \mathbf{b}$, where $\mathbf{A} = \mathbf{LU}$ exists and

> $\mathbf{L}$ is unit lower triangular with lower bandwidth p
>
> $\mathbf{U}$ is **nonsingular** upper triangular with upper bandwidth q

1. Solve $\mathbf{Lz} = \mathbf{b}$ for $\mathbf{z}$ using band forward substitution

```
for j=1:n
    for i=j+1:min(j+p,n)     % L(i,j)=0 for i>j+p
        b(i)=b(i)-L(i,j)*b(j);
    end
end     % Overwrite b with z
```

Complexity: $\mathcal{O}(2np) \ll \mathcal{O}(n^2)$ if $p \ll n$

Solving Band Triangular Systems (cont'd)

2. Solve $\mathbf{U}\mathbf{x} = \mathbf{z}$ for $\mathbf{x}$ using band backward substitution

```
for j=n:-1:1      % b is z after applying band
forward substitution
    b(j)=b(j)/U(j,j);
    for i=max(1,j-q):j-1      % U(i,j)=0 for j>i+q
        b(i)=b(i)-U(i,j)*b(j);
    end
end      % Overwrite b with x
```

Complexity: $\mathcal{O}(2nq) \ll \mathcal{O}(n^2)$ if $q \ll n$

Read Chapter 4 of textbook for more on special linear systems and their decompositions

Solving Band Triangular Systems (cont'd)

2. Solve $\mathbf{U}\mathbf{x} = \mathbf{z}$ for $\mathbf{x}$ using band backward substitution

```
for j=n:-1:1      % b is z after applying band
forward substitution
    b(j)=b(j)/U(j,j);
    for i=max(1,j-q):j-1      % U(i,j)=0 for j>i+q
        b(i)=b(i)-U(i,j)*b(j);
    end
end      % Overwrite b with x
```

Complexity: $\mathcal{O}(2nq) \ll \mathcal{O}(n^2)$ if $q \ll n$

Read Chapter 4 of textbook for more on special linear systems and their decompositions