# Matrix Computations
# Chapter 5: Positive Semidefinite Matrices
## Section 5.3 Matrix Inequalities and Schur Complement

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# PSD Matrix Inequalities

- Inequalities for matrices are defined based on the notion of PSD matrices
- PSD matrix inequalities are frequently used in topics like semidefinite programming
- Definitions:
  - $\mathbf{A} \succeq \mathbf{B}$ means that $\mathbf{A} - \mathbf{B}$ is PSD
  - $\mathbf{A} \succ \mathbf{B}$ means that $\mathbf{A} - \mathbf{B}$ is PD
  - $\mathbf{A} \not\succeq \mathbf{B}$ means that $\mathbf{A} - \mathbf{B}$ is indefinite
- Consequences immediately from the definitions: For any $\mathbf{A}, \mathbf{B}, \mathbf{C} \in \mathbb{S}^n$,
  - $\mathbf{A} \succeq \mathbf{0}, \alpha \geq 0$ (resp. $\mathbf{A} \succ \mathbf{0}, \alpha > 0$) $\Longrightarrow \alpha\mathbf{A} \succeq \mathbf{0}$ (resp. $\alpha\mathbf{A} \succ \mathbf{0}$)
  - $\mathbf{A}, \mathbf{B} \succeq \mathbf{0}$ (resp. $\mathbf{A} \succ \mathbf{0}, \mathbf{B} \succ \mathbf{0}$) $\Longrightarrow \mathbf{A} + \mathbf{B} \succeq \mathbf{0}$ (resp. $\mathbf{A} + \mathbf{B} \succ \mathbf{0}$)
  - $\mathbf{A} \succeq \mathbf{B}, \mathbf{B} \succeq \mathbf{C}$ (resp. $\mathbf{A} \succ \mathbf{B}, \mathbf{B} \succ \mathbf{C}$) $\Longrightarrow \mathbf{A} \succeq \mathbf{C}$ (resp. $\mathbf{A} \succ \mathbf{C}$)
  - $\mathbf{A} \not\succeq \mathbf{B}$ does not imply $\mathbf{B} \succeq \mathbf{A}$

# Properties of PSD Matrix Inequalities

Let $\mathbf{A}, \mathbf{B} \in \mathbb{S}^n$

- $\mathbf{A} \succeq \mathbf{B} \implies \lambda_k(\mathbf{A}) \geq \lambda_k(\mathbf{B})$ for all $k$; the converse is not always true

- $\mathbf{A} \succeq \mathbf{I}$ (resp. $\mathbf{A} \succ \mathbf{I}$) $\iff \lambda_k(\mathbf{A}) \geq 1$ for all $k$ (resp. $\lambda_k(\mathbf{A}) > 1$ for all $k$)
- $\mathbf{I} \succeq \mathbf{A}$ (resp. $\mathbf{I} \succ \mathbf{A}$) $\iff \lambda_k(\mathbf{A}) \leq 1$ for all $k$ (resp. $\lambda_k(\mathbf{A}) < 1$ for all $k$)

- If $\mathbf{A}, \mathbf{B} \succ \mathbf{0}$ then $\mathbf{A} \succeq \mathbf{B} \iff \mathbf{B}^{-1} \succeq \mathbf{A}^{-1}$

# Properties of PSD Matrix Inequalities (cont'd)

- For $\mathbf{A} \succeq \mathbf{B} \succeq \mathbf{0}$, $\det(\mathbf{A}) \geq \det(\mathbf{B})$
- For $\mathbf{A} \succeq \mathbf{B}$, $\mathrm{tr}(\mathbf{A}) \geq \mathrm{tr}(\mathbf{B})$
- For $\mathbf{A} \succeq \mathbf{B} \succ \mathbf{0}$, $\mathrm{tr}(\mathbf{A}^{-1}) \leq \mathrm{tr}(\mathbf{B}^{-1})$

# Schur Complement

Let $\mathbf{X} \in \mathbb{R}^{n \times n}$ be partitioned as

$$\mathbf{X} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}$$

where $\mathbf{A} \in \mathbb{R}^{m \times m}$ and $\mathbf{D} \in \mathbb{R}^{(n-m) \times (n-m)}$

Assume $\mathbf{D}$ is nonsingular and solve the linear system

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \mathbf{y} \end{bmatrix} = \begin{bmatrix} \mathbf{c} \\ \mathbf{d} \end{bmatrix}$$

If $\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C}$ is nonsingular, we obtain

$$\begin{aligned} \mathbf{x} &= (\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1}(\mathbf{c} - \mathbf{B}\mathbf{D}^{-1}\mathbf{d}) \\ \mathbf{y} &= \mathbf{D}^{-1}(\mathbf{d} - \mathbf{C}(\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1}(\mathbf{c} - \mathbf{B}\mathbf{D}^{-1}\mathbf{d})) \end{aligned}$$

The matrix $\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C}$ is called the Schur Complement of $\mathbf{D}$ in $\mathbf{X}$

Similarly, when $\mathbf{A}$ is nonsingular, the matrix $\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}$ is the Schur Complement of $\mathbf{A}$ in $\mathbf{X}$

# Schur Complement (cont'd)

Suppose $\mathbf{D}$ and the Schur complement $\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C}$ are nonsingular
Rewrite the solution of the linear system as

$$\mathbf{x} = (\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1}\mathbf{c} - (\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1}\mathbf{B}\mathbf{D}^{-1}\mathbf{d}$$

$$\mathbf{y} = -\mathbf{D}^{-1}\mathbf{C}(\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1}\mathbf{c} + (\mathbf{D}^{-1} + \mathbf{D}^{-1}\mathbf{C}(\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1}\mathbf{B}\mathbf{D}^{-1})\mathbf{d}$$

Then, we derive the inverse of $\mathbf{X}$ as

$$\begin{aligned}
\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}^{-1} &= \begin{bmatrix} (\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1} & -(\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1}\mathbf{B}\mathbf{D}^{-1} \\ -\mathbf{D}^{-1}\mathbf{C}(\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1} & \mathbf{D}^{-1} + \mathbf{D}^{-1}\mathbf{C}(\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1}\mathbf{B}\mathbf{D}^{-1} \end{bmatrix} \\
&= \begin{bmatrix} (\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1} & \mathbf{0} \\ -\mathbf{D}^{-1}\mathbf{C}(\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1} & \mathbf{D}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{I} & -\mathbf{B}\mathbf{D}^{-1} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \\
&= \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{D}^{-1}\mathbf{C} & \mathbf{I} \end{bmatrix} \begin{bmatrix} (\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{D}^{-1} \end{bmatrix} \begin{bmatrix} \mathbf{I} & -\mathbf{B}\mathbf{D}^{-1} \\ \mathbf{0} & \mathbf{I} \end{bmatrix}
\end{aligned}$$

It follows that

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix} = \begin{bmatrix} \mathbf{I} & \mathbf{B}\mathbf{D}^{-1} \\ \mathbf{0} & \mathbf{I} \end{bmatrix} \begin{bmatrix} \mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C} & \mathbf{0} \\ \mathbf{0} & \mathbf{D} \end{bmatrix} \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ \mathbf{D}^{-1}\mathbf{C} & \mathbf{I} \end{bmatrix}$$

# Schur Complement (cont'd)

Previously, if $\mathbf{D}$ and the Schur complement $\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C}$ are nonsingular,

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}^{-1} = \begin{bmatrix} (\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1} & -(\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1}\mathbf{B}\mathbf{D}^{-1} \\ -\mathbf{D}^{-1}\mathbf{C}(\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1} & \mathbf{D}^{-1} + \mathbf{D}^{-1}\mathbf{C}(\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1}\mathbf{B}\mathbf{D}^{-1} \end{bmatrix}$$

Now suppose $\mathbf{A}$ and the Schur complement $\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}$ are nonsingular. Likewise,

$$\begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{C} & \mathbf{D} \end{bmatrix}^{-1} = \begin{bmatrix} \mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{B}(\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1}\mathbf{C}\mathbf{A}^{-1} & -\mathbf{A}^{-1}\mathbf{B}(\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1} \\ -(\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1}\mathbf{C}\mathbf{A}^{-1} & (\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1} \end{bmatrix}$$

Compare the above two expressions of $\mathbf{X}^{-1}$. If $\mathbf{A}, \mathbf{D}$ and both Schur complements $\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C}$, $\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B}$ are nonsingular, then

$$(\mathbf{A} - \mathbf{B}\mathbf{D}^{-1}\mathbf{C})^{-1} = \mathbf{A}^{-1} + \mathbf{A}^{-1}\mathbf{B}(\mathbf{D} - \mathbf{C}\mathbf{A}^{-1}\mathbf{B})^{-1}\mathbf{C}\mathbf{A}^{-1}$$

By setting $\mathbf{D} = \mathbf{I}$ and $\mathbf{B}' = -\mathbf{B}$, the above equation leads to the *matrix inversion lemma*

$$(\mathbf{A} + \mathbf{B}'\mathbf{C})^{-1} = \mathbf{A}^{-1} - \mathbf{A}^{-1}\mathbf{B}'(\mathbf{I} + \mathbf{C}\mathbf{A}^{-1}\mathbf{B}')^{-1}\mathbf{C}\mathbf{A}^{-1}$$

# Schur Complement of PSD Matrices

Let $\mathbf{X} \in \mathbb{S}^n$ and partition it as

$$\mathbf{X} = \begin{bmatrix} \mathbf{A} & \mathbf{B} \\ \mathbf{B}^T & \mathbf{C} \end{bmatrix}$$

where $\mathbf{A} \in \mathbb{S}^n$ and $\mathbf{C} \in \mathbb{S}^{n-m}$

The Schur complements are $\mathbf{A} - \mathbf{B}\mathbf{C}^{-1}\mathbf{B}^T$ and $\mathbf{C} - \mathbf{B}^T\mathbf{A}^{-1}\mathbf{B}$

**Properties**:

- With nonsingular $\mathbf{C}$, $\mathbf{X} \succ \mathbf{0} \iff \mathbf{C} \succ \mathbf{0}$ and $\mathbf{A} - \mathbf{B}\mathbf{C}^{-1}\mathbf{B}^T \succ \mathbf{0}$
- With $\mathbf{C} \succ \mathbf{0}$, $\mathbf{X} \succeq \mathbf{0} \iff \mathbf{A} - \mathbf{B}\mathbf{C}^{-1}\mathbf{B}^T \succeq \mathbf{0}$

- With nonsingular $\mathbf{A}$, $\mathbf{X} \succ \mathbf{0} \iff \mathbf{A} \succ \mathbf{0}$ and $\mathbf{C} - \mathbf{B}^T\mathbf{A}^{-1}\mathbf{B} \succ \mathbf{0}$
- With $\mathbf{A} \succ \mathbf{0}$, $\mathbf{X} \succeq \mathbf{0} \iff \mathbf{C} - \mathbf{B}^T\mathbf{A}^{-1}\mathbf{B} \succeq \mathbf{0}$

**Example**: For any $\mathbf{b} \in \mathbb{R}^n$ and any symmetric and PD $\mathbf{C}$,

$$1 - \mathbf{b}^T\mathbf{C}^{-1}\mathbf{b} \geq 0 \quad \iff \quad \mathbf{C} - \mathbf{b}\mathbf{b}^T \succeq \mathbf{0}$$

# Important Facts for Proving the Properties of Schur Complement

Let $\mathbf{Y} = \begin{bmatrix} \mathbf{I} & \mathbf{0} \\ -\mathbf{C}^{-1}\mathbf{B}^T & \mathbf{I} \end{bmatrix}$, which is nonsingular. Then consider $\mathbf{Y}^T\mathbf{X}\mathbf{Y}$