

Convex Optimization Problems

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Outline

- 1 Optimization Problems
- 2 Convex Optimization
- 3 Quasi-Convex Optimization
- 4 Classes of Convex Problems: LP, QP, SOCP, SDP

Optimization Problems in Standard Form I

$$\begin{array}{ll}\underset{\mathbf{x}}{\text{minimize}} & f_0(\mathbf{x}) \\ \text{subject to} & f_i(\mathbf{x}) \leq 0 \quad i = 1, \dots, m \\ & h_i(\mathbf{x}) = 0 \quad i = 1, \dots, p\end{array}$$

• $\mathbf{x} = (x_1, \dots, x_n)$ is the optimization variable

• $f_0 : \mathbb{R}^n \rightarrow \mathbb{R}$ is the objective function

• $f_i : \mathbb{R}^n \rightarrow \mathbb{R} \quad i = 1, \dots, m$ are the inequality constraint functions

• $h_i : \mathbb{R}^n \rightarrow \mathbb{R} \quad i = 1, \dots, p$ are the equality constraint functions

Optimization Problems in Standard Form II

Feasibility:

- a point $\mathbf{x} \in \text{dom } f_0$ is feasible if it satisfies all the constraints and infeasible otherwise
- a problem is feasible if it has at least one feasible point and infeasible otherwise.

Optimal value:

$$p^* = \inf\{f_0(\mathbf{x}) \mid f_i(\mathbf{x}) \leq 0, i = 1, \dots, m, h_i(\mathbf{x}) = 0, i = 1, \dots, p\}$$

- $p^* = \infty$ if problem is infeasible (no \mathbf{x} satisfies the constraints)
- $p^* = -\infty$ if problem is unbounded below

Optimal solution: \mathbf{x}^* such that $f(\mathbf{x}^*) = p^*$ (and \mathbf{x}^* feasible).

Global and Local Optimality

- A feasible x is **optimal** if $f_0(x) = p^*$; X_{opt} is the set of optimal points.
- A feasible x is **locally optimal** if it is optimal within a ball, i.e., there is an $R > 0$ such that x is optimal for

$$\begin{array}{ll}\underset{z}{\text{minimize}} & f_0(z) \\ \text{subject to} & f_i(z) \leq 0 \quad i = 1, \dots, m \\ & h_i(z) = 0 \quad i = 1, \dots, p \\ & \|z - x\|_2 \leq R\end{array}$$

Example:

- $f_0(x) = 1/x$, $\text{dom } f_0 = \mathbb{R}_{++}$: $p^* = 0$, no optimal point
- $f_0(x) = x^3 - 3x$: $p^* = -\infty$, local optimum at $x = 1$.

Implicit Constraints

- The standard form optimization problem has an explicit constraint:

$$\mathbf{x} \in \mathcal{D} = \bigcap_{i=0}^m \text{dom } f_i \cap \bigcap_{i=1}^p \text{dom } h_i$$

- \mathcal{D} is the domain of the problem
- The constraints $f_i(\mathbf{x}) \leq 0, h_i(\mathbf{x}) = 0$ are the explicit constraints
- A problem is unconstrained if it has no explicit constraints
- Example:

$$\underset{\mathbf{x}}{\text{minimize}} \quad \log(b - \mathbf{a}^T \mathbf{x})$$

is an unconstrained problem with implicit constraint $b > \mathbf{a}^T \mathbf{x}$

Feasibility Problem

- ⚡ Sometimes, we don't really want to minimize any objective, just to find a feasible point:

$$\begin{array}{ll}\underset{\mathbf{x}}{\text{find}} & \mathbf{x} \\ \text{subject to} & f_i(\mathbf{x}) \leq 0 \quad i = 1, \dots, m \\ & h_i(\mathbf{x}) = 0 \quad i = 1, \dots, p\end{array}$$

- ⚡ This feasibility problem can be considered as a special case of a general problem:

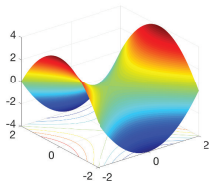
$$\begin{array}{ll}\underset{\mathbf{x}}{\text{minimize}} & 0 \\ \text{subject to} & f_i(\mathbf{x}) \leq 0 \quad i = 1, \dots, m \\ & h_i(\mathbf{x}) = 0 \quad i = 1, \dots, p\end{array}$$

where $p^* = 0$ if constraints are feasible and $p^* = \infty$ otherwise.

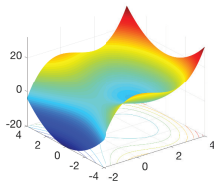
Stationary Points

Given a smooth function $f : \mathbb{R}^n \rightarrow \mathbb{R}$, a point $x \in \mathbb{R}^n$ is called

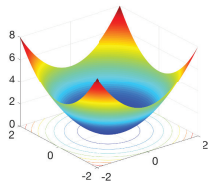
- A **stationary point**, if $\nabla f(x) = 0$;
- A **local minimum**, if x is a stationary point and there exists a neighborhood $\mathcal{B} \subseteq \mathbb{R}^n$ of x such that $f(x) \leq f(y)$ for any $y \in \mathcal{B}$;
- A **global minimum**, if x is a stationary point and $f(x) \leq f(y)$ for any $y \in \mathbb{R}^n$;
- **Saddle point**, if x is a stationary point and for any neighborhood $\mathcal{B} \subseteq \mathbb{R}^n$ of x , there exist $y, z \in \mathcal{B}$ such that $f(z) \leq f(x) \leq f(y)$ and $\lambda_{\min}(\nabla^2 f(x)) \leq 0$.



(a) strict saddle



(b) local minimum



(c) global minimum

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Convex Optimization Problem

- Convex optimization problem in standard form:

$$\begin{array}{ll}\underset{\mathbf{x}}{\text{minimize}} & f_0(\mathbf{x}) \\ \text{subject to} & f_i(\mathbf{x}) \leq 0 \quad i = 1, \dots, m \\ & \mathbf{Ax} = \mathbf{b}\end{array}$$

where f_0, f_1, \dots, f_m are convex and equality constraints are affine.

- **Local and global optima:** any locally optimal point of a convex problem is globally optimal
- Most problems are not convex when formulated
- Reformulating a problem in convex form is an art, there is no systematic way

Example

- The following problem is nonconvex (why not?):

$$\begin{array}{ll}\underset{x}{\text{minimize}} & x_1^2 + x_2^2 \\ \text{subject to} & x_1/(1 + x_2^2) \leq 0 \\ & (x_1 + x_2)^2 = 0\end{array}$$

- The objective is convex.
- The equality constraint function is not affine; however, we can rewrite it as $x_1 = -x_2$ which is then a linear equality constraint.
- The inequality constraint function is not convex; however, we can rewrite it as $x_1 \leq 0$ which again is linear.
- We can rewrite it as

$$\begin{array}{ll}\underset{x}{\text{minimize}} & x_1^2 + x_2^2 \\ \text{subject to} & x_1 \leq 0 \\ & x_1 = -x_2\end{array}$$

Global and Local Optimality

Any locally optimal point of a convex problem is globally optimal.

Proof: Suppose \mathbf{x} is locally optimal (around a ball of radius R) and \mathbf{y} is optimal with $f_0(\mathbf{y}) < f_0(\mathbf{x})$. We will show this cannot be.

Just take the segment from \mathbf{x} to \mathbf{y} : $\mathbf{z} = \theta\mathbf{y} + (1 - \theta)\mathbf{x}$.

Obviously the objective function is strictly decreasing along the segment since $f_0(\mathbf{y}) < f_0(\mathbf{x})$:

$$\theta f_0(\mathbf{y}) + (1 - \theta)f_0(\mathbf{x}) < f_0(\mathbf{x}) \quad \theta \in (0, 1]$$

Using now the convexity of the function, we can write

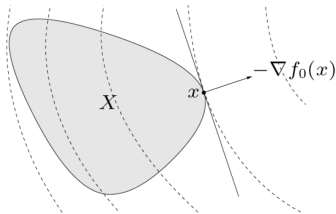
$$f_0(\theta\mathbf{y} + (1 - \theta)\mathbf{x}) < f_0(\mathbf{x}) \quad \theta \in (0, 1]$$

Finally, just choose θ sufficiently small such that the point \mathbf{z} is in the ball of local optimality of \mathbf{x} , arriving at a contradiction.

Optimality Criterion for Differentiable f_0 I

Minimum Principle: A feasible point x is optimal if and only if

$$\nabla f_0(x)^T(y - x) \geq 0 \quad \text{for all feasible } y$$



Optimality Criterion for Differentiable f_0 II

- **Unconstrained problem:** x is optimal iff

$$x \in \text{dom } f_0, \quad \nabla f_0(x) = 0$$

- **Equality constrained problem:** $\min_x f_0(x)$ s.t. $Ax = b$
 x is optimal iff

$$x \in \text{dom } f_0, \quad Ax = b, \quad \nabla f_0(x) + A^T \nu = 0$$

- **Minimization over nonnegative orthant:** $\min_x f_0(x)$ s.t. $x \succeq 0$
 x is optimal iff

$$x \in \text{dom } f_0, \quad x \succeq 0, \quad \begin{cases} \nabla_i f_0(x) \geq 0 & x_i = 0 \\ \nabla_i f_0(x) = 0 & x_i > 0 \end{cases}$$

Equivalent Reformulations I

🔊 Eliminating/introducing equality constraints:

$$\begin{array}{ll}\underset{\mathbf{x}}{\text{minimize}} & f_0(\mathbf{x}) \\ \text{subject to} & f_i(\mathbf{x}) \leq 0 \quad i = 1, \dots, m \\ & \mathbf{Ax} = \mathbf{b}\end{array}$$

is equivalent to

$$\begin{array}{ll}\underset{\mathbf{z}}{\text{minimize}} & f_0(\mathbf{Fz} + \mathbf{x}_0) \\ \text{subject to} & f_i(\mathbf{Fz} + \mathbf{x}_0) \leq 0 \quad i = 1, \dots, m\end{array}$$

where \mathbf{F} and \mathbf{x}_0 are such that $\mathbf{Ax} = \mathbf{b} \iff \mathbf{x} = \mathbf{Fz} + \mathbf{x}_0$ for some \mathbf{z} .

Equivalent Reformulations II

☛ Introducing slack variables for linear inequalities:

$$\begin{array}{ll}\underset{\mathbf{x}}{\text{minimize}} & f_0(\mathbf{x}) \\ \text{subject to} & \mathbf{a}_i^T \mathbf{x} \leq b_i \quad i = 1, \dots, m\end{array}$$

is equivalent to

$$\begin{array}{ll}\underset{\mathbf{x}, \mathbf{s}}{\text{minimize}} & f_0(\mathbf{x}) \\ \text{subject to} & \mathbf{a}_i^T \mathbf{x} + s_i = b_i \quad i = 1, \dots, m \\ & s_i \geq 0\end{array}$$

Equivalent Reformulations III

• **Epigraph form:** a standard form convex problem is equivalent to

$$\begin{array}{ll}\underset{\mathbf{x}, t}{\text{minimize}} & t \\ \text{subject to} & f_0(\mathbf{x}) - t \leq 0 \\ & f_i(\mathbf{x}) \leq 0 \quad i = 1, \dots, m \\ & \mathbf{Ax} = \mathbf{b}\end{array}$$

Equivalent Reformulations IV

• Minimizing over some variables:

$$\begin{array}{ll}\underset{\mathbf{x}, \mathbf{y}}{\text{minimize}} & f_0(\mathbf{x}, \mathbf{y}) \\ \text{subject to} & f_i(\mathbf{x}) \leq 0 \quad i = 1, \dots, m\end{array}$$

is equivalent to

$$\begin{array}{ll}\underset{\mathbf{x}}{\text{minimize}} & \tilde{f}_0(\mathbf{x}) \\ \text{subject to} & f_i(\mathbf{x}) \leq 0 \quad i = 1, \dots, m\end{array}$$

where $\tilde{f}_0(\mathbf{x}) = \inf_{\mathbf{y}} f_0(\mathbf{x}, \mathbf{y})$

Outline

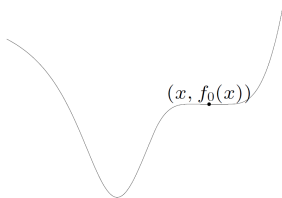
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Quasiconvex Optimization

$$\begin{array}{ll}\underset{x}{\text{minimize}} & f_0(x) \\ \text{subject to} & f_i(x) \leq 0 \quad i = 1, \dots, m \\ & Ax = b\end{array}$$

where $f_0 : \mathbb{R}^n \rightarrow \mathbb{R}$ is quasiconvex and f_1, \dots, f_m are convex

- 🦉 Observe that it can have locally optimal points that are not (globally) optimal:



Quasiconvex Optimization

- **Convex representation** of sublevel sets of a quasiconvex function f_0 : there exists a family of convex functions $\phi_t(\mathbf{x})$ for fixed t such that

$$f_0(\mathbf{x}) \leq t \iff \phi_t(\mathbf{x}) \leq 0$$

- **Example:**

$$f_0(\mathbf{x}) = \frac{p(\mathbf{x})}{q(\mathbf{x})}$$

with p convex, q concave, and $p(\mathbf{x}) \geq 0$, $q(\mathbf{x}) > 0$ on $\text{dom } f_0$. We can choose:

$$\phi_t(\mathbf{x}) = p(\mathbf{x}) - tq(\mathbf{x})$$

- for $t \geq 0$, $\phi_t(\mathbf{x})$ is convex in \mathbf{x}
- $p(\mathbf{x})/q(\mathbf{x}) \leq t$ if and only if $\phi_t(\mathbf{x}) \leq 0$

Quasiconvex Optimization

Solving a quasiconvex problem via convex feasibility problems: the idea is to solve the epigraph form of the problem with a sandwich technique in t :

- for fixed t the epigraph form of the original problem reduces to a feasibility convex problem

$$\phi_t(\mathbf{x}) \leq 0, \quad f_i(\mathbf{x}) \leq 0 \forall i, \quad \mathbf{Ax} \leq \mathbf{b}$$

- if t is too small, the feasibility problem will be infeasible
- if t is too large, the feasibility problem will be feasible
- start with upper and lower bounds on t (termed u and l , resp.) and use a sandwich technique (bisection method): at each iteration use $t = (l + u)/2$ and update the bounds according to the feasibility or infeasibility of the problem.

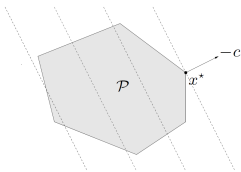
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Linear Programming (LP)

$$\begin{array}{ll}\underset{x}{\text{minimize}} & c^T x + d \\ \text{subject to} & Gx \leq h \\ & Ax = b\end{array}$$

- Convex problem: affine objective and constraint functions.
- Feasible set is a polyhedron:



ℓ_1 - and ℓ_∞ - Norm Problems as LPs I

☛ ℓ_∞ -norm minimization:

$$\begin{array}{ll}\underset{x}{\text{minimize}} & \|x\|_\infty \\ \text{subject to} & Gx \leq h \\ & Ax = b\end{array}$$

is equivalent to the LP

$$\begin{array}{ll}\underset{t, x}{\text{minimize}} & t \\ \text{subject to} & -t\mathbf{1} \preceq x \preceq t\mathbf{1} \\ & Gx \leq h \\ & Ax = b\end{array}$$

ℓ_1 - and ℓ_∞ - Norm Problems as LPs II

☛ ℓ_1 -norm minimization:

$$\begin{array}{ll}\text{minimize}_{\mathbf{x}} & \|\mathbf{x}\|_1 \\ \text{subject to} & \mathbf{G}\mathbf{x} \leq \mathbf{h} \\ & \mathbf{A}\mathbf{x} = \mathbf{b}\end{array}$$

is equivalent to the LP

$$\begin{array}{ll}\text{minimize}_{\mathbf{t}, \mathbf{x}} & \sum_i t_i \\ \text{subject to} & -\mathbf{t} \preceq \mathbf{x} \preceq \mathbf{t} \\ & \mathbf{G}\mathbf{x} \leq \mathbf{h} \\ & \mathbf{A}\mathbf{x} = \mathbf{b}\end{array}$$

Examples: Chebyshev Center of a Polyhedron I

- Chebyshev center of a polyhedron

$$\mathcal{P} = \{\mathbf{x} \mid \mathbf{a}_i^T \mathbf{x} \leq b_i, i = 1, \dots, m\}$$

is center of largest inscribed ball

$$\mathcal{B} = \{\mathbf{x}_c + \mathbf{u} \mid \|\mathbf{u}\| \leq r\}$$

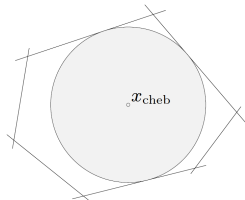
- Let's solve the problem

$$\begin{array}{ll} \text{maximize} & r \\ & r, \mathbf{x}_c \end{array}$$

$$\text{subject to} \quad \mathbf{x} \in \mathcal{P} \quad \text{for all} \quad \mathbf{x} = \mathbf{x}_c + \mathbf{u} \text{ with } \|\mathbf{u}\| \leq r$$

- Observe that $\mathbf{a}_i^T \mathbf{x} \leq b_i$ for all $\mathbf{x} \in \mathcal{B}$ if and only if

$$\sup_{\mathbf{u}} \{\mathbf{a}_i^T (\mathbf{x}_c + \mathbf{u}) \mid \|\mathbf{u}\| \leq r\} \leq b_i$$



Examples: Chebyshev Center of a Polyhedron II

- Using Schwartz inequality, the supremum condition can be rewritten as

$$\mathbf{a}_i^T \mathbf{x}_c + r \|\mathbf{a}_i\|_2 \leq b_i$$

- Hence, the Chebyshev center can be obtained by solving:

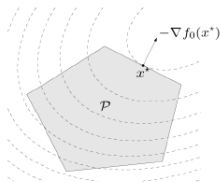
$$\begin{array}{ll} \underset{r, \mathbf{x}_c}{\text{maximize}} & r \\ \text{subject to} & \mathbf{a}_i^T \mathbf{x}_c + r \|\mathbf{a}_i\|_2 \leq b_i, \quad i = 1, \dots, m \end{array}$$

which is an LP.

Quadratic Programming (QP)

$$\begin{array}{ll}\underset{x}{\text{minimize}} & (1/2) x^T P x + q^T x + r \\ \text{subject to} & Gx \leq h \\ & Ax = b\end{array}$$

- Convex problem (assuming $P \in \mathbb{S}_+^n \succeq 0$): convex quadratic objective and affine constraint functions.
- Minimization of a convex quadratic function over a polyhedron:



Quadratically Constrained QP (QCQP)

$$\begin{array}{ll}\underset{\mathbf{x}}{\text{minimize}} & (1/2) \mathbf{x}^T \mathbf{P}_0 \mathbf{x} + \mathbf{q}_0^T \mathbf{x} + r_0 \\ \text{subject to} & (1/2) \mathbf{x}^T \mathbf{P}_i \mathbf{x} + \mathbf{q}_i^T \mathbf{x} + r_i \leq 0 \quad i = 1, \dots, m \\ & \mathbf{A} \mathbf{x} = \mathbf{b}\end{array}$$

- Convex problem (assuming $\mathbf{P}_i \in \mathbb{S}_+^n \succeq \mathbf{0}$): convex quadratic objective and constraint functions.

Second-Order Cone Programming (SOCP)

$$\begin{array}{ll}\underset{\mathbf{x}}{\text{minimize}} & \mathbf{f}^T \mathbf{x} \\ \text{subject to} & \|\mathbf{A}_i \mathbf{x} + \mathbf{b}_i\| \leq \mathbf{c}_i^T \mathbf{x} + d_i \quad i = 1, \dots, m \\ & \mathbf{F} \mathbf{x} = \mathbf{g}\end{array}$$

- Convex problem: linear objective and second-order cone constraints
- For \mathbf{A}_i row vector, it reduces to an LP
- For $\mathbf{c}_i = \mathbf{0}$, it reduces to a QCQP
- More general than QCQP and LP

Robust LP as an SOCP

🐼 Sometimes, the parameters of an optimization problem are imperfect

🐼 Consider the robust LP:

$$\begin{array}{ll}\underset{\mathbf{x}}{\text{minimize}} & \mathbf{c}^T \mathbf{x} \\ \text{subject to} & \mathbf{a}_i^T \mathbf{x} \leq b_i \quad \forall \mathbf{a}_i \in \mathcal{E}_i, \quad i = 1, \dots, m\end{array}$$

where $\mathcal{E}_i = \{\bar{\mathbf{a}}_i + \mathbf{P}_i \mathbf{u} \mid \|\mathbf{u}\| \leq 1\}$

🐼 It can be rewritten as the SOCP:

$$\begin{array}{ll}\underset{\mathbf{x}}{\text{minimize}} & \mathbf{c}^T \mathbf{x} \\ \text{subject to} & \bar{\mathbf{a}}_i^T \mathbf{x} + \|\mathbf{P}_i^T \mathbf{x}\|_2 \leq b_i \quad i = 1, \dots, m\end{array}$$

Generalized Inequality Constraints

- Convex problem with generalized inequality constraints:

$$\begin{array}{ll}\underset{x}{\text{minimize}} & f_0(x) \\ \text{subject to} & f_i(x) \preceq_{K_i} \mathbf{0} \quad i = 1, \dots, m \\ & Ax = b\end{array}$$

where f_0 is convex and f_i are K_i -convex w.r.t. proper cone K_i

- It has the same properties as a standard convex problem
- **Conic form problem:** special case with affine objective and constraints:

$$\begin{array}{ll}\underset{x}{\text{minimize}} & c^T x \\ \text{subject to} & Fx + g \preceq_K \mathbf{0} \\ & Ax = b\end{array}$$

Semidefinite Programming (SDP)

$$\begin{array}{ll}\underset{x}{\text{minimize}} & \mathbf{c}^T \mathbf{x} \\ \text{subject to} & x_1 \mathbf{F}_1 + x_2 \mathbf{F}_2 + \cdots + x_n \mathbf{F}_n \preceq \mathbf{G} \\ & \mathbf{A}\mathbf{x} = \mathbf{b}\end{array}$$

- Inequality constraint is called linear matrix inequality (LMI)
- Convex problem: linear objective and linear matrix inequality (LMI) constraints
- Observe that multiple LMI constraints can always be written as a single one

SDP I

• LP and equivalent SDP:

$$\begin{array}{ll} \underset{x}{\text{minimize}} & \mathbf{c}^T \mathbf{x} \\ \text{subject to} & \mathbf{A}\mathbf{x} \preceq \mathbf{b} \end{array} \qquad \begin{array}{ll} \underset{x}{\text{minimize}} & \mathbf{c}^T \mathbf{x} \\ \text{subject to} & \text{diag}(\mathbf{A}\mathbf{x} - \mathbf{b}) \preceq \mathbf{0} \end{array}$$

• SOCP and equivalent SDP:

$$\begin{array}{ll} \underset{x}{\text{minimize}} & \mathbf{f}^T \mathbf{x} \\ \text{subject to} & \|\mathbf{A}_i \mathbf{x} + \mathbf{b}_i\| \leq \mathbf{c}_i^T \mathbf{x} + d_i, \quad i = 1, \dots, m \end{array}$$

$$\begin{array}{ll} \underset{x}{\text{minimize}} & \mathbf{f}^T \mathbf{x} \\ \text{subject to} & \begin{bmatrix} (\mathbf{c}_i^T \mathbf{x} + d_i) \mathbf{I} & \mathbf{A}_i \mathbf{x} + \mathbf{b}_i \\ \mathbf{A}_i \mathbf{x} + \mathbf{b}_i & \mathbf{c}_i^T \mathbf{x} + d_i \end{bmatrix} \succeq \mathbf{0}, \quad i = 1, \dots, m \end{array}$$

SDP II

• Eigenvalue minimization:

$$\underset{\mathbf{x}}{\text{minimize}} \quad \lambda_{\max}(\mathbf{A}(\mathbf{x}))$$

where $\mathbf{A}(\mathbf{x}) = \mathbf{A}_0 + x_1\mathbf{A}_1 + \cdots + x_n\mathbf{A}_n$, is equivalent to SDP

$$\begin{array}{ll} \underset{\mathbf{x}, t}{\text{minimize}} & t \\ \text{subject to} & \mathbf{A}(\mathbf{x}) \preceq t\mathbf{I} \end{array}$$

• It follows from

$$\lambda_{\max}(\mathbf{A}(\mathbf{x})) \leq t \iff \mathbf{A}(\mathbf{x}) \preceq t\mathbf{I}$$

Reference

Chapter 4 of:

- Stephen Boyd and Lieven Vandenberghe, *Convex Optimization*. Cambridge, U.K.: Cambridge University Press, 2004.