**EE150-Signals and System, FALL 2024**

Homework Set #4

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**Problem 1.** (20 points) Determine the Fourier transform of the following signals:

(a) $x(t) = \begin{cases} \cos(\pi t), & |t| \le 1 \\ 0, & |t| > 1 \end{cases}$

(b) $x(t) = \cos(6t + \frac{\pi}{4})$

(c) As shown in the Figure 1, $x(t)$ is a continuous periodic signal with fundamental period T = 6:

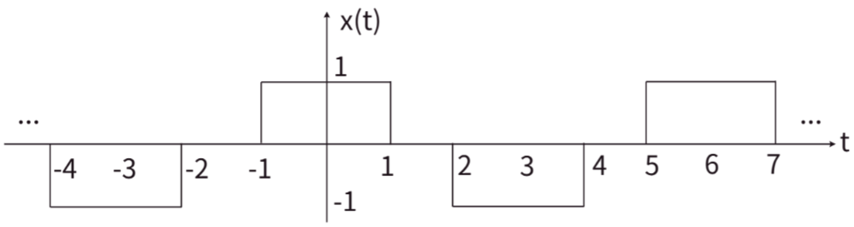


Figure 1

**Solution:**

(a) $X(j\omega) = \int_{-\infty}^{\infty} x(t) e^{-j\omega t} dt = \int_{-1}^{1} \cos(\pi t)\, e^{-j\omega t} dt \ = \frac{1}{2}\int_{-1}^{1} (e^{j\pi t} + e^{-j\pi t}) e^{-j\omega t} dt$

$$= \frac{1}{2j(\pi-\omega)}(e^{j(\pi-\omega)} - e^{-j(\pi-\omega)}) + \frac{1}{2j(\pi+w)}(e^{j(\pi+\omega)} - e^{-j(\pi+\omega)})$$

$$= \frac{\sin(\pi-\omega)}{\pi-\omega} + \frac{\sin(\pi+\omega)}{\pi+\omega} = \frac{\sin\omega}{\pi-\omega} - \frac{\sin\omega}{\pi+\omega}$$

(b) Method 1:

$$\cos\left(6t + \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}\cos(6t) - \frac{\sqrt{2}}{2}\sin(6t)$$

And $\cos(6t) = \frac{e^{j6t}+e^{-j6t}}{2}$, $\sin(6t) = \frac{e^{j6t}-e^{-j6t}}{2j}$

So $\cos\left(6t + \frac{\pi}{4}\right) = \frac{\sqrt{2}\ +j\sqrt{2}}{4} e^{j6t} + \frac{\sqrt{2}\ -j\sqrt{2}}{2} e^{-j6t}$

$\because 1 \overset{\mathcal{F}}{\leftrightarrow} 2\pi\delta(\omega)$, $\therefore$ $e^{j6t} \cdot 1 \overset{\mathcal{F}}{\leftrightarrow} 2\pi\delta(\omega-6)$ and $e^{-j6t} \cdot 1 \overset{\mathcal{F}}{\leftrightarrow} 2\pi\delta(\omega+6)$ frequency shifting properties:

So $X(j\omega) = \frac{\sqrt{2}\ \pi + j\sqrt{2}\ \pi}{2}\delta(\omega - 6) + \frac{\sqrt{2}\ \pi - j\sqrt{2}\ \pi}{2}\delta(\omega + 6)$

Method 2:

$$\cos\left(6t + \frac{\pi}{4}\right) = \frac{\sqrt{2}}{2}\cos(6t) - \frac{\sqrt{2}}{2}\sin(6t)$$

Let $x_1(t) = \frac{\sqrt{2}}{2}\cos(6t)$, considering FS: $a_1 = a_{-1} = \frac{\sqrt{2}}{4}$, and $a_k = 0$ when $k \neq \pm 1$

So $X_1(j\omega) = \sum_{k=-\infty}^{\infty} a_k\, 2\pi\delta(\omega - k\omega_0) = \frac{\sqrt{2}\ \pi}{2}\delta(\omega-6) + \frac{\sqrt{2}\ \pi}{2}\delta(\omega+6)$

Let $x_2(t) = \frac{\sqrt{2}}{2}\sin(6t)$, $a_1 = \frac{\sqrt{2}}{4j}, a_{-1} = -\frac{\sqrt{2}}{4j}$, and $a_k = 0$ when $k \neq \pm 1$

So $X_2(j\omega) = \sum_{k=-\infty}^{\infty} a_k\, 2\pi\delta(\omega - k\omega_0) = \frac{-j\sqrt{2}\ \pi}{2}\delta(\omega-6) + \frac{j\sqrt{2}\ \pi}{2}\delta(\omega+6)$

$$X(j\omega) = X_1(j\omega) - X_2(j\omega) = \frac{\sqrt{2}\ \pi + j\sqrt{2}\ \pi}{2}\delta(\omega-6) + \frac{\sqrt{2}\ \pi - j\sqrt{2}\ \pi}{2}\delta(\omega+6)$$

(c) Let $x_1(t)$ be a periodic signal as shown in the Figure 2, then $x(t) = x_1(t) - x_1(t-3)$

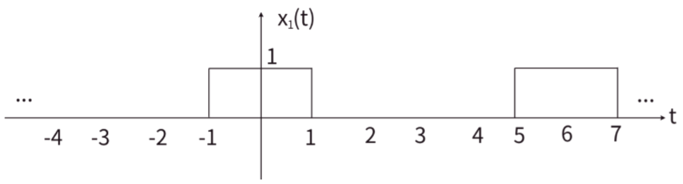


Figure 2

Considering Fourier Series, then $x_1(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}$, where $\omega_0 = \frac{2\pi}{T} = \frac{\pi}{3}$

$$a_k = \frac{1}{T}\int_{-1}^{1} x_1(t)e^{-jk\omega_0 t}dt = -\frac{1}{jk\omega_0 T}\left(e^{-jk\omega_0} - e^{jk\omega_0}\right) = \frac{2}{k\omega_0 T}\left(\frac{e^{jk\omega_0} - e^{-jk\omega_0}}{2j}\right)$$

$$= \frac{2\sin(k\omega_0)}{k\omega_0 T} = \frac{\sin\left(\frac{k\pi}{3}\right)}{k\pi}$$

So $X_1(j\omega) = 2\pi\sum_{k=-\infty}^{\infty} \frac{\sin\left(\frac{k\pi}{3}\right)}{k\pi}\delta\left(\omega - \frac{k\pi}{3}\right)$

$x(t) = x_1(t) - x_1(t-3)$

So $X(j\omega) = X_1(j\omega) - X_1(j\omega)e^{-3j\omega} = 2\pi(1 - e^{-3j\omega})\sum_{k=-\infty}^{\infty} \frac{\sin\left(\frac{k\pi}{3}\right)}{k\pi}\delta\left(\omega - \frac{k\pi}{3}\right)$

**Problem 2.** (15 points) Determine the inverse Fourier transform of $X(j\omega)$:

(a) $X(j\omega) = u(\omega-2) - u(\omega-4)$

(b) $X(j\omega) = 2\cos(3\omega)$

(c) $X(j\omega)$ as shown in the Figure 3:

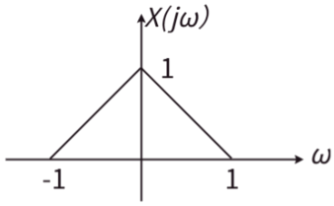


Figure 3

**Solution:**

(a) Let $G(j\omega)=u(\omega+1)-u(\omega-1)$, then $X(j\omega)=G(j(\omega-3))$

$$g(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}G(j\omega)e^{j\omega t}d\omega=\frac{1}{2\pi}\int_{-1}^{1}e^{j\omega t}d\omega=\frac{1}{2\pi}\frac{e^{jt}-e^{-jt}}{jt}=\frac{1}{\pi}\frac{\text{sint}}{t}$$

Then using frequency shifting properties:

$$\because X(j\omega)=G(j(\omega-3)) \qquad \therefore\ x(t)=e^{j3t}g(t)=e^{j3t}\ \frac{1}{\pi}\frac{\text{sint}}{t}$$

(b) $X(j\omega)=2\cos(3\omega)=e^{j3\omega}+e^{-j3\omega}$

$\because \delta(\text{t})\overset{\mathcal{F}}{\leftrightarrow}1 \quad \therefore \delta(t+3)+\delta(t-3)\ \overset{\mathcal{F}}{\leftrightarrow}\ e^{j3\omega}+e^{-j3\omega}$ (time shifting property)

So $X(j\omega)\overset{\mathcal{F}^{-1}}{\longleftrightarrow}\ \delta(t+3)+\delta(t-3)$

(c) Method 1: using the Fourier inverse transform

$$x(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}X(j\omega)e^{j\omega t}d\omega=\frac{1}{2\pi}\int_{-1}^{0}(\omega+1)e^{j\omega t}d\omega+\frac{1}{2\pi}\int_{0}^{1}(1-\omega)e^{j\omega t}d\omega$$

$$=\frac{1}{2\pi}\left(\frac{1-e^{-jt}}{jt}+\frac{e^{-jt}}{jt}+\frac{1-e^{-jt}}{t^2}\right)+\frac{1}{2\pi}\left(\frac{e^{jt}-1}{jt}-\frac{e^{jt}}{jt}+\frac{1-e^{jt}}{t^2}\right)$$

$$=\frac{1}{2\pi}\left(\frac{2-e^{-jt}-e^{jt}}{t^2}\right)=\frac{1}{\pi}\left(\frac{1-\cos t}{t^2}\right)$$

Method 2:

Let $X_1(j\omega)=\frac{d}{d\omega}X(j\omega)$, $X_1(j\omega)$ as shown in the Figure 4.

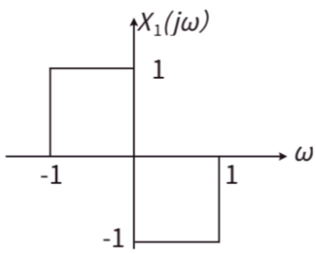


Figure 4

$$x_1(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X_1(j\omega)e^{j\omega t}d\omega = \frac{1}{2\pi}\left(\int_{-1}^{0} e^{j\omega t}d\omega - \int_{0}^{1} e^{j\omega t}d\omega\right)$$

$$= \frac{2-(e^{jt}+e^{-jt})}{2\pi jt}$$

$$= \frac{1-\cos t}{j\pi t}$$

Using frequency domain integrals properties, then $x(t) = -\frac{x_1(t)}{jt} + \pi x_1(0)\delta(t)$

$x_1(0) = \lim_{t\to 0}\frac{1-\cos t}{j\pi t} = 0$, so $x(t) = -\frac{x_1(t)}{jt} = -\frac{\frac{1-\cos t}{j\pi t}}{jt} = \frac{1}{\pi}\left(\frac{1-\cos t}{t^2}\right)$

**Problem 3.** (15 points)

(a) Determine the Fourier transform of the following signal:

$$x(t) = te^{-2|t|}$$

(b) Use the result from part (a), along with the duality property, to determine the Fourier transform of the following signal:

$$f(t) = \frac{8t}{(4+t^2)^2}$$

**Solution:**

(a) $e^{-2|t|} \overset{\mathcal{F}}{\leftrightarrow} \frac{4}{4+\omega^2}$ $\quad\therefore te^{-2|t|} \overset{\mathcal{F}}{\leftrightarrow} j\frac{d}{d\omega}\left(\frac{4}{4+\omega^2}\right) = \frac{-8j\omega}{(4+\omega^2)^2}$ (frequency domain differential property)

(b) $f(t) = \frac{8t}{(4+t^2)^2} \xrightarrow{t=\omega} \frac{8\omega}{(4+\omega^2)^2}$

From part (a), $\frac{8\omega}{(4+\omega^2)^2} = j\cdot\frac{-8j\omega}{(4+\omega^2)^2} \overset{\mathcal{F}^{-1}}{\longleftrightarrow} jte^{-2|t|}$

Then using duality properties, $F(j\omega) = 2\pi\cdot(-j\omega e^{-2|\omega|}) = -2j\pi\omega e^{-2|\omega|}$

**Problem 4.** (15 points) Let $X(j\omega)$ denotes the Fourier transform of the signal x(t) depicted in the Figure 5:

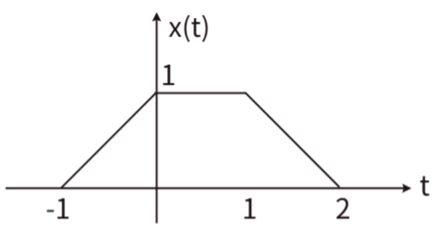


Figure 5

(a) Determine $\int_{-\infty}^{\infty} X(j\omega)d\omega$

(b) Sketch the inverse Fourier transform of $Re\{X(j\omega)\}$ and $Im\{X(j\omega)\}$

**Solution:**

(a) Considering Fourier inverse transform: $x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$

$$x(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega 0}d\omega == \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)d\omega$$

So $\int_{-\infty}^{\infty} X(j\omega)d\omega = \ 2\pi x(0) = 2\pi$

(b) $Re\{X(j\omega)\} \overset{\mathcal{F}^{-1}}{\longleftrightarrow} \frac{1}{2}(x(t) + x(-t)$, so the inverse Fourier transform of $Re\{X(j\omega)\}$ is shown in the Figure 6:

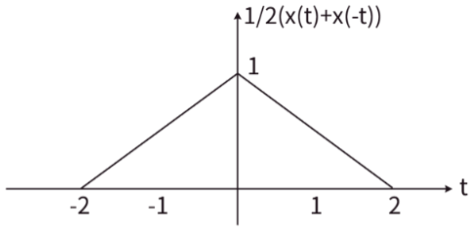


Figure 6

$Im\{X(j\omega)\} \overset{\mathcal{F}^{-1}}{\longleftrightarrow} \frac{1}{2j}(x(t) - x(-t))$, so the inverse Fourier transform of $Im\{X(j\omega)\}$ is shown in the Figure 7:

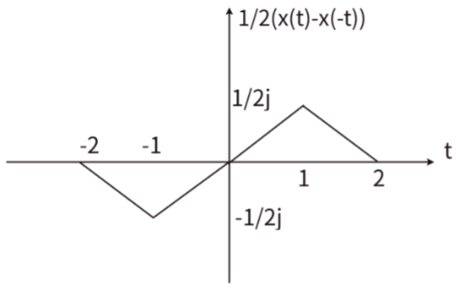


Figure 7

**Problem 5.** (15 points) Consider a signal x(t) with Fourier transform $X(j\omega)$. Suppose we are given the following facts:

(a) $x(t)$ is real and nonnegative

(b) $Ae^{-t}u(t) \overset{\mathcal{F}}{\leftrightarrow} (1 + j\omega)X(j\omega)$

(c) $\int_{-\infty}^{\infty} |\ X(j\omega)\ |^2\ d\omega = \ 2\pi$

Determine a closed-form expression for x(t)

**Solution:**

$Ae^{-t}u(t) \overset{\mathcal{F}}{\leftrightarrow} \frac{A}{j\omega+1}$

From condition (b): $Ae^{-t}u(t) \overset{\mathcal{F}}{\leftrightarrow} (1+j\omega)X(j\omega)$

So $\frac{A}{j\omega+1} = (1+j\omega)X(j\omega)$ $\quad\quad$ $\therefore\ X(j\omega) = \frac{A}{(j\omega+1)^2} = A \cdot j\frac{d}{d\omega}\frac{1}{j\omega+1}$

$\mathcal{F}^{-1}\{X(j\omega)\} = At \cdot \mathcal{F}^{-1}\left\{\frac{1}{j\omega+1}\right\} = Ate^{-t}u(t)$ frequency domain differential propoerty

From condition (c), then using Parseval's relation:

$$\int_{-\infty}^{\infty} |\, x(t)\,|^2\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\, X(j\omega)\,|^2\, d\omega = \frac{1}{2\pi}\cdot 2\pi = 1$$

So $\int_{-\infty}^{\infty} |\, x(t)\,|^2\, dt = \int_{0}^{\infty} |\, Ate^{-t}\,|^2\, dt = \frac{2!}{2^3}A^2 = \frac{A^2}{4} = 1,$ $\quad\therefore$ A=±2

From condition (a): $x(t)$ is real and nonnegative, then A = 2, $x(t) = 2te^{-t}u(t)$

PS:

$$\int_{0}^{\infty} t^n\, e^{-\alpha t} dt = \frac{n!}{\alpha^{n+1}}$$

**Problem 6.** (20 points) A causal and stable LTI system has the frequency response:

$$H(j\omega) = \frac{j\omega - 1}{(j\omega)^2 + 5j\omega + 6}$$

(a) Determine a differential equation relating the input x(t) and output y(t) of the LTI system

(b) What is the output of the LTI system when the input is $x(t) = e^{-t}u(t)$

(c) What is the output of the LTI system when the input is $x(t) = \sqrt{3}\sin\left(t + \frac{\pi}{4}\right)$

**Solution:**

(a) $H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{j\omega-1}{(j\omega)^2+5j\omega+6}$, so $((j\omega)^2 + 5j\omega + 6)\cdot Y(j\omega) = (j\omega - 1)\cdot X(j\omega)$

Using differentia property, $\frac{d}{dt}y(t) \overset{\mathcal{F}}{\leftrightarrow} j\omega Y(j\omega)$, $\frac{d^2}{dt^2}y(t) \overset{\mathcal{F}}{\leftrightarrow} (j\omega)^2 Y(j\omega)$

So the differential equation is $\frac{d^2}{dt^2}y(t) + 5\frac{d}{dt}y(t) + 6y(t) = \frac{d}{dt}x(t) - x(t)$

(b) $x(t) = e^{-t}u(t) \overset{\mathcal{F}}{\leftrightarrow} X(j\omega) = \frac{1}{(j\omega+1)}$, then $Y(j\omega) = X(j\omega)\cdot H(j\omega) = \frac{1}{(jw+1)}\cdot\frac{j\omega-1}{(j\omega)^2+5j\omega+6}$

$$Y(j\omega) = \frac{j\omega - 1}{(j\omega+1)(j\omega+2)(j\omega+3)} = \frac{A}{(j\omega+1)} + \frac{B}{(j\omega+2)} + \frac{C}{(j\omega+3)}$$

And $A = \frac{-1-1}{(-1+2)(-1+3)} = -1,\ B = \frac{-2-1}{(-2+1)(-2+3)} = 3,\ C = \frac{-3-1}{(-3+1)(-3+2)} = -2$

So $y(t) = -e^{-t}u(t) + 3e^{-2t}u(t) - 2e^{-3t}u(t)$

(c) Let $\omega$=1, $H(j1) = \frac{j-1}{-1+5j+6} = \frac{j-1}{5j+5} = \frac{1}{5}\frac{j-1}{j+1}$

$| H(j1) |= \frac{1}{5}\frac{\sqrt{1+1}}{\sqrt{1+1}} = \frac{1}{5}$ , $\angle H(j1) = \text{arctang}\left(\frac{1}{-1}\right) - \text{arctang}\left(\frac{1}{1}\right) = -\frac{\pi}{4} - \frac{\pi}{4} = -\frac{\pi}{2}$

So $y(t) = \frac{1}{5}\cdot\sqrt{3}\sin\left(t + \frac{\pi}{4} - \frac{\pi}{2}\right) = \frac{\sqrt{3}}{5}\sin\left(t - \frac{\pi}{4}\right)$