Lecture 10 Image Reconstruction

(Problem Definition)

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SIST Building-3 420

Outline

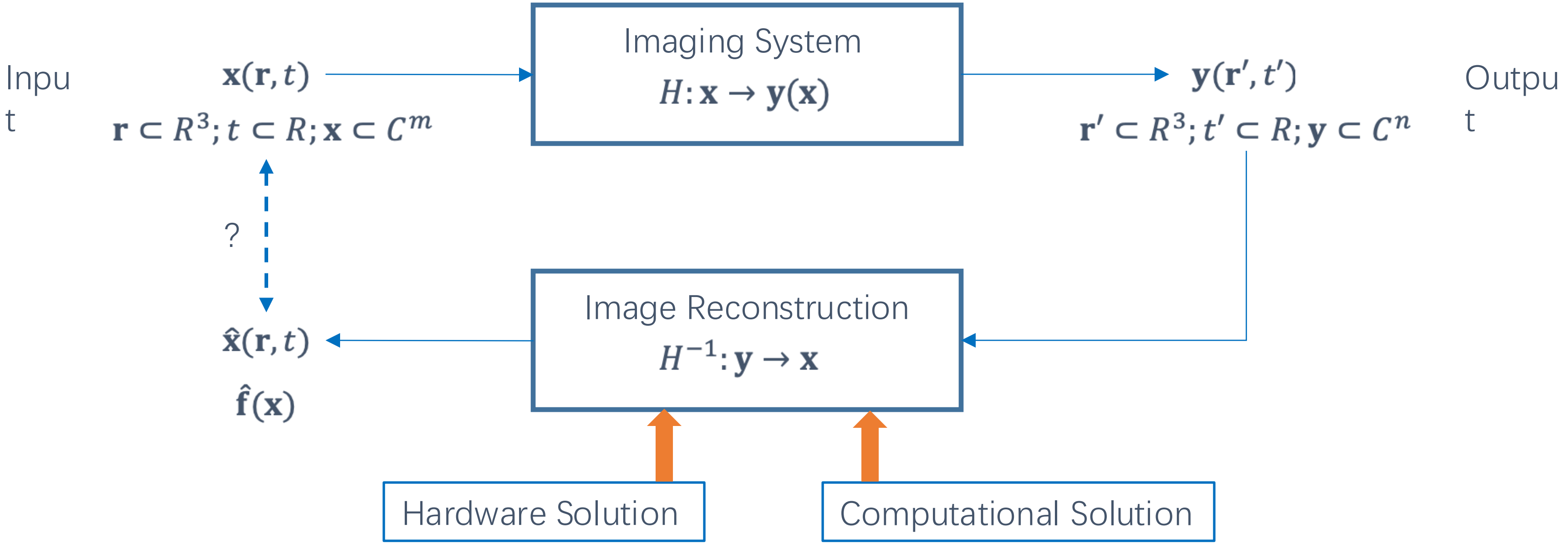
– Projection and back-projection

– Radon transform

– Fourier-Slice Theorem

– Filtered back-projection

Image Reconstruction: A System's View



1. Is H even known?

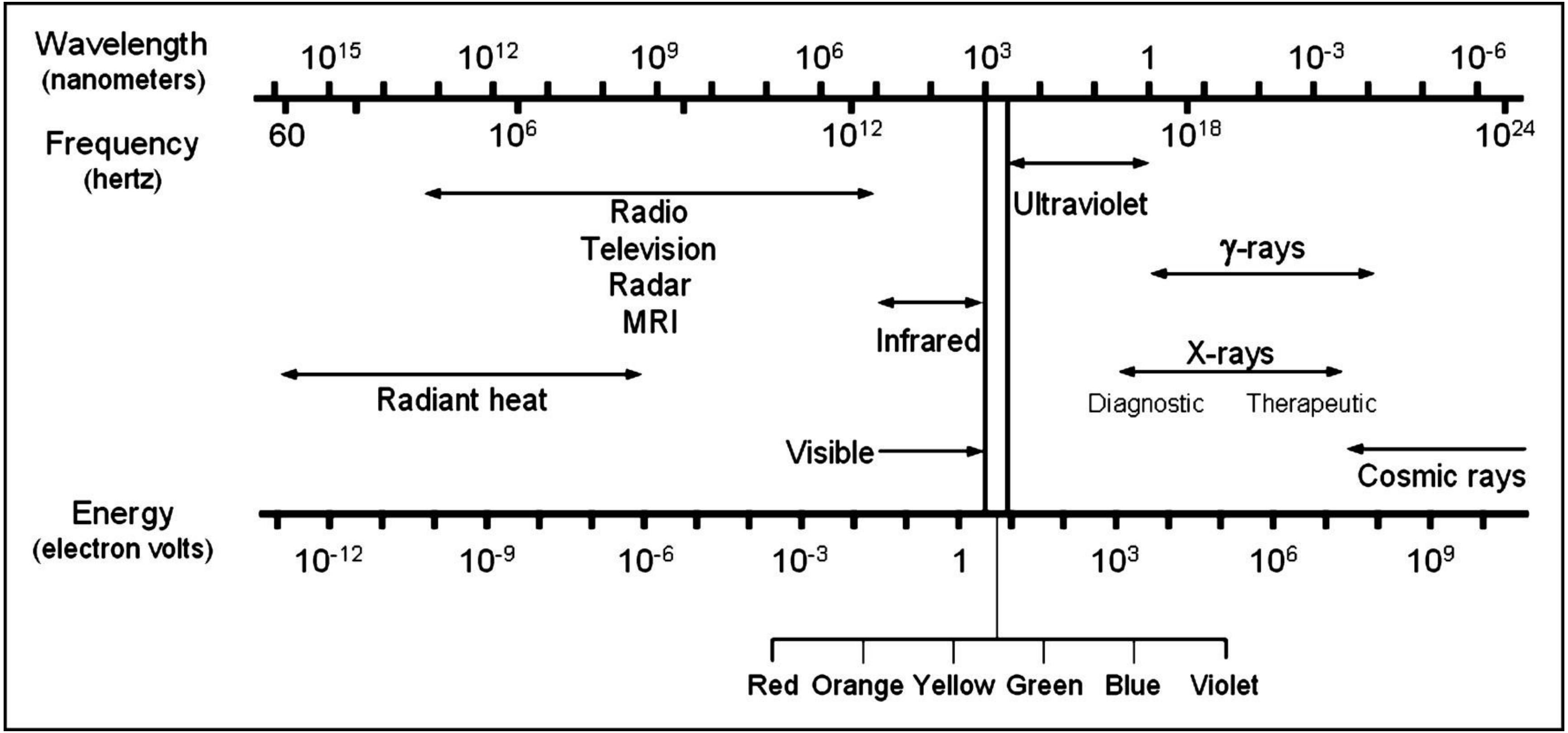
2. What's the evaluation criteria for reconstruction?

Reconstruction Evaluation Criteria

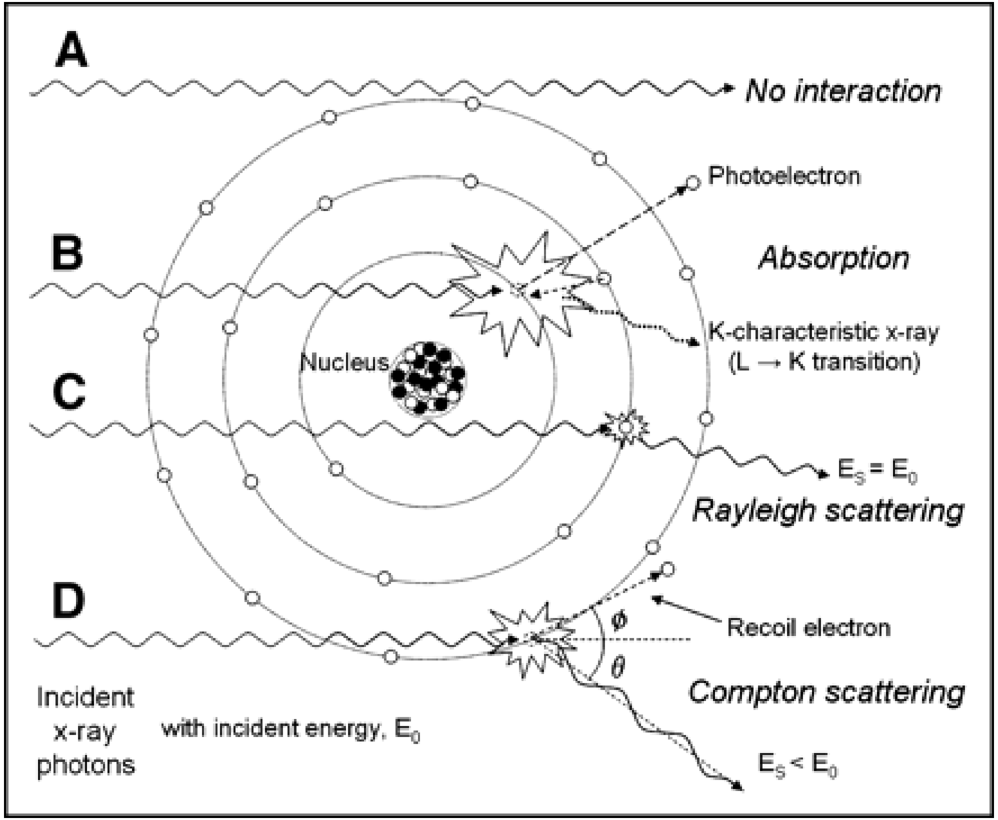
- Least Mean Square Errors: needs to know the ground truth. Typically done in numerical simulation;
- Point Spread Functions: evaluating image blurring;
- Signal-to-Noise (SNR): sensitivity to noise amplification;
- Contrast-to-Noise (CNR) ratio: e.g. sampling density correction in gridding;
- Ultimate test: real applications, e.g. by clinical practice.

X-ray Tomography Imaging Principle

• Electromagnetic spectrum



• X-ray interactions with matter



Illustrative summary of x-ray and γ-ray interactions.

(A) Primary, unattenuated beam does not interact with material.

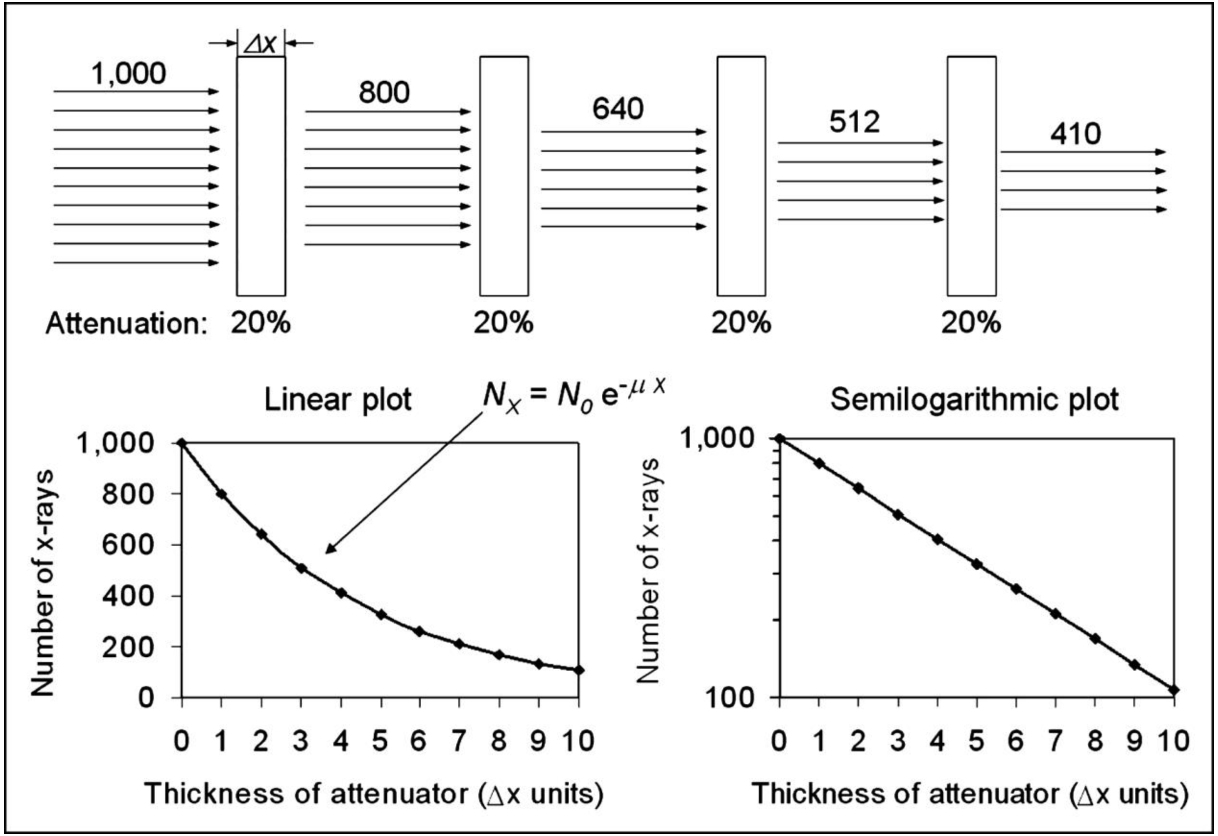
(B) Photoelectric absorption.

(C) Rayleigh scattering.

(D) Compton scattering.

[1] Seibert JA, Boone JM. X-ray imaging physics for nuclear medicine technologists. Part 2: X-ray interactions and image formation. J Nucl Med Technol. 2005 Mar;33(1):3-18. PMID: 15731015.

• Linear Attenuation Coefficient



For a ***monoenergetic beam*** of N_0 photons incident on a thin slab of material of thickness x , the intensity attenuation of X-rays can be expressed by the following equation:

$$\mathrm{N}_x = \mathrm{N}_0 * e^{-\mu x}$$

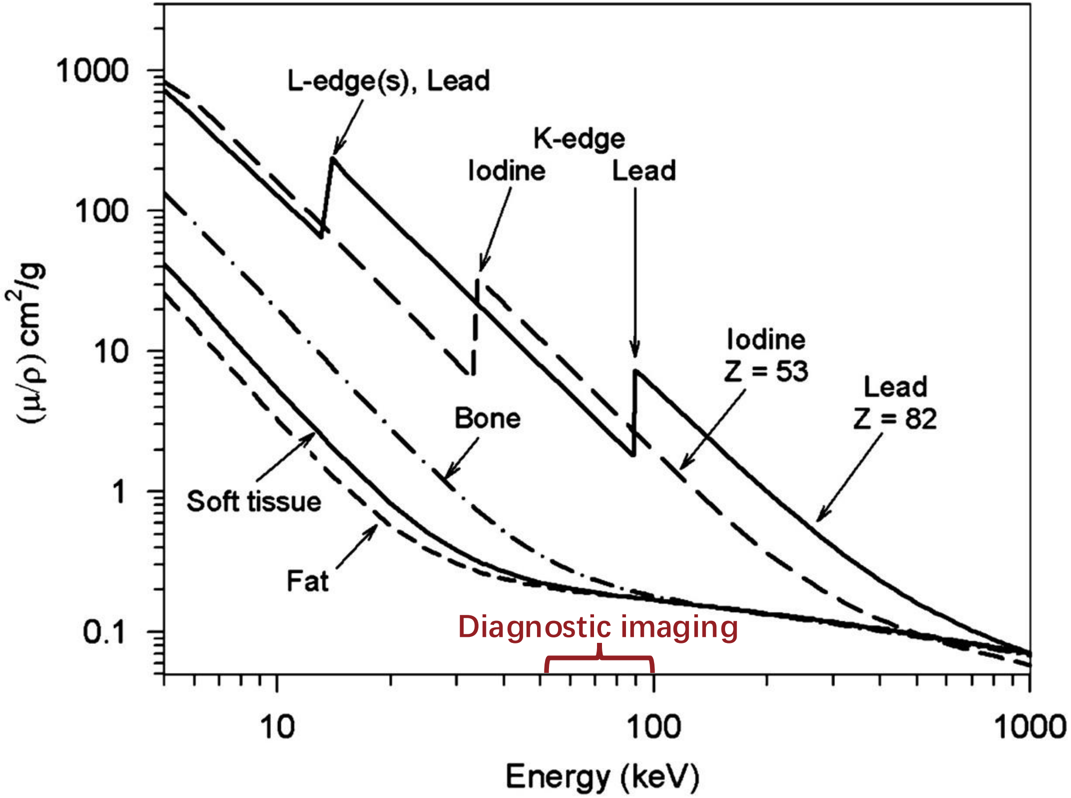
•N_x is the intensity of X-rays after passing through a material with thickness x;

•N_0 is the initial intensity of the incident X-rays;

•μ the linear attenuation coefficient, describing the material's ability to absorb and scatter X-rays;

•x is the thickness of the material the X-rays pass through.

• Mass attenuation coefficient



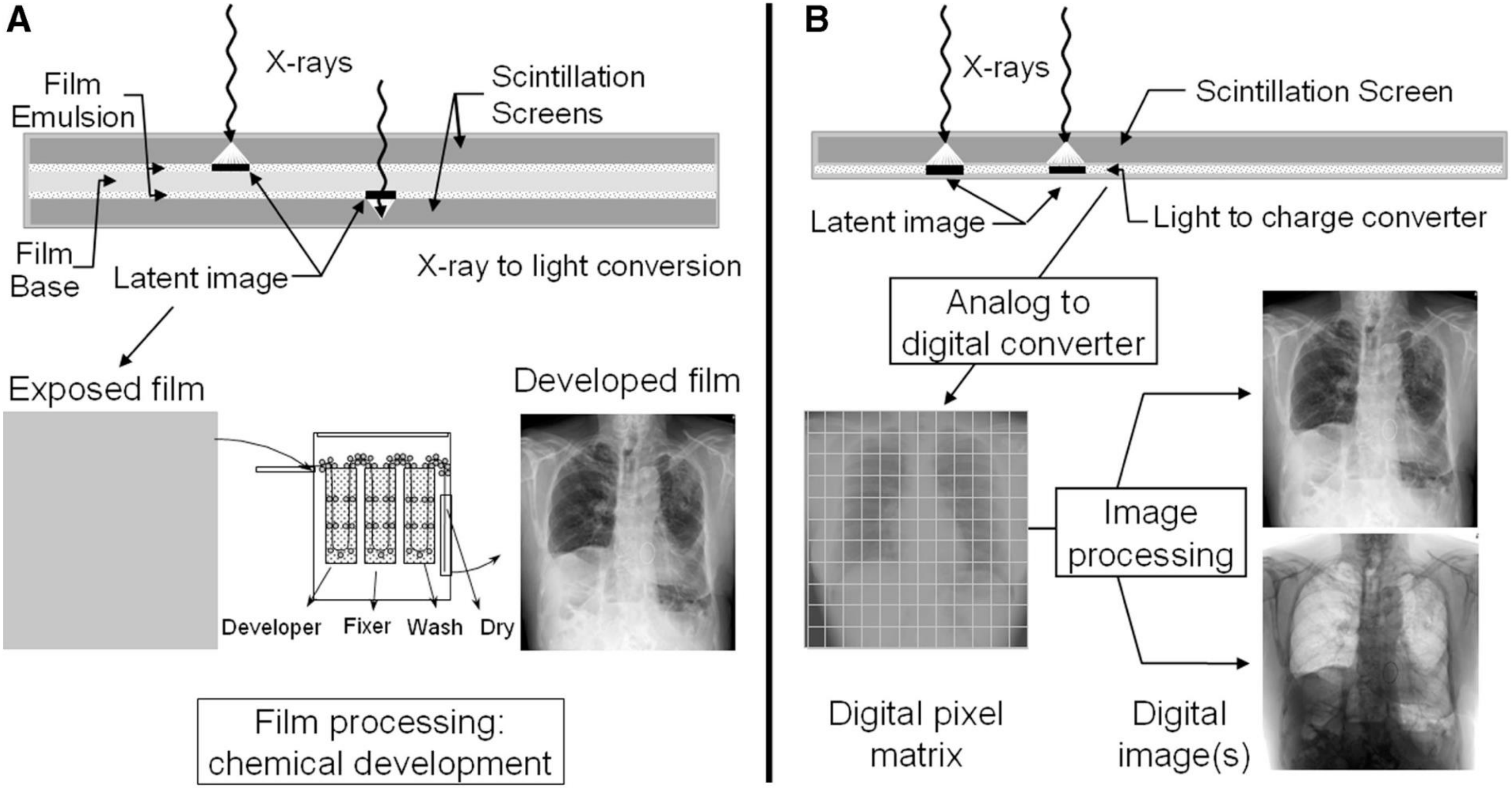
- Mass attenuation coefficient (μ/ρ) of several materials are illustrated as function of energy.

- The X-ray beams used in diagnostic imaging are typically **low-energy X-rays**, usually in the range of **50 to 150 kiloelectron volts (keV)**.

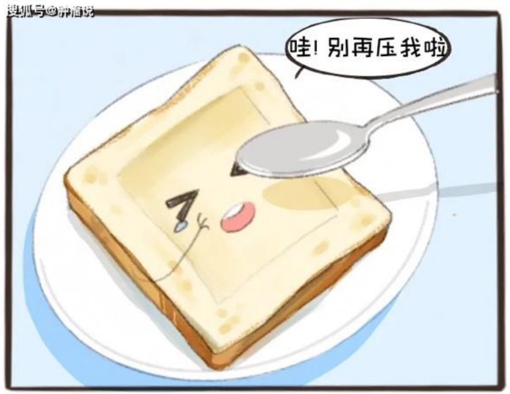
Q: The amount of energy loss of the X-ray photon depends on what conditions?

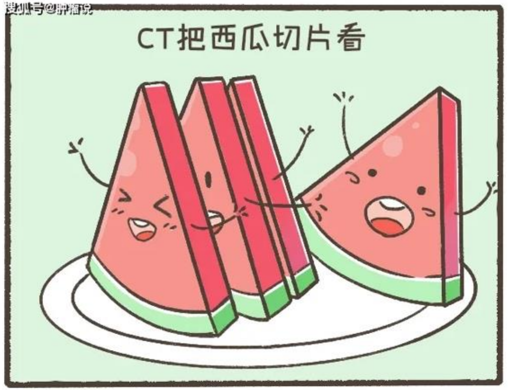
A. Density of the matter.
B. The types of atoms contained in this matter.
C. The energy level carried by X-ray photons.
~~D. The time to be traversed through the object.~~
E. The distance to be traversed through the object.

• X-ray image acquisition

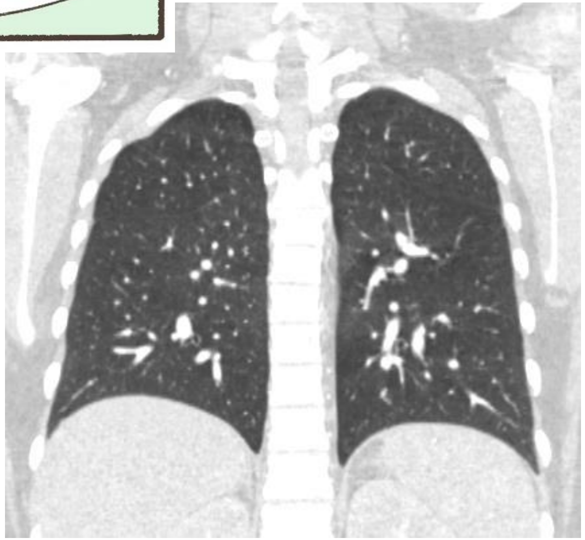


• Different between X-ray image and CT





X-ray image



Computed Tomography (CT)

A brief timeline of CT

Nobel Prize in Physics

1901

Röntgen's discovery of X-rays

Maturation of Mathematical Theory in CT Reconstruction

1956

First CT scanner

1974

The first CT scanner was invented by the British engineer Sir Godfrey Hounsfield.

Nobel Prize in Physiology or Medicine

1979

CT is approved for clinical use

Single-slice helical CT

1998

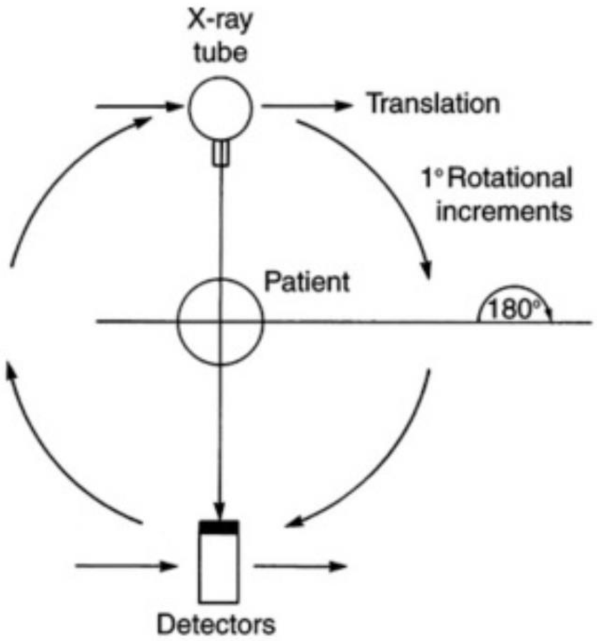
Multi-slice helical CT

2008

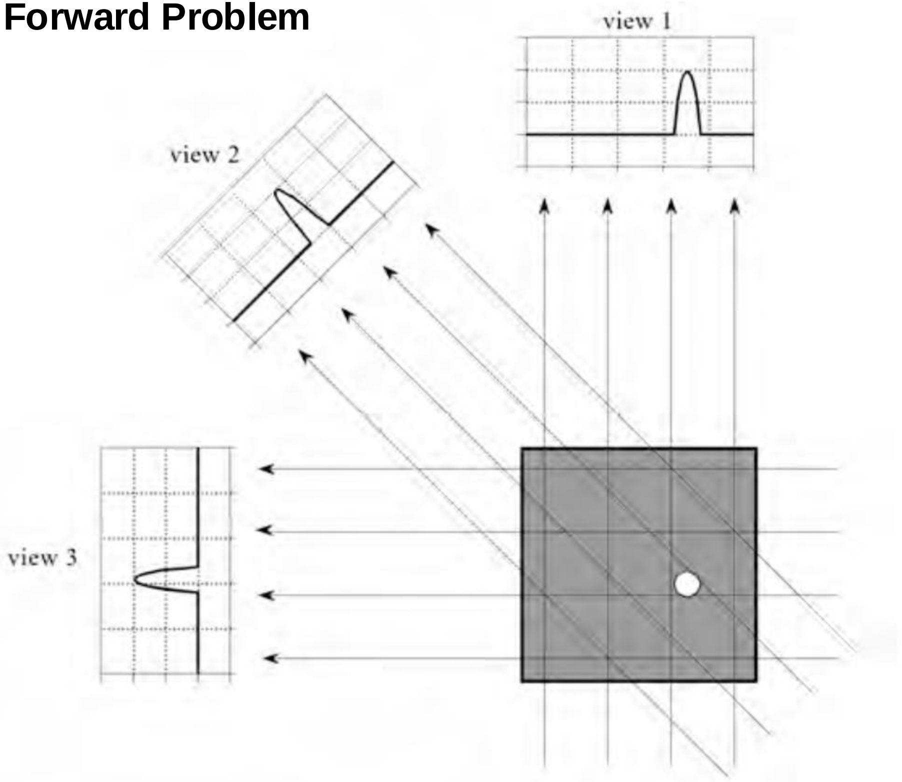
Dual-energy CT
Cone Beam CT
Photo-counting CT

Nowdays

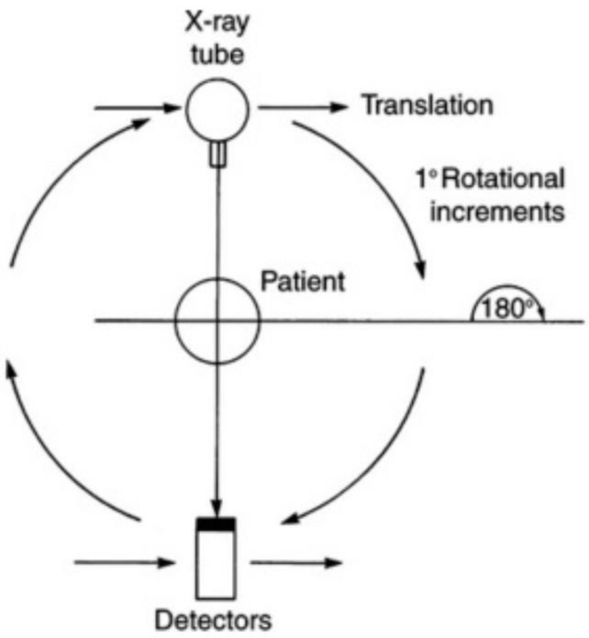
• X-ray Computed Tomography (CT)



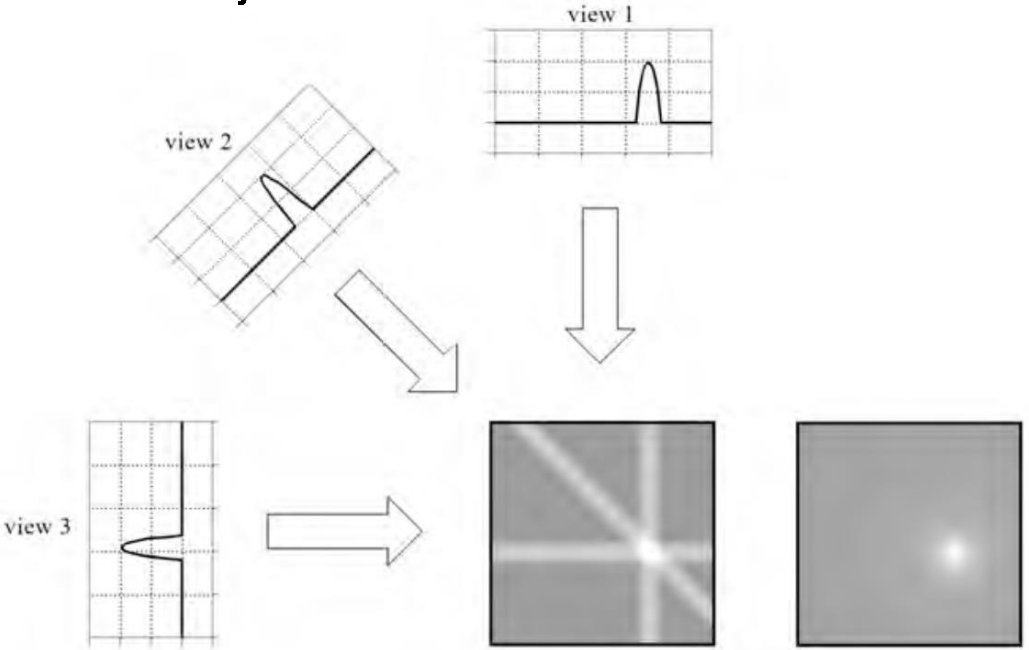
Forward Problem



• CT reconstruction: Back Projection

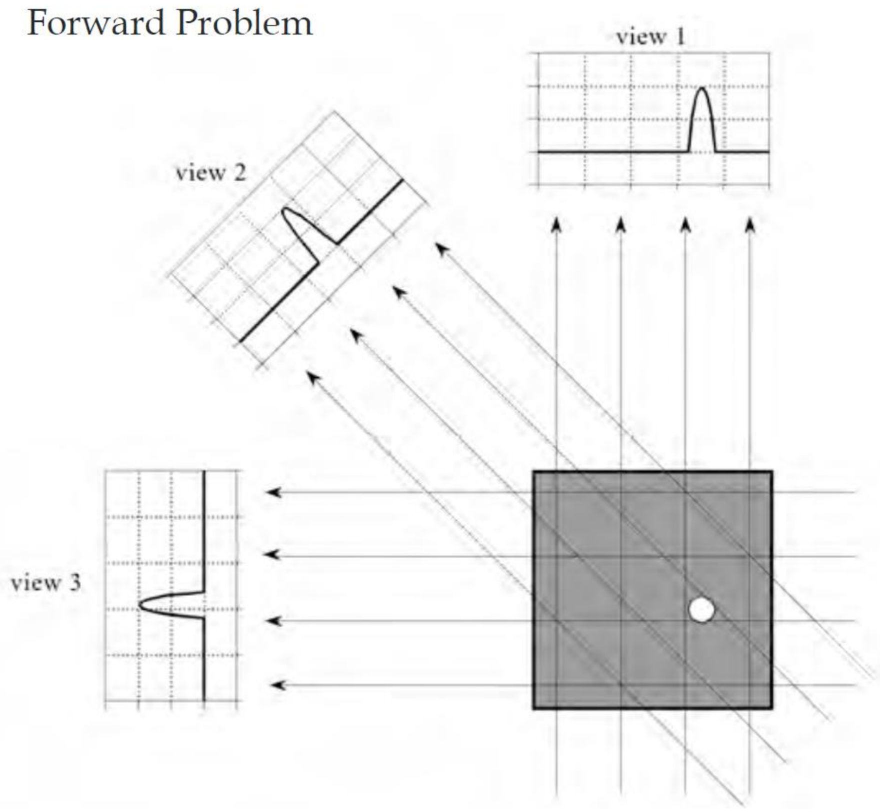


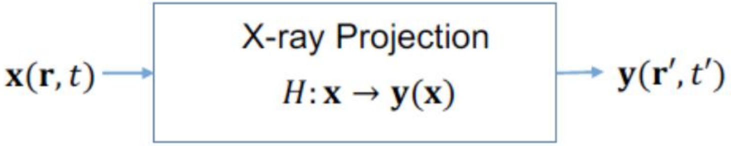
Inverse Problem
Back Projection



a. Using 3 views   b. Using many views

• CT reconstruction: Discrete Optimization





$$\mathbf{y} = \mathbf{A}\mathbf{x}$$

$$\hat{\mathbf{x}} = \underset{\mathbf{x}}{\operatorname{argmin}} \|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2$$

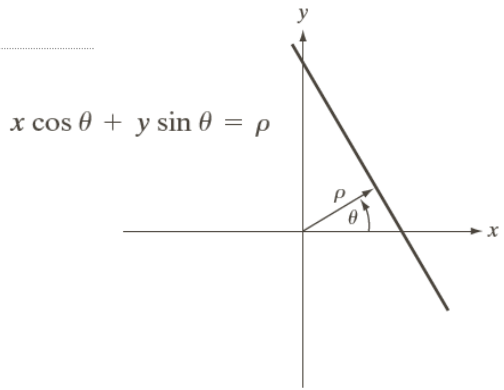
$$s.t. \ \ some\ constraints$$

Radon Transform &
Fourier-Slice Theorem

• Radon Transform

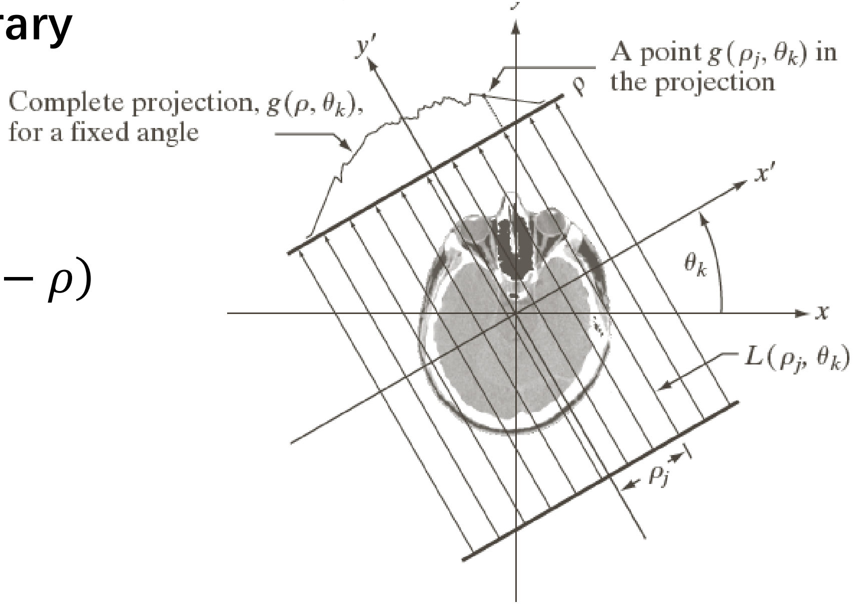
- Normal representation for a line:

$$x \cos\theta + y \sin\theta = \rho$$



- The projection of $f(x, y)$ along an arbitrary line in the xy-plane:

$$g(\rho, \theta) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x, y)\delta(x \cos\theta + y \sin\theta - \rho)$$



• Radon Transform

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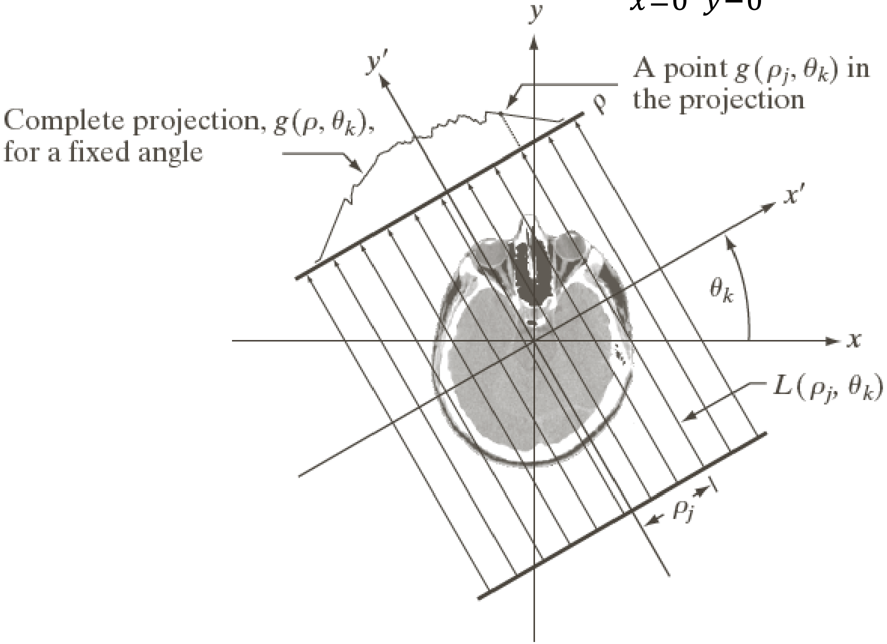
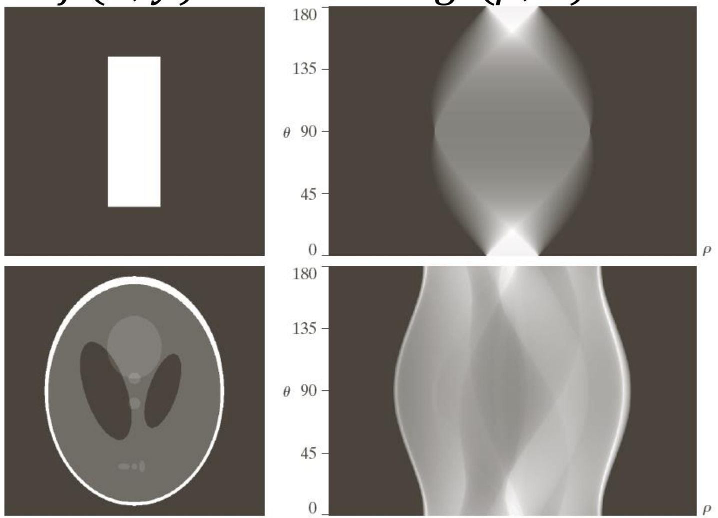


Image domain $f(x, y)$

Sensory domain/Sinogram $g\ (\rho, \theta)$



• Fourier-Slice Theorem

The 1D FT of a projection with respect of ρ:

$$G(\omega,\theta)=\int_{-\infty}^{\infty} g(\rho,\theta)e^{-j2\pi\omega\rho}d\rho$$

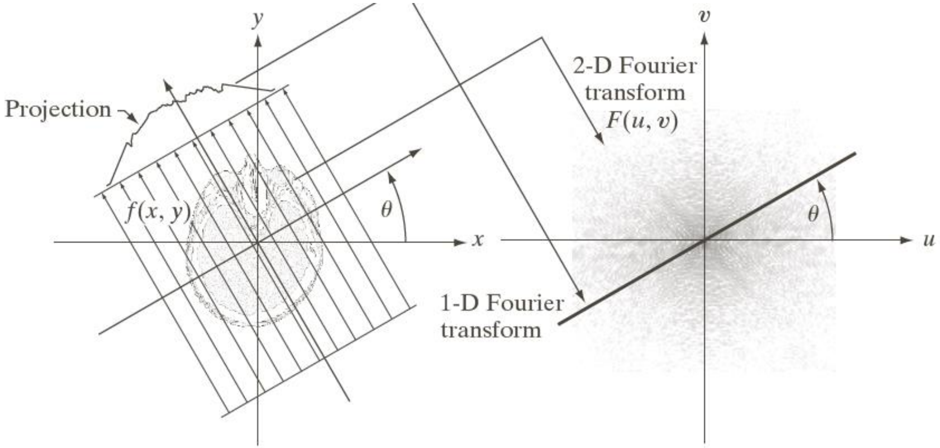
where projection $g(\rho,\theta)$ is

$$g(\rho,\theta)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\delta(x\cos\theta+y\sin\theta-\rho)dxdy$$

then

$$G(\omega,\theta)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi\omega(x\cos\theta+y\sin\theta)}dxdy$$

$$=\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)e^{-j2\pi(ux+vy)}dxdy\right]_{u=\omega\cos\theta;v=\omega\sin\theta}$$



Therefore $G(\omega,\theta)=F(\omega\cos\theta\,,\omega\sin\theta)$

Filtered Back Projection

• Back Projection from Radon Transform

- For a fixed value of rotation θ_k:

$$f_{\theta_k}(x,y) = g(\rho,\theta_k) = g(x\cos\theta_k + y\sin\theta_k\,,\theta_k)$$

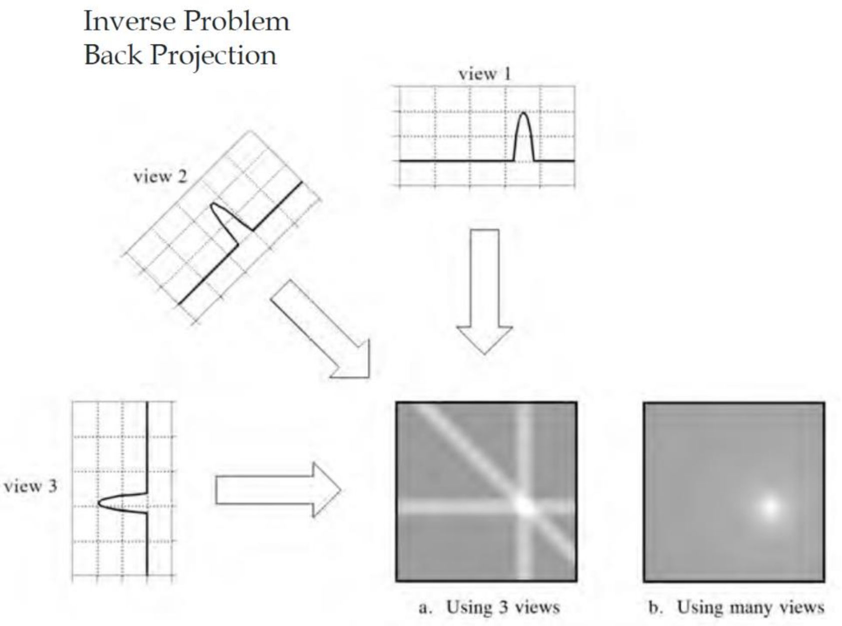
- Then a single backprojection obtained at an angle θ :

$$f_\theta(x,y) = g(x\cos\theta + y\sin\theta\,,\theta)$$

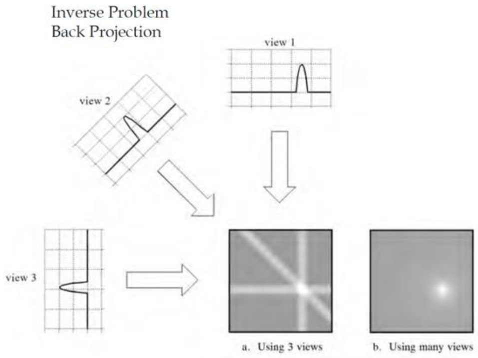
where $g(\rho,\theta)$ is the projection value.

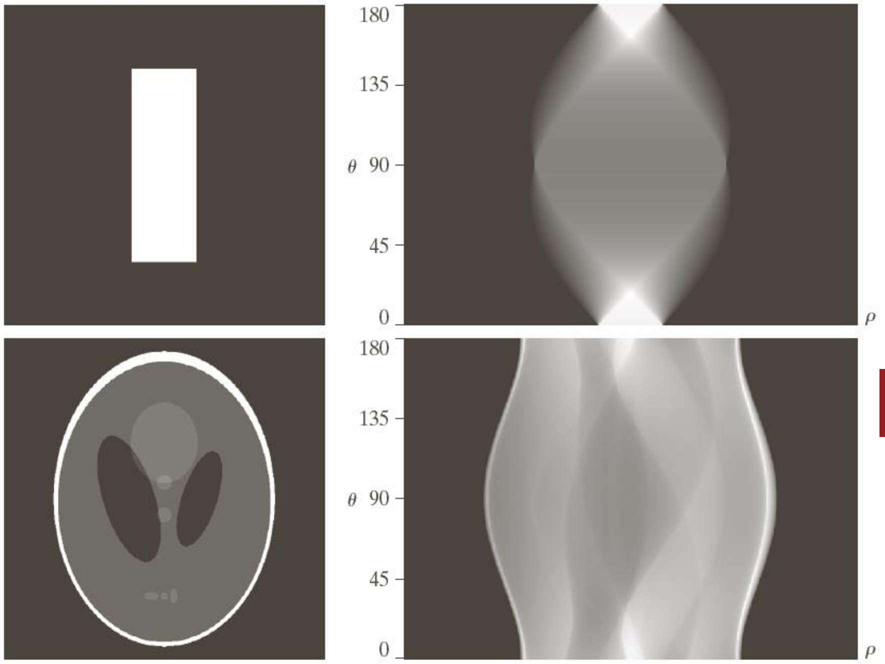
- The final image by summing over all the back-projected images

$$f(x,y) = \sum_{\theta=0}^{\pi} f_\theta(x,y)$$

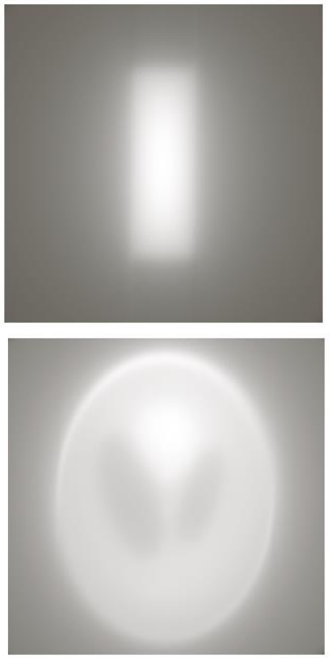


• Back Projection from Radon Transform





BP



Rafael C. Gonzalez and Richard E. Woods. Digital Image Processing (3rd Edition). Prentice Hall, August 2007

• Parallel-Beam Filtered Back projections

- **The 2D IFT of $F(u,v)$ with Fourier-slice theorem:**

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(u,v)e^{j2\pi(ux+vy)}dudv$$

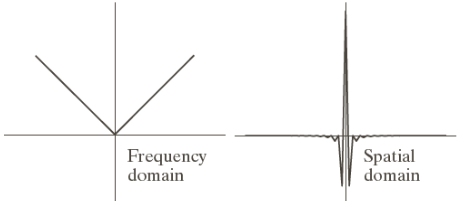
$$= \int_{0}^{2\pi}\int_{0}^{\infty} G(\omega,\theta)e^{j2\pi\omega(x\cos\theta+y\sin\theta)}\omega d\omega d\theta$$

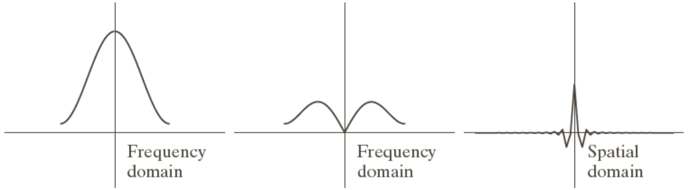
$$= \int_{0}^{\pi}\left[\int_{-\infty}^{\infty} |\omega| G(\omega,\theta)e^{j2\pi\omega\rho}d\omega\right]_{\rho=x\cos\theta+y\sin\theta} d\theta$$

- **Convolution backprojection**

$$f(x,y) = \int_{0}^{\pi}[s(\rho)\otimes g(\rho,\theta)]_{\rho=x\cos\theta+y\sin\theta}d\theta$$

Where $s(\rho) = \mathrm{IFT}(|\omega|)$, $g(\rho,\theta) = \mathrm{IFT}[G(\omega,\theta)]$





a b
c d e

FIGURE 5.42 (a) Frequency domain plot of the filter $|\omega|$ after band-limiting it with a box filter. (b) Spatial domain representation. (c) Hamming windowing function. (d) Windowed ramp filter, formed as the product of (a) and (c). (e) Spatial representation of the product (note the decrease in ringing).

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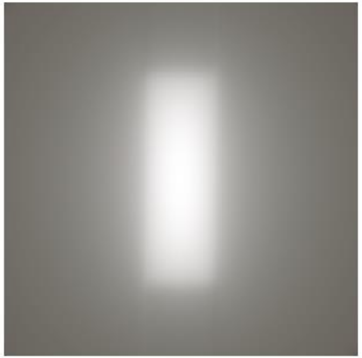
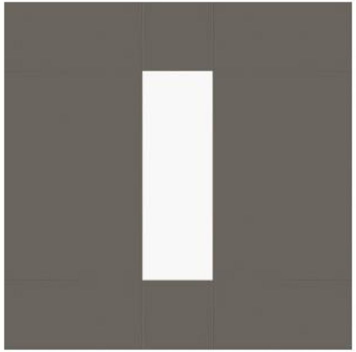
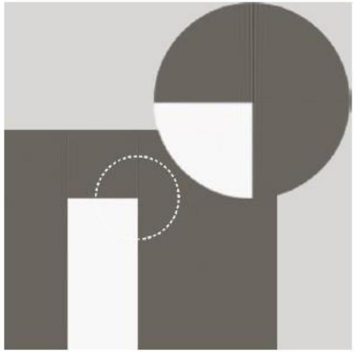
• Parallel-Beam Filtered Back projections

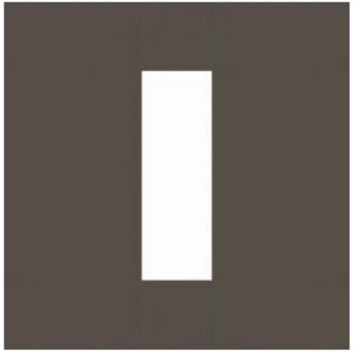


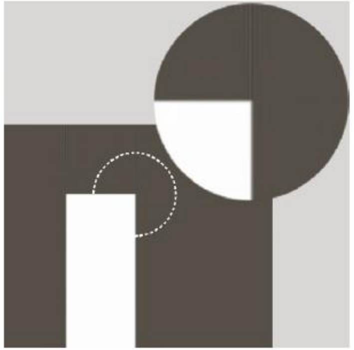
CT recom via BP



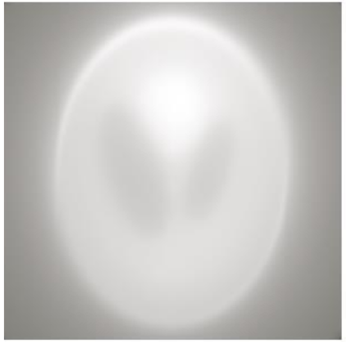


FBP via slope filter + square window





FBP via slope filter + Hamming window



CT recom via BP

FBP via slope filter + square window

FBP via slope filter + Hamming window

Summery and challenges

- Basic assumptions in current X-ray CT imaging process.
  - ✓ Linear Attenuation Coefficient：radiation hardening，metal artifact， calcification artifact
  - ✓ Full observations on sensory data： Limited angle image reconstruction，Sparse view image reconstruction
  - ✓ Subject stability during image scanning： Motion artifact，Breath holding during scanning

Thank you!