SI231B: Matrix Computations, 2024 Fall

Homework Set #2

Acknowledgements:

1) Deadline: **2024-11-14 23:59:59**
2) Please submit the PDF file to gradescope. Course entry code: 8KJ345.
3) You have 5 “free days” in total for all late homework submissions.
4) If your homework is handwritten, please make it clear and legible.
5) All your answers are required to be in English.
6) Please include the main steps in your answer; otherwise, you may not get the points.

Problem 1. (**Pivoting in LU decomposition**, 12 points)

Consider the matrix

$$\mathbf{A}=\begin{bmatrix}2&4&6&8\\3&1&7&2\\5&9&2&6\\8&3&5&7\end{bmatrix}$$

1) Use partial pivoting to find a permutation matrix **P**, a unit lower triangular matrix **L** and an upper triangular matrix **U** such that **PA=LU**. Also, please represent the permutation matrix **P** as a product of interchange permutations.(e.g. **P**=$\Pi_3\Pi_2\Pi_1$). (5 points)
2) Use complete pivoting to find permutation matrices **P** and **Q**, a unit lower triangular matrix **L** and an upper triangular matrix **U** such that $\mathbf{PAQ}^T=\mathbf{LU}$. (7 points)

Solution:

1) $\Pi_1=\begin{bmatrix}0&0&0&1\\0&1&0&0\\0&0&1&0\\1&0&0&0\end{bmatrix}$, $\Pi_1A=\Pi_1A^{(0)}=\begin{bmatrix}8&3&5&7\\3&1&7&2\\5&9&2&6\\2&4&6&8\end{bmatrix}$, $\mathbf{M}_1=\begin{bmatrix}1&0&0&0\\-\frac{3}{8}&1&0&0\\-\frac{5}{8}&0&1&0\\-\frac{1}{4}&0&0&1\end{bmatrix}$

$A^{(1)}=M_1\Pi_1A^{(0)}=\begin{bmatrix}8&3&5&7\\0&-\frac{1}{8}&\frac{41}{8}&-\frac{5}{8}\\0&\frac{57}{8}&-\frac{9}{8}&\frac{13}{8}\\0&\frac{13}{4}&\frac{19}{4}&\frac{25}{4}\end{bmatrix}$, $\Pi_2=\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}$, $\Pi_2A^{(1)}=\begin{bmatrix}8&3&5&7\\0&\frac{57}{8}&-\frac{9}{8}&\frac{13}{8}\\0&-\frac{1}{8}&\frac{41}{8}&-\frac{5}{8}\\0&\frac{13}{4}&\frac{19}{4}&\frac{25}{4}\end{bmatrix}$

$\mathbf{M}_2=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&\frac{1}{57}&1&0\\0&-\frac{26}{57}&0&1\end{bmatrix}$, $A^{(2)}=M_2\Pi_2A^{(1)}=\begin{bmatrix}8&3&5&7\\0&\frac{57}{8}&-\frac{9}{8}&\frac{13}{8}\\0&0&\frac{97}{19}&-\frac{34}{57}\\0&0&\frac{100}{19}&\frac{314}{57}\end{bmatrix}$, $\Pi_3=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}$,

$\Pi_3A^{(2)}=\begin{bmatrix}8&3&5&7\\0&\frac{57}{8}&-\frac{9}{8}&\frac{13}{8}\\0&0&\frac{100}{19}&\frac{314}{57}\\0&0&\frac{97}{19}&-\frac{34}{57}\end{bmatrix}$, $\mathbf{M}_3=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&-\frac{97}{100}&1\end{bmatrix}$, $\mathbf{U}=A^{(3)}=M_3\Pi_3A^{(2)}=\begin{bmatrix}8&3&5&7\\0&\frac{57}{8}&-\frac{9}{8}&\frac{13}{8}\\0&0&\frac{100}{19}&\frac{314}{57}\\0&0&0&-\frac{297}{50}\end{bmatrix}$,

$\mathbf{P}=\Pi_3\Pi_2\Pi_1=\begin{bmatrix}0&0&0&1\\0&0&1&0\\1&0&0&0\\0&1&0&0\end{bmatrix}$, $\tau_1=\begin{bmatrix}0\\\frac{3}{8}\\\frac{5}{8}\\\frac{1}{4}\end{bmatrix}$, $\tau_2=\begin{bmatrix}0\\0\\-\frac{1}{57}\\\frac{26}{57}\end{bmatrix}$, $\tau_3=\begin{bmatrix}0\\0\\0\\\frac{97}{100}\end{bmatrix}$, $\widetilde{\tau_1}=\Pi_3\Pi_2\tau_1=\begin{bmatrix}0\\\frac{5}{8}\\\frac{1}{4}\\\frac{3}{8}\end{bmatrix}$, $\widetilde{\tau_2}=\Pi_3\tau_2=\begin{bmatrix}0\\0\\\frac{26}{57}\\-\frac{1}{57}\end{bmatrix}$,

$\widetilde{\tau_3}=\tau_3$, $\mathbf{L}=\begin{bmatrix}1&0&0&0\\\frac{5}{8}&1&0&0\\\frac{1}{4}&\frac{26}{57}&1&0\\\frac{3}{8}&-\frac{1}{57}&\frac{97}{100}&1\end{bmatrix}$ $\mathbf{LU}=\begin{bmatrix}1&0&0&0\\\frac{5}{8}&1&0&0\\\frac{1}{4}&\frac{26}{57}&1&0\\\frac{3}{8}&-\frac{1}{57}&\frac{97}{100}&1\end{bmatrix}\begin{bmatrix}8&3&5&7\\0&\frac{57}{8}&-\frac{9}{8}&\frac{13}{8}\\0&0&\frac{100}{19}&\frac{314}{57}\\0&0&0&-\frac{297}{50}\end{bmatrix}=\begin{bmatrix}8&3&5&7\\5&9&2&6\\2&4&6&8\\3&1&7&2\end{bmatrix}$ =**PA**

2) $\Pi_1=\begin{bmatrix}0&0&1&0\\0&1&0&0\\1&0&0&0\\0&0&0&1\end{bmatrix}$, $\Gamma_1=\begin{bmatrix}0&1&0&0\\1&0&0&0\\0&0&1&0\\0&0&0&1\end{bmatrix}$, $\Pi_1A^{(0)}\Gamma_1=\begin{bmatrix}9&5&2&6\\1&3&7&2\\4&2&6&8\\3&8&5&7\end{bmatrix}$, $\mathbf{M}_1=\begin{bmatrix}1&0&0&0\\-\frac{1}{9}&1&0&0\\-\frac{4}{9}&0&1&0\\-\frac{1}{3}&0&0&1\end{bmatrix}$

$A^{(1)}$**=**$\mathbf{M}_1\Pi_1A^{(0)}\Gamma_1=\begin{bmatrix}9&5&2&6\\0&\frac{22}{9}&\frac{61}{9}&\frac{4}{3}\\0&-\frac{2}{9}&\frac{46}{9}&\frac{16}{3}\\0&\frac{19}{3}&\frac{13}{3}&5\end{bmatrix}$, $\Pi_2=\mathbf{I}$, $\Gamma_2=\begin{bmatrix}1&0&0&0\\0&0&1&0\\0&1&0&0\\0&0&0&1\end{bmatrix}$, $\Pi_2A^{(1)}\Gamma_2=\begin{bmatrix}9&2&5&6\\0&\frac{61}{9}&\frac{22}{9}&\frac{4}{3}\\0&\frac{46}{9}&-\frac{2}{9}&\frac{16}{3}\\0&\frac{13}{3}&\frac{19}{3}&5\end{bmatrix}$

$\mathbf{M}_2=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&-\frac{46}{61}&1&0\\0&-\frac{39}{61}&0&1\end{bmatrix}$, $A^{(2)}$**=**$\mathbf{M}_2\Pi_2A^{(1)}\Gamma_2=\begin{bmatrix}9&2&5&6\\0&\frac{61}{9}&\frac{22}{9}&\frac{4}{3}\\0&0&-\frac{126}{61}&\frac{264}{61}\\0&0&\frac{291}{61}&\frac{253}{61}\end{bmatrix}$, $\Pi_3=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}$, $\Gamma_3=\mathbf{I}$,

$\Pi_3A^{(2)}\Gamma_3=\begin{bmatrix}9&2&5&6\\0&\frac{61}{9}&\frac{22}{9}&\frac{4}{3}\\0&0&\frac{291}{61}&\frac{253}{61}\\0&0&-\frac{126}{61}&\frac{264}{61}\end{bmatrix}$ $\mathbf{M}_3=\begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&\frac{42}{97}&1\end{bmatrix}$, $\mathbf{U}=A^{(3)}$**=**$\mathbf{M}_3\Pi_3A^{(2)}\Gamma_3=\begin{bmatrix}9&2&5&6\\0&\frac{61}{9}&\frac{22}{9}&\frac{4}{3}\\0&0&\frac{291}{61}&\frac{253}{61}\\0&0&0&\frac{594}{97}\end{bmatrix}$

$\mathbf{P}=\Pi_3\Pi_2\Pi_1=\begin{bmatrix}0&0&1&0\\0&1&0&0\\0&0&0&1\\1&0&0&0\end{bmatrix}$, $\mathbf{Q}=\Gamma_3\Gamma_2\Gamma_1=\begin{bmatrix}0&1&0&0\\0&0&1&0\\1&0&0&0\\0&0&0&1\end{bmatrix}$, $\tau_1=\begin{bmatrix}0\\\frac{1}{9}\\\frac{4}{9}\\\frac{1}{3}\end{bmatrix}$, $\tau_2=\begin{bmatrix}0\\0\\\frac{46}{61}\\\frac{39}{61}\end{bmatrix}$, $\tau_3=\begin{bmatrix}0\\0\\0\\-\frac{42}{97}\end{bmatrix}$,

$\widetilde{\tau_1}=\Pi_3\Pi_2\tau_1=\begin{bmatrix}0\\\frac{1}{9}\\\frac{1}{3}\\\frac{4}{9}\end{bmatrix}$, $\widetilde{\tau_2}=\Pi_3\tau_2=\begin{bmatrix}0\\0\\\frac{39}{61}\\\frac{46}{61}\end{bmatrix}$, $\widetilde{\tau_3}=\tau_3$, $\mathbf{L}=\begin{bmatrix}1&0&0&0\\\frac{1}{9}&1&0&0\\\frac{1}{3}&\frac{39}{61}&1&0\\\frac{4}{9}&\frac{46}{61}&-\frac{42}{97}&1\end{bmatrix}$,

$$\mathbf{LU}=\begin{bmatrix}1&0&0&0\\\frac{1}{9}&1&0&0\\\frac{1}{3}&\frac{39}{61}&1&0\\\frac{4}{9}&\frac{46}{61}&-\frac{42}{97}&1\end{bmatrix}\begin{bmatrix}9&2&5&6\\0&\frac{61}{9}&\frac{22}{9}&\frac{4}{3}\\0&0&\frac{291}{61}&\frac{253}{61}\\0&0&0&\frac{594}{97}\end{bmatrix}=\begin{bmatrix}9&2&5&6\\1&7&3&2\\3&5&8&7\\4&6&2&8\end{bmatrix}=\mathbf{PAQ}^T$$

Problem 2. (**Cholesky decomposition**, 13 points)

Let **A** be a $n \times n$ symmetric positive definite matrix:

1) Assume **A** is a banded matrix with bandwidth b. Write a pseudo code for A's Cholesky decomposition, and then analyze its complexity (flops). (5 points)
2) Assume **A** is a 8×8 positive definite symmetric matrix, and its Cholesky decomposition is $\mathbf{A}$=$\mathbf{G}\mathbf{G}^T$. Denote a_{ij} and g_{ij} as the (i, j)-th element of **A** and **G**. Suppose $a_{11} = 25$, $a_{21} = -20$, $a_{31} = 15$, $a_{15} = -10$, $a_{22} = 17$, $a_{32} = -8$, $a_{25} = 5$, $a_{33} = 26$, $a_{35} = -11$. Find the g_{53}. (8 points)

Solution:

1) Pseudo code:

```
function L = banded_Cholesky(A, b)
    n = size(A, 1);
    L = zeros(n, n);
    for i = 1:n
        sum_diag = 0;
        for k = max(1, i - b):(i - 1)
            sum_diag = sum_diag + L(i, k)^2;
        end
        L(i, i) = sqrt(A(i, i) - sum_diag);
        for j = (i + 1):min(n, i + b)
            sum_offdiag = 0;
            for k = max(1, i - b):(i - 1)
                sum_offdiag = sum_offdiag + L(i, k) * L(j, k);
            end
            L(j, i) = (A(j, i) - sum_offdiag) / L(i, i);
        end
    end
end
```

Time complexity: $O(nb^2)$. Proof: there are three loop, 1st loop run n time, 2nd and 3rd loop run at most b time, so time complexity is $O(nb^2)$

2) Value of a_{ij} satisfied $a_{ij} = \sum_{k=1}^{n}[\mathbf{G}]_{ik}[\mathbf{G}^T]_{kj} = \sum_{k=1}^{n}[\mathbf{G}]_{ik}[\mathbf{G}]_{jk}$=$\sum_{k=1}^{\min(i,j)} g_{ik}g_{jk}$. From $25 = a_{11} = g_{11}^2$ we have $g_{11} = 5$; from $-20 = a_{21} = g_{11}g_{21}$ we have $g_{21} = -4$; from $15 = a_{31} = g_{11}g_{31}$ we have $g_{31} = 3$; from $-10 = a_{15} = g_{11}g_{51}$ we have $g_{51} = -2$; from $17 = a_{22} = g_{21}^2 + g_{22}^2$ we have $g_{22} = 1$; from $-8 = a_{32} = g_{21}g_{31} + g_{22}g_{32}$ we have $g_{32} = 4$; from $5 = a_{25} = g_{21}g_{51} + g_{22}g_{52}$ we have $g_{52} = -3$; from $26 = a_{33} = g_{31}^2 + = g_{32}^2 = g_{33}^2$ we have $g_{33} = 1$; from $-11 = a_{35} = g_{31}g_{51} + g_{32}g_{52} + g_{33}g_{53}$, we have $g_{53} = 7$

Problem 3. (**Applications of banded matrix**, 25 points)

Banded matrix can be used in solving differential equations. For example, we use finite difference method to solve 1-D heat conduction equation in range $x \in [0,1]$ with Dirichlet boundary condition at $x = 0$ and Neumann boundary condition at $x = 1$:

$$\begin{cases} \frac{\partial}{\partial t}u = k\frac{\partial^2}{\partial x^2}u + f(t,x) \\ u(t,0) = T_0 \\ \frac{\partial}{\partial x}u(t,1) = 0 \end{cases} \tag{1}$$

The equations above are the heat conduction equation and its boundary conditions, where $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a function to describe the temperature, $u(t,x)$ is the temperature at time t and at spatial coordinate x, $f(t,x)$ is a function to describe time and spatial distribution of the heat source , k is the thermal diffusivity and T_0 is the temperature at point $x = 0$. We can use Taylor expansion to discretize the system:

$$\begin{cases} \frac{u(t+\Delta t,x)-u(t,x)}{\Delta t} = k\frac{u(t,x-\Delta x)-2u(t,x)+u(t,x+\Delta x)}{\Delta x^2} + f(t,x) \\ u(t,0) = T_0 \\ \frac{u(t,1)-u(t,1-\Delta x)}{\Delta x} = 0 \end{cases} \tag{2}$$

By discretizing the space with n+1 points (i.e. $\Delta x = \frac{1}{n}$, $x_i = i\Delta x$, $i = 0, \cdots, n$), we can represent temperature distribution u at time $t_k = k\Delta t$ by $\mathbf{u}^{(k)}$, where $u_i^{(k)} = [\mathbf{u}^{(k)}]_i = u(k\Delta t, i\Delta x)$. As a result, we may solve the difference equation (2) through an iterative formula: $\mathbf{u}^{(k+1)} = \mathbf{A}\mathbf{u}^{(k)} + \mathbf{f}$

1) Give the expression of the $(n+1) \times (n+1)$ matrix $\mathbf{A}$ and the vector $\mathbf{f}$. Please organize matrix $\mathbf{A}$ such that $\mathbf{A}[2:n, 2:n]$ is a banded matrix. (10 points)
2) Given $n = 8$, $k = \frac{1}{640}$, $T_0 = 5$, and$f(t,x) = \begin{cases} 1 & x = 0.5 \\ 0 & \text{otherwise} \end{cases}$, find the stable solution of $\mathbf{u}$ numerically using LU decomposition. (Note: when the solution $\mathbf{u}$ is stable, $u(t+\Delta t, x) = u(t,x)$, so you need to derive another (but similar) linear equations) (15 points)

Solution:

1) First, according to the equation, we have

$$u_i^{(k+1)} = \frac{k\Delta t}{\Delta x^2}u_{i-1}^{(k)} + (1 - 2\frac{k\Delta t}{\Delta x^2})u_i^{(k)} + \frac{k\Delta t}{\Delta x^2}u_{i+1}^{(k)} + \Delta t f(k\Delta t, i\Delta x)$$

Then the Dirichlet boundary condition gives

$$u_0^{(k+1)} = T_0$$

Then the Neumann boundary condition gives

$$u_n^{(k+1)} = u_{n-1}^{(k+1)} = \frac{k\Delta t}{\Delta x^2}u_{n-2}^{(k)} + (1 - 2\frac{k\Delta t}{\Delta x^2})u_{n-1}^{(k)} + \frac{k\Delta t}{\Delta x^2}u_n^{(k)} + \Delta t f(k\Delta t, (n-1)\Delta x)$$

As a result, we have **A**= $\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \frac{k\Delta t}{\Delta x^2} & 1-2\frac{k\Delta t}{\Delta x^2} & \frac{k\Delta t}{\Delta x^2} & 0 & 0 & 0 & \cdots & 0 \\ 0 & \frac{k\Delta t}{\Delta x^2} & 1-2\frac{k\Delta t}{\Delta x^2} & \frac{k\Delta t}{\Delta x^2} & 0 & 0 & \cdots & 0 \\ 0 & 0 & \frac{k\Delta t}{\Delta x^2} & 1-2\frac{k\Delta t}{\Delta x^2} & \frac{k\Delta t}{\Delta x^2} & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & \frac{k\Delta t}{\Delta x^2} & 1-2\frac{k\Delta t}{\Delta x^2} & \frac{k\Delta t}{\Delta x^2} & 0 \\ 0 & 0 & \cdots & 0 & 0 & \frac{k\Delta t}{\Delta x^2} & 1-2\frac{k\Delta t}{\Delta x^2} & \frac{k\Delta t}{\Delta x^2} \\ 0 & 0 & \cdots & 0 & 0 & \frac{k\Delta t}{\Delta x^2} & 1-2\frac{k\Delta t}{\Delta x^2} & \frac{k\Delta t}{\Delta x^2} \end{bmatrix}$

and **f**=$\Delta t \begin{bmatrix} T_0/\Delta t \\ f(k\Delta t, \Delta x) \\ f(k\Delta t, 2\Delta x) \\ \vdots \\ f(k\Delta t, (n-1)\Delta x) \\ f(k\Delta t, (n-1)\Delta x) \end{bmatrix}$

2) When the temperature is stable, the equation is **Bu**=$\widetilde{\mathbf{f}}$,

where **B**= $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.1 & 0.2 & -0.1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -0.1 & 0.2 & -0.1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -0.1 & 0.2 & -0.1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -0.1 & 0.2 & -0.1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -0.1 & 0.2 & -0.1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -0.1 & 0.2 & -0.1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -0.1 & 0.2 & -0.1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \end{bmatrix}$ and $\widetilde{\mathbf{f}}$= $\begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

Apply LU decomposition with partial pivoting to **B**, we have

PB $=$ **LU**, where **L**= $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{10} & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{2} & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{2}{3} & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -\frac{3}{4} & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\frac{4}{5} & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{5}{6} & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{6}{7} & -\frac{4}{35} & 1 \end{bmatrix}$,

$$\mathbf{U}=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{5} & -\frac{1}{10} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{3}{20} & -\frac{1}{10} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{2}{15} & -\frac{1}{10} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{8} & -\frac{1}{10} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{3}{25} & -\frac{1}{10} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{7}{60} & -\frac{1}{10} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{70} \end{bmatrix}, \mathbf{P}=\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix},$$

so $\mathbf{Bu} = \mathbf{P}^T\mathbf{LUu} = \widetilde{\mathbf{f}}$, solve $\mathbf{u}=\begin{bmatrix} 5 \\ 15 \\ 25 \\ 35 \\ 45 \\ 45 \\ 45 \\ 45 \\ 45 \end{bmatrix}$

Problem 4. (**Least-squares**, 15 points)

1) Prove that if $\epsilon = \mathbf{A}\hat{\mathbf{x}} - \mathbf{b}$, where $\hat{\mathbf{x}}$ is a least-squares solution to $\mathbf{A}\mathbf{x} = \mathbf{b}$, then $\|\epsilon\|_2^2 = \|\mathbf{b}\|_2^2 - \|\Pi_{\mathcal{R}(\mathbf{A})}\mathbf{b}\|_2^2$. (**Hint**: A least-squares solution $\hat{\mathbf{x}}$ satisfies $\mathbf{A}\hat{\mathbf{x}} = \Pi_{\mathcal{R}(\mathbf{A})}\mathbf{b}$) (5 points)
2) For $\mathbf{A} \in \mathbb{R}^{m\times n}$ and $\mathbf{b} \in \mathbb{R}^m$, prove that $\mathbf{x}_2$ is a least-squares solution to $\mathbf{A}\mathbf{x} = \mathbf{b}$ if and only if $\mathbf{x}_2$ is part of a solution to the following augmented system
$$\begin{bmatrix} \mathbf{I}_{m\times m} & \mathbf{A} \\ \mathbf{A}^T & \mathbf{0}_{n\times n} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ 0 \end{bmatrix}. \tag{3}$$
(5 points)
3) Researchers have studied a certain type of cancer and raised hypothesizes that in the short run the number y of malignant cells in a particular tissue grows exponentially with time t, i.e., $y = \alpha_0 e^{\alpha_1 t}$ for some $\alpha_0, \alpha_1 > 0$. Formulate the problem of estimating the parameters α_0 and α_1 into a least-squares problem and solve it using the researchers' observed data given below.

t (days)	1	2	3	4	5
y (cells)	16	27	45	74	122

(**Hint:** Transform the exponential function into a linear function) (5 points)

Solution:

1) Since $\Pi_{\mathcal{R}(\mathbf{A})} = \Pi_{\mathcal{R}(\mathbf{A})}^T = \Pi_{\mathcal{R}(\mathbf{A})}^2$, we obtain
$$\begin{aligned} \|\epsilon\|_2^2 &= (\mathbf{b} - \Pi_{\mathcal{R}(\mathbf{A})}\mathbf{b})^T(\mathbf{b} - \Pi_{\mathcal{R}(\mathbf{A})}\mathbf{b}) \\ &= \mathbf{b}^T\mathbf{b} - \mathbf{b}^T\Pi_{\mathcal{R}(\mathbf{A})}^T\mathbf{b} - \mathbf{b}^T\Pi_{\mathcal{R}(\mathbf{A})}\mathbf{b} + \mathbf{b}^T\Pi_{\mathcal{R}(\mathbf{A})}^T\Pi_{\mathcal{R}(\mathbf{A})}\mathbf{b} \\ &= \mathbf{b}^T\mathbf{b} - \mathbf{b}^T\Pi_{\mathcal{R}(\mathbf{A})}\mathbf{b} \\ &= \|\mathbf{b}\|_2^2 - \|\Pi_{\mathcal{R}(\mathbf{A})}\mathbf{b}\|_2^2. \end{aligned}$$
2) If $\mathbf{x}_2$ is a least-squares solution $\Longrightarrow \mathbf{A}^T\mathbf{A}\mathbf{x}_2 = \mathbf{A}^T\mathbf{b} \Longrightarrow \mathbf{0} = \mathbf{A}^T(\mathbf{b} - \mathbf{A}\mathbf{x}_2)$. Let $\mathbf{x}_1 = \mathbf{b} - \mathbf{A}\mathbf{x}_2$, then we have
$$\begin{bmatrix} \mathbf{I}_{m\times m} & \mathbf{A} \\ \mathbf{A}^T & \mathbf{0}_{n\times n} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{I}_{m\times m} & \mathbf{A} \\ \mathbf{A}^T & \mathbf{0}_{n\times n} \end{bmatrix} \begin{bmatrix} \mathbf{b} - \mathbf{A}\mathbf{x}_2 \\ \mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ 0 \end{bmatrix}$$
The converse is true because
$$\begin{aligned} \begin{bmatrix} \mathbf{I}_{m\times m} & \mathbf{A} \\ \mathbf{A}^T & \mathbf{0}_{n\times n} \end{bmatrix} \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{b} \\ 0 \end{bmatrix} \quad &\Longrightarrow \quad \mathbf{A}\mathbf{x}_2 = \mathbf{b} - \mathbf{x}_1 \quad \text{and} \quad \mathbf{A}^T\mathbf{x}_1 = \mathbf{0} \\ &\Longrightarrow \quad \mathbf{A}^T\mathbf{A}\mathbf{x}_2 = \mathbf{A}^T\mathbf{b} - \mathbf{A}^T\mathbf{x}_1 = \mathbf{A}^T\mathbf{b}. \end{aligned}$$
3) First, transform the exponential function into $\ln y = \ln\alpha_0 + \alpha_1 t$.

Let

$$\mathbf{x} = \begin{bmatrix} \ln \alpha_0 \\ \alpha_1 \end{bmatrix}, \quad \mathbf{A} = \begin{bmatrix} 1 & t_1 \\ 1 & t_2 \\ \vdots & \vdots \\ 1 & t_5 \end{bmatrix}, \quad \text{and} \quad \mathbf{b} = \begin{bmatrix} \ln y_1 \\ \ln y_2 \\ \vdots \\ \ln y_5 \end{bmatrix}.$$

By solving the following least-squares problem:

$$\min_{\mathbf{x}} \frac{1}{2}\|\mathbf{A}\mathbf{x} - \mathbf{b}\|_2^2,$$

We obtain $\mathbf{x} = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T\mathbf{b} = [2.2753, 0.5071]^T$, and thus $\alpha_0 = 9.7308, \alpha_1 = 0.5071$.

Problem 5. (**QR decomposition**, 20 points)

Consider the following matrices:

$$\mathbf{A}_1 = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 2 \\ 1 & 2 & -4 \\ 1 & 4 & -2 \end{bmatrix}, \quad \mathbf{A}_2 = \begin{bmatrix} 1 & -7 & 15 \\ 2 & -14 & -3 \\ -2 & 14 & 0 \\ 4 & -3 & 4 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 5 \\ -15 \\ 0 \\ 30 \end{bmatrix}. \tag{4}$$

1) Apply the Gram-Schmidt procedure to find the QR decomposition of the matrix $\mathbf{A}_1$. (7 points)
2) Use Householder reflection to find an orthonormal basis for $\mathcal{R}(\mathbf{A}_2)$. Note that the sign in the expression of $\mathbf{v} = \mathbf{x} \mp \|\mathbf{x}\|_2 \mathbf{e}_1$ is determined to be the one that maximizes $\|\mathbf{v}\|_2$. (8 points)
3) Determine the least-squares solution for $\mathbf{A}_2\mathbf{x} = b$ using Householder QR. (5 points)

Solution:

1) Let $\mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{a}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 4 \end{bmatrix}$, $\mathbf{a}_3 = \begin{bmatrix} 0 \\ 2 \\ -4 \\ -2 \end{bmatrix}$. First, we choose $\tilde{\mathbf{q}}_1 = \mathbf{a}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, then $\mathbf{q}_1 = \frac{\tilde{\mathbf{q}}_1}{\|\tilde{\mathbf{q}}_1\|_2} = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

   The second vector would be

$$\tilde{\mathbf{q}}_2 = \mathbf{a}_2 - (\mathbf{q}_1^T \mathbf{a}_2)\mathbf{q}_1 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 4 \end{bmatrix} - 4\frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 0 \\ 2 \end{bmatrix}, \quad \mathbf{q}_2 = \frac{\tilde{\mathbf{q}}_2}{\|\tilde{\mathbf{q}}_2\|_2} = \frac{1}{\sqrt{6}}\begin{bmatrix} -1 \\ -1 \\ 0 \\ 2 \end{bmatrix}.$$

   The third vector would be

$$\tilde{\mathbf{q}}_3 = \mathbf{a}_3 - (\mathbf{q}_1^T \mathbf{a}_3)\mathbf{q}_1 - (\mathbf{q}_2^T \mathbf{a}_3)\mathbf{q}_2 = \begin{bmatrix} 0 \\ 2 \\ -4 \\ -2 \end{bmatrix} - \frac{-4}{2}\frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} - \frac{-6}{\sqrt{6}}\frac{1}{\sqrt{6}}\begin{bmatrix} -11 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ -3 \\ 1 \end{bmatrix}, \quad \mathbf{q}_3 = \frac{\tilde{\mathbf{q}}_3}{\|\tilde{\mathbf{q}}_3\|_2} = \frac{1}{\sqrt{14}}\begin{bmatrix} 0 \\ 2 \\ -3 \\ 1 \end{bmatrix}.$$

   Hence, we obtain the QR decomposition as

$$\mathbf{A}_1 = \mathbf{QR} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{\sqrt{6}} & 0 \\ \frac{1}{2} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{14}} \\ \frac{1}{2} & 0 & -\frac{3}{\sqrt{14}} \\ \frac{1}{2} & \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{14}} \end{bmatrix} \begin{bmatrix} 2 & 4 & -2 \\ 0 & \sqrt{6} & -\sqrt{6} \\ 0 & 0 & \sqrt{14} \end{bmatrix}$$

2) (a) **Way 1:**

Let $\mathbf{H}_1 \in \mathbb{R}^{4\times 4}$ be the Householder reflection w.r.t. $\mathbf{a}_1 = [1,2,-2,4]^T$. From $\mathbf{v}_1 = \mathbf{a}_1 - \|\mathbf{a}_1\|_2\mathbf{e}_1 = [-4,2,-2,4]^T$, we have

$$\mathbf{H}_1 = \mathbf{I} - \frac{2}{\|\mathbf{v}_1\|_2^2}\mathbf{v}_1\mathbf{v}_1^T = \frac{1}{5}\begin{bmatrix} 1 & 2 & -2 & 4 \\ 2 & 4 & 1 & -2 \\ -2 & 1 & 4 & 2 \\ 4 & -2 & 2 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{H}_1\mathbf{A}_2 = \begin{bmatrix} 5 & -15 & 5 \\ 0 & -10 & 2 \\ 0 & 10 & -5 \\ 0 & 5 & 14 \end{bmatrix}$$

Let $\mathbf{H}_2 \in \mathbb{R}^{3\times 3}$ be the Householder reflection w.r.t. $\mathbf{a}_2 = [-10,10,5]^T$. From $\mathbf{v}_2 = \mathbf{a}_2 - \|\mathbf{a}_2\|_2\mathbf{e}_1 = [-25,10,5]^T$, we have $\hat{\mathbf{H}}_2 = \mathbf{I} - \frac{2}{\|\mathbf{v}_2\|_2^2}\mathbf{v}_2\mathbf{v}_2^T = \frac{1}{15}\begin{bmatrix} -10 & 10 & 5 \\ 10 & 11 & -2 \\ 5 & -2 & 14 \end{bmatrix}$ and $\mathbf{H}_2 = \frac{1}{15}\begin{bmatrix} 15 & 0 & 0 & 0 \\ 0 & -10 & 10 & 5 \\ 0 & 10 & 11 & -2 \\ 0 & 5 & -2 & 14 \end{bmatrix}$.

Then,

$$\mathbf{H}_2\mathbf{H}_1\mathbf{A}_2 = \begin{bmatrix} 5 & -15 & 5 \\ 0 & 15 & 0 \\ 0 & 0 & -\frac{21}{5} \\ 0 & 0 & \frac{72}{5} \end{bmatrix}.$$

Let $\mathbf{H}_3 \in \mathbb{R}^{2\times 2}$ be the Householder reflection w.r.t. $\mathbf{a}_3 = [-\frac{21}{5}, \frac{72}{5}]^T$. From $\mathbf{v}_3 = \mathbf{a}_3 - \|\mathbf{a}_3\|_2\mathbf{e}_1 = [-\frac{96}{5}, \frac{72}{5}]^T$, we have

$$\hat{\mathbf{H}}_3 = \mathbf{I} - \frac{2}{\|\mathbf{v}_3\|_2^2}\mathbf{v}_3\mathbf{v}_3^T = \frac{1}{25}\begin{bmatrix} -7 & 24 \\ 24 & 7 \end{bmatrix} \quad \text{and} \quad \mathbf{H}_3 = \frac{1}{25}\begin{bmatrix} 25 & 0 & 0 & 0 \\ 0 & 25 & 0 & 0 \\ 0 & 0 & -7 & 24 \\ 0 & 0 & 24 & -7 \end{bmatrix}.$$

Then,

$$\mathbf{H}_3\mathbf{H}_2\mathbf{H}_1\mathbf{A}_2 = \begin{bmatrix} 5 & -15 & 5 \\ 0 & 15 & 0 \\ 0 & 0 & 15 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore, we have the QR decomposition as

$$\mathbf{Q} = \mathbf{H}_1\mathbf{H}_2\mathbf{H}_3 = \frac{1}{15}\begin{bmatrix} 3 & -4 & 14 & -2 \\ 6 & -8 & -5 & -10 \\ -6 & 8 & 2 & -11 \\ 12 & 9 & 0 & 0 \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} 5 & -15 & 5 \\ 0 & 15 & 0 \\ 0 & 0 & 15 \\ 0 & 0 & 0 \end{bmatrix}$$

The columns of matrix $\mathbf{Q}$ are an orthonormal basis for $\mathcal{R}(\mathbf{A}_2)$.

(b) **Way 2 (recommended):** Let $\mathbf{H}_1 \in \mathbb{R}^{4\times 4}$ be the Householder reflection w.r.t. $\mathbf{a}_1 = [1, 2, -2, 4]^T$. From $\mathbf{v}_1 = \mathbf{a}_1 + \|\mathbf{a}_1\|_2 \mathbf{e}_1 = [6, 2, -2, 4]^T$, we have

$$\mathbf{H}_1 = \mathbf{I} - \frac{2}{\|\mathbf{v}_1\|_2^2}\mathbf{v}_1\mathbf{v}_1^T = \frac{1}{15}\begin{bmatrix} -3 & -6 & 6 & -12 \\ -6 & 13 & 2 & -4 \\ 6 & 2 & 13 & 4 \\ -12 & -4 & 4 & 7 \end{bmatrix} \quad \text{and} \quad \mathbf{H}_1\mathbf{A}_2 = \begin{bmatrix} -5 & 15 & -5 \\ 0 & -\frac{20}{3} & -\frac{29}{3} \\ 0 & \frac{20}{3} & \frac{20}{3} \\ 0 & \frac{35}{3} & -\frac{28}{3} \end{bmatrix}$$

Let $\mathbf{H}_2 \in \mathbb{R}^{3\times 3}$ be the Householder reflection w.r.t. $\mathbf{a}_2 = [-\frac{20}{3}, \frac{20}{3}, \frac{35}{3}]^T$. From $\mathbf{v}_2 = \mathbf{a}_2 - \|\mathbf{a}_2\|_2\mathbf{e}_1 = [-\frac{65}{3}, \frac{20}{3}, \frac{35}{3}]^T$, we have $\hat{\mathbf{H}}_2 = \mathbf{I} - \frac{2}{\|\mathbf{v}_2\|_2^2}\mathbf{v}_2\mathbf{v}_2^T = \begin{bmatrix} -\frac{4}{9} & \frac{4}{9} & \frac{7}{9} \\ \frac{4}{9} & \frac{101}{117} & -\frac{28}{117} \\ \frac{7}{9} & -\frac{28}{117} & \frac{68}{117} \end{bmatrix}$ and $\mathbf{H}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -\frac{4}{9} & \frac{4}{9} & \frac{7}{9} \\ 0 & \frac{4}{9} & \frac{101}{117} & -\frac{28}{117} \\ 0 & \frac{7}{9} & -\frac{28}{117} & \frac{68}{117} \end{bmatrix}$.

Then,

$$\mathbf{H}_2\mathbf{H}_1\mathbf{A}_2 = \begin{bmatrix} -5 & 15 & -5 \\ 0 & 15 & 0 \\ 0 & 0 & \frac{48}{13} \\ 0 & 0 & -\frac{189}{13} \end{bmatrix}.$$

Let $\mathbf{H}_3 \in \mathbb{R}^{2\times 2}$ be the Householder reflection w.r.t. $\mathbf{a}_3 = [\frac{48}{13}, -\frac{189}{13}]^T$. From $\mathbf{v}_3 = \mathbf{a}_3 + \|\mathbf{a}_3\|_2\mathbf{e}_1 = \frac{1}{13}[243, -189]^T$, we have

$$\hat{\mathbf{H}}_3 = \mathbf{I} - \frac{2}{\|\mathbf{v}_3\|_2^2}\mathbf{v}_3\mathbf{v}_3^T = \frac{1}{65}\begin{bmatrix} -16 & 63 \\ 63 & 16 \end{bmatrix} \quad \text{and} \quad \mathbf{H}_3 = \frac{1}{65}\begin{bmatrix} 65 & 0 & 0 & 0 \\ 0 & 65 & 0 & 0 \\ 0 & 0 & -16 & 63 \\ 0 & 0 & 63 & 16 \end{bmatrix}.$$

Then,

$$\mathbf{H}_3\mathbf{H}_2\mathbf{H}_1\mathbf{A}_2 = \begin{bmatrix} -5 & 15 & -5 \\ 0 & 15 & 0 \\ 0 & 0 & -15 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore, we have the QR decomposition as

$$\mathbf{Q} = \mathbf{H}_1\mathbf{H}_2\mathbf{H}_3 = \frac{1}{15}\begin{bmatrix} -3 & -4 & -14 & 2 \\ -6 & -8 & 5 & 10 \\ 6 & 8 & -2 & 11 \\ -12 & 9 & 0 & 0 \end{bmatrix}, \quad \mathbf{R} = \begin{bmatrix} -5 & 15 & -5 \\ 0 & 15 & 0 \\ 0 & 0 & -15 \\ 0 & 0 & 0 \end{bmatrix}.$$

The columns of matrix $\mathbf{Q}$ are an orthonormal basis for $\mathcal{R}(\mathbf{A}_2)$.

3) Since $\mathbf{A}_2$ is full column rank, there is a unique least-squares solution $\mathbf{x} = (\mathbf{A}_2^T\mathbf{A}_2)^{-1}\mathbf{A}_2^T\mathbf{b}$, i.e., $\mathbf{A}_2^T\mathbf{A}_2\mathbf{x} = \mathbf{A}_2^T\mathbf{b}$. Substituting $\mathbf{A}_2 = \mathbf{QR}$ into it gives $(\mathbf{QR})^T\mathbf{QR}\mathbf{x} = \mathbf{R}^T\mathbf{R}\mathbf{x} = \mathbf{R^T}\mathbf{Q^T}\mathbf{b}$, where the first quality comes

from $\mathbf{Q}^T\mathbf{Q} = \mathbf{I}$. Due to $\mathbf{R}^T$ is nonsingular, we have $\mathbf{R}\mathbf{x} = \mathbf{Q}^T\mathbf{b}$. This is just an upper-triangular system that is efficiently solved by back substitution. The solution is

$$\mathbf{x} = \frac{1}{45}\begin{bmatrix} 364 \\ 74 \\ 29 \end{bmatrix}.$$

Problem 6. (Givens Rotation, 15 points)

1) Perform the following sequence of rotations in $\mathbb{R}^3$ beginning with

$$\mathbf{v}_0 = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}.$$

Rotate $\mathbf{v}_0$ clockwise by 90° around the y-axis to produce $\mathbf{v}_1$. Then, rotate $\mathbf{v}_1$ counterclockwise by 30° around the z-axis to produce $\mathbf{v}_2$. Determine the coordinates of $\mathbf{v}_1$, $\mathbf{v}_2$ and the orthogonal matrix $\mathbf{Q}$ such that $\mathbf{Q}\mathbf{v}_0 = \mathbf{v}_2$. Note that a vector $\mathbf{v} \in \mathbb{R}^3$ can be rotated clockwise by an angle θ around the x-axis by means of a multiplication $J(2,3,\theta)\mathbf{v}$ in which $J(2,3,\theta)$ is an appropriate orthogonal matrix as described below. (5 points)

$$J(2,3,\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}$$

2) Does it matter in which order the rotations in $\mathbb{R}^3$ are performed? For example, we introduce the following operations:

(a) Rotate clockwise around the x-axis by an angle θ.

(b) Rotate clockwise around the y-axis by an angle ϕ.

Suppose that a vector $\mathbf{v} \in \mathbb{R}^3$ conducts the above two operations in the following ways:

- $\mathbf{v} \overset{(a)}{\Longrightarrow} \mathbf{v}' \overset{(b)}{\Longrightarrow} \mathbf{v}''$
- $\mathbf{v} \overset{(b)}{\Longrightarrow} \mathbf{v}^* \overset{(a)}{\Longrightarrow} \mathbf{v}^{**}$

Is the result $\mathbf{v}''$ the same as $\mathbf{v}^{**}$? (5 points)

3) Extend the vector

$$\mathbf{x} = \frac{1}{3}\begin{bmatrix} -1 \\ 2 \\ 0 \\ -2 \end{bmatrix}$$

to an orthonormal basis for $\mathbb{R}^4$ using Givens rotations. (**Hint**: Sequentially annihilate the second and fourth elements of $\mathbf{x}$ to construct a QR decomposition) (5 points)

Solution:

1) First, rotating $\mathbf{v}_0$ clockwise 90° around the y-axis produces

$$\mathbf{v}_1 = J(3,1,90^\circ)\mathbf{v}_0 = \begin{bmatrix} \cos 90^\circ & 0 & -\sin 90^\circ \\ 0 & 1 & 0 \\ \sin 90^\circ & 0 & \cos 90^\circ \end{bmatrix} \mathbf{v}_0 = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Next, rotating $\mathbf{v}_1$ counterclockwise $30°$ around the z-axis yields

$$\mathbf{v}_2 = J(1,2,-30°)\mathbf{v}_1 = \begin{bmatrix} \cos(-30°) & \sin(-30°) & 0 \\ -\sin(-30°) & \cos(-30°) & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathbf{v}_1 = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}-1}{2} \\ \frac{\sqrt{3}+1}{2} \\ 1 \end{bmatrix}$$

Hence,

$$\mathbf{Q} = J(1,2,-30°)J(3,1,90°) = \begin{bmatrix} 0 & -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 0 & \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ 1 & 0 & 0 \end{bmatrix}$$

2) It matters because the rotation matrices generally do not commute with each other in matrix multiplication (this is easily verified by direct multiplication). For the rotation operation (a), we derive the rotation matrix as:

$$J(2,3,\theta) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix}.$$

For the rotation operation (b), we derive the rotation matrix as:

$$J(3,1,\phi) = \begin{bmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{bmatrix}.$$

Hence, we have

$$\mathbf{v}'' = J(3,1,\phi)J(2,3,\theta)\mathbf{v} = \begin{bmatrix} \cos\phi & \sin\theta\sin\phi & -\cos\theta\sin\phi \\ 0 & \cos\theta & \sin\theta \\ \sin\phi & -\sin\theta\cos\phi & \cos\theta\cos\phi \end{bmatrix} \mathbf{v}.$$

$$\mathbf{v}^{**} = J(2,3,\theta)J(3,1,\phi)\mathbf{v} = \begin{bmatrix} \cos\phi & 0 & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta & \sin\theta\cos\phi \\ \cos\theta\sin\phi & -\sin\theta & \cos\theta\cos\phi \end{bmatrix} \mathbf{v}.$$

This means $\mathbf{v}''$ is not necessarily equal to $\mathbf{v}^{**}$.

3) First, we annihilate the second element of $\mathbf{x}$. For $\theta = \tan^{-1}(\frac{2}{-1})$, $\cos\theta = \frac{-1}{\sqrt{5}}$, $\sin\theta = \frac{2}{\sqrt{5}}$:

$$\mathbf{x}' = J(1,2,\tan^{-1}(-2))\mathbf{x} = \begin{bmatrix} -\frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 & 0 \\ -\frac{2}{\sqrt{5}} & \frac{-1}{\sqrt{5}} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \frac{1}{3} \begin{bmatrix} -1 \\ 2 \\ 0 \\ -2 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} \sqrt{5} \\ 0 \\ 0 \\ -2 \end{bmatrix},$$

Next, we annihilate the fourth element of $\mathbf{x}$. For $\theta = \tan^{-1}(\frac{-2}{\sqrt{5}})$, $\cos\theta = \frac{\sqrt{5}}{3}$, $\sin\theta = \frac{-2}{3}$:

$$\mathbf{x}'' = J(1,4,\tan^{-1}(-2/\sqrt{5}))\mathbf{x}' = \begin{bmatrix} \frac{\sqrt{5}}{3} & 0 & 0 & -\frac{2}{3} \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \frac{2}{3} & 0 & 0 & \frac{\sqrt{5}}{3} \end{bmatrix} \frac{1}{3} \begin{bmatrix} \sqrt{5} \\ 0 \\ 0 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Therefore, the columns of

$$\mathbf{Q} = \Big(J(1,4,\tan^{-1}(-2/\sqrt{5}))J(1,2,\tan^{-1}(-2))\Big)^T = J(1,2,\tan^{-1}(-2))^T J(1,4,\tan^{-1}(-2/\sqrt{5}))^T$$
$$= \begin{bmatrix} -\frac{1}{3} & -\frac{2}{\sqrt{5}} & 0 & -\frac{2}{3\sqrt{5}} \\ 2/3 & -\frac{1}{\sqrt{5}} & 0 & \frac{4}{3\sqrt{5}} \\ 0 & 0 & 1 & 0 \\ -\frac{2}{3} & 0 & 0 & \frac{\sqrt{5}}{3} \end{bmatrix}$$

are an orthonormal set containing the specified vector $\mathbf{x}$.