



# Lecture 9

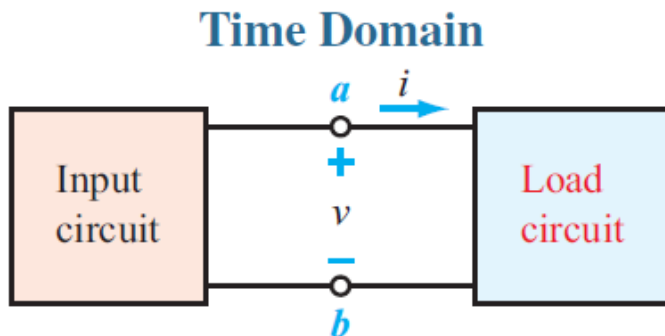
## - AC Power Calculation



# Outline

- Instantaneous power
- Average power
- Apparent power
- Power factor
- Complex power

# AC Power in Time Domain: Instantaneous



$$v(t) = V_m \cos(\omega t + \theta_v) \quad i(t) = I_m \cos(\omega t + \theta_i)$$

Instantaneous power:  
power at any instant of time.

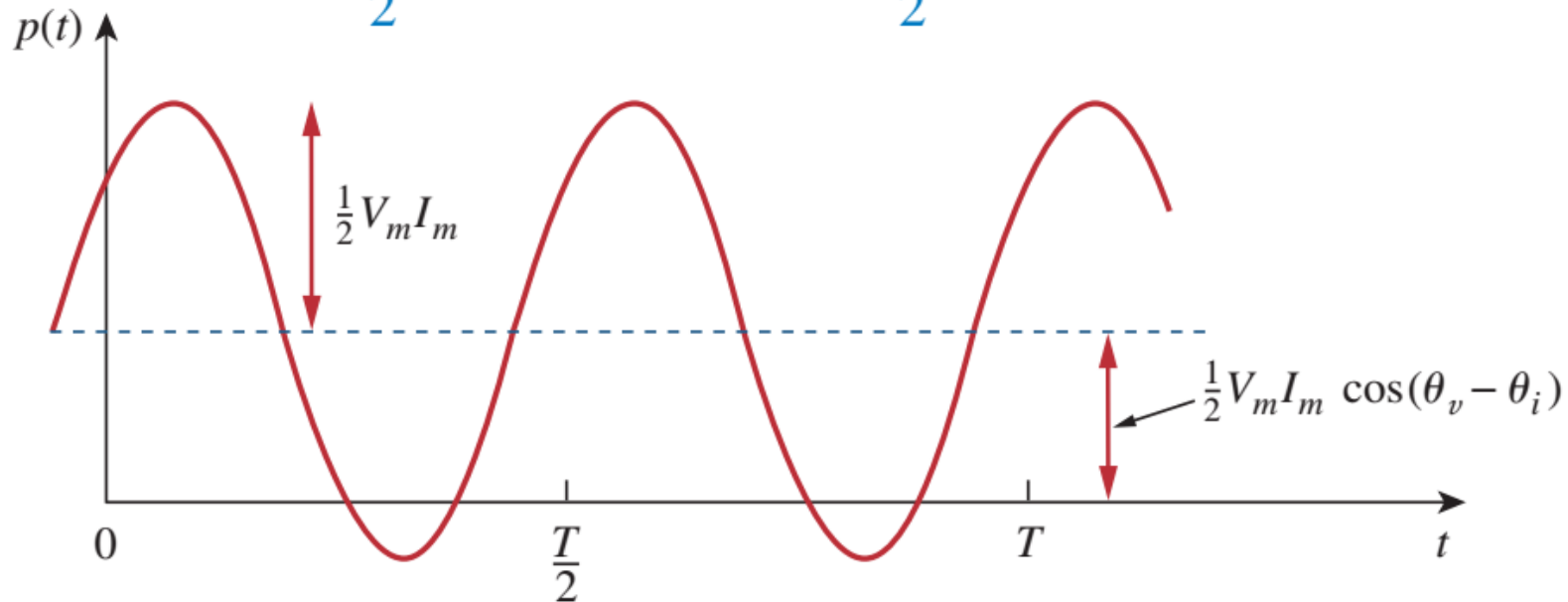
$$p(t) = v(t)i(t) = V_m I_m \cos(\omega t + \theta_v) \cos(\omega t + \theta_i)$$

$$p(t) = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) + \frac{1}{2} V_m I_m \cos(2\omega t + \theta_v + \theta_i)$$



# AC Power in Time Domain: Instantaneous

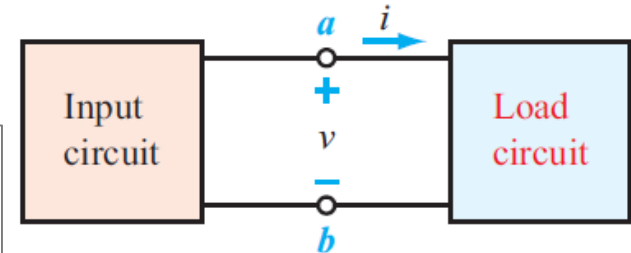
$$p(t) = \frac{1}{2}V_m I_m \cos(\theta_v - \theta_i) + \frac{1}{2}V_m I_m \cos(2\omega t + \theta_v + \theta_i)$$



## Average Power $P$ (Capitalized)

$$v(t) = V_m \cos(\omega t + \theta_v) \quad i(t) = I_m \cos(\omega t + \theta_i)$$

$$p(t) = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) + \frac{1}{2} V_m I_m \cos(2\omega t + \theta_v + \theta_i)$$



### Average (or real) power (unit: watts)

The **average power**, in watts, is the average of the instantaneous power over one period.

$$P = \frac{1}{T} \int_0^T p(t) dt$$



## Average Power $P$ (time domain)

$$\begin{aligned} P &= \frac{1}{T} \int_0^T \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) dt \\ &\quad + \frac{1}{T} \int_0^T \frac{1}{2} V_m I_m \cos(2\omega t + \theta_v + \theta_i) dt \\ &= \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) \frac{1}{T} \int_0^T dt \\ &\quad + \frac{1}{2} V_m I_m \frac{1}{T} \int_0^T \cos(2\omega t + \theta_v + \theta_i) dt \end{aligned}$$

$$P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i)$$



## Average Power $P$ (phasor domain)

$$v(t) = V_m \cos(\omega t + \theta_v) \quad i(t) = I_m \cos(\omega t + \theta_i)$$

$$\mathbf{V} = V_m \angle \theta_v \text{ and } \mathbf{I} = I_m \angle \theta_i,$$

$$\frac{1}{2} \mathbf{V} \mathbf{I}^* = \frac{1}{2} V_m I_m \angle \theta_v - \theta_i$$

$$= \frac{1}{2} V_m I_m [\cos(\theta_v - \theta_i) + j \sin(\theta_v - \theta_i)]$$

$$P = \frac{1}{2} \text{Re}[\mathbf{V} \mathbf{I}^*] = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i)$$



## Two special cases for average power $P$

- For a purely resistive load  $R$ :

$$P = \frac{1}{2} V_m I_m = \frac{1}{2} I_m^2 R = \frac{1}{2} |\mathbf{I}|^2 R \quad \text{where } |\mathbf{I}|^2 = \mathbf{I} \times \mathbf{I}^*$$

- For a purely reactive load:

$$P = \frac{1}{2} V_m I_m \cos 90^\circ = 0$$

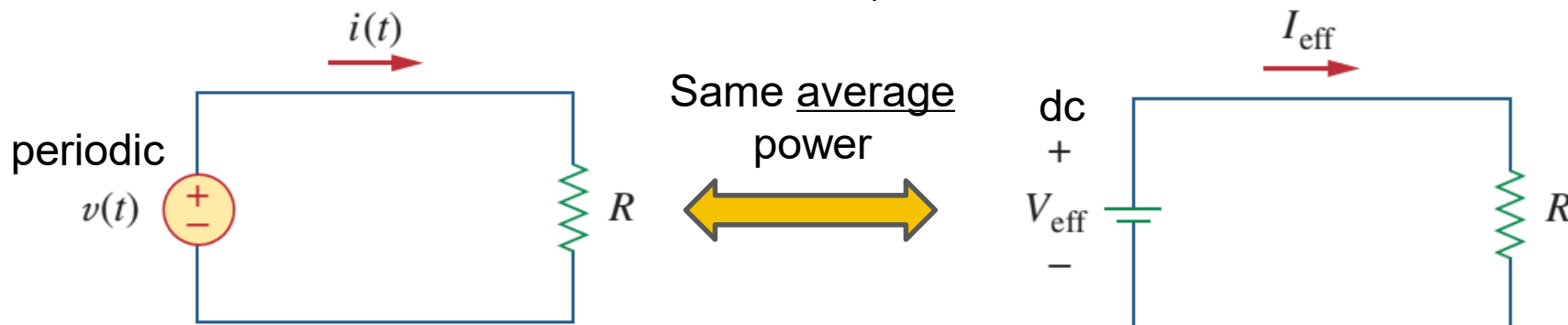
A resistive load ( $R$ ) absorbs power ~~at all times~~, while a reactive load ( $L$  or  $C$ ) absorbs zero average power.



## Effective Value (RMS)

- For any periodic function  $x(t)$  in general, its rms value is

$$X_{\text{eff}} = X_{\text{rms}} = \sqrt{\frac{1}{T} \int_0^T x^2(t) dt}$$



$$V_{\text{eff}} = \sqrt{\frac{1}{T} \int_0^T v^2(t) dt}$$

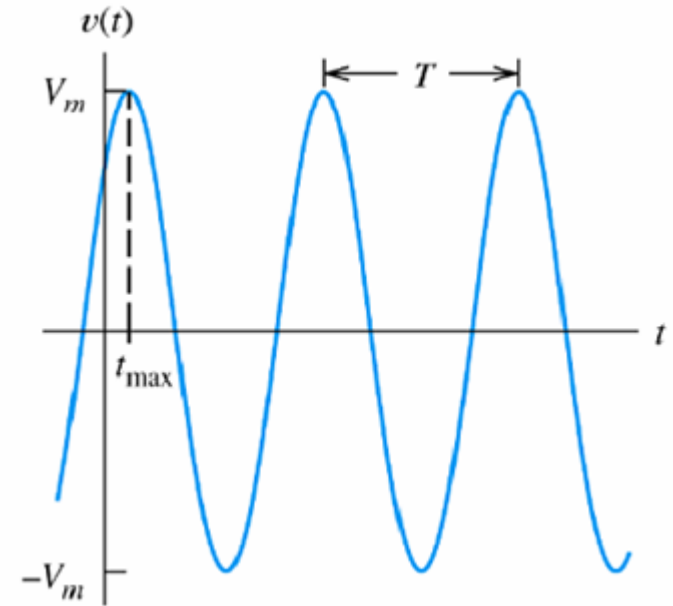
Similarly:

$$I_{\text{eff}} = \sqrt{\frac{1}{T} \int_0^T i^2(t) dt}$$

## RMS of a sinusoidal signal

- The RMS value of  $v(t) = V_m \cos(\omega t + \phi)$  is

$$\begin{aligned} V_{\text{rms}} &= \sqrt{\frac{1}{T} \int_{t_0}^{t_0+T} v^2(t) dt} \\ &= \sqrt{\frac{1}{T} \int_{t_0}^{t_0+T} V_m^2 \cdot \cos^2(\omega t + \phi) dt} \\ &= \frac{V_m}{\sqrt{2}} \end{aligned}$$



Average  
Power

$$\begin{aligned} P &= \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) = \frac{V_m}{\sqrt{2}} \frac{I_m}{\sqrt{2}} \cos(\theta_v - \theta_i) \\ &= V_{\text{rms}} I_{\text{rms}} \cos(\theta_v - \theta_i) \end{aligned}$$



# Outline

- Instantaneous power
- Average power
- Apparent power
- Power factor
- Complex power



# Apparent Power

$$P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) = \frac{V_m}{\sqrt{2}} \frac{I_m}{\sqrt{2}} \cos(\theta_v - \theta_i) \\ = V_{\text{rms}} I_{\text{rms}} \cos(\theta_v - \theta_i)$$

$$S \text{ or } S_a = V_{\text{rms}} I_{\text{rms}}$$

Unit: volt-amp (VA)

It seems apparent that the power should be the voltage-current product, *by analogy with dc resistive circuits.*



# Power Factor

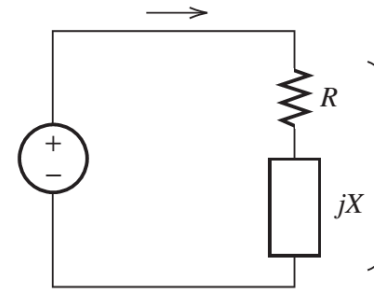
$$P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) = \frac{V_m}{\sqrt{2}} \frac{I_m}{\sqrt{2}} \cos(\theta_v - \theta_i) \\ = V_{\text{rms}} I_{\text{rms}} \cos(\theta_v - \theta_i)$$

- The power factor

$$\text{pf} = \frac{P}{S} = \cos(\theta_v - \theta_i)$$

- $(\theta_v - \theta_i)$  is called power factor angle.
  - $>0$  means a *lagging* pf (current lags voltage)
  - $<0$  means a *leading* pf (current leads voltage)
- pf ranges from 0 to 1.

## Power Factor-2



Power factor leading and lagging relationships for a load  $\mathbf{Z} = R + jX$ .

Load Type	$\phi_z = (\theta_v - \theta_i)$	I-V Relationship	$pf$
Purely Resistive ( $X = 0$ )	$\phi_z = 0$	$\mathbf{I}$ in-phase with $\mathbf{V}$	1
Inductive ( $X > 0$ )	$0 < \phi_z \leq 90^\circ$	$\mathbf{I}$ lags $\mathbf{V}$	lagging
Purely Inductive ( $X > 0$ and $R = 0$ )	$\phi_z = 90^\circ$	$\mathbf{I}$ lags $\mathbf{V}$ by $90^\circ$	lagging
Capacitive ( $X < 0$ )	$-90^\circ \leq \phi_z < 0$	$\mathbf{I}$ leads $\mathbf{V}$	leading
Purely Capacitive ( $X < 0$ and $R = 0$ )	$\phi_z = -90^\circ$	$\mathbf{I}$ leads $\mathbf{V}$ by $90^\circ$	leading



## Power Factor-3

$$P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) = \frac{V_m}{\sqrt{2}} \frac{I_m}{\sqrt{2}} \cos(\theta_v - \theta_i) \\ = V_{\text{rms}} I_{\text{rms}} \cos(\theta_v - \theta_i)$$

- The power factor

$$\text{pf} = \frac{P}{S} = \cos(\theta_v - \theta_i)$$

- $(\theta_v - \theta_i)$  is called power factor angle.
- $(\theta_v - \theta_i)$  is equal to the angle of the load impedance

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = \frac{V_m \angle \theta_v}{I_m \angle \theta_i} = \frac{V_m}{I_m} \angle \theta_v - \theta_i$$

Also

$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = \frac{V_{\text{rms}}}{I_{\text{rms}}} \angle \theta_v - \theta_i$$



# Outline

- Instantaneous power
- Average power
- Apparent power
- Power factor
- **Complex power**





# Complex Power

$$v(t) = V_m \cos(\omega t + \theta_v) \Rightarrow \mathbf{V} = V_m \angle \theta_v$$

$$i(t) = I_m \cos(\omega t + \theta_i) \Rightarrow \mathbf{I} = I_m \angle \theta_i$$

We observe that:

$$\begin{aligned} \frac{1}{2} \mathbf{V} \mathbf{I}^* &= \frac{1}{2} V_m I_m \angle (\theta_v - \theta_i) \\ &= \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) + j \frac{1}{2} V_m I_m \sin(\theta_v - \theta_i) \end{aligned}$$

- Define a ***single*** power metric

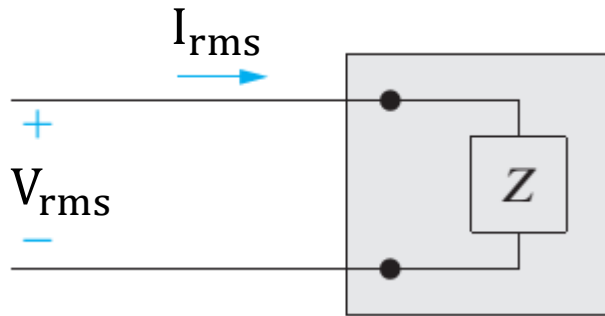
$$\mathbf{S} = \frac{1}{2} \mathbf{V} \mathbf{I}^* = \mathbf{V}_{\text{rms}} \mathbf{I}_{\text{rms}}^* = V_{\text{rms}} I_{\text{rms}} \angle (\theta_v - \theta_i) = P + jQ$$

Unit: volt-amp (VA)

Complex power is important in power analysis because it contains ***all*** the information pertaining to the power absorbed by a given load.



## Another Way to Calculate Complex Power using impedance



$$\mathbf{V}_{\text{rms}} = \mathbf{I}_{\text{rms}} \mathbf{Z}$$

$$\begin{aligned} \mathbf{S} &= \mathbf{V}_{\text{rms}} \mathbf{I}_{\text{rms}}^* \\ &= \mathbf{V}_{\text{rms}} \left( \frac{\mathbf{V}_{\text{rms}}}{\mathbf{Z}} \right)^* \\ &= \frac{|\mathbf{V}_{\text{rms}}|^2}{\mathbf{Z}^*} \end{aligned}$$

$$\begin{aligned} \mathbf{S} &= \mathbf{V}_{\text{rms}} \mathbf{I}_{\text{rms}}^* \\ &= \mathbf{I}_{\text{rms}} \mathbf{Z} \mathbf{I}_{\text{rms}}^* \\ &= |\mathbf{I}_{\text{rms}}|^2 \mathbf{Z} \\ &= |\mathbf{I}_{\text{rms}}|^2 (R + jX) \\ &= |\mathbf{I}_{\text{rms}}|^2 R + j |\mathbf{I}_{\text{rms}}|^2 X \\ &= I_{\text{rms}}^2 R + j I_{\text{rms}}^2 X \end{aligned}$$

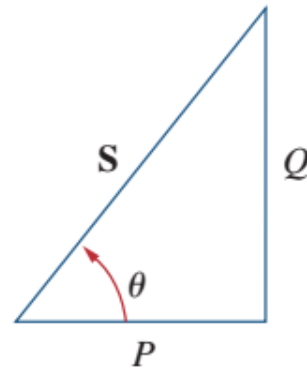
$$\mathbf{S} = I_{\text{rms}}^2 \mathbf{Z} = \frac{V_{\text{rms}}^2}{\mathbf{Z}^*} = \mathbf{V}_{\text{rms}} \mathbf{I}_{\text{rms}}^*$$

$$\begin{aligned} P &= \text{Re}(\mathbf{S}) = I_{\text{rms}}^2 R \\ Q &= \text{Im}(\mathbf{S}) = I_{\text{rms}}^2 X \end{aligned}$$

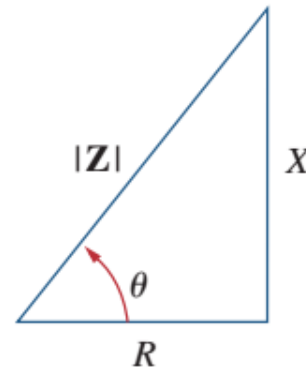
# Power Triangle

$$P = \text{Re}(S) = I_{\text{rms}}^2 R$$

$$Q = \text{Im}(S) = I_{\text{rms}}^2 X$$



(a)



(b)

**Figure 11.21**

(a) Power triangle, (b) impedance triangle.

Quantity	Units
Complex power	volt-amps
Average power	watts
Reactive power	var



$$\mathbf{S} = \frac{1}{2} \mathbf{V} \mathbf{I}^* = \mathbf{V}_{\text{rms}} \mathbf{I}_{\text{rms}}^* = V_{\text{rms}} I_{\text{rms}} \angle(\theta_v - \theta_i) = P + jQ$$

$$P = \text{Re}(\mathbf{S}) = I_{\text{rms}}^2 R$$

$$Q = \text{Im}(\mathbf{S}) = I_{\text{rms}}^2 X$$

- Average (or real) power

$$P = \text{Re}\left[\frac{1}{2} \mathbf{V} \mathbf{I}^*\right]$$

Unit: W

- Reactive power

$$Q = \text{Im}\left[\frac{1}{2} \mathbf{V} \mathbf{I}^*\right]$$

Unit: Volt Amperes Reactive (VAR)

- Apparent power

$$S = |\mathbf{S}| = \frac{1}{2} V_m I_m = V_{\text{rms}} I_{\text{rms}}$$

Unit: volt-amp (VA)



$$\text{Complex Power} = \mathbf{S} = P + jQ = \mathbf{V}_{\text{rms}}(\mathbf{I}_{\text{rms}})^*$$

$$= |\mathbf{V}_{\text{rms}}| |\mathbf{I}_{\text{rms}}| \angle \theta_v - \theta_i$$

$$\text{Apparent Power} = S = |\mathbf{S}| = |\mathbf{V}_{\text{rms}}| |\mathbf{I}_{\text{rms}}| = \sqrt{P^2 + Q^2}$$

$$\text{Real Power} = P = \text{Re}(\mathbf{S}) = S \cos(\theta_v - \theta_i)$$

$$\text{Reactive Power} = Q = \text{Im}(\mathbf{S}) = S \sin(\theta_v - \theta_i)$$

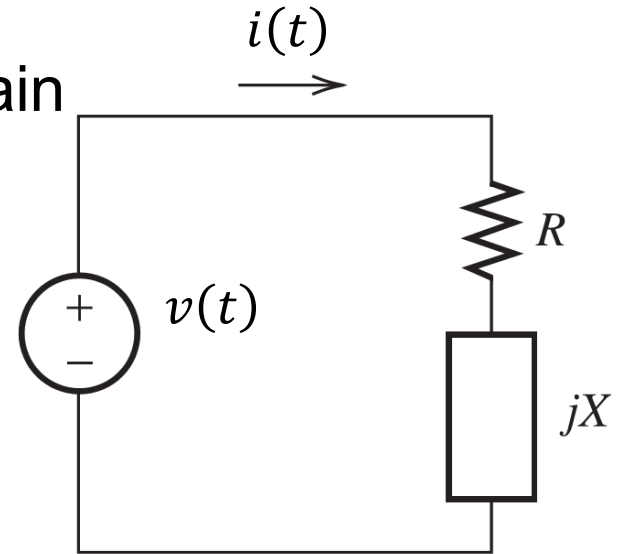
$$\text{Power Factor} = \frac{P}{S} = \cos(\theta_v - \theta_i)$$



# Reactive Power $Q$

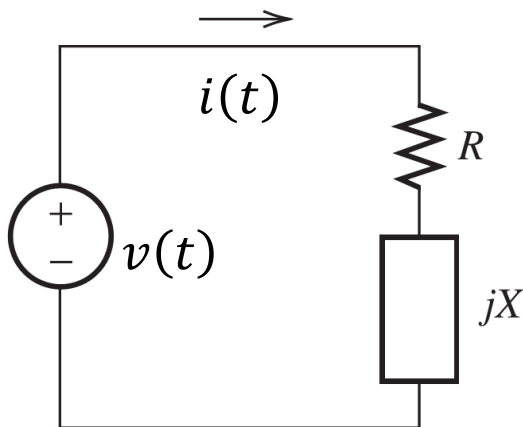
Let us look at Instantaneous power again

$$\begin{aligned} p(t) &= v(t)i(t) \\ p(t) &= p_R(t) + p_X(t) \\ p_R(t) &= \\ p_X(t) &= \end{aligned}$$



## Reactive Power $Q$ : Peak Exchanged Power

- Definition: The peak instantaneous power associated with the energy storage elements contained in a general load.



$$Q = \frac{1}{2} V_m I_m \sin(\theta_v - \theta_i) = V_{rms} I_{rms} \sin(\theta_v - \theta_i)$$

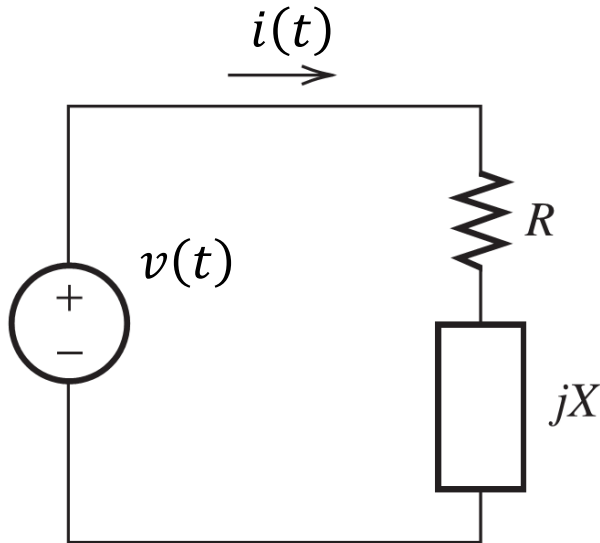
$$Q = \begin{cases} 0 & \text{for resistive } (\theta_v - \theta_i = 0^\circ) \\ \frac{1}{2} V_m I_m & \text{for inductive } (\theta_v - \theta_i = 90^\circ) \\ -\frac{1}{2} V_m I_m & \text{for capacitive } (\theta_v - \theta_i = -90^\circ) \end{cases}$$

- Reactive power is still of concern to power-system engineers
  - Transmission lines/transformers/fuses et al. must be capable of withstanding the current associated with reactive power.



## Example

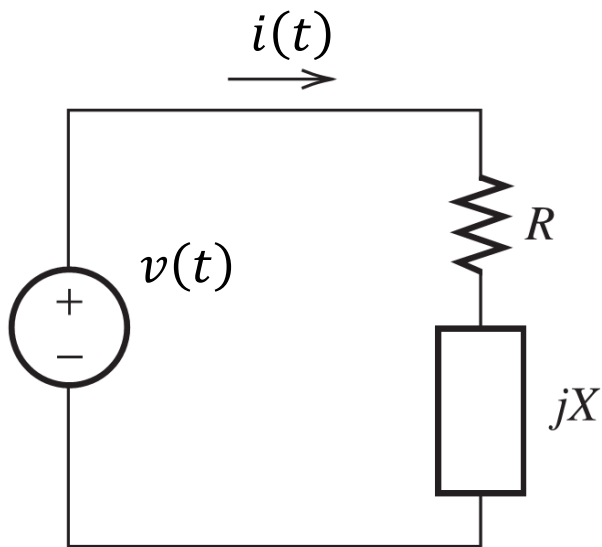
- Find the average power and reactive power absorbed by an impedance  $Z = 30 - j70\Omega$ , when a voltage  $V_m = 120\angle 0^\circ$  is applied across it.





## Example

- Find the average power and reactive power absorbed by an impedance  $Z = 30 - j70\Omega$ , when a voltage  $V_m = 120\angle 0^\circ$  is applied across it.



$$\begin{aligned} \mathbf{I}_m &= \frac{\mathbf{V}_m}{Z} = \frac{120\angle 0^\circ}{30 - j70} = \frac{120\angle 0^\circ}{76.16\angle -66.8^\circ} \\ &= 1.576\angle 66.8^\circ \text{ A} \end{aligned}$$

$$P = \frac{1}{2} V_m I_m \cos(\theta_v - \theta_i) = 37.24 \text{ W}$$

$$Q = \frac{1}{2} V_m I_m \sin(\theta_v - \theta_i) = -86.91 \text{ VAR}$$



## Exercise

- The voltage across a load is  $v(t) = 60\cos(\omega t - 10^\circ)V$ , and the current through the load is  $i(t) = 1.5\cos(\omega t + 50^\circ)$ . Find
  - The complex and apparent powers.
  - The real and reactive powers.
  - The power factor and the load impedance.



## Exercise

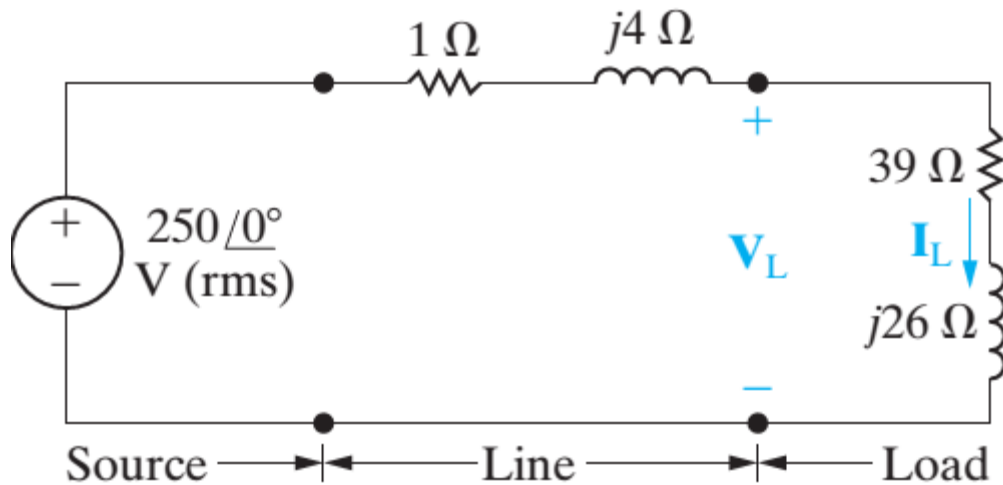
- The voltage across a load is  $v(t) = 60\cos(\omega t - 10^\circ)\text{V}$ , and the current through the load is  $i(t) = 1.5\cos(\omega t + 50^\circ)$ . Find
  - The complex and apparent powers.
  - The real and reactive powers.
  - The power factor and the load impedance.

$$\mathbf{S} = \mathbf{V}_{\text{rms}} \mathbf{I}_{\text{rms}}^* = 45 \angle -60^\circ \text{ VA}$$

$$\text{pf} = 0.5 \text{ (leading)}$$

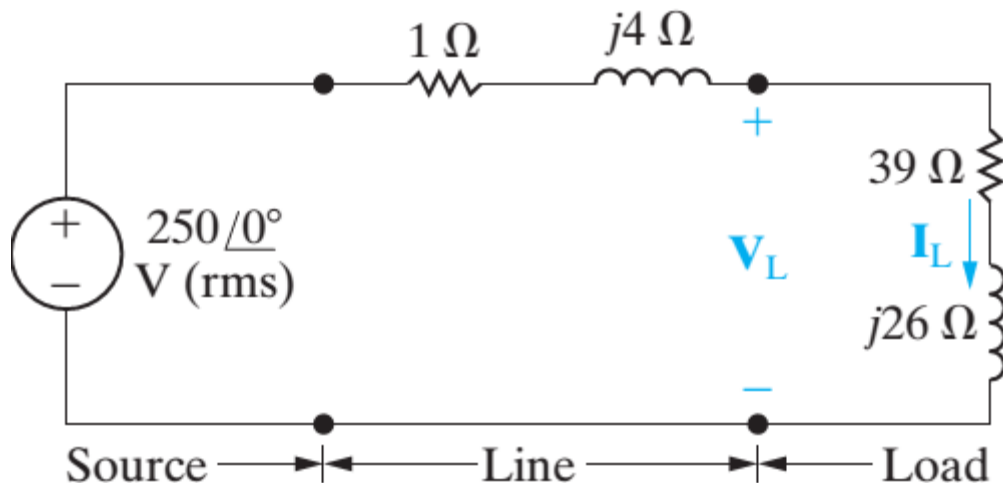
$$\mathbf{Z} = \frac{\mathbf{V}}{\mathbf{I}} = 40 \angle -60^\circ \Omega$$

## Example



- Find  $V_L$  and  $I_L$ .
- Find the average and reactive power
  - Delivered to the load
  - Delivered to the line
  - Supplied by the source

## Example



- Find  $V_L$  and  $I_L$ .
- Find the average and reactive power
  - Delivered to the load
  - Delivered to the line
  - Supplied by the source

$$\begin{aligned} I_L &= \frac{250\angle 0^\circ}{40 + j30} = 4 - j3 \\ &= 5\angle -36.87^\circ \text{ (rms)} \end{aligned}$$

$$\begin{aligned} V_L &= I_L(39 + j26) \\ &= 234 - j13 \\ &= 234.36\angle -3.18^\circ \end{aligned}$$

Load:

$$V_L I_L^* = 975 + j650 \text{ VA}$$

Line:

$$P = (5)^2(1) = 25 \text{ W}$$

$$Q = (5)^2(4) = 100 \text{ VAR}$$

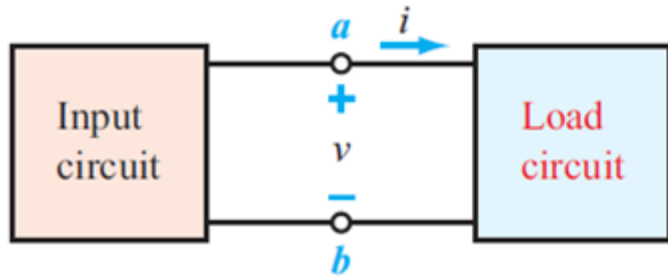
Source:

$$250\angle 0^\circ I_L^* = 1000 + j750 \text{ VA}$$



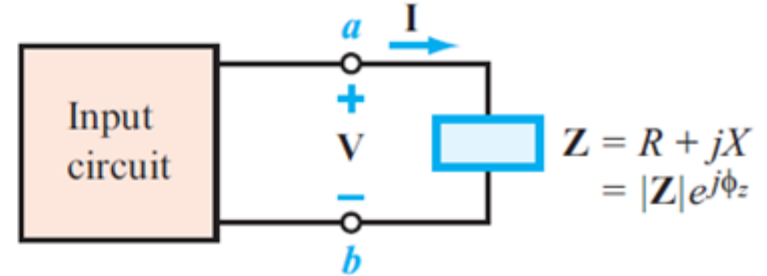
# Complex Power

## Time Domain



$$\begin{aligned} v(t) &= V_m \cos(\omega t + \phi_v) \\ i(t) &= I_m \cos(\omega t + \phi_i) \\ V_{\text{rms}} &= V_m / \sqrt{2} \\ I_{\text{rms}} &= I_m / \sqrt{2} \end{aligned}$$

## Phasor Domain



$$\begin{aligned} V &= V_m e^{j\phi_v} \\ I &= I_m e^{j\phi_i} \\ V_{\text{rms}} &= V_{\text{rms}} e^{j\phi_v} \\ I_{\text{rms}} &= I_{\text{rms}} e^{j\phi_i} \end{aligned}$$

## Complex Power

$$S = \frac{1}{2} \mathbf{V} \mathbf{I}^* = V_{\text{rms}} \mathbf{I}_{\text{rms}}^* = P + jQ$$

### Real Average Power

$$\begin{aligned} P &= \Re[S] \text{ [W]} \\ &= V_{\text{rms}} I_{\text{rms}} \cos(\phi_v - \phi_i) \\ &= I_{\text{rms}}^2 R \end{aligned}$$

### Reactive Power

$$\begin{aligned} Q &= \Im[S] \text{ [VAr]} \\ &= V_{\text{rms}} I_{\text{rms}} \sin(\phi_v - \phi_i) \\ &= I_{\text{rms}}^2 X \end{aligned}$$

### Apparent Power

$$\begin{aligned} S &= |S| = \sqrt{P^2 + Q^2} \\ &= V_{\text{rms}} I_{\text{rms}} \\ &= I_{\text{rms}}^2 |Z| \end{aligned}$$

### Power Factor

$$\begin{aligned} pf &= \frac{P}{S} \\ &= \cos(\phi_v - \phi_i) \\ &= \cos \phi_z \end{aligned}$$

$$\begin{aligned} S &= S e^{j\phi_s} \\ \phi_s &= \phi_v - \phi_i = \phi_z \end{aligned}$$



- Maximum average power transfer

