SI231B: Matrix Computations, 2024 Fall

Homework Set #4

Acknowledgements:

1) Deadline: **2024-12-15 23:59:59**
2) Please submit the PDF file to gradescope. Course entry code: 8KJ345.
3) You have 5 "free days" in total for all late homework submissions.
4) If your homework is handwritten, please make it clear and legible.
5) All your answers are required to be in English.
6) Please include the main steps in your answer; otherwise, you may not get the points.

Problem 1. (**Properties of Positive Semidefinite Matrices**, 15 points)

Let $\mathbf{A}$ be a $n \times n$ symmetric positive semidefinite matrix, and its eigendecomposition is $\mathbf{A}$=$\mathbf{V}\Lambda\mathbf{V}^T$

1) Given a set $S = \{(i,j)|\mathbf{A}[i:j, i:j]$ is positive semidefinite, $i \leq j\}$, find $|S|$, the cardinality of S (i.e. the number of elements in S), and explain why (Note: if $\mathbf{A}_{ii} \geq 0$, then $(i,i) \in S$) (5 points)
2) Denote $\mathbf{A}^{1/2}$=$\mathbf{V}\Lambda^{1/2}\mathbf{V}^T$, then prove $\mathbf{A}^{1/2}$ is the unique positive semidefinite factor for the factorization $\mathbf{A}$=$\mathbf{B}^T\mathbf{B}$ (10 points)

Solution:

1) Actually, all of $\mathbf{A}[i:j, i:j]$ is positive semidefinite, since $\mathbf{A}[i:j, i:j]$ is principle submatrix. So the number is $C_n^2 + n = n*(n+1)/2$
2) Assume there exist another positive semidefinite matrix $\widetilde{\mathbf{B}}$ such that $\mathbf{A} = \widetilde{\mathbf{B}}^T\widetilde{\mathbf{B}}$, then denote its eigendecomposition is $\widetilde{\mathbf{B}} = \widetilde{\mathbf{V}}\mathbf{D}\widetilde{\mathbf{V}}^T$, so we have $\mathbf{B}^T\mathbf{B} = \widetilde{\mathbf{B}}^T\widetilde{\mathbf{B}}$, hence $\widetilde{\mathbf{V}}\mathbf{D}^2\widetilde{\mathbf{V}}^T = \mathbf{V}\Lambda\mathbf{V}^T$, so $\mathbf{D}^2 = \widetilde{\mathbf{V}}^T\mathbf{V}\Lambda\mathbf{V}^T\widetilde{\mathbf{V}}$. Denote $\mathbf{E} = \widetilde{\mathbf{V}}^T\mathbf{V}$, so we have $\mathbf{D}^2 = \mathbf{E}\Lambda\mathbf{E}^T$. It is obvious that $\mathbf{E}$ is orthogonal, so $\mathbf{D}^2$ have same eigenvalue with Λ, and $\mathbf{D}$ have same eigenvalue with $\Lambda^{1/2}$

   We notice that for a interchange matrix Π_{ij},

   Π_{ij}Diag$(d_1, \cdots, d_n)\Pi_{ij}$=Diag$(d_1, \cdots, d_{i-1}, d_j, d_{i+1}, \cdots, d_{j-1}, d_i, d_{j+1}, \cdots, d_n)$. So $\mathbf{D}$ and $\Lambda^{1/2}$ can be connected with permutation matrix: $\mathbf{D} = \Pi_k \cdots \Pi_1 \Lambda^{1/2} \Pi_i \cdots \Pi_k$.Denote $\mathbf{P} = \Pi_k \cdots \Pi_1$, so $\mathbf{D} = \mathbf{P}\Lambda^{1/2}\mathbf{P}^T$, substitute into $\mathbf{D}^2 = \mathbf{E}\Lambda\mathbf{E}^T$, we have $\mathbf{P}\Lambda\mathbf{P}^T = \mathbf{E}\Lambda\mathbf{E}^T$, hence $\Lambda = \mathbf{P}^T\mathbf{E}\Lambda\mathbf{E}^T\mathbf{P}$. Denote $\mathbf{Q} = \mathbf{P}^T\mathbf{E}$, so we have $\Lambda = \mathbf{Q}\Lambda\mathbf{Q}^T$. We notice that $\mathbf{Q}$ is also orthogonal.

   Denote Λ=Diag$(\lambda_1, \cdots, \lambda_n)$, then$[\mathbf{Q}\Lambda\mathbf{Q}^T]_{ij} = \sum_{k=1}^n \lambda_k \mathbf{Q}_{ik}\mathbf{Q}_{jk}$, so $\lambda_i = [\mathbf{Q}\Lambda\mathbf{Q}^T]_{ii} = \sum_{k=1}^n \lambda_k \mathbf{Q}_{ik}^2$, we can write a linear equations: $\begin{bmatrix} \mathbf{Q}_{11}^2 & \cdots & \mathbf{Q}_{1n}^2 \\ \vdots & \vdots & \vdots \\ \mathbf{Q}_{n1}^2 & \cdots & \mathbf{Q}_{nn}^2 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{bmatrix}$, since $\mathbf{Q}$ is orthogonal, $0 \leq \mathbf{Q}_{ij}^2 \leq 1$ and $\sum_{k=1}^n \mathbf{Q}_{ik}^2 = 1$. Since $\lambda_i \geq 0$, then min$(\lambda_1, \cdots, \lambda_n) \leq \sum_{k=1}^n \lambda_k \mathbf{Q}_{ik}^2 \leq$max$(\lambda_1, \cdots, \lambda_n)$.

   Denote λ_m=max$(\lambda_1, \cdots, \lambda_n)$, then it means $\sum_{k=1}^n \lambda_k \mathbf{Q}_{mk}^2 \leq \lambda_m$, the only way to achieve $\sum_{k=1}^n \lambda_k \mathbf{Q}_{mk}^2 = \lambda_m$ is $\mathbf{Q}_{mm} = 1$, which means $\mathbf{Q}_{mk} = 0$ for $m \neq k$. Also, since $\mathbf{Q}$ is orthogonal, then $\mathbf{Q}_{km} = 0$ for $m \neq k$, so we can take the minor part of the linear equations:

$$\begin{bmatrix} \mathbf{Q}_{11}^2 & \cdots & \mathbf{Q}_{1,(m-1)} & \mathbf{Q}_{1,(m+1)} & \cdots & \mathbf{Q}_{1n}^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{Q}_{(m-1)1}^2 & \cdots & \mathbf{Q}_{(m-1),(m-1)} & \mathbf{Q}_{(m-1),(m+1)} & \cdots & \mathbf{Q}_{(m-1),n}^2 \\ \mathbf{Q}_{(m+1)1}^2 & \cdots & \mathbf{Q}_{(m+1),(m-1)} & \mathbf{Q}_{(m+1),(m+1)} & \cdots & \mathbf{Q}_{(m+1),n}^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \mathbf{Q}_{n1}^2 & \cdots & \mathbf{Q}_{n,(m-1)} & \mathbf{Q}_{n,(m+1)} & \cdots & \mathbf{Q}_{n,n}^2 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_{m-1} \\ \lambda_{m+1} \\ \vdots \\ \lambda_n \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ \vdots \\ \lambda_{m-1} \\ \lambda_{m+1} \\ \vdots \\ \lambda_n \end{bmatrix}.$$

   Then we can repeat the process, until we find that $\mathbf{Q}_{ii} = 1$ for $i = 1, \cdots, n$. As a result, $\mathbf{Q} = \mathbf{I}$, so $\mathbf{P} = \mathbf{E}$. Substitute $\mathbf{P} = \mathbf{E}$, $\mathbf{D} = \mathbf{P}\Lambda^{1/2}\mathbf{P}^T$ and $\mathbf{E} = \widetilde{\mathbf{V}}^T\mathbf{V}$ into $\widetilde{\mathbf{B}} = \widetilde{\mathbf{V}}\mathbf{D}\widetilde{\mathbf{V}}^T$, we have $\widetilde{\mathbf{B}} = \mathbf{V}\Lambda^{1/2}\mathbf{V}^T = \mathbf{B}$, which means $\mathbf{A}^{1/2}$=$\mathbf{V}\Lambda^{1/2}\mathbf{V}^T$ is the unique positive semidefinite factor for factorization $\mathbf{A}$=$\mathbf{B}^T\mathbf{B}$

Problem 2. (**Principle Component Analysis**, 15 points)

Principal Component Analysis (PCA) is a technique for dimensionality reduction that projects high-dimensional data onto a new set of orthogonal axes, called principal components, to extract the main features of the data. Each principal component is a linear combination of the original variables and is ranked based on the amount of variance it explains, with the first principal component retaining the most information. PCA reduces dimensionality, eliminates correlations between features, and preserves the primary structure of the original data as much as possible. It is widely used in pattern recognition, data compression, and visualization.

Main process of PCA is: 1. Standardize the data 2. Compute the covariance matrix, 3. Solve for eigenvalues and eigenvectors, 4. Select the principal components and 5. Project the data.

Given data matrix **X**= $\begin{bmatrix} 12 & -13 & -23 & 18 \\ -33 & 12 & 22 & -22 \\ 10 & -11 & -6 & 10 \\ 20 & -11 & -3 & 0 \\ 6 & -4 & -8 & -29 \\ 9 & -23 & 6 & -50 \\ -18 & 8 & 13 & -3 \\ 39 & -23 & -24 & -2 \\ 6 & -10 & 3 & -50 \\ 14 & 6 & -11 & 22 \end{bmatrix}$, where the sample number is $n = 10$ and the dimension of data is 4.

1) Find its covariance matrix $\mathbf{C}$. (Note: $\mathbf{C} = \frac{1}{n-1}\mathbf{X}_{\text{centered}}^T\mathbf{X}_{\text{centered}}$, where $\mathbf{X}_{\text{centered}} = \mathbf{X} - \mathbf{1}\mathbf{x}_{\text{average}}^T$, and $\mathbf{x}_{\text{average}} = \frac{1}{n}\mathbf{X}^T\mathbf{1}$) (5 points)
2) Apply principle component analysis to the data (i.e. apply eigendecomposition to the covariance matrix $\mathbf{C}$). (5 points)
3) Keep the first two main component, which are the eigenvectors associated with the largest two eigenvalue of $\mathbf{C}$, and find a projection matrix $\mathbf{P} \in \mathbb{R}^{4\times 4}$ that project each data sample onto the span of the two main components, so the data is compressed and the data matrix after projection have rank 2. (i.e. $\text{rank}(\mathbf{P}\mathbf{X}_{\text{centered}}^T) = 2$) (5 points)

All calculation results are rounded to four significant figures. It is recommended to use a calculator for calculation.

Solution:

1) $\mathbf{x}_{\text{average}} = [6.5, -6.9, -3.1, -10.6]^T$, $\mathbf{X}_{\text{centered}}$= $\begin{bmatrix} 5.5000 & -6.1000 & -19.9000 & 28.6000 \\ -39.5000 & 18.9000 & 25.1000 & -11.4000 \\ 3.5000 & -4.1000 & -2.9000 & 20.6000 \\ 13.5000 & -4.1000 & 0.1000 & 10.6000 \\ -0.5000 & 2.9000 & -4.9000 & -18.4000 \\ 2.5000 & -16.1000 & 9.1000 & -39.4000 \\ -24.5000 & 14.9000 & 16.1000 & 7.6000 \\ 32.5000 & -16.1000 & -20.9000 & 8.6000 \\ -0.5000 & -3.1000 & 6.1000 & -39.4000 \\ 7.5000 & 12.9000 & -7.9000 & 32.6000 \end{bmatrix}$,

$\mathbf{C} = \mathbf{X}_{\text{centered}}^T \mathbf{X}_{\text{centered}}$= $\begin{bmatrix} 389.3889 & -186.8333 & -246.7222 & 121.2222 \\ -186.8333 & 150.3222 & 100.2333 & 64.5111 \\ -246.7222 & 100.2333 & 215.2111 & -193.0667 \\ 121.2222 & 64.5111 & -193.0667 & 680.2667 \end{bmatrix}$

2) $\mathbf{C} = \mathbf{V}\Lambda\mathbf{V}^T$, where $\mathbf{V}$= $\begin{bmatrix} 0.6465 & 0.0045 & -0.5987 & 0.4728 \\ 0.5271 & -0.6973 & 0.4715 & -0.1169 \\ 0.5515 & 0.6654 & 0.2609 & -0.4301 \\ -0.0090 & 0.2664 & 0.5925 & 0.7602 \end{bmatrix}$,

Λ=Diag(24.9250,31.2217,524.0862,854.9559)

3) First two main component are $\mathbf{v}_4 = [0.4728, -0.1169, -0.4301, 0.7602]^T$ and $\mathbf{v}_3 = [-0.5987, 0.4715, 0.2609, 0.5925]^T$,let $\mathbf{B} = [\mathbf{v}_4, \mathbf{v}_3]$, so projection matrix $\mathbf{P} = \mathbf{B}(\mathbf{B}^T\mathbf{B})^{-1}\mathbf{B}^T$= $\begin{bmatrix} 0.5820 & -0.3376 & -0.3595 & 0.0046 \\ -0.3376 & 0.2360 & 0.1733 & 0.1905 \\ -0.3595 & 0.1733 & 0.2530 & -0.1723 \\ 0.0046 & 0.1905 & -0.1723 & 0.9289 \end{bmatrix}$.

Problem 3. (**Matrix Inequalities and Schur Complement**, 15 points)

Given symmetric positive definite matrices $\mathbf{A}$ and $\mathbf{B}$:

1) Prove that $\mathbf{A} \succeq \mathbf{B}$ if and only if $\mathbf{B}^{-1} \succeq \mathbf{A}^{-1}$(10 points)
2) For the symmetric matrix $\mathbf{X}$= $\begin{bmatrix} \mathbf{A} & \mathbf{C} \\ \mathbf{C}^T & \mathbf{D} \end{bmatrix}$, let its Schur Complement of $\mathbf{A}$ in $\mathbf{X}$ be $\mathbf{S}$. Prove that $\mathbf{S} \succeq \mathbf{0}$ is equivalent to "If $\mathbf{A} \succ \mathbf{0}$, then $\mathbf{X} \succeq \mathbf{0}$"

   (5 points)

Solution:

1) Forward:

   Since $\mathbf{A} \succ \mathbf{0}$, denote its eigendecomposition is $\mathbf{A} = \mathbf{V}\Lambda\mathbf{V}^T$, then $\mathbf{A}^{-1/2} = \mathbf{V}\Lambda^{-1/2}\mathbf{V}^T$ is symmetric positive definite. Since $\mathbf{A} \succeq \mathbf{B}$, then $\mathbf{x}^T\mathbf{A}\mathbf{x} \geq \mathbf{x}^T\mathbf{B}\mathbf{x}$ for any $\mathbf{x} \in \mathbb{R}$. Denote $\mathbf{y} = \mathbf{A}^{-1/2}\mathbf{x}$, then $\{\mathbf{y} = \mathbf{A}^{-1/2}\mathbf{x} | \mathbf{x} \in \mathbb{R}\} = \mathbb{R}$, so $0 < \mathbf{x}^T\mathbf{A}^{-1/2}\mathbf{B}\mathbf{A}^{-1/2}\mathbf{x} = \mathbf{y}^T\mathbf{B}\mathbf{y} \leq \mathbf{y}^T\mathbf{A}\mathbf{y} = \mathbf{x}^T\mathbf{I}\mathbf{x}$ for any $\mathbf{x} \in \mathbb{R}$, as a result, $\mathbf{A}^{-1/2}\mathbf{B}\mathbf{A}^{-1/2} \preceq \mathbf{I}$,which means all eigenvalues of $\mathbf{A}^{-1/2}\mathbf{B}\mathbf{A}^{-1/2}$ is less or equal than 1. Since $\mathbf{A}^{-1/2}\mathbf{B}\mathbf{A}^{-1/2}$ is similar to $\mathbf{B}\mathbf{A}^{-1}$, then all eigenvalues of $\mathbf{B}\mathbf{A}^{-1}$ is less or equal than 1. So all eigenvalues of $(\mathbf{B}\mathbf{A}^{-1})^{-1} = \mathbf{A}\mathbf{B}^{-1}$ is equal or greater than 1. Similarly, its similar matrix $\mathbf{A}^{1/2}\mathbf{B}^{-1}\mathbf{A}^{1/2} \succeq \mathbf{I}$, so for any $\mathbf{x} \in \mathbb{R}$,
   $\mathbf{x}^T\mathbf{B}^{-1}\mathbf{x} = \mathbf{y}^T\mathbf{A}^{1/2}\mathbf{B}^{-1}\mathbf{A}^{1/2}\mathbf{y} \geq \mathbf{y}^T\mathbf{y} = \mathbf{x}^T\mathbf{A}^{-1}\mathbf{x}$, as a result, $\mathbf{B}^{-1} \succeq \mathbf{A}^{-1}$.

   Backward: Define $\mathbf{D} = \mathbf{B}^{-1}$ and $\mathbf{C} = \mathbf{A}^{-1}$, then just copy the derivations above, only replace $\mathbf{A}$ with $\mathbf{D}$ and $\mathbf{B}$ with $\mathbf{C}$, then we will get $\mathbf{A} = \mathbf{C}^{-1} \succeq \mathbf{D}^{-1} = \mathbf{B}$.
2) Martix $\mathbf{X}$ can be factorization as $\mathbf{X} = \mathbf{N}\mathbf{P}\mathbf{N}^T$, where $\mathbf{N}$= $\begin{bmatrix} \mathbf{I} & 0 \\ \mathbf{C}^T\mathbf{A}^{-1} & \mathbf{I} \end{bmatrix}$ and $\mathbf{P}$= $\begin{bmatrix} \mathbf{A} & \mathbf{0} \\ \mathbf{0} & \mathbf{S} \end{bmatrix}$.

   Forward: If both $\mathbf{S} \succeq \mathbf{0}$ and $\mathbf{A} \succ \mathbf{0}$, then $\mathbf{P} \succ \mathbf{0}$. Also, since $\mathbf{A} \succ \mathbf{0}$, $\mathbf{N}$ is invertible , so $\mathbf{P} \succeq \mathbf{0} \Longrightarrow \mathbf{X} \succeq \mathbf{0}$.

   Backward: If both $\mathbf{X} \succeq \mathbf{0}$ and $\mathbf{A} \succ \mathbf{0}$, then means $\mathbf{N}$ is invertible , so $\mathbf{X} \succeq \mathbf{0} \Longrightarrow \mathbf{P} \succeq \mathbf{0}$, since $\mathbf{S}$ is principle submatrix of $\mathbf{P}$, then $\mathbf{S} \succeq \mathbf{0}$.

Problem 4. (**Singular Value Decomposition (SVD)**, 20 points)

Consider the matrices

$$\mathbf{A} = \begin{bmatrix} -4 & -2 & -4 & -2 \\ 2 & -2 & 2 & 1 \\ -4 & 1 & -4 & -2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}.$$

1) Find the singular value decomposition of $\mathbf{A}$, i.e., $\mathbf{A} = \mathbf{U}\boldsymbol{\Sigma}\mathbf{V}^T$. (10 points)

2) Use the SVD above to decompose $\mathbf{b}$ as $\mathbf{b} = \hat{\mathbf{b}} + \mathbf{z}$, where $\hat{\mathbf{b}} \in \mathcal{R}(\mathbf{A})$ and $\mathbf{z} \in \mathcal{N}(\mathbf{A}^T)$. (5 points)

3) Use the SVD above to determine $\mathbf{A}^\dagger$ and solve the least-squares problem $\mathbf{A}\mathbf{x} = \mathbf{b}$. (5 points)

Solution:

1) We have $\mathbf{A}\mathbf{A}^T = \begin{bmatrix} 40 & -14 & 34 \\ -14 & 13 & -20 \\ 34 & -20 & 37 \end{bmatrix}$, so the characteristic polynomial is

$$\det \begin{bmatrix} 40-x & -14 & 34 \\ -14 & 13-x & -20 \\ 34 & -20 & 37-x \end{bmatrix} = -(x-81)(x-9)x$$

Hence the eigenvalues of $\mathbf{A}\mathbf{A}^T$ are $\lambda_1 = 81$, $\lambda_2 = 9$ and $\lambda_3 = 0$ with, respectively, unit eigenvectors

$$\mathbf{q}_1 = \frac{1}{3}\begin{bmatrix} -2 \\ 1 \\ -2 \end{bmatrix}, \quad \mathbf{q}_2 = \frac{1}{3}\begin{bmatrix} -2 \\ -2 \\ 1 \end{bmatrix}, \quad \mathbf{q}_3 = \frac{1}{3}\begin{bmatrix} 1 \\ -2 \\ -2 \end{bmatrix}.$$

It follows that

$$\mathbf{U} = \frac{1}{3}\begin{bmatrix} -2 & -2 & 1 \\ 1 & -2 & -2 \\ -2 & 1 & -2 \end{bmatrix}$$

Similarly, we have $\mathbf{A}^T\mathbf{A} = \begin{bmatrix} 36 & 0 & 36 & 18 \\ 0 & 9 & 0 & 0 \\ 36 & 0 & 36 & 18 \\ 18 & 0 & 18 & 9 \end{bmatrix}$ and the characteristic polynomial is $(x-81)(x-9)x^2$. Hence, we obtain the eigenvalues of $\mathbf{A}^T\mathbf{A}$ as $\lambda_1 = 81, \lambda_2 = 9, \lambda_3 = \lambda_4 = 0$ with, respectively, unit eigenvectors

$$\hat{\mathbf{q}}_1 = \frac{1}{3}\begin{bmatrix} 2 \\ 0 \\ 2 \\ 1 \end{bmatrix}, \quad \hat{\mathbf{q}}_2 = \frac{1}{3}\begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \end{bmatrix}, \quad \hat{\mathbf{q}}_3 = \frac{1}{3}\begin{bmatrix} 2 \\ 0 \\ -1 \\ -2 \end{bmatrix} \quad \hat{\mathbf{q}}_4 = \frac{1}{3}\begin{bmatrix} 1 \\ 0 \\ -2 \\ 2 \end{bmatrix}$$

$$\mathbf{V}^T = \frac{1}{3}\begin{bmatrix} 2 & 0 & 2 & 1 \\ 0 & 3 & 0 & 0 \\ 2 & 0 & -1 & -2 \\ 1 & 0 & -2 & 2 \end{bmatrix}.$$

The singular values are $\sigma_1 = 9, \sigma_2 = 3, \sigma_3 = 0$.

$$\tilde{\mathbf{\Sigma}} = \begin{bmatrix} 9 & 0 \\ 0 & 3 \end{bmatrix}, \quad \mathbf{\Sigma} = \begin{bmatrix} 9 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

2) Since $\mathbf{q}_1, \mathbf{q}_2 \in \mathcal{R}(\mathbf{A})$ and $\mathbf{q}_3 \in \mathcal{N}(\mathbf{A}^T)$, we have

$$\hat{\mathbf{b}} = \mathbf{U}_{*,1:2}\mathbf{U}_{*,1:2}^T\mathbf{b} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix},$$

$$\mathbf{z} = \mathbf{U}_{*,3}\mathbf{U}_{*,3}^T\mathbf{b} = \begin{bmatrix} -1 \\ 2 \\ 2 \end{bmatrix}.$$

3)

$$\mathbf{A}^\dagger = \mathbf{V}_{*,1:2}\tilde{\mathbf{\Sigma}}^{-1}\mathbf{U}_{*,1:2}^T = \frac{1}{81}\begin{bmatrix} -4 & 2 & -4 \\ -18 & -18 & 9 \\ -4 & 2 & -4 \\ -2 & 1 & -2 \end{bmatrix}.$$

Hence, the solution is $\mathbf{x} = \mathbf{A}^\dagger\mathbf{b} = -\frac{1}{27}\begin{bmatrix} 4 & 9 & 4 & 2 \end{bmatrix}^T$.

Problem 5. (**Sensitivity in Linear Systems**, 15 points)

1) Let $\mathbf{A}\mathbf{x} = \mathbf{b}$ be a nonsingular linear system in which the nonsingular $\mathbf{A}$ is known exactly but $\mathbf{b}$ is subject to an uncertainty $\mathbf{e}$, and consider $\mathbf{A}\tilde{\mathbf{x}} = \mathbf{b} - \mathbf{e} = \tilde{\mathbf{b}}$. Show that
$$\kappa^{-1}\frac{\|\mathbf{e}\|_2}{\|\mathbf{b}\|_2} \leq \frac{\|\mathbf{x}-\tilde{\mathbf{x}}\|_2}{\|\mathbf{x}\|_2} \leq \kappa\frac{\|\mathbf{e}\|_2}{\|\mathbf{b}\|_2}, \quad \text{where} \quad \kappa = \|\mathbf{A}\|_2\|\mathbf{A}^{-1}\|_2. \tag{1}$$
(5 points)

2) When does the equalities in (1) hold, i.e.,
$$\kappa^{-1}\frac{\|\mathbf{e}\|_2}{\|\mathbf{b}\|_2} = \frac{\|\mathbf{x}-\tilde{\mathbf{x}}\|_2}{\|\mathbf{x}\|_2} = \kappa\frac{\|\mathbf{e}\|_2}{\|\mathbf{b}\|_2} \quad ? \tag{2}$$
(5 points)

3) Now let $\mathbf{A}, \mathbf{b}, \mathbf{x}$ be differentiable functions of a variable t, i.e., $\mathbf{A} = \mathbf{A}(t), \mathbf{b} = \mathbf{b}(t), \mathbf{x} = \mathbf{x}(t)$. Differentiate $\mathbf{b} = \mathbf{A}\mathbf{x}$ to obtain $\mathbf{b}' = (\mathbf{A}\mathbf{x})' = \mathbf{A}'\mathbf{x} + \mathbf{A}\mathbf{x}'$ (with $*'$ denoting $d*/dt$). Prove that
$$\frac{\|\mathbf{x}'\|_2}{\|\mathbf{x}\|_2} \leq \kappa(\frac{\|\mathbf{b}'\|_2}{\|\mathbf{b}\|_2} + \frac{\|\mathbf{A}'\|_2}{\|\mathbf{A}\|_2}).$$
(5 points)

Solution:

1) Use $\|\mathbf{b}\|_2 = \|\mathbf{A}\mathbf{x}\|_2 \leq \|\mathbf{A}\|_2\|\mathbf{x}\|_2$ with $\mathbf{x} - \tilde{\mathbf{x}} = \mathbf{A}^{-1}\mathbf{e}$ to write
$$\frac{\|\mathbf{x}-\tilde{\mathbf{x}}\|_2}{\|\mathbf{x}\|_2} = \frac{\|\mathbf{A}^{-1}\mathbf{e}\|_2}{\|\mathbf{x}\|_2} \leq \frac{\|\mathbf{A}\|_2\|\mathbf{A}^{-1}\|_2\|\mathbf{e}\|_2}{\|\mathbf{b}\|_2} = \kappa\frac{\|\mathbf{e}\|_2}{\|\mathbf{b}\|_2},$$
where $\kappa = \|\mathbf{A}\|_2\|\mathbf{A}^{-1}\|_2$ is a condition number. Furthermore, $\|\mathbf{e}\|_2 = \|\mathbf{A}(\mathbf{x}-\tilde{\mathbf{x}})\|_2 \leq \|\mathbf{A}\|_2\|(\mathbf{x}-\tilde{\mathbf{x}})\|_2$ and $\|\mathbf{x}\|_2 \leq \|\mathbf{A}^{-1}\|_2\|\mathbf{b}\|_2$ imply
$$\frac{\|\mathbf{x}-\tilde{\mathbf{x}}\|_2}{\|\mathbf{x}\|_2} \geq \frac{\|\mathbf{e}\|_2}{\|\mathbf{A}\|_2\|\mathbf{x}\|_2} \geq \frac{\|\mathbf{e}\|_2}{\|\mathbf{A}\|_2\|\mathbf{A}^{-1}\|_2\|\mathbf{b}\|_2} = \frac{1}{\kappa}\frac{\|\mathbf{e}\|_2}{\|\mathbf{b}\|_2}.$$
Combining the two inequalities above yields
$$\kappa^{-1}\frac{\|\mathbf{e}\|_2}{\|\mathbf{b}\|_2} \leq \frac{\|\mathbf{x}-\tilde{\mathbf{x}}\|_2}{\|\mathbf{x}\|_2} \leq \kappa\frac{\|\mathbf{e}\|_2}{\|\mathbf{b}\|_2}, \quad \text{where} \quad \kappa = \|\mathbf{A}\|_2\|\mathbf{A}^{-1}\|_2.$$

2) Let $\mathbf{A} = \mathbf{U}\mathbf{\Sigma}\mathbf{V}^T$ be an SVD, so $\mathbf{A}\mathbf{V}_{*k} = \sigma_k\mathbf{U}_{*k}$ for each k. If $\mathbf{b}$ and $\mathbf{e}$ are directed along left-hand singular vectors associated with σ_i and σ_n, respectively, i.e., $\mathbf{b} = \beta\mathbf{U}_{*1}$ and $\mathbf{e} = \epsilon\mathbf{U}_{*n}$, then
$$\mathbf{x} = \mathbf{A}^{-1}\mathbf{b} = \mathbf{A}^{-1}(\beta\mathbf{U}_{*1}) = \frac{\beta\mathbf{V}_{*1}}{\sigma_1} \quad \text{and} \quad \mathbf{x} - \tilde{\mathbf{x}} = \mathbf{A}^{-1}\mathbf{e} = \mathbf{A}^{-1}(\epsilon\mathbf{U}_{*\mathbf{n}}) = \frac{\epsilon\mathbf{V}_{*n}}{\sigma_n}.$$
Then, we have
$$\frac{\|\mathbf{x}-\tilde{\mathbf{x}}\|_2}{\|\mathbf{x}\|_2} = (\frac{\sigma_1}{\sigma_n})\frac{|\epsilon|}{|\beta|} = \kappa\frac{\|\mathbf{e}\|_2}{\|\mathbf{b}\|_2} \quad \text{when} \quad \mathbf{b} = \beta\mathbf{U}_{*1} \quad \text{and} \quad \mathbf{e} = \epsilon\mathbf{U}_{*n}.$$
Thus the upper bound in (1) is attainable for all $\mathbf{A}$.

The lower bound is realized in the opposite situation when $\mathbf{b}$ and $\mathbf{e}$ are directed along $\mathbf{U}_{*n}$ and $\mathbf{U}_{*1}$, respectively. If $\mathbf{b} = \beta\mathbf{U}_{*n}$ and $\mathbf{e} = \epsilon\mathbf{U}_{*1}$, then the same argument yields $\mathbf{x} = \sigma_n^{-1}\beta\mathbf{V}_{*n}$ and $\mathbf{x} - \tilde{\mathbf{x}} = \sigma_1^{-1}\epsilon\mathbf{V}_{*1}$. Then, we have
$$\frac{\|\mathbf{x}-\tilde{\mathbf{x}}\|_2}{\|\mathbf{x}\|_2} = (\frac{\sigma_n}{\sigma_1})\frac{|\epsilon|}{|\beta|} = \kappa^{-1}\frac{\|\mathbf{e}\|_2}{\|\mathbf{b}\|_2} \quad \text{when} \quad \mathbf{b} = \beta\mathbf{U}_{*n} \quad \text{and} \quad \mathbf{e} = \epsilon\mathbf{U}_{*1}.$$

3) From $\mathbf{b}' = (\mathbf{A}\mathbf{x})' = \mathbf{A}'\mathbf{x} + \mathbf{A}\mathbf{x}'$, we have

$$\begin{aligned}\|\mathbf{x}'\|_2 &= \|\mathbf{A}^{-1}\mathbf{b}' - \mathbf{A}^{-1}\mathbf{A}'\mathbf{x}\|_2 \leq \|\mathbf{A}^{-1}\mathbf{b}'\|_2 + \|\mathbf{A}^{-1}\mathbf{A}'\mathbf{x}\|_2 \\ &\leq \|\mathbf{A}^{-1}\|_2\|\mathbf{b}'\|_2 + \|\mathbf{A}^{-1}\|_2\|\mathbf{A}'\|_2\|\mathbf{x}\|_2.\end{aligned}$$

Consequently,

$$\begin{aligned}\frac{\|\mathbf{x}'\|_2}{\|\mathbf{x}\|_2} &\leq \frac{\|\mathbf{A}^{-1}\|_2\|\mathbf{b}'\|_2}{\|\mathbf{x}\|_2} + \|\mathbf{A}^{-1}\|_2\|\mathbf{A}'\|_2 \\ &\leq \|\mathbf{A}\|_2\|\mathbf{A}^{-1}\|_2\frac{\|\mathbf{b}'\|_2}{\|\mathbf{A}\|_2\|\mathbf{x}\|_2} + \|\mathbf{A}\|_2\|\mathbf{A}^{-1}\|_2\frac{\|\mathbf{A}'\|_2}{\|\mathbf{A}\|_2} \\ &\leq \kappa\frac{\|\mathbf{b}'\|_2}{\|\mathbf{b}\|_2} + \kappa\frac{\|\mathbf{A}'\|_2}{\|\mathbf{A}\|_2} = \kappa(\frac{\|\mathbf{b}'\|_2}{\|\mathbf{b}\|_2} + \frac{\|\mathbf{A}'\|_2}{\|\mathbf{A}\|_2}).\end{aligned}$$

Problem 6. (Lower-Rank Matrices, Matrix Norms, 20 points)

1) Suppose $\mathbf{A} \in \mathbb{R}^{m\times n}$ has an SVD as $\mathbf{A} = \mathbf{U}\begin{bmatrix}\mathbf{D} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\mathbf{V}^T$ with $\mathbf{D} = \operatorname{diag}(\sigma_1, \dots, \sigma_r)$.
   (a) Let $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r$ be the nonzero singular values of $\mathbf{A}$. Then for each $k < r$, show that the distance from $\mathbf{A}$ to the closest matrix of rank k is

$$\sigma_{k+1} = \min_{\operatorname{rank}(\mathbf{B})=k} \|\mathbf{A} - \mathbf{B}\|_2. \tag{3}$$

   (**Hint**: partition $\mathbf{V} = (\mathbf{F}_{n\times(k+1)}|\mathbf{G})$ and construct $\mathbf{x} \in \mathcal{N}(\mathbf{BF})$) (5 points)

   (b) For $p = \min\{m, n\}$, let $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_p$ and $\beta_1 \geq \beta_2 \geq \cdots \geq \beta_p$ be all the singular values (including the zero ones) for $\mathbf{A}$ and $\mathbf{A}+\mathbf{E}$, respectively. Prove that

$$|\sigma_k - \beta_k| \leq \|\mathbf{E}\|_2 \quad \text{for each} \quad k = 1, 2, \dots, p. \tag{4}$$

   (5 points)

2) Let $\mathbf{U}_{m\times r}$ be a matrix with orthonormal columns, and let $\mathbf{V}_{k\times n}$ be a matrix with orthonormal rows. Show that for any $\mathbf{A} \in \mathbb{R}^{r\times k}$,

$$\|\mathbf{UAV}\|_2 = \|\mathbf{A}\|_2. \tag{5}$$

   (**Hint:** Start with $\|\mathbf{UA}\|_2$) (5 points)

3) Prove that if σ_r is the smallest nonzero singular value of $\mathbf{A}_{m\times n}$, then

$$\sigma_r = \min_{\substack{\|\mathbf{x}\|_2=1\\ \mathbf{x}\in\mathcal{R}(\mathbf{A}^T)}} \|\mathbf{Ax}\|_2 = \frac{1}{\|\mathbf{A}^\dagger\|_2} \tag{6}$$

   (5 points)

Solution:

1) (a) Suppose $\operatorname{rank}(\mathbf{B}_{m\times n}) = k$, define $\mathbf{S} = \operatorname{diag}(\sigma_1, \dots, \sigma_{k+1})$, and partition $\mathbf{V} = (\mathbf{F}_{n\times(k+1)}|\mathbf{G})$. Since $\operatorname{rank}(\mathbf{BF}) \leq \operatorname{rank}(\mathbf{B}) = k$, $\dim\mathcal{N}(\mathbf{BF}) = k + 1 - \operatorname{rank}(\mathbf{BF}) \geq 1$, so there is an $\mathbf{x} \in \mathcal{N}(\mathbf{BF})$ with $\|\mathbf{x}\|_2 = 1$. Consequently, $\mathbf{BFx} = 0$ and

$$\mathbf{AFx} = \mathbf{U}\begin{bmatrix}\mathbf{D} & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\mathbf{V}^T\mathbf{Fx} = \mathbf{U}\begin{bmatrix}\mathbf{S} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & * & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0}\end{bmatrix}\begin{bmatrix}\mathbf{x}\\ 0\\ 0\end{bmatrix} = \mathbf{U}\begin{bmatrix}\mathbf{Sx}\\ 0\\ 0\end{bmatrix}.$$

   Since $\|\mathbf{A}-\mathbf{B}\|_2 = \max_{\|\mathbf{y}\|_2=1} \|(\mathbf{A}-\mathbf{B})\mathbf{y}\|_2$, and since $\|\mathbf{Fx}\|_2 = \|\mathbf{x}\|_2 = 1$,

$$\|\mathbf{A}-\mathbf{B}\|_2^2 \geq \|(\mathbf{A}-\mathbf{B})\mathbf{Fx}\|_2^2 = \|\mathbf{Sx}\|_2^2 = \sum_{i=1}^{k+1} \sigma_i^2 x_i^2 \geq \sigma_{k+1}^2 \sum_{i=1}^{k+1} x_i^2 = \sigma_{k+1}^2.$$

   Equality holds for $\mathbf{B}_k = \mathbf{U}\begin{bmatrix}\mathbf{D}_k & \mathbf{0}\\ \mathbf{0} & \mathbf{0}\end{bmatrix}\mathbf{V}^T$ with $\mathbf{D}_k = \operatorname{diag}(\sigma_1, \dots, \sigma_k)$, and thus (3) is proven.

(b) From $\mathbf{A} = \sum_{i=1}^{p} \sigma_i \mathbf{u}_i \mathbf{v}_i^T$, we set $\mathbf{A}_{k-1} = \sum_{i=1}^{k-1} \sigma_i \mathbf{u}_i \mathbf{v}_i^T$, then

$$\begin{aligned}\sigma_k &= \|\mathbf{A} - \mathbf{A}_{k-1}\|_2 = \|\mathbf{A} + \mathbf{E} - \mathbf{A}_{k-1} - \mathbf{E}\|_2 \\ &\geq \|\mathbf{A} + \mathbf{E} - \mathbf{A}_{k-1}\|_2 - \|\mathbf{E}\|_2 \\ &\geq \beta_k - \|\mathbf{E}\|_2\end{aligned}$$

Combine this with the observation that

$$\begin{aligned}\sigma_k &= \min_{\mathrm{rank}(\mathbf{B})=k-1} \|\mathbf{A} - \mathbf{B}\|_2 = \min_{\mathrm{rank}(\mathbf{B})=k-1} \|\mathbf{A} + \mathbf{E} - \mathbf{B} - \mathbf{E}\|_2 \\ &\leq \min_{\mathrm{rank}(\mathbf{B})=k-1} \|\mathbf{A} + \mathbf{E} - \mathbf{B}\|_2 + \|\mathbf{E}\|_2 = \beta_k + \|\mathbf{E}\|_2\end{aligned}$$

to conclude that $|\sigma_k - \beta_k| \leq \|\mathbf{E}\|_2$.

2) First show that $\|\mathbf{UA}\|_2 = \|\mathbf{A}\|_2$ by

$$\|\mathbf{UA}\|_2^2 = \max_{\|\mathbf{x}\|_2=1} \|\mathbf{UAx}\|_2^2 = \max_{\|\mathbf{x}\|_2=1} \mathbf{x}^T\mathbf{A}^T\mathbf{U}^T\mathbf{UAx} = \max_{\|\mathbf{x}\|_2=1} \mathbf{x}^T\mathbf{A}^T\mathbf{Ax} = \max_{\|\mathbf{x}\|_2=1} \|\mathbf{Ax}\|_2^2 = \|\mathbf{A}\|_2^2.$$

Now use this together with $\|\mathbf{A}\|_2 = \|\mathbf{A}^T\|_2^2$ to observe that

$$\|\mathbf{AV}\|_2 = \|\mathbf{V}^T\mathbf{A}^T\|_2 = \|\mathbf{A}^T\|_2 = \|\mathbf{A}\|_2.$$

Therefore, $\|\mathbf{UAV}\|_2 = \|\mathbf{U}(\mathbf{AV})\|_2 = \|\mathbf{AV}\|_2 = \|\mathbf{A}\|_2$.

3) If $\mathbf{A} = \mathbf{U}\begin{bmatrix}\mathbf{D} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}\end{bmatrix}\mathbf{V}^T$ and $\mathbf{A}^\dagger_{n\times m} = \mathbf{V}\begin{bmatrix}\mathbf{D}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{0}\end{bmatrix}\mathbf{U}^T$ are SVDs in which $\mathbf{V} = (\mathbf{V}_1|\mathbf{V}_2)$, then the columns of $\mathbf{V}_1$ are an orthonormal basis for $\mathcal{R}(\mathbf{A}^T)$, so $\mathbf{x} \in \mathcal{R}(\mathbf{A}^T)$ and $\|\mathbf{x}\|_2 = 1$ if and only if $\mathbf{x} = \mathbf{V}_1\mathbf{y}$ with $\|\mathbf{y}\|_2 = 1$. Since the 2-norm is unitarily invariant in (5),

$$\min_{\substack{\|\mathbf{x}\|_2=1 \\ \mathbf{x}\in\mathcal{R}(\mathbf{A}^T)}} \|\mathbf{Ax}\|_2 = \min_{\|\mathbf{y}\|_2=1} \|\mathbf{AV}_1\mathbf{y}\|_2 = \min_{\|\mathbf{y}\|_2=1} \|\mathbf{Dy}\|_2 = \frac{1}{\|\mathbf{D}^{-1}\|_2} = \sigma_r = \frac{1}{\|\mathbf{A}^\dagger\|_2}.$$