



# Lecture 13

## - Laplace Transform



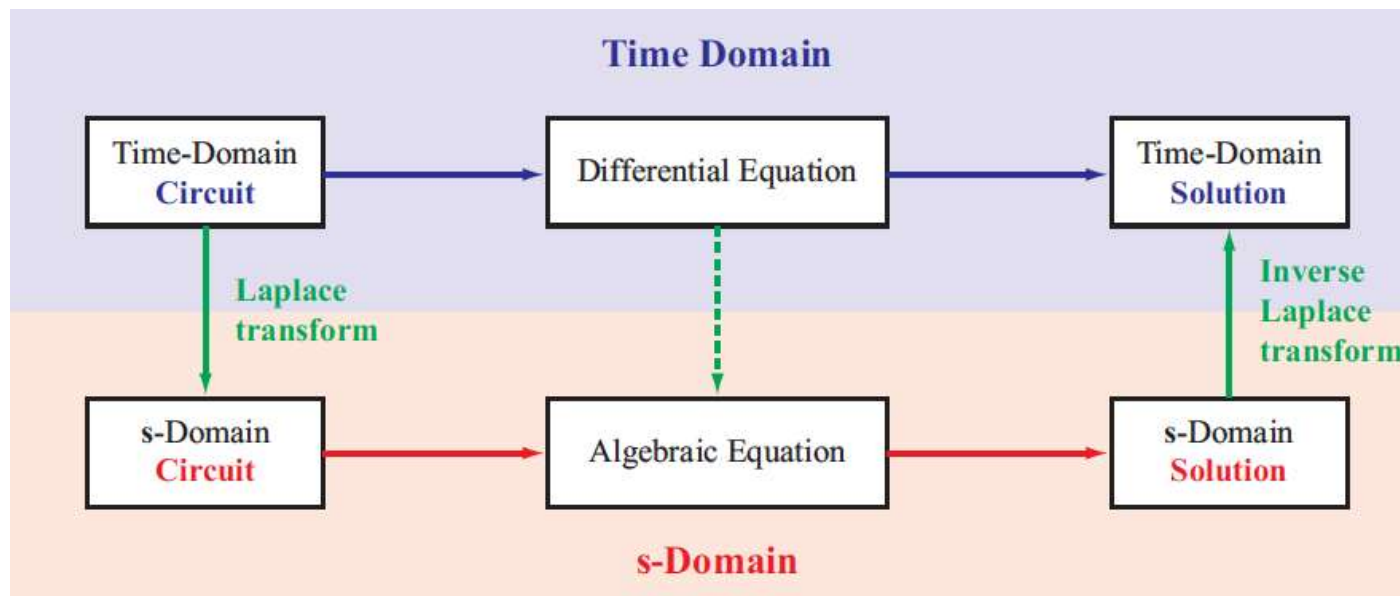
## Analysis Techniques

Circuit Excitation	Method of Solution
dc (w/ switches)	DC/Transient analysis <b>(transient + steady state)</b>
A.C.	Phasor-domain analysis (Steady state only)
<i>Periodic</i> waveform	<i>Fourier series + Phasor-domain</i> (Steady state only)
<b>Waveform</b>	<b>Laplace transform</b> <b>(transient + steady state)</b>

[Source: Berkeley]



# Laplace Transform Technique



[Source: Berkeley]



## The French Newton Pierre-Simon Laplace (Late 1700)

- Developed mathematics in astronomy, physics, and statistics
- Began work in calculus which led to the Laplace Transform
- Focused later on celestial mechanics
  - One of the first scientists to suggest the existence of black holes



[https://en.wikipedia.org/wiki/Pierre-Simon\\_Laplace](https://en.wikipedia.org/wiki/Pierre-Simon_Laplace)



## What are Laplace Transforms?

$$F(s) = \mathcal{L}[f(t)] = \int_{0_-}^{\infty} f(t)e^{-st} dt$$

- $f(t) \rightarrow F(s)$ ,
- $t$  is real, being integrated
- $s$  is variable **complex**;  $s = \sigma + j\omega$ .
- Note integral starts from  $0_-$
- Assume  $f(t)=0$  for all  $t < 0$



## Inverse Laplace Transforms

$$f(t) = \mathcal{L}^{-1}[F(s)] = \frac{1}{2\pi j} \int_{\sigma - j\infty}^{\sigma + j\infty} F(s) e^{st} ds$$

- Conversely,  $F(s) \rightarrow f(t)$ ,  $t$  is variable and  $s$  is integrated.



**TABLE 12.1** An Abbreviated List of Laplace Transform Pairs

Type	$f(t) \ (t > 0-)$	$F(s)$
(step)	$u(t)$	$\frac{1}{s}$
(ramp)	$t$	$\frac{1}{s^2}$
(exponential)	$e^{-at}$	$\frac{1}{s + a}$
(sine)	$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$
(cosine)	$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$
(damped ramp)	$te^{-at}$	$\frac{1}{(s + a)^2}$
(damped sine)	$e^{-at} \sin \omega t$	$\frac{\omega}{(s + a)^2 + \omega^2}$
(damped cosine)	$e^{-at} \cos \omega t$	$\frac{s + a}{(s + a)^2 + \omega^2}$











TABLE 12.2 An Abbreviated List of Operational Transforms

Operation	$f(t)$	$F(s)$
Multiplication by a constant	$Kf(t)$	$KF(s)$
Addition/subtraction	$f_1(t) + f_2(t) - f_3(t) + \dots$	$F_1(s) + F_2(s) - F_3(s) + \dots$
First derivative (time)	$\frac{df(t)}{dt}$	$sF(s) - f(0^-)$
Second derivative (time)	$\frac{d^2f(t)}{dt^2}$	$s^2F(s) - sf(0^-) - \frac{df(0^-)}{dt}$
$n$ th derivative (time)	$\frac{d^n f(t)}{dt^n}$	$s^n F(s) - s^{n-1}f(0^-) - s^{n-2}\frac{df(0^-)}{dt} - s^{n-3}\frac{d^2f(0^-)}{dt^2} - \dots - \frac{d^{n-1}f(0^-)}{dt^{n-1}}$
Time integral	$\int_0^t f(x) dx$	$\frac{F(s)}{s}$
Translation in time	$f(t - a)u(t - a), a > 0$	$e^{-as}F(s)$
Translation in frequency	$e^{-at}f(t)$	$F(s + a)$
Scale changing	$f(at), a > 0$	$\frac{1}{a}F\left(\frac{s}{a}\right)$
First derivative (s)	$tf(t)$	$-\frac{dF(s)}{ds}$
$n$ th derivative (s)	$t^n f(t)$	$(-1)^n \frac{d^n F(s)}{ds^n}$
$s$ integral	$\frac{f(t)}{t}$	$\int_s^\infty F(u) du$



## Homogeneity and Additivity

$$\mathcal{L}[a_1 f_1(t)] = a_1 \mathcal{L}[f_1(t)] = a_1 F_1(s)$$

$$\mathcal{L}[a_1 f_1(t) + a_2 f_2(t)] = a_1 \mathcal{L}[f_1(t)] + a_2 \mathcal{L}[f_2(t)] = a_1 F_1(s) + a_2 F_2(s)$$

here  $a_1$  and  $a_2$  are constants

Important implication:

$$\sum_{k=1}^k i_k(t) = 0 \quad \longleftrightarrow \quad \sum_{k=1}^k I_k(s) = 0$$

$$\sum_{k=1}^k u_k(t) = 0 \quad \longleftrightarrow \quad \sum_{k=1}^k U_k(s) = 0$$



## Time Differentiation

$$\mathcal{L}\left[\frac{d}{dt}f(t)\right] = sF(s) - f(0_-)$$



## Initial and final value

$$\mathcal{L}\left[\frac{d}{dt}f(t)\right] = sF(s) - f(0_-)$$



## Time integral

$$\mathcal{L}\left[\int_{0_-}^t f(\tau)d\tau\right] = \frac{1}{s}F(s)$$



## Translation in the Time Domain

$$\mathcal{L}[f(t-a) u(t-a)] = e^{-sa} F(s)$$

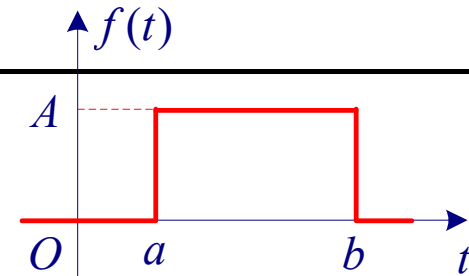




- Example

$$f(t) = A[u(t-a) - u(t-b)]$$

$$F(s) = A\mathcal{L}[u(t-a) - u(t-b)] = \frac{A}{s}(e^{-as} - e^{-bs})$$

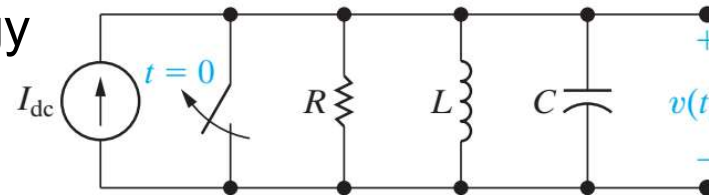






## Applying the Laplace Transform

- We assume no initial energy stored at  $t=0$



$$\frac{v(t)}{R} + \frac{1}{L} \int_0^t v(x) dx + C \frac{dv(t)}{dt} = I_{dc} u(t).$$

$$\frac{V(s)}{R} + \frac{1}{L} \frac{V(s)}{s} + C[sV(s) - v(0^-)] = I_{dc} \left( \frac{1}{s} \right)$$

$$V(s) \left( \frac{1}{R} + \frac{1}{sL} + sC \right) = \frac{I_{dc}}{s}$$

$$V(s) = \frac{I_{dc}/C}{s^2 + (1/RC)s + (1/LC)} \quad \longrightarrow \quad v(t) = \mathcal{L}^{-1}\{V(s)\}.$$



## Inverse Transforms

In principle, we can recover  $f$  from  $F$  via

$$f(t) = \frac{1}{2\pi j} \int_{\sigma-j\infty}^{\sigma+j\infty} F(S) e^{st} ds$$

Surprisingly, this formula isn't really useful!

What is more common/useful as follows:

$$F(s) = \frac{P(s)}{Q(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0}$$



Generally

$$F(s) = \frac{P(s)}{Q(s)} = \frac{b_m s^m + b_{m-1} s^{m-1} + \dots + b_1 s + b_0}{a_n s^n + a_{n-1} s^{n-1} + \dots + a_1 s + a_0}$$

$a_i$  and  $b_i$  are real constants, and the exponents  $m, n$  are positive integers

- If  $m < n$ , proper rational function
- If  $m > n$ , improper rational function



## Partial Fraction Expansion with Real Distinct Roots

- Let  $F(s)$  be proper rational function, then

$$F(s) = \frac{P(s)}{Q(s)} = \frac{b_ms^m + b_{m-1}s^{m-1} + \dots + b_1s + b_0}{a_ns^n + a_{n-1}s^{n-1} + \dots + a_1s + a_0}$$

Case I: If the roots are real,  $p_i \neq p_j$  for  $\forall i \neq j$

$$F(s) = \frac{P(s)}{Q(s)} = \frac{K_1}{s - p_1} + \frac{K_2}{s - p_2} + \dots + \frac{K_n}{s - p_n} = \sum_{j=1}^n \frac{K_j}{s - p_j}$$

$p_j(j=1, 2, \dots, n)$  are *the roots* of equation  $Q(s)=0$

$K_j(j=1, 2, \dots, n)$  are unknown constants



## Partial Fraction Expansion **with Real Distinct Roots**

$$F(s) = \frac{P(s)}{Q(s)} = \frac{K_1}{s - p_1} + \frac{K_2}{s - p_2} + \cdots + \frac{K_n}{s - p_n} = \sum_{j=1}^n \frac{K_j}{s - p_j}$$

Case I:

If the roots are real,  $p_i \neq p_j$  for  $\forall i \neq j$

$$K_j = \lim_{s \rightarrow p_j} (s - p_j)F(s) = (s - p_j)F(s) \Big|_{s=p_j}$$



## Exercise

$$F(s) = \frac{s^2 + 3s + 5}{s^3 + 6s^2 + 11s + 6}$$

$$F(s) = \frac{s^2 + 3s + 5}{(s+1)(s+2)(s+3)} = \frac{K_1}{s+1} + \frac{K_2}{s+2} + \frac{K_3}{s+3}$$





## Partial Fraction Expansion **with Multiple Roots**

- Case II:
- If  $Q(s)$  has multiple roots

$$F(s) = \frac{K_{11}}{s - p_1} + \frac{K_{12}}{(s - p_1)^2} + \cdots + \frac{K_{1r}}{(s - p_1)^r} + \frac{K_{r+1}}{s - p_{r+1}} \cdots + \frac{K_n}{s - p_n}$$

$$\begin{aligned} K_{1r} &= (s - p_1)^r F(s) \Big|_{s=p_1} \\ K_{1(r-1)} &= \frac{d}{ds} [(s - p_1)^r F(s)] \Big|_{s=p_1} \\ K_{1(r-2)} &= \frac{1}{2!} \frac{d^2}{ds^2} [(s - p_1)^r F(s)] \Big|_{s=p_1} \\ &\vdots \\ K_{11} &= \frac{1}{(r-1)!} \frac{d^{r-1}}{ds^{r-1}} [(s - p_1)^r F(s)] \Big|_{s=p_1} \end{aligned}$$





## Exercise

$$F(s) = \frac{10s^2 + 4}{s(s+1)(s+2)^2}$$

$$F(s) = \frac{K_{11}}{s} + \frac{K_{21}}{s+1} + \frac{K_{31}}{s+2} + \frac{K_{32}}{(s+2)^2}$$

$$f(t) = [1 - 14e^{-t} + (13 + 22t)e^{-2t}]u(t)$$



## Partial Fraction Expansion **with Complex Roots**

Case III:

If  $F(s)$  has a pole of  $p_1$  expressed by a complex number, then it must have a complex root  $P_2$  as a conjugate of  $P_1$

$$p_1 = \alpha + j\omega \quad p_2 = p_1^* = \alpha - j\omega$$

$$F(s) = \frac{K_1}{s - (\alpha + j\omega)} + \frac{K_2}{s - (\alpha - j\omega)}$$

$$K_1 = [s - (\alpha + j\omega)] F(s) \big|_{s=\alpha+j\omega}$$

$$K_2 = [s - (\alpha - j\omega)] F(s) \big|_{s=\alpha-j\omega} \quad K_2 = K_1^* = |K_1| e^{-j\phi_K}$$



$$\begin{aligned} f(t) &= K_1 e^{(\alpha+j\omega)t} + K_2 e^{(\alpha-j\omega)t} = |K_1| e^{\alpha t} [e^{j(\omega t+\varphi_K)} + e^{-j(\omega t+\varphi_K)}] \\ &= 2 |K_1| e^{\alpha t} \cos(\omega t + \varphi_K) \end{aligned}$$



## Partial Fraction Expansion **with Complex Roots**

• Example: 
$$F(s) = \frac{s^2 + 3s + 7}{(s^2 + 4s + 13)(s + 1)}$$

$$p_1 = -2 + j3, \quad p_2 = -2 - j3, \quad p_3 = -1$$

$$F(s) = \frac{K_1}{s - (-2 + j3)} + \frac{K_1^*}{s - (-2 - j3)} + \frac{K_3}{s + 1}$$

$$K_3 = \left. \frac{s^2 + 3s + 7}{s^2 + 4s + 13} \right|_{s=-1} = 0.5$$



## EXAMPLE:

$$F(s) = \frac{2s^3 + 33s^2 + 93s + 54}{s(s + 1)(s^2 + 5s + 6)}.$$

$$F(s) = \frac{14s^2 + 56s + 152}{(s + 6)(s^2 + 4s + 20)}.$$