# Signals and Systems

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# Chapter 6: Time and frequency characterization of signals and systems

- The magnitude-phase representation of Fourier Transform
- The magnitude-phase representation of the frequency response of LTI systems
- Time-domain properties of ideal frequency-selective filters
- Time-domain and frequency-domain aspects of non-ideal filters
- First-order system

# The magnitude-phase representation of FT

## *Magnitude and phase spectrum*

- Continuous FT $\quad x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} X(j\omega) \qquad X(j\omega) = |X(j\omega)|e^{j\angle X(j\omega)}$

- Discrete FT $\quad x[n] \longleftrightarrow X(e^{j\omega}) \qquad X(e^{j\omega}) = |X(e^{j\omega})|e^{j\angle X(e^{j\omega})}$

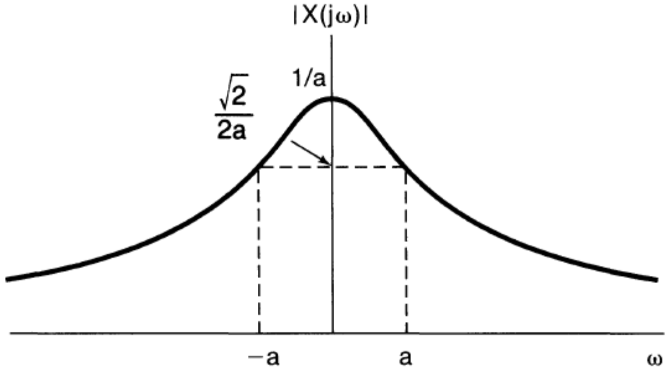
- Magnitude spectrum: $|X(j\omega)|$ and $|X(e^{j\omega})|$

- Phase spectrum (angle): $\angle X(j\omega)$and $\angle X(e^{j\omega})$

# The magnitude-phase representation of FT

## *Magnitude spectrum*

Continuous time as an example




$$\text{IFT: } x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega$$

- IFT: decomposition of the signal $x(t)$ into a "sum" of complex exponentials at different frequencies

- $|X(j\omega)|$: describes the basic frequency content of a signal, and the relative magnitude of the each frequency (complex exponential)

- $|X(j\omega)|^2$: energy-density spectrum of $x(t)$

- $|X(j\omega)|^2 d\omega/2\pi$: energy in the signal between $\omega$ and $\omega + d\omega$

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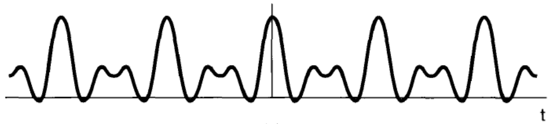
# The magnitude-phase representation of FT

## *Phase spectrum*

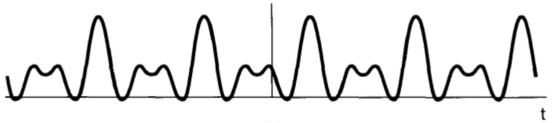
- $\angle X(j\omega)$
  - relative phase of the each complex exponential
  - significant effect on the nature of the signal
  - changes in $\angle X(j\omega)$ lead to phase distortion

- Example 1: $x(t) = 1 + \frac{1}{2}\cos(2\pi t + \varphi_1) + \frac{1}{2}\cos(4\pi t + \varphi_2) + \frac{1}{2}\cos(6\pi t + \varphi_3)$



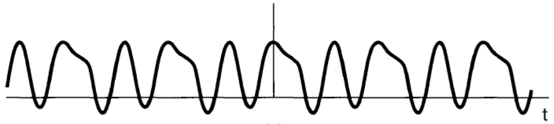
(a)

$\varphi_1 = \varphi_2 = \varphi_3 = 0$



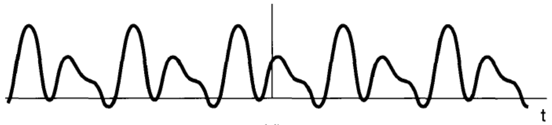
(b)

$\varphi_1 = 4\,rad, \varphi_2 = 8\,rad, \varphi_3 = 12\,rad$



(c)

$\varphi_1 = 6\,rad, \varphi_2 = -2.7\,rad, \varphi_3 = 0.93\,rad$



(d)

$\varphi_1 = 1.2\,rad, \varphi_2 = 4.1\,rad, \varphi_3 = -7.02\,rad$

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- First-order system

# The magnitude-phase representation of LTI

## *Gain and phase shift*

- For LTI system

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = x(t) * h(t)$$

$$X(j\omega) \longrightarrow \boxed{H(j\omega)} \longrightarrow Y(j\omega) = H(j\omega)X(j\omega)$$

- The frequency response $H(j\omega) = |H(j\omega)|e^{j\angle H(j\omega)}$

- $|H(j\omega)|$: Gain of the LTI system; $\angle H(j\omega)$: phase shift of the LTI system

$$Y(j\omega) = H(j\omega)X(j\omega) = |H(j\omega)||X(j\omega)|e^{j(\angle H(j\omega)+\angle X(j\omega))}$$

$$|Y(j\omega)| = |H(j\omega)||X(j\omega)| \quad \angle Y(j\omega) = \angle H(j\omega) + \angle X(j\omega)$$

# The magnitude-phase representation of LTI

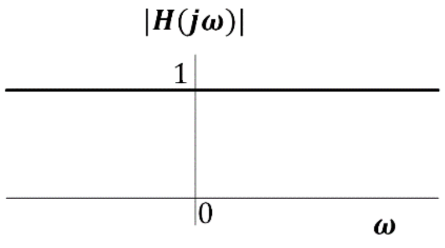
## *Linear phase system*

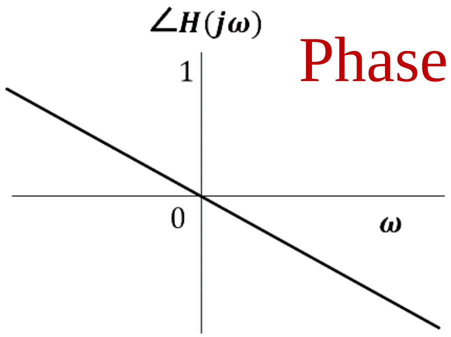
$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

For $H(j\omega) = e^{-j\omega t_0}$

$|H(j\omega)| = 1 \qquad \angle H(j\omega) = -\omega t_0$

All-pass system



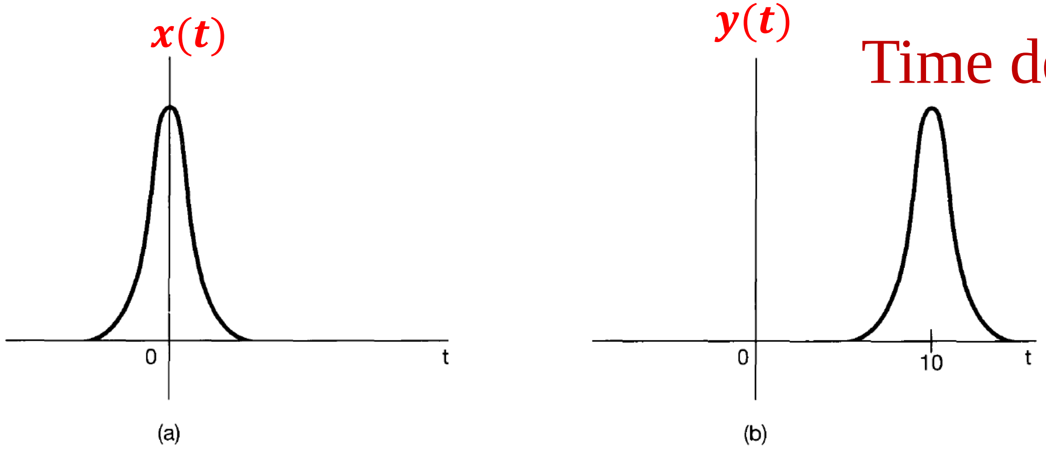

Phase shift

Output of system:

$$Y(j\omega) = H(j\omega)X(j\omega)$$

$$= X(j\omega)e^{-j\omega t_0}$$

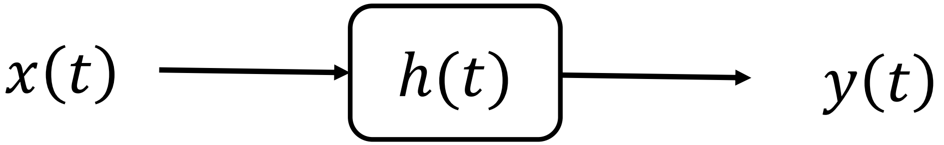
$$y(t) = x(t - t_0)$$




Time delay

# The magnitude-phase representation of LTI

## *Non-linear phase system*




For $H(j\omega) = H_1(j\omega)H_2(j\omega)$

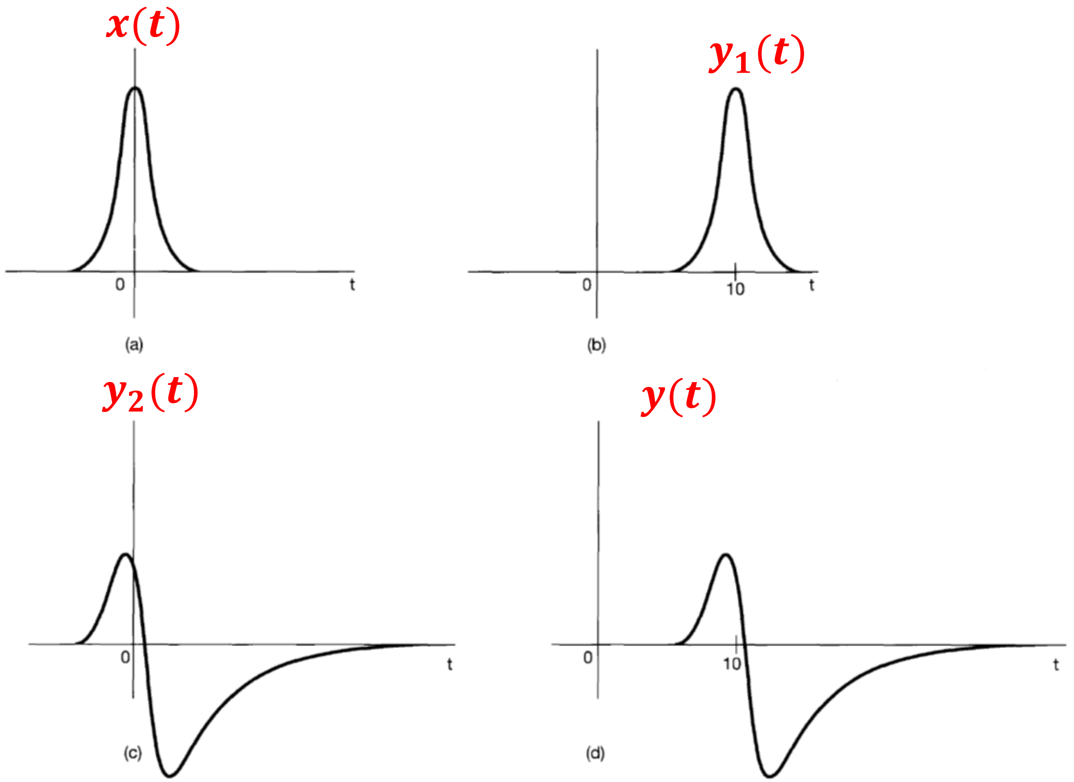
$$H_1(j\omega) = e^{-j\omega t_0}$$

$$H_2(j\omega) = e^{\angle H_2(j\omega)}$$

$\angle H_2(j\omega)$ is a nonlinear function of $\omega$

$$|H(j\omega)| = 1$$

$$\angle H(j\omega) = -\omega t_0 + \angle H_2(j\omega)$$

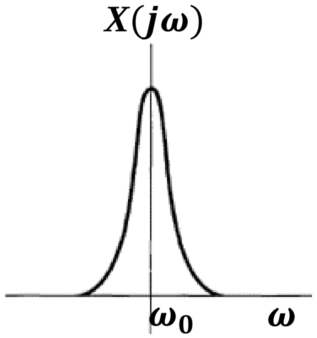


# The magnitude-phase representation of LTI

## *Group delay*

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

- Consider a system with $\angle H(j\omega)$ (a nonlinear function of $\omega$)




- For a narrow band input $x(t)$, $\angle H\,(j\omega) \simeq -\phi - \alpha\omega$

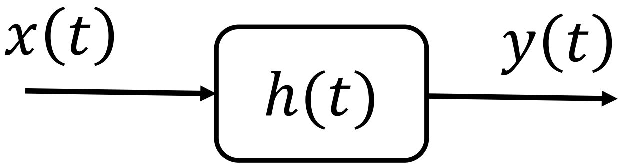
$$Y(j\omega) \simeq X(j\omega)|H(j\omega)|e^{-j\phi}e^{-j\alpha\omega}$$

- The time delay $\alpha$ is referred to as the group delay at $\omega = \omega_0$

$$\tau(\omega) = -\frac{d}{d\omega}\{\angle H\,(j\omega)\}$$

# The magnitude-phase represe

## *Group delay: example*




- Consider

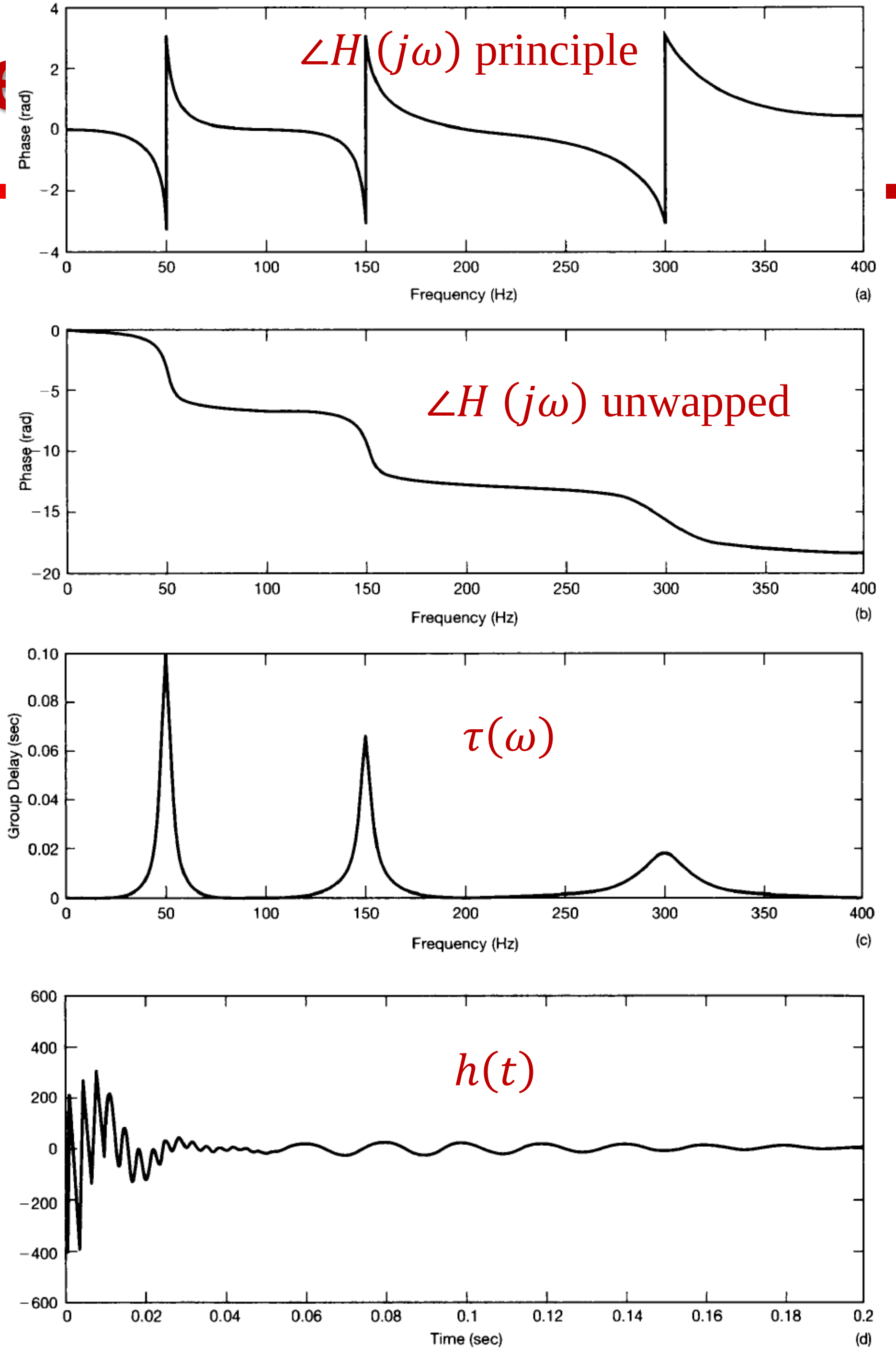
$$H(j\omega) = \prod_{i=1}^{3} H_i(j\omega) \qquad H_i(j\omega) = \frac{1 + (j\omega/\omega_i)^2 - 2j\zeta_i(\omega/\omega_i)}{1 + (j\omega/\omega_i)^2 + 2j\zeta_i(\omega/\omega_i)}$$

$$\begin{cases} \omega_1 = 315 \text{ rad/sec and } \zeta_1 = 0.066, \\ \omega_2 = 943 \text{ rad/sec and } \zeta_2 = 0.033, \\ \omega_3 = 1888 \text{ rad/sec and } \zeta_3 = 0.058. \end{cases}$$

$$|H_i(j\omega)| = 1 \Rightarrow |H(j\omega)| = 1$$

$$\angle H_i\,(j\omega) = -2\arctan\left[\frac{2\zeta_i(\omega/\omega_i)}{1-(\omega/\omega_i)^2}\right]$$

$$\angle H(j\omega) = \sum_{i=1}^{3} \angle H_i\,(j\omega) \qquad \tau(\omega) = -\frac{d}{d\omega}\{\angle H\,(j\omega)\}$$



# The magnitude-phase representation of LTI

## ***Log-Magnitude and Bode Plots***

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

Time domain: $y(t) = x(t) * h(t)$ Convolution

Frequency domain: $Y(j\omega) = H(j\omega)X(j\omega)$ Multiplication

$$|Y(j\omega)| = |H(j\omega)||X(j\omega)|$$

$$\angle Y(j\omega) = \angle H(j\omega) + \angle X(j\omega)$$

Logarithmic amplitude: $\log|Y(j\omega)| = \log|H(j\omega)| + \log|X(j\omega)|$ Summation

Logarithmic amplitude scale: $20\log_{10}$ , referred to as *decibels* (dB).

Bode plots: Plots of $20\log_{10}|H(j\omega)|$and $\angle H(j\omega)$ versus $\log_{10}(\omega)$

# The magnitude-phase representation of LTI

## *Log-Magnitude and Bode Plots*

Plot of $20\log_{10}|H(j\omega)|$ vs $\log_{10}(\omega)$

Plot of $\angle H(j\omega)$ vs. $\log_{10}(\omega)$

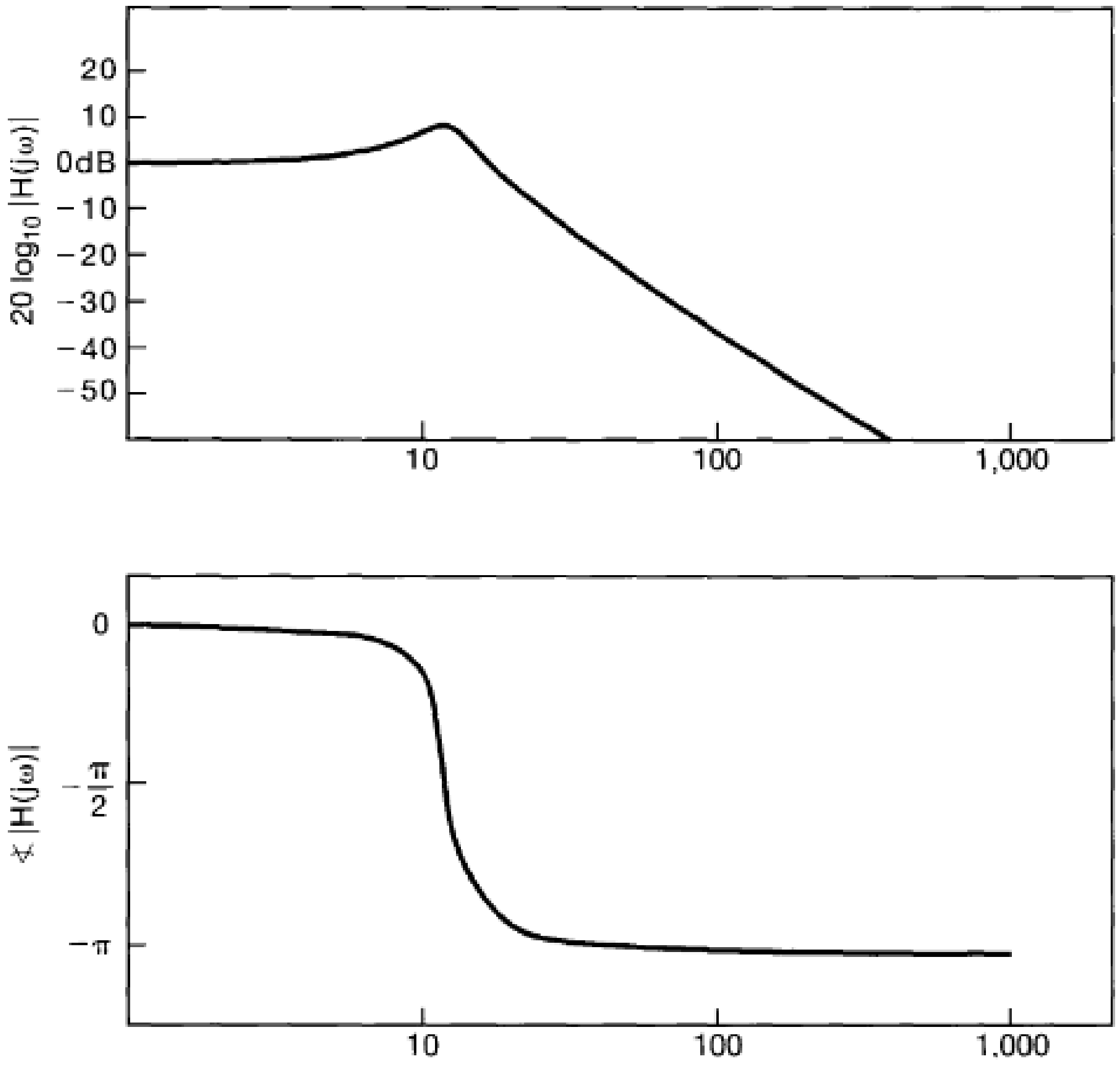



**Figure 6.8** A typical Bode plot. (Note that $\omega$ is plotted using a logarithmic scale.)

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# Time-domain properties of ideal frequency-selective filters

## *Frequency-selective filters*

Low-pass filter

High-pass filter

Band-pass filter

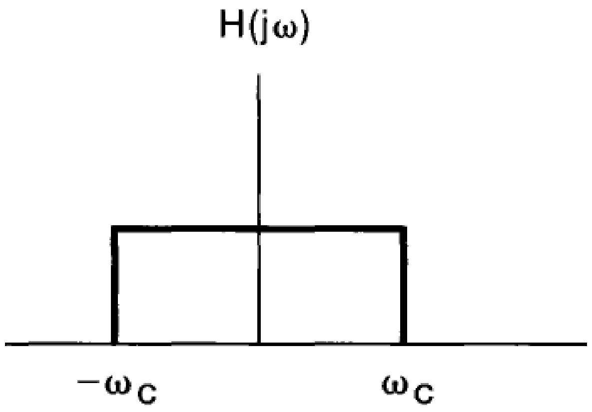
We focus on low-pass filter, similar concepts and results hold for high-pass and band pass filter.

# Time-domain properties of ideal frequency-selective filters

## *Ideal low-pass filters: zero phase*

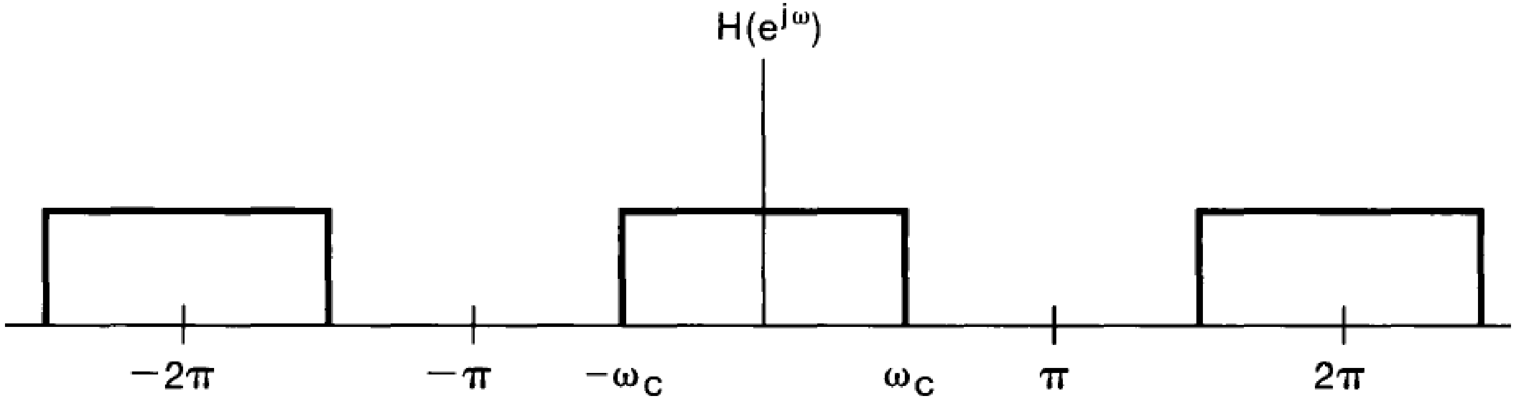
CT

$$H(j\omega) = \begin{cases} 1, & |\omega| \le \omega_c \\ 0, & |\omega| > \omega_c \end{cases}$$




DT

$$H(e^{j\omega}) = \begin{cases} 1, & |\omega| \le \omega_c \\ 0, & \omega_c < |\omega| < \pi \end{cases}$$



# Time-domain properties of ideal frequency-selective filters

## *Ideal low-pass filters: zero phase*

- Impulse response:

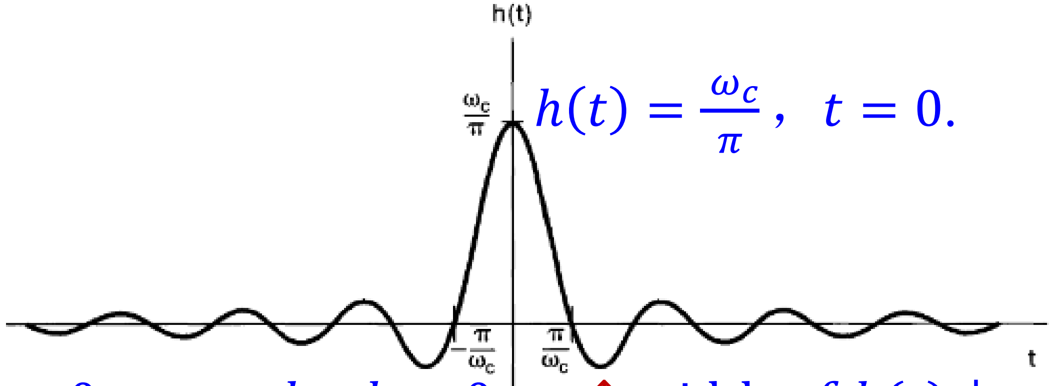
$$h(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{H}(j\omega)e^{j\omega t}d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} 1\cdot e^{j\omega t}d\omega$$

$$= \frac{1}{2\pi}\cdot\frac{1}{jt}e^{j\omega t}\Big|_{-\omega_c}^{\omega_c}$$

$$= \frac{1}{2\pi}\cdot\frac{1}{jt}\cdot 2j\sin(\omega_c t) = \frac{\sin\omega_c t}{\pi t}$$

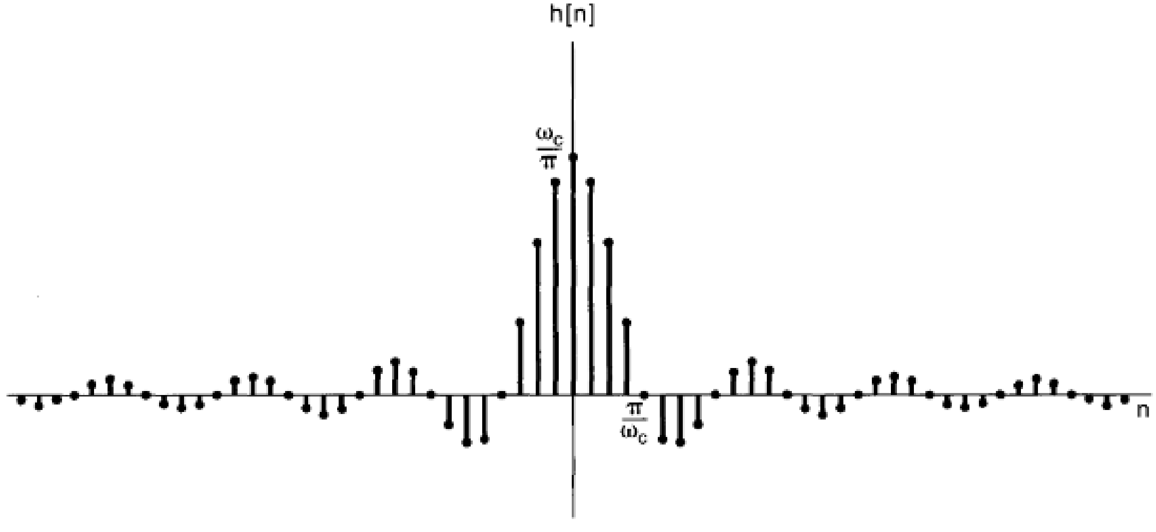
$$h(n) = \frac{\sin\omega_c n}{\pi n}$$




$h(t) = \frac{\omega_c}{\pi},\ \ t = 0.$

$h(t) = 0,\ \omega_c t = k\pi, k \neq 0.$ $\omega_c\uparrow$, width of $h(t)\downarrow$

(a)




(b)

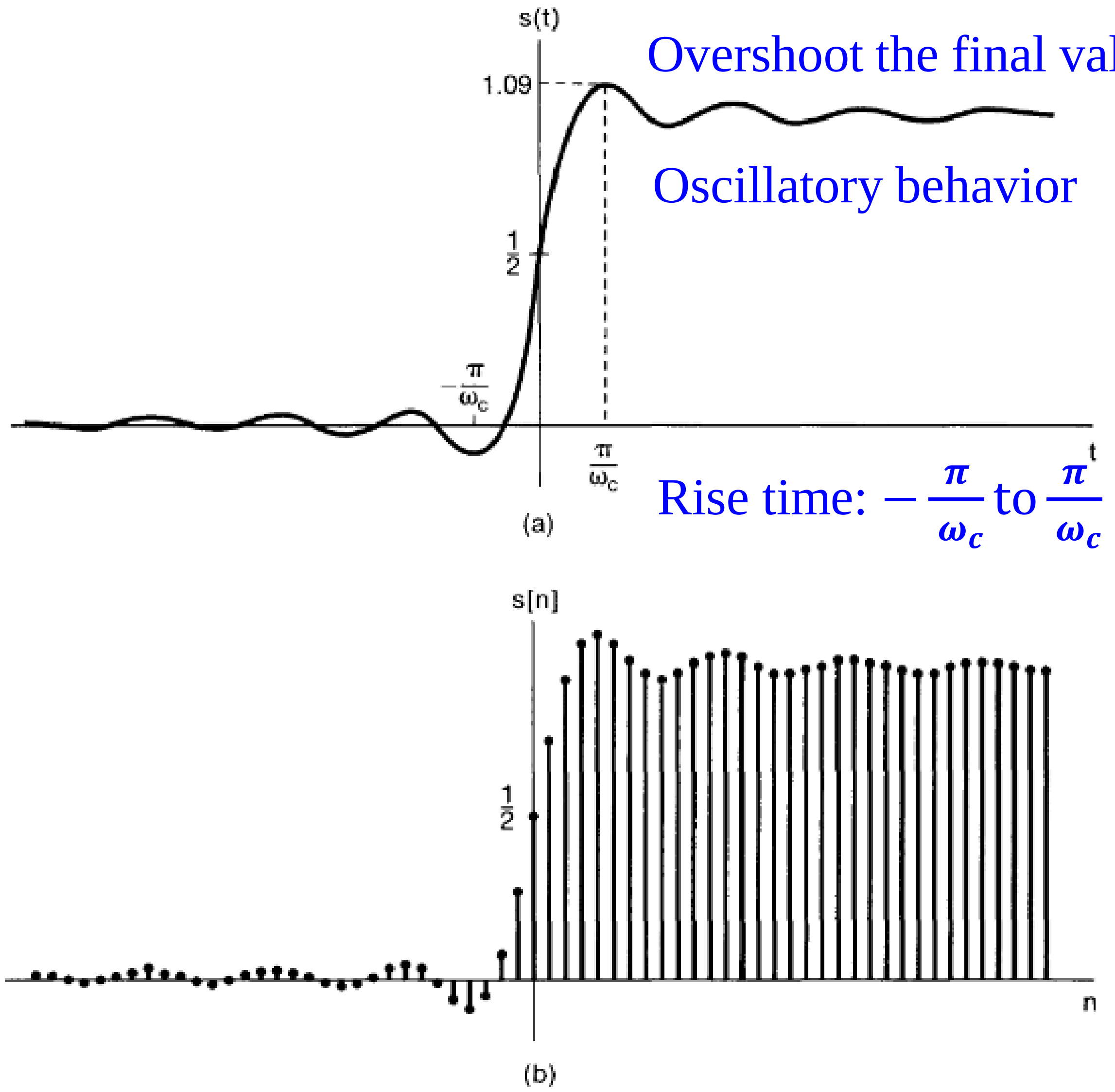
# Time-domain properties of ideal frequency-selective filters

## *Ideal low-pass filters: zero phase*

- Step response:

$$s(t) = \int_{-\infty}^{t} h(\tau)d\tau$$

$$s(n) = \sum_{m=-\infty}^{n} h(m)$$




Overshoot the final value

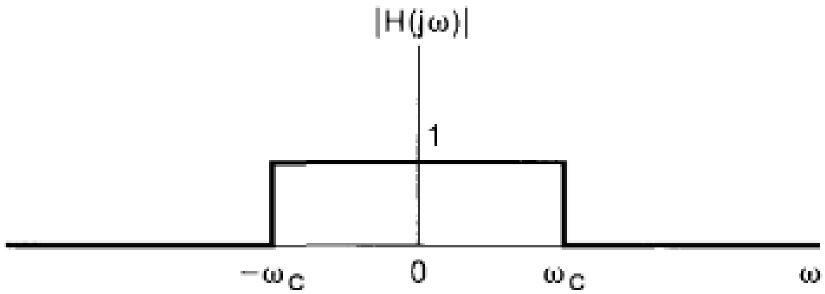
Oscillatory behavior

Rise time: $-\frac{\pi}{\omega_c}$ to $\frac{\pi}{\omega_c}$

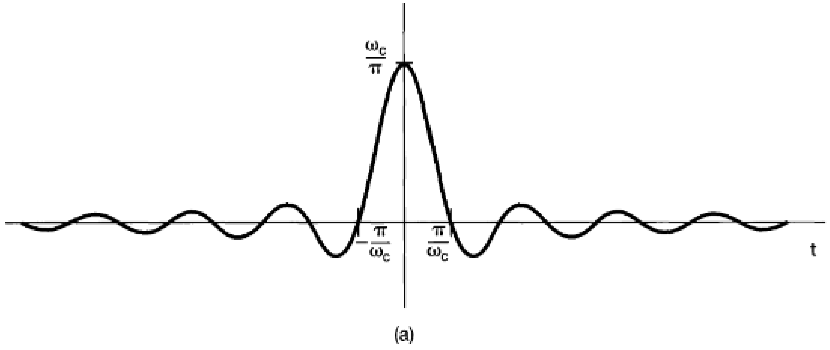
# Time-domain properties of ideal frequency-selective filters

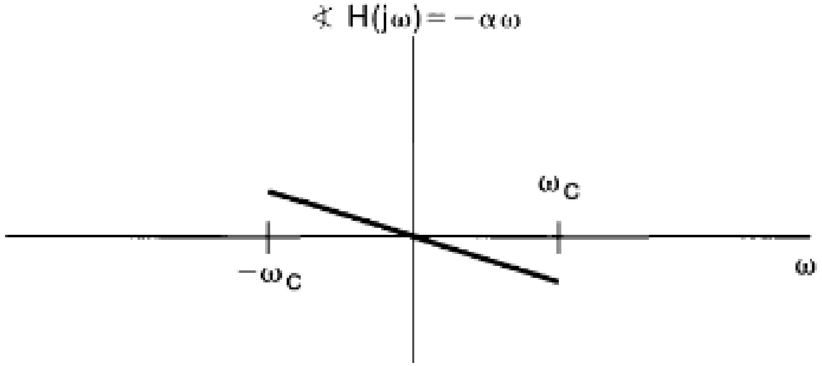
## *Ideal low-pass filters: linear phase*

- Impulse response:

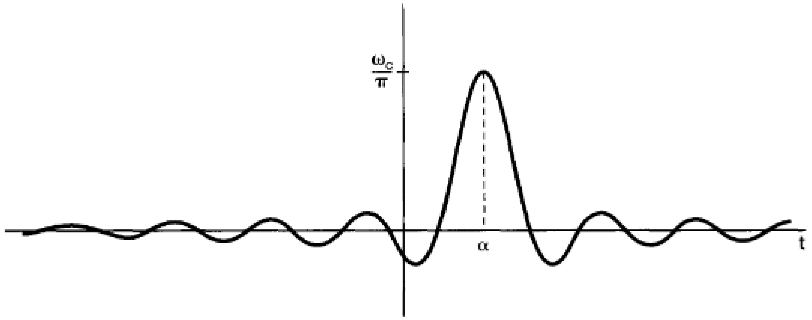



h(t)




$h(t-\alpha)$



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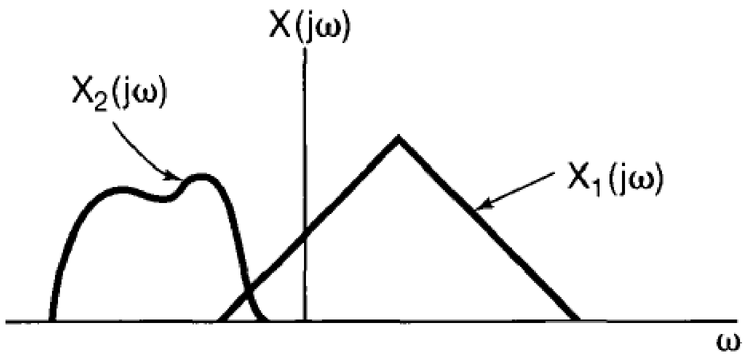
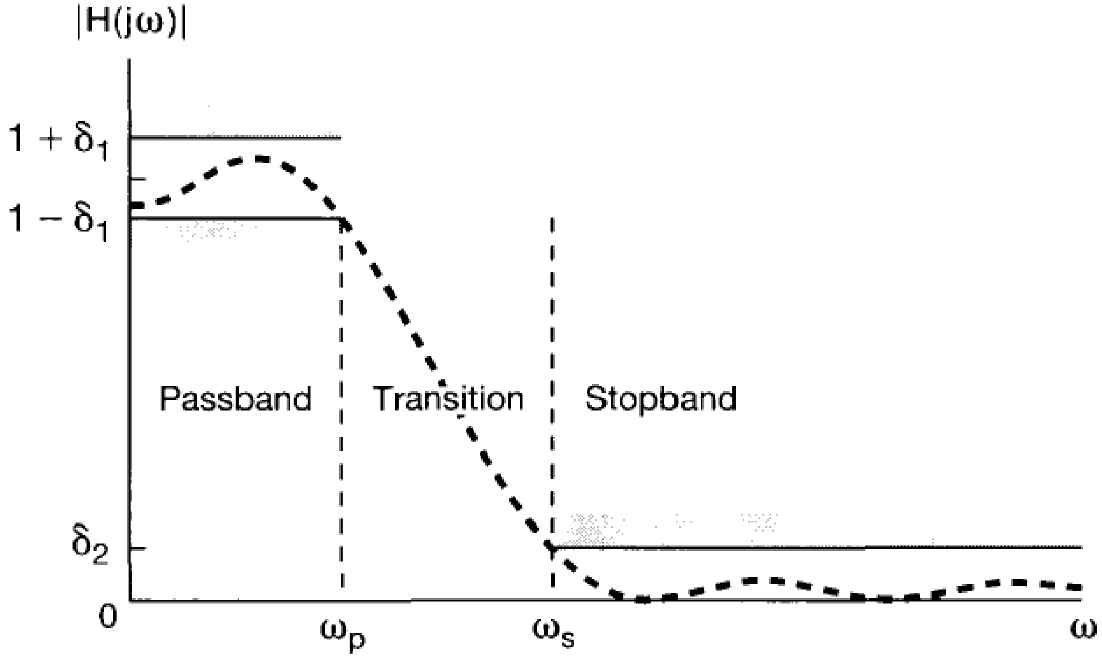
# Non-ideal filters

## *Frequency domain (low-pass)*

- Idea Low-pass filter is not implementable

- Gradual transition band is sometimes preferable

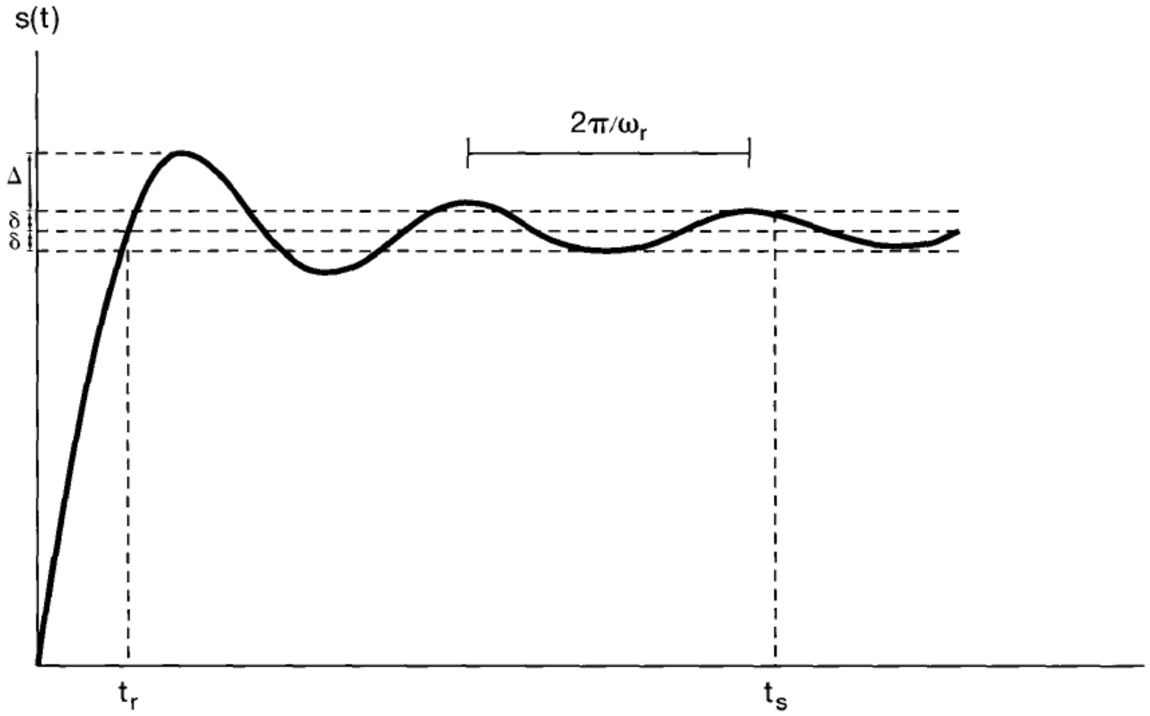



- Pass band: $0 - \omega_p$, stop band: $\omega > \omega_s$, transition: $\omega_s$ - $\omega_p$
- Pass-band ripple: $\delta_1$, stop-band ripple: $\delta_2$
- Linear (nearly) linear phase over the passband is desirable.

# Non-ideal filters

## *Time domain (low-pass)*




Step response of a CT low-pass filter

- Rise time: $t_r$
- Overshoot: $\Delta$
- Ringing frequency: $\omega_r$
- Settling time: $t_s$

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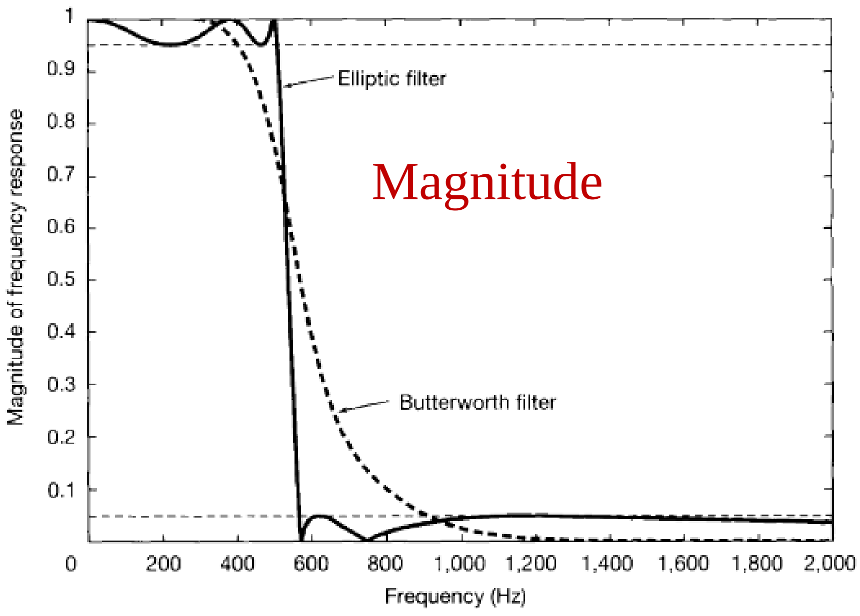
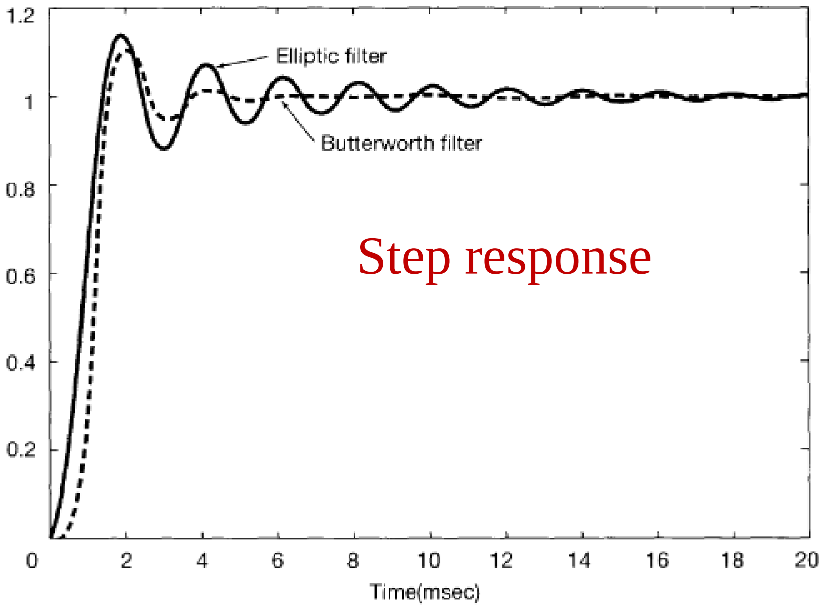
# Non-ideal filters

## *An example*




- Fifth-order Butterworth filter and a fifth-order elliptic filter
- Same cutoff frequency (500 Hz)
- Same passband and stopband ripple

Trade-off between time-domain ($t_s$) and frequency-domain ($\omega_s$ - $\omega_p$).

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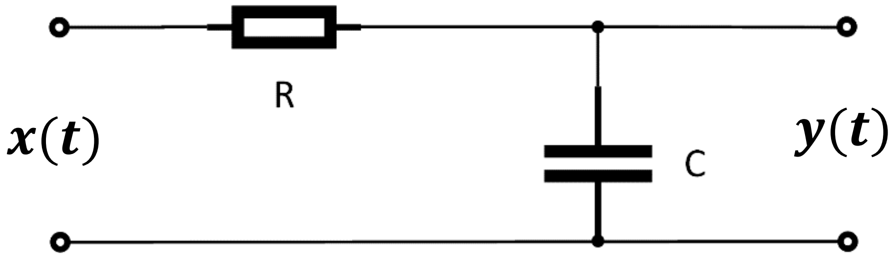
- First-order and second-order systems

# First-order systems

## *First-order system (Continuous time)*




- Differential equation: $$C\frac{dy(t)}{dt}=\frac{x(t)-y(t)}{R}$$

$$\tau\frac{dy(t)}{dt}+y(t)=x(t),\tau=RC$$

- Frequency response: $$\tau j\omega Y(j\omega)+Y(j\omega)=X(j\omega)$$

$$H(j\omega)=\frac{Y(j\omega)}{X(j\omega)}=\frac{1}{j\omega\tau+1}$$

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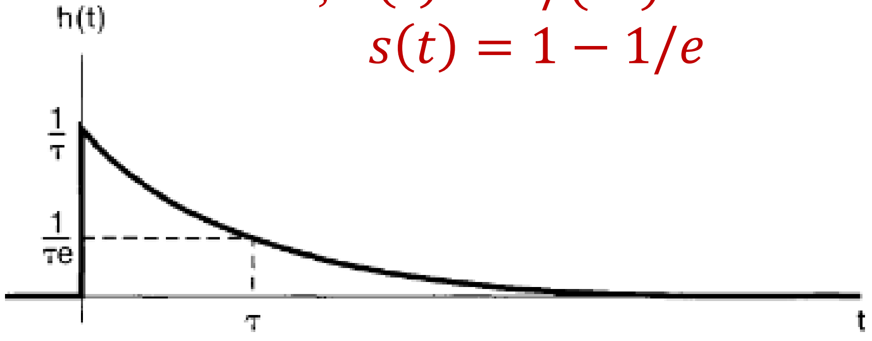
# First-order systems

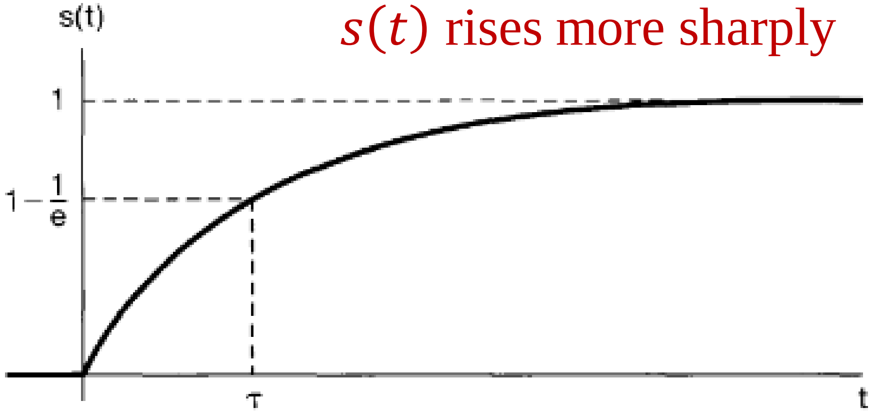
## *First-order system (Continuous time)*

- $\tau$: time constant
- $t = \tau$, $h(t) = 1/(\tau e)$
  $s(t) = 1 - 1/e$




(a)

- $\tau \downarrow$, $h(t)$ decays more sharply
  $s(t)$ rises more sharply




(b)

❑ Impulse response

$$H(j\omega) = \frac{1}{j\omega\tau + 1} = \frac{1/\tau}{j\omega + 1/\tau}$$

$$e^{-at}u(t), a > 0 \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{j\omega + a}$$

$$h(t) = \frac{1}{\tau} e^{-t/\tau} u(t)$$

❑ Step response

$$s(t) = \int_{-\infty}^{t} h(t')\,dt' = \frac{1}{\tau}\int_{0}^{t} e^{-t'/\tau}\,dt' = \begin{cases} 0, t < 0 \\ 1 - e^{-t/\tau}), t \geq 0 \end{cases}$$

$$s(t) = (1 - e^{-t/\tau})\,u(t)$$

# First-order systems

## *Bold Plots (Continuous time)*

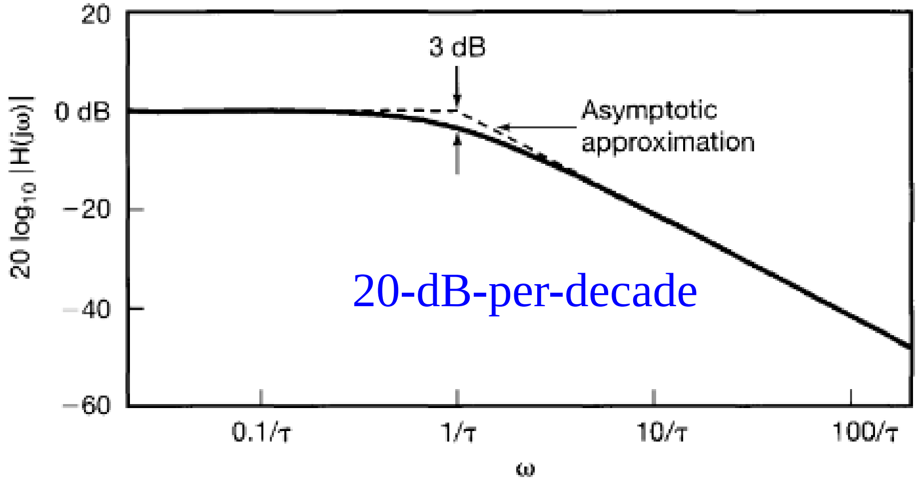
$$H(j\omega) = \frac{1}{j\omega\tau + 1}$$

- **$20\log_{10}|H(j\omega)|$** $= -10\log_{10}[(\omega\tau)^2 + 1]$

$$\simeq \begin{cases} 0, & \omega \ll 1/\tau \\ -20\log_{10}(\omega) - 20\log_{10}(\tau), & \omega \gg 1/\tau \end{cases}$$

$$\omega = 1/\tau, 20\log_{10}|H(j\omega)| = -10\log_{10}(2) \simeq -3dB$$

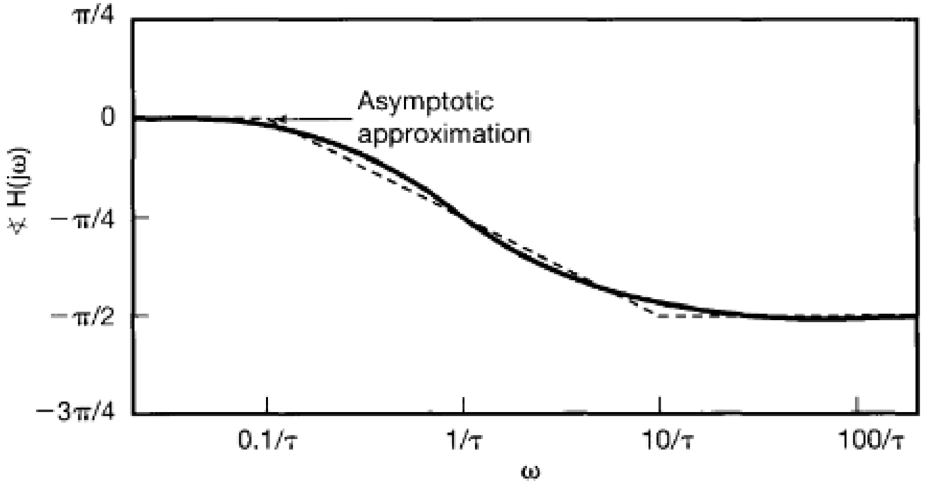
$\omega = 1/\tau$： break frequency




- **$\angle H(j\omega)$** $= -\tan^{-1}(\omega\tau)$

$$\simeq \begin{cases} 0, & \omega \le 0.1/\tau \\ -\frac{\pi}{4}[\log_{10}(\omega\tau) + 1], & 0.1/\tau \le \omega \le 10/\tau \\ -\pi/2, & \omega \ge 10/\tau \end{cases}$$

$$\omega = 1/\tau, \angle H(j\omega) = -\pi/4$$

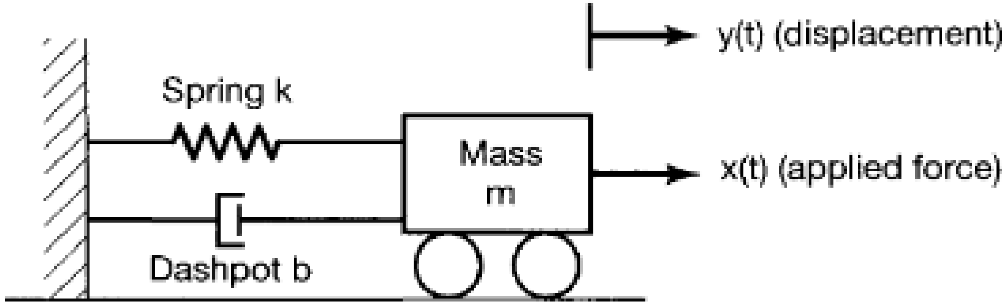



$\tau \downarrow$, $h(t)$ and $s(t)$ more sharply, break frequency $\uparrow$.

# Second-order systems

## *Differential equation*

$$m\frac{d^2y(t)}{dt} = x(t) - ky(t) - b\frac{dy(t)}{dt}$$




$$\Downarrow$$

$$\frac{d^2y(t)}{dt} + \left(\frac{b}{m}\right)\frac{dy(t)}{dt} + \left(\frac{k}{m}\right)y(t) = \frac{1}{m}x(t)$$

$$\Downarrow$$

$$\omega_n{}^2 = \frac{k}{m} \qquad \omega_n = \sqrt{\frac{k}{m}} \qquad \zeta = \frac{b}{2\sqrt{km}} \qquad 2\zeta\omega_n = \frac{b}{m}$$

$$\frac{d^2y(t)}{dt} + 2\zeta\omega_n\frac{dy(t)}{dt} + \omega_n{}^2y(t) = \omega_n{}^2x(t)$$

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# Second-order systems

- Frequency response: $$\frac{d^2y(t)}{dt} + 2\zeta\omega_n\frac{dy(t)}{dt} + \omega_n{}^2y(t) = \omega_n{}^2x(t)$$

$$(j\omega)^2Y(j\omega) + 2\zeta\omega_n(j\omega)Y(j\omega) + \omega_n{}^2Y(j\omega) = \omega_n{}^2X(j\omega)$$

$$H(j\omega) = \frac{\omega_n{}^2}{(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n{}^2}$$

- Impulse response: $$H(j\omega) = \frac{\omega_n{}^2}{(j\omega - c_1)(j\omega - c_2)} = \frac{M_1}{(j\omega - c_1)} - \frac{M_2}{(j\omega - c_2)}$$

$\zeta \neq 1$

$c_1, c_2$: roots of $(j\omega)^2 + 2\zeta\omega_n(j\omega) + \omega_n{}^2 = 0$

$$c_1 = -\zeta\omega_n + \omega_n\sqrt{\zeta^2 - 1}, \quad c_2 = -\zeta\omega_n - \omega_n\sqrt{\zeta^2 - 1}$$

$$M_1 = M_2 = M = \frac{\omega_n}{2\sqrt{\zeta^2 - 1}} \Longrightarrow h(t) = M[e^{c_1t} - e^{c_2t}]u(t)$$

# Second-order systems

- Impulse response:

$\zeta = 1$ $\quad c_1 = c_2 = -\omega_n$ $\quad H(j\omega) = \dfrac{{\omega_n}^2}{(j\omega + \omega_n)^2}$

Critically damped $\quad te^{-at}u(t) \xrightarrow{\mathcal{F}} H(j\omega) = \dfrac{1}{(j\omega + a)^2}$ $\quad \therefore h(t) = {\omega_n}^2 te^{-\omega_n t}u(t)$

- Recall $\zeta \neq 1$ $\quad h(t) = M[e^{c_1 t} - e^{c_2 t}]u(t)$

$0 < \zeta < 1$

$$h(t) = \frac{\omega_n}{2\sqrt{\zeta^2 - 1}}\left[e^{\left(-\zeta\omega_n + \omega_n\sqrt{\zeta^2-1}\right)t} - e^{\left(-\zeta\omega_n - \omega_n\sqrt{\zeta^2-1}\right)t}\right]u(t)$$

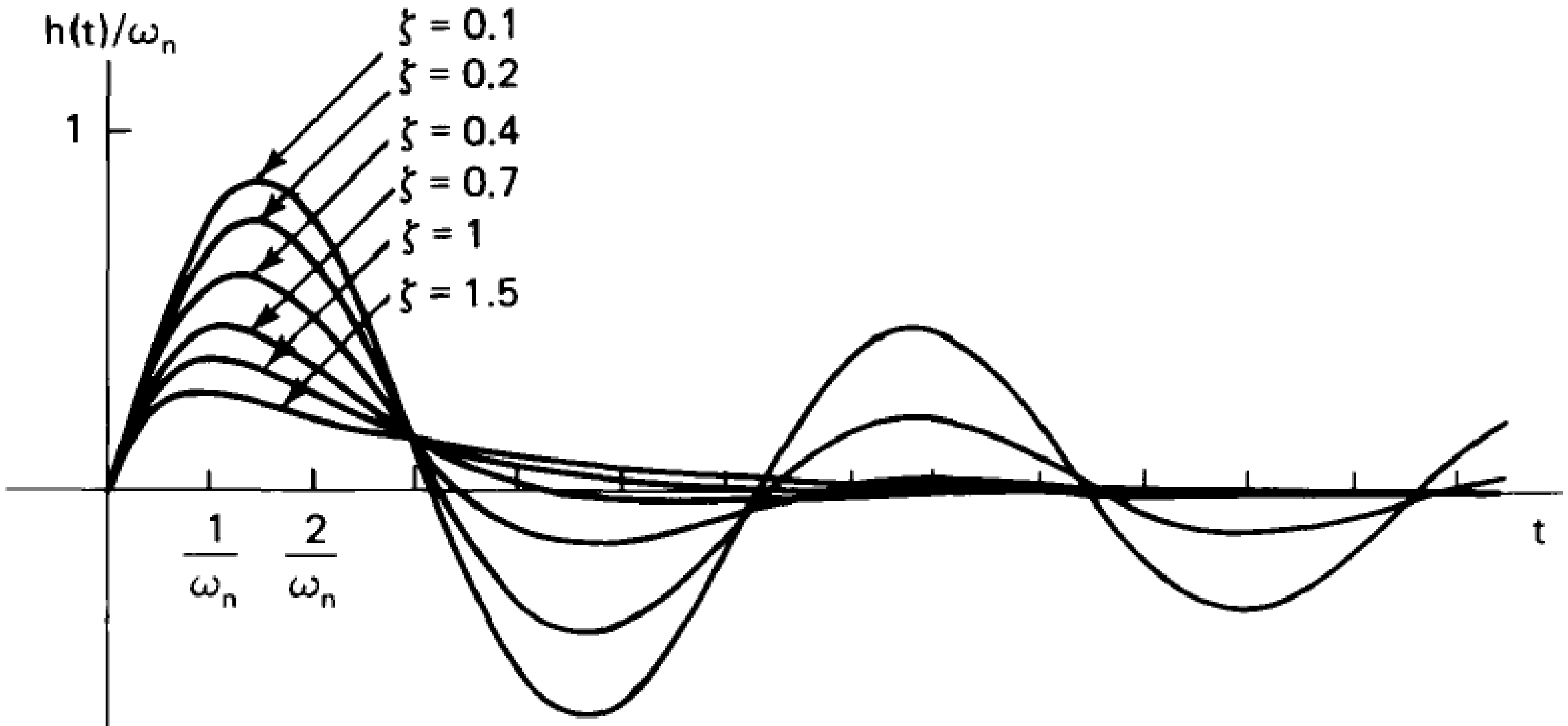
$$= \frac{\omega_n e^{-\zeta\omega_n t}}{2\sqrt{\zeta^2 - 1}}\left[e^{j\omega_n\sqrt{1-\zeta^2}t} - e^{-j\omega_n\sqrt{1-\zeta^2}t}\right]u(t)$$

$\zeta > 1$: over damped

$$= \frac{\omega_n e^{-\zeta\omega_n t}}{2\sqrt{\zeta^2 - 1}}[2j\sin(\omega_n\sqrt{1 - \zeta^2}\, t)]u(t) = \frac{\omega_n e^{-\zeta\omega_n t}}{\sqrt{1 - \zeta^2}}[\sin(\omega_n\sqrt{1 - \zeta^2}\, t)]u(t)$$

# Second-order systems

- Impulse response:




$0 < \zeta < 1$: damped

$\zeta > 1$: overdamped

$\zeta = 1$: critically damped

$\omega_n$: un-damped natural frequency

$\zeta$: damping ratio

# Second-order systems

- Step response

$\zeta \neq 1$

$$s(t) = \int_{-\infty}^{t} h(t')\,dt' = M\int_{0}^{t} e^{c_1 t'} - e^{c_2 t'}\,dt'$$

$$h(t) = M[e^{c_1 t} - e^{c_2 t}]u(t)$$

$$= \begin{Bmatrix} 0, t < 0 \\ M\left(\frac{e^{c_1 t'}}{c_1} - \frac{e^{c_2 t'}}{c_2}\right)\Big|_0^t = 1 + M\left[\frac{e^{c_1 t}}{c_1} - \frac{e^{c_2 t}}{c_2}\right], t \geq 0 \end{Bmatrix} = \left\{1 + M\left[\frac{e^{c_1 t}}{c_1} - \frac{e^{c_2 t}}{c_2}\right]\right\}u(t)$$

$\zeta = 1$

$$h(t) = {\omega_n}^2 t e^{-\omega_n t}u(t)$$

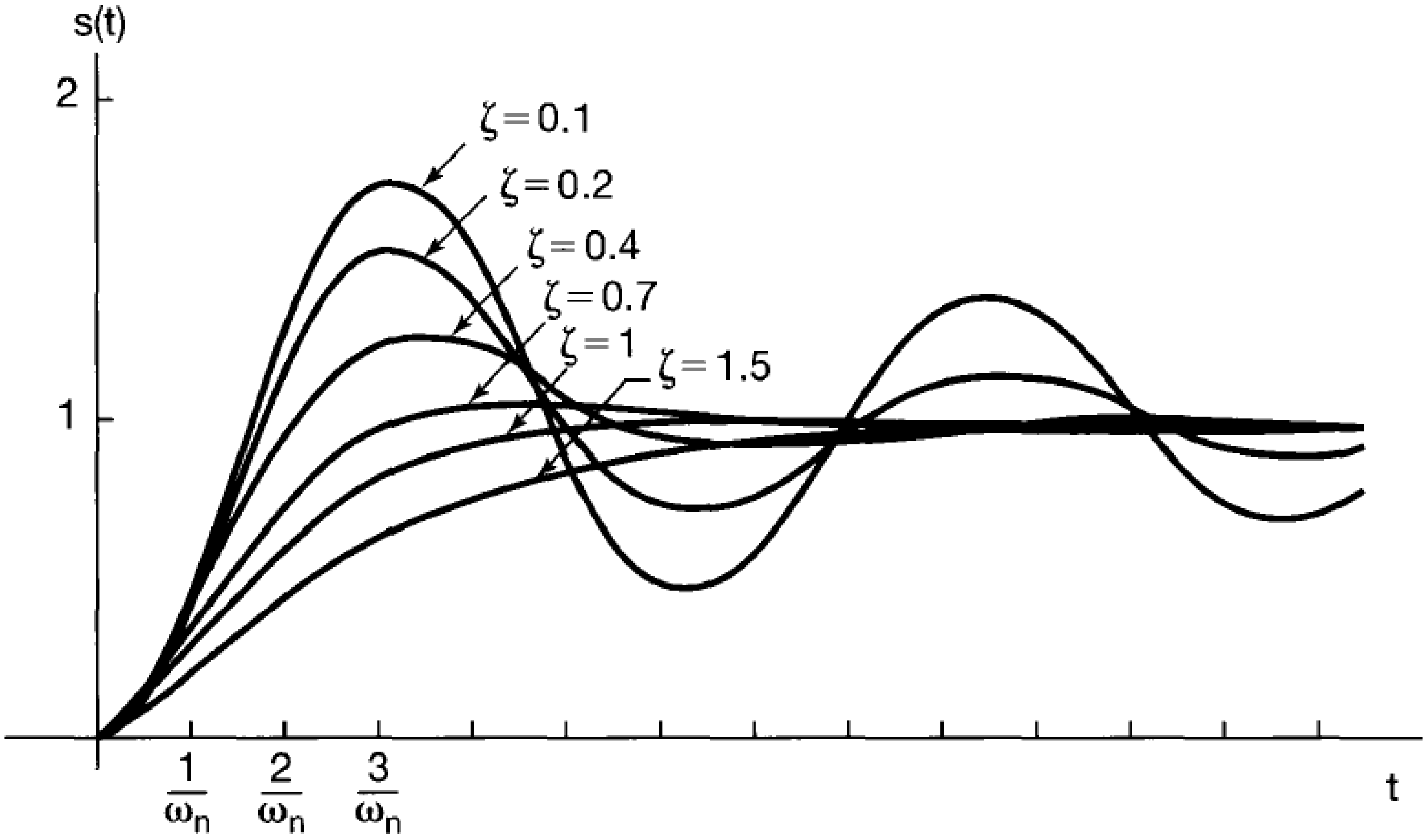
$$s(t) = \int_{0}^{t} {\omega_n}^2 t' e^{-\omega_n t'}\,dt' = -\omega_n \int_{0}^{t} t'\,de^{-\omega_n t'}$$

$$= \begin{Bmatrix} 0, t < 0 \\ -\omega_n t' e^{-\omega_n t'}\Big|_0^t - \int_{0}^{t} e^{-\omega_n t'}\,d(-\omega_n t') = 1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t}, t \geq 0 \end{Bmatrix}$$

$$s(t) = [1 - e^{-\omega_n t} - \omega_n t e^{-\omega_n t}]u(t)$$

# Second-order systems

- Step response



# Second-order systems

- Bold plots

$$H(j\omega)=\frac{{\omega_n}^2}{(j\omega)^2+2\zeta\omega_n(j\omega)+{\omega_n}^2}=\frac{1}{(j\omega/\omega_n)^2+2\zeta(j\omega/\omega_n)+1}$$

$$\mathbf{20log_{10}}|\boldsymbol{H(j\omega)}|=-20\log_{10}|(j\omega/\omega_n)^2+2\zeta(j\omega/\omega_n)+1|$$

$$=-10\log_{10}\left\{\left[1-\left(\frac{\omega}{\omega_n}\right)^2\right]^2+4\zeta^2\left(\frac{\omega}{\omega_n}\right)^2\right\}$$

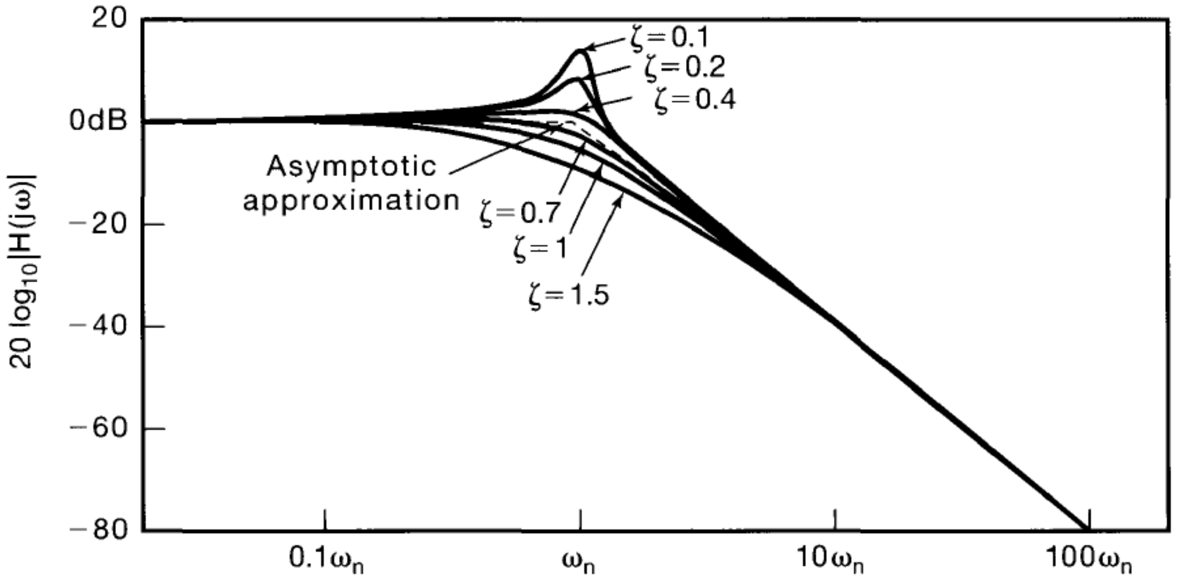
$$\simeq\begin{cases}0, & \omega\ll\omega_n\\ -40\log_{10}\omega+40\log_{10}\omega_n, & \omega\gg\omega_n\end{cases}$$

$$\angle\boldsymbol{H(j\omega)}=-\tan^{-1}\left[\frac{2\zeta(\omega/\omega_n)}{1-(\omega/\omega_n)^2}\right]\simeq\begin{cases}0, & \omega\le 0.1\omega_n\\ -\frac{\pi}{2}\left[\log_{10}\left(\frac{\omega}{\omega_n}\right)+1\right], & 0.1\omega_n\le\omega\le 10\omega_n\\ -\pi, & \omega\ge 10\omega_n\end{cases}$$

# Second-order systems

- Bold plots

$20\log_{10}|H(j\omega)|$




$\angle H(j\omega)$