Ch.4 *The Continuous-Time Fourier Transform (CTFT)*

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Part IV *Systems Characterized by Differential Equations*

Differential Equations

- Use differential equations to represent the input and output of continuous-time LTI systems:

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

Also consider the LTI system is characterized by

$$Y(j\omega) = H(j\omega)X(j\omega) \Longrightarrow H(j\omega) = \frac{Y(j\omega)}{X(j\omega)}$$

- Apply Fourier transform to both sides of the equation:

$$\mathcal{F}\left\{\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k}\right\} = \mathcal{F}\left\{\sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}\right\}$$

- Use the linearity property:

$$\sum_{k=0}^{N} a_k \mathcal{F}\left\{\frac{d^k y(t)}{dt^k}\right\} = \sum_{k=0}^{M} b_k \mathcal{F}\left\{\frac{d^k x(t)}{dt^k}\right\}$$

Differential Equations

- Use the differentiation property

$$\sum_{k=0}^{N} a_k \, (j\omega)^k Y(j\omega) = \sum_{k=0}^{N} b_k \, (j\omega)^k X(j\omega)$$

- Or equivalently:

$$Y(j\omega)\left[\sum_{k=0}^{N} a_k \, (j\omega)^k\right] = X(j\omega)\left[\sum_{k=0}^{N} b_k \, (j\omega)^k\right]$$

- Therefore

$$\boxed{H(j\omega) = \frac{Y(j\omega)}{X(j\omega)} = \frac{\sum_{k=0}^{N} b_k \, (j\omega)^k}{\sum_{k=0}^{N} a_k \, (j\omega)^k}}$$

Differential Equations

- Example 1. Consider a stable LTI system characterized by the differential equation

$$\frac{dy(t)}{dt} + ay(t) = x(t), a > 0$$

please find the impulse response of this system.

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- Solution:

$$\mathcal{F}\left\{\frac{dy(t)}{dt} + ay(t)\right\} = \mathcal{F}\{x(t)\}$$

$$j\omega Y(j\omega) + aY(j\omega) = X(j\omega)$$

$$H(j\omega) = \frac{1}{j\omega + a} \Longrightarrow h(t) = e^{-at}u(t)$$

Differential Equations

- Example 2. Consider a stable LTI system characterized by the differential equation

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + 2x(t)$$

please find the impulse response of this system.

Differential Equations

- Example 2. Consider a stable LTI system characterized by the differential equation

$$\frac{d^2y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + 2x(t)$$

please find the impulse response of this system.

- Solution:

$$H(j\omega) = \frac{(j\omega)+2}{(j\omega)^2 + 4(j\omega) + 3} = \frac{\frac{1}{2}}{j\omega + 1} + \frac{\frac{1}{2}}{j\omega + 3}$$

$$h(t) = \frac{1}{2}e^{-t}u(t) + \frac{1}{2}e^{-3t}u(t)$$

Summary

- Systems Characterized by Differential Equations

- Reference in textbook:
  - 4.7