# Numerical Optimization, Fall 2024
# Homework 6

Due 23:59 (CST), Dec. 1, 2024

## Problem 1: Projection Calculations [25pts]

Compute the projection of a point onto the following sets:

1. Projection onto the $L_2$ ball:
$$\min_x \|x - c\|_2^2 \quad \text{s.t. } \|x\|_2 \leq r. \tag{1}$$

2. Projection onto the $L_\infty$ ball:
$$\min_x \|x - c\|_2^2 \quad \text{s.t. } \|x\|_\infty \leq r. \tag{2}$$

3. Projection onto a hyperplane:
$$\min_x \|x - c\|_2^2 \quad \text{s.t. } a^\top x = b. \tag{3}$$

4. Projection onto a half-space:
$$\min_x \|x - c\|_2^2 \quad \text{s.t. } a^\top x \leq b. \tag{4}$$

5. Projection onto the intersection of hyperplanes (assume $A^\top A$ is invertible):
$$\min_x \|x - c\|_2^2 \quad \text{s.t. } Ax = b. \tag{5}$$

## Problem 2: Frank-Wolfe Subproblem Calculations [15pts]

Solve the Frank-Wolfe subproblem for the following constraint sets:

1. $L_1$ ball:
$$\min_s \nabla f(x)^\top s \quad \text{s.t. } \|s\|_1 \leq r. \tag{6}$$

2. $L_2$ ball:
$$\min_s \nabla f(x)^\top s \quad \text{s.t. } \|s\|_2 \leq r. \tag{7}$$

3. $L_\infty$ ball:
$$\min_s \nabla f(x)^\top s \quad \text{s.t. } \|s\|_\infty \leq r. \tag{8}$$

### Problem 3: Write the KKT Conditions [30pts]

Write the Karush-Kuhn-Tucker (KKT) conditions for the following problems and calculate the stationary points:

1. Linear Programming (LP):

$$\begin{aligned} \min_x \quad & -2x_1 - 3x_2, \\ \text{s.t.} \quad & x_1 + x_2 \leq 4, \\ & x_1 - 2x_2 \leq 1, \\ & x_1, x_2 \geq 0. \end{aligned} \tag{9}$$

2. Quadratic Programming (QP):

$$\begin{aligned} \min_x \quad & \frac{1}{2}\left(x_1^2 + 2x_1x_2 + 2x_2^2\right) - 4x_1 - 6x_2, \\ \text{s.t.} \quad & x_1 + x_2 \leq 5, \\ & x_1, x_2 \geq 0. \end{aligned} \tag{10}$$

3. Nonlinear Problem (NLP):

$$\begin{aligned} \min_x \quad & (x_1 - 1)^2 + (x_2 - 2)^2 + x_1x_2, \\ \text{s.t.} \quad & x_1^2 + x_2^2 \leq 4, \\ & x_1 - x_2 = 1. \end{aligned} \tag{11}$$

### Problem 4: Algorithm Selection and Implementation [30pts]

1. Consider the problem:

$$\min_x \|Ax - b\|_2^2 \quad \text{s.t.} \quad \|x\|_2 \leq r. \tag{12}$$

   Analyze whether Gradient Descent (GD) or Frank-Wolfe (FW) is more suitable for this problem. Provide a detailed explanation.

2. Consider the problem:

$$\min_x \|Ax - b\|_2^2 \quad \text{s.t.} \quad \|x\|_1 \leq r. \tag{13}$$

   Analyze whether Gradient Descent (GD) or Frank-Wolfe (FW) is more suitable for this problem. Provide a detailed explanation.

3. Write a program to randomly generate $A$ and $b$, compute the solutions under both constraints ($L_2$ and $L_1$ balls), and compare the performance and results of GD and FW.

## Submission Requirements

- Submit a PDF file with detailed derivations and explanations.
- Include the program code in the PDF (e.g., Python or MATLAB).