CS243: Introduction to Algorithmic Game Theory

Matching II (Dengji ZHAO)

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Recap: Matching

Matching (Mechanism Design without Money)

- Agents in two sides.
- A matching: each agent is assigned to at most one agent on the other side.

Two-sided matching

- Agent in one set has strict preferences over agents in other set, e.g. students to schools

One-sided matching

- Only one side has strict preference on the other side, e.g. house allocation

Recap: One-sided Matching: House Allocation

- Without initial allocation
  - Serial dictatorship mechanism: pareto optimal
- With initial allocation
  - Top-trading-cycle (TTC) mechanism: pareto optimal, truthful

Two-sided Matching

Definition

A stable matching is a matching with no blocking pair, a blocking pair is two agents who prefer to match with each other.

Stable Matchings:

- Boy-Proposing Deferred Acceptance: stable
- Girl-Proposing Deferred Acceptance: stable

Question

Is Deferred Acceptance truthful?

Truthful Stable Matching

Theorem

The direct mechanism associated with the male propose algorithm is truthful for the males.

Question

Is there a mechanism that is both stable and truthful for both the males and females?

$\succ_{m_1}$	$\succ_{m_2}$	$\succ_{m_3}$	$\succ_{w_1}$	$\succ_{w_2}$	$\succ_{w_3}$
w_2	w_1	w_1	m_1	m_3	m_1
w_1	w_3	w_2	m_3	m_1	m_3
w_3	w_2	w_3	m_2	m_2	m_2

No Truthful and Stable Matching Mechanism

Theorem

There exists no mechanism that is both stable and truthful (in two-sided matching).

Proof.

- Consider two boys and two girls with the following preference profile:
  - $b_1 : g_1 \succ_{b_1} g_2 \succ_{b_1} b_1; b_2 : g_2 \succ_{b_2} g_1 \succ_{b_2} b_2$
  - $g_1 : b_2 \succ_{g_1} b_1 \succ_{g_1} g_1; g_2 : b_1 \succ_{g_2} b_2 \succ_{g_2} g_2$
- Only two stable matchings: $(b_1, g_1), (b_2, g_2)$ and $(b_1, g_2), (b_2, g_1)$, if the mechanism chooses the first matching, then g_1 will misreport $b_2 \succ_{g_1} g_1 \succ_{g_1} b_1$ to force the mechanism to choose the other matching.

$\square$

Kidney Disease

- Kidney failure: a serious medical problem
- Preferred treatment: kidney transplant
  - Cadaver kidneys
  - Donation from live healthy people/relatives
  - Must be blood- and tissue-type compatible

Kidney Disease

http://optn.transplant.hrsa.gov

118,241

people need a lifesaving organ transplant (total waiting list candidates). Of those, **75,814** people are active waiting list candidates. Totals as of today 9:58am

5,367

organ transplants performed so far in 2017

Total Transplants January - February 2017

as of 03/19/2017

2,553

donors

Total Donors January - February 2017

as of 03/19/2017

Organ donation and transplantation can save lives

Every ten minutes, someone is added to the national transplant waiting list.

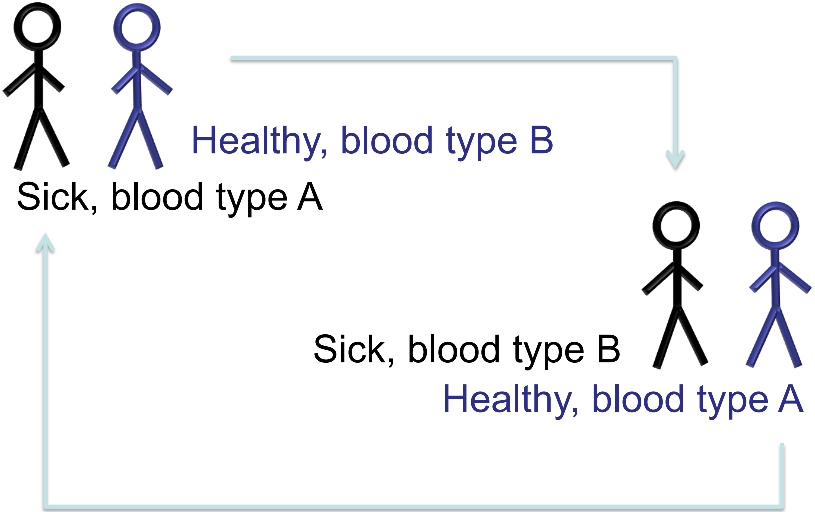
On average, 22 people die each day while waiting for a transplant.

One organ donor can save eight lives. Sign up to be a donor in your state.

Kidney Donation and Kidney Exchange: One-sided Matching

- Incompatible pairs
  - a patient and a donor (they are incompatible)
- Kidney exchanges
  - incompatible pairs participate in swaps

2-cycle Swap



Healthy, blood type B
Sick, blood type A
Sick, blood type B
Healthy, blood type A

Top Trading Cycle for Kidney Exchange?

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- Solution?
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  - Find vertex-disjoint cycles of length $\leq k$ that cover as many vertices as possible
- What will happen if there is one extra donor without patient?

Many-to-One Matching: College Admissions

Many-to-One Matching: College Admissions

- A set of colleges C, each $c \in C$ has a capacity q_c.
- A set of students I, each $i \in I$ has a preference $\succeq_i$ over C.
- Each college $c \in C$ has a preference $\succeq_c$ over 2^I.

Matching in College Admissions

Definition

A matching for college admissions is $\mu : C \cup I \Rightarrow 2^{C \cup I}$ such that:

- $\mu(c) \subseteq I$ such that $|\mu(c)| \leq q_c$ for all $c \in C$,
- $\mu(i) \subseteq C$ such that $|\mu(i)| \leq 1$ for all $i \in I$, and
- $i \in \mu(c)$ if and only if $\mu(i) = \{c\}$ for all $c \in C$ and $i \in I$.

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A matching μ is

- **blocked by a college** $c \in C$ if there exists $i \in \mu(c)$ such that $\emptyset \succ_c \{i\}$.
- **blocked by a student** $i \in I$ if $\emptyset \succ_i \mu(i)$.
- **individually rational** if it is not blocked by any college or student.

Stable Matching

A matching μ is **blocked by a pair** $(c, i) \in C \times I$ if

1. $c \succ_i \mu(i)$, and
2. either there exists $j \in \mu(c)$ such that $\{i\} \succ_c \{j\}$, or $|\mu(c)| < q_c$ and $\{i\} \succ_c \emptyset$.

Definition

A matching is stable if it is not blocked by any student, college or pair.

College-Proposing Deferred Acceptance Algorithm

1. Each college c proposes to its top q_c acceptable students.
2. Each student rejects any unacceptable proposals and, if more than one acceptable proposal is received, she "holds" the most-preferred and rejects the rest.
3. Repeat until no more rejections. Each student is matched with the college she has been holding in the last step.

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Quiz

Does the college-proposing deferred acceptance algorithm give a stable matching?

Stable and Truthful

Theorem

There exists no mechanism that is stable and truthful.

Theorem

Truth-telling is a weakly dominant strategy for all students under the Student-Proposing Deferred Acceptance mechanism.

Theorem

There exists no stable mechanism where truth-telling is a weakly dominant strategy for all colleges.

Not Truthful for Colleges

There are 2 colleges c_1, c_2 with $q_{c_1} = 2, q_{c_2} = 1$, and 2 students i_1, i_2. The preferences are as follows:

- $\succeq_{i_1}$: $\{c_1\} \succeq_{i_1} \{c_2\} \succeq_{i_1} \emptyset$;
- $\succeq_{i_2}$: $\{c_2\} \succeq_{i_2} \{c_1\} \succeq_{i_2} \emptyset$;
- $\succeq_{c_1}$: $\{i_1, i_2\} \succeq_{c_1} \{i_2\} \succeq_{c_1} \{i_1\} \succeq_{c_1} \emptyset$;
- $\succeq_{c_2}$: $\{i_1\} \succeq_{c_2} \{i_2\} \succeq_{c_2} \emptyset$

The only stable matching is $(c_1, i_1), (c_2, i_2)$.

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- $\succeq_{c_1}: \{i_1, i_2\} \succeq_{c_1} \{i_2\} \succeq_{c_1} \{i_1\} \succeq_{c_1} \emptyset$;
- $\succeq_{c_2}: \{i_1\} \succeq_{c_2} \{i_2\} \succeq_{c_2} \emptyset$

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Question

Is there any way for college c_1 to manipulate to receive a better matching?

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- $\succeq_{c_1}$: $\{i_1, i_2\} \succeq_{c_1} \{i_2\} \succeq_{c_1} \{i_1\} \succeq_{c_1} \emptyset$;
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The only stable matching is $(c_1, i_1), (c_2, i_2)$.

Question

Is there any way for college c_1 to manipulate to receive a better matching?

- Yes, e.g. $\succeq'_{c_1}$: $\{i_2\} \succeq'_{c_1} \emptyset \succeq'_{c_1} \{i_1, i_2\} \succeq'_{c_1} \{i_1\}$

Advanced Reading

- *Matching Markets: Theory and Practice* by Atila Abdulkadirog and Tayfun Sonmez