TA Lecture 06 - Continuous Random Variables

Nov 19-20

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Outline

HW 6

Challenge Questions of HW6-7

Problem 1

Suppose a fair coin is tossed repeatedly, and we obtain a sequence of H and T(H denotes Head and T denotes Tail). Let N denote the number of tosses to observe the first occurrence of the pattern "HH". Find the PMF of N.

Problem 1 Solution

$P_0 = P_1 = 0$

$P_2 = \frac{1}{4}$

$P_3 = \frac{1}{8} = P_4$

$$P_k = P(N=k, S_1 = H) + P(N=k, S_1 = T)$$

$k \geq 3$

$$= \frac{1}{4} P_{k-2} + \frac{1}{2} P_{k-1}$$

P_k 特征方程

$$\cancel{x^2 + \frac{1}{2}x = 2} \qquad x^2 = \frac{1}{4} + \frac{1}{2}x$$

$$x = \frac{1 \pm \sqrt{5}}{4}$$

$$P_k = m\left(\frac{1-\sqrt{5}}{4}\right)^k + n\left(\frac{1+\sqrt{5}}{4}\right)^k$$

$$m = \frac{5-\sqrt{5}}{10} \qquad n = \frac{5+\sqrt{5}}{10}$$

$$P_k =$$

Problem 2

The Cauchy distribution has PDF

$$f(x) = \frac{1}{\pi\left(1 + x^2\right)}$$

for all x. Find the CDF of a random variable with the Cauchy PDF. Hint: Recall that the derivative of the inverse tangent function $\tan^{-1}(x)$ is $\frac{1}{1+x^2}$.

$$F(x) = \frac{1}{\pi}\tan^{-1}(x) + \frac{1}{2}$$

Problem 2 Solution

Problem 3

The Pareto distribution with parameter $a > 0$ has PDF

$$f(x) = \frac{a}{x^{a+1}}$$

for $x \geq 1$ (and 0 otherwise). This distribution is often used in statistical modeling. Find the CDF of a Pareto r.v. with parameter a; check that it is a valid CDF.

$$F_X(x) = 1 - x^{-a}$$

Problem 3 Solution

Problem 4

The *Beta distribution* with parameters $a = 3, b = 2$ has PDF

$$f(x) = 12x^2(1-x), \text{ for } 0 < x < 1$$

Let X have this distribution.

(a) Find the CDF of X. $4x^3 - 3x^4 \quad 0 < x < 1$

(b) Find $P(0 < X < 1/2)$. $\frac{5}{16}$

(c) Find the mean and variance of X (without quoting results about the Beta distribution).

$E(X) = \frac{3}{5}$

$Var(X) = \frac{1}{25}$

Problem 4 Solution

The Exponential is the analog of the Geometric in continuous time. This problem explores the connection between Exponential and Geometric in more detail, asking what happens to a Geometric in a limit where the Bernoulli trials are performed faster and faster but with smaller and smaller success probabilities.

Suppose that Bernoulli trials are being performed in continuous time; rather than only thinking about first trial, second trial, etc., imagine that the trials take place at points on a timeline. Assume that the trials are at regularly spaced times $0, \Delta t, 2\Delta t, \ldots$, where Δt is a small positive number. Let the probability of success of each trial be $\lambda \Delta t$, where λ is a positive constant. Let G be the number of failures before the first success (in discrete time), and T be the time of the first success (in continuous time).

(a) Find a simple equation relating G to T. Hint: Draw a timeline and try out a simple example. $T = G\Delta t$

(b) Find the CDF of T. Hint: First find $P(T > t)$.

(c) Show that as $\Delta t \to 0$, the CDF of T converges to the Expo(λ) CDF, evaluating all the CDFs at a fixed $t \geq 0$.

(b) $\cancel{P}\ F_X(T) = 1 - (1-\lambda\Delta t)^{\lfloor \frac{t}{\Delta t} \rfloor + 1}$

(c) $n \to \infty,\ (1+\frac{x}{n})^n \to e^x$

Problem 5 Solution

Problem 6

The *Gumbel distribution* is the distribution of $-\log X$ with $X \sim \text{Expo}(1)$.

$e^{-e^{-t}}$

(a) Find the CDF of the Gumbel distribution.

(b) Let $X_1, X_2, \ldots$ be i.i.d. Expo(1) and let $M_n = \max(X_1, \ldots, X_n)$. Show that as $n \to \infty$, the CDF of $M_n - \log n$ converges to the Gumbel CDF.

(b) $X_i \leq x$

$P(M_n \leq m) = (1 - e^{-m})^n$

$M_n - \log n \leq t$

~~$m = \log n$~~ $+t$

$m = \log n + t$

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(HW7) Optional Challenging Problem

Let $X \sim \mathcal{N}(0,1)$, its corresponding CDF is denoted as Φ and the corresponding PDF is denoted as φ.

(a) If $x > 0$, show the following inequality holds:

$$\frac{x}{x^2+1}\varphi(x) \leq 1 - \Phi(x) \leq \frac{1}{x}\varphi(x)$$

(b) Define the function $g(x)$ as follows:

$$g(x) = \frac{2}{\sqrt{\pi}} \int_x^{\infty} e^{-t^2} dt, \forall x \geq 0$$

Show the following inequality holds:

$$g(x) \leq e^{-x^2}, \forall x \geq 0$$