EE150 -Signals and Systems, Fall 2024

Homework Set #3

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Problem 1 (10 pt)

The impulse response of a LTI system is:

$$h(t) = e^{-3t}u(t)$$

(a) Find the system function H(s)

(b) Find the output signal of the system if the input signal is

$$x(t) = 5\sin(3t) + 3\cos(5t)$$

Solution:

1. $y(t) = h(t) * e^{st} =$

$$\int_{-\infty}^{+\infty} h(\tau)e^{s(t-\tau)}d\tau = e^{st}\int_{-\infty}^{+\infty} e^{-3\tau}u(\tau)e^{-s\tau}d\tau = e^{st}\int_{0}^{+\infty} e^{-(s+3)\tau}d\tau$$

$$H(S) = \int_{0}^{+\infty} e^{-(s+3)\tau}d\tau = \frac{1}{s+3}$$

2. $x(t) = 5\sin(3t) + 3\cos(5t) = \frac{5}{2j}e^{j3t} - \frac{5}{2j}e^{-j3t} + \frac{3}{2}e^{j5t} + \frac{3}{2}e^{-j5t}$

$$y(t) = H(3j)\left(\frac{5}{2j}e^{j3t}\right) + H(-3j)\left(-\frac{5}{2j}e^{-j3t}\right) + H(5j)\left(\frac{3}{2}e^{j5t}\right) + H(-5j)\left(\frac{3}{2}e^{-j5t}\right)$$
$$= \frac{5e^{j3t}}{-6+6j} - \frac{5e^{-j3t}}{6+6j} + \frac{3e^{-j5t}}{6-10j} + \frac{3e^{j5t}}{6+10j}$$

Problem 2 (15 pt)

(a) A continuous-time periodic signal x(t) is real valued and has a fundamental period T=6. The nonzero Fourier series coefficients for x(t) are

$$a_1 = a_{-1}^* = 3\,j \quad , \quad a_3 = a_{-3} = 5$$

Express x(t) in the form:

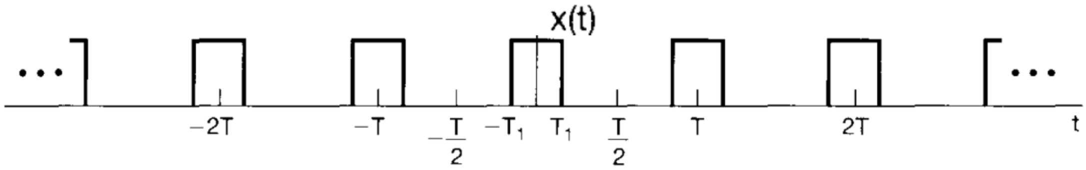
$$x(t) = \sum_{k=0}^{\infty} A_k \cos(w_k t + \emptyset_k)$$

(b) Find the Fourier series coefficients for the following signal:

$$x(t) = 4 + 2\cos\left(\frac{2\pi}{3}t\right) + 4\sin\left(\frac{4\pi}{5}t\right) + 6\cos\left(\frac{8\pi}{15}t\right)$$

(Hint: Calculate the basic signal period first)

(c) $T_1 = 1, T = 4$, The magnitude of x(t) is 1. Find the Fourier series coefficients of x(t).



Solution:

1. $x(t) = 6\cos[(\pi/3)t + \pi/2] + 10\cos(\pi t)$

2. $T_1 = \frac{2\pi}{\frac{2\pi}{3}} = 3\,, T_1 = \frac{2\pi}{\frac{4\pi}{5}} = \frac{5}{2},\ T_3 = \frac{2\pi}{\frac{8\pi}{15}} = \frac{15}{4}\quad T_0 = SCM(T_1, T_1, T_3) = 15\ \ w_0 = \frac{2\pi}{15}$

$$x(t) = 4 + 2\cos\left(\frac{2\pi}{3}t\right) + 4\sin\left(\frac{4\pi}{5}t\right) + 6\cos\left(\frac{8\pi}{15}t\right)$$

$$= 4 + e^{j\frac{2\pi}{3}t} + e^{-j\frac{2\pi}{3}t} - 2je^{j\frac{4\pi}{5}t} + 2je^{-j\frac{4\pi}{5}t} + 3e^{j\frac{8\pi}{15}t} + 3e^{-j\frac{8\pi}{15}t}$$

$$= 4 + e^{j\frac{2\pi}{15}*5t} + e^{-j\frac{2\pi}{15}*5t} - 2je^{j\frac{2\pi}{15}*6t} + 2je^{-j\frac{2\pi}{15}*6t} + 3e^{j\frac{2\pi}{15}*4t} + 3e^{-j\frac{2\pi}{15}*4t}$$

$$a_0 = 4, a_5 = a_{-5} = 1, a_6 = -2j, a_{-6} = 2j, a_4 = a_{-4} = 3$$

3. $a_0 = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)dt = \frac{1}{T}\int_{-T_1}^{T_1} 1dt = \frac{2T_1}{T} = \frac{1}{2}$

$$a_k = \frac{1}{T}\int_{-T_1}^{T_1} e^{-jkw_0t}dt = -\frac{1}{jkw_0T}e^{-jkw_0t}\Big|_{-T_1}^{T_1} = \frac{2T_1}{\mathrm{T}}\frac{\sin(kw_0T_1)}{kw_0T_1} = \frac{\sin(\frac{\pi}{2}k)}{\pi k}\quad k \neq 0$$

Problem 3 (15 pt)

Suppose that we are given the following information about a signal x[n]:

(a) x[n] is a real and odd signal.

(b) x[n] has period N = 10 and Fourier series coefficients a_k

(c) $a_{21} = 5j$

(d) $\frac{1}{10}\sum_{n=0}^{9}|x[n]|^2 = 50$

Show that $x[n] = A\sin(Bn + C)$, and specify numerical values for the constants A, B, and C.

Solution:

x[n] is a real and odd signal.$x[-n] = x[n]$ and$a_{-k} = a_k^*$

x[n] has period N = 10 $a_1 = a_{21} = 5j$ $a_{-1} = -a_1 = -5j$ $a_{-1} = a_9 = -5j$

$$\frac{1}{10}\sum_{n=0}^{9} |x[n]|^2 = \sum_{k=0}^{9} |a_k|^2 = 50$$

$\sum_{k=2}^{8} |a_k|^2 = 0$

$$x[n] = \sum_{k=<n>} a_k e^{j\left(\frac{2\pi}{N}\right)kn} = 5j\, e^{j\frac{\pi}{5}n} - 5je^{j\frac{9\pi}{5}n} = -10\sin\left[\frac{\pi}{5}n\right] = -10\sin\left[\frac{\pi}{5}n + 2\pi k\right]$$

$$A = -10, B = \frac{\pi}{5},$$

$$C = 2\pi k, \text{k be any integer}$$

Problem 4 (10 pt)

Let x(t) be a periodic signal with fundamental period T and Fourier series coefficients a_k. Derive the Fourier series coefhcients of each of the following signals in terms of a_k:

(a) Re{x(t)}

(b) x(2t-1) [for this part, first determine the period of x(2t-1)]

Solution:

1. $\mathrm{Re}\{x(t)\} = \frac{x(t)+x^*(t)}{2}$ $c_k = \frac{a_k + a_{-k}^*}{2}$

2. $T' = \frac{T}{2}$ The Fourier series coefficient of x(2t) is a_k, $b_k = a_k e^{-j\left(\frac{2\pi}{T}\right)k}$

Problem 5 (10 pt)

(a) Let $x_1(t)$ be a continuous-time periodic signal with fundamental frequency w_1, and Fourier series coefficients a_k. Given that

$$x_2(t) = x_1(2-t) + x_1(t-3)$$

how is the fundamental frequency w_2 of $x_2(t)$ related to w_1? Also, find a relationship between the Fourier series coefficients b_k of $x_2(t)$ and the coefficients a_k.

(b) Each of the two sequences $x_1[n]$ and $x_2[n]$ has a period N = 4, and the corresponding Fourier series coefficients are specified as $x_1[n] \overset{FS}{\leftrightarrow} a_k,\ x_2[n] \overset{FS}{\leftrightarrow} b_k$

Where $a_0 = 1, a_1 = 2, a_2 = 3, a_3 = 4$, $b_0 = 4, b_1 = 3, b_2 = 2, b_3 = 1$

Determine the Fourier series coefficients c_k for the signal $g[n] = x_1[n]x_2[n]$.

Solution:

1.

$$w_2 = w_1$$

$$x_1(t) \overset{FS}{\leftrightarrow} a_k,\ x_1(t+2) \overset{FS}{\leftrightarrow} a_k e^{-jkw_1(-2)} = a_k e^{j2kw_1}$$

$$x_1(-t+2) \overset{FS}{\leftrightarrow} a_{-k} e^{-j2kw_1}$$

$$x_1(t-3) \overset{FS}{\leftrightarrow} a_k e^{-j3kw_1}$$

$$x_2(t) = x_1(2-t) + x_1(t-3) \overset{FS}{\leftrightarrow} a_{-k} e^{-j2kw_1} + a_k e^{-j3kw_1}$$

$$b_k = a_{-k} e^{-j2kw_1} + a_k e^{-j3kw_1}$$

2. $c_k = \sum_{l=-\infty}^{\infty} a_k\, b_{k-l} = \sum_{l=0}^{3} a_k\, b_{k-l}$

$c_3 = 4*4+3*3+2*2+1*1 = 30$

$c_2 = 1*2+2*3+3*4+4*1 = 24$

$c_1 = 1*3+2*4+3*1+4*2 = 22$

$c_0 = 1*4+2*1+3*2+4*3 = 24$

$c_0 = 24 \quad c_1 = 22 \quad c_2 = 24 \quad c_3 = 30$

Problem 6 (20 pt)

Let

$$x(t) = \begin{cases} -2t & 0 \le t \le 1 \\ 2t-4 & 1 \le t \le 2 \end{cases}$$

be a periodic signal with fundamental period T = 2 and Fourier series coefficients a_k.

(a) Determine the value of a_0.

(b) Determine the Fourier series representation of d x(t)/dt .

(c) Use the result of part (b) and the differentiation property of the continuous-time Fourier series to help determine the Fourier series coefficients of x(t).

Solution:

1. $a_0 = -1$

2. $g(t) = \frac{dx(t)}{dt} = \begin{cases} -2 & 0 \le t \le 1 \\ 2 & 1 \le t \le 2 \end{cases}$

$$b_0 = 0 \quad b_k = \frac{2(e^{-jk\pi} - 1)}{jk\pi} \quad k \neq 0$$

$$\frac{dx(t)}{dt} = \sum_{\substack{-\infty, \\ k \neq 0}}^{+\infty} \frac{2(e^{-jk\pi} - 1)}{jk\pi} e^{jk\pi t}$$

3. $g(t) = \frac{dx(t)}{dt} \overset{FS}{\leftrightarrow} b_k = jkw_0 a_k = jk\pi a_k$

$$a_k = \frac{b_k}{jk\pi} = \frac{2(1 - e^{-jk\pi})}{k^2\pi^2}$$

Problem 7 (20 pt)

Consider a causal discrete-time LTI system :

$$h[n] = \left(\frac{1}{2}\right)^n u(n)$$

Find the Fourier series representation of the output y[n] for each of the following inputs:

(a) $x[n] = \cos(\frac{5\pi}{6} n)$

(b) $x[n] = \sin\left(\frac{\pi}{3} n\right) + \cos\left(\frac{4\pi}{3} n\right) + \sin\ (\frac{4\pi}{3} n)$

Solution:

1. $H(e^{jw}) = \frac{1}{1 - \frac{1}{2}e^{-jw}}$

$$x[n] = \cos\left(\frac{5\pi}{6} n\right) = \frac{1}{2}(e^{j\frac{5}{6}\pi n} + e^{-j\frac{5}{6}\pi n})$$

$$y[n] = \sum_{k=<N>} a_k\, H(e^{jkw_0}) e^{jkw_0 n} = \sum_{k=<N>} b_k\, e^{jk(\frac{2\pi}{N})n}$$

$$y[n] = \frac{1}{2} H\left(e^{j\frac{5\pi}{6}}\right) e^{j\frac{5\pi}{6}n} + \frac{1}{2} H\left(e^{j\frac{-5\pi}{6}}\right) e^{j\frac{-5\pi}{6}n} = \frac{1}{2} \frac{1}{1 - \frac{1}{2}e^{-j\frac{5\pi}{6}}} e^{j\frac{5\pi}{6}n} + \frac{1}{2} \frac{1}{1 - \frac{1}{2}e^{j\frac{5\pi}{6}}} e^{j\frac{-5\pi}{6}n}$$

2. $x[n] = \sin\left(\frac{\pi}{3} n\right) + \cos\left(\frac{4\pi}{3} n\right) + \sin\left(\frac{4\pi}{3} n\right)$

$= \frac{1}{2j}\left(e^{j\frac{1}{3}\pi n} - e^{-j\frac{1}{3}\pi n}\right) + \frac{1}{2}(e^{j\frac{4}{3}\pi n} + e^{-j\frac{4}{3}\pi n}) + \frac{1}{2j}(e^{j\frac{4}{3}\pi n} - e^{-j\frac{4}{3}\pi n})$

$y[n] = \frac{1}{2j} H\left(e^{j\frac{1}{3}\pi}\right) e^{j\frac{1}{3}\pi n} - \frac{1}{2j} H\left(e^{j-\frac{1}{3}\pi}\right) e^{-j\frac{1}{3}\pi n} + \left(\frac{1}{2} + \frac{1}{2j}\right) H\left(e^{j\frac{4}{3}\pi}\right) e^{j\frac{4}{3}\pi n}$

$$+\left(\frac{1}{2}-\frac{1}{2j}\right) H\left(e^{-j\frac{4}{3}\pi}\right)e^{-j\frac{4}{3}\pi n}$$

$$=\frac{1}{2j}\frac{1}{1-\frac{1}{2}e^{-j\frac{1}{3}\pi}}e^{j\frac{1}{3}\pi n}-\frac{1}{2j}\frac{1}{1-\frac{1}{2}e^{j\frac{1}{3}\pi}}e^{-j\frac{1}{3}\pi n}+\left(\frac{1}{2}+\frac{1}{2j}\right)\frac{1}{1-\frac{1}{2}e^{-j\frac{4}{3}\pi}}e^{j\frac{4}{3}\pi n}$$

$$+\left(\frac{1}{2}-\frac{1}{2j}\right)\frac{1}{1-\frac{1}{2}e^{j\frac{4}{3}\pi}}e^{-j\frac{4}{3}\pi n}$$

P.S.

The teaching assistants responsible for checking the answers are Li Jiaqi and Wang Xingbei.