# Homework 1

Due: 2025-03-16
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## 1 Game Playing

Alice and Bob are playing a game. They both have a coin. In this game, they each will chose a side of the coin and show it to each other simultaneously. The result is decided with the following rules:

1. If they show different sides, Bob wins $3 from Alice.
2. If they both show head, Alice wins $1 from Bob. If they both show tails, Alice wins $5 from Bob.

### 1.1 Mixed Strategy(1.5pt)

Suppose Alice shows head with probability $p$, Bob shows head with probability $q$. Calculate Alice's supposed payoff in terms of $p$ and $q$.

### 1.2 Nash Equilibrium(1.5pt)

Calculate the Nash equilibrium of this game. What are Alice and Bob's expected payoffs in the equilibrium? Do you think this game is fair?

## 2 True or False

Consider whether the following statements are true or false. If true, give the proof, otherwise give a counter-example

### 2.1 (1pt)

Given a two player game where the action space of both players is $\{A, B\}$. Suppose $(A, A)$ is the unique pure strategy Nash equilibrium, then action $A$ is a dominant strategy for at least one of the players.

### 2.2 (1pt)

Given a two player game where the action space of both players is $\{A, B\}$. Suppose $(A, A)$ is the unique Nash equilibrium, then action $A$ is a dominant strategy for both players.

## 3 Auctions

### 3.1 Truthful Auctions (1pt)

Explain whether the flowing auctions are truthful:

1. First price auctions.
2. Fixed price auctions.

### 3.2 Second Price Auction with Budget (2pt)

Consider a second price auction for a single indivisible item. Suppose each bidder $i$ has a value $v_i > 0$ and a budget $c_i > 0$. If a bidder wins the object and has to pay higher than the budget, the bidder will simply drop out from the auction but is charged with a small penalty $\epsilon > 0$. Compute a bid in the auction for each player $i$ which will be a weakly dominant strategy for the player.

## 4 Weighted Vickrey-Clarke-Groves Mechanism (2pt)

A mechanism $(f, p_1, \ldots, p_n)$ is called a weighted VCG mechanism if

1. $f(v_1, \ldots, v_n) \in \arg\max_{a \in A} \left(c_a + \sum_i w_i v_i(a)\right)$, where $c_a, w_1, \ldots, w_n \in \mathbb{R}^+$;
2. for some functions $h_1, \ldots, h_n$, where $h_i : V_{-i} \mapsto \mathbb{R}$, we have that for all $v_1 \in V_1, \ldots, v_n \in V_n$ : $p_i(v_1, \ldots, v_n) = h_i(v_{-i}) - c_a/w_i - \sum_{j \neq i}(w_j/w_i)v_j(a)$

Prove that the weighted VCG mechanism is incentive compatible.