Ch.1 *Overview*

Part III *Systems Classification and Properties*

Lecturer: Yijie Mao

Outline

- **Continuous-time and Discrete-time Systems**
  - Continuous-time and Discrete-time Systems
  - Interconnections of systems

- **Basic System Properties**
  - Systems with and without memory
  - Invertibility and inverse system
  - *Causal and Non-causal Systems*
  - Stability
  - Time-Invariance
  - Linearity

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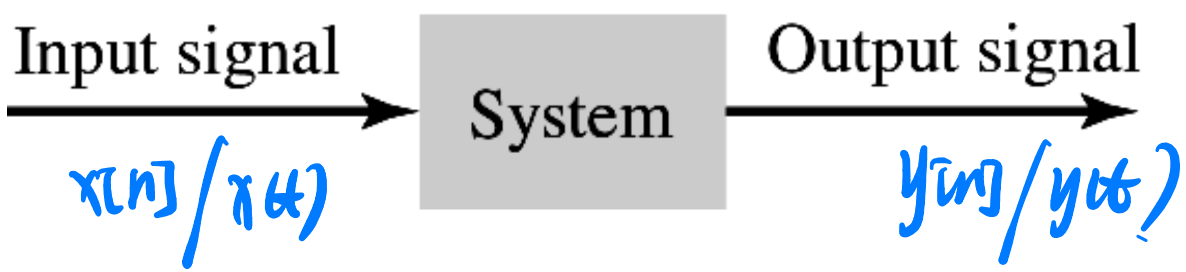
System Representation

- A system is formally defined as an entity that manipulates one or more signals to accomplish a function, thereby yielding new signals.

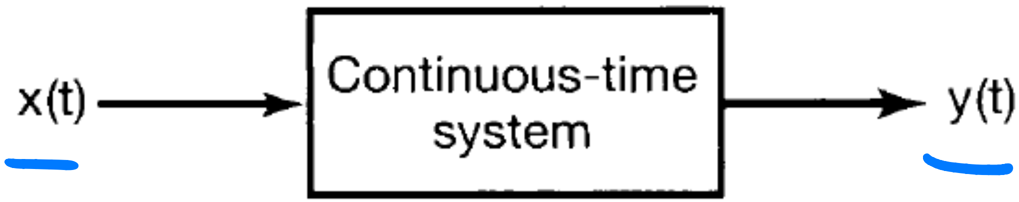
- The system can be viewed as an **interconnection of operations** that transforms an input signal *x* into an output signal *y* with properties different from those of *x*.



Continuous-time and Discrete-time Systems

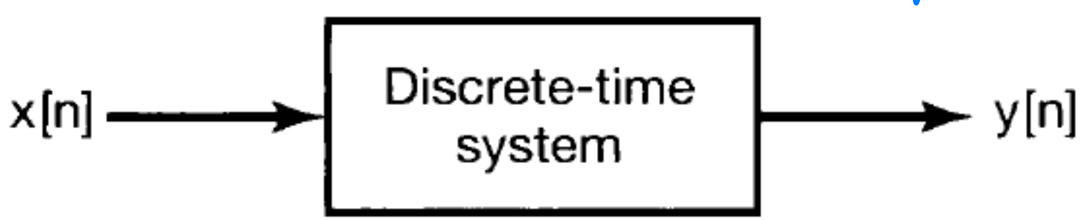
- ***Continuous-time system***: the input x and output y are continuous-time signals.

$x(t) \rightarrow y(t)$



- ***Discrete-time system***: the input x and output y are discrete-time signals

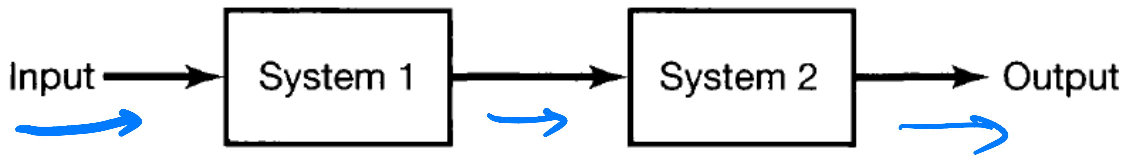
$x[n] \rightarrow y[n]$



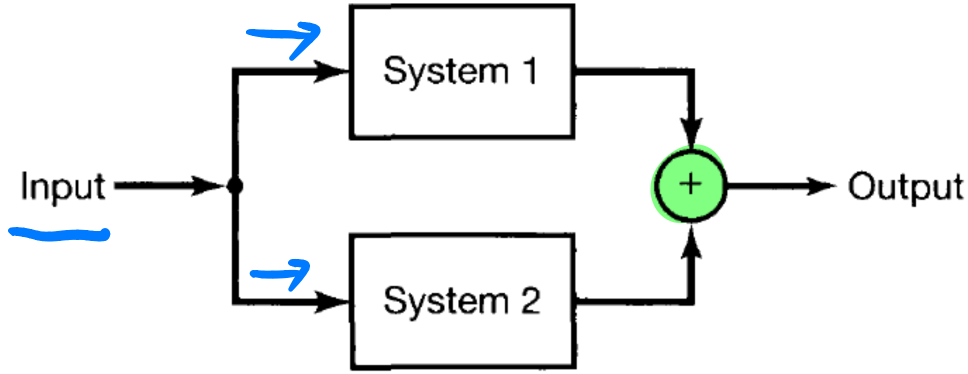
Continuous-time and Discrete-time Systems

Interconnections of systems

- **Cascade (Series):** the output of System 1 is the input of System 2.



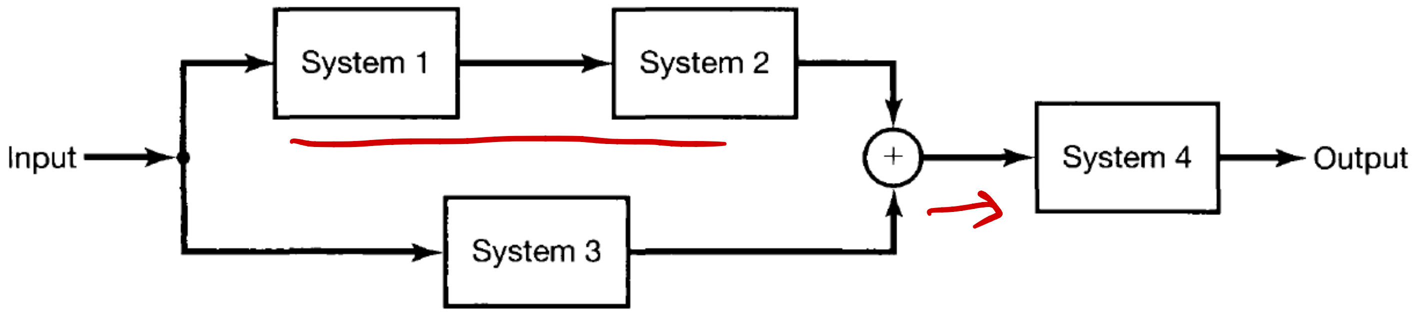
- **Parallel:** the same input is applied to Systems 1 and 2; the final output is the sum of the outputs of Systems 1 and 2.



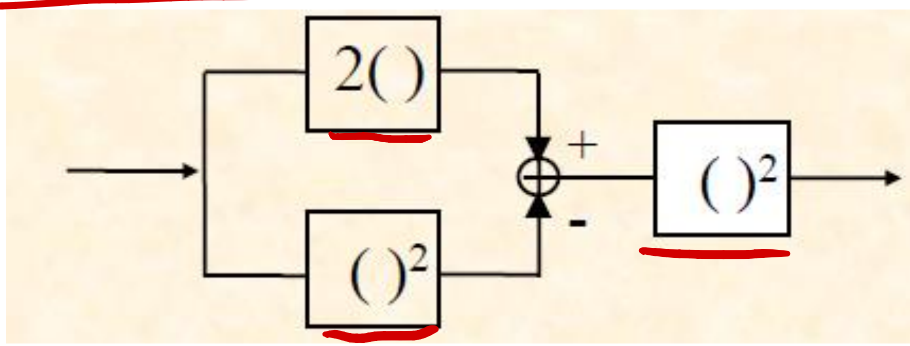
Continuous-time and Discrete-time Systems

Interconnections of systems

- **Series/Parallel**



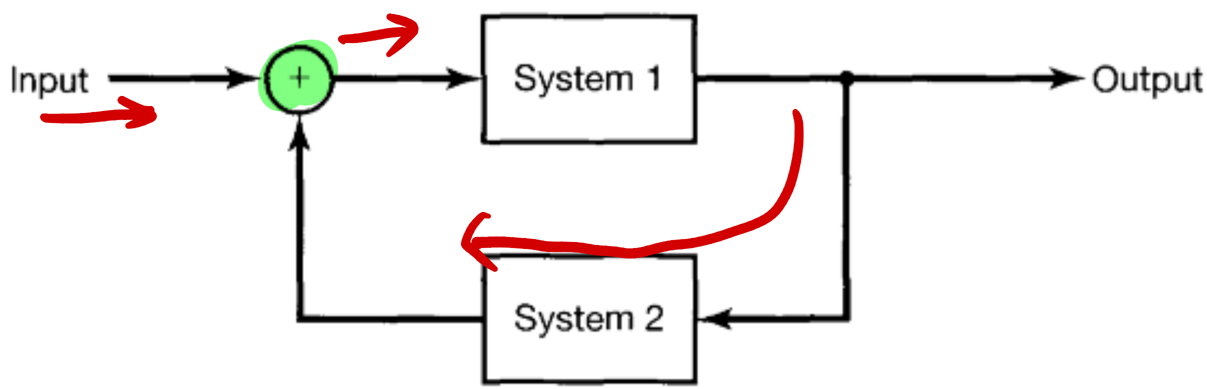
Ex. $y[n] = (2x[n] - x[n]^2)^2$

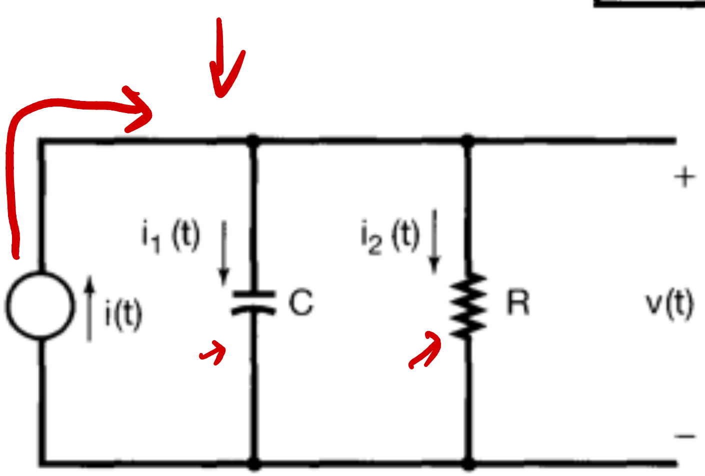


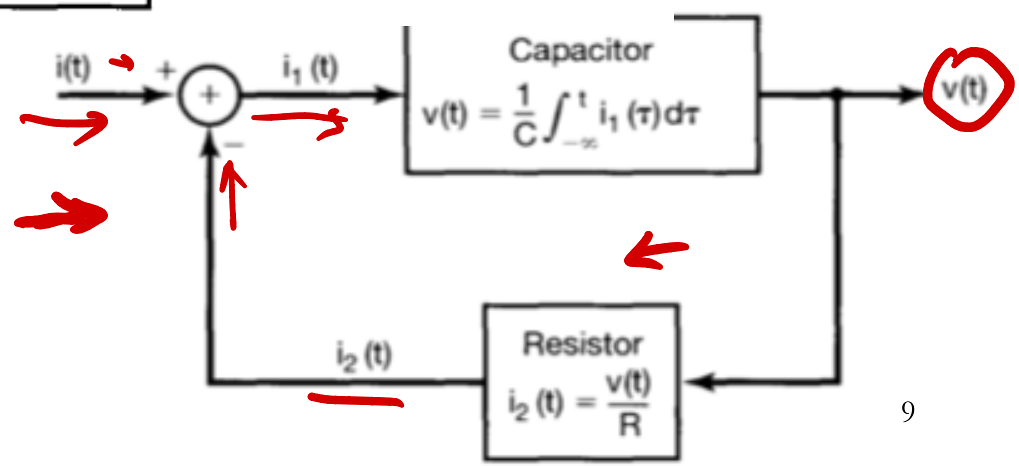
Continuous-time and Discrete-time Systems

Interconnections of systems

- **Feedback:** Output of System 1 is the input to System 2; Output of System 2 is fed back and added to the external input to produce the actual input to System 1.







Outline

- **Continuous-time and Discrete-time Systems**
  - Continuous-time and Discrete-time Systems
  - Interconnections of systems

- **Basic System Properties**
  - Systems with and without memory
  - Invertibility and inverse system
  - *Causal and Non-causal Systems*
  - Stability
  - Time-Invariance
  - Linearity

Systems with and without memory

- **A system is said to be memoryless if the output at any time depends on only the input at that same time. Otherwise, the system is said to have memory.**

  - System without memory:
    - Output is dependent only on the current input
    - Examples:

$$y[n] = (2x[n] - x^2[n])^2$$

$$y(t) = Rx(t) \quad \text{Resistor}$$

$$y(t) = x(t)$$

$$y[n] = x[n]$$

Identity System

Systems with and without memory

- **A system is said to be memoryless if the output at any time depends on only the input at that same time. Otherwise, the system is said to have memory.**

  - System with memory:

    - Output is dependent on the current and past/future inputs and outputs.
    - Examples:

$$y[n] = \sum_{k=-\infty}^{n} x[k] \leftarrow \text{accumulator}$$

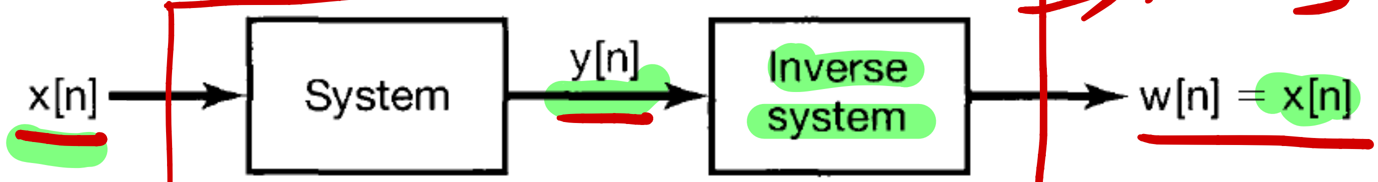
$$y[n] = x[n-1] \leftarrow \text{delay}$$

$$y(t) = \frac{1}{C}\int_{-\infty}^{t} x(\tau)\,d\tau \leftarrow \text{Capacitor}$$

Invertibility and inverse system

- A system is invertible if distinct inputs lead to distinct outputs.

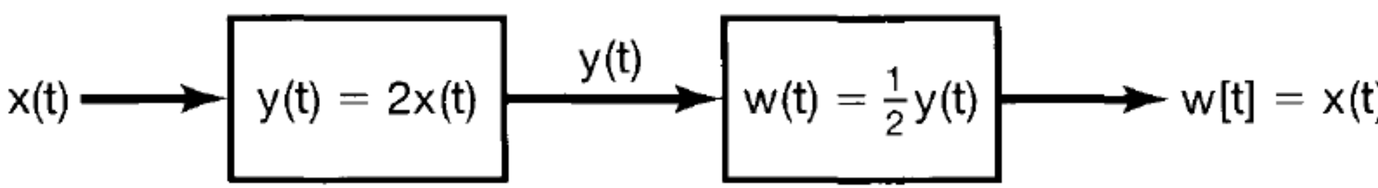
identity System



If w[n]=x[n], then system 2 is the inverse system of system 1.

- Example

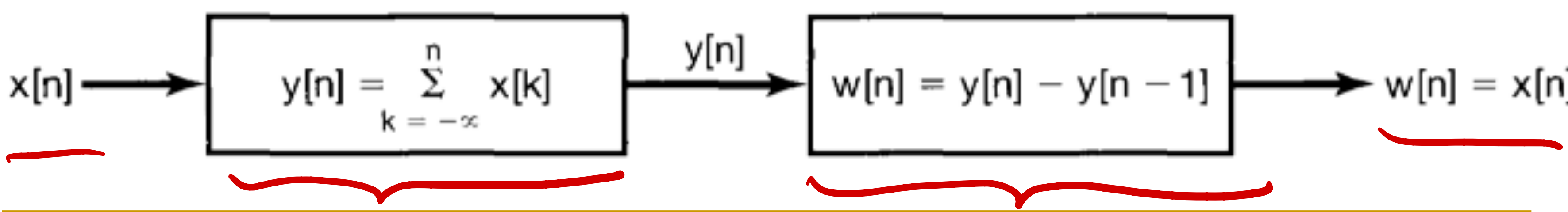
$$\boxed{y(t) = 2x(t)} \qquad w(t) = \frac{1}{2}y(t)$$



Invertibility and inverse system

- **Invertible Example:**
  - **Accumulator:** $y[n] = \sum_{k=-\infty}^{n} x[k]$
  - The difference between two successive outputs is precisely the inputs:

$$\boxed{y[n] - y[n-1]} = x[n]$$



Invertibility and inverse system

- **Noninvertible Example:**

$$y[n] = 0$$

All $x[n]$ leads to the same $y[n]$

$$y(t) = x^2(t)$$

Cannot determine the sign of the inputs

Causal and Non-causal Systems

- A system is said to be ***causal*** if the output at any time only depends on the input at the present time and before.
- A system is said to be ***non-causal*** if its output signal depends on one or more future values of the input signal.

$$y(t) = Rx(t)$$ memoryless → Causal

$$y[n] = \sum_{k=-\infty}^{n} x[k]$$ Causal.

$$y(t) = \frac{1}{C}\int_{-\infty}^{t} x(\tau)\,d\tau$$ Causal.

$$y[n] = \underline{x[n]} - \underline{x[n+1]}$$ non-causal.

$$y(t) = x(t+1)$$ non-causal.

Causal and Non-causal Systems

- **Causality Example:**

$$y[n] = x[-n]$$

non-causal.

$$y(t) = x(t)\boxed{\cos(t+1)}$$

causal

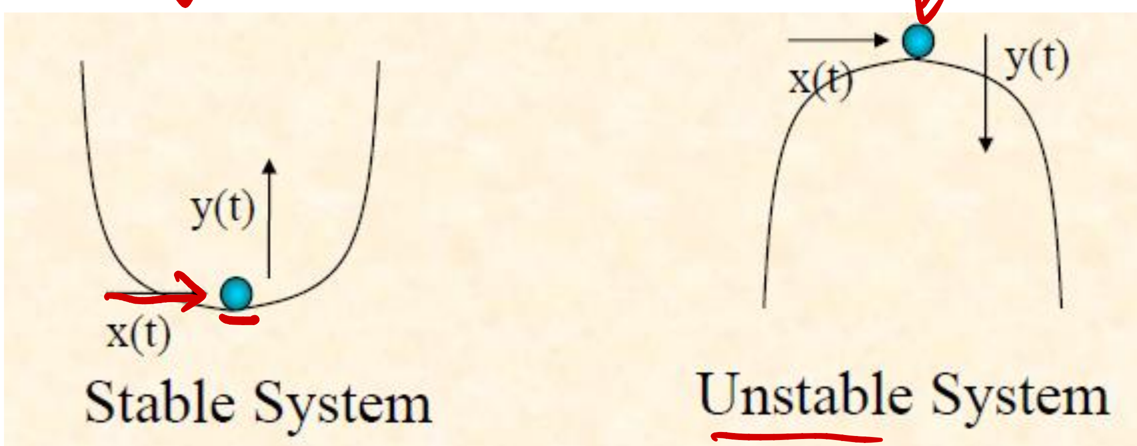
Stability

$|x(t)| < B \longrightarrow |y(t)| \le B$

- A system is stable if bounded input gives bounded output.

bounded-input bounded-output (BIBO) stable

- E.g. x(t): the horizontal force; y(t): vertical displacement



Stable System

Unstable System

Stability

- **Stability Example:**

$$y(t) = tx(t)$$

$|x(t)| < B$

$|y(t)| = \underline{|t \cdot B|}$

unstable.

$$y(t) = \underline{e^{x(t)}}$$

$|x(t)| < B \rightarrow -B < x(t) < B$

$e^{-B} < y(t) < e^{B}$

stable

Time-Invariance

- A system is time-invariant if a time-shift (advance or delay) at the input causes an identical shift at the output.
- For a continuous-time system, time-invariance exists if:

$$\text{If } \boxed{x(t) \rightarrow y(t)} \qquad \text{Then } x(t-t_0) \rightarrow y(t-t_0)$$

- For a discrete-time system, the system is time-invariant if

$$\text{If } x[n] \rightarrow y[n] \qquad \text{Then } x[n-n_0] \rightarrow y[n-n_0]$$

- A system not satisfying equation above equations is time-varying.
- time-invariance can be tested by correlating the shifted output with the output produced by a shifted input.

Time-Invariance

- Time-Invariance Example

$$y(t) = \sin[x(t)]$$

$$x_1(t) \to \boxed{\sin[x(t)]} \to y_1(t)$$

$$x_2(t) \to \boxed{\sin[x(t)]} \to y_2(t)$$

Let

$$y_1(t) = \sin[x_1(t)]$$
$$x_2(t) = x_1(t - t_0)$$

Then

$$y_2(t) = \sin[x_2(t)] = \sin[x_1(t - t_0)] = y_1(t - t_0)$$

Hence, $y(t)$ is time-invariant (T.I.)

Time-Invariance

- Time-Variance Example

$$y[n] = nx[n]$$

$$x_1[n] \rightarrow \boxed{nx[n]} \rightarrow y_1[n]$$

$$x_2[n] \rightarrow \boxed{nx[n]} \rightarrow y_2[n]$$

Let

$$y_1[n] = n \cdot x_1[n]$$

$$x_2[n] = x_1[n - n_0]$$

Then

$$y_2[n] = n \cdot x_2[n] = n \cdot x_1[n - n_0]$$

However

$$y_1[n - n_0] = (n - n_0) \cdot x_1[n - n_0] \neq y_2[n]$$

Hence, $y[n]$ is not time-invariant (T.I.)

Time-Invariance

- Time-Variance Example

$$y(t) = x(2t)$$

$x_1(2t)$

$$x_1(t) \rightarrow \boxed{x(2t)} \rightarrow y_1(t)$$

$$x_2(t) \rightarrow \boxed{x(2t)} \rightarrow y_2(t)$$

$y_1(t-t_0)$

Let

$$y_1(t) = x_1(2t)$$

$$x_2(t) = \boxed{x_1(t - t_0)}$$

Then

$$y_2(t) = \underline{x_2(2t)} = x_1(2t - t_0)$$

However

$$\boxed{y_1(t - t_0)} = x_1(2(t - t_0)) \neq \underline{y_2(t)}$$

Hence, $y(t)$ is not time-invariant (T.I.)

Linearity

- If a system is ***linear***, it has to satisfy the following two conditions:
  - **Additivity**

    The response to $x_1(t) + x_2(t)$ is $y_1(t) + y_2(t)$
  - **Scaling/Homogeneity**

    complex constant

    The response to $a \cdot x_1(t)$ is $a \cdot y_1(t)$
- Combine the above two properties, we can conclude

$$ax_1(t) + bx_2(t) \rightarrow ay_1(t) + by_2(t)$$

Superposition property (additivity and homogeneity)

- For discrete-time:

$$ax_1[n] + bx_2[n] \rightarrow ay_1[n] + by_2[n]$$

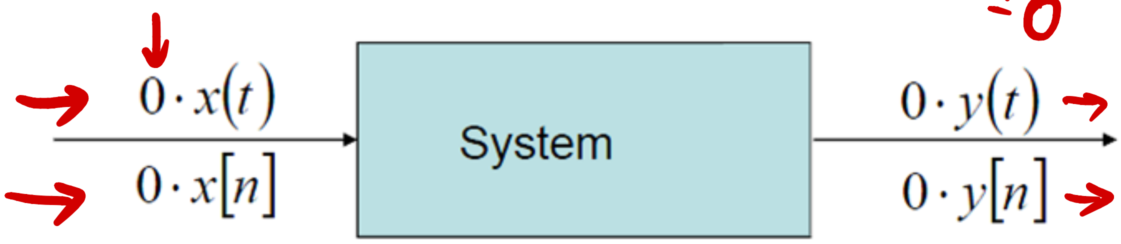
Linearity

- If linear, zero input gives zero output.

$a x_1(t) \rightarrow a y_1(t)$

$= 0$

zero in = zero out



Linear-system

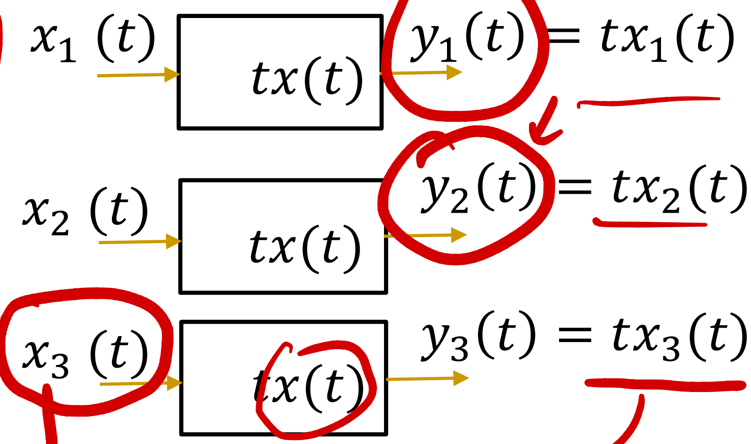
- Question: Is $y[n] = 2x[n] + 3$ linear? X
- Answer: No, because it violates zero-in zero-out property.
- However, this system is an "incremental linear system": difference of output is a linear function of difference of input.

$$y_1[n] - y_2[n] = 2x_1[n] + 3 - (2x_2[n] + 3) = 2(x_1[n] - x_2[n])$$

Linear and Nonlinear Systems

- Linearity Example

$$y(t) = tx(t)$$



Let $\quad y_1(t) = tx_1(t)$

$$y_2(t) = tx_2(t)$$

$$x_3(t) = ax_1(t) + bx_2(t)$$

Then $\quad y_3(t) = f\{x_3(t)\} = t[ax_1(t) + bx_2(t)]$

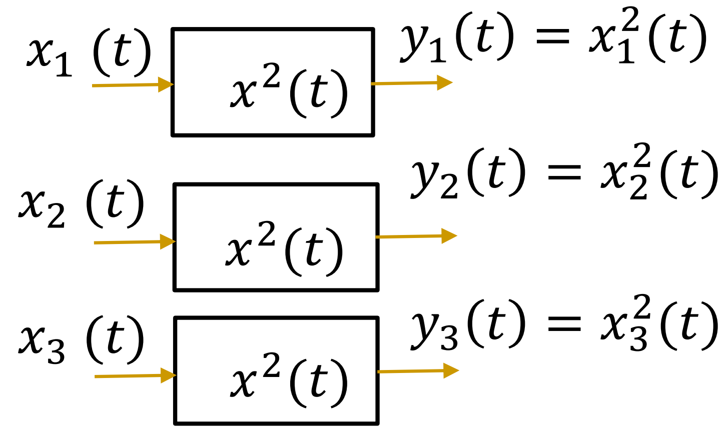
Since

$$y_3'(t) = [ay_1(t) + by_2(t)] = atx_1(t) + btx_1(t) = y_3(t)$$

Hence, $y[n]$ is linear

Linear and Nonlinear Systems

- Linearity Example



$$y(t) = x^2(t)$$

Let
$$y_1(t) = x_1^2(t)$$
$$y_2(t) = x_2^2(t)$$
$$x_3(t) = ax_1(t) + bx_2(t)$$

Then $\quad y_3\ (t) = f\{x_3(t)\} = [ax_1(t) + bx_2(t)]^2$

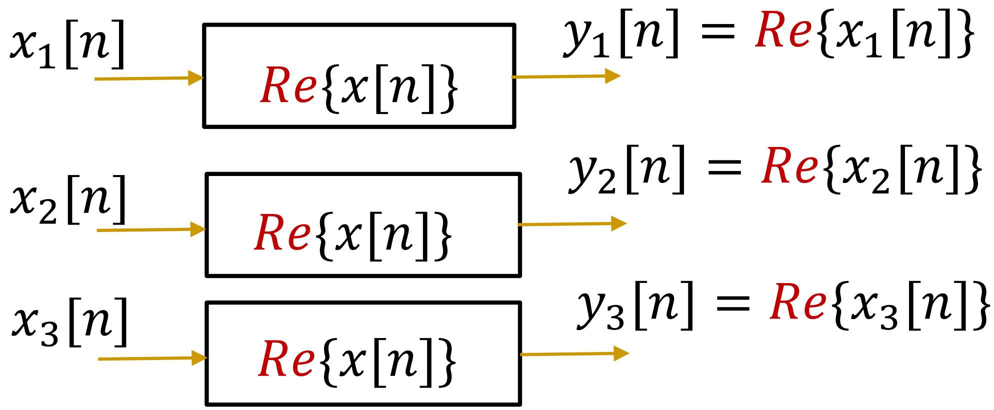
Since
$$y_3'(t) = ay_1(t) + by_2(t) = ax_1^2(t) + \mathrm{b}x_2^2(t) \neq y_3\ (t)$$

Hence, $y(t)$ is non-linear

Linear and Nonlinear Systems

- Linearity Example

$$y[n] = Re\{x[n]\}$$



Let $y_1[n] = Re\{x_1[n]\}$

$y_2[n] = Re\{x_2[n]\}$

$x_3[n] = ax_1[n] + bx_2[n]$

Then $y_3[n] = f\{x_3[n]\} = Re\{ax_1[n] + bx_2[n]\}$

Since

$$y_3'[n] = ay_1[n] + by_2[n] = aRe\{x_1[n]\} + bRe\{x_2[n]\} \neq y_3[n]$$

Hence, $y[n]$ is non-linear

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Reference in textbook: 1.5, 1.6