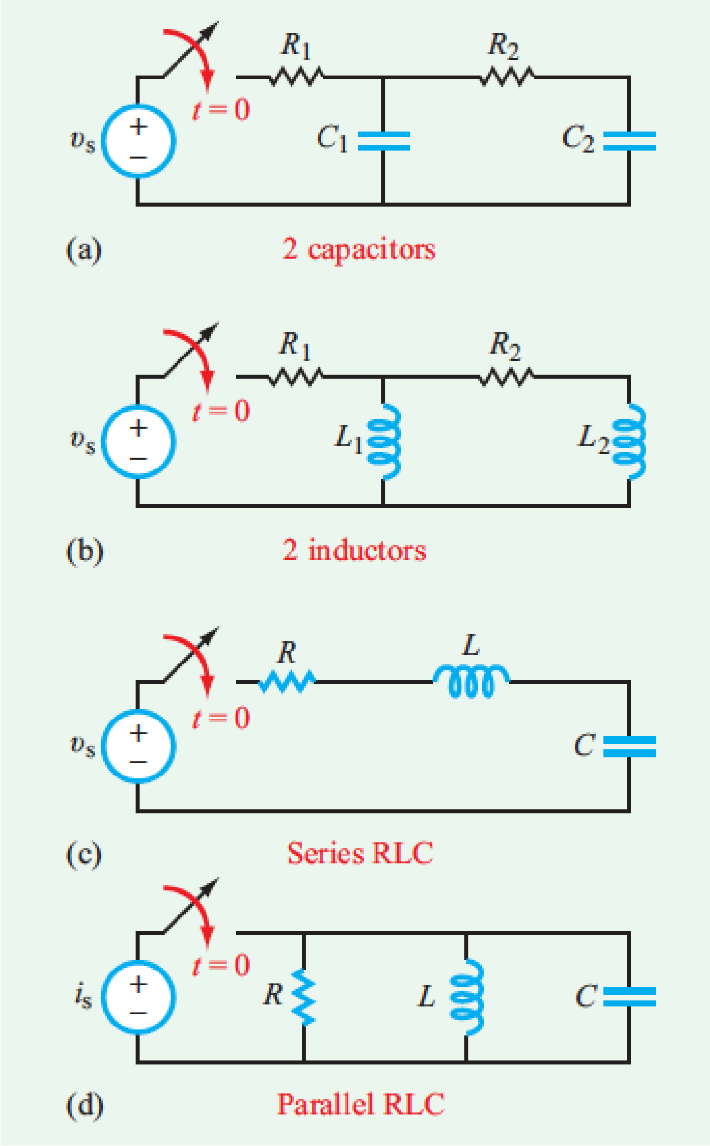
Lecture 6

- Second-Order Circuits

Second-Order Circuits

- **Two** energy storage elements
- **Analysis**: basically determine voltage or current as a function of time
- A second-order circuit is characterized by a second-order differential equation.



[Source: Berkeley]

Outline

- Natural Response Series/Parallel RLC circuit

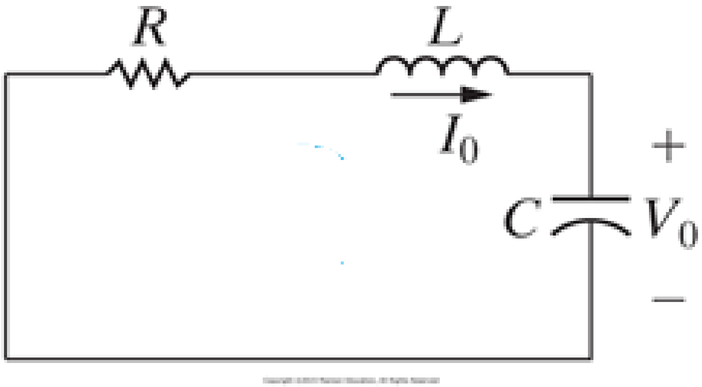
  Source-free

- Step Response of a Series/Parallel RLC Circuit

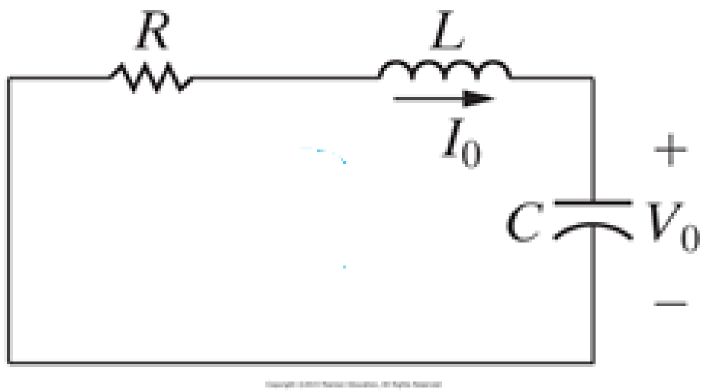
  With Independent Source

- General 2nd-order circuits

Source-Free Series RLC Circuit



Source-Free Series RLC Circuit



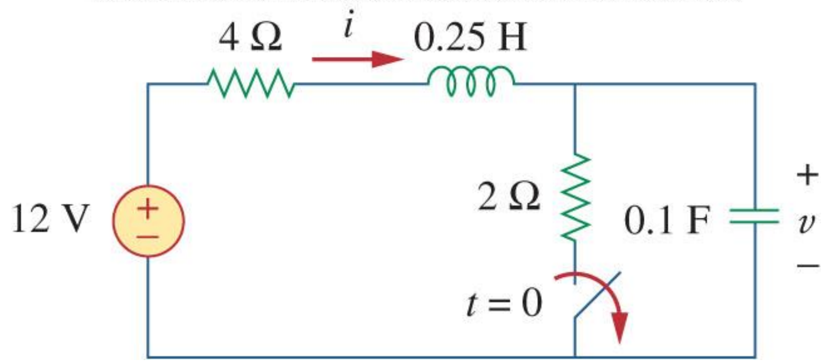
$$\frac{d^2v}{d^2t}+\frac{R}{L}\frac{dv}{dt}+\frac{1}{\mathrm{LC}}v=0$$

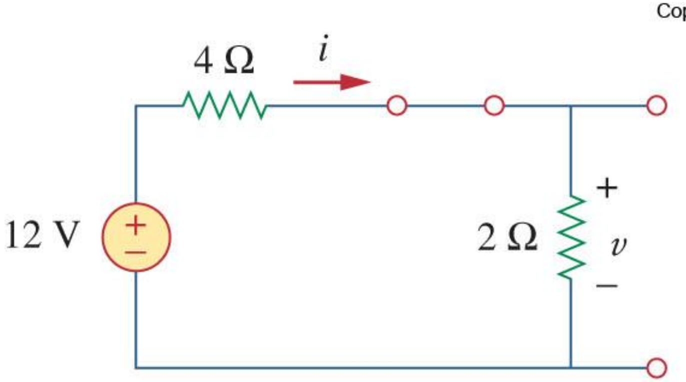
$$\boxed{s^2+\frac{R}{L}s+\frac{1}{LC}=0}$$

$$s=-\frac{R}{2L}\pm\sqrt{\left(\frac{R}{2L}\right)^2-\frac{1}{LC}}$$

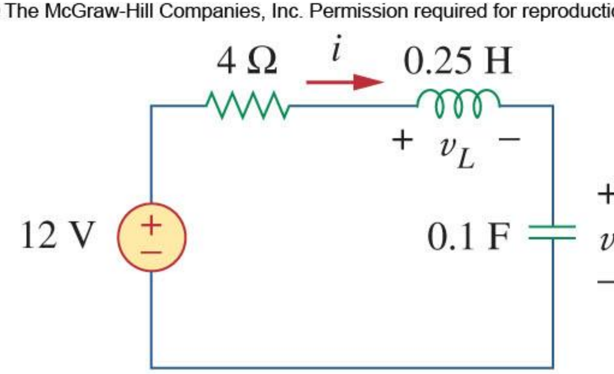
Example

- The switch has been closed for a long time. It is open at $t = 0$. Find
  - $v(0^+), i(0^+)$
  - $dv(0^+)/dt$
  - $di(0^+)/dt$





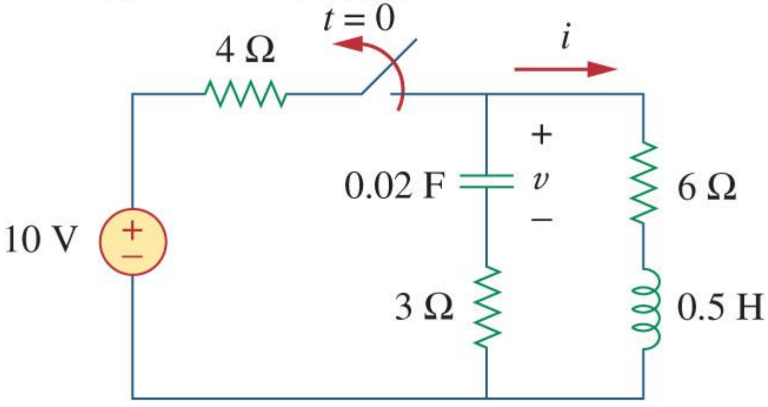
(a)



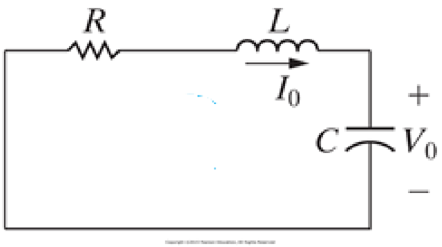
(b)

Exercise

- Assume the circuit has reached steady state at $t = 0^-$. Find
  - $v(0^+), dv(0^+)/dt$
  - $i(0^+), di(0^+)/dt$



Case 1: Overdamped ($\alpha > \omega_0$)



$$\frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{1}{LC}v = 0$$

$$s = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

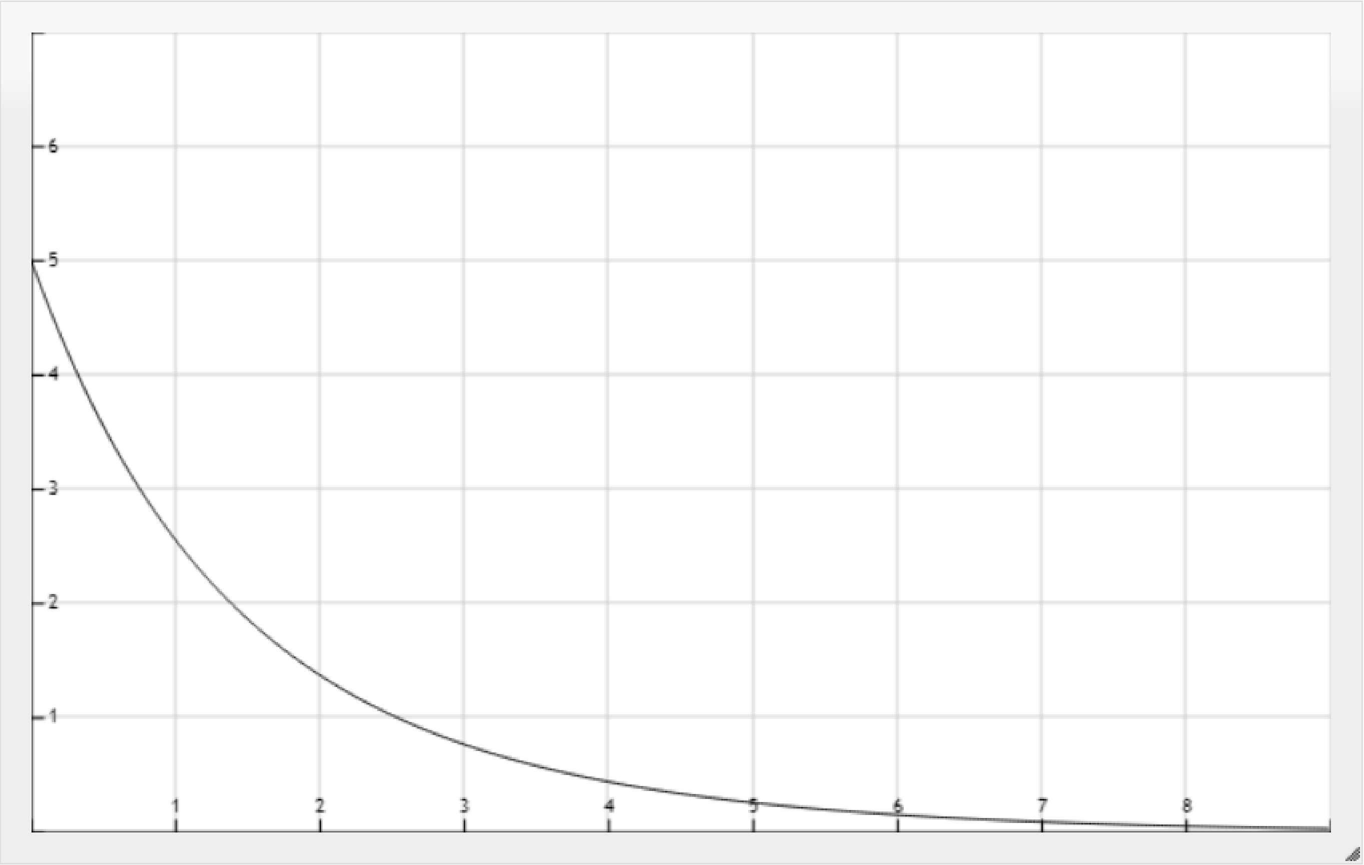
$$\alpha = \frac{R}{2L} \quad \omega_0 = \frac{1}{\sqrt{LC}}$$

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_o^2} \qquad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_o^2}$$

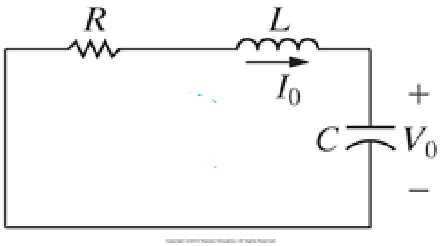
$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

An example

$V_c =$ 2e^(-t)+3e^(-t/2)



Case 2: Critically Damped ($\alpha=\omega_0$)



$$\frac{d^2v}{dt^2} + \frac{R}{L}\frac{dv}{dt} + \frac{1}{LC}v = 0 \qquad s = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

$$\alpha = \frac{R}{2L} \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_o^2} \qquad s_2 = -\alpha - \sqrt{\alpha^2 - \omega_o^2}$$

$$s_1 = s_2 = -\alpha = -\frac{R}{2L}$$

Case 2: Critically Damped ($\alpha = \omega_0$)

$$v(t) = (A_1 t + A_2)e^{-\alpha t}$$

Case 3: Underdamped ($\alpha<\omega_0$)

$$\frac{d^2v}{dt^2}+\frac{R}{L}\frac{dv}{dt}+\frac{1}{\mathrm{LC}}v=0$$

$$\alpha=\frac{R}{2L} \qquad \omega_0=\frac{1}{\sqrt{LC}}$$

$$\boxed{s=-\alpha\pm\sqrt{\alpha^2-\omega_0^2}}$$

$$s_1=-\alpha+\sqrt{\alpha^2-\omega_0^2}=-\alpha+\sqrt{-(\omega_0^2-\alpha^2)}=-\alpha+j\omega_d$$

$$s_2=-\alpha-\sqrt{\alpha^2-\omega_0^2}=-\alpha-\sqrt{-(\omega_0^2-\alpha^2)}=-\alpha-j\omega_d$$

where $j=\sqrt{-1}$ and $\omega_d=\sqrt{\omega_0^2-\alpha^2}$.

- ω_0 is often called the resonant frequency;
- ω_d is called the damping frequency.

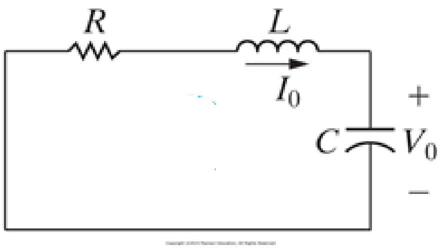
The natural response

$$v(t)=A_1e^{s_1t}+A_2e^{s_2t}$$

becomes

The natural response

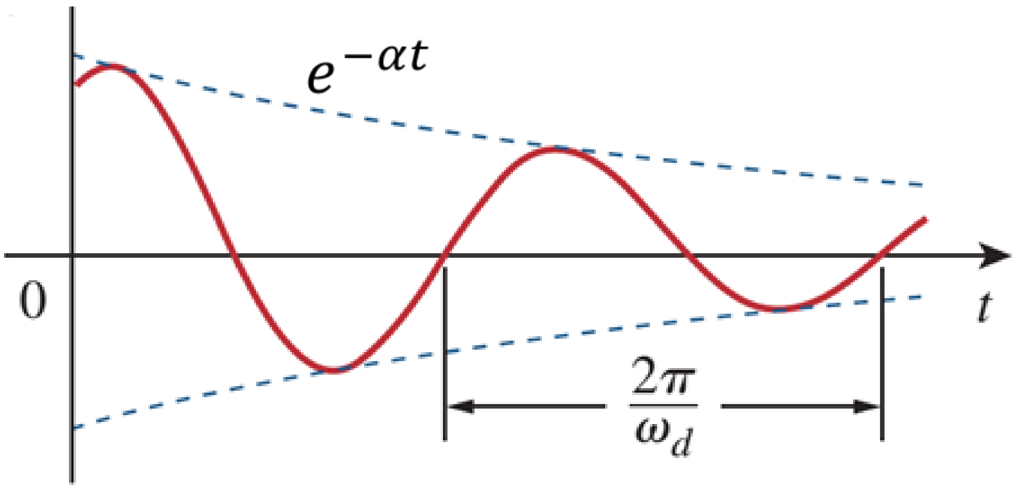
$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$



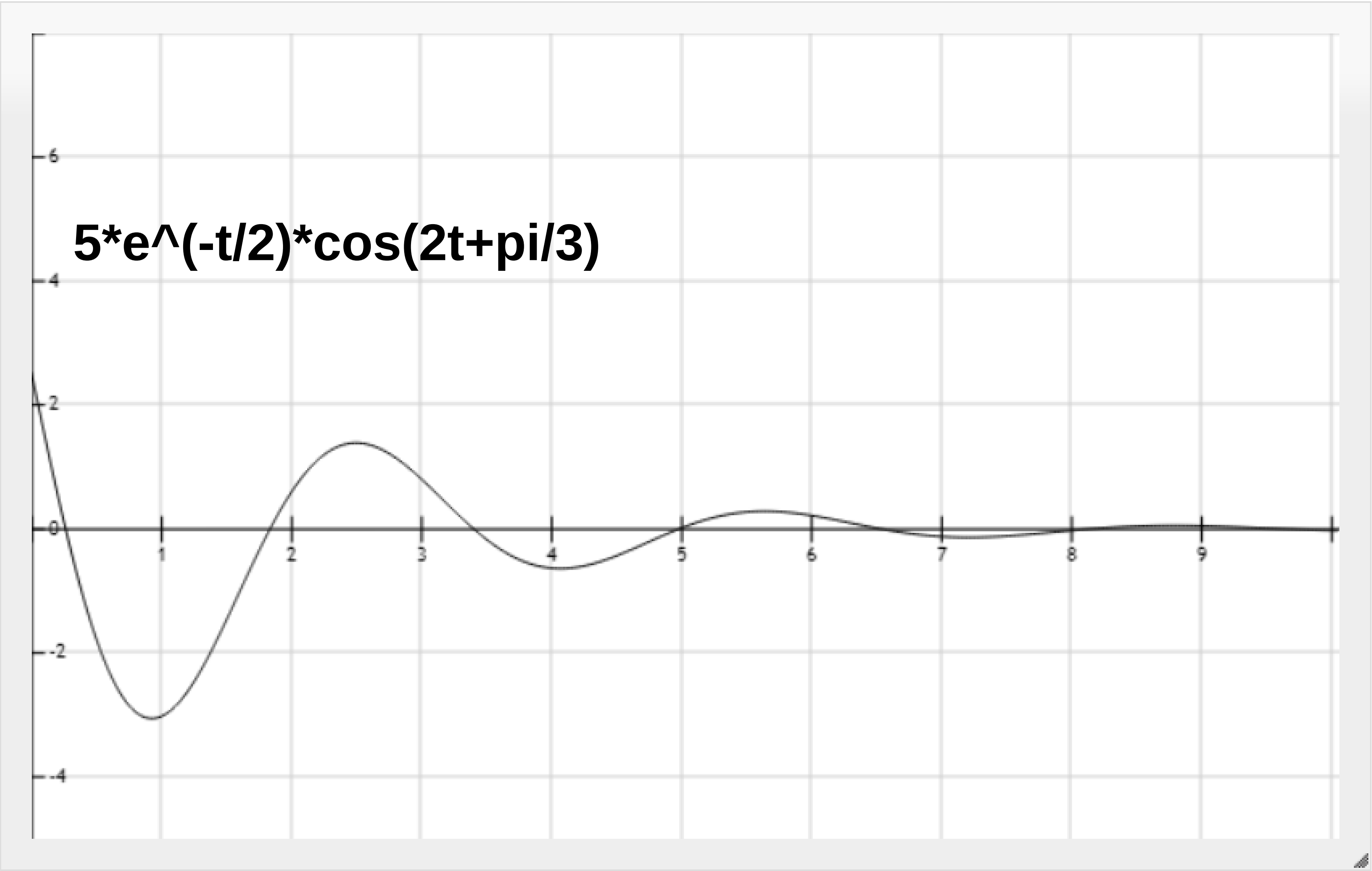
becomes

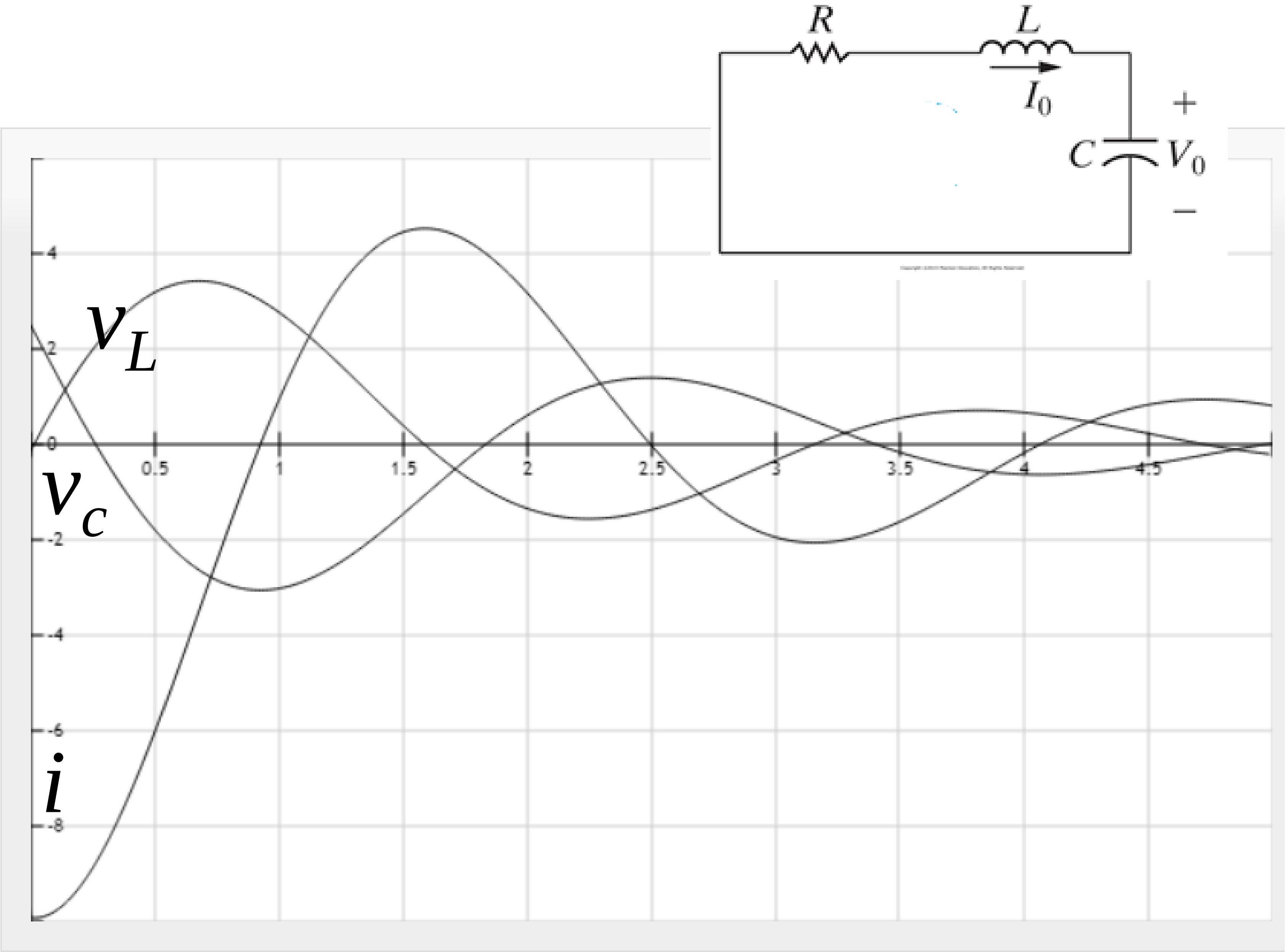
$$v(t) = e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t)$$

- Exponential $e^{-\alpha t}$ * Sine/Cosine term
  - Exponentially damped, time constant = $1/\alpha$
  - Oscillatory, period $T = \frac{2\pi}{\omega_d}$

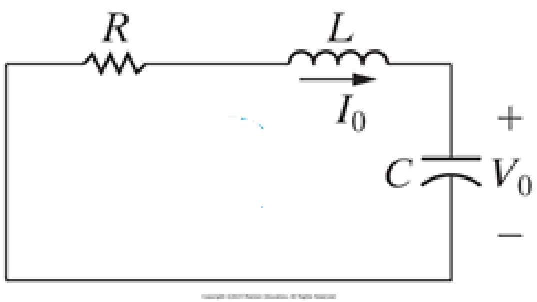


Example




R
L
I_0
+
C
V_0
−
4
2
0
-2
-4
-6
-8
0.5
1
1.5
2
2.5
3
3.5
4
4.5
v_L
v_C
i

Properties of Series RLC Network - *v(t)*



- Behavior captured by damping
  - Gradual loss of the initial stored energy
  - α determines the rate of damping

$$\alpha = \frac{R}{2L} \qquad \omega_0 = \frac{1}{\sqrt{LC}}$$

- ◆ $\alpha > \omega_0$, overdamped

$$v(t) = A_1 e^{s_1 t} + A_2 e^{s_2 t}$$

- ◆ $\alpha = \omega_0$, critically damped

$$v(t) = (A_1 t + A_2) e^{-\alpha t}$$

- ◆ $\alpha < \omega_0$, underdamped

$$v(t) = e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t)$$

[Source: Berkeley]

Example

- Find $v(t)$ in the circuit below. Assume the circuit has reached steady state at $t = 0^{-}$.