Ch.4 *The Continuous-Time Fourier Transform (CTFT)*

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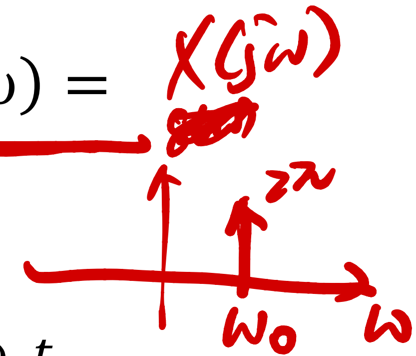
Part II *Fourier Transform for Periodic Signals*

Fourier Transform for Periodic Signal

- A period signal can be represented by a FS, but also a FT:

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t} \longrightarrow x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{X(j\omega)}_{2\pi\delta(\omega)} e^{j\omega t}\, d\omega$$

- Consider $x(t)$ and its FT, $X(j\omega)$. Assume $X(j\omega) = 2\pi\delta(\omega - \omega_0)$, find $x(t)$.



$$x(t) = \frac{1}{\cancel{2\pi}}\int_{-\infty}^{\infty} \cancel{2\pi} \cdot \underbrace{\delta(\omega - \omega_0)e^{j\omega t}}_{\omega_0}\, d\omega = e^{j\omega_0 t}$$

$$\boxed{e^{j\omega_0 t} \overset{F}{\longleftrightarrow} 2\pi\delta(\omega - \omega_0)}$$

Fourier Transform for Periodic Signal

aX(t)+

- Now for more general case

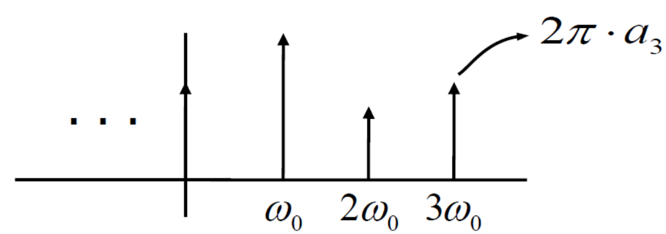
$$\underline{X(j\omega)=\sum_{k=-\infty}^{\infty}a_k\underbrace{2\pi\delta(\omega-k\omega_0)}_{\Delta}}\rightarrow x(t)=\sum_{k=-\infty}^{\infty}a_k\underbrace{e^{jk\omega_0 t}}_{\Delta}$$

exactly Fourier Series representation of a periodic signal.

- We can find the FT for a periodic signal by

$$x(t)\xleftrightarrow{\mathcal{FS}}a_k \quad \Rightarrow \quad X(j\omega)=\sum_{k=-\infty}^{\infty}a_k 2\pi\delta(\omega-k\omega_0)$$

- Note: If $x(t)$ is periodic with period T $\rightarrow X(j\omega)$ is discrete, with frequency spacing $\omega_0=\frac{2\pi}{T}$

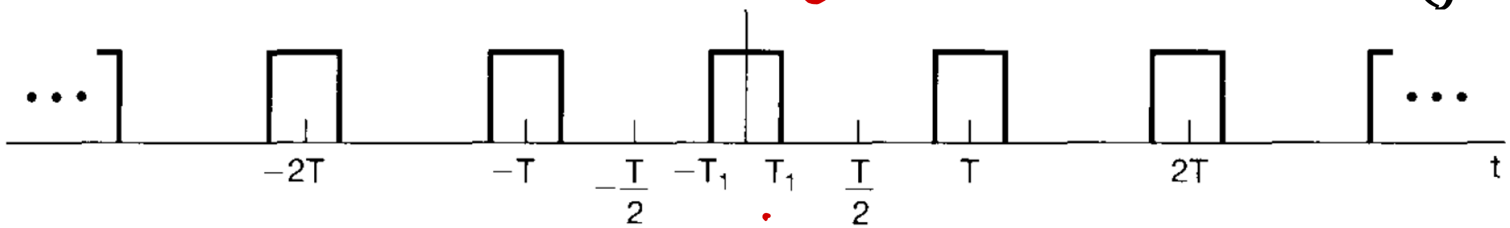


Fourier Transform for Periodic Signal

- Example 1:

$= \sum a_k e^{jk\omega_0 t}$

$X(j\omega)$?

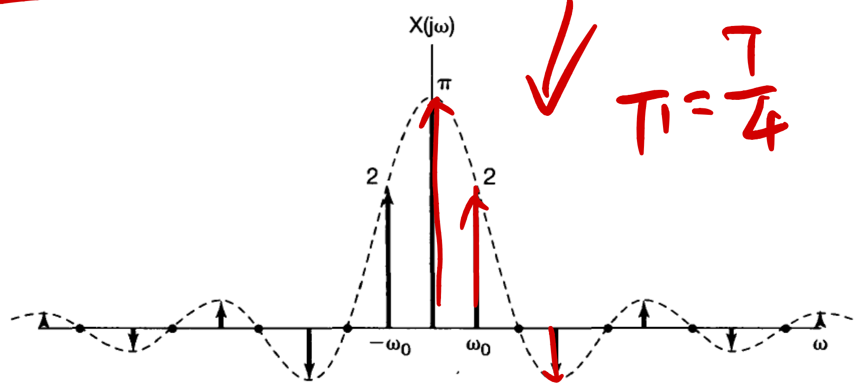


- Solution:

$$\boxed{a_k = \frac{\sin(k\omega_0 T_1)}{\pi k}}$$

$$X(j\omega) = \sum_{k=-\infty}^{\infty} a_k 2\pi\delta(\omega - k\omega_0) = \sum_{k=-\infty}^{\infty} \frac{2\sin(k\omega_0 T_1)}{k}\delta(\omega - k\omega_0)$$

$T_1 = \frac{T}{4}$



Fourier Transform for Periodic Signal

- Example 2: $x_1(t) = \sin \omega_0 t$ and $x_2(t) = \cos \omega_0 t$
  Please find $X_1(j\omega)$ and $X_2(j\omega)$.

$a_1 = \frac{1}{2j}$ $\quad a_{-1} = -\frac{1}{2j}$ $\quad k=0,\ k=\pm 2 \text{ -- } a_k = 0$

- Solution:

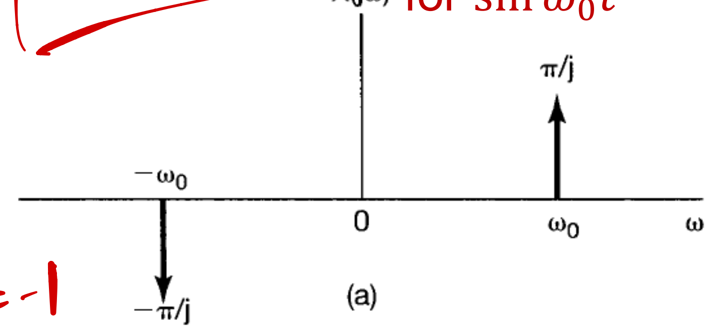
$$x_1(t) = \sin \omega_0 t = \frac{1}{2j} e^{j\omega_0 t} - \frac{1}{2j} e^{-j\omega_0 t}$$

$$X_1(j\omega) = \sum_{k=-\infty}^{\infty} a_k 2\pi\delta(\omega - k\omega_0)$$

$k=1,\ k=-1$

$$= \frac{\pi}{j}\delta(\omega - \omega_0) - \frac{\pi}{j}\delta(\omega + \omega_0)$$

for $\sin \omega_0 t$

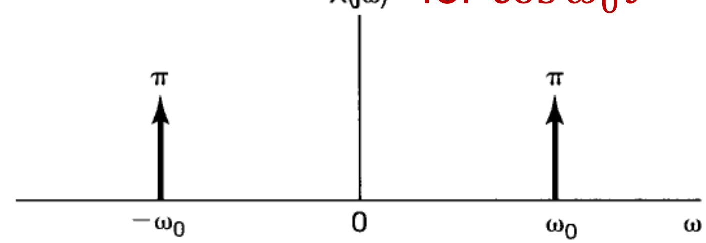


$a_1 = \frac{1}{2}$ $\quad a_{-1} = \frac{1}{2}$

$$x_2(t) = \cos \omega_0 t = \frac{1}{2} e^{j\omega_0 t} + \frac{1}{2} e^{-j\omega_0 t}$$

$$X_2(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

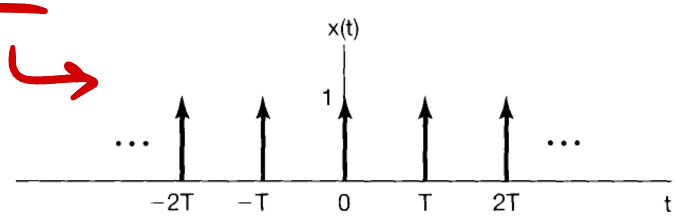
for $\cos \omega_0 t$



Fourier Transform for Periodic Signal

- Example 3: $x(t) = \sum_{k=-\infty}^{\infty} \delta(t - kT)$. Please find $X(j\omega)$.

$T,\ w_0 = \frac{2\pi}{T}$



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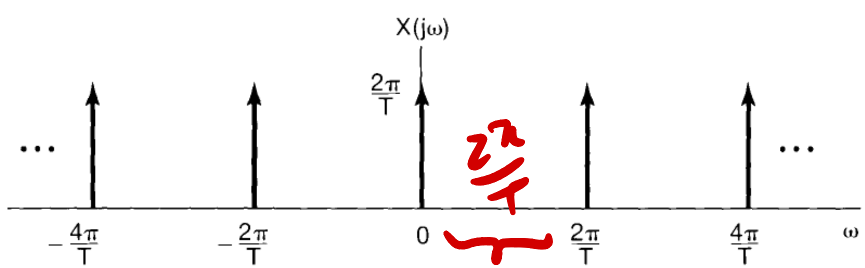
- Solution:

$$a_k = \frac{1}{T}\int_{-T/2}^{T/2} \delta(t) e^{-jk\omega_0 t} dt = \frac{1}{T}$$

$$X(j\omega) = \sum_{k=-\infty}^{\infty} a_k 2\pi\delta(\omega - k\omega_0)$$

$$= \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta(\omega - k\omega_0) = \frac{2\pi}{T}\sum_{k=-\infty}^{\infty} \delta\left(\omega - \frac{2k\pi}{T}\right)$$



Summary

- Fourier Transform for Periodic Signals

- Reference in textbook:
  - 4.2