Ch.4 *The Continuous-Time Fourier Transform (CTFT)*

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Part III *Properties of Continuous-Time Fourier Transform*

Outline

- Properties of Continuous-Time Fourier Transform
- The Convolution Property
- The Multiplication Property

Outline

- Properties of Continuous-Time Fourier Transform
    - Linearity
    - Time Shifting
    - Conjugation and Conjugate Symmetry
    - Time Reversing
    - Differentiation and Integration
    - Time and Frequency Scaling
    - Duality
    - Parseval's Relation

Notation for FT Pairs

- FT pairs:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)e^{j\omega t}d\omega \qquad X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

- Notation:

$$x(t) \xleftrightarrow{\mathcal{F}} X(j\omega)$$

$$x(t) = \mathcal{F}^{-1}\{X(j\omega)\} \qquad X(j\omega) = \mathcal{F}\{x(t)\}$$

Linearity

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega) \qquad y(t) \overset{\mathcal{F}}{\longleftrightarrow} Y(j\omega)$$

$$ax(t) + by(t) \overset{\mathcal{F}}{\longleftrightarrow} aX(j\omega) + bY(j\omega)$$

- Proof:

$$\begin{aligned} ax(t) + by(t) &= \frac{a}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega + \frac{b}{2\pi}\int_{-\infty}^{+\infty} Y(j\omega)e^{j\omega t}d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} \left(aX(j\omega)e^{j\omega t} + bY(j\omega)e^{j\omega t}\right)d\omega \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty} (aX(j\omega) + bY(j\omega))e^{j\omega t}d\omega \end{aligned}$$

Time Shifting

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega)$$

$$\Downarrow$$

$$x(t-t_0) \overset{\mathcal{F}}{\longleftrightarrow} e^{-j\omega t_0}X(j\omega)$$

- Proof:

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega$$

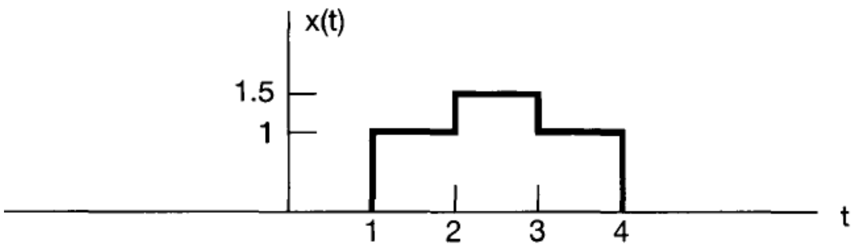
$$x(t-t_0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega(t-t_0)}d\omega = \frac{1}{2\pi}\int_{-\infty}^{+\infty} (e^{-j\omega t_0}X(j\omega))e^{j\omega t}d\omega$$

$$\mathcal{F}\{x(t)\} = X(j\omega) = |X(j\omega)|e^{j\sphericalangle X(j\omega)}$$

$$\mathcal{F}\{x(t-t_0)\} = e^{-j\omega t_0}X(j\omega) = |X(j\omega)|e^{j[\sphericalangle X(j\omega)-\omega t_0]}$$

Time Shifting

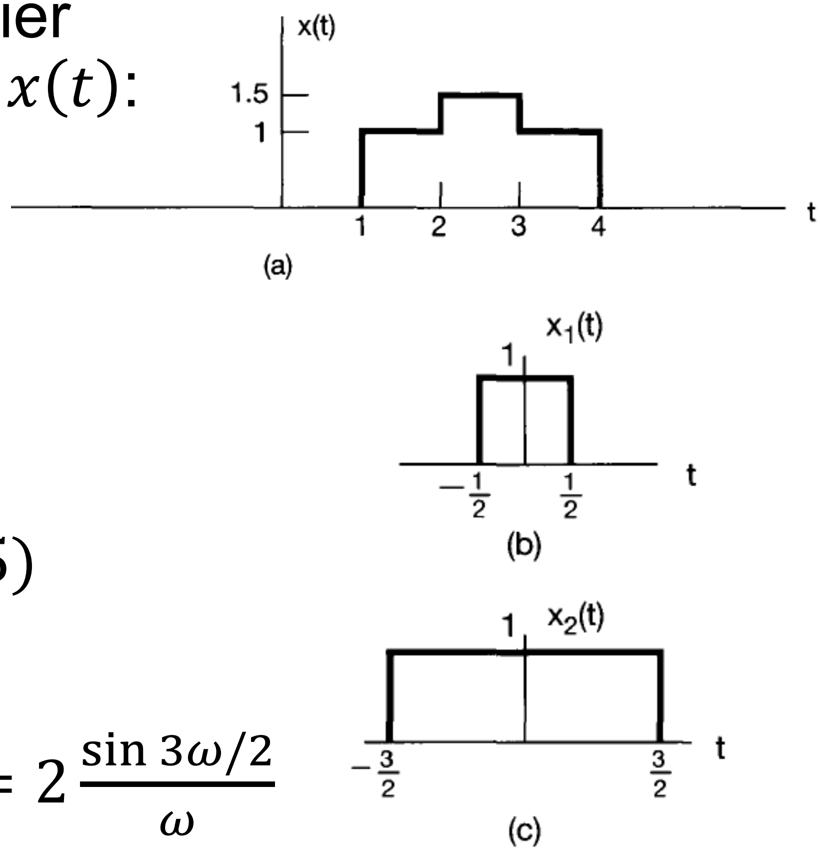
- Example 1. Please find the Fourier transform of the following signal $x(t)$:



(a)

Time Shifting

- Example 1. Please find the Fourier transform of the following signal $x(t)$:



- Solution:

$x(t)$ can be expressed as a linear combination of $x_1(t)$ and $x_2(t)$

$$x(t) = \frac{1}{2}x_1(t-2.5) + x_2(t-2.5)$$

Since:

$$X_1(j\omega) = 2\frac{\sin \omega/2}{\omega} \text{ and } X_2(j\omega) = 2\frac{\sin 3\omega/2}{\omega}$$

We have:

$$X(j\omega) = e^{-j5\omega/2}\left(\frac{\sin \omega/2 + 2\sin 3\omega/2}{\omega}\right)$$

Conjugation and Conjugate Symmetry

- Conjugation Property

$$x(t) \xleftrightarrow{\mathcal{F}} X(j\omega) \Rightarrow x^*(t) \xleftrightarrow{\mathcal{F}} X^*(-j\omega)$$

- Conjugation Symmetry if $x(t)$ is real

$$X(-j\omega) = X^*(j\omega) \Leftrightarrow \begin{gathered} \mathcal{R}e\{X(j\omega)\} = \mathcal{R}e\{X(-j\omega)\}, \\ \mathcal{I}m\{X(j\omega)\} = -\mathcal{I}m\{X(-j\omega)\} \end{gathered}$$

  - For a real-valued signal, the FT need only to be specified for positive frequencies.

Time Reversing

$$x(t) \xleftrightarrow{\mathcal{F}} X(j\omega) \Rightarrow x(-t) \xleftrightarrow{\mathcal{F}} X(-j\omega)$$

- $x(t)$ even $\Longrightarrow X(j\omega) = X(-j\omega)$, $x(t)$ real $\Longrightarrow X(-j\omega) = X^*(j\omega)$
- $x(t)$ real and even $\Longrightarrow X(j\omega)$ real and even
- $x(t)$ real and odd $\Longrightarrow X(j\omega)$ purely imaginary and odd

- If $x(t)$ real

$$\begin{aligned} x(t) &= x_e(t) + x_o(t) \\ \mathcal{F}\{x(t)\} &= \mathcal{F}\{x_e(t)\} + \mathcal{F}\{x_o(t)\} \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} \mathcal{E}v\{x(t)\} &\xleftrightarrow{\mathcal{F}} \mathcal{R}e\{X(j\omega)\} \\ \mathcal{O}d\{x(t)\} &\xleftrightarrow{\mathcal{F}} j\mathcal{I}m\{X(j\omega)\} \end{aligned}$$

Time Reversing

- Example 2. Please find the Fourier transform of the signal $x(t) = e^{-a|t|}$, where $a > 0$.

Time Reversing

- Example 2. Please find the Fourier transform of the signal $x(t) = e^{-a|t|}$, where $a > 0$.

- Solution:

Use FT properties

$$e^{-a|t|} = e^{-at}u(t) + e^{at}u(-t) = 2\mathcal{E}v\{e^{-at}u(t)\}$$

$$\mathcal{E}v\{e^{-at}u(t)\} \xleftrightarrow{\mathcal{F}} \mathcal{R}e\left\{\frac{1}{a+j\omega}\right\}$$

$$\mathcal{F}\{e^{-a|t|}\} = 2\mathcal{R}e\left\{\frac{1}{a+j\omega}\right\} = \frac{2a}{a^2+\omega^2}$$

Differentiation and Integration

- Differentiation Property

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega) \Rightarrow \frac{dx(t)}{dt} \overset{\mathcal{F}}{\longleftrightarrow} j\omega X(j\omega)$$

➢ Proof:

$$\frac{dx(t)}{dt} = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\frac{d(e^{j\omega t})}{dt}d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega)\cdot j\omega\cdot e^{j\omega t}d\omega$$

Differentiation and Integration

- Example 3. Please find the Fourier transform of the signal $x(t) = u(t)$.

Differentiation and Integration

- Integration Property

$$x(t) \xleftrightarrow{\mathcal{F}} X(j\omega)$$

$$\Downarrow$$

$$\int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{\mathcal{F}} \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$$

DC component

- Proof:

$$\mathcal{F}\left\{\int_{-\infty}^{t} x(\tau)d\tau\right\} = \int_{-\infty}^{\infty}\int_{-\infty}^{t} x(\tau)d\tau\, e^{-j\omega t}dt = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\tau)u(t-\tau)d\tau\, e^{-j\omega t}dt$$

$$= \int_{-\infty}^{\infty} x(\tau) \int_{-\infty}^{\infty} u(t-\tau)e^{-j\omega t}dt\, d\tau = \int_{-\infty}^{\infty} x(\tau)e^{-j\omega\tau}\left(\frac{1}{j\omega} + \pi\delta(\omega)\right)d\tau$$

$$= \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$$

Differentiation and Integration

- Example 3. Please find the Fourier transform of the signal $x(t) = u(t)$.

- Solution:

$$x(t) = \int_{-\infty}^{t} g(\tau)d\tau\,, \qquad \text{since} \quad g(t) = \delta(t) \xleftrightarrow{\mathcal{F}} G(jw) = 1$$

Therefore, we can use the integration property

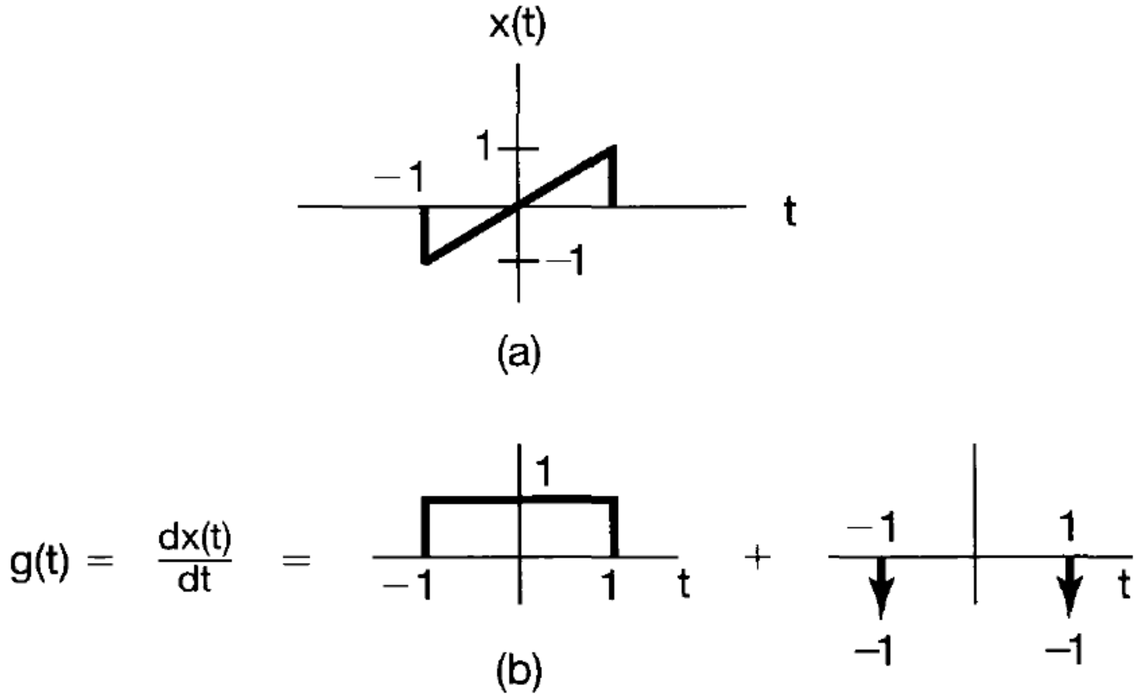
$$X(j\omega) = \frac{G(jw)}{j\omega} + \pi G(0)\delta(\omega) = \frac{1}{j\omega} + \pi\delta(\omega)$$

We can also recover $G(j\omega)$ by differential property

$$\delta(t) = \frac{du(t)}{dt} \xleftrightarrow{\mathcal{F}} j\omega\left[\frac{1}{j\omega} + \pi\delta(\omega)\right] = 1$$

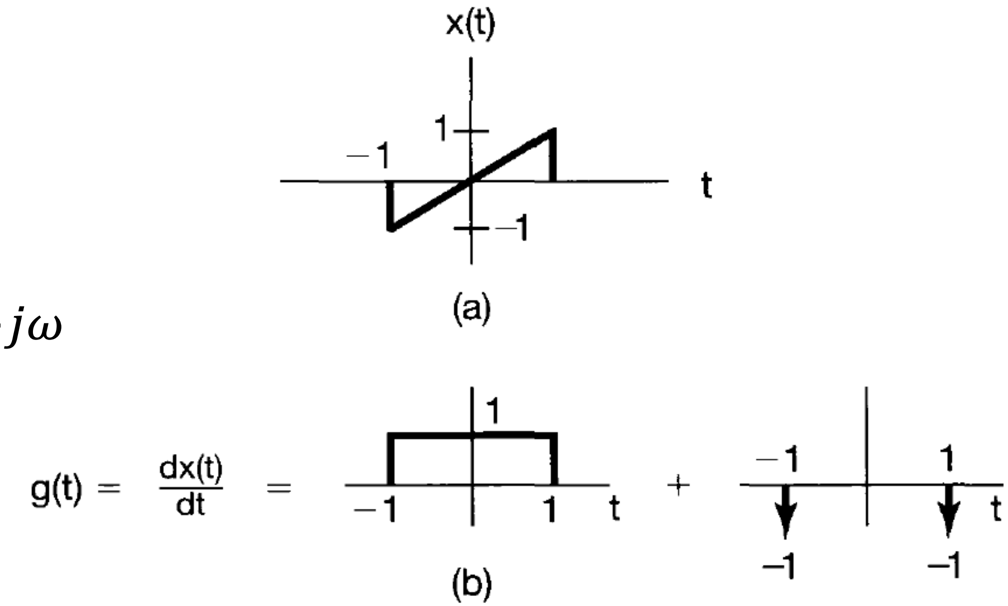
Differentiation and Integration

- Example 4. Please find the Fourier transform of the following signal $x(t)$:



Differentiation and Integration

- Example 4. Please find the Fourier transform of the following signal $x(t)$:



- Solution:

$$g(t) = \frac{d}{dt}x(t)$$

$$G(j\omega) = \frac{2\sin\omega}{\omega} - e^{j\omega} - e^{-j\omega}$$

- Use FT properties

$$X(j\omega) = \frac{G(j\omega)}{j\omega} + \pi G(0)\delta(\omega)$$
$$= \frac{2\sin\omega}{j\omega^2} - \frac{2\cos\omega}{j\omega}$$

Time and Frequency Scaling

$$x(t) \overset{\mathcal{F}}{\longleftrightarrow} X(j\omega) \Rightarrow x(at) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|} X\left(\frac{j\omega}{a}\right)$$

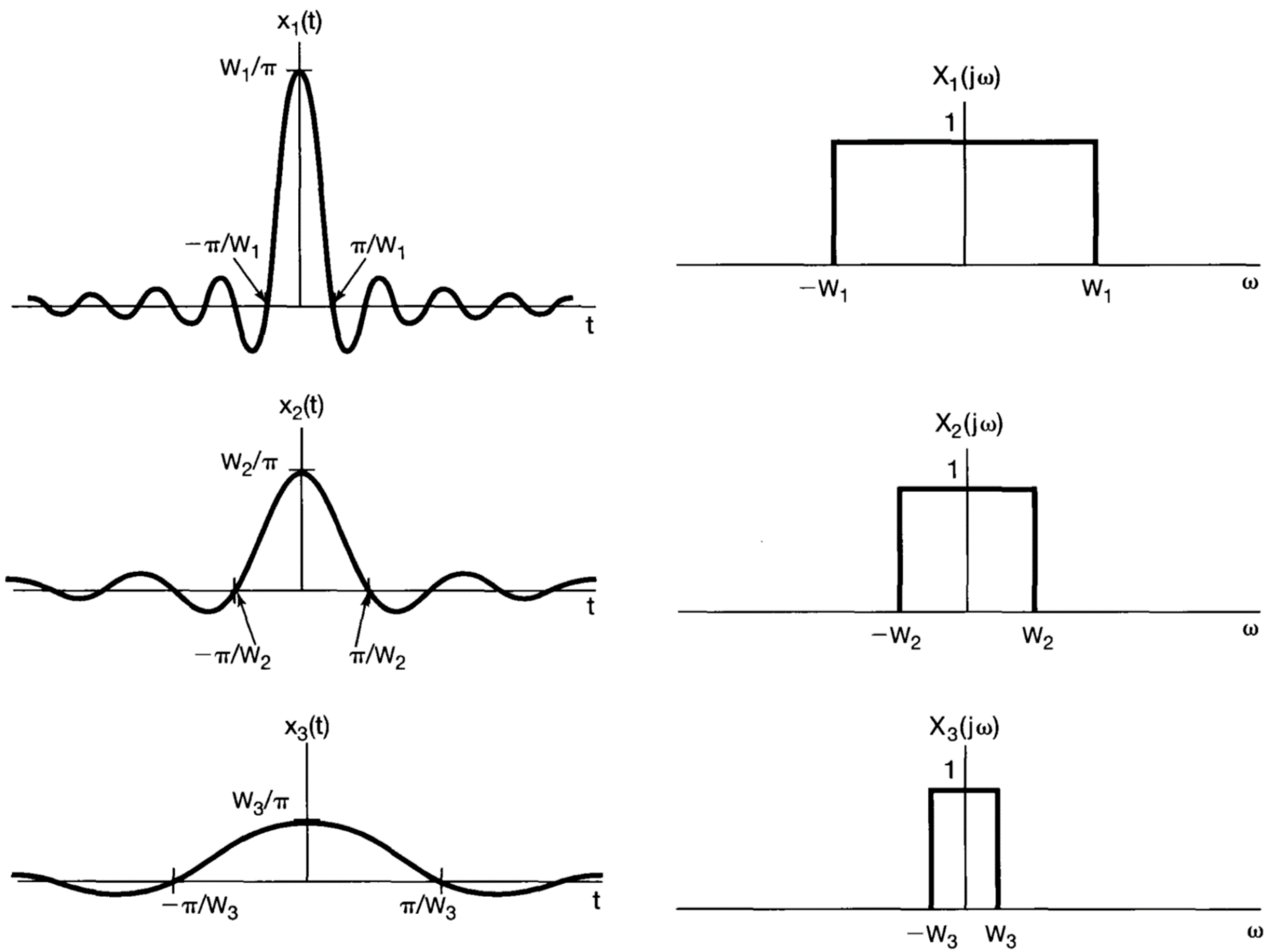
$$a \neq 0$$

- Proof:

$$\mathcal{F}\{x(at)\} = \int_{-\infty}^{+\infty} x(at) e^{-j\omega t} dt$$

$$\mathcal{F}\{x(at)\} = \begin{cases} \dfrac{1}{a} \displaystyle\int_{-\infty}^{+\infty} x(\tau) e^{-j(\omega/a)\tau} d\tau \, , a > 0 \\ -\dfrac{1}{a} \displaystyle\int_{-\infty}^{+\infty} x(\tau) e^{-j(\omega/a)\tau} d\tau , a < 0 \end{cases}$$

Time and Frequency Scaling



Duality

$$x(t) \xleftrightarrow{\mathcal{F}} X(j\omega) \quad \Rightarrow \quad X(t) \xleftrightarrow{\mathcal{F}} 2\pi x(-j\omega)$$

- Proof:

$$x(t) = \mathcal{F}^{-1}\{X(j\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega$$

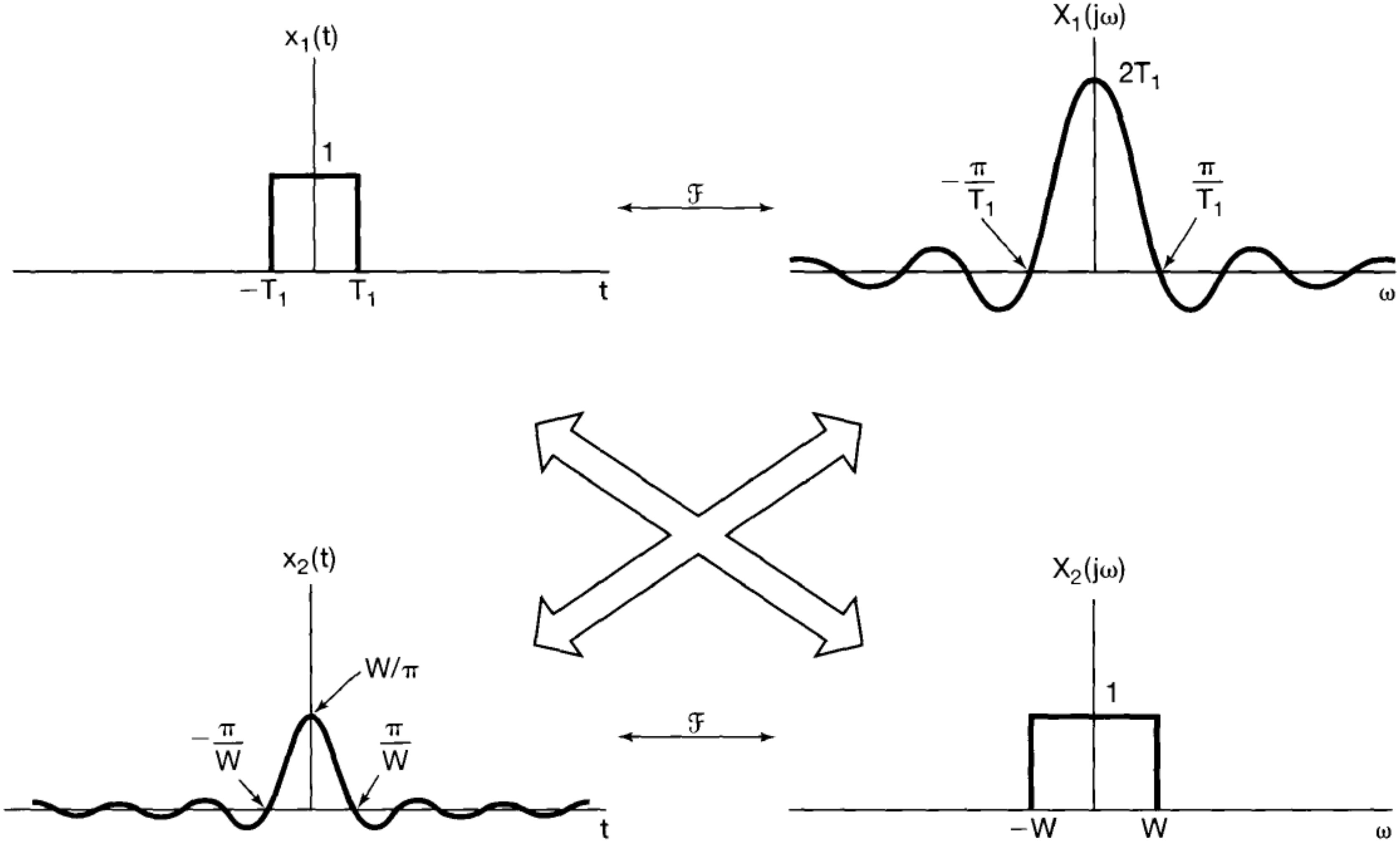
$t \rightarrow -t$

$$2\pi x(-t) = \int_{-\infty}^{+\infty} X(j\omega)e^{-j\omega t}d\omega$$

$t \rightarrow \omega$

$$2\pi x(-j\omega) = \int_{-\infty}^{+\infty} X(t)e^{-j\omega t}dt = \mathcal{F}\{X(t)\}$$

Duality



Duality

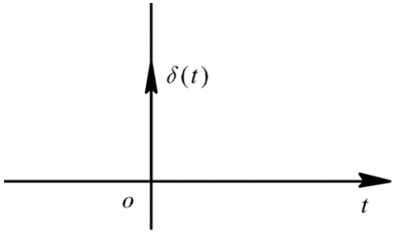
- Example 5. Please find the Fourier transform of the following signal $x(t) = \delta(t)$ and $x(t) = 1$.

- Solution:

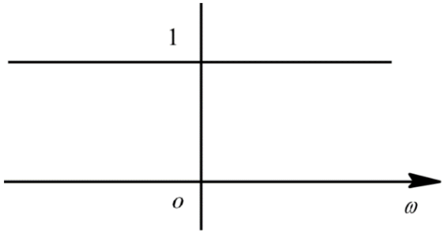
$$x(t) = \delta(t) \xleftrightarrow{\mathcal{F}} X(j\omega) = 1$$

$$x(t) = 1 \xleftrightarrow{\mathcal{F}} X(j\omega) = 2\pi\delta(\omega)$$

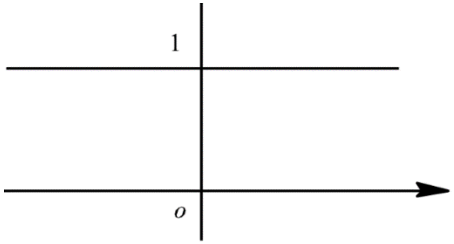
$x(t)$



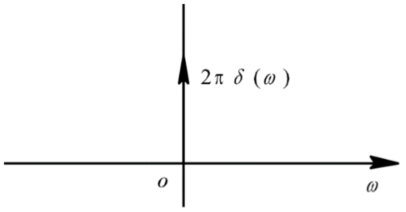
$X(j\omega)$



$x(t)$



$X(j\omega)$



- Principle

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(j\omega) \cdot e^{j\omega t}\, d\omega$$

$$x(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(t) \cdot e^{j\omega t}\, dt$$

$$2\pi \cdot x(-j\omega) = \int_{-\infty}^{\infty} X(t) \cdot e^{-j\omega t}\, dt$$

Duality

- Example 6. Please find the Fourier transform of the following signal $g(t) = \frac{2}{1+t^2}$.

- Solution:

  Calculate $G(j\omega)$ is difficult; use duality property

$$e^{-a|t|} \overset{\mathcal{F}}{\longleftrightarrow} \frac{2a}{a^2+\omega^2}$$

$$e^{-|t|} = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2}{1+\omega^2} \cdot e^{j\omega t}\, d\omega$$

$$2\pi e^{-|\omega|} = \int_{-\infty}^{\infty} \frac{2}{1+t^2} \cdot e^{-j\omega t}\, dt$$

$$\therefore G(j\omega) = 2\pi e^{-|\omega|}$$

Duality

- Duality property can determine or suggest other FT properties

$$\boxed{\frac{dx(t)}{dt} \xleftrightarrow{\mathcal{F}} j\omega X(j\omega)} \Leftrightarrow \boxed{-jtx(t) \xleftrightarrow{\mathcal{F}} \frac{dX(j\omega)}{d\omega}}$$

$$\boxed{\int_{-\infty}^{t} x(\tau)d\tau \xleftrightarrow{\mathcal{F}} \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)} \Leftrightarrow \boxed{-\frac{1}{jt}x(t) + \pi x(0)\delta(t) \xleftrightarrow{\mathcal{F}} \int_{-\infty}^{\omega} x(\eta)d\eta}$$

$$\boxed{x(t-t_0) \xleftrightarrow{\mathcal{F}} e^{-j\omega t_0}X(j\omega)} \Leftrightarrow \boxed{e^{j\omega_0 t}x(t) \xleftrightarrow{\mathcal{F}} X(j(\omega-\omega_0))}$$

Parseval's Relation

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega$$

- Proof:

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \int_{-\infty}^{+\infty} x(t)x^*(t)dt$$

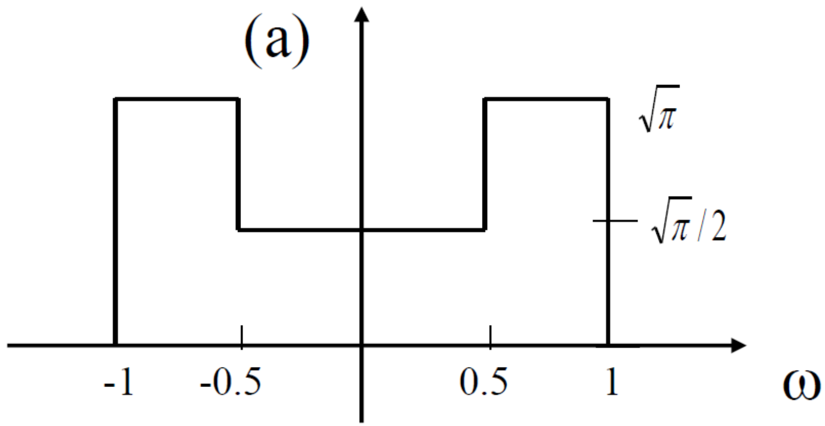
$$= \int_{-\infty}^{+\infty} x(t)\left[\frac{1}{2\pi}\int_{-\infty}^{+\infty} X^*(j\omega)e^{-j\omega t}d\omega\right]dt$$

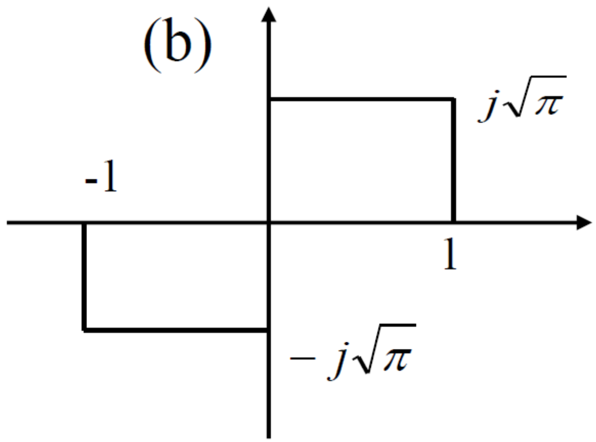
$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} X^*(j\omega)\left[\int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt\right]d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega$$

Parseval's Relation

- Example 7. Find $E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$ and $D = \left.\frac{dx(t)}{dt}\right|_{t=0}$ for the following two $X(j\omega)$:

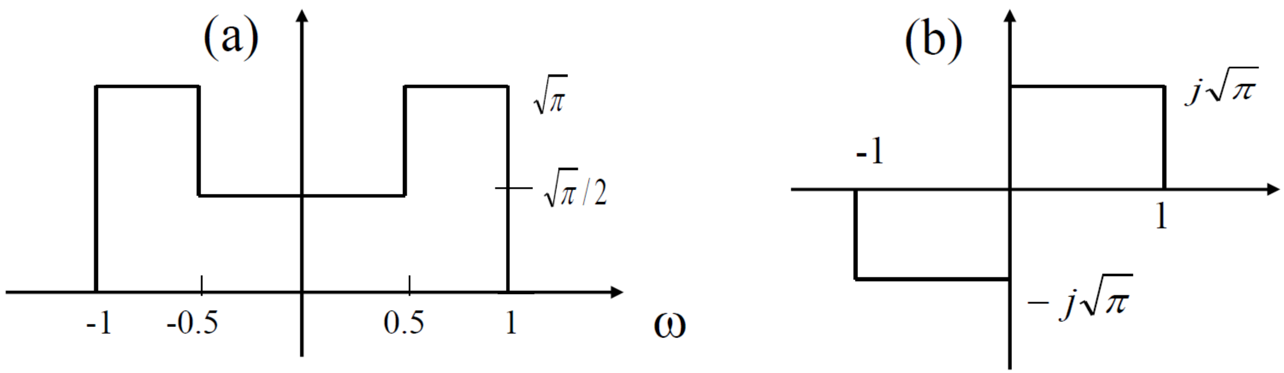




Parseval's Relation

- Example 7. Find $E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$ and $D = \left.\frac{dx(t)}{dt}\right|_{t=0}$ for the following two $X(j\omega)$:

- Solution:



$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt = \frac{1}{2\pi} \int_{-\infty}^{+\infty} |X\,(j\omega)|^2 d\omega = \begin{cases} \frac{5}{8} & for\ (a) \\ 1 & for\ (b) \end{cases}$$

Parseval's Relation

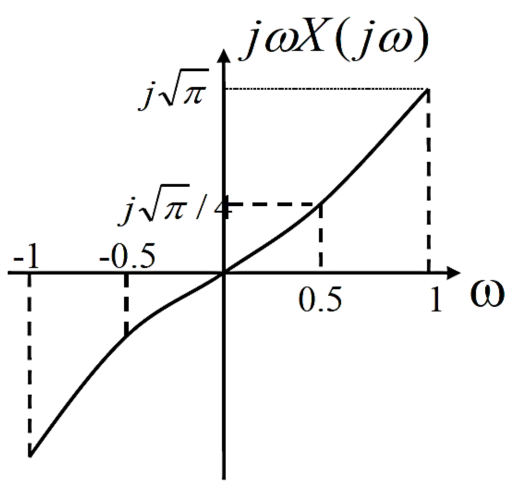
- Solution:

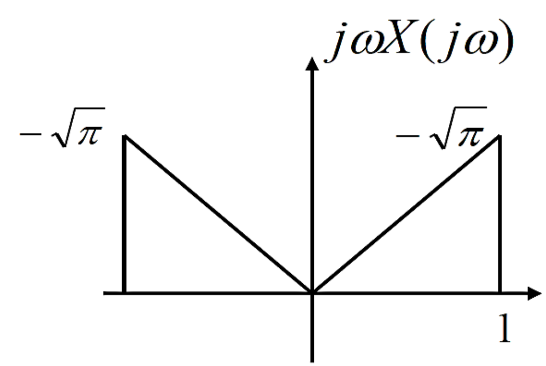
For D, remember $g(t) = \dfrac{dx(t)}{dt} \overset{\mathcal{F}}{\longleftrightarrow} j\omega X(j\omega) = G(j\omega)$

Also note that

$$g(0) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} G\,(j\omega)d\omega = D$$

$$\Longrightarrow D = \frac{1}{2\pi}\int_{-\infty}^{+\infty} j\omega X\,(j\omega)d\omega = \begin{cases} 0 & for\ (a) \\ -\dfrac{\sqrt{\pi}}{2\pi} & for\ (b) \end{cases}$$





Outline

- Properties of Continuous-Time Fourier Transform
- The Convolution Property
- The Multiplication Property

The Convolution Property

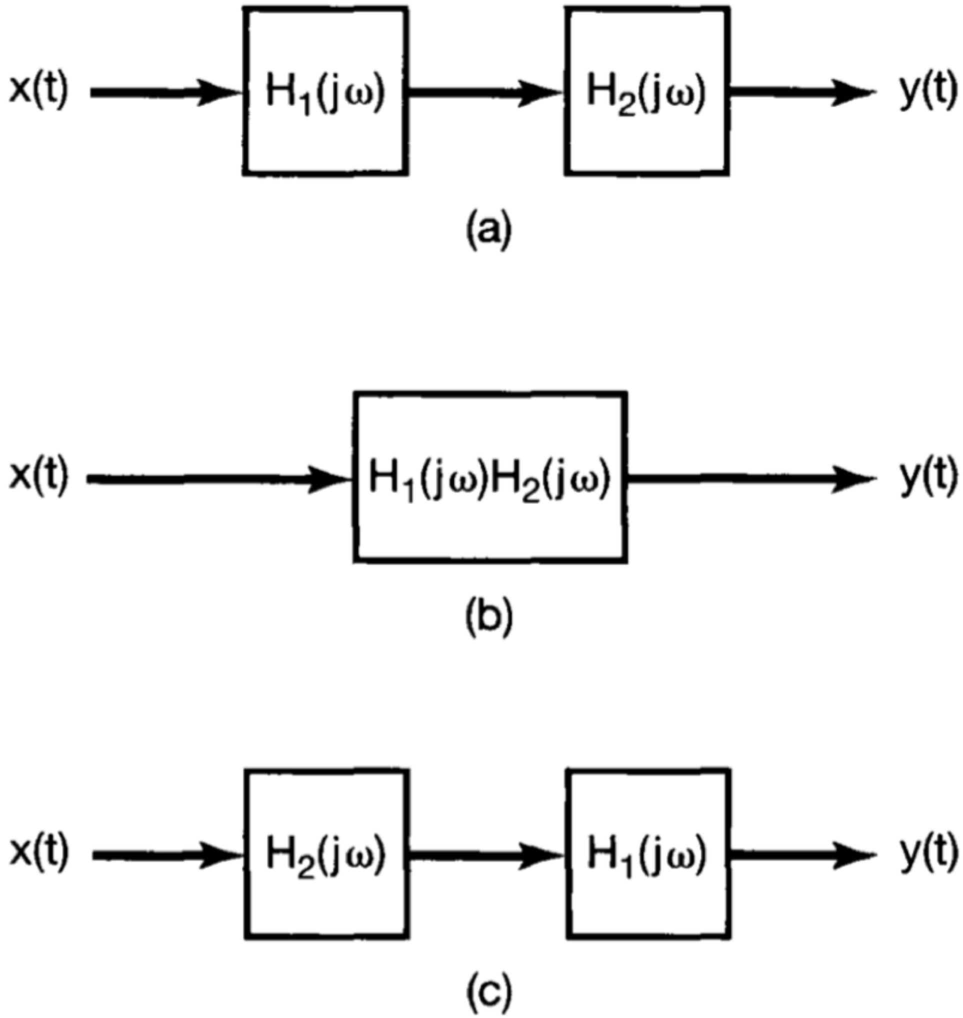
$$y(t) = h(t) * x(t) \xleftrightarrow{\mathcal{F}} Y(j\omega) = H(j\omega)X(j\omega)$$

where $h(t)$ is system impulse response, $H(j\omega)$ is the frequency response.

- Proof:

$$\begin{aligned}
Y(j\omega) = \mathcal{F}\{y(t)\} &= \int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} x(\tau)h(t-\tau)d\tau\right]e^{-j\omega t}dt \\
&= \int_{-\infty}^{+\infty} x(\tau)\left[\int_{-\infty}^{+\infty} h(t-\tau)e^{-j\omega t}dt\right]d\tau \\
&= \int_{-\infty}^{+\infty} x(\tau)e^{-j\omega\tau}H(j\omega)d\tau = H(j\omega)\int_{-\infty}^{+\infty} x(\tau)e^{-j\omega\tau}d\tau \\
&= H(j\omega)X(j\omega)
\end{aligned}$$

The Convolution Property



The Convolution Property

- Example 8. Consider a continuous-time LTI system with impulse response $h(t) = \delta(t - t_0)$. Let $\mathcal{F}\{x(t)\} = X(j\omega)$, determine $Y(j\omega)$.

$$x(t) \longrightarrow \boxed{\quad h(t) \quad} \longrightarrow y(t)$$

The Convolution Property

- Example 8. Consider a continuous-time LTI system with impulse response $h(t) = \delta(t - t_0)$. Let $\mathcal{F}\{x(t)\} = X(j\omega)$, determine $Y(j\omega)$.

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

- Solution 1:

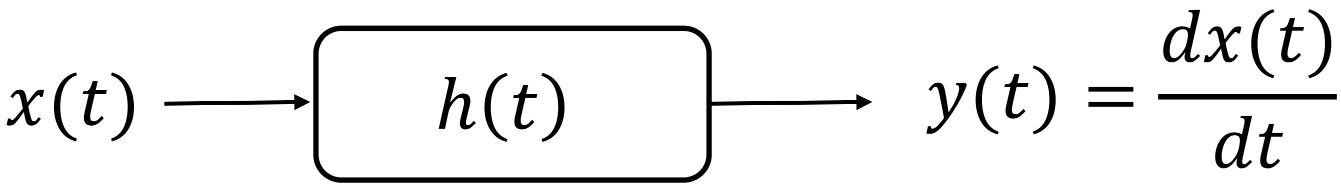
$$H(j\omega) = e^{-j\omega t_0} \qquad Y(j\omega) = H(j\omega)X(j\omega) = e^{-j\omega t_0}X(j\omega)$$

- Solution 2:

$$y(t) = x(t - t_0) \qquad Y(j\omega) = e^{-j\omega t_0}X(j\omega)$$

The Convolution Property

- Example 9. Consider a continuous-time LTI system with $y(t) = \frac{dx(t)}{dt}$, determine its frequency response $H(j\omega)$.



The Convolution Property

- Example 9. Consider a continuous-time LTI system with $y(t) = \frac{dx(t)}{dt}$, determine its frequency response $H(j\omega)$.

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = \frac{dx(t)}{dt}$$

- Solution:

  Differentiation property $\Rightarrow$ $Y(j\omega) = j\omega X(j\omega)$

  Convolution property $\Rightarrow$ $Y(j\omega) = H(j\omega)X(j\omega)$

  Therefore, $H(j\omega) = j\omega$

The Convolution Property

- Example 10. Consider a continuous-time LTI system with $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$. Let $\mathcal{F}\{x(t)\} = X(j\omega)$, determine $Y(j\omega)$ for any input $x(t)$.

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = \int_{-\infty}^{t} x(\tau)d\tau$$

The Convolution Property

- Example 10. Consider a continuous-time LTI system with $y(t) = \int_{-\infty}^{t} x(\tau)d\tau$. Let $\mathcal{F}\{x(t)\} = X(j\omega)$, determine $Y(j\omega)$ for any input $x(t)$.

$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = \int_{-\infty}^{t} x(\tau)d\tau$$

- Solution:

  Unit impulse resonse: $h(t) = \int_{-\infty}^{t} \delta(\tau)d\tau = u(t)$

  Frequency response: $H(j\omega) = \frac{1}{j\omega} + \pi\delta(\omega)$

  Convolution property $Y(j\omega) = H(j\omega)X(j\omega)$

  $$Y(j\omega) = \frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$$

Consistent with integration property

The Convolution Property

- Example 11. Consider a continuous-time LTI system with impulse response $h(t) = e^{-at}u(t), \mathrm{a} > 0$ to the input signal $x(t) = e^{-bt}u(t), b > 0$. Determine $y(t)$.

$$x(t) = e^{-bt}u(t) \rightarrow \boxed{h(t) = e^{-at}u(t)} \rightarrow y(t) = ?$$

The Convolution Property

- Example 11. Consider a continuous-time LTI system with impulse response $h(t) = e^{-at}u(t), \text{a} > 0$ to the input signal $x(t) = e^{-bt}u(t), b > 0$. Determine $y(t)$.

$$x(t) = e^{-bt}u(t) \rightarrow \boxed{h(t) = e^{-at}u(t)} \rightarrow y(t) = ?$$

- Solution:

$$e^{-at}u(t) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{a+j\omega}, \quad \rightarrow \quad Y(j\omega) = X(j\omega)H(j\omega) = \frac{1}{(a+j\omega)(b+j\omega)}$$

  - If $a \neq b$, $Y(j\omega) = \frac{1}{b-a}\left(\frac{1}{a+j\omega} - \frac{1}{b+j\omega}\right)$

$$y(t) = \frac{1}{b-a}\left(e^{-at}u(t) - e^{-bt}u(t)\right)$$

  - If $a = b$, $Y(j\omega) = \frac{1}{(a+j\omega)^2} = j\frac{d}{d\omega}\left[\frac{1}{a+j\omega}\right]$

Since $te^{-at}u(t) \overset{\mathcal{F}}{\longleftrightarrow} j\frac{d}{d\omega}\left[\frac{1}{a+j\omega}\right]$,

$$y(t) = te^{-at}u(t)$$

Outline

- Properties of Continuous-Time Fourier Transform
- The Convolution Property
- The Multiplication Property

The Multiplication Property

$$r(t) = s(t)p(t) \xleftrightarrow{\mathcal{F}} R(j\omega) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} S(j\theta)P(j(\omega - \theta))d\theta$$

- multiplication in time corresponds to convolution in frequency
- multiplication of two signals is often referred to as amplitude modulation

The Multiplication Property

- Proof:

$$s(t)p(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} S(j\theta)e^{j\theta t}\, d\theta \, \frac{1}{2\pi}\int_{-\infty}^{\infty} P(j\omega')e^{j\omega' t}\, d\omega'$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{2\pi}\int_{-\infty}^{\infty} S(j\theta)\, P(j\omega')e^{j(\theta+\omega')t}\, d\theta d\omega'$$

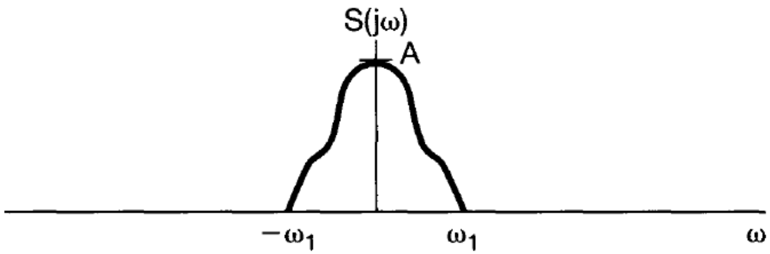
$$\omega' = \omega - \theta$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{1}{2\pi}\int_{-\infty}^{\infty} S[j(\theta)]\, P(j(\omega - \theta))e^{j\omega t}\, d\theta d\omega$$

$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \underbrace{\boxed{\frac{1}{2\pi}\int_{-\infty}^{\infty} S(j\theta)P(j(\omega - \theta))d\theta}}_{R(j\omega)} e^{j\omega t}\, d\omega$$

The Multiplication Property

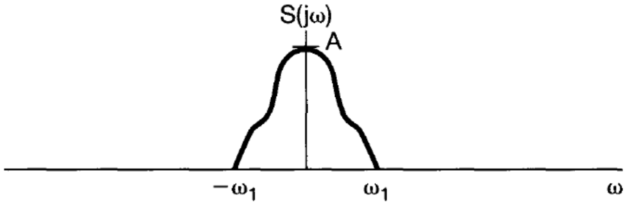
- Example 12. Consider a signal $p(t) = \cos\omega_0 t$ and a signal $s(t)$ with spectrum $S(j\omega)$, determine the FT of $r(t) = p(t)s(t)$.



(a)

The Multiplication Property

- Example 12. Consider a signal $p(t) = \cos\omega_0 t$ and a signal $s(t)$ with spectrum $S(j\omega)$, determine the FT of $r(t) = p(t)s(t)$.

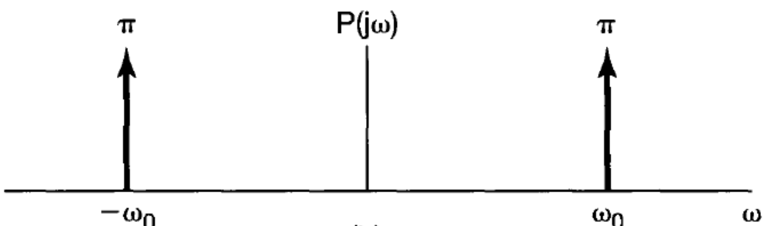


(a)

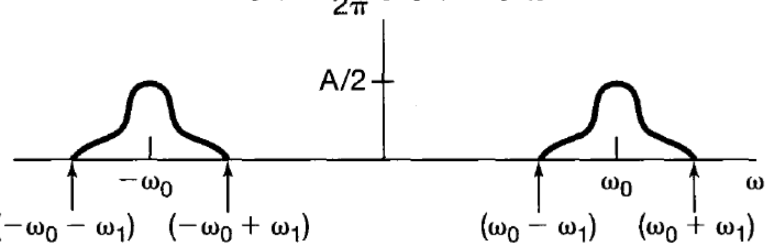
- Solution:

$$P(j\omega) = \pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)$$

$$R(j\omega) = 1/2\pi \cdot S(j\omega) * P(j\omega)$$
$$= 1/2\pi \cdot S(j\omega) * [\pi\delta(\omega - \omega_0) + \pi\delta(\omega + \omega_0)]$$
$$= 1/2[S[j(\omega - \omega_0)] + S[j(\omega + \omega_0)]]$$



(b)

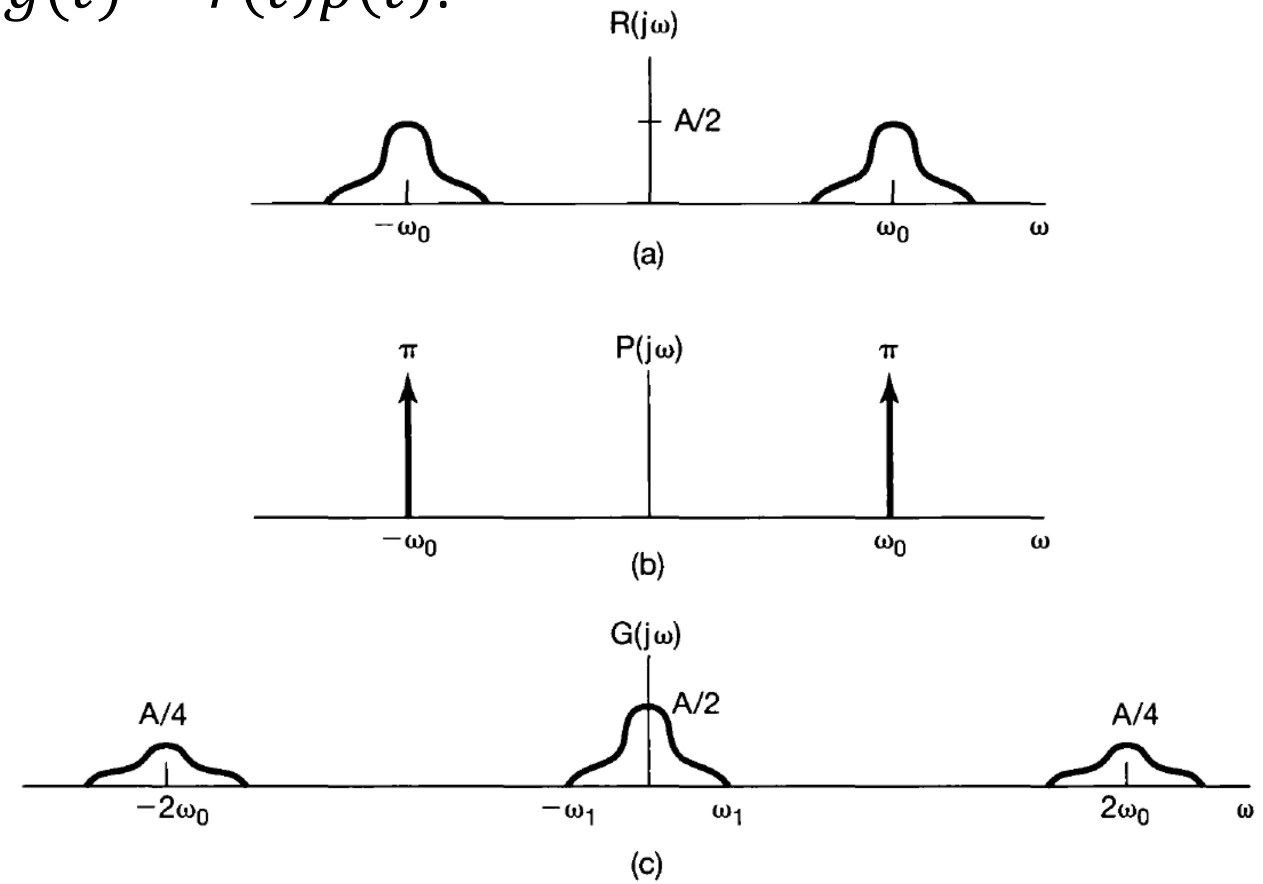


(c)

(This problem illustrates the "modulation process" that is discussed in Principle Comm.)

The Multiplication Property

- Example 13. Consider a signal $p(t) = \cos \omega_0 t$ and a signal $r(t)$ as obtained in Example 12. Determine the FT of $g(t) = r(t)p(t)$.



(This problem illustrates the “demodulation process" that is discussed in Principle Comm.)

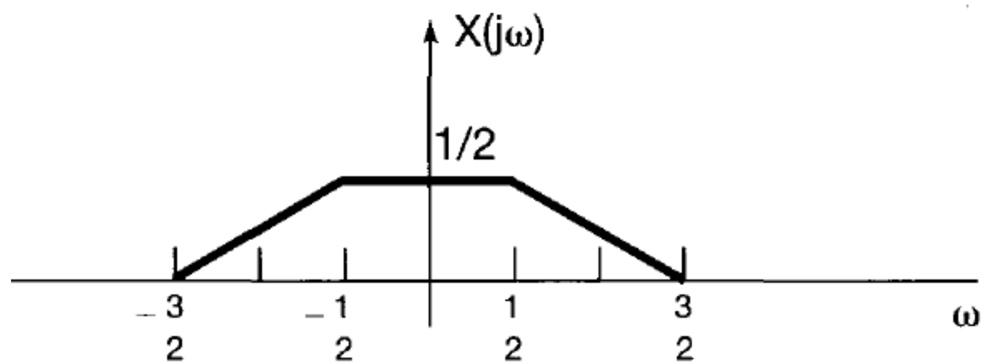
The Multiplication Property

- Example 14. Determine the FT of $x(t) = \dfrac{\sin(t)\sin(t/2)}{\pi t^2}$.

- Solution:

$$x(t) = \pi\left(\frac{\sin(t)}{\pi t}\right)\left(\frac{\sin(t/2)}{\pi t}\right)$$

$$X(j\omega) = \frac{1}{2}\mathcal{F}\left\{\frac{\sin(t)}{\pi t}\right\} * \mathcal{F}\left\{\frac{\sin(t/2)}{\pi t\cancel{/2}}\right\}$$



Properties of CTFT

TABLE 4.1 PROPERTIES OF THE FOURIER TRANSFORM

Section	Property	Aperiodic signal	Fourier transform
		$x(t)$ $y(t)$	$X(j\omega)$ $Y(j\omega)$
4.3.1	Linearity	$ax(t) + by(t)$	$aX(j\omega) + bY(j\omega)$
4.3.2	Time Shifting	$x(t - t_0)$	$e^{-j\omega t_0} X(j\omega)$
4.3.6	Frequency Shifting	$e^{j\omega_0 t} x(t)$	$X(j(\omega - \omega_0))$
4.3.3	Conjugation	$x^*(t)$	$X^*(-j\omega)$
4.3.5	Time Reversal	$x(-t)$	$X(-j\omega)$
4.3.5	Time and Frequency Scaling	$x(at)$	$\frac{1}{\|a\|} X\left(\frac{j\omega}{a}\right)$
4.4	Convolution	$x(t) * y(t)$	$X(j\omega)Y(j\omega)$
4.5	Multiplication	$x(t)y(t)$	$\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\theta)Y(j(\omega-\theta))d\theta$
4.3.4	Differentiation in Time	$\frac{d}{dt}x(t)$	$j\omega X(j\omega)$
4.3.4	Integration	$\int_{-\infty}^{t} x(t)dt$	$\frac{1}{j\omega}X(j\omega) + \pi X(0)\delta(\omega)$
4.3.6	Differentiation in Frequency	$tx(t)$	$j\frac{d}{d\omega}X(j\omega)$
4.3.3	Conjugate Symmetry for Real Signals	$x(t)$ real	$\begin{cases} X(j\omega) = X^*(-j\omega) \\ \Re e\{X(j\omega)\} = \Re e\{X(-j\omega)\} \\ \Im m\{X(j\omega)\} = -\Im m\{X(-j\omega)\} \\ \|X(j\omega)\| = \|X(-j\omega)\| \\ \sphericalangle X(j\omega) = -\sphericalangle X(-j\omega) \end{cases}$
4.3.3	Symmetry for Real and Even Signals	$x(t)$ real and even	$X(j\omega)$ real and even
4.3.3	Symmetry for Real and Odd Signals	$x(t)$ real and odd	$X(j\omega)$ purely imaginary and odd
4.3.3	Even-Odd Decomposition for Real Signals	$x_e(t) = \mathcal{E}v\{x(t)\}$ [$x(t)$ real] $x_o(t) = \mathcal{O}d\{x(t)\}$ [$x(t)$ real]	$\Re e\{X(j\omega)\}$ $j\Im m\{X(j\omega)\}$

4.3.7 Parseval's Relation for Aperiodic Signals

$$\int_{-\infty}^{+\infty} |x(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{+\infty} |X(j\omega)|^2 d\omega$$

Summary

- Properties of Continuous-Time Fourier Transform
- The Convolution Property
- The Multiplication Property
- Reference in textbook:
  - 4.3, 4.4, 4.5, 4.6