Some Properties

Let $\mathbf{A} \in \mathbb{R}^{n \times n}$

$$\mathbf{A}\mathbf{v} = \lambda \mathbf{v}$$

$\Updownarrow$ Hermitian transpose

$$\mathbf{v}^H \mathbf{A}^H = \lambda^* \mathbf{v}^H$$

$*$ conjugate

A^T

$\Updownarrow$ transpose, $\mathbf{A}^H = \mathbf{A}^T$

$$\mathbf{A}\mathbf{v}^* = \lambda^* \mathbf{v}^*$$

$$\mathbf{w}\mathbf{A} = \lambda \mathbf{w}$$

$$\Updownarrow$$

$$\mathbf{A}^H \mathbf{w}^H = \lambda^* \mathbf{w}^H$$

$$\Updownarrow$$

$$\mathbf{A}^T \mathbf{w}^H = \lambda^* \mathbf{w}^H$$

- $\mathbf{v}^*$ is an eigenvector associated with eigenvalue λ^*
- Complex eigenvalues appear in conjugate pairs
- $\mathbf{w}^H$ is an eigenvector associated with eigenvalue λ^* of $\mathbf{A}^T$
- $\mathbf{w}^T$ is an eigenvector associated with eigenvalue λ of $\mathbf{A}^T$
- $\mathbf{A}$ and $\mathbf{A}^T$ have the same set of eigenvalues because $\det(\lambda \mathbf{I} - \mathbf{A}) = \det(\lambda \mathbf{I} - \mathbf{A})^T = \det(\lambda \mathbf{I} - \mathbf{A}^T)$
- The set of eigenvalues corresponding to (right) eigenvectors is the set of eigenvalues corresponding to left eigenvectors

Some Properties (cont'd)

Fact: The eigenvalues of any triangular matrix are its diagonal entries

$$A = \begin{bmatrix} a_{11} & & * \\ & \ddots & \\ 0 & & a_{nn} \end{bmatrix}, \quad \lambda I - A = \begin{bmatrix} \lambda - a_{11} & & -* \\ & \ddots & \\ 0 & & \lambda - a_{nn} \end{bmatrix}$$

$$\det(\lambda I - A) = \prod_{i=1}^{n} (\lambda - a_{ii})$$

Fact: $\mathbf{A} \in \mathbb{R}^{n \times n}$ is nonsingular if and only if all its eigenvalues are nonzero

A has an eigenvalue at $0 \iff 0 = \det(0 \cdot I - A) = (-1)^n \det(A)$

$\iff \det(A) = 0 \iff$ A singular

Fact: Suppose $(\mathbf{v}, \lambda)$ is an eigen-pair of $\mathbf{A}$, then $(\mathbf{v}, \lambda^k)$ is an eigen-pair of $\mathbf{A}^k$ for any positive integer k

$$A^k v = A^{k-1} (\underbrace{Av}_{= \lambda v}) = \lambda A^{k-1} v = \lambda A^{k-2} (Av) = \cdots = \lambda^{k-1} (Av) = \lambda^k v$$

Repeated Eigenvalues

- Let $\lambda_1, \ldots, \lambda_n$ be the n eigenvalues of $\mathbf{A} \in \mathbb{R}^{n \times n}$
- WLOG, order $\lambda_1, \ldots, \lambda_n$ so that $\{\lambda_1, \ldots \lambda_k\}$, $k \leq n$ is the set of all **distinct** eigenvalues of $\mathbf{A}$: $\lambda_i \neq \lambda_j$ for all $i, j \in \{1, \ldots, k\}$, $i \neq j$ and $\lambda_i \in \{\lambda_1, \ldots \lambda_k\}$ for all $i \in \{1, \ldots, n\}$
- Define the algebraic multiplicity of eigenvalue λ_i as the multiplicity of λ_i as root of $p(\lambda)$, denoted by μ_i
- Every λ_i may have multiple eigenvectors (scaling not counted)
- If $\dim \mathcal{N}(\lambda_i \mathbf{I} - \mathbf{A}) = r$, we can find r linearly independent $\mathbf{v}_i$'s
- Define the geometric multiplicity of eigenvalue λ_i as the maximum number of linearly independent eigenvectors associated with λ_i, denoted by γ_i
  - $\gamma_i = \dim \mathcal{N}(\lambda_i \mathbf{I} - \mathbf{A}) = n - \text{rank}(\lambda_i \mathbf{I} - \mathbf{A})$

dim $\mathcal{N}(\cdot)$

nullity

Null$(\cdot)$

Repeated Eigenvalues (cont'd)

Fact: For every eigenvalue λ_i of $\mathbf{A}$, $\mu_i \geq \gamma_i \geq 1$

Proof: Let $q_1, \cdots, q_{\gamma_i}$ be an orthonormal basis of $\mathcal{N}(\lambda_i I - A)$

and let $q_{\gamma_i+1}, \cdots, q_n$ be s.t. $Q = [\underbrace{q_1 \cdots q_{\gamma_i}}_{Q_1} \vdots \underbrace{q_{\gamma_i+1} \cdots q_n}_{Q_2}]$ is unitary

$$Q^H (AQ) = \begin{bmatrix} Q_1^H \\ Q_2^H \end{bmatrix} [AQ_1 \quad AQ_2] = \begin{bmatrix} Q_1^H A Q_1 & Q_1^H A Q_2 \\ Q_2^H A Q_1 & Q_2^H A Q_2 \end{bmatrix} \quad (*)$$

For $i = 1, \cdots, \gamma_i$, $Aq_i = \lambda_i q_i \Rightarrow AQ_1 = \lambda_i Q_1$

Also, $Q_1^H Q_1 = I$, $Q_2^H Q_2 = I$

$$(*) = \begin{bmatrix} \lambda_i I_{\gamma_i} & Q_1^H A Q_2 \\ 0 & Q_2^H A Q_2 \end{bmatrix}$$

$$\lambda I - Q^H A Q = \begin{bmatrix} (\lambda - \lambda_i) I_{\gamma_i} & -Q_1^H A Q_2 \\ 0 & \lambda I - Q_2^H A Q_2 \end{bmatrix}$$

$$\det(\lambda I - A) = \det(Q^H (\lambda I - A) Q) = \det(\lambda I - Q^H A Q)$$

$$\overset{(*)}{=} \det((\lambda - \lambda_i) I_{\gamma_i}) \det(\lambda I_{n-\gamma_i} - Q_2^H A Q) = (\lambda - \lambda_i)^{\gamma_i} \cdot \underbrace{\det(\cdots)}_{\text{a polynomial of } \lambda}$$

$\Rightarrow \det(\lambda I - A)$ has at least γ_i roots at λ_i

Repeated Eigenvalues (cont'd)

upper triangular

Example: $\mathbf{A} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$

$\det(\lambda I - A) = (\lambda - 3)(\lambda - 2)^2$

$\lambda_1 = 3 \quad \mu_1 = 1 \quad \gamma_1 = 1$

$\lambda_2 = 2 \quad \mu_2 = 2 \quad \gamma_2 = ?$

$$Av = 2v \iff \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 2v_1 \\ 2v_2 \\ 2v_3 \end{bmatrix} \iff \begin{cases} 3v_1 = 2v_1 \\ 2v_2 + v_3 = 2v_2 \\ 2v_3 = 2v_3 \end{cases}$$

$$\iff v = \begin{bmatrix} 0 \\ * \\ 0 \end{bmatrix}$$

eigenspace span $\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\} \Rightarrow \gamma_2 = 1$

or

$$2I - A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

rank 2 $\Rightarrow \gamma_2 = 3 - 2 = 1$

Repeated Eigenvalues (cont'd)

Example: $\mathbf{A} = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$\lambda_1 = 3 \quad \mu_1 = 1 \quad \gamma_1 = 1$

$\lambda_2 = 2 \quad \mu_2 = 2 \quad \gamma_2 = ?$

$$Av = 2v \iff \begin{cases} 3v_1 = 2v_1 \\ 2v_2 = 2v_2 \\ 2v_3 = 2v_3 \end{cases} \iff v = \begin{bmatrix} 0 \\ * \\ * \end{bmatrix}$$

eigenspace $\quad \text{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\} \Rightarrow \gamma_2 = 2$

or

$$2I - A = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{rank } 1 \Rightarrow \gamma_2 = 3 - 1 = 2$$

Similarity Transformation

- Let $\mathbf{Q} = [\mathbf{q}_1 \ \cdots \ \mathbf{q}_n] \in \mathbb{R}^{n\times n}$ (or $\mathbb{C}^{n\times n}$) be a nonsingular matrix
  - The columns of Q form a basis of $\mathbb{R}^n$ (or $\mathbb{C}^n$)
- Let $A \in \mathbb{R}^{n\times n}$ (or $\mathbb{C}^{n\times n}$). We call $\tilde{\mathbf{A}} = \mathbf{Q}^{-1}\mathbf{A}\mathbf{Q}$ a similarity transformation
- A and $\tilde{A}$ are said to be similar
- Similar matrices represent the same linear map under two (possibly) different bases, with $\mathbf{Q}$ being the change of basis matrix
- **Interpretation**: Consider a linear system $\mathbf{A}\mathbf{x} = \mathbf{y}$ and let $\mathbf{x} = \mathbf{Q}\tilde{\mathbf{x}}$, $\mathbf{y} = \mathbf{Q}\tilde{\mathbf{y}}$

$$x = \tilde{x}_1 q_1 + \cdots + \tilde{x}_n q_n$$

$$\mathbf{A}\mathbf{x} = \mathbf{y} \Leftrightarrow \mathbf{A}\mathbf{Q}\tilde{\mathbf{x}} = \mathbf{Q}\tilde{\mathbf{y}} \Leftrightarrow \mathbf{Q}^{-1}\mathbf{A}\mathbf{Q}\tilde{\mathbf{x}} = \tilde{\mathbf{y}} \Leftrightarrow \tilde{\mathbf{A}}\tilde{\mathbf{x}} = \tilde{\mathbf{y}}$$

$$= x_1 e_1 + \cdots + x_n e_n$$

Similarity Transformation (cont'd)

- Every **square** matrix is similar to itself $I^{-1} A I = A$
- If **A** is similar to **B** and **B** is similar to **C**, then **A** is similar to **C**

$\exists$ nonsingular Q, P s.t. $B = Q^{-1} A Q, \quad C = P^{-1} B P$

$C = P^{-1} (Q^{-1} A Q) P = (QP)^{-1} A (QP), \quad QP$ nonsingular

- If A, B are invertible and similar, then A^{-1} and B^{-1} are also similar

$B = Q^{-1} A Q \iff B^{-1} = (Q^{-1} A Q)^{-1} = Q^{-1} A^{-1} Q$

- Similar matrices have the same characteristic polynomial, determinant, rank, nullity, trace, eigenvalues, algebraic multiplicity, geometric multiplicity, etc.

$\tilde{A} = Q^{-1} A Q$

$\det(\lambda I - \tilde{A}) = \det(\lambda I - Q^{-1} A Q) = \det\left(Q^{-1} (\lambda I - A) Q\right)$

$= \underbrace{\det(Q^{-1}) \det(\lambda I - A) \det(Q)} = \det(\lambda I - A)$

Eigendecomposition

A matrix $\mathbf{A} \in \mathbb{R}^{n\times n}$ (or $\mathbb{C}^{n\times n}$) is said to be diagonalizable, or admit an eigendecomposition, if there exists a nonsingular $\mathbf{V} \in \mathbb{C}^{n\times n}$ s.t.

$$\mathbf{A} = \mathbf{V}\Lambda\mathbf{V}^{-1}$$

where $\Lambda = \mathrm{Diag}(\lambda_1, \ldots, \lambda_n)$, or, $\mathbf{A}$ is similar to a diagonal matrix

- In this definition, we didn't say that $(\mathbf{v}_i, \lambda_i)$ is an eigen-pair of $\mathbf{A}$, but it indeed has to be

$$\begin{aligned} \mathbf{A} = \mathbf{V}\Lambda\mathbf{V}^{-1} &\iff \mathbf{AV} = \mathbf{V}\Lambda,\ \mathbf{V} \text{ nonsingular} \\ &\iff \mathbf{A}\mathbf{v}_i = \lambda_i \mathbf{v}_i,\ i = 1, \ldots, n, \\ &\qquad \mathbf{v}_1, \ldots, \mathbf{v}_n \text{ linearly independent} \end{aligned}$$

- The key lies in finding n linearly independent eigenvectors to form $\mathbf{V}$

Eigendecomposition (cont'd)

$A = V \Lambda V^{-1}$

A and Λ similar

$\det(A) = \det(\Lambda) = \prod_{i=1}^{n} \lambda_i$

Facts: Suppose $\mathbf{A}$ admits an eigendecomposition

1. $\det(\mathbf{A}) = \prod_{i=1}^{n} \lambda_i$

   including repeated eigenvalues

2. $\mathrm{tr}(\mathbf{A}) = \sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} \lambda_i$

   $\mathrm{rank}(A) = \mathrm{rank}(\Lambda)$

3. $\mathrm{rank}(\mathbf{A})$ = number of nonzero eigenvalues of $\mathbf{A}$

4. Suppose $\mathbf{A}$ is also nonsingular. Then, $\mathbf{A}^{-1} = \mathbf{V}\Lambda^{-1}\mathbf{V}^{-1}$

   $\Lambda^{-1} = \begin{bmatrix} \frac{1}{\lambda_1} & & \\ & \ddots & \\ & & \frac{1}{\lambda_n} \end{bmatrix}$

Note: Facts 1–2 are indeed true for any $\mathbf{A}$; Facts 3–4 may not hold when $\mathbf{A}$ does not admit an eigendecomposition