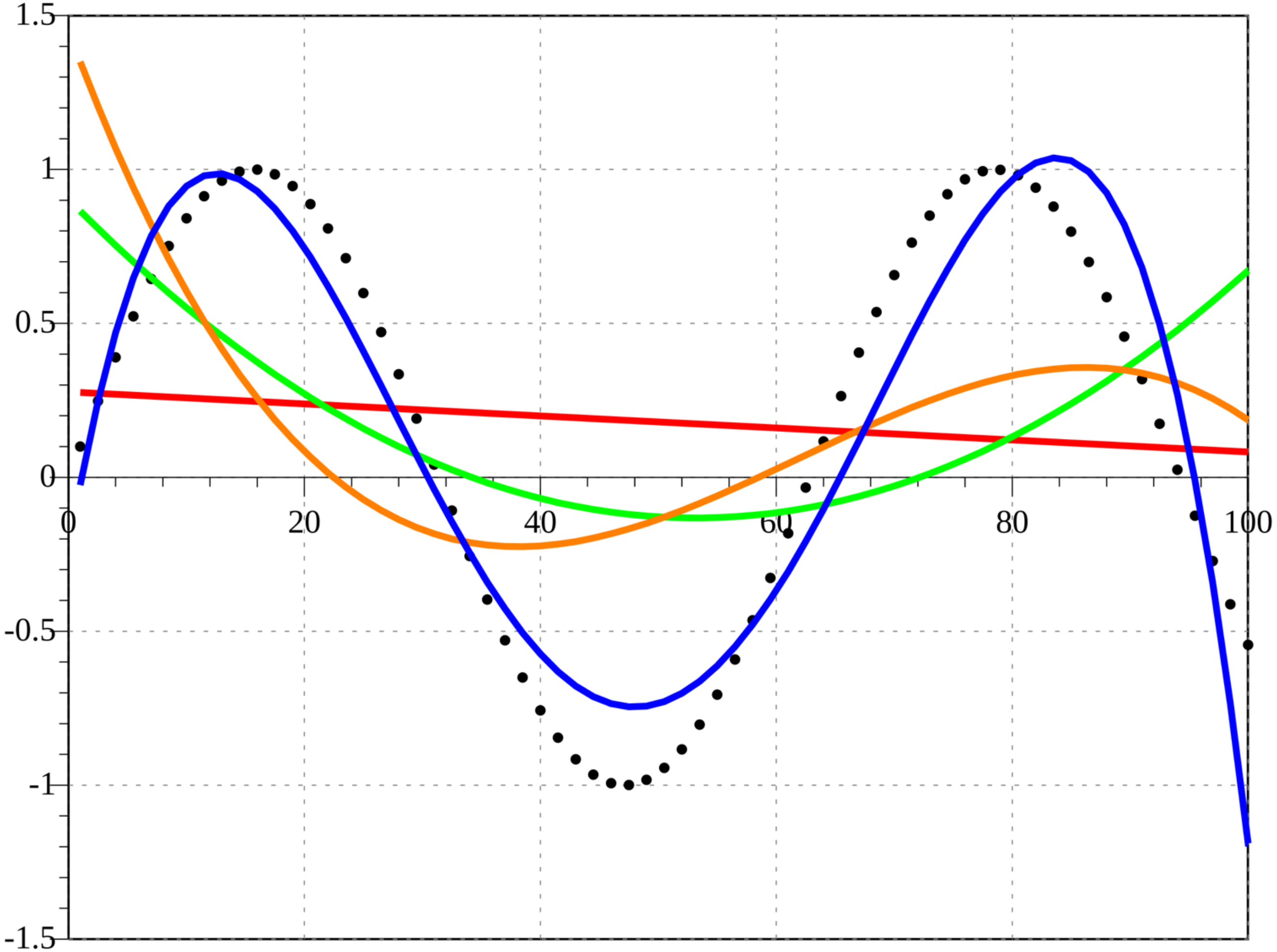
Fitting



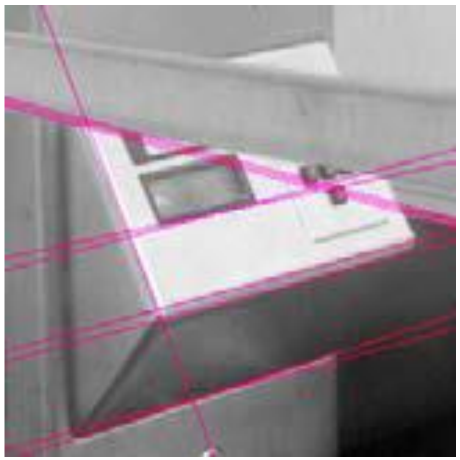
From [wiki]

Fitting

- We've learned how to detect edges, corners, blobs. Now what?
- We would like to form a higher-level, more compact representation of the features in the image by grouping multiple features according to a simple model

Fitting

- Choose a *parametric model* to represent a set of features



simple model: lines

simple model: circles

complicated model: car

Source: K. Grauman

Application: Line Detection

3D Wireframe detection

(a) Input image (b) 3D wireframe

Lane detection

Learning to Reconstruct 3D Manhattan Wireframes from a Single Image

Image credit: [End-to-end Lane Detection](#)

Application: Room Plan

Apple [roomplan]

Fitting: Issues

Case study: Line detection

- **Noise** in the measured feature locations
- **Extraneous data:** clutter (outliers), multiple lines
- **Missing data:** occlusions

Fitting: Overview

- If we know which points belong to the line, how do we find the “optimal” line parameters?
  - Least squares

- What if there are outliers?
  - Robust fitting, RANSAC

- What if there are many lines?
  - Voting methods: RANSAC, Hough transform

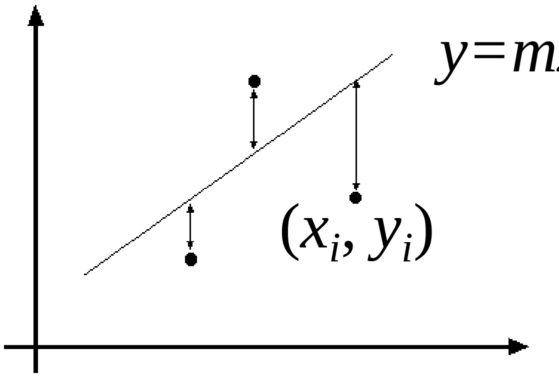
- What if we’re not even sure it’s a line?
  - Model selection (not covered)

Least squares line fitting

Data: $(x_1, y_1), \ldots, (x_n, y_n)$

Line equation: $y_i = m x_i + b$

Find (m, b) to minimize



$$\boxed{E = \sum_{i=1}^{n} (y_i - m x_i - b)^2}$$

$$E = \|Y - XB\|^2 \quad \text{where} \quad Y = \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix} \qquad X = \begin{bmatrix} x_1 & 1 \\ \vdots & \vdots \\ x_n & 1 \end{bmatrix} \qquad B = \begin{bmatrix} m \\ b \end{bmatrix}$$

$$E = \|Y - XB\|^2 = (Y - XB)^T (Y - XB) = Y^T Y - 2(XB)^T Y + (XB)^T (XB)$$

$$\frac{dE}{dB} = 2X^T XB - 2X^T Y = 0$$

http://faculty.bicmr.pku.edu.cn/~wenzw/bigdata/matrix-cook-book.pdf **sec 2.4**

$$\boxed{X^T XB = X^T Y}$$

Normal equations: least squares solution to $XB=Y$

Normal Equation

Wolfram MathWorld FROM THE MAKERS OF MATHEMATICA AND WOLFRAM|ALPHA

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Algebra › Linear Algebra › Matrices › Matrix Operations ›

Normal Equation

Given a matrix equation

$$\mathsf{A}\,\mathbf{x} = \mathbf{b},$$

the normal equation is that which minimizes the sum of the square differences between the left and right sides:

$$\mathsf{A}^{\mathrm{T}}\,\mathsf{A}\,\mathbf{x} = \mathsf{A}^{\mathrm{T}}\,\mathbf{b}.$$

It is called a normal equation because $\mathbf{b} - \mathsf{A}\,\mathbf{x}$ is normal to the range of A.

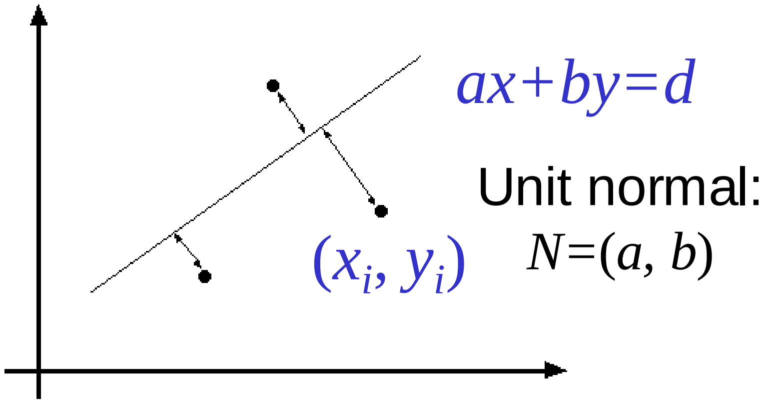
Here, $\mathsf{A}^{\mathrm{T}}\,\mathsf{A}$ is a normal matrix.

Problem with “vertical” least squares

- Not rotation-invariant
- Fails completely for vertical lines

Total least squares

Distance between point (x_i, y_i) and line $ax+by=d$ $(a^2+b^2=1)$: $|ax_i + by_i - d|$

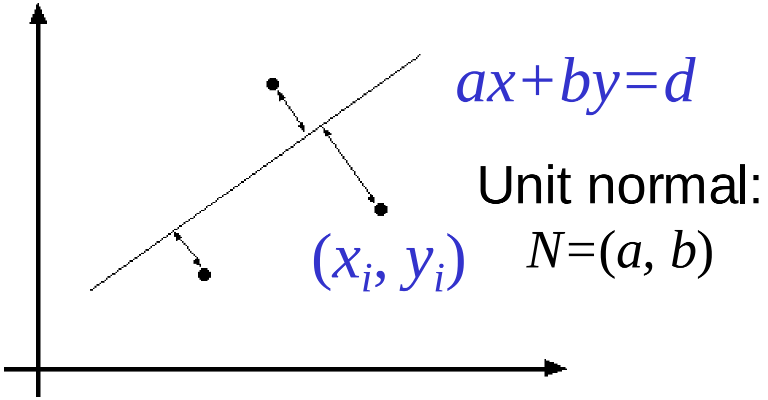


Total least squares

Distance between point (x_i, y_i) and line $ax+by=d$ $(a^2+b^2=1)$: $|ax_i + by_i - d|$

Find (a, b, d) to minimize the sum of squared perpendicular distances

$$\boxed{E = \sum_{i=1}^{n} (ax_i + by_i - d)^2}$$



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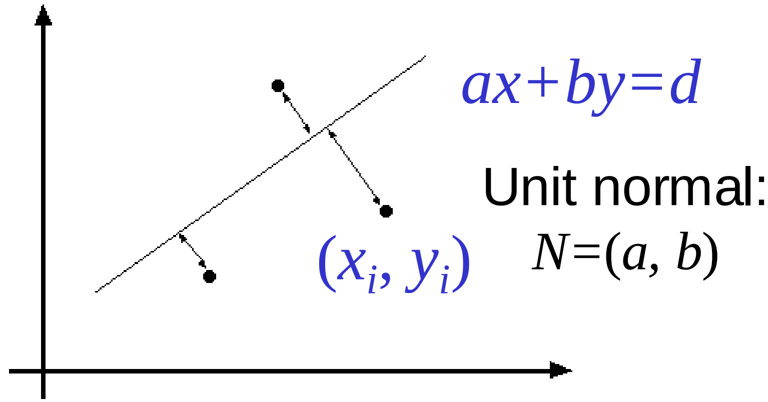


$$\boxed{E = \sum_{i=1}^{n} (ax_i + by_i - d)^2}$$

$$\frac{\partial E}{\partial d} = \sum_{i=1}^{n} -2(ax_i + by_i - d) = 0 \qquad d = \frac{a}{n}\sum_{i=1}^{n} x_i + \frac{b}{n}\sum_{i=1}^{n} y_i = a\bar{x} + b\bar{y}$$

$$E = \sum_{i=1}^{n} (a(x_i - \bar{x}) + b(y_i - \bar{y}))^2 = \left\| \begin{bmatrix} x_1 - \bar{x} & y_1 - \bar{y} \\ \vdots & \vdots \\ x_n - \bar{x} & y_n - \bar{y} \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} \right\|^2 = (UN)^T (UN)$$

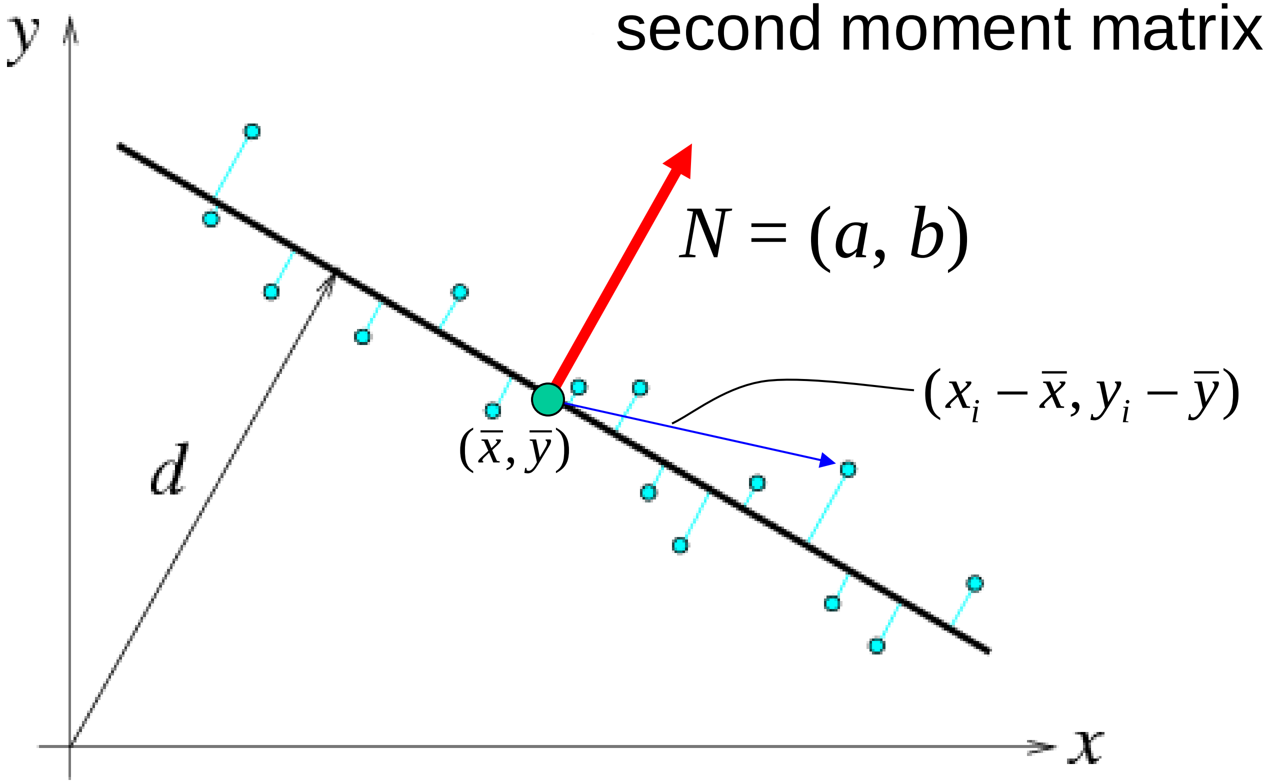
$$\frac{dE}{dN} = 2(U^T U)N = 0$$

Solution to $(U^T U)N = 0$, subject to $\|N\|^2 = 1$: eigenvector of $U^T U$ associated with the smallest eigenvalue (least squares solution to *homogeneous linear system* $UN = 0$)

Total least squares

$$U = \begin{bmatrix} x_1 - \bar{x} & y_1 - \bar{y} \\ \vdots & \vdots \\ x_n - \bar{x} & y_n - \bar{y} \end{bmatrix} \qquad U^T U = \begin{bmatrix} \sum_{i=1}^{n} (x_i - \bar{x})^2 & \sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y}) \\ \sum_{i=1}^{n} (x_i - \bar{x})(y_i - \bar{y}) & \sum_{i=1}^{n} (y_i - \bar{y})^2 \end{bmatrix}$$

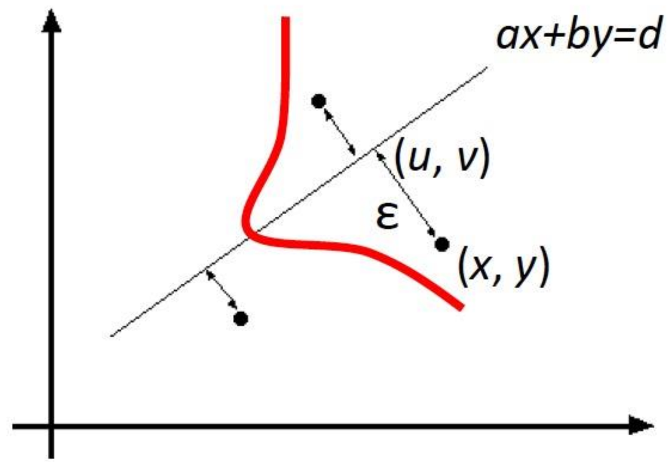
second moment matrix



F&P (2nd ed.) sec. 22.1

Least squares as likelihood maximization

- **Generative model**: line points are sampled independently and corrupted by Gaussian noise in the direction perpendicular to the line



$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix} + \varepsilon \begin{pmatrix} a \\ b \end{pmatrix}$$

Likelihood of points given line parameters (*a*, *b*, *d*):

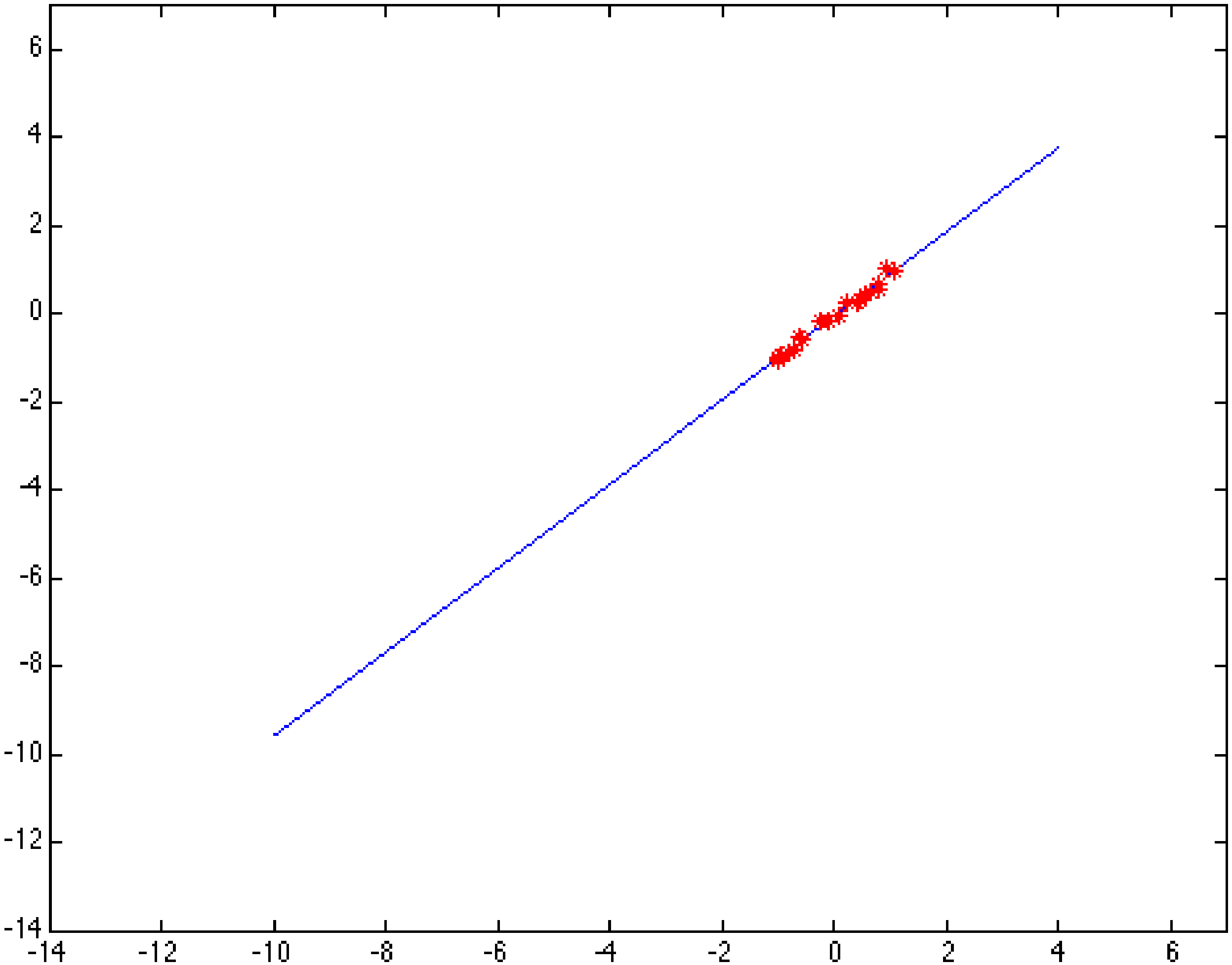
$$P(x_1, y_1, \ldots, x_n, y_n \mid a, b, d) = \prod_{i=1}^{n} P(x_i, y_i \mid a, b, d) \propto \prod_{i=1}^{n} \exp\left(-\frac{(ax_i + by_i - d)^2}{2\sigma^2} \right)$$

Log-likelihood: $L(x_1, y_1, \ldots, x_n, y_n \mid a, b, d) = -\frac{1}{2\sigma^2} \sum_{i=1}^{n} (ax_i + by_i - d)^2$

Source： S. Lazebnik

Least squares: Robustness to noise

Least squares fit to the red points:



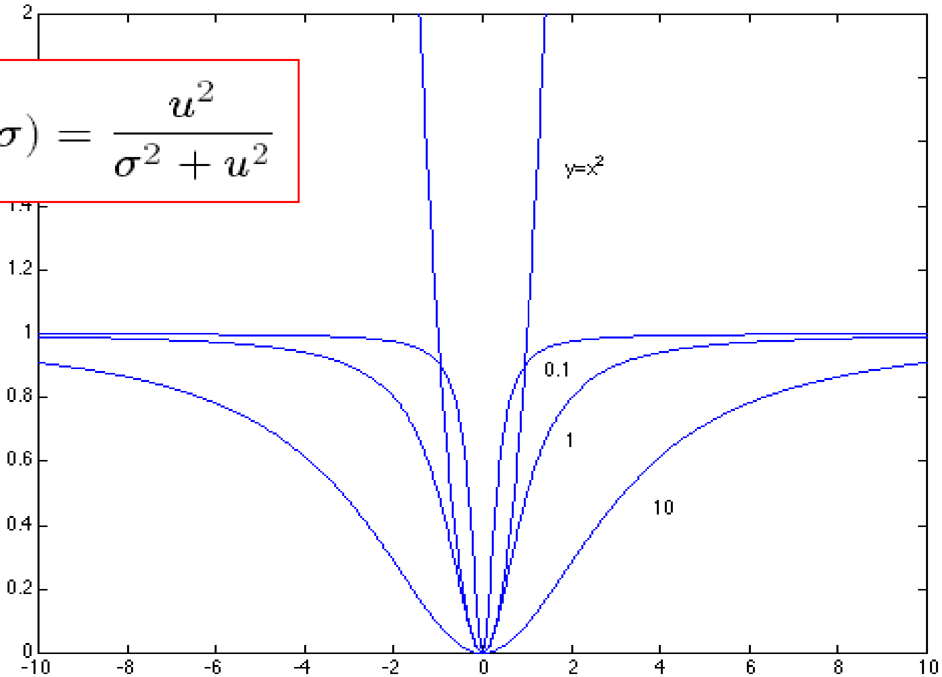
Robust estimators

- General approach: find model parameters θ that minimize

$$\sum_i \rho\left(r_i\left(x_i, \theta\right); \sigma\right)$$

$r_i(x_i, \theta)$ – residual of ith point w.r.t. model parameters θ
ρ – robust function with scale parameter σ

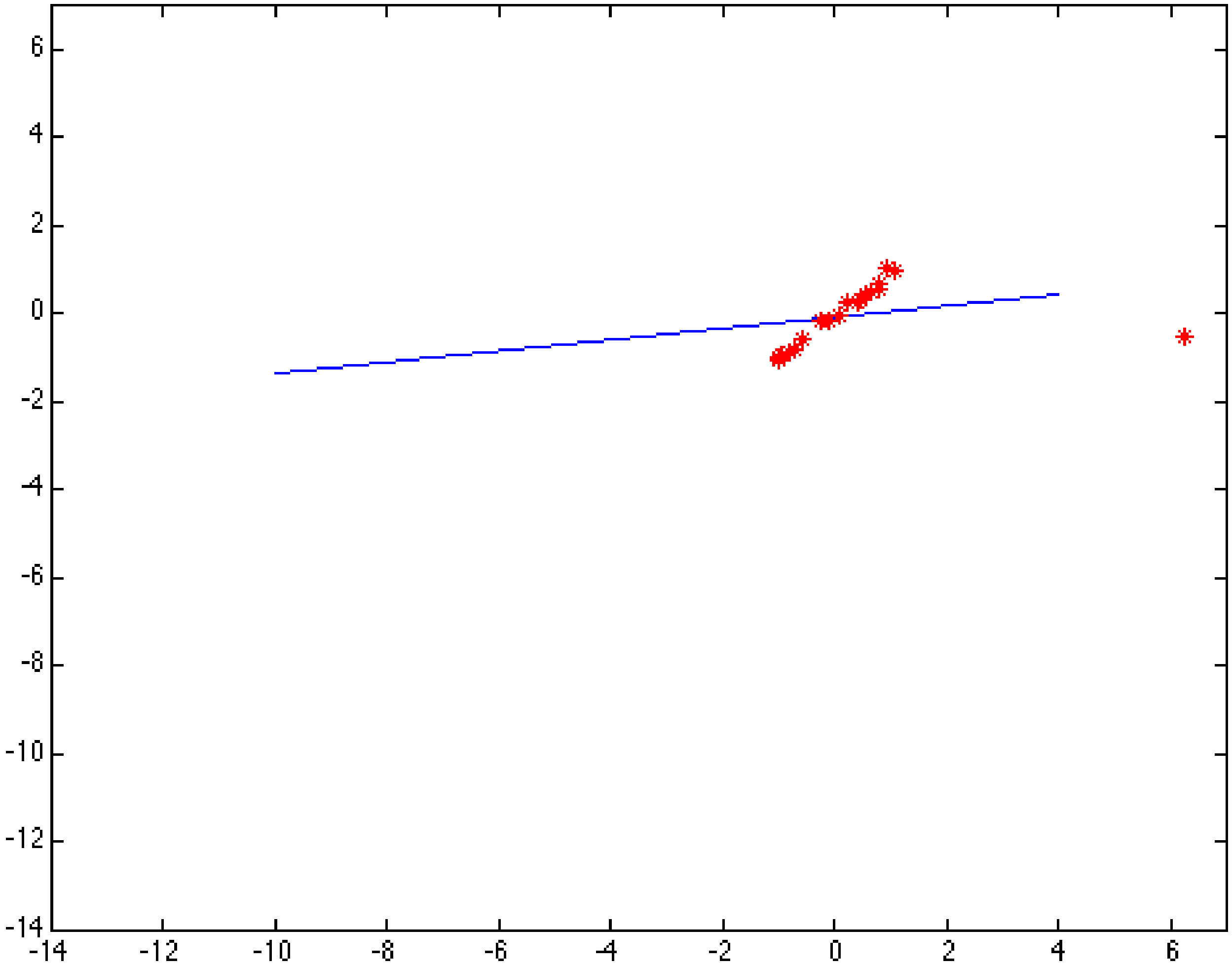
$$\rho(u;\sigma) = \frac{u^2}{\sigma^2 + u^2}$$



The robust function ρ behaves like squared distance for small values of the residual *u* but saturates for larger values of *u*

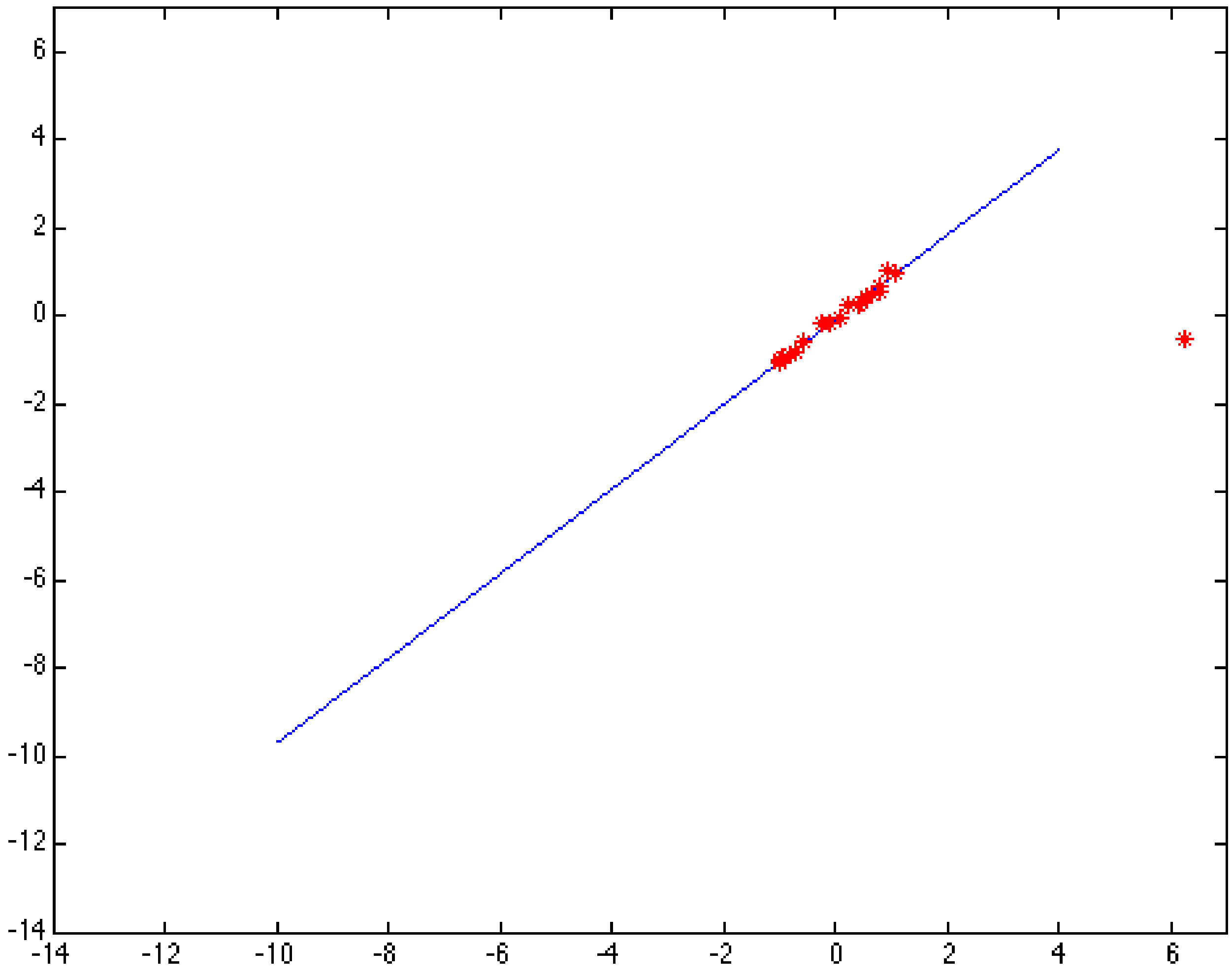
Least squares: Robustness to noise

Least squares fit with an outlier:



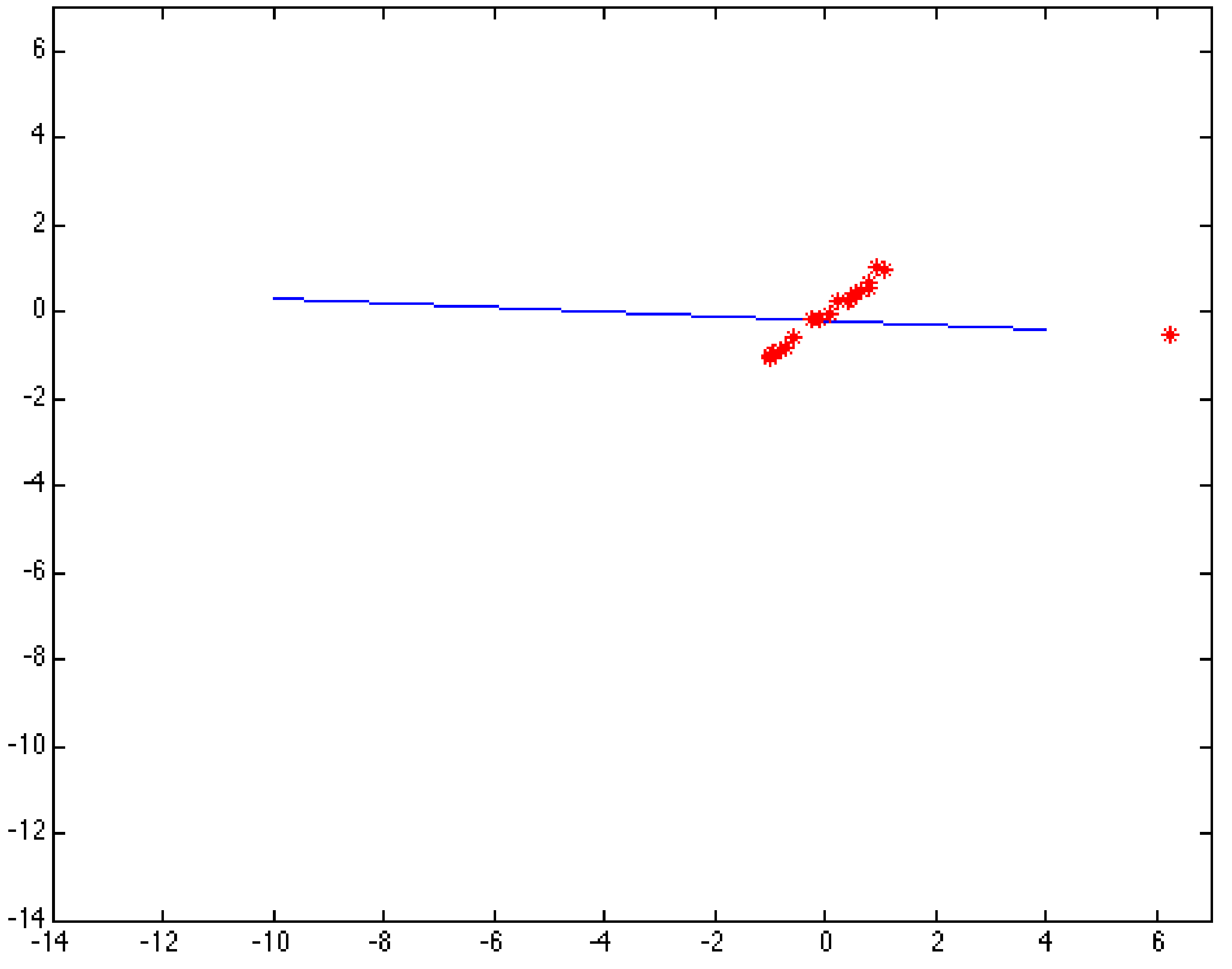
Problem: squared error heavily penalizes outliers

Choosing the scale: Just right



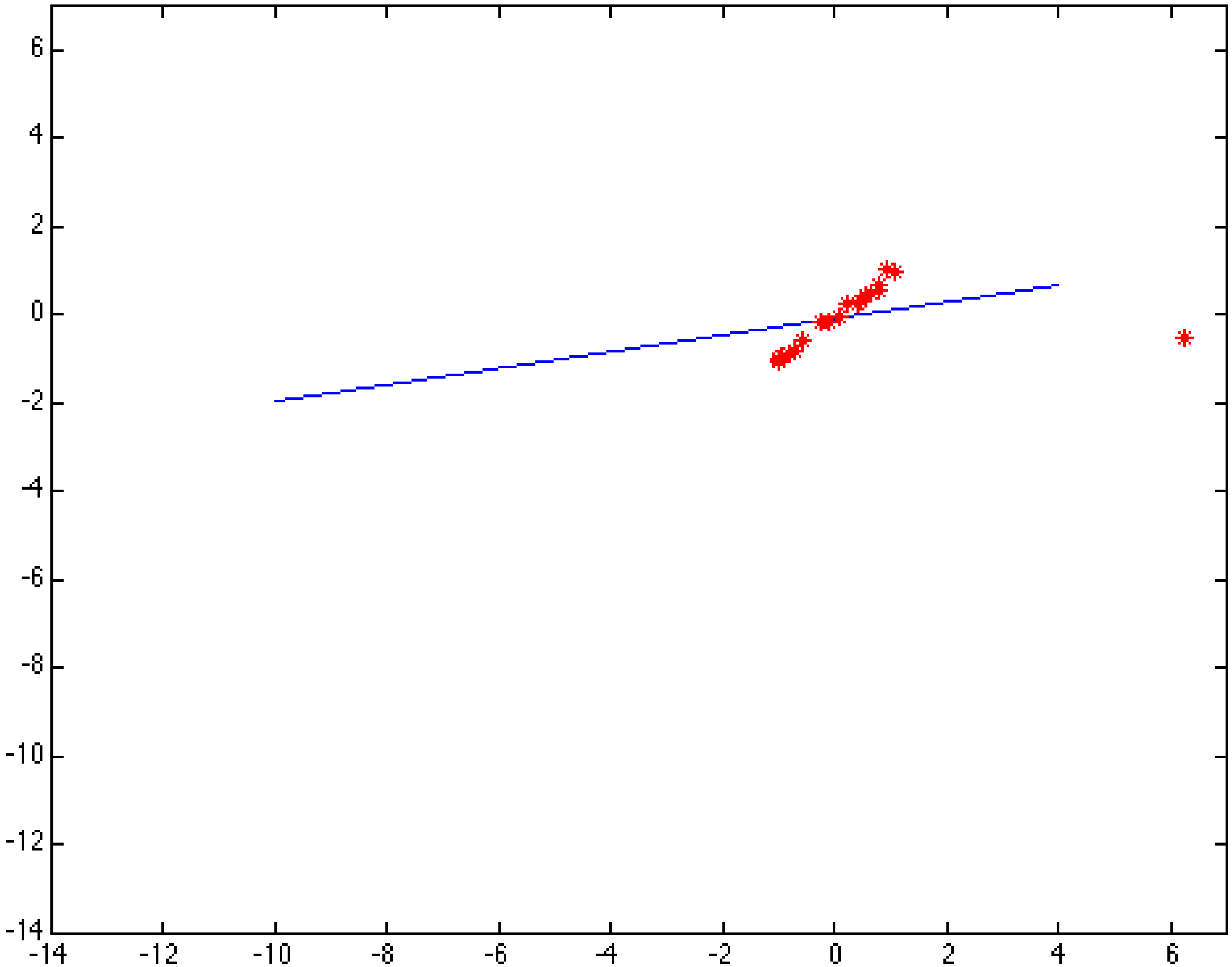
The effect of the outlier is minimized

Choosing the scale: Too small



The error value is almost the same for every point and the fit is very poor

Choosing the scale: Too large



Behaves much the same as least squares

Robust estimation: Details

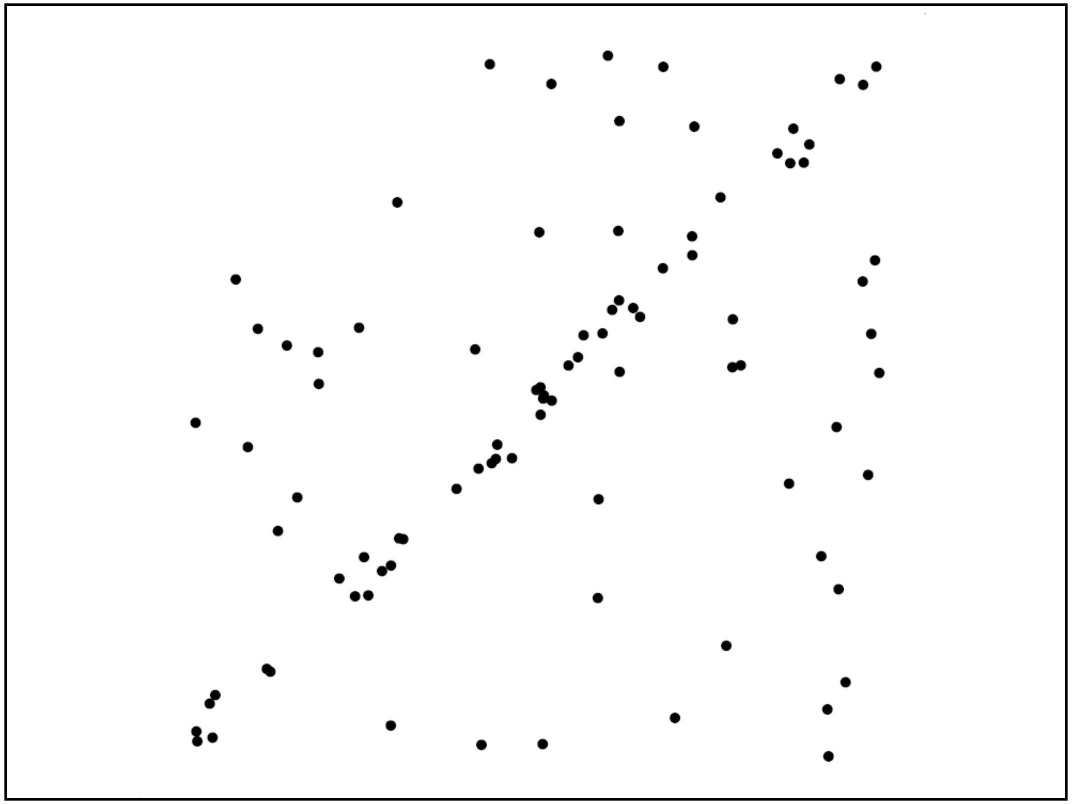
- Robust fitting is a nonlinear optimization problem that must be solved iteratively
- Least squares solution can be used for initialization
- Scale of robust function should be chosen adaptively based on median residual

RANSAC

- Robust fitting can deal with a few outliers – what if we have very many?
- Random sample consensus (RANSAC): Very general framework for model fitting in the presence of outliers
- Outline
  - Choose a small subset of points uniformly at random
  - Fit a model to that subset
  - Find all remaining points that are “close” to the model and reject the rest as outliers
  - Do this many times and choose the best model

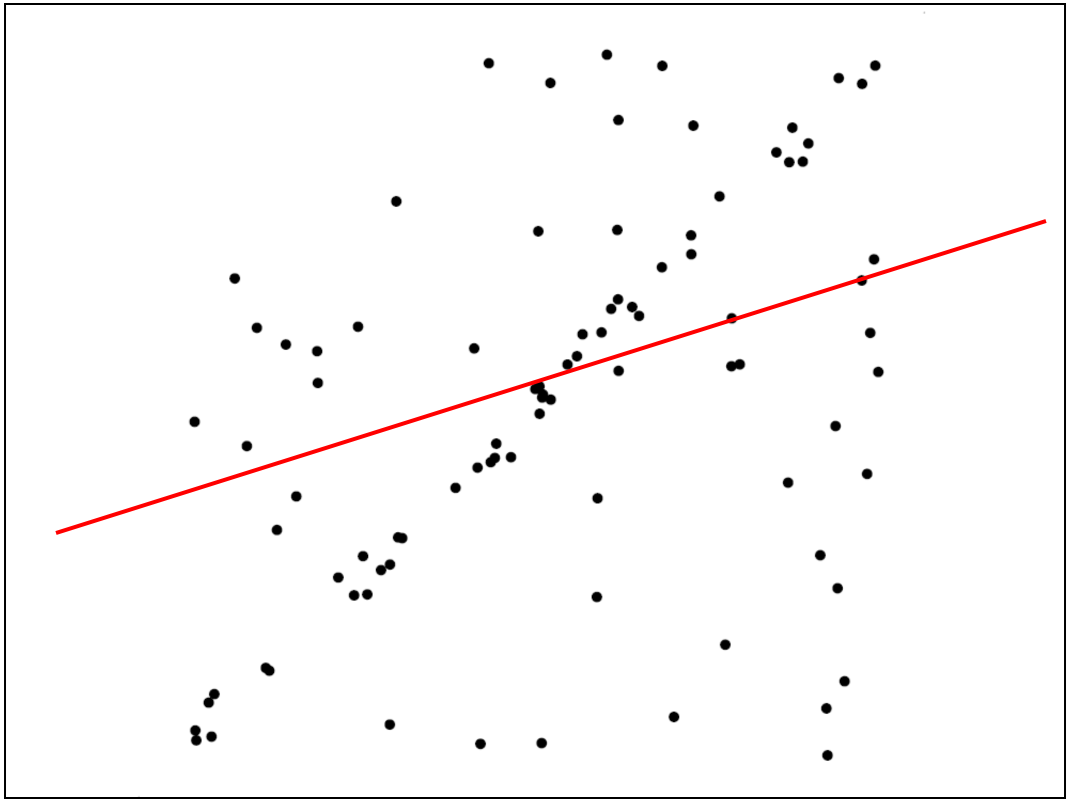
M. A. Fischler, R. C. Bolles. Random Sample Consensus: A Paradigm for Model Fitting with Applications to Image Analysis and Automated Cartography. Comm. of the ACM, Vol 24, pp 381-395, 1981.

RANSAC for line fitting example



Source: R. Raguram

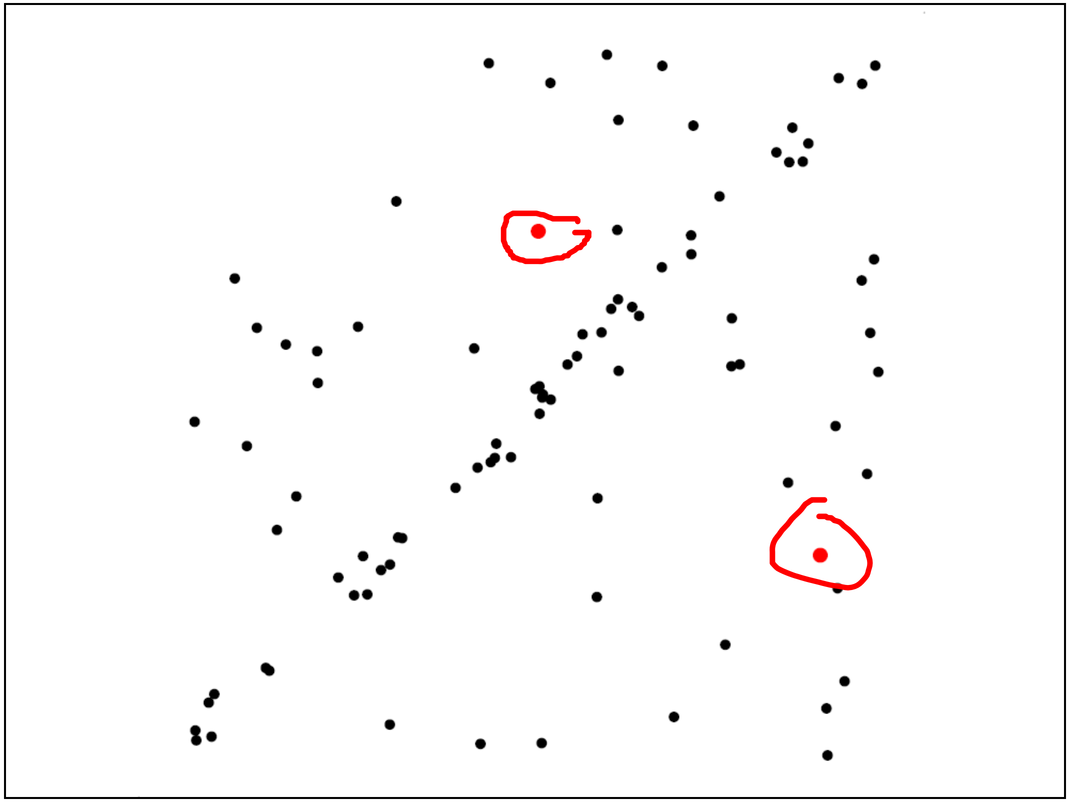
RANSAC for line fitting example



Least-squares fit

Source: R. Raguram

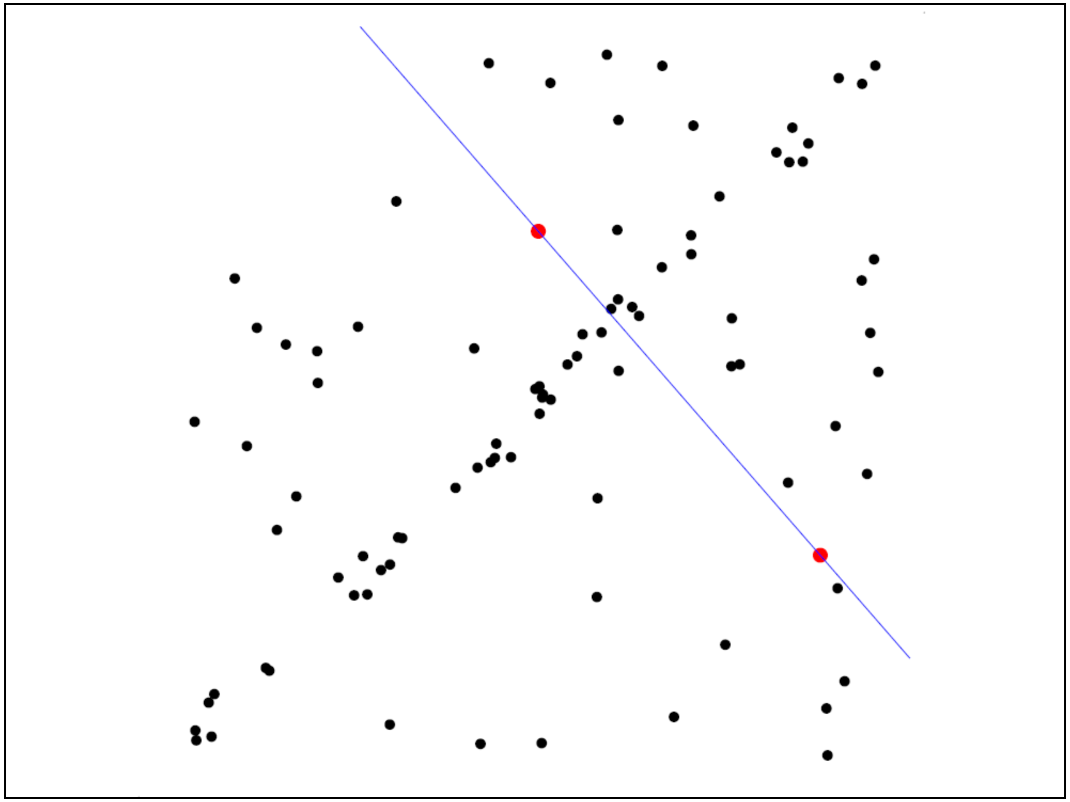
RANSAC for line fitting example



1. Randomly select minimal subset of points

Source: R. Raguram

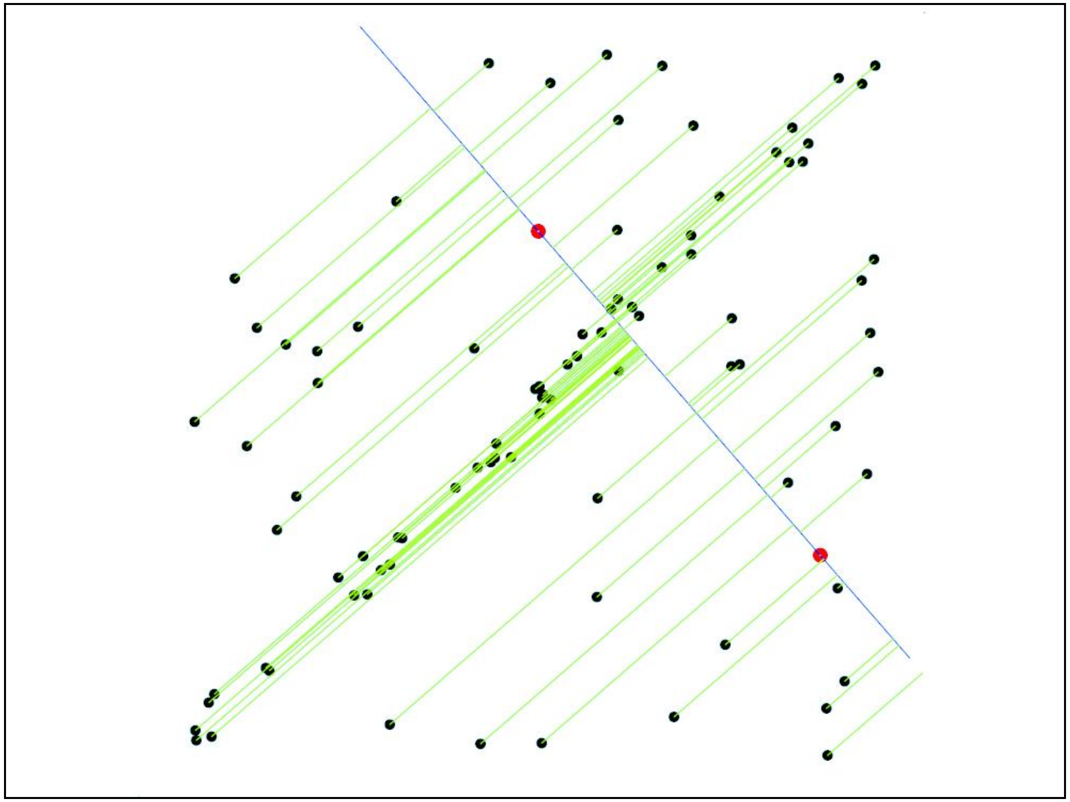
RANSAC for line fitting example



1. Randomly select minimal subset of points
2. Hypothesize a model

Source: R. Raguram

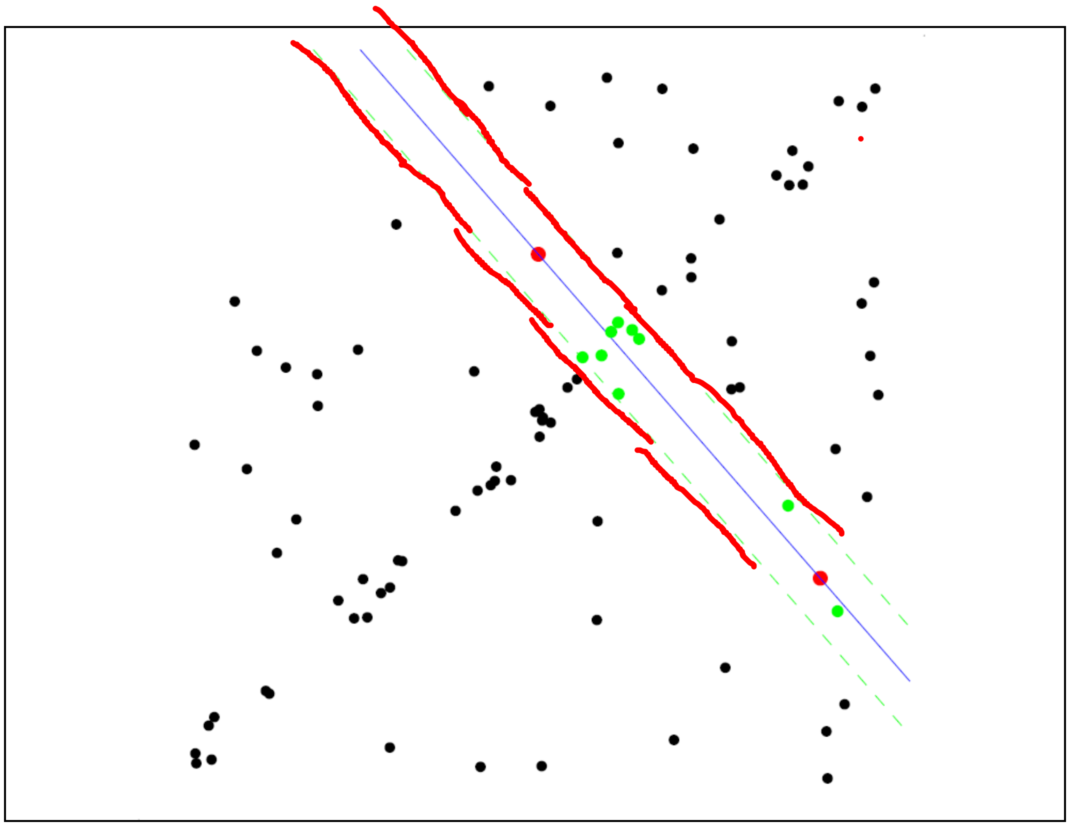
RANSAC for line fitting example



1. Randomly select minimal subset of points
2. Hypothesize a model
3. Compute error function

Source: R. Raguram

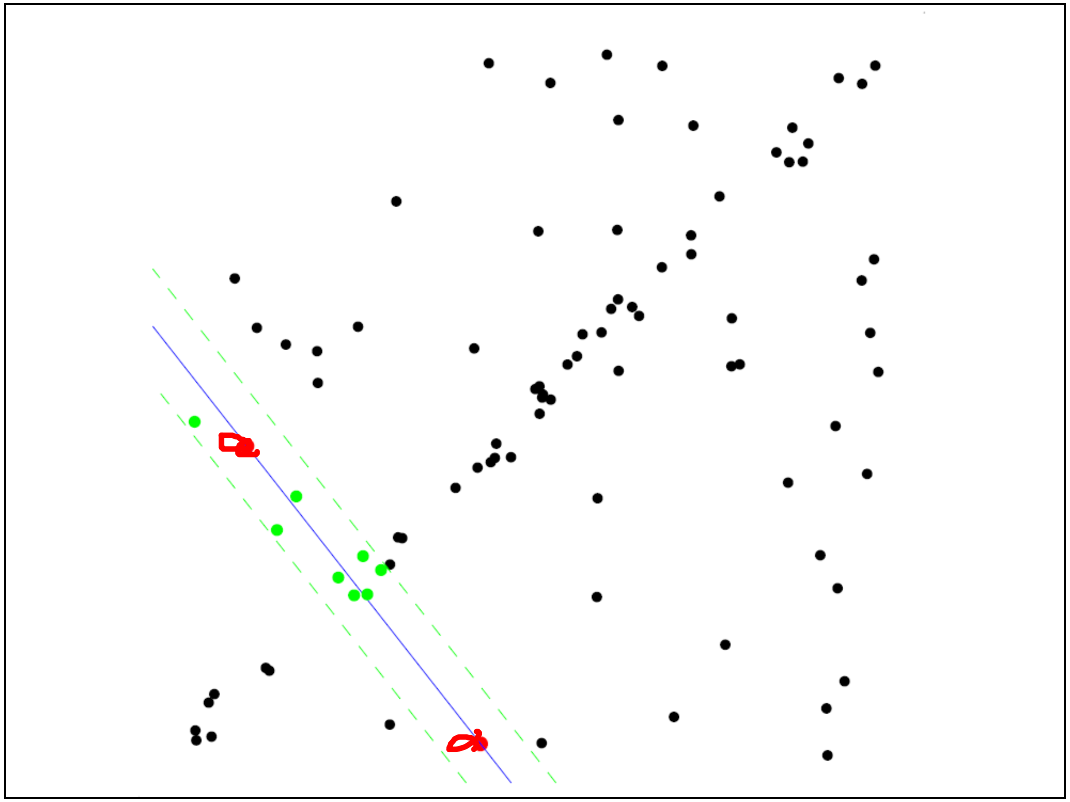
RANSAC for line fitting example



1. Randomly select minimal subset of points
2. Hypothesize a model
3. Compute error function
4. Select points consistent with model

Source: R. Raguram

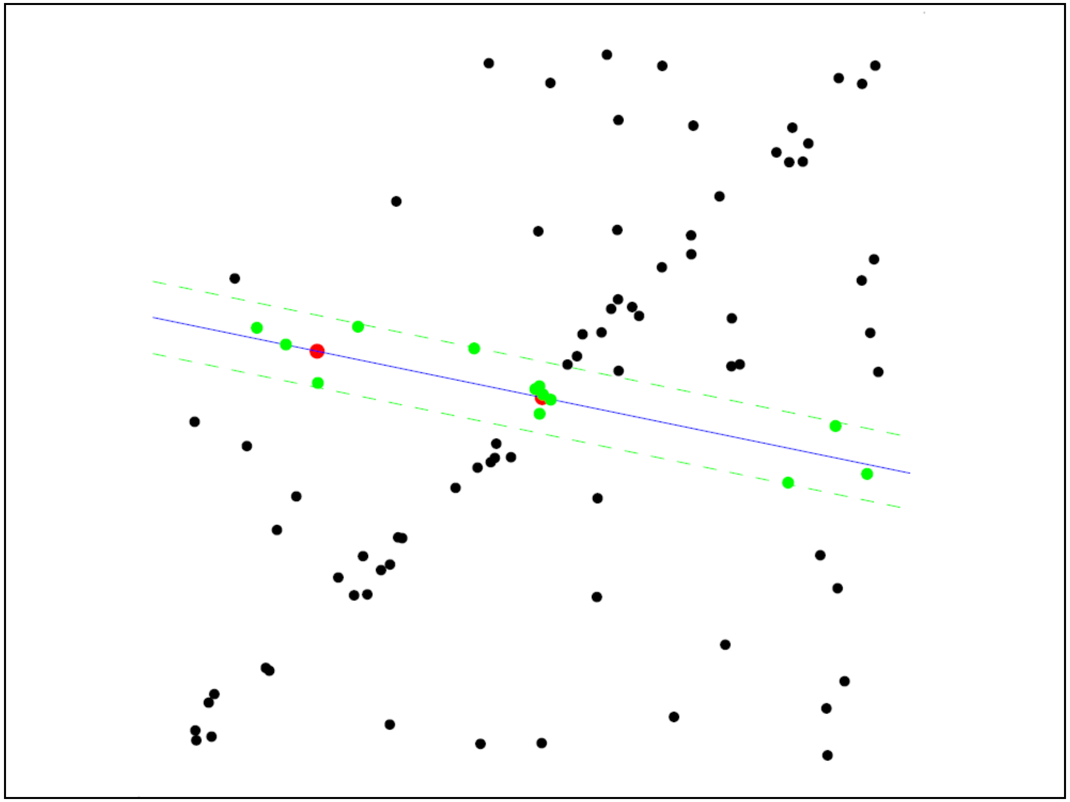
RANSAC for line fitting example



1. Randomly select minimal subset of points
2. Hypothesize a model
3. Compute error function
4. Select points consistent with model
5. Repeat *hypothesize-and-verify* loop

Source: R. Raguram

RANSAC for line fitting example



1. Randomly select minimal subset of points
2. Hypothesize a model
3. Compute error function
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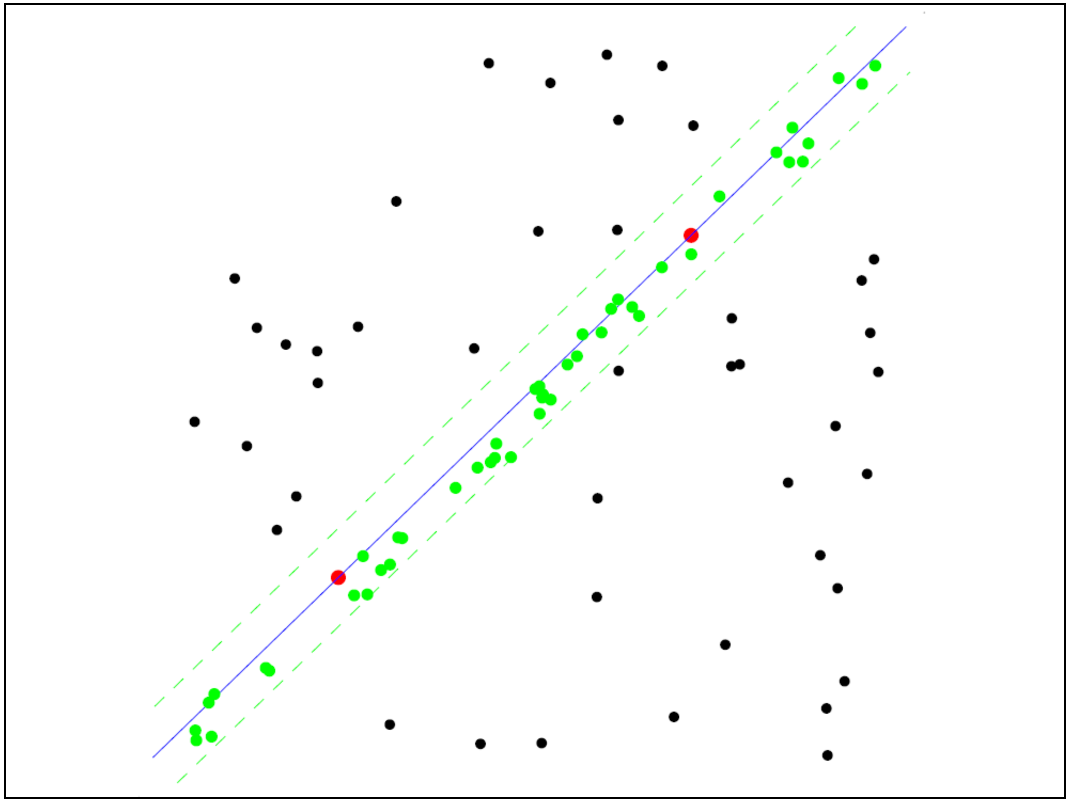
Source: R. Raguram

RANSAC for line fitting example

Uncontaminated sample



1. Randomly select minimal subset of points
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3. Compute error function
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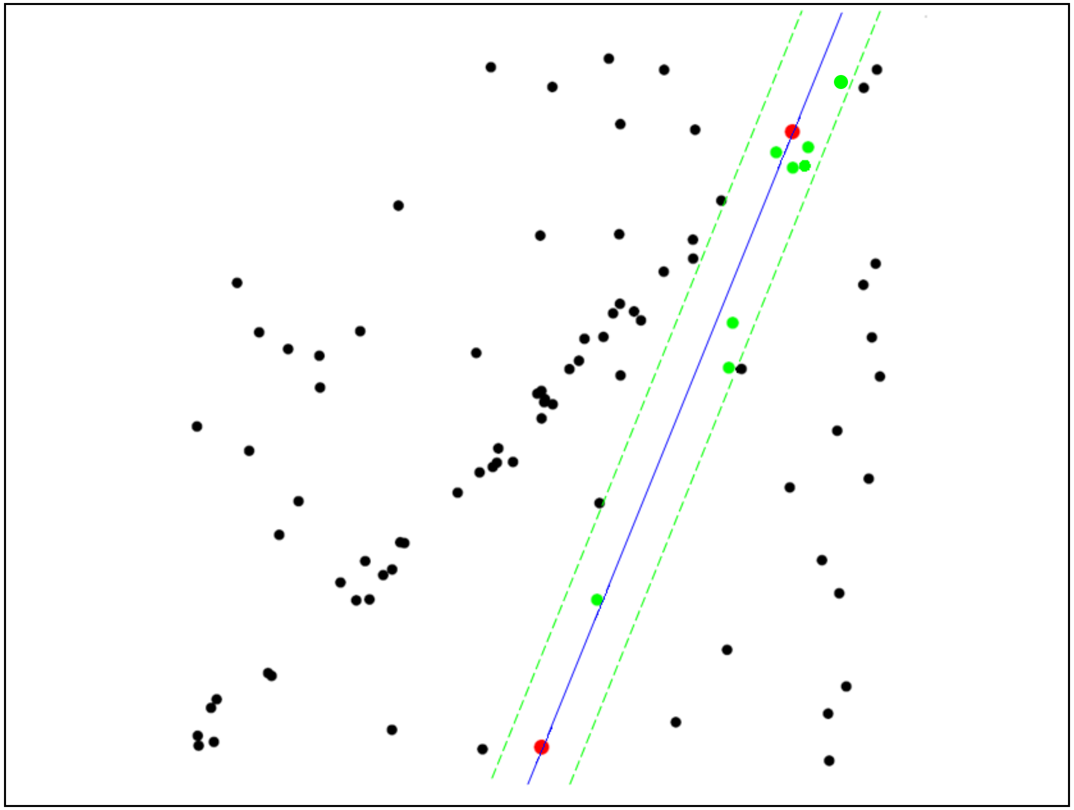
5. Repeat *hypothesize-and-verify* loop

Source: R. Raguram

RANSAC for line fitting example



1. Randomly select minimal subset of points
2. Hypothesize a model
3. Compute error function
4. Select points consistent with model
5. Repeat *hypothesize-and-verify* loop

Source: R. Raguram

RANSAC for line fitting

Repeat ***N*** times:

- Draw ***s*** points uniformly at random
- Fit line to these ***s*** points
- Find *inliers* to this line among the remaining points (i.e., points whose distance from the line is less than ***t***)
- If there are ***d*** or more inliers, accept the line and refit using all inliers

Choosing the parameters

- Initial number of points ***s***
  - Typically minimum number needed to fit the model
- Distance threshold ***t***
  - Choose ***t*** so probability for inlier is *p* (e.g. 0.95)
  - Zero-mean Gaussian noise with std. dev. σ
  - $t^2=3.84\sigma^2$ follows χ^2 distribution for 2D line (emperically)
- Number of samples ***N***
  - Choose ***N*** so that, with probability *p*, at least one random sample is free from outliers (e.g. *p*=0.99) (outlier ratio: *e*)

Source: M. Pollefeys

Choosing the parameters

- Initial number of points ***s***
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- Number of samples ***N***
  - Choose ***N*** so that, with probability *p*, at least one random sample is free from outliers (e.g. *p*=0.99) (outlier ratio: *e*)

$$\boxed{\left(1-\left(1-e\right)^s\right)^N = 1-p}$$

$$N = \log\left(1-p\right) / \log\left(1-\left(1-e\right)^s\right)$$

	proportion of outliers *e*						
s	5%	10%	20%	25%	30%	40%	50%
2	2	3	5	6	7	11	17
3	3	4	7	9	11	19	35
4	3	5	9	13	17	34	72
5	4	6	12	17	26	57	146
6	4	7	16	24	37	97	293
7	4	8	20	33	54	163	588
8	5	9	26	44	78	272	1177

Source: M. Pollefeys

Choosing the parameters

- Initial number of points ***s***
  - Typically minimum number needed to fit the model
- Distance threshold ***t***
  - Choose ***t*** so probability for inlier is *p* (e.g. 0.95)
  - Zero-mean Gaussian noise with std. dev. σ: $t^2=3.84\sigma^2$
- Number of samples ***N***
  - Choose ***N*** so that, with probability *p*, at least one random sample is free from outliers (e.g. *p*=0.99) (outlier ratio: *e*)
- Consensus set size ***d***
  - Should match expected inlier ratio

Source: M. Pollefeys

Adaptively determining the number of samples

- Outlier ratio *e* is often unknown a priori, so pick worst case, e.g. 50%, and adapt if more inliers are found, e.g. 80% would yield *e*=0.2
- Adaptive procedure:
  - $N=\infty$, *sample_count* =0
  - While *N* >*sample_count*
    - – Choose a sample and count the number of inliers
    - – If inlier ratio is highest of any found so far, set e = 1 – (number of inliers)/(total number of points)
    - – Recompute *N* from *e:*

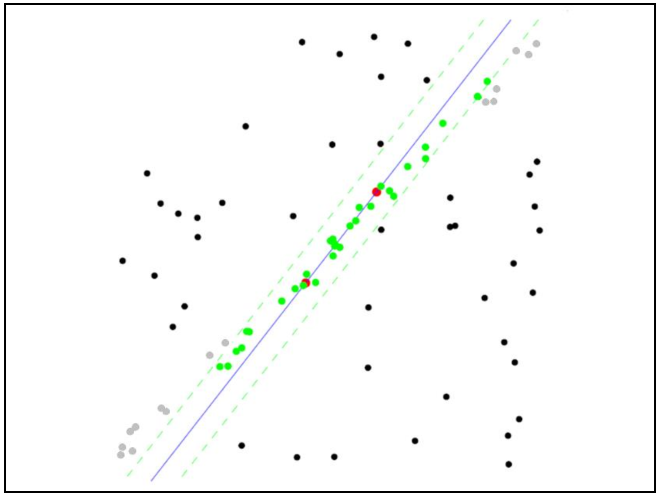
$$N = \log(1-p)/\log\left(1-(1-e)^s\right)$$

    - – Increment the *sample_count* by 1

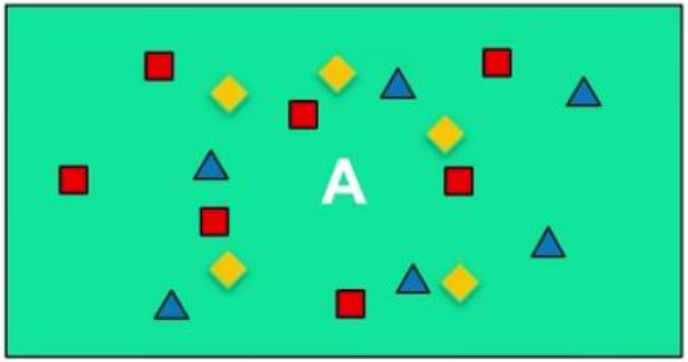
Source: M. Pollefeys

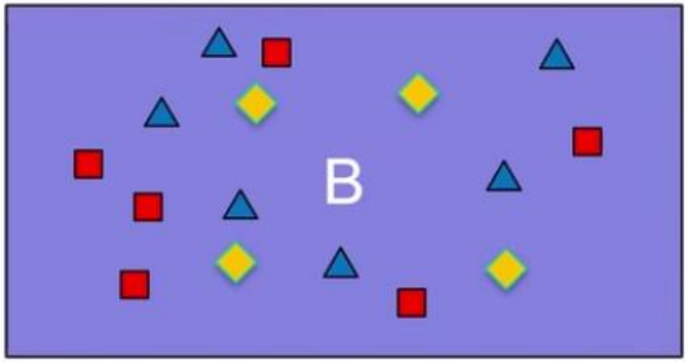
RANSAC pros and cons

- Pros
  - Simple and general
  - Applicable to many different problems
  - Often works well in practice
- Cons
  - Lots of parameters to tune
  - Doesn’t work well for low inlier ratios (too many iterations, or can fail completely)
  - Can’t always get a good initialization of the model based on the minimum number of samples



Exercise: Fitting Affine Transformation





$$\begin{bmatrix} x_a \\ y_a \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix} \begin{bmatrix} x_b \\ y_b \\ 1 \end{bmatrix}$$

Fitting: Review

- ✓ If we know which points belong to the line, how do we find the “optimal” line parameters?
  - ✓ Least squares

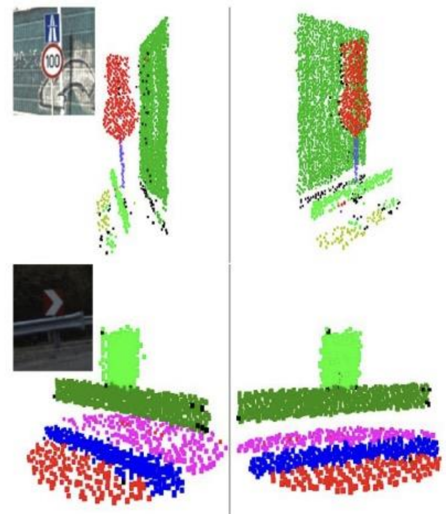
- ✓ What if there are outliers?
  - ✓ Robust fitting, RANSAC

- What if there are many lines?
  - Voting methods: RANSAC, Hough transform

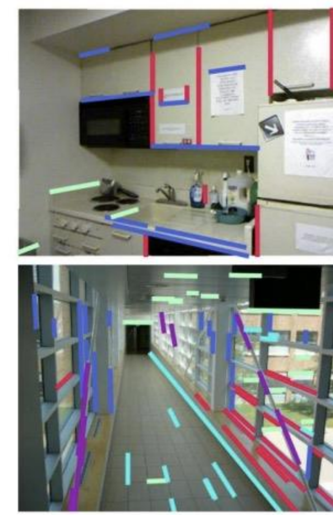
Exploiting the Spatial Coherence of Geometric Data

Motivation: In vision, we usually have geometric data, e.g., 3D points, where the points often originate from spatially coherent structures.

Two-view geometry. (*Left*) Rigid motions in two views. (*Right*) 1st images of image pairs with the inliers of homographies.



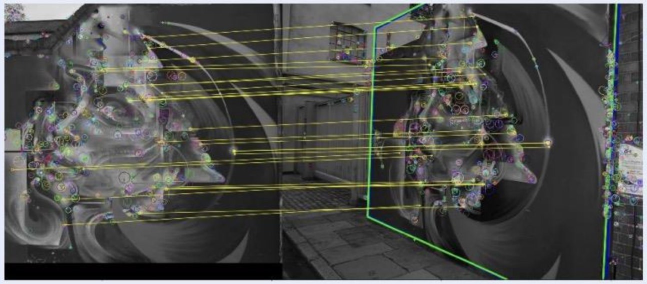
Planes in LiDAR data.



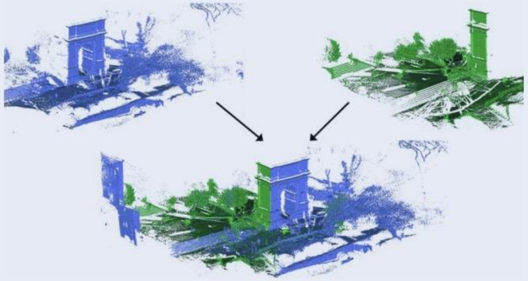
Vanishing points (similar line directions)

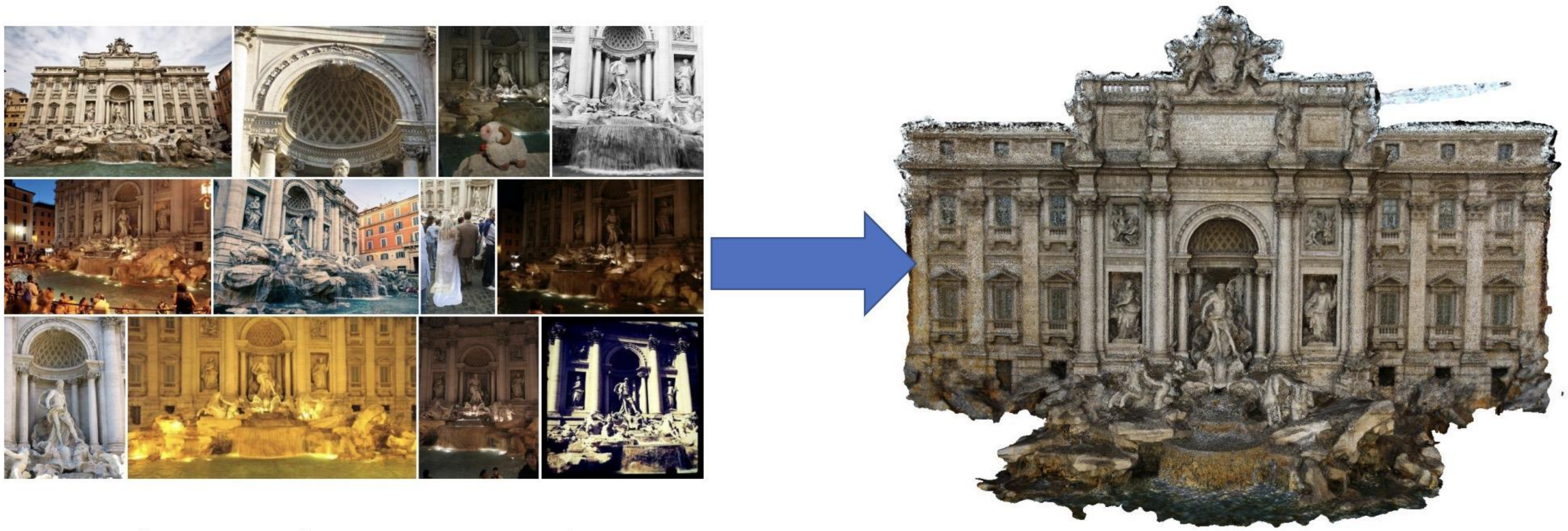
Fundamental
&
Essential matrices

Homography



Rigid point cloud
registration





- 30k images from YCC100M dataset, in 26 scenes
- “Ground truth” established by COLMAP reconstruction
- The basis of Image Matching Competitions 2019 & 2020

https://www.cs.ubc.ca/research/image-matching-challenge/current/