Lecture 12

- Frequency Response

Outline

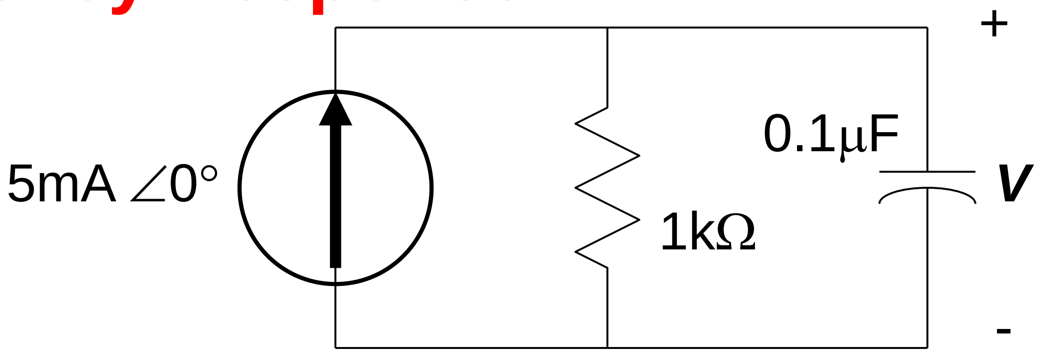
- Frequency response

-Transfer function

~~*-Bode plots (or diagram)*~~

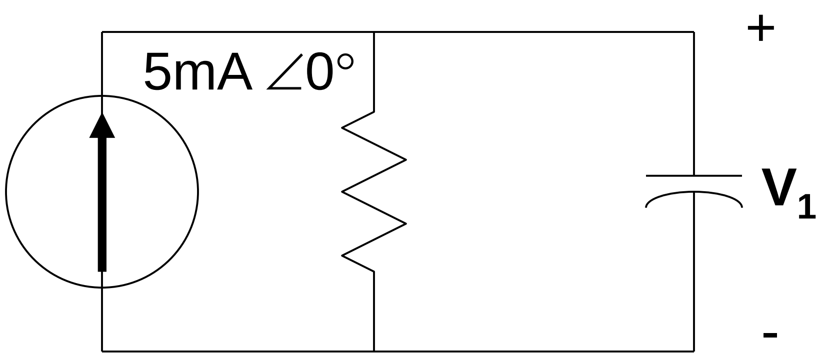
-Resonance

Frequency Response



Find ***V***

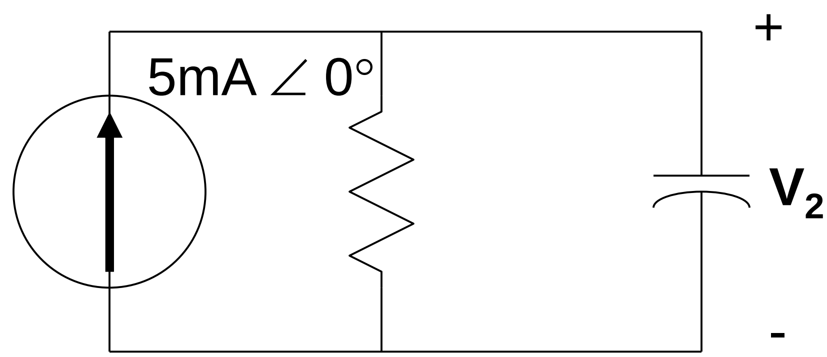
Case 1: ω=2π x 3000



$$\mathbf{Z}_{eq} = 468.2\angle -62.1°\Omega$$

$$\mathbf{V}_1 = 2.34\angle -62.1°\mathrm{V}$$

Case 2: ω=2π x 455000

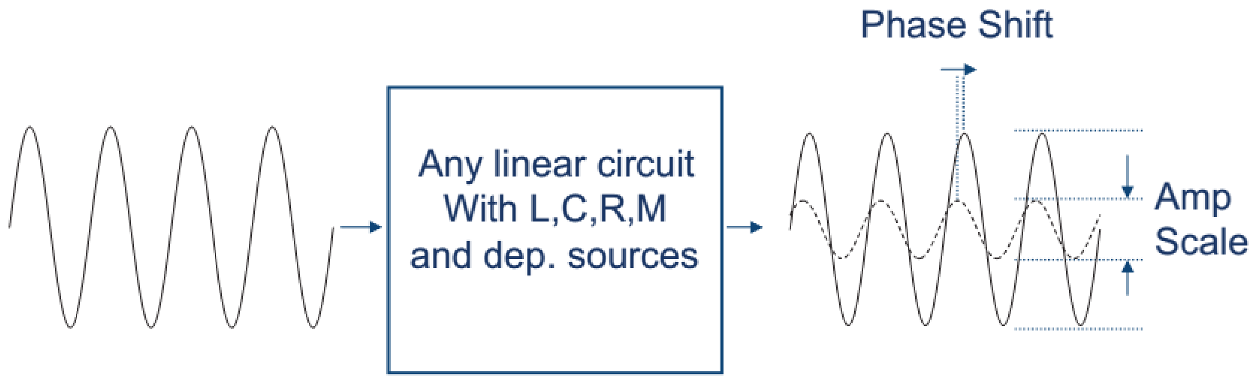


$$\mathbf{Z}_{eq} = 3.5\angle -89.8°\Omega$$

$$\mathbf{V}_2 = 17.5\angle -89.8°\mathrm{mV}$$

[Source: Berkeley]

Frequency Response



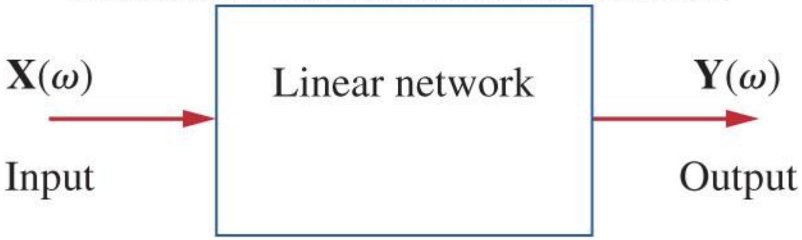
- The "Frequency Response" is a characterization of the input-output relation for sinusoidal inputs at **all** frequencies.

- Its output is also a sinusoid at the ***same*** frequency.
- Only the magnitude and phase of the output differ from the input.
- Significant for applications, esp. in communications and control systems.

[Source: Berkeley]

Transfer Function

- The transfer function *H(ω)* is the frequency-dependent ratio of a forced function *Y(ω)* to the forcing function *X(ω)*.



$$H(\omega) = \frac{Y(\omega)}{X(\omega)}$$

$$H(\omega) = \text{Voltage gain} = \frac{V_o(\omega)}{V_i(\omega)}$$

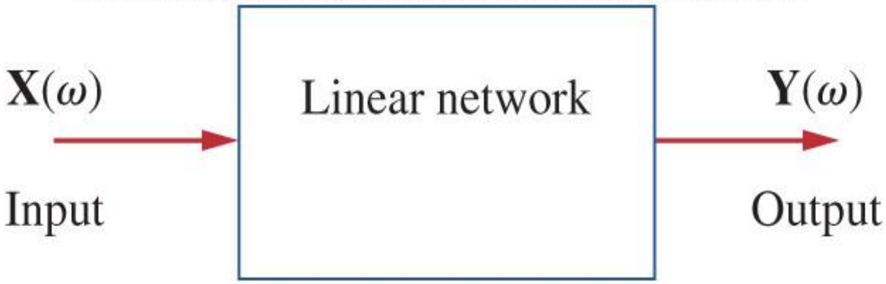
$$H(\omega) = \text{Current gain} = \frac{I_o(\omega)}{I_i(\omega)}$$

$$H(\omega) = \text{Transfer impedance} = \frac{V_o(\omega)}{I_i(\omega)}$$

$$H(\omega) = \text{Transfer admittance} = \frac{I_o(\omega)}{V_i(\omega)}$$

Transfer Function

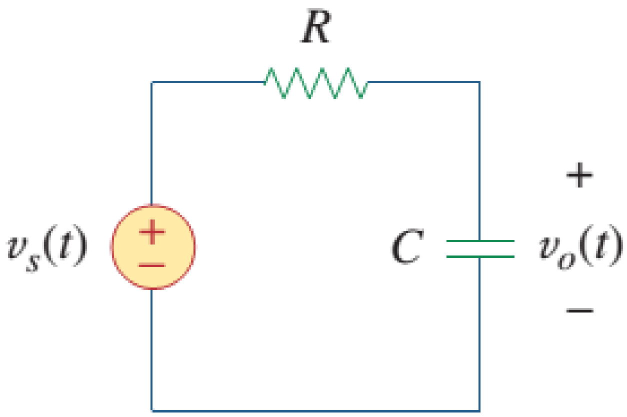
- Complex quantity
- Both *magnitude and phase* are functions of frequency

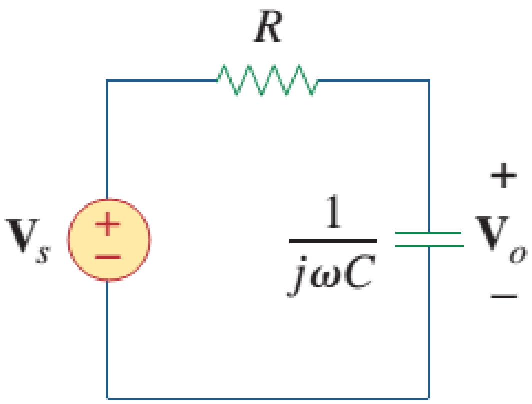


For example:

$$\frac{\mathbf{V_{out}}}{\mathbf{V_{in}}} = \frac{V_{out}}{V_{in}} \angle\left(\theta_{out} - \theta_{in}\right)$$

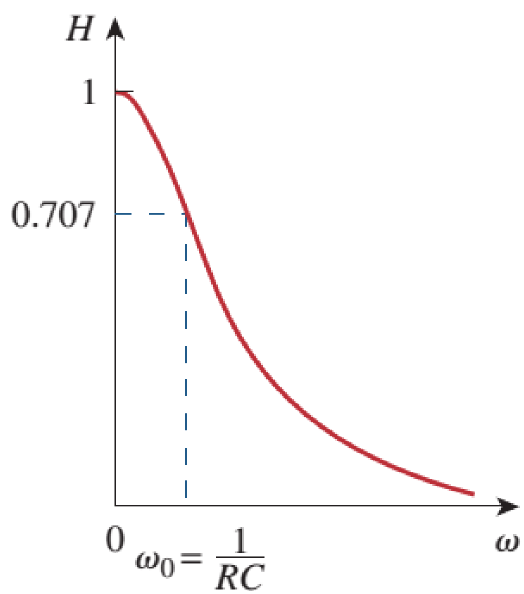
Example

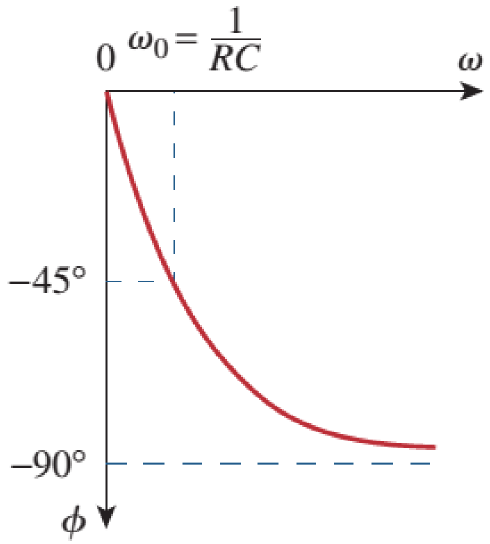




$$H = \frac{1}{\sqrt{1 + (\omega/\omega_0)^2}}, \qquad \phi = -\tan^{-1}\frac{\omega}{\omega_0}$$

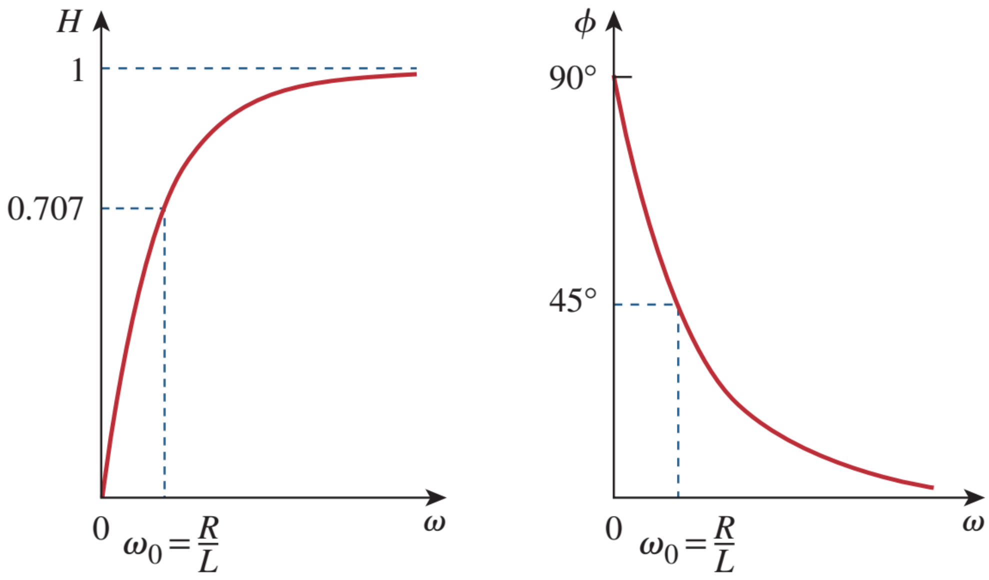
ω/ω_0	H	ϕ	ω/ω_0	H	ϕ
0	1	0	10	0.1	$-84°$
1	0.71	$-45°$	20	0.05	$-87°$
2	0.45	$-63°$	100	0.01	$-89°$
3	0.32	$-72°$	∞	0	$-90°$

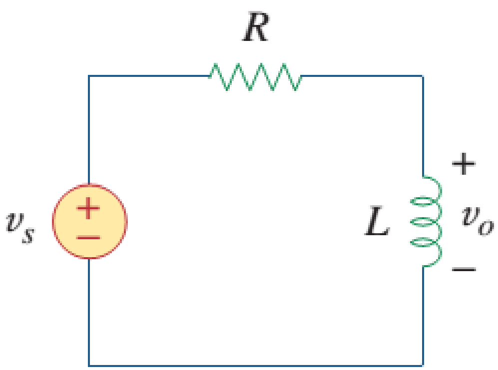




Exercise

- Obtain the transfer function $\mathbf{V_o}/\mathbf{V_s}$ of the RL circuit.

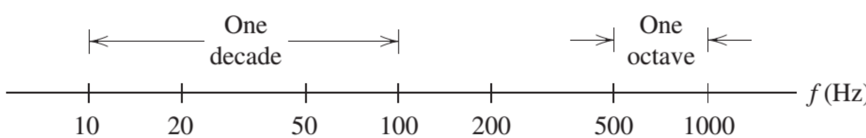




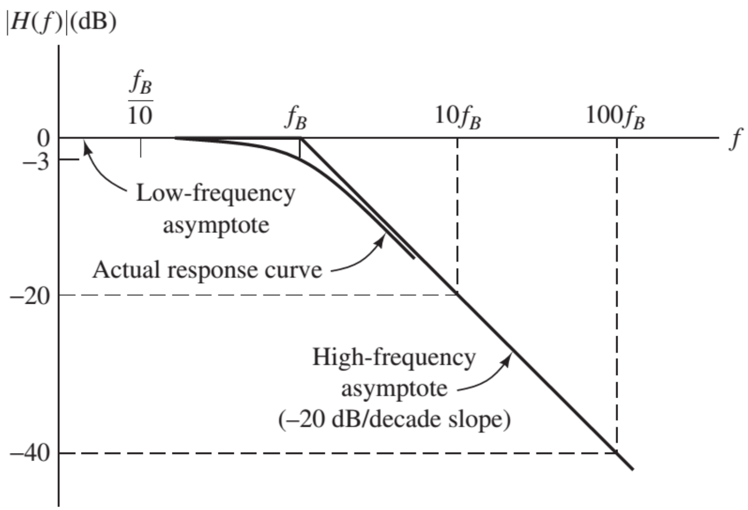
~~Bode Plots~~

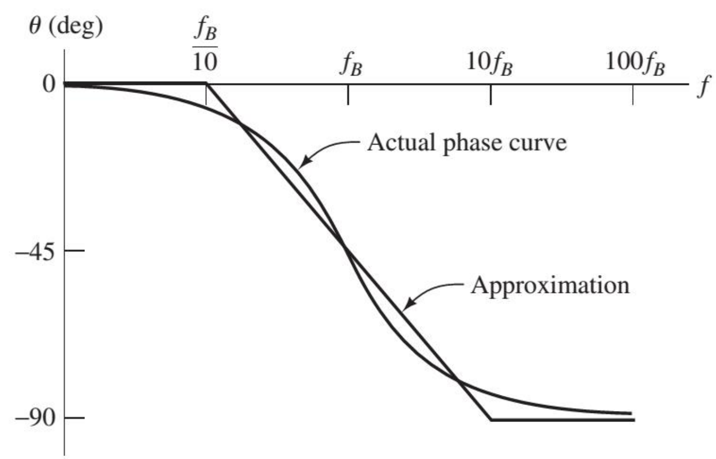
Plotting the frequency response, magnitude & phase, on plots with

- Frequency X in log scale



- Y scale in dB (for magnitude) & degree (for phase)





Bel and Decibel (dB)

- A **bel** (symbol **B**) is a unit of measure of ratios of power levels, i.e. relative power levels.
  - The name was coined in the early 20th century in honor of Alexander Graham Bell, a telecommunication pioneer.
  - Definition of bel:

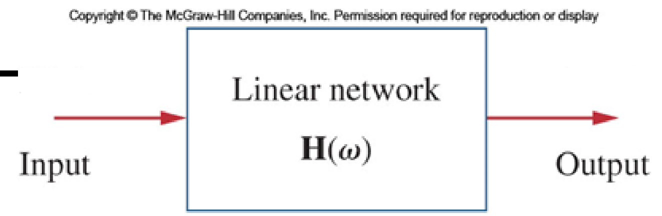
$$\text{Ratio with a unit of B} = \log_{10}(P_1/P_2)$$

  where P_1 and P_2 are power levels.

- One bel is too large for everyday use, so the **decibel (dB)**, equal to 0.1B, is more commonly used.

$$\text{Ratio with a unit of dB} = 10\log_{10}(P_1/P_2)$$

  - used to measure electric power, gain or loss of amplifiers, etc.



dB for Voltage or Current

- We can similarly relate the reference voltage or current to the reference power, as

$$P = (V)^2/R \text{ or } P = (I)^2R$$

Hence,

$$\text{Voltage, V in decibels} = \mathbf{20}\log_{10}(V_1/V_2)$$
$$\text{Current, I, in decibels} = \mathbf{20}\log_{10}(I_1/I_2)$$

Question: How many decibels larger is the voltage of a 9-volt transistor battery than that of a 1.5-volt AA battery?

Question: **The voltage gain** of an amplifier with input = 0.2 mV and output = 0.5 V is ?

[Source: Berkeley]

Summary of dB

If G is defined as the power gain,

$$G = \frac{P}{P_0},$$

then the corresponding gain in dB is defined as

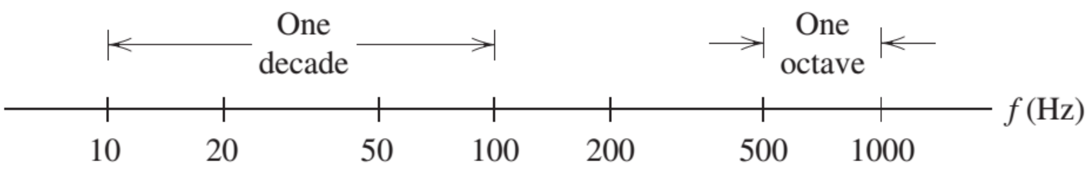
$$G\ [\text{dB}] = 10\log G = 10\log\left(\frac{P}{P_0}\right) \qquad (\text{dB}).$$

$$G\ [\text{dB}] = 10\log\left(\frac{\frac{1}{2}|\mathbf{V}|^2/R}{\frac{1}{2}|\mathbf{V}_0|^2/R}\right) = 20\log\left(\frac{|\mathbf{V}|}{|\mathbf{V}_0|}\right)$$

$\frac{P}{P_0}$	dB
10^N	$10N$ dB
10^3	30 dB
100	20 dB
10	10 dB
4	$\simeq 6$ dB
2	$\simeq 3$ dB
1	0 dB
0.5	$\simeq -3$ dB
0.25	$\simeq -6$ dB
0.1	-10 dB
10^{-N}	$-10N$ dB

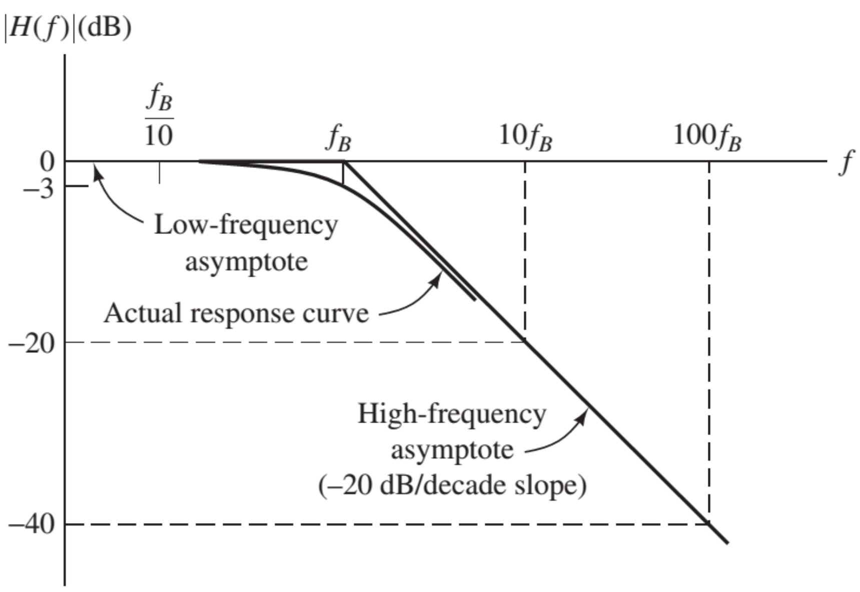
$\left\lvert\frac{\mathbf{V}}{\mathbf{V}_0}\right\rvert$ or $\left\lvert\frac{\mathbf{I}}{\mathbf{I}_0}\right\rvert$	dB
10^N	$20N$ dB
10^3	60 dB
100	40 dB
10	20 dB
4	$\simeq 12$ dB
2	$\simeq 6$ dB
1	0 dB
0.5	$\simeq -6$ dB
0.25	$\simeq -12$ dB
0.1	-20 dB
10^{-N}	$-20N$ dB

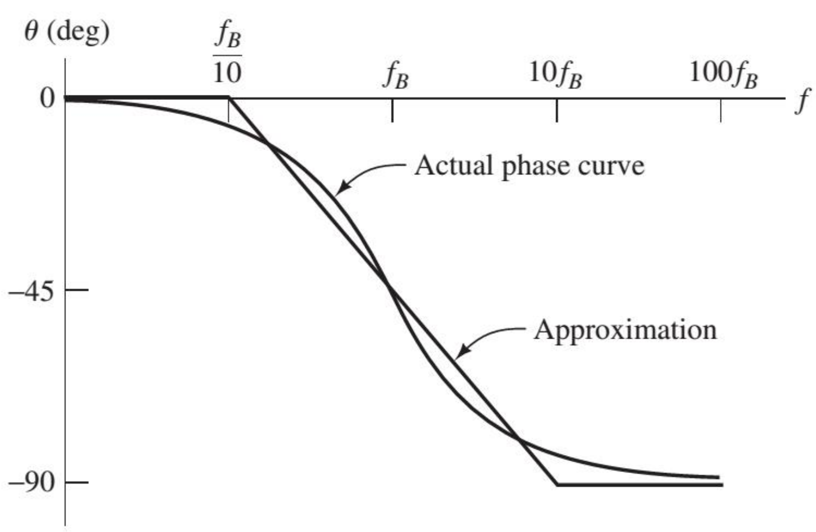
Bode Plots



Plotting the frequency response, magnitude or phase, on plots with

- Frequency X in log scale
- Y scale in dB (for magnitude) or degree (for phase)





Bode Plots

- Bode plot is particularly useful for displaying transfer function-- a general form is displayed as:

$$\mathbf{H}(\omega) = \frac{K(j\omega)^{\pm 1}(1 + j\omega/z_1)[1 + j2\zeta_1\omega/\omega_k + (j\omega/\omega_k)^2]\cdots}{(1 + j\omega/p_1)[1 + j2\zeta_2\omega/\omega_n + (j\omega/\omega_n)^2]\cdots}$$

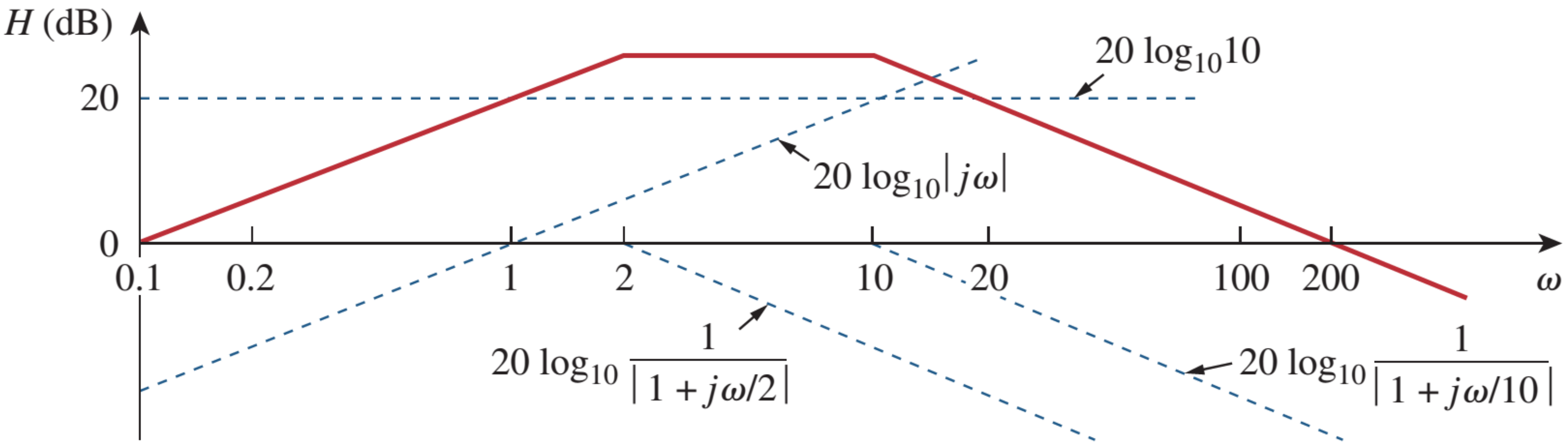
In constructing a Bode plot, we plot each factor separately and then add them graphically. The factors can be considered one at a time and then combined additively because of the logarithms involved. It is this mathematical convenience of the logarithm that makes Bode plots a powerful engineering tool.

Example--Standard Form

$$\mathbf{H}(\omega) = \frac{200\,j\omega}{(j\omega + 2)(j\omega + 10)}$$

$$\mathbf{H}(\omega) = \frac{10\,j\omega}{(1 + j\omega/2)(1 + j\omega/10)}$$

Example - Magnitude



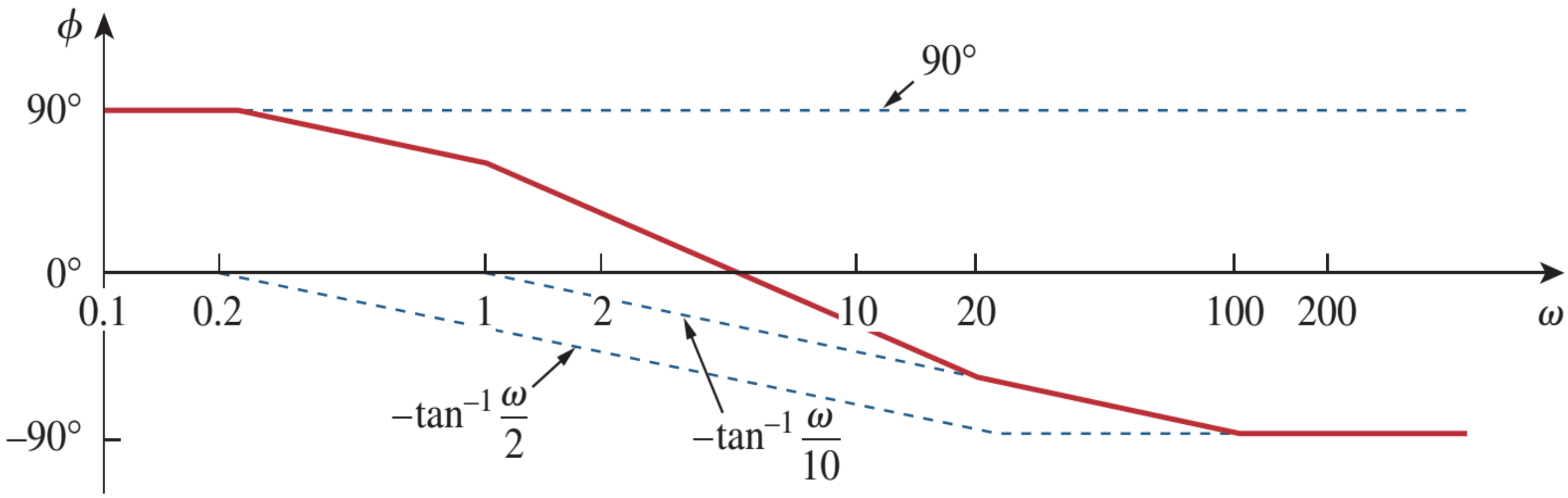
$$H_{\text{dB}} = 20\log_{10}10 + 20\log_{10}|j\omega| - 20\log_{10}\left|1 + \frac{j\omega}{2}\right| - 20\log_{10}\left|1 + \frac{j\omega}{10}\right|$$

Example - Phase

$$\mathbf{H}(\omega) = \frac{10j\omega}{(1 + j\omega/2)(1 + j\omega/10)}$$

$$= \frac{10|j\omega|}{|1 + j\omega/2||1 + j\omega/10|} \underline{/90^\circ - \tan^{-1}\omega/2 - \tan^{-1}\omega/10}$$

$$\phi = 90^\circ - \tan^{-1}\frac{\omega}{2} - \tan^{-1}\frac{\omega}{10}$$



Outline

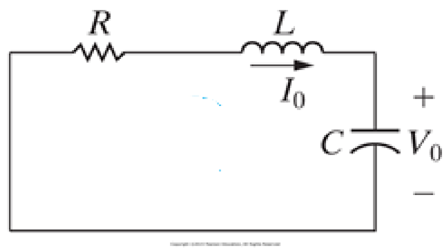
- Frequency response

-Transfer function

~~*-Bode plots (or diagram)*~~

-Resonance

Resonance in Series RLC Network (Underdamped)



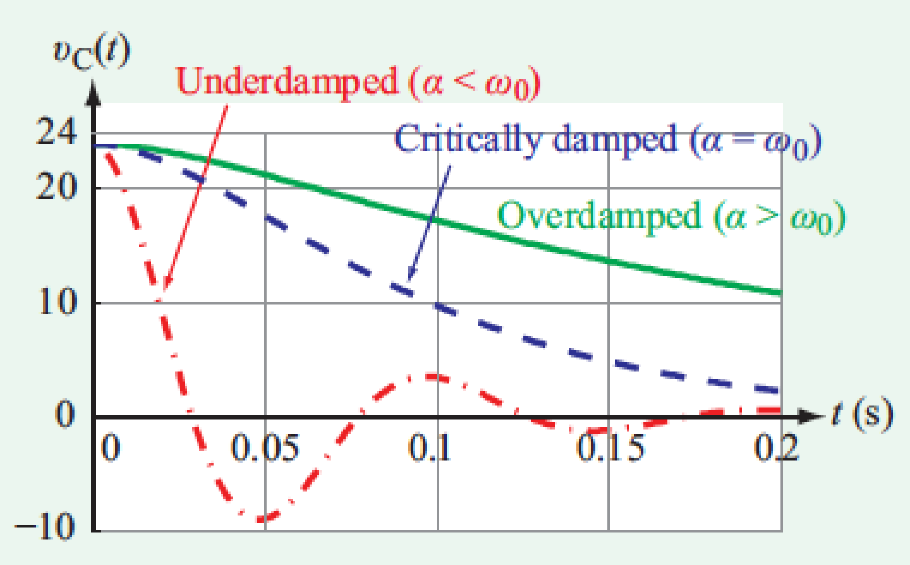
- $\alpha < \omega_0$ (i.e., $R < 2\sqrt{\frac{L}{C}}$)

$$v(t) = e^{-\alpha t}(B_1 \cos \omega_d t + B_2 \sin \omega_d t)$$

$$\frac{d^2 v}{d^2 t} + \frac{R}{L}\frac{dv}{dt} + \frac{1}{\mathrm{LC}} v = 0$$

$$s = -\frac{R}{2L} \pm \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$
$$= -\alpha \pm \sqrt{-(\omega_0^2 - \alpha^2)}$$
$$= -\alpha \pm j\omega_d$$

$$\alpha = \frac{R}{2L} \quad \omega_0 = \frac{1}{\sqrt{LC}} \quad \omega_d = \sqrt{\omega_0^2 - \alpha^2}$$

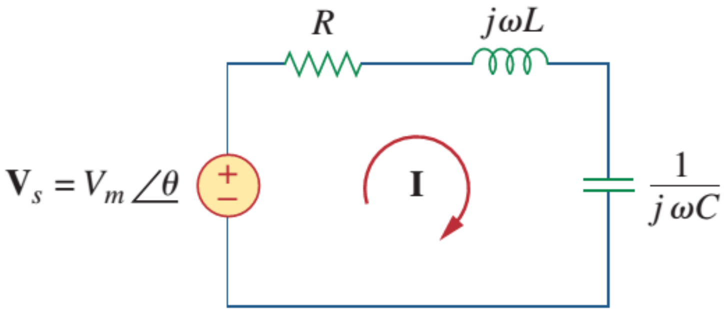


[Source: Berkeley]

Series Resonance

- A series resonant circuit consists of an inductor and capacitor in series.

$$H(\omega) = \frac{\boldsymbol{V}}{\boldsymbol{I}} = \mathbf{Z} = R + j\left(\omega L - \frac{1}{\omega C}\right)$$

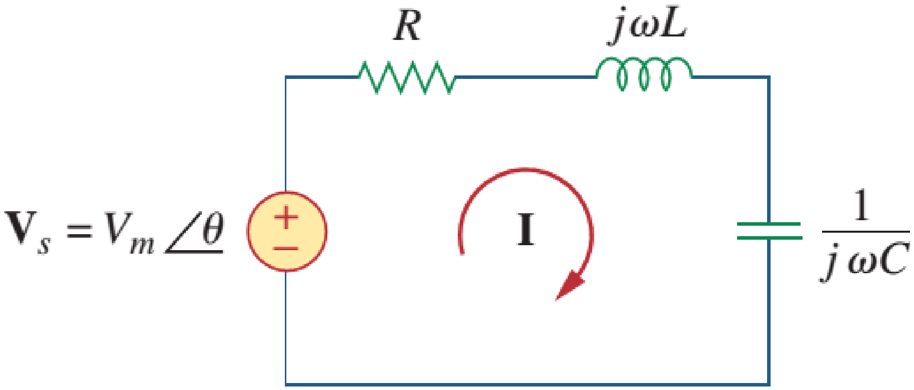


  - Resonance occurs when the imaginary part of *Z* is zero.
  - The value of ω that satisfies this is called the resonant frequency.

$$\boxed{\omega_0 = \frac{1}{\sqrt{LC}} \text{ rad/s}}$$

Series Resonance

- At resonance:
  - The impedance is purely resistive
  - The voltage $\boldsymbol{V_s}$ and the current $\boldsymbol{I}$ are in phase
  - The magnitude of the transfer function is minimum
  - The inductor and capacitor voltages can be much higher than the source voltage

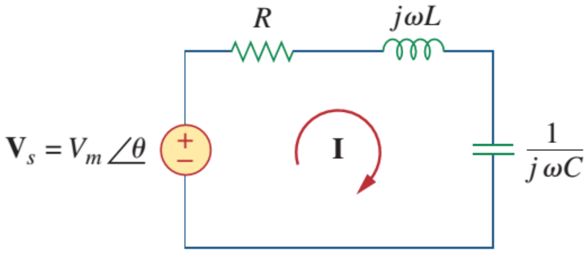


$$|\mathbf{V}_L| = \frac{V_m}{R}\omega_0 L$$

$$|\mathbf{V}_C| = \frac{V_m}{R}\frac{1}{\omega_0 C}$$

$$H(\omega) = \frac{\boldsymbol{V}}{\boldsymbol{I}} = \mathbf{Z} = R + j\left(\omega L - \frac{1}{\omega C}\right)$$

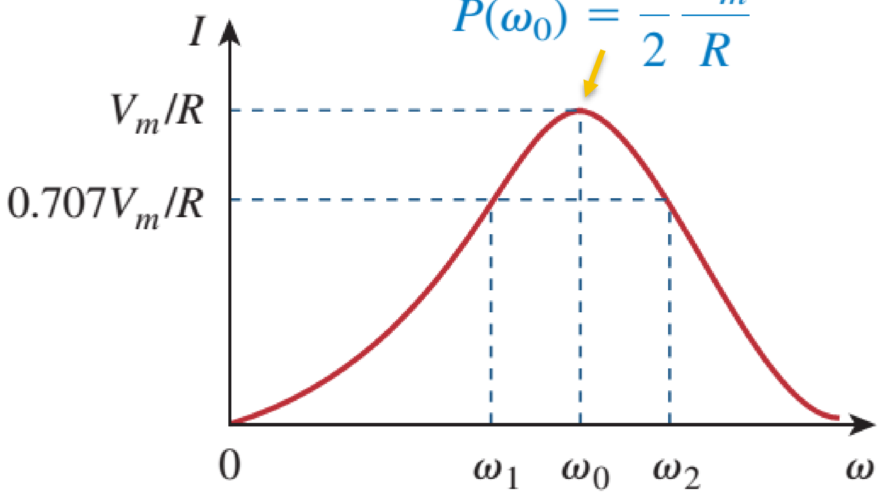
Half-Power Frequencies



- the current magnitude:

$$I = |\mathbf{I}| = \frac{V_m}{\sqrt{R^2 + (\omega L - 1/\omega C)^2}}$$

$$P(\omega_0) = \frac{1}{2}\frac{V_m^2}{R}$$



$$\omega_0 = \frac{1}{\sqrt{LC}} \text{ rad/s}$$

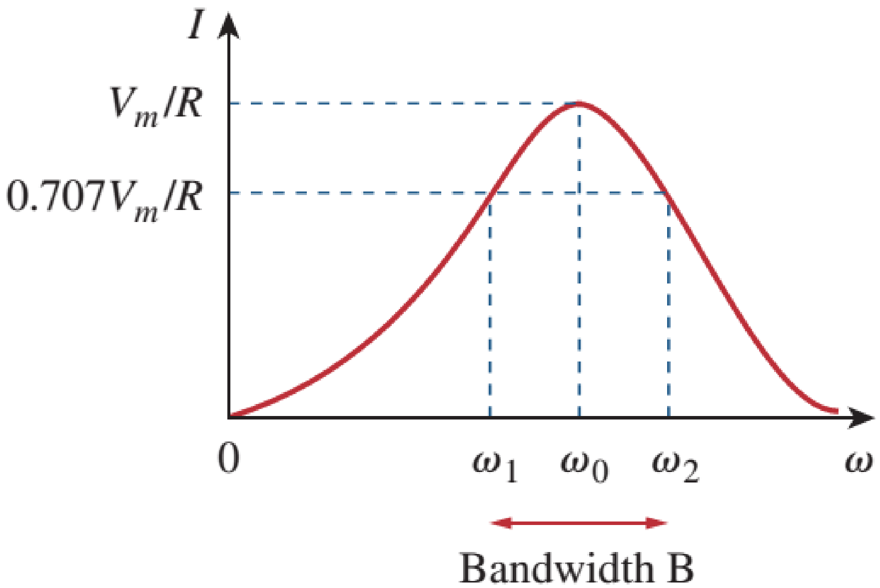
$$\boxed{P(\omega_1) = P(\omega_2) = \frac{1}{2}P(\omega_0)}$$

$$\omega_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

$$\omega_2 = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

$$\omega_0 = \sqrt{\omega_1 \omega_2}$$

Bandwidth



$$\omega_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

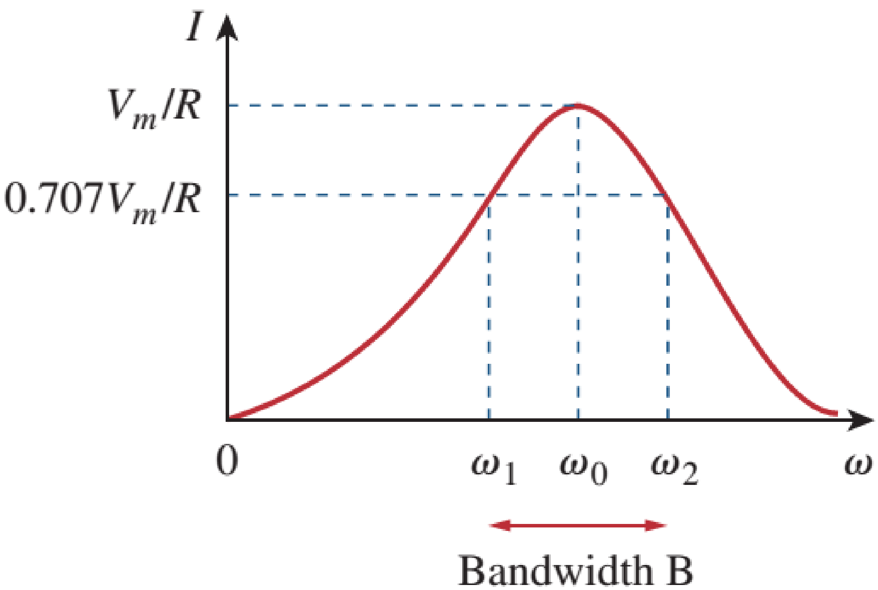
$$\omega_2 = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

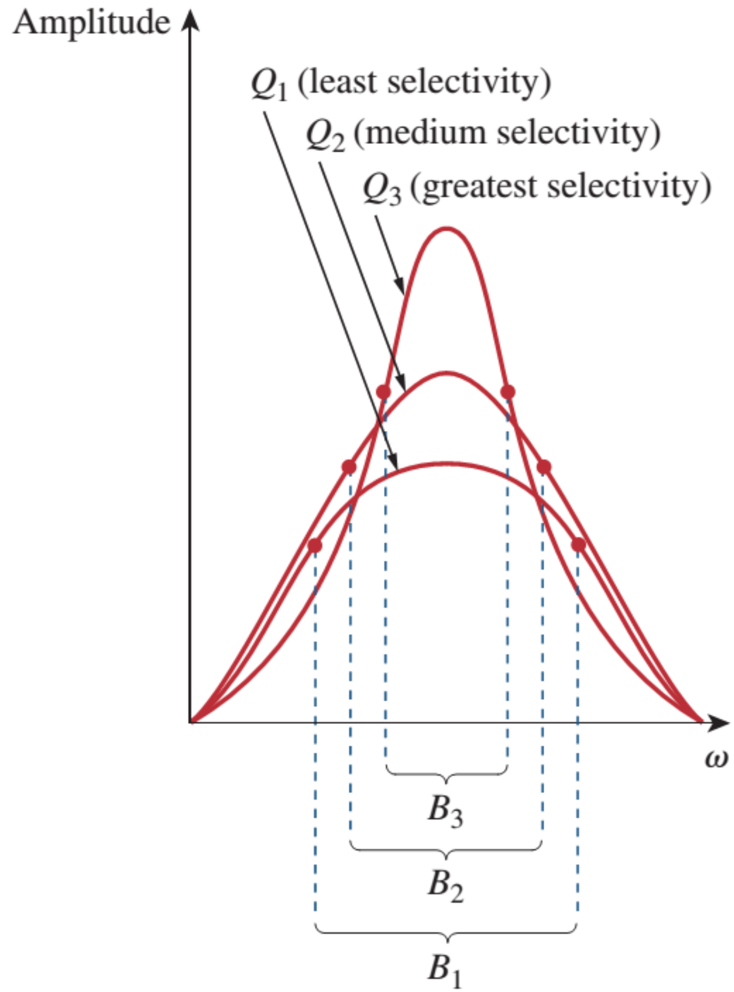
$$B = \omega_2 - \omega_1 = \frac{R}{L}$$

- Bandwidth: the difference between the two half-power frequencies

Quality Factor Q

- Quality factor Q: measure the "sharpness" of the resonance.



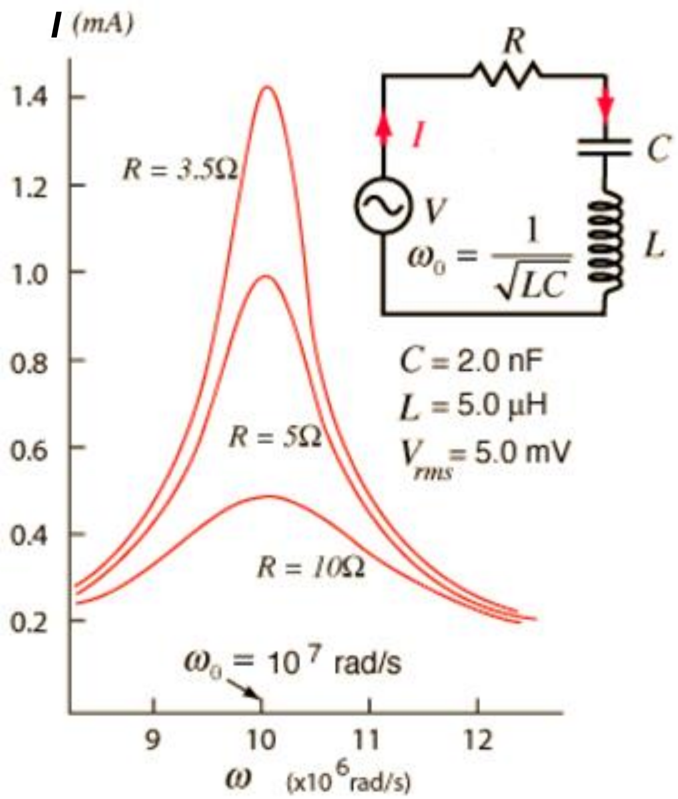


The smaller the B, the higher the Q.

$$Q = \frac{\omega_0}{B} \qquad \omega_0 = \frac{1}{\sqrt{LC}} \qquad B = \omega_2 - \omega_1 = \frac{R}{L}$$

$$\boxed{Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 CR}}$$

Quality Factor *Q*

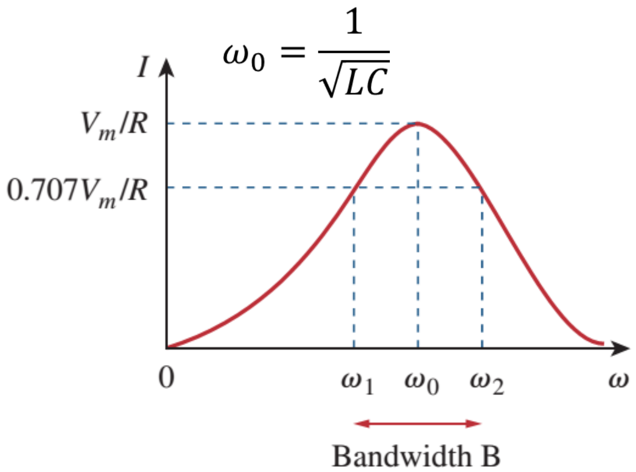


[Source: Georgia State U]

$$Q = \frac{\omega_0}{B}$$

$$\boxed{Q = \frac{\omega_0 L}{R} = \frac{1}{\omega_0 C R}}$$

Approximation of Half-Power Frequencies



$$\omega_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

$$\omega_2 = \frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 + \frac{1}{LC}}$$

$$\frac{R}{L} = B \qquad B = \frac{\omega_0}{Q}$$

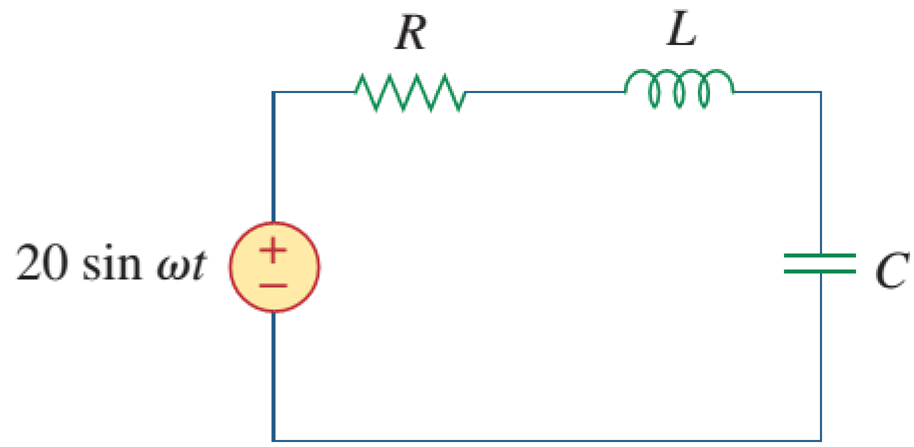
$$\omega_1 = -\frac{\omega_0}{2Q} + \omega_0\sqrt{1 + \frac{1}{4Q^2}}$$

$$\omega_2 = \frac{\omega_0}{2Q} + \omega_0\sqrt{1 + \frac{1}{4Q^2}}$$

- For high-Q ($Q \geq 10$) circuits, half-power frequencies can be approximated as

$$\omega_1 \simeq \omega_0 - \frac{B}{2}, \qquad \omega_2 \simeq \omega_0 + \frac{B}{2}$$

Example



In the circuit, $R = 2\Omega$, $L = 1\text{mH}$ and $C = 0.4\mu\text{F}$

- Find resonant frequency ω_0.
- Calculate *Q* and bandwidth *B.*
- Find half-power frequencies.
- Determine the amplitude of the current at $\omega_{0,}$ ω_1 and ω_2.

$$\omega_0 = \frac{1}{\sqrt{LC}} = \frac{1}{\sqrt{10^{-3} \times 0.4 \times 10^{-6}}} = 50 \text{ krad/s}$$

$$B = \omega_2 - \omega_1 = 2 \text{ krad/s}$$

$$Q = \frac{\omega_0}{B} = \frac{50}{2} = 25$$

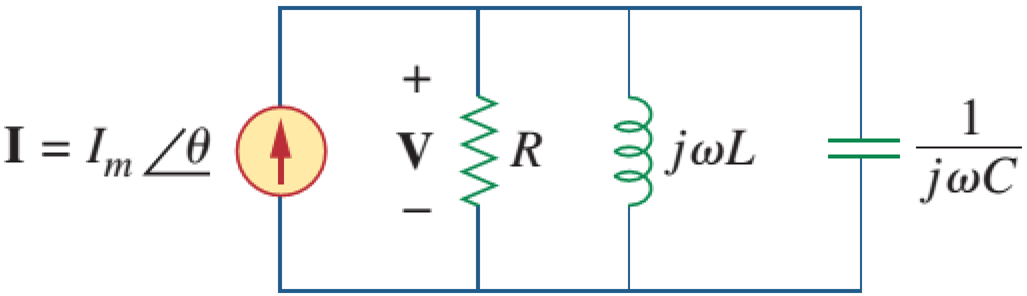
At $\omega = \omega_0$,

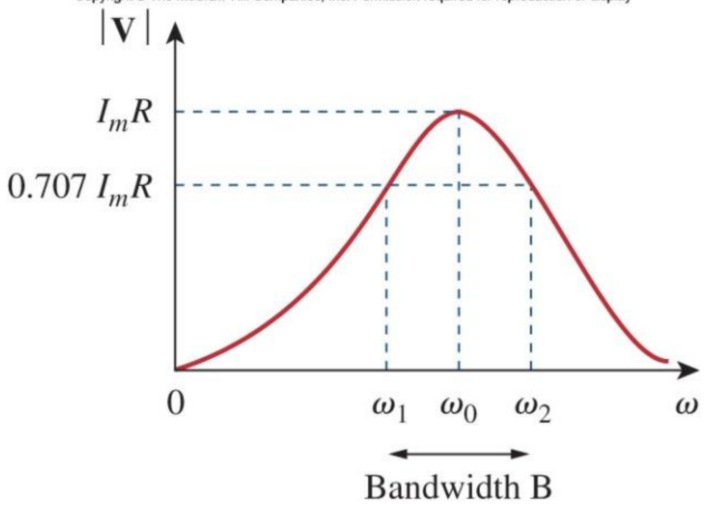
$$I = \frac{V_m}{R} = \frac{20}{2} = 10 \text{ A}$$

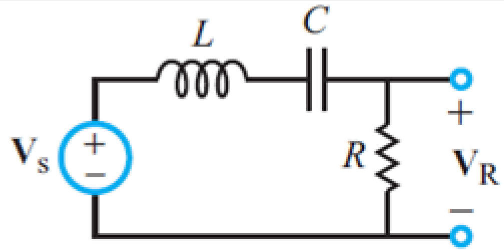
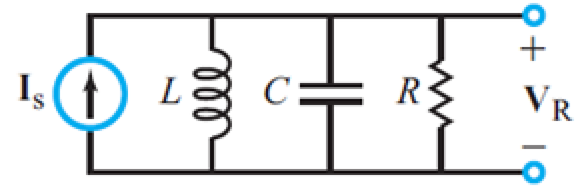
At $\omega = \omega_1, \omega_2$,

$$I = \frac{V_m}{\sqrt{2}R} = \frac{10}{\sqrt{2}} = 7.071 \text{ A}$$

Parallel resonance





RLC Circuit		
Transfer Function	$\mathbf{H} = \dfrac{\mathbf{V}_R}{\mathbf{V}_s}$	$\mathbf{H} = \dfrac{\mathbf{V}_R}{\mathbf{I}_s}$
Resonant Frequency, ω_0	$\dfrac{1}{\sqrt{LC}}$	$\dfrac{1}{\sqrt{LC}}$
Bandwidth, B	$\dfrac{R}{L}$	$\dfrac{1}{RC}$
Quality Factor, Q	$\dfrac{\omega_0}{B} = \dfrac{\omega_0 L}{R}$	$\dfrac{\omega_0}{B} = \dfrac{R}{\omega_0 L}$
Lower Half-Power Frequency, ω_1	$\left[-\dfrac{1}{2Q} + \sqrt{1 + \dfrac{1}{4Q^2}}\right]\omega_0$	$\left[-\dfrac{1}{2Q} + \sqrt{1 + \dfrac{1}{4Q^2}}\right]\omega_0$
Upper Half-Power Frequency, ω_2	$\left[\dfrac{1}{2Q} + \sqrt{1 + \dfrac{1}{4Q^2}}\right]\omega_0$	$\left[\dfrac{1}{2Q} + \sqrt{1 + \dfrac{1}{4Q^2}}\right]\omega_0$

Notes: (1) The expression for Q of the series RLC circuit is the inverse of that for Q of the parallel circuit. (2) For $Q \geq 10$, $\omega_1 \simeq \omega_0 - \frac{B}{2}$, and $\omega_2 \simeq \omega_0 + \frac{B}{2}$. [Source: Berkeley]