Adversarial Bandits

CS245: Online Optimization and Learning

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Review of Online Learning with Full Information

Online Learning with Full Information

Initialization: $x_1 \in \mathcal{K}$.

For $t = 1, \cdots, T$:

- **Learner:** Submit x_t.
- **Environment:** Observe the full loss $f_t(\cdot)$.
- **Update:** $x_{t+1} = \text{Alg}(x_1, x_2, \cdots, x_t, f_1, f_2, \cdots, f_t)$.

Online learning with full information:

- We know the complete information of loss functions $f_t(\cdot)$.

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- We know the complete information of loss functions $f_t(\cdot)$.
- We studied OMD and FTRL and obtain $O(\sqrt{T})$ regret.

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Online learning with full information:

- We know the complete information of loss functions $f_t(\cdot)$.
- We studied OMD and FTRL and obtain $O(\sqrt{T})$ regret.
- We studied some variants such as online learning with the prediction and delayed feedback, which can be addressed with "Optimistic OMD/FTRL".

Online Learning with Bandit Feedback

Online Learning with Bandit Feedback

Initialization: $x_1 \in \mathcal{K}$.

For $t = 1, \cdots, T$:

- **Learner:** Submit x_t.
- **Environment:** Observe the loss $f_t(x_t)$ only at x_t.
- **Update:** $x_{t+1} = \text{Alg}(f_1(x_1), \nabla \hat{f}_1(x_1), \cdots, f_t(x_t), \nabla \hat{f}_t(x_t))$.

Online learning with bandit feedback:

- We know the bandit information of loss functions at the decision point $f_t(x_t)$.

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Online learning with bandit feedback:

- We know the bandit information of loss functions at the decision point $f_t(x_t)$.
- We need to use these bandit feedback to estimate and the loss function or the gradient.

From Expert Problem to (Adversarial) Bandits problem

Expert problem:

Initialization: N experts/models.

For each day $t = 1, \cdots, T$:

- **Learner:** Obtain predictions from N experts/models and sample an expert i from a probability simplex x_t.
- **Environment:** Observe the loss of each model ℓ_t.

Bandit problem:

Initialization: K arms.

For each round $t = 1, \cdots, T$:

- **Learner:** Pull an arm $i \in [K]$.
- **Environment:** Observe the reward of the chosen arm $r_t(i)$.

(Adversarial) Bandits problem

Stochastic Bandit problem:

Initialization: K arms.

For each round $t = 1, \cdots, T$:

- **Learner:** Pull an arm $a_t \in [K]$.
- **Environment:** Observe the reward of the arm $r_t(a_t)$, which is stochastic from some unknown distribution.

Adversarial Bandit problem:

Initialization: K arms.

For each round $t = 1, \cdots, T$:

- **Learner:** Pull an arm $a_t \in [K]$.
- **Environment:** Observe the reward of the arm $r_t(a_t)$, which could be arbitrary and adversarial.

(Adversarial) Bandits problem

We define the regret of adversarial bandit given a sequence of actions $\{a_t\}$ by an algorithm

$$\text{Regret}(\{a_t\}) = \max_i \sum_{t=1}^{T} r_t(i) - \sum_{t=1}^{T} r_t(a_t).$$

The expected reward of an algorithm is

$$\text{Regret}(T) = \mathbb{E}\left[\max_i \sum_{t=1}^{T} r_t(i) - \sum_{t=1}^{T} r_t(a_t)\right].$$

Online Mirrored Descent for Expert Problem

Hedge as Online Mirrored Descent:

Initialization: $x_1 = [1/K, \cdots, 1/K]$ and η.
For each day $t = 1, \cdots, T$:

- **Learner:** Sample an expert i from x_t.
- **Environment:** Observe the full loss ℓ_t.
- **Update:** $x_{t+1} = \arg\min_{\mathcal{K}} \langle x, \ell_t \rangle + \frac{1}{\eta} B_\psi(x; x_t)$.

Hedge $\longrightarrow$ Exponentiated Gradient $\longrightarrow$ OMD!

OMD is a strong and general framework to design online algorithms with full information. Can it be used to solve adversarial bandit problems?

Online Mirrored Descent for Adversarial Bandit Problems

Online Mirrored Descent for Adversarial Bandits:

Initialization: $x_1 = [1/K, \cdots, 1/K]$ and η.

For each day $t = 1, \cdots, T$:

- **Learner:** Sample an arm a_t from x_t.
- **Environment:** Observe the reward of arm a_t : $r_t(a_t)$.
- **Update:** $x_{t+1} = \arg\min_{\mathcal{K}} \langle x, -\hat{r}_t \rangle + \frac{1}{\eta} B_\psi(x; x_t)$.

As discussed, we only observed the reward of the selected arm i, which is arbitrary and adversarial.

In adversarial bandits, the reward is linear!

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As discussed, we only observed the reward of the selected arm i, which is arbitrary and adversarial.

In adversarial bandits, the reward is linear!

In OMD, we use the reward estimator of $\hat{r}_t$ to replace true reward or loss (r_t or ℓ_t). The estimator is super important!

Importance Estimator for Reward

The estimator $\hat{r}_t$ is super important! A naive way is to just consider what we have observed as the estimator

$$\hat{r}_t(i) = r_t(i), \quad \text{if } a_t = i.$$

Does it work?

Importance Estimator for Reward

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Does it work?

a_t is action at time t

Another possible way is to do the importance estimator:

$$\hat{r}_t(i) = \frac{r_t(i)}{x_t(i)}, \text{ if action } a_t = i.$$

$$r_t(i) = \frac{\mathbb{1}(a_t = i)\, r_t(i)}{x_t(i)}$$

or

$$\hat{r}_t(i) = \mathbb{I}(a_t = i)\frac{r_t(i)}{x_t(i)}.$$

Importance Estimator for Reward

Are the Importance Estimators unbiased?

$$E[\hat{r}_t(i)] = E\left[\frac{\mathbb{1}(a_t = i)\, r_t(i)}{x_t(i)}\right] = r_t(i)$$

What are the variances of the Importance Estimator?

$$E[\hat{r}_t^2(i)] = E\left[\frac{\mathbb{1}(a_t = i)\, r_t^2(i)}{x_t^2(i)}\right] = \frac{r_t^2(i)}{x_t(i)}$$

can be very small and results in large variance

Importance Estimator for Reward

We have two estimators:

$$\hat{r}_t(i) = 1 - \frac{1 - r_t(i)}{x_t(i)}, \text{ if action } a_t = i,$$
$$\hat{r}_t(i) = 1 - \mathbb{I}(a_t = i)\frac{1 - r_t(i)}{x_t(i)}.$$

which one is unbiased? and why?

Online Mirrored Descent for Adversarial Bandit Problems

Online Mirrored Descent for Adversarial Bandits:

Initialization: $x_1 = [1/K, \cdots, 1/K]$ and η.

For each day $t = 1, \cdots, T$:

- **Learner:** Sample an arm i from x_t.
- **Environment:** Observe the reward of arm i : $r_t(i)$.
- **Reward Estimator:** $\hat{r}_t(i) = r_t(i)/x_t(i)$ and 0 otherwise.
- **Update:** $x_{t+1} = \arg\min_{\mathcal{K}} \langle x, -\hat{r}_t \rangle + \frac{1}{\eta} B_\psi(x; x_t)$.

OMD for adversarial bandit is quite straightforward: replace r_t with its unbiased estimator $\hat{r}_t$.

In adversarial bandits, it seems we only update x with each individual coordinate (arm).

B_ψ is KL divergence with ψ being the negative entropy.

FTRL for Adversarial Bandit Problems

FTRL for Adversarial Bandits:

Initialization: $x_1 = [1/K, \cdots, 1/K]$ and η.

For each day $t = 1, \cdots, T$:

- **Learner:** Sample an arm i from x_t.
- **Environment:** Observe the reward of arm i : $r_t(i)$.
- **Reward Estimator:** $\hat{r}_t(i) = r_t(i)/x_t(i)$ and 0 otherwise.
- **Update:** $x_{t+1} = ???$.

Exp3 Algorithm:

Initialization: $x_1 = [1/K, \cdots, 1/K]$ and η.
For each day $t = 1, \cdots, T$:

- **Learner:** Sample an arm i from x_t.
- **Environment:** Observe the reward $r_t(i)$.
- **Reward Estimator:** $\hat{r}_t(i) = r_t(i)/x_t(i)$ and 0 otherwise.
- **Update:** $x_{t+1,i} = e^{\eta \sum_{s=1}^{t} \hat{r}_s(i)} / \sum_i e^{\eta \sum_{s=1}^{t} \hat{r}_s(i)}$.

Exp3 represents "exponential-weight algorithm for exploration and exploitation".

Exp3 is very similar with exponential gradient except using the total estimated rewards $\sum_{s=1}^{t} \hat{r}_s(i), \forall i$.

Exp3 Algorithm – Regret and Possible Issue

Since Exp3 is viewed as OMD with bandit feedback, we could do the "reduction" from bandit to full feedback. Recall the regret of OMD with full information to be

Theorem 1 (OMD with Full Info)

Let ψ be the negative entropy function in B_ψ. Let fixed learning rate $\eta_t = \eta$. Online mirrored descent algorithm achieves

$$Regret(T) \leq \frac{\log K}{\eta} + \frac{\eta}{2} \sum_{t=1}^{T} \|r_t\|^2.$$

Let $\ell_t = -r_t$. According to the pushback lemma, suppose x_{t+1} minimizes the function $F(x)$ such that

$$F(x) := \langle x, \ell_t \rangle + \frac{1}{\eta} B(x; x_t).$$

For any x, we have

$$F(x_{t+1}) \leq F(x) - \frac{1}{\eta} B(x; x_{t+1}).$$

Therefore, we have

$$\eta \langle x_{t+1}, \ell_t \rangle + B(x_{t+1}; x_t) \leq \eta \langle x^*, \ell_t \rangle + B(x^*; x_t) - B(x^*; x_{t+1}).$$

which implies

$$\langle x_t - x^*, \ell_t \rangle + \langle x_{t+1} - x_t, \ell_t \rangle + \frac{B(x_{t+1}; x_t)}{\eta} \leq \frac{B(x^*; x_t) - B(x^*; x_{t+1})}{\eta}.$$

Review: OMD with Full Info – Proof

Next, we use the definition of Bregman divergence, where $\psi(x)$ is 1-strongly convex to show $\frac{1}{2}\|x_{t+1} - x_t\|^2 \leq B(x_{t+1}; x_t)$ such that

$$\begin{aligned} &\langle x_{t+1} - x_t, \ell_t \rangle + \frac{B(x_{t+1}; x_t)}{\eta} + \frac{\eta \|\ell_t\|^2}{2} \\ \geq &\langle x_{t+1} - x_t, \ell_t \rangle + \frac{\|x_{t+1} - x_t\|^2}{2\eta} + \frac{\eta \|\ell_t\|^2}{2} \end{aligned}$$

Therefore, we have

$$\begin{aligned} &\langle x_t - x^*, \ell_t \rangle + \frac{1}{2}\left\|\frac{x_{t+1} - x_t}{\sqrt{\eta}} + \sqrt{\eta}\ell_t\right\|^2 \\ \leq &\frac{B(x^*; x_t) - B(x^*; x_{t+1})}{\eta} + \frac{\eta \|\ell_t\|^2}{2}. \end{aligned}$$

Exp3 Algorithm – Regret and Possible Issue

$$\text{Regret}(T) \le \frac{\log K}{\eta} + \frac{\eta}{2} E\left[\sum_{t=1}^{T} \|\hat{r}_t\|_\infty^2\right]$$

$$= \frac{\log K}{\eta} + \frac{\eta}{2} \sum_{t=1}^{T} \frac{r_t^2(i)}{x_t(i)}$$

$$\le \frac{\log K}{\eta} + \frac{\eta}{2} \sum_{t=1}^{T} \sum_{i=1}^{K} \frac{r_t^2(i)}{x_t(i)}$$

We are in trouble when $x_t(i)$ is small.

Exp3 Algorithm – Regret and Refined Analysis

Exp3 is motivated by EG with full information and it is supposed to work! Indeed, we need a refined analysis.

Theorem 2

Suppose $\eta = \sqrt{K \log K / T}$. Exp3 algorithm achieves the regret

$$\begin{aligned} Regret(T) \leq & \frac{\log K}{\eta} + \frac{\eta}{2}\mathbb{E}\left[\sum_{t=1}^{T} \|r_t\|^2\right]. \\ = & O(\sqrt{TK \log K}). \end{aligned}$$

Exp3 returns the regret $O(\sqrt{T})$! Moreover, Exp3 with bandit feedback only has $O(\sqrt{K})$ loss because EG with full info $O(\sqrt{T \log K})$.

Exp3 Algorithm – Regret and Refined Analysis

Intuitively, a small action probability $x_t(i)$ for arm i would cancel its large variance $\frac{(r_t(i))^2}{x_t(i)}$!

Exp3 Algorithm – Regret and Refined Analysis

For OMD, we have a local and strong version of regret analysis as follows.

Lemma 3

Let ψ be twice-differentiable convex function in B_ψ. Let fixed learning rate $\eta_t = \eta$. Online mirrored descent algorithm achieves

$$\begin{aligned}\langle x_t - x, \ell_t \rangle \leq & \frac{1}{\eta}(B(x, x_t) - B(x, x_{t+1})) \\ & + \frac{\eta}{2} \min\{\|\ell_t\|^2_{(\nabla^2\psi(z_t))^{-1}}, \|\ell_t\|^2_{(\nabla^2\psi(z'_t))^{-1}}\}.\end{aligned}$$

where z_t is between x_t and x_{t+1}; z'_t is between x_t and x'_{t+1} with $x'_{t+1} = \arg\min \langle x, \ell_t \rangle + \frac{1}{\eta} B_\psi(x; x_t)$.

The lemma can be proved by using Pushback Lemma.

Exp3 Algorithm – Regret and Refined Analysis

By refined analysis, we have

$$\sum_{t=1}^{T} \langle x_t - x^*, \ell_t \rangle \leq \frac{B(x^*, x_1)}{\eta} + \frac{\eta}{2} \sum_{t=1}^{T} \|\ell_t\|^2_{(\nabla^2 \psi(x_t))^{-1}}.$$

Proof of Lemma 3 : we need to prove part 1 and part 2.

$$\langle x_{t+1}, l_t \rangle + \frac{1}{\eta} B(x_{t+1}; x_t)$$

$$\leq \langle x, l_t \rangle + \frac{1}{\eta} B(x; x_t) - \frac{1}{\eta} B(x; x_{t+1})$$ pushback Lemma

$$\Rightarrow \quad \langle x_t, l_t \rangle + \boxed{\frac{1}{\eta} B(x_{t+1}; x_t) + \langle x_{t+1} - x_t, l_t \rangle}$$ key terms

$$\leq \langle x, l_t \rangle + \frac{1}{\eta} B(x; x_t) - \frac{1}{\eta} B(x; x_{t+1})$$

Recall the definition of Bregman divergence :

$$B(x_{t+1}; x_t) = \psi(x_{t+1}) - \psi(x_t) - \langle x_{t+1} - x_t, \nabla \psi(x_t) \rangle$$

$$= \frac{1}{2} (x_{t+1} - x_t)^T \nabla^2 \psi(z) (x_{t+1} - x_t)$$ z is between x_t and x_{t+1}

now we have

$$\langle x_t, l_t\rangle + \frac{1}{2\eta}(x_{t+1}-x_t)^T\nabla\psi^2(z)(x_{t+1}-x_t) + \langle x_{t+1}-x_t, l_t\rangle$$

$$\leq \langle x, l_t\rangle + \frac{1}{\eta}B(x; x_t) - \frac{1}{\eta}B(x; x_{t+1})$$

Since

$$\frac{1}{2\eta}(x_{t+1}-x_t)^T\nabla\psi^2(z)(x_{t+1}-x_t) + \langle x_{t+1}-x_t, l_t\rangle + \frac{\eta}{2} l_t^T(\nabla\psi^2(z))^{-1} l_t$$

is positive, like $\frac{a^2}{2\eta} + ab + \frac{\eta}{2}b^2$. We prove part 1.

For part 2,

$$\frac{1}{\eta}B(x_{t+1}; x_t) + \langle x_{t+1}-x_t, l_t\rangle$$

$$\geq \min_x \frac{1}{\eta}B(x; x_t) + \langle x - x_t, l_t\rangle$$

$$= \frac{1}{\eta}B(x'_{t+1}; x_t) + \langle x'_{t+1} - x_t, l_t\rangle \quad \text{by definition of } x'_{t+1}$$

Follow the exact steps in part 1, we prove part 2.

You can check Lemma 6.14 in the book of

A Modern Introduction to Online Learning.

Now we prove our main results of Theorem 2.

By Lemma 3, we have

$$\langle x_t - x, \ell_t \rangle \leq \frac{1}{\eta}\left(B(x, x_t) - B(x, x_{t+1})\right)$$

$$+ \frac{\eta}{2} \min\left(\|\hat{r}_t\|^2_{(\nabla\psi^2(z_t))^{-1}}, \|\hat{r}_t\|^2_{(\nabla\psi^2(z_t'))^{-1}} \right)$$

We verify $z_t' \leq x_t$. Recall z_t' is between x_t and x_{t+1}'

where $x_{t+1}' = \arg\min \langle x, \ell_t \rangle + \frac{1}{\eta} B(x; x_t)$

$$= \arg\min \underbrace{\langle x, \ell_t \rangle + \frac{1}{\eta} \sum_i x_i \log \frac{x_i}{x_{t,i}}}_{\phi(x)}$$

$$\frac{\partial \phi(x)}{\partial x_i} = \eta \ell_{t,i} + \log \frac{x_i}{x_{t,i}} + 1 = 0$$

$$\Rightarrow \quad x_{t+1,i}' = x_{t,i}\, e^{-\eta \ell_{t,i} - 1} \leq x_{t,i}, \quad \forall i$$

Therefore, we have $z_t' \leq x_t$, which implies that

$$\langle x_t - x, \ell_t \rangle \leq \frac{1}{\eta}\left(B(x, x_t) - B(x, x_{t+1})\right) + \frac{\eta}{2} \|\hat{r}_t\|^2_{(\nabla\psi^2(x_t))^{-1}}$$

$$= \frac{1}{\eta}\left(B(x, x_t) - B(x, x_{t+1})\right) + \frac{\eta}{2}\sum_{i=1}^{K} x_{t,i}\, \hat{r}_t^2(i)$$

$$\text{Regret}(T) \leq \frac{B(x^*; x_1)}{\eta} + \frac{\eta}{2}\, E\left[\sum_{t=1}^{T}\sum_{i=1}^{K} x_{t,i}\, \hat{r}_t^2(i)\right]$$

$$E[\hat{r}_t^2(i)] = \frac{r_t^2(i)}{x_{t,i}}$$

$$= \frac{B(x^*; x_1)}{\eta} + \frac{\eta}{2}\, E\left[\sum_{t=1}^{T}\sum_{i=1}^{K} r_t^2(i)\right]$$

Can we do other/better regularization?

Online Mirrored Descent for Adversarial Bandits:

Initialization: $x_1 = [1/K, \cdots, 1/K]$ and η.

For each day $t = 1, \cdots, T$:

- **Learner:** Sample an arm i from x_t.
- **Environment:** Observe the reward of arm i : $r_t(i)$.
- **Reward Estimator:** $\hat{\ell}_t(i) = \ell_t(i)/x_t(i)$ and 0 otherwise.
- **OMD Update:** $x_{t+1} = \arg\min_{\mathcal{K}} \langle x, \hat{\ell}_t \rangle + \frac{1}{\eta} B_{\psi}(x; x_t)$.
- **FTRL Update:** $x_{t+1} = \arg\min_{\mathcal{K}} \sum_{s=1}^{t} \langle x, \hat{\ell}_s \rangle + \eta \psi(x)$.

Can we design better ψ?

- "minimize" the stability term $\|\ell_t\|^2_{(\nabla^2 \psi(x_t))^{-1}}$

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Can we design better ψ?

- "minimize" the stability term $\|\ell_t\|^2_{(\nabla^2 \psi(x_t))^{-1}}$
- not introduce too much bias $\frac{B(x^*, x_1)}{\eta}$

Can we do other/better regularization?

$O(\sqrt{TK\log K})$

$$\sum_{t=1}^{T}\langle x_t - x^*, \ell_t\rangle \le \frac{B_\psi(x^*, x_1)}{\eta} + \frac{\eta}{2}\sum_{t=1}^{T}\|\hat{\ell}_t\|^2_{(\nabla^2\psi(x_t))^{-1}}.$$

$\psi(x) = x^2$ $\quad \frac{K}{\eta} \quad \frac{\eta}{2}\sum_t\sum_i \frac{\ell_t^2(i)}{x_t(i)}$

$\psi(x) = \sum_i x(i)\log x(i)$ $\quad \frac{\log K}{\eta} \quad \frac{\eta}{2}\sum_t\sum_i \frac{\ell_t^2(i)}{x_t(i)} \cdot x_t(i)$

$\psi(x) = -\sum_i \log x(i)$ $\qquad \frac{\log T}{\eta}$ $\qquad \frac{\eta}{2} \boxed{\sum_t \sum_i \ell_t(i) \cdot x_t(i)}$

$\psi(x) = -\sum_i \sqrt{x_i}$ $\qquad \frac{\sqrt{K}}{\eta}$ $\qquad \frac{\eta}{2} T \sqrt{K}$

$\frac{\eta}{2} \sum_t \sum_i \frac{\ell_t^2(i)}{x_t(i)} \cdot x_t^{\frac{3}{2}}(i)$

$B(y;x) = \psi(y) - \psi(x) - \langle y-x, \nabla\psi(x) \rangle$ $\qquad \frac{\eta}{2} T.$

$= -\log x^*(i) + \log x_1(i) - \langle x^*(i) - x_1(i), -\frac{1}{x_1(i)} \rangle$

$= -\log x^*(i) + \log x_1(i) + \langle x^*(i) - x_1(i), \frac{1}{x_1(i)} \rangle$

$\hat{x}^*(i) = \varepsilon \cdot 1 + (1-\varepsilon) \cdot x^*(i)$

$T\varepsilon \qquad \varepsilon \leq \frac{1}{\sqrt{T}}$