Cooperative Games in Social Network

Dengji Zhao

ShanghaiTech University, Shanghai, China

Outline

- The Shapley Value
- Invitation Incentive Mechanisms

Coalitional/Cooperative Game

- A set of agents N.
- Each subset of agents(coalition) $S \subseteq N$ cooperate together can generate some value $v(S) \in R$. Assume $v(\emptyset) = 0$. N is called grand coalition. $v : 2^N \rightarrow R$ is called the characteristic function of the game. v is often assumed to be monotonic: $S \subseteq T \rightarrow v(S) \leq v(T)$.
- The possible outcomes of the game is defined by $V(S) = \{x \in R^S : \sum_{i \in S} x_i \leq v(S)\}$.

Example

- Three agents $\{1, 2, 3\}$.
- $v(\{1\}) = v(\{2\}) = 10, v(\{3\}) = 1;$
  $v(\{1, 2\}) = 20, v(\{1, 3\}) = v(\{2, 3\}) = 12;$
  $v(\{1, 2, 3\}) = 22.$

Shapley Value: a Fair Distribution of Payoffs

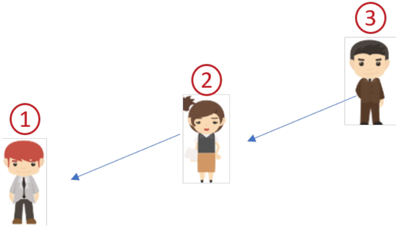
Given a coalitional game (N, v), the Shapley value of each player i is:

$$\phi_i(v) = \sum_{S \subset N \setminus \{i\}} \frac{|S|!(n - |S| - 1)!}{n!}(v(S \cup \{i\}) - v(S))$$

- Three agents $\{1, 2, 3\}$.
- $v(\{1\}) = v(\{2\}) = 10, v(\{3\}) = 1;$
  $v(\{1, 2\}) = 20, v(\{1, 3\}) = v(\{2, 3\}) = 12;$
  $v(\{1, 2, 3\}) = 22.$

Shapley Value: a Fair Distribution of Payoffs

$v(S \cup \{i\}) - v(S)$:marginal contribution for agent i,which is denoted as c_i



- Initially, the coalition is empty and agent 1 comes. $c_1 = v(\{1\}) - v(\emptyset) = 10$
- Agent 2 comes. $c_2 = v(\{1,2\}) - v(\{1\}) = 10$
- Agent 3 comes. $c_3 = v(\{1,2,3\}) - v(\{1,2\}) = 2$

Shapley Value: a Fair Distribution of Payoffs

- List all the permutations and get respective marginal contributions.

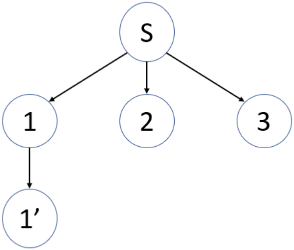
Permutation	c_1	c_2	c_3
1,2,3	10	10	2
1,3,2	10	10	2
2,3,1	10	10	2
2,1,3	10	10	2
3,1,2	11	10	1
3,2,1	10	11	1

Get the final payoff: $\phi_i = \frac{\sum c_i}{6}$

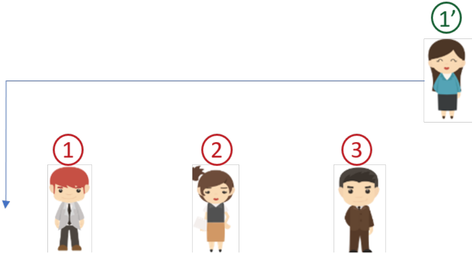
- $\phi_1 = \frac{61}{6}, \phi_2 = \frac{61}{6}, \phi_3 = \frac{10}{6}$

To Attract More Participants

- We want to incentivize participants to invite more participants.
- What if agent 1 invites agent $1'$ who can complete the same task?
  (i.e. $v(\{1\} \cup S) = v(\{1'\} \cup S) = v(\{1, 1'\} \cup S)$)

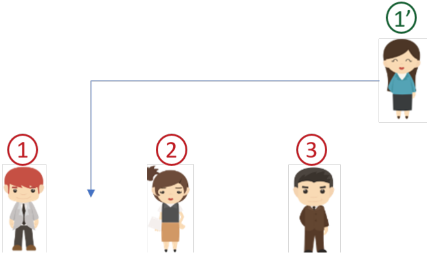


To Attract More Participants



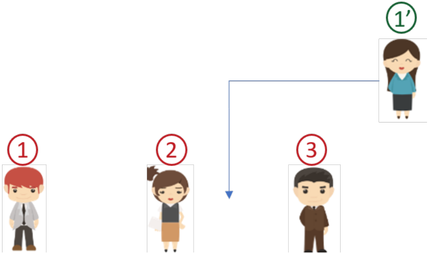
- Without agent 1': $c_1 = 10$, $c_2 = 10$, $c_3 = 2$
- With agent 1': $c'_{1'} = 10$, $c'_1 = 0$, $c'_2 = 10$, $c'_3 = 2$

To Attract More Participants



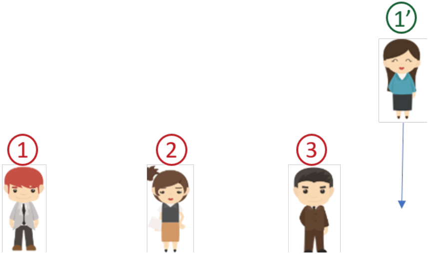
- Without agent 1’:$c_1 = 10$, $c_2 = 10$, $c_3 = 2$
- With agent 1’:$c'_1 = 10$, $c'_{1'} = 0$, $c'_2 = 10$, $c'_3 = 2$

To Attract More Participants



- Without agent 1’:$c_1 = 10, c_2 = 10, c_3 = 2$
- With agent 1’:$c_1' = 10, c_2' = 10, c_{1'}' = 0, c_3' = 2$

To Attract More Participants



- Without agent 1’: $c_1 = 10, c_2 = 10, c_3 = 2$
- With agent 1’: $c_1' = 10, c_2' = 10, c_3' = 2, c_{1'}' = 0$

To Attract More Participants

- Similarly, we can get all the permutations and marginal contributions.

Permutation	c_1	c_2	c_3	$c_{1'}$
1,2,3,1'	$10 \rightarrow 10$	10	2	0
1,3,2,1'	$10 \rightarrow 10$	10	2	0
...	.	.	.	.
$3,1',1,2$	$11 \rightarrow 0$	10	1	11
$1',3,2,1$	$10 \rightarrow 0$	10	2	10

- For the permutation where agent $1'$ is before 1, $c_1 = 0$ since agent $1'$ performs the same as agent 1.

Outline

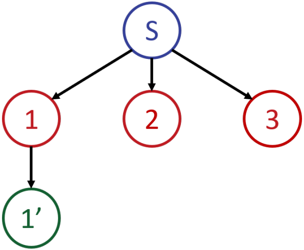
- The Shapley Value
- Invitation Incentive Mechanisms

A Layer-based Solution

Main idea

- Divide the agents into layers in terms of distance
- Lower layers always join the game earlier than higher layers

Layered Shapley Value



- Recalling the previous example, agents 1,2,3 are in the first layer while agent $1'$ is in the second layer.
- We only retain the permutation where $1'$ is after the set $\{1,2,3\}$ (i.e 1,3,2,$1'$) and remove others

Layered Shapley Value

1′ 1 2 3	1 1′ 2 3	1 2 1′ 3	1 2 3 1′
1′ 1 3 2	1 1′ 3 2	1 3 1′ 2	1 3 2 1′
1′ 2 1 3	2 1′ 1 3	2 1 1′ 3	2 1 3 1′
1′ 2 3 1	2 1′ 3 1	2 3 1′ 1	2 3 1 1′
1′ 3 1 2	3 1′ 1 2	3 1 1′ 2	3 1 2 1′
1′ 3 2 1	3 1′ 2 1	3 2 1′ 1	3 2 1 1′

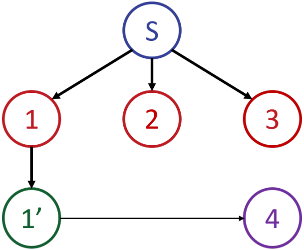
- The permutations in blue dotted box are retained, others are removed.

Layered Shapley Value

- Similarly, we can get marginal contributions for the remaining permutation.

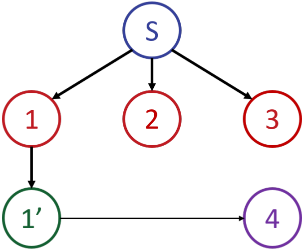
Permutation	c_1	c_2	c_3	$c_{1'}$
1,2,3,1'	10	10	2	0
1,3,2,1'	10	10	2	0
2,1,3,1'	10	10	2	0
2,3,1,1'	10	2	10	0
3,1,2,1'	11	10	1	0
3,2,1,1'	10	11	1	0

A General Solution: Permission Shapley Value



- In the previous example, suppose agent $1'$ invites agent 4.
- $v(\{1\}) = v(\{2\}) = 10.$
  $v(\{3\}) = 1;$
  $v(\{1,2\}) = 20;$
  $v(\{1,3\}) = v(\{2,3\}) = 12;$
  $v(\{1,2,3\}) = 22.$
- $v(\{1'\} \cup S) = v(\{1\} \cup S) = v(\{1,1'\} \cup S)$
- $v(\{4\} \cup S) = v(S) + 100$

A General Solution: Permission Shapley Value



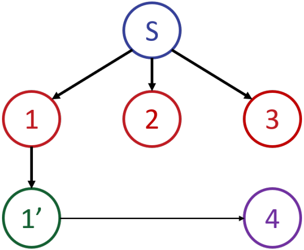
Permission structure:

- A permission structure on N is an asymmetric mapping p:$N \rightarrow 2^N$, i.e., $j \in p(i)$ implies that $i \notin p(j)$
- $p(i)$: the set of players who invited i into the coalition.
- In the graph, $p(1') = \{1\}, p(4) = \{1'\}$.

Applying Permission Shapley Value

Autonomous

A coalition $S \subseteq N$ is autonomous in a permission structure p if for all $i \in S, p(i) \subseteq S$.

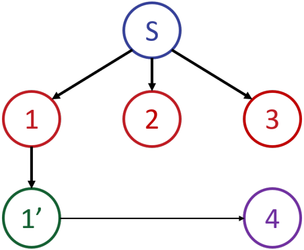


- In this graph, $S = \{1, 2, 1'\}$ is autonomous, but $S = \{1, 2, 4\}$ is not autonomous since $p(4) = \{1'\} \not\subseteq S$.

Applying Permission Shapley Value

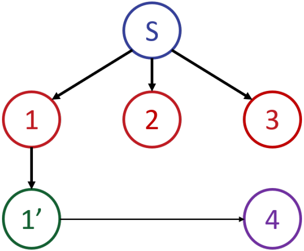
Largest Autonomous Part

Let p be a permission structure on N. Then the largest automonous part of a coalition $S \subset N$ is defined by
$\alpha(S) = \cup\{T | T \subseteq S \text{ and } T \in A_p\}$
where A_p denotes the collection of all autonomous coalitions in p.



- In the graph, $\alpha(\{2, 3, 1', 4\}) = \{2, 3\}$

Applying Permission Shapley Value



Permission Shapley value $v^p(S)$.

- $v^p(S) = v(\alpha(S))$

- In the graph, e.g.$v^p(\{1, 2, 3, 4\}) = v(\{1, 2, 3\}) = 22$ since $\alpha(\{1, 2, 3, 4\}) = \{1, 2, 3\}$.

Applying Permission Shapley Value

- Use permutation{1,2,4,3,1'} as an example.

Set S	$\alpha(S)$	$v(S)$	$v^p(S)$
1	1	10	10
1,2	1,2	20	20
1,2,4	1,2	120	20
1,2,4,3	1,2,3	122	22
$1{,}2{,}4{,}3{,}1'$	$1{,}2{,}4{,}3{,}1'$	122	122

- $\phi_1 = \frac{61}{6} + \frac{100}{3}, \phi_2 = \frac{61}{6}, \phi_3 = \frac{10}{6}$
- $\phi_{1'} = \frac{100}{3}, \phi_4 = \frac{100}{3}$