CS243: Introduction to Algorithmic Game Theory

Social Choice (Dengji ZHAO)

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Recap

Answer Yes/No for the following questions:

Q1 In a simultaneous move game, given a strategy vector $s \in S$, if for each player i, and each alternate strategy vector $s' \in S$, we have $u_i(s_i, s'_{-i}) \geq u_i(s'_i, s'_{-i})$, then s is a Nash equilibrium?

Q2 A mechanism is truthful means that reporting valuation function truthfully is a dominant strategy for all players?

Q3 The second price auction for selling one item is truthful, efficient and individually rational?

Q4 Following mechanism $(f, p_1, ..., p_n)$ is truthful and efficient?

- given players' valuation function report profile $(v_1, ..., v_n)$

$$f(v_1, ..., v_n) \in argmax_{a \in A} \sum_i v_i(a)$$

$$p_i(v_1, ..., v_n) = \sum_{j \neq i} v_j(f(v_{-i})) - \sum_{j \neq i} v_j(f(v_1, ..., v_n))$$

Social Choice

- In a mechanism design setting, each player i has a valuation function $v_i : A \to \mathbb{R}$
- Now assume that a player does not have a value for each outcome $a \in A$, instead she has a preference $\succ_i$ over all outcomes in A. $\succ_i$ is a linear order on A, e.g. $a_1 \succ_i a_2 \succ_i a_3 \ldots$.
- We want to study the following:
  1. How to choose one outcome from A?
  2. What is the overall ranking on A by aggregating all players' preferences?

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Social Choice (Voting)

General Social Choice Setting

- A set of n players/voters.
- A set of alternatives A (the candidates).
- Let L be the set of all linear orders on A.
- Each voter i has a preference $\succ_i \in L$, a *total order* on A (antisymmetric, transitive). $a \succ_i b$ means i prefers a to b.

Definition

- A function $f : L^n \to A$ is called a social choice function.
- A function $F : L^n \to L$ is called a social welfare function.

Voting Methods: Social Choice Functions

- Majority vote
  - among two candidates, selects the candidate which has a majority vote, that is, more than half of the votes
- Plurality
  - the candidate that was placed first by the largest number of voters wins
- Borda count
  - each candidate among the n candidates gets $n - i$ points for every voter who ranked him in place i, and the candidate with most points wins

Majority Vote

- Consider three candidates $\{a, b, c\}$ and three voters $\{1, 2, 3\}$.
- Their preferences are the following:
  - $a \succ_1 b \succ_1 c$
  - $b \succ_2 c \succ_2 a$
  - $c \succ_3 a \succ_3 b$

Question

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- $a \succ b \succ c \succ a$ (Condorcet's Paradox)

Plurality

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The candidate that was placed first by the largest number of voters wins: which candidate wins?

Borda Count

- Consider three candidates $\{a, b, c\}$ and three voters $\{1, 2, 3\}$.
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  - $c \succ_3 a \succ_3 b$

Question

Each candidate among the n candidates gets $n - i$ points for every voter who ranked him in place i, and the candidate with most points wins: which candidate wins?

Strategic Manipulations

- $a \succ_1 b \succ_1 c$
- $b \succ_2 c \succ_2 a$
- $c \succ_3 a \succ_3 b$

Assume that under tie-breaking, a is preferred to b and b is preferred to c, can a voter manipulate (misreport preference) to change the outcome in his/her favour?

- *Plurality*: The candidate that was placed first by the largest number of voters wins (a wins).
- *Borda count*: Each candidate among the n candidates gets $n - i$ points for every voter who ranked him in place i, and the candidate with most points wins (a wins).

Truthful (Incentive Compatible) Social Choice Function

Definition 9.4 A social choice function f can be *strategically manipulated* by voter i if for some $\prec_1, \ldots, \prec_n \in L$ and some $\prec_i' \in L$ we have that $a \prec_i a'$ where $a = f(\prec_1, \ldots, \prec_i, \ldots, \prec_n)$ and $a' = f(\prec_1, \ldots, \prec_i', \ldots, \prec_n)$. That is, voter i that prefers a' to a can ensure that a' gets socially chosen rather than a by strategically misrepresenting his preferences to be $\prec_i'$ rather than $\prec_i$. f is called *incentive compatible* if it cannot be manipulated.

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Definition 9.7 Voter i is a *dictator* in social choice function f if for all $\prec_1$, $\ldots, \prec_n \in L$, $\forall b \neq a$, $a \succ_i b \Rightarrow f(\prec_1, \ldots, \prec_n) = a$. f is called a *dictatorship* if some i is a dictator in it.

Theorem 9.8 (Gibbard–Satterthwaite) *Let f be an incentive compatible social choice function onto A, where $|A| \geq 3$, then f is a dictatorship.*

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Theorem 9.8 (Gibbard–Satterthwaite) *Let f be an incentive compatible social choice function onto A, where $|A| \geq 3$, then f is a dictatorship.*

Gibbard-Satterthwaite Theorem is an implication of Arrow's Theorem

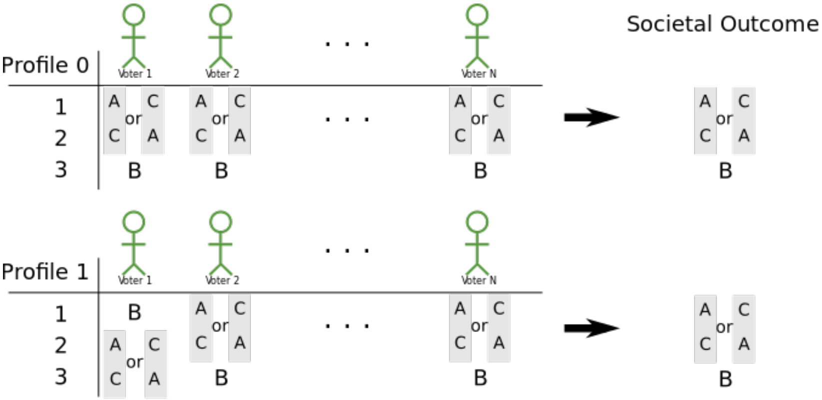
Arrow's Theorem

Theorem (Arrow's Theorem)

Every social welfare function over a set of more than 2 candidates ($|A| \geq 3$) that satisfies unanimity and independence of irrelevant alternatives is a dictatorship.

- F satisfies unanimity if for every $\succ \in L$, $F(\succ, \ldots, \succ) = \succ$.
- F satisfies independence of irrelevant alternatives if for every $a, b \in A$, every $\succ_1, \ldots, \succ_n, \succ'_1, \ldots, \succ'_n \in L$, if $\succ = F(\succ_1, \ldots, \succ_n)$ and $\succ' = F(\succ'_1, \ldots, \succ'_n)$, then $a \succ_i b \Leftrightarrow a \succ'_i b$ for all i implies $a \succ b \Leftrightarrow a \succ' b$.
- Voter i is a dictator in F if for all $\succ_1, \ldots, \succ_n \in L$, $F(\succ_1, \ldots, \succ_n) = \succ_i$. F is not a dictatorship if no i is a dictator in F.

To Prove Arrow's Theorem



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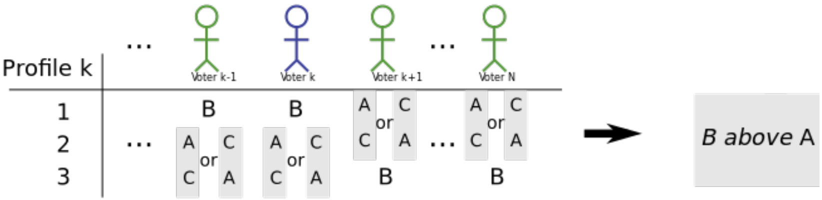

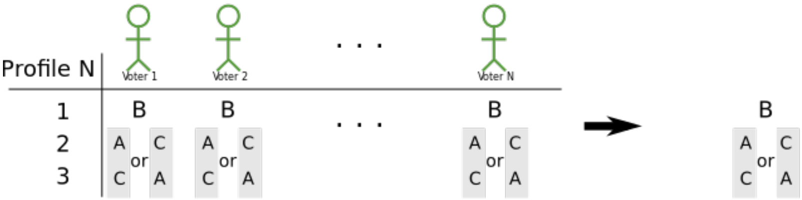

To Prove Arrow's Theorem

- There is a pivotal voter k for B over A.
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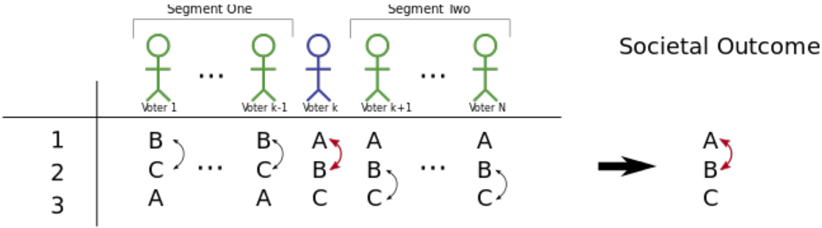
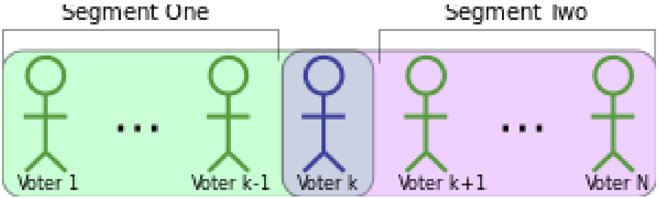
- There is a pivotal voter *k* for B over A.
- The pivotal voter for B over A is a dictator for B over C.



To Prove Arrow's Theorem

- There is a pivotal voter k for B over A.
- The pivotal voter for B over A is a dictator for B over C.
- There can be at most one dictator.



Since voter k is the dictator for B over C, the pivotal voter for B over C must appear among the first k voters. That is, outside of segment two. Likewise, the pivotal voter for C over B must appear among voters k through N. That is, outside of Segment One.

Advanced Reading

- Social Choice [AGT Chapter 9.2]