EE160 Control Theory

Lab 3 Time-Domain Analysis of Linear Systems (2)

Objectives

1. Mastering the dynamic performance indicators of second-order systems.
2. Understand the relationship between ζ/ω and the system response.

2 Time-Domain Analysis for Second-Order System (Dynamic Performance Indicators)

Second-order systems are widely used in control engineering, such as RLC networks, electric motors, and the motion of objects. And they are easy to analyze mathematically and can be approximated to replace higher-order systems for study under certain conditions. In this section, we will take second-order systems as an example to explore the dynamic performance of the system.

2.1 Mathematical Model of a Second-Order System

The mathematical model of a second-order system is typically described as a second-order linear differential equation, which can be represented as:

$$\frac{d^2c(t)}{dt^2} + 2\zeta\omega_n\frac{dc(t)}{dt} + \omega_n^2 c(t) = r(t)$$

The corresponding closed-loop transfer function of a typical second-order system is given by:

$$T(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Where:

- $c(t)$ is the system's output or response.
- $\frac{d^2c(t)}{dt^2}$ and $\frac{dc(t)}{dt}$ are the second and first derivatives of the output.
- ω_n is the system's natural frequency (in radians per second).
- ζ is the damping ratio, describing the degree of damping in the system's response.
- $r(t)$ is the external force or input signal.
- $G(s)$ is the closed-loop transfer function.
- s is the Laplace variable.

From the closed-loop transfer function of the system, we can derive the characteristic equation of the system as:

$$s^2 + 2\zeta\omega_n s + \omega_n^2 = 0$$

Further, from the characteristic equation, we can find the characteristic roots (also known as eigenvalues or poles) of the system:

$$s_{1,2} = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}$$

It's easy to know that ζ and ω determine the location of the system's poles. The dynamic characteristics of the second-order system can be described by ζ and ω. Since ω is the natural frequency of the system which should not be less than zero, let's discuss according to the different value of ζ.

ζ	Poles	Step Responses
$\zeta < 0$ unstable	Roots with positive real parts located in the right half of the s-plane.	Divergent oscillation.
$\zeta = 0$ undamped	Conjugate imaginary roots located on the imaginary axis of the s-plane.	Constant amplitude oscillation.
$0 < \zeta < 1$ underdamped	Conjugate complex roots with negative real parts located in the left half of the s-plane.	Damped oscillation.
$\zeta = 1$ critically damped	Equal negative real roots located on the real axis of the left half of the s-plane.	Monotonically increasing and converging.
$\zeta > 1$ overdamped	Unequal negative real roots located on the real axis of the left half of the s-plane.	Increasing slower relative to $\zeta = 1$.

Table 1: The effect of ζ on the position of the poles and the characterises of the step responses

For $\zeta \geq 0$, the relationship is shown in Figure1

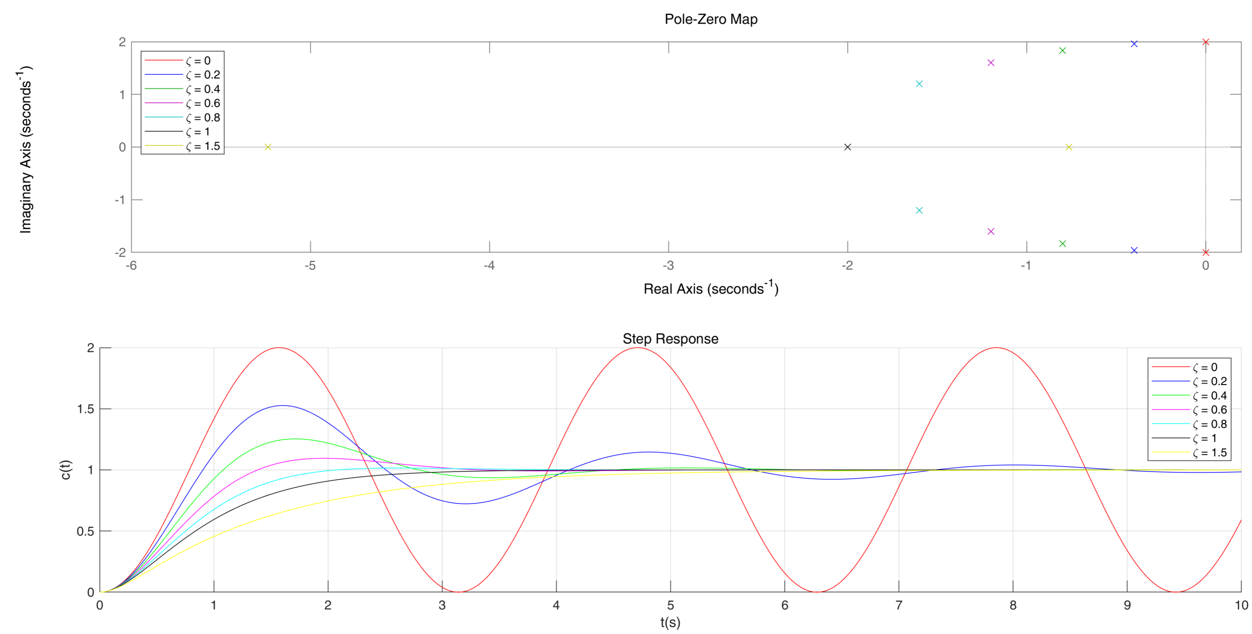

Figure 1: ζ, Poles and Step Response

2.2 Dynamic Performance Indicators of a Second-Order System

With a step signal as the input to the second-order system, the dynamic performance indicators are shown in Figure2

The dynamic performance indicators include:

- T_d: **Delay time** refers to the time required for the response curve to first reach half of its final value.

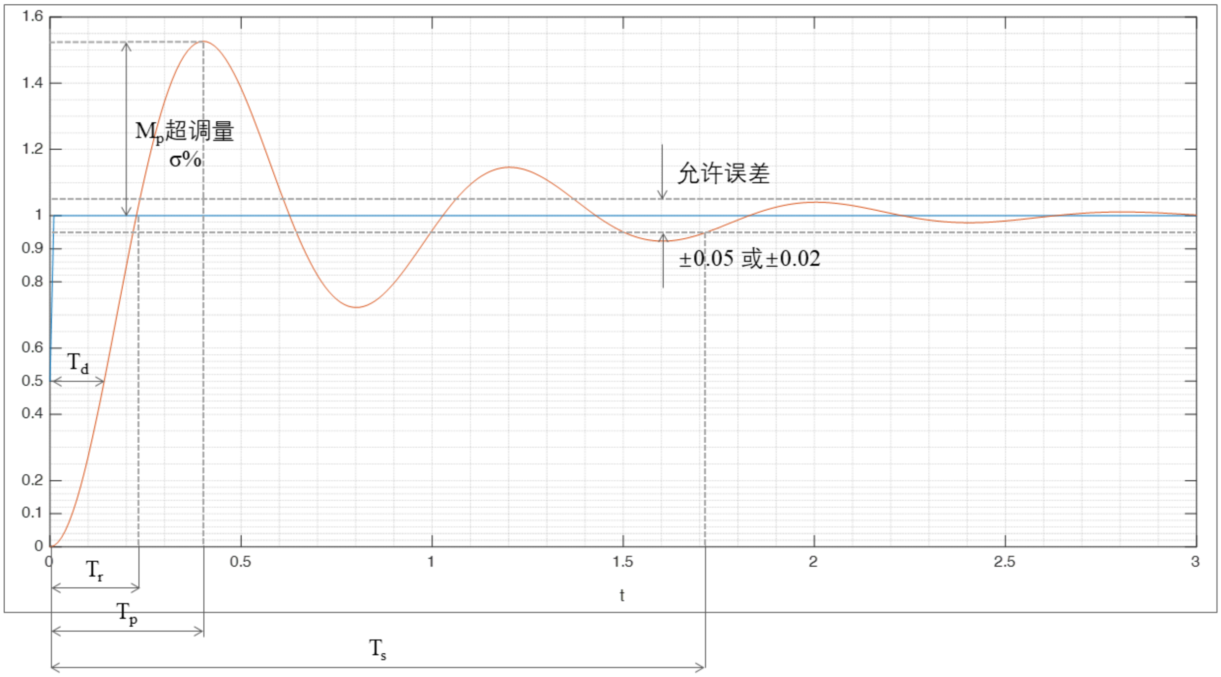

Figure 2: Dynamic Performance Indicators of a Second-order System

- T_r: **Rise time** refers to the time required for the response to rise from 10% to 90% of its final value. For systems with oscillations, it can also be defined as the time taken from zero to first reach the final value.
- T_p: **Peak time** refers to the time required for the response to exceed its final value and reach the first peak.
- T_s: **Settling time** refers to the time required for the response to reach and remain within ±5% (or ±2%) the allowable error range.
- M_p: **Maximum Overshoot** is defined as the percentage by which the response exceeds its final steady-state value before it settles down to that value. Use $c(T_p)$ to represent the peak value, and $c(\infty)$ to represent the steady-state value, we have

$$M_p = \frac{c(T_p) - c(\infty)}{c(\infty)} \times 100\%$$

2.2.1 How Do ζ and ω_n Affect the Dynamic Performance of an Underdamped System

Not all second-order systems possess all of the aforementioned dynamic performance indicators. Underdamped systems provide a good environment for studying these dynamic performance indicators. We will then take the underdamped system as an example to investigate how ζ and ω_n affect the dynamic response performance of the second-order system.

For an underdamped second-order system, the step response in s domain is:

$$\begin{aligned} C(s) &= R(s) \cdot G(s) \\ &= \frac{1}{s} \cdot \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} \\ &= \frac{1}{s} - \frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} \\ &= \frac{1}{s} - \frac{s + \zeta\omega_n}{(s+\zeta\omega_n)^2 + \omega_d^2} - \frac{\zeta\omega_n}{(s+\zeta\omega_n)^2 + \omega_d^2} \end{aligned} \tag{1}$$

$\omega_d = \omega_n\sqrt{1-\zeta^2}$ is called **damped frequency**. It is the frequency at which the system oscillates under the influence of damping.

Apply the inverse Laplace transform on $C(s)$, we can get the system response:

$$c(t) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} sin(\omega_d t + \beta), t \geq 0 \tag{2}$$

$\beta = arccos\zeta$ is called **damping angle**. Refer to Figure3 to get more information.

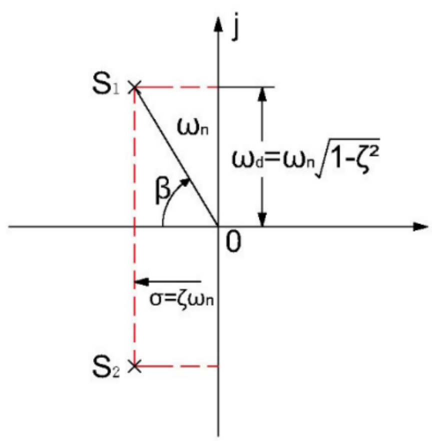

Figure 3: The Relationship between $\omega_n, \omega_d, \beta$

Let's find out the dynamic performance indicators by 2

1. $c(\infty)$: $c(\infty) = \lim_{t\to\infty} c(t) = 1$
2. T_r: The first of t that makes $c(t) = 1$, that is:

$$c(t_r) = 1 - \frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n T_r} sin(\omega_d T_r + \beta) = 1$$

$$\frac{1}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n T_r} sin(\omega_d T_r + \beta) = 0$$

Since $e^{-\zeta\omega_n t} \neq 0$, so

$$sin(\omega_d T_r + \beta) = 0$$

$$T_r = \frac{k\pi - \beta}{\omega_d}, k = 1$$

$$T_r = \frac{\pi - \beta}{\omega_d} = \frac{\pi - arccos\zeta}{\omega_n\sqrt{1-\zeta^2}}$$

3. T_p: Obtaining extremum points by differentiation.

$$\left.\frac{dc(t)}{dt}\right|_{t=T_p} = 0$$

$$\frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n T_p} \cdot \omega_n \cdot sin(\omega_d T_p + \beta - \beta) = 0$$

$$sin(\omega_d T_p + \beta - \beta) = 0$$

$$T_p = \frac{k\pi}{\omega_d}, k = 1$$

$$T_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

4. M_p: The value of the step response at T_p.

$$\begin{aligned} c(T_p) &= 1 - \frac{1}{\sqrt{1-\zeta^2}}e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} sin(\pi + \beta) \\ &= 1 - \frac{1}{\sqrt{1-\zeta^2}}e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \cdot (-\sqrt{1-\zeta^2}) \\ &= 1 + e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \\ \sigma\% &= \frac{c(T_p) - c(\infty)}{c(\infty)} \cdot 100\% \\ &= e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}} \end{aligned}$$

5. T_s: The step response deviation is:

$$\begin{aligned} e(t) &= r(t) - c(t) \\ &= 1 - (1 - \frac{1}{\sqrt{1-\zeta^2}}e^{-\zeta\omega_n t} sin(\omega_d + \beta)) \\ &= \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} sin(\omega_d t + \beta) \\ \Delta &= \frac{e^{-\zeta\omega_n T_s}}{\sqrt{1-\zeta^2}} sin(\omega_d T_s + \beta) \end{aligned}$$

Ignore the effect of the sine function, we heve:

$$\Delta = \frac{e^{-\zeta\omega_n T_s}}{\sqrt{1-\zeta^2}} = 0.05/0.02$$

$$\Delta = 0.05, T_s \approx \frac{3}{\zeta\omega_n}, 0 < \zeta < 1$$

$$\Delta = 0.02, T_s \approx \frac{4}{\zeta\omega_n}, 0 < \zeta < 1$$

2.2.2 Dynamic Performance for Stable System

The performance indicators of the step response for a stable second-order system in underdamped, critically damped, and overdamped conditions are shown in Table 2:

Indicator	$0<\zeta<1$	$\zeta=1$	$\zeta>1$
$c(\infty)$	1		
$c(T_p)$	$1+e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}$	\	\
$\sigma\%$	$e^{-\frac{\pi\zeta}{\sqrt{1-\zeta^2}}}$	\	\
T_p	$T_p=\frac{\pi}{\omega_d}=\frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$	\	\
T_s	$\Delta=0.05, T_s\approx\frac{3}{\zeta\omega_n}, 0<\zeta<1$	\	\
	$\Delta=0.02, T_s\approx\frac{4}{\zeta\omega_n}, 0<\zeta<1$		

Table 2: Dynamic Performance Indicator for Stable System

2.3 Analysis of a Typical Second-Order System

Figure 4 shows a typical second-order system circuit.

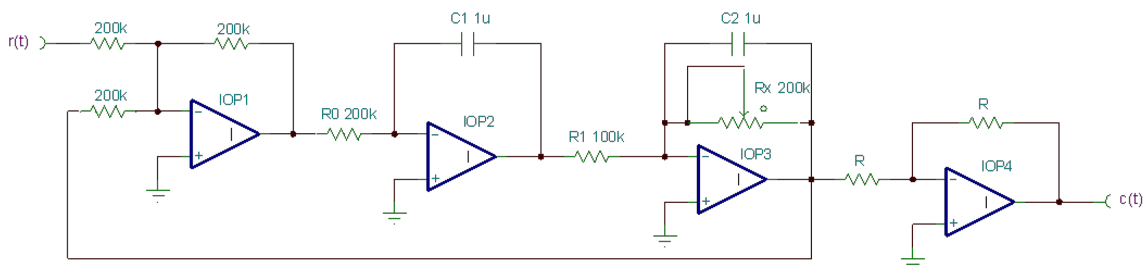

Figure 4: A Second-Order System Circuit

Its corresponding structure diagram can be represented as:

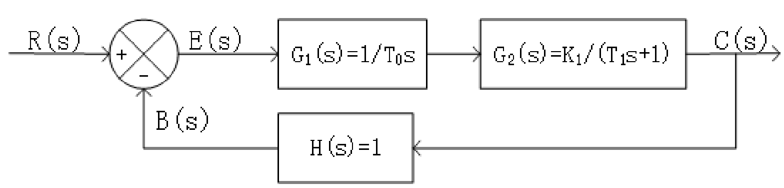

Figure 5: Schematic Diagram of the Second-Order System

Its open-loop transfer function is:

$$G(s)H(s)=\frac{B(s)}{R(s)}=\frac{C(s)}{E(s)}=\frac{1}{T_0s}\cdot\frac{K_1}{T_1s+1}\cdot 1=\frac{K_1}{T_0s\cdot(T_1s+1)}=\frac{K}{s(T_1s+1)}$$

$$T_0=R_0C_1, T_1=R_xC_2, K1=\frac{R_x}{R_1}, K=\frac{K_1}{T_0}$$

The closed-loop transfer function of the system is:

$$T(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1+G(s)} = \frac{K_1}{T_0 s \cdot (T_1 s + 1) + K_1} = \frac{K_1}{T_0 T_1 s^2 + T_0 s + K_1}$$

Convert it to the typical form:

$$T(s) = \frac{\frac{K_1}{T_0 T_1}}{s^2 + \frac{1}{T_1} s + \frac{K_1}{T_0 T_1}} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

so

$$\omega_n = \sqrt{\frac{K_1}{T_0 T_1}}, \zeta = \frac{1}{2}\sqrt{\frac{T_0}{K_1 T_1}}$$

3 Time-Domain Analysis for Third-Order System (System Stability)

For a second-order system, it is relatively easy to find the roots of the characteristic equation of the closed-loop system, and thus judge the system's stability based on the position of the poles. However, for higher-order systems, solving high-degree equations becomes somewhat difficult, and we need to use other methods to judge the system's stability. A typical third-order system circuit is shown in Figure 6

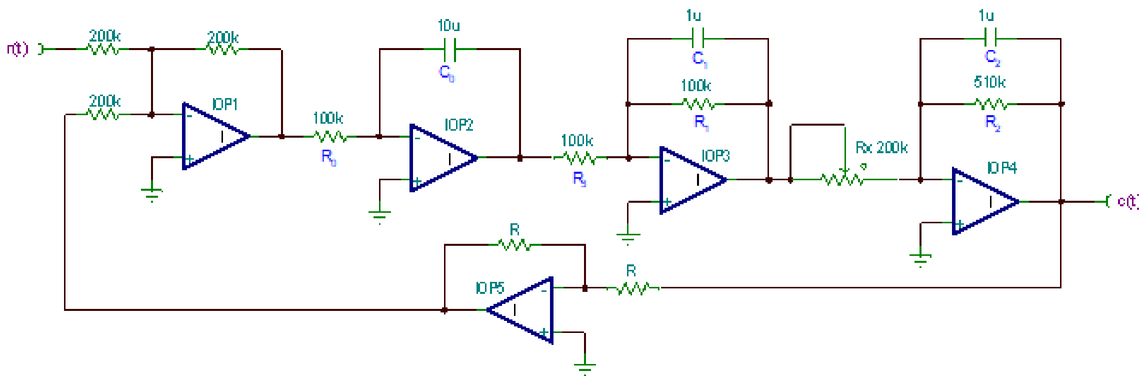

Figure 6: A Thire-Order System Circuit

Its corresponding structure diagram can be represented as:

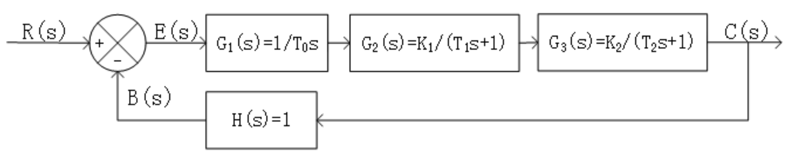

Figure 7: Schematic Diagram of the Third-Order System

Its open-loop transfer fuction is:

$$G(s)H(s) = \frac{B(s)}{R(s)} = \frac{C(s)}{E(s)} = \frac{1}{T_0 s} \cdot \frac{K_1}{T_1 s + 1} \cdot \frac{K_2}{T_2 s + 1} \cdot 1 = \frac{K}{s \cdot (T_1 s + 1) \cdot (T_2 s + 1)}, K = \frac{K_1 K_2}{T_0}$$

$$T_0 = R_0 \times C_0 = 100k \times 10\mu F = 1$$

$$T_1 = R_1 \times C_1 = 100k \times 1\mu F = 0.1$$

$$T_2 = R_2 \times C_2 = 510k \times 100\mu F \approx 0.5$$

$$K_1 = \frac{R_3}{R_1} = \frac{100k}{100k} = 1, K_2 = \frac{R_2}{R_x} = \frac{500k}{R_x}$$

$$K = \frac{K_1 K_2}{T_0}$$

So the closed-loop transfer function of the system is:

$$T(s) = \frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)} = \frac{K}{s \cdot (0.1s + 1) \cdot (0.5s + 1) + K} = \frac{20K}{s^3 + 12s^2 + 20s + 20K}$$

The characteristic equation is:

$$s^3 + 12s^2 + 20s + 20K = 0$$

At this point, solving the equation becomes difficult, so we consider using the Routh-Hurwitz criterion to determine the system's stability. For

$$D(s) = a_0S^n + a_1S^{n-1} + \cdots + a_{n-1}S + a_n = 0$$

Construct the Routh array.

$$\begin{array}{cccccc}
s^n & a_0 & a_2 & a_4 & a_6 & \cdots \\
s^{n-1} & a_1 & a_3 & a_5 & a_7 & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdots \\
s^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdots \\
\vdots & & & & & \\
s^2 & e_1 & e_2 & & & \\
s^1 & f_1 & & & & \\
1^0 & g_1 & & & &
\end{array}$$

$$b_1 = \frac{\begin{vmatrix} a_0 & a_2 \\ a_1 & a_3 \end{vmatrix}}{-a_1}, b_2 = \frac{\begin{vmatrix} a_0 & a_4 \\ a_1 & a_5 \end{vmatrix}}{-a_1}, b_3 = \frac{\begin{vmatrix} a_0 & a_6 \\ a_1 & a_7 \end{vmatrix}}{-a_1}$$

$$c_1 = \frac{\begin{vmatrix} a_1 & a_3 \\ b_1 & b_2 \end{vmatrix}}{-b_1}, c_2 = \frac{\begin{vmatrix} a_1 & a_5 \\ b_1 & b_3 \end{vmatrix}}{-b_1}, c_3 = \frac{\begin{vmatrix} a_1 & a_7 \\ b_1 & b_4 \end{vmatrix}}{-b_1}$$

$$\cdots$$

Check the Routh table, if all elements of the first column are positive, the system is stable. If there are negative numbers in the first column, the system is unstable, and the number of sign changes in the first column is equal to the number of system characteristic roots in the right half of the complex plane.