Numerical Optimization, Fall 2024
Homework 2

Due 23:59 (CST), Oct. 17, 2024

1. Show there exits 1-1 correspondence between the extreme points of the two problems

$$\boldsymbol{S}_1 = \{\boldsymbol{x} \in \mathbb{R}^n : \boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}, \boldsymbol{x} \geq 0\}.$$
$$\boldsymbol{S}_2 = \{(\boldsymbol{x}, \boldsymbol{y}) \in \mathbb{R}^n \times \mathbb{R}^m : \boldsymbol{A}\boldsymbol{x} + \boldsymbol{y} = \mathrm{b}, \boldsymbol{x} \geq 0, \boldsymbol{y} \geq 0\},$$

where $\boldsymbol{A} \in \mathbb{R}^{m\times n}, \boldsymbol{b} \in \mathbb{R}^m$.[10pts]
Solution:

Suppose that the extreme points of $\boldsymbol{S}_1$ compose the set $\boldsymbol{P}_1$, and the extreme points of $\boldsymbol{S}_2$ compose the set $\boldsymbol{P}_2$. We can construct the mapping from $\boldsymbol{P}_1$ to $\boldsymbol{P}_2$. For all $\boldsymbol{x} \in \boldsymbol{P}_1$, we have $\boldsymbol{x} \in \mathbb{R}^n$, $\boldsymbol{x} \geq 0$, and $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$. Hence, there exists $\boldsymbol{y} \in \mathbb{R}^m$ such that $\boldsymbol{A}\boldsymbol{x} + \boldsymbol{y} = \boldsymbol{b}$, where $y_i = b_i - (\boldsymbol{A}\boldsymbol{x})_i$. Therefore, we map $\boldsymbol{x}$ to $(\boldsymbol{x}, \boldsymbol{y})$.
Now, we need to prove that this mapping is bijective.

Surjective:
1. First, we prove that the mapping result is in $\boldsymbol{S}_2$. Since $\boldsymbol{A}\boldsymbol{x} \leq \boldsymbol{b}$, i.e., $(\boldsymbol{A}\boldsymbol{x})_i \leq b_i$, we have $y_i = b_i - (\boldsymbol{A}\boldsymbol{x})_i \geq 0$ for all $i = 1, \ldots, m$. Therefore, $(\boldsymbol{x}, \boldsymbol{y}) \in \boldsymbol{S}_2$.

2. Next, we prove that $(\boldsymbol{x}, \boldsymbol{y})$ is also an extreme point of $\boldsymbol{S}_2$. Suppose $(\boldsymbol{x}, \boldsymbol{y})$ is not an extreme point of $\boldsymbol{S}_2$. Then there exists $\lambda \in (0, 1)$ and $(\boldsymbol{x}_1, \boldsymbol{y}_1), (\boldsymbol{x}_2, \boldsymbol{y}_2) \in \boldsymbol{S}_2$ with $(\boldsymbol{x}_1, \boldsymbol{y}_1) \neq (\boldsymbol{x}_2, \boldsymbol{y}_2)$ such that $\boldsymbol{x} = \lambda\boldsymbol{x}_1 + (1 - \lambda)\boldsymbol{x}_2$. But since $\boldsymbol{x}$ is an extreme point of $\boldsymbol{S}_1$, this leads to a contradiction. Hence, $(\boldsymbol{x}, \boldsymbol{y}) \in \boldsymbol{P}_2$.

Injective:
If $\boldsymbol{x}_1 = \boldsymbol{x}_2 \in \boldsymbol{P}_1$, then $\boldsymbol{A}\boldsymbol{x}_1 = \boldsymbol{A}\boldsymbol{x}_2$ implies $\boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}_1 = \boldsymbol{b} - \boldsymbol{A}\boldsymbol{x}_2$, so $\boldsymbol{y}_1 = \boldsymbol{y}_2$. Thus, $(\boldsymbol{x}_1, \boldsymbol{y}_1) = (\boldsymbol{x}_2, \boldsymbol{y}_2)$, and the mapping is injective.

Therefore, the mapping is bijective, and the extreme points of $\boldsymbol{S}_1$ and $\boldsymbol{S}_2$ are in one-to-one correspondence.

2. Does $\boldsymbol{P} = \{\boldsymbol{x} \in \mathbb{R}^2 | 0 \leq x_1 \leq 1\}$ have extreme points? What is its standard form? Does it have extreme points? Find an extreme point if there exits one, and explain why.[20pts]

Solution:
Since the set $\boldsymbol{P}$ consists of two parallel lines, it has no extreme points.
Now, let's write $\boldsymbol{P}$ in standard form. Since x_2 is unbounded, we can express it as $x_2 = u - v$ where $u, v \geq 0$. Additionally, since $x_1 \leq 1$, we introduce a slack variable s such that $x_1 + s = 1$. Thus, the standard form of $\boldsymbol{P}$ is $\{(x_1, s, u, v) \in \mathbb{R}^4 | x_1 + s = 1, x_1, s, u, v \geq 0\}$.
In this standard form, P has extreme points. One of the extreme points is $(x_1, s, u, v) = (1, 0, 0, 0)$. This is because the constraints $x_1 + s = 1$, $s \geq 0$, $v \geq 0$, and $u \geq 0$ are all active, and these four independent active constraints match the number of variables in the standard form. Hence, $(x_1, s, u, v) = (1, 0, 0, 0)$ is a BFS and also an extreme point.

3. Rewrite the following problems as LP and convert the LP to standard form.[30pts]
Given $\boldsymbol{A} \in \mathbb{R}^{m\times n}, \boldsymbol{b} \in \mathbb{R}^m, m < n$.

   (a) AVE (absolute value error) linear regression:

$$\min_{\boldsymbol{x}} \|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}\|_1.$$

   (b) Robust linear regression:

$$\min_{\boldsymbol{x}} \|\boldsymbol{A}\boldsymbol{x} - \boldsymbol{b}\|_\infty.$$

   (c)

$$\begin{aligned} \min \quad & \max_{i=1,\ldots,k} (\boldsymbol{c}_i^T \boldsymbol{x} + d_i) \\ \text{s.t.} \quad & \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}. \end{aligned}$$

Solution:

(a) LP Form:
Introduce auxiliary variables $\boldsymbol{z} \in \mathbb{R}^m$ where each z_i represents $|(\boldsymbol{A}\boldsymbol{x})_i - b_i|$.

$$\begin{aligned} \min \quad & \sum_{i=1}^m z_i \\ \text{s.t.} \quad & z_i \geq (\boldsymbol{A}\boldsymbol{x})_i - b_i, \quad \forall i = 1, \ldots, m \\ & z_i \geq -((\boldsymbol{A}\boldsymbol{x})_i - b_i), \quad \forall i = 1, \ldots, m \end{aligned}$$

Standard Form (LP):

$$\begin{aligned} \min \quad & \sum_{i=1}^m z_i \\ \text{s.t.} \quad & \boldsymbol{z}_i - (\boldsymbol{A}(\boldsymbol{u} - \boldsymbol{v}))_i + b_i = s_i, \quad \forall i = 1, \ldots, m \\ & z_i + (\boldsymbol{A}(\boldsymbol{u} - \boldsymbol{v}))_i - b_i = t_i, \quad \forall i = 1, \ldots, m \\ & z_i, u_i, v_i, s_i, t_i \geq 0, \quad \forall i = 1, \ldots, m \end{aligned}$$

(b) LP Form:

Introduce an auxiliary variable z that represents $\max_{i=1,\dots,k}|(\boldsymbol{Ax})_i - b_i|$.

$$\begin{aligned} \min \quad & z \\ \text{s.t.} \quad & z \geq (\boldsymbol{Ax})_i - b_i, \quad \forall i = 1, \dots, m \\ & z \geq -((\boldsymbol{Ax})_i - b_i), \quad \forall i = 1, \dots, m \end{aligned}$$

Standard Form (LP):

$$\begin{aligned} \min \quad & z \\ \text{s.t.} \quad & z - (\boldsymbol{A}(\boldsymbol{u} - \boldsymbol{v}))_i + b_i = s_i, \quad \forall i = 1, \dots, m \\ & z + (\boldsymbol{A}(\boldsymbol{u} - \boldsymbol{v}))_i - b_i = t_i, \quad \forall i = 1, \dots, m \\ & z, u_i, v_i, s_i, t_i \geq 0, \quad \forall i = 1, \dots, m \end{aligned}$$

(c) LP Form:

Introduce an auxiliary variable t that represents $\max_{i=1,\dots,k}(\boldsymbol{c}_i^T\boldsymbol{x}+d_i)$.

$$\begin{aligned} \min \quad & t \\ \text{s.t.} \quad & \boldsymbol{c}_i^T\boldsymbol{x} + d_i \leq t, \quad \forall i = 1, \dots, k \\ & \boldsymbol{Ax} \geq \boldsymbol{b} \end{aligned}$$

Standard Form (LP):

$$\begin{aligned} \min \quad & m - n \\ \text{s.t.} \quad & m - n - \boldsymbol{c}_i^T(\boldsymbol{u} - \boldsymbol{v}) - d_i = p_i, \quad \forall i = 1, \dots, k \\ & \boldsymbol{A}(\boldsymbol{u} - \boldsymbol{v}) - \boldsymbol{b} = \boldsymbol{q} \\ & m, n, u_i, v_i, p_i, q_i \geq 0, \quad \forall i = 1, \dots, k \end{aligned}$$

4. In a standard form LP problem, assuming an optimal solution exists and every basic feasible solution is nondegenerate. Show that the objective function monotonically decreases after each pivot and the simplex method terminates after a finite number of iterations. [20pts]

Solution:

Suppose the current chosen basic set: $A = \begin{bmatrix} B & N \end{bmatrix}$, $x = \begin{bmatrix} x_B \\ x_N \end{bmatrix}$, $c = \begin{bmatrix} c_B \\ c_N \end{bmatrix}$,

Then we have $x_B = B^{-1}b - B^{-1}Nx_N$

$$\begin{aligned} f = c^T x = \begin{bmatrix} c_B^T & c_N^T \end{bmatrix} \begin{bmatrix} x_B \\ x_N \end{bmatrix} &= c_B^T x_B + c_N^T x_N \\ &= c_B^T(B^{-1}b - B^{-1}Nx_N) + c_N^T x_N \\ &= c_B^T B^{-1}b + (c_N^T - c_B^T B^{-1}N)x_N. \end{aligned}$$

After pivot, x_q become positive from zero. Therefore, we have the objective f decreases after pivot by $r_q < 0$.

Simplex method moves from current BFS to its adjacent BFS after every iterations. Since every BFS is nondegenerate, no BFS can be visited twice. There is a finite number of BFS, the algorithm must eventually terminate after a finite number of iterations.

5. Use simplex method to solve the following problem. You should write every pivot.[20pts]

$$\begin{aligned}
\min \quad & x_1 - 2x_2 + x_3 \\
\text{s.t.} \quad & x_1 + x_2 - 2x_3 + x_4 = 10, \\
& 2x_1 - x_2 + 4x_3 \leq 8, \\
& -x_1 + 2x_2 - 4x_3 \leq 4, \\
& x_j \geq 0, \quad j = 1, \cdots, 4.
\end{aligned}$$

Solution:
Convert it to standard form:

$$\begin{aligned}
\min \quad & x_1 - 2x_2 + x_3 \\
\text{s.t.} \quad & x_1 + x_2 - 2x_3 + x_4 = 10, \\
& 2x_1 - x_2 + 4x_3 + x_5 = 8, \\
& -x_1 + 2x_2 - 4x_3 + x_6 = 4, \\
& x_j \geq 0, \quad j = 1, \cdots, 6.
\end{aligned}$$

The simplex tableau is:

	x_1	x_2	x_3	x_4	x_5	x_6	b
x_4	1	1	-2	1	0	0	10
x_5	2	-1	4	0	1	0	8
x_6	-1	$\boxed{2}$	-4	0	0	1	4
	1	-2	1	0	0	0	0

	x_1	x_2	x_3	x_4	x_5	x_6	b
x_4	$\frac{3}{2}$	0	0	1	0	$-\frac{1}{2}$	8
x_5	$\frac{3}{2}$	0	$\boxed{2}$	0	1	$\frac{1}{2}$	10
x_2	$-\frac{1}{2}$	1	-2	0	0	$\frac{1}{2}$	2
	0	0	-3	0	0	1	4

	x_1	x_2	x_3	x_4	x_5	x_6	b
x_4	$\frac{3}{2}$	0	0	1	0	$-\frac{1}{2}$	8
x_3	$\frac{3}{4}$	0	1	0	$\frac{1}{2}$	$\frac{1}{4}$	5
x_2	1	1	0	0	1	1	12
	$\frac{9}{4}$	0	0	0	$\frac{3}{2}$	$\frac{7}{4}$	19

Optimal solution: $(x_1, x_2, x_3, x_4) = (0, 12, 5, 8)$.
Optimal value: $f_{\min} = -19$.