Matrix Computations
Chapter 6: Singular value, Singular Vectors, and Singular Value Decomposition

Section 6.3 SVD for Linear Systems

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Linear Systems: Sensitivity Analysis

Given nonsingular $\mathbf{A} \in \mathbb{R}^{n \times n}$ and $\mathbf{y} \in \mathbb{R}^n$, let $\mathbf{x}$ be the solution to

$$\mathbf{y} = \mathbf{A}\mathbf{x}$$

Consider a perturbed version of the above system:

$$\hat{\mathbf{A}} = \mathbf{A} + \Delta\mathbf{A}, \quad \hat{\mathbf{y}} = \mathbf{y} + \Delta\mathbf{y}$$

where $\Delta\mathbf{A}$ and $\Delta\mathbf{y}$ are errors (e.g., floating point errors, measurement errors, etc.)

Let $\hat{\mathbf{x}}$ be a solution to the perturbed system

$$\hat{\mathbf{y}} = \hat{\mathbf{A}}\hat{\mathbf{x}}$$

Problem: Analyze how the solution error $\|\hat{\mathbf{x}} - \mathbf{x}\|_2$ scales with $\Delta\mathbf{A}$ and $\Delta\mathbf{y}$

Remark: We have already studied sensitivity analysis of linear systems in Section 1.3. Here, we focus on its relation with SVD

Condition Number

The condition number of matrix $\mathbf{A}$ is defined as

$$\kappa(\mathbf{A}) = \|\mathbf{A}\| \cdot \|\mathbf{A}^{-1}\|$$

Let the above norm be 2-norm. Then, $\|\mathbf{A}\|_2 = \sigma_{\max}(\mathbf{A})$, $\|\mathbf{A}^{-1}\|_2 = 1/\sigma_{\min}(\mathbf{A})$, and

$$\kappa(\mathbf{A}) = \frac{\sigma_{\max}(\mathbf{A})}{\sigma_{\min}(\mathbf{A})}$$

For nonsingular $\mathbf{A}$, $\sigma_{\max}(\mathbf{A}) \geq \sigma_{\min}(\mathbf{A}) > 0$

Thus, $\kappa(\mathbf{A}) \geq 1$, and $\kappa(\mathbf{A}) = 1$ if $\mathbf{A}$ is orthogonal

- $\mathbf{A}$ is said to be ill-conditioned if $\kappa(\mathbf{A})$ is very large, referring to the cases where $\mathbf{A}$ is close to singular

$A = U \Sigma V^T$

$\Sigma = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{bmatrix}$

$A^{-1} = (U \Sigma V^T)^{-1} = V \Sigma^{-1} U^T$

SVD of A^{-1}

$\Sigma^{-1} = \begin{bmatrix} \frac{1}{\sigma_1} & & \\ & \ddots & \\ & & \frac{1}{\sigma_n} \end{bmatrix}$

$\|A^{-1}\|_2 = \frac{1}{\sigma_n}$

$\sigma_{max}(A^{-1})$

If A is orthogonal, it has SVD

$A = \underbrace{A}_{U} \cdot \underbrace{I}_{\Sigma} \cdot \underbrace{I^T}_{V^T}$

singular values of A are all 1.

Sensitivity Analysis

Theorem

Let $\varepsilon > 0$ *be s.t.*

$$\frac{\|\Delta \mathbf{A}\|_2}{\|\mathbf{A}\|_2} \leq \varepsilon, \qquad \frac{\|\Delta \mathbf{y}\|_2}{\|\mathbf{y}\|_2} \leq \varepsilon.$$

If ε *is sufficiently small s.t.* $\varepsilon \kappa(\mathbf{A}) < 1$, *then*

$$\frac{\|\hat{\mathbf{x}} - \mathbf{x}\|_2}{\|\mathbf{x}\|_2} \leq \frac{2\varepsilon \kappa(\mathbf{A})}{1 - \varepsilon \kappa(\mathbf{A})}$$

Implications:

- For small errors and in the worst-case sense, the relative error $\|\hat{\mathbf{x}} - \mathbf{x}\|_2 / \|\mathbf{x}\|_2$ tends to increase with the condition number
- In particular, for $\varepsilon \kappa(\mathbf{A}) \leq \frac{1}{2}$, the error bound is simplified to

$$\frac{\|\hat{\mathbf{x}} - \mathbf{x}\|_2}{\|\mathbf{x}\|_2} \leq 4\varepsilon \kappa(\mathbf{A})$$

Proof

Let $\Delta x = \hat{x} - x$. Then, $\hat{y} = \hat{A}\hat{x}$ can be written as

$$y + \Delta y = (A + \Delta A)(x + \Delta x)$$

$$\Downarrow \quad y = Ax$$

$$A \Delta x = \Delta y - \Delta A\, x - \Delta A \Delta x$$

$$\Downarrow \quad A \text{ nonsingular}$$

$$\Delta x = A^{-1}(\Delta y - \Delta A\, x - \Delta A \Delta x)$$

$$\Downarrow$$

$$\|\Delta x\|_2 \le \|A^{-1}\|_2 \cdot \|\Delta y - \Delta A x - \Delta A \Delta x\|_2$$

$$\le \|A^{-1}\|_2 \Big(\underbrace{\|\Delta y\|_2}_{1} + \underbrace{\|\Delta A\, x\|_2}_{2} + \underbrace{\|\Delta A \Delta x\|_2}_{3} \Big)$$

Proof (cont'd)

$$\|\Delta y\|_2 \leq \varepsilon \|y\|_2 = \varepsilon \|Ax\|_2 \leq \varepsilon \|A\|_2 \cdot \|x\|_2 \quad ①$$

$$\|\Delta A x\|_2 \leq \|\Delta A\|_2 \cdot \|x\|_2 \leq \varepsilon \|A\|_2 \|x\|_2 \quad ②$$

$$\|\Delta A \Delta x\|_2 \leq \|\Delta A\|_2 \cdot \|\Delta x\|_2 \leq \varepsilon \|A\|_2 \|\Delta x\|_2 \quad ③$$

① + ② + ③ $\Rightarrow$

$$\|\Delta x\|_2 \leq \|A^{-1}\|_2 \, \varepsilon \|A\|_2 \left(2\|x\|_2 + \|\Delta x\|_2\right)$$

$$= 2\varepsilon \kappa(A) \|x\|_2 + \varepsilon \kappa(A) \|\Delta x\|_2$$

$$\Rightarrow \frac{\|\Delta x\|_2}{\|x\|_2} \leq \frac{2\varepsilon \kappa(A)}{1 - \varepsilon \kappa(A)}$$

Interpretation of Linear Systems under SVD

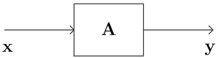
Consider the linear system

$$\mathbf{y} = \mathbf{A}\mathbf{x}$$

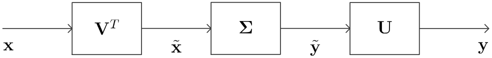
where $\mathbf{A} \in \mathbb{R}^{m \times n}$ is the system matrix, $\mathbf{x} \in \mathbb{R}^{n}$ is the system input, and $\mathbf{y} \in \mathbb{R}^{m}$ is the system output

Using SVD $\mathbf{A} = \mathbf{U}\Sigma\mathbf{V}^{T}$, we can write

$$\mathbf{y} = \mathbf{U}\tilde{\mathbf{y}}, \qquad \tilde{\mathbf{y}} = \Sigma\tilde{\mathbf{x}}, \qquad \tilde{\mathbf{x}} = \mathbf{V}^{T}\mathbf{x}$$

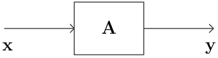


(a) linear system

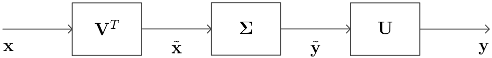


(b) equivalent system

Interpretation of Linear Systems under SVD (cont'd)

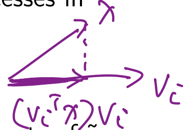


(a) linear system



(b) equivalent system

Implication: *All* linear systems work by performing three processes in cascade



- $\tilde{\mathbf{x}} = \mathbf{V}^T\mathbf{x}$: Let $\mathbf{x}$ resolve into $\mathbf{v}_1, \ldots, \mathbf{v}_n$ (rotate by $\mathbf{V}^T$)
- $\tilde{\mathbf{y}} = \Sigma\tilde{\mathbf{x}}$: Element-wise scale the first $p = \min\{m, n\}$ elements of $\tilde{\mathbf{x}}$ by $\sigma_i \geq 0$, $i = 1, \ldots, p$, and then either truncate or zero-pad to obtain the m-dimensional $\tilde{\mathbf{y}}$
- $\mathbf{y} = \mathbf{U}\tilde{\mathbf{y}}$: Reconstitute with basis $\mathbf{u}_1, \ldots, \mathbf{u}_m$ (rotate by $\mathbf{U}$)

Solution of Linear Systems via SVD

Problem: Given general $\mathbf{A} \in \mathbb{R}^{m \times n}$ and $\mathbf{y} \in \mathbb{R}^m$, determine

- whether $\mathbf{y} = \mathbf{A}\mathbf{x}$ has a solution (more precisely, whether there exists an $\mathbf{x}$ such that $\mathbf{y} = \mathbf{A}\mathbf{x}$)
- what is the solution

thin SVD $A = U_1 \tilde{\Sigma} V_1^T$

It can be shown via SVD that

$y \in \mathcal{R}(U_1) = \mathcal{R}(A)$

$$
\begin{aligned}
\mathbf{y} = \mathbf{A}\mathbf{x} \quad &\Longleftrightarrow \quad \mathbf{y} = \mathbf{U}_1 \tilde{\Sigma} \mathbf{V}_1^T \mathbf{x} \\
&\Longleftrightarrow \quad \mathbf{U}_1^T \mathbf{y} = \tilde{\Sigma} \mathbf{V}_1^T \mathbf{x}, \;\; \mathbf{U}_2^T \mathbf{y} = \mathbf{0} \\
&\Longleftrightarrow \quad \mathbf{V}_1^T \mathbf{x} = \tilde{\Sigma}^{-1} \mathbf{U}_1^T \mathbf{y}, \;\; \mathbf{U}_2^T \mathbf{y} = \mathbf{0}
\end{aligned}
$$

$$
\Longleftrightarrow \quad \boxed{\begin{array}{l} \mathbf{x} = \mathbf{V}_1 \tilde{\Sigma}^{-1} \mathbf{U}_1^T \mathbf{y} + \boldsymbol{\eta}, \\ \text{for any } \boldsymbol{\eta} \in \mathcal{R}(\mathbf{V}_2) = \mathcal{N}(\mathbf{A}), \\ \mathbf{U}_2^T \mathbf{y} = \mathbf{0} \end{array}}
$$

$= \mathrm{Null}(V_1^T) = \mathcal{R}(V_1)^\perp$

existence of solution

If $\tilde{z}$ is a solution to $Cz = b$,

$z + \eta$, $\forall \eta \in \mathrm{Null}(C)$ is also a solution

Solution of Linear Systems via SVD (cont'd)

$$\mathbf{y} = \mathbf{A}\mathbf{x} \quad \Longleftrightarrow \quad \begin{array}{l} \mathbf{x} = \mathbf{V}_1 \tilde{\Sigma}^{-1} \mathbf{U}_1^T \mathbf{y} + \boldsymbol{\eta}, \\ \text{for any } \boldsymbol{\eta} \in \mathcal{R}(\mathbf{V}_2) = \mathcal{N}(\mathbf{A}), \\ \mathbf{U}_2^T \mathbf{y} = \mathbf{0} \end{array}$$

Case (a): Full-column rank $\mathbf{A}$, i.e., $r = n \leq m$

$\Rightarrow$ $Null(A) = \{0\}$

- There is no $\mathbf{V}_2$, and $\mathbf{U}_2^T \mathbf{y} = \mathbf{0}$ is equivalent to $\mathbf{y} \in \mathcal{R}(\mathbf{U}_1) = \mathcal{R}(\mathbf{A})$
- **Result**: The linear system has a solution if and only if $\mathbf{y} \in \mathcal{R}(\mathbf{A})$, and the solution, if exists, is uniquely given by $\mathbf{x} = \mathbf{V}_1 \tilde{\Sigma}^{-1} \mathbf{U}_1^T \mathbf{y}$

Case (b): Full-row rank $\mathbf{A}$, i.e., $r = m \leq n$

- There is no $\mathbf{U}_2$ $\Rightarrow \mathcal{R}(A) = \mathbb{R}^m$
- **Result**: The linear system always has a solution, and the solution is given by $\mathbf{x} = \mathbf{V}_1 \tilde{\Sigma}^{-1} \mathbf{U}^T \mathbf{y} + \boldsymbol{\eta}$ for any $\boldsymbol{\eta} \in \mathcal{N}(\mathbf{A})$

Least Squares via SVD

Consider the LS problem: Given general $\mathbf{A} \in \mathbb{R}^{m \times n}$,

$$\min_{\mathbf{x} \in \mathbb{R}^n} \|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2^2$$

U orthogonal

For any $\mathbf{x} \in \mathbb{R}^n$,

$$\|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2^2 = \|\mathbf{y} - \mathbf{U}\Sigma \underbrace{\mathbf{V}^T\mathbf{x}}_{=\tilde{\mathbf{x}}}\|_2^2 = \|\underbrace{\mathbf{U}^T\mathbf{y}}_{=\tilde{\mathbf{y}}} - \Sigma\tilde{\mathbf{x}}\|_2^2$$

$$= \sum_{i=1}^{r} |\tilde{y}_i - \sigma_i \tilde{x}_i|^2 + \sum_{i=r+1}^{p} |\tilde{y}_i|^2$$

$$\geq \sum_{i=r+1}^{p} |\tilde{y}_i|^2$$

where the equality can be attained if $\tilde{\mathbf{x}}$ satisfies $\tilde{y}_i = \sigma_i \tilde{x}_i$ for $i = 1, \ldots, r$

$U^T y = \begin{bmatrix} U_1^T y \\ U_2^T y \end{bmatrix}$ $\quad U_1^T y = \begin{bmatrix} \tilde{y}_1 \\ \vdots \\ \tilde{y}_r \end{bmatrix}$ $\quad U_2^T y = \begin{bmatrix} \tilde{y}_{r+1} \\ \vdots \\ \tilde{y}_p \end{bmatrix}$

Least Squares via SVD (cont'd)

It can be shown that such a $\tilde{\mathbf{x}}$ corresponds to

$$\mathbf{x} = \mathbf{V}_1 \tilde{\Sigma}^{-1} \mathbf{U}_1^T \mathbf{y} + \mathbf{V}_2 \mathbf{x}_2' \text{ for any } \mathbf{x}_2' \in \mathbb{R}^{n-r} \qquad (\star)$$

which is the desired LS solution $\quad \in \mathcal{R}(V_2) = \mathcal{N}ull\,(A)$

Verification:

$$\|y - Ax\|_2^2 \overset{(\star)}{=} \| y - \underbrace{U_1 \tilde{\Sigma} V_1^T}_{\text{thin SVD of } A} (V_1 \tilde{\Sigma}^{-1} U_1^T y + V_2 x_2') \|_2$$

$$\underset{V_1^T V_2 = 0}{=} \| y - U_1 U_1^T y \|_2 \underset{U \text{ orthogonal}}{=} \| U^T y - U^T U_1 U_1^T y \|_2$$

$$U^T U_1 = \begin{bmatrix} U_1^T \\ U_2^T \end{bmatrix} U_1 = \begin{bmatrix} I \\ 0 \end{bmatrix} \Rightarrow U^T U_1 U_1^T y = \begin{bmatrix} U_1^T y \\ 0 \end{bmatrix}$$

Then, $U^T y - U^T U_1 U_1^T y = \begin{bmatrix} U_1^T y \\ U_2^T y \end{bmatrix} - \begin{bmatrix} U_1^T y \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ U_2^T y \end{bmatrix}$

$$\Rightarrow \|y - Ax\|_2^2 = \|U_2^T y\|_2^2 = \sum_{i=r+1}^{?} |\tilde{y}_i|^2$$

Pseudo-Inverse

Given $\mathbf{A} \in \mathbb{R}^{m \times n}$, a pseudo-inverse of $\mathbf{A}$ is defined as a matrix $\mathbf{A}^\dagger \in \mathbb{R}^{n \times m}$ satisfying the Moore-Penrose conditions:

(i) $\mathbf{A}\mathbf{A}^\dagger\mathbf{A} = \mathbf{A}$; (ii) $\mathbf{A}^\dagger\mathbf{A}\mathbf{A}^\dagger = \mathbf{A}^\dagger$; (iii) $\mathbf{A}\mathbf{A}^\dagger$ is symmetric (iv) $\mathbf{A}^\dagger\mathbf{A}$ is symmetric

Given the thin SVD $\mathbf{A} = \mathbf{U}_1\tilde{\Sigma}\mathbf{V}_1^T$,

$$\mathbf{A}^\dagger = \mathbf{V}_1\tilde{\Sigma}^{-1}\mathbf{U}_1^T$$

- $\mathbf{x}_{\mathrm{LS}} = \mathbf{A}^\dagger\mathbf{y} + \boldsymbol{\eta}$ for any $\boldsymbol{\eta} \in \mathcal{R}(\mathbf{V}_2)$
- The same applies to the linear system $\mathbf{y} = \mathbf{A}\mathbf{x}$ that has a solution
- When $\mathbf{A}$ has full column rank
  - $\mathbf{A}^\dagger = (\mathbf{A}^T\mathbf{A})^{-1}\mathbf{A}^T$
  - $\mathbf{A}^\dagger\mathbf{A} = \mathbf{I}$
- when $\mathbf{A}$ has full row rank
  - $\mathbf{A}^\dagger = \mathbf{A}^T(\mathbf{A}\mathbf{A}^T)^{-1}$
  - $\mathbf{A}\mathbf{A}^\dagger = \mathbf{I}$

Orthogonal Projections

- With SVD, the orthogonal projections of $\mathbf{y}$ onto $\mathcal{R}(\mathbf{A})$ and $\mathcal{R}(\mathbf{A})^{\perp}$ are given by

$$\Pi_{\mathcal{R}(\mathbf{A})}(\mathbf{y}) = \mathbf{A}\mathbf{x}_{\text{LS}} = \mathbf{A}\mathbf{A}^{\dagger}\mathbf{y} = \mathbf{U}_1\mathbf{U}_1^T\mathbf{y}$$
$$\Pi_{\mathcal{R}(\mathbf{A})^{\perp}}(\mathbf{y}) = \mathbf{y} - \mathbf{A}\mathbf{x}_{\text{LS}} = (\mathbf{I} - \mathbf{A}\mathbf{A}^{\dagger})\mathbf{y} = \mathbf{U}_2\mathbf{U}_2^T\mathbf{y}$$

- The orthogonal projector and orthogonal complement projector of $\mathbf{A}$ are given by

$$\mathbf{P}_{\mathbf{A}} = \mathbf{U}_1\mathbf{U}_1^T, \qquad \mathbf{P}_{\mathbf{A}}^{\perp} = \mathbf{U}_2\mathbf{U}_2^T$$

- Properties:
  - $\mathbf{P}_{\mathbf{A}}$ is idempotent, i.e., $\mathbf{P}_{\mathbf{A}}\mathbf{P}_{\mathbf{A}} = \mathbf{P}_{\mathbf{A}}$
  - $\mathbf{P}_{\mathbf{A}}$ is symmetric
  - The eigenvalues of $\mathbf{P}_{\mathbf{A}}$ are either 0 or 1
  - $\mathcal{R}(\mathbf{P}_{\mathbf{A}}) = \mathcal{R}(\mathbf{A})$
  - The same properties above apply to $\mathbf{P}_{\mathbf{A}}^{\perp}$, and $\mathbf{I} = \mathbf{P}_{\mathbf{A}} + \mathbf{P}_{\mathbf{A}}^{\perp}$

Minimum 2-Norm Solution to Underdetermined Linear Systems

Consider solving the linear system $\mathbf{y} = \mathbf{A}\mathbf{x}$ with fat $\mathbf{A} \in \mathbb{R}^{m \times n}$, $m < n$

- This is an underdetermined linear system: more unknowns n than the number of equations m

Assume $\mathbf{A}$ has full row rank. We already know that $\mathbf{x} = \mathbf{A}^{\dagger}\mathbf{y} + \boldsymbol{\eta}$ for any $\boldsymbol{\eta} \in \mathcal{R}(\mathbf{V}_2)$ is a solution

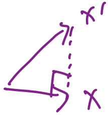
Now discard $\boldsymbol{\eta}$ and take $\mathbf{x} = \mathbf{A}^{\dagger}\mathbf{y}$ as one particular solution. This is the *unique* minimum 2-norm solution to $\mathbf{y} = \mathbf{A}\mathbf{x}$, i.e., it uniquely solves

$$\min_{\mathbf{x} \in \mathbb{R}^n} \|\mathbf{x}\|_2^2 \qquad \text{s.t. } \mathbf{y} = \mathbf{A}\mathbf{x}$$

A full row rank $\Rightarrow$ $x = A^{\dagger} y = A^T (AA^T)^{-1} y$

Let x' be another solution. Clearly, $A(x - x') = y - y = 0$

Note that $(x' - x)^T x = (x' - x)^T A^T (AA^T)^{-1} y = 0$



$$\|x'\|_2^2 = \|(x' - x) + x\|_2^2 = \|x' - x\|_2^2 + \|x\|_2^2$$

$$\geq \|x\|_2^2$$

Equality holds iff $x = x'$.

Matrix Computations
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Section 6.4 Application of SVD

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Low-Rank Matrix Approximation

Aim: Given $\mathbf{A} \in \mathbb{R}^{m \times n}$ with $\mathrm{rank}(\mathbf{A}) = r$ and $k \in \{1, \ldots, r-1\}$, find $\mathbf{B} \in \mathbb{R}^{m \times n}$ s.t. $\mathrm{rank}(\mathbf{B}) \leq k$ and $\mathbf{B}$ best approximates $\mathbf{A}$

- Closely related to the matrix factorization problem in Section 3.4
- Applications: PCA, dimensionality reduction, etc.

$$A = U \Sigma V^T = \sum_{i=1}^{r} \sigma_i u_i v_i^T$$

$$\sigma_1 \geq \cdots \geq \sigma_r > 0$$

Truncated SVD: Denote

$$\mathbf{A}_k = \sum_{i=1}^{k} \sigma_i \mathbf{u}_i \mathbf{v}_i^T$$

Perform the aforementioned approximation by choosing $\mathbf{B} = \mathbf{A}_k$

Application Example: Image Compression

- Let $\mathbf{A} \in \mathbb{R}^{m \times n}$ be a matrix whose (i,j)th entry a_{ij} stores the (i,j)th pixel of an image

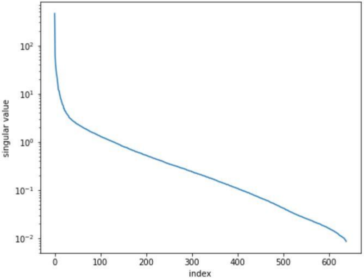
- Memory size for storing $\mathbf{A}$: mn

- Truncated SVD: store $\{\mathbf{u}_i, \sigma_i \mathbf{v}_i\}_{i=1}^k$ instead of the full $\mathbf{A}$, and recover the image by $\mathbf{B} = \mathbf{A}_k$

- Memory size for truncated SVD: $(m+n)k$
  - Much less than mn if $k \ll \min\{m, n\}$

Application Example: Image Compression (cont'd)

original image, size: 639 x 853



Application Example: Image Compression (cont'd)

k=10

k=15

k=20

k=30

Low-Rank Matrix Approximation

Truncated SVD provides the best approximation in the LS sense

Theorem (Eckart-Young-Mirsky)

Given $\mathbf{A} \in \mathbb{R}^{m\times n}$ *and* $k \in \{1, \ldots, r\}$, *The truncated SVD* $\mathbf{A}_k$ *is an optimal solution to*

$$\min_{\mathbf{B}\in\mathbb{R}^{m\times n},\ \mathrm{rank}(\mathbf{B})\leq k} \|\mathbf{A} - \mathbf{B}\|_F^2$$

The matrix 2-norm version of the Eckart-Young-Mirsky theorem

Theorem

Given $\mathbf{A} \in \mathbb{R}^{m\times n}$ *and* $k \in \{1, \ldots, r\}$, *the truncated SVD* $\mathbf{A}_k$ *is an optimal solution to*

$$\min_{\mathbf{B}\in\mathbb{R}^{m\times n},\ \mathrm{rank}(\mathbf{B})\leq k} \|\mathbf{A} - \mathbf{B}\|_2$$

Low-Rank Matrix Approximation

Recall the matrix factorization problem in Section 3.4

$$\min_{\mathbf{A}\in\mathbb{R}^{m\times k},\mathbf{B}\in\mathbb{R}^{k\times n}} \|\mathbf{Y}-\mathbf{A}\mathbf{B}\|_F^2$$

where $k \le \min\{m,n\}$, $\mathbf{A}$ is a basis matrix, and $\mathbf{B}$ is a coefficient matrix

The matrix factorization problem may be reformulated as

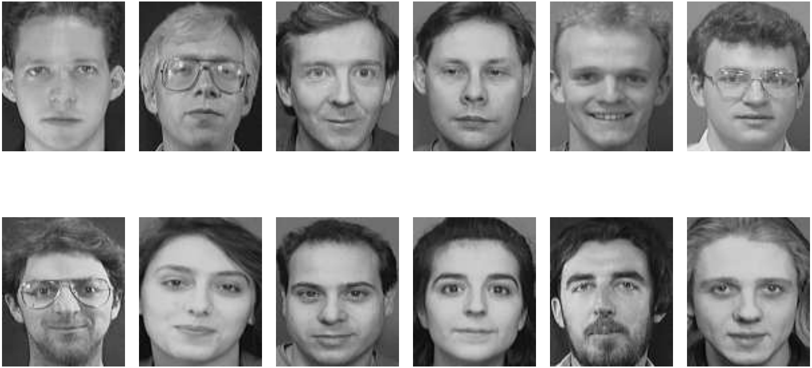
$$\min_{\mathbf{Z}\in\mathbb{R}^{m\times n},\mathrm{rank}(\mathbf{Z})\le k} \|\mathbf{Y}-\mathbf{Z}\|_F^2$$

The truncated SVD $\mathbf{Y}_k = \sum_{i=1}^{k}\sigma_i\mathbf{u}_i\mathbf{v}_i^T$, where $\mathbf{Y}=\mathbf{U}\Sigma\mathbf{V}^T$ denotes the SVD of $\mathbf{Y}$, is an optimal solution by the Eckart-Young-Mirsky Theorem

An optimal solution to the matrix factorization problem is

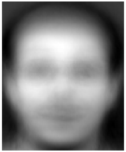
$$\mathbf{A} = [\ \mathbf{u}_1,\ldots,\mathbf{u}_k\ ], \qquad \mathbf{B} = [\ \sigma_1\mathbf{v}_1,\ldots,\sigma_k\mathbf{v}_k\ ]^T$$

Dimensionality Reduction of a Face Image Dataset

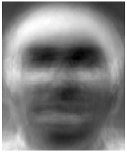


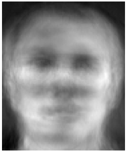
A face image dataset. Image size = 112×92, number of face images = 400. Each $\mathbf{x}_i$ is the vectorization of one face image, leading to $m = 112 \times 92 = 10304$, $n = 400$.

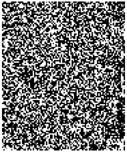
Dimensionality Reduction of a Face Image Dataset

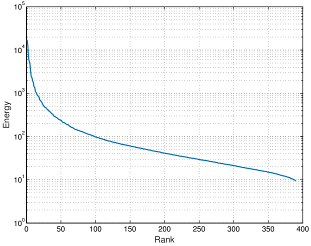

Mean face


1st principal left singular vector


2nd principal left singular vector


3rd principal left singular vector


400th left singular vector



Energy Concentration

Singular Value Inequalities

Similar to variational characterization for eigenvalues of real symmetric matrices, there have been a collection of variational characterization results for singular values

- Courant-Fischer characterization: Let $\mathbf{A} \in \mathbb{R}^{m \times n}$. For each $k = 1, \ldots, p = \min\{m, n\}$,

$$\sigma_k(\mathbf{A}) = \min_{\substack{\mathcal{S} \subseteq \mathbb{R}^n: \\ \dim \mathcal{S} = n-k+1}} \max_{\mathbf{x} \in \mathcal{S}, \ \|\mathbf{x}\|_2 = 1} \underbrace{\|\mathbf{Ax}\|_2}_{= \sqrt{x^T A^T A x}}$$

$\sigma_k^2(A)$ eigenvalu of $A^T A$

- Weyl's inequality: For any $\mathbf{A}, \mathbf{B} \in \mathbb{R}^{m \times n}$,

$$\sigma_{k+\ell-1}(\mathbf{A}+\mathbf{B}) \leq \sigma_k(\mathbf{A}) + \sigma_\ell(\mathbf{B}), \qquad k, \ell \in \{1, \ldots, p\}, \ k + \ell - 1 \leq p.$$

- Corollaries:
  - $\sigma_k(\mathbf{A}+\mathbf{B}) \leq \sigma_k(\mathbf{A}) + \sigma_1(\mathbf{B}), \ k = 1, \ldots, p$
  - $|\sigma_k(\mathbf{A}+\mathbf{B}) - \sigma_k(\mathbf{A})| \leq \sigma_1(\mathbf{B}), \ k = 1, \ldots, p$

Computing the SVD via the Power Method

Apply the power method to compute the thin SVD

- Assume $m \geq n$ and $\sigma_1 > \sigma_2 > \ldots \sigma_n > 0$

$m \leq n$

$\Downarrow$

AA^T

- Apply the power method to $\mathbf{A}^T\mathbf{A}$ to obtain $\mathbf{v}_1$
- Obtain $\mathbf{u}_1 = \mathbf{A}\mathbf{v}_1 / \|\mathbf{A}\mathbf{v}_1\|_2$, $\sigma_1 = \|\mathbf{A}\mathbf{v}_1\|_2$

$$Av_1 = [u_1 \cdots u_n] \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_n \end{bmatrix} \underbrace{\begin{bmatrix} v_1^T \\ \vdots \\ v_n^T \end{bmatrix} v_1}_{[1\ 0 \cdots 0]^T} = \sigma_1 u_1$$

$$\Rightarrow u_1 = \frac{Av_1}{\sigma_1} = \frac{Av_1}{\|Av_1\|_2}$$

- Do deflation $\mathbf{A} := \mathbf{A} - \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T$, and repeat the above steps until all singular components are found