# EE150 Signals and Systems – Part 5: Discrete-time Fourier Transform (DTFT)

# Continuous-time Fourier Transform

Continuous-time Fourier transform of $x(t)$ can be interpreted as

$$e^{j\omega t} \longrightarrow \boxed{x(t)} \longrightarrow X(j\omega)e^{j\omega t}$$

LTI system with impulse response $x(t)$

$X(j\omega)$: eigenvalue of $e^{j\omega t}$

$$X(j\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t}dt$$

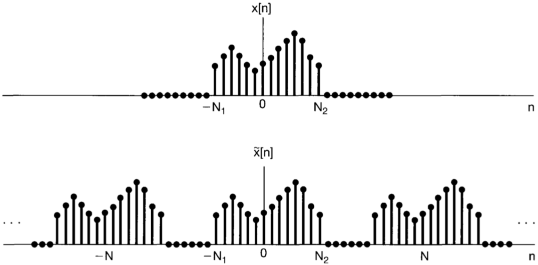
# Discrete-time Fourier Transform

Discrete-time LTI system with impulse response $x[n]$

$$e^{j\omega n} \longrightarrow \boxed{x[n]} \longrightarrow X(e^{j\omega})e^{j\omega n}$$

$$\begin{aligned} X(e^{j\omega})e^{j\omega n} &= e^{j\omega n} * x[n] \\ &= \sum_{m=-\infty}^{\infty} x[m]e^{j\omega(n-m)} \\ &= e^{j\omega n} \sum_{m=-\infty}^{\infty} x[m]e^{-j\omega m} \end{aligned}$$

# Fourier Series to Fourier Transform




$$\tilde{x}[n] = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n} \qquad a_k = \frac{1}{N}\sum_{n=\langle N\rangle} \tilde{x}[n]e^{-jk(2\pi/N)n}$$

As $x[n] = \tilde{x}[n]$ for $-N_1 \leq n \leq N_2$, we have

$$a_k = \frac{1}{N}\sum_{n=-N_1}^{N_2} x[n]\mathrm{e}^{-jk(2\pi/N)n} = \frac{1}{N}\sum_{n=-\infty}^{\infty} x[n]\mathrm{e}^{-jk(2\pi/N)n}$$

# Fourier Series to Fourier Transform cont.

1. Define $X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$
2. As $a_k = \frac{1}{N}X(e^{jk\omega_0})$, then

$$\tilde{x}[n] = \sum_{k=<N>} \frac{1}{N} X(e^{jk\omega_0})e^{jk\omega_0 n}$$
$$= \frac{1}{2\pi} \sum_{k=<N>} X(e^{jk\omega_0})e^{jk\omega_0 n}\omega_0$$

3. As $N \to \infty$, we have $\tilde{x}[n] \to x[n]$.

# Discrete-time Fourier Transform

- Forward and inverse transforms

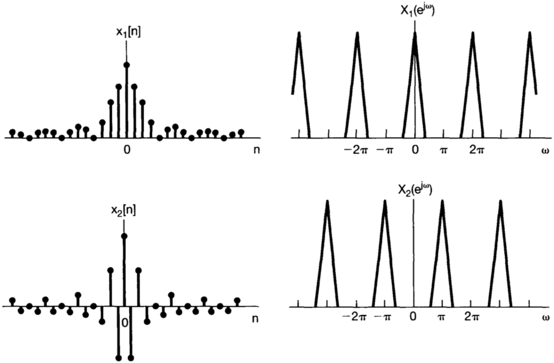
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n} \qquad \text{(DTFT)}$$

$$x[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega \qquad \text{(Inverse DTFT)}$$

- $x[n]$: linear combination of complex exponential infinitesimally close in frequency with amplitudes $X(e^{j\omega})(d\omega/2\pi)$
- $X(e^{j\omega})$ is the spectrum of $x[n]$

# Discrete-time Fourier Transform

1. $e^{j\omega n}$ is periodic in terms of $\omega$: $e^{j\omega n} = e^{j(\omega + k2\pi)n}$, $k \in \mathbb{Z}$
2. Hence, $X(e^{j\omega})$ is periodic (with period $2\pi$)



# Discrete-time Complex Exponential

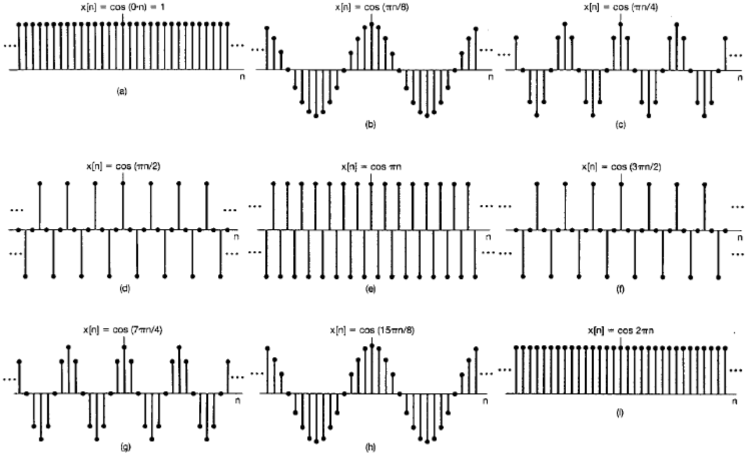


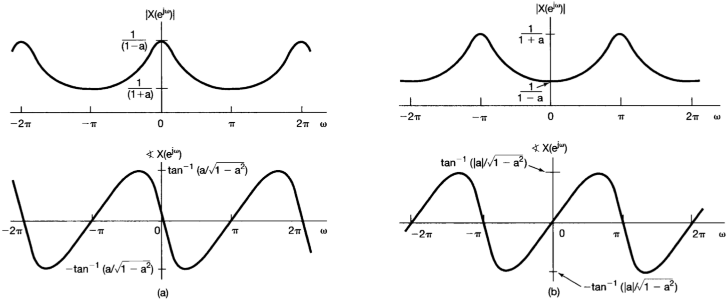

**Figure 1.27** Discrete-time sinusoidal sequences for several different frequencies.

# Example 5.1

Calculate the DTFT of signal $x[n] = a^n u[n]$, $|a| < 1$

Solution:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^n u[n] e^{-j\omega n} = \sum_{n=0}^{\infty} (ae^{-j\omega})^n = \frac{1}{1 - ae^{-j\omega}}$$



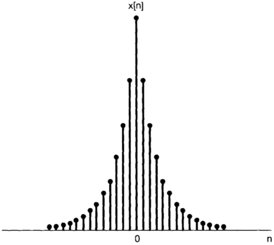
# Example 5.2

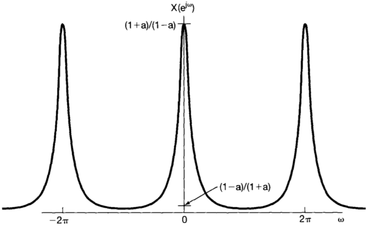
Calculate the DTFT of signal $x[n] = a^{|n|}$, $|a| < 1$.

Solution:

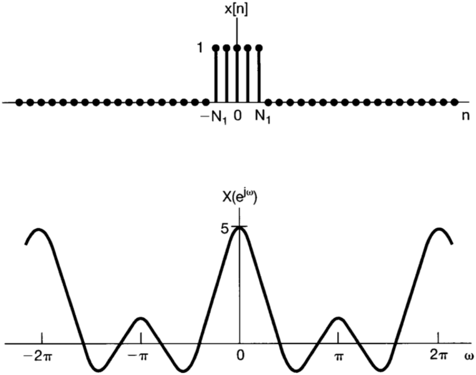
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} a^{|n|} e^{-j\omega n} = \sum_{n=0}^{\infty} a^n e^{-j\omega n} + \sum_{n=-\infty}^{-1} a^{-n} e^{-j\omega n}$$

$$= \frac{1}{1 - ae^{-j\omega}} + \frac{ae^{j\omega}}{1 - ae^{j\omega}} = \frac{1 - a^2}{1 - 2a\cos\omega + a^2}$$



# Example 5.3




$$x[n] = \begin{cases} 1, & |n| \leq N_1 \\ 0, & |n| > N_1 \end{cases}$$

$$X(e^{j\omega}) = \sum_{n=-N_1}^{N_1} e^{-j\omega n} = \frac{\sin \omega \left(N_1 + \frac{1}{2}\right)}{\sin(\omega/2)}$$

# Convergence Issues of DTFT

1. $x[n]$ is absolutely summable

$$\sum_{n=-\infty}^{\infty} |x[n]| < \infty$$

2. OR $x[n]$ has finite energy

$$\sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$$

# Fourier Transform for Periodic Signals

Fourier transform can be applied to periodic signal

Consider $x(t)$ and its FT $X(\mathrm{e}^{j\omega})$

Assume $X(\mathrm{e}^{j\omega}) = \sum_{l=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 - 2\pi l)$

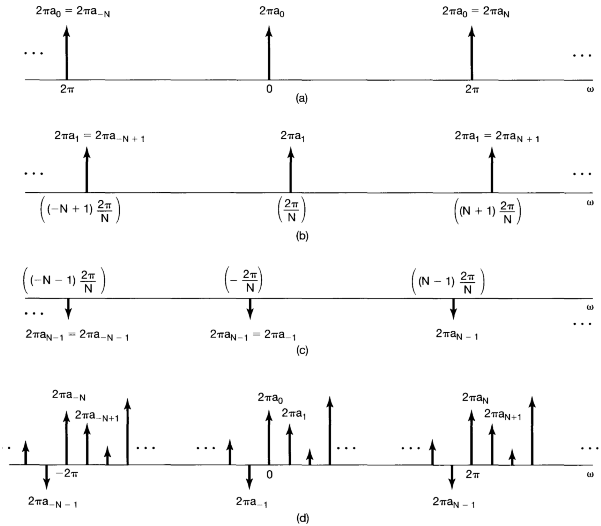
$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(\mathrm{e}^{j\omega})\mathrm{e}^{j\omega n}\mathrm{d}\omega = \frac{1}{2\pi}\int_{2\pi}\sum_{l=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 - 2\pi l)\mathrm{e}^{j\omega n}\mathrm{d}\omega$$
$$= \mathrm{e}^{j(\omega_0 + 2\pi r)n} = \mathrm{e}^{j\omega_0 n}$$

Hence,

$$x[n] = \sum_{k=<N>} a_k \mathrm{e}^{jk(2\pi/N)n} \xleftrightarrow{FT} X(\mathrm{e}^{j\omega}) = \sum_{k=-\infty}^{\infty} 2\pi a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$$

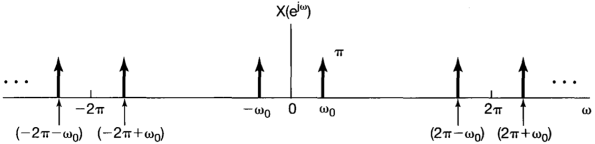
# Fourier Transform for Periodic Signals cont.

$$x[n] = a_0 + a_1 e^{j(2\pi/N)n} + a_2 e^{j2(2\pi/N)n} + \cdots + a_{N-1} e^{j(N-1)(2\pi/N)n}$$



# Example 5.5

1. Consider the periodic signal
$x[n] = \cos \omega_0 n = \frac{1}{2}\mathrm{e}^{j\omega_0 n} + \frac{1}{2}\mathrm{e}^{-j\omega_0 n}$ with $\omega_0 = \frac{2\pi}{5}$
2. Hence, $X(\mathrm{e}^{j\omega}) =$
$\sum_{l=-\infty}^{\infty} \pi\delta\left(\omega - \frac{2\pi}{5} - 2\pi l\right) + \sum_{l=-\infty}^{\infty} \pi\delta\left(\omega + \frac{2\pi}{5} - 2\pi l\right)$
3. It equals $X(\mathrm{e}^{j\omega}) = \pi\delta\left(\omega - \frac{2\pi}{5}\right) + \pi\delta\left(\omega + \frac{2\pi}{5}\right), -\pi \leq \omega \leq \pi$



# Properties of DTFT

$$x[n] \xleftrightarrow{FT} X(e^{j\omega})$$

1. Periodicity: DTFT is always periodic in $\omega$ with period $2\pi$, i.e., $X(e^{j(\omega+2\pi)}) = X(e^{j\omega})$.
2. Linearity: $ax_1[n] + bx_2[n] \xleftrightarrow{FT} aX_1(e^{j\omega}) + bX_2(e^{j\omega})$
3. Time shifting: $x[n-n_0] \xleftrightarrow{FT} e^{-j\omega n_0}X(e^{j\omega})$
4. Frequency shifting: $e^{j\omega_0 n}x[n] \xleftrightarrow{FT} X(e^{j(\omega-\omega_0)})$ CT也有 但是没写
5. Conjugation and conjugate symmetry:
   $x^*[n] \xleftrightarrow{FT} X^*(e^{-j\omega})$
   If $x[n]$ is real valued, then its transform $X(e^{j\omega})$ is conjugate symmetric, i.e., $X(e^{j\omega}) = X^*(e^{-j\omega})$

# Properties of DTFT cont.

6. Differencing

$$x[n] - x[n-1] \xleftrightarrow{FT} \left(1 - \mathrm{e}^{-j\omega}\right) X(\mathrm{e}^{j\omega})$$

**Proof.**

$$x[n] - x[n-1] = x[n] * [\delta[n] - \delta[n-1]]$$

$$\delta[n-k] \xleftrightarrow{FT} e^{-j\omega k}$$

$\square$

7. Accumulation: $y[n] = \sum_{m=-\infty}^{n} x[m]$

$$\sum_{m=-\infty}^{n} x[m] \xleftrightarrow{FT} \frac{1}{1 - \mathrm{e}^{-j\omega}} X(\mathrm{e}^{j\omega}) + \pi X(\mathrm{e}^{j0}) \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k)$$

# Example 5.8

Derive the Fourier transform of unit step $x[n] = u[n]$

Solution: $g[n] = \delta[n] \xleftrightarrow{FT} G(e^{j\omega}) = 1$

Unit step function is the running sum of the unit impulse

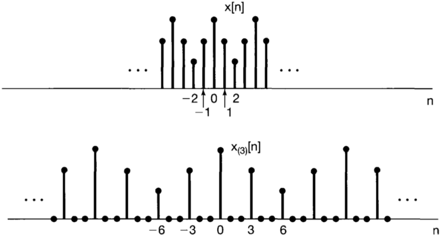
$$x[n] = \sum_{m=-\infty}^{n} g[m]$$

Taking the Fourier transform of both sides and using accumulation yield

$$\begin{aligned} X(e^{j\omega}) &= \frac{1}{1 - e^{-j\omega}} G(e^{j\omega}) + \pi G(e^{j0}) \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k) \\ &= \frac{1}{1 - e^{-j\omega}} + \pi \sum_{k=-\infty}^{\infty} \delta(\omega - 2\pi k) \end{aligned}$$

# Properties of DTFT cont.

8. Time reversal: $x[-n] \xleftrightarrow{FT} X(\mathrm{e}^{-j\omega})$
9. Differentiation in frequency: $nx[n] \xleftrightarrow{FT} j\frac{\mathrm{d}X(\mathrm{e}^{j\omega})}{\mathrm{d}\omega}$
10. Parseval's Relation: $\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi}\int_{2\pi} |X(\mathrm{e}^{j\omega})|^2 \mathrm{d}\omega$
11. Time expansion: $x_{(k)}[n] \xleftrightarrow{FT} X(\mathrm{e}^{jk\omega})$



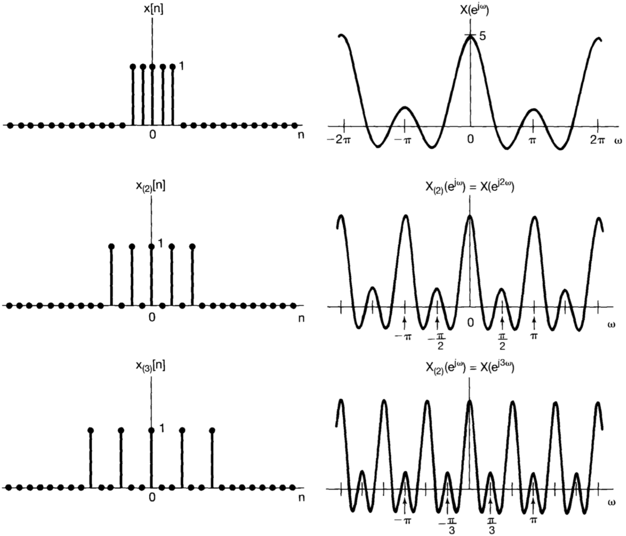

$$x_{(k)}[n] = \begin{cases} x[n/k], & \text{if } n \text{ is a multiple of } k \\ 0, & \text{if } n \text{ is not a multiple of } k \end{cases}$$

$$X_{(k)}(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x_{(k)}[n]e^{-j\omega n} = \sum_{r=-\infty}^{+\infty} x_{(k)}[rk]e^{-j\omega rk}$$

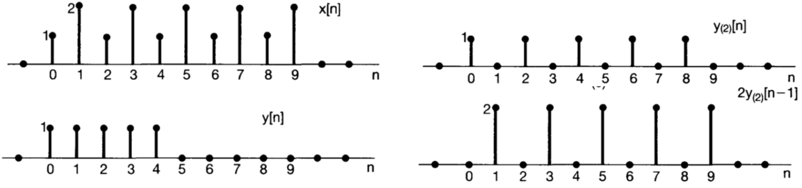
since $x_{(k)}[rk] = x[r]$

$$X_{(k)}(e^{j\omega}) = \sum_{r=-\infty}^{+\infty} x[r]e^{-j(k\omega)r} = X(e^{jk\omega})$$

# Time Expansion



# Example 5.9




$$x[n] = y_{(2)}[n] + 2y_{(2)}[n-1],$$

$$y_{(2)}[n] = \begin{cases} y[n/2], & \text{if } n \text{ is even} \\ 0, & \text{if } n \text{ is odd} \end{cases}$$

$$Y(e^{j\omega}) = e^{-j2\omega}\frac{\sin(5\omega/2)}{\sin(\omega/2)}$$

$$y_{(2)}[n] \overset{\mathfrak{F}}{\longleftrightarrow} e^{-j4\omega}\frac{\sin(5\omega)}{\sin(\omega)}$$

$$2y_{(2)}[n-1] \overset{\mathfrak{F}}{\longleftrightarrow} 2e^{-j5\omega}\frac{\sin(5\omega)}{\sin(\omega)}$$

$$X(e^{j\omega}) = e^{-j4\omega}(1 + 2e^{-j\omega})\left(\frac{\sin(5\omega)}{\sin(\omega)}\right)$$

# Convolution Property

- $y[n] = h[n] * x[n]$

$$
\begin{aligned}
Y(j\omega) &= \sum_n y[n]e^{-j\omega n} \\
&= \sum_n \sum_m x[m]h[n-m]e^{-j\omega n} \\
&= \sum_m \left( \sum_n h[n-m]e^{-j\omega(n-m)} \right) x[m]e^{-j\omega m} \\
&= \sum_m H(j\omega)x[m]e^{-j\omega m} \\
&= H(j\omega)X(j\omega)
\end{aligned}
$$

- Frequency response $H(e^{j\omega})$ captures the change in complex amplitude of the Fourier Transform of the input at each frequency $\omega$

# Example 5.11

1. Consider an LTI system with impulse response $h[n] = \delta[n - n_0]$
2. Frequency response is

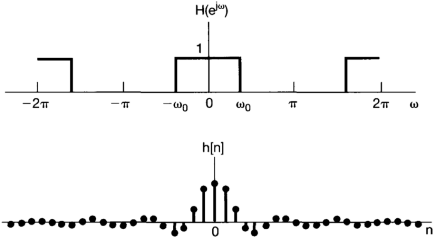
$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} \delta[n - n_0] e^{-j\omega n} = e^{-j\omega n_0}$$

3. For any input $x[n]$ with Fourier transform $X(e^{j\omega})$, the Fourier transform of the output is $Y(e^{j\omega}) = e^{-j\omega n_0} X(e^{j\omega})$
4. Frequency response $H(e^{j\omega}) = e^{-j\omega n_0}$ has unity magnitude at all frequencies and a phase characteristic $-\omega n_0$ that is linear with frequency

# Example 5.12

1. Consider the discrete-time ideal lowpass filter $H(e^{j\omega})$
2. Impulse response and frequency response of an LTI system are a Fourier transform pair

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}d\omega = \frac{1}{2\pi}\int_{-\omega_0}^{\omega_0} e^{j\omega n}d\omega = \frac{\sin\omega_0 n}{\pi n}$$



# Example 5.13

Consider an LTI system with impulse response $h[n] = \alpha^n u[n]$ with $|\alpha| < 1$, and suppose that the input to this system is $x[n] = \beta^n u[n]$ with $|\beta| < 1$. Evaluate the output $y[n]$.

1. Time-domain convolution: $y[n] = x[n] * h[n]$
2. Convolution property:

$$Y(e^{j\omega}) = X(e^{j\omega})H(e^{j\omega}) = \frac{1}{(1-\alpha e^{-jw})(1-\beta e^{-j\omega})}$$

# Multiplication Property

- $y[n] = x[n]h[n]$

$$y[n] = \frac{1}{(2\pi)^2}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}d\omega \int_{-\pi}^{\pi} H(e^{j\rho})e^{j\rho n}d\rho$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi}\left(\frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})H(e^{j(\nu-\omega)})d\omega\right)e^{j\nu n}d\nu$$

- So

$$Y(e^{j\nu}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})H(e^{j(\nu-\omega)})d\omega$$

(periodic convolution)

# Example 5.15

Calculate the Fourier transform $X(e^{j\omega})$ of signal $x[n] = x_1[n]x_2[n]$, where $x_1[n] = \frac{\sin(3\pi n/4)}{\pi n}$ and $x_2[n] = \frac{\sin(\pi n/2)}{\pi n}$

Solution: $X(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta$
Define $\hat{X}_1(e^{j\omega})$ equals $X_1(e^{j\omega})$ for $-\pi < \omega \leq \pi$, and equals 0 otherwise. Hence,

$$\begin{aligned} X(e^{j\omega}) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{X}_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{X}_1(e^{j\theta})X_2(e^{j(\omega-\theta)})d\theta \end{aligned}$$

# Properties of DTFT

**TABLE 5.1** PROPERTIES OF THE DISCRETE-TIME FOURIER TRANSFORM

| Section | Property | Aperiodic Signal | Fourier Transform |
|---|---|---|---|
| | | $x[n]$<br>$y[n]$ | $\left.\begin{array}{l}X(e^{j\omega})\\Y(e^{j\omega})\end{array}\right\}$ periodic with period $2\pi$ |
| 5.3.2 | Linearity | $ax[n]+by[n]$ | $aX(e^{j\omega})+bY(e^{j\omega})$ |
| 5.3.3 | Time Shifting | $x[n-n_0]$ | $e^{-j\omega n_0}X(e^{j\omega})$ |
| 5.3.3 | Frequency Shifting | $e^{j\omega_0 n}x[n]$ | $X(e^{j(\omega-\omega_0)})$ |
| 5.3.4 | Conjugation | $x^*[n]$ | $X^*(e^{-j\omega})$ |
| 5.3.6 | Time Reversal | $x[-n]$ | $X(e^{-j\omega})$ |
| 5.3.7 | Time Expansion | $x_{(k)}[n]=\begin{cases}x[n/k], & \text{if } n=\text{multiple of } k\\ 0, & \text{if } n\neq\text{multiple of } k\end{cases}$ | $X(e^{jk\omega})$ |
| 5.4 | Convolution | $x[n]*y[n]$ | $X(e^{j\omega})Y(e^{j\omega})$ |
| 5.5 | Multiplication | $x[n]y[n]$ | $\frac{1}{2\pi}\int_{2\pi}X(e^{j\theta})Y(e^{j(\omega-\theta)})d\theta$ |
| 5.3.5 | Differencing in Time | $x[n]-x[n-1]$ | $(1-e^{-j\omega})X(e^{j\omega})$ |
| 5.3.5 | Accumulation | $\sum_{k=-\infty}^{n}x[k]$ | $\frac{1}{1-e^{-j\omega}}X(e^{j\omega})$<br>$+\pi X(e^{j0})\sum_{k=-\infty}^{+\infty}\delta(\omega-2\pi k)$ |
| 5.3.8 | Differentiation in Frequency | $nx[n]$ | $j\frac{dX(e^{j\omega})}{d\omega}$ |
| 5.3.4 | Conjugate Symmetry for Real Signals | $x[n]$ real | $\begin{cases}X(e^{j\omega})=X^*(e^{-j\omega})\\ \mathcal{Re}\{X(e^{j\omega})\}=\mathcal{Re}\{X(e^{-j\omega})\}\\ \mathcal{Im}\{X(e^{j\omega})\}=-\mathcal{Im}\{X(e^{-j\omega})\}\\ \lvert X(e^{j\omega})\rvert=\lvert X(e^{-j\omega})\rvert\\ \sphericalangle X(e^{j\omega})=-\sphericalangle X(e^{-j\omega})\end{cases}$ |
| 5.3.4 | Symmetry for Real, Even Signals | $x[n]$ real an even | $X(e^{j\omega})$ real and even |
| 5.3.4 | Symmetry for Real, Odd Signals | $x[n]$ real and odd | $X(e^{j\omega})$ purely imaginary and odd |
| 5.3.4 | Even-odd Decomposition of Real Signals | $x_e[n]=\mathcal{Ev}\{x[n]\}$ [$x[n]$ real]<br>$x_o[n]=\mathcal{Od}\{x[n]\}$ [$x[n]$ real] | $\mathcal{Re}\{X(e^{j\omega})\}$<br>$j\mathcal{Im}\{X(e^{j\omega})\}$ |
| 5.3.9 | Parseval's Relation for Aperiodic Signals | | |

$$\sum_{n=-\infty}^{+\infty}|x[n]|^2=\frac{1}{2\pi}\int_{2\pi}|X(e^{j\omega})|^2d\omega$$

# Basic DTFT Pairs

| Signal | Fourier Transform | Fourier Series Coefficients (if periodic) |
|---|---|---|
| $\sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$ | $2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$ | $a_k$ |
| $e^{j\omega_0 n}$ | $2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - \omega_0 - 2\pi l)$ | (a) $\omega_0 = \frac{2\pi m}{N}$<br>$a_k = \begin{cases} 1, & k = m, m \pm N, m \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$<br>(b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic |
| $\cos \omega_0 n$ | $\pi \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) + \delta(\omega + \omega_0 - 2\pi l)\}$ | (a) $\omega_0 = \frac{2\pi m}{N}$<br>$a_k = \begin{cases} \frac{1}{2}, & k = \pm m, \pm m \pm N, \pm m \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$<br>(b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic |
| $\sin \omega_0 n$ | $\frac{\pi}{j} \sum_{l=-\infty}^{+\infty} \{\delta(\omega - \omega_0 - 2\pi l) - \delta(\omega + \omega_0 - 2\pi l)\}$ | (a) $\omega_0 = \frac{2\pi r}{N}$<br>$a_k = \begin{cases} \frac{1}{2j}, & k = r, r \pm N, r \pm 2N, \ldots \\ -\frac{1}{2j}, & k = -r, -r \pm N, -r \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$<br>(b) $\frac{\omega_0}{2\pi}$ irrational $\Rightarrow$ The signal is aperiodic |
| $x[n] = 1$ | $2\pi \sum_{l=-\infty}^{+\infty} \delta(\omega - 2\pi l)$ | $a_k = \begin{cases} 1, & k = 0, \pm N, \pm 2N, \ldots \\ 0, & \text{otherwise} \end{cases}$ |
| Periodic square wave<br>$x[n] = \begin{cases} 1, & \lvert n\rvert \le N_1 \\ 0, & N_1 < \lvert n\rvert \le N/2 \end{cases}$<br>and<br>$x[n+N] = x[n]$ | $2\pi \sum_{k=-\infty}^{+\infty} a_k \delta\left(\omega - \frac{2\pi k}{N}\right)$ | $a_k = \frac{\sin[(2\pi k/N)(N_1 + \frac{1}{2})]}{N \sin[2\pi k/2N]}, \; k \ne 0, \pm N, \pm 2N, \ldots$<br>$a_k = \frac{2N_1 + 1}{N}, \; k = 0, \pm N, \pm 2N, \ldots$ |
| $\sum_{k=-\infty}^{+\infty} \delta[n - kN]$ | $\frac{2\pi}{N} \sum_{k=-\infty}^{+\infty} \delta\left(\omega - \frac{2\pi k}{N}\right)$ | $a_k = \frac{1}{N}$ for all $k$ |

# Basic DTFT Pairs cont.

| Signal | Fourier Transform | Fourier Series Coefficients (if periodic) |
|---|---|---|
| $a^n u[n], \quad \lvert a\rvert < 1$ | $\frac{1}{1 - ae^{-j\omega}}$ | — |
| $x[n]\begin{cases} 1, & \lvert n\rvert \le N_1 \\ 0, & \lvert n\rvert > N_1 \end{cases}$ | $\frac{\sin[\omega(N_1 + \frac{1}{2})]}{\sin(\omega/2)}$ | — |
| $\frac{\sin Wn}{\pi n} = \frac{W}{\pi}\operatorname{sinc}\left(\frac{Wn}{\pi}\right)$<br>$0 < W < \pi$ | $X(\omega) = \begin{cases} 1, & 0 \le \lvert\omega\rvert \le W \\ 0, & W < \lvert\omega\rvert \le \pi \end{cases}$<br>$X(\omega)$ periodic with period $2\pi$ | — |
| $\delta[n]$ | $1$ | — |
| $u[n]$ | $\frac{1}{1 - e^{-j\omega}} + \sum_{k=-\infty}^{+\infty} \pi\delta(\omega - 2\pi k)$ | — |
| $\delta[n - n_0]$ | $e^{-j\omega n_0}$ | — |
| $(n+1)a^n u[n], \quad \lvert a\rvert < 1$ | $\frac{1}{(1 - ae^{-j\omega})^2}$ | — |
| $\frac{(n+r-1)!}{n!(r-1)!}a^n u[n], \quad \lvert a\rvert < 1$ | $\frac{1}{(1 - ae^{-j\omega})^r}$ | — |

# Duality in Discrete Time Fourier Series

1. $f[k]$ is the Fourier series coefficient of signal $g[n]$

$$g[n] \xleftrightarrow{FS} f[k], \qquad f[k] = \frac{1}{N} \sum_{n=<N>} g[n] e^{-jk(2\pi/N)n}$$

2. $f[k]$ is a periodic sequence with period $N$. Its Fourier series expansion is

$$f[n] = \sum_{k=<N>} \frac{1}{N} g[-k] e^{jk(2\pi/N)n}$$

3. Hence,

$$f[n] \xleftrightarrow{FS} \frac{1}{N} g[-k]$$

# Example 5.16

Consider the following periodic signal with a period of $N = 9$

$$x[n] = \begin{cases} \frac{1}{9}\frac{\sin(5\pi n/9)}{\sin(\pi n/9)}, & n \neq \text{multiple of } 9 \\ \frac{5}{9}, & n = \text{multiple of } 9 \end{cases}$$

From previous example, we know that

$$g[n] = \begin{cases} 1, & |n| \leq 2 \\ 0, & 2 < |n| \leq 4 \end{cases} \xleftrightarrow{FS} b_k = \begin{cases} \frac{1}{9}\frac{\sin(5\pi k/9)}{\sin(\pi k/9)}, & k \neq \text{multiple of } 9 \\ \frac{5}{9}, & k = \text{multiple of } 9 \end{cases}$$

# Discrete-time FT and FS

- Let $x(t)$ be periodic with period $2\pi$ ($\omega_0 = 1$)
- FS and DTFT

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jkt} \qquad X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n}$$

$$a_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} x(t) e^{-jkt} dt \qquad x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega}) e^{j\omega n} d\omega$$

- Really one and the same (time-frequency exchange)

# Example 5.17

1. Determine the Fourier transform of sequence $x[n] = \frac{\sin(\pi n/2)}{\pi n}$
2. From Example 3.5, we know that $g(t) = 1$, if $|t| \leq T_1$, and $g(t) = 0$, if $T_1 \leq |t| \leq \pi$. And $g(t)$ is a periodic square with period $2\pi$. Its Fourier coefficients are $a_k = \frac{\sin(kT_1)}{k\pi}$
3. If $T_1 = \frac{\pi}{2}$, then $a_k = x[k]$. Hence,

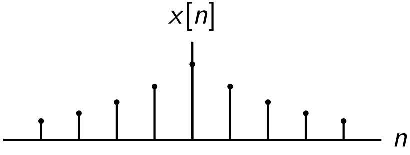
$$\frac{\sin(\pi k/2)}{\pi k} = \frac{1}{2\pi}\int_{-\pi}^{\pi} g(t)\mathrm{e}^{-jkt}\mathrm{d}t = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} \mathrm{e}^{-jkt}\mathrm{d}t$$

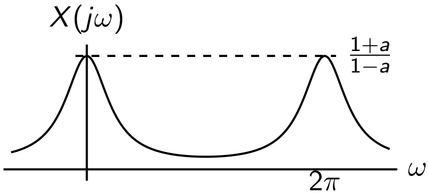
4. $$\frac{\sin(\pi n/2)}{\pi n} = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} \mathrm{e}^{jn\omega}\mathrm{d}\omega$$

$$\frac{\sin(\pi n/2)}{\pi n} = \frac{1}{2\pi}\int_{-\pi/2}^{\pi/2} \mathrm{e}^{jn\omega}\mathrm{d}\omega \to X(\mathrm{e}^{j\omega}) = \begin{cases} 1, & |\omega| \leq \frac{\pi}{2} \\ 0, & \frac{\pi}{2} < |\omega| \leq \pi \end{cases}$$

# Discrete-time FT and FS

Example (1). (Aperiodic)




$$x[n] = a^{|n|}, \quad 0 < a < 1$$

$$\begin{aligned} X(j\omega) &= \sum_n a^{|n|} e^{-j\omega n} \\ &= \sum_{n=0}^{\infty} a^n e^{-j\omega n} + \sum_{n=1}^{\infty} a^n e^{j\omega n} \\ &= \frac{1}{1 - ae^{-j\omega}} + \frac{ae^{j\omega}}{1 - ae^{j\omega}} \\ &= \frac{1 - a^2}{1 - a \cdot 2\cos\omega + a^2} \end{aligned}$$

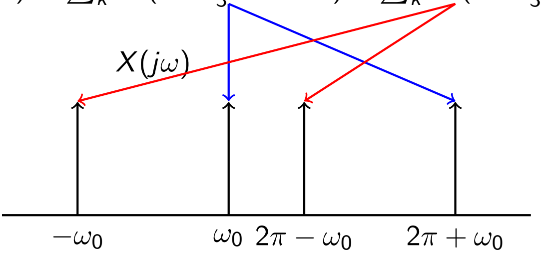
# Discrete-time FT and FS

Example (2). (Periodic)

$$x[n] = \cos(\omega_0 n) = \frac{1}{2}e^{j\omega_0 n} + \frac{1}{2}e^{-j\omega_0 n} \quad \text{with} \quad \omega_0 = \frac{2\pi}{3}$$

From table: $e^{j\omega_0 n} \longleftrightarrow \sum_{k=-\infty}^{\infty} 2\pi\delta(\omega - \omega_0 - 2\pi k)$

$$X(j\omega) = \sum_k \pi\delta(\omega - \tfrac{2}{3}\pi - 2\pi k) + \sum_k \pi\delta(\omega + \tfrac{2}{3}\pi - 2\pi k)$$



# Summaries of Fourier Series and Fourier Transform

| | Continuous time | | Discrete time | |
|---|---|---|---|---|
| | **Time domain** | **Frequency domain** | **Time domain** | **Frequency domain** |
| Fourier Series | $x(t) = \sum_{k=-\infty}^{+\infty} a_k e^{jk\omega_0 t}$<br>continuous time<br>periodic in time | $a_k = \frac{1}{T_0}\int_{T_0} x(t)e^{-jk\omega_0 t}$<br>discrete frequency<br>aperiodic in frequency | $x[n] = \sum_{k=\langle N\rangle} a_k e^{jk(2\pi/N)n}$<br>discrete time<br>periodic in time | $a_k = \frac{1}{N}\sum_{k=\langle N\rangle} x[n]e^{-jk(2\pi/N)n}$<br>discrete frequency<br>periodic in frequency |
| Fourier Transform | $x(t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega$<br>continuous time<br>aperiodic in time | $X(j\omega) = \int_{-\infty}^{+\infty} x(t)e^{-j\omega t}dt$<br>continuous frequency<br>aperiodic in frequency | $x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}$<br>discrete time<br>aperiodic in time | $X(e^{j\omega}) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j\omega n}$<br>continuous frequency<br>periodic in frequency |

duality

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duality

# Discrete Fourier Transform (DFT)

Let $x[n]$ be a finite-length sequence of length $N$.
Suppose $x[n] = 0$ for $n \notin [0 : N-1]$.

The DFT of $x[n]$, denoted as $X[k]$, is defined as

$$X[k] = \sum_{n=0}^{N-1} x[n] W_N^{kn} \qquad k = 0, 1, \ldots, N-1$$

$$\text{where} \quad W_N = e^{-j(2\pi/N)}$$

The inverse DFT (IDFT) of $X[k]$ is given by

$$x[n] = \frac{1}{N} \sum_{n=0}^{N-1} X[k] W_N^{-kn} \qquad n = 0, 1, \ldots, N-1$$

# Discrete Fourier Transform (DFT)

$$\begin{bmatrix} X[0] \\ X[1] \\ \vdots \\ X[N-1] \end{bmatrix} = \begin{bmatrix} W_N^{0\cdot 1} & W_N^{0\cdot 2} & \cdots & W_N^{0\cdot N} \\ W_N^{1\cdot 1} & W_N^{1\cdot 2} & \cdots & W_N^{1\cdot N} \\ \vdots & \vdots & \ddots & \vdots \\ W_N^{(N-1)\cdot 1} & W_N^{(N-1)\cdot 2} & \cdots & W_N^{(N-1)\cdot N} \end{bmatrix} \begin{bmatrix} x[0] \\ x[1] \\ \vdots \\ x[N-1] \end{bmatrix}$$

# Important Features of DFT

- one-to-one correspondence between $x[n]$ and $X[k]$.
- an extreme fast algorithm for its calculation, called Fast Fourier Transform (FFT)
- DFT is closely related to discrete Fourier series and Fourier Transform.
- DFT is an appropriate representation for digital computer realization as it is discrete and of finite length in both time and frequency domain.