Quiz1

1 Taylor Series expansion

(20pt) In mathematics, the Taylor series of a function is an infinite sum of terms expressed in terms of the function's derivatives at a single point. For most common functions, the function and the sum of its Taylor series are equal near this point. Please provide the Taylor expansion of sin(x) at $x = 0$:

Solution:

$$\sin(\mathrm{x}) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$$

2 Vector Norm

(12pt) Assuming a vector $\mathrm{x} = [\mathrm{x_1}, \mathrm{x_2}, \dots, \mathrm{x_m}]$, please provide the explanation of several commonly used norm calculations in mathematical notation or text:

1. Manhattan (L1) Norm: $\|x\|_1 = \sum_{i=1}^{n} |x_i|$

2. Euclidean (L2) Norm: $\|x\|_2 = (\sum_{i=1}^{n} |x_i|^2)^{\frac{1}{2}}$

3. Zero Norm: 向量 x 中非零元素的个数

4. Infinity (L∞) Norm: $\|x\|_\infty = \max_{1 \le i \le m} |x_i|$

(8pt) Consider a function $f(x) = \frac{1}{2}\|Ax - b\|_2^2$. Please provide the first derivative, $f'(x)$.

$f'(x) = A^T Ax - A^T b = A^T(Ax - b)$

3 Expectation

Consider two random variables X, Y. Assume that we have $P(X = x) = \frac{1}{2^x}$ for $x \in \mathbb{Z}_{\ge 1}$ (integers greater than or equal to 1) and $P(Y = y|X = x) = \frac{1}{n}$ for $y \in \{1, 2, ..., n\}$. Assume n is a fixed positive integer constant. What is $\mathbb{E}[Y]$?

Solution: X and Y are independent of each other, so $P(Y = y|X = x) = P(Y = y) = \frac{1}{n}$, then $\mathbb{E}[Y] = \frac{1}{n}\sum_{y=1}^{n} y = \frac{n+1}{2}$.

4 Matrix

(10pt) Please determine the rank of the following matrix:

$$\begin{pmatrix} 2 & 2 & 1 & 3 & 4 \\ -2 & 1 & 1 & 0 & 2 \\ 1 & 1 & 1 & 0 & 2 \\ 3 & 3 & 1.5 & 4.5 & 6 \end{pmatrix}$$

Solution: 3

(10pt) Matrix A is an $n \times n$ matrix and satisfies the following equation:

$$A^2 = A.$$

What are the possible values of $\det(A)$? Please provide a proof.

Solution: Recalling that $\det AB = (\det A)(\det B)$ for $n \times n$ matrices A and B, we see that $A^2 = A$ implies that $(\det A)^2 = (\det A)$ Thus $(\det A)$ can be equal to zero or one.

5 Gaussian distribution

(20pt) The Gaussian distribution, commonly known as the normal distribution, is the most prevalent probability distribution in statistics.

(1) Please provide the probability density function for a 1-dimensional continuous random variable x following the Gaussian distribution. Solution:

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)$$

(2) Please provide the derivative in (1) Solution:

$$f^{'}(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} \left[-\frac{(x-\mu)}{\sigma^2} \right]$$

(3) Following the Gaussian distribution, Please provide the probability density function for a p-dimensional continuous random variable x. Solution:

$$p(x \mid \mu, \Sigma) = \frac{1}{(2\pi)^{p/2}|\Sigma|^{1/2}} e^{-\frac{1}{2}(x-\mu)^T \Sigma^{-1}(x-\mu)}$$

其中 x, $\mu \in \mathbb{R}^p$, $\Sigma \in \mathbb{R}^{p\times p}$, Σ 为协方差矩阵

(4) Please compute the partial derivative of (3). Solution:

$$\frac{\partial p(\mathbf{x})}{\partial \mathbf{x}} = -p(\mathbf{x})\mathbf{\Sigma}^{-1}(\mathbf{x}-\mu)$$