EE160: Introduction to Control

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Lecture 5 The Stability of Feedback Linear System

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Introduction

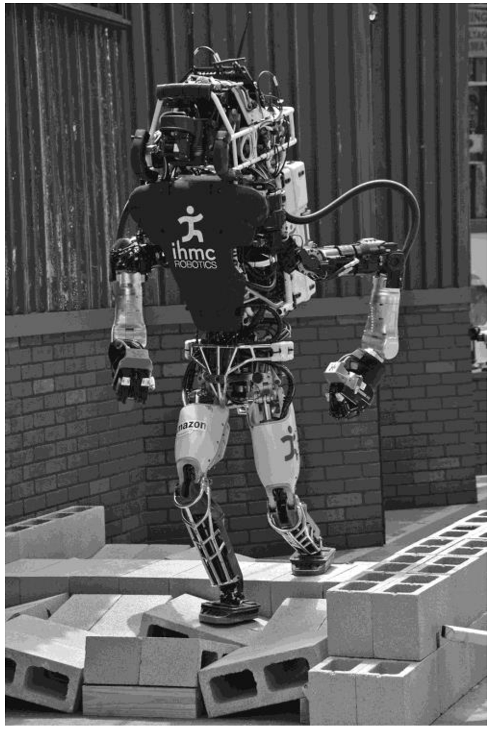
Stable is the pass score, the bottom line of controller design !

Introduction

Stability of closed-loop feedback systems is central and fundamental to control system design.

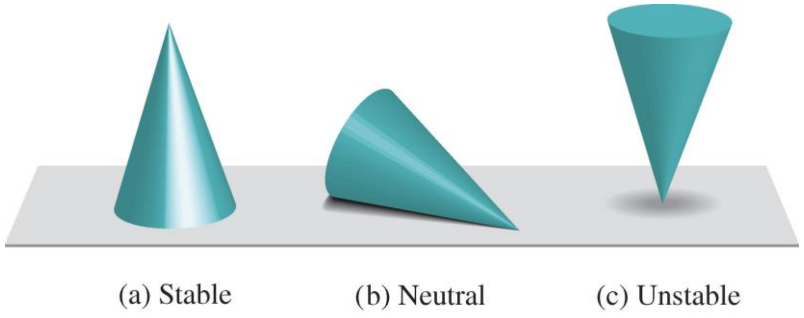
Motivation：

- From a practical point of view, a closed-loop feedback system that is unstable is of minimal value.
- Many physical systems are inherently open loop unstable, e.g. airplane

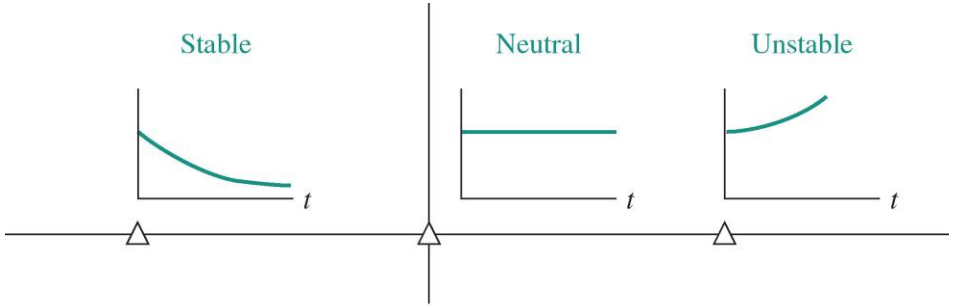


Introduction

The definition of stability



- The response to a displacement, or initial condition, will result in either a decreasing, neutral, or increasing response.



- The location in the s-plane of the poles of a system indicates the resulting transient response.

Different type of stability

- BIBO
- Absolute and relative stability
- Internal Stability， Lyapunov stable
- L-2/L-infinity stable， Input to state stability
- Asymptotical stable， Exponential stable
- Practical ISS， Almost Lyapunov stable…

Outcome

❑ Concept of BIBO stability .

❑ Absolute and relative stability.

❑ Understand the relationship of the s-plane pole locations to system stability.

❑ Routh–Hurwitz stability criterion to determine stability.

Stability Definition

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1. In terms of Input-Output relation

A stable system is defined as a system with a bounded (limited) system response. That is, if the system is subjected to a bounded input or disturbance and the response is bounded in magnitude, the system is said to be (BIBO) stable.

2. In terms of the location of the poles of the closed-loop transfer function

$$T(s) = \frac{p(s)}{q(s)} = \frac{K\prod_{i=1}^{M}(s + z_i)}{s^N \prod_{k=1}^{Q}(s + \sigma_k)\prod_{m=1}^{R}[s^2 + 2\alpha_m s + (\alpha_m^2 + \omega_m^2)]},$$

where $q(s) = \Delta(s) = 0$ is the characteristic equation whose roots are the poles of the closed-loop system.

The output response for a step function input

$$y(t) = \sum_{i=1}^{N} C_i t^N + \sum_{k=1}^{Q} A_k e^{-\sigma_k t} + \sum_{m=1}^{R} D_m e^{-\alpha_m t}\sin(\omega_m t + \theta_m)$$

a necessary and sufficient condition for a feedback system to be stable is that all the poles of the system transfer function are located at the left half of the complex plane.

Routh–Hurwitz stability criterion

a necessary and sufficient condition for a feedback system to be stable is that all the poles of the system transfer function are located at the left half of the complex plane.

Roots of the characteristic equation have negative real parts.

$$\Delta(s) = q(s) = a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0 = 0.$$

- Write the equation in factored form

$$a_n\,(s - r_1)(s - r_2)\cdots(s - r_n) = 0,$$

- Ask your PC's help:
  Matlab :pole, pzmap, roots
  Python: numpy.roots

- In the late 1800s, A. Hurwitz and E. J. Routh independently published a method of investigating the stability of a linear system **without solving the characteristic equation.**

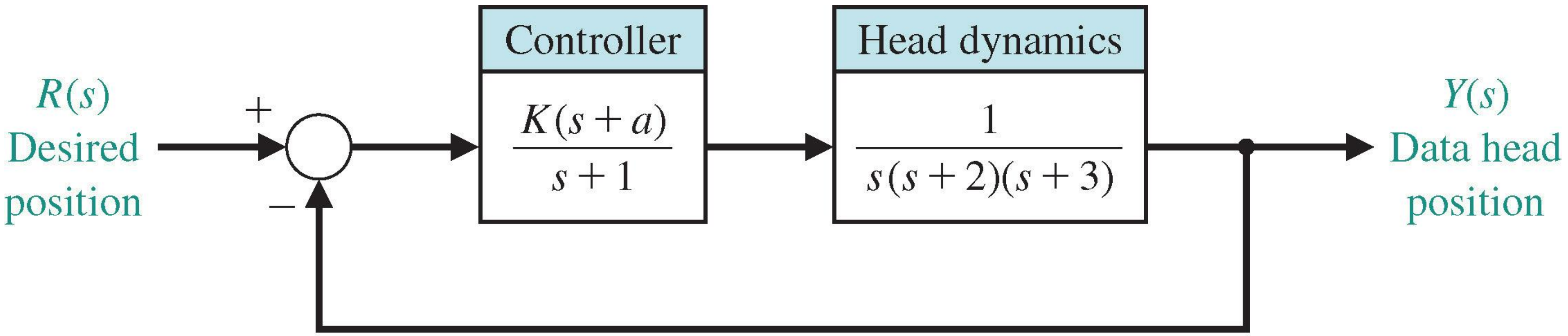
```
% This script computes the roots of the characteristic
% equation q(s) = s^3 + 2 s^2 + 4 s + K for 0<K<20
%
K=[0:0.5:20];
for i=1:length(K)
   q=[1 2 4 K(i)];
   p(:,i)=roots(q);
end
plot(real(p),imag(p),'x'), grid
xlabel('Real axis'), ylabel('Imaginary axis')
```

Loop for roots as a function of K

(b)

Routh–Hurwitz stability criterion

Example 5：Welding control



The characteristic equation is

$$1 + G(s) = 1 + \frac{K(s + a)}{s(s + 1)(s + 2)(s + 3)} = 0.$$

We desire to determine the range of K and a for which the system is stable.

Routh–Hurwitz stability criterion

The Routh–Hurwitz criterion is based on ordering the coefficients of the characteristic equation

$$a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \cdots + a_1 s + a_0 = 0$$

into an array as follows

$$\begin{array}{c|cccc}
s^n & a_n & a_{n-2} & a_{n-4} & \cdots \\
s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \cdots \\
\hline
s^{n-2} & b_{n-1} & b_{n-3} & b_{n-5} & \cdots \\
s^{n-3} & c_{n-1} & c_{n-3} & c_{n-5} & \cdots \\
\vdots & \vdots & \vdots & \vdots & \\
s^0 & h_{n-1} & & &
\end{array}$$

where

$$b_{n-1} = \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}} = \frac{-1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-2} \\ a_{n-1} & a_{n-3} \end{vmatrix},$$

$$b_{n-3} = -\frac{1}{a_{n-1}} \begin{vmatrix} a_n & a_{n-4} \\ a_{n-1} & a_{n-5} \end{vmatrix}, \ldots$$

$$c_{n-1} = \frac{-1}{b_{n-1}} \begin{vmatrix} a_{n-1} & a_{n-3} \\ b_{n-1} & b_{n-3} \end{vmatrix}, \quad \ldots$$

and so on.

Routh–Hurwitz stability criterion

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$$a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \cdots + a_1 s + a_0 = 0$$

into an array as follows

$$\begin{array}{l|llll} s^n & a_n & a_{n-2} & a_{n-4} & \cdots \\ s^{n-1} & a_{n-1} & a_{n-3} & a_{n-5} & \cdots \\ s^{n-2} & b_{n-1} & b_{n-3} & b_{n-5} & \cdots \\ s^{n-3} & c_{n-1} & c_{n-3} & c_{n-5} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \\ s^0 & h_{n-1} & & & \end{array}$$

Conclusions:

- The Routh–Hurwitz criterion states that the number of roots of characteristic equation with positive real parts is equal to the number of changes in sign of the first column of the Routh array.
- This criterion requires that there be no changes in sign in the first column for a stable system.
- This requirement is both necessary and sufficient.

Q: what about zero?

Ming-Tzu Ho, A. Datta and S. P. Bhattacharyya, "An elementary derivation of the Routh-Hurwitz criterion," in IEEE Transactions on Automatic Control, vol. 43, no. 3, pp. 405-409, March 1998, doi: 10.1109/9.661607.

Routh–Hurwitz stability criterion

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Example 1 : The characteristic polynomial of a third-order system is

$$q(s) = a_3 s^3 + a_2 s^2 + a_1 s + a_0.$$

The Routh array is

$$\begin{array}{c|cc} s^3 & a_3 & a_1 \\ s^2 & a_2 & a_0 \\ s^1 & b_1 & 0 \\ s^0 & c_1 & 0 \end{array},$$

where $b_1 = \dfrac{a_2 a_1 - a_0 a_3}{a_2}$ and $c_1 = \dfrac{b_1 a_0}{b_1} = a_0.$

Without loss of generality, we assume $a_3 > 0$.
For the third-order system to be stable, it is necessary and sufficient that the coefficients be positive and

$$a_2 a_1 > a_0 a_3.$$

$$q(s) = (s - 1 + j\sqrt{7})(s - 1 - j\sqrt{7})(s + 3)$$
$$= s^3 + s^2 + 2s + 24.$$

$$\begin{array}{c|cc} s^3 & 1 & 2 \\ s^2 & 1 & 24 \\ s^1 & -22 & 0 \\ s^0 & 24 & 0 \end{array}.$$

Two changes in sign appear in the first column,
Means that two roots lie in the right-half plane

Q: what about $a_2 a_1 = a_0 a_3$?

Routh–Hurwitz stability criterion

Example 2：Consider the following characteristic polynomial:

$$q(s) = s^5 + 2s^4 + 2s^3 + 4s^2 + 11s + 10.$$

The Routh array is then

$$\begin{array}{c|ccc} s^5 & 1 & 2 & 11 \\ s^4 & 2 & 4 & 10 \\ s^3 & 0 & & \\ s^2 & & & \\ s^1 & & & \\ s^0 & & & \end{array}$$

Special Case I: If only one element in the array is zero, it may be replaced with a small positive number, ϵ, that is allowed to approach zero after completing the array.

where

$$c_1 = \frac{4\epsilon - 12}{\epsilon} \quad \text{and} \quad d_1 = \frac{6c_1 - 10\epsilon}{c_1}.$$

When $0 < \epsilon \ll 1$, we find that $c_1 < 0$ and $d_1 > 0$.

Therefore, there are two sign changes in the first column; hence the system is unstable with two roots in the right half-plane.

Routh–Hurwitz stability criterion

Example 3： Marginally Stable

$$q(s) = s^3 + 2s^2 + 4s + K,$$

where K is an adjustable loop gain.

$$\begin{array}{c|cc} s^3 & 1 & 4 \\ s^2 & 2 & K \\ s^1 & \dfrac{8-K}{2} & 0 \\ s^0 & K & 0 \end{array}$$

Note that we obtain a row of zeros

For a stable system, we require that

$$0 < K < 8.$$

When K=8 we have two roots on the jw-axis and a marginal stability case.

Special case II occurs when the polynomial contains singularities that are symmetrically located about the origin of the s-plane.

Special Case II. There is a zero in the first column, and the other elements of the row containing the zero are also zero.

Routh–Hurwitz stability criterion

Example 3： Marginally Stable

$$q(s) = s^3 + 2s^2 + 4s + K,$$

where K is an adjustable loop gain.

$$\begin{array}{c|cc} s^3 & 1 & 4 \\ s^2 & 2 & K \\ s^1 & \dfrac{8-K}{2} & 0 \\ s^0 & K & 0 \end{array}$$

The order of the auxiliary polynomial is always even and indicates the number of symmetrical root pairs

When K=8, we need to introduce the auxiliary polynomial, U(s)

$$U(s) = 2s^2 + Ks^0 = 2s^2 + 8 = 2(s^2 + 4) = 2(s + j2)(s - j2).$$

which is the equation of the row preceding the row of zeros.

Since the jw-axis roots of the characteristic equation are simple, the system is neither stable nor unstable; it is instead called marginally stable, since it has an undamped sinusoidal mode.

Special Case II. There is a zero in the first column, and the other elements of the row containing the zero are also zero.

Routh–Hurwitz stability criterion

Example 4： Repeated roots jw-axis.

$$q(s) = (s+1)(s+j)(s-j)(s+j)(s-j) = s^5 + s^4 + 2s^3 + 2s^2 + s + 1.$$

The Routh array is

$$\begin{array}{c|ccc} s^5 & 1 & 2 & 1 \\ s^4 & 1 & 2 & 1 \\ s^3 & \epsilon & \epsilon & 0 \\ s^2 & 1 & 1 & \\ s^1 & \epsilon & 0 & \\ s^0 & 1 & & \end{array}.$$

When $0 < \epsilon \ll 1$, we note the absence of sign changes in the first column.

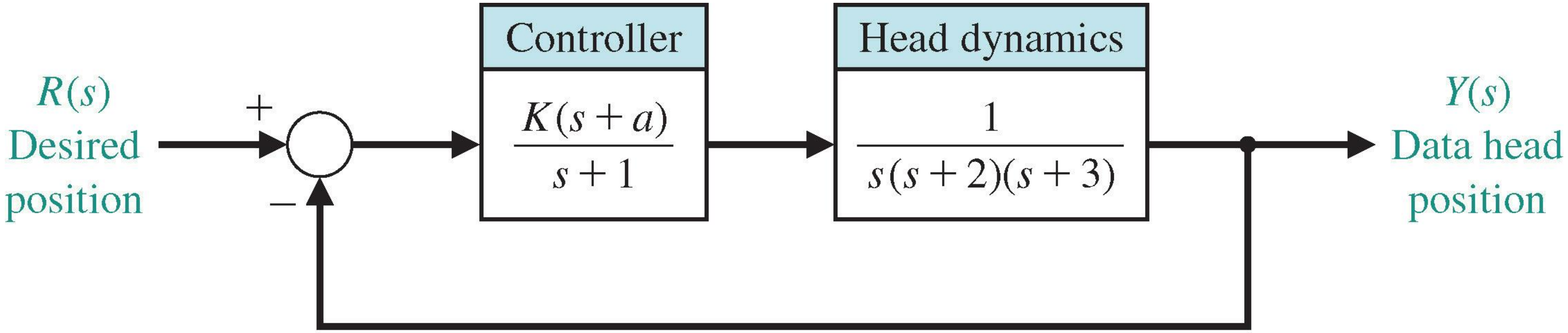
However, as $\epsilon \to 0$, we obtain a row of zero at s^3 and s^1 , while the auxiliary polynomial are

$$s^4 + 2s^2 + 1 = (s^2 + 1)^2,$$

indicating the repeated roots on the jw-axis. Hence, the system is unstable.

Routh–Hurwitz stability criterion

Example 5： Welding control



To determine the range of K and a for which the system is stable.
The characteristic equation is rewritten as

$$q(s) = s^4 + 6s^3 + 11s^2 + (K + 6)s + Ka = 0.$$

The Routh array

$$\begin{array}{c|ccc} s^4 & 1 & 11 & Ka \\ s^3 & 6 & K+6 & \\ s^2 & b_3 & Ka & \\ s^1 & c_3 & & \\ s^0 & Ka & & \end{array},$$

Routh–Hurwitz stability criterion

Example 5：Welding control

The Routh array

$$
\begin{array}{c|ccc}
s^4 & 1 & 11 & Ka \\
s^3 & 6 & K+6 & \\
s^2 & b_3 & Ka & \\
s^1 & c_3 & & \\
s^0 & Ka & &
\end{array},
$$

where

$$b_3 = \frac{60 - K}{6} \quad \text{and} \quad c_3 = \frac{b_3(K+6) - 6Ka}{b_3}.$$

a) $b_3 \geq 0$ requires that K be less than 60

b) $c_3 \geq 0$ requires

$$(K - 60)(K + 6) + 36Ka \leq 0.$$

$$\Downarrow$$

$$a \leq \frac{(60 - K)(K + 6)}{36K}$$

Routh–Hurwitz stability criterion

Suppose we write the characteristic equation of an nth-order system as

$$s^n + a_{n-1}s^{n-1} + a_{n-2}s^{n-2} + \cdots + a_1 s + \omega_n{}^n = 0.$$

We divide through by $\omega_n{}^n$ and use $\overset{*}{s} = s/\omega_n$ to obtain the normalized form of the characteristic equation:

$$\overset{*}{s}{}^n + b\overset{*}{s}{}^{n-1} + c\overset{*}{s}{}^{n-2} + \cdots + 1 = 0.$$

We summarize the stability criterion for up to a sixth-order characteristic equation, as follows

Table 6.1 The Routh–Hurwitz Stability Criterion

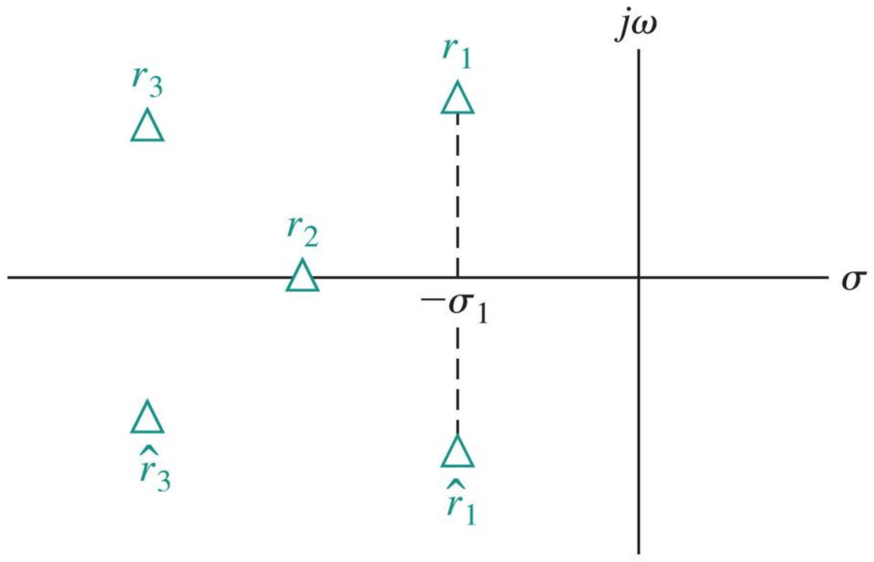
n	Characteristic Equation	Criterion
2	$s^2 + bs + 1 = 0$	$b > 0$
3	$s^3 + bs^2 + cs + 1 = 0$	$bc - 1 > 0$
4	$s^4 + bs^3 + cs^2 + ds + 1 = 0$	$bcd - d^2 - b^2 > 0$
5	$s^5 + bs^4 + cs^3 + ds^2 + es + 1 = 0$	$bcd + b - d^2 - b^2e > 0$
6	$s^6 + bs^5 + cs^4 + ds^3 + es^2 + fs + 1 = 0$	$(bcd + bf - d^2 - b^2e)e + b^2c - bd - bc^2f - f^2 + bfe + cdf > 0$

Note: The equations are normalized by $(\omega_n)^n$.

Relative Stability

The verification of stability using the Routh–Hurwitz criterion provides only a partial answer to the question of stability, but how stable?

The relative stability of a system can be defined as the property that is measured by the relative real part of each root or pair of roots.



utilizing a change of variable, which shifts the s-plane axis in order to utilize the Routh–Hurwitz criterion to determine the relative stability.

Example: Axis shift

$$q(s) = s^3 + 4s^2 + 6s + 4.$$

Setting the shifted variable s_n equal to $s + 1$, we obtain

$$(s_n - 1)^3 + 4(s_n - 1)^2 + 6(s_n - 1) + 4 = s_n^{\ 3} + s_n^{\ 2} + s_n + 1.$$

Then the Routh array is established as

$$\begin{array}{c|cc} s_n^{\ 3} & 1 & 1 \\ s_n^{\ 2} & 1 & 1 \\ s_n^{\ 1} & 0 & 0 \\ s_n^{\ 0} & 1 & 0 \end{array}$$

All zero row

There are roots on the shifted imaginary axis that can be obtained from the auxiliary polynomial

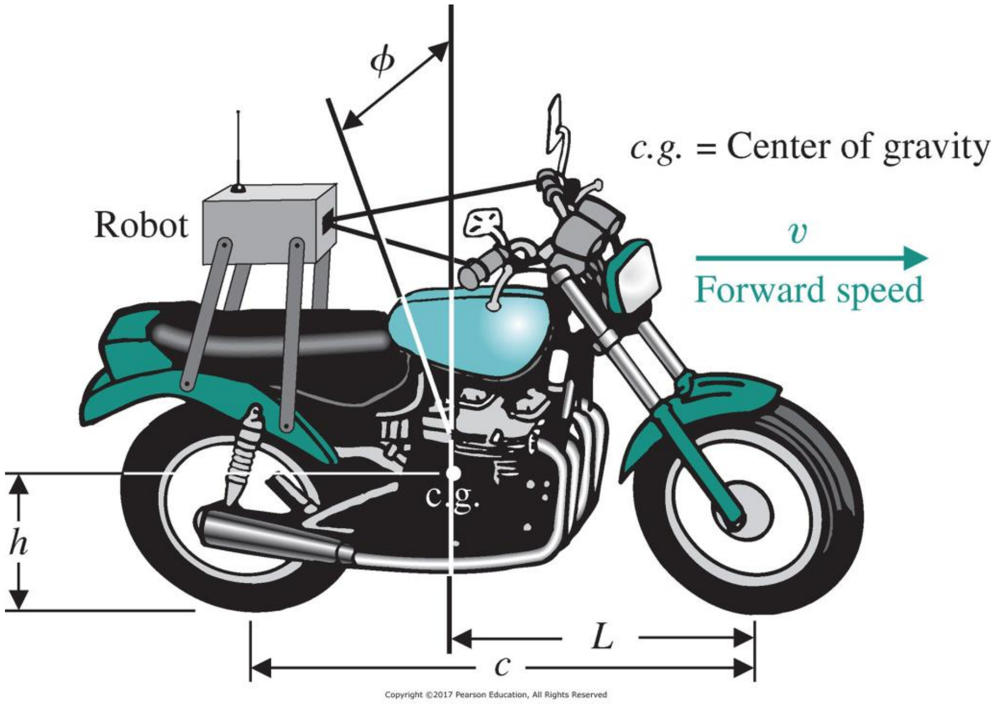
$$U(s_n) = s_n^{\ 2} + 1 = (s_n + j)(s_n - j) = (s + 1 + j)(s + 1 - j).$$

Design Example

Robot-controlled motorcycle

Scenario

The motorcycle move in a straight line at constant forward speed



Control Goal

Control the motorcycle in the vertical position, and maintain the prescribed position in the presence of disturbances

Variable to Be Controlled:

$\phi(t)$, the angle between the plane of symmetry of the motorcycle and the vertical

Design Specification

DS1 The closed-loop system must be stable

DS2 $\phi(t) = \phi_d(s) = 0$

Since our focus here is on stability rather than transient response characteristics, the control specifications will be related to stability only; transient performance is an issue that we need to address once we have investigated all the stability issues.

Design Example

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Modelling

The motorcycle model is given by

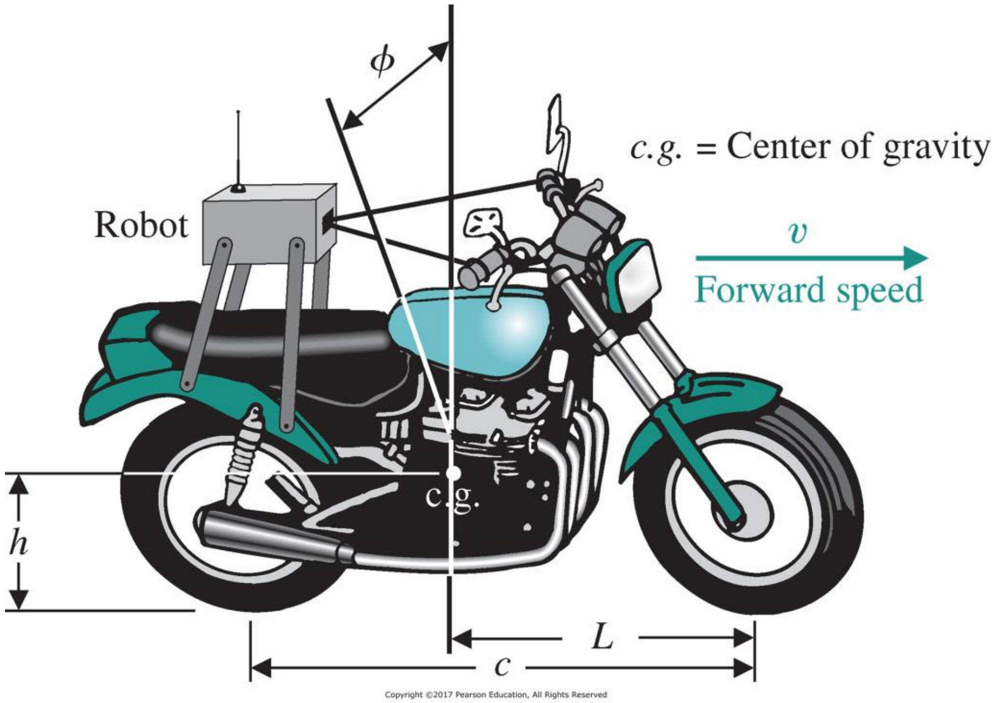
$$G(s) = \frac{1}{s^2 - \alpha_1},$$

The controller is given by

$$G_c(s) = \frac{\alpha_2 + \alpha_3 s}{\tau s + 1},$$

where

$$\alpha_1 = g/h, \qquad \alpha_2 = v^2/(hc) \qquad \alpha_3 = vL/(hc).$$



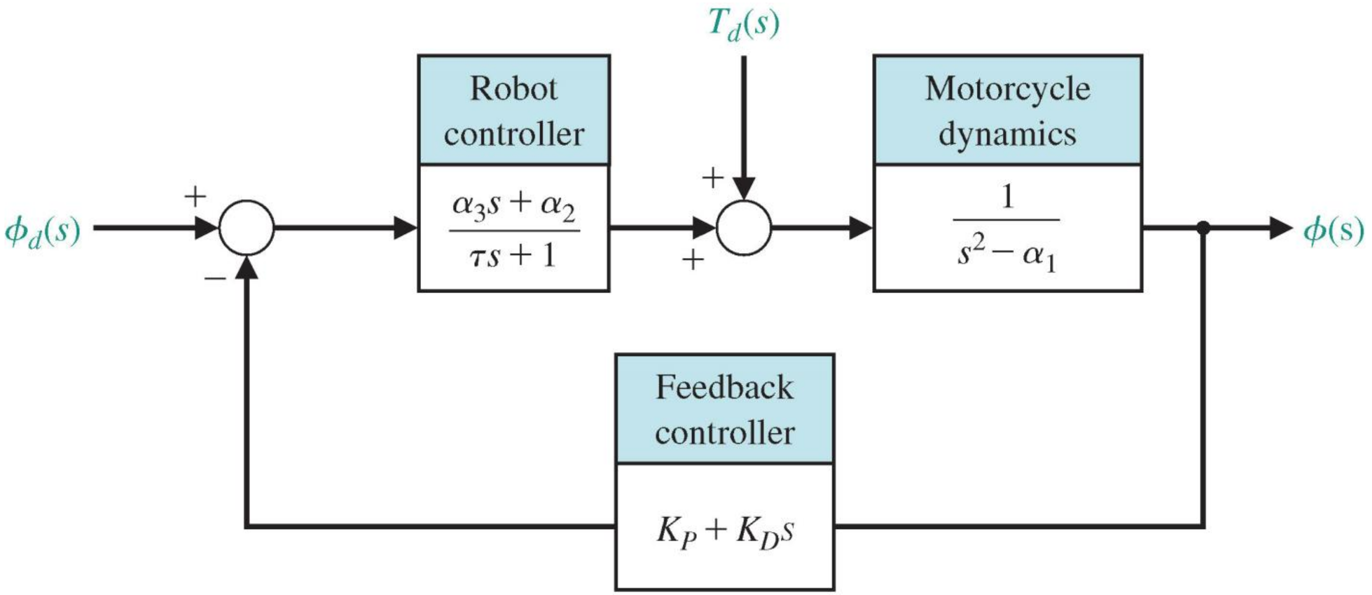
and time- constant of the controller is denoted by τ.

Approximations &Simplifications:
Control is accomplished by turning the handlebar.
The front wheel rotation about the vertical is not evident in the transfer functions.
The transfer functions assume a constant forward speed.

Reference :L. Hatvani, “Adaptive Control: Stabilization,” Applied Control, edited by Spyros G. Tzafestas, Marcel Decker, New York, 1993, pp. 273–287.

Design Example

Open-loop system is not a BIBO stable one, hence we add a feedback as:



We want to use the Routh–Hurwitz technique to analyze the closed-loop system stability.

Table 6.2 Physical Parameters

τ	0.2 s
α_1	$9.1/s^2$
α_2	$2.7\ 1/s^2$
α_3	1.35 1/s

Given above physical parameter, what values of Kp and KD lead to closed-loop stability?

Design Example

The closed-loop transfer function from $\phi(t)$ to $\phi_d(s)$

$$T(s) = \frac{\alpha_2 + \alpha_3 s}{\Delta(s)},$$

$$\Delta(s) = \tau s^3 + (1 + K_D\alpha_3)s^2 + (K_D\alpha_2 + K_P\alpha_3 - \tau\alpha_1)s + K_P\alpha_2 - \alpha_1.$$

set up the following Routh array

$$\begin{array}{c|cc} s^3 & \tau & K_D\alpha_2 + K_P\alpha_3 - \tau\alpha_1 \\ s^2 & 1 + K_D\alpha_3 & K_P\alpha_2 - \alpha_1 \\ s & a & \\ 1 & K_P\alpha_2 - \alpha_1 & \end{array}$$

$$a = \frac{(1 + K_D\alpha_3)(K_D\alpha_2 + K_P\alpha_3 - \tau\alpha_1) - \tau(\alpha_2 K_P - \alpha_1)}{1 + K_D\alpha_3}.$$

By inspecting column 1, we determine that for stability we require

$$K_D > -1/\alpha_3,\ K_P > \alpha_1/\alpha_2, \text{ and } a > 0.$$

$$K_D > 0 \text{ and } K_P > 3.33.$$

Design Example

For example, selecting

$$K_P = 10 \text{ and } K_D = 5$$

yields a stable closed-loop system. The closed-loop poles are

$$s_1 = -35.2477,\ s_2 = -2.4674, \text{ and } s_3 = -1.0348.$$

Can you estimate the transient performance, e.g. Tp, Ts, P.O.?

For this robot-controlled motorcycle,

1. we do not expect to have to respond to nonzero command inputs, hence $\phi_d(t) = 0$ and

$$E(s) = T(s)[0] \qquad \lim_{t\to\infty} e(t) = s\,E(s) = 0$$

2. we certainly want to remain upright in the presence of external disturbances

Open-Loop

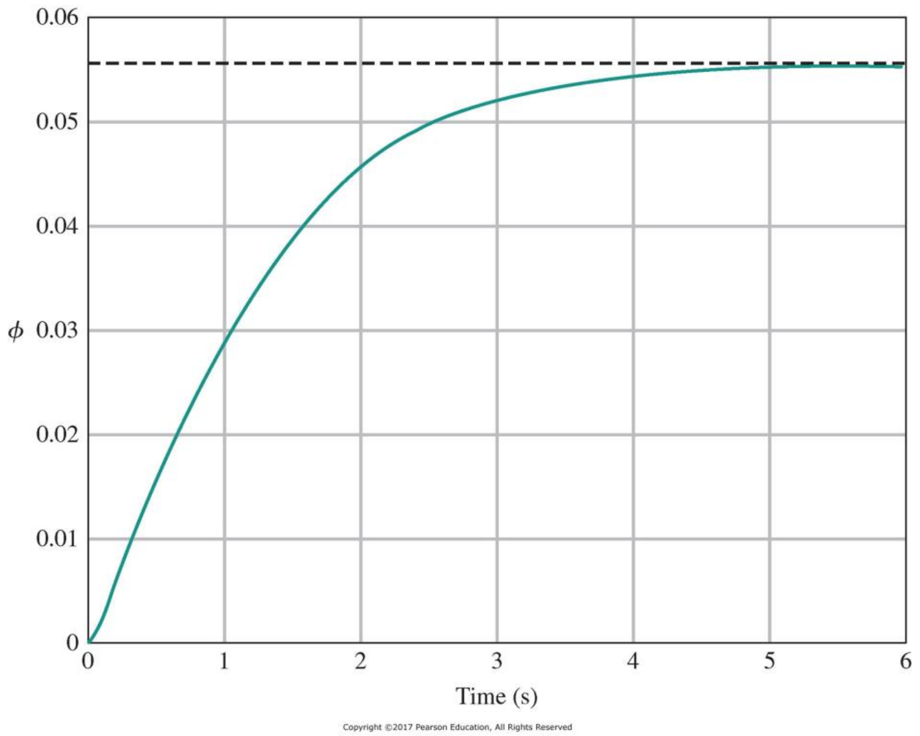
$$\phi(s) = \frac{1}{s^2 - \alpha_1} T_d(s).$$

Thus we see that the motorcycle is unstable.

Design Example

With the feedback and robot controller in the loop, the closed-loop transfer function from the disturbance to the output is

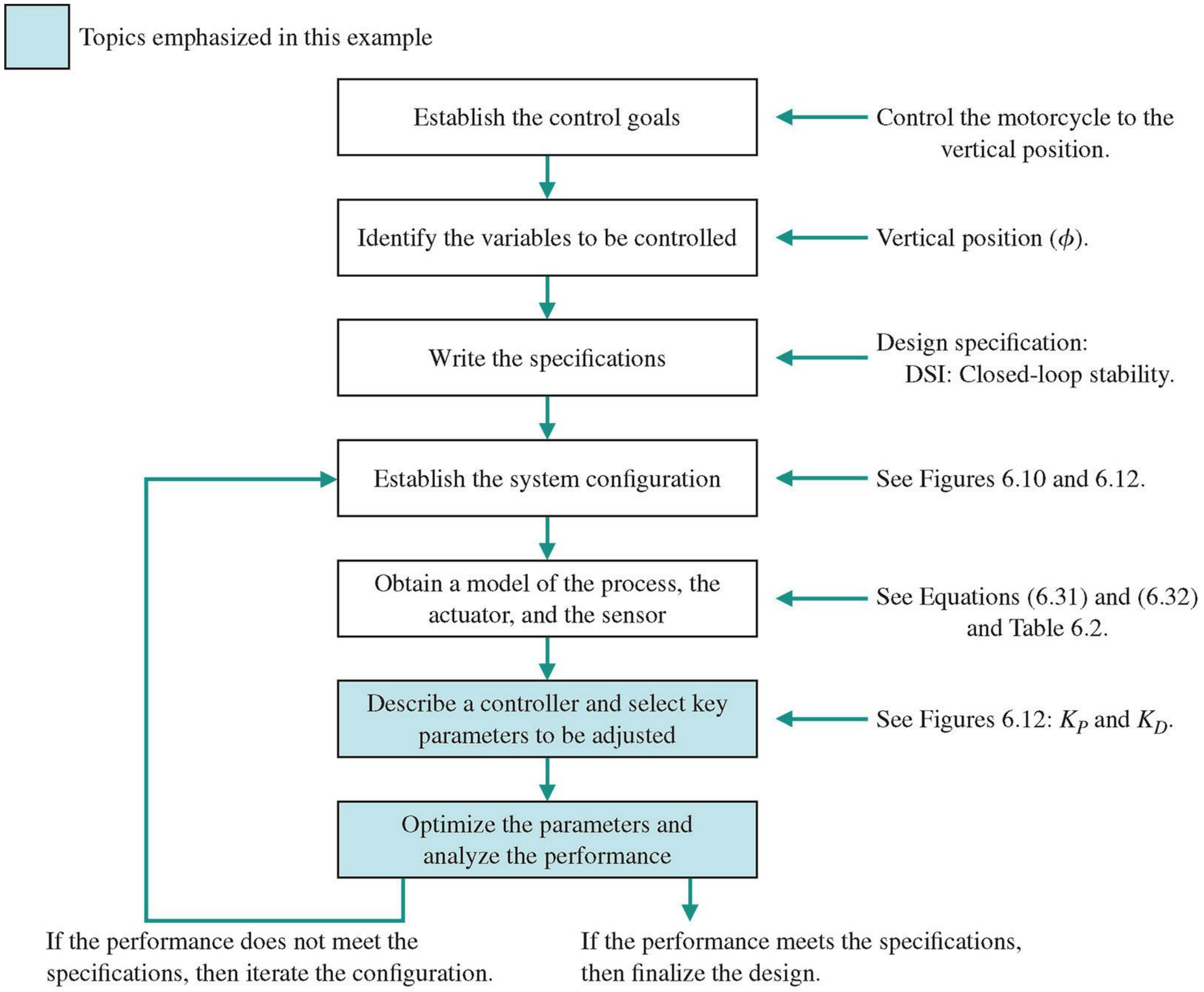
$$\frac{\phi(s)}{T_d(s)} = \frac{\tau s + 1}{\tau s^3 + (1 + K_D\alpha_3)s^2 + (K_D\alpha_2 + K_P\alpha_3 - \tau\alpha_1)s + K_P\alpha_2 - \alpha_1}.$$



Is it possible to change the feedback loop to achieve zero steady state error?

The response to a step disturbance is shown above ; the response is stable. However, the control system manages to keep the motorcycle upright, although it is tilted at about ϕ = 0.055 rad = 3.18 deg.

Summary



Key words List:

Poles and Zeros
Characteristic equation
Stable
BIBO
Marginally Stable
Relative Stable
RH criterion
Auxiliary polynomial
Axis shift

Homework

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CP6.4 Consider the closed-loop transfer function

$$T(s) = \frac{1}{s^5 + 2s^4 + 2s^3 + 4s^2 + s + 2}.$$

(a) Using the Routh–Hurwitz method, determine whether the system is stable. If it is not stable, how many poles are in the right half-plane? (b) Compute the poles of $T(s)$ and verify the result in part (a). (c) Plot the unit step response, and discuss the results.

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Thanks!

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