: Succesive Derivative :

$$y = \sin x$$
$$\frac{dy}{dx} = \cos x$$
$$y_2 = -\sin x$$
$$y_3 = -\cos x$$
$$y_4 = \sin x$$

This is called succsive Derivative

☑ D operator:

Let, $D = \frac{d}{dx}$, $D^2 = \frac{d^2}{dx^2}$

$\frac{dy}{dx} = Dy$ $\qquad \frac{d^2y}{dx^2} = D^2y$

$y_n = D^ny$, where $D^n \equiv \frac{d^n}{dx^n}$

properties : (i) $D^m(\phi(x)) = \frac{d^m}{dx^m}(\phi(x)) = \phi^m(x)$

(ii) Suppose $\phi(x) = \phi_1(x) + \phi_2(x) + \dots + \phi_n(x)$

then,

$$D^m(\phi(x)) = \frac{d^m}{dx^m}(\phi_1(x)) + \frac{d^m}{dx^m}(\phi_2(x)) + \dots + \frac{d^m}{dx^m}(\phi_n(x))$$

(iii) $D^m D^n(\phi(x)) = D^{m+n}(\phi(x))$

Problem, find y_n, where $y = x^n$

$\therefore y_1 = n \cdot x^{n-1}$

$y_2 = n(n-1) \cdot x^{(n-2)}$

$y_3 = n \cdot (n-1)(n-2) x^{(n-3)}$

$y_n = n!$

Problem, $y = \frac{x}{x+1}$ show that $y_5(0) = 5!$

Given $y = \frac{x}{x+1}$ [u by v method]

$$y_1 = \frac{(x+1) \cdot 1 - x \cdot (1)}{(x+1)^2} \Rightarrow y_1 = \frac{1}{(x+1)^2}$$

$\cancel{y_2 = -2(}$

$y_1 = (x+1)^{-2}$

$y_2 = -2 \times (x+1)^{-3} \times 1$

$y_3 = -2 \times -3 \times (x+1)^{-4}$

$y_n = (-1)^{n+1} n! \, (x+1)^{-(n+1)}$

$y_5(0) = (-1)^{5+1} \times 5! \times (0+1)^{-(5+1)}$

$= 5!$ [proved]

Problem if $y = 2\cos x(\sin x - \cos x)$, S.T. $(y_{10})_0 = 2^{10}$

Given,

$y = 2\cos x(\sin x - \cos x)$

$y_1 = 2[\cos x(\sin x + \cos x) \mp (\sin x - \cos x)\sin x]$

$y = 2\sin x\cos x - 2\cos^2 x$

$y = \sin 2x - (1 + \cos 2x)$

$y_1 = 2(\cos 2x - \sin 2x)$

$y_2 = -2 \times 2(+\sin 2x + \cos 2x)$

$y_3 = 2 \times 2 \times 2(\cos 2x - \sin 2x)$

$y_n = 2^n \times (-1)^{n+1} \times (\cos 2x + (-1)^n \sin 2x)$

$(y_{10})_0 = 2^{10} \times 1 \times (1 + 0)$

$= 2^{10}$ ~~(Ans)~~ ~~(R.H.S)~~

$=$ R.H.S [P. ~~b~~]

Problem

$y = x^{(n-1)}\log x$, P.T. $y_n = \frac{(n-1)!}{x}$

$y_1 = (n-1) \cdot x^{(n-2)} \cdot \ln x + \frac{1}{x} \times x^{(n-1)}$

$y_1 = x^{(n-2)} \left[(n-1)\ln x + 1 \right] =$

$y_2 = (n-1)(n-2) \cdot x^{(n-3)} \cdot \ln x + (n-1) x^{(n-2)} \times \frac{1}{x} + (n-2)\, x^{(n-3)}$

$y_2 = (n-1)(n-2)\, x^{(n-3)} \cdot \ln x + (n-1)\, x^{(n-3)} + (n-2)\, x^{(n-3)}$

$y_2 = x^{(n-3)} \left[(n-1)(n-2) \ln x + (n-1) + (n-2) \right]$

$= x^{(n-3)} \left[(n-1)(n-2) \ln x + 2n - 3 \right]$

$y_2 = x^{(n-3)} \cdot (n-2) \cdot \left[(n-1)\ln x + 1 \right] + \left[(n-1) \times \frac{1}{x} \cdot x^{(n-2)} \right]$

$= \frac{1}{x} \left[x^{(n-2)} \left[(n-2)(n-1) \cdot \ln x + n - 2 + n - 1 \right] \right]$

$y_3 = (n-3)\, x^{(n-4)} \left[(n-1)(n-2) \ln x + 2n - 3 \right] + x \times (n-1)(n-2) \times \frac{1}{x} + 2$

$= x^{(n-4)} \left[(n-3)(n-2)(n-1) \ln x + 2n^2 - 3n - 6n - 9 + (n-1)(n-2) + \right.$

Given, $y = x^{(n-1)} \ln x$

~~$y = \frac{x^n}{x} \ln x$~~

~~$y_1 =$~~ Now, $y_n = D^n \left(x^{(n-1)} \ln x \right)$ where ~~mead~~ $D \equiv \frac{d}{dx}$

$y_n = D^{(n-1)} \cdot D \left(x^{n-1} \ln x \right)$

$y_n = D^{n-1} \cdot \left[x^{n-2} + (n-1)\, x^{(n-2)} \ln x \right]$

Basic Rule

$\frac{d^2}{dx^2} (x^2) = 2$

$\frac{d^3}{dx^3} (x^2) = 0$. ie $D^3(x^2) = 0$

$3 > 2$

~~let, $\phi(x) = \left[x^{n-2} + (n-1)x^{(n-2)} \cdot \ln x\right]$~~

$$y_n = D^{n-1}\left[x^{n-2} + (n-1)x^{(n-2)} \cdot \ln x\right]$$

$$y_n = (n-1)D^{n-1}\left(x^{(n-2)} \cdot \ln x\right)$$

$$= (n-1)\, D^{(n-2)} \cdot D\left(x^{(n-2)} \cdot \ln x\right)$$

$$= (n-1)\, D^{(n-2)} \cdot \left[(n-2)\, x^{n-3} \ln x + x^{(n-3)}\right]$$

$$= (n-1)\, D^{n-2}\left[x^{(n-3)} + (n-2)\, x^{(n-3)} \ln x\right]$$

$$= (n-1)(n-2) \cdot D^{n-2}\left(x^{(n-3)} \ln x\right)$$

$$= (n-1)(n-2)(n-3) \ldots D^{(n-3)} \ldots$$

$$= (n-1)(n-2)(n-3) \ldots 3 \cdot 2 \cdot 1 \times D^{1}\left(x^{\cancel{n-n}\,0} \times \ln x\right)$$

$$= \frac{(n-1)!}{x}$$

$$= R.H.S \quad [\text{P-2}]$$

Leibnitz's theorem :

$$\frac{d}{dx}(u \cdot v) = \frac{dv}{dx} \cdot u + \frac{du}{dx} \cdot v$$

$$\therefore (u \cdot v)_1 = u \cdot v_1 + v \cdot u_1$$

$$\Rightarrow (uv)_2 = \frac{d}{dx}(uv_1 + u_1 v)$$

$$= u \cdot v_2 + v_1 u_1 + v u_2 + u_1 v_1$$

$$= uv_2 + 2u_1 v_1 + v u_2$$

$$= uv_2 + {}^2C_1 u_1 v_1 + u_2 v$$

$$(uv)_3 = uv_3 + {}^3C_1 u_1 v_2 + {}^3C_2 u_2 v_1 + v u_3$$

$$(uv)_n = u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 u_{(n-2)} \cdot v_2 + \ldots + u \cdot v_n$$

Statement of Leibniz's theorem:

u and v be are two functions of x such as they are derivable upto n times, then their product is also derivable n times and $(uv)_n = u_n v + {}^nC_1 u_{n-1} v_1 + {}^nC_2 \cdot u_{(n-2)} \cdot v_2 + \dots + uv_n$, where the subscripts denotes the order of differensiation.

Problem

If $y = \sin(m \sin^{-1}x)$ show that:

(i) $(1-x^2)y_2 - xy_1 + m^2 y = 0$

(ii) $(1-x^2)\, y_{n+2} - (2n+1)xy_{n+1} + (m^2-n^2)y_n = 0$

Hence, P.T. $y_n(0) = 0$ if n is even.

Given, $y = \sin(m \sin^{-1}x)$ — (1)

Diff. (1) w.r.t x we get,

$$y_1 = \frac{dy}{dx} = m \cos(m \sin^{-1}x) \cdot \frac{1}{\sqrt{1-x^2}} \quad \text{— (2)}$$

$$\Rightarrow y_1\sqrt{1-x^2} = m\cos(m\sin^{-1}x)$$

Diff. WRT 'x' we get...

$$\Rightarrow y_2\sqrt{1-x^2} + y_1 \cdot \frac{1}{2} \times (1-x^2)^{\frac{1}{2}-1} \cdot (-2x) = -m^2 \sin(m\sin^{-1}x) \cdot \frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow y_2\sqrt{1-x^2} - \frac{x}{\sqrt{1-x^2}} y_1 = -m^2 y \frac{1}{\sqrt{1-x^2}}$$

$$\Rightarrow y_2(1-x^2) - xy_1 + m^2 y = 0 \quad \text{[(i) Proved]}$$

— (iii)

$\Rightarrow$

Now, Differentiating (iii) n times w.r.t x by leibnitz's rule,

$$y_{n+2}(1-x^2) + {}^nC_1 \cdot y_{n+1} \cdot (-2x) + {}^nC_2\, y_n(-2)$$ farther terms will be 0.

$$x y_{n+1} + {}^nC_1\, y_n \cancel{x} 1 \cancel{+}$$

So, from, (iv)

$$y_{n+2}(1-x^2) + {}^nC_1 \cdot y_{n+1}(-2x) + {}^nC_2\, y_n(-2) + x y_{n+1} + {}^nC_1 y_n + m^2 y_n = 0$$

$$\Rightarrow (1-x^2) y_{n+2} - 2x n y_{n+1} - \cancel{2}\, \frac{n(n-1)}{\cancel{2}} y_n - x y_{n+1} - \cancel{n y_n} + m^2 y$$

; ${}^nC_1 = n$

; ${}^nC_2 = \frac{n(n-1)}{2}$

$$\Rightarrow (1-x^2) y_{n+2} - (2n+1) x y_{n+1} + (m^2-n^2) y_n = 0 \quad [\text{(ii) proved}] \quad \text{--- (5)}$$

at $x=0$, $\Rightarrow$ $y_{n+2}(0) + (m^2-n^2) y_n(0) = 0$

$y(0) = 0$

$y_1(0) = m$

~~$y_2(0) =$~~ $y_2(0)(1-0^2) - 0 \times y_1(0) + m^2 \cdot y(0) = 0$

$y_2(0) = 0$

in (5) puting $n = \cancel{2}\, 1$

$$(1-x^2) y_3 - (2+1) \cdot x y_2 + (m^2-n^2) y_1 = 0$$

puting $x = 0$,

$$1 \cdot y_3(0) - 3 \cdot 0 \cdot y_2(0) + (m^2-n^2) \cdot m = 0$$

$$y_3(0) = m(\cancel{m}1 - m^2)$$

Puting $n=2$ in (5) and setting $x=0$;

$$(1-0^2) \cdot y_4(0) - (4+1) \cdot \cancel{0 \cdot y_3(0)}^{\,0} + \cancel{(m^2-4)\, y_2(0)}^{\,0} = 0$$

$$\Rightarrow \quad y_4(0) = 0$$

Thus $y_n(0) = 0$ when we concider n is even.

But the Interesting Part is finding the pattern in odd n's.

Let's put $n=3$ to find $y_5(0)$, setting $x=0$,

$$1 \cdot y_5(0) - 0 + (m^2-9) \cdot y_3(0) = 0$$

$$\Rightarrow \quad y_5(0) = m \cdot (\cancel{m}\ 1^2 - m^2) \cdot (3^2 - m^2)$$

So, $\quad y_7(0) = m \cdot (1^2 - m^2) \cdot (3^2 - m^2) \cdot (5^2 - m^2)$

~~So,~~ $\cancel{y_{n+2}(0) = m \prod_{i=1}}$

thanks to GPT,

$$\boxed{y_n(0) = m \prod_{k=1}^{\frac{n-1}{2}} \left((2k-1)^2 - m^2\right)}$$

problem if $f(x) = \tan x$ then by using lebnitz's rule Prove That,

$$f^n(0) - {}^nC_2 f^{n-2}(0) + {}^nC_4 f^{n-4}(0) - \cdots = \sin\left(\frac{n\pi}{2}\right)$$

given, $f(x) = \tan x$

$= \sin x \cdot \frac{1}{\cos x}$

$\Rightarrow \cos x\, f(x) = \sin x$

Now, Diff. n times with respect to x by lebnitz rule,

$$\Rightarrow f^n(x) \cdot \cos x + {}^nC_1 f^{n-1}(x) \cdot (-\sin x) + {}^nC_2 f^{n-2}(x) \cdot (-\cos x)$$
$$+ {}^nC_3 f^{(n-3)}(x)(\sin x) + {}^nC_4 f^{n-4}(x)(\cos x) + \cdots + f(x) \cdot \frac{d^n}{dx^n}(\cos x)$$
$$= \frac{d^n}{dx^n} \sin x \quad — (1)$$

Now, lets find the nth differencision of $f(x) = \sin x$,

$f'(x) = \cos x = \sin\left(\frac{n\pi}{2} + x\right)$

$f^2(x) = -\sin x = \sin(\pi + x)$

$f^3(x) = -\cos x = \sin\left(\frac{3\pi}{2} + x\right)$

$f^4(x) = \sin x = \sin x = \sin(2\pi + x)$

$f^n(x) = \sin\left(\frac{n\pi}{2} + x\right)$

Now, lets find the nth differentiation of $f(x) = (\cos x)$

$f(x) = \cos x =$

$f'(x) = -\sin x = \cos\left(\frac{\pi}{2} + x\right)$

$f^2(x) = -\cos x = \cos(\pi + x)$

$f^3(x) = \sin x = \cos\left(\frac{3\pi}{2} + x\right)$

$f^4(x) = \cos x = \cos(2\pi + x)$

$f^n(x) = \cos\left(\frac{n\pi}{2} + x\right)$

(1) ...

$$f^n(x)\cdot\cos x + {}^nC_1 f^{n-1}(x)\cdot(-\sin x) + {}^nC_2 f^{n-2}(x)\cdot(-\cos x) + {}^nC_3 f^{n-3}(x)\cdot(\sin x) + {}^nC_4 f^{n-4}(x)\cos x$$
$$+ \cdots + \cos\left(\frac{n\pi}{2}+x\right) = \sin\left(\frac{n\pi}{2}+x\right)$$

puting $x=0$,

$$f^n(0) - {}^nC_2 f^{n-2}(0) + {}^nC_4 f^{n-4}(0) - \cdots + \cos\left(\frac{n\pi}{2}\right) = \sin\left(\frac{n\pi}{2}\right)$$

$$[L.H.S = R.H.S]$$

Problem: If $y^{1/m} + y^{-1/m} = 2x$, show that

(i) $(x^2-1)y_2 + xy_1 - m^2y = 0$

(ii) $(x^2-1)y_{n+2} + (2n+1)xy_{n+1} + (n^2-m^2)y_n = 0$

Given,

$$\cancel{y^{1/m} + \frac{1}{y^{1/m}} = 2x}$$

$$\cancel{\Rightarrow y^{2/m} + 1 = 2x\cdot y^{1/m}}$$

$$y^{1/m} + y^{-1/m} = 2x$$

$$\frac{1}{m}y^{\frac{1}{m}-1}\cdot\frac{dy}{dx} - \frac{1}{m}\left(y^{-1-\frac{1}{m}}\right)\frac{dy}{dx} = 2$$

$$\frac{dy}{dx}\times\frac{1}{m}\left(y^{\frac{1-m}{m}} - y^{-\frac{m+1}{m}}\right) = 2$$

$$y^{1/m} + y^{-1/m} = 2x$$

$$\Rightarrow \left(y^{1/m} + y^{-1/m}\right)^2 = 4x^2$$

$$\Rightarrow \left(y^{1/m} - y^{-1/m}\right)^2 + 4\cdot y^{1/m}\cdot\frac{1}{y^{1/m}} = 4x^2$$

$$\Rightarrow \left(y^{1/m} - y^{-1/m}\right)^2 = 4x^2 - 4$$

$$\Rightarrow \left(y^{1/m} - y^{-1/m}\right) = \pm 2\sqrt{x^2-1} \quad \text{—— (11)}$$

(1) + (11)

$$2y^{1/m} = 2\left(x \pm \sqrt{x^2-1}\right)$$

$$\Rightarrow \frac{1}{m}\ln y = \ln\left|x \pm \sqrt{x^2-1}\right|$$

$$\Rightarrow \frac{y_1}{my} = \frac{1}{\left|x\pm\sqrt{x^2-1}\right|}\times\left(1 \pm \frac{2x}{2\sqrt{x^2-1}}\right)$$

$$\Rightarrow \frac{y_1}{my} = \frac{1}{(x \pm \sqrt{x^2-1})} \times \frac{\pm(x \pm \sqrt{x^2-1})}{\sqrt{x^2-1}}$$

$$\Rightarrow y_1(\sqrt{x^2-1}) = my$$

$$\Rightarrow y_1^2(x^2-1) = m^2y^2$$

$$\Rightarrow 2y_1y_2(x^2-1) + y_1^2(2x) = 2m^2yy_1$$

$$\Rightarrow y_2(x^2-1) + xy_1 = m^2y$$

$$\Rightarrow y_2(x^2-1) + xy_1 - m^2y = 0$$

Now, using leibnitz here,

$$y_{n+2}(x^2-1) + {}^nC_1 y_{n+1}(2x) + {}^nC_2 y_n(2) + y_{n+1}x + {}^nC_1 y_n - m^2y_n = 0$$

$$\Rightarrow y_{n+2}(x^2-1) + y_{n+1}(2nx) + y_n\left(\frac{\cancel{2}n(n-1)}{\cancel{2}}\right) + y_{n+1}x + ny_n - m^2y_n = 0$$

$$\Rightarrow y_{n+2}(x^2-1) + xy_{n+1}(2n+1) + y_n(n^2-n+n-m^2) = 0$$

$$\boxed{\Rightarrow y_{n+2}(x^2-1) + xy_{n+1}(2n+1) + y_n(n^2-m^2) = 0}$$

Reduction Formula

It says that express integral of a function involving a generic power term of another integral with similar structure, but involving a lower power.

Eg: $I_n = \int \sin^n x \, dx$

$$I_n = \frac{-\sin^{n-1} x \cos x}{n} + \left(\frac{n-1}{n}\right) I_{n-2}$$

$\int uv \, dx$ [where u, v are the functions of x]

DI Method for By Parts.

$$\int x^4 \ln x \, dx = \frac{x^5}{5}\ln x - \frac{x^5}{25} + C$$

	D	I
+	$\ln x$	x^4
−	$\frac{1}{x}$	$\frac{x^5}{5}$
+		

2nd stop

DI Method

$$\int x^2 \sin 3x \, dx = -\frac{x^2}{3}\cos 3x + \frac{2x}{9}\sin 3x + \frac{2}{27}\cos 3x + C$$

D	I
$+x^2$	$\sin 3x$
$-2x$	$-\cos 3x \cdot \frac{1}{3}$
$+2$	$-\sin 3x \cdot \frac{1}{9}$
-0	$\cos 3x \cdot \frac{1}{27}$

1st stop

$$\int e^x \sin x \, dx = -e^x \cos x + e^x \sin x - \int e^x \sin x \, dx$$

$$= \frac{1}{2} e^x [\sin x - \cos x] + C$$

	D	I
+	e^x	$\sin x$
−	e^x	$-\cos x$
+	e^x	$-\sin x$

Q) lets Find the reduction formula for $\int \sin^n x \, dx$, n being a positive integer >1.

Let, $I_n = \int \sin^n x \, dx$

$= \int \sin^{(n-1)} x \cdot \sin x \, dx$

	D	I
$+$	$\sin^{(n-1)} x$	$\sin x$
$-$	$(n-1)\sin^{(n-2)} x \cdot \cos x$	$-\cos x$
$+$		

$$I_n = -\sin^{(n-1)} x \cdot \cos x + (n-1)\int \sin^{(n-2)} x \cdot \cos^2 x \, dx$$

$$= -\sin^{(n-1)} x \cdot \cos x + (n-1)\int \sin^{(n-2)} x \cdot (1-\sin^2 x) \, dx$$

$$= -\sin^{(n-1)} x \cos x + (n-1)\left[\int \sin^{(n-2)} x \, dx - \int \sin^n x \, dx\right]$$

$$\cancel{I_n + (n-1)} I_n = (n-1)\int \sin^{(n-2)} x \, dx - \sin^{(n-1)} x \cos x$$

$$I_n = \frac{n-1}{n} I_{n-2} - \frac{\sin^{(n-1)} x \cos x}{n} \quad [\text{Ans}]$$

Q) Find the reduction formula for $\int \tan^n x \, dx$, n being positive integer >1.

	D	I
$+$	$\tan^{(n-2)} x$	
$-$		
$+$		
$-$		

$$I_n = \int \tan^n x \, dx$$

$$= \int \tan^{(n-2)} x \tan^2 x \, dx$$

$$= \int \tan^{(n-2)} x (\sec^2 x - 1) \, dx$$

$$= \int \tan^{(n-2)} x \sec^2 x \, dx - \int \tan^{(n-2)} x \, dx$$

$$I_n = \int \tan^{(n-2)}x \sec^2 x - I_{n-2}$$

	D	I
+	$\tan^{(n-2)}x$	$\sec^2 x$
-	$(n-2)\tan^{(n-3)}x \cdot \sec^2 x \tan x$	

$$= \tan^{(n-2)}x \cdot \tan x - \int (n-2)\tan^{(n-2)}x\,(1+\tan^2 x) - I_{n-2}$$

$$= \tan^{(n-1)}x - (n-2)\left[\int \tan^{(n-2)}x\,dx + \int \tan^n x\,dx\right] - I$$

$$I_n = \tan^{(n-1)}x - (n-2)I_{n-2} - (n-2)I_n - I_{n-2}$$

$$\cancel{I_n} + nI_n - \cancel{2}I_n = \tan^{(n-1)}x - (n-2)I_{n-2} - I_{n-2}$$

$$I_n(n-1) = \tan^{(n-1)}x - I_{(n-2)}(n-2+1)$$

$$I_n = \frac{\tan^{(n-1)}x}{(n-1)} - I_{n-2}$$

(1) Find the reduction formula for $\int \sec^n x\,dx$, $n > 1$, if

let $I_n = \int \sec^n x\,dx$

	D	I
+	$\sec^{(n-2)}x$	$\sec^2 x$
+	$(n-2)\sec^{(n-3)}x \cdot \sec x \cdot \tan x$	$\tan x$

$$I_n = \int \sec^{(n-2)}x \cdot \sec^2 x\,dx$$

$$I_n = \tan x \sec^{(n-2)}x - (n-2)\int \sec^{(n-2)}x \tan^2 x$$

$\hookrightarrow (\sec^2 x - 1)$

$$I_n = \tan x \sec^{(n-2)}x + (n-2)\int \sec^{(n-2)}x\,dx \mp (n-2)\int \sec^n x\,dx$$

$$I_n = \tan x \sec^{(n-2)}x + (n-2)I_{n-2} \mp (n-2)I_n$$

$$I_n \cancel{(n-1)} = \frac{\tan x \sec^{(n-2)}x}{(n-1)} + \frac{n-2}{(n-1)}I_{n-2}$$

Q) find the Reduction formula for $\int \cos^n x\,dx$ evaluate $\int_0^{\pi/2} \cos^n x\,dx$

Let, $I_n = \int \cos^n x\,dx$

$$= \int \cos^{(n-1)} x \cdot \cos x\,dx$$

	D	I
+	$\cos^{(n-1)} x$	$\cos x$
+	$(n-1)\cos^{(n-2)} x \cdot \sin x$	$\sin x$

$$I_n = \sin x \cdot \cos^{(n-1)} x + (n-1)\int \cos^{(n-2)} x \cdot (1-\cos^2 x)\,dx$$

$$I_n = \sin x \cdot \cos^{(n-1)} x + (n-1)\int \cos^{(n-2)} x\,dx - (n-1)\int \cos^n x\,dx$$

$$\cancel{I_n(1 + n - 1)} = \frac{\sin x \cdot \cos^{(n-1)} x}{n} + \frac{n-1}{n} I_{n-2}$$

$$\int_0^{\pi/2} \cos^n x\,dx = \left[\cancel{\frac{\sin\frac{\pi}{2} \cdot \cos^{(n-1)}\frac{\pi}{2}}{n}} - 0\right]$$

$$\int_0^{\frac{\pi}{2}} \cos^n x\,dx = \frac{n-1}{n} I_{n-2}$$

$$= \begin{cases} \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\cdots\left(\frac{2}{3}\right) I_1 & n \text{ is Odd} \\ \left(\frac{n-1}{n}\right)\left(\frac{n-3}{n-2}\right)\cdots\left(\frac{3}{4}\right) I_2 & n \text{ is even} \end{cases}$$

Find the reduction formula for $\int \cos^m x \cos nx \, dx$ where m, n being +ve integer.

let, $I_{m,n} = \int \cos^m x \cos nx \, dx$

$$I_{m,n} = \frac{\cos^m x \cdot \sin nx}{n} + \frac{m}{n}\int \cos^{(m-1)} x \cdot \sin x \cdot \sin nx \, dx$$

D	I
$\cos^m x$	$\cos nx$
$+ \; m\cos^{(m-1)} x \cdot \sin x$	$\frac{\sin nx}{n}$

We know,

$$\sin x \cdot \sin nx + \cos x \cos nx = \cos(nx - x)$$

$$\sin x \cdot \sin nx = \cos(n-1)x - \cos x \cos nx$$

$$\therefore I_{m,n} = \frac{\cos^m x \sin nx}{n} + \frac{m}{n}\int \cos^{(m-1)} x \cdot [\cos(n-1)x - \cos x \cos nx] \, dx$$

$$I_{m,n} = \frac{\cos^m x \sin nx}{n} + \frac{m}{n}\int \cos^{(m-1)} x \cdot \cos(n-1)x \, dx - \frac{m}{n}\int \underbrace{\cos^{\cancel{(m-1)}} x \cdot \cancel{\cos x} \cdot \cos nx \, dx}_{I_{m,n}}$$

$$\left(1 + \frac{m}{n}\right) I_{m,n} = \frac{\cos^m x \sin nx}{n} + \frac{m}{n} I_{(m-1),(n-1)}$$

$$\therefore I_{m,n} = \frac{\cos^m x \sin nx}{m+n} + \frac{m}{m+n} I_{(m-1),(n-1)}$$

Q Find the reduction formula for $\int \sin^m x \cos^n x\,dx$ being a positive integer.

$\sin^2 a + (\cos a)^2$

$1 + 2\sin a \cos b$

let, $I_{m,n} = \int \sin^m x \cdot \cos^n x \, dx$

$= \int \sin^m x \cdot \cos^{n-1} x \cdot \cos x \, dx$

$\frac{x^3}{3}$

$\int z^n dz$

$$I_{m,n} = \frac{\sin^{m+1} \cdot \cos^{n-1} x}{m+1} + \frac{n-1}{m+1} \int \cos^{n-2} x \cdot \sin^{m+2} x \, dx$$

D   I

$+ \quad \cos^{n-1} x \qquad \sin^m x \cdot \cos x$

$+ (n-1) \cos^{n-2} x \sin x \qquad \frac{\sin^{m+1} x}{m+1}$

$$I_{m,n} = \frac{\sin^{m+1} x \cdot \cos^{n-1} x}{m+1} + \frac{n-1}{m+1} \int \cos^{n-2} x \cdot (1 - \cos^2 x) \sin^m x \, dx$$

$$I_{m,n} = \frac{\sin^{m+1} x \cdot \cos^{n-1} x}{m+1} + \frac{n-1}{m+1} \int \cos^{(n-2)} x \cdot \sin^m x \, dx - \frac{n-1}{m+1} \int \cos^n x \cdot \sin^m x \, dx$$

$$\left(1 + \frac{n-1}{m+1}\right) I_{m,n} = \frac{\sin^{m+1} x \cdot \cos^{n-1} x}{m+1} + \frac{n-1}{m+1} \int$$

$$I_{m,n} = \frac{\sin^{m+1} x \cdot \cos^{n-1} x}{\cancel{m+1+n}\; m+n} + \frac{n-1}{m+n} I_{m,(n-2)} \quad (\text{Ans})$$

Q $I_{m,n} = \int_0^{\pi/2} \sin^m x \cos^n x \, dx$; m,n being a positive integer > 1,

P.T. $I_{m,n} = \left(\frac{n-1}{m+n}\right) I_{m,n-2}$

$$\begin{array}{ccc} & D & I \\ + & \cos^{n-1}x & \int \sin^m x \cos x \, dx \\ + & (n-1)\cos^{n-2}x \cdot \sin x & \frac{\sin^{m+1}}{m+1} \\ + & & \end{array}$$

$$I_{m,n} = \int_0^{\pi/2} \sin^m x \cos^n x \, dx$$

$$= \int_0^{\pi/2} \sin^m x \cdot \cos x \cdot \cos^{n-1} x \, dx$$

$$= \frac{\sin^{(m+1)} x \cdot \cos^{(n-1)} x}{m+1} + \frac{n-1}{m+1} \int \sin^{m+2} x \cdot \cos^{n-2} x \, dx$$

$$I_{m,n} = \frac{\sin^{(m+1)} x \cdot \cos^{(n-1)} x}{m+1} + \frac{n-1}{m+1} \int \sin^m x \cdot \cos^{n-2} x (1-\cos^2 x) dx$$

$$I_{m,n} = \quad " \quad + \frac{n-1}{m+1} \left[I_{m,n-2} - I_{m,n} \right]$$

$$I_{m,n} \left[1 + \frac{n-1}{m+1} \right] = \frac{\sin^{m+1} x \cdot \cos^{n-1} x}{m+1} + \frac{n-1}{m+1} I_{m,n-2}$$

$$I_{m,n} \Big|_0^{\pi/2} = \left[\left[\frac{\sin^{m+1} x \cdot \cos^{n-1} x}{m+n} \right]_0^{\pi/2} + \frac{n-1}{m+n} I_{m,n-2} \right]_0^{\pi/2}$$

$$= \left[\left(\frac{\sin^{m+1}(\pi/2) \cdot \cos^{n-1}(\pi/2)}{m+n} \right) + \frac{n-1}{m+n} I_{m,n-2} \right]$$

$$I_{m,n} = 0 + \frac{n-1}{m+n} I_{m,n-2}$$

L.H.S = R.H.S

Q find the reduction formula for $\int x^m e^x \, dx$ and use the formula to evaluate $\int_0^1 x^3 e^x \, dx$

Let, $I_m = \int x^m e^x \, dx$

	D	I
+	x^m	e^x
	$(-1)^1 (m) x^{m-1}$	e^x
	$\vdots$	$\vdots$
	$(-1)^m \; m!$	e^x

$$= x^m e^x - m! \, I_{m-1}$$

$$e^x\left[x^m + \sum_{n=1}^{m} x^{(m-n)} (-1)^n \left[\prod_{k=1}^{n} (m-k+1)\right]\right]$$

$m = 3$,

$$e^x\left[x^3 + x^{3-1} (-1)^1 \cdot (3-1+1) + x^{3-2} (-1)^2 \cdot (3-1+1)\cdot(3-2+1) + x^{3-3} (-1)^3 \cdot (3-1+1)(3-2+1)(3-3+1)\right]$$

$$e^x(x^3 - 3x^2 + 6x^2 - 6) \leftarrow$$

$$-e \times 0 - 1$$

+	x^3	e^x
−	$3x^2$	e^x
+	$3 \cdot 2x$	e^x
−	$3 \cdot 2 \cdot 1$	e^x

$$e^x\left[x^3 - 3x^2 + 6x - 6\right]$$

$$-2e + 6$$

$$\int x^m e^x \, dx = e^x\left[x^m + \sum_{n=1}^{m} x^{m-n} (-1)^n \prod_{k=1}^{n} (m-k+1)\right]$$

Q Find the reduction formula for $\int e^x x^m dx$

find $\int_0^1 e^x x^3 dx$

Let, $I_m = \int x^m e^x dx$

$= e^x x^m - m I_{m-1}$

So, $I_3 = e^x x^3 - 3 \cdot (e^x x^2 - 2(e^x \cdot x - 1(e^x)))$

$= \left[e^x(x^3 - 3x^2 + 6x - 6)\right]_0^1$

$= 6 - 2e$

Q If $I_n = \int_0^{\pi/2} x^n \sin x \, dx$; $n > 1$ prove that

$I_n + n(n-1) I_{n-2} = n \left(\frac{\pi}{2}\right)^{n-1}$

Given, $I_n = \int_0^{\pi/2} x^n \sin x \, dx$

D	I
$+x^n$	$\sin x$
$-n(x^{n-1})$	$-\cos x$
$+n(n-1)x^{n-2}$	$-\sin x$

$= -x^n \cos x + n x^{n-1} \sin x - n(n-1)\int_0^{\pi/2} x^{n-2} \sin x \, dx$

$[I_n]_0^{\pi/2} = \left[n x^{n-1} \sin x - x^n \cos x\right]_0^{\pi/2} - n(n-1) I_{n-2}$

$I_n + n(n-1) I_{n-2} = \left[n x^{n-1} \sin x - x^n \cos x\right]_0^{\pi/2}$

L.H.S

$= n\left(\frac{\pi}{2}\right)^{n-1} 1 - \left(\frac{\pi}{2}\right)^n \cdot \cancel{\cos\left(\frac{\pi}{2}\right)}^{0}$

$- n(0)^{n-1} \sin 0 + 0^n \cos 0$

$= n \left(\frac{\pi}{2}\right)^{n-1}$

L.H.S = R.H.S (proved)

Q Find the reduction formula for $\int \tan^n x \, dx$, $n>1$, evaluate $\int_0^{\pi/4} \tan^4 x \, dx$

$\tan^2 x = \sec^2 x - 1$

Let,

$$I_n = \int \tan^n x \, dx$$

D I

$\tan^{n-2} x$ $\sec^2 x$

$-(n-2)\tan^{n-3} \cdot \sec^2 x$ $\tan x$

$$= \int \tan^{(n-2)} x \cdot (\sec^2 x - 1) \, dx$$

$$= -I_{n-2} + \left[\tan^{n-2} x \cdot \tan x - (n-2) \int \tan^{n-2} x \sec^2 x \, dx\right]$$

$$= -I_{n-2} + \left[\tan^{n-1} x - (n-2) \int \tan^{n-2} x \, (1+\tan^2 x) \, dx\right]$$

$$I_n = -I_{n-2} + \left[\tan^{n-1} x - (n-2) I_{n-2} - (n-2) I_n\right]$$

$$I_n = -I_{n-2} + \tan^{n-1} x - (n-2) I_{n-2} - (n-2) I_n$$

$$I_n = \frac{1}{2}\left[(n-1) I_{n-2} + \tan^{n-1} x\right]$$

$$- \frac{\cancel{n-1}}{\cancel{n-1}} \times I_{n-2} + \frac{\tan^{n-1} x}{n-1}$$

$$I_n = \frac{\tan^{n-1} x}{n-1} - I_{n-2}$$

$$I_4 = \frac{\tan^3 x}{3} - I_2$$

$$= \left[\frac{\tan^3 x}{3} - \left[\frac{\tan x}{1} - x\right]\right]_0^{\pi/4}$$

$$= \frac{1}{3} - \left(1 - \frac{\pi}{4}\right) \qquad \frac{\pi}{4} - \frac{2}{3} \text{ (Ans)}$$

Q. If $\int_0^{\frac{\pi}{2}} x^n \sin x \, dx$, $n > 1$, S.T. $I_n + n(n-1)I_{n-2} = n\left(\frac{\pi}{2}\right)^{n-1}$

find the value of $\int_0^{\pi/2} x^5 \sin x \, dx$

$$I_n = \int_0^{\frac{\pi}{2}} x^n \sin x \, dx$$

D	I
$+ x^n$	$\sin x$
$-nx^{n-1}$	$-\cos x$
$+n(n-1)x^{n-2}$	$-\sin x$

$$\cancel{= -x^n \cos x + n\int x^{n-1}}$$

$$= \left[-x^n \cos x + n x^{n-1} \sin x - n(n-1)\int x^{n-2} \sin x\right]$$

$$\therefore I_n = \left[n x^{n-1} \sin x - x^n \cos x - n(n-1) I_{n-2}\right]_0^{\pi/2} \quad \text{--- (1)}$$

$$I_n + n(n-1)I_{n-2} = \left[n x^{n-1} \sin x - x^n \cos x\right]_0^{\pi/2}$$

$$= n\left(\frac{\pi}{2}\right)^{n-1} \sin\left(\frac{\pi}{2}\right) - \left(\frac{\pi}{2}\right)^n \cos\frac{\pi}{2}$$

$$- n(0)^{n-1} \sin 0 + (0)^n \cos 0$$

$$= n\left(\frac{\pi}{2}\right)^{n-1}$$

$$\therefore I_n + n(n-1)I_{n-2} = n\left(\frac{\pi}{2}\right)^{n-1}$$

$$\therefore \text{L.H.S} = \text{R.H.S} \quad [\text{Proved}]$$

from (1)

$$I_5 = \left[5x^4 \sin x - x^5 \cos x - 20\left[3x^2 \sin x - x^3 \cos x - 6\left[\sin x - x\cos x\right]\right]\right]_0^{\pi/2}$$

$$= 5 \times \left(\frac{\pi}{2}\right)^4 \times 1 - 60\left(\frac{\pi}{2}\right)^2 + \cancel{120} + 120 \quad \cancel{= 120}$$

$$= 5 \times \left(\frac{\pi}{2}\right)^4 - 60\left(\frac{\pi}{2}\right)^2 + 120 \approx 2.397 \ldots$$

(Ans)

Q. If $I_n = \int_0^{\frac{\pi}{2}} x \sin^n x \, dx$, $n>1$, S.T,

$$I_n = \frac{n-1}{n} I_{n-2} + \frac{1}{n^2},$$ evaluate I_5

~~Let,~~ Given,

$$I_n = \int_0^{\pi/2} x \sin^n x \, dx$$

D	I
$\sin^{n-1} x$	$x \sin x$
$n \sin^{n-1} x \cdot \cos x$	$\frac{x^2}{2}$

$$= \int_0$$

D	I
$+\sin^{n-1} x$	$\sin x \cdot x$
$-(n-1) \sin^{n-2} x \cdot \cos x$	$-x\cos x + \sin x$

$$I_n = \Big[\sin^{n-1} x(-x\cos x + \sin x) + (n-1)\int x \sin^{n-2} x (1-\sin^2 x) dx - (n-1)\int \cancel{x} \sin^{n-1} x \cdot \cos x \, dx\Big]_0^{\pi/2}$$

$$\int z^{(n-1)} dz \quad \frac{z^n}{n}$$

$$I_n = \Big[\sin^{n-1} x(-x\cos x + \sin x) + (n-1) I_{n-2} - (n-1) I_n - (n-1)\frac{(\sin x)^n}{n}\Big]_0^{\pi/2}$$

$$I_n = \left[\frac{x \sin^{n-1} x \cos x + \sin^n x}{n}\right]_0^{\pi/2} + \frac{n-1}{n} I_{n-2} - \left[\frac{n-1}{n^2} \sin^n x\right]_0^{\pi/2}$$

$$I_n = \frac{0+1}{n} + \frac{n-1}{n} I_{n-2} - \frac{n-1}{n^2}$$

$$I_n = \cancel{\frac{n}{n}} + \frac{n-1}{n} I_{n-2} + \frac{\cancel{n}-\cancel{n}+1}{n^2}$$

$$I_n = \frac{n-1}{n} I_{n-2} + \frac{1}{n^2}$$

LHS = R.H.S [Proved]

$$\therefore \quad I_5 = \frac{4}{5} I_3 + \frac{1}{25}$$

$$= \left[\frac{4}{5}\left[\frac{2}{3}(-x\cos x + \sin x) + \frac{1}{9}\right] + \frac{1}{25}\right]_0^{\pi/2}$$

$$= \left[-\frac{8}{15} x\cos x + \frac{8}{15}\sin x + \frac{4}{45} + \frac{1}{25}\right]_0^{\frac{\pi}{2}}$$

$$= \frac{8}{15} + \frac{4}{45} + \frac{1}{25} - \cancel{\frac{4}{45}} - \cancel{\frac{1}{25}}$$

$$\cancel{= \frac{8}{15}(\ldots)} = 0.6622222$$

$$\int \frac{x\sin x}{1+\sin x}\,dx = \int \frac{x\sin x(1-\sin x)}{\cos^2 x}\,dx$$

$$\cancel{= -\int x\tan^2 x\,dx} \quad +$$

$$(1-\cos^2 x)$$

$$= -\int \frac{x\sin^2 x}{\cos^2 x}\,dx + \int \frac{x\sin x}{\cos^2 x}\,dx$$

$$= -\int \frac{x}{\cos^2 x}\,dx + \int x\,dx + \int x\tan x\sec x$$

$$= -x\tan x + \int \tan x\,dx + \int x\,dx + \int x\tan x\sec x$$

D	I
$+x$	$\sec^2 x$
-1	$\tan x$
$+x$	$\cancel{\tan x\sec x}$
-1	$\sec x$

$$= -x\tan x + \frac{x^2}{2} - \int \frac{dz}{z} + x\sec x - \int \sec x \quad \left[\begin{aligned} z &= \cos x \\ dz &= -\sin x\,dx \end{aligned}\right]$$

$$= \frac{x^2}{2} - x\tan x - \ln|\cos x| + x\sec x - \ln|\sec x + \tan x| + C$$

2D Geometry

Transformation of axes:

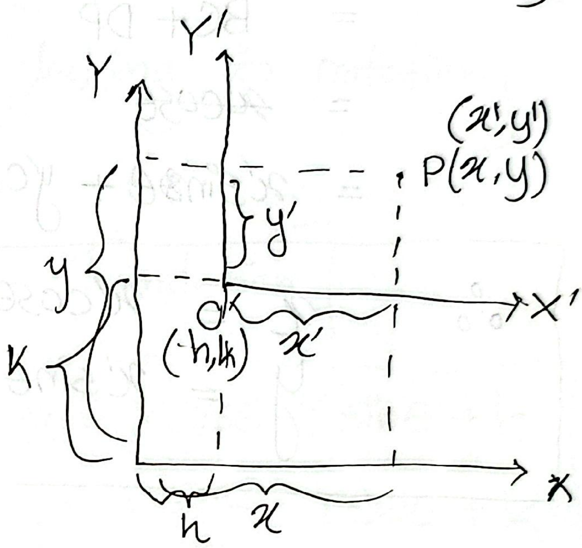
there are 3 types.

(i) Translation (Change of origin without changing the direction of axes)

(ii) Rotation (Change of direction of axes without the origin

(iii) Rigid Motion (combination of translation and rotation)

Translation:



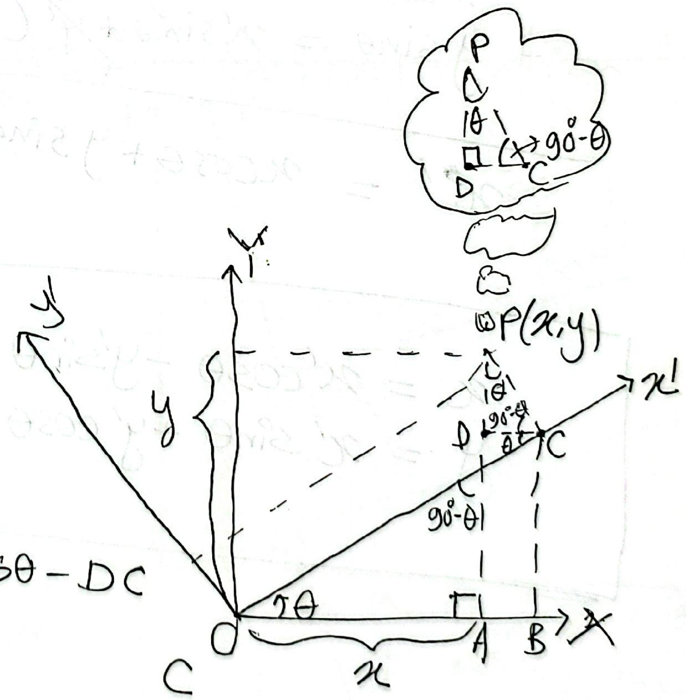
$P \equiv (x,y)$ in xy-system

$O' = (h,k)$

$$\boxed{\begin{aligned} \therefore\ x &= h + x' \\ y &= k + y' \\ x' &= x - h \\ y' &= y - k \end{aligned}}$$

} formulas for translation

Rotation of Axis:



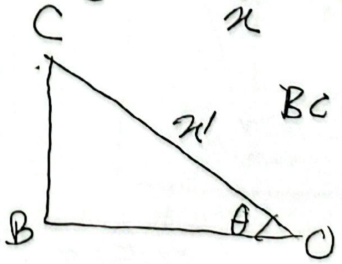
$P \equiv (x,y)$ in XY-system

$x = OA$

$= OB - AB$

$= OB - DC$

$\rightarrow x = x'\cos\theta - DC$

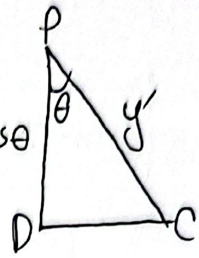


$BC - x'\sin\theta$

$\cos\theta = \dfrac{OB}{x'} \Rightarrow OB = x'\cos\theta$

~~$x' = \theta$~~

$$x = OA = OB + \cancel{BA} - AB$$
$$= OB - DC$$
$$= x'\cos\theta - y'\sin\theta$$



$DP = y'\cos\theta$

$DC = y'\sin\theta$

$$y = AP$$
$$= AD + DP$$
$$= BC + DP \quad [DP =$$
$$= \cancel{x'\cos\theta}$$
$$= x'\sin\theta + y\cos\theta$$

$$\boxed{\begin{aligned} \therefore\quad x &= x'\cos\theta - y'\sin\theta \\ y &= x'\sin\theta + y'\cos\theta \end{aligned}}$$

Formula

$$\begin{aligned} x\cos\theta &= x'\cos^2\theta \mp y'(\ ) \\ +\ y\sin\theta &= x'\sin^2\theta \pm y'(\ ) \end{aligned}$$
$$x' = x\cos\theta + y\sin\theta$$

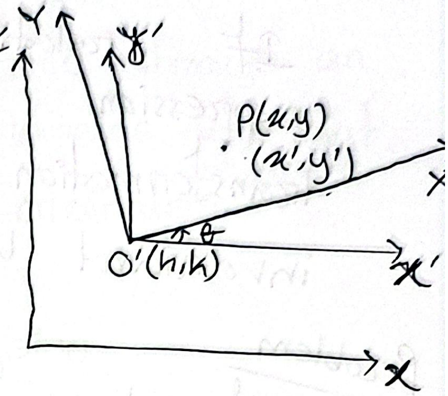
$$\begin{aligned} x\sin\theta &= x'(\ ) - y'\sin^2\theta \\ y\cos\theta &= x'(\ ) + y'\cos^2\theta \\ (-)\ & \quad\ (-) \qquad (-) \end{aligned}$$
$$y' = x\sin\theta - y\cos\theta$$

$$\boxed{\begin{aligned} x &= x'\cos\theta - y'\sin\theta & \quad x' &= x\cos\theta + y\sin\theta \\ y &= x'\sin\theta + y'\cos\theta & \quad y' &= x\sin\theta - y\cos\theta \end{aligned}}$$

Rigid Motion:

~~Rotation followed by Translation~~

Case 1 : ~~Translation followed by rotation~~
Tran the rot



$P \equiv (x, y)$ in ~~x-y~~ system

$P \equiv (x', y')$ " x'-y' "

$P \equiv (X, Y)$ in X-Y "

$$x' = X\cos\theta \cancel{+} - Y\sin\theta$$
$$y' = X\sin\theta + Y\cos\theta$$

[just looking to rotation]

$$x = x' + h$$
$$y = y' + k$$

[------ Translation]

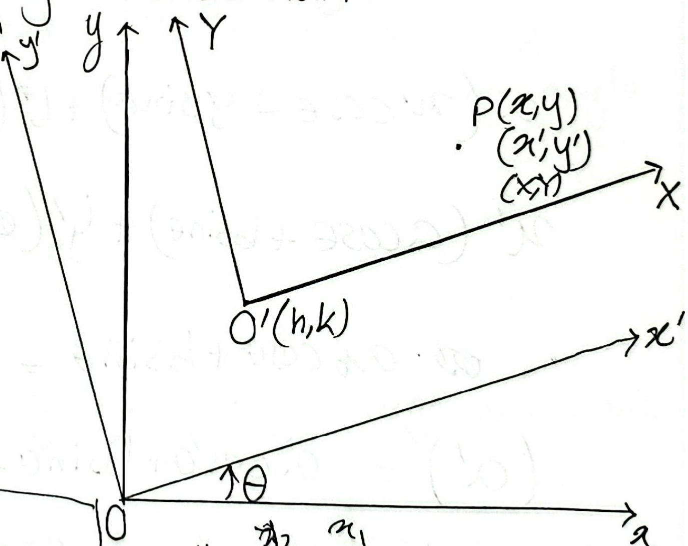
$$x - h = X\cos\theta - Y\sin\theta \Rightarrow$$
$$y - k = X\sin\theta + Y\cos\theta \Rightarrow$$

$$\boxed{\begin{aligned} x &= X\cos\theta - Y\sin\theta + h \\ y &= X\sin\theta + Y\cos\theta + k \end{aligned}}$$

Formula

Case 2 : ~~Trans~~ ~~rotation~~ ~~followed by~~ ~~Translation~~ rotation
rot then tran

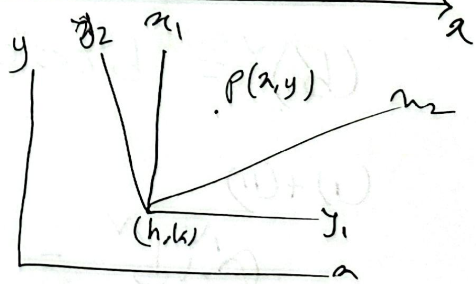
$P \equiv (x, y)$ in xy system



$$x = x'\cos\theta - y'\sin\theta$$
$$y = x'\sin\theta + y'\cos\theta$$

$$x' = X + h$$
$$y' = Y + k$$

$$\boxed{\begin{aligned} x &= (X+h)\cos\theta - (Y+k)\sin\theta \\ y &= (X+h)\cancel{\cos}\sin\theta + (Y+k)\cos\theta \end{aligned}}$$



$$x_1 = x_2\cos\theta - y_2\sin\theta$$
$$y_1 = x_2\sin\theta + y_2\cos\theta$$

$$x_1 = x - h$$
$$y_1 = y - k$$

Invariant :

If relation connecting the coefficients of an expressions remain unchanged under orthogonal transformation ~~whe~~ then we called they are invariant under that orthogonal transformation

Problem

If by a transformation of one rectangular axis to another with the same origin, the expression $ax+by$ changes to $a'x'+b'y'$ PT

$a^2+b^2 = a'^2+b'^2$

Let the axes be rotated through an angle ~~theta~~ θ, the due to rotation of axes we have

$$\left.\begin{aligned} x &= x'\cos\theta - y'\sin\theta \\ y &= x'\sin\theta + y'\cos\theta \end{aligned}\right\}$$

where $P \equiv (x,y)$ in xy System

$P \equiv (x',y')$ in $x'y'$ sys

$$a(x'\cos\theta - y'\sin\theta) + b(x'\sin\theta + y'\cos\theta) = a'x'+b'y'$$

$$x'(a\cos\theta + b\sin\theta) + y'(b\cos\theta - a\sin\theta) = a'x'+b'y'$$

$$\therefore a\cos\theta + b\sin\theta = a' \quad , \quad b\cos\theta - a\sin\theta = b'$$

$$(a')^2 = a^2\cos^2\theta + b^2\sin^2\theta + \cancel{2ab\sin\theta\cos\theta} \quad \text{--- (i)}$$

$$(b')^2 = b^2\cos^2\theta + a^2\sin^2\theta - \cancel{2ab\sin\theta\cdot\cos\theta} \quad \text{--- (ii)}$$

(i) + (ii)

$$a'^2+b'^2 = a^2(\underbrace{\cos^2\theta+\sin^2\theta}_{1}) + b^2(\underbrace{\sin^2\theta+\cos^2\theta}_{1})$$

$$a'^2+b'^2 = a^2+b^2 \qquad L.H.S = R.H.S$$

$x = x' + h$ $\quad$ $x = x'\cos\theta - y'\sin\theta$

$y = y' + k$ $\quad$ $y = x'\sin\theta + y'\cos\theta$

Q. The Coordinate ~~are~~ axes are rotated through an angle $\frac{\pi}{3}$ if the transformed coordinates of a Point are $(2\sqrt{3}, -6)$, find it's original co-ordinates.

Given,

$x' = 2\sqrt{3}$

$y' = -6$

$\theta = \frac{\pi}{3}$

$$x = x'\cos\theta - y'\sin\theta$$
$$= 2\sqrt{3} \times \cos\frac{\pi}{3} - (-6) \times \sin\frac{\pi}{3}$$
$$= \cancel{2}\sqrt{3} \times \frac{1}{\cancel{2}} + \cancel{6}^{3} \times \frac{\sqrt{3}}{\cancel{2}}$$
$$= 4\sqrt{3}$$

So, it's original Coordinates are,

$(4\sqrt{3}, 0)$

$$y = x'\sin\theta + y'\cos\theta$$
$$= 2\sqrt{3} \times \frac{\sqrt{3}}{2} + (-6) \times \frac{1}{2}$$
$$= 3 - 3$$
$$= 0$$

Q. What does the equations $11x^2 + 16xy - y^2 = 0$ becomes on turning the axes through an angle $\tan^{-1}\left(\frac{1}{2}\right)$

Grenarel eq^n of 2^{nd} degree

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

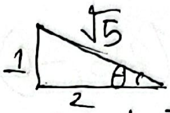
now if we apply rotation on this eq^n xy term will vanish in reduce eq^n

if we apply translation x, y term " "

Due to rotation of the angle $\theta = \tan^{-1}\frac{1}{2}$, we have,

$$x = x'\cos\theta - y'\sin\theta$$
$$y = x'\sin\theta + y'\cos\theta$$

where,
$P \equiv (x,y)$ in xy-system before rotation
$P \equiv (x',y')$ " $x'y'$ " after "



$\theta = \tan^{-1}\frac{1}{2}$

$\sin\theta = \frac{1}{\sqrt5}$

$\cos\theta = \frac{2}{\sqrt5}$

$\cancel{11\left(x'\times\frac{2}{\sqrt5} - y'\times\frac{1}{\sqrt5}\right)^2 + 16\left(x'\times\frac{2}{\sqrt5}\right)}$

$$11\left(\frac{2x'}{\sqrt5} - \frac{y'}{\sqrt5}\right)^2 + 16\left(\frac{2x'}{\sqrt5} - \frac{y'}{\sqrt5}\right)\left(\frac{x'}{\sqrt5} + \frac{2y'}{\sqrt5}\right) - \left(\frac{x'}{\sqrt5} + \frac{2y'}{\sqrt5}\right)^2 = 0$$

$$\Rightarrow 11 \times \frac{4x'^2 - 4x'y' + y'^2}{5} + 16\,\frac{2x'^2 + 4x'y' - x'y' - 2y'^2}{5} - \frac{x'^2 + 4x'y' + 4y'^2}{5} = 0$$

$$\Rightarrow 44x'^2 - \cancel{44x'y'} + 11y'^2 + 32x'^2 + \cancel{64x'y'} - \cancel{16x'y'} - 32y'^2 - x'^2 - \cancel{4x'y'} - 4y'^2 = 0$$

$$\Rightarrow 75x'^2 - 25y'^2 = 0$$

$$\Rightarrow 3x'^2 - y'^2 = 0$$

, which is the ~~eqⁿ~~ required ~~of the~~ transform eqⁿ.

$\propto$

Q. Determine the θ so that the eq^n $lx+my+n=0$ $(m\neq 0)$ may be reduced to the form $by+c=0$.

$x = ---$; $y = ---$; [where, $P\equiv(x,y)---$, $P\equiv(x',y')---$]

$$l.(x'\cos\theta - y'\sin\theta) + m(x'\sin\theta + y'\cos\theta) + n = 0$$

$$x'(l\cos\theta + m\sin\theta) + y'(\cancel{ls}\ m\cos\theta - l\sin\theta) + n = 0$$

the co-efficient of x' should be 0 & co-efficient of y' should be b.

$$l\cos\theta + m\sin\theta = 0 \quad ; \quad \cancel{x'\sin\theta}\ m\cos\theta - l\sin\theta = b$$

$$\cos\theta = -\frac{m}{l}\sin\theta \quad -(1)$$

putting (1)

$$m\cdot\left(-\frac{m}{l}\sin\theta\right) - l\sin\theta = b$$

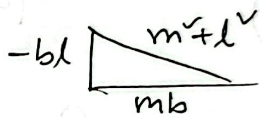
$$\frac{m^2\sin\theta + l^2\sin\theta}{l} = -b$$

$$\sin\theta = \frac{-bl}{m^2+l^2}$$

$$= -\frac{m}{\cancel{l}} \times \frac{-b\cancel{l}}{m^2+l^2}$$

$$= \frac{mb}{m^2+l^2}$$

~~so, $\theta = \cos^{-1}\left(\frac{mb}{m^2+l^2}\right) = \sin^{-1}\left(\frac{-bl}{m^2+l^2}\right)$~~



so, $\tan\theta = -\frac{\cancel{b}l}{\cancel{m^2+l^2}} \times \frac{\cancel{m^2+l^2}}{m\cancel{b}}$

$$\theta = \tan^{-1}\left(\frac{-l}{m}\right) \quad [m\neq 0]$$

Q. Find the angle of rotation for which $ax^2+2hxy+by^2$ becomes the expression of AX^2+BY^2

Let the angle of the rotation $=\theta$

$$x = X\cos\theta - Y\sin\theta$$
$$y = X\sin\theta + Y\cos\theta$$

[where, $P\equiv(x,y)$ in xy-system, $P\equiv(X,Y)$ in XY-system]

so, $ax^2+2hxy+by^2 = a(X\cos\theta - Y\sin\theta)^2 + 2h(X\cos\theta - Y\sin\theta)(X\sin\theta + Y\cos\theta) + b(X\sin\theta + Y\cos\theta)^2$

$= aX^2\cos^2\theta + aY^2\sin^2\theta - 2aXY\sin\theta\cdot\cos\theta + bX^2\sin^2\theta + bY^2\cos^2\theta$
$+ 2bXY\sin\theta\cdot\cos\theta + 2hX^2\sin\theta\cos\theta + 2hXY\cos^2\theta$
$- 2hXY\sin^2\theta - 2hY^2\sin\theta\cos\theta$

$$= X^2(a\cos^2\theta + b\sin^2\theta + 2h\sin\theta\cos\theta) + Y^2(a\sin^2\theta + b\cos^2\theta$$

$$+ 2h\sin\theta\cdot\cos\theta) + XY(-2a\sin\theta\cdot\cos\theta + 2b\sin\theta\cdot\cos\theta + 2h\cos^2\theta - 2h\sin^2\theta)$$

now comparing this eq^n with $AX^2 + BY^2$, we can say that,

$$a\cos^2\theta + b\sin^2\theta + 2h\sin\theta\cdot\cos\theta = A$$

$$a\sin^2\theta + b\cos^2\theta + 2h\sin\theta\cdot\cos\theta = B$$

$$-2a\sin\theta\cdot\cos\theta + 2b\sin\theta\cdot\cos\theta + 2h\cos^2\theta - 2h\sin^2\theta = 0$$

$$\Rightarrow \sin 2\theta(a+b) + 2h(\cos 2\theta) = 0$$

$$\Rightarrow \tan 2\theta = \frac{-2h}{(a-b)}$$

$$\cancel{\Rightarrow \theta = \frac{1}{2}\tan^{-1}\left(\frac{2h}{a+b}\right)} \quad \theta = \frac{1}{2}\tan^{-1}\left(\frac{-2h}{a-b}\right)$$

$$\cancel{\Rightarrow \frac{2\tan\theta}{1-\tan^2\theta} = \frac{2h}{a+b}}$$

$$\cancel{\Rightarrow (a+b)\tan\theta = h - h\tan^2\theta}$$

so the angle of axes rotation is $\frac{1}{2}\tan^{-1}\left(\frac{-2h}{a-b}\right)$

Q. Find the angle θ in the transformation of axes without change of origin which reduces the eq^n ~~to~~ $8x^2+4xy+8y^2-30=0$ to the form $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

$$x = x'\cos\theta - y'\sin\theta$$
$$y = x'\sin\theta + y'\cos\theta$$

[where θ is the angle at which the axes rotated.
$P\equiv(x,y)$ in xy system
$P\equiv(x',y')$ " $x'y'$ "]

$$8(x'\cos\theta - y'\sin\theta)^2 + 4(x'\cos\theta - y'\sin\theta)(x'\sin\theta + y'\cos\theta) + 8(x'\sin\theta + y'\cos\theta)^2 - 30 = 0$$

$$x'^2(8\cos^2\theta + 4\cos\theta\cdot\sin\theta + 8\sin^2\theta) + y'^2(8\sin^2\theta - 4\sin\theta\cdot\cos\theta + 8\cos^2\theta) + x'y'(\cancel{-16\sin\theta\cos\theta} + 4\cos^2\theta - 4\sin^2\theta + \cancel{16\sin\theta\cdot\cos\theta}) - 30 = 0$$

$$x'^2(8+4\cos\theta\cdot\sin\theta) + y'^2(8-4\cos\theta\cdot\sin\theta) + 4x'y'(\cos 2\theta) - 29 = 1$$

now comparing this eq^n with $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ we get,

$\frac{1}{a^2} = 8+4\cos\theta\cdot\sin\theta \Rightarrow 1 = 2a^2(4+\sin 2\theta)$

$\frac{1}{b^2} = 8-4\cos\theta\cdot\sin\theta \Rightarrow a = \frac{1}{\sqrt{2(4+\sin 2\theta)}}$

$b = \frac{1}{\sqrt{2(4-\sin 2\theta)}}$

~~$4(\cos 2\theta) - 29 = 0$~~

~~$\Rightarrow \cos 2\theta = \frac{29}{4}$~~

~~$\theta = \frac{1}{2}\cos^{-1}\left(\frac{29}{4}\right)$~~

$4\cdot\cos 2\theta = 0$

$\therefore \theta = \frac{\pi}{4}$

$a = \frac{1}{\sqrt{10}}$ $\quad \frac{\sqrt{10}}{10}$

$b = \frac{1}{\sqrt{6}}$

$$x'^2\left(8+2\sin\left(\tfrac{\pi}{2}\right)\right)+y'^2\left(8-2\sin\left(\tfrac{\pi}{2}\right)\right)-30=\frac{x'^2}{a^2}+\frac{y'^2}{b^2}-1$$

$$10x'^2+6y'^2-30=\frac{x'^2}{a^2}+\frac{y'^2}{b^2}-1$$

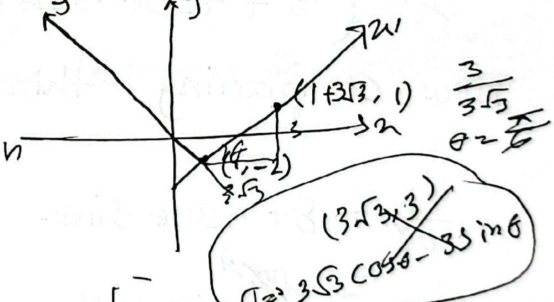
$$\cancel{x'^2\left(10-\frac{1}{a^2}\right)+y'^2\left(6-\frac{1}{b^2}\right)-29=0}$$

$$10x_1'^2+6y'^2-30=\frac{x'^2}{a^2/30}+\frac{y'^2}{b^2/30}-30$$

$$10=\frac{30}{a^2} \qquad b=\sqrt{5}$$

$$a=\sqrt{3}$$

Q. The co-ordinates of two points are $(1,-2)$ and $(1+3\sqrt{3},1)$ the origin shifted to the point $(1,-2)$ and the new x axis in the line joining the given points find the formula for rigid motion.



due to rigid motion (translation followed by rotation) we have,

$$x=x'\cos\theta-y'\sin\theta+h \quad -(1)$$

$$y=x'\sin\theta+y'\cos\theta+k$$

where $P\equiv(x,y)$ in xy sys
$P\equiv(x',y')$ in $x'y'$ "
(h,k) in the new origin

Given, $(h,k)\equiv(1,-2)$

$$\left.\begin{aligned}x&=\frac{\sqrt{3}x'}{2}-\frac{y'}{2}+1\\ y&=\frac{x'}{2}+\frac{\sqrt{3}y'}{2}-2\end{aligned}\right\}\ (Ans)$$

Q Show that there is only one fixed point in a rigid motion.

We know,

$$x = x'\cos\theta - y'\sin\theta + h$$
$$y = x'\sin\theta + y'\cos\theta + k$$

[where, (h,k) is the new origin in x',y' system
$P \equiv (x,y)$ in xy system
$P \equiv (x',y')$ " $x'y'$ "]

Lets assume the poin (α, β) is fixed in this rigid motion.

So,
$$\alpha = \alpha\cos\theta - \beta\sin\theta + h$$
$$\beta = \alpha\sin\theta + \beta\cos\theta + k$$

$\alpha(1-\cos\theta) + \beta(\sin\theta) - h = 0$

$$\left.\begin{aligned} \alpha(1-\cos\theta) + \beta\sin\theta - h &= 0 \\ \beta(1-\cos\theta) - \alpha\sin\theta - k &= 0 \end{aligned}\right\} (2)$$

$\rightarrow -\alpha(\sin\theta) + \beta(1-\cos\theta) - k = 0$

$$\begin{vmatrix} 1-\cos\theta & \sin\theta \\ -\sin\theta & (1-\cos\theta) \end{vmatrix}$$

$$\boxed{\alpha = \frac{h \times (1-\cos\theta) - k(\sin\theta)}{(1-\cos\theta)^2 + \sin^2\theta}}$$

$= (1-\cos\theta)^2 + \sin^2\theta$

$= 1 + 1 - 2\cos\theta$

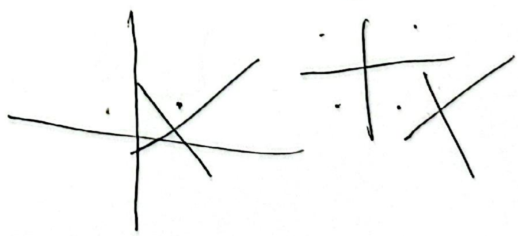
$= 2(1-\cos\theta)$

$D_x = \begin{vmatrix} c_1 & \sin\theta \\ c_2 & b_2\cos\theta \end{vmatrix}$

$D_y = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}$

as $\theta \neq 0$ [if rigid motion happens]

then, $2(1-\cos\theta) \neq 0$ means (2) eqns have unique solutions.



Real Valued Function

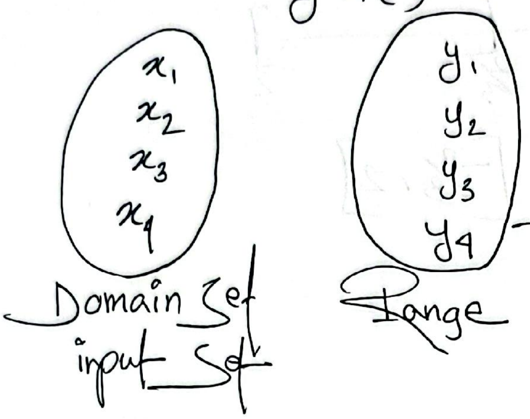
Domain and Range of a function:

$f: A \to B$ be a real valued function

$y = f(x)$

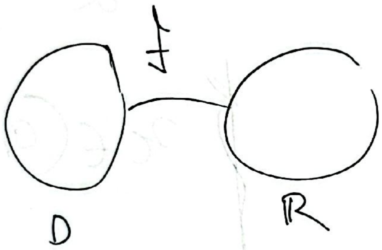
$f(x_i) = y_i$ for i 1 to 4

y is the image of x



→ it's actually co domain when only having the outputs, it's range.

Let, $D \subset \mathbb{R}$ and $f: D \to \mathbb{R}$



$f(x) \leq M \quad \forall x \in D$

$f(x) \geq m \quad \forall x \in D$

M is the upper bound on the Domain D

m — — lower —

If our function is both upper & lower bounded we will say our function is bounded function.

Find the Domain:

(i)

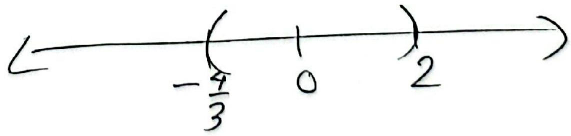
$$f(x) = \sqrt{8 + 2x - 3x^2}$$

$$8 + 2x - 3x^2 \geq 0$$

$$8 + 6x - 4x - 3x^2 \geq 0$$

$$2(4+3x) - x(4+3x) \geq 0$$

$$(2-x)(4+3x) \geq 0$$



$x \in [-4/3, 2]$

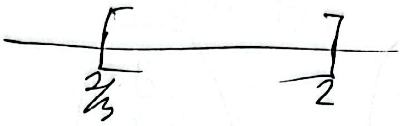
(ii) $\sin^{-1}\left(\frac{3x-2}{4}\right)$

$-1 \leq \frac{3x-2}{4} \leq 1$

$-4 \leq 3x-2 \leq 4$

$-2 \leq 3x \leq 6$

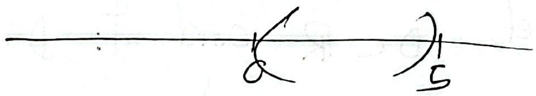
$-\frac{2}{3} \leq x \leq 2$



$x \in \left[-\frac{2}{3}, 2\right]$

(iii) $\sqrt{\ln\left(\frac{5x-x^2}{4}\right)}$

$\ln\left(\frac{5x-x^2}{4}\right) \geq 0$



$\frac{5x-x^2}{4} \geq 1$

$x \in (0,5)$

$5x-x^2 \geq 4$

$\cancel{x(5-x) > 0}$

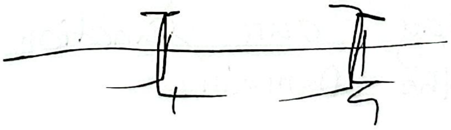
$5x-x^2-4 \geq 0$

$x^2-5x+4 \leq 0$

$x^2 - x - 4x + 4 \leq 0$

$x(x-1)-4(x-1) \leq 0$

$(x-4)(x-1) \leq 0$



$x \in \cancel{(-\infty, 1] \cup [4, \infty)}$

$x \in [1,4]$

(iv) $\frac{1}{\sqrt{|x|-x}}$ $\qquad x \in (-\infty, 0)$

$|x|-x > 0$

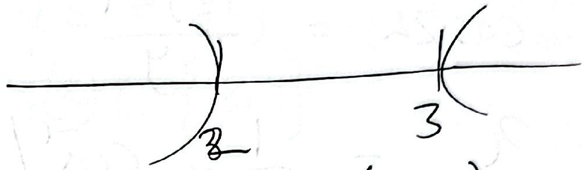
$\cancel{x+x > 0}$

(v) $\log(x^2-5x+6)$

$x^2-5x+6 > 0$

$x^2-3x-2x+6 > 0$

$(x-3)(x-2) > 0$



$x \in (-\infty, 2) \cup (3, \infty)$

Find the range

(i) $\frac{x}{x^2+1}$

Let, $y = \frac{x}{x^2+1}$

$\Rightarrow x^2y + y = x$

$\Rightarrow x^2y - x + y = 0$

$\Rightarrow x^2y - x + y = 0$

$x = \frac{-b \pm \sqrt{\cancel{4ac}\; b^2-4ac}}{2a}$

$x = \frac{-(-1) \pm \sqrt{\cancel{4.y.y}\; 1-4y^2}}{2 \times y}$

$\cancel{x = \frac{1 \pm 2y}{2y}}$

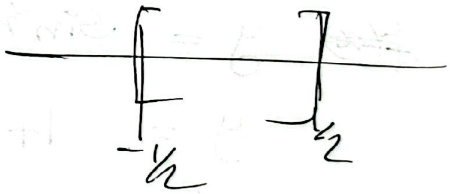
$x =$

$1-4y^2 \geq 0$

$\cancel{2y \neq 0}$

$\cancel{y \neq 0}$

$(1+2y)(1-2y) \geq 0$



(ii) $\frac{1}{2-\cos 2x} = y$

$1 = 2y - y\cos 2x$

$y\cos 2x = 2y-1$

$\cos 2x = \frac{2y-1}{y}$

$x = \frac{1}{2} \times \cos^{-1}\left(\frac{2y-1}{y}\right)$

$-1 \leq \frac{2y-1}{y} \leq +1$ $\qquad y \neq 0$

$-3 \leq -\frac{1}{y} \leq -1$

$3 \geq \frac{1}{y} \geq 1 \quad \times \rightarrow \quad \frac{1}{3} \leq y \leq 1$

$\frac{1}{3} \geq y \geq 1 \quad \times$

$f(x) \in [1/3, 1]$

(iii) $\sin^2 x + \cos^2 x$

$y = \sin^2 x + \cos^2 x$

$y = 1 + \cos^2 x$ $\qquad y \in [2, 1]$

$\cos^2 x = y-1$

$x = \cos^{-1}\sqrt{y-1}$

$-1 \leq \sqrt{y-1} \leq +1$

$y-1 \geq 0$

$y \geq 1$

$\sqrt{y-1} \leq 1$

$y \leq 2$

Find the domain & Range both

(i) $f(x) = \frac{|x|}{x}$

$x \in \mathbb{R} - \{0\}$

~~$f(x) \in \{1\}$~~

$f(x) \in \{1, -1\}$

(ii) $f(x) = \frac{|x|}{x} + 2$

$x \in \mathbb{R} - \{0\}$

$f(x) \in \{3, 1\}$

(iii) $f(x) = \frac{1}{\sqrt{x - [x]}}$

$x \in \mathbb{R} - \mathbb{Z}$

$\sqrt{x - [x]} \neq 0$

$x - [x] \neq -ve$ or ≥ 0

$x \notin \mathbb{Z}$

$y = \frac{1}{\sqrt{x - [x]}}$

$y^2 (x - [x]) = 1$

$y^2 x - y[x] - 1 = 0$

$y^2 x - 2y[x] + y[x] - 1 = 0$

~~$y(x - 2[x]) + y$~~

Delta Epsilon Defination of limit.

$y = f(x)$ defined in a certain neighorhood of $x = c$

neighbhhood

$(c-\delta \quad c \quad c+\delta)$ $\quad \delta > 0$

$\therefore$ $y = f(x)$ approaches the limit l as x approaches c. if for preassigned $\varepsilon (>0)$ however small may be, $\exists$ a $\delta (>0)$ such that for all $x (\neq c) \in |x-c| < \delta \Rightarrow |f(x) - l| < \varepsilon$ we write $\lim f(x) = l$.

BPRP ♡

$\lim\limits_{x \to a} f(x) = L$ means

$\forall \varepsilon > 0, \exists \delta > 0$. S.T.

$0 < |x-a| < \delta \Rightarrow |f(x) - L| < \varepsilon$

$\lim\limits_{x \to 4} \sqrt{2x+1} = \sqrt{2(4)+1} = 3$ means,

$\forall \varepsilon > 0, \exists \delta > 0$, st.

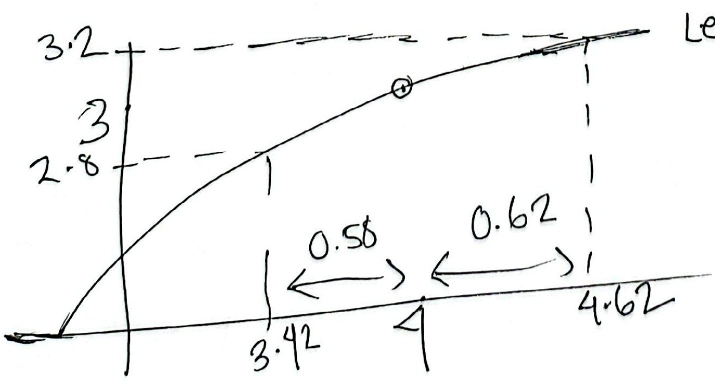
$0 < |x-4| < \delta \Rightarrow |\sqrt{2x+1} - 3| < \varepsilon$

$\sqrt{2x+1} = 3.2$

$x = 4.62$

$\sqrt{2x+1} = 2.8$

$= 3.42$



Let $\varepsilon = 0.2$

$\delta = 0.58$

proof

Given, $\varepsilon > 0$

Choose $\delta = \frac{\varepsilon}{2}$

Suppose,

$0 < |x-4| < \delta$

Check

$|\sqrt{2x+1} - 3|$ $\quad 2x+1-9$

$= \left| \frac{(\sqrt{2x+1} - 3)(\sqrt{2x+1} + 3)}{(\sqrt{2x+1} + 3)} \right|$

$= \frac{2|x-4|}{\sqrt{(2x+1)} + 3}$

$\hookrightarrow > 1$

$\cancel{8} < 2|x-4|$

$< 2\delta$

$< \cancel{2} \cdot \frac{\cancel{2}\varepsilon}{\cancel{2}}$

$< \varepsilon$

Proof

Given,

$\varepsilon > 0$

choose $\delta = \frac{\varepsilon}{|x+3|}$

Suppose,

$\forall \varepsilon > 0 \; \exists \delta > 0$ st

$0 < |x-3| < \delta$

check

$|x^2 - 9|$

$= |x+3||x-3|$

$< |x+3| \times \delta$

$< \cancel{|x+3|} \times \frac{\varepsilon}{\cancel{|x+3|}}$

$< \varepsilon$

$\varepsilon - \delta$ defination Math

$$\lim_{x \to 3} \frac{x^2 - 9}{x - 3}$$

given, $\forall \varepsilon > 0 \ \exists \ \delta > 0$ st. $0 < |x-3| < \delta$

Choose $\delta = \varepsilon$

suppose $\left| \frac{x^2-9}{x-3} - 6 \right|$

$= |x+3-6|$

$= |x-3|$

$< \delta$

$< \varepsilon$

continuity of a function:

if $\lim_{x \to c} f(x) = f(c)$ then $f(x)$ will be continuous at $x = c$.

$\varepsilon > 0 \to$ we assume $0 < |f(x) - f(c)| < \varepsilon$

$\delta > 0 \to$ we show $0 < |x - c| < \delta \to$ it implies $\to |f(x) - f(c)| < \varepsilon$

if $\lim_{x \to c^-} f(x) \neq \lim_{x \to c^+} f(x)$ but both exists, this is jump discontinuity

if $\lim_{x \to c^-} f(x)$ & $\lim_{x \to c^+} f(x)$ do not exist or tends to $\pm\infty$ it's called peak discontinuity

if $\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) \neq f(c)$ it's called hole discontinuity.

Q. What value of 'a' is the function $f(x) = \begin{cases} x^n - 1 & x < 3 \\ 2ax & x \geq 3 \end{cases}$ continues everywhere.

$$\lim_{x \to 3^-} (x^n - 1) = \lim_{x \to 3^+} 2ax$$

$$3^n - 1 = 2 \times a \times 3$$

$$a = \frac{3^n - 1}{6} \text{ (Ans)}$$

Q. A function $f(x)$ defined as,

$$f(x) \begin{cases} -2\sin x & -\pi \leq x < -\frac{\pi}{2} \\ a\sin x + b & -\frac{\pi}{2} \leq x < \frac{\pi}{2} \\ \cos x & \frac{\pi}{2} \leq x < \pi \end{cases}$$

— this function is continuous on $[-\pi, \pi]$ find a, b

$$\lim_{x \to (-\frac{\pi}{2})^-} (-2\sin x) = \lim_{x \to (-\frac{\pi}{2})^+} a\sin x + b$$

$$-2\sin\left(-\frac{\pi}{2}\right) = a\{\sin\left(-\frac{\pi}{2}\right) + b$$

$$2 = b - a \quad \text{—— (1)}$$

$$\lim_{x\to \frac{\pi}{2}^-} a\sin x+b = \lim_{x\to\frac{\pi}{2}^+} \cos x$$

$$\Rightarrow a\sin\left(\frac{\pi}{2}\right)+b = \cos\frac{\pi}{2}$$

$$\Rightarrow a+b = 0 \quad \text{——— (ii)}$$

(i) + (ii)

$$\begin{array}{r} b-a = 2 \\ a+b = 0 \\ \hline 2b = 2 \end{array}$$

$\therefore\ a = -1,\ b = 1$ (Ans)

$$\Rightarrow b = 1$$

$$\therefore 1-a = 2$$

$$\Rightarrow a = -1$$

Q $f(x) \begin{cases} 5x-4 & 0<x<1 \\ 4x^2-3x & 1\le x<2 \\ 3x+4 & x\ge 2 \end{cases}$ Examine the continuity $x=1$

$\lim_{x\to 1^-} f(x)$	$\lim_{x\to 1^+} f(x)$
$= \lim_{x\to 1^-} (5x-4)$	$\cancel{\neq} \lim_{x\to 1^+} (4x^2-3x)$
$= 5\times(1)-4$	$\cancel{\neq}\ 4\times 1^2 - 3\times 1$
$= 1$	$= 1$

$\therefore f(x)$ is continuous on the point $x=1$.

Q. Show that $f(x) = |x-1|$ is continuous at $x=1$

Given,

$$f(x) = |x-1|$$

$$\therefore f(x) = \begin{cases} -x+1 & x < 1 \\ x-1 & x \geq 1 \end{cases}$$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} (-x+1)$$

$$= 0$$

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (x-1)$$

$$= 0$$

$$f(1) = 1-1$$

$$= 0$$

$\therefore \lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$

hense the function $f(x)$ is continuous on the point $x = 1$

Q $f(x) = \begin{cases} x^2 \sin(1/x) , & x \neq 0 \\ 0 , & x = 0 \end{cases}$ S.T. it's continuous at $x=0$.

$$\lim_{x\to 0^-} f(x)$$

$$= \lim_{x\to 0^-} x^2 \sin 1/x$$

$$= 0$$

$$\lim_{x\to 0^+} \cancel{x^2 \sin} f(x)$$

$$= \lim_{x\to 0^+} x^2 \sin 1/x$$

$$= 0$$

$$f(0)$$

$$= 0$$

this are oscilatting so do not matter

$$\therefore \lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0)$$

$\therefore f(x)$ is continuous at $x=0$.

Q Given, $f(x) = \begin{cases} \sin\frac{1}{x} + 2x - [x], & x \neq 0 \\ 0, & x = 0 \end{cases}$

$\therefore \; f(x) \begin{cases} \sin\frac{1}{x} + 2x + 1, & -1 \le x < 0 \\ \sin\frac{1}{x} + 2x & 0 \le x \le 1 \\ 0 & x = 0 \end{cases}$

$\cancel{\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} \sin\frac{1}{x} + 2x + 1 = \in}$

when $x \to 0^-$,

$-1 \le \sin\frac{1}{x} \le +1$, as sin itself is a oscilating fn.

so,

$-1+1 \le \sin\frac{1}{x} + 1 \le 2$

$0 \le \sin\frac{1}{x} + 1 \le 2$

See $\sin\frac{1}{x}$ will oscilate between -1 to $+1$ as $x \to 0$ and for $x < 0$ $\lfloor x \rfloor = -1$ and for $x > 0$ $\lfloor x \rfloor = 0$ so it's not continuous fn.

Indeterminant form

$\frac{0}{0}, \frac{\infty}{\infty}, \infty^0, 0^0, \infty-\infty, 1^\infty, 0\times\infty$

$\lim_{x\to 2} \frac{x^2-4}{x-2} = \lim_{x\to 2}(x+2)$

$\frac{0}{0}$ form $= 4$

L'Hopitals Rule

$\lim_{x\to 2} \frac{x^2-4}{x-2}$

$= \lim_{x\to 2} \frac{2x-0}{1-0}$

$= 4$

$\lim_{x\to 2 \to a} \frac{x^2-4 \to f(a)=0}{x-2 \to g(a)=0}$

$\lim_{x\to 2 \to a} \frac{f'(a)\neq 0}{g'(a)\neq 0} = \frac{2x}{1}$

Let,

$f,g:[a,b]\to \mathbb{R}$ S.T.

$f(a)=g(a)=0$

$g(x)\neq 0$ when $a<x<b$

f,g both are diffable at $x=a$ and $g(a)$

then $\lim_{x\to a}\frac{f(x)}{g(x)}$ exists and $f(a)\neq 0$ is equal to $\frac{f'(a)}{g'(a)}$

Q. $\lim_{x\to 0} \frac{\tan x - x}{x-\sin x}$ $\left(\equiv \frac{0}{0} \text{ form}\right)$

$= \lim_{x\to 0}\frac{\sec^2 x - 1}{1-\cos x}$ $(\equiv$ " $)$ by LH

$= \lim_{x\to 0}\frac{2\sec^2 x\tan x}{\sin x}$ $(\equiv$ " $)$

$= 2\lim_{x\to 0}\frac{2\sec^2 x\tan^2 x+\sec^4 x}{\cos x}$

$= 2\times 1$

$= 2$

Q. $\lim\limits_{x\to 0} \dfrac{e^x - e^{\sin x}}{x - \sin x}$ $\left(\equiv \dfrac{0}{0} \text{ form}\right)$

$\lim\limits_{x\to 0} \dfrac{e^x - e^{\sin x}\cdot \cancel{\sin} \cos x}{1-\cos x}$ $(\equiv 4)$ by L-

$\lim\limits_{x\to 0} \dfrac{e^x + e^{\sin x}\sin x - \cos^2 x\, e^{\sin x}}{\sin x}$

$\lim\limits_{x\to 0} \dfrac{e^x + e^{\sin x}\cdot \sin x\cdot\cos x + \cancel{\sin x}\, e^{\sin x}\cos x + e^{\sin x}\, 2\cos x - e^{\sin x}\cos^2 x}{\cos x}$

$= \cancel{8} - 1$

Q. $\lim\limits_{x\to 0} \dfrac{e^x - e^{-x} - 2x}{x - \sin x} \equiv \dfrac{0}{0}$

$= \lim\limits_{x\to 0} \dfrac{e^x + e^{-x} - 2}{1-\cos x}$

$= \lim\limits_{x\to 0} \dfrac{e^x - e^{-x}}{\sin x}$

$= \lim\limits_{x\to 0} \dfrac{e^x + e^{-x}}{\cos x}$

$= \cancel{8}\ \dfrac{2}{1}$

Q $\lim_{x\to 1} (1-x)\tan\frac{\pi x}{2}$ $\left(\equiv 0\times\infty \text{ form}\right]$

$= \lim_{x\to 1} \frac{(1-x)(\sin\frac{\pi x}{2})}{\cos(\frac{\pi x}{2})}$ $\left(\equiv \frac{0}{0} \text{ form}\right)$

$= \lim_{x\to 1} \frac{\cancel{(-1)}^{(1-x)} \cos\frac{\pi x}{2}\times\frac{\pi}{2} \oplus -\cdot\sin\frac{\pi x}{2}}{\cancel{+}\sin(\frac{\pi x}{2})\times\frac{\pi}{2}}$

$= \cancel{\frac{0}{\pi/2}} \lim_{x\to 1} \frac{(1-x)(\cos\frac{\pi x}{2})\cancel{\frac{\pi}{2}}^{0}}{-\cancel{\sin(\frac{\pi x}{2})}\times\cancel{\frac{\pi}{2}}} - \frac{\cancel{\sin\frac{\pi x}{2}}}{\cancel{\sin\frac{\pi x}{2}}\times\frac{\pi}{2}}$

$=0$

$= \frac{2}{\pi}$ (ans)

Q. $\lim_{x\to 1^+} (1-x)^{\cos\frac{\pi x}{2}} \equiv 0^0 \text{ form}$

~~lim~~ let, $y = (1-x)^{\cos\frac{\pi x}{2}}$

$\ln y = \cos\frac{\pi x}{2}\ln(1-x) \left(\equiv 0\times\infty \text{ form}\right)$

$\lim_{x\to +1} \ln y = \lim_{x\to 1} \cos\frac{\pi x}{2}\ln(1-x)$

$= \lim_{x\to 1} \frac{\ln(1-x)}{\sec\frac{\pi x}{2}}$ $\left(\equiv \frac{\infty}{\infty} \text{ form}\right)$

$\frac{\cancel{+}\frac{1}{1-x}}{\cancel{+}\frac{\sin\frac{\pi x}{2}\times\frac{\pi}{2}}{\cos^2\frac{\pi x}{2}}}$

$= \frac{-\frac{1}{1-x}}{\sec\frac{\pi x}{2}\cdot\tan\frac{\pi x}{2}\times\frac{\pi}{2}}$

$\frac{2}{\pi}\ \frac{\cos^2\frac{\pi x}{2}}{\sin\frac{\pi x}{2} - x\sin\frac{\pi x}{2}}$

$\frac{+1\times(-1)}{(1-x)^2}$

$\frac{\pi}{2}\left(\sec\frac{\pi x}{2}\tan^2\frac{\pi x}{2}\cancel{\frac{\pi}{2}} + \tan\sec^3\frac{\pi x}{2}\right)$

$\cancel{\frac{2}{\pi}}\times \frac{2\cos\frac{\pi x}{2}^{0}(-\sin\frac{\pi x}{2})\times\cancel{\frac{\pi}{2}}}{\underbrace{\cos\frac{\pi x}{2}}_{0}\times\frac{\pi}{2} - x\underbrace{\cos\frac{\pi x}{2}}_{0}\times\frac{\pi}{2} - \sin\frac{\pi x}{2}} \to -1$

$=0$

$\ln y = 0 \quad y = e^0 = 1$

$$(x\sec^2 x - 1)\sec^2 x$$

$$\tan^2 x$$

$$= \frac{x\sec^2 x - 1}{\tan x} \Rightarrow \frac{\tan x\,(2x\sec^2 x\tan x + \sec^2 x)}{\tan^2 x}$$

$$= \frac{2x\sec^2 x\tan^2 x + \sec^2 x\tan x + \sec^2 x - x\sec^2 x}{\tan^2 x}$$

$$\lim_{x\to 0} \frac{dx}{x}$$

$$\lim_{x\to x} \frac{\frac{d}{dx}(x)}{x}$$

$$\lim_{x\to 0} \frac{\sec^2 x}{\tan x} - \lim_{x\to 0} \frac{1}{x}$$

$$\lim_{x\to 0} \frac{1+\tan^2 x}{\tan x} - \text{———}$$

$$+ \lim_{x\to 0} \frac{1}{x} + \lim_{x\to 0} \tan x \text{———}$$

$$0$$

$$y = e^0$$

$$= 1$$

$$\lim_{x\to 0}\left(\frac{2^x+3^x}{2}\right)^{1/x} \equiv 1^\infty \text{ for}$$

$$y = \frac{1}{x}\left\{\ln\left(2^x+3^x\right) - \ln(2)\right\}$$

$$\frac{2^x\ln 2 + 3^x\ln 3}{2^x+3^x} \qquad \cancel{\ln(2)\lim_{x\to 0}\frac{1}{x}}$$

$$e^{\frac{\ln 2+\ln 3=\ln(6)}{2}} \quad \cancel{e}\ \cancel{e} \qquad \frac{\sqrt{6}}{\cancel{e^2}} \qquad \sqrt{e^{\ln 6}} = \sqrt{6}\ (\text{Ans})$$

Determine the value of a, b such that

$$\lim_{x\to 0}\frac{x(1+a\cos x)-b\sin x}{x^3} = 1 \quad \left(\equiv \frac{0}{0}\text{ form}\right)$$

$$\cancel{\lim_{x\to 0}\frac{x(1+}{}}$$

$$\lim_{x\to 0}\frac{(1+a\cos x)+x(-a\sin x)-b\cos x}{3x^2}$$

$$\lim_{x\to 0}\frac{1+a\cancel{\cos x}^{\,1+a} - b\cancel{\cos x}^{\,1-b} - a\cancel{x\sin x}^{\,0}}{3x^2}$$

here denominator $3x^2 \to 0$ thus we must have nominator $\to 0$ as $x \to 0$ since the limit exists finitely.

$$1+a-b=0$$

$$\text{thus } \lim_{x\to 0}\frac{1+a\cos x-b\cos x-ax\sin x}{3x^2} \quad \left(\equiv \frac{0}{0}\text{ form}\right)$$

$$= \frac{-\cancel{a\sin x}^{\,0}+\cancel{b\sin x}^{\,0}-\cancel{a\sin x}^{\,0}-\cancel{ax\cos x}^{\,-ax}}{6x \to 0}$$

$$\cancel{-ax=0}$$

$$= \frac{-a\cos x + b\cos x - a\cos x + \underbrace{ax\sin x}_{\to 0} - a\cos x}{6}$$

$$-a + b - a - a = 6$$

$$-3a + b = 6$$

$$1 + a - b = 0$$

$$1 - 2a = 6$$

$$a = -\frac{5}{2}$$

$$\cancel{b = \frac{3}{2}} \quad b = -\frac{3}{2}$$

$$\lim_{x \to 0} \frac{e^{ax} - e^{x} - x}{x^2}$$

$$\frac{e^{ax} \times a - e^{x} - 1}{2x}$$

as denominator $2x \to 0$ so $x \to 0$

$$e^{ax} \times a - e^{x} - 1 \to 0 \text{ as } x \to 0$$

$$a - 2 = 0$$

$$a = 2$$

determine a, b & c if $\lim_{x \to 0} \frac{(a+b\cos x)x - 3c\sin x}{x^5} = 1$

$$\frac{a+b(\cos x - x\sin x) - c\cos x}{5x^4}$$

$a+b-c=0$

$$\frac{-b\sin x - b(x\cos x + \sin x) + c\sin x}{20x^3}$$

$$\frac{-b\cos x - b\cos x - b(-x\sin x + \cos x) + c\cos x}{60x^2}$$

$-3b+c=0$

$$\frac{+b\sin x + b\sin x + b(x\cos x + \sin x) + b\sin x - c\sin x}{120 \cdot x}$$

$b\cos \quad + b\cos \quad + b(\cos x + b\cos x - c\cos x = 120$

$5b - c = 120$

$2b = 120$

$b = 60$

$c = 180$

$a = 120$

If α, β be the roots of the eq^n $ax^2+bx+c=0$ then show that $\lim\limits_{x\to\alpha} \frac{1-\cos(ax^2+bx+c)}{(x-\alpha)^2} = \frac{1}{2}a^2(\alpha-\beta)^2$

A/Q,

$$\alpha = \frac{-b \pm \sqrt{b^2-4ac}}{2a}$$

$$\beta = \frac{-b-\sqrt{b^2-4ac}}{2a}$$

$$\alpha-\beta = \frac{-\cancel{b}+\sqrt{--} + \cancel{b}+\sqrt{-}}{2a}$$

$$= \frac{\pm\sqrt{b^2-4ac}}{a} \quad \left[\pm \text{ bcz if } \begin{matrix}\alpha = --\\ \beta = --\end{matrix}\right]$$

$$(\alpha-\beta)^2 = \frac{b^2-4ac}{a}$$

$$R.H.S = \frac{1}{2}a^2(\alpha-\beta)^2$$

$$= \frac{1}{2}a^2 \frac{b^2-4ac}{\cancel{a}}$$

$$= \cancel{\frac{ab^2-4a^2c}{2}} \quad \frac{b^2-4ac}{2} \text{ ---- (i)}$$

$$L.H.S = \lim_{x\to\alpha} \frac{1-\cos(ax^2+bx+c)}{(x-\alpha)^2} \quad \left(\equiv \frac{0}{0} \text{ form}\right)$$

by appl L ~

$$= \lim_{x\to\alpha} \frac{\sin(ax^2+bx+c)\times(2ax+b)}{2(x-\alpha)\times 1} \quad (\equiv \text{ u })$$

$$= \lim_{x\to\alpha} = \frac{\cos(ax^2 \sim)\times(2ax+b)^2 + \cancel{\sin(\cdots)}\times 2a}{3}$$

$$= \frac{(2ax+b)}{2}$$

$$= \left\{ \frac{2a\cancel{a}\left(\frac{-b \pm \sqrt{b^2-4ac}}{2\cancel{a}}\right)+b}{2} \right\}$$

$$= \frac{2\cancel{a}\left(\pm\sqrt{b^2-4ac}\right)}{2}$$

$$= \frac{}{2a}$$

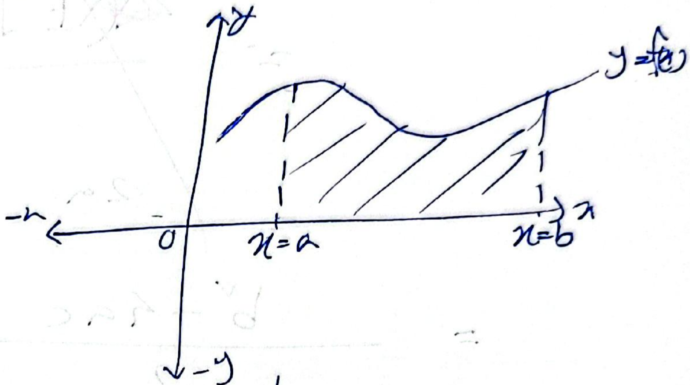
$$= \frac{b^2-4ac}{2} \quad (h) \quad (n = d\,1)$$

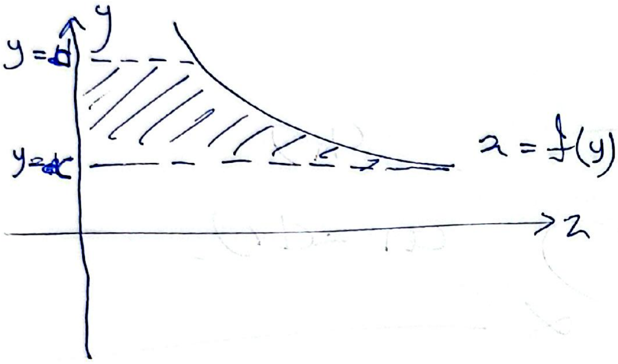
$$= \cdot$$

Application of Integral Calculus (Area Under a curve):

Quadrature: process of finding the are bounded by the region of a plane curve is called quadrature.

Area $= \int_a^b y\,dx$

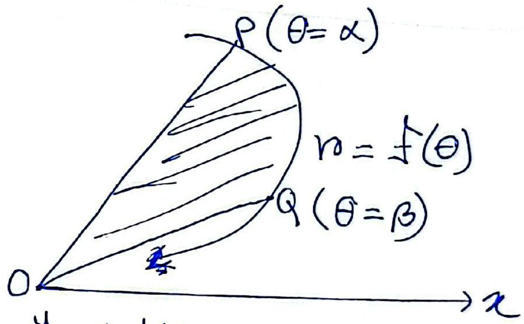




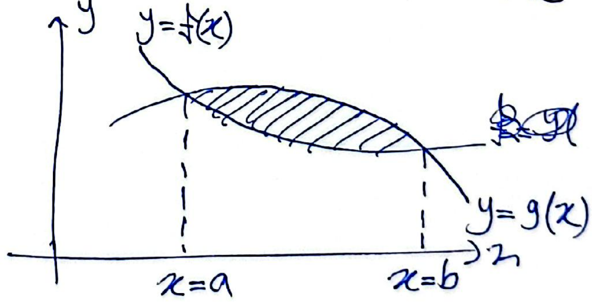
Area $= \int_c^d x\,dy$.

polar curve

Area $= \frac{1}{2}\int_\alpha^\beta r^2\,d\theta$

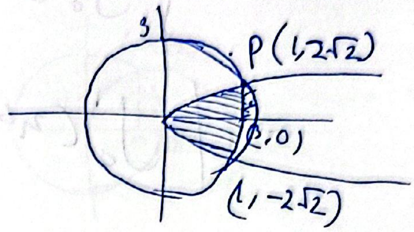


Area bet^n two curves



area $= \int_a^b \{g(x) - f(x)\}\,dx$

Q. F.T. area of the smaller portion enclosed by the curves $x^2+y^2=9$, $y^2=8x$



$x^2+8x=9$

$\cancel{x(x+8)=9}$

$x^2+8x-9=0$

$x^2+(9-1)x-9=0$

$x(x+9)-1(x+9)=0$

$x=1,\ x=-9$

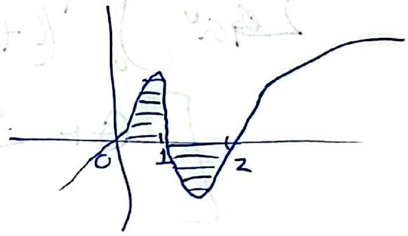
$y=\pm 2\sqrt{2}$

$$\text{area} = \int_0^1 \sqrt{8x}\,dx + \int_1^3 \sqrt{9-x^2}\,dx$$

$$= 2\sqrt{2}\left[\frac{x^{3/2}}{3/2}\right]_0^1 + \left[\frac{x}{2}\sqrt{9-x^2}+\frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right)\right]_1^3$$

$$= \frac{4\sqrt{2}}{3} + \frac{9\pi}{4} - \frac{\sqrt{8}}{2} - \frac{9}{2}\sin^{-1}\frac{1}{3}$$

Q. F.T. A of the region bounded by $y=x(x-1)(x-2)$ and x axis



Given,

$y=x(x-1)(x-2)$

$= x^3-3x^2+2x$

if $y=0$, $x=1$ or $x=2$, or $x=0$

So the given curve meets x axis at $(0,0),(1,0),(2,0)$

Now, $\frac{dy}{dx} = 3x^2-6x+2$

$\frac{d^2y}{dx^2} = 6x-6$

$= 6(x-1)$

When, $\frac{d^2y}{dx^2} > 0$ if $x>1$

" <0 " $x<0$

" $=0$ " $x=1$

$\frac{d^3y}{dx^3} = 6 \neq 0$

So the curve is conclave upwards in $[0,1]$ & downwards $[1,2]$ and has a point of intersection at $(1,0)$

again $y<0$ for $x<0$

and $y\to-\infty$ as $x\to\infty$

$y>0 \to$ $0<x<1$

$y<0$ $1<x<2$

$x>2, y>0$ and $y\to\infty$ as $x\to\infty$

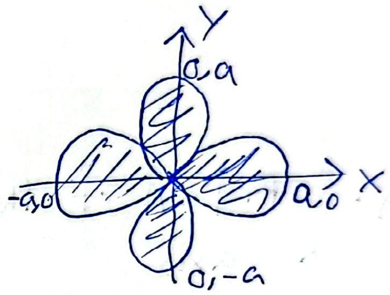
$$\text{Area} = \left|\int_0^1 y\,dx\right| + \left|\int_1^2 y\,dx\right|$$

$$= \left|\int_0^1 (x^3-3x^2+2x)\,dx\right| + \left|\int_1^2 x^3-3x^2+2x\,dx\right|$$

$$= \frac{1}{2} \text{ sq unit}$$

F.T. entire area enclosed by the curve $r = a\cos 2\theta$

$$8\times \int_0^{\pi/4} \frac{1}{2} r^2\,d\theta$$



$$8\times\frac{1}{2}\int_0^{\pi/4} a^2 \cos^2 2\theta$$

$$2a^2 \int_0^{\pi/4} (1+\cos 4\theta)\,d\theta$$

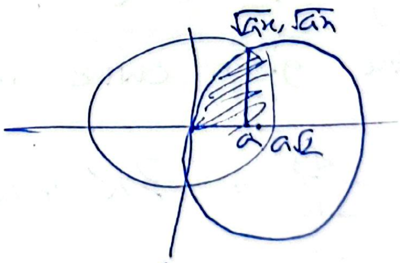
$$\left[\theta + \frac{\sin 4\theta}{4}\right]_0^{\pi/4}$$

F.T. Area common to the circles $r = a\sqrt{2}$ & $r = 2a\cos\theta$

$a\sqrt{2} = 2a\cos\theta$

$\cos\theta = \frac{1}{\sqrt{2}}$

$\theta = \frac{\pi}{4}$



$x^2+y^2 = r^2$

$x^2+y^2 = 2a^2$

parametric form of circle $\quad 2\times \int_0^a$

$x = r\cos\theta \qquad y = r\sin\theta$

$\frac{x}{r} = \cos\theta \qquad y =$

$(x-a)^2+y^2 = a^2$

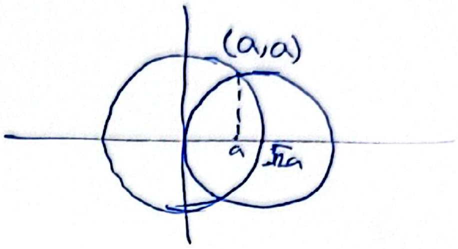
$r = 2a\times\frac{x}{r}$

$r^2 = 2ax$

$\Rightarrow x^2+y^2 = 2ax$

point of intersection of these two circles

$$\text{(i)} - \text{(ii)} \Rightarrow (a, a)$$



$$2\times\left[\int_0^a \sqrt{\cancel{2ax-x^2}\; a^2-(x-a)^2} + \int_a^{\sqrt{2}a}\sqrt{2a^2-x^2}\right]$$

$$= 2\times\left[\left[\frac{x-a}{2}\sqrt{\cancel{2ax}\,a^2-(x-a)^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x-a}{a}\right)\right]_0^a + \left[\frac{x}{2}\sqrt{2a^2-x^2} + \frac{2a^2}{2}\sin^{-1}\left(\frac{x}{\sqrt{2}a}\right)\right]_a^{\sqrt{2}a}\right]$$

$$= 2\times\left[\left[\cancel{\frac{a}{\sqrt{2}}\sqrt{0}} + \frac{2a^2}{2}\frac{\pi}{2}\cancel{\sin^{-1}(1)}\right) + \frac{a^2}{2}\times\frac{\pi}{2} - \frac{a^2}{2} - a^2\frac{\pi}{4}$$

$$= \cancel{2a^2}$$

$$= a^2(\pi - 1) \text{ sq unit}$$

Vector Analysis

scalar triple product or box product:

$$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

$$\vec{b} = b_2\hat{i} + b_2\hat{j} + b_3\hat{k}$$

$$\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + a_3b_3 = |\vec{a}||\vec{b}|\cos\theta$$

now,

$$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$$

$$\vec{a}\cdot(\vec{b}\times\vec{c}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = [\vec{a}\ \vec{b}\ \vec{c}]$$

= scalar quantity

property

(i) $\vec{a}\cdot(\vec{b}\times\vec{c}) = -a\cdot(\vec{c}\times\vec{b}) = b\ (\vec{c}\times\vec{a})$

ie, $[\vec{a}\ \vec{b}\ \vec{c}] = -[\vec{a}\ \vec{c}\ \vec{b}] = [\vec{b}\ \vec{c}\ \vec{a}]$

(i) $\vec{a}\cdot(\vec{b}\times\vec{c}) = \cancel{\vec{a}\times(\vec{b})\times\vec{c}}\ (\vec{a}\times\vec{b})\cdot\vec{c}$

(iii) $[\vec{a}\ \vec{b}\ \vec{c}] = 0 \Rightarrow$ $\vec{a}, \vec{b}$ & $\vec{c}$ are ~~colinear~~ coplaner and two of them are equal or colinear.

$$\boxed{\text{remember } [\vec{a}\ \vec{b}\ \vec{b}] = 0 \text{ as } \vec{a}, \vec{b} \text{ will always colinear}}$$

Vector Triple Product :

Let, $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$

$\vec{c} = c_1\hat{i} + c_2\hat{j} + c_3\hat{k}$

$\vec{a} \times (\vec{b} \times \vec{c})$

দুয়ের element এর মধ্যে কোনটা ∵ প্রথমটি হবে।

$$= \underbrace{(\vec{a}\cdot\vec{c})\vec{b}} - (\vec{a}\cdot b)\vec{c}$$

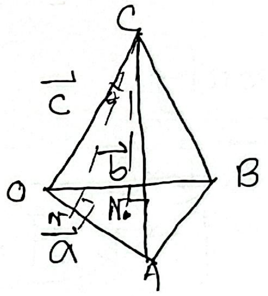
multiplication of scalar $(\vec{a}\cdot\vec{c})$ with vector $\vec{b}$

Theorem

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b}) \times \vec{c}$$

$$(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{b}\cdot\vec{c})\vec{a}$$

Volume of Tetrahedron



$OA = \vec{a}$
$OC = \vec{c}$
$OB = \vec{b}$

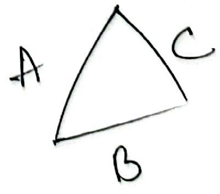
$CN \perp$ to the plane OAB

$$= \frac{1}{3}(\text{Area of triangle } \Delta OAB) \times (\text{height of } CN)$$

$$= \frac{1}{6}\left|\overline{OA} \times \overline{OB}\right| |OC| \cos\theta$$

~~$= \frac{1}{6}(\vec{a} \times \vec{b}) \cdot OC \cos\theta$~~

bcz the area of triangle $= \frac{1}{2}|A \times B|$



$$= \frac{1}{6}\, \vec{a} \times (\vec{b} \cdot \vec{c})$$

$$= \frac{1}{6}\left[\vec{a}\ \vec{b}\ \vec{c}\right]$$

Q. Show that the four points $A(2,1,4)$, $B(3,-1,7)$, $C(0,4,0)$ and $D(2,0,6)$ are coplaner.

Let's asume $O \equiv (0,0,0)$ is the origin

~~So the positions vectors of A, B, C & D are —~~

~~$OA = 2\hat{i}+\hat{j}+4\hat{k}$; $OB = 3\hat{i}-\hat{j}+7\hat{k}$; OC~~

~~So~~ Let,

~~$\vec{AB}$ = vector which with initia~~

~~$\vec{AB} = \vec{B}-\vec{A}$~~

$\vec{AB}$ = " with initial point A and final point B

$\vec{AC}$ = " " " " A " " C

$\vec{AD}$ = " " " " A " " D

So, $\vec{AB} = \hat{i} - 2\hat{j} + 3\hat{k}$

$\vec{AC} = -2\hat{i} + 3\hat{j} - 4\hat{k}$

$\vec{AD} = -\hat{j} + 2\hat{k}$

~~So,~~ if A, B, C & D are coplaner then $\vec{AB}$, $\vec{AC}$ & $\vec{AD}$ shall also be "

So, $[\vec{AB}\ \vec{AC}\ \vec{AD}]$ should be 0

Now, $[\vec{AB}\ \vec{AC}\ \vec{AD}]$

$$= \cancel{(AB \times AC).AD} \begin{vmatrix} 1 & -2 & +3 \\ -2 & +3 & -4 \\ 0 & -1 & +2 \end{vmatrix}$$

$= 1(6-4)+2(-4)+3(2)$

$= 2-8+6$

$= -6+6$

$= 0$

So, it proves that $\vec{AB}$ $\vec{AC}$ & $\vec{AD}$ are coplanear and A, B, C & D are also " " ( )

Q. For any vector $\vec{a}$ P.T. $\hat{i}\times(\vec{a}\times\hat{i})+\hat{j}\times(\vec{a}\times\hat{j})+\hat{k}\times(\vec{a}\times\hat{k})=2\vec{a}$

~~So~~, Let, $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$

$L.H.S = \hat{i}\times(a\times\hat{i})+\hat{j}\times(\vec{a}\times\hat{j})+\hat{k}\times(\vec{a}\times\hat{k})$

$= \cancel{(\hat{i}\cdot\hat{i})\vec{a}-(\hat{i}\cdot\vec{a})\times\hat{i}+}$

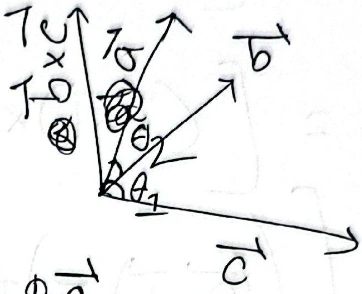
$= (\hat{i}\cdot\hat{i})\vec{a}-(\hat{i}\cdot\vec{a})\hat{i}+(\hat{j}\cdot\hat{j})\vec{a}-(\hat{j}\cdot\vec{a})\hat{j}+(\hat{k}\cdot\hat{k})\cdot\vec{a} - (\hat{k}\cdot\vec{a})\hat{i}$

$= 3\vec{a} - (\hat{i}\cdot(a_1\hat{i}+a_2\hat{j}+a_3\hat{k}))\hat{i}$
$- (\hat{j}(a_1\hat{j}+a_2\hat{j}+a_3\hat{k}))\hat{j}$
$- (\hat{k}(a_1\hat{i}+a_2\hat{j}+a_3\hat{k}))\hat{k}$

$= 3\vec{a} - a_1\hat{i}-a_2\hat{j}-a_3\hat{k}$

$= 2\vec{a} = R.H.S$

Q. If $\vec{a}, \vec{b}, \vec{c}$ three unit vector s.T. $\vec{a} \times (\vec{b} \times \vec{c}) = \frac{1}{2}\vec{b}$ Find the angles which $\vec{a}$ makes with $\vec{b}$ and $\vec{c}$ where $\vec{b}$ and $\vec{c}$ are non-~~polar~~ parallel.



We know,

$$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{b})\vec{c}$$

$$= \left(|\vec{a}||\vec{c}|\cos\theta_1\right)\vec{b} - \left[|\vec{a}||\vec{b}|\cos\theta_2\right]\vec{c}$$

~~So this expression~~

$$= \frac{1}{2}\vec{b} \quad \left[\text{Given}\right]$$

so, $|\vec{a}||\vec{c}|\cos\theta_1 = \frac{1}{2} \Rightarrow \cos\theta_1 = \frac{1}{2|\vec{a}||\vec{c}|}$

$|\vec{a}||\vec{b}|\cos\theta_2 = 0 \Rightarrow \cos\theta_2 = 0 \Rightarrow \theta_2 = 90^\circ$

~~now, as $\vec{a}, \vec{b}$ & $\vec{c}$ are vectors and has some magnitude and not a zero vector.~~

~~so, $\cos\theta_2$~~

$$\theta_1 = \cos^{-1}\left(\frac{1}{2|\vec{a}||\vec{c}|}\right) = \cos^{-1}\left(\frac{1}{2}\right) = \frac{\pi}{3}$$ as $\vec{a}, \vec{c}$ are unit vec

$\theta_2 = 90^\circ$

θ_1 is the angle betwen $\vec{a}$ & $\vec{c}$

θ_2 — — — $\vec{a}$ & $\vec{b}$

Q. $[\vec{a}\times\vec{b} \quad \vec{b}\times\vec{c} \quad \vec{c}\times\vec{a}] = [\vec{a}\,\vec{b}\,\vec{c}]^2$

$LHS = [\vec{a}\times\vec{b} \quad \vec{b}\times\vec{c} \quad \vec{c}\times\vec{a}]$

$= [(\vec{a}\times\vec{b})\times(\vec{b}\times\vec{c})]\cdot(\vec{c}\times\vec{a})$

$= \left[\left((\vec{a}\times\vec{b})\cdot\vec{c}\right)\vec{b} - \left((\vec{a}\times\vec{b})\cdot\vec{b}\right)\vec{c}\right]\cdot(\vec{c}\times\vec{a})$

$= \left[[\vec{a}\,\vec{b}\,\vec{c}]\vec{b} - [\vec{a}\,\vec{b}\,\vec{b}]\vec{c}\right]\cdot(\vec{c}\times\vec{a})$

$= [\vec{a}\,\vec{b}\,\vec{c}]\;\vec{b}\cdot(\vec{c}\times\vec{a})$

$= [\vec{a}\,\vec{b}\,\vec{c}]\,[\vec{b}\,\vec{c}\,\vec{a}]$

$= [\vec{a}\,\vec{b}\,\vec{c}]\,[\vec{a}\,\vec{b}\,\vec{c}]$

$= [\vec{a}\,\vec{b}\,\vec{c}]^2$

$= R.H.S$ (Proved)

Q. P.T. $\left[\vec{\alpha}+\vec{\beta} \quad \vec{\beta}+\vec{\gamma} \quad \vec{\gamma}+\vec{\alpha}\right] = 2\left[\vec{\alpha} \ \vec{\beta} \ \vec{\gamma}\right]$

$$L.H.S = \left[\vec{\alpha}+\vec{\beta} \quad \vec{\beta}+\vec{\gamma} \quad \vec{\gamma}+\vec{\alpha}\right]$$

$$= (\vec{\alpha}+\vec{\beta}) \cdot \left[(\vec{\beta}+\vec{\gamma}) \times (\vec{\gamma}+\vec{\alpha})\right]$$

$$= (\vec{\alpha}+\vec{\beta}) \cdot \left[(\vec{\beta}+\vec{\gamma}) \times \vec{\gamma} + (\vec{\beta}+\vec{\gamma}) \times \vec{\alpha}\right]$$

$$= (\vec{\alpha}+\vec{\beta}) \cdot \left[\vec{\beta}\times\vec{\gamma} + \vec{\gamma}\times\vec{\gamma} + \vec{\alpha}\times\vec{\beta} + \vec{\gamma}\times\vec{\alpha}\right]$$

$$= \vec{\alpha}\cdot(\vec{\beta}\times\vec{\gamma}) + \vec{\alpha}\cdot(\vec{\alpha}\times\vec{\beta}) + \vec{\alpha}\cdot(\vec{\gamma}\times\vec{\alpha})$$
$$+ \vec{\beta}(\vec{\beta}\times\vec{\gamma}) + \vec{\beta}(\vec{\alpha}\times\vec{\beta}) + \vec{\beta}(\vec{\gamma}\times\vec{\alpha})$$

$$= \left[\vec{\alpha} \ \vec{\beta} \ \vec{\gamma}\right] + \left[\vec{\alpha} \ \vec{\alpha} \ \vec{\beta}\right] + \left[\vec{\alpha} \ \vec{\gamma} \ \vec{\alpha}\right] + \left[\vec{\beta} \cdot \vec{\beta} \ \vec{\gamma}\right]$$
$$+ \left[\vec{\beta} \ \vec{\alpha} \ \vec{\beta}\right] + \left[\vec{\beta} \ \vec{\gamma} \ \vec{\alpha}\right]$$

$$= 2\left[\vec{\alpha} \ \vec{\beta} \ \vec{\gamma}\right]$$

$$= R.H.S$$

Q

~~According~~

A/Q $\frac{1}{6}\left[\vec{AB}\ \vec{AC}\ \vec{AP}\right] = 5$

$\Rightarrow \left[\vec{AB}\ \vec{AC}\ \vec{AP}\right] = 30$

$\Rightarrow \cancel{6\hat{i}}\left(\vec{AB}\times\vec{AC}\right)\cdot\vec{AP} = 30$

$\Rightarrow (x-3)\times(-2)+(y-2)(-6)+(z-1)(+6) = 30$

$\Rightarrow \cancel{+6}-2x-6y+12+6z-\cancel{6} = 30$

$\Rightarrow -2x-6y+6z = 18$

$\boxed{x+3y-3z+9=0}$ → this is the locus

Let $P = x\hat{i}+y\hat{j}+z\hat{k}$

$\vec{AB} = -6\hat{i}-2\hat{j}-4\hat{k}$

$\vec{AC} = -3\hat{i}-2\hat{j}-3\hat{k}$

$$\vec{AB}\times\vec{AC} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -6 & -2 & -4 \\ -3 & -2 & -3 \end{vmatrix}$$

$= \hat{i}(-2)+\hat{j}(-6)+\hat{k}(6$

$= -2\hat{i}-6\hat{j}+6\hat{k}$

$AP = (x-3)\hat{i}+(y-2)\hat{j}+(z-1)\hat{k}$

Q. P.T. $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = [\vec{a}\ \vec{c}\ \vec{d}]\vec{b}-[\vec{b}\ \vec{c}\ \vec{d}]\vec{a}$

$= [\vec{a}\ \vec{b}\ \vec{d}]\vec{c}-[\vec{a}\ \vec{b}\ \vec{c}]\vec{d}$

L.H.S $= (\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})$

$= \left[(\vec{a}\times\vec{b})\cdot\vec{d}\right]\vec{c} - \left[(\vec{a}\times\vec{b})\cdot\vec{c}\right]\vec{d}$

$= [a\ b\ d]\vec{c} - [a\ b\ c]\vec{d}$

$=$ R.H.S

Q. P.T. $[\vec{a}\times\vec{b} \;\; \vec{c}\times\vec{d} \;\; \vec{e}\times\vec{f}] = [\vec{a}\;\vec{b}\;\vec{c}][\vec{c}\;\vec{d}\;\vec{f}] - [\vec{a}\;\vec{b}\;\vec{f}][\vec{c}\;\vec{d}\;\vec{e}]$

$$L.H.S = [\vec{a}\times\vec{b} \;\; \vec{c}\times\vec{d} \;\; \vec{e}\times\vec{f}]$$

$$= (\vec{a}\times\vec{b})\cdot\{(\vec{c}\times\vec{d})\times(\vec{e}\times\vec{f})\}$$

$$= (\vec{a}\times\vec{b})\cdot\left[\{(\vec{c}\times\vec{d})\cdot\vec{f}\;\vec{e}\} - \{(\vec{c}\times\vec{d})\cdot\vec{e}\;\vec{f}\}\right]$$

$$= (\vec{a}\times\vec{b})\{[\vec{f}\;\vec{c}\;\vec{d}]\vec{e} - [\vec{e}\;\vec{c}\;\vec{d}]\vec{f}\}$$

$$= \vec{e}(\vec{a}\times\vec{b})[\vec{f}\;\vec{c}\;\vec{d}] - \vec{f}(\vec{a}\times\vec{b})[\vec{e}\;\vec{c}\;\vec{d}]$$

$$= [\vec{e}\;\vec{a}\;\vec{b}][\vec{f}\;\vec{c}\;\vec{d}] - [\vec{f}\;\vec{a}\;\vec{b}][\vec{e}\;\vec{c}\;\vec{d}]$$

$$= [\vec{a}\;\vec{b}\;\vec{e}][\vec{c}\;\vec{d}\;\vec{f}] - [\vec{a}\;\vec{b}\;\vec{f}][\vec{c}\;\vec{d}\;\vec{e}]$$

$$= R.H.S \quad [P-E]$$

Q. S.T the four points whose position vector is $\vec{\alpha}, \vec{\beta}, \vec{\gamma}, \vec{\delta}$ are coplanar iff $[\vec{\alpha}\;\vec{\beta}\;\vec{\gamma}] = [\vec{\beta}\;\vec{\gamma}\;\vec{\delta}] + [\vec{\gamma}\;\vec{\delta}\;\vec{\alpha}] + [\vec{\alpha}\;\vec{\beta}\;\vec{\delta}]$

~~L.H.S = $[\vec{\alpha}\;\vec{\beta}\;\vec{\gamma}]$~~

as $\vec{\alpha}, \vec{\beta}, \vec{\gamma}, \vec{\delta}$ are coplanar then,

$$[\vec{\alpha}-\vec{\beta},\; \vec{\alpha}-\vec{\gamma},\; \vec{\alpha}-\vec{\delta}] = 0 \qquad [(\vec{\delta}-\vec{\alpha})\cdot(\vec{\delta}-\vec{\beta})(\vec{\delta}-\vec{\gamma})] = 0$$

$$\Rightarrow (\vec{\alpha}-\vec{\beta})\cdot\{(\vec{\alpha}-\vec{\gamma})\times(\vec{\alpha}-\vec{\delta})\} = 0$$

$$\Rightarrow (\vec{\alpha}-\vec{\beta})\cdot\{-(\vec{\alpha}\times\vec{\delta}) - (\vec{\gamma}\times\vec{\alpha}) + (\vec{\gamma}\times\vec{\delta})\} = 0$$

$$\cancel{= -[(\vec{\alpha}-\vec{\beta})\;\vec{\alpha}\;\vec{\delta}] - [(\vec{\alpha}-\vec{\beta})\;\vec{\gamma}\;\vec{\alpha}] + [(\vec{\alpha}-\vec{\beta})\;\vec{\gamma}\;\vec{\delta}] = 0}$$

$$\Rightarrow [\vec{\beta}\;\vec{\alpha}\;\vec{\delta}] + [\vec{\beta}\;\vec{\gamma}\;\vec{\alpha}] + [\vec{\alpha}\;\vec{\gamma}\;\vec{\delta}] = [\vec{\beta}\;\vec{\gamma}\;\vec{\delta}]$$

Q If $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ s.t $\vec{a}+\vec{b}+\vec{c}+\vec{d} = 0$ Then s.t.

$$\frac{|\vec{a}|}{[\hat{\beta}\ \hat{\gamma}\ \hat{\delta}]} = \frac{-|\vec{b}|}{[\hat{\gamma}\ \hat{\delta}\ \hat{\alpha}]} = \frac{|\vec{c}|}{[\hat{\delta}\ \hat{\alpha}\ \hat{\beta}]} = \frac{-|\vec{d}|}{[\hat{\alpha}\ \hat{\beta}\ \hat{\gamma}]}$$, where

$\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta}$ are the unit vectors along $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ respectively.

$$\frac{|\vec{a}|}{[\hat{\beta}\ \hat{\gamma}\ \hat{\delta}]} = -\frac{|\vec{b}|}{[\gamma\ \delta\ \alpha]}$$

$$\Rightarrow \frac{|\vec{a}|}{|\vec{b}|} \times \frac{\hat{\gamma}\cdot(\delta\times\alpha)}{\hat{\beta}\cdot(\gamma\times\delta)} = \frac{|\vec{c}|}{|\vec{d}|} \times \frac{\hat{\alpha}\cdot(\hat{\beta}\times\hat{\gamma})}{\hat{\delta}\cdot(\hat{\alpha}\times\hat{\beta})}$$

$$\Rightarrow \frac{|\vec{a}|\,|\vec{d}|}{|\vec{b}|\,|\vec{c}|} = \frac{\hat{\alpha}\cdot(\hat{\beta}\times\hat{\gamma})\ \cancel{\hat{\beta}\cdot(\hat{\gamma}\times\hat{\delta})}\ \hat{\gamma}\cdot(\hat{\delta}\times\hat{\beta})}{\cancel{\hat{\delta}\cdot(\hat{\alpha}\times\hat{\beta})}\ \hat{\gamma}(\hat{\delta}\times\hat{\alpha})}$$

$$\hat{\alpha}\cdot(\hat{\beta}\times\delta)$$

$$\Rightarrow \frac{|\vec{a}|\,|\vec{d}|}{|\vec{b}|\,|\vec{c}|} = \frac{(\hat{\beta}\times\hat{\gamma})\ (\hat{\delta}\times\hat{\beta})}{}$$

Given, $\vec{a}+\vec{b}+\vec{c}+\vec{d} = \vec{0}$

$$\Rightarrow |\vec{a}|\cdot\hat{\alpha} + |\vec{b}|\cdot\hat{\beta} + |\vec{c}|\hat{\gamma} + |\vec{d}|\hat{\delta} = 0$$

multiplying both side with $(\hat{\gamma}\times\hat{\delta})$

$$|\vec{a}|[\hat{\alpha}\ \hat{\gamma}\ \hat{\delta}] + |\vec{b}|[\hat{\beta}\ \hat{\gamma}\ \hat{\delta}] = 0$$

$$\frac{|\vec{a}|}{[\hat{\beta}\ \hat{\gamma}\ \hat{\delta}]} = -\frac{|\vec{b}|}{[\hat{\gamma}\ \hat{\delta}\ \hat{\alpha}]}$$

Multiplying both side by $(\hat{\alpha}\times\hat{\beta})$

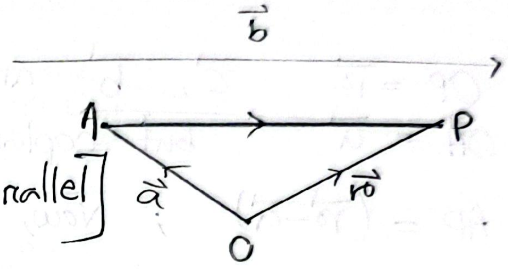
$$|\vec{c}|[\hat{\gamma}\ \hat{\alpha}\ \hat{\beta}] + |\vec{d}|[\hat{\delta}\ \hat{\alpha}\ \hat{\beta}] = 0$$

$$= \frac{\vec{c}}{[\hat{\delta}\ \hat{\alpha}\ \hat{\beta}]} = \frac{-|\vec{d}|}{[\alpha\ \beta\ \gamma]}$$

$[\beta\ \ \alpha]$

Lecture-2 Tapasree Ma'am (Polmaths)

(i) Eq^n of a straight line through a point and parallel to a given vector



$AP = (\vec{r} - \vec{a})$

$(\vec{r} - \vec{a}) \times \vec{b} = 0$ [as they are parallel]

[non parametric form]

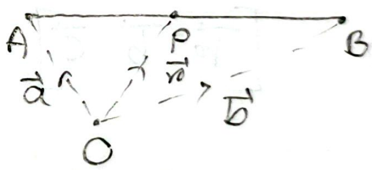
$\vec{r} - \vec{a} = t\, \vec{b}$

$\Rightarrow \vec{r} = \vec{a} + t\vec{b}$ [parametric form]

(ii) Eq^n of st line passing throug two given point

$OA = \vec{a}$, $OB = \vec{b}$, $OP = \vec{r}$

P is an arbitory point.



$AP = (\vec{r} - \vec{a})$

$AB = (\vec{b} - \vec{a})$

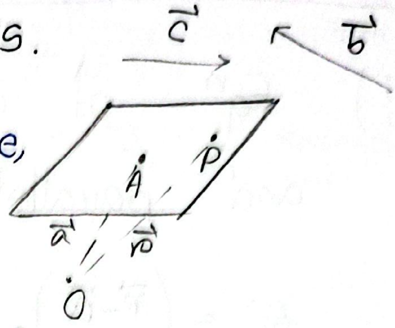
$AP \parallel AB \Rightarrow (\vec{r} - \vec{a}) \times (\vec{b} - \vec{a}) = 0$

[non para]

$(\vec{r} - \vec{a}) = t\,(\vec{b} - \vec{a})$

$\vec{r} = \vec{a} + t\,(\vec{b} - \vec{a})$ [para form]

(III) Eq^n of a plane through a given point and $\parallel$ to two non-collinear vectors.



P is an arbitrary point on the plane,

$OP = \vec{r}$ $\vec{c}, \vec{b}$ are two non-collinear vectors
$OA = \vec{a}$ but coplaner to the plane.

$AP = (\vec{r} - \vec{a})$; Now, AP, $\vec{c}$ & $\vec{b}$ are coplanner,

So, $[\vec{r} - \vec{a} \quad \vec{c} \quad \vec{b}] = 0$

$\Rightarrow (\vec{r} - \vec{a}) \cdot \vec{c} \times \vec{b} = 0$

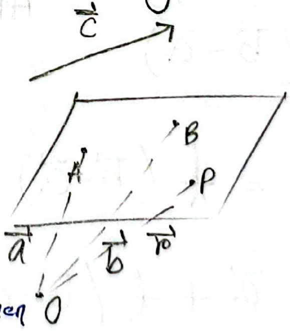
$\Rightarrow (\vec{r} \cdot \vec{c} - \vec{a} \cdot \vec{c}) \times \vec{b}$

$\Rightarrow \vec{r} \cdot \vec{c} \times \vec{b} - \vec{a} \cdot \vec{c} \times \vec{b} = 0$

$\Rightarrow [r \; c \; b] = [a \; c \; b]$

$\Rightarrow [\vec{r} \; \vec{b} \; \vec{c}] = [\vec{a} \; \vec{b} \; \vec{c}]$ [[illegible]]

(IV) Eq^n ~~to~~ of a plane throug two given points and parallel to a given vector.



$AB = (\vec{b} - \vec{a})$

$\cancel{(\vec{b} - \vec{a}) \times \vec{c} = 0}$ [it's not possible as $\vec{c}$ is coplanner not collinear]

$AB = (\vec{b} - \vec{a})$
$AP = (\vec{r} - \vec{a})$ — these two and $\vec{c}$ is coplanner,

$[(\vec{b} - \vec{a}) \quad (\vec{r} - \vec{a}) \quad \vec{c}] = 0$

$$(\vec b-\vec a)\cdot\{(\vec r-\vec a)\times\vec c\}=0$$

$$(\vec b-\vec a)\cdot\{\vec r\times\vec c-\vec a\times\vec c\}=0$$

$$\cancel{\vec b\cdot(\vec r\times\vec c)-\vec b\cdot(\vec a\times\vec c)-\vec a\cdot(\vec r\times\vec c)+\vec a\cdot(\vec a\times\vec c)=0}$$

$$\cancel{[\vec b\ \vec r\ \vec c]-[\vec b\ \vec a\ \vec c]-[\vec a\ \vec r\ \vec c]=0}$$

$$(\vec b-\vec a)\cdot\vec r\times\vec c=(\vec b-\vec a)\cdot\vec a\times\vec c$$

$$[\vec b-\vec a\quad\vec r\quad\vec c]=[(\vec b-\vec a)\quad\vec a\quad\vec c]$$

$$[\vec r\quad(\vec b-\vec a)\quad\vec c]=[\vec a\quad\vec b-\vec a\quad\vec c]$$

$$=\vec a\cdot\{(\vec b-\vec a)\times c\}$$

$$=\vec a\cdot\vec b\times\vec c-\vec a\cdot\vec a\times\vec c$$

$$=[\vec a\ \vec b\ \vec c]$$

$\therefore [\vec r\quad\vec b-\vec a\quad\vec c]=[\vec a\quad\vec b\quad\vec c]$ [প্যারা নাই]

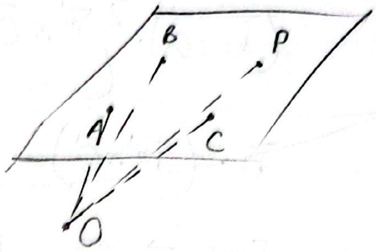
$$(\vec r-\vec a)=s(\vec b-\vec a)+t\vec c$$

$\vec r=\vec a+s(\vec b-\vec a)+t\vec c$ [প্যারা নেওয়া]

(v) Eq^n of a plane through three given points.

$OA = \vec{a}$
$OB = \vec{b}$
$OC = \vec{c}$
$OP = \vec{r}$

$AB = (\vec{b}-\vec{a})$
$AC = (\vec{c}-\vec{a})$
$AP = (\vec{r}-\vec{a})$



$$[(\vec{b}-\vec{a}) \quad (\vec{c}-\vec{a}) \quad (\vec{r}-\vec{a})] = 0$$

$$\Rightarrow (\vec{b}-\vec{a}) \cdot \{(\vec{c}-\vec{a}) \times (\vec{r}-\vec{a})\} = 0$$

$$\Rightarrow (\vec{b}-\vec{a}) \cdot \{\vec{c}\times\vec{r} - \vec{c}\times\vec{a} - \vec{a}\times\vec{r}\} = 0$$

$$\Rightarrow [(\vec{b}-\vec{a}) \ \vec{c} \ \vec{r}] - [(\vec{b}-\vec{a}) \ \vec{c} \ \vec{a}] - [(\vec{b}-\vec{a}) \ \vec{a} \ \vec{r}] = 0$$

$$\Rightarrow [\vec{b} \ \vec{c} \ \vec{r}] - [\vec{b} \ \vec{c} \ \vec{a}] - [\vec{b} \ \vec{a} \ \vec{r}] - [\vec{a} \ \vec{c} \ \vec{r}] = 0$$

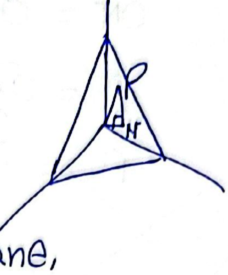
$$\Rightarrow [\vec{r} \ \vec{b} \ \vec{c}] - [\vec{a} \ \vec{b} \ \vec{c}] - [\vec{r} \ \vec{b} \ \vec{a}] - [\vec{r} \ \vec{a} \ \vec{c}] = 0$$

$$\Rightarrow [r \ b \ c] - [r \ b \ a] - [r \ a \ c] = [a \ b \ c]$$

[দরকার নাই]

$\cancel{(\vec{r}-\vec{a})}$ $\vec{r} = \vec{a} + s(\vec{b}-\vec{a}) + t(\vec{c}-\vec{a})$

(vi) Eq^n of plane in Normal form.



P an arbitory point on the plane,

$P \equiv (x, y, z)$

$OP = x\hat{i} + y\hat{j} + z\hat{k} = \vec{r}$

$ON = \vec{n}$ = normal vector

$\because NP \cdot ON = 0$

$(\vec{r} - \vec{n}) \cdot \vec{n} = 0$

$\vec{r} \cdot \vec{n} = |\vec{n}|^2 \Rightarrow \vec{r} \cdot |\vec{n}| \hat{n} = |\vec{n}|^2$

$\Rightarrow \vec{r} \cdot \hat{n} = |\vec{n}|$

$\boxed{\cancel{\Rightarrow \vec{r} \cdot |\vec{n}| \hat{n} = |\vec{n}|^2} \\ \cancel{\Rightarrow \vec{r} \cdot \hat{n} = |\vec{n}|}}$

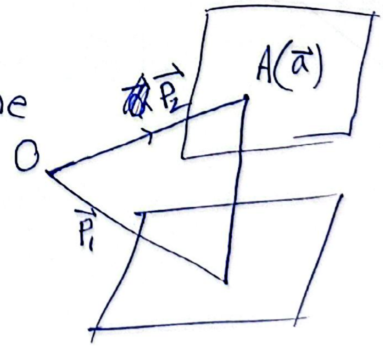
(vii) Angle between two intersecting plane

if θ be the angle between two intersecting plane

$r \cdot \hat{n}_1 = q_1$ & $\vec{r} \cdot \hat{n}_2 = q_2$ is $\theta = \cos^{-1}(\hat{n} \cdot \hat{n})$

if, $r \cdot \vec{n}_1 = q_1$ & $r \cdot \vec{n}_2 = q_2$ then $\theta = \cos^{-1}\left(\dfrac{\vec{n}_1 \cdot \vec{n}_2}{|\vec{n}_1| |\vec{n}_2|}\right)$

(viii) Distance of a point from a plane



$r \cdot \hat{n}_1 = P_1$

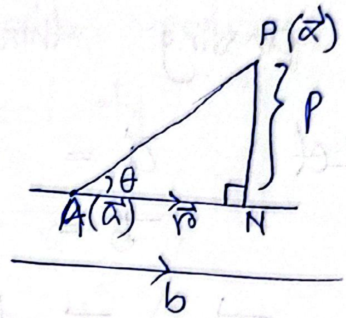
$r \cdot \hat{n}_2 = P_2$

$P_1 - P_2$

$\boxed{P_1 - a \cdot \hat{n}_2}$

this is the distance

(IX) Distance of a point from a straight line.



$\vec{r} = \vec{a} + t\vec{b}$

$PN \perp AN$

$AP = \vec{\alpha} - \vec{a}$

$\angle PAN = \theta$

$\Delta PAN \quad p = AP\sin\theta$

$= |\vec{\alpha} - \vec{a} \times \vec{b}|$

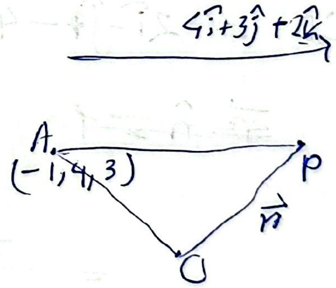
(X) Equation of plane passing through the intersecting line of two planes.

$\vec{r}\cdot\hat{n}_1 = P_1$

$\vec{r}\cdot\hat{n}_2 = P_2$

$(\vec{r}\cdot\hat{n}_1 - P_1) + \lambda(\vec{r}\cdot\hat{n}_2 - P_2) = 0$ where λ is ~~scaler~~ scaler.

Q. Find the vector Eqn of ~~s~~ the straight line passing through the point $(-1,4,3)$ and parallel to the vector $4\hat{i} + 3\hat{j} + 2\hat{k}$



Let,

$A \equiv (-1,4,3)$

$AP = (\vec{r} + \hat{i} - 4\hat{j} - 3\hat{k})$

$(\vec{r} + \hat{i} - 4\hat{j} - 3\hat{k}) \times (4\hat{i} + 3\hat{j} + 2\hat{k}) = 0$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x+1 & y-4 & z-3 \\ 4 & 3 & 2 \end{vmatrix}$$

$(2y - 8 - 3z + 9)\hat{i} + (4z - 12 - 2x - 2)\hat{j} + (3x + 3 - 4y + 16)\hat{k} = 0$

$2y - 3z + 1 = 0 \quad | \quad 4z - 2x - 14 = 0 \quad | \quad 3x - 4y + 19 = 0$

Q Find the vector equation of the straight line passing through two points $(2,3,4)$ & $(-1,3,-2)$.

Let $\vec{a} = 2\hat{i}+3\hat{j}+4\hat{k} = \vec{OA}$ where $A \equiv (2,3,4)$

$\vec{b} = -\hat{i}+3\hat{j}-2\hat{k} = \vec{OB}$ " $B \equiv (-1,3,-2)$

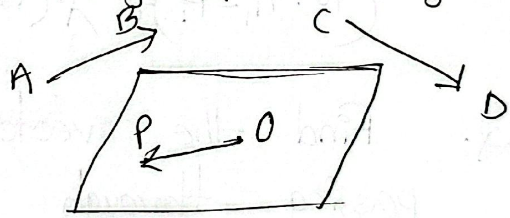
$\vec{r} = \vec{a}+t(\vec{b}-\vec{a})$ [where t is the parameter]

$\vec{r} = (2\hat{i}+3\hat{j}+4\hat{k})+t(-3\hat{i}-6\hat{k})$

Q ST three points with posn vector $\hat{i}-2\hat{j}+3\hat{k}$, $-2\hat{i}+\hat{j}$ & $2\hat{i}-3\hat{j}+4\hat{k}$ lie on the straight line

do as you like ---

Q Find the vector Eqn of a plane passing throug the origin and $\parallel$ to $\hat{i}+2\hat{j}+3\hat{k}$ & $2\hat{i}-\hat{j}-4\hat{k}$



~~$OP = x\hat{i}+y\hat{j}+z\hat{k}$~~ $OP = \vec{r}$

$AB = \hat{i}+2\hat{j}+3\hat{k}$

$CD = 2\hat{i}-\hat{j}-4\hat{k}$

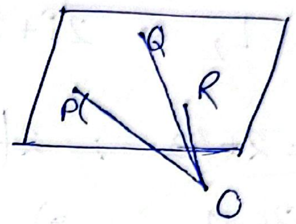
$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ 2 & -1 & -4 \end{vmatrix}$$

~~$\vec{r}\cdot\hat{n}=1$~~ $[\vec{r}\ AB\ CD]$

$= \vec{r}\cdot(AB \times CD)$

$= \vec{r}\cdot(-5\hat{i}+10\hat{j}-5\hat{k}) = 0$

Q. Write the vector eqn of a plane through the points P, Q & R having position vectors $2\hat{i}+2\hat{j}-\hat{k}$, $3\hat{i}+4\hat{j}+2\hat{k}$ & $7\hat{i}+6\hat{k}$ respectively.



$OP(\vec{a}) = 2\hat{i}+2\hat{j}-\hat{k}$

$OQ(\vec{b}) = 3\hat{i}+4\hat{j}+2\hat{k}$

$OR(\vec{c}) = 7\hat{i}+6\hat{k}$

$$(\vec{r}-\vec{a}) = t(\vec{b}-\vec{a}) + s(\vec{c}-\vec{a})$$

$$\vec{r} = \vec{a} + t(\vec{b}-\vec{a}) + s(\vec{c}-\vec{a})$$

Q. F.T. V. Eqn of straight line pass throu point $\hat{i}-2\hat{j}+\hat{k}$ $\perp$ $2\hat{i}+\hat{j}-\hat{k}$ & $\hat{i}-2\hat{j}+\hat{k}$

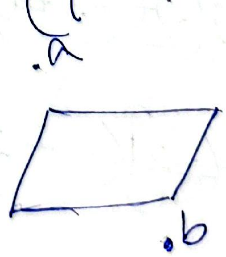
This straight line which is perpendicular to $2\hat{i}+\hat{j}-\hat{k}$ & $\hat{i}-2\hat{j}+\hat{k}$ so, it should parallal to $\vec{b} = (2\hat{i}+\hat{j}-k)\times(\hat{i}-2\hat{j}+\hat{k})$

$$\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 1 & -1 \\ 1 & -2 & +1 \end{vmatrix}$$

$$= -\hat{i} - 3\hat{j} - 5\hat{k}$$

$$\vec{r} = (\hat{i}-2\hat{j}+\hat{k}) + t(-\hat{i}-3\hat{j}-5\hat{k}) \quad [t \text{ is scaler}]$$

Dis from a to plane



$$\frac{1}{|\vec{n}|}(q - \vec{a}.\vec{n}) \quad [\vec{a} = \text{pos vec of } a]$$

$$\frac{1}{|\vec{n}|}(q - \vec{b}.\vec{n})$$

Q. Find the Point of intersection of $\vec{r} = 3\hat{i}+2\hat{j}-\hat{k}+t(\hat{i}+2\hat{k})$

& $\vec{r} = 2\hat{i}+2\hat{j}-\hat{k}+s(3\hat{i}-2\hat{k})$

Let, $L_1 = 3\hat{i}+2\hat{j}-\hat{k}+t(\hat{i}+2\hat{k}) \Rightarrow \hat{i}(3+t)+\hat{j}(2)+\hat{k}(2t-1)$

$L_2 = 2\hat{i}+2\hat{j}-\hat{k}+s(3\hat{i}-2\hat{k}) \Rightarrow \hat{i}(2+3s)+\hat{j}(2)+\hat{k}(-2s-1)$

$3+t = 2+3s$; $2t-1 = -2s-1$

$\Rightarrow t = 3s-1$ $\qquad \Rightarrow t = -s$

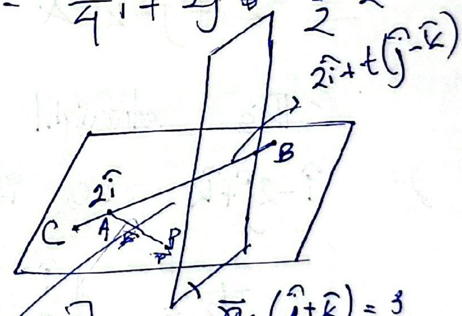
$\Rightarrow -s = 3s-1$ $\qquad \Rightarrow t = -\frac{1}{4}$

$\Rightarrow s = \frac{1}{4}$

$\therefore$ the co-ordinate of intersecting point

$= (\frac{11}{4},$

the pos vec of int point $= \frac{11}{4}\hat{i} + 2\hat{j} - \frac{3}{2}\hat{k}$



$\vec{AP} = \vec{r}-\vec{a}$

$\cancel{\vec{AB}}\ \vec{CB} = (2+t)\hat{i} - t\hat{k}$

$[\vec{r}\ \cancel{\vec{a}}\ \ \vec{CB}\ \ (\hat{i}+\hat{k})] = [\vec{a}\ \ \vec{CB}\ \ (\hat{i}+\hat{k})]$

$$\begin{vmatrix} i & j & k \\ (2+t) & 0 & -t \\ +1 & 0 & +1 \end{vmatrix}$$

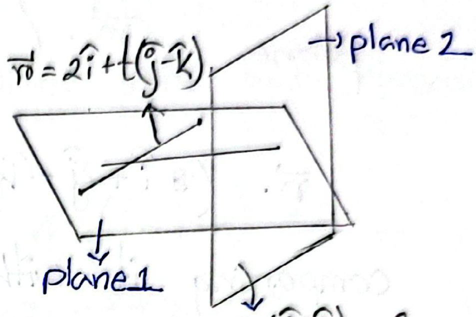
~~$\vec{r}.\{2t$~~

$\vec{r}.\{\hat{j}(-2t-2)\} = 2\hat{i}.\{-2\hat{j}(t+1)\}$

$$\begin{matrix} i & j & k \\ 2 & 0 & 0 \end{matrix}$$

$\vec{r}.\ \cancel{(-2\hat{j}(}\ -2(t+1)\hat{j} = 0.$

Q. Find the eq^n of the plane, which contain the line $\vec{r} = 2\hat{i} + t(\hat{j} - \hat{k})$ & is perpendicular to the plane $\vec{r}\cdot(\hat{i}+\hat{k}) = 3$. Find also the position vector of the point where this plane meets the line $\vec{r} = t(2\hat{i}+3\hat{j}+\hat{k})$.



the normal vector of the,

$$= \left\{ (\hat{j}-\hat{k}) \times (\hat{i}+\hat{k}) \right\}$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 1 & -1 \\ 1 & 0 & 1 \end{vmatrix}$$

$$\vec{n} = \hat{i} - \hat{j} - \hat{k}$$

$$\vec{r} = 2\hat{i} + t(\hat{j}-\hat{k}) \quad \text{— (I)}$$

$$\vec{r} = \vec{a} + t\vec{b} \quad \text{— (II)}$$

by comparing, (I) & (II)

$\vec{a} = 2\hat{i}$, its the position ~~of~~ vector of a point exists on the line and the plane also.

$$(\vec{r}-\vec{a})\cdot\vec{n} = 0$$

$$\boxed{\vec{r}\cdot\vec{n} = 2} \rightarrow Eq^n \text{ of the plane.}$$

$$\vec{r}\cdot(\hat{i}-\hat{j}-\hat{k}) = 2$$

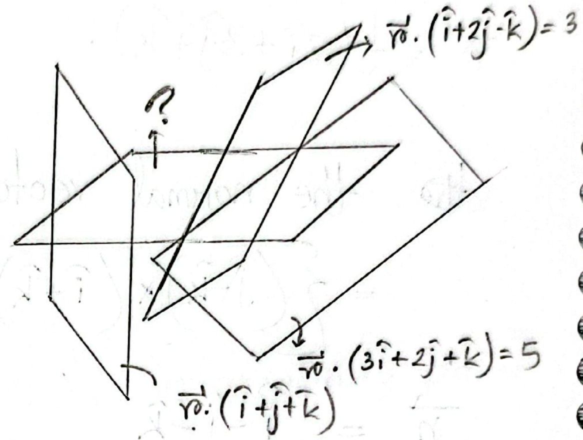
$$t(2\hat{i}+3\hat{j}+\hat{k})\cdot(\hat{i}-\hat{j}-\hat{k}) = 2$$

$$2t - 3t - t = 2$$

$$t = -1$$

So the position vec of the inter- point $= -2\hat{i} - 3\hat{j} - \hat{k}$.

Q. Find the equation of plane, which ~~contain the line~~ ~~$\vec{r} = 2\hat{i} + t(\hat{j}-\hat{k})$~~ Passing through the line of intersection of the planes $\vec{r}\cdot(\hat{i}+2\hat{j}-\hat{k}) = 3$ and $\vec{r}\cdot(3\hat{i}+2\hat{j}+\hat{k}) = 5$ and perpendicular to the plane $\vec{r}\cdot(\hat{i}+\hat{j}+\hat{k}) = 4$.



$\vec{r}\cdot(\hat{i}+2\hat{j}-\hat{k}) = 3$

comparing it with

$\vec{r}\cdot\vec{n}_1 = q_1$

we get, $n_1 = (\hat{i}+2\hat{j}-\hat{k})$, $q_1 = 3$

$\vec{r}\cdot(3\hat{i}+\hat{j}+\hat{k}) = 5$

Comparing it with,

$\vec{r}\cdot\vec{n}_2 = q_2$

$\vec{n}_2 = (3\hat{i}+\hat{j}+\hat{k})$, $q_2 = 5$

$\left(\vec{r}\cdot(\hat{i}+2\hat{j}-\hat{k}) - 3\right) + \lambda\left\{\vec{r}\cdot(3\hat{i}+2\hat{j}+\hat{k}) - 5\right\} = 0$

$x + 2y - z - 3 + 3\lambda x + 2\lambda y + \lambda z - \lambda 5 = 0$

$x(1+3\lambda) + y(2+2\lambda) + z(-1+\lambda) + (-\lambda 5+3) = 0$

~~$x + 3\lambda x = 1$; $2y + 2\lambda y =$~~

$1+3\lambda = 1$; $2+2\lambda = 1$; $-1+\lambda = 1$, $\lambda 5 + 3 = 4$

$\Rightarrow \lambda = 0$; $\lambda = -\frac{1}{2}$; $\lambda = 2$; $\lambda = \frac{1}{5}$

~~we~~ let,

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$\left\{ x + 2y - z - 3 + 3x\lambda + 2y\lambda + z\lambda - 5\lambda \right\} = 0$$

$$\Rightarrow \quad x(1+3\lambda) + y(2+2\lambda) + z(\lambda-1) - (3+5\lambda) = 0 \quad \text{—(I)}$$

— this is the eq^n of genarel form of plane with that intersection line.

now this plane is perpendicular with $\vec{r}\cdot(\hat{i}+\hat{j}+\hat{k}) = 4$ —(II)

~~or, $x\hat{i}+y\hat{j}+z = 4$~~

or, ~~$x+y+z-4=0$~~ —(II)

~~Comparing (I) & (II)~~

So, the ~~normal of~~ plane (I) & (II) ~~will be~~ perpendicular to each-other, so their normal $\left((1+3\lambda)\hat{i} + (2+2\lambda)\hat{j} + (\lambda-1)\hat{k}\right.$

~~$(1+3\lambda)\cdot 1 + (2+2\lambda)\cdot 1$~~ & $(\hat{i}+\hat{j}+\hat{k})$ will be $\perp$ also.

So, $\quad (1+3\lambda)\cdot 1 + (2+2\lambda)\cdot 1 + (\lambda-1)\cdot 1 = 0$

$$\Rightarrow \cancel{1} + 3\lambda + 2 + 2\lambda + \lambda - \cancel{1} = 0$$

$$\Rightarrow \lambda = -\frac{1}{3}$$

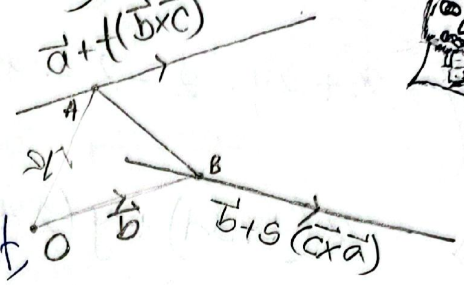
So, the eq^n of the plane (I),

$$\vec{r}\cdot\left(\frac{4}{3}\hat{j} - \frac{4}{3}\hat{k}\right) = \frac{4}{3}$$

$$\vec{r}\cdot(\hat{j}-\hat{k}) = 1$$

Q. S.T. the lines $\vec{r} = \vec{a} + t(\vec{b}\times\vec{c})$ will intersect
$\vec{r} = \vec{b} + s(\vec{c}\times\vec{a})$

if $\vec{a}\cdot\vec{c} = \vec{b}\cdot\vec{c}$.



$OA = \vec{a}$ $\quad AB = \vec{b} - \vec{a}$

$OB = \vec{b}$

Now, lets assume ~~see if~~ the intersect, means these two lines are coplanerr,

$$\left[AB \quad (\vec{b}\times\vec{c}) \quad (\vec{c}\times\vec{a})\right] = 0$$

$$= \left\{(\vec{b}-\vec{a}) \times (b\times c)\right\} \cdot (\vec{c}\times\vec{a})$$

$$= \left[\left\{(\vec{b}-\vec{a})\cdot\vec{c}\right\}\vec{b} - \left\{(\vec{b}-\vec{a})\cdot\vec{b}\right\}\vec{c}\right] \cdot (\vec{c}\times\vec{a})$$

$$= (\vec{b}\cdot\vec{c}) - (\vec{a}\cdot\vec{c})$$

$$= (\vec{b}\times\vec{c}) \cdot \left\{(\vec{b}-\vec{a}) \times (\vec{c}\times\vec{a})\right\}$$

$$= (\vec{b}\times\vec{c}) \cdot \left\{(\vec{b}\cdot\vec{a})\vec{c} - (\vec{b}\cdot\vec{c})\vec{a} - (\vec{a}\cdot\vec{a})\vec{c} + (\vec{a}\cdot\vec{c})\vec{a}\right\}$$

$$= (\vec{b}-\vec{a}) \cdot \left\{(\vec{b}\times\vec{c}) \times (\vec{c}\times\vec{a})\right\}$$

$$= (\vec{b}-\vec{a}) \cdot \left\{ \cancel{[\vec{b}\ \vec{c}\ \vec{a}]\,\vec{c} - [c\ c\ a]\,\vec{b}} \quad [\vec{b}\ \vec{c}\ \vec{a}]\,\vec{c} - [b\ c\ \cancel{c}]\,\vec{a} \right\} \to 0$$

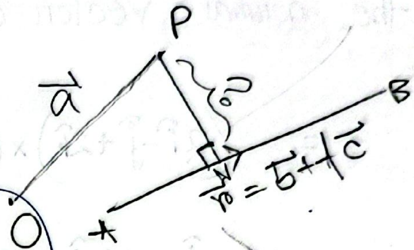
$$\Rightarrow (\vec{b}\cdot\vec{c})\,[\vec{b}\ \vec{c}\ \vec{a}] - (\vec{a}\cdot\vec{c})\,[\vec{b}\ \vec{c}\ \vec{a}] = 0$$

$$\vec{b}\cdot\vec{c} = \vec{a}\cdot\vec{c} \ (\longleftrightarrow)$$

Q. If $\vec{a}$ be the position vector of a point P. Find the distance of P from the straight line $\vec{r} = \vec{b} + t\vec{c}$ where $\vec{a} = 5\hat{i} - 6\hat{j} + 2\hat{k}$

$\vec{b} = \hat{i} - \hat{j} + 2\hat{k}$ & $\vec{c} = -4\hat{j} - 3\hat{k}$

Let $\overline{AB} = \vec{r} = \vec{b} + t\vec{c}$

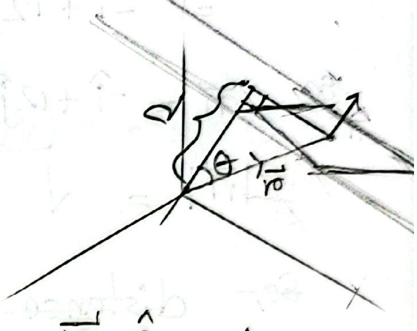


$\therefore$ ~~PN $\perp$ AB~~

~~Let, $N = (x, y, z)$~~

$PN = \cancel{(x\hat{i} + y\hat{j} + z\hat{k}) - \vec{a}}$ $\quad \vec{a} + s$

$= (x-5)\hat{i} + (y+6)\hat{j} + (z-2)\hat{k}$



$\vec{r} \cdot \hat{n} = d$

Now,

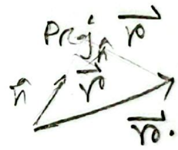
$AB = (\hat{i} - \hat{j} + 2\hat{k}) + t(-4\hat{j} - 3\hat{k})$

$= \hat{i} + (-1-4t)\hat{j} + (2-3t)\hat{k}$

$\text{proj}_a b = \frac{a \cdot b}{|a|}$

As, PN $\perp$ AB

$(x-5) \times 1 + (y+6)$



$\text{Proj}_{\hat{n}} \vec{r} \quad \frac{\hat{n} \cdot \vec{r}}{|\hat{n}|}$

$\vec{r} \cdot \hat{n} = |\vec{r}| \cos\theta$

$=$

asume, PN $\perp$ AB

$ON = \vec{b} + \lambda\vec{c}$

$PN = \vec{b} - \vec{a} + \lambda\vec{c} =$

as, $\overline{PN} \perp \overline{AB}$, $(\vec{b} - \vec{a} + \lambda\vec{c}) \cdot \vec{c} = 0$

$\vec{b} \cdot \vec{c} - \vec{a} \cdot \vec{c} + \lambda\, \vec{c} \cdot \vec{c} = 0$

$\lambda = \frac{(\vec{a} - \vec{b}) \cdot \vec{c}}{\vec{c} \cdot \vec{c}}$

$\vec{b} - \vec{a} + \frac{(\vec{a} - \vec{b}) \cdot \vec{c}}{|\vec{c}||\vec{c}|} \vec{c}$

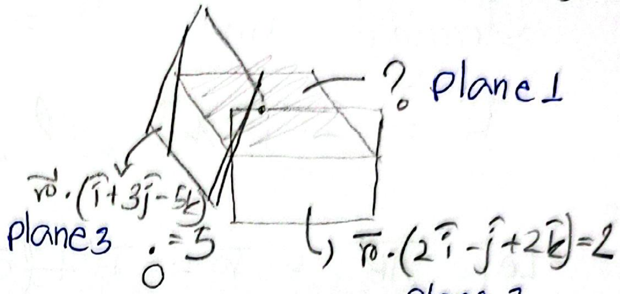
$= -4\hat{i} + 5\hat{j} + \frac{(-20)}{25_5} (-4\hat{j} - 3\hat{k})$

$= -4\hat{i} + \frac{9}{5}\hat{j} + \frac{12}{5}\hat{k}$

$\sqrt{16 + \frac{81}{25} + \frac{144}{25}}$

$= \sqrt{25} = 5$ (Ans)

Q. Find the eq^n of plane passing through the point $5\hat{i}+2\hat{j}-3\hat{k}$ and $\perp$ to $\vec{r}\cdot(2\hat{i}-\hat{j}+2\hat{k})=2$ & $\vec{r}\cdot(\hat{i}+3\hat{j}-5\hat{k})=5$ also.



the normal vector of plane 1,

$$= (2\hat{i}-\hat{j}+2\hat{k})\times(\hat{i}+3\hat{j}-5\hat{k})$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 2 \\ 1 & 3 & -5 \end{vmatrix}$$

$$= -\hat{i}+12\hat{j}+7\hat{k}$$

so, $\vec{n} = -\hat{i}+12\hat{j}+7\hat{k}$

$|\vec{n}| = \sqrt{194}$

~~So, distance from origin to~~ eq^n of ~~plane 1~~,

$$d = \left|\frac{(5\hat{i}+2\hat{j}-3\hat{k})\cdot(-\hat{i}+12\hat{j}+7\hat{k})}{\sqrt{194}}\right|$$

$$= \left|\frac{-5+24-21}{\sqrt{194}}\right|$$

$$= \left|-\frac{2}{\sqrt{194}}\right|$$

$$= \frac{2}{\sqrt{194}} \text{ units}$$

So,

$$(5\hat{i}+2\hat{j}-3\hat{k})\cdot(-\hat{i}+12\hat{j}+7\hat{k})$$

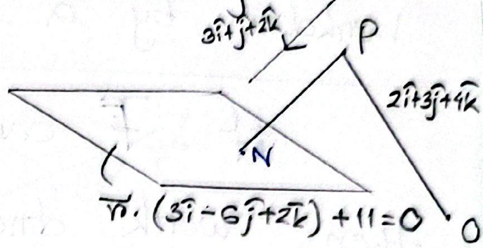
$$= -5+24-21$$

$$= -2$$

So, the eq^n of the plane 1

$$\vec{r}\cdot(-\hat{i}+12\hat{j}+7\hat{k})+2=0$$

Q. Find the distance of the point $2\hat{i}+3\hat{j}+4\hat{k}$ to the plane $\vec{r}\cdot(3\hat{i}-6\hat{j}+2\hat{k})+11=0$, measure along the vector $3\hat{i}+\hat{j}+2\hat{k}$.



$$\cancel{\dfrac{\{\vec{r}-(2\hat{i}+3\hat{j}+4\hat{k})\}\cdot(3\hat{i}+\hat{j}+2\hat{k})}{\sqrt{9+1+4}}}$$

$$(2\hat{i}+3\hat{j}+4\hat{k})+t(3\hat{i}+\hat{j}+2\hat{k})$$

$$\{(2+3t)\hat{i}+(3+t)\hat{j}+(4+2t)\hat{k}\}\cdot(3\hat{i}-6\hat{j}+2\hat{k})+11=0$$

$$(6+9t)-18-6t+8+4t=-11$$

$$7t=-11+18-8-6$$

$$7t=-7$$

$$t=-1$$

So,

$$ON=(2\hat{i}+3\hat{j}+4\hat{k})-(3\hat{i}+\hat{j}+2\hat{k})$$

$$OP=(2\hat{i}+3\hat{j}+4\hat{k})$$

So, $PN=-3\hat{i}-\hat{j}-2\hat{k}$

distance of $PN=\sqrt{(-3)^2+(-1)^2+(-2)^2}$

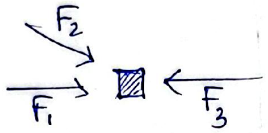
$=\sqrt{14}$ units

Application to Mechanics

Workdone by a force :

if $\vec{F}$ causing displacement ~~in~~ $\vec{d}$ in a direction then work done by $\vec{F}$ is defined as, $W = \vec{F} \cdot \vec{d}$

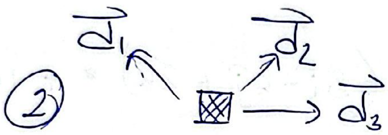
Note : (1)



$$\vec{F} = \vec{F_1} + \vec{F_2} + \vec{F_3} + \ldots + \vec{F_n}$$

$$W = \vec{F} \cdot \vec{d} = \vec{F_1} \cdot \vec{d} + \vec{F_2} \cdot \vec{d} + \cdots + \vec{F_n} \cdot \vec{d}$$

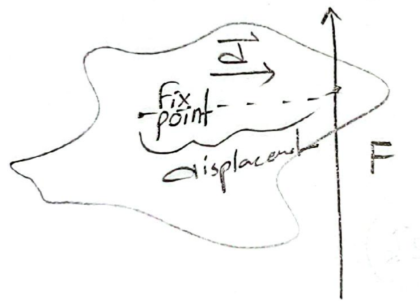
(2)



$$\vec{d} = \vec{d_1} + \vec{d_2} + \vec{d_3} + \cdots + \vec{d_n}$$

$$W = \vec{F} \cdot \vec{d} = \vec{F} \cdot \vec{d_1} + \vec{F} \cdot \vec{d_2} + \cdots + \vec{F} \cdot \vec{d_n}$$

Moment or torque of a force

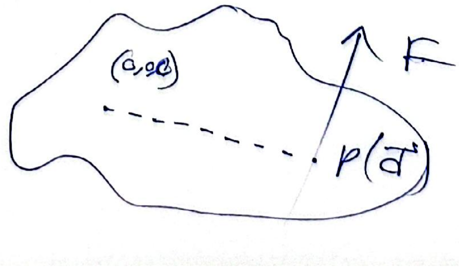


$\cancel{\vec{F} \times \vec{d}}$

$\vec{d} \times \vec{F}$

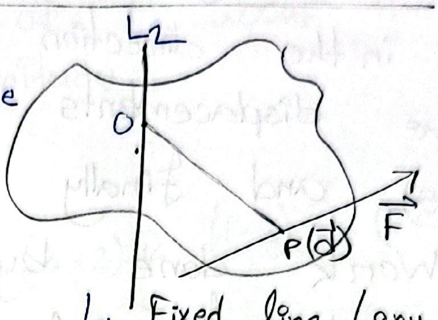
The moment or torque of a force $\vec{F}$ about a point O is denoted by the vector $\vec{M}$ & is ~~denoted~~ defined, by $\vec{M} = \vec{d} \times \vec{F}$ where $P(\vec{d})$ be any point on the line of ~~ve~~ action $\vec{F}$

i.e. $\vec{OP} = \vec{d}$ where O is origin



Moment of a force about a line.

$\vec{r} = \vec{a} + t\,\vec{b}$ → Fixed line



$(\vec{F} \times \vec{d}) \cdot \hat{b}$

$= [F\ d\ b]$

Fixed line (any point on this line

although for some fucking reason we use $\hat{n}$ here not $\hat{b}$

The Moment of a force $\vec{F}$ about a line L, L_2 is defin~~ded~~ as the moment of $\vec{M}$ along the line L_1, L_2, where $\vec{M}$ is the moment of $\vec{F}$ about Origin, Thus the moment of $\vec{F}$ about the line $L_1 L_2$ is $\vec{M} \cdot \hat{b} = \vec{d} \times \vec{F} \cdot \hat{b} = [\vec{d}\ \vec{F}\ \vec{b}]$, where $\hat{b}$ is the unit vector along the line $L_1 L_2$ and $P(\vec{d})$ any point on the line of action of $\vec{F}$ that is $\vec{OP} = \vec{d}$, where O is the origin.

Q. A particle acted on by constant forces $4\hat{i} + 2\hat{j} + 3\hat{k}$ and $\hat{i} + \hat{j} + \hat{k}$ is displaced from $\hat{i} + 2\hat{j}$ to $2\hat{i} - \hat{j} + 3\hat{k}$ find the work done by the force.

$$\vec{F} = (4\hat{i} + 2\hat{j} + 3\hat{k}) + (\hat{i} + \hat{j} + \hat{k})$$

$$= 5\hat{i} + 3\hat{j} + 4\hat{k}$$

$$\vec{d} = (2\hat{i} - \hat{j} + 3\hat{k}) - (\hat{i} + 2\hat{j})$$

$$= \hat{i} - 3\hat{j} + 3\hat{k}$$

$$W = \vec{F} \cdot \vec{d}$$

$$= 5 - 9 + 12$$

$$= 8 \text{ units}$$

Q A force of magnitude 5unit act on a particle A(1,−1,3) in the direction $(2\hat{i}+\hat{j}+3\hat{k})$ causing three successive displacements from A to B (2,−1,7) to C (1,3,5) and finally from C to D (2,7,4) F.T. Work done by the force in bringing the particle from A to D.

So,

$\cancel{\vec{d}} =$ Total displacement $(\vec{d}) =$ AB+BC+CD

$$\vec{d} = (\hat{i}+4\hat{k}) + (-\hat{i}+4\hat{j}-2\hat{k}) + (\hat{i}+4\hat{j}-\hat{k})$$

$$= \hat{i}+8\hat{j}+\hat{k}$$

~~the~~ the direction of force $\hat{F} = \dfrac{2\hat{i}+\hat{j}+3\hat{k}}{\sqrt{2^2+1^2+3^2}}$

$$= \frac{2\hat{i}+\hat{j}+3\hat{k}}{\sqrt{14}}$$

A.Q.,

$$|F| = 5$$

So, $$\vec{F} = \frac{10\hat{i}+5\hat{j}+15\hat{k}}{\sqrt{14}}$$

we know,

$$W = \vec{F}\cdot\vec{d}$$

$$= \cancel{\frac{10}{\sqrt{14}}}\ \frac{1}{\sqrt{14}}\left(10+40+15\right)$$

$$= \frac{65}{\sqrt{14}}\ \text{units}$$

Q. A Force $= 3\hat{i}+2\hat{j}-4\hat{k}$ is applied at the point $(1,-1,2)$. Find the moment of $\vec{F}$ about the point $(2,-1,3)$. Also find it's magnitude.

$$\vec{F} = 3\hat{i}+2\hat{j}-4\hat{k}$$

$$\vec{d} = \cancel{(2-1)\hat{i}+(-1+1)\hat{j}+(3-2)\hat{k}} \quad (1-2)\hat{i}+(-1+1)\hat{j}+(2-3)\hat{k}$$

$$\vec{M} = \cancel{(3\hat{i}+2\hat{j}-4\hat{k})\times(\hat{i}+\hat{k})}$$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ -1 & 0 & -1 \\ 3 & 2 & -4 \end{vmatrix}$$

$$= \vec{d}\times\vec{F}$$

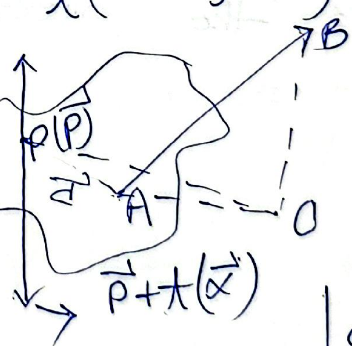
$$= (-\hat{i}-\hat{k})\times(3\hat{i}+2\hat{j}-4\hat{k})$$

$$= +2\hat{i}-7\hat{j}-2\hat{k}$$

$$|\vec{M}| = \sqrt{57} \text{ units}$$

Q. Find the moment of a force $\overline{AB}$ about a line passing through the point $(1,2,3)$ along the vector $\vec{\alpha} = 6\hat{i}+3\hat{j}+2\hat{k}$, where the position vector of A & B $=(3\hat{i}+\hat{j}+2\hat{k})$ & $(3\hat{i}+4\hat{j}+5\hat{k})$

axis of rotation $:= (\hat{i}+2\hat{j}+3\hat{k}) + t(6\hat{i}+3\hat{j}+2\hat{k})$



moment of F about the

this is the point P

$OP = \hat{i}+2\hat{j}+3\hat{k}$

$OA = 3\hat{i}+\hat{j}+2\hat{k}$

$OB = 3\hat{i}+4\hat{j}+5\hat{k}$

$AB = 3\hat{j}+3\hat{k}$

$$\begin{vmatrix} 2 & -1 & -1 \\ 0 & 3 & 3 \end{vmatrix}$$

$\vec{F} = 3\hat{j}+3\hat{k}$

$\vec{d} = 2\hat{i}-\hat{j}-\hat{k}$

$$\vec{M} = \vec{d}\times\vec{F}$$

$$= (2\hat{i}-\hat{j}-\hat{k})\times(3\hat{j}+3\hat{k})$$

$$= -6\hat{j}+6\hat{k}$$

about the line

$\vec{p} + t(\vec{\alpha})$ the moment would be,

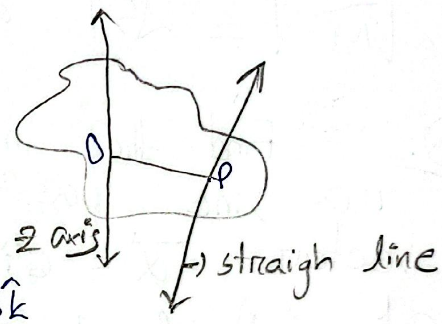
$$\vec{M} = (\vec{d} \times \vec{F}) \cdot \hat{\alpha}$$

$$= (-6\hat{j} + 6\hat{k}) \cdot \frac{1}{7}(6\hat{i} + 3\hat{j} + 2\hat{k})$$

$$= -\frac{18}{7} + \frac{12}{7}$$

$$= -\frac{6}{7} \text{ units}$$

Q. A force $\vec{F}$ of mag 10 units act along the line $\frac{x-2}{5} = \frac{y-1}{4} = \frac{z-3}{3}$. Find the torque of the force $\vec{F}$ about Z axis.



$(2, 1, 3)$ is a point on the line $\frac{x-2}{5} = \frac{y-1}{4} = \frac{z-3}{3}$ and this line is parallel to $5\hat{i} + 4\hat{j} + 3\hat{k}$

So, the direction of this line, $(\hat{F})$

$$= \frac{1}{\sqrt{5^2 + 4^2 + 3^2}} (5\hat{i} + 4\hat{j} + 3\hat{k})$$

$$= \frac{1}{\sqrt{50}} (5\hat{i} + 4\hat{j} + 3\hat{k})$$

$$\vec{F} = |F| \cdot \hat{F}$$

$$= 10 \times \frac{1}{\sqrt{50}} (5\hat{i} + 4\hat{j} + 3\hat{k})$$

$$= \sqrt{2}\, \cancel{2(5}\,(5\hat{i} + 4\hat{j} + 3\hat{k})$$

$\vec{d} = 2\hat{i}+\hat{j}+3\hat{k}$

$M = [\vec{d}\ \vec{F}\ \hat{k}]$

$= 8\sqrt{2} - 5\sqrt{2}$

$= 3\sqrt{2}$ units

$$\begin{vmatrix} 2 & 1 & 3 \\ 5\sqrt{2} & 4\sqrt{2} & 3\sqrt{2} \end{vmatrix}$$

Rotation of a rigid ~~motion~~ body about a fixed axis:

$\vec{V} = (\vec{\omega} \times \vec{r})$. [$\vec{r}$ = is the radius]

Q. A Rigid body is spinning with an angular velocity of 4 radians per second about an axis through the point $\hat{i}+\hat{j}-\hat{k}$ and $\parallel$ to $3\hat{i}+2\hat{j}+\hat{k}$. Find the velocity of the particle at the point $4\hat{i}+\hat{j}-\hat{k}$ of the body and also find the speed of the particle at this point.

A.Q,

$|\vec{\omega}| = 4$ rad/s

$\hat{\omega} \parallel 3\hat{i}+2\hat{j}+\hat{k}$

$\hat{\omega} = \dfrac{3\hat{i}+2\hat{j}+\hat{k}}{\sqrt{14}}$

$\vec{W}$: $\vec{\omega} = \dfrac{4}{\sqrt{14}}(3\hat{i}+2\hat{j}+\hat{k})$

$\vec{v} = \vec{\omega} \times 4\hat{i}+\hat{j}-\hat{k}$

$\vec{v} = \left\{\dfrac{4}{\sqrt{14}}(3\hat{i}+2\hat{j}+\hat{k})\right\} \times (4\hat{i}+\hat{j}-\hat{k} - \hat{i}+\hat{j}+\hat{k})$

$\sqrt{\frac{12}{14}+\frac{24}{14}}$

$\frac{1}{\sqrt{14}} \times$ $\sqrt{14} \times \sqrt{15}$

$\frac{12}{\sqrt{14}} \times \sqrt{5}$

$\frac{4}{\sqrt{14}}$ | $+3$ $+2$ $+1$; 3 1 9

$= \dfrac{4}{\sqrt{14}}(-3\hat{i}+7\hat{j}-5\hat{k})$

$r = \sqrt{\left(\frac{13}{14}\times 3-4\right)^2 + \left(\frac{13}{14}\times 2-1\right)^2 + \left(\frac{13}{14}\times 1+1\right)^2}$

$\dfrac{4}{\sqrt{14}}(3\hat{j} - 6\hat{k})$

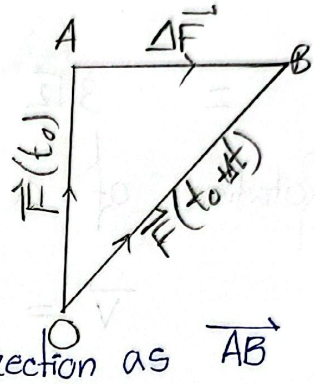
-72

$\frac{14}{7} = -\frac{216}{7}$

Vector Calculus

vector Derivatives

1/ Derivative of a vector Function:



$$\Delta\vec{F} = \vec{F}(t_0+\Delta t) - F(t_0)$$

Since, $\dfrac{\Delta\vec{F}}{\Delta t} = \dfrac{\overrightarrow{AB}}{\Delta t}$ and Δt is scaler this ratio has the same direction as $\overrightarrow{AB}$

~~as~~, if, $\Delta t \to 0 \Rightarrow \dfrac{\Delta\vec{F}}{\Delta t} \to$ an unique finite number.

then we will say $\vec{F}$ is derivavable at $t = t_0$

and the value of it will be $\lim\limits_{\Delta t\to 0} \dfrac{\Delta\vec{F}}{\Delta t} = \lim\limits_{\Delta t = 0} \dfrac{\vec{F}(t_0+\Delta t) - F(t_0)}{\Delta t}$

$$= \frac{d\vec{F}(t_0)}{dt} = \vec{F}'(t_0)$$

2/ Derivative of ~~sot~~ scaler product:

$$\frac{d}{dt}(F\cdot G) = \frac{dF}{dt}\cdot G + F\cdot\frac{dG}{dt} \quad [F \,\&\, G \text{ are scaler function of } t]$$

Proof,

$\phi = \vec{F}\cdot\vec{G}$, ϕ = scaler, making it a scaler f^n of t

We have, $\lim\limits_{\Delta t\to 0} \dfrac{\phi(t_0+\Delta t) - \phi(t_0)}{\Delta t}$

$$= \lim_{\Delta t\to 0} \frac{\vec{F}\cdot\vec{G}(t_0+\Delta t) - \vec{F}\cdot\vec{G}(t_0)}{\Delta t}$$

$$= \lim_{\Delta t\to 0} \frac{F(t_0+\Delta t)\cdot\vec{G}(t_0+\Delta t) - \vec{F}(t_0)\cdot\vec{G}(t_0+\Delta t) + \vec{F}(t_0)\cdot G(t_0+\Delta t) -}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\vec{F}(t_0+\Delta t) - \vec{F}(t_0)}{\Delta t} \cdot \vec{G}(t_0+\Delta t) + F(t_0) \cdot \frac{\vec{G}(t_0+\Delta t) - \vec{G}(t_0)}{\Delta t}$$

$$= \vec{F}'(t_0) \cdot \vec{G} + \vec{F} \cdot \vec{G}'(t_0) \quad [\text{As } \Delta t \to 0]$$

$$\frac{d}{dt}(\vec{F} \cdot \vec{F}) = 2\vec{F} \cdot \frac{d\vec{F}}{dt} = 2F \cdot \frac{dF}{dt}$$

$$\vec{F} \cdot \vec{F} = F^2$$

$$\frac{d}{dt}(\text{''}) = 2F \frac{dF}{dt}$$

3/ Derivative of vector product:

$$\frac{d}{dt}(\vec{F} \times \vec{G}) = \frac{d\vec{F}}{dt} \times \vec{G} + \vec{F} \times \frac{d\vec{G}}{dt}$$

proof:

$$\frac{d}{dt}(\vec{F} \times \vec{G}) = \lim_{\Delta t \to 0} \frac{\vec{F} \times \vec{G}(t_0+\Delta t) - \vec{F} \times \vec{G}(t_0)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\vec{F}(t_0+\Delta t) \times \vec{G}(t_0+\Delta t) - \vec{F}(t_0) \times \vec{G}(t_0+\Delta t) + \vec{F}(t_0) \times \vec{G}(t_0+\Delta t) - \vec{F}(t_0) \times \vec{G}(t_0)}{\Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\vec{F}(t_0+\Delta t) - \vec{F}(t_0)}{\Delta t} \times \vec{G}(t_0+\Delta t) + \vec{F}(t_0) \times \frac{\vec{G}(t_0+\Delta t) - \vec{G}(t_0)}{\Delta t}$$

$$= \vec{F}'(t_0) \times \vec{G}(t_0) + \vec{F}(t_0) \times \vec{G}'(t_0)$$

Q if $\vec{F}$ & $\frac{d\vec{F}}{dt}$ are diff-able P.T. $\frac{d}{dt}\left(\vec{F} \times \frac{d\vec{F}}{dt}\right) = \vec{F} \times \frac{d^2\vec{F}}{dt^2}$

$$\underbrace{\cancel{\vec{F}' \times \vec{F}'}}_{0} + \vec{F} \times \vec{F}'' = R.H.S (L-S)$$

4/ Derivative of scaler triple product :

$$\frac{d}{dt}[\vec{F}\,\vec{G}\,\vec{H}] = [\vec{F}'\,\vec{G}\,\vec{H}] + [\vec{F}\,\vec{G}'\,\vec{H}] + [\vec{F}\,\vec{G}\,\vec{H}']$$

proof:

$$L.H.S = \frac{d}{dt}[\vec{F}\,\vec{G}\,\vec{H}]$$

$$= \frac{d}{dt}\,\vec{F}\cdot(\vec{G}\times\vec{H})$$

$$= \vec{F}'\cdot(\vec{G}\times\vec{H}) + \vec{F}\cdot\left\{\vec{G}'\times\vec{H} + \vec{G}\times\vec{H}'\right\}$$

$$= [\vec{F}'\,\vec{G}\,\vec{H}] + [\vec{F}\,\vec{G}'\,\vec{H}] + [\vec{F}\,\vec{G}\,\vec{H}']$$

$$= R.H.S$$ [Proved]

5/ Derivative of vector Triple Product :

$$\frac{d}{dt}[\vec{F}\times(\vec{G}\times\vec{H})] = \frac{d\vec{f}}{dt}\times(\vec{G}\times\vec{H}) + \vec{F}(\vec{G}'\times\vec{H}) + \vec{F}(\vec{G}\times\vec{H}')$$

Proof:

$$L.H.S = \frac{d}{dt}[\vec{F}\times(\vec{G}\times\vec{H})]$$

$$\cancel{= \frac{d}{dt}\{\vec{F}\cdot H}$$

$$= \vec{F}'\times(\vec{G}\times\vec{H}) + \vec{F}\times\left\{\vec{G}'\times\vec{H} + \vec{G}\times\vec{H}'\right\}$$

$$= \vec{F}'\times(\vec{G}\times\vec{H}) + \vec{F}\times(\vec{G}'\times\vec{H}) + \vec{F}\times(\vec{G}\times\vec{H}')$$

$$= R.H.S$$ (Proved)

Q. vector derivative of a vector function $\vec{r}$ is zero iff $\vec{r}$ is a constant vector

Ans as $\vec{r}$ is a const vector,

for $t = t_0$ & $t = t_0 + \Delta t$,

$\vec{r} = \vec{r}$

So, L.H.S $= \frac{d}{dt}(\vec{r}) = \lim_{\Delta t \to 0} \frac{\vec{r}(t_0+\Delta t) - \vec{r}(t_0)}{\Delta t}$

$= \lim_{\Delta t \to 0} \frac{\vec{r} - \vec{r}}{\Delta t}$

$= 0$

$=$ R.H.S (proved)

Q. If $\vec{r} = 3t\hat{i} + 3t^2\hat{j} + 2t^3\hat{k}$ F.T. $[\vec{r}' \; \vec{r}'' \; \vec{r}''']$

$\vec{r}' = 3\hat{i} + 6t\hat{j} + 6t^2\hat{k}$

$\vec{r}'' = 6\hat{j} + 12t\hat{k}$

$\vec{r}''' = 12\hat{k}$

$$\begin{vmatrix} 0 & 6 & 12t \\ 0 & 0 & 12 \end{vmatrix}$$

$\cancel{72\hat{j}}$ $72\hat{i}$

So, $[\vec{r}' \; \vec{r}'' \; \vec{r}'''] = \left[(3\hat{i}+6t\hat{j}+6t^2\hat{k}) \cdot \{(6\hat{j}+12t\hat{k}) \times 12\hat{k}\}\right]$

$= \cancel{432}$ 216 (Ans)

Q. If $\vec{r_1} = t^2\hat{i} - t\hat{j} + (2t+1)\hat{k}$ at, $t = 1$, $\vec{r_2} = (2t^2-3)\hat{i} + t^2\hat{j} - t\hat{k}$, $\vec{r_3} = t\hat{i} + t^2\hat{j} + t^3\hat{k}$ F.T. (i) $\frac{d}{dt}[\vec{r_1} \times \vec{r_2}]$

(ii) $\vec{r_3} \times \vec{r_2}$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ t^2 & -t & (2t+1) \\ 4t & 2t & -1 \end{vmatrix}$$

$\vec{r_2}' = 4t\hat{i} + 2t\hat{j} - \hat{k}$

$\vec{r_3}''' = (\hat{i} + 2t\hat{j} + 3t^2\hat{k})'$

$= 2\hat{j} + 6t\hat{k}$

(i) $\left[(-t-4t^2)\hat{i} + (9t^2+4t)\hat{j} + (2t^3+4t^2)\hat{k}\right]'$

$= \left[(-1-8t)\hat{i} + (18t+4)\hat{j} + (6t^2+8t)\hat{k}\right]$

$= -9\hat{i} + 22\hat{j} + 14\hat{k}$ (Ans)

(ii) → H.W

Q. If $\frac{d\vec{a}}{dt} = \vec{\omega} \times \vec{a}$ & $\frac{d\vec{b}}{dt} = \vec{\omega} \times \vec{b}$ S.T. $\frac{d}{dt}(\vec{a} \times \vec{b}) = \vec{\omega} \times (\vec{a} \times \vec{b})$

$$L.H.S = \frac{d}{dt}(\vec{a} \times \vec{b})$$

$$= \vec{a}' \times \vec{b} + \vec{a} \times \vec{b}'$$

$$= (\vec{\omega} \times \vec{a}) \times \vec{b} + \vec{a} \times (\vec{\omega} \times \vec{b})$$

$$= (\vec{\omega} \cdot \vec{b})\vec{a} - (\vec{a} \cdot \vec{b})\vec{\omega} + (\vec{a} \cdot \vec{b})\vec{\omega} - (\vec{a} \cdot \vec{\omega})\vec{b}$$

$$= (\omega \cdot b)\vec{a} - (a \cdot \omega)\vec{b}$$

$$= \vec{\omega} \times (\vec{a} \times \vec{b})$$

$$= R.H.S$$

Q. P.T. necessary and sufficient condition for the vector $\vec{a}(t)$ to be const magnitude is $\vec{a} \cdot \vec{a}' = 0$.

Let,
$$\vec{a}(t) = x(t)\hat{i} + y(t)\hat{j} + z(t)\hat{k}$$
$$\vec{a}'(t) = x'(t)\hat{i} + y'(t)\hat{j} + z'(t)\hat{k}$$

$$|\vec{a}(t)| = \sqrt{x^2(t) + y^2(t) + z^2(t)}$$

$$|\vec{a}'(t)| = \sqrt{x'^2(t) + y'^2(t) + z'^2(t)}$$

if $\vec{a}$ is const mag vec,

$$x^2(t) + y^2(t) + z^2(t) = x'^2(t) + y'^2(t) + z'^2(t)$$

$$|\vec{a}(t)|^2 = a^2$$

$$\vec{a}(t) \cdot \vec{a}(t) = a^2$$

$$\vec{a}' \cdot \vec{a} + \vec{a} \cdot \vec{a}' = 0$$

$$2\,\vec{a} \cdot \vec{a}' = 0$$

$$\vec{a} \cdot \vec{a}' = 0$$

Q. P.T. necessary and sufficient condition for $\vec{a}(t)$ to be const dir $\rightarrow$ $\vec{a}\times\vec{a}'=0$

if, $\vec{a}$ is const dir. $\Rightarrow$ $\hat{a}=\frac{\vec{a}}{|\vec{a}|}=$ Const

$\frac{d}{dt}\hat{a}=0$

$\cancel{\frac{d^2}{dt^2}\hat{a}=0}$

$\frac{d}{dt}(\vec{a}\times\vec{a}')$

$=\cancel{\vec{a}'\times\vec{a}'+\vec{a}\times\vec{a}''}$

$=\cancel{0+\vec{a}\times\vec{a}''}$

$=\cancel{\vec{a}\times\vec{a}''}\ \cancel{\hat{a}|\vec{a}|\times\hat{a}''|\vec{a}''|}$

$=$

$=\hat{a}|\vec{a}|\times\frac{d}{dt}\hat{a}|\vec{a}|$

$=\hat{a}|\vec{a}|\times\left\{\cancel{\frac{d}{dt}\hat{a}}|\vec{a}|+\frac{d}{dt}|\vec{a}|\hat{a}\right\}$

$=\hat{a}|\vec{a}|\times\frac{d}{dt}|\vec{a}|\hat{a}$

$=0$ = R.H.S.

Q. At any instant t, the position vector of a moving particle on plane $\vec{r}=\cos\omega t\,\hat{i}+\sin\omega t\,\hat{j}$, where ω is const, show $\vec{v}\perp\vec{r}$, $\hat{a}$ is towards O and $|\vec{a}|\propto\vec{r}$, $\vec{r}\times\vec{v}$ is const vector.

$\vec{r}=\cos\omega t\,\hat{i}+\sin\omega t\,\hat{j}$

$\vec{v}=-\omega\sin\omega t\,\hat{i}+\omega\cos\omega t\,\hat{j}$

$\vec{a}=-\omega^2\cos\omega t\,\hat{i}-\omega^2\sin\omega t\,\hat{j}$

$$\begin{array}{ccc} i & j & k \\ \cos\omega t & \sin\omega t & 0 \\ -\omega\sin\omega t & \omega\cos\omega t & 0 \end{array}$$

$\omega(\cos^2\omega t+$

ω

$\vec{v}\cdot\vec{r}=-\omega\sin\omega t\cos\omega t+\omega\sin\omega t\cos\omega t$

$=0$

$\therefore\ \vec{v}\perp\vec{r}$

$\vec{a}=-\omega^2(\vec{r})$

$|\vec{a}|\propto|\vec{r}|$

$\hat{a}=-\cos\omega t\,\hat{i}-\sin\omega t\,\hat{j}$

$\hat{a}=-\vec{r}$

$\vec{r}\times\vec{v}=\omega\cos^2\omega t+\omega\sin^2\omega t$

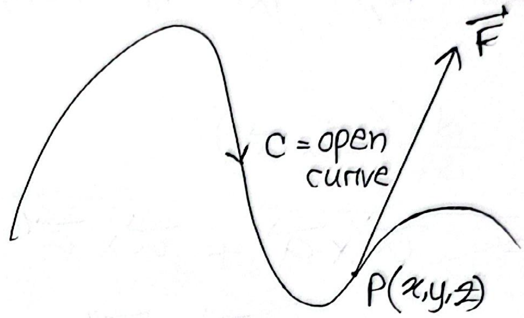
$=\omega$

$=$ const

Vector Integrations :

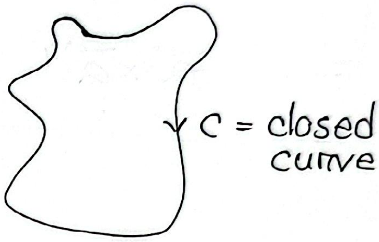
Vector line integral

$$\int_C \vec{F} \cdot d\vec{r}$$



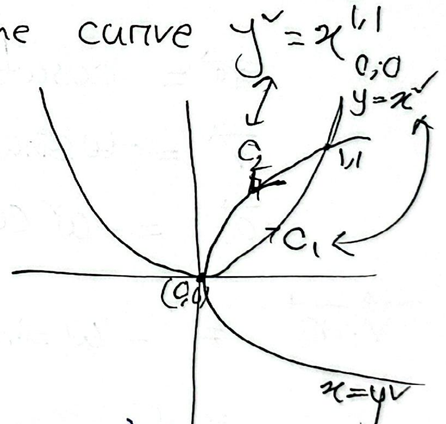
$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$
$$d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$$

<u>Note:</u> $\oint_C \vec{F} \cdot d\vec{r} \rightarrow$ For closed curve



$\int_C \vec{F} \cdot d\vec{r} \rightarrow$ for open curve

Q F.T circulation of $\vec{F}$ round the curve C, where $\vec{F} = (2x+y^2)\hat{i} + (3y-4x)\hat{j}$ and C is the curve $y = x^2$ from (0,0) to (1,1) and the curve $y^2 = x$ 1,1 0,0



Given,

$$\vec{F} = (2x+y^2)\hat{i} + (3y-4x)\hat{j}$$

Let, $\vec{r} = x\hat{i} + y\hat{j}$

$$d\vec{r} = dx\hat{i} + dy\hat{j}$$

Now, $\vec{F} \cdot d\vec{r} = (2x+y^2)\,dx + (3y-4x)\,dy$

then, for the curve C_1 : the path along $y = x^2$ from (0,0) to (1,1), we have

$$\int_{C_1} \vec{F} \cdot d\vec{r} =$$

$y = x^2$

$dy = 2x\,dx$

$$\int_{C_1} \vec{F}\cdot d\vec{r} = \int_0^1 (2x+\cancel{x^4})\,dx + (3x^2-4x)\,2x\,dx$$

$$= \left[x^2 + \frac{x^5}{5} + \frac{6x^4}{4} - \frac{8x^3}{3} \right]_0^1$$

$$= \cancel{\frac{2\cdot 20+4+30-80}{120}} \quad \frac{60+12+90-160}{60}$$

$$= \cancel{-\frac{8}{}}\;\cancel{2} \quad \frac{1}{30}$$

$y^2 = x$

$2y\,dy = dx \Rightarrow dy = \frac{dx}{2y}$

$$\int_{C_2} \vec{F}\cdot d\vec{r} = \cancel{\int_0 (2x+x)\,dx} \quad \int_1^0 (2y^2+y^4)\,2y\,dy + (3y-4y^2)\,dy$$

$$\cancel{= \left[\frac{6y^4}{4} + \frac{3y^2}{2} - \frac{4y^3}{3} \right]_1^0}$$

$$\cancel{= \frac{-36+45+40}{30} \quad \frac{-18-18+16}{12}}$$

$$\cancel{= \frac{41}{30}}$$

$$= \int_1^0 (6y^3+3y-4y^2)\,dy$$

$$= \left[6\frac{y^4}{4} + \frac{3y^2}{2} - \frac{4y^3}{3} \right]_1^0$$

$$= -\frac{3}{2} - \frac{3}{2} + \frac{4}{3}$$

$$= -3 + \frac{4}{3} = -\frac{5}{3}$$

Hence the circulation of $\vec{F}$ round the curve C

$$\oint_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r}$$

$$= \frac{1}{30} - \frac{5}{3}$$

$$= \frac{-49}{30}$$

Q. If $\vec{F} = (5xy - 6x^2)\hat{i} + (2y - 4x)\hat{j} + 2z^2\hat{k}$

Find $\int \vec{F}\cdot d\vec{r}$ along the curve C

(i) the straight lines from $(0,0,0)$ to $(0,0,1)$ then to $(0,1,1)$ and then to $(1,1,1)$

(ii) $x = t^2$, $y = t$, $z = t^3$

(iii) the line from $(0,0,0)$ to $(1,1,1)$

(i) $C_1 =$ ~~x~~, $x=0, y=0, z=0$ to 1, $dx=0, dy=0$

$C_2 = x=0, y=0$ to 1, $z=0 \Rightarrow dx=0=dz$, $y=0$ to 1

$C_3 = x = 0$ to 1, $y=1$, $z=1$

Given, $\vec{F} = (5xy - 6x^2)\hat{i} + (2y-4x)\hat{j} + 2z^2\hat{k}$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

$$\vec{F}\cdot d\vec{r} = (5xy - 6x^2)dx + (2y-4x)dy + 2z^2dz$$

for, C_1 $\int \vec{F}\cdot d\vec{r} = (\cancel{5xy - 6x^2})^{\to 0}dx + 0 + 2z^2dz$

$$= \left[2\frac{z^3}{3}\right]_0^1$$

$$= \frac{2}{3}$$

for, C_2 $\int_0^1 2y\,dy \Rightarrow \left[2\frac{y^2}{2}\right]_0^1$

$\Rightarrow 1$

for C_3 $\int_0^1 \vec{F}\cdot d\vec{r} = \int_0^1 \cancel{5x} 6x^2\, dx$

$= \frac{5}{2} - 6\times\frac{x^3}{3}$

$= \cancel{2} \quad \frac{15-12}{6}$

$= \frac{1}{2}$

$\therefore$ for C,

$$\int_C \vec{F}\cdot d\vec{r} = \cancel{C_1+C_2+C_3} \quad \cancel{\tfrac{2}{3}+1+\tfrac{1}{2}} \quad \int_{C_1} + \int_{C_2} + \int_{C_3}$$

$$= \cancel{\tfrac{-1}{3}} \quad \frac{4+6+3}{6}$$

$$= \frac{13}{6}$$

(ii) Given,

$$\vec{F} = (5xy - 6x^2)\hat{i} + (2y - 4x)\hat{j} + 2z^2\hat{k}$$

$x = t^2 \Rightarrow dx = 2t\,dt$ $\qquad d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

$y = t \Rightarrow dx = dt$

$z = t^3 \Rightarrow dz = 3t^2 dt$

$$\int_C \vec{F}\cdot d\vec{r} = \int_0^1 (5\times t^2\times t - 6\times t^4)2t\,dt + (2\times t - 4\times t^2)dt + 2(t^6)3t^2 dt$$

$$\cancel{= 5\frac{t^4}{4} - 6\frac{t^5}{5}}$$

$$= \int_0^1 10t^4 - 12t^5 + 2t - 4t^2 + 6t^8\, dt$$

$$\frac{1}{3} \Leftarrow = \left[\cancel{10}^{2}\times\frac{t^5}{\cancel{5}} - \cancel{12}^{2}\times\frac{t^6}{\cancel{6}} + \cancel{2}\frac{t^2}{\cancel{2}} - 4\times\frac{t^3}{3} + \cancel{6}^{2}\times\frac{t^9}{\cancel{9}_3}\right]_0^1$$

(iii) Given,

$$\vec{F} = (5xy - 6x^2)\hat{i} + (2y - 4x)\hat{j} + 2z^2\hat{k}$$

~~Let,~~

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$$

~~x, y, z all are 0 to~~

~~dx = dy = dz = 1~~

$$\int_C \vec{F} \cdot d\vec{r} = 5$$

The parametrisation of line segment from (0,0,0) to (1,1,1)

$$\vec{r}(t) = (1-t)(0,0,0) + t(1,1,1)$$

$$= (t, t, t) \text{ when } 0 \le t \le 1$$

$x(t) = t$

$y(t) = t$

$z(t) = t$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$$\int_C \vec{F} \cdot d\vec{r}$$

$$= \int_{t=0}^{1} (5t^2 - 6t^2)\,dt + (2t - 4t)\,dt + 2t^2\,dt$$

$$= \int_0^1 (t^2 - 2t)\,dt$$

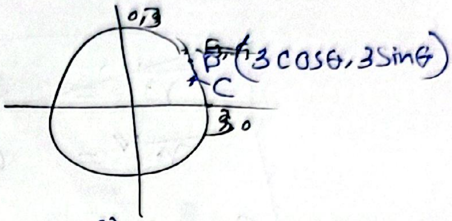
$$= \left[\frac{t^3}{3} - t^2 \right.$$

$$= -\frac{2}{3} \text{ (Ans)}$$

Q F.T. circulation of $\vec{F}$ around the curve C, where $\vec{F} = (2x-y+4z)\hat{i} + (x+y-z)\hat{j} + (3x-2y+4z^3)\hat{k}$ and C is the circle $x^2+y^2=9, z=0$

$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$d\vec{r} = dx\,\hat{i} + dy\,\hat{j} + dz\,\hat{k}$ [$dz=0$



$$\oint_C \vec{F}\cdot d\vec{r} = \oint_0^{2\pi} (2x-y+4z)\,dx + (x+y-z)\,dy + (3x-2y+4z^3)\,dz$$

Curve ~~x^2+y^2~~ closed curve $x^2+y^2=9$, $z=0$ we have,

$x = 3\cos\theta \qquad y = 3\sin\theta$

$dx = -3\sin\theta\, d\theta \qquad dy = 3\cos\theta\, d\theta \qquad dz = 0$

$$\oint_C \vec{F}\cdot d\vec{r} = \int_0^{2\pi} (6\cos\theta - 3\sin\theta)\times -3\sin\theta\, d\theta + 9(\cos\theta+\sin\theta)\cos\theta\, d\theta$$

~~$= \int_c^{2\pi} (18\cos\theta - 3\sin\theta$~~

$$= \int_0^{2\pi} -18\sin\theta\cdot\cos\theta + 9\sin^2\theta + 9\cos^2\theta + 9\sin\theta\cos\theta\, d\theta$$

$$= \int_0^{2\pi} 9 - 9\sin\theta\cdot\cos\theta\, d\theta$$

$$= \left[9\theta + \frac{9}{4}\times\cos 2\theta\right]_0^{2\pi}$$

$= 18\pi + \cancel{\frac{9}{2}\times 4\pi^2} + \cancel{\frac{9}{4}} - \cancel{\frac{9}{4}}$

~~$= 18(\pi+\pi^2)$~~

$= 18\pi\ \cancel{(1+\pi)}$

Vector Integration of single variable

Q. Evaluate $\int_2^3 \left(\vec{r} \times \frac{d^2\vec{r}}{dt^2}\right) dt$ where $\vec{r} = t^3\hat{i} + 2t^2\hat{j} + 3t\hat{k}$

$$\cancel{\vec{r}' = 3t^2dt\,\hat{i} + 4tdt\,\hat{j} + 3dt\,\hat{k}}$$
$$\cancel{\vec{r}'' = 6tdt\,\hat{i} + 4dt^2\,\hat{j} + 3dt^2\,0\hat{k}}$$
~~So,~~ $\cancel{\vec{r} \times \vec{r}''}$

$$\vec{r}' = 3t^2\hat{i} + 4t\hat{j} + t\hat{k}$$
$$\vec{r}'' = 6t\hat{i} + 4\hat{j} + \hat{k}$$

So, $\vec{r} \times \vec{r}''$

$$\begin{matrix} i & j & k \\ t^3 & 2t^2 & 3t \\ 6t & 4 & 1 \end{matrix}$$

$$= (2t^2 - 12t)\hat{i} + (18t^2 - t^3)\hat{j} + \hat{k}(4t^3 - 12t^3)$$

$$\int_2^3 (\vec{r} \times \vec{r}'')\, dt = \left[(2t^2 - 12t)\hat{i} + (18t^2 - t^3)\hat{j} + (4t^3 - 12t^3)\hat{k}\right]_2^3$$

$$= \left[\frac{2t^3}{3}\hat{i} - \frac{12t^2}{2}\hat{i} + \frac{18t^3}{3}\hat{j} - \frac{t^4}{4}\hat{j} + \frac{4t^4}{4}\hat{k} - \frac{12t^4}{4}\hat{k}\right]_2^3$$

$$= \left[\hat{i}\left(\frac{4t^3 - 36t^2}{6}\right) + \hat{j}\left(\frac{72t^3 - 3t^4}{12}\right) + \hat{k}\left(\cancel{-8t^4}\,-2t^4\right)\right]_2^3$$

$$= -17.33\hat{i} + 97.75\hat{j} - 130\hat{k}$$

by another method

$$\frac{d}{dt}\left(\vec{r} \times \frac{d\vec{r}}{dt}\right)$$
$$= \vec{r} \times \frac{d^2\vec{r}}{dt^2} + \cancel{\frac{d\vec{r}}{dt} \times \frac{d\vec{r}}{dt}}^{\,0}$$
$$= \vec{r} \times \frac{d^2\vec{r}}{dt^2}$$

$$d\left(\vec{r}\times\frac{d\vec{r}}{dt}\right)=\left(\vec{r}\times\frac{d^2\vec{r}}{dt^2}\right)dt$$

$$=\int_2^3 d\left(\vec{r}\times\frac{d\vec{r}}{dt}\right)=\int_2^3\left(\vec{r}\times\frac{d^2\vec{r}}{dt^2}\right)dt$$

So, $$\int_2^3\left(\vec{r}\times\frac{d^2\vec{r}}{dt^2}\right)dt=\left[\vec{r}\times\frac{d\vec{r}}{dt}\right]_2^3$$

my nme

$j\left(\frac{24t^3}{4}\right)$

my na

my name is

$x+y^2=1$

Given, $\vec{r}(t)=2\hat{i}-\hat{j}+2\hat{k}$

at $t=2$ ~~$r(t)=$~~

and $\vec{r}(t)=4\hat{i}-2\hat{j}+3\hat{k}$

at $t=3$

$$\frac{d(\vec{r}\cdot\vec{r})}{dt}=\vec{r}\cdot\frac{dr}{dt}+\frac{dr}{dt}\cdot r$$

$$\int_2^3 d(\vec{r}\cdot\vec{r})=\int_2^3 2\,\vec{r}\cdot\frac{d\vec{r}}{dt}\,dt$$

$$\left[\vec{r}\cdot\vec{r}\right]_2^3$$

$$\frac{1}{2}\left[|\vec{r}|^2\right]_2^3=\cancel{\tfrac{3}{2}}\int_2^3\vec{r}\cdot\frac{d\vec{r}}{dt}$$

$$\int_2^3\vec{r}\cdot\frac{d\vec{r}}{dt}=\frac{1}{2}\left[29-9\right]$$

$$=10\ \text{(proved)}$$

Classification of Conics :

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$$

↳ General Eq^n of 2nd degree in x, y variables.

Notations

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = abc + 2fgh - af^2 - bg^2 - ch^2$$

$$D = \begin{vmatrix} a & h \\ h & b \end{vmatrix} = ab - h^2$$

Q. Reduce eq^n $x^2 + 2xy + y^2 - 4x + 8y - 6 = 0$ to its canonical form and the conic. [Non-cetral conic]

knowlegde

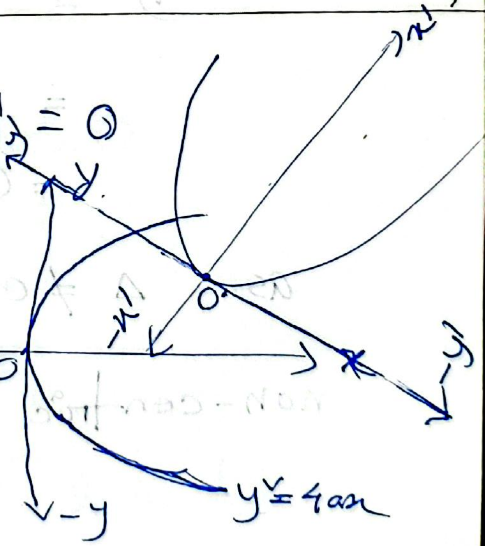
$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

if $\Delta \neq 0$ & $D = 0$ ⇒ non-central conic (parabola for our syll)

parabola = $y^2 = 4ax$ → canonical form / normal form

$$\boxed{ax^2 + 2hxy + by^2} + \boxed{2gx + 2fy + c} = 0$$

$y^2 = 4ax$



(i) Rotation
(ii) Translation

} steps to convert gn eq to standard eq^n.

Solution

comparing the given eq^n

$$x^2 + 2xy + y^2 - 4x + 8y - 6 = 0 \quad -(i)$$

with

$$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0 \quad -(ii)$$

$a = 1,\ h = 1,\ b = 1,\ g = -2,\ f = 4,\ c = -6$

Now,

$$\Delta = \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1 & -2 \\ 1 & 1 & 4 \\ -2 & 4 & -6 \end{vmatrix}$$

$$= abc + 2fgh - af^2 - bg^2 - ch^2$$

$$= -36 \neq 0$$

$$D = ab - h^2$$

$$= 1 - 1$$

$$= 0$$

as $\because \Delta \neq 0$ & $D = 0$ we will get a non-centric ~~parabola~~ conic (parabola).

Now,

Ratating the axes through an angle θ :

$$\left.\begin{aligned} x &= x'\cos\theta - y'\sin\theta \\ y &= x'\sin\theta + y'\cos\theta \end{aligned}\right\} \text{ where } \theta = \frac{1}{2}\tan^{-1}\left(\frac{2h}{a-b}\right)$$

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{1}{0}\right) = \frac{\pi}{4}$$

$$\left.\begin{aligned} x &= \frac{x'-y'}{\sqrt{2}} \\ y &= \frac{x'+y'}{\sqrt{2}} \end{aligned}\right\} \text{——— (ii)}$$

using ~~f~~ (ii) to the given eqⁿ (i) reduces to

$$\left(\frac{x'-y'}{\sqrt{2}}\right)^2 + 2\left(\frac{x'-y'}{\sqrt{2}}\right)\left(\frac{x'+y'}{\sqrt{2}}\right) + \left(\frac{x'+y'}{\sqrt{2}}\right)^2 - 4\times\left(\frac{x'-y'}{\sqrt{2}}\right) + 8\left(\frac{x'+y'}{\sqrt{2}}\right) - 6 = 0$$

$$\Rightarrow \frac{x'^2 - 2x'y' + y'^2}{2} + x'^2 - y'^2 + \frac{x'^2 + 2x'y' + y'^2}{2} + \frac{8x' + 8y' - 4x' + 4y'}{\sqrt{2}} - 6 = 0$$

$$\Rightarrow \frac{x'^2 - 2x'y' + y'^2 + 2x'^2 - 2y'^2 + x'^2 + 2x'y' + y'^2 + 8\sqrt{2}x' + 8\sqrt{2}y' - 4\sqrt{2}x' + 4\sqrt{2}y' - 12}{2} = 0$$

$$\Rightarrow 4x'^2 + 4\sqrt{2}x' + 12\sqrt{2}y' - 12 = 0$$

$$\Rightarrow x'^2 + \sqrt{2}x' + 3\sqrt{2}y' - 3 = 0$$

$$\Rightarrow 2x'^2 + \sqrt{2}(2x' + 6y') - 6 = 0$$

Now for translation I should look for something wholesquare.

$\Rightarrow (\sqrt{2}x'+1)^2 + 6\sqrt{2}y' - 7 = 0$

this will eat all x' and x'^2

$$\Rightarrow (\sqrt{2}x'+1)^2 = \cancel{7 - 6\sqrt{2}y'} - 6\sqrt{2}\left(y' - \frac{7}{6\sqrt{2}}\right)$$

Translation

$$\left.\begin{aligned} X &= x'+\alpha \\ Y &= y'+\beta \end{aligned}\right\}$$ no coeff. in front of ~~X & Y~~ x' & y'

$\Rightarrow$ so $\sqrt{2}x'$ → disturbing

$$\Rightarrow 2\left(x' + \frac{1}{\sqrt{2}}\right)^2 = -6\sqrt{2}\left(y' - \frac{7}{6\sqrt{2}}\right)$$

$$\Rightarrow \left(x' + \frac{1}{\sqrt{2}}\right)^2 = -\frac{6\sqrt{2}}{\sqrt{2}}\left(y' - \frac{7}{6\sqrt{2}}\right)$$

$$\left.\begin{aligned} x' + \frac{1}{\sqrt{2}} &\equiv X = x'+\alpha \\ y' - \frac{7}{6\sqrt{2}} &\equiv Y = y'+\beta \end{aligned}\right\} \quad \text{---(4)}$$

Translating the oringin to the point

$$O' \equiv (\alpha, \beta)$$
$$\equiv \left(\frac{1}{\sqrt{2}}, -\frac{7}{6\sqrt{2}}\right)$$ by (4) and thus

the eqn (3) becomes

$$\boxed{X^2 = \frac{6\sqrt{2}}{2}Y}$$

reduced canonical form of gn. eqn (1)

Q. $7x^2 - 6xy \mp y^2 - 4x - 4y - 2 = 0$, reduce and ~~ex~~ say the nature of the conic.

Comparing
$$7x^2 - 6xy - y^2 + 4x - 4y - 2 = 0 \quad - 41$$
with $ax^2 + 2hxy + by^2 + 2gx \mp 2fy + c = 0$

we get
$$a = 7, \quad b = -1, \quad h = -3, \quad g = +2, \quad f = -2, \quad c = -2$$

So, $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$

$= 14 + 24 \cancel{\oplus} - 28 + 4 + 18$

$= 32$

$D = -7 - 9$

$= -16$

So, this general eq^n of 2nd degree is ~~a~~ non-centric, hyperbola.

to reduce this eq^n we need to rotate that by an angle θ:

$$\theta = \frac{1}{2}\tan^{-1}\left(\frac{2h}{a-b}\right)$$
$$= \frac{1}{2}\tan^{-1}\left(\frac{-6}{8}\right)$$
$$= \frac{1}{2}\tan^{-1}\left(-\frac{3}{4}\right)$$
$$= -18.435$$

$$x = x' \cos\theta - y'\sin\theta$$

$$y = x' \overset{\sin\theta}{\cancel{\cos\theta}} + y'\cancel{\sin\theta}\cos\theta$$

$$\left.\begin{aligned} x &= 0.915x' + 0.4y' \\ y &= -0.4x' + 0.915y' \end{aligned}\right\} \text{—(ii)}$$

putting (ii) to (i)

$\Rightarrow ($

Sphere

$(x-\alpha)^2+(y-\beta)^2=r^2$ eq^n of circle where (α,β) is the center r is the radius.

if (α,β,γ) be the center of the sphere and r is the radius

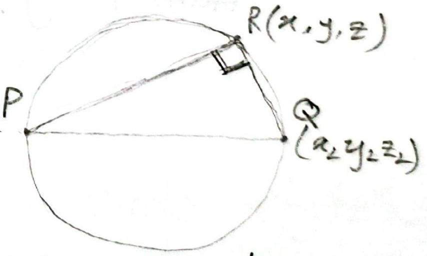
$$(x-\alpha)^2+(y-\beta)^2+(z-\gamma)^2=r^2 \longrightarrow \text{Basic Form}$$

General form:

$$x^2+y^2+z^2+2gx+2fy+2hz+c=0$$

where, $(-g,-f,-h)$ is the center and radius is $\sqrt{g^2+f^2+h^2-c}$

Eq^n of a sphere on a diameter with give Extremities



$(x-x_1),(y-y_1),(z-z_1) \rightarrow$ direction Ratio $(x_1y_1z_1)P$ of PR

$(x-x_2),(y-y_2),(z-z_2)$ " " " of RQ

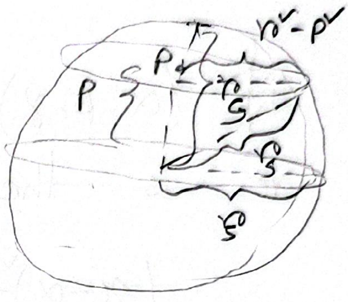
$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)+(z-z_1)(z-z_2)=0$ [dot product zero]

—— this is the eq^n

Eq^n of a circle from a sphere

$$x^2+y^2+z^2+2gx+2fy+2hz+C=0, \quad lx+my+nz=\phi$$

Types of circle



(i) greed circle

$P=0$

$r_c = r_s$

(ii) small circle

when

$p < r$

$r_c = \sqrt{r_s^2 - p^2}$

Radical Plane

$$S_1 = x^2+y^2+z^2+2g_1x+2f_1y+2h_1z+c_1=0$$

$$S_2 = x^2+y^2+z^2+2g_2x+2f_2y+2h_2z+C_2=0$$

points of intersection will satisfy the eq^{n}s S_1 and S_2

$$S_1 - S_2 \equiv 2(g_1-g_2)x+2(f_1-f_2)y+2(h_1-h_2)z+C_1-C_2$$

— this is the eq^n of radical plane. $= 0$

~~agin~~ $(-g,-f,-h)$ is the center.

Q. Obtain the equation of the sphere having the circle $x^2+y^2+z^2+10y-4z-8=0$, $x+y+z=3$ as a great circle.

$$x^2+y^2+z^2+10y-4z-8+\lambda(x+y+z-3)=0$$

$$\Rightarrow x^2+y^2+z^2+\lambda x+(10+\lambda)y+(\lambda-4)z-(8+3\lambda)=0$$

comparing it with

$$x^2+y^2+z^2+2gx+2fy+2hz+c=0$$

$$\Rightarrow g=-\frac{\lambda}{2},\ f=-\frac{10+\lambda}{2},\ h=\frac{-(\lambda-4)}{2}$$

center exists on $x+y+z=3$ so,

$$-\frac{\lambda}{2}-\frac{10+\lambda}{2}-\frac{\lambda-4}{2}=3$$

$$\Rightarrow \lambda+10+\lambda+\lambda-4=-6$$

$$\Rightarrow \lambda=-4$$

So, the eqn,

$$x^2+y^2+z^2-4x+6y-8z+4=0$$

Q. F.T. Eqn of sphere through the circle $x^2+y^2+z^2=25$, $x+2y-z+2=0$ and the point $(1,1,1)$

$$x^2+y^2+z^2+\lambda(x+2y-z+2)-25=0 \quad -(1)$$

Comp ---

$$f=-\frac{\lambda}{2},\ g=-\lambda,\ h=\frac{\lambda}{2}$$

Since (1) passes through (1,1,1)

$$1^2+1^2+1^2-25 \quad \lambda(1+2-1+2)=0$$

$$\lambda = \frac{11}{2}$$

$$x^2+y^2+z^2+\frac{11}{2}x+11y-\frac{11}{2}z \overset{-14}{\cancel{+11-25}}=0 \quad \text{(Ans)}$$

Q. F.T. center and the radius of the circle given by $2x-3y+6z=62$, $x^2+y^2+z^2-4x+2y-2z-58=0$

$\cancel{2x-3y}$ $x^2+y^2+z^2-4x+2y-2z-58=0$

Com_r ____

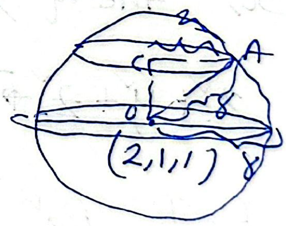
$f=-2$, $g=+1$, $h=-1$, $c=-58$

So the center of the sphere,

$\equiv(2,-1,1)$ and radius of the

sphere $\sqrt{4+1+1+58}$

$=8$



$$\overline{OC}=\left|\frac{2\times2+(-3)\times(-1)+6\times1-62}{\sqrt{4+9+36}}\right|$$

$$=\left|\frac{-49}{7}\right|$$

$$=|+7|=7$$

$r^2=64-49$

$r=\sqrt{15}$

$O \equiv (2,-1,1)$

$2x - 3y + 6z = 62$ the eqn of OC in symmetric form $\frac{x-2}{2} = \frac{y+1}{-3} = \frac{z-1}{6} = r$

(say) where r is non zero

so anypoint on this ~~plane~~ line,

$(2r+2), (-3r-1), (6r+1)$

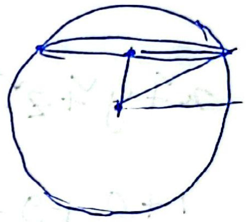
$2(2r+2) - 3(-3r-1) + 6(6r+1) = 62$ [as C lies on the line and the plane also]

$49r + 13 = 62$

$r = \frac{49}{49}$

$r = 1$

~~so~~ $(4,-4,7)$ ( )



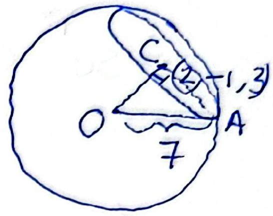
$x^2 + y^2 + z^2 - 2y - 4z - 11 = 0$

. center $= (0, 1, 2)$

$r_s = \sqrt{16}$

Q F.T. eq^n of the circle on the sphere $x^2+y^2+z^2=49$ whose center is at $(2,-1,3)$.

the center of the sphere $\equiv (0,0,0)$
and the radius $= 7$ unit.



So, $\overline{OC} = \sqrt{2^2+(-1)^2+3^2}$

$= \sqrt{14}$

So, $CA = \sqrt{49-14}$

$= \sqrt{35}$

So, if the eq^n of the circle is

$(x-2)^2+(y+1)^2+(z-3)^2 = 35$

$x^2+y^2+z^2-4x-6y+2z-16+\overset{1/2}{\lambda}(3x+y+3z-4)=0$

$1+0+9-4+6-16+\lambda(3+9-4)=0$

$+8\lambda = +4$

$\lambda = \frac{1}{2}$

$x^2+y^2+z^2$