

7.3 The Jacobi and Gauss-Seidel Iterative Methods

The Jacobi Method

Two assumptions made on Jacobi Method:

1. The system given by

$$\begin{aligned}a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n &= b_1 \\a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n &= b_2 \\&\vdots \\a_{n1}x_1 + a_{n2}x_2 + \cdots a_{nn}x_n &= b_n\end{aligned}$$

Has a unique solution.

2. The coefficient matrix A has no zeros on its main diagonal, namely, $a_{11}, a_{22}, \dots, a_{nn}$ are nonzeros.

Main idea of Jacobi

To begin, solve the 1st equation for x_1 , the 2nd equation for x_2 and so on to obtain the rewritten equations:

$$\begin{aligned}x_1 &= \frac{1}{a_{11}}(b_1 - a_{12}x_2 - a_{13}x_3 - \cdots a_{1n}x_n) \\x_2 &= \frac{1}{a_{22}}(b_2 - a_{21}x_1 - a_{23}x_3 - \cdots a_{2n}x_n) \\&\vdots \\x_n &= \frac{1}{a_{nn}}(b_n - a_{n1}x_1 - a_{n2}x_2 - \cdots a_{n,n-1}x_{n-1})\end{aligned}$$

Then make an initial guess of the solution $\mathbf{x}^{(0)} = (x_1^{(0)}, x_2^{(0)}, x_3^{(0)}, \dots, x_n^{(0)})$. Substitute these values into the right hand side the of the rewritten equations to obtain the *first approximation*, $(x_1^{(1)}, x_2^{(1)}, x_3^{(1)}, \dots, x_n^{(1)})$.

This accomplishes one **iteration**.

In the same way, the *second approximation* $(x_1^{(2)}, x_2^{(2)}, x_3^{(2)}, \dots, x_n^{(2)})$ is computed by substituting the first approximation's x -vales into the right hand side of the rewritten equations.

By repeated iterations, we form a sequence of approximations $\mathbf{x}^{(k)} = (x_1^{(k)}, x_2^{(k)}, x_3^{(k)}, \dots, x_n^{(k)})^t$, $k = 1, 2, 3, \dots$

The Jacobi Method. For each $k \geq 1$, generate the components $x_i^{(k)}$ of $\mathbf{x}^{(k)}$ from $\mathbf{x}^{(k-1)}$ by

$$x_i^{(k)} = \frac{1}{a_{ii}} \left[\sum_{\substack{j=1, \\ j \neq i}}^n (-a_{ij}x_j^{(k-1)}) + b_i \right], \quad \text{for } i = 1, 2, \dots, n$$

Example. Apply the Jacobi method to solve

$$\begin{aligned} 5x_1 - 2x_2 + 3x_3 &= -1 \\ -3x_1 + 9x_2 + x_3 &= 2 \\ 2x_1 - x_2 - 7x_3 &= 3 \end{aligned}$$

Continue iterations until two successive approximations are identical when rounded to three significant digits.

Solution To begin, rewrite the system

$$\begin{aligned} x_1 &= \frac{-1}{5} + \frac{2}{5}x_2 - \frac{3}{5}x_3 \\ x_2 &= \frac{2}{9} + \frac{3}{9}x_1 - \frac{1}{9}x_3 \\ x_3 &= -\frac{3}{7} + \frac{2}{7}x_1 - \frac{1}{7}x_2 \end{aligned}$$

Choose the initial guess $x_1 = 0, x_2 = 0, x_3 = 0$

The first approximation is

$$\begin{aligned} x_1^{(1)} &= \frac{-1}{5} + \frac{2}{5}(0) - \frac{3}{5}(0) = -0.200 \\ x_2^{(1)} &= \frac{2}{9} + \frac{3}{9}(0) - \frac{1}{9}(0) = 0.222 \\ x_3^{(1)} &= -\frac{3}{7} + \frac{2}{7}(0) - \frac{1}{7}(0) = -0.429 \end{aligned}$$

Continue iteration, we obtain

n	$k = 0$	$k = 1$	$k = 2$	$k = 3$	$k = 4$	$k = 5$	$k = 6$
$x_1^{(k)}$	0.000	-0.200	0.146	0.192			
$x_2^{(k)}$	0.000	0.222	0.203	0.328			
$x_2^{(k)}$	0.000	-0.429	-0.517	-0.416			

The Jacobi Method in Matrix Form

Consider to solve an $n \times n$ size system of linear equations $A\mathbf{x} = \mathbf{b}$ with $A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nn} \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$ for $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$.

We split A into

$$A = \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & a_{nn} \end{bmatrix} - \begin{bmatrix} 0 & \dots & 0 & 0 \\ -a_{21} & \dots & 0 & 0 \\ \vdots & & \ddots & \vdots \\ -a_{n1} & \dots & -a_{n,n-1} & 0 \end{bmatrix} - \begin{bmatrix} 0 & -a_{12} & \dots & -a_{1n} \\ 0 & 0 & & \vdots \\ \vdots & \vdots & \ddots & -a_{n-1,n} \\ 0 & 0 & \dots & 0 \end{bmatrix} = D - L - U$$

$A\mathbf{x} = \mathbf{b}$ is transformed into $(D - L - U)\mathbf{x} = \mathbf{b}$

$$D\mathbf{x} = (L + U)\mathbf{x} + \mathbf{b}$$

Assume D^{-1} exists and $D^{-1} = \begin{bmatrix} \frac{1}{a_{11}} & 0 & \dots & 0 \\ 0 & \frac{1}{a_{22}} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \frac{1}{a_{nn}} \end{bmatrix}$

Then

$$\mathbf{x} = D^{-1}(L + U)\mathbf{x} + D^{-1}\mathbf{b}$$

The matrix form of Jacobi iterative method is

$$\mathbf{x}^{(k)} = D^{-1}(L + U)\mathbf{x}^{(k-1)} + D^{-1}\mathbf{b} \quad k = 1, 2, 3, \dots$$

Define $T = D^{-1}(L + U)$ and $\mathbf{c} = D^{-1}\mathbf{b}$, Jacobi iteration method can also be written as

$$\mathbf{x}^{(k)} = T\mathbf{x}^{(k-1)} + \mathbf{c} \quad k = 1, 2, 3, \dots$$

Numerical Algorithm of Jacobi Method

Input: $A = [a_{ij}]$, \mathbf{b} , $\mathbf{XO} = \mathbf{x}^{(0)}$, tolerance TOL , maximum number of iterations N .

Step 1 Set $k = 1$

Step 2 while $(k \leq N)$ do Steps 3-6

Step 3 For for $i = 1, 2, \dots, n$

$$x_i = \frac{1}{a_{ii}} \left[\sum_{\substack{j=1, \\ j \neq i}}^n (-a_{ij}\mathbf{XO}_j) + b_i \right],$$

Step 4 If $\|\mathbf{x} - \mathbf{XO}\| < TOL$, then OUTPUT $(x_1, x_2, x_3, \dots, x_n)$;
STOP.

Step 5 Set $k = k + 1$.

Step 6 For for $i = 1, 2, \dots, n$

Set $\mathbf{XO}_i = x_i$.

Step 7 OUTPUT $(x_1, x_2, x_3, \dots, x_n)$;
STOP.

Another stopping criterion in Step 4: $\frac{\|\mathbf{x}^{(k)} - \mathbf{x}^{(k-1)}\|}{\|\mathbf{x}^{(k)}\|}$