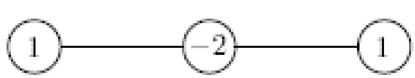
# Lecture 9: Numerical Partial Differential Equations(Part 2)

# Finite Difference Method to Solve Poisson's Equation

- Poisson's equation in 1D:

$$\begin{cases} -\frac{d^2u}{dx^2} = f(x), \ x \in (0,1) \\ u(0) = u(1) = 0 \end{cases}.$$

- Spatial Discretization: $0 = x_0 < \cdots < x_M = 1$. Define $\Delta x = \frac{1}{M}$. Then $x_i = i\Delta x$.
- $\frac{d^2u(x_i)}{dx^2} \sim \frac{u(x_{i-1}) - 2u(x_i) + u(x_{i+1})}{\Delta x^2}$
- Stencil of finite difference approximation



- Finite difference equations: for $i = 1, \dots, M-1$

$$-u_{i-1} + 2u_i - u_{i+1} = \Delta x^2 f_i$$
$$u_0 = 0$$
$$u_M = 0$$

with $f_i = f(x_i)$

- Put into matrix equation format:

  Let $\boldsymbol{u} = (u_1, u_2, \dots, u_{M-1})^T$, $\boldsymbol{f} = (f_1, f_2, \dots, f_{M-1})^T$

$$A\boldsymbol{u} = \Delta x^2 \boldsymbol{f}$$

$$A = \begin{bmatrix} 2 & -1 & & & & & \\ -1 & 2 & -1 & & & & \\ & -1 & 2 & -1 & & & \\ & & \ddots & \ddots & \ddots & & \\ & & -1 & 2 & -1 & & \\ & & & -1 & 2 & -1 & \\ & & & & -1 & 2 & \end{bmatrix}$$

# 2D Poisson's Equation

Consider to solve

$$\begin{cases} -\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = f(x,y), \ (x,y) \in \Omega \\ \qquad u(x,y) = 0 \quad on \quad \partial\Omega \end{cases}$$

with $\Omega$ is rectangle $(0,1) \times (0,1)$ and $\partial\Omega$ is its boundary.

- Define $h = \frac{1}{M}$.
- Spatial Discretization: $0 = x_0 < \cdots < x_M = a$ with $x_i = ih$ and $0 = y_0 < \cdots < y_M = 1$ with $y_j = jh$.

Finite difference equation at grid point $(i, j)$:

$$-\left(\frac{u_{i-1,j}-2u_{i,j}+u_{i+1,j}}{h^2}+\frac{u_{i,j-1}-2u_{i,j}+u_{i,j+1}}{h^2}\right)=f(x_i, y_j) \text{ or}$$

$$-u_{i,j-1}-u_{i-1,j}+4u_{i,j}-u_{i+1,j}-u_{i,j+1}=h^2 f(x_i, y_j)$$

- Five-point stencil of the finite difference approximation