Vector Subspace

A non-null subset S of a vector space V is subspace, iff (i) $\alpha + \beta \in S \quad \forall \alpha, \beta$ in S

(ii) $c\alpha \in S \quad \forall c \in \mathbb{R}$ and $\alpha \in S$

The two conditions (i) and (ii) can be expressed as the single condition $a\alpha + b\beta \in S \quad \forall \alpha, \beta \in S, \ a, b \in \mathbb{R}$

Ex: Let S be the subset of $\mathbb{R}^3$ defined by

$$S = \{(x, y, z) \in \mathbb{R}^3 : y = z = 0\}$$

$\Rightarrow$ S is a nonempty subset of $\mathbb{R}^3$ since $(0,0,0) \in S$

Let $\alpha = (x_1, 0, 0)$, $\beta = (x_2, 0, 0)$

(i) $\alpha + \beta = (x_1 + x_2, 0, 0) \in S$

(ii) $c\alpha = c(x_1, 0, 0) = (cx_1, 0, 0) \in S$ since $cx_1 \in \mathbb{R}$

This proves that S is subspace of $\mathbb{R}^3$

Ex: $S = \{(x, y, z) \in \mathbb{R}^3 : x^2 + y^2 = z^2\}$

$\Rightarrow$ Then S is nonempty subset of $\mathbb{R}^3$, since $(0,0,0) \in S$

Let $\alpha = (x_1, y_1, z_1) \in S \Rightarrow x_1^2 + y_1^2 = z_1^2$

$\beta = (x_2, y_2, z_2) \in S \Rightarrow x_2^2 + y_2^2 = z_2^2$

~~Therefore~~ then $\alpha + \beta = (x_1 + x_2, y_1 + y_2, z_1 + z_2)$

$$= (x_1 + x_2)^2 + (y_1 + y_2)^2$$

$$= x_1^2 + x_2^2 + 2x_1x_2 + y_1^2 + y_2^2 + 2y_1y_2$$

$$= (x_1^2 + y_1^2) + (x_2^2 + y_2^2) + 2x_1x_2 + 2y_1y_2$$

$$= z_1^2 + z_2^2 + 2x_1x_2 + 2y_1y_2$$

$$\neq (z_1 + z_2)^2$$

S is not a subspace as $\alpha + \beta \notin S$

Th: Intersection of two subspace is subspace.

Note : union of two subspace may not be a subspace

$\Rightarrow$ Let $S = \{(0, x) : x \text{ is real}\}$

$T = \{(x, 0) : x \text{ is real}\}$

Let $\alpha_1 = (0,1) \in$
$\alpha_2 = (0,0) \in$

$\Rightarrow$ Here we see $\alpha = (0,1) \in S \cup T$

$\beta = (1,0) \in S \cup T$

Then $\alpha = \alpha_1 + \alpha_2$
$= (0,1) \in S \cup T$

But $\alpha + \beta = (0,1) + (1,0) = (1,1) \notin S \cup T$

becous it neither belongs to S nor to T

So, $S \cup T$ is not a subspace.

Linear Combination

Let V be a vector space over a field $\mathbb{R}$.
Let $\alpha_1, \alpha_2, \ldots \alpha_n \in V$, a vector space β in V is said to be a linear combination of the vectors if β can be expressed as

$$\beta = c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n$$

Ex: Let $\alpha_1 = (3,5)$, $\alpha_2 = (4,-8)$ be the vectors of $\mathbb{R}^2$ or E_2 $(\mathbb{R} \times \mathbb{R})$

Then $2\alpha_1 - 3\alpha_2 = 2(3,5) - 3(4,-3)$
$= (6,10) - (12,-9)$
$= (-6,19) = \beta \in \mathbb{R}^2$

$*$ $S_1 = \{(x,y,z) \in \mathbb{R}^3 ; 2x - 4y + z = 0\}$

Let $\alpha = (x_1, y_1, z_1) \in \mathbb{R}^3 \Rightarrow 2x_1 - 4y_1 + z_1 = 0$

$\beta = (x_2, y_2, z_2) \in \mathbb{R}^3 \Rightarrow 2x_2 - 4y_2 + z_2 = 0$

(i) Then $\alpha + \beta = (x_1 + x_2,\ y_1 + y_2,\ z_1 + z_2)$

$\Rightarrow 2(x_1 + x_2) - 4(y_1 + y_2) + (z_1 + z_2)$

$= (2x_1 - 4y_1 + z_1) + (2x_2 - 4y_2 + z_2) = 0 + 0 = 0$

$\therefore \alpha + \beta \in S_1$

(ii) $c\alpha = (cx_1, cy_1, cz_1)$ where c is an arbitrary const

$\Rightarrow 2cx_1 - 4cy_1 + cz_1 = c(2x_1 - 4y_1 + z_1)$

$= c \times 0 = 0$

$\therefore c\alpha \in S_1$

$\therefore S_1 = \{(x,y,z) \in \mathbb{R}^3 : 2x - 4y + z = 0\}$ is a subspace.

$*$ $S_2 = \{(x,y,z) \in \mathbb{R}^3 : 2x - 4y + z = 1\}$

Let $\alpha = (x_1, y_1, z_1) \in \mathbb{R}^3 \Rightarrow 2x_1 - 4y_1 + z_1 = 1$

$\beta = (x_2, y_2, z_2) \in \mathbb{R}^3 \Rightarrow 2x_2 - 4y_2 + z_2 = 1$

Then $\alpha + \beta = (x_1 + x_2,\ y_1 + y_2,\ z_1 + z_2)$

$\Rightarrow 2(x_1 + x_2) - 4(y_1 + y_2) + (z_1 + z_2)$

$= (2x_1 - 4y_1 + z_1) + (2x_2 - 4y_2 + z_2)$

$= 1 + 1 = 2 \neq 1$

$\therefore \alpha + \beta \notin S_2$

$\therefore S_2 = \{(x,y,z) \in \mathbb{R}^3 : 2x - 4y + z = 1\}$ is not a subspace.

Linear dependence and Linear independence of vectors

Let V be a vector space over $\mathbb{R}$. The set of vectors $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ are called linearly dependent, if it is possible to get n scalars $c_1, c_2, \dots, c_n$ in $\mathbb{R}$, at least one nonzero, such that $c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n = \theta$

And $\{\alpha_1, \alpha_2, \dots, \alpha_n\}$ are linearly independent if

$$c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n = \theta$$

$$\Rightarrow c_1 = c_2 = \cdots = c_n = 0 \text{ only}.$$

Examine the set of vectors

Ex: $V = \{(1,2,3), (3,-1,4), (4,1,7)\} \in \mathbb{R}^3$ (or F_3) is L.D in $\mathbb{R}^3$

$$c_1(1,2,3) + c_2(3,-1,4) + c_3(4,1,7) = (0,0,0)$$

$$\Rightarrow \left.\begin{aligned} c_1 + 3c_2 + 4c_3 &= 0 \\ 2c_1 - c_2 + c_3 &= 0 \\ 3c_1 + 4c_2 + 7c_3 &= 0 \end{aligned}\right\} \Rightarrow \begin{pmatrix} 1 & 3 & 4 \\ 2 & -1 & 1 \\ 3 & 4 & 7 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Now

$$\begin{pmatrix} 1 & 3 & 4 \\ 2 & -1 & 1 \\ 3 & 4 & 7 \end{pmatrix} \xrightarrow[R_3' = R_3 - 3R_1]{R_2' = R_2 - 2R_1} \begin{pmatrix} 1 & 3 & 4 \\ 0 & -7 & -7 \\ 0 & -5 & -5 \end{pmatrix} \xrightarrow{R_2' = -\frac{1}{7}R_2} \begin{pmatrix} 1 & 3 & 4 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

$$\rightarrow \begin{pmatrix} 1 & 3 & 4 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix} \rightarrow \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Downarrow$$

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$c_1 + c_3 = 0 \Rightarrow c_1 = -c_3$

$c_2 + c_3 = 0 \Rightarrow c_2 = -c_3$

if we take $c_3 = -1$

$\Rightarrow c_1 = c_2 = 1$

$\therefore 1(1,2,3) + (3,-1,4) + (-1)(4,1,7) = (0,0,0)$

So, dimension of that space is 3,

$$\Delta = \begin{vmatrix} 1 & 2 & 3 \\ 3 & -1 & 4 \\ 4 & 1 & 7 \end{vmatrix} = 1(-7-4) - 2(21-16) + 3(3+4)$$
$$= -11 - 10 + 21 = 0$$

As determinant value of vectors are zero.
So, the vectors are Lineary dependent.

Ex: $V = \{(1,0,0), (0,1,0), (0,0,1)\} \in \mathbb{R}^3$

$c_1(1,0,0) + c_2(0,1,0) + c_3(0,0,1) = (0,0,0)$

$\Rightarrow c_1 = 0$
$c_2 = 0$
$c_3 = 0$

So, the vectors $\{(1,0,0), (0,1,0), (0,0,1)\}$ are L.I

$$\Delta = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \neq 0$$

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Note: Any part of a collection of linearly independent vectors is linearly independent

Ex: $\{(1,0,0,0), (0,1,0,0), (0,0,1,0)$ and $(0,0,0,1)\}$ are lineary independent
So, the vectors $\{(1,0,0,0), (0,1,0,0), (0,0,1,0)\}$ are also L.I.

Th: The n number of vectors $a_1 = (a_{11}, a_{12}, \ldots, a_{1n})$, $a_2 = (a_{21}, a_{22}, \ldots, a_{2n})$, $\ldots$, $a_n = (a_{n1}, a_{n2}, \ldots a_{nn})$ will be independent iff

$$\begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} \neq 0$$

$\rightarrow a_0 = a_1 = \cdots = a_4 = 0$

Note: All the vectors $\alpha_1, \alpha_2, \ldots \alpha_m$ are L.I iff Rank of $A = m$

If Rank of $A < m$, then the vectors $\alpha_1, \alpha_2, \ldots, \alpha_m$ are dependent.

Ex: Find the value of K, so that the vectors $(1,3,1)$, $(2,K,0)$ and $(0,4,1)$ are Linearly dependent in $\mathbb{R}^3$

$\Rightarrow$ Since the given vectors are Linearly dependent

So, $$\begin{vmatrix} 1 & 3 & 1 \\ 2 & k & 0 \\ 0 & 4 & 1 \end{vmatrix} = 0$$

$$\Rightarrow 1(K-0) - 2(3-4) = 0 \Rightarrow K - 2(-1) = 0$$
$$\Rightarrow K = -2$$

Ex: we can check whether the four vectors of $\mathbb{R}^5$ $\alpha_1 = (2,4,8,12,8)$, $\alpha_2 = (1,2,4,6,4)$, $\alpha_3 = (2,2,2,2,2)$ and $\alpha_4 = (-1,0,2,4,2)$ are L.I or L.D.

$$\Rightarrow A = \begin{pmatrix} 2 & 4 & 8 & 12 & 8 \\ 1 & 2 & 4 & 6 & 4 \\ 2 & 2 & 2 & 2 & 2 \\ -1 & 0 & 2 & 4 & 2 \end{pmatrix} \xrightarrow[\frac{1}{2}R_3]{\frac{1}{2}R_1} \begin{pmatrix} 1 & 2 & 4 & 6 & 4 \\ 1 & 2 & 4 & 6 & 4 \\ 1 & 1 & 1 & 1 & 1 \\ -1 & 0 & 2 & 4 & 2 \end{pmatrix}$$

$$\xrightarrow{R_2' = R_2 - R_1} \begin{pmatrix} 1 & 2 & 4 & 6 & 4 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 5 & 3 \end{pmatrix} \xrightarrow{R_1' = R_1 - R_3 - R_4} \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & 5 & 3 \end{pmatrix} \longleftarrow$$

Ex: prove that the set of vectors $\{(1,2,2), (2,1,2), (2,2,1)\}$ is L.I in $\mathbb{R}^3$

~~maximum no of independent vectors~~

So, dimension of that space is 3,

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Ex: P.T the set of all polynomial of degree less or equal to 2 is a vector space w.r.t usual addition between two polynomial and usual multiplication between a number and a poly.

$\Rightarrow$ $P_2(x) = \{ f(x) : f(x) = a_0 + a_1 x + a_2 x^2 \}$

Let $f(x)$ & $g(x) \in P_2$

then $f(x) + g(x) \in P_n$

Let $h(x) = c_0 + c_1 x + c_2 x^2$

II : Let $f(x) = a_0 + a_1 x + a_2 x^2$, c be a real number

Then $c \cdot f(x) = ca_0 + ca_1 x + ca_2 x^2 \in P_2$

is also a polynomial of degree 2

Ex: $S = \{ (x,y,z) ; x+y+z = 1 \}$ is not a subspace

because $\alpha = (x_1, y_1, z_1)$, $\beta = (x_2, y_2, z_2)$

But $\alpha + \beta = (x_1 + x_2) + (y_1 + y_2) + (z_1 + z_2) = 2 \neq 1$

$a_0 = a_1 = \cdots = a_4 = 0$

So, $1, x, x^2, x^3, x^4$ are L.I

they form a basis.

Q: Find whether the vector $(3,1,2,-1)$ in $\mathbb{R}^4$ can be expressed as linear combination of three vectors $(1,3,2,1)$, $(2,-1,-2,-1)$ and $(-1,2,3,1)$

$\Rightarrow$ $(3,1,2,-1) = c_1(1,3,2,1) + c_2(2,-1,-2,-1) + c_3(-1,2,3,1)$

$$c_1 + 2c_2 - c_3 = 3$$
$$3c_1 - c_2 + 2c_3 = 1$$
$$2c_1 - 2c_2 + 3c_3 = 2$$
$$c_1 - c_2 + c_3 = -1$$

$$\begin{pmatrix} 1 & 2 & -1 \\ 3 & -1 & 2 \\ 2 & -2 & 3 \\ 1 & -1 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 2 \\ -1 \end{pmatrix}$$

$\downarrow$ $R_4 - R_1$ $R_2 - 3R_1$ $R_3 - 2R_1$

$$\begin{pmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & -6 & 5 \\ 0 & -3 & 2 \end{pmatrix}$$

$\downarrow$

$$\begin{pmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & 1 & 0 \\ 0 & -3 & 2 \end{pmatrix}$$

$\downarrow$

$$\begin{pmatrix} 1 & 2 & -1 \\ 0 & -7 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 2 & -1 \\ 0 & 0 & 5 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} c \end{pmatrix}$$

$\downarrow$

$$\begin{pmatrix} 1 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$

$$c_1 + 2c_2 - c_3 = 3$$
$$c_2 = 2$$
$$c_3 = +1$$

$(-2, -6, -4, -2)$
$(4, -2, -4, -2)$
$(1, 2, 3, -1)$
$= 3, 0, -1, -3$

$c_1 = -1$
$c_2 = 4$
$c_3 = 4$ } Ans

$c_1 + 4 - 1 = 3$
$c_1 = 0$
$c_1 + c_2 = c_3$
$c_1 = -c_2$

maximum no of independent vectors

So, dimension of that space is 3.

Generator or spanning vector

Let V be a vector space over $\mathbb{R}$. Also let $\alpha_1, \alpha_2, \ldots$ be the elements of V and S be a subspace of V (may be equal to V). If every element of S can be expressed as a linear combination of $\alpha_1, \alpha_2, \ldots \alpha_n$, then we say $\alpha_1, \alpha_2, \ldots \alpha_n$ generator or span the subspace S.

Ex: $(1,0,0), (0,1,0), (0,0,1) \in \mathbb{R}^3$

are generators of the entire vector space $\mathbb{R}^3$

For any element of $\mathbb{R}^3$, say (x_1, x_2, x_3) can be expressed

as $(x_1, x_2, x_3) = x_1(1,0,0) + (0,1,0) + (0,0,1)$

$(1,1,1) = (1,0,0) + (0,1,0) + (0,0,1)$

$(2,1,0) = 2(1,0,0) + 1(0,1,0) + 0.(0,0,1)$

Generator or spanning vector

~~Let V be a vector space over IR.~~ Also let $\alpha_1, \alpha_2, \ldots \alpha_n$ be ~~n number of vectors of V~~

Basis : Let V be a vector space over a field F. A set S of vectors in V is said to be a basis of V, if

(i) S is linearly independent in V, and

(ii) S generators V

Ex: prove that the set $S = \{(1,0,1), (0,1,1), (1,1,0)\}$ is a basis of $\mathbb{R}^3$.

Let $\alpha_1 = (1,0,1)$, $\alpha_2 = (0,1,1)$, $\alpha_3 = (1,1,0)$

If we take the determinant value of the vectors

$$\Rightarrow \left| \begin{matrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{matrix} \right| = 1(-1) + 1(-1) = -2 \neq 0$$

$\Rightarrow$ That the set is L.I

let $\xi = (a,b,c)$ be an arbitrary vector of $\mathbb{R}^3$. let us examine if $\xi \in L(S)$

If possible let $(a,b,c) = r_1\alpha_1 + r_2\alpha_2 + r_3\alpha_3$

$\Rightarrow r_1(1,0,1) + r_2(0,1,1) + r_3(1,1,0)$

$\Rightarrow r_1 + r_2 = a, \; r_2 + r_3 = b, \; r_1 + r_2 = c$

This is a nonhomogenous system of three equations in r_1, r_2, r_3

The Coefficient determinant $= \left| \begin{matrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \end{matrix} \right| \neq 0$

maximum no of independent vectors

So, dimension of that space is 3,

Therefore, by Cramer's rule, there exists a unique solution for r_1, r_2, r_3.

This proven that $\xi \in L(S) \Rightarrow \mathbb{R}^3 \subset L(S)$

Again $S \subset \mathbb{R}^3 \Rightarrow L(S) \subset \mathbb{R}^3$

$\Rightarrow L(S) = \mathbb{R}^3$

Thus the set S fulfils both the conditions for a basis of $\mathbb{R}^3$

Ex: Find a basis for the vector space $\mathbb{R}^3$, that contains the vectors $(1,2,1)$ and $(3,6,2)$

$\Rightarrow$ The standard basis of $\mathbb{R}^3 = \{(1,0,0), (0,1,0), (0,0,1)\}$

If we select the first vector $(1,0,0)$ from this and then find whether the vectors $\{(1,0,0), (1,2,1), (3,6,2)\}$ are independent

we see
$$\begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 1 \\ 3 & 6 & 2 \end{vmatrix} = 1(4-6) \neq 0$$

$\therefore$ The set $\{(1,0,0), (1,2,1), (3,6,2)\}$ are L.I

Since $\dim(\mathbb{R}^3) = 3$ and ~~therefore~~ the set contains 3 independent vectors.

So, the set is a basis.

In the set of vectors basis

Let

$$A = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \right\} \in M_{2\times2}(x),\ x \in \mathbb{R}$$

and c_1, c_2, c_3, and c_4 are arbitrary const or scalar

$$c_1\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + c_2\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} + c_3\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} + c_4\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} c_1+c_2+c_3+c_4 & c_2+c_3+c_4 \\ c_3+c_4 & c_4 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$\Rightarrow \left.\begin{aligned} c_1+c_2+c_3+c_4 &= 0 \rightarrow (i) \\ c_2+c_3+c_4 &= 0 \rightarrow (ii) \\ c_3+c_4 &= 0 \rightarrow (iii) \\ c_4 &= 0 \rightarrow (iv) \end{aligned}\right\}$$

(iv) $\Rightarrow c_4 = 0$

(iii) $c_3 + c_4 = 0$

$\Rightarrow c_3 + 0 = 0 \Rightarrow c_3 = 0$

Ily $c_2 = 0$ & $c_1 = 0$

As $c_1 = c_2 = c_3 = c_4 = 0$

So, the set of vector elements are Linearly independent.

These four vectors $\left\{\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}\right\}$ span of $M_{2\times2}$

So the set of vector space of $M_{2\times2}(x)$ is a basis

* $\dim(A) = 4$

maximum no of independent vectors

So, dimension of that space is 3,

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First Method.

$\alpha = (3,1)$, $\beta = (2,-1)$

Any vector in $\mathbb{R}^2$ can be written as (x_1, y_1)

$c_1(3,1) + c_2(2,-1) = (x_1, y_1)$, where c_1 & c_2 are Scalar

$\Rightarrow (3c_1 + 2c_2, c_1 - c_2) = (x_1, y_1)$

$\Rightarrow$ $3c_1 + 2c_2 = x_1 \longrightarrow$ (i)

$c_1 - c_2 = y_1 \rightarrow$ (ii) $\Rightarrow$ $\boxed{c_1 = c_2 + y_1} \longrightarrow$ (iii)

(i) $3c_1 + 2c_2 = x_1$

$\Rightarrow 3(c_2 + y_1) + 2c_2 = x_1$

$\Rightarrow 3c_2 + 3y_1 + 2c_2 = x_1$

$\Rightarrow 5c_2 = x_1 - 3y_1$

$\Rightarrow \boxed{c_2 = \frac{1}{5}(x_1 - 3y_1)} \rightarrow$ (iv)

using (iv) in (iii)

$\Rightarrow c_1 = \frac{1}{5}(x_1 - 3y_1) + y_1$

$\boxed{c_1 = \frac{1}{5}(x_1 + 2y_1)}$

So, for any $(x_1, y_1) \in \mathbb{R}^2$, there exists a linear combination

$\frac{1}{5}(x_1 + 2y_1)(3,1) + \frac{1}{5}(x_1 - 3y_1)(2,-1) = (x_1, y_1)$

$\Rightarrow$ $\alpha = (3,1)$ and $\beta = (2,-1)$ span $\mathbb{R}^2$

Another method

determinant value of $\{\alpha, \beta\}$

$\Rightarrow \begin{vmatrix} 3 & 1 \\ 2 & -1 \end{vmatrix} = -3 - 2 = -5 \neq 0$

So the vectors $\{(3,1), (2,-1)\}$ are linearly independent

So vectors are span the entire $\mathbb{R}^2$

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26. Express $(5,2,1)$ as a linear combination of $(1,4,0)$, $(2,2,1)$ and $(3,0,1)$

$\Rightarrow (5,2,1) = c_1(1,4,0) + c_2(2,2,1) + c_3(3,0,1)$

$\Rightarrow (c_1 + 2c_2 + 3c_3, 4c_1 + 2c_2, c_2 + c_3) = (5,2,1)$

$c_1 + 2c_2 + 3c_3 = 5$

$\Rightarrow c_1 + 2c_2 + 6c_1 = 5$

$\Rightarrow 7c_1 + 2c_2 = 5$

$4c_1 + 2c_2 = 2$ & $c_2 + c_3 = 1$

$$\begin{array}{r} 4c_1 + 2c_2 = 2 \\ -\ 2c_3 + 2c_2 = 2 \\ \hline 4c_1 - 2c_3 = 0 \end{array}$$

$\Rightarrow 2c_1 = c_3$

$\boxed{c_3 = 2}$

$\Rightarrow 2c_1 + c_2 = 1$

$2 + c_2 = 1$

$\boxed{c_2 = -1}$

$$\begin{array}{r} 7c_1 + 2c_2 = 5 \\ 4c_1 + 2c_2 = 2 \\ \hline 3c_1 = 3 \end{array} \Rightarrow \boxed{c_1 = 1}$$

$$\boxed{(5,2,1) = (1,4,0) - (2,2,1) + 2(3,0,1)}$$

Ans

Alternative

$$\begin{vmatrix} 1 & 2 & 3 \\ 4 & 2 & 0 \\ 0 & 1 & 1 \end{vmatrix}$$

$= 1(2) - 2(4) + 3(4)$

$= 2 - 8 + 12$

$= 6 \neq 0$

Ex: In the set $\{(1,0,0), (0,1,0), (0,0,1), (1,1,1)\}$ a basis of $\mathbb{R}^3$

$$\Rightarrow A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \xrightarrow{R_4' = R_4 - R_1 - R_2 - R_3} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} = B$$

Rank $(A) = 3 <$ number of vectors

$\Rightarrow$ the ~~set~~ set of vectors are L.D

So the set is not a basis.

where B is an Echelon matrix having 3 nonzero rows

Here $\{(1,0,0), (0,1,0), (0,0,1)\}$ are the ~~two~~ three maximum no of independent vectors

So, dimension of that space is 3.

Ex: Extend the set $\{(2,1,1),(1,1,1)\}$ to a basis of $\mathbb{R}^3$.

$\Rightarrow$ we know $\{(1,0,0),(0,1,0),(0,0,1)\}$ is a the standard basis of $\mathbb{R}^3$.

we Select the first vector $(1,0,0)$ from this and then find whether the three vectors $(2,1,1),(1,1,1),(1,0,0)$ are independent

We see $\begin{vmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 0 \end{vmatrix} = 1(1-1) = 0$

$\therefore$ $\{(2,1,1),(1,1,1),(1,0,0)\}$ are not independent.

So, we replace the selected vector $(1,0,0)$ by $(0,1,0)$ and find whether the vectors $\{(2,1,1),(1,1,1),(0,1,0)\}$ are independent

We see $\begin{vmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 0 \end{vmatrix} = -1(2-1) = -1 \neq 0$

$\begin{pmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\therefore$ The set $\{(2,1,1),(1,1,1),(0,1,0)\}$ are L.I.

$\Rightarrow \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

Since $\dim(\mathbb{R}^3) = 3$ and there are 3 vectors in this set So this set $\{(2,1,1),(1,1,1),(0,1,0)\}$ is a basis of $\mathbb{R}^3$.

$c_1 = 0$

$c_2 + c_3 = 0 \Rightarrow c_2 = -c_3$

Extension theorem

A Linearly independent set of vectors ~~can be extended to a basis if it is not a basis itself.~~ in a finite dimensional vector space V over a field F is a basis of V, or it can be extended to a basis of V.

Ex: Find a basis of the vector space of all poly with real coefficient having degree ≤ 4

$\Rightarrow$ $P_4(x) = a_0 + a_1x + \cdots + a_4x^4$

which is noting but a linear Combination of $1, x, \ldots, x^4$

Now, $a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 = 0$

$\Rightarrow a_0 = a_1 = \cdots = a_4 = 0$

So, $1, x, x^2, x^3, x^4$ are L.I

they form a basis.

Dimension or Rank of a vector space

The number of vectors present in a basis of a vector space V is called the dimension of V. It is denoted by dim(V)

A vector space is called finite dimensional if it has a basis consisting of a finite number of vectors.

A vector space that is not finite dimensional is called infinite dimensional.

EX: (i) The vector space $\{0\}$ has dimension zero.

(ii) the vector space $F^n / \mathbb{R}^n$ has dimension n.

(iii) The vector space $M_{m\times n}(F)$ has dimension mn

(iv) The vector space $P_n(F)$ has dimension $(n+1)$

EX: Find a basis and the dimension of the subspace S of $\mathbb{R}^3$, where $S = \{(x,y,z) \in \mathbb{R}^3 ; x+y+z=0\}$

$\Rightarrow$ Let $\alpha = (x, y, z)$ be arbitrary element of S

$$x+y+z=0 \Rightarrow z = -x-y$$

$$\alpha = (x, y, -x-y)$$
$$= (x,0,-x) + (0,y,-y) = x(1,0,-1) + y(0,1,-1)$$

Every element of S can be expressed as linear combination of the two vectors $\alpha_1 = (1,0,-1)$ & $\alpha_2 = (0,1,-1)$

Now $c_1\alpha_1 + c_2\alpha_2 = \theta$

$$\Rightarrow c_1(1,0,-1) + c_2(0,1,-1) = (0,0,0)$$

$$\Rightarrow c_1 = 0 \qquad -c_1-c_2=0$$
$$c_2 = 0$$

So, dimension of that space is 3,

Hence α_1 and α_2 are independent.

$\therefore \{\alpha_1, \alpha_2\}$ is a basis of S

Since a basis of S Contains two vectors, so $\dim(S) = 2$

2nd process: The coefficient matrix $A = (1\ 1\ 1)$

$$\text{Rank}(A) = 1$$

No of independent Solution = No of unknowns − Rank of A

$$= 3-1 = 2$$

$$\dim(S) = 2$$

The equ is reduced to $x+y+z=0$

$$\Rightarrow z = -x-y$$

$\therefore (x, y, -x-y)$ is Solution

$$= x(1,0,-1) + y(0,1,-1)$$

$\therefore \{(1,0,-1), (0,1,-1)\}$ form a basis of S.