Gamma function

$$\boxed{\Gamma(n) = \int_0^{\infty} e^{-x} x^{n-1}\, dx}$$

The integral is convergent iff $n > 0$

* In many problem in the application of Integral Calculus, the use of the Beta & Gamma functions often facilitates calculations.

properties of Gamma function

(i) $\Gamma(1) = 1$.

$$\int_0^{\infty} e^{-x} x^{0}\, dx$$

$$= \lim_{X\to\infty} \int_0^{X} e^{-x}\, dx$$

$$= \lim_{X\to\infty} \left[-e^{-x}\right]_0^{X} = \lim_{X\to\infty} (-e^{-X} + e^{0})$$

$$= -e^{-\infty} + 1 = -0 + 1 = 1$$

(ii) $\Gamma(n+1) = n\Gamma(n)$

LIATE

$$\Rightarrow \Gamma(n+1) = \int_0^{\infty} e^{-x} x^{n}\, dx$$

$$= \lim_{X\to\infty} \int_0^{X} e^{-x} x^{n}\, dx = \lim_{X\to\infty} \left[x^n \int e^{-x}dx - \int\left\{\frac{d}{dx}(x^n)\int e^{-x}dx\right\}dx\right]$$

$$= \lim_{X\to\infty} \left[-x^n e^{-x}\Big|_0^X + \int_0^X n x^{n-1} e^{-x}\, dx\right]$$

$$= \lim_{X\to\infty} \left[(-X^n e^{-X} - 0) + n\int_0^X x^{n-1} e^{-x}\, dx\right]$$

$$= 0 + n \lim_{X\to\infty} \int_0^X e^{-x} x^{n-1}\, dx$$

$$= n\int_0^{\infty} e^{-x} x^{n-1}\, dx = n\Gamma(n)$$

$$\therefore \boxed{\Gamma(n+1) = n\Gamma(n)}$$ (proved)

(iii) $\Gamma(n+1) = n!$ when n is a positive integer

$\Rightarrow$ we know $\Gamma(n+1) = n\Gamma(n)$

$= n(n-1)\Gamma(n-1)$

$= n(n-1)(n-2)\Gamma(n-2)$

$\cdots\cdots\cdots$

$= n(n-1)(n-2)\cdots 1\,\Gamma(1)$

$= n(n-1)(n-2)\cdots 1 = n!$

(iv) $\boxed{\Gamma(n)\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}}$, $0 < n < 1$

(v) $\boxed{\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}}$

we know $\Gamma(n)\Gamma(1-n) = \frac{\pi}{\sin n\pi}$

putting $n = \frac{1}{2}$, we get

$\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right) = \frac{\pi}{\sin \pi/2}$

$\Rightarrow \left\{\Gamma\left(\frac{1}{2}\right)\right\}^2 = \pi$

$\Rightarrow \Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

Ex: Show that $\Gamma\left(\frac{7}{2}\right) = \frac{15}{8}\sqrt{\pi}$

$\Gamma\left(\frac{7}{2}\right) = \Gamma\left(\frac{5}{2}+1\right)$

$= \frac{5}{2}\Gamma\left(\frac{5}{2}\right)$

$= \frac{5}{2}\Gamma\left(\frac{3}{2}+1\right)$

$= \frac{5}{2}\cdot\frac{3}{2}\Gamma\left(\frac{1}{2}+1\right)$

$= \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$

$= \frac{15}{8}\sqrt{\pi}$

Ex: Show that

$\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right) = \frac{2}{\sqrt{3}}\pi$

$\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{2}{3}\right) = \Gamma\left(\frac{1}{3}\right)\Gamma\left(1-\frac{1}{3}\right)$

$= \frac{\pi}{\sin\frac{\pi}{3}} = \frac{\pi}{\sqrt{3}/2} = \frac{2\pi}{\sqrt{3}}$

Beta function

It can be shown that the integral $\int_0^1 x^{m-1}(1-x)^{n-1}\,dx$ is convergent iff $m>0$, $n>0$. Then if $m>0$, $n>0$ the integral has a definite value, which is denoted by $\beta(m,n)$.

$$\boxed{\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx}\ , \quad m>0,\ n>0$$

* $\beta(1,2) = \int_0^1 x^0 (1-x)^1\,dx = \int_0^1 (1-x)\,dx = \left[x-\frac{x^2}{2}\right]_0^1 = \frac{1}{2}$ have a value.

* $\beta(2,0) = \int_0^1 x(1-x)^{-1}\,dx = \int_0^1 \frac{x}{1-x}\,dx$ has no value.

$$= -\int_0^1 \frac{(1-x)}{(1-x)}\,dx + \int_0^1 \frac{dx}{1-x} = -\int_0^1 dx + \lim_{\epsilon_2\to 0^+}\int_0^{1-\epsilon_2}\frac{dx}{(1-x)}$$

$$= -1 + \lim_{\epsilon_2\to 0^+}\left[-\log(1-x)\right]_0^{1-\epsilon_2}$$

$$= -1 + \lim_{\epsilon_2\to 0^+}\left[-\log\epsilon_2 + \log 0\right] = -1+\infty = \infty$$

Properties of Beta function

* $\beta(m,n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta \cos^{2n-1}\theta\,d\theta$ | * $\beta(m,n) = \beta(n,m)$

* $\beta(m,n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}\,dx$

* $\beta\left(\frac{1}{2},\frac{1}{2}\right) = \pi$

using $\beta(m,n) = 2\int_0^{\pi/2}\sin^{2m-1}\theta\cos^{2n-1}\theta\,d\theta$

$$\beta\left(\tfrac{1}{2},\tfrac{1}{2}\right) = 2\int_0^{\pi/2}\sin^0\theta\cos^0\theta\,d\theta$$

$$= 2\int_0^{\pi/2} d\theta = 2\cdot\frac{\pi}{2} = \pi$$

Relation between Beta & Gamma Function

$$\beta(m,n) = \frac{\Gamma(m)\,\Gamma(n)}{\Gamma(m+n)}$$

Ex: Show that $\int_0^{\pi/2} \sin^4 x \cos^5 x \, dx = \frac{8}{315}$

we know $2\int_0^{\pi/2} \sin^{2m-1} x \cos^{2n-1} x \, dx = \beta(m,n)$

Comparing with given problem

$$\int_0^{\pi/2} \sin^{2\cdot\frac{5}{2}-1} x \cos^{2\cdot 3-1} x \, dx = \frac{1}{2}\beta\left(\frac{5}{2},3\right)$$

$$= \frac{1}{2}\,\frac{\Gamma\left(\frac{5}{2}\right)\Gamma(3)}{\Gamma\left(\frac{5}{2}+3\right)}$$

$$= \frac{1}{2}\,\frac{\Gamma\left(\frac{3}{2}+1\right)\Gamma(2+1)}{\Gamma\left(\frac{9}{2}+1\right)}$$

$$= \frac{1}{2}\,\frac{\Gamma\left(\frac{5}{2}\right)\cdot 2\cdot 1}{\frac{9}{2}\cdot\frac{7}{2}\frac{5}{2}\Gamma\left(\frac{5}{2}\right)}$$

$$= \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{\frac{9}{2}\cdot\frac{7}{2}\cdot\frac{5}{2}} = \frac{8}{315}$$

$\Gamma(n) = \int_0^\infty e^{-x} x^{n-1} \, dx$

Ex: prove that $\int_0^\infty e^{-x^4} \cdot x^2 \, dx \times \int_0^\infty e^{-x^4} \, dx = \frac{\pi}{8\sqrt{2}}$

$\Rightarrow$ put $x^4 = z \Rightarrow 4x^3 dx = dz \Rightarrow dx = \frac{1}{4x^3} dz = \frac{1}{4} x^{-3} dz = \frac{1}{4} z^{-3/4} dz$

x	∞	0
z	∞	0

$$= \int_0^\infty e^{-z} \cdot z^{1/2} \cdot \frac{1}{4} z^{-3/4} dz \times \int_0^\infty e^{-z} \cdot \frac{1}{4} z^{-3/4} dz$$

$$= \frac{1}{16}\int_0^\infty e^{-z} z^{-1/4} dz \times \int_0^\infty e^{-z} z^{-3/4} dz$$

$$= \frac{1}{16}\int_0^\infty e^{-z} z^{3/4-1} dz \times \int_0^\infty e^{-z} z^{1/4-1} dz$$

$$= \frac{1}{16}\Gamma\left(\frac{3}{4}\right)\Gamma\left(\frac{1}{4}\right) = \frac{1}{16}\Gamma\left(1-\frac{1}{4}\right)\Gamma\left(\frac{1}{4}\right) = \frac{1}{16}\frac{\pi}{\sin \pi/4} = \frac{\pi}{16\cdot\frac{1}{\sqrt{2}}} = \frac{\pi\sqrt{2}}{16}$$

x: using definition of Beta function evaluate $\int_0^{\pi/2} \cos^4 x \, dx$

$$\Rightarrow \int_0^{\pi/2} \cos^4 x \, dx = \frac{1}{2} \cdot 2\int_0^{\pi/2} \sin^{2\cdot\frac{1}{2}-1} x \cos^{2\cdot\frac{5}{2}-1} x \, dx$$

$$= \frac{1}{2} \beta\left(\frac{1}{2}, \frac{5}{2}\right)$$

$$= \frac{1}{2} \frac{\Gamma(\frac{1}{2})\Gamma(\frac{5}{2})}{\Gamma(3)} \qquad \Gamma(n+1) = n\Gamma(n)$$

$$= \frac{1}{2} \frac{\sqrt{\pi} \cdot \frac{3}{2} \cdot \frac{1}{2}\sqrt{\pi}}{2.1}$$

$$= \frac{1}{2} \cdot \frac{3\sqrt{\pi} \cdot \sqrt{\pi}}{8} = \frac{3\pi}{16}$$

Ex: $\int_0^1 x^{3/2}(1-x)^{3/2} dx = \frac{3\pi}{128}$ (Show that)

$$\Rightarrow \int_0^1 x^{3/2}(1-x)^{3/2} dx = \int_0^1 x^{5/2-1}(1-x)^{\frac{5}{2}-1} dx$$

$$= \beta(5/2, 5/2)$$

$$= \frac{\Gamma(\frac{5}{2})\Gamma(5/2)}{\Gamma(5)} = \frac{\{\Gamma(\frac{5}{2})\}^2}{\Gamma(5)}$$

$$= \frac{\{\Gamma(\frac{3}{2}+1)\}^2}{\Gamma(5)} = \frac{\{\frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi}\}^2}{4.3.2}$$

$$= \frac{1}{24} \cdot \frac{3^2}{(4)^2} \pi = \frac{9}{24 \times 16} \pi = \frac{3\pi}{128}$$

Ex: prove that $\int_0^{\infty} e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$

$\Rightarrow$ Let $z = x^2 \Rightarrow dz = 2x\,dx = 2z^{1/2} dx \Rightarrow dx = \frac{1}{2} z^{-1/2} dz$

$$\therefore \int_0^{\infty} e^{-x^2} dx = \int_0^{\infty} e^{-z} \cdot \frac{1}{2} z^{-1/2} dz = \frac{1}{2}\int_0^{\infty} z^{-1/2} e^{-z} dz$$

$$= \frac{1}{2}\int_0^{\infty} e^{-z} z^{\frac{1}{2}-1} dz$$

$$= \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2} \quad \text{Ans.}$$

Ex: Express $\int_a^b (x-a)^m (b-x)^n dx$ in terms of Beta. Hence evaluate $\int_3^7 (x-3)^4 (7-x)^{1/3} dx$

$\Rightarrow \int_a^b (x-a)^m (b-x)^n dx$ putting $z = x-a$, $dz = dx$

x	b	a
z	$b-a$	0

$$= \int_0^{b-a} z^m (b-(z+a))^n dz$$

$$= \int_0^{b-a} z^m (b-a-z)^n dz = \int_0^{b-a} z^m \{(b-a)-z\}^n dz$$

$$= (b-a)^n \int_0^{b-a} z^m \left(1-\frac{z}{b-a}\right)^n dz$$

Again putting $t = \frac{z}{b-a}$

$\Rightarrow z = t(b-a)$

z	$b-a$	0
t	1	0

$dz = dt(b-a)$

$$= (b-a)^n \int_0^1 t^m (b-a)^m (1-t)^n (b-a)dt$$

$$= (b-a)^{m+n+1} \int_0^1 t^m (1-t)^n dt$$

$$= (b-a)^{m+n+1} \beta(m+1, n+1)$$

$$\therefore \int_3^7 (x-3)^4 (7-x)^{1/3} dx = (7-3)^{(4+\frac{1}{3}+1)} \beta\left(4+1, \frac{1}{3}+1\right)$$

$$= 4^{\frac{16}{3}} \beta\left(5, \frac{4}{3}\right)$$

$$= 4^{\frac{16}{3}} \frac{\Gamma(5)\Gamma(\frac{4}{3})}{\Gamma(5+\frac{4}{3})} = 4^{16/3} \cdot \frac{4!\,\Gamma(\frac{4}{3})}{(4+\frac{4}{3})(3+\frac{4}{3})(2+\frac{4}{3})(1+\frac{4}{3})\frac{4}{3}\Gamma(\frac{4}{3})}$$

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