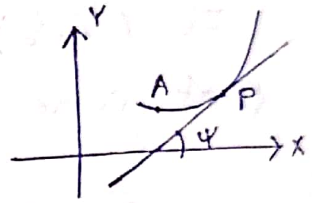
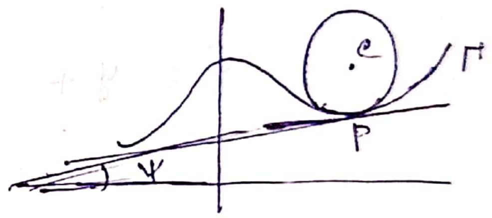
Curvature : Defined as the rate of change of the angle made by the tangent w.r.t the length of arc of the curve

$$K = \left(\frac{d\psi}{ds}\right)_{at\ P}$$



Note : Curvature at every pt on a St Line = 0

* The circle lies on the same side of the tangent as the curve and the circle has same curvature as Γ at P.



This circle is called circle of curvature.

Its centre C is called centre of curvature of Γ at P.

The radius of circle of curvature at P is called radius of curvature

$$\rho = \frac{1}{K}$$

Formula of Radius of curvature (ρ) in polar Coordinate system ($r = f(\theta)$)

$$\rho = \left| \frac{(r^2 + r_1^2)^{3/2}}{r^2 + 2r_1^2 - rr_2} \right|$$

where $r_1 = \frac{dr}{d\theta}$, $r_2 = \frac{d^2r}{d\theta^2}$

Curvature

The curvature of a curve at a particular point which will give a definite numerical measure of bending which the curve undergoes at the point.

Formula • For $y = f(x)$

$\rho \rightarrow$ radius of curvature

$$\rho = \left| \frac{\left\{ 1 + \left(\frac{dy}{dx}\right)^2 \right\}^{3/2}}{\frac{d^2y}{dx^2}} \right| \quad \left(\text{where } \frac{d^2y}{dx^2} \neq 0 \right)$$

The positive root is taken in the numerator (i.e $y_2 > 0$)
The radius of curvature, will, therefore be positive
i.e the curve is concave upwards.

and radius of curvature will be negative, when $y_2 < 0$
i.e the curve is concave downwards.

• For the curve $x = f(y)$

$$\rho = \frac{\{1 + x_1^2\}^{3/2}}{x_2} \quad (\text{where } x_2 \neq 0)$$

• For $f(x, y) = 0$

$$\rho = \frac{(f_x^2 + f_y^2)^{3/2}}{f_{xx} f_y^2 - 2 f_{yx} f_x f_y + f_{yy} f_x^2}$$

(assuming that the denominator $\neq 0$)

• **Parametric Equation:**

$x = f(t), \quad y = \phi(t)$

$$\rho = \frac{(x'^2 + y'^2)^{3/2}}{x'y'' - y'x''}, \quad (\text{where } x'y'' - y'x'' \neq 0)$$

& x' & y' denotes differentiations w.r.t 't'.

Ex: Find the radius of curvature of the cycloid.

Prove

$$x = a(\theta + \sin\theta), \quad y = a(1 - \cos\theta)$$

$$\Rightarrow \frac{dx}{d\theta} = a(1+\cos\theta), \quad \frac{dy}{d\theta} = +a\sin\theta$$

$$\frac{dy}{dx} = \frac{a\sin\theta}{a(1+\cos\theta)} = \frac{\sin\theta}{1+\cos\theta} = \frac{2\sin\theta/2 \cos\theta/2}{2\cos^2\theta/2} = \tan\theta/2$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\cdot\frac{dy}{dx} = \frac{d}{d\theta}\frac{dy}{dx}\cdot\frac{d\theta}{dx} = \frac{d}{d\theta}(\tan\theta/2)\cdot\frac{1}{a(1+\cos\theta)}$$

$$= \frac{\sec^2\theta/2}{2\cdot 2a\cos^2\theta/2}$$

$$= \frac{1}{4a\cos^4\theta/2}$$

$$\therefore \rho = \frac{(1+y_1^2)^{3/2}}{y_2}$$

$$= \frac{(1+\tan^2\theta/2)^{3/2}}{\frac{1}{4a\cos^4\theta/2}} = \sec^3\theta/2 \times 4a\cos^4\theta/2$$

$$= 4a\cos\theta/2$$

$$\therefore \rho = 4a\cos\theta/2 \quad \text{Ans.}$$

Scanned with OKEN Scanner

Formula of Centre of curvature

If $(\bar{x}, \bar{y})$ be the coordinate of the centre of curvature of each curve at any pt P then

(i)
$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2} \qquad (y_2 \neq 0)$$

$$\bar{y} = y + \frac{1+y_1^2}{y_2} \qquad y_1 = \frac{dy}{dx}, \quad y_2 = \frac{d^2y}{dx^2}$$

(ii)
$$\bar{x} = x + \frac{1+x_1^2}{x_2}$$

$$\bar{y} = y - \frac{x_1(1+x_1^2)}{x_2}$$

Evolute : Γ be a curve and $(\bar{x}, \bar{y})$ be the centre of curvature of Γ at P, the locus of centre of curvature as the pt P travels on Γ is called the Evolute of the curve Γ.

If Γ_1 be evolute of Γ then Γ is called Involute of the curve Γ_1

Similarly we get $\sin\theta = \left(\dfrac{by}{a^2 - b^2}\right)^{\frac{1}{3}}$

Squaring and adding, $(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = (a^2 - b^2)^{\frac{2}{3}}$

which is the required evolute.

Example 11. Find the evolute of the curve (astroid)

$$x^{2/3} + y^{2/3} = a^{2/3}$$

Solution. For convenience we use parametric equation

$$x = a\cos^3\theta, \quad y = a\sin^3\theta$$

$$\frac{dx}{d\theta} = 3a\cos^2\theta(-\sin\theta) = -3a\sin\theta\cos^2\theta.$$

$$\frac{dy}{d\theta} = 3a\sin^2\theta\cos\theta.$$

$$\therefore \frac{dy}{dx} = \frac{dy}{d\theta}\Big/\frac{dx}{d\theta} = \frac{3a\sin^2\theta\cos\theta}{-3a\sin\theta\cos^2\theta} = -\frac{\sin\theta}{\cos\theta} = -\tan\theta.$$

1st Process.

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}(-\tan\theta)$$

$$= -\sec^2\theta \cdot \frac{d\theta}{dx} = -\sec^2\theta\left(-\frac{1}{3a\sin\theta\cos^2\theta}\right) = \frac{1}{3a\sin\theta\cos^4\theta}$$

If $(\bar{x}, \bar{y})$ be centre of curvature of the curve at (x, y) then

$$\bar{x} = x - \frac{y_1(1+y_1^2)}{y_2} = a\cos^3\theta - \frac{-\tan\theta(1+\tan^2\theta)}{\dfrac{1}{3a\sin\theta\cos^4\theta}}$$

$$= a\cos^3\theta + \tan\theta\sec^2\theta \cdot 3a\sin\theta\cos^4\theta$$

$$= a\cos^3\theta + 3a\frac{\sin^2\theta}{\cos^3\theta}\cos^4\theta$$

$$= a\cos^3\theta + 3a\sin^2\theta\cos\theta$$

$$= a\cos^3\theta + 3a(1-\cos^2\theta)\cos\theta$$

or, $\bar{x} = 3a\cos\theta - 2a\cos^3\theta$ ... (1)

and $\bar{y} = y + \dfrac{1+y_1^2}{y_2} = a\sin^3\theta + \dfrac{1+\tan^2\theta}{\dfrac{1}{3a\sin\theta\cos^4\theta}}$

$$= a\sin^3\theta + \sec^2\theta \cdot 3a\sin\theta\cos^4\theta$$

$$= a\sin^3\theta + 3a\frac{\sin\theta}{\cos^2\theta} \cdot \cos^4\theta$$

$$= a\sin^3\theta + 3a\sin\theta\cos^2\theta$$

$$= a\sin^3\theta + 3a\sin\theta(1-\sin^2\theta).$$

or, $\bar{y} = 3a\sin\theta - 2a\sin^3\theta$ ... (2)

We shall eliminate θ (as it represent (x, y)) from (1) and (2).

Adding (1) and (2),

$$\bar{x} + \bar{y} = 3a(\cos\theta + \sin\theta) - 2a(\cos^3\theta + \sin^3\theta)$$

$$= a\left[3(\cos\theta+\sin\theta) - 2\left\{(\cos\theta+\sin\theta)^3 - 3\cos\theta\sin\theta(\cos\theta+\sin\theta)\right\}\right]$$

$$= a(\cos\theta+\sin\theta)\left[3 - 2\left\{(\cos\theta+\sin\theta)^2 - 3\cos\theta\sin\theta\right\}\right]$$

$$= a(\cos\theta+\sin\theta)(3 - 2(1-\sin\theta\cos\theta))$$

$$= a(\cos\theta+\sin\theta)(1+2\sin\theta\cos\theta)$$

$$= a(\cos\theta+\sin\theta)^3$$

or, $(\bar{x}+\bar{y})^{\frac{2}{3}} = a^{\frac{2}{3}}(\cos\theta+\sin\theta)^2$ ... (3)

On subtraction (2) from (1),

$$\bar{x} - \bar{y} = 3a(\cos\theta - \sin\theta) - 2a(\cos^3\theta - \sin^3\theta)$$

or, $(\bar{x}-\bar{y})^{\frac{2}{3}} = a^{\frac{2}{3}}(\cos\theta-\sin\theta)^2$ ... (4)

as above.

Adding (3) and (4) we get $(\bar{x}+\bar{y})^{\frac{2}{3}} + (\bar{x}-\bar{y})^{\frac{2}{3}} = a^{\frac{2}{3}}.2$

$\therefore$ the locus of the centre of curvature, that is the equation of the Evolute is $(x+y)^{\frac{2}{3}} + (x-y)^{\frac{2}{3}} = 2a^{\frac{2}{3}}$

Example 4. Find the radius of curvature of the curve

$$x = a\cos^3\theta, \quad y = a\sin^3\theta \text{ at } \theta = \frac{\pi}{4}.$$

Solution. $\dfrac{dx}{d\theta} = -3a\cos^2\theta\sin\theta$; $\dfrac{dy}{d\theta} = 3a\sin^2\theta\cos\theta$

$$\therefore \frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta} = -\frac{\sin\theta}{\cos\theta} = -\tan\theta.$$

$$\therefore \text{ at } \theta = \frac{\pi}{4}, \quad y_1 = -\tan\frac{\pi}{4} = -1.$$

Now, $\dfrac{d^2y}{dx^2} = \dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d}{dx}(-\tan\theta)$

$$= -\sec^2\theta \cdot \frac{d\theta}{dx} = \sec^2\theta \cdot \frac{1}{3a\cos^2\theta\sin\theta} = \frac{1}{3a\cos^4\theta\sin\theta}$$

$$\therefore \text{ at } \theta = \frac{\pi}{4}, \quad y_2 = \frac{1}{3a\left(\frac{1}{\sqrt{2}}\right)^4 \cdot \frac{1}{\sqrt{2}}} = \frac{1}{3a \cdot \frac{1}{4\sqrt{2}}} = \frac{4\sqrt{2}}{3a}.$$

$$\therefore \text{ at } \theta = \frac{\pi}{4}, \quad \rho = \frac{(1+y_1^2)^{3/2}}{y_2} = \frac{(1+(-1)^2)^{3/2}}{\frac{4\sqrt{2}}{3a}} = \frac{3a}{4\sqrt{2}} \cdot 2^{3/2} = \frac{3a \cdot 2}{4} = \frac{3a}{2}.$$