Technical Report: Reverse Poisson Counting Process With Random Observations

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Abstract

This paper introduces the Reverse Poisson Counting Process (RPCP), a stochastic model derived from the Poisson counting process with limited and random capacity, characterized by counting backward from a defined maximum level. Explicit analytical formulas are developed for key stochastic properties, including the probability mass function and mean, specifically for scenarios involving random capacity. The model is shown to represent extreme cases of established stochastic processes, such as the M/M/1 queue with instant service completion and the death-only process. The explicit formulas facilitate straightforward analysis and optimization of system functionals.

Keywords: Reverse Poisson counting process, network security, accountability, availability, duality, stochastic optimization, computer networks.

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1. Preliminaries

This section extends the classical process to encompass scenarios with capacity constraints, providing a foundational understanding for subsequent analyses. This background is critical for understanding the deviations and novel aspects of the RPCP. The variations of the Poisson counting process, particularly those incorporating limitations on capacity, are thoroughly examined to set the stage for the introduction and analysis of the RPCP in the next section.

1.1. *Poisson Counting Process*

A counting process is a stochastic process characterized by a sequence of random variables that quantify the number of events occurring within a specified time interval which is represented as **Model-1**. The number of counting events $N(t)$ within a time interval $(0, t]$ could be defined as $\{N(t); t \geq 0\}$, with the initial condition that $N(0) = 0$. The Poisson counting process is a stochastic process wherein the inter-arrival times follow independent exponential distributions, all characterized by the same rate λ. The probability of observing k counting events within the time interval $(0, t]$ could be determined as follows:

$$P\{N(t) = k\} = \frac{e^{-\lambda t} (\lambda t)^k}{k!}, k = 0, 1, \ldots \tag{1}$$

and the probability generating function (PGF) for (1) is given by:

$$G(z, t) = \mathbb{E}[N(t)] = e^{\lambda(z-1)t}. \tag{2}$$

1.2. *Memoryless Observation Process*

This subsection establishes the observation process with the memoryless properties for detailing the Poisson counting process. The counting process is observed at random time instances, consistent with a point process equivalent to one transaction approval time (TAT).

$$\mathcal{T} := \sum_{i \geq 0} \varepsilon_{\tau_i}, \tau_0 (= 0), \tau_1, \ldots, \tag{3}$$

and the duration between observation points is stochastically equivalent (i.e., $\Delta_1 = \tau_n - \tau_{n-1}$). Let us consider the number of counts Q_t within each observation interval $[\tau_{n-1}, \tau_n)$. The duration of each observation interval is treated as a random variable Δ_1 from (3). The functional of Poisson counting measure within the random observation interval $\varphi(z, \theta)$ has been defined as follows:

$$\varphi(z, \theta) = \mathbb{E}\left[z^{Q_{\Delta_1}} e^{-\theta \Delta_1}\right], |z| < 1, \mathrm{Re}(\theta) \geq 0, \tag{4}$$

which could be calculated from (2) and (4) as follows:

$$\varphi(z,\theta) = \mathbb{E}\left[z^{-\lambda(1-z)\Delta_1} e^{-\theta\Delta_1}\right] = \alpha(\lambda(1-z)+\theta), \tag{5}$$

where

$$\alpha(\theta) = \mathbb{E}\left[e^{-\theta\Delta_1}\right].$$

Given that the observation duration possesses a memoryless property, the random variable representing this duration follows an exponential distribution. From (4) and (5), the following relationships are derived:

$$\varphi(z,\theta) = \frac{1}{(1+\lambda a + a\theta) - \lambda a z} = \left[1 - \left(\frac{\lambda a}{1+\lambda a+\lambda\theta}\right) z\right]^{-1}, \tag{6}$$

where a is the mean of the observation duration (i.e., $a = \mathbb{E}[\Delta_1]$). From (6), the PGF of Q_{Δ_1} simplifies to $\varphi(z,0)$ which is expressed as follows:

$$\begin{aligned}\varphi(z,0) &= \left(\frac{1}{1+\lambda a}\right)\left[1 - \left(\frac{\lambda a}{1+\lambda a}\right) z\right]^{-1} \\ &= \left(\frac{1}{1+\lambda a}\right)\sum_{k\geq 0}\left\{\left(\frac{\lambda a}{1+\lambda a}\right) z\right\}^{k}, k = 0, 1, \ldots.\end{aligned} \tag{7}$$

Based on (7), the probability (denoted as P_k) of the Poisson counting process given an exponentially distributed observation time can be expressed as:

$$P_k = \boldsymbol{P}\{Q_{\Delta_1} = k\} = (1-\beta)\beta^k, k = 0, 1, \ldots, \tag{8}$$

where

$$\beta = \frac{\lambda a}{1+\lambda a}, P_0 = 1 - \beta = \frac{1}{1+\lambda a}. \tag{9}$$

Let Q_t^0 represent the counting number with a limited capacity M at each observation at time τ_n (i.e., $t = \tau_n$) as represented by **Model-2**. It is noted that each observation process resets upon reaching time τ_n from the starting moment τ_{n-1}, and only the duration between these observation moments is considered (i.e., $\tau_{n-1} = 0$; $\tau_n - \tau_{n-1} = \tau_1 - \tau_0 = \Delta_1$). Compared to the original Poisson counting process $N(t)$ (i.e., **Model-1**), counts exceeding M are considered losses and the loss probability P_{Loss} could be calculated as follows:

$$P_{\text{Loss}} = \sum_{k \geq M+1} P_k = 1 - \sum_{k=0}^{M} P_k, \tag{10}$$

where

$$\sum_{k=0}^{M} P_k = 1 - \beta^{M+1}.$$

From (9)-(10), the loss probability is determined as follows:

$$P_{\text{Loss}} = \left(\frac{\lambda a}{1 + \lambda a} \right)^{M+1}. \tag{11}$$

The functional $\widehat{\varphi}(z, \theta)$ of this truncated counting process could be defined as follows:

$$\widehat{\varphi}(z, \theta) = \mathbb{E}\left[z^{Q^0_{\Delta_1}} e^{-\theta \Delta_1} \right], |z| < 1, \text{Re}(\theta) \geq 0, \tag{12}$$

and $\widehat{\varphi}(z, 0)$ becomes the PGF of Q^0_t with the exponential observation duration Δ_1. From (8)-(9) and (12), the probability of the truncated Poisson counting process with the exponential observation time could be found as follows:

$$\pi^0_k = \boldsymbol{P}\left\{ Q^0_{\Delta_1} = k \right\} = A^{-1} P_k, k = 0, 1, \ldots, M, \tag{13}$$

where

$$\frac{P_0}{\pi^0_0} = \frac{P_1}{\pi^0_1} = \cdots = \frac{P_M}{\pi^0_M} = A,$$

$$A = 1 - P_{\text{Loss}} = 1 - \left(\frac{\lambda a}{1 + \lambda a} \right)^{M+1}. \tag{14}$$

2. Reverse Poisson Counting Process

This section presents detailed proofs and formulations that define the behavior of the RPCP which referred as **Model-3**. Additionally, the analysis extends to a variant of the RPCP that incorporates random capacity, providing a more flexible and realistic model for various applications. The RPCP is fundamentally a Poisson counting process with both a limited capacity and a memoryless observation process.

However, unlike typical counting processes, the RPCP counts backward from a defined maximum level M, representing the full capacity. Let Q_t^1 be the number of negative counting from the full capacity M at the observation time τ_n (i.e., $t = \tau_n$). Each observation process shall be reset once the duration reaches τ_n from the starting moment τ_{n-1} and the capacity is fully restored to M. From (12) and (13), the probability of the RPCP with the exponential observation time could be found as follows:

$$\pi_k^1 = \boldsymbol{P}\left\{Q_{\Delta_1}^1 = k\right\} = \begin{cases} 1 - \left(\sum_{k=1}^{M} \pi_k^1\right), & k = 0, \\ \pi_{M-k}^0, & k = 1, 2, \ldots, M, \end{cases} \tag{15}$$

and

$$\begin{aligned} \left(\sum_{k=1}^{M} \pi_k^1\right) &= \sum_{k=1}^{M} \pi_{M-k}^0 = \sum_{j=1}^{M-1} \pi_j^0 = \left(\sum_{j=1}^{M-1} A^{-1} P_j\right) \\ &= \left(\frac{1-\beta}{1-\beta^{M+1}}\right)\left(\frac{1-\beta^M}{1-\beta}\right) = \frac{1-\beta^M}{1-\beta^{M+1}}. \end{aligned} \tag{17}$$

From (14), we have

$$\pi_0^1 = 1 - \frac{1-\beta^M}{1-\beta^{M+1}} = \xi\left(M\right), \tag{16}$$

$$\pi_k^1 = A^{-1} P_{M-k} = \xi\left(M\right)\left(\frac{1}{\beta}\right)^k, k = 1, \ldots, M,$$

where

$$\xi\left(M\right) = \frac{\beta^M\left(1-\beta\right)}{1-\beta^{M+1}}, \beta = \frac{\lambda a}{1+\lambda a}, \tag{17}$$

because we have

$$\pi_k^1 = A^{-1} P_{M-k} = \frac{\left(1-\beta\right)}{\left(1-\beta^{M+1}\right)} \beta^{M-k},$$

$$= \left\{ \frac{\beta^{M}(1-\beta)}{1-\beta^{M+1}} \right\} \beta^{-k} = \xi(M) \left(\frac{1}{\beta} \right)^{k}, \beta \leq 1.$$

From (15)-(17), the probability of the RPCP with the exponential observation duration finally determined as follows:

$$\pi_k^1 = \xi(M) \left(1 + \frac{1}{\lambda a} \right)^{k}, k = 0, \ldots, M, \tag{18}$$

where

$$\xi(M) = \frac{\beta^{M+1}}{\lambda a \left(1-\beta^{M+1}\right)}, \frac{1}{\lambda a} = \left(\frac{1}{\beta} - 1 \right). \tag{19}$$

The functional $\widehat{\psi}(z,\theta)$ of this truncated counting process could be defined as follows:

$$\widehat{\psi}(z,\theta) = \mathbb{E}\left[z^{Q^1_{\Delta_1}} e^{-\theta \Delta_1} \right], |z| < 1, \mathrm{Re}(\theta) \geq 0, \tag{20}$$

and the PGF of RPCP Q_t^1 with the exponential observation duration Δ_1 becomes $\widehat{\varphi}(z,0)$ which could be calculated as follows:

$$\widehat{\psi}(z,0) = \mathbb{E}\left[z^{Q^1_{\Delta_1}} \right] = \frac{\xi(M) \left\{ \left(\frac{z}{\beta} \right)^{M+1} - 1 \right\}}{\frac{z}{\beta} - 1}, |z| < 1, \tag{21}$$

and the mean of the RPCP is as follows:

$$\mathbb{E}\left[Q^1_{\Delta_1} \right] = \frac{d}{dz} \widehat{\psi}(z,0) \bigg|_{z=1},$$

which yields

$$\mathbb{E}\left[Q^1_{\Delta_1} \right] = \xi(M)(1+\lambda a) \left\{ (M - \lambda a) \left(1 + \frac{1}{\lambda a} \right)^{M} + \lambda a \right\}. \tag{22}$$

■

2.1. Random Capacity of Nodes

The concept of random capacity is targeted for extend extending the RPCP model to accommodate scenarios where the capacity M is not fixed rather varies randomly. This enhancement allows the model to better represent real-world systems with fluctuating resources or capacities, adding a layer of complexity and realism to the analysis. Let Q_t^1 denote the number of counts from the random capacity $\boldsymbol{M}$ representing the random full capacity at the initial moment (i.e., $t = \tau_0 = 0$). The probability mass function (PMF) of $\boldsymbol{M}$ representing the system's full capacity or the number of full network nodes) is given by $b(n) := P\{\boldsymbol{M} = n\}, n = 0, 1, \ldots, M_{\max}$. The mean of $\boldsymbol{M}$ is denoted as $\widetilde{m} = \mathbb{E}[\boldsymbol{M}]$ and $M_{\max}$ is the maximum capacity of system or the number of maximum nodes on the network security. From (18), the probability of the RPCP under the random capacity could be as follows:

$$\pi_k^1 = \mathbb{E}[\xi(\boldsymbol{M})] \cdot \left(1 + \frac{1}{\lambda a}\right)^k, k = 0, \ldots, M_{\max}, \tag{23}$$

where

$$\mathbb{E}[\xi(\boldsymbol{M})] = \left(\frac{1}{\lambda a}\right) \cdot \mathbb{E}\left[\frac{\beta^{\boldsymbol{M}+1}}{(1 - \beta^{\boldsymbol{M}+1})}\right],$$

and

$$\mathbb{E}[\xi(\boldsymbol{M})] \simeq \xi(\mathbb{E}[\boldsymbol{M}]) = \frac{\beta^{\{\widetilde{m}+1\}}}{\lambda a\left(1 - \beta^{\{\widetilde{m}+1\}}\right)}, \tag{24}$$

which is assumed that $\mathbb{E}[G(\boldsymbol{X})] \simeq G(\mathbb{E}[\boldsymbol{X}])$.