Session 2 : Fall 2016 Linear Algebra Writtens' Workshop

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1 Problems

1.1 Determinants, Trace and Rank

January 1995 Problem 2

Suppose A is an $n \times n$ matrix and that

$$x = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}, y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix}, y^* = \begin{pmatrix} y_1 \dots y_n \end{pmatrix}$$

Suppose that all the entries of A, x and y are real.

Part 1

Show that there exist numbers a and b so that $\det(A + sxy^*) = a + bs$.

Part 2

Show that if $\det(A) \neq 0$ then $a = \det(A)$ and $b = \det(A)y^*A^{-1}x$.

Part 3

Is it true that $a = 0$ if $\det(A) = 0$?

January 2012 Problem 2

Let $H = (h_{ij})$ be an $n \times n$ matrix such that (i) All coefficients $h_{ij} \in \{+1, -1\}$ and (ii) The row vectors $h_i = (h_{i1}, \dots h_{in})$ are mutually orthogonal

Part 1

Find a simple expression for HH^T.

Part 2

Find $det(H)$.

Part 3

Let $u = (u_1, \dots u_n)$ with all $u_j = \pm 1$. Prove that at least one coordinate of Hu^T has absolute value at least $\sqrt{n}$. (Hint: Find the Euclidean norm of Hu^T.

January 2014 Problem 1

Consider $n \times n$ real matrices A, B and C with $ABC = 0$.

- What can be maximal possible rank of CBA?
- What is a maximal possible rank of CBA if we assume that C, A are diagonal?
- What is a maximal possible rank of CBA if we assume that A, B, C are symmetric?

1.2 Projection and Orthogonal Matrix

September 2014 Problem 3

A complex $n \times n$ matrix U is unitary if it satisfies $U^H U = I$. A real unitary matrix is called orthogonal.

1. If U is $n \times n$ and unitary, show that

$$\max_{j,k} |U_{j,k}| \geq \frac{1}{\sqrt{n}}$$

with equality if and only if all of the elements of U are equal in magnitude.

2. Give an example of a 3×3 unitary matrix U for which all of the elements of U are equal in magnitude.

3. Prove the existence for every positive integer n of an $n \times n$ unitary matrix U for which all of the elements of U are equal in magnitude.

4. Now consider orthogonal matrices. For which $n \in \{2, 3, 4, 5, 6\}$ do there exist orthogonal matrices with all of their (real) elements equal in magnitude? For those $n \in \{2, 3, 4, 5, 6\}$ for which such matrices exist, construct on; and for the remaining $n \in \{2, 3, 4, 5, 6\}$, prove that no such orthogonal matrix exists.

5. Find an infinite set S of positive integers such that for every $n \in S$ there exists an $n \times n$ orthogonal matrix all of whose elements are equal in magnitude.

September 2012 Problem 2

Find 3×3 matrices A, B and C that correspond to the following three linear operations. In all steps, explain your answers.

Part 1

Let A represent the orthogonal projection onto the plane $x - y + z = 0$.

Part 2

Let B represent the reflection across the plane $x - y + z = 0$.

Part 3

Let C represent rotation by π about the line $x = -y = z$. (Fortunately, it doesn't matter the sign of rotation, as either direction is the same.)

Part 4

Evaluate A^4, B^2 and C^3.

Part 5

Evaluate $AB - BA$, $AC - CA$, and $BC - CB$.

September 2015 Problem 3

In this problem, $\mathcal{M}_n(\mathbb{C})$ is the set of square matrices with size $n \times n$ with coefficients in $\mathbb{C}$, and for any matrix A, A^* is its conjugate transpose, defined such that $A^*_{ij} = \overline{A_{ji}}$. $\mathcal{U}_n(\mathbb{C})$ is the set of unitary matrices in $\mathcal{M}_n(\mathbb{C})$, i.e. matrices U whose adjoints U^* are also their inverses: $U^*U = UU^* = I_n$

Let k be an integer in $[\![1, n]\!]$. We consider a matrix $P \in \mathcal{M}_n(\mathbb{C})$ that satisfies the conditions $(\mathcal{P}_k)$ given by:

$$(\mathcal{P}_k) \qquad P^2 = P = P^* , \qquad \mathrm{rk}(P) = k$$

1. Show that the entries of P satisfy

$$\begin{cases} \forall i \in \mathbb{N} \ s.t. 1 \leq i \leq n \, , \ 0 \leq P_{ii} \leq 1 \\ \sum_{i=1}^{n} P_{ii} = k \end{cases}$$

2. Let $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ be n real numbers and D the diagonal matrix such that $D_{ii} = \lambda_i$ for all integers i such that $1 \leq i \leq n$. Show that for any matrix P satisfying $(\mathcal{P}_k)$

$$\text{Trace}(PD) \leq \sum_{i=1}^{k} \lambda_i$$

Find a matrix Q that satisfies $(\mathcal{P}_k)$ and such that $\text{Trace}(PD) = \sum_{i=1}^{k} \lambda_i$

3. Show that if P_1 and P_2 are two matrices satisfying $(\mathcal{P}_k)$, there exists $U \in \mathcal{U}_n(\mathbb{C})$ such that $P_2 = UP_1U^*$. Show then that

$$\sum_{i=1}^{k} \lambda_i = \max_{U \in \mathcal{U}_n(\mathbb{C})} \text{Trace}(UPU^*D)$$

where P is a matrix satisfying $(\mathcal{P}_k)$.

1.3 Inner Product and Norms

January 2016 Problem 4

4. Introduction: you might recall the double-angle formulas $2\sin^2\theta = (1 - \cos 2\theta)$ and $2\cos^2\theta = (1 + \cos 2\theta)$. These formulae, along with the change-of-variables $x = \sin\theta$, can be used to show:

$$\begin{aligned} \int_{-1}^{+1} \frac{1}{\sqrt{1-x^2}} dx &= \pi, \text{ and } \int_{-1}^{+1} \frac{x^2}{\sqrt{1-x^2}} \text{dx} = \frac{\pi}{2}, \\ \text{and } \int_{-1}^{+1} \frac{x^4}{\sqrt{1-x^2}} dx &= \frac{3\pi}{8}, \text{ and } \int_{-1}^{+1} \frac{x^6}{\sqrt{1-x^2}} dx = \frac{5\pi}{16} \end{aligned}$$

Now consider the vector-space P_n of polynomials with real coefficients of degree n or less defined on the interval $x \in [-1, 1]$. Endow this space with the following inner product:

$$\langle f, g \rangle_T = \int_{-1}^{+1} f(x)\, g(x) \frac{dx}{\sqrt{1-x^2}}.$$

(a) Find a basis $\{t_0(x), t_1(x), t_2(x)\}$ for P_2 that is orthonormal with respect to $\langle \cdot, \cdot \rangle_T$, where t_0 is constant, $t_1(x)$ is linear, and $t_2(x)$ is quadratic.

(b) Express the integration operator $\int_{-1}^{+1} (\cdot)\, dx$ as a linear operator on P_2 in terms of the basis $\{t_0, t_1, t_2\}$.

(c) Now consider the polynomial

$$a(x) = \sqrt{\frac{2}{\pi}} \left[4x^3 + 2x^2 - 3x + x - 1 - \frac{1}{\sqrt{2}} \right] \in P_3.$$

Find the polynomial $b(x) \in P_2$ that is 'closest' to $a(x)$ in the T-norm. That is, find the $b(x)$ that minimizes:

$$\langle a - b, a - b \rangle_T = \int_{-1}^{+1} (a(x) - b(x))^2 \frac{dx}{\sqrt{1-x^2}}.$$

(d) Now, with n arbitrary, find a map from P_n to P_2 that sends a polynomial $a(x) \in P_n$ to the polynomial $b(x) \in P_2$ which is closest to $a(x)$ in the T-norm.

1.4 Jordan Form

January 2014 Problem 3

Find $\lim_{N\to\infty} \|A^N(x)\|/\|B^N(x)\|$ as a function of $x \in \mathbb{R}^2$ if the matrices A, B are:

$$A = \begin{pmatrix} 3 & -1 \\ 1 & 5 \end{pmatrix},$$
$$B = \begin{pmatrix} 6 & -2 \\ 2 & 2 \end{pmatrix}.$$

1.5 Related Problems for practice

S97Q5, J98Q2, J99Q3, J00Q4, S00Q2, J01Q5, S03Q1, S03Q2, J03Q5, S04Q5, J05Q1, J05Q2-Q3, J06Q3, J07Q2, J08Q3, J08Q5, J09Q3, S10Q1, S11Q3, S12Q2, S12Q4