Counting & Probability: Practice & Review

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MATHCOUNTS Fermat Class

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Warmup Problems

1. If a fair 4-sided die is rolled 7 times, what is the probability that no two consecutive rolls are the same?
2. In a certain town, the weather each day depends on the previous day. If it rained yesterday, there is a 50% of rain today. If it didn't rain yesterday, there is a 20% chance of rain today. Given that it rained on January 1st, 2022, find the probability that
   - it rained yesterday (January 3rd)?
   - it will rain today (January 4th)?
3. How many ways can ten identical apples be distributed among four people, if each person must have at least one apple?

Warmup Solutions

1. For every roll after the first one, there is a $3/4$ chance that it will not be the same as the previous roll. Since the last six rolls are all independent events, the probability that no two consecutive rolls are equal is $\left(\frac{3}{4}\right)^6 = \frac{729}{4096}$.

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2. First, note that the probability it rained on January 2nd is $1/2$. Then, by adding up the cases for the weather on January 2nd, the probability that it rained on January 3rd is $\frac{1}{2} \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{5} = \frac{7}{20}$.

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   Similarly, the probability of rain today is therefore $\frac{7}{20} \cdot \frac{1}{2} + \frac{13}{20} \cdot \frac{1}{5} = \frac{61}{200}$.
3. Let the number of apples each person gets be $a+1, b+1, c+1,$ and $d+1$, such that a, b, c, d are all nonnegative integers. Then,

$$(a+1)+(b+1)+(c+1)+(d+1) = 10 \implies a+b+c+d = 6.$$

   By Stars and Bars, the number of solutions to this is $\binom{9}{3} = 84$.

Expected Value

- Given Any event X with possible outcomes $x_1, x_2, \ldots, x_n$, each having probability $p_1, p_2, \ldots, p_n$, the expected value of X (which can be represented by $\mathbb{E}(X)$) is equal to $x_1 p_1 + x_2 p_2 + \cdots + x_n p_n$.

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  - For example, if X is the roll of a standard die, then $\mathbb{E}(X) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + 3 \cdot \frac{1}{6} + 4 \cdot \frac{1}{6} + 5 \cdot \frac{1}{6} + 6 \cdot \frac{1}{6} = \frac{7}{2}$. In this case, all probabilities are the same, so the expected value is just the average of the possible outcomes.

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  - If X is the sum of the rolls of two standard dice, then

$$\begin{aligned}\mathbb{E}(X) = 2 \cdot \frac{1}{36} + 3 \cdot \frac{2}{36} + 4 \cdot \frac{3}{36} + 5 \cdot \frac{4}{36} + 6 \cdot \frac{5}{36} + 7 \cdot \frac{6}{36} + 8 \cdot \frac{5}{36} \\ + 9 \cdot \frac{4}{36} + 10 \cdot \frac{3}{36} + 11 \cdot \frac{2}{36} + 12 \cdot \frac{1}{36} = 7.\end{aligned}$$

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- These two examples have an interesting connection: the expected value of the roll of two dice (7) is twice the expected value of the roll of one die (7/2). This is called **Linearity of Expectation**.
- Linearity of expectation means that for events $X_1, X_2, \ldots, X_k$, the expected value of the sum of these events is the sum of the expected value of each event, for both independent and dependent events. This can be written as

$$\mathbb{E}(X_1 + X_2 + \cdots + X_k) = \mathbb{E}(X_1) + \mathbb{E}(X_2) + \cdots + \mathbb{E}(X_k).$$

1. On the number line, the points $1, 2, 3, \ldots, 9, 10$ are marked. Of these ten points, two distinct points are chosen randomly, with each point being equally likely to be chosen. What is the expected value of the distance between the two points?

2. There is a $5 \times 5 \times 5$ cube, which is painted red on all six of its faces. The cube is then cut into 125 unit cubes. One of these unit cubes is chosen at random, with each of them being equally likely to be chosen. What is the expected number of red faces on this unit cube?

3. Stacia has five pairs of gloves, each pair being a different color. She washed them all and now wants to match up the pairs. She randomly pairs each left glove with a right glove. If all five pairs do not match, she randomly pairs them all up again, and repeats until all five pairs do match. What is the expected number of times she will pair the gloves until all of them match?

Expected Value Practice Solutions

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   If the two points have a distance d, then we can let the smaller number be x and the larger number be $x+d$. Then, $1 \leq x$ and $x+d \leq 10$, so $1 \leq x \leq 10-d$. This has $10-d$ integer solutions for x. The total number of pairs of distinct points that can be chosen is $\binom{10}{2} = 45$, so the probability that the two points have a distance d between each other is $\frac{10-d}{45}$. Using the formula for expected value $\mathbb{E}(X) = x_1p_1 + x_2p_2 + \cdots + x_np_n$, the expected value of d is the sum of $d \cdot \frac{10-d}{45}$ as d ranges from 1 to 9. This can be rewritten as

$$\frac{1\cdot 9}{45} + \frac{2\cdot 8}{45} + \frac{3\cdot 7}{45} + \cdots + \frac{8\cdot 2}{45} + \frac{9\cdot 1}{45}$$

$$= \frac{1}{45}(9+16+21+24+25+24+21+16+9) = \frac{165}{45} = \frac{11}{3}.$$

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   With these values, we can find the probability of each case, and thus the expected value. The answer is

$$0 \cdot \frac{27}{125} + 1 \cdot \frac{54}{125} + 2 \cdot \frac{36}{125} + 3 \cdot \frac{8}{125} = \frac{54 + 72 + 24}{125} = \frac{150}{125} = \frac{6}{5}.$$

3. Let x be the expected number of times that she will need to pair the gloves until they match. First, we will find the probability that she gets the gloves to match on her first try. If we let the colors of the gloves be red, orange, yellow, green, and blue, this is the same as the probability that two random permutations of ROYGB match. This probability is $\frac{1}{5!}$, because no matter what the first permutation is, there are $5!$ possibilities for the second permutation, of which one will match the first one. This also means that there is a probability of $1 - \frac{1}{5!}$ that her first pairing of the gloves doesn't result in a perfect match. At this point, she is back to square one, because each time, her next pairing is completely independent of the previous pairings. The only difference is, she has already tried one pairing.

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$$x = \left(\frac{1}{5!}\right)(1) + \left(1 - \frac{1}{5!}\right)(x+1)$$

$$x = \frac{1}{120} + \frac{119}{120}(x+1)$$

$$x = \frac{1}{120} + \frac{119x}{120} + \frac{119}{120}$$

$$\frac{x}{120} = 1 \implies x = 120.$$

More Practice Problems

1. Al and Bo are planning to both go to a restaurant between 7:00 p.m. and 8:00 p.m., and both arrive at a random time within that hour. Find the probability that they meet each other if
   - both Al and Bo will wait 20 minutes before leaving the restaurant
   - Al will wait 30 minutes before leaving, but Bo will leave immediately after arriving if Al isn't there.
2. How many five-digit numbers are there with a digit-sum of 37? How about 35? (Keep in mind that the first digit must be nonzero.)
3. If there are n rocks, of which exactly m are blue and the rest are gray, how many ways are there to place the n rocks in a line such that no blue rocks are next to each other? (All gray rocks are indistinguishable and all blue rocks are indistinguishable).

1. Instead of the hour, we can just focus on the number of minutes a that Al arrives after 7:00 and the number of minutes b that Bo arrives after 7:00. This means that $0 \leq a, b \leq 60$. Because there are an infinite number of times the two could arrive at, we need to use geometric probability in the form of a graph to model this.

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   The area of the entire square (all possibilities for a and b) is 60^2. The area of the gray region can be found by subtracting the two white triangles, meaning the gray region has area $60^2 - 2\left(\frac{40 \cdot 40}{2}\right) = 60^2 - 40^2$. The answer is therefore

$$\frac{60^2 - 40^2}{60^2} = \frac{3600 - 1600}{3600} = \frac{2000}{3600} = \frac{5}{9}.$$

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   The area of the entire square (all possibilities for a and b) is 60^2. The area of the gray region can be found by subtracting the two white triangles, meaning the gray region has area $60^2 - \frac{30 \cdot 30}{2} - \frac{60 \cdot 60}{2} = 3600 - 450 - 1800 = 1350$. The answer is therefore
$$\frac{1350}{3600} = \frac{3}{8}.$$

2. Consider a five-digit number $\overline{abcde}$ with digit sum $a+b+c+d+e=37$. At first, we will ignore the fact that $a \neq 0$. Notice that

$$
\begin{aligned}
a+b+c+d+e=37 &\implies 0 = 37 - a - b - c - d - e \\
&\implies 8 = 45 - a - b - c - d - e \\
&\implies 8 = (9-a)+(9-b)+(9-c)+(9-d)+(9-e).
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Let $a' = 9-a$, $b' = 9-b$, and so on until $e' = 9-e$. Because each digit must be between 0 and 9, a', b', c', d', e' are nonnegative integers which are also between 0 and 9. Furthermore, they add up to 8. By Stars and Bars, the number of solutions to this is $\binom{12}{4} = 495$.

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Now, we must subtract the cases where $a = 0$ because these would result in a five-digit integer where the leading digit is zero. If $a = 0$, then $b+c+d+e=37$. However, because none of the digits can exceed 9, $b+c+d+e \leq 9 \cdot 4 = 36$. Therefore, there are actually no solutions where $a = 0$, and our original answer of 495 is correct.

More Practice Solutions

2. Now, consider a five-digit number $\overline{abcde}$ with digit sum $a+b+c+d+e=35$. Like last time, we will first ignore the fact that $a \neq 0$. Notice that

$$\begin{aligned} a+b+c+d+e=35 &\implies 0 = 35-a-b-c-d-e \\ &\implies 10 = 45-a-b-c-d-e \\ &\implies 10 = (9-a)+(9-b)+(9-c)+(9-d)+(9-e). \end{aligned}$$

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$$\begin{aligned} a+b+c+d+e=35 &\implies 0=35-a-b-c-d-e \\ &\implies 10=45-a-b-c-d-e \\ &\implies 10=(9-a)+(9-b)+(9-c)+(9-d)+(9-e). \end{aligned}$$

Let $a'=9-a$, $b'=9-b$, and so on until $e'=9-e$. Because each digit must be between 0 and 9, a', b', c', d', e' are nonnegative integers which are also between 0 and 9. Furthermore, they add up to 10. By Stars and Bars, the number of solutions to this is $\binom{14}{4}=1001$, but ...

2. Now, consider a five-digit number $\overline{abcde}$ with digit sum $a+b+c+d+e=35$. Like last time, we will first ignore the fact that $a \neq 0$. Notice that

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Let $a' = 9-a$, $b' = 9-b$, and so on until $e' = 9-e$. Because each digit must be between 0 and 9, a', b', c', d', e' are nonnegative integers which are also between 0 and 9. Furthermore, they add up to 10. By Stars and Bars, the number of solutions to this is $\binom{14}{4} = 1001$, but this ignores the fact that none of a', b', c', d', e' can be greater than 9. (If any of a', b', c', d', e' were greater than 9, it would mean that one of a, b, c, d, e was less than zero, which is impossible.) This means we counted solutions $(0,0,0,0,10)$, $(0,0,0,10,0)$, and so on. There are five of these, so the actual number of solutions is $1001-5=996$.

2. Now, we must subtract the cases where $a = 0$ because these would result in a five-digit integer where the leading digit is zero. If $a = 0$, then

$$\begin{aligned} b + c + d + e = 35 &\implies 0 = 35 - b - c - d - e \\ &\implies 1 = (9 - b) + (9 - c) + (9 - d) + (9 - e) \\ &\implies 1 = b' + c' + d' + e'. \end{aligned}$$

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Since b', c', d', e' are all nonnegative integers, by Stars and Bars, there are $\binom{4}{3} = 4$ solutions to this. This means there are 4 possibilities where $a = 0$ and the number still has a digit sum of 35.

Subtracting these 4 cases from our total count gives an answer of $996 - 4 = 992$.

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   This means there are $n-m+1$ possible places for the blue rocks, so since we must place m in these places, the number of possible ways to arrange the rocks is $\binom{n-m+1}{m}$.

   This means that if $m > n-m+1$, which translates to $2m-1 > n$, then there are no possible arrangements of the rocks.

1. If three real numbers are uniformly and randomly chosen from between 0 and 3, what is the probability that their sum is less than 3?
2. Al and Bo are playing a game. Each of them has an unfair coin; Al's coin has a $2/3$ chance of coming up heads and Bo's coin has a $2/5$ chance of coming up heads. The game is as follows: Al and Bo alternate flipping their coin until both coins match, with Al going first. The winning is whoever flipped their coin most recently. For example, a game could go as follows: Al starts by flipping a tails, then Bo flips a heads, then Al flips tails again, and Bo flips a tails, winning the game. What is the probability of Al winning?

1. Let the three numbers be $x + y + z$, where $0 \leq x, y, z \leq 3$. We want to find the probability that $x + y + z \leq 3$. There are infinitely many possibilities for each of $x, y,$ and z, so to find this probability, we can use geometric probability. Just like how we need two dimensions to describe two unknowns, we need three to describe three unknowns.

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   The total region we are concerned with is the set of points where $0 \leq x \leq 3$, $0 \leq y \leq 3$, and $0 \leq z \leq 3$. This forms a $3 \times 3 \times 3$ cube. The equation $x + y + z = 3$ forms a plane in 3D space, which slices through this cube and forms a pyramid. Here are three ways of viewing this:

1. To find the probability that $x + y + z \leq 3$, we want to find the ratio of the volume of this pyramid to the volume of the whole cube. The general volume of a pyramid is $\frac{1}{3} \times$ base $\times$ height. In this case, if we choose the triangle formed by $(0, 0, 0)$, $(3, 0, 0)$, and $(0, 3, 0)$ to be the base, the base is an isosceles right triangle with legs that are 3 units long. The apex points $(0, 0, 3)$ is 3 units above the base, so the height is 3. Thus,

$$\begin{aligned} \text{Volume of Pyramid} &= \frac{1}{3} \times \text{base} \times \text{height} \\ &= \frac{1}{3} \times \left(\frac{1}{2} \cdot 3 \cdot 3\right) \times 3 \\ &= \frac{27}{6} = \frac{9}{2}. \end{aligned}$$

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The entire cube of points where $0 \le x, y, z \le 3$ is a cube with edge length 3, so its volume is 27. Finally, the probability that $x + y + z \le 3$ is $\frac{9/2}{27} = \frac{1}{6}$.

2. In order for Al to win, Al and Bo's flips must be an alternating sequence of heads and tails (or tails and heads) until Al finally matches Bo's last flip. Let a string of H's and T's represent their flips; for instance, HTHTHTHTHH would represent Al flipping heads, then Bo flipping tails, and so on until Bo wins by flipping heads. We will use casework on Al's first flip.

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   If Al's first flip is heads, then Bo's first flip must be tails, and Al and Bo must keep alternating heads and tails until Al flips a tail to win. This can be represented by the strings HTT, HTHTT, HTHTHTT, ... More generally, the string of flips if Al starts with heads and wins is some number of HT's and a T at the end. The probability of HT (Al flipping H and Bo flipping T) is $\frac{2}{3} \cdot \frac{3}{5} = \frac{2}{5}$, and the probability of Al flipping the final T is $\frac{1}{3}$.

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   If Al's first flip is heads, then Bo's first flip must be tails, and Al and Bo must keep alternating heads and tails until Al flips a tail to win. This can be represented by the strings HTT, HTHTT, HTHTHTT, ... More generally, the string of flips if Al starts with heads and wins is some number of HT's and a T at the end. The probability of HT (Al flipping H and Bo flipping T) is $\frac{2}{3} \cdot \frac{3}{5} = \frac{2}{5}$, and the probability of Al flipping the final T is $\frac{1}{3}$. Thus, the probability of this case is the sum of the probabilities for each possible string, which is

$$\begin{aligned}
&\left(\frac{2}{5}\right)\left(\frac{1}{3}\right) + \left(\frac{2}{5}\right)^2\left(\frac{1}{3}\right) + \left(\frac{2}{5}\right)^3\left(\frac{1}{3}\right) + \cdots \\
&= \frac{1}{3}\left(\left(\frac{2}{5}\right) + \left(\frac{2}{5}\right)^2 + \cdots\right) \\
&= \frac{1}{3}\left(\frac{2/5}{1 - 2/5}\right) = \frac{2}{9}.
\end{aligned}$$

2. Similarly, if Al's first flip is a tails, then the string of flips is one of THH, THTHH, THTHTHH, ... More generally, the string of flips if Al starts with tails and wins is some number of TH's and a H at the end. The probability of TH (Al flipping T and Bo flipping H) is $\frac{1}{3} \cdot \frac{2}{5} = \frac{2}{15}$, and the probability of Al flipping the final H is $\frac{2}{3}$.

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$$\left(\frac{2}{15}\right)\left(\frac{2}{3}\right) + \left(\frac{2}{15}\right)^2\left(\frac{2}{3}\right) + \left(\frac{2}{15}\right)^3\left(\frac{2}{3}\right) + \cdots$$

$$= \frac{2}{3}\left(\left(\frac{2}{15}\right) + \left(\frac{2}{15}\right)^2 + \cdots\right)$$

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2. Similarly, if Al's first flip is a tails, then the string of flips is one of THH, THTHH, THTHTHH, ... More generally, the string of flips if Al starts with tails and wins is some number of TH's and a H at the end. The probability of TH (Al flipping T and Bo flipping H) is $\frac{1}{3} \cdot \frac{2}{5} = \frac{2}{15}$, and the probability of Al flipping the final H is $\frac{2}{3}$. Then, adding up the probabilities like last case gives us

$$\left(\frac{2}{15}\right)\left(\frac{2}{3}\right) + \left(\frac{2}{15}\right)^2\left(\frac{2}{3}\right) + \left(\frac{2}{15}\right)^3\left(\frac{2}{3}\right) + \cdots$$

$$= \frac{2}{3}\left(\left(\frac{2}{15}\right) + \left(\frac{2}{15}\right)^2 + \cdots\right)$$

$$= \frac{2}{3}\left(\frac{2/15}{1 - 2/15}\right) = \frac{4}{39}.$$

Adding up the two cases, we deduce that Al's chance of winning is

$$\frac{2}{9} + \frac{4}{39} = \frac{38}{117}.$$