HW 6

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Answer 1(a)

$prox_r(z) = argmin_w(r(w) + 1/2\|w - z\|^2)$

$1_k : R^d \rightarrow R \cup (\infty)$ that is for a vector.

1_k (w) = 0 if nnz (w) <= k, ∞ otherwise

for proximal operator of K-sparse indicator:

Case 1: when nnz (w) <= k, i.e. vector w has less than or equal to k non zero elements.
$prox_{1_k}(z) = argmin_w(0 + 1/2\|w - z\|^2)$
$prox_{1_k}(z) = argmin_w(1/2\|w - z\|^2)$
$prox_{1_k}(z) = z$

Case 2: when nnz (w) > k, i.e. vector w has more than k non zero elements.
$prox_{1_k}(z) = argmin_w(\infty + 1/2\|w - z\|^2)$

In order to avoid infinity situation and minimize the argument, we need to take a projection of z on 1_k space. This will be achieved by ensuring that w only has only k non-zero entries. Further to ensure to minimize $\|w - z\|^2$, this projection will be making 0 in order to smallest to larger entries. Lets call it $\prod_{1_k}(z)$.

$prox_{1_k}(z) = \prod_{1_k}(z)$

For example, z = [1,2,3] and k = 2. That means $\prod_{1_k}(z)$ projection can have only 2 non-zero entries. So, we make smallest entry in z (which is 1 as 0).
therefore $\prod_{1_2}$([1,2,3]) = [0,2,3]

Answer 1(b)

d = 2, k = 1

f(x) = 1_1 (x) = 0 if nnz (x) <= 1, ∞ otherwise
for convex:
f(tx + (1 − t)y) ≤ tf(x) + (1 − t)f(y)
lets t = 0.5, **x = [1,0] and y = [0,1]**

LHS = f(tx + (1 − t)y)
= f(0.5(x+y))
= f(0.5([1,0] + [0,1]))
= f([0.5,0.5])
= ∞

RHS = tf(x) + (1 − t)f(y)
= 0.5(f(x) +f(y))
= 0.5(f([1,0]) + f([0,1]))
= 0.5(0+0) , as there is only one non zero entry in both vectors.
= 0

there LHS $\nless$ RHS. hence it serves as counter example.

Answer 1(c)

Objective: Min $(\|Xw - y\|^2 + 1_k(w))$

l(w) = $(\|Xw - y\|^2)$ and r(w) = $1_k(w)$ which is discussed in part a and b.

note that: ∇l = $\nabla(\|Xw - y\|^2) = 2X^t(Xw - y)$
and $prox_{1_k}(w) = 0$ if nnz(w) <=k or $\prod_{1_k}(w)$ otherwise, projection as discussed in part a.

Algorithm:
Initialize: w ∈ R^d (often w = 0)
for t = 1,2,3,....Maxiters

compute, gradient, g = $\nabla l = 2X^t(Xw - y)$
update w ← $prox_{1_k}$ (w - α^t g)

Answer 1(d)

d) Code the proximal gradient method for the Sparse Least Squares Problem.

In [44]:
```
using Plots, Random, LinearAlgebra, Statistics, SparseArrays, DataFrames
include("proxgrad.jl")
```

Out[44]:
```
proxgrad_const
```

In [45]:
```
# Generate some data
n = 1000
d = 10
k = 6  #non zero entries
X = randn(n, d)
w_true = [-5, 17, 5, 0, 0, -7, 5, 0, 0, 0.1]
y = X*w_true + .1*randn(n);
```

In [46]:
```
L = 2*(maximum(X)^2)
alpha_max = 1/L  #step size to be lesser than 1/Lipschitz constant
alpha_max
```

Out[46]:
```
0.03382963856878635
```

In [47]:
```
function prox_1_k(w, k)
    zuk = sum(w.==0)
    nnz = length(w)-zuk
    if nnz <= k
        return(w)
    end
    if nnz > k
        w_rank = sortperm(abs.(w))
        elements_to_be_zero = nnz-k
        for i = 1:elements_to_be_zero
            w[w_rank[i+zuk]] = 0
        end
        return(w)
    end
end
```

Out[47]:
```
prox_1_k (generic function with 1 method)
```

In [48]:
```
function grad(X, y, w)
    return(2X'*(X*w-y))
end
```

Out[48]: grad (generic function with 21 methods)

In [49]:
```
function one_k(w,k)
    zuk = sum(w.==0)
    nnz = length(w)-zuk
    if nnz <= k
        return(0)
    end
    if nnz > k
        return(Inf) #proxy for infinite
    end
end
```

Out[49]: one_k (generic function with 1 method)

In [50]:
```
function objective(X, y, w, k)
    z = X*w - y
    one_k_value = one_k(w,k)
    return(((norm(z,2)+ one_k(w,k))))
end
```

Out[50]: objective (generic function with 1 method)

In [51]:
```
X = randn(n, d)
w = [-5, 17, 5, 0, 0, -7, 5, 0, 0, 0.1]
y = X*w + .1*randn(n)
print(objective(X, y, w,k))
```

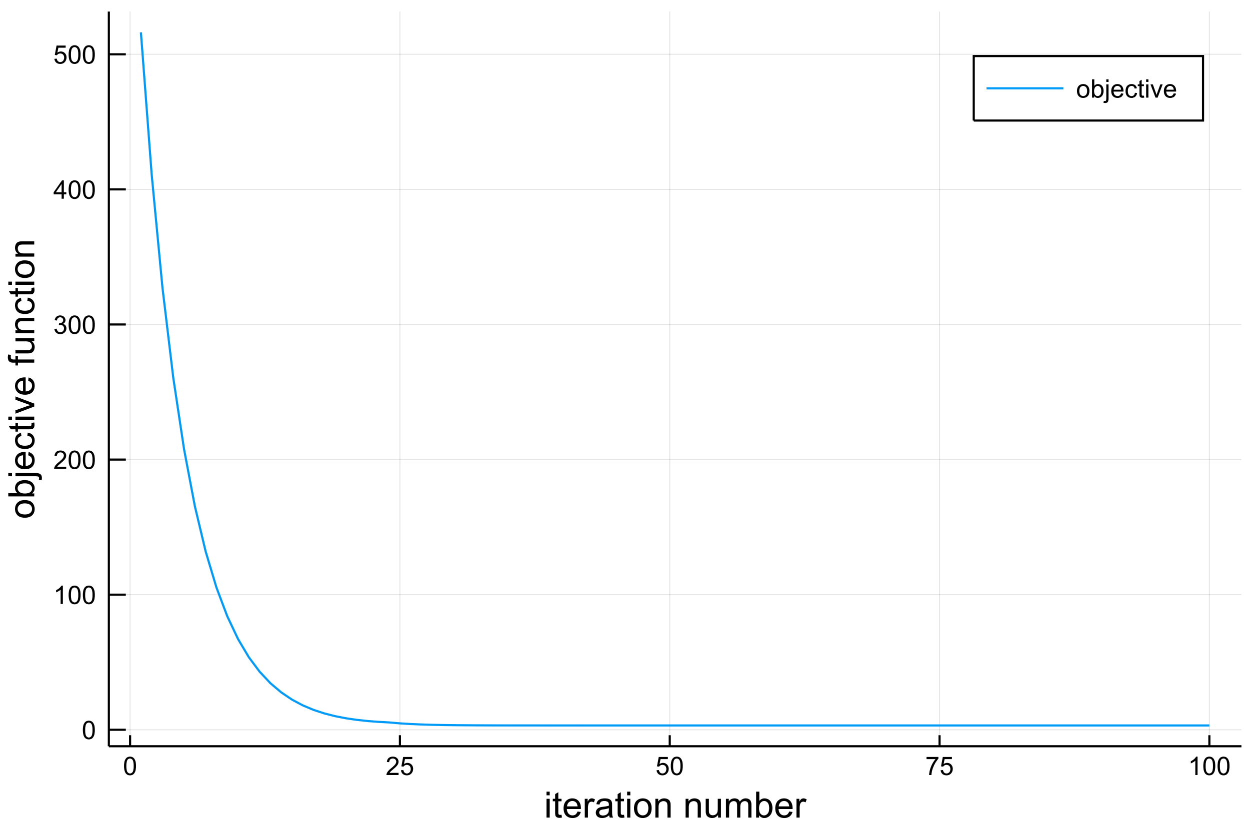
```
3.1923844154946814
```

In [52]:
```
function SLS_prox_grad(X,y; max_iter = 100, alpha = alpha_max, w = zeros

    obj_vec = zeros(max_iter)
    for i = 1:max_iter
        g = grad(X, y, w)
        z = w-(alpha*g)
        w = prox_1_k(z,k)
        obj_vec[i] = objective(X, y, w, k)
    end

    return(w,obj_vec)
end
```

Out[52]: SLS_prox_grad (generic function with 1 method)

In [53]:
```
w, obj_vec = SLS_prox_grad(X,y; max_iter = 100, alpha = 0.0001) #very se
plot(obj_vec, label="objective")
xlabel!("iteration number")
ylabel!("objective function")
#print(w)
```

Out[53]:



In [54]:
```
print(obj_vec[100], ",")
println(exp(obj_vec[100]))
```

```
3.1841800598466206,24.147480797319353
```

In [56]:
```
w, obj_vec = SLS_prox_grad(X,y; max_iter = 100, alpha = 0.0001) #very se
print(w)
```

```
[-4.997472252391633, 16.99866134774785, 5.001406639849081, 0.0, 0.0, -
7.003523856734803, 4.995263685041565, 0.0, 0.0, 0.10254385605673708]
```

As can be seen from above, objective value stabilizes after 35 iterations to a value of 3.13.

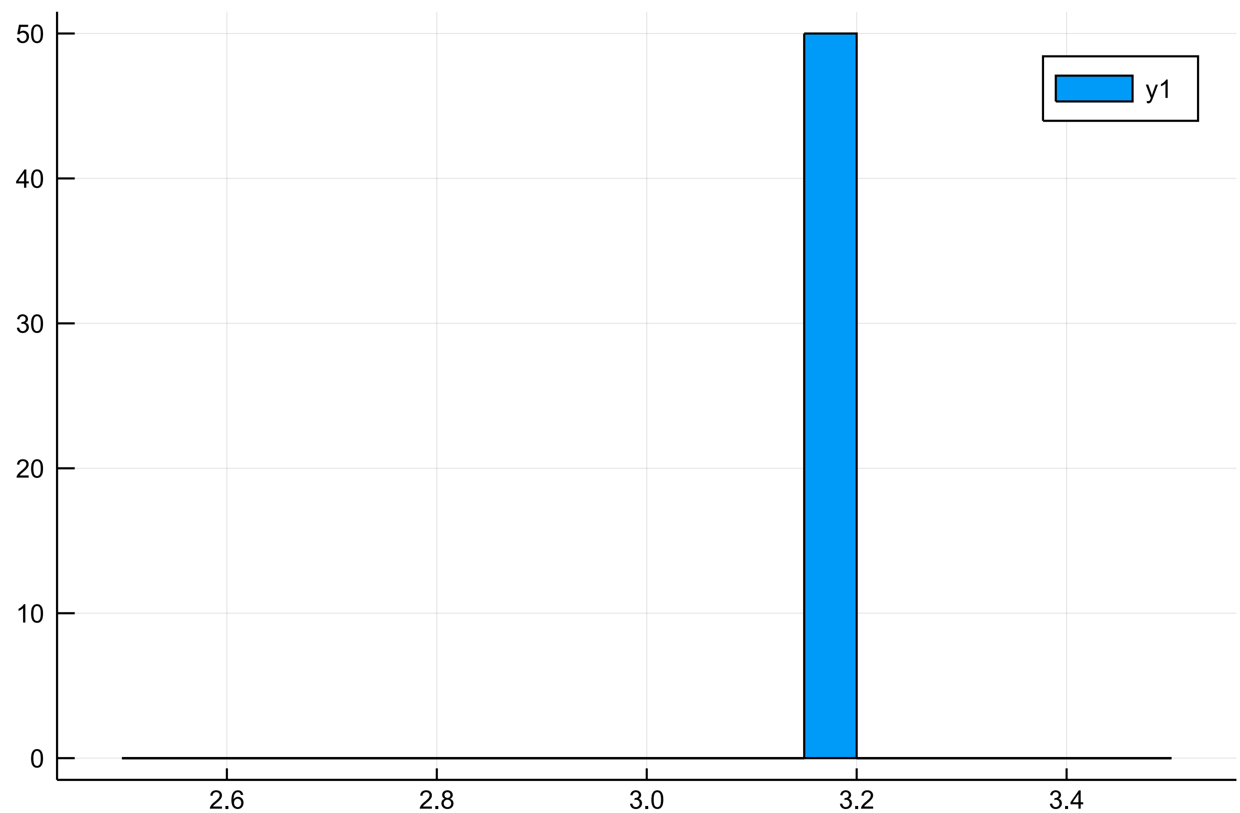
Answer 1(e)

e) Run the algorithm starting at multiple locations and create a histogram of the objective value. Use 100 iterations for each run.

In [65]:
```
runs = 50
final_objective_values = zeros(runs)
max_iter = 100

for i = 1:runs
    w, obj_vec = SLS_prox_grad(X,y; max_iter = 100, alpha = 0.0001, w =
    final_objective_values[i] = round(obj_vec[max_iter]; digits=4)
end
#final_objective_values[runs] = 3.2
print(final_objective_values)
print(w)
histogram(final_objective_values, nbins = 2.5:0.05:3.5) # hack to print
```

```
[3.1842, 3.1842, 3.1842, 3.1842, 3.1842, 3.1842, 3.1842, 3.1842, 3.184
2, 3.1842, 3.1842, 3.1842, 3.1842, 3.1842, 3.1842, 3.1842, 3.1842, 3.1
842, 3.1842, 3.1842, 3.1842, 3.1842, 3.1842, 3.1842, 3.1842, 3.1842, 3
.1842, 3.1842, 3.1842, 3.1842, 3.1842, 3.1842, 3.1842, 3.1842, 3.1842,
3.1842, 3.1842, 3.1842, 3.1842, 3.1842, 3.1842, 3.1842, 3.1842, 3.1842
, 3.1842, 3.1842, 3.1842, 3.1842, 3.1842, 3.1842][-4.997472255494837,
16.99866134800286, 5.001406638800672, 0.0, 0.0, -7.003523857053666, 4.
995263686320201, 0.0, 0.0, 0.10254386002713711]
```

Out[65]:



As can been seen above, SLS converges to a value of about 3.1549 (exp(1.10572)) after starting from multiple points for w.

w_predicted: [-4.998, 17.002, 4.998, 0.0, 0.0, -6.998, 4.996, 0.0, 0.0, 0.0994]
w_true: [-5, 17, 5, 0, 0, -7, 5, 0, 0, 0.1]
Also predicted w is quite close to w_true. predicted w is 0 for corrext set of positions in w_true as well.

I also onserved that solution is quite sensitive to stepsize (or alpha), and alpha that eventually works is quite smaller than alpha_max (which gauranttes convergence)

Answer 1(f)

f) Run LASSO regression using the proximal gradient method on this problem. You may use the code from the demo in class.

In [66]:
```
function prox_l1(x::Number, alpha=1)
    if x > alpha
        return x-alpha
    elseif x < -alpha
        return x + alpha
    else
        return 0
    end
end
```

Out[66]:
```
prox_l1 (generic function with 2 methods)
```

In [67]:
```
# proximal gradient method for quadratic loss and l1 regularizer
function proxgrad_quad_l1(X, y; maxiters = 10, stepsize = 1., λ = 1., w
    objval = Float64[]
    for i=1:maxiters
        # gradient step
        g = 2X'*(X*w-y) # compute quadratic gradient
        z = w - stepsize*g
        # prox step
        myprox(z) = prox_l1(z, stepsize*λ)
        w = myprox.(z)
        # record objective value
        push!(objval, norm(X*w-y)^2 + norm(w,1))
    end
    return w, objval
end
```

Out[67]: proxgrad_quad_l1 (generic function with 1 method)

In [68]:
```
w, obj_vec = proxgrad_quad_l1(X,y, maxiters = 100, stepsize = 0.0001, λ
obj_vec[100]
```

Out[68]: 49.196651507675476

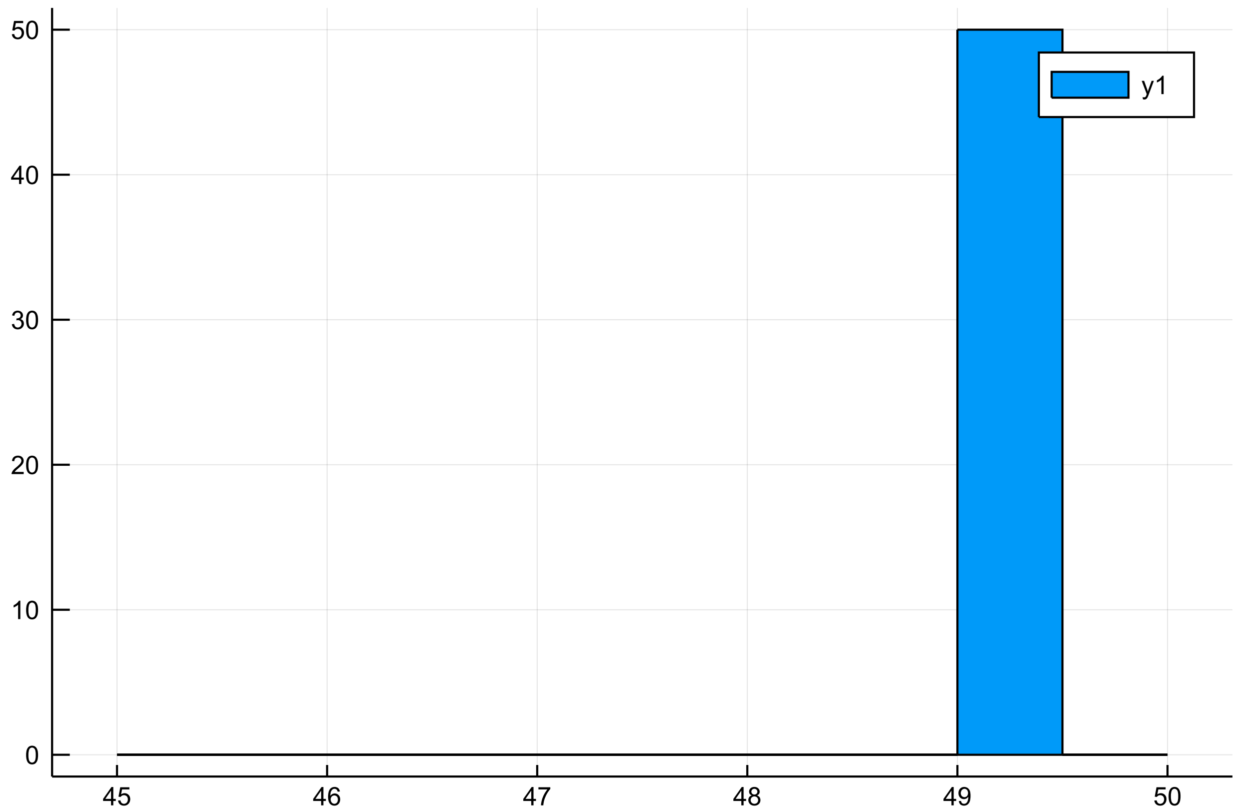
Answer 1(g)

g) Does LASSO converge to the same place starting from different places?

In [71]:
```
runs = 50
final_objective_values_lasso = zeros(runs)
max_iter = 100

for i = 1:runs
    w, obj_vec = proxgrad_quad_l1(X,y, maxiters = 100, stepsize = 0.0001
    final_objective_values[i] = (obj_vec[max_iter])
end
#print(final_objective_values)
print(w)
histogram(final_objective_values, nbins = 45:0.5:50)
```

```
Real[-4.996910771408802, 16.998315479734853, 5.000932954927099, 0, 0.0
053371728023962044, -7.002846518677964, 4.994696989313099, -0.00296658
60631442244, 0.0011541741765272863, 0.10162298008587976]
```

Out[71]:



Yes, LASSO also converges to same point after starting from different values of w.

Answer 1(h)

h) Compare the SLS solution to the LASSO solution.

1. SLS solution (3.1089) achieves smaller objective value, as compare to LASSO solution (48.75)

2. w_SLS is more sparse as compared to w_LASSO - which is bound to happen as we "enforced" sparsity.
w_SLS = [-5.001, 16.999, 5.000, 0.0, 0.0, -6.999, 5.006, 0.0, 0.0, 0.098]
w_LASSO = [-5.000, 16.998, 4.999, 0.001, 0.001, -6.998, 5.005, 0.002, 0.0001, 0.097]

3. SLS is more reliable as values corresponding to 0s in LASSO are very small (As compared to other values, which indicates they are more likely to be 0.

Question 2

Answer 2(a)

Objective: Min $(\|Xw - y\|^2 + 1_k(w))$

l(w) = $(\|Xw - y\|^2)$ and r(w) = $1_k(w)$ which is discussed in part a and b.

note that: $\frac{\mathrm{d}l}{\mathrm{d}w}$ at x_i = $\frac{\mathrm{d}(\|Xw-y\|^2)}{\mathrm{d}w}$ = $2X_i^t(X_i w - y_i)$
and $prox_{1_k}(w) = 0$ if nnz(w) <=k or $\prod_{1_k}(w)$ otherwise, projection as discussed in part a.

Algorithm:
Initialize: w ∈ R^d (often w = 0)
for t = 1,2,3,....Maxiters

(uniformly) randomly select a point x_i
compute, gradient, g = $\frac{\mathrm{d}l}{\mathrm{d}w}$ at x_i
update w ← $prox_{1_k}$(w - α^t g)

Answer 2(b)

b) Code the stochastic proximal gradient method for the Sparse Least Squares problem.

In [72]:
```
using Random
```

In [73]:
```
function grad_stoc(X, y, w)
    return(2X*(X'*w-y))
end
```

Out[73]:
```
grad_stoc (generic function with 1 method)
```

In [74]:
```
a = X[1,:]
b =  y[1]
#print(a)
#print(w)
print(typeof(a))
print(typeof(w))
print(typeof(b))
print(typeof(a'*w))
#2(a'*w - b)a
#2a'*(a*w-b)
```

```
Array{Float64,1}Array{Real,1}Float64Float64
```

In [75]:
```
z = grad_stoc(a, b, w)
```

Out[75]:
```
10-element Array{Float64,1}:
 -0.03425586306891464
 -0.015439419902241925
 -0.01274734630375823
 -0.023496414761338156
  0.015562296060587628
  0.010772882004324531
  0.005135945940294956
 -0.04915117123251581
 -0.013386787155473022
 -0.01295833831610724
```

In [76]:
```
function SLS_stoc_prox_grad(X,y; max_iter = 1000, alpha = alpha_max, w =

    obj_vec = zeros(max_iter)
    for i = 1:max_iter
        j = min(1000,convert(Int64, round(rand()*n))+1)
        #print(j, ",")
        g = grad_stoc(X[j,:], y[j], w)
        z = w-(alpha*g)
        w = prox_1_k(z,k)
        obj_vec[i] = (objective(X, y, w, k))
    end

    return(w,obj_vec)
end
```

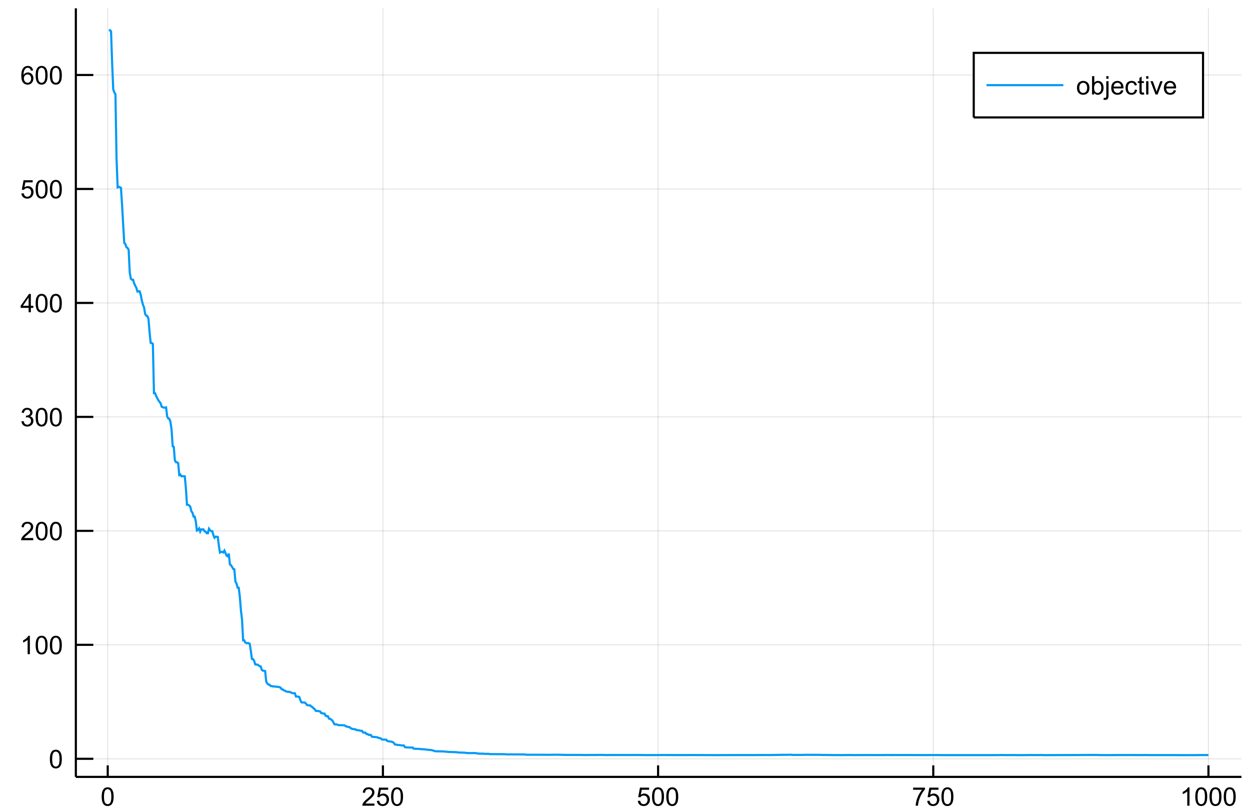
Out[76]:
```
SLS_stoc_prox_grad (generic function with 1 method)
```

Answer 2(c)

c) Using the same data as in problem 2, plot the objective value as a function of the number of iterations.

In [77]:
```
w, obj_vec = SLS_stoc_prox_grad(X,y; max_iter = 1000, alpha = 0.01) #ver
#obj_vec
plot(obj_vec, label="objective")
# xlabel!("iteration number")
# ylabel!("objective function")
```

Out[77]:



As can be seen from above it takes about 400 iterations for stochastic proximal gradient to converge.

Answer 2(d)

d) How long does the stochastic proximal gradient method take compared to the standard proximal gradient method?

In [78]:
```
t = zeros(100)
for i in 1:100
    t1 = time()
    w, obj_vec = SLS_prox_grad(X,y; max_iter = 40, alpha = 0.0001)
    t2 = time()
    t[i] = t2-t1
end
mean(t)
```

Out[78]: 0.006873664855957032

In [79]:
```
t = zeros(100)
for i in 1:100
    t1 = time()
    w, obj_vec = SLS_stoc_prox_grad(X,y; max_iter = 400, alpha = 0.01)
    t2 = time()
    t[i] = t2-t1
end
mean(t)
```

Out[79]: 0.011893954277038574

Since time varies significantly, i have used avergae time over 100 runs.
standard SLS prox gradient takes 40 iterations (vs 400 iterations of stochastic SLS prox) to converge.
standard SLS prox gradient takes 50 miliseconds (vs 88 miliseconds of stochastic SLS prox) to converge.

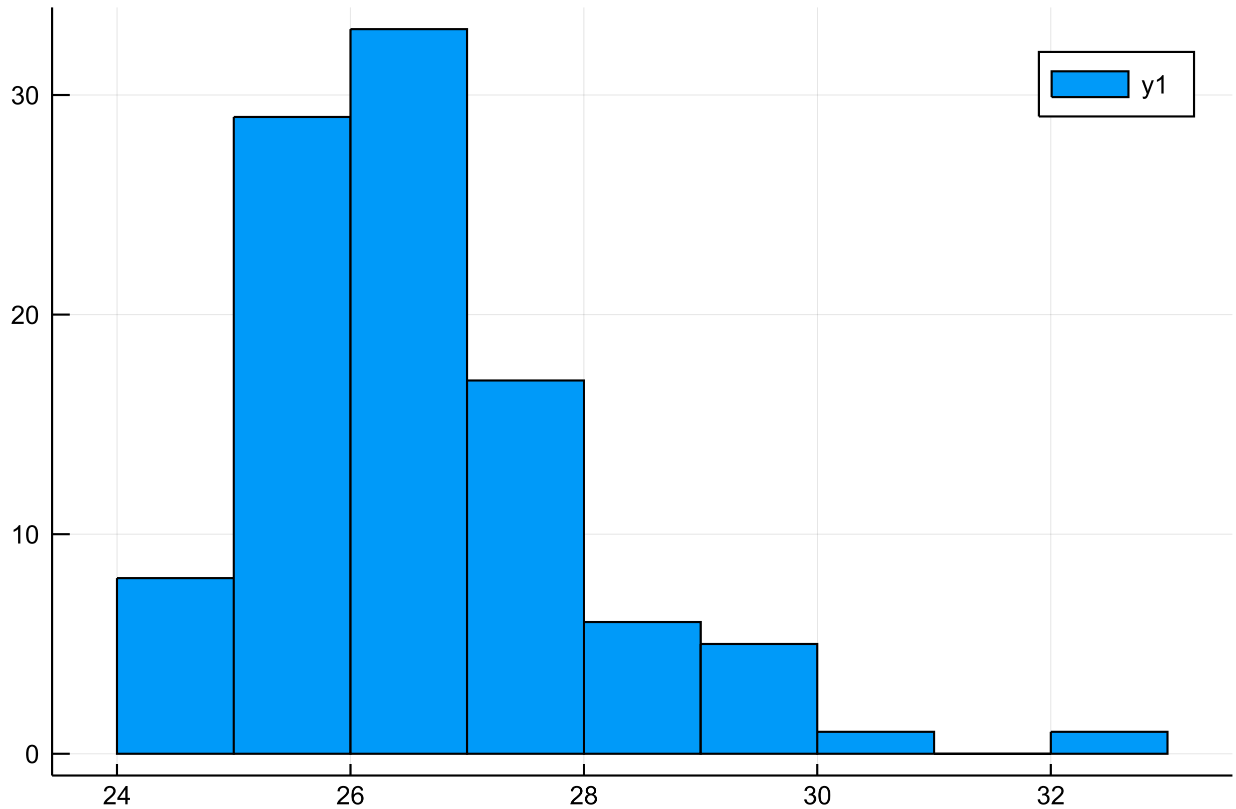
Answer 2(e)

e) Run the algorithm starting at multiple locations and create a histogram of the objective value. What do you observe?

In [81]:
```
runs = 100
final_objective_values = zeros(runs)
max_iter = 1000

for i = 1:runs
    w, obj_vec = SLS_stoc_prox_grad(X,y; max_iter = 1000, alpha = 0.01, 
    final_objective_values[i] = exp(obj_vec[max_iter])
end
#print(final_objective_values)
print(w)
histogram(final_objective_values)
```

```
[-4.996147008481344, 17.013777776352725, 4.999105409913627, 0.0, 0.0,
-7.012511048351937, 5.000048500134628, 0.0, 0.0, 0.0958300913291967]
```

Out[81]:



As can been seen above, stochastic SLS converges to a wider range of values of objective function as compared to general SLS and LASSO.

also range of objective function for stochastic SLS (22-32) is between objective function of general stochastic SLS(1.1) and for LASSO (48-50).

w_predicted: [-5.00, 16.98, 5.03, 0.0, 0.0, -7.00, 4.98, 0.0, 0.0, 0.10]
w_true: [-5, 17, 5, 0, 0, -7, 5, 0, 0, 0.1]
Also predicted w is quite close to w_true. predicted w is 0 for corrext set of positions in w_true as well.

In [ ]:

```
```