PARUL UNIVERSITY
FACULTY OF ENGINEERING AND TECHNOLOGY

DEPARTMENT OF APPLIED SCIENCE AND HUMANITIES

SUBJECT : BASIC MATHEMATICS

Chapter-2 – Determinant and Matrices

2.1 INTRODUCTION

Consider $\begin{aligned} a_1x + b_1y = 0, \\ a_2x + b_2y = 0; \end{aligned}$

Multiplying the first equation by b_2,the second by b_1, subtracting and dividing by x,we obtained two homogeneous linear equations

$$a_1b_2 - a_2b_1 = 0$$

This result is sometimes written as

$$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0$$

and the expression on the left is called the determinant.

A determinant also is an arrangements of numbers in rows and columns but it always has a square form and can be reduced to a single value. Therefore, a determinant is distinct from matrix in the sense that the determinant is always in square shape and it has a numerical value. The arrangements of the numbers of a determinant is enclosed within two vertical parallel lines.

2.2 EXPANSION OF THE DETERMINANT

Determinants can be represented as linear combination of order two with co-efficients from second row and third row or in terms of elements of any columns. The only thing to remember is that 2x2 determinant acccompaying any co-efficient can be obtained by deleting the row and column contating the co-efficient in the original determinant.

Further, the signs accompanying the co-efficient in the original determinant will follow the following checker board pattern:

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Example : Give the determinants with co-efficients from first column in the following co-efficients of the determinant. $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$

Solution : $\Delta = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}$

$$= a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$$

$$= a_1b_2c_3 - a_1b_3c_2 - a_2b_1c_3 + a_2b_3c_1 + a_3b_1c_2 + a_3b_2c_1$$

ORDER OF A DETERMINANT

The determinant of a square matrix of order n is known as determinant of order n.

Remark

For an determinant of order 2 i.e., $\begin{vmatrix} a & b \\ c & d \end{vmatrix}$ consisting 2 rows, 2 columns and having 2x2=4 elements. The Row 1 consisting the elements a and b, Row 2 consisting the elements c and d

Similarly, The Column 1 consisting the elements a and c and Column 2 consisting the elements b and d

2.3 DETERMINANT OF ORDER TWO

Let $a, b, c,$ and d be any four number (real or complex). Then

$$|A| = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

Represent the number $ad - bc$ and is called a determinant of order two.

For example
$$|A| = \begin{vmatrix} 5 & 2 \\ 3 & -7 \end{vmatrix} = (5)(-7) - (3)(2)$$
$$= -35 - 6 = -41$$

Q. Solve the following by using determinant and find the values of x.

1) $\begin{vmatrix} x-2 & 3 \\ 4 & x+2 \end{vmatrix} = 0$

Solution:

$$\begin{vmatrix} x-2 & 3 \\ 4 & x+2 \end{vmatrix} = 0 \Rightarrow (x-2)(x+2)-(4)(3)=0 \Rightarrow (x^2-4)-12=0 \Rightarrow x^2-16=0 \Rightarrow x^2=16 \Rightarrow x=\pm 4$$

2) $\begin{vmatrix} x+1 & 2 \\ 2 & x-2 \end{vmatrix} = 0$

Solution:

$$\begin{vmatrix} x+1 & 2 \\ 2 & x-2 \end{vmatrix} = 0$$

$$\Rightarrow (x+1)(x-2)-4=0 \Rightarrow x^2-x-2-4=0 \Rightarrow x^2-x-6=0 \Rightarrow (x-3)(x+2)=0 \Rightarrow x=3 \vee -2$$

Q. Solve the following determinants

1) $\begin{vmatrix} 2 & 1 \\ 3 & 5 \end{vmatrix}$ 2) $\begin{vmatrix} -2 & 5 \\ -3 & 2 \end{vmatrix}$ 3) $\begin{vmatrix} 3 & 8 \\ -2 & 0 \end{vmatrix}$ 4) $\begin{vmatrix} x & -y \\ y & x \end{vmatrix}$

2.4 <u>DETERMINANT OF ORDER THREE</u>

$$|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

is called a determinant of order 3 because it is consisting 3 rows and 3 columns. Its value can be obtained as follows:

$$|A| = a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$
$$= a_{11}(a_{22}a_{33} - a_{32}a_{23}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{31}a_{22})$$

For example,

$$|A| = \begin{vmatrix} 2 & 3 & 5 \\ -1 & 2 & 3 \\ 4 & -2 & 1 \end{vmatrix} = 2\begin{vmatrix} 2 & 3 \\ -2 & 1 \end{vmatrix} - 3\begin{vmatrix} -1 & 3 \\ 4 & 1 \end{vmatrix} + 4\begin{vmatrix} -1 & 2 \\ 4 & -2 \end{vmatrix}$$
$$= 2(2+6) - 3(-1-12) + 4(2-8)$$
$$= 2(8) - 3(-13) + 4(-6) = 16 + 39 - 24 = 31$$

Q. Solve the following determinants

1) $\begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$ (Ans:- 0) 2) $\begin{vmatrix} -1 & -3 & 2 \\ 3 & -2 & 1 \\ -5 & 4 & 5 \end{vmatrix}$ (Ans:- 78) 3) $\begin{vmatrix} 3 & 4 & -2 \\ 1 & 2 & 0 \\ 2 & 6 & -3 \end{vmatrix}$ (Ans:- -10)

4) $\begin{vmatrix} 4 & -5 & -3 \\ 3 & -3 & 2 \\ 8 & -10 & -6 \end{vmatrix}$ (Ans:- 0)

<u>Remarks</u>

- The value of a determinant is not changed if it expanded along any row or column.
- When no reference of the corresponding matrix is needed, we may denote a determinant by D.
- The determinant of a square zero matrix is zero.

SOLVED EXAMPLES

Example 1: Solve for $x: \begin{vmatrix} x & 3 \\ 5 & 2x \end{vmatrix} = \begin{vmatrix} 5 & -4 \\ 5 & 3 \end{vmatrix}$

Solution: We have

$$\begin{vmatrix} x & 3 \\ 5 & 2x \end{vmatrix} = \begin{vmatrix} 5 & -4 \\ 5 & 3 \end{vmatrix}$$
$$\Rightarrow 2x^2 - 15 = 15 + 20$$
$$\Rightarrow 2x^2 = 50 \Rightarrow x^2 = 25 \Rightarrow x = \pm 5$$

Example 2 : Expand $\begin{vmatrix} x+1 & x-2 \\ x+2 & x-1 \end{vmatrix}$

Solution:

$$\begin{vmatrix} x+1 & x-2 \\ x+2 & x-1 \end{vmatrix} = (x-1)(x+1) - (x-2)(x+2)$$
$$= (x^2 - 1) - (x^2 - 4) = x^2 - 1 - x^2 + 4 = 3$$

Example 3: Expand $\begin{vmatrix} a+b & a-b \\ -a+b & -a-b \end{vmatrix}$

Solution:

$$\begin{vmatrix} a+b & a-b \\ -a+b & -a-b \end{vmatrix} = (a+b)(-a-b) - (a-b)(-a+b)$$
$$= (-a^2 - ab - ab - b^2) - (-a^2 + ab + ab - b^2)$$
$$= -a^2 - ab - ab - b^2 + a^2 - ab - ab + b^2 = -4ab$$

2.5 <u>PROPERTIES OF DETERMINANT</u>

Property 1: The value of a determinant does not change when rows and columns are interchanged.

For example: Let us take

$$A = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix}$$

Then by considering Row 1

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2\begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3\begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix}$$
$$= 1(45 - 48) - 2(36 - 42) + 3(32 - 35)$$
$$= 1(-3) - 2(-6) + 3(-3) = -3 + 12 - 9 = 0$$

Now the same procedure we will do by considering Column 1

$$|A| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1\begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 4\begin{vmatrix} 2 & 3 \\ 8 & 9 \end{vmatrix} + 7\begin{vmatrix} 2 & 3 \\ 5 & 6 \end{vmatrix}$$

$$= 1(45-48) - 4(18-24) + 7(12-15)$$

$$= 1(-3) - 4(-6) + 7(-3) = -3 + 24 - 21 = 0$$

Hence, we can say that determinant value doesn't change when Rows and Columns are interchanged.

Property 2: If any two rows (or columns) of a determinant are interchanged, the sign of the determinant is changed.

For example: Let us take $A = \begin{vmatrix} 1 & 2 & 0 \\ 4 & 2 & -1 \\ 3 & 1 & 1 \end{vmatrix}$ then

$$|A| = \begin{vmatrix} 1 & 2 & 0 \\ 4 & 2 & -1 \\ 3 & 1 & 1 \end{vmatrix} = 1(2+1) - 2(4+3) + 0(4-6)$$

$$= 1(3) - 2(7) + 0(-2) = 3 - 14 = -11$$

Now by interchanging Row1 and Row2 we have another matrix say $B = \begin{vmatrix} 4 & 2 & -1 \\ 1 & 2 & 0 \\ 3 & 1 & 1 \end{vmatrix}$

Then,

$$|B| = \begin{vmatrix} 4 & 2 & -1 \\ 1 & 2 & 0 \\ 3 & 1 & 1 \end{vmatrix} = 4(2-0) - 2(1-0) - 1(1-6)$$

$$= 4(2) - 2(1) - 1(-5) = 8 - 2 + 5 = 11$$

Hence, we can see that after changing Row 1 and Row 2 the sign of a determinant has changed.

Property 3: If two rows and two columns of the determinant are identical(same), then the value of a determinant is zero.

For example: Let us take a matrix having Row 1 and Row 2 is identical

$A = \begin{vmatrix} 1 & -1 & 3 \\ 1 & -1 & 3 \\ 2 & 4 & 6 \end{vmatrix}$ then $|A| = \begin{vmatrix} 1 & -1 & 3 \\ 1 & -1 & 3 \\ 2 & 4 & 6 \end{vmatrix} = 1(-6-12)+1(6-6)+3(4+2)$

$$= 1(-18)+1(0)+3(6) = -18+18 = 0$$

Hence, we can see that in above example by taking two rows identical the value of a determinant is zero.

Property 4: If all the elements of any Row or any column of a determinant are multiplied by same number then the value of a determinant is also multiplied by same number.

For example: Let us take $A = \begin{vmatrix} 1 & 2 & 0 \\ 4 & 2 & -1 \\ 3 & 1 & 1 \end{vmatrix}$ then from Property 2 we know that $|A| = -11$

Now let us select any number, say 2 , which is multiplie with all the elements of any arow, say Row 1, then the resultant determinant, say $B = \begin{vmatrix} 2 & 4 & 0 \\ 4 & 2 & -1 \\ 3 & 1 & 1 \end{vmatrix}$

Now, $|B| = \begin{vmatrix} 2 & 4 & 0 \\ 4 & 2 & -1 \\ 3 & 1 & 1 \end{vmatrix} = 2(2+1)-4(4+3)+0(4-6)$

$$= 2(3)-4(7)+0(-2) = 6-28-0 = -22 = 2(-11) = 2|A|$$

Hence, we can see that after multiplying 2 with the first Row of a matrix A the value of a determinant is also multiplied by 2.

Property 5: If all the elements of any Row or any column of a determinant are zero then the value of a determinant is also zero.

For example: Let us take $A = \begin{vmatrix} 0 & 0 & 0 \\ 4 & 2 & -1 \\ 3 & 1 & 1 \end{vmatrix}$ then

$$|A| = \begin{vmatrix} 0 & 0 & 0 \\ 4 & 2 & -1 \\ 3 & 1 & 1 \end{vmatrix} = 0(2+1)-0(4+3)+0(4-6) = 0$$

Hence, we can see that in the above examples all the elements of Row 1 are zero hence the value of a determinant is also zero.

SOLVED EXAMPLES

Example 1: Show that $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+y \end{vmatrix} = xy$

Solution: Before expanding the above determinant let us apply column or row operations to reduce the computational process

We have L.H.S.= $\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1+x & 1 \\ 1 & 1 & 1+y \end{vmatrix}$

On applying $C_2 \to C_2 - C_1 \text{ and } C_3 \to C_3 - C_1$ in above determinant, we get

$$\begin{vmatrix} 1 & 0 & 0 \\ 1 & x & 0 \\ 1 & 0 & y \end{vmatrix} = 1(xy-0) - 0(y-0) + 0(0-x) = xy = \text{R.H.S}$$

Example 2: Prove that $\begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix} = abc(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c})$

Solution: We have $\text{L.H.S} = \begin{vmatrix} 1+a & 1 & 1 \\ 1 & 1+b & 1 \\ 1 & 1 & 1+c \end{vmatrix}$

On applying $C_1 \to C_1 - C_2 \text{ and } C_2 \to C_2 - C_3$ in above determinant, we get

$$\begin{vmatrix} a & 0 & 1 \\ -b & b & 1 \\ 0 & -c & 1+c \end{vmatrix} = a[b(1+c)+c]-0+1[bc-0]$$

$$= a[(b+bc+c)]+bc = ab+abc+ac+bc = abc+bc+ac+ab$$

$$= abc(1+\frac{1}{a}+\frac{1}{b}+\frac{1}{c}) = \text{R.H.S}$$

Q. Prove the following by using determinant:

1) Prove that $\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix} = (x-y)(y-z)(z-x)$

2) Prove that $\begin{vmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ac & bc & -c^2 \end{vmatrix} = 4a^2b^2c^2$

2.6 MINORS AND COFACTORS OF AN ELEMENT

- **Minor** of an element is defined as the determinant obtained by deleting the row and column in which that element lies. For example, in the determinant $|A| = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$,

The minor of a_{11} can be obtained by deleting the 1st Row and 1St Columnn and it is denoted by M_{11}. Therefore, $M_{11} = \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$ and so on.

- **Cofactor** of an element a_{ij} is related to its minor as $C_{ij} = (-1)^{i+j} M_{ij}$, where $'i'$ denotes the i^{th} row and $'j'$ denotes the j^{th} column to which the element a_{ij} belongs.

SOLVED EXAMPLES

Example 1: Find the minor and cofactor of elements of the determinant $\begin{vmatrix} 5 & -2 \\ 3 & 7 \end{vmatrix}$

Solution:

Minorof the element a_{11} is $M_{11} = |7| = 7$

Minor of the element a_{12} is $M_{12} = |3| = 3$

Minor of the element a_{21} is $M_{21} = |-2| = -2$

Minor of the element a_{22} is $M_{22} = |5| = 5$

Therefore, $M_{ij} = \begin{vmatrix} 7 & 3 \\ -2 & 5 \end{vmatrix}$

Hence, Cofactor's are given by

$$C_{11} = (-1)^{1-1} M_{11} = +1(7) = 7$$

$$C_{12} = (-1)^{1-2} M_{12} = -1(3) = -3$$

$$C_{21} = (-1)^{2-1} M_{21} = -1(-2) = 2$$

$$C_{22} = (-1)^{2-2} M_{22} = +1(5) = 5$$

Therefore, $C_{ij} = \begin{vmatrix} 7 & -3 \\ 2 & 5 \end{vmatrix}$

Example 2: Find the minor and cofactor of elements of the determinant $\begin{vmatrix} 4 & 3 & 1 \\ 1 & 3 & 2 \\ 2 & 1 & 5 \end{vmatrix}$

Solution: Minor's of an each element is given below,

$M_{11} = \begin{vmatrix} 3 & 2 \\ 1 & 5 \end{vmatrix} = 15 - 2 = 13$ $\quad M_{12} = \begin{vmatrix} 1 & 2 \\ 2 & 5 \end{vmatrix} = 5 - 4 = 1$ $\quad M_{13} = \begin{vmatrix} 1 & 3 \\ 2 & 1 \end{vmatrix} = 1 - 6 = -5$

$M_{21} = \begin{vmatrix} 3 & 1 \\ 1 & 5 \end{vmatrix} = 15 - 1 = 14$ $\quad M_{22} = \begin{vmatrix} 4 & 1 \\ 2 & 5 \end{vmatrix} = 20 - 2 = 18$ $\quad M_{23} = \begin{vmatrix} 4 & 3 \\ 2 & 1 \end{vmatrix} = 4 - 6 = -2$

$M_{31} = \begin{vmatrix} 3 & 1 \\ 3 & 2 \end{vmatrix} = 6 - 3 = 3$ $\quad M_{32} = \begin{vmatrix} 4 & 1 \\ 1 & 2 \end{vmatrix} = 8 - 1 = 7$ $\quad M_{33} = \begin{vmatrix} 4 & 3 \\ 1 & 3 \end{vmatrix} = 12 - 3 = 9$

Therefore, $M_{ij} = \begin{vmatrix} 13 & 1 & -5 \\ 14 & 18 & -2 \\ 3 & 7 & 9 \end{vmatrix}$

Now, the cofactors are

$C_{11} = (-1)^{1+1} M_{11} = +1(13) = 13$ $\qquad C_{12} = (-1)^{1+2} M_{12} = -1(1) = -1$

$C_{13} = (-1)^{1+3} M_{13} = +1(-5) = -5$ $\qquad C_{21} = (-1)^{2+1} M_{21} = -1(14) = -14$

$C_{22} = (-1)^{2+2} M_{22} = +1(18) = 18$ $\qquad C_{23} = (-1)^{2+3} M_{23} = -1(-2) = 2$

$C_{31} = (-1)^{3+1} M_{31} = +1(3) = 3$ $\qquad C_{32} = (-1)^{3+2} M_{32} = -1(7) = -7$

$C_{33} = (-1)^{3+3} M_{33} = +1(9) = 9$ $\qquad$ Therefore, $C_{ij} = \begin{vmatrix} 13 & -1 & -5 \\ -14 & 18 & 2 \\ 3 & -7 & 9 \end{vmatrix}$

Q. Find the Minor's and cofactor's of the following

1) $\begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix}$ $\quad$ 2) $\begin{vmatrix} -2 & 3 \\ -5 & 6 \end{vmatrix}$ $\quad$ 3) $\begin{vmatrix} 5 & 3 & -1 \\ 4 & -3 & 0 \\ 6 & 1 & 2 \end{vmatrix}$ $\quad$ 4) $\begin{vmatrix} 5 & 6 & 7 \\ -8 & 2 & -1 \\ 3 & -4 & 10 \end{vmatrix}$

2.7 <u>CRAMMER'S RULE</u>

Consider the system of linear equations

$$a_1x + b_1y + c_1z = d_1$$
$$a_2x + b_2y + c_2z = d_2$$
$$a_3x + b_3y + c_3z = d_3 \qquad (1)$$

Then ,We define

Δ = determinant coefficient

$$= \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Now we define Δ_x which is obtained by supperessing the coloum of coefficients of x and replacing it by the coloumn of constant terms d_1, d_2, d_3 on right and side.

$$\Delta_x = \begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}$$

Similarly, we obtained

$$\Delta_y = \begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix} \text{ and } \Delta_z = \begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}$$

Case 1: If $\Delta \neq 0$ solution of system (1) is given by

$$x = \frac{\Delta_x}{\Delta}, y = \frac{\Delta_y}{\Delta} \text{ and } z = \frac{\Delta_z}{\Delta}$$

And system is called consistent.

Case 2: If $\Delta = 0$, also $\Delta_x = \Delta_y = \Delta_z \neq 0$, then system does not possess any common solution and system is called inconsistent.

SOLVED EXAMPLES

Example 1: Using the crammer's rule, solve the following system of equations

$$x + y - 4 = 0,\ 2x - 3y - 8 = 0$$

Solution: The given equation is

$$x + y - 4 = 0,$$
$$2x - 3y - 8 = 0$$

Here, $\Delta = \begin{vmatrix} 1 & 1 \\ 2 & -3 \end{vmatrix} = -3 - 2 = -5 \neq 0$, $\Delta_x = \begin{vmatrix} 4 & 1 \\ 3 & -3 \end{vmatrix} = -12 - 3 = -15$, $\Delta_y = \begin{vmatrix} 1 & 4 \\ 2 & 3 \end{vmatrix} = 3 - 8 = -5$

$\therefore$ By Crammer's rule,

$$x = \frac{\Delta_x}{\Delta} = \frac{-15}{-5} = 3, \quad y = \frac{\Delta_y}{\Delta} = \frac{-5}{-5} = 1$$

Example 2: Solve the following by Crammer's rule

$$x + y + z = 6$$
$$x - y + z = 2$$
$$3x + 2y - 4z = -5$$

Solution: We have

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 3 & 2 & -4 \end{vmatrix} = 1(4 - 2) - 1(-4 - 3) + 1(2 + 3)$$

$$= 1(2) - 1(-7) + 1(5) = 2 + 7 + 5 = 14 \neq 0$$

$$\Delta_x = \begin{vmatrix} 6 & 1 & 1 \\ 2 & -1 & 1 \\ -5 & 2 & -4 \end{vmatrix} = 6(4 - 2) - 1(-8 + 5) + 1(4 - 5) = 6(2) - 1(-3) + 1(-1) = 12 + 3 - 1 = 14$$

$$\Delta_y = \begin{vmatrix} 1 & 6 & 1 \\ 1 & 2 & 1 \\ 3 & -5 & -4 \end{vmatrix} = 1(-8 + 5) - 6(-4 - 3) + 1(-5 - 6) = 1(-3) - 6(-7) + 1(-11) = -3 + 42 - 11 = 28$$

$$\Delta_z = \begin{vmatrix} 1 & 1 & 6 \\ 1 & -1 & 2 \\ 3 & 2 & -5 \end{vmatrix} = 1(5 - 4) - 1(-5 - 6) + 6(2 + 3) = 1(1) - 1(-11) + 6(5) = 1 + 11 + 30 = 42$$

$$x = \frac{\Delta_x}{\Delta} = \frac{14}{14} = 1, \; y = \frac{\Delta_y}{\Delta} = \frac{28}{14} = 2, \; z = \frac{\Delta_z}{\Delta} = \frac{42}{14} = 3$$

Hence, the solution is given by $x = 1, y = 2, z = 3.$

Example 3: Solve using Crammer's rule

$$x+y=5,\ y+z=3,\ x+z=4$$

Solution: Given system can be rewrite as

$$x+y+0z=5$$
$$0x+y+z=3$$
$$x+0y+z=4$$

$$\Delta=\begin{vmatrix}1 & 1 & 0\\ 0 & 1 & 1\\ 1 & 0 & 1\end{vmatrix}=1(1-0)-1(0-1)+0(0-1)=1+1+0=2\neq 0$$

$$\Delta_x=\begin{vmatrix}5 & 1 & 0\\ 3 & 1 & 1\\ 4 & 0 & 1\end{vmatrix}=5(1-0)-1(3-4)+0(0-4)=5+1+0=6$$

$$\Delta_y=\begin{vmatrix}1 & 5 & 0\\ 0 & 3 & 1\\ 1 & 4 & 1\end{vmatrix}=1(3-4)-5(0-1)+0(0-3)=-1+5+0=4$$

$$\Delta_z=\begin{vmatrix}1 & 1 & 5\\ 0 & 1 & 3\\ 1 & 0 & 4\end{vmatrix}=1(4-0)-1(0-3)+5(0-1)=4+3-5=2$$

$$x=\frac{\Delta_x}{\Delta}=\frac{6}{2}=3,\quad y=\frac{\Delta_y}{\Delta}=\frac{4}{2}=2,\quad z=\frac{\Delta_z}{\Delta}=\frac{2}{2}=1$$

Q. Solve the following by using Crammer's rule.

1) $x+2y=-1,\ 2x+y=1$ 2) $3x-2y=1,\quad x+2y=3$

3) $x-2y+3z=2,\ 2x-3z=3,\ x+y+z=6$

4) $x+2y+3z=6,\ \ 2x+4y+z=17,\ \ 3x+2y+9z=2$

2.8 INTRODUCTION OF MATRICES

'Matrices' is a powerful tool of modern mathematics. The study of 'Matrices' is essential in almost every important branch of science like mathematics and physics.
The word 'matrix' was used by J.J. Sylvester in 1850 and developed by 'Arthur Cayley' in 1858.

- **Definition**
  A set having mn numbers either real or complex, arranged in the form of rectangular array in which there are m rows and n columns. This rectangular arrangement is called a matrix of order $m \times n$ which is denoted by $\left[a_{ij}\right]_{m\times n}$, where $i = 1, 2, 3, \ldots m$ and $j = 1, 2, 3, \ldots n$ and a matrix of order $m \times n$ is usually written as

$$\left[a_{ij}\right]_{m\times n} = \left|\begin{matrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{matrix}\right|_{m\times n}$$

Q. Write down the difference between determinant and matrices.

Determinant	**Matrices**
1) Determinant has a specific value	1) Matrix has no specific value. It is just an arrangement of numbers.
2) In a determinant, number of rows and columns are equal.	2)In a matrix, number of rows and columns may or may not be equal
3) Elements of a determinant are enclosed in $\lvert\ \rvert$	3) Elements of a matrix are enclosed $[\ ]$ or $(\ )$
4) If a determinant is multiplied by any scalar k, then that scalar will be multiplied to only one specific row.	4) If a matrix is multiplied by any scalar k, then that scalar will be multiplied to each element of a matrix.

2.9 <u>TYPES OF MATRICES</u>

- <u>**Row matrix:-**</u> A matrix of order $1 \times n$ is called a row matrix(i.e., a matrix having only 1 row and n number of columns).
  Example:- (1) $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}_{1\times 3}$ (2) $\begin{bmatrix} 5 & 6 & 7 & 8 \end{bmatrix}_{1\times 4}$

- <u>**Column matrix:-**</u> A matrix of order $m \times 1$ is called a column matrix(i.e., a matrix having only 1 column and m number of rows).

Example:- (1) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{3\times1}$ (2) $\begin{bmatrix} 5 \\ 6 \\ 7 \\ 8 \end{bmatrix}_{4\times1}$

- **Null or Zero matrix:-** If all the elements of the given matrix are zero then it is known as Null or Zero matrix.

  Example:- (1) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}_{2\times2}$ (2) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}_{2\times3}$

- **Square matrix:-** A matrix having an equal number of rows and columns is known as square matrix.

  Example:- (1) $\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times2}$ (2) $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}_{3\times3}$

Remark:- In the square matrix $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$,the elements (1 5 9) are known as diagonal elements and (3 5 7) are known as subsidiary diagonal elements.

- **Diagonal matrix:-** A **squrae** matrix is said to be an diagonal matrix if main diagonal elements are non-zero and rest of all the elements are zero is known as a diagonal matrix.

  Example:- (1) $\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}_{2\times2}$ (2) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}_{3\times3}$

- **Scalar matrix:-** A **diagonal** matrix, in which all the elements on the main diagonal are same.

  Example:- (1) $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}_{2\times2}$ (2) $\begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 3 \end{bmatrix}_{3\times3}$

- **Identity matrix:-** An identity matrix is an special case of scalar matrix In which all the elements on the main diagonal are same and equals to 1.

Example:- (1) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}_{2\times2}$ (2) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times3}$

- **Upper-triangular matrix:-**A **square** matrix in which all the elements <u>below</u> to the diagonal elements are zero then it is known as Upper-triangular matrix.

Example:- (1) $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}_{3\times3}$ (2) $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 5 & -1 & 2 \\ 0 & 0 & 6 & -3 \\ 0 & 0 & 0 & 9 \end{bmatrix}_{4\times4}$

- **Lower-triangular matrix:-**A **square** matrix in which all the elements <u>above</u> to the diagonal elements are zero then it is known as Lower -triangular matrix.

Example:- (1) $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 4 & 0 \\ 3 & 5 & 6 \end{bmatrix}_{3\times3}$ (2) $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 5 & 0 & 0 \\ 3 & -1 & -2 & 0 \\ 4 & 2 & 3 & 9 \end{bmatrix}_{4\times4}$

- **Transpose of an matrix:-**By converting rows into columns or columns into rows of any matrix A is known as a transpose of an matrix A and it is denoted by A^T or A'.

Example:- (1) $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2\times3} \Rightarrow A^T = \begin{bmatrix} 1 & 4 \\ 2 & 5 \\ 3 & 6 \end{bmatrix}_{3\times2}$

(2) $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}_{3\times3} \Rightarrow A^T = \begin{bmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{bmatrix}_{3\times3}$

- **Symmetric matrix:-** For any **square** matrix A, if $A = A^T$ then it is known as symmetric matrix.

Example:- (1) $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 6 & 5 \end{bmatrix} \Rightarrow A^T = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 5 & 6 \\ 3 & 6 & 5 \end{bmatrix}$ Here, we can see that $A = A^T$ so A is symmetric matrix.

- **skew-symmetric matrix:-** For any **square** matrix A, if $A = -A^T$ then it is known as Skew symmetric matrix.

Example:- (1) $A=\begin{bmatrix}0 & 3 & 5\\ -3 & 0 & -2\\ -5 & 2 & 0\end{bmatrix} \Rightarrow A^T=\begin{bmatrix}0 & -3 & -5\\ 3 & 0 & 2\\ 5 & -2 & 0\end{bmatrix}=-\begin{bmatrix}0 & 3 & 5\\ -3 & 0 & -2\\ -5 & 2 & 0\end{bmatrix}=-A$

Here, we can see that $A=-A^T$ so A is skew-symmetric matrix.

- **<u>Singular and non-singular matrix:-</u>**For any **square** matrix A, if $|A|\neq 0,$ then it is known as non-singular matrix and if $|A|=0,$ then it is known as singular matrix.

Example:- (1) If $A=\begin{bmatrix}2 & 1\\ 8 & 4\end{bmatrix} \Rightarrow |A|=8-8=0 \Rightarrow \text{Singular matrix.}$

(2) If $A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix} \Rightarrow |A|=4-6=-2\neq 0 \Rightarrow \text{non-singular matrix}$

2.10 <u>ALGEBRA OF MATRICES</u>

(i) **<u>Addition of Matrices:-</u>** Let $A \text{ and } B$ be two matrices of $m\times n$ order. Then the sum of $A \text{ and } B$ i.e., $(A+B)$ is defined to be the matrix of the same order $m\times n$ obtained by adding the corresponding elements of $A \text{ and } B$.

Let $A=\left[a_{ij}\right]_{m\times n} \text{ and } B=\left[b_{ij}\right]_{m\times n} \text{ then } A+B=\left[a_{ij}+b_{ij}\right]_{m\times n}$

(ii) **<u>Substraction of two matrices:-</u>** Let $A \text{ and } B$ be two matrices of $m\times n$ order. Then the sum of $A \text{ and } B$ i.e., $(A-B)$ is defined to be the matrix of the same order $m\times n$ obtained by substracting the corresponding elements of $A \text{ and } B$.

<u>Remark</u>

- For addition and substraction of any two matrices, the order of both the matrices should be same.

SOLVED EXAMPLES

Example 1: If $A=\begin{bmatrix}1 & 2 & 3\\ -1 & 4 & 5\end{bmatrix}_{2\times 3} \text{ and } B=\begin{bmatrix}5 & 2 & 6\\ 4 & -2 & 3\end{bmatrix}_{2\times 3}$ then find $A+B$ and $A-B$

Solution: We have $A=\begin{bmatrix}1 & 2 & 3\\ -1 & 4 & 5\end{bmatrix}_{2\times 3} \text{ and } B=\begin{bmatrix}5 & 2 & 6\\ 4 & -2 & 3\end{bmatrix}_{2\times 3}$ then

$$A+B=\begin{bmatrix}6 & 4 & 9\\ 3 & 2 & 8\end{bmatrix}_{2\times 3} \text{ and } A-B=\begin{bmatrix}-4 & 0 & -3\\ -5 & 2 & 2\end{bmatrix}_{2\times 3}$$

Example 2: If $A=\begin{bmatrix}1 & 4\\ 3 & 2\\ 2 & 5\end{bmatrix}_{3\times 2}$ $and\ B=\begin{bmatrix}-1 & -2\\ 0 & 5\\ 3 & 1\end{bmatrix}_{3\times 2}$ then find (i) $A+B$ (ii) $A-B$ (iii) $3A-2B$ (iv) $2A+3B$

Solution: We have $A=\begin{bmatrix}1 & 4\\ 3 & 2\\ 2 & 5\end{bmatrix}_{3\times 2}$ $and\ B=\begin{bmatrix}-1 & -2\\ 0 & 5\\ 3 & 1\end{bmatrix}_{3\times 2}$ then

(i) $$A+B=\begin{bmatrix}1 & 4\\ 3 & 2\\ 2 & 5\end{bmatrix}_{3\times 2}+\begin{bmatrix}-1 & -2\\ 0 & 5\\ 3 & 1\end{bmatrix}_{3\times 2}=\begin{bmatrix}0 & 2\\ 3 & 7\\ 5 & 6\end{bmatrix}_{3\times 2}$$

(ii) $$A-B=\begin{bmatrix}1 & 4\\ 3 & 2\\ 2 & 5\end{bmatrix}_{3\times 2}-\begin{bmatrix}-1 & -2\\ 0 & 5\\ 3 & 1\end{bmatrix}_{3\times 2}=\begin{bmatrix}2 & 6\\ 3 & -3\\ -1 & 4\end{bmatrix}_{3\times 2}$$

(iii) $$3A-2B=3\begin{bmatrix}1 & 4\\ 3 & 2\\ 2 & 5\end{bmatrix}_{3\times 2}-2\begin{bmatrix}-1 & -2\\ 0 & 5\\ 3 & 1\end{bmatrix}_{3\times 2}=\begin{bmatrix}3 & 12\\ 9 & 6\\ 6 & 15\end{bmatrix}_{3\times 2}-\begin{bmatrix}-2 & -4\\ 0 & 10\\ 6 & 2\end{bmatrix}_{3\times 2}=\begin{bmatrix}5 & 16\\ 9 & -4\\ 0 & 13\end{bmatrix}_{3\times 2}$$

(iv) $$2A+3B=2\begin{bmatrix}1 & 4\\ 3 & 2\\ 2 & 5\end{bmatrix}_{3\times 2}+3\begin{bmatrix}-1 & -2\\ 0 & 5\\ 3 & 1\end{bmatrix}_{3\times 2}=\begin{bmatrix}2 & 8\\ 6 & 4\\ 4 & 10\end{bmatrix}_{3\times 2}+\begin{bmatrix}-3 & -6\\ 0 & 15\\ 9 & 3\end{bmatrix}_{3\times 2}=\begin{bmatrix}-1 & 2\\ 6 & 19\\ 13 & 13\end{bmatrix}_{3\times 2}$$

Example 3: If $A=\begin{bmatrix}1 & 2 & 1\\ 3 & 4 & 2\end{bmatrix}_{2\times 3}$ $\text{and } B=\begin{bmatrix}3 & -2 & 4\\ 1 & 5 & 0\end{bmatrix}_{2\times 3}$ then find matrix $X \text{ from } X+A+B=0$

Solution: Here

$$X+A+B=0 \Rightarrow X+\begin{bmatrix}1 & 2 & 1\\ 3 & 4 & 2\end{bmatrix}_{2\times 3}+\begin{bmatrix}3 & -2 & 4\\ 1 & 5 & 0\end{bmatrix}_{2\times 3}=0$$

$$\Rightarrow X+\begin{bmatrix}4 & 0 & 5\\ 4 & 9 & 2\end{bmatrix}_{2\times 3}=0$$

$$\Rightarrow X=-\begin{bmatrix}4 & 0 & 5\\ 4 & 9 & 2\end{bmatrix}_{2\times 3}=\begin{bmatrix}-4 & 0 & -5\\ -4 & -9 & -2\end{bmatrix}_{2\times 3}$$

$$\Rightarrow X=\begin{bmatrix}-4 & 0 & -5\\ -4 & -9 & -2\end{bmatrix}_{2\times 3}$$

Q. Solving the following questions by using matrix addition and subtraction

1) If $A=\begin{bmatrix}1 & 4\\ 7 & 9\end{bmatrix}$ and $B=\begin{bmatrix}5 & 6\\ 2 & 3\end{bmatrix}$ then find $(i)\ A+B\ (ii)\ A-B\ (iii)\ 2A+4B\ (iv)\ 3A-B$

2) If $A=\begin{bmatrix}2 & -1 & 0\\ 3 & 2 & -4\\ 5 & 1 & 9\end{bmatrix}, B=\begin{bmatrix}17 & -1 & 3\\ -24 & -1 & -16\\ -7 & 1 & 1\end{bmatrix}$ and $4A+3C=B$ then find C

3) If $A=\begin{bmatrix}1 & 2 & -3\\ 5 & 0 & 2\\ 1 & -1 & 1\end{bmatrix}, B=\begin{bmatrix}3 & 1 & -2\\ 0 & 1 & 4\\ -2 & 0 & -1\end{bmatrix}$ and $A+2C=B$ then find C

2.11 PROPERTIES OF MATRIX ADDITION

Property 1:- Addition of matrices is commutative.

(i.e., If A and B be two matrices of $m\times n$ order then $A+B=B+A$)

Property 2:- Addition of matrices is associative.

(i.e., If A, B and C be three matrices of $m\times n$ order then $(A+B)+C=A+(B+C)$)

SOLVED EXAMPLES

Example 1: If $A=\begin{bmatrix}3 & 7\\ 9 & 8\end{bmatrix} \text{and } B=\begin{bmatrix}-1 & 2\\ 0 & 4\end{bmatrix}$ then prove the commutative property of addition.

Solution: Here,

$$A+B=\begin{bmatrix}3 & 7\\ 9 & 8\end{bmatrix}+\begin{bmatrix}-1 & 2\\ 0 & 4\end{bmatrix}=\begin{bmatrix}2 & 9\\ 9 & 12\end{bmatrix},\quad B+A=\begin{bmatrix}-1 & 2\\ 0 & 4\end{bmatrix}+\begin{bmatrix}3 & 7\\ 9 & 8\end{bmatrix}=\begin{bmatrix}2 & 9\\ 9 & 12\end{bmatrix}$$

It is clear that $A+B=B+A$. Hence commutative property satisfied.

Example 2: Check the associative property of addition for given matrices

$$A=\begin{bmatrix}1 & 5 & 9\\ 8 & 6 & 4\\ 3 & 2 & 0\end{bmatrix}, B=\begin{bmatrix}-1 & 0 & -2\\ 4 & 8 & 3\\ -5 & -2 & 1\end{bmatrix} \, and \, C=\begin{bmatrix}2 & 5 & 4\\ -3 & 6 & -8\\ -1 & 2 & 4\end{bmatrix}$$

Solution:

$$L.H.S.=(A+B)+C=\left(\begin{bmatrix}1 & 5 & 9\\ 8 & 6 & 4\\ 3 & 2 & 0\end{bmatrix}+\begin{bmatrix}-1 & 0 & -2\\ 4 & 8 & 3\\ -5 & -2 & 1\end{bmatrix}\right)+\begin{bmatrix}2 & 5 & 4\\ -3 & 6 & -8\\ -1 & 2 & 4\end{bmatrix}$$

$$=\begin{bmatrix}0 & 5 & 7\\ 12 & 14 & 7\\ -2 & 0 & 1\end{bmatrix}+\begin{bmatrix}2 & 5 & 4\\ -3 & 6 & -8\\ -1 & 2 & 4\end{bmatrix}$$

$$=\begin{bmatrix}2 & 10 & 11\\ 9 & 20 & -1\\ -3 & 2 & 5\end{bmatrix}$$

$$R.H.S. = A+(B+C) = \begin{bmatrix} 1 & 5 & 9 \\ 8 & 6 & 4 \\ 3 & 2 & 0 \end{bmatrix} + \left(\begin{bmatrix} -1 & 0 & -2 \\ 4 & 8 & 3 \\ -5 & -2 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 5 & 4 \\ -3 & 6 & -8 \\ -1 & 2 & 4 \end{bmatrix} \right)$$

$$= \begin{bmatrix} 1 & 5 & 9 \\ 8 & 6 & 4 \\ 3 & 2 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 5 & 2 \\ 1 & 14 & -5 \\ -6 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 10 & 11 \\ 9 & 20 & -1 \\ -3 & 2 & 5 \end{bmatrix}$$

Hence, $(A+B)+C = A+(B+C)$ so, associative property satisfied.

Q. Solve the following by using properties

(1) Verify the associative property for the given matrices

$$A = \begin{bmatrix} 1 & 4 \\ 6 & 8 \end{bmatrix}, B = \begin{bmatrix} 2 & 3 \\ 5 & 7 \end{bmatrix} \text{and } C = \begin{bmatrix} -1 & 2 \\ 3 & 8 \end{bmatrix}$$

(2) Check the commutative property for the given matrices

$$A = \begin{bmatrix} 2 & 6 & 7 \\ -1 & 5 & 3 \\ 1 & 0 & 5 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & -2 \\ -4 & 8 & -3 \\ 5 & -2 & 1 \end{bmatrix}$$

2.12 MULTIPLICATION OF TWO MATRICES

Let $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{n\times p}$ be two matrices such that the number of columns in A is equal to the number of rows in B then the product of A and B denoted by AB is defined as a matrix $C = [c_{ik}]_{m\times p}$ where $c_{ik} = \sum a_{ij} b_{jk}$ or

The product AB is defined as the matrix whose element in the i^{th} row and k^{th} column is $a_{i1}b_{1k} + a_{i2}b_{2k} + a_{i3}b_{3k} + \ldots + a_{in}b_{nk}$ thus we conduct that :

If A is an $m\times n$ matrix and B is an $n\times k$ matrix then the product matrix AB, is an

$m \times k$ matrix.

Note:-The product AB can be calculated only if the number of columns in A is equal to the number of rows in B.

- **Remark:**

(1) If $AB = BA$, then the matrices A and B are called commutative and if $AB = -BA$ then the matrices A and B are called anticommutative.
(2) The product of two non-zero matrices may be a zero matrix.

SOLVED EXAMPLES

Example 1: If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2\times 3}$ and $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}_{2\times 2}$ Can we find product AB and BA?

If yes then find AB and BA.

Solution: -For product AB, $\text{the number of columns of } A = 3 \neq 2 = \text{the number of rows of } B$

So product AB can't define.

Now, for product BA, $\text{the number of columns of } B = 2 = \text{the number of rows of } A$

So we can find product BA

$$BA = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 1.1+2.4 & 1.2+2.5 & 1.3+2.6 \\ 3.1+4.4 & 3.2+4.5 & 3.3+4.6 \end{bmatrix} = \begin{bmatrix} 9 & 12 & 15 \\ 19 & 26 & 33 \end{bmatrix}_{2\times 3}$$

Example 2: If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2\times 3}$, $B = \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 2 \end{bmatrix}_{3\times 2}$ then find AB and BA if possible.

Solution: -For product AB, $\text{the number of columns of } A = 3 = \text{the number of rows of } B$

$$AB = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}_{2\times 3} \begin{bmatrix} 1 & 2 \\ 2 & 1 \\ 1 & 2 \end{bmatrix}_{3\times 2} = \begin{bmatrix} 1.1+2.2+3.1 & 1.2+2.1+3.2 \\ 1.4+5.2+6.1 & 4.2+5.1+6.2 \end{bmatrix} = \begin{bmatrix} 8 & 10 \\ 20 & 25 \end{bmatrix}_{2\times 2}$$

Now, for product BA, $\text{the number of columns of } B = 2 = \text{the number of rows of } A$

$$BA=\begin{bmatrix}1&2\\2&1\\1&2\end{bmatrix}_{3\times2}\begin{bmatrix}1&2&3\\4&5&6\end{bmatrix}_{2\times3}=\begin{bmatrix}1.1+2.4&1.2+2.5&1.3+2.6\\2.1+1.4&2.2+1.5&2.3+1.6\\1.1+2.4&1.2+2.5&1.3+2.6\end{bmatrix}_{3\times3}=\begin{bmatrix}9&12&15\\6&9&12\\9&12&15\end{bmatrix}_{3\times3}$$

Example 3: If $A=\begin{bmatrix}1&-1\\-1&1\end{bmatrix}$, then show that $A^3=4A$

Solution: -$L.H.S=A^3=A^2.A$

Now, $A^2=A.A=\begin{bmatrix}1&-1\\-1&1\end{bmatrix}\begin{bmatrix}1&-1\\-1&1\end{bmatrix}=\begin{bmatrix}1.1+(-1).(-1)&1.(-1)+(-1).(1)\\(-1).1+1.(-1)&(-1).(-1)+1.1\end{bmatrix}=\begin{bmatrix}2&-2\\-2&2\end{bmatrix}$

$$L.H.S=A^3=A^2.A=\begin{bmatrix}2&-2\\-2&2\end{bmatrix}\begin{bmatrix}1&-1\\-1&1\end{bmatrix}=\begin{bmatrix}2.1+(-2).(-1)&2.(-1)+(-2).(1)\\(-2).1+2.(-1)&(-2).(-1)+(2).(1)\end{bmatrix}=\begin{bmatrix}4&-4\\-4&4\end{bmatrix}$$

$R.H.S=4A=4\begin{bmatrix}1&-1\\-1&1\end{bmatrix}=\begin{bmatrix}4&-4\\-4&4\end{bmatrix}$ Hence, we can see that $L.H.S=R.H.S$

Example 4: Show that $A=\begin{bmatrix}3&1\\-1&2\end{bmatrix}$ is a solution of the matrix equation $A^2-5A+7I=O$

Solution: -Here,

$$L.H.S.=A^2-5A+7I=A.A-5A+7I=\begin{bmatrix}3&1\\-1&2\end{bmatrix}\begin{bmatrix}3&1\\-1&2\end{bmatrix}-5\begin{bmatrix}3&1\\-1&2\end{bmatrix}+7\begin{bmatrix}1&0\\0&1\end{bmatrix}$$

$$=\begin{bmatrix}3.3+1.(-1)&3.1+1.2\\(-1).3+2.(-1)&(-1).1+2.2\end{bmatrix}-\begin{bmatrix}15&5\\-5&10\end{bmatrix}+\begin{bmatrix}7&0\\0&7\end{bmatrix}$$

$$=\begin{bmatrix}8&5\\-5&3\end{bmatrix}-\begin{bmatrix}15&5\\-5&10\end{bmatrix}+\begin{bmatrix}7&0\\0&7\end{bmatrix}=\begin{bmatrix}8-15+7&5-5+0\\-5+5+0&3-10+7\end{bmatrix}$$

$$=\begin{bmatrix}0&0\\0&0\end{bmatrix}=O=R.H.S$$

Q. solve the following by using Matrix Multiplication.

1) If $A=\begin{bmatrix}1&2\\3&4\end{bmatrix}$, then show that $A^2-5A=2I$

2) If $A=\begin{bmatrix}1 & -2 & 3\\ 2 & 3 & -1\\ -3 & 1 & 2\end{bmatrix}, B=\begin{bmatrix}1 & 0 & 2\\ 0 & 1 & 2\\ 1 & 2 & 0\end{bmatrix}$ then find AB and BA .Also check $AB \neq BA$

3) If $A=\begin{bmatrix}1 & 2 & 2\\ 2 & 1 & 2\\ 2 & 2 & 1\end{bmatrix}$ then prove that $A^2-4A-5I=O$

2.13 ADJOINT OF A MATRIX

Definition:- Let $A=\left[a_{ij}\right]_{n\times n}$ be a square matrix of order $n\times n$. Then the adjoint of A is a matrix of the same order $n\times n$ which is obtained by the transpose of a matrix whose elements are cofactor of the element of A in the determinant A. That is if $B=\left[A_{ij}\right]_{n\times n}$ where A_{ij} are the cofactors of the elements a_{ij} in the determinant $|A|$. Then B is called of A. It is denoted by $adjA.$

Remark

- Sometimes the adjoint of a matrix is also called the adjugated of the matrix.
- For finding $adjA.$ of a matrix A of order 2×2 we will apply the given short-cut method (Replace the diagonal elements and change the sign of subsidiary diagonal elements).

Property 1: If A is a square matrix of order n then

$A.(adjA)=(adjA).A=|A|\,I_n$

Example 1: Find the adjoint of the matrix $A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}_{2\times 2}$

Solution: Here, the matrix A is of order 2×2 we will apply the shortcut method [that is replace the diagonal elements (1, 4) and change the sign of the subsidiary diagonal elements (2, 3)] Therefore,

$$A=\begin{bmatrix}1 & 2\\ 3 & 4\end{bmatrix}_{2\times 2}\Rightarrow adjA=\begin{bmatrix}4 & -2\\ -3 & 1\end{bmatrix}$$

Example 2: Find the adjoint of the matrix

$$A=\begin{bmatrix}1 & 2 & 4\\ 5 & 7 & 8\\ 9 & 10 & 12\end{bmatrix}$$

Solution: For the given matrix A, we have

$$A_{11}=\begin{bmatrix}7 & 8\\ 10 & 12\end{bmatrix}=4 \qquad A_{12}=-\begin{bmatrix}5 & 8\\ 9 & 12\end{bmatrix}=12 \qquad A_{13}=\begin{bmatrix}5 & 7\\ 9 & 10\end{bmatrix}=-13$$

$$A_{21}=-\begin{bmatrix}2 & 4\\ 10 & 12\end{bmatrix}=16 \qquad A_{22}=\begin{bmatrix}1 & 4\\ 9 & 12\end{bmatrix}=-24 \qquad A_{23}=-\begin{bmatrix}1 & 2\\ 9 & 10\end{bmatrix}=8$$

$$A_{31}=\begin{bmatrix}2 & 4\\ 7 & 8\end{bmatrix}=-12 \qquad A_{32}=-\begin{bmatrix}1 & 4\\ 5 & 8\end{bmatrix}=12 \qquad A_{33}=\begin{bmatrix}1 & 2\\ 5 & 7\end{bmatrix}=-3$$

Therefore the matrix B formed by the cofactors of the elements of $|A|$ is:

$$B=\begin{bmatrix}4 & 12 & -13\\ 16 & -24 & 12\\ -13 & 8 & -3\end{bmatrix}$$

Now,

$$adjA=\text{transpose of the matrix } B=\begin{bmatrix}4 & 16 & -12\\ 12 & -24 & 8\\ -12 & 12 & -3\end{bmatrix}$$

Example 3*:* - find the adjoint of the matrix $A=\begin{bmatrix}1 & 2\\ 3 & -5\end{bmatrix}$ and verify the property $A.(adjA)=(adjA).A=|A|\,I_n$

Sollution:- Here, $A=\begin{bmatrix}1 & 2\\ 3 & -5\end{bmatrix}\Rightarrow adjA=\begin{bmatrix}-5 & -2\\ -3 & 1\end{bmatrix}$

$$A.(adjA)=\begin{bmatrix}1 & 2\\ 3 & -5\end{bmatrix}\begin{bmatrix}-5 & -2\\ -3 & 1\end{bmatrix}=\begin{bmatrix}1.(-5)+2.(-3) & 1.(-2)+2.1\\ 3.(-5)+(-5).(-3) & 3.(-2)+(-5).1\end{bmatrix}=\begin{bmatrix}-11 & 0\\ 0 & -11\end{bmatrix} \qquad (1)$$

$$(adjA).A=\begin{bmatrix}-5 & -2\\ -3 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\ 3 & -5\end{bmatrix}=\begin{bmatrix}(-5).1+(-2).3 & (-5).2+(-2).(-5)\\ (-3).1+1.3 & (-3).2+1.(-5)\end{bmatrix}=\begin{bmatrix}-11 & 0\\ 0 & -11\end{bmatrix} \qquad (2)$$

$$|A|.I_n = |A|.I_2 = \begin{vmatrix} 1 & 2 \\ 3 & -5 \end{vmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = (-5-6)\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = -11\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -11 & 0 \\ 0 & -11 \end{bmatrix} \qquad (3)$$

From (1), (2) and (3) we can see that $A.(adjA) = (adjA).A = |A|\,I_n$

Hence, property is satisfied.

Q. Solve the following:

1) Find the adjoint of the given matrix $\begin{bmatrix} 1 & 4 \\ -2 & 5 \end{bmatrix}$

2) Find the adjoint of the given matrix $\begin{bmatrix} 2 & -1 & 3 \\ 1 & 0 & 5 \\ -1 & 3 & 4 \end{bmatrix}$

3) find the adjoint of the matrix $A = \begin{bmatrix} 4 & 3 \\ 1 & -2 \end{bmatrix}$ and verify the property $A.(adjA) = (adjA).A = |A|\,I_n$

2.14 <u>INVERSE OF AN MATRIX</u>

Definition: Let A be a square matrix of order $n \times n$ and there exists a square matrix of the same order such that AB = BA = I_n , where I_n is a unit matrix of order $n \times n$.Then the matrix B is called the inverse of a matrix A.

Remarks

- A matrix A is in invertible if it is non-singular (i.e., $|A| \neq 0$)
- $A^{-1} = \dfrac{adjA}{|A|}, |A| \neq 0$

SOLVED EXAMPLES

Example 1: - Can we find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$

Solution: - For finding the inverse of any matrix A we need $|A| \neq 0$

But here, $|A| = \begin{vmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{vmatrix} = 1(45-48) - 2(36-42) + 3(32-35) = -3 + 12 - 9 = 0$

Therefore, we can't find the inverse of the given matrix A.

Example 1: Find A^{-1} where $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$

Solution: - Here, $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \Rightarrow |A| = 4 - 6 = -2 \neq 0$

Now, $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \Rightarrow adjA = \begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$

Hence,

$$A^{-1} = \frac{adjA}{|A|} = \frac{1}{-2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}$$

Example 2: - Find A^{-1} where $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$

Solution: - Here, $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \Rightarrow |A| = 1(1-4) - 2(2-4) + 2(4-2) = -3 + 4 + 4 = 5 \neq 0$

Now, we will find the adjoint of a given matrix A

$A_{11} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = -3$ $\quad A_{12} = -\begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix} = 2$ $\quad A_{13} = \begin{bmatrix} 2 & 1 \\ 2 & 2 \end{bmatrix} = 2$

$A_{21} = -\begin{bmatrix} 2 & 2 \\ 2 & 1 \end{bmatrix} = 2$ $\quad A_{22} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = -3$ $\quad A_{23} = -\begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} = 2$

$A_{31} = \begin{bmatrix} 2 & 2 \\ 1 & 2 \end{bmatrix} = 2$ $\quad A_{32} = -\begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix} = 2$ $\quad A_{33} = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} = -3$

Therefore the matrix B formed by the cofactors of the elements of $|A|$ is:

$$B = \begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}$$

Now, $$adjA = \text{transpose of the matrix } B = \begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}$$

Hence,

$$A^{-1} = \frac{adjA}{|A|} = \frac{1}{5}\begin{bmatrix} -3 & 2 & 2 \\ 2 & -3 & 2 \\ 2 & 2 & -3 \end{bmatrix}$$

Example 3: - Show that the matrix $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$ satisfies the equation $A^3 - 6A^2 + 9A - 4I = O$. Hence find A^{-1}

Solution: - Here, $A = \begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & -1 \\ 1 & -1 & 2 \end{bmatrix}$

By using matrix multiplication, we have $A^2 = \begin{bmatrix} 6 & -5 & 5 \\ -5 & 6 & -5 \\ 5 & -5 & 6 \end{bmatrix}$ and $A^3 = \begin{bmatrix} 22 & -21 & 21 \\ -21 & 22 & -21 \\ 21 & -21 & 22 \end{bmatrix}$

$$A^3-6A^2+9A-4I=\begin{bmatrix}22&-21&21\\-21&22&-21\\21&-21&22\end{bmatrix}-6\begin{bmatrix}6&-5&5\\-5&6&-5\\5&-5&6\end{bmatrix}+9\begin{bmatrix}2&-1&1\\-1&2&-1\\1&-1&2\end{bmatrix}-4\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}$$

$$=\begin{bmatrix}22&-21&21\\-21&22&-21\\21&-21&22\end{bmatrix}-\begin{bmatrix}36&-30&30\\-30&36&-30\\30&-30&36\end{bmatrix}+\begin{bmatrix}18&-9&9\\-9&18&-9\\9&-9&18\end{bmatrix}-\begin{bmatrix}4&0&0\\0&4&0\\0&0&4\end{bmatrix}$$

$$=\begin{bmatrix}0&0&0\\0&0&0\\0&0&0\end{bmatrix}=O$$

Now,

Hence, it satisfies the equation.

Now, multiplying A^{-1} both the sides

$$A^{-1}\left(A^3-6A^2+9A-4I\right)=A^{-1}(O)$$

$$A^2-6A+9I-4A^{-1}=0\Rightarrow 4A^{-1}=A^2-6A+9I$$

$$\Rightarrow A^{-1}=\frac{1}{4}\left(A^2-6A+9I\right)$$

$$\Rightarrow A^{-1}=\frac{1}{4}\left(\begin{bmatrix}6&-5&5\\-5&6&-5\\5&-5&6\end{bmatrix}-6\begin{bmatrix}2&-1&1\\-1&2&-1\\1&-1&2\end{bmatrix}+9\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}\right)$$

$$\Rightarrow A^{-1}=\frac{1}{4}\begin{bmatrix}3&1&-1\\1&3&1\\-1&1&3\end{bmatrix}$$

Example 4: - Show that $A^{-1}=A$ where $A=\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$

Solution: - Here, $A=\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}\Rightarrow |A|=0-0+1(0-1)=-1\neq 0$

Now, we will find the adjoint of a given matrix A

$$A_{11}=\begin{bmatrix}1&0\\0&0\end{bmatrix}=0 \qquad A_{12}=-\begin{bmatrix}0&0\\1&0\end{bmatrix}=0 \qquad A_{13}=\begin{bmatrix}0&1\\1&0\end{bmatrix}=-1$$

$$A_{21}=-\begin{bmatrix}0&1\\0&0\end{bmatrix}=0 \qquad A_{22}=\begin{bmatrix}0&1\\1&0\end{bmatrix}=-1 \qquad A_{23}=-\begin{bmatrix}0&0\\1&0\end{bmatrix}=0$$

$$A_{31}=\begin{bmatrix}0&1\\1&0\end{bmatrix}=-1 \qquad A_{32}=-\begin{bmatrix}0&1\\0&0\end{bmatrix}=0 \qquad A_{33}=\begin{bmatrix}0&0\\0&1\end{bmatrix}=0$$

Therefore the matrix B formed by the cofactors of the elements of $|A|$ is:

$$B=\begin{bmatrix}0&0&-1\\0&-1&0\\-1&0&0\end{bmatrix}$$

$$adjA=\text{transpose of the matrix } B=\begin{bmatrix}0&0&-1\\0&-1&0\\-1&0&0\end{bmatrix}$$

Now,

Hence,

$$A^{-1}=\frac{adjA}{|A|}=\frac{1}{-1}\begin{bmatrix}0&0&-1\\0&-1&0\\-1&0&0\end{bmatrix}=\begin{bmatrix}0&0&1\\0&1&0\\1&0&0\end{bmatrix}$$

We can see that $A^{-1}=A$.

Q. Solve the following

1) Find the inverse of the matrix $A=\begin{bmatrix}5&2\\7&3\end{bmatrix}$

2) Find the inverse of the matrix $A=\begin{bmatrix}2&-2&4\\2&3&2\\-1&1&-1\end{bmatrix}$

3) Find the inverse of the matrix $A=\begin{bmatrix}1&0&2\\2&1&0\\3&2&1\end{bmatrix}$

2.15 PROPERTIES OF TRANSPOSE AND INVERSE

Let $A=\left[a_{ij}\right]_{m\times n}$ and $B=\left[b_{ij}\right]_{m\times n}$ be any two square matrices then

Property 1: - $(A+B)^T = A^T + B^T$

Property 2: - $(AB)^{-1}=B^{-1}A^{-1}$

SOLVED EXAMPLES

Example 1: - For giver matrices $A=\begin{bmatrix}1&2\\3&4\end{bmatrix}$ and $B=\begin{bmatrix}4&5\\6&7\end{bmatrix}$ then prove that $(A+B)^T = A^T + B^T$

Solution: - Here, $A=\begin{bmatrix}1&2\\3&4\end{bmatrix}$ and $B=\begin{bmatrix}4&5\\6&7\end{bmatrix}$

$$A+B=\begin{bmatrix}1&2\\3&4\end{bmatrix}+\begin{bmatrix}4&5\\6&7\end{bmatrix}=\begin{bmatrix}5&7\\9&11\end{bmatrix}\Rightarrow(A+B)^T=\begin{bmatrix}5&9\\7&11\end{bmatrix} \quad (1)$$

Now, $A^T=\begin{bmatrix}1&3\\2&4\end{bmatrix}$ and $B^T=\begin{bmatrix}4&6\\5&7\end{bmatrix}\Rightarrow A^T+B^T=\begin{bmatrix}1&3\\2&4\end{bmatrix}+\begin{bmatrix}4&6\\5&7\end{bmatrix}=\begin{bmatrix}5&9\\7&11\end{bmatrix}$ (2)

From (1) and (2), we can say that $(A+B)^T = A^T + B^T$.

Example 2: - For giver matrices $A=\begin{bmatrix}2&1\\-1&3\end{bmatrix}$ and $B=\begin{bmatrix}2&3\\1&2\end{bmatrix}$ then prove that $(AB)^{-1}=B^{-1}A^{-1}$

Solution: - $AB=\begin{bmatrix}2&1\\-1&3\end{bmatrix}\begin{bmatrix}2&3\\1&2\end{bmatrix}=\begin{bmatrix}2.2+1.1&2.3+1.2\\(-1).2+3.1&(-1).3+3.2\end{bmatrix}=\begin{bmatrix}5&8\\1&3\end{bmatrix}$

Now, $AB=\begin{bmatrix}5 & 8\\ 1 & 3\end{bmatrix} \Rightarrow (AB)^{-1}=\frac{1}{|AB|}adjAB=\frac{1}{7}\begin{bmatrix}3 & -8\\ -1 & 5\end{bmatrix}$ (1)

Next, $A=\begin{bmatrix}2 & 1\\ -1 & 3\end{bmatrix} \Rightarrow A^{-1}=\frac{1}{|A|}adjA=\frac{1}{7}\begin{bmatrix}3 & -1\\ 1 & 2\end{bmatrix}$ and

$B=\begin{bmatrix}2 & 3\\ 1 & 2\end{bmatrix} \Rightarrow B^{-1}=\frac{1}{|B|}adjB=\frac{1}{1}\begin{bmatrix}2 & -3\\ -1 & 2\end{bmatrix}=\begin{bmatrix}2 & -3\\ -1 & 2\end{bmatrix}$

Lastly $B^{-1}A^{-1}=\begin{bmatrix}2 & -3\\ -1 & 2\end{bmatrix}\frac{1}{7}\begin{bmatrix}3 & -1\\ 1 & 2\end{bmatrix}=\frac{1}{7}\begin{bmatrix}2.3+(-3).1 & 2.(-1)+(-3).2\\ (-1).3+2.1 & (-1).(-1)+2.2\end{bmatrix}=\frac{1}{7}\begin{bmatrix}3 & -8\\ -1 & 5\end{bmatrix}$ (2)

From (1) and (2), we can say that $(AB)^{-1}=B^{-1}A^{-1}$

Q. Solve the following:

(1)For giver matrices $A=\begin{bmatrix}1 & 2 & 5\\ 3 & 4 & -2\\ -1 & 0 & 6\end{bmatrix}$ and $B=\begin{bmatrix}2 & -1 & 5\\ 4 & 2 & -3\\ 3 & 1 & 0\end{bmatrix}$ then prove that

$(A+B)^T=A^T+B^T$

(2) For giver matrices $A=\begin{bmatrix}1 & 2\\ 0 & 3\end{bmatrix}$ and $B=\begin{bmatrix}1 & 4\\ 3 & 1\end{bmatrix}$ then prove that

$(AB)^{-1}=B^{-1}A^{-1}$

2.16 <u>MATRIX INVERSION METHOD</u>

In this section, we shall express the system of linear equations as matrix equation and solve them using inverse of the coefficient matrix.
Consider the system of equations.

$$a_1x+b_1y+c_1z=d_1$$
$$a_2x+b_2y+c_2z=d_2$$
$$a_3x+b_3y+c_3z=d_3$$

Let $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$

Then, the system of equations can be expressed in the form

$$\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$

i.e., $AX = B$

If A is non-singular matrix, then its inverse exists.
Hence, we have

$$A^{-1}(AX) = A^{-1}B$$
$$X = A^{-1}B$$

This matrix equation provides solution for the variables $x, y \text{ and } z.$

SOLVED EXAMPLES

Example 1: - Solve for $x \text{ and } y$ by inverting the matrix in the following

$$\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$$

Solution: - Here, $A = \begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \end{bmatrix}$, $B = \begin{bmatrix} 4 \\ 7 \end{bmatrix}$

Now by matrix inversion method, we have $X = A^{-1}B$

Therefore, $A^{-1} = \frac{1}{|A|} adjA = \frac{1}{5}\begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}$

Now, $X = A^{-1}B = \frac{1}{5}\begin{bmatrix} 3 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 7 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 3.4 + (-1).7 \\ (-1).4 + 2.7 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 5 \\ 10 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$

Example 2: - *Solve using matrix inversion method, the following equations*

$$x + y = 0, y + z = 1, z + x = 3$$

Solution: - Here, $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$, $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}$

Now, $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \Rightarrow |A| = 1(1-0) - 1(0-1) + 0 = 2 \neq 0 \Rightarrow A^{-1}\text{exists}$

Let A_{ij} be the cofactor of a_{ij} is IAI. Then we have

$$A_{11} = 1, \quad A_{12} = 1, \quad A_{13} = -1$$

$$A_{21} = -1, \quad A_{22} = 1, \quad A_{23} = 1$$

$$A_{31} = -1, \quad A_{32} = -1, \quad A_{33} = 1$$

Now,

$$adjA = \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$$

$$A^{-1} = \frac{1}{|A|} adjA = \frac{1}{2} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix}$$

We have, $X = A^{-1}B = \frac{1}{2} \begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1.0+(-1).1+1.3 \\ 1.0+1.1+(-1).1 \\ (-1).0+1.1+1.3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$

Q. Solve the following by using matrix inversion method

1) $5x + 2y = 4, \quad 7x + 3y = 5$

2) $2x - y + 3z = 1, \quad x + 2y - z = 2, \quad 5y - 5z = 3$

3) $2x - y + 3z = 9, \quad x + y + z = 6, \quad x - y + z = 2$