PARUL UNIVERSITY - FACULTY OF ENGINEERING & TECHNOLOGY
Department of Applied Science & Humanities
1st Semester B.C.A
Basic Mathematics

Chapter : 5 : Trigonometry

5.1 INTRODUCTION

The word trigonometry is derived from two Greek word "trigon" and “ metron ", means "triangle" and "to measure" respectively. Therefore, trigonometry means to measure a triangle.i.e., “Trigonometry is that branch of Mathematics which deals with angles, whether of a triangle or any other figure “.

Trigonometry specifically deals with the relationships between the sides and the angles of triangles, that is, on the trigonometric functions, and with calculations based on these functions.

5.2 ANGLES AND QUADRANTS

Consider the Fig. 5.1, the angle is obtained by rotating a given ray about its end points. The original ray is called the initial side and the ray into which the initial sides rotates is called the terminal side of the angle

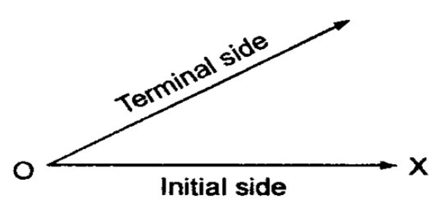


Fig. 5.1

Remarks

- The measure of an angle is the amount of rotation required to get the terminal side from initial side.

- If the revolving line revolves in anticlockwise direction, then add the angle is positive and if revolving line in clockwise direction, then it is called negative angle. This may be clear in the following Fig. 5.2.

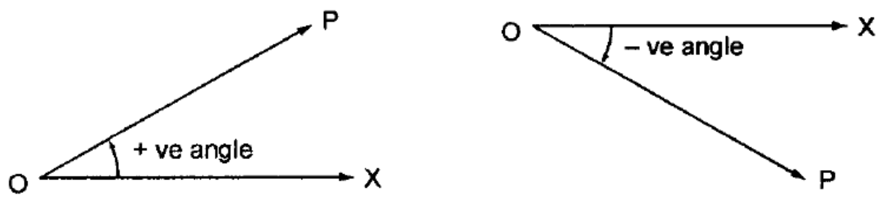


Fig. 5.2.

5.3 <u>Measurements of an angle</u>

Generally, we measure the angles in degree or in radian, which are defined as follows :

1) **Degree Measure:** We can divide the right angle into 90 equal parts and each part is known as a degree. Thus, a right angle is equal to 90 degrees.

Similarly, we can say that the circumference of a circle can be divided into 360 equal parts. One degree is denoted by $1°$.

Again, we can divide a degree into sixty equal parts. Each part is known as a minute and is denoted by $1'$.

i.e., $1° = 60'$ (sixty minutes)

A minute can also be divided into sixty equal parts and each small part is known as seconds and is denoted by $1''$

i.e., $1' = 60''$(seconds)

2) **Radian measure:**Let us take a circle of radius ‘a’ then ‘’a radian’’ is an angle subtended at the center of a circle by an arc equal in length to the radius of the circle. One radian angle is denoted by as 1^c in the Fig. 5.3.

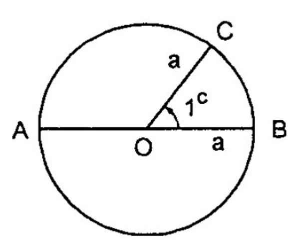


Fig. 5.3.

- **Various types of angles**

  (a) An angle θ is to be *acute angle* if it remains in quadrant number 1

  i.e., $0^0 \le \theta < 90^0$

  (b) An angle θ is said to be *right angle* if $\theta = 90^0$

  (c) An angle is said to be *obtuse angle* ifit remains in quadrant number 2

  i.e., $90^0 \le \theta < 180^0$

  (d) An angle θ is said to be *straight angle,* if $\theta = 180^0$

  (e) An angle θ is said to be reflexive angle if it remains in quadrant number 3 and 4.

  i.e., $180^0 < \theta < 360^0$

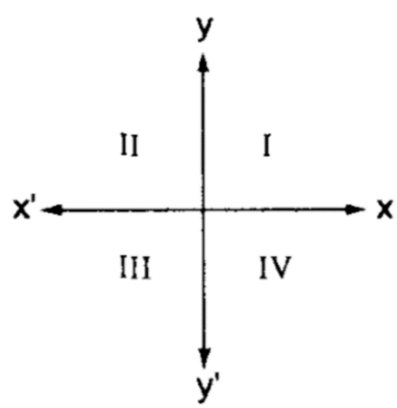


Fig. 5.4.

- **<u>Remarks</u>**

  Relation between degree and radian:

$$\pi \text{ radian} = 180 \text{ degree}$$

Therefore, $$1 \text{ radian} = \frac{180}{\pi} \text{ degree}$$

Similarly, $$180 \text{ degree} = \pi \text{ radian}$$

Therefore, $$1 \text{ degree} = \frac{\pi}{180} \text{ radian}$$

SOLVED EXAMPLES

Example 1: - Find the degree measure of the following measure

(a) $\left(\frac{5\pi}{12}\right)^c$ **(b)** $\left(\frac{\pi}{15}\right)^c$ **(c)** $\left(\frac{\pi}{32}\right)^c$

Solution: - (a) Given $\left(\frac{5\pi}{12}\right)^c = \left(\frac{5\pi}{12}\times\frac{180}{\pi}\right)^\circ = 75^\circ$

(b) $\left(\frac{\pi}{15}\right)^c = \left(\frac{\pi}{15}\times\frac{180}{\pi}\right)^\circ = 12^\circ$

(c)

$$\left(\frac{\pi}{32}\right)^c = \left(\frac{\pi}{32}\times\frac{180}{\pi}\right)^0 = \left(\frac{45}{8}\right)^0$$

$$= 45^0\left(\frac{1}{8}\right)^0$$

$$= 45^0\left(\frac{1}{8}\times 60\right)'$$

$$= 45^0\left(\frac{15}{2}\right)'$$

$$= 45^0 15'\left(\frac{1}{2}\times 60\right)''$$

$$= 45^0 15' 30''$$

Exercise: - Find the degree measure of the following measure

(a) $\left(\frac{7\pi}{12}\right)^c$ **(b)** $\left(\frac{\pi}{75}\right)^c$ **(c)** $\left(\frac{1}{4}\right)^c$

Example 2: - Find the radian measure of the given degree measure:

(a) 210^0 **(b)** $-22^0 30'$ **(c)** $40^0 20'$

Solution: - (a) Given $210^0 = \left(210\times\frac{\pi}{180}\right)^c = \left(\frac{7\pi}{6}\right)^c$ **(b)**Given

$$-22^0 30' = -\left(22+\frac{30}{60}\right)^0 = -\left(\frac{45}{2}\right)^0$$

$$= -\left(\frac{45}{2}\times\frac{\pi}{180}\right)^c = -\left(\frac{\pi}{8}\right)^c$$

(c) Given

$$40°20' = 40° + \left(\frac{20}{60}\right)^{\circ}$$

$$¿\left(40+\frac{1}{3}\right)^{\circ}$$

$$¿\left(\frac{121}{3}\right)^{\circ}$$

$$¿\left(\frac{121}{3}\times\frac{\pi}{180}\right)^c$$

$$¿\left(\frac{121\pi}{540}\right)^c$$

Exercise: Find the radian measure of the given degree measure:

(a) 240^0 **(b)** 15^0 **(c)** $5^0 37' 30''$

Example : Find the circular measure of 40° 27’ 30”.

Solution : we know that ; $30'' = \frac{30'}{60} = \frac{1'}{2}$ and $\left(\frac{1}{2}\right)' = \left(\frac{1}{120}\right)^{\circ}$

And $27' = \left(\frac{27}{60}\right)^{\circ} = \left(\frac{9}{20}\right)^{\circ}$

Therefore ; $40^{\circ} 27' 30'' = 40° + \left(\frac{9}{20}\right)^{\circ} + \left(\frac{1}{120}\right)^{\circ}$

$$¿\left(\frac{4800+54+1}{120}\right)^{\circ}$$

$$¿\left(\frac{4855}{120}\right)^{\circ}$$

$$¿\left(\frac{971}{24}\right)^{\circ}$$

$$¿\left(\frac{971}{24}\times\frac{\pi}{180}\right)^{C}=0.7058\,radian$$

- **Pythagoras Theorem: -**
  In a right-angle triangle, the area of the square whose side is equal to the sum of the areas of the squares whose sides are the two legs.

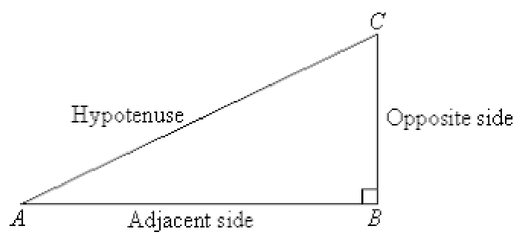


Fig. 5.5

In the Fig. 5.5, $AB^2 + BC^2 = AC^2$

5.4 Trigonometric Ratios or Functions

- In a right-angle triangle ABC, if $\angle CAB = \theta$
  $\Rightarrow BC = \text{Opposite side}$
  AB=Adjacent side
  AC=Hypotenuse
  Trigonometric ratios are defined for acute angles as the ratio of the sides of a right-angled triangle. The extension of trigonometric ratios to any angle in terms of radian measure (real numbers) are called trigonometric functions.
- The six trigonometric functions are sine(sin), cosine(cos), tangent(tan), cosecant(csc), secant(sec), and cotangent(cot). Now we will define the ratios formed by these functions at angle θ.
- $sin\theta = \dfrac{\text{Opposite side}}{\text{Hypotenuse}}$, $cos\theta = \dfrac{\text{Adjacent side}}{\text{Hypotenuse}}$, $\tan\theta = \dfrac{\text{Opposite side}}{\text{Adjacent side}}$
  $\text{cosec}\theta = \dfrac{\text{Hypotenuse}}{\text{Opposite side}}$, $sec\theta = \dfrac{\text{Hypotenuse}}{\text{Adjacent side}}$, $\cot\theta = \dfrac{\text{Adjacent side}}{\text{Opposite side}}$

From last definitions, it follows some definitions:

(a) cosecθ is the reciprocal of $sin\theta$ i.e., $\text{cosec}\theta = \frac{1}{\sin\theta}$

(b) $sec\theta$ is the reciprocal of $cos\theta$ i.e., $\sec\theta = \frac{1}{\cos\theta}$

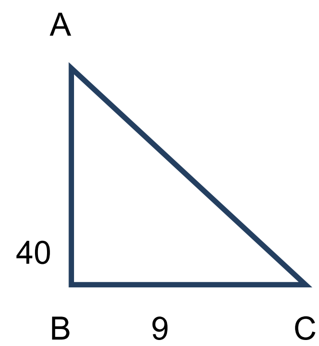
(c) $\cot\theta$ is the reciprocal of $\tan\theta$ i.e., $\cot\theta = \frac{1}{\tan\theta}$

SOLVED EXAMPLES

Example 1: -If $\cos\theta = \frac{9}{41}$ determine the values of other five trigonometric ratios.

Solution: - Given $\cos\theta = \frac{9}{41} = \frac{\text{Adjacent side}}{\text{Hypoteneous}}$

By Pythagoras theorem,



$$AC^2 = AB^2 + BC^2$$
$$(41)^2 = AB^2 + (9)^2$$
$$1681 = AB^2 + 81$$
$$AB^2 = 1600 \Rightarrow AB = 40$$

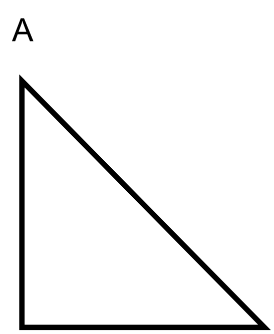
Now,

$$\sin\theta = \frac{\text{Opposite side}}{\text{Hypoteneous}} = \frac{40}{41}, \tan\theta = \frac{\text{Opposite side}}{\text{adjacent side}} = \frac{40}{9}, \cos ec\theta = \frac{\text{Hypoteneous}}{\text{Opposite side}} = \frac{41}{40}$$
$$\sec\theta = \frac{\text{Hypoteneous}}{\text{adjacent side}} = \frac{41}{9}, \cot\theta = \frac{\text{adjacent side}}{\text{opposite side}} = \frac{9}{40}$$

Example 2: -In ΔABC if $\cos B = \frac{3}{5}$, then find $\sin B, \tan B, \sin A, \cos A, \tan A$.

Solution: - Given $\cos B = \frac{3}{5} = \frac{\text{Adjacent side}}{\text{Hypoteneous}}$

By Pythagoras theorem,



$AB^2 = AC^2 + BC^2$

$(5)^2 = AC^2 + (3)^2$

$25 = AC^2 + 9$

$AC^2 = 16 \Rightarrow AC = 4$

4 5

C 3 B

$\sin B = \frac{4}{5}, \tan B = \frac{4}{3}, \sin A = \frac{3}{5}$

$\cos A = \frac{4}{5}, \tan A = \frac{3}{4}$

Exercise: -1) If $\sin\theta = \frac{12}{13}$ determine the values of other five trigonometric ratios.

2) In ΔABC if $\cos A = \frac{15}{17}$, then find $\sin A, \tan A, \operatorname{cosec} A, \sec A, \cot A$.

- **Remarks: -**

(a) Sine and cosine functions are called primary functions whereas tangent, cotangent, secant and cosecant functions are called secondary trigonometric functions.

(b) When the terminal ray coincides with x-axis, $\operatorname{cosec}\theta$ and $\cot\theta$ are not defined.

(c) When the terminal ray coincides with y-axis, $sec\theta$ and $\tan\theta$ are not defined.

(d) The domain of the sine function is the set of real number, whereas its range is the set of real numbers from -1 to 1.

(e) The domain of the cosine function is the set of all real numbers and range is the set of real numbers from -1 to 1.

(f) The domain of the tangent function is the set of all real numbers except odd multiples of $\frac{\pi}{2}$ and its range is the set of all real numbers.

(g) The student should not commit the mistake of regarding sine as $\sin X\theta$; $\sin\theta$ means the sine of angle θ, it is absolutely wrong to perform such operations as:

$$\sin(A+B) = \sin A + \sin B$$
$$\sin 2x + \sin x = \sin(2x + x)$$

(h) Power notation for trigonometric function $(\sin\theta)^2$ is written as $\sin^2\theta$ and is read as sin square θ, $(\sin\theta)^3$ is written as $\sin^3\theta$ and is read as sin cube θ

5.5 GRAPHS OF CIRCULAR FUNCTIONS

(1) Sine : The Sine Function has up-down curve (which repeats every 2π radians, or 360°).It starts at 0, heads up to 1 by π/2 radians (90°) and then heads down to −1.

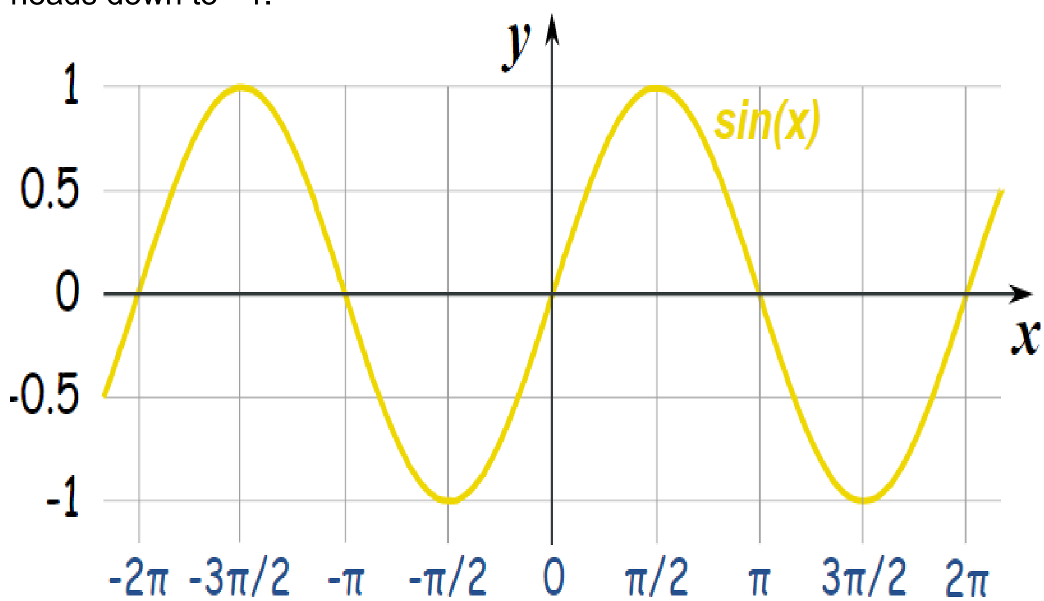


Fig. 5.6.

(2) Cosine : Cosine is just like Sine, but it starts at 1 and heads down until π radians (180°) and then heads up again

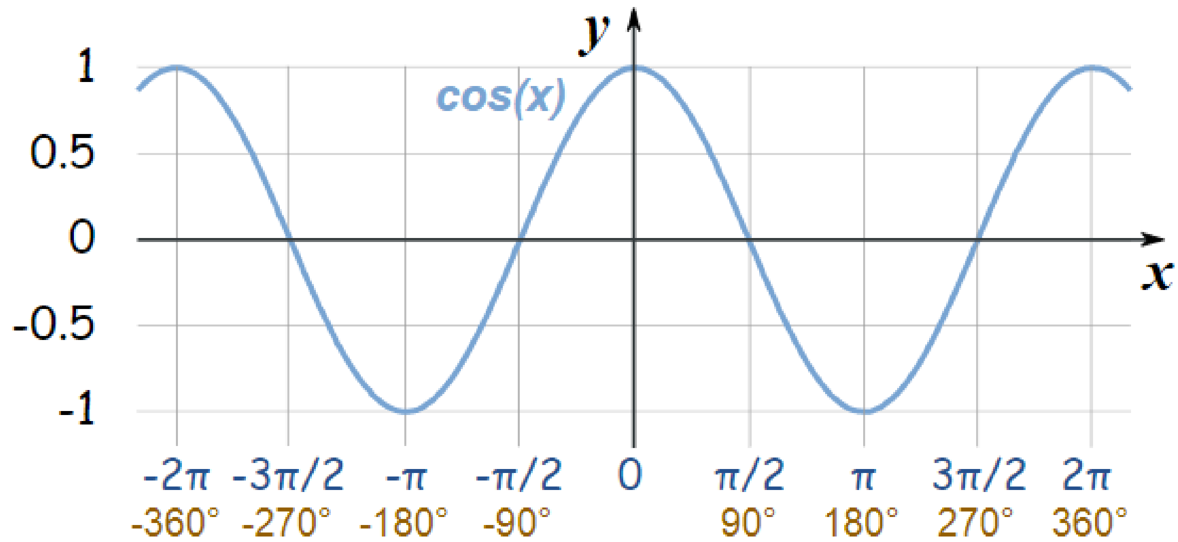


Fig. 5.7.

(3) Tangent :The Tangent function has a completely different shape . it goes between negative and positive Infinity ,crossing through 0, and at every π radians (180°), as shown on this plot. At π/2 radians (90°), and at −π/2 (−90°), 3π/2 (270°), etc, the function is officially undefined, because it could be positive Infinity or negative Infinity.

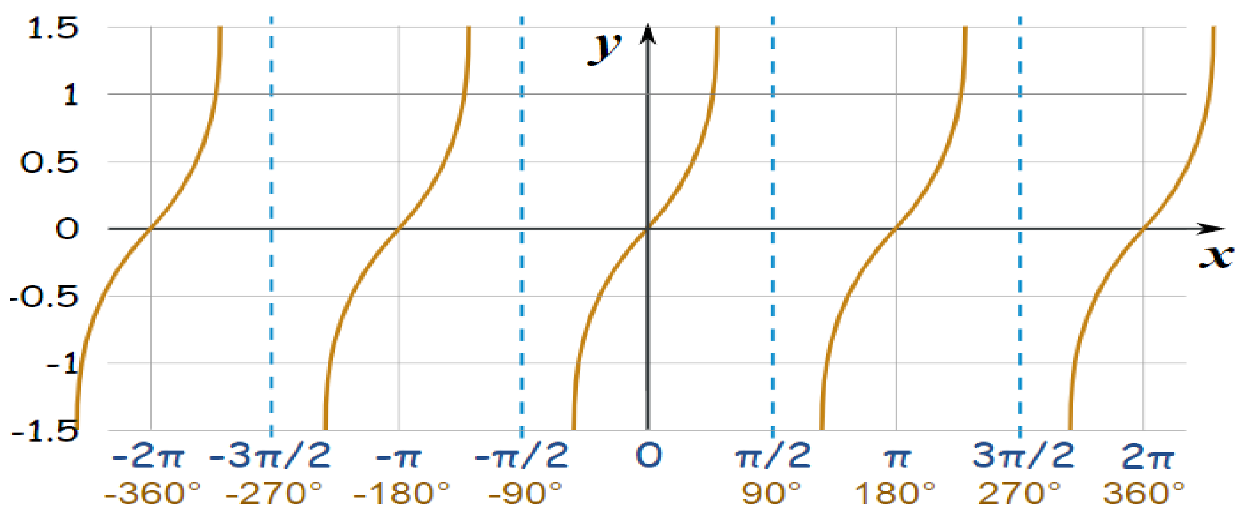


Fig. 5.8.

5.6 <u>Trigonometric identities and equations</u>

An expression involving trigonometric function which is true for all those values of θ for which the functions are defined is called a trigonometric identity. Otherwise, it is a trigonometric equation.

Reciprocal Identities:

$$csc\theta = \frac{1}{sin\theta}$$

$$sec\theta = \frac{1}{cos\theta}$$

$$cot\theta = \frac{1}{tan\theta}$$

Quotient Identities:

$$tan\theta = \frac{sin\theta}{cos\theta}$$

$$cot\theta = \frac{cos\theta}{sin\theta}$$

Other Identities

$$\sin^2\theta + \cos^2\theta = 1$$
$$\tan^2\theta + 1 = sec^2\theta$$
$$1 + \cot^2\theta = csc^2\theta$$

example:

(i) $\sec\theta = \frac{1}{\cos\theta}$ and $\cos ec\theta = \frac{1}{\sin\theta}$ are true for all values of θ except those for which $\cos\theta = 0, \sin\theta = 0$ so these are called trigonometric identities.

(ii) $\sin\theta = \cos\theta$ is an expression which does not hold for all values of θ. So it is an equation and not identity

(iii) The values of trigonometric ratio of trigonometric angles of

$\left(30^0\right)\frac{\pi}{6}, \left(45^0\right)\frac{\pi}{4}, \left(60^0\right)\frac{\pi}{3}$ and $\left(90^0\right)\frac{\pi}{2}$

θ	0^0	30^0	45^0	60^0	90^0
$\sin\theta$	0	$\frac{1}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{\sqrt{3}}{2}$	1
$\cos\theta$	1	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{2}$	0
$\tan\theta$	0	$\frac{1}{\sqrt{3}}$	1	$\sqrt{3}$	∞
$\mathrm{cosec}\theta$	∞	2	$\sqrt{2}$	$\frac{2}{\sqrt{3}}$	1
$sec\theta$	1	$\frac{2}{\sqrt{3}}$	$\sqrt{2}$	2	∞
$\cot\theta$	∞	$\sqrt{3}$	1	$\frac{1}{\sqrt{3}}$	0

SOLVED EXAMPLES

Example 1: - Prove that $\sin^2\theta\cot\theta\sec\theta = \sin\theta$
Solution: -

$$\text{L.H.S.} = \sin^2\theta \cot\theta \sec\theta = \sin^2\theta\left(\frac{\cos\theta}{\sin\theta}\right)\left(\frac{1}{\cos\theta}\right)$$

$$= \sin\theta$$

$$= \text{R.H.S.}$$

Example 2: Prove that $\dfrac{1+\cot\theta}{1+\tan\theta} = \cot\theta$

Solution:

$$\text{L.H.S.} = \frac{1+\cot\theta}{1+\tan\theta}$$

$$= \frac{1+\dfrac{\cos\theta}{\sin\theta}}{1+\dfrac{\sin\theta}{\cos\theta}} = \frac{\dfrac{\sin\theta+\cos\theta}{\sin\theta}}{\dfrac{\cos\theta+\sin\theta}{\cos\theta}} = \frac{\cos\theta}{\sin\theta} = \cot\theta = \text{R.H.S.}$$

Example 3: -Prove that $\sqrt{\dfrac{1+\cos\theta}{1-\cos\theta}} = \cos ec\theta + \cot\theta$

Solution: -

$$\text{L.H.S.} = \sqrt{\frac{1+\cos\theta}{1-\cos\theta}} = \sqrt{\frac{1+\cos\theta}{1-\cos\theta}\times\frac{1+\cos\theta}{1+\cos\theta}}$$

$$= \sqrt{\frac{(1+\cos\theta)^2}{1-\cos^2\theta}}$$

$$= \sqrt{\frac{(1+\cos\theta)^2}{\sin^2\theta}} = \frac{1+\cos\theta}{\sin\theta}$$

$$= \frac{1}{\sin\theta} + \frac{\cos\theta}{\sin\theta}$$

$$= \cos ec\theta + \cot\theta$$

$$= \text{R.H.S.}$$

Example 4: - Prove that $\dfrac{\tan\theta - \cot\theta}{\sin\theta\cos\theta} = \sec^2\theta - \cos ec^2\theta$

Solution: -

$$\text{L.H.S.} = \frac{\tan\theta - \cot\theta}{\sin\theta\cos\theta} = \frac{\dfrac{\sin\theta}{\cos\theta} - \dfrac{\cos\theta}{\sin\theta}}{\sin\theta\cos\theta}$$

$$= \frac{\dfrac{\sin^2\theta - \cos^2\theta}{\sin\theta\cos\theta}}{\sin\theta\cos\theta}$$

$$= \frac{\sin^2\theta - \cos^2\theta}{(\sin\theta\cos\theta)^2}$$

$$= \frac{\sin^2\theta}{\sin^2\theta\cos^2\theta} - \frac{\cos^2\theta}{\sin^2\theta\cos^2\theta}$$

$$= \frac{1}{\cos^2\theta} - \frac{1}{\sin^2\theta}$$

$$= \sec^2\theta - \cos ec^2\theta$$

$$= \text{R.H.S.}$$

Example 5: -Find the value of $2\sin^2\frac{\pi}{4} + 2\cos^2\frac{\pi}{4} + \sec^2\frac{\pi}{3}$

Solution: -Given

$$2\sin^2\frac{\pi}{4} + 2\cos^2\frac{\pi}{4} + \sec^2\frac{\pi}{3}$$

$$= 2\left(\frac{1}{\sqrt{2}}\right)^2 + 2\left(\frac{1}{\sqrt{2}}\right)^2 + (2)^2$$

$$= 2\left(\frac{1}{2}\right) + 2\left(\frac{1}{2}\right) + 4 = 1 + 1 + 4 = 6$$

Example 6: - Find the value of $\left(3\cos\frac{\pi}{3}\sec\frac{\pi}{3} - 4\sin\frac{\pi}{6}\tan\frac{\pi}{4}\right)\cos 2\pi$

Solution: -Given

$$\left(3\cos\frac{\pi}{3}\sec\frac{\pi}{3}-4\sin\frac{\pi}{6}\tan\frac{\pi}{4}\right)\cos 2\pi$$
$$=\left(3\left(\frac{1}{2}\right)(2)-4\left(\frac{1}{2}\right)(1)\right)(1) \qquad (\because \cos 2\pi = 1)$$
$$=(3-2)(1)$$
$$=1$$

Exercise: 1) Show that $(\sec A-\tan A)^2=\frac{1-\sin A}{1+\sin A}$

2) Show that $\frac{1}{\sec\theta+\tan\theta}-\frac{1}{\cos\theta}=\frac{1}{\cos\theta}-\frac{1}{\sec\theta-\tan\theta}$

3) Show that $(1+\cot\theta+\tan\theta)(\sin\theta-\cos\theta)=\frac{\sec\theta}{\operatorname{cosec}^2\theta}-\frac{\operatorname{cosec}\theta}{\sec^2\theta}$

4) Show that $\sqrt{\frac{1-\sin\theta}{1+\sin\theta}}=\sec\theta-\tan\theta$

5) Show that $\tan^2\theta-\sin^2\theta=(\tan^2\theta)(\sin^2\theta)$

6) Find the value of $3\sin^2\left(\frac{\pi}{6}\right)+2\cos^2\left(\frac{\pi}{3}\right)+\tan^2\left(\frac{\pi}{4}\right)$

7) Find the value of $4\sec^2 30^0+5\cot^2 30^0+\operatorname{cosec}^2 90^0$

5.7 Signs of trigonometry identities

(a) Functions of negative angles:

Let θ be any angle. Then

$$\sin(-\theta) = -\sin\theta$$
$$\cos(-\theta) = \cos\theta$$
$$\tan(-\theta) = -\tan\theta$$
$$\cot(-\theta) = -\cot\theta$$
$$\sec(-\theta) = \sec\theta$$
$$\operatorname{cosec}(-\theta) = -\operatorname{cosec}\theta$$

(b) Quadrant system

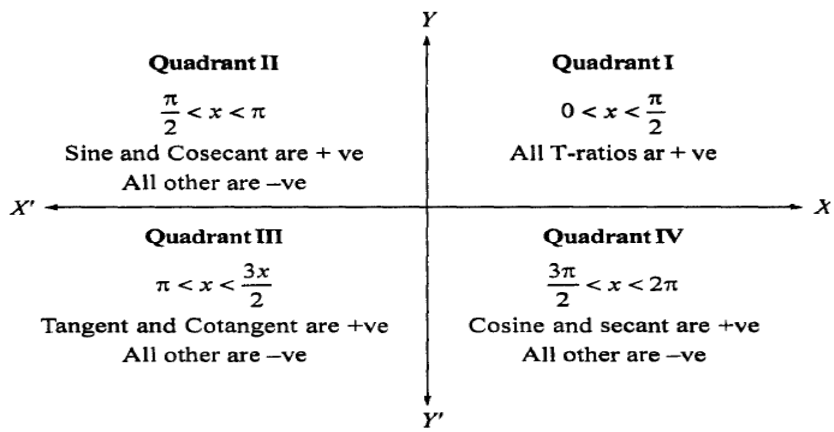


Fig. 5.9

Remarks:

In the last quadrant system, some observations can be discussed:

(a) In the 1st quadrant all the trigonometric ratios are positive and $\theta \in \left[0, \frac{\pi}{2}\right]$

(b) In the 2nd quadrant only $\sin\theta$ and $\operatorname{cosec}\theta$ are positive and rest of the remaining ratios are negative and $\theta \in \left[\frac{\pi}{2}, \pi\right]$

(c) In the 3rd quadrant only $\tan\theta$ and $\cot\theta$ are positive and rest of the remaining ratios are negative and $\theta \in \left[\pi, \frac{3\pi}{2}\right]$

(d) In the 4th quadrant only $\cos\theta$ and $\sec\theta$ are positive and rest of the remaining ratios are negative and $\theta \in \left[\frac{3\pi}{2}, 2\pi\right]$

- **Some important formulas**:

1) When n is an integer then

$$\cos(2n\pi + \theta) = \cos\theta$$
$$\sin(2n\pi + \theta) = \sin\theta$$

2) For all values of θ

$$\cos(-\theta) = \cos\theta$$
$$\sin(-\theta) = \sin\theta$$

3) For all real number x,

$$\cos\left(\frac{\pi}{2} - x\right) = \sin x, \quad \cos\left(\frac{\pi}{2} + x\right) = -\cos x$$
$$\sin\left(\frac{\pi}{2} - x\right) = \cos x, \quad \sin\left(\frac{\pi}{2} + x\right) = \cos x$$

4) For all real number x,

$$\cos(x + 2\pi) = \cos x$$
$$\sin(x + 2\pi) = \sin x$$

For example: (i)

$$\cos(30 + 2\pi) = \cos 30 = \frac{\sqrt{3}}{2}$$
$$\sin(60 + 2\pi) = \sin 60 = \frac{\sqrt{3}}{2}$$

5.8 <u>Trigonometric ratios of compound angles (To be used directly)</u>

1. (a) $\sin(A + B) = \sin A \cos B + \cos A \sin B$

(b) $\sin(A-B)=\sin A\cos B-\cos A\sin B$

(c) $\cos(A+B)=\cos A\cos B-\sin A\sin B$

(d) $\cos(A-B)=\cos A\cos B+\sin A\sin B$

(e) $\tan(A+B)=\dfrac{\tan A+\tan B}{1-\tan A\tan B}$

(f) $\tan(A-B)=\dfrac{\tan A-\tan B}{1+\tan A\tan B}$

(g) $\cot(A+B)=\dfrac{\cot A\cot B-1}{\cot A\cot B}$

(h) $\cot(A-B)=\dfrac{\cot A\cot B+1}{\cot B-\cot A}$

2. (a) $\sin 2A=2\sin A\cos A=\dfrac{2\tan A}{1+\tan^2 A}$

(b) $\cos 2A=\cos^2 A-\sin^2 A=1-2\sin^2 A=2\cos^2 A-1=\dfrac{1-\tan^2 A}{1+\tan^2 A}$

(c) $\tan 2A=\dfrac{2\tan A}{1-\tan^2 A}$

3. (a) $\sin 3A=3\sin A-4\sin^3 A$

(b) $\tan 3A=\dfrac{3\tan A-\tan^2 A}{1-3\tan^2 A}$

(c) $\cos A+\cos B=2\cos\left(\dfrac{A+B}{2}\right)\cos\left(\dfrac{A-B}{2}\right)$

(d) $\cos A-\cos B=2\sin\left(\dfrac{A+B}{2}\right)\sin\left(\dfrac{B-A}{2}\right)$

(e) $\sin A + \sin B = 2\sin\left(\frac{A+B}{2}\right)\cos\left(\frac{A-B}{2}\right)$

(f) $\sin A - \sin B = 2\cos\left(\frac{A+B}{2}\right)\sin\left(\frac{A-B}{2}\right)$

4. (a) $2\sin A\cos B = \sin(A+B) + \sin(A-B)$

(b) $2\cos A\sin B = \sin(A+B) - \sin(A-B)$

(c) $2\cos A\cos B = \cos(A+B) + \cos(A-B)$

(d) $2\sin A\sin B = \cos(A-B) - \cos(A+B)$

SOLVED EXAMPLES

Example 1: - Find the exact value of $\sin 15^{\circ}$

Solution: -Given

$$\sin 15^{\circ} = \sin\left(45^{\circ} - 30^{\circ}\right)$$
$$= \sin 45^{\circ}\cos 30^{\circ} - \cos 45^{\circ}\sin 30^{\circ}$$
$$= \frac{1}{\sqrt{2}}\frac{\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\frac{1}{2}$$
$$= \frac{\sqrt{2}}{2}\frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2}\frac{1}{2}$$
$$= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}$$

Example 2: -Find the exact value of $\tan 75^{0}$

Solution: -Given

$$\tan 75^\circ = \tan\left(45^\circ + 30^\circ\right) = \frac{\tan 45^\circ + \tan 30^\circ}{1 - \tan 45^\circ \tan 30^\circ}$$

$$= \frac{1 + \frac{1}{\sqrt{3}}}{1 - 1\left(\frac{1}{\sqrt{3}}\right)} = \frac{\sqrt{3}+1}{\sqrt{3}-1}$$

$$= \frac{\sqrt{3}+1}{\sqrt{3}-1} \times \frac{\sqrt{3}-1}{\sqrt{3}-1}$$

$$= \frac{2}{2 - 2\sqrt{3}} = \frac{1}{1 - \sqrt{3}}$$

Example 3: -Use the double angle formula to find the exact value of each expression $\sin 120^0$ and $\cos 120^0$

Solution: -Here,

$$\sin 120^\circ = \sin 2\left(60^\circ\right)$$

$$= 2\sin 60^\circ \cos 60^\circ$$

$$= 2\left(\frac{1}{2}\right)\left(\frac{\sqrt{3}}{2}\right) = \frac{\sqrt{3}}{2}$$

Similarly,

$$\cos 120^\circ = \cos 2\left(60^\circ\right) = \cos^2 60^\circ - \sin^2 60^\circ$$

$$= \left(\frac{1}{2}\right)^2 - \left(\frac{\sqrt{3}}{2}\right)^2$$

$$= \frac{1}{4} - \frac{3}{4} = \frac{-2}{4} = -\frac{1}{2}$$

Example 4: -Show that $\sin 75^0 - \sin 15^0 = \cos 105^0 + \cos 15^0$

Solution: -

$$\text{L.H.S}=\sin 75^0 - \sin 15^0$$
$$= 2\cos\left(\frac{75^0+15^0}{2}\right)\sin\left(\frac{75^0-15^0}{2}\right)$$
$$= 2\cos 45^0 \sin 30^0 = 2\left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right) = \frac{1}{\sqrt{2}}$$

$$\text{R.H.S.} = \cos 105^0 + \cos 15^0$$
$$= 2\cos\left(\frac{105^0+15^0}{2}\right)\cos\left(\frac{105^0-15^0}{2}\right)$$
$$= 2\cos 60^0 \cos 45^0 = 2\left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{2}}\right) = \frac{1}{\sqrt{2}}$$

Exercise: -(1) Calculate

(a) $\cos 15^0$ (b) $\cos 75^0$

(2) Find the value of $\cot 75^0$, $\sin 15^0$, $\tan 15^0$

(3) Solve $\sin 75^0 + \cos 75^0 + \sin 15^0 + \cos 15^0$

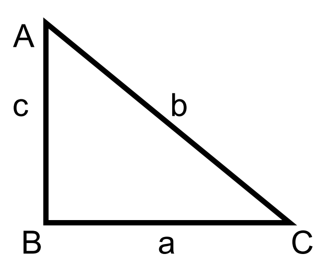
(4) Prove that $\sin 105^0 + \cos 105^0 = \cos 45^0$

5.9 <u>Properties of Triangle and solution of Triangle:</u>

- **<u>Properties of Triangle:-</u>**

(a) The measure of sides AB, BC and CA will be denoted by c, a and b

i.e., $AB = c, BC = a$ and $CA = b.$



(b) The measure of the angles of ΔABC will be denoted by A, B and C

Note that: $A + B + C = 180$

(c) The perimeter of ΔABC will be denoted by $2S$

i.e., $2S = a+b+c$

$$\therefore S = \frac{a+b+c}{2}$$

(d) The circumradius of ΔABC will be denoted by $'R'$

(e) The inradius of ΔABC will be denoted by $'r'$

(f) The area of ΔABC will be denoted by Δ.

- **<u>Solution of Triangles: -</u>**

By using below formulas, we can find the perimeter, inradius, circumference and area of a ΔABC

(a) $S = \dfrac{a+b+c}{2}$

(b) $\Delta = \sqrt{S(S-a)(S-b)(S-c)}$

(c) $\Delta = \dfrac{abc}{4R} \Rightarrow R = \dfrac{abc}{4\Delta}$

(d) $\Delta = rs \Rightarrow r = \dfrac{\Delta}{s}$

SOLVED EXAMPLES

Example 1: - Prove that $\dfrac{1}{ab}+\dfrac{1}{bc}+\dfrac{1}{ca}=\dfrac{1}{2Rr}$

Solution: -

$$\text{L.H.S.} = \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} = \frac{c}{abc} + \frac{a}{abc} + \frac{b}{abc}$$
$$= \frac{a+b+c}{abc}$$
$$= \frac{2S}{4R\Delta} \qquad (2S = a+b+c)$$
$$= \frac{1}{2Rr} \qquad \left(r = \frac{\Delta}{s}\right)$$
$$= \text{R.H.S.}$$

Example 2: - In a ΔABC, $a = 4, b = 5 \text{ and } c = 7$ then find the area of a triangle.

Solution: -Given $a = 4, b = 5 \text{ and } c = 7$

Then $S = \frac{a+b+c}{2} = \frac{4+5+7}{2} = 8$

Now,

$$\Delta = \sqrt{S(S-a)(S-b)(S-c)}$$
$$= \sqrt{8(8-4)(8-5)(8-7)}$$
$$= \sqrt{8(4)(3)(1)}$$
$$= \sqrt{96} = \sqrt{16 \times 6} = 4\sqrt{6}$$

Example 3: -In a ΔABC, $a = 5, b = 6 \text{ and } c = 7$ the find the value of R.

Solution: -Given $a = 5, b = 6 \text{ and } c = 7$

Then $S = \frac{a+b+c}{2} = \frac{5+6+7}{2} = 9$

Now,

$$\Delta = \sqrt{S(S-a)(S-b)(S-c)}$$
$$= \sqrt{9(9-5)(9-6)(9-7)}$$
$$= \sqrt{9(4)(3)(2)}$$
$$= \sqrt{216} = \sqrt{36\times 6} = 6\sqrt{6}$$

Therefore, $$R = \frac{abc}{4\Delta} = \frac{(5)(6)(7)}{4\left(6\sqrt{6}\right)} = \frac{35}{4\sqrt{6}}$$

Example 4: -In a ΔABC, $a = 2, b = 3 \text{ and } c = 3$ the find the value of r.

Solution: - Given $a = 2, b = 3 \text{ and } c = 3$

Then $S = \frac{a+b+c}{2} = \frac{2+3+3}{2} = 4$

Now,

$$\Delta = \sqrt{S(S-a)(S-b)(S-c)}$$
$$= \sqrt{4(4-2)(4-3)(4-3)}$$
$$= \sqrt{4(2)(1)(1)}$$
$$= \sqrt{8} = \sqrt{4\times 2} = 2\sqrt{2}$$

Therefore, $$r = \frac{\Delta}{s} = \frac{2\sqrt{2}}{4} = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

Exercise: - 1) In a ΔABC, $a = 1, b = 3 \text{ and } c = 4$ the find the area of a triangle, $r \text{ and } R$

2) In a ΔABC, $a = 5, b = 8 \text{ and } c = 5$ the find the area of a triangle, $r \text{ and } R$.

5.10 <u>APPLICATIONS OF TRIGONOMETRY</u>

- Main use of trigonometry is in Construction or else this field of mathematics can be applied in astronomy, navigation, acoustics medical imaging, civil engineering, seismology, electrical engineering phonetics, chemistry, number theory and many more.
- Also It is used in oceanography in calculating the height of   tides in oceans.
- The sine and cosine functions are fundamental to the theory of periodic functions, those that describe the sound and light waves. Calculus is made up of Trigonometry and Algebra.

Trigonometry can be used to roof a house, to make the roof inclined

(in the case of single individual bungalows) and the height of the roof in buildings etc.and it used in cartography (creation of maps).

Real life example

- We can find height of the mountain or hill using trigonometry.

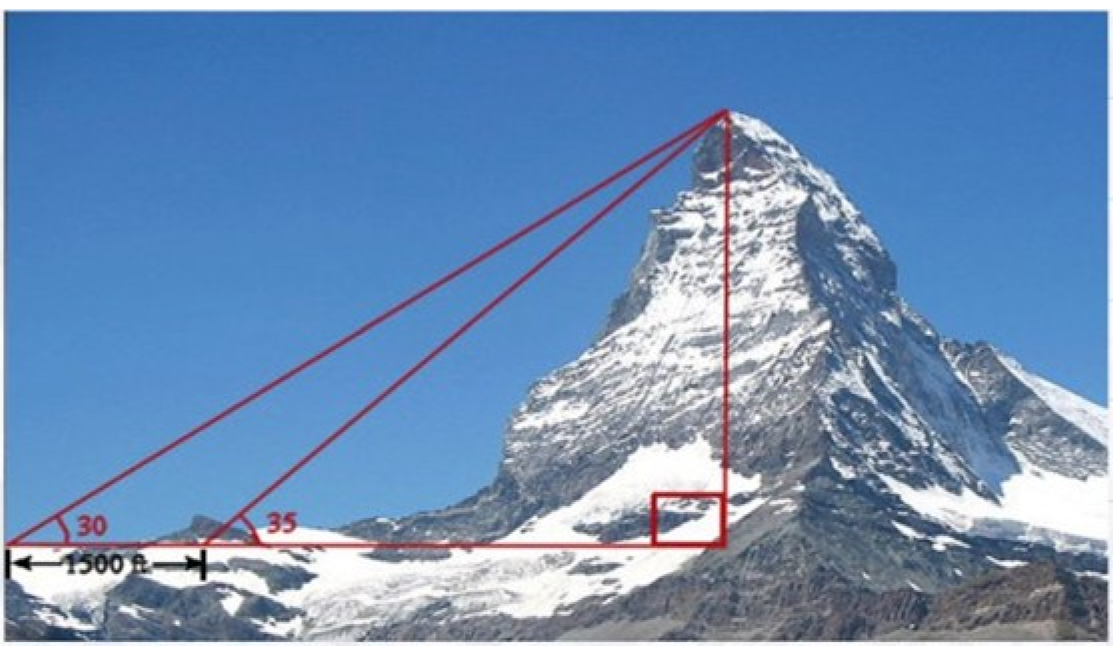


Fig. 5.10

- Primitive forms of trigonometry were used in construction of the wonder of world "GIZA" Pyramid.

Fig. 5.11