PARUL UNIVERSITY
FACULTY OF ENGINEERING AND TECHNOLOGY

DEPARTMENT OF APPLIED SCIENCE AND HUMANITIES

SUBJECT: BASIC MATHEMATICS

Chapter- 4 - Permutation and Combination

4.1 <u>INTRODUCTION :</u> Permutations refer to different arrangements of things from a given lot taken one or more at a time whereas combinations refer to different sets or groups made out of a given lot, without repeating an element, taking one or more of them at a time.The distinction will be clear from the following illustration of combinations and permutations made out of a set of three elements {a,b,c}.

Combinations	Permutations
(i) one at a time :{a},{b},{c}	{a},{b},{c}
(ii) two at a time : {a,b} {b,c} {a,c}	{a,b} {b,a}
	{b,c} {c,b}
	{a,c} {c,a}

FACTORIAL REPRESENTATION

$$10! = 1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10 = ¿$$
$$10! = 9! \times 10 = 5! \times 6 \times 7 \times 8 \times 9 \times 10$$

Factorial notation is denoted by !. The Product of the first n natural numbers 1,2,3,……,n is called n factorial and it is written as n!.

i.e. $n! = 1 \times 2 \times 3 \ldots\ldots(n-2)(n-1)n$

For example $5! = 5 \times 4 \times 3 \times 2 \times 1$

Note 0! = 1

Also, we can write

5! = 5 × 4! = 5 × 4 × 3! = 5 × 4 × 3 × 2! = 5 × 4 × 3 × 2 × 1!

SOLVED EXAMPLES

Example 1 : Find the value of 7!.

Solution: $7! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 = 5040$

Example 2: Evaluate $\frac{8!}{3! \cdot 5!}$

$$\frac{8!}{3!.5!} = \frac{5! \times 6 \times 7 \times 8}{3!.5!} = 56$$

$$¿\frac{40320}{6 \times 120} = 56$$

Solution: $\frac{8!}{3! \cdot 5!} = \frac{8 \cdot 7 \cdot 6 \cdot 5!}{(3 \cdot 2 \cdot 1)(5!)}$

$= 8 \cdot 7$

$= 56$

Example 3: show that $\frac{10!}{8!} = 90$

$$\frac{10!}{8!} = \frac{8! \times 9 \times 10}{8!} = 90$$

Solution: $10! = 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

And $8! = 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$

$$\frac{10!}{8!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}{8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1}$$

$\Rightarrow$

$= 90$

Example 4 If $\frac{1}{8!}+\frac{1}{9!}=\frac{x}{10!}, \text{ find } x$

Solution :

We have

$$\frac{1}{8!}+\frac{1}{9\times 8!}=\frac{x}{10\times 9\times 8!}$$

$$\frac{1}{8!}\left(1+\frac{1}{9}\right)=\frac{x}{8!\times 9\times 10}$$

$$\therefore \frac{10}{9}=\frac{x}{90}$$

$$\therefore \frac{10\times 90}{9}=x$$

$$\therefore x=100$$

Example 1: Evaluate $\frac{7!}{5!}$

Example 2: Evaluate $\frac{n!}{r!(n-r)!}, \text{ when } n=5, r=2.$

$$\frac{5!}{2!\times 3!}=\frac{3!\times 4\times 5}{2\times 3!}=10$$

{A,B,C,D,E} = {A,B},{B,C},{C,D},{D,E},{A,E},{A,C},{A,D},{B,D},{B,E},{C,E}

$$Combination = n_C r = \frac{n!}{r!\times(n-r)!} \quad PERMUTATION = n_P r = \frac{n!}{(n-r)!}$$

$$¿ \text{out of } n \text{ total objects at a time selection of } r \text{ objects (order doesnt matter)}$$

$$\frac{5!}{2! \times 3!} = \frac{3! \times 4 \times 5}{2 \times 3!} = 10$$

$$¿ \text{it means } (a, b) \wedge (b, a) \text{ will be considered as a same case}$$

$$Permutation = n_P r = \frac{n!}{(n-r)!}$$

$$¿ \text{out of } n \text{ total objects at a time selection of } r \text{ objects (order matters)}$$

$$¿ \text{it means } (a, b) \wedge (b, a) \text{ will be considered as different cases}$$

Example:

Find the probability of 53 Sundays in a leap year

LEAP YEAR = 366 DAYS =52 WEEKS 2 DAYS

olution: total number of days in a leap year =366

52 complete weeks=364 days

2 days =

Monday, Tuesday

Tuesday, Wednesday

Wednesday, Thursday

Thursday, Friday

Friday, Saturday

Saturday, Sunday

Sunday, Monday

FUNDAMENTAL COUNTING PRINCIPLE

Fundamental Counting Principle can be used to determine the number of possible outcomes when there are two or more characteristics.

Principal of Multiplication: ***"If an event can occur in m different ways, following which another event can occur in n different ways, then the total number of occurrence of the events in the given order is m×n."***

Let us consider the following problem. Mohan has 3 pants and 2 shirts. How many different pairs of a pant and a shirt, can he dress up with? There are 3 ways in which a pant can be chosen, because there are 3 pants available. Similarly, a shirt can be chosen in 2 ways. For every choice of a pant, there are 2 choices of a shirt. Therefore, there are $3 \times 2 = 6$ pairs of a pant and a shirt. Let us name the three pants as P_1, P_2 , P_3 and the two shirts as S_1, S_2. Then, these six possibilities can be illustrated in the Fig.1.

Let us consider another problem of the same type. Sabnam has 2 school bags, 3 tiffin boxes and 2 water bottles. In how many ways can she carry these items (choosing one each).

A school bag can be chosen in 2 different ways. After a school bag is chosen, a tiffin box can be chosen in 3 different ways. Hence, there are 2 × 3 = 6 pairs of school bag and a tiffin box. For each of these pairs a water bottle can be chosen in 2 different ways. Hence, there are 6 × 2 = 12 different ways in which, Sabnam can carry these items to school. If we name the 2 school bags as B1, B2, the three tiffin boxes as T1, T2, T3 and the two water bottles as W1, W2, these possibilities can be illustrated in the Fig. 2.

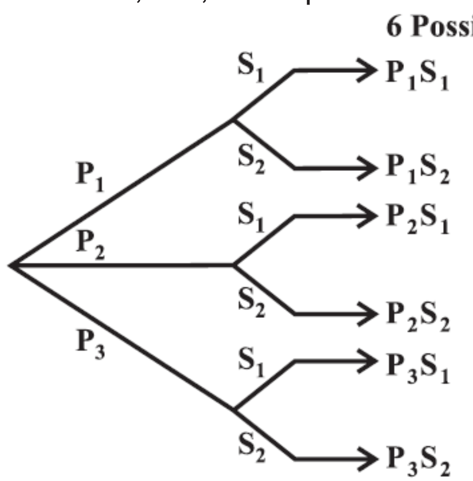


Fig 1

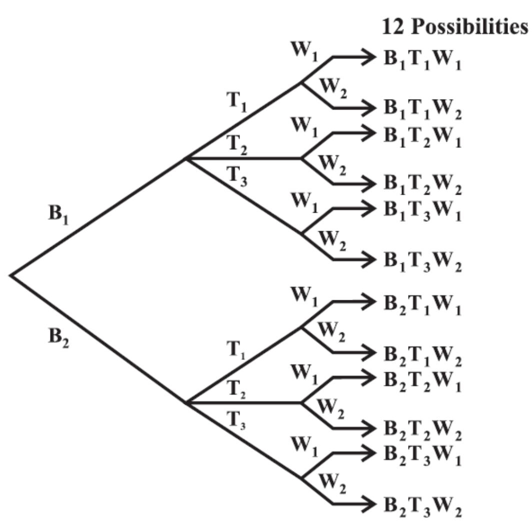


Fig 2

In fact, the problems of the above types are solved by applying the following principle known as the ***fundamental principle of counting*,** or, simply, the ***multiplication principle***, which states that

"If an event can occur in m different ways, following which another event can occur in n different ways, then the total number of occurrence of the events in the given order is m×n."

The above principle can be generalized for any finite number of events. For example, for 3 events, the principle is as follows:

'If an event can occur in m different ways, following which another event can occur in n different ways, following which a third event can occur in p different ways, then the total number of occurrence to 'the events in the given order is $m \times n \times p$."

In the first problem, the required number of ways of wearing a pant and a shirt was the number of different ways of the occurrence of the following events in succession:

(i) the event of choosing a pant

(ii) the event of choosing a shirt.

In the second problem, the required number of ways was the number of different ways of the occurrence of the following events in succession:

(i) the event of choosing a school bag

(ii) the event of choosing a tiffin box

(iii) the event of choosing a water bottle.

Here, in both the cases, the events in each problem could occur in various possible orders. But, we have to choose any one of the possible orders and count the number of different ways of the occurrence of the events in this chosen order.

SOLVED EXAMPLES

Example 1 Find the number of 4 letter words, with or without meaning, which can be formed out of the letters of the word ROSE, where the repetition of the letters is not allowed.

Solution There are as many words as there are ways of filling in 4 vacant places by the 4 letters, keeping in mind that the repetition is not allowed. The first place can be filled in 4 different ways by anyone of the 4 letters R,O,S,E. Following which, the second place can be filled in by anyone of the remaining 3 letters in 3 different ways, following which the third place can be filled in 2 different ways; following which, the fourth place can be filled in 1 way. Thus, the number of ways in which the

4 places can be filled, by the multiplication principle, is 4 × 3 × 2 × 1 = 24. Hence, the required number of words is 24.

NOTE :- If the repetition of the letters was allowed, how many words can be formed? One can easily understand that each of the 4 vacant places can be filled in succession in 4 different ways. Hence, the required number of words = 4 × 4 × 4 × 4 = 256.

Example 2 : Three persons go into a railway carriage , where there are 8 seats. In how many ways can they seat themselves?
Solution : Since there are 8 vacant seats , the first men can choose any one of these 8 seats. There are thus 8 ways of filling the first seat, when that one is occupied 7 seats are left , therefore the second men can occupy any one of the 7 seats. The last men can now seat himself in one of the remaining 6 seats.
Therefore, number of ways in which three person can occupy 8 seats is
8×7×6=336

Example 3 How many 2 digit even numbers can be formed from the digits 1, 2, 3, 4, 5 if the digits can be repeated?

Solution There will be as many ways as there are ways of filling 2 vacant places $\boxed{}\ \boxed{}$ in succession by the five given digits. Here, in this case, we start filling in unit's place, because the options for this place are 2 and 4 only and this can be done in 2 ways; following which the ten's place can be filled by any of the 5 digits in 5 different ways as the digits can be repeated. Therefore, by the multiplication principle, the required number of two digits even numbers is 2 × 5, i.e., 10.

Example: In how many ways can a chairman and a vice-chairman of a board of 6 members can occupy their seats?

PERMUTATIONS VS. COMBINATIONS

Permutation and combination both are ways to count the possibilities .The difference between them is whether order matters or not.

PERMUTATIONS

In Example 1 of the previous Section, we are actually counting the different possible arrangements of the letters such as ROSE, REOS, ..., etc. Here, in this list, each arrangement is different from other. In other words, the order of writing the letters is important. Each arrangement is called a ***permutation of* 4 *different letters taken all at a time*.** Now, if we have to determine the number of 3-letter words, with or without meaning, which can be formed out of the letters of the word NUMBER, where the repetition of the letters is not allowed, we need to count the arrangements NUM,NMU, MUN, NUB, ..., etc. Here, we are counting the permutations of 6 different letters taken 3 at a time. The required number of words = 6 × 5 × 4 = 120 (by using multiplication principle). If the repetition of the letters was allowed, the required number of words would be 6 × 6 × 6 = 216.

Permutations : A permutation is an arrangement of objects in a definite order.
Note : order matters in permutations
To find the number of Permutations of n items, we can use the Fundamental Counting Principle or factorial notation.

Permutation of n different things ,taken r at a time is denoted by the symbol
$p(n,r)$ or ${}_nP_r$ or ${}^nP_r$.

Theorem 1: To find the number of Permutations of n items chosen r at a time, you can use the formula (**When repetition is not allowed):**

$$ {}_np_r = \frac{n!}{(n-r)!} \quad \text{where } 0 \le r \le n . $$

where n = number of objects
r = number of positions

For example : $$ {}_5p_3 = \frac{5!}{(5-3)!} = \frac{5!}{2!} = 5*4*3 = 60 $$

<u>SOLVED EXAMPLES</u>

Example 1 : From a club of 24 members, a President, Vice President, Secretary, Treasurer and Historian are to be elected. In how many ways can the offices be filled?

Solution :There are 24 members. Therefore, n=24.and r=4.
Hence, The Office will be filled in following ways:

$$_{24}p_5 = \frac{24!}{(24-5)!} = \frac{24!}{19!}$$
$$= 24 * 23 * 22 * 21 * 20$$
$= 5,100,480$ ways

$$¿\frac{24!}{19!} = \frac{19! \times 20 \times 21 \times 22 \times 23 \times 24}{19!}$$

NOTE: The number of permutations of *n objects* taken all at a time, denoted by the symbol ^nP_n and *is given by* $^nP_n = n!$

Theorem 2 : When repetition of objects is allowed :The number of permutations of n things taken all at a time, when repetition of objects is allowed is n^n .
And the number of permutations of *n objects, taken r at a time, when repetition* ofobjects is allowed, is n^r .

For example : The number of 3-letter words which can be formed by the letters of the word NUMBER = $^nP_r = \frac{6!}{3!} =$ 4 × 5 × 6 = 120. Here, in this case also, the repetition is not allowed. If the repetition is allowed, the required number of words would be $6 \times 6 \times 6 = ¿$ $6^3 = 216$.

<u>**SOLVED EXAMPLES**</u>

Example 1: How Many new Permutations of all letters of the word TUESDAY are Possible?
Solution: The Number of arrangements is $^7P_7 = 7!$
Now, 7! = 7×6×5×4×3×2×1= 5040
Hence, 5040 arrangements exist.

Example 2 : How many 3 letter words with or without meaning can be formed by word NUTS when repetition is allowed?
Solution: Here: n= 4 (no of letters we can choose from)
r = 3 (no of letters in the required word)
Thus by Theorem 2: $n^r = 4^3 = 64$
Thus 64 words are possible.

Permutations when all the objects are not distinct objects

Suppose we have to find the number of ways of rearranging the letters of the word ROOT.
In this case, the letters of the word are not all different. There are 2 Os, which are of the same kind.
Let us treat, temporarily, the 2 Os as different, say, O1 and O2 .The number of permutations of 4-different letters, in this case, taken all at a time is 4!. Consider one of these permutations say, RO1O2T☜ Corresponding to this permutation, we have 2 ! permutations RO1O2T and RO2O1T which will be exactly the same permutation if O1 and O2 are not treated as different, i.e., if O1 and O2 are the same O at both places.

Therefore, the required number of permutations $^4P_2 = \frac{4!}{2!} = 4\times 3 = 12.$

Permutations when O1, O2 are when O1, O2 are different. | Permutations When O are same

$$\left.\begin{matrix} RO_1O_2T \\ RO_2O_1T \end{matrix}\right] \longrightarrow ROOT$$

$$\left.\begin{matrix} TO_1O_2R \\ TO_2O_1R \end{matrix}\right] \longrightarrow TOOR$$

$$\left.\begin{matrix} R O_1 T O_2 \\ R O_2 T O_1 \end{matrix}\right] \longrightarrow R\,O\,T\,O$$

$$\left.\begin{matrix} T O_1 R O_2 \\ T O_2 R O_1 \end{matrix}\right] \longrightarrow T\,O\,R\,O$$

$$\left.\begin{matrix} R T O_1 O_2 \\ R T O_2 O_1 \end{matrix}\right] \longrightarrow R\,T\,O\,O$$

$$\left.\begin{matrix} T R O_1 O_2 \\ T R O_2 O_1 \end{matrix}\right] \longrightarrow T\,R\,O\,O$$

$$\left.\begin{matrix} O_1\ O_2\ R T \\ O_2\ O_1\ T R \end{matrix}\right] \longrightarrow O\,O\,R\,T$$

$$\left.\begin{matrix} O_1 R O_2 T \\ O_2 R O_1 T \end{matrix}\right] \longrightarrow O\,R\,O\,T$$

$$\left.\begin{matrix} O_1\ T O_2\ R \\ O_2\ T O_1\ R \end{matrix}\right] \longrightarrow O\,T\,O\,R$$

$$\left.\begin{matrix} O_1\ R T O_2 \\ O_2\ R T O_1 \end{matrix}\right] \longrightarrow O\,R\,T\,O$$

$$\left.\begin{matrix} O_1\ T R O_2 \\ O_2\ T R O_1 \end{matrix}\right] \longrightarrow O\,T\,R\,O$$

$$\left.\begin{matrix} O_1\ O_2 T R \\ O_2\ O_1 T R \end{matrix}\right] \longrightarrow O\,O\,T\,R$$

Theorem 3 : The number of permutations of n objects or things of which p things are of one kind,q things are of second kind , r things are of third kind and all the rest are different is given by

$$\frac{n!}{p!\times q!\times r!}$$

SOLVED EXAMPLES

Example 1 : Find number of permutations of word ALLAHABAD.
Solution:Here total number of word (n) = 9
Number of repeated A's =p1= 4

Number of repeated L's =p2= 2
Rest all letters are different.
Thus applying theorem 3, we have:

$$\frac{n!}{p_1!p_2!}$$

$$= \frac{9!}{4!2!}$$

= 7560 Ways

Example 2 ; Find the number of words, with or without meaning, that can be formed with the letters of the word 'SWIMMING?

Solution:
The word 'SWIMMING contains 8 letters. Of which, I occurs twice and M occurs twice.

Therefore, the number of words formed by this word = 8! / (2!*2!) = 10080.

$$\frac{8!}{2!\times 2!} = \frac{2!\times 3\times 4\times 5\times 6\times 7\times 8}{2\times 2} = ¿$$

Example : Find the number of words, with or without meaning, that can be formed with the letters of the word 'INDIA'.

Example 3 : How many numbers greater than a million can be formed with the digits 4,5,5,0,4,5,3?

--million -→ 4554530 -→ 420
-- number starting with zero → 455453

Solution : Each number must consist of 7 or more digits. There are 7digits in all of which there are 2 four , 3 fives and the rest are different.

Therefore, the total numbers are $\frac{7!}{2!3!}$ = 420

Of these numbers , some begin with zero and are less than one million which must be rejected.

The number beginning with zero are $\frac{6!}{2!3!}$ = 60

Therefore, the required numbers are 420 − 60=360

NOTE : $^{n}P_{r} =$ n (n-1) (n-2) (n-3) ……. (n-r+1)

$$n_P r = \frac{n!}{(n-r)!} = \frac{(n-r)! \times (n-r+1) \times (n-r+2) \ldots (n-2)(n-1)n}{(n-r)!}$$
$$¿ n(n-1)(n-2)(n-3)\ldots(n-r+1)$$

Example Find the value of *n* such that

$(i)\ {}^{n}P_{5} = 42\,{}^{n}P_{3}, n > 4$ $\qquad (ii)\ \frac{{}^{n}P_{4}}{{}^{n-1}P_{4}} = \frac{5}{3}, n > 4$

Solution (i) Given that ${}^{n}P_{5} = 42\,{}^{n}P_{3}$

$$n_p 5 = n(n-1)(n-2)(n-3)\ldots(n-5+1)$$
$$¿ n(n-1)(n-2)(n-3)(n-4)$$
$$n_p 3 = n(n-1)(n-2)$$
$$now, n_p 5 = 42 n_p 3$$
$$\therefore n(n-1)(n-2)(n-3)(n-4) = 42 n(n-1)(n-2)$$
$$n^2 - 4n - 3n + 12 = 42$$

$$n^2 - 7n + 12 - 42 = 0$$
$$n^2 - 7n - 30 = 0$$
$$n^2 - 10n + 3n - 30 = 0$$
$$n(n-10) + 3(n-10) = 0$$
$$(n+3)(n-10) = 0$$
$$n+3 = 0 \vee n-10 = 0$$
$$n = -3 \vee n = 10$$

(ii) Given that $\frac{^{n}P_4}{^{n-1}P_4} = \frac{5}{3}$

Therefore,

$$3n(n-1)(n-2)(n-3) = 5(n-1)(n-2)(n-3)(n-4)$$
$$3n = 5(n-4) \qquad [as\,(n-1)(n-2)(n-3) \neq 0, n > 4]$$
$$n = 10.$$

RESTRICTED PERMUTATIONS

(i) Number of Permutations of n different things taken all at a time, when m specified things always come together is m! × (n-m+1)!

(II) Number of Permutations of n different things taken all at a time, when m specified things always never come together is n! - m! × (n-m+1)!

MIXED EXAMPLES

Example 1: In how many different ways can the letters of the word THERAPY be arranged so that the vowels never come together.

Solution :Here n= 7 and m=2.

So, using special case n! - m! × (n-m+1)!

= 7! - 2! × (7-2+1)!

= 7! – (2! × (7-2+1)!)

= 7! - (2! × (6!))

= 7! - (2 × 720)

= 7! - 1440 = 5040 - 1440= 3600 ways

Example 2 : How many different words can be formed with the letters of the word 'SUPER' such that the vowels always come together?

Solution:

2! × (5-2+1)!
$¿ 2 \times 4! = 24 \times 2 = 48$

The word ‘SUPER’ contains 5 letters.

In order to find the number of permutations that can be formed where the two vowels U and E come together.

In these cases, we group the letters that should come together and consider that group as one letter.

So, the letters are S,P,R, (UE). Now the number of words are 4.

Therefore, the number of ways in which 4 letters can be arranged is 4!

In U and E, the number of ways in which U and E can be arranged is 2!

Hence, the total number of ways in which the letters of the ‘SUPER’ can be arranged such that vowels are always together are

4! * 2! = 48 ways.

Example 3 : In how many ways can 5 children be arranged in a line such that (i) two particular children of them are always together (ii) two particular children of them are never together.
Solution:(i)
We consider the arrangements by taking 2 particular children together as one and hence the remaining 4 can be arranged in 4! = 24 ways.
Again two particular children taken together can be arranged in two ways.
Therefore, there are 24 × 2 = 48 total ways of arrangement.
or m! × (n-m+1)! = 2! × (5-2+1)! = 2! × 4! = 48

(ii) two particular children of them are
never together. n=5; m=2
n! - m! × (n-m+1)!$¿ 5! - (2! \times 4!) = 120 - 48 = 72$

ii) Among the 5! = 120 permutations of 5 children, there are 48 in which two children are together. In the remaining 120 – 48 = 72 permutations, two particular children are never together.
or n! - m! × (n-m+1)! = 5!-48= 120 – 48 = 72

Example 4 : In How many ways can the letters of word " ARRANGE" be arranged so that two 'R' do not come together? And The two Rs and the two As come together ?
Solution : Here n=7 . And in word ARRANGE , letters A and R both are repeated 2 times. So,

$$\frac{7!}{2!2!} = \frac{7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{2\cdot1\cdot2\cdot1}$$
$$= 1260$$

The number of arrangement in which the Rs come together can be obtained by treating the two Rs as one letter. Thus , there are 6 letters of which two (the two As) are similar and so the total number of arrangements =

$$\frac{6!}{2!} = \frac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{2\cdot1}$$
$$= 360$$

So, number of ways In which the two letters do not come together is
= 1260 − 360 = 900.

(ii) The two Rs and the two As come together ?
The number of arrangements in which the two Rs and the two As come together can be obtained by treating the two Rs and the two As as a single letters. Thus , there are 5 letters which are all different and so the number of arrangements is 5 ! = 120

Example : Find the number of permutations of the letters of the word 'REMAINS' such that the vowels always occur in odd places.

<u>Solution</u>:
The word 'REMAINS' has 7 letters.

There are 4 consonants and 3 vowels in it.

Writing in the following way makes it easier to solve these type of questions.

(1) (2) (3) (4) (5) (6) (7)

No. of ways 3 vowels can occur in 4 different places = ${}^4P_3$ = 24 ways.

After 3 vowels take 3 places, no. of ways 4 consonants can take 4 places = ${}^4P_4$ = 4! = 24 ways.

Therefore, total number of permutations possible = 24*24 = 576 ways.

Example 5 : In how many ways can 4 red, 3 yellow and 2 green discs be arranged in a row if the discs of the same color are similar?
Solution : Total number of discs are 4+3+2=9.Out of 9 discs , 4 are of the first kind (red),3 are of the second kind (yellow) and 2 are of the third kind(green).

Thus, number of permutation is ;

$$= \frac{9!}{4!\cdot 3!\cdot 2!}$$
$$= \frac{9\cdot 8\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2\cdot 1}{4\cdot 3\cdot 2\cdot 1\cdot 3\cdot 2\cdot 1\cdot 2\cdot 1}$$
$= 1260$ ways

Example 6 : In how many ways can 5 boys and 4 girls be arranged on a bench if
a)there are no restrictions?
Solution: Here , n=9.Then the arrangement of boys and girls (if there is no restriction) is : 9! or ${}^9p_9$

b) boys and girls alternate?
Solution : A boy will be on each end. i.e. BGBGBGBGB = 5 × 4 × 4 × 3 × 3 × 2 × 2 × 1 ×1 = 5! × 4! or ${}^5p_5 \times {}^4p_4$ = 2880

c) boys and girls are in separate groups?
Solution: The arrangement should be in following way:

Boys & Girls or Girls& Boys
= (5! × 4!)+(4! × 5!) = (5! × 4!) × 2 or $^5p_5 \times {}^4p_4 \times 2$
= 5760

Example 7 : In how many ways can the letters of the word STRANGE be arranged so that (i) the vowels are never separated.
(ii) the vowels occupy only the odd places.

Solution: (i) there are 7 letters. Since the vowels are not to be separated we may regard them as forming one letter.so there are six letters S,T,R,A,N,G and E. They can be arrange among themselves in 6! Ways.The two Vowels can again be arrange in 2! ways.

$\therefore$The total number of arrangements = 6! $\times$2! = 1440.

(ii) Since the number of letters in the word STRANGE are 7, the total number of places are 7, and the number of odd places are 4(1,3,5,7).The two vowel A and E are to occupy two of these four odd places which they can occupy in $^4P_2$ ways.

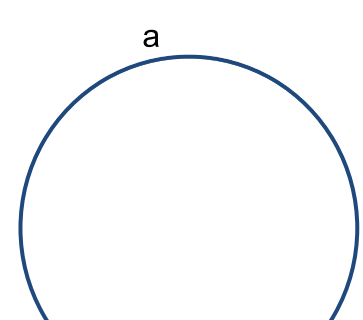
When the vowels have been placed in one way,there remain five places to be filled up by the remaining 5 consonants which can be done in $^5P_5$ ways of arranging the consonants.

The total number of arrangements = $^4P_2 \times {}^5P_5$
= 12 $\times$120 =1440

4.6 CIRCULAR PERMUTATIONS

Circular permutations in which objects are arranged in a **circle**.

Consider arranging 5 objects (a, b, c, d, e) around a circular table. The arrangements are abcde bcdea cdeab deabc eabcd different in a line, but are identical around a circle.



e b
c

d c

- To calculate the number of ways in which n objects can be arranged in a circle, we arbitrarily fix the position of one object, so the remaining (n-1) objects can be arranged as if they were on a straight line in (n-1)! ways.

  i.e. the number of arrangements = (n – 1) ! in a circle.

<u>SOLVED EXAMPLES</u>

Example 1 : At a dinner party 6 men and 6 women sit at a round table. In how many ways can they sit if:

a) there are no restrictions

Solution : (12 – 1)! = 11!

b) men and women alternate

Solution : (6 – 1)! × 6! = 5! × 6!

C) Ted and Carol must sit together

Solution : (TC) × other 10 = 2! × 10!

Example 2 : In how many ways 5 boys and 5 girls be seated around a table if there are no restrictions ?

REMARK : Number of arrangements of n beads for forming a necklace $= \frac{(n-1)!}{2}$.
(In case of necklace, anticlockwise and clockwise arrangements are same.)

Example 3 : In how many ways can 8 differently coloured beads be threaded on a strings?

Solution : As necklace can be turned over, clockwise and anti-clockwise arrangements are the same $=\frac{(n-1)!}{2}$.

$=\frac{(8-1)!}{2}$

$=\frac{7!}{2}$

$=2520$

4.7 COMBINATIONS

A combination is an arrangement of items in which order does not matter.

NOTICE: order does not matter!

Since the order does not matter in combinations, there are fewer combinations than Permutations. The combinations are a “subset” of the permutations.

To find the number of combinations of n items chosen r at a time, you can use the formula; ${}^{n}C_{r}=\frac{n!}{r!(n-r)!}$.

An alternative (and more common) way to denote an r-combination is $\binom{n}{r}$ and $\binom{n}{r}=\frac{n!}{r!(n-r)!}$.

Solved Examples

Example 1 : Find the value of $\binom{6}{4}$.

Solution : Here, n=6 and r=4.

${}^{6}C_{4}=\binom{6}{4}=\frac{6!}{4!(6-4)!}$

$=\frac{6\cdot5\cdot4\cdot3\cdot2\cdot1}{4\cdot3\cdot2\cdot1\cdot2\cdot1}$

$=15$

Example 2 : A student must answer 3 out of 6 questions on a test.In how many different ways can student select the questions?

Solution : The required number of ways

$${}^{6}C_{3} = \frac{6!}{3!(6-3)!}$$

¿ $\frac{6!}{3!(3)!}$

¿ $\frac{6.5.4.3!}{3!.3!}$

= 20

Example 3 : In How many ways 4 white and 3 black balls be selected from a box containing 20 white and 15 black balls.

Solution : 4 out of 20 white balls can be selected in

$${}^{20}C_{4} = \frac{20!}{4!(20-4)!}$$

$= 4845$ ways

And 3 out of 15 black balls can be selected in

$${}^{15}C_{3} = \frac{15!}{3!(15-3)!}$$

$= 455$ ways

Therefore, the two processes can be carried out together so in $4845 \times 455 = 22,04,475$ ways.

Example 4 : From 6 boys and 4 girls , 5 are to be selected for admissions for a particular course. In how many ways can't this be done if there must be exactly 2 girls ?

Solution : There has to be exactly 2 girls and there should be 3 boys , the possible combination would be

$${}^{4}C_{2} \times {}^{6}C_{3} = \frac{4 \cdot 3}{2 \cdot 1} \times \frac{6 \cdot 5 \cdot 4}{3 \cdot 2 \cdot 1}$$

$= 6 \times 20$

$= 120$ ways

Example 5 :A party of 3 ladies and 4 gentlemen is to be formed from 8 ladies and 7 gentlemen. In howmany different ways can the party be formed if Mrs.X and Mr. Y refuse to join the same party?

Solution : 3 ladies can be selected out of 8 ladies in $^8C_3$ ways and 4 gentlemen can be selected out of 7 gentlemen in $^7C_4$ ways.

The numberof ways of choosing the committee

$= {}^8C_3 \times {}^7C_4 = \frac{8!}{3!5!} \times \frac{7!}{4!3!}$ = 1960 ways

If both Mrs. X and Mr.Y are members,there remain to be selected 2 ladies from 7 ladies and 3gentlemen from 6 gentlemen.This can be done in

$= {}^7C_2 \times {}^6C_3 = \frac{7!}{2!4!} \times \frac{6!}{3!3!}$ = 420 ways

The number of ways of forming the party in which Mrs.X and Mr.Y refuse to join is

= 1960-420

= 1540

Example : Suppose we have 4 different objects A, B, C and D. Taking 2 at a time, if we have to make combinations, these will be AB, AC, AD, BC, BD, CD. Here, AB and BA are the same combination as order does not alter the combination. This is why we have not included BA, CA, DA, CB, DB and DC in this list. There are as many as 6 combinations of 4 different objects taken 2 at a time, i.e., $^4C_2 = 6.$

Corresponding to each combination in the list, we can arrive at 2! permutations as 2 objects in each combination can be rearranged in 2! ways. Hence, the number of permutations = $^4C_2 \times 2!$

On the other hand, the number of permutations of 4 different things taken 2 at a time = $^4P_2$.Therefore $^4P_2$ = $^4C_2 \times 2!$ *or* $\frac{4!}{2!(4-2)!} = {}^4C_2$

Now, let us suppose that we have 5 different objects A, B, C, D, E. Taking 3 at a time, if we have to make combinations, these will be ABC, ABD, ABE, BCD, BCE, CDE, ACE, ACD, ADE, BDE.

Corresponding to each of these $^5C_3$ combinations, there are 3! permutations, because, the three objects in each combination can be rearranged in 3 ! ways. Therefore, the total of permutations = $^5C_3 \times 3!$

Therefore $^5P_3 = {^5C_3} \times 3!$ *or* $\frac{5!}{3!(5-3)!} = {^5C_3}$

These examples suggest the following theorem showing relationship between permutation and combination:

Theorem $^nP_r = {^nC_r}\, r!, 0 < r \leq n.$

COMPLEMENTARY THEOREM

The number of combinations of n different things taken r at a time, is same as the number of combinations of n different things taken (n-r) at a time.
$^nC_r = {^nC_{n-r}}$,where $0 \leq r \leq n$

Corollary : If in the formula $^nC_{n-r} = {^nC_r}$,we put
(i)r=n,then $^nC_0 = {^nC_n} = 1$
(ii) r=n-1 ,then $^nC_1 = {^nC_{n-1}} = n$, etc.

SOLVED EXAMPLES

Example 1 : Find the value of r if $^{18}C_r = {^{18}C_{r+2}}$
Solution : Since $^nC_r = {^nC_{n-r}}$, we have $^{18}C_r = {^{18}C_{18-r}}$
But, we are given $^{18}C_{18-r} = {^{18}C_{r+2}}$
$\Rightarrow$ 18-r=r+2
$\Rightarrow$ 18-2=r+r
$\Rightarrow$ 16=2r
$\Rightarrow$ r = 8

Example 2 : Find n, for $^nC_6 : {^{n-3}C_3} = 91:4$.

Solution :we know that

$^nC_6 = \frac{n!}{6!(n-6)!}$

and ${}^{n-3}C_3$ = $\frac{(n-3)!}{3!(n-3-3)!}$

$$\Rightarrow \frac{{}^{n}C_6}{{}^{n-3}C_3} = \frac{n!}{6!(n-6)!} \times \frac{3!(n-6)!}{(n-3)!}$$

$$= \frac{n!}{(n-3)!} \times \frac{1}{6.5.4}$$

$$= \frac{n(n-1)(n-2)}{6.5.4}$$

Also, it is given $\frac{n(n-1)(n-2)}{6.5.4} = \frac{91}{4}$

⟹ n(n-1)(n-2)=5.6.91=5.6.7.13 = 15.14.13

Expressing the three consecutive integers in descending order,
we get n=15.

APPLICATIONS

Application are of combinatorics

Communication networks, cryptography and network security Permutations are frequently used in communication networks and parallel and distributed systems. Routing different permutations on a network for performance evaluation is a Common problem in these fields.

Many communication networks require secure transfer of information, which drives development in cryptography and network security .This area has recently become particularly significant because of the increased use of internet information transfers. Associated problems include protecting the privacy of transactions and other confidential data transfers preserving the network security from attacks by viruses and hacker.

Encryption process involves manipulations of sequences of codes such asdigits, characters, and words. Hence, they are closely related to combinatorics, possibly with intelligent encryption process.

For example, one common type of encryption process is interchanging .i.e., permuting parts of a sequence.

Computer architecture

Design of computer chips involves consideration of possible permutations of input to outputpins.Field-programmable interconnection chips provide user programmable interconnectionfor a desired permutation. Arrangement of logic gates is a basic element for computer architecture design.

Languages

Both natural and computer languages are closely related to combinatorics .This is becausethe components of these languages, such as sentences, paragraphs, programs,and blocks, are arrangements of smaller elements, such as words, characters, and atoms. For example, a string dearchnig algorithm may rely on combinatorics of words and characters.
Direct applications of this can include word processing and databases.
Another important application area is performance analysis of these string searching algorithms. The study of computability what we can compute and how it is accomplished draws heavily on combinatorics.

Accumulation of electronic communication data

Since combinatorics are extensively applied to these intelligent computing techniques, there is a wide spectrum of potentials for the national security issue. Some specific examples may include string searching algorithms and their performance analysis in communication data , pre and post analysis of combinatorial sequence of information elements , and combinatorial pattern matching.