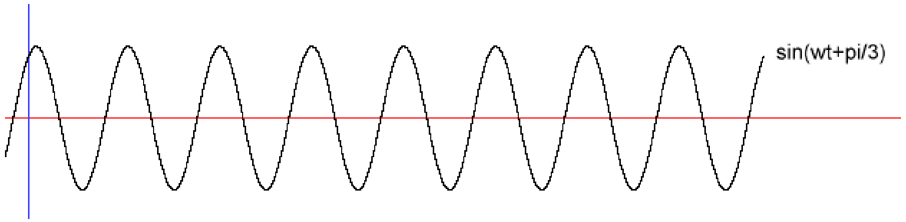
Image Enhancement:

Frequency representation

Dr. Tushar Sandhan

Introduction

- Signal decomposition



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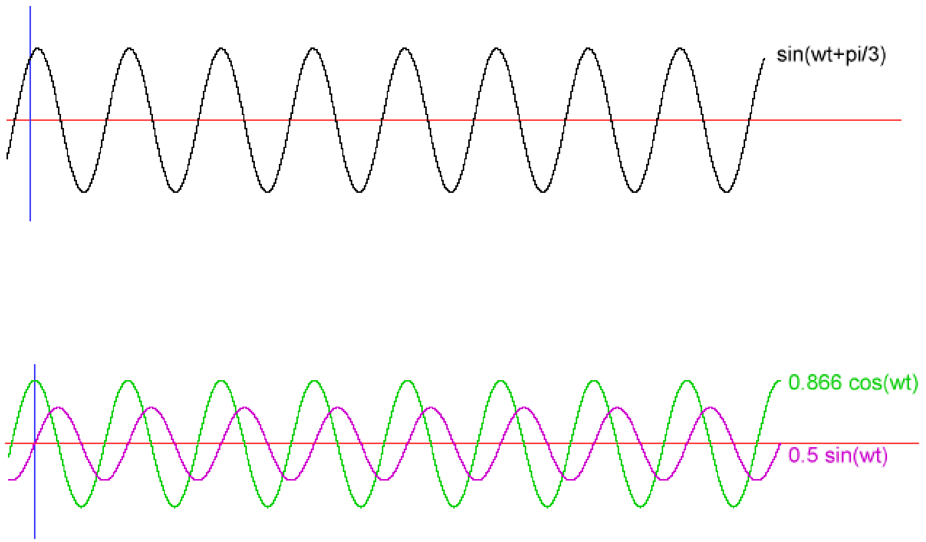

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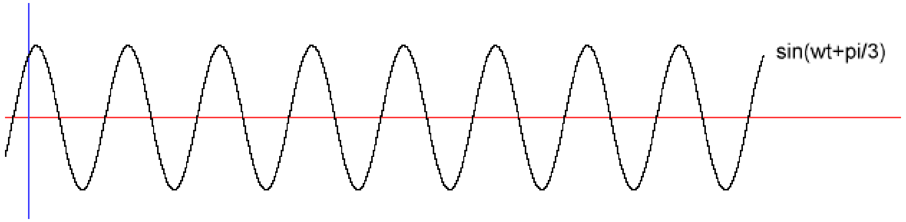

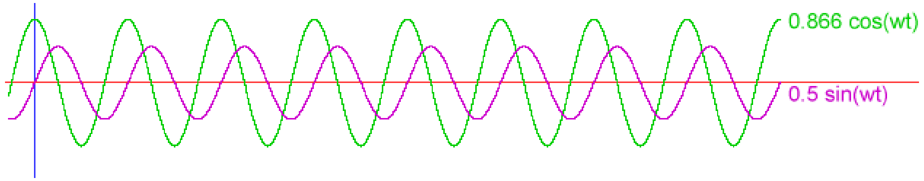

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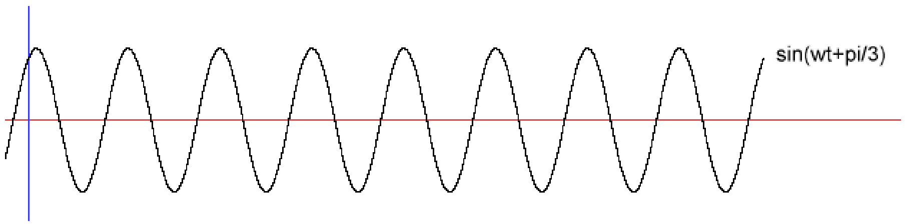

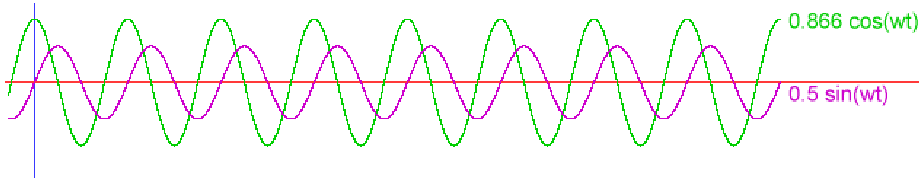

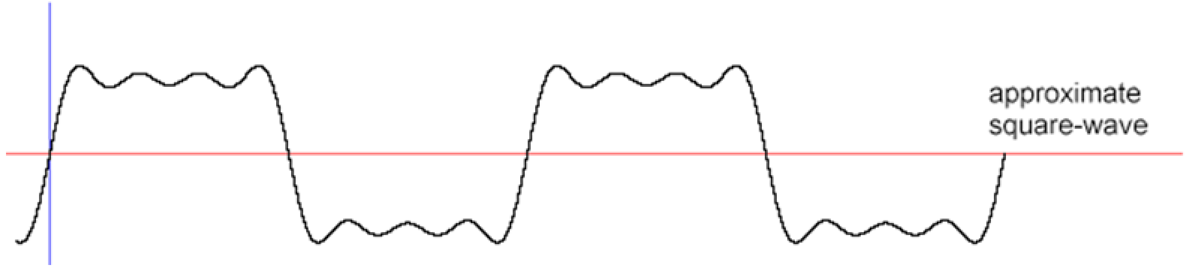

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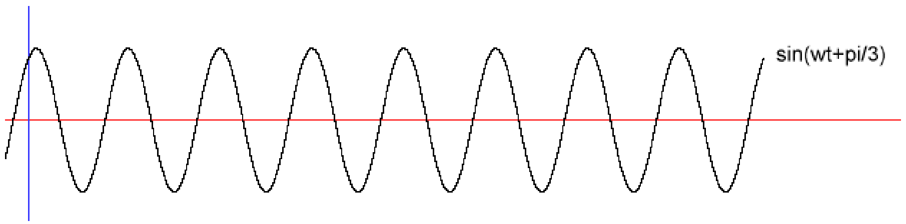

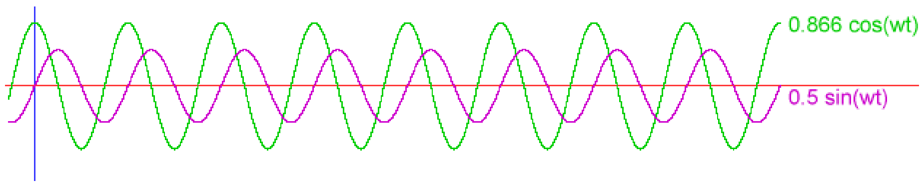

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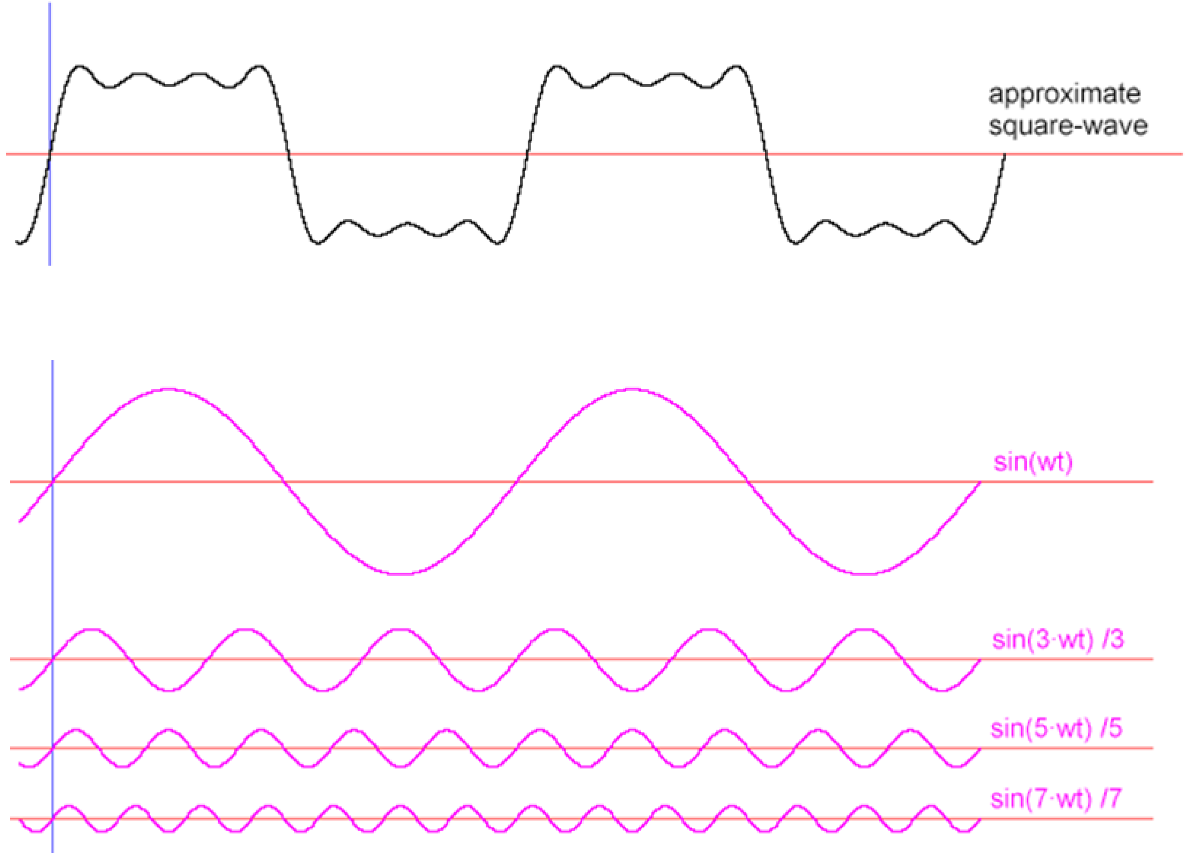

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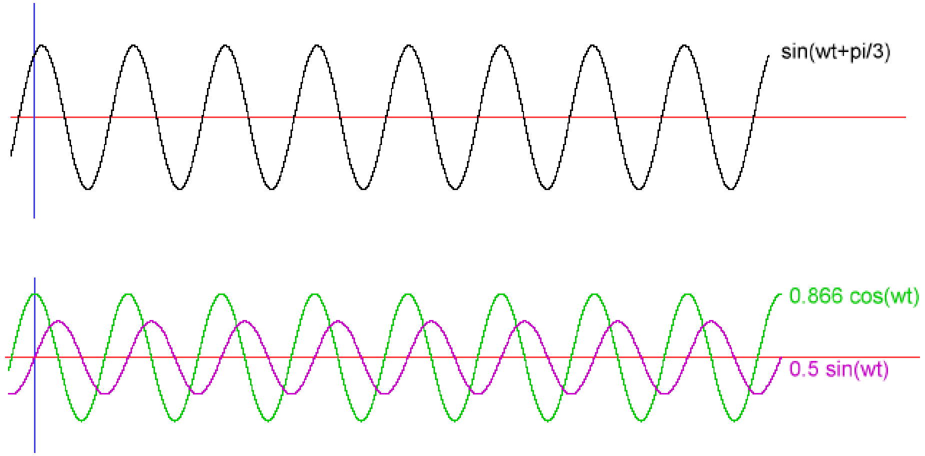

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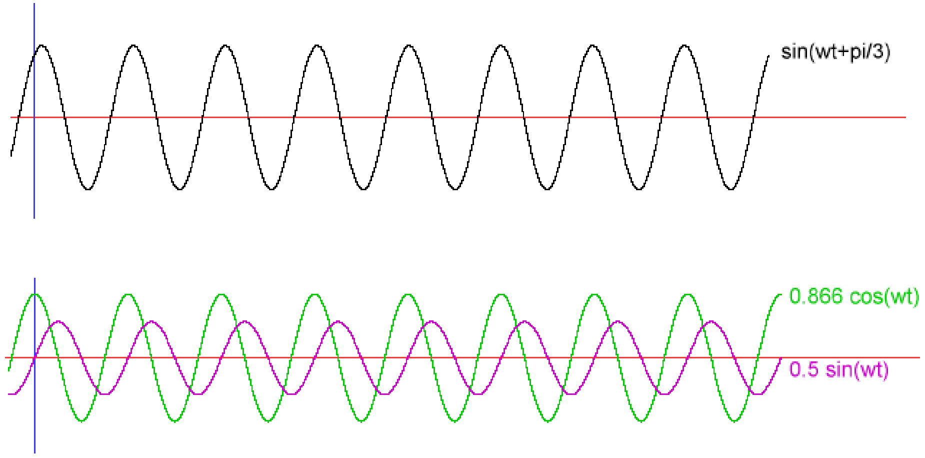

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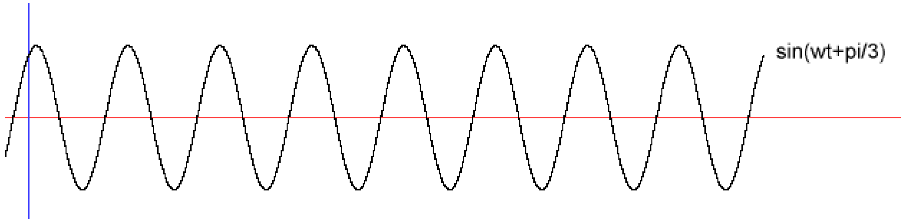

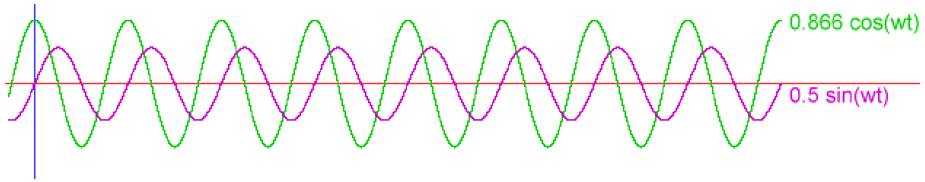

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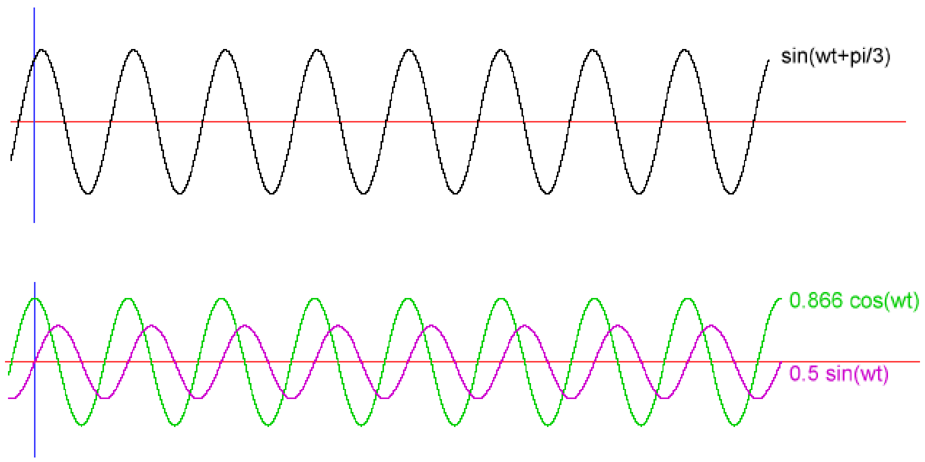

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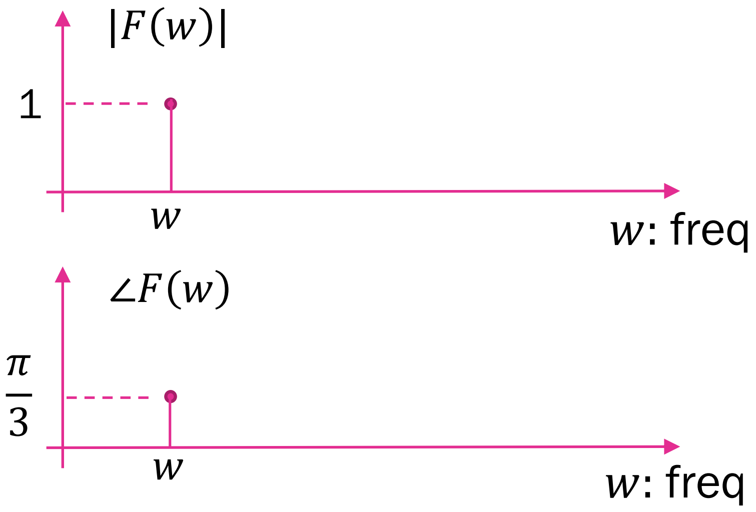

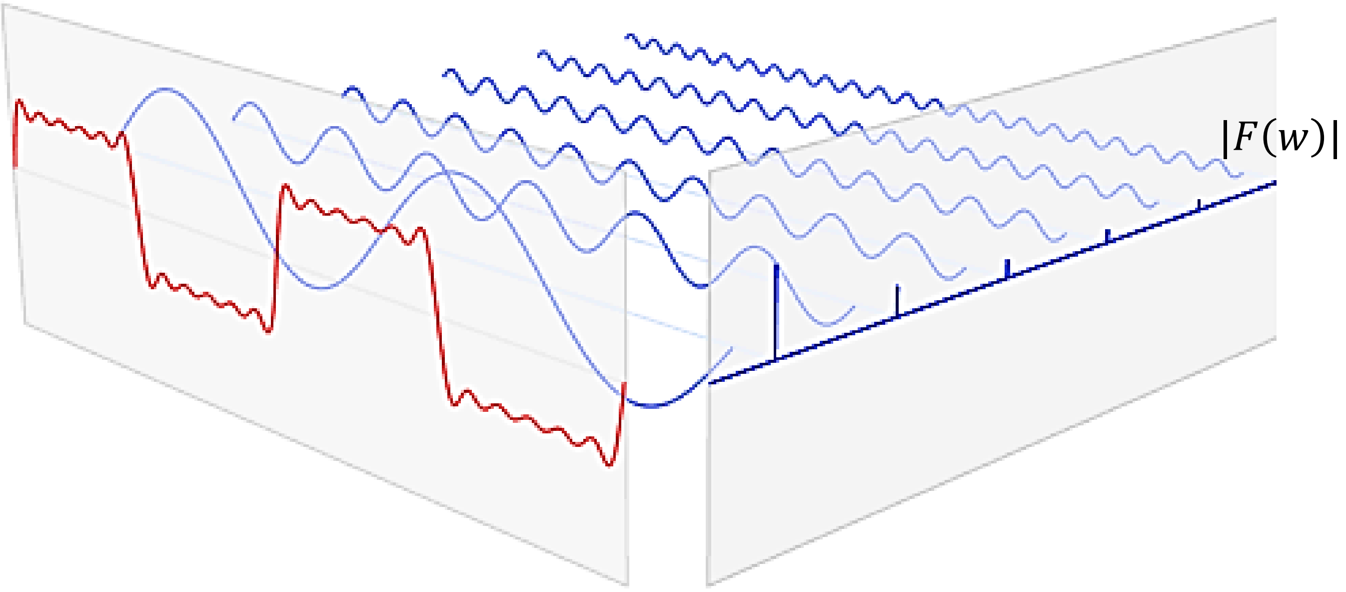

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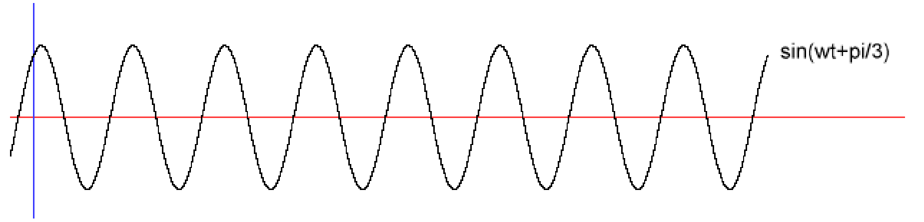

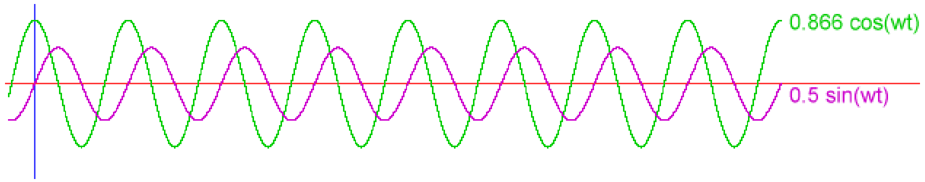

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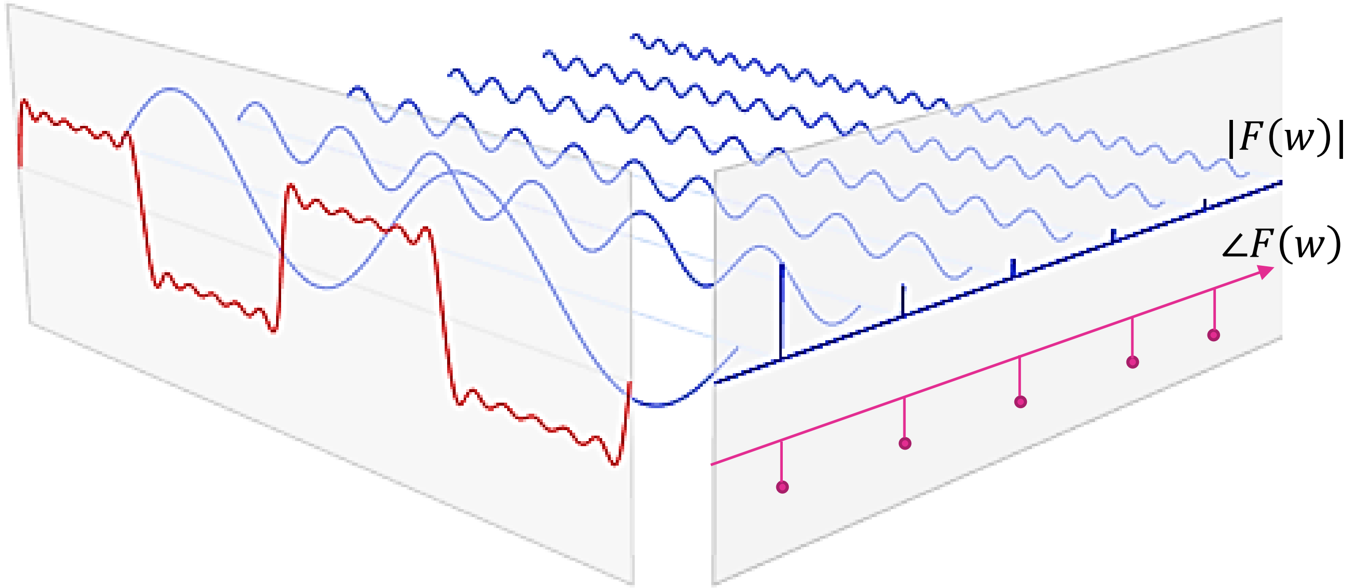

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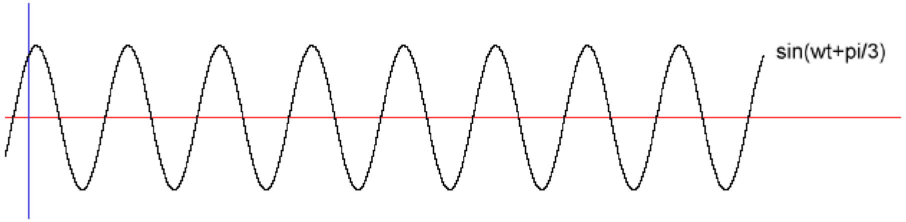

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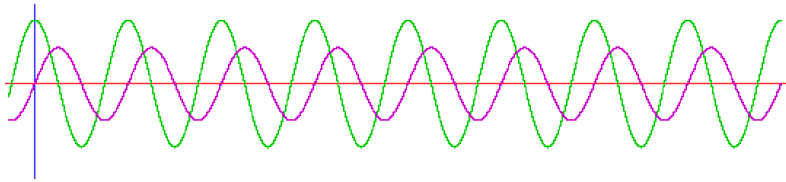

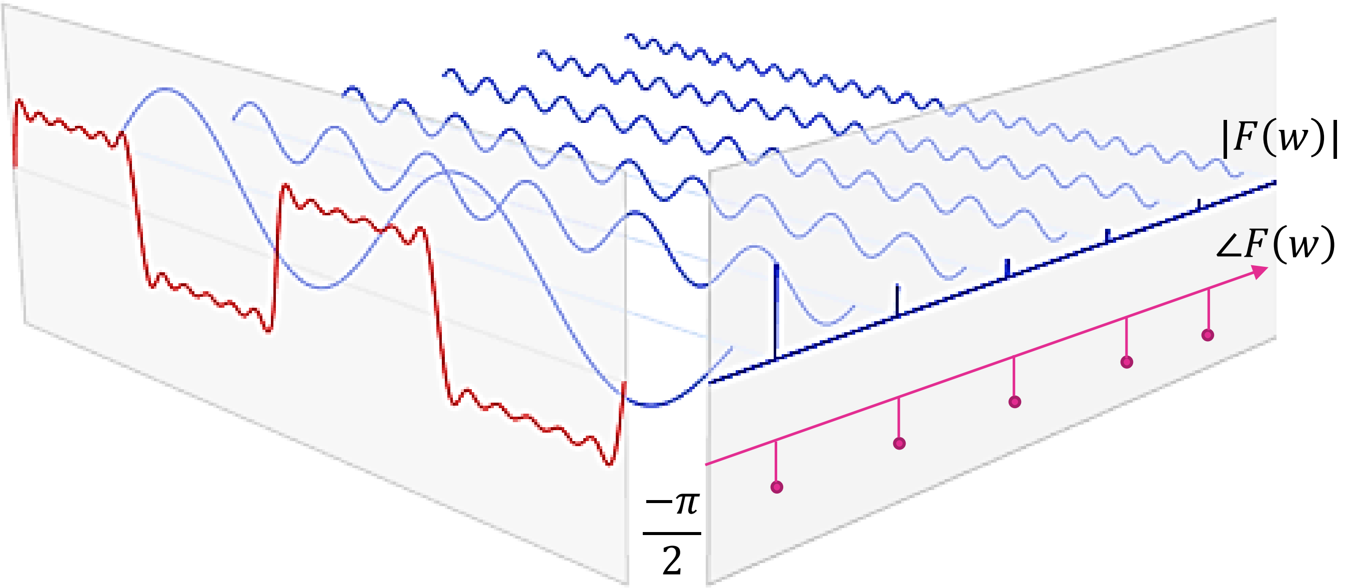

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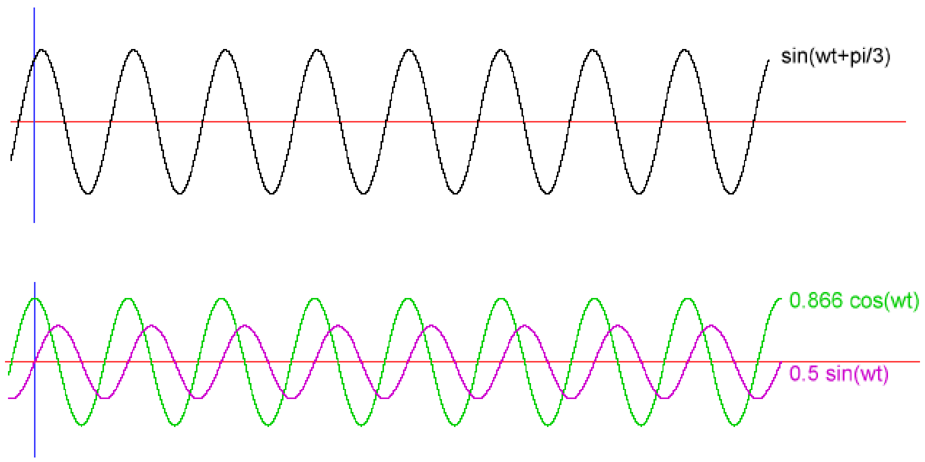

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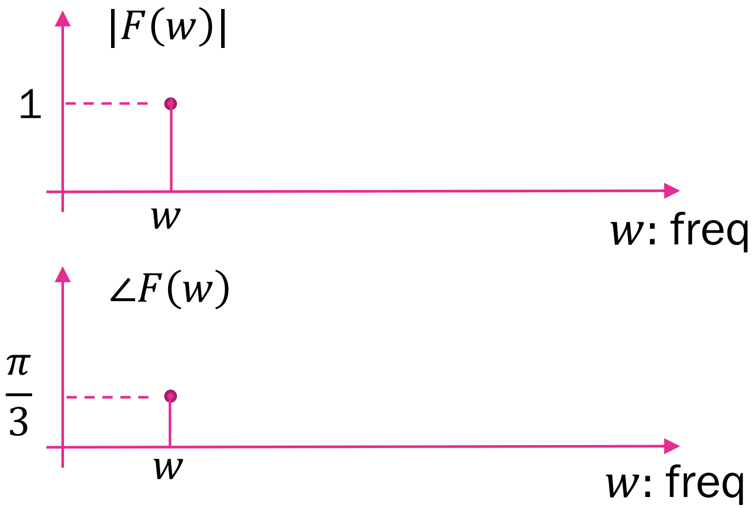
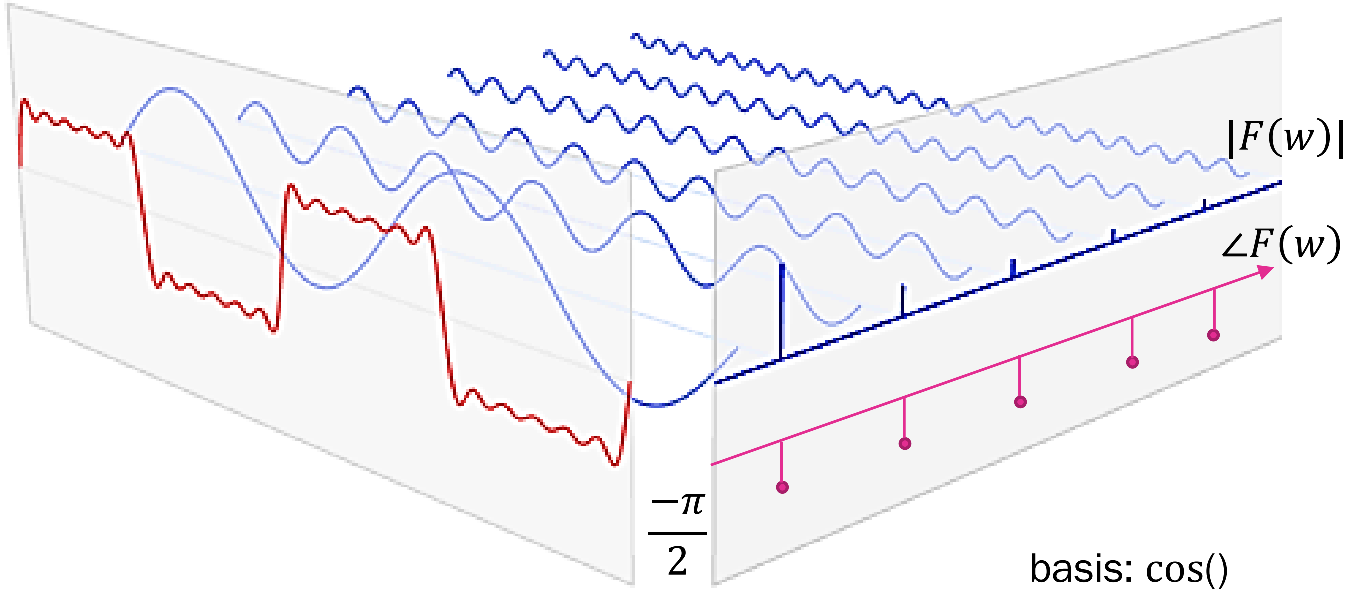


basis: sin()



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- 2D harmonics
  - $v_o = 0?$

$$f(x, y) = A \cos(u_0 x + v_0 y + \varphi)$$

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    - spatial period $P_x = \frac{2\pi}{u_0}$
    - vertical stripes in the image

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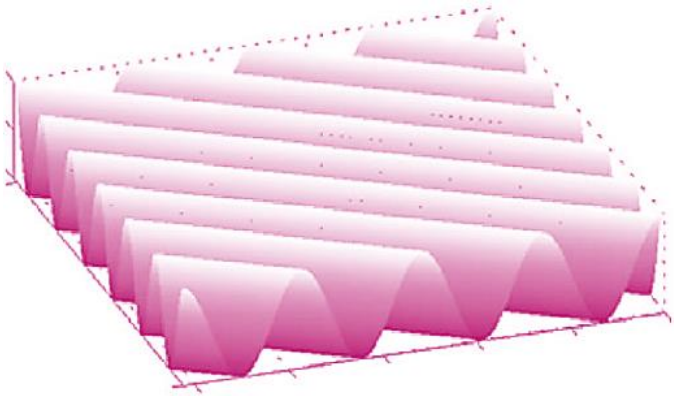

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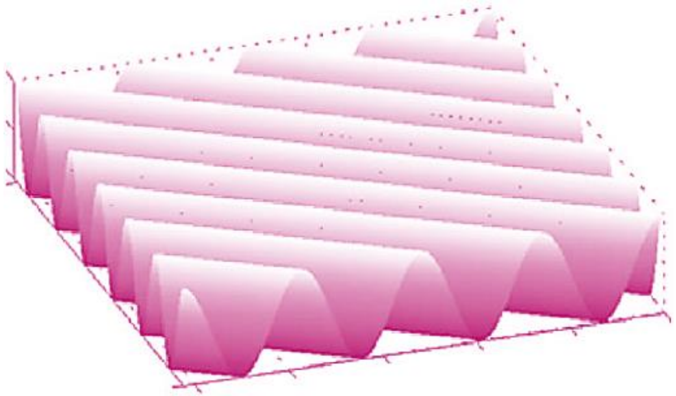

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    - in both x, y directions
    - x direction: $\frac{u_0}{2\pi}(m^{-1})$
    - number of stripes per meter in x direction

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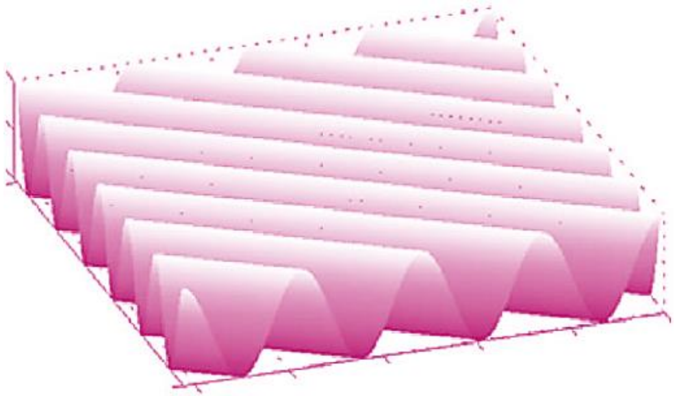

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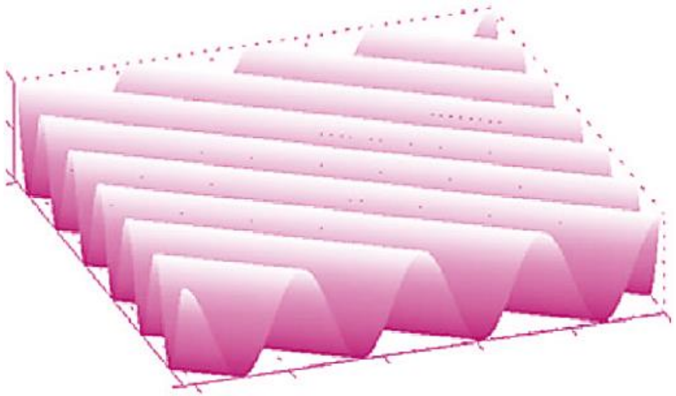

  - space frequency
    - in both x, y directions
    - x direction: $\frac{u_0}{2\pi}(m^{-1})$
    - number of stripes per meter in x direction
  - phase
    - determines shift d_x of ridges from origin of coordinates
    - d_x is from origin of coordinates as a fraction of harmonic's period
    - $d_x = P_x \frac{\phi}{2\pi} = \frac{\phi}{u_0}$

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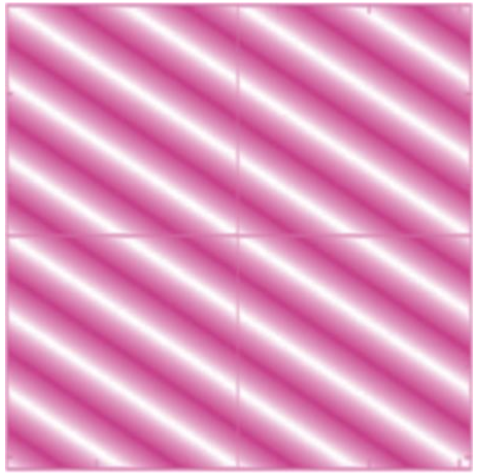

Frequency

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  - $u_0 \neq 0$ & $v_o \neq 0$?
  - stripes become oblique
    - They make angel ϑ with x axis
    - $\vartheta = \arctan(\frac{v_0}{u_0})$
    - ratio of both frequencies determines orientation
    - ridges of stripes are characterized by
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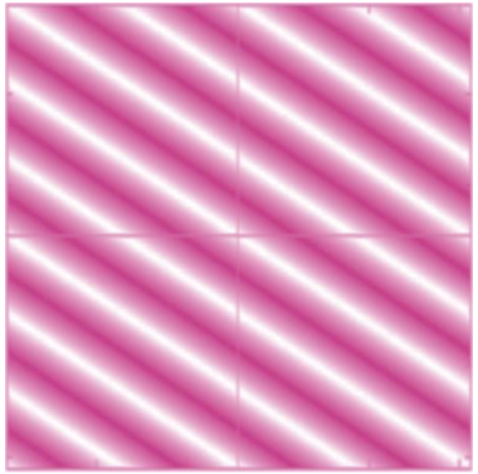

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    - family of linear equations
    - oblique parallel lines
    - distance between them = period P of harmonics
    - $P = \frac{2\pi}{\sqrt{u_0^2+v_0^2}}, \;\; w_0 = \sqrt{u_0^2 + v_0^2}$
    - shifting of strides by distance d wrt origin & $\perp$ to ridges
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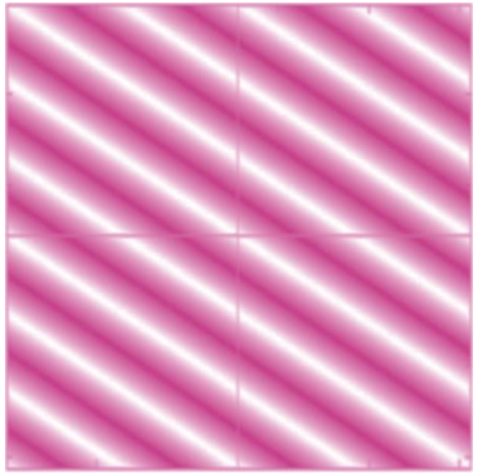

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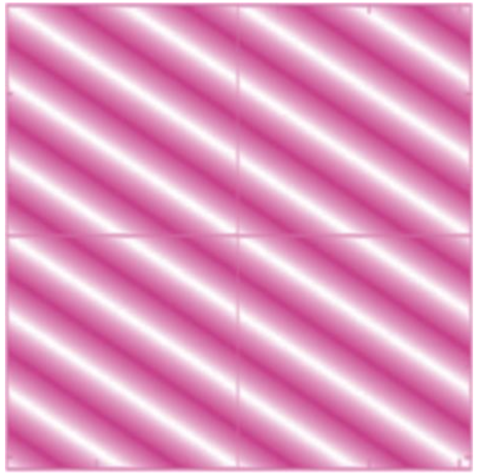

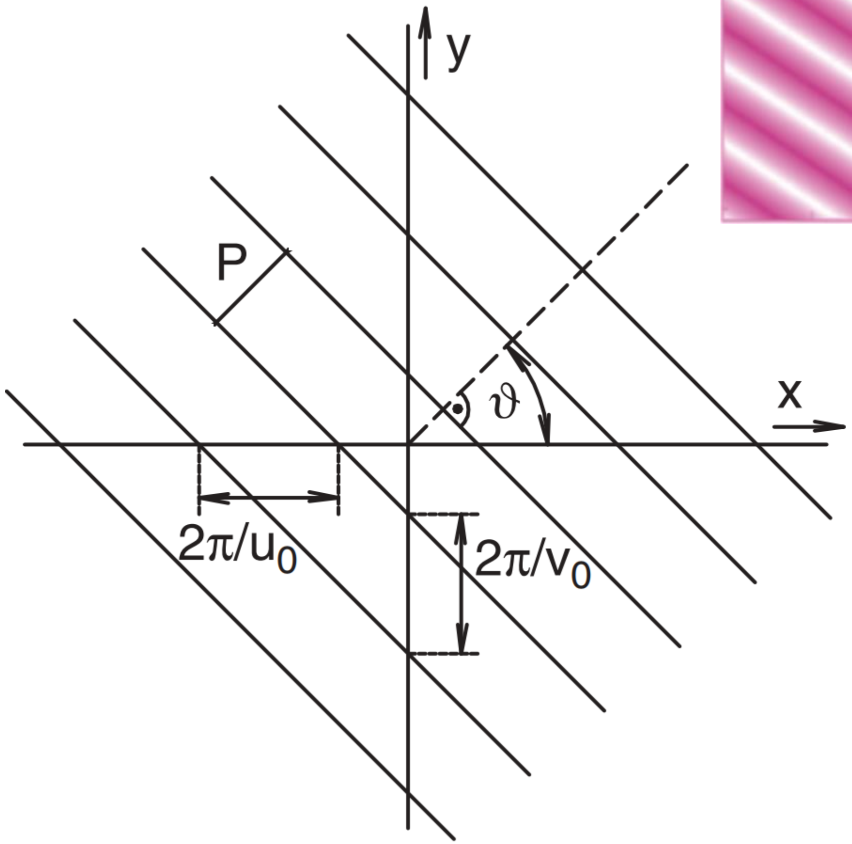

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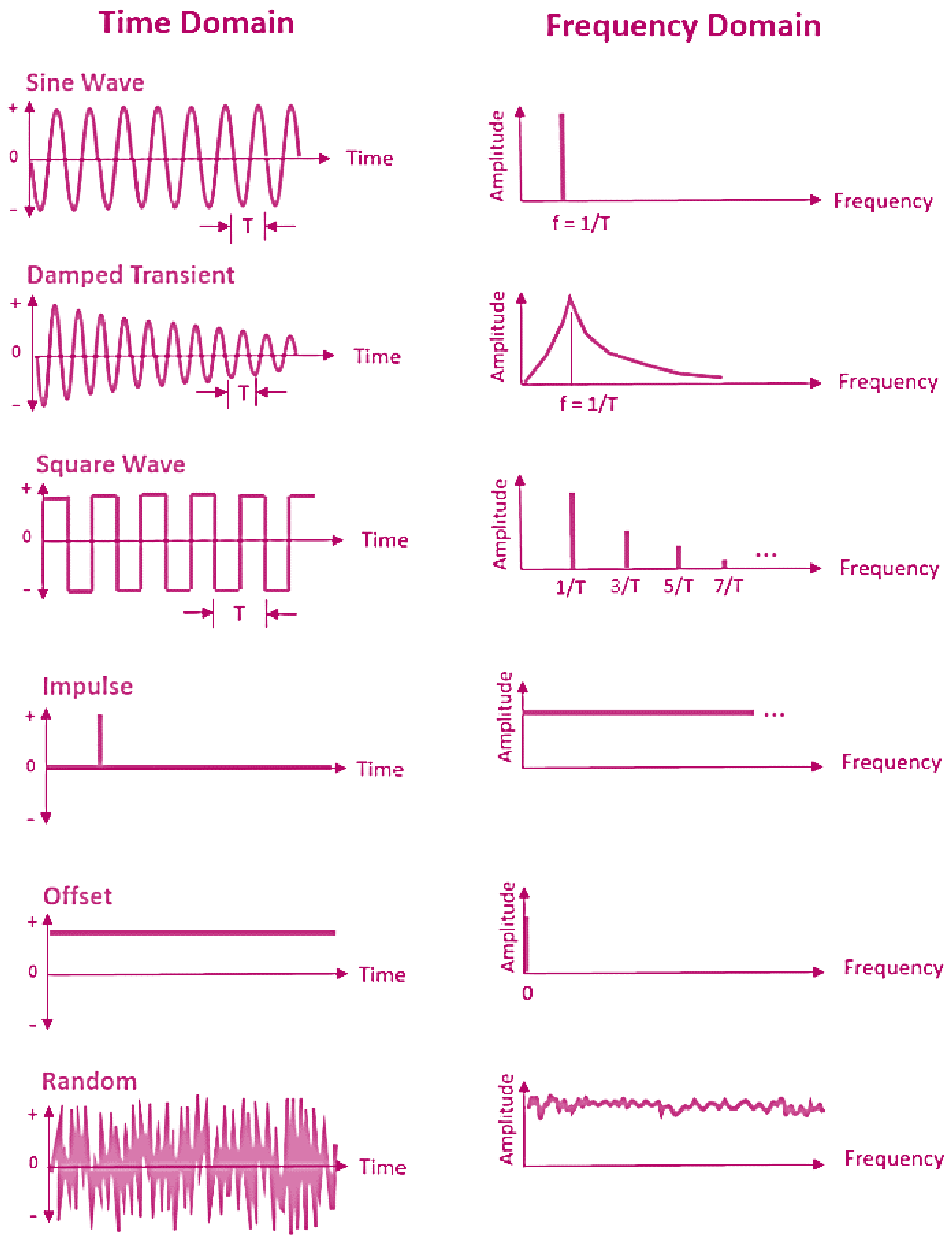

Time Domain
Frequency Domain
Sine Wave
Time
T
Amplitude
Frequency
f = 1/T
Damped Transient
Time
T
Amplitude
Frequency
f = 1/T
Square Wave
Time
T
Amplitude
Frequency
1/T
3/T
5/T
7/T
Impulse
Time
Amplitude
Frequency
Offset
Time
Amplitude
Frequency
0
Random
Time
Amplitude
Frequency

Frequency

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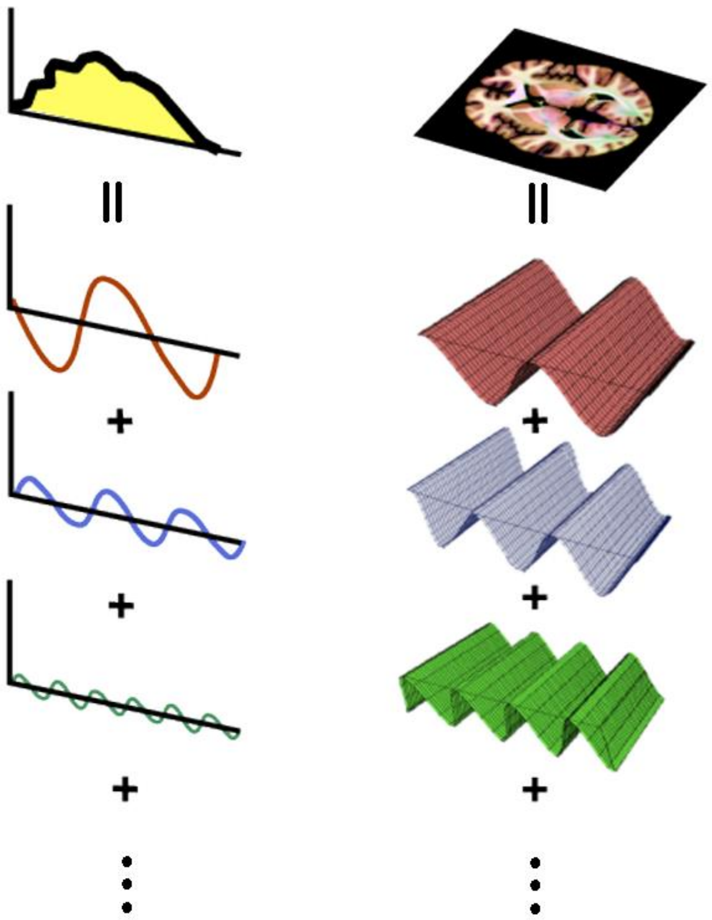
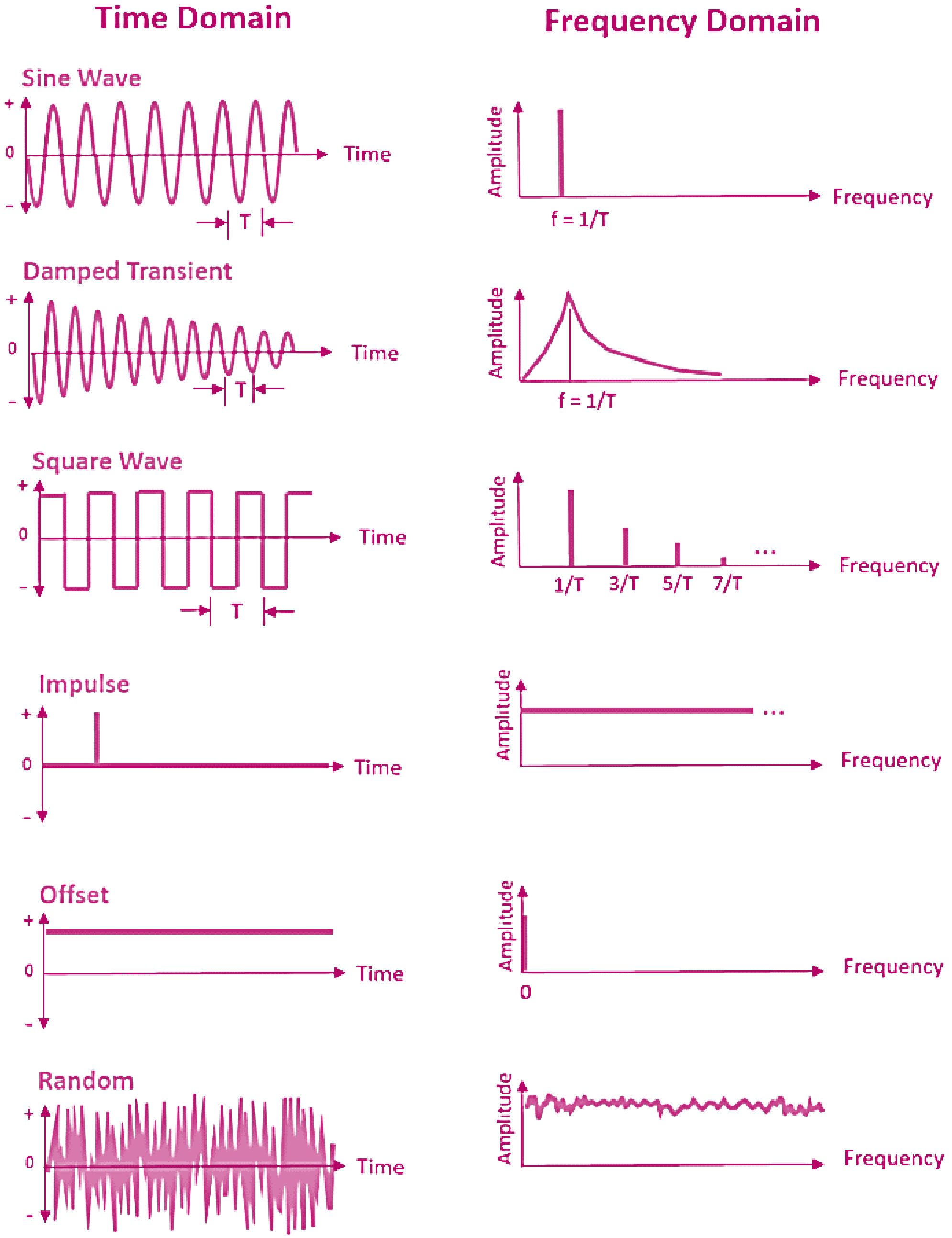
- 2D





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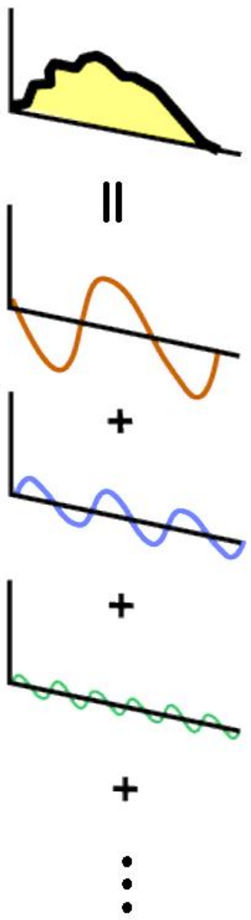
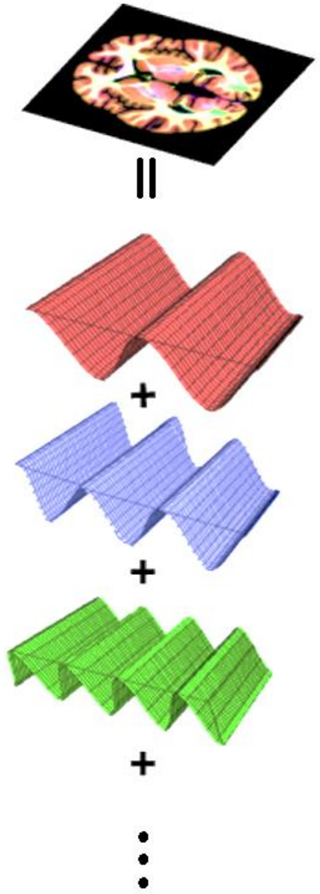

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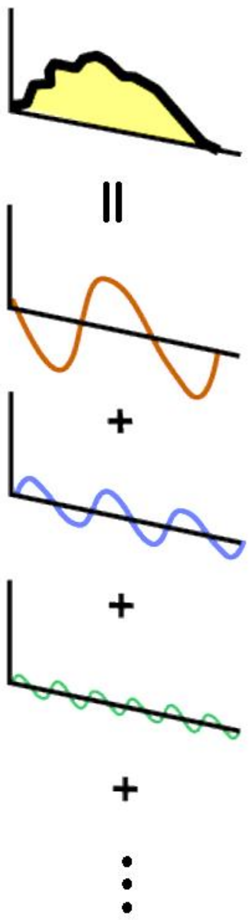

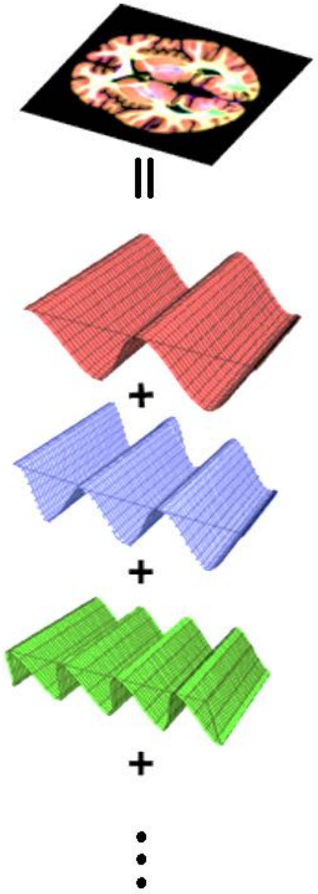

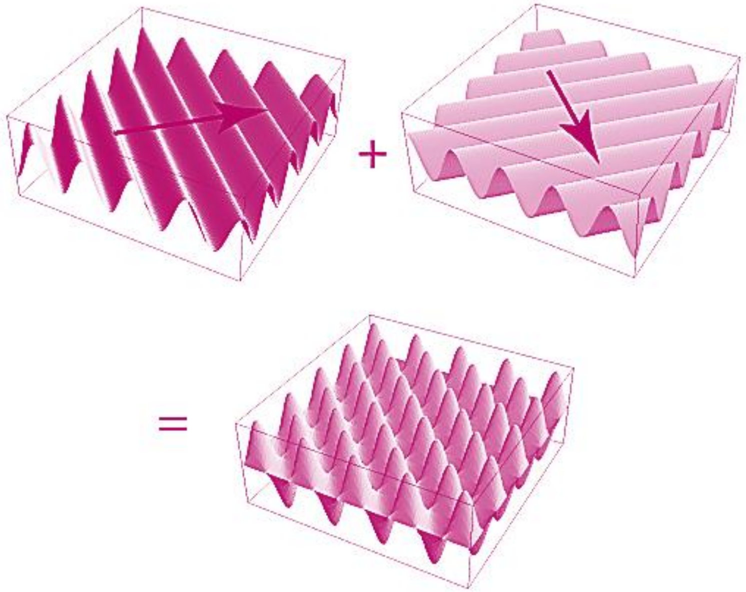

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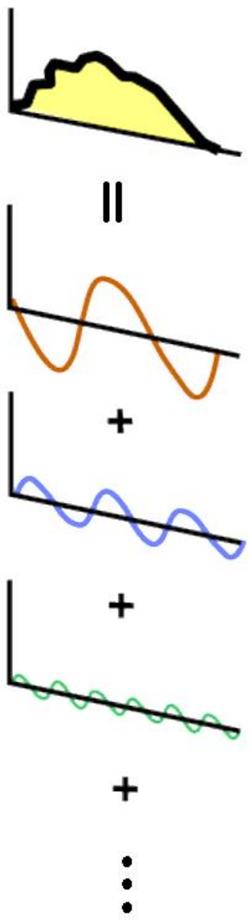

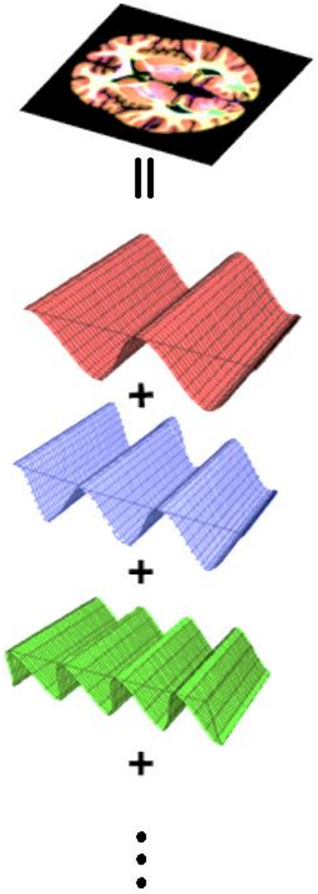

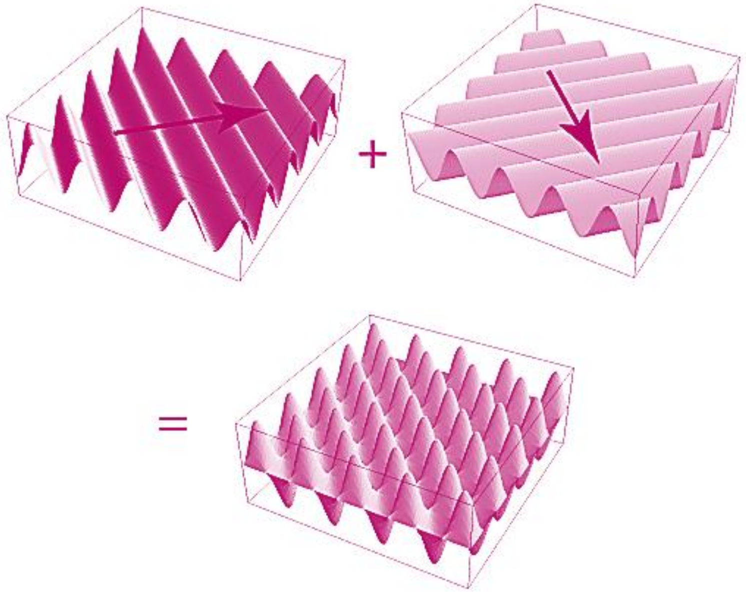

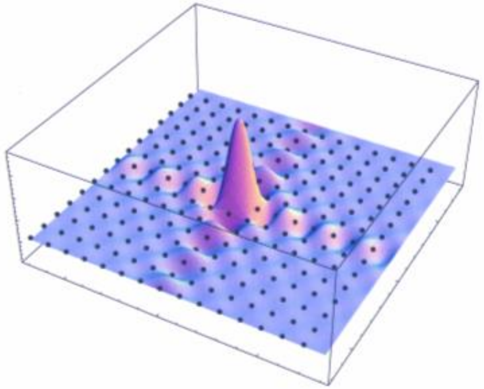

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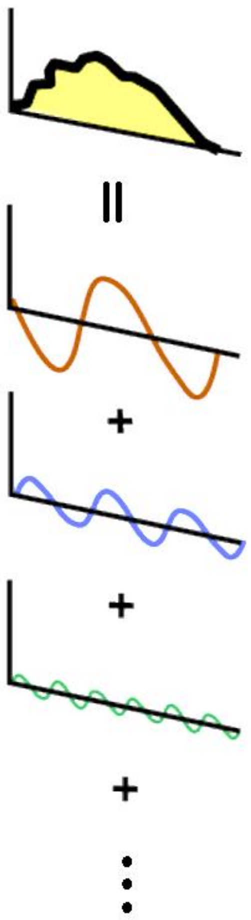
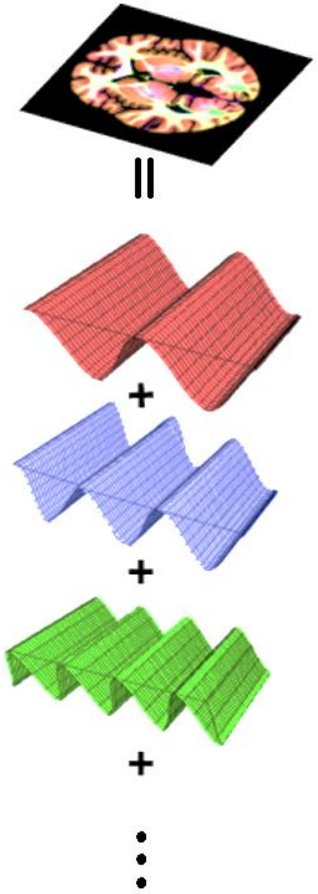
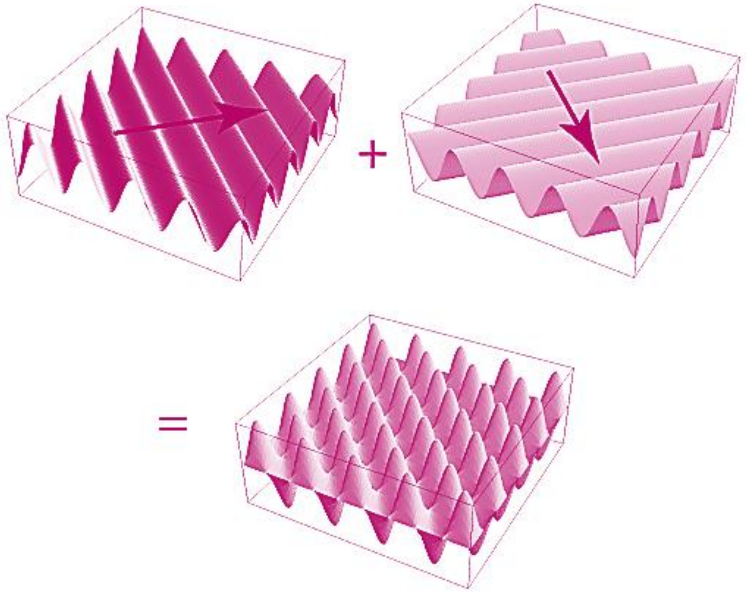
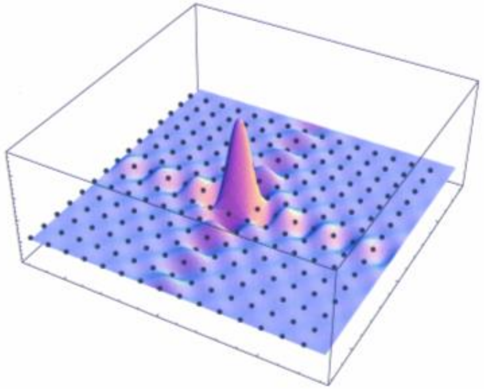
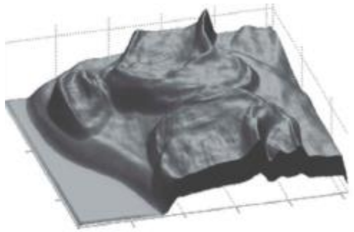


- 2D









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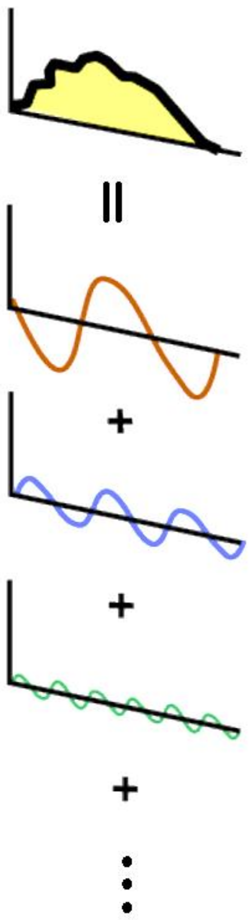

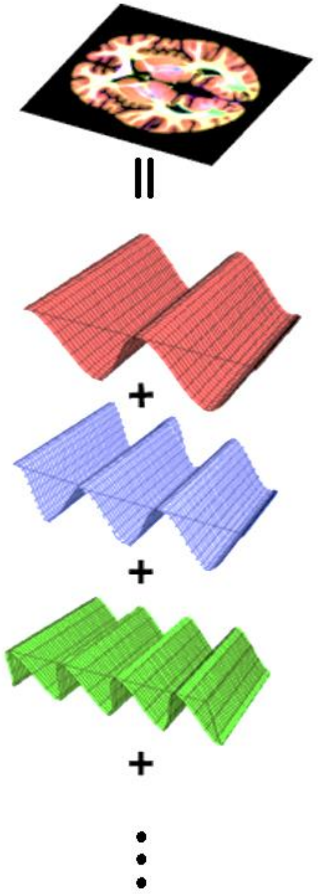

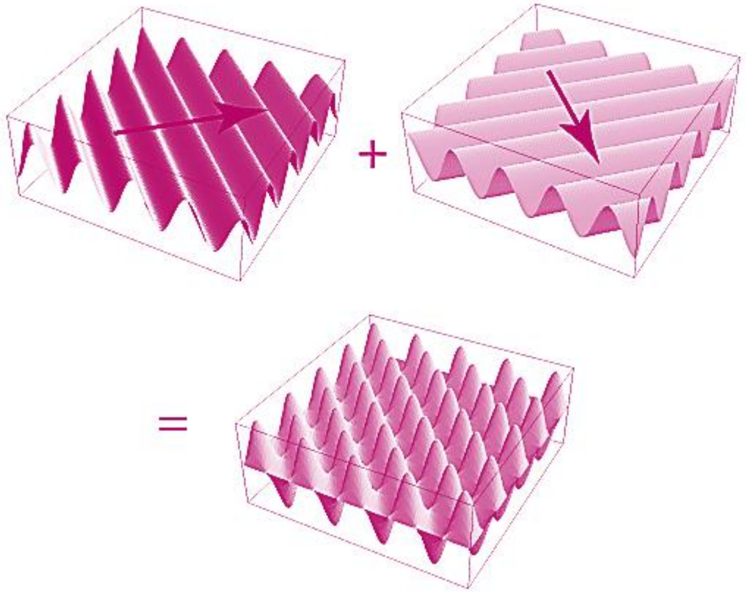

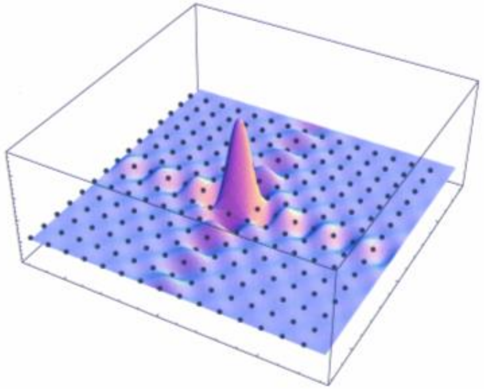

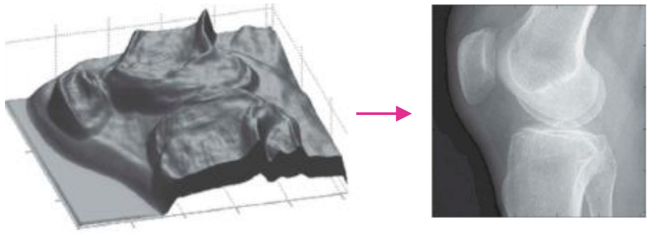

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  - sinusoids are considered as harmonics
  - $f(x)$ with period L

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$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos \frac{2\pi nx}{L} + b_n \sin \frac{2\pi nx}{L})$$

Fourier

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$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\frac{2\pi nx}{L} + b_n \sin\frac{2\pi nx}{L}\right)$$

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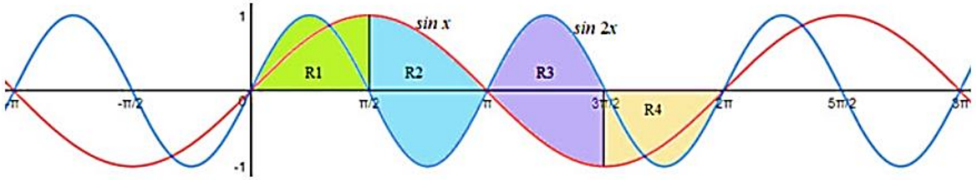

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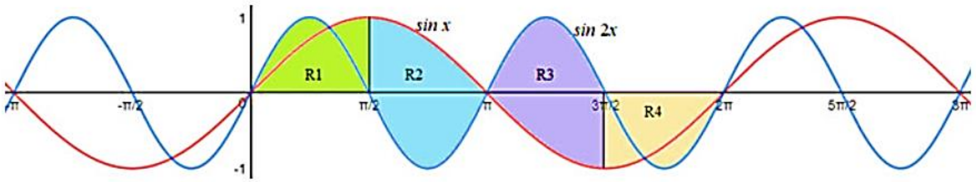

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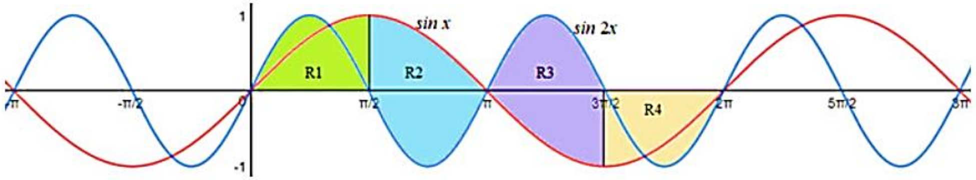

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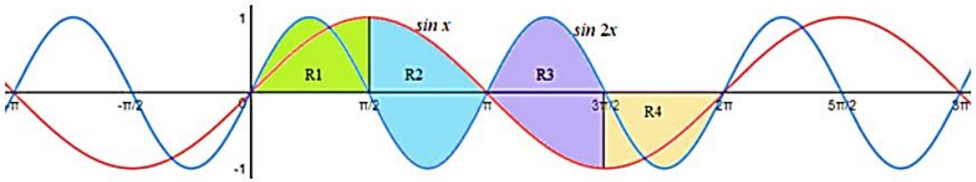

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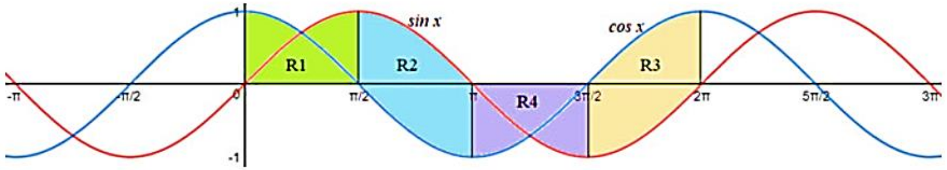

Fourier

- Fourier series
  - orthogonality

$$f(x) = a_0 + \sum_{n=1}^{\infty} (a_n \cos \frac{2\pi nx}{L} + b_n \sin \frac{2\pi nx}{L})$$

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$$\int_{-L/2}^{L/2} \cos\frac{2\pi\, nx}{L} \cos\frac{2\pi\, mx}{L} dx = \begin{cases} L/2 & for\, n = m \\ 0 & for\, n \neq m \end{cases}$$

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Fourier

- Fourier series
  - equation

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Fourier

- Fourier series
  - Complex
  - $e^{i\theta} = ?$

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Fourier

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$$\int_{-\frac{L}{2}}^{\frac{L}{2}} e^{-\frac{i2\pi mx}{L}} e^{\frac{i2\pi nx}{L}} = \begin{cases} L & for\, m = n \\ 0 & otherwise \end{cases}$$

Fourier

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Fourier

- Fourier Transform
  - any signal
  - finite discontinuities in finite interval
  - $\int_{-\infty}^{\infty} |f(t)|\, \mathrm{d}t < \infty$

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$L \rightarrow \infty$ Frequency n/L becomes continuous

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$$n/L \longrightarrow u$$

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$$L\, c_n = \int_{-\frac{L}{2}}^{\frac{L}{2}} f(x) e^{-j\frac{2\pi nx}{L}} dx \longrightarrow F(u) = \int_{-\infty}^{\infty} f(x) \exp(-j2\pi ux)\, dx$$

$L \rightarrow \infty$

Frequency n/L becomes continuous

$$n/L \rightarrow u$$

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Fourier

- Discrete Fourier Transform
  - any digital signal
  - N samples in [0, L]
  - Δx sample step in x direction
  - $L = ?$

$$c_n = \frac{1}{L} \int_{-L/2}^{L/2} f(x) e^{-j\frac{2\pi n x}{L}} dx$$

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$$c_n = \frac{\Delta x}{N\Delta x}\sum_{k=0}^{N-1} f(k) e^{-j\frac{2\pi n k \Delta x}{N\Delta x}}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} f(k) e^{-j\frac{2\pi n k}{N}} n = 0, 1, 2, \ldots, N-1$$

Fourier

- Discrete Fourier Transform
  - DFT
  - N samples in [0, L]
  - $\frac{1}{N}$ outside scaling constant is interchangeably used either in IDFT or in DFT (like below)

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Fourier

- 2D DFT
  - MxN samples
    $0 \le x < M$
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  - image $f(x, y)$

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❑ DFT: $$F(u,v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) e^{-j2\pi(ux/M + vy/N)}$$

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❑ DFT: $$F(u,v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) e^{-j2\pi(ux/M + vy/N)}$$

$u = x = 0, 1, 2, \ldots, M-1$

Fourier

- 2D DFT
  - MxN samples
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    $0 \leq y < N$
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Fourier

- 2D DFT
  - separability

$$F(u,v) = \sum_{x=0}^{M-1} \sum_{y=0}^{N-1} f(x,y) e^{-j2\pi(ux/M + vy/N)}$$

Fourier

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$$F(u,v) = \sum_{x=0}^{M-1} \left[\sum_{y=0}^{N-1} f(x,y) e^{-j2\pi vy/N} \right] e^{-j2\pi ux/M}$$

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FT_x and FT_y are the 1D FTs on row and column, respectively.

Fourier

- 2D DFT
  - image
  - use grayscale image for DFT

Fourier

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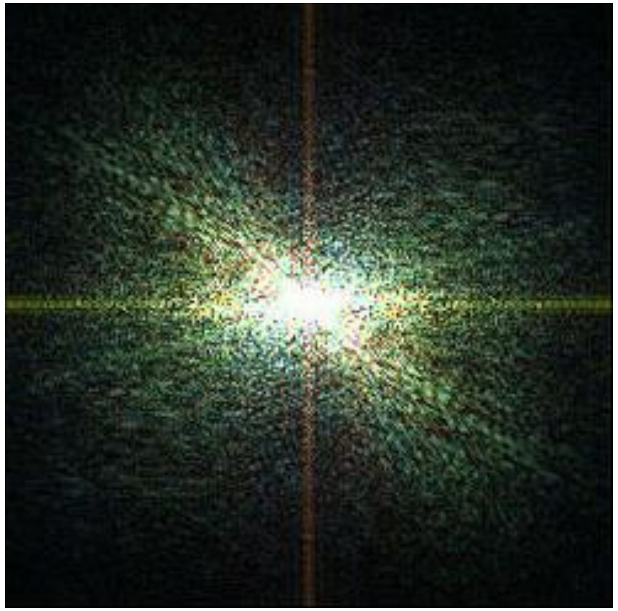
→



Fourier

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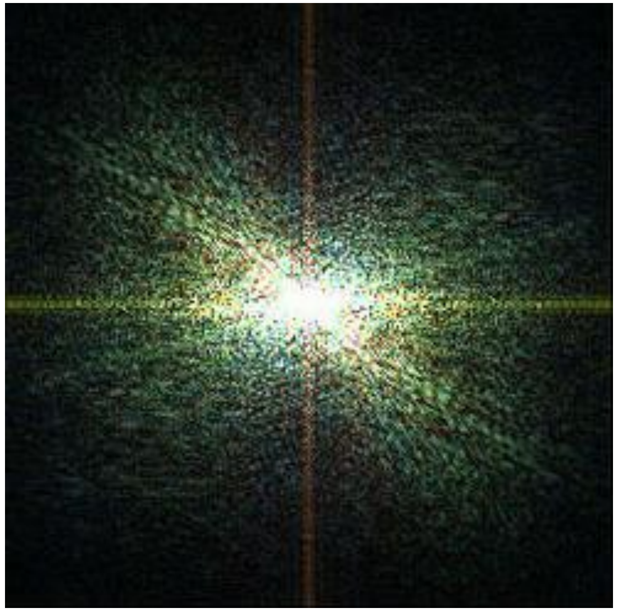
  - image
  - use grayscale image for DFT

$f(x, y)$

$\longrightarrow$



Fourier

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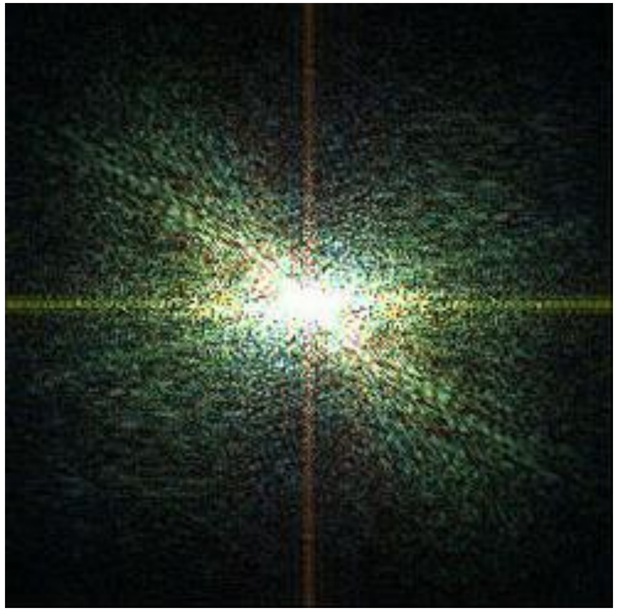
$f(x, y)$

$\rightarrow$

Enhanced($|F(u, v)|$)



Fourier

- 2D DFT
  - image

Fourier

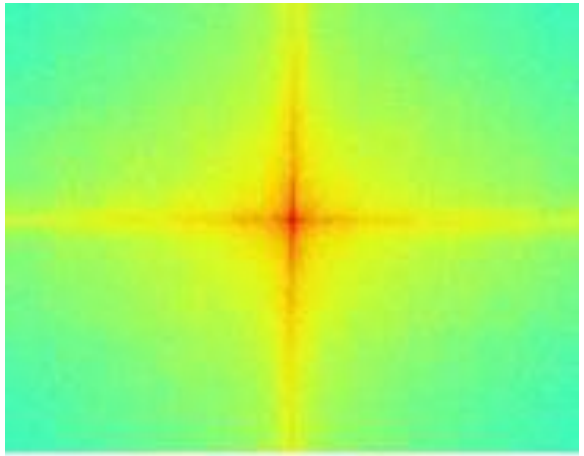
- 2D DFT
  - image

$f(x,y)$

Fourier

- 2D DFT
  - image

$f(x, y)$



Fourier

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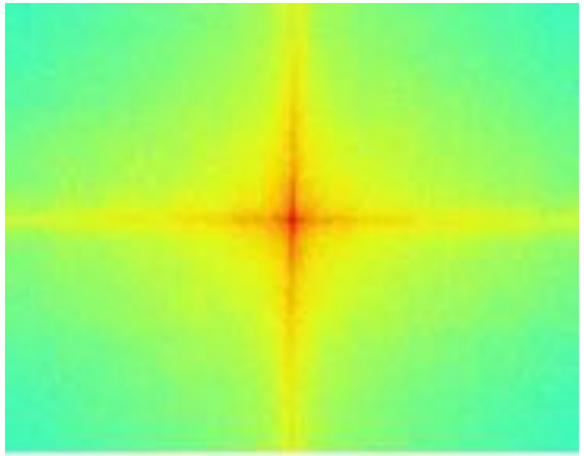
  - image

$f(x, y)$

$|F(u, v)|$



Fourier

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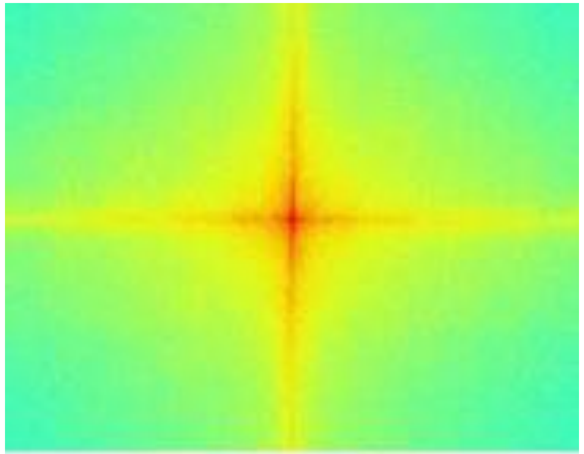

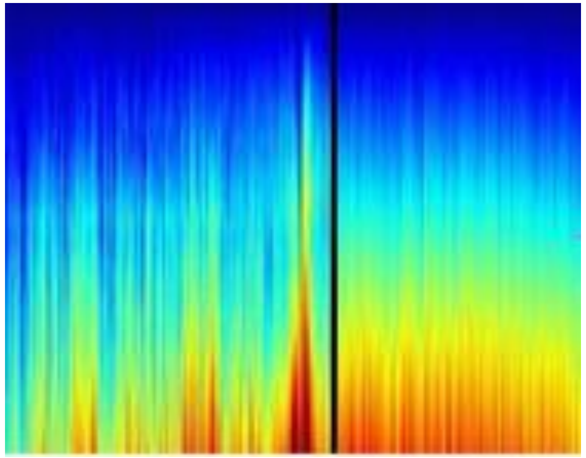

Fourier

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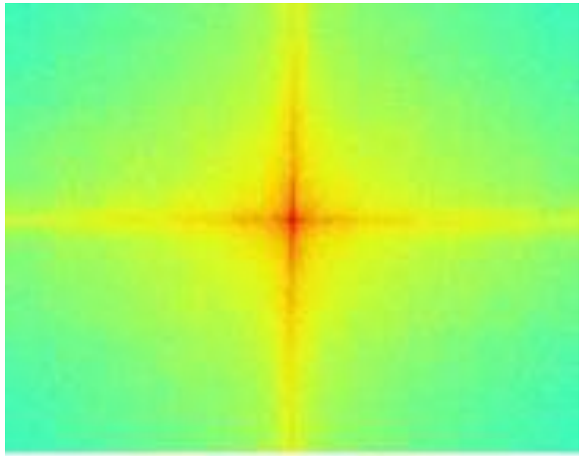
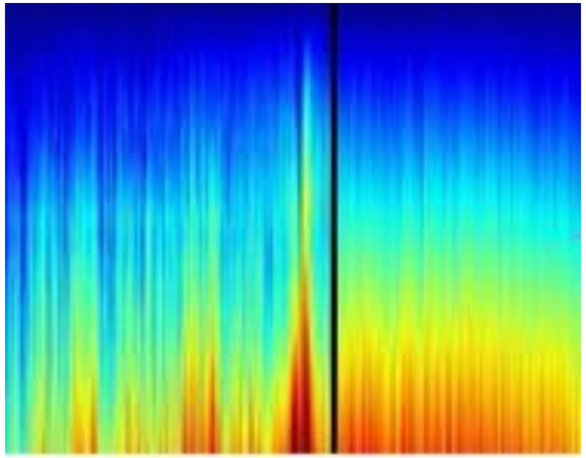
  - image

$f(x, y)$

$|F(u, v)|$



$\angle F(u, v)$



Conclusion

- Frequency representation
- Fourier
  - Series
  - Transform
  - 2D DFT

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"Four years at IIT

transform $\left(f \overset{*}{_F} T\right)$

life phenomenally"

-TS

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Lagrange, Laplace, Monge