Multi-Resolution Analysis

Dr. Tushar Sandhan

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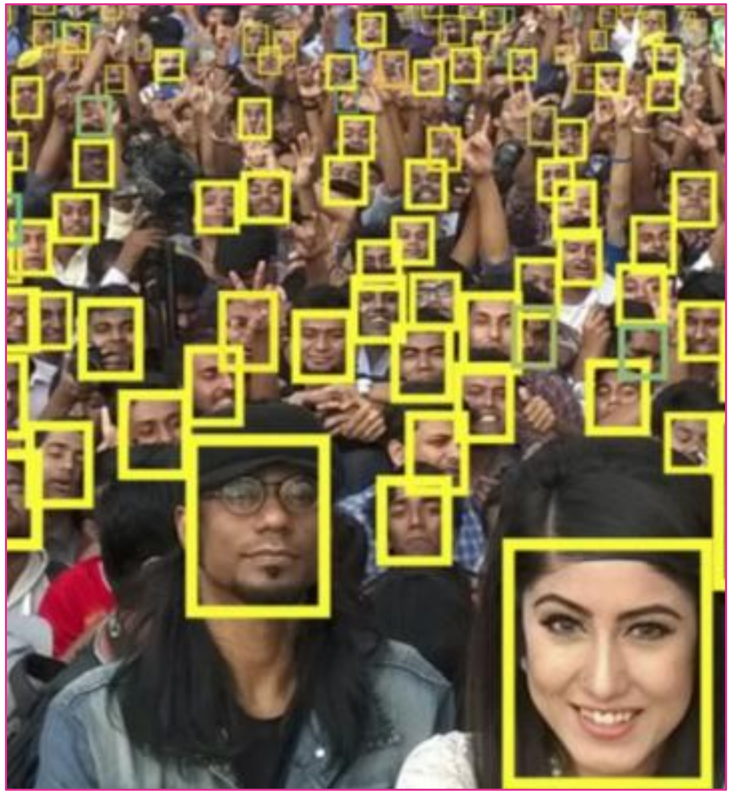
  - info at various scales (resolutions)
  - vary the window size (W)
  - vary the image size itself & keeping W fixed



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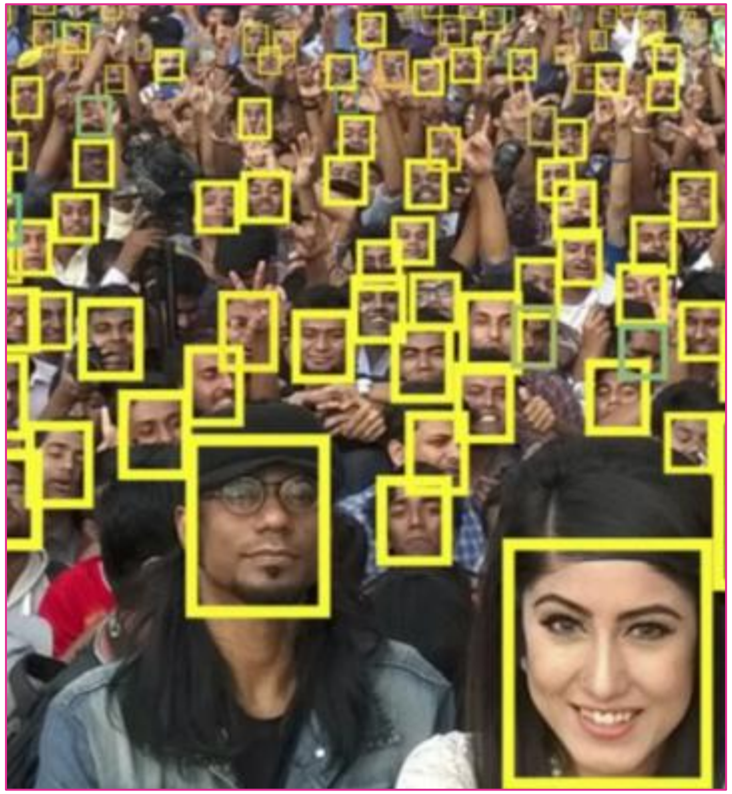

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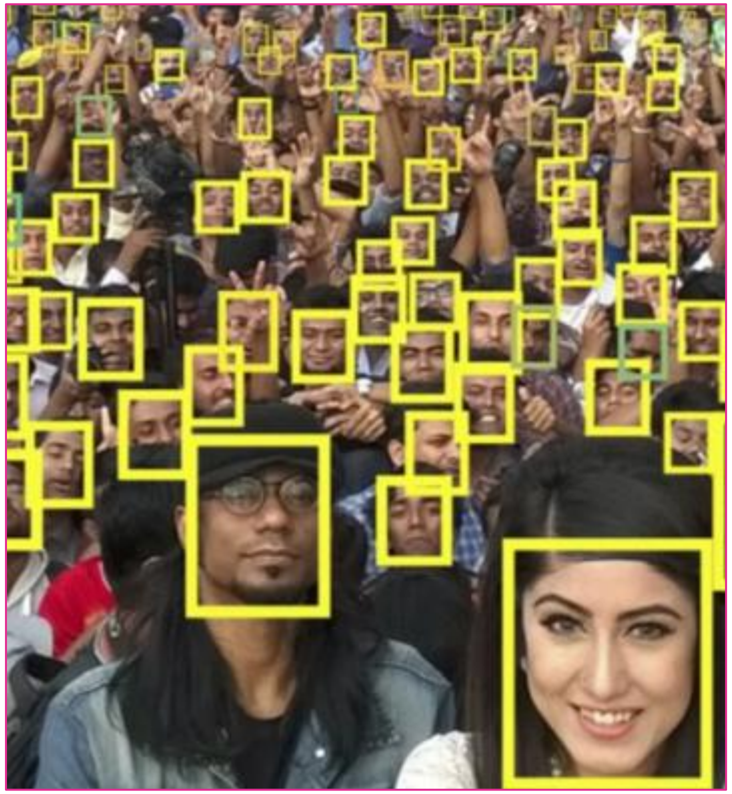
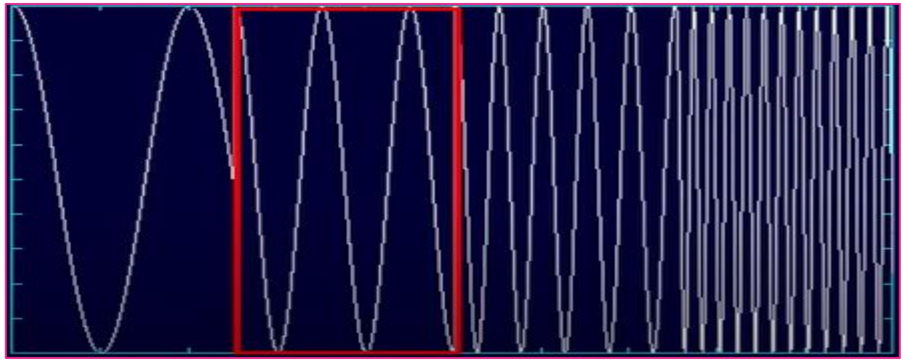


- Larger objects can be analyzed @ _____ resolution
- Smaller ones can be analyzed @ _____ resolution

Resolution relations

- Fourier analysis
  - info abt freq content in a signal
  - can we find out at what time the freq has occurred?
    - STFT (Gabor’1946)
    - if the time window is short: can assume stationarity
  - can’t locate sudden signal changes (edges)
  - FT is ideal for stationary signals



upchirp

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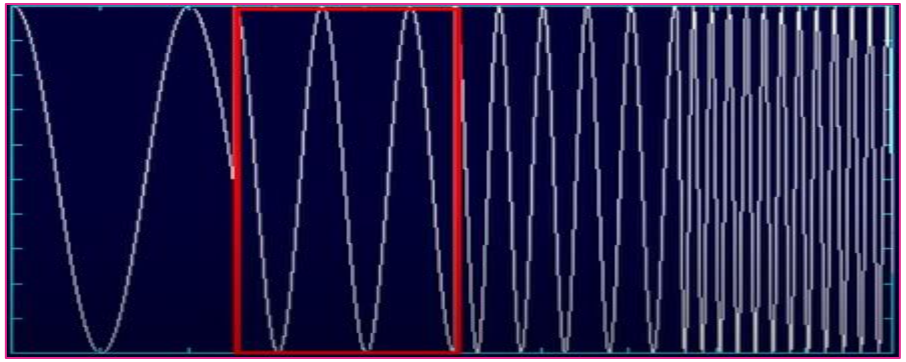

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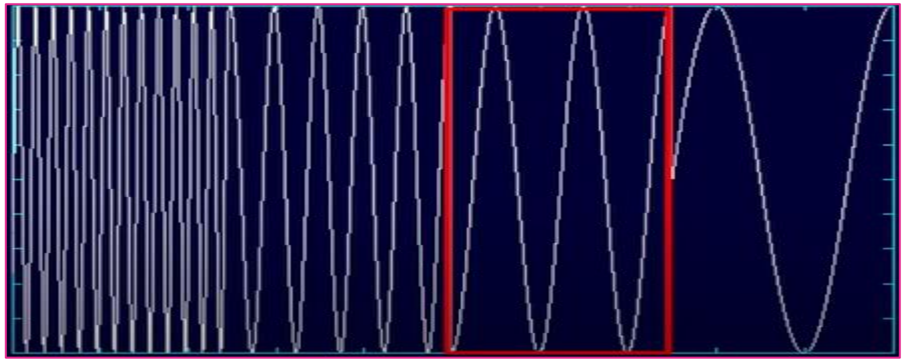

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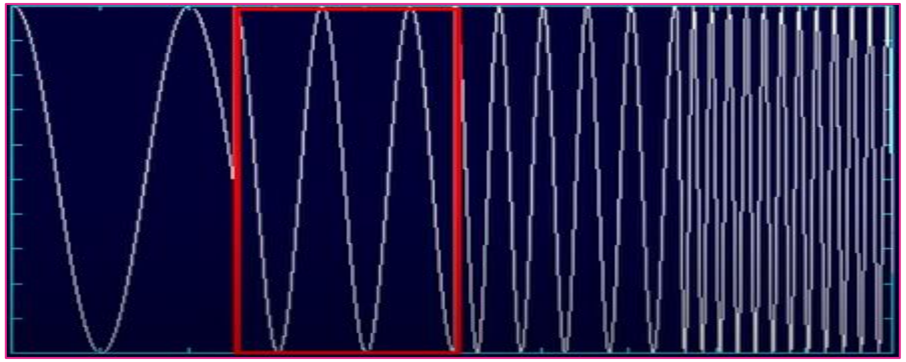

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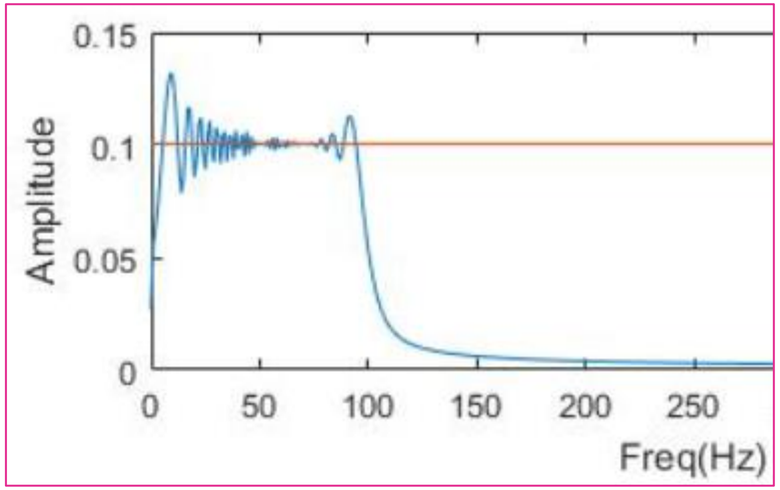

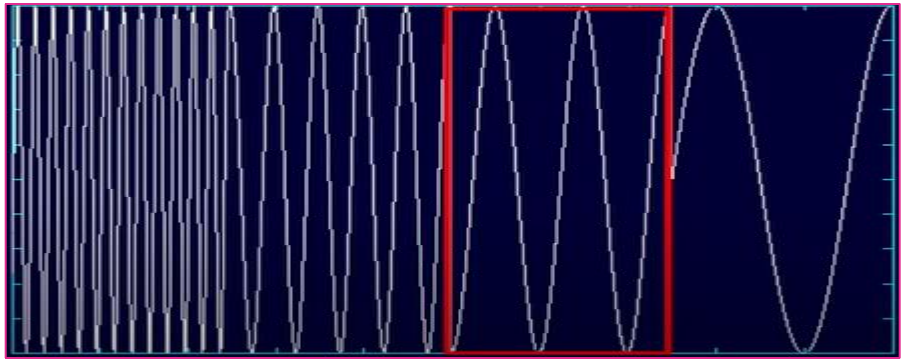

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  - low resolution
  - mid res.
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- Do high freq. components always contain useful info?
  - noisy image
  - high freq is good near abrupt transitions (e.g. edgy regions)

MRA: Multi-Resolution Analysis

- Find a space which can be fragmented in to subspaces
  - basis should be independent
  - basis functions: orthonormal
  - a function in L^2 to control scale (freq): $(s\ or\ n)$ and space (time): $(\tau\ or\ m)$

$$\psi \in L^2 \longrightarrow \psi_{m,n}$$

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$$(f_1, f_2) = \int_{-1}^{1} x^2 \cdot x^3 \, dx$$

MRA: represent a function

- Wavelet is the unified parametrized family of basis functions
  - wavelet ψ is called orthogonal, if the family of functions $\psi_{m,n}$ are orthogonal
  - then a given signal can be uniquely expressed as sum of infinite series:

$$f(t) = \sum_{n,m=-\infty}^{\infty} c_{m,n} \psi_{m,n}(t)$$

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$$c_{m,n} = (f, \psi_{m,n})$$ … (from orthogonality)

MRA: space properties

- sequence of embedded subspaces of L^2 : $\{V_m\}$ where $m \in I$ (integers)

1. Assimilation
    - Subspaces can be embedded into each other

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$$\cup_{m=-\infty}^{\infty} V_m \qquad \text{is dense in } L^2$$

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$$f(t) \in V_m$$

$$\text{iff } f(2t) \in V_{m+1}, \ \forall m \in I$$

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   - $\exists$ a function $\phi \in V_0$ such that $\{ \phi_{0,n}(x) = \phi(x-n) \}$ is orthogonal basis for V_0 so that

   - ϕ is called scaling func (or father wavelet)
   - ϕ when contracted, then its integer translates can span any V_m:

$$V_m = span\{ \phi(2^m x - n) \}$$

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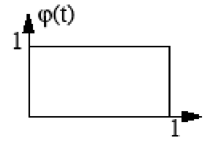

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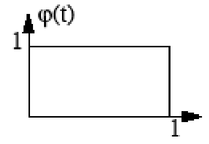

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- ϕ generates MRA: if $\{V_m\}$ is a multiresolution of L^2 and if V_0 is closed subspace generated by translates of a single function ϕ

MRA: refinement equation

- we have to reflect $\phi(x)$ in to the higher resolution or refined space V_1
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$$c_n = (\phi, \phi_{1,n}) \qquad \qquad \sum_{-\infty}^{\infty} |c_n|^2 = 1$$

Scaling function in orthogonal complement?

- If U & W are subspaces then their sum is the subspace

$$U + W = \{\mathbf{u} + \mathbf{w} \colon \mathbf{u} \in U, \mathbf{w} \in W\}$$

- The direct sum is the sum of independent subspaces

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Orthogonal complementary space

$$\begin{aligned}
V_{m+1} &= V_m \oplus W_m \\
&= (V_{m-1} \oplus W_{m-1}) \oplus W_m \\
&\vdots \\
&= V_0 \oplus W_0 \oplus W_1 \oplus \cdots \oplus W_m \\
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V_0 is just a reference space (reference can be any index, i.e. resolution can scale-up or down)

$$\oplus_{n=-\infty}^{\infty} W_n = L^2$$

MRA: mother wavelet

- complementary space also spans L^2
  - $\exists$ a function $\psi \in W_0$ such that $\{ \psi_{0,n}(x) = \psi(x-n) \}$ is orthogonal basis for W_0
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- ψ spans orthogonal complement subset $\mathrm{W_m}$, while scaling function ϕ spans the subsets V_m

MRA: WT

- wavelet

$$\psi_{a,b}(t) = \frac{1}{\sqrt{b}}\psi(\frac{t-a}{b})$$

- decomposition

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$$W(a,b) = K\int_{-\infty}^{+\infty}\psi^*(\frac{x-a}{b})f(x)dx$$

Haar

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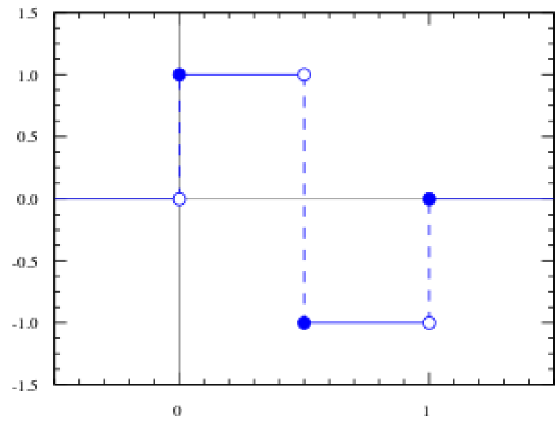

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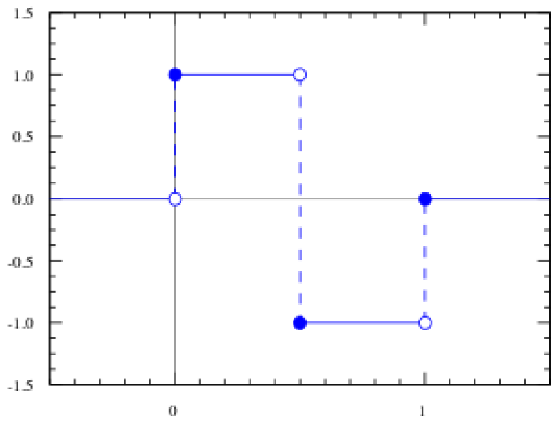

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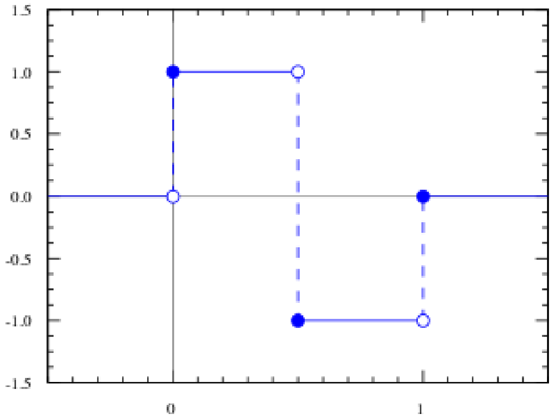

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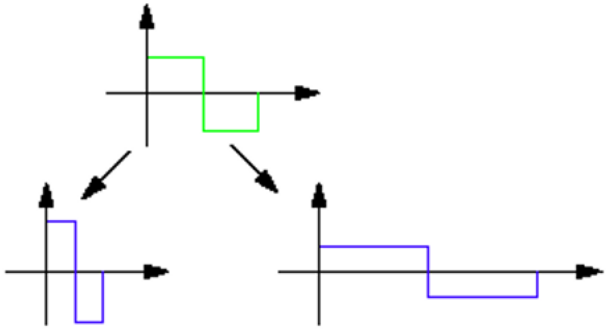

Conclusion

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- Image analysis in multiple resolutions

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  - Spaces
  - Basis functions
  - Wavelets