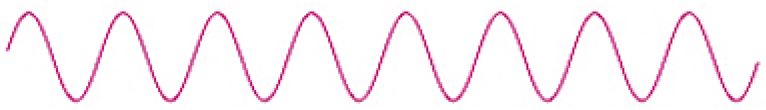
MRA wavelet transform

Dr. Tushar Sandhan

Introduction

- Sinewaves
  - basis FT



Introduction

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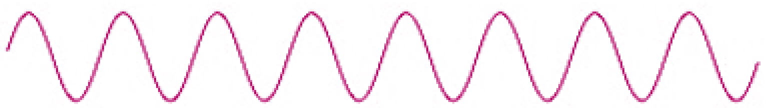

Introduction

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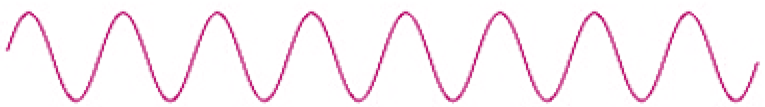


- Wavelets
  - basis WT

Mother wavelets

- complementary space also spans L^2
  - ∃ a function $\psi \in W_0$ such that $\{ \psi_{0,n}(x) = \psi(x-n) \}$ is orthogonal basis for W_0
  - Orthogonal basis for W_m are given as:

$$\psi_{m,n}(x) = 2^{m/2}\psi(2^m x - n)$$

  - ψ is called mother wavelet

- ψ spans orthogonal complement subset $\mathrm{W_m}$, while scaling function ϕ spans the subsets V_m

Wavelet Transform (WT)

- wavelet

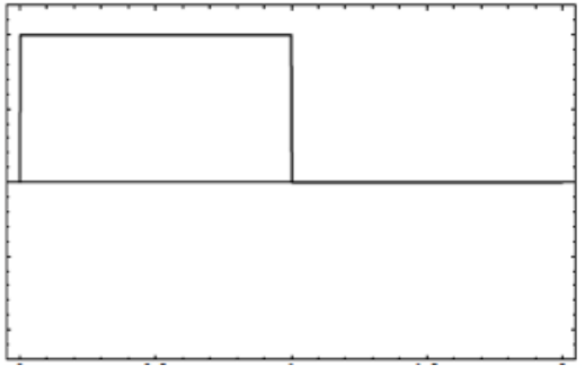
$$\psi_{a,b}(t) = \frac{1}{\sqrt{b}}\psi(\frac{t-a}{b})$$

- decomposition

$$W(a,b) = K\int_{-\infty}^{+\infty} \psi^*(\frac{x-a}{b})f(x)dx$$

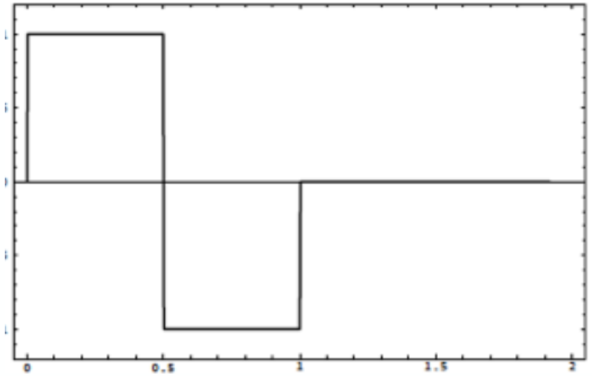
MRA: Haar example

Haar scaling function



$$\varphi(t) = \begin{cases} 1 & 0 \le t < 1 \\ 0 & \text{otherwise} \end{cases}$$

Haar (mother) wavelet

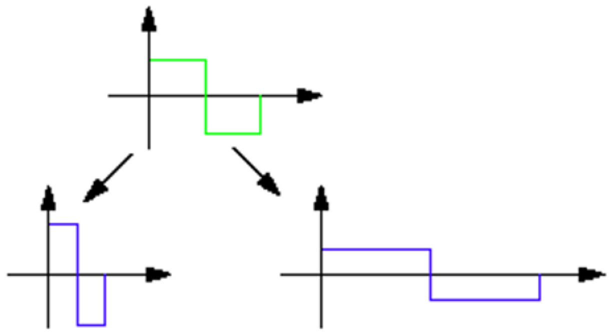


$$\psi(t) = \begin{cases} 1 & 0 \le t < \frac{1}{2} \\ -1 & \frac{1}{2} \le t < 1 \\ 0 & \text{otherwise} \end{cases}$$

MRA: Haar

- Basis functions

$$\psi(t) = \begin{cases} 1 & 0 \le t < 0.5 \\ -1 & 0.5 \le t < 1 \\ 0 & \text{else} \end{cases}$$



$$\psi_{m,n}(x) = 2^{m/2}\psi(2^m x - n)$$

- Compute WT on discrete grid

MRA: Haar

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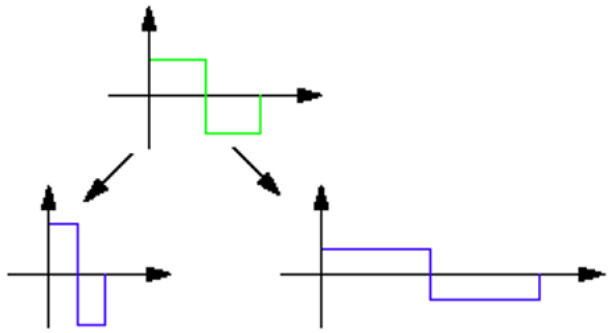

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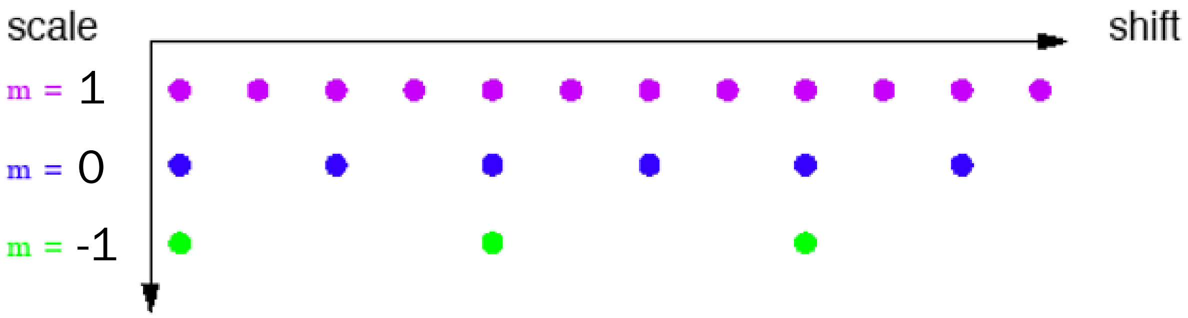

Haar

$$\boxed{x(t) = \begin{cases} 9 & \text{if } t \in [0, 1/4) \\ 7 & \text{if } t \in [1/4, 1/2) \\ 3 & \text{if } t \in [1/2, 3/4) \\ 5 & \text{if } t \in [3/4, 1) \end{cases}}$$

Scaling const. in ϕ is ignored for simplicity

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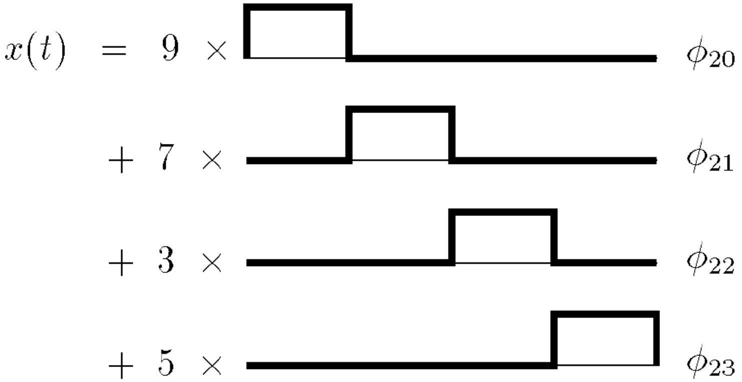

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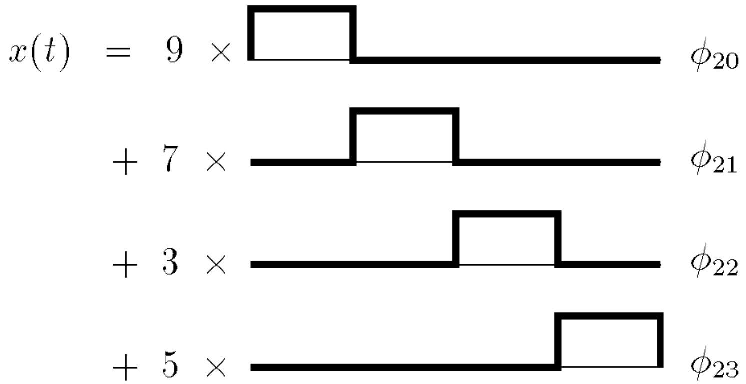

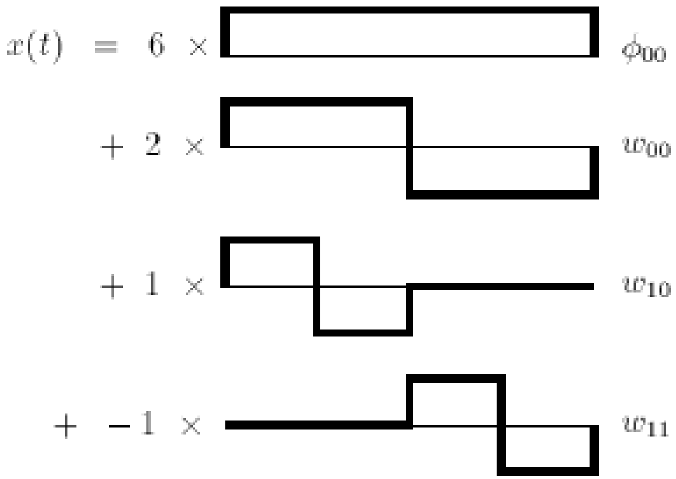

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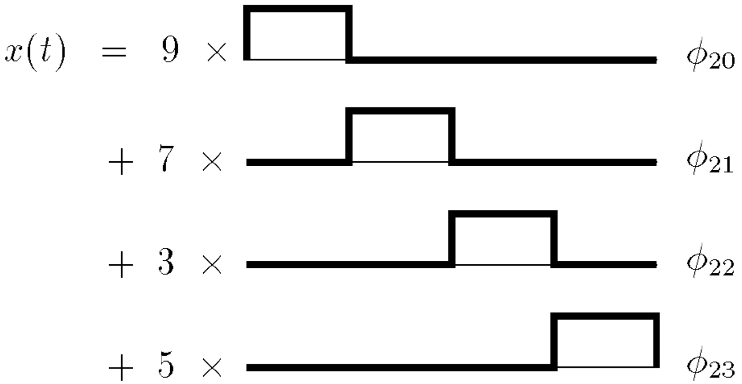
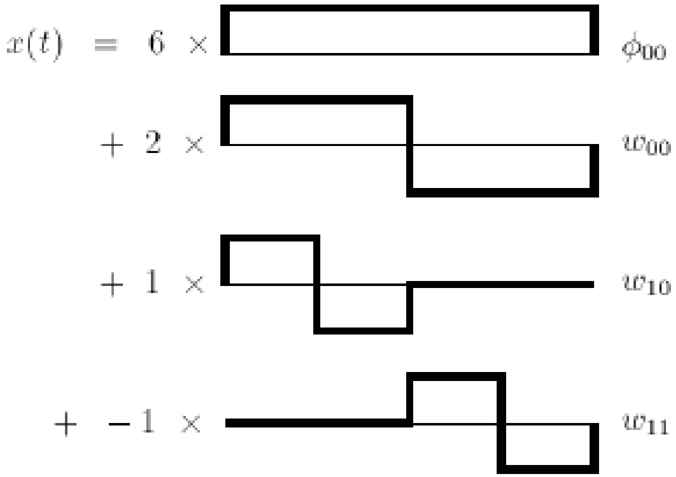
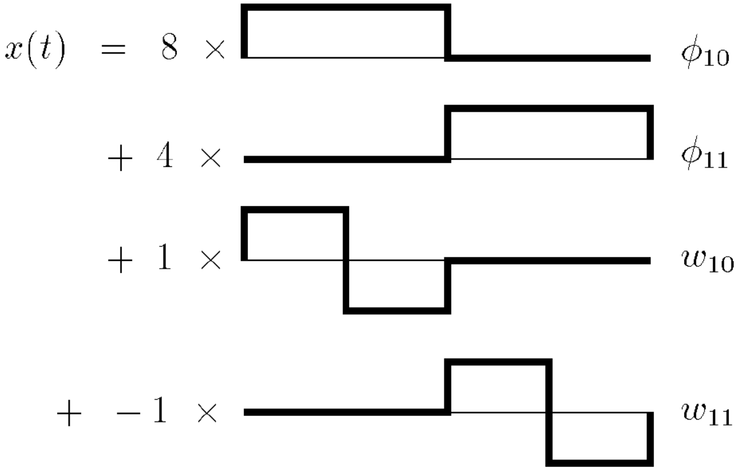






Scaling const. in ϕ is ignored for simplicity

Haar

- A sequence of embedded approximation subsets of $L_2(R)$:

$$\{0\} \leftarrow \ldots V_{j-1} \subset V_j \subset V_{j+1} \ldots \rightarrow \mathbf{L}^2(\mathbf{R})$$

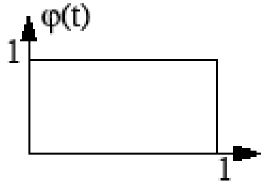
with : $f(x) \in V_j$ iff $f(2x) \in V_{j+1}$

$$f(t) \in V_0 \Leftrightarrow f(t-k) \in V_0,\ k \in \mathbb{Z}$$

$\{\phi(t-n)\}_{n \in I}$ forms a orthonormal basis for V_0

- And a sequence of orthogonal complements:

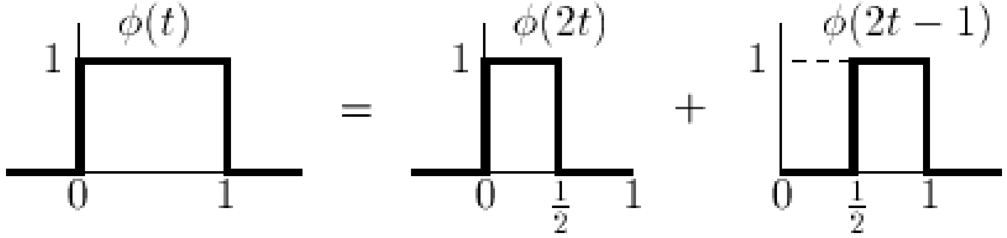
W_n such that $V_{n+1} = V_n \oplus W_n$



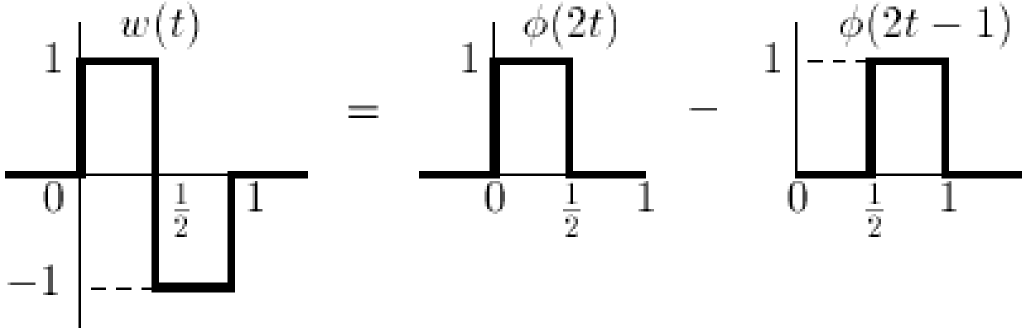
- ϕ is the scaling function. It's a low pass filter.
  - a basis in V_m is given by

$$\phi_{m,n}(t) = 2^{\frac{m}{2}} \phi(2^m t - n),\ n \in I$$

Haar: refinement equation



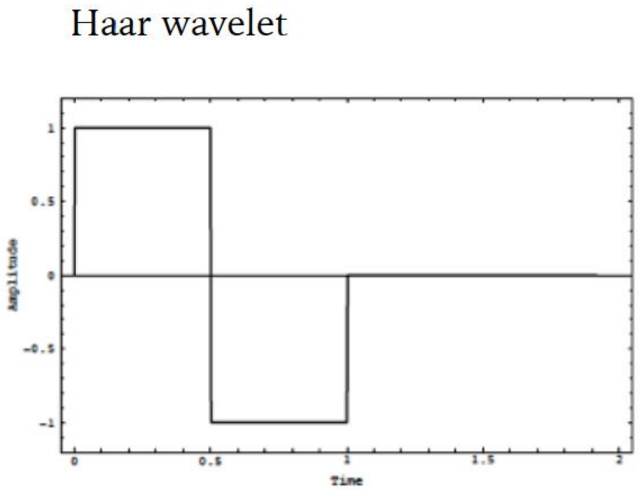
$$\varphi(t) = \sum_{n \in \mathbb{Z}} h_n \varphi(2t-n) = \varphi(2t) + \varphi(2t-1)$$

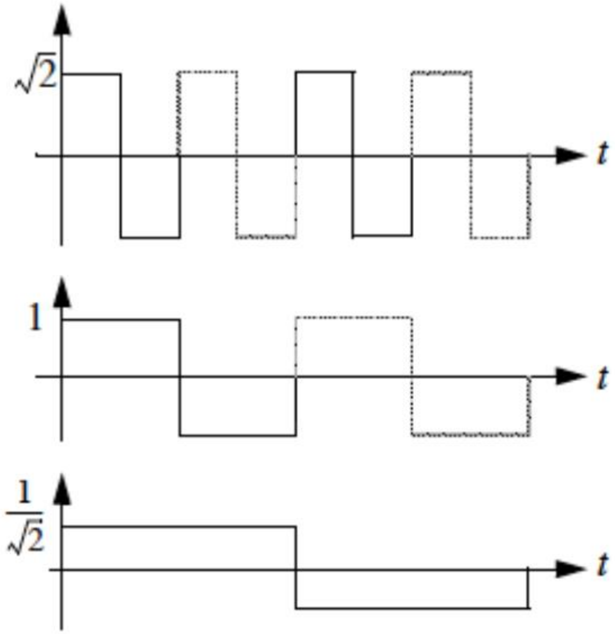


$$\psi(t) = \sum_{n \in \mathbb{Z}} g_n \varphi(2t-n)$$

Defines the wavelet function

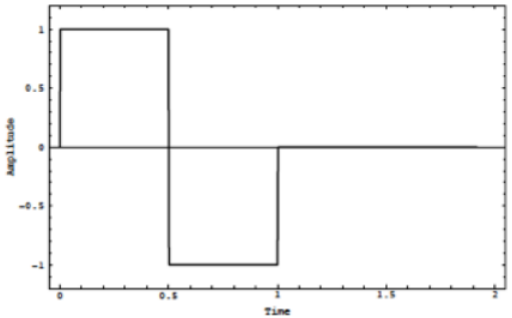
MRA: Haar matrix

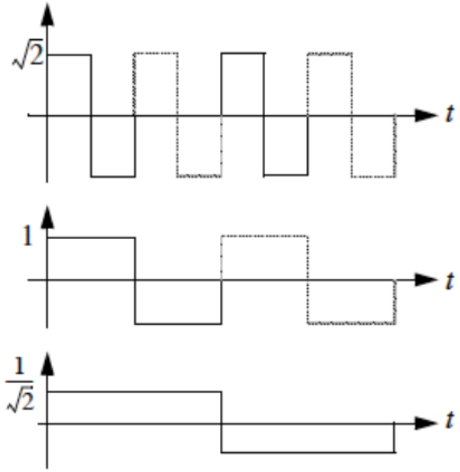




MRA: Haar matrix

- Haar wavelet





$$
\begin{array}{rl}
a{=}2 & \begin{bmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{bmatrix} \\
a{=}1 & \begin{bmatrix} 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \end{bmatrix} \\
a{=}0 & \begin{bmatrix} 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \end{bmatrix} \\
\text{scaling function:} & \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}
\end{array}
= \mathbf{H}^T
$$

Haar Transform (HT)

- Haar transform of a signal
  - Find the coefficients **w** given the input signal **f**
  - also use orthonormality and normalize matrix **H**

$$\mathbf{f} = \mathbf{H}\mathbf{w}$$

$$\mathbf{w} = \mathbf{H}^{-1}\mathbf{f} = \mathbf{H}^{T}\mathbf{f}$$

  - e.g.:

$$\frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ \sqrt{2} & -\sqrt{2} & 0 & 0 \\ 0 & 0 & \sqrt{2} & -\sqrt{2} \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 5 \\ -2 \\ -1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$$

MRA: continuous wavelets

- Fourier transform
  - Variable of frequency (ω)

$$\boxed{X(\omega) = \int_{-\infty}^{-\infty} x(t) e^{-j\omega t} dt}$$

- Wavelet transform
  - Variable of scale (s, σ, a, m) & time (τ, t, b, n)

$$\boxed{X(a,b) = \int_{-\infty}^{-\infty} x(t) \psi_{a,b}^{*}(t) dt}$$

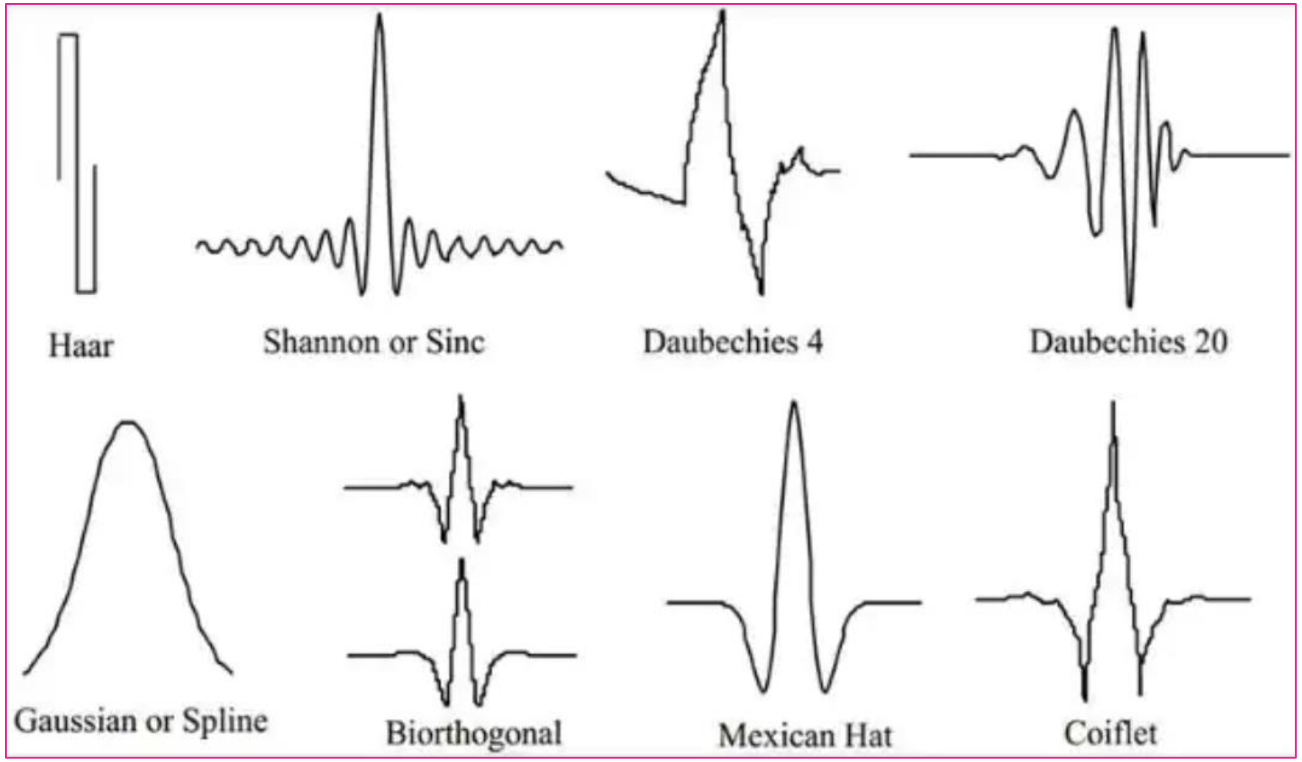
MRA: continuous wavelets

- WT: decompose a signal over a set of basis functions in complementary orthogonal spaces W
  - these basis functions we call wavelets (e.g. mother wavelets)
  - Wavelets have compact (finite) support

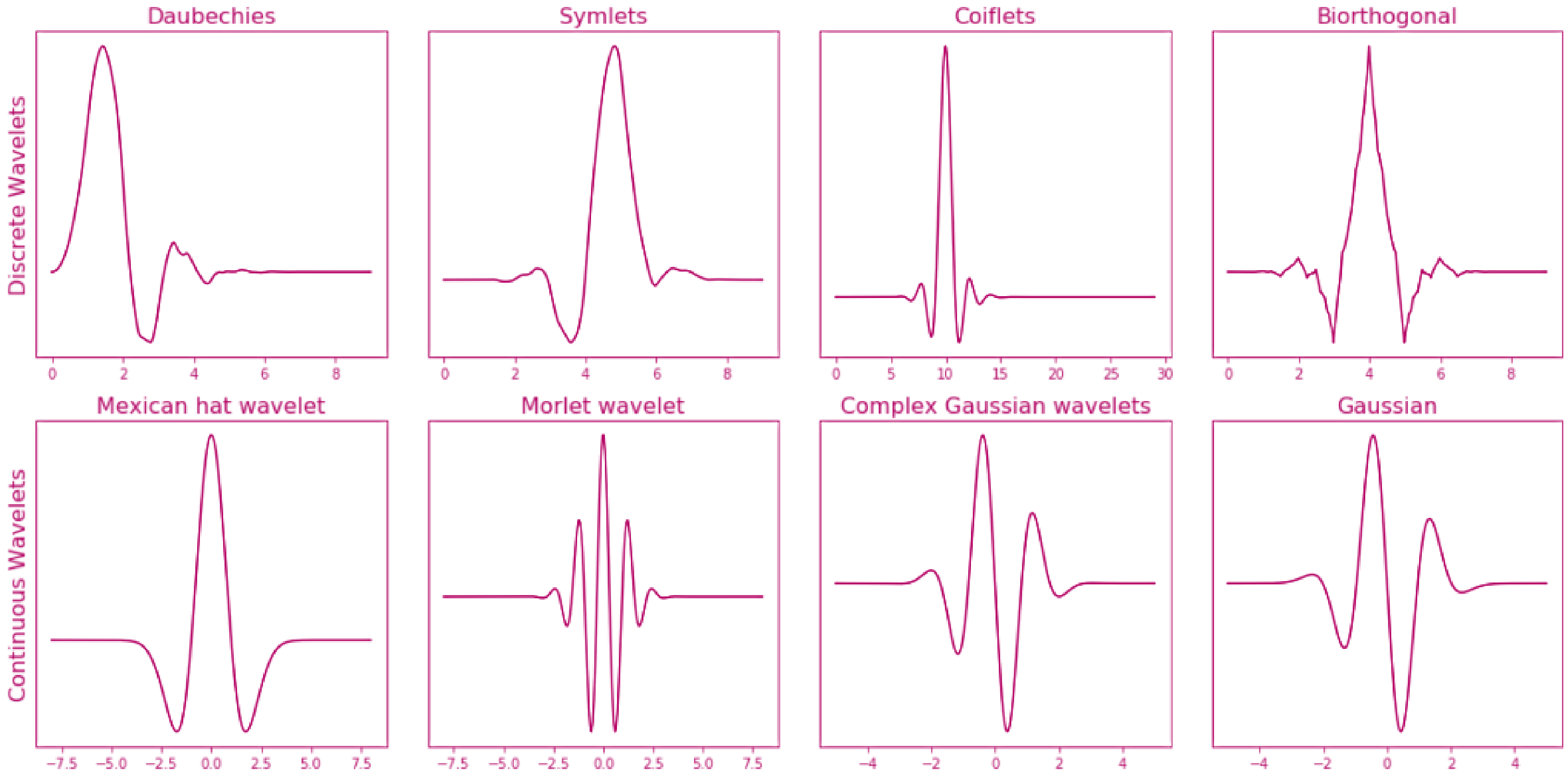
  - they should have zero mean and follow energy/info. preservation principal

Wavelets

- Some examples of mother wavelets



Wavelets

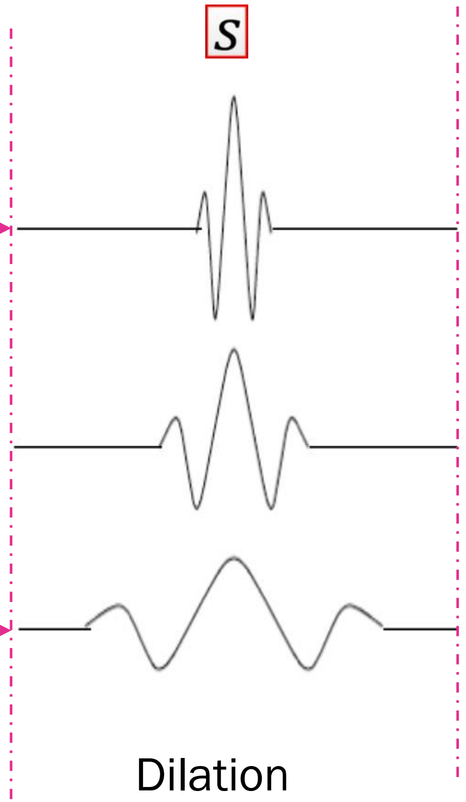

Daubechies
Symlets
Coiflets
Biorthogonal
Discrete Wavelets
Mexican hat wavelet
Morlet wavelet
Complex Gaussian wavelets
Gaussian
Continuous Wavelets

WT: creating new wavelets

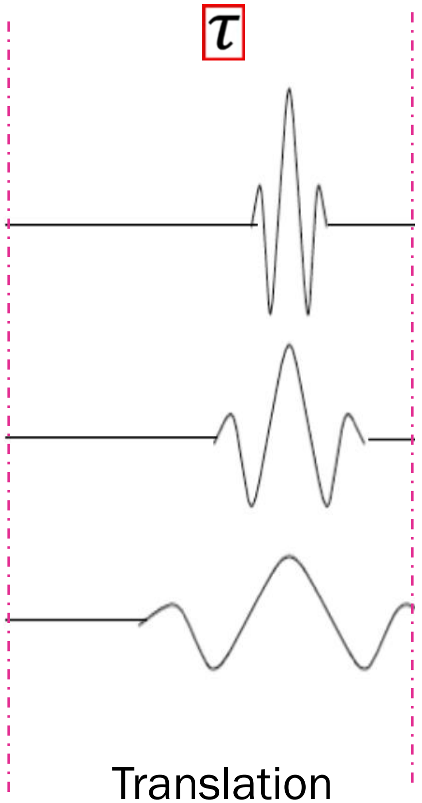
$$F(\tau, s) = \frac{1}{\sqrt{|s|}} \int_{-\infty}^{+\infty} f(t)\, \psi^*\left(\frac{t-\tau}{s}\right) dt$$

Good at resolving high freq component with good time resolution

Good at resolving low freq component with bad time resolution



Dilation

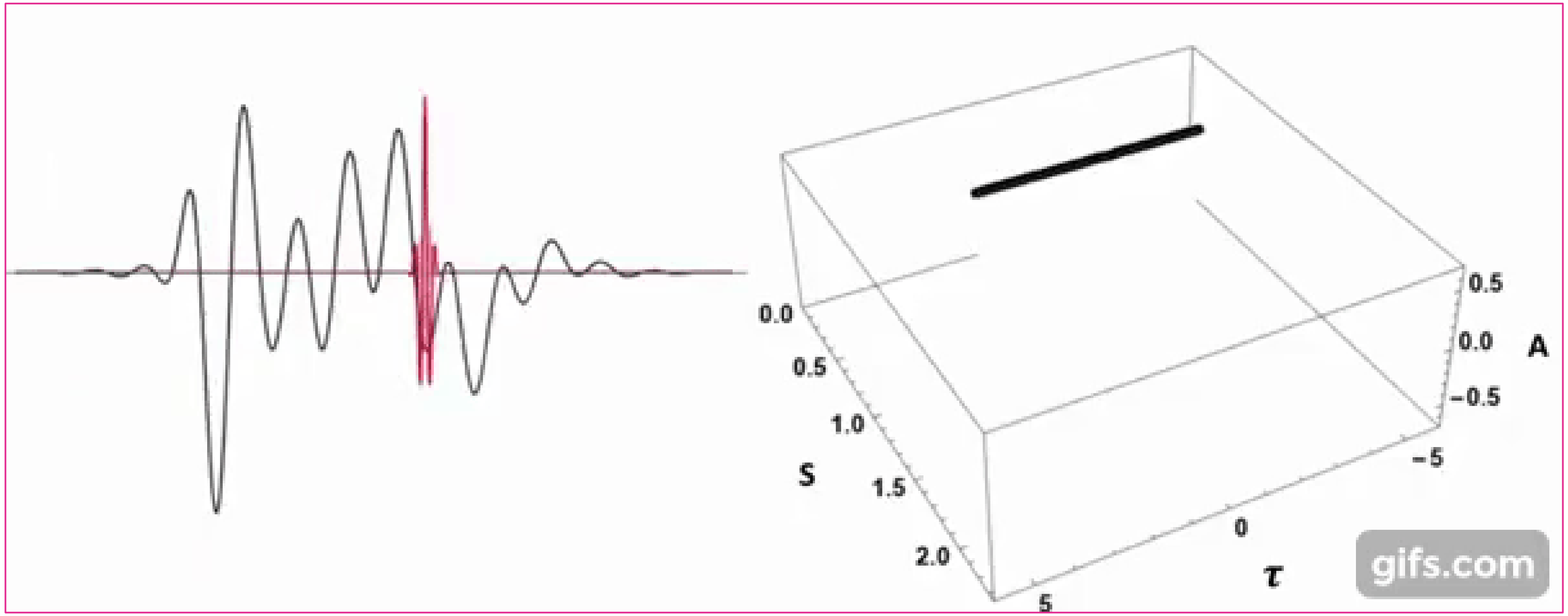


Translation

WT

$$F(\tau, s) = \frac{1}{\sqrt{|s|}} \int_{-\infty}^{+\infty} f(t)\, \psi^* \left(\frac{t-\tau}{s}\right) dt$$

s: scale = 1/freq



courtesy: A. Nicoll

DWT

- Discrete:

$$D[a,b] = \frac{1}{\sqrt{b}} \sum_{m=0}^{p-1} f[t_m] \psi \left[\frac{t_m - a}{b} \right]$$

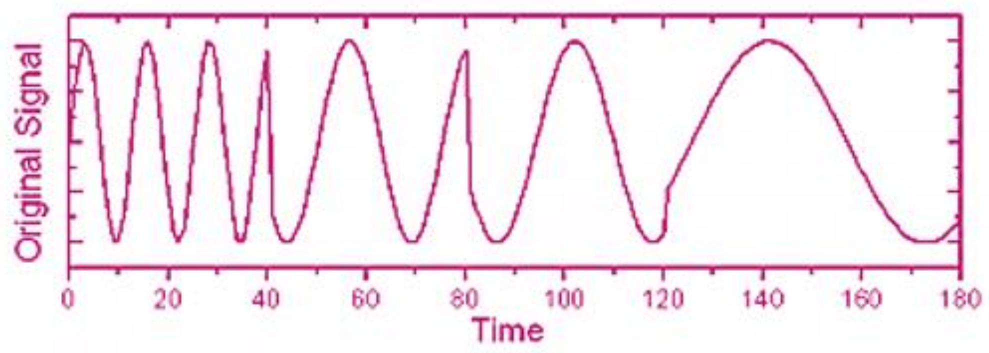
a = τ
b = s

$a = k2^{-j} \quad b = 2^{-j}$

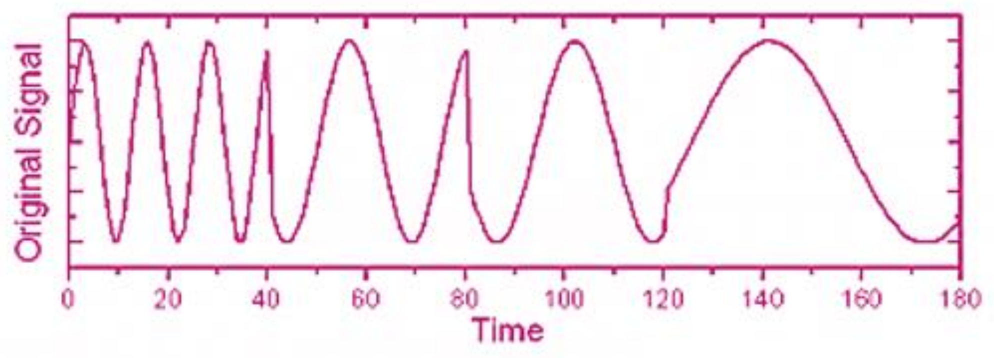
- Continuous:

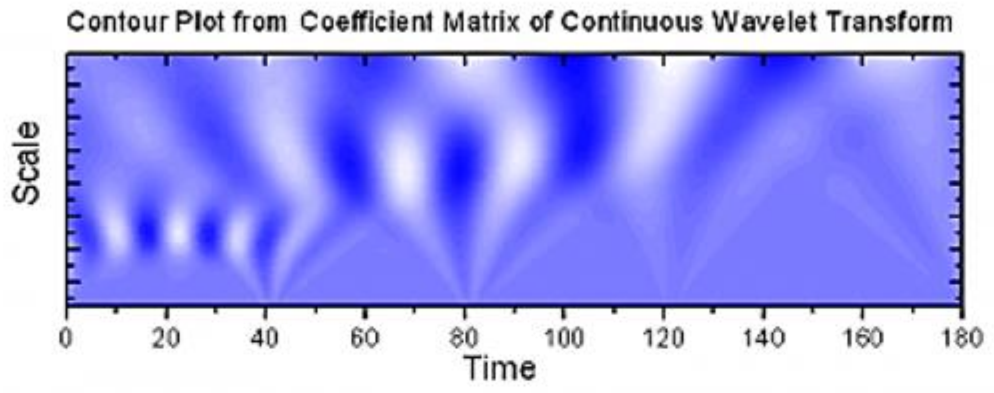
$$F(\tau, s) = \frac{1}{\sqrt{|s|}} \int_{-\infty}^{+\infty} f(t)\, \psi^* \left(\frac{t - \tau}{s} \right) dt$$

Continuous WT



Continuous WT





Conclusion

- Wavelet Transform

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- Wavelet Transform

- MRA
  - Examples
  - Basis functions
  - Wavelet transform

- Applications
  - Denoising
  - Compression