

Mathematical Reasoning

STATEMENT : or Mathematical statement.

A sentence which is either true or false but cannot be both are called a statement. A sentence which is an exclamatory or a wish or an imperative or an interrogative can not be a statement.

If a statement is true then its truth value is T and if it is false then its truth value is F

For ex.

- (i) "New Delhi is the capital of India", a true statement
- (ii) " $3 + 2 = 6$ ", a false statement
- (iii) "Where are you going ?" not a statement because it cannot be defined as true or false

Note : A statement cannot be both true and false at a time

SIMPLE STATEMENT :

Any statement whose truth value does not depend on other statement are called simple statement

For ex. (i) " $\sqrt{2}$ is an irrational number" (ii) "The set of real number is an infinite set"

COMPOUND STATEMENT :

A statement which is a combination of two or more simple statements are called compound statement

Here the simple statements which form a compound statement are known as its sub statements

For ex.

- (i) "If x is divisible by 2 then x is even number"
- (ii) " $\triangle ABC$ is equilateral if and only if its three sides are equal"

LOGICAL CONNECTIVES :

The words or phrases which combined simple statements to form a compound statement are called logical connectives.

In the following table some possible connectives, their symbols and the nature of the compound statement formed by them

S.N.	Connectives	symbol	use	operation
1.	and	\wedge	$p \wedge q$	conjunction
2.	or	\vee	$p \vee q$	disjunction
3.	not	\sim or ' '	$\sim p$ or p'	negation
4.	If then	\Rightarrow or \rightarrow	$p \Rightarrow q$ or $p \rightarrow q$	Implication or conditional
5.	If and only if (iff)	\Leftrightarrow or \leftrightarrow	$p \Leftrightarrow q$ or $p \leftrightarrow q$	Equivalence or Bi-conditional

Explanation :

(i) $p \wedge q \equiv$ statement p and q

($p \wedge q$ is true only when p and q both are true otherwise it is false)

(ii) $p \vee q \equiv$ statement p or q

($p \vee q$ is true if at least one from p and q is true i.e. $p \vee q$ is false only when p and q both are false)

(iii) $\sim p \equiv$ not statement p

($\sim p$ is true when p is false and $\sim p$ is false when p is true)

(iv) $p \Rightarrow q \equiv$ statement p then statement q

($p \Rightarrow q$ is false only when p is true and q is false otherwise it is true for all other cases)

(v) $p \Leftrightarrow q \equiv$ statement p if and only if statement q

($p \Leftrightarrow q$ is true only when p and q both are true or false otherwise it is false)

TRUTH TABLE :

A table which shows the relationship between the truth value of compound statement $S(p, q, r, \dots)$ and the truth values of its sub statements p, q, r, \dots is said to be truth table of compound statement S

If p and q are two simple statements then truth table for basic logical connectives are given below

Conjunction

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Disjunction

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Negation

p	$(\sim p)$
T	F
F	T

Conditional

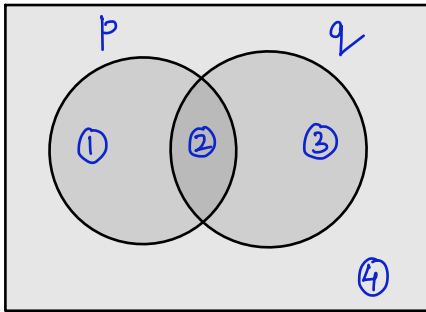
p	q	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

Biconditional

p	q	$p \rightarrow q$	$q \rightarrow p$	$(p \rightarrow q) \wedge (q \rightarrow p)$ or $p \leftrightarrow q$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

Note : If the compound statement contain n sub statements then its truth table will contain 2^n rows.

Use of set theory :-



$$p \vee q \rightarrow ①, ②, ③$$

$$p \wedge q \rightarrow ②$$

$$p \rightarrow q \rightarrow ②, ③, ④$$

$$(p \rightarrow q) \wedge (q \rightarrow p)$$

Region	p	q	$p \vee q$	$p \wedge q$	$p \rightarrow q$	$q \rightarrow p$	$p \leftrightarrow q$
①	T	F	T	F	F	T	F
②	T	T	T	T	T	T	T
③	F	T	T	F	T	F	F
④	F	F	F	F	T	T	T

Rem

De-morgan's law :-

$$\overline{A \cap B} \equiv \bar{A} \cup \bar{B}$$
$$\overline{A \cup B} \equiv \bar{A} \cap \bar{B}$$

**

$$p \rightarrow q \equiv \sim p \vee q$$

$$p \rightarrow q \equiv \sim ① \equiv \sim(p \wedge \sim q)$$
$$\downarrow$$
$$\sim p \vee q$$

$$\sim(A \cap B) \equiv \sim A \cup \sim B$$

$$\sim(A \cup B) \equiv \sim A \cap \sim B$$

LOGICAL EQUIVALENCE :

Two compound statements $S_1(p, q, r, \dots)$ and $S_2(p, q, r, \dots)$ are said to be logically equivalent or simply equivalent if they have same truth values for all logically possibilities

Two statements S_1 and S_2 are equivalent if they have identical truth table i.e. the entries in the last column of their truth table are same. If statements S_1 and S_2 are equivalent then we write $S_1 \equiv S_2$

For ex. The truth table for $(p \rightarrow q)$ and $(\sim p \vee q)$ given as below

p	q	$(\sim p)$	$p \rightarrow q$	$\sim p \vee q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

We observe that last two columns of the above truth table are identical hence compound statements $(p \rightarrow q)$ and $(\sim p \vee q)$ are equivalent

i.e.

$$p \rightarrow q \equiv \sim p \vee q$$

TAUTOLOGY AND CONTRADICTION :

- (i) **Tautology** : A statement is said to be a tautology if it is true for all logical possibilities
i.e. its truth value always T. **it is denoted by t.**

For ex. the statement $p \vee \sim (p \wedge q)$ is a tautology

p	q	$p \wedge q$	$\sim (p \wedge q)$	$p \vee \sim (p \wedge q)$
T	T	T	F	T
T	F	F	T	T
F	T	F	T	T
F	F	F	T	T

Clearly, The truth value of $p \vee \sim (p \wedge q)$ is T for all values of p and q. so $p \vee \sim (p \wedge q)$ is a tautology

- (ii) **Contradiction** : A statement is a contradiction if it is false for all logical possibilities.

i.e. its truth value always F. **It is denoted by c.**

Contradiction = Fallacy.

For ex. The statement $(p \vee q) \wedge (\sim p \wedge \sim q)$ is a contradiction

p	q	$\sim p$	$\sim q$	$p \vee q$	$(\sim p \wedge \sim q)$	$(p \vee q) \wedge (\sim p \wedge \sim q)$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	F

Clearly, then truth value of $(p \vee q) \wedge (\sim p \wedge \sim q)$ is F for all value of p and q. So $(p \vee q) \wedge (\sim p \wedge \sim q)$ is a contradiction.

Note : The negation of a tautology is a contradiction and negation of a contradiction is a tautology

DUALITY :

Two compound statements S_1 and S_2 are said to be duals of each other if one can be obtained from the other by replacing \wedge by \vee and \vee by \wedge

If a compound statement contains the special variable t (tautology) and c (contradiction) then obtain its dual we replaced t by c and c by t in addition to replacing \wedge by \vee and \vee by \wedge .

Note :

(i) the connectives \wedge and \vee are also called dual of each other.

For ex. The duals of the following statements

(i) $(p \wedge q) \vee (r \vee s)$ (ii) $(p \vee t) \wedge (p \vee c)$

(iii) $\sim(p \wedge q) \vee [p \wedge \sim(q \vee \sim s)]$

are as given below

(i) $(p \vee q) \wedge (r \wedge s)$

(ii) $(p \wedge c) \vee (p \wedge t)$

(iii) $\sim(p \vee q) \wedge [p \vee \sim(q \wedge \sim s)]$

CONVERSE, INVERSE AND CONTRAPOSITIVE OF THE CONDITIONAL

STATEMENT ($p \rightarrow q$):

- (i) **Converse** : The converse of the conditional statement $p \rightarrow q$ is defined as $q \rightarrow p$
- (ii) **Inverse** : The inverse of the conditional statement $p \rightarrow q$ is defined as $\sim p \rightarrow \sim q$
- (iii) **Contrapositive** : The contrapositive of conditional statement $p \rightarrow q$ is defined as $\sim q \rightarrow \sim p$

Same

NEGATION OF COMPOUND STATEMENTS :

If p and q are two statements then

(i) **Negation of conjunction** : $\sim(p \wedge q) \equiv \sim p \vee \sim q$

p	q	$\sim p$	$\sim q$	$(p \wedge q)$	$\sim(p \wedge q)$	$(\sim p \vee \sim q)$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

(ii) **Negation of disjunction** : $\sim(p \vee q) \equiv \sim p \wedge \sim q$

p	q	$\sim p$	$\sim q$	$(p \vee q)$	$\sim(p \vee q)$	$(\sim p \wedge \sim q)$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

(iii) **Negation of conditional** : $\sim(p \rightarrow q) \equiv p \wedge \sim q$

p	q	$\sim q$	$(p \rightarrow q)$	$\sim(p \rightarrow q)$	$(p \wedge \sim q)$
T	T	F	T	F	F
T	F	T	F	T	T
F	T	F	T	F	F
F	F	T	T	F	F

(iv) **Negation of biconditional** : $\sim(p \leftrightarrow q) \equiv (p \wedge \sim q) \vee (q \wedge \sim p)$

we know that $p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$

$\therefore \sim(p \leftrightarrow q) \equiv \sim[(p \rightarrow q) \wedge (q \rightarrow p)]$

$\equiv \sim(p \rightarrow q) \vee \sim(q \rightarrow p)$

$\equiv (p \wedge \sim q) \vee (q \wedge \sim p)$

Note : The above result also can be proved by preparing truth table for $\sim(p \leftrightarrow q)$ and $(p \wedge \sim q) \vee (q \wedge \sim p)$

ALGEBRA OF STATEMENTS :

If p, q, r are any three statements then the some law of algebra of statements are as follow

(i) Idempotent Laws :

$$(a) p \wedge p \equiv p \quad (b) p \vee p \equiv p$$

$$\text{i.e. } p \wedge p \equiv p \equiv p \vee p$$

p	$(p \wedge p)$	$(p \vee p)$
T	T	T
F	F	F

(ii) Comutative laws :

$$(a) p \wedge q \equiv q \wedge p \quad (b) p \vee q \equiv q \vee p$$

p	q	$(p \wedge q)$	$(q \wedge p)$	$(p \vee q)$	$(q \vee p)$
T	T	T	T	T	T
T	F	F	F	T	T
F	T	F	F	T	T
F	F	F	F	F	F

(iii) Associative laws :

$$(a) (p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$$

$$(b) (p \vee q) \vee r \equiv p \vee (q \vee r)$$

p	q	r	$(p \wedge q)$	$(q \wedge r)$	$(p \wedge q) \wedge r$	$p \wedge (q \wedge r)$
T	T	T	T	T	T	T
T	T	F	T	F	F	F
T	F	T	F	F	F	F
T	F	F	F	F	F	F
F	T	T	F	T	F	F
F	T	F	F	F	F	F
F	F	T	F	F	F	F
F	F	F	F	F	F	F

Similarly we can proved result (b)

(iv) Distributive laws :

$$(a) p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r) \quad (c) p \wedge (q \wedge r) \equiv (p \wedge q) \wedge (p \wedge r)$$

$$(b) p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r) \quad (d) p \vee (q \vee r) \equiv (p \vee q) \vee (p \vee r)$$

p	q	r	$(q \vee r)$	$(p \wedge q)$	$(p \wedge r)$	$p \wedge (q \vee r)$	$(p \wedge q) \vee (p \wedge r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	F	T	T
T	F	T	T	F	T	T	T
T	F	F	F	F	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F

Similarly we can prove result (b), (c), (d)

- (v) **De Morgan Laws** : (a) $\sim(p \wedge q) \equiv \sim p \vee \sim q$
 (b) $\sim(p \vee q) \equiv \sim p \wedge \sim q$

p	q	$\sim p$	$\sim q$	$(p \wedge q)$	$\sim(p \wedge q)$	$(\sim p \vee \sim q)$
T	T	F	F	T	F	F
T	F	F	T	F	T	T
F	T	T	F	F	T	T
F	F	T	T	F	T	T

Similarly we can prove result (b)

- (vi) **Involution laws (or Double negation laws)** :

$$\sim(\sim p) \equiv p$$

p	$\sim p$	$\sim(\sim p)$
T	F	T
F	T	F

- (vii) **Identity Laws** : If p is a statement and t and c are tautology and contradiction respectively then

$$(a) p \wedge t \equiv p \quad (b) p \vee t \equiv t \quad (c) p \wedge c \equiv c \quad (d) p \vee c \equiv p$$

p	t	c	$(p \wedge t)$	$(p \vee t)$	$(p \wedge c)$	$(p \vee c)$
T	T	F	T	T	F	T
F	T	F	F	T	F	F

- (viii) **Complement Laws** :

$$(a) p \wedge (\sim p) \equiv c \quad (b) p \vee (\sim p) \equiv t \quad (c) (\sim t) \equiv c \quad (d) (\sim c) \equiv t$$

p	$\sim p$	$(p \wedge \sim p)$	$(p \vee \sim p)$
T	F	F	T
F	T	F	T