Complex Numbers

$x^2+1=0 \longrightarrow$ $x \in R$, $x \in \phi$

$\downarrow$

$x^2 = -1$

$x^2 = i^2$ $\quad i = \sqrt{-1}$

$\boxed{x = \pm i}$

Any C.N. $z = a + ib$

$a = Re(z)$ $\quad b = Im(z)$

$a, b \in R$

If $a=0$, $b \neq 0$, $z = ib$ = purely img.

If $b=0$, $a \neq 0$, $z = a$ = purely real

If $a=0=b$ $\quad z = 0 \Rightarrow$ purely real as well as purely img.

Any Real number is C.N. with img. part $=0$ $\quad 5 = 5 + i0$

Algebra of C.N.

$z_1 = x_1 + iy_1 \qquad z_2 = x_2 + iy_2$

① Addition $\Rightarrow z_1 + z_2 = (x_1 + x_2) + i(y_1 + y_2)$

② Subtraction $\Rightarrow z_1 - z_2 = (x_1 - x_2) + i(y_1 - y_2)$

③ multiplication $\Rightarrow z_1 \cdot z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)$

④ division $\Rightarrow \dfrac{z_1}{z_2} \quad (z_2 \neq 0)$

$$\frac{x_1 + iy_1}{x_2 + iy_2} \times \frac{x_2 - iy_2}{x_2 - iy_2}$$

$$\frac{z_1}{z_2} = \frac{(x_1 x_2 + y_1 y_2) + i(y_1 x_2 - x_1 y_2)}{x_2^2 - i^2 y_2^2}$$

$$\frac{z_1}{z_2} = \left(\frac{x_1 x_2 + y_1 y_2}{x_2^2 + y_2^2}\right) + i\left(\frac{y_1 x_2 - y_2 x_1}{x_2^2 + y_2^2}\right)$$

⑤ Equality of C.N. If $z_1 = z_2$ then

$Re(z_1) = Re(z_2)$ & $Im(z_1) = Im(z_2)$

If $(x-3)+i(y-5)=3+2i$, $x, y \in R$

then find (x, y)

$\Rightarrow x-3=3$ & $y-5=2$

$x=6$ $\quad$ $y=7$

$(x-6)+i(y-7)=0$

$x-6=0$ & $y-7=0$

$x=6$, $y=7$

(6) Additive inverse of $z = -z$

$$z+(-z)=0 \leftarrow \text{Additive identity}$$

(7) multiplicative inverse of $z = \frac{1}{z}$ $\quad (z \neq 0)$

$$z\left(\frac{1}{z}\right)=1 \leftarrow \text{multiplictive identity}$$

Powers of i

i	$i^2=-1$	$i^3=-i$	$i^4=1$
$i^5=i$	$i^6=i^2=-1$	$i^7=-i$	$i^8=1$
$i^{4I+1}=i$	$i^{4I+2}=-1$	$i^{4I+3}=-i$	$i^{4I}=1$

$$i^{4I}+i^{4I+1}+i^{4I+2}+i^{4I+3}=0$$

$i^{562} = i^{560+2} = (i^4)^{140}\, i^2 = -1$

$i^{565} + i^{566} + i^{567} + i^{568} = 0$

Ex:- $\sum_{r=0}^{102} i^r = ??$ $(i = \sqrt{-1})$

$1 + \underbrace{i^1 + i^2 + i^3 + i^4}_{0} + \underbrace{\quad\cdot\quad}_{0} + \underbrace{i^{97} + i^{98} + i^{99} + i^{100}}_{0} + i^{101} + i^{102}$

$= 1 + i^{100}\, i + i^{100}\, i^2 = 1 + i - 1 = i$

Ex:- If $\left(\frac{1+i}{1-i}\right)^n = \text{Real}$, $n \in N$ find n

$\left(\frac{1+i}{1-i} \times \frac{1+i}{1+i}\right)^n = \text{Real}$

$\left(\frac{(1+i)^2}{1-i^2}\right)^n$

$\left(\frac{\cancel{1} + \cancel{i^2} + 2i}{1+1}\right)^n = i^n = \text{Real}$ $\quad n \in N$

$n = 2, 4, 6, 8 \ldots\ldots$

$n = 2k$, $k \in N$

$\left(\frac{1+i}{1-i}\right)^n = 1$ find $n \in N$

$i^n = 1$ $\quad \boxed{n = 4K, \ K \in N}$

Ex:- find roots of $z^2 + 2(1+2i)z - (11+2i) = 0$

$$z = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$az^2 + bz + c = 0$

$a, b, c \in R$

$D > 0$

$D = 0$

$D < 0$

$$z = \frac{-2(1+2i) \pm \sqrt{4(1+2i)^2 + 4(11+2i)}}{2}$$

$$z = -(1+2i) \pm \sqrt{1 + 4i^2 + 4i + 11 + 2i}$$

$$z = -(1+2i) \pm \sqrt{8+6i}$$

$$z = -(1+2i) \pm (3+i)$$

(+) $z = 2 - i$

(−) $z = -4 - 3i$

Let

$\sqrt{8+6i} = a + ib$

$8 + 6i = a^2 - b^2 + 2abi$

$\left.\begin{array}{l} a^2 - b^2 = 8 \\ ab = 3 \end{array}\right\} \begin{array}{l} a = 3 \\ b = 1 \end{array}$

Ex:- find square root of $(9 + 40i)$

Let $\sqrt{9+40i} = a+ib$ $\quad$ $a, b \in R$

$9+40i = a^2-b^2+2abi$

$9 = a^2-b^2$ $\quad\Big|\quad$ $40 = 2ab \Rightarrow \boxed{b = \frac{20}{a}}$

$9 = a^2 - \frac{400}{a^2} \Rightarrow (a^2)^2 - 9a^2 - 400 = 0$

$(a^2-25)(a^2+16) = 0$

$a = \pm 5$

$a=5,\ b=4$ or $a=-5,\ b=-4$

Square root of $9+40i = \pm(5+4i)$

Ex:- If $2z^2+2(i-1) = z-10$ has a purely img. root then find other root.

Let $z = iy$ $\quad$ $y \in R$

$2i^2y^2+2(i-1) = iy-10$

$-2y^2-2+2i = iy-10$

Compare both sides

$-2y^2 - 2 = -10$ & $2 = y$

$y^2 = 4$

$y = \pm 2 \quad \boxed{y = 2}$

$z = 2i$ is a root

$2z^2 - z + 2i + 8 = 0$

sum of roots $= +\frac{1}{2}$ product $= \frac{2i+8}{2}$

$2i + \beta = \frac{1}{2}$

$\beta = \frac{1}{2} - 2i$

Exe $z^2 - (3+i)z + m + 2i = 0$ | $z = C.N$, $m \in R$

If eqn has a real root then find m and other root.

Let $z = x \in R$ is root

$z^2 - (3+i)z + m + 2i = 0$

$x^2 - (3+i)x + m + 2i = 0$

$(x^2 - 3x + m) + i(2 - x) = 0$

$x^2 - 3x + m = 0$

$4 - 6 + m = 0$

$m = 2$

$2 - x = 0$

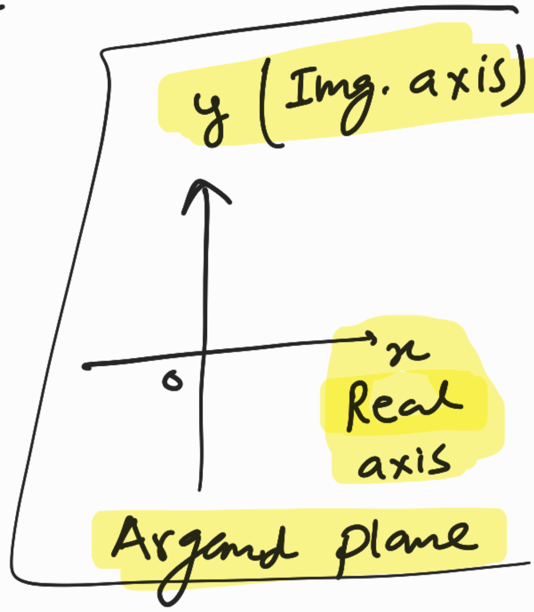
$x = 2$ Root

$z^2 - (3+i)z + 2 + 2i = 0$

$z = 2,\quad 2 + \beta = 3 + i$

$\beta = 1 + i$

Important terms

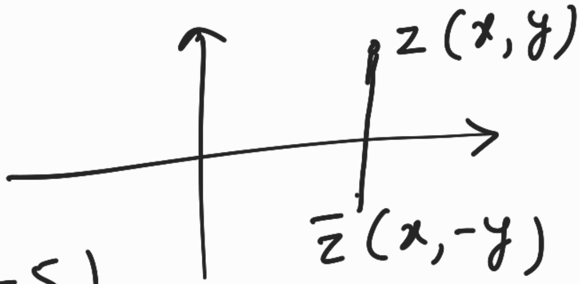


① Conjugate of $z = \bar{z}$

$z = x + iy = (x, y)$

$\bar{z} = x - iy = (x, -y)$

$\bar{z}$ is image of z w.r.t. Real axis



$z = 3 - 5i$

$z = 3 + i(-5) \equiv (3, -5)$

$\bar{z} = 3 - i(-5) = 3 + i5 = (3, 5)$

$\mathbb{C}$ = set of C.N.

properties $z, z_1, z_2 \in \mathbb{C}$

① $\overline{(\bar{z})} = z$

$$\left.\begin{aligned} z &= x+iy \\ \bar{z} &= x-iy \end{aligned}\right\} \quad \boxed{\begin{aligned} Re(z) &= x \\ &= Re(\bar{z}) \end{aligned}}$$

② $z+\bar{z} = 2\,Re(z)$

$x_1+iy_1 = z_1$

$z-\bar{z} = 2i\,Im(z)$

$x_2+iy_2 = z_2$

③ $\overline{z_1 \pm z_2} = \bar{z}_1 \pm \bar{z}_2$

$\overline{z_1+z_2+\cdots+z_n} = \bar{z}_1+\bar{z}_2+\cdots+\bar{z}_n$

④ $\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$

⑤ $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2} \quad (z_2 \neq 0)$

⑥ $z\,\bar{z} = x^2+y^2$

$(x+iy)(x-iy) = x^2-i^2y^2$

$$\boxed{\begin{aligned} &\text{If } z_1 = z_2 \\ &\text{then} \\ &\bar{z}_1 = \bar{z}_2 \end{aligned}}$$

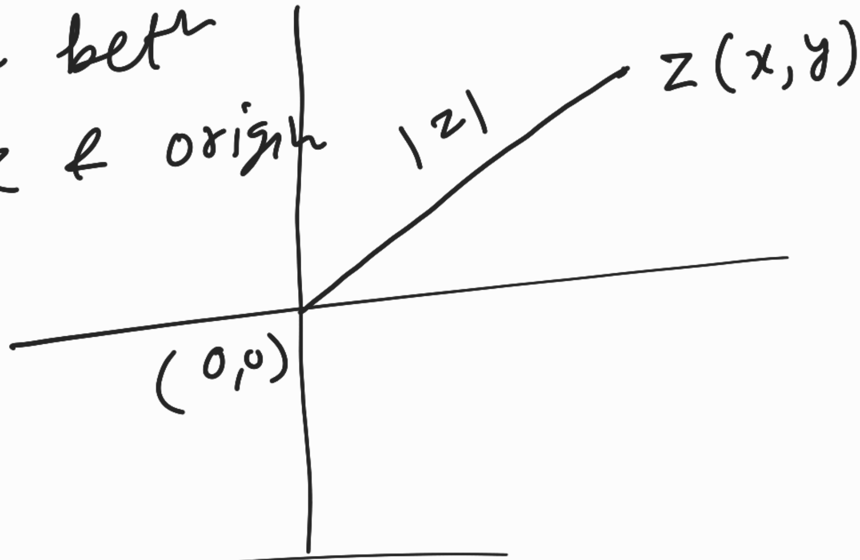
⑦ If $f(a+ib) = x+iy$

then $f(a-ib) = x-iy$

Modulus of $z = |z|$

$z = x + iy$ $\quad |z| = \sqrt{x^2 + y^2}$

$|z|$ = distance betn point z & origin



$$|z| = \sqrt{(\mathrm{Re}\,z)^2 + (\mathrm{Im}\,z)^2}$$

Properties

1. $|z| \geq 0$
2. $|z| \geq \mathrm{Re}(z)$

   $|z| \geq \mathrm{Im}(z)$
3. $|z| = |\bar{z}| = |-z| = |-\bar{z}|$
4. $z\bar{z} = |z|^2$
5. $|z_1 z_2| = |z_1|\,|z_2|$

$$|z_1 z_2 \cdots z_n| = |z_1||z_2| \cdots |z_n|$$

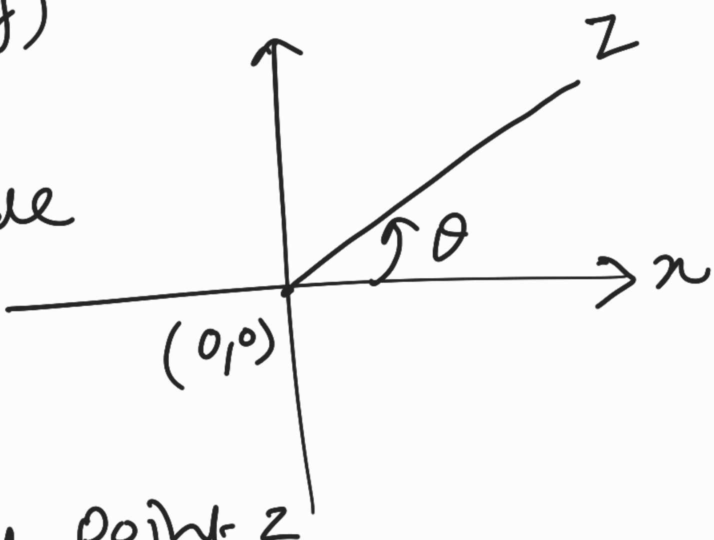
6. $\left|\frac{z_1}{z_2}\right| = \frac{|z_1|}{|z_2|} \quad (z_2 \neq 0)$

7. $|z^n| = |z|^n$

Argument of z / Amplitude of z

$z = x + iy = (x, y)$

arg(z) = angle made by line segment joining origin and point z with +x axis

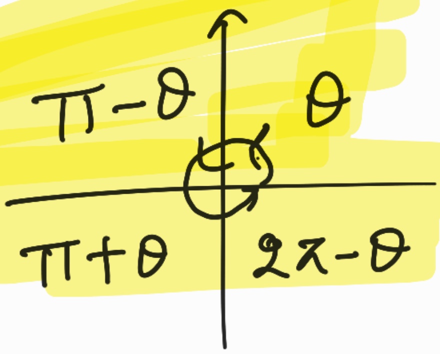


anticlock

$z = x + iy$

$\tan\theta = |y/x|$

arg(z) ⇒



$\theta = \tan^{-1}\left|\frac{y}{x}\right|$ = acute angle

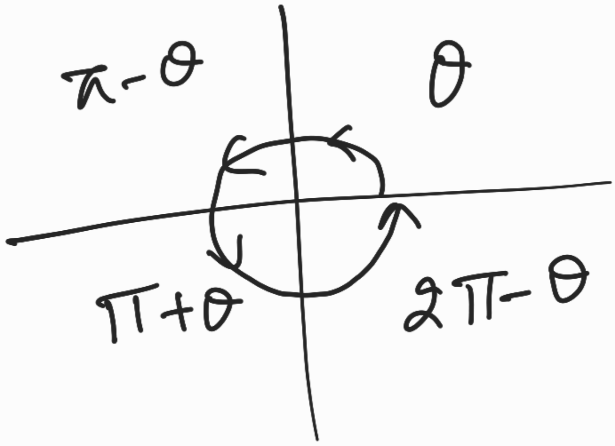
arg(z) depends on quadrant

① Least +ve argument

$$\boxed{0 < \arg(z) \leq 2\pi}$$

$z = (x+iy)$

$\tan^{-1}\left|\frac{y}{x}\right| = \theta$



② General value of argument

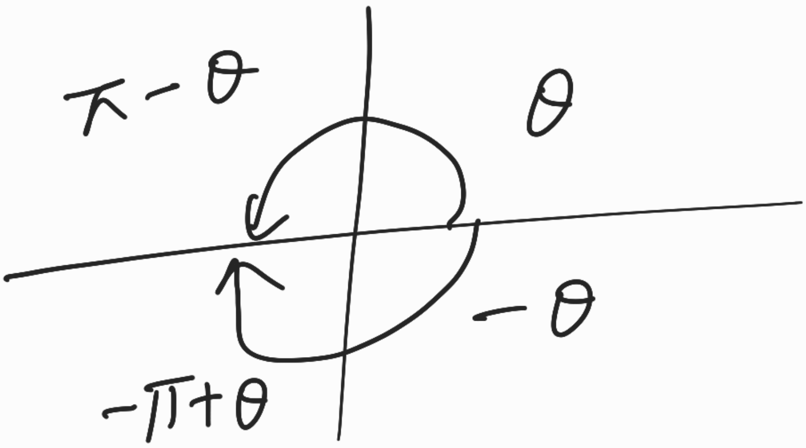
$$\boxed{2n\pi \pm \alpha}$$

$n \in I$

α = Least +ve arg(z)

③ Principal value of argument = Amplitude

$$-\pi < \operatorname{Amp}(z) \le \pi$$



$$\theta = \tan^{-1}\left|\frac{y}{x}\right|$$

Ex:-

	Least +ve arg(z)	general value arg(z)	amp(z)
① $z = 1+\sqrt{3}i$ $= (1, \sqrt{3})$ I^{st} quadrant $\theta = \pi/3$	$\frac{\pi}{3}$ $\pi-\theta$ | θ $\pi+\theta$ | $2\pi-\theta$	$2n\pi \pm \theta$	$\frac{\pi}{3}$ $\pi-\theta$ | θ $-\pi+\theta$ | $-\theta$
② $z = \sqrt{3}-i$	$2\pi - \frac{\pi}{6} = \frac{11\pi}{6}$	$2n\pi \pm 11\frac{\pi}{6}$	$-\pi/6$

$z = (\sqrt{3}, -1)$ $\in$ 4th quadrant

$$\tan\theta = \left|\frac{-1}{\sqrt{3}}\right| \Rightarrow \theta = \pi/6$$

③ $z = -4\sqrt{3} - 4i \equiv (-4\sqrt{3}, -4) \in$ 3rd quad.

$$\tan\theta = \left|\frac{-4}{-4\sqrt{3}}\right| = \frac{1}{\sqrt{3}} \Rightarrow \theta = \pi/6$$

Least +ve $\arg(z) = \pi + \theta = \frac{7\pi}{6}$

general $\arg(z) = 2n\pi \pm 7\pi/6$

$\text{amp}(z) = -\pi + \theta = -\pi + \frac{\pi}{6} = \frac{-5\pi}{6}$

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④ $z = -1 + i = (-1, 1) \in$ 2nd quad

$$\tan\theta = \left|\frac{1}{-1}\right| = 1 \Rightarrow \theta = \pi/4$$

Least +ve $= \pi - \pi/4 = \frac{3\pi}{4} = \text{amp}(z)$

general $\arg(z) = 2n\pi \pm 3\pi/4$

prop. $\Rightarrow$ ① $\underbrace{\arg(\bar{z})}_{\operatorname{amp}(\bar{z})} = -\arg(z)$

$\operatorname{amp}(\bar{z}) = -\operatorname{amp}(z)$

② $\operatorname{amp}(z_1 \cdot z_2) = \operatorname{amp}(z_1) + \operatorname{amp}(z_2) + 2k\pi, \; k \in I$

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③ $\operatorname{amp}\left(\frac{z_1}{z_2}\right) = \operatorname{amp}(z_1) - \operatorname{amp}(z_2) + 2k\pi$

④ $\operatorname{amp}(z^n) = n \operatorname{amp}(z) + 2k\pi$

$k \in I$