CONTINUITY - (L)

Complete Solutions Of DYS & Exercises

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Subject ________

$\boxed{DYS \Rightarrow 1}$

(i) If $f(x) = \begin{cases} \cos x & ; x \geq 0 \\ x + K, & x < 0 \end{cases}$

find the value of K if $f(x)$ is continuous at $x = 0$

$\because f(x)$ is continuous at $x=0$ So at $x=0$ both left and Right limits must exist and both must be equal to $\cos x$.

$$\lim_{x \to 0^-} x + K \Rightarrow 0 + K \Rightarrow K$$

$$\lim_{x \to 0} \cos x \Rightarrow 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$\cos(0) = 1$$

$$\lim_{x \to 0^+} \cos x \Rightarrow \cos(0) = 1$$

$$\therefore \boxed{K = 1}$$

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(ii) If $f(x) = \begin{cases} \dfrac{|x+2|}{\tan^{-1}(x+2)} & ; \quad x \neq -2 \\ 2 & ; \quad x = -2 \end{cases}$

then discuss the continuity of $f(x)$ at $x = -2$

$\begin{cases} \dfrac{x+2}{\tan^{-1}(x+2)} \\ \dfrac{-(x+2)}{\tan^{-1}(x+2)} & ; \quad x < -2 \end{cases}$

$\lim\limits_{x \to -2^-} \dfrac{-(x+2)}{\tan^{-1}(x+2)}$

$x \to -2 - h$, $h \to 0$ $\therefore \lim\limits_{h \to 0} \dfrac{-(-2-h+2)}{\tan^{-1}(-2-h+2)}$

$\lim\limits_{h \to 0} \dfrac{-(-h)}{\tan^{-1}(-h)} = -1$

$\lim\limits_{x \to -2^+} \dfrac{x+2}{\tan^{-1}(x+2)}$

$x \to -2 + h$, $h \to 0$ $\therefore \lim\limits_{h \to 0} \dfrac{-2+h+2}{\tan^{-1}(-2+h+2)}$

$\lim\limits_{h \to 0} \dfrac{h}{\tan^{-1} h} \Rightarrow 1$

$\therefore$ L.H.L $\neq$ R.H.L

$\therefore$ $\boxed{\text{Discontinuous at } x = -2}$

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DYS ⇒ 2

(i) If $f(x) = \begin{cases} \dfrac{x^2}{9} & : 0 \le x < 1 \\ -1 & : 1 \le x < \sqrt{2} \\ \dfrac{2b^2 - 4b}{x^2} & : \sqrt{2} \le x < \infty \end{cases}$

then find the value of a & b if f(x) is continuous in [0,∞)

∵ f(x) is continuous in [0,∞)

∴ we check continuty at 1 & $\sqrt{2}$

$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^-} f(x) = f(1)$

$\lim_{x \to \sqrt{2}^+} f(x) = \lim_{x \to \sqrt{2}^-} f(x) = f(\sqrt{2})$

$\therefore -1 = \dfrac{x^2}{9} \Rightarrow -1 = \dfrac{1}{9} \Rightarrow \boxed{9 = -1}$

& $\dfrac{2b^2 - 4b}{x^2} = -1 \Rightarrow \dfrac{2b^2 - 4b}{(\sqrt{2})^2} = -1$

$2b^2 - 4b = -2 \Rightarrow b^2 - 2b = -1$

$b^2 - 2b + 1 = 0$

$(b-1)^2 = 0$

$\boxed{b = 1}$

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(ii) Discuss the continuity of $f(x) = \begin{cases} |x-3| & : 0 \le x < 1 \\ \sin x & : 1 \le x \le \frac{\pi}{2} \\ \log_{\frac{\pi}{2}} x & : \frac{\pi}{2} < x < 3 \end{cases}$ in $[0,3)$

$$\therefore f(x) = \begin{cases} -(x-3) & ; 0 \le x < 1 \\ \sin x & ; 1 \le x \le \pi/2 \\ \log_{\frac{\pi}{2}} x & ; \frac{\pi}{2} < x < 3 \end{cases}$$

Now we check continuity at 1 & $\frac{\pi}{2}$

$$\lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \sin x \Rightarrow \sin 1$$

$$\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} -(x-3) \Rightarrow -(1-3) \Rightarrow 2$$

$$\lim_{x \to \frac{\pi}{2}^+} f(x) = \lim_{x \to \frac{\pi}{2}^+} \log_{\frac{\pi}{2}} x \Rightarrow \log_{\frac{\pi}{2}} \frac{\pi}{2} \Rightarrow 1$$

$$\lim_{x \to \frac{\pi}{2}^-} f(x) = \lim_{x \to \frac{\pi}{2}^-} \sin x \Rightarrow \sin \frac{\pi}{2} \Rightarrow 1$$

$$\lim_{x \to \frac{\pi}{2}} \sin x = \sin \frac{\pi}{2} = 1$$

$\therefore$ $\boxed{\text{Discontinuous at } x = 1 \quad \text{Continuous at } x = \pi/2}$

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Do yourself -3 :

(i) If $f(x)=\begin{cases} \dfrac{1}{x-1} & ; \quad 1<x<2 \\ x^2-3 & ; \quad 2\le x<4 \\ 5 & ; \quad x=4 \\ 14-\dfrac{x^{1/2}}{2} & ; \quad x>4 \end{cases}$, then discuss the types of discontinuity for the function.

Solution:

$$\lim_{x\to 2^-} f(x) = 1 \quad \text{and} \quad f(2)=1$$

$$\lim_{x\to 2^+} f(x) = 2^2-3=1 \;;\; \text{Continuous at } x=2.$$

$$\lim_{x\to 4^-} f(x) = 4^2-3=13 \;\&\; f(4)=13$$

$$\lim_{x\to 4^+} f(x) = 14-\frac{4^{1/2}}{2} = 14-1=13$$

Discontinuous at $x=4$

And it is removable type of discontinuity as $f(4^+)=f(4^-)\neq f(4)$

So, change $f(4)=13$ to make continuous.

Do yourself -4 :

(i) Discuss the type of discontinuity for $f(x)=\begin{cases} -1 & ; \quad x \le -1 \\ |x| & ; \quad -1 < x < 1 \\ (x+1) & ; \quad x \ge 1 \end{cases}$

Solution:

$$\left.\begin{aligned} &\lim_{x \to -1^-} f(x) = -1 \\ &f(-1) = -1 \\ &\lim_{x \to -1^+} f(x) = 1 \end{aligned}\right\}$$ Irremovable Type of discontinuity at $x = -1$

$$\lim_{x \to 1^-} f(x) = 1$$

$$\lim_{x \to 1^+} f(x) = 2 \quad \& \quad f(1) = 2$$

Irremovable Type of discontinuity.

at $x = 1$.

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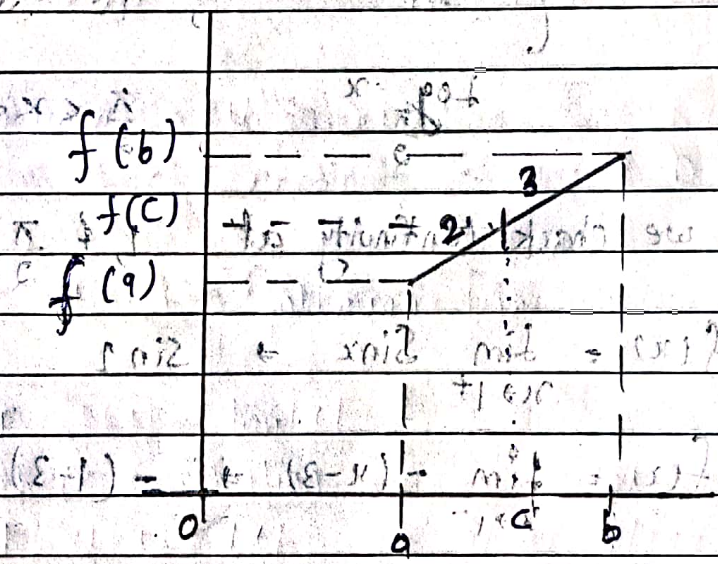
$\boxed{DYS \Rightarrow 5.}$

(i) If $f(x)$ is continuous in $[a,b]$ such that

$f(c) = \dfrac{2f(a)+3f(b)}{5}$ then prove that

$c \in (a,b)$



By Internal division $\dfrac{a\,f(a)+b\,(f(b))}{a+b}$

$\dfrac{2f(a)+(3)f(b)}{5}$

$\therefore$ In internal division $\boxed{c \in (a,b)}$

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DYS ⇒ 6.

(i) Let $f(x) = [x]$ & $g(x) = sgn(x)$ [Where $[\cdot]$ denotes G.I.F.] then discuss the continuity of $f(x) \pm g(x)$, $f(x) \cdot g(x)$ & $\frac{f(x)}{g(x)}$ at $x=0$

$f(x) = [x]$ & $g(x) = sgn(x)$

~~$h(x) = f(x) + g(x) = [x] + sgn(x)$~~

~~$\lim_{x \to 0^+} h(x) = 0 + 1 \Rightarrow 1$~~

~~$\lim_{x \to 0^-} h(x) =$~~

∵ we know that G.I.F is discontinuous at all integers.

∴ $f(x)$ is discontinuous at $x=0$

∵ Sgn function is not defined at $x=0$

∵ Also we know that sum, difference, product, quotient (defined in its domain) of two continuous functions is continuous.

∴ $f(x) \pm g(x)$, $f(x) \cdot g(x)$, $\frac{f(x)}{g(x)}$ are all

$\boxed{\text{discontinuous at } x=0}$

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(ii) If $f(x) = \sin|x|$ & $g(x) = \tan|x|$ then discuss the continuity of $f(x) \pm g(x)$, $\frac{f(x)}{g(x)}$ & $f(x) \cdot g(x)$

$f(x) = \sin|x|$

which is composition of functions

let $U(x) = |x|$ & $V(x) = \sin x$

$(V \circ U)(x) = \sin|x| = f(x)$

Now Here $R(U) \subseteq D(V)$ $\quad R(U) = R$

$D(V) = R$

$(V \circ U)(x)$ is continuous function

$\therefore f(x)$ is continuous function

Now $g(x) = \tan|x|$

which is Composition of functions

Let $U(x) = |x|$ & $V(x) = \tan x$

$(V \circ U)(x) = \tan|x| = g(x)$

Here $R(U) \not\subseteq D(V)$ $\quad R(U) = R$

$D(V) = R - (2n+1)\frac{\pi}{2}$

$(V \circ U)$ is Not Continuous function.

$\therefore g(x)$ is not continuous function

$\therefore f(x) \pm g(x)$, $f(x) \cdot g(x)$ will be discontinuous at odd multiple of $\frac{\pi}{2}$ i.e. $[(2n+1)\pi/2]$; $n \in Z$

& $\frac{f(x)}{g(x)}$ is discontinuous in $\frac{n\pi}{2}$, $n \in Z$

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DYS ⇒ 7.

(1) If $g(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ 0 & \text{if } x \notin \mathbb{Q} \end{cases}$ then find the points where function is continuous.

x if $x \in \mathbb{Q}$, set of Rational point $\left(\frac{p}{q}\right)$

0 if $x \notin \mathbb{Q}$ set of irrational points

$\therefore$ at $x = 0$

given point function is continuous.

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(ii) If $f(x) = \begin{cases} x^2 ; x \in Q \\ 1-x^2 ; x \notin Q \end{cases}$

then find the points where function is continuous.

for continuous function we check.

$$x^2 = 1 - x^2$$

$$2x^2 = 1$$

$$x^2 = \frac{1}{2}$$

$$\boxed{x = \pm \frac{1}{\sqrt{2}}}$$

$\therefore$ For these points given function is continuous.

EXERCISE (O-1)

CONTINUITY
EXERCISE (O -1)
Solutions

1. Let $f(x)=\begin{cases} ax+1 & \text{if} \quad x<1 \\ 3 & \text{if} \quad x=1 \\ bx^2+1 & \text{if} \quad x>1 \end{cases}$. If $f(x)$ is continuous at x = 1 then (a – b) is equal to-

(A) 0 (B) 1 (C) 2 (D) 4

Solution:

For continuity at $x=1$

- $\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1^-}(ax+1) = \underline{a+1}$

- $\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1^+}(bx^2+1) = \underline{b+1}$

For continuity at $x=1$

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x) = f(1)$$

$$\Rightarrow a+1 = b+1 = 3$$

$$\Rightarrow \boxed{a-b=0}$$

$\boxed{1-(A)}$.

2. The function $f(x)=\dfrac{4-x^2}{4x-x^3}$, is-

(A) discontinuous at only one point in its domain.
(B) discontinuous at two points in its domain.
(C) discontinuous at three points in its domain.
(D) continuous everywhere in its domain.

Solution:

$$f(x) = \frac{4-x^2}{4x-x^3}, \quad x \in \mathbb{R} - \{0, 2, -2\}$$

$$= \frac{1}{x}, \quad x \in \mathbb{R} - \{0, 2, -2\}$$

At any point 't' in domain, $t \in \mathbb{R} - \{0, 2, -2\}$

$$\lim_{x \to t} f(x) = \lim_{x \to t} \frac{1}{x} = \frac{1}{t} = f(t) \qquad (t \neq 0)$$

$\Rightarrow f(x)$ is continuous at all $t \in$ Domain

$\boxed{3-(D)}$.

3. If $f(x)=\begin{cases} -4\sin x+\cos x & \text{for } x\le -\frac{\pi}{2} \\ a\sin x+b & \text{for } -\frac{\pi}{2}<x<\frac{\pi}{2} \\ \cos x+2 & \text{for } x\ge \frac{\pi}{2} \end{cases}$ is continuous then :

(A) a = –1, b = 3 (B) a = 1, b = –3

(C) a = 1, b = 3 (D) a = –1, b = –3

Solution:

- for $x \in (-\infty, -\pi/2]$

$f(x) = -4\sin x + \cos x$

$\sin x$ & $\cos x$ are continuous everywhere

$\Rightarrow f(x)$ is continuous for $x \in (-\infty, -\pi/2]$.

Similarly, • $f(x)$ is continuous for $x \in (-\frac{\pi}{2}, \pi/2)$

& also continuous for $x \in [\pi/2, \infty)$

- For continuity of $f(x)$ over $x \in \mathbb{R}$

$f(x)$ must be continuous at $x = -\pi/2$ & $\pi/2$.

- For continuity at $x = -\pi/2$

$\lim_{x\to -\frac{\pi}{2}^+} f(x) = f(-\pi/2) \Rightarrow \lim_{x\to -\frac{\pi}{2}^+} (a\sin x + b) = -4$

$\Rightarrow -a+b=-4$ — ①

- For continuity at $x = \pi/2$

$\lim_{x\to \frac{\pi}{2}^-} f(x) = f(\pi/2) \Rightarrow \lim_{x\to \frac{\pi}{2}^-} (a\sin x + b) = 2$

$\Rightarrow a+b=2$ — ②

From ① and ②

$\boxed{a=-1}$, $\boxed{b=3}$

$\boxed{4-(A)}$

4. The function $f(x)=\begin{cases}\frac{1}{4}(3x^2+1) & -\infty < x \le 1 \\ 5-4x & 1 < x < 4 \\ 4-x & 4 \le x < \infty\end{cases}$ is -

(A) continuous at x = 1 & x = 4
(B) continuous at x = 1, discontinuous at x= 4
(C) continuous at x = 4, discontinuous at x = 1
(D) discontinuous at x = 1 & x = 4

Solution:

- At $x=1$

$\rightarrow$ $f(x)$ is continuous at $x=1$ from Left hand side.

$\rightarrow \lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} (5-4x) = 1$

, $f(1) = \frac{1}{4}(3\times 1^2+1) = 1$

$\Rightarrow \lim_{x\to 1^+} f(x) = f(1)$

So $f(x)$ is continuous at $x=1$.

- At $x=4$

$\rightarrow$ $f(x)$ is continuous from right hand side.

$f(4)=0$

$\rightarrow \lim_{x\to 4^-} f(x) = \lim_{x\to 4^-} 5-4x = -15$

$\Rightarrow \lim_{x\to 4^-} f(x) \neq f(4)$

So $f(x)$ is not continuous at $x=4$.

$\boxed{5-(B)}$.

5. If $f(x) = \frac{x^2 - bx + 25}{x^2 - 7x + 10}$ for $x \neq 5$ and f is continuous at $x = 5$, then $f(5)$ has the value equal to-

(A) 0 (B) 5 (C) 10 (D) 25

Solution:

$f(x)$ is continuous at $x = 5$

$\Rightarrow \lim_{x \to 5} f(x) = f(5)$ (finite)

- $\lim_{x \to 5} f(x) = \lim_{x \to 5} \frac{x^2 - bx + 25}{x^2 - 7x + 10}$

since $\lim_{x \to 5} x^2 - 7x + 10 = 0$

for existence of $\lim_{x \to 5} f(x)$ finitely

$\lim_{x \to 5} x^2 - bx + 25$ must be equal to 0

$\Rightarrow 25 - 5b + 25 = 0 \Rightarrow \boxed{b = 10}$

$\lim_{x \to 5} f(x) = \lim_{x \to 5} \frac{x^2 - 10x + 25}{x^2 - 7x + 10} = \lim_{x \to 5} \frac{(x-5)(x-5)}{(x-5)(x-2)}$

$= \lim_{x \to 5} \frac{x-5}{x-2} = 0$

So $\boxed{f(5) = 0}$.

$\boxed{6 - (A)}$.

6. $y = f(x)$ is a continuous function such that its graph passes through (a,0). Then $\lim_{x \to a} \frac{\log_e(1+3f(x))}{2f(x)}$ is-

(A) 1 (B) 0 (C) $\frac{3}{2}$ (D) $\frac{2}{3}$

Solution:

Given $f(x)$ is continuous function, & its graph passes through $(a,0)$

$\Rightarrow \lim_{x \to a} f(x) = f(a) = 0.$

$$\lim_{x \to a} \left(\frac{\ln(1+3f(x))}{3f(x)} \times \frac{3}{2} \right)$$

$= 3/2$ (Ans)

$\boxed{8 - (C)}$

7. Given $f(x) = \begin{cases} |x+1| & \text{if} \quad x < -2 \\ 2x+3 & \text{if} \quad -2 \le x < 0 \\ x^2+3 & \text{if} \quad 0 \le x < 3 \\ x^3-15 & \text{if} \quad x \ge 3 \end{cases}$. Then number of point(s) of discontinuity of $f(x)$ is-

(A) 0 (B) 1 (C) 2 (D) 3

Solution:

i) $x < -2$, $f(x) = |x+1|$, $f(x)$ is continuous for all $x < -2$.

ii) $x = -2$.

$\lim_{x \to -2^-} f(x) = \lim_{x \to -2^-} |x+1| = 1$.

$f(-2) = -1$

$\Rightarrow \lim_{x \to -2^-} f(x) \neq f(-2)$

$\Rightarrow f(x)$ is discontinuous at $x = -2$

iii) for $x \in [-2, 0)$, $f(x) = 2x+3$ is continuous.

iv) at $x = 0$,

$\lim_{x \to 0^-} f(x) = \lim_{x \to 0^-} (2x+3) = 3$

$f(0) = 3 = \lim_{x \to 0^+} f(x)$

So $f(x)$ is continuous at $x = 0$.

v) for $x \in [0, 3)$, $f(x) = x^2+3$, is continuous

vi) at $x = 3$

$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} (x^2+3) = 12$

$\lim_{x \to 3^+} f(x) = f(3) = 3^3 - 15 = 12$

$\Rightarrow f(x)$ is continuous at $x = 3$

vii) for $x \in [3, \infty)$, $f(x) = x^3 - 15$, is continuous.

So. Number of point of discont. is $\boxed{1}$.

$\boxed{11 - (B)}$.

8. If $f(x)$ is continuous and $f\left(\frac{9}{2}\right)=\frac{2}{9}$, then the value of $\lim_{x\to 0} f\left(\frac{1-\cos 3x}{x^2}\right)$ is-

(A) $\frac{2}{9}$ (B) $\frac{9}{2}$ (C) 0 (D) data insufficient

Solⁿ:

$$\lim_{x\to 0} f\left(\frac{1-\cos 3x}{x^2}\right) = \lim_{x\to 0} f\left(\frac{1-\cos 3x}{9x^2}\cdot 9\right)$$

$$= f\left(\frac{1}{2}\cdot 9\right) = f(9/2) = 2/9$$

9. f is a continuous function on the real line. Given that $x^2 + (f(x) - 2)x - \sqrt{3} \cdot f(x) + 2\sqrt{3} - 3 = 0$. Then the value of $f(\sqrt{3})$

(A) can not be determined

(B) is $2(1 - \sqrt{3})$

(C) is zero

(D) is $\frac{2(\sqrt{3}-2)}{\sqrt{3}}$

Solution:

$f(x)$ is continuous function on $\mathbb{R}$.

Given $x^2 + (f(x)-2)x - \sqrt{3}f(x) + 2\sqrt{3} - 3 = 0$

$\Rightarrow f(x)(x-\sqrt{3}) + x^2 - 2x + 2\sqrt{3} - 3 = 0$

for $x \neq \sqrt{3}$ $\quad f(x) = -\dfrac{x^2-2x+2\sqrt{3}-3}{x-\sqrt{3}}$

$$= -\frac{(x^2-3) - 2(x-\sqrt{3})}{(x-\sqrt{3})}, \quad x \neq \sqrt{3}$$

$$= -x - \sqrt{3} + 2, \quad x \neq \sqrt{3}$$

Since $f(x)$ is continuous,

$$f(\sqrt{3}) = \lim_{x\to\sqrt{3}} f(x) = \lim_{x\to\sqrt{3}} (-x-\sqrt{3}+2)$$

$$= 2(1-\sqrt{3}).$$

$\boxed{13-(B)}$.

EXERCISE (O-2)

EXERCISE (O-2)

[SINGLE CORRECT CHOICE TYPE]

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function $\forall x \in \mathbb{R}$ and $f(x) = 5 \ \forall x \in$ irrational. Then the value of $f(3)$ is -

(A) 1 (B) 2 (C) 5 (D) cannot determine

Soln. (c).

Given $f(x) = 5 \quad \forall x \in$ irrational.

Let $f(x) = \begin{cases} 5 & \forall x \in \text{irrational} \\ a & \forall x \in \text{rational.} \end{cases}$

$\because$ $f(x)$ is continous for all $x \in R$.

$\Rightarrow \quad a = 5$.

$\therefore \quad f(x) = 5 \quad \forall x \in R$.

$\Rightarrow \quad f(3) = 5$ Ans.

2. If $f(x)=\dfrac{1}{(x-1)(x-2)}$ and $g(x)=\dfrac{1}{x^2}$, then set of points in domain of $fog(x)$ at which $fog(x)$ is discontinuous.

(A) $\left\{-1,0,1,\frac{1}{\sqrt{2}}\right\}$ (B) ϕ (C) $\{0,1\}$ (D) $\left\{0,1,\frac{1}{\sqrt{2}}\right\}$

Sol^n: (B).

$$f(g(x)) = \frac{1}{\left(\frac{1}{x^2}-1\right)\left(\frac{1}{x^2}-2\right)} = \frac{x^4}{(1-x^2)(1-2x^2)}.$$

$\therefore$ Domain of $f(g(x)) = R - \left\{-1, -\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 1\right\}$.

The function $f(g(x))$ is continous in its domain.

$\therefore$ Set of points where $f(g(x))$ is discontinous in its domain is ϕ.

3. The function $f(x) = [x].\cos\frac{2x-1}{2}\pi$, where [•] denotes the greatest integer function, is discontinuous at :-

(A) all x (B) all integer points (C) no x (D) x which is not an integer

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Sol^n: (A). $f(x) = [x]\cos\left(x-\frac{1}{2}\right)\pi.$

$$= [x]\sin(\pi x).$$

$\because$ $[x]$ is continous at all pts other than integers and $\sin\pi x$ is always continous

$\therefore$ $f(x)$ will be continous at all pts other than integers

Let $x = a$ be any integer.

$$\lim_{x\to a^-} f(x) = (a-1)\sin(a\pi) = 0.$$

$$\lim_{x\to a^+} f(x) = f(a) = a\sin a\pi = 0.$$

$$\Rightarrow \lim_{x\to a^-} f(x). = f(a) = \lim_{x\to a^+} f(x).$$

$\Rightarrow$ $f(x)$ is continous at all integral pts.

$\therefore$ $f(x)$. is continous $\forall\, x\in R$.

[MULTIPLE CORRECT CHOICE TYPE]

4. Which of the following function(s) is/are discontinuous at x = 0 ?

(A) $f(x) = \sin\frac{\pi}{2x}$, $x \neq 0$ and $f(0) = 1$

(B) $g(x) = x\sin\left(\frac{\pi}{x}\right)$, $x \neq 0$ and $g(0) = \pi$

(C) $h(x) = \frac{|x|}{x}$, $x \neq 0$ and $h(0) = 1$

(D) $k(x) = \frac{1}{1+e^{\cot x}}$, $x \neq 0$ and $k(0) = 0$.

Sol^n. A, B, C, D.

A). The value of $\sin\left(\frac{\pi}{2x}\right)$ oscillates b/w -1 and 1 as x approaches 0.

$\therefore \lim\limits_{x\to 0} \sin\left(\frac{\pi}{2x}\right)$ does not exist.

$\therefore f(x)$ is discontinous at $x = 0$.

B). $g(x) = \begin{cases} x\sin(\pi/x) & x \neq 0 \\ \pi & x = 0 \end{cases}$

$\lim\limits_{x\to 0} g(x) = \lim\limits_{x\to 0} x\sin\left(\frac{\pi}{x}\right) = 0 \neq g(0)$.

$\therefore g(x)$ is discontinous at $x = 0$.

C). $h(x) = \begin{cases} \frac{|x|}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$

$h(0^-) = -1$; $h(0^+) = 1$.

$\Rightarrow \lim\limits_{x\to 0} h(x)$ does not exist.

$\therefore h(x)$ discontinous at $x = 0$.

D). $k(x) = \begin{cases} \dfrac{1}{1+e^{\cot x}} & x \neq 0 \\ 0 & x = 0 \end{cases}$

$$\lim_{x \to 0^-} k(x) = \lim_{x \to 0^-} \frac{1}{1+e^{\cot x}} = 1.$$

$$\lim_{x \to 0^+} k(x) = \lim_{x \to 0^+} \frac{1}{1+e^{\cot x}} = 0.$$

$\because$ $\lim\limits_{x \to 0} k(x)$ does not exist.

$\therefore$ $k(x)$ is discontinous at $x = 0$.

5. A function $f(x)$ is defined as $f(x)=\dfrac{A\sin x+\sin 2x}{x^3},(x\neq 0)$. If the function is continuous at $x = 0$, then -

(A) $A = -2$ (B) $f(0) = -1$ (C) $A = 1$ (D) $f(0) = 1$

Solⁿ. A, B.

$$\lim_{x\to 0.} \frac{A\sin x+\sin 2x}{x^3}.$$

$$= \lim_{x\to 0} \frac{\sin x\,(A+2\cos x)}{x^3}.$$

for limit to exist $A=-2$.

$$\therefore \lim_{x\to 0} f(x) = \lim_{x\to 0} -2\,\frac{\sin x\,(1-\cos x)}{x^3}. = -2\times\frac{1}{2} = -1.$$

6. Which of the following function(s) can be defined continuously at x = 0 ?

(A) $f(x)=\dfrac{1}{1+2^{\cot x}}$

(B) $f(x)=\cos\left(\dfrac{|\sin x|}{x}\right)$

(C) $f(x)= x\sin\dfrac{\pi}{x}$

(D) $f(x)=\dfrac{1}{\ell n|x|}$

Soln. B, C, D :

A). $f(0^-) = \lim\limits_{x\to 0^-} \dfrac{1}{1+2^{\cot x}} = 1.$

$f(0^+) = \lim\limits_{x\to 0^+} \dfrac{1}{1+2^{\cot x}} = 0.$

$\because$ limit does not exist $\therefore$ can't be defined continously at $x=0$.

B). $f(0^-) = \lim\limits_{x\to 0^-} \cos\left(-\dfrac{\sin x}{x}\right) = \cos(-1) = \cos(1).$

$f(0^+) = \lim\limits_{x\to 0^+} \cos\left(\dfrac{\sin x}{x}\right) = \cos(1)$

$\therefore$ for $f(x)$ to be continous at $x=0$ $f(0)=\cos 1.$

C). $f(0^-) = f(0^+) = \lim\limits_{x\to 0} x\sin\dfrac{\pi}{x} = 0.$

$\therefore$ for $f(x)$ to be continous at $x=0$, $f(0)=0.$

D). $f(0^-) = f(0^+) = \lim\limits_{x\to 0} \dfrac{1}{\ln|x|} = 0.$

$\therefore$ for $f(x)$ to be continous at $x=0$, $f(0)=0.$

7. If f is defined on an interval [a, b]. Which of the following statement(s) is/are INCORRECT ?

(A) If f(a) and f(b), have opposite sign, then there must be a point $c \in (a, b)$ such that $f(c) = 0$.

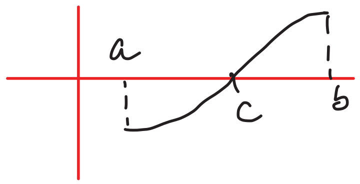
(B) If f is continuous on [a, b], $f(a) < 0$ and $f(b) > 0$, then there must be a point $c \in (a, b)$ such that $f(c)=0$.

(C) If f is continuous on [a, b] and there is a point c in (a, b) such that $f(c) = 0$, then f(a) and f(b) have opposite sign.

(D) If f has no zeroes on [a, b], then f(a) and f(b) have the same sign.

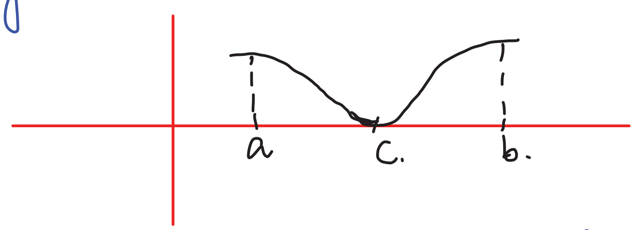
Soln. A, C, D.

A) for this to be always true f(x) must be continous.

B). 

Correct statement as can be inferred by the graph.

c). Incorrect statement as can be explained by graph below.



D). Correct only if f(x) is continous.

8. Which of the following functions can be defined at indicated point so that resulting function is continuous -

(A) $f(x)=\dfrac{x^2-2x-8}{x+2}$ at $x=-2$

(B) $f(x)=\dfrac{x-7}{|x-7|}$ at $x=7$

(C) $f(x)=\dfrac{x^3+64}{x+4}$ at $x=-4$

(D) $f(x)=\dfrac{3-\sqrt{x}}{9-x}$ at $x=9$

Solⁿ. A, C, D.

A). $\lim\limits_{x\to -2} f(x) = \lim\limits_{x\to -2} \dfrac{(x-4)(x+2)}{(x+2)} = -6.$

$\therefore$ for $f(x)$ to be continous at $x=-2$. $f(-2)=-6$.

B). $f(7^-) = \lim\limits_{x\to 7^-} \dfrac{x-7}{-(x-7)} = -1.$

$f(7^+) = \lim\limits_{x\to 7^+} \dfrac{x-7}{x-7} = 1.$

$\therefore$ limit does not exist.

$\therefore$ $f(x)$ cannot be continously defined at $x=7$.

C). $\lim\limits_{x\to -4} f(x) = \lim\limits_{x\to -4} \dfrac{(x+4)(x^2-4x+16)}{(x+4)} = 48$

for $f(x)$ to be continous at $x=-4$, $f(-4)=48$.

D). $\lim\limits_{x\to 9} f(x) = \lim\limits_{x\to 9} \dfrac{3-\sqrt{x}}{(3-\sqrt{x})(3+\sqrt{x})} = \dfrac{1}{6}.$

for $f(x)$ to be continous at $x=9$, $f(9)=\dfrac{1}{6}$.

9. In which of the following cases the given equations has atleast one root in the indicated interval ?

(A) $x - \cos x = 0$ in $(0, \pi/2)$

(B) $x + \sin x = 1$ in $(0, \pi/6)$

(C) $\dfrac{a}{x-1} + \dfrac{b}{x-3} = 0$, $a, b > 0$ in $(1, 3)$

(D) $f(x) - g(x) = 0$ in (a, b) where f and g are continuous on $[a, b]$ and $f(a) > g(a)$ and $f(b) < g(b)$.

Soln: A, B, C, D.

A). $f(x) = x - \cos x$.

$f(0) = -1$; $f(\pi/2) = \pi/2$.

$\because$ $f(0) \cdot f(\pi/2) < 0 \Rightarrow f(x) = 0$ has at least one root in $(0, \pi/2)$.

B). $f(x) = x + \sin x - 1$.

$f(0) = -1$; $f(\pi/6) = \pi/6 + \frac{1}{2} - 1 = \frac{\pi - 3}{6}$ (>0).

$\because$ $f(0) \cdot f(\pi/6) < 0 \Rightarrow f(x) = 0$ has at least one root in $(0, \pi/6)$.

---

c). $f(x) = \dfrac{a}{x-1} + \dfrac{b}{x-3}$.

$f(1^+) \to +\infty$. $f(3^-) \to -\infty$.

$\Rightarrow f(x) = 0$ has at least one root in $(1, 3)$.

D). $h(x) = f(x) - g(x)$.

$h(a) = f(a) - g(a) > 0$; $h(b) = f(b) - g(b) < 0$.

$\Rightarrow$ $h(x) = 0$ has at least one root in (a, b).

[MATRIX TYPE]

10.

Column-I		Column-II	
(A)	$\lim\limits_{x\to 1}\dfrac{x^3-1}{\ell nx}$ is	(P)	2
(B)	$\lim\limits_{x\to 0}\dfrac{x(\cos x-\cos 2x)}{2\sin x-\sin 2x}$ is	(Q)	3
(C)	$\lim\limits_{x\to 0}\dfrac{\tan x\sqrt{\tan x}-\sin x\sqrt{\sin x}}{x^3.\sqrt{x}}$ is	(R)	$\dfrac{3}{2}$
(D)	If $f(x) = \cos(x\cos\frac{1}{x})$ and $g(x) = \dfrac{\ell n(\sec^2 x)}{x\sin x}$ are both continuous at x = 0 then $f(0) + g(0)$ equals	(S)	$\dfrac{3}{4}$

Sol^n. $A\to Q$, $B\to R$, $C\to S$, $D\to P$.

A) $$\lim_{x\to 1}\frac{x^3-1}{\ln x} = \lim_{x\to 1}\frac{\frac{x^3-1}{x-1}}{\frac{\ln(1+(x-1))}{(x-1)}} = \frac{3}{1} = 3.$$

B) $$\lim_{x\to 0}\frac{x(\cos x-\cos 2x)}{2\sin x-\sin 2x} = \lim_{x\to 0}\frac{x\cdot 2\sin\frac{3x}{2}\cdot\sin\frac{x}{2}}{2\sin x(1-\cos x)}$$

$$= \lim_{x\to 0}\frac{\frac{\sin\frac{3x}{2}}{x}\cdot\frac{\sin x/2}{x}}{\frac{\sin x}{x}\left(\frac{1-\cos x}{x^2}\right)} = \frac{3/2\times 1/2}{1/2} = \frac{3}{2}.$$

C) $$\lim_{x\to 0}\frac{(\tan x)^{3/2}-(\sin x)^{3/2}}{x^3\sqrt{x}} = \lim_{x\to 0}\frac{\tan^3 x-\sin^3 x}{x^3\sqrt{x}\left((\tan x)^{3/2}+(\sin x)^{3/2}\right)}$$

$$= \lim_{x\to 0}\frac{\left(\frac{\tan x-\sin x}{x^3}\right)\left(\frac{\tan^2 x}{x^2}+\frac{\tan x\sin x}{x^2}+\frac{\sin^2 x}{x^2}\right)}{\frac{(\tan x)^{3/2}}{x^{3/2}}+\frac{(\sin x)^{3/2}}{x^{3/2}}}$$

$$= \frac{\frac{1}{2}\times 3}{2} = \frac{3}{4}.$$

D) $$f(0) = \lim_{x\to 0}\cos\left(x\cos\frac{1}{x}\right) = \cos(0) = 1.$$

$$g(0) = \lim_{x\to 0}\frac{\ln(1+\tan^2 x)}{\tan^2 x}\times\frac{\frac{\tan^2 x}{x^2}}{\frac{\sin x}{x}} = 1.$$

$\therefore\ f(0)+g(0) = 2.$

11. Match the function in Column-I with its behaviour at x = 0 in column-II, where [.] denotes greatest integer function & sgn(x) denotes signum function.

Column-I	Column-II
(A) $f(x) = [x][1 + x]$	(P) LHL exist at $x = 0$
(B) $f(x) = [-x][1 + x]$	(Q) RHL exist at $x = 0$
(C) $f(x) = (\text{sgn}(x))[2 - x][1 + \lvert x\rvert]$	(R) Continuous at $x = 0$
(D) $f(x) = [\cos x]$	(S) $\lim_{x\to 0} f(x)$ exists but function is discontinuous at $x = 0$
	(T) $\lim_{x\to 0} f(x)$ does not exist

Sol^n: $A \rightarrow (P, Q, R)$; $B \rightarrow (P, Q, T)$; $C \rightarrow (P, Q, T)$; $D \rightarrow (P, Q, S)$.

A). $f(x) = [x][1+x]$

$f(0^-) = (-1) \times 0 = 0.$

$f(0^+) = 0 \times 1 = 0.$

$f(0) = 0.$

$\therefore$ continous at $x=0$.

B). $f(x) = [-x][1+x]$.

$f(0^-) = 0 \times 0 = 0.$

$f(0^+) = -1 \times 1 = -1.$

$\lim_{x\to 0} f(x)$ does not exist.

(C) $f(x) = \text{sgn}(x).[2-x][1+|x|]$.

$f(0^-) = -1 \times 2 \times 1 = -2.$

$f(0^+) = 1 \times 1 \times 1 = 1.$

$\Rightarrow \lim_{x\to 0} f(x)$ does not exist.

D) $f(x) = [\cos x]$.

$f(0^-) = 0$.

$f(0^+) = 0$.

$f(0) = 1$.

$\lim_{x \to 0} f(x)$ exists but $f(x)$ is discontinous. at $x=0$.

EXERCISE (S-1)

1. If the function $f(x)=\dfrac{3x^2+ax+a+3}{x^2+x-2}$, $(x \neq -2)$ is continuous at $x=-2$. Find $f(-2)$.

Solⁿ) $\because$ $f(x)$ is continuous at $x=-2$

So $f(-2)=\lim\limits_{x\to -2} f(x)$

$$f(-2)=\lim_{x\to -2}\frac{3x^2+ax+a+3}{x^2+x-2}=\left(\frac{15-a}{0}\right)$$

$(15-a)$ must be zero

So $15-a=0 \Rightarrow a=15$

$$f(-2)=\lim_{x\to -2}\frac{3x^2+15x+18}{x^2+x-2}$$

$$=\lim_{x\to -2}\frac{(x+2)(3x+9)}{(x+2)(x-1)}$$

$$=\lim_{x\to -2}\left(\frac{3x+9}{x-1}\right)=-1$$

2. Find all possible values of a and b so that f(x) is continuous for all $x \in \mathbb{R}$ if

$$f(x) = \begin{cases} |ax+3| & \text{if } x \le -1 \\ |3x+a| & \text{if } -1 < x \le 0 \\ \dfrac{b\sin 2x}{x} - 2b & \text{if } 0 < x < \pi \\ \cos^2 x - 3 & \text{if } x \ge \pi \end{cases}$$

Sol^n) $\because$ f(x) is continuous $\forall\, x \in \mathbb{R}$

so it must be continuous at

$x = -1, 0, \pi$

at $x = \pi$

$$f(\pi) = f(\pi^-) = f(\pi^+)$$

$$\cos^2\pi - 3 = \frac{b\sin 2\pi}{\pi} - 2b$$

$$-2 = -2b \Rightarrow \boxed{b = 1}$$

at $x = 0$

$$f(0) = f(0^-) = f(0^+)$$

$$|a| = \lim_{x\to 0^+}\left(\frac{b\sin 2x}{x} - 2b\right)$$

$$|a| = \lim_{x\to 0^+} 2b\left(\frac{\sin 2x}{2x}\right) - 2b = 0$$

$\boxed{a = 0}$. Also If $a = 0$ then f(x) is continuous at $x = -1$

3. The function $f(x) = \begin{cases} \left(\dfrac{6}{5}\right)^{\frac{\tan 6x}{\tan 5x}} & \text{if } 0 < x < \dfrac{\pi}{2} \\ b+2 & \text{if } x = \dfrac{\pi}{2} \\ (1+|\cos x|)^{\left(\frac{a|\tan x|}{b}\right)} & \text{if } \dfrac{\pi}{2} < x < \pi \end{cases}$

Determine the values of 'a' & 'b', if f is continuous at $x = \pi/2$.

Sol^n)

$$f\left(\frac{\pi}{2}\right) = b+2 \quad \text{---(i)}$$

LHL at $x = \frac{\pi}{2}$

$$f\left(\frac{\pi}{2}^-\right) = \lim_{x \to \frac{\pi}{2}^-} \left(\frac{6}{5}\right)^{\frac{\tan 6x}{\tan 5x}}$$

replace $x = \frac{\pi}{2} - h$

$$= \lim_{h \to 0} \left(\frac{6}{5}\right)^{\frac{\tan(3\pi - 6h)}{\tan\left(\frac{5\pi}{2} - 5h\right)}}$$

$$= \lim_{h \to 0} \left(\frac{6}{5}\right)^{\frac{\tan 6h}{\cot 5h}} = \left(\frac{6}{5}\right)^0 = 1$$

$$f\left(\frac{\pi}{2}^-\right) = 1 \quad \text{—— (ii)}$$

<u>RHL at $x = \frac{\pi}{2}$</u>

$$f\left(\frac{\pi}{2}^+\right) = \lim_{x \to \frac{\pi}{2}^+} f(x)$$

$$= \lim_{x \to \frac{\pi}{2}^+} \left(1+|\cos x|\right)^{\frac{a|\tan x|}{b}}$$

replace $x = \frac{\pi}{2} + h$

$$= \lim_{h \to 0} \left(1+|\sin h|\right)^{\frac{a|\cot h|}{b}} \quad (1^\infty \text{ form})$$

$$= e^{\lim_{h \to 0} \frac{a}{b|\tan h|}\left(1+|\sin h| - 1\right)}$$

$$= e^{\lim_{h \to 0} \frac{a|\cos h|}{b}} = e^{\frac{a}{b}}$$

$$f\left(\frac{\pi}{2}^{+}\right) = e^{a/b} \quad \text{——— (iii)}$$

$\because$ $f(x)$ is continuous at $x = \frac{\pi}{2}$

So $f\left(\frac{\pi}{2}\right) = f\left(\frac{\pi}{2}^{-}\right) = f\left(\frac{\pi}{2}^{+}\right)$

$$b + 2 = 1 = e^{a/b}$$

$\boxed{b = -1}$ $\boxed{a = 0}$

4. Suppose that $f(x) = x^3 - 3x^2 - 4x + 12$ and $h(x) = \begin{cases} \dfrac{f(x)}{x-3} & , \quad x \neq 3 \\ K & , \quad x = 3 \end{cases}$ then

(a) find all zeroes of f(x).

(b) find the value of K that makes h continuous at x = 3.

(c) using the value of K found in (b), determine whether h is an even function.

(a) $f(x) = x^3 - 3x^2 - 4x + 12$

$f(x) = (x-2)(x+2)(x-3)$

$f(x) = 0$ at $x = 2, -2, 3$

(b) $h(x)$ is continuous at $x = 3$

$\Rightarrow h(3) = \lim_{x \to 3} h(x)$

$$K = \lim_{x \to 3} \frac{f(x)}{x-3}$$

$$K = \lim_{x \to 3} (x-2)(x+2) = 5$$

(c) $h(x) = \begin{cases} \dfrac{f(x)}{x-3} & x \neq 3 \\ K & x = 3 \end{cases}$

$\because K = 5$, $f(x) = (x^2-4)(x-3)$

$$h(x) = \begin{cases} x^2 - 4, & x \neq 3 \\ 5 & x = 3 \end{cases}$$

$h(-x) = h(x) \quad \forall \; x \in R$

So $h(x)$ is an even function

5. Let $f(x) = \begin{cases} \dfrac{1-\sin \pi x}{1+\cos 2\pi x}, & x < \dfrac{1}{2} \\ p, & x = \dfrac{1}{2} \\ \dfrac{\sqrt{2x-1}}{\sqrt{4+\sqrt{2x-1}}-2}, & x > \dfrac{1}{2} \end{cases}$. Determine the value of p, if possible, so that the function is continuous at x = 1/2.

Sol^n) $f\left(\frac{1}{2}\right) = p$ — (i)

LHL at $x = \frac{1}{2}$

$$f\left(\frac{1}{2}^-\right) = \lim_{x \to \frac{1}{2}^-} \frac{1-\sin \pi x}{1+\cos 2\pi x}$$

put $x = \frac{1}{2} - h$

$$= \lim_{h \to 0} \frac{1-\sin\left(\frac{\pi}{2}-\pi h\right)}{1+\cos\left(\pi - 2\pi h\right)}$$

$$= \lim_{h \to 0} \frac{(1-\cos \pi h)}{(1-\cos 2\pi h)}$$

$$= \lim_{h \to 0} \frac{2\sin^2\left(\frac{\pi h}{2}\right)}{2\sin^2(\pi h)} = \frac{1}{4} \quad \text{---(ii)}$$

RHL at $x = \frac{1}{2}$

$$f\left(\frac{1}{2}^+\right) = \lim_{x \to \frac{1}{2}^+} \frac{\sqrt{2x-1}}{\sqrt{4+\sqrt{2x-1}} - 2}$$

$$= \lim_{x \to \frac{1}{2}^+} \frac{\left(\sqrt{2x-1}\right)\left(\sqrt{4+\sqrt{2x-1}}+2\right)}{\left(\sqrt{2x-1}\right)}$$

$$= \lim_{x \to \frac{1}{2}^+} \left(\sqrt{4+\sqrt{2x-1}} + 2\right) = 4 \quad \text{---(iii)}$$

$\therefore$ RHL $\neq$ LHL at $x = \frac{1}{2}$

So $f(x)$ can not be continuous

$\Rightarrow$ No value of p exists

6. Given the function $g(x) = \sqrt{6-2x}$ and $h(x) = 2x^2 - 3x + a$. Then

(a) evaluate $h(g(2))$

(b) If $f(x) = \begin{cases} g(x), & x \le 1 \\ h(x), & x > 1 \end{cases}$, find 'a' so that f is continuous.

Soln

(a) $h(g(x)) = 2g^2(x) - 3g(x) + a$

$g(2) = \sqrt{2}$

$h(g(2)) = 4 - 3\sqrt{2} + a$

(b)

$f(x) = \begin{cases} \sqrt{6-2x}, & x \le 1 \\ 2x^2 - 3x + a, & x > 1 \end{cases}$

$\because f(x)$ is continuous so

$f(1) = f(1^-) = f(1^+)$

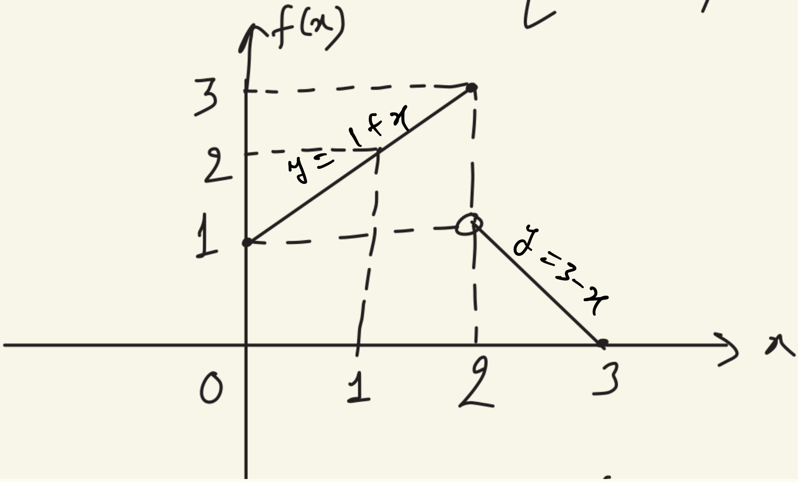
$f(1^-) = f(1) = 2$, $f(1^+) = 2 - 3 + a$

$\Rightarrow 2 = a - 1 \Rightarrow a = 3$

7. Let $f(x) = \begin{cases} 1+x, & 0 \le x \le 2 \\ 3-x, & 2 < x \le 3 \end{cases}$. Determine the form of $g(x) = f[f(x)]$ & hence find the point of discontinuity of g, if any.

Sol^n

$$g(x) = f(f(x)) = \begin{cases} 1+f(x), & 0 \le f(x) \le 2 \\ 3-f(x), & 2 < f(x) \le 3 \end{cases}$$



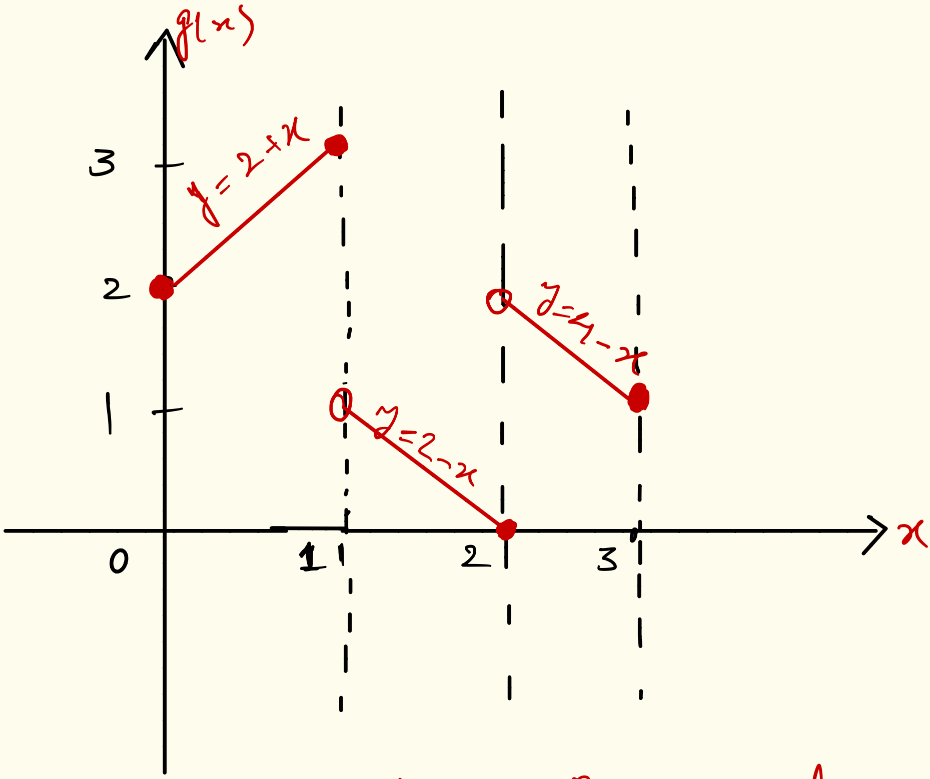
$\hookrightarrow 0 \le f(x) < 1 \Rightarrow f(x) = 3-x \; ; \; 2 < x \le 3$

$\hookrightarrow 1 \le f(x) \le 2 \Rightarrow f(x) = 1+x \; ; \; 0 \le x \le 1$

$\hookrightarrow 2 < f(x) \le 3 \Rightarrow f(x) = 1+x \; ; \; 1 < x \le 2$

$$\therefore g(x) = f(f(x)) = \begin{cases} 1+(3-x) & ; \; 2 < x \le 3 \\ 1+(1+x) & ; \; 0 \le x \le 1 \\ 3-(1+x) & ; \; 1 < x \le 2 \end{cases}$$

$$\therefore\ g(x) = \begin{cases} 2+x & ; \quad 0 \le x \le 1 \\ 2-x & ; \quad 1 < x \le 2 \\ 4-x & ; \quad 2 < x \le 3 \end{cases}$$



$\therefore$ $g(x)$ is discontinuous at $x = 1$ and at $x = 2$.

8. Determine a & b so that f is continuous at $x = \frac{\pi}{2}$ where $f(x) = \begin{cases} \frac{1-\sin^3 x}{3\cos^2 x} & \text{if} \quad x < \frac{\pi}{2} \\ a & \text{if} \quad x = \frac{\pi}{2} \\ \frac{b(1-\sin x)}{(\pi-2x)^2} & \text{if} \quad x > \frac{\pi}{2} \end{cases}$

Soln) $f\left(\frac{\pi}{2}\right) = a$ —— (1)

$$f\left(\frac{\pi}{2}^+\right) = \lim_{x \to \frac{\pi}{2}^+} \frac{b(1-\sin x)}{(\pi - 2x)^2}$$

$$x = \frac{\pi}{2} + h$$

$$= \lim_{h \to 0} \frac{b \cdot (1-\cos h)}{(-2h)^2}$$

$$= \lim_{h \to 0} \frac{b}{4} \frac{1-\cos h}{h^2} = \frac{b}{8}$$

$$f\left(\frac{\pi}{2}^+\right) = \frac{b}{8} \text{ —— (2)}$$

$$f\left(\frac{\pi}{2}^-\right) = \lim_{x \to \frac{\pi}{2}^-} \frac{1 - \sin^3 x}{3\cos^2 x}$$

$$= \lim_{x \to \frac{\pi}{2}^-} \frac{\cancel{(1-\sin x)}(1+\sin^2 x + \sin x)}{3\cancel{(1-\sin x)}(1+\sin x)}$$

$$f\left(\frac{\pi}{2}^-\right) = \frac{1}{2} \quad \text{--- } (3)$$

$\because f(x)$ is continuous at $x = \frac{\pi}{2}$

So $f\left(\frac{\pi}{2}^-\right) = f\left(\frac{\pi}{2}^+\right) = f\left(\frac{\pi}{2}\right)$

$$a = \frac{1}{2} = \frac{b}{8}$$

$$\boxed{a = \frac{1}{2}} \quad \boxed{b = 4}$$

9. Determine the values of a, b & c for which the function $f(x) = \begin{cases} \dfrac{\sin(a+1)x + \sin x}{x} & \text{for } x < 0 \\ c & \text{for } x = 0 \\ \dfrac{(x+bx^2)^{1/2} - x^{1/2}}{bx^{3/2}} & \text{for } x > 0 \end{cases}$ is continuous at x = 0.

Sol^n) $f(0) = c$ —— (1)

$$f(0^+) = \lim_{x \to 0^+} \frac{\sqrt{bx^2+x} - \sqrt{x}}{bx\sqrt{x}} \quad (b \neq 0)$$

$$= \lim_{x \to 0^+} \frac{(bx^2 + x - x)}{bx\sqrt{x}\left(\sqrt{bx^2+x} + \sqrt{x}\right)}$$

$$= \lim_{x \to 0^+} \frac{b\,x^2}{bx\sqrt{x}\,\sqrt{x}\left(\sqrt{bx+1}+1\right)}$$

$$f(0^+) = \frac{1}{2} \quad \text{—— (2)}$$

$$f(0^-) = \lim_{x \to 0} \frac{\sin((a+1)x) + \sin x}{x}$$

$$= \lim_{x \to 0} (a+1) \left\{ \frac{\sin (a+1)x}{(a+1)x} \right\} + \left\{ \frac{\sin x}{x} \right\}$$

$$f(0^-) = (a+1) + 1 = a+2 \quad \text{—(3)}$$

$\because f(x)$ is continuous at $x=0$

So $\quad f(0) = f(0^+) = f(0^-)$

$$c = \frac{1}{2} = a+2$$

$$\boxed{c = \frac{1}{2}} \qquad \boxed{a = \frac{-3}{2}}$$

$$b \in R - \{0\}$$

EXERCISE (S-2)

1. If $f(x) = \dfrac{\sin 3x + A\sin 2x + B\sin x}{x^5}$ $(x \neq 0)$ is cont. at $x = 0$. Find A & B. Also find f(0).

$$f(0) = \lim_{x\to 0} \frac{A\sin 2x + B\sin x + \sin 3x}{x^5}$$

$$f(0) = \lim_{x\to 0} \frac{A\left(2x - \frac{(2x)^3}{6} + \frac{(2x)^5}{120}\right) + B\left(x - \frac{x^3}{6} + \frac{x^5}{120}\right) + \left(3x - \frac{(3x)^3}{6} + \frac{(3x)^5}{120}\right)}{x^5}$$

$$f(0) = \lim_{x\to 0} \frac{(2A + B + 3)^{=0} + \frac{x^3}{6}(-8A - B - 27)^{=0} + \frac{x^5}{120}(32A + B + 243)}{x^5}$$

$2A + B + 3 = 0$, $\quad 8A + B + 27 = 0$

$(A, B) \equiv (-4, 5)$

$$l = \frac{32A + B + 243}{120} = f(0) = 1$$

2. Find the locus of (a, b) for which the function $f(x) = \begin{cases} ax - b & \text{for} \quad x \le 1 \\ 3x & \text{for} \quad 1 < x < 2 \\ bx^2 - a & \text{for} \quad x \ge 2 \end{cases}$

is continuous at x = 1 but discontinuous at x = 2.

Cont at $x=1$, $\quad a-b=3$

discont at $x=2$, $\quad 4b-a \neq 6$ $\quad$ Now

so locus is

$4b-(3+b) \neq 6 \Rightarrow b \neq 3$

$x-y=3$

& $y \neq 3$

3. Let $f(x) = \begin{cases} (\sin x + \cos x)^{\operatorname{cosec} x} & ; \quad \frac{-\pi}{2} < x < 0 \\ a & ; \quad x = 0 \\ \dfrac{e^{1/x} + e^{2/x} + e^{3/|x|}}{ae^{2/x} + be^{3/|x|}} & ; \quad 0 < x < \frac{\pi}{2} \end{cases}$

If f(x) is continuous at x = 0, find the value of $(a^2 + b^2)$.

RHL =

$$f(0^+) = \lim_{x \to 0} \frac{e^{1/x} + e^{2/x} + e^{3/x}}{ae^{2/x} + be^{3/x}}$$

$$f(0^+) = \lim_{x \to 0^+} \frac{e^{3/x}}{be^{3/x}} = \frac{1}{b}$$

$$f(0) = a$$

LHL

$$= f(0^-) = e^{\lim_{x \to 0^-} \operatorname{cosec} x (\sin x + \cos x - 1)} \quad (1^\infty \text{ form})$$

$$= e^{\lim_{x \to 0} \left(\frac{\sin x + \cos x - 1}{\sin x} \right)}$$

$$= e^{\lim_{x \to 0} \left(1 - \left(\frac{1 - \cos x}{\sin x} \right) \right)}$$

$$= e^{1-0} = e^1$$

$$\therefore \quad f(0) = f(0^+) = f(0^-)$$

$$\Rightarrow \quad a = \frac{1}{b} = e$$

$$\Rightarrow \quad (a, b) = \left(e, \frac{1}{e} \right)$$

$$\therefore \quad a^2 + b^2 = e^2 + 1/e^2 \quad //$$

4. Let $f(x) = x^3 - x^2 - 3x - 1$ and $h(x) = \dfrac{f(x)}{g(x)}$, where h is a rational function such that

(a) Domain of h(x) is $\mathbb{R} - \{-1\}$

(b) $\lim\limits_{x\to\infty} h(x) = \infty$ and (c) $\lim\limits_{x\to -1} h(x) = \dfrac{1}{2}$.

Find $\lim\limits_{x\to 0} (3h(x) + f(x) - 2g(x))$

$$f(x) = (x+1)(x^2-2x-1)$$

$$h(x) = \frac{(x+1)(x^2-2x-1)}{\lambda(x+1)^k} \qquad \text{Domain } x \in R - \{-1\}$$

$\lim\limits_{x\to -1} h(x)$ exist only when $k = 1$

$$\lim_{x\to -1} \frac{(x+1)(x^2-2x-1)}{\lambda(x+1)} = \frac{1}{2}$$

$$\frac{2}{\lambda} = \frac{1}{2} \Rightarrow \lambda = 4 \quad , \; g(x) = 4(x+1)$$

$$\lim_{x\to\infty} \frac{(x+1)(x^2-2x-1)}{4(x+1)} \to \infty$$

$$g(x) = 4(x+1) \quad , \quad h(x) = \frac{x^2-2x-1}{4}$$

$$f(x) = (x+1)(x^2-2x-1)$$

$\therefore \lim\limits_{x\to 0} (3h(x) + f(x) - 2g(x))$

$$= 3\underbrace{\lim_{x\to 0} h(x)}_{= -1/4} + \underbrace{\lim_{x\to 0} f(x)}_{= -1} - 2\underbrace{\lim_{x\to 0} g(x)}_{= 4}$$

$$= -\frac{3}{4} - 1 - 8 = -\frac{39}{4} \quad // \text{Ans}$$

5. (a) Let f be a real valued continuous function on $\mathbb{R}$ and satisfying $f(-x) - f(x) = 0 \ \forall \in \mathbb{R}$. If $f(-5) = 5$, $f(-2) = 4$, $f(3) = -2$ and $f(0) = 0$ then find the minimum number of zero's of the equation $f(x) = 0$.

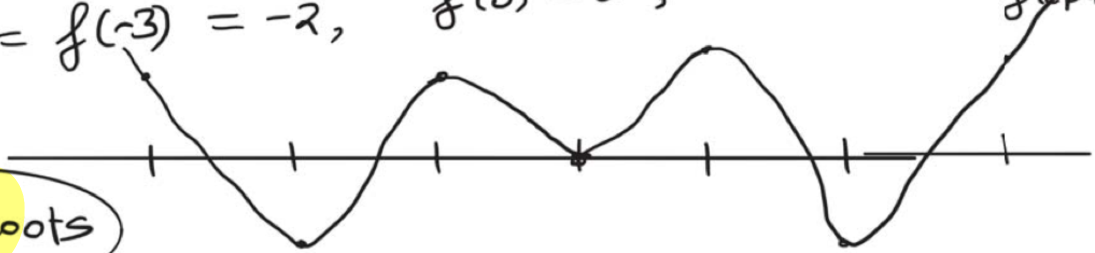
Sol^n

(a)

$$f(x) = f(-x)$$

$f(-5) = f(5) = 5$, $f(-2) = f(2) = 4$

$f(3) = f(-3) = -2$, $f(0) = 0$, now plot the graph



min 5 roots

5. (b) Find the number of points of discontinuity of the function f(x) = [5x] + {3x} in [0, 5] where [y] and {y} denote largest integer less than or equal to y and fractional part of y respectively.

Sol:

(b) $f(x) = [5x] + \{3x\}$

$f(x)$ is continuous at all integer points

$[5x]$: $5x = n \Rightarrow x = \frac{n}{5}$

$x = \underbrace{0, \frac{1}{5}, \frac{2}{5}, \ldots\ldots, \frac{24}{5}, \frac{25}{5}}$ Total = 26 − 6 = 20 point

$\{3x\}$: $3x = n \Rightarrow x = \frac{n}{3}$

$x = 0, \frac{1}{3}, \frac{2}{3}, \ldots\ldots, \frac{15}{3}$ Total = 16 − 6 = 10 point

One Cont + other DC = DC

Total = 30 point discontinuous

They both are discontinuous at $x = 0, 1, \ldots, 6$

$f(n) = 5n$ $f(n^+) = 5n$, $f(n^-) = 5n - 1 + 1 = 5n$

Total = 30 points

6. (a) If g : [a, b] → [a, b] is continuous & onto function, then show that there is some c ∈ [a, b] such that g(c) = c.

Sol :

10 (a) $h(x) = g(x) - x$

Range of $g(x)$ is $[a, b]$

$g(a), g(b) \in [a, b]$

Apply IMVT in $x \in [a, b]$

$h(a) = g(a) - a \geq 0$

$h(b) = g(b) - b \leq 0$

$h(a)\, h(b) \leq 0 \Rightarrow h(x) = 0$ for atleast one $x \in [a, b]$

$\Rightarrow g(x) = x$ for atleast one $x \in [a, b]$

6.

~~(b)~~ Let f be continuous on the interval [0, 1] to $\mathbb{R}$ such that f(0) = f(1). Prove that there exists a point c in $\left[0,\frac{1}{2}\right]$ such that $f(c)=f\left(c+\frac{1}{2}\right)$

Sol :-

(b) $g(x) = f\left(x+\frac{1}{2}\right) - f(x)$

Apply IMVT in $x \in \left[0, \frac{1}{2}\right]$

$$g(0) = f\left(\tfrac{1}{2}\right) - f(0)$$

$$g\left(\tfrac{1}{2}\right) = f(1) - f\left(\tfrac{1}{2}\right) = -\left(f\left(\tfrac{1}{2}\right) - f(0)\right)$$

$g(0)\cdot g\left(\frac{1}{2}\right) \leq 0$ so $g(x)=0$ for atleast one c such that $f\left(c+\frac{1}{2}\right) = f(c)$

EXERCISE (JM)

1. If f(x) is continuous and f(9/2) = 2/9, then $\lim_{x\to 0} f\left(\frac{1-\cos 3x}{x^2}\right)$ is equal to:

[JEE Mains Offline-2014]

(1) 9/2 (2) 0 (3) 2/9 (4) 8/9

Sol:- $\underset{x\to 0}{Lt} f\left(\frac{1-\cos 3x}{9x^2}\cdot 9\right) = \underset{x\to 0}{Lt} f\left(\frac{9}{2}\right) = \frac{2}{9}$.

2. If the function $f(x)=\begin{cases}\dfrac{\sqrt{2+\cos x}-1}{(\pi-x)^2}, & x\neq\pi\\ k \quad\quad\quad\quad\quad , & x=\pi\end{cases}$ is continuous at x = π, then k equals -

[JEE Mains Offline-2014]

(1) $\frac{1}{4}$ (2) $\frac{1}{2}$ (3) 2 (4) 0

let $\pi - x = h$ then $x = \pi - h$

$$\lim_{x\to\pi}\frac{\sqrt{2+\cos x}-1}{(\pi-x)^2}=\lim_{h\to 0}\frac{\sqrt{2-\cos h}-1}{h^2}$$

$$=\lim_{h\to 0}\frac{2-\cos h-1}{h^2}\cdot\frac{1}{\sqrt{2-\cos h}+1}$$

$$=\lim_{h\to 0}\left(\frac{1-\cos h}{h^2}\right)\cdot\frac{1}{2}=\frac{1}{4}.$$

3. Let $f : R \to R$ be a function defined as : $f(x) = \begin{cases} 5, & \text{if } x \le 1 \\ a + bx, & \text{if } 1 < x < 3 \\ b + 5x, & \text{if } 3 \le x < 5 \\ 30, & \text{if } x \ge 5 \end{cases}$. Then, f is :

[JEE(Main)-2019]

(1) continuous if a = 5 and b = 5

(2) continuous if a = –5 and b = 10

(3) continuous if a = 0 and b = 5

(4) not continuous for any values of a and b

$a + b = 5$ at $x = 1$.

$a + 3b = b + 15 \Rightarrow a + 2b = 15$ at $x = 3$.

$b + 25 = 30$ at $x = 5$

$\Rightarrow b = 5 \Rightarrow a = 0$

But these values not hold for

$a + 2b = 15$.

So, not continious for any values of a & b.

4. Let $f : [-1,3] \to R$ be defined as $f(x)=\begin{cases} |x|+[x] & , \quad -1 \le x < 1 \\ x+|x| & , \quad 1 \le x < 2 \\ x+[x] & , \quad 2 \le x \le 3 \end{cases}$,

where [t] denotes the greatest integer less than or equal to t. Then, f is discontinuous at:

[JEE(Main)-2019]

(1) four or more points
(2) only one point
(3) only two points
(4) only three points

<u>at $x = -1$.</u>

R.H.L. = 1 − 1 = 0 & $f(-1) = 1 - 1 = 0$

<u>at $x = 0$</u>

L.H.L. = 0 − 1 = −1 $\quad f(0) = 0$

R.H.L. = 0 + 0 = 0 <u>so</u>, discontinuous.

<u>at $x = 1$.</u>

L.H.L. = 1 + 0 = 1

$f(1) = 2$

R.H.L. = 1 + 1 = 2 <u>so</u>, discontinuous

<u>at $x = 2$</u>

L.H.L. = 2 + 2 = 4. $\quad f(2) = 4$.

R.H.L. = 2 + 2 = 4. <u>so</u>, continuous

<u>at $x = 3$</u>

L.H.L. = 3 + 2 = 5 $\quad f(3) = 3 + 3 = 6$

<u>so</u>, discontinuous ⇒ total 3 points in [−1, 3]

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5. If the function f defined on $\left(\frac{\pi}{6}, \frac{\pi}{3}\right)$ by $f(x) = \begin{cases} \frac{\sqrt{2}\cos x - 1}{\cot x - 1}, & x \neq \frac{\pi}{4} \\ k, & x = \frac{\pi}{4} \end{cases}$ is continuous, then k is equal to

[JEE(Main)-2019]

(1) $\frac{1}{2}$ (2) 1 (3) $\frac{1}{\sqrt{2}}$ (4) 2

Sol. $f(x) = \begin{cases} \dfrac{\sqrt{2}\cos x - 1}{\cot x - 1} & x \neq \pi/4 \\ k & x = \pi/4 \end{cases}$

$$\therefore \lim_{x \to \pi/4} \frac{\sqrt{2}\left(\cos x - \frac{1}{\sqrt{2}}\right)}{(\cos x - \sin x)} \cdot \sin x$$

$$\Rightarrow \lim_{x \to \pi/4} \frac{\cos x - \cos \pi/4}{\frac{1}{\sqrt{2}}\cos x - \frac{1}{\sqrt{2}}\sin x} \cdot \sin x = \frac{\cos x - \cos \pi/4}{\sin(\pi/4 - x)} \cdot \sin x$$

$$\Rightarrow \lim_{x \to \pi/4} \frac{2\sin\left(\frac{x + \pi/4}{2}\right)\sin\left(\frac{\pi/4 - x}{2}\right)}{2\sin\left(\frac{\pi/4 - x}{2}\right)\cos\left(\frac{\pi/4 - x}{2}\right)} \cdot \sin x$$

$$\Rightarrow \lim_{x \to \pi/4} \frac{1}{\sqrt{2}} \cdot \frac{1}{\sqrt{2}} = \frac{1}{2} = k$$

6. If $f(x)=[x]-\left[\frac{x}{4}\right], x \in R$, where $[x]$ denotes the greatest integer function, then :

[JEE(Main)-2019]

(1) Both $\lim_{x\to 4-} f(x)$ and $\lim_{x\to 4+} f(x)$ exist but are not equal

(2) $\lim_{x\to 4-} f(x)$ exists but $\lim_{x\to 4+} f(x)$ does not exist

(3) $\lim_{x\to 4+} f(x)$ exists but $\lim_{x\to 4-} f(x)$ does not exist

(4) f is continuous at $x = 4$

$$\underset{x\to 4^-}{\text{lt}} f(x) = \underset{x\to 4^-}{\text{lt}} [x] - \left[\frac{x}{4}\right] = 3 - 0 = 3.$$

$$\underset{x\to 4^+}{\text{lt}} f(x) = \underset{x\to 4^+}{\text{lt}} [x] - \left[\frac{x}{4}\right] = 4 - 1 = 3.$$

Sol L.H.L. = R.H.L. at $x = 4$.

f is continuous at $x = 4$.

7. If the function $f(x)=\begin{cases} a|\pi-x|+1, x\le 5 \\ b|x-\pi|+3, x>5 \end{cases}$ is continuous at x = 5, then the value of a – b is :-

[JEE(Main)-Apr 19]

(1) $\frac{2}{5-\pi}$ (2) $\frac{2}{\pi-5}$ (3) $\frac{2}{\pi+5}$ (4) $\frac{-2}{\pi+5}$

at $x=5$

$$L.H.L. = \lim_{x\to 5^-} a|\pi-x|+1 = a(5-\pi)+1$$

$$R.H.L. = \lim_{x\to 5^+} b|x-\pi|+3 = b(5-\pi)+3$$

$$a(5-\pi)+1 = b(5-\pi)+3$$

$$\Rightarrow a(5-\pi)-b(5-\pi) = 2$$

$$\Rightarrow a-b = \frac{2}{5-\pi}$$

8. If $f(x) = \begin{cases} \dfrac{\sin(p+1)+\sin x}{x} & , \quad x<0 \\ q & , \quad x=0 \\ \dfrac{\sqrt{x+x^2}-\sqrt{x}}{x^{3/2}} & , \quad x>0 \end{cases}$ is continuous at x = 0, then the ordered pair (p,q) is equal to :

[JEE(Main)-Apr 19]

(1) $\left(\frac{5}{2}, \frac{1}{2}\right)$ (2) $\left(-\frac{3}{2}, -\frac{1}{2}\right)$ (3) $\left(-\frac{1}{2}, \frac{3}{2}\right)$ (4) $\left(-\frac{3}{2}, \frac{1}{2}\right)$

$$L.H.L. = \lim_{x \to 0} \frac{\sin(p+1)x + \sin x}{x} \quad \text{using L'HP}$$

$$= \lim_{x \to 0} \cos(p+1)x \cdot (p+1) + \cos x$$

$$= p+1+1 = p+2.$$

$$R.H.L. = \lim_{x \to 0} \frac{\sqrt{1+x}-1}{x} \times \frac{\sqrt{1+x}+1}{\sqrt{1+x}+1}$$

$$= \lim_{x \to 0} \frac{1+x-1}{x} \cdot \frac{1}{\sqrt{1+x}+1} = \frac{1}{2}$$

<u>Sol</u> $q = \frac{1}{2}$ and $p+2 = \frac{1}{2} \Rightarrow p = -\frac{3}{2}$

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EXERCISE (JA)

EXERCISE (JA)

1. Discuss the continuity of the function $f(x)=\begin{cases}\dfrac{e^{1/(x-1)}-2}{e^{1/(x-1)}+2}, & x\neq 1\\ 1, & x=1\end{cases}$

at x = 1. **[REE 2001 (Mains), 3 out of 100]**

Soln! - $\lim\limits_{h\to 0} f(1+h) = \lim\limits_{h\to 0} \dfrac{e^{1/h}-2}{e^{1/h}+2} = 1$

$\lim\limits_{h\to 0} f(1-h) = \lim\limits_{h\to 0} \dfrac{e^{-1/h}-2}{e^{-1/h}+2} = \dfrac{0-2}{0+2} = -1$

So function is discontinouse at $x=1$ as L.H.L $\neq$ R.H.L

2. For every integer n, let a_n and b_n be real numbers. Let function $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} a_n + \sin \pi x, & \text{for } x \in [2n, 2n+1] \\ b_n + \cos \pi x, & \text{for } x \in (2n-1, 2n) \end{cases}, \text{ for all integers n.}$$

If f is continuous, then which of the following holds(s) for all n ? **[JEE 2012, 4]**

(A) $a_{n-1} - b_{n-1} = 0$ (B) $a_n - b_n = 1$ (C) $a_n - b_{n+1} = 1$ (D) $a_{n-1} - b_n = -1$

f is continuous on R

$f(2n) = a_n + \sin(2n\pi) = a_n$

At $x = 2n$

$$\lim_{x \to 2n^-} f(x) = \lim_{x \to 2n^-} b_n + \cos \pi x = b_n + 1$$

$$\lim_{x \to 2n^+} f(x) = \lim_{x \to 2n^+} a_n + \sin \pi x = a_n$$

$\Rightarrow \quad a_n = b_n + 1$

Thus $a_n - b_n = 1$

Again at $x = 2n+1$

$$f(2n+1) = a_n + \sin(\pi(2n+1))$$

$$f(2n+1) = a_n$$

$$\lim_{x \to 2n+1^-} f(x) = \lim_{x \to 2n+1^-} a_n + \sin \pi x = a_n$$

Also $\lim_{x \to 2n+1^+} f(x)$

$$= \lim_{x \to 2n+1^+} b_{n+1} + \cos \pi x = b_{n+1} - 1$$

$$\Rightarrow \quad a_n = b_{n+1} - 1$$

Now replace n by $n-1$

$$\boxed{a_{n-1} = b_n - 1}$$

Ans

3. For every pair of continuous function $f, g : [0, 1] \to \mathbb{R}$ such that

$$\max\{f(x) : x \in [0, 1]\} = \max\{g(x) : x \in [0,1]\},$$

the correct statement(s) is(are) :

(A) $(f(c))^2 + 3f(c) = (g(c))^2 + 3g(c)$ for some $c \in [0,1]$

(B) $(f(c))^2 + f(c) = (g(c))^2 + 3g(c)$ for some $c \in [0,1]$

(C) $(f(c))^2 + 3f(c) = (g(c))^2 + g(c)$ for some $c \in [0,1]$

(D) $(f(c))^2 = (g(c))^2$ for some $c \in [0,1]$ **[JEE(Advanced)-2014, 3]**

Soln:– $f, g \in [0,1] \to R$

We take two cases.

$\boxed{\text{Case I}}$ Let f and g attain their common maximum value at p

$\Rightarrow f(p) = g(p)$ where $p \in [0,1]$

$\boxed{\text{Case II}}$ Let f and g attain their common maximum value at different points

$\Rightarrow f(a) = M$ & $g(b) = M$

$\Rightarrow f(a) - g(a) > 0$ & $f(b) - g(b) < 0$

$\Rightarrow f(c) - g(c) = 0$ for some $c \in [0,1]$

as f and g are continuous functions

$\Rightarrow f(c) - g(c) = 0$ for some $c \in [0,1]$ ---- I

Option (A) $\Rightarrow (f(c))^2 - (g(c))^2 + 3(f(c) - g(c)) = 0$

which is true from I

Option (D) $\Rightarrow$

$(f(c))^2 = (g(c))^2$ which is true from (I)

Now if we take $f(x) = 1$; $g(x) = 1$

B & D does not hold

4. Let [x] be the greatest integer less than or equal to x. Then, at which of the following point(s) the function $f(x) = x\cos(\pi(x + [x]))$ is discontinuous ? **[JEE(Advanced)-2017, 4(–2)]**

(A) $x = -1$ (B) $x = 0$ (C) $x = 2$ (D) $x = 1$

Soln :-

$$f(x) = x\cos(\pi[x] + x\pi)$$

$$\Rightarrow f(x) = (-1)^{[x]} x\cos(x\pi)$$

Which is discontinuous at all integers except

$x = 0$