COMPLEX NUMBER

Do yourself

&

Elementary

Do yourself - 1 :

(i) Determine least positive value of n for which $\left(\frac{1+i}{1-i}\right)^n = 1$

(ii) Find the value of the sum $\sum_{n=1}^{5}(i^n + i^{n+2})$, where $i = \sqrt{-1}$.

Sol (i)

$$\left(\frac{1+i}{1-i}\right)^n = 1$$

$$\Rightarrow \left(\frac{(1+i)^2}{2}\right)^n = (i)^n$$

$$\Rightarrow n = 4$$

(ii)

$$\sum_{n=1}^{5} i^n + i^{n+2}$$

$$\Rightarrow \frac{i(1-i^5)}{1-i} + \frac{i^3(1-i^5)}{1-i}$$

$$\Rightarrow i + i^3 = 0$$

Do yourself - 2 :

(i) Find the value of $x^3 + 7x^2 - x + 16$, where $x = 1 + 2i$.

(ii) If $a + ib = \dfrac{c+i}{c-i}$, where c is a real number, then prove that : $a^2 + b^2 = 1$ and $\dfrac{b}{a} = \dfrac{2c}{c^2-1}$.

(iii) Find square root of $-15 - 8i$

(I) $x = 1 + 2i$

$\Rightarrow (x-1) = 2i$

$\Rightarrow (x-1)^2 = -4$

$\Rightarrow x^2 - 2x + 5 = 0$

$\Rightarrow x^3 + 7x^2 - x + 16 = (x^2 - 2x + 5)(x + 9)$
$+ 12x - 29$

$\Rightarrow 12x + 29 = 12(1+2i) - 29$
$= -17 + 24i$

(II) $a + ib = \dfrac{c+i}{c-i} = \dfrac{c^2 - 1 + 2ci}{c^2+1}$

$a = \dfrac{c^2-1}{c^2+1}$, $b = \dfrac{2c}{c^2+1}$

$\therefore$ $b/a = 2c/c^2-1$

(iii) Let $\sqrt{-15-8i} = x+iy$

$\Rightarrow -15-8i = x^2-y^2+2ixy$

$\Rightarrow x^2-y^2 = -15 \rightarrow (1)$

& $2xy = -8 \rightarrow (2)$

$\Rightarrow x^2+y^2 = 17 \rightarrow (3)$

by (1) & (3)

$x = \pm 1, \ y = \pm 4$

$\therefore \sqrt{-15-8i} = \pm(1-4i)$

Do yourself - 3 :

Find the modulus and amplitude of following complex numbers :

(i) $-2+2\sqrt{3}i$ (ii) $-\sqrt{3}-i$ (iii) $-2i$ (iv) $\dfrac{1+2i}{1-3i}$ (v) $\dfrac{2+6\sqrt{3}i}{5+\sqrt{3}i}$

(i) $|-2+2\sqrt{3}i| = \sqrt{4+12} = 4 = |z|$

$amp(-2+2\sqrt{3}i) = \pi - \tan^{-1}(\sqrt{3})$

$= \pi - \pi/3 = 2\pi/3$

(ii) $|z| = |-\sqrt{3}-i| = \sqrt{3+1} = 2$

$amp(-\sqrt{3}-i) = -(\pi - \tan^{-1}(1/\sqrt{3}))$

$= -5\pi/6$

(iii) $|-2i| = 2$

$amp(-2i) = -\pi/2$

(iv) $|z| = \left|\dfrac{1+2i}{1-3i}\right| = \sqrt{\dfrac{5}{10}} = \sqrt{\dfrac{1}{2}}$

$$\arg\left(\frac{1+2i}{1-3i}\right) = \arg(1+2i) - \arg(1-3i)$$

$$\Rightarrow \tan^{-1}(2/1) - (-\tan^{-1}(3/1))$$

$$\Rightarrow \tan^{-1}(2) + \tan^{-1}(3)$$

$$= 3\pi/4$$

(v) $|z| = \left|\dfrac{2+6\sqrt{3}i}{5+\sqrt{3}i}\right| = \sqrt{\dfrac{112}{28}}$

$$\Rightarrow |z| = 2$$

Do yourself - 4 :

(i) Find the distance between two complex numbers $z_1 = 2+3i$ & $z_2 = 7-9i$ on the complex plane.

(ii) Find the locus of $|z-2-3i| = 1$.

(iii) If z is a complex number, then $z^2 + \bar{z}^2 = 2$ represents -

(A) a circle (B) a straight line (C) a hyperbola (D) an ellipse

(i) $|z_1 - z_2| = d$

$\Rightarrow \sqrt{(2-7)^2 + (3+9)^2} = \sqrt{25+144} = 13$

(ii) let $z = x+iy$

$\Rightarrow |(x-2)+(y-3)i| = 1$

$\Rightarrow (x-2)^2 + (y-3)^2 = 1$ which is circle

(iii) let $z = x+iy$

$z^2 + \bar{z}^2 = 2$

$\Rightarrow (x^2 - y^2 + 2xyi) + (x^2 - y^2 - 2ixy) = 2$

$\Rightarrow x^2 - y^2 = 1$ which Hyperbola

Do yourself - 5 :

Express the following complex number in polar form and exponential form :

(i) $-2+2i$ (ii) $-1-\sqrt{3}i$ (iii) $\dfrac{(1+7i)}{(2-i)^2}$ (iv) $(1-\cos\theta+i\sin\theta), \theta \in (0,\pi)$

(1) $z = -2+2i$

$|z| = \sqrt{4+4} = 2\sqrt{2} = r$

$\theta = \pi - \tan^{-1}(1) \Rightarrow 3\pi/4$

$\Rightarrow$ $z = 2\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) = 2\sqrt{2}e^{i3\pi/4}$

(ii) $z = -1-\sqrt{3}i$

$|z| = 2 = r$

$\theta = -(\pi - \tan^{-1}(\sqrt{3})) = -2\pi/3$

or $4\pi/3$

$\therefore$ $z = 2\left(\cos 4\pi/3 + i\sin 4\pi/3\right) = 2e^{i4\pi/3}$

(iii) $z = \dfrac{1+7i}{(2-i)^2} = \dfrac{(1+7i)}{(3-4i)} = \dfrac{-25+25i}{25}$

$= (-1+i)$

$|z| = \sqrt{2}$, $\theta = \pi - \pi/4 = 3\pi/4$

$z = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) = \sqrt{2}\, e^{i3\pi/4}$

(iv) $z = (1 - \cos\theta + i\sin\theta)$ $\theta \in (0, \pi)$

$z = \underbrace{2\sin\theta/2}_{>0}\left[\sin\theta/2 + i\cos\theta/2\right]$

$\therefore |z| = 2\sin\theta/2$,

$z = 2\sin\theta/2\left[\cos\left(\pi/2 - \theta/2\right) + i\sin\left(\frac{\pi}{2} - \theta/2\right)\right]$

$z = 2\sin\theta/2\; e^{i\left(\pi/2 - \theta/2\right)}$

Do yourself - 6 :

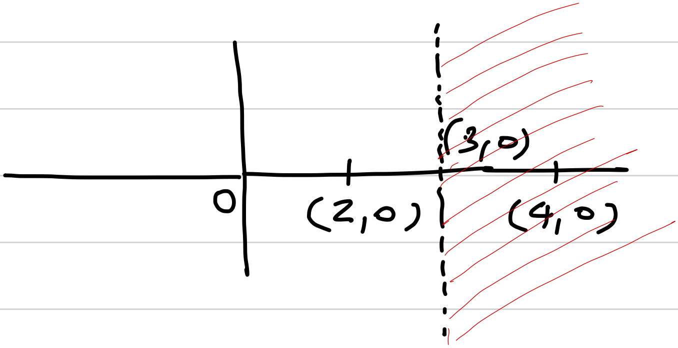
(i) The inequality $|z-4|<|z-2|$ represents region given by -

(A) $Re(z)>0$ (B) $Re(z)<0$ (C) $Re(z)>3$ (D) none

(ii) If $z=re^{i\theta}$, then the value of $|e^{iz}|$ is equal to -

(A) $e^{-r\cos\theta}$ (B) $e^{r\cos\theta}$ (C) $e^{r\sin\theta}$ (D) $e^{-r\sin\theta}$

① $|z-4| < |z-2|$



so $Re(z) > 3$

② $z = re^{i\theta} = r(\cos\theta + i\sin\theta)$

$e^{iz} = e^{i(r\cos\theta + ir\sin\theta)}$

$= \underbrace{e^{-r\sin\theta}} \cdot e^{ir\cos\theta}$

$|e^{iz}| = e^{-r\sin\theta}$

Do yourself - 7 :

(i) A complex number $z = 3 + 4i$ is rotated about another fixed complex number $z_1 = 1 + 2i$ in anticlockwise direction by 45° angle. Find the complex number represented by new position of z in argand plane.

(ii) If A, B, C are three points in argand plane representing the complex number z_1, z_2, z_3 such that $z_1 = \dfrac{\lambda z_2 + z_3}{\lambda + 1}$, where $\lambda \in R$, then find the distance of point A from the line joining points B and C.

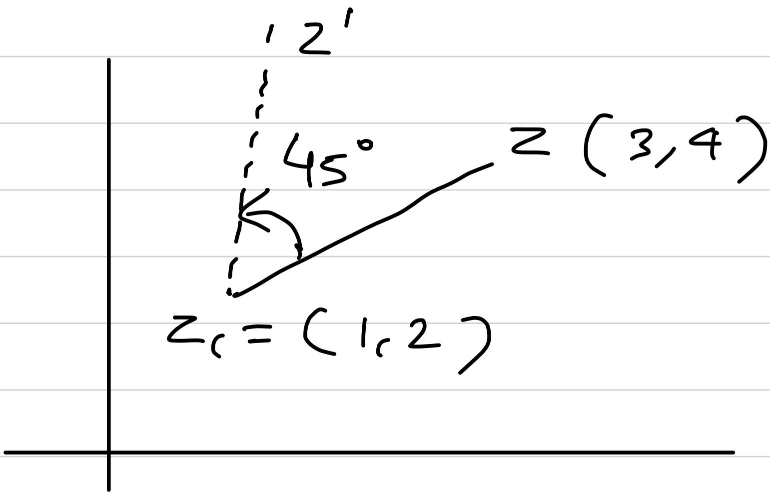
(iii) If $A(z_1), B(z_2), C(z_3)$ are vertices of ΔABC in which $\angle ABC = \dfrac{\pi}{4}$ and $\dfrac{AB}{BC} = \sqrt{2}$, then find z_2 in terms of z_1 and z_3.

(iv) If a & b are real numbers between 0 and 1 such that the points $z_1 = a + i$, $z_2 = 1 + bi$ and $z_3 = 0$ form an equilateral triangle then a and b are equal to :-

(A) $a = b = 1/2$ (B) $a = b = 2 - \sqrt{3}$ (C) $a = b = -2 + \sqrt{3}$ (D) $a = b = \sqrt{2} - 1$

(v) If $\arg\left(\dfrac{z-1}{z+1}\right) = \dfrac{\pi}{4}$, find locus of z.

(1)



by Rotation

$$(z - z_1)\, e^{i\pi/4} = (z' - z_1)$$

or $$z' = (z - z_1)\, e^{i\pi/4} + z_1$$

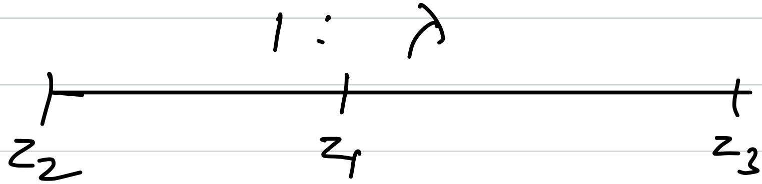
$$= \left((3,4) - (1,2)\right)\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) + (1,2)$$

$$= (2,2)(1+i)\frac{1}{\sqrt{2}} + (1,2)$$

$$= (2\sqrt{2}\, i) + (1 + 2i)$$

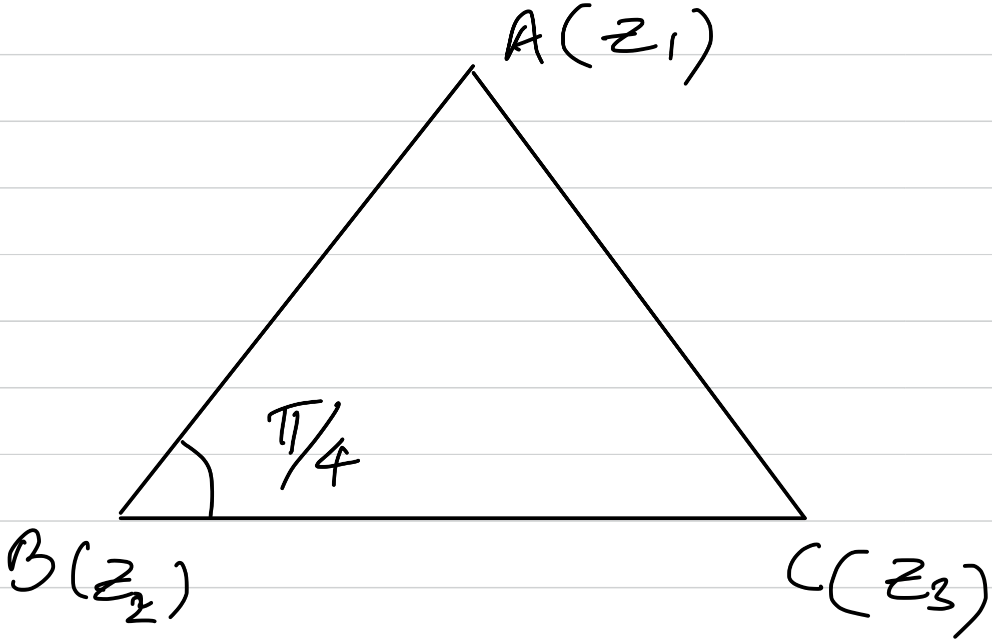
$$= 1 + (2\sqrt{2} + 2)\, i$$

(ii)



$\therefore$ z_1, z_2, z_3 are collinear so distance of z_1 from line is zero.

(11)



$$\frac{AB}{BC} = \sqrt{2}$$, By rotation about point B

$$\frac{(z_1 - z_2)}{|z_1 - z_2|} = \frac{z_3 - z_2}{|z_3 - z_2|} \cdot e^{i\pi/4}$$

$$\Rightarrow (z_1 - z_2) = \sqrt{2}(z_3 - z_2)\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$$

$$\Rightarrow (z_1 - z_2) = (z_3 - z_2)(1+i)$$

$$\Rightarrow z_1 - z_3(1+i) = z_2 - z_2(1+i)$$

$$\Rightarrow (z_1 - z_3) - iz_3 = -iz_2$$

$$\Rightarrow z_2 = z_3 + i(z_1 - z_3)$$

(iv) for equilateral Δ

$$z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$$

$$z_1 = a+i, \quad z_2 = 1+bi, \quad z_3 = 0$$

$$(a+i)^2 + (1+bi)^2 = (a+i)(1+bi)$$

$\Rightarrow (a^2 - b^2) + (2a + 2b)i = (a-b) + (ab+1)i$

$\Rightarrow \quad a^2 - b^2 = a - b \quad$ and $\quad 2a + 2b = ab + 1$

$\Rightarrow \quad a = b \quad$ or $\quad a + b = 1$

If $a = b$ then

$\Rightarrow \quad 4a = a^2 + 1$

$\Rightarrow \quad a^2 - 4a + 1 = 0 \quad \Rightarrow \quad a = b = 2 - \sqrt{3}$

again If $a+b=1$

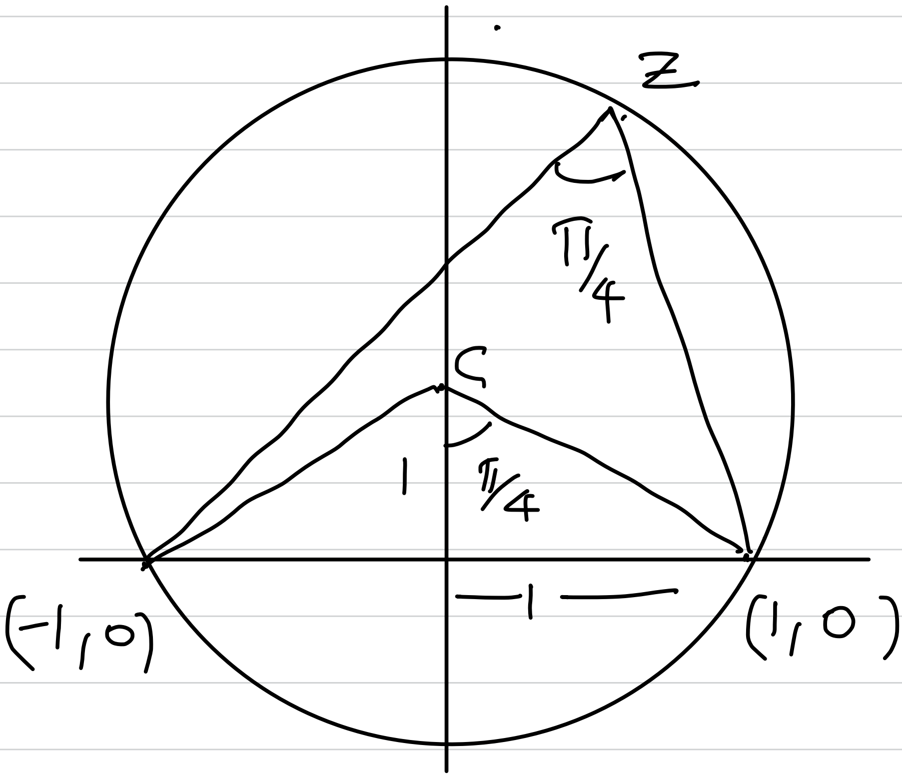
then

$2 = ab + 1$

or $\quad a^2 - a + 1 = 0$ which

is complex.

(v) $\arg\left(\frac{z-1}{z+1}\right) = \pi/4$



by diagrame centre of circle is c (0,1) and z represent major arc of circle.

Do yourself - 8 :

(i) If $z_r = \cos\frac{2r\pi}{5} + i\sin\frac{2r\pi}{5}$, $r = 0,1,3,4,........$, then $z_1z_2z_3z_4z_5$ is equal to -

(A) –1 (B) 0 (C) 1 (D) none of these

(ii) If $(x-1)^4 - 16 = 0$, then the sum of non-real complex values of x is -

(A) 2 (B) 0 (C) 4 (D) none of these

(iii) If $(\sqrt{3}-i)^n = 2^n$, $n \in Z$, then n is a multiple of -

(A) 6 (B) 10 (C) 9 (D) 12

(1) $Z_r = \cos\frac{2r\pi}{5} + i\sin\frac{2r\pi}{5}$

$\Rightarrow z_1 . z_2 . z_3 z_4 z_5$

$\Rightarrow \cos\left(\frac{2\pi}{5}(1+2+3+4+5)\right) + i\sin\left(\frac{2\pi}{5}(1+2+\ldots 5)\right)$

$\Rightarrow \cos(6\pi) + i\sin 6\pi$

$\Rightarrow 1$

(ii) $(x-1)^4 = 16$

$(x-1) = (16)^{1/4} = \pm 2$

$x = 3, \quad x = -1 \quad$ so sum $= 2$

(iii) $(\sqrt{3}-i)^n = 2^n$

$\Rightarrow \left(\frac{\sqrt{3}-i}{2}\right)^n = 1$

$\Rightarrow \left(\cos(-\pi/6) + i\sin(-\pi/6)\right)^n$

$\Rightarrow \cos\left(-\frac{n\pi}{6}\right) + i\sin\left(-\frac{n\pi}{6}\right)$

$\Rightarrow$ for $n = 12$

$\Rightarrow \cos(-2\pi) + i\sin(-2\pi)$

$\Rightarrow \underline{1}$

Do yourself - 9 :

(i) If ω is an imaginary cube root of unity, then $(1+\omega-\omega^2)^2$ equals : -

(A) ω (B) -4ω (C) ω^2 (D) 4ω

(ii) If ω is a non real cube root of unity, then the expression $(1-\omega)(1-\omega^2)(1+\omega^4)(1+\omega^8)$ is equal to :-

(A) 0 (B) 3 (C) 1 (D) 2

(i) we know $1+\omega+\omega^2=0$

so $(1+\omega-\omega^2)^2 = (-2\omega^2)^2$

$= 4\omega$

(ii) $(1-\omega)(1-\omega^2)(1+\omega^4)(1+\omega^8)$

$\Rightarrow (1-\omega)(1-\omega^2)(1+\omega)(1+\omega^2)$

$\Rightarrow (1-\omega^2)(1-\omega^4)$

$\Rightarrow (1-\omega^2)(1-\omega)$

$\Rightarrow 1-(\omega+\omega^2)+\omega^3$

$\Rightarrow 3$

ELEMENTARY EXERCISE

1. Simplify and express the result in the form of a + bi

(a) $\left(\frac{1+2i}{2+i}\right)^2$ (b) $-i(9+6i)(2-i)^{-1}$ (c) $\left(\frac{4i^3-i}{2i+1}\right)^2$ (d) $\frac{(2+i)^2}{2-i}-\frac{(2-i)^2}{2+i}$

(a) $\left(\frac{1+2i}{2+i}\right)^2 = \left(\frac{(1+2i)(2-i)}{5}\right)^2$

$= \frac{7}{5} + \frac{24}{5}i$

(b) $\frac{-i(9+6i)}{(2-i)} = \frac{-i(9+6i)(2+i)}{5}$

$= \frac{21}{5} - \frac{12}{5}i$

(c) $\left(\frac{4i^3-i}{2i+1}\right)^2 = \frac{(-5i)(1-2i)}{(1+2i)(1-2i)}$

$= 3+4i$

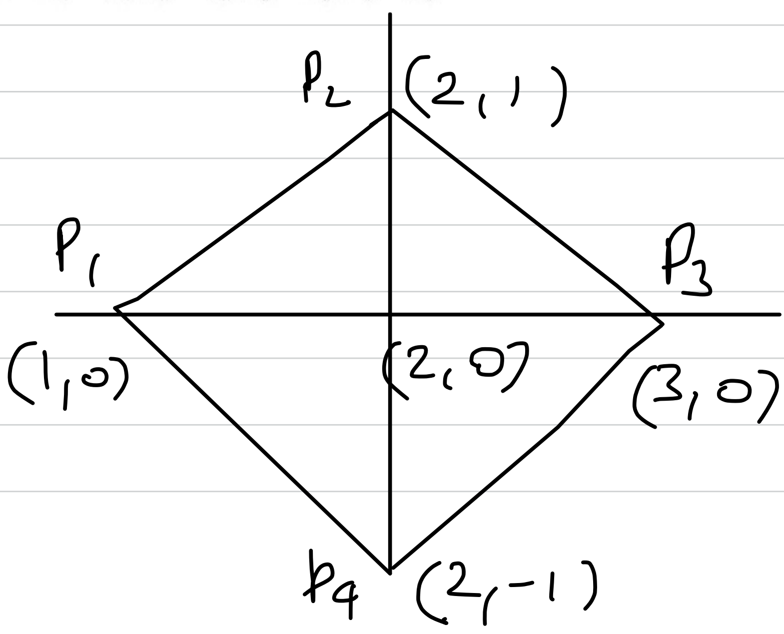
(d) $$\frac{(2+i)^2}{(2-i)} - \frac{(2-i)^2}{(2+i)}$$

$$\Rightarrow \frac{(3+4i)}{2-i} - \frac{(3-4i)}{(2+i)}$$

$$\Rightarrow \left[(3+4i)(2+i) - (3-4i)(2-i)\right]\frac{1}{5}$$

$$\Rightarrow 22/5\, i$$

(e) A square $P_1P_2P_3P_4$ is drawn in the complex plane with P_1 at (1, 0) and P_3 at (3, 0). Let P_n denotes the point (x_n, y_n) n = 1, 2, 3, 4. Find the numerical value of the product of complex numbers $(x_1 + i\,y_1)(x_2 + i\,y_2)(x_3 + i\,y_3)(x_4 + i\,y_4)$.



$P_1 = (1, 0)$ $P_2 (2, 1)$

$P_3 = (3, 0)$ $P_4 (2, -1)$

$P_1 \cdot P_2 \cdot P_3 \cdot P_4 = (1)(3)(2+i)(2-i)$

$= 15$

2. Given that x, y $\in$ R, solve :

(a) $(x + 2y) + i(2x - 3y) = 5 - 4i$

(b) $(x + iy) + (7 - 5i) = 9 + 4i$

(c) $x^2 - y^2 - i(2x + y) = 2i$

(a)

$$\left.\begin{aligned} x + 2y &= 5 \\ 2x - 3y &= -4 \end{aligned}\right\} \quad x = 1, \; y = 2$$

(b)

$$\left.\begin{aligned} x + 7 &= 9 \\ y - 5 &= 4 \end{aligned}\right\} \quad \begin{aligned} x &= 2 \\ y &= 9 \end{aligned}$$

(c)

$$\left.\begin{aligned} x^2 - y^2 &= 0 \\ 2x + y &= -2 \end{aligned}\right\} \quad \begin{gathered} (-2, 2) \\ \text{or} \\ \left(-\tfrac{2}{3}, -\tfrac{2}{3}\right) \end{gathered}$$

3. Find the square root of :

(a) $9 + 40i$ (b) $-11 - 60i$ (c) $50i$

(a) $\sqrt{9+40i} = x+iy$

$\Rightarrow x^2 - y^2 + 2ixy = 9+40i$

$\Rightarrow x^2 - y^2 = 9 \rightarrow (1)$

$2xy = 40 \rightarrow (2)$

$\Rightarrow x^2 + y^2 = 41 \rightarrow (3)$

$\Rightarrow x = \pm 5, \quad y = \pm 4$

$\Rightarrow \pm(5+4i)$

(b) $x^2 - y^2 = -11 \rightarrow (1)$

$2xy = -60 \rightarrow (2)$

$\therefore x^2 + y^2 = 61$ $\left\{ \begin{array}{l} x = \pm 5 \\ y = \mp 6 \\ \pm(5-6i) \end{array} \right.$

(c) $\sqrt{50i} = x + iy$

$x^2 - y^2 = 0$

$2xy = 50$

$x^2 + y^2 = 50$

$x = y = \pm 5$

$\therefore \sqrt{30i} = \pm 5(1+i)$

4. (a) If $f(x) = x^4 + 9x^3 + 35x^2 - x + 4$, find $f(-5 + 4i)$

(b) If $g(x) = x^4 - x^3 + x^2 + 3x - 5$, find $g(2 + 3i)$

Sol. (a) $x = -5 + 4i$

$\Rightarrow x + 5 = 4i$

$\Rightarrow x^2 + 10x + 41 = 0$

$x^4 + 9x^3 + 35x^2 - x + 4$

$= (x^2 + 10x + 41)(x^2 - x + 4) - 160$

$= -160$

(b) $x = 2 + 3i$

$(x-2)^2 = 3i \Rightarrow x^2 - 4x + 13 = 0$

$x^4 - x^3 + x^2 + 3x - 5 =$

$(x^2 - 4x + 13)(x^2 + 3x) - 36x - 5$

$\Rightarrow -36(2 + 3i) - 5 = -77 - 108i$

5. Solve the following equations over C and express the result in the form a + ib, a, b ∈ R.

(a) $ix^2 - 3x - 2i = 0$ (b) $2(1+i)x^2 - 4(2-i)x - 5 - 3i = 0$

(a) $ix^2 - 3x - 2i = 0$

$\Rightarrow x^2 + 3ix - 2 = 0$

$\Rightarrow x = \dfrac{-3i \pm \sqrt{-9+8}}{2}$

$\Rightarrow x = \dfrac{-3i \pm i}{2} = -i, -2i$

(b) $x = \dfrac{4(2-i) \pm \sqrt{16(2-i)^2 + 8(5+3i)}}{4(1+i)}$

$x = \dfrac{4(2-i) \pm 8}{4(1+i)}$

$x = \dfrac{3-5i}{2}, -\left(\dfrac{1+i}{2}\right)$

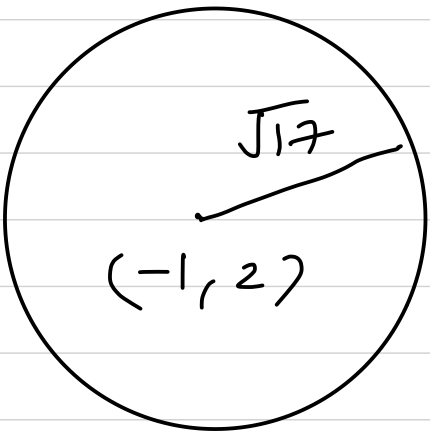
6. Locate the points representing the complex number z on the Argand plane :

(a) $|z+1-2i| = \sqrt{7}$ (b) $|z-1|^2 + |z+1|^2 = 4$

(c) $\left|\frac{z-3}{z+3}\right| = 3$ (d) $|z-3| = |z-6|$

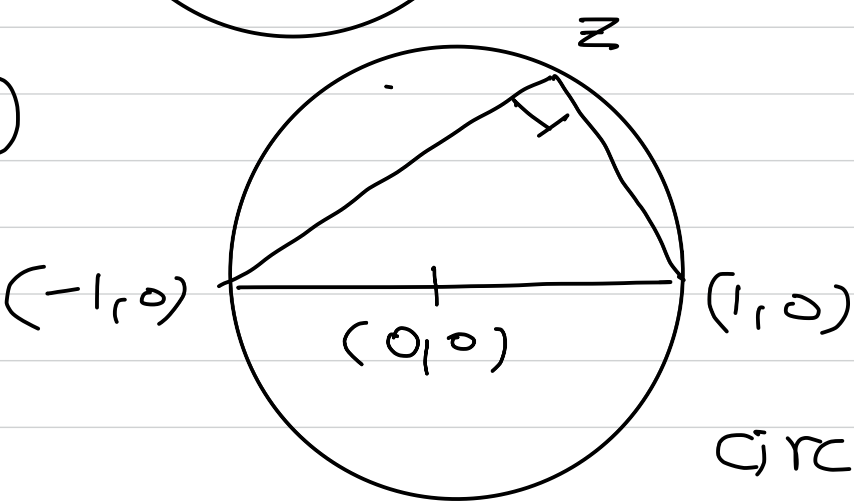
ⓐ $|z+1-2i| = \sqrt{7}$

$|z-(-1+2i)| = \sqrt{7}$



Circle

ⓑ



circle

(c) $\left|\frac{z-3}{z+3}\right| = 3$

let $z = x + iy$

$\Rightarrow ((x-3)^2 + y^2) = ((x+3)^2 + y^2)9$

$\Rightarrow x^2 + y^2 - 6x + 9 = (x^2 + y^2 + 6x + 9)9$

$\Rightarrow 8x^2 + 8y^2 + 60x + 72 = 0$

$\Rightarrow x^2 + y^2 + \frac{15}{2}x + 9 = 0$

Circle : centre $\left(-\frac{15}{4}, 0\right)$

Radius $9/4$

(d) $|z-3| = |z-6|$

$z = x + iy$

$\Rightarrow |(x-3) + iy| = |(x-6) + iy|$

$\Rightarrow (x-3)^2 + y^2 = (x-6)^2 + y^2$

$\Rightarrow x^2 + 9 - 6x = x^2 + 36 - 12x$

$\Rightarrow 6x = 27$

$\Rightarrow x = 9/2$ straight line

7. If a & b are real numbers between 0 & 1 such that the points $z_1 = a + i$, $z_2 = 1 + bi$ & $z_3 = 0$ form an equilateral triangle, then find the values of 'a' and 'b'.

Sol. $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$

$(a+i)^2 + (1+bi)^2 = (a+i)(1+bi)$

$\Rightarrow a^2 - 1 + 2ai + 1 - b^2 + 2bi$

$= a + abi + i - b$

$\Rightarrow a^2 - b^2 = (a-b) \rightarrow (1)$

$2a + 2b = ab + 1 \rightarrow (2)$

by (1) & (2)

$a = b = (2 - \sqrt{3})$

8. Let $z_1 = 1 + i$ and $z_2 = -1 - i$. Find $z_3 \in C$ such that triangle $z_1z_2z_3$ is equilateral.

Sol. for equilaletral Δ

$$|z_1 - z_2| = |z_2 - z_3| = |z_3 - z_1|$$

Let $z_3 = x + iy$

$$2\sqrt{2} = \sqrt{(x+1)^2 + (y+1)^2} = \sqrt{(x-1)^2 + (y-1)^2}$$

$$\Rightarrow 4x + 4y = 0 \rightarrow (1)$$

$$x = \pm\sqrt{3}, \quad y = \sqrt{3}$$

$$\therefore \sqrt{3}(1-i) \ ; \ \sqrt{3}(-1+i)$$

9. For what real values of x & y are the numbers $-3 + ix^2 y$ & $x^2 + y + 4i$ conjugate comp

Sol. $-3 + ix^2 y = z_1$

$x^2 + y + 4i = z_2$

Given $\bar{z}_1 = z_2$

$\therefore -3 - ix^2 y = x^2 + y + 4i$

$\Rightarrow \left.\begin{array}{l} x^2 + y = -3 \\ -x^2 y = 4 \end{array}\right\}$

$\left.\begin{array}{l} x = 1, \ y = -4 \\ x = -1, \ y = -4 \end{array}\right\}$

10. If $(x + iy)^{1/3} = a + bi$, then prove that $4(a^2 - b^2) = \frac{x}{a} + \frac{y}{b}$.

$$(x+iy)^{1/3} = a+ib$$

$$\Rightarrow (x+iy) = a^3 - ib^3 + 3iab(a+ib)$$

$$\Rightarrow \left.\begin{aligned} x &= a^3 - 3ab^2 \\ y &= -b^3 + 3a^2b \end{aligned}\right\}$$

$$\Rightarrow \frac{x}{a} = a^2 - 3b^2$$

$$\& \quad \frac{y}{b} = -b^2 + 3a^2$$

$$\frac{x}{a} + \frac{y}{b} = 4(a^2 - b^2)$$ Hence pro

11. (a) Prove the identity, $|1-z_1\bar{z}_2|^2 - |z_1 - z_2|^2 = (1-|z_1|^2)(1-|z_2|^2)$

(b) Prove the identity, $|1+z_1\bar{z}_2|^2 + |z_1 - z_2|^2 = (1+|z_1|^2)(1+|z_2|^2)$

(c) For any two complex numbers, prove that $|z_1+z_2|^2 + |z_1-z_2|^2 = 2\left[|z_1|^2 + |z_2|^2\right]$. Also give the geometrical interpretation of this identity.

(a) $|1 - z_1\bar{z}_2|^2 - |z_1 - z_2|^2$

$\Rightarrow (1 - z_1\bar{z}_2)(1 - \bar{z}_1 z_2) - (z_1 - z_2)(\bar{z}_1 - \bar{z}_2)$

$\Rightarrow 1 - \bar{z}_1 z_2 - z_1\bar{z}_2 + |z_1|^2|z_2|^2 - |z_1|^2 - |z_2|^2$
$+ \bar{z}_1 z_2 + z_1\bar{z}_2$

$\Rightarrow 1 - |z_1|^2 - |z_2|^2 + |z_1|^2|z_2|^2$

$\Rightarrow (1 - |z_1|^2)(1 - |z_2|^2)$

(b) $(1 + z_1\bar{z}_2)(1 + \bar{z}_1 z_2) + (z_1 - z_2)(\bar{z}_1 - \bar{z}_2)$

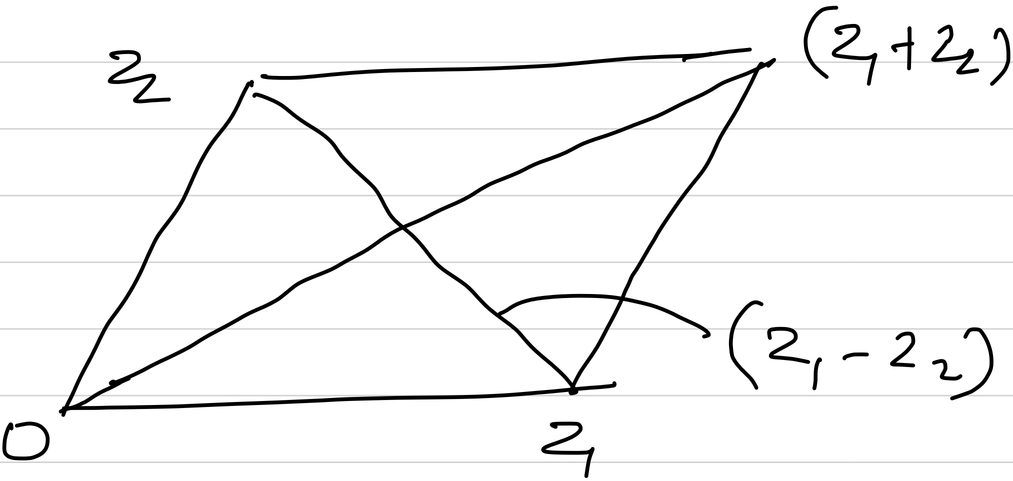
$\Rightarrow 1 + |z_1|^2 + |z_2|^2 + |z_1|^2|z_2|^2$

$\Rightarrow (1 + |z_1|^2)(1 + |z_2|^2)$

(c) $|z_1+z_2|^2 + |z_1-z_2|^2$

$\Rightarrow (z_1+z_2)(\overline{z_1+z_2}) + (z_1-z_2)(\overline{z_1-z_2})$

$\Rightarrow 2\left[|z_1|^2 + |z_2|^2\right]$



sum of squares of diagonals is equal to sum of squares of sides

12. Find the Cartesian equation of the locus of 'z' in the complex plane satisfying, $|z-4|+|z+4|=16$.

$|z - z_1| + |z - z_2| = k$

If $k > |z_1 - z_2|$

then locus of z is ellipse

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

So $2a = 16$

$a = 8$

$2ae = 8 \quad (|z_1 - z_2| = 8)$

$ae = 4$

$e = 1/2, \quad b^2 = a^2(1-e^2)$

$\therefore b = 4\sqrt{3} \qquad \therefore \boxed{\frac{x^2}{64} + \frac{y^2}{48} = 1}$

Paragraph for question nos. 13 to 15

Consider a complex number $w = \frac{z-i}{2z+1}$, where $z = x + iy$ and $x, y \in R$.

13. If the complex number w is purely imaginary then locus of z is -

(A) a straight line

(B) a circle with centre $\left(-\frac{1}{4}, \frac{1}{2}\right)$ and radius $\frac{\sqrt{5}}{4}$.

(C) a circle with centre $\left(\frac{1}{4}, -\frac{1}{2}\right)$ and passing through origin.

(D) neither a circle nor a straight line.

<u>Sol</u>. $z = x + iy$

$$\Rightarrow w = \frac{(x+iy) - i}{2(x+iy)+1} = \frac{x + (y-1)i}{(2x+1) + i2y}$$

$$\Rightarrow w = \frac{(x+(y-1)i)((2x+1) - 2iy)}{((2x+1)+2iy)((2x+1)-2iy)}$$

$$\Rightarrow w = \frac{x(2x+1) + 2y(y-1) + i(\ldots)}{(2x+1)^2 + 4y^2}$$

$\mathrm{Re}(z) = 0$

$\Rightarrow 2x^2 + 2y^2 + x - 2y = 0$

$\Rightarrow x^2 + y^2 + \frac{1}{2}x - y = 0$

centre $(-1/4, 1/2)$, $R = \sqrt{5}/4$

14. If the complex number w is purely real then locus of z is

(A) a straight line passing through origin

(B) a straight line with gradient 3 and y intercept (–1)

(C) a straight line with gradient 2 and y intercept 1.

(D) none

$$w = \frac{i(2x+1)(y-1) - 2ixy}{(2x+1)^2 + 4y^2} + \ldots$$

$$Im(z) = 0$$

$$\Rightarrow \cancel{2xy} - 2x + y - 1 - \cancel{2xy} = 0$$

$$\Rightarrow 2x - y + 1 = 0$$

15. If $|w| = 1$ then the locus of $P(z)$ is

(A) a point circle

(B) an imaginary circle

(C) a real circle

(D) not a circle.

$|w| = 1$

$\Rightarrow |z - i|^2 = |2z + 1|^2$

$\Rightarrow (z - i)(\bar{z} + i) = (2z + 1)(2\bar{z} + 1)$

$\Rightarrow 3|z|^2 + 2(z + \bar{z}) - i(z - \bar{z}) = 0$

$\Rightarrow 3(x^2 + y^2) + 4x + 2y = 0$

$\Rightarrow x^2 + y^2 + \frac{4}{3}x + \frac{2}{3}y = 0$

which is real circle.

O-1

EXERCISE (O-1)

1. If $z+z^3=0$ then which of the following must be true on the complex plane?

(A) $\mathrm{Re}(z)<0$ (B) $\mathrm{Re}(z)=0$ (C) $\mathrm{Im}(z)=0$ (D) $z^4=1$

$z+z^3=0 \Rightarrow z(1+z^2)=0$

$\Rightarrow z=0$ or $z^2+1=0$

$z=\pm i$

in both situation $\mathrm{Re}(z)=0$

$\Rightarrow$ option (B)

(2) Let $i = \sqrt{-1}$. The product of the real part of the roots of $z^2 - z = 5 - 5i$ is

(A) – 25 (B) – 6 (C) – 5 (D) 25

$$z^2 - z + (5i - 5) = 0$$

$$\therefore z = \frac{1 \pm \sqrt{1 - 4(5i - 5)}}{2}$$

$$= \frac{1 \pm \sqrt{21 - 20i}}{2} = \frac{1 \pm (5 - 2i)}{2}$$

$$= \frac{1 + 5 - 2i}{2}, \quad \frac{1 - 5 + 2i}{2}$$

$$= 3 - i, \quad -2 + i$$

$\therefore$ Product of real part $= -6$

(B) option

(3) The complex number z satisfying $z + |z| = 1 + 7i$ then the value of $|z|^2$ equals

(A) 625 (B) 169 (C) 49 (D) 25

$z = x + iy$

$\Rightarrow z + |z| = x + iy + \sqrt{x^2 + y^2} = 1 + 7i$

$\Rightarrow x + \sqrt{x^2 + y^2} = 1, \quad y = 7.$

$\Rightarrow x + \sqrt{x^2 + 49} = 1.$

$\Rightarrow x^2 + 49 = 1 + x^2 - 2x$

$\Rightarrow 2x = -48 \Rightarrow x = -24.$

$\therefore |z|^2 = x^2 + y^2 = 625 \Rightarrow$ (A) option

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4 If $\frac{x-3}{3+i}+\frac{y-3}{3-i}=i$ where x, y $\in$ R then

(A) x = 2 & y = – 8 (B) x = – 2 & y = 8 (C) x = – 2 & y = – 6 (D) x = 2 & y = 8

$$\frac{x-3}{3+i}+\frac{y-3}{3-i}=i$$

$$\Rightarrow (x-3)(3-i)+(y-3)(3+i)=i(10)$$

$$\Rightarrow (3x-9+3y-9)+i(y-3-x+3)=10i$$

$$\Rightarrow (3x+3y-18)+i(y-x)=10i$$

$3x+3y-18=0$ $\quad$ $y-x=10$

$\Rightarrow x+y=6$ — (1) $\quad$ $y-x=10$ — (2)

$$\left.\begin{array}{l} (1)+(2) \Rightarrow y=8 \\ (1)-(2) \Rightarrow x=-2 \end{array}\right\} \Rightarrow \text{option } (B)$$

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3. Number of complex numbers z satisfying $z^3 = \bar{z}$ is

(A) 1 (B) 2 (C) 4 (D) 5

$$z^3 = \bar{z}$$

$$\Rightarrow |z|^3 = |\bar{z}|$$

$$\Rightarrow |z| = 0, 1$$

$$\Rightarrow z = 0$$

OR $\quad |z| = 1$

$$\Rightarrow z^4 = z\bar{z} = |z|^2 = 1$$

$$\Rightarrow z = (1)^{\frac{1}{4}} = 1, -1, i, -i$$

$\Rightarrow$ There are 5 solutions.

Ans. D

(6) The value of sum $\sum_{n=1}^{13}(i^n + i^{n+1})$, where $i = \sqrt{-1}$, equals

(A) i (B) $i - 1$ (C) -1 (D) 0

$$\sum_{n=1}^{13} i^n + \sum_{n=1}^{13} i^{n+1} = (i + i^2 + \cdots + i^{13}) + (i^2 + i^3 + \cdots + i^{14})$$

$$= i^{13} + i^2 = -1 + i$$

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⑦ If $z = x + iy$ & $\omega = \dfrac{1 - iz}{z - i}$ then $|\omega| = 1$ implies that, in the complex plane

(A) z lies on the imaginary axis

(B) z lies on the real axis

(C) z lies on the unit circle

(D) none

$$\omega = \frac{1 - iz}{z - i} \Rightarrow |\omega| = 1$$

$$\Rightarrow |1 - iz| = |z - i|$$

$$\Rightarrow |-i(i + z)| = |z - i|$$

$$\Rightarrow |z + i| = |z - i|$$

$\Rightarrow$ z is $\perp^r$ bisector of i & $-i$

$\Rightarrow$ (B) option

8 On the complex plane locus of a point z satisfying the inequality

$2 \leq |z-1| < 3$ denotes

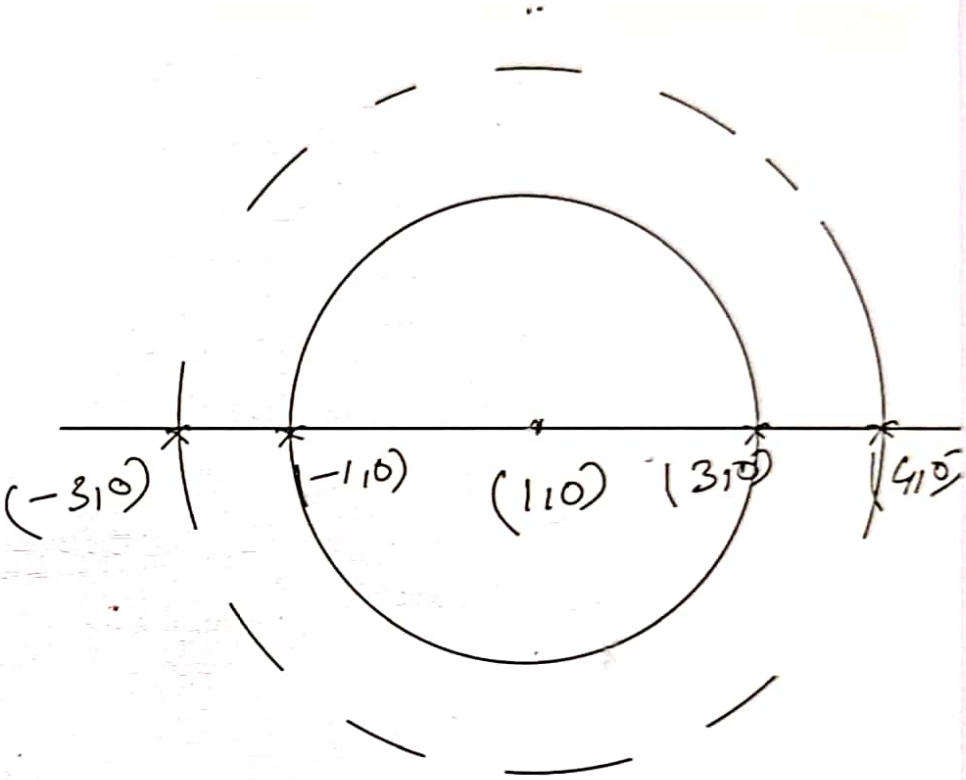
(A) region between the concentric circles of radii 3 and 1 centered at (1, 0)

(B) region between the concentric circles of radii 3 and 2 centered at (1, 0) excluding the inner and outer boundaries.

(C) region between the concentric circles of radii 3 and 2 centered at (1, 0) including the inner and outer boundaries.

(D) region between the concentric circles of radii 3 and 2 centered at (1, 0) including the inner boundary and excluding the outer boundary.

⇒ option D



(9) The locus of z, for arg $z = -\pi/3$ is

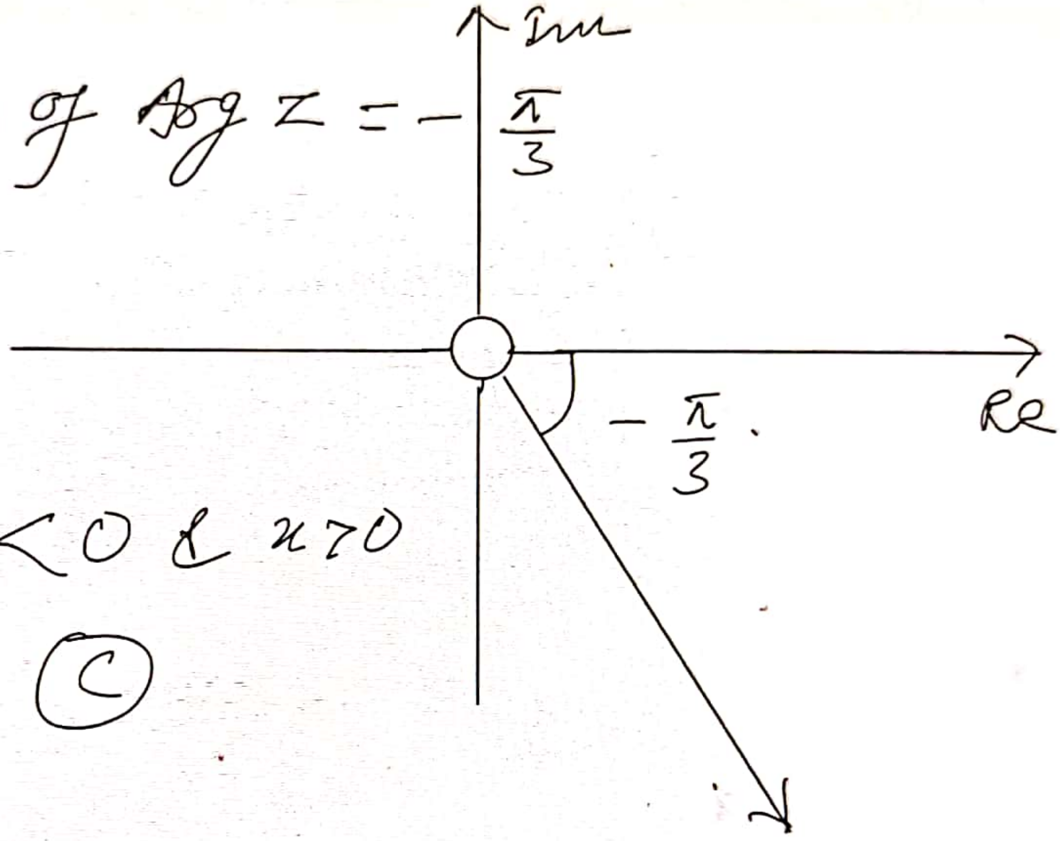
(A) same as the locus of z for arg $z = 2\pi/3$

(B) same as the locus of z for arg $z = \pi/3$

(C) the part of the straight line $\sqrt{3}x + y = 0$ with $(y < 0, x > 0)$

(D) the part of the straight line $\sqrt{3}x + y = 0$ with $(y > 0, x < 0)$

locus of Arg $z = -\frac{\pi}{3}$

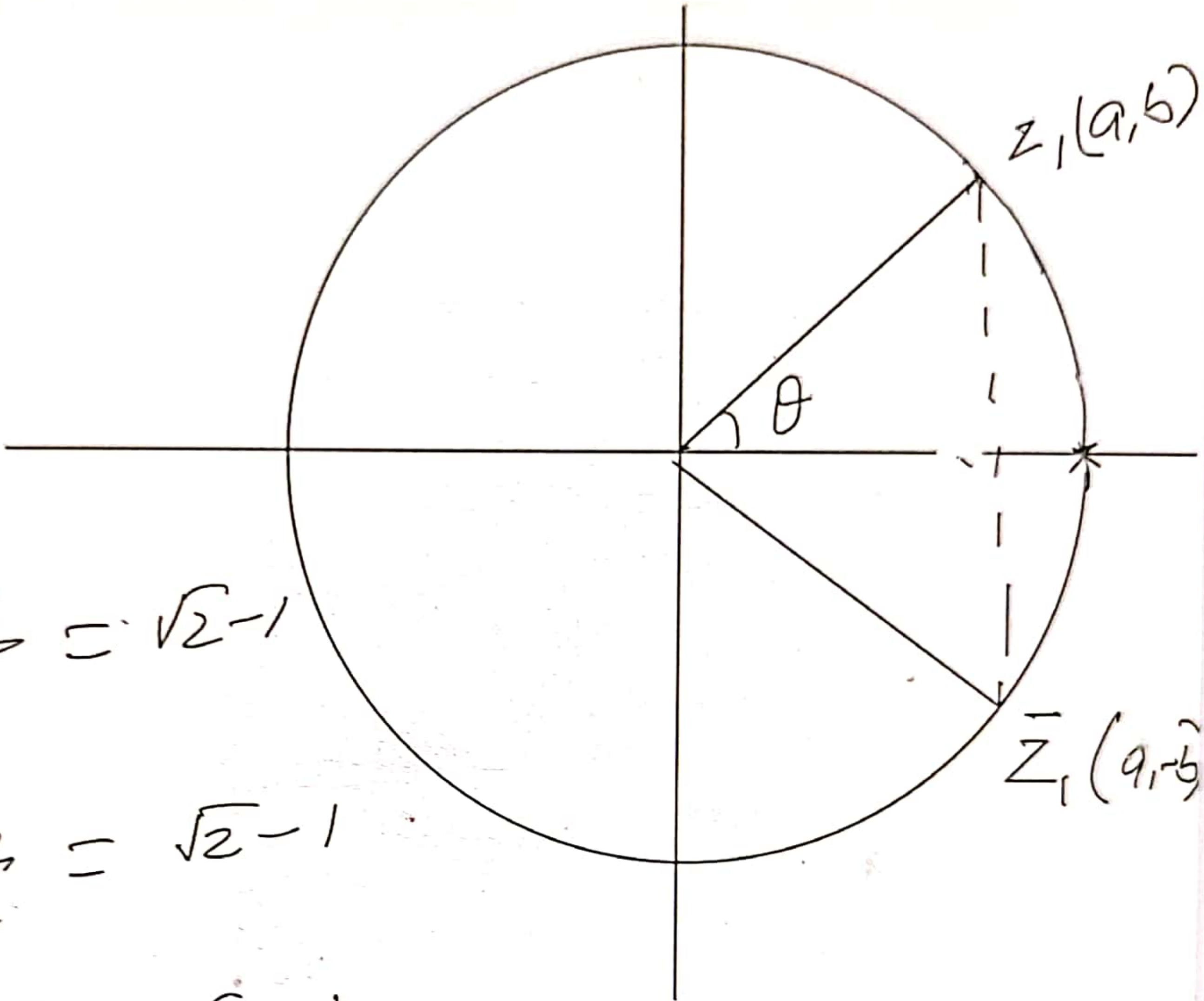


Here, $y < 0$ & $x > 0$

$\Rightarrow$ option C

10 If z_1 & $\bar{z}_1$ represent adjacent vertices of a regular polygon of n sides with centre at the origin & if $\frac{\operatorname{Im} z_1}{\operatorname{Re} z_1} = \sqrt{2} - 1$ then the value of n is equal to :

(A) 8 (B) 12 (C) 16 (D) 24



$$\frac{\operatorname{Im} z_1}{\operatorname{Re} z_1} = \sqrt{2}-1$$

$$\Rightarrow \frac{b}{a} = \sqrt{2}-1$$

$$\Rightarrow \tan\theta = \sqrt{2}-1$$

$$\Rightarrow \theta = \frac{45}{2}$$

$$\Rightarrow \frac{360}{n} = 45 \Rightarrow n = \frac{\cancel{360}^{\,8}}{\cancel{45}}$$

$\therefore$ $n = 8 \Rightarrow$ option (A)

(11) For $Z_1 = \sqrt[6]{\frac{1-i}{1+i\sqrt{3}}}$; $Z_2 = \sqrt[6]{\frac{1-i}{\sqrt{3}+i}}$; $Z_3 = \sqrt[6]{\frac{1+i}{\sqrt{3}-i}}$ which of the following holds good?

(A) $\sum |Z_1|^2 = \frac{3}{2}$

(B) $|Z_1|^4 + |Z_2|^4 = |Z_3|^{-8}$

(C) $\sum |Z_1|^3 + |Z_2|^3 = |Z_3|^{-6}$

(D) $|Z_1|^4 + |Z_2|^4 = |Z_3|^8$

$$|Z_1| = \left(\frac{\sqrt{2}}{2}\right)^{1/6} = \left(\frac{1}{2}\right)^{1/12}$$

$$|Z_2| = \left(\frac{1}{2}\right)^{\frac{1}{12}} \quad \& \quad |Z_3| = \left(\frac{1}{2}\right)^{\frac{1}{12}}$$

(A) $\sum |Z_1|^2 = 3\cdot\left(\frac{1}{2}\right)^{\frac{1}{6}}$

(B) $|Z_1|^4 + |Z_2|^4 = \left(\frac{1}{2}\right)^{\frac{1}{3}} + \left(\frac{1}{2}\right)^{\frac{1}{3}} = \frac{2}{2^{1/3}}$

$$= 2^{2/3}$$

$$|Z_3|^{-8} = \left(\frac{1}{2}\right)^{\frac{-8}{12}} = 2^{2/3}.$$

$\Rightarrow$ option (B)

12 Number of real or purely imaginary solution of the equation, $z^3 + iz - 1 = 0$ is :

(A) zero (B) one (C) two (D) three

Let z is purely imaginary. then

$z = i\alpha$

$\Rightarrow (i\alpha)^3 + i(i\alpha) - 1 = 0$

$\Rightarrow -i\alpha^3 - \alpha - 1 = 0$

$\Rightarrow \alpha = 0, \quad \alpha = -1$

Not possible simultaneously

Let z is real. so. $z = k$.

$k^3 + ik - 1 = 0$.

$\Rightarrow k = 0, \quad k^3 - 1 = 0 \Rightarrow k = 1$.

Again not possible simultaneously

so. no solution $\Rightarrow$ (A)

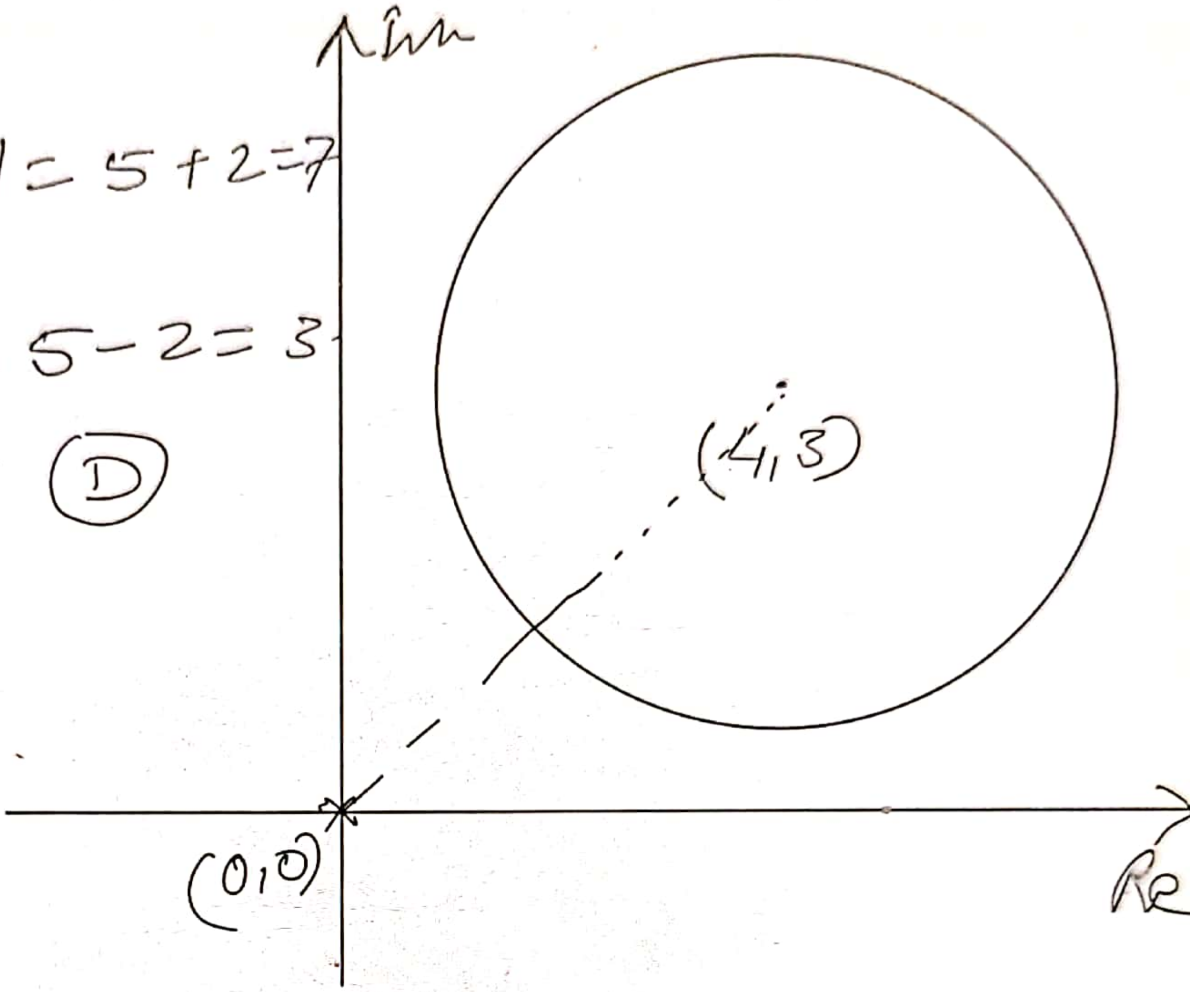
73) A point 'z' moves on the curve $|z-4-3i|=2$ in an argand plane. The maximum and minimum values of $|z|$ are

(A) 2, 1 (B) 6, 5 (C) 4, 3 (D) 7, 3

Max $|z| = 5+2=7$

Min $|z| = 5-2=3$

$\Rightarrow$ option D



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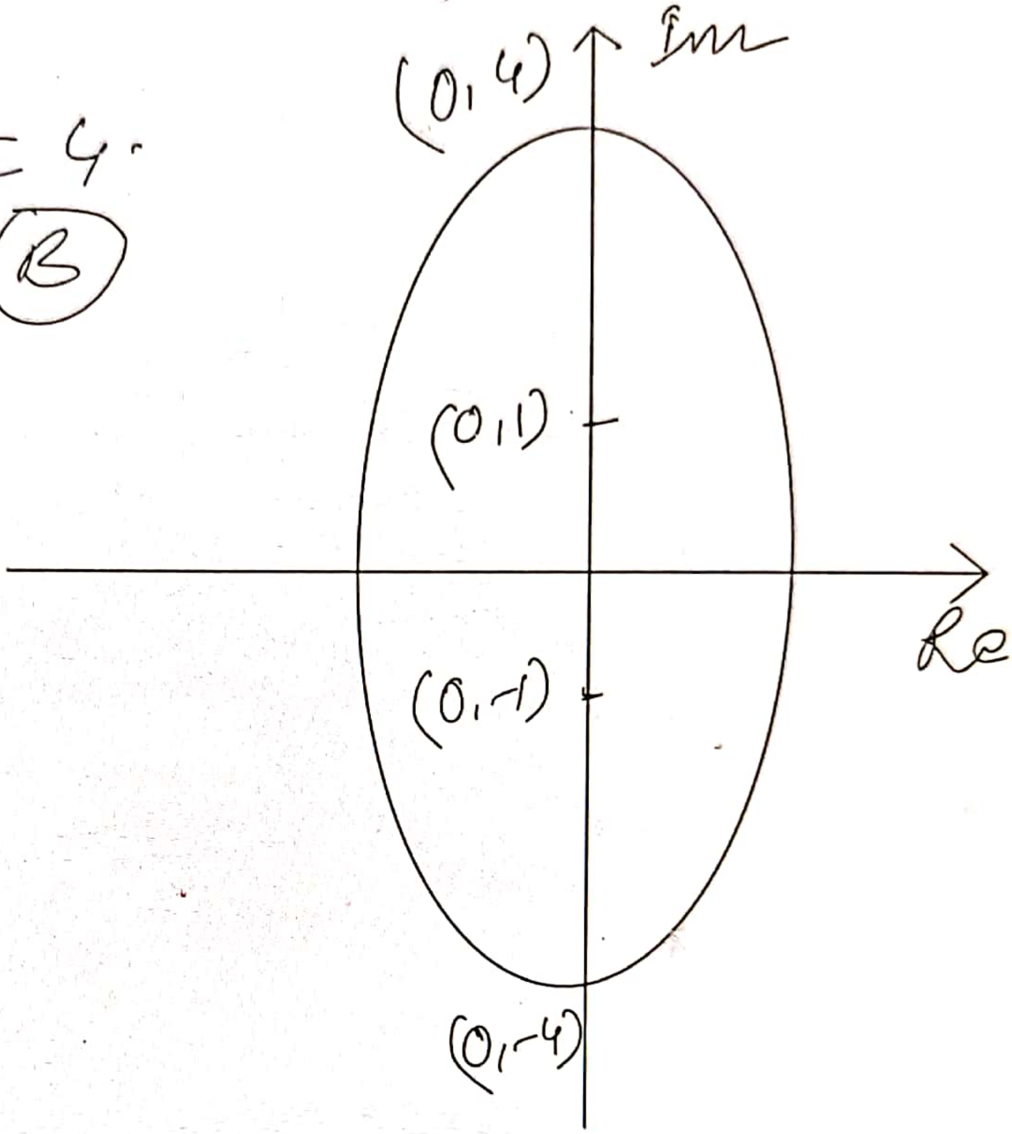
(14) If z is a complex number satisfying the equation $|z+i|+|z-i|=8$, on the complex plane then maximum value of $|z|$ is

(A) 2 (B) 4 (C) 6 (D) 8

Max $|z| = 4$.

$\Rightarrow$ option (B)



$$\Rightarrow k = \frac{60}{\sqrt{120}} = \sqrt{\frac{60}{2}} = \sqrt{30}$$

$$= |z_1 \cdot z_2 \, z_4| \Rightarrow \text{(D) option.}$$

(15) Let Z be a complex number satisfying the equation $(Z^3 + 3)^2 = -16$ then $|Z|$ has the value equal to

(A) $5^{1/2}$ (B) $5^{1/3}$ (C) $5^{2/3}$ (D) 5

$(z^3+3)^2 = -16$

$\Rightarrow z^3 + 3 = 4i$

$\Rightarrow z^3 = 4i - 3$

$\Rightarrow |z^3| = 5$

$\Rightarrow |z| = 5^{1/3} \Rightarrow$ option (B)

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16 If z_1, z_2, z_3 are 3 distinct complex numbers such that $\frac{3}{|z_2 - z_3|} = \frac{4}{|z_3 - z_1|} = \frac{5}{|z_1 - z_2|}$,

then the value of $\frac{9}{z_2 - z_3} + \frac{16}{z_3 - z_1} + \frac{25}{z_1 - z_2}$ equals

(A) 0 (B) 3 (C) 4 (D) 5

let $\frac{3}{|z_2 - z_3|} = k \Rightarrow |z_2 - z_3| = \frac{3}{k}$

$\Rightarrow (z_2 - z_3)(\bar{z}_2 - \bar{z}_3) = \frac{9}{k^2}$

$\Rightarrow (\bar{z}_2 - \bar{z}_3) k^2 = \frac{9}{z_2 - z_3}$

similarly. $\frac{16}{z_3 - z_1} = k^2(\bar{z}_3 - \bar{z}_1)$

$\frac{25}{z_1 - z_2} = k^2(\bar{z}_1 - \bar{z}_2)$

So, $\frac{9}{z_2 - z_3} + \frac{16}{z_3 - z_1} + \frac{25}{z_1 - z_2} =$

$k^2(\bar{z}_2 - \bar{z}_3 + \bar{z}_3 - \bar{z}_1 + \bar{z}_1 - \bar{z}_2) = 0$

$\Rightarrow$ option (A)

(17) The area of the triangle whose vertices are the roots $z^3 + iz^2 + 2i = 0$ is

(A) 2 (B) $\frac{3}{2}\sqrt{7}$ (C) $\frac{3}{4}\sqrt{7}$ (D) $\sqrt{7}$

$$z^3 + iz^2 + 2i = 0$$

$$(z-i)(z^2+2iz-2)=0$$

$$z = i \qquad z^2 + 2iz - 2 = 0$$

$$z = \frac{-2i \pm \sqrt{-4+8}}{2}$$

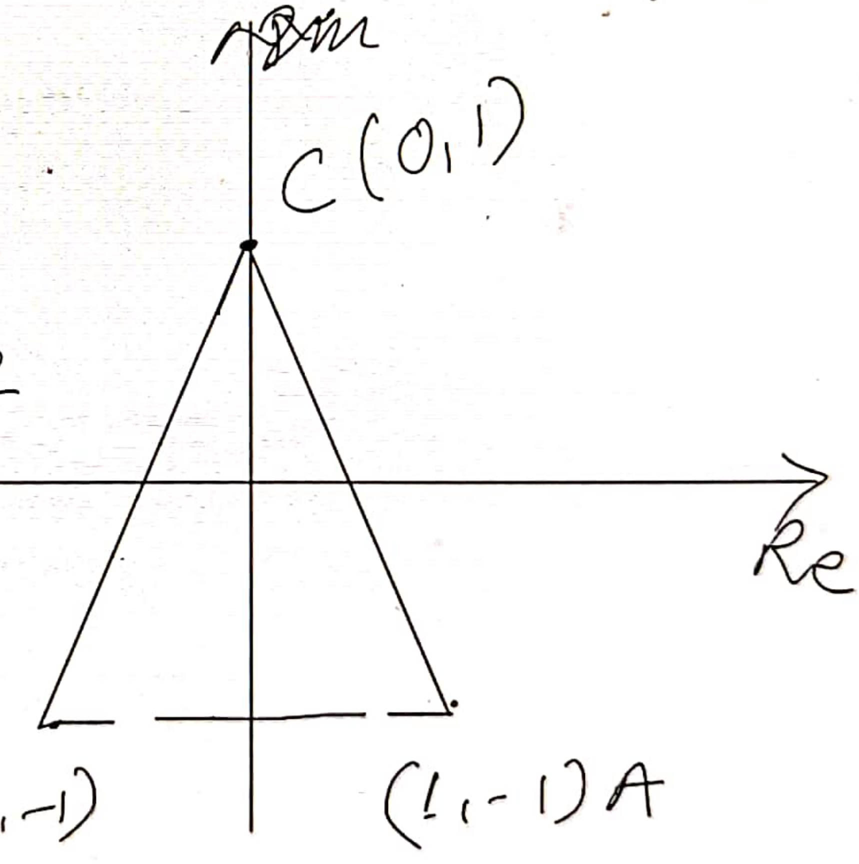
$$= \frac{-2i \pm 2}{2} = -1-i$$

$$1-i$$

Area of (ΔABC)

$= \frac{1}{2} \times 2 \times 2 = 2$

$\Rightarrow$ option (A)



(18) Consider two complex numbers α and β as

$$\alpha = \left(\frac{a+bi}{a-bi}\right)^2 + \left(\frac{a-bi}{a+bi}\right)^2, \text{ where } a, b \in R \text{ and } \beta = \frac{z-1}{z+1}, \text{ where } |z| = 1, \text{ then}$$

(A) Both α and β are purely real
(B) Both α and β are purely imaginary
(C) α is purely real and β is purely imaginary
(D) β is purely real and α is purely imaginary

sol:-

Note that $\alpha = \overline{\alpha} \Rightarrow \alpha$ is real

and $\beta + \overline{\beta} = \frac{z-1}{z+1} + \frac{\overline{z}-1}{\overline{z}+1} = \frac{(z-1)(\overline{z}+1)+(z+1)(\overline{z}-1)}{(z+1)(\overline{z}+1)} = \frac{2z\overline{z}-2}{D^r} = 0$; so 'β' is purely imaginary

as $z\overline{z} = |z|^2 = 1$ (given)]

19 Let Z is complex satisfying the equation $z^2 - (3 + i)z + m + 2i = 0$, where $m \in R$. Suppose the equation has a real root. The additive inverse of non real root, is

(A) $1 - i$ (B) $1 + i$ (C) $-1 - i$ (D) -2

Sol:-

Let α be the real root

$\alpha^2 - (3 + i)\alpha + m + 2i = 0$

$(\alpha^2 - 3\alpha + m) + i(2 - \alpha) = 0$

$\therefore \alpha = 2$ (real root)

$\therefore \quad 4 - 6 + m = 0 \Rightarrow m = 2$

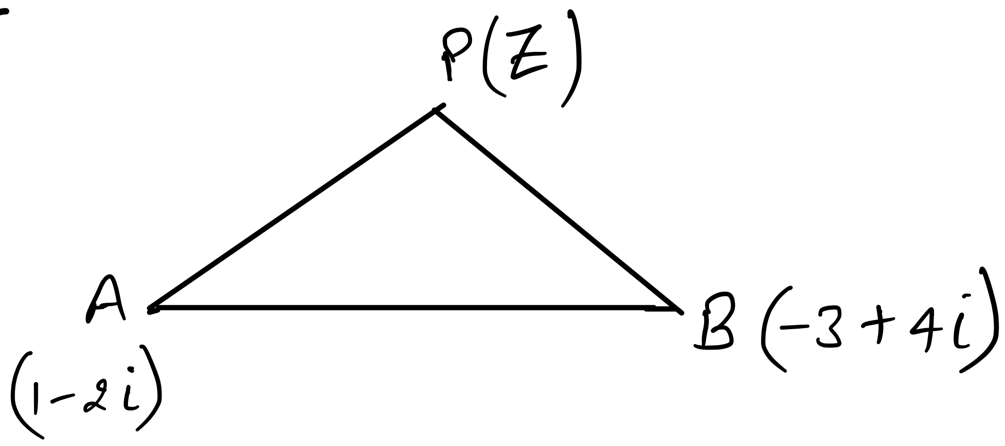
Product of the roots $= 2(1 + i)$ with one root as 2

non real root $= 1 + i$, addivitve inverse is $-1 - i$ **Ans**]

20 The minimum value of $|z-1+2i|+|4i-3-z|$ is

(A) $\sqrt{5}$ (B) 5 (C) $2\sqrt{13}$ (D) $\sqrt{15}$

Sol:-



$$PA + PB > AB$$

$$\therefore (PA+PB)_{min.} = AB = \sqrt{(1+3)^2+(-2-4)^2}$$

$$= \sqrt{52}$$

$$= \boxed{2\sqrt{13}}$$

Ans.

21) If $i = \sqrt{-1}$, then $4 + 5\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{334} + 3\left(-\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)^{365}$ is equal to

(A) $1 - i\sqrt{3}$ (B) $-1 + i\sqrt{3}$ (C) $i\sqrt{3}$ (D) $-i\sqrt{3}$

Sol.

$$4 + 5\omega^{334} + 3\omega^{365}$$

$$= 4 + 5\omega^{333}.\omega + 3.\omega^{363}.\omega^2$$

$$= 4 + 5\omega + 3\omega^2$$

$$= 4 + 4\omega + 4\omega^2 + \omega - \omega^2$$

$$= 4(1 + \omega + \omega^2) + \omega - \omega^2 \qquad \{\because 1 + \omega + \omega^2 = 0\}$$

$$= \omega - \omega^2$$

$$= \omega - (-1 - \omega)$$

$$= \omega + 1 + \omega$$

$$= 1 + 2\omega$$

$$= 1 + 2\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$$

$$= 1 - 1 + i\sqrt{3}$$

$$= \boxed{i\sqrt{3}}$$

Ans.

22. $z_1 = \dfrac{a}{1-i}$; $z_2 = \dfrac{b}{2+i}$; $z_3 = a - bi$ for $a, b \in \mathbb{R}$

if $z_1 - z_2 = 1$ then the centroid of the triangle formed by the points z_1, z_2, z_3 in the argand's plane is given by

(A) $\frac{1}{9}(1+7i)$ ✓ (B) $\frac{1}{3}(1+7i)$ (C) $\frac{1}{3}(1-3i)$ (D) $\frac{1}{9}(1-3i)$

Sol:-

$\because z_1 = \dfrac{a}{1-i} \Rightarrow z_1 = \dfrac{a(1+i)}{2}$

$z_2 = \dfrac{b}{2+i} \Rightarrow z_2 = \dfrac{b(2-i)}{5}$

Now $\because z_1 - z_2 = 1$

$$\Rightarrow \left(\frac{a}{2} - \frac{2b}{5}\right) + i\left(\frac{a}{2} + \frac{b}{5}\right) = 1$$

$$\therefore \frac{a}{2} - \frac{2b}{5} = 1 \text{ \& } \frac{a}{2} + \frac{b}{5} = 0$$

$$\Rightarrow a = 2/3, \quad b = -5/3$$

$\therefore z_3 = a - ib$

$$\Rightarrow \boxed{z_3 = \frac{2}{3} + i\frac{5}{3}}$$

$$\text{\&} \boxed{z_1 = \frac{1+i}{3}}$$

$$\text{\&} \boxed{z_2 = \frac{-2+i}{3}}$$

$$\therefore \text{Centroid} = \frac{z_1 + z_2 + z_3}{3}$$

$$= \frac{\frac{1}{3} + i\frac{7}{3}}{3}$$

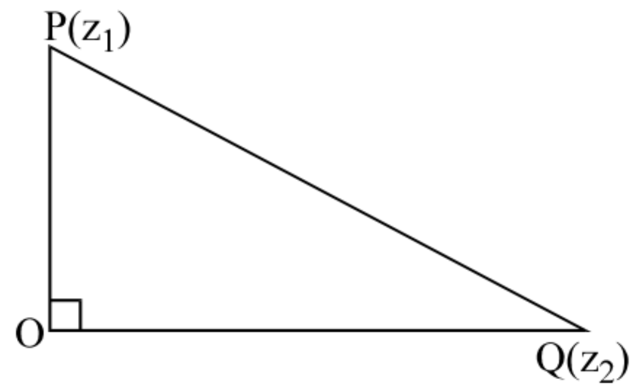
$$= \boxed{\frac{1+7i}{9}}$$

Ans.

23 If P and Q are respectively by the complex numbers z_1 and z_2 such that $\left|\frac{1}{z_1}+\frac{1}{z_2}\right|=\left|\frac{1}{z_1}-\frac{1}{z_2}\right|$, then the circumcentre of ΔOPQ (where O is the origin) is

(A) $\frac{z_1-z_2}{2}$ (B) $\frac{z_1+z_2}{2}$ (C) $\frac{z_1+z_2}{3}$ (D) z_1+z_2

[Sol. We have $\left|\frac{1}{z_2}+\frac{1}{z_1}\right|=\left|\frac{1}{z_2}-\frac{1}{z_1}\right|$



$\Rightarrow |z_1+z_2|=|z_1-z_2|$ $\Rightarrow z_1\bar{z}_2+z_2\bar{z}_1=0$

$\Rightarrow \frac{z_1}{z_2}$ is purely imaginary.

Hence ΔPQR is right angled at O. **[12th, 20-12-2009, complex]**

$\therefore$ Circumcentre of ΔPOQ is the mid point of PQ i.e. $\frac{1}{2}(z_1+z_2)$]

24

A particle starts from a point $z_0 = 1 + i$, where $i = \sqrt{-1}$. It moves horizontally away from origin by 2 units and then vertically away from origin by 3 units to reach a point z_1. From z_1 particle moves $\sqrt{5}$ units in the direction of $2\hat{i} + \hat{j}$ and then it moves through an angle of $\text{cosec}^{-1}\sqrt{2}$ in anticlockwise direction of a circle with centre at origin to reach a point z_2. The arg z_2 is given by

(A) $\sec^{-1}2$ (B) $\cot^{-1}0$ (C) $\sin^{-1}\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)$ (D) $\cos^{-1}\left(\frac{-1}{2}\right)$

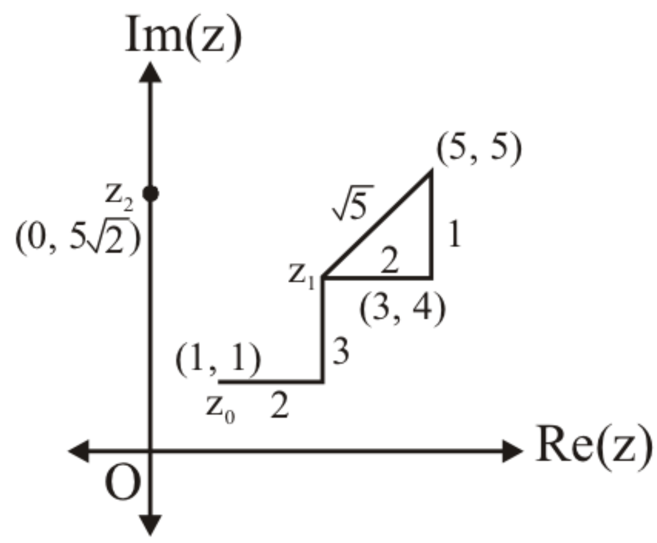
[Sol. Clearly $z_1 = 3 + 4i$ **[Online test-4, P-2]**

After moving by $\sqrt{5}$ distance in direction of $2\hat{i} + \hat{j}$, particle will reach at point $(5\hat{i} + 5\hat{j})$

If particle moves by an angle $\frac{\pi}{4}$ then it will reach at y-axis

At $z_2 = 0 + 5\sqrt{2}\,i$

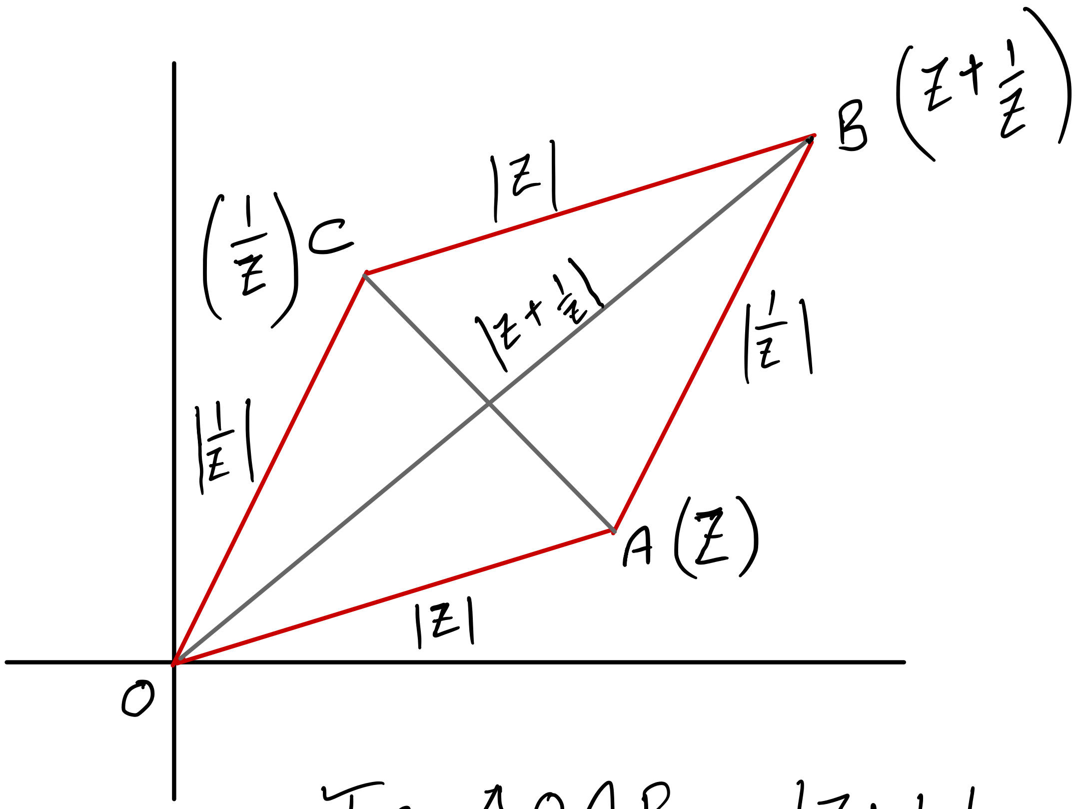
Hence $\text{amp}(z_2) = \frac{\pi}{2} = \cot^{-1}0$]



Q5 If the complex number z satisfies the condition $|z| \geq 3$, then the least value of $\left|z + \frac{1}{z}\right|$ is equal to :

(A) 5/3 (B) 8/3 (C) 11/3 (D) none of these

Sol :-



In $\triangle OAB$ use $\left|z+\frac{1}{z}\right| \geq |z| - \left|\frac{1}{z}\right|$

$$\because \quad \left|z + \frac{1}{z}\right| \geq |z| - \frac{1}{|z|}$$

$$\left|z + \frac{1}{z}\right|_{\text{least}} \geq 3 - \frac{1}{3} \geq \frac{8}{3} \qquad]$$

26. Given $z_p = \cos\left(\frac{\pi}{2^P}\right) + i\sin\left(\frac{\pi}{2^P}\right)$, then $\lim_{n\to\infty}(z_1 z_2 z_3 \ldots z_n) =$

(A) 1 (B) -1 ✓ (C) i (D) $-i$

Sol:- $\because Z_p = e^{i\frac{\pi}{2^p}}$

$$Z_1 Z_2 Z_3 \cdots Z_n = e^{i\frac{\pi}{2}} \cdot e^{i\frac{\pi}{2^2}} \cdot e^{i\cdot\frac{\pi}{2^3}} \cdots\cdots e^{i\frac{\pi}{2^n}}$$

$$= e^{i\pi\left(\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots\cdots+\frac{1}{2^n}\right)}$$

$$= e^{i\pi\left[\frac{1}{2}\left(\frac{1-(1/2)^n}{1-1/2}\right)\right]}$$

$$= e^{i\pi\left[1-\frac{1}{2^n}\right]}$$

$$\therefore \lim_{n\to\infty}(Z_1 \cdot Z_2 \cdot Z_3 \cdots Z_n)$$

$$= \lim_{n\to\infty} e^{i\pi\left[1-\frac{1}{2^n}\right]}$$

$$= e^{i\pi}$$

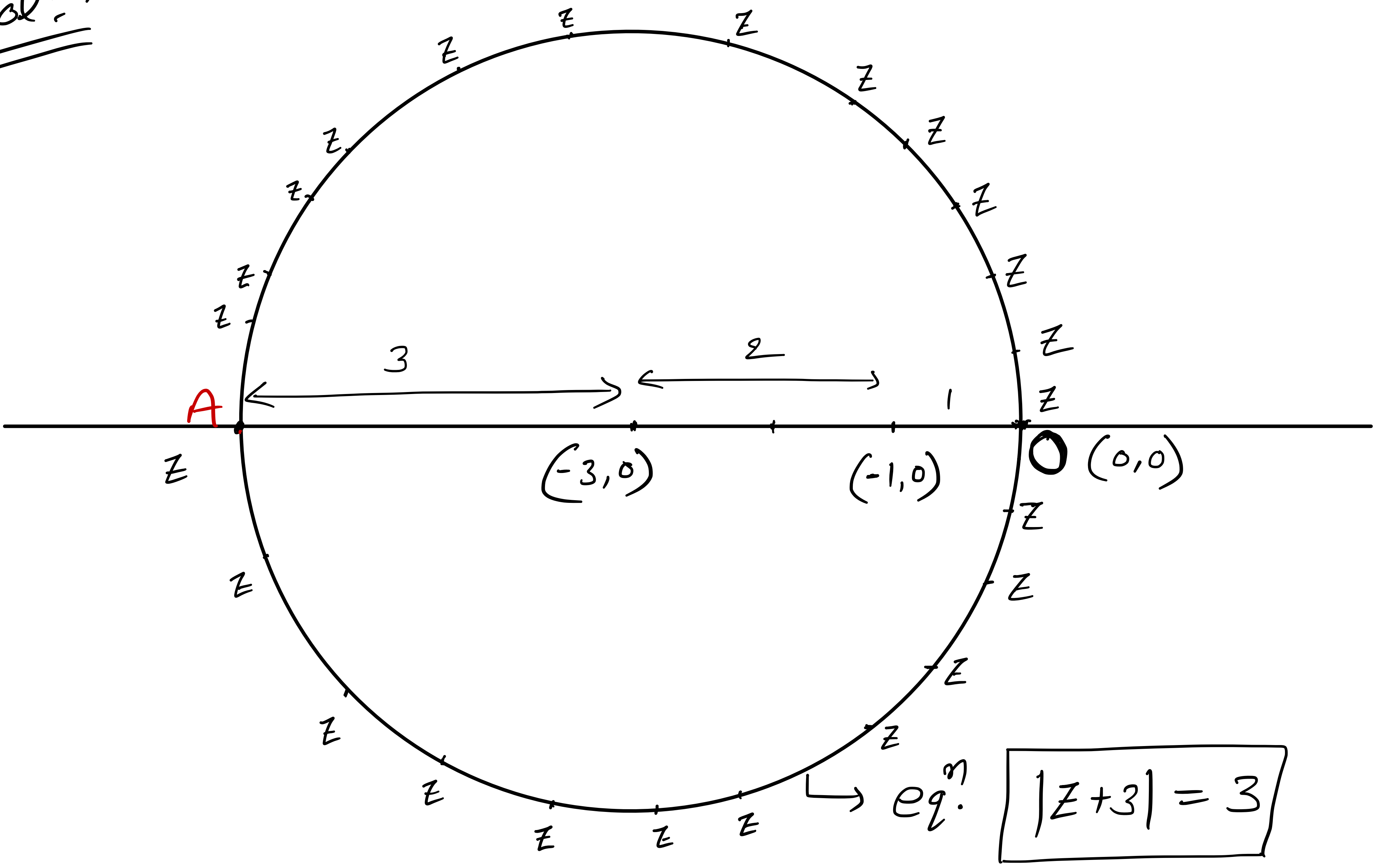
$$= \cos\pi + i\sin\pi$$

$$= -1 \text{ Ans.}$$

(27) The maximum & minimum values of $|z+1|$ when $|z+3| \le 3$ are :

~~(A)~~ $(5,0)$ (B) $(6,0)$ (C) $(7,1)$ (D) $(5,1)$

Sol:-



$\hookrightarrow$ eqⁿ $\boxed{|Z+3| = 3}$

for $|Z+3| \le 3$ all 'Z' lie on boundary or inside the Circle.

* $|Z+1|$ = distance b/w Z & Point $(-1,0)$

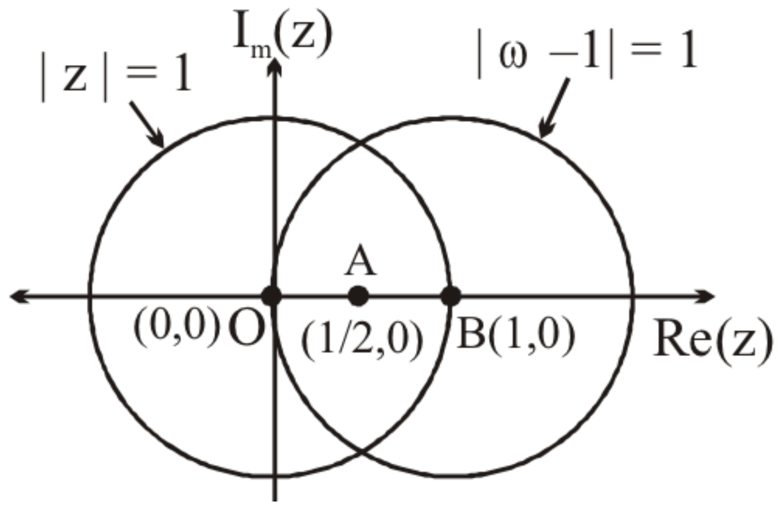
$\therefore |Z+1|_{min.} = 0$ {when Z lie at $(-1,0)$}

$\therefore |Z+1|_{max.} = 3+2 = 5$ {when Z lie at 'A'}

28. If $|z| = 1$ and $|\omega - 1| = 1$ where $z, \omega \in C$, then the largest set of values of $|2z-1|^2 + |2\omega - 1|^2$ equals

(A) [1, 9] (B) [2, 6] (C) [2, 12] (D) [2, 18]

[Sol. Least distance and greatest distance of any z and ω from the point $\left(\frac{1}{2}, 0\right)$ are $\frac{1}{2}$ and $\frac{3}{2}$ respectively.



$$\therefore \quad \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^2 \le \left|z - \frac{1}{2}\right|^2 + \left|\omega - \frac{1}{2}\right|^2 \le \left(\frac{3}{2}\right)^2 + \left(\frac{3}{2}\right)^2$$

Hence $2 \le |2z-1|^2 + |2\omega - 1|^2 \le 18$

Alternatively: $(2z-1)(2\bar{z}-1) + (2\omega - 1)(2\bar{\omega} - 1)$

$4 + 1 - 2(z + \bar{z}) + 4 - 2(\omega + \bar{\omega}) + 1$

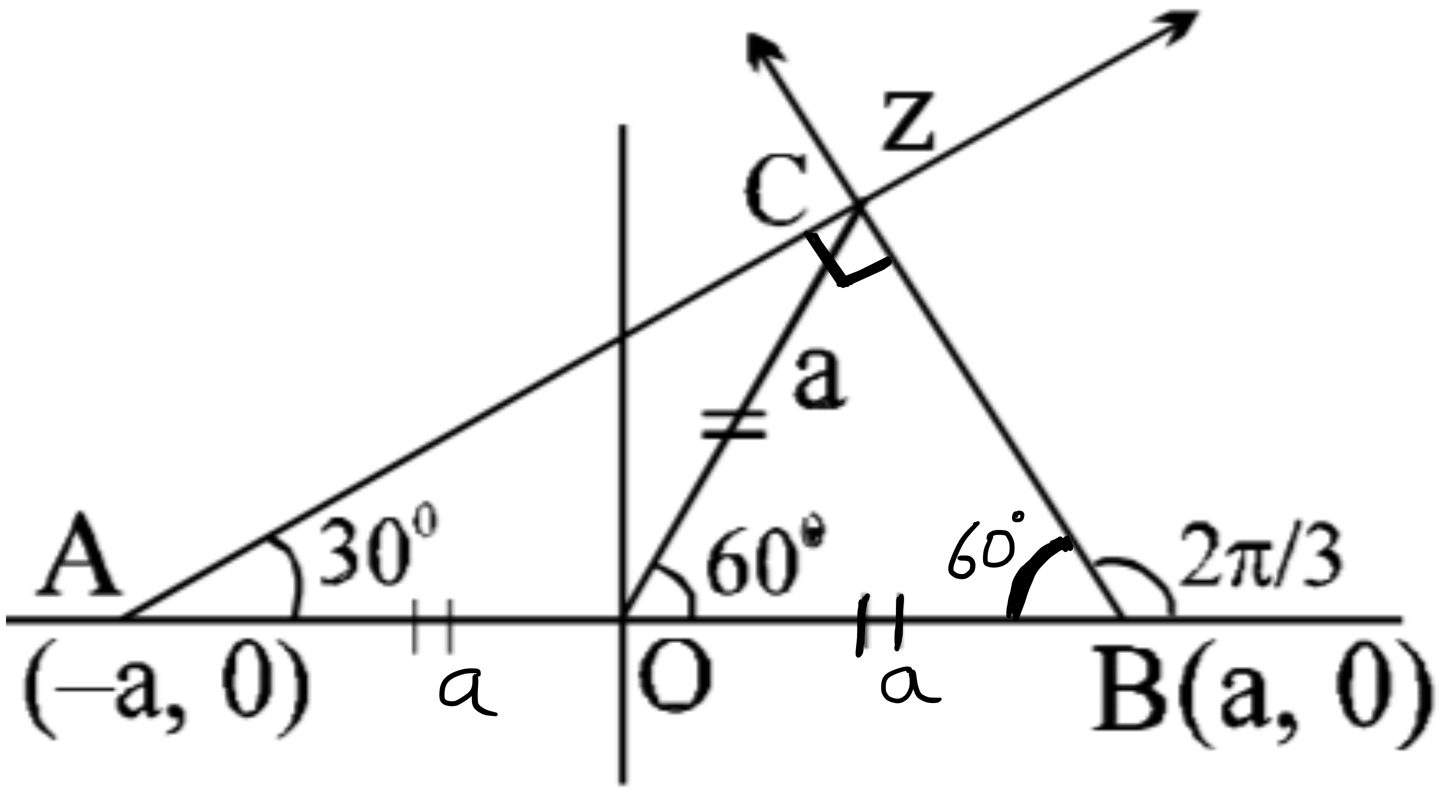
$10 - 2[2 \text{ Re } z + 2 \text{ Re } \omega]$

$10 - 4[\text{Re } z + \text{Re } \omega]$]

29 If $\text{Arg}(z+a) = \frac{\pi}{6}$ and $\text{Arg}(z-a) = \frac{2\pi}{3}$; $a \in R^+$, then

(A) z is independent of a (B) $|a| = |z+a|$ (C) $z = a\text{ Cis }\frac{\pi}{6}$ (D) $z = a\text{ Cis }\frac{\pi}{3}$

Sol:-



Refer the figure z lies on the point of intersection of the rays from A and B. ΔACB is a right angle and OBC is an equilateral triangle

$\Rightarrow OC = a \Rightarrow z = a\text{ Cis }\frac{\pi}{3} \Rightarrow$ (D)

(30) If z_1, z_2, z_3 are the vertices of the ΔABC on the complex plane which are also the roots of the equation, $z^3 - 3\alpha z^2 + 3\beta z + x = 0$, then the condition for the ΔABC to be equilateral triangle is

(A) $\alpha^2 = \beta$ (B) $\alpha = \beta^2$ (C) $\alpha^2 = 3\beta$ (D) $\alpha = 3\beta^2$

Sol :-

$$z_1 + z_2 + z_3 = 3\alpha \; ; \; \sum z_1 z_2 = 3\beta$$

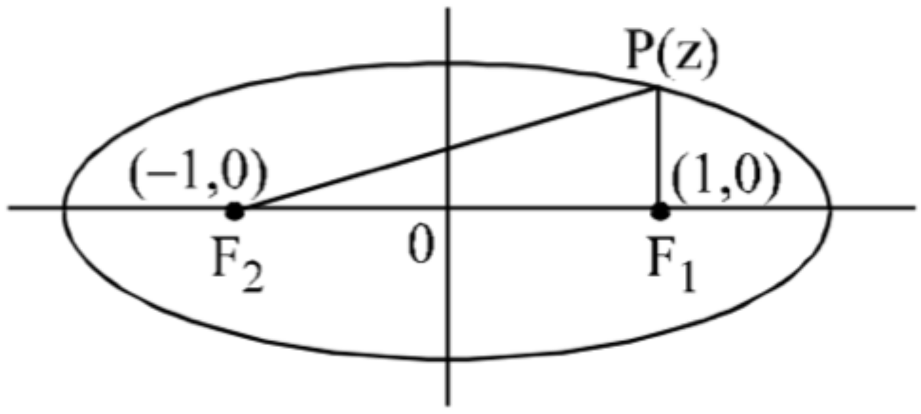
If ABC is equilateral $\quad z_1^2 + z_2^2 + z_3^2 = z_1 z_2 + z_2 z_3 + z_3 z_1$

$$(z_1 + z_2 + z_3)^2 = 3\sum z_1 z_2$$

$$9\alpha^2 = 3 \cdot 3\beta = 9\beta \Rightarrow \alpha^2 = \beta \quad]$$

(3) The locus represented by the equation, $|z-1|+|z+1|=2$ is :
(A) an ellipse with focii $(1,0)$; $(-1,0)$
(B) one of the family of circles passing through the points of intersection of the circles $|z-1|=1$ & $|z+1|=1$
(C) the radical axis of the circles $|z-1|=1$ and $|z+1|=1$
(D) the portion of the real axis between the points $(1,0)$; $(-1,0)$ including both.

Sol :-



Note that $|z-1|+|z+1|$ denotes the sum of the distances of P from F_1 and F_2

since $|z_1+1|+|z_1-1|=2$

hence locus will not be the ellipse]

[MATCH THE COLUMN]

32 Match the equation in z, in **Column-I** with the corresponding values of arg(z) in **Column-II**.

Column-I (equations in z)		**Column-II** (principal value of arg (z))	
(A)	$z^2 - z + 1 = 0$	(P)	$-2\pi/3$
(B)	$z^2 + z + 1 = 0$	(Q)	$-\pi/3$
(C)	$2z^2 + 1 + i\sqrt{3} = 0$	(R)	$\pi/3$
(D)	$2z^2 + 1 - i\sqrt{3} = 0$	(S)	$2\pi/3$

[Sol. (A) $z = \dfrac{1 \pm \sqrt{-3}}{2} = \dfrac{1 + i\sqrt{3}}{2}$ or $\dfrac{1 - i\sqrt{3}}{2}$ **[12th, 07-12-2008, P-2]**

$\text{amp } z = \dfrac{\pi}{3}$ or $\text{amp } z = -\dfrac{\pi}{3} \Rightarrow$ **Q, R**

(B) $z = \dfrac{-1 \pm \sqrt{3}\,i}{2} = \dfrac{-1 + i\sqrt{3}}{2}$ or $\dfrac{-1 - i\sqrt{3}}{2}$

$\text{amp } z = \dfrac{2\pi}{3}$ or $-\dfrac{2\pi}{3} \Rightarrow$ **P, S**

(C) $2z^2 = -1 - i\sqrt{3} \Rightarrow z^2 = \dfrac{-1 - i\sqrt{3}}{2} = \cos\left(-\dfrac{2\pi}{3}\right) + i\sin\left(-\dfrac{2\pi}{3}\right)$

$$z = \cos\left(\frac{2m\pi - (2\pi/3)}{2}\right) + i\sin\left(\frac{2m\pi - (2\pi/3)}{2}\right)$$

$m = 0$, $z = \cos\left(-\dfrac{\pi}{3}\right) + i\sin\left(-\dfrac{\pi}{3}\right)$

$m = 1$, $z = \cos\left(\dfrac{2\pi}{3}\right) + i\sin\left(\dfrac{2\pi}{3}\right) \Rightarrow \text{amp } z = -\dfrac{\pi}{3}$ or $\dfrac{2\pi}{3} \Rightarrow$ **Q, S**

(D) $2z^2 + 1 - i\sqrt{3} = 0$

$$z^2 = \frac{-1 + i\sqrt{3}}{2} = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)$$

$$z = \cos\left(\frac{2m\pi + (2\pi/3)}{2}\right) + i\sin\left(\frac{2m\pi + (2\pi/3)}{2}\right)$$

$m = 0$, $z = \cos\left(\dfrac{\pi}{3}\right) + i\sin\left(\dfrac{\pi}{3}\right)$

$m = 1$, $\cos\left(\dfrac{4\pi}{3}\right) + i\sin\left(\dfrac{4\pi}{3}\right)$ or $\cos\left(-\dfrac{2\pi}{3}\right) + i\sin\left(-\dfrac{2\pi}{3}\right) \Rightarrow$ **P, R]**

O-2

1. Let z_1 & z_2 be non zero complex numbers satisfying the equation, $z_1^2 - 2z_1z_2 + 2z_2^2 = 0$. The geometrical nature of the triangle whose vertices are the origin and the points representing z_1 & z_2 is :

(A) an isosceles right angled triangle
(B) a right angled triangle which is not isosceles
(C) an equilateral triangle
(D) an isosceles triangle which is not right angled.

$$z_1^2 - 2z_1z_2 + 2z_2^2 = 0$$

$$\left(\frac{z_1}{z_2}\right)^2 - 2\left(\frac{z_1}{z_2}\right) + 2 = 0$$

$$t^2 - 2t + 2 = 0$$

$$(t-1)^2 = -1$$

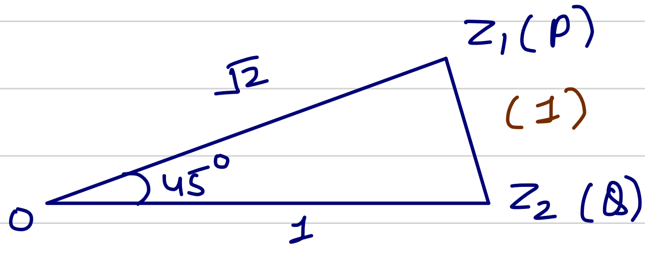
$$t - 1 = \pm i$$

$$\frac{z_1}{z_2} = 1 \pm i$$

$$\frac{z_1}{z_2} = 1+i \quad , \quad \frac{z_1}{z_2} = 1-i$$

$$\frac{z_1}{z_2} = \sqrt{2}\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right) \quad , \quad \frac{z_1}{z_2} = \sqrt{2}\, e^{-i\pi/4}$$

$$= \sqrt{2}\, e^{i\pi/4}$$



Let $|z_2| = 1 \Rightarrow |z_1| = \sqrt{2}$.

$$\cos 45^\circ = \frac{|z_1|^2 + |z_2|^2 - |z_1 - z_2|^2}{2|z_1||z_2|} = \frac{1}{\sqrt{2}}$$

$$1 + 2 - |z_1 - z_2|^2 = 2$$

$|z_1 - z_2| = 1 \Rightarrow$ triangle is isosceles Right angled.

2. Let P denotes a complex number z on the Argand's plane, and Q denotes a complex number $\sqrt{2|z|^2}\,\text{CiS}\left(\frac{\pi}{4}+\theta\right)$ where $\theta = \text{amp } z$. If 'O' is the origin, then the Δ OPQ is :

(A) isosceles but not right angled

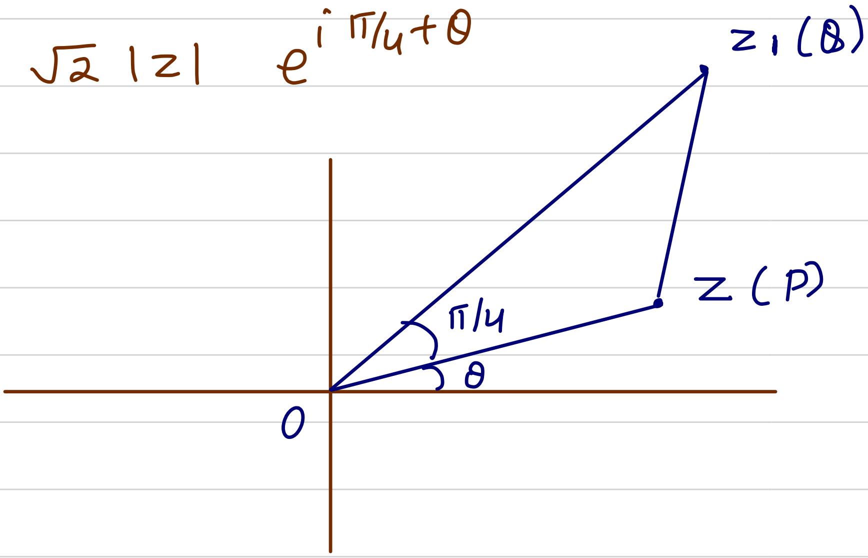
(B) right angled but not isosceles

(C) right isosceles

(D) equilateral.

$$z_1 = \sqrt{2|z|^2}\; e^{i\left(\frac{\pi}{4}+\theta\right)}$$

$$z_1 = \sqrt{2}\,|z|\; e^{i\,\pi/4+\theta}$$



Let $|OP| = 1 \Rightarrow |OQ| = |z_1| = \sqrt{2}$, $\angle POQ = 45^\circ$

$$\cos 45^\circ = \frac{OP^2 + OQ^2 - PQ^2}{2\,OP \cdot OQ} = \frac{1}{\sqrt{2}} \Rightarrow PQ = 1$$

$\therefore\ OP^2 + PQ^2 = OQ^2 \rightarrow$ triangle is isosceles Right angled triangle.

3. If $z=\frac{\pi}{4}(1+i)^4\left(\frac{1-\sqrt{\pi}\,i}{\sqrt{\pi}+i}+\frac{\sqrt{\pi}-i}{1+\sqrt{\pi}\,i}\right)$, then $\left(\frac{|z|}{\text{amp}\, z}\right)$ equals

(A) 1 (B) π (C) 3π (D) 4

$$z = \frac{\pi}{4}(1+i)^4\left(\frac{1-\sqrt{\pi}\,i}{\sqrt{\pi}+i} + \frac{\sqrt{\pi}-i}{1+\sqrt{\pi}\,i}\right)$$

$$z = \frac{\pi}{4}(1+i)^4\left[\frac{(1+\pi)+(\pi+1)}{\sqrt{\pi}+\pi i+i-\sqrt{\pi}}\right]$$

$$z = \frac{\pi}{4}(1+i)^4\, 2\left[\frac{\pi+1}{(\pi+1)i}\right]$$

$$z = -\frac{\pi}{2}(1+i)^4\, i$$

$$|z| = \frac{\pi}{2}(\sqrt{2})^4 = 2\pi$$

$$\text{amp}\, z = -\frac{\pi}{2} + 4\left(\frac{\pi}{4}\right) = \frac{\pi}{2}$$

$$\frac{|z|}{\text{amp}\, z} = \underline{\underline{4}}$$

4. z is a complex number such that $z + \frac{1}{z} = 2\cos 3°$, then the value of $z^{2000} + \frac{1}{z^{2000}} + 1$ is equal to

(A) 0 (B) -1 (C) $\sqrt{3}+1$ (D) $1-\sqrt{3}$

$$z + \frac{1}{z} = 2\cos 3°,$$

$$z^2 - 2z\cos 3° + 1 = 0$$

$$z = \frac{2\cos 3° \pm \sqrt{4\cos^2 3° - 4}}{2}$$

$$z = \cos 3° \pm i\sin 3°$$

Let $z = \cos 3° \pm i\sin 3°$

$$\therefore \; z^{2000} + \frac{1}{z^{2000}} + 1 = (\cos 3° + i\sin 3°)^{2000} + (\cos 3° - i\sin 3°)^{2000} + 1$$

$$= \cos 6000° + \cos 6000° + 1$$

$$= 2\cos(6000°) + 1$$

$$= 2\cos 240° + 1$$

$$= 0$$

If $z^4 + 1 = \sqrt{3}\,i$

(A) z^3 is purely real

(B) z represents the vertices of a square of side $2^{1/4}$

(C) z^9 is purely imaginary

(D) z represents the vertices of a square of side $2^{3/4}$.

$$z^4 = -1 + \sqrt{3}\,i$$

$$z^4 = 2\left(-\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)$$

$$z^4 = 2\left(\cos(\pi - \pi/3) + i \sin(\pi - \pi/3)\right)$$

$$z = 2^{1/4}\left(\cos\left(\frac{\frac{2\pi}{3} + 2k\pi}{4}\right) + i \sin\left(\frac{2k\pi + 2\pi/3}{4}\right)\right)$$

$$\boxed{z = 2^{1/4} \cdot e^{i(3k+1)\frac{\pi}{6}}} \qquad k = 0, 1, 2, 3$$

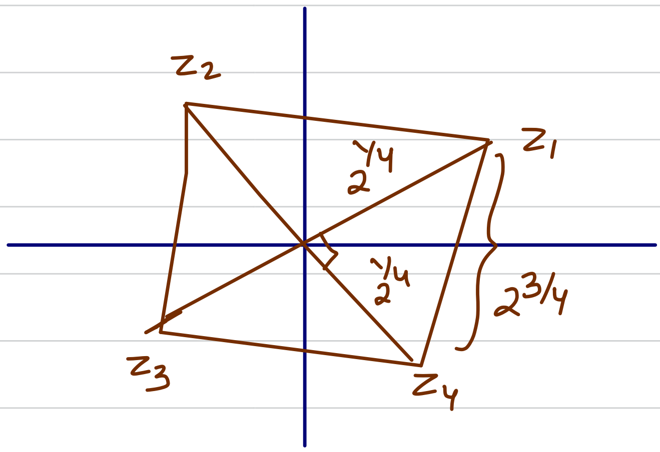
$\therefore$

$$z_1 = 2^{1/4} \cdot e^{i\pi/6}$$

$$z_2 = 2^{1/4}\, e^{i\,2\pi/3}$$

$$z_3 = 2^{1/4}\, e^{i\,7\pi/6}$$

$$z_4 = 2^{1/4}\, e^{i\frac{5\pi}{3}}$$



Square of side $2^{3/4}$.

Let z is a complex number satisfying the equation $Z^6 + Z^3 + 1 = 0$. If this equation has a root $re^{i\theta}$ with $90° < \theta < 180°$ then the value of 'θ' is

(A) 100° (B) 110° (C) 160° (D) 170°

$z^6 + z^3 + 1 = 0$

$z^3 = \omega$ or $z^3 = \omega^2$

$z^3 = e^{i\frac{2\pi}{3}}$

$z = e^{i\left(2k\pi + \frac{2\pi}{3}\right)\frac{1}{3}}$, $k = 0, 1, 2$

$z_1 = e^{i\,2\pi/9}$

$z_2 = e^{i\,8\pi/9} \longrightarrow \theta = \frac{8\pi}{9} = \underline{\underline{160°}}$ Ans

$z_3 = e^{i\,14\pi/9}$

If A and B be two complex numbers satisfying $\frac{A}{B}+\frac{B}{A}=1$. Then the two points represented by A and B and the origin form the vertices of

(A) an equilateral triangle
(B) an isosceles triangle which is not equilateral
(C) an isosceles triangle which is not right angled
(D) a right angled triangle

Let $A \equiv (z_1)$, $B \equiv (z_2)$

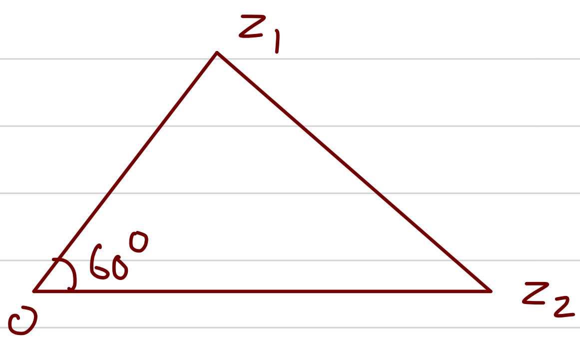
$\therefore \quad \frac{z_1}{z_2}+\frac{z_2}{z_1}=1$

$z_1^2+z_2^2-z_1 z_2=0$

$\left(\frac{z_1}{z_2}\right)^2-\left(\frac{z_1}{z_2}\right)+1=0$

$t^2-t+1=0$

$\Rightarrow \quad t=-\omega \quad \Rightarrow \quad \frac{z_1}{z_2}=-\omega \quad \Rightarrow \quad \frac{z_1}{z_2}=\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$



$\therefore$ Triangle is equilateral

If $1, \alpha_1, \alpha_2 \ldots\ldots, \alpha_{2008}$ are $(2009)^{th}$ roots of unity, then the value of $\sum_{r=1}^{2008} r(\alpha_r + \alpha_{2009-r})$ equals

(A) 2009 (B) 2008 (C) 0 (D) – 2009

$$z^{2009} = 1$$

$$z = \cos\frac{2k\pi}{2009} + i\sin\frac{2k\pi}{2009}, \quad k = 0, 1, 2, ---- 2008$$

$$\therefore \quad \alpha_1 = e^{i2\pi/2009}$$
$$\alpha_2 = e^{i4\pi/2009}$$
$$\alpha_3 = e^{i6\pi/2009}$$
$$\vdots$$
$$\alpha_{2008} = e^{i4016\pi/2009}$$

$$\sum_{r=1}^{2008} r(\alpha_r + \alpha_{2009-r})$$

$$\Rightarrow \sum_{r=1}^{2008} r\left(e^{i\frac{2r\pi}{2009}} + e^{i\frac{2\pi}{2009}(2009-r)}\right)$$

$$\sum_{r=1}^{2008} r\left(e^{i2r\pi/2009} + e^{-i2r\pi/2009}\right)$$

$$\sum_{r=1}^{2008} r\left(2\cos\frac{2\pi r}{2009}\right) \qquad \frac{2\pi}{2009} = \theta$$

$$\Rightarrow 2[\cos\theta + 2\cos 2\theta + 3\cos 3\theta + -- + 2008\cos 2008\theta)$$

here $\cos(2008\theta) = \cos\theta$

$\cos(2007\theta) = \cos 2\theta$ $\left\{ \text{as } \theta = \frac{2\pi}{2009} \right\}$

$\cos(2006\theta) = \cos 3\theta$

$\vdots \qquad \vdots$

$$\Rightarrow 4018\left[\cos\theta + \cos 2\theta + \cos 3\theta + \cdots + \cos(1004\theta)\right]$$

$$4018 \cdot \left(\frac{\sin 1004\,\theta/2}{\sin\theta/2}\right) \cdot \left(\cos\left(\frac{\theta + 1004\theta}{2}\right)\right)$$

$$4018 \cdot \left(\frac{\sin \frac{1004\pi}{2009}}{\sin \pi/2009}\right) \cdot \cos\left(\frac{1005\pi}{2009}\right)$$

$\left\{ \cos\frac{1005\pi}{2009} = -\cos\frac{1004\pi}{2009} \right\}$

$$-2009 \left(\frac{\sin \frac{2008\pi}{2009}}{\sin \pi/2009}\right)$$

$$-2009\,(1)$$

$$= -2009$$

If $x = \frac{1+\sqrt{3}i}{2}$ then the value of the expression, $y = x^4 - x^2 + 6x - 4$, equals

(A) $-1 + 2\sqrt{3}i$ (B) $2 - 2\sqrt{3}i$ (C) $2 + 2\sqrt{3}i$ (D) none

$x = \frac{1}{2} + \frac{\sqrt{3}}{2} i \Rightarrow x = \cos 60^\circ + i \sin 60^\circ .$

$\therefore x^4 - x^2 + 6x - 4 = (\cos 240^\circ + i \sin 240^\circ) - (\cos 120^\circ + i \sin 120^\circ)$
$+ 6(\cos 60^\circ + i \sin 60^\circ) - 4$

$= -1 + 2\sqrt{3} i$

$$(1+\omega)' = A + B\omega$$

$$\left({}^7C_0\,\omega^7 + {}^7C_1\,\omega^6 + {}^7C_2\,\omega^5 + {}^7C_3\,\omega^4 + {}^7C_4\,\omega^3 + {}^7C_5\,\omega^2 + {}^7C_6\,\omega + {}^7C_7\cdot 1\right) = A + B\omega$$

$$\omega + 7 + 21\omega^2 + 35\omega + 35 + 21\omega^2 + 7\omega + 1 = A + B\omega$$

$$42\omega^2 + 43\omega + 43 = A + B\omega$$

$$42(-1-\omega) + 43\omega + 43 = A + B\omega$$

$$\omega + 1 = A + B\omega$$

$$\underline{\underline{A = B = 1}}$$

(b) If ($w \neq 1$) is a cube root of unity then $\begin{vmatrix} 1 & 1+i+w^2 & w^2 \\ 1-i & -1 & w^2-1 \\ -i & -i+w-1 & -1 \end{vmatrix} =$

(A) 0 (B) 1 (C) i (D) w

$$1\left(1-(\omega^2-1)(\omega-1-i)\right) - (1+i+\omega^2)\left((-1+i)+i(\omega^2-1)\right) + \omega^2\left((1-i)(-i+\omega-1) - i\right)$$

$$\left[\cancel{1} - \cancel{\omega^3} + \omega^2 + \omega^2 i + \omega - 1 - i\right] - (1+\omega^2+i)\left[-1+i\omega^2\right] + \omega^2\left[-i+\omega-1-1-i\omega+\cancel{1}-\cancel{1}\right]$$

$$(-2+i(\omega^2-1)) - (-1-\omega^2-i+i\omega^2+i\omega-\omega^2) + \omega^2(\omega-2-i(\omega+1))$$

$$-2+i(\omega^2-1) - (-1-2\omega^2) - i(-2) + (1-2\omega^2) - i(1+\omega^2)$$

$$(-2+1+2\omega^2+1-2\omega^2) + i(\omega^2-1+2-1-\omega^2) = 0$$

$\therefore \quad (1+2i)z - (2i-1)z = 10i$

$z(1+2i-2i+1) = 10i$

$z = 5i$

intercept on $\operatorname{Im} z$ axis $= 5$

If z is a complex number which simultaneously satisfies the equations $3|z-12| = 5|z-8i|$ and $|z-4| = |z-8|$ then the Im(z) can be

(A) 15 (B) 16 (C) 17 (D) 8

$3|z-12| = 5|z-8i|$

$9((x-12)^2+y^2) = 25(x^2+(y-8)^2)$

$9x^2-216x+1296+9y^2 = 25x^2+25y^2+1600-400y$

$16x^2+16y^2+216x-400y+304 = 0$

$x^2+y^2+\frac{27}{2}x-25y+19=0$ —— (1)

$|z-4| = |z-8|$

$(x-4)^2+y^2 = (x-8)^2+y^2 \Rightarrow x^2-8x+16 = x^2+64-16x$

$8x = 48$

$\boxed{x=6}$ —— (2)

from (1) and (2):

$x=6$, $y=17$ or 8

Ans

Let z_1, z_2, z_3 be non-zero complex numbers satisfying the equation $z^4 = iz$. Which of the following statement(s) is/are correct?

(A) The complex number having least positive argument is $\left(\frac{\sqrt{3}}{2}, \frac{1}{2}\right)$.

(B) $\sum_{k=1}^{3} \text{Amp}(z_k) = \frac{\pi}{2}$

(C) Centroid of the triangle formed by z_1, z_2 and z_3 is $\left(\frac{1}{\sqrt{3}}, \frac{-1}{3}\right)$

(D) Area of triangle formed by z_1, z_2 and z_3 is $\frac{3\sqrt{3}}{2}$

$z^4 = iz$

For Non-zero 'z', $z^3 = i$

$z^3 = \cos\left(2k\pi + \frac{\pi}{2}\right) + i \sin\left(2k\pi + \frac{\pi}{2}\right)$

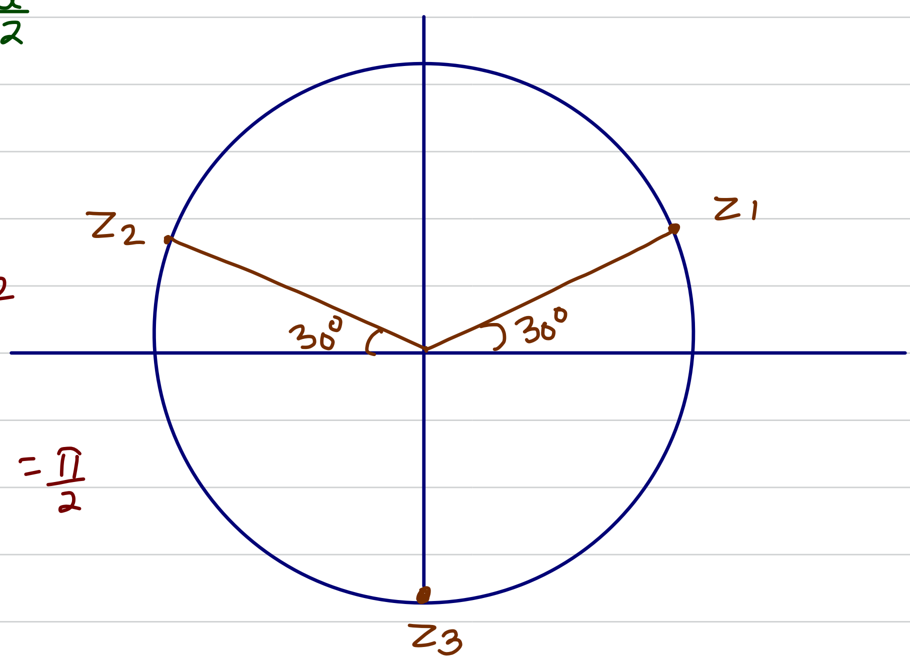
$z = \cos\left(\frac{2k\pi + \pi/2}{3}\right) + i \sin\left(\frac{2k\pi + \pi/2}{3}\right)$

$k = 0, 1, 2$

$z_1 = \frac{\sqrt{3}}{2} + \frac{i}{2}$

$z_2 = -\frac{\sqrt{3}}{2} + \frac{i}{2}$

$z_3 = -i$



least argument $= \frac{\sqrt{3}}{2} + i/2$

$\sum_{k=1}^{3} \text{amp}\, z = \frac{\pi}{6} + \pi - \frac{\pi}{6} - \frac{\pi}{2} = \frac{\pi}{2}$

(A, B)

If $\alpha_1, \alpha_2, \alpha_3, \ldots\ldots., \alpha_{n-1}$ are the imaginary n^{th} roots of unity then the product $\prod_{r=1}^{n-1}(i-\alpha_r)$ (where $i=\sqrt{-1}$) can take the value equal to

(A) 0 (B) 1 (C) i (D) $(1+i)$

$$z^n = 1$$

$$z = \cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n}, \quad k = 0, 1, 2, ---(n-1)$$

$$\alpha_1 = e^{i\frac{2\pi}{n}}, \quad \alpha_2 = e^{i\frac{4\pi}{n}}, \quad \alpha_3 = e^{i\frac{6\pi}{n}}, ----, \quad \alpha_{n-1} = e^{i(n-1)\frac{2\pi}{n}}$$

$$z^n - 1 = (z-1)(z-\alpha_1)(z-\alpha_2) --- (z-\alpha_{n-1})$$

$$1+z+z^2+\cdots+z^{n-1} = (z-\alpha_1)(z-\alpha_2) -- (z-\alpha_{n-1})$$

put $z = i$

$$\frac{i^n - 1}{i-1} = (i-\alpha_1)(i-\alpha_2) --- (i-\alpha_{n-1})$$

If

$n = 4k \Rightarrow 0$

$n = 4k+1 \Rightarrow 1$

$n = 4k+2 \Rightarrow i+1$

$n = 4k+3 \Rightarrow \dfrac{-i-1}{i-1} = i$

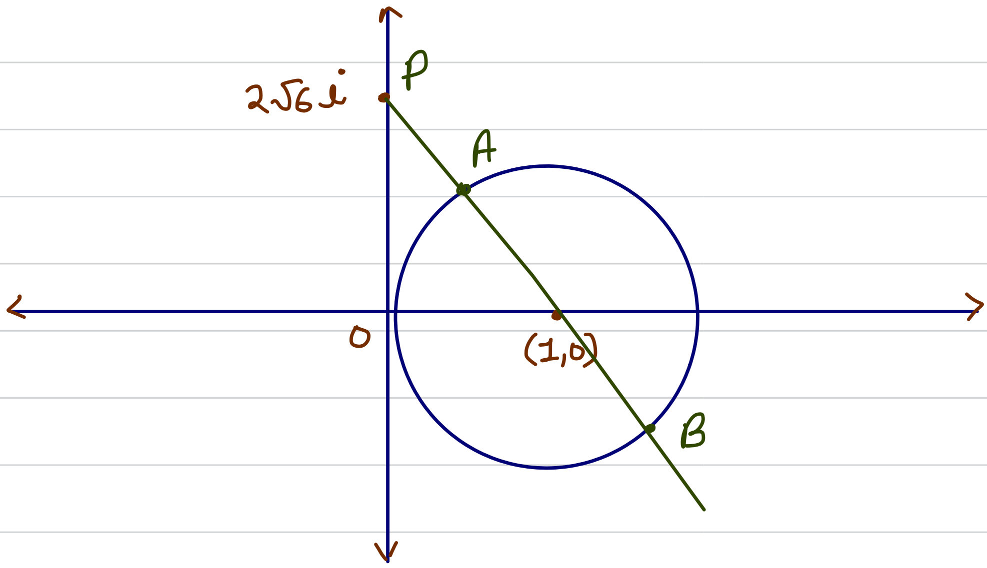
ˈet point z moves on $|z - 1| = 1$ such that minimum & maximum value of $|z - 2\sqrt{6}i|$ are m & M ɛspectively, then-

ʌ) $m + M = 10$

(B) $m^2 + M^2 = 52$

(C) $m + M = 8$

(D) $m^2 + M^2 = 16$



max. value of $|z - 2\sqrt{6}i| = BP = 6$

nin value " " $= AP = 4$

$A = \{z : |z+1| \le 2 + \text{Re}(z)\}$, $B = \{z : |z-1| \ge 1\}$ and $C = \left\{z : \frac{|z-1|}{|z+1|} \ge 1\right\}$

The number of point(s) having integral coordinates in the region $A \cap B \cap C$ is

(A) 4 (B) 5 (C) 6 (D) 10

The area of region bounded by $A \cap B \cap C$ is

(A) $2\sqrt{3}$ (B) $\sqrt{3}$ (C) $4\sqrt{3}$ (D) 2

The real part of the complex number in the region $A \cap B \cap C$ and having maximum amplitude

(A) –1 (B) $\frac{-3}{2}$ (C) $\frac{1}{2}$ (D) – 2

. For A, $|z+1| \le 2 + \text{Re}(z)$ **[12th, 20-12-2009, complex]**

$\Rightarrow \quad (x+1)^2 + y^2 \le 4 + 4x + x^2$

$\Rightarrow \quad y^2 \le 3 + 2x$

$\Rightarrow \quad y^2 \le 2\left(x + \frac{3}{2}\right)$ .....(1)

For B, $|z-1| \ge 1$

$\Rightarrow \quad (x-1)^2 + y^2 \ge 1$ ......(2)

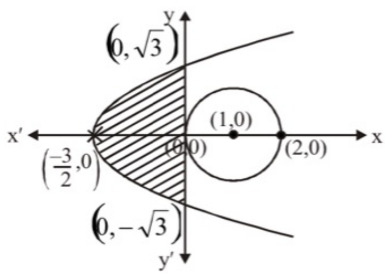
For C, $|z-1|^2 \ge |z+1|^2$

$\Rightarrow \quad (z-1)(\bar{z}-1) \ge (z+1)(\bar{z}+1)$

$\Rightarrow \quad (z\bar{z} - \bar{z} - z + 1) \ge (z\bar{z} + \bar{z} + z + 1)$

$\Rightarrow \quad z + \bar{z} \le 0$

i.e. $\quad x \le 0$ .....(3)



(i) (–1,0), (–1, 1) , (–1,–1), (0,0), (0,1), (0,–1) but z = – 1 is not in the domain in set C

$\therefore$ Total number of point(s) having integral coordinates in the region $A \cap B \cap C$ is 5.

(ii) Required area $= 2\int_{-3/2}^{0} \sqrt{2\left(x + \frac{3}{2}\right)}\, dx = 2\sqrt{3}$ (square units)

(iii) Clearly $z = \frac{-3}{2} + i0$ is the complex number in the region $A \cap B \cap C$ and having maximum amp

$\therefore \quad \text{Re}(z) = \frac{-3}{2}$]

S-1

Exercise (S-1)

(1) Find the modulus, argument and the principal argument of the complex numbers

(i) $6\left(\cos(-310^\circ)+i\sin(-310^\circ)\right)$

$= 6\left(\cos 50^\circ + i\sin 50^\circ\right)$

$= r\left(\cos\theta + i\sin\theta\right)$

modulus $= 6$ and principal argument $= 50^\circ$

argument $= 50^\circ + 2k\pi = \frac{5\pi}{18} + 2k\pi$

(ii) $-2\left(\cos 30^\circ + i\sin 30^\circ\right)$

$= 2\left(\cos(-180^\circ+30^\circ) + i\sin(-180^\circ+30^\circ)\right)$

$= 2\left(\cos(-150^\circ) + i\sin(-150^\circ)\right)$

modulus $= 2$ pri. argument $= -\frac{5\pi}{6}$

argument $= -\frac{5\pi}{6} + 2k\pi$

(iii) $\frac{2+i}{4i+(1+i)^2} = \frac{2+i}{4i+1-1+2i} = \frac{2+i}{6i}$

$= \frac{-1+2i}{-6} = \frac{1}{6}\left[1-2i\right]$

modulus $= \frac{\sqrt{1+4}}{6} = \frac{\sqrt{5}}{6}$

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(2)

polynomial, $P(z) = 2z^4 + az^3 + bz^2 + cz + 3$

If two roots of the equation $P(z) = 0$ are 2 and i, then find the value of 'a'.

Solⁿ:- two roots are 2 & i

then $-i$ must be a root of the equation

Now product of roots

$= (2)(i)(-i)(\alpha) = 3/2$

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(3)

in the form of $(a+ib)$

Sol^n:-

$$z^2 - (3-2i)z + 5 - 5i = 0$$

$$\Rightarrow z = \frac{(3-2i) \pm \sqrt{(3-2i)^2 - 4(5-5i)}}{2}$$

$$\Rightarrow z = \frac{3-2i \pm \sqrt{9-4-12i-20+20i}}{2}$$

$$\Rightarrow z = \frac{3-2i \pm \sqrt{-15+8i}}{2}$$

$$\Rightarrow z = \frac{3-2i \pm \sqrt{(1+4i)^2}}{2}$$

$$\Rightarrow z = \frac{3-2i \pm (1+4i)}{2}$$

$$\Rightarrow z = \frac{4+2i}{2} \quad \text{OR} \quad \frac{2-6i}{2}$$

$$= 2+i \qquad = 1-3i$$

(8). If $(1-i)$ is a root of the equation

$$z^3 - 2(2-i)z^2 + (4-5i)z - 1 + 3i = 0$$

then find the other 2 roots.

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$$z^2-(3-i)z+2-i$$

$$z-1+i \overline{\smash{\big)}\, z^3-2(2-i)z^2+(4-5i)z-1+3i}$$

$$z^3-(1-i)z^2$$

$$(-3+i)z^2+(4-5i)z-1+3i$$

$$(-3+i)z^2+(2-4i)z$$

$$(2-i)z-1+3i$$

$$(2-i)z-1+3i$$

$$\times$$

$$(z-1+i)\left(z^2-(3-i)z+2-i\right)=0$$

$$\Rightarrow z^2-(3-i)z+2-i=0$$

$$\Rightarrow z^2-\left(1+(2-i)\right)z+2-i=0$$

$$\Rightarrow z^2-z-(2-i)(z-1)=0$$

$$\Rightarrow z(z-1)-(2-i)(z-1)=0$$

$$\Rightarrow (z-1)\left(z-(2-i)\right)=0$$

$$\Rightarrow z=1,\ z=2-i$$

(5) (a) if $iz^3+z^2-z+i=0$

$$\Rightarrow iz^2(z-i)-(z-i)=0$$

$$\Rightarrow z=i \quad \text{OR} \quad iz^2=1$$

$$\Rightarrow |z|=1 \quad \text{OR} \quad |i||z|^2=1$$

$$\Rightarrow |z|=1$$

(b). $\left|\dfrac{z_1 - 2z_2}{2 - z_1\bar{z}_2}\right| = 1$ and $|z_2| \neq 1$, find $|z_1|$

$$\left|\frac{z_1 - 2z_2}{2 - z_1\bar{z}_2}\right| = 1$$

$\Rightarrow$ $|z_1 - 2z_2|^2 = |2 - z_1\bar{z}_2|^2$

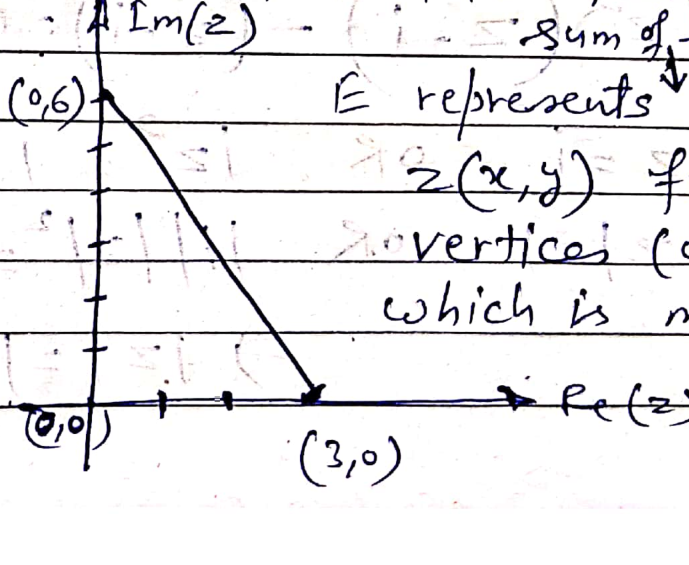
$\Rightarrow$ $(z_1 - 2z_2)(\bar{z}_1 - 2\bar{z}_2) = (2 - z_1\bar{z}_2)(2 - \bar{z}_1 z_2)$

$\Rightarrow$ $|z_1|^2 - \cancel{2z_1\bar{z}_2} - \cancel{2z_2\bar{z}_1} + 4|z_2|^2$
$= 4 - \cancel{2\bar{z}_1 z_2} - \cancel{2z_1\bar{z}_2} + |z_1|^2|z_2|^2$

$\Rightarrow$ $(|z_1|^2 - 4)(|z_2|^2 - 1) = 0$

$\Rightarrow$ $|z_1| = 2$

(c) Find the minimum value of the expression $E = |z|^2 + |z-3|^2 + |z-6i|^2$



sum of squares

E represents distances of $z(x,y)$ from three vertices $(0,0)$, $(3,0)$ & $(0,6)$ which is minimum if (x,y) is centroid of the Δ

$\Rightarrow E_{min.} = |1+2i|^2 + |-2+2i|^2 + |1-4i|^2$

$= 5 + 8 + 17 = 30$

(7). Let z be a complex number such that $z \in C \setminus R$ and $\frac{1+z+z^2}{1-z+z^2} \in R$, then prove that $|z| = 1$

Sol^n:-

(if z is real then $z = \bar{z}$)

$$\Rightarrow \frac{1+z+z^2}{1-z+z^2} = \frac{1+\bar{z}+\bar{z}^2}{1-\bar{z}+\bar{z}^2}$$

$$\Rightarrow 1-\bar{z}+\bar{z}^2+z-|z|^2+|z|^2\bar{z}+z^2$$
$$-|z|^2 z+|z|^4 = 1-z+z^2+\bar{z}-|z|^2$$
$$+|z|^2 z+\bar{z}^2-\bar{z}|z|^2+|z|^4$$

$$\Rightarrow 2\left[z-\bar{z}+|z|^2\bar{z}-|z|^2 z\right]=0$$

$$\Rightarrow (z-\bar{z})(|z|^2-1)=0$$

but $z \neq \bar{z}$ because $z \in C \setminus R$

$$\Rightarrow |z|^2 = 1 \Rightarrow |z| = 1$$

8). Let $z = (0,1) \in \mathbb{C}$, Express $\sum_{k=0}^{n} z^k$ in terms of the positive integer n.

Sol^n:- here $z = i$

$$\Rightarrow \sum_{k=0}^{n} z^k = \sum_{k=0}^{n} i^k = i^0 + i^1 + i^2 + i^3 + \cdots + i^n$$

if $n = 4m$ $\Rightarrow$ then answer is 1

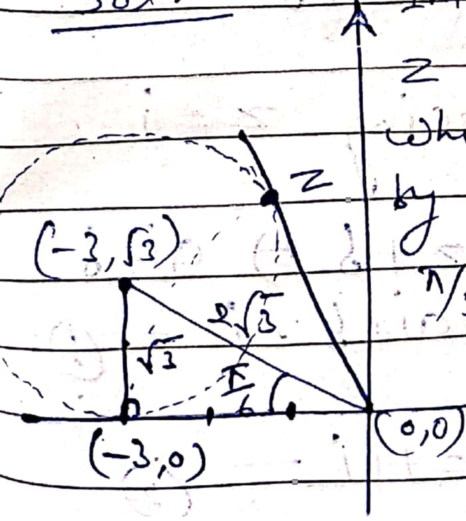
if $n = 4m+1$ $\Rightarrow$ " " " $1+i$

if $n = 4m+2$ $\Rightarrow$ " " " i

if $n = 4m+3$ $\Rightarrow$ " " " 0

(9). Among the complex numbers z satisfying the condition $|z+3-\sqrt{3}i| = \sqrt{3}$, find the number having the least positive argument.

Sol^n:-



In this figure z is the required point which can be obtained by rotating $(-3,0)$ by $\pi/3$ in clockwise direction

$|z-(-3+\sqrt{3}i)| = \sqrt{3}$ is a circle having $(-3,\sqrt{3})$ as centre and $\sqrt{3}$ as radius

so $z = -3 \cdot e^{-i\pi/3}$

$$\Rightarrow z = -3\left[\frac{1}{2} - i\frac{\sqrt{3}}{2}\right]$$

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(10). If A, B and C are the angles of a triangle

$$D = \begin{vmatrix} e^{-2iA} & e^{iC} & e^{iB} \\ e^{iC} & e^{-2iB} & e^{iA} \\ e^{iB} & e^{iA} & e^{-2iC} \end{vmatrix}$$ where $i = \sqrt{-1}$

the find the value of D

Solution:- open the Determinant by Ist row

(put $A+B = \pi - C$

& $e^{2ik\pi} = 1$)

$$e^{-2iA}\left[e^{2iA} - e^{2iA}\right] - e^{iC}\left[e^{-iC} - e^{+i(\pi - C)}\right] + e^{iB}\left[e^{i(\pi - B)} - e^{-iB}\right]$$

$$= -\left[1 - e^{i\pi}\right] + \left[e^{i\pi} - 1\right] = -2 - 2 = -4$$

(11).

let $f(z) = g(z)(z^2+1) + az + b$

Dividing by $z - i$ means replacing $z = i$

the remainder comes to be $az + b \Rightarrow ai + b$

but according to the question $ai + b = i$ —— (1)

similarly for $z + i \Rightarrow -ai + b = 1 + i$ —— (2)

upon dividing by z^2+1, the remainder is

$$\Rightarrow az+b = \frac{i}{2}z + \frac{1}{2}+i$$

Q. Let $|z| = 2$ and $\omega = \frac{z+1}{z-1}$ where z

If M & m respectively be the greatest & least modulus of ω, then find the value

$(2010m + M)$

Sol^n:- Let $z = 2(\cos\theta + i\sin\theta)$

$$\omega = \frac{2\cos\theta + 1 + 2i\sin\theta}{2\cos\theta - 1 + 2i\sin\theta}$$

$$= \sqrt{\frac{(2\cos\theta+1)^2 + 4\sin^2\theta}{(2\cos\theta-1)^2 + 4\sin^2\theta}}$$

$$= \sqrt{\frac{5+4\cos\theta}{5-4\cos\theta}}$$

although, there are many methods to identify the minimum & maximum val of $\frac{5+4\cos\theta}{5-4\cos\theta}$

(i.e. replace $\cos\theta = \frac{1-\tan^2\theta/2}{1+\tan^2\theta/2}$ then put $\tan^2\theta/_2 \geq 0$ OR $\frac{d}{d\theta} = 0$)

but most easy method seen here

put, $\cos\theta = 1$ for maximum value

& $\cos\theta = -1$ for the minimum value

S-2

EXERCISE (S-2)

1. Find the sum of the series $1(2-\omega)(2-\omega^2)+2(3-\omega)(3-\omega^2)\,.......\,(n-1)(n-\omega)(n-\omega^2)$ where ω is one of the imaginary cube root of unity.

Required sum =

$0+1(2-\omega)(2-\omega^2)+2(3-\omega)(3-\omega^2)$
$+\,\ldots\ldots\ldots+(n-1)(n-\omega)(n-\omega^2)$

$$=\sum_{r=1}^{n}(r-1)(r-\omega)(r-\omega^2)$$

$$=\sum_{r=1}^{n}(r-1)(r^2+r+1)$$

$$=\sum_{r=1}^{n}(r^3-1)$$

$$=\sum_{r=1}^{n}r^3-\sum_{r=1}^{n}(1)$$

$$=\left(\frac{n(n+1)}{2}\right)^2-n$$

2 Let $z = x + iy$ be a complex number, where x and y are real numbers. Let A and B be the sets defined by

$A = \{z \mid |z| \le 2\}$ and $B = \{z \mid (1-i)z + (1+i)\bar{z} \ge 4\}$. Find the area of the region $A \cap B$.

$|z| \le 2 \Rightarrow x^2 + y^2 \le 4 \cdots\cdots 1$

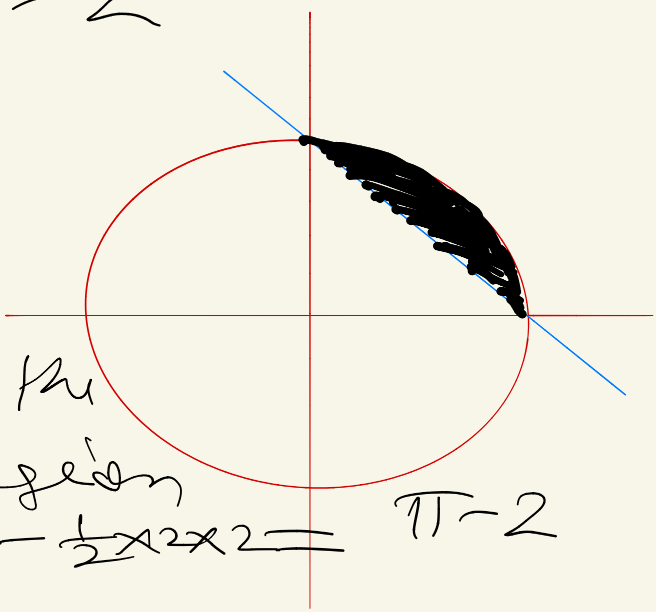
$(1-i)z + (1+i)\bar{z} \ge 4$

$\Rightarrow (1-i)(x+iy) + (1+i)(x-iy) \ge 4$

$\Rightarrow x + \cancel{iy} - \cancel{ix} + y + x - \cancel{iy} + \cancel{ix} + y \ge 4$

$\Rightarrow 2x + 2y \ge 4$

$\Rightarrow x + y \ge 2 \cdots\cdots 2$



Required area
= area of the shaded region
$= \frac{1}{4}\pi \times 4 - \frac{1}{2} \times 2 \times 2 = \pi - 2$

3. Interpret the following locii in $z \in \mathbb{C}$.

(a) $1 < |z - 2i| < 3$

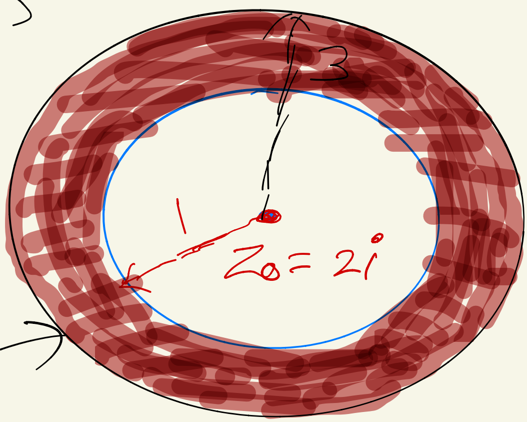
(b) $\mathrm{Re}\left(\dfrac{z+2i}{iz+2}\right) \leq 4 \quad (z \neq 2i)$

3(a). $|Z - Z_0| < R$ represents interior of a circle with Centre at Z_0 and radius R

$R < |Z - Z_0|$ represents exterior of a circle with centre at Z_0 and radius R

$1 < |Z - 2i| < 3$

represents Shaded region



(3) (b) $\operatorname{Re}\left(\dfrac{z+2i}{iz+2}\right) \le 4 \quad (z \neq 2i)$

put $z = x + iy$

$$\frac{z+2i}{iz+2} = \frac{x+iy+2i}{ix-y+2}$$

$$= \frac{x+(y+2)i}{(2-y)+ix}$$

$$= \frac{\{x+(y+2)i\}\{2-y-ix\}}{(2-y)^2+x^2}$$

$$= \frac{(2-y)x + x(y+2) + i(4-y^2-x^2)}{(2-y)^2+x^2}$$

Now $\operatorname{Re}\left(\dfrac{z+2i}{iz+2}\right) \le 4$

$\Rightarrow (2-y)x + x(y+2) \le 4\{x^2+(2-y)^2\}$

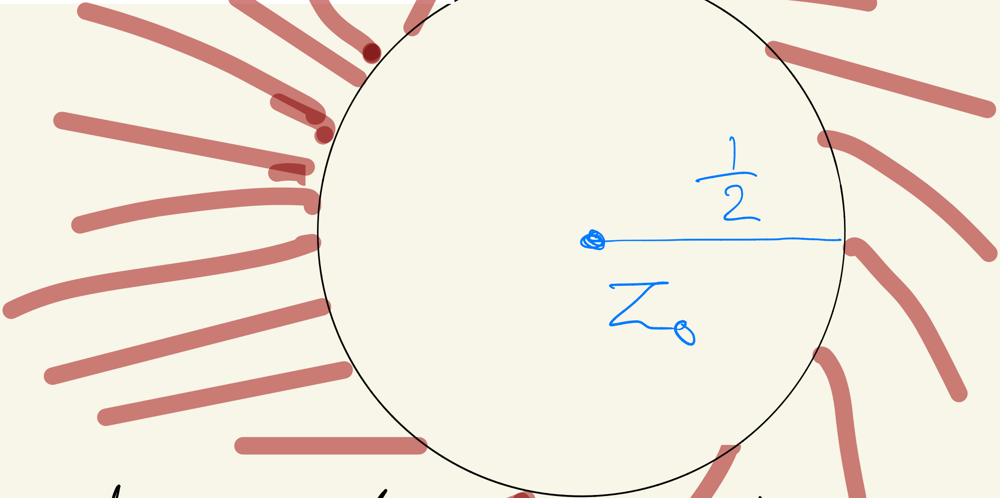
$\Rightarrow 2x - \cancel{xy} + \cancel{xy} + 2x \le 4x^2 + 4(2-y)^2$

$\Rightarrow x^2 - x + (y-2)^2 \ge 0$

$\Rightarrow \left(x-\frac{1}{2}\right)^2 + (y-2)^2 \ge \frac{1}{4}$

$\operatorname{Re}\left(\dfrac{z+2i}{iz+2}\right) \le 4$ represents

region outside or on the circle with centre $\frac{1}{2} + 2i$ and radius $\frac{1}{2}$.



Centre at $z_0 = \frac{1}{2} + 2i$

Radius $= \frac{1}{2}$

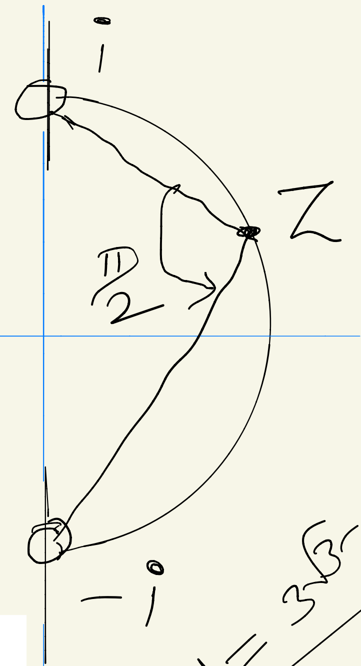
(3) (c) $\operatorname{Arg}(z + i) - \operatorname{Arg}(z - i) = \pi/2$

$\Rightarrow \operatorname{Arg}\left(\frac{z+i}{z-i}\right) = \frac{\pi}{2}$

represents semi-circle

Centre at $(0, 0)$

& Radius $= 1$

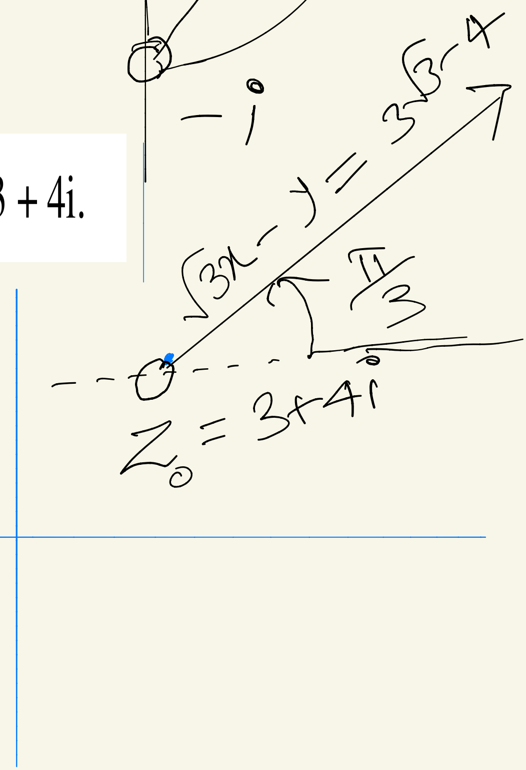


(3) (d) $\operatorname{Arg}(z - a) = \pi/3$ where $a = 3 + 4i$.

$\operatorname{Arg}(z-a) = \pi/3$

represents

a ray.



4. If the equation $(z+1)^7 + z^7 = 0$ has roots $z_1, z_2, \ldots, z_7$, find the value of

(a) $\sum_{r=1}^{7} \text{Re}(z_r)$ and (b) $\sum_{r=1}^{7} \text{Im}(z_r)$

$(Z+1)^7 = -Z^7$

$\Rightarrow |Z+1|^7 = |-Z^7|$

$\Rightarrow |Z-(-1)| = |Z-0|$

$\Rightarrow$ Z lies on $\perp^r$ bisector of segment joining $Z=0$ & $Z=-1$

$\Rightarrow$ Z lies on $x = \frac{-1}{2}$

$\Rightarrow \text{Re}(Z) = -\frac{1}{2}$

$\Rightarrow \sum_{r=1}^{7} \text{Re}(Z_r) = \frac{-1}{2} \times 7 = \frac{-7}{2}$

$Z = -\frac{1}{2} + \beta i$ is a root of equation $(Z+1)^7 + Z^7 = 0$

$\Rightarrow \left(-\frac{1}{2} + \beta i + 1\right)^7 + \left(-\frac{1}{2} + \beta i\right)^7 = 0$

$\Rightarrow \left(\frac{1}{2} + i\beta\right)^7 + \left(-\frac{1}{2} + \beta i\right)^7 = 0$

Take conjugate both the sides

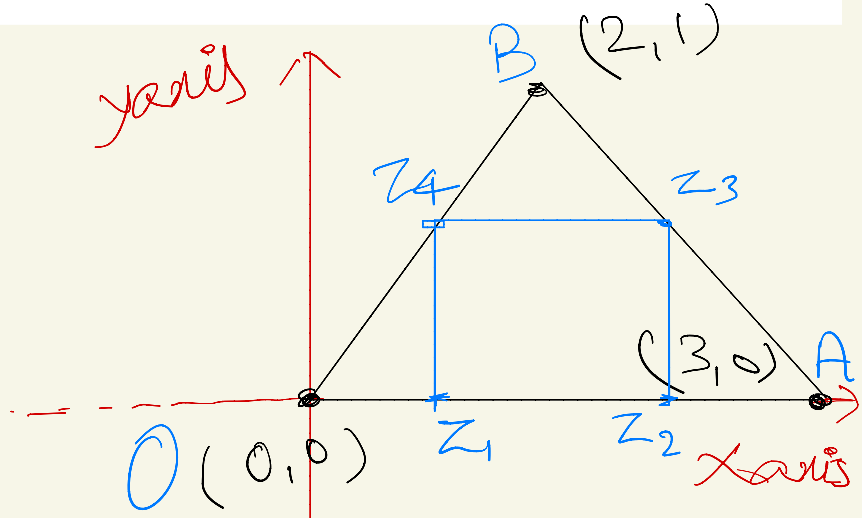
$\Rightarrow \left(\frac{1}{2} - i\beta\right)^7 + \left(-\frac{1}{2} - \beta i\right)^7 = 0$

$\Rightarrow \left(-\frac{1}{2} - i\beta + 1\right)^7 + \left(-\frac{1}{2} - \beta i\right)^7 = 0$

$\Rightarrow Z = -\frac{1}{2} - \beta i$ will also be a root of $(Z+1)^7 + Z^7 = 0$

$$\Rightarrow \sum_{r=1}^{7} \operatorname{Im}(Z_r) = 0$$

5 Let $z_i (i = 1, 2, 3, 4)$ represent the vertices of a square all of which lie on the sides of the triangle with vertices (0, 0), (2, 1) and (3, 0). If z_1 and z_2 are purely real, then area of triangle formed by z_3, z_4 and origin is $\frac{m}{n}$ (where m and n are in their lowest form). Find the value of (m + n).



Let a be the side length of the square

Suppose $Z_1(\alpha, 0)$

$\Rightarrow Z_2(\alpha + a, 0)$, $Z_3(\alpha + a, a)$

$Z_4(\alpha, a)$

equation of AB $\Rightarrow x + y - 3 = 0$

Z_3 lies on AB $\Rightarrow \alpha + a + a = 3$ --- ①

equation of OA $\Rightarrow y = \frac{1}{2}x$

Z_4 lies on OA

$\Rightarrow \quad a = \frac{1}{2}\alpha \cdots$ (2)

From (1) & (2)

$$a = \frac{3}{4}$$

$\Rightarrow$ Area of $\triangle O Z_3 Z_4$

$$= \frac{1}{2} \times \frac{9}{16} = \frac{9}{32} \Rightarrow \begin{array}{l} m = 9 \\ n = 32 \end{array}$$

$\Rightarrow \quad m + n = 41$

6 Let $f(x) = ax^3 + bx^2 + cx + d$ be a cubic polynomial with real coefficients satisfying $f(i) = 0$ and $f(1 + i) = 5$. Find the value of $a^2 + b^2 + c^2 + d^2$. (where $i = \sqrt{-1}$)

$$f(x) = 0 \begin{cases} i \\ -i \\ \alpha \end{cases}$$

$$\Rightarrow \quad f(x) = a(x-i)(x+i)(x-\alpha)$$

$$f(x) = a(x^2+1)(x-\alpha)$$

put $x = 1+i$

$$\Rightarrow \quad 5 = a(1+2i)(1-\alpha+i)$$

$$\Rightarrow \quad 5 = a\{(1-\alpha) + i(2-2\alpha+1) - 2\}$$

$$= a\{-1-\alpha + i(3-2\alpha)\}$$

Equate real & imaginary part

$$\Rightarrow \quad \alpha = \frac{3}{2}, \quad a = \frac{5\times 2}{-5} = -2$$

Now $i - i + \alpha = \dfrac{-b}{a}$

$\Rightarrow \dfrac{3}{2} = \dfrac{-b}{-2} \Rightarrow b = 3$

$(i - i)\alpha + i(-i) = \dfrac{c}{a}$

$\boxed{c = -2}$

$i(-i)\alpha = \dfrac{-d}{a}$

$\Rightarrow \dfrac{3}{2} = \dfrac{d}{5} \Rightarrow d = 3$

Now $a^2 + b^2 + c^2 + d^2$

$= 4 + 9 + 4 + 9 = \underline{\underline{26}}$

7. A particle starts to travel from a point P on the curve $C_1 : |z - 3 - 4i| = 5$, where $|z|$ is maximum. From P, the particle moves through an angle $\tan^{-1}\frac{3}{4}$ in anticlockwise direction on $|z - 3 - 4i| = 5$ and reaches at point Q. From Q, it comes down parallel to imaginary axis by 2 units and reaches at point R. Find the complex number corresponding to point R in the Argand plane.

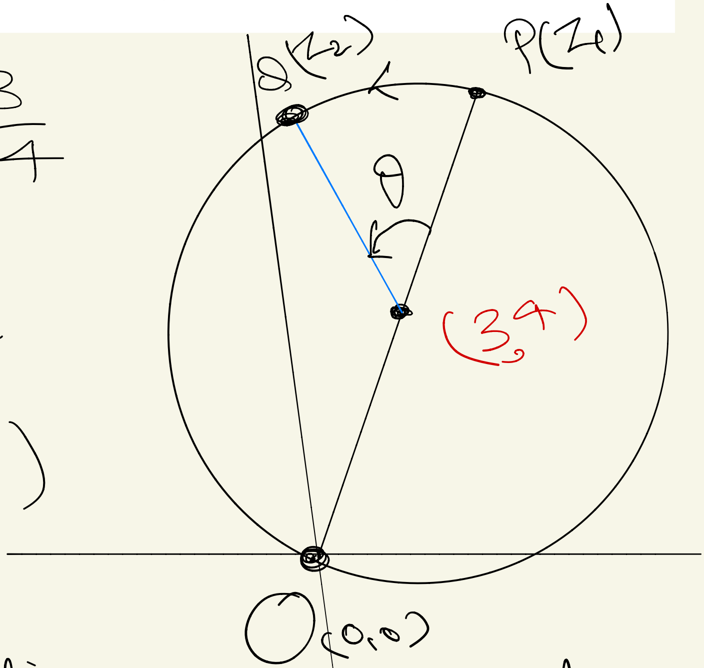
$z_1 = 6 + 8i$

$\tan\theta = \frac{3}{4}$

$\sin\theta = \frac{3}{5}$

$\cos\theta = \frac{4}{5}$

$e^{i\theta} = \left(\frac{4+3i}{5}\right)$



using Rotation concept about centre of the circle

$z_2 - (3+4i) = (6+8i - 3 - 4i)e^{i\theta}$

$z_2 = 3 + 4i + (3+4i)\left(\frac{4+3i}{5}\right)$

$= 3 + 9i$

$\Rightarrow R(z_3), \quad z_3 = \underline{\underline{3 + 7i}}$

8 If the biquadratic $x^4 + ax^3 + bx^2 + cx + d = 0$ $(a, b, c, d \in R)$ has 4 non real roots, two with sum $3 + 4i$ and the other two with product $13 + i$. Find the value of 'b'.

let z_1, z_2, z_3 & z_4 be the roots of

$$x^4 + ax^3 + bx^2 + cx + d = 0$$

$$b = (z_1+z_2)(z_3+z_4) + z_1z_2 + z_3z_4$$

<u>Now</u> $z_1+z_2 = 3+4i$, $z_3z_4 = 13+i$

As co-efficients are real, non-real roots must be in conjugate pair

$$z_3 = \bar{z}_1, \quad z_4 = \bar{z}_2$$

$$13+i = z_3z_4 \Rightarrow 13-i = \bar{z}_3\bar{z}_4$$

$$\Rightarrow z_1z_2 = 13-i$$

$$3-4i = \bar{z}_1+\bar{z}_2 = z_3+z_4$$

So, $b = (3+4i)(3-4i) + 13-i + 13+i$

$$= 25+26 = 51$$

JM

1. If $\left|Z-\frac{4}{Z}\right|=2$, then the maximum value of $|Z|$ is equal to :- **[AIEEE-2009]**

(1) 2 (2) $2+\sqrt{2}$ (3) $\sqrt{3}+1$ (4) $\sqrt{5}+1$

Sol.

(2)

$$|Z|=\left|\left(Z-\frac{4}{Z}\right)+\frac{4}{Z}\right| \Rightarrow |Z|=\left|Z-\frac{4}{Z}+\frac{4}{Z}\right|$$

$$\Rightarrow |Z| \le \left|Z-\frac{4}{Z}\right|+\frac{4}{|Z|} \Rightarrow |Z| \le 2+\frac{4}{|Z|}$$

$$\Rightarrow |Z|^2-2|Z|-4 \le 0$$

$$\left(|Z|-\left(\sqrt{5}+1\right)\right)\left(|Z|-\left(1-\sqrt{5}\right)\right) \le 0 \Rightarrow 1-\sqrt{5} \le |Z| \le \sqrt{5}+1$$

2 The number of complex numbers z such that $|z - 1| = |z + 1| = |z - i|$ equals :- **[AIEEE-2010**

(1) 0 (2) 1 (3) 2 (4) ∞

Sol. Solve $|z-1| = |z+1| \rightarrow 1$

$\Rightarrow x = 0$

$|z+1| = |z-i| \Rightarrow \rightarrow 2$

$y = -x$

$|z-1| = |z-i| \Rightarrow \rightarrow 3$

$y = x$

by 1, 2 & 3

$z = (0,0)$

(circumcentre)

3 If α and β are the roots of the equation $x^2 - x + 1 = 0$, then $\alpha^{2009} + \beta^{2009} =$ **[AIEEE-2010]**

(1) –2 (2) –1 (3) 1 (4) 2

$$x^2 - x + 1 = 0 \begin{cases} -\omega \\ -\omega^2 \end{cases}$$

$$(-\omega)^{2009} + (-\omega^2)^{2009}$$

$$(-1)^{2009}\left[\omega^2 + \omega\right] = 1$$

4. Let α, β be real and z be a complex number. If $z^2 + \alpha z + \beta = 0$ has two distinct roots on the line Re $z = 1$, then it is necessary that :- **[AIEEE-2011]**

(1) $|\beta| = 1$ (2) $\beta \in (1, \infty)$ (3) $\beta \in (0,1)$ (4) $\beta \in (-1, 0)$

Sol: Suppose roots are $1+pi,\ 1+qi$

Sum of roots $1+pi+1+qi = -\alpha$ which is real

$\Rightarrow$ roots of $1+pi,\ 1-pi$

Product of roots $= \beta = 1+p^2 \in (1, \infty)$

$p \neq 0$ since roots are distinct.

✓

5 If $\omega(\neq 1)$ is a cube root of unity, and $(1+\omega)^7 = A + B\omega$. Then (A, B) equals :- **[AIEEE-2011]**

(1) (1, 0) (2) (–1, 1) (3) (0, 1) (4) (1, 1)

Sol.

$$(1+w)^7 = A + Bw$$

$$= (-w^2)^7 = A + Bw$$

$$= -w^2 = A + Bw$$

$$= 1 + w = A + Bw$$

$$\Rightarrow A = 1, \; B = 1$$

EXERCISE (JM)

6 If $z \neq 1$ and $\frac{z^2}{z-1}$ is real, then the point represented by the complex number z lies : **[AIEEE-2012]**

(1) on the imaginary axis.

(2) either on the real axis or on a circle passing through the origin.

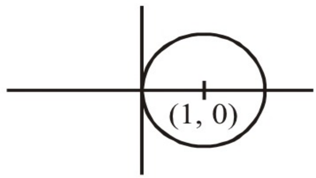
(3) on a circle with centre at the origin.

(4) either on the real axis or on a circle not passing through the origin.

Ans. (2)

Sol. $\frac{z^2}{z-1}$ is purely real where $(Z \neq 1)$

so $\frac{\bar{z}^2}{\bar{z}-1} = \frac{z^2}{z-1}$



$$z\bar{z}^2 - \bar{z}^2 = \bar{z}z^2 - z^2$$

$$z\bar{z}(z-\bar{z}) = z^2 - \bar{z}^2$$

$$z\bar{z}(z-\bar{z}) = (z+\bar{z})(z-\bar{z})$$

$$\Rightarrow \bar{z} - z = 0 \text{ or } z + \bar{z} = z\bar{z}$$

$$\Rightarrow \bar{z} = z \quad \text{or} \quad x^2 + y^2 - 2x = 0$$

$$(x-1)^2 + y^2 = 1$$

so either lie on z real axis or on a circle passing through the origin.

7 If z is a complex number of unit modulus and argument θ, then $\arg\left(\frac{1+z}{1+\bar{z}}\right)$ equals

[JEE (Main)-2013]

(1) $-\theta$ (2) $\frac{\pi}{2}-\theta$ (3) θ (4) $\pi-\theta$∴

Solⁿ

$$|z|=1 \Rightarrow |z|^2=1 \Rightarrow \boxed{z\bar{z}=1}$$

Now $$\frac{1+z}{1+\bar{z}} = \frac{z\bar{z}+z}{1+\bar{z}} = \frac{z(1+\bar{z})}{1+\bar{z}}$$

$$= z$$

$$\Rightarrow \arg\left(\frac{1+z}{1+\bar{z}}\right) = \arg(z) = \theta$$

option (3)

Ans.

8 If z is a complex number such that $|z| \geq 2$, then the minimum value of $\left|z+\frac{1}{2}\right|$:

[JEE(Main)-2014]

(1) is equal to $\frac{5}{2}$

(2) lies in the interval (1, 2)

(3) is strictly greater than $\frac{5}{2}$

(4) is strictly greater than $\frac{3}{2}$ but less than $\frac{5}{2}$

Sol. $\left|z+\frac{1}{2}\right| \geq \left||z|-\frac{1}{2}\right|$

Min. value of $\left|z+\frac{1}{2}\right|$ occurs at $|z| = 2$

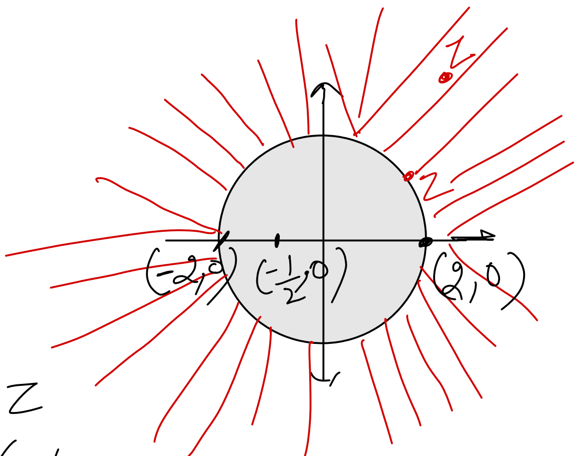
$\because\ |z| \geq 2$

$\therefore\ \left|z+\frac{1}{2}\right|_{Min} = \left|2-\frac{1}{2}\right| = \frac{3}{2}$

option (2) Ans.

Alter:- $|z| \geq 2$

z lies on on the circle or outside the circle.



$\left|z+\frac{1}{2}\right|$ = distance of z from point $\left(-\frac{1}{2}, 0\right)$

$\Rightarrow \left|z+\frac{1}{2}\right|_{min} = 3/2$

9 A complex number z is said to be unimodular if $|z| = 1$. Suppose z_1 and z_2 are complex numbers such that $\dfrac{z_1 - 2z_2}{2 - z_1\bar{z}_2}$ is unimodular and z_2 is not unimodular. Then the point z_1 lies on a : **[JEE(Main)-2015]**

(1) circle of radius 2

(2) circle of radius $\sqrt{2}$

(3) straight line parallel to x-axis

(4) straight line parallel to y-axis

Ans. (1)

Sol. $\dfrac{|z_1 - 2z_2|}{|2 - z_1\bar{z}_2|} = 1 \Rightarrow |z_1 - 2z_2|^2 = |2 - z_1\bar{z}_2|^2$

$\Rightarrow (z_1 - 2z_2)(\bar{z}_1 - 2\bar{z}_2) = (2 - z_1\bar{z}_2)(2 - \bar{z}_1 z_2)$

$\Rightarrow |z_1|^2 + 4|z_2|^2 - 4 - |z_1|^2\,|z_2|^2 = 0$

$\Rightarrow 4(|z_2|^2 - 1) - |z_1|^2\,(|z_2|^2 - 1) = 0$

$\Rightarrow |z_1|^2 - 4 = 0 \quad \Rightarrow |z_1| = 2$ is a circle of radius 2 and centre at origin.

10 A value of θ for which $\frac{2+3i\sin\theta}{1-2i\sin\theta}$ is purely imaginary, is : **[JEE(Main)-2016]**

(1) $\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (2) $\frac{\pi}{3}$ (3) $\frac{\pi}{6}$ (4) $\sin^{-1}\left(\frac{\sqrt{3}}{4}\right)$

Ans. (1)

Sol. $Z = \dfrac{2+3i\sin\theta}{1-2i\sin\theta}$

$$\Rightarrow Z = \frac{(2+3i\sin\theta)(1+2i\sin\theta)}{1+4\sin^2\theta}$$

$$= \frac{(2-6\sin^2\theta)+7i\sin\theta}{1+4\sin^2\theta}$$

for purely imaginary Z, Re(Z) = 0

$$\Rightarrow 2 - 6\sin^2\theta = 0 \Rightarrow \sin\theta = \pm\frac{1}{\sqrt{3}}$$

$$\Rightarrow \theta = \pm\sin^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

1

Let ω be a complex number such that $2\omega + 1 = z$ where $z = \sqrt{-3}$. If $\begin{vmatrix} 1 & 1 & 1 \\ 1 & -\omega^2 - 1 & \omega^2 \\ 1 & \omega^2 & \omega^7 \end{vmatrix} = 3k$, then k is equal to :- **[JEE(Main)-2017]**

(1) 1 (2) –z (3) z (4) –1

Ans. (2)

Sol. Here ω is complex cube root of unity

$R_1 \to R_1 + R_2 + R_3$

$$= \begin{vmatrix} 3 & 0 & 0 \\ 1 & -\omega^2 - 1 & \omega^2 \\ 1 & \omega^2 & \omega \end{vmatrix}$$

$$= 3(-1 - \omega - \omega) = -3z \quad \Rightarrow \quad k = -z$$

(12) If α, β ∈ C are the distinct roots of the equation $x^2 - x + 1 = 0$, then $\alpha^{101} + \beta^{107}$ is equal to-

[JEE(Main)-2018]

(1) 0 (2) 1 (3) 2 (4) –1

Solⁿ roots of the eqⁿ $x^2 - x + 1 = 0$ are '$-\omega$' and '$-\omega^2$'.

$$\Rightarrow \alpha^{101} + \beta^{107} = (-\omega)^{101} + (-\omega^2)^{107}$$

$$= -\left\{\omega^{101} + \omega^{214}\right\}$$

$$= -\left\{\omega^2 + \omega\right\} = 1 \text{ Ans.}$$

option (2)

13 Let z be a complex number such that |z| + z = 3 + i (where i = $\sqrt{-1}$). Then |z| is equal to :-

[JEE(Main)-2019]

(1) $\frac{5}{4}$ (2) $\frac{\sqrt{41}}{4}$ (3) $\frac{\sqrt{34}}{3}$ (4) $\frac{5}{3}$

Sol^n Let $z = x + iy$

given $|z| + z = 3 + i \Rightarrow \sqrt{x^2+y^2} + x + iy = 3 + i$

$\Rightarrow \boxed{y = 1}$ and $\sqrt{x^2+y^2} + x = 3$

$\Rightarrow \sqrt{x^2+1} + x = 3$

$\Longrightarrow \sqrt{x^2+1} = 3 - x$

$\Rightarrow x^2 + 1 = 9 + x^2 - 6x \Rightarrow 6x = 8$

$\Rightarrow \boxed{x = \frac{4}{3}}$

$|z| = \sqrt{x^2+y^2} = \sqrt{1 + \frac{16}{9}} = \frac{5}{3}$

option (4)

14 If $\frac{z-\alpha}{z+\alpha}(\alpha \in R)$ is a purely imaginary number and $|z| = 2$, then a value of α is : **[JEE(Main)-2019]**

(1) 1 (2) 2 (3) $\sqrt{2}$ (4) $\frac{1}{2}$

Ans. (2)

Sol. $$\frac{z-\alpha}{z+\alpha}+\frac{\bar{z}-\alpha}{\bar{z}+\alpha}=0$$

$$z\bar{z}+z\alpha-\alpha\bar{z}-\alpha^2+z\bar{z}-z\alpha+\bar{z}\alpha-\alpha^2=0$$

$|z|^2 = \alpha^2$, a $= \pm 2$

15) Let Z_1 and Z_2 be two complex numbers satisfying $|Z_1| = 9$ and $|Z_2 - 3 - 4i| = 4$. Then the minimum value of $|Z_1 - Z_2|$ is : **[JEE(Main)-2019]**

(1) 0 (2) 1 (3) $\sqrt{2}$ (4) 2

Ans. (1)

$$|z_1| = 9 \quad , \quad |z_2 - (3+4i)| = 4$$

C_1 (0, 0) radius $r_1 = 9$

C_2(3, 4), radius $r_2 = 4$

$C_1C_2 = |r_1 - r_2| = 5$

$\therefore$ Circle touches internally

$$\therefore \; |z_1 - z_2|_{\min} = 0$$

16. If $z=\frac{\sqrt{3}}{2}+\frac{i}{2}\left(i=\sqrt{-1}\right)$, then $\left(1+iz+z^5+iz^8\right)^9$ is equal to **[JEE(Main)-2019]**

(1) -1 (2) 1 (3) 0 (4) $\left(-1+2i\right)^9$

Sol^n

$$z=\frac{\sqrt{3}}{2}+\frac{i}{2}=-i\underbrace{\left(-\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)}_{\omega}=-i\omega$$

$$\left(1+iz+z^5+iz^8\right)^9$$

$$=\left\{1+i(-i\omega)+(-i\omega)^5+i(-i\omega)^8\right\}^9$$

$$=\left\{1+\omega-i\omega^2+i\omega^2\right\}^9$$

$$=\left\{1+\omega\right\}^9=\left(-\omega^2\right)^9=-\omega^{18}=\underline{\underline{-1}}\text{ Ans.}$$

Option (1)

17 Let $z \in C$ be such that $|z| < 1$. If $\omega = \frac{5+3z}{5(1-z)}$, then :- **[JEE(Main)-2019]**

(1) $5\mathrm{Im}(\omega) < 1$ (2) $4\mathrm{Im}(\omega) > 5$ (3) $5\mathrm{Re}(\omega) > 1$ (4) $5\mathrm{Re}(\omega) > 4$

Solⁿ

$$5\omega - 5\omega z = 5 + 3z$$

$$\Rightarrow 5\omega - 5 = z(5\omega + 3) \Rightarrow z = \frac{5\omega - 5}{5\omega + 3}$$

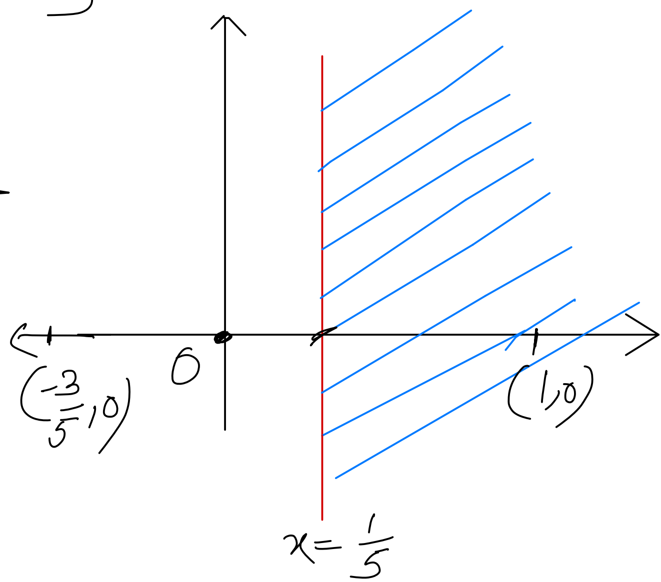
$$\Rightarrow |z| < 1 \Rightarrow \frac{|5\omega - 5|}{|5\omega + 3|} < 1$$

$$\Rightarrow |\omega - 1| < \left|\omega + \frac{3}{5}\right|$$

ω lies on the right side of the line $x = \frac{1}{5} \Rightarrow \mathrm{Re}(\omega) > \frac{1}{5}$.

$$\Rightarrow \boxed{5\mathrm{Re}(\omega) > 1}$$

option (3)



(18) If z and w are two complex numbers such that $|zw| = 1$ and $\arg(z) - \arg(w) = \frac{\pi}{2}$, then :

[JEE(Main)-2019]

(1) $\bar{z}w = i$ (2) $\bar{z}w = -i$ (3) $z\bar{w} = \frac{1-i}{\sqrt{2}}$ (4) $z\bar{w} = \frac{-1+i}{\sqrt{2}}$

Sol^n

$$\arg(z) = \frac{\pi}{2} + \arg(\omega)$$

and $|z|.|\omega| = 1 \Rightarrow |z| = \frac{1}{|\omega|}$

$$\Rightarrow z = \frac{i\omega}{|\omega|^2} \Rightarrow z = \frac{i\omega}{\omega\bar{\omega}}$$

$$\Rightarrow \boxed{z\bar{\omega} = i} \Rightarrow \boxed{\bar{z}\omega = -i}$$

option (2)

19 The equation $|z-i| = |z-1|$, $i = \sqrt{-1}$, represents: **[JEE(Main)-2019]**

(1) the line through the origin with slope -1.

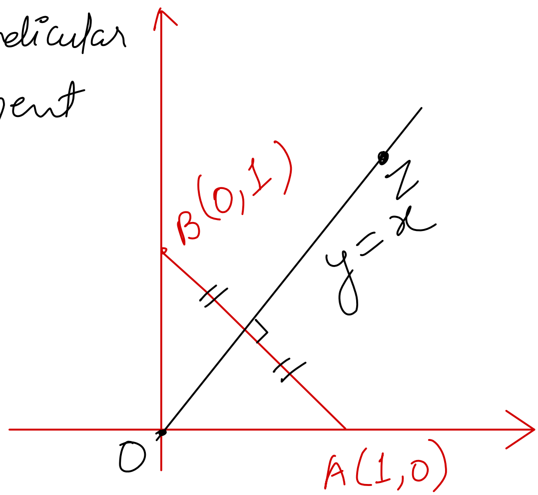
(2) a circle of radius 1.

(3) a circle of radius $\frac{1}{2}$.

(4) the line through the origin with slope 1.

sol^n

z lies on the perpendicular bisector of line segment joining AB.



option (4)

JA

EXERCISE (JA)

1. Let z_1 and z_2 be two distinct complex numbers and let $z = (1-t)z_1 + tz_2$ for some real number t with $0 < t < 1$. If Arg(w) denotes the principal argument of a nonzero complex number w, then

[JEE 2010, 3M]

(A) $|z - z_1| + |z - z_2| = |z_1 - z_2|$

(B) $\text{Arg}(z - z_1) = \text{Arg}(z - z_2)$

(C) $\begin{vmatrix} z - z_1 & \bar{z} - \bar{z}_1 \\ z_2 - z_1 & \bar{z}_2 - \bar{z}_1 \end{vmatrix} = 0$

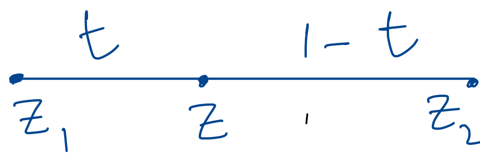
(D) $\text{Arg}(z - z_1) = \text{Arg}(z_2 - z_1)$

Sol^n

$$z = \frac{(1-t)\,z_1 + t\,z_2}{(1-t) + t}$$

[Section formula for internal division as $t \in (0,1)$]

So z_1, z & z_2 (in order) are collinear.



$\Rightarrow \quad |z - z_1| + |z - z_2| = |z_1 - z_2|$

And, $\text{Arg}(z - z_1) = \text{Arg}(z_2 - z) = \text{Arg}(z_2 - z_1)$

Since, they are collinear. So,

$$\begin{vmatrix} z & \bar{z} & 1 \\ z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \end{vmatrix} = 0 \Rightarrow \begin{vmatrix} z - z_1 & \bar{z} - \bar{z}_1 & 0 \\ z_1 & \bar{z}_1 & 1 \\ z_2 - z_1 & \bar{z}_2 - \bar{z}_1 & 0 \end{vmatrix} \begin{bmatrix} R_1 \to R_1 - R_2 \\ R_3 \to R_3 - R_2 \end{bmatrix} = 0$$

$$\Rightarrow \begin{vmatrix} z - z_1 & \bar{z} - \bar{z}_1 \\ z_2 - z_1 & \bar{z}_2 - \bar{z}_1 \end{vmatrix} = 0 \quad \text{[Expanding along } C_3]$$

2. Let ω be the complex number $\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$. Then the number of distinct complex numbers z satisfying $\begin{vmatrix} z+1 & \omega & \omega^2 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0$ is equal to **[JEE 2010, 3M]**

Sol^n

$$\begin{vmatrix} z & z & z \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} [R_1 \to R_1 + R_2 + R_3] = 0$$

$$\Rightarrow z \begin{vmatrix} 1 & 1 & 1 \\ \omega & z+\omega^2 & 1 \\ \omega^2 & 1 & z+\omega \end{vmatrix} = 0$$

$$\Rightarrow z \begin{vmatrix} 1 & 1-\omega & 1-\omega^2 \\ \omega & z & 0 \\ \omega^2 & 0 & z \end{vmatrix} \begin{bmatrix} C_2 \to C_2 - C_1\omega \\ C_3 \to C_3 - C_1\omega^2 \end{bmatrix} = 0$$

$$\Rightarrow z\left[\omega^2(\omega^2-1)\, z + z\,(z - \omega(1-\omega))\right] = 0$$

$$\Rightarrow z^2\left[\omega - \omega^2 + z - \omega + \omega^2\right] = 0$$

$$\Rightarrow z^3 = 0 \Rightarrow \text{only } \boxed{\text{one}} \text{ distinct root}$$

3. Match the statements in **Column-I** with those in **Column-II.**

[**Note :** Here z takes values in the complex plane and Im z and Re z denote, respectively, the imaginary part and the real part of z.]

Column I	**Column II**
(A) The set of points z satisfying $\lvert z - i\lvert z\rvert \rvert = \lvert z + i\lvert z\rvert \rvert$ is contained in or equal to	(p) an ellipse with eccentricity $\frac{4}{5}$
(B) The set of points z satisfying $\lvert z + 4\rvert + \lvert z - 4\rvert = 10$ is contained in or equal to	(q) the set of points z satisfying Im z = 0
(C) If $\lvert w\rvert = 2$, then the set of points $z = w - \frac{1}{w}$ is contained in or equal to	~~(t)~~ r the set of points z satisfying $\lvert \text{Im } z\rvert \le 1$
(D) If $\lvert w\rvert = 1$, then the set of points $z = w + \frac{1}{w}$ is contained in or equal to	(s) the set of points z satisfying $\lvert \text{Re } z\rvert \le 2$
	(t) the set of points z satisfying $\lvert z\rvert \le 3$

[JEE 10, 3+3+8]

Sol^n (A.) Let $z_1 = z$ & $z_2 = i|z|$

so, $|z_1 - z_2| = |z_1 + z_2| \Rightarrow \overrightarrow{Oz_1} \perp \overrightarrow{Oz_2}$

$\Rightarrow z_1$ lies on x-axis as z_2 lies on +ve Y-axis

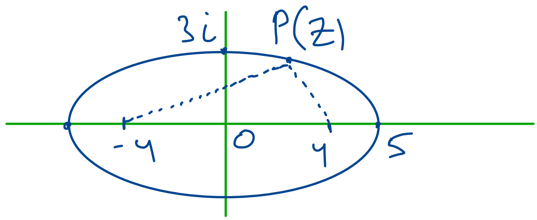
$\Rightarrow z = z_1 \in \mathbb{R} \Rightarrow \boxed{A \rightarrow q, r}$

(B.) $PS + PS' = 10 = 2a$

Now, $ae = 4 \Rightarrow e = \frac{4}{5}$

Also, $b^2 = 25\left(1 - \frac{16}{25}\right) \Rightarrow b = 3$

$\Rightarrow \boxed{B \rightarrow p}$



(C) Let $w = 2e^{i\theta}$

So, $z = w - \frac{1}{w} = 2e^{i\theta} - \frac{1}{2e^{i\theta}}$

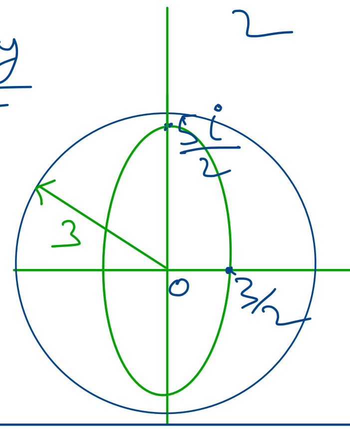
$\Rightarrow z = x + iy = 2e^{i\theta} - \frac{e^{-i\theta}}{2}$

$\Rightarrow x = 2\cos\theta - \frac{\cos\theta}{2}$ & $y = 2\sin\theta + \frac{\sin\theta}{2}$

$\Rightarrow \cos\theta = \frac{2x}{3}$ & $\sin\theta = \frac{2y}{5}$

$\Rightarrow \frac{4x^2}{9} + \frac{4y^2}{25} = 1 \Rightarrow e = \frac{4}{5}$

$\Rightarrow \boxed{C \to p, s, t}$



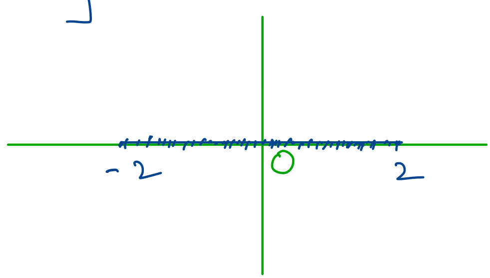
(D) Again let $w = e^{i\theta}$

So, $z = x + iy = e^{i\theta} + e^{-i\theta}$

$\Rightarrow x = 2\cos\theta$ & $y = 0$

$\Rightarrow z = 2\cos\theta \in [-2, 2]$

$\Rightarrow \boxed{D \to q, r, s, t}$



4. Comprehension (3 questions together)

Let a,b and c be three real numbers satisfying

$$\begin{bmatrix} a & b & c \end{bmatrix}\begin{bmatrix} 1 & 9 & 7 \\ 8 & 2 & 7 \\ 7 & 3 & 7 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \qquad \text{...(E)}$$

(i) If the point P(a,b,c), with reference to (E), lies on the plane 2x + y + z = 1, then the value of 7a + b + c is

(A) 0 (B) 12 (C) 7 (D) 6

(ii) Let ω be a solution of $x^3 - 1 = 0$ with $\text{Im}(\omega) > 0$. If a = 2 with b and c satisfying (E), then the value of $\frac{3}{\omega^a}+\frac{1}{\omega^b}+\frac{3}{\omega^c}$ is equal to -

(A) –2 (B) 2 (C) 3 (D) –3

(iii) Let b = 6, with a and c satisfying (E). If α and β are the roots of the quadratic equation $ax^2 + bx + c = 0$, then $\sum_{n=0}^{\infty}\left(\frac{1}{\alpha}+\frac{1}{\beta}\right)^n$ is -

(A) 6 (B) 7 (C) $\frac{6}{7}$ (D) ∞

[JEE 2011, 3+3+3]

Sol^n

$a + 8b + 7c = 0$ ..... (I)

$9a + 2b + 3c = 0$ ..... (II)

$7a + 7b + 7c = 0$ ..... (III)

(I) - (III) :— $-6a + b = 0 \Rightarrow b = 6a$

$\Rightarrow c = -7a$ (from (I))

Now, (II) is always satisfied for such b & c

Hence $(a, b, c) \equiv (K, 6K, -7K);\ K \in \mathbb{R}$

(i) $2K + 6K - 7K = 1 \Rightarrow K = 1$

$\Rightarrow (a, b, c) \equiv (1, 6, -7)$

$\Rightarrow 7a + b + c = 6$

(ii) $a = 2 \Rightarrow K = 2 \Rightarrow b = 12,\ c = -14$

So, $\frac{3}{\omega^a} + \frac{1}{\omega^b} + \frac{3}{\omega^c} = \frac{3}{\omega^2} + \frac{1}{\omega^{12}} + \frac{3}{\omega^{-14}}$

$= 3(\omega + \omega^2) + 1 = 1 - 3 = -2$

(iii) $b = 6 \Rightarrow K = 1 \Rightarrow (a, b, c) = (1, 6, -7)$

So, $x^2 + 6x - 7 = 0$; $x = \alpha, \beta$

Now, $\sum_{n=0}^{\infty} \left(\frac{\alpha+\beta}{\alpha\beta}\right)^n = \sum_{n=0}^{\infty} \left(\frac{-6}{-7}\right)^n$

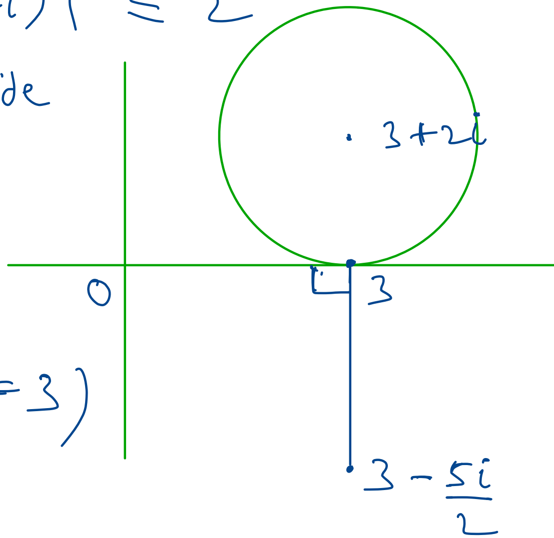
$= \frac{1}{1 - \frac{6}{7}} = 7$

---

5. If z is any complex number satisfying $|z - 3 - 2i| \le 2$, then the minimum value of $|2z - 6 + 5i|$ is **[JEE 2011, 4M]**

Sol^n Given $|z - (3+2i)| \le 2$

So, z lies on or inside the circle shown.



Now, $2\left|z - \left(3 - \frac{5i}{2}\right)\right|$

$\ge 2 \times \frac{5}{2}$ (when $z = 3$)

$= \boxed{5}$

6. Let $\omega = e^{2i\pi/3}$, and a, b, c, x, y, z be non-zero complex numbers such that

$$a + b + c = x$$
$$a + b\omega + c\omega^2 = y$$
$$a + b\omega^2 + c\omega = z.$$

Note the correction (otherwise it is bonus)

Then the value of $\dfrac{|x|^2 + |y|^2 + |z|^2}{|a|^2 + |b|^2 + |c|^2}$ is **[JEE 2011, 4M]**

Sol

$$|x|^2 = x\bar{x} = (a+b+c)(\bar{a}+\bar{b}+\bar{c})$$

$$= |a|^2 + |b|^2 + |c|^2 + 2\,\mathrm{Re}(a\bar{b} + b\bar{c} + c\bar{a}) \quad \text{--- (I)}$$

Also, $|y|^2 = y\bar{y} = (a + b\omega + c\omega^2)(\bar{a} + \bar{b}\omega^2 + \bar{c}\omega)$

$$= |a|^2 + |b|^2 + |c|^2 + 2\,\mathrm{Re}(a\bar{b}\omega^2 + b\bar{c}\omega^2 + c\bar{a}\omega^2) \quad \text{--- (II)}$$

And, $|z|^2 = z\bar{z} = (a + b\omega^2 + c\omega)(\bar{a} + \bar{b}\omega + \bar{c}\omega^2)$

$$= |a|^2 + |b|^2 + |c|^2 + 2\,\mathrm{Re}(a\bar{b}\omega + b\bar{c}\omega + c\bar{a}\omega) \quad \text{--- (III)}$$

(I) + (II) + (III) :—

$$|x|^2 + |y|^2 + |z|^2 = 3(|a|^2 + |b|^2 + |c|^2)$$

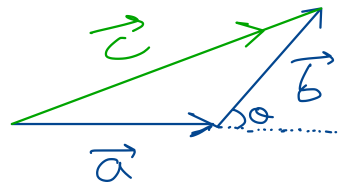
$$\Rightarrow \frac{|x|^2 + |y|^2 + |z|^2}{|a|^2 + |b|^2 + |c|^2} = \boxed{3}$$

7. Match the statements given in **Column I** with the values given in **Column II**

Column I	**Column II**
(A) If $\vec{a} = \hat{j} + \sqrt{3}\hat{k}$, $\vec{b} = -\hat{j} + \sqrt{3}\hat{k}$ and $\vec{c} = 2\sqrt{3}\hat{k}$ form a triangle, then the internal angle of the triangle between $\vec{a}$ and $\vec{b}$ is	(p) $\frac{\pi}{6}$
(B) If $\int_a^b (f(x) - 3x)dx = a^2 - b^2$, then the value of $f\left(\frac{\pi}{6}\right)$ is	(q) $\frac{2\pi}{3}$
(C) The value of $\frac{\pi^2}{\ell n 3}\int_{7/6}^{5/6} \sec(\pi x)dx$ is	(r) $\frac{\pi}{3}$
(D) The maximum value of $\left|\text{Arg}\left(\frac{1}{1-z}\right)\right|$ for	(s) π
$\lvert z\rvert = 1$, $z \neq 1$ is given by	(t) $\frac{\pi}{2}$

[JEE 2011, 2+2+2+2M]

Sol^n (A.) $\vec{a} + \vec{b} = \vec{c}$



Now, $\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|} = \frac{1}{2}$

$\Rightarrow \theta = \frac{\pi}{3} \Rightarrow$ internal angle $= \frac{2\pi}{3} \Rightarrow \boxed{A \to q}$

(B.) $f(x) - 3x = -2x \Rightarrow f(x) = x$

$\Rightarrow f\left(\frac{\pi}{6}\right) = \frac{\pi}{6} \Rightarrow \boxed{B \to p}$ (if $a \neq b$)

But, if $a = b$ then LHS = RHS = 0 $\forall f(x)$

So, $f(x)$ can be any function $\Rightarrow \boxed{B \to \text{All}}$

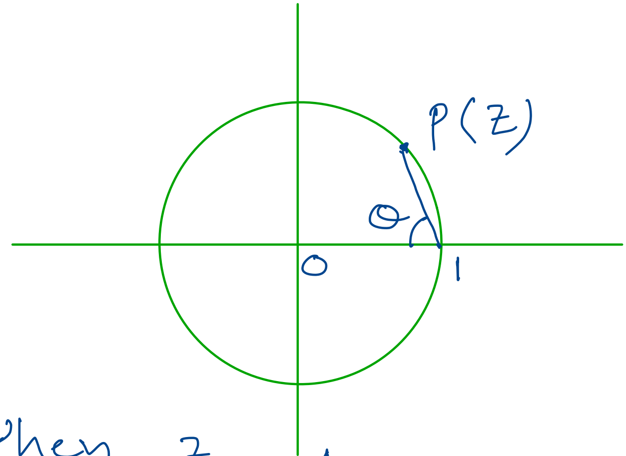
(C.) $I = \dfrac{\frac{\pi^2}{\ln 3}\left[\ln|\sec\pi x + \tan\pi x|\right]_{7/6}^{5/6}}{\pi}$

$= \frac{\pi}{\ln 3}\left[\ln\sqrt{3} - \ln\frac{1}{\sqrt{3}}\right] = \pi \Rightarrow \boxed{C \to s}$

(D.)

$$\left|Arg\left(\frac{1}{1-z}\right)\right|$$

$$= \left|Arg\left(\frac{0-1}{z-1}\right)\right|$$



$$= |\theta| < \frac{\pi}{2} \quad (\text{when } z \to 1 \text{ on circle})$$

$$\Rightarrow \boxed{D \to t}$$ (Best suitable choice as equality does not hold)

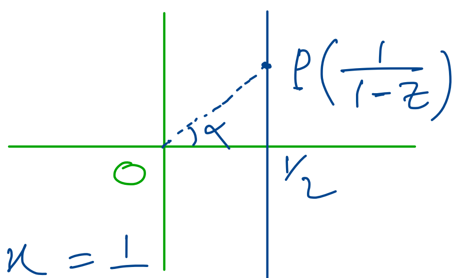
<u>Aliter:</u>

Let $z = e^{i\theta}$; $\theta \neq 2n\pi$

$$\frac{1}{1-z} = \frac{1}{1-\cos\theta - i\sin\theta} = \frac{1}{2\sin\frac{\theta}{2}\left(\sin\frac{\theta}{2} - i\cos\frac{\theta}{2}\right)}$$

$$= \frac{1}{-2i\sin\frac{\theta}{2}\, e^{i\theta/2}} = \frac{\cos\frac{\theta}{2} - i\sin\frac{\theta}{2}}{-2i\sin\theta/2}$$

$$= \frac{1}{2} + \frac{i\cot\theta/2}{2}$$



So, locus of $\frac{1}{1-z}$ is line $x = \frac{1}{2}$

Hence, $\left|Arg\left(\frac{1}{1-z}\right)\right| = |\alpha| < \frac{\pi}{2}$

8. Match the statements given in **Column I** with the intervals/union of intervals given in **Column II**

Column I	Column II
(A) The set $\left\{\text{Re}\left(\frac{2iz}{1-z^2}\right) : z \text{ is a complex number, } \|z\|=1,\ z \neq \pm 1\right\}$ is	(p) $(-\infty,-1)\cup(1,\infty)$
(B) The domain of the function $f(x)=\sin^{-1}\left(\frac{8(3)^{x-2}}{1-3^{2(x-1)}}\right)$ is	(q) $(-\infty,0)\cup(0,\infty)$
(C) If $f(\theta)=\begin{vmatrix} 1 & \tan\theta & 1 \\ -\tan\theta & 1 & \tan\theta \\ -1 & -\tan\theta & 1 \end{vmatrix}$, then the set $\left\{f(\theta) : 0 \le \theta < \frac{\pi}{2}\right\}$ is	(r) $[2,\infty)$
	(s) $(-\infty,-1]\cup[1,\infty)$
(D) If $f(x) = x^{3/2}(3x-10)$, $x \ge 0$, then $f(x)$ is increasing in	(t) $(-\infty,0]\cup[2,\infty)$

[JEE 2011, 2+2+2+2M]

Solⁿ (A) Let $z = e^{i\theta}$; $\theta \neq n\pi$

So, $\text{Re}\left(\frac{2iz}{1-z^2}\right) = \text{Re}\left(\frac{2i}{\frac{1}{z}-z}\right) = \text{Re}\left(\frac{2i}{e^{-i\theta}-e^{i\theta}}\right)$

$= \text{Re}\left(\frac{2i}{-2i\sin\theta}\right) = \text{Re}(-\text{cosec}\,\theta)$

$= -\text{cosec}\,\theta \in (-\infty,-1]\cup[1,\infty) \Rightarrow \boxed{A \to s}$

(B) $\frac{8\times 3^{(x-2)}}{1-3^{(2x-2)}} \in [-1,1]$

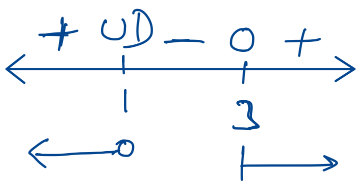
$\Rightarrow -1 \le \frac{\frac{8}{3}\times 3^{(x-1)}}{1-\left(3^{(x-1)}\right)^2} \le 1$

$$\Rightarrow -1 \le \frac{\frac{8t}{3}}{1-t^2} \le 1 \quad \left[\text{Let } 3^{(x-1)} = t > 0\right]$$

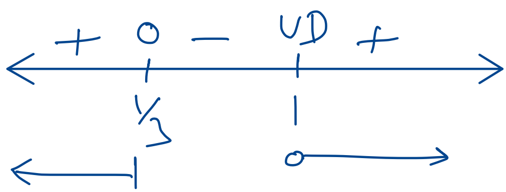
$$\Rightarrow \frac{3t^2-8t-3}{3(t^2-1)} \ge 0 \quad \& \quad \frac{3t^2+8t-3}{3(1-t^2)} \le 0$$

$$\Rightarrow \frac{(t-3)\left(t+\frac{1}{3}\right)}{(t-1)(t+1)} \ge 0 \quad \& \quad \frac{\left(t-\frac{1}{3}\right)(t+3)}{(t-1)(t+1)} \ge 0$$

$$\Rightarrow \frac{t-3}{t-1} \ge 0 \quad \& \quad \frac{t-\frac{1}{3}}{t-1} \ge 0 \quad [\text{as } t>0]$$



&



$$\Rightarrow t \in \left(0, \tfrac{1}{3}\right] \cup [3, \infty)$$

$$\Rightarrow 3^{(x-1)} \in \left(0, 3^{-1}\right] \cup [3, \infty)$$

$$\Rightarrow x-1 \in (-\infty, -1] \cup [1, \infty)$$

$$\Rightarrow x \in (-\infty, 0] \cup [2, \infty)$$

$$\Rightarrow \boxed{B \rightarrow t}$$

(C.) $f(\theta) = \begin{vmatrix} 0 & 0 & 2 \\ -\tan\theta & 1 & \tan\theta \\ -1 & -\tan\theta & 1 \end{vmatrix} \quad [R_1 \to R_1 + R_3]$

$\Rightarrow f(\theta) = 2(1 + \tan^2\theta) = 2\sec^2\theta \in [2, \infty)$

$\Rightarrow \boxed{C \to s}$

(D.) $f(x) = 3x^{\left(\frac{5}{2}\right)} - 10x^{\left(\frac{3}{2}\right)}$; $x \geq 0$

$\Rightarrow f'(x) = \dfrac{15}{2} x^{\left(\frac{3}{2}\right)} - 15x^{\left(\frac{1}{2}\right)}$

$\Rightarrow f'(x) = \dfrac{15\sqrt{x}}{2}(x - 2)$

$f'(x) =$ $\quad - \quad +$

$x = 0 \quad \downarrow \quad 2 \quad \uparrow$

So, $f(x)$ is increasing in $[2, \infty)$

$\Rightarrow \boxed{D \to s}$

9. Let z be a complex number such that the imaginary part of z is nonzero and $a = z^2 + z + 1$ is real. Then a **cannot** take the value - **[JEE 2012, 3M, –1M]**

(A) -1 (B) $\frac{1}{3}$ (C) $\frac{1}{2}$ (D) $\frac{3}{4}$ ✓

Sol^n Let $z = x + iy$; $y \neq 0$

Now, $a = (x^2 - y^2 + x + 1) + i(y + 2xy)$ is real $\Rightarrow$ $y + 2xy = 0$

$\Rightarrow$ $1 + 2x = 0$ (as $y \neq 0$)

$\Rightarrow$ $x = -\frac{1}{2}$

$\Rightarrow$ $a = \frac{3}{4} - y^2 \in \left(-\infty, \frac{3}{4}\right)$

10. Let complex numbers α and $\frac{1}{\bar{\alpha}}$ lie on circles $(x - x_0)^2 + (y - y_0)^2 = r^2$ and $(x - x_0)^2 + (y - y_0)^2 = 4r^2$ respectively. If $z_0 = x_0 + iy_0$ satisfies the equation $2|z_0|^2 = r^2 + 2$, then $|\alpha| =$

[JEE(Advanced) 2013, 2M]

(A) $\frac{1}{\sqrt{2}}$ (B) $\frac{1}{2}$ (C) $\frac{1}{\sqrt{7}}$ (D) $\frac{1}{3}$

Sol^n Given $|\alpha - z_0| = r \Rightarrow |\alpha - z_0|^2 = r^2$

$\Rightarrow |\alpha|^2 + |z_0|^2 - 2\mathrm{Re}(\bar{\alpha} z_0) = r^2$ ... (I)

Also, $\left|\frac{1}{\bar{\alpha}} - z_0\right| = 2r$

$\Rightarrow |1 - \bar{\alpha} z_0| = 2r|\alpha|$

$\Rightarrow 1 + |\bar{\alpha} z_0|^2 - 2\mathrm{Re}(\bar{\alpha} z_0) = 4r^2|\alpha|^2$

$\Rightarrow 1 + |\alpha|^2|z_0|^2 - 2\mathrm{Re}(\bar{\alpha} z_0) = 4r^2|\alpha|^2$ ... (II)

Now, (I) − (II) :—

$|\alpha|^2 + |z_0|^2 - 1 - |\alpha|^2|z_0|^2 = r^2(1 - 4|\alpha|^2)$

$\Rightarrow (1 - |\alpha|^2)(|z_0|^2 - 1) = r^2(1 - 4|\alpha|^2)$

$\Rightarrow (1 - |\alpha|^2)\left(\frac{r^2}{2}\right) = r^2(1 - 4|\alpha|^2)$

[Given $2(|z_0|^2 - 1) = r^2$]

$\Rightarrow |\alpha|^2 = \frac{1}{7} \Rightarrow |\alpha| = \frac{1}{\sqrt{7}}$

11. Let ω be a complex cube root of unity with $\omega \neq 1$ and $P = [p_{ij}]$ be a $n \times n$ matrix with $p_{ij} = \omega^{i+j}$. Then $P^2 \neq 0$, when $n =$ **[JEE(Advanced) 2013, 3, (–1)]**

(A) 57 ✓(B) 55 ✓(C) 58 ✓(D) 56

Sol^n

$$[P^2]_{ij} = [P \times P]_{ij} = \sum_{K=1}^{n} p_{iK} p_{Kj}$$

$$= \sum_{K=1}^{n} \omega^{i+K} \omega^{K+j} = \omega^{(i+j)} \sum_{K=1}^{n} \omega^{2K}$$

$$= \omega^{(i+j)} (\omega^2 + \omega^4 + \dots + \omega^{2n})$$

$$= \omega^{(2+i+j)} \left[\frac{\omega^{2n} - 1}{\omega^2 - 1}\right]$$

Now, if n is multiple of 3, then

$\omega^{2n} = 1 \Rightarrow [P^2]_{ij} = 0 \quad \forall\ i, j$

$\Rightarrow P^2 = 0$

Also, if n is not multiple of 3, then

$\omega^{2n} = \omega$ or $\omega^2 \Rightarrow \dfrac{\omega^{2n} - 1}{\omega^2 - 1} \neq 0$

$\Rightarrow [P^2]_{ij} \neq 0 \quad \forall\ i, j$ $\quad [\text{as } \omega^{(2+i+j)} \neq 0]$

$\Rightarrow P^2 \neq 0$ (if n is not multiple of 3)

12. Let $w = \frac{\sqrt{3}+i}{2}$ and $P = \{w^n : n = 1, 2, 3, \ldots\}$. Further $H_1 = \left\{z \in C : \text{Re}\, z > \frac{1}{2}\right\}$ and $H_2 = \left\{z \in C : \text{Re}\, z < \frac{-1}{2}\right\}$, where C is the set of all complex numbers. If $z_1 \in P \cap H_1$, $z_2 \in P \cap H_2$ and O represents the origin, then $\angle z_1Oz_2 =$ **[JEE-Advanced 2013, 4, (–1)]**

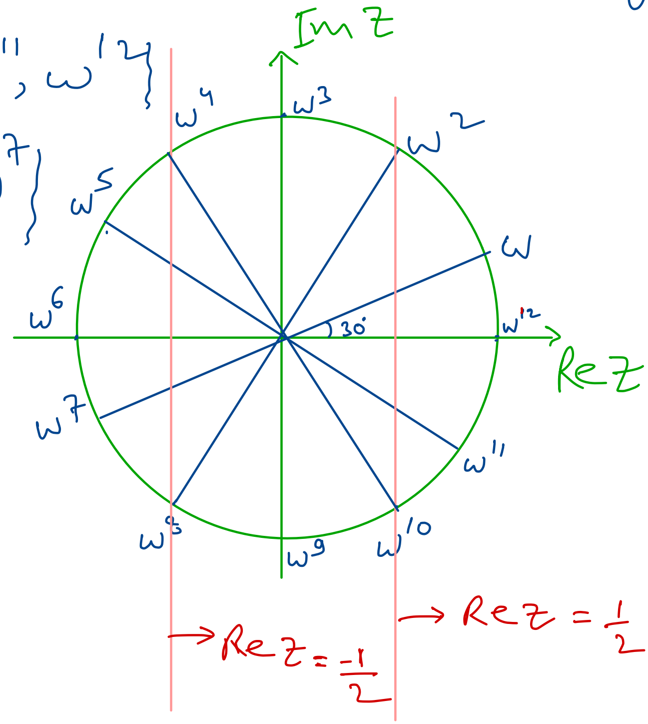
(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{6}$ (C) $\frac{2\pi}{3}$ (D) $\frac{5\pi}{6}$

Sol^n $w = e^{i\pi/6} = e^{i\left(\frac{2\pi}{12}\right)}$

$\Rightarrow P$ = set of 12th roots of unity

Now, $P \cap H_1 = \{w, w^{11}, w^{12}\}$

& $P \cap H_2 = \{w^5, w^6, w^7\}$



Hence, $\angle z_1 O z_2$ can be $\frac{2\pi}{3}$, $\frac{5\pi}{6}$ or π

Let $S = S_1 \cap S_2 \cap S_3$, where $S_1 = \{z \in C : |z| < 4\}$, $S_2 = \left\{z \in C : \operatorname{Im}\left[\frac{z-1+\sqrt{3}i}{1-\sqrt{3}i}\right] > 0\right\}$ and $S_3 = \{z \in C : \operatorname{Re} z > 0\}$.

13. $\min\limits_{z \in S} |1-3i-z| =$ **[JEE(Advanced) 2013, 3, (–1)]**

(A) $\frac{2-\sqrt{3}}{2}$ (B) $\frac{2+\sqrt{3}}{2}$ (C) $\frac{3-\sqrt{3}}{2}$ (D) $\frac{3+\sqrt{3}}{2}$

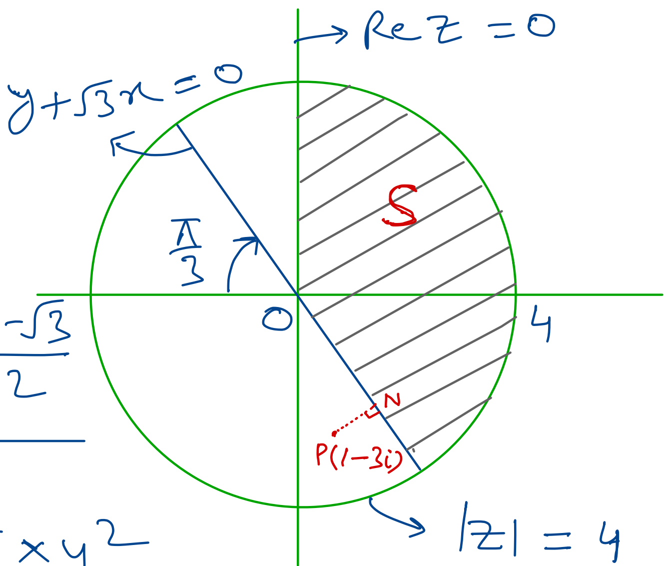
14. Area of S = **[JEE(Advanced) 2013, 3, (–1)]**

(A) $\frac{10\pi}{3}$ (B) $\frac{20\pi}{3}$ (C) $\frac{16\pi}{3}$ (D) $\frac{32\pi}{3}$

Sol^n

$$\operatorname{Im}\left[\frac{z-1+\sqrt{3}i}{1-\sqrt{3}i}\right] > 0 \quad (\text{Put } z = x+iy)$$

$$\Rightarrow \sqrt{3}x + y > 0$$



(13) $\min\limits_{z \in S} |1-3i-z|$

$$= PN = \frac{|\sqrt{3}-3|}{2} = \frac{3-\sqrt{3}}{2}$$

(14) Area of $S = \frac{\pi \times 4^2}{4} + \frac{1}{2} \times 4^2 \times \frac{\pi}{3}$

$$= \frac{20\pi}{3}$$

15. The quadratic equation $p(x) = 0$ with real coefficients has purely imaginary roots. Then the equation $p(p(x)) = 0$ has

(A) only purely imaginary roots

(B) ~~all real roots~~

(C) two real and two purely imaginary roots

(D) neither real nor purely imaginary roots.

[JEE(Advanced) 2014, 3(–1)]

Sol^n Since $p(x)$ has real roots, so complex roots occur in conjugate.

Let $p(x) = ax^2 + bx + c = 0$; $x = \pm ki$, $k > 0$

Put $x = ki$; $-ak^2 + bki + c = 0$

$\Rightarrow c = ak^2$ & $b = 0$

Hence, $p(x) = ax^2 + ak^2 = a(x^2 + k^2)$

Now, $p(p(x)) = 0 \Rightarrow p(x) = \pm ki$

$\Rightarrow a(x^2 + k^2) = \pm ki$

obviously, x cannot be real.

If possible, let $x = \alpha i$, $\alpha \in R - \{0\}$

So, $a(-\alpha^2 + k^2) = \pm ki$ (Not possible)

Hence, all roots of $p(p(x)) = 0$ are imaginary but not purely imaginary.

16. Let $z_k = \cos\left(\frac{2k\pi}{10}\right) + i\sin\left(\frac{2k\pi}{10}\right)$; $k = 1, 2, \ldots\ldots 9$.

List-I		**List-II**
P.	For each z_k there exists a z_j such that $z_k \cdot z_j = 1$	1. True
Q.	There exists a $k \in \{1, 2, \ldots\ldots, 9\}$ such that $z_1 \cdot z = z_k$ has no solution z in the set of complex numbers.	2. False
R.	$\frac{\lvert 1-z_1\rvert\lvert 1-z_2\rvert \ldots \lvert 1-z_9\rvert}{10}$ equals	3. 1
S.	$1 - \sum_{k=1}^{9} \cos\left(\frac{2k\pi}{10}\right)$ equals	4. 2

Codes :

	P	Q	R	S
(A)	1	2	4	3
(B)	2	1	3	4
(C)	1	2	3	4
(D)	2	1	4	3

[JEE(Advanced) 2014, 3(–1)]

Sol^n $z_K = \alpha^K$; $K = 1, 2, 3, \ldots, 9$

where $\alpha = e^{i\left(\frac{2\pi}{10}\right)}$; Such that $\alpha^{10} = 1$

P.) $z_K z_j = 1 \Rightarrow \alpha^K \alpha^j = \alpha^{10}$

$\Rightarrow \alpha^j = \alpha^{(10-K)} \Rightarrow j = 10 - K$

$\Rightarrow z_j = \alpha^{(10-K)} \Rightarrow \boxed{P \rightarrow 1}$

Q.) $z_1 \cdot z = z_K \Rightarrow \alpha z = \alpha^K$

$\Rightarrow z = \alpha^{(K-1)}$ (Solution exist for all $K = 1, 2, \ldots, 9$) $\Rightarrow \boxed{Q \rightarrow 2}$

R.) z_K ; $K = 1, 2, \ldots, 9$ are 10th roots of unity.

So, $z^{10} - 1 = (z-1)(z-z_1)\ldots(z-z_9)$

$$\Rightarrow \frac{z^{10}-1}{z-1} = (z-z_1)\ldots(z-z_9)$$

$$\Rightarrow 1 + z + z^2 + \ldots + z^9 = (z-z_1)\ldots(z-z_9)$$

Put $z = 1$ & thereafter taking modulus both sides :—

$$10 = |1-z_1|\,|1-z_2|\ldots|1-z_9| \Rightarrow \boxed{R \rightarrow 3}$$

S.) we have, sum of all roots of unity vanishes.

So, $1 + z_1 + z_2 + \ldots + z_9 = 0$

$$\Rightarrow \sum_{K=1}^{9} z_K = -1$$

$$\Rightarrow \sum_{K=1}^{9} \left(\cos\frac{2\pi K}{10} + i\sin\frac{2\pi K}{10}\right) = -1$$

$$\Rightarrow \sum_{K=1}^{9} \cos\left(\frac{2K\pi}{10}\right) = -1 \quad \Rightarrow \boxed{S \rightarrow 4}$$

17.

Column-I	Column-II
(A) In $\mathbb{R}^2$, if the magnitude of the projection vector of the vector $\alpha\hat{i}+\beta\hat{j}$ on $\sqrt{3}\hat{i}+\hat{j}$ is $\sqrt{3}$ and if $\alpha = 2+\sqrt{3}\beta$, then possible value(s) of $\lvert\alpha\rvert$ is (are)	(P) 1
(B) Let a and b be real numbers such that the function $f(x)=\begin{cases}-3ax^2-2, & x<1\\ bx+a^2, & x\ge 1\end{cases}$ is differentiable for all $x \in \mathbb{R}$. Then possible value(s) of a is (are)	(Q) 2
(C) Let $\omega \neq 1$ be a complex cube root of unity. If $(3-3\omega+2\omega^2)^{4n+3} + (2+3\omega-3\omega^2)^{4n+3} + (-3+2\omega+3\omega^2)^{4n+3} = 0$ then possible value(s) of n is (are)	(R) 3
(D) Let the harmonic mean of two positive real number a and b be 4, If q is a positive real number such that a, 5, q, b is an arithmetic progression, then the value(s) of $\lvert q-a\rvert$ is (are)	(S) 4
	(T) 5

[JEE 2015, 8(Each 2M, –1M)]

Solⁿ

(A.) given $\dfrac{\left|(\alpha\hat{i}+\beta\hat{j})\cdot(\sqrt{3}\hat{i}+\hat{j})\right|}{\left|\sqrt{3}\hat{i}+\hat{j}\right|} = \sqrt{3}$

$\Rightarrow \left|\sqrt{3}\alpha+\beta\right| = 2\sqrt{3}$ --- (I)

Also, given that $\alpha = 2+\sqrt{3}\beta$ --- (II)

Hence, $\alpha = 2$ or -1 $\Rightarrow$ $\boxed{A \to P, Q}$

(B.) $f(x)$ must be continuous at $x=1$

So, $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x)$

$\Rightarrow -3a-2 = b+a^2$ ..... (I)

Also, LHD & RHD are equal at $x=1$

So, $(-6ax)_{x=1} = b \Rightarrow -6a = b$ ---- (ii)

from (i) & (ii) :— $a = 1$ or $2 \Rightarrow \boxed{B \to P, Q}$

(C.) Let $\alpha = 3 - 3\omega + 2\omega^2 = 1 - 5\omega \neq 0$

So, given $\alpha^{4n+3} + (\alpha\omega)^{4n+3} + (\alpha\omega^2)^{4n+3} = 0$

$\Rightarrow \alpha^{(4n+3)}\left(1^{4n+3} + \omega^{(4n+3)} + (\omega^2)^{(4n+3)}\right) = 0$

$\Rightarrow 1^{(4n+3)} + \omega^{(4n+3)} + (\omega^2)^{(4n+3)} = 0$

[as $\alpha \neq 0$]

$\Rightarrow 4n+3$ should not be multiple of 3

$\Rightarrow n$ " " " " " "

$\Rightarrow \boxed{C \to P, Q, S, T}$

(D.) given $\dfrac{2ab}{a+b} = 4$ ---- (i)

Also, $5 = a + \dfrac{b-a}{3} = \dfrac{b+2a}{3}$ ---- (ii)

from (i) & (ii) :— $2a^2 - 17a + 30 = 0$

$\Rightarrow a = 6$ or $\dfrac{5}{2} \Rightarrow b = 3$ or 10

$\Rightarrow q - a = 2\left(\dfrac{b-a}{3}\right) = -2$ or 5

$\Rightarrow \boxed{D \to Q, T}$

18. For any integer k, let $\alpha_k = \cos\left(\frac{k\pi}{7}\right) + i\sin\left(\frac{k\pi}{7}\right)$, where $i = \sqrt{-1}$. The value of the expression

$\dfrac{\sum_{k=1}^{12}|\alpha_{k+1}-\alpha_k|}{\sum_{k=1}^{3}|\alpha_{4k-1}-\alpha_{4k-2}|}$ is **[JEE 2015, 4M, –0M]**

Solⁿ

$$\alpha_k = e^{i\left(\frac{k\pi}{7}\right)} \Rightarrow \alpha_{k+1} - \alpha_k = e^{\frac{ik\pi}{7}}\left(e^{\frac{i\pi}{7}} - 1\right)$$

$$\Rightarrow |\alpha_{k+1} - \alpha_k| = \left|e^{\left(\frac{i\pi}{7}\right)} - 1\right| = c \text{ (constant)}$$

Also, $$|\alpha_{4k-1} - \alpha_{4k-2}| = \left|e^{\left(\frac{i(4k-1)\pi}{7}\right)} - e^{i\left(\frac{(4k-2)\pi}{7}\right)}\right|$$

$$= \left|e^{\left(\frac{i(4k-2)\pi}{7}\right)}\left(e^{\frac{i\pi}{7}} - 1\right)\right| = \left|e^{\left(\frac{i\pi}{7}\right)} - 1\right| = c$$

Hence, $$N_r. = \sum_{k=1}^{12} c = 12c \; ; \; c \neq 0$$

& $$D_r. = \sum_{k=1}^{3} c = 3c$$

So, $$\frac{N_r.}{D_r.} = \frac{12c}{3c} = \boxed{4}$$

19. Let $z = \frac{-1+\sqrt{3}i}{2}$, where $i = \sqrt{-1}$, and $r, s \in \{1, 2, 3\}$. Let $P = \begin{bmatrix} (-z)^r & z^{2s} \\ z^{2s} & z^r \end{bmatrix}$ and I be the identity matrix of order 2. Then the total number of ordered pairs (r, s) for which $P^2 = -I$ is

[JEE(Advanced)-2016, 3(0)]

Solⁿ

$z = \omega$. So, $P = \begin{bmatrix} (-\omega)^r & \omega^{2s} \\ \omega^{2s} & \omega^r \end{bmatrix}$

Now, $P^2 = \begin{bmatrix} \omega^{2r} + \omega^{4s} & ((-1)^r + 1)\omega^{(r+2s)} \\ ((-1)^r + 1)\omega^{(r+2s)} & \omega^{4s} + \omega^{2r} \end{bmatrix}$

Given, $P^2 = -I$

$\Rightarrow \omega^{2r} + \omega^{4s} = -1$ & $((-1)^r + 1)\omega^{(r+2s)} = 0$

$\Rightarrow (-1)^r + 1 = 0$ & $\omega^{2r} + \omega^{4s} = -1$

$\Rightarrow r = 1$ or 3

If $r = 1$:-

$\omega^{4s} = -1 - \omega^2 = \omega$

$\Rightarrow \omega^s = \omega \Rightarrow s = 1$

If $r = 3$:- $\omega^{4s} = -2$ (Not Possible)

Hence, there is only $\boxed{\text{one}}$ pair $(r, s) \equiv (1, 1)$

20. Let a,b $\in \mathbb{R}$ and $a^2 + b^2 \neq 0$. Suppose $S = \left\{ z \in \mathbb{C} : z = \frac{1}{a+ibt}, t \in \mathbb{R}, t \neq 0 \right\}$, where $i = \sqrt{-1}$. If z = x + iy and $z \in S$, then (x,y) lies on

(A) the circle with radius $\frac{1}{2a}$ and centre $\left(\frac{1}{2a}, 0\right)$ for $a > 0, b \neq 0$

(B) the circle with radius $-\frac{1}{2a}$ and centre $\left(-\frac{1}{2a}, 0\right)$ for $a < 0, b \neq 0$

(C) the x-axis for $a \neq 0, b = 0$

(D) the y-axis for $a = 0, b \neq 0$

[JEE(Advanced)-2016, 4(–2)]

Solⁿ

$$x + iy = \frac{1}{a+ibt} = \frac{a-ibt}{a^2+b^2t^2}$$

$\Rightarrow x = \frac{a}{a^2+b^2t^2}$ (Note $\frac{x}{a} > 0$) & $y = \frac{-bt}{a^2+b^2t^2}$

Case I :- If $a = 0 \Rightarrow b \neq 0$

$x = 0$ & $y = \frac{-1}{bt} \in R - \{0\} \Rightarrow$ Y-axis

Case II :- If $b = 0 \Rightarrow a \neq 0$

$y = 0$ & $x = \frac{1}{a} \Rightarrow (x, y) \equiv \left(\frac{1}{a}, 0\right)$ (A point lying on X-axis)

Case III :- $a \neq 0, b \neq 0$

Now, $x^2 + y^2 = \frac{1}{a^2+b^2t^2}$ & $\frac{y}{x} = \frac{-bt}{a}$

Hence, $x^2 + y^2 = \frac{1}{a^2 + \frac{a^2y^2}{x^2}} = \frac{x^2}{a^2(x^2+y^2)}$

$\Rightarrow (x^2+y^2)^2 = \frac{x^2}{a^2} \Rightarrow x^2 + y^2 = \frac{x}{a}$ [As $\frac{x}{a} > 0$]

$\Rightarrow$ center $\left(\frac{1}{2a}, 0\right)$ & radius $= \frac{1}{2|a|}$

21. Let a, b, x and y be real numbers such that a – b = 1 and y ≠ 0. If the complex number z = x + iy satisfies $\text{Im}\left(\frac{az+b}{z+1}\right) = y$, then which of the following is(are) possible value(s) of x ?

[JEE(Advanced)-2017, 4(–2)]

(A) $-1-\sqrt{1-y^2}$ (B) $1+\sqrt{1+y^2}$ (C) $1-\sqrt{1+y^2}$ (D) $-1+\sqrt{1-y^2}$

Sol^n $\text{Im}\left(\frac{az+b}{z+1}\right) = y$

$\Rightarrow \text{Im}\left(\frac{a(x+iy)+b}{x+1+iy}\right) = y$

$\Rightarrow \text{Im}\left[\frac{((ax+b)+aiy)(x+1-iy)}{(x+1)^2+y^2}\right] = y$

$\Rightarrow y(a-b) = y\left[(x+1)^2+y^2\right]$

$\Rightarrow 1-y^2 = (x+1)^2$ [as $y \neq 0$ & $a-b=1$]

$\Rightarrow x+1 = \pm\sqrt{1-y^2}$

$\Rightarrow x = -1 \pm \sqrt{1-y^2}$

22. For a non-zero complex number z, let arg(z) denotes the principal argument with $-\pi < \arg(z) \le \pi$.

Then, which of the following statement(s) is (are) **FALSE** ? **[JEE(Advanced)-2018, 4(–2)]**

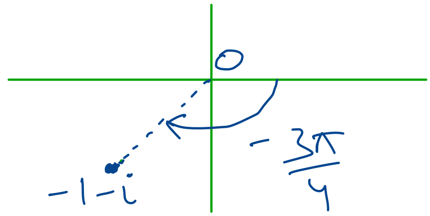
(A) $\arg(-1-i) = \frac{\pi}{4}$, where $i = \sqrt{-1}$

(B) The function $f : \mathbb{R} \to (-\pi, \pi]$, defined by $f(t) = \arg(-1 + it)$ for all $t \in \mathbb{R}$, is continuous at all points of $\mathbb{R}$, where $i = \sqrt{-1}$

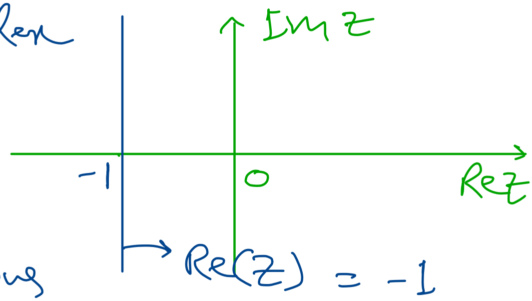
(C) For any two non-zero complex numbers z_1 and z_2, $\arg\left(\frac{z_1}{z_2}\right) - \arg(z_1) + \arg(z_2)$ is an integer multiple of 2π

(D) For any three given distinct complex numbers z_1, z_2 and z_3, the locus of the point z satisfying the condition $\arg\left(\frac{(z-z_1)(z_2-z_3)}{(z-z_3)(z_2-z_1)}\right) = \pi$, lies on a straight line

Sol^n (A) $\arg(-1-i) = -\frac{3\pi}{4}$



(B.) The variable complex no. $-1+it$ lies on the line $\text{Re}\,z = -1$



So, $f(t)$ is discontinuous at $t=0$ because

$$LHL = \lim_{t \to 0^-} f(t) = -\pi$$

& $$RHL = \lim_{t \to 0^+} f(t) = \pi = f(0)$$

(C.) This is the well known result. For proof we can consider

$z_1 = r_1 e^{i\theta_1}$ & $z_2 = r_2 e^{i\theta_2}$

So, $\arg\left(\frac{z_1}{z_2}\right) - \arg(z_1) + \arg(z_2)$

$= \arg\left(\frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}\right) - (\theta_1 + 2K_1\pi) + (\theta_2 + 2K_2\pi)$

[Here $\arg z$ denotes general argument]

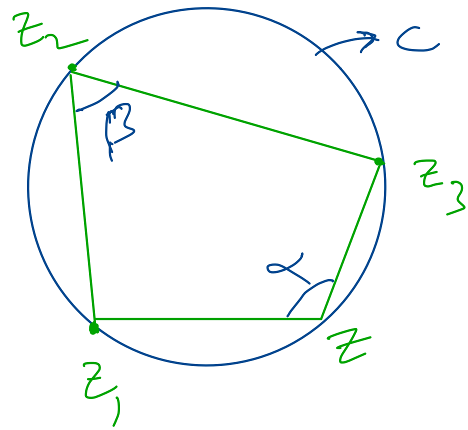
$= (\theta_1 - \theta_2 + 2K_3\pi) - (\theta_1 + 2K_1\pi) + (\theta_2 + 2K_2\pi)$

$= 2K\pi$ = integral multiple of π

(d.) $\arg\left(\frac{z - z_1}{z - z_3}\right) + \arg\left(\frac{z_2 - z_3}{z_2 - z_1}\right) = \pi$

$\Rightarrow \arg\left(\frac{z_1 - z}{z_3 - z}\right) + \arg\left(\frac{z_3 - z_2}{z_1 - z_2}\right) = \pi$

$\Rightarrow \alpha + \beta = \pi$



Let c be the circumcircle of three given complex numbers z_1, z_2 & z_3

Thus, locus of z is a minor arc of the circle c.

23. Let s, t, r be the non-zero complex numbers and L be the set of solutions $z = x + iy \left(x, y \in \mathbb{R}, i = \sqrt{-1}\right)$ of the equation $sz + t\bar{z} + r = 0$, where $\bar{z} = x - iy$. Then, which of the following statement(s) is (are) TRUE ? **[JEE(Advanced)-2018, 4(–2)]**

(A) If L has exactly one element, then $|s| \neq |t|$

(B) If $|s| = |t|$, then L has infinitely many elements

(C) The number of elements in $L \cap \{z : |z - 1 + i| = 5\}$ is at most 2

(D) If L has more than one element, then L has infinitely many elements

Solⁿ

$$sz + t\bar{z} + r = 0 \quad \cdots\cdots (I)$$

Taking conjugate both sides :—

$$\bar{s}\bar{z} + \bar{t}z + \bar{r} = 0 \quad \cdots\cdots (II)$$

$(I) \times \bar{s} - (II) \times t$:—

$$z\left(|s|^2 - |t|^2\right) = \bar{r}t - r\bar{s}$$

case I : $|s| \neq |t|$

$$z = \frac{\bar{r}t - r\bar{s}}{|s|^2 - |t|^2} \quad \text{(unique solution)}$$

case II : $|s| = |t|$

$$z \times 0 = \bar{r}t - r\bar{s}$$

So, if $\bar{r}t \neq r\bar{s} \Rightarrow$ No solution

and if $\bar{r}t = r\bar{s} \Rightarrow t = \dfrac{r\bar{s}}{\bar{r}}$ (as $r \neq 0$)

$$\Rightarrow sz + \frac{r\bar{s}}{\bar{r}}\bar{z} + r = 0 \quad \text{(from (I))}$$

$\Rightarrow \quad \bar{r}_2 s z + r_2 \bar{s} \bar{z} + |r_2|^2 = 0$

$\Rightarrow \quad Re(\bar{r}_2 s z) = -\dfrac{|r_2|^2}{2}$

$\Rightarrow \quad \bar{r}_2 s z = -\dfrac{|r_2|^2}{2} + ki, \; k \in R$

$\Rightarrow z = \dfrac{-|r_2|^2 + 2ki}{2\bar{r}_2 s}$ [Infinite solution as $r_2, s \neq 0$ & $k \in R$]

Thus, unique solution exist only when $|s| \neq |t|$

option (c)

In case of infinite solution, we find locus of z. Let $\dfrac{1}{2\bar{r}_2 s} = \alpha + i\beta$, where α & β are fixed non-zero reals

So, $z = x + iy = (\alpha + i\beta)(-|r_2|^2 + 2ki)$

$\Rightarrow x = -\alpha |r_2|^2 - 2k\beta$
& $y = 2k\alpha - \beta |r_2|^2$

{ Eliminating k we get locus as straight line.

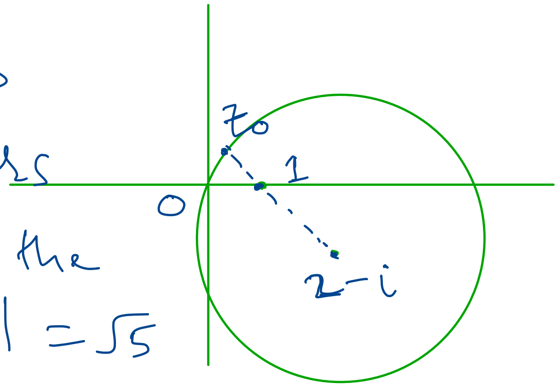
Now, $|z - 1 + i| = 5$ represents a circle. So, line and circle can intersect atmost two points.

24. Let S be the set of all complex numbers z satisfying $|z-2+i| \geq \sqrt{5}$. If the complex number z_0 is such that $\frac{1}{|z_0-1|}$ is the maximum of the set $\left\{\frac{1}{|z-1|} : z \in S\right\}$, then the principal argument of $\frac{4-z_0-\bar{z}_0}{z_0-\bar{z}_0+2i}$ is **[JEE(Advanced)-2019, 3(–1)]**

(1) $\frac{\pi}{4}$ (2) $-\frac{\pi}{2}$ (3) $\frac{3\pi}{4}$ (4) $\frac{\pi}{2}$

Sol^n

Set S contains all complex numbers lying on or outside the circle $|z-2+i| = \sqrt{5}$



Now, If $\frac{1}{|z-1|}$ is maximum, then $|z-1|$ is minimum.

From figure, we find $z_0 = x_0 + iy_0$ lies in the 1st quadrant, $0 < x_0 < 1$

Now, $\frac{4-(z_0+\bar{z}_0)}{z-\bar{z}_0+2i} = \frac{4-2\,\mathrm{Re}\,z_0}{2i(\mathrm{Im}\,z_0)+2i}$

$= \frac{2(2-x_0)}{2i(y+1)} = -i\left(\frac{2-x_0}{y+1}\right)$; which lies on −ve Y-axis (as $x_0 \in (0,1)$)

Hence, required argument $= -\frac{\pi}{2}$

25. Let $\omega \neq 1$ be a cube root of unity. Then the minimum of the set

$$\{|a + b\omega + c\omega^2|^2 : a,\ b,\ c \text{ distinct non-zero integers}\}$$

equals ____ **[JEE(Advanced)-2019, 3(0)]**

<u>Sol</u>

Solⁿ

$$|a+b\omega+c\omega^2|^2 = (a+b\omega+c\omega^2)\overline{(a+b\omega+c\omega^2)}$$

$$= (a+b\omega+c\omega^2)(a+b\omega^2+c\omega)$$

$$= a^2+b^2+c^2-ab-bc-ca$$

$$= \frac{(a-b)^2+(b-c)^2+(c-a)^2}{2};$$

which is minimum when a, b & c are three consecutive integers.

Hence, $|a+b\omega+c\omega^2|^2 \geq \frac{1^2+1^2+2^2}{2} = \boxed{3}$