Race - 05: Integration:

Q1 : $\int \frac{dx}{6x^3-7x^2-3x}$ OR $\int \frac{dx}{x(2x-3)(3x+1)}$

$$\frac{1}{x(2x-3)(3x+1)} = \frac{A}{x} + \frac{B}{2x-3} + \frac{C}{3x+1}$$ [using Partial fraction]

$$1 = A(2x-3)(3x+1) + Bx(3x+1) + Cx(2x-3)$$

put $x = 0$ then $A = -\frac{1}{3}$

$x = \frac{3}{2}$ then $B = \frac{4}{33}$; $x = -\frac{1}{3}$ then $C = \frac{9}{11}$

Thus.

$$\int \frac{dx}{x(2x-3)(3x+1)} = \int -\frac{1\,dx}{3x} + \int \frac{4\,dx}{33(2x-3)} + \int \frac{9\,dx}{11(3x+1)}$$

$$\Rightarrow -\frac{1}{3}\ln|x| + \frac{2}{33}\ln|2x-3| + \frac{3}{11}\ln(3x+1) + C$$

Q2. $\int \frac{x\,dx}{x^4 - 3x^2 + 2}$

put $x^2 = t$

$2x\,dx = dt$

$\therefore \frac{1}{2}\int \frac{dt}{t^2 - 3t + 2} = \frac{1}{2}\int \frac{dt}{(t-2)(t-1)} = \frac{1}{2}\left[\int \frac{(t-1)-(t-2)}{(t-2)(t-1)}\,dt\right.$

$\Rightarrow \frac{1}{2}\left[\int \frac{dt}{t-2} - \int \frac{dt}{t-1}\right]$

$\Rightarrow \frac{1}{2}\left[\ln(t-2) - \ln(t-1)\right] + C$

$\Rightarrow \frac{1}{2}\ln\left(\frac{t-2}{t-1}\right) + c = \ln\sqrt{\frac{x^2-2}{x^2-1}} + c$

③ $\int \frac{dx}{x^4 - x^2} \Rightarrow \int \frac{dx}{(x^2-1)x^2}$

$\Rightarrow \int \frac{dx}{x^2-1} - \int \frac{dx}{x^2}$

$\Rightarrow \frac{1}{2} \ln\left(\frac{x-1}{x+1}\right) + \frac{1}{x} + C$

(4) $\int \frac{x^2 dx}{1-x^4} = \int \frac{-x^2 dx}{(x^2-1)(x^2+1)}$

$- \left[\frac{1}{2}\right] \int \frac{(x^2+1) + (x^2-1)}{(x^2-1)(x^2+1)} dx \Big]$

$= -\frac{1}{2} \left[\int \frac{dx}{x^2-1} + \int \frac{dx}{x^2+1} \right]$

$\Rightarrow -\frac{1}{2} \left[\frac{1}{2} \ln \left| \frac{x-1}{x+1} \right| + \tan^{-1} x \right]$

$\Rightarrow -\frac{1}{4} \ln \left(\frac{x-1}{x+1} \right) - \frac{1}{2} \tan^{-1} x + C$

$\Rightarrow \frac{1}{4} \ln \left(\frac{x+1}{x-1} \right) - \frac{1}{2} \tan^{-1} x + C$

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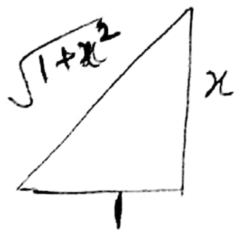
5) $\int \frac{dx}{(1+x^2)^4}$ put $x = \tan\theta$

$dx = \sec^2\theta \, d\theta$

$$I = \int \frac{\sec^2\theta \, d\theta}{(1+\tan^2\theta)^4}$$



$$I = \int \cos^6\theta \, d\theta$$

Now using Reduction formula

$$I = \int \cos^6\theta \, d\theta = \frac{\sin\theta \cdot \cos^5\theta}{6} + \frac{5}{6}\left\{ \frac{\sin\theta \cos^3\theta}{4} + \frac{3}{4}\left\{ \frac{\sin\theta \cos\theta}{2} + \frac{1}{2}\theta \right\}\right\}$$

$$I = \frac{\sin\theta \cos^5\theta}{6} + \frac{5}{24}\sin\theta \cos^3\theta + \frac{15}{48}\sin\theta \cos\theta + \frac{15}{48}\theta$$

From triangle substitute $\sin\theta$, $\cos\theta$ and θ

$$I = \frac{1}{6}\left(\frac{x}{\sqrt{1+x^2}} \cdot \frac{1}{(1+x^2)^{\frac{1}{5}}}\right) + \frac{5}{24} \cdot \frac{x}{\sqrt{1+x^2}} \cdot \frac{1}{(1+x^2)^{\frac{3}{2}}} + \frac{15}{48} \cdot \frac{x}{1+x^2} + \frac{15}{48}\tan^{-1}x + c$$

simplifying

$$I = \frac{15x^5 + 40x^3 + 33x}{48(1+x^2)^3} + \frac{15}{48}\tan^{-1}x + C$$

(6) $\int \frac{x^3-1}{4x^3-x}$

First step : $\frac{x^3-1}{4x^3-x} = \frac{1}{4} + \frac{\frac{1}{4}x-1}{4x^3-x} = \frac{1}{4} + \frac{x-4}{4[x(2x-1)(2x+1)]}$

second step : [Partial fraction]

$$\frac{x-4}{x(2x-1)(2x+1)} = \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{2x+1}$$

$$x-4 = A(2x-1)(2x+1) + B(x)(2x+1) + Cx(2x-1)$$

$x=0$ then $-A=-4$ or $A=4$

$x=\frac{1}{2}$ then $B=-\frac{7}{2}$

$x=-\frac{1}{2}$ then $-C=-\frac{9}{2}$ or $C=\frac{9}{2}$

$$\frac{x-4}{x(2x-1)(2x+1)} = \frac{4}{x} - \frac{7}{2(2x-1)} + \frac{9}{2(2x+1)}$$

Now $\int \frac{x^3-1}{4x^3-x}\,dx = \int \frac{1}{4}dx + \int \frac{1}{x}dx - \int \frac{7\,dx}{8(2x-1)} + \int \frac{9\,dx}{8(2x+1)}$

$= \frac{1}{4}x + \ln|x| - \frac{7}{16}\ln|2x+1) + \frac{9}{16}\ln|2x+1|$

(7) $\displaystyle\int \frac{dx}{\sqrt[4]{(x-1)^3 \cdot (x+2)^5}}$

$$\int \frac{dx}{\sqrt[4]{\left(\frac{x-1}{x+2}\right)^3 \cdot (x+2)^8}} \Rightarrow \int \frac{dx}{(x+2)^2 \cdot \sqrt[4]{\left(\frac{x-1}{x+2}\right)^3}}$$

$$\frac{x-1}{x+2} = t$$

$$\frac{(x+2)\cdot 1 - (x-1)\cdot 1}{(x+2)^2} \cdot dx = dt$$

$$\frac{dx}{(x+2)^2} = \frac{dt}{3}$$

Thus $\displaystyle\int \frac{dt}{3\sqrt[4]{t^3}} \Rightarrow \int \frac{dt}{3 \cdot t^{\frac{3}{4}}} = \int \frac{t^{-\frac{3}{4}} dt}{3}$

$$\Rightarrow \frac{1}{3} \cdot \left[\frac{t^{-\frac{3}{4}+1}}{-\frac{3}{4}+1} \right] = \frac{4}{3} \cdot t^{\frac{1}{4}} = \frac{4}{3}\left(\frac{x-1}{x+2}\right)^{\frac{1}{4}}$$

(8) $\int \frac{dx}{1-\sin^4 x} = \int \frac{dx}{\cos^2 x\,(1+\sin^2 x)}$

$\Rightarrow \int \frac{dx}{(1-\sin^2 x)(1+\sin^2 x)}$

$\frac{1}{2}\left[\int \frac{1\,dx}{1-\sin^2 x} + \int \frac{1\,dx}{1+\sin^2 x}\right]$

$\frac{1}{2}\left[\int \sec^2 x\,dx + \int \frac{\sec^2 x\,dx}{\sec^2 x + \tan^2 x}\right]$

$\frac{1}{2}\left[\int \sec^2 x\,dx + \int \frac{\sec^2 x\,dx}{1+2\tan^2 x}\right]$

$\frac{1}{2}\left[\tan x + \frac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}\tan x)\right] + c$

$\left\{ \int \frac{\sec^2 x\,dx}{1+2\tan^2 x} \right.$

put $\tan x = t$

$\sec^2 x\,dx = dt$

$\frac{1}{2}\int \frac{dt}{t^2 + \left(\frac{1}{\sqrt{2}}\right)^2}$

$\frac{1}{2} \times \sqrt{2}\tan^{-1}(\sqrt{2}t)$

$\left. \frac{1}{\sqrt{2}}\tan^{-1}(\sqrt{2}\tan x) \right\}$

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(9) $\int \sqrt[3]{\frac{\sin^2 x}{\cos^{14} x}}\, dx$

$$\int \frac{dx}{\sin^{-\frac{2}{3}} x \cdot \cos^{\frac{14}{3}} x}$$

$$\int \frac{dx}{(\tan x)^{-\frac{2}{3}} \cos^4 x} \Rightarrow \int \frac{\sec^4 x\, dx}{(\tan x)^{-\frac{2}{3}}}$$

$$\Rightarrow \int \frac{\sec^2 x\,(1+\tan^2 x)}{(\tan x)^{-\frac{2}{3}}}\, dx$$

put $\tan x = t$

$\sec^2 x\, dx = dt$

$$\because \int \frac{1+t^2}{t^{-\frac{2}{3}}}\, dt \Rightarrow \int \left(t^{\frac{2}{3}} + t^{\frac{8}{3}}\right) dt$$

$$\Rightarrow \frac{t^{\frac{2}{3}+1}}{\frac{2}{3}+1} + \frac{t^{\frac{8}{3}+1}}{\frac{8}{3}+1} + C$$

$$\Rightarrow \frac{3}{5} t^{\frac{5}{3}} + \frac{3}{11} t^{\frac{11}{3}} + C$$

$$\Rightarrow \frac{3}{55} t^{\frac{5}{3}} \left[11 + 5t^2\right]$$

$$\Rightarrow \frac{3}{55} \sqrt[3]{\tan^5 x}\, (5\tan^2 x + 11)$$

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(10) $\int \frac{1+x^4}{(1-x^4)^{\frac{3}{2}}} dx = \int \frac{1+x^4}{x^3\left(\frac{1}{x^2}-x^2\right)^{\frac{3}{2}}}$

$\Rightarrow \int \frac{\left(\frac{1}{x^3}+x\right) dx}{\left(\frac{1}{x^2}-x^2\right)^{\frac{3}{2}}}$

put $\left[\begin{array}{l} \frac{1}{x^2}-x^2=t^2 \\ \left(\frac{-2}{x^3}-2x\right)dx=2t\,dt \\ \left(\frac{1}{x^3}+x\right)dx=t\,dt \end{array}\right]$

$\Rightarrow \int \frac{-t\,dt}{t^3} = \int -\frac{1}{t^2} dt$

$\Rightarrow +\frac{1}{t}+c \Rightarrow \frac{+1}{\sqrt{\frac{1}{x^2}-x^2}} + c$

$\Rightarrow c + \frac{x}{\sqrt{1-x^4}}$

(11) $\int \frac{dx}{\cos x \sin^3 x} \Rightarrow \int \frac{(\sin^2 x + \cos^2 x)\,dx}{\cos x \cdot \sin^3 x}$

$\Rightarrow \int \frac{dx}{\cos x \sin x} + \int \frac{\cos x}{\sin^3 x}\,dx = I_1 + I_2$

Now

$I_1 = \int \frac{dx}{\sin x \cos x} = \int \frac{2}{\sin 2x}\,dx = \int 2 \operatorname{cosec} 2x\,dx$

$\Rightarrow \frac{2 \cdot \ln \tan x}{2} = \ln \tan x$

$I_2 = \int \frac{\cos x\,dx}{\sin^3 x}$ put $\sin x = t$

$\cos x\,dx = dt$

$\therefore \int \frac{dt}{t^3} \Rightarrow \frac{t^{-3+1}}{-3+1} + c$

$I_2 \Rightarrow \frac{1}{-2t^2} = \frac{-1}{2\sin^2 x}$

Thus: $\ln \tan x - \frac{1}{2\sin^2 x} + c$

(12) $\int \frac{dx}{\cos^3 x \sin^3 x} \Rightarrow \int \frac{(\sin^2 x + \cos^2 x)^2 dx}{\cos^3 x \sin^3 x}$

$\Rightarrow \int \left(\frac{\sin^4 x + \cos^4 x + 2\sin^2 x \cos^2 x}{\cos^3 x \cdot \sin^3 x} \right) dx$

$\Rightarrow \int \frac{\sin x \, dx}{\cos^3 x} + \int \frac{\cos x \, dx}{\sin^3 x} + \int \frac{2 \, dx}{\sin x \cos x}$

$\int \tan x \cdot \sec^2 x + \int \cot x \cdot \operatorname{cosec}^2 x + \int 4 \operatorname{cosec} 2x \, dx$

$\frac{\tan^2 x}{2} - \frac{\cot^2 x}{2} + 2 \ln|\tan x| + C$

$\Rightarrow \frac{1}{2}(\tan^2 x - \cot^2 x) + 2 \ln|\tan x| + C$

(13) $\int \frac{\sin x \, dx}{(1-\cos x)^2}$

put $\cos x = t$

$-\sin x dx = dt$

$\int \frac{-dt}{(1-t)^2} = \frac{1}{1-t} \times -1 = \frac{1}{t-1} + c$

$\Rightarrow \frac{1}{\cos x - 1} + c$

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(14) $\int \frac{dx}{\sqrt[4]{\sin^3 x \cos^5 x}} \Rightarrow \int \frac{dx}{\sin^{\frac{3}{4}} x \cdot \cos^{\frac{5}{4}} x}$

$\Rightarrow \int \frac{dx}{(\tan x)^{\frac{3}{4}} \cdot \cos^2 x} \Rightarrow \int \frac{\sec^2 x \, dx}{(\tan x)^{\frac{3}{4}}}$

put $\tan x = t$

$\sec^2 x \, dx = dt$

$\int \frac{dt}{t^{\frac{3}{4}}} \Rightarrow 4 t^{\frac{1}{4}} \Rightarrow 4 (\tan x)^{\frac{1}{4}} + c$

(15) $\int \frac{x^4\,dx}{x^{15}-1} \Rightarrow \int \frac{x^4\,dx}{(x^5)^3-1}$

put $x^5 = z$

$5x^4 dx = dz$

$$\int \frac{dz}{5(z^3-1)} = \frac{1}{5}\int \frac{dz}{(z-1)(z^2+z+1)} \quad \text{--- (1)}$$

Now $\frac{1}{(z-1)(z^2+z+1)} = \frac{A}{z-1} + \frac{Bz+C}{z^2+z+1}$

$1 = A(z^2+z+1) + (Bz+C)(z-1)$

$z=1$ then $3A = 1$ or $A = \frac{1}{3}$

$z=0$ then $A - C = 1$ or $C = -\frac{2}{3}$

$z=-1$ then $A + (C-B)(-2) = 1$

$A - 2C + 2B = 1$

$2B = 1 - \frac{1}{3} - \frac{4}{3} \Rightarrow \frac{-2}{3}$ OR $B = -\frac{1}{3}$

$$\int \frac{1\,dz}{(z-1)(z^2+z+1)} = \int \frac{1}{3(z-1)} + \int \frac{-\frac{1}{3}z - \frac{2}{3}}{z^2+z+1}$$

$$= \int \frac{dz}{3(z-1)} - \frac{1}{3}\int \frac{z+2}{z^2+z+1}$$

$I_1 - I_2$

continue (15)

$$I_1 = \int \frac{dz}{3(z-1)} = \frac{1}{3}\ln|z-1|$$

$$I_2 = -\frac{1}{3}\int \frac{z+2}{z^2+z+1} = -\frac{1}{3}\left[\int \frac{\frac{1}{2}(2z+1)}{z^2+z+1} + \int \frac{3\ dz}{2(z^2+z+1)}\right]$$

$$I_2 = -\frac{1}{3}\left[\frac{1}{2}\ln(z^2+z+1) + \frac{3}{2}\times\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2z+1}{\sqrt{3}}\right)\right]$$

Thus: $I_1 - I_2$

$$\frac{1}{3}\ln|z-1| - \frac{1}{3}\left[\frac{1}{2}\ln(z^2+z+1) + \sqrt{3}\tan^{-1}\left(\frac{2z+1}{\sqrt{3}}\right)\right]$$

put in ①

$$\frac{1}{5}\left[\frac{1}{3}\ln|z-1| - \frac{1}{3}\left[\frac{1}{2}\ln(z^2+z+1) + \sqrt{3}\tan^{-1}\left(\frac{2z+1}{\sqrt{3}}\right)\right]\right]$$

$$\frac{1}{15}\left[\frac{1}{2}\ln\frac{(z-1)^2}{z^2+z+1} - \sqrt{3}\tan^{-1}\left(\frac{2z+1}{\sqrt{3}}\right)\right] + c$$

where $z = t^5$

(16) $\int \frac{\sin 2x \, dx}{\sin^4 x + \cos^4 x}$ $\Rightarrow$ Divide N^r and D^r by $\cos^4 x$

$$\Rightarrow \int \frac{2\tan x \sec^2 x}{1 + \tan^4 x} dx$$

$\tan^2 x = t$

$2\tan x \sec^2 x \, dx = dt$

$\therefore \int \frac{dt}{1+t^2} \Rightarrow \tan^{-1} t + c \Rightarrow \tan^{-1}(\tan^2 x) + c$

(17) $\int \frac{(x^2-1)\,dx}{x\cdot\sqrt{x^4+3x^2+1}}$

$\Rightarrow \int \frac{x^2-1}{x^2\cdot\sqrt{x^2+3+\frac{1}{x^2}}}$

$\Rightarrow \int \frac{\left(1-\frac{1}{x^2}\right)dx}{\sqrt{\left(x+\frac{1}{x}\right)^2+1}} \Rightarrow$ put $x+\frac{1}{x}=t$

$\left(1-\frac{1}{x^2}\right)dx = dt$

$\Rightarrow \int \frac{dt}{\sqrt{t^2+1}} = \log\left(t+\sqrt{t^2+1}\right)$

$\Rightarrow \log\left(x+\frac{1}{x}+\sqrt{\left(x+\frac{1}{x}\right)^2+1}\right)+c$

$\Rightarrow \log\left(\frac{x^2+1+\sqrt{x^4+3x^2+1}}{x}\right)+c$

(18)

$$\int \frac{(e^{3x}+e^{x})\,dx}{e^{4x}-e^{2x}+1}$$ Divide N^r and D^r by e^{2x}

$$\Rightarrow \int \frac{e^{x}+e^{-x}}{e^{2x}-1+e^{-2x}}\,dx$$

$$\Rightarrow \int \frac{(e^{x}+e^{-x})\,dx}{(e^{x}-e^{-x})^2+1}$$

put $e^{x}-e^{-x}=t$

$$(e^{x}+e^{-x})\,dx=dt$$

$$\Rightarrow \int \frac{dt}{t^2+1} = \tan^{-1}t + c \Rightarrow \tan^{-1}(e^{x}-e^{-x})+c$$

$$\Rightarrow \tan^{-1}\left(e^{x}-\frac{1}{e^{x}}\right)+c$$

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(19) $\int \frac{x+\sin x}{1+\cos x}\,.dx$

$\Rightarrow \int \frac{x+2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\,dx$

$\Rightarrow \int \frac{x}{2}\cdot\sec^2\frac{x}{2} + \int \tan\frac{x}{2}\,dx$

using By parts.

$\Rightarrow \int \frac{x}{2}\cdot\sec^2\frac{x}{2}\,dx + \left[x\tan\frac{x}{2} - \int \frac{1}{2}\sec^2\frac{x}{2}\cdot x\,dx\right]$

$\Rightarrow \cancel{\int \frac{x}{2}\sec^2\frac{x}{2}\,dx} + x\tan\frac{x}{2} - \cancel{\int \frac{x}{2}\sec^2\frac{x}{2}\,dx} + c$

$\Rightarrow x\tan\frac{x}{2} + c$

(20) $\displaystyle\int \frac{x^2-1}{x^2+1} \cdot \frac{dx}{\sqrt{1+x^4}}$

$\Rightarrow \displaystyle\int \frac{(x^2-1)\,dx}{x\left(x+\frac{1}{x}\right) \cdot x \cdot \sqrt{x^2+\frac{1}{x^2}}}$

$I \Rightarrow \displaystyle\int \frac{\left(1-\frac{1}{x^2}\right)dx}{\left(x+\frac{1}{x}\right) \cdot \sqrt{\left(x+\frac{1}{x}\right)^2-2}}$

Put $x+\frac{1}{x} = t$ $\therefore$

$\left(1-\frac{1}{x^2}\right)dx = dt$

$I \therefore= \displaystyle\int \frac{dt}{t \cdot \sqrt{t^2-2}}$ $\left\{ \displaystyle\int \frac{1}{|x|\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\left(\frac{x}{a}\right)+c \right\}$

$I = \frac{1}{\sqrt{2}} \sec^{-1}\left(\frac{t}{\sqrt{2}}\right) + c$

$I = \frac{1}{\sqrt{2}} \sec^{-1}\left(\frac{x^2+1}{\sqrt{2}\,x}\right) + c$

OR

$I = \frac{1}{\sqrt{2}} \cos^{-1}\left(\frac{\sqrt{2}\,x}{x^2+1}\right) + c$

(21) $$\int \frac{\left(x+\sqrt{1+x^2}\right)^{15}}{\sqrt{1+x^2}}\,dx$$

$$x+\sqrt{1+x^2}=t$$

$$\left(1+\frac{2x}{2\sqrt{1+x^2}}\right)dx=dt$$

$$\frac{\left(x+\sqrt{1+x^2}\right)}{\sqrt{1+x^2}}\,dx=dt$$

$$\frac{dx}{\sqrt{1+x^2}}=\frac{dt}{x+\sqrt{1+x^2}}$$

$$\frac{dx}{\sqrt{1+x^2}}=\frac{dt}{t}$$

Thus: $$\int \frac{t^{15}}{}\times\frac{dt}{t}=\int t^{14}\,dt$$

$$\Rightarrow \frac{t^{15}}{15}+c$$

$$\Rightarrow \frac{\left(x+\sqrt{1+x^2}\right)^{15}}{15}+c$$

(22) $\int \frac{dx}{5-4\sin x+3\cos x}$

put $\sin x = \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}$ and $\cos x = \frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$

$$5-4\sin x+3\cos x = 5-4\left(\frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)+3\left(\frac{1-\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}\right)$$

$$= \frac{5+5\tan^2\frac{x}{2}-8\tan\frac{x}{2}+3-3\tan^2\frac{x}{2}}{1+\tan^2\frac{x}{2}}$$

$$5-4\sin x+3\cos x = \frac{2\tan^2\frac{x}{2}-8\tan\frac{x}{2}+8}{1+\tan^2\frac{x}{2}}$$

Thus:

$$\int \frac{dx}{5-4\sin x+3\cos x} = \int \frac{1+\tan^2\frac{x}{2}}{2\tan^2\frac{x}{2}-8\tan\frac{x}{2}+8}$$

$$= \int \frac{\sec^2\frac{x}{2}\,dx}{2\left(\tan\frac{x}{2}-2\right)^2}$$

put $\tan\frac{x}{2}-2 = t$

$\Rightarrow \frac{1}{2}\sec^2\frac{x}{2}\,dx = dt$

$\Rightarrow \int \frac{dt}{t^2} = -\frac{1}{t}+c$

$\Rightarrow \frac{-1}{\left(\tan\frac{x}{2}-2\right)}+c$

$\Rightarrow \frac{1}{\left(2-\tan\frac{x}{2}\right)}+c$

(23) $$\int \frac{dx}{4 - 3\cos^2 x + 5\sin^2 x}$$

Divide N^r and D^r by $\cos^2 x$

$$\int \frac{\sec^2 x \, dx}{4\sec^2 x - 3 + 5\tan^2 x}$$

$$\int \frac{\sec^2 x \, dx}{9\tan^2 x + 1}$$

put $\tan x = t$

$\sec^2 x \, dx = dt$

$$\int \frac{dt}{9t^2+1} = \frac{1}{9} \int \frac{dt}{t^2 + (\frac{1}{3})^2}$$

$$\Rightarrow \frac{1}{9}\left[3 \tan^{-1}(3t)\right] + C$$

$$\Rightarrow \frac{1}{3} \tan^{-1}(3\tan x) + C$$

(24) $$\int \frac{2x^2+41x-91}{(x-1)(x+3)(x-4)}\,dx$$

using partial fraction

$$\frac{2x^2+41x-91}{(x-1)(x+3)(x-4)} = \frac{A}{x-1} + \frac{B}{x+3} + \frac{C}{x-4}$$

$$2x^2+41x-91 = A(x+3)(x-4) + B(x-1)(x-4) + C(x-1)(x+3)$$

$x=4$ then $21C = 105$ $\therefore C=5$

$x=-3$ then $28B = -196$ $\therefore B=-7$

$x=1$ then $-12A = -48$ $\therefore A=4$

$$\int \frac{2x^2+41x-91}{(x-1)(x+3)(x-4)} = \int \frac{4\,dx}{x-1} - \int \frac{7\,dx}{x+3} + \int \frac{5\,dx}{x-4}$$

$$= 4\ln|x-1| - 7\ln|x+3| + 5\ln|x-4| + c$$

$$= \ln\left(\frac{(x-1)^4\cdot(x-4)^5}{(x+3)^7}\right) + c$$

(25) $\int \frac{dx}{\sin\frac{x}{2}\cdot\sqrt{\cos^3\frac{x}{2}}}$

$\int \frac{\sqrt{\cos\frac{x}{2}}}{\sin\frac{x}{2}\cdot\cos^2\frac{x}{2}}$

$I = \int \frac{\sqrt{\cos\frac{x}{2}}\cdot\sin\frac{x}{2}}{\sin^2\frac{x}{2}\cdot\cos^2\frac{x}{2}} \Rightarrow \int \frac{\sqrt{\cos\frac{x}{2}}\cdot\sin\frac{x}{2}\,dx}{(1-\cos^2\frac{x}{2})\cos^2\frac{x}{2}}$

put $\cos\frac{x}{2} = t^2$

$-\sin\frac{x}{2}\cdot\frac{1}{2}\,dx = 2t\,dt$

$\Rightarrow \sin\frac{x}{2}\,dx = -4t\,dt$

$I = \int \frac{t\cdot(-4t\,dt)}{(1-t^4)t^4} = \int \frac{4t^2\,dt}{(t^4-1)t^4}$

$I = \int \frac{4\,dt}{(t^2-1)(t^2+1)t^2}$

using partial fraction

$\frac{4}{(t^2-1)t^2(t^2+1)} = \frac{A}{t^2-1} + \frac{B}{t^2} + \frac{C}{t^2+1}$

$4 = A(t^2)(t^2+1) + B(t^4-1) + Ct^2(t^2-1)$

$t=0$ then $B=-4$

$t=1$ then $2A = 4 \therefore A=2$

$t=2$ then $20A + 15B + 12C = 4$

$40 - 60 + 12C = 4$

$C = 2$

$\int \frac{4}{(t^2-1)t^2(t^2+1)} = \int \frac{2\,dt}{t^2-1} - \int \frac{4\,dt}{t^2} + \int \frac{2\,dt}{t^2+1}$

$= \log\frac{t-1}{t+1} + \frac{4}{t} + 2\tan^{-1}t$

$\Rightarrow \frac{4}{t} + 2\tan^{-1}t - \log\left|\left(\frac{1+t}{1-t}\right)\right| + C$

$\Rightarrow \frac{4}{\sqrt{\cos\frac{x}{2}}} + 2\tan^{-1}\sqrt{\cos\frac{x}{2}} - \log\left|\frac{1+\sqrt{\cos\frac{x}{2}}}{1-\sqrt{\cos\frac{x}{2}}}\right| + C$