BINOMIAL THEOREM

Factorial Notation

$n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot \ldots \ldots (n-1) \cdot n$

$1! = 1$, $2! = 1 \times 2 = 2$, $3! = 1 \times 2 \times 3 = 6$

$4! = 1 \times 2 \times 3 \times 4 = 24$, $5! = 120$, $6! = 720$

$7! = 5040$

$\boxed{0! = 1}$

${}^nC_r$: $n \in N$, $r \in W$, $\boxed{n \geq r}$

$${}^nC_r = \frac{n!}{(n-r)!\, r!}$$

eg $${}^nC_0 = \frac{n!}{(n-0)!\, 0!} = \frac{n!}{n! \cdot 1} = 1$$

$${}^nP_r = \frac{n!}{(n-r)!}$$, $n \in N$, $r \in W$, $n \geq r$

① $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \ldots (n-2)(n-1) \times n$

$n! = (n-1)! \times n = n(n-1)(n-2)!$

$$7! = 7\times 6! = 7\cdot 6\cdot 5! = 7\cdot 6\cdot 5\cdot 4!$$

$$n! = n(n-1)! = n(n-1)(n-2)!$$

$$^7C_5 = \frac{7!}{(7-5)!\,5!} = \frac{7!}{5!\,2!} = \frac{7\times 6\times \cancel{5!}}{\cancel{5!}\times 2} = 21$$

② $$^nC_r = \frac{n}{r}\,{}^{n-1}C_{r-1} = \frac{n}{r}\cdot\frac{(n-1)}{(r-1)}\,{}^{n-2}C_{r-2} = \cdots$$

$$^{19}C_8 = \frac{19}{8}\cdot {}^{18}C_7 = \frac{19}{8}\cdot\frac{18}{7}\cdot {}^{17}C_6 = \frac{19}{8}\cdot\frac{18}{7}\cdot\frac{17}{6}\,{}^{16}C_5$$

$$= \cdots$$

③ $$\frac{^nC_r}{^nC_{r-1}} = \frac{n-(r-1)}{r} = \frac{n-r+1}{r}$$

$$\frac{^{17}C_6}{^{17}C_5} = \frac{17-5}{6} = \frac{12}{6} = 2$$

$$\frac{^nC_r}{^nC_{r-1}} = \frac{\dfrac{\cancel{n!}}{r!\,(n-r)!}}{\dfrac{\cancel{n!}}{(r-1)!\,(n-r+1)!}} = \frac{(n-r+1)!\,(r-1)!}{r!\,(n-r)!}$$

$$= \frac{(n-r+1)\cancel{(n-r)!}\,\cancel{(r-1)!}}{r\cancel{(r-1)!}\,\cancel{(n-r)!}} = \frac{n-r+1}{r}$$

$$\frac{^{18}C_7}{^{18}C_8} = \frac{1}{\dfrac{^{18}C_8}{^{18}C_7}} = \frac{1}{\dfrac{18-7}{8}} = \frac{8}{11}$$

④ $^{n}C_{r} + {^{n}C_{r-1}} = {^{n+1}C_{r}}$

eg: $^{2}C_{2} + {^{3}C_{2}} + {^{4}C_{2}} + \cdots + {^{19}C_{2}} = ?$

$^{3}C_{3} + {^{3}C_{2}} + {^{4}C_{2}} + {^{5}C_{2}} + \cdots + {^{18}C_{2}} + {^{19}C_{2}}$

$^{4}C_{3} + {^{4}C_{2}} + {^{5}C_{2}} + \cdots + {^{18}C_{2}} + {^{19}C_{2}}$

$^{5}C_{3} + {^{5}C_{2}} + \cdots + {^{18}C_{2}} + {^{19}C_{2}}$

$\vdots$

$^{18}C_{3} + {^{18}C_{2}} + {^{19}C_{2}}$

$^{19}C_{3} + {^{19}C_{2}} = {^{20}C_{3}}$

⑤ $^{n}C_{x} = {^{n}C_{y}} \Rightarrow x = y$ or $x + y = n$

$^{n}C_{x} = {^{n}C_{n-x}}$ $\quad ({^{n}C_{0}} = {^{n}C_{n-0}},\ {^{n}C_{1}} = {^{n}C_{n-1}})$

eg Find the Min Value of n if n! is divided by 81 ($n \in N$)

Sol^n: $\frac{n!}{3^4} \in I \Rightarrow \frac{9!}{3^4} \in I$

$9! = 3^4 \times \lambda$, $n = 9$

1. BINOMIAL EXPRESSION :

An algebraic expression consisting of two different terms is called a **Binomial Expression.**

e.g. (i) $x + y$ (ii) $x - y$ (iii) $3x - 2y$ (iv) $x^2 + \frac{1}{x^2}$ (v) $(x + y)^n$

(vi) $x+x^2$

Note :

(i) $x + 3x$ is not a binomial; it is a monomial

(ii) $x + y + z$ is trinomial.

(iii) The expression containing more than two different terms is multinomial.

(vii) $1+x$

(viii) $xy+x$

× $y+4y$ (Not two dissimilar terms)

$x+y+x^2$ ✓ (multinomial), $x-y^2+3x$ (Binomial)

2. BINOMIAL THEOREM :

The formula by which any positive integral index (power) of a binomial expression can be expanded in the form of a series is known as **Binomial Theorem**. (This theorem was given by Newton)

Binomial Theorem

- $n \in N$ (JEE)
- Any index (not in JEE) (negative or fractional index)

$(x+y)^n \longrightarrow$ Expansion of $(x+y)^n$

GENERAL EXPANSION OF BINOMIAL THEOREM :

Statement : $n \in N$

$$(x+y)^n = {}^nC_0\, x^n + {}^nC_1\, x^{n-1}\, y + {}^nC_2\, x^{n-2}y^2 + \ldots\ldots + {}^nC_r\, x^{n-r}y^r + \ldots\ldots + {}^nC_n y^n = \sum_{r=0}^{n} {}^nC_r\, x^{n-r}y^r.$$

$$(x+y)^n = {}^nC_0 \cdot x^n y^0 + {}^nC_1 \cdot x^{n-1} y^1 + {}^nC_2 \cdot x^{n-2} y^2 + \ldots\ldots + \ldots + {}^nC_{n-1}\, x^{n-(n-1)} \cdot y^{n-1} + {}^nC_n\, x^0 y^n$$

$$(x+y)^n = {}^nC_0\, x^n + {}^nC_1 \cdot x^{n-1} \cdot y + {}^nC_2\, x^{n-2} \cdot y^2 + \ldots + {}^nC_n\, x^0 y^n$$

- The number of terms in the expansion of $\boxed{(x+y)^n}$ is $(n+1)$ i.e. one more than the index.
- The sum of the indices of x & y in each term is n.
- Power of first variable (x) decreases while of second variable (y) increases.
- Binomial coefficients are also called **combinatorial coefficients.**
- Binomial coefficients of the terms equidistant from the begining and end are equal.
- r^{th} term from the beginning in the expansion of $(x+y)^n$ is same as r^{th} term from end in the expansion of $(y+x)^n$.

eg: Expand $(2x-3y)^4$

Soln: $${}^4C_0(2x)^4 + {}^4C_1(2x)^3(-3y) + {}^4C_2(2x)^2(-3y)^2 + {}^4C_3(2x)(-3y)^3 + {}^4C_4(-3y)^4$$

eg. Expand $\left(x+\frac{2}{x^2}\right)^8$

Soln: $${}^8C_0\, x^8 + {}^8C_1\, x^7\left(\frac{2}{x^2}\right) + \cdots + {}^8C_7\, x\left(\frac{2}{x^2}\right)^7 + {}^8C_8\left(\frac{2}{x^2}\right)^8$$

$(x+y)^2$: No. of terms $= 3$

$^nC_0,\ ^nC_1,\ ^nC_2,\ \ldots,\ ^nC_n$ — Binomial Coefficient or Combinatorial coeffi.

$$(x+y)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1} y + {}^nC_2 x^{n-2} y^2 + \cdots + {}^nC_{n-1} x y^{n-1} + {}^nC_n y^n$$

$$(y+x)^n = {}^nC_0 y^n + {}^nC_1 y^{n-1} x + {}^nC_2 y^{n-2} x^2 + \cdots + {}^nC_{n-1} y x^{n-1} + {}^nC_n x^n$$

eg Find the 7^{th} term from the end in $\left(\frac{x}{3} - x^2\right)^{10}$

$=$ 7^{th} term from the beginning in $\left(-x^2 + \frac{x}{3}\right)^{10}$

3. GENERAL TERM :

General term in the expansion of $(x + y)^n$ is $\mathbf{T_{r+1} = {}^nC_r \cdot x^{n-r} \cdot y^r}$, $0 \le r \le n$

$$(x+y)^n = \underbrace{{}^nC_0 x^n}_{T_1} + \underbrace{{}^nC_1 x^{n-1} y}_{T_2} + \underbrace{{}^nC_2 x^{n-2} y^2}_{T_3} + \dots + \underbrace{{}^nC_{n-1} x y^{n-1}}_{T_n} + \underbrace{{}^nC_n y^n}_{T_{n+1}}$$

$$\boxed{T_{r+1} = {}^nC_r \, x^{n-r} \cdot y^r}$$

1) Find the 7th term in the expansion of $\left(\frac{4x}{5} - \frac{5}{2x}\right)^9$

2) $\left(\frac{3}{x^2} - \frac{x^3}{6}\right)^7$ find 4th term from the end

Sol^n: $T_{r+1} = {}^9C_r \left(\frac{4x}{5}\right)^{9-r} \left(-\frac{5}{2x}\right)^r$

$r+1 = 7 \Rightarrow r = 6$, $T_7 = {}^9C_6 \left(\frac{4x}{5}\right)^3 \left(\frac{-5}{2x}\right)^6$

Ans

2) $\left(-\frac{x^3}{6} + \frac{3}{x^2}\right)^7$

$T_{r+1} = {}^7C_r \left(-\frac{x^3}{6}\right)^{7-r} \cdot \left(\frac{3}{x^2}\right)^r$, $r+1 = 4$

$r = 3$

3

Find the coefficient of x^{32} in the expansion of $\left(x^4 - \frac{1}{x^3}\right)^{15}$ [**Ans.** ${}^{15}C_4$]

4

Find the term independent of 'x' in the expansion of $\left(\frac{3x^2}{2} - \frac{1}{3x}\right)^9$ [**Ans.** T_7]

Solⁿ: $\left(x^4 - \frac{1}{x^3}\right)^{15}$

$$T_{r+1} = {}^{15}C_r \left(x^4\right)^{15-r} \left(-\frac{1}{x^3}\right)^r$$

$$T_{r+1} = {}^{15}C_r \cdot x^{60-4r} (-1)^r \cdot x^{-3r}$$

$$T_{r+1} = {}^{15}C_r \cdot x^{60-7r} (-1)^r$$

$$60-7r = 32 \Rightarrow r = \frac{28}{7} = 4$$

$$T_5 = {}^{15}C_4 (-1)^4 \cdot x^{32} = {}^{15}C_4 \cdot x^{32}$$

4 $$T_{r+1} = {}^9C_r \left(\frac{3x^2}{2}\right)^{9-r} \left(-\frac{1}{3x}\right)^r$$

$$T_{r+1} = {}^9C_r \cdot \left(\frac{3}{2}\right)^{9-r} \left(-\frac{1}{3}\right)^r \cdot \frac{x^{18-2r}}{x^r} \cdot x^{18-3r}$$

$$18-3r = 0 \Rightarrow r = 6$$

Ans: $T_{6+1} = T_7$

E(5) $\left(\sqrt[3]{\frac{a}{\sqrt{b}}}+\sqrt{\frac{b}{\sqrt[3]{a}}}\right)^{21}$ find the term contain same power of a and b. Find the power also. [**Ans.** 10th, 5/2]

$$T_{r+1} = {}^{21}C_r \left(\frac{a}{\sqrt{b}}\right)^{\frac{21-r}{3}} \cdot \left(\frac{b}{\sqrt[3]{a}}\right)^{r/2}$$

$$T_{r+1} = {}^{21}C_r \cdot \frac{a^{\frac{21-r}{3}}}{b^{\frac{21-r}{6}}} \cdot \frac{b^{r/2}}{a^{r/6}}$$

$$\boxed{T_{r+1} = {}^{21}C_r \cdot a^{\frac{21-r}{3}-\frac{r}{6}} \cdot b^{\frac{r}{2}-\left(\frac{21-r}{6}\right)}}$$

$$\frac{21-r}{3}-\frac{r}{6} = \frac{r}{2}-\left(\frac{21-r}{6}\right)$$

$$(21-r)\left(\frac{1}{3}+\frac{1}{6}\right) = r\left(\frac{1}{2}+\frac{1}{6}\right)$$

$$\frac{21-r}{2} = \frac{2r}{3} \Rightarrow 63 = 7r$$

$$r = 9$$

$$\boxed{T_{10}^{th},\ T_{10} = {}^{21}C_9 \cdot a^{5/2} \cdot b^{5/2}}$$

Ans

E(6) Find the coefficient of x^2 in the expansion of $(1 - x + 2x^2)\left(x + \frac{1}{x}\right)^{10}$

$x^2: \quad (1-x+2x^2)\left(x+\frac{1}{x}\right)^{10}$

$x^2: \quad (1-x+2x^2)\ {}^{10}C_r \cdot \underbrace{x^{10-r}\left(\frac{1}{x}\right)^r}$

$= \alpha$

$x^2: \quad (1-x+2x^2) \cdot {}^{10}C_r \cdot x^{10-2r}$

$x^2: \quad \alpha\left(x^{10-2r} - x^{11-2r} + 2x^{12-2r}\right)$

$10-2r=2 \qquad 11-2r=2 \qquad 12-2r=2$

$r=4 \qquad r=\frac{9}{2}\ \times \qquad r=5$

$x^2: \quad {}^{10}C_4 + 2\ {}^{10}C_5$

E(8) If 6^{th}, 7^{th}, 8^{th} terms in the expansion of $(x + y)^n$ are respectively 112,7 and $\frac{1}{4}$. Find x, y, n $\boxed{(x, y > 0)}$

$T_6 = 112 = {}^nC_5\, x^{n-5} y^5 \quad (r=5) \quad$ — (1)

$T_7 = 7 = {}^nC_6\, x^{n-6} y^6 \quad (r=6) \quad$ — (2)

$T_8 = \frac{1}{4} = {}^nC_7 \cdot x^{n-7} \cdot y^7 \quad (r=7) \quad$ — (3)

(2) ÷ (1)

$$\frac{{}^nC_6}{{}^nC_5} \cdot \frac{x^{n-6} y^6}{x^{n-5} y^5} = \frac{7}{112} = \frac{1}{16}$$

$$\frac{n-5}{6} \cdot \frac{y}{x} = \frac{1}{16} \quad \text{— (4)}$$

(3) ÷ (2)

$$\frac{{}^nC_7\, x^{n-7} y^7}{{}^nC_6\, x^{n-6} y^6} = \frac{1}{28} \Rightarrow \frac{n-6}{7} \cdot \left(\frac{y}{x}\right) = \frac{1}{28} \quad \text{— (5)}$$

Divide (4) ÷ (5)

$$\frac{(n-5).7}{(n-6).6} = \frac{\cancel{28}\,4}{\cancel{16}\,4}$$

$$4n - 20 = 6n - 36 \Rightarrow 2n = 16$$

$$\Rightarrow n = 8 \quad \text{— (6)}$$

From (4) $\left(\frac{3}{6}\right) \cdot \frac{y}{x} = \frac{1}{16} \Rightarrow x = 8y \quad$ — (7)

From (1) $112 = {}^nC_5\, x^{n-5} \cdot y^5$

$$112 = {}^8C_5 \cdot x^3 \cdot y^5 \qquad \left({}^8C_5 = \frac{8!}{5!\,3!} = \frac{8 \times 7 \times 6 \times \cancel{5!}}{\cancel{5!} \times 3!} = 56\right)$$

$$2 = (8y)^3 \cdot y^5$$

$$2 = 8^3 \cdot y^8 \Rightarrow y^8 = \frac{2}{2^9} = \left(\frac{1}{2}\right)^8 \Rightarrow y = \frac{1}{2}$$

$$x = 8y = 4$$

$$\left(x = 4, \quad y = 1/2, \quad n = 8\right)$$

NOTE:

$\rightarrow (x+y)^n = {}^nC_0 x^n + {}^nC_1 x^{n-1} y + \cdots + {}^nC_n y^n$

$$(x+y)^n = \sum_{r=0}^{n} {}^nC_r \, x^{n-r} y^r$$

$\rightarrow (1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \cdots + {}^nC_n x^n$

$$(1+x)^n = \sum_{r=0}^{n} {}^nC_r \, x^r$$

$$\boxed{T_{r+1} = {}^nC_r (1)^{n-r} \cdot x^r}$$

Coefficient of x^r in $(1+x)^n$ is ${}^nC_r$

E(9) Find the coefficient of x^m in $(1 + x)^m + (1 + x)^{m+1} + \ldots\ldots + (1 + x)^n$, $m \le n$.

Solⁿ:

$${}^{m}C_m + {}^{m+1}C_m + {}^{m+2}C_m + \cdots + {}^{n}C_m$$

$$\downarrow$$

$$\underbrace{{}^{m+1}C_{m+1} + {}^{m+1}C_m} + {}^{m+2}C_m + \cdots + {}^{n-1}C_m + {}^{n}C_m$$

$$\underbrace{{}^{m+2}C_{m+1} + {}^{m+2}C_m} + \cdots + {}^{n-1}C_m + {}^{n}C_m$$

$$\vdots$$

$$\underbrace{{}^{n-1}C_{m+1} + {}^{n-1}C_m} + {}^{n}C_m$$

$${}^{n}C_{m+1} + {}^{n}C_m = {}^{n+1}C_{m+1}$$

Find the sum of the series $\sum_{r=0}^{n}(-1)^r\cdot{}^nC_r\left[\frac{1}{2^r}+\frac{3^r}{2^{2r}}+\frac{7^r}{2^{3r}}+\frac{15^r}{2^{4r}}+\ldots\ldots\text{up to m terms}\right]$

Sol^n:

$$\sum_{r=0}^{n}(-1)^r\cdot{}^nC_r\left(\frac{1}{2}\right)^r+\sum_{r=0}^{n}(-1)^r\cdot{}^nC_r\cdot\left(\frac{3}{4}\right)^r+\sum_{r=0}^{n}(-1)^r\cdot{}^nC_r\left(\frac{7}{8}\right)^r+\ldots$$

$$\sum_{r=0}^{n}{}^nC_r\left(-\frac{1}{2}\right)^r+\sum_{r=0}^{n}{}^nC_r\left(-\frac{3}{4}\right)^r+\sum_{r=0}^{n}{}^nC_r\left(-\frac{7}{8}\right)^r+\ldots + \text{m-terms}$$

$$\downarrow$$

$$\left(1-\frac{1}{2}\right)^n+\left(1-\frac{3}{4}\right)^n+\left(1-\frac{7}{8}\right)^n+\ldots+\text{m-terms}$$

$$\left(\frac{1}{2}\right)^n+\left(\frac{1}{4}\right)^n+\left(\frac{1}{8}\right)^n+\ldots+\text{m-terms}$$

$$a=\left(\frac{1}{2}\right)^n,\quad r=\frac{\left(\frac{1}{4}\right)^n}{\left(\frac{1}{2}\right)^n}=\left(\frac{1}{2}\right)^n,\ \text{no of terms}=m$$

$$S_m=\frac{\left(\frac{1}{2}\right)^n\left(1-\left(\left(\frac{1}{2}\right)^n\right)^m\right)}{\left(1-\left(\frac{1}{2}\right)^n\right)}\qquad\left(S_n=\frac{a(1-r^n)}{1-r}\right)$$

$$\frac{\left(\frac{1}{2}\right)^n\left(1-\frac{1}{2^{mn}}\right)}{\left(1-\frac{1}{2^n}\right)}\ \text{Ans}$$

eg Find the Coefficient of x^7 in $(1+x^2)^5(1+2x)^4$

$$x^7: \quad (1+x^2)^5 \, (1+2x)^4$$

$$0 \le r_1 \le 5$$
$$0 \le r_2 \le 4$$

$$\left({}^5C_{r_1} (x^2)^{r_1} \right) \left({}^4C_{r_2} (2x)^{r_2} \right)$$

$${}^5C_{r_1} \cdot {}^4C_{r_2} \cdot 2^{r_2} \cdot x^{2r_1 + r_2}$$

$$2r_1 + r_2 = 7$$

r_1	0	1	2	3
r_2	7	5	3	1
	X	X		

$${}^5C_2 \cdot {}^4C_3 \cdot 2^3 + {}^5C_3 \cdot {}^4C_1 \cdot 2^1$$

(b) Find the coefficient of the term independent of x in the expansion

of $\left(\dfrac{x+1}{x^{2/3}-x^{1/3}+1}-\dfrac{x-1}{x-x^{1/2}}\right)^{10}$

Q. Find the number of rational terms in the expansion of $\left(2^{1/3} + 5\cdot 8^{\frac{1}{5}}\right)^{41}$

$$T_{r+1} = {}^{41}C_r \cdot 2^{\frac{41-r}{3}} \cdot \left(5\cdot 8^{\frac{1}{5}}\right)^r$$

$$T_{r+1} = {}^{41}C_r \cdot 2^{\frac{41-r}{3}} \cdot 5^r \cdot 2^{\frac{3r}{5}}$$

$\frac{r}{5} \in I$, $r = 0, 5, 10, 15, 20, 25, 30, 35, 40$

$\frac{41-r}{3} \in I \Rightarrow r = 5, 20, 35$

$r \in \{5, 20, 35\}$

T_6, T_{21}, T_{36} = rational terms