Area Under the Curve

Do yourself - 1 :

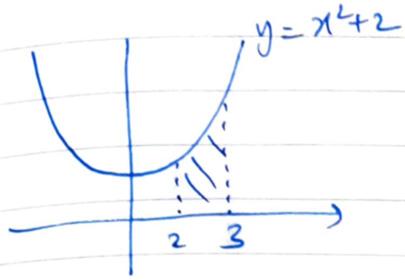
(i) Find the area bounded by $y = x^2 + 2$ above x-axis between $x = 2$ & $x = 3$.

(ii) Using integration, find the area of the curve $y = \sqrt{1-x^2}$ with co-ordinate axes bounded in first quadrant.

(iii) Find the area bounded by the curve $y = 2\cos x$ and the x-axis from $x = 0$ to $x = 2\pi$.

(iv) Find the area bounded by the curve $y = x|x|$, x-axis and the ordinates $x = -\frac{1}{2}$ and $x=1$.

(i). $y = x^2+2$, x axis b/w $x=2$ and $x=3$

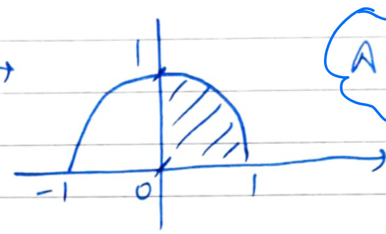


$$A = \int_2^3 x^2+2\, dx = \frac{x^3}{3}+2x\Big|_2^3$$

$$\Rightarrow A = \frac{25}{3} \text{ Sq. units. } \underline{\text{Ans}}.$$

(ii) $y = \sqrt{1-x^2}$, coordinate axes in I quadrant.

$y = \sqrt{1-x^2} \Rightarrow y \geq 0$.
$x^2+y^2 = 1$
$\therefore$ semicircle



$A = \frac{\pi(1)^2}{4} = \frac{\pi}{4}$ $\underline{\text{Ans}}$

using integration: $A = \int_0^1 \sqrt{1-x^2}\, dx$

$$= \frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}\frac{x}{1}\Big|_0^1$$

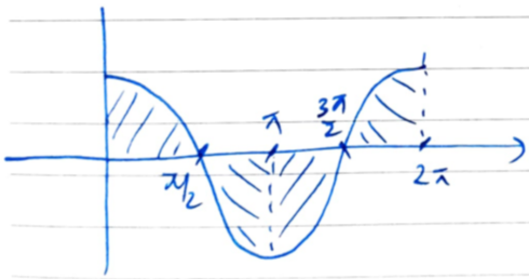
$$= 0 + \frac{1}{2}\left(\pi/2\right) - (0+0) = \pi/4 \text{ Sq. units}$$

first quadrant.

(iii) Find the area bounded by the curve y = 2cosx and the x-axis from x = 0 to x = 2π.

(iv) Find the area bounded by the curve y = x|x|, x-axis and the ordinates $x = -\frac{1}{2}$ and x=1.

(iii). $y = 2\cos x$, x axis b/w $x = 0$ and $x = 2\pi$

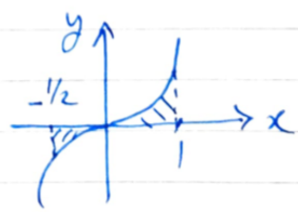


We have four symmetric parts of equal area.

$$\therefore A = 4 \cdot \int_0^{\pi/2} 2\cos x \, dx.$$

$$\Rightarrow A = 8 \cdot \int_0^{\pi/2} \cos x \, dx = 8 \text{ Sq. units Ans}$$

(iv) $y = x|x| = \begin{cases} x^2 & x \geq 0 \\ -x^2 & x < 0 \end{cases}$



$$A = -\int_{-1/2}^{0} (-x^2)\,dx + \int_0^1 x^2\,dx$$

$$\Rightarrow A = \frac{x^3}{3}\Big|_{-1/2}^{0} + \frac{x^3}{3}\Big|_0^1 = \frac{1}{24} + \frac{1}{3} = \frac{9}{24} = \frac{3}{8} \text{ Sq. units Ans}$$

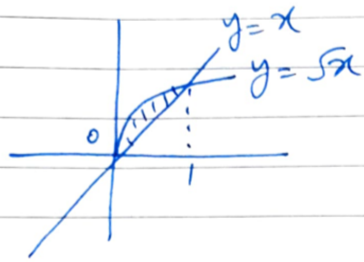
Do yourself - 2 :

(i) Find the area bounded by $y = \sqrt{x}$ and $y = x$.

(ii) Find the area bounded by the curves $x = y^2$ and $x = 3 - 2y^2$.

(iii) Find the area of the region bounded by the curves $x = \frac{1}{2}$, $x = 2$, $y = \log x$ and $y = 2^x$.

(i) b/w $y = \sqrt{x}$ and $y = x$.



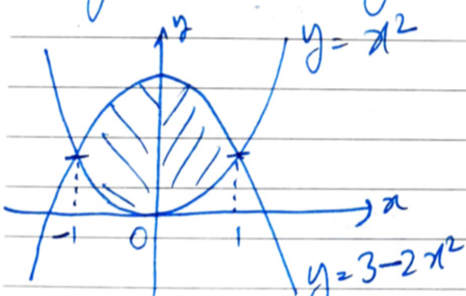
$x = \sqrt{x} \Rightarrow x = 0,\ x = 1$

$A = \int_0^1 \sqrt{x} - x\,dx \Rightarrow A = \frac{2}{3}x^{3/2} - \frac{x^2}{2}\Big|_0^1$

$\Rightarrow A = \frac{2}{3} - \frac{1}{2} = \frac{1}{6}$ Sq. units Ans

(ii) $x = y^2$ and $x = 3 - 2y^2$

Area bounded between these curves will be equal to area bounded between $y = x^2$ and $y = 3 - 2x^2$



$y = x^2 = 3 - 2x^2 \Rightarrow 3x^2 = 3$

$\Rightarrow x = 1,\ x = -1$

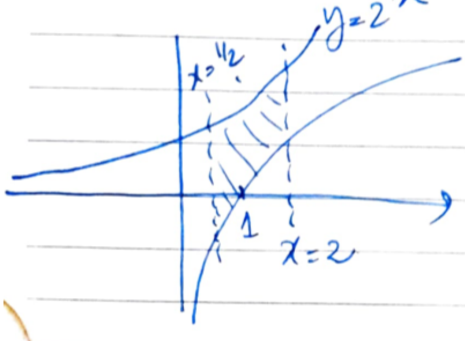
$\therefore A = 2 \cdot \int_0^1 3 - 2x^2 - x^2\,dx$

$\Rightarrow A = 2\left(\int_0^1 3 - 3x^2\,dx\right)$

$\Rightarrow A = 2\left(3x - x^3\right)\Big|_0^1 = 4$ Sq. units.

(iii) Find the area of the region bounded by the curves $x = \frac{1}{2}$, $x = 2$, $y = \log x$ and $y = 2^x$.

(iii) $x = 1/2, \quad x = 2, \quad y = \ln x, \quad y = 2^x.$



$$A = \int_{1/2}^{2} 2^x - \ln x \, dx$$

$$\Rightarrow A = \frac{2^x}{\ln 2} - (x \cdot \ln x - x)\Big|_{1/2}^{2}$$

$$\Rightarrow A = \frac{4}{\ln 2} - \frac{\sqrt{2}}{\ln 2} - \left(2\ln 2 - 2 - \left(\frac{1}{2}\ln\frac{1}{2} - \frac{1}{2}\right)\right)$$

$$\Rightarrow A = \frac{4 - \sqrt{2}}{\ln 2} - \frac{5}{2}\ln 2 + \frac{3}{2} \text{ sq. units } \underline{\text{Ans}}.$$

Do yourself : 3

(i) Find the area inside the circle $x^2-2x + y^2 - 4y + 1 = 0$ and outside the ellipse $x^2-2x+4y^2-16y+13= 0$

Sol^n Circle: $x^2-2x+y^2-4y+1=0$

$\Rightarrow (x-1)^2+(y-2)^2=4 \longrightarrow (\mathrm{I})$

Ellipse: $x^2-2x+4y^2-16y+13=0$

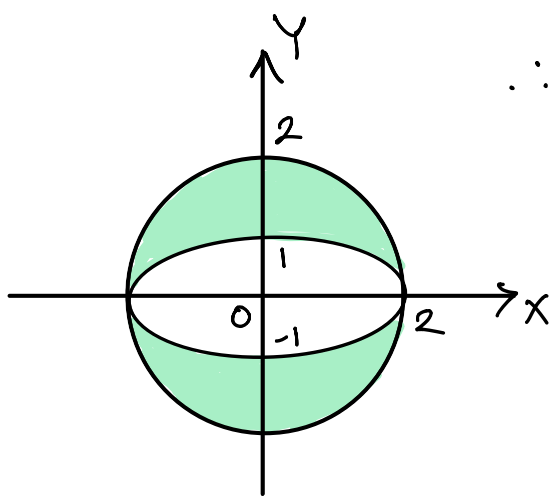
$\Rightarrow (x^2-2x)+4(y^2-4y)+13=0$

$\Rightarrow (x-1)^2+4(y-2)^2=4$

$\Rightarrow \dfrac{(x-1)^2}{4}+\dfrac{(y-2)^2}{1}=1 \longrightarrow (\mathrm{II})$

Shifting origin to $(1,2)$, we get

circle: $X^2+Y^2=4$; Ellipse: $\dfrac{X^2}{4}+\dfrac{Y^2}{1}=1$



$\therefore$ Required Area

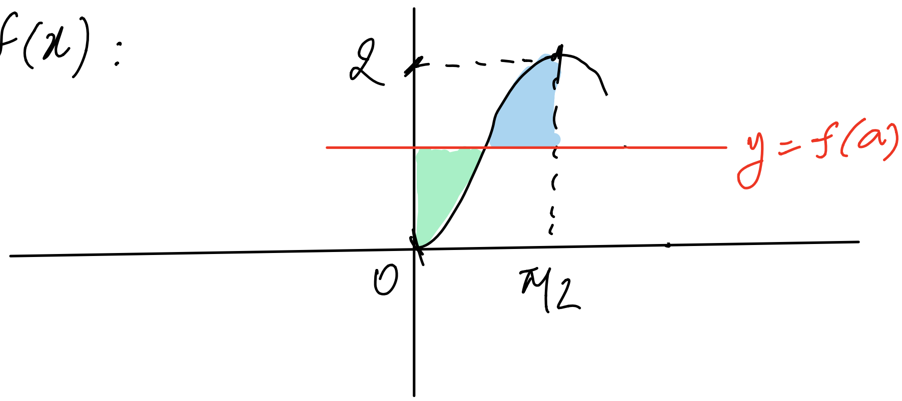
$= \pi(2)^2-\pi(2)(1)$

$= 2\pi$ Sq. units Ans.

Do yourself - 4 :

(i) Find the value of 'a' $(0 < a < \frac{\pi}{2})$ for which the area bounded by the curve $f(x) = \sin^3 x + \sin x$, $y = f(a)$ between $x = 0$ & $x = \frac{\pi}{2}$ is minimum.

Sol^n $f(x)$:

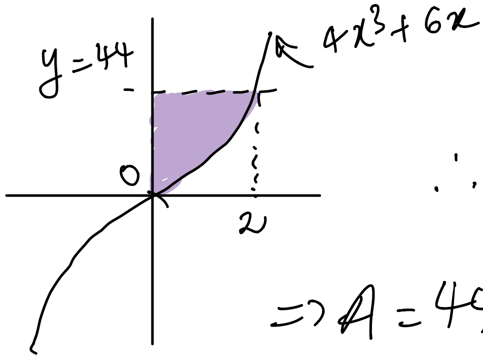


for Amin : $a = \frac{0+\pi/2}{2} = \pi/4$ Ans

Do yourself - 5 :

(i) Find the area bounded by the inverse of bijective function $f(x) = 4x^3 + 6x$, the x-axis and the ordinates $x = 0$ & $x = 44$.

Sol^n Reqd. Area = Area b/w $y = 4x^3 + 6x$, y axis b/w $y = 0$ and $y = 44$.



At $y = 44 = 4x^3 + 6x$
$\Rightarrow x = 2$

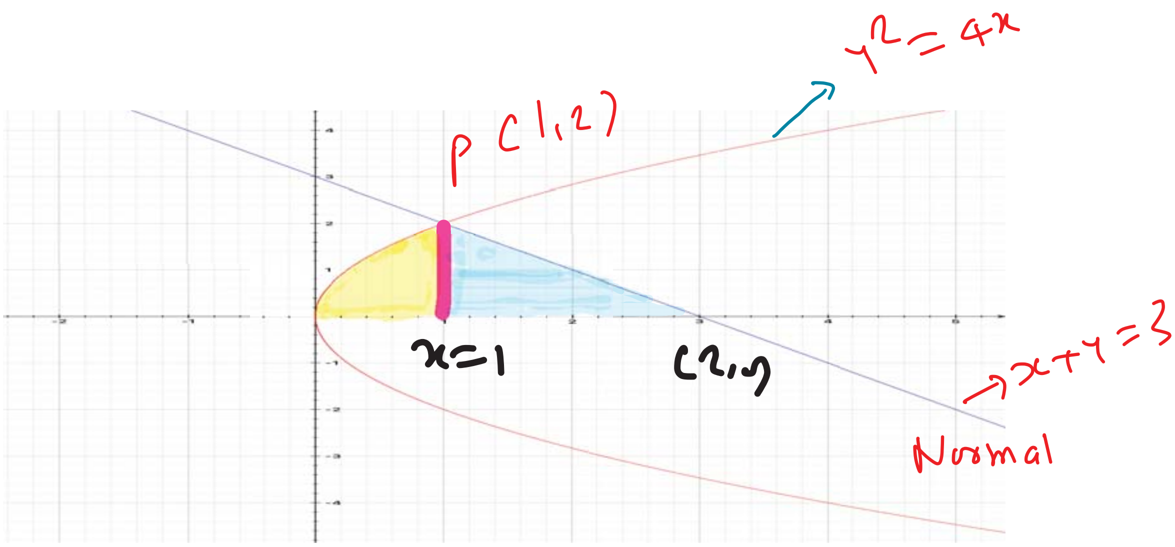
$\therefore A = \int_0^2 44 - (4x^3 + 6x)\,dx$

$\Rightarrow A = 44x - x^4 - 3x^2 \Big|_0^2 = 60$ sq. units

EXERCISE (O-1).

1. The area bounded in the first quadrant by the normal at $(1, 2)$ on the curve $y^2 = 4x$, x-axis & the curve is given by:

(A) $\frac{10}{3}$ (B) $\frac{7}{3}$ (C) $\frac{4}{3}$ (D) $\frac{9}{2}$



$y^2 = 4x$; Slope of tangent at $(1,2)$ is

$$\frac{dy}{dx} = \frac{2}{y} = 1$$

$\Rightarrow$ Slope of Normal at $(1,2) = -1$

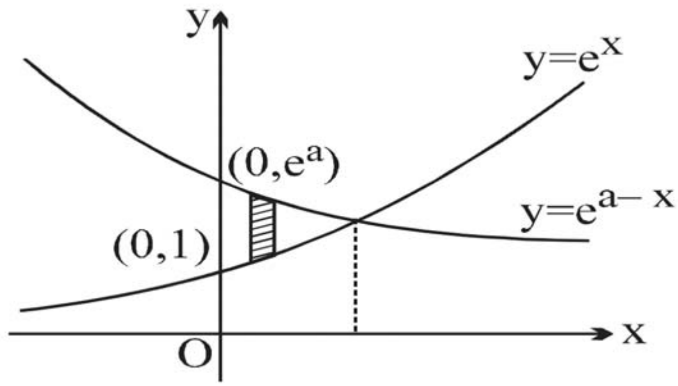
Eq^n of Normal $(y-2) = -1(x-1)$; $x+y=3$

$$\text{Required area} = \int_0^1 2\sqrt{x}\,dx + \underbrace{\frac{1}{2} \times 2 \times 2}_{\text{Area of triangle}}$$

$$A = 2\frac{\left(x^{3/2}\right)_0^1}{3/2} + 1 = \frac{4}{3} + 2 = \frac{10}{3} \text{ Ans}$$

2. Let 'a' be a positive constant number. Consider two curves $C_1: y=e^x$, $C_2: y=e^{a-x}$. Let S be the area of the part surrounding by C_1, C_2 and the y-axis, then $\lim\limits_{a\to 0} \frac{S}{a^2}$ equals

(A) 4 (B) 1/2 (C) 0 (D) 1/4



Solving

$e^x = a^{a-x}$

We get

$e^{2x} = a$

$\Rightarrow \quad x = \frac{a}{2}$

$$S = \int_0^a \left(e^{a-x} - e^x\right) dx = \left[-\left(e^a \cdot e^{-x} + e^x\right)\right]_0^{a/2}$$

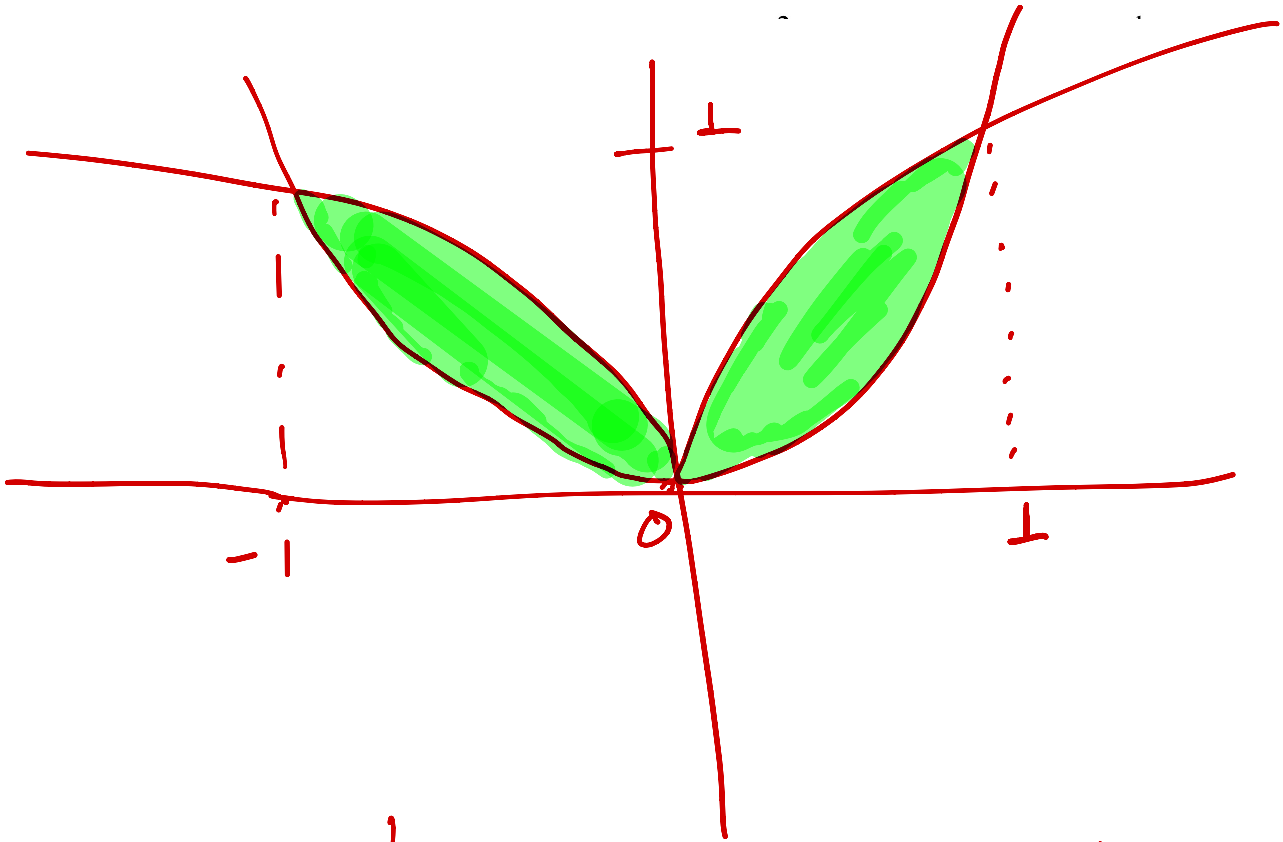
$$= \left(e^a + 1\right) - \left(e^{a/2} + e^{a/2}\right)$$

$$= e^a - 2 \cdot e^{a/2} + 1 = \left(e^{a/2} - 1\right)^2$$

$$\lim_{a\to 0} \frac{S}{a^2} = \lim_{a\to 0} \left(\frac{e^{a/2} - 1}{a}\right)^2 = \frac{1}{4} \text{ Ans}$$

3. The area of the region(s) enclosed by the curves $y = x^2$ and $y = \sqrt{|x|}$ is

(A) 1/3 (B) 2/3 (C) 1/6 (D) 1



$$\text{Area} = 2\int_0^1 \left(\sqrt{x} - x^2\right) dx$$

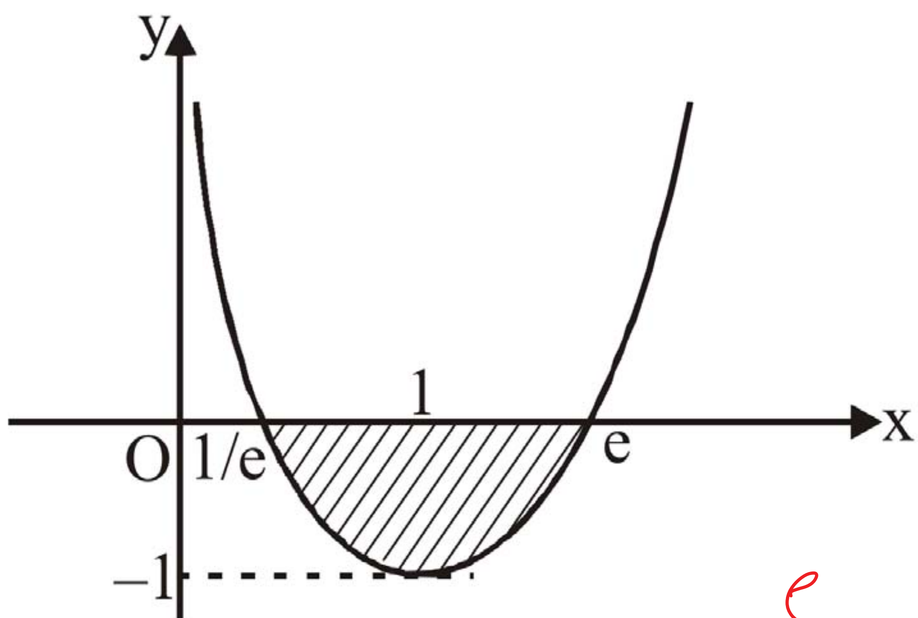
$$= 2\left[\frac{x^{3/2}}{3/2} - \frac{x^3}{3}\right]_0^1$$

$$= 2\left[\frac{2}{3} - \frac{1}{3}\right]$$

$$= \frac{2}{3}$$

4. Area enclosed by the graph of the function $y = ln^2x - 1$ lying in the 4^{th} quadrant is

(A) $\frac{2}{e}$ (B) $\frac{4}{e}$ (C) $2\left(e+\frac{1}{e}\right)$ (D) $4\left(e-\frac{1}{e}\right)$



$$A = \left| \int_{1/e}^{e} \left((\ln x)^2 - 1\right) dx \right| = \left| \int_{1/e}^{e} \underset{\mathrm{I}}{1} \cdot \underset{\mathrm{II}}{(\ln x)^2}\, dx - \left(e - \frac{1}{e}\right) \right|$$

Apply integration by parts

$$A = \left| \left[x(\ln x)^2\right]_{1/e}^{e} - 2\int_{1/e}^{e} \frac{(\ln x)}{\cancel{x}} \cdot \cancel{x}\, dx - \left(e - \frac{1}{e}\right) \right|$$

$$A = \frac{4}{e} \text{ Ans}$$

[Note
$\int (\ln x) dx = x \ln x - x$]

5. The area bounded by the curve $y = f(x)$ (where $f(x) \geq 0$), the co-ordinate axes & the line $x = x_1$ is given by $x_1 \cdot e^{x_1}$. Therefore $f(x)$ equals :

(A) e^x (B) xe^x (C) $xe^x - e^x$ (D) $xe^x + e^x$

$$A = \int_0^x f(x)\,dx$$

Given $\int_0^x f(x)\,dx = xe^x$

Differentiating

$$f(x) = e^x + xe^x$$

Ans.

6. The slope of the tangent to a curve $y=f(x)$ at $(x, f(x))$ is $2x+1$. If the curve passes through the point $(1, 2)$ then the area of the region bounded by the curve, the x-axis and the line $x=1$ is

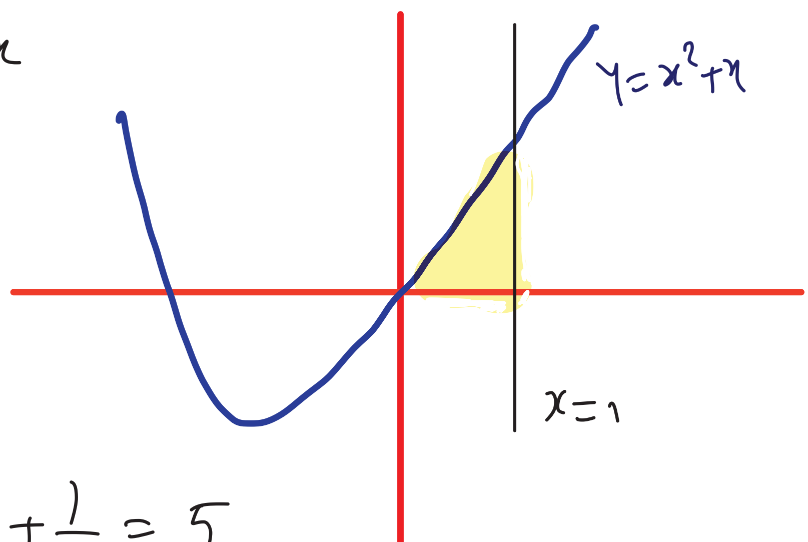
(A) $\frac{5}{6}$ (B) $\frac{6}{5}$ (C) $\frac{1}{6}$ (D) 1

$f'(x) = 2x+1$

Integrating $\Rightarrow f(x) = x^2 + x + c$

Curve passes through $(1,2) \Rightarrow 2 = 1+1+c$; $c = 0$

$\Rightarrow f(x) = x^2 + x$

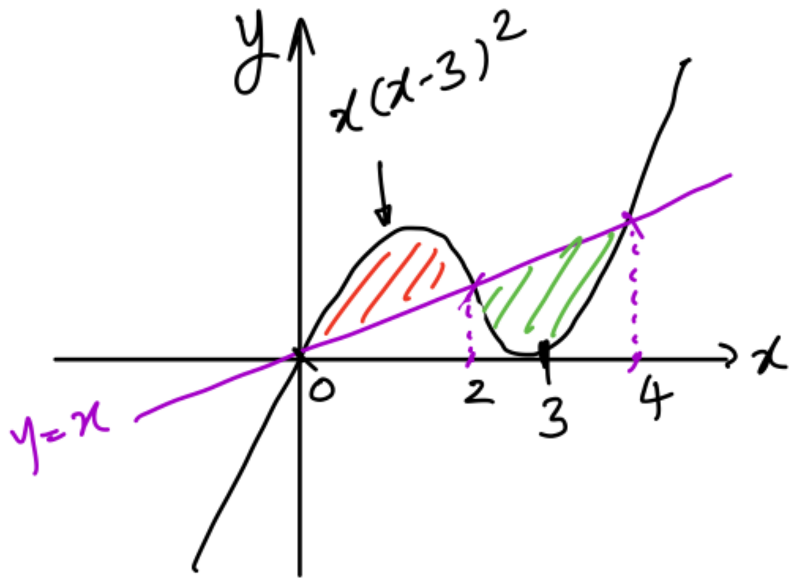


$A = \int_0^1 (x^2 + x)\, dx = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}$

$A = \frac{5}{6}$ Ans

7. The area bounded by the curves $y = x(x-3)^2$ and $y = x$ is (in sq. units) :

(A) 28 (B) 32 (C) 4 (D) 8



$x(x-3)^2 = x$

$\Rightarrow x = 0, 2, 4$

Required Area

$$= \int_0^2 x(x-3)^2 - x \, dx + \int_2^4 x - x(x-3)^2 \, dx$$

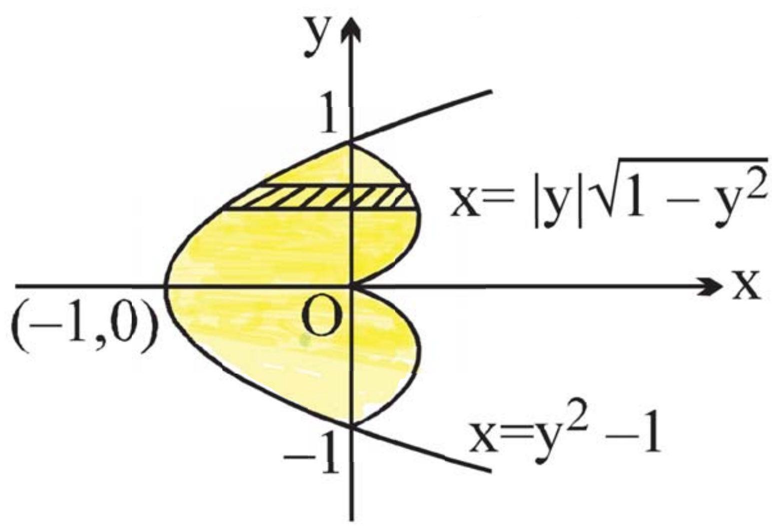
$$= \int_0^2 x^3 - 6x^2 + 8x \, dx - \int_2^4 x^3 - 6x^2 + 8x \, dx$$

$$= \frac{x^4}{4} - 2x^3 + 4x^2 \Big|_0^2 - \left(\frac{x^4}{4} - 2x^3 + 4x^2\right)\Big|_2^4$$

$= 4 + 4 = 8$ sq. units.

8. Area of the region enclosed between the curves $x = y^2 - 1$ and $x = |y|\sqrt{1-y^2}$ is

(A) 1 (B) 4/3 (C) 2/3 (D) 2



$$A = 2\int_0^1 \left[y\sqrt{1-y^2} - (y^2-1) \right] dy$$

$\hookrightarrow$ Put $1 - y^2 = t^2$

$A = 2$ ____

9. The curve $y = ax^2 + bx + c$ passes through the point $(1, 2)$ and its tangent at origin is the line $y = x$. The area bounded by the curve, the ordinate of the curve at minima and the tangent line is

(A) $\frac{1}{24}$ (B) $\frac{1}{12}$ (C) $\frac{1}{8}$ (D) $\frac{1}{6}$

Solⁿ $y = ax^2 + bx + c \longrightarrow$ Passes through $(1, 2)$

$\Rightarrow \quad 2 = a + b + c \longrightarrow (1)$

Also Tangent at origin : $y = x$

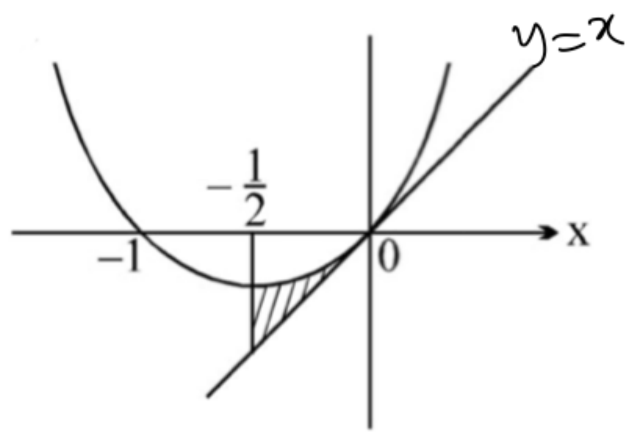
$\Rightarrow$ Curve passes through origin $\Rightarrow c = 0$

$\left.\frac{dy}{dx}\right|_{(0,0)} = 1 \Rightarrow \left.2ax + b\right|_{(0,0)} = 1 \Rightarrow b = 1$

$\therefore$ by Eqⁿ (1) : $a = 1$

$\Rightarrow$ Curve will be

$y = x^2 + x$



Minima at $\frac{dy}{dx} = 0$

$\Rightarrow 2x + 1 = 0 \Rightarrow x = -1/2$

$\therefore$ Reqd. Area $= \int_{-1/2}^{0} x^2 + x - x \, dx$

$= 1/24$ sq. units Ans

EXERCISE (0-2).

1. The area bounded by the curve $y = x\,e^{-x}$; $xy = 0$ and $x = c$ where c is the x-coordinate of the curve's inflection point, is

(A) $1 - 3e^{-2}$ (B) $1 - 2e^{-2}$ (C) $1 - e^{-2}$ (D) 1

Sol^n

$$y = x e^{-x}$$

$$y' = e^{-x} - x e^{-x}$$

$$y'' = -2e^{-x} + xe^{-x} = e^{-x}(x-2)$$

$y'' = 0 \Rightarrow x = 2$ is point of inflection $\Rightarrow c = 2$

$$\text{Area} = \int_0^2 \underset{\text{I}}{x}\, \underset{\text{II}}{e^{-x}}\, dx$$

$$= \left(-x e^{-x} - e^{-x}\right)_0^2 = 1 - 3e^{-2}$$

2. A function $y = f(x)$ satisfies the differential equation $\frac{dy}{dx} - y = \cos x - \sin x$, with initial condition that y is bounded when $x \to \infty$. The area enclosed by $y = f(x)$, $y = \cos x$ and the y-axis in the 1st quadrant

(A) $\sqrt{2} - 1$ (B) $\sqrt{2}$ (C) 1 (D) $\frac{1}{\sqrt{2}}$

Sol^n

$$\frac{dy}{dx} - y = \cos x - \sin x$$

$$I.F. = e^{\int -1 \cdot dx} = e^{-x}$$

Solution of D.E. is

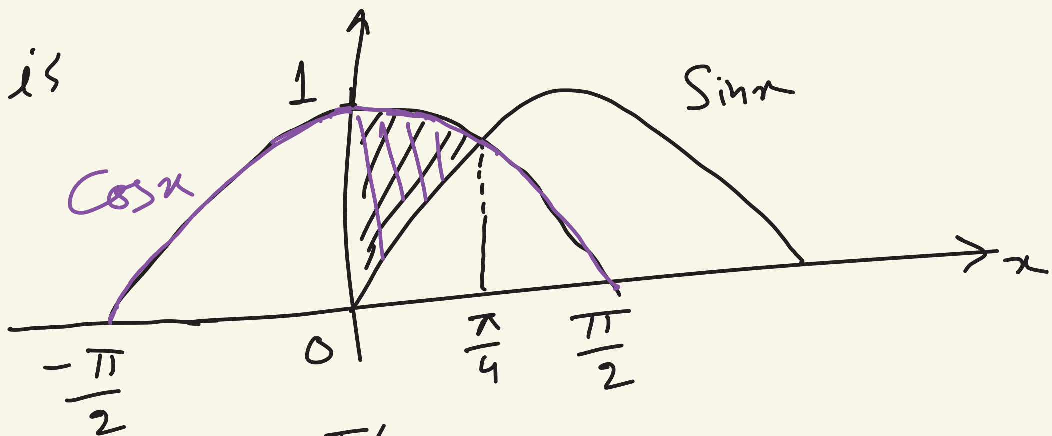
$$y e^{-x} = \int e^{-x}(\cos x - \sin x)\, dx$$

$$\boxed{y e^{-x} = e^{-x} \sin x + c}$$

as $x \to \infty$, $e^{-x} \to 0$ and y is bounded so 'c' must be zero

So $y = \sin x$

Now Area bounded by $f(x) = \sin x$, $y = \cos x$, y axis and I^{st} quad. is



$$\text{Area} = \int_0^{\pi/4} (\cos x - \sin x)\,dx$$

$$= (\sin x + \cos x)_0^{\pi/4} = \sqrt{2} - 1$$

3. If the area bounded between x-axis and the graph of $y = 6x - 3x^2$ between the ordinates $x = 1$ and $x = a$ is 19 square units then 'a' can take the value

(A) 4 or – 2

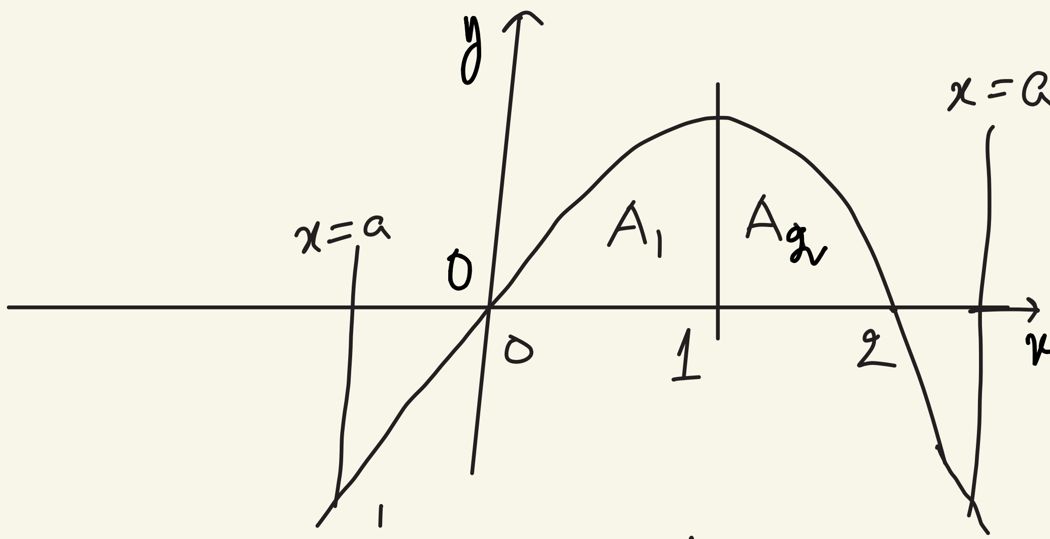
(B) two values are in (2, 3) and one in (–1, 0)

(C) two values one in (3,4) and one in (–2,–1)

(D) none of these

Solⁿ

$$y = 6x - 3x^2$$



$$\text{Area } A_1 = \int_0^1 (6x - 3x^2)\,dx = 2 \text{ Sq. unit}$$

$$\text{Area } A_2 = \int_1^2 (6x - 3x^2)\,dx = 2 \text{ Sq. unit}$$

So either $\boxed{a < 0 \text{ OR } a > 2}$ as $19 > A_1$ & $19 > A_2$

now
$$\int_a^0 (6x-3x^2)\,dx = -17 \quad \text{—(i)}$$

and
$$\int_2^a (6x-3x^2)\,dx = -17 \quad \text{—(ii)}$$

from (i) $\Rightarrow a^3 - 3a^2 + 17 = 0$

Let $g(a) = a^3 - 3a^2 + 17$

$g(-1)\,g(-2) < 0 \Rightarrow$ one root of $g(a) = 0$ lies in $(-2,-1)$

from (ii) $\Rightarrow a^3 - 3a^2 - 13 = 0$

Let $H(a) = a^3 - 3a^2 - 13$

$H(3)\,H(4) < 0 \Rightarrow$ one root of $H(a) = 0$ lies in $(3,4)$

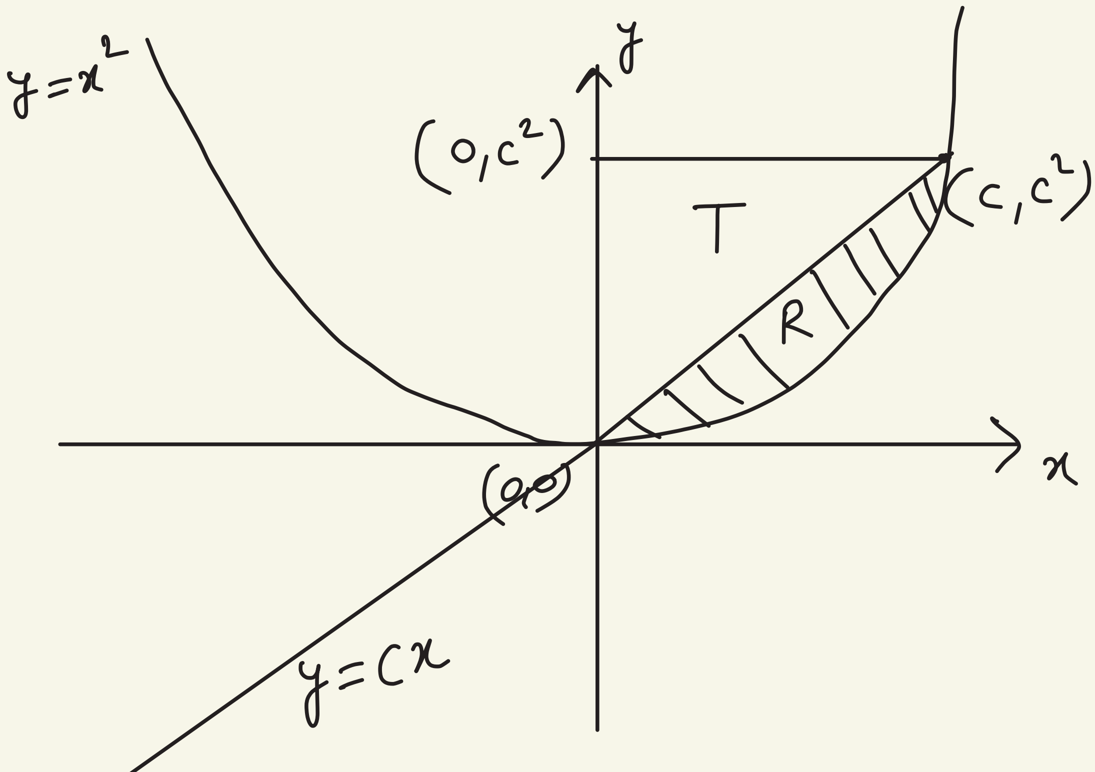
4. Let T be the triangle with vertices (0, 0), $(0, c^2)$ and (c, c^2) and let R be the region between $y = cx$ and $y = x^2$ where $c > 0$ then

(A) Area (R) $= \dfrac{c^3}{6}$

(B) Area of R $= \dfrac{c^3}{3}$

(C) $\lim\limits_{c\to 0^+} \dfrac{\text{Area (T)}}{\text{Area (R)}} = 3$

(D) $\lim\limits_{c\to 0^+} \dfrac{\text{Area (T)}}{\text{Area (R)}} = \dfrac{3}{2}$



$$\text{Area (T)} = \frac{1}{2} c^2 \cdot c = \frac{c^3}{2} \text{ sq. unit}$$

$$\text{Area (R)} = \int_0^c (cx - x^2)\,dx = \frac{c^3}{6} \text{ sq. unit}$$

5. Suppose f is defined from $R \to [-1, 1]$ as $f(x) = \dfrac{x^2-1}{x^2+1}$ where R is the set of real number. Then the statement which does not hold is

(A) f is many one onto

(B) f increases for $x > 0$ and decrease for $x < 0$

(C) minimum value is not attained even though f is bounded

(D) the area included by the curve $y = f(x)$ and the line $y = 1$ is π sq. units.

Soln $f : R \to [-1, 1]$

$$f(x) = \frac{x^2-1}{x^2+1} = 1 - \frac{2}{x^2+1}$$

$$1 \le x^2+1 < \infty \Rightarrow \boxed{-1 \le f(x) < 1}$$

$f(x)$ is even function $\Rightarrow$ Many one

Codomain $= [-1, 1]$ Range $= [-1, 1)$

$f(x)$ is many one and into

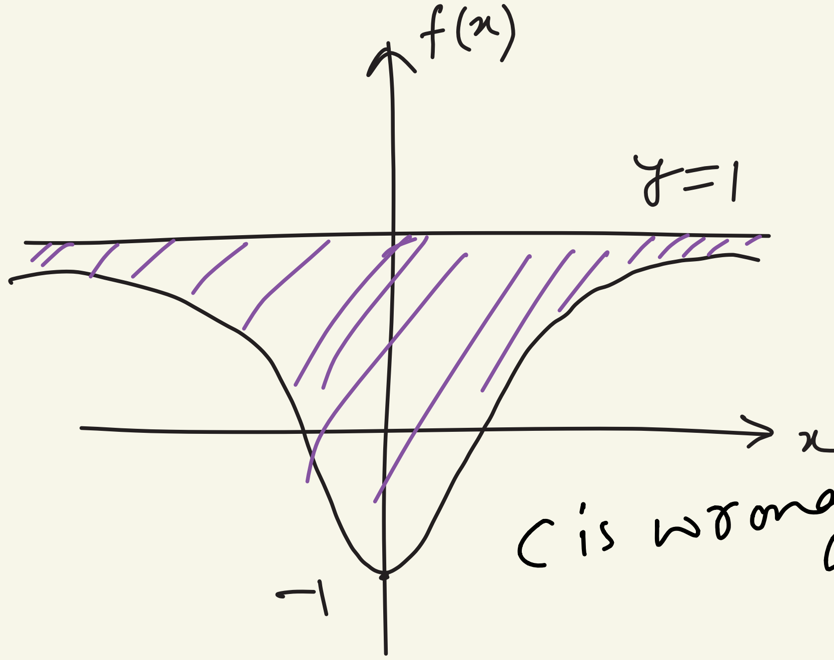
$\therefore$ A is wrong

$$f'(x) = \frac{4x}{(x^2+1)^2}$$

B is correct

$-\infty \leftarrow \; - \; 0 \; + \; \rightarrow \infty$

now FOR option D



C is wrong

Area between $y=f(x)$ & $y=1$

$$A = \int_{-\infty}^{\infty} 1 - \left(\frac{x^2-1}{x^2+1}\right) dx = \int_{-\infty}^{\infty} \frac{2\,dx}{x^2+1}$$

$$A = 2 \cdot \int_{-\infty}^{\infty} \frac{dx}{1+x^2} \Rightarrow A = 2\tan^{-1}x \Big|_{-\infty}^{\infty}$$

$$\Rightarrow A = 2\pi \text{ Sq. units}$$

D is wrong

$\therefore$ Ans: A, C, D

6. Which of the following statement(s) is/are True for the function

$f(x) = (x-1)^2(x-2) + 1$ defined on $[0, 2]$?

(A) Range of f is $\left[\frac{23}{27}, 1\right]$.

(B) The coordinates of the turning point of the graph of $y = f(x)$ occur at $(1, 1)$ and $\left(\frac{5}{3}, \frac{23}{27}\right)$.

(C) The value of p for which the equation $f(x) = p$ has 3 distinct solutions lies in interval $\left(\frac{23}{27}, 1\right)$.

(D) The area enclosed by $y = f(x)$, the lines $x = 0$ and $y = 1$ as x varies from 0 to 1 is $\frac{7}{12}$.

Sol^n $f(x) = (x-1)^2(x-2) + 1, \quad x \in [0,2]$

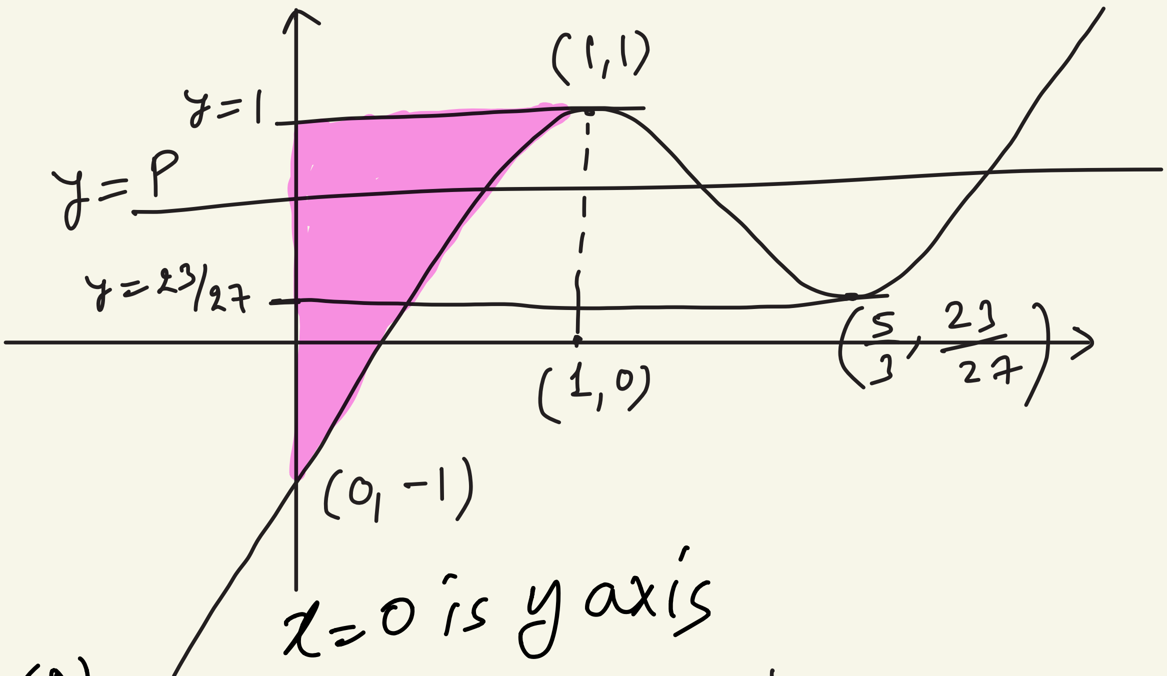
$$f(x) = (x-1)^3 - (x-1)^2 + 1$$

$$f'(x) = 3(x-1)^2 - 2(x-1)$$

$$f'(x) = (x-1)[3x-5]$$

$$f'(x) = 0 \text{ at } x = 1, 5/3$$

$$f(1) = 1 \quad f(5/3) = \frac{23}{27}$$



$x=0$ is y axis

(D)

$$\int_0^1 \left(1-f(x)\right)dx = \int_0^1 \left[(x-1)^2-(x-1)^3\right]dx$$

$$= \left[\frac{(x-1)^3}{3}-\frac{(x-1)^4}{4}\right]_0^1 = \frac{7}{12}$$

Range : $[-1,1]$

Ans: B, C, D

7. Let $f(x) = 2 - |x - 1|$ & $g(x) = (x - 1)^2$, then -

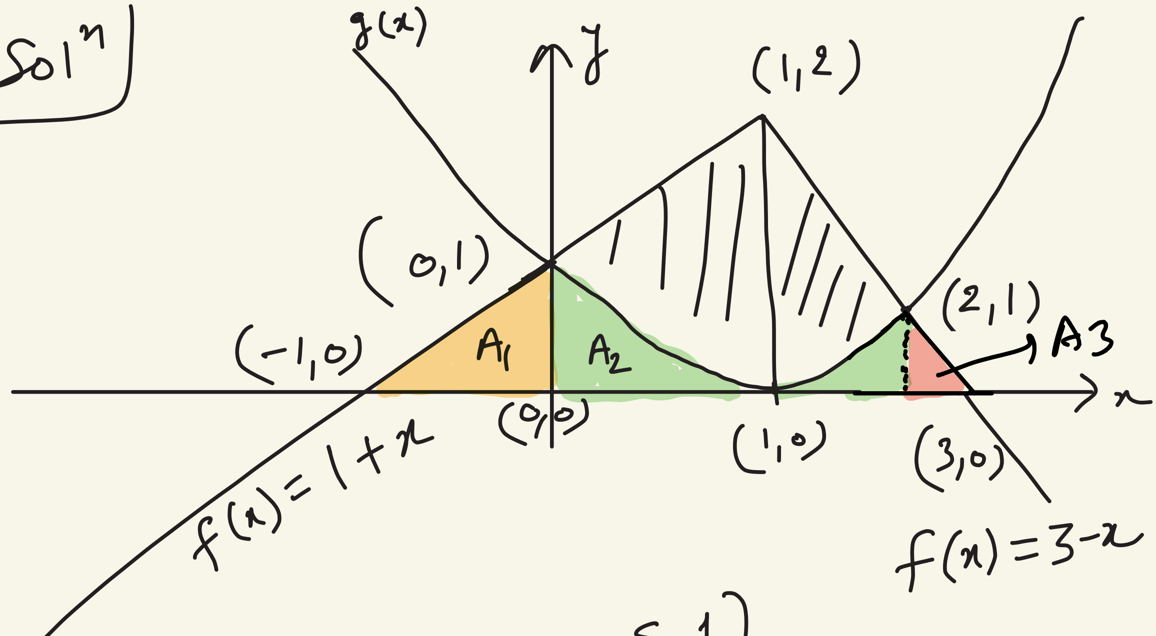
(A) area bounded by $f(x)$ & $g(x)$ is $\frac{7}{6}$

(B) area bounded by $f(x)$ & $g(x)$ is $\frac{7}{3}$

(C) area bounded by $f(x)$, $g(x)$ & x-axis is $\frac{5}{3}$

(D) area bounded by $f(x)$, $g(x)$ & x-axis is $\frac{5}{6}$

Sol^n



$$f(x) = \begin{cases} 1+x, & x \leq 1 \\ 3-x, & x \geq 1 \end{cases}$$

Area bounded b/w $f(x)$ & $g(x)$

$$= \int_0^1 \{(1+x) - (x-1)^2\} dx + \int_1^2 \{(3-x) - (x-1)^2\} dx$$

$$= \frac{7}{3} \text{ Sq. unit}$$

Area bounded by $f(x)$, $g(x)$ and x axis $= A_1 + A_2 + A_3$

$$= \int_{-1}^{0} (1+x)\,dx + \int_{0}^{2} (x-1)^2\,dx + \int_{2}^{3} (3-x)\,dx$$

$$= \frac{5}{3} \text{ sq. unit}$$

8. If A_1 denotes area of the region bounded by the curves $C_1 : y = (x-1)e^x$, tangent to C_1 at $(1,0)$ & y-axis and A_2 denotes the area of the region bounded by C_1 and co-ordinate axes in fourth quadrant, then -

(A) $A_1 > A_2$ (B) $A_1 < A_2$ (C) $2A_1 + A_2 = 2$ (D) $A_1 + 2A_2 = 4$

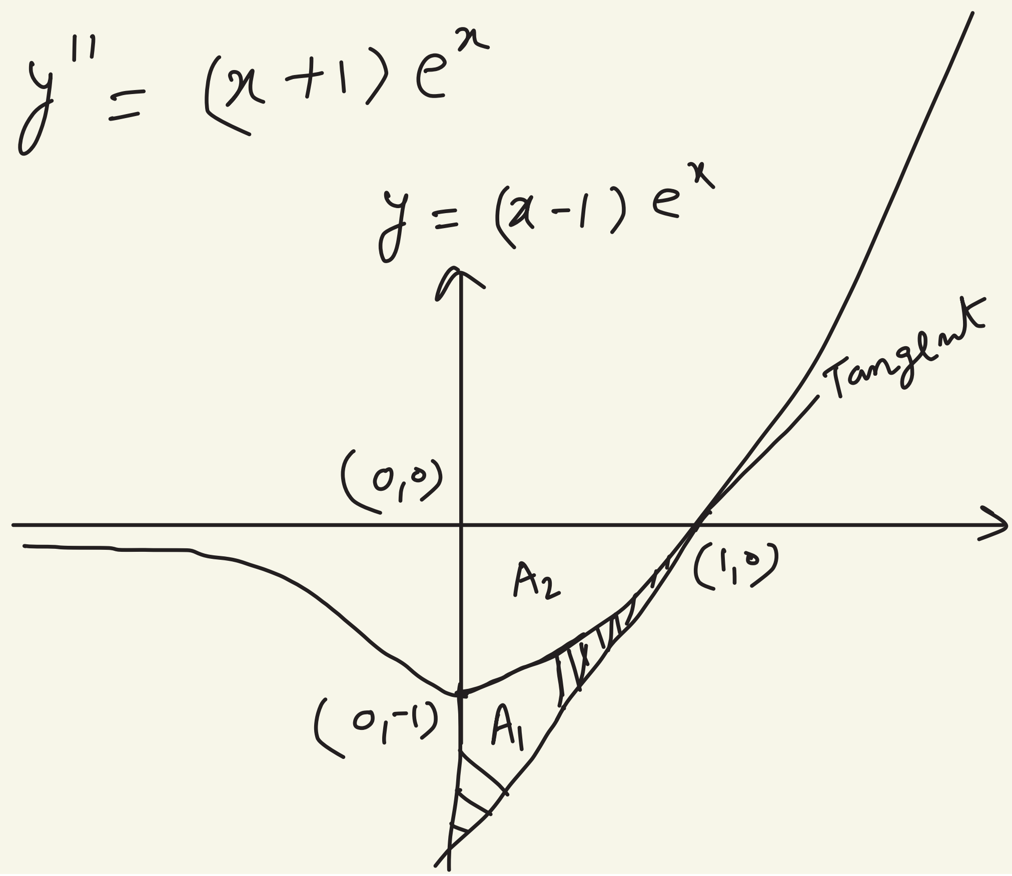
Sol^n $y = (x-1)e^x$ $(x \to \infty, y \to \infty)$

$y = xe^x - e^x$ $(x \to -\infty, y \to 0)$

$y' = xe^x$ (− before 0, + after 0)

$y'' = (x+1)e^x$

$y = (x-1)e^x$



$$A_2 = \left| \int_0^1 \left(\underset{I}{x}\, \underset{II}{e^x} - e^x \right) dx \right|$$

$$A_2 = \left| \left(xe^x - 2e^x \right)_0^1 \right| = e - 2 \approx .71$$

tangent to c_1 at $(1,0)$ is

$$y - 0 = e\,(x-1) \Rightarrow y = ex - e$$

$$A_1 = \int_0^1 \left\{ \left(xe^x - e^x\right) - \left(ex - e\right) \right\} dx$$

$$A_1 = \int_0^1 xe^x\,dx - \int_0^1 e^x\,dx - \left(e\,\frac{x^2}{2}\right)_0^1 + \left(ex\right)_0^1$$

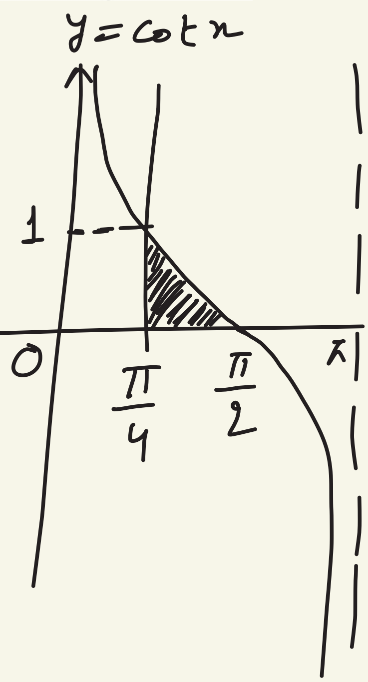
$$A_1 = \underbrace{2 - \frac{e}{2}}_{\approx .64} \Rightarrow 2A_1 = 4 - e$$

$$A_2 > A_1$$

9. Area bounded by the curve $y = \cot x$, $x = \frac{\pi}{4}$ and $y = 0$ is-

(A) $\int_0^{\pi/4} \tan\left(\frac{\pi}{4} - x\right)dx$ (B) $\frac{\pi}{4} - \int_0^1 \tan^{-1} x\,dx$ (C) $1 - \int_0^1 \tan^{-1} x\,dx$ (D) $\int_0^{\pi/4} \tan^{-1} x\,dx$

Soln



Area w.r. to y axis

$$= \int_0^1 (\cot^{-1} y)\,dy - \frac{\pi}{4}$$

$$= \int_0^1 \left(\frac{\pi}{2} - \tan^{-1} y\right) dy - \frac{\pi}{4}$$

$$= \frac{\pi}{2} - \int_0^1 \tan^{-1} y\,dy - \frac{\pi}{4}$$

$$= \frac{\pi}{4} - \int_0^1 \tan^{-1} y\,dy$$ — option (B)

Area w.r. to x axis $= \int_{\frac{\pi}{4}}^{\pi/2} \cot x \, dx$

Let $x = \frac{\pi}{2} - t$

$$= \int_{\frac{\pi}{4}}^{0} -\cot\left(\frac{\pi}{2} - t\right) dt$$

$$= \int_{0}^{\pi/4} \tan t \, dt$$

use $t \longrightarrow \frac{\pi}{4} - t$

$$= \int_{0}^{\pi/4} \tan\left(\frac{\pi}{4} - t\right) dt$$

$\longrightarrow$ option (A)

EXERCISE (S-1)

Area under the Curves
EXERCISE (S-1)
Solutions

1. Find the area bounded on the right by the line $x+y=2$, on the left by the parabola $y=x^2$ and below by the x-axis.

Solution:

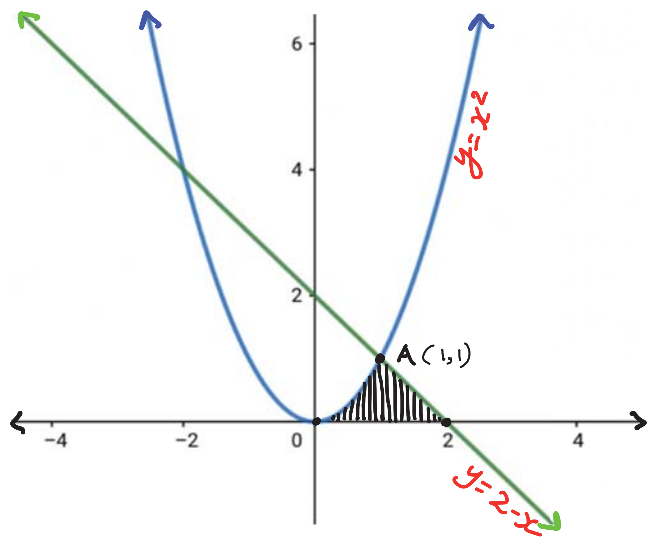
For point A

$y=x^2$ & $y=2-x$

$\Rightarrow 2-x=x^2$

$\Rightarrow x^2+x-2=0$

$\Rightarrow x=1$ and -2



Required Area:

$$= \int_0^1 x^2\,dx + \int_1^2 (2-x)\,dx$$

$$= \left[\frac{x^3}{3}\right]_0^1 + \left[2x-\frac{x^2}{2}\right]_1^2$$

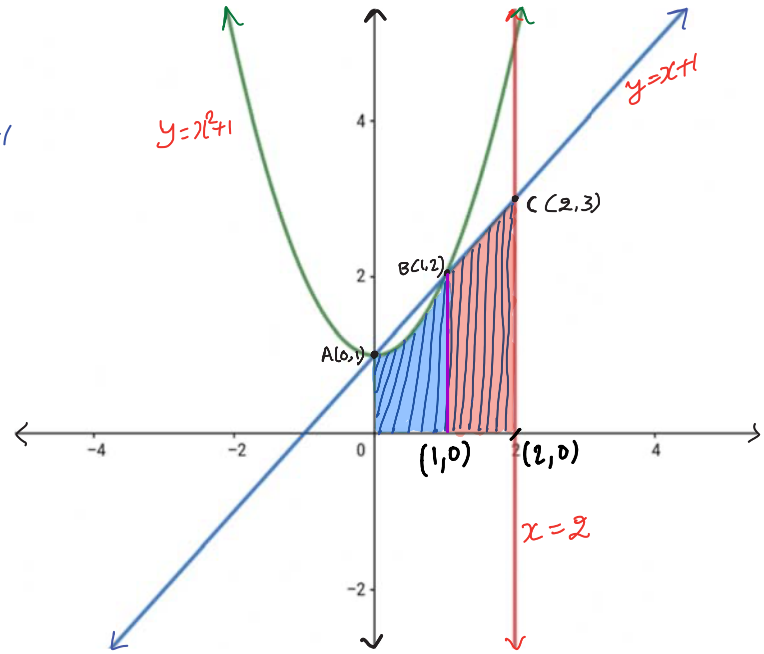
$$= \frac{1}{3} + (4-2)-\left(2-\frac{1}{2}\right)$$

$$= \frac{1}{3}+2-\frac{3}{2} = \frac{2+12-9}{6}$$

$$= \boxed{\frac{5}{6}} \text{ sq. units.}$$ Ans.

2. Find the area of the region $\{(x, y) : 0 \le y \le x^2 + 1, 0 \le y \le x + 1, 0 \le x \le 2\}$.

Solution:



For A, & B

$y = x^2+1$ and $y = x+1$

$\Rightarrow x+1 = x^2+1$

$\Rightarrow x^2 - x = 0$

$\Rightarrow x = 0$ & 1

$\Rightarrow A(0,1)$ & $B(1,2)$

For C

$y = x+1, x = 2$

$\Rightarrow y = 3$

$\Rightarrow C \equiv (2,3)$

Required Area:

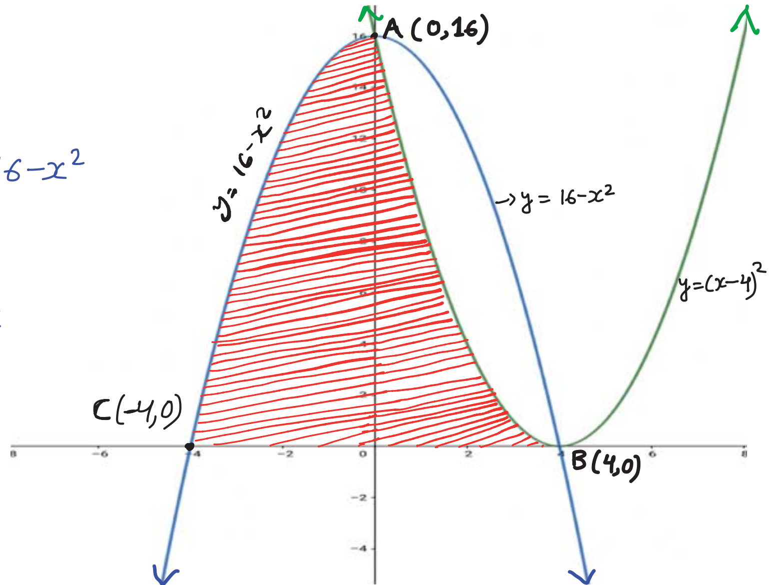
$$= \int_0^1 (x^2+1)\,dx + \int_1^2 (x+1)\,dx$$

$$= \left[\frac{x^3}{3} + x\right]_0^1 + \left[\frac{x^2}{2} + x\right]_1^2 = \frac{1}{3} + 1 + \left[4 - 3/2\right] = \frac{4}{3} + \frac{5}{2}$$

$$= \boxed{\frac{23}{6}} \text{ sq. units}$$

3. Find the area of the region bounded by curves $f(x) = (x - 4)^2$, $g(x) = 16 - x^2$ and the x-axis.

Solution:



For A & B

$y = (x-4)^2$ and $y = 16 - x^2$

$16 - x^2 = (x-4)^2$

$\Rightarrow 16 - x^2 = x^2 + 16 - 8x$

$\Rightarrow 2x^2 - 8x = 0$

$\Rightarrow 2x(x-4) = 0$

$\Rightarrow x = 0$ & 4

For C

$16 - x^2 = 0 \Rightarrow x = -4, 4$

$\Rightarrow C \cong (-4, 0)$

The area of the region bounded by curves $f(x) = (x-4)^2$ & $g(x) = 16 - x^2$ and the x-axis is:

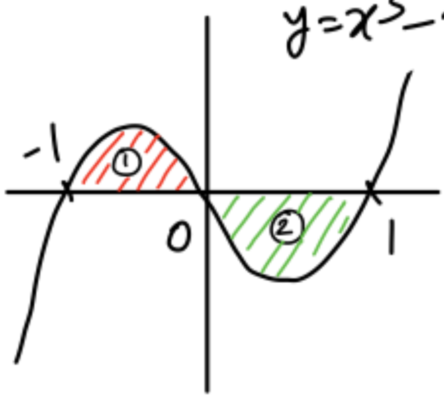
$$\int_{-4}^{0} (16 - x^2)\,dx + \int_{0}^{4} (x-4)^2\,dx$$

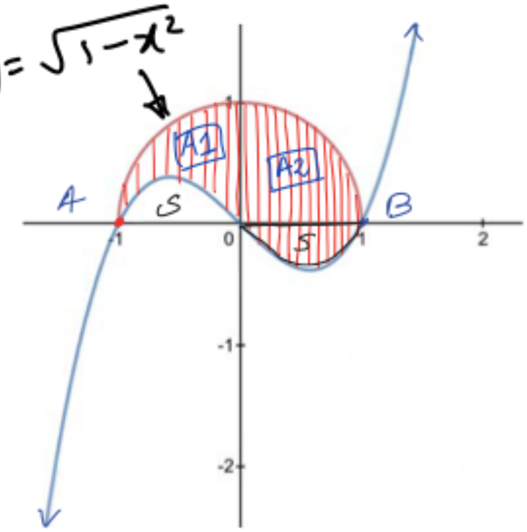
$$= \left[16x - \frac{x^3}{3}\right]_{-4}^{0} + \left[\frac{(x-4)^3}{3}\right]_{0}^{4}$$

$$= \left[0 - \left(-64 + \frac{64}{3}\right)\right] + \left[0 - \frac{(-4)^3}{3}\right] = \frac{128}{3} + \frac{64}{3}$$

$$= \frac{192}{3} = \boxed{64} \text{ Ans.}$$

4. Find the area bounded by the curves $y=\sqrt{1-x^2}$ and $y=x^3-x$. Also find the ratio in which the y-axis divided this area.





$y=f(x)=x^3-x$ is an odd function, which is symmetric about origin. Hence areas ① & ② are equal. Let this area be S

$$\therefore S=\int_{-1}^{0} x^3-x\,dx = \frac{x^4}{4}-\frac{x^2}{2}\Big|_{-1}^{0} = 1/4$$

$y=\sqrt{1-x^2}$ represents a semicircle of radius 1, above x axis as shown

(i) Area bounded by the curves $= \dfrac{\pi(1)^2}{2} - S + S$

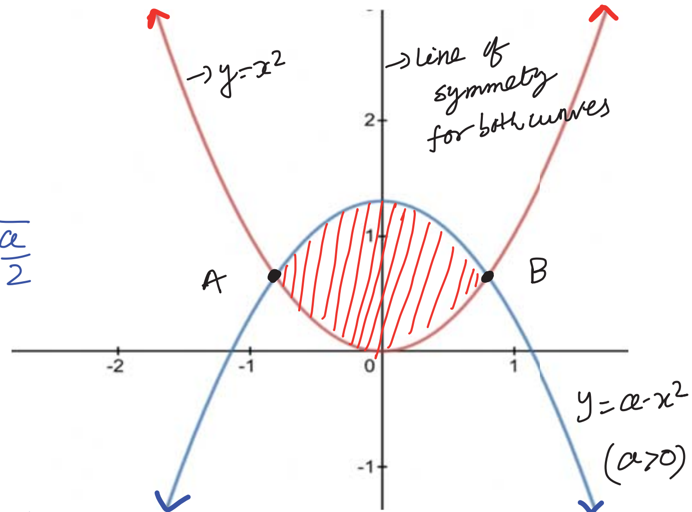
$= \pi/2$ Sq. units (Ans)

(ii) Ratio of Areas

$$\frac{A_1}{A_2} = \frac{\frac{\pi(1)^2}{4}-S}{\frac{\pi(1)^2}{4}+S} = \frac{\frac{\pi}{4}-1/4}{\pi/4+1/4} = \frac{\pi-1}{\pi+1}$$ Ans.

5. If the area enclosed by the parabolas $y = a - x^2$ and $y = x^2$ is $18\sqrt{2}$ sq. units. Find the value of 'a'.

Solution :



For point A & B

$x^2 = a - x^2$

$\Rightarrow 2x^2 = a \Rightarrow x = \pm\sqrt{\frac{a}{2}}$

Area enclosed by the curves $y = a - x^2$ & $y = x^2$

$$\therefore \int_{-\sqrt{a/2}}^{\sqrt{a/2}} \left[(a-x^2)-(x^2)\right] dx = 2\int_0^{\sqrt{a/2}} (a-2x^2)\,dx$$

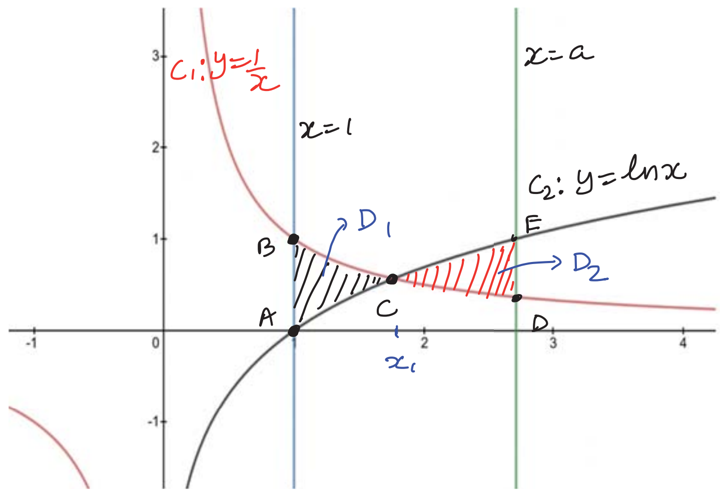
$$= 2\left[ax - \frac{2x^3}{3}\right]_0^{\sqrt{a/2}} = 2\left[\frac{a^{3/2}}{\sqrt{2}} - \frac{2a^{3/2}}{3\times 2\sqrt{2}}\right]$$

$$= \frac{2\sqrt{2}\,a^{3/2}}{3} \quad \text{\& it is given } 18\sqrt{2}$$

$$\Rightarrow \frac{2\sqrt{2}\,a^{3/2}}{3} = 18\sqrt{2} \Rightarrow a^{3/2} = 27 \Rightarrow \boxed{a=9} \quad \text{Ans.}$$

6. Consider two curves $C_1 : y = \frac{1}{x}$ and $C_2 : y = \ln x$ on the xy plane. Let D_1 denotes the region surrounded by C_1, C_2 and the line $x = 1$ and D_2 denotes the region surrounded by C_1, C_2 and the line $x = a$. If $D_1 = D_2$. Find the value of 'a'.

Solution



Let point of intersection of the curves $y = \frac{1}{x}$ and $y = \ln x$ is x_1

Given $D_1 = D_2$

$$\Rightarrow \int_1^{x_1} \left(\frac{1}{x} - \ln x\right) dx = \int_{x_1}^{a} \left(\ln x - \frac{1}{x}\right) dx$$

$$\Rightarrow \int_1^{x_1} \left(\frac{1}{x} - \ln x\right) dx + \int_{x_1}^{a} \left(\frac{1}{x} - \ln x\right) dx = 0$$

$$\Rightarrow \int_1^{a} \left(\frac{1}{x} - \ln x\right) dx = 0 \Rightarrow \Big[\ln|x| - x\ln x + x\Big]_1^a = 0$$

$$\Rightarrow (\ln|a| - a\ln a + a) - (1) = 0$$

$$\Rightarrow (1-a)\ln a = 1 - a \Rightarrow \ln a = 1 \quad (\text{if } a \neq 1)$$

$$\Rightarrow \boxed{a = e} \text{ Ans.}$$

7. Find the area enclosed between the curves : $y = \log_e(x+e)$, $x = \log_e(1/y)$ & the x-axis.

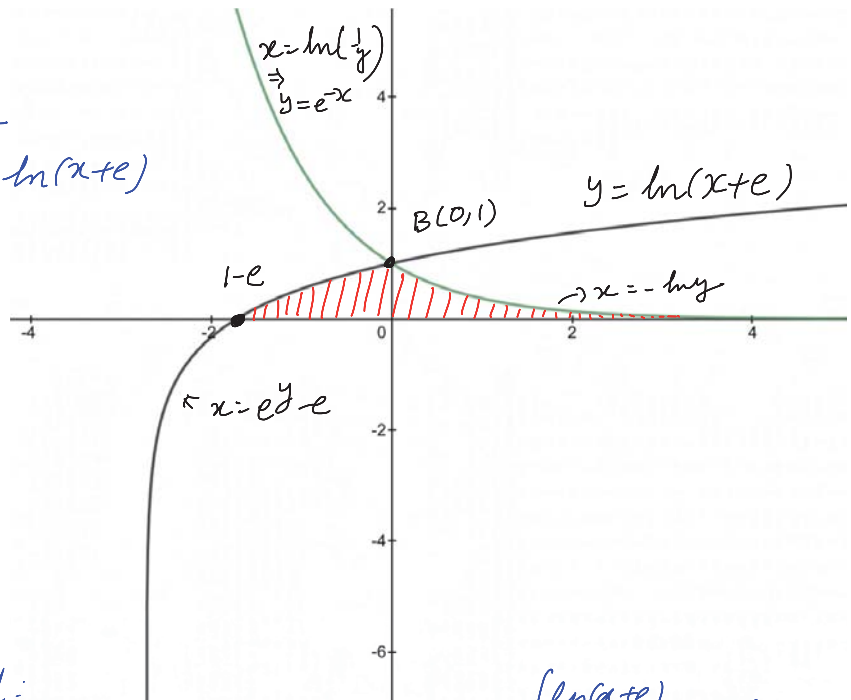
Solution:

For point B

$x = \ln\frac{1}{y}$ & $y = \ln(x+e)$

$\boxed{e^{-x} = \ln(x+e)}$

$x = 0$ & $y = 1$



Area enclosed:

$$\int_{1-e}^{0} \ln(x+e)\,dx + \int_{0}^{\infty} e^{-x}\,dx$$

$$\int \ln(x+e) = x\ln(x+e) - \int \frac{x}{x+e}\,dx$$

$$= x\ln(x+e) - x + e\ln(x+e) + C$$

$$= (x+e)\ln(x+e) - x + C$$

$$= \Big[(x+e)\ln(x+e) - x\Big]_{1-e}^{0} + \Big[-e^{-x}\Big]_{0}^{\infty}$$

$$= \big[e - (1-e)\big] + 1 = \boxed{2} \quad \textbf{Ans.}$$

Or

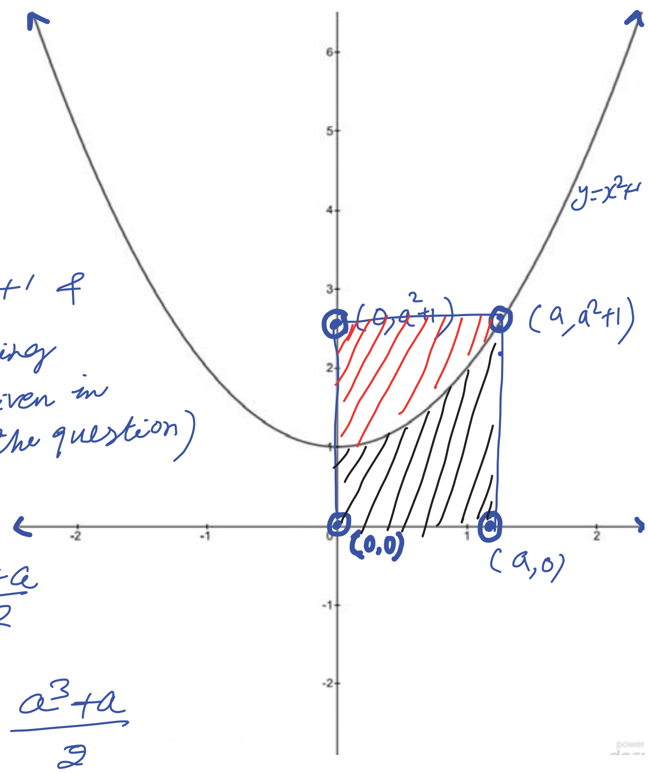
using horizontal strip

$$\text{Area} = \int_0^1 \Big(\big(-\ln y\big) - \big(e^y - e\big)\Big)dy = -\Big[y\ln y - y\Big]_0^1 - \Big[e^y - ey\Big]_0^1$$

$$= 1 - \big[(e - e) - 1\big] = \boxed{2} \text{ sq. unit}$$

8. Find the positive value of 'a' for which the parabola $y = x^2 + 1$ bisects the area of the rectangle with vertices $(0, 0)$, $(a, 0)$, $(0, a^2 + 1)$ and $(a, a^2 + 1)$.

Solution:



Area of rectangle

$= a \times (a^2+1) = a^3 + a$

$\Rightarrow$ Area betn $y = a^2+1$, $y = x^2+1$ & $x = 0$ is $\frac{a^3+a}{2}$ (According to given in the question)

$$\Rightarrow \int_0^a \left((a^2+1) - (x^2+1)\right) dx = \frac{a^3+a}{2}$$

$$\Rightarrow \left[(a^2+1)x - \frac{x^3}{3} - x\right]_0^a = \frac{a^3+a}{2}$$

$$\Rightarrow a^3 + a - \frac{a^3}{3} - a = \frac{a^3+a}{2}$$

$$\Rightarrow 4a^3 = 3a^3 + 3a \Rightarrow a^3 - 3a = 0$$

$$\Rightarrow a = 0 \text{ or } a = \pm\sqrt{3}$$

The positive value of a is $\boxed{\sqrt{3}}$.

9. Find the value of 'c' for which the area of the figure bounded by the curve, $y = 8x^2 - x^5$, the straight lines $x = 1$ & $x = c$ & the abscissa axis is equal to 16/3.

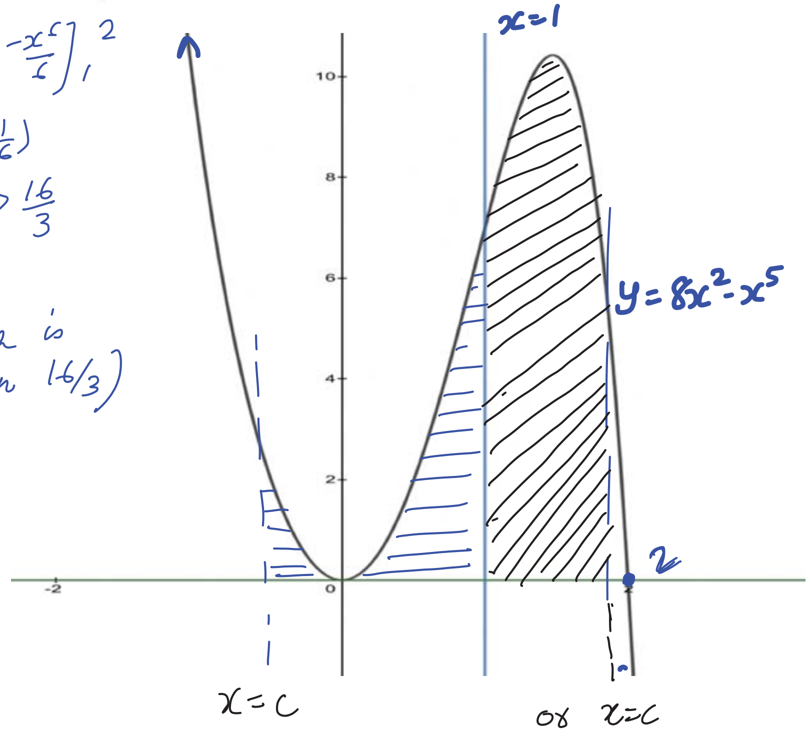
$$\int_1^2 (8x^2 - x^5)\,dx = \left[\frac{8x^3}{3} - \frac{x^6}{6}\right]_1^2$$

$$= \left(\frac{64}{3} - \frac{32}{3}\right) - \left(\frac{8}{3} - \frac{1}{6}\right)$$

$$= \frac{32}{3} - \frac{15}{6} = \frac{49}{6} > \frac{16}{3}$$

$\Rightarrow c < 2$

($\because$ If $c \geq 2$ Area is greater than $16/3$)



For $c < 2$

$$\Rightarrow \left|\int_1^c (8x^2 - x^5)\,dx\right| = \frac{16}{3}$$

$$\Rightarrow \left|\left[\frac{8x^3}{3} - \frac{x^6}{6}\right]_1^c\right| = \frac{16}{3} \Rightarrow \left|\frac{8c^3}{3} - \frac{c^6}{6} - \frac{15}{6}\right| = \frac{16}{3}$$

$\Rightarrow |16c^3 - c^6 - 15| = 32$, let $c^3 = t$, $t < 8$

$16t - t^2 - 15 = \pm 32$

or $t^2 - 16t - 17 = 0$

$\Rightarrow t = -1,\ t = 17$ (rejected)

$\Rightarrow c^3 = -1$

$\Rightarrow \boxed{c = -1}$ Ans

$\Rightarrow t^2 - 16t + 47 = 0$

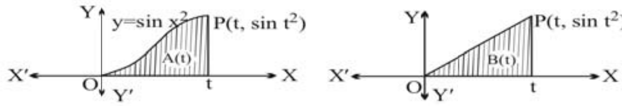
$t = \frac{16 \pm \sqrt{68}}{2}$

$t = 8 \pm \sqrt{17}$, $t = 8 + \sqrt{17}$ rejected

or $c^3 = 8 - \sqrt{17}$

$\Rightarrow \boxed{c = (8 - \sqrt{17})^{1/3}}$ Ans.

10. The figure shows two regions in the first quadrant.



A(t) is the area under the curve $y = \sin x^2$ from 0 to t and B(t) is the area of the triangle with vertices O, P and M(t, 0). Find $\lim\limits_{t \to 0} \dfrac{A(t)}{B(t)}$.

Solution:

$$A(t) = \int_0^t \sin t^2\, dt$$

$$B(t) = \frac{1}{2}\, OM \times MP = \frac{1}{2}\, t \cdot \sin t^2$$

$$\lim_{t\to 0} \frac{A(t)}{B(t)} = \lim_{t\to 0} \frac{\int_0^t \sin t^2\, dt}{\frac{1}{2} t \sin t^2}$$

$$= \lim_{t\to 0} \frac{2\frac{d}{dt}\left(\int_0^t \sin t^2\, dt\right)}{\frac{d}{dt}\left(t \sin t^2\right)}$$ (By using L. Hospital)

$$= \lim_{t\to 0} \frac{2 \times \sin t^2}{2t^2 \cos t^2 + \sin t^2}$$

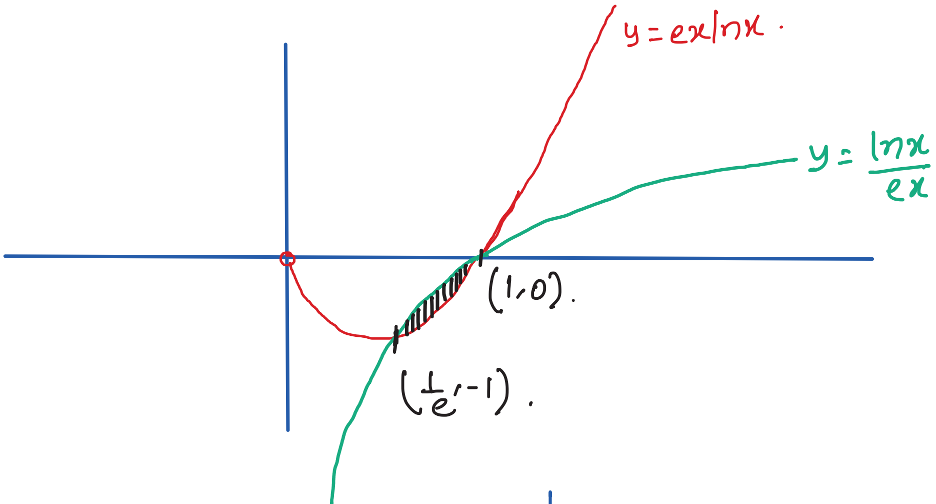
$$= \lim_{t\to 0} \frac{2\,\frac{\sin t^2}{t^2}}{(2\cos t^2) + \frac{\sin t^2}{t^2}}$$

$$= \frac{2 \times 1}{2 \times 1 + 1} = \boxed{\frac{2}{3}}$$ **Ans.**

EXERCISE (S-2)

EXERCISE (S-2)

1. Compute the area of the region bounded by the curves y = e. x. $ln\,x$ & y = $ln\,x/(e.x)$ where ln e=1.



Area bounded $= \displaystyle\int_{1/e}^{1} \left(\frac{\ln x}{ex} - ex\ln x\right) dx.$

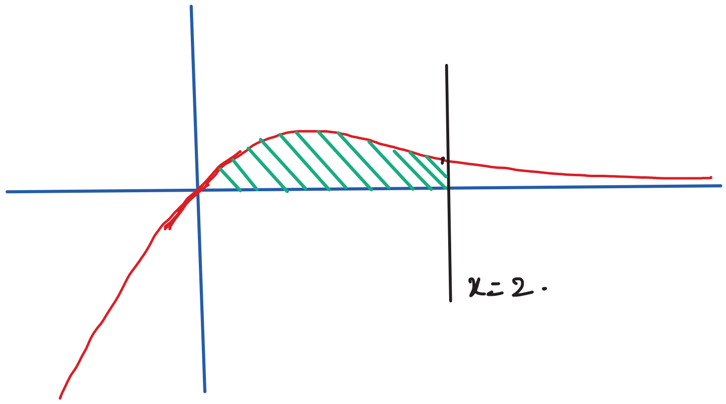
$$= \left[\frac{1}{2e}(\ln x)^2 - \left(\frac{ex^2}{2}\ln x - \frac{ex^2}{4}\right)\right]_{1/e}^{1}$$

$$= \left[\frac{e}{4} - \left(\frac{1}{2e} + \frac{1}{2e} + \frac{1}{4e}\right)\right]$$

$$= \frac{e^2 - 5}{4e} \text{ sq. units.}$$

2. Find the area bounded by the curve $y = x\,e^{-x}$; $xy = 0$ and $x = c$ where c is the x-coordinate of the curve's inflection point.

<u>Solⁿ.</u>



$$y = xe^{-x}$$

$$\frac{dy}{dx} = e^{-x}(1-x).$$

$$\frac{d^2y}{dx^2} = e^{-x}(x-2).$$

∴ curve inflection pt occurs at $x = 2$.

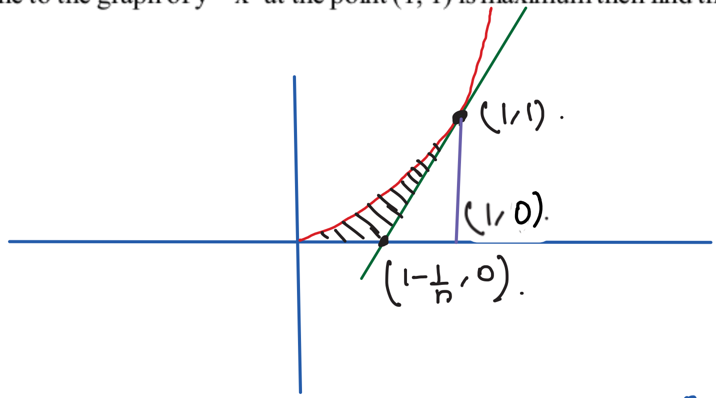
$$\therefore \text{Required Area} = \int_0^2 xe^{-x}\,dx.$$

$$= \left[-xe^{-x} - e^{-x}\right]_0^2.$$

$$= 1 - 3e^{-2}.$$

3. Consider the curve $y = x^n$ where $n > 1$ in the 1^{st} quadrant. If the area bounded by the curve, the x-axis and the tangent line to the graph of $y = x^n$ at the point $(1, 1)$ is maximum then find the value of n.

Soln.



Equation of tangent to $y = x^n$ at $(1,1)$. is.

$$y - 1 = n(x-1)$$

Required area $= \left(\int_0^1 x^n dx\right) - \frac{1}{2n} = \frac{1}{n+1} - \frac{1}{2n}$.

$$\therefore A(n) = \frac{1}{n+1} - \frac{1}{2n}.$$

$$\Rightarrow A'(n) = -\frac{1}{(n+1)^2} + \frac{1}{2n^2} = \frac{(n+1)^2 - 2n^2}{2n^2(n+1)^2}.$$

$$= \frac{\left(1-(\sqrt{2}-1)n\right)\left(1+(\sqrt{2}+1)n\right)}{2n^2(n+1)^2}.$$

sign scheme of $A'(n)$:

$$\underset{-}{\quad} \quad -\frac{1}{\sqrt{2}+1} \quad \underset{+}{\quad} \quad \frac{1}{\sqrt{2}-1} \quad \underset{-}{\quad}$$

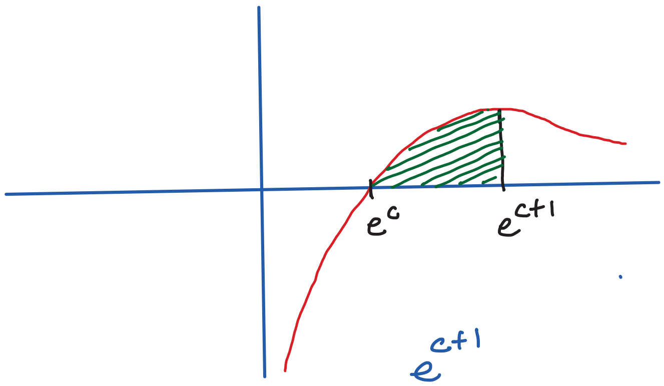
$\therefore$ $A(n)$ becomes maximum at $n = \frac{1}{\sqrt{2}-1} = \sqrt{2}+1$. Ans.

4. Show that the area bounded by the curve $y=\dfrac{\ln x - c}{x}$, the x-axis and the vertical line through the maximum point of the curve is independent of the constant c.

Solⁿ.

$$y = \frac{\ln x - c}{x}$$

$$\frac{dy}{dx} = \frac{(c+1) - \ln x}{x^2} = 0 \Rightarrow x = e^{c+1}$$

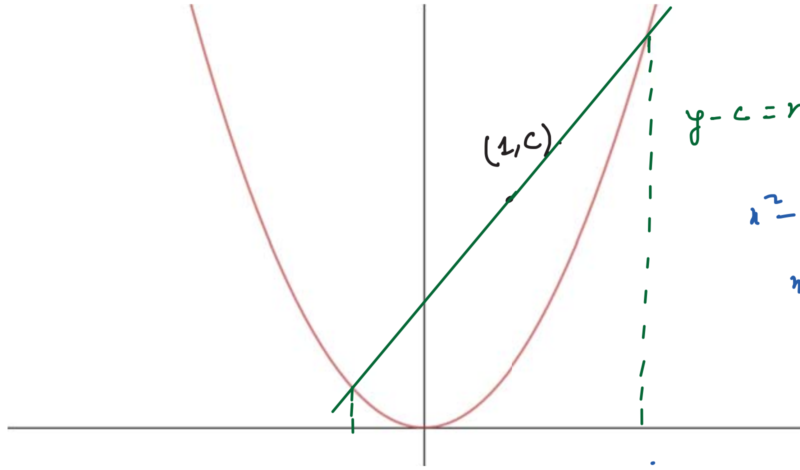


Required Area $= \displaystyle\int_{e^c}^{e^{c+1}} \frac{\ln x - c}{x}\, dx$

$$\left[\frac{1}{2}(\ln x - c)^2\right]_{e^c}^{e^{c+1}} = \frac{1}{2} \text{ (independant of c).}$$

5. Let 'c' be the constant number such that $c > 1$. If the least area of the figure given by the line passing through the point $(1, c)$ with gradient 'm' and the parabola $y = x^2$ is 36 sq. units find the value of $(c^2 + m^2)$.

Solⁿ.



$y - c = m(x-1)$

$x^2 - c = mx - m.$

x^2

$$y = x^2. \quad \text{—(i)}$$

$$y = c + m(x-1). \quad \text{—(ii)}$$

Solving (i) and (ii)

$$x^2 - mx + m - c = 0. \quad \text{—(iii)}$$

x_1 and x_2 are roots of above eqⁿ.

$\Rightarrow x_1 + x_2 = m \; ; \; x_1 x_2 = m - c \; ; \; x_2 - x_1 = \sqrt{m^2 - 4(m-c)}$

$$\text{Area} = \int_{x_1}^{x_2} \left((c-m) + mx - x^2\right) dx$$

$$= (c-m)(x_2 - x_1) + \frac{m}{2}\cdot(x_2^2 - x_1^2) - \frac{1}{3}(x_2^3 - x_1^3).$$

$$= (x_2 - x_1)\left[(c-m) + \frac{m}{2}\cdot(x_2 + x_1) - \frac{1}{3}(x_2^2 + x_1 x_2 + x_1^2)\right]$$

$$= \sqrt{m^2 - 4m + 4c}\left[c - m + \frac{m^2}{2} - \frac{1}{3}\left(m^2 - (m-c)\right)\right].$$

$$= \frac{\left(m^2 - 4m + 4c\right)^{3/2}}{6}. = \frac{\left((m-2)^2 + 4(c-1)\right)^{3/2}}{6}.$$

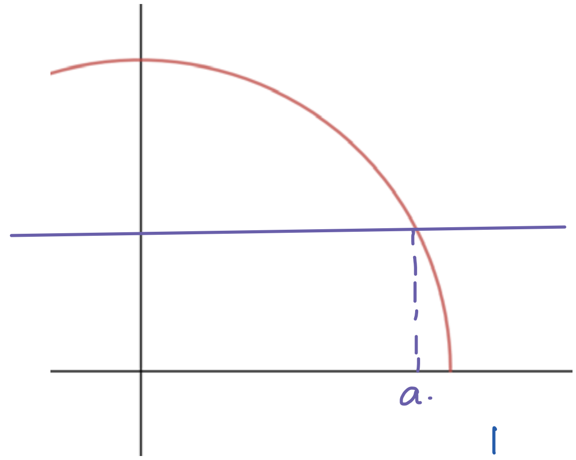
Area will be minimum when $m = 2$.

$$\therefore \frac{\left(4(c-1)\right)^{3/2}}{6} = 36. \Rightarrow c = 10.$$

$$\therefore \boxed{c^2 + m^2 = 104.}$$

6. For what values of $a \in [0,1]$ does the area of the figure bounded by the graph of the function $y = f(x)$ and the straight lines $x = 0$, $x = 1$ & $y = f(a)$ is at a minimum & for what values it is at a maximum if $f(x) = \sqrt{1-x^2}$. Find also the maximum & the minimum areas.

Sol^n.



$$\text{Area} = \int_0^a \left(f(x) - f(a)\right) dx + \int_a^1 \left(f(a) - f(x)\right) dx.$$

$$A(a) = \int_0^a f(x)\,dx + \int_1^a f(x)\,dx - f(a)\,(2a-1).$$

$$A'(a) = f(a) + f(a) - 2f(a) - f'(a)\,(2a-1).$$

$$= -f'(a)\,(2a-1).$$

$+$ (before 0), $-$ (between 0 and $\frac{1}{2}$), $+$ (after $\frac{1}{2}$).

$\therefore$ for area to be min^m. $\boxed{a = \frac{1}{2}.}$

$$A_{min} = \int_0^{1/2} \sqrt{1-x^2}\,dx - \int_{1/2}^{1} \sqrt{1-x^2}\,dx = \boxed{\frac{3\sqrt{3}-\pi}{12}.}$$

$$A(0) = \text{local maxima} = -\int_0^1 \sqrt{1-x^2} + f(0) = 1 - \frac{\pi}{4}.$$

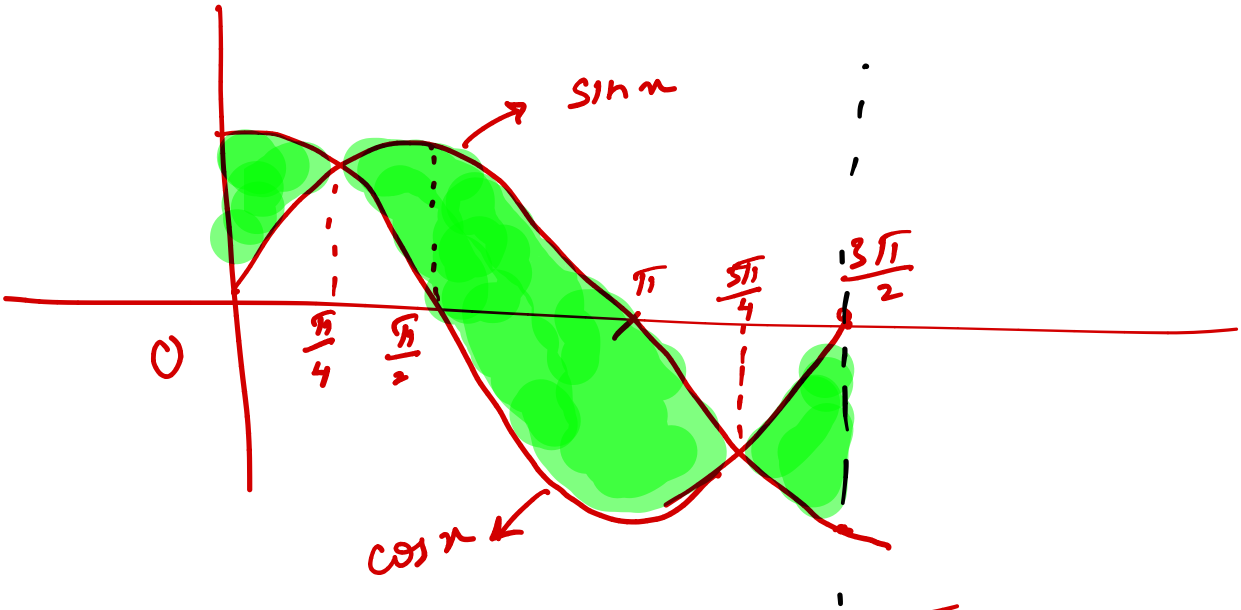
$$A(1) = \int_0^1 \sqrt{1-x^2}\,dx - f(1) = \frac{\pi}{4} = A_{max}.$$

EXERCISE (JM)

EXERCISE (JM)

1. The area bounded by the curves y = cos x and y = sin x between the ordinates x = 0 and x = $\frac{3\pi}{2}$ is :- **[AIEEE-2010]**

(1) $4\sqrt{2} - 2$ (2) $4\sqrt{2} + 2$ (3) $4\sqrt{2} - 1$ (4) $4\sqrt{2} + 1$



$$\text{Area} = \int_0^{\pi/4} (\cos x - \sin x)\,dx + \int_{\pi/4}^{5\pi/4} (\sin x - \cos x)\,dx$$

$$+ \int_{5\pi/4}^{3\pi/2} (\cos x - \sin x)\,dx$$

$$= \left[\sin x + \cos x\right]_0^{\pi/4} + \left[-\cos x - \sin x\right]_{\pi/4}^{5\pi/4}$$

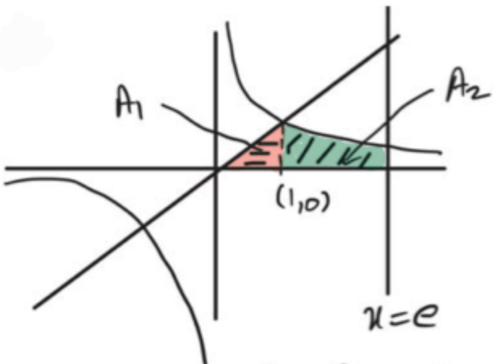
$$+ \left[\sin x + \cos x\right]_{5\pi/4}^{3\pi/2}$$

$$= 4\sqrt{2} - 2$$

2. The area of the region enclosed by the curves y = x, x = e, $y = \frac{1}{x}$ and the positive x-axis is:-

[AIEEE-2011]

(1) $\frac{3}{2}$ square units (2) $\frac{5}{2}$ square units (3) $\frac{1}{2}$ square units (4) 1 square units



$A = A_1 + A_2 = \frac{3}{2}$

$A_1 + A_2 = \text{area}$

$A_1 = \frac{1}{2}$

$A_2 = \int_1^e \frac{1}{x}\,dx$

$A_2 = (\log_e x)_1^e = 1$

$\therefore A_1 + A_2 = 3/2$ Ans 1

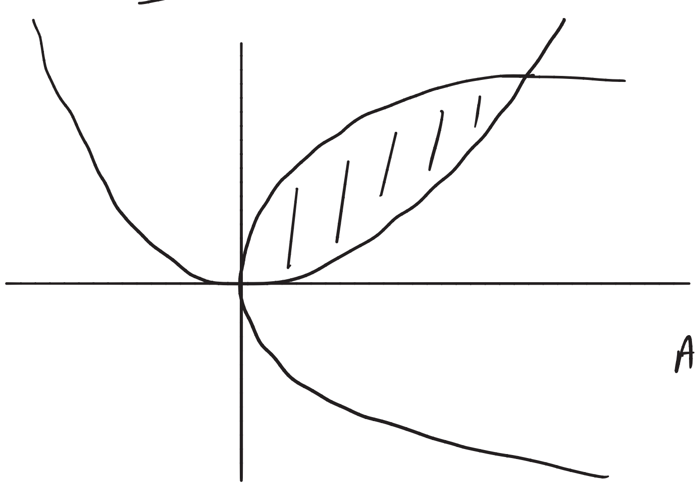
3. The area bounded by the curves $y^2 = 4x$ and $x^2=4y$ is :- **[AIEEE-2011]**

(1) 0 (2) $\frac{32}{3}$ (3) $\frac{16}{3}$ (4) $\frac{8}{3}$

$y^2 = 4x$ and $x^2 = 4y$

Area bounded $= 16/3 = \frac{16ab}{3}$

<u>OR</u>



Point of intersection

$= (4,4)$

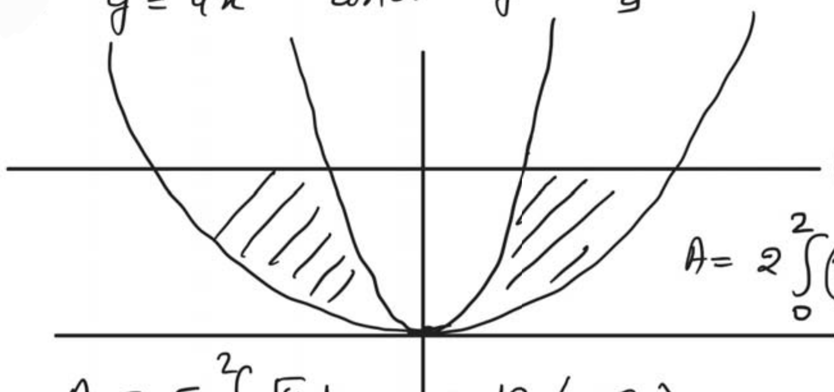
$$A = \int_0^4 \left(2\sqrt{x} - \frac{x^2}{4}\right) dx = \frac{16}{3}$$

4. The area bounded between the parabolas $x^2 = \frac{y}{4}$ and $x^2 = 9y$, and the straight line y = 2 is :

[AIEEE-2012]

(1) $10\sqrt{2}$ (2) $20\sqrt{2}$ (3) $\frac{10\sqrt{2}}{3}$ (4) $\frac{20\sqrt{2}}{3}$

$y = 4x^2$ and $y = \frac{x^2}{9}$ and $y = 2$



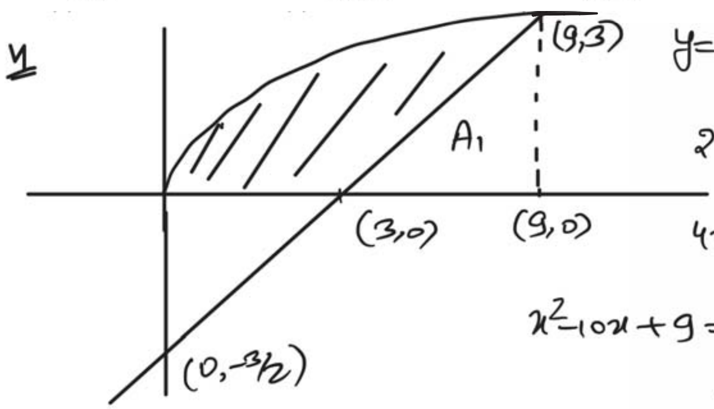
$y = 2$

$$A = 2\int_0^2 \left(3\sqrt{y} - \frac{\sqrt{y}}{2}\right)dy$$

$$A = 5\int_0^2 \sqrt{y}\,dy = \frac{10}{3}(2\sqrt{2})$$

5. The area (in square units) bounded by the curves $y = \sqrt{x}$, $2y - x + 3 = 0$, x-axis and lying in the first quadrant is : **[JEE (Main)-2013]**

(1) 9 (2) 36 (3) 18 (4) $\frac{27}{4}$



$y = \sqrt{x}$ and $2y = x - 3$

$2\sqrt{x} = x - 3$

$4x = x^2 - 6x + 9$

$x^2 - 10x + 9 = 0 \Rightarrow x = 1, 9$

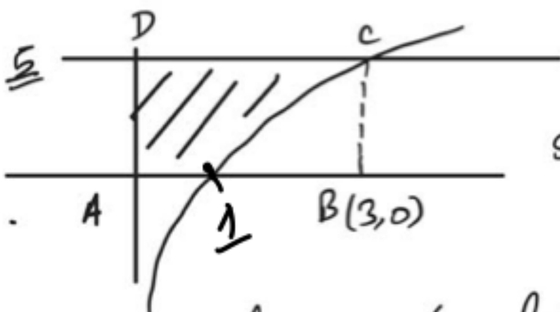
$A_1 = \frac{1}{2} \times 3 \times 6 = 9$, Area $= \int_0^9 \sqrt{x}\,dx = \frac{2}{3}\left(x^{3/2}\right)_0^9$

Area $= \frac{2}{3} \cdot 27 = 18$ (Area $- A_1 =$ Req Area

6. The area bounded by the curve $y = ln(x)$ and the lines $y = 0$, $y = ln\,(3)$ and $x = 0$ is equal to :

[JEE-Main (On line)-2013]

(1) $3\,ln\,(3) - 2$ (2) 3 (3) 2 (4) $3\,ln\,(3) + 2$



$y = \log_e 3$

shaded Area $= \underbrace{3\log_e 3}_{Ar(ABCD)} - \int_1^3 \log_e x\,dx$

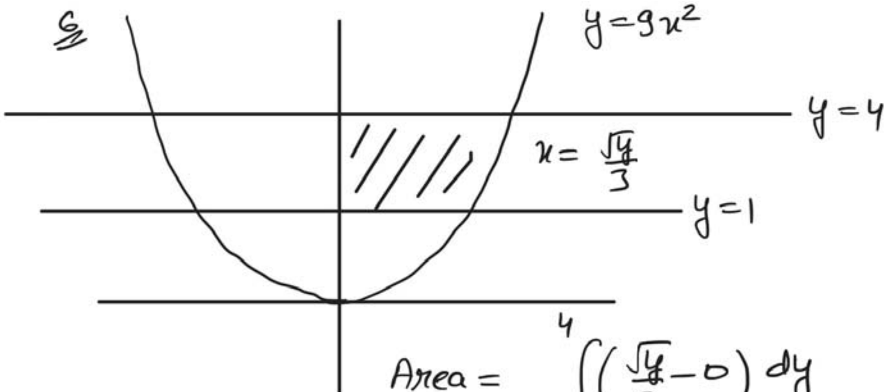
$3\log_e 3 - (x\log_e x - x)_1^3$

$3\log_e 3 - (3\log_e 3 - 2) = 2$

7. The area of the region (in sq. units), in the first quadrant, bounded by the parabola $y = 9x^2$ and the lines $x = 0$, $y = 1$ and $y = 4$, is :- **[JEE-Main (On line)-2013]**

(1) 7/9 (2) 14/3 (3) 14/9 (4) 7/3

Sol



$$\text{Area} = \int_1^4 \left(\frac{\sqrt{y}}{3} - 0\right) dy$$

$$A = \frac{2}{9}\left(y^{3/2}\right)_1^4 = \frac{2}{9}(8-1) = \frac{14}{9}$$

8. The area under the curve $y = |\cos x - \sin x|$, $0 \le x \le \frac{\pi}{2}$, and above x-axis is :

[JEE-Main (On line)-2013]

(1) $2\sqrt{2}$ (2) $2\sqrt{2} + 2$ (3) 0 (4) $2\sqrt{2} - 2$

$$A = \int_0^{\pi/2} |\cos x - \sin x|\,dx = \int_0^{\pi/4} (\cos x - \sin x)\,dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x)\,dx \Rightarrow (\sin x + \cos x)_0^{\pi/4} + (\sin x + \cos x)_{\pi/2}^{\pi/4}$$

$$A = (\sqrt{2}-1) + (\sqrt{2}-1) = 2\sqrt{2}-2$$

9. Let $f : [-2, 3] \to [0, \infty)$ be a continuous function such that $f(1-x) = f(x)$ for all $x \in [-2, 3]$. If R_1 is the numerical value of the area of the region bounded by $y = f(x)$, $x = -2$, $x = 3$ and the axis of x and $R_2 = \int_{-2}^{3} x f(x)\,dx$, then : **[JEE-Main (On line)-2013]**

(1) $2R_1 = 3R_2$ (2) $R_1 = R_2$ (3) $3R_1 = 2R_2$ (4) $R_1 = 2R_2$

S $f : [-2,3] \to [0,\infty)$ $R_1 = \int_{-2}^{3} f(x)\,dx$

$R_2 = \int_{-2}^{3} x\, f(x)\,dx$

(king) $R_2 = \int_{-2}^{3} (1-x) f(1-x)\,dx$

Add

$2R_2 = \int_{-2}^{3} f(x)\,dx = R_1$

10. The area of the region described by $A = \{(x, y) : x^2 + y^2 \le 1 \text{ and } y^2 \le 1 - x\}$ is :

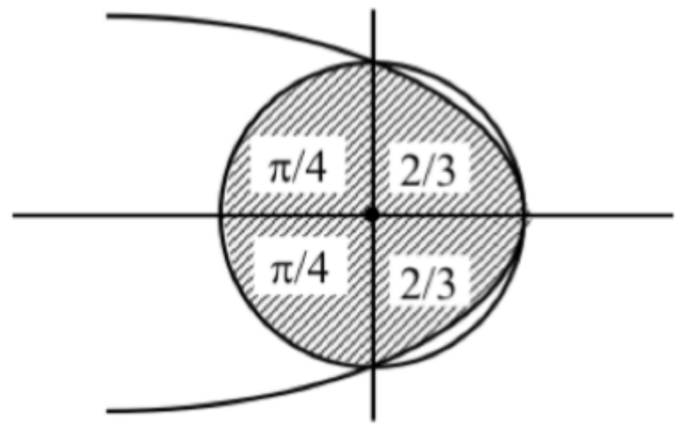
[JEE(Mair

(1) $\frac{\pi}{2}+\frac{4}{3}$ (2) $\frac{\pi}{2}-\frac{4}{3}$ (3) $\frac{\pi}{2}-\frac{2}{3}$ (4) $\frac{\pi}{2}+\frac{2}{3}$

Sol. **1**

$$A = \frac{1}{2}\times\pi + 2\int_0^1 \sqrt{1-x}\,dx$$

$$= \frac{\pi}{2}+\frac{4}{3}.$$

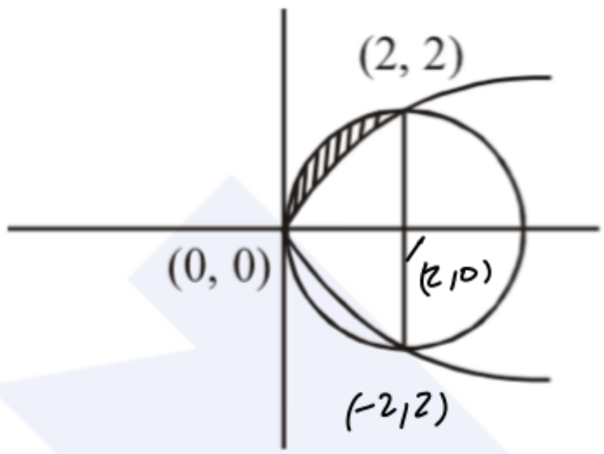


11. The area (in sq.units) of the region $\{(x, y) : y^2 \geq 2x \text{ and } x^2 + y^2 \leq 4x,\ x \geq 0,\ y \geq 0\}$ is :-

[JEE(Main)

(1) $\frac{\pi}{2} - \frac{2\sqrt{2}}{3}$ (2) $\pi - \frac{4}{3}$ (3) $\pi - \frac{8}{3}$ (4) $\pi - \frac{4\sqrt{2}}{3}$

Sol.



$y^2 = 2x$
$x^2 + y^2 = 4x$

$x = 2, y = 2$
$x = 0, y = 0$
$x = 2, y = -2$

(2,0) is center of circle

Reqd A = Ar. of Semicircle − Ar. inside Parabola

$$= \frac{\pi(2)^2}{4} - \sqrt{2}\int_0^2 \sqrt{x}\,dx$$

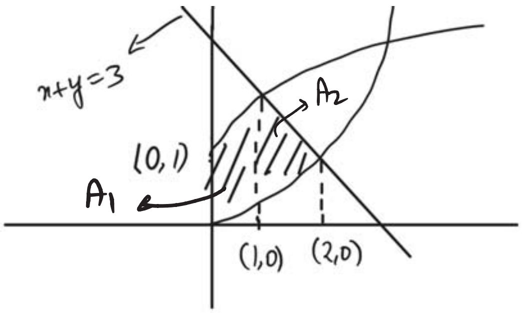
$$= \pi - \sqrt{2} \cdot \frac{2}{3}\ 2\sqrt{2}$$

$$= \pi - 8/3$$

12. The area (in sq. units) of the region $\{(x, y) : x \geq 0, x + y \leq 3, x^2 \leq 4y \text{ and } y \leq 1 + \sqrt{x}\}$ is :

[JEE(Main)-2017]

(1) $\frac{5}{2}$ (2) $\frac{59}{12}$ (3) $\frac{3}{2}$ (4) $\frac{7}{3}$



Point of intersection of $y = 1 + \sqrt{x}$ and

$x + y = 3$

$x + \sqrt{x} = 2 \Rightarrow x = 1$

Point of intersection of $y = \frac{x^2}{4}$ and $x + y = 3$

$x + \frac{x^2}{4} = 3 \Rightarrow x^2 + 4x - 12 = 0 \Rightarrow x = 2, -6$

$A = A_1 + A_2$

$A_1 = \int_0^1 \left((1+\sqrt{x}) - \frac{x^2}{4}\right) dx = \left(x + \frac{2}{3}x^{2/3} - \frac{x^3}{12}\right)_0^1$

$A_1 = 1 + \frac{2}{3} - \frac{1}{12} = \frac{12 + 8 - 1}{12} = \frac{19}{12}$

$A_2 = \int_1^2 \left((3-x) - \frac{x^2}{4}\right) dx$

$A_2 = \left(3x - \frac{x^2}{2} - \frac{x^3}{12}\right)_1^2 = 3 - \frac{3}{2} - \frac{7}{12}$

$A_2 = \frac{36 - 18 - 7}{12} = \frac{11}{12}$

$A = A_1 + A_2 = \frac{19}{12} + \frac{11}{12} = \frac{5}{2}$

13. Let $g(x) = \cos x^2$, $f(x) = \sqrt{x}$ and α, β $(\alpha < \beta)$ be the roots of the quadratic equation $18x^2 - 9\pi x + \pi^2 = 0$. Then the area (in sq. units) bounded by the curve $y = (gof)(x)$ and the lines $x = \alpha$, $x = \beta$ and $y = 0$ is- **[JEE(Main)-2018]**

(1) $\frac{1}{2}(\sqrt{3}+1)$ (2) $\frac{1}{2}(\sqrt{3}-\sqrt{2})$ (3) $\frac{1}{2}(\sqrt{2}-1)$ (4) $\frac{1}{2}(\sqrt{3}-1)$

$$18x^2 - 9\pi x + \pi^2 = 0 \Rightarrow (6x-\pi)(3x-\pi) = 0$$

$$\beta = \frac{\pi}{3}, \quad \alpha = \frac{\pi}{6}, \quad g[f(x)] = \cos x$$

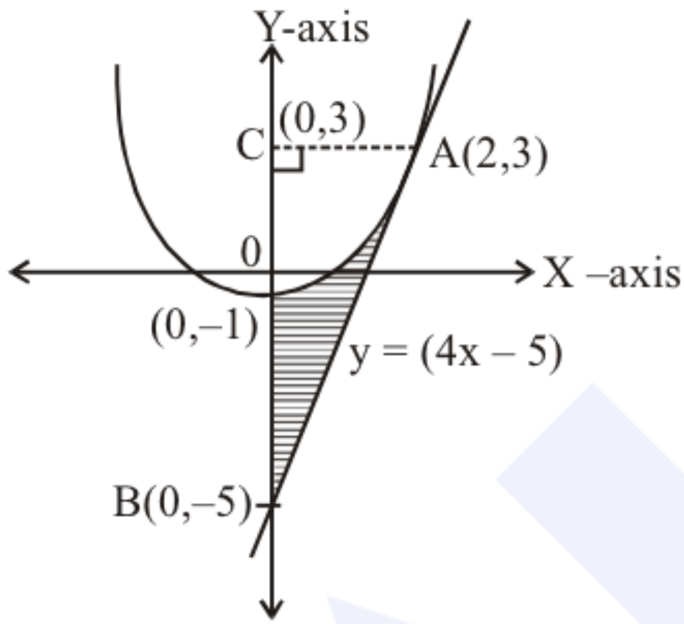
$$A = \int_{\pi/3}^{\pi/6} \cos x \, dx = \frac{\sqrt{3}-1}{2}$$

14. The area (in sq. units) bounded by the parabola $y = x^2 - 1$, the tangent at the point (2, 3) to it and the y-axis is : **[JEE(Main)-2019]**

(1) $\frac{14}{3}$ (2) $\frac{56}{3}$ (3) $\frac{8}{3}$ (4) $\frac{32}{3}$

Ans. (3)

Sol.



Equation of tangent at (2,3) on

$y = x^2 - 1$, is $y = (4x - 5)$ ....(i)

$\therefore$ Required shaded area

$$= \text{ar}\ (\Delta ABC) - \int_{-1}^{3} \sqrt{y+1}\ dy$$

$$= \frac{1}{2}.(8).(2) - \frac{2}{3}\left((y+1)^{3/2}\right)_{-1}^{3}$$

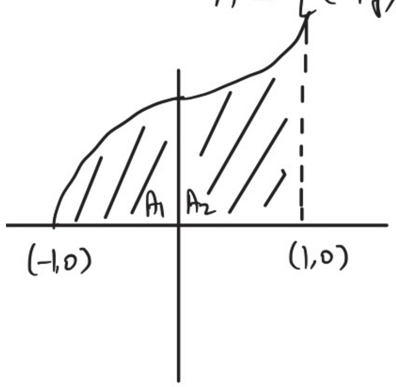
$$= 8 - \frac{16}{3} = \frac{8}{3} \text{ (square units)}$$

15. The area of the region $A = \left[(x, y) : 0 \le y \le x|x| + 1 \text{ and } -1 \le x \le 1\right]$ in sq. units, is : **[JEE(Main)-2019]**

(1) $\frac{2}{3}$ (2) $\frac{1}{3}$ (3) 2 (4) $\frac{4}{3}$

14

$$A = \left[(x, y) : 0 \le y \le x|x| + 1 , \; x \in [-1, 1] \right]$$



$$y = \begin{cases} 1 - x^2 & x \in [-1, 0) \\ y = 1 + x^2 & x \in [0, 1] \end{cases}$$

$$y \le 1 - x^2 \quad \text{and} \quad y \le x^2 + 1$$

$$A = A_1 + A_2 = \int_{-1}^{0} (1 - x^2)\, dx + \int_{0}^{1} (1 + x^2)\, dx$$

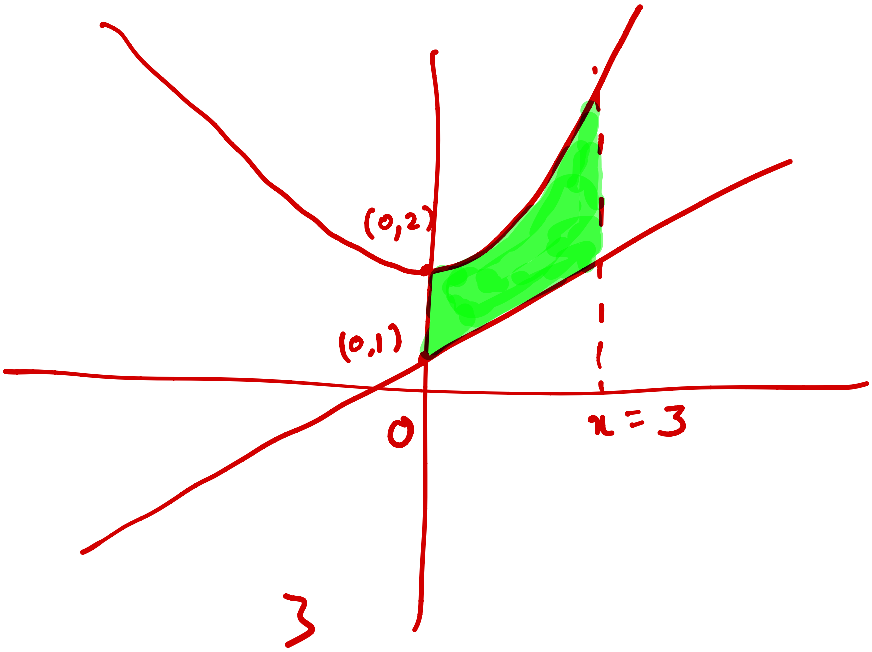
$$A_1 = \left(x - \frac{x^3}{3} \right)_{-1}^{0} = \frac{2}{3}$$

$$A_2 = \left(x + \frac{x^3}{3} \right)_{0}^{1} = \frac{4}{3}$$

$$A = 2$$

16. The area (in sq. units) of the region bounded by the parabola, $y = x^2 + 2$ and the lines, $y = x + 1$, $x = 0$ and $x = 3$, is : **[JEE(Main)-2019]**

(1) $\frac{15}{4}$ (2) $\frac{15}{2}$ (3) $\frac{21}{2}$ (4) $\frac{17}{4}$



required area $= \int_0^3 [(x^2+2) - (x+1)]\,dx$

$= \int_0^3 (x^2 - x + 1)\,dx$

$= \left[\frac{x^3}{3} - \frac{x^2}{2} + x\right]_0^3$

$= 9 - \frac{9}{2} + 3 - 0$

$= \frac{15}{2}$

17. The area (in sq. units) of the region $A = \{(x, y) : x^2 \le y \le x + 2\}$ is **[JEE(Main)-2019]**

(1) $\frac{10}{3}$ (2) $\frac{9}{2}$ (3) $\frac{31}{6}$ (4) $\frac{13}{6}$

Sol. $x^2 \le y \le x + 2$

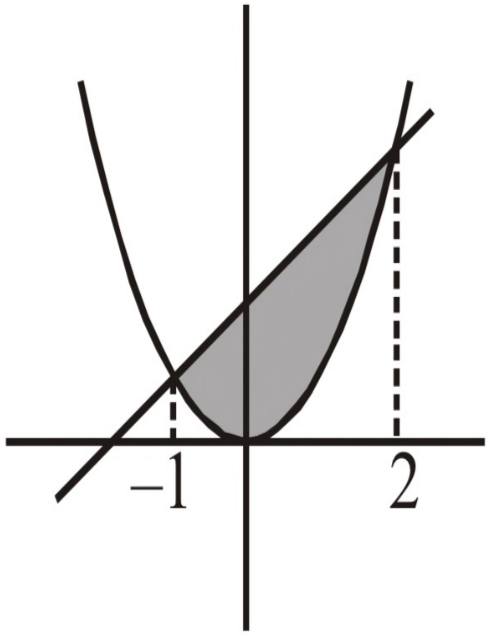
$x^2 = y \; ; \; y = x + 2$

$x^2 = x + 2$

$x^2 - x - 2 = 0$

$(x - 2)(x - 1) = 0$

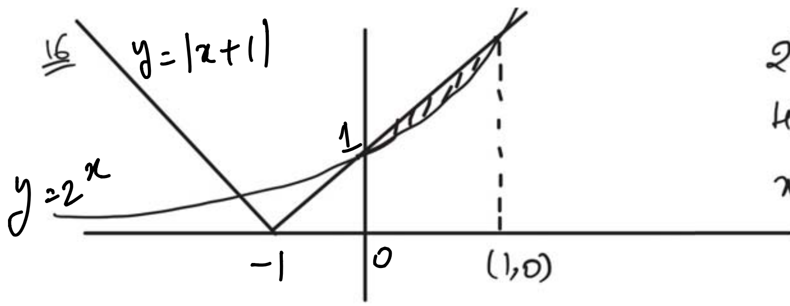
$x = 2, -1$



$$\text{Area} = \int_{-1}^{2} (x + 2) - x^2 \, dx = \frac{9}{2}$$

18. The area (in sq. units) of the region bounded by the curves $y = 2^x$ and $y = |x + 1|$, in the first quadrant is : **[JEE(Main)-2019]**

(1) $\frac{3}{2}-\frac{1}{\log_e 2}$ (2) $\frac{1}{2}$ (3) $\log_e 2+\frac{3}{2}$ (4) $\frac{3}{2}$



$2^x = x+1$

Hit & trial at

$x = 0, 1$

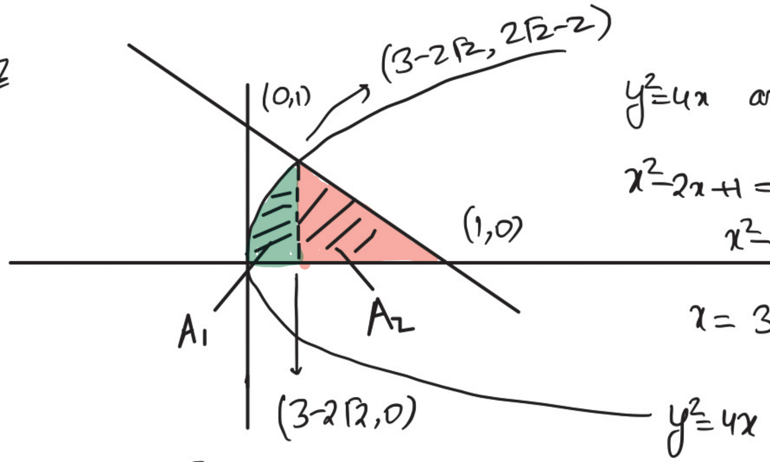
$$A = \int_0^1 (x+1-2^x)\,dx = \left(\frac{x^2}{2} + x - \frac{2^x}{\log_e 2}\right)_0^1$$

$$A = \left(\frac{3}{2} - \frac{2}{\log_e 2}\right) - \left(\frac{-1}{\log_e 2}\right) = \frac{3}{2} - \frac{1}{\log_e 2}$$

19. If the area (in sq. units) of the region $\{(x, y) : y^2 \leq 4x,\ x + y \leq 1,\ x \geq 0,\ y \geq 0\}$ is $a\sqrt{2} + b$, then $a - b$ is equal to : **[JEE(Main)-2019]**

(1) $\frac{8}{3}$ (2) $\frac{10}{3}$ (3) 6 (4) $-\frac{2}{3}$

17



$y^2 = 4x$ and $x + y = 1$

$x^2 - 2x + 1 = 4x$

$x^2 - 6x + 1 = 0$

$x = 3 \pm 2\sqrt{2}$

$a = 8/3$

$b = -10/3$

$$A = \underbrace{\int_0^{3-2\sqrt{2}} 2\sqrt{x}\, dx}_{A_1} + \underbrace{\frac{1}{2}\cdot(2\sqrt{2}-2)(2\sqrt{2}-2)}_{A_2}$$

$$A = \frac{4}{3}\left(x^{3/2}\right)_0^{3-2\sqrt{2}} + (2-\sqrt{2})^2 \Rightarrow \frac{4}{3}(\sqrt{2}-1)^3 + 2(\sqrt{2}-1)^2$$

$$= (3-2\sqrt{2})\left(\frac{4\sqrt{2}}{3} - \frac{4}{3} + 2\right) = \frac{8\sqrt{2}-10}{3}$$

EXERCISE (JA)

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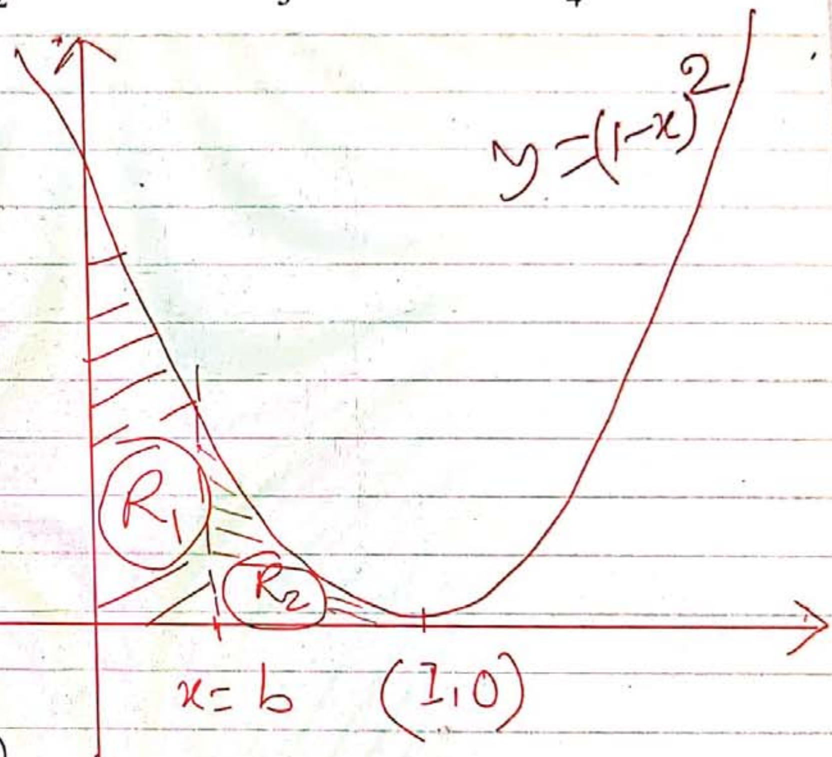
1. **(a)** Let the straight line x = b divide the area enclosed by $y = (1-x)^2$, y = 0 and x = 0 into two parts $R_1(0 \le x \le b)$ d and $R_2(b \le x \le 1)$ such that $R_1 - R_2 = \frac{1}{4}$. Then b equals

(A) $\frac{3}{4}$ (B) $\frac{1}{2}$ (C) $\frac{1}{3}$ (D) $\frac{1}{4}$

$$R_1 = \int_0^b (1-x)^2\,dx$$



$$= \left.\frac{(x-1)^3}{3}\right|_0^b$$

$$= \frac{(b-1)^3}{3} + \frac{1}{3}$$

$$= \frac{1}{3}\left((b-1)^3 + 1\right)$$

$$R_2 = \int_b^1 (1-x)^2\,dx = \left(\frac{(x-1)^3}{3}\right)_b^1$$

$$= 0 - \left(\frac{(b-1)^3}{3}\right)$$

$$R_1 - R_2 = \frac{1}{4} \Rightarrow \frac{1}{3}\left((b-1)^3 + 1 + (b-1)^3\right) = \frac{1}{4}$$

$$\Rightarrow 2(b-1)^3 = \frac{3}{4} - 1 = -\frac{1}{4} \Rightarrow (b-1)^3 = -\frac{1}{8}$$

$$\Rightarrow b = \frac{1}{2}$$

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(b) Let $f:[-1,2] \to [0,\infty)$ be a continuous function such that $f(x) = f(1-x)$ for all $x \in [-1,2]$.

Let $R_1 = \int_{-1}^{2} x f(x)dx$, and R_2 be the area of the region bounded by $y = f(x)$, $x=-1$, $x=2$, and the x-axis. Then -

(A) $R_1 = 2R_2$ (B) $R_1 = 3R_2$ (C) $2R_1 = R_2$ (D) $3R_1 = R_2$

[JEE 2011, 3+3]

$R_1 = \int_{-1}^{2} x f(x) \cdot dx$ (using king)

$= \int_{-1}^{2} (2-1-x) f(2-1-x) dx = \int_{-1}^{2} (1-x) f(1-x) dx$

$R_1 = \int_{-1}^{2} (1-x) \cdot f(x) \cdot dx = \int_{-1}^{2} f(x) dx - \int_{-1}^{2} x f(x) dx$

$2R_1 = R_2$. (Ans)

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2. The area enclosed by the curve $y = \sin x + \cos x$ and $y = |\cos x - \sin x|$ over the interval $\left[0, \frac{\pi}{2}\right]$ is

[JEE(Advanced) 2013, 2M]

(A) $4(\sqrt{2}-1)$ (B) $2\sqrt{2}(\sqrt{2}-1)$ (C) $2(\sqrt{2}+1)$ (D) $2\sqrt{2}(\sqrt{2}+1)$

$$y = |\cos x - \sin x| = \begin{cases} \cos x - \sin x, & x \in \left[0, \frac{\pi}{4}\right] \\ \sin x - \cos x, & x \in \left[\frac{\pi}{4}, \frac{\pi}{2}\right] \end{cases}$$

So, required area

$$= \int_0^{\pi/4} (\sin x + \cos x) - (\cos x - \sin x)\, dx$$

$$+ \int_{\pi/4}^{\pi/2} (\sin x + \cos x) - (\sin x - \cos x) \cdot dx$$

$$= \int_0^{\pi/4} 2 \sin x \, dx + \int_{\pi/4}^{\pi/2} 2 \cos x \, dx$$

$$= 2 \left(-\cos x\right)_0^{\pi/4} + 2 \left(\sin x\right)_{\pi/4}^{\pi/2}$$

$$= -2\left(\frac{1}{\sqrt{2}} - 1\right) + 2\left(1 - \frac{1}{\sqrt{2}}\right)$$

$$= 2(2 - \sqrt{2}) = 2\sqrt{2}(\sqrt{2} - 1)$$ <u>Ans.</u>

3. Let $F(x)=\int_{x}^{x^2+\frac{\pi}{6}} 2\cos^2 t\,dt$ for all $x \in \mathbb{R}$ and $f:\left[0,\frac{1}{2}\right]\to[0,\infty)$ be a continuous function. For $a\in\left[0,\frac{1}{2}\right]$, if $F'(a)+2$ is the area of the region bounded by $x=0$, $y=0$, $y=f(x)$ and $x=a$, then $f(0)$ is **[JEE 2015, 4M, –0M]**

$$F'(x) = 2\cos^2\left(x^2+\frac{\pi}{6}\right).2x - 2\cos^2 x.$$

$$F'(a)+2 = 4.a\cos^2\left(a^2+\frac{\pi}{6}\right) - 2\cos^2 a+2$$

$$= \int_0^a f(x).dx$$

On differentiation we get:

$$f(a) = 4a\,2\cos\left(a^2+\frac{\pi}{6}\right)\left(-\sin\left(a^2+\frac{\pi}{6}\right)\right)(2a) +$$

$$4\cos^2\left(a^2+\frac{\pi}{6}\right) + 4\cos a\sin a$$

$$\Rightarrow f(0) = 4\cos^2\frac{\pi}{6} = 4.\frac{3}{4} = 3.\text{(Ans)}$$

4. If the line $x = \alpha$ divides the area of region $R = \{(x, y) \in \mathbb{R}^2 : x^3 \le y \le x, 0 \le x \le 1\}$ into two equal parts, then **[JEE(Advanced)-2017, 3(–2)]**

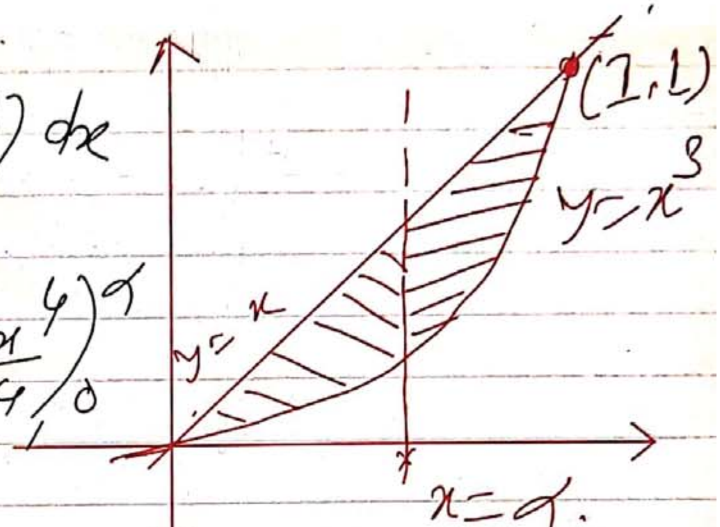
(A) $\frac{1}{2} < \alpha < 1$

(B) $\alpha^4 + 4\alpha^2 - 1 = 0$

(C) $0 < \alpha \le \frac{1}{2}$

(D) $2\alpha^4 - 4\alpha^2 + 1 = 0$

$$\frac{1}{2}\int_0^1 (x - x^3)\,dx = \int_0^{\alpha} (x - x^3)\,dx$$



$$\Rightarrow \frac{1}{2}\left(\frac{x^2}{2} - \frac{x^4}{4}\right)_0^1 = \left(\frac{x^2}{2} - \frac{x^4}{4}\right)_0^{\alpha}$$

$$\Rightarrow \frac{1}{2}\left(\frac{1}{2} - \frac{1}{4}\right) = \left(\frac{\alpha^2}{2} - \frac{\alpha^4}{4}\right)$$

$$\Rightarrow \frac{1}{8} = \frac{\alpha^2}{2} - \frac{\alpha^4}{4}$$

$$\Rightarrow 2\alpha^4 - 4\alpha^2 + 1 = 0 \Rightarrow \text{option (D)}$$

let $f(\alpha) = 2\alpha^4 - 4\alpha^2 + 1$

$f(\frac{1}{2}) = 2 \cdot \frac{1}{16} - 4 \cdot \frac{1}{4} + 1 = \frac{1}{8}$

$f(1) = -1$ & $f(0) = 1$ so, $\alpha \in (\frac{1}{2}, 1)$

option (A)

5. Let $f : [0, \infty) \to \mathbb{R}$ be a continuous function such that $f(x) = 1 - 2x + \int_0^x e^{x-t} f(t)dt$ for all $x \in [0, \infty)$. Then, which of the following statement(s) is (are) TRUE ?

(A) The curve $y = f(x)$ passes through the point $(1, 2)$ **[JEE(Advanced)-2018, 4(–2)]**

(B) The curve $y = f(x)$ passes through the point $(2, -1)$

(C) The area of the region $\{(x, y) \in [0, 1] \times \mathbb{R} : f(x) \le y \le \sqrt{1-x^2}\}$ is $\frac{\pi-2}{4}$

(D) The area of the region $\{(x, y) \in [0, 1] \times \mathbb{R} : f(x) \le y \le \sqrt{1-x^2}\}$ is $\frac{\pi-1}{4}$

Sol

$f(x) = 1-2x + e^x \cdot \int_0^x e^{-t} f(t)\,dt$

$\Rightarrow f(x)\cdot e^{-x} = (1-2x)e^{-x} + \int_0^x e^{-t} f(t)\,dt$

Differentiate and Simplify:

$f'(x) - f(x) = -(1-2x) - 2 + f(x)$

$\Rightarrow f'(x) = 2f(x) + 2x - 3$

If $y = f(x)$, $f'(x) = dy/dx$

$\Rightarrow \frac{dy}{dx} = 2y + 2x - 3 \Rightarrow \frac{dy}{dx} = 2(x+y) - 3$

put $x+y = t \Rightarrow 1 + \frac{dy}{dx} = \frac{dt}{dx}$

$\Rightarrow \frac{dt}{dx} - 1 = 2t - 3 \Rightarrow \frac{dt}{dx} = 2(t-1)$

$\Rightarrow \frac{dt}{t-1} = 2\,dx \Rightarrow \ln(t-1) = 2x + c$

$\Rightarrow \ln(x+y-1) = 2x + C$

$\Rightarrow x+y-1 = e^{2x+C} = e^{2x}\cdot e^{C} = K\cdot e^{2x}$

$\Rightarrow y = K\cdot e^{2x} - x + 1 \longrightarrow (1)$

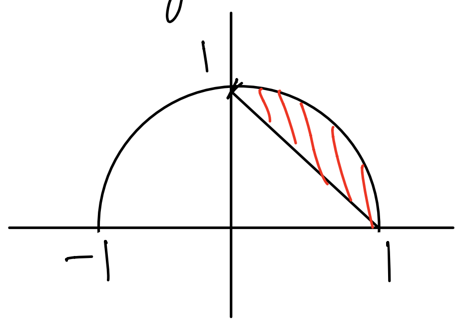
$\because f(x) = 1-2x + e^{x}\cdot\int_0^x e^{-t} f(t)\,dt$

At $x=0$: $f(0) = 1-0+0 = 1 \Rightarrow$ At $x=0, y=1$

Put in (1): $1 = K - 0 + 1 \Rightarrow K = 0$

$\therefore y = 1-x = f(x) \Rightarrow$ B is correct

Also, Region: $f(x) \le y \le \sqrt{1-x^2}$



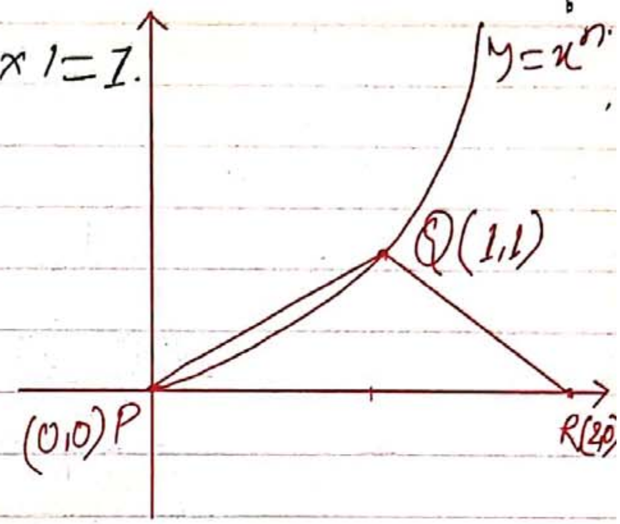
$$A = \frac{\pi(1)^2}{4} - \frac{1}{2}\cdot 1\cdot 1$$

$$= \frac{\pi-2}{4} \text{ Sq. units}$$

$\Rightarrow$ (C) is correct

6. A farmer F_1 has a land in the shape of a triangle with vertices at P(0, 0), Q(1, 1) and R(2, 0). From this land, a neighbouring farmer F_2 takes away the region which lies between the side PQ and a curve of the form $y = x^n (n > 1)$. If the area of the region taken away by the farmer F_2 is exactly 30% of the area of ΔPQR, then the value of n is _____ **[JEE(Advanced)-2018, 3(0)]**

Area of $\Delta PQR = \frac{1}{2} \times 2 \times 1 = 1.$



So, $\int_0^1 (x - x^n)\, dx = \frac{3}{10}$

$\left(\frac{x^2}{2} - \frac{x^{n+1}}{n+1}\right)_0^1 = \frac{3}{10}$

$\Rightarrow \frac{1}{2} - \frac{1}{n+1} = \frac{3}{10}$

$\Rightarrow \frac{1}{n+1} = \frac{1}{2} - \frac{3}{10} = \frac{1}{5}$

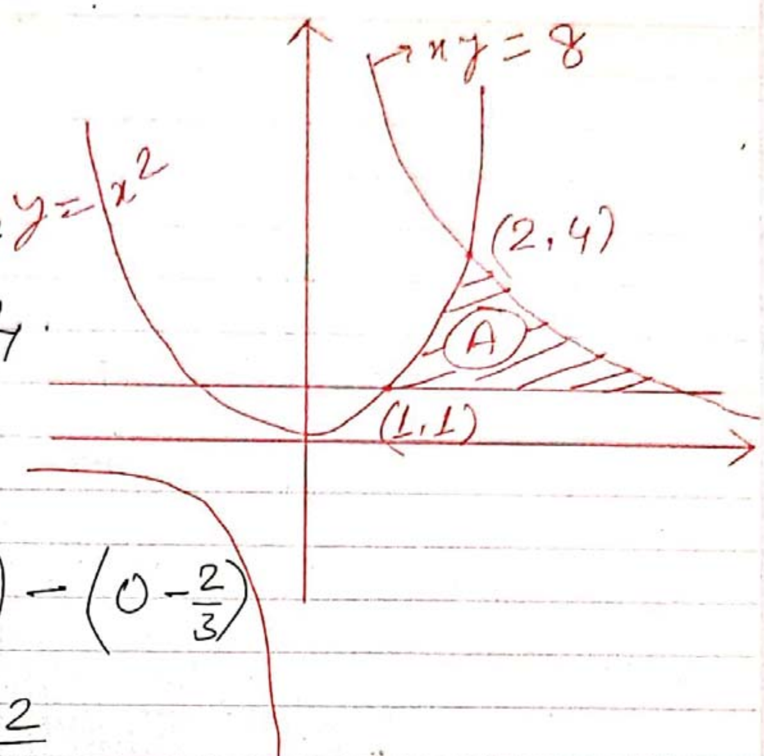
$\Rightarrow n = 4.$ Ans.

7. The area of the region $\{(x, y) : xy \le 8,\ 1 \le y \le x^2\}$ is **[JEE(Advanced)-2019, 3(–1)]**

(1) $8\log_e 2 - \frac{14}{3}$ (2) $16\log_e 2 - \frac{14}{3}$ (3) $16\log_e 2 - 6$ (4) $8\log_e 2 - \frac{7}{3}$

required area

$$= A = \int_1^4 \left(\frac{8}{y} - \sqrt{y}\right) dy$$



$$= \left(8 \ln y - \frac{2}{3} y^{3/2}\right)_1^4$$

$$= \left(8 \ln 4 - \frac{2}{3} 8\right) - \left(0 - \frac{2}{3}\right)$$

$$= 8 \ln 4 - \frac{16}{3} + \frac{2}{3}$$

$$= 8 \ln 4 - \frac{14}{3} = 16 \ln 2 - \frac{14}{3}.$$

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