Parabola - (L)

Complete Solutions Of DYS & Exercises

DO YOURSELF

Do yourself - 1 :

(i) Name the conic represented by the equation $\sqrt{ax} + \sqrt{by} = 1$, where a, b $\in$ R, a, b, > 0.

$(\sqrt{by})^2 = (1 - \sqrt{ax})^2$

$\Rightarrow by = (1 - \sqrt{ax})^2 \Rightarrow$ Parabola

(ii) Find the vertex, axis, focus, directrix, latus rectum of the parabola $4y^2 + 12x - 20y + 67 = 0$.

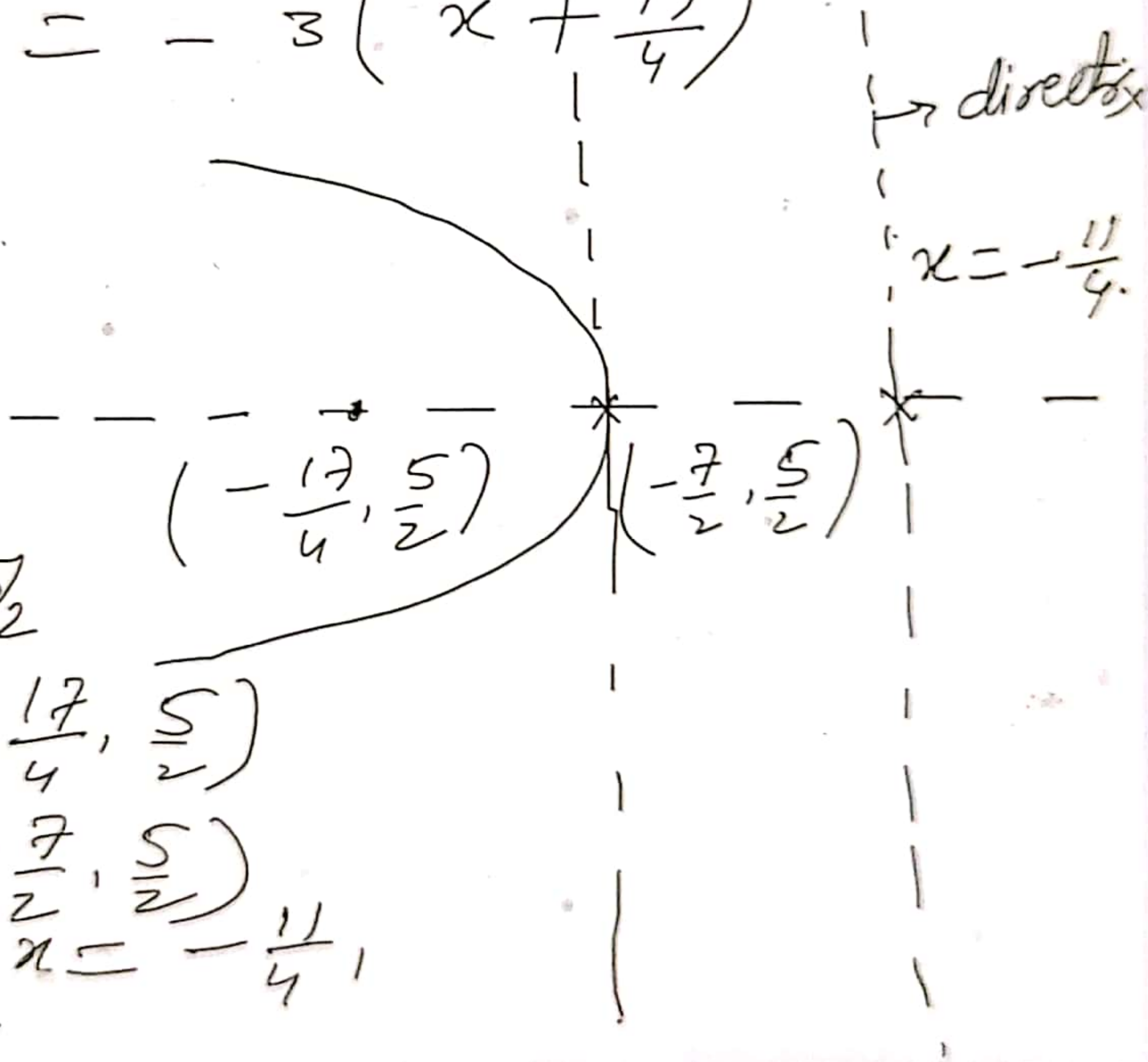
$$4(y^2 - 5y) + 12x + 67 = 0$$

$$\Rightarrow 4\left(\left(y - \frac{5}{2}\right)^2 - \frac{25}{4}\right) + 12x + 67 = 0$$

$$\Rightarrow \left(y - \frac{5}{2}\right)^2 = -\left(3x + \frac{67}{4}\right) + \frac{25}{4}$$

$$\Rightarrow \left(y - \frac{5}{2}\right)^2 = -\left(3x + \frac{42}{4}\right)$$

$$\Rightarrow \left(y - \frac{5}{2}\right)^2 = -3\left(x + \frac{14}{4}\right)$$



Axis is $y = 5/2$

Focus $\left(-\frac{17}{4}, \frac{5}{2}\right)$

vertex $\left(-\frac{7}{2}, \frac{5}{2}\right)$

directrix $x = -\frac{11}{4}$,

$L \cdot R = 3$.

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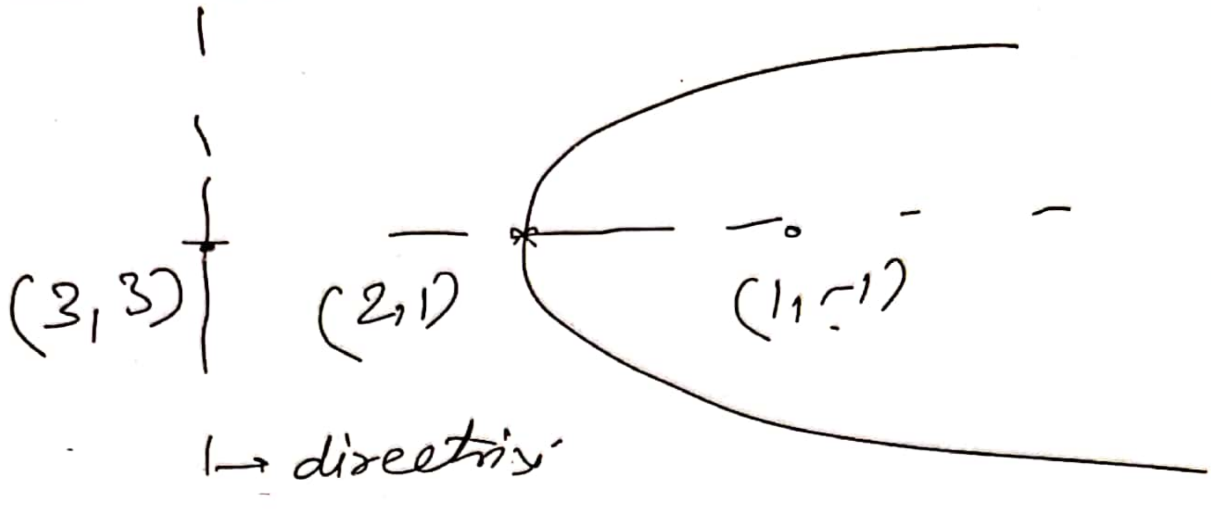
(iii) Find the equation of the parabola whose focus is (1, –1) and whose vertex is (2, 1). Also find its axis and latus rectum.



$$y-3 = -\frac{1}{4}\times 2(x-3)$$

$$y-3 = -\frac{1}{2}(x-3)$$

$$\Rightarrow 2y-6 = -x+3 \Rightarrow x+2y-9=0$$

Equation of parabola is:

$$\sqrt{(y+1)^2+(x-1)^2} = \left|\frac{x+2y-9}{\sqrt{5}}\right|$$

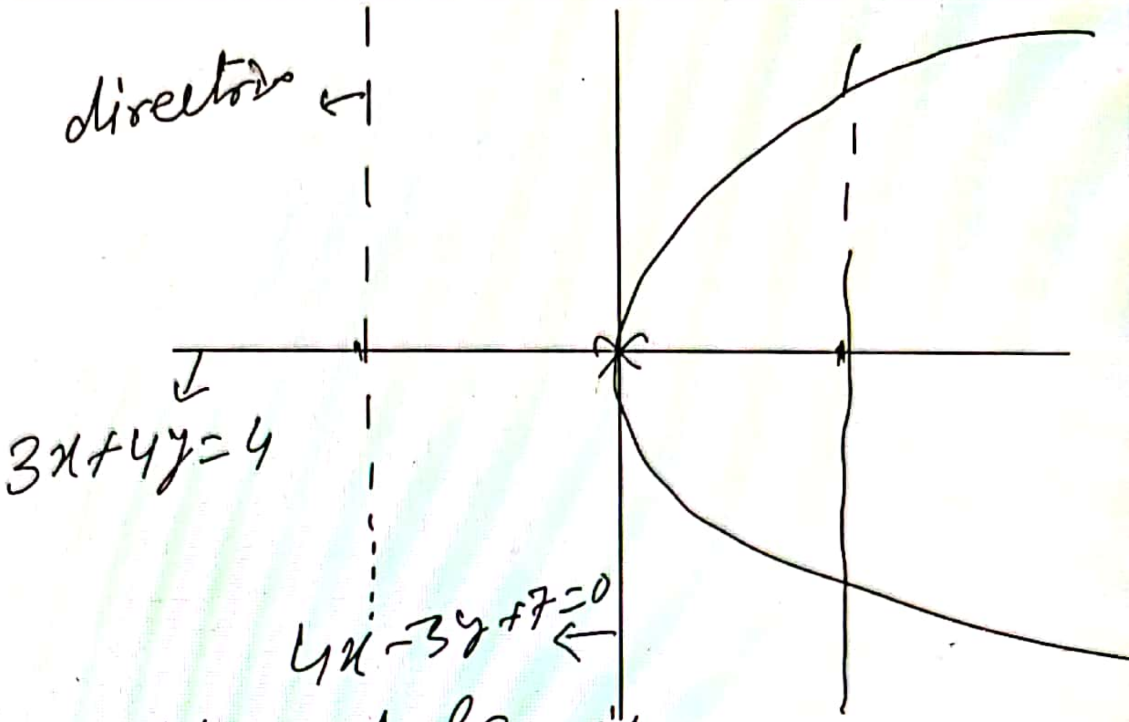
$$\Rightarrow 5(x^2+y^2-2x+2y+2) = x^2+4y^2+81+4xy-36y-18x$$

$$\Rightarrow 4x^2+y^2+8x+46y+10-81-4xy=0$$

$$\Rightarrow 4x^2+y^2+8x+46y-4xy-71=0$$

Ans.

(iv) Find the equation of the parabola whose latus rectum is 4 units, axis is the line $3x + 4y = 4$ and the tangent at the vertex is the line $4x - 3y + 7 = 0$.



equation of parabola is.

$$\left|\frac{3x+4y-4}{\sqrt{5}}\right|^2 = 4(4x-3y+7)$$

$$\Rightarrow (3x+4y-4)^2 = 20(4x-3y+7)$$

Do yourself - 2 :

(i) Find the value of 'a' for which the point $(a^2 - 1, a)$ lies inside the parabola $y^2 = 8x$.

$\Rightarrow y^2 - 8x = 0$

$\Rightarrow a^2 - 8(a^2 - 1) < 0$

$\Rightarrow -7a^2 + 8 < 0$

$\Rightarrow \left(a^2 - \frac{8}{7}\right) > 0$

$\Rightarrow \left(a - \sqrt{\frac{8}{7}}\right)\left(a + \sqrt{\frac{8}{7}}\right) > 0$

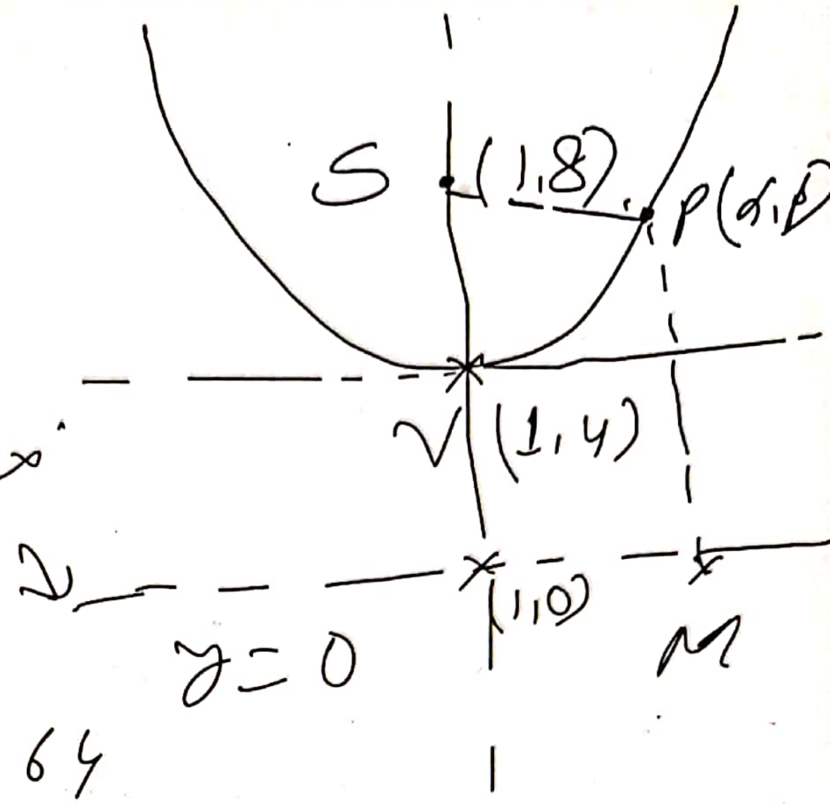
$\therefore a \in \left(-\infty, -\sqrt{\frac{8}{7}}\right) \cup \left(\sqrt{\frac{8}{7}}, \infty\right)$ Ans.

(ii) The focal distance of a point on the parabola $(x-1)^2 = 16(y-4)$ is 8. Find the co-ordinates.

For parabola

$PS = PM$

$\Rightarrow \beta = 8$



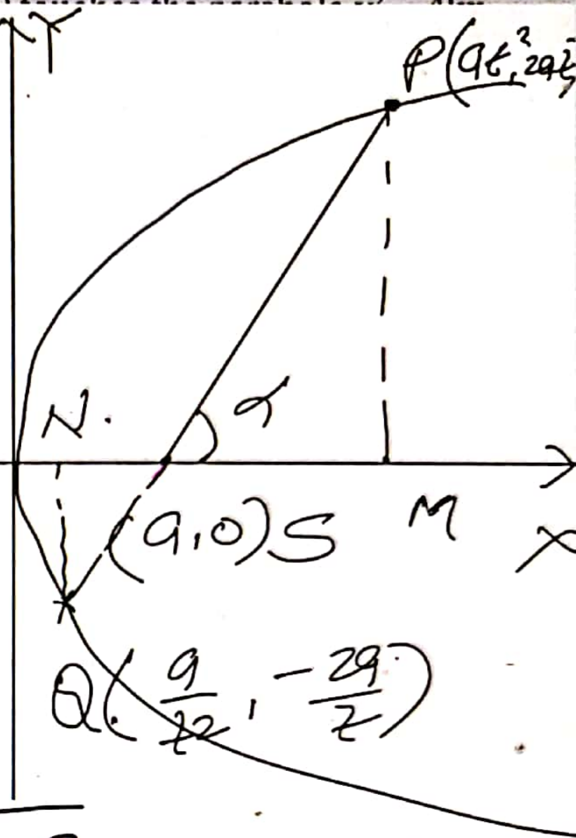
So $(\alpha-1)^2 + (8-8)^2 = 64$

$\Rightarrow \alpha - 1 = 8, -8$

$\Rightarrow \alpha = -7, 9$

So Points are $(-7, 8)$ and $(9, 8)$

(iii) Show that the focal chord of parabola $y^2 = 4ax$ makes an angle α with x-axis is of length $4a\,\text{cosec}^2\alpha$.



$$\tan\alpha = \frac{2a\left(t + \frac{1}{t}\right)}{a\left(t^2 - \frac{1}{t^2}\right)}$$

$$\tan\alpha = \frac{2}{t - \frac{1}{t}}$$

$$t - \frac{1}{t} = 2\cot\alpha \quad \text{— (1)}$$

length PQ

$$= \sqrt{a^2\left(t^2 - \frac{1}{t^2}\right)^2 + 4a^2\left(t + \frac{1}{t}\right)^2}$$

$$= a\sqrt{(2\cot\alpha)^2\left(t + \frac{1}{t}\right)^2 + 4\left(t + \frac{1}{t}\right)^2}$$

$$= a\sqrt{(4\cot^2\alpha + 4)\left(t + \frac{1}{t}\right)^2}$$

$$= 2a\,\text{cosec}\,\alpha \cdot \sqrt{\left(t - \frac{1}{t}\right)^2 + 4}$$

$$= 2a\,\text{cosec}\,\alpha \cdot \sqrt{4\cot^2\alpha + 4}$$

$$= 4a\,\text{cosec}^2\alpha.$$ Proved

(iv) Find the condition that the straight line $ax + by + c = 0$ touches the parabola $y^2 = 4kx$.

$y^2 = 4kx$ and line be $y = -\frac{a}{b}x - \frac{c}{b}$.

$m = -\frac{a}{b}$ and Cost $= -\frac{c}{b}$

for tangent

$y = mx + \frac{a}{m}$

$\Rightarrow y = -\frac{a}{b}x + \frac{k}{-a/b}$

$\Rightarrow \frac{kb}{-a} = -\frac{c}{b}$

$\Rightarrow kb^2 = ac$ Ans.

(v) Find the length of the chord of the parabola $y^2 = 8x$, whose equation is $x + y = 1$.

length of chord $= \frac{4}{m^2}\sqrt{a(1+m^2)(a-mc)}$

$a = 2, \quad m = -1, \quad c = 1$

$\therefore$ length $= \frac{4}{1}\sqrt{2(1+1)(2+1)}$

$= 4\sqrt{12} = 8\sqrt{3}$

Do yourself - 3 :

(i) Find the equation of the tangent to the parabola $y^2 = 12x$, which passes through the point (2, 5). Find also the co-ordinates of their points of contact.

Equation of tangent is

$y = mx + \frac{3}{m}$ passes through (2,5)

$\Rightarrow 5 = 2m + \frac{3}{m}$

$\Rightarrow 2m^2 - 5m + 3 = 0$

$\Rightarrow 2m^2 - 2m - 3m + 3 = 0$

$\Rightarrow 2m(m-1) - 3(m-1) = 0$

$\therefore m = \frac{3}{2}, 1$

$\therefore$ tangent is $y = x + 3$

and $y = \frac{3}{2}x + 2$.

let point of contact is (α, β) so:

$\beta y = 6(x+\alpha)$

$\Rightarrow 6x - \beta y + 6\alpha = 0$ on comparing

$(3, 6)$ for $x - y + 3 = 0$ and $\left(\frac{4}{3}, 4\right)$ for $3x - 2y + 4 = 0$

(ii) Find the equation of the tangents to the parabola $y^2 = 16x$, which are parallel and perpendicular respectively to the line $2x - y + 5 = 0$. Find also the co-ordinates of their points of contact.

$y^2 = 16x$ $\quad a = 4.$

for // tangent $\quad m = 2$

$\therefore$ eq.^n of tangent is $y = 2x + \frac{4}{2}$

$y = 2x + 2$

$\beta y = 8x + 8\alpha$

$$\frac{\beta}{1} = \frac{8}{2} = \frac{8\alpha}{2}$$

$\Rightarrow \beta = 4, \quad \alpha = 1$

for ⊥ tangent: $\quad m = -\frac{1}{2}.$

$y = -\frac{1}{2}x + \frac{4}{-1/2} = -\frac{x}{2} - 8$

$\Rightarrow x + 2y + 16 = 0$

on Comparing

$\frac{8}{1} = -\frac{\beta}{2} = \frac{8\alpha}{16} \Rightarrow \alpha = 16$

$\beta = -16$

(iii) Prove that the locus of the point of intersection of tangents to the parabola $y^2 = 4ax$ which meet at an angle θ is $(x + a)^2 \tan^2\theta = y^2 - 4ax$.

equation of tangent in slope form is

$y = mx + \frac{a}{m}$ let $P(h, k)$ be the point of intersection so.

$m^2 h - km + a = 0$

$m_1 + m_2 = \frac{k}{h}, \quad m_1 \cdot m_2 = \frac{a}{h}$

$\tan\theta = \left| \frac{m_1 - m_2}{1 + m_1 \cdot m_2} \right|$

$\Rightarrow \tan^2\theta = \frac{(m_1 + m_2)^2 - 4 m_1 \cdot m_2}{(1 + m_1 m_2)^2}$

$\Rightarrow \left(1 + \frac{a}{h}\right)^2 \tan^2\theta = \frac{k^2}{h^2} - \frac{4a}{h}$

$\Rightarrow (h + a)^2 \tan^2\theta = (k^2 - 4ah)$

$\Rightarrow (x + a)^2 \tan^2\theta = y^2 - 4ax$

Proved

Do yourself - 4 :

(i) If three distinct and real normals can be drawn to $y^2 = 8x$ from the point (a, 0), then -

(A) $a > 2$ (B) $a \in (2, 4)$ (C) $a > 4$ (D) none of these

Equation of normal is

$y = mx - 2am - am^3$ where a = 2

$\Rightarrow y = mx - 4m - 2m^3$ also $P(a, 0)$

$\Rightarrow 0 = am - 4m - 2m^3$

$\Rightarrow m(2m^2 + 4 - a) = 0$

$\Rightarrow m = 0,\quad m^2 = \frac{a-4}{2} \geq 0$

$\Rightarrow a \geq 4 \Rightarrow$ option (C)

(ii) Find the number of distinct normal that can be drawn from (–2, 1) to the parabola $y^2 - 4x - 2y - 3 = 0$.

$(y-1)^2 = 4x+3+1 = 4(x+1)$

Equation of normal is

$y-1 = m(x+1) - 2m - m^3$.

passes through $(-2,1)$ so,

$0 = m(-1) - 2m - m^3$.

$\Rightarrow m^3 + 3m = 0 \Rightarrow m(m^2+3)=0$

$\Rightarrow m = 0$

So, only one normal.

(iii) If $2x + y + k = 0$ is a normal to the parabola $y^2 = -16x$, then find the value of k.

$y^2 = 4(-4)x$, $m = -2$

equation of normal is

$y = mx - 2(-4)m - (-4)m^3$

$\Rightarrow y = -2x + 8(-2) + 4(-2)^3$

$\Rightarrow y = -2x - 16 - 32$

$\Rightarrow y + 2x + 48 = 0$

given $y + 2x + k = 0$

$\Rightarrow k = 48$

(iv) Three normals are drawn from the point (7, 14) to the parabola $x^2 - 8x - 16y = 0$. Find the co-ordinates of the feet of the normals.

$$x^2 - 8x = 16y \Rightarrow (x-4)^2 = 16y + 16$$

$$\Rightarrow (x-4)^2 = 16(y+1) \quad \text{---} \quad (1)$$

Equation of normal

$$x - 4 = \frac{y+1}{m} - \frac{8}{m} - \frac{4}{m^3} \quad \cdots (2)$$

$$3 = \frac{15-8}{m} - \frac{4}{m^3} \Rightarrow 3m^3 - 7m^2 + 4 = 0$$

$$\Rightarrow (m-1)(m-2)(3m+2) = 0 \Rightarrow m = 1, 2, -\frac{2}{3}$$

$\underline{m = 1} \Rightarrow (2) \Rightarrow (x-4) = (y+1) - 12$

$$\Rightarrow (x-4) = y - 11$$

$$(1) \Rightarrow (y-11)^2 = 16(y+1)$$

on solving $(-4, 3)$

$\underline{\text{for } m = 2}$: $(2) \Rightarrow x - 4 = \frac{y+1}{2} - \frac{8}{2} - \frac{4}{8}$

$$\Rightarrow x - 4 = \frac{y+1-9}{2} = \frac{y-8}{2}$$

$$(1) \Rightarrow \left(\frac{y-8}{2}\right)^2 = 16(y+1) \Rightarrow (0, 0)$$

$\underline{\text{for } m = -2/3}$ $(2) \Rightarrow x - 4 = \frac{-3(y+1)}{2} + 12 + \frac{27}{2}$

on solving $(16, 8)$

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Do yourself - 5 :

Find the angle between the tangents drawn from the origin to the parabola, $y^2 = 4a(x - a)$.

Equation of tangent is

$$y = m(x-a) + \frac{a}{m} \qquad P(0,0)$$

$$\Rightarrow 0 = -am + \frac{a}{m}$$

$$\Rightarrow a\left(\frac{1}{m} - m\right) = 0$$

$$m^2 - 1 = 0 \Rightarrow m_1 \cdot m_2 = -1.$$

So, angle is $\pi/2$

Do yourself - 6 :

(i) Find the equation of the chord of contacts of tangents drawn from a point (2, 1) to the parabola $x^2 = 2y$.

Equation of chord of contact : $T = 0$

$\Rightarrow 2x = y + 1$ Ans.

(ii) Find the co-ordinates of the middle point of the chord of the parabola $y^2 = 16x$, the equation of which is $2x - 3y + 8 = 0$

$P(\alpha, \beta)$ then required equation is.

$\beta y - 8(x+\alpha) = \beta^2 - 16\alpha.$

$\Rightarrow \quad 8x - \beta y + 8\alpha + \beta^2 - 16\alpha = 0$

$\Rightarrow \quad 8x - \beta y + \beta^2 - 8\alpha = 0 \quad — (1)$

$$\frac{8}{2} = \frac{-\beta}{-3} = \frac{\beta^2 - 8\alpha}{8}$$

$\Rightarrow \quad \beta = 12, \qquad \beta^2 - 8\alpha = 32$

$8\alpha = 144 - 32$

$8\alpha = 112$

$\therefore \alpha = 14$

So required point is $(14, 12)$

(iii) Find the locus of the mid-point of the chords of the parabola $y^2 = 4ax$ such that tangent at the extremities of the chords are perpendicular.

$P(\alpha, \beta)$ so, chord with given mid point is $T = S_1$

$\Rightarrow \beta y - 2a(x+\alpha) = \beta^2 - 4a\alpha$

This chord passes through $(a, 0)$ so,

$0 - 2a(a+\alpha) = \beta^2 - 4a\alpha$

$\Rightarrow -2a^2 - 2a\alpha = \beta^2 - 4a\alpha$

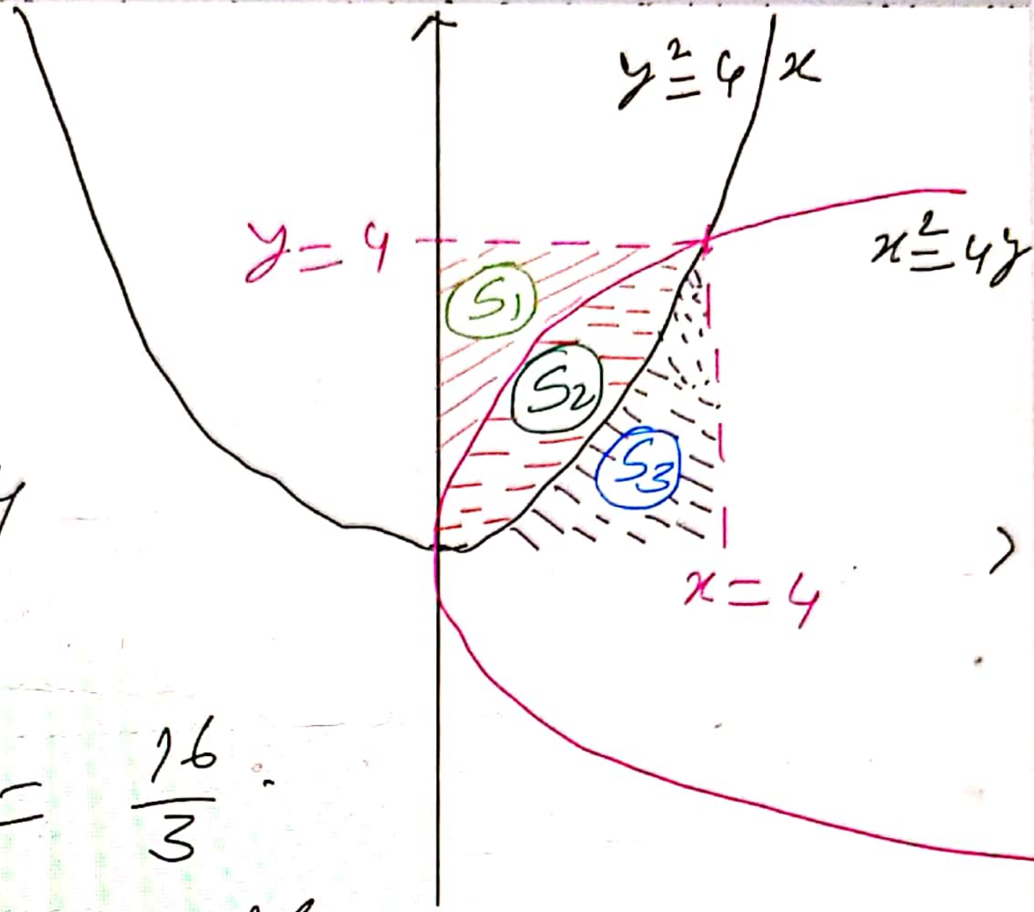
$\Rightarrow y^2 = 2ax - 2a^2 = 2a(x-a)$

required eq^n is $y^2 = 2a(x-a)$

Do yourself - 7 :

(i) The parabola $y^2 = 4x$ and $x^2 = 4y$ divide the square region bounded by the line $x = 4$, $y = 4$ and the co-ordinates axes. If S_1, S_2, S_3 are respectively the areas of these parts numbered from top to bottom; then find $S_1 : S_2 : S_3$.

Area S_1 is



$$= \int_0^4 \frac{x^2}{4}\,dx$$

$$= \frac{1}{4}\left(\frac{x^3}{3}\right)_0^4$$

$$= \frac{64}{4\times 3} = \frac{16}{3}.$$

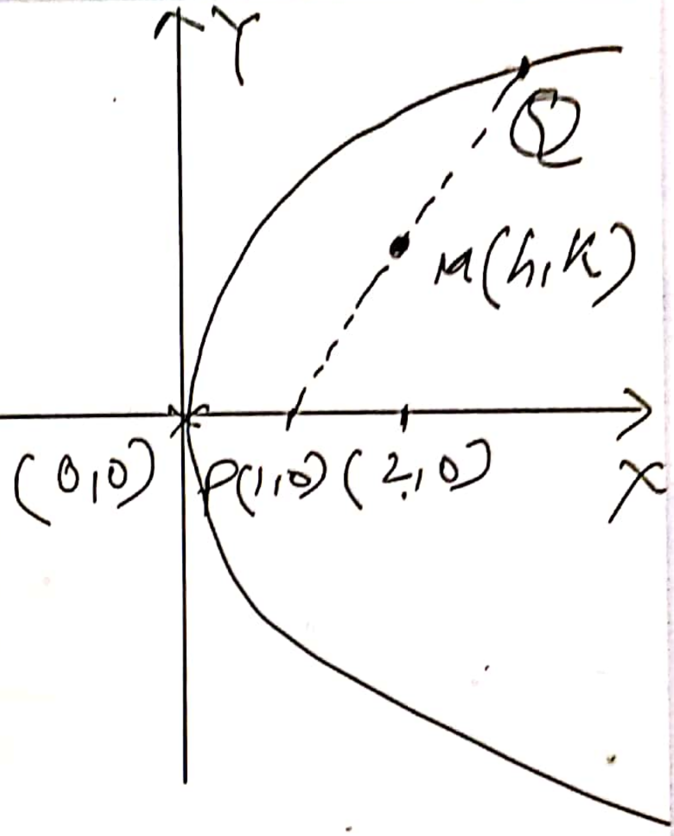
Similarly $S_3 = \frac{16}{3}$.

and $S_2 = 16 - \left(\frac{16}{3} + \frac{16}{3}\right) = \frac{16}{3}$.

$\therefore S_1 : S_2 : S_3 = 1 : 1 : 1$

(ii) Let P be the point (1, 0) and Q a point on the parabola $y^2 = 8x$, then find the locus of the mid point of PQ.

$Q(\alpha, \beta)$ let <u>So.</u>



$h = \frac{\alpha+1}{2}, \quad k = \frac{\beta}{2}$

$\Rightarrow \alpha = 2h - 1$

$\& \ \beta = 2k$

<u>So.</u> $(2k)^2 = 8(2h-1)$

$\Rightarrow \cancel{4}y^2 = \overset{2}{\cancel{8}}(2x-1)$

$\Rightarrow y^2 = 2(2x-1) \Rightarrow y^2 - 4x + 2 = 0$

0-1

1. The equation of the directrix of the parabola, $y^2 + 4y + 4x + 2 = 0$ is -

(A) $x = -1$ (B) $x = 1$ (C) $x = -3/2$ (D) $x = 3/2$

<u>Sol.</u> $y^2 + 4y + 4x + 2 = 0$

$\Rightarrow y^2 + 4y + 4 = -4x + 2$

$\Rightarrow (y+2)^2 = -4(x - 1/2)$

$\Rightarrow$ Directrix $\Rightarrow x - 1/2 = 1$

$\Rightarrow x = 3/2$

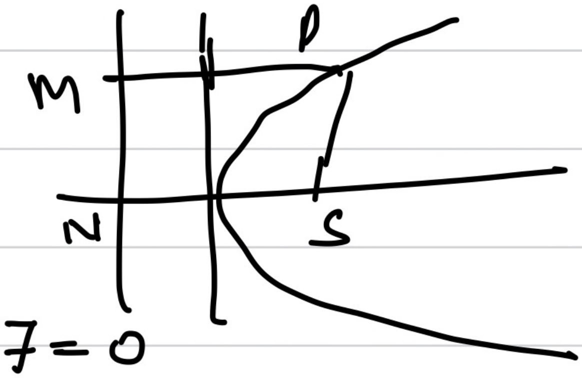
2. Length of the latus rectum of the parabola $25[(x-2)^2+(y-3)^2]=(3x-4y+7)^2$ is-

(A) 4 (B) 2 (C) 1/5 (D) 2/5

Sol.

$$(x-2)^2+(y-3)^2=\left(\frac{3x-4y+7}{5}\right)^2$$

$\Rightarrow PS^2 = PM^2$



$\therefore$ Focus $=(2,3)$

Directrix $= 3x-4y+7=0$

$\therefore$ Length of latusrectum $= 2\,PN$

$$= 2\left|\frac{6-12+7}{5}\right| = \frac{2}{5}$$

3. If the line $x - 1 = 0$ is the directrix of the parabola $y^2 - kx + 8 = 0$ then one of the values of 'k' is -

(A) 1/8 (B) 8 (C) 4 (D) 1/4

Sol. $y^2 = Kx + 8$

$\Rightarrow y^2 = Kx - 8$

$\Rightarrow y^2 = K(x - 8/K)$ $\quad 4a = K$

$\therefore$ eqn of directrix

$= x - \frac{8}{K} = -K/4$ which satisfy

$x = 1$ so $1 - \frac{8}{K} + \frac{K}{4} = 0$

$\Rightarrow K = 4$

4. The length of the intercept on y – axis cut off by the parabola, $y^2 - 5y = 3x - 6$ is

(A) 1 (B) 2 (C) 3 (D) 5

Sol. for y intercept, $x = 0$

so $y^2 - 5y + 6 = 0$

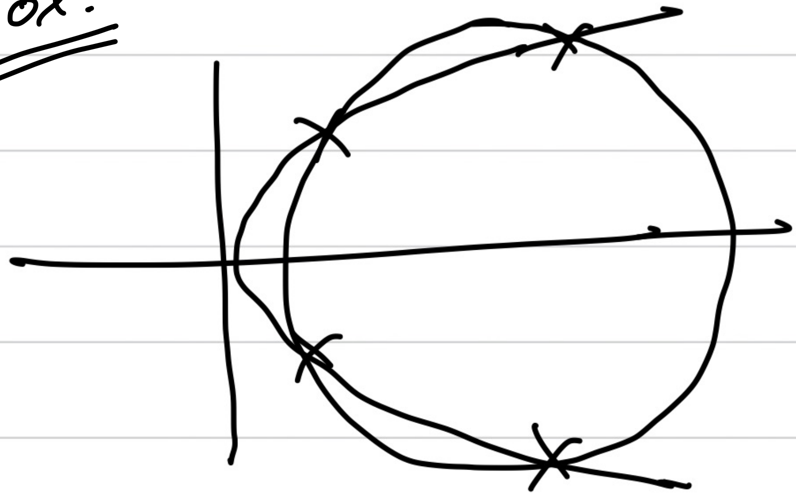
or $y = 3, 2$

$\therefore |y_1 - y_2| = 1$

5. Maximum number of common chords of a parabola and a circle can be equal to

(A) 2 (B) 4 (C) 6 (D) 8

Sol:-



MAX. NO. of common chords $= {}^4C_2$

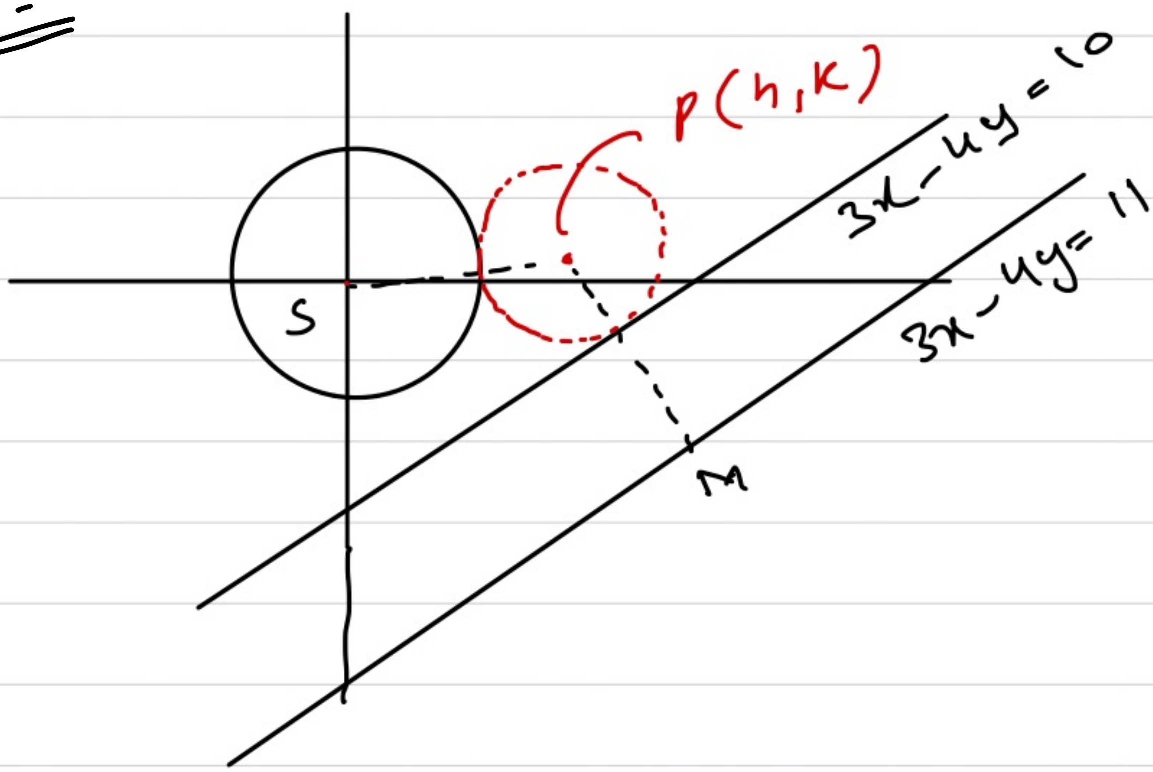
$= 6$

6. A variable circle is drawn to touch the line $3x - 4y = 10$ and also the circle $x^2 + y^2 = 1$ externally then the locus of its centre is -

(A) straight line (B) circle

(C) pair of real, distinct straight lines (D) parabola

<u>sol:-</u>



By diagrame PS = PM

$$\sqrt{h^2 + k^2} = \left|\frac{3h - 4k - 11}{5}\right|$$

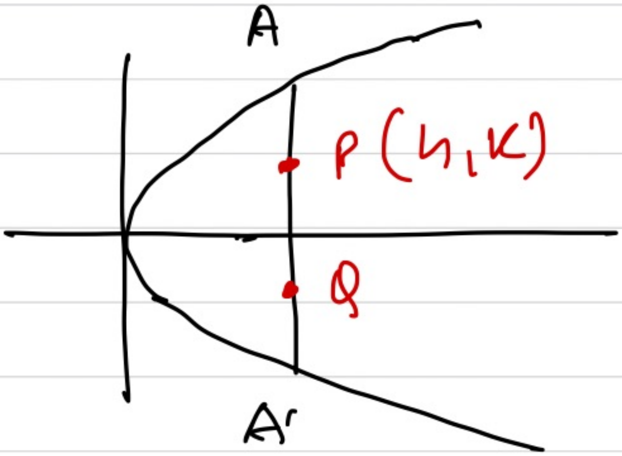
or $25(h^2 + k^2) = (3h - 4k - 11)^2$

which is parabola.

7. The locus of the point of trisection of all the double ordinates of the parabola $y^2 = \ell x$ is a parabola whose latus rectum is -

(A) $\frac{\ell}{9}$ (B) $\frac{2\ell}{9}$ (C) $\frac{4\ell}{9}$ (D) $\frac{\ell}{36}$

Sol.



Let $y^2 = 4ax$

$4a = \ell$

$A(at^2, 2at)$, $A'(at^2, -2at)$

$P(h, k)$

P divides AA' in 1 : 2 ratio

So $P\left(at^2, \frac{2}{3}at\right) \equiv (h, k)$

$y^2 = (\ell/9)\, x$

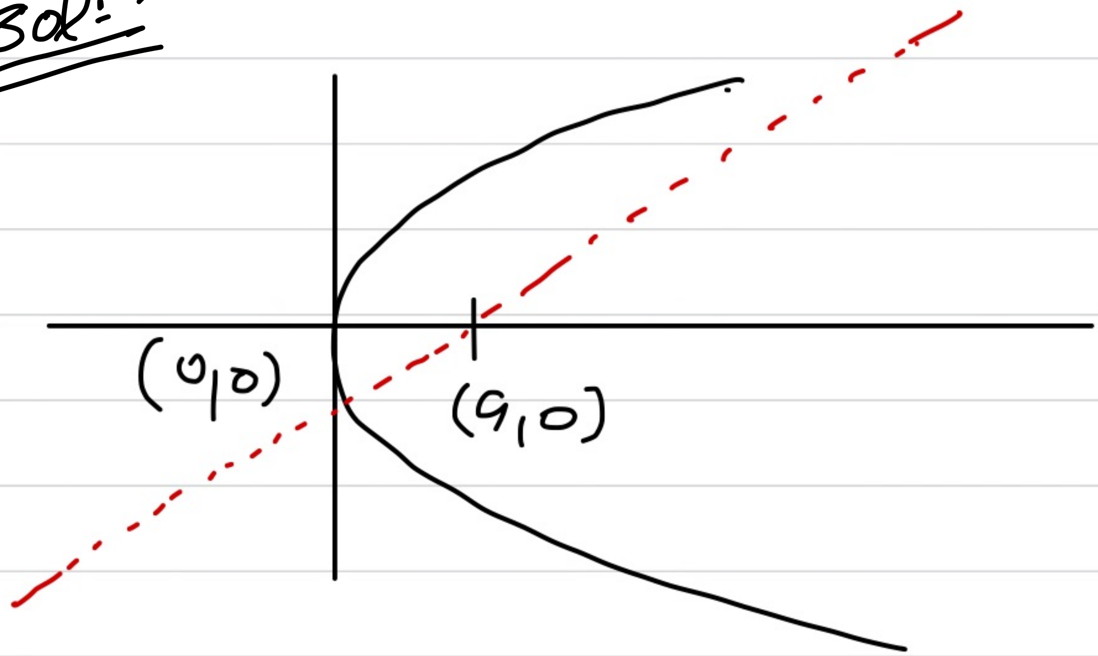
$\therefore$ L.R. $= \boxed{\frac{\ell}{9}}$ Ans.

8. The straight line $y = m(x - a)$ will meet the parabola $y^2 = 4ax$ in two distinct real points if

(A) $m \in R$ (B) $m \in [-1, 1]$

(C) $m \in (-\infty, 1] \cup [1, \infty)$ (D) $m \in R - \{0\}$

Sol:-



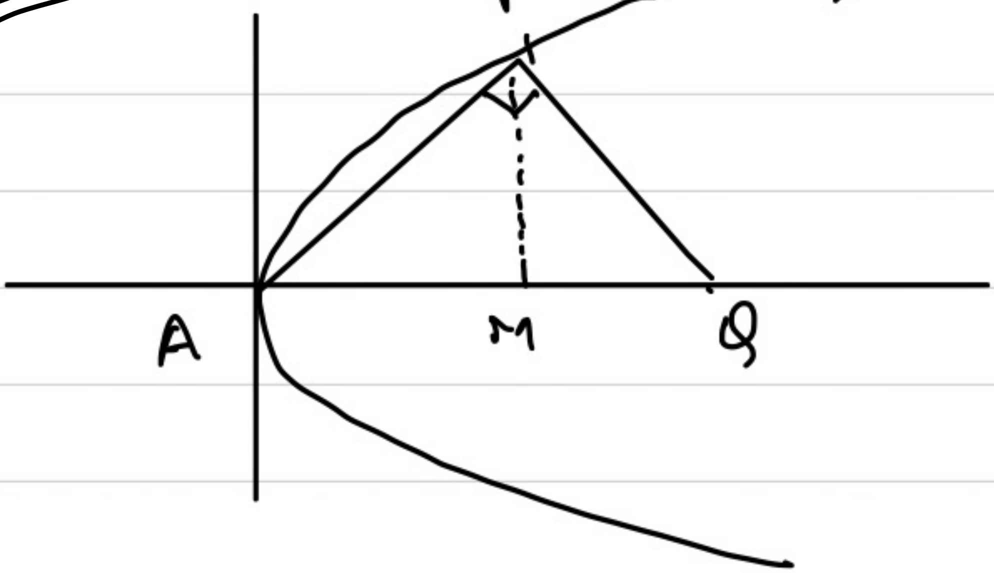
The line always cuts the parabola two distinct points except ~~for~~ $m = 0$ (By diagram)

9. The vertex A of the parabola $y^2 = 4ax$ is joined to any point P on it and PQ is drawn at right angles to AP to meet the axis in Q. Projection of PQ on the axis is equal to

(A) twice the length of latus rectum (B) the latus length of rectum

(C) half the length of latus rectum (D) one fourth of the length of latus rectum

Sol :-



Eqⁿ of AP = $y = \frac{2}{t}(x) \Rightarrow 2x - ty = 0$

Eqⁿ of PQ = $tx + 2y + c = 0$

$= tx + 2y - (at^3 + 4at) = 0$

$\therefore M(at^2, 0)$

$Q(at^2 + 4a, 0)$

$\therefore MQ = \text{projection} = 4a$

10. The equation of the circle drawn with the focus of the parabola $(x - 1)^2 - 8y = 0$ as its centre and touching the parabola at its vertex is :

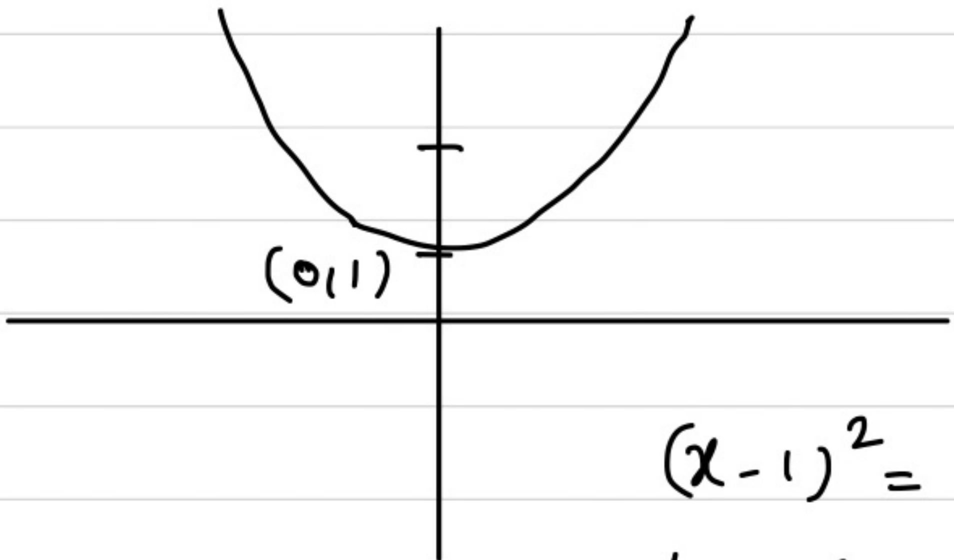
(A) $x^2 + y^2 - 4y = 0$

(B) $x^2 + y^2 - 4y + 1 = 0$

(C) $x^2 + y^2 - 2x - 4y = 0$

(D) $x^2 + y^2 - 2x - 4y + 1 = 0$

Sol:-



$(x-1)^2 = 8y$

$4a = 8, \Rightarrow a = 2$

focus $\Rightarrow$ $\left.\begin{array}{l} x-1 = 0 \\ y = 2 \end{array}\right\} (1,2)$

Eqn of circle whose centre is (1,2) & passing through centre so radius will be 2

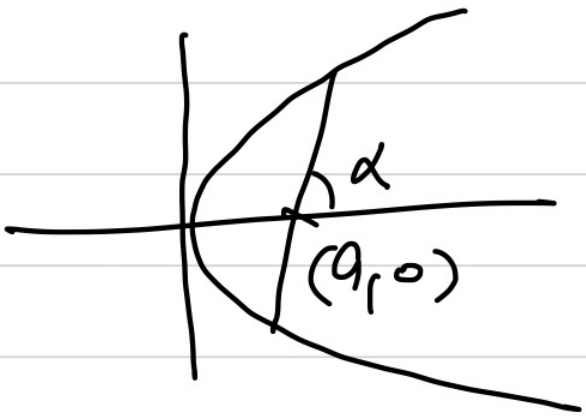
$\Rightarrow (x-1)^2 + (y-2)^2 = 4$

$\Rightarrow x^2 + y^2 - 2x - 4y + 1 = 0$

11. If a focal chord of $y^2 = 4x$ makes an angle $\alpha, \alpha \in \left(0, \frac{\pi}{4}\right]$ with the positive direction of x-axis, then minimum length of this focal chord is -

(A) $2\sqrt{2}$ (B) $4\sqrt{2}$ (C) 8 (D) 16

Sol.



Let angle of focal chord from x axis is α then length of focal chord is $4a\,\mathrm{cosec}^2\alpha$

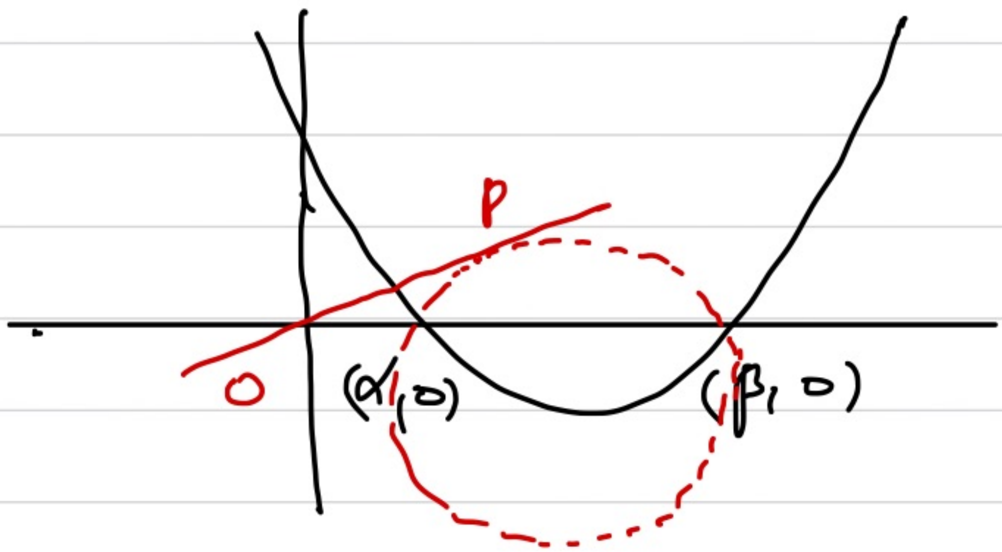
$\therefore$ min length $= 4a(\mathrm{cosec}^2 \pi/4)$

$= 4 \times 2 = 8$

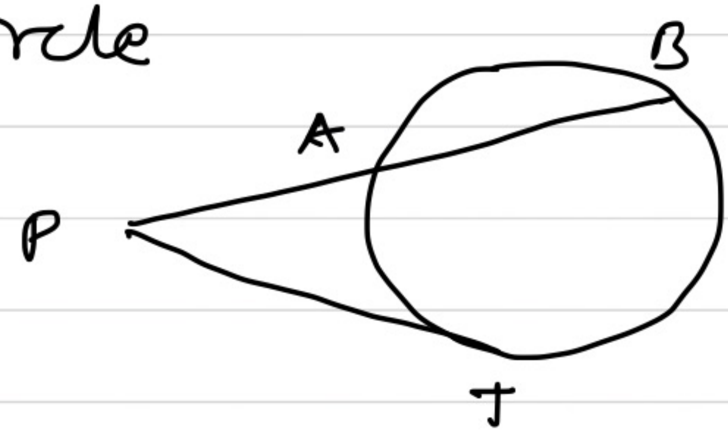
12. A parabola $y = ax^2 + bx + c$ crosses the x-axis at $(\alpha,0)$ $(\beta,0)$ both to the right of the origin. A circle also passess through these two points. The length of a tangent from the origin to the circle is :

(A) $\sqrt{\frac{bc}{a}}$ (B) ac^2 (C) $\frac{b}{a}$ (D) $\sqrt{\frac{c}{a}}$

Sol.



we know in circle



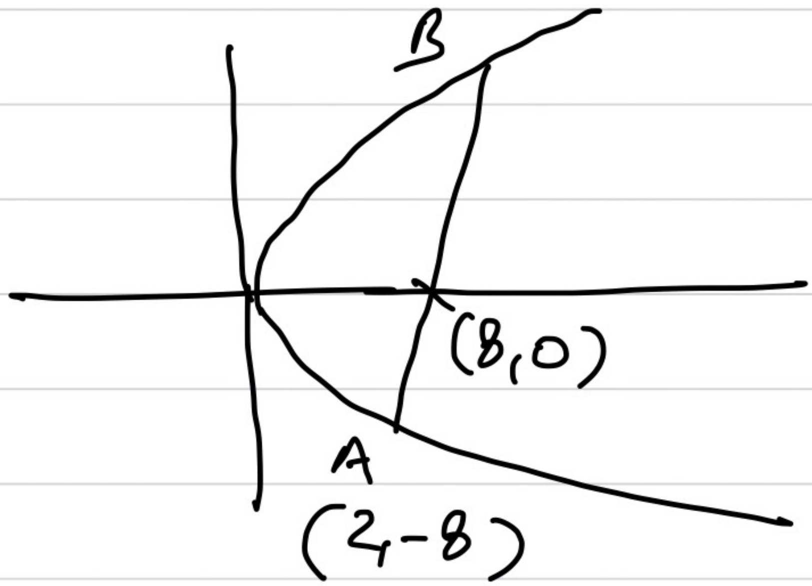
$$PA \cdot PB = PT^2$$

$$OP = \sqrt{\alpha \cdot \beta} = \sqrt{c/a}$$

13. If (2,–8) is one end of a focal chord of the parabola $y^2 = 32x$, then the other end of the focal chord, is-

(A) (32,32) (B) (32,–32) (C) (–2,8) (D) (2,8)

Sol.



$y^2 = 32x$

$\underline{a = 8}$

$\boxed{t_1 t_2 = -1}$

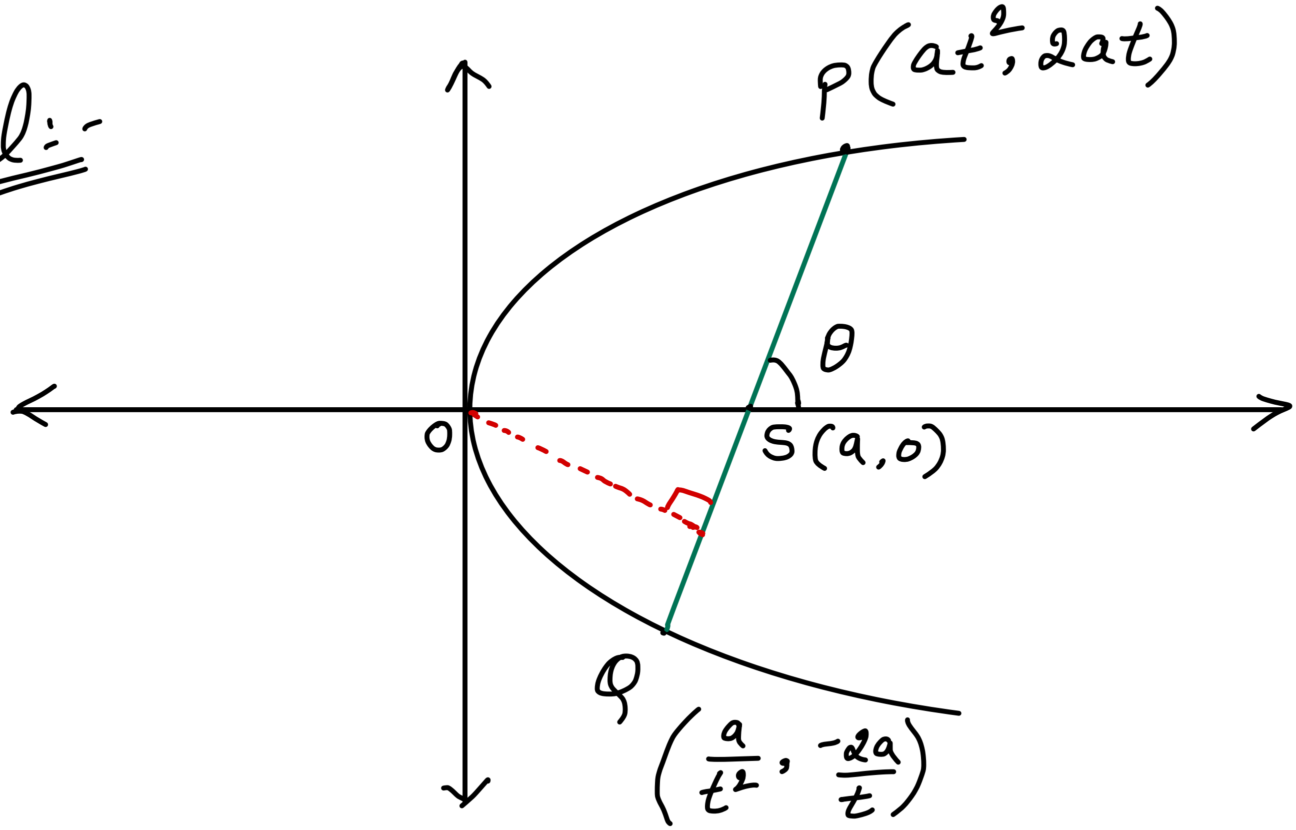
$A\ (at_1^2, 2at_1) = (2, -8) \Rightarrow t = -\frac{1}{2}$

$B\ \left(\frac{a}{t_1^2}, \frac{-2a}{t_1}\right) = (32, 32)$

14. The length of a focal chord of the parabola $y^2 = 4ax$ at a distance b from the vertex is c, then

(A) $2a^2 = bc$ (B) $a^3 = b^2c$ (C) $ac = b^2$ (D) $b^2c = 4a^3$

Sol:-



OM = b

focal length $PQ = 4a \operatorname{cosec}^2\theta$ —— (1)

From right angled triangle OMS,

$$\sin\theta = \frac{OM}{OS} = \frac{b}{a}$$

$\therefore \operatorname{cosec}\theta = \frac{a}{b}$ —— (2)

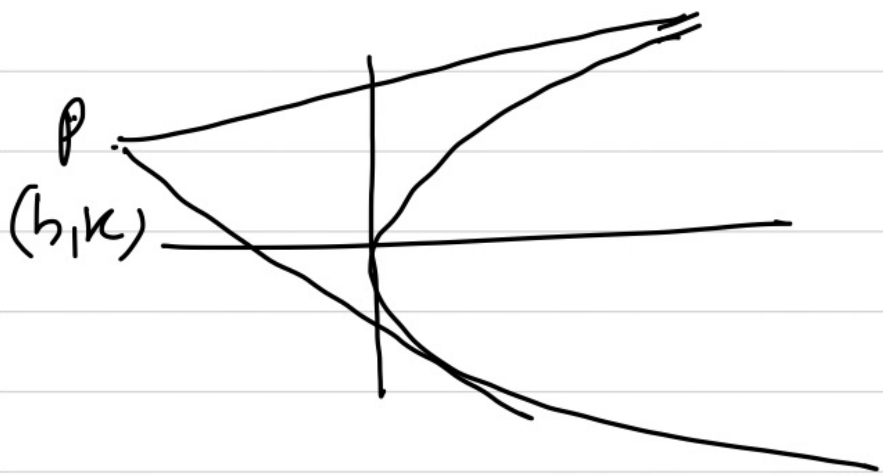
from (1) & (2), $PQ = c = 4a \cdot \left(\frac{a}{b}\right)^2$

$\Rightarrow b^2c = 4a^3$ Ans.

15. From an external point P, pair of tangent lines are drawn to the parabola, $y^2 = 4x$. If θ_1 & θ_2 are the inclinations of these tangents with the axis of x such that, $\theta_1 + \theta_2 = \frac{\pi}{4}$, then the locus of P is :

(A) $x - y + 1 = 0$ (B) $x + y - 1 = 0$ (C) $x - y - 1 = 0$ (D) $x + y + 1 = 0$

Sol:-



Eqn of tangent for $y^2 = 4x$

$\Rightarrow y = mx + \frac{1}{m}$

$\Rightarrow k = mh + \frac{1}{m}$

$\Rightarrow m^2h - mk + 1 = 0 < \begin{matrix} m_1 \\ m_2 \end{matrix}$

$\theta_1 + \theta_2 = \pi/4$ (Given)

$$\tan(\theta_1+\theta_2) = 1$$

$$\Rightarrow \frac{m_1+m_2}{1-m_1 m_2} = 1$$

$$\Rightarrow \frac{k/h}{1-1/h} = 1$$

$$\Rightarrow \frac{k}{h-1} = 1$$

$$\Rightarrow \boxed{x-y-1=0}$$

16. y-intercept of the common tangent to the parabola $y^2 = 32x$ and $x^2 = 108y$ is

(A) – 18 (B) – 12 (C) – 9 (D) – 6

<u>Sol.</u> Let eqn of tangent

for $y^2 = 32x$ is

$$y = mx + 8/m$$

also tangent for

$$x^2 = 108y$$

$\Rightarrow x^2 = 108(mx + 8/m)$

$\Rightarrow D = 0$ (for tangent)

$\Rightarrow m = -2/3$

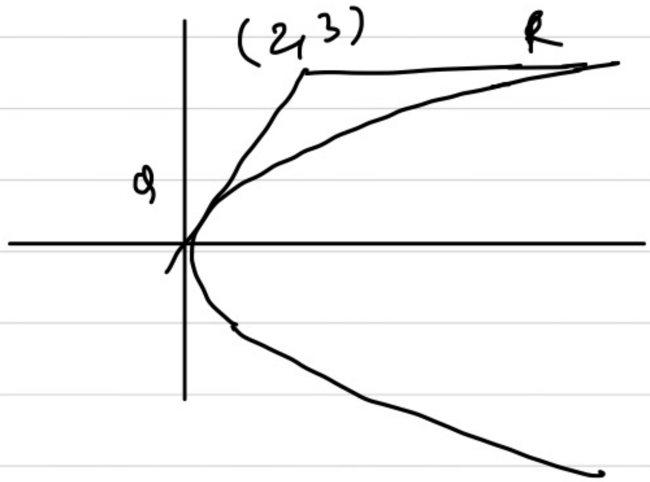
Eqn of tangent $y = \frac{2}{3}x - 12$

$\therefore \quad y = -12$

17. The points of contact Q and R of tangent from the point P (2, 3) on the parabola $y^2 = 4x$ are

(A) (9, 6) and (1, 2) (B) (1, 2) and (4, 4) (C) (4, 4) and (9, 6) (D) (9, 6) and $(\frac{1}{4}, 1)$

Sol.



Let Eqn of tangent is

$y = mx + 1/m$ which

passes through (2,3)

$\Rightarrow 3 = 2m + \frac{1}{m}$

$\Rightarrow m = 1, \ 1/2$

So point of contacts are

$(a/m^2, 2a/m)$

$\therefore (1,2)$ & $(4,4)$

18. If the lines $(y - b) = m_1(x + a)$ and $(y - b) = m_2(x + a)$ are the tangents to the parabola $y^2 = 4ax$, then

(A) $m_1 + m_2 = 0$ (B) $m_1m_2 = 1$ (C) $m_1m_2 = -1$ (D) $m_1 + m_2 = 1$

Sol: Both lines are passing through $(-a, b)$ & tangent to parabola so by property tangent are perpendicular from Directrix of parabola.

$$\therefore \boxed{m_1 m_2 = -1}$$

19 The equation of the common tangent touching the circle $(x-3)^2+y^2=9$ and the parabola $y^2=4x$ above the x-axis is -

(A) $\sqrt{3}y = 3x+1$ (B) $\sqrt{3}y = -(x+3)$ (C) $\sqrt{3}y = x+3$ (D) $\sqrt{3}y = -(3x+1)$

any tangent of given parabola is :

$$ty = x + t^2$$

If it also touch given circle $(x-3)^2+y^2=9$,

then :

$\perp^r$ distance = radius

$$\left|\frac{3+t^2}{\sqrt{1+t^2}}\right| = 3$$

$$\therefore \quad 9+t^4+6t^2 = 9+9t^2$$

$$t^4-3t^2 = 0 \Rightarrow t=0, \quad t=\pm\sqrt{3}$$

since tangent touches above x-axis $\Rightarrow t=\sqrt{3}$.

eq^n of tangent :

$$\sqrt{3}y = x+3$$

20 If $x + y = k$ is normal to $y^2 = 12x$, then 'k' is-

(A) 3 (B) 9 (C) –9 (D) –3

any Normal :

$$y = mx - 6m - 3m^3$$

$\therefore \quad m = -1$

$\therefore \quad y = -x + 6 + 3$

$$x + y = 9$$

$\therefore \quad K = 9$

21 Equation of the other normal to the parabola $y^2 = 4x$ which passes through the intersection of those at (4,–4) and (9a, –6a) is -

(A) $5x - y + 115 = 0$ (B) $5x + y - 135 = 0$ (C) $5x - y - 115 = 0$ (D) $5x + y + 115 = 0$

eqⁿ of Normal at $(4,-4)$: $y+4 = 2(x-4)$

$y = 2x - 12$ —— ①

eqⁿ of Normal at $(9,-6)$: $y+6 = 3(x-9)$

$y = 3x - 33$ —— ②

by Solving ① and ② : $x = 21,\ y = 30$

eqⁿ of any Normal :

$y = mx - 2m - m^3$, If it passes through $(21, 30)$

$30 = 21m - 2m - m^3$

$\Rightarrow m = -5$

$\therefore$ eqⁿ of Normal : $\boxed{5x + y - 135 = 0}$ Ans.

22 Length of the normal chord of the parabola, $y^2 = 4x$, which makes an angle of $\frac{\pi}{4}$ with the axis of x is:

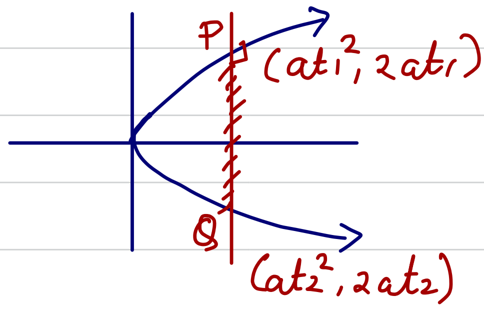
(A) 8 (B) $8\sqrt{2}$ (C) 4 (D) $4\sqrt{2}$

eqⁿ of Normal at $P(at_1^2, 2at_1)$:

$$y = mx - 2m - m^3$$

$$m = \tan\frac{\pi}{4} \Rightarrow m = 1 \Rightarrow t_1 = -1$$

$P \equiv (1, -2)$



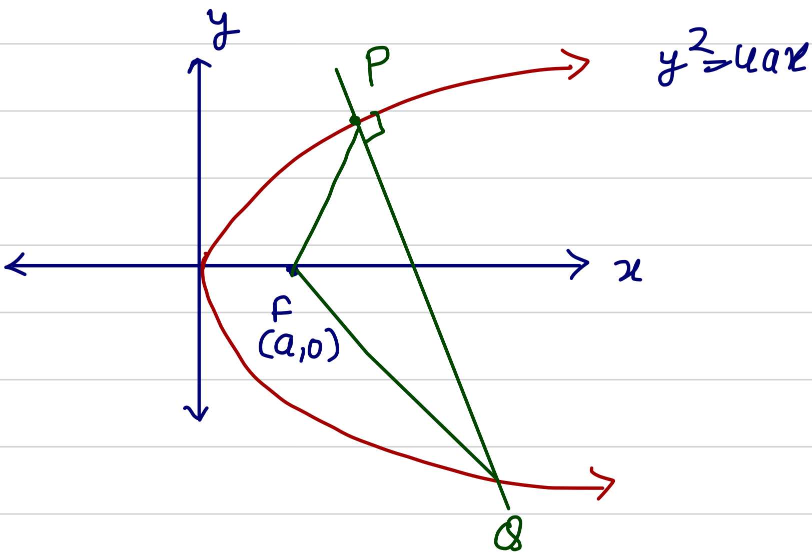
$$t_2 = -t_1 - \frac{2}{t_1} = 1 + 2 = 3$$

$Q \equiv (9, 6)$

$$PQ = \sqrt{64 + 64} = 8\sqrt{2}$$

23 The normal chord of a parabola $y^2 = 4ax$ at the point whose ordinate is equal to the abscissa, then angle subtended by normal chord at the focus is :

(A) $\frac{\pi}{4}$ (B) $\tan^{-1}\sqrt{2}$ (C) $\tan^{-1}2$ (D) $\frac{\pi}{2}$



Let P is $(at^2, 2at)$

since $at^2 = 2at$

$t = 2$

Point P is $(4a, 4a)$

Let Q is $(at_2^2, 2at_2)$:

$$t_2 = -t - \frac{2}{t}$$

$$= -2 - \frac{2}{2} = -3$$

$$Q \equiv (9a, -6a)$$

$M_1 = M_{PF} = \frac{4}{3}$, $M_2 = M_{QF} = -\frac{3}{4}$

$M_1 \cdot M_2 = -1 \Rightarrow \theta = 90^\circ$

24 Which one of the following lines cannot be the normals to $x^2 = 4y$?

(A) $x - y + 3 = 0$ (B) $x + y - 3 = 0$ (C) $x - 2y + 12 = 0$ (D) $x + 2y + 12 = 0$

equation of Parabola is $x^2 = 4y$

eq^n of Normal:

$$ty = -x + 2t + t^3$$

(A) Normal at $t = -1$

(B) Normal at $t = 1$

(C) Normal at $t = -2$

(d) Not a Normal

25 Tangents are drawn from the points on the line $x - y + 3 = 0$ to parabola $y^2 = 8x$. Then the variable chords of contact pass through a fixed point whose coordinates are :

(A) (3,2) (B) (2,4) (C) (3,4) (D) (4,1)

any point on line '$x-y+3=0$' is $P(x_1, x_1+3)$

then equation of chord of contact wrt to P.

$$y(x_1+3) = 4(x+x_1)$$

$$yx_1+3y = 4x+4x_1$$

$$\underbrace{4x-3y}_{L_1} + x_1\underbrace{(4-y)}_{L_2} = 0$$

$L_1 + \lambda L_2 = 0$, familly of lines always passes through fixed point, which is intersection of

$4x-3y=0$ & $y=4$ is $(3,4)$

= Ans

26 Tangents are drawn from the point $(-1, 2)$ on the parabola $y^2 = 4x$. The length, these tangents will intercept on the line $x = 2$ is :

(A) 6 (B) $6\sqrt{2}$ (C) $2\sqrt{6}$ (D) none of these

any tangent of $y^2=4x$:

$$ty = x + t^2$$

If it passes through $(-1,2)$.

$$2y = -1 + t^2 \Rightarrow t^2 - 2y = 1$$

$$(t-1)^2 = 2$$

$$t = 1 \pm \sqrt{2}$$

eqⁿ of tangents:

$$(1-\sqrt{2})y = x + (1-\sqrt{2})^2$$

&

$$(1+\sqrt{2})y = x + (1+\sqrt{2})^2$$

Intercept on line $x=2$:

$$|y_1 - y_2| = \left| \frac{2+(1-\sqrt{2})^2}{1-\sqrt{2}} - \frac{2+(1+\sqrt{2})^2}{1+\sqrt{2}} \right| = 6\sqrt{2}$$

27 If the locus of the middle points of the chords of the parabola $y^2 = 2x$ which touches the circle $x^2 + y^2 - 2x - 4 = 0$ is given by $(y^2 + 1 - x)^2 = \lambda(1 + y^2)$, then the value of λ is equal to-

(A) 3 (B) 4 (C) 5 (D) 6

equation of chord of $y^2=2x$, whose mid point is (h,k).

$$ky-(x+h) = k^2-2h$$

$$x-ky+k^2-h=0$$

If it touches circle $x^2+y^2-2x-4=0$.

$\therefore$ $\perp$r distance from center $(1,0)$ = Radius $=\sqrt{5}$

$$\left|\frac{1+k^2-h}{\sqrt{1+k^2}}\right| = \sqrt{5}$$

$$(1-h+k^2)^2 = 5(1+k^2)$$

Locus is :

$$(y^2+1-x)^2 = 5(1+y^2)$$

$$\therefore \lambda = 5$$

28 The locus of the mid point of the focal radii of a variable point moving on the parabola, $y^2 = 8x$ is a parabola whose-

(A) Latus rectum is half the latus rectum of the original parabola

(B) Vertex is (1,0)

(C) Directrix is y-axis

(D) Focus has the co-ordinates (2,0)

Let any point $P(2t^2, 4t)$, focus is $(2,0)$

Let mid point of S & P is $Q(h,k)$.

$$h = \frac{2t^2+2}{2}, \quad k = \frac{4t+0}{2}$$

$$\Rightarrow \quad h = t^2 + 1, \quad k = 2t$$

$$h - 1 = \frac{k^2}{4} \Rightarrow \quad y^2 = 4(x-1)$$

Latusrectum $= 4$

Vertex $= (1,0)$

directrix $\equiv x = 0$

focus $\equiv (2,0)$

29 Consider a circle with its centre lying on the focus of the parabola, $y^2 = 2px$ such that it touches the directrix of the parabola. Then a point of intersection of the circle & the parabola is

(A) $\left(\frac{p}{2}, p\right)$ (B) $\left(\frac{p}{2}, -p\right)$ (C) $\left(-\frac{p}{2}, p\right)$ (D) $\left(-\frac{p}{2}, -p\right)$

center of circle $= \left(\frac{p}{2}, 0\right)$

Radius $= p$

eqn of circle: $\left(x - \frac{p}{2}\right)^2 + y^2 = p^2$

$\therefore$ $\left(x - \frac{p}{2}\right)^2 + 2px = p^2$

$x^2 + \frac{p^2}{4} - px + 2px = p^2$

$4x^2 + p^2 + 4px = 4p^2$

$4x^2 + 4px - 3p^2 = 0$

$4x^2 + 6px - 2px - 3p^2 = 0$

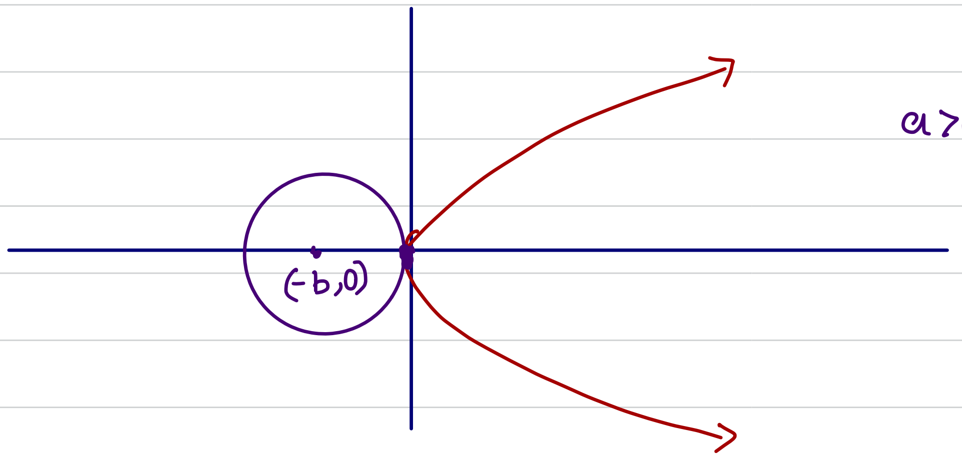
$2x(2x + 3p) - p(2x + 3p) = 0$

$x = \frac{p}{2}, \quad x = \frac{-3p}{2}$

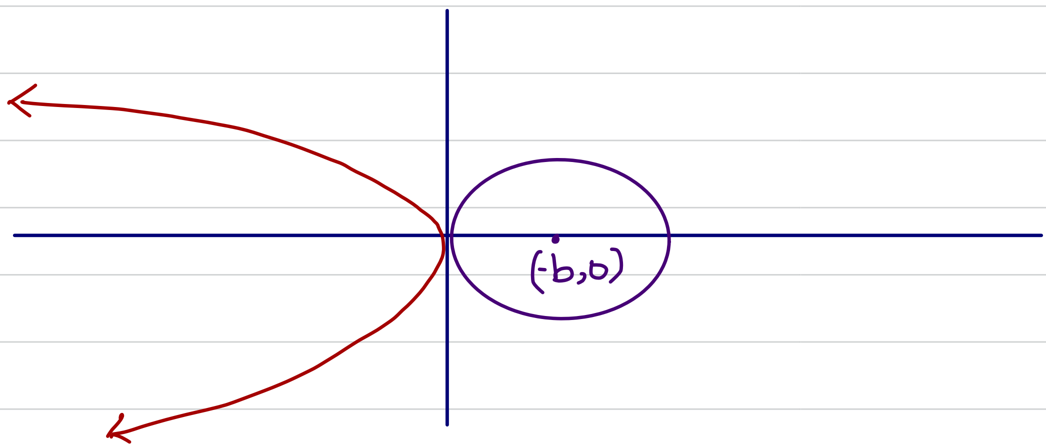
Point of Intersections are: $\left(\frac{p}{2}, p\right)$ $\left(\frac{p}{2}, -p\right)$

30 Let $y^2 = 4ax$ be a parabola and $x^2 + y^2 + 2bx = 0$ be a circle. If parabola and circle touch each other externally then :

(A) $a > 0, b > 0$ (B) $a > 0, b < 0$ (C) $a < 0, b > 0$ (D) $a < 0, b < 0$



$a > 0$ & $b > 0$



$a < 0$ & $b < 0$

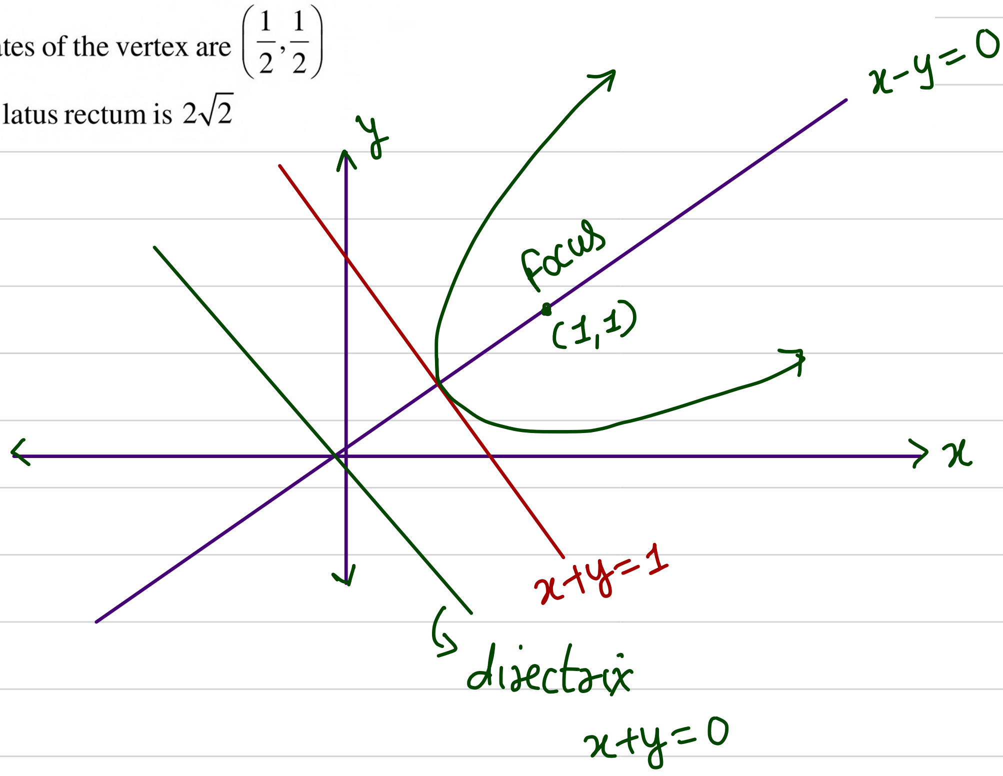
31 The focus of the parabola is (1,1) and the tangent at the vertex has the equation x + y = 1. Then :

(A) equation of the parabola is $(x-y)^2 = 2(x+y-1)$

(B) equation of the parabola is $(x-y)^2 = 4(x+y-1)$

(C) the co-ordinates of the vertex are $\left(\frac{1}{2}, \frac{1}{2}\right)$

(D) length of the latus rectum is $2\sqrt{2}$



eq^n of Parabola :

$$\left(\frac{x-y}{\sqrt{2}}\right)^2 = \frac{4}{\sqrt{2}}\left(\frac{x+y-1}{\sqrt{2}}\right)$$

$$\therefore \quad (x-y)^2 = 4(x+y-1)$$

vertex is $\left(\frac{1}{2}, \frac{1}{2}\right)$

$$L\cdot R = \frac{4}{\sqrt{2}} = 2\sqrt{2}$$

32 The straight line $y + x = 1$ touches the parabola

(A) $x^2 + 4y = 0$ (B) $x^2 - x + y = 0$ (C) $4x^2 - 3x + y = 0$ (D) $x^2 - 2x + 2y = 0$

Solve the given equations with all option's.

(a) $x^2+4y=0$

$x^2+4(1-x)=0 \Rightarrow x^2-4x+4=0 \Rightarrow (x-2)^2=0$

$\therefore$ given Line is tangent.

(b) $x^2-x+y=0$

$x^2-x+1-x=0 \Rightarrow x^2-2x+1=0 \Rightarrow (x-1)^2=0$

$\therefore$ given line is tangent.

(c) $4x^2-3x+1-x=0$

$4x^2-4x+1=0, \Rightarrow (2x-1)^2=0$

$\therefore$ given line is tangent.

(d) $x^2-2x+2y=0$

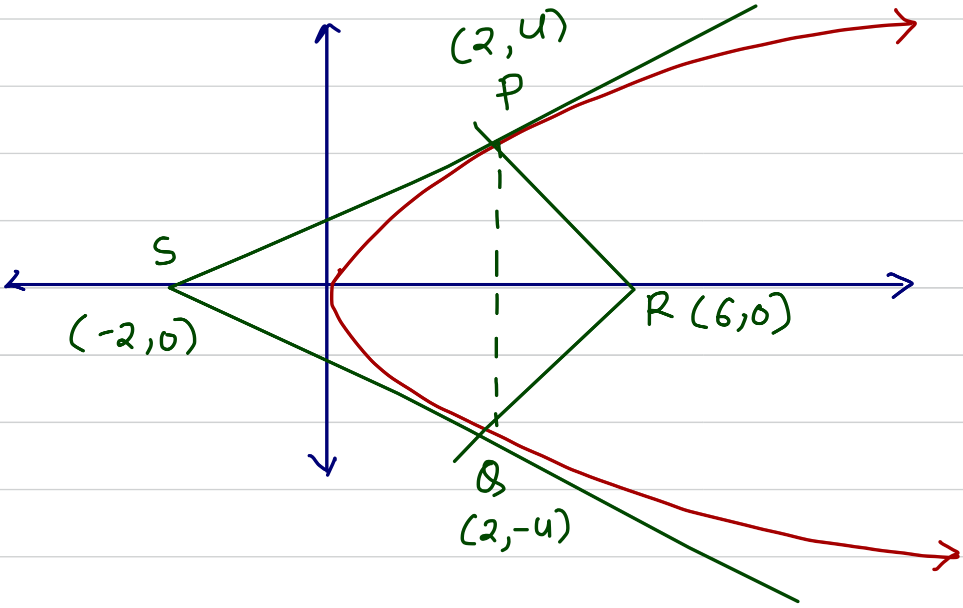
$x^2-2x+2(1-x)=0 \Rightarrow x^2-4x+2=0$

$\therefore$ given line is Not tangent.

33 Consider the parabola $y^2 = 8x$

Area of the figure formed by the tangents and normals drawn at the extremities of its latus rectum is

(A) 8 (B) 16 (C) 32 (D) 64



tangent at P(2,4) : $y = x + 2$

Normal at P(2,4) : $y = -x + 6$

area of 'PSQR' $\equiv 2\left(\frac{1}{2}(8)(4)\right) = 32$

34 Distance between the tangent to the parabola and a parallel normal inclined at 30° with the x-axis, is

(A) $\frac{16}{3}$ (B) $\frac{16\sqrt{3}}{9}$ (C) $\frac{2}{3}$ (D) $\frac{16}{\sqrt{3}}$

eqn of tangent whose slope is $\frac{1}{\sqrt{3}}$: $y = \frac{x}{\sqrt{3}} + 2\sqrt{3}$

eqn of Normal " " " : $y = \frac{1}{\sqrt{3}}x - \frac{14}{3\sqrt{3}}$

distance b/w them : $\dfrac{2\sqrt{3} + \frac{14}{3\sqrt{3}}}{\sqrt{1 + \frac{1}{3}}} = \dfrac{16}{3}$

35

Column-I (Equation of a conic)		Column-II (Nature of conic)	
(A)	$xy + a^2 = a(x + y)$	(P)	Ellipse
(B)	$2x^2 - 72xy + 23y^2 - 4x - 28y - 48 = 0$	(Q)	Hyperbola
(C)	$6x^2 - 5xy - 6y^2 + 14x + 5y + 4 = 0$	(R)	Parabola
(D)	$14x^2 - 4xy + 11y^2 - 44x - 58y + 71 = 0$	(S)	line pair
(E)	$4x^2 - 4xy + y^2 - 12x + 6y + 9 = 0$		

(A) $xy - ax - ay + a^2 = 0$

$A=0,\ B=0,\ C=a^2,\ H=\frac{1}{2},\ g=-\frac{a}{2},\ f=-\frac{a}{2}$

$\Delta = 0 + 2\left(-\frac{a}{2}\right)\left(-\frac{a}{2}\right)\left(\frac{1}{2}\right) - a^2\left(\frac{1}{4}\right) = 0 \Rightarrow$ line Pair

(B) $2x^2 - 72xy + 23y^2 - 4x - 28y - 48 = 0$

$A=2,\ B=23,\ C=-48,\ H=-36,\ g=-2,\ f=-14$

$\Delta = 2(23)(-48) + 2(-14)(-2)(-36) - 2(28) - 23(4) + 48(36)^2 \neq 0$

$H^2 - AB = 1296 - 46 > 0 \Rightarrow$ Hyperbola

(C) $6x^2 - 5xy - 6y^2 + 14x + 5y + 4 = 0$

$A=6,\ B=-6,\ C=4,\ H=-\frac{5}{2},\ g=7,\ f=\frac{5}{2}$

$\Delta = 6(-6)(4) + 2\left(\frac{5}{2}\right)(7)\left(-\frac{5}{2}\right) - 6\left(\frac{25}{4}\right) + 6(49) - 4\left(\frac{25}{4}\right) = 0$

line pair

(D) $14x^2 - 4xy + 11y^2 - 44x - 58y + 71 = 0$

$A=14,\ B=11,\ C=71,\ H=-2,\ g=-22,\ f=-29$

$\Delta = 14(11)(71) + 2(-29)(-22)(-2) - 14(-29)^2 - 11(22)^2 - 71(2)^2 \neq 0$

$H^2 - AB = 4 - 14\times 11 < 0$, ellipse

(E) $\Delta = 0$, line pair

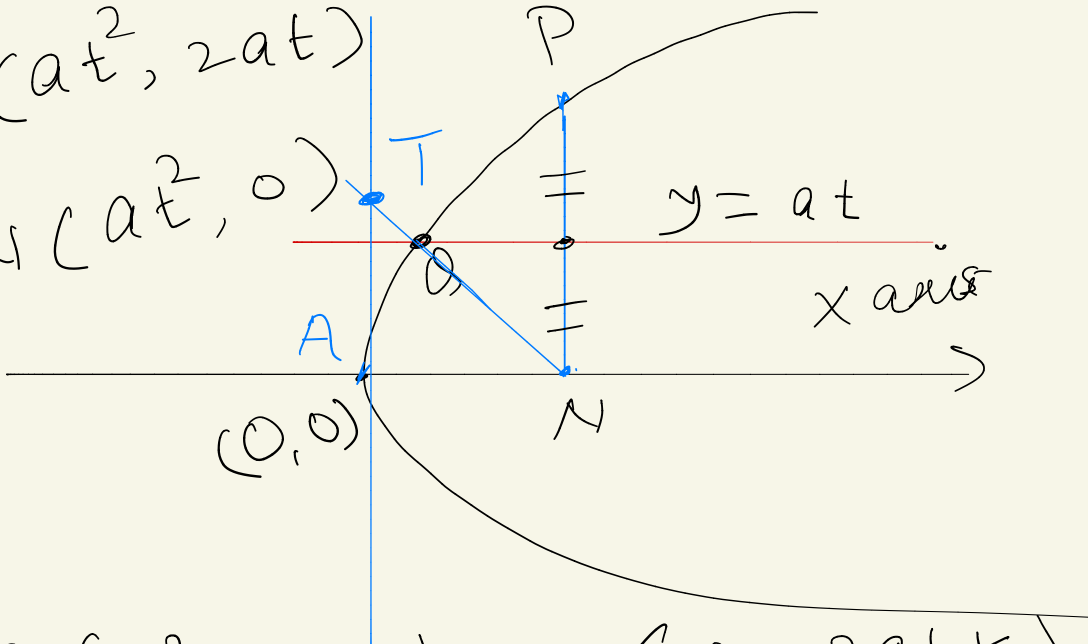
0-2

1 PN is an ordinate of the parabola $y^2 = 4ax$ (P on $y^2 = 4ax$ and N on x-axis). A straight line is drawn parallel to the axis to bisect NP and meets the curve in Q. NQ meets the tangent at the vertex in a point T such that AT = kNP, then the value of k is (where A is the vertex)

(A) 3/2 (B) 2/3 (C) 1 (D) none

$P(at^2, 2at)$

$N(at^2, 0)$



$Q\left(\frac{at^2}{4}, at\right)$, $T(0, 2atk)$

equation of TN

$\Rightarrow \frac{x}{at^2} + \frac{y}{2atk} = 1$

Q lies on it

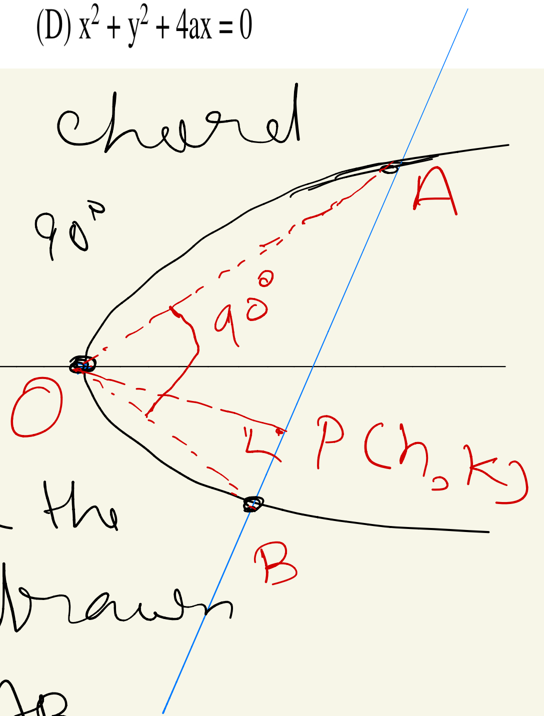
$\Rightarrow \frac{\cancel{at^2}}{4\cancel{at^2}} + \frac{\cancel{at}}{2\cancel{at}k} = 1 \Rightarrow \frac{1}{2k} = \frac{3}{4}$

$\Rightarrow k = 2/3$

2 Locus of the feet of the perpendiculars drawn from vertex of the parabola $y^2 = 4ax$ upon all such chords of the parabola which subtend a right angle at the vertex is

(A) $x^2 + y^2 - 4ax = 0$ (B) $x^2 + y^2 - 2ax = 0$

(C) $x^2 + y^2 + 2ax = 0$ (D) $x^2 + y^2 + 4ax = 0$



Let AB is a chord that subtends 90° at Vertex O.

Let P(h, k) be the foot of ⊥r drawn from O to AB.

Slope of chord AB $= -\frac{h}{k}$

equation of AB is

$y - k = (x-h)\left(-\frac{h}{k}\right) \Rightarrow ky + hx = h^2 + k^2$

$\Rightarrow \frac{ky + hx}{h^2 + k^2} = 1 \quad \cdots \quad (1)$

using homogenization

$y^2 - 4ax \cdot 1 = 0$

$\Rightarrow y^2 - 4ax\left(\frac{ky + hx}{h^2 + k^2}\right) = 0$

As AB subtends 90° at Vertex

$\Rightarrow h^2 + k^2 - 4ah = 0$

$\Rightarrow$ locus of P is

$$\boxed{x^2 + y^2 - 4ax = 0}$$

Alternative: $A(at_1^2, 2at_1)$

$B(at_2^2, 2at_2)$

slope of $OA = \frac{2}{t_1}$

slope of $OB = \frac{2}{t_2}$

As OA $\perp^r$ OB

$\Rightarrow \boxed{-4 = t_1 t_2}$

equation of AB

$\Rightarrow y(t_1+t_2) = 2x + 2a t_1 t_2$

Slope of AB $= \dfrac{2}{t_1+t_2}$

$\Rightarrow \dfrac{2}{t_1+t_2} \times \dfrac{k}{h} = -1$

$\Rightarrow \boxed{t_1+t_2 = -\dfrac{2k}{h}}$

P(h, k) lies on AB

$\Rightarrow k(t_1+t_2) = 2h + 2a t_1 t_2$

$\Rightarrow k\left(-\dfrac{2k}{h}\right) = 2h + 2a \times (-4)$

$\Rightarrow k^2 + h^2 - 4ah = 0$

$\Rightarrow \boxed{x^2+y^2-4ax=0}$

3 Through the focus of the parabola $y^2 = 2px (p > 0)$ a line is drawn which intersects the curve at $A(x_1, y_1)$ and $B(x_2, y_2)$. The ratio $\frac{y_1 y_2}{x_1 x_2}$ equals-

(A) 2 (B) –1 (C) –4 (D) some function of p

Suppose $A(at_1^2, 2at_1)$

$B(at_2^2, 2at_2)$

$\Rightarrow \quad x_1 x_2 = a^2 t_1^2 t_2^2$

$y_1 y_2 = 4a^2 t_1 t_2$

$$\frac{y_1 y_2}{x_1 x_2} = \frac{4}{t_1 t_2} = ?$$

As AB is a focal chord

$\Rightarrow \quad t_1 t_2 = -1$

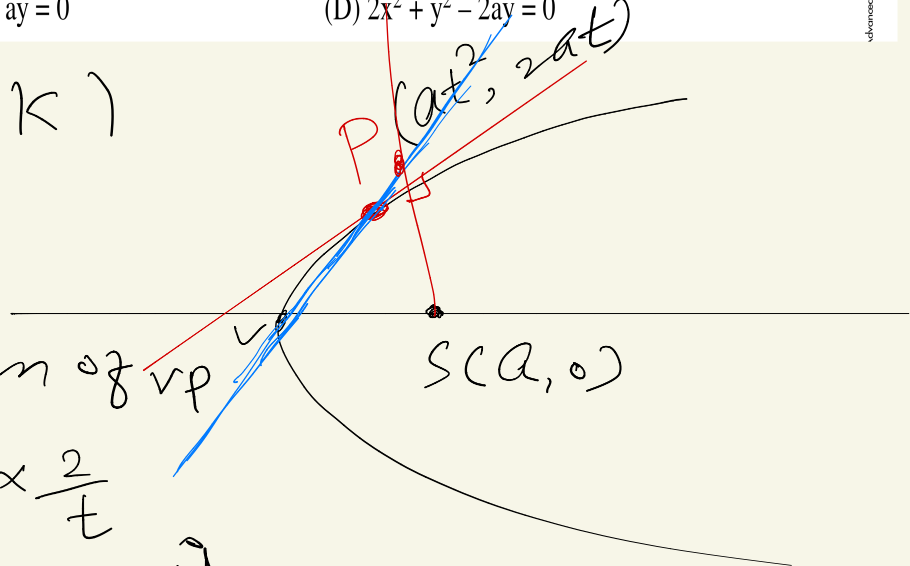
$$\Rightarrow \quad \frac{y_1 y_2}{x_1 x_2} = -4$$

4 The straight line joining any point P on the parabola $y^2 = 4ax$ to the vertex and perpendicular from the focus to the tangent at P, intersect at R, then the equation of the locus of R is

(A) $x^2 + 2y^2 - ax = 0$ (B) $2x^2 + y^2 - 2ax = 0$

(C) $2x^2 + 2y^2 - ay = 0$ (D) $2x^2 + y^2 - 2ay = 0$

$R(h, k)$



Equation of VP

$\Rightarrow y = x \times \frac{2}{t}$

R lies on it

$\Rightarrow k = \frac{2h}{t} \Rightarrow \boxed{t = \frac{2h}{k}}$ ...①

RS $\perp^r$ to tangent

$\Rightarrow -t = \frac{k-0}{h-a}$

using ①

$\frac{2h}{k} = \frac{k}{a-h} \Rightarrow 2ah - 2h^2 = k^2$

$\Rightarrow 2x^2 + y^2 - 2ax = 0$

5 If two normals to a parabola $y^2 = 4ax$ intersect at right angles then the chord joining their feet pass through a fixed point whose co-ordinates are :

(A) $(-2a,0)$ (B) $(a,0)$ (C) $(2a,0)$ (D) none

$P(at_1^2, 2at_1)$, $Q(at_2^2, 2at_2)$

Equation of chord joining

$PQ \Rightarrow y(t_1+t_2) = 2x + 2at_1t_2$

Slope of the normal at $P = -t_1$

Slope of the normal at $Q = -t_2$

Normals at $P \perp^r$ normal at Q

$\Rightarrow t_1t_2 = -1$

$\Rightarrow$ chord joining PQ

passes through $(a, 0)$

6 If the normal to a parabola $y^2 = 4ax$ at P meets the curve again in Q and if PQ and the normal at Q makes angles α and β respectively with the x-axis then $\tan\alpha(\tan\alpha + \tan\beta)$ has the value equal to

(A) 0 (B) – 2 (C) $-\frac{1}{2}$ (D) – 1

$P(at_1^2, 2at_1)$, $Q(at_2^2, 2at_2)$

Normal at P meets the curve at Q

$\Rightarrow t_2 = -t_1 - \frac{2}{t_1} \Rightarrow t_2 + t_1 = -\frac{2}{t_1}$ ... (1)

Slope of PQ = Slope of the normal at P = $-t_1$

Slope of the normal at Q = $-t_2$

$\Rightarrow -t_1 = \tan\alpha$, $-t_2 = \tan\beta$

Now $\tan\alpha(\tan\alpha + \tan\beta)$

$= (-t_1)(-t_1 - t_2) = t_1(t_1 + t_2)$

$= -2$

7 Normal to the parabola $y^2 = 8x$ at the point P (2, 4) meets the parabola again at the point Q. If C is the centre of the circle described on PQ as diameter then the coordinates of the image of the point C in the line y = x are

(A) (– 4, 10) (B) (– 3, 8) (C) (4, – 10) (D) (– 3, 10)

$P(2t^2, 4t)$, $Q(2T^2, 4T)$

$t = 1$

Normal at P meets curve at

$Q \Rightarrow T = -t - \frac{2}{t} = -3$

$\Rightarrow Q(18, -12)$

$\Rightarrow C(10, -4)$

$\Rightarrow$ Image of C w.r.t. $y = x$

is $(-4, 10)$

(D) equal to the distance of the point P from the directrix.

8 Normals are drawn at points A, B, and C on the parabola $y^2 = 4x$ which intersect at P(h, k). The locus of the point P if the slope of the line joining the feet of two of them is 2, is

(A) $x + y = 1$ (B) $x - y = 3$ (C) $y^2 = 2(x - 1)$ (D) $y^2 = 2\left(x - \frac{1}{2}\right)$

Equation of any normal

$$y = mx - 2m - m^3$$

If this normal passes through P(h, k), then

$$k = mh - 2m - m^3$$

$$\Rightarrow m^3 + (2 - h)m + k = 0 \quad m_1, m_2, m_3$$

$A(m_1^2, -2m_1)$, $B(m_2^2, -2m_2)$,

$C(m_3^2, -2m_3)$

$$2 = \frac{2(m_2 - m_1)}{m_1^2 - m_2^2} = \frac{-2}{m_1 + m_2}$$

$$\Rightarrow m_1 + m_2 = -1$$

Also we have

$$m_1 + m_2 + m_3 = 0$$

$$\Rightarrow m_3 = 1$$

$$\Rightarrow k = h - 3$$

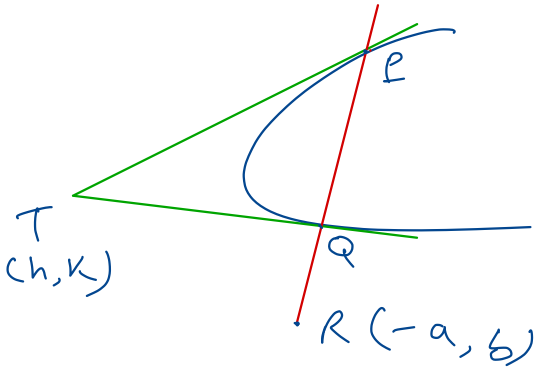
$\Rightarrow$ locus of P is

$$\boxed{x - y - 3 = 0}$$

9 TP & TQ are tangents to the parabola, $y^2 = 4ax$ at P & Q. If the chord PQ passes through the fixed point $(-a, b)$ then the locus of T is :

(A) $ay = 2b(x - b)$ (B) $bx = 2a(y - a)$ (C) $by = 2a(x - a)$ (D) $ax = 2b(y - b)$

Soln



PQ is the chord of contact of pair of tangents drawn from T to $y^2 = 4ax$.

Hence, $\overleftrightarrow{PQ}$:- $yk = 2a(x+h)$; which passes through $(-a, b)$.

Hence, $bk = 2a(-a+h)$

So, locus is $by = 2a(x-a)$

10 P is a point on the parabola $y^2 = 4ax (a > 0)$ whose vertex is A. PA is produced to meet the directrix in D and M is the foot of the perpendicular from P on the directrix. If a circle is described on MD as a diameter then it intersects the x-axis at a point whose co-ordinates are :

(A) $(-3a,0)$ (B) $(-a,0)$ (C) $(-2a,0)$ (D) $(a,0)$

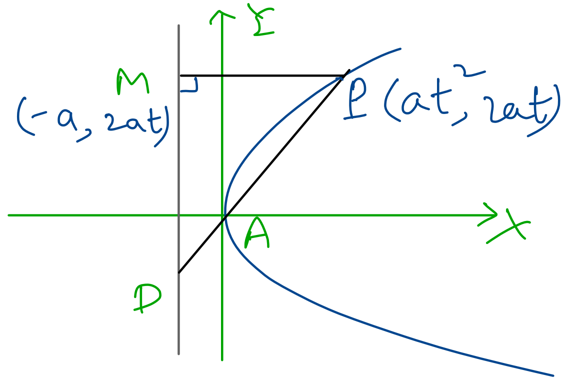
Sol^n

$\overleftrightarrow{AP}$:-

$$y = \frac{2at}{at^2}x$$

Put $x = -a$;

$$y = \frac{-2a}{t} \Rightarrow D\left(-a, -\frac{2a}{t}\right)$$



A circle with MD as diameter is :-

$$(x+a)^2 + (y-2at)\left(y+\frac{2a}{t}\right) = 0$$

Put $y = 0$;

$$(x+a)^2 = 4a^2 \Rightarrow x+a = \pm 2a$$

$$\Rightarrow x = -3a, a$$

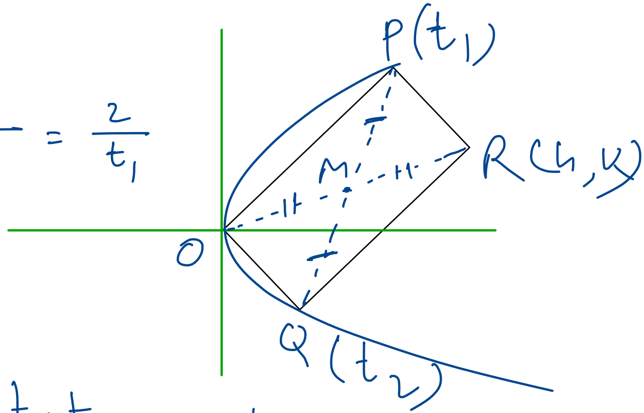
Hence, required points of intersection are :- $(a,0)$ and $(-3a,0)$

11 If from the vertex of a parabola $y^2 = 4x$ a pair of chords be drawn at right angles to one another and with these chords as adjacent sides a rectangle be made, then the locus of the further end of the rectangle is -

(A) an equal parabola
(B) a parabola with focus at (9,0)
(C) a parabola with directrix as $x - 7 = 0$
(D) a parabola having tangent at its vertex $x = 8$

Sol^n

slope of OP $= \dfrac{2t_1}{t_1^2} = \dfrac{2}{t_1}$



Since, $\angle POQ = 90^\circ$

$\Rightarrow \dfrac{2}{t_1} \times \dfrac{2}{t_2} = -1 \Rightarrow t_1 t_2 = -4 \quad \ldots\ (1)$

Now, M is the mid-point of PQ as well as OR.

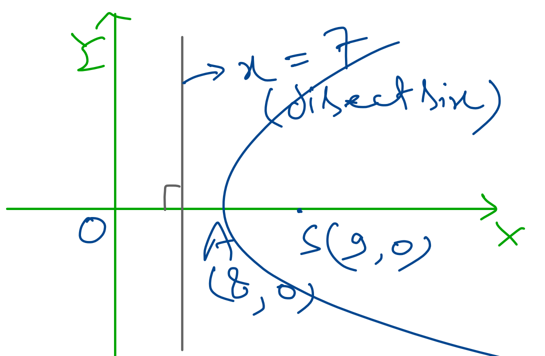
So, $\dfrac{t_1^2 + t_2^2}{2} = \dfrac{h+0}{2}$ & $\dfrac{2t_1 + 2t_2}{2} = \dfrac{k+0}{2}$

$\Rightarrow h = (t_1+t_2)^2 - 2t_1 t_2$ & $k = 2(t_1+t_2)$

$\Rightarrow h = \left(\dfrac{k}{2}\right)^2 - 2\times(-4)$ (from (1))

$\Rightarrow y^2 = 4(x-8)$

So, $a = 1$



Now, both parabolas have same length of latus rectum.

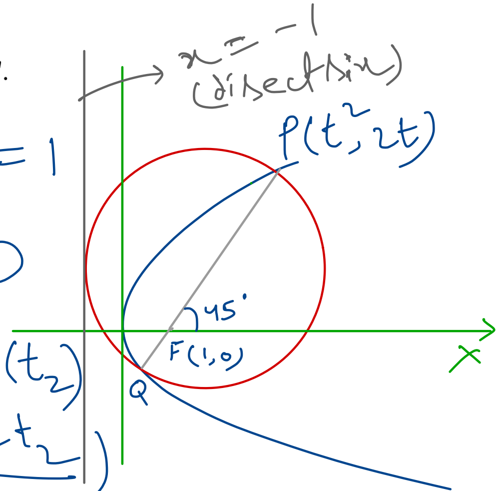
12 A circle 'S' is described on the focal chord of the parabola $y^2 = 4x$ as diameter. If the focal chord is inclined at an angle of 45° with axis of x, then which of the following is/are true ?

(A) Radius of the circle is 4.

(B) Centre of the circle is (3,2)

(C) The line $x + 1 = 0$ touches the circle

(D) The circle $x^2 + y^2 + 2x - 6y + 3 = 0$ is orthogonal to 'S'.



Sol^n Slope of $PF = \dfrac{2t}{t^2-1} = 1$

$\Rightarrow t^2 - 2t - 1 = 0$; ---- (1)

where $t = t_1, t_2$ and $P(t_1)$, $Q(t_2)$

So, center $\equiv \left(\dfrac{t_1^2 + t_2^2}{2}, \dfrac{2t_1 + 2t_2}{2}\right)$

$\equiv \left(\dfrac{(t_1+t_2)^2 - 2t_1t_2}{2}, t_1 + t_2\right)$

$\equiv \left(\dfrac{4+2}{2}, 2\right) \equiv (3, 2)$ (from (1))

Also, Diameter = length of focal chord

$= PQ = 4 \times 1 \operatorname{cosec}^2(45°) = 8 \Rightarrow r = 4$

Now, $S: (x-3)^2 + (y-2)^2 = 16$

Put $x = -1$; $y = 2, 2$ (equal roots)

$\Rightarrow x = -1$ is tangent to S

Again $S: x^2 + y^2 - 6x - 4y - 3 = 0$

Since, $2(-3 \times 1 + (-2)(-3)) \neq (-3 + 3) \Rightarrow$ (D) is false.

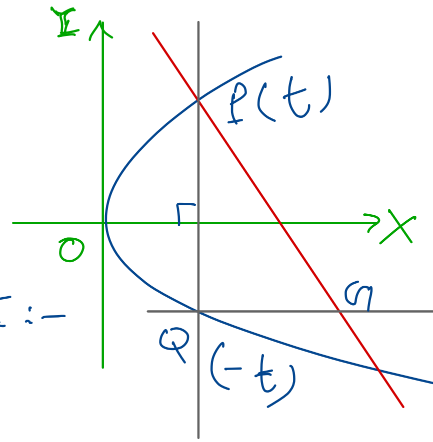
13 PQ is a double ordinate of the parabola $y^2 = 4ax$. If the normal at P intersect the line passing through Q and parallel to axis of x at G, then locus of G is a parabola with -

(A) length of latus rectum equal to 4a (B) vertex at (4a,0)

(C) directrix as the line $x - 3a = 0$ (D) focus at (5a,0)

Sol^n

$\overleftarrow{PQ}$:—



$$y + xt = 2at + at^3$$

For point G, put $y = -2at$:-

$$x = 4a + at^2$$

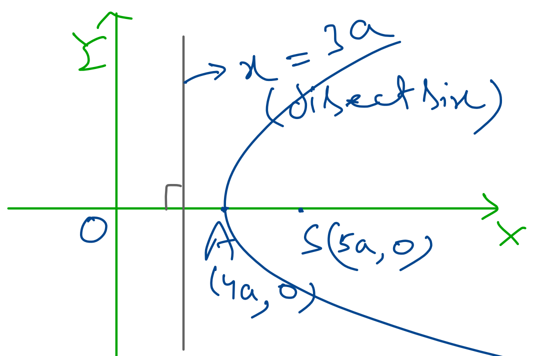
So, $G(x, y) \equiv (4a + at^2, -2at)$

$\Rightarrow t = \frac{-y}{2a} \Rightarrow x = 4a + \frac{ay^2}{4a^2}$

$\Rightarrow y^2 = 4a(x - 4a)$ [Locus of G]

So, Length of latus rectum = 4a

& vertex $\equiv (4a, 0)$



Paragraph for question nos. 14 to 16

Tangents are drawn to the parabola $y^2 = 4x$ from the point $P(6, 5)$ to touch the parabola at Q and R. C_1 is a circle which touches the parabola at Q and C_2 is a circle which touches the parabola at R. Both the circles C_1 and C_2 pass through the focus of the parabola.

14 Area of the ΔPQR equals

(A) $\frac{1}{2}$ (B) 1 (C) 2 (D) $\frac{1}{4}$

15 Radius of the circle C_2 is

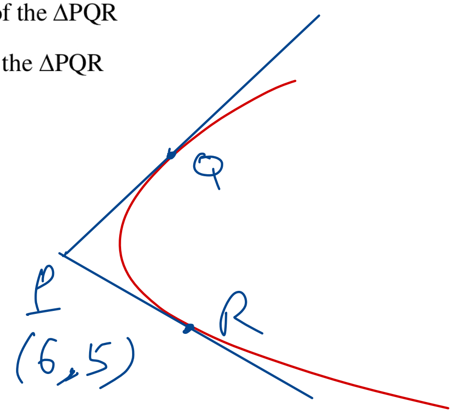
(A) $5\sqrt{5}$ (B) $5\sqrt{10}$ (C) $10\sqrt{2}$ (D) $\sqrt{210}$

16 The common chord of the circles C_1 and C_2 passes through the

(A) incentre of the ΔPQR (B) circumcenter of the ΔPQR

(C) centroid of the ΔPQR (D) orthocenter of the ΔPQR

Soln



Any tangent is

$y = mx + \frac{1}{m}$

It passes through (6, 5).

So, $5 = 6m + \frac{1}{m} \Rightarrow 6m^2 - 5m + 1 = 0$

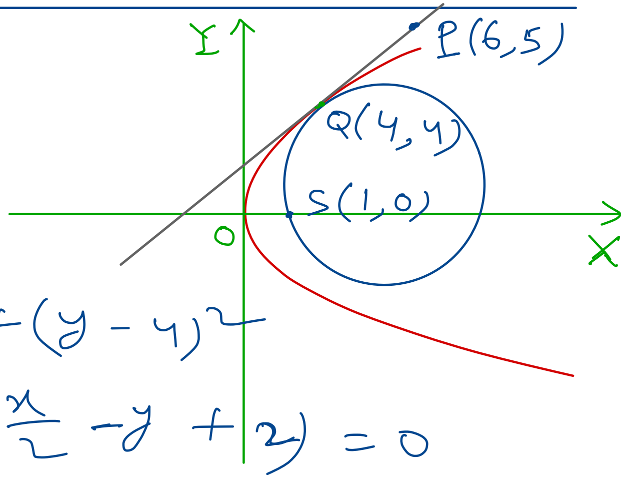
$\Rightarrow m = \frac{1}{2}, \frac{1}{3}$ $\Rightarrow Q\left(\frac{1}{(1/2)^2}, \frac{2}{(1/2)}\right)$

& $R\left(\frac{1}{(1/3)^2}, \frac{2}{(1/3)}\right)$

Thus, $Q(4, 4)$ & $R(9, 6)$

14.) $[PQR] = \frac{1}{2}\begin{vmatrix} 6 & 5 & 1 \\ 4 & 4 & 1 \\ 9 & 6 & 1 \end{vmatrix} = \frac{1}{2}$

15.) $\overleftrightarrow{PQ}$:-



$y = \frac{x}{2} + 2$

So, $c_1 : (x-4)^2 + (y-4)^2 + \lambda\left(\frac{x}{2} - y + 2\right) = 0$

It passes through $S(1,0)$. So, $\lambda = -10$

Hence, $c_1 : x^2 + y^2 - 13x + 2y + 12 = 0$

Now, $\overleftrightarrow{PR}$:- $y = \frac{x}{3} + 3$ & $R \equiv (9,6)$

Similarly, we get $c_2 : x^2 + y^2 - 28x + 18y + 27 = 0$

Now, $r_1 = \frac{5\sqrt{5}}{2}$ & $r_2 = 5\sqrt{10}$

16.) The common chord is :-

$15x - 16y - 15 = 0$;

which passes through $\left(\frac{19}{3}, 5\right)$ i.e. centroid of $\triangle PQR$

S-1

EXERCISE (S-1)

1 'O' is the vertex of the parabola $y^2 = 4ax$ & L is the upper end of the latus rectum. If LH is drawn perpendicular to OL meeting OX in H, prove that the length of the double ordinate through H is $4a\sqrt{5}$.

Sol. $m_{OL} = \frac{2a}{a} = 2$

$m_{LH} = -\frac{1}{2}$

eq^n of LH $\Rightarrow y - 2a = -\frac{1}{2}(x - a)$

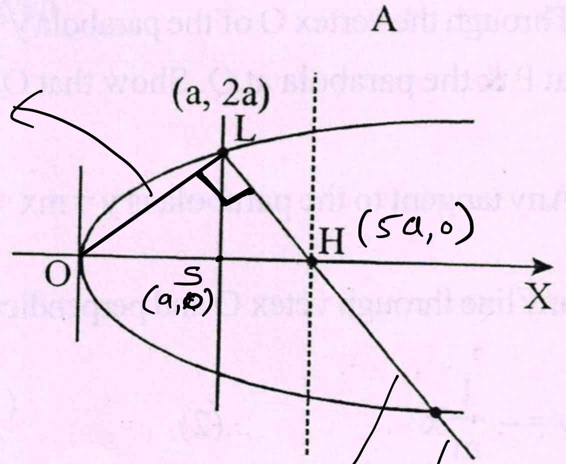
H:(5a, 0)

Double ordinate = x = 5a

$y^2 = 4ax = 4a(5a) = 20a^2$

$y = 2\sqrt{5}\,a \Rightarrow \boxed{\text{length} = 4\sqrt{5}}$

$m = 2$



$m = -\frac{1}{2}$

eq^n:-

$y - 2a = -\frac{1}{2}(x - a)$

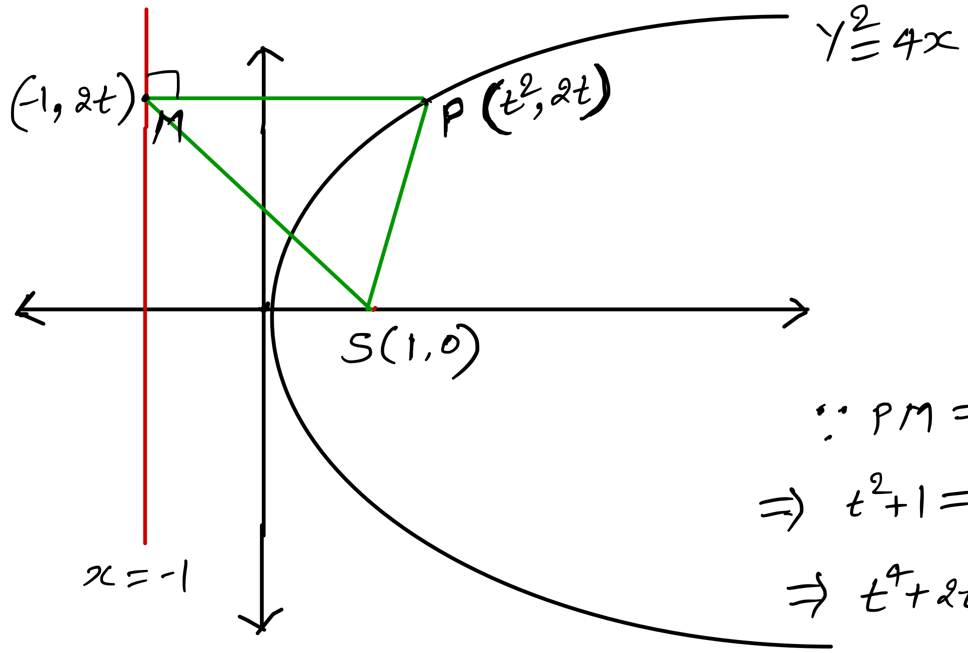
Put $x = 0$

then $y = 5a$

1

2 A point P on a parabola $y^2 = 4x$, the foot of the perpendicular from it upon the directrix, and the focus are the vertices of an equilateral triangle, find the area of the equilateral triangle.

Sol:



$\because PM = MS$

$\Rightarrow t^2 + 1 = \sqrt{(1+1)^2 + (0-2t)^2}$

$\Rightarrow t^4 + 2t^2 + 1 = 4 + 4t^2$

$\Rightarrow t^4 - 2t^2 - 3 = 0$

$\Rightarrow (t^2 - 3)(t^2 + 1) = 0$

$\Rightarrow t = \sqrt{3}$ or $-\sqrt{3}$

take any value of 't'

$\therefore M(-1, 2\sqrt{3})$ & $S(1,0)$

length $MS = \sqrt{4 + 12} = 4$

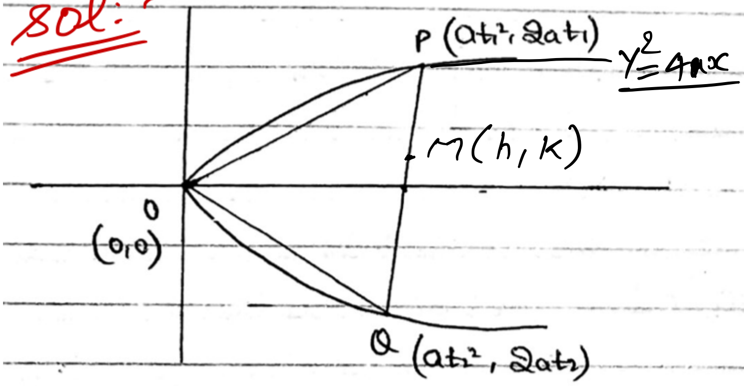
So Area of equilateral $\Delta = \frac{\sqrt{3}}{4} \cdot (4)^2$

$= \boxed{4\sqrt{3}}$ Ans.

2

3 Through the vertex O of a parabola $y^2 = 4x$, chords OP & OQ are drawn at right angles to one another. Show that for all positions of P, PQ cuts the axis of the parabola at a fixed point. Also find the locus of the middle point of PQ.

Sol:-



eq^n of PQ =

$(t_1+t_2)y = 2(x+at_1t_2)$

Put $y=0$, $t_1t_2=-4$

$\Rightarrow 0 = 2(x+(-4a))$

$\Rightarrow x = 4a$

Point on the axis $\Rightarrow (4a, 0)$ $\because$ Given $a=1$

$\therefore (4,0) \Rightarrow$ fixed point 1.

$\because$ PQ : $T = S_1$

$\Rightarrow yk - 2(x+h) = k^2 - 4h$

$\Rightarrow ky = 2x - 2h + k^2$ —— (1)

$\because$ PQ : $y - 0 = m(x-4)$

$\Rightarrow y = mx - 4m$ —— (2)

compare (1) & (2)

$$\frac{k}{1} = \frac{2}{m} = \frac{k^2-2h}{-4m}$$

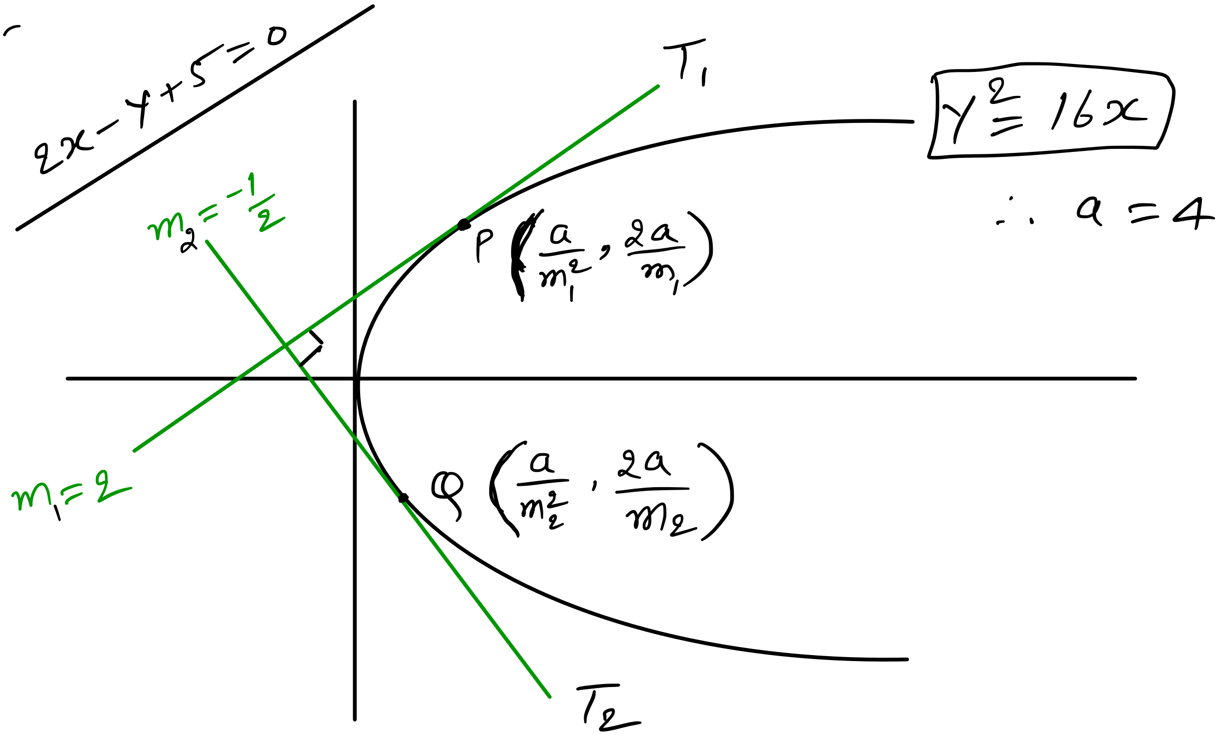
$-8 = k^2 - 2h$

$\Rightarrow \boxed{y^2 = 2(x-4)}$ 2.

3

4 Find the equations of the tangents to the parabola $y^2 = 16x$, which are parallel & perpendicular respectively to the line $2x - y + 5 = 0$. Find also the coordinates of their points of contact.

Sol:-



$\because\ T_1 : \quad y = m_1 x + \frac{a}{m_1} \Rightarrow y = 2x + \frac{4}{2} \Rightarrow \boxed{2x - y + 2 = 0}$ — ①

$\because\ T_2 : \quad y = m_2 x + \frac{a}{m_2} \Rightarrow y = -\frac{1}{2}x + \frac{4}{-1/2} \Rightarrow \boxed{2y + x + 16 = 0}$ — ②

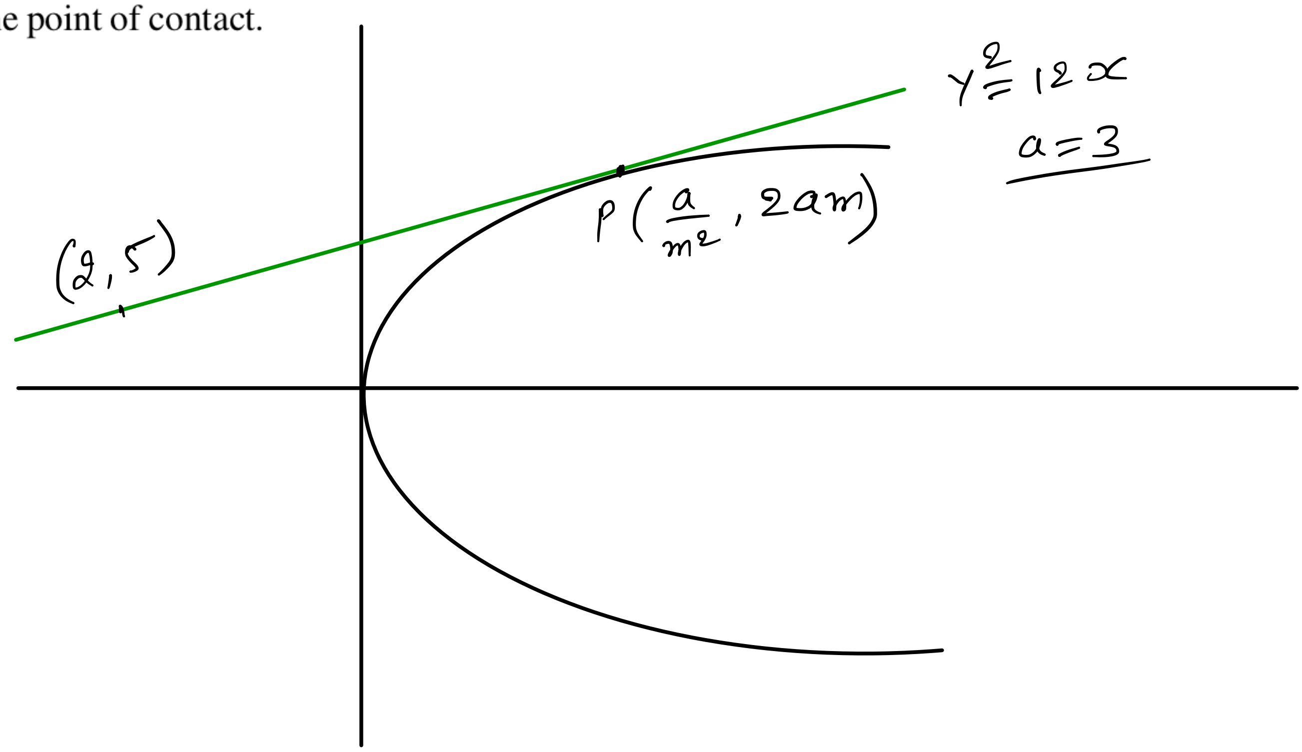
$* \ P\left(\frac{a}{m_1^2}, \frac{2a}{m_1}\right) \equiv P\left(\frac{4}{4}, \frac{8}{2}\right) \Rightarrow P(1, 4)$ $\underline{1}$.

$* \ Q\left(\frac{a}{m_2^2}, \frac{2a}{m_2}\right) \equiv Q\left(\frac{4}{\left(-\frac{1}{2}\right)^2}, \frac{8}{\left(-\frac{1}{2}\right)}\right) \Rightarrow \boxed{Q(16, -16)}$ $\underline{1}$.

4

5 Find the equations of the tangents of the parabola $y^2 = 12x$, which passes through the point (2,5). Also find the point of contact.

Sol:-



$T: \quad y = mx + \frac{3}{m}$ pass through $(2,5)$

$\Rightarrow \quad 5 = 2m + \frac{3}{m}$

$\Rightarrow \quad 2m^2 - 5m + 3 = 0$

$\Rightarrow \quad (2m-3)(m-1) = 0$

$\Rightarrow \quad m = \frac{3}{2}$ or 1

$\therefore$ Tangent : $\boxed{y = \frac{3}{2}x + 2}$

& Tangent : $\boxed{y = x + 3}$

P.O.C.

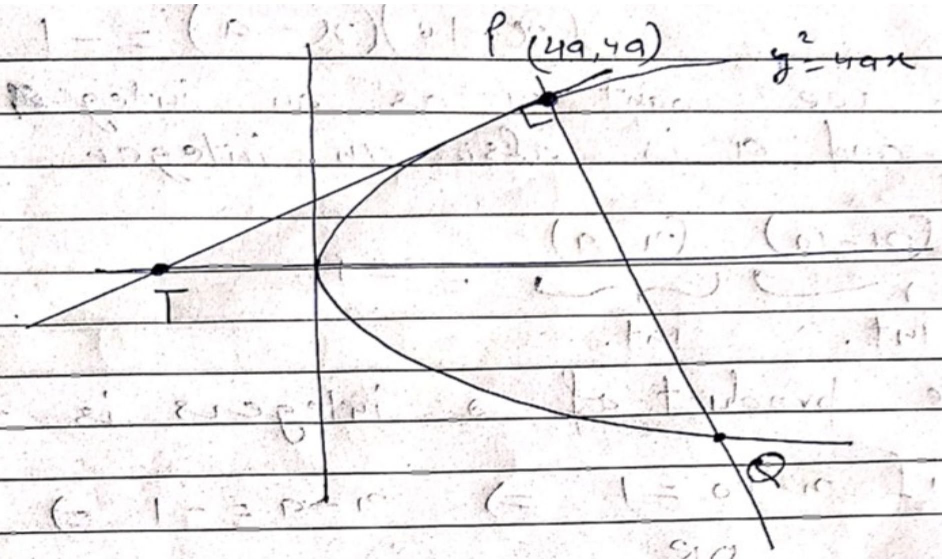
$P\left(\frac{a}{m^2}, 2am\right)$

$a = 3$ & $m = 3/2$ | $a = 3$ & $m = 1$

$\boxed{P\left(\frac{4}{3}, 9\right)}$ $\qquad$ $\boxed{Q(3,6)}$

6 In the parabola $y^2 = 4ax$, the tangent at the point P, whose abscissa is equal to the latus rectum meets the axis in T & the normal at P cuts the parabola again in Q. Prove that PT : PQ = 4 : 5.

Sol^n



eq^n of tangent at P (4a, 4a)

$$yy_1 = 2a(x + x_1)$$

$$\Rightarrow 4ay = 2a(x + 4a)$$

$$\Rightarrow 2y = x + 4a$$

$$T \equiv (-4a, 0)$$

$$m_{PQ} = \left(-\frac{dx}{dy}\right)_{(4a,4a)} = -2$$

eq^n of PQ $\Rightarrow$ $y - 4a = -2(x - 4a)$

$$Q \equiv (9a, -6a)$$

(we can also use $t_2 = -\left(t_1 + \frac{2}{t_1}\right)$, means normal drawn at t_1 meets parabola again at t_2)

So $PT = \sqrt{(8a)^2 + (4a)^2}$

$= \sqrt{80}\, a$

& $PQ = \sqrt{(5a)^2 + (10a)^2} = \sqrt{125}\, a$

So $\frac{PT}{PQ} = \sqrt{\frac{80}{125}} = \sqrt{\frac{16}{25}} = \frac{4}{5}$

7 Show that the normals at the points (4a, 4a) & at the upper end of the latus rectum of the parabola $y^2 = 4ax$ intersect on the same parabola.

Sol. $y^2 = 4ax$ ...(1)

normal at (4a, 4a)

$$y - y_1 = -\frac{y}{2a}(x - x_1)$$

$$\text{or } y - 4a = -\frac{4a}{2a}(n - ya)$$

$$y - 4a = -2x + 8a$$

$$\text{or } \boxed{2x + y = 12a} \quad ...(2)$$

upper end of latus rectum is (a, 2a)

now

normal at (a, 2a) ⇒

$$y - 2a = -\frac{2a}{2a}(x - a)$$

$$y - 2a = -x + a$$

$$\text{or } \boxed{x + y = 3a} \quad ...(3)$$

intersection poi nt of equation (2) & (3)

$$2x + y = 12a$$
$$x + y = 3a$$
$$- \quad - \quad -$$

$$\boxed{x = 9a} \ \& \ \boxed{y = -6a}$$

8 Three normals to $y^2 = 4x$ pass through the point (15, 12). Show that if one of the normals is given by $y = x - 3$ & find the equations of the others.

Sol. $y^2 = 4ax$ standard parabola here we have to given

$$y^2 = 4x$$

$$\therefore a = 1$$

Any normal to the parabola is

$$y = mx - 2am - am^3$$

$$\because a = 1$$

$$\therefore y = mx - 2m - m^3 \quad ...(1)$$

If it passes through the point (15, 12), then

$$12 = 15m - 2m - m^3$$

or $$m^3 - 13m + 12 = 0 \quad ...(2)$$

Above equation (2) being a cubic in m gives us three values of m showing that there will be three normals to the parabola through the point (15, 12).

Clearly $m = 1$ satisfies the equation (2) and hence it can be written as

$$(m - 1)(m^2 + m - 12) = 0$$

$$\text{or } (m - 1)(m + 4)(m - 3) = 0$$

$$\therefore m = 1, -4, 3 \qquad \text{and } a = 1$$

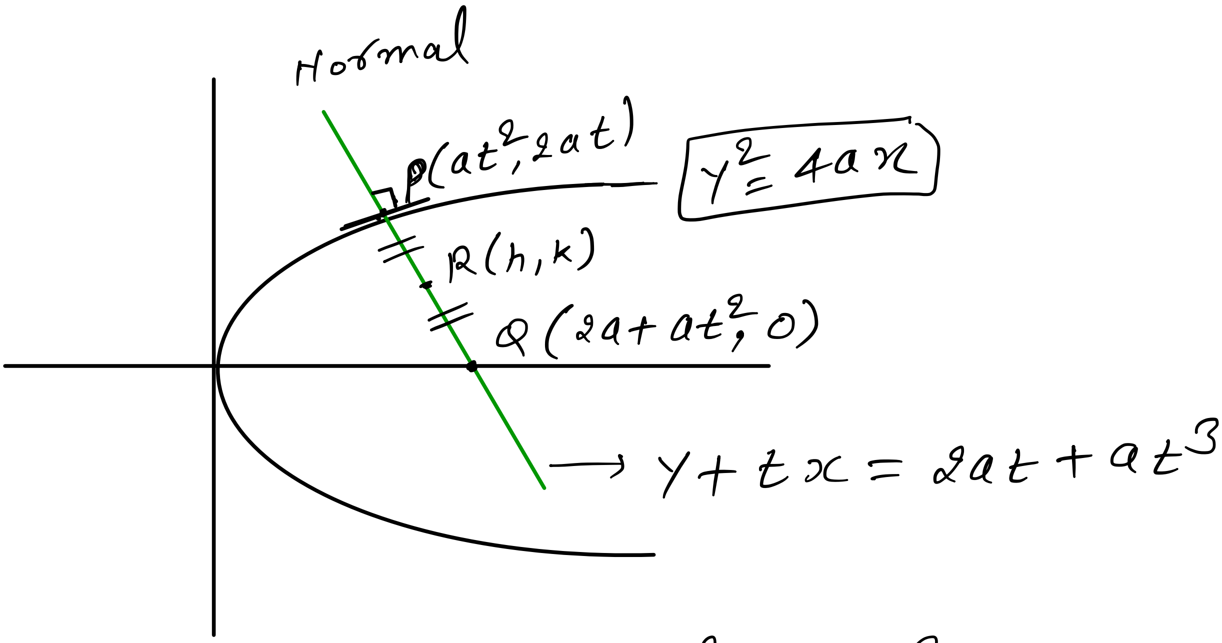
putting these values in equation (1), we get

$y = x - 3$ there are the

$y = -4x + 72$ equations of

& $y = 3x - 33$ three normals.

g Prove that the locus of the middle point of portion of a normal to $y^2 = 4ax$ intercepted between the curve & the axis is another parabola. Find the vertex & the latus rectum of the second parabola.



$$\therefore \frac{2a + at^2 + at^2}{2} = h \quad \& \quad \frac{2at + 0}{2} = k$$

$$\Rightarrow a + at^2 = h \quad \text{①}$$

$at = k$

$\boxed{t = \frac{k}{a}}$

put in ①

$$\therefore a + a\left(\frac{k}{a}\right)^2 = h$$

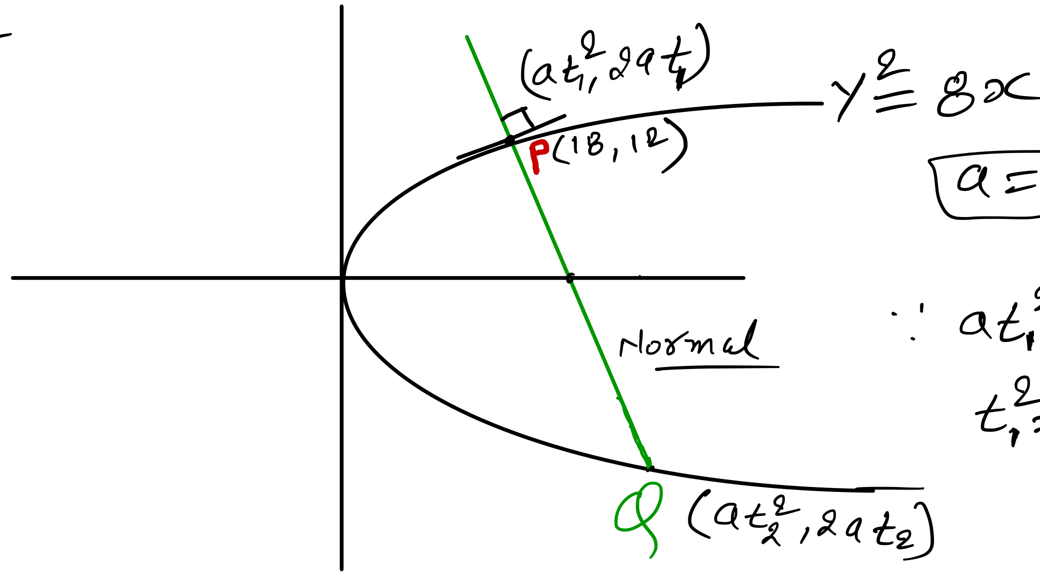
$$\Rightarrow \boxed{y^2 = a(x-a)} \quad \text{Parabola}$$

$$\boxed{\begin{array}{c} \text{Vertex} : (a, 0) \\ L \cdot R = a \end{array}}$$

1

10 If the normal at P(18, 12) to the parabola $y^2 = 8x$ cuts it again at Q, show that $9PQ = 80\sqrt{10}$

Sol:-



$\boxed{a=2}$

$\because at_1^2 = 18$ & $2at_1 = 12$

$t_1^2 = 9$ | $\boxed{t_1 = 3}$

$\because\ t_2 = -t_1 - \frac{2}{t_1}$

$\Rightarrow t_2 = -3 - \frac{2}{3} \Rightarrow \boxed{t_2 = \frac{-11}{3}}$

$\therefore Q(at_2^2, 2at_2) \equiv \left(2\cdot\frac{121}{9},\ 2\cdot 2\cdot\left(\frac{-11}{3}\right)\right)$

$\equiv \left(\frac{242}{9},\ -\frac{44}{3}\right)$

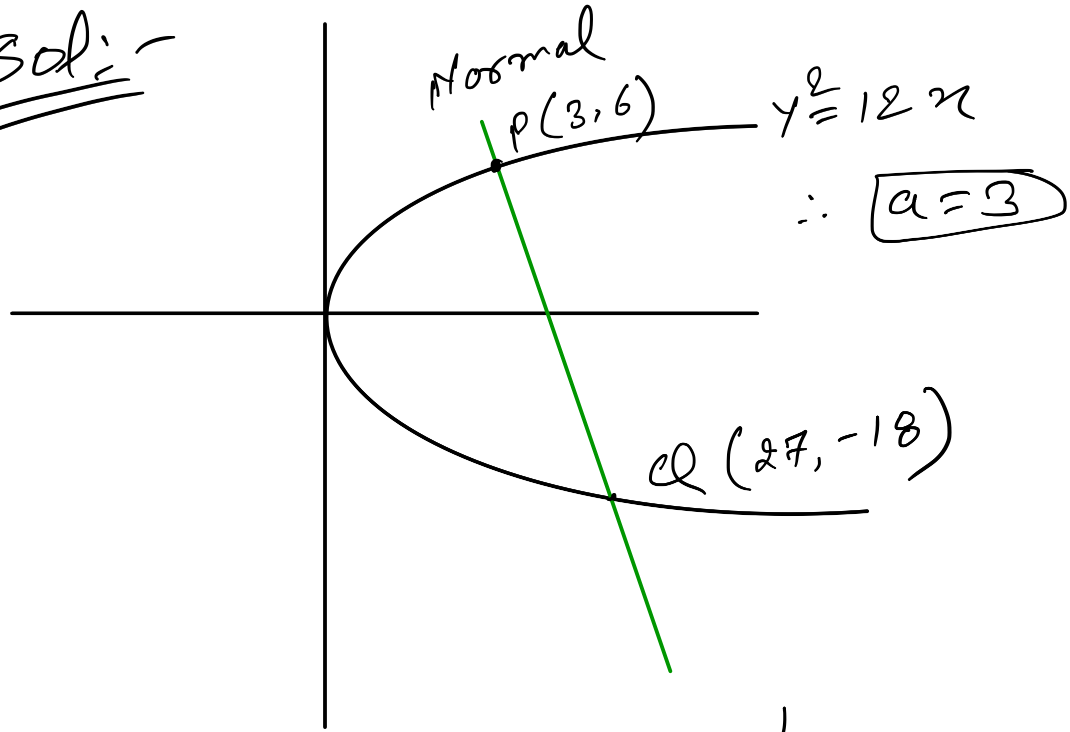
$\therefore PQ = \sqrt{\left(\frac{242}{9} - 18\right)^2 + \left(-\frac{44}{3} - 12\right)^2}$

$\Rightarrow PQ = \sqrt{\frac{1600}{81} + \frac{1600}{9}} = 80\sqrt{\frac{10}{81}}$

$\Rightarrow PQ = \frac{80}{9}\sqrt{10} \Rightarrow \boxed{9.PQ = 80\sqrt{10}}$

11 Prove that, the normal to $y^2 = 12x$ at $(3, 6)$ meets the parabola again in $(27, -18)$ & circle on this normal chord as diameter is $x^2 + y^2 - 30x + 12y - 27 = 0$.

Sol:-



$P(at_1^2, 2at_1) \equiv (3, 6)$

$\therefore 2at_1 = 6$

$\Rightarrow \boxed{t_1 = 1}$

$\because t_2 = -t_1 - \frac{2}{t_1}$

$\Rightarrow t_2 = -1 - \frac{2}{1} = -3$

$\Rightarrow \boxed{t_2 = -3}$

$\therefore Q(at_2^2, 2at_2) \equiv (3\cdot(-3)^2, 2\cdot 3(-3))$

$Q = (27, -18)$

Circle : PQ as diameter

$\Rightarrow (x-3)(x-27) + (y-6)(y+18) = 0$

$\Rightarrow \boxed{x^2 + y^2 - 30x + 12y - 27 = 0}$

12 Show that the normals at two suitable distinct real points on the parabola $y^2 = 4ax$ $(a > 0)$ intersect at a point on the parabola whose abscissa $> 8a$.

Sol^n

if normals drawn at t_1 & t_2 meets on the parabola again then

$$t_1 t_2 = 2 \quad \text{—— (1)}$$

& $$t_3 = -\left(t_1 + \frac{2}{t_1}\right) \quad \text{—— (2)}$$

& $$t_3 = -\left(t_2 + \frac{2}{t_2}\right) \quad \text{—— (3)}$$

$$\Rightarrow t_3 = -(t_1 + t_2) \quad \text{—— (4)}$$

Now $$\left|t_1 + \frac{2}{t_1}\right| \geq 2\sqrt{t_1 \cdot \frac{2}{t_1}} = 2\sqrt{2}$$

(A.M. $\geq$ G.M.)

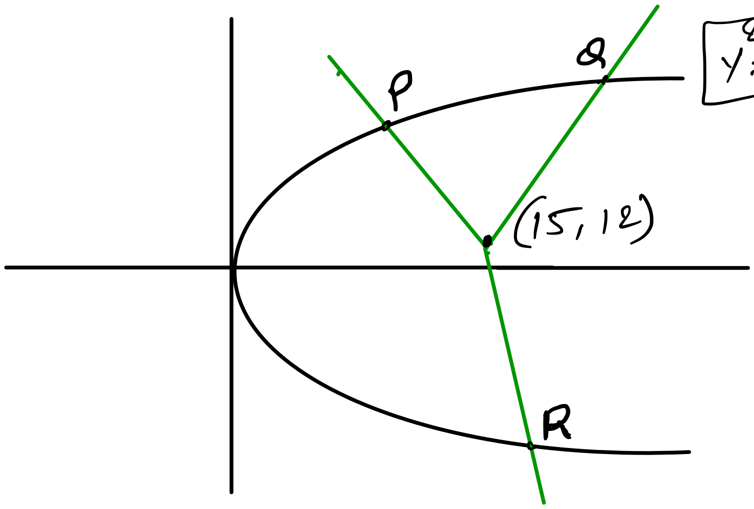
(but equality can not hold because in this case $t_1 = t_2$)

$$\Rightarrow |t_3| = \left|t_1 + \frac{2}{t_1}\right| > 2\sqrt{2}$$

so abscissa of $t_3 = a t_3^2 > 8a$

13 The normal to the parabola $y^2 = 4x$ at the point P,Q & R are concurrent at the point(15,12). Find
(a) the equation of the circle circumscribing the triangle PQR
(b) the co-ordinates of the centroid of the triangle PQR.

Sol:-



$\boxed{y^2 = 4x}$; $a = 1$

$P(am_1^2, -2am_1)$

$Q(am_2^2, -2am_2)$

$R(am_3^2, -2am_3)$

Normal: $y = mx - 2am - am^3$

pass through $(15, 12)$; $a = 1$

$\Rightarrow 12 = 15m - 2m - m^3$

$\Rightarrow m^3 - 13m + 12 = 0$

$\Rightarrow (m-1)(m^2 + m - 12) = 0$

$\Rightarrow (m-1)(m+4)(m-3) = 0$

$\Rightarrow \boxed{m_1 = 1}$ & $\boxed{m_2 = 3}$ & $\boxed{m_3 = -4}$

$\therefore P(am_1^2, -2am_1) \equiv (1, -2)$

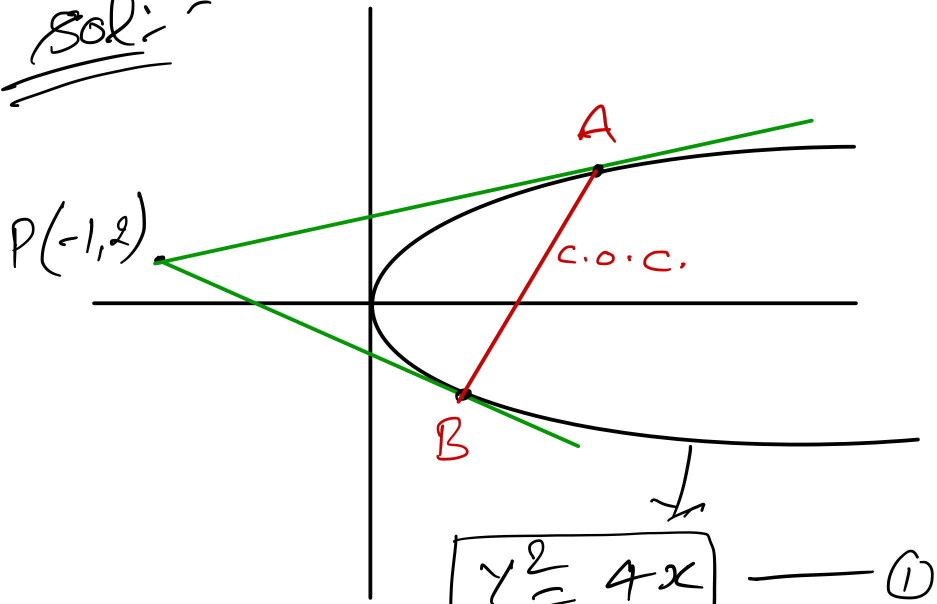
$Q(am_2^2, -2am_2) \equiv (9, -6)$

$R(am_3^2, -2am_3) \equiv (16, 8)$

Now D.Y.S.

14 From the point $(-1, 2)$ tangent lines are drawn to the parabola $y^2 = 4x$. Find the equation of the chord of contact. Also find the area of the triangle formed by the chord of contact & the tangents.

Sol:-



$\boxed{y^2 = 4x}$ —— ①

C.O.C.

$T = 0$

$\Rightarrow yy_1 - 2(x + x_1) = 0$

$\Rightarrow 2y - 2(x - 1) = 0$

$\Rightarrow \boxed{x - y = 1}$ —— ②

* $\because$ Area of ΔPAQ

$$= \frac{(S_1)^{3/2}}{2a}$$

$$= \frac{(8)^{3/2}}{2 \times 1} = \frac{(2\sqrt{2})^3}{2}$$

$$= \boxed{8\sqrt{2}}$$

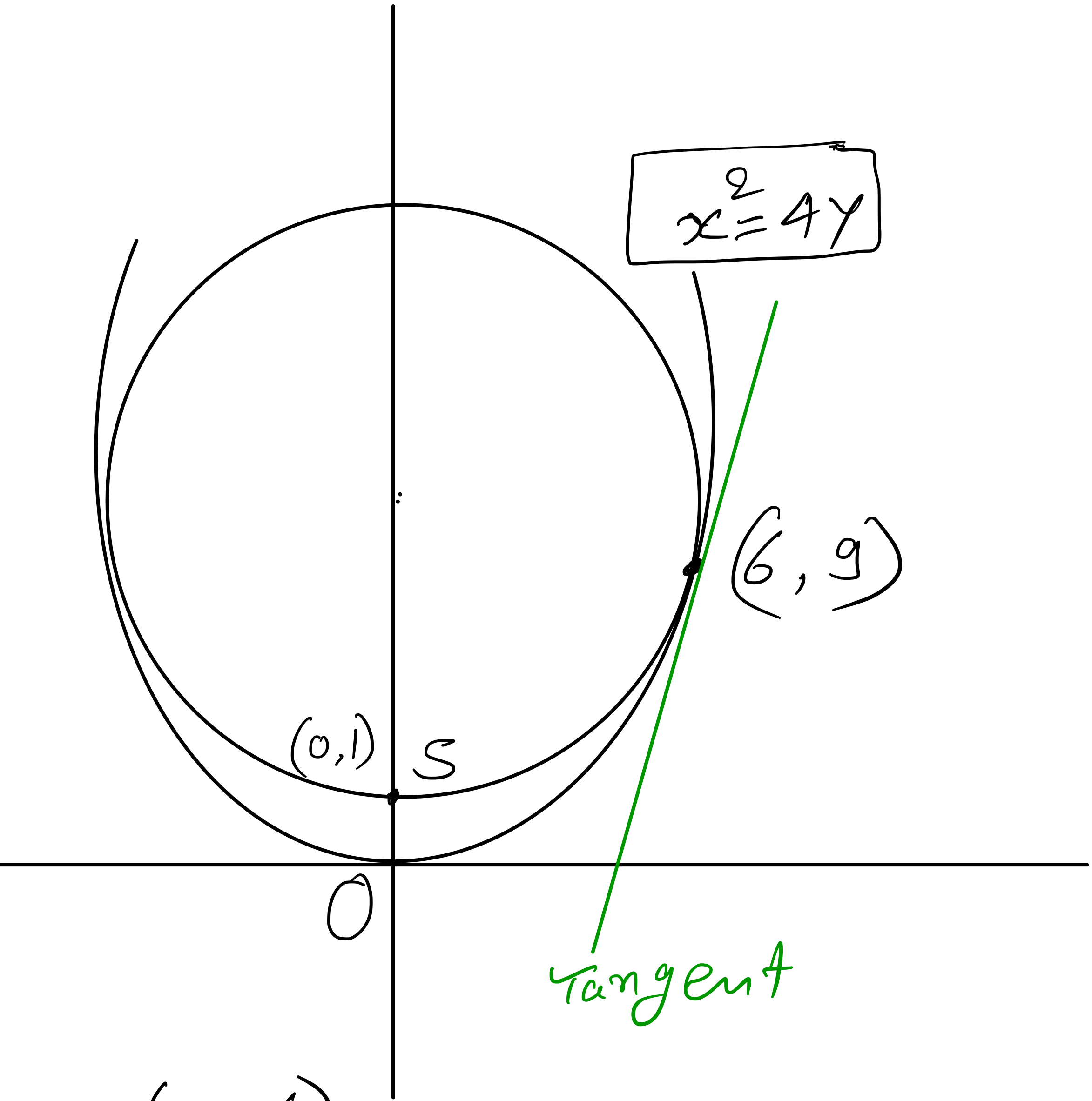
15 Find the equation of the circle which passes through the focus of the parabola $x^2 = 4y$ & touches it at the point $(6,9)$.

Sol:-

$\because x^2 = 4y$

$2x = 4y'$

$\left.\frac{dy}{dx}\right)_{(6,9)} = 3$



$T: \quad y-9 = 3(x-6)$

$\Rightarrow \boxed{3x - y = 9}$

$\therefore$ circle : $S + \lambda L = 0$

$\Rightarrow (x-6)^2 + (y-9)^2 + \lambda(3x-y-9) = 0$

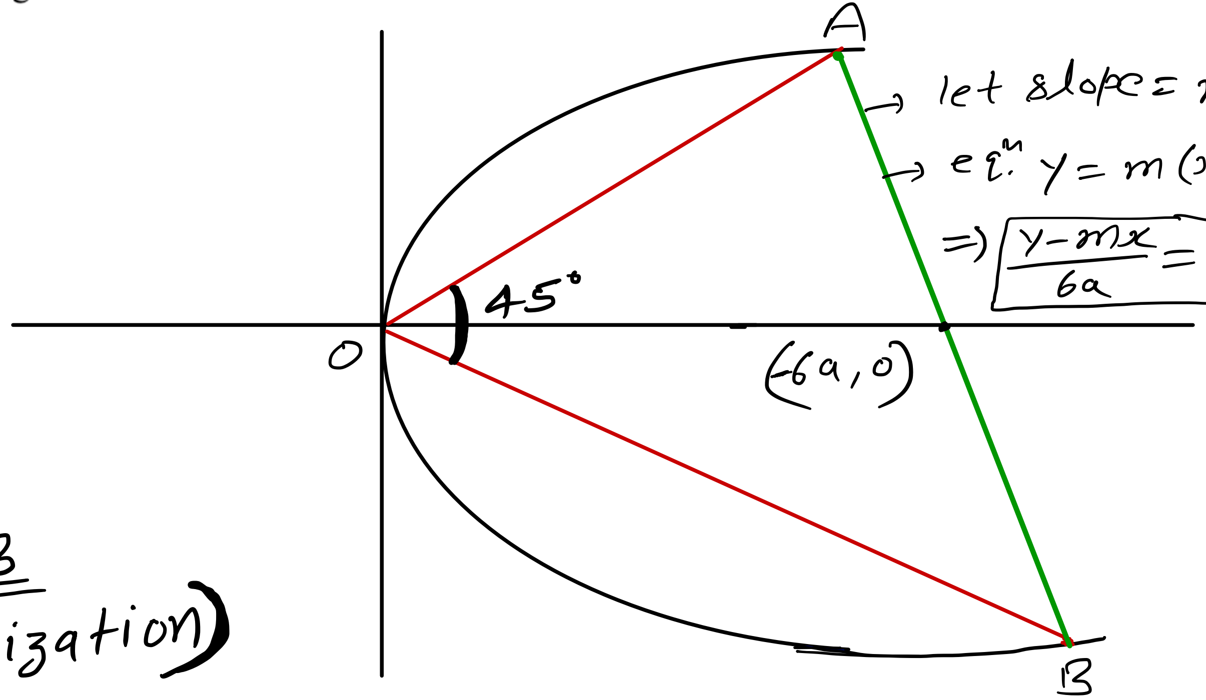
pass through $(0,1)$ then $\lambda = 10$

$\therefore$ circle : $(x-6)^2 + (y-9)^2 + 10(3x-y-9) = 0$

$\Rightarrow \boxed{x^2 + y^2 + 18x - 28y + 27 = 0}$

16 Find the equations of the chords of the parabola $y^2 = 4ax$ which pass through the point $(-6a, 0)$ and which subtends an angle of $45°$ at the vertex.

Sol:-



let slope = m

eqⁿ: $y = m(x+6a)$

$\Rightarrow \boxed{\frac{y - mx}{6a} = 1}$

OA & OB (Homogenization)

$$y^2 - 4ax\left(\frac{y-mx}{6a}\right) = 0$$

$$\Rightarrow y^2 - \frac{2}{3m}xy + \frac{2}{3}x^2 = 0$$

$$\because \quad \tan(45°) = \left|\frac{2\sqrt{\left(\frac{-2}{3m}\right)^2 - \frac{2}{3}}}{1 + \frac{2}{3}}\right|$$

$$\Rightarrow \frac{5}{6} = \sqrt{\frac{4}{9m^2} - \frac{2}{3}} \Rightarrow \boxed{m = \pm\frac{2}{7}}$$

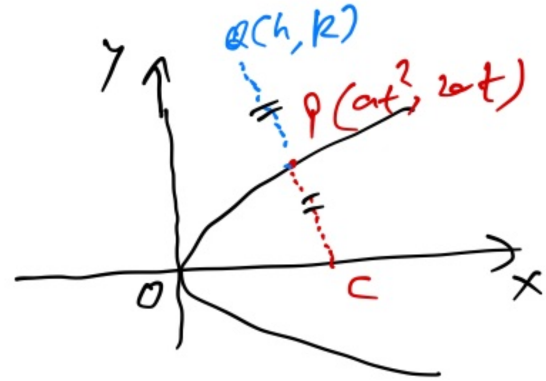
$\therefore$ eqⁿ of AB $\Rightarrow$ $y - 0 = \pm\frac{2}{7}(x + 6a)$

$$\Rightarrow \boxed{7y \pm 2(x+6a) = 0}$$

Ans.

S-2

1. PC is the normal at P to the parabola $y^2 = 4ax$, C being on the axis. CP is produced outwards to Q so that PQ = CP; show that the locus of Q is a parabola.



Normal at P to the parabola is

$$y + xt = 2at + at^3$$

at C, $y = 0$

$$\Rightarrow \quad x = 2a + at^2$$

$$\therefore \quad C(2a + at^2, 0)$$

we have to find locus of $Q(h, k)$ s.t.

$$PQ = CP$$

$\Rightarrow$ P is mid point of Q & C

$$\therefore \quad at^2 = \frac{h + 2a + at^2}{2} \quad \& \quad 2at = \frac{k + 0}{2}$$

$$\Rightarrow \boxed{t = \frac{k}{4a}}$$

$$h + 2a = at^2$$

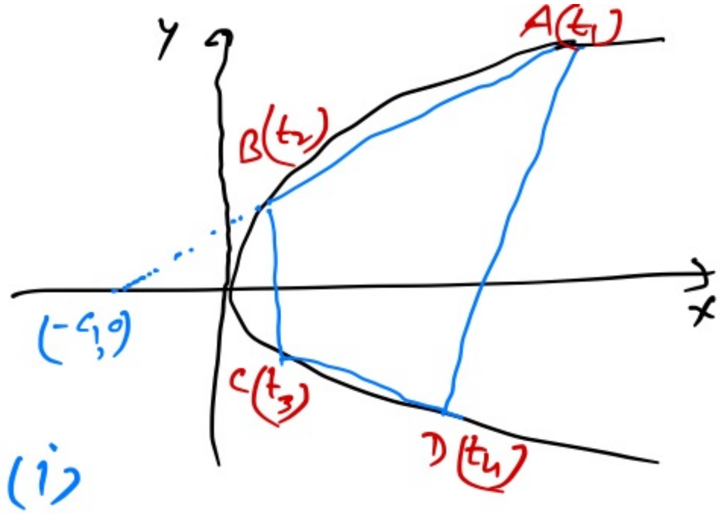
$$= a \cdot \frac{k^2}{16a^2}$$

$$h + 2a = \frac{k^2}{16a}$$

$\therefore$ locus of $P(h, k)$ is

$$\boxed{y^2 = 16a(x + 2a)} \quad \text{Ans}$$

2. A quadrilateral is inscribed in a parabola $y^2 = 4ax$ and three of its sides pass through fixed points on the axis. Show that the fourth side also passes through fixed point on the axis of the parabola.



Suppose side AB of quadrilateral passes through fixed point $(c_1, 0)$ on the x-axis

$\therefore \quad t_1 t_2 = -\dfrac{c_1}{a} \quad$ — (i)

Similarly side BC & CD also passes through fixed point on x-axis at $(c_2, 0)$ & $(c_3, 0)$ respectively

$\therefore \quad t_2 t_3 = -\dfrac{c_2}{a} \quad$ — (ii)

& $\quad t_3 t_4 = -\dfrac{c_3}{a} \quad$ — (iii)

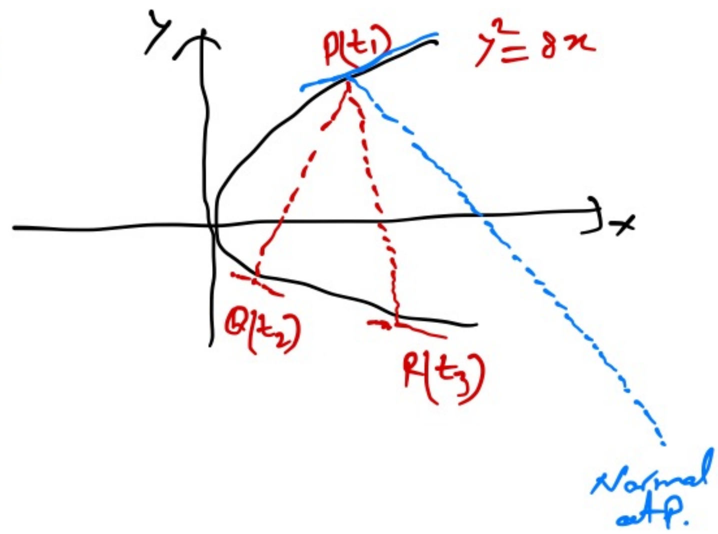
Now $\dfrac{(i) \times (iii)}{(ii)}$ gives $\quad t_1 t_4 = -\left(\dfrac{c_1 c_3}{c_2}\right) \cdot \dfrac{1}{a}$

or $\quad t_1 t_4 = -\dfrac{c_4}{a}$

Hence side AD will also pass through fixed point $(c_4, 0)$ where $c_4 = \dfrac{c_1 c_3}{c_2}$

Ans

3. Let P(a,b) and Q(c,d) are the two points on the parabola $y^2 = 8x$ such that the normals at them meet in (18,12). Find the product (abcd).



$P(t_1) \equiv P(2t_1^2, 4t_1) \equiv (18, 12)$

$\therefore t_1 = 3$

so $t_2 t_3 = 2$ — (1)

& $t_2 + t_3 = -t_1$

or $t_2 + t_3 = -3$ — (2)

$\therefore t_2 = -1, \quad t_3 = -2$

$\therefore Q(2, -4)$ & $R(8, -8)$

$\therefore a = 2, \quad b = -4$

$c = 8, \quad d = -8$

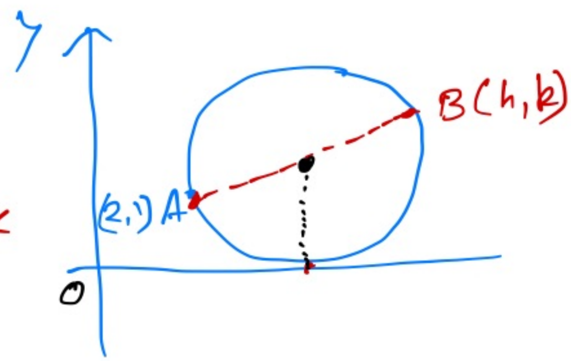
$\therefore abcd = 512$ Ans

4. A variable circle passes through the point A(2,1) and touches the x-axis. Locus of the other end of the diameter through A is a parabola.

(a) Find the length of the latus rectum of the parabola.

(b) Find the co-ordinates of the foot of the directrix of the parabola.

(c) The two tangents and two normals at the extremities of the latus rectum of the parabola constitutes a quadrilateral. Find area of quadrilateral.



AB is diameter, so diametric form of eqn of circle is

$$(x-2)(x-h)+(y-1)(y-k)=0$$

$$x^2+y^2-x(h+2)-y(1+k)+(2h+k)=0$$

Circle touches x-axis

$$\Rightarrow \quad g^2=c$$

$$\Rightarrow \left(-\frac{h+2}{2}\right)^2=(2h+k)$$

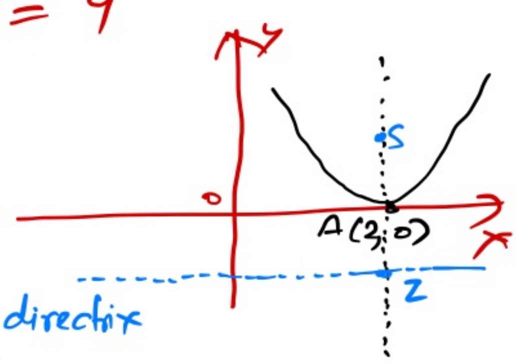
After simplification we get

$$(h-2)^2=4k$$

$\therefore$ Locus of B(h,k) ie other end of diameter is

$$(x-2)^2=4y$$

(i) length of latus rectum $=4$



(ii) $4a=4 \Rightarrow a=1$

$\therefore S(2,1)$

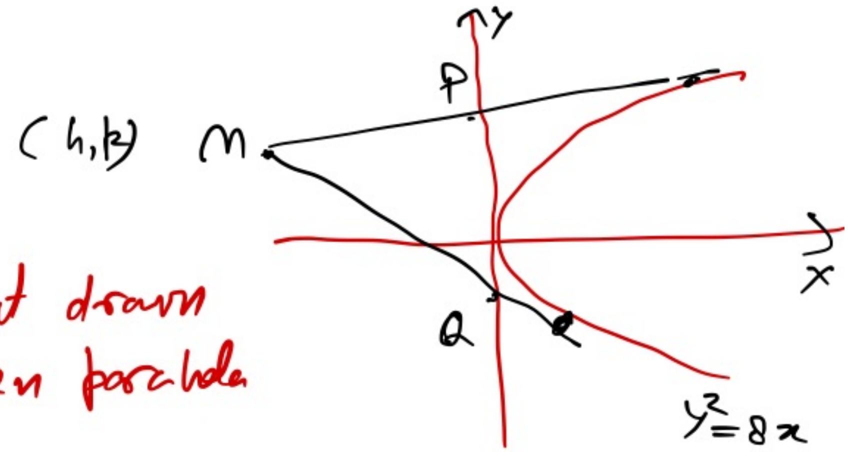
& foot of directrix is $(2,-1)$

(iii) Area of quadrilateral (square)

$$=\frac{1}{2}\cdot 4a\cdot 4a=8a^2=8 \text{ Ans} \quad (\because a=1)$$

5. Two tangents to the parabola $y^2 = 8x$ meet the tangent at its vertex in the points P & Q. If PQ = 4 units, prove that the locus of the point of the intersection of the two tangents is $y^2 = 8(x+2)$.

$l(PQ) = 4$



eqⁿ of pair of tangent drawn from $M(h,k)$ to given parabola is $SS_1 = T^2$

ie $(y^2 - 8x)(k^2 - 8h) = [yk - 4(x+h)]^2$

Pair of tangent meet y-axis at P & Q, so
Put $x = 0$ in above eqⁿ, we get

$$y^2 - yk + 2h = 0 \quad \begin{matrix} y_1 \\ y_2 \end{matrix}$$

$y_1 + y_2 = k$ & $y_1 y_2 = 2h$

where $P(0, y_1)$ & $Q(0, y_2)$

where $|y_1 - y_2| = 4 \rightarrow$ Given

$\because (y_1 - y_2)^2 = (y_1 + y_2)^2 - 4y_1y_2$

$16 = k^2 - 8h$

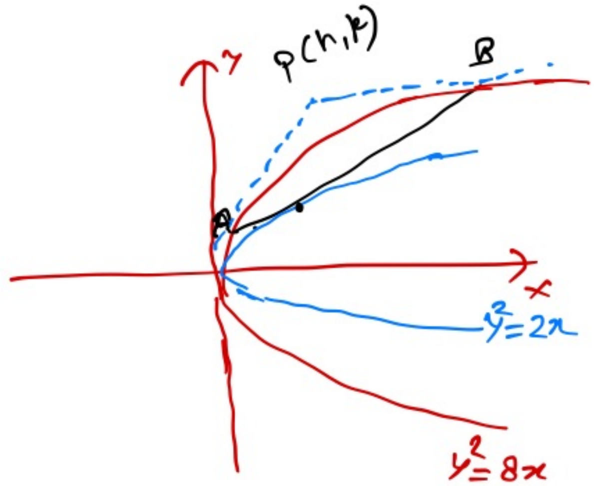
$k^2 = 8(h+2)$

$\therefore$ locus of $M(h,k)$ is

$$\boxed{y^2 = 8(x+2)}$$

Ans

6. A variable chords of the parabola $y^2 = 8x$ touches the parabola $y^2 = 2x$. The locus of the point of intersection of the tangent at the end of the chord is a parabola. Find its latus rectum.



variable tangent to the parabola $y^2 = 2x$ is

$$y = mx + \frac{1}{2m} \quad \text{— (1)}$$

Above represent chord AB of parabola $y^2 = 8x$

which is also chord of contact to the parabola $y^2 = 8x$ wrt point $P(h,k)$, therefore eq^n of AB can be re-written as

$$T = 0$$

ie $\quad ky = 4(x+h)$

$$ky = 4x + 4h \quad \text{— (2)}$$

(1) & (2) represent same line (Identical)

$$\Rightarrow \quad \frac{k}{1} = \frac{4}{m} = \frac{4h}{1/2m}$$

$k = 4h \cdot 2m$

$k = 8hm$

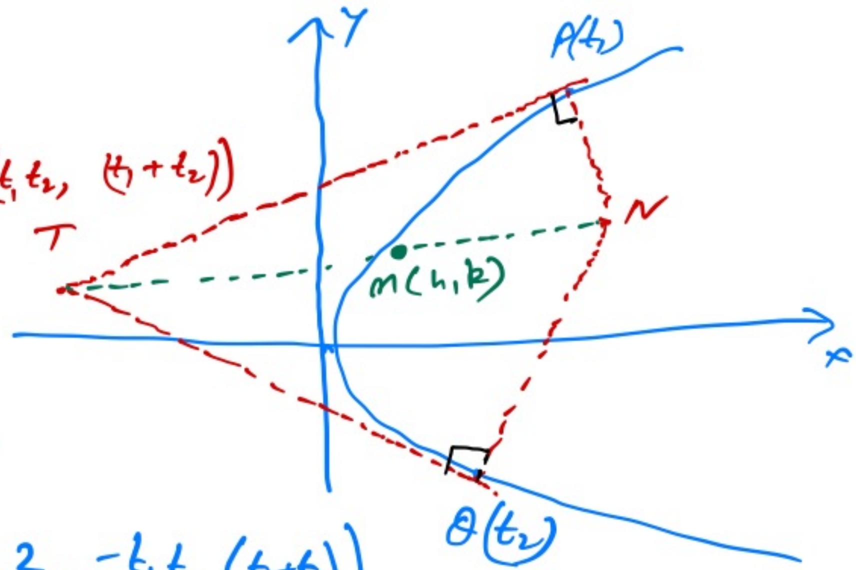
$m = \dfrac{4}{k}$ $\qquad$ $m = \dfrac{k}{8h}$

equate value of m, we get $\dfrac{4}{k} = \dfrac{k}{8h}$

$\therefore$ Locus of $P(h,k)$ is $\boxed{y^2 = 32x}$

$\therefore$ length of latus-rectum is $= 32$ Ans

7. PQ, a variable chord of the parabola $y^2 = 4x$ subtends a right angle at the vertex. The tangents at P and Q meet at T and the normals at those points meet at N. If the locus of the mid point of TN is a parabola, then find its latus rectum.



Chord PQ subtend 90° at vertex of Parabola

$\Rightarrow t_1 t_2 = -4$ — (i)

Normal at P & Q meet at $N\left(t_1^2 + t_2^2 + t_1 t_2 + 2, -t_1 t_2 (t_1 + t_2)\right)$

mid point of T & N is $m(h, k)$

$\therefore h = \dfrac{t_1^2 + t_2^2 - 6}{2}$ & $k = \dfrac{5(t_1 + t_2)}{2}$

$\therefore \left(\dfrac{2k}{5}\right)^2 = t_1^2 + t_2^2 + 2t_1 t_2$
$= (2h+6) - 8$

$\therefore$ locus of $m(h, k)$ is

$y^2 = \dfrac{25}{2}(x - 1)$

$\therefore$ length of latus rectum $= \dfrac{25}{2}$ Ans

J. M

EXERCISE (JM)

1. If two tangents drawn from a point P to the parabola $y^2 = 4x$ are at right angles then the locus of P is :-

[AIEEE-2010]

(1) $x = 1$ (2) $2x + 1 = 0$ (3) $x = -1$ (4) $2x - 1 = 0$

Sol^n The locus of point of intersection of two perpendicular tangents is directrix

Here directrix $x = -1$

Ans. option (3)

2. Given : A circle, $2x^2 + 2y^2 = 5$ and a parabola, $y^2 = 4\sqrt{5}\,x$.

Statement–I : An equation of a common tangent to these curves is $y = x + \sqrt{5}$.

Statement–II : If the line, $y = mx + \frac{\sqrt{5}}{m}$ $(m \neq 0)$ is their common tangent, then m satisfies $m^4 - 3m^2 + 2 = 0$. **[JEE (Main)-2013]**

(1) Statement-I is true, Statement-II is true; statement-II is a **correct** explanation for Statement-I.

(2) Statement-I is true, Statement-II is true; statement-II is **not** a correct explanation for Statement-I.

(3) Statement-I is true, Statement-II is false.

(4) Statement-I is false, Statement-II is true.

Solⁿ option (2)

eqⁿ of tangent to the parabola

$$y = mx + \frac{\sqrt{5}}{m} \quad \text{— (I)}$$

$$\Rightarrow \boxed{mx - y + \frac{\sqrt{5}}{m} = 0}$$

this line touches the circle.

$$\frac{\left|\frac{\cancel{\sqrt{5}}}{m}\right|}{\sqrt{m^2+1}} = \frac{\cancel{\sqrt{5}}}{\sqrt{2}}$$

$$\Rightarrow m^4 + m^2 = 2 \Rightarrow (m^2+2)(m^2-1) = 0$$

$$\Rightarrow m^2 = -2 \text{ (Not possible)}$$

So, $m^2 = 1 \Rightarrow \boxed{m = \pm 1}$

it satisfies $\underline{m^4 - 3m^2 + 2 = 0}$

3. Statement 1 : The slope of the tangent at any point P on a parabola, whose axis is the axis of x and vertex is at the origin, is inversely proportional to the ordinate of the point P.

Statement 2 : The system of parabolas $y^2 = 4ax$ satisfies a differential equation of degree 1 and order 1.

(1) Statement 1 is True Statement 2 is True, Statement 2 is a correct explanation for Statement 1.

(2) Statement 1 is True, Statement 2 is False.

~~(3)~~ Statement 1 is True, Statement 2 is True statement 2 is not a correct explanation for statement 1.

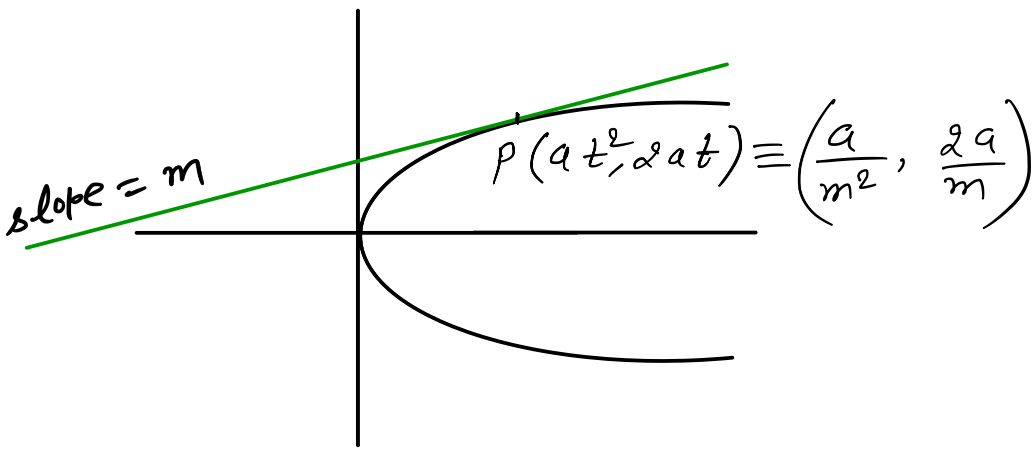
(4) Statement 1 is False, Statement 2 is True **[JEE-Main (On line)-2013]**

Sol:-

St. 1.

T

for tangent $t = 1/m$



$\therefore$ slope of Tangent is Inversely proportional to Ordinate of 'P'.

St. 2.

T

$\because y^2 = 4ax$

$$\Rightarrow \boxed{\frac{dy}{dx} = \frac{2a}{y}}$$

order = 1

degree = 1

Ans $\rightarrow$ option(3)

4. **Statement 1 :** The line $x - 2y = 2$ meets the parabola, $y^2 + 2x = 0$ only at the point $(-2, -2)$

Statement 2 : The line $y = mx - \frac{1}{2m}$ $(m \neq 0)$ is tangent to the parabola, $y^2 = -2x$ at the point $\left(-\frac{1}{2m^2}, -\frac{1}{m}\right)$.

(1) Statement 1 is false; Statement 2 is true.

(2) Statement 1 is true; Statement 2 is true; Statement 2 is not a correct explanation for Statement 1.

(3) Statement 1 is true; Statement 2 is false.

(4) Statement 1 is true; Statement 2 is true; Statement 2 is a correct explanation for Statement 1.

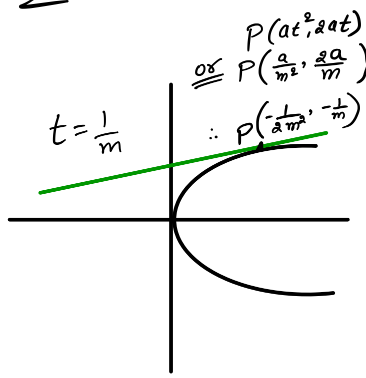
[JEE-Main (On line)-2013]

Sol:- St. 2. $\because$ parabola : $\boxed{y^2 = -2x}$

$\therefore a = -\frac{1}{2}$

Tangent : $y = mx + \frac{a}{m}$

$\Rightarrow \boxed{y = mx - \frac{1}{2m}}$



St. 1 this is also correct same as St. 2.

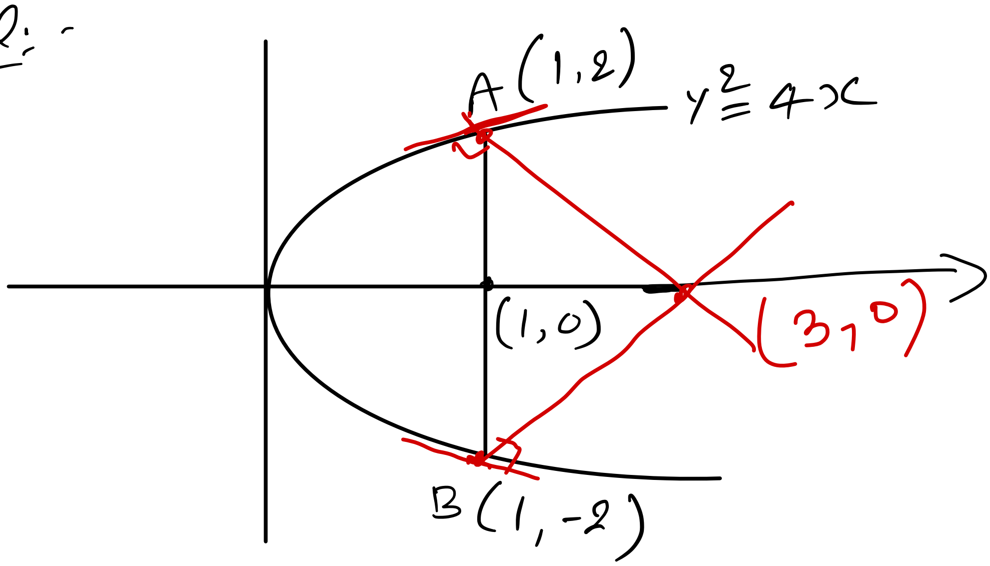
Ans $\rightarrow$ option (4)

5. The point of intersection of the normals tothe parabola $y^2 = 4x$ at the ends of its latus rectum is :

[JEE-Main (On line)-2013]

(1) (0, 3) (2) (2, 0) (3) (3, 0) (4) (0, 2)

Sol :



Normal at A : $x + y = 3$

Normal at B : $x - y = 3$

$\Rightarrow$ Pt. of Intersection

$\left.\begin{array}{l} x = 3 \\ y = 0 \end{array}\right]$ A.

Ans ⇒ Option (3)

6. The slope of the line touching both, the parabolas $y^2 = 4x$ and $x^2 = -32y$ is : **[JEE(Main)-2014]**

(1) $\frac{1}{2}$ (2) $\frac{3}{2}$ (3) $\frac{1}{8}$ (4) $\frac{2}{3}$

Sol:-

Now tangent with slope 'm' to

$y^2 = 4x$ is $y = mx + \frac{1}{m}$ ...(1)

Also tangent with slope 'm' to

$x^2 = -32y$ is $y = mx + 8m^2$ ...(2)

If (1) and (2) are equations of the same line then their coefficients should match.

$\Rightarrow 8m^2 = \frac{1}{m} \quad \Rightarrow m = \frac{1}{2}$

Ans ⇒ option (1)

7. Let O be the vertex and Q be any point on the parabola, $x^2 = 8y$. If the point P divides the line segment OQ internally in the ratio 1 : 3, then the locus of P is :- **[JEE(Main)-2015]**

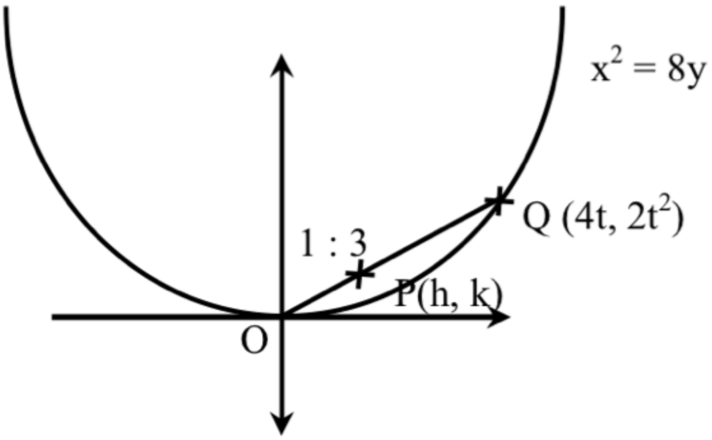
(1) $y^2 = 2x$ (2) $x^2 = 2y$ (3) $x^2 = y$ (4) $y^2 = x$

Sol:- Let $Q(4t, 2t^2)$

$$h = \frac{4t}{4} = t$$

$$k = \frac{2t^2}{4} = \frac{t^2}{2}$$

$$\Rightarrow \quad x^2 = 2y$$



Ans⇒ option (2)

8. Let P be the point on the parabola, $y^2 = 8x$ which is at a minimum distance from the cente C of the circle, $x^2 + (y + 6)^2 = 1$. Then the equation of the circle, passing through C and having its centre at P is : **[JEE(Main)-2016]**

(1) $x^2 + y^2 - 4x + 9y + 18 = 0$
(2) $x^2 + y^2 - 4x + 8y + 12 = 0$

(3) $x^2 + y^2 - x + 4y - 12 = 0$
(4) $x^2 + y^2 - \frac{x}{4} + 2y - 24 = 0$

Sol: -

Normal at $P(at^2, 2at)$ is $y + tx = 2at + at^3$

Given it passes $(0, -6)$

$\Rightarrow -6 = 2at + at^3 \qquad (a = 2)$

$-6 = 4t + 2t^3$

$t^3 + 2t + 3 = 0$

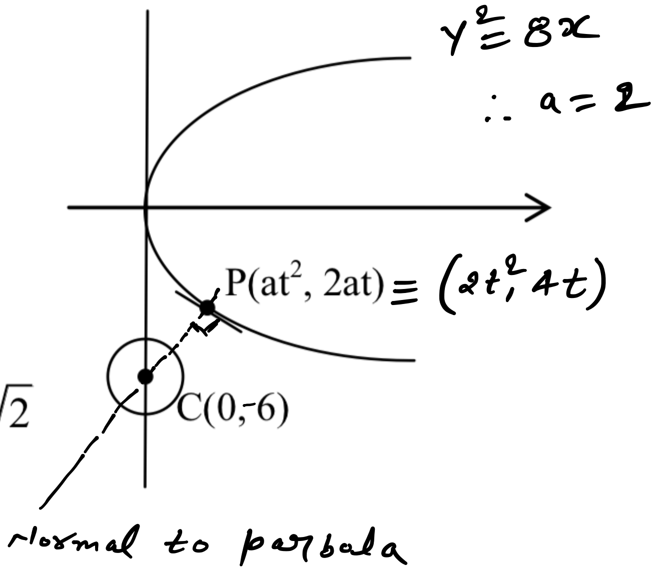
$t = -1$

so, $P(a, -2a) = (2, -4)$. $[a = 2]$

radius of circle = CP = $\sqrt{2^2 + (-4+6)^2} = 2\sqrt{2}$

Circle is $(x-2)^2 + (y+4)^2 = \left(2\sqrt{2}\right)^2$

$x^2 + y^2 - 4x + 8y + 12 = 0$



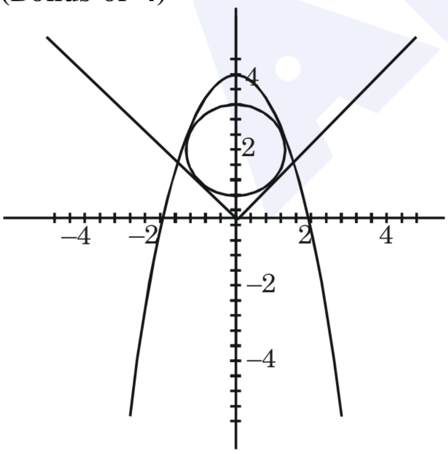
Ans: option (2)

9. The radius of a circle, having minimum area, which touches the curve $y = 4 - x^2$ and the lines, $y = |x|$ is :- **[JEE-Main 2017]**

(1) $4(\sqrt{2}+1)$ (2) $2(\sqrt{2}+1)$ (3) $2(\sqrt{2}-1)$ (4) $4(\sqrt{2}-1)$

Ans. (Bonus or 4)

Sol.



$x^2 + (y - \beta)^2 = r^2$

$x - y = 0$

$$\left|\frac{0-\beta}{\sqrt{2}}\right| = r \Rightarrow \beta = r\sqrt{2}$$

$$x^2 + (y-\beta)^2 = \frac{\beta^2}{2}$$

$$\Rightarrow 4 - y + (y-\beta)^2 = \frac{\beta^2}{2}$$

$$\Rightarrow y^2 - y(2\beta+1) + \frac{\beta^2}{2} + 4 = 0$$

D = 0

$$\Rightarrow (2\beta+1)^2 - 4\left(\frac{\beta^2}{2}+4\right) = 0$$

$$4\beta^2 + 4\beta + 1 - 2\beta^2 - 16 = 0$$

$$\Rightarrow 2\beta^2 + 4\beta - 15 = 0$$

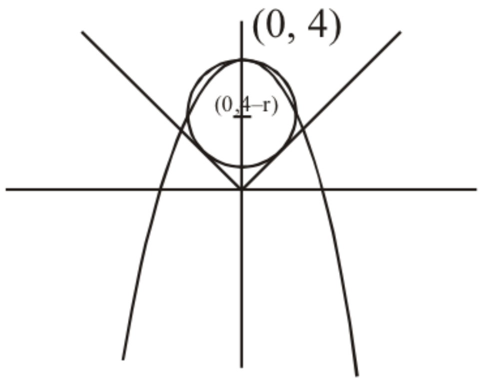
$$\beta = \frac{-4 \pm \sqrt{16+120}}{4} = \frac{-4 \pm 2\sqrt{34}}{4}$$

$$= \frac{-2 \pm \sqrt{34}}{2} \Rightarrow \frac{\sqrt{34}-2}{2}$$

$$r = \frac{\sqrt{34}-2}{2\sqrt{2}}$$

which is not in options therefore it must be bonus. But according to history of JEE-Mains it seems they had following line of thinking.

Given curves are $y = 4 - x^2$ and $y = |x|$



There are two circles satisfying the given conditions. The circle shown is of least area.

Let radius of circle is 'r'

$\therefore$ co-ordinates of centre = $(0, 4 - r)$

$\because$ circle touches the line $y = x$ in first quadrant

$$\therefore \left|\frac{0-(4-r)}{\sqrt{2}}\right| = r \Rightarrow r - 4 = \pm r\sqrt{2}$$

$$\therefore r = \frac{4}{\sqrt{2}+1} = 4(\sqrt{2}-1)$$

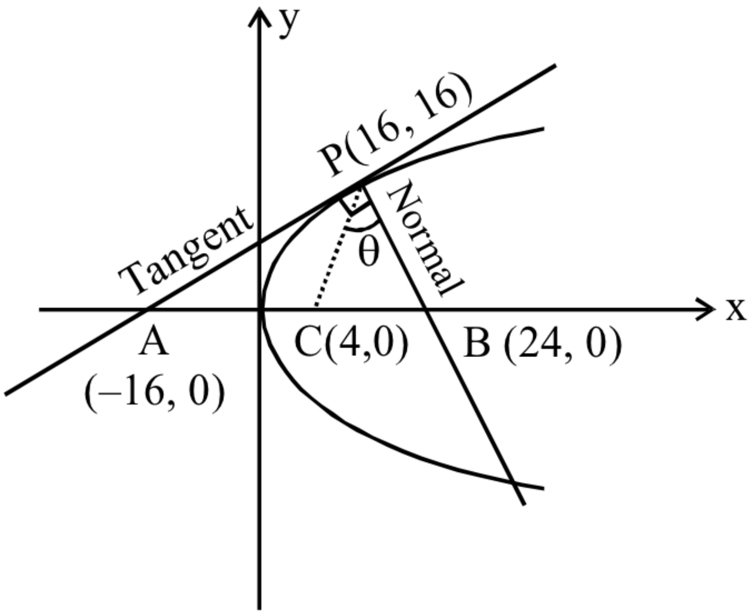
which is given in option 4.

10. Tangent and normal are drawn at P(16, 16) on the parabola $y^2 = 16x$, which intersect the axis of the parabola at A and B, respectively. If C is the centre of the circle through the points P, A and B and $\angle CPB = \theta$, then a value of $\tan\theta$ is - **[JEE-Main 2018]**

(1) 2 (2) 3 (3) $\frac{4}{3}$ (4) $\frac{1}{2}$

Sol. Equation of tangent at P(16, 16)

is $x - 2y + 16 = 0$



$$m_{PC} = \frac{4}{3}$$

$$m_{PB} = -2$$

Hence, $\tan\theta = \left|\frac{m_{PC} - m_{PB}}{1 + m_{PC} \cdot m_{PB}}\right|$

$\tan\theta = 2$

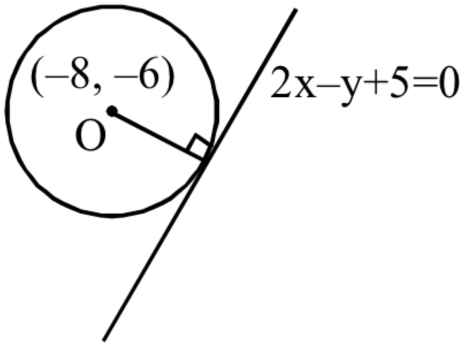
Ans. option (1)

11. If the tangent at (1, 7) to the curve $x^2 = y - 6$ touches the circle $x^2 + y^2 + 16x + 12y + c = 0$ then the value of c is : **[JEE-Main 2018]**

(1) 185 (2) 85 (3) 95 (4) 195

Sol. Equation of tangent at (1, 7) to $x^2 = y - 6$ is

$$2x - y + 5 = 0.$$



Now, perpendicular from centre O(–8, –6) to $2x - y + 5 = 0$ should be equal to radius of the circle

$$\left|\frac{-16+6+5}{\sqrt{5}}\right| = \sqrt{64+36-C}$$

$$\sqrt{5} = \sqrt{100-C}$$

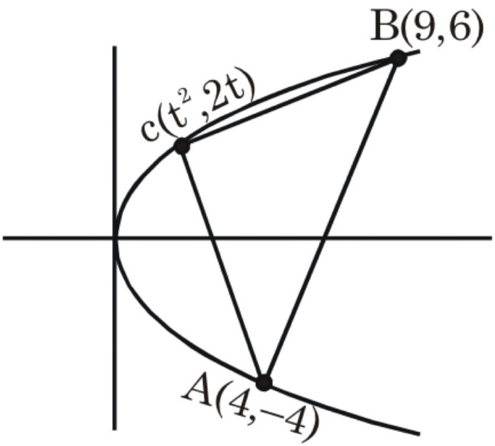
then, C = 95

Ans→ option(3)

12. Let A(4,–4) and B(9,6) be points on the parabola, $y^2 = 4x$. Let C be chosen on the arc AOB of the parabola, where O is the origin, such that the area of ΔACB is maximum. Then, the area (in sq. units) of ΔACB, is: **[JEE (Main)-Jan 19]**

(1) $31\frac{3}{4}$ (2) 32 (3) $30\frac{1}{2}$ (4) $31\frac{1}{4}$

Sol.

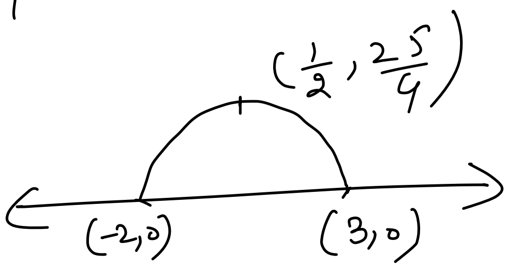


$\boxed{-2 < t < 3}$

(as C lies on arcAOB)

$$\text{Area} = \left| \frac{1}{2} \begin{vmatrix} t^2 & 2t & 1 \\ 9 & 6 & 1 \\ 4 & -4 & 1 \end{vmatrix} \right| = \frac{1}{2}\left|10t^2 - 10t - 60\right|$$

$$= 5\left|t^2 - t - 6\right|$$



max. value

$$= 5\left(\frac{25}{4}\right) = \frac{125}{4}$$

$$= 31\frac{1}{4}$$

Ans option(4)

13. If the parabolas $y^2=4b(x-c)$ and $y^2=8ax$ have a common normal, then which one of the following is a valid choice for the ordered triad (a,b,c) **[JEE (Main)-Jan 19]**

(1) (1, 1, 0) (2) $\left(\frac{1}{2},2,3\right)$ (3) $\left(\frac{1}{2},2,0\right)$ (4) (1, 1, 3)

Ans. (1,2,3,4)

Sol. Normal to these two curves are

$y = m(x - c) - 2bm - bm^3,$

$y = mx - 4am - 2am^3$

Compare

If they have a common normal

$(c + 2b)\, m + bm^3 = 4am + 2am^3$

Now $(4a - c - 2b)\, m = (b - 2a)m^3$

We get all options are correct for m = 0

(common normal x-axis)

Ans. (1), (2), (3), (4)

Remark :

If we consider question as

If the parabolas $y^2 = 4b(x - c)$ and $y^2 = 8ax$ have a common normal other than x-axis, then which one of the following is a valid choice for the ordered triad (a, b, c) ?

When $m \neq 0$: $(4a - c - 2b) = (b - 2a)m^2$

$$m^2 = \frac{c}{2a-b} - 2 > 0 \Rightarrow \frac{c}{2a-b} > 2$$

Now according to options, option 4 is correct

Ans ⇒ options 1, 2, 3, 4

14. If the tangent to the parabola $y^2 = x$ at a point (α, β), $(\beta > 0)$ is also a tangent to the ellipse, $x^2 + 2y^2 = 1$, then α is equal to : **[JEE (Main)-Apr 19]**

(1) $2\sqrt{2}+1$ (2) $\sqrt{2}-1$ (3) $\sqrt{2}+1$ (4) $2\sqrt{2}-1$

Sol^n $\boxed{\beta^2 = \alpha}$

Now eq^n of tangent to the parabola at (α, β) is

$$y(\beta) = \frac{1}{2}(x+\alpha) \Rightarrow y(\beta) = \frac{1}{2}(x+\beta^2)$$

$$\Rightarrow \boxed{y = \frac{1}{2\beta}\cdot x + \frac{\beta}{2}} \quad \text{——} \quad (1)$$

slope of these tangent $m = \frac{1}{2\beta}$

eq^n of tangent to the ellipse with slope $\frac{1}{2\beta}$ is

$$y = \frac{1}{2\beta}\cdot x \pm \sqrt{\left(\frac{1}{2\beta}\right)^2 + \frac{1}{2}} \quad \text{——} \quad (2)$$

Comparing (1) and (2)

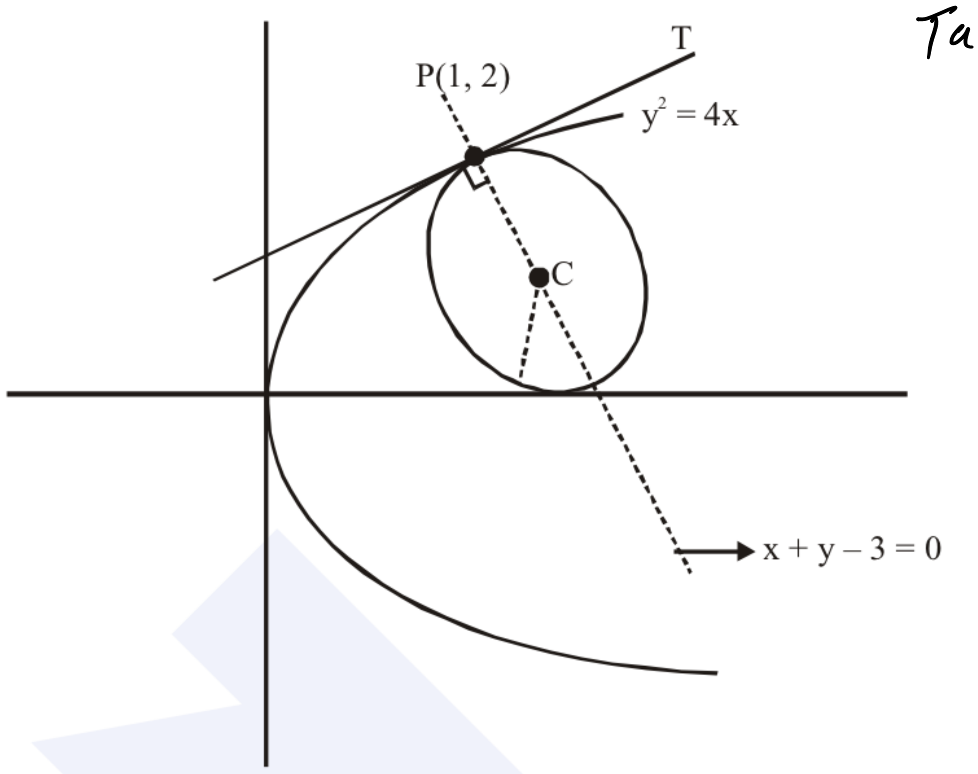
$$\left(\frac{\beta}{2}\right)^2 = \left(\frac{1}{2\beta}\right)^2 + \frac{1}{2} \Rightarrow \boxed{\beta^2 = \sqrt{2}+1 = \alpha}$$

Ans $\rightarrow$ option (3)

15. The area (in sq. units) of the smaller of the two circles that touch the parabola, $y^2 = 4x$ at the point (1, 2) and the x-axis is :- **[JEE (Main)-Apr 19]**

(1) $4\pi(2-\sqrt{2})$ (2) $8\pi(3-2\sqrt{2})$ (3) $4\pi(3+\sqrt{2})$ (4) $8\pi(2-\sqrt{2})$

Sol.



Tangent of Parabola at P(1,2)

$yy_1 = 2(x+x_1)$

$2y = 2(x+1)$

$\Rightarrow \boxed{x - y + 1 = 0}$

Equation of circle is

$(x-1)^2 + (y-2)^2 + \lambda(x - y + 1) = 0$

$\Rightarrow x^2 + y^2 + x(\lambda - 2) + y(-4 - \lambda) + (5 + \lambda) = 0$

{ $S + \lambda L = 0$ }
{ $S = 0$ is point circle

As cirlce touches x axis then $g^2 - c = 0$

$$\frac{(\lambda-2)^2}{4} = (5+\lambda)$$

$\lambda^2 + 4 - 4\lambda = 20 + 4\lambda$

$\lambda^2 - 8\lambda - 16 = 0$

$$\lambda = \frac{8 \pm \sqrt{128}}{2}$$

$$\lambda = 4 \pm 4\sqrt{2}$$

$$\text{Radius} = \left|\frac{(-4-\lambda)}{2}\right|$$

Put λ and get least radius.

Ans $\Rightarrow$ option (2)

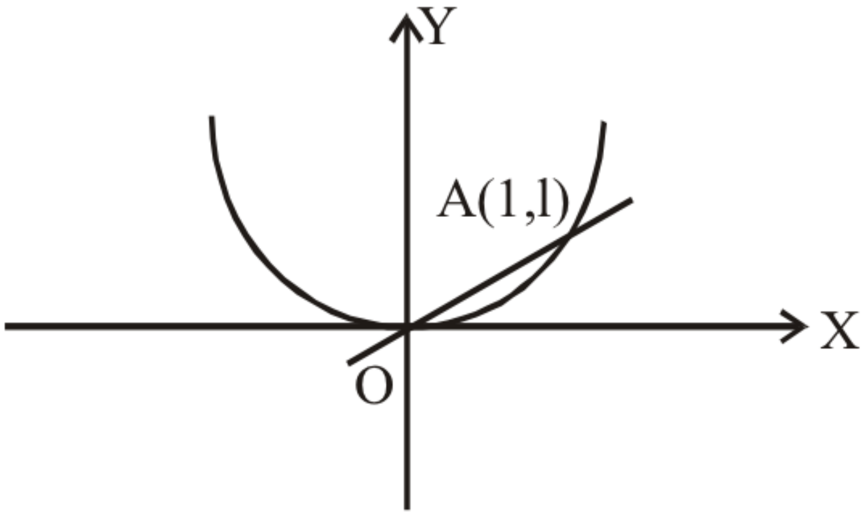
16. The tangents to the curve $y = (x-2)^2 - 1$ at its points of intersection with the line $x - y = 3$, intersect at the point : **[JEE (Main)-Apr 19]**

$\hookrightarrow (y+1) = (x-2)^2$

(1) $\left(-\frac{5}{2},-1\right)$ (2) $\left(-\frac{5}{2},1\right)$ (3) $\left(\frac{5}{2},-1\right)$ (4) $\left(\frac{5}{2},1\right)$

Sol. Put $x - 2 = X$ & $y + 1 = Y$

$\therefore$ given curve becomes $Y = X^2$ and $Y = X$



tangent at origin is X-axis

and tangent at A(1,1) is $Y + 1 = 2X$

Put Y = 0

$\therefore$ there intersection is $\left(\frac{1}{2},0\right)$

$$\therefore \quad x-2=\frac{1}{2} \;\&\; y+1=0$$

therefore $x=\frac{5}{2}, y=-1$

Ans $\Rightarrow$ option (3)

17. The equation of a common tangent to the curves, $y^2 = 16x$ and $xy = -4$ is : **[JEE (Main)-Apr 19]**

(1) $x + y + 4 = 0$ (2) $x - 2y + 16 = 0$ (3) $2x - y + 2 = 0$ (4) $x - y + 4 = 0$

Sol. tangent to the parabola $y^2 = 16x$ is $y = mx + \frac{4}{m}$

solve it by curve $xy = -4$

$$\text{i.e. } mx^2 + \frac{4}{m}x + 4 = 0$$

condition of common tangent is $D = 0$

$$\therefore m^3 = 1$$

$$\Rightarrow m = 1$$

$\therefore$ equation of common tangent is $y = x + 4$

Ans option (4)

J-A

EXERCISE (JA)

1. The tangent PT and the normal PN to the parabola $y^2 = 4ax$ at a point P on it meet its axis at points T and N, respectively. The locus of the centroid of the triangle PTN is a parabola whose

(A) vertex is $\left(\frac{2a}{3}, 0\right)$ (B) directrix is $x = 0$ **[JEE 2009, 4]**

(C) latus rectum is $\frac{2a}{3}$ (D) focus is $(a, 0)$

Sol. **(A, D)**

$G \equiv (h, k)$

$\Rightarrow h = \dfrac{2a + at^2}{3}, \; k = \dfrac{2at}{3}$

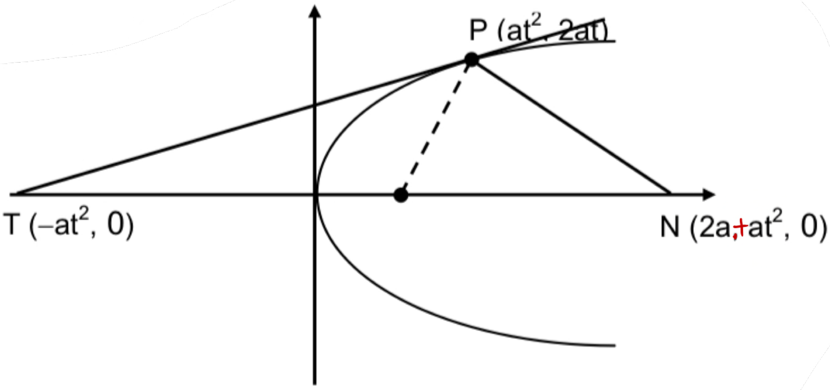
$\Rightarrow \left(\dfrac{3h - 2a}{a}\right) = \dfrac{9k^2}{4a^2}$

$\Rightarrow$ required parabola is

$\dfrac{9y^2}{4a^2} = \dfrac{(3x - 2a)}{a} = \dfrac{3}{a}\left(x - \dfrac{2a}{3}\right)$

$\Rightarrow y^2 = \dfrac{4a}{3}\left(x - \dfrac{2a}{3}\right)$

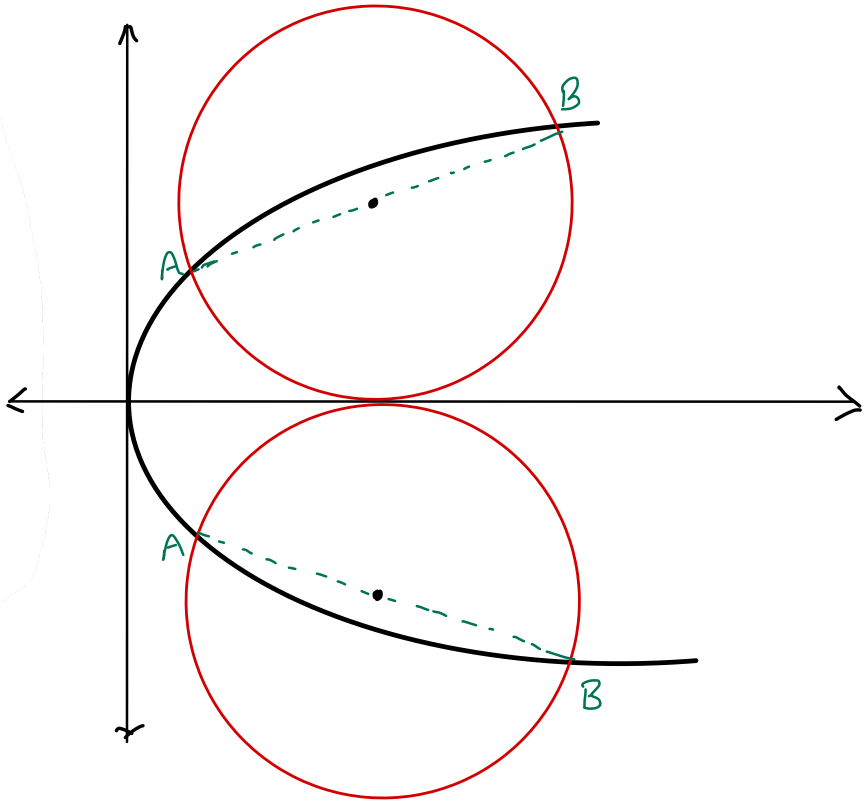
Vertex $\equiv \left(\dfrac{2a}{3}, 0\right)$; Focus $\equiv (a, 0)$



2. Let A and B be two distinct point on the parabola $y^2 = 4x$. If the axis of the parabola touches a circle of radius r having AB as its diameter, then the slope of the line joining A and B can be -

(A) $\frac{-1}{r}$ (B) $\frac{1}{r}$ (C) $\frac{2}{r}$ (D) $\frac{-2}{r}$ **[JEE 2010,3]**

Sol:-



(C), (D)

$A = (t_1^2,\ 2t_1),\ B = (t_2^2,\ 2t_2)$

$$\text{Centre} = \left[\frac{t_1^2 + t_2^2}{2},\ (t_1 + t_2)\right]$$

$t_1 + t_2 = \pm r$

$$m = \frac{2(t_1 - t_2)}{t_1^2 - t_2^2} = \frac{2}{t_1 + t_2} = \pm\frac{2}{r}.$$

3. Consider the parabola $y^2 = 8x$. Let Δ_1 be the area of the triangle formed by the end points of its latus rectum and the point $P\left(\frac{1}{2}, 2\right)$ on the parabola, and Δ_2 be the area of the triangle formed by drawing tangents at P and at the end points of the latus rectum. Then $\frac{\Delta_1}{\Delta_2}$ is **[JEE 2011,4]**

Sol:-

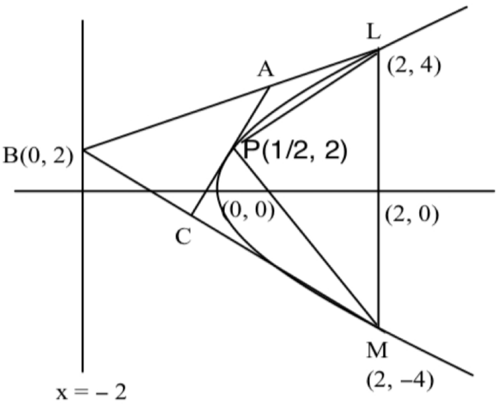
Consider the parabola $y^2 = 8x$. Let Δ_1 be the area of the triangle formed by the end points of its latus rectum and the point $P\left(\frac{1}{2}, 2\right)$ on the parabola, and Δ_2 be the area of the triangle formed by drawing tangents at P and at the end points of the latus rectum. Then $\frac{\Delta_1}{\Delta_2}$ is

(2)

$y^2 = 8x = 4.2.x$

$\frac{\Delta LPM}{\Delta ABC} = 2$

$\frac{\Delta_1}{\Delta_2} = 2$

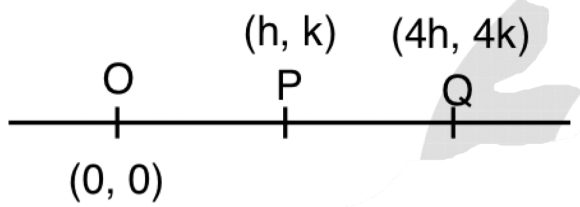


4. Let (x,y) be any point on the parabola $y^2 = 4x$. Let P be the point that divides the line segment from (0,0) to (x,y) in the ratio 1 : 3. Then the locus of P is- **[JEE 2011,3]**

(A) $x^2 = y$ (B) $y^2 = 2x$ (C) $y^2 = x$ (D) $x^2 = 2y$

Sol:-

(C)



$y^2 = 4x$

and Q will lie on it

$\Rightarrow (4k)^2 = 4 \times 4h$

$\Rightarrow k^2 = h$

$\Rightarrow y^2 = x$ (replacing h by x and k by y)

5. Let L be a normal to the parabola $y^2 = 4x$. If L passes through the point (9,6), then L is given by - **[JEE 2011,4]**

(A) $y - x + 3 = 0$ (B) $y + 3x - 33 = 0$ (C) $y + x - 15 = 0$ (D) $y - 2x + 12 = 0$

Sol: -

(A, B, D)

$y^2 = 4x$

Equation of normal is $y = mx - 2m - m^3$.

It passes through (9, 6)

$\Rightarrow m^3 - 7m + 6 = 0$

$\Rightarrow m = 1, 2, -3$

$\Rightarrow y - x + 3 = 0, y + 3x - 33 = 0, y - 2x + 12 = 0.$

6. Let S be the focus of the parabola $y^2 = 8x$ & let PQ be the common chord of the circle $x^2 + y^2 - 2x - 4y = 0$ and the given parabola. The area of the triangle PQS is **[JEE 2012, 4M]**

Sol :

(4)

The parabola is $x = 2t^2$, $y = 4t$

Solving it with the circle we get :

$4t^4 + 16t^2 - 4t^2 - 16t = 0$

$\Rightarrow t^4 + 3t^2 - 4t = 0 \Rightarrow t = 0, 1$

so, the points P and Q are (0, 0) and (2, 4) which are also diametrically opposite points on the circle. The focus is $S \equiv (2, 0)$.

The area of $\Delta PQS = \frac{1}{2} \times 2 \times 4 = 4$.

Paragraph for Question 7 and 8

Let PQ be a focal chord of the parabolas $y^2 = 4ax$. The tangents to the parabola at P and Q meet at a point lying on the line $y = 2x + a$, $a > 0$.

7. If chord PQ subtends an angle θ at the vertex of $y^2 = 4ax$, then $\tan\theta =$

[JEE(Advanced) 2013, 3, (–1)]

(A) $\frac{2}{3}\sqrt{7}$ (B) $\frac{-2}{3}\sqrt{7}$ (C) $\frac{2}{3}\sqrt{5}$ (D) $\frac{-2}{3}\sqrt{5}$

8. Length of chord PQ is **[JEE(Advanced) 2013, 3, (–1)]**

(A) 7a (B) 5a (C) 2a (D) 3a

Sol:- 7.

(D)

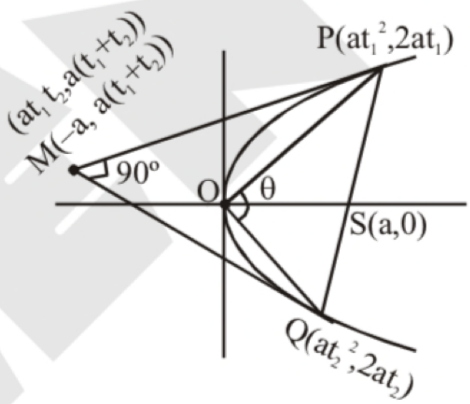
Single tangent at the extrimities of a focal chord will intersect on directrix.

$\therefore$ $M(-a, a(t_1 + t_2))$

lies on $y = 2x + a$

$a(t_1 + t_2) = -2a + a \Rightarrow t_1 + t_2 = -1$

& $t_1t_2 = -1$



$$\tan\theta = \left(\frac{\frac{2}{t_1} - \frac{2}{t_2}}{1 + \frac{4}{t_1t_2}}\right) = \left(\frac{2(t_2 - t_1)}{3}\right)$$

$\because$ $(t_2 - t_1)^2 = (t_2 + t_1)^2 - 4t_1t_2 = 5$

$t_2 - t_1 = \pm\sqrt{5}$

$\therefore$ $\tan\theta = \pm\frac{2\sqrt{5}}{3}$

but θ is obtuse because O is the interior point of the circle for which PQ is diameter.

$\therefore$ $\tan\theta = \frac{-2\sqrt{5}}{3}$

Sol:- 8.

(B)

Length of focal chord $PQ = a(t_1 - t_2)^2$

$= a[(t_1 + t_2)^2 - 4t_1t_2]$

$= a[1 + 4] = 5a$

9. A line L : y = mx + 3 meets y – axis at E(0,3) and the arc of the parabola $y^2 = 16x$, $0 \le y \le 6$ at the point $F(x_0,y_0)$. The tangent to the parabola at $F(x_0,y_0)$ intersects the y-axis at $G(0,y_1)$. The slope m of the line L is chosen such that the area of the triangle EFG has a local maximum. Match List-I with List-II and select the correct answer using the code given below the lists.

	List-I		List-II
P.	m =	1.	$\frac{1}{2}$
Q.	Maximum area of ΔEFG is	2.	4
R.	y_0 =	3.	2
S.	y_1 =	4.	1

Codes :

	P	Q	R	S
(A)	4	1	2	3
(B)	3	4	1	2
(C)	1	3	2	4
(D)	1	3	4	2

[JEE(Advanced) 2013, 3, (–1)]

Ans. (A)

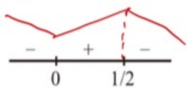
Sol. Let $F(4t^2, 8t)$

where $0 \le 8t \le 6 \Rightarrow \quad 0 \le t \le \frac{3}{4}$

$$\Delta EFG = \frac{1}{2}(3-4t)4t^2$$

$$\Delta = (6t^2 - 8t^3)$$

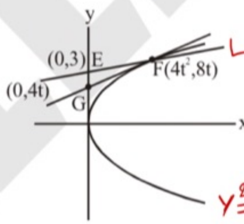
$$\frac{d\Delta}{dt} = 12t - 24t^2 = 0 \begin{cases} t = 0(\text{minima}) \\ t = \frac{1}{2}(\text{maxima}) \end{cases}$$



$$\Rightarrow \quad m = \frac{8t-3}{4t^2-0} = \frac{4-3}{1} = 1$$

$$(\Delta EFG)_{max} = \frac{6}{4} - 1 = \frac{1}{2}$$

$y_0 = 8t = 4$ & $y_1 = 4t = 2$



L

$y^2 = 16x$
$a = 2$

$F(x_0, y_0) \equiv (4t^2, 8t)$
$G(0, y_1) \equiv (0, 4t)$

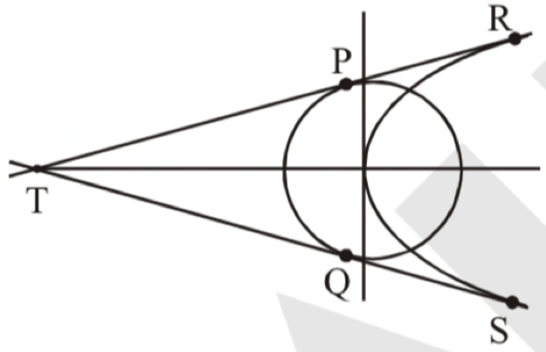
$\therefore y_0 = 8t$
$\& \ y_1 = 4t$

10 The common tangents to the circle $x^2 + y^2 = 2$ and the parabola $y^2 = 8x$ touch the circle at the point P, Q and the parabola at the points R,S. Then the area of the quadrilateral PQRS is -

(A) 3 (B) 6 (C) 9 (D) 15

[JEE(Advanced)-2014, 3(–1)]

Sol. Ans. (D)



$$y = mx + \frac{2}{m}$$

$$\frac{\left|0-0+\frac{2}{m}\right|}{\sqrt{1+m^2}} = \sqrt{2} \Rightarrow 2 = m^2(1 + m^2) \Rightarrow m = \pm 1$$

TP : $-x + y = 2$

So P(–1, 1) & Q(–1, –1)

& $R\left(\frac{2}{m}, \frac{4}{m}\right) \equiv R(2,4)$ & $S(2,-4)$

So $\Delta = \frac{1}{2} 10.3 = 15$

Paragraph For Questions 11 and 12

Let a,r,s,t be nonzero real numbers. Let $P(at^2, 2at)$, Q, $R(ar^2, 2ar)$ and $S(as^2, 2as)$ be distinct points on the parabola $y^2 = 4ax$. Suppose that PQ is the focal chord and lines QR and PK are parallel, where K is the point (2a,0).

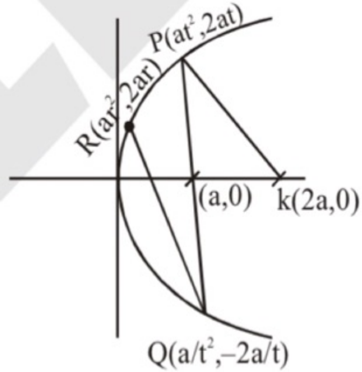
11. The value of r is- **[JEE(Advanced)-2014, 3(–1)]**

(A) $-\frac{1}{t}$ (B) $\frac{t^2+1}{t}$ (C) $\frac{1}{t}$ (D) $\frac{t^2-1}{t}$

Sol. Ans. (D)

$\because$ PQ is a focal chord

$\therefore$ co-ordinates of point Q are $=\left(\frac{a}{t^2}, -\frac{2a}{t}\right)$



$$m_{QR} = \frac{2a\left(r+\frac{1}{t}\right)}{a\left(r^2-\frac{1}{t^2}\right)} = \frac{2}{\left(r-\frac{1}{t}\right)}$$

$$m_{PK} = \frac{2at-0}{a\left(t^2-2\right)} = \frac{2t}{t^2-2}$$

Given $m_{QR} = m_{PK}$

$$\Rightarrow \frac{2}{r-\frac{1}{t}} = \frac{2t}{t^2-2} \Rightarrow r = \frac{t^2-2}{t} + \frac{1}{t}$$

$$\Rightarrow r = t - \frac{2}{t} + \frac{1}{t} \Rightarrow r = \frac{t^2-1}{t}$$

12. If st = 1, then the tangent at P and the normal at S to the parabola meet at a point whose ordinate is-

[JEE(Advanced)-2014, 3(–1)]

(A) $\frac{\left(t^2+1\right)^2}{2t^3}$ (B) $\frac{a\left(t^2+1\right)^2}{2t^3}$ (C) $\frac{a\left(t^2+1\right)^2}{t^3}$ (D) $\frac{a\left(t^2+2\right)^2}{t^3}$

Ans. (B)

Equation of tangent at point P is $ty = x + at^2$ ....(i)

Equation of normal at point S is $\frac{1}{t}x + y = \frac{2a}{t} + \frac{a}{t^3}$

$\Rightarrow$ $t^2x + t^3y = 2at^2 + a$ ....(ii)

Multiply equation (i) by t^2 and then subtract from equation (ii),

we get,

$2t^3y = 2at^2 + at^4 + a$

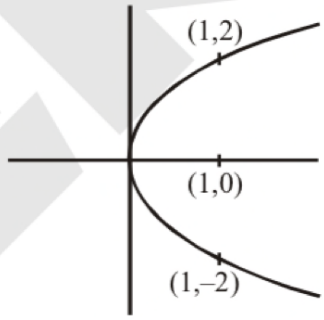
$\Rightarrow$ $2t^3y = a(1 + t^4 + 2t^2)$

$\Rightarrow$ $\boxed{y = \frac{a(1+t^2)^2}{2t^3}}$

13 If the normals of the parabola $y^2 = 4x$ drawn at the end points of its latus rectum are tangents to the circle $(x - 3)^2 + (y + 2)^2 = r^2$, then the value of r^2 is **[JEE 2015, 4M, –0M]**

Ans. 2

Sol. The co-ordinates of latus rectum are (1,2) and (1,–2)



clearly slope of tangent is given by $\frac{dy}{dx} = \frac{2}{y}$

$\therefore$ At y = 2 slope of normal = –1

and At y = –2 slope of normal = 1

$\therefore$ Equation of normal at (1,2)

$(y - 2) = -1(x - 1) \Rightarrow x + y = 3$

Now, this line is tangent to circle

$(x - 3)^2 + (y + 2)^2 = r^2$

$\therefore$ perpendicular distance from centre to line = Radius of circle

$\therefore \frac{|3-2-3|}{\sqrt{2}} = r \Rightarrow r^2 = 2$

$\therefore$ Ans. is (2)

14 Let the curve C be the mirror image of the parabola $y^2 = 4x$ with respect to the line $x + y + 4 = 0$. If A and B are the points of intersection of C with the line $y = -5$, then the distance between A and B is **[JEE 2015, 4M, –0M]**

Ans. **4**

Sol. Let there be a point $(t^2, 2t)$ on $y^2 = 4x$

Clearly its reflection in $x + y + 4 = 0$ is given by

$$\frac{x-t^2}{1} = \frac{y-2t}{1} = \frac{-2(t^2+2t+4)}{2}$$

$\therefore$ $x = -(2t + 4)$ & $y = -(t^2 + 4)$

Now, $y = -5 \Rightarrow t = \pm 1$

$\therefore$ $x = -6$ or $x = -2$

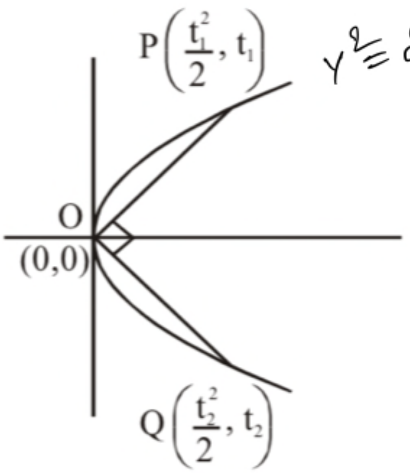
$\therefore$ Distance between A & B = 4

15 Let P and Q be distinct points on the parabola $y^2 = 2x$ such that a circle with PQ as diameter passes through the vertex O of the parabola. If P lies in the first quadrant and the area of the triangle ΔOPQ is $3\sqrt{2}$, then which of the following is(are) the coordinates of P ? **[JEE 2015, 4M, –2M]**

(A) $\left(4, 2\sqrt{2}\right)$ (B) $\left(9, 3\sqrt{2}\right)$ (C) $\left(\frac{1}{4}, \frac{1}{\sqrt{2}}\right)$ (D) $\left(1, \sqrt{2}\right)$

Ans. (A,D)

Sol.



$y^2 = 2x$; $\because a = \frac{1}{2}$

$\because \quad \angle POQ = \frac{\pi}{2} \Rightarrow \quad t_1t_2 = -4$

$\therefore \quad \left|\frac{1}{2}\begin{vmatrix} 0 & 0 & 1 \\ \frac{t_1^2}{2} & t_1 & 1 \\ \frac{t_2^2}{2} & t_2 & 1 \end{vmatrix}\right| = 3\sqrt{2}$

$\Rightarrow \quad \left|\frac{t_1^2t_2 - t_1t_2^2}{2}\right| = 6\sqrt{2}$

$\Rightarrow \quad |t_1 - t_2| = 3\sqrt{2}$

$\Rightarrow \quad t_1 + \frac{4}{t_1} = 3\sqrt{2} \qquad (\because \quad t_1 > 0)$

We get $t_1 = 2\sqrt{2}, \sqrt{2}$

$P(4, 2\sqrt{2})$ or $(1, \sqrt{2})$

16 The circle C_1: $x^2 + y^2 = 3$, with centre at O, intersects the parabola $x^2 = 2y$ at the point P in the first quadrant. Let the tangent to the circle C_1 at P touches other two circles C_2 and C_3 at R_2 and R_3, respectively. Suppose C_2 and C_3 have equal radii $2\sqrt{3}$ and centres Q_2 and Q_3, respectively. If Q_2 and Q_3 lie on the y-axis, then- **[JEE(Advanced)-2016, 4(–2)]**

(A) $Q_2Q_3 = 12$ (B) $R_2R_3 = 4\sqrt{6}$

(C) area of the triangle OR_2R_3 is $6\sqrt{2}$ (D) area of the triangle PQ_2Q_3 is $4\sqrt{2}$

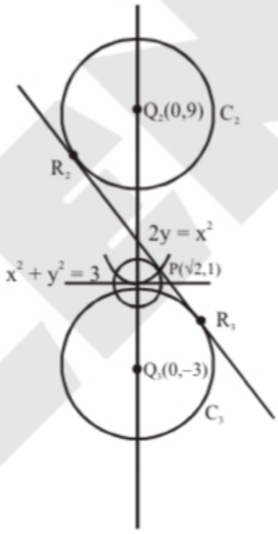
Ans. **(A,B,C)**

Sol. On solving $x^2 + y^2 = 3$ and $x^2 = 2y$ we get point $P(\sqrt{2},1)$

Equation of tangent at P

$\sqrt{2}.x + y = 3$

Let Q_2 be (0,k) and radius is $2\sqrt{3}$



$$\therefore \left|\frac{\sqrt{2}(0)+k-3}{\sqrt{2+1}}\right| = 2\sqrt{3}$$

$\therefore$ k = 9, – 3

$Q_2(0,9)$ and $Q_3(0, -3)$

hence $Q_2Q_3 = 12$

R_2R_3 is internal common tangent of circle C_2 and C_3

$$\therefore R_2R_3 = \sqrt{(Q_2Q_3)^2 - (2\sqrt{3}+2\sqrt{3})^2}$$

$$= \sqrt{12^2 - 48} = \sqrt{96} = 4\sqrt{6}$$

Perpendicular distance of origin O from R_2R_3 is equal to radius of circle $C_1 = \sqrt{3}$

Hence area of $\Delta OR_2R_3 = \frac{1}{2}\times(R_2R_3)\sqrt{3} = \frac{1}{2}.4\sqrt{6}.\sqrt{3} = 6\sqrt{2}$

Perpendicular Distance of P from $Q_2Q_3 = \sqrt{2}$

$\therefore$ Area of $\Delta PQ_2Q_3 = \frac{1}{2}\times 12\times\sqrt{2} = 6\sqrt{2}$

17 Let P be the point on the parabola $y^2 = 4x$ which is at the shortest distance from the center S of the circle $x^2 + y^2 - 4x - 16y + 64 = 0$. Let Q be the point on the circle dividing the line segment SP internally. Then-

(A) $SP = 2\sqrt{5}$

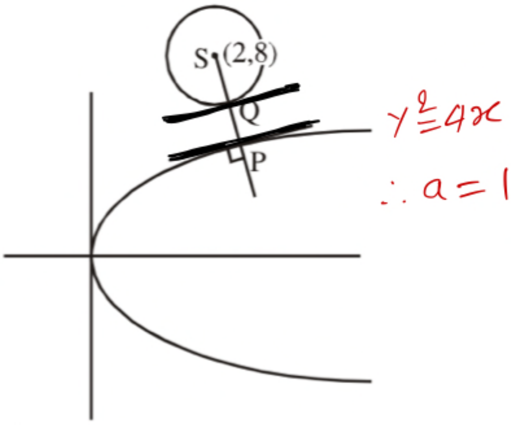
(B) $SQ : QP = (\sqrt{5}+1) : 2$

(C) the x-intercept of the normal to the parabola at P is 6

(D) the slope of the tangent to the circle at Q is $\frac{1}{2}$ **[JEE(Advanced)-2016, 4(–2)]**

Ans. (A,C,D)

Sol..



$y^2 = 4x$

point P lies on normal to parabola passing through centre of circle

$y + tx = 2t + t^3$ ....(i)

$8 + 2t = 2t + t^3$

$t = 2$

P(4, 4)

$SP = \sqrt{(4-2)^2 + (4-8)^2}$

$SP = 2\sqrt{5}$

$SQ = 2$

$\Rightarrow$ $PQ = 2\sqrt{5} - 2$

$\frac{SQ}{QP} = \frac{1}{\sqrt{5}-1} = \frac{\sqrt{5}+1}{4}$

To find x intercept

put y = 0 in (i)

$\Rightarrow$ $x = 2 + t^2$

$x = 6$

$\because$ Slope of common normal $= -t = -2$

$\therefore$ Slope of tangent $= \frac{1}{2}$

18 If a chord, which is not a tangent, of the parabola $y^2 = 16x$ has the equation $2x + y = p$, and midpoint (h, k), then which of the following is(are) possible value(s) of p, h and k ?

[JEE(Advanced)-2017, 4(–2)]

(A) $p = 5, h = 4, k = -3$
(B) $p = -1, h = 1, k = -3$
(C) $p = -2, h = 2, k = -4$
(D) $p = 2, h = 3, k = -4$

Ans. (D)

Sol. Equation of chord with mid point (h, k) :

$$k.y - 16\left(\frac{x+h}{2}\right) = k^2 - 16h$$

$\Rightarrow \quad 8x - ky + k^2 - 8h = 0$

Comparing with $2x + y - p = 0$, we get

$k = -4;\ 2h - p = 4$

only (D) satisfies above relation.