BINOMIAL THEOREM

SOLUTION

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Do yourself - 1 :

(i) Expand $\left(3x^2 - \frac{x}{2}\right)^5$ **(ii)** Expand $(y + x)^n$

Solution.

(i). $\left(3x^2 - \frac{x}{2}\right)^5 = {}^5C_0 \cdot (3x^2)^5 - {}^5C_1 \cdot (3x^2)^{5-1} \cdot \left(\frac{x}{2}\right)^1 + {}^5C_2 \cdot (3x^2)^{5-2} \cdot \left(\frac{x}{2}\right)^2 - {}^5C_3 (3x^2)^{5-3} \cdot \left(\frac{x}{2}\right)^3 + {}^5C_4 (3x^2)^{5-4} \cdot \left(\frac{x}{2}\right)^4 - {}^5C_5 \cdot \left(\frac{x}{2}\right)^5$

(ii). $(y+x)^n = {}^nC_0\, y^n + {}^nC_1\, y^{n-1} \cdot x^1 + {}^nC_2\, y^{n-2} \cdot x^2 + - - - - + {}^nC_{n-1}\, y^{n-(n-1)} \cdot x^{n-1} + {}^nC_n\, x^n$

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Do yourself - 2 :

(i) Find the 7^{th} term of $\left(3x^2 - \frac{1}{3}\right)^{10}$

(ii) Find the term independent of x in the expansion : $\left(2x^2 - \frac{3}{x^3}\right)^{25}$

(iii) Find the middle term in the expansion of : (a) $\left(\frac{2x}{3} - \frac{3}{2x}\right)^6$ (b) $\left(2x^2 - \frac{1}{x}\right)^7$

Solution.

(i). $T_{r+1} = {}^{10}C_r \cdot (3x^2)^{10-r} \left(-\frac{1}{3}\right)^r$

Put r = 6,

$\therefore T_7 = {}^{10}C_6 (3x^2)^4 \left(-\frac{1}{3}\right)^6$

$= \frac{10!}{6!\,4!} 3^4 x^8 \cdot \frac{1}{3^6} = \frac{70}{3} x^8$ **Ans.**

(ii). $T_{r+1} = {}^{25}C_r (2x^2)^{25-r} \cdot \left(-\frac{3}{x^3}\right)^r$

$= {}^{25}C_r\, 2^{25-r} \cdot (-3)^r\, x^{50-5r}$

For term independent of x, power of x equal to zero.

$\Rightarrow r = 10$

$\therefore T_{11} = {}^{25}C_{10}\, 2^{15} \cdot (-3)^{10}$ **Ans.**

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(iii).

a). Middle term :- $T_{(n+2)/2} = {}^nC_{n/2} . x^{n/2} . y^{n/2}$

$$T_4 = {}^6C_3\left(\frac{2x}{3}\right)^3 . \left(\frac{-3}{2x}\right)^3 = -20 \quad \textbf{Ans.}$$

b). 2 Middle term :- $T_{(n+1)/2}$ & $T_{[(n+1)/2]+1}$

T_4 & T_5 are two middle terms.

$$\therefore \quad T_4 = {}^7C_3(2x^2)^4\left(-\frac{1}{x}\right)^3 = -560x^5 \quad \textbf{Ans.}$$

$$\& \quad T_5 = {}^7C_4(2x^2)^3\left(-\frac{1}{x}\right)^4 = 280x^2 \quad \textbf{Ans.}$$

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Do yourself -3 :

(i) Find the numerically greatest term in the expansion of $(3-2x)^9$, when $x = 1$.

(ii) In the expansion of $\left(\frac{1}{2}+\frac{2x}{3}\right)^n$ when $x = -\frac{1}{2}$, it is known that 3^{rd} term is the greatest term. Find the possible integral values of n.

Solution.

(i). Using $\frac{n+1}{1+\left|\frac{a}{b}\right|}-1 \le r \le \frac{n+1}{1+\left|\frac{a}{b}\right|}$

$$\Rightarrow \frac{9+1}{1+\left|\frac{3}{-2}\right|}-1 \le r \le \frac{9+1}{1+\left|\frac{3}{-2}\right|} \qquad \{x=1\}$$

$$\Rightarrow \frac{10}{1+3/2}-1 \le r \le \frac{10}{1+3/2}$$

$$\Rightarrow 4-1 \le r \le 4$$

$$\Rightarrow 3 \le r \le 4$$

$\therefore$ $r = 3$ and 4

$$T_4 = {}^9C_3\,(3)^6(-2x)^3 \quad \& \quad T_5 = {}^9C_4\,(3)^5(-2x)^4$$

i.e. 4,89,888 **Ans.**

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(ii). $\left(\frac{1}{2} + \frac{2x}{3}\right)^n$

It is given that T_3 is greatest, so r=2.

Using $\dfrac{n+1}{1+\left|\frac{a}{b}\right|} - 1 \le r \le \dfrac{n+1}{1+\left|\frac{a}{b}\right|}$ {x=-1/2}

$$\Rightarrow \frac{n+1}{1+\left|\frac{1/2}{-1/3}\right|} - 1 \le 2 \le \frac{n+1}{1+\left|\frac{1/2}{-1/3}\right|}$$

Solving, we get $4 \le n \le \frac{13}{2}$

$\therefore\ n = 4, 5$ and 6 **Ans.**

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Do yourself - 4 :

(i) $\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+........+\binom{n}{n}=$

(A) 2^{n-1} (B) ${}^{2n}C_n$ (C) 2^n (D) 2^{n+1}

(ii) If $(1+x)^n = C_0 + C_1x + C_2x^2 + ... + C_nx^n$, $n \in N$. Prove that

(a) $3C_0 - 8C_1 + 13C_2 - 18C_3 + ...$ upto $(n+1)$ terms $= 0$, if $n \geq 2$.

(b) $2C_0 + 2^2\frac{C_1}{2} + 2^3\frac{C_2}{3} + 2^4\frac{C_3}{4} + ... + 2^{n+1}\frac{C_n}{n+1} = \frac{3^{n+1}-1}{n+1}$

(c) $C_0^2 + \frac{C_1^2}{2} + \frac{C_2^2}{3} + ... + \frac{C_n^2}{n+1} = \frac{(2n+1)!}{((n+1)!)^2}$

Solution.

(i). ${}^nC_0 + {}^nC_1 + {}^nC_2 + ----- + {}^nC_n = 2^n$

Option C

As, $(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + ... + {}^nC_nx^n$ ....(A)

Put x=1, in eq.(A) to get the required result.

(ii).

a). $S = 3C_0 - 8C_1 + 13C_2 - 18C_3 + ----$ upto $(n+1)$ terms

$$S = \sum_{r=0}^{n+1} t_r = \sum_{r=0}^{n+1} (-1)^r.(5r+3)\,{}^nC_r$$

$$S = 5\sum_{r=0}^{n+1}(-1)^r r.\,{}^nC_r + 3\sum_{r=0}^{n+1}(-1)^r.\,{}^nC_r$$

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$$= 5\sum_{r=0}^{n+1}(-1)^r . r . \frac{n!}{r!\,n-r!} + 3\sum_{r=0}^{n+1}(-1)^r . {}^nC_r$$

$$= 5\sum_{r=0}^{n+1}(-1)^r . \cancel{r} . \frac{n(n-1)!}{\cancel{r}.(r-1)!\,n-r!} + 3\sum_{r=0}^{n+1}(-1)^r . {}^nC_r$$

$$= 5n\sum_{r=1}^{n+1}(-1)^r\, {}^{n-1}C_{r-1} + 3\sum_{r=0}^{n+1}(-1)^r . {}^nC_r$$

$\Downarrow$ 0 $\qquad\qquad$ $\Downarrow$ 0

$\sum_{r=0}^{n}(-1)^r\,{}^nC_r = 0$ $\qquad\qquad$ $\sum_{r=0}^{n}(-1)^r\,{}^nC_r = 0$

b). $$S = 2.C_0 + 2^2.\frac{C_1}{2} + 2^3.\frac{C_2}{3} + 2^4.\frac{C_3}{4} + \text{----} + 2^{n+1}.\frac{C_n}{n+1}$$

$$(1 + x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + \ldots\ldots. + {}^nC_nx^n \qquad \ldots\ldots\ldots.(A)$$

Aliter : (Using method of integration)

Integrating (A), we get

$$\frac{(1+x)^{n+1}}{n+1} + C = C_0x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + \ldots\ldots.. + \frac{C_nx^{n+1}}{n+1} \qquad \text{(where C is a constant)}$$

Put x = 0, we get, $C = -\frac{1}{n+1}$

$$\therefore \quad \frac{(1+x)^{n+1} - 1}{n+1} = C_0x + \frac{C_1x^2}{2} + \frac{C_2x^3}{3} + \ldots\ldots.. + \frac{C_nx^{n+1}}{n+1} \qquad \textbf{......(1)}$$

Put x=2 , in eq.(1) to get the required result.

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c). $$S = \sum_{r=0}^{n} \frac{C_r^2}{r+1}$$

$$= \sum_{r=0}^{n} \frac{1}{n+1}\left[\left(\frac{(n+1)}{r+1}\cdot {}^{n}C_r\right)\times\left({}^{n}C_r\right)\right]$$

$$= \sum_{r=0}^{n} \frac{1}{n+1}\left[{}^{n+1}C_{r+1} \times {}^{n}C_r\right]$$

$$= \sum_{r=0}^{n} \frac{1}{n+1}\left[{}^{n+1}C_{r+1} \times {}^{n}C_{n-r}\right]$$

Now, addition of suffixes is constant.

$$\therefore S = \frac{1}{n+1}\left[{}^{2n+1}C_{n+1}\right]$$

$$= \frac{1}{n+1} \times \frac{(2n+1)!}{n!\,(n+1)!}$$

$$S = \frac{(2n+1)!}{((n+1)!)^2}$$

Hence Proved

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Do yourself - 5 :

(i) Find the coefficient of x^2y^5 in the expansion of $(3 + 2x - y)^{10}$.

Solution.

General term = $\dfrac{10!}{r!\,s!\,t!}\, 3^r.(2x)^s.(-y)^t$

where r+S+t = 10

$= \dfrac{10!}{r!\,s!\,t!}\, 3^r.\, 2^s.(-1)^t.\, x^s.\, y^t$

For coefficient, x^2y^5 we must have s=2, t=5 & r=3

$\therefore$ Coefficient of x^2y^5 will be, $\dfrac{10!}{3!\,2!\,5!}\,.\,3^3.\,2^2.(-1)^5$

$\Rightarrow -272160$ **Ans.**

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Do yourself - 6 :

(i) Prove that $5^{25} - 3^{25}$ is divisible by 2.

(ii) Find the remainder when the number 9^{100} is divided by 8.

(iii) Find last three digits in 19^{100}.

(iv) Let $R = (8+3\sqrt{7})^{20}$ and [.] denotes greatest integer function, then prove that :

(a) [R] is odd (b) $R-[R] = 1 - \frac{1}{(8+3\sqrt{7})^{20}}$

(v) Find the digit at unit's place in the number $17^{1995} + 11^{1995} - 7^{1995}$.

Solution.

(i). $5^{25} - 3^{25} \Rightarrow (1+4)^{25} - (1+2)^{25}$

$\Rightarrow \left({}^{25}C_0 + {}^{25}C_1 \cdot 4^1 + {}^{25}C_2 4^2 + \cdots + {}^{25}C_{25} 4^{25}\right)$

$- \left({}^{25}C_0 + {}^{25}C_1 \cdot 2^1 + {}^{25}C_2 \cdot 2^2 + \cdots + {}^{25}C_{25} \cdot 2^{25}\right)$

$\Rightarrow (\cancel{1} + 4K_1) - (\cancel{1} + 2K_2)$, where K_1 & $K_2 \in I$

$\Rightarrow$ Divisible by 2

Hence Proved

Alternatively:-

$(x^n - y^n)$ $n \in N$ is always divisible by $(x - y)$

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(ii).

$$9^{100} = (1+8)^{100}$$

$$= {}^{100}C_0 + {}^{100}C_1 . 8^1 + {}^{100}C_2 8^2 + --- + {}^{100}C_{100} . 8^{100}$$

$$= 1 + 8K \quad , \text{where } K \in I$$

$\therefore$ Remainder $= 1$ **Ans.**

(iii). To find last 3 digits,we need to find remainder when it is divided by 1000.

$$19^{100} = (1-20)^{100}$$

$$= {}^{100}C_0 - {}^{100}C_1 . 20^1 + {}^{100}C_2 . 20^2 + {}^{100}C_3 . 20^3 + ---$$

$$= 1 - 100 \times 20^1 + \frac{100 \times 99}{2} \times 20^2 + \frac{100 \times 99 \times 98}{6} \times 20^3 + ---$$

$$= 1 - 2000 + 198 \times 10^3 + ----$$

When the above exp. is divided by 1000, we get remainder 1.

$\therefore$ Last 3 digits $= 001$ **Ans.**

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(iv). Let $R = (8+3\sqrt{7})^{20} = I + f$ { f represent fractional part of R }

Also let, $f' = (8-3\sqrt{7})^{20}$

a). Adding, $I + f + f' = (8+3\sqrt{7})^{20} + (8-3\sqrt{7})^{20}$

$\boxed{I + f + f' = I_E}$ ....(1) { Even Integral values }

As we know, $0 < f < 1$ & $0 < f' < 1$

$\Rightarrow \boxed{0 < f + f' < 2}$

$\therefore$ $I + f + f' = I_E$ is possible only when $f + f'$ is also an integer. i.e $\boxed{f + f' = 1}$

From equation (1),

$$I + 1 = I_E$$

or $I = I_E - 1$

$\boxed{I = I_O}$ { Odd Integral values }

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b). $R - [R] = \{R\}$ { fractional part of R }

$\therefore \{R\} = f \Rightarrow 1 - f'$ $\boxed{\because f + f' = 1}$

$$= 1 - (8 - 3\sqrt{7})^{20}$$

$$= 1 - \frac{1}{(8 + 3\sqrt{7})^{20}}$$ Hence Proved

(v). To find out last digit, we need to find remainder when it is divided by 10.

$$\underbrace{17^{1995} - 7^{1995}}_{\text{is divisible by } 10} + 11^{1995}$$

$$11^{1995} \Downarrow (1+10)^{1995}$$

$(x^n - y^n)$ $n \in N$ is always divisible by $(x - y)$

$$\Rightarrow 10K_1 + (1 + 10K_2), \text{ where } K_1 \,\&\, K_2 \in I$$

When the above exp. is divided by 10, we get remainder 1.

$\therefore$ Last digit $= \underline{1}$ **Ans.**

EXERCISE 01

1. If the coefficients of x^7 & x^8 in the expansion of $\left[2+\frac{x}{3}\right]^n$ are equal, then the value of n is :

(A) 15 (B) 45 (C) 55 (D) 56

Coefficients of $x^7 = {}^nC_7 (2)^{n-7}\left(\frac{1}{3}\right)^7$

Coefficients of $x^8 = {}^nC_8\, 2^{n-8}\left(\frac{1}{3}\right)^8$

$\because$ Given that: coeff. of x^7 = coeff. of x^8

$\Rightarrow \quad {}^nC_7\, 2^{n-7}\left(\frac{1}{3}\right)^7 = {}^nC_8\, 2^{n-8}\left(\frac{1}{3}\right)^8$

$\Rightarrow \quad \frac{\cancel{n!}}{7!\,(n-7)!}\cdot\frac{2^{n-7}}{2^{n-8}} = \frac{\cancel{n!}}{8!\,(n-8)!}\times\frac{3^7}{3^8}$

$$2 = \frac{7!}{8!}\,\frac{(n-7)!}{(n-8)!}\times\frac{1}{3}$$

$$2 = \frac{1}{8}\times(n-7)\times\frac{1}{3}$$

$\Rightarrow \quad n-7 = 48$

$\Rightarrow \quad \boxed{n = 55}$

2. If the constant term of the binomial expansion $\left(2x-\frac{1}{x}\right)^n$ is -160, then n is equal to -

(A) 4 (B) 6 (C) 8 (D) 10

General term $T_{r+1} = {}^nC_r (2x)^{n-r} \left(-\frac{1}{x}\right)^r$

$$= {}^nC_r\, 2^{n-r} \frac{x^{n-r}}{x^r} (-1)^r$$

$$T_{r+1} = {}^nC_r\, 2^{n-r} x^{n-2r} (-1)^r$$

Constant term: $n-2r=0$

$\Rightarrow r = n/2$

$$T_{n/2+1} = {}^nC_{n/2}\, 2^{n-n/2}\, x^{n-2(n/2)} (-1)^{n/2}$$

$\because$ Given constant term $= -160$

$\Rightarrow {}^nC_{n/2}\, 2^{n/2} (-1)^{n/2} = -160$

From observation $n/2 =$ odd integer

$\Rightarrow n = 2 \times$ odd integer.

$\Rightarrow$ Check for options (B) & (D)

Option B:- $n=6$, $\Rightarrow {}^6C_3\, 2^3 (-1)^3 = -160$

Hence option (B) is correct.

3) The coefficient of x^{49} in the expansion of $(x-1)\left(x-\frac{1}{2}\right)\left(x-\frac{1}{2^2}\right)....\left(x-\frac{1}{2^{49}}\right)$ is equal to -

(A) $-2\left(1-\frac{1}{2^{50}}\right)$

(B) +ve coefficient of x

(C) –ve coefficient of x

(D) $-2\left(1-\frac{1}{2^{49}}\right)$

coefficient of x^{49} in the expansion

$$(x-1)\left(x-\frac{1}{2}\right)\left(x-\frac{1}{2^2}\right) ---- \left(x-\frac{1}{2^{49}}\right)$$

$$= x^{49}\left(-1-\frac{1}{2}-\frac{1}{2^2} ---- -\frac{1}{2^{49}}\right)$$

$$\left(\because a+ar+ar^2+ar^3+ ---- +ar^{n-1} = \frac{a(1-r^n)}{1-r}\right)$$

$$\Rightarrow -1-\frac{1}{2}-\frac{1}{2^2} ---- -\frac{1}{2^{49}} = -\frac{1\left(1-\left(\frac{1}{2}\right)^{50}\right)}{1-\frac{1}{2}}$$

$$= -2\left(1-\frac{1}{2^{50}}\right)$$

4 Number of rational terms in the expansion of $\left(\sqrt{2}+\sqrt[4]{3}\right)^{100}$ is :

(A) 25 (B) 26 (C) 27 (D) 28

General term in the expansion $\left(\sqrt{2}+3^{1/4}\right)^{100}$:

$$T_{r+1} = {}^{100}C_r\left(\sqrt{2}\right)^{100-r}\left(3^{1/4}\right)^{r}$$

$$= {}^{100}C_r\left(2^{1/2}\right)^{100-r} 3^{r/4}$$

$$T_{r+1} = {}^{100}C_r\, 2^{50-r/2}\, 3^{r/4}$$

$\Rightarrow$ Rational terms: $\frac{r}{4} = I$

$\Rightarrow$ $r = 0, 4, 8, 12, \text{------}, 100$

$\Rightarrow$ Number of terms $= 26$

5. The expression $[x + (x^3 - 1)^{1/2}]^5 + [x - (x^3 - 1)^{1/2}]^5$ is a polynomial of degree

(A) 5 (B) 6 (C) 7 (D) 8

$\because (a+b)^5 = {}^5C_0 a^5 + {}^5C_1 a^4 b^1 + {}^5C_2 a^3 b^2 + {}^5C_3 a^2 b^3 + {}^5C_4 a b^4 + {}^5C_5 b^5$

& $(a-b)^5 = {}^5C_0 a^5 - {}^5C_1 a^4 b + {}^5C_2 a^3 b^2 - {}^5C_3 a^2 b^3 + {}^5C_4 a b^4 - {}^5C_5 b^5$

$\Rightarrow (a+b)^5 + (a-b)^5 = 2[{}^5C_0 a^5 + {}^5C_2 a^3 b^2 + {}^5C_4 a b^4]$

$\Rightarrow [x + (x^3-1)^{1/2}]^5 + [x - (x^3-1)^{1/2}]^5$

$= 2[{}^5C_0 x^5 + {}^5C_2 x^3 [(x^3-1)^{1/2}]^2 + {}^5C_4 x [(x^3-1)^{1/2}]^4]$

$= 2[x^5 + 10x^3(x^3-1) + {}^5C_4 x (x^3-1)^2]$

$= 2[x^5 + 10x^6 - 10x^3 + 5x(x^6 - 2x^3 + 1)]$

$= 2[x^5 + 10x^6 - 10x^3 + 5x^7 - 10x^4 + 1]$

$\Rightarrow$ Highest power of $x = 7$.

6 Given $(1-2x+5x^2-10x^3)(1+x)^n = 1 + a_1x + a_2x^2 + ...$ and that $a_1^2 = 2a_2$ then the value of n is-

(A) 6 (B) 2 (C) 5 (D) 3

$(1-2x+5x^2-10x^3)(1+x)^n = 1+a_1x+a_2x^2+......$

Let $E = (1-2x+5x^2-10x^3)(1+x)^n$

$E = (1+x)^n - 2x(1+x)^n + 5x^2(1+x)^n - 10x^3(1+x)^n$

In the expression E, coefficient of x is a_1 and coefficient of x^2 is a_2.

$\Rightarrow a_1 =$ coefficient of x in $(1+x)^n - 2 \times$ coefficient of x^0 in $(1+x)^n$

($\because$ coefficient of x is zero in terms $5x^2(1+x)^n$ and $10x^3(1+x)^n$)

$\Rightarrow a_1 = {}^nC_1 - 2\,{}^nC_0 = n-2 \quad (\because {}^nC_1 = n)$

$a_2 =$ coefficient of x^2 in $(1+x)^n - 2 \times$ coeff. of x in $(1+x)^n + 5 \times$ coefficient of x^0 in $(1+x)^n$

$\Rightarrow a_2 = {}^nC_2 - 2 \times {}^nC_1 + 5 \times {}^nC_0$

$\Rightarrow a_2 = \frac{n(n-1)}{2} - 2n + 5$

$\because$ Given that $a_1^2 = 2a_2$

$$\Rightarrow (n-2)^2 = 2\left(\frac{n(n-1)}{2} - 2n + 5\right)$$

$$\Rightarrow n^2 - 4n + 4 = n(n-1) - 4n + 10$$

$$\Rightarrow n^2 - 4n + 4 = n^2 - n - 4n + 10$$

$$\Rightarrow \boxed{n = 6}$$

7 The sum of the co-efficients of all the even powers of x in the expansion of $(2x^2 - 3x + 1)^{11}$ is -

(A) 2.6^{10} (B) 3.6^{10} (C) 6^{11} (D) none

Let

$$(2x^2 - 3x + 1)^{11} = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \dots + a_{22} x^{22} \quad \text{—(1)}$$

put $x = 1$ in equation (1) $\Rightarrow$

$$(2 - 3 + 1)^{11} = a_0 + a_1 + a_2 + \dots + a_{22} \quad \text{—(2)}$$

Now put $x = -1$ in equation (1) $\Rightarrow$

$$(2 + 3 + 1)^{11} = a_0 - a_1 + a_2 \dots + a_{22} \quad \text{—(3)}$$

Question is asking the sum of even powers in the expansion of (1) $\Rightarrow a_0 + a_2$

On adding (2) + (3) $\Rightarrow$

$$0 + 6^{11} = 2(a_0 + a_2 + a_4 + \dots + a_{22})$$

$$\Rightarrow a_0 + a_2 + a_4 + \dots + a_{22} = \frac{6^{11}}{2} = 3.6^{10}$$

$\Rightarrow$ Answer is option (B)

8. Let $\binom{n}{k}$ represents the combination of 'n' things taken 'k' at a time, then the value of the sum

$\binom{99}{97}+\binom{98}{96}+\binom{97}{95}+\ldots\ldots+\binom{3}{1}+\binom{2}{0}$ equals -

(A) $\binom{99}{97}$ (B) $\binom{100}{98}$ (C) $\binom{99}{98}$ (D) $\binom{100}{97}$

$$^{99}C_{97} + {}^{98}C_{96} + {}^{97}C_{95} + \ldots + {}^{3}C_{1} + {}^{2}C_{0} =$$

$(\because {}^{2}C_{0} = {}^{3}C_{0})$

$$^{99}C_{97} + {}^{98}C_{96} + {}^{97}C_{95} + \ldots\ldots + \underbrace{{}^{3}C_{1} + {}^{3}C_{0}}_{=}$$

$$= {}^{99}C_{97} + {}^{98}C_{96} + {}^{97}C_{95} \ldots\ldots + {}^{4}C_{2} + {}^{4}C_{1}$$

$(\because {}^{n}C_{r} + {}^{n}C_{r-1} = {}^{n+1}C_{r})$

$$= {}^{99}C_{97} + {}^{98}C_{96} + {}^{97}C_{95} + \ldots + {}^{5}C_{3} + {}^{5}C_{2}$$

$$= {}^{99}C_{97} + {}^{99}C_{96} = {}^{100}C_{97}$$

Hence option (D) is the answer.

If $n \in N$ and if $(1 + 4x + 4x^2)^n = \sum_{r=0}^{2n} a_r x^r$, where $a_0, a_1, a_2, \ldots\ldots, a_{2n}$ are real numbers.

9 The value of $2\sum_{r=0}^{n} a_{2r}$, is

(A) $9^n - 1$ (B) $9^n + 1$ (C) $9^n - 2$ (D) $9^n + 2$

Given:- $(1+4x+4x^2)^n = a_0 + a_1 x + a_2 x^2 + \ldots + a_{2n} x^{2n}$ — (1)

put $x = 1$

$\Rightarrow (1+4+4)^n = a_0 + a_1 + a_2 + \ldots + a_{2n}$

$\Rightarrow a_0 + a_1 + a_2 + \ldots + a_{2n} = 9^n$ — (2)

Now put $x = -1$

$\Rightarrow (1-4+4)^n = a_0 - a_1 + a_2 + \ldots + a_{2n}$

$\Rightarrow a_0 - a_1 + a_2 + \ldots + a_{2n} = 1$ — (3)

(2) + (3) $\Rightarrow$

$2(a_0 + a_2 + \ldots + a_{2n}) = 9^n + 1$

Hence correct answer is option (B)

(10) The value of $2\sum_{r=1}^{n} a_{2r-1}$, is-

(A) $9^n - 1$ (B) $9^n + 1$ (C) $9^n - 2$ (D) $9^n + 2$

Given:- $(1+4x+4x^2)^n = a_0 + a_1 x + a_2 x^2 + \dots + a_{2n} x^{2n}$. — (1)

put $x = 1$

$\Rightarrow (1+4+4)^n = a_0 + a_1 + a_2 + \dots + a_{2n}$

$\Rightarrow a_0 + a_1 + a_2 + \dots + a_{2n} = 9^n$ — (2)

Now put $x = -1$

$\Rightarrow (1-4+4)^n = a_0 - a_1 + a_2 + \dots + a_{2n}$

$\Rightarrow a_0 - a_1 + a_2 + \dots + a_{2n} = 1$ — (3)

(2) − (3) $\Rightarrow$

$\Rightarrow 2(a_1 + a_3 + a_5 + \dots + a_{2n-1}) = 9^n - 1$

$\Rightarrow$ Correct answer is (A)

(11) The value of a_{2n-1} is -

(A) 2^{2n} (B) $n. 2^{2n}$ (C) $(n-1)2^{2n}$ (D) $(n+1)2^{2n}$

1 1 1 1

Given :-

$$(1+4x+4x^2)^n = a_0 + a_1 x + a_2 x^2 + \cdots + a_{2n-1} x^{2n-1} + a_{2n} x^{2n} \quad \text{--- (1)}$$

$$(1+4x+4x^2)^n = (1+2x)^{2n}$$

$$a_{2n-1} = {}^{2n}C_{2n-1} (2)^{2n-1}$$

$$a_{2n-1} = (2n) \cdot 2^{2n-1} = n \cdot 2^{2n}$$

Correct option is (B)

EXERCISE (O-2)

[ONE OR MORE THAN ONE CORRECT CHOICE TYPE]

1 Greatest term in the binomial expansion of $(a + 2x)^9$ when $a = 1$ & $x = \frac{1}{3}$ is :

(A) 3^{rd} & 4^{th} (B) 4^{th} & 5^{th} (C) only 4^{th} (D) only 5^{th}

Solution:

Greatest Term in $(a+2x)^9$ when $a=1$, $x=\frac{1}{3}$

using $\frac{n+1}{1+\left|\frac{a}{b}\right|}-1 \le r \le \frac{n+1}{1+\left|\frac{a}{b}\right|}$

Here $a=1$, $b=2x=\frac{2}{3}$, $n=9$

$$\frac{9+1}{1+\left|\frac{3}{2}\right|}-1 \le r \le \frac{9+1}{1+\left|\frac{1}{\frac{2}{3}}\right|}$$

$$3 \le r \le 4$$

$r = 3, 4$ Thus

Greatest Term will be T_4 and T_5

2 If $\sum_{r=1}^{10} r(r-1)\ {}^{10}C_r = k.\ 2^9$, then k is equal to-

(A) 10 (B) 45 (C) 90 (D) 100

Solution:

$$\sum_{r=1}^{10} r(r-1)\ {}^{10}C_r = k \cdot 2^9$$

LHS

$$\sum_{r=1}^{10} r(r-1) \cdot {}^{10}C_r$$

$$\sum_{r=1}^{10} r(r-1) \cdot \frac{10}{r} \cdot \frac{9}{r-1} \cdot {}^{8}C_{r-2}$$

$\left\{ \text{using } {}^{n}C_r = \frac{n}{r}\ {}^{n-1}C_{r-1} \right\}$

$$90 \sum_{r=1}^{10} {}^{8}C_{r-2}$$

$$90\left[{}^{8}C_0 + {}^{8}C_1 \cdots + {}^{8}C_8\right]$$

$$90\left[2^8\right] = 45 \cdot 2^9$$

Thus $k = 45$

3 The sum $\frac{\binom{11}{0}}{1}+\frac{\binom{11}{1}}{2}+\frac{\binom{11}{2}}{3}+..........+\frac{\binom{11}{11}}{12}$ equals $\left(\text{where} \binom{n}{r} \text{denotes } {}^{n}C_{r}\right)$

(A) $\frac{2^{11}}{12}$ (B) $\frac{2^{12}}{12}$ (C) $\frac{2^{11}-1}{12}$ (D) $\frac{2^{12}-1}{12}$

Solution:

$$\frac{{}^{11}C_0}{1}+\frac{{}^{11}C_1}{2}+\frac{{}^{11}C_2}{3}\cdots+\frac{{}^{11}C_{11}}{12}$$

$$\sum_{r=0}^{11}\frac{{}^{11}C_r}{r+1} \qquad \left\{ {}^{n}C_r=\frac{r+1}{n+1}\cdot {}^{n+1}C_{r+1} \right\}$$

$$\sum_{r=0}^{11}\frac{1}{r+1}\cdot\frac{r+1}{12}\cdot {}^{12}C_{r+1}$$

$$\frac{1}{12}\sum_{r=0}^{11} {}^{12}C_{r+1}=\frac{1}{12}\left[{}^{12}C_1+{}^{12}C_2\cdots+{}^{12}C_{12}\right]$$

$$=\frac{1}{12}\left[2^{12}-1\right]=\frac{2^{12}-1}{12}$$

4. In the expansion of $\left(\frac{x+1}{x^{2/3}-x^{1/3}+1}-\frac{x-1}{x-x^{1/2}}\right)^{10}$, the term which does not contain x is-

(A) ${}^{11}C_4-{}^{10}C_3$ (B) ${}^{10}C_7$ (C) ${}^{10}C_4$ (D) ${}^{11}C_5-{}^{10}C_5$

Solution:

$$\left(\frac{x+1}{x^{\frac{2}{3}}-x^{\frac{1}{3}}+1}-\frac{x-1}{x-x^{\frac{1}{2}}}\right)^{10}$$

simplifying:

Think $\frac{x+1}{x^{\frac{2}{3}}-x^{\frac{1}{3}}+1}=\frac{(x^{\frac{1}{3}})^3+1}{x^{\frac{2}{3}}-x^{\frac{1}{3}}+1}=\frac{(x^{\frac{1}{3}}+1)\cancel{(x^{\frac{2}{3}}-x^{\frac{1}{3}}+1)}}{\cancel{x^{\frac{2}{3}}-x^{\frac{1}{3}}+1}}$

$$\boxed{\frac{x+1}{x^{\frac{2}{3}}-x^{\frac{1}{3}}+1}=x^{\frac{1}{3}}+1} \text{ —— (1)}$$

$$\frac{x-1}{x-x^{\frac{1}{2}}}=\frac{(x^{\frac{1}{2}}-1)(x^{\frac{1}{2}}+1)}{x^{\frac{1}{2}}(x^{\frac{1}{2}}-1)}=\frac{x^{\frac{1}{2}}+1}{x^{\frac{1}{2}}}$$

$$\boxed{\frac{x-1}{x-x^{\frac{1}{2}}}=1+x^{-\frac{1}{2}}} \text{ —— (2)}$$

Now $\left((x^{\frac{1}{3}}+1)-(1+x^{-\frac{1}{2}})\right)^{10}$

$(x^{\frac{1}{3}}-x^{-\frac{1}{2}})^{10}$

$T_{r+1}={}^{10}C_r(x^{\frac{1}{3}})^{10-r}\cdot(x^{-\frac{1}{2}})^r$

$={}^{10}C_r\, x^{\frac{20-5r}{6}}$ For x^0; $\frac{20-5r}{6}=0$

$r=4$

$T_5={}^{10}C_4$ (Hence C)

Also ${}^{11}C_5-{}^{10}C_5={}^{10}C_4$ (Hence D)

Also ${}^{11}C_4-{}^{10}C_3={}^{10}C_4$ (Hence A)

$\boxed{\text{So A, C, D}}$

5

Let $(1 + x^2)^2 (1 + x)^n = A_0 + A_1x + A_2x^2 + \ldots\ldots$ If A_0, A_1, A_2 are in A.P. then the value of n is-

(A) 2 (B) 3 (C) 5 (D) 7

Solution:

$$(1+x^2)^2(1+x)^n = A_0 + A_1x + A_2x^2 \ldots$$

Expanding $(1+x^2)^2(1+x)^n$

$$(1+2x^2+x^4)({}^nC_0 + {}^nC_1x + {}^nC_2x^2 \ldots)$$

$$\Rightarrow \quad {}^nC_0 + {}^nC_1x + ({}^nC_2 + 2\cdot{}^nC_0)x^2 \ldots$$

Thus on comparing.

$$A_0 = {}^nC_0 = 1$$

$$A_1 = {}^nC_1 = n$$

$$A_2 = {}^nC_2 + 2\cdot{}^nC_0 = \frac{n(n-1)}{2} + 2$$

A_0, A_1, A_2 in AP

$$2n = 1 + \frac{n(n-1)}{2} + 2$$

$$4n = 2 + n^2 - n + 4$$

$$n^2 - 5n + 6 = 0$$

$$(n-2)(n-3) = 0$$

$$\boxed{n = 2, \quad n = 3}$$

A, B

6. In the expansion of $\left(x^3 + 3.2^{-\log_{\sqrt{2}}\sqrt{x^3}}\right)^{11}$

(A) there appears a term with the power x^2
(B) there does not appear a term with the power x^2
(C) there appears a term with the power x^{-3}
(D) the ratio of the co-efficient of x^3 to that of x^{-3} is 1/3

Sol:

$$\left(x^3 + 3\cdot 2^{-\log_{\sqrt{2}}\sqrt{x^3}}\right)^{11}$$

$$\Rightarrow \left(x^3 + \frac{3}{x^3}\right)^{11} \qquad \left\{ 2^{-\log_{\sqrt{2}}\sqrt{x^3}} \quad 2^{-\log_2 x^3} = \frac{1}{x^3} \right.$$

$$\Rightarrow T_{r+1} = {}^{11}C_r (x^3)^{11-r} \cdot \left(\frac{3}{x^3}\right)^r$$

$$= {}^{11}C_r\, x^{33-6r} \cdot 3^r$$

For Term containg x^2; $33-6r=2$
$r = \frac{31}{6}$

Hence No term containing x^2 [Hence B]

for term containing x^{-3}; $33-6r=-3$
$r=6$

① —— $T_7 = {}^{11}C_6\, x^{-3} \cdot 3^6$ (Hence C)

for term containing x^3; $33-6r=3$
$r=5$

② —— $T_6 = {}^{11}C_5\, x^3 \cdot 3^5$

Ratio of term containing x^3 to term containing x^{-3}

$$\frac{T_6}{T_7} = \frac{{}^{11}C_5 (3^5)}{{}^{11}C_6 (3^6)} = \frac{1}{3} \text{ (Hence D)}$$

7 If it is known that the third term of the binomial expansion $\left(x + x^{\log_{10} x}\right)^5$ is 10^6 then x is equal to-

(A) 10 (B) $10^{-5/2}$ (C) 100 (D) 5

Solution:

$$\left(x + x^{\log_{10} x}\right)^5$$

$$T_3 = {}^5C_2 \,(x)^3 \left(x^{\log_{10} x}\right)^2 = 10^6$$

$$T_3 = 10\, x^3\, x^{2\log_{10} x} = 10^6$$

$$x^3 \cdot x^{2\log_{10} x} = 10^5$$

Taking Logarithm both sides

$$\log_{10} x^3 + \log_{10}\left(x^{2\log_{10} x}\right) = \log_{10} 10^5$$

$$3\log_{10} x + 2\log_{10} x\,(\log_{10} x) = 5$$

$$2(\log_{10} x)^2 + 3\log_{10} x - 5 = 0$$

put $\log_{10} x = y$ $\therefore 2y^2 + 3y - 5 = 0$

$$(y-1)(2y+5) = 0$$

$$y = 1, \quad y = \frac{-5}{2}$$

$$\log_{10} x = 1, \quad \log_{10} x = \frac{-5}{2}$$

$$x = 10 \;;\quad x = 10^{-\frac{5}{2}}$$

Hence A and B.

EXERCISE (S-1)

1. (a) If the coefficients of $(2r+4)^{th}$, $(r-2)^{th}$ terms in the expansion of $(1+x)^{18}$ are equal, find r.

(b) If the coefficients of the r^{th}, $(r+1)^{th}$ & $(r+2)^{th}$ terms in the expansion of $(1+x)^{14}$ are in AP, find r.

(c) If the coefficients of 2^{nd}, 3^{rd} & 4^{th} terms in the expansion of $(1+x)^{2n}$ are in AP, show that $2n^2 - 9n + 7 = 0$.

Solution (a) In $(1+x)^{18}$ coeff. of T_{2r+4} = coeff of T_{r-2}

$\Rightarrow \quad {}^{18}C_{2r+3} = {}^{18}C_{r-3}$

$\Rightarrow \quad 2r+3 = r-3$ or $2r+3+r-3 = 18$

$\Rightarrow \quad r = -6$ or $3r = 18$

$\Rightarrow$ does not exist or $\boxed{r = 6}$

(b) In $(1+x)^{14}$ coeff r^{th}, $(r+1)^{th}$ and $(r+2)^{th}$ in A.P.

$\Rightarrow \quad {}^{14}C_{r-1}, \ {}^{14}C_r, \ {}^{14}C_{r+1}$ in A.P.

$$\Rightarrow \quad 2 \cdot {}^{14}C_r = {}^{14}C_{r-1} + {}^{14}C_{r+1}$$

$$\Rightarrow \quad \frac{2 \cdot 14!}{r!\,(14-r)!} = \frac{14!}{(r-1)!\,(15-r)!} + \frac{14!}{(r+1)!\,(13-r)!}$$

$$\Rightarrow \quad \frac{2}{r(14-r)} = \frac{1}{(15-r)(14-r)} + \frac{1}{(r+1)r}$$

$$\Rightarrow \quad \frac{2}{r(14-r)} = \frac{r(r+1) + (15-r)(14-r)}{r(r+1)(15-r)(14-r)}$$

$\Rightarrow 2(r+1)(15-r) = r(r+1) + (15-r)(14-r)$

$\Rightarrow r^2 - 14r + 45 = 0$

$\Rightarrow (r-5)(r-9) = 0$

$\Rightarrow \boxed{r = 5 \quad \text{or} \quad r = 9}$

(c) In $(1+x)^{2n}$ coeff. of T_2, T_3, T_4 in A.P.

i.e. ${}^{2n}C_1, {}^{2n}C_2, {}^{2n}C_3$ in A.P.

$$\Rightarrow 2 \cdot {}^{2n}C_2 = {}^{2n}C_1 + {}^{2n}C_3$$

$$\Rightarrow \frac{2 \cdot (2n)!}{2!\,(2n-2)!} = \frac{(2n)!}{1!\,(2n-1)!} + \frac{(2n)!}{3!\,(2n-3)!}$$

$$\Rightarrow \frac{1}{(2n-2)} = \frac{1}{(2n-1)(2n-2)} + \frac{1}{6}$$

$$\Rightarrow \frac{(2n-2)}{(2n-1)(2n-2)} = \frac{1}{6}$$

$$\Rightarrow 6(n-1) = (2n-1)(n-1)$$

$$\Rightarrow (n-1)(2n-7) = 0$$

$$\Rightarrow \boxed{2n^2 - 9n + 7 = 0}$$

2. Find the term independent of x in the expansion of (i) $\left[\sqrt{\frac{x}{3}}+\frac{\sqrt{3}}{2x^2}\right]^{10}$ (ii) $\left[\frac{1}{2}x^{1/3}+x^{-1/5}\right]^8$

(i) $T_{r+1} = {}^{10}C_r \left(\frac{x}{3}\right)^{\frac{10-r}{2}} \cdot \frac{3^{r/2}}{(2x^2)^r} = {}^{10}C_r\, 3^{r-5}\, 2^{-r}\, x^{\frac{10-5r}{2}}$

For constant Term $\Rightarrow \frac{10-5r}{2} = 0 \Rightarrow r=2$

$\therefore \boxed{T_3 = {}^{10}C_2 \cdot \frac{1}{3^3\, 2^2} = \frac{5}{12}}$

(ii) $T_{r+1} = {}^8C_r \frac{x^{\frac{8-r}{3}}}{2^{8-r}} \cdot x^{-\frac{r}{5}} = {}^8C_r \frac{1}{2^{8-r}}\, x^{\frac{40-8r}{15}}$

For Constant term $\Rightarrow \frac{40-8r}{15} = 0 \Rightarrow r=5$

$\therefore \boxed{T_6 = {}^8C_5 \cdot \frac{1}{2^3} = 7}$

3. Prove that the ratio of the coefficient of x^{10} in $(1 - x^2)^{10}$ & the term independent of x in $\left(x - \frac{2}{x}\right)^{10}$ is 1 : 32.

$T_{r+1} = {}^{10}C_r (-x^2)^r$

For x^{10} $\Rightarrow 2r = 10$

$\Rightarrow r = 5$

$T_6 = {}^{10}C_5 \, x^{10} \cdot (-1)^5$

$\therefore$ coeff. of $x^{10} = -{}^{10}C_5$

$T_{r+1} = {}^{10}C_r (x)^{10-r} \left(-\frac{2}{x}\right)^r$

$= {}^{10}C_r (-2)^r \cdot x^{10-2r}$

For constant term

$10 - 2r = 0$

$\Rightarrow r = 5$

$\therefore$ Coeff of constant term $= {}^{10}C_5 (-2)^5$

$\therefore$ Required Ratio $= \dfrac{-{}^{10}C_5}{-{}^{10}C_5 (2)^5} = \dfrac{1}{32}$

$= \boxed{1 : 32}$

4 Find the sum of the series $\sum_{r=0}^{n}(-1)^r \cdot {}^nC_r\left[\frac{1}{2^r}+\frac{3^r}{2^{2r}}+\frac{7^r}{2^{3r}}+\frac{15^r}{2^{4r}}+\text{.........up to m terms}\right]$

$$= \underbrace{\sum_{r=0}^{n}(-1)^r\, {}^nC_r\left(\frac{1}{2}\right)^r}_{} + \underbrace{\sum_{r=0}^{n}(-1)^r \cdot {}^nC_r\left(\frac{3}{4}\right)^r}_{} + \sum_{r=0}^{n}(-1)^r \cdot {}^nC_r \cdot \left(\frac{7}{8}\right)^r + \text{--- m terms}$$

$$= \left(1-\frac{1}{2}\right)^n + \left(1-\frac{3}{4}\right)^n + \left(1-\frac{7}{8}\right)^n + \text{—— m terms}$$

$$= \left(\frac{1}{2}\right)^n + \left(\frac{1}{4}\right)^n + \left(\frac{1}{8}\right)^n + \text{————} + \text{ m terms}$$

$$= \left(\frac{1}{2^1}\right)^n + \left(\frac{1}{2^2}\right)^n + \left(\frac{1}{2^3}\right)^n + \text{————} + \text{ m terms}$$

This is a sum of G.P. whose first term is $\left(\frac{1}{2}\right)^n$ common ratio $\left(\frac{1}{2}\right)^n$ and number of terms is m

$$= \frac{\left(\frac{1}{2}\right)^n\left(1-\left[\left(\frac{1}{2}\right)^n\right]^m\right)}{1-\left(\frac{1}{2}\right)^n} \qquad (\because r<1)$$

$$= \frac{\left(1-\frac{1}{2^{nm}}\right)}{(2^n-1)}$$

$$= \boxed{\frac{2^{nm}-1}{2^{nm}(2^n-1)}}$$

5. Find numerically greatest term in the expansion of :

(i) $(2+3x)^9$ when $x=\frac{3}{2}$ (ii) $(3-5x)^{15}$ when $x=\frac{1}{5}$

(i) $$\frac{9+1}{\left|\frac{2}{3x}\right|+1}-1 \le r \le \frac{9+1}{\left|\frac{2}{3x}\right|+1}$$

$$\Rightarrow \frac{10}{\frac{4}{9}+1}-1 \le r \le \frac{10}{\frac{4}{9}+1}$$

$$\Rightarrow \frac{77}{13} \le r \le \frac{90}{13}$$

$$\Rightarrow 5.9 \le r \le 6.9$$

$$\Rightarrow r = 6$$

$$\therefore T_7 = T_{6+1} = {}^9C_6\,(2)^3.\left(3.\frac{3}{2}\right)^6 = \boxed{\frac{3^{13}.7}{2}}$$

(ii) $(3-5x)^{15}$ when $x = 1/5$

$$\frac{15+1}{\left|\frac{3}{-5x}\right|+1}-1 \le r \le \frac{16}{\left|\frac{3}{-5x}\right|+1}$$

$$\Rightarrow 3 \le r \le 4$$

$$\Rightarrow r = 3 \text{ or } r = 4$$

$$\therefore T_4 = {}^{15}C_3.3^{12}.\left(-5.\frac{1}{5}\right)^3 = -{}^{15}C_3.3^{12}$$
$$= -455\times 3^{12}$$

$$T_5 = {}^{15}C_4.3^{11}.(-1)^4 = {}^{15}C_4.3^{11} = 455\times 3^{12}$$

$$\therefore |T_4| = |T_5| = \boxed{455\times 3^{12}}$$

(6) Find the term independent of x in the expansion of $(1+x+2x^3)\underbrace{\left(\frac{3x^2}{2}-\frac{1}{3x}\right)^9}_{P}$.

$$= 1.(x^0 \text{ in } P) + 1.(x^{-1} \text{ in } P) + 2.(x^{-3} \text{ in } P)$$

$$T_{r+1} = {}^9C_r \cdot \left(\frac{3}{2}\right)^{9-r} \cdot x^{18-2r} \cdot \left(-\frac{1}{3}\right)^r \cdot x^{-r}$$

$$= {}^9C_r \cdot \frac{3^{9-2r}}{2^{9-r}} \cdot (-1)^r \cdot x^{18-3r}$$

(x^0) for constant term $\quad 18-3r=0 \Rightarrow r=6$

$$\therefore T_7 = {}^9C_6 \cdot \frac{3^{-3}}{2^3} x^0$$

(x^{-1}) For constant term $\quad 18-3r=-1 \Rightarrow r=\frac{19}{3}$

Not Possible

(x^{-3}) For constant term $\quad 18-3r=-3 \Rightarrow r=7$

$$\therefore T_8 = {}^9C_7 \frac{3^{-5}}{2^2} (-1)^7 x^{-3}$$

$$\Rightarrow 1.\left({}^9C_6 \cdot \frac{1}{2^3 . 3^3}\right) + 2.\left(-{}^9C_7 \cdot \frac{1}{2^2 3^5}\right)$$

$$= \frac{9.8.7}{3.2} \cdot \frac{1}{2^3 3^3} - 2 \cdot \frac{9.8}{2.1} \cdot \frac{1}{2^2 3^5}$$

$$= \boxed{17/54}$$

(7) Let $(1+x^2)^2 \cdot (1+x)^n = \sum_{K=0}^{n+4} a_K \cdot x^K$. If a_1, a_2 & a_3 are in AP, find n.

L.H.S. $= (1+x^2)^2 \cdot (1+x)^n$

$= (1+2x^2+x^4)\cdot(1+{}^nC_1 x + {}^nC_2 x^2 + {}^nC_3 x^3 + \text{———})$

$= 1 + nx + (2+{}^nC_2)x^2 + (2n+{}^nC_3)x^3 + \text{———}$

R.H.S. $a_0 x^0 + a_1 x^1 + a_2 x^2 + a_3 x^3 + \text{———}$

Now compair L.H.S and R.H.S.

$a_0 = 1$, $a_1 = n$, $a_2 = 2 + {}^nC_2$, $a_3 = 2n + {}^nC_3$

a_1, a_2, a_3 are in A.P. (Given)

$\Rightarrow \quad a_1 + a_3 = 2a_2$

$\Rightarrow \quad n + 2n + {}^nC_3 = 4 + 2\cdot {}^nC_2$

$\Rightarrow \quad 3n + \frac{n(n-1)(n-2)}{6} = 4 + 2\cdot\left(\frac{n(n-1)}{2}\right)$

$\Rightarrow \quad 6(n^2 - 4n + 4) = n(n-1)(n-2)$

$\Rightarrow \quad (n-2)[n^2 - 7n + 12] = 0$

$\Rightarrow \quad (n-2)(n-3)(n-4) = 0$

$\Rightarrow \quad \boxed{n = 2 \text{ or } 3 \text{ or } 4}$

8) Let $f(x) = 1 - x + x^2 - x^3 + \ldots\ldots + x^{16} - x^{17} = a_0 + a_1(1+x) + a_2(1+x)^2 + \ldots\ldots + a_{17}(1+x)^{17}$, find the value of a_2.

$$1 - x + x^2 - x^3 + \text{———} + x^{16} - x^{17}$$

$$= \frac{1 \cdot (1 - (-x)^{18})}{1 - (-x)} = \frac{1 - x^{18}}{1 + x}$$

$$a_0 + a_1(1+x) + a_2(1+x)^2 + \text{-----} + a_{17}(1+x)^{17}$$

Let $(1+x) = t \Rightarrow -x = 1 - t$

$$a_0 + a_1 t + a_2 t^2 + \text{———} + a_{17} t^{17}$$

$$= \frac{1 - (1-t)^{18}}{t}$$

$a_2 =$ coeff of t^2 in $\dfrac{1 - (1-t)^{18}}{t}$

$=$ coeff of t^3 in $1 - (1-t)^{18}$

$$= 1 - \left({}^{18}C_0 - {}^{18}C_1(t) + {}^{18}C_2(t)^2 - {}^{18}C_3(t)^3 + \text{-----}\right)$$

$$= 1 - 1 + {}^{18}C_1(t) - {}^{18}C_2(t)^2 + {}^{18}C_3(t)^3 - \text{----}$$

$\therefore a_2 =$ coeff of t^3 $= {}^{18}C_3 = \boxed{816}$

(9) Let $N = {}^{2000}C_1 + 2 \cdot {}^{2000}C_2 + 3 \cdot {}^{2000}C_3 + \ldots\ldots + 2000 \cdot {}^{2000}C_{2000}$. Prove that N is divisible by 2^{2003}.

$$N = {}^{2000}C_1 + 2 \cdot {}^{2000}C_2 + 3 \cdot {}^{2000}C_3 + \text{———} + 2000 \cdot {}^{2000}C_{2000}$$

$$N = \sum_{r=1}^{2000} r \cdot {}^{2000}C_r \qquad \left(\because r \cdot {}^{n}C_r = n \cdot {}^{n-1}C_{r-1}\right)$$

$$N = \sum_{r=1}^{2000} 2000 \cdot {}^{1999}C_{r-1}$$

$$N = 2000 \sum_{r=1}^{2000} {}^{1999}C_{r-1} \qquad \left(\because {}^{n}C_0 + {}^{n}C_1 + \text{—} + {}^{n}C_n = 2^n\right)$$

$$N = 2000 \cdot 2^{1999}$$

$$N = 2^4 \cdot 125 \cdot 2^{1999} = 125 \cdot \left(2^{2003}\right)$$

So N is divisible by $\boxed{2^{2003}}$

(10) Find the coefficient of

(a) $x^2 y^3 z^4$ in the expansion of $(ax - by + cz)^9$.

(b) $a^2 b^3 c^4 d$ in the expansion of $(a - b - c + d)^{10}$.

(a) $$(ax - by + cz)^9 = \sum_{p+q+r=9} \frac{9!}{p!\,q!\,r!} (ax)^p (-by)^q (cz)^r$$

We want to get $x^2y^3z^4$ this implies that $p=2, q=3, r=4$

$\therefore$ Coeff of $x^2y^3z^4$ is $\dfrac{9!}{2!\,3!\,4!} (a)^2 \cdot (-b)^3 \cdot (c)^4$

$$= \boxed{-1260\, a^2 b^3 c^4}$$

(b) $$(a-b-c+d)^{10} = \sum_{p+q+r+s=10} \frac{10!}{p!\,q!\,r!\,s!} (a)^p (-b)^q (-c)^r (d)^s$$

We want to get $a^2b^3c^4d^1$ this implies that $p=2, q=3, r=4, s=1$

$\therefore$ Coefficient of $a^2b^3c^4d^1$ in $(a-b-c+d)^{10}$

$$= \frac{10!}{2!\,3!\,4!\,1!} (1)^2 \cdot (-1)^3 \cdot (-1)^4 \cdot (1)^4$$

$$= \boxed{-12600}$$

(11) Find the coefficient of

(a) x^4 in the expansion of $(1 + x + x^2 + x^3)^{11}$

(b) x^4 in the expansion of $(2 - x + 3x^2)^6$

(a) $(1+x+x^2+x^3)^{11} = [(1+x) + x^2(1+x)]^{11}$

$= (1+x)^{11} . (1+x^2)^{11}$

For coeff of x^4

$(1+x)^{11}$	$(1+x^2)^{11}$
4	0
2	2
0	4

$= x^4 \text{ in } (1+x)^{11} + (x^2 \text{ in } (1+x)^{11})(x^2 \text{ in } (1+x^2)^{11})$
$+ x^4 \text{ in } (1+x^2)^{11}$

$= {}^{11}C_4 + {}^{11}C_2 . {}^{11}C_1 + {}^{11}C_2 = \boxed{990}$

(b) $(2-x+3x^2)^6 = \sum_{n_1+n_2+n_3=6} \frac{6!}{n_1!\, n_2!\, n_3!} (2)^{n_1} . (-x)^{n_2} . (3x^2)^{n_3}$

$= \sum_{n_1+n_2+n_3=6} \frac{6!}{n_1!\, n_2!\, n_3!} (2)^{n_1} . (-1)^{n_2} . (3)^{n_3} (x)^{n_2+2n_3}$

$n_1 + n_2 + n_3 = 6$

$n_2 + 2n_3 = 4$

n_1	n_2	n_3
2	4	0
3	2	1
4	0	2

$\therefore$ Coeff of x^4 $= \frac{6!}{2!\,4!\,0!} \times 2^2 . (-1)^4 . (3)^0 + \frac{6!}{3!\,2!\,1!} \times 2^3 . (-1)^2 . (3)^1$

$+ \frac{6!}{4!\,0!\,2!} \times 2^4 . (-1)^0 . (3)^2 = \boxed{3660}$

(12) Find the coefficient of x^r in the expression :

$$(x+3)^{n-1} + (x+3)^{n-2}(x+2) + (x+3)^{n-3}(x+2)^2 + \ldots\ldots + (x+2)^{n-1}$$

Let $(x+3) = a$ & $(x+2) = b$

Now $a^{n-1} + a^{n-2}b + a^{n-3}b + \text{———} + b^{n-1}$

$$= a^{n-1}\left[1 + \left(\frac{b}{a}\right) + \left(\frac{b}{a}\right)^2 + \text{———} + \left(\frac{b}{a}\right)^{n-1}\right]$$

$$= a^{n-1}\frac{\left(1 - \left(\frac{b}{a}\right)^n\right)}{1 - \left(\frac{b}{a}\right)} = a^{n-1}\left(\frac{a^n - b^n}{a^n}\right) \times \frac{a}{a-b}$$

$$= \frac{(x+3)^n - (x+2)^n}{(x+3) - (x+2)} = (x+3)^n - (x+2)^n$$

Coeff of x^r in $(x+3)^n$ − Coeff of x^r in $(x+2)^n$

$$= {}^nC_r \cdot 3^{n-r} - {}^nC_r \cdot 2^{n-r}$$

$$= \boxed{{}^nC_r\left(3^{n-r} - 2^{n-r}\right)}$$

13. Given that $(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + ... + a_{2n}x^{2n}$, find the values of :

(i) $a_0 + a_1 + a_2 + ... + a_{2n}$;

(ii) $a_0 - a_1 + a_2 - a_3 ..... + a_{2n}$;

$$(1+x+x^2)^n = a_0 + a_1x + a_2x^2 + - + a_nx^n + - + a_{2n}x^{2n} \longrightarrow (1)$$

(i) Put $x=1$

$$a_0 + a_1 + a_2 + \text{———} + a_{2n} = \boxed{3^n}$$

(ii) Put $x=-1$

$$a_0 - a_1 + a_2 - a_3 + - \cdots + a_{2n} = (1-1+1)^n = \boxed{1}$$

14. Prove the following identities using the theory of permutation where $C_0, C_1, C_2, \ldots, C_n$ are the combinatorial coefficients in the expansion of $(1+x)^n, n \in N$:

$${}^{100}C_{10} + 5.\,{}^{100}C_{11} + 10.\,{}^{100}C_{12} + 10.\,{}^{100}C_{13} + 5.\,{}^{100}C_{14} + {}^{100}C_{15} = {}^{105}C_{90}$$

(e) ${}^{100}C_{10} + 5.\,{}^{100}C_{11} + 10.\,{}^{100}C_{12} + 10.\,{}^{100}C_{13} + 5.\,{}^{100}C_{14} + {}^{100}C_{15}$

$= {}^5C_0.\,{}^{100}C_{10} + {}^5C_1.\,{}^{100}C_{11} + {}^5C_2.\,{}^{100}C_{12} + {}^5C_2.\,{}^{100}C_{13} + {}^5C_1\,{}^{100}C_{14} + {}^5C_5\,{}^{100}C_{15}$

$= {}^5C_0.\,{}^{100}C_{90} + {}^5C_1.\,{}^{100}C_{89} + {}^5C_2.\,{}^{100}C_{88} + {}^5C_3.\,{}^{100}C_{87} + {}^5C_4\,{}^{100}C_{86} + {}^5C_5\,{}^{100}C_{85}$

Now Sum of suffixes is constant

$= \boxed{{}^{105}C_{90} \text{ or } {}^{105}C_{15}}$

15 If $C_0, C_1, C_2, \ldots, C_n$ are the combinatorial coefficients in the expansion of $(1+x)^n$, $n \in N$, then prove the following :

(a) $C_1 + 2C_2 + 3C_3 + \ldots + n \cdot C_n = n \cdot 2^{n-1}$

(b) $C_0 + 2C_1 + 3C_2 + \ldots + (n+1)C_n = (n+2)2^{n-1}$

(c) $C_0 + 3C_1 + 5C_2 + \ldots + (2n+1)C_n = (n+1)\, 2^n$

(d) $(C_0+C_1)(C_1+C_2)(C_2+C_3) \ldots (C_{n-1}+C_n) = \dfrac{C_0 . C_1 . C_2 \ldots C_{n-1} (n+1)^n}{n!}$

(e) $1 . C_o^2 + 3 . C_1^2 + 5 . C_2^2 + \ldots + (2n+1)\, C_n^2 = \dfrac{(n+1)(2n)!}{n!\, n!}$

(a) $C_1 + 2C_2 + 3C_3 + \ldots + n . C_n$

Method 1

$$\sum_{r=1}^{n} r . {}^nC_r \qquad \left(\because r . {}^nC_r = n . {}^{n-1}C_{r-1}\right)$$

$$= n \sum_{r=1}^{n} {}^{n-1}C_{r-1}$$

$$= n . \left({}^{n-1}C_0 + {}^{n-1}C_1 + \text{---} + {}^{n-1}C_{n-1}\right)$$

$$= \boxed{n . 2^{n-1}}$$

Method 2
By Derivative Method

$$(1+x)^n = C_0 + C_1 x + C_2 x^2 + \text{---} + C_n x^n$$

differentiating with respect of x

$$n . (1+x)^{n-1} = C_1 + 2C_2 x + 3 . C_3 x^2 + \text{--} + n . C_n x^{n-1}$$

Now Put $x = 1$

$$n . (1+1)^{n-1} = C_1 + 2 . C_2 + 3 . C_3 + \text{---} + n . C_n$$

$$\therefore C_1 + 2C_2 + 3 . C_3 + \text{---} + n . C_n = \boxed{n . 2^{n-1}}$$

(b) $C_0 + 2C_1 + 3C_2 + \text{———} + (n+1)C_n$

Method 1

$= \sum_{r=0}^{n} (r+1)\,{}^nC_r$

$= \sum_{r=0}^{n} r.\,{}^nC_r + \sum_{r=0}^{n} {}^nC_r$

$= n \sum_{r=0}^{n} {}^{n-1}C_{r-1} + 2^n$ $\quad \left(\because r.\,{}^nC_r = n.\,{}^{n-1}C_{r-1}\right)$

$= n.\,2^{n-1} + 2^n = \boxed{2^{n-1}(n+2)}$

Method 2

$(1+x)^n = C_0 + C_1x + C_2x^2 + \text{———} + C_nx^n$

$x.(1+x)^n = C_0x + C_1x^2 + C_2x^3 + \text{———} + C_nx^{n+1}$

Now Differentiate with respect to x

$x.n.(1+x)^{n-1} + (1+x)^n = C_0 + 2C_1x + 3C_2x^2 + \text{——} + (n+1)C_nx^n$

Now put $x=1$

$C_0 + 2C_1 + 3C_2 + \text{———} + (n+1)C_n = n.2^{n-1} + 2^n$

$= \boxed{2^{n-1}(n+2)}$

(c) $C_0 + 3C_1 + 5C_2 + \text{-----} + (2n+1)C_n$

Method 1

$= \sum_{r=0}^{n} (2r+1)\,{}^nC_r$

$= 2\sum_{r=0}^{n} r.\,{}^nC_r + \sum_{r=0}^{n} {}^nC_r$

$= 2\left(n.\,2^{n-1}\right) + 2^n = n.2^n + 2^n = \boxed{2^n(n+1)}$

Method 2 $C_0 + 3C_1 + 5C_2 + \cdots + (2n+1)C_n$

$= (C_0 + C_1 + C_2 + \cdots + C_n) + (2C_1 + 4C_2 + \cdots + 2nC_n)$

$= (C_0 + C_1 + C_2 + \cdots + C_n) + 2\underbrace{(C_1 + 2C_2 + \cdots + n.C_n)}_{\text{Question (a)}}$

$= 2^n + 2(n.2^{n-1})$

$= 2^n + n.2^n$

$= \boxed{2^n(n+1)}$

(d) $(C_0 + C_1)(C_1 + C_2)(C_2 + C_3) \cdots (C_{n-1} + C_n)$

$= C_0\left(1 + \frac{C_1}{C_0}\right).C_1\left(1 + \frac{C_2}{C_1}\right).C_2\left(1 + \frac{C_3}{C_2}\right) \cdots C_{n-1}\left(1 + \frac{C_n}{C_{n-1}}\right)$

$= C_0.C_1.C_2 \cdots C_{n-1}\left(1 + \frac{n}{1}\right)\left(1 + \frac{n-1}{2}\right)\left(1 + \frac{n-2}{3}\right) \cdots \left(1 + \frac{1}{n}\right)$

$= C_0.C_1.C_2 \cdots C_{n-1}\left(\frac{n+1}{1}\right).\left(\frac{n+1}{2}\right).\left(\frac{n+1}{3}\right) \cdots \left(\frac{n+1}{n}\right)$

$= \boxed{C_0.C_1.C_2 \cdots C_{n-1}\,\frac{(n+1)^n}{n!}}$

(e) $1.C_0^2 + 3.C_1^2 + 5.C_2^2 + \cdots + (2n+1)C_n^2$

M1 $= \sum_{r=0}^{n} (2r+1)({}^nC_r)^2$

$= 2\sum_{r=0}^{n} r.{}^nC_r\,{}^nC_r + \sum_{r=0}^{n} ({}^nC_r)^2 \quad (\because r.{}^nC_r = n.{}^{n-1}C_{r-1})$

$= 2n\sum_{r=0}^{n} {}^{n-1}C_{r-1}.{}^nC_r + \sum_{r=0}^{n} {}^nC_r.{}^nC_{n-r}$

$= 2n\sum_{r=0}^{n} {}^{n-1}C_{r-1}\,{}^nC_{n-r} + \sum_{r=0}^{n} {}^nC_r.{}^nC_{n-r}$

$= 2n\left({}^{2n-1}C_{n-1}\right) + {}^{2n}C_n = 2n.\frac{(2n-1)!}{(n-1)!\,n!} + \frac{(2n)!}{n!\,n!}$

$= \frac{(2n)!}{n!\,(n-1)!}\left(1 + \frac{1}{n}\right)$

$= \boxed{\frac{(2n)!\,(n+1)}{n!\;n!}}$

M2 $S = 1.C_0^2 + 3.C_1^2 + 5.C_2^2 + \cdots + (2n+1)C_n^2$

$S = (2n+1)C_0^2 + (2n-1)C_1^2 + (2n-3)C_2^2 + \cdots + 1.C_n^2$

Adding

$2S = (2n+2)C_0^2 + (2n+2)C_1^2 + (2n+2)C_2^2 + \cdots + (2n+2)C_n^2$

$2S = (2n+2)(C_0^2 + C_1^2 + \cdots + C_n^2)$

$2S = 2(n+1)\,{}^{2n}C_n \Rightarrow S = \boxed{(n+1)\frac{(2n)!}{n!\,n!}}$

** M2 method Can also be used for part (a), (b) & (c)

16 Prove that

(a) $\frac{C_1}{C_0}+\frac{2C_2}{C_1}+\frac{3C_3}{C_2}+\ldots\ldots+\frac{n.C_n}{C_{n-1}}=\frac{n(n+1)}{2}$ (b) $C_0+\frac{C_1}{2}+\frac{C_2}{3}+\ldots\ldots+\frac{C_n}{n+1}=\frac{2^{n+1}-1}{n+1}$

(c) $2 . C_0+\frac{2^2.C_1}{2}+\frac{2^3.C_2}{3}+\frac{2^4.C_3}{4}+\ldots\ldots\frac{2^{n+1}.C_n}{n+1}=\frac{3^{n+1}-1}{n+1}$

(d) $C_0-\frac{C_1}{2}+\frac{C_2}{3}-\ldots\ldots+(-1)^n\frac{C_n}{n+1}=\frac{1}{n+1}$

(a) $\frac{C_1}{C_0}+2\frac{C_2}{C_1}+3\frac{C_3}{C_2}+\ldots\ldots+\frac{nC_n}{C_{n-1}}$

$=\left(\frac{n-0}{1}\right)+2\left(\frac{n-1}{2}\right)+3\left(\frac{n-2}{3}\right)+\ldots\ldots+n\left(\frac{n-(n-1)}{n}\right)$

$=(n)+(n-1)+(n-2)+\ldots\ldots+(1)$

$=\boxed{\frac{n(n+1)}{2}}$

$\left\{\text{Using } \begin{aligned} {}^nC_1 &= n \\ {}^nC_2 &= \frac{n(n-1)}{2!} \\ {}^nC_3 &= \frac{n(n-1)(n-2)}{3!} \end{aligned}\right\}$

(b) Method 1

$C_0+\frac{C_1}{2}+\frac{C_2}{3}+\ldots\ldots+\frac{C_n}{n+1}$

$=\sum_{r=0}^{n}\frac{{}^nC_r}{r+1}$ $\left(\because \frac{{}^nC_r}{r+1}=\frac{{}^{n+1}C_{r+1}}{n+1}\right)$

$=\frac{1}{n+1}\sum_{r=0}^{n}{}^{n+1}C_{r+1}$

$=\frac{1}{n+1}\left({}^{n+1}C_1+{}^{n+1}C_2+\text{———}+{}^{n+1}C_{n+1}\right)$

$=\frac{1}{n+1}\left(2^{n+1}-{}^{n+1}C_0\right)=\boxed{\frac{2^{n+1}-1}{n+1}}$

Method 2 By Integration Method

$$(1+x)^n = C_0 + C_1 x + C_2 x^2 + \cdots\cdots + C_n x^n$$

Integration with respect to x

$$\frac{(1+x)^{n+1}}{n+1} + K = C_0 x + C_1 \frac{x^2}{2} + C_2 \frac{x^3}{3} + \text{——} + C_n \frac{x^{n+1}}{n+1}$$

if $x=0 \Rightarrow K = -\frac{1}{n+1}$

$$\frac{(1+x)^{n+1}}{n+1} - \frac{1}{n+1} = C_0 x + C_1 \frac{x^2}{2} + C_2 \frac{x^3}{3} + \text{——} + C_n \frac{x^{n+1}}{n+1} \longrightarrow (1)$$

Now Put $x=1$ in eq^n (1)

$$C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \cdots\cdots + \frac{C_n}{n+1} = \boxed{\frac{2^{n+1}-1}{n+1}}$$

or

$$\int_0^1 (1+x)^n \, dx = \int_0^1 \left(C_0 + C_1 x + C_2 x^2 + \text{——} + C_n x^n\right) dx$$

$$\left.\frac{(1+x)^{n+1}}{n+1}\right|_{x=0}^{x=1} = C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \text{——} + \frac{C_n}{n+1}$$

$$= \frac{2^{n+1}-1}{n+1}$$

(c) $2.C_0 + \frac{2^2 . C_1}{2} + \frac{2^3 . C_2}{3} + \text{-----} + \frac{2^{n+1} . C_n}{n+1}$

$$= \sum_{r=0}^{n} \frac{2^{r+1} . {}^{n}C_r}{r+1} \qquad \left(\because \frac{{}^{n}C_r}{r+1} = \frac{{}^{n+1}C_{r+1}}{n+1}\right)$$

$$= \frac{1}{n+1} \sum_{r=0}^{n} 2^{r+1} . {}^{n+1}C_{r+1}$$

$$= \frac{1}{n+1}\left[(1+2)^{n+1} - {}^{n+1}C_0\, 2^0\right]$$

$$= \boxed{\frac{3^{n+1} - 1}{n+1}}$$

(d) $C_0 - \frac{C_1}{2} + \frac{C_2}{3} - \text{......} + (-1)^n . \frac{C_n}{n+1}$

$$= \sum_{r=0}^{n} \frac{(-1)^r . {}^{n}C_r}{r+1} \qquad \left(\because \frac{{}^{n}C_r}{r+1} = \frac{{}^{n+1}C_{r+1}}{n+1}\right)$$

$$= \frac{1}{n+1} \sum_{r=0}^{n} (-1)^r\, {}^{n+1}C_{r+1}$$

$$= \frac{1}{n+1}\left[{}^{n+1}C_1 - {}^{n+1}C_2 + {}^{n+1}C_3 - {}^{n+1}C_4 \text{ -----} + (-1)^n\, {}^{n+1}C_{n+1}\right]$$

$$= \frac{1}{n+1}\left[{}^{n+1}C_0 - \left({}^{n+1}C_0 - {}^{n+1}C_1 + {}^{n+1}C_2 - \text{-----} - (-1)^n\, {}^{n+1}C_{n+1}\right)\right]$$

$$= \frac{1}{n+1}\left[1 - 0\right] = \boxed{\frac{1}{n+1}}$$

** $\boxed{\text{Integration method can also be used for (c) \& (d) part.}}$

EXERCISE (S-2)

1. Prove that $\sum_{K=0}^{n} {}^nC_K \sin Kx.\cos(n-K)x = 2^{n-1}\sin nx.$

Solution: Let

$$S_n = {}^nC_0 \sin 0x \cdot \cos nx + {}^nC_1 \sin x \cdot \cos(n-1)x + \text{------} + {}^nC_{n-1} \sin(n-1)x \cos x + {}^nC_n \sin nx \cdot \cos 0x$$

Applying Carl Friedrich Gauss Method

https://en.wikipedia.org/wiki/Carl_Friedrich_Gauss

$$S_n = {}^nC_n \sin nx \cdot \cos 0x + {}^nC_{n-1} \sin(n-1)x \cdot \cos x + \text{.....} + {}^nC_1 \sin x \cdot \cos(n-1)x + {}^nC_0 \sin 0x \cos nx$$

Add

$$2S_n = {}^nC_0 \sin(0x + nx) + {}^nC_1 \sin(x + (n-1)x) + \ldots + {}^nC_n \sin(nx + 0x)$$

$$2S_n = \sin nx \cdot ({}^nC_0 + {}^nC_1 + \ldots + {}^nC_n) = 2^n \cdot \sin nx$$

$$\Rightarrow \quad S_n = 2^{n-1} \sin nx \quad [\text{Hence Proved}]$$

(2) Let $a = \left(4^{1/401} - 1\right)$ and let $b_n = {}^nC_1 + {}^nC_2 \cdot a + {}^nC_3 \cdot a^2 + \dots\dots\dots + {}^nC_n \cdot a^{n-1}$.

Find the value of $(b_{2006} - b_{2005})$

Solution:

$$\boxed{a+1 = 4^{\frac{1}{401}}}$$

$$b_n = {}^nC_1 + {}^nC_2 a + {}^nC_3 a^2 + \dots + {}^nC_n a^{n-1}$$

$$\Rightarrow b_n \cdot a = {}^nC_1 a + {}^nC_2 a^2 + {}^nC_3 a^3 + \dots + {}^nC_n a^n$$

$$\Rightarrow b_n \cdot a + 1 = 1 + {}^nC_1 a + {}^nC_2 a^2 + {}^nC_3 a^3 + \dots + {}^nC_n a^n$$

$$= {}^nC_0 + {}^nC_1 a + {}^nC_2 a^2 + \dots + {}^nC_n a^n$$

$$\Rightarrow b_n a + 1 = (1+a)^n \Rightarrow \boxed{b_n = \frac{(1+a)^n - 1}{a}}$$

Now, $$b_{2006} - b_{2005} = \left[\frac{(1+a)^{2006} - 1}{a} - \frac{(1+a)^{2005} - 1}{a}\right]$$

$$= \frac{4^{\frac{2006}{401}} - 4^{\frac{2005}{401}}}{a}$$

$$= 4^{\frac{2005}{401}} \frac{\left[4^{\frac{1}{401}} - 1\right]}{a} = 4^5 = 2^{10}.$$

EXERCISE (S-2)

(3) Let a and b be the coefficient of x^3 in $(1 + x + 2x^2 + 3x^3)^4$ and $(1 + x + 2x^2 + 3x^3 + 4x^4)^4$ respectively. Find the value of $(a - b)$.

Solution:

Coefficient of x^3 in $(1+x+2x^2+3x^3)^4 = a$

$\Rightarrow$ Coefficient of x^3 in $(f(x))^4 = a$

Now,

$b =$ Coefficient of x^3 in $(1+x+2x^2+3x^3+4x^4)^4$

$=$ coefficient of x^3 in $(f(x)+4x^4)^4$

[where $f(x) = 1+x+2x^2+3x^3$]

Now, $b =$ Coefficient of x^3 in $\left[(f(x))^4 + {}^4C_1 (f(x))^3 (4x^4) + \ldots\right]$

$=$ Coefficient of x^3 in $(f(x))^4$

$= a$

$\Rightarrow \boxed{b - a = 0}$

4. Find the sum of the roots (real or complex) of the equation $x^{2001} + \left(\frac{1}{2} - x\right)^{2001} = 0$.

Solution:

$$x^{2001} + (-1)^{2001}\left(x - \frac{1}{2}\right)^{2001} = 0$$

$$\Rightarrow x^{2001} = \left({}^{2001}C_0 x^{2001} - {}^{2001}C_1 x^{2000}\frac{1}{2} + {}^{2001}C_2 x^{1999}\left(\frac{1}{2}\right)^2 + \ldots\right)$$

$$\Rightarrow {}^{2001}C_1 x^{2000}\frac{1}{2} - {}^{2001}C_2 x^{1999}\left(\frac{1}{2}\right)^2 + \ldots\ldots = 0$$

$$\text{Sum of roots} = -\frac{\text{Coefficient of } x^{1999}}{\text{Coefficient of } x^{2000}}$$

$$= -\frac{\left(-{}^{2001}C_2\left(\frac{1}{2}\right)^2\right)}{{}^{2001}C_1\left(\frac{1}{2}\right)}$$

$$= \frac{2001 \times 2000}{2 \times 2 \times 2001}$$

$$= \boxed{500}$$

5. Find the coefficient of x^{49} in the polynomial

$$\left(x-\frac{C_1}{C_0}\right)\left(x-2^2\cdot\frac{C_2}{C_1}\right)\left(x-3^2\cdot\frac{C_3}{C_2}\right)\ldots\ldots\ldots\ldots\ldots\left(x-50^2\cdot\frac{C_{50}}{C_{49}}\right),\text{ where } C_r={}^{50}C_r.$$

Solution:

We know that, $\frac{{}^nC_r}{{}^nC_{r-1}}=\frac{n-(r-1)}{r}=\frac{n-r+1}{r}$

Further, $r^2\cdot\frac{{}^{50}C_r}{{}^{50}C_{r-1}}=r(50-r+1)=r(51-r)$

For a polynomial if expressed as:

$$x^{50}+a_1x^{49}+a_2x^{48}+\ldots+a_{49}x^0=(x-\alpha_1)(x-\alpha_2)\ldots\ldots(x-\alpha_{50})$$

Coefficient of $x^{49}=a_1=-$ (sum of zeroes)

$$\Rightarrow S=-(\alpha_1+\alpha_2+\ldots+\alpha_{50})$$

$$\Rightarrow S=-\left[\sum_{r=1}^{50}r(51-r)\right]$$

$$\Rightarrow S=-\left[51\times\frac{50\times51}{2}-\frac{50\times51\times101}{6}\right]$$

$$\Rightarrow S=-\frac{50\times51}{6}\left[153-101\right]=-25\times17\times52\;\boxed{=-22100}$$

6 If $\binom{n}{r}$ denotes ${}^nC_r$, then

(a) Evaluate : $2^{15}\binom{30}{0}\binom{30}{15}-2^{14}\binom{30}{1}\binom{29}{14}+2^{13}\binom{30}{2}\binom{28}{13}\ldots\ldots-\binom{30}{15}\binom{15}{0}$

Solution: (a)

$$\text{Sum, } S=\sum_{r=0}^{15} 2^{15-r}\cdot {}^{30}C_r\ {}^{30-r}C_{15-r}(-1)^r$$

$$=\sum_{r=0}^{15}(-1)^r 2^{15-r}\cdot\frac{30!}{r!\,(30-r)!}\times\frac{(30-r)!}{(15-r)!\,15!}$$

$$=\sum_{r=0}^{15}(-1)^r 2^{15-r}\frac{30!}{15!}\times\frac{1}{r!\,(15-r)!}$$

$$=\sum_{r=0}^{15}(-1)^r 2^{15-r}\frac{30!}{15!\,15!}\times\frac{15!}{r!\,(15-r)!}$$

$$=\sum_{r=0}^{15}(-1)^r 2^{15-r}\cdot {}^{30}C_{15}\cdot {}^{15}C_r$$

$$S={}^{30}C_{15}\sum_{r=0}^{15}(-1)^r\,{}^{15}C_r\,2^{15-r}={}^{30}C_{15}(2-1)^{15}$$

$$={}^{30}C_{15}$$

(b) Prove that : $\sum_{r=1}^{n}\binom{n-1}{n-r}\binom{n}{r}=\binom{2n-1}{n-1}$

Solution: $S=\sum_{r=1}^{n} {}^{n-1}C_{n-r}\,{}^{n}C_{r}$

$S={}^{n-1}C_{n-1}\cdot{}^{n}C_{1}+{}^{n-1}C_{n-2}\,{}^{n}C_{2}+{}^{n-1}C_{n-3}\,{}^{n}C_{3}+\dots+{}^{n-1}C_{0}\,{}^{n}C_{n}$

Observe:

$(1+x)^{n-1}={}^{n-1}C_{n-1}+{}^{n-1}C_{n-2}\,x+{}^{n-1}C_{n-3}\,x^{2}+\dots+{}^{n-1}C_{0}\,x^{n-1}$

$(x+1)^{n}={}^{n}C_{0}\,x^{n}+{}^{n}C_{1}\,x^{n-1}+{}^{n}C_{2}\,x^{n-2}+\dots\dots+{}^{n}C_{n}\,x^{0}$

$S=$ Coefficient of x^{n-1} in the product $(1+x)^{n-1}\cdot(x+1)^{n}$

$=$ Coefficient of x^{n-1} in $(1+x)^{2n-1}$

$S={}^{2n-1}C_{n-1}$ [Hence Proved]

$S=$ number of ways to select 'n' students out of $(n-1)$ Boys & 'n' girls in a class room $={}^{2n-1}C_{n}={}^{2n-1}C_{n-1}$

(c) Prove that : $\binom{n}{r}\binom{r}{k} = \binom{n}{k}\binom{n-k}{r-k}$

Solution:

$$^{n}C_{r} \cdot {}^{r}C_{k} = \frac{n!}{r!(n-r)!} \times \frac{r!}{k!(r-k)!}$$

$$= \frac{n!}{k!} \times \frac{1}{(n-r)!(r-k)!}$$

$$= \frac{n!}{k!(n-k)!} \times \frac{(n-k)!}{(n-r)!(r-k)!}$$

$$= {}^{n}C_{k} \cdot {}^{n-k}C_{r-k}$$

[Hence Proved]

EXERCISE (JM)

1. Let $S_1 = \sum_{j=1}^{10} j(j-1)\,{}^{10}C_j$, $S_2 = \sum_{j=1}^{10} j\,{}^{10}C_j$ and $S_3 = \sum_{j=1}^{10} j^2\,{}^{10}C_j$. **[AIEEE-2010]**

Statement–1 : $S_3 = 55 \times 2^9$.

Statement–2 : $S_1 = 90 \times 2^8$ and $S_2 = 10 \times 2^8$.

(1) Statement–1 is true, Statement–2 is true ; Statement–2 is a correct explanation for Statement–1.

(2) Statement–1 is true, Statement–2 is true ; Statement–2 is not a correct explanation for Statement–1.

✓ (3) Statement–1 is true, Statement–2 is false.

(4) Statement–1 is false, Statement–2 is true.

Soln:

$$\Rightarrow S_1 = \sum_{j=2}^{10} \cancel{j(j-1)} \times \frac{10}{\cancel{j}} \times \frac{9}{\cancel{j-1}} \times {}^8C_{j-2}$$

$$= 90 \times 2^8$$

$$S_2 = \sum_{j=1}^{10} \cancel{j} \cdot \frac{10}{\cancel{j}} \times {}^9C_{j-1}$$

$$= 10 \cdot 2^9$$

$$S_3 = \sum_{j=1}^{10} j^{\cancel{2}} \cdot \frac{10}{\cancel{j}} \cdot {}^9C_{j-1}$$

$$= 10 \sum_{j=1}^{10} (j-1+1)\,{}^9C_{j-1}$$

$$= 10\left[\sum_{j=1}^{10} (j-1)\cdot {}^9C_{j-1} + \sum_{j=1}^{10} {}^9C_{j-1}\right]$$

$$= 10\left[9 \cdot \sum_{j=2}^{10} {}^8C_{j-2} + 2^9\right]$$

$$= 10 \times 9 \times 2^8 + 10 \times 2^9$$

$$= 55 \times 2^9$$

EXERCISE (JM)

2. The coefficient of x^7 in the expansion of $(1 - x - x^2 + x^3)^6$ is :- **[AIEEE 2011]**

- (1) – 144 (2) 132 (3) 144 (4) – 132

Solt^n: $\Rightarrow (1-x)^6 (1-x^2)^6$

Coeff of x^7 can be found by expanding and then multiplying above binomial expressions.

Thus

$$(1-x)^6 \times (1-x^2)^6$$

Coeff of x^1 × Coeff of x^6

+ Coeff of x^3 × Coeff of x^4

+ coeff. of x^5 × Coeff of x^2

$$\Rightarrow -{}^6C_1 \times -{}^6C_3 + (-{}^6C_3) \times ({}^6C_2) + (-{}^6C_5) \times (-{}^6C_1)$$

$$\Rightarrow 120 - 300 + 36$$

$$\Rightarrow -144 \text{ Ans}$$

3. If n is a positive integer, then $\left(\sqrt{3}+1\right)^{2n}-\left(\sqrt{3}-1\right)^{2n}$ is : **[AIEEE 2012]**

(1) a rational number other than positive integers (2) an irrational number

(3) an odd positive integer (4) an even positive integer

Soln :-

$$(\sqrt{3}+1)^{2n} = {}^{2n}C_0(\sqrt{3})^{2n} + {}^{2n}C_1(\sqrt{3})^{2n-1}(1)^1 + \cdots {}^{2n}C_{2n}(1)^{2n}.$$

$$(\sqrt{3}-1)^{2n} = {}^{2n}C_0(\sqrt{3})^{2n} - {}^{2n}C_1(\sqrt{3})^{2n-1}(1)^1 + \cdots {}^{2n}C_{2n}(1)^{2n}$$

Substracting both

$$(\sqrt{3}+1)^{2n} - (\sqrt{3}-1)^{2n} = 2\left[{}^{2n}C_1(\sqrt{3})^{2n-1} + {}^{2n}C_3(\sqrt{3})^{2n-3} + \cdots {}^{2n}C_{2n-1}(\sqrt{3})^1\right]$$

= an irrational no.

EXERCISE (JM)

4. The term independent of x in expansion of $\left(\frac{x+1}{x^{2/3}-x^{1/3}+1}-\frac{x-1}{x-x^{1/2}}\right)^{10}$ is : **[JEE-Main 2013]**

(1) 4 (2) 120 (3) 210 (4) 310

Soln :- Given expression can be written as

$$\left[\frac{(x^{1/3})^3+(1)^3}{x^{2/3}-x^{1/3}+1}-\frac{(\sqrt{x})^2-(1)^2}{\sqrt{x}(\sqrt{x}-1)}\right]^{10}$$

$$\Rightarrow \left[\frac{(x^{1/3}+1)\cancel{(x^{2/3}-x^{1/3}+1)}}{\cancel{(x^{2/3}-x^{1/3}+1)}}-\frac{\cancel{(\sqrt{x}-1)}(\sqrt{x}+1)}{\sqrt{x}\,\cancel{(\sqrt{x}-1)}}\right]^{10}$$

$$\Rightarrow \left[x^{1/3}+1-\left(1+\frac{1}{\sqrt{x}}\right)\right]^{10}$$

$$\Rightarrow \left(x^{1/3}-x^{-1/2}\right)^{10}$$

General term (T_{r+1}) $= {}^{10}C_r\,(x^{1/3})^{10-r}\,(x^{-1/2})^{r}$

$\Rightarrow$ power of x $= \frac{10-r}{3}-\frac{r}{2}=0$

$\Rightarrow r=4$

$\therefore$ Coeff of $T_5 = {}^{10}C_4$

$= 210$ Ans

EXERCISE (JM)

5. If the coefficients of x^3 and x^4 in the expansion of $(1 + ax + bx^2)(1 - 2x)^{18}$ in powers of x are both zero, then (a, b) is equal to :- **[JEE(Main)-2014]**

(1) $\left(16, \frac{251}{3}\right)$ (2) $\left(14, \frac{251}{3}\right)$ (3) $\left(14, \frac{272}{3}\right)$ (4) $\left(16, \frac{272}{3}\right)$

Solⁿ :- $(1-2x)^{18} + ax(1-2x)^{18} + bx^2(1-2x)^{18}$

Coeff of x^3

$\Rightarrow {}^{18}C_3(-2)^3 + a\cdot{}^{18}C_2(-2)^2 + b\cdot{}^{18}C_1(-2)^1$

$\Rightarrow -6528 + 612a - 36b = 0$

$\Rightarrow 17a - b = \frac{34\times16}{3}$ —— (1)

Coeff of x^4

$\Rightarrow {}^{18}C_4(-2)^4 + a\,{}^{18}C_3(-2)^3 + b\,{}^{18}C_2(-2)^2$

$\Rightarrow 48960 - 6528a + 612b = 0$

$\Rightarrow 32a - 3b = 240$ —— (2)

Solving (1) & (2)

$a = 16, \quad b = \frac{272}{3}.$

6. The sum of coefficients of integral powers of x in the binomial expansion of $\left(1-2\sqrt{x}\right)^{50}$ is : **[JEE(Main)-2015]**

(1) $\frac{1}{2}\left(3^{50}-1\right)$ (2) $\frac{1}{2}\left(2^{50}+1\right)$ (3) $\frac{1}{2}\left(3^{50}+1\right)$ (4) $\frac{1}{2}\left(3^{50}\right)$

Solt^n

$$(1-2\sqrt{x})^{50} = 1 - {}^{50}C_1(2\sqrt{x}) + {}^{50}C_2(2\sqrt{x})^2 + \ldots + {}^{50}C_{50}(2\sqrt{x})^{50}$$

$$(1+2\sqrt{x})^{50} = 1 + {}^{50}C_1(2\sqrt{x}) + {}^{50}C_2(2\sqrt{x})^2 + \ldots + {}^{50}C_{50}(2\sqrt{x})^{50}.$$

adding both to get integral pow of x

$$(1-2\sqrt{x})^{50} + (1+2\sqrt{x})^{50} = 2\left[1 + {}^{50}C_2(2\sqrt{x})^2 + \ldots + \ldots + {}^{50}C_{50}(2\sqrt{x})^{50}\right] \quad \text{------ (1)}$$

Now to get the sum of coefficient put $x=1$ in above eq^n.

$$\Rightarrow \frac{(-1)^{50}+(3)^{50}}{2} = \text{Sum of coeff of integral power of } x$$

$$\Rightarrow \frac{3^{50}+1}{2} \quad \underline{\text{Ans}}$$

EXERCISE (JM)

7. If the number of terms in the expansion of $\left(1-\frac{2}{x}+\frac{4}{x^2}\right)^n, x \neq 0$, is 28, then the sum of the coefficients of all the terms in this expansion, is :- **[JEE(Main)-2016]**

(1) 729 (2) 64 (3) 2187 (4) 243

Soln :- No. of terms in the expansion of $\left(1-\frac{2}{x}+\frac{4}{x^2}\right)^n$ is ${}^{n+2}C_2$ (considering $\frac{1}{x}$ & $\frac{1}{x^2}$ distinct)

$\therefore$ ${}^{n+2}C_2 = 28 \Rightarrow \underline{n=6}$

$\therefore$ Sum of Coeff $= (1-2+4)^6 = 729$

But number of dissimilar terms actually will be $(2n+1)$, because in exp. of $\left(1-\frac{2}{x}+\frac{4}{x^2}\right)^n$ minimum pow on x is 0 & max is $\frac{1}{x^{2n}}$

So total $(2n+1)$ dissimilar terms, therefore

$\Rightarrow$ $2n+1 = 28$

$\Rightarrow$ $n = \frac{27}{2}$, a fraction which is not possible

Hence 'bonus'.

EXERCISE (JM)

8. The value of $({}^{21}C_1 - {}^{10}C_1) + ({}^{21}C_2 - {}^{10}C_2) + ({}^{21}C_3 - {}^{10}C_3) + ({}^{21}C_4 - {}^{10}C_4) + \dots + ({}^{21}C_{10} - {}^{10}C_{10})$ is :- **[JEE(Main)-2017]**

(1) $2^{20} - 2^{10}$ (2) $2^{21} - 2^{11}$ (3) $2^{21} - 2^{10}$ (4) $2^{20} - 2^{9}$

Solⁿ :-

$$\Rightarrow ({}^{21}C_1 + {}^{21}C_2 + {}^{21}C_3 + \dots + {}^{21}C_{10}) - ({}^{10}C_1 + {}^{10}C_2 + \dots {}^{10}C_{10}) \quad \text{——— (1)}$$

We know that

$$({}^{21}C_0 + {}^{21}C_1 + \dots + {}^{21}C_{10}) + \underbrace{{}^{21}C_{11} + \dots + {}^{21}C_{21}}_{{}^{21}C_{10} + \dots + {}^{21}C_0} = 2^{21}$$

$$\Rightarrow \therefore \quad {}^{21}C_1 + {}^{21}C_2 + \dots + {}^{21}C_{10} = 2^{20} - 1$$

So from (1)

$$\Rightarrow (2^{20} - 1) - (2^{10} - 1)$$

$$\Rightarrow 2^{10}(2^{10} - 1) \quad \underline{\underline{\text{Ans}}}$$

EXERCISE (JM)

9. The sum of the co-efficients of all odd degree terms in the expansion of $\left(x+\sqrt{x^3-1}\right)^5+\left(x-\sqrt{x^3-1}\right)^5$, $(x > 1)$ is - **[JEE(Main)-2018]**

(1) 0 (2) 1 (3) 2 (4) –1

Soln

$$\left(x+\sqrt{x^3-1}\right)^5 = x^5 + {}^5C_1 x^4\sqrt{x^3-1} + \text{——} + {}^5C_5\left(\sqrt{x^3-1}\right)^5$$

$$\left(x-\sqrt{x^3-1}\right)^5 = x^5 - {}^5C_1 \cdot x^4 \cdot \sqrt{x^3-1} + \ldots\ldots - {}^5C_5\left(\sqrt{x^3-1}\right)^5$$

adding both

$$\Rightarrow 2\left[x^5 + {}^5C_2 \cdot x^3 \cdot (x^3-1) + {}^5C_4 \cdot x\,(x^3-1)^2\right]$$

$$\Rightarrow 2\left[x^5 + 10 \cdot x^6 - 10x^3 + 5x^7 - 10x^4 + 5x\right]$$

Now sum of odd degree terms will be

$$\Rightarrow 2\left[1-10+5+5\right] = 2 \text{ Ans}$$

EXERCISE (JM)

10. If the fractional part of the number $\frac{2^{403}}{15}$ is $\frac{k}{15}$, then k is equal to : **[JEE(Main)- 2019]**

(1) 14 (2) 6 (3) 4 (4) 8

Solⁿ:-

$$\Rightarrow \frac{2^{403}}{15} = \frac{2^3\left(2^4\right)^{100}}{15}$$

$$\Rightarrow \frac{8}{15}(1+15)^{100}$$

$$\Rightarrow \frac{8}{15}\left[1 + {}^{100}C_1(15) + {}^{100}C_2\cdot(15)^2 + \cdots\ {}^{100}C_{100}(15)^{100}\right]$$

$$\Rightarrow \frac{8}{15}(1+15\lambda) = \frac{8}{15} + 8\lambda$$

$\because$ 8λ is integer

$\therefore$ fractional part of $\frac{2^{403}}{15}$ is $\frac{8}{15} \Rightarrow k=8$

11. The coefficient of t^4 in the expansion of $\left(\frac{1-t^6}{1-t}\right)^3$ is **[JEE(Main)- 2019]**

(1) 12 (2) 15 (3) 10 (4) 14

Ans. (2)

Sol. $(1 - t^6)^3 (1 - t)^{-3}$

$(1 - t^{18} - 3t^6 + 3t^{12}) (1 - t)^{-3}$

$\Rightarrow$ cofficient of t^4 in $(1 - t)^{-3}$ is

${}^{3+4-1}C_4 = {}^6C_2 = 15$

12. If $\sum_{r=0}^{25}\left\{ {}^{50}C_r \cdot {}^{50-r}C_{25-r} \right\} = K\left({}^{50}C_{25}\right)$, then K is equal to : **[JEE(Main)- 2019]**

(1) $2^{25}-1$ (2) $(25)^2$ (3) 2^{25} (4) 2^{24}

Ans. (3)

Sol. $\sum_{r=0}^{25} {}^{50}C_r \cdot {}^{50-r}C_{25-r}$

$$=\sum_{r=0}^{25}\frac{50!}{r!\,(50-r)!}\times\frac{(50-r)!}{(25)!\,(25-r)!}$$

$$=\sum_{r=0}^{25}\frac{50!}{25!\,25!}\times\frac{25!}{(25-r)!\,(r!)}$$

$$= {}^{50}C_{25}\sum_{r=0}^{25} {}^{25}C_r = \left(2^{25}\right)\,{}^{50}C_{25}$$

$\therefore\ K = 2^{25}$

13. The sum of the real values of x for which the middle term in the binomial expansion of $\left(\frac{x^3}{3}+\frac{3}{x}\right)^8$ equals 5670 is : **[JEE(Main)- 2019]**

(1) 6 (2) 8 (3) 0 (4) 4

Ans. (3)

Sol. $T_5 = {}^8C_4 \frac{x^{12}}{81} \times \frac{81}{x^4} = 5670$

$\Rightarrow 70x^8 = 5670$

$\Rightarrow x = \pm\sqrt{3}$

14. The value of r for which $^{20}C_r\,^{20}C_0 + ^{20}C_{r-1}\,^{20}C_1 + ^{20}C_{r-2}\,^{20}C_2 + ... ^{20}C_0\,^{20}C_r$ is maximum, is **[JEE(Main)- 2019]**

(1) 20 (2) 15 (3) 11 (4) 10

Ans. (1)

Sol. Given sum = coefficient of x^r in the expansion of $(1 + x)^{20}(1 + x)^{20}$,

which is equal to $^{40}C_r$

It is maximum when r = 20

15. Let $(x + 10)^{50} + (x - 10)^{50} = a_0 + a_1x + a_2x^2 + ... + a_{50}x^{50}$, for all $x \in R$, then $\frac{a_2}{a_0}$ is equal to:-

[JEE(Main)- 2019]

(1) 12.50 (2) 12.00 (3) 12.75 (4) 12.25

Ans. (4)

Sol. $(10 + x)^{50} + (10 - x)^{50}$

$\Rightarrow a_2 = 2.{}^{50}C_2 10^{48},\ a_0 = 2.10^{50}$

$$\frac{a_2}{a_0} = \frac{{}^{50}C_2}{10^2} = 12.25$$

EXERCISE (JM)

16. Let $S_n = 1 + q + q^2 + \ldots\ldots + q^n$ and $T_n = 1 + \left(\frac{q+1}{2}\right) + \left(\frac{q+1}{2}\right)^2 + \ldots\ldots + \left(\frac{q+1}{2}\right)^n$, where q is a real number and $q \neq 1$. If ${}^{101}C_1 + {}^{101}C_2.S_1 + \ldots\ldots + {}^{101}C_{101}.S_{100} = \alpha T_{100}$, then α is equal to :-**[JEE(Main)- 2019]**

(1) 2^{100} (2) 200 (3) 2^{99} (4) 202

Soln:-

$$S_n = \frac{q^{n+1}-1}{q-1},$$

$$T_n = \frac{\left(\frac{q+1}{2}\right)^{n+1}-1}{\frac{q+1}{2}-1} = \frac{1}{2^n}\frac{\left\{(q+1)^{n+1}-2^{n+1}\right\}}{(q-1)}$$

Now given expression can be written as

$$\Rightarrow \sum_{r=1}^{101} {}^{101}C_r \cdot S_{r-1} =$$

$$\Rightarrow \frac{1}{q-1}\sum_{1}^{101} {}^{101}C_r \cdot (q^r-1)$$

$$\Rightarrow \frac{1}{q-1}\left[\left\{(1+q)^{101}-1\right\}-\left(2^{101}-1\right)\right]$$

$$\Rightarrow \frac{1}{q-1}\left((1+q)^{101}-2^{101}\right) \times \frac{2^{100}}{2^{100}}$$

$$\Rightarrow 2^{100} \cdot T_{100}$$

$$\therefore \alpha = 2^{100}$$

Altier:

Ans. (1)

Sol. ${}^{101}C_1 + {}^{101}C_2S_1 + \ldots.. + {}^{101}C_{101}S_{100} = \alpha T_{100}$

${}^{101}C_1 + {}^{101}C_2(1 + q) + {}^{101}C_3(1 + q + q^2) + \ldots.. + {}^{101}C_{101}(1 + q + \ldots.. + q^{100})$

$$= 2\alpha \frac{\left(1-\left(\frac{1+q}{2}\right)^{101}\right)}{(1-q)}$$

$\Rightarrow {}^{101}C_1(1 - q) + {}^{101}C_2(1 - q^2) + \ldots.. + {}^{101}C_{101}(1 - q^{101})$

$$= 2\alpha\left(1-\left(\frac{1+q}{2}\right)^{101}\right)$$

$\Rightarrow (2^{101} - 1) - ((1 + q)^{101} - 1)$

$$= 2\alpha\left(1-\left(\frac{1+q}{2}\right)^{101}\right)$$

$$\Rightarrow 2^{101}\left(1-\left(\frac{1+q}{2}\right)^{101}\right) = 2\alpha\left(1-\left(\frac{1+q}{2}\right)^{101}\right)$$

$\Rightarrow \alpha = 2^{100}$

EXERCISE (JM)

17. The total number of irrational terms in the binomial expansion of $\left(7^{1/5}-3^{1/10}\right)^{60}$ is :

[JEE(Main)- 2019]

(1) 55 (2) 49 (3) 48 (4) 54

Ans. (4)

General term $T_{r+1} = {}^{60}C_r \; 7^{\frac{60-r}{5}} \; 3^{\frac{r}{10}}$

$\therefore$ for rational term, $r = 0, 10, 20, 30, 40, 50, 60$

$\Rightarrow$ no of rational terms $= 7$

$\therefore$ number of irrational terms $= 54$

18. If some three consecutive in the binomial expansion of $(x + 1)^n$ is powers of x are in the ratio 2 : 15 : 70, then the average of these three coefficient is :- **[JEE(Main)- 2019]**

(1) 964 (2) 625 (3) 227 (4) 232

Sol. $$\frac{{}^nC_{r-1}}{{}^nC_r} = \frac{2}{15}$$

$$\frac{\frac{n!}{(r-1)!(n-r+1)!}}{\frac{n!}{r!(n-r)!}} = \frac{2}{15}$$

$$\frac{r}{n-r+1} = \frac{2}{15}$$

$15r = 2n - 2r + 2$

$$\boxed{17r = 2n+2}$$

$$\frac{{}^nC_r}{{}^nC_{r+1}} = \frac{15}{70}$$

$$\frac{\frac{n!}{r!(n-r)!}}{\frac{n!}{(r+1)!(n-r-1)!}} = \frac{3}{14}$$

$$\frac{r+1}{n-r} = \frac{3}{14}$$

$14r + 14 = 3n - 3r$

$$\begin{array}{c} 3n - 17r = 14 \\ 2n - 17r = -2 \\ \hline n = 16 \end{array}$$

$17r = 34$, $r = 2$

${}^{16}C_1$, ${}^{16}C_2$, ${}^{16}C_3$

$$\frac{{}^{16}C_1 + {}^{16}C_2 + {}^{16}C_3}{3} = \frac{16+120+560}{3}$$

$$\frac{680+16}{3} = \frac{696}{3} = 232$$

19. The coefficient of x^{18} in the product $(1+x)(1-x)^{10}(1 + x + x^2)^9$ is : **[JEE(Main)- 2019]**

(1) –84 (2) 84 (3) 126 (4) –126

Sol. $(1 + x)(1 - x)^{10}(1 + x + x^2)^9$

$(1 - x^2)(1 - x^3)^9$

$\Rightarrow \quad 1 \times (1-x^3)^9 - x^2(1-x^3)^9$

$\Rightarrow \quad {}^9C_6 - 0$

(as there is no x^{16} in $(1-x^3)^9$)

$\Rightarrow \quad 84$ Ans

20. If $^{20}C_1 + (2^2)\ ^{20}C_2 + (3^2)\ ^{20}C_3 + ... + (20^2)^{20}C_{20} = A(2^\beta)$, then the ordered pair (A, β) is equal to: **[JEE(Main)- 2019]**

(1) (420, 18) (2) (380, 19) (3) (380, 18) (4) (420, 19)

Sol. $(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + ... + {}^nC_nx^n$

Diff. w.r.t. x

$\Rightarrow n(1+x)^{n-1} = {}^nC_1 + {}^nC_2(2x) + ...+ {}^nC_n n(x)^{n-1}$

Multiply by x both side

$\Rightarrow nx(1+x)^{n-1} = {}^nC_1 x + {}^nC_2(2x^2) +....+ {}^nC_n(n x^n)$

Diff w.r.t. x

$\Rightarrow n[(1+x)^{n-1} + (n-1)x(1+x)^{n-2}]$

$= {}^nC_1 + {}^nC_2 2^2x + ... {}^nC_n (n^2)x^{n-1}$

Put x = 1 and n = 20

$\Rightarrow {}^{20}C_1 + 2^2\ {}^{20}C_2 + 3^2\ {}^{20}C_3 + ... + (20)^2\ {}^{20}C_{20}$

$= 20 \times 2^{18}[2 + 19] = 420(2^{18}) = A(2^\beta)$

Altier:

General term of given exp.

$$\sum_{r=1}^{20} r^2 \cdot {}^{20}C_r$$

$$\Rightarrow \sum_{r=1}^{20} r\, \cancel{r} \cdot \frac{20}{\cancel{r}} \cdot {}^{19}C_{r-1}$$

$$\Rightarrow 20 \sum_{r=1}^{20} (r-1+1)\ {}^{19}C_{r-1}$$

$$\Rightarrow 20 \left[\sum_{r=1}^{20} (r-1)\ {}^{19}C_{r-1} + \sum_{r=1}^{20} {}^{19}C_{r-1}\right]$$

$$\Rightarrow 20 \left[19 \times \sum_{r=2}^{20} {}^{18}C_{r-2} + \sum_{r=1}^{20} {}^{19}C_{r-1}\right]$$

$$\Rightarrow 20 \left[19 \times 2^{18} + 2^{19}\right]$$

$$\Rightarrow 20 \times 2^{18}(19+2)$$

$$\Rightarrow 420 \times 2^{18}.$$

EXERCISE (JM)

21. The term independent of x in the expansion of $\left(\frac{1}{60}-\frac{x^8}{81}\right)\cdot\left(2x^2-\frac{3}{x^2}\right)^6$ is equal to :

[JEE(Main)- 2019]

(1) 36 (2) – 108 (3) – 72 (4) – 36

Sol. $\frac{1}{60}\left(2x^2-\frac{3}{x^2}\right)^6-\frac{1}{81}\cdot x^8\left(2x^2-\frac{3}{x^2}\right)^6$

its general term

$$\frac{1}{60}\,{}^6C_r 2^{6-r}(-3)^r(x)^{12-4r}-\frac{1}{81}\,{}^6C_r 2^{6-r}(-3)^r(x)^{20-4r}$$

for term independent of x, r for I[st] expression is 3 and r for second expression is 5

$\therefore$ term independent of x = – 36

$$\Rightarrow \frac{1}{60}\cdot{}^6C_3\cdot 2^3(-3)^3-\frac{1}{81}\cdot{}^6C_5\cdot 2\cdot(-3)^5$$

$$\Rightarrow -72+36$$

$\therefore$ term independent of $x = -36$.

EXERCISE (JA)

1. For r = 0, 1,....,10, let A_r, B_r and C_r denote, respectively, the coefficient of x^r in the expansions of $(1 + x)^{10}$, $(1 + x)^{20}$ and $(1 + x)^{30}$. Then $\sum_{r=1}^{10} A_r(B_{10}B_r - C_{10}A_r)$ is equal to -

(A) $B_{10} - C_{10}$ (B) $A_{10}\left(B_{10}^2 - C_{10}A_{10}\right)$ (C) 0 (D) $C_{10} - B_{10}$

[JEE 2010, 5]

$A_r = {}^{10}C_r$, $B_r = {}^{20}C_r$

$C_r = {}^{30}C_r$

$B_{10} = {}^{20}C_{10}$

$C_{10} = {}^{30}C_{10}$

$\sum_{r=1}^{10} A_r B_r = \sum_{r=1}^{10} {}^{10}C_r \, {}^{20}C_r$

$= {}^{10}C_1 \, {}^{20}C_1 + {}^{10}C_2 \, {}^{20}C_2 + \cdots + {}^{10}C_{10} \, {}^{20}C_{10}$

$= {}^{10}C_{10} \, {}^{20}C_0 + {}^{10}C_9 \, {}^{20}C_1 + {}^{10}C_8 \, {}^{20}C_2 + \cdots + {}^{10}C_0 \, {}^{20}C_{10} - 1$

= selection of 10 objects

from $(a_1, a_2, \ldots a_{10})(b_1, b_2, \ldots b_{20}) - 1$

$= {}^{30}C_{10} - 1$

$$\sum_{r=1}^{10} A_r A_r$$

$$= ({}^{10}C_1)^2 + ({}^{10}C_2)^2 + \cdots + ({}^{10}C_{10})^2$$

$$= {}^{10}C_0\,{}^{10}C_{10} + {}^{10}C_1\,{}^{10}C_9 + {}^{10}C_2\,{}^{10}C_8 + \cdots + {}^{10}C_{10}\,{}^{10}C_0 - 1$$

= selection of 10 objects from $(a_1, a_2, \ldots a_{10})(b_1, b_2, \ldots b_{10}) - 1$

$$= {}^{20}C_{10} - 1$$

$$\sum_{r=1}^{10} A_r (B_{10} B_r - C_{10} A_r)$$

$$= B_{10} \sum A_r B_r - C_{10} \sum A_r A_r$$

$$= B_{10}({}^{30}C_{10} - 1) - C_{10}({}^{20}C_{10} - 1)$$

$$= C_{10} - B_{10}$$

Ans- D

2. The coefficients of three consecutive terms of $(1 + x)^{n+5}$ are in the ratio 5 : 10 : 14. Then n =

[JEE (Advanced) 2013, 4M, –1M]

$^{n+5}C_{r-1}$, $^{n+5}C_r$, $^{n+5}C_{r+1}$

are 3 consecutive co-efficients

Now $$\frac{^{n+5}C_{r-1}}{^{n+5}C_r} = \frac{5}{10}$$

$$\Rightarrow \frac{\cancel{(n+5)!} \times r!\,\cancel{(n+5-r)!}}{\cancel{(r-1)!}\,(n+6-r)!\,\cancel{(n+5)!}} = \frac{1}{2}$$

$$\Rightarrow 2r = n+6-r$$

$$\Rightarrow 3r = n+6 \quad \text{---- (1)}$$

$$\frac{^{n+5}C_r}{^{n+5}C_{r+1}} = \frac{10}{14}$$

$$\Rightarrow 5n - 12r + 18 = 0 \quad \text{-- (2)}$$

on solving (1) & (2)

$$\Rightarrow \underline{n=6}$$

3. Coefficient of x^{11} in the expansion of $(1 + x^2)^4(1 + x^3)^7(1 + x^4)^{12}$ is -

(A) 1051 (B) 1106 (C) 1113 (D) 1120

[JEE(Advanced)-2014, 3(–1)]

Co-eff. of x^{11} in

$(1 + 4x^2 + 6x^4 + 4x^6 + x^8) \times$

$(1 + 7x^3 + {}^7C_2 x^6 + {}^7C_3 x^9 + {}^7C_4 x^{12} + \cdots) \times$

$(1 + 12x^4 + {}^{12}C_2 x^8 + {}^{12}C_3 x^{12} + \cdots)$

$= 7 \times {}^{12}C_2 + 4 \times {}^7C_3 + 6 \times 7 \times 12 + 1 \times 7$

$= 462 + 140 + 504 + 7$

$= 1113$

Ans. C

4. The coefficient of x^9 in the expansion of $(1+x)(1+x^2)(1+x^3)\ldots(1+x^{100})$ is **[JEE 2015, 4M, –0M]**

co-eff. of x^9 in

$(1+x)(1+x^2)\text{-----}(1+x^{100})$

$=$ co-eff. of x^9 in

$(1+x)(1+x^2)(1+x^3)(1+x^4)\text{---}(1+x^9)$

$= 1\cdot x^9 + x\cdot x^8 + x^2\cdot x^7 + x^3\cdot x^6 + x^4\cdot x^5$

$+ x\times x^2\times x^6 + x\times x^3\times x^5 + x\,x^2 x^3 x^4$

$= 8$

5. Let m be the smallest positive integer such that the coefficient of x^2 in the expansion of $(1 + x)^2 + (1 + x)^3 + \ldots\ldots + (1 + x)^{49} + (1 + mx)^{50}$ is $(3n + 1)\,{}^{51}C_3$ for some positive integer n. Then the value of n is **[JEE(Advanced)-2016, 3(0)]**

co-eff. of x^2 in

$$(1+x)^2 + (1+x)^3 + \cdots + (1+x)^{49} + (1+mx)^{50}$$

$$= {}^2C_2 + {}^3C_2 + {}^4C_2 + \cdots + {}^{49}C_2 + {}^{50}C_2 m^2$$

$$= {}^3C_3 + {}^3C_2 + {}^4C_2 + \cdots + {}^{49}C_2 + {}^{50}C_2 m^2$$

$$\left\{ \begin{array}{l} {}^2C_2 = 1 = {}^3C_3 \\ {}^nC_r + {}^nC_{r-1} = {}^{n+1}C_r \end{array} \right\}$$

$$= {}^4C_3 + {}^4C_2 + \cdots + {}^{49}C_2 + {}^{50}C_2 m^2$$

$$= {}^{50}C_3 + {}^{50}C_2 m^2$$

from question

$$(3n+1)\,{}^{51}C_3 = {}^{50}C_3 + {}^{50}C_2 m^2$$

$$\Rightarrow (3n)\,{}^{51}C_3 = \left({}^{50}C_3 - {}^{51}C_3\right) + {}^{50}C_2 m^2$$

$$= {}^{50}C_2 m^2 - {}^{50}C_2$$

$$\frac{(\cancel{3}n)(51)!}{3!\,\cancel{(48)!}} = \frac{(m^2-1)\times\cancel{(50)!}}{\cancel{2!}\,\cancel{(48)!}}$$

$\cancel{3}$

$$n \times 51 = m^2 - 1$$

$$\Rightarrow m^2 = (51)n + 1$$

Least possible value of $m = 16$

$$\Rightarrow \underline{\underline{n = 5}}$$

6. Let $X = \left({}^{10}C_1\right)^2 + 2\left({}^{10}C_2\right)^2 + 3\left({}^{10}C_3\right)^2 + \ldots + 10\left({}^{10}C_{10}\right)^2$, where ${}^{10}C_r$, $r \in \{1, 2, \ldots, 10\}$ denote binomial coefficients. Then, the value of $\frac{1}{1430}X$ is _____ . **[JEE(Advanced)-2018, 3(0)]**

$$(1+x)^n = C_0 + C_1 x + C_2 x^2 + \cdots + C_n x^n$$

On differentiating with respect to x

$$\Rightarrow n\cdot(1+x)^{n-1} = C_1 + 2C_2 x + 3C_3 x^2 + \cdots + n\cdot C_n x^{n-1} \quad \text{—— (1)}$$

$$(1+x)^n = C_0 x^n + C_1 x^{n-1} + \cdots + C_{n-1} x + C_n \quad \cdots (2)$$

$$(1) \times (2)$$

$$\Rightarrow n\cdot(1+x)^{2n-1} = \{C_1 + 2C_2 x + \cdots + n\cdot C_n x^{n-1}\} \times \{C_0 x^n + C_1 x^{n-1} + \cdots + C_{n-1} x + C_n\}$$

Equate co-eff. of x^{n-1}

$$\Rightarrow C_1^2 + 2C_2^2 + \cdots + n\cdot C_n^2$$
$$= \text{co-eff. of } x^{n-1} \text{ in } n\cdot(1+x)^{2n-1}$$
$$= n \times {}^{2n-1}C_{n-1}$$

put $n = 10$

$$X = 10 \times {}^{19}C_9 = 2860 \times 323$$

$$\frac{1}{1430} \times X = \underline{646}$$

7. Suppose $\det\begin{bmatrix} \sum_{k=0}^{n} k & \sum_{k=0}^{n} {}^{n}C_k k^2 \\ \sum_{k=0}^{n} {}^{n}C_k k & \sum_{k=0}^{n} {}^{n}C_k 3^k \end{bmatrix} = 0$, holds for some positive integer n. Then $\sum_{k=0}^{n} \frac{{}^{n}C_k}{k+1}$ equals

[JEE(Advanced)-2019, 3(0)]

$$\sum_{K=0}^{n} K = \sum_{K=1}^{n} K = \frac{n(n+1)}{2}$$

$$\sum_{K=0}^{n} K \cdot {}^{n}C_K = \sum_{K=1}^{n} K \cdot {}^{n}C_K$$

$$= \sum_{K=1}^{n} n \cdot {}^{n-1}C_{K-1} = n \sum_{K=1}^{n} {}^{n-1}C_{K-1}$$

$$= n \cdot 2^{n-1}$$

$$\sum_{K=0}^{n} K^2\, {}^{n}C_K = \sum_{K=1}^{n} K \cdot (K \cdot {}^{n}C_K)$$

$$= \sum_{K=1}^{n} K \cdot (n \cdot {}^{n-1}C_{K-1})$$

$$= n \cdot \sum_{K=1}^{n} K \cdot {}^{n-1}C_{K-1} = n \cdot \sum_{K=1}^{n} (K-1+1)\, {}^{n-1}C_{K-1}$$

$$= n \sum_{K=1}^{n} {}^{n-1}C_{K-1} + n \sum_{K=1}^{n} (K-1)\, {}^{n-1}C_{K-1}$$

$$= n \cdot 2^{n-1} + n \cdot (n-1) \sum_{K=2}^{n} {}^{n-2}C_{K-2}$$

$$= n \cdot 2^{n-1} + n(n-1)\, 2^{n-2}$$

$$= n \cdot 2^{n-2} \left[2 + n - 1\right]$$

$$= n(n+1)\, 2^{n-2}$$

$$\sum_{K=0}^{n} {}^{n}C_{K}\, 3^{K} = (1+3)^{n} = 4^{n}$$

Now $\begin{vmatrix} \frac{n(n+1)}{2} & n(n+1)\, 2^{n-2} \\ n \cdot 2^{n-1} & 4^{n} \end{vmatrix} = 0$

$$\Rightarrow \begin{vmatrix} n+1 & n(n+1) \\ 1 & 4 \end{vmatrix} = 0$$

$$\Rightarrow 4n + 4 - n^2 - n = 0$$

$$\Rightarrow n^2 - 3n - 4 = 0 \Rightarrow \underline{\underline{n = 4}}$$

$$\sum_{K=0}^{4} \frac{{}^4C_K}{(K+1)}$$

$$\int_0^1 (C_0 + C_1 x + C_2 x^2 + C_3 x^3 + C_4 x^4)\,dx = \int_0^1 (1+x)^4\,dx$$

$$\Rightarrow C_0 + \frac{C_1}{2} + \frac{C_2}{3} + \frac{C_3}{4} + \frac{C_4}{5}$$

$$= \frac{2^5 - 1}{5}$$

$$\Rightarrow \sum_{K=0}^{4} \frac{{}^4C_K}{K+1} = \frac{31}{5}$$

$$= 6.20$$