Complex Number

Complete development of the number system can be summarised as

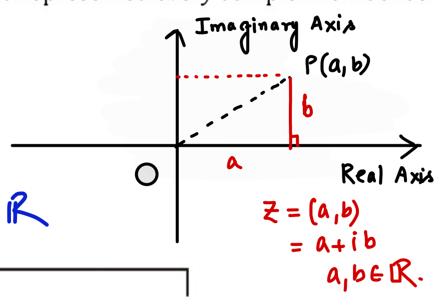
$$N \subset W \subset I \subset Q \subset \text{Real numbers} \subset C$$

Real numbers: Rational, Irrational

Every complex number z can be written as

$z = x + iy$ where $x, y \in R$ and $i = \sqrt{-1}$. x is called the real part of z and y is the imaginary part of complex. Note that the sign + does not indicate addition as normally understood, nor does the symbol i denote a number. These things are parts of the scheme used to express numbers of a new class and they signify the pair of real numbers (x, y) to form a single complex number.

Master Argand had done a systematic studies on complex numbers and represented every complex number as a set of ordered pair (x, y) on a plane called complex plane. All complex number lying on the real axis were called as purely real and these lying on imaginary axis as purely imaginary. In fact every complex can be classified as :



$z = x + iy$; $x, y \in R$

- Purely real / Real — if $y = 0$
- Purely imaginary — if $x = 0$
- Imaginary — if $y \neq 0$

Hence 0 + 0i is both a purely real as well as purely imaginary but not imaginary.'

Note : With every complex $z = x + iy$, $x, y \in R$ we associate the point P(x, y) in the complex plane. Conversely if Q(u, v) is any point in the complex plane, we associate the complex number $w = u + iv$ with Q.

Note that :

(i) The symbol i combines itself and with real numbers as per the rules of algebra together with $i^2 = -1$; $i^3 = -i$; $i^4 = 1$. Infact $i^{4n} = 1$, $n \in I$, $i^{4n+1} = i$, $i^{4n+2} = -1$, $i^{4n+3} = -i$

(ii) Every real number can also be treated as complex with its imaginary part zero. Hence there is one-one mapping between the set of complex numbers and the set of points on the complex plane.

$$\boxed{i + i^2 + i^3 + i^4 = 0.} \Rightarrow \left(i^{4n} + i^{4n+1} + i^{4n+2} + i^{4n+3}\right) = 0$$

$n \in I$

Q $\sum_{r=0}^{2021} (i)^r = \left(i^0 + i^1 + i^2 + \cdots + i^{2021}\right)$

$(i = \sqrt{-1})$

$$i^0 + \underbrace{(\qquad\qquad)}_{0} + i^{2021}$$

$i^{2021} = i^{2020} \cdot i$

$1 \times i$

$= i$

$(1 + i)$

Q $\sum_{r=1}^{11} (i)^{r!} = \underbrace{(i)^{\lfloor 1} + (i)^{\lfloor 2} + (i)^{\lfloor 3}}_{} + \underbrace{(i)^{\lfloor 4} + \cdots + (i)^{\lfloor 11}}_{}$

$= \left(i + i^2 + i^6\right) + (1 + 1 + \cdots + 1)$

$= (i - 1 - 1) + 8 = (6 + i)$ Ans

Q The equation $x(x^2+1)(x^2-5x+6)=0$ has

(A) Three real roots.

(B) Three imaginary roots.

(C) " purely imaginary roots.

(D) Five complex roots.

Soln

$$x(x^2+1)(x^2-5x+6)=0$$

$x=0$; $x^2+1=0$; $x^2-5x+6=0 \rightarrow \boxed{x=2;3}$

$\boxed{x=0}$; $x^2=-1 \Rightarrow \boxed{x=\pm i}$

ALGEBRA OF COMPLEX NUMBERS :

Fundamental operations with complex numbers :

(a) Addition $(a + bi) + (c + di) = (a + c) + (b + d)i$

(b) Subtraction $(a + bi) - (c + di) = (a - c) + (b - d)i$

(c) Multiplication $(a + bi)(c + di) = (ac - bd) + (ad + bc)i$

(d) Division $\frac{a+bi}{c+di} = \frac{a+bi}{c+di}\cdot\frac{c-di}{c-di} = \frac{ac+bd}{c^2+d^2} + \frac{bc-ad}{c^2+d^2}i$

Additive inverse : If $z + w = 0$, then w is additive inverse of z.

Multiplicative inverse : If $zw = 1$, then w is called multiplicative inverse (or reciprocal) of z and denoted by z^{-1}.

Note :

(a) Inequality in complex numbers are never talked. If $a + ib > c + id$ has to be meaningful $\Rightarrow b = d = 0$.

(b) In real number system if

$a^2 + b^2 = 0 \Rightarrow a = 0 = b$ but if z_1 and z_2 one complex numbers then $z_1^2 + z_2^2 = 0$ does not imply $z_1 = z_2 = 0$

$\underbrace{z_1 = 1+i}$; $\underbrace{z_2 = 1-i}$ $z_1^2 + z_2^2 = \cancel{1} + \cancel{i^2} + \cancel{2i} + \cancel{1} + \cancel{i^2} - \cancel{2i} = 0$.

(c) $\sqrt{a}\,\sqrt{b} = \sqrt{ab}$ only if atleast one of either a or b is non-negative.

(If a and b are positive real then $\sqrt{-a}\sqrt{-b} = -\sqrt{ab}$)

* $\sqrt{-2}\cdot\sqrt{-7} \neq \sqrt{14}$.

$(\sqrt{2}i)(\sqrt{7}i) = \sqrt{14}i^2 = -\sqrt{14}$.

(d) In case x is real then $|x| = \begin{bmatrix} x & \text{if} & x \ge 0 \\ -x & \text{if} & x < 0 \end{bmatrix}$ but in case of complex, $|z|$ altogether has a different meaning.

(e) If $z_1 z_2 = 0 \not\Rightarrow z_1 = 0$ OR $z_2 = 0$

EQUALITY IN COMPLEX NUMBERS :

Equalities however in complex numbers are meaningful. Two complex numbers z_1 and z_2 are said to be equal if

$\text{Re } z_1 = \text{Re } z_2$ and $\text{Im}(z_1) = \text{Im}(z_2)$

(i.e. they occupy the same position of complex plane)

eg $2 + ib = a - 3i$ where $a, b \in \mathbb{R}$

$a = 2$ & $b = -3$

$\underline{Q}$ $\left(i^{19} + \frac{1}{i^{25}}\right)^2 = \left(i^3 + \frac{1}{i}\right)^2 = (-i-i)^2$

$= (-2i)^2$

$= 2^2 i^2$

$= -4.$

$i = \sqrt{-1}$

$\underline{Q}$ $\left(\frac{1+i}{1-i}\right)^{16} + \left(\frac{1-i}{1+i}\right)^8 = \left(\frac{1+i}{1-i} \times \frac{1+i}{1+i}\right)^{16} + \left(\frac{1-i}{1+i} \times \frac{1-i}{1-i}\right)^8$

$= \left(\frac{(1+i)^2}{2}\right)^{16} + \left(\frac{(1-i)^2}{2}\right)^8$

$= \left(\frac{\cancel{1} + \cancel{i^2} + 2i}{2}\right)^{16} + \left(\frac{\cancel{1} + \cancel{i^2} - 2i}{2}\right)^8$

$= i^{16} + (-i)^8 = 2.$

Q If $\frac{(1+i)x-2i}{3+i}+\frac{(2-3i)y+i}{3-i}=i$, $x,y \in R$, find (x, y).

Sol^n

$$\left(\frac{(1+i)x-2i}{10}\right) \times (3-i) + \frac{((2-3i)y+i)}{10} \times (3+i) = i$$

$$3(1+i)x - 6i - i(1+i)x + 2i^2 + 3(2-3i)y + 3i + i(2-3i)y + i^2 = 10i$$

$$3x + x - 2 + 6y + 3y - 1 = 0.$$

$$3x - 6 - x - 9y + 3 + 2y = 10.$$

Q Find the least positive n $\in$ N if $\left(\frac{1+i}{1-i}\right)^n = 1$

Sol^n

$$\left(\frac{1+i}{1-i} \times \frac{1+i}{1+i}\right)^n = 1 \Rightarrow \left(\frac{(1+i)^2}{2}\right)^n = 1$$

$$i^n = 1.$$

$$n_{\text{smallest}} = 4.$$

Q If the equation $2z^2 + 2(i-1) = z - 10$ has a purely imaginary root, then find the other root.

Sol^n Let $z = i\alpha$; $\alpha \in \mathbb{R}$

$$2i^2\alpha^2 + 2(i-1) = i\alpha - 10.$$

$$-2\alpha^2 + 2i - 2 = i\alpha - 10.$$

$$\left.\begin{aligned} -2\alpha^2 - 2 &= -10. \\ 2 &= \alpha \end{aligned}\right\} ** \quad \Omega \quad \boxed{\alpha = 2}$$

$$2z^2 - z + (2i - 2 + 10) = 0 \quad \begin{matrix} \beta \\ i\alpha \end{matrix}$$

Ans

$$S.O.R = \frac{1}{2} = (2i + \beta) \Rightarrow \boxed{\beta = \frac{1}{2} - 2i}$$

Q. A,B,C are the points representing the complex numbers z_1, z_2, z_3 respectively. If z_1, z_2, z_3 are roots of equation $z^3 + z^2 + z + 1 = 0$, then find area of ΔABC.

Sol^n

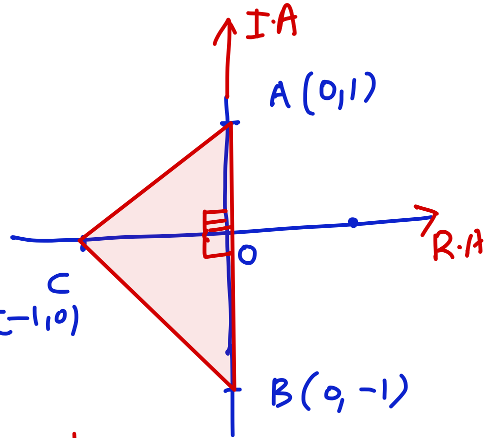
$$z^3 + z^2 + z + 1 = 0$$

$$(z^2+1)(z+1) = 0 \Rightarrow \quad z = i; -i; -1$$

A (0, 1)

B (0, −1)

C (−1, 0)



$$\Delta_{ABC} = \frac{1}{2} \times 2 \times 1 = 1. \text{ sq. units}$$

Q (a) If $z^2 + 2(1 + 2i)z - (11 + 2i) = 0$, find z in the form of a + ib.

(b) If $f(x) = x^4 - 4x^3 + 4x^2 + 8x + 44$, find $f(3 + 2i)$.

(a)

$$z = \frac{-\cancel{2}(1+2i) \pm \sqrt{\cancel{4}(1+2i)^2 + \cancel{4}(11+2i)}}{\cancel{2}}$$

$$\boxed{z = -(1+2i) \pm \sqrt{8+6i}}$$

$\sqrt{4} = +2$

$\sqrt{x^2} = |x|$

$\sqrt{2+3i} = \pm(\quad)$

HOW TO FIND SQUARE ROOT OF COMPLEX NO.

$\sqrt{8+6i} = a+ib$; $a, b \in \mathbb{R}$

$8+6i = a^2 - b^2 + 2abi$

$a^2 - b^2 = 8$; $ab = 3 \Rightarrow a, b > 0$; $a, b < 0$.

$(a^2+b^2)^2 = (a^2-b^2)^2 + 4a^2b^2$

$(a^2+b^2)^2 = 64 + 4 \times 9 = 100$

$a^2 + b^2 = 10.$

$a^2 - b^2 = 8$

$\left.\begin{array}{l} a^2 = 9 \Rightarrow a = \pm 3 \\ b^2 = 1 \Rightarrow b = \pm 1 \end{array}\right\}$

$\sqrt{8+6i} = \pm(3+i)$

* $\sqrt{8+6i} = \sqrt{(a+ib)^2}$ ($a, b \in \mathbb{R}$

$\boxed{8 = a^2 - b^2}$

$6 = 2ab$

$ab = 3 \Rightarrow$ $a = 3$, $b = 1$

$\sqrt{(3+i)^2}$

$\pm(3+i)$

if $a = 3$ then $b = 1$

if $a = -3$ then $b = -1$

(b) If $f(x) = x^4 - 4x^3 + 4x^2 + 8x + 44$, find $f(3 + 2i)$.

$\boxed{\text{Ans} \rightarrow 5}$

Sol^n

$x = 3+2i \Rightarrow (x-3)^2 = (2i)^2$

$x^2 + 9 - 6x = -4$

$\boxed{x^2 - 6x + 13 = 0}$ —(1)—

$x^4 - 4x^3 + 4x^2 + 8x + 44 = Q\,\underbrace{(x^2 - 6x + 13)}_{0} + R$

$\boxed{\text{Dividend} = \text{Divisor} \times \text{Quotient} + \text{Remainder}}$

$Q = x^2 + 2x + 3$

$x^2 - 6x + 13 \;\big)\; x^4 - 4x^3 + 4x^2 + 8x + 44$

$R = 5$

Q If sum of reciprocal of roots of equation $x^2 + ipx = 4x - i$ is $2 - qi$, $p, q \in R$, then find q.

Sol^n

$$x^2 + (ip-4)x + i = 0 \quad \begin{matrix} \alpha \\ \beta \end{matrix} \quad p, q \in \mathbb{R}$$

$$\alpha + \beta = -(ip-4)$$

$$\alpha\beta = i$$

$$\frac{1}{\alpha} + \frac{1}{\beta} = 2 - qi$$

$$\frac{\alpha+\beta}{\alpha\beta} = 2 - qi$$

$$\frac{-(ip-4)}{i} = 2 - qi$$

$$-ip + 4 = 2i + q$$

$$\left.\begin{matrix} p = -2 \\ q = 4 \end{matrix}\right\}$$

Q Let z be a Complex number satisfying equ $z^2 - (3+i)z + m + 2i = 0$ where $m \in \mathbb{R}$. Suppose the equation has a real root then find the non-real root?

Sol^n Let 'α' be the real root.

$$\alpha^2 - (3+i)\alpha + m + 2i = 0.$$

$$(\alpha^2 - 3\alpha + m) + i(2-\alpha) = 0 + 0i$$

$$2 - \alpha = 0 \quad \text{and} \quad \alpha^2 - 3\alpha + m = 0$$

$$\alpha = 2 \quad \& \quad m = 2$$

$$z^2 - (3+i)z + 2 + 2i = 0 \quad \begin{matrix} \nearrow \alpha \\ \searrow \beta \end{matrix}$$

$$\alpha\beta = 2(1+i)$$

$$2\beta = 2(1+i) \Rightarrow \beta = 1+i$$

Q. If $(4-3i)$ is one root of equation $(1+i)x^2 - (7+3i)x + 6+8i = 0$ then find the other root ?

Sol^n

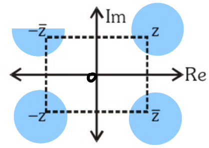
$$(4-3i)\alpha = \frac{6+8i}{(1+i)} \times \frac{(1-i)}{(1-i)}$$

$$(4-3i)\alpha = \frac{(6+8i)(1-i)}{2}$$

$$\alpha = \frac{(3+4i)(1-i)(4+3i)}{(4-3i)\times(4+3i)}$$

THREE IMPORTANT TERMS : CONJUGATE/MODULUS/ARGUMENT :

(a) CONJUGATE :



If $z = a + ib$ then its conjugate complex is obtained by changing the sign of its imaginary part and denoted by $\bar{z}$. i.e. $\bar{z} = a - ib$.

$\bar{z}$ can be treated as image of z in real axis.

$z = a + ib \;;\; \bar{z} = a - ib$

Note that :

(i) $z + \bar{z} = 2\,\text{Re}(z)$

(ii) $z - \bar{z} = 2i\,\text{Im}(z)$

$z + \bar{z} = 2a = 2\,\text{Re}(z)$

$z - \bar{z} = 2ib = 2i\,\text{Im}(z)$

** (iii) $z\bar{z} = a^2 + b^2 = |z|^2$

(iv) If z lies in 1^{st} quadrant then $\bar{z}$ lies in 4^{th} quadrant and $-\bar{z}$ in the 2^{nd} quadrant.

→ Plot $\left(\frac{1}{z}\right)$ on argand plane

$z\bar{z} = |z|^2$

$\frac{1}{z} = \frac{\bar{z}}{|z|^2}$

Note: ① If $z + \bar{z} = 0 \Rightarrow z$ is purely imaginary.

② If $z - \bar{z} = 0 \Rightarrow z$ is purely real./real.

*** Important Properties of Conjugate :**

If $z, z_1, z_2 \in C$, then ;

(i) $\overline{(\bar{z})} = Z$ **(ii)** $z + \bar{z} = 2\text{Re}(z)$ **(iii)** $z - \bar{z} = 2i\,\text{Im}(z)$ **(iv)** $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$

(v) $\overline{z_1 - z_2} = \bar{z}_1 - \bar{z}_2$ **(vi)** $\overline{z_1 z_2} = \bar{z}_1 \cdot \bar{z}_2$ **(vii)** $\overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}$; $z_2 \neq 0$

(viii) $z_1\bar{z}_2 + \bar{z}_1 z_2 = 2\,\text{Re}(z_1\bar{z}_2)$ **(ix)** $z_1\bar{z}_2 - \bar{z}_1 z_2 = 2i\,\text{Im}(z_1\bar{z}_2)$

$z_1 = x_1 + iy_1$

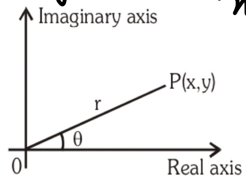
$z_2 = x_2 + iy_2$

$\bar{z}_2 = x_2 - iy_2$

$\bar{z}_1 = x_1 - iy_1$

(b) MODULUS :

$|z-0| =$ distance betⁿ origin & complex no. z



If P denotes a complex number $z = x + iy$ then $OP = |z| = \sqrt{x^2 + y^2}$

NOTE THAT $|z| \geq 0$. & $|z| = 0 \Rightarrow z = 0 + 0i$

All complex numbers having the same modulus lie on a circle with centre as origin and $r = |z|$.

$|z|$ represents the distance betⁿ z and origin

$|z_1 - z_2| =$ distance betⁿ 2 complex nos z_1 & z_2 on Argand plane.

Important Properties of Modulus :

(i) $|z| \geq 0$

(ii) $|z| \geq \text{Re}(z)$

(iii) $|z| \geq \text{Im}(z)$

(iv) $|z| = |\bar{z}| = |-z| = |-\bar{z}|$

* **(v)** * $z\bar{z} = |z|^2$

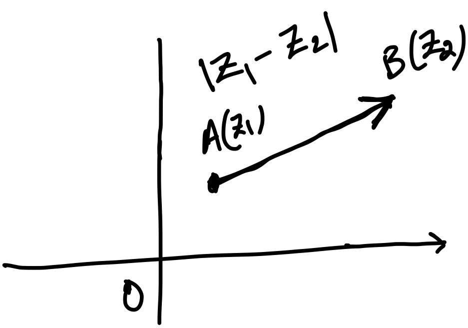
(vi) $|z_1 z_2| = |z_1| . |z_2|$

(vii) $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$, $z_2 \neq 0$

(viii) $|z^n| = |z|^n$

Complex Number	Conjugate Complex Number	Modulus of Complex Number
$1 + i\sqrt{3}$	$1 - i\sqrt{3}$	2
$-2 + 3i$	$-2 - 3i$	$\sqrt{13}$
$5i$	$-5i$	5
$\cos\theta + i\sin\theta$	$\cos\theta - i\sin\theta$	1
$\left(\dfrac{1+2i}{1-i}\right)$	$\dfrac{-1}{2} - \dfrac{3i}{2}$	$\sqrt{\dfrac{5}{2}}$

$$z = \frac{1+2i}{1-i} \times \frac{1+i}{1+i} = \frac{(1+i) + (2i + 2i^2)}{2} = \frac{-1+3i}{2} \rightarrow \bar{z} = \frac{-1-3i}{2}$$



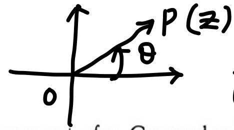
$z = a + ib$
$\bar{z} = a - ib$
$-z = -a - ib$
$-\bar{z} = -a + ib$.

Gold :-

$z\bar{z} = |z|^2$

* If $|z| = 1$ (unimodular) then $z\bar{z} = 1 \Rightarrow \bar{z} = \dfrac{1}{z}$ or $z = \dfrac{1}{\bar{z}}$
* If $|z| = K$ $(K > 0)$ then $z\bar{z} = K^2 \Rightarrow \bar{z} = \dfrac{K^2}{z}$.

ARGUMENT :



$\vec{OP} = z$

(I) General value of argument

If OP makes a directed angle θ with positive real axis then θ is called one of the argument of z. General values of argument of z are given by $2n\pi + \theta$, $n \in \mathbb{N}$.

Note :

(i) any two arguments differ by a multiple of 2π.

* (ii) by specifying the modulus and argument, a complex number is completely defined.

(iii) for the complex number $0 + 0i$ the argument is not defined and this is the only complex number which is completely defined by talking in terms of its modulus. i.e. $|z| = 0 \Rightarrow z = 0 + i0$

(iv) argument impart direction & modulus impart distance from origin.

* (II) Amplitude (Principal value of argument) :

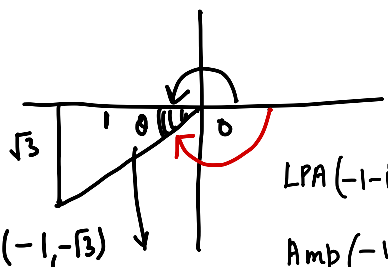
The unique value of θ such that $-\pi < \theta \le \pi$ is called principal value of argument. Unless otherwise stated, amp z refers to the principal value of argument.

(III) Least positive argument :

The value of θ such that $0 < \theta \le 2\pi$ is called the least positive argument.

There is one least positive argument in $(0, 2\pi]$

Complex Number	Principle value of argument	Least value of argument	General value of argument
$1+i\sqrt{3}$	$\pi/3$	$\pi/3$	$2n\pi+\theta$
$-2+3i$	$\pi-\tan^{-1}3/2$	$\pi-\tan^{-1}3/2$	
$-1-i\sqrt{3}$	$-2\pi/3$	$4\pi/3$	
$\sqrt{3}-i$	$-\pi/6$	$11\pi/6$	
4	0	2π	

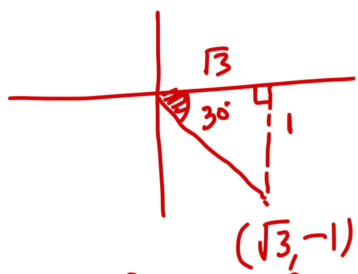


$\tan\theta = \sqrt{3}$

$LPA(-1-i\sqrt{3}) = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$

$Amp(-1-i\sqrt{3}) = -\frac{2\pi}{3}$

$z = \sqrt{3} - i$



$LPA(\sqrt{3}-i) = 330°$.

$Amp(\sqrt{3}-i) = -30°$.

Note :-

If $z = a + ib$; $a, b \in \mathbb{R}$

$$\boxed{\theta = \tan^{-1}\left|\frac{b}{a}\right|}$$

① 1^{st} Quad : $Amp(z) = \theta$.

② 2^{nd} Quad : $Amp(z) = \pi - \theta$.

③ 3^{rd} Quad : $Amp(z) = -(\pi - \theta)$

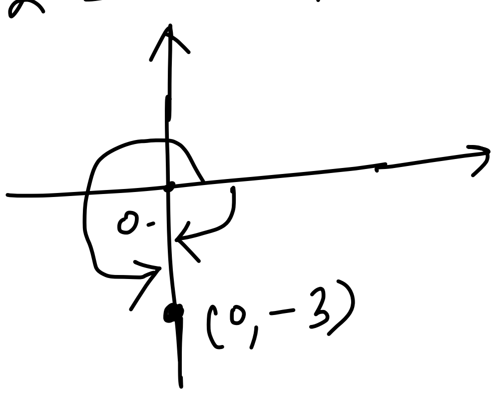
④ 4^{th} Quad : $Amp(z) = -\theta$

eg: $z = 5i$ $\qquad Amp(z) = \pi/2$

$LPA(z) = \pi/2$

$z = -3i$ $\qquad Amp(z) = -\pi/2$

$\angle PA(z) = 3\pi/2$



(IV) Important Properties of Amplitude :

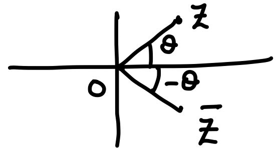
(i) $\text{amp}(z_1 . z_2) = \text{amp } z_1 + \text{amp } z_2 + 2k\pi.\ k \in \mathbb{Z}$

(ii) $\text{amp}\left(\dfrac{z_1}{z_2}\right) = \text{amp } z_1 - \text{amp } z_2 + 2k\pi;\ k \in \mathbb{Z}$

(iii) $\text{amp}(z^n) = n\ \text{amp}(z) + 2k\pi.$

where proper value of k must be chosen so that RHS lies in $(-\pi, \pi]$.

(iv) $\arg(\bar{z}) = -\arg(z)$



Note : For any complex number

$\text{amp } z + \text{amp of } (-\bar{z}) = \pi$ or $\text{amp } z + \text{amp}(-1) + \text{amp } \bar{z} = \pi$

Q Find principal value argument of $\dfrac{(1+i)(1+i\sqrt{3})}{i^5}$

Sol^n

$$z = \frac{z_1 z_2}{z_3}$$

$z_1 = 1+i$
$z_2 = 1+i\sqrt{3}$
$z_3 = i$

M-1

$$\text{amp } z = \text{amp } z_1 + \text{amp } z_2 - \text{amp } z_3 + 2K\pi.$$

$$= \left(\frac{\pi}{4} + \frac{\pi}{3} - \frac{\pi}{2}\right) + 2K\pi.$$

$$= \left(\frac{\pi}{4} - \frac{\pi}{6}\right) + 2K\pi$$

$$= \frac{\pi}{12} + 2K\pi \rightarrow \boxed{K=0}$$

$\frac{\pi}{12} \rightarrow (-\pi, \pi]$

M-2

$$z = \frac{(1+i)(1+i\sqrt{3})}{i}$$

$$z = \frac{1+i\sqrt{3}+i-\sqrt{3}}{i}$$

$$= \frac{(1-\sqrt{3})+i(\sqrt{3}+1)}{i} \times \left(\frac{i}{i}\right)$$

Q If $\arg(z_1 z_2) = \frac{5\pi}{6}$ and $\arg(z_1^2 \bar{z}_2) = \pi/6$ then $\arg(z_1) = ?$

Sol^n

$$\arg z_1 + \arg z_2 + 2K\pi = \frac{5\pi}{6}$$

$$2\arg z_1 - \arg z_2 + 2\lambda\pi = \pi/6$$

Add

$$3\arg(z_1) + 2(K+\lambda)\pi = \pi.$$

integer

$$\arg z_1 = \frac{\pi}{3} - \frac{2\pi}{3}(K+\lambda)$$

$\arg z_1 = \pi/3$ $\quad K+\lambda=0$

$\arg z_1 = -\pi/3$ $\quad K+\lambda=1$

$\arg z_1 = \pi$ $\quad K+\lambda=-1$

$(-\pi, \pi]$

Q If $z = \dfrac{(1+i)(1+2i)(1+3i)}{(1-i)(2-i)(3-i)}$. Find amp z & | z |.

Sol^n

$$z = \frac{z_1 z_2 z_3}{z_4 z_5 z_6} \Rightarrow |z| = \left|\frac{z_1 z_2 z_3}{z_4 z_5 z_6}\right|$$

M-1

By property

$$= \frac{|z_1||z_2||z_3|}{|z_4||z_5||z_6|}$$

$$= \frac{\sqrt{2}\cdot\sqrt{5}\cdot\sqrt{10}}{\sqrt{2}\,\sqrt{5}\cdot\sqrt{10}}$$

$$\boxed{|z| = 1} \ast\ast$$

M-2

$$z = \frac{(1+i)\cancel{(1+2i)}\cancel{(1+3i)}}{\cancel{(1+i)}\cancel{(2i+1)}(3i+1)} \cdot (i^3).$$

$$\boxed{z = -i}$$

$$|z| = 1$$

$$\text{amp}(z) = -\frac{\pi}{2}.$$

Q $|z_1+z_2|^2 + |z_1-z_2|^2 = 2\left[|z_1|^2+|z_2|^2\right]$ Give proof and its geometrical interpretation.

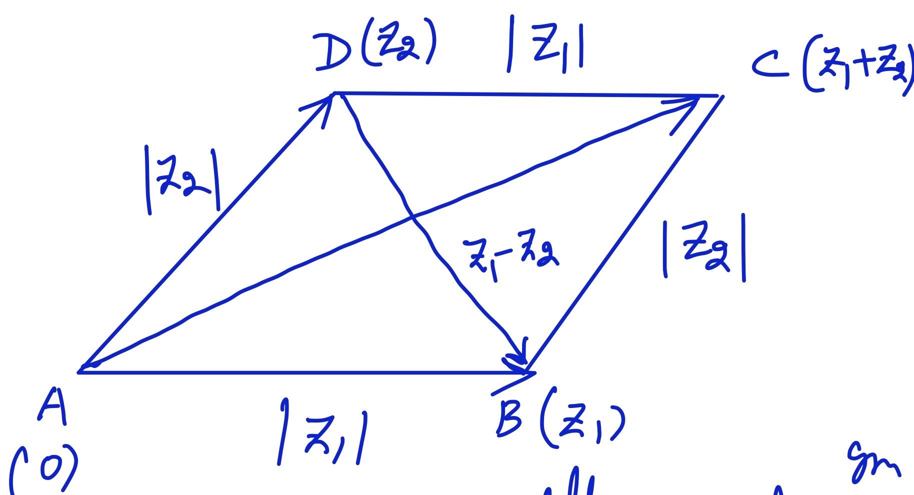
Solⁿ

$$\boxed{|z|^2 = z\bar{z}}$$

LHS: $(z_1+z_2)(\bar{z}_1+\bar{z}_2) + (z_1-z_2)(\bar{z}_1-\bar{z}_2)$

$z_1\bar{z}_1 + \cancel{z_1\bar{z}_2} + \cancel{z_2\bar{z}_1} + z_2\bar{z}_2 + z_1\bar{z}_1 - \cancel{z_1\bar{z}_2} - \cancel{z_2\bar{z}_1} + z_2\bar{z}_2$

$2\left[|z_1|^2+|z_2|^2\right] = \text{RHS}$ (H.P)



Sum of all the squares of the sides of ||gm is equal sum of squares of its diagonals.

Q If $|z_1| = |z_2| = |z_3| = ... = |z_n| = 1$, then prove that $|z_1 + z_2 + ... + z_n| = \left|\frac{1}{z_1}+\frac{1}{z_2}+....+\frac{1}{z_n}\right|$

Soln

$$z\bar{z} = |z|^2$$

$$z\bar{z} = 1 \Rightarrow \boxed{\bar{z} = \frac{1}{z}}$$

RHS:

$$\left|\frac{1}{z_1}+\frac{1}{z_2}+\cdots+\frac{1}{z_n}\right| = \left|\bar{z}_1+\bar{z}_2+\cdots+\bar{z}_n\right|$$

$$= \left|\overline{z_1+z_2+\cdots+z_n}\right|$$

We know

$$|z| = |\bar{z}|$$

$$= |z_1+z_2+\cdots+z_n|$$

$$= \text{LHS}$$

Q If z_1 & z_2 are unimodular and $z_1 z_2 \neq -1$

then $z = \dfrac{z_1 + z_2}{1 + z_1 z_2}$ is

~~(A)~~ Purely real.

(B) Purely imaginary.

(C) Imaginary.

~~(D)~~ Real.

Soln

$$\left.\begin{aligned} |z_1| = 1 &\Rightarrow z_1 \overline{z_1} = 1 \\ |z_2| = 1 &\Rightarrow z_2 \overline{z_2} = 1. \end{aligned}\right\}$$

$$z = \frac{z_1 + z_2}{1 + z_1 z_2}$$

$$\overline{z} = \overline{\left(\frac{z_1 + z_2}{1 + z_1 z_2}\right)} = \frac{\overline{z_1} + \overline{z_2}}{1 + \overline{z_1}\,\overline{z_2}}$$

$$\overline{z} = \frac{\frac{1}{z_1} + \frac{1}{z_2}}{1 + \frac{1}{z_1} \cdot \frac{1}{z_2}} = \frac{z_1 + z_2}{1 + z_1 z_2} = z$$

$\overline{z} = z \Rightarrow z$ is Real

Q If $|z_1| = |z_2| = |z_3| = 1$ and $z_1 + z_2 + z_3 = 0$ then find value of $(\underbrace{z_1^2 + z_2^2 + z_3^2}) = ?$

Soln

$$(z_1 + z_2 + z_3)^2 = 0^2$$

$$\underbrace{z_1^2 + z_2^2 + z_3^2}_{E} + 2(\overbrace{z_1 z_2 + z_2 z_3 + z_3 z_1}^{0}) = 0.$$

$$E + 2 z_1 z_2 z_3 \left(\frac{1}{z_3} + \frac{1}{z_1} + \frac{1}{z_2} \right) = 0$$

$$E + 2 z_1 z_2 z_3 \left(\bar{z}_3 + \bar{z}_1 + \bar{z}_2 \right) = 0$$

$$E + 2 z_1 z_2 z_3 \left(\underbrace{\overline{z_1 + z_2 + z_3}}_{0} \right) = 0$$

$$\overline{z_1 + z_2 + z_3} = \bar{0} = 0$$

$$E + 0 = 0$$

$$\therefore \boxed{E = 0} \text{ Ans}$$

HW

Q If $(a + ib)^5 = \alpha + i\beta$ (where $a, b, \alpha, \beta \in R$), then show that $(b + ia)^5 = \beta + i\alpha$

Q If $\left|\frac{6z-i}{2+3iz}\right| \le 1$ then find $|z|_{max} = ?$

Sol^n

$$\frac{|6z-i|}{|2+3iz|} \le 1 \Rightarrow |6z-i|^2 \le |2+3iz|^2$$

$$(6z-i)(\overline{6z-i}) \le (2+3iz)(\overline{2+3iz})$$

$$(6z-i)(6\bar{z}+i) \le (2+3iz)(2-3i\bar{z})$$

$$36|z|^2 + \cancel{6iz} - \cancel{6i\bar{z}} - i^2 \le 4 - \cancel{6i\bar{z}} + \cancel{6iz} + 9|z|^2$$

$$27|z|^2 \le 3 \Rightarrow |z|^2 \le \frac{1}{9}$$

$$\boxed{|z| \le \frac{1}{3}}$$

$$|z|_{max} = \frac{1}{3} \quad \text{Ans}$$

Q If z_1, z_2, z_3 are distinct complex numbers such that

$$\frac{3}{|z_1|}=\frac{4}{|z_2|}=\frac{5}{|z_3|} \text{ and } \left|\frac{9}{z_1}+\frac{16}{z_2}+\frac{25}{z_3}\right|=k\left|\frac{z_1+z_2+z_3}{z_1^2}\right|, \text{ then find k.}$$

Sol^n

$$\frac{3}{|z_1|}=\frac{4}{|z_2|}=\frac{5}{|z_3|}=\lambda \text{ (say)}$$

$$|z_1|=\frac{3}{\lambda} \quad ; |z_2|=\frac{4}{\lambda} \; ; \quad |z_3|=\frac{5}{\lambda}.$$

$$\downarrow$$

$$|z_1|^2=\frac{9}{\lambda^2}$$

$$z_1\bar{z}_1=\frac{9}{\lambda^2} \Rightarrow \frac{1}{z_1}=\frac{\lambda^2\bar{z}_1}{9}$$

$$\|^{ly} \quad \frac{1}{z_2}=\frac{\lambda^2\bar{z}_2}{16}$$

$$\frac{1}{z_3}=\frac{\lambda^2\bar{z}_3}{25}$$

$$\left|\cancel{9}\cdot\frac{\lambda^2\bar{z}_1}{\cancel{9}}+\cancel{16}\cdot\frac{\lambda^2\bar{z}_2}{\cancel{16}}+\cancel{25}\cdot\frac{\lambda^2\bar{z}_3}{\cancel{25}}\right|=k\frac{|z_1+z_2+z_3|}{|z_1|^2}$$

$$\lambda^2\cancel{\left|\overline{z_1+z_2+z_3}\right|}=k\frac{\cancel{|z_1+z_2+z_3|}}{|z_1|^2}$$

$$k=\lambda^2|z_1|^2=\cancel{\lambda^2}\cdot\frac{9}{\cancel{\lambda^2}} \Longrightarrow \boxed{k=9} \text{ Ans}$$

Representation of Complex no in different forms :-

(i) Ordered pair (x, y) where $x, y \in \mathbb{R}$.

(ii) **Cartesian form / Algebric form:**

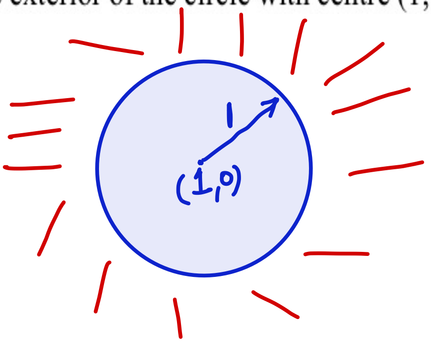
$z = x + iy$; Here $|z| = \sqrt{x^2+y^2}$; $\bar{z} = x - iy$ $\theta = \tan^{-1}\dfrac{y}{x}$

Generally this form is used in locus problems or while solving equations.

Example : $z = x+iy \Rightarrow \dfrac{1}{z} = \dfrac{1}{x+iy} \times \dfrac{x-iy}{x-iy} = \dfrac{x-iy}{x^2+y^2}$

(a) $\text{Re}\left(\dfrac{1}{z}\right) < \dfrac{1}{2}$; $\text{Re}\,\dfrac{x-iy}{x^2+y^2} < \dfrac{1}{2}$; $\dfrac{x}{x^2+y^2} < \dfrac{1}{2}$ $\Rightarrow$ $x^2 + y^2 - 2x > 0$.

Locus is the exterior of the circle with centre (1, 0) and radius = 1.



(b) Find the set of points on the complex plane for which $z^2 + z + 1$ is real and positive.

Sol^n Let $z = x+iy$; $x, y \in \mathbb{R}$

$z^2 = x^2 - y^2 + 2xyi$

$z^2+z+1 = x^2-y^2+2xyi + x+iy+1$

$= (x^2-y^2+x+1) + i\,(\underbrace{2xy+y}_{0})$.

Real & +ve

$(2x+1)y = 0$ and $\underbrace{x^2-y^2+x+1}_{E} > 0$

C-I $y = 0$

C-II $x = -1/2$

C-I If $y=0$

$E>0 \Rightarrow x^2-y^2+x+1>0$

$\underbrace{x^2+x+1}>0 \quad \forall x\in\mathbb{R}$

C-II If $x=-\frac{1}{2}$

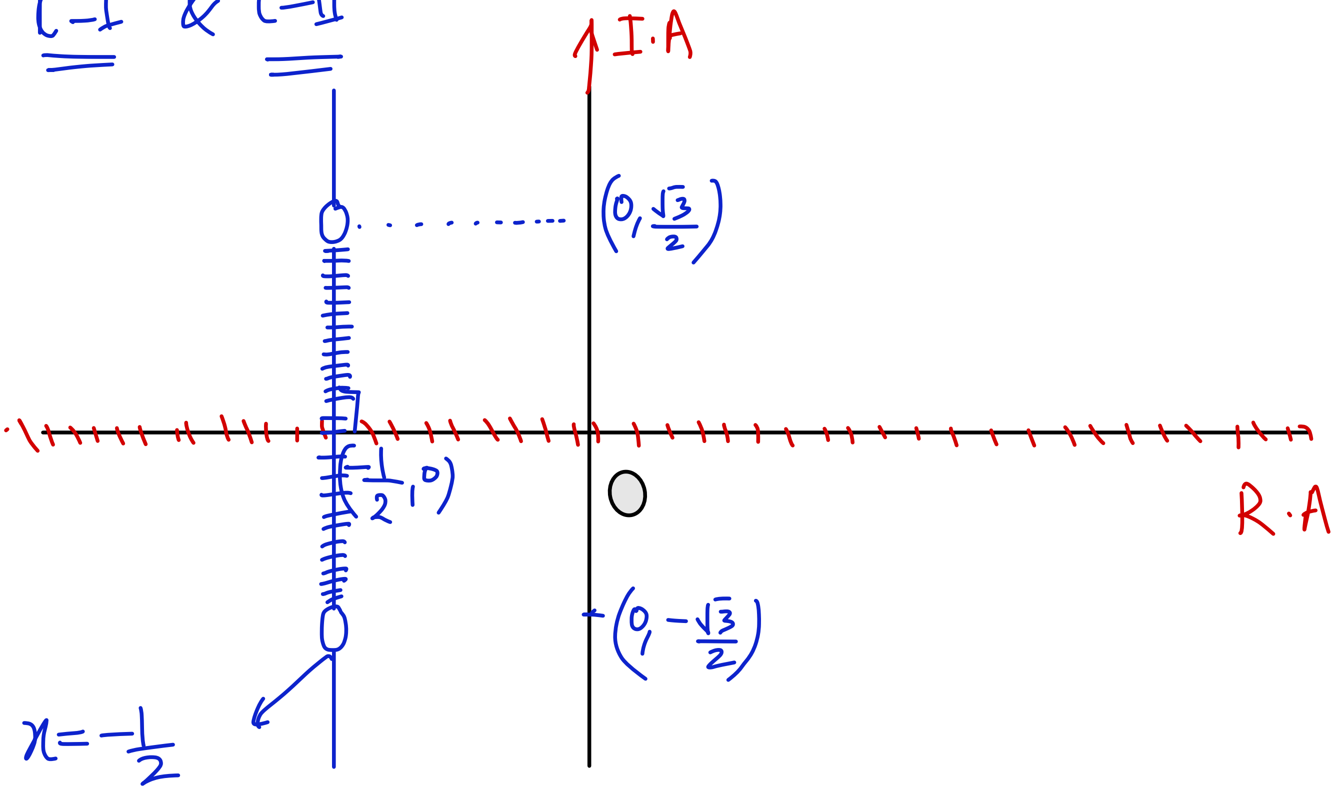
$E>0 \Rightarrow \left(-\frac{1}{2}\right)^2-y^2+\left(-\frac{1}{2}\right)+1>0$

$\frac{1}{4}-y^2-\frac{1}{2}+1>0.$

$\frac{3}{4}-y^2>0 \Rightarrow y^2-\left(\frac{\sqrt{3}}{2}\right)^2<0$

$\therefore y\in\left(-\frac{\sqrt{3}}{2},\frac{\sqrt{3}}{2}\right)$

From C-I & C-II



HW

Q If $(a + ib)^5 = \alpha + i\beta$ (where $a, b, \alpha, \beta \in R$), then show that $(b + ia)^5 = \beta + i\alpha$

Solⁿ

$$(a+ib)^5 = \alpha + i\beta$$

$$i^5 (a+ib)^5 = i^5 (\alpha + i\beta)$$

$$(ai + i^2 b)^5 = (i\alpha + i^2\beta)$$

$$(ai - b)^5 = (i\alpha - \beta)$$

$$(b - ia)^5 = \beta - i\alpha .$$

Take conjugate :-

$$(b+ia)^5 = \beta + i\alpha \qquad \underline{\underline{(H.P.)}}$$

Q Find locus represented by :-

(i) $|z| = 5$

$\downarrow$

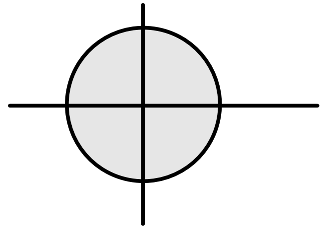
$|z-0| = 5$

$z = h+ik \Rightarrow |z| = \sqrt{h^2+k^2}$

$|z| = 5$

$h^2+k^2 = 25$

$x^2+y^2 = 25$

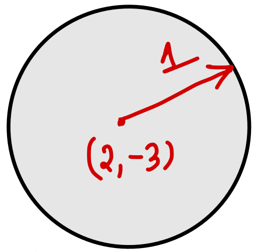


(ii) $|z-2+3i| = 1$

$z = h+ik$

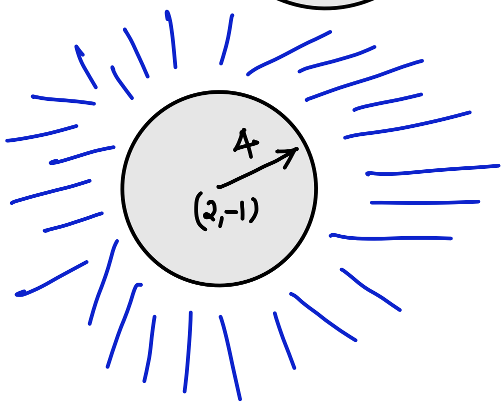
$\downarrow$

$|z-(2-3i)| = 1$



(iii) $|z-2+i| > 4$.

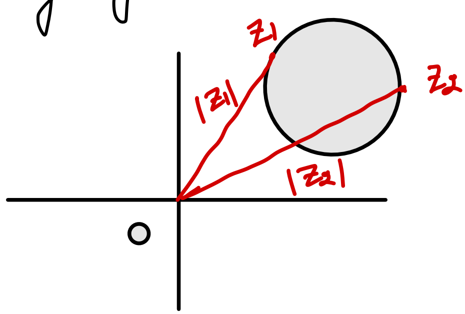
$|z-(2-i)| > 4$



True / False :-

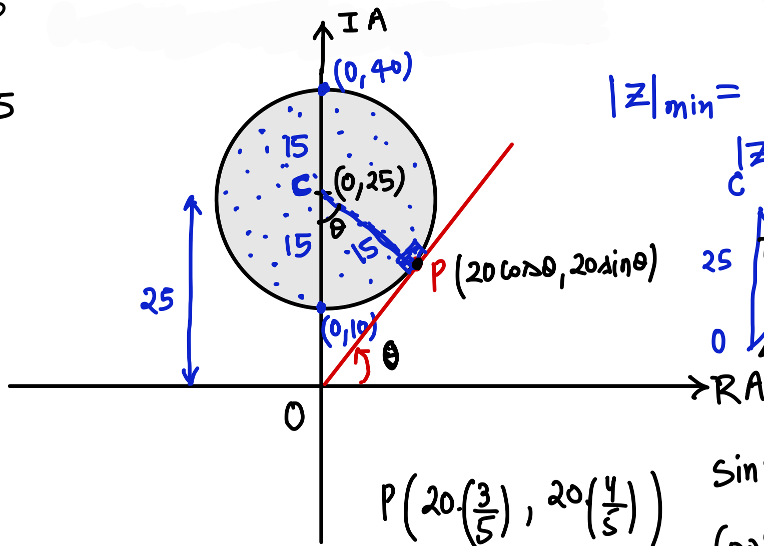
All Complex Numbers lying on the same Circle have same modulus.

FALSE



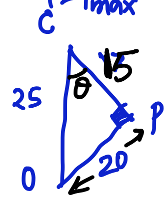
Q $|z - 25i| \le 15$, then find maximum and minimum value of $|z|$ and find complex number z having least positive argument. ?

$|z - (25i)| \le 15$



$|z|_{min} = 10$

$|z|_{max} = 40.$



$P\left(20\cdot\left(\frac{3}{5}\right), 20\left(\frac{4}{5}\right)\right)$

$\sin\theta = \frac{20}{25}$

$\cos\theta = \frac{15}{25}$

$P(12, 16)$

$\hookrightarrow z = 12 + 16i$

$\hookrightarrow$ LPA

(iii) **Trigonometric form / Polar form:**

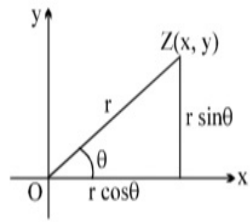
$z = x + iy = r(\cos\theta + i\sin\theta) = r\,\text{CiS}\,\theta$

where $|z| = r$; amp $z = \theta \in (-\pi, \pi]$

note that $(\text{CiS}\,\alpha)(\text{CiS}\,\beta) = \text{CiS}(\alpha + \beta)$

$(\text{CiS}\,\alpha)(\text{CiS}\,(-\beta) = \text{CiS}(\alpha - \beta)$

$$\frac{1}{(\text{CiS})\alpha} = (\text{CiS}\,\alpha)^{-1} = \text{CiS}(-\alpha)$$



Q write the complex number in polar form :-

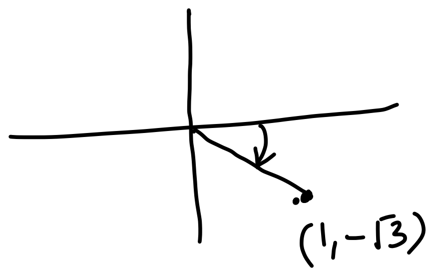
(1) $z = 1 - i\sqrt{3}$

(2) $z = -2(\cos 300^\circ + i\sin 300^\circ)$

(3) HW $z = 1 + \cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}$.

① $z = 1 - i\sqrt{3}$

$r = |z| = \sqrt{1+3} = 2$



$\text{amp}(z) = -\pi/3$

$z = 2\,\text{Cis}(-\pi/3) = 2(\cos(-\pi/3) + i\sin(-\pi/3))$

② $z = -2(\cos 300^\circ + i\sin 300^\circ)$

$z = -2\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) = -1 + i\sqrt{3}$

$\left.\begin{array}{l} |z| = 2 \\ \text{amp}(z) = \frac{2\pi}{3}. \end{array}\right\}$

$\boxed{z = 2\,\text{Cis}\left(\frac{2\pi}{3}\right)}$

(iv) **Exponential form:** $r = |z| \geq 0$; $\theta = amp(z)$ $(-\pi, \pi]$

Since $e^{ix} = \cos x + i \sin x$ hence $z = re^{i\theta}$ is the exponential representation.

Note that (a) $\cos x = \dfrac{e^{ix} + e^{-ix}}{2}$ and $\sin x = \dfrac{e^{ix} - e^{-ix}}{2i}$ are known as Eulers identities.

(b) $\cos ix = \dfrac{e^{x} + e^{-x}}{2} = \cosh x$ is always positive real $\forall\, x \in R$ and is ≥ 1.

and $\sin ix = \dfrac{e^{x} - e^{-x}}{2}\, i = i \sinh x$ is always purely imaginary.

$$\boxed{\bar{z} = re^{-i\theta}}$$

① $\cos\theta - i\sin\theta = e^{-i\theta}$

② $\sin\theta - i\cos\theta = -ie^{i\theta}$.

$\frac{1}{i}(i\sin\theta - i^2\cos\theta) = \frac{1}{i}(\cos\theta + i\sin\theta)$

Q Simplify :-

$$\left(\frac{\cos x + i\sin x}{\sin x - i\cos x}\right)^4$$

Sol^n

$$\left(\frac{\cancel{e^{ix}}}{-i\,\cancel{e^{ix}}}\right)^4 = 1$$

$e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\cdots$

$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \cdots$

$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right)$

$+ i\left(x - \frac{x^3}{3!} + \cdots\right)$

$\underbrace{e^{ix} = \cos x + i\sin x}$

Q. Convert into exponential form :-

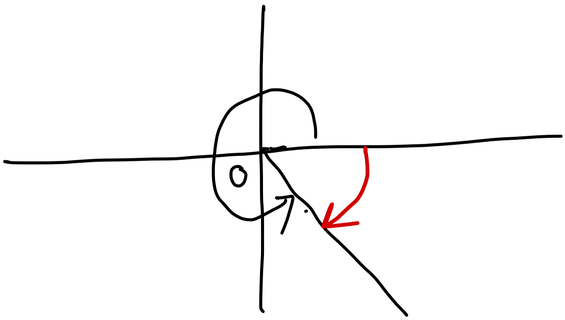
$$z = \sin\frac{\pi}{7} - i\cos\frac{\pi}{7}.$$

$$|z| = \sqrt{\sin^2(\,) + \cos^2(\,)} = 1$$

$$r = 1$$

Sol^n

$$z = \cos\left(\frac{3\pi}{2} + \frac{\pi}{7}\right) + i\sin\left(\frac{3\pi}{2} + \frac{\pi}{7}\right)$$



$$\text{amp}(z) = -\left(2\pi - \left(\frac{3\pi}{2} + \frac{\pi}{7}\right)\right)$$

$$\text{amp}(z) = -\left(\frac{\pi}{2} - \frac{\pi}{7}\right) = -\frac{5\pi}{14}. \checkmark$$

$$\boxed{z = 1.e^{-i\frac{5\pi}{14}}.}$$

$$z = re^{i\theta}$$

Q If $z = e^{e^{i\theta}}$ find $|z|$ & amp(z)

Sol^n

$$z = e^{e^{i\theta}} = e^{\cos\theta + i\sin\theta} = e^{\cos\theta} \cdot e^{i\sin\theta}$$

$$z = \underbrace{\left(e^{\cos\theta}\right)}_{r = |z|} \cdot \left(e^{i(\sin\theta)}\right)$$

$$\text{amp}(z) = \underbrace{\sin\theta}_{[-1, 1]}.$$

Note:-

Rem

$$\boxed{\begin{aligned}\sqrt{i} &= \pm\frac{1}{\sqrt{2}}(1+i)\\ \sqrt{-i} &= \pm\frac{1}{\sqrt{2}}(1-i)\end{aligned}}$$

$$\left.\begin{aligned} i &= e^{i\pi/2} \\ -i &= e^{-i\pi/2}\end{aligned}\right\}$$

$\sqrt{i} = a+ib \quad ; \quad a, b \in \mathbb{R}$

$i = a^2 - b^2 + 2abi$

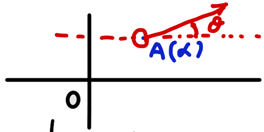
$a^2 - b^2 = 0 \quad \& \quad 2ab = 1.$

$(a^2+b^2)^2 = (a^2-b^2)^2 + 4a^2b^2$

$$\left.\begin{aligned} a^2+b^2 &= 1 \\ a^2-b^2 &= 0\end{aligned}\right\} \quad a^2 = \frac{1}{2} \; ; \; b^2 = \frac{1}{2}$$

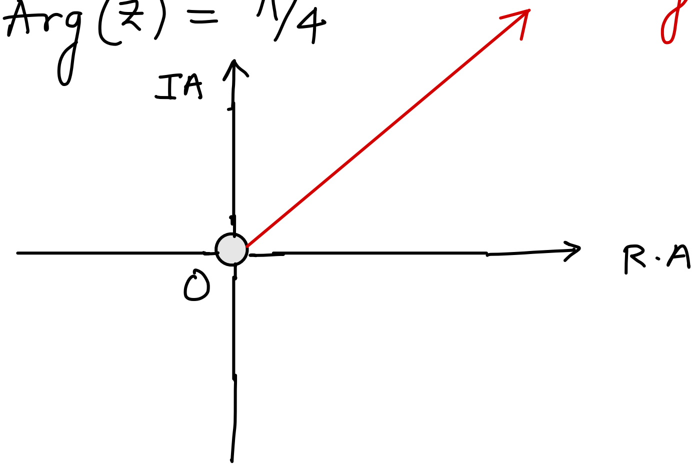
$$\left.\begin{aligned} a &= \frac{1}{\sqrt{2}} \Rightarrow b = \frac{1}{\sqrt{2}} \\ a &= -\frac{1}{\sqrt{2}} \Rightarrow b = -\frac{1}{\sqrt{2}}.\end{aligned}\right\}$$

Note:-

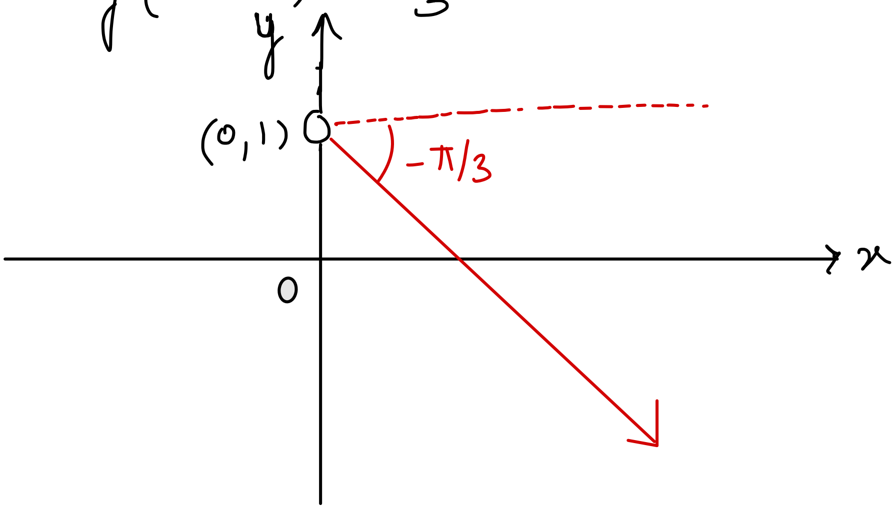


$Arg(z-\alpha)=\theta$ denotes a ray emanating from the point $A(\alpha)$ moving away from A.

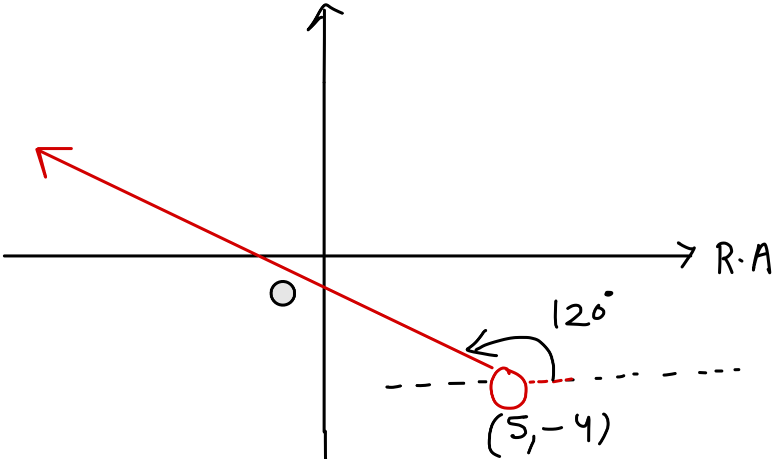
eg: $Arg(z)=\pi/4$ $\qquad$ $Arg(z-0)=\frac{\pi}{4}$.



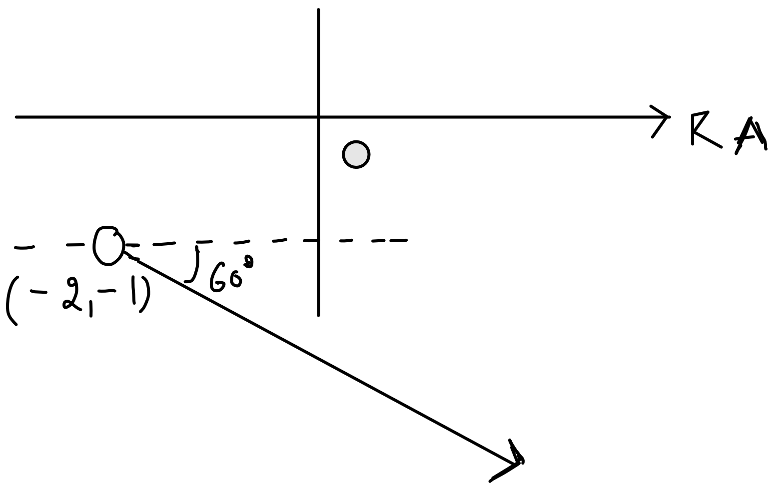
eg: $Arg(z-i)=\frac{-\pi}{3}$



eg: $\text{Arg}(z-5+4i)=\frac{2\pi}{3}$ → $\text{Arg}(z-(5-4i))=\frac{2\pi}{3}$

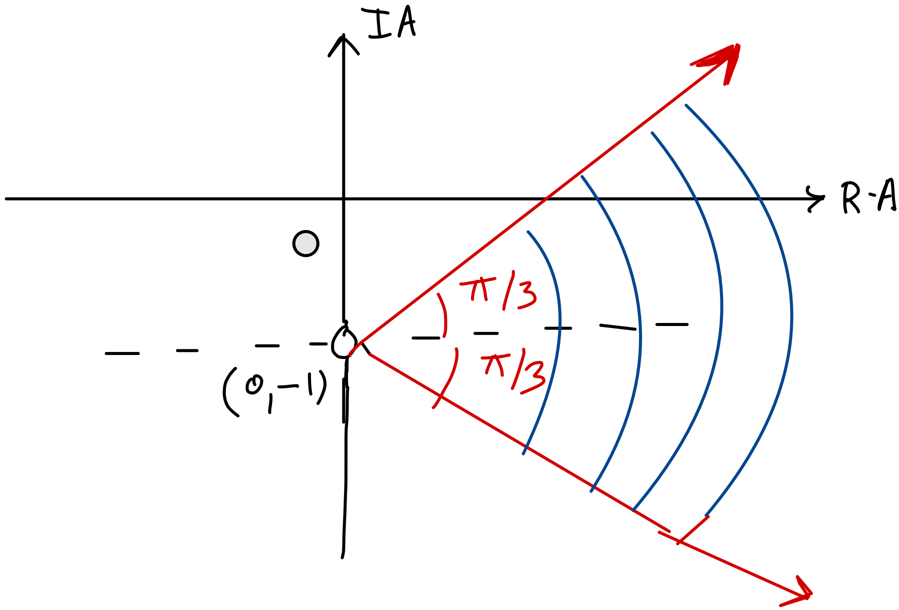


eg $\text{Arg}(z+2+i)=-\pi/3$ → $\text{Arg}(z-(-2-i))=-\frac{\pi}{3}$



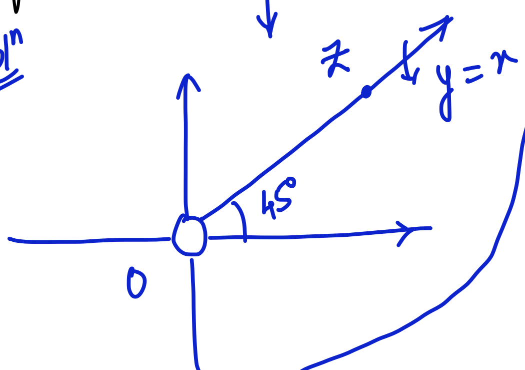
eg

$\left|Amp(z+i)\right| \le \frac{\pi}{3}$. $\quad \Rightarrow \quad -\frac{\pi}{3} \le Amp(z-(-i)) \le \frac{\pi}{3}$



Q If $\arg z = \frac{\pi}{4}$ and $|z+3-i| = 4$ then find z.

Soln



$\boxed{z = \lambda(1+i)}$; $\lambda > 0$

$|\lambda + \lambda i + 3 - i| = 4$

$\sqrt{(\lambda+3)^2 + (\lambda-1)^2} = 4$

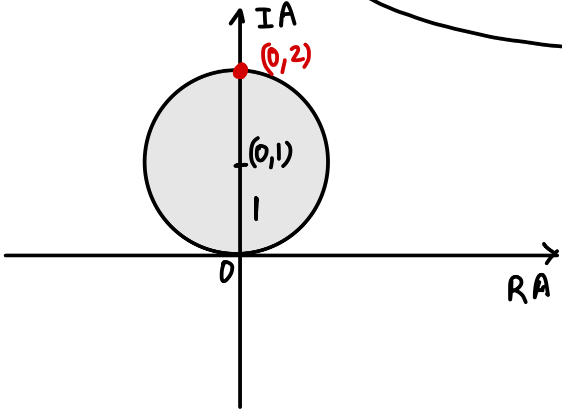
$2\lambda^2 + 10 + 4\lambda = 4^2$

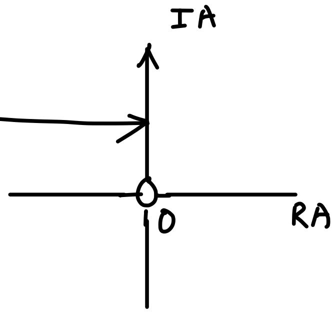
$\Leftarrow \lambda^2 + 2\lambda - 3 = 0$

$\boxed{\lambda = 1}$; $\cancel{-3}$

Q If $|z - i| = 1$ and $\text{Arg } z = \frac{\pi}{2}$, find the number of complex number.

Sol^n



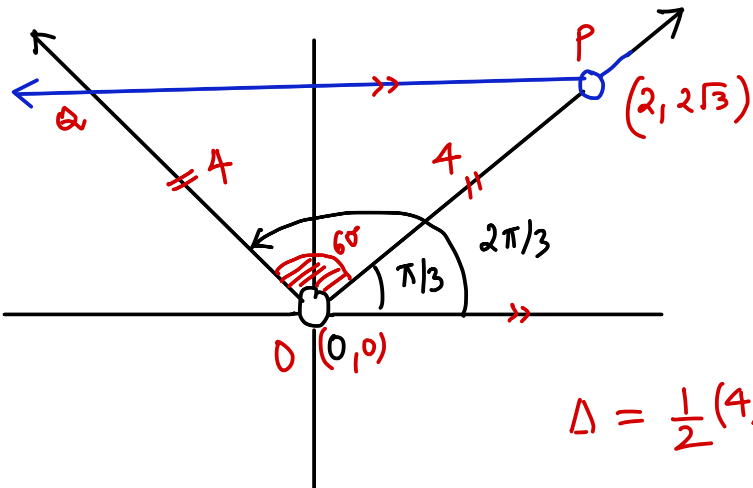


$\therefore \boxed{z = 2i}$

Q Find the area bounded by the curves $\text{Arg } z = \frac{\pi}{3}$,

$\text{Arg } z = \frac{2\pi}{3}$ & $\text{Arg }(z - 2 - 2\sqrt{3}i) = \pi$ on the complex plane.

$\downarrow$ $\text{Arg}(z - (2+2\sqrt{3}i)) = \pi$



$OP = \sqrt{4+12}$
$= 4$

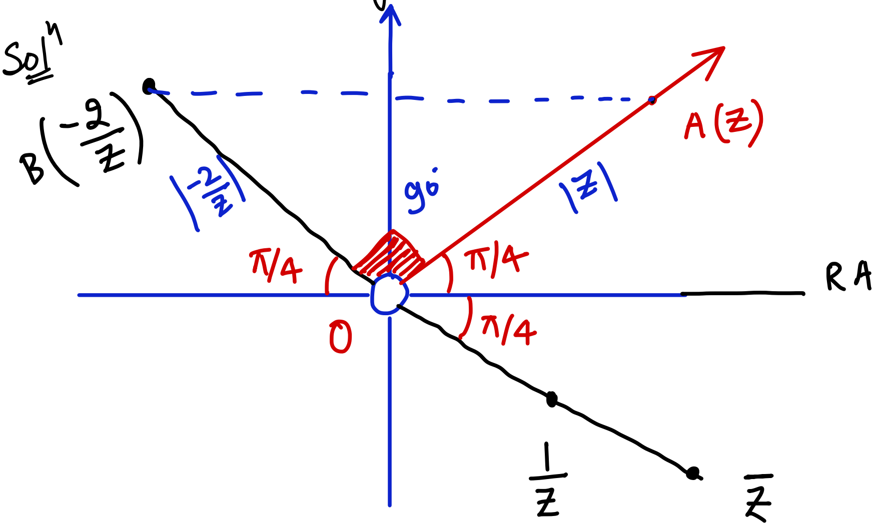
$\Delta = \frac{1}{2}(4)(4) \cdot \sin 60°$

$= 8 \cdot \left(\frac{\sqrt{3}}{2}\right) = 4\sqrt{3}$ sq. units

Q If $amp(z) = \frac{\pi}{4}$, then find the area of Δ formed by z, $\frac{-2}{z}$ and origin?

$$z \rightarrow \frac{1}{z} \rightarrow -\frac{2}{z}$$

Sol^n



$$z\bar{z} = |z|^2$$

$$\frac{1}{z} = \frac{\bar{z}}{|z|^2}$$

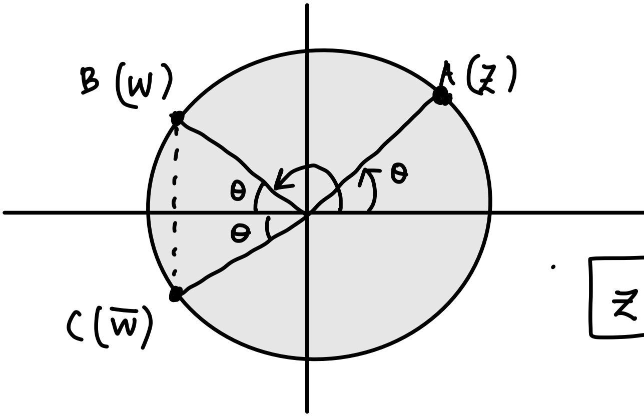
$$\Delta = \frac{1}{2}\cancel{|z|}\frac{2}{\cancel{|z|}} = 1 \text{ sq. units.}$$

Q If z and w are two non zero complex number such that $|z| = |w|$ and $\text{amp } z + \text{amp } w = \pi$, then prove that $z = -\bar{w}$.

$\text{amp}(z) = \theta$

$\text{amp}(w) = \pi - \theta.$

Sol^n



$$\boxed{z = -\bar{w}}$$

Q If $|\bar{z}| - 2z = 2i$ then find $|z|$ & $amp(z)$

Sol^n

$z = x + iy$

$\bar{z} = x - iy$

$$\sqrt{x^2+y^2} - 2(x+iy) = 2i$$

$$\sqrt{x^2+y^2} = 2x \quad \text{and} \quad -2y = 2$$

$x > 0$

$$\boxed{y = -1}$$

$$x^2 + y^2 = 4x^2$$

$$3x^2 = y^2$$

$$3x^2 = 1 \Rightarrow x = \frac{1}{\sqrt{3}} \text{ or } x = -\frac{1}{\sqrt{3}}$$

✓ xx

$$z = x + iy = \frac{1}{\sqrt{3}} + i(-1)$$

Q Prove that $|z_1+z_2|^2 = |z_1|^2 + |z_2|^2 + 2\,\mathrm{Re}(z_1\bar{z}_2)$

Sol^n

$$|z_1+z_2|^2 = (z_1+z_2)\overline{(z_1+z_2)}$$

LHS: $(z_1+z_2)(\bar{z}_1+\bar{z}_2) = z_1\bar{z}_1 + z_1\bar{z}_2 + z_2\bar{z}_1 + z_2\bar{z}_2$

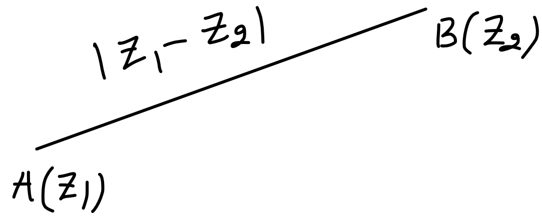
$= |z_1|^2 + |z_2|^2 + (z_1\bar{z}_2 + z_2\bar{z}_1)$

conjugate

$= |z_1|^2 + |z_2|^2 + 2\,\mathrm{Re}(z_1\bar{z}_2)$

$=$ RHS.

Distance :-



$z_1 (1-3i)$

$z_2 (-2+i)$

$z_1 = x_1 + i y_1$ & $z_2 = x_2 + i y_2$

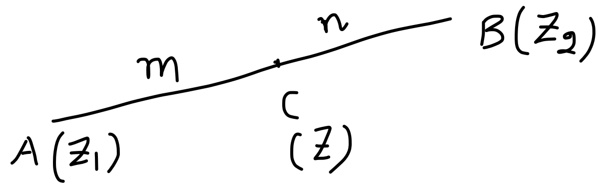
$$AB = |z_1 - z_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}.$$

$$|z_1 - z_2|$$

$$= \sqrt{3^2 + 4^2}$$

$$= 5.$$

Section Formula :-



$$\boxed{z = \frac{m z_2 + n z_1}{m+n}}$$

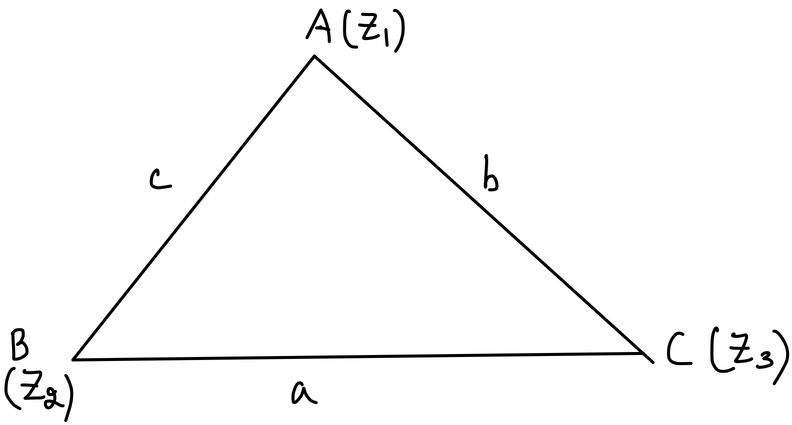
CENTROID, INCENTRE, ORTHOCENTRE & CIRCUMCENTRE OF A TRIANGLE ON A COMPLEX PLANE :

(i) Centroid 'G' $= \dfrac{z_1 + z_2 + z_3}{3} = z_G$

(ii) Incentre 'I' $= \dfrac{az_1 + bz_2 + cz_3}{a+b+c}$

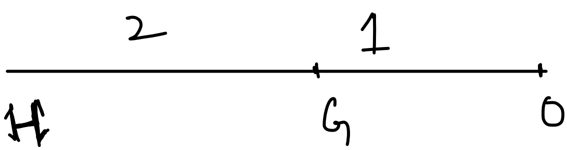
(iii) Orthocentre : $Z_H = \dfrac{z_1 \tan A + z_2 \tan B + z_3 \tan C}{\Sigma \tan A \text{ OR } \left(\prod \tan A\right)}$

(iv) Circumcentre : $Z_O = \dfrac{z_1 \sin 2A + z_2 \sin 2B + z_3 \sin 2C}{\Sigma \sin 2A \text{ OR } 4\left(\prod \sin A\right)}$



$a = |z_2 - z_3|$ $\qquad$ $c = |z_1 - z_2|$

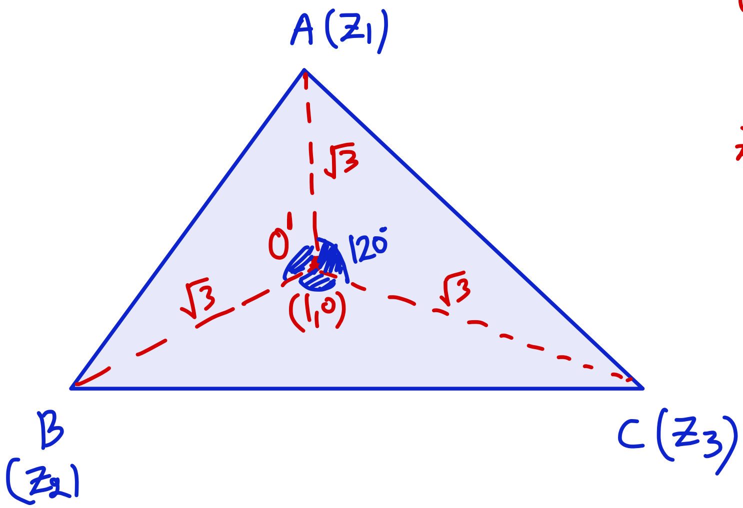
$b = |z_1 - z_3|$



$$\boxed{\frac{HG}{GO} = \frac{2}{1}}$$

Q If z_1, z_2, z_3 are vertices of Δ such that $|z_1 - 1| = |z_2 - 1| = |z_3 - 1| = \sqrt{3}$ and $z_1 + z_2 + z_3 = 3$, find area of Δ?

Sol^n



$O'(1,0)$ → Circumcentre

$$z_G = \frac{z_1 + z_2 + z_3}{3}$$

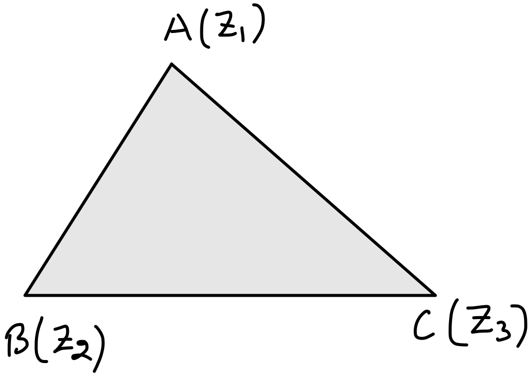
$z_G = 1$.

∴ Δ must be equilateral.

$$\Delta_{ABC} = 3\left(\frac{1}{2} \cdot \sqrt{3} \cdot \sqrt{3} \cdot \sin 120^\circ\right)$$

$$= 3 \cdot \frac{3}{2} \cdot \frac{\sqrt{3}}{2} = \frac{9\sqrt{3}}{4} \text{ sq. units}$$

Area of Δ :-



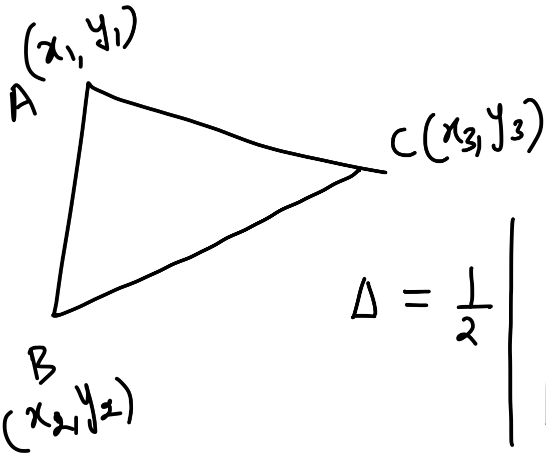
Rem

$$\Delta = \left| \frac{i}{4} \begin{vmatrix} 1 & 1 & 1 \\ z_1 & z_2 & z_3 \\ \bar{z}_1 & \bar{z}_2 & \bar{z}_3 \end{vmatrix} \right|$$

$z_1 = (x_1, y_1)$

$z_2 = (x_2, y_2)$

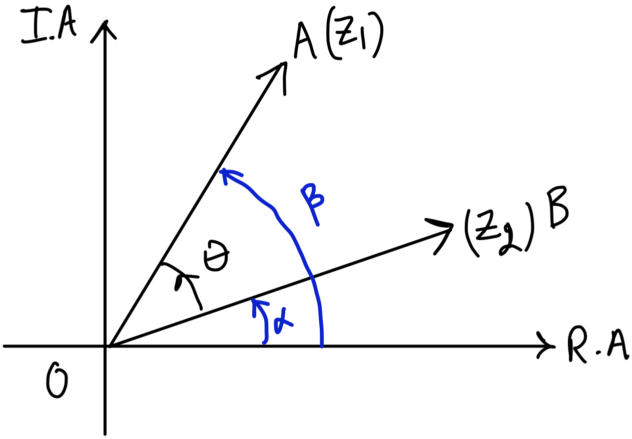
$z_3 = (x_3, y_3)$



$$\Delta = \frac{1}{2} \left| \begin{vmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{vmatrix} \right|$$

$$\left. \begin{aligned} x_1 &= \frac{z_1 + \bar{z}_1}{2} \\ y_1 &= \frac{z_1 - \bar{z}_1}{2i} \end{aligned} \right\}$$

Angle between 2 complex nos. :-



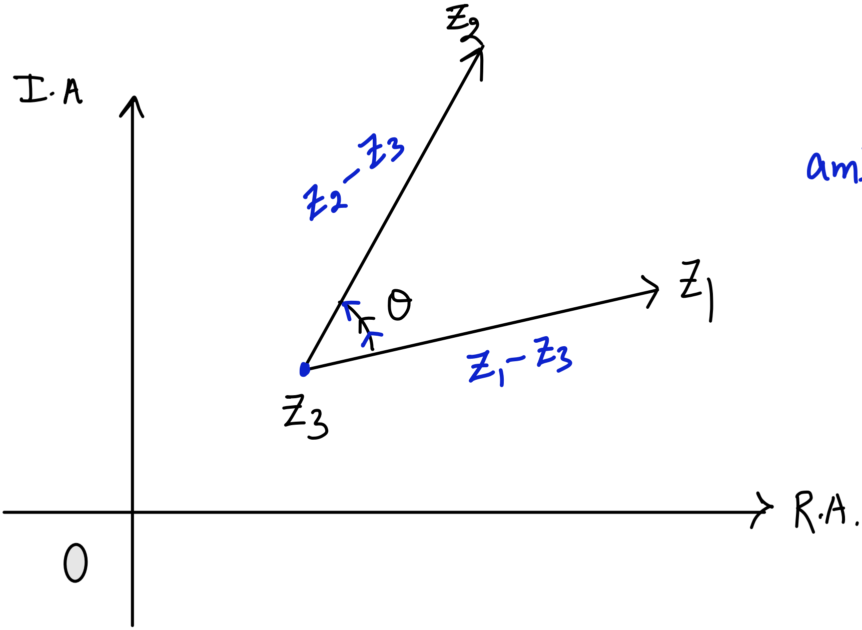
amp$(z_1) = \beta$
amp$(z_2) = \alpha$

$\beta - \alpha = \theta$.

amp z_1 − amp z_2 = θ.

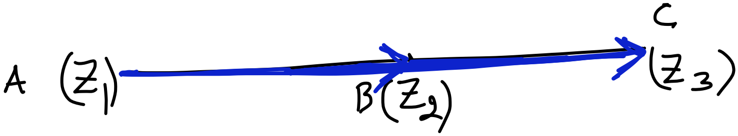
$$\boxed{\text{amp}\left(\frac{z_1}{z_2}\right) = \theta}$$

Rem



$$\text{amp}\left(\frac{z_2 - z_3}{z_1 - z_3}\right) = \theta.$$

Condition of Collinearity :-



M-1

$$AB + BC = AC$$

$$|z_1 - z_2| + |z_2 - z_3| = |z_1 - z_3|.$$

M-2

$$\begin{vmatrix} 1 & 1 & 1 \\ z_1 & z_2 & z_3 \\ \bar{z}_1 & \bar{z}_2 & \bar{z}_3 \end{vmatrix} = 0.$$

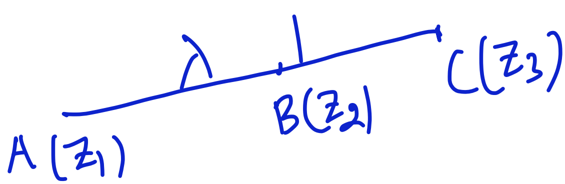
M-3

$$\operatorname{Arg}\left(\frac{z_3 - z_1}{z_2 - z_1}\right) = 0.$$

$\Rightarrow \quad \dfrac{z_3 - z_1}{z_2 - z_1}$ is P.R.

$$\Rightarrow \quad \frac{z_3 - z_1}{z_2 - z_1} = \overline{\left(\frac{z_3 - z_1}{z_2 - z_1}\right)}$$

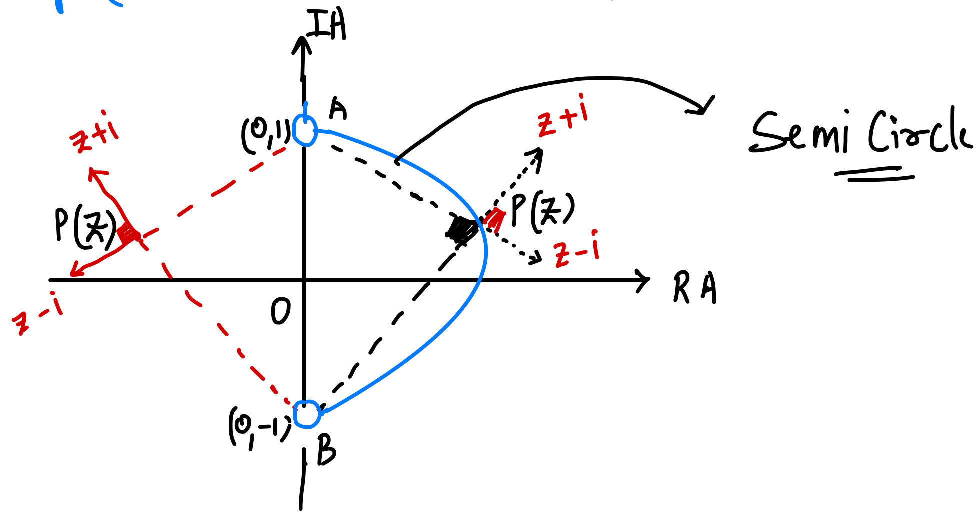
M-4



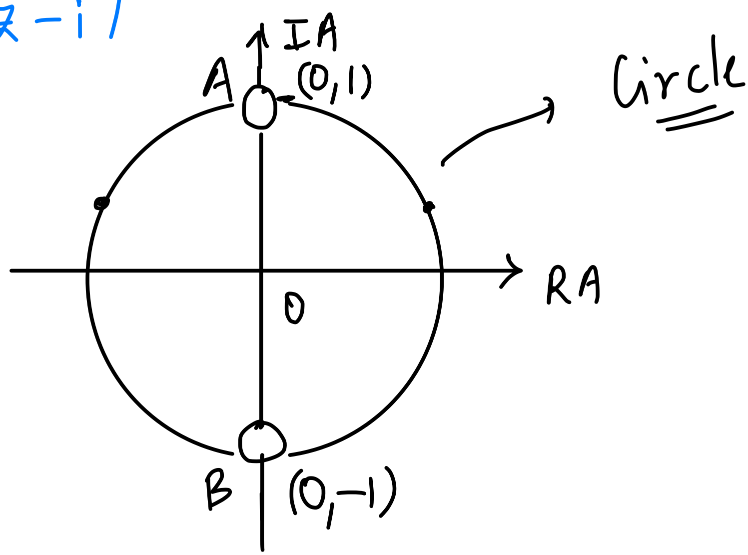
$$\boxed{z_2 = \frac{\lambda z_3 + z_1}{\lambda + 1}}$$

Q $Arg\left(\frac{z+i}{z-i}\right) = \frac{\pi}{2}$ Find Locus of z ?

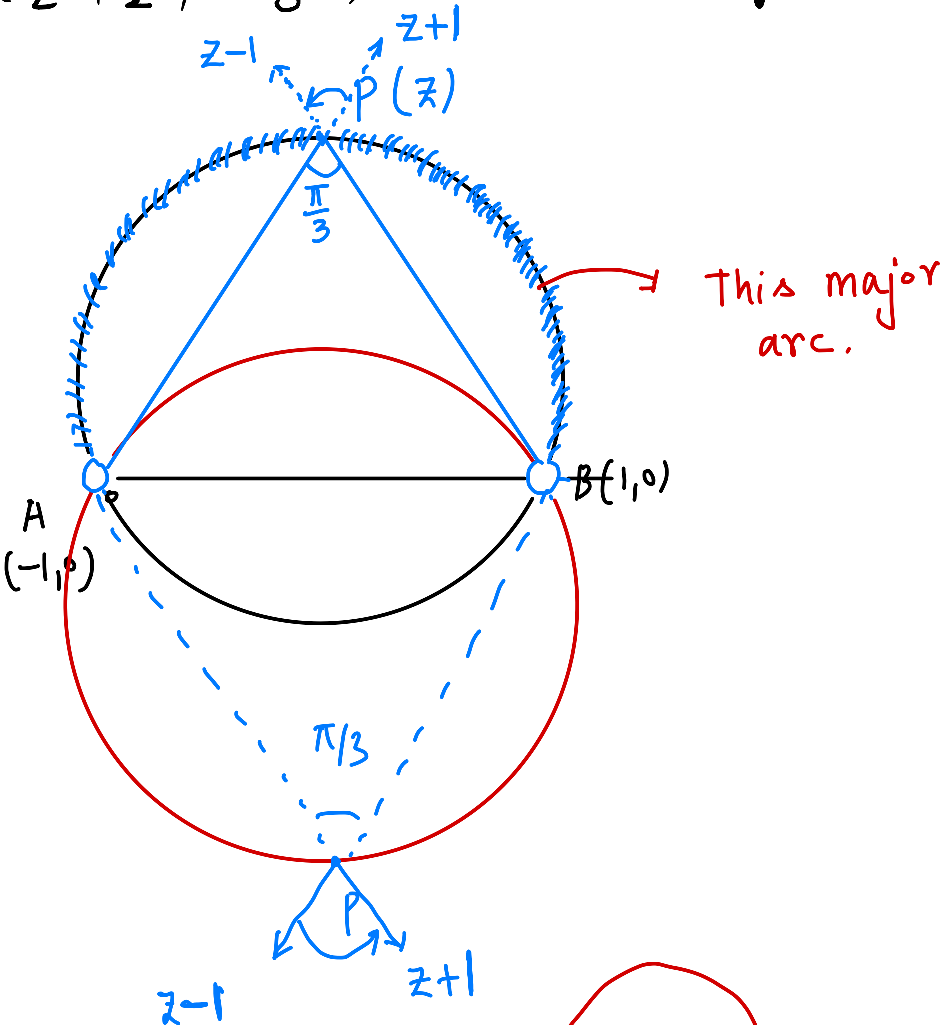
Sol^n $Arg(z+i) - Arg(z-i) = \frac{\pi}{2}$.



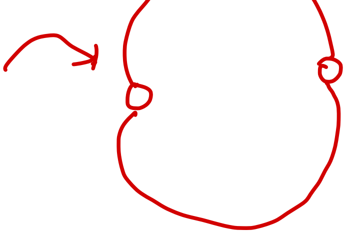
$Arg\left(\frac{z+i}{z-i}\right) = \pm \pi/2$



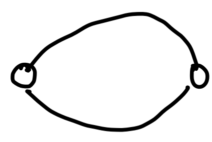
Q $\operatorname{Arg}\left(\frac{z-1}{z+1}\right) = \frac{\pi}{3}$, find locus of z ?



$\operatorname{Arg}\left(\frac{z-1}{z+1}\right) = \pm\frac{\pi}{3}$



$\operatorname{Arg}\left(\frac{z-1}{z+1}\right) = \pm\frac{2\pi}{3}$



Q $\left| \operatorname{Arg}\left(\frac{z-1}{z+1}\right) \right| = \frac{2\pi}{3}$

already done (at back)

Note :-

If $\arg\left(\frac{z-z_1}{z-z_2}\right)=\theta$ then locus of 'z' (where z_1 & z_2 are fixed).

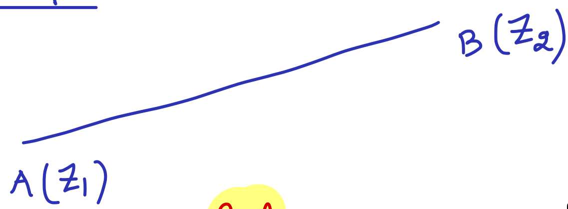
is a circular arc.

(i) If θ is acute then its major arc.

(ii) If θ is obtuse " " minor arc.

(iii) If θ is $\frac{\pi}{2}$ " " semi-circle.

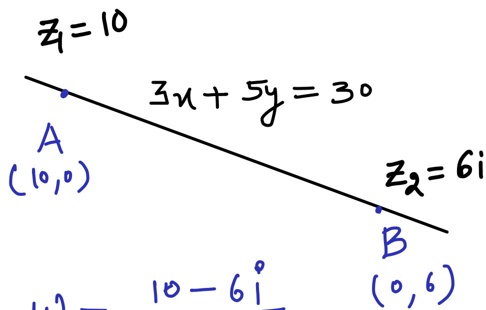
Complex slope :-



Rem

$$\boxed{\omega = \frac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2}}$$

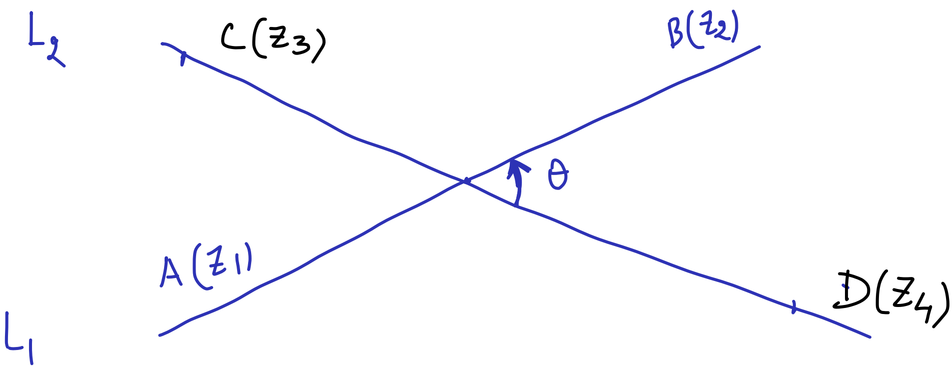
Complex slope



$$\omega = \frac{10 - 6i}{10 + 6i}$$

Complex slope

Condition for lines to be parallel or perpendicular :-



$$\operatorname{Arg}\left(\frac{z_2 - z_1}{z_4 - z_3}\right) = \theta.$$

$\boxed{\text{① If } L_1 \parallel L_2 \text{ then } \theta = 0 \text{ or } \pi.}$

$$\operatorname{Arg}\left(\frac{z_2 - z_1}{z_4 - z_3}\right) = 0 \text{ or } \pi.$$

$$\Rightarrow \quad \frac{z_2 - z_1}{z_4 - z_3} \text{ is P.R.}$$

$$\Rightarrow \quad \frac{z_2 - z_1}{z_4 - z_3} = \frac{\bar{z}_2 - \bar{z}_1}{\bar{z}_4 - \bar{z}_3}$$

Rem

$$\frac{z_2 - z_1}{\bar{z}_2 - \bar{z}_1} = \frac{z_4 - z_3}{\bar{z}_4 - \bar{z}_3} \Rightarrow \boxed{\omega_1 = \omega_2}$$

$\boxed{\text{② If } L_1 \perp L_2 \text{ then } \theta = \pm \pi/2.}$

$$\operatorname{Arg}\left(\frac{z_2 - z_1}{z_4 - z_3}\right) = \pm\frac{\pi}{2}$$

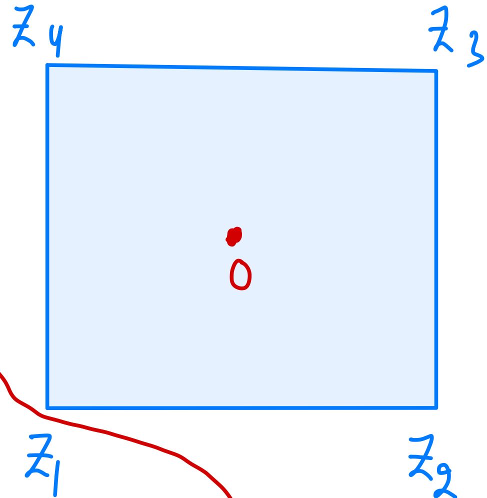
$$\Rightarrow \quad \frac{z_2 - z_1}{z_4 - z_3} \text{ is P.I i.e } \frac{z_2 - z_1}{z_4 - z_3} + \frac{\bar{z}_2 - \bar{z}_1}{\bar{z}_4 - \bar{z}_3} = 0$$

Rem

$$\Rightarrow \quad \boxed{\omega_1 + \omega_2 = 0}$$

Q

If z_1, z_2, z_3, z_4 in order are the vertices of the square taken in order then which of the following is(are) TRUE?



(a) $\dfrac{z_1 - z_3}{z_2 - z_4}$ is purely imaginary

(b) $\dfrac{z_1 - z_2}{z_3 - z_4}$ is purely real

(c) $\dfrac{z_4 - z_3}{z_2 - z_3}$ is purely real.

(d) $\dfrac{z_2 + z_4}{z_1 + z_3}$ is purely imaginary

(e) $$\frac{z_1 - z_3}{z_2 - z_4} + \frac{\bar{z}_1 - \bar{z}_3}{\bar{z}_2 - \bar{z}_4} = 0.$$

(f) $$\frac{z_4 - z_3}{\bar{z}_4 - \bar{z}_3} + \frac{z_2 - z_3}{\bar{z}_2 - \bar{z}_3} = 0.$$

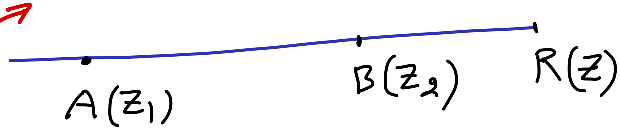
$$\frac{\cancel{\left(\frac{z_2 + z_4}{2}\right)}}{\cancel{\left(\frac{z_1 + z_3}{2}\right)}} = 1$$

Equation of straight line in various forms :

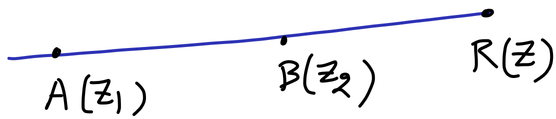
(1) Two point form :-

$$\boxed{\frac{z - z_1}{\bar{z} - \bar{z}_1} = \frac{z_2 - z_1}{\bar{z}_2 - \bar{z}_1}}$$

$w_1 = w_2$



(2) Parametric form :-

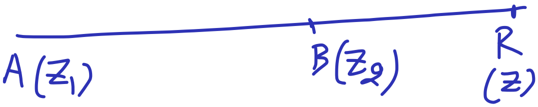


$$\boxed{z = z_1 + \lambda (z_2 - z_1) \; ; \lambda \in \mathbb{R}}$$

$$\frac{z - z_1}{z_2 - z_1} = \lambda \quad ; \lambda \in \mathbb{R}$$

$$\frac{z - z_1}{z_2 - z_1} = \frac{\bar{z} - \bar{z}_1}{\bar{z}_2 - \bar{z}_1} \Rightarrow \frac{z - z_1}{\bar{z} - \bar{z}_1} = \frac{z_2 - z_1}{\bar{z}_2 - \bar{z}_1}$$

(3) Determinant form :-



$$\begin{vmatrix} 1 & 1 & 1 \\ z & z_1 & z_2 \\ \bar{z} & \bar{z}_1 & \bar{z}_2 \end{vmatrix} = 0.$$

$$z_1 \bar{z}_2 - \bar{z}_1 z_2 - 1(z\bar{z}_2 - \bar{z} z_2) + (z\bar{z}_1 - \bar{z} z_1) = 0$$

$$z(\underbrace{\bar{z}_1 - \bar{z}_2}_{\bar{\alpha}}) + \bar{z}(\underbrace{z_2 - z_1}_{(-\alpha)}) + \underbrace{z_1 \bar{z}_2 - \bar{z}_1 z_2}_{\beta \in \mathbb{R}} = 0.$$

$$iz(\bar{z}_1 - \bar{z}_2) + i\bar{z}(z_2 - z_1) + i(z_1\bar{z}_2 - \bar{z}_1 z_2) = 0$$

$$i(z_1 - z_2) = \alpha$$

$$-i(\bar{z}_1 - \bar{z}_2) = \bar{\alpha}$$

$$z\bar{\alpha} + \bar{z}\alpha - i(z_1\bar{z}_2 - \bar{z}_1 z_2) = 0$$

$$\boxed{z\bar{\alpha} + \bar{z}\alpha + \beta = 0}$$

β is R.R

$\alpha \to$ Complex No.

Q Find x-intercept of line

$$z(3-4i) + \bar{z}(3+4i) + 5 = 0$$

M-1

$z = x + iy$

$ax + by + c = 0$

$(x+iy)(3-4i) + (x-iy)(3+4i) + 5 = 0$

$3x - \cancel{4xi} + \cancel{3yi} + 4y + 3x + \cancel{4xi} - \cancel{3yi} + 4y + 5 = 0.$

$6x + 8y + 5 = 0$

$x_{int} = -5/6.$

M-2

$z = \bar{z}$ (on the R.A)

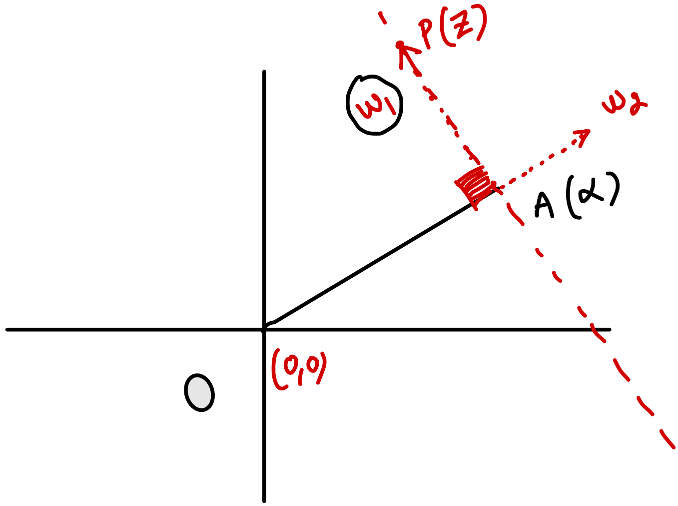
$z(3-4i) + z(3+4i) + 5 = 0$

$6z + 5 = 0$

$\boxed{z = -5/6.}$

Q Find the equation of a line on complex plane which passes through a point A denotes by complex number α and is perpendicular to the vector $\overrightarrow{OA}$.

Sol^n



$\overrightarrow{OA} = \alpha$

$w_1 + w_2 = 0.$

$$\frac{z-\alpha}{\bar{z}-\bar{\alpha}} + \frac{\alpha-0}{\bar{\alpha}-0} = 0.$$

$$\left(\frac{z-\alpha}{\alpha}\right) + \left(\frac{\bar{z}-\bar{\alpha}}{\bar{\alpha}}\right) = 0.$$

$$\left(\frac{z-\alpha}{\alpha}\right) \text{ is P.I.}$$

i.e $\quad \dfrac{z-\alpha}{\alpha} = \lambda i \;;\; \lambda \in \mathbb{R}$

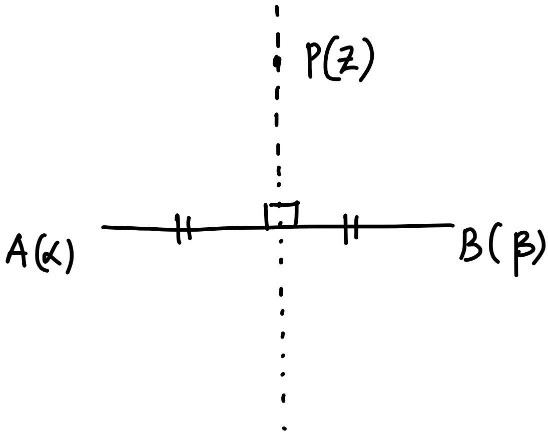
$$z = \alpha + (\lambda\alpha) i \;;\; \lambda \in \mathbb{R}$$

Alt :-

$$\operatorname{Arg}\left(\frac{z-\alpha}{\alpha}\right) = \frac{\pi}{2}$$

$$\frac{z-\alpha}{\alpha} \text{ is P.I}$$

Note : It is to be noted that the equation $|z-\alpha| = |z-\beta|$ denotes the equation of the perpendicular bisector of the line joining the points α & β.



Q If $|z| = 1$ and $w = \dfrac{z-1}{z+1}$, then find locus of w?

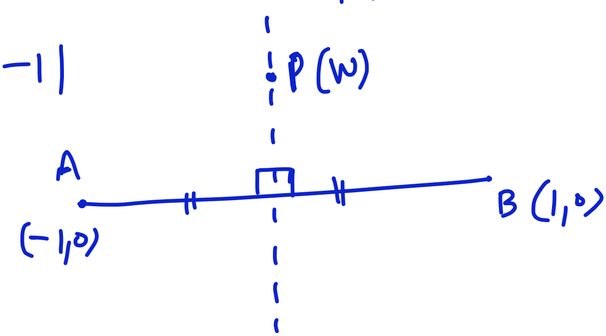
Sol

$$wz + w = z - 1$$

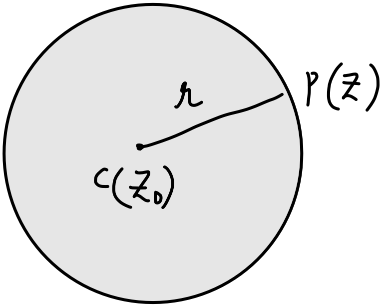
$$z = \frac{-1-w}{w-1} \Rightarrow z = \frac{w+1}{1-w}.$$

$$|z| = 1 \Rightarrow \left|\frac{w+1}{1-w}\right| = 1$$

$$|w+1| = |w-1|$$



Equation of Circle :-



$$\underset{\text{variable}}{z} - \underset{\text{fixed}}{z_0}$$

$$|z - z_0| = r$$

$$|z - z_0|^2 = r^2$$

$$(z - z_0)(\bar{z} - \bar{z}_0) = r^2$$

$$z\bar{z} - z\bar{z}_0 - z_0\bar{z} + \underbrace{z_0\bar{z}_0 - r^2}_{\beta} = 0.$$

$\beta \rightarrow$ Real quantity

$\alpha = -z_0$

$\bar{\alpha} = -\bar{z}_0$

Rem

$$\boxed{z\bar{z} + \alpha\bar{z} + \bar{\alpha}z + \beta = 0}$$

** Coeff of $z\bar{z}$ is '1'

where centre is $-$ Coeff of $\bar{z}$

radius $= r = \sqrt{\alpha\bar{\alpha} - \beta}$

Q $2z\bar{z} + (6-8i)\bar{z} + (6+8i)z + 32 = 0$

Find centre & radius of Circle ?

Sol^n

M-1

$$z\bar{z} + (\underbrace{3-4i}_{\alpha})\bar{z} + (3+4i)z + 16 = 0.$$

Centre $= -$ Coeff of $\bar{z}$

$= -(3-4i) = -3+4i$

$$\text{radius} = \sqrt{\alpha\bar{\alpha} - \beta}$$

$$= \sqrt{25-16} = 3.$$

M-2

$z = x+iy.$

$\bar{z} = x-iy.$

$$(x+3)^2 + (y-4)^2 = 9.$$

Q Find Locus of 'z' where $|2z-3| = 2$

Sol^n $\left|z - \frac{3}{2}\right| = 1$

Circle whose centre is $\left(\frac{3}{2}, 0\right)$

and rad = 1.

Alt :- $z = x + iy$

Q If $|z| = 1$ & $|w-3| = 1$ then find the maximum and minimum value of : ① $|z-w|$

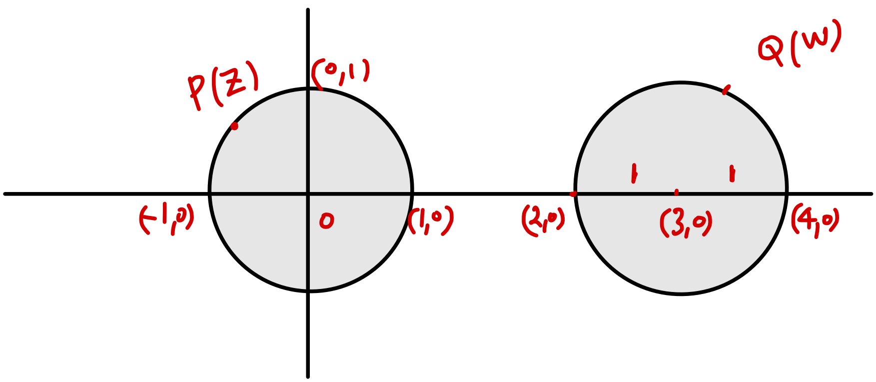
② $|2z-3w|$

Sol^n $|z| = 1$ & $|w-3| = 1$

① $|z-w|$

least = 1

greatest = 5

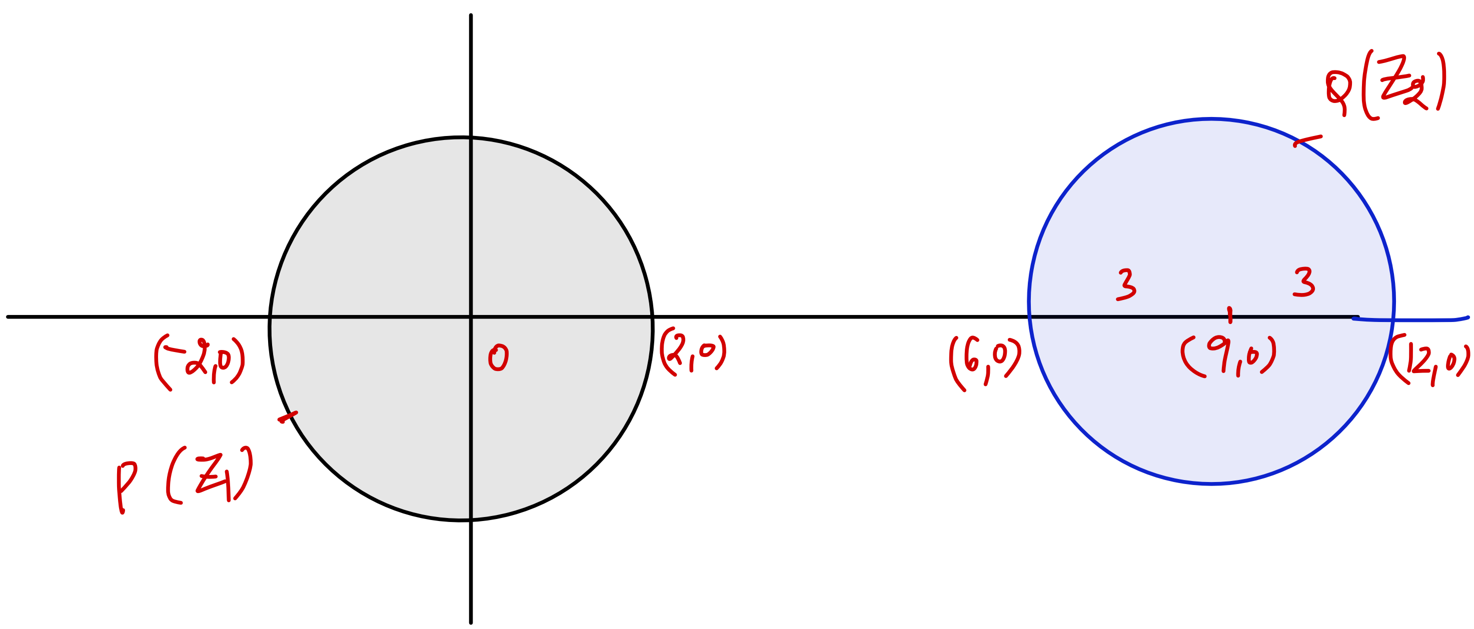


(b) $|z| = 1$ & $|w-3| = 1$

$|\underbrace{2z}_{z_1} - \underbrace{3w}_{z_2}| = \underbrace{|z_1 - z_2|}$; where $z_1 = 2z$, $z_2 = 3w$.

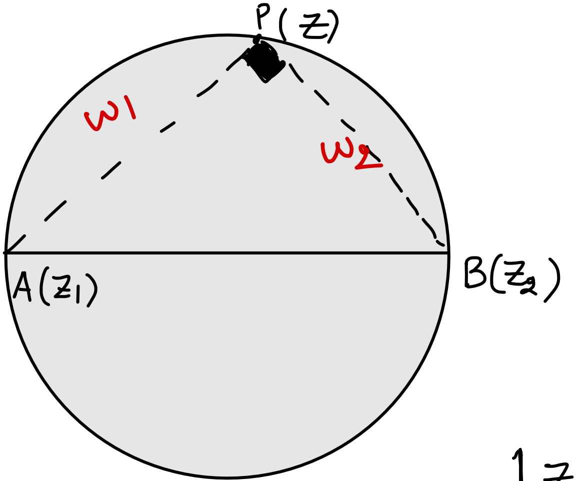
$|2z| = 2 \Rightarrow |z_1| = 2$ ✓

$|3w - 9| = 3 \Rightarrow |z_2 - 9| = 3$. ✓



$|2z - 3w|$ → least = 4.
→ greatest = 14.

Diamatrical form of Circle :-

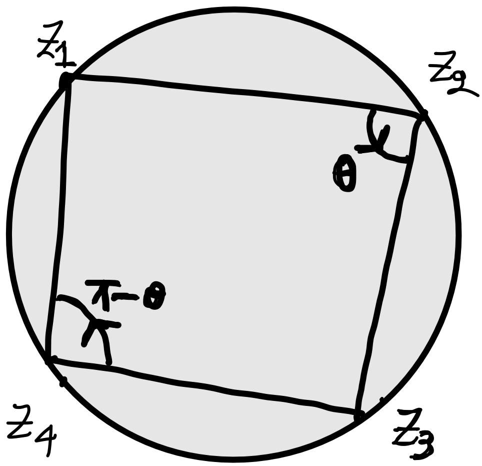


$$\frac{z-z_1}{\bar{z}-\bar{z}_1}+\frac{z_2-z}{\bar{z}_2-\bar{z}}=0.$$

$$\frac{z-z_1}{\bar{z}-\bar{z}_1}+\frac{z-z_2}{\bar{z}-\bar{z}_2}=0.$$

$$|z-z_1|^2+|z-z_2|^2=|z_1-z_2|^2.$$

Condition for 4 points to be con-cyclic :-



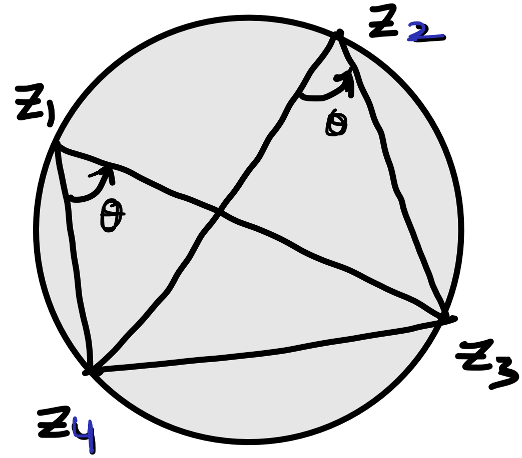
$$\arg\left(\frac{z_3-z_2}{z_1-z_2}\right)+\arg\left(\frac{z_1-z_4}{z_3-z_4}\right)=\pi$$

$$\arg\left(\frac{z_3-z_2}{z_1-z_2}\cdot\frac{z_1-z_4}{z_3-z_4}\right)=\pi.$$

$\frac{z_3-z_2}{z_1-z_2}\cdot\frac{z_1-z_4}{z_3-z_4}$ is P.R. (-ve)

$$\boxed{\frac{z_3-z_2}{z_1-z_2}\cdot\frac{z_1-z_4}{z_3-z_4}=\frac{\bar{z}_3-\bar{z}_2}{\bar{z}_1-\bar{z}_2}\cdot\frac{\bar{z}_1-\bar{z}_4}{\bar{z}_3-\bar{z}_4}.}$$

$$\arg\left(\frac{z_3 - z_1}{z_4 - z_1}\right) = \arg\left(\frac{z_3 - z_2}{z_4 - z_2}\right)$$



$$\arg\left(\frac{z_3 - z_1}{z_4 - z_1} \cdot \frac{z_4 - z_2}{z_3 - z_2}\right) = 0$$

$$\left(\frac{z_3 - z_1}{z_4 - z_1} \cdot \frac{z_4 - z_2}{z_3 - z_2}\right) \text{ is P.R. } \underline{\underline{(+ve)}}$$

$$\boxed{\frac{z_3 - z_1}{z_4 - z_1} \cdot \frac{z_4 - z_2}{z_3 - z_2} = \frac{\bar{z}_3 - \bar{z}_1}{\bar{z}_4 - \bar{z}_1} \cdot \frac{\bar{z}_4 - \bar{z}_2}{\bar{z}_3 - \bar{z}_2}}$$

Equation of Circle . . .

If we replace z_4 by z then we will obtain the equation of Circle through 3 non-Collinear points z_1, z_2, & z_3.

NOTE :– ✱✱

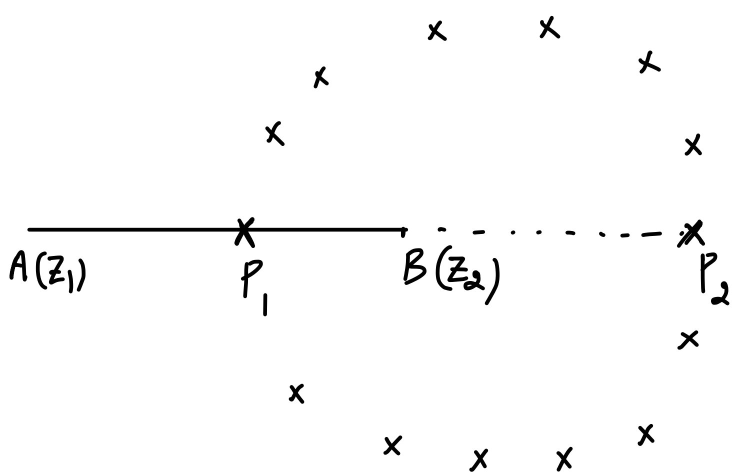
A point P(z) moves in argand plane in such a way that its distances from two fixed points $A(z_1)$ & $B(z_2)$ are such that $\frac{PA}{PB} = k$

$\Rightarrow \left|\frac{z - z_1}{z - z_2}\right| = k \quad (k > 0)$

(i) $k = 1$: locus is the perpendicular bisector of the line joining z_1 & z_2.

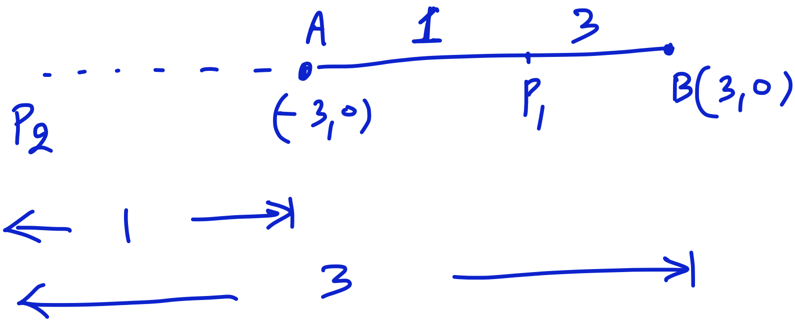
(ii) $k \neq 1$, locus is a circle whose diametric ends (P_1 & P_2) are the points which divides line joining z_1 and z_2 internally & externally in the ratio $k : 1$

$\therefore$ Centre of the circle = mid point of P_1P_2



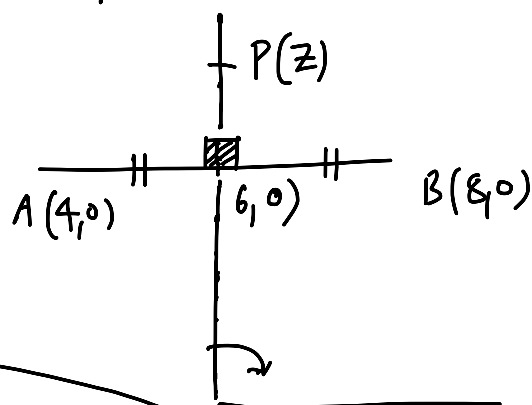
Q $\left|\dfrac{z-3}{z+3}\right| = 3.$

Circle.



Q If $\dfrac{|z-12|}{|z-8i|}=\dfrac{5}{3}$ and $\left|\dfrac{z-4}{z-8}\right|=1$, then the value of z is

$$|z-4| = |z-8|$$



$$\boxed{z = 6+\lambda i}$$

$\lambda \in \mathbb{R}.$

$$3|6+\lambda i-12| = 5|6+\lambda i-8i|$$

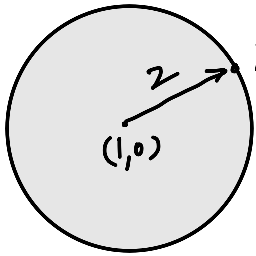
$$3\sqrt{\lambda^2+36} = 5\sqrt{36+(\lambda-8)^2}.$$

Q. Let P(z) is a point in complex plane which satisfies $|z-1| = 2$

Locus of point Q(w) which satisfies $w = z + \frac{2}{z-1}$ is a conic, whose eccentricity is ___

Sol^n

$|z-1| = 2$



$P(\underbrace{1+2\cos\theta, 2\sin\theta}_{z})$

$$W = z + \frac{2}{z-1}$$

$$h+ik = (1+2\cos\theta + i2\sin\theta) + \frac{\cancel{2}}{\cancel{2}\cos\theta + i\cancel{2}\sin\theta} \times \frac{\cos\theta - i\sin\theta}{\cos\theta - i\sin\theta}$$

$$h+ik = 1+2\cos\theta + (2\sin\theta)i + (\cos\theta - i\sin\theta)$$

$$h = 3\cos\theta + 1 \quad ; \quad k = \sin\theta.$$

$$\left(\frac{h-1}{3}\right)^2 + k^2 = 1.$$

$$\frac{(x-1)^2}{3^2} + y^2 = 1. \quad \text{Ans}$$

Locus

$$|z - z_1| + |z - z_2| = K \text{ (constant)}$$

z_1 & $z_2 \rightarrow$ fixed Complex No.

(1) If $K > |z_1 - z_2|$ then locus of 'z' is an ellipse whose foci are z_1 and z_2 and length of major axis is 'K'.

$$\boxed{e = \frac{|z_1 - z_2|}{K}}$$

$e \rightarrow$ eccentricity

(2) If $K = |z_1 - z_2|$ then locus of 'z' is a line segment joining z_1 & z_2.

(3) If $K < |z_1 - z_2|$ then NO LOCUS.

Q1 $|z - 4i| + |z + 4i| = 10$

$10 \rightarrow$ major axis $= 10$

$K = 10$

✓ $A(0, 4)$

✓ $B(0, -4)$

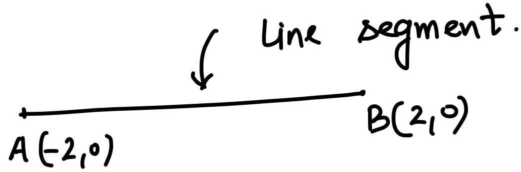
$AB = 8$

$K > AB$

locus will be ellipse.

✓ $e = \frac{8}{10} = \frac{4}{5}$.

Q2 $$|z-2| + |z+2| = 4.$$



Q3 $$|z-1| + |z+1| = 1$$

$AB = 2$

No Locus

$$\left|\, |z - z_1| - |z - z_2| \,\right| = K \text{ (Constant)}$$; z_1 & z_2 are fixed Complex No.

(1) If $K > |z_1 - z_2|$ then No Locus.

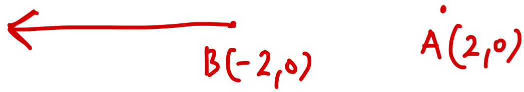
(2) If $K < |z_1 - z_2|$ then Locus of 'z' is hyperbola whose foci are z_1 & z_2 and length of transverse axes is equal to 'K'

$$\boxed{e = \frac{|z_1 - z_2|}{K}}$$

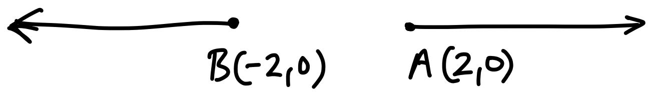
(3) If $K = |z_1 - z_2|$ then Locus of 'z' is pair of rays emanating from z_1 & z_2 and moving away from it.

① $|z-2| - |z+2| = 4.$

$AB = 4$

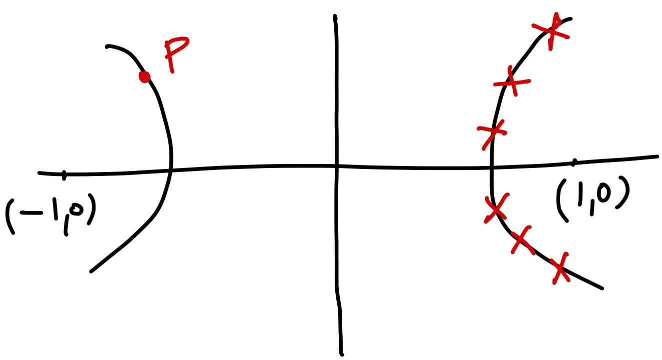


$$\big||z-2| - |z+2|\big| = 4$$

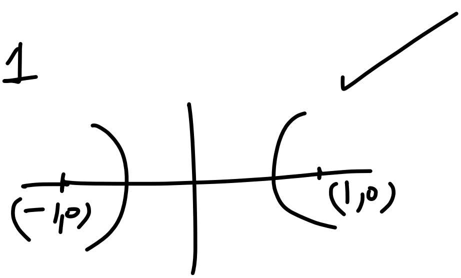


② $|z-1| - |z+1| = 1$ $\quad K = 1$

$AB = 2$ $\quad K < AB$



$$\big||z-1| - |z+1|\big| = 1$$

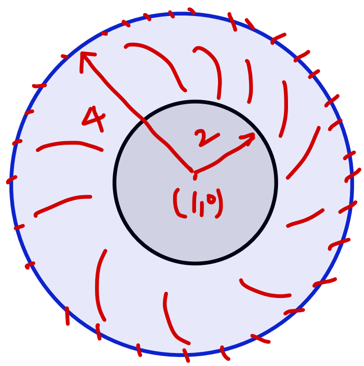


③ $|z-4i| - |z+4i| = 10$ $\quad$ $K = 10$

$\underbrace{\qquad\qquad}_{AB = 8}$ $\quad$ $K > AB$

NO LOCUS

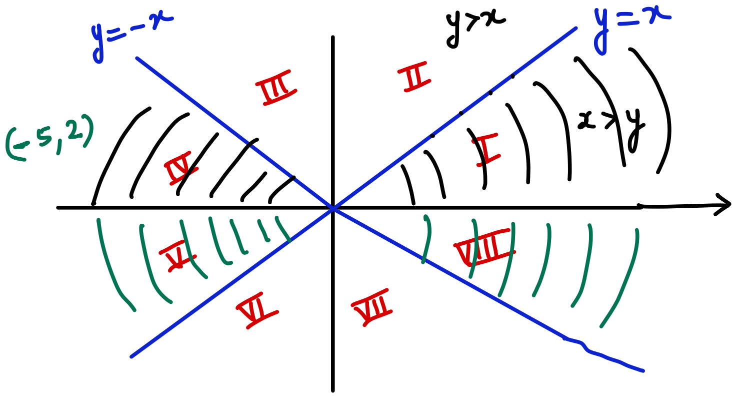
Q $2 < |z-1| \le 4$



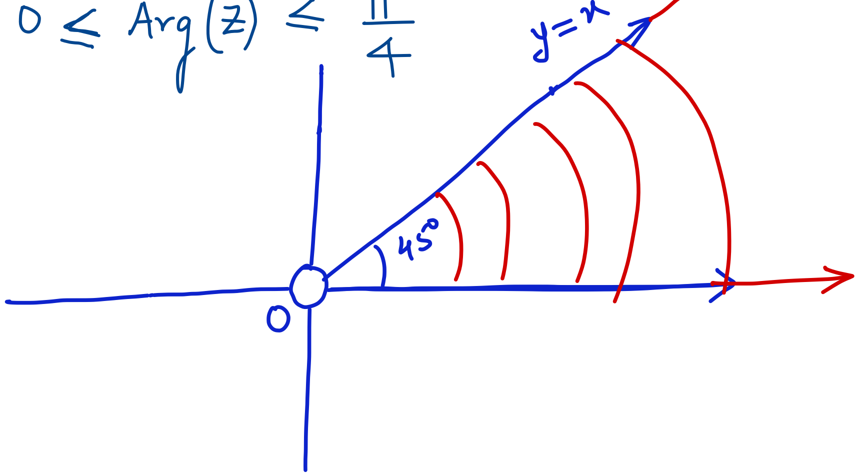
Q $\operatorname{Re}(z^2) > 0$.

$z = x + iy$; $z^2 = x^2 - y^2 + 2xyi$

$\operatorname{Re}(z^2) = x^2 - y^2 > 0 \Rightarrow \boxed{(x-y)(x+y) > 0}$



Q $0 \leq \operatorname{Arg}(z) \leq \frac{\pi}{4}$



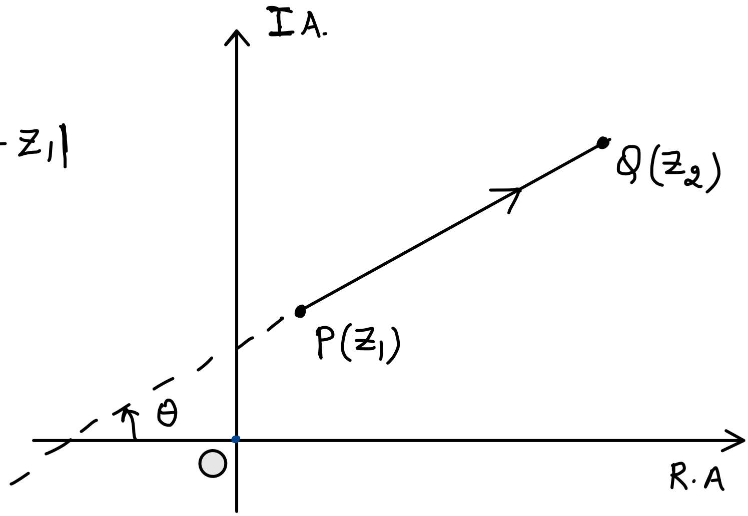
Vectorial representation of complex number

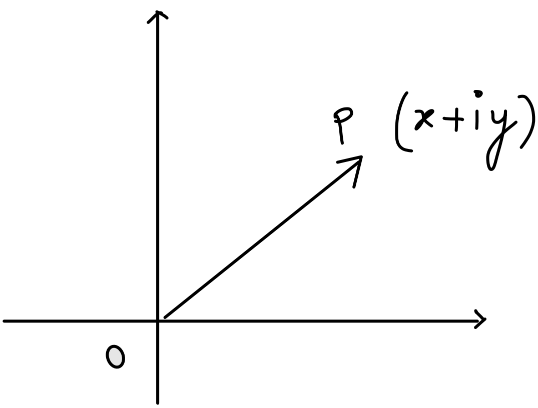
Complex Number 'z_1' corresponding to point 'p' can be treated as the position vector of point 'p'.

$$\boxed{\overrightarrow{PQ} = z_2 - z_1}$$

such that $PQ = |z_2 - z_1|$

& $\operatorname{Arg}(z_2 - z_1) = \theta$.

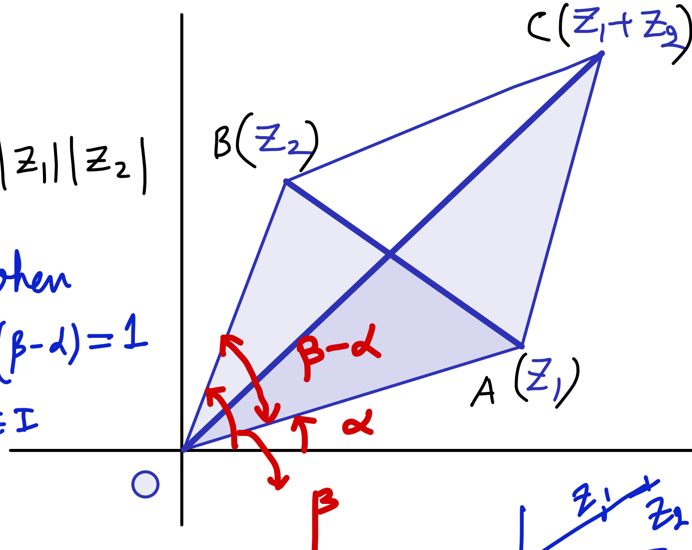




$$\overrightarrow{OP} = x\hat{i} + y\hat{j}$$

Note :-

$$|z_1+z_2|^2 = \overbrace{|z_1|^2+|z_2|^2+2|z_1||z_2|\cos(\beta-\alpha)}^{E}.$$

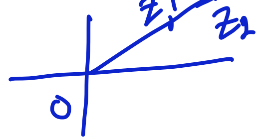


$$E_{max} \le |z_1|^2+|z_2|^2+2|z_1||z_2|$$

$$E_{max} \le (|z_1|+|z_2|)^2 \text{; when}$$

$$\cos(\beta-\alpha)=1$$

$$\beta-\alpha = 2n\pi; \ n\in I$$

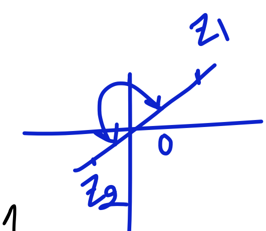


$$\therefore \boxed{|z_1+z_2| \le |z_1|+|z_2|}^{\star} \Rightarrow \cos(\beta-\alpha)=1$$

$$\arg(z_1)=\arg(z_2)$$

$||^{ly}$

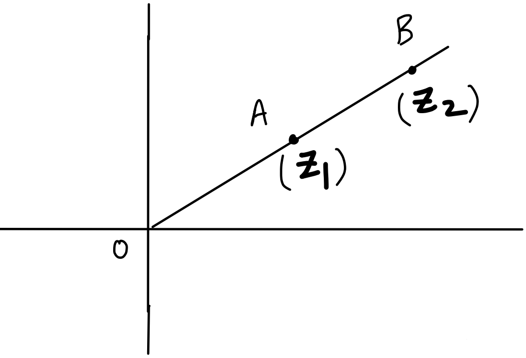
$$E \geq |z_1|^2+|z_2|^2-2|z_1||z_2|$$

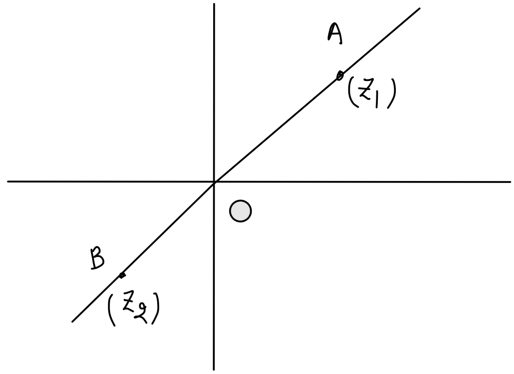


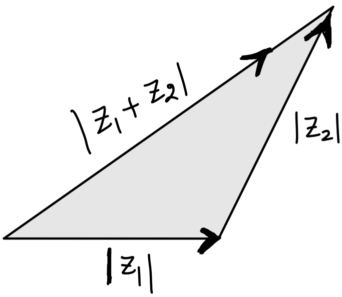
$$E \geq (|z_1|-|z_2|)^2 \ \star$$

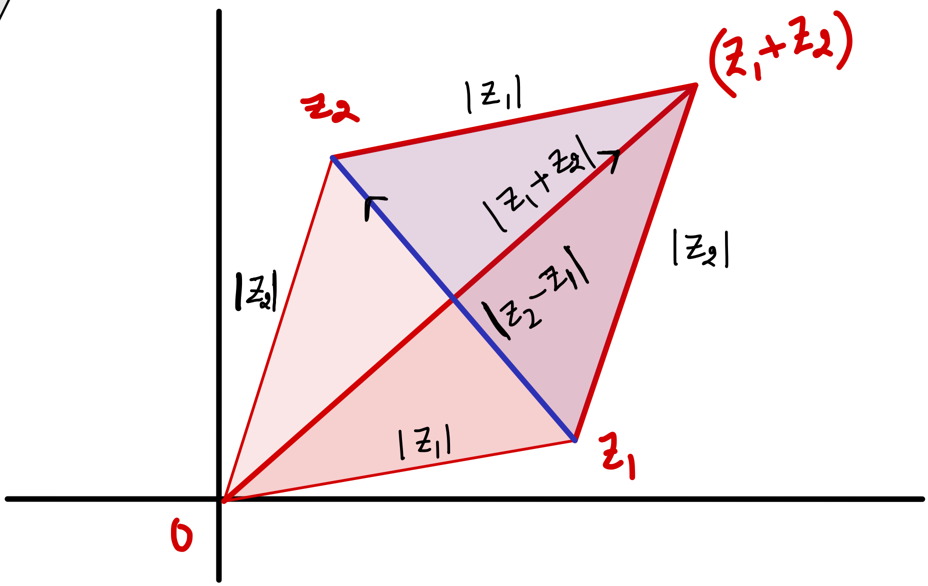
$$\boxed{|z_1+z_2| \geq \big||z_1|-|z_2|\big|} \Rightarrow \cos(\beta-\alpha)=-1$$

$$\therefore |\arg(z_1)-\arg(z_2)| = \pi.$$


B
A
(z_2)
(z_1)
O


A
(z_1)
B
(z_2)


$|z_1+z_2|$
$|z_2|$
$|z_1|$


(z_1+z_2)
$|z_1|$
z_2
$|z_1+z_2|$
$|z_2|$
$|z_2|$
$|z_2-z_1|$
$|z_1|$
z_1
O

$$\big||z_1| - |z_2|\big| \le |z_1 + z_2| \le |z_1| + |z_2|$$ [TRIANGLE INEQUALITY]

Imp

$$\big||z_1| - |z_2|\big| \le |z_1 - z_2| \le |z_1| + |z_2|$$

Q1 Find least value of $|z| + |1 - z|$?

Sol^n

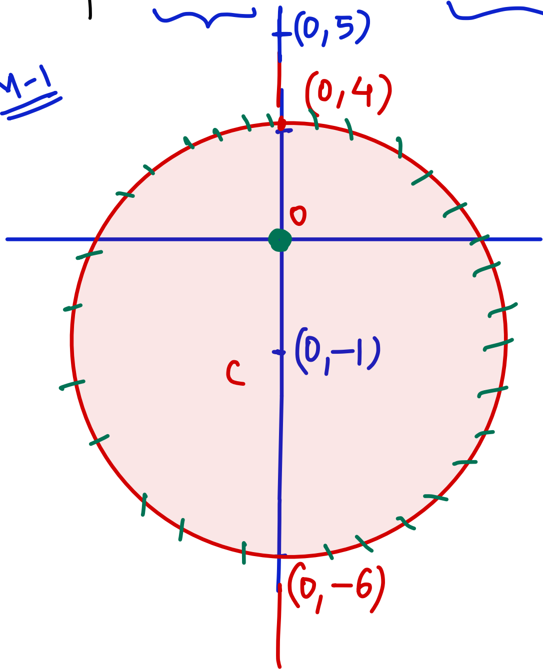
$$|\cancel{z} + 1 - \cancel{z}| \le |z| + |1 - z|$$

$$1 \le |z| + |1 - z|$$

$\therefore$ least value is '1'.

Q If $|z+i| = 5$ then find max & minimum value of $\underbrace{|z|}$ and $\underbrace{|z-5i|}$?

Sol^n M-1



① $|z|_{min} = 4$ ✓

$|z|_{max} = 6$. ✓

② $|z-5i|$

$\underbrace{min = 1}$ $\underbrace{max = 11}$.

M-2 $\underbrace{||z|-|i|| \leq |z+i|} \leq |z|+|i|$

$||z|-1| \leq \underbrace{5 \leq |z|+1}$

$\boxed{|z| \geq 4.}$

$-5 \leq |z|-1 \leq 5$

$\boxed{-4 \leq |z| \leq 6}$ $\cap$

Ans $\boxed{4 \leq |z| \leq 6}$

(ii)

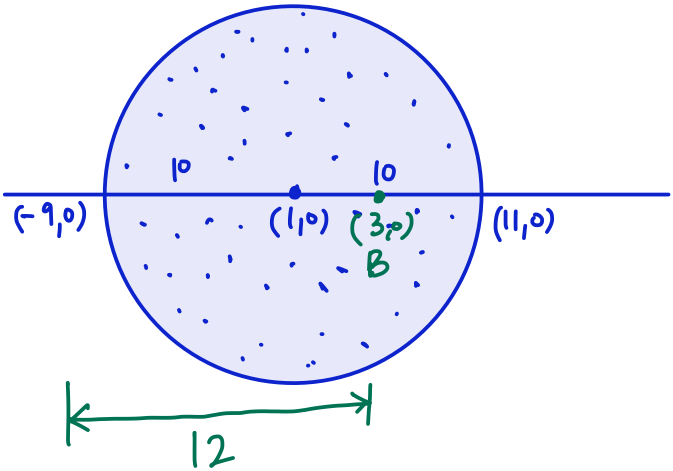
$||z-5i|-|6i|| \leq |z-5i+6i| \leq |z-5i|+|6i|$

$||z-5i|-6| \leq 5 \leq |z-5i|+6$

Q If $|z-1| \leq 10$ find $|z-3|_{max}$

$|z-3|_{min}$?

Solⁿ



$|z-3|_{min} = 0.$

$|z-3|_{max} = 12.$

Q Find the greatest and least values of $|z|$ if z satisfies $\left|z - \dfrac{4}{z}\right| = 2$.

$$\boxed{z \neq 0}$$

$$\Big||z| - \left|\tfrac{4}{z}\right|\Big| \leq \left|z + \left(-\tfrac{4}{z}\right)\right| \leq |z| + \left|\tfrac{4}{z}\right|$$

$|z| = r$

$r > 0$

$$\left|r - \frac{4}{r}\right| \leq 2 \leq r + \frac{4}{r}$$

$$\left|r - \frac{4}{r}\right| \leq 2$$

$$\boxed{-2 \leq r - \frac{4}{r} \leq 2}$$

$$\cap$$

$$\boxed{2 \leq r + \frac{4}{r}}$$

HW

Let z_1 lies on $|z| = 1$ and z_2 lies on $|z| = 2$, then which of the following is/are true -

(A) maximum value of $|z_1 + z_2|$ is 3

(B) minimum value of $|2z_1 - z_2|$ is 0

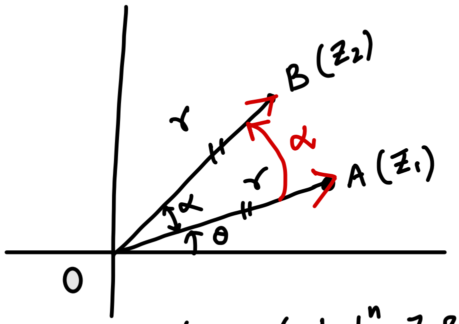
(C) maximum value of $|2z_1 + z_2|$ is 4

(D) minimum value of $|2z_1 - 3z_2|$ is 5

HW

Q. Let z is a complex number such that $|z| = 1$, then maximum value of $|z+1| + |z^2 - z + 1|$

Rotation of complex number:



$\alpha \rightarrow \angle$ betⁿ z_1 & z_2

$z_1 = r e^{i\theta}$

$z_2 = r e^{i(\theta+\alpha)}$

$\dfrac{z_2}{z_1} = e^{i\alpha} \Rightarrow \boxed{z_2 = z_1 \cdot e^{i\alpha}}$

$\angle$ rotate

final vector

initial vector

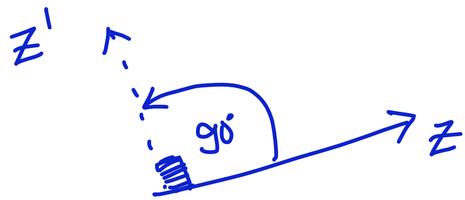
$e^{i\alpha} = \cos\alpha + i\sin\alpha$

$|e^{i\alpha}| = 1.$

$z_2 e^{-i\alpha} = z_1$

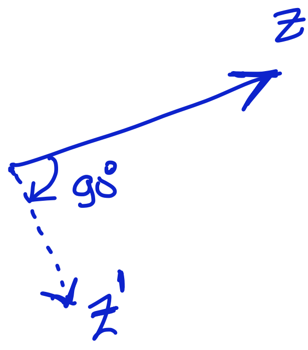
Note: ① $i = e^{i\pi/2}$

$z' = z i = z e^{i\pi/2}$



② $-i = e^{-i\pi/2}$

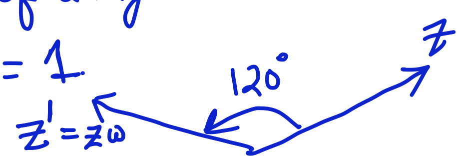
$z' = z(-i) = z e^{-i\pi/2}$



③ $\omega = e^{i2\pi/3}$; $\omega \rightarrow$ cube root of unity

$z' = z\omega = z e^{i2\pi/3}$

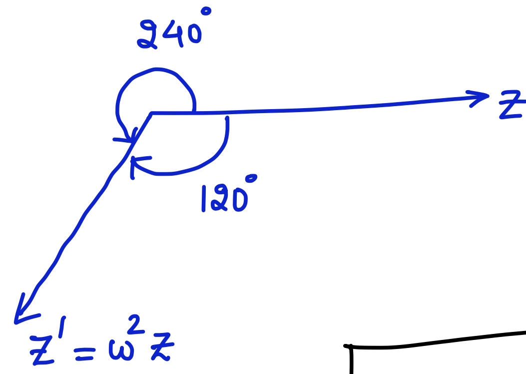
$|\omega| = 1$



(4) $\omega^2 = e^{i4\pi/3}$

$z' = \omega^2 z = z e^{i4\pi/3}$

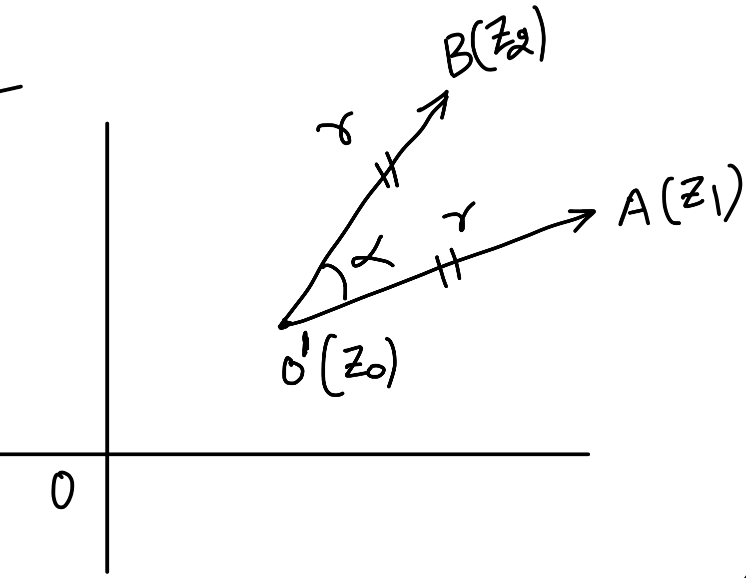
$\omega^2 \to$ cube root of unity



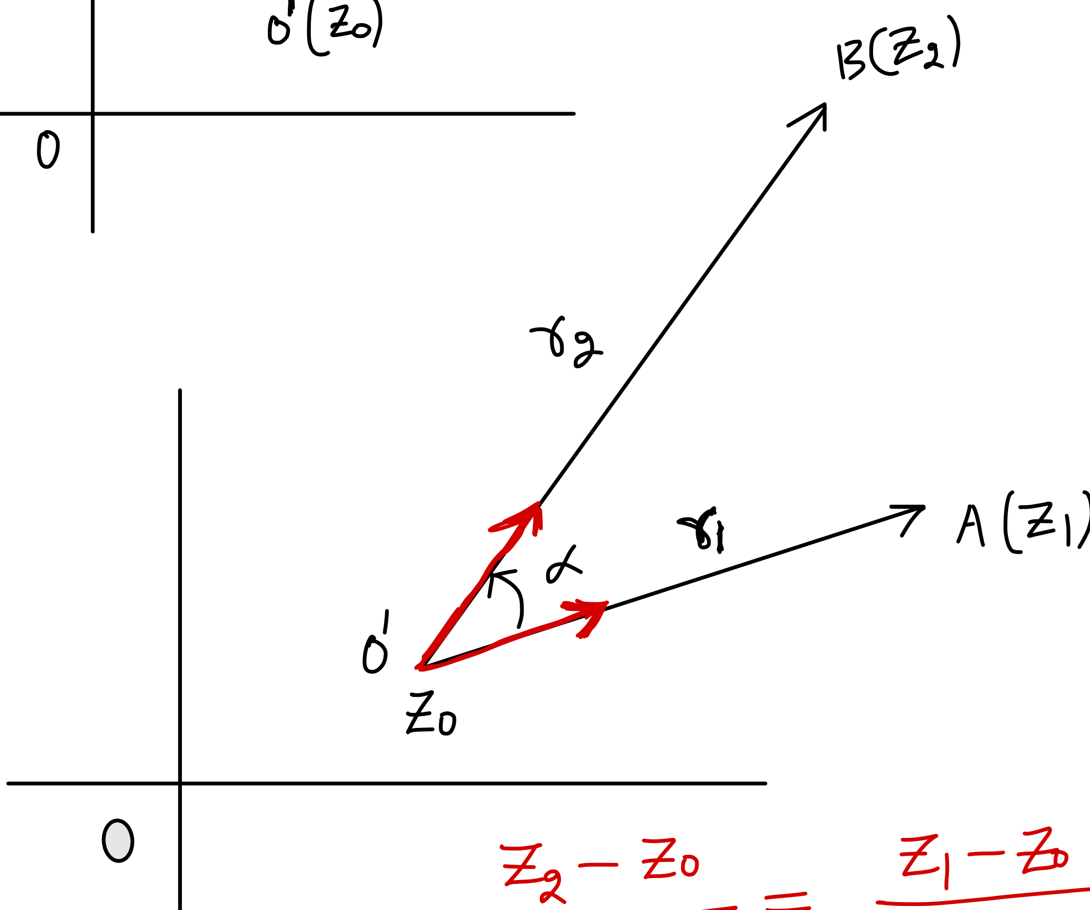
$$\boxed{z_2 - z_0 = e^{i\alpha}(z_1 - z_0)}$$

Note :-

①



②



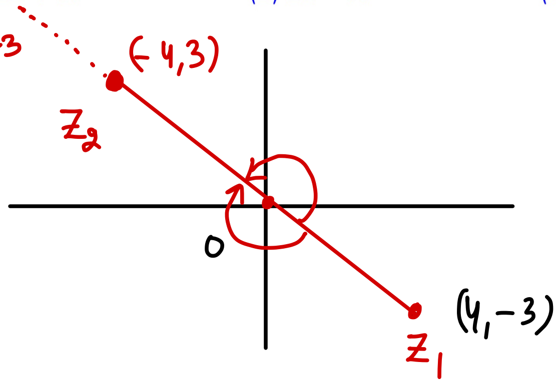
$$\frac{z_2 - z_0}{|z_2 - z_0|} = \frac{z_1 - z_0}{|z_1 - z_0|} e^{i\alpha}$$

$$\boxed{\frac{z_2 - z_0}{z_1 - z_0} = \frac{r_2}{r_1} e^{i\alpha}}$$ *

Q In the Argand plane, the vector $z = 4 - 3i$ is turned in the clockwise sense through $180°$ and stretched three times. The complex number represented by the new vector is

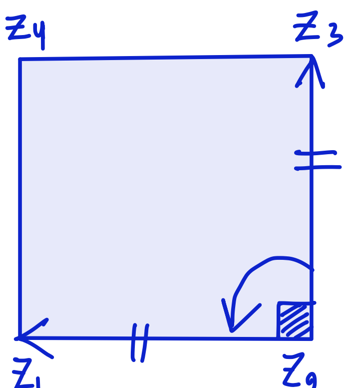
(A) $12 + 9i$ (B) $12 - 9i$ (C) $-12 - 9i$ (D) $-12 + 9i$ ✓

Sol^n



Q If z_1, z_2, z_3, z_4 are the vertices of square taken in order, then prove that $2z_2 = (1 + i)z_1 + (1 - i)z_3$.

Sol^n



$$z_1 - z_2 = (z_3 - z_2)\, i$$

$$z_1 - i z_3 = z_2 (1 - i)$$

$$z_2 = \frac{z_1}{1-i} - \frac{i z_3}{1-i}$$

$$z_2 = \frac{z_1(1+i)}{2} - \frac{i(1+i) z_3}{2}$$

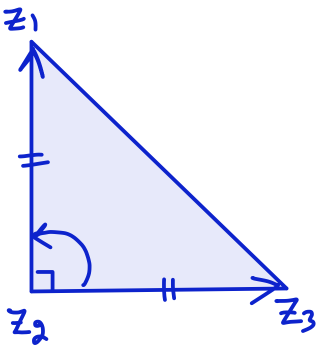
$$2 z_2 = (1+i) z_1 + (1-i) z_3$$

(H.P)

Q If z_1, z_2, z_3 are the vertices of an isosceles triangle right angled at z_2,

then prove that $z_1^2 + 2z_2^2 + z_3^2 = 2z_2(z_1 + z_3)$

Sol^n

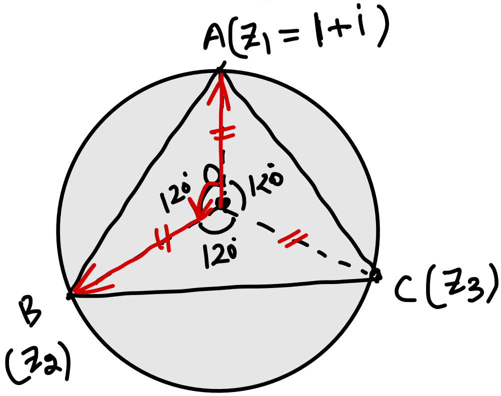


$$z_1 - z_2 = (z_2 - z_3)\, i$$

$$(z_1 - z_2)^2 = (z_2 - z_3)^2\, i^2$$

$$z_1^2 + z_2^2 - 2z_1z_2 = -(z_2^2 + z_3^2 - 2z_2z_3)$$

Q If z_1, z_2, z_3 are the vertices of a equilateral triangle with circumcentre at the origin. If $z_1 = (1 + i)$ then find z_2 and z_3.



$$z_2 - 0 = (z_1 - 0)\, e^{i(2\pi/3)}$$

$$z_2 = (1+i)\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right)$$

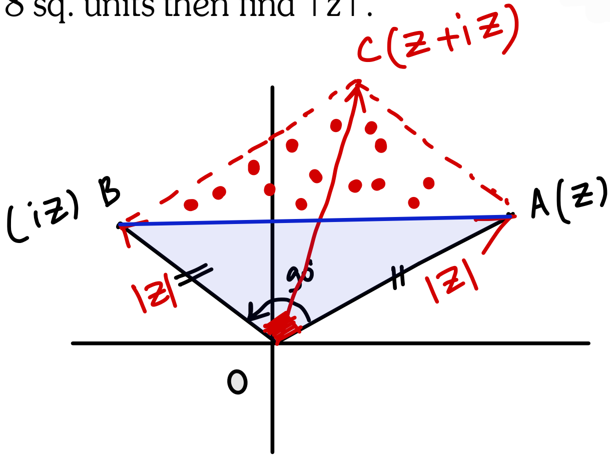
$$\frac{z_1 + z_2 + z_3}{3} = 0.$$

Alt:-

$$z_3 = z_1 e^{-i(2\pi/3)}$$

Q If the area of the triangle formed by z, iz and $\underbrace{z + iz}$ is 8 sq. units then find $|z|$.

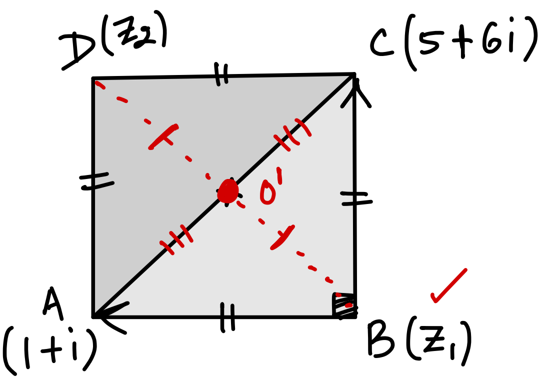
Sol^n



$$|iz| = |i||z|$$
$$= 1\ |z| = |z|$$

$$Ar(\Delta ABC) = Ar(\Delta AOB)$$
$$8 = \frac{1}{2}|z||z|$$
$$|z|^2 = 16$$
$$\boxed{|z| = 4}$$

Q If 1 + i and 5 + 6i are the extremities of diagonals of square, then find the remaining vertices.



$$(5+6i - z_1)\, i = (1+i - z_1)$$

Rem :-

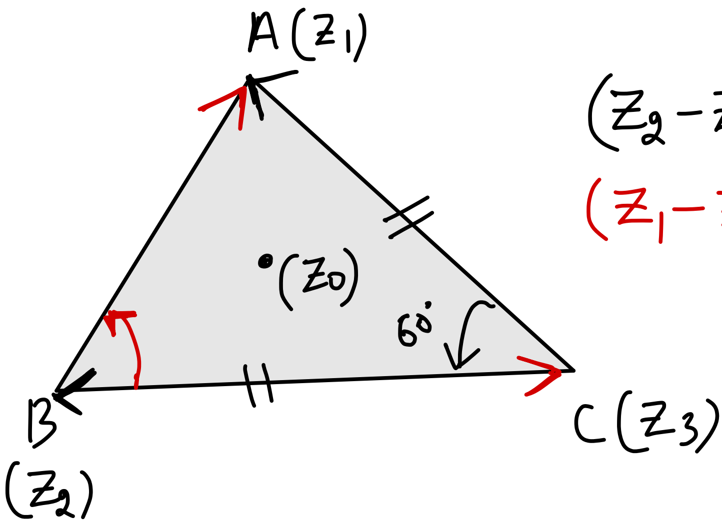
Q If z_1, z_2, z_3 are vertices of an equilateral Δ & 'z_0' is circumcentre of Δ then :-

(i) $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$

i.e $\sum z_1^2 = \sum z_1 z_2$.

(ii) $z_1^2 + z_2^2 + z_3^2 = 3z_0^2$.

(iii) $\dfrac{1}{z_1 - z_2} + \dfrac{1}{z_2 - z_3} + \dfrac{1}{z_3 - z_1} = 0$.



$(z_2 - z_3) = (z_1 - z_3)e^{i\pi/3}$

$(z_1 - z_2) = (z_3 - z_2)e^{i\frac{\pi}{3}}$

divide :-

$$\frac{z_2 - z_3}{z_1 - z_2} = \frac{z_1 - z_3}{z_3 - z_2}$$

$z_2z_3 - z_2^2 - z_3^2 + \cancel{z_2z_3} = z_1^2 - z_1z_3$

$- z_1z_2 + \cancel{z_2z_3}$

(ii) $\dfrac{z_1+z_2+z_3}{3} = z_0$

$(z_1+z_2+z_3)^2 = (3z_0)^2$

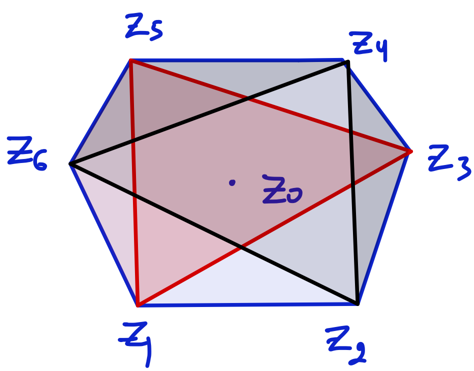
$\sum z_1^2 + 2\underbrace{\sum z_1 z_2} = 9z_0^2$

$3\sum z_1^2 = 9z_0^2$

$\boxed{\sum z_1^2 = 3z_0^2}$ (H.P)

$\Big(\underbrace{\sum z_1^2 = \sum z_1 z_2}_{\text{(i) result}}\Big)$

Q If z_r (r = 1, 2, ... 6) are the vertices of a regular hexagon then $\sum_{r=1}^{6} z_r^2 = 6z_0^2$, where z_0 is the circumcentre.



$z_1^2 + z_3^2 + z_5^2 = 3z_0^2$

Add $\underline{z_2^2 + z_4^2 + z_6^2} = \underline{3z_0^2}$

Q HW

Let z_1 and z_2 be roots of the equation $z^2 + pz + q = 0$, where the coefficients p and q may be complex numbers. Let A and B represent z_1 and z_2 in the complex plane. If $\angle AOB = \alpha \neq 0$ and $OA = OB$, where O is the origin, prove that $p^2 = 4q\cos^2\alpha/2$.

Q HW A, B, C are the points representing the complex numbers z_1, z_2 z_3 respectively and the circumcentre of the triangle ABC lies at the origin. If the altitudes of the triangle through the opposite vertices meets the circumcircle at D, E, F respectively. Find the complex numbers corresponding to the D, E, F in terms of

z_1, z_2, z_3.

Q Let z_1 lies on $|z| = 1$ and z_2 lies on $|z| = 2$, then which of the following is/are true -

(A) maximum value of $|z_1 + z_2|$ is 3 (B) minimum value of $|2z_1 - z_2|$ is 0

(C) maximum value of $|2z_1 + z_2|$ is 4 (D) minimum value of $|2z_1 - 3z_2|$ is 5

Sol^n (ABC)

$|z| = 1 \leftarrow z_1 \Rightarrow |z_1| = 1$

$|z| = 2 \leftarrow z_2 \Rightarrow |z_2| = 2$

(A) $|z_1 + z_2| \le |z_1| + |z_2|$

$|z_1| + |z_2| \le 3.$

(B) $\big||2z_1| - |z_2|\big| \le |2z_1 + (-z_2)|$

$|2 - 2| \le |2z_1 - z_2| \Rightarrow |2z_1 - z_2| \geq 0.$

(C) $|2z_1 + z_2| \le |2z_1| + |z_2|$

$\le 2 + 2$

≤ 4

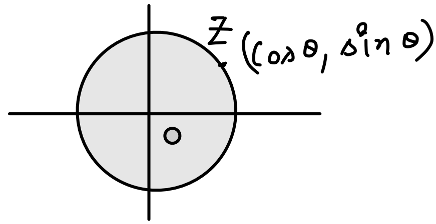
HW

Q Let z is a complex number such that $|z| = 1$, then maximum value of $|z+1| + |z^2 - z + 1|$

Sol^n $|z|^2 = 1 \Rightarrow z\bar{z} = 1$.

$E = |z+1| + |z^2 - z + 1| = |z+1| + |z^2 - z + z\bar{z}|$

$E = |z+1| + \underbrace{|z|}_{1}\,|z - 1 + \bar{z}|$



$E = |\cos\theta + 1 + i\sin\theta| + |2\cos\theta - 1|$

$E = \sqrt{(\cos\theta+1)^2 + \sin^2\theta} + |2\cos\theta - 1|$

$E = \sqrt{4\cos^2\frac{\theta}{2}} + \left|4\cos^2\frac{\theta}{2} - 3\right| = 2\left|\cos\frac{\theta}{2}\right| + \left|4\cos^2\frac{\theta}{2} - 3\right|$

Let $\left|\cos\frac{\theta}{2}\right| = t$; $t \in [0, 1]$

$E = 2t + |4t^2 - 3|$

$-4t^2 + 2t + 3$

$t \in \left[0, \frac{\sqrt{3}}{2}\right]$

max value at $t = \frac{1}{4}$

equal to $\frac{13}{4}$

$4t^2 + 2t - 3$

$t \in \left[\frac{\sqrt{3}}{2}, 1\right]$

max value at $t = 1$ equal to

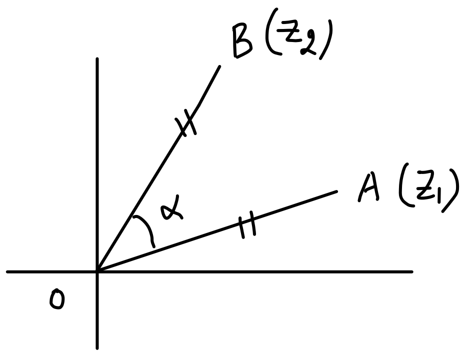
3

Hence, max value $= \frac{13}{4}$ Ans

Q HW Let z_1 and z_2 be roots of the equation $z^2 + pz + q = 0$, where the coefficients p and q may be complex numbers. Let A and B represent z_1 and z_2 in the complex plane. If $\angle AOB = \alpha \neq 0$ and $OA = OB$, where O is the origin, prove that $p^2 = 4q\cos^2\alpha/2$.

Sol^n

$$z_1 + z_2 = -p \quad \& \quad z_1 z_2 = q$$



$$\boxed{z_2 = z_1 e^{i\alpha}}$$

$$1 + \frac{z_2}{z_1} = e^{i\alpha} + 1$$

$$\frac{z_2 + z_1}{z_1} = 1 + e^{i\alpha}$$

$$\frac{-p}{z_1} = 1 + \cos\alpha + i\sin\alpha$$

$$\frac{-p}{z_1} = 2\cos^2\frac{\alpha}{2} + 2i\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$$

$$\frac{-p}{z_1} = 2\cos\frac{\alpha}{2}\, e^{i\frac{\alpha}{2}}$$

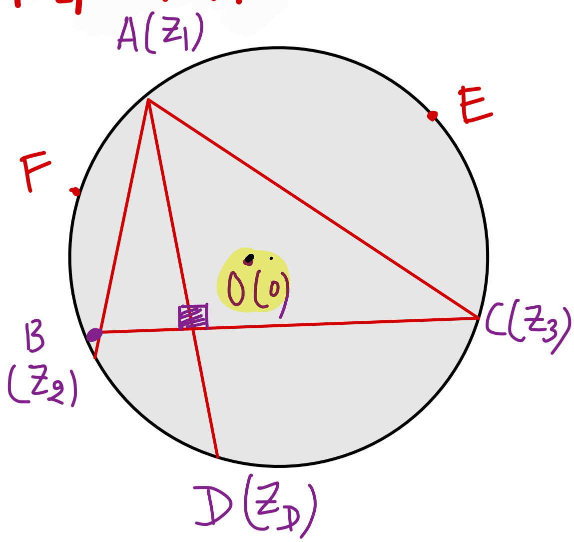
$$\frac{p^2}{z_1^2} = 2^2\cos^2\frac{\alpha}{2} \cdot \underbrace{e^{i\alpha}}$$

$$p^2 = 4\cos^2\frac{\alpha}{2} \cdot z_1^{\cancel{2}} \cdot \left(\frac{z_2}{\cancel{z_1}}\right)$$

$$p^2 = 4\cos^2\frac{\alpha}{2} \cdot (q) \qquad \text{(H.P)}$$

Q HW A, B, C are the points representing the complex numbers z_1, z_2 z_3 respectively and the circumcentre of the triangle ABC lies at the origin. If the altitudes of the triangle through the opposite vertices meets the circumcircle at D, E, F respectively. Find the complex numbers corresponding to the D, E, F in terms of z_1, z_2, z_3.

$$|z_1|^2 = |z_2|^2 = |z_3|^2 = r^2.$$



Solⁿ

$$z_1 \bar{z}_1 = r^2$$

$$\boxed{\bar{z}_1 = \frac{r^2}{z_1}}$$

$$\omega_{AD} + \omega_{BC} = 0.$$

$$\frac{z_D - z_1}{\bar{z}_D - \bar{z}_1} + \frac{z_2 - z_3}{\bar{z}_2 - \bar{z}_3} = 0.$$

$$\frac{z_D - z_1}{\frac{r^2}{z_D} - \frac{r^2}{z_1}} + \frac{z_2 - z_3}{\frac{r^2}{z_2} - \frac{r^2}{z_3}} = 0$$

$$\left(\frac{z_D - z_1}{z_1 - z_D}\right) \cdot z_1 z_D + \left(\frac{z_2 - z_3}{z_3 - z_2}\right) z_2 z_3 = 0.$$

$$\boxed{z_D = -\frac{z_2 z_3}{z_1}} \; ; \; z_E = -\frac{z_1 z_3}{z_2}.$$

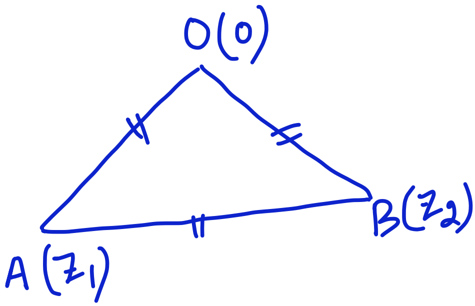
$$z_F = -\frac{z_1 z_2}{z_3}.$$

Alt :- Use Rotation.

Q z_1 and z_2 are roots of $3z^2+3z+b=0$.
If O (origin), A(z_1), B(z_2) form an equilateral Δ then find 'b'?

Sol^n

$z_1+z_2=-1$ & $z_1 z_2=\frac{b}{3}$.



$0^2+z_1^2+z_2^2 = 0(z_1)+0(z_2)+z_1z_2$

$\underbrace{z_1^2+z_2^2} = z_1z_2$

$(z_1+z_2)^2 = 3z_1z_2$

$1 = 3\left(\frac{b}{3}\right)$

$\therefore \boxed{b=1}$

DEMOIVRE'S THEOREM : **

Statement :

If $z = (\cos\theta + i\sin\theta)^n$, then

(i) $(\cos n\theta + i\sin n\theta)$ is the value of z, if $n \in \mathbb{Z}$ integer

(ii) $(\cos n\theta + i\sin n\theta)$ is one of the value of z, if $n = p/q$, where p & $q \in \mathbb{Z}$, $q \neq 0$ and are coprime.

$z = (\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$. is the only value of z.

$z = (\cos\theta + i\sin\theta)^{\frac{1}{5}} = \left(\cos\frac{\theta}{5} + i\sin\frac{\theta}{5}\right)$ one of the value of z.

$x^3 = 1 \Rightarrow x = 1 \in \mathbb{R}$

$x = (1)^{1/3} = (\cos 0 + i\sin 0)^{\frac{1}{3}} = (\cos 2n\pi + i\sin 2n\pi)^{\frac{1}{3}}$

$= \left(\cos\frac{2n\pi}{3} + i\sin\frac{2n\pi}{3}\right)$

$n = 0, 1, 2, 3, 4, 5$

4 basic steps to determine the roots of a complex number are :

(a) Write the complex number whose roots are to be determined in polar form/ exponential form.

(b) Add $2m\pi$ to the argument

(c) Apply DMT

(d) Put $m = 0, 1, 2, 3, \ldots.(n-1)$ to get all then n ; n^{th} roots.

Q

(i) $z=\left(1+i\sqrt{3}\right)^{1/4}$ (ii) $z=(-8)^{2/3}$

(i) $z = 2^{\frac{1}{4}}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^{1/4}$

$$z = (2)^{1/4}\left(\cos\left(2m\pi + \frac{\pi}{3}\right) + i\sin\left(2m\pi + \frac{\pi}{3}\right)\right)^{\frac{1}{4}}$$

$$z = (2)^{1/4}\left(\cos\left(\frac{2m\pi + \pi/3}{4}\right) + i\sin\left(\frac{2m\pi + \pi/3}{4}\right)\right)$$

$m = 0;\ 1;\ 2;\ 3$

if $m=0$ then $z_1 = (2)^{1/4}\cdot \operatorname{Cis}\left(\frac{\pi}{12}\right) = (2)^{1/4} e^{i\left(\frac{\pi}{12}\right)}$.

if $m=1$ then $z_2 = (2)^{1/4} \operatorname{Cis}\left(\frac{7\pi}{12}\right) = (2)^{1/4} e^{i\left(\frac{7\pi}{12}\right)}$.

if $m=2$ then $z_3 = (2)^{1/4} \operatorname{Cis}\left(\frac{13\pi}{12}\right)$

if $m=3$ " $z_4 = (2)^{1/4} \operatorname{Cis}\left(\frac{19\pi}{12}\right)$

(ii) $z = (-8)^{2/3} = \left((-8)^2\right)^{1/3} = (64)^{\frac{1}{3}}$

$$z = 4\,(1)^{1/3} = 4\left(\cos\frac{2m\pi}{3} + i\sin\frac{2m\pi}{3}\right)$$

$m = 0;\ 1;\ 2.$

Q $2\sqrt{2}z^4 = \left(\sqrt{3}-1\right)+i\left(\sqrt{3}+1\right)$

$$z^4 = \frac{\sqrt{3}-1}{2\sqrt{2}} + i\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)$$

$$z = \left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right)^{\frac{1}{4}}$$

$$z = \cos\left(\frac{2m\pi + \frac{5\pi}{12}}{4}\right) + i\sin\left(\frac{\frac{5\pi}{12} + 2m\pi}{4}\right)$$

$m = 0; 1; 2; 3$

Q Find the roots of the equation $z^5 + z^4 + z^3 + z^2 + z + 1 = 0$ having the least positive argument.

Sol^n

1. $\dfrac{(z^6 - 1)}{(z-1)} = 0 \Rightarrow z^6 = 1 \; ; \; \boxed{z \neq 1}$

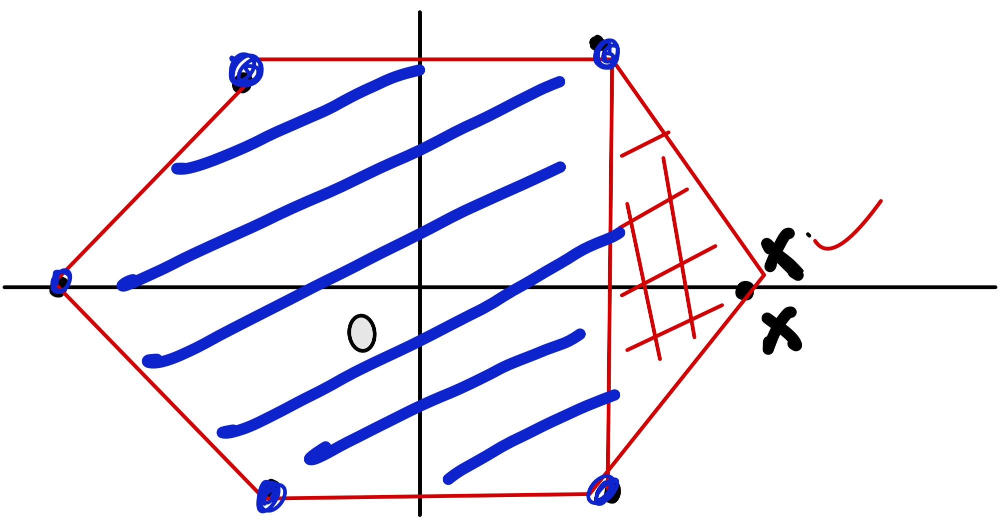
$z = (1)^{1/6} = \left(\cos \frac{2m\pi}{6} + i \sin \frac{2m\pi}{6}\right)$

$m = 0; 1; 2; 3; 4; 5$

XX

$\frac{2\pi}{6}, \frac{4\pi}{6}, \frac{6\pi}{6}, \frac{8\pi}{6}, \frac{10\pi}{6}$

$m = 1 \rightarrow \boxed{\angle PA = \pi/3}$



Q Find the number of roots of the equation $z^{10} - z^5 - 992 = 0$ with real part −ve.

Soln

$$z^5 = t$$

$$t^2 - t - 992 = 0.$$

$$t^2 - 32t + 31t - 992 = 0.$$

$$(t-32)(t+31) = 0.$$

$$z^5 = 32$$

$$z = 2(1)^{1/5}$$

$$= 2\left(\cos\frac{2m\pi}{5} + i\sin\frac{2m\pi}{5}\right)$$

$m = 0;\ 1;\ 2;\ 3;\ 4$

✗ ✗ ✓ ✓ ✗

(2) soln with −ve real part

$$z^5 = -31.$$

$$z = (31)^{1/5}(-1)^{1/5}$$

$$z = \lambda\left(\cos\left(\frac{\pi+2m\pi}{5}\right) + i\sin\left(\frac{\pi+2m\pi}{5}\right)\right)$$

$m = 0;\ 1;\ 2;\ 3;\ 4$

✗ ✓ ✓ ✓ ✗

(3) soln with −ve real part

Hence, 5 Complex Nos with −ve Real part.

Q Solve the following equations : (i) $\overline{z} = iz^2$ HW (ii) $z^5 = \overline{z}$ (iii) $z^4 = |z|$ HW

Rem

$$\boxed{z^n = \overline{z}}$$

$(n+2)$ sol^n

$$|z^n| = |\overline{z}| = |z|$$

$$|z|^n = |z| \Rightarrow |z|\left(|z|^{n-1} - 1\right) = 0$$

$|z| = 0$ OR $|z| = 1$

$\boxed{z = 0}$ * 1 sol^n

$$z^n \cdot \overline{z} = z\overline{z} = |z|^2$$

$$\boxed{z^{n+1} = 1.} \longrightarrow z = (1)^{\frac{1}{n+1}}$$

$(n+1)$ sol^n

Total $(n+2)$ sol^n.

Cube roots of unity : Imp

$$z^3 = 1 \Rightarrow z = (1)^{1/3} = \left(\cos\frac{2m\pi}{3} + i\sin\frac{2m\pi}{3}\right)$$

$m = 0;\ 1;\ 2$

$m=0 \Rightarrow z_1 = \text{Cis}\, 0 = 1 = e^{i0}$

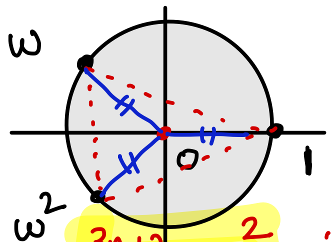
$m=1 \Rightarrow z_2 = \text{Cis}\,\frac{2\pi}{3} = e^{i2\pi/3} = \omega = \frac{-1+i\sqrt{3}}{2}$

$m=2 \Rightarrow z_3 = \text{Cis}\,\frac{4\pi}{3} = e^{i4\pi/3} = \omega^2 = \frac{-1-i\sqrt{3}}{2}$

$\omega = \overline{\omega^2}$

$\omega^2 = \overline{\omega}$

$1 = |\omega| = |\omega^2|$



Note:-

Rem

① $\boxed{\begin{aligned} -1+i\sqrt{3} &= 2\omega \\ -1-i\sqrt{3} &= 2\omega^2 \end{aligned}}$

$*\ \boxed{\omega^{3n} = 1}\ ;\ \omega^{3n+1} = \omega\ ;\ \omega^{3n+2} = \omega^2\ ;\ n \in I.$

② $1 + \omega + \omega^2 = 0$

③ $1^r + \omega^r + (\omega^2)^r \longrightarrow$
- $0\ ;\ r$ is not integral multiple of 3, i.e. $r \neq 3\lambda;\ \lambda \in I$
- $3\ ;\ r = 3\lambda;\ \lambda \in I$

eg: $1^4 + \omega^4 + (\omega^2)^4 = \underbrace{1 + \omega + \omega^2}_{0} = 0.$

$\omega^8 = \underbrace{\omega^6}_{1} \cdot \omega^2 = \omega^2$

$1+w=-w^2$
$1+w^2=-w$

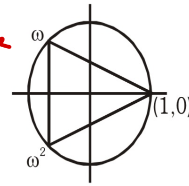
Note :

(i) $\omega = \overline{\omega^2}$, $\omega^{3n} = 1$, $\omega^{3n+1} = \omega$, $\omega^{3n+2} = \omega^2$, $\frac{1}{\omega} = \omega^2$, $\frac{1}{\omega^2} = \omega$, $1.\omega.\omega^2 = 1$

(ii) If w is one of the imaginary cube roots of unity then $1 + w + w^2 = 0$.
In general $1 + w^r + w^{2r} = 0$; when $r \neq 3\lambda$
$1 + w^r + w^{2r} = 3$; when $r = 3\lambda$

(iii) $|1| = |\omega| = |\omega^2|$

(iv) The three cube roots of unity when plotted on the argand plane constitute the vertices of an equilateral triangle. and they lie on a Unit Circle centered at origin.



(v) The following factorisation should be remembered :
(a, b, c $\in$ R & ω is the cube root of unity)

Rem

$a^3 - b^3 = (a - b)(a - \omega b)(a - \omega^2 b)$; $x^2 + x + 1 = (x - \omega)(x - \omega^2)$;

$a^3 + b^3 = (a + b)(a + \omega b)(a + \omega^2 b)$; $x^2+1 = (x-i)(x+i)$

$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a + \omega b + \omega^2 c)(a + \omega^2 b + \omega c)$

$x^2 - x + 1 = (x+w)(x+w^2)$

$$a^3 - b^3 = (a-b)(a^2+ab+b^2) = (a-b)\, b^2\left(\underbrace{\left(\frac{a}{b}\right)}_{t}^2 + \left(\frac{a}{b}\right) + 1\right)$$

$$= (a-b)\, b^2 \left(\frac{a}{b} - w\right)\left(\frac{a}{b} - w^2\right)$$

$$= (a-b)(a-bw)(a-bw^2).$$

Alt:

$$z^3 = 1 \Rightarrow (z-1)\underbrace{(z^2+z+1)}_{D<0} = 0$$

$$z^2+z+1 = (z-w)(z-w^2)$$

$$z^3 + 0z^2 + 0z - 1 = 0 \quad \begin{cases} 1 \\ w \\ w^2 \end{cases}$$

$1+w+w^2 = 0.$ & $1 \cdot w \cdot w^2 = 1 \Rightarrow w^3 = 1.$

$$z^3 = -1 \Rightarrow z = (-1)^{1/3} = -\underbrace{(1)^{1/3}}$$

$$\underbrace{-1 \quad -\omega \quad -\omega^2}$$

$$z^3 + 1 = 0 \Rightarrow (z+1)\underbrace{(z^2 - z + 1)}_{D<0} = 0$$

$$-\omega \ \& \ -\omega^2$$

$$\boxed{z^2 - z + 1 = (z+\omega)(z+\omega^2)}$$

eg:

$$z^3 + 27 = 0 \Rightarrow z = (-27)^{1/3} \to -3, \ -3\omega, \ -3\omega^2.$$

$$z = -3\,(1)^{1/3}$$

$$z = (-8)^{2/3} = (64)^{1/3} \to 4, \ 4\omega, \ 4\omega^2.$$

Note:- $a, b, c \in \mathbb{R}$

$$a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$$

$$= \left(\frac{a+b+c}{2}\right)\left((a-b)^2+(b-c)^2+(c-a)^2\right)$$

$$= (a+b+c)\left((a+b+c)^2-3\sum ab\right).$$

$$= (a+b+c)\underbrace{(a+b\omega+c\omega^2)}\underbrace{(a+b\omega^2+c\omega)}$$

conjugate

Rem

$$|a+b\omega+c\omega^2| = |a+b\omega^2+c\omega| = \sqrt{a^2+b^2+c^2-ab-bc-ca}$$

$$= \sqrt{\frac{(a-b)^2+(b-c)^2+(c-a)^2}{2}}$$

$$\left.\begin{aligned} z &= a+b\omega+c\omega^2 \\ \bar{z} &= a+b\omega^2+c\omega \end{aligned}\right\}$$

$$z\bar{z} = |z|^2$$

$$a^2+b^2+c^2-ab-bc-ca$$

E(1) If ω is imaginary cube root of unity, then find the value of

(a) $(1 + \omega - \omega^2)^7$.

(b) $\sum_{r=0}^{10} (1 + \omega^r + \omega^{2r})$

(c) $(1 + 2\omega + 3\omega^2)^{10} + (2 + 3\omega + \omega^2)^{10} + (3 + \omega + 2\omega^2)^{10}$

(a)

$$1+\omega = -\omega^2$$

$$(1+\omega-\omega^2)^7 = (-2\omega^2)^7 = -128\,\omega^{14}$$

$$= -128.\,\omega^2.$$

(b)

$$\sum_{r=0}^{10}(1+\omega^r+\omega^{2r}) = 3+3+3+3$$

$$= 12 \text{ Ans}$$

$r = 0;\ 1;\ 2;\ 3;\ 4;\ 5;\ 6;\ 7;\ 8;\ 9;\ 10.$

(c)

$$(1+2\omega+3\omega^2)^{10} + (2+3\omega+\omega^2)^{10} + (3+\omega+2\omega^2)^{10}$$

$$\omega^{10}\underbrace{\left(\frac{1}{\omega}+2+3\omega\right)^{10}} + \underbrace{(2+3\omega+\omega^2)^{10}} + (\omega^2)^{10}\underbrace{\left(\frac{3}{\omega^2}+\frac{1}{\omega}+2\right)^{10}}$$

$$(\omega+1+\omega^2)(2+3\omega+\omega^2)^{10} = 0 \text{ Ans}$$

HW

E(2) If ω is non-real cube root of unity, then find the value of $\dfrac{1+2\omega+3\omega^2}{2+3\omega+\omega^2}+\dfrac{2+3\omega+\omega^2}{3+\omega+2\omega^2}$.

HW

Q Value of $\left(\sqrt{3}-i\right)^{100}+\left(\sqrt{3}+i\right)^{100}$

E(5) α & β are roots of $x^2 - x + 1 = 0$, then find value of $\alpha^{2013} + \beta^{2013}$.

HW

HW

E(6) If ω is non-real cube root of unity, then prove that z, ωz, $\omega^2 z$ are vertices of equilateral triangle, where $z \neq 0$.

E(7) Find the solutions of given equations : (i) $z^3 + 27 = 0$ (ii) $z^3 - 27 = 0$ (iii) $4z^2 + 2z + 1 = 0$

HW

E(8) If α be a complex number satisfying $z^4 + z^3 + 2z^2 + z + 1 = 0$, then find $|\alpha|$.

HW

HW

E(9) If the area of the triangle in the Argand diagram, formed by Z, ωZ and $Z + \omega Z$ were ω is the usual complex cube root of unity is $16\sqrt{3}$ square units, then $|Z|$ is -

(A) 16 (B) 4 (C) 8 (D) 3

Q Find all the complex numbers z satisfying $z^2 + z|z| + |z^2| = 0$.

HW

E(2) If ω is non-real cube root of unity, then find the value of $\dfrac{1+2\omega+3\omega^2}{2+3\omega+\omega^2}+\dfrac{2+3\omega+\omega^2}{3+\omega+2\omega^2}$.

Solⁿ

$$\frac{\omega\left(\frac{1}{\omega}+2+3\omega\right)}{(2+3\omega+\omega^2)}+\frac{\omega\left(\frac{2}{\omega}+3+\omega\right)}{(3+\omega+2\omega^2)}$$

$$= 2\omega.$$

Q Value of $(\sqrt{3}-i)^{100}+(\sqrt{3}+i)^{100} = 2^{100}(\omega+\omega^2) = -2^{100}$

$$2\omega = -1+i\sqrt{3}$$

$$2\omega = i(i+\sqrt{3}) \Rightarrow (\sqrt{3}+i)^{100} = \left(\frac{2\omega}{i}\right)^{100}$$

$$(\sqrt{3}+i)^{100} = 2^{100}\cdot\omega^{100} = 2^{100}\cdot\omega$$

$$2\omega^2 = -1-i\sqrt{3}$$

$$i(2\omega^2) = -i+\sqrt{3} \Rightarrow (\sqrt{3}-i)^{100} = i^{100}(2\omega^2)^{100}$$

$$= 2^{100}\cdot\omega^{200}$$

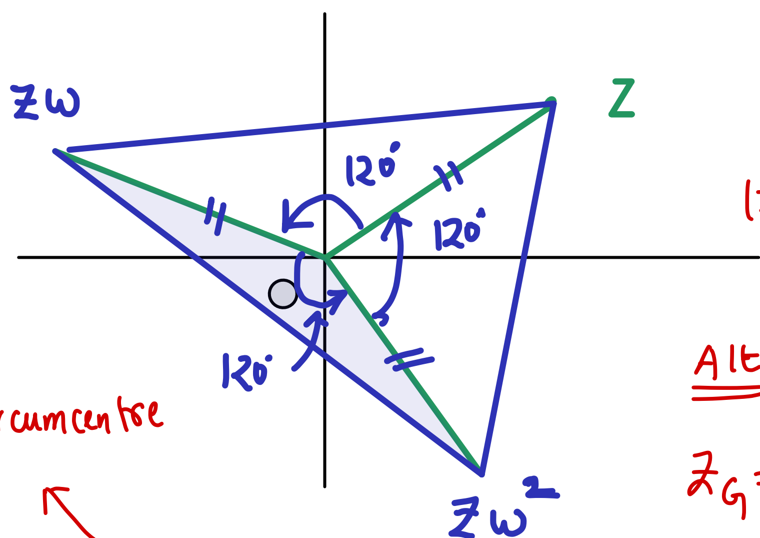
$$= 2^{100}\omega^2$$

E(5) α & β are roots of $x^2 - x + 1 = 0$, then find value of $\alpha^{2013} + \beta^{2013}$.

$x^2 - x + 1 = 0 \rightarrow -\omega, -\omega^2$

$(-\omega)^{2013} + (-\omega^2)^{2013} = -1 + (-1) = -2$. Ans

E(6) If ω is non-real cube root of unity, then prove that z, ωz, $\omega^2 z$ are vertices of equilateral triangle, where $z \neq 0$.



$|z\omega| = |z||\omega|$ ($|\omega| \to 1$)

$|z\omega| = |z|$

$|z\omega^2| = |z||\omega^2|$ ($|\omega^2| \to 1$)

'O' → Circumcentre

Alt:

$z_G = \dfrac{z + z\omega + z\omega^2}{3}$

$z_G = 0$

Centroid $= 0$.

E(7) Find the solutions of given equations : (i) $z^3 + 27 = 0$ (ii) $z^3 - 27 = 0$ (iii) $4z^2 + 2z + 1 = 0$

(i) $z = (-27)^{1/3}$

$-3;\ -3\omega;\ -3\omega^2$

(ii) $z^3 = 27$

$z = (27)^{1/3}$ → 3, 3ω, $3\omega^2$

(iii) $4z^2 + 2z + 1 = 0$

Let $2z = t$

$t^2 + t + 1 = 0$ → ω, ω^2

$2z = \omega;\ \omega^2$

$\therefore z = \dfrac{\omega}{2};\ \dfrac{\omega^2}{2}.$

E(8) If α be a complex number satisfying $z^4 + z^3 + 2z^2 + z + 1 = 0$, then find $|\alpha|$.

Sol^n

$$z^4 + z^3 + z^2 + z^2 + z + 1 = 0$$

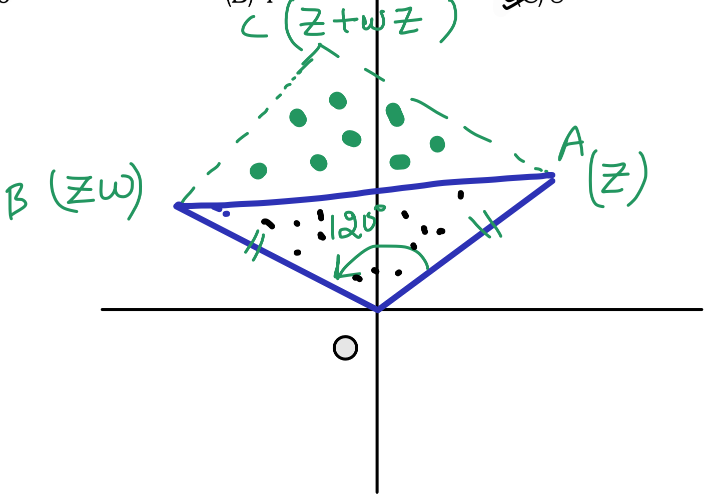
$$(z^2+1)\,(z^2+z+1) = 0$$

$$i \; ; \; -i \qquad\qquad \omega \; ; \; \omega^2$$

$$\therefore \; |\alpha| = 1.$$

E(9) If the area of the triangle in the Argand diagram, formed by Z, ωZ and Z + ωZ were ω is the usual complex cube root of unity is $16\sqrt{3}$ square units, then $|Z|$ is -

(A) 16 (B) 4 ~~(C)~~ 8 (D) 3



$$\frac{1}{2}\,|z|\,|z\omega|\,\sin 120^\circ = 16\sqrt{3}$$

$$|z|^2 = 64 \Rightarrow |z| = 8.$$

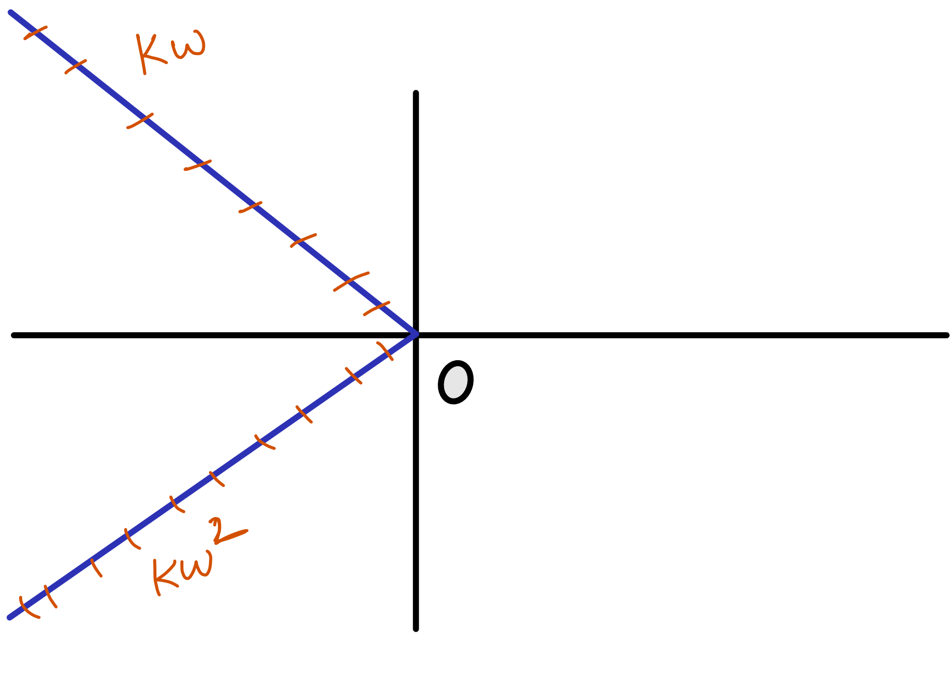
Q Find all the complex numbers z satisfying $z^2 + z|z| + |z^2| = 0$.

$$\left(\frac{z}{|z|}\right)^2 + \frac{z}{|z|} + 1 = 0 \qquad \text{Let } \frac{z}{|z|} = t$$

$$t^2 + t + 1 = 0$$

$$t = \omega ; \quad \omega^2$$

$$z = |z|\omega ; \quad z = |z|\omega^2 . \qquad |z| = k ; \underline{\underline{k > 0}}$$



$$z^3 = 1 \begin{cases} 1 = e^{i0} \\ \omega = e^{i2\pi/3} \\ \omega^2 = e^{i4\pi/3} \end{cases}$$

$$z^3 + 0z^2 + 0z - 1 = 0$$

$S.O.R = 0$

$P.O.R = 1$

$$1^r + \omega^r + (\omega^2)^r \begin{cases} 0\ ;\ r \neq 3\lambda \\ 3\ ;\ r = 3\lambda \end{cases}$$

$\lambda \in I$

$$z^4 = 1 \begin{cases} 1 \\ -1 \\ i \\ -i \end{cases}$$

$$z^4 + 0z^3 + 0z^2 + 0z - 1 = 0$$

$S.O.R = 0$

$P.O.R = -1$

$$1^r + (-1)^r + (i)^r + (-i)^r \begin{cases} 0\ ;\ r \neq 4\lambda \\ 4\ ;\ r = 4\lambda \end{cases}$$

$\lambda \in I.$

$$\boxed{x^n - a = 0.}$$

$\rightarrow x^n + \underline{\underline{0}}x^{n-1} + \cdots\cdots - a = 0$

$x = (a)^{1/n}$ with roots $x_1, x_2, x_3, \ldots, x_n$

1. Total No. of roots $= n$.
2. $|x_1| = |x_2| = \cdots |x_n| = |a^{\frac{1}{n}}|$
3. All roots lie on circle centered at origin.
4. $\sum_{i=1}^{n} x_i = 0$. (S.O.R)
5. $\prod_{i=1}^{n} x_i \to \begin{cases} a; & n \in \text{odd.} \\ -a; & n \in \text{even.} \end{cases}$
6. $\sum_{i=1}^{n} x_i^r \to \begin{cases} 0; & r \neq n\lambda, \ \lambda \in I \\ n(a)^{r/n}; & r = n\lambda \end{cases}$ **

$x^n - a = 0$ with roots $x_1, x_2, \ldots, x_n$

$$\boxed{x_1^n = a.}$$

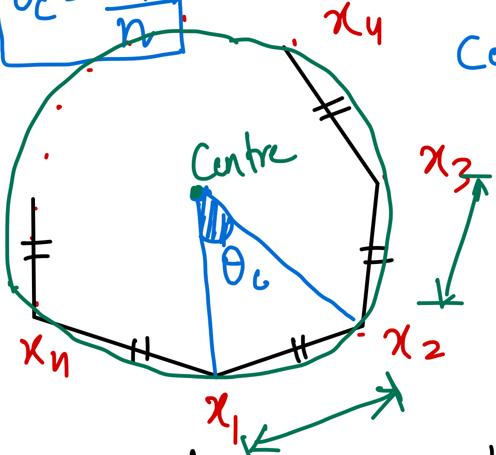
$x_1^r = ?$

$x_1^{n\lambda} = (x_1^n)^\lambda = a^\lambda$

$x_1^r + x_2^r + \cdots + x_n^r = a^\lambda + a^\lambda + \cdots + a^\lambda$

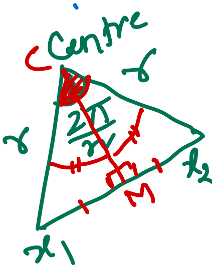
7. All the roots are in GP with Common ratio $= e^{i(2\pi/n)}$.

$$\boxed{\theta_c = \frac{2\pi}{n}}$$



All sides are equal & all interior $\angle$'s are also equal.

8. $\underline{\underline{\text{side of } n\text{-gon.}}}$



$r = |x_1| = |x_2| = \cdots$

$r = |a^{1/n}|$

$$\boxed{\begin{aligned} &\text{Area of } \Delta x_1x_2C \\ &= \tfrac{1}{2}\cdot r\cdot r\cdot \sin\left(\tfrac{2\pi}{n}\right) \\ &\text{Ar. of } n\text{-gon} = \tfrac{nr^2}{2}\sin\left(\tfrac{2\pi}{n}\right) \end{aligned}}$$

$\frac{x_1M}{r} = \sin\left(\frac{\pi}{n}\right)$

$x_1x_2 = 2r\sin\left(\frac{\pi}{n}\right)$

Perimeter of n-gon $= 2nr\sin\left(\frac{\pi}{n}\right)$

eg:

$$Z = (\sqrt{3} + i)^{1/6}$$ $z_1, z_2, \ldots, z_6$

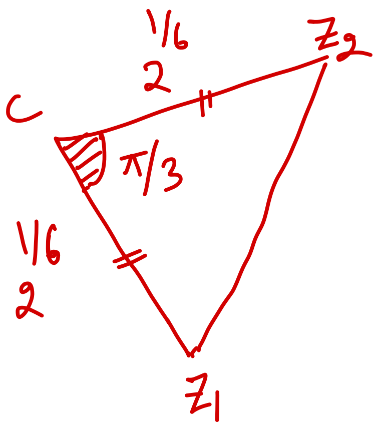
$$|z| = (2)^{1/6}.$$

① $\Sigma z_1 = 0$

② $\Pi z_1 = -(\sqrt{3}+i)$

$$z^6 = (\sqrt{3}+i)$$

$$z^6 + 0z^5 + 0z^4 + 0z^3 + 0z^2 + 0z - (\sqrt{3}+i) = 0$$



$$\text{Ar of hexagon} = 6\left(\frac{1}{2} \cdot (2)^{1/6} \cdot (2)^{1/6} \sin 60^\circ\right)$$

eg: $z = (\cos\alpha + i\sin\alpha)^{3/5}$. where $(\alpha \in \mathbb{R})$

Product of roots = ?

$$z = \left((\cos\alpha + i\sin\alpha)^{3}\right)^{\frac{1}{5}} = (\cos 3\alpha + i\sin 3\alpha)^{\frac{1}{5}}$$ x_1, x_2, x_3, x_4, x_5

① $x_1 x_2 x_3 x_4 x_5 = (\cos 3\alpha + i\sin 3\alpha)$.

② $x_1^4 + x_2^4 + x_3^4 + x_4^4 + x_5^4 = 0.$

n^{th} ROOTS OF UNITY :

$z^n = 1$ → $1, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}$

If $1, \alpha_1, \alpha_2, \alpha_3 \ldots\ldots \alpha_{n-1}$ are the n, n^{th} root of unity then :

(i) They are in G.P. with common ratio $e^{i(2\pi/n)}$ OR $\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$ and lie on standard unit circle on argand plane. n^{th} root of unity represents vertices of n sided polygon.

(ii) $1^p + \alpha_1^p + \alpha_2^p + \ldots\ldots + \alpha_{n-1}^p = 0$ if p is not an integral multiple of n .

$1^p + (\alpha_1)^p + (\alpha_2)^p + \ldots\ldots + (\alpha_{n-1})^p = n$ if p is an integral multiple of n .

* **(iii)** $(1-\alpha_1)(1-\alpha_2)\ldots\ldots(1-\alpha_{n-1}) = n$.

* **(iv)** $(1+\alpha_1)(1+\alpha_2)\ldots\ldots(1+\alpha_{n-1}) = 0$ if n is even and 1 if n is odd.

(v) $1 . \alpha_1 . \alpha_2 . \alpha_3 \ldots\ldots\ldots \alpha_{n-1} = 1$ or -1 according as n is odd or even.

* **(vi)** $(w-\alpha_1)(w-\alpha_2)\ldots\ldots(w-\alpha_{n-1}) = \begin{cases} 0 & \text{if } n = 3k \\ 1 & \text{if } n = 3k+1 \\ 1+w & \text{if } n = 3k+2 \end{cases}$

*

(vii) $\underbrace{\sum_{m=0}^{n-1}\left(\cos\frac{2m\pi}{n} + i\sin\frac{2m\pi}{n}\right)}_{} = 0$ → $\sum_{m=1}^{n-1} \text{Cis}\left(\frac{2m\pi}{n}\right) = -1$

$S.O.R = 1 + \alpha_1 + \alpha_2 + \ldots + \alpha_{n-1} = 0.$

$Z = (1)^{\frac{1}{n}} = e^{i\left(\frac{2m\pi}{n}\right)}$; $m = 0; 1; \ldots\ldots, n-1$

$m = 0 \Rightarrow Z_1 = e^{i0} = 1$ → 1

$m = 1 \Rightarrow Z_2 = e^{i\frac{2\pi}{n}} = \text{Cis}\left(\frac{2\pi}{n}\right) = \alpha_1 = \alpha$

$m = 2 \Rightarrow Z_3 = e^{i\frac{4\pi}{n}} = \text{Cis}\left(\frac{4\pi}{n}\right) = \alpha_2 = \alpha^2$

$\ldots\ldots \quad \alpha_3 = \alpha^3$

$\ldots\ldots \quad \alpha_4 = \alpha^4$

$\vdots$

$m = n-1 \Rightarrow Z_n = e^{i\frac{2(n-1)\pi}{n}} = \text{Cis}\frac{2(n-1)}{n}\pi = \alpha_{n-1} = \alpha^{n-1}$

α_1 & α_{n-1} → Conjugate of each other $\Rightarrow \alpha_1 + \alpha_{n-1} = 2\cos(2\pi/n)$

α_2 & α_{n-2} " " " " $\Rightarrow \alpha_2 + \alpha_{n-2} = 2\cos(4\pi/n)$

Rem

$$\boxed{1+x+x^2+\cdots+x^{n-1} = (x-\alpha_1)(x-\alpha_2)\cdots\cdots(x-\alpha_{n-1})}$$

$$z^n-1 = (z-1)(z-\alpha_1)(z-\alpha_2)\cdots\cdots(z-\alpha_{n-1})$$

$$\left(\frac{z^n-1}{z-1}\right) = (z-\alpha_1)(z-\alpha_2)\cdots\cdots(z-\alpha_{n-1})$$

$$\underbrace{1+z+z^2+z^3+\cdots+z^{n-1}}_{} = (z-\alpha_1)(z-\alpha_2)\cdots\cdots(z-\alpha_{n-1}) \quad -(1)-$$

put $z=\omega$; complex cube root of unity

$$\frac{\omega^n-1}{\omega-1} = (\omega-\alpha_1)(\omega-\alpha_2)\cdots\cdots(\omega-\alpha_{n-1}).$$

ω^{n}:
- $1 \Rightarrow n = 3\lambda$
- $\omega \Rightarrow n = 3\lambda+1$ $\quad$ $\lambda \in I$
- $\omega^2 \Rightarrow n = 3\lambda+2$

Rem.

$$\boxed{\sin\frac{\pi}{n}\cdot\sin\frac{2\pi}{n}\,\sin\frac{3\pi}{n}\cdots\sin\left(\frac{n-1}{n}\right)\pi=\frac{n}{2^{n-1}}}$$

$$\sin\frac{\pi}{5}\,\sin\frac{2\pi}{5}\,\sin\frac{3\pi}{5}\,\sin\frac{4\pi}{5}=\frac{5}{2^4}=\frac{5}{16}.$$

$z^n=1 \;\to\; 1,\ \alpha_1,\ \alpha_2,\ \ldots,\ \alpha_{n-1}$

$\alpha_1=\operatorname{Cis}\left(\frac{2\pi}{n}\right)$

$\alpha_2=\operatorname{Cis}\left(\frac{4\pi}{n}\right)$

$\vdots$

We know,

$$(1-\alpha_1)(1-\alpha_2)\cdots(1-\alpha_{n-1})=n$$

$$1-\alpha_1=1-e^{i2\pi/n}=1-\cos\frac{2\pi}{n}-i\sin\frac{2\pi}{n}$$

$$|1-\alpha_1|=\sqrt{\left(1-\cos\frac{2\pi}{n}\right)^2+\left(-\sin\frac{2\pi}{n}\right)^2}$$

$$=\sqrt{2\left(1-\cos\frac{2\pi}{n}\right)}=2\sin\frac{\pi}{n}$$

$$|1-\alpha_2|=2\sin\frac{2\pi}{n}.$$

$\vdots$

$$|1-\alpha_{n-1}|=$$

$$\overbrace{|1-\alpha_1|}^{2\sin\frac{\pi}{n}}\;\overbrace{|1-\alpha_2|}^{2\sin\frac{2\pi}{n}}\cdots\underbrace{|1-\alpha_{n-1}|}=n\Rightarrow\sin\frac{\pi}{n}\cdot\sin\frac{2\pi}{n}\cdots\sin\left(\frac{n-1}{n}\right)\pi=\frac{n}{2^{n-1}}$$

Q① Evaluate $\sum_{\lambda=0}^{12}\left(\sin\frac{2\lambda\pi}{13}-i\cos\frac{2\lambda\pi}{13}\right)$

$$z=(1)^{1/13}$$

$$\frac{1}{i}\sum_{\lambda=0}^{12}\left(i\sin\frac{2\lambda\pi}{13}-i^2\cos\frac{2\lambda\pi}{13}\right)=\frac{1}{i}\underbrace{\sum_{\lambda=0}^{12}\text{Cis}\left(\frac{2\lambda\pi}{13}\right)}_{0.}=0$$

② Evaluate $\sum_{\lambda=1}^{12}\left(\sin\frac{2\lambda\pi}{13}-i\cos\frac{2\lambda\pi}{13}\right)$ $=\frac{1}{i}\underbrace{\sum_{\lambda=1}^{12}\text{Cis}\left(\frac{2\lambda\pi}{13}\right)}_{-1}=\frac{-1}{i}$

$$=i$$

Ans

③ Evaluate $\sum_{\lambda=1}^{11}\left(\sin\frac{2\lambda\pi}{13}-i\cos\frac{2\lambda\pi}{13}\right)$

$$\frac{1}{i}\sum_{\lambda=1}^{11}\text{Cis}\left(\frac{2\lambda\pi}{13}\right)$$

$$\frac{1}{i}\left(\underbrace{\sum_{\lambda=1}^{12}\text{Cis}\,\frac{2\lambda\pi}{13}}-\text{Cis}\left(\frac{2\times12\pi}{13}\right)\right)$$

$$\frac{1}{i}\left(-1-\text{Cis}\left(\frac{24\pi}{13}\right)\right)$$

Q $\displaystyle\sum_{K=1}^{12} \cos\frac{2K\pi}{13} = ?$

-1

$z = (1)^{1/13}$

$$\sum_{m=1}^{12}\left(\text{Cis}\,\frac{2m\pi}{13}\right) = -1$$

$$\boxed{\begin{aligned}&\left(\cos\frac{2\pi}{13} + i\sin\frac{2\pi}{13}\right)\\ &+\left(\cos\frac{4\pi}{13} + i\sin\frac{4\pi}{13}\right)\\ &\quad\vdots \qquad \vdots\\ &+\left(\cos\frac{24\pi}{13} + i\sin\frac{24\pi}{13}\right)\end{aligned} = -1}$$

$\text{Re}\,(\text{LHS}) = \text{Re}\,(\text{RHS})$

Q If $1, \alpha, \alpha^2, \alpha^3, \dots, \alpha^{10}$ are the roots of equation $z^{11} - 1 = 0$ then $\underbrace{\prod_{K=1}^{10} (1+\alpha^K)}$ $= ?$

$$(1+\alpha)(1+\alpha^2)\dots(1+\alpha^{10})$$

$$(1+\alpha_1)(1+\alpha_2)\dots\dots(1+\alpha_{10}) = 1$$

Ans

Q If 'α' $(\alpha \neq 1)$ be the fifth root of unity then find value of $\log_{\sqrt{3}} \left| 1+\alpha+\alpha^2+\alpha^3 - \frac{2}{\alpha} \right| = ?$

Sol^n

$z^5 = 1$: $1, \alpha, \alpha^2, \alpha^3, \alpha^4$

$\boxed{\alpha^5 = 1}$ *

$|\alpha^4| = 1$

$$\log_{\sqrt{3}} \left| \underbrace{1+\alpha+\alpha^2+\alpha^3} - \frac{2}{\alpha} \right|$$

$$\log_{\sqrt{3}} \left| -\alpha^4 - \frac{2\alpha^4}{\alpha^5} \right|$$

$$\log_{\sqrt{3}} \left| -3\alpha^4 \right| = \log_{\sqrt{3}}(3)$$

$$= 2 \text{ Ans}$$

Q Let z is a complex number and $\alpha_1, \alpha_2, \ldots\ldots\alpha_{17}$ are 17^{th} roots of unity, then $\dfrac{\left(\sum_{k=1}^{17}|z+\alpha_k|^2\right)-z-\bar{z}}{17\left(|z|^2+1\right)-z-\bar{z}}$ is

1

Sol^n

$z^{17}=1$ — $\alpha_1, \alpha_2, \ldots, \alpha_{17}$

$|\alpha_i|=1$

$$\boxed{\alpha_1+\alpha_2+\cdots+\alpha_{17}=0} \; *$$

$$|z+\alpha_k|^2=(z+\alpha_k)(\bar{z}+\bar{\alpha}_k)$$
$$= z\bar{z}+z\bar{\alpha}_k+\bar{z}\alpha_k+\alpha_k\bar{\alpha}_k$$
$$= |z|^2+|\alpha|^2+z\bar{\alpha}_k+\bar{z}\alpha_k$$

$$\sum_{k=1}^{17}|z+\alpha_k|^2=\sum_{k=1}^{17}\left(|z|^2+\underbrace{|\alpha|^2}_{1}+z\bar{\alpha}_k+\bar{z}\alpha_k\right)$$

$$=17|z|^2+17+z\underbrace{\cancel{\left(\sum_{k=1}^{17}\bar{\alpha}_k\right)}}_{0}+\bar{z}\underbrace{\cancel{\left(\sum_{k=1}^{17}\alpha_k\right)}}_{0}$$

Q Prove that all roots of the equation $\left(\frac{z+1}{z}\right)^n = 1$ are collinear on the complex plane.

Sol^n

$$(z+1)^n = z^n$$

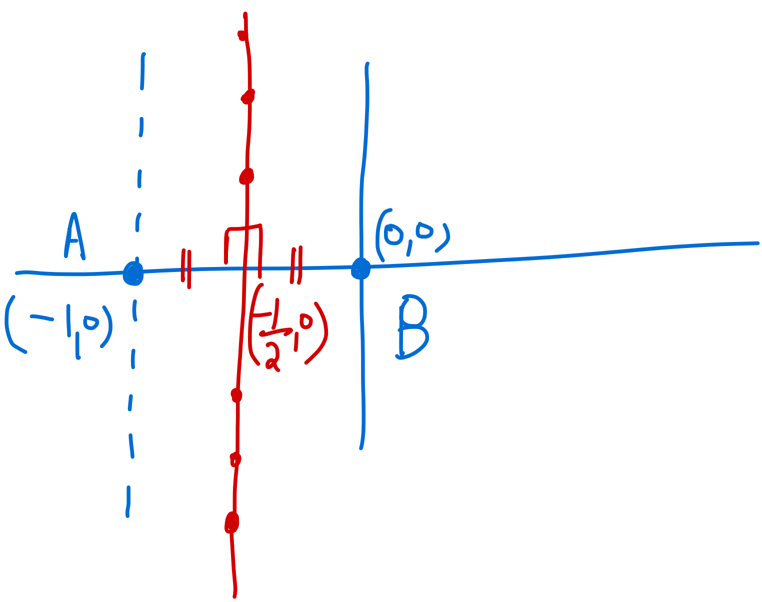
$$|(z+1)^n| = |z^n|$$

$$|z+1|^n = |z|^n$$

$$\boxed{|z+1| = |z|}$$

'z' lies on $\perp$ bisector joining

A $(-1,0)$ & B $(0,0)$

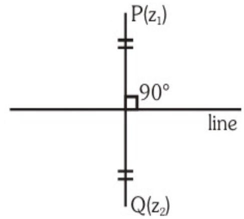


Reflection points for A line :

Q Prove that two points $P(z_1)$ & $Q(z_2)$ will be reflection points of each other in the straight line $\bar{\alpha}z + \alpha\bar{z} + r = 0$, iff $\bar{\alpha}z_1 + \alpha\bar{z}_2 + r = 0$, where r is real and α is non zero complex constant. **[IIT-1997]**

OR

$\bar{\alpha} z_2 + \alpha \bar{z}_1 + r = 0$



Proof :

Let $P(z_1)$ is the reflection point of $Q(z_2)$ then the perpendicular bisector of z_1 & z_2 must be the line

$\bar{\alpha}z + \alpha\bar{z} + r = 0$ .... (i)

Now perpendicular bisector of z_1 & z_2 is, $|z - z_1| = |z - z_2|$

or $(z - z_1)(\bar{z} - \bar{z}_1) = (z - z_2)(\bar{z} - \bar{z}_2)$

$-z\bar{z}_1 - z_1\bar{z} + z_1\bar{z}_1 = -z\bar{z}_2 - z_2\bar{z} + z_2\bar{z}_2$ ($z\bar{z}$ cancels on either side)

or $(\bar{z}_2 - \bar{z}_1)z + (z_2 - z_1)\bar{z} + z_1\bar{z}_1 - z_2\bar{z}_2 = 0$ .... (ii)

Comparing (i) & (ii) $\dfrac{\bar{\alpha}}{\bar{z}_2 - \bar{z}_1} = \dfrac{\alpha}{z_2 - z_1} = \dfrac{r}{z_1\bar{z}_1 - z_2\bar{z}} = \lambda$

$\therefore$ $\bar{\alpha} = \lambda(\bar{z}_2 - \bar{z}_1)$ .... (iii) $\alpha = \lambda(z_2 - z_1)$ .... (iv)

and $r = \lambda(z_1\bar{z}_1 - z_2\bar{z}_2)$ .... (v)

Multiply (iii) by z_1 ; (iv) by $\bar{z}_2$ and adding

$\bar{\alpha}z_1 + \alpha\bar{z}_2 + r = \lambda\,[(\cancel{z_1\bar{z}_2} - \cancel{z_1\bar{z}_1}) + \cancel{z_2\bar{z}_2} - \cancel{\bar{z}_2 z_1} + \cancel{z_1\bar{z}_1} - \cancel{z_2\bar{z}_2}] = 0$

Note that we could also multiply (iii) by z_2 & (4) by $\bar{z}_1$ & add to get the same result.

Hence the $\bar{\alpha}z_1 + \alpha\bar{z}_2 + r = 0$.

Again, let $\bar{\alpha}z_1 + \alpha\bar{z}_2 + r = 0$ is true w.r.t. the line $\bar{\alpha}z + \alpha\bar{z} + r = 0$.

Subtracting $\bar{\alpha}(z - z_1) + \alpha(\bar{z} - \bar{z}_2) = 0$

or $|(z - z_1)|\,|\cancel{\bar{\alpha}}| = |\cancel{\alpha}|\,|(\bar{z} - \bar{z}_2)|$

Hence 'z' lies on the perpendicular bisector of joins of z_1 & z_2.