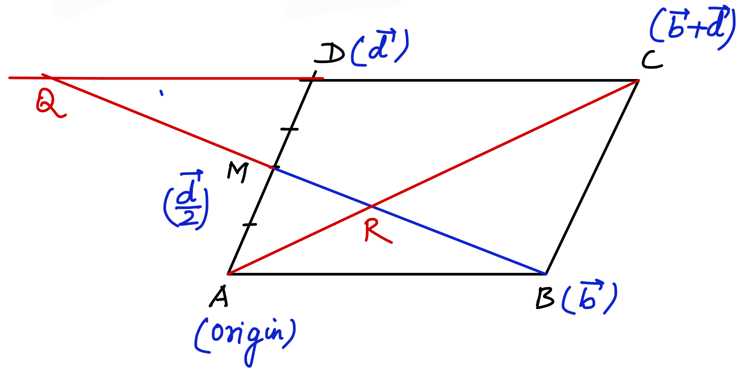
Q Through the middle point M of the side AD of || gm ABCD, a straight line BM is drawn intersecting AC at R and CD produced at Q. use vectors to prove that QR = 2RB.

Sol^n



Vector eqn of line AC:

$\vec{r} = \vec{0} + \lambda(\vec{b}+\vec{d})$ — ① —

Vector eqn of line BM:

$\vec{r} = \vec{b} + \mu\left(\vec{b} - \frac{\vec{d}}{2}\right)$ — ② —

Solve ① & ② to get pv of pt R.

$\left.\begin{array}{l}\lambda = 1+\mu. \\ \lambda = -\frac{\mu}{2}\end{array}\right\}$ $\mu = -\frac{2}{3}$, $\lambda = \frac{1}{3}$ $\therefore$ pv of R: $\frac{1}{3}(\vec{b}+\vec{d})$.

Vector eqn of line CD: $\vec{r} = \vec{d} + t(\vec{b})$ — ③ —

Solve ② & ③ to get pv of Q:

$1+\mu = t$

$-\frac{\mu}{2} = 1 \Rightarrow \boxed{\mu = -2}$ $\therefore$ pv of Q $(-\vec{b}+\vec{d})$.

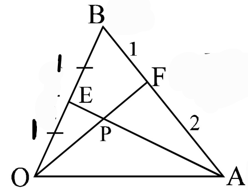
$\vec{QR}$ = pv of R − pv of Q $= \frac{4\vec{b} - 2\vec{d}}{3}$

$\vec{RB}$ = pv of B − pv of R $= \frac{2\vec{b} - \vec{d}}{3}$

$|\vec{QR}| = \frac{2}{3}|2\vec{b} - \vec{d}|$; $|\vec{RB}| = \frac{|2\vec{b} - \vec{d}|}{3}$ (H.P.)

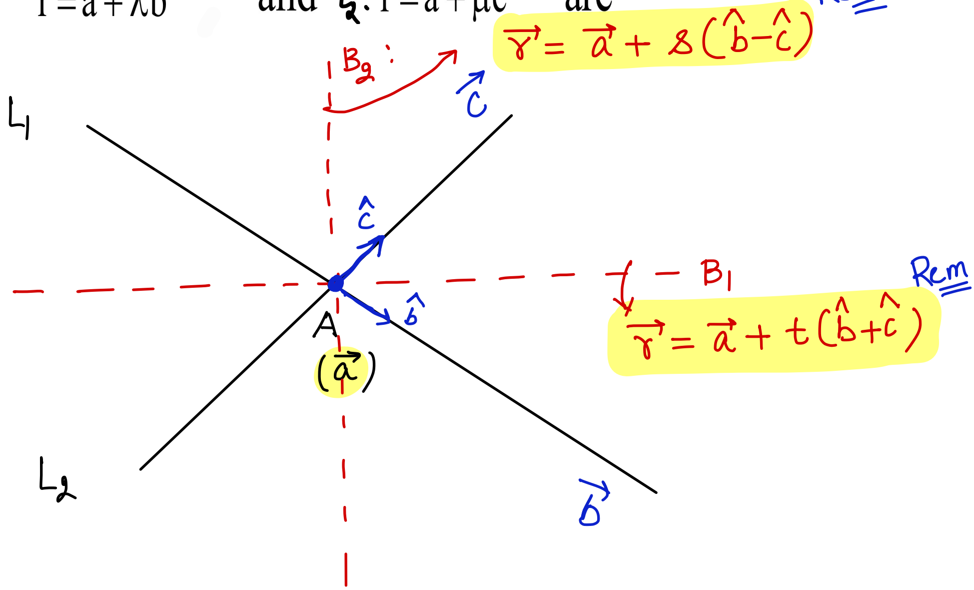
Q HW

In ΔAOB , E is the mid point of OB and F divides BA in the ratio 1 : 2 use vectors to prove that $\frac{OP}{PF}=\frac{3}{2}$.



Vector equation of the bisectors of the angles between the lines :-

$L_1: \vec{r}=\vec{a}+\lambda\vec{b}$ and $L_2: \vec{r}=\vec{a}+\mu\vec{c}$ are



TEST OF COLLINEARITY :

Three points A,B,C with position vectors $\vec{a},\vec{b},\vec{c}$ respectively are collinear, if and only if there exist scalars x , y , z not all zero simultaneously such that: $x\vec{a}+y\vec{b}+z\vec{c}=0$, where $x+y+z=0$.

[**Proof : (i) Necessary :**

Let A, B and C collinear $\overrightarrow{AB}=\vec{b}-\vec{a}$; $\overrightarrow{AC}=\vec{c}-\vec{a}$

$\therefore$ A, B, C are collinear $\Rightarrow$ $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are collinear

$\Rightarrow$ $\vec{b}-\vec{a}$ and $\vec{c}-\vec{a}$ are collinear

$\therefore$ $\vec{b}-\vec{a}=\lambda(\vec{c}-\vec{a})$

or $(\lambda-1)\vec{a}+\vec{b}-\lambda\vec{c}=0$

$x\vec{a}+y\vec{b}+z\vec{c}=0$ where $x=\lambda-1$, $y=1$, $z=-\lambda$,
obviously, $x+y+z=0$

(ii) Sufficient :

$\therefore$ Again let $x\vec{a}+y\vec{b}+z\vec{c}=0$ where $x+y+z=0$, x, y, z not all zero.
T.P.T. A,B, C are collinear, let $y\neq 0$

now $\vec{b}=-\dfrac{x\vec{a}+z\vec{c}}{y}=\dfrac{x\vec{a}+z\vec{c}}{x+z}$

Hence $B(\vec{b})$ divides the join of $A(\vec{a})$ and $C(\vec{c})$ in the ratio of z and x

$\Rightarrow$ A, B, C are collinear.

Q Find whether the following points are collinear or not

(i) $2\hat{i}+5\hat{j}-4\hat{k}$; $\hat{i}+4\hat{j}-3\hat{k}$; $4\hat{i}+7\hat{j}-6\hat{k}$

(ii) $3\hat{i}-4\hat{j}+3\hat{k}$; $-4\hat{i}+5\hat{j}-6\hat{k}$; $4\hat{i}-7\hat{j}+6\hat{k}$

Note: Collinearly can also be checked by first finding the equation of line through two points and satisfying the third point.

(i) $A(2,5,-4)$
$B(1,4,-3)$
$C(4,7,-6)$

$\overrightarrow{AB}=-\hat{i}-\hat{j}+\hat{k}$

$\overrightarrow{BC}=3\hat{i}+3\hat{j}-3\hat{k}$

$-\frac{1}{3}=-\frac{1}{3}=-\frac{1}{3}$

$\overrightarrow{AB}\parallel\overrightarrow{BC}$

Hence collinear

Alt:

$|\overrightarrow{AB}|=\sqrt{3}$

$|\overrightarrow{BC}|=3\sqrt{3}$

$|\overrightarrow{AC}|=2\sqrt{3}$

$AB+AC=BC$

Q

HW

Vectors $\vec{P}$, $\vec{Q}$, act at 'O' (origin) have a resultant $\vec{R}$. If any transversal cuts their line of action at A, B, C respectively, then show that $\frac{OP}{OA}+\frac{OQ}{OB}=\frac{OR}{OC}$.

Scalar Product (Dot Product) :-

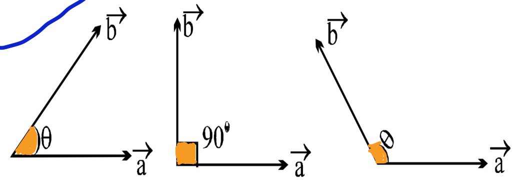
(1) $\vec{a}.\vec{b}=|\vec{a}||\vec{b}|\cos\theta$ for two non zero vectors $\vec{a}\,\&\,\vec{b}$ if

where $\vec{a}\,\&\,\vec{b}$ are not Collinear

(i) $\vec{a}.\vec{b}>0 \quad \Rightarrow \quad \theta$ is acute and $\vec{a}\neq\lambda\vec{b}$

(ii) $\vec{a}.\vec{b}=0 \quad \Rightarrow \quad \theta=\pi/2$

(iii) $\vec{a}.\vec{b}<0 \quad \Rightarrow \quad \theta$ is obtuse and $\vec{a}\neq\lambda\vec{b}$



$\vec{a}\,\&\,\vec{b}$ are not Collinear

① $\vec{a}.\vec{b}>0$

$\Rightarrow \theta\in[0,\pi/2)$

② $\vec{a}.\vec{b}<0$

$\Rightarrow \theta\in(\frac{\pi}{2},\pi]$

③ $\vec{a}.\vec{b}=0$

$\theta=\pi/2$ or $\vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$

* Dot product is commutative i.e. $\vec{a}.\vec{b}=\vec{b}.\vec{a}$

Conventionally $\vec{a}.\vec{a}=|\vec{a}|^2=\vec{a}^2$

$\hat{i}.\hat{i}=\hat{j}.\hat{j}=\hat{k}.\hat{k}=1$; $\hat{i}.\hat{j}=\hat{j}.\hat{k}=\hat{k}.\hat{i}=0$

* Dot product is distributive

$\vec{a}.(\vec{b}+\vec{c})=\vec{a}.\vec{b}+\vec{a}.\vec{c}$

$(\vec{a}+\vec{b})^2=(\vec{a}+\vec{b}).(\vec{a}+\vec{b})$

Rem

Simple identities to rembember are ,

(i) $(\vec{a}+\vec{b}).(\vec{a}-\vec{b})=\vec{a}^2-\vec{b}^2$

(ii) $(\vec{a}+\vec{b})^2=\vec{a}^2+2\vec{a}.\vec{b}+\vec{b}^2=(\vec{a}+\vec{b}).(\vec{a}+\vec{b})$

(iii) $(\vec{a}-\vec{b})^2=\vec{a}^2-2\vec{a}.\vec{b}+\vec{b}^2$

(iv) $(\vec{a}+\vec{b})^2=(\vec{a}-\vec{b})^2+4\vec{a}.\vec{b}$

(v) $(\vec{a}+\vec{b}+\vec{c})^2=\vec{a}^2+\vec{b}^2+\vec{c}^2-2\sum(\vec{a}.\vec{b})$

(vi) $\vec{a}.\vec{b}=\frac{1}{4}\left((\vec{a}+\vec{b})^2-(\vec{a}-\vec{b})^2\right)$

Q(1) $|\vec{a}|=11$; $|\vec{b}|=23$ and $|\vec{a}-\vec{b}|=30$, find $|\vec{a}+\vec{b}|$

Q(2) If $\vec{a}+\vec{b}+\vec{c}=\vec{0}$, $|\vec{a}|=3$; $|\vec{b}|=1$ and $|\vec{c}|=4$. Find $\sum(\vec{a}.\vec{b})=-13$

② $\vec{a}+\vec{b}+\vec{c}=\vec{0}$

$\underbrace{a^2}_{9}+\underbrace{b^2}_{+1}+\underbrace{c^2}_{+16}+2\underbrace{\sum\vec{a}.\vec{b}}=0.$

① $|\vec{a}-\vec{b}|=30 \Rightarrow |\vec{a}-\vec{b}|^2=900 \Rightarrow (\vec{a}-\vec{b}).(\vec{a}-\vec{b})=900$

$a^2-2\vec{a}.\vec{b}+b^2=900 \Rightarrow \vec{a}.\vec{b}=-125.$

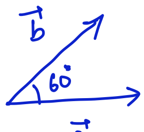
$|\vec{a}+\vec{b}|=?$ $\quad |\vec{a}+\vec{b}|^2=a^2+b^2+2\vec{a}.\vec{b}$

$=121+529-250$

$|\vec{a}+\vec{b}|^2=400 \Rightarrow |\vec{a}+\vec{b}|=20$ Ans

Q If $\vec{a}, \vec{b}, \vec{c}$ are unit vectors such that $\vec{a}$ is perpendicular to plane of $\vec{b}$ and $\vec{c}$ and angle between $\vec{b}$ and $\vec{c}$ is $\frac{\pi}{3}$. Find $|\vec{a}+\vec{b}+\vec{c}| = ?$

Sol^n $|\vec{a}| = |\vec{b}| = |\vec{c}|$ & $\vec{a}\cdot\vec{b} = 0$ & $\vec{a}\cdot\vec{c} = 0$



$$\vec{b}\cdot\vec{c} = |\vec{b}||\vec{c}|\cos 60° = 1.1.\frac{1}{2} = \frac{1}{2}$$

$$|\vec{a}+\vec{b}+\vec{c}|^2 = a^2+b^2+c^2+2(\vec{a}\cdot\vec{b}+\vec{b}\cdot\vec{c}+\vec{c}\cdot\vec{a})$$
$$= 1+1+1+2(0+\frac{1}{2}+0)$$
$$|\vec{a}+\vec{b}+\vec{c}|^2 = 4 \Rightarrow |\vec{a}+\vec{b}+\vec{c}| = 2. \text{ Ans}$$

* Q Given $a^2+b^2+c^2 = 9$ then find **max** value of $(a+2b+c)$?

Sol^n $\vec{V_1} = a\hat{i}+b\hat{j}+c\hat{k} \Rightarrow |\vec{V_1}| = \sqrt{a^2+b^2+c^2} = 3.$

$$\underbrace{a+2b+c} = \underbrace{(a\hat{i}+b\hat{j}+c\hat{k})}_{\vec{V_1}}\cdot\underbrace{(\hat{i}+2\hat{j}+\hat{k})}_{\vec{V_2}} = \underbrace{\sqrt{a^2+b^2+c^2}}_{3}\ \underbrace{\sqrt{1^2+2^2+1^2}}_{\sqrt{6}}\ \underbrace{\cos\theta}$$

$$\underbrace{a+2b+c} = (3)(\sqrt{6})\ \underline{\cos\theta}$$

$$\therefore (a+2b+c)_{max} = 3\sqrt{6} \quad \text{when } \cos\theta = 1.$$

Q If $a>0$ and A, B, C are variable angles such that $\left(\sqrt{a^2-4}\right)\tan A + a\tan B + \left(\sqrt{a^2+4}\right)\tan C = 6a$ then find min value of $\sum \tan^2 A$?

Sol^n $\vec{V_1} = (\tan A)\hat{i} + (\tan B)\hat{j} + (\tan C)\hat{k}$.
$\vec{V_2} = \left(\sqrt{a^2-4}\right)\hat{i} + a\hat{j} + \left(\sqrt{a^2+4}\right)\hat{k}$.
$\vec{V_1}\cdot\vec{V_2}$

$$\vec{V_1}\cdot\vec{V_2} = |\vec{V_1}||\vec{V_2}|\cos\theta$$

$$6a = \sqrt{\sum \tan^2 A}\,\sqrt{a^2\cancel{-4}+a^2+a^2\cancel{+4}}\cdot\cos\theta$$

$$\frac{6a}{\sqrt{3a^2}\cos\theta} = \sqrt{\sum\tan^2 A}. \Rightarrow \left(\sqrt{\sum\tan^2 A}\right)_{min} = \frac{6}{\sqrt{3}}$$

$$\left(\sum \tan^2 A\right)_{min} = \frac{36}{3} = 12 \quad \text{Ans}$$

Q If two points P and Q are on curve $y = 2^{x+2}$, such that $\vec{OP}\cdot\hat{i} = -1$ and $\vec{OQ}\cdot\hat{i} = 2$, where $\hat{i}$ is unit vector along x-axis, $|\vec{OQ} - 4\vec{OP}| = ?$

('O' is origin)

Sol^n pv of $P(\alpha\hat{i}+\beta\hat{j})$
pv of $Q(a\hat{i}+b\hat{j})$

$\vec{OQ}\cdot\hat{i} = a = 2$

$b = 2^{a+2} \Rightarrow b = 2^4 = 16$

$\vec{OP} = \alpha\hat{i}+\beta\hat{j}$

$\vec{OP}\cdot\hat{i} = -1 \Rightarrow \boxed{\alpha=-1}$

$\beta = 2^{\alpha+2} \Rightarrow \boxed{\beta = 2}$

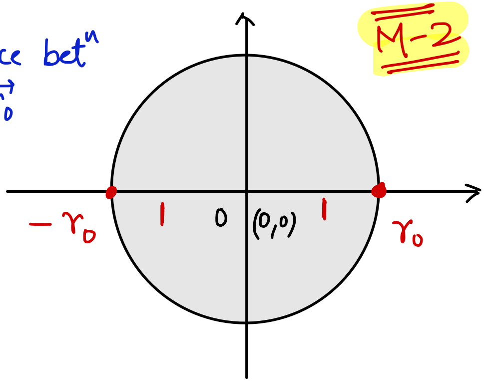
$\vec{OQ} = 2\hat{i}+16\hat{j}$
$\vec{OP} = -\hat{i}+2\hat{j}$

$|\vec{OQ} - 4\vec{OP}| = |6\hat{i}+8\hat{j}| = 10$

Ans

$|\vec{r}-\vec{r}_0| \Rightarrow$ distance betn $\vec{r}$ & $\vec{r}_0$

$|\vec{r}+\vec{r}_0| \Rightarrow$ distance betn $\vec{r}$ & $-\vec{r}_0$



M-2

Geometrically

Q Let $\vec{r}_0$ be constant vector and $\vec{r}$ is a variable vector in x-y plane such that both lie on the curve $|\vec{r}|=1$. If a and b are the least and greatest values of $\left(|\vec{r}-\vec{r}_0|+|\vec{r}+\vec{r}_0|\right)$, then (b^2-a^2) is equal to

Soln

$$\vec{r}_0 = (\cos\alpha)\hat{i} + (\sin\alpha)\hat{j}$$

$$\vec{r} = (\cos\theta)\hat{i} + (\sin\theta)\hat{j}$$

$$\underbrace{|\vec{r}-\vec{r}_0| + |\vec{r}+\vec{r}_0|}_{E} = \sqrt{(\cos\alpha-\cos\theta)^2+(\sin\alpha-\sin\theta)^2} + \sqrt{(\cos\alpha+\cos\theta)^2+(\sin\alpha+\sin\theta)^2}$$

$$E = \sqrt{2-2\cos(\alpha-\theta)} + \sqrt{2+2\cos(\alpha-\theta)}$$

$$E^2 = 2-\cancel{2\cos(\alpha-\theta)} + 2+\cancel{2\cos(\alpha-\theta)} + 2\sqrt{4-4\cos^2(\alpha-\theta)}$$

$$E^2 = 4+2\sqrt{4-4\cos^2(\alpha-\theta)}$$

$$E^2_{min} = 4 \Rightarrow E_{min} = 2 = a$$

$$E^2_{max} = 4+2\sqrt{4} \Rightarrow E_{max} = 2\sqrt{2} = b$$

$$\therefore\ b^2-a^2 = 4$$

Ans

General Expression for dot product

If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ & $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ then

$\vec{a}.\vec{b}=a_1b_1+a_2b_2+a_3b_3$ & $\cos\theta=\dfrac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}=\dfrac{a_1b_1+a_2b_2+a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}.\sqrt{b_1^2+b_2^2+b_3^2}}$

Projection of vector :-

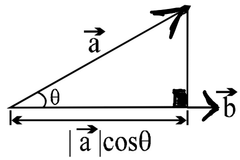
Rem

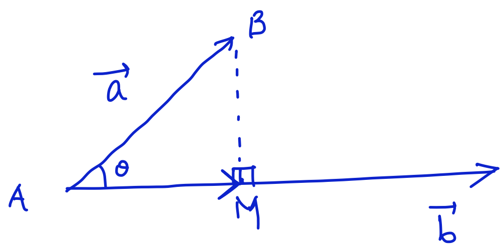
Projection of $\vec{a}$ on $\vec{b}$ $=\dfrac{\vec{a}.\vec{b}}{|\vec{b}|}$. It can be +ve, – ve or zero.

$\vec{a}.\hat{b}$

$|\vec{a}|\cos\theta$

Note that vector component of $\vec{a}$ along $\vec{b}=\left(\dfrac{\vec{a}.\vec{b}}{\vec{b}^2}\right)\vec{b}$ and vector component of $\vec{a}$ perpendicular to $\vec{b}=\vec{a}-\left(\dfrac{\vec{a}.\vec{b}}{\vec{b}^2}\right)\vec{b}$





$\vec{AM}=(|\vec{a}|\cos\theta)\,\hat{b}$

$\vec{AM}+\vec{MB}=\vec{AB}$

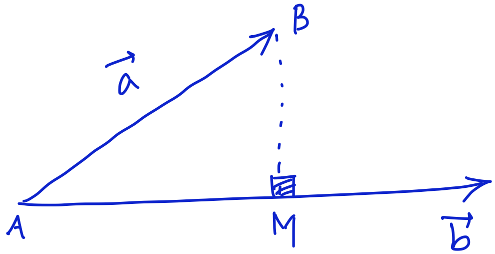
$\vec{MB}=\vec{a}-\vec{AM}.$

Q If $\vec{a}=2\hat{i}-3\hat{j}+6\hat{k}$ and $\vec{b}=-2\hat{i}+5\hat{j}-14\hat{k}$. If $\lambda=\dfrac{\text{Projection of } \vec{a} \text{ on } \vec{b}}{\text{Projection of } \vec{b} \text{ on } \vec{a}}$. Find λ.

$$\lambda=\frac{|\vec{a}|\cancel{\cos\theta}}{|\vec{b}|\cancel{\cos\theta}}=\frac{|\vec{a}|}{|\vec{b}|}$$

Q Express the vector $\vec{a} = 5\hat{i} - 2\hat{j} + 5\hat{k}$ as the sum of two vectors such that one is parallel to $\vec{b} = 3\hat{i} + \hat{k}$ and the other perpendicular to $\vec{b}$.

Sol



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General note:

(i) Maximum value of $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|$

(ii) Minimum value of $\vec{a}.\vec{b} = -|\vec{a}||\vec{b}|$

(iii) Any vector $\vec{a}$ can be expressed as $(\vec{a}.\hat{i})\hat{i} + (\vec{a}.\hat{j})\hat{j} + (\vec{a}.\hat{k})\hat{k}$

**

$$\boxed{\vec{a} = (\vec{a}.\hat{i})\hat{i} + (\vec{a}.\hat{j})\hat{j} + (\vec{a}.\hat{k})\hat{k}}$$

$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$\vec{a}.\hat{i} = a_1$

$\vec{a}.\hat{j} = a_2$

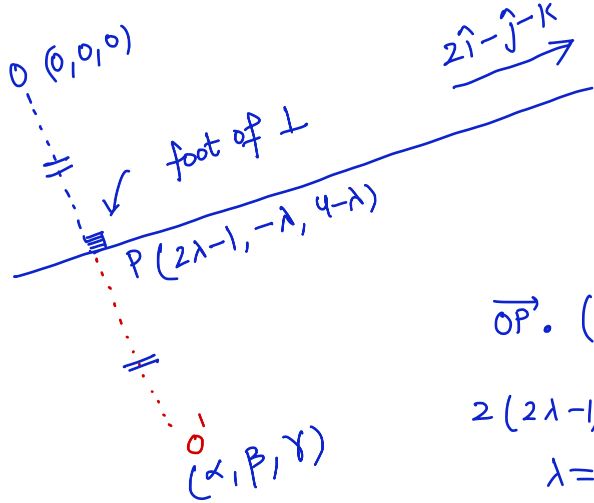
$\vec{a}.\hat{k} = a_3$

Q1 Find the foot of the perpendicular from the origin on the line $\vec{r} = -\hat{i} + 4\hat{k} + \lambda(2\hat{i} - \hat{j} - \hat{k})$. Also find the p.v. of its image in the line.

Q2 A line passes through a point with p.v. $\hat{i} - 2\hat{j} - \hat{k}$ and is parallel to the vector $\hat{i} - 2\hat{j} + 2\hat{k}$. Find the distance of a point P (5, 0, – 4) from the line.

HW

Sol^n



$\vec{r} = (-\hat{i} + 4\hat{k}) + \lambda(2\hat{i} - \hat{j} - \hat{k})$

general pt :

$(2\lambda - 1, -\lambda, 4 - \lambda)$

$\vec{OP} \cdot (2\hat{i} - \hat{j} - \hat{k}) = 0.$

$2(2\lambda - 1) + \lambda + \lambda - 4 = 0.$

$\lambda = 1$

$\therefore$ pv of $P(1, -1, 3)$

$\frac{\alpha + 0}{2} = 1 \; ; \; \frac{\beta + 0}{2} = -1 \; ; \; \frac{\gamma + 0}{2} = 3.$

② Ans = 5.