DIRECTION COSINES OF LINE : * *

If line makes angles α, β, γ with x, y & z axis respectively then $\pi - \alpha, \pi - \beta$ & $\pi - \gamma$ is another set of angle that line makes with principle axes. Hence if ℓ, m & n are direction cosines of line then $-\ell$, $-m$ & $-n$ are also direction cosines of the same line.

Q Find D.C. of line passing through $\underset{A}{(1,1,1)}$ & $\underset{B}{(3, 2, 3)}$

Sol^n $\vec{AB} = 2\hat{i} + \hat{j} + 2\hat{k}$; $|\vec{AB}| = 3$

dc's of line $\pm\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$

DIRECTION RATIOS OF A LINE :

3 numbers which are proportional to direction cosines of a line are called the direction ratio's of a line. Hence if a, b, c are d.r. of a line, then $\frac{\ell}{a} = \frac{m}{b} = \frac{n}{c}$

RELATIONSHIP BETWEEN DIRECTION COSINE & DIRECTION RATIOS :

If $\frac{\ell}{a} = \frac{m}{b} = \frac{n}{c} = \lambda$ (say) $\Rightarrow$ $\ell = a\lambda;\ m = b\lambda;\ n = c\lambda$

Now $(a^2 + b^2 + c^2)\lambda^2 = 1$ (as $\ell^2 + m^2 + n^2 = 1$)

$\Rightarrow$ $\lambda = \frac{\pm 1}{\sqrt{a^2 + b^2 + c^2}}$

$\ell = \pm\frac{a}{\sqrt{a^2 + b^2 + c^2}}$; $m = \pm\frac{b}{\sqrt{a^2 + b^2 + c^2}}$; $n = \pm\frac{c}{\sqrt{a^2 + b^2 + c^2}}$

Note :

(i) Direction ratios of a line joining two points A (x_1, y_1, z_1) and B (x_2, y_2, z_2) are proportional to $x_2 - x_1$; $y_2 - y_1$; $z_2 - z_1$

Since $\vec{AB} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$.

Hence the direction ratios of a vector $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ are proportional to the numbers a_1, a_2 and a_3.

(ii) If a line is having direction cosines ℓ, m, n $\Rightarrow$ it is travelling along the vector $\ell\hat{i} + m\hat{j} + n\hat{k}$. and $-\ell\hat{i} - m\hat{j} - n\hat{k}$

Hence angle between two lines with direction cosines ℓ_1, m_1, n_1 and ℓ_2, m_2, n_2 is

$$\cos\theta = \ell_1\ell_2 + m_1m_2 + n_1n_2$$

$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2} \times \sqrt{a_2^2 + b_2^2 + c_2^2}} \text{ (in terms of direction ratios)}$$

(a) If L_1 is perpendicular to L_2 then $\ell_1\ell_2 + m_1m_2 + n_1n_2 = 0$ or $a_1a_2 + b_1b_2 + c_1c_2 = 0$

(b) If L_1 is parallel to L_2 then $\frac{\ell_1}{\ell_2} = \frac{m_1}{m_2} = \frac{n_1}{n_2}$ or $\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2}$

(iii) Direction cosine of axes :

Since the positive x-axes makes angle 0°, 90°, 90° with axes of x, y and z respectively,

$\therefore$ D.C.'s of x axes are 1, 0, 0.

D.C.'s of y-axis are 0, 1, 0

D.C.'s of z-axis are 0, 0, 1

Q Find the direction cosines of a line perpendicular to two lines whose dr's are 1, 2, 3 and –2, 1, 4.

M–1

$\vec{a} = \hat{i} + 2\hat{j} + 3\hat{k}$

$\vec{b} = -2\hat{i} + \hat{j} + 4\hat{k}$

$\vec{a} \times \vec{b} = 5(\hat{i} - 2\hat{j} + \hat{k})$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ -2 & 1 & 4 \end{vmatrix} = \hat{i}(5) - \hat{j}(10) + \hat{k}(5)$$

$= 5\hat{i} - 10\hat{j} + 5\hat{k}$

$= 5(\hat{i} - 2\hat{j} + \hat{k})$

dc's of line $\pm\left(\frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$

M–2

Req. dir^n ratios of line be a, b, c

$\vec{v} = a\hat{i} + b\hat{j} + c\hat{k}$ now $\vec{v}\cdot\vec{a} = 0$ & $\vec{v}\cdot\vec{b} = 0$

$a + 2b + 3c = 0$

$-2a + b + 4c = 0$

$$\frac{a}{2(4) - 1(3)} = \frac{b}{-2(3) - 1(4)} = \frac{c}{1(1) - (-2\times2)}$$

Q The direction cosines ℓ, m, n of two lines are connected by the relations $\ell + m + n = 0$ and $2\ell m + 2\ell n - mn = 0$. Find them and the angles between them.

*

Sol^n

$$\boxed{l = -m-n} \quad -(1)-$$

$$2(-m-n)m + 2(-m-n)n - mn = 0$$

$$2m^2 + \underbrace{5mn} + 2n^2 = 0 \Rightarrow (2m+n)(m+2n) = 0$$

C-I

$$2m + n = 0$$

$$\boxed{n = -2m}$$

$$l = -m - (-2m)$$

$$l = m$$

$$l : m : n \equiv m : m : -2m$$

$$\equiv \underbrace{1 : 1 : -2}$$

dc's : $\pm\left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}\right)$

$$\vec{v_1} = \hat{i} + \hat{j} - 2\hat{k}$$

C-II

$$2n + m = 0$$

$$m = -2n$$

$$n = -\frac{m}{2}$$

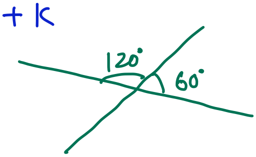
$$l = -m + \frac{m}{2} = -\frac{m}{2}$$

$$l : m : n \equiv -\frac{m}{2} : m : -\frac{m}{2}$$

$$\equiv \underbrace{1 : -2 : 1}$$

dc's : $\pm\left(\frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$

$$\vec{v_2} = \hat{i} - 2\hat{j} + \hat{k}$$



$$\vec{v_1} \cdot \vec{v_2} = |v_1||v_2| \cos\theta$$

$$1 - 2 - 2 = \sqrt{6}\sqrt{6} \cos\theta \Rightarrow \cos\theta = -\frac{3}{6} = -\frac{1}{2}$$

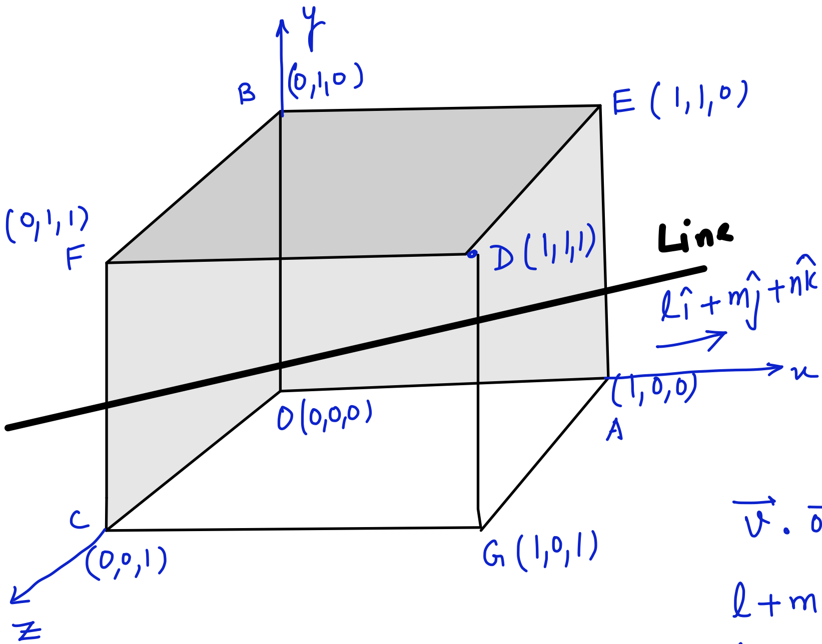
$$\theta = 120° \text{ or } 60°$$

Q

HW

A variable line has dc's ℓ, m, n and $\ell + \delta\ell$, $m + \delta m$, $n + \delta n$ in two adjacent positions. If $\delta\theta$ be the angle between the lines in these two positions then prove that $(\delta\theta)^2 = (\delta\ell)^2 + (\delta m)^2 + (\delta n)^2$.

Q A line makes angle $\alpha, \beta, \gamma, \delta$ with four diagonals of a cube. Prove that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \frac{4}{3}$.



Body diagonals.

$$\vec{OD} = \hat{i} + \hat{j} + \hat{k}$$
$$\vec{BG} = \hat{i} - \hat{j} + \hat{k}$$
$$\vec{AF} = -\hat{i} + \hat{j} + \hat{k}$$
$$\vec{CE} = \hat{i} + \hat{j} - \hat{k}.$$

$$\vec{v} = l\hat{i} + m\hat{j} + n\hat{k}$$

$$\vec{v} \cdot \vec{OD} = |v|\,|\vec{OD}| \cos\alpha$$

$$\left.\begin{aligned} l+m+n &= 1 \cdot \sqrt{3} \cos\alpha \\ l-m+n &= 1 \cdot \sqrt{3} \cos\beta \\ -l+m+n &= 1 \cdot \sqrt{3} \cos\gamma \\ l+m-n &= 1 \cdot \sqrt{3} \cos\delta. \end{aligned}\right\}$$

$l^2 + m^2 + n^2 = 1$

$\parallel^{ly}$

square & add

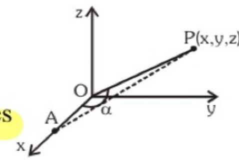
$$4 = 3\left(\Sigma \cos^2\alpha\right)$$

Important point :

Direction cosines of a line have two sets but direction ratios of a line have infinite possible sets.

PROJECTIONS :

(a) Projection of line segment OP on co-ordinate axes :



Let line segment make angle α with x-axis

Thus, the projections of line segment OP on axes are the absolute values of the co-ordinates of P. i.e.

Projection of OP on x-axis = $|x|$

Projection of OP on y-axis = $|y|$

Projection of OP on z-axis = $|z|$

Now, in ΔOAP, angle A is a right angle and OA = x

$$OP = \sqrt{x^2 + y^2 + z^2}$$

$$\therefore \quad \cos\alpha = \frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{|OP|}$$

if $|OP| = r$, then $x = |OP|\cos\alpha = \ell r$

Similarly $y = |OP|\cos\beta = mr$, $z = nr$, where ℓ, m, n are DC's of line

(b) Projection of a line segment AB on coordinate axes :

Projection of the point $A(x_1, y_1, z_1)$ on x-axis is $E(x_1, 0, 0)$. Projection of point $B(x_2, y_2, z_2)$ on x-axis is $F(x_2, 0, 0)$.

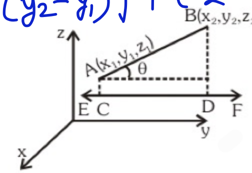
Hence projection of AB on x-axis is $EF = |x_2 - x_1|$.

Similarly, projection of AB on y and z-axis are $|y_2 - y_1|$, $|z_2 - z_1|$ respectively.

* **(c) Projection of line segment AB on a line having direction cosines ℓ, m, n :**

$\vec{AB} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$

Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$.



Now projection of AB on EF = CD = AB $\cos\theta$

$$= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2} \times \frac{|(x_2-x_1)\ell+(y_2-y_1)m+(z_2-z_1)n|}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}}$$

$$= |(x_2-x_1)\ell+(y_2-y_1)m+(z_2-z_1)n|$$

Projection of $\vec{AB}$ on $\vec{CD}$ is

$= |\vec{b_1} \cdot \vec{b_2}|$ where $\vec{b_1}$ = d.r of AB

$\vec{b_2}$ = d.c of CD

Q Find the length of projection of the line segment joining the points (–1, 0, 3) and (2, 5, 1) on the line whose direction ratios are 6, 2, 3.

Sol^n

The direction cosines ℓ, m, n of the line are given by $\frac{\ell}{6}=\frac{m}{2}=\frac{n}{3}=\frac{\sqrt{\ell^2+m^2+n^2}}{\sqrt{6^2+2^2+3^2}}=\frac{1}{\sqrt{49}}=\frac{1}{7}$

$$\therefore \quad \ell=\frac{6}{7},\ m=\frac{2}{7},\ n=\frac{3}{7}$$

The required length of projection is given by

$$=|\ell(x_2-x_1)+m(y_2-y_1)+n(z_2-z_1)|=\left|\frac{6}{7}[2-(-1)]+\frac{2}{7}(5-0)+\frac{3}{7}(1-3)\right|$$

$$=\left|\frac{6}{7}\times 3+\frac{2}{7}\times 5+\frac{3}{7}\times -2\right|=\left|\frac{18}{7}+\frac{10}{7}-\frac{6}{7}\right|=\left|\frac{18+10-6}{7}\right|=\frac{22}{7}$$ **Ans.**

**

Q Find the intercept made by lines

$l_1: \vec{r}=\hat{i}+\hat{j}+\hat{k}+\lambda(3\hat{i}-\hat{j})$

$l_2: \vec{r}=4\hat{i}-\hat{k}+\mu(2\hat{i}+3\hat{k})$

on a line with dr's (2, 1, 2) ?

PLANES :

DEFINITION OF PLANE :

A plane is a surface such that a line segment joining any two points on the surface lies wholly on it.

A linear equation in three variables of the type $Ax + By + Cz + D = 0$, denotes the general equation of a plane.

Where A, B, C are not simultaneously zero. Dividing by A we get $x + \frac{B}{A}y + \frac{C}{A}z + \frac{D}{A} = 0$. Thus equation of the plane involves only 3 arbitrary constants, hence in order to determine a unique plane 3 independent conditions are needed.

Note :

(i) Equation of y-z plane is $x = 0$.

(ii) Equation of z-x plane is $y = 0$.

(iii) Equation of x-y plane is $z = 0$.

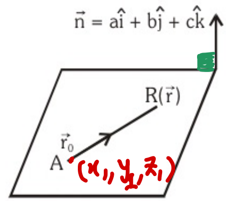
(iv) Equation of the plane parallel to x-y plane at a distance c is $z = c$ or $z = -c$.

(v) Equation of the plane parallel to y-z plane at a distance c is $x = c$ or $x = -c$

(vi) Equation of the plane parallel to z-x plane at a distance c is $y = c$ or $y = -c$.

DIFFERENT FORMS OF THE EQUATIONS OF PLANES

(a) Equation of a plane passing through a fixed point



If $\vec r_0$ is p.v. of point on the plane & $\vec n$ be vector normal to the plane, then

$\overrightarrow{AR}.\vec n = 0$

$\left\{ \begin{array}{l} \vec r = x\hat i + y\hat j + z\hat k \\ \vec r_0 = x_1\hat i + y_1\hat j + z_1\hat k \end{array} \right.$

$(\vec r - \vec r_0).\vec n = 0$ .... (i)

$\vec n = a\hat i + b\hat j + c\hat k$

$\vec r.\vec n = \vec r_0.\vec n$

$a(x-x_1) + b(y-y_1) + c(z-z_1) = 0$

Hence $\vec r.\vec n = d$ is the general equation of a plane in vector form

$ax + by + cz \underbrace{-ax_1 - by_1 - cz_1}_{d} = 0$

If $\vec r = x\hat i + y\hat i + z\hat k$ and $\vec r_0 = x_1\hat i + y_1\hat i + z_1\hat k$

then (i) becomes $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

$\boxed{P:\ ax + by + cz + d = 0}$

This is the equation of the plane containing the point $(x_1 y_1 z_1)$, where $a\hat i + b\hat j + c\hat k$ is a vector normal to it, where a, b, c are the dr's of a normal to the plane.

eg: $P:\quad 2(x+1) - 3(y+2) + 4(1-3z) = 0.$

$2(x-(-1)) - 3(y-(-2)) - 12\left(z - \frac{1}{3}\right) = 0$

$A\left(-1, -2, \frac{1}{3}\right)$; $\vec n = 2\hat i - 3\hat j - 12\hat k.$

Plane Parallel to the Coordinate Planes :

(i) Equation of yz plane is $x = 0$.

(ii) Equation of zx plane is $y = 0$.

(iii) Equation of xy plane is $z = 0$.

(iv) Equation of the plane parallel to xy plane at a distance c is $z = c$ or $z = -c$.

(v) Equation of the plane parallel to yz plane at a distance c is $x = c$ or $x = -c$

(vi) Equation of the plane parallel to zx plane at a distance c is $y = c$ or $y = -c$.

Equations of Planes Parallel to the Axes :

If $a = 0$, the plane is parallel to x-axis i.e. equation of the plane parallel to x-axis is $\mathbf{by + cz + d = 0}$.

Similarly, equations of planes parallel to y-axis and parallel to z-axis are $\mathbf{ax + cz + d = 0}$ and $\mathbf{ax + by + d = 0}$, respectively.

Q (a) Convert the plane $3x + 4y + z = 9$ into vector form.

$\vec{r}.(3\hat{i}+4\hat{j}+\hat{k}) = 9$

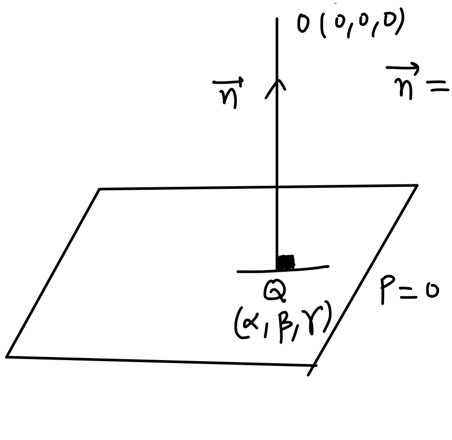
(b) Convert $\vec{r}.(\hat{i}+\hat{j}+\hat{k}) = 3$ into cartesian form

(b) $\vec{r} = x\hat{i}+y\hat{j}+z\hat{k}$

$$\boxed{x+y+z=3}$$

Q Find the equation of the plane if the feet of normal from origin on a plane is α, β and γ.

Sol^n



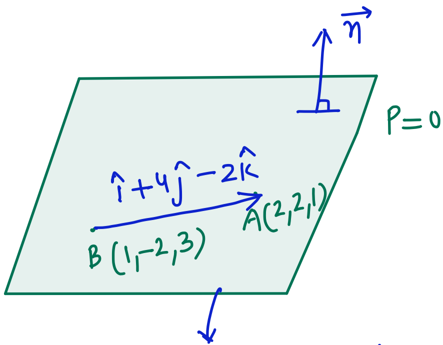
$\vec{n} = -\alpha\hat{i} - \beta\hat{j} - \gamma\hat{k}$

Plane:–

$-\alpha(x-\alpha) - \beta(y-\beta) - \gamma(z-\gamma) = 0$

Q Find the equation of the plane passing through the points A(2, 2, 1) and B(1, –2, 3) and perpendicular to the plane x – 2y + 3z + 4 = 0.

Solⁿ



$P_1: x-2y+3z+4=0$

$\vec{n_1} = \hat{i}-2\hat{j}+3\hat{k}$

$\left.\begin{array}{l} \vec{n} \perp \vec{n_1} \\ \vec{n} \perp \vec{BA} \end{array}\right\} \therefore \vec{n} = \vec{n_1} \times \vec{BA}$

$\vec{n} = a\hat{i}+b\hat{j}+c\hat{k}$

$P: a(x-2)+b(y-2)+c(z-1)=0.$

Q Find the equation of the plane through the point (2, –3, 1) and || to the plane 3x – 4y + 2z = 5.

Solⁿ $P: 3(x-2)-4(y+3)+2(z-1)=0$

Q Find the equation of the plane through the point (1, 0, –2) and perpendicular to the planes 2x + y – z = 2 and x – y – z = 3.

Solⁿ

$\vec{n} = \vec{n_1} \times \vec{n_2} = (2\hat{i}+\hat{j}-\hat{k}) \times (\hat{i}-\hat{j}-\hat{k}) = a\hat{i}+b\hat{j}+c\hat{k}$

$P: a(x-1)+b(y-0)+c(z+2)=0$

Q Two planes are given by equations x + 2y – 3z = 0 and 2x + y + z + 3 = 0. Find

$\vec{n}_1 = \hat{i} + 2\hat{j} - 3\hat{k}$ $\quad \vec{n}_2 = 2\hat{i} + \hat{j} + \hat{k}$

(a) DC's of their normals and the acute angle between them.

dc's : $\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{-3}{\sqrt{14}}$

$\vec{n}_1 \cdot \vec{n}_2 = |n_1||n_2| \cos\theta$

$2 + 2 - 3 = \sqrt{14}\sqrt{6}\cos\theta \Rightarrow \cos\theta = \frac{1}{\sqrt{84}}$

$\theta = \cos^{-1}\left(\frac{1}{\sqrt{84}}\right)$

(b) DC's of the their line of intersection.

(c) Equation of the plane perpendicular to both of them through the point (2, 2, 1)

(b) $\vec{n}_1 \times \vec{n}_2 = 5\hat{i} - 7\hat{j} - 3\hat{k}$

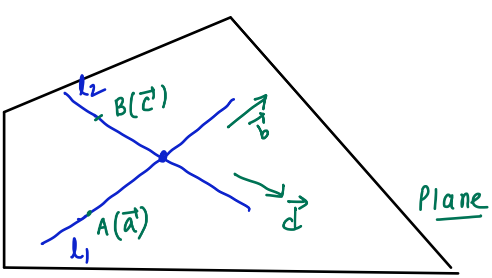
dc's $\pm\left(\frac{5}{\sqrt{83}}, \frac{-7}{\sqrt{83}}, \frac{-3}{\sqrt{83}}\right)$

(c) $\vec{n} = \vec{n}_1 \times \vec{n}_2$

$P: \quad 5(x-2) - 7(y-2) - 3(z-1) = 0$

Q Find the condition for coplanarity of two lines

$\vec{r} = \vec{a} + \lambda\vec{b}$ and $\vec{r} = \vec{c} + \mu\vec{d}$



$[\vec{BA} \;\; \vec{b} \;\; \vec{d}] = 0$

$[\vec{a} - \vec{c} \;\; \vec{b} \;\; \vec{d}] = 0$

$\Downarrow$

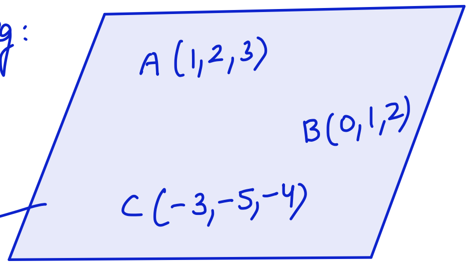
$[\vec{a} \;\; \vec{b} \;\; \vec{d}] - [\vec{c} \;\; \vec{b} \;\; \vec{d}] = 0.$

Equation of a Plane through three points :

The equation of the plane through three non-collinear points (x_1, y_1, z_1), (x_2, y_2, z_2) (x_3, y_3, z_3) is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$

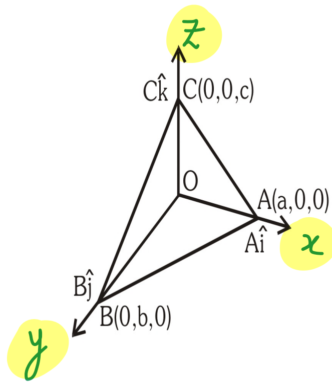
eg:



$$P: \begin{vmatrix} x-1 & y-2 & z-3 \\ -1 & -1 & -1 \\ -4 & -7 & -7 \end{vmatrix} = 0.$$

Intercept form of the plane :

$\boxed{\dfrac{x}{a}+\dfrac{y}{b}+\dfrac{z}{c}=1}$ Rem where A(a, 0, 0); B(0, b, 0) and C(0, 0, c)



Note : (1) Vector area of $\Delta ABC = \dfrac{1}{2}\left[(ab(\hat{i}\times\hat{j}) + bc(\hat{j}\times\hat{k}) + ca(\hat{k}\times\hat{i})\right]$

$$= \frac{1}{2}\left[(bc\,\hat{i} + ca\,\hat{j} + ab\,\hat{k})\right]$$

i.e. $\boxed{\text{Area of the } \Delta ABC = \dfrac{1}{2}\sqrt{a^2b^2 + b^2c^2 + c^2a^2}}$ Rem

HW

Q Let ABCD be a tetrahedron such that edges AB, AC, AD are mutually $\perp$. Let the areas of $\triangle ABC$, $\triangle ACD$, $\triangle ADB$ are 3, 4, 5 sq. units respectively. then find the area of $\triangle BCD$?

HW :-

JM Q1 to 19

JA Q1 to 3