Ellipse - (L)

Complete Solutions Of DYS & Exercises

DO
YOURSELF

Do yourself - 1 :

(i) If LR of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $(a<b)$ is half of its major axis, then find its eccentricity.

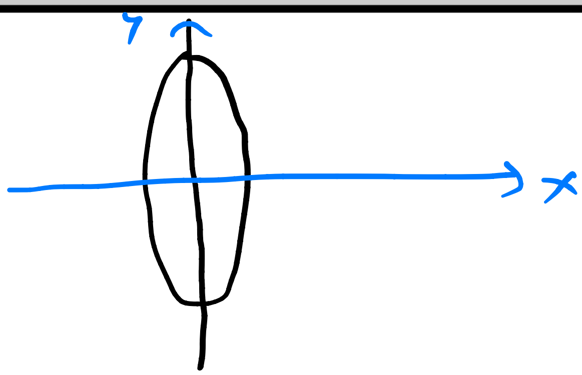
(ii) Find the equation of the ellipse whose foci are (4, 6) & (16, 6) and whose semi-minor axis is 4.

(iii) Find the eccentricity, foci and the length of the latus-rectum of the ellipse $x^2+4y^2+8y-2x+1=0$.

Sol^n

1/(i)

$\ell(L.R) = \frac{2a^2}{b}$



$\ell(\text{major axis}) = 2b$

According to question

$$\frac{2a^2}{b} = \frac{1}{2}\cdot 2b \Rightarrow 2a^2 = b^2$$

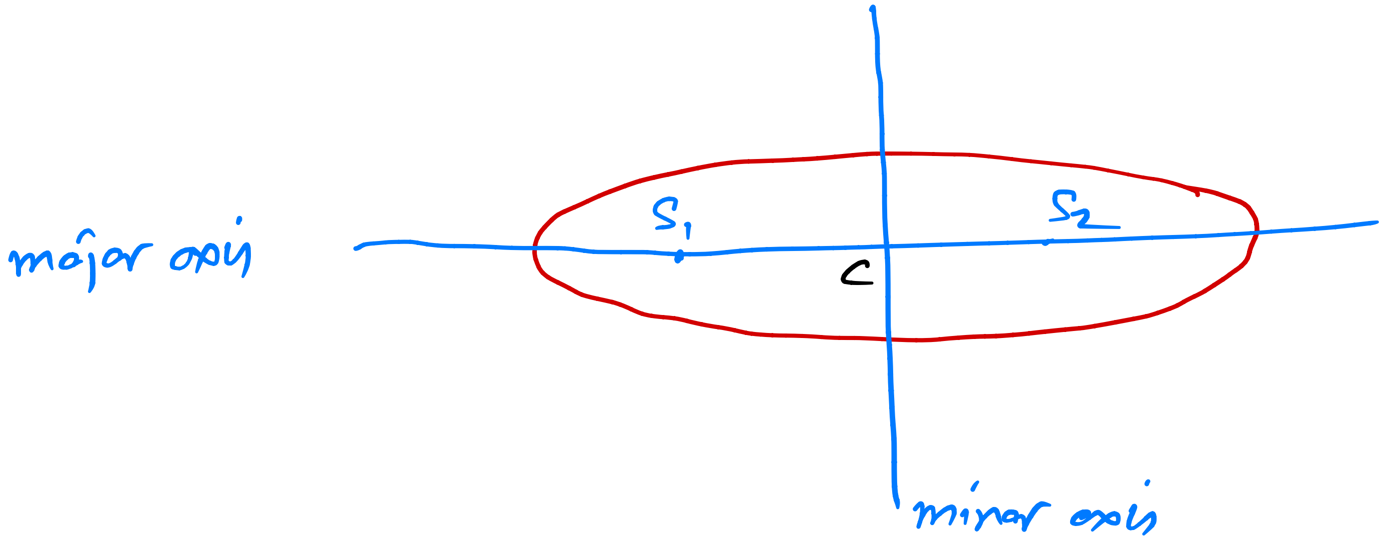
& $a^2 = b^2(1-e^2)$ $(\because a<b)$

$a^2 = 2a^2(1-e^2)$

$1-e^2 = \frac{1}{2}$

$\boxed{e = \frac{1}{\sqrt{2}}}$ Ans

1(ii) $S_1(4,6)$ & $S_2(16,6)$ & $b=4$



distance between both foci $= S_1S_2$

$$2ae = 12$$

$$ae = 6$$

$$\therefore \quad b^2 = a^2(1-e^2)$$

$$= a^2 - a^2e^2$$

$$16 = a^2 - 36$$

$$a^2 = 52$$

Centre of ellipse is mid point of focus.

$$\therefore \quad C\left(\frac{4+16}{2}, \frac{6+6}{2}\right) = C(10,6)$$

$\therefore$ equation of ellipse

$$\frac{(x-10)^2}{52} + \frac{(y-6)^2}{16} = 1 \qquad Ans$$

1(iii) Equation of ellipse

$$x^2+4y^2-2x+8y+1=0$$

$$x^2-2x+1+4(y^2+2y+1)=4$$

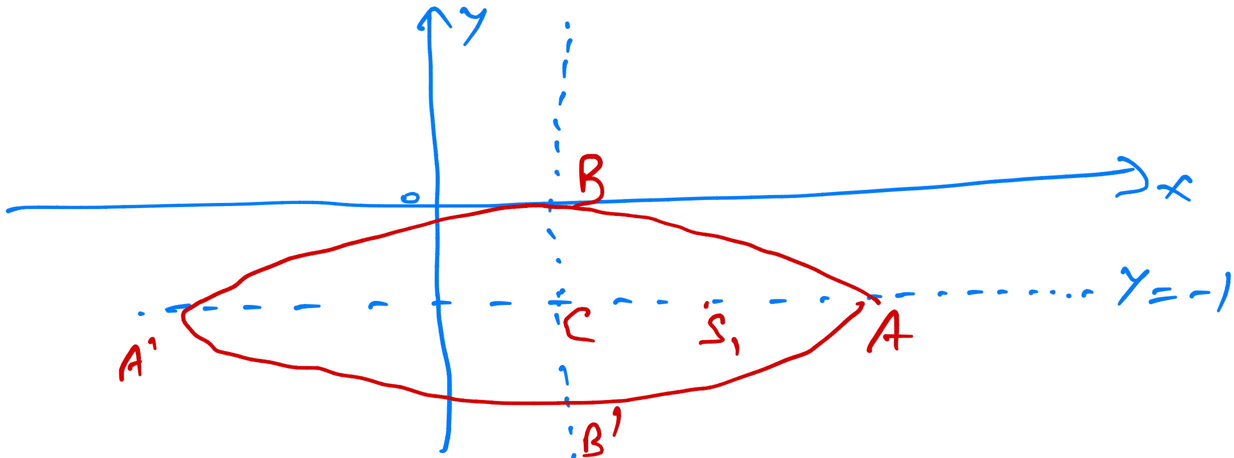
$$(x-1)^2+4(y+1)^2=4$$

$$\boxed{\frac{(x-1)^2}{4}+\frac{(y+1)^2}{1}=1}$$

Centre $=(1,-1)$, $a=2$, $b=1$ $(a>b)$

$\therefore$ $b^2=a^2(1-e^2)$

$1=4(1-e^2) \Rightarrow e=\frac{\sqrt{3}}{2}$



$CS=ae=\sqrt{3}$

$\therefore$ foci $\equiv(1\pm ae,-1)$

$\equiv(1+\sqrt{3},-1)$ & $(1-\sqrt{3},-1)$

& length of latus rectum $=\frac{2b^2}{a}=\frac{2.1}{2}=1$

Ans

Do yourself - 2 :

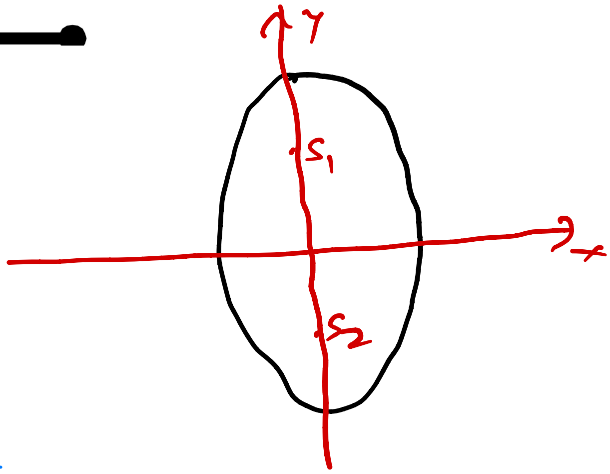
(i) The foci of an ellipse are (0, ±2) and its eccentricity is $\frac{1}{\sqrt{2}}$. Find its equation

(ii) Find the centre, the length of the axes, eccentricity and the foci of ellipse
$12x^2 + 4y^2 + 24x - 16y + 25 = 0$

(iii) The equation $\frac{x^2}{8-t} + \frac{y^2}{t-4} = 1$, will represent an ellipse if

(A) $t \in (1, 5)$ (B) $t \in (2, 8)$ (C) $t \in (4, 8) - \{6\}$ (D) $t \in (4, 10) - \{6\}$

Sol^n 2(i) $S_1(0, 2)$ & $S_2(0, -2)$



mid point of focus S_1 & S_2 is $(0, 0)$ & it is centre of ellipse.

$be = 2$ & $e = \frac{1}{\sqrt{2}}$ $(\because a < b)$

$b = 2\sqrt{2}$

$\because a^2 = b^2 - b^2e^2$

$a^2 = 8 - 4 = 4$

$\therefore$ eq^n of ellipse is

$$\frac{(x-0)^2}{4} + \frac{(y-0)^2}{8} = 1$$

Ans

2/(ii) $12x^2+4y^2+24x-16y+25=0$

$12(x^2+2x+1)+4(y^2-4y+4)+25=12+16$

$12(x+1)^2+4(y-2)^2=3$

$$\frac{(x+1)^2}{1/4}+\frac{(y-2)^2}{3/4}=1$$

$a=\frac{1}{2}$ & $b=\frac{\sqrt{3}}{2}$ $\boxed{a<b}$

Centre $x+1=0$ & $y-2=0 \Rightarrow (-1,2)$

$a^2=b^2(1-e^2)$

$\frac{1}{4}=\frac{3}{4}(1-e^2) \Rightarrow 1-e^2=\frac{1}{3}$

$\therefore e=\sqrt{\frac{2}{3}}$

length of major axis $=2b=\sqrt{3}$

length of minor axis $=2a=1$

foci $(-1, 2\pm be)=\left(-1, 2\pm\frac{\sqrt{3}}{2}\cdot\sqrt{\frac{2}{3}}\right)$

$=\left(-1, 2\pm\frac{1}{\sqrt{2}}\right)$

Solⁿ 2(iii) $\frac{x^2}{8-t} + \frac{y^2}{t-4} = 1$

represent ellipse, if

$8-t>0$

ie $t<8$ — (i)

$t-4>0$

ie $t>4$ — (ii)

from (i) & (ii) gives

$\boxed{4<t<8}$ — (iii)

But when $8-t = 4-t$

$\Rightarrow$ $t=6$

in this case, equation represent circle

$\therefore$ for ellipse

$t \in (4,8) - \{6\}$

Do yourself - 3 :

(i) Find the position of the point (4, 3) relative to the ellipse $2x^2 + 9y^2 = 113$.

(ii) A tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $(a > b)$ having slope -1 intersects the axis of x & y in point A & B respectively. If O is the origin then find the area of triangle OAB.

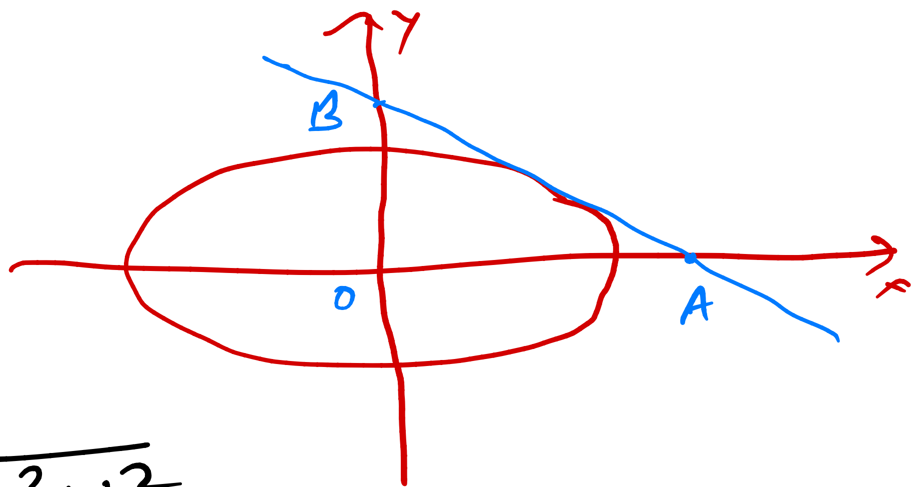
(iii) Find the condition for the line $x\cos\theta + y\sin\theta = P$ to be a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Sol^n 3(i)

$$S(x,y) \equiv 2x^2+9y^2-113$$
$$S(4,3) \equiv 2.4^2+9.3^2-113$$
$$\equiv 32+81-113$$
$$=0$$

∴ point lie on the ellipse.

3(ii)



Any tangent to the ellipse is

$$y = mx \pm \sqrt{a^2m^2+b^2}$$

where $m=-1$ (given in the question)

so $y = -x + \sqrt{a^2+b^2}$ (take +ve sign)

∴ $A(\sqrt{a^2+b^2}, 0)$ & $B(0, \sqrt{a^2+b^2})$

Area of $\Delta OAB = \frac{1}{2} \cdot OA \cdot OB = \frac{1}{2}(a^2+b^2)$

Ans

3(iii) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

its tangent is given as $x\cos\theta + y\sin\theta = p$ —(i)

its tangent in slope form is

$$y = mx + \sqrt{a^2m^2 + b^2}$$

or $-mx + y = \sqrt{a^2m^2 + b^2}$ — (ii)

Now (i) & (ii) represent identical lines

$$\Rightarrow \quad \frac{\cos\theta}{-m} = \frac{\sin\theta}{1} = \frac{p}{\sqrt{a^2m^2+b^2}}$$

$m = -\cot\theta$

$\sqrt{a^2m^2+b^2} = \frac{p}{\sin\theta}$

eliminate m from these two, we get

$$a^2\cot^2\theta + b^2 = \frac{p^2}{\sin^2\theta}$$

$$a^2\frac{\cos^2\theta}{\sin^2\theta} + b^2 = \frac{p^2}{\sin^2\theta}$$

multiplied by $\sin^2\theta$

$$\therefore \boxed{a^2\cos^2\theta + b^2\sin^2\theta = p^2}$$

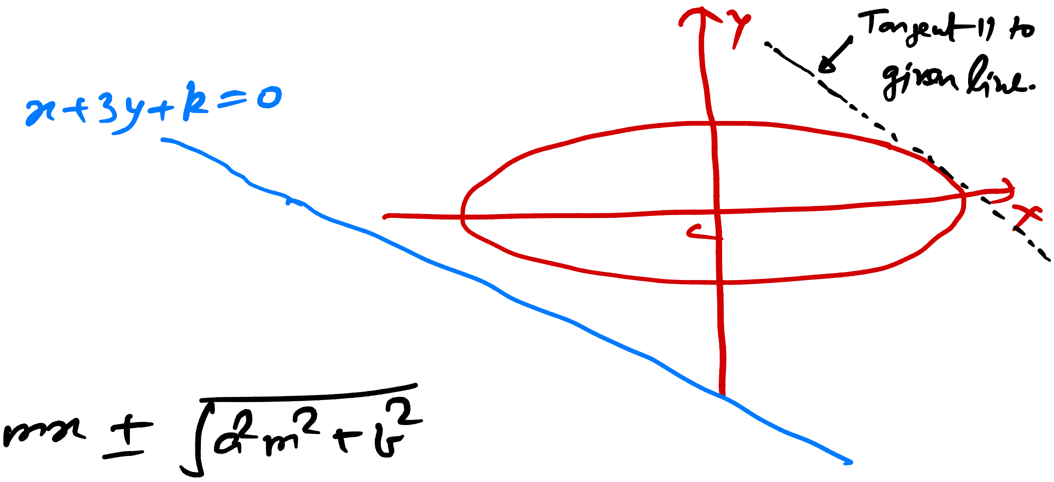
Ans

Do yourself - 4 :

(i) Find the equation of the tangents to the ellipse $9x^2 + 16y^2 = 144$ which are parallel to the line $x + 3y + k = 0$.

(ii) Find the equation of the tangent to the ellipse $7x^2 + 8y^2 = 100$ at the point $(2, -3)$.

Solⁿ 4(i)



$$\frac{x^2}{16} + \frac{y^2}{9} = 1$$

its tangent is

$$y = mx \pm \sqrt{a^2m^2 + b^2}$$

Here $a^2 = 16$, $b^2 = 9$ & $m = -\frac{1}{3}$

$\therefore$ equation of tangent is

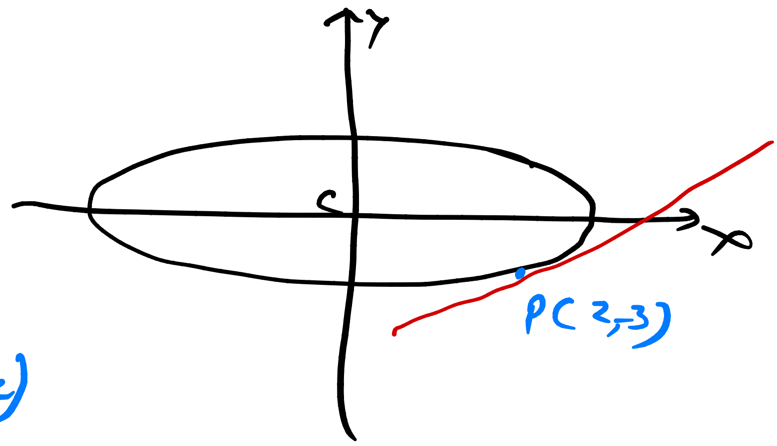
$$y = -\frac{1}{3}x \pm \sqrt{\frac{16}{9} + 9}$$

$$3y + x = \pm\sqrt{16 + 81}$$

ie $\quad 3y + x \pm \sqrt{97} = 0$ Ans

4/(ii) $7x^2+8y^2=100$ & $P(2,-3)$

Point $P(2,-3)$ lie on the ellipse (Because point satisfy equation of given ellipse)



So we can use standard substitution to get equation of tangent

ie x^2 replaced by $2x$

y^2 " $-3y$

100 " 100

$\therefore$ equation of tangent is

$$7\cdot 2x + 8\cdot(-3y) = 100$$

ie $7x - 12y = 50$ Ans

Do yourself - 5 :

(i) Find the equation of the normal to the ellipse $9x^2 + 16y^2 = 288$ at the point (4, 3)

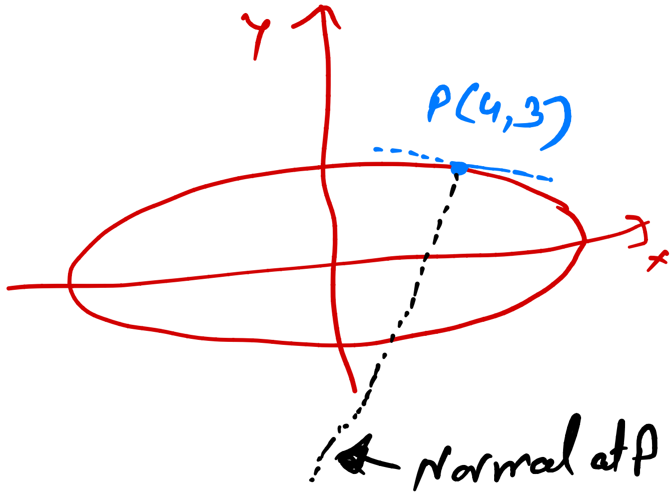
(ii) Let P be a variable point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with foci F_1 and F_2. If A is the area of the triangle PF_1F_2, then find maximum value of A.

(iii) If the normal at the point $P(\theta)$ to the ellipse $\frac{x^2}{3}+\frac{y^2}{2}=1$ intersects it again at the point $Q(2\theta)$, then find $\cos\theta$.

(iv) Show that for all real values of 't' the line $2tx + y\sqrt{1-t^2} = 1$ touches a fixed ellipse. Find the eccentricity of the ellipse.

Solution

5/(i) P(4,3) lie on the ellipse, equation of normal to the ellipse



$$\frac{x^2}{16}+\frac{y^2}{9}=1 \text{ is}$$

$$\frac{x-x_1}{x_1/a^2}=\frac{y-y_1}{y_1/b^2}$$

where $a=4$, $b=3$, $x_1=4$, $y_1=3$, so

$$\frac{x-4}{4/16}=\frac{y-3}{3/9}$$

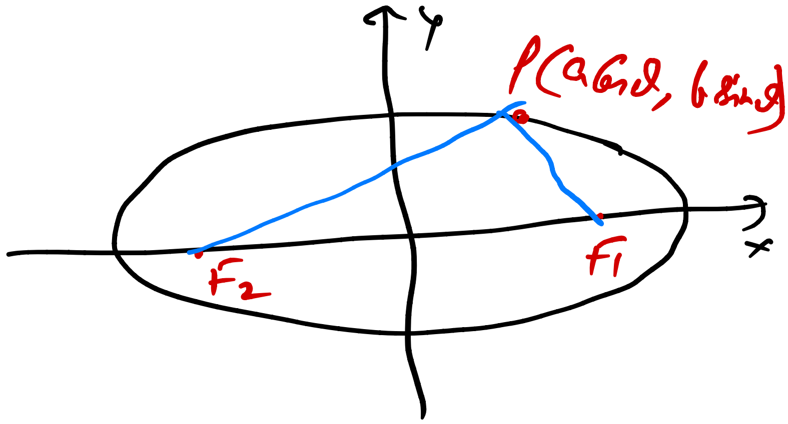
$$\Rightarrow \quad 4(x-4)=3(y-3)$$

ie $\boxed{4x-3y=7}$ Ans

5/(ii)

Any point on the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \text{ is}$$



$P(a\cos\theta,\ b\sin\theta)$

where $F_1(ae, 0)$ & $F_2(-ae, 0)$

Area of triangle $PF_1F_2 = A = \frac{1}{2}\cdot(F_1F_2)\cdot b\sin\theta$

(Area $= \frac{1}{2}\cdot$ base $\cdot$ height)

$\therefore\ A = \frac{1}{2}\cdot 2ae\cdot b\sin\theta$

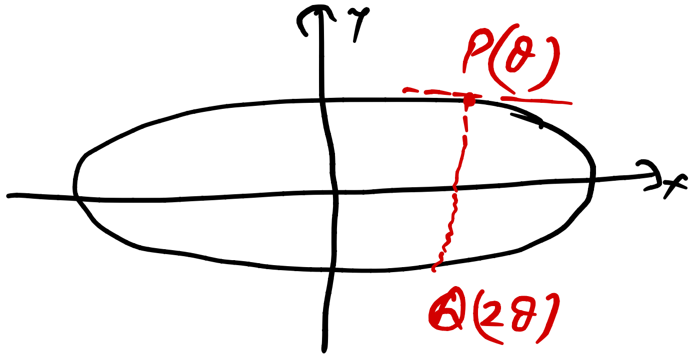
$= abe\sin\theta$

Here θ is variable

$\therefore\ A_{max} = abe$

(when $\theta = \pi/2$)

5 (iii) $\dfrac{x^2}{3}+\dfrac{y^2}{2}=1$



$a=\sqrt{3},\ b=\sqrt{2}$

any point P on the ellipse

is $P(\sqrt{3}\cos\theta,\ \sqrt{2}\sin\theta)\equiv P(\theta)$

& $Q(\sqrt{3}\cos 2\theta,\ \sqrt{2}\sin 2\theta)\equiv Q(2\theta)$

Normal at $P(\theta)$ in parametric form is

$$ax\sec\theta - by\,\mathrm{cosec}\,\theta = a^2-b^2$$

ie $\sqrt{3}\,x\sec\theta - \sqrt{2}\,y\,\mathrm{cosec}\,\theta = 3-2$ —(1)

above normal passes through $Q(2\theta)$, so

$Q(\sqrt{3}\cos 2\theta,\ \sqrt{2}\sin 2\theta)$ will satisfy eq^n (1)

$$\therefore\ \sqrt{3}\cdot\sqrt{3}\cos 2\theta\cdot\sec\theta - \sqrt{2}\cdot\sqrt{2}\sin 2\theta\cdot\mathrm{cosec}\,\theta = 1$$

$$\frac{3\cos 2\theta}{\cos\theta} - \frac{2\sin 2\theta}{\sin\theta} = 1$$

$$\frac{3(2\cos^2\theta-1)}{\cos\theta} - \frac{2\cdot 2\sin\theta\cos\theta}{\sin\theta} = 1$$

Put $\cos\theta = t$

$\Rightarrow\ 2t^2 - t - 3 = 0$

$(t+1)(2t-3)=0$

$\therefore\ \boxed{t=\cos\theta=-1}$ (∵ $\cos\theta = t = \frac{3}{2}$ is rejected)

Ans

Sol^n
5(iv)

Let eq^n of ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

equation of its tangent, in slope form, is

$$y=mx+\sqrt{a^2m^2+b^2} \quad -(i)$$

But according to question, line

$2xt + y\sqrt{1-t^2} = 1$ is tangent

i.e. $$y=-2x\frac{t}{\sqrt{1-t^2}}+\frac{1}{\sqrt{1-t^2}} \quad -(ii)$$

Because given line (eq^n (ii)) is similar to it tangent equation in slope form (eq^n(i)) so given line will touch the ellipse.

Compare (i) & (ii)

$$\therefore m=-\frac{2t}{\sqrt{1-t^2}} \quad \& \quad a^2m^2+b^2=\frac{1}{1-t^2} \quad -(iii)$$

Put value of m in eq^n (iii)

$$a^2 \cdot \frac{4t^2}{1-t^2} + b^2 = \frac{1}{1-t^2}$$

$$4a^2 t^2 + b^2(1-t^2) = 1$$

$$(b^2-1) + t^2(4a^2-b^2) = 0$$

$\because \forall t$, it hold

$$\Rightarrow \quad b^2 = 1 \ \& \ 4a^2 - b^2 = 0$$

$$\therefore \quad b^2 = 1 \ \& \ a^2 = \frac{1}{4}$$

$(a < b)$

$$\therefore \quad a^2 = b^2(1-e^2)$$

$$\therefore \quad \boxed{e = \frac{\sqrt{3}}{2}} \quad \text{Ans}$$

ka b a

Do yourself - 6 :

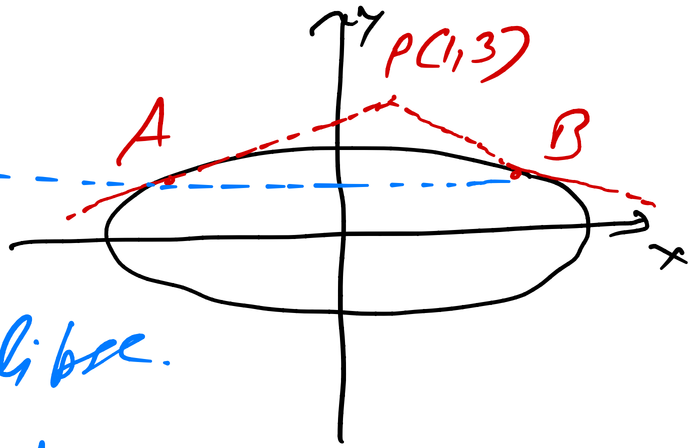
(i) Find the equation of chord of contact to the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ at the point (1, 3).

(ii) If the chord of contact of tangents from two points (x_1, y_1) and (x_2, y_2) to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ are at right angles, then find $\frac{x_1x_2}{y_1y_2}$.

(iii) If a line $3x - y = 2$ intersects ellipse $\frac{x^2}{8}+\frac{y^2}{4}=1$ at points A & B, then find co-ordinates of point of intersection of tangents at points A & B.

PAIR OF TANGENTS :

Sol^n 6 (i)



AB is chord of contact of point P(1,3) wrt given ellipse.

so eq^n of chord of contact is

$$T = 0$$

ie $$1\cdot\frac{x}{16} + 3\cdot\frac{y}{9} = 1$$

ie $$\boxed{\frac{x}{16} + \frac{y}{3} = 1}$$

Ans

Solⁿ 6(ii)

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Chord of Contact from $P(x_1, y_1)$ wrt ellipse

is $T=0$

ie $$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1 \quad \text{— (i)}$$

& chord of contact from $Q(x_2, y_2)$ wrt ellipse

is $T=0$

ie $$\frac{xx_2}{a^2}+\frac{yy_1}{b^2}=1 \quad \text{— (ii)}$$

$\because$ Line represented by eqⁿ (i) & (ii) are perpendicular

$$\Rightarrow m_1 m_2 = -1$$

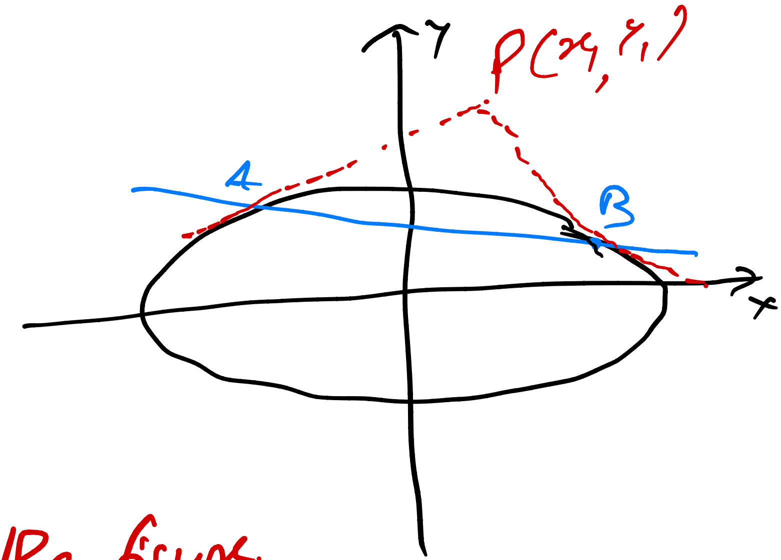
$$\Rightarrow \left(-\frac{x_1 b^2}{a^2 y_1}\right)\cdot\left(-\frac{x_2 b^2}{y_1 a^2}\right)=-1$$

$$\therefore \frac{x_1 x_2}{y_1 y_2}=-\frac{a^4}{b^4}$$

Ans

6(iii)

$$\frac{x^2}{8}+\frac{y^2}{4}=1$$



Line $3x - y = 2$ — (i)

intersect ellipse at A & B, as shown in the figure.

Tangent at A & B meet at Point $P(x_1, y_1)$

So $P(x_1, y_1)$ is the point wrt to ellipse, line AB is its chord of contact.

So chord of contact of $P(x_1, y_1)$ wrt ellipse is

$$\frac{xx_1}{8}+\frac{yy_1}{4}=1 \quad \text{— (ii)}$$

So (i) & (ii) are identical

$$\Rightarrow \quad \frac{x_1/8}{3} = \frac{y_1/4}{-1} = \frac{1}{2}$$

$\therefore x_1 = 12, \quad y_1 = -2$

$\therefore P(12, -2)$

Ans

Do yourself - 7 :

(i) Find the equation of chord of the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ whose mid point be $(-1, 1)$.

Sol^n $P(-1, 1)$ is mid point of chord, whose equation will be given by

$$T = S_1$$

ie $$\frac{(-1).x}{16} + \frac{1.y}{9} - 1 = \frac{(-1)^2}{16} + \frac{1}{9} - 1$$

ie $$-\frac{x}{16} + \frac{y}{9} = \frac{1}{16} + \frac{1}{9}$$

ie $$-9x + 16y = 25$$ Ans

Do yourself - 8 :

(i) A man running round a racecourse note that the sum of the distance of two flag-posts from him is always 20 meters and distance between the flag-posts is 16 meters. Find the area of the path be encloses in square meters

(ii) If chord of contact of the tangent drawn from the point (α, β) to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ touches the circle $x^2+y^2=k^2$, then find the locus of the point (α, β).

Solution

8(i) Here two flag post represent foci of the ellipse, where

$$2ae = 16$$
$$ae = 8 \quad -(i)$$

Sum of distance from man to two flag post

$$= 20$$

$$2a = 20$$
$$a = 10 \quad -(ii)$$

$$b^2 = a^2(1-e^2)$$
$$= 10^2 - 8^2 = 36$$
$$b = 6$$

$\therefore$ Area of path enclosed

$$= \text{Area of ellipse}$$
$$= \pi ab$$
$$= 60\pi$$

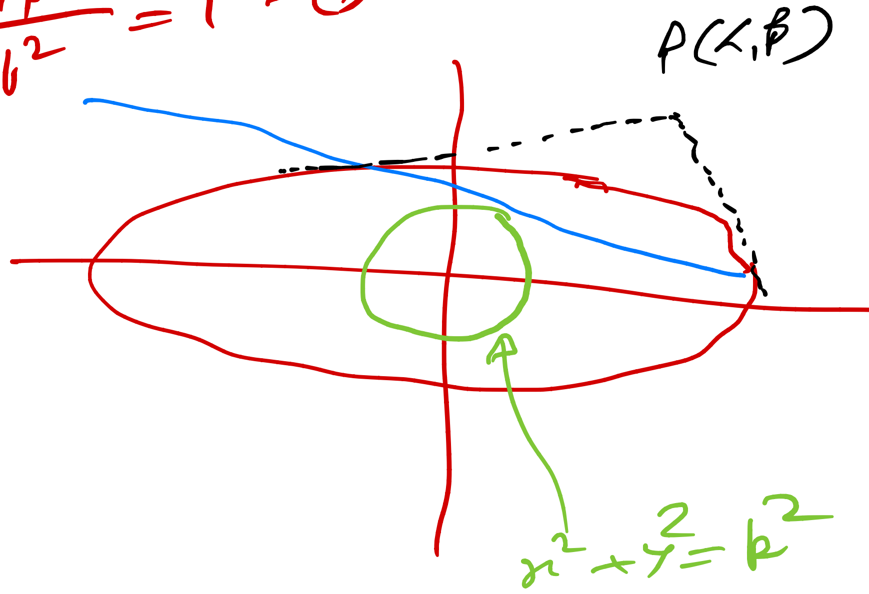
Ans

Solⁿ 8/(ii)

Chord of Contact $P(\alpha,\beta)$ wrt ellipse is

$$T = 0$$

ie $$\frac{\alpha x}{a^2} + \frac{y\beta}{b^2} = 1 \quad -(i)$$



Chord of Contact (eqⁿ(i)) touches given circle

$\Rightarrow$ $$p = r$$

$$\Rightarrow \left| \frac{0+0-1}{\sqrt{\frac{\alpha^2}{a^4}+\frac{\beta^2}{b^2}}} \right| = k$$

Square both side

$$\therefore \boxed{\frac{\alpha^2}{a^4} + \frac{\beta^2}{b^2} = \frac{1}{k^2}}$$

Ans

0-1

EXERCISE (O-1)

[STRAIGHT OBJECTIVE TYPE]

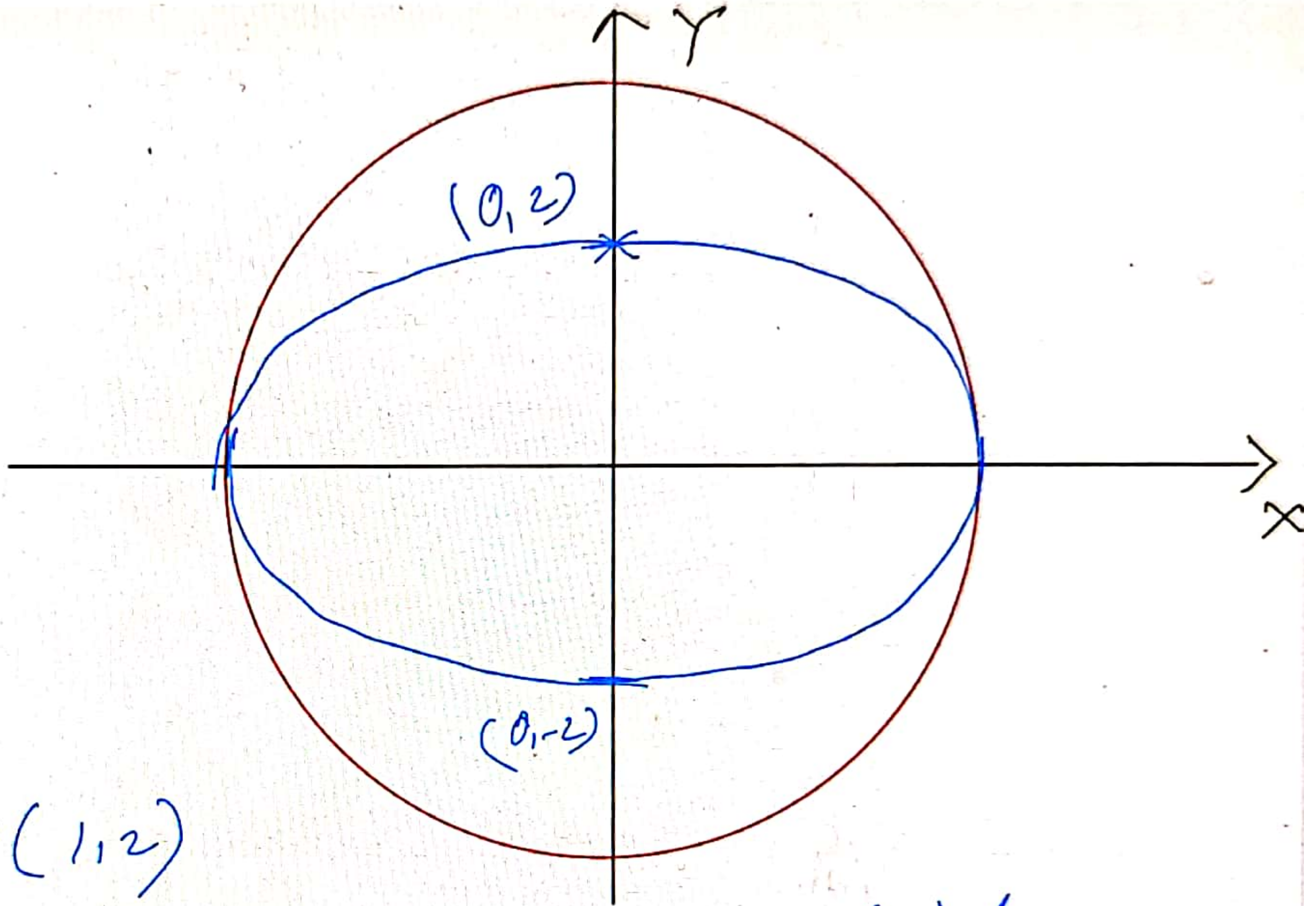
1. Let 'E' be the ellipse $\frac{x^2}{9} + \frac{y^2}{4} = 1$ & 'C' be the circle $x^2 + y^2 = 9$. Let P & Q be the points (1 , 2) and (2, 1) respectively. Then :

(A) Q lies inside C but outside E

(B) Q lies outside both C & E

(C) P lies inside both C & E

(D) P lies inside C but outside E.



for P(1,2)

E : $\frac{1}{9} + \frac{4}{4} - 1 > 0 \Rightarrow$ outside.

C : $1 + 4 - 9 < 0 \Rightarrow$ Inside

for Q (2,1)

E : $\frac{4}{9} + \frac{1}{4} - 1 < 0 \Rightarrow$ Inside.

C : $4 + 1 - 9 < 0 \Rightarrow$ Inside

$\Rightarrow$ option D

2. The eccentricity of the ellipse $(x-3)^2 + (y-4)^2 = \frac{y^2}{9}$ is

(A) $\frac{\sqrt{3}}{2}$ (B) $\frac{1}{3}$ (C) $\frac{1}{3\sqrt{2}}$ (D) $\frac{1}{\sqrt{3}}$

$$(x-3)^2 + y^2 - 8y + 16 - \frac{y^2}{9} = 0$$

$$\Rightarrow (x-3)^2 + \frac{8}{9}y^2 - 8y + 16 = 0$$

$$\Rightarrow (x-3)^2 + \frac{8}{9}(y^2 - 9y) + 16 = 0$$

$$\Rightarrow (x-3)^2 + \frac{8}{9}\left(y - \frac{9}{2}\right)^2 - \frac{\cancel{8}^{2}}{\cancel{9}} \times \frac{\cancel{81}^{9}}{\cancel{4}} + 16 = 0$$

$$\Rightarrow (x-3)^2 + \frac{8}{9}\left(y - \frac{9}{2}\right)^2 = 2$$

$$\Rightarrow \frac{(x-3)^2}{2} + \frac{\left(y - \frac{9}{2}\right)^2}{\frac{9}{4}} = 1.$$

$$a^2 = 2, \quad b^2 = \frac{9}{4}.$$

$$e^2 = 1 - \frac{2 \times 4}{9} = \frac{1}{9}.$$

$$e = \frac{1}{3}.$$

3. The equation, $2x^2 + 3y^2 - 8x - 18y + 35 = K$ represents

(A) no locus if $K > 0$

(B) an ellipse if $K < 0$

(C) a point if $K = 0$

(D) a hyperbola if $K > 0$

$2(x^2 - 4x) + 3(y^2 - 6y) + 35 = K.$

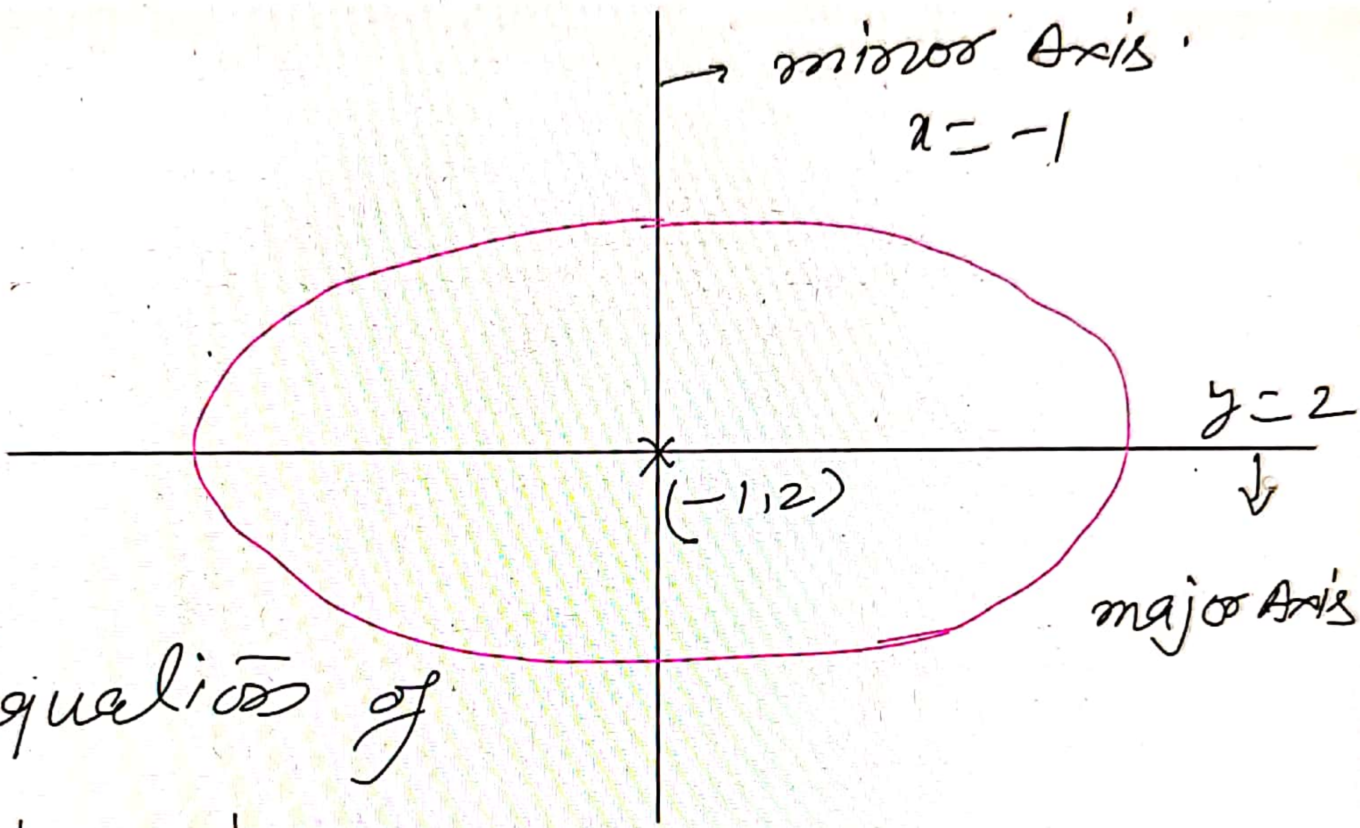
$\Rightarrow 2(x-2)^2 + 3(y-3)^2 + 35 - 8 - 27 = K$

$\Rightarrow 2(x-2)^2 + 3(y-3)^2 = K$

If $K = 0$ then equation is a point

4. If the ellipse $\frac{(x-h)^2}{M}+\frac{(y-k)^2}{N}=1$ has major axis on the line y = 2, minor axis on the line x = –1, major axis has length 10 and minor axis has length 4. The number h,k,M,N (in this order only) are -

(A) –1,2,5,2 (B) –1,2,10,4 (C) 1,–2,25,4 (D) –1,2,25,4



∴ equation of ellipse is:

$$\frac{(x+1)^2}{25}+\frac{(y-2)^2}{4}=1$$

$h=-1$

$k=2$

$M=25$

$N=4$

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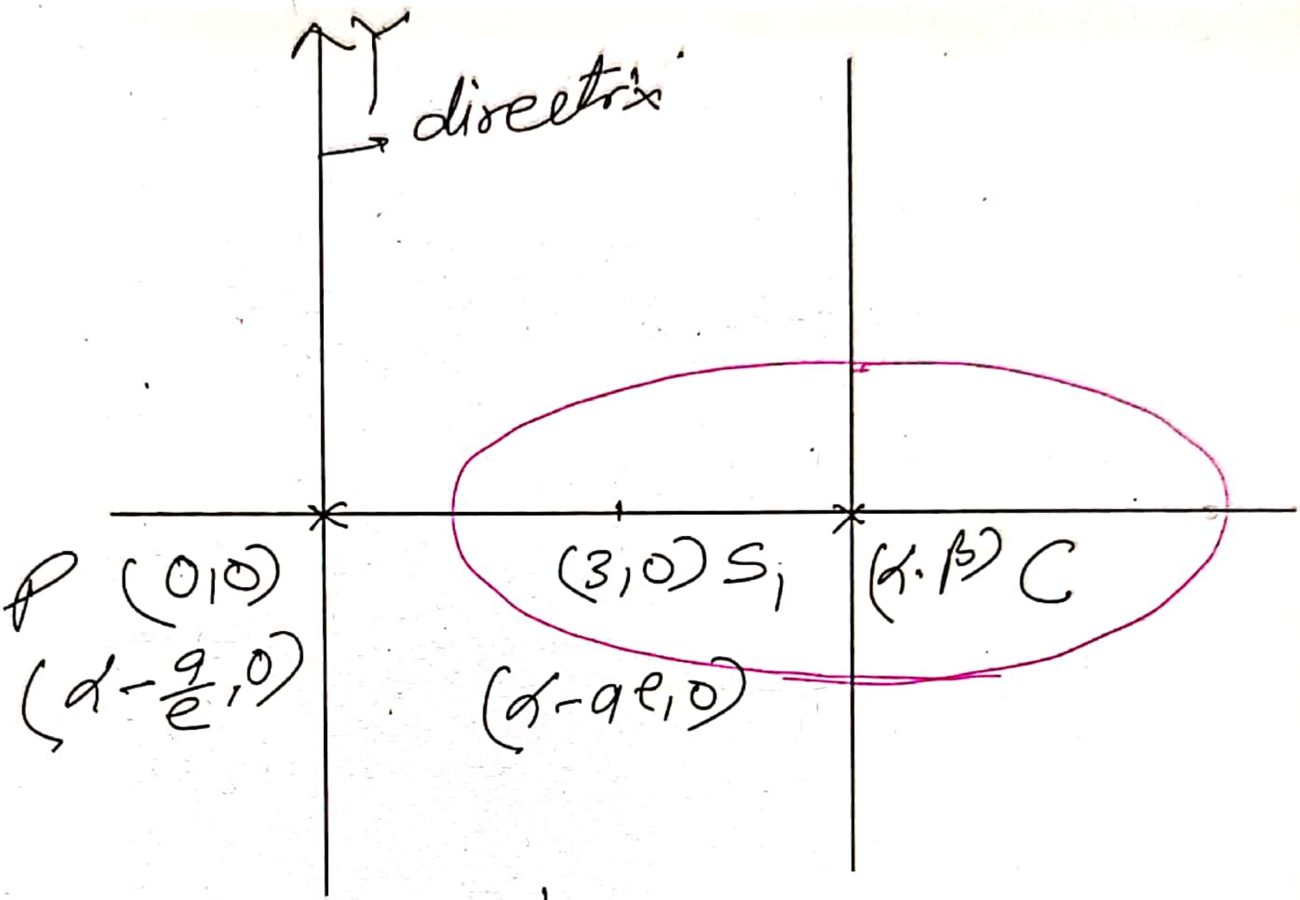
5. The y-axis is the directrix of the ellipse with eccentricity e = 1/2 and the corresponding focus is at (3, 0), equation to its auxiliary circle is

(A) $x^2 + y^2 - 8x + 12 = 0$ (B) $x^2 + y^2 - 8x - 12 = 0$

(C) $x^2 + y^2 - 8x + 9 = 0$ (D) $x^2 + y^2 = 4$



$PS_1 = \left|\frac{a}{e} - ae\right| = 3$

$\Rightarrow a\left|2 - \frac{1}{2}\right| = 3 \Rightarrow a = 2.$

$CS_1 = ae$

$\Rightarrow \alpha = 3 + ae = 3 + 1 = 4.$

∴ Equation of Auxiliary circle is

$(x-4)^2 + y^2 = 4$

$\Rightarrow x^2 + y^2 - 8x + 12 = 0$

6 The latus rectum of a conic section is the width of the function through the focus. The positive difference between the length of the latus rectum of $3y = x^2 + 4x - 9$ and $x^2 + 4y^2 - 6x + 16y = 24$ is-

(A) $\frac{1}{2}$ (B) 2 (C) $\frac{3}{2}$ (D) $\frac{5}{2}$

$$3y = x^2 + 4x - 9 = (x+2)^2 - 13$$

$$\Rightarrow (x+2)^2 = 3y + 13 = 3\left(y + \frac{13}{3}\right)$$

$$L \cdot R = 3$$

$$x^2 - 6x + 4y^2 + 16y = 24$$

$$\Rightarrow (x-3)^2 + 4(y^2 + 4y) = 24 + 9$$

$$\Rightarrow (x-3)^2 + 4(y+2)^2 = 24 + 9 + 16 = 49$$

$$\Rightarrow \frac{(x-3)^2}{7^2} + \frac{(y+2)^2}{\left(\frac{7}{2}\right)^2} = 1$$

$$L \cdot R = \frac{2b^2}{a} = \frac{2 \cdot \frac{49}{4}}{7} = \frac{7}{2}$$

$$\therefore \text{difference} = \frac{7}{2} - 3 = \frac{1}{2}$$

7 Let S(5,12) and S'(–12,5) are the foci of an ellipse passing through the origin. The eccentricity of ellipse equals -

(A) $\frac{1}{2}$ (B) $\frac{1}{\sqrt{3}}$ (C) $\frac{1}{\sqrt{2}}$ (D) $\frac{2}{3}$

$PS + PS' = 2a$ $\quad P(0,0)$

$\Rightarrow 13 + 13 = 2a \Rightarrow a = 13$

$SS' = 2ae = \sqrt{(17)^2 + 7^2}$

$\Rightarrow 2ae = \sqrt{289 + 49} = \sqrt{338}$

$e = \frac{\sqrt{338}}{26} = \frac{13\sqrt{2}}{26} = \frac{1}{\sqrt{2}}$

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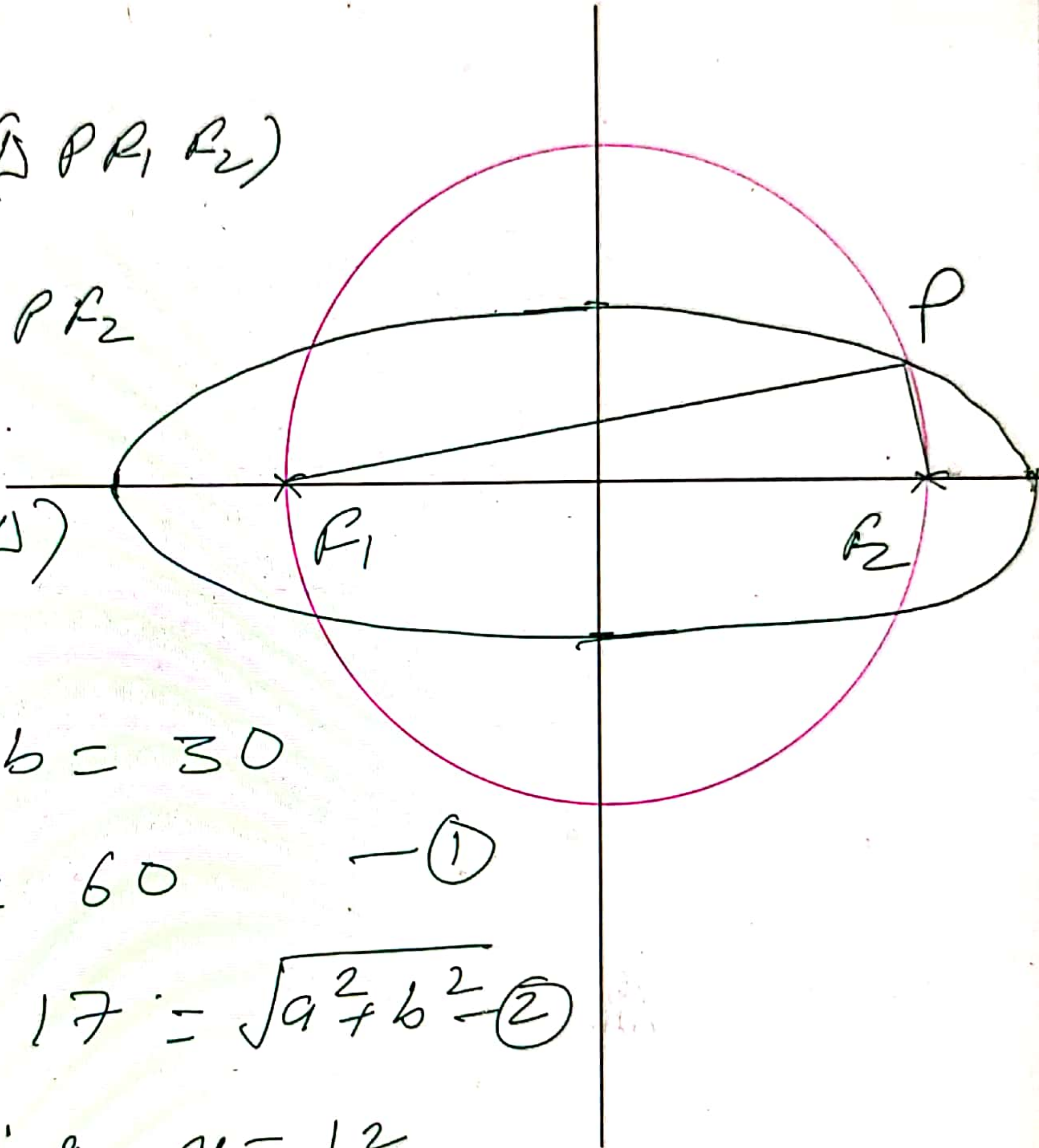
8 A circle has the same centre as an ellipse & passes through the foci F_1 & F_2 of the ellipse, such that the two curves intersect in 4 points. Let 'P' be any one of their point of intersection. If the major axis of the ellipse is 17 & the area of the triangle PF_1F_2 is 30, then the distance between the foci is :

(A) 11 (B) 12 (C)13 (D) none



Area of (ΔPF_1F_2)

$= \frac{1}{2} PF_1 \cdot PF_2$

(ΔPF_1F_2 is right angle Δ)

$= \frac{1}{2} a \cdot b = 30$

$\Rightarrow ab = 60$ — (1)

$F_1F_2 = 17 = \sqrt{a^2+b^2}$ — (2)

on solving $a = 12$
$b = 5$

$\therefore F_1F_2 = 13$

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9 (a) Which of the following is an equation of the ellipse with centre (–2,1), major axis running from (–2,6) to (–2,–4) and focus at (–2,5) ?

(A) $\frac{(x-2)^2}{25}+\frac{(y+1)^2}{16}=1$ (B) $\frac{(x+2)^2}{25}+\frac{(y-1)^2}{9}=1$

(C) $\frac{(x-2)^2}{9}+\frac{(y+1)^2}{25}=1$ (D) $\frac{(x+2)^2}{9}+\frac{(y-1)^2}{25}=1$

(b) Which of the following statement(s) is/are correct for the ellipse of 11(a) ?

(A) auxiliary circle is $(x + 2)^2 + (y - 1)^2 = 25$

(B) director circle is $(x + 2)^2 + (y - 1)^2 = 34$

(C) Latus rectum $=\frac{18}{5}$

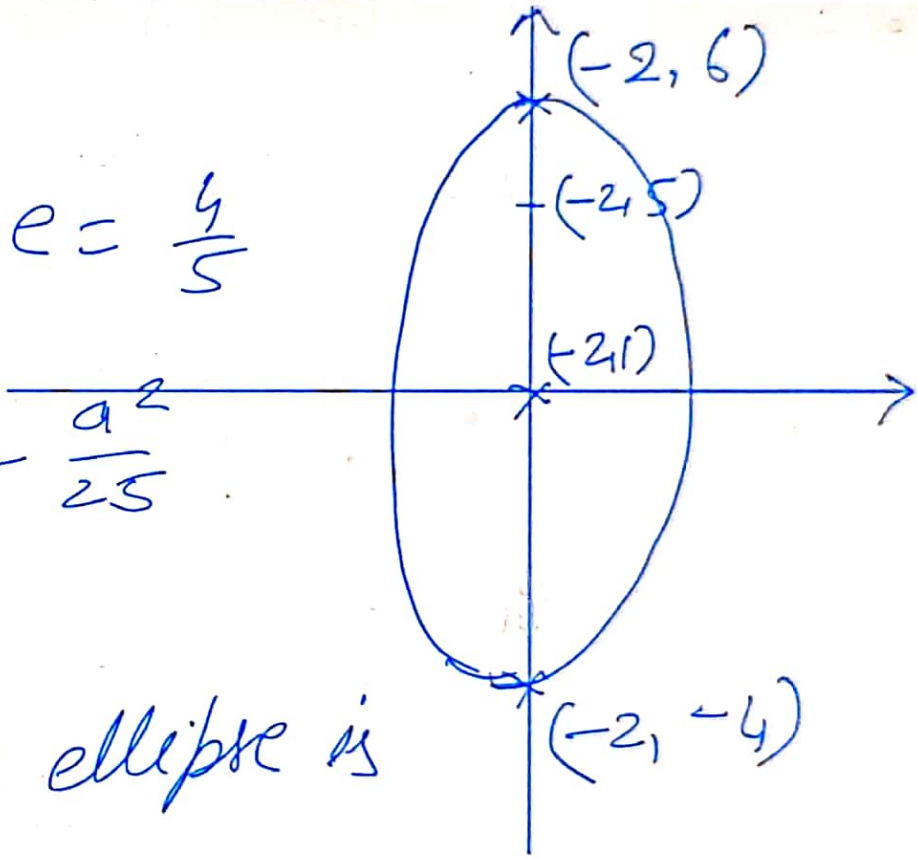
(D) eccentricity $=\frac{4}{5}$

(a) $b = 5$

$be = 4 \Rightarrow e = \frac{4}{5}$

$e^2 = \frac{16}{25} = 1 - \frac{a^2}{25}$

$a^2 = 9$



∴ equation of ellipse is

$\frac{(x+2)^2}{9} + \frac{(y-1)^2}{25} = 1 \Rightarrow$ (D)

(b) Auxilliary circle. $(x+2)^2+(y-1)^2 = 25$

D.C. is $(x+2)^2+(y-1)^2 = 34$

L.R $= \frac{2a^2}{b} = 18/5$

10 $x - 2y + 4 = 0$ is a common tangent to $y^2 = 4x$ & $\frac{x^2}{4} + \frac{y^2}{b^2} = 1$. Then the value of b and the other common tangent are given by :

(A) $b = \sqrt{3}$; $x + 2y + 4 = 0$

(B) $b = 3$; $x + 2y + 4 = 0$

(C) $b = \sqrt{3}$; $x + 2y - 4 = 0$

(D) $b = \sqrt{3}$; $x - 2y - 4 = 0$

Tangent to $y^2 = 4x$ is $y = mx + \frac{1}{m}$

and for $\frac{x^2}{4} + \frac{y^2}{b^2} = 1$ is $y = mx \pm \sqrt{a^2m^2 + b^2}$

$$y = mx \pm \sqrt{4m^2 + b^2}$$

given $y = \frac{x}{2} + 2$

$\Rightarrow \sqrt{4 \cdot \frac{1}{4} + b^2} = 2 \Rightarrow b^2 = 3.$

$\Rightarrow b = \sqrt{3}$

equation of tangent is

$$y = mx \pm \sqrt{4m^2 + 3}$$

& $y = mx + \frac{1}{m}$

$\Rightarrow 4m^2 + 3 = \frac{1}{m^2} \Rightarrow 4m^4 + 3m^2 - 1 = 0$

$\Rightarrow 4m^4 + 4m^2 - m^2 - 1 = 0.$

$\Rightarrow 4m^2(m^2+1) - 1(m^2+1) = 0 \Rightarrow m = \pm \frac{1}{2}.$

$\therefore y = -\frac{x}{2} - 2 \Rightarrow 2y + x + 4 = 0$

11 Consider the particle travelling clockwise on the elliptical path $\frac{x^2}{100} + \frac{y^2}{25} = 1$. The particle leaves the orbit at the point (–8, 3) and travels in a straight line tangent to the ellipse. At what point will the particle cross the y-axis?

(A) $\left(0, \frac{25}{3}\right)$ (B) $\left(0, \frac{23}{3}\right)$ (C) (0, 9) (D) $\left(0, \frac{26}{3}\right)$

Equation of tangent to the ellipse.

$$\frac{-8x}{100} + \frac{3y}{25} = 1.$$

Cross the Y-axis at $x=0$

$$\Rightarrow y = \frac{25}{3}.$$

[MULTIPLE OBJECTIVE TYPE]

12 Consider the ellipse $\frac{x^2}{\tan^2\alpha} + \frac{y^2}{\sec^2\alpha} = 1$ where $\alpha \in (0, \pi/2)$.

Which of the following quantities would vary as α varies ?

(A) degree of flatness
(B) ordinate of the vertex
(C) coordinates of the foci
(D) length of the latus rectum

Solution

Ans ⇒ A, B, D

$$e^2 = 1 - \frac{\tan^2\alpha}{\sec^2\alpha} = \cos^2\alpha \quad (\text{as } \sec^2\alpha > \tan^2\alpha)$$

hence $e = \cos\alpha$; vertex $(0, \pm \sec\alpha)$

$$\text{foci} = (0, 1)\,;\ l(LR) = \frac{2b^2}{a} = \frac{2\tan^2\alpha}{\sec\alpha} = 2\sin\alpha \cdot \tan\alpha \quad]$$

13 The equation of the common tangents of the parabola $y^2 = 4x$ and an ellipse $\frac{x^2}{4} + \frac{y^2}{3} = 1$ are -

(A) $x - 2y + 4 = 0$ (B) $x + 2y + 4 = 0$ (C) $2x - y + 1 = 0$ (D) $2x + y + 1 = 0$

Soln:- Tangent to parabola is

$$y = mx + \frac{1}{m} \text{ ------ I}$$

For ellipse

$$y = mx \pm \sqrt{a^2m^2 + b^2} = mx \pm \sqrt{4m^2 + 3} \text{ - II}$$

Comparing both $4m^2 + 3 = \frac{1}{m^2}$

$$4m^4 + 3m^2 - 1 = 0$$

$$4m^4 + 4m^2 - m^2 - 1 = 0$$

$$4m^2(m^2+1) - (m^2+1) = 0$$

$$(4m^2 - 1)(m^2+1) = 0$$

$$m = \frac{1}{2};\ m = -\frac{1}{2}$$

Put in I^{st}

$$\boxed{x - 2y + 4 = 0 \ \& \ x + 2y + 4 = 0}$$

14 If length of perpendicular drawn from origin to any normal of the ellipse $\frac{x^2}{16}+\frac{y^2}{25}=1$ is ℓ, then ℓ cannot be -

(A) 4 (B) $\frac{5}{2}$ (C) $\frac{1}{2}$ (D) $\frac{2}{3}$

Equation of the normal at the point (acos θ, bsin θ) is :

$ax \cdot \sec\theta - by \cdot \operatorname{cosec}\theta = (a^2 - b^2)$.

$$\frac{4x}{\cos\theta} - \frac{5y}{\sin\theta} + 9 = 0$$

$$l = \frac{9}{\sqrt{16\sec^2\theta + 25\operatorname{cosec}^2\theta}}$$

$$a^2\sec^2\theta + b^2\operatorname{cosec}^2\theta = (a+b)^2_{min}$$

$$l_{max} = \frac{9}{(4+5)} = 1$$

So only A & B are possible

[COMPREHENSION TYPE]

Paragraph for question nos. 15 to 17

Consider the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ and the parabola $y^2 = 2x$. They intersect at P and Q in the first and fourth quadrants respectively. Tangents to the ellipse at P and Q intersect the x-axis at R and tangents to the parabola at P and Q intersect the x-axis at S.

15 The ratio of the areas of the triangles PQS and PQR, is

(A) 1 : 3 (B) 1 : 2 (C) 2 : 3 (D) 3 : 4

16 The area of quadrilateral PRQS, is

(A) $\frac{3\sqrt{15}}{2}$ (B) $\frac{15\sqrt{3}}{2}$ (C) $\frac{5\sqrt{3}}{2}$ (D) $\frac{5\sqrt{15}}{2}$

17 The equation of circle touching the parabola at upper end of its latus rectum and passing through its vertex, is

(A) $2x^2 + 2y^2 - x - 2y = 0$ (B) $2x^2 + 2y^2 + 4x - \frac{9}{2}y = 0$

(C) $2x^2 + 2y^2 + x - 3y = 0$ (D) $2x^2 + 2y^2 - 7x + y = 0$

Soln:—

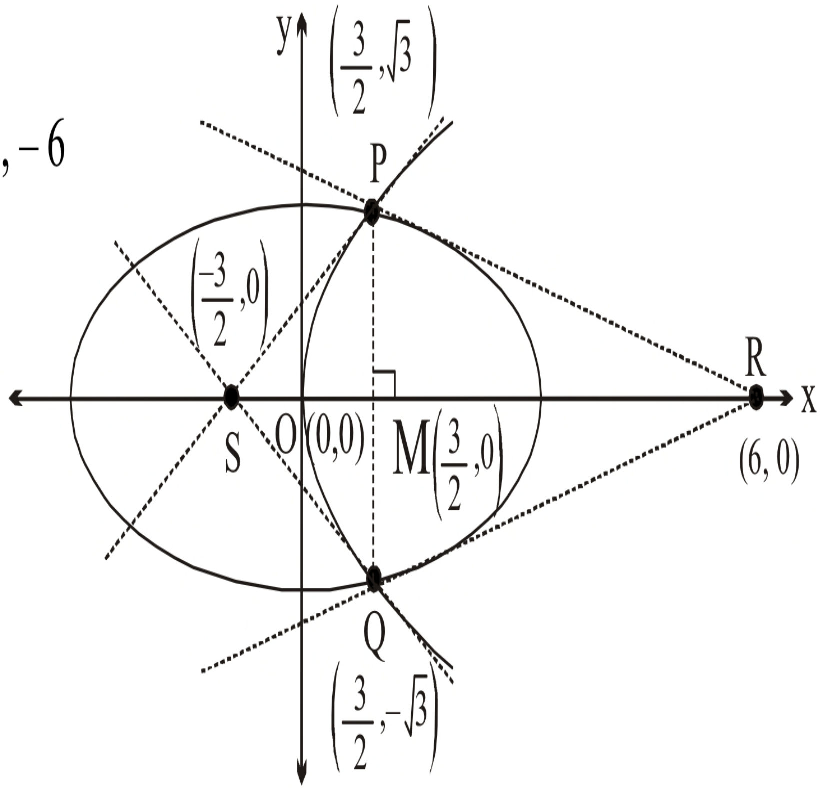
Solving the curves $y^2 = 2x$ and $\frac{x^2}{9}+\frac{y^2}{4}=1$ for the points of intersection, we have

$$4x^2 + 18x - 36 = 0 \Rightarrow x = \frac{3}{2}, -6$$

But from $y^2 = 2x$ we have $x > 0$

$$\therefore \quad x = \frac{3}{2}$$

at which $y^2 = 2 \cdot \frac{3}{2}$



$\Rightarrow \qquad y = \pm\sqrt{3}$

$\therefore \qquad P\left(\frac{3}{2}, \sqrt{3}\right)$ and $Q\left(\frac{3}{2}, -\sqrt{3}\right)$

Now equation of tangents at P and Q to ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ is $\frac{x}{9}\left(\frac{3}{2}\right)+\frac{y}{4}\left(\pm\sqrt{3}\right)=1$ which intersect at R(6, 0)

Equation of tangents at P and Q to parabola $y^2 = 2x$ will be $y\left(\pm\sqrt{3}\right)=x+\frac{3}{2}$ which cut x-axis $S\left(\frac{-3}{2}, 0\right)$

$$\therefore \qquad \frac{\text{Area }\Delta PQS}{\text{Area }\Delta PQR} = \frac{\frac{1}{2}PQ\cdot MS}{\frac{1}{2}PQ\cdot MR} = \frac{MS}{MR} = \frac{\frac{3}{2}-\left(\frac{-3}{2}\right)}{6-\frac{3}{2}} = \frac{3}{\frac{9}{2}} = \frac{2}{3} \textbf{ Ans.(i)}$$

Area of quadrilateral PRQS $= \frac{1}{2}PQ(MS+MR) = \frac{1}{2}\cdot 2\sqrt{3}\,(6-(-3/2)) = \frac{15\sqrt{3}}{2}$ **Ans.(ii)**

(iii) Clearly upper end of latus rectum of parabola is $\left(\frac{1}{2},1\right)$.

And equation of tangent at $\left(\frac{1}{2},1\right)$ to

$y^2 = 2x$ is $y = x + \frac{1}{2}$

$\therefore$ The equation of circle is

$$\left(x-\frac{1}{2}\right)^2 + (y-1)^2 + \lambda\left(y-x-\frac{1}{2}\right) = 0$$

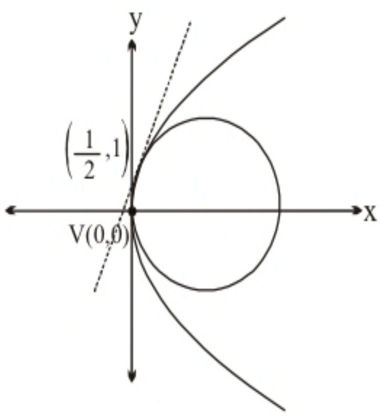
As above circle passes through V (0,0), so

$$\frac{1}{4}+1-\frac{\lambda}{2} = 0 \quad \Rightarrow \lambda = \frac{5}{2}$$

$\Rightarrow$ The equation of required circle is

$$\left(x-\frac{1}{2}\right)^2 + (y-1)^2 + \frac{5}{2}\left(y-x-\frac{1}{2}\right) = 0$$

$\Rightarrow 2x^2 + 2y^2 - 7x + y = 0$]



[MATRIX MATCH TYPE]

18 **Column-I** **Column-II**

(A) The eccentricity of the ellipse which meets the straight line $2x - 3y = 6$ on the X-axis and the straight line $4x + 5y = 20$ on the Y-axis and whose principal axes lie along the coordinate axes, is (P) $\frac{1}{2}$

(B) A bar of length 20 units moves with its ends on two fixed straight lines at right angles. A point P marked on the bar at a distance of 8 units from one end describes a conic whose eccentricity is (Q) $\frac{1}{\sqrt{2}}$

(C) If one extremity of the minor axis of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the foci form an equilateral triangle, then its eccentricity, is (R) $\frac{\sqrt{5}}{3}$

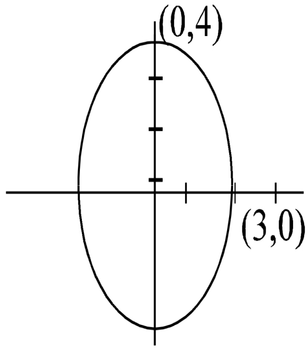
(D) There are exactly two points on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ whose distance from the centre of the ellipse are greatest and equal to $\sqrt{\frac{a^2+2b^2}{2}}$. Eccentricity of this ellipse is equal to (S) $\frac{\sqrt{7}}{4}$

[Ans. (A) S; (B) R ; (C) P ; (D) Q]

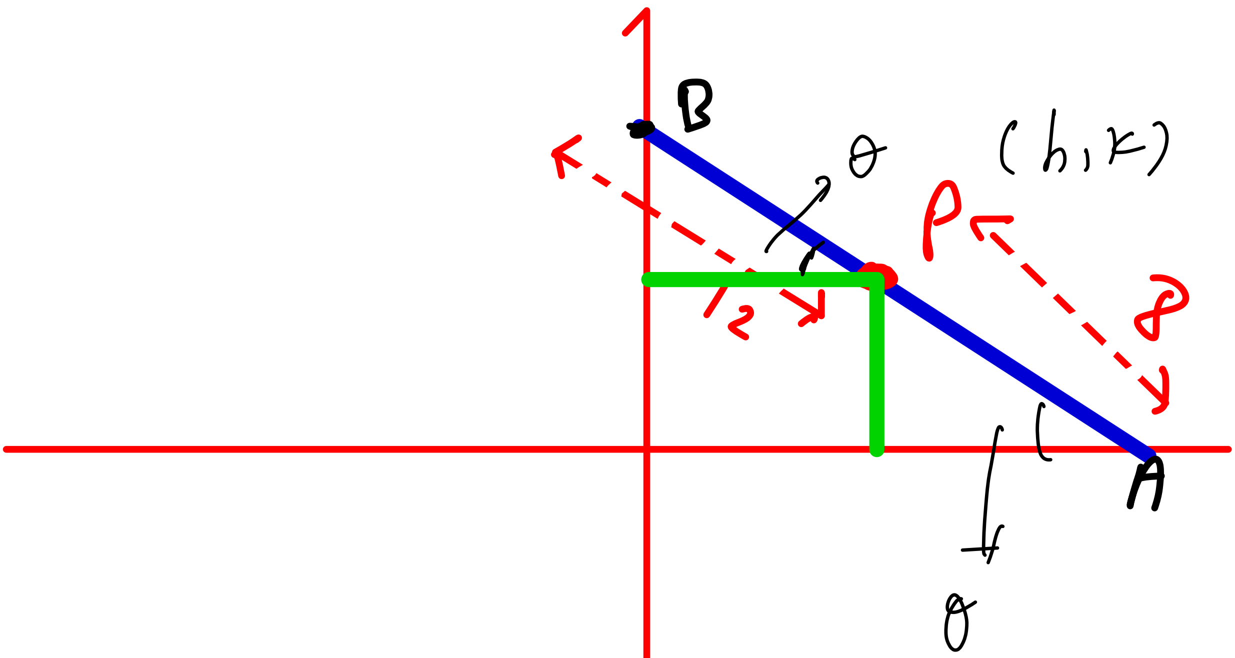
[Sol.

(A) $e^2 = 1 - \frac{9}{16} = \frac{7}{16}$ ellipse touch the x-axis at (3, 0) and y-axis at (0,4)

$e = \frac{\sqrt{7}}{4}$ **Ans.**



(B)



$h = 12\cos\theta, \quad k = 8\sin\theta$

$\Rightarrow$ Locus is ellipse

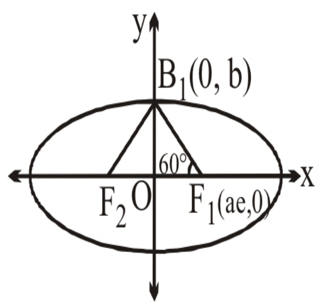
$$\frac{x^2}{144} + \frac{y^2}{64} = 1$$

$$\begin{array}{r} 144 \\ 64 \\ \hline 80 \end{array}$$

$$e = \sqrt{1 - \frac{64}{144}} = \frac{4\sqrt{5}}{12}$$

$$\boxed{e = \frac{\sqrt{5}}{3}}$$

(C) We have $\frac{b}{ae} = \sqrt{3} \Rightarrow a^2(1-e^2) = 3a^2e^2$ (As $b^2 = a^2(1-e^2)$)



$\Rightarrow e^2 = \frac{1}{4}$

Hence $e = \frac{1}{2}$ **Ans.**]

(D) The given distance is clearly the length of semi major axis

Thus, $\sqrt{\frac{a^2+2b^2}{2}} = a \Rightarrow 2b^2 = a^2 \Rightarrow 2a^2(1-e^2) = a^2$

$\Rightarrow e^2 = \frac{1}{2} \Rightarrow e = \frac{1}{\sqrt{2}}$ **Ans.**]

0-2

1 Equation of the common tangent to the ellipses, $\frac{x^2}{a^2+b^2}+\frac{y^2}{b^2}=1$ and $\frac{x^2}{a^2}+\frac{y^2}{a^2+b^2}=1$ is -

(A) $ay = bx + \sqrt{a^4 - a^2b^2 + b^4}$

(B) $by = ax - \sqrt{a^4 + a^2b^2 + b^4}$

(C) $ay = bx - \sqrt{a^4 + a^2b^2 + b^4}$

(D) $by = ax + \sqrt{a^4 - a^2b^2 + b^4}$

Soln

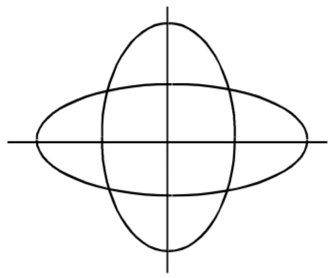
[Sol. Equation of a tangent to $\frac{x^2}{a^2+b^2}+\frac{y^2}{b^2}=1$

$$y = mx \pm \sqrt{(a^2+b^2)m^2+b^2} \qquad \text{....(1)}$$

If (1) is also a tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{a^2+b^2}=1$ then

$(a^2+b^2)m^2 + b^2 = a^2m^2 + a^2 + b^2$ (using $c^2 = a^2m^2 + b^2$)

$b^2m^2 = a^2 \Rightarrow m^2 = \frac{a^2}{b^2} \Rightarrow m = \pm\frac{a}{b}$



$y = \pm\frac{a}{b}x \pm \sqrt{(a^2+b^2)\frac{a^2}{b^2}+b^2}$

$by = \pm ax \pm \sqrt{a^4 + a^2b^2 + b^4}$

Note : Although there can be four common tangents but only one of these appears in B]

$$h = \frac{\frac{A\cos\theta}{a} + 0}{2}$$

$$2h = \frac{A}{a}\cos\theta$$

$$\left(\frac{2a}{A}\right)h = \cos\theta \quad \text{---- I}$$

Similarly

$$\left(\frac{2b}{A}\right)k = -\sin\theta \quad \text{---- II}$$

Square & add

$$\frac{4a^2}{A^2}h^2 + \frac{4b^2}{A^2}k^2 = 1$$

$$\frac{x^2}{\left(\frac{A}{2a}\right)^2} + \frac{y}{\left(\frac{A}{2b}\right)^2} = 1 \quad \text{---- II}$$

Let $a > b$

$\Rightarrow e = \sqrt{1 - \frac{b^2}{a^2}}$ in given ellipse

Let e' is eccentricity of Eqⁿ II

then it must be vertical ellipse

$$e' = \sqrt{1 - \frac{A^2/4a^2}{A^2/4b^2}}$$

$$e' = \sqrt{1 - \frac{b^2}{a^2}}$$

$$\Rightarrow \boxed{e = e'}$$

2 The area of the rectangle formed by the perpendiculars from the centre of the standard ellipse to the tangent and normal at its point whose eccentric angle is $\pi/4$, is :

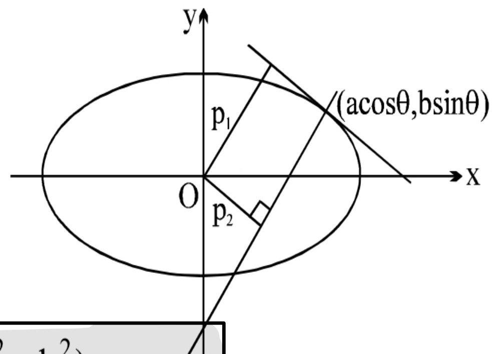
(A) $\dfrac{\left(a^2-b^2\right)ab}{a^2+b^2}$ (B) $\dfrac{\left(a^2-b^2\right)}{\left(a^2+b^2\right)ab}$ (C) $\dfrac{\left(a^2-b^2\right)}{ab\left(a^2+b^2\right)}$ (D) $\dfrac{a^2+b^2}{\left(a^2-b^2\right)ab}$

Soln:—

$$T: \frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$$

$$p_1=\left|\frac{ab}{\sqrt{b^2\cos^2\theta+a^2\sin^2\theta}}\right| \quad ....(1)$$

$$N_1: \frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2$$



$$p_2=\left|\frac{(a^2-b^2)\sin\theta\cos\theta}{\sqrt{a^2\sin^2\theta+b^2\cos^2\theta}}\right| \quad ....(2)$$

$$p_1p_2=\frac{ab(a^2-b^2)}{2\left(\frac{a^2}{2}+\frac{b^2}{2}\right)} \quad \text{when } \theta=\pi/4; \quad \boxed{p_1p_2==\frac{ab(a^2-b^2)}{a^2+b^2} \text{ Ans]}}$$

3 The locus of the middle point of chords of an ellipse $\frac{x^2}{16}+\frac{y^2}{25}=1$ passing through P(0, 5) is another ellipse E. The coordinates of the foci of the ellipse E, is

(A) $\left(0, \frac{3}{5}\right)$ and $\left(0, \frac{-3}{5}\right)$

(B) (0, – 4) and (0, 1)

(C) (0, 4) and (0, 1)

(D) $\left(0, \frac{11}{2}\right)$ and $\left(0, \frac{-1}{2}\right)$

Soln.

Consider mid Point (h, k)

Equation of chord

$$T = S_1$$

$$\frac{hx}{16} + \frac{ky}{25} - 1 = \frac{h^2}{16} + \frac{k^2}{25} - 1$$

$$\| (0, 5)$$

$$\frac{k}{5} = \frac{h^2}{16} + \frac{k^2}{25}$$

locus of (h, k)

Vertical ellipse

$$\frac{x^2}{(2)^2} + \frac{(y - 5/2)^2}{(5/2)^2} = 1$$

$$e^2 = 1 - \frac{a^2}{b^2} = 1 - \frac{4}{25/4} = \frac{9}{25}$$

$$e = 3/5$$

$$\text{foci} = \left(0, \tfrac{5}{2} \pm be\right) = \left(0, \tfrac{5}{2} \pm 3/2\right)$$

$$= (0, 4), \quad (0, 1)$$

Ans C

4 Extremities of the latus rectum of the ellipses $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1\ (a>b)$ having a given major axis 2a lies on-

(A) $x^2 = a(a - y)$ (B) $x^2 = a(a + y)$ (C) $y^2 = a(a + x)$ (D) $y^2 = a(a - x)$

Soln.

extrimities of latus rectum $(\pm ae, \pm b^2/a)$

now check options

A) $x^2 = a(a-y)$

$\parallel\ (ae, b^2/a)$

$a^2e^2 = a\left(a - \frac{b^2}{a}\right)$

$a^2\left(1-\frac{b^2}{a^2}\right) = a^2-b^2$ ✓

C) $y^2 = a(a+x)$

$(ae, \pm\frac{b^2}{a})$ $(-ae, \pm\frac{b^2}{a})$

$\frac{b^4}{a^2} = a(a+ae)$

$1+e = \frac{b^4}{a^4}$ ✗

$1-e = \frac{b^4}{a^2}$ ✗ ✗

Similarly B and D check

Ans A, B

5 If a number of ellipse (whose axes are x & y axes) be described having the same major axis 2a but a variable minor axis then the tangents at the ends of their latus rectum pass through fixed points which can be -

(A) (0,a) (B) (0,0) (C) (0,–a) (D) (a,a)

2 2

end Points of latus rectum $(\pm ae, \pm b^2/a)$

tangent at $(\pm ae, \pm b^2/a)$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$T = 0$

$$\pm \frac{x ae}{a^2} \pm \frac{y b^2/a}{b^2} = 1$$

$$\pm \frac{ex}{a} \pm \frac{y}{a} = 1$$

A (0, a) B (0, 0) (0, -a) (a, a)

$0 \pm 1 = 1$ $0 + 0 = 1$ $0 \mp 1 = 1$ $\pm e \pm 1 = 1$

$+1 = 1$ ✓ × $+1 = 1$ ×

Ans A, C

6 Tangents are drawn from any point on the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ to the circle $x^2+y^2=1$ and respective chord of contact always touches a conic 'C', then -

(A) minimum distance between 'C' & ellipse is $\frac{3}{2}$

(B) maximum distance between 'C' & ellipse is $\frac{10}{3}$

(C) eccentricity of 'C' is $\frac{\sqrt{5}}{3}$

(D) product of eccentricity of 'C' & ellipse is 1

$\frac{x^2}{9}+\frac{y^2}{4}=1$ $\quad e_1 = 1-\frac{4}{9}=\frac{5}{9}$ $\quad e_A=\frac{\sqrt{5}}{3}$

$P(3\cos\theta, 2\sin\theta)$

COC of P to the circle

$T=0$

$3x\cos\theta + 2y\sin\theta = 1$

$\frac{x\cos\theta}{\frac{1}{3}} + \frac{y\sin\theta}{1/2} = 1$

by observation this line is tangent to ellipse (C)

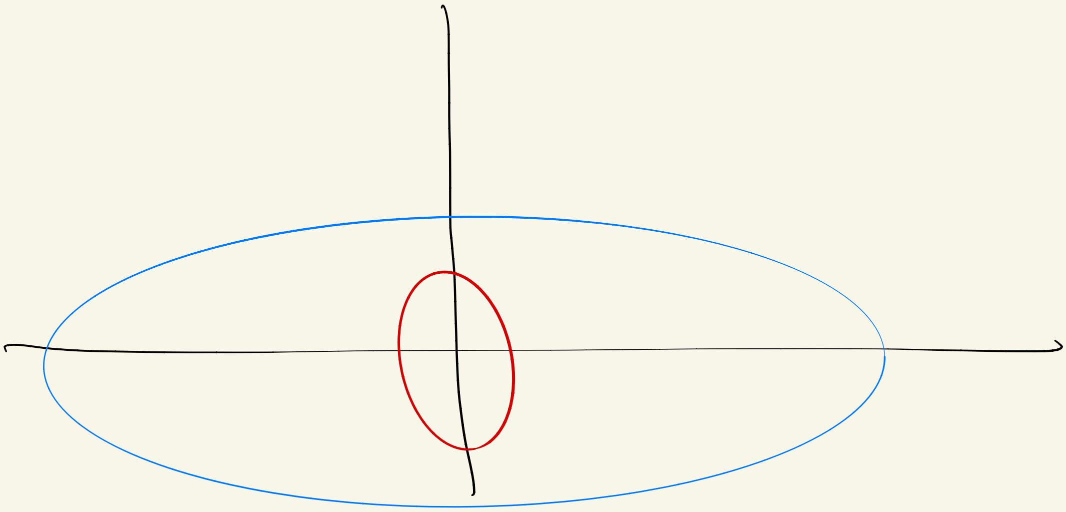
$\frac{x^2}{(1/3)^2}+\frac{y^2}{(1/2)^2}=1$

$e^2 = 1-\frac{(1/3)^2}{(1/2)^2} = 1-\frac{4}{9}=\frac{5}{9}$

$e_2 = \sqrt{5}/3$

Ans C

$e_1 e_2 = \frac{5}{9} \neq$ not D



Min distance $= 2 - \frac{1}{2} = 3/2$ Ans A

Max " $= 3 + \frac{1}{3} = \frac{10}{3}$ Ans B

Ans A, B, C

7 Two lines are drawn from point $P(\alpha, \beta)$ which touches $y^2 = 8x$ at A, B and touches $\frac{x^2}{4}+\frac{y^2}{6}=1$ at C, D, then -

(A) $\alpha + \beta = -4$

(B) $\alpha\beta = 4$

(C) Area of triangle PAB is $128\sqrt{2}$

(D) Area of triangle PAB is $32\sqrt{2}$

$y^2 = 8x$

$\frac{x^2}{4} + \frac{y^2}{6} = 1$

Common tangent

$y = mx + 2/m$

$y = mx \pm \sqrt{4m^2 + 6}$

Compare

$$\frac{1}{1} = \frac{m}{m} = \frac{2/m}{\pm\sqrt{4m^2+6}}$$

$$4m^4 + 6m^2 = 4$$

$$2m^4 + 3m^2 - 2 = 0$$

$$\left(m^2 + \frac{4}{2}\right)\left(m^2 - 1/2\right) = 0$$

$m^2 = -2$ ×

$m^2 = 1/2$

$m = 1/\sqrt{2}$

$m = -1/\sqrt{2}$

tangent

$y = \frac{x}{\sqrt{2}} + 2\sqrt{2}$

$y = \frac{-x}{\sqrt{2}} - 2\sqrt{2}$

POI $(-4, 0)$ $\quad \alpha + \beta = -4$

$P(\alpha, \beta)$ $\quad$ Ms A

$A\left(\frac{a}{m_1^2}, \frac{2a}{m_1}\right)$ $\quad B(4, -4\sqrt{2})$

$m_1 = 1/\sqrt{2}$

$a = 2$

$(4, 4\sqrt{2})$

$$\triangle PAB = \frac{1}{2} \begin{vmatrix} -4 & 0 & 1 \\ 4 & 4\sqrt{2} & 1 \\ 4 & -4\sqrt{2} & 1 \end{vmatrix} =$$

$$= \frac{1}{2}\left(-4\, 8\sqrt{2} - 0 + 1.(-32\sqrt{2})\right)$$

$$= |-32\sqrt{2}| = 32\sqrt{2}$$

Ms A, D

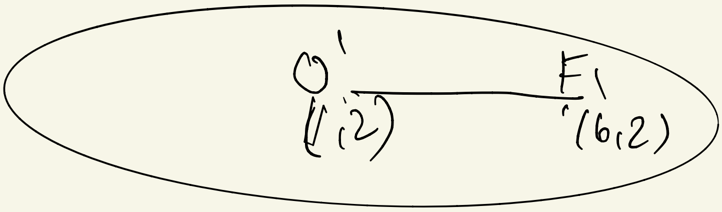
S-1

EXERCISE (S-1)

1. **(a)** Find the equation of the ellipse with its centre (1, 2), focus at (6, 2) and passing through the point (4, 6).

   (b) An ellipse passes through the points (– 3, 1) & (2, –2) & its principal axis are along the coordinate axes in order. Find its equation.

a)



$$\frac{(x-1)^2}{a^2} + \frac{(y-2)^2}{b^2} = 1$$

$O'F_1$

$$ae = \sqrt{(6-1)^2 + 0}$$

$||\ (4,6)$

$$a^2e^2 = 25$$

$$\frac{9}{a^2} + \frac{16}{b^2} = 1$$

$$a^2\left(1 - \frac{b^2}{a^2}\right) = 25$$

$$a^2 - b^2 = 25$$

got a and b

$$a^2 = 45 \quad b^2 = 20$$

$$\frac{(x-1)^2}{45} + \frac{(y-2)^2}{20} = 1$$

b)

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$(-3,1)$

$(-2,2)$

$$\frac{9}{a^2}+\frac{1}{b^2}=1$$

$$\frac{4}{a^2}+\frac{4}{b^2}=1$$

get a, b

$$a^2=\frac{32}{3} \qquad b^2=\frac{32}{5}$$

$$\frac{3x^2}{32}+\frac{5y^2}{32}=1$$

$+ \tan \alpha/2 \tan \beta/2 = \dfrac{d-a}{d+a}$ —(I)

similarly for γ, δ and $(-d, 0)$ $\quad \tan \gamma/2 \tan \delta/2 = \dfrac{-d-a}{-d+a}$ —(II)

$$= \frac{d+a}{d-a}$$

multiply (I) and (II)

$$\tan \alpha/2 \tan \beta/2 \tan \frac{\gamma}{2} \tan \frac{\delta}{2} = 1$$

2 Suppose x and y are real numbers and that $x^2 + 9y^2 - 4x + 6y + 4 = 0$ then find the maximum value of $(4x - 9y)$.

$$(x-2)^2 + 9(y+1/3)^2 = 1$$

$$\frac{(x-2)^2}{1^2} + \frac{(y+1/3)^2}{(1/3)^2} = 1$$

$x - 2 = 1 \cdot \cos\theta \qquad x = 2 + \cos\theta$

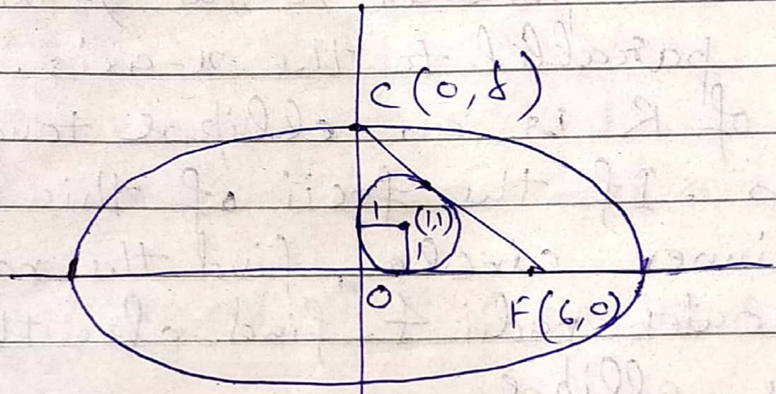
$y + 1/3 = \frac{1}{3}\sin\theta \qquad y = \frac{-1}{3} + \frac{\sin\theta}{3}$

$$4x - 9y = 8 + 4\cos\theta + 3 - 3\sin\theta$$
$$= 11 + (4\cos\theta - 3\sin\theta)_{max}$$
$$= 11 + 5 = 16$$

3

Q:- Point 'O' is the centre of the ellipse with major axis AB & minor axis CD. Point F is one focus of the ellipse. If OF = 6 & the diameter of the inscribed circle of triangle OCF is 2, The find the product (AB)(CD)

Sol^n:-



here $OF = ae = 6$ —— ①

equation of FC $\Rightarrow$ $\dfrac{x}{6} + \dfrac{y}{b} = 1$

$\Rightarrow bx + 6y - 6b = 0$

Now length of $\perp$ from $(1,1)$ is 1

$\Rightarrow \dfrac{-b - 6 + 6b}{\sqrt{36 + b^2}} = 1 \Rightarrow b = \dfrac{5}{2}$ —— ②

Now $\Rightarrow e^2 = 1 - \dfrac{b^2}{a^2}$

from ① & ②

$e^2 = 1 - \dfrac{25e^2}{144}$

$\Rightarrow e = \dfrac{12}{13} \Rightarrow a = \dfrac{13}{2}$

So $(AB)(CD)$

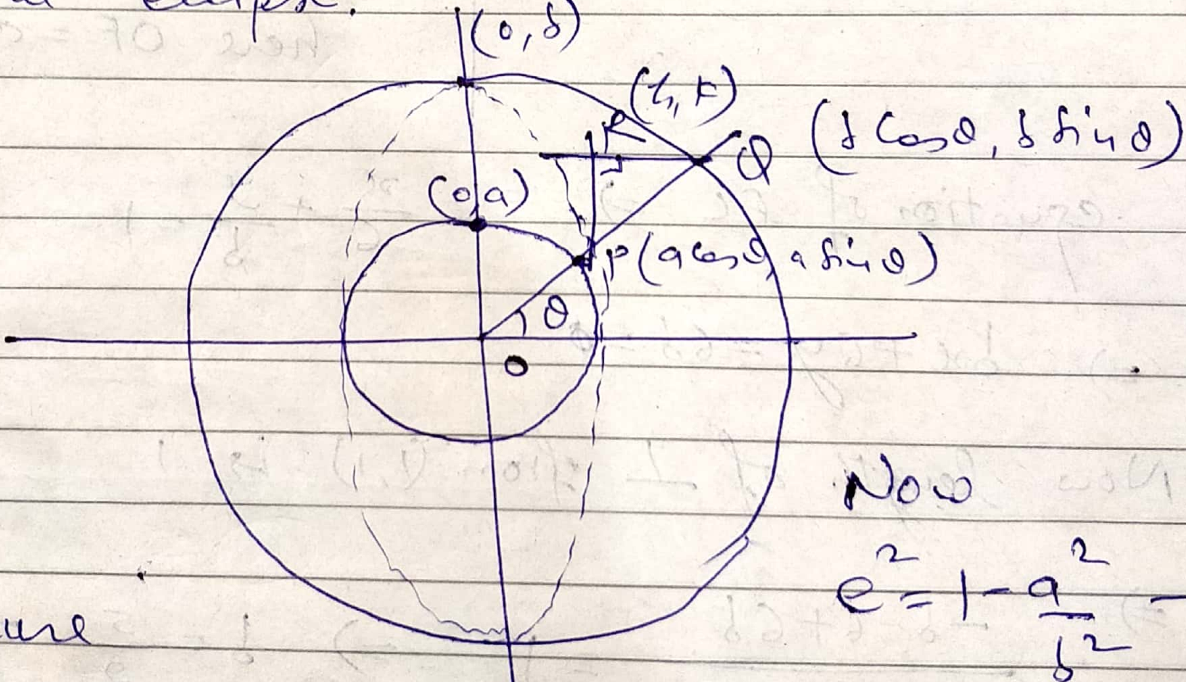
$= (2a)(2b) = \underline{\underline{65}}$

final answer.

4

Q:- 'O' is the origin & also the centre of 2 concentric circles having radii of the inner & the outer circle as 'a' & 'b' respectively. A line OPQ is drawn to cut the inner circle in P & the outer circle in Q. PR is drawn parallel to the y-axis & QR is drawn parallel to the x-axis. Prove that the locus of R is an ellipse touching the 2 circles. If the focii of this ellipse lie on the inner circle, find the ratio of inner and outer radii & find also the eccentricity of the ellipse.

Sol^n:-



from figure

$h = a\cos\theta, \quad k = a\sin\theta$

$\Rightarrow$ eq^n of ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \quad (\text{where } b > a)$$

which can be shown as figure

Now

$$e^2 = 1 - \frac{a^2}{b^2} \quad \text{--- (1)}$$

& (0, a) is the focus

$$\Rightarrow a = be \quad \text{--- (2)}$$

from (1) & (2)

$$e = \frac{1}{\sqrt{2}} = \frac{a}{b}$$

Q:- Find the condition so that the line $px+qy=r$ intersects the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ in points whose eccentric angles differ by $\pi/4$.

Solution:-

we know that equation of line joining points α & β is given by

$$\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right)+\frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha-\beta}{2}\right)$$

Now $\beta=\alpha+\pi/4$ (according to question)

$$\Rightarrow \frac{x}{a}\cos(\alpha+\pi/8)+\frac{y}{b}\sin\left(\alpha+\frac{\pi}{8}\right)=\cos\pi/8 \quad \text{——} (1)$$

which coincides with

$$px+qy=r \quad \text{——} (2)$$

Comparing (1) & (2)

$$\underset{(\mathrm{I})}{\frac{\cos(\alpha+\pi/8)}{ap}}=\underset{(\mathrm{II})}{\frac{\sin(\alpha+\pi/8)}{bq}}=\underset{(\mathrm{III})}{\frac{\cos\pi/8}{r}}$$

from (I) & (II) $\Rightarrow \cos(\alpha+\pi/8)=\frac{ap}{r}\cos\pi/8$ —— (3)

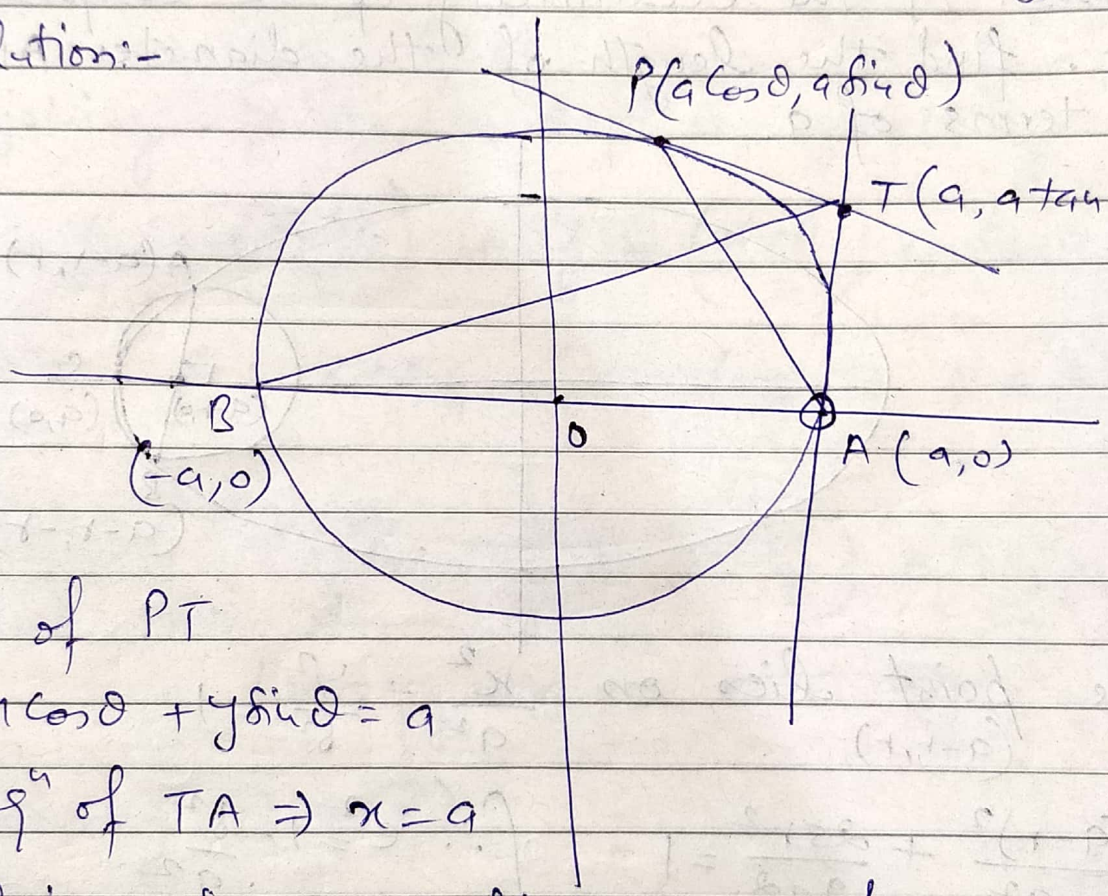
from (II) & (III) $\Rightarrow \sin(\alpha+\pi/8)=\frac{bq}{r}\cos\pi/8$ —— (4)

$(3)^2+(4)^2 \Rightarrow \frac{a^2p^2\cos^2\pi/8}{r^2}+\frac{b^2q^2\cos^2\pi/8}{r^2}=1$

6

The tangent at any point P of a circle $x^2+y^2=a^2$ meets the tangent at a fixed point A(a, 0) in T and T is joined to B, the other end of the diameter through A. Prove that the locus of the intersection of AP and BT is an ellipse whose eccentricity is $1/\sqrt{2}$

Solution:-



eqn of PT

$x\cos\theta + y\sin\theta = a$

& eqn of TA $\Rightarrow x = a$

solving above equations $x = a$ & $y = a\tan\theta/_2$

$\Rightarrow T \equiv (a, a\tan\theta/_2)$

Now eqn of BT $\Rightarrow y = \frac{\tan\theta/_2}{2}(x+a)$ —— (1)

eqn of PA $\Rightarrow y = -\cot\theta/_2 (x-a)$ —— (2)

multiply (1) & (2) $\Rightarrow$ $\frac{x^2}{a^2} + \frac{y^2}{\left(\frac{a}{\sqrt{2}}\right)^2} = 1 \Rightarrow e = \sqrt{1 - \frac{a^2}{2a^2}}$

$= 1/\sqrt{2}$

7 :- Find equations of lines with equal intercepts on the axes & which touch the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$

Sol:-

let the equation of tangent be

$$y = mx \pm \sqrt{16m^2 + 9}$$

but $m = -1$
(according to ques)

$\Rightarrow \quad x + y \pm 5 = 0$

8 A tangent having slope $-4/3$ to the ellipse $\frac{x^2}{18} + \frac{y^2}{32} = 1$, intersects the axis of x & y in points A & B respectively. If O is the origin, find the area of triangle OAB.

Sol^n:- eq^n of tangent

$$y = -\frac{4}{3}x \pm \sqrt{32 + 32} \quad \left[\because y = mx \pm \sqrt{a^2m^2 + b^2}\right]$$

$\Rightarrow \frac{x}{6} + \frac{y}{8} = \pm 1 \quad \Rightarrow x \text{ intercept} = \pm 6$

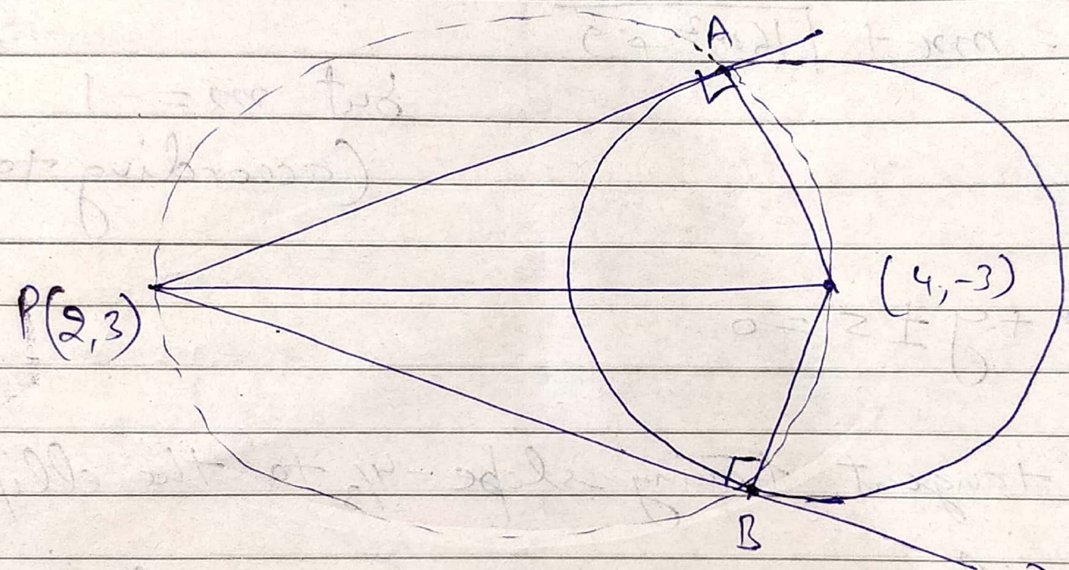
y intercept $= \pm 8$

$\Rightarrow$ area of $\Delta AOB = \frac{1}{2} \times 6 \times 8 = 24$ sq. units.

9

Tangents drawn from the point $P(2,3)$ to the circle $x^2+y^2-8x+6y+1=0$ touch the circle at the points A & B. The circumcircle of the $\triangle PAB$ cuts the director circle of ellipse $\frac{(x+5)^2}{9}+\frac{(y-3)^2}{b^2}=1$ orthogonally, Find the value of b^2.

Solution:-



It clearly shows that (2,3) & (4,-3) will be ends of diameter of circle PAB

so eqn of circle is $(x-2)(x-4)+(y-3)(y+3)=0$

$\Rightarrow x^2+y^2-6x-1=0$

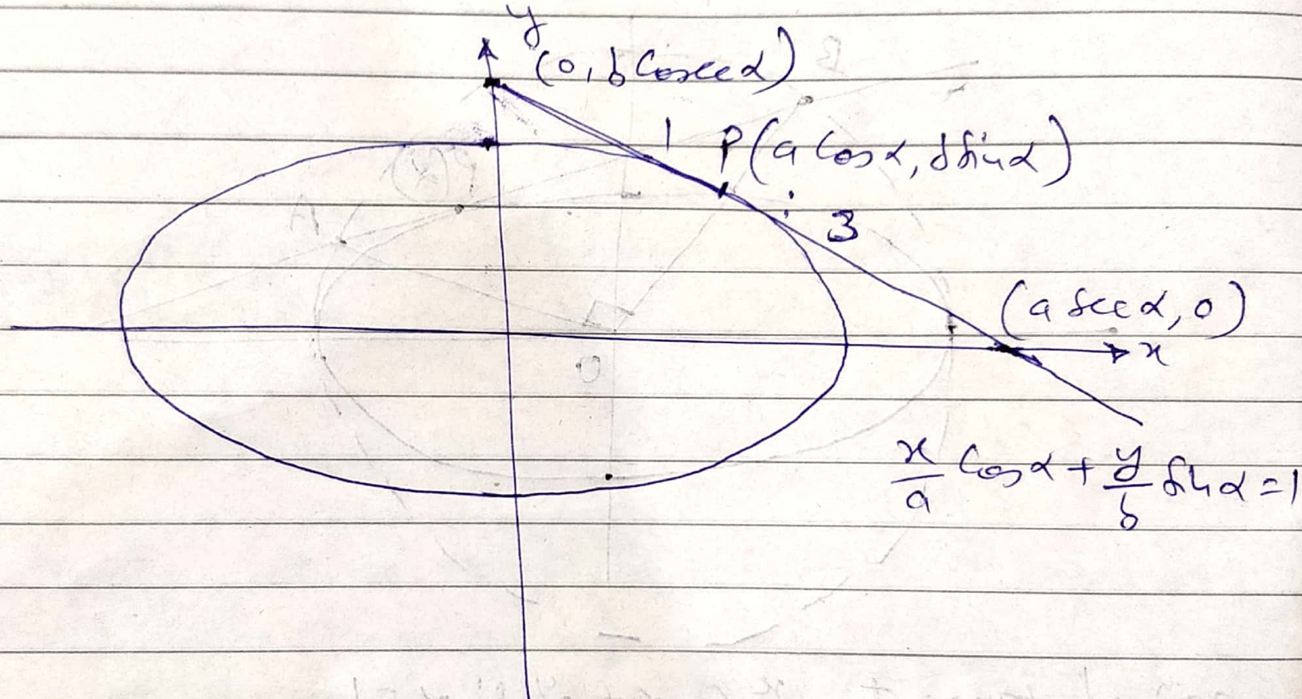
as it intersects the circle $(x+5)^2+(y-3)^2=9+b^2$ orgonally then $(2g_1g_2+2f_1f_2=c_1+c_2)$

$2(-3)(5)+2(0)(-3)=-1+25-b^2$

$\Rightarrow b^2=54$

Q:- A tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ touches at the point P on it in the first quadrant & meets the coordinate axes in A & B respectively. If P divides AB in the ratio 3:1 reckoning from the x-axis find the equation of tangent.

Sol^n:-



Now from section formula

$a\cos\alpha=\frac{a\sec\alpha}{4} \Rightarrow \cos\alpha=\frac{1}{2}$ $\left(0<\alpha<\frac{\pi}{2}\right)$

$\Rightarrow \sin\alpha=\frac{\sqrt{3}}{2}$

Now the eq^n of tangent

$$\frac{x}{2a}+\frac{\sqrt{3}\,y}{2b}=1$$

11. If the normal at the point P(θ) to the ellipse $\frac{x^2}{14}+\frac{y^2}{5}=1$, intersects it again at the point Q(2θ), show that cos θ = – (2/3).

Sol. $a^2 = 14,\ b^2 = 5$

Eqn of Normal

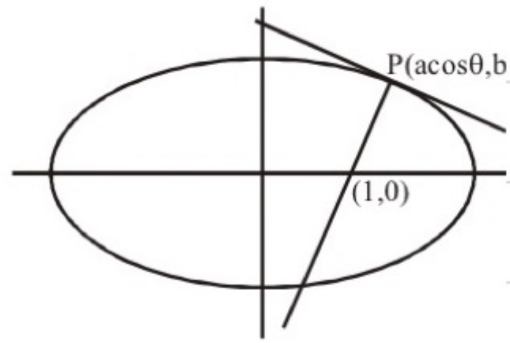
$$\Rightarrow \frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = 9$$

$$\Rightarrow 18\cos^2\theta - 9\cos\theta - 14 = 0$$

$$\Rightarrow \cos\theta = -\frac{2}{3}$$

12. Find the equation of the largest circle with centre (1, 0) that can be inscribed in the ellipse $x^2 + 4y^2 = 16$.

Sol:-



$$\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = 12 \; ; \; a = 4, b = 2$$

$$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = 12$$

$$\frac{4x}{\cos\theta} - \frac{2y}{\sin\theta} = 12$$

It passes through (1, 0)

$2x\sec\theta - y\,\text{cosec}\theta = 6 \Rightarrow 2\sec\theta = 6 \Rightarrow \sec\theta = 3$

$\cos\theta = \frac{1}{3}$, $\quad \sin^2\theta = 1 - \frac{1}{9} = \frac{8}{9}$

$$\sin\theta = \frac{2\sqrt{2}}{3}$$

$$r^2 = (a\cos\theta - 1)^2 + b^2\sin^2\theta$$
$$= a^2\cos^2\theta + b^2\sin^2\theta + 1 - 2a\cos\theta$$

$$= 16\cdot\frac{1}{9} + 4\cdot\frac{8}{9} + 1 - \frac{8}{3}$$

$$= \frac{16}{9} + \frac{32}{9} + 1$$

$$= \frac{48}{9} + 1 - \frac{8}{3} = \frac{59}{9} - \frac{24}{9} = \frac{33}{9} = \frac{11}{3}$$

Hence circle in $(x-1)^2 + y^2 = \frac{11}{3}$]

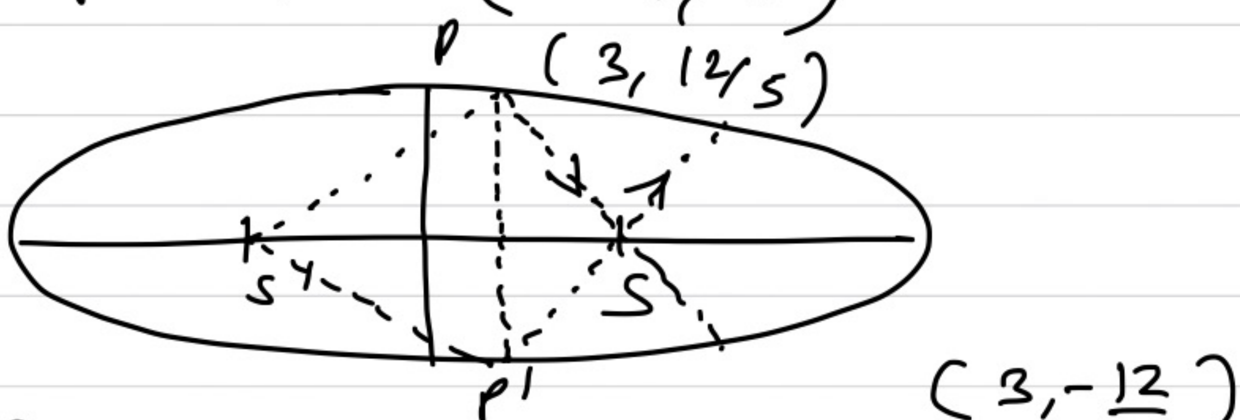
13. A ray emanating from the point $(-4, 0)$ is incident on the ellipse $9x^2 + 25y^2 = 225$ at the point P with abscissa 3. Find the equation of the reflected ray after first reflection.

Sol. $\frac{x^2}{25} + \frac{y^2}{9} = 1$

$e = 4/5$, $= (\pm ae, 0)$

focus $= (\pm 5 \times \frac{4}{5}, 0)$

focus $= (\pm 4, 0)$

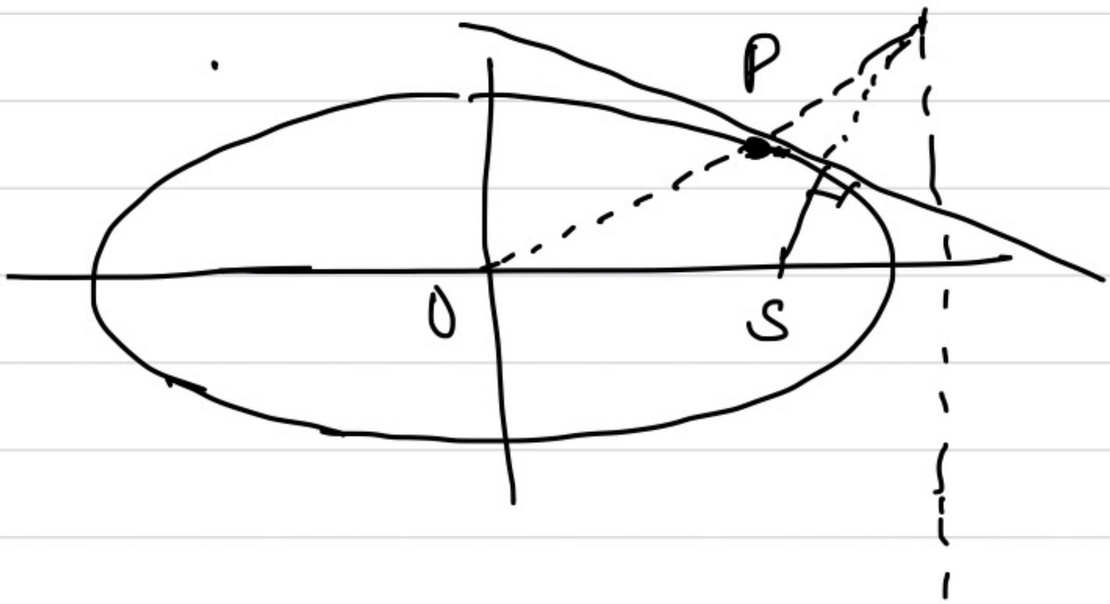


P & $P'(3, \pm 12/5)$ $S(4, 0)$

$\left(y \pm \frac{12}{5}\right) = \frac{12}{5}(x - 4)$

$\Rightarrow 12x \pm 5y = 48$

14. Prove that, in an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, the perpendicular from a focus upon any tangent and the line joining the centre of the ellipse to the point of contact meet on the corresponding directrix.



Eqn of tangent at (x_1, y_1)

$$\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=0 \rightarrow (1)$$

Eqn of $\perp$ (1) which passes through $S(ae, 0)$

$$\frac{xy_1}{b^2}-\frac{x_1}{a^2}y=\frac{aey_1}{b^2} \rightarrow (2)$$

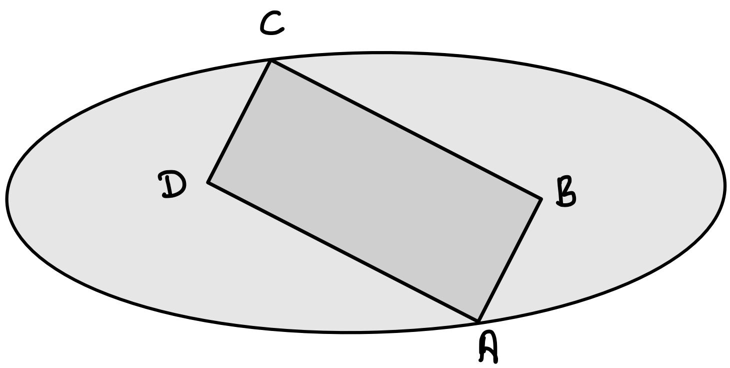
Eqn of OP : $y=\frac{y_1}{x_1}x \rightarrow (3)$

by (2) & (3) $x=a/e$ on directrix.

S-2

1. Rectangle ABCD has area 200. An ellipse with area 200π passes through A and C and has foci at B and D. Find the perimeter of the rectangle.

Sol:-



Let $2a$, $2b$, e are major axis, minor axis & eccentricity of ellipse.

Area of ellipse $= \pi ab = 200\pi \Rightarrow ab = 200$ ... ①

Let sides of ABCD are p, q respectively.

As B & D are foci, so $BA + DA = 2a$ (major Axis)

$\Rightarrow \quad p + q = 2a$ .... ②

Also area of ABCD $= pq = 200$ ... ③

$BD = 2ae \Rightarrow BD^2 = 4a^2e^2 = 4(a^2 - b^2)$

$\Rightarrow p^2 + q^2 = 4a^2 - 4b^2$

$\Rightarrow (p+q)^2 - 2pq = 4a^2 - 4b^2$

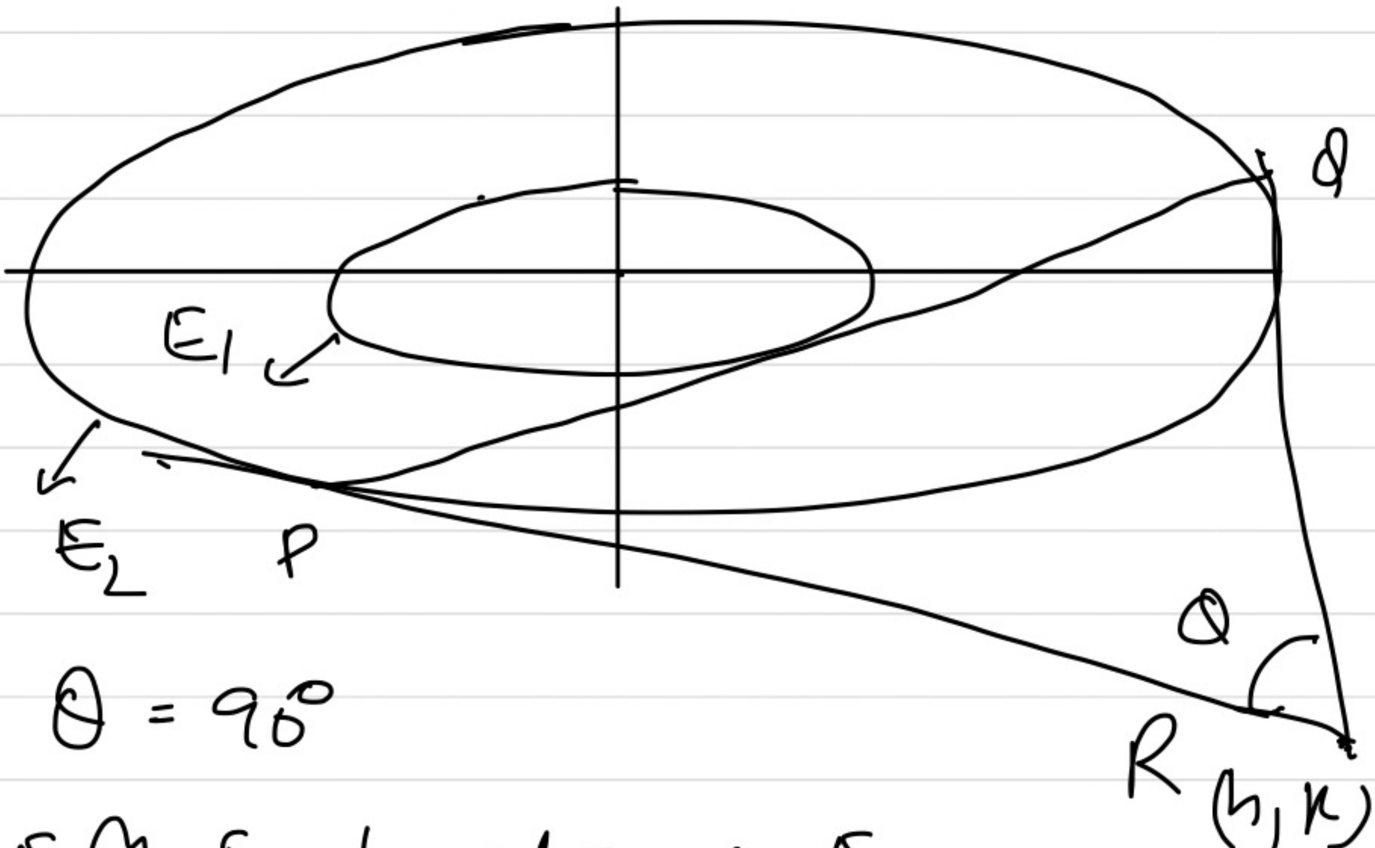
$\Rightarrow (2a)^2 - 2 \times 200 = 4a^2 - 4b^2 \Rightarrow b = 10$

as $b = 10$, so, $a = 20$

$\therefore$ perimeter of rectangle $= 2(p+q) = 2(2a)$
$= 4a = 80$

2. A tangent to the ellipse $x^2 + 4y^2 = 4$ meets the ellipse $x^2 + 2y^2 = 6$ at P & Q. Prove that the tangents at P & Q of the ellipse $x^2 + 2y^2 = 6$ are at right angles.

$E_1 = \frac{x^2}{4} + \frac{y^2}{1} = 1$ & $E_2 = \frac{x^2}{6} + \frac{y^2}{3} = 1$



$\theta = 90^\circ$

Eqn of tangent at E_1

$\frac{x\cos\theta}{2} + y\sin\theta = 1 \longrightarrow$ (1)

from R PQ is chord of contact

$$\frac{xh}{6} + \frac{yk}{3} = 1 \rightarrow (2)$$

(1) & (2) are identical

$$\frac{\cos\theta/2}{h/6} = \frac{\sin\theta/1}{k/3} = 1$$

$$\cos\theta = h/3 \quad , \quad \sin\theta = k/3$$

or $h^2 + k^2 = 9$

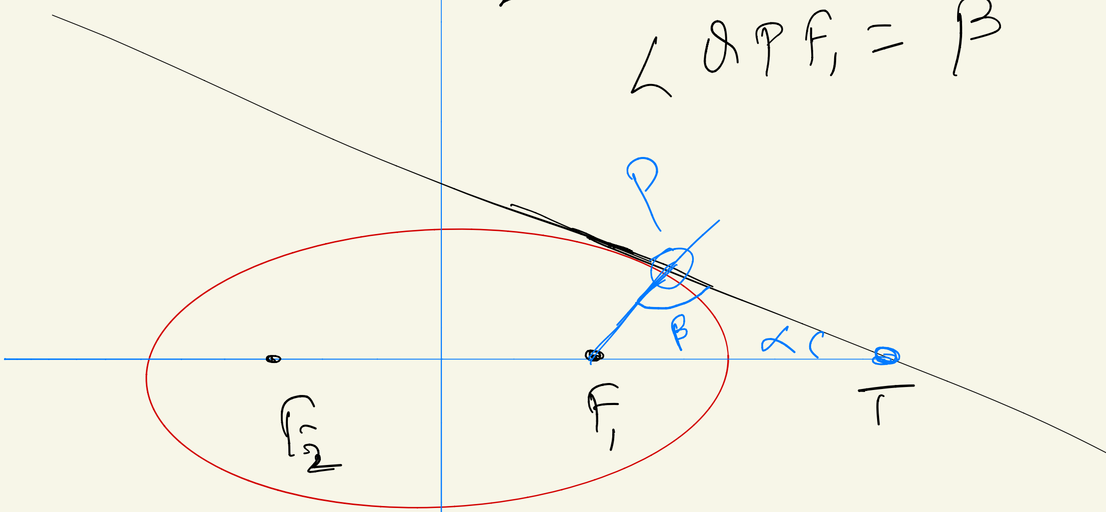
Which is director circle

for E_2 $\therefore$ $\theta = 90^\circ$

3 If the tangent at any point of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ makes an angle α with the major axis and an angle β with the focal radius of the point of contact then show that the eccentricity 'e' of the ellipse is given by the absolute value of $\frac{\cos\beta}{\cos\alpha}$.

$\angle PQF_1 = \alpha$

$\angle QPF_1 = \beta$



Equation of ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1, \quad a>b$$

F_1 and F_2 are foci and e is eccentricity of the ellipse

$P(a\cos\theta, b\sin\theta)$ be a point

on the ellipse
Equation of tangent at
P is $\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$

Slope $= -\frac{\cos\theta}{\sin\theta}\left(\frac{b}{a}\right)$

$\Rightarrow \tan\alpha = \left(\frac{b}{a}\right)\frac{\cos\theta}{(\sin\theta)}$ ... (1)

$T\left(\frac{a}{\cos\theta}, 0\right)$, $F_1(ae, 0)$

$\Rightarrow F_1T = a\sec\theta - ae$

$PF_1 = a - ae\cos\theta$

Apply Sine Rule in ΔF_1PT

$\frac{F_1T}{\sin\beta} = \frac{F_1P}{\sin\alpha}$

$\Rightarrow \frac{a(1-e\cos\theta)}{\cos\theta \cdot \sin\beta} = \frac{a(1-e\cos\theta)}{\sin\alpha}$

$\Rightarrow \cos\theta = \frac{\sin\alpha}{\sin\beta}$ --- (2)

From ① & ②

$$\tan\alpha = \left(-\frac{b}{a}\right)\frac{\sin\alpha \cdot \cancel{\sin\beta}}{\cancel{\sin\beta} \cdot \sqrt{\sin^2\beta - \sin^2\alpha}}$$

$$\Rightarrow \sin^2\beta - \sin^2\alpha = \frac{b^2}{a^2}\cos^2\alpha$$

$$\Rightarrow \cancel{1} - \cos^2\beta \cancel{-1} + \cos^2\alpha = \frac{b^2}{a^2}\cos^2\alpha$$

$$\Rightarrow \left(1 - \frac{b^2}{a^2}\right)\cos^2\alpha = \cos^2\beta$$

$$\Rightarrow e^2 = \frac{\cos^2\beta}{\cos^2\alpha}$$

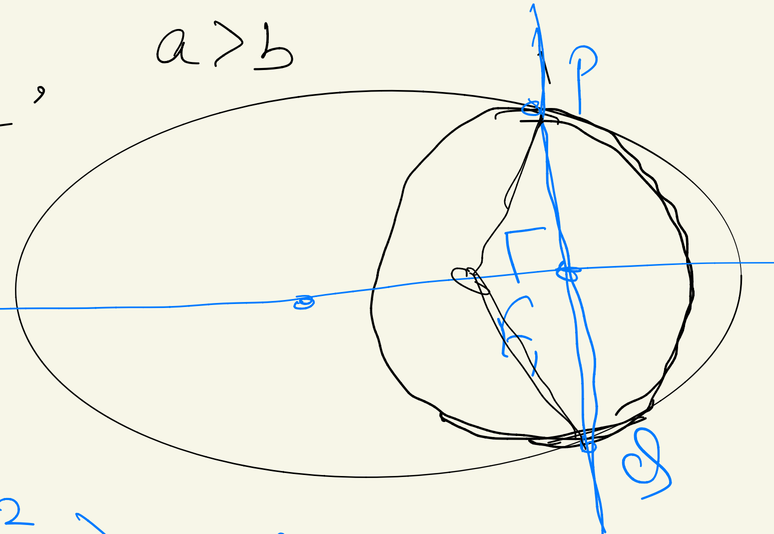
$$\Rightarrow e = \left|\frac{\cos\beta}{\cos\alpha}\right|$$

4 Prove that the equation to the circle, having double contact with the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ (with eccentricity e) at the ends of a latus rectum, is $x^2 + y^2 - 2ae^3x = a^2(1 - e^2 - e^4)$.

$e^2 = 1 - \frac{b^2}{a^2}$, $a > b$

$a^2e^2 = a^2 - b^2$

$\Rightarrow b^2 = a^2 - a^2e^2$



$P\left(ae, \frac{b^2}{a}\right)$, $Q\left(ae, -\frac{b^2}{a}\right)$

Equation of normal at P

$\frac{xa^2}{ae} - \frac{yb^2a}{b^2} = a^2 - b^2$

$\Rightarrow \frac{ax}{e} - ay = a^2e^2$ ---- (1)

equation of normal at Q

$\Rightarrow \frac{ax}{e} + ay = a^2e^2$ - - - (2)

Solving (1) & (2) $\Rightarrow$ centre of the circle at $(ae^3, 0)$

Equation of circle

$$\Rightarrow (x - ae^3)^2 + y^2 = (ae^3 - ae)^2 + \left(\frac{b^2}{a}\right)^2$$

$$\Rightarrow x^2 + y^2 - 2ae^3x + \cancel{a^2e^6}$$
$$= \cancel{a^2e^6} + a^2e^2 - 2a^2e^4 + \left(\frac{a^2 - a^2e^2}{a}\right)^2$$

$$\Rightarrow x^2 + y^2 - 2ae^3x = a^2\left[e^2 - 2e^4 + 1 + e^4 - 2e^2\right]$$

$$\Rightarrow x^2 + y^2 - 2ae^3x = a^2(1 - e^2 - e^4)$$

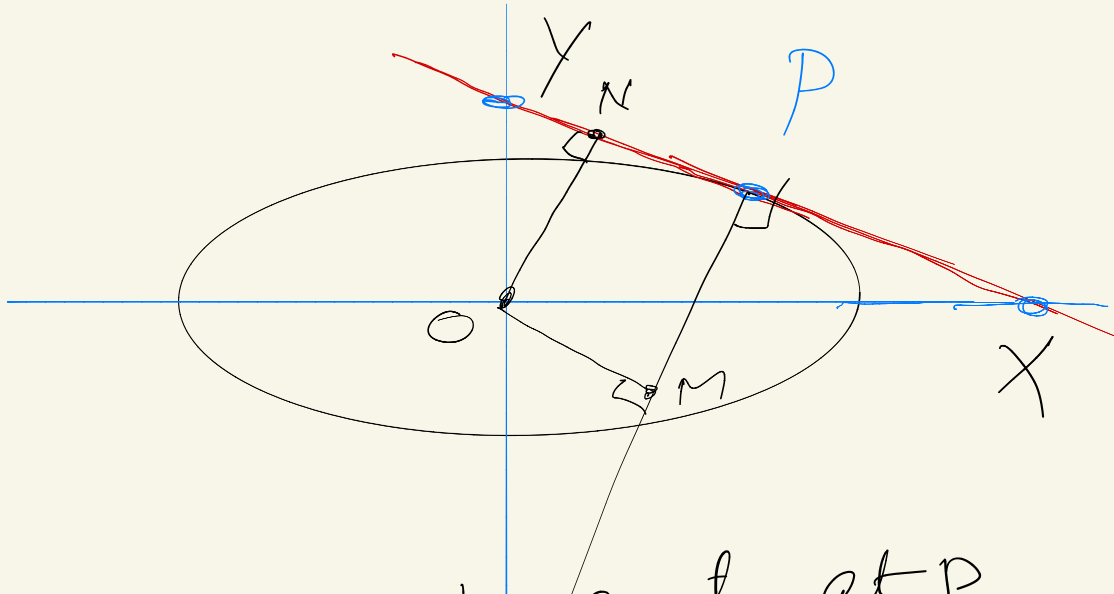
5 Consider the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with centre 'O' where $a > b > 0$. Tangent at any point P on the ellipse meets the coordinate axes at X and Y and N is the foot of the perpendicular from the origin on the tangent at P. Minimum length of XY is 36 and maximum length of PN is 4.

(a) Find the eccentricity of the ellipse.

(b) Find the maximum area of an isosceles triangle inscribed in the ellipse if one of its vertex coincides with one end of the major axis of the ellipse.

(c) Find the maximum area of the triangle OPN.

$P(a\cos\theta, b\sin\theta)$



Equation of tangent at P

$\Rightarrow \frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$

$\Rightarrow X(a\sec\theta, 0),\ Y(0, b\csc\theta)$

$XY=\sqrt{a^2\sec^2\theta+b^2\csc^2\theta}$

$\Rightarrow xy = \sqrt{a^2 + a^2\tan^2\theta + b^2 + b^2\cot^2\theta}$

$= \sqrt{a^2 + b^2 + a^2\tan^2\theta + b^2\cot^2\theta}$

using A.M. $\geq$ G.M

$\Rightarrow a^2\tan^2\theta + b^2\cot^2\theta \geq 2ab$

$\Rightarrow a^2 + b^2 + a^2\tan^2\theta + b^2\cot^2\theta \geq a^2 + b^2 + 2ab$

$\Rightarrow xy \geq a+b$

$\Rightarrow$ Minimum of $xy = a+b$

$\Rightarrow a+b = 36 \cdots (1)$

Equation of normal at P

$\Rightarrow x a \sec\theta - y b \operatorname{cosec}\theta = a^2 - b^2$

From sketch $PN = OM$

$= \dfrac{a^2 - b^2}{\sqrt{a^2\sec^2\theta + b^2\operatorname{cosec}^2\theta}}$

Now $a^2\sec^2\theta + b^2\csc^2\theta$

$\geq (a+b)^2$

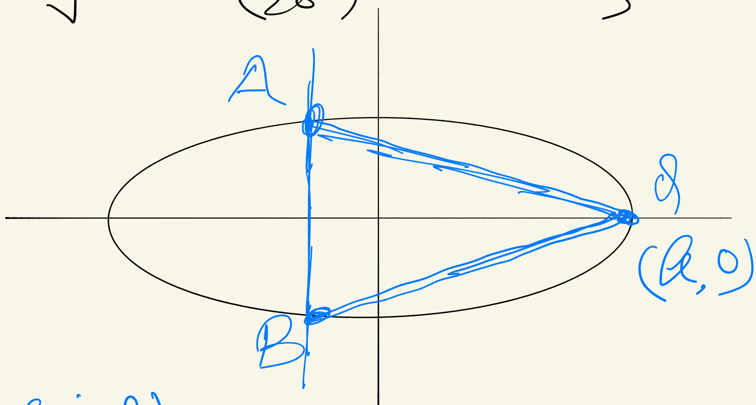
$\Rightarrow$ Minimum of $PN = a-b$

$\Rightarrow \quad a-b = 4 \text{ --- } ②$

from ① & ② $\quad a=20,\ b=16$

(a) $e = \sqrt{1-\left(\frac{16}{20}\right)^2} = \frac{3}{5}$

(b)



$A(a\cos\theta,\ b\sin\theta)$ $\quad S(a, 0)$

$B(a\cos\theta,\ -b\sin\theta)$

Area of $\Delta ABS = \frac{1}{2}(2b\sin\theta)(a-a\cos\theta)$

$\Rightarrow$ Area $= ab\sin\theta(1-\cos\theta)$

$= 320\,(\sin\theta - \sin\theta\cdot\cos\theta)$

To have maximum area,

$\cos\theta - \cos^2\theta + \sin^2\theta = 0$

$\Rightarrow \cos 2\theta - \cos\theta = 0$

$\Rightarrow \sin\frac{3\theta}{2}\cdot\sin\left(-\frac{\theta}{2}\right) = 0$

$\Rightarrow \frac{3\theta}{2} = \pi \Rightarrow \theta = \frac{2\pi}{3}$

$\{$Here $0 \le \theta < 2\pi\}$

maximum area $= 320 \times \frac{\sqrt{3}}{2} \times \frac{3}{2}$

$= 240\sqrt{3}$

(c) Area of $\triangle OPN$

$= \frac{1}{2}(PN)(ON)$

$$PN = OM = \frac{a^2 - b^2}{\sqrt{a^2\sec^2\theta + b^2\operatorname{cosec}^2\theta}}$$

$$ON = \frac{1}{\sqrt{\frac{\cos^2\theta}{a^2} + \frac{\sin^2\theta}{b^2}}}$$

$\Rightarrow$

$$(PN)(ON) = \frac{36 \times 4}{\sqrt{1 + 1 + \frac{a^2}{b^2}\tan^2\theta + \frac{b^2}{a^2}\cot^2\theta}}$$

using A.M. $\geq$ G.M.

$$\Rightarrow \quad \frac{a^2}{b^2}\tan^2\theta + \frac{b^2}{a^2}\cot^2\theta \geq 2$$

$\Rightarrow$ maximum value of $(PN)(ON)$

$= 72$

maximum area of $\triangle OPN = \underline{\underline{36}}$

JM

EXERCISE (JM)

1. The ellipse $x^2 + 4y^2 = 4$ is inscribed in a rectangle aligned with the coordinate axes, which in turn is inscribed in another ellipse that passes through the point (4, 0). Then the equation of the ellipse is :- **[AIEEE-2009]**

(1) $4x^2 + 48y^2 = 48$ (2) $4x^2 + 64y^2 = 48$ (3) $x^2 + 16y^2 = 16$ (4) $x^2 + 12y^2 = 16$

Sol:-

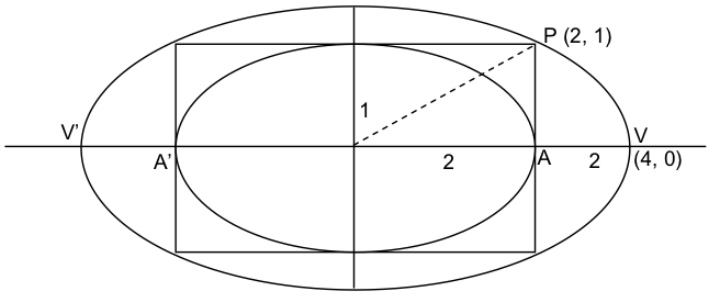
$x^2 + 4y^2 = 4 \Rightarrow \frac{x^2}{4} + \frac{y^2}{1} = 1 \Rightarrow a = 2, b = 1 \Rightarrow P = (2, 1)$

Required Ellipse is $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \Rightarrow \frac{x^2}{4^2} + \frac{y^2}{b^2} = 1$

(2, 1) lies on it

$\Rightarrow \frac{4}{16} + \frac{1}{b^2} = 1 \Rightarrow \frac{1}{b^2} = 1 - \frac{1}{4} = \frac{3}{4} \Rightarrow b^2 = \frac{4}{3}$

$\therefore \frac{x^2}{16} + \frac{y^2}{\left(\frac{4}{3}\right)} = 1 \Rightarrow \frac{x^2}{16} + \frac{3y^2}{4} = 1 \Rightarrow x^2 + 12y^2 = 16$



Ans → option(4)

2. Equation of the ellipse whose axes are the axes of coordinates and which passes through the point $(-3, 1)$ and has eccentricity $\sqrt{2/5}$ is :- **[AIEEE-2011]**

(1) $3x^2 + 5y^2 - 15 = 0$ (2) $5x^2 + 3y^2 - 32 = 0$

(3) $3x^2 + 5y^2 - 32 = 0$ (4) $5x^2 + 3y^2 - 48 = 0$

Sol:-

$$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$$

$$\frac{9}{a^2}+\frac{1}{b^2} = 1 \text{ .... (1)}$$

case - 1 when $a > b$

$b^2 = a^2 (1 - e^2)$

$b^2 = a^2 (1 - 2/5)$

$5b^2 = 3a^2$ .... (2)

from (1) & (2)

$$\frac{9\times3}{5b^2}+\frac{1}{b^2} = 1 \quad \Rightarrow \quad b^2 = \frac{32}{5}$$

$$\therefore\ a^2 = \frac{32}{3}$$

$\Rightarrow$ eqn is $3x^2 + 5y^2 - 32 = 0$

Case II when $b > a$

$a^2 = b^2 (1 - e^2)$

$= \frac{3}{5} b^2$ .... (3)

from (1) & (3)

$$a^2 = \frac{48}{5}, \ b^2 = 16$$

$$\therefore\ \frac{5x^2}{48}+\frac{y^2}{16} = 1$$

$\Rightarrow 5x^2 + 3y^2 - 48 = 0$

Ans $\rightarrow$ option (3, 4)

3. An ellipse is drawn by taking a diameter of the circle $(x-1)^2 + y^2 = 1$ as its semi-minor axis and a diameter of the circle $x^2 + (y-2)^2 = 4$ as its semi-major axis. If the centre of the ellipse is at the origin and its axes are the coordinate axes, then the equation of the ellipse is : **[AIEEE-2012]**

(1) $x^2 + 4y^2 = 16$ (2) $4x^2 + y^2 = 4$ (3) $x^2 + 4y^2 = 8$ (4) $4x^2 + y^2 = 8$

Sol :-

Semi minor axis b = 2
Semi major axis a = 4

Equation of ellipse = $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ $\Rightarrow \dfrac{x^2}{16} + \dfrac{y^2}{4} = 1$

$\Rightarrow x^2 + 4y^2 = 16.$

Ans-) option (1)

4. **Statement–1 :** An equation of a common tangent to the parabola $y^2 = 16\sqrt{3}x$ and the ellipse $2x^2 + y^2 = 4$ is $y = 2x + 2\sqrt{3}$.

Statement–2 : If the line $y = mx + \frac{4\sqrt{3}}{m}$, $(m \neq 0)$ is a common tangent to the parabola $y^2 = 16\sqrt{3}x$ and the ellipse $2x^2 + y^2 = 4$, then m satisfies $m^4 + 2m^2 = 24$. **[AIEEE-2012]**

(1) Statement–1 is true, Statement–2 is false.

(2) Statement–1 is false, Statement–2 is true.

✓(3) Statement–1 is true, Statement–2 is true ; Statement–2 is a correct explanation for Statement–1.

(4) Statement–1 is true, Statement–2 is true ; Statement–2 is not a correct explanation for Statement–1.

Sol:-

eqn of tangent to the parabola $y^2 = 16\sqrt{3}x$ is

$$y = mx + \frac{4\sqrt{3}}{m} \quad \text{——— (1)}$$

and eqn of tangent to the given ellipse is $y = mx \pm \sqrt{2m^2+4}$ — (2)

from (1) and (2) $\frac{4\sqrt{3}}{m} = \pm\sqrt{2m^2+4}$

$$\Rightarrow \boxed{m^4 + 2m^2 = 24}$$

$\Rightarrow m^2 = 4, -6$ but '-6' is not possible.

$$\Rightarrow \boxed{m = \pm 2}$$

for $m = 2$, tangent is

$$\boxed{y = 2x + 2\sqrt{3}}$$ Ans

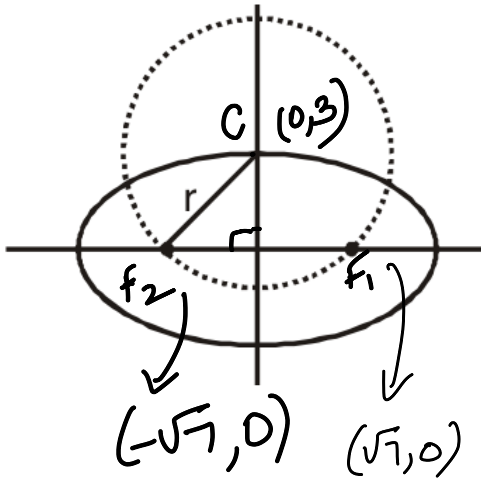
5. The equation of the circle passing through the foci of the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ and having centre at (0, 3) is : **[JEE (Main)-2013]**

(1) $x^2 + y^2 - 6y - 7 = 0$ (2) $x^2 + y^2 - 6y + 7 = 0$

(3) $x^2 + y^2 - 6y - 5 = 0$ (4) $x^2 + y^2 - 6y + 5 = 0$

Sol:-

$a = 4,\ b = 3,\ e = \sqrt{1-\frac{9}{16}} \quad \Rightarrow e = \frac{\sqrt{7}}{4}$



Focii is $(\pm ae, 0) \quad \Rightarrow \quad (\pm\sqrt{7}, 0)$

$r = \sqrt{(ae)^2+b^2}$

$\sqrt{7+9}$

$= 4$

Now equation of circle is $(x-0)^2 + (y-3)^2 = 16$

$x^2 + y^2 - 6y - 7 = 0$

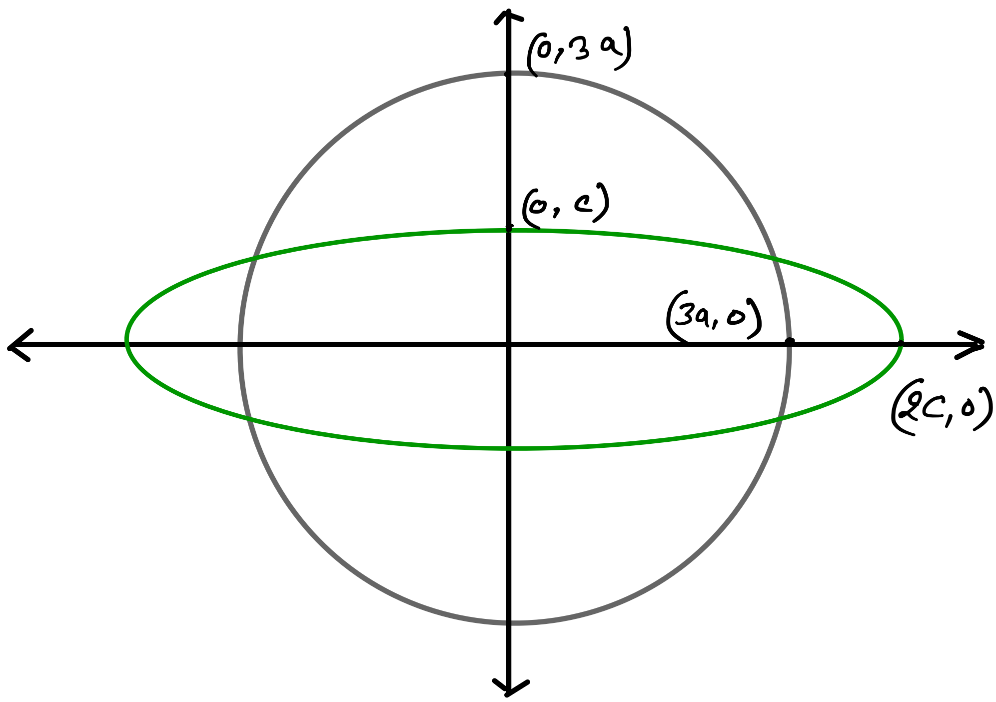
Ans → option (1)

6. If a and c are positive real number and the ellipse $\frac{x^2}{4c^2}+\frac{y^2}{c^2}=1$ has four distinct points in common with the circle $x^2 + y^2 = 9a^2$, then **[JEE-Main (On line)-2013]**

(1) $6ac + 9a^2 - 2c^2 > 0$ (2) $6ac + 9a^2 - 2c^2 < 0$

(3) $9ac - 9a^2 - 2c^2 < 0$ ✓(4) $9ac - 9a^2 - 2c^2 > 0$

Sol:-



$\therefore \boxed{c < 3a < 2c}$

$3a - c > 0$

$3a - 2c < 0$

$\therefore (3a-c)(3a-2c) < 0$

$\Rightarrow 9a^2 - 9ac + 2c^2 < 0$

$\Rightarrow \boxed{9ac - 9a^2 - 2c^2 > 0}$

Option (4) Ans

NOW, we have $c < 3a < 2c$ —— (1)

on squaring $c^2 < 9a^2 < 4c^2$ —— (2)

Now multiplying (1) by $2c$

$$2c^2 < 6ac < 4c^2 \quad \text{—— (3)}$$

NOW, By (2) + (3)

$$3c^2 < 9a^2 + 6ac < 8c^2$$

$$\Rightarrow c^2 < \underbrace{9a^2 + 6ac - 2c^2}_{\text{lies between two positive number}} < 6c^2$$

$$\Rightarrow 9a^2 + 6ac - 2c^2 > 0$$

$\Rightarrow$ option (1) is also correct

SO, option (1) and (4) both are correct.

7. Equation of the line passing through the points of intersection of the parabola $x^2 = 8y$ and the ellipse $\frac{x^2}{3} + y^2 = 1$ is - **[JEE-Main (On line)-2013]**

(1) $y + 3 = 0$ (2) $3y + 1 = 0$ (3) $3y - 1 = 0$ (4) $y - 3 = 0$

soln $x^2 = 8y$ —— (1) $(y \geq 0)$

$\frac{x^2}{3} + y^2 = 1$ —— (2)

$\frac{8y}{3} + y^2 = 1 \Rightarrow 3y^2 + 8y - 3 = 0$

$\Rightarrow 3y^2 + 9y - y - 3 = 0 \Rightarrow 3y(y+3) - (y+3) = 0$

$\Rightarrow (y+3)(3y-1) = 0 \Rightarrow \boxed{y = \frac{1}{3}, -3}$

but $y = -3$ not possible, so $y = 1/3$

$\Rightarrow x^2 = \frac{8}{3} \Rightarrow x = \pm \frac{2\sqrt{2}}{\sqrt{3}}$

$\Rightarrow$ Points of intersection $\equiv \left(\frac{2\sqrt{2}}{\sqrt{3}}, \frac{1}{3}\right)$

and $\left(\frac{-2\sqrt{2}}{\sqrt{3}}, \frac{1}{3}\right)$

eqn of line through these two points is $y = \frac{1}{3} \Rightarrow \boxed{3y - 1 = 0}$

option (3) Ans.

8. Let the equations of two ellipses be $E_1 : \frac{x^2}{3}+\frac{y^2}{2}=1$ and $E_2 : \frac{x^2}{16}+\frac{y^2}{b^2}=1$. If the product of their eccentricities is $\frac{1}{2}$, then the length of the minor axis of ellipse E_2 is :- **[JEE-Main (On line)-2013]**

(1) 9 (2) 8 (3) 2 (4) 4

Sol. :

$\because$ $\boxed{e_1 \cdot e_2 = \frac{1}{2}}$ Given

$\because$ $e_1^2 \cdot e_2^2 = \left(1-\frac{2}{3}\right)\left(1-\frac{b^2}{16}\right)$

$\frac{1}{4} = \frac{1}{3} \cdot \left(1-\frac{b^2}{16}\right)$

$\frac{3}{4} = 1-\frac{b^2}{16}$

$\frac{b^2}{16} = 1-\frac{3}{4}$

$\Rightarrow b^2 = 4$

$\Rightarrow b = 2$

length of minor axis of ellipse $E_2 = 2b = 4$ Ans.

option (4) Ans.

Case II: if $b > 4$

$$e_1^2 e_2^2 = \left(1 - 2/3\right)\left(1 - \frac{16}{b^2}\right) = \frac{1}{4}$$

$$\Rightarrow 1 - \frac{16}{b^2} = 3/4 \Rightarrow \frac{16}{b^2} = 1/4$$

$$\Rightarrow b^2 = 64 \Rightarrow \boxed{b = 8}$$

length of minor axis $= 2a$

$$= 2(4) = 8$$

option (2) is also correct:

Although, in this case we do not need to find 'b' as we can directly say that length of minor axis is equal to $2a$ i.e. 8.

So option (2) & option (4) both are correct. Ans.

9. If the curves $\frac{x^2}{\alpha}+\frac{y^2}{4}=1$ and $y^3 = 16x$ intersect at right angles, then a value of α is : **[JEE-Main (On line)-2013]**

(1) $\frac{4}{3}$ (2) $\frac{3}{4}$ (3) $\frac{1}{2}$ (4) 2

Sol:-

Let the point of intersection be (x_1, y_1). Now:-

$\because \frac{x^2}{\alpha}+\frac{y^2}{4}=1$ & $y^3 = 16x$

$\Rightarrow \frac{2x_1}{\alpha}+\frac{2y_1 \cdot y'}{4}=0$ | $\Rightarrow 3y_1^2 \cdot y' = 16$

| $\Rightarrow y' = \frac{16}{3y_1^2}$

$\Rightarrow m_1 = -\frac{x_1 \cdot 4}{\alpha \cdot y_1}$ | $\Rightarrow m_2 = \frac{16}{3y_1^2}$

$\because m_1 \cdot m_2 = -1$

$\frac{-4x_1}{\alpha \cdot y_1} \cdot \frac{16}{3y_1^2} = -1$ {since (x_1, y_1) lies on $y^3 = 16x$}

$\Rightarrow \frac{64 \cdot x_1}{3\alpha \cdot y_1^3} = -1$ $y_1^3 \rightarrow 16x_1$

$\Rightarrow \boxed{\alpha = \frac{4}{3}}$

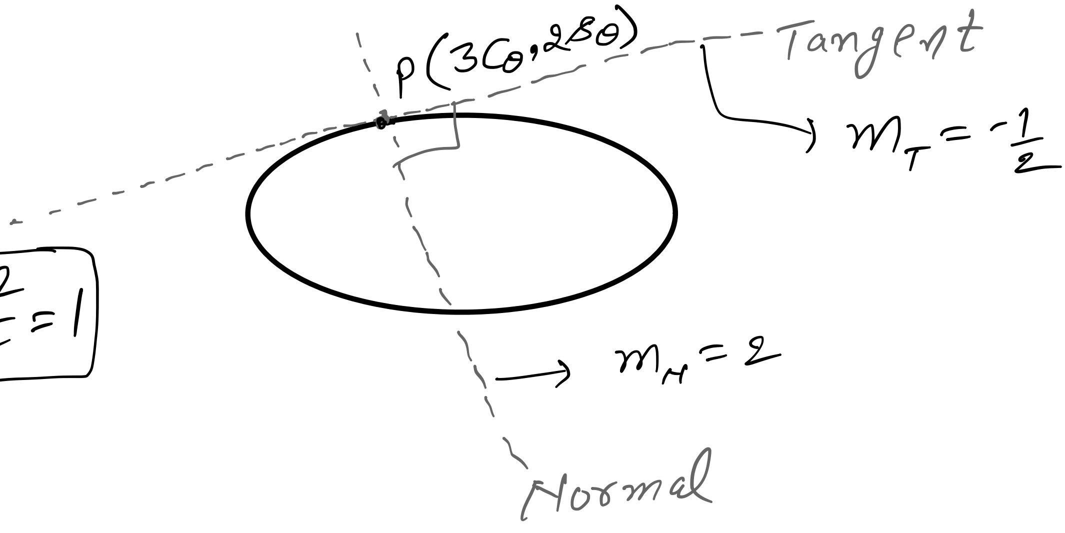
Ans: option (1)

10. A point on the ellipse, $4x^2 + 9y^2 = 36$, where the normal is parallel to the line, $4x - 2y - 5 = 0$, is :

[JEE-Main (On line)-2013]

(1) $\left(\frac{8}{5}, -\frac{9}{5}\right)$ (2) $\left(-\frac{9}{5}, \frac{8}{5}\right)$ (3) $\left(\frac{8}{5}, \frac{9}{5}\right)$ ✓(4) $\left(\frac{9}{5}, \frac{8}{5}\right)$

Sol:-



$$E: \boxed{\frac{x^2}{9} + \frac{y^2}{4} = 1}$$

Tangent : $\dfrac{x\cos\theta}{3} + \dfrac{y\sin\theta}{2} = 1$

$\because m_T = -\frac{1}{2}$

$\Rightarrow -\dfrac{2\cos\theta}{3\sin\theta} = -1/2$

$$\tan\theta = 4/3$$

$\Rightarrow \sin\theta = 4/5$ and $\cos\theta = 3/5$

or

$\sin\theta = -4/5$ and $\cos\theta = -3/5$

$\Rightarrow P\left(9/5, 8/5\right)$ or $\left(-9/5, \frac{-8}{5}\right)$

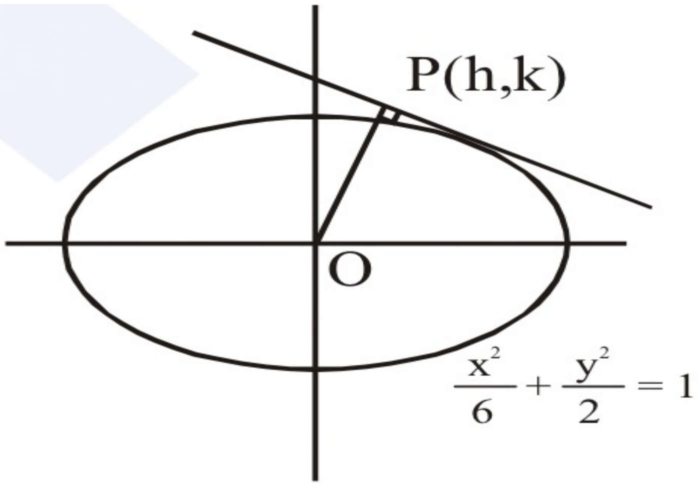
Ans $\rightarrow$ option (4)

11. The locus of the foot of perpendicular drawn from the centre of the ellipse $x^2 + 3y^2 = 6$ on any tangent to it is : **[JEE(Main)-2014]**

(1) $(x^2 - y^2)^2 = 6x^2 + 2y^2$ (2) $(x^2 - y^2)^2 = 6x^2 - 2y^2$

(3) $(x^2 + y^2)^2 = 6x^2 + 2y^2$ (4) $(x^2 + y^2)^2 = 6x^2 - 2y^2$

Sol.



Let the foot of perpendicular be (h, k)

then $m_{op} = \frac{k}{h}$

equation of tangent is

$y = mx \pm \sqrt{a^2m^2 + b^2}$

$y = mx \pm \sqrt{6m^2 + 2}$

satisfied by (h, k) and $m = -\frac{1}{m_{op}} = -\frac{h}{k}$

$$\left(k + \frac{h^2}{k}\right)^2 = \frac{6h^2}{k^2} + 2$$

multiply by k^2

$(k^2 + h^2)^2 = 6h^2 + 2k^2$

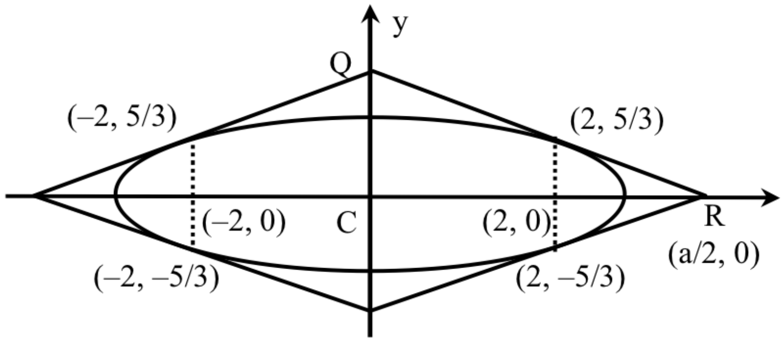
$\Rightarrow (x^2 + y^2)^2 = 6x^2 + 2y^2$

Ans → option (3)

12. The area (in sq. units) of the quadrilateral formed by the tangents at the end points of the latera recta to the ellipse $\frac{x^2}{9}+\frac{y^2}{5}=1$ is : **[JEE(Main)-2015]**

(1) $\frac{27}{2}$ (2) 27 (3) $\frac{27}{4}$ (4) 18

Sol:-



$a = 3,\ b = \sqrt{5}$

$e = \sqrt{1-\frac{5}{9}} = \frac{2}{3}$

foci = $(\pm 2, 0)$

tangent at P $\Rightarrow \frac{2x}{9}+\frac{5y}{3.5}=1$

$\frac{2x}{9}+\frac{y}{3}=1$

$2x + 3y = 9$

Area of quadrilateral

= 4 × (area of triangle QCR)

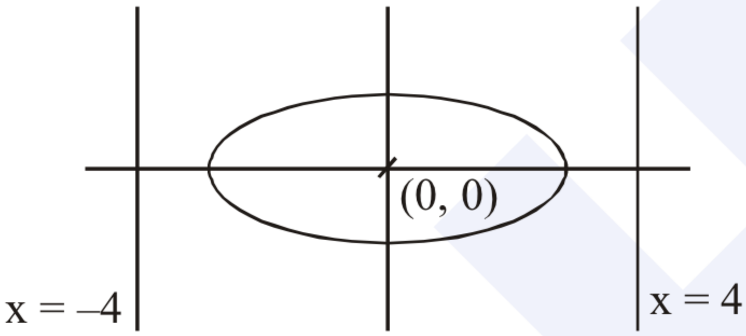
$=\left(\frac{1}{2}\times\frac{9}{2}\times 3\right)\times 4 = 27$

Ans → option (2)

13. The eccentricity of an ellipse whose centre is at the origin is $\frac{1}{2}$. If one of its directices is $x = -4$, then the equation of the normal to it at $\left(1, \frac{3}{2}\right)$ is :- **[JEE-Main 2017]**

(1) $x + 2y = 4$ (2) $2y - x = 2$ (3) $4x - 2y = 1$ (4) $4x + 2y = 7$

Sol. Eccentricity of ellipse $= \frac{1}{2}$



Now, $-\frac{a}{e} = -4 \Rightarrow a = 4 \times \frac{1}{2} = 2$

$\therefore\ b^2 = a^2(1 - e^2) = a^2\left(1 - \frac{1}{4}\right) = 3$

$\therefore$ Equation of ellipse

$$\frac{x^2}{4} + \frac{y^2}{3} = 1$$

$$\Rightarrow \frac{x}{2} + \frac{2y}{3} \times y' = 0 \Rightarrow \boxed{y' = -\frac{3x}{4y}}$$

$$y'|_{(1,3/2)} = -\frac{3}{4} \times \frac{2}{3} = -\frac{1}{2}$$ $\Rightarrow$ slope of normal $= 2$

$\therefore$ Equation of normal at $\left(1, \frac{3}{2}\right)$

$$y - \frac{3}{2} = 2(x - 1) \Rightarrow 2y - 3 = 4x - 4$$

$\therefore\ 4x - 2y = 1$

Ans $\rightarrow$ option (3)

JA

1. The line passing through the extremity A of the major axis and extremity B of the minor axis of the ellipse $x^2 + 9y^2 = 9$ meets its auxiliary circle at the point M. Then the area of the triangle with vertices at A, M and the origin O is **[JEE 2009, 3]**

(A) $\frac{31}{10}$ (B) $\frac{29}{10}$ (C) $\frac{21}{10}$ (D) $\frac{27}{10}$

Sol: -

Equation of line AM is $x + 3y - 3 = 0$

Perpendicular distance of line from origin $= \frac{3}{\sqrt{10}}$

Length of AM $= 2\sqrt{9 - \frac{9}{10}} = 2 \times \frac{9}{\sqrt{10}}$

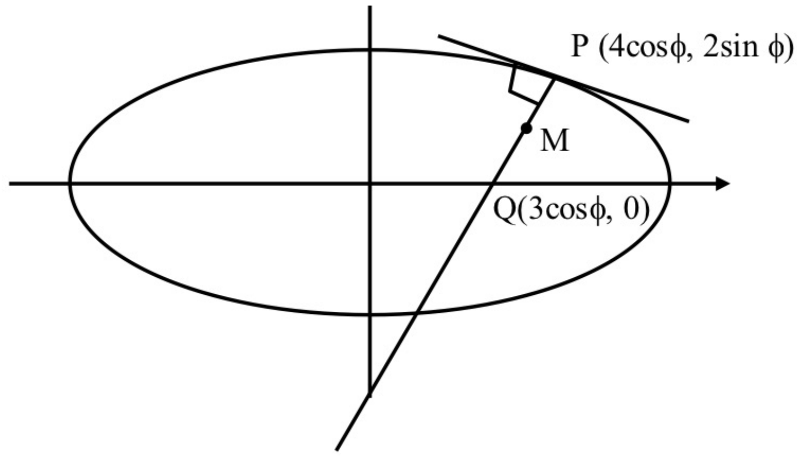
$\Rightarrow$ Area $= \frac{1}{2} \times 2 \times \frac{9}{\sqrt{10}} \times \frac{3}{\sqrt{10}} = \frac{27}{10}$ sq. units.

2. The normal at a point P on the ellipse $x^2 + 4y^2 = 16$ meets the x-axis at Q. If M is the mid point of the line segment PQ, then the locus of M intersects the latus rectums of the given ellipse at the point **[JEE 2009, 3]**

(A) $\left(\pm\frac{3\sqrt{5}}{2}, \pm\frac{2}{7}\right)$ (B) $\left(\pm\frac{3\sqrt{5}}{2}, \pm\frac{\sqrt{19}}{4}\right)$ (C) $\left(\pm 2\sqrt{3}, \pm\frac{1}{7}\right)$ (D) $\left(\pm 2\sqrt{3}, \pm\frac{4\sqrt{3}}{7}\right)$

Sol:

$E: \frac{x^2}{16} + \frac{y^2}{4} = 1$



Normal is $4x \sec\phi - 2y \operatorname{cosec}\phi = 12$

$Q \equiv (3\cos\phi, 0)$

$M \equiv (\alpha, \beta)$

$\alpha = \frac{3\cos\phi + 4\cos\phi}{2} = \frac{7}{2}\cos\phi$

$\Rightarrow \cos\phi = \frac{2}{7}\alpha$

$\beta = \sin\phi$

$\cos^2\phi + \sin^2\phi = 1$

$\Rightarrow \frac{4}{49}\alpha^2 + \beta^2 = 1 \Rightarrow \frac{4}{49}x^2 + y^2 = 1$

$\Rightarrow$ latus rectum $x = \pm 2\sqrt{3}$

$\frac{48}{49} + y^2 = 1 \Rightarrow y = \pm\frac{1}{7}$

$(\pm 2\sqrt{3}, \pm 1/7)$.

PARAGRAPH : 3 to 5

Tangents are drawn from the point P(3,4) to the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ touching the ellipse at points A and B. **[JEE 2010, 3+3+3]**

3. The coordinates of A and B are

(A) (3,0) and (0,2)

(B) $\left(-\frac{8}{5},\frac{2\sqrt{261}}{15}\right)$ and $\left(-\frac{9}{5},\frac{8}{5}\right)$

(C) $\left(-\frac{8}{5},\frac{2\sqrt{161}}{15}\right)$ and (0,2)

(D) (3,0) and $\left(-\frac{9}{5},\frac{8}{5}\right)$

Sol:-

$y = mx+\sqrt{9m^2+4}$ {passing through P(3,4)}

$4-3m=\sqrt{9m^2+4}$

$16+9m^2-24m = 9m^2+4 \Rightarrow m = \frac{12}{24}=\frac{1}{2}$

* Equation is $y-4=\frac{1}{2}(x-3)$

$2y-8 = x-3 \Rightarrow x-2y+5=0$ ①

Let $B=(\alpha,\beta) \Rightarrow \frac{x\alpha}{9}+\frac{y\beta}{4}-1=0$ ② $\Rightarrow \frac{\alpha/9}{1}=\frac{\beta/4}{-2}=\frac{-1}{5} \Rightarrow \alpha=-\frac{9}{5}, \beta=\frac{8}{5}$

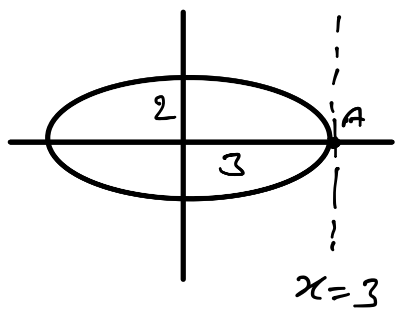
① ② } Compare ① & ②

⇓

$B\equiv\left(-\frac{9}{5},\frac{8}{5}\right)$.

* Eqn of second Tagent

$y-4=\frac{1}{0}(x-3) \Rightarrow \boxed{x=3}$



$\therefore A(3,0)$

4. The orthocenter of the triangle PAB is

(A) $\left(5, \frac{8}{7}\right)$

(B) $\left(\frac{7}{5}, \frac{25}{8}\right)$

(C) $\left(\frac{11}{5}, \frac{8}{5}\right)$

(D) $\left(\frac{8}{25}, \frac{7}{5}\right)$

Sol:-

Slope of BD must be 0

$$\Rightarrow y - \frac{8}{5} = 0\left(x + \frac{9}{5}\right) \Rightarrow y = \frac{8}{5}$$

Hence y coordinate of D is 8/5.

5. The equation of the locus of the point whose distances from the point P and the line AB are equal, is -

(A) $9x^2 + y^2 - 6xy - 54x - 62y + 241 = 0$ (B) $x^2 + 9y^2 + 6xy - 54x + 62y - 241 = 0$

(C) $9x^2 + 9y^2 - 6xy - 54x - 62y - 241 = 0$ (D) $x^2 + y^2 - 2xy + 27x + 31y - 120 = 0$

Sol:-

Locus is parabola

Equation of AB Is $\frac{3x}{9} + \frac{4y}{4} = 1 \Rightarrow \frac{x}{3} + y = 1 \Rightarrow x + 3y - 3 = 0$

$$(x - 3)^2 + (y - 4)^2 = \frac{(x+3y-3)^2}{10}$$

$$10x^2 + 90 - 60x + 10y^2 + 160 - 80y = x^2 + 9y^2 + 9 + 6xy - 6x - 18y$$

$$\Rightarrow 9x^2 + y^2 - 6xy - 54x - 62y + 241 = 0.$$

6. The ellipse $E_1 : \frac{x^2}{9}+\frac{x^2}{4}=1$ is inscribed in a rectangle R whose sides are parallel to the coordinate axes. Another ellipse E_2 passing through the point (0,4) circumscribes the rectangle R. The eccentricity of the ellipse E_2 is - **[JEE 2012, 3M, –1M]**

(A) $\frac{\sqrt{2}}{2}$ (B) $\frac{\sqrt{3}}{2}$ (C) $\frac{1}{2}$ (D) $\frac{3}{4}$

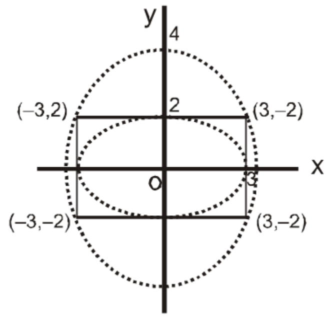
Sol:-

Let required ellipse is

$E_2 : \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

It passes thorugh (0, 4)

$0 + \frac{16}{b^2} = 1 \quad \Rightarrow \quad b^2 = 16$



It also passes through (±3, ±2)

$\frac{9}{a^2}+\frac{4}{b^2} = 1$

$\frac{9}{a^2}+\frac{1}{4} = 1$

$\frac{9}{a^2}=\frac{3}{4} \quad \Rightarrow \quad a^2 = b^2 (1 – e^2)$

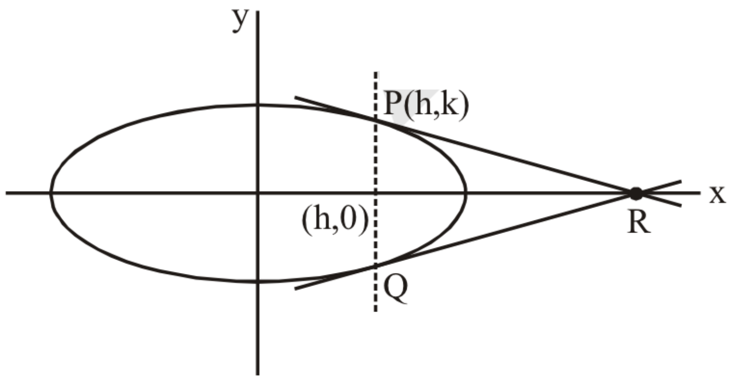
$\frac{12}{16} = 1– e^2$

$e^2 = 1 – \frac{12}{16} = \frac{4}{16} = \frac{1}{16}$

$e = \frac{1}{2}$

7. A vertical line passing through the point (h,0) intersects the ellipse $\frac{x^2}{4}+\frac{y^2}{3}=1$ at the points P and Q. Let the tangents to the ellipse at P and Q meet at the point R. If $\Delta(h)$ = area of the triangle PQR, $\Delta_1 = \max\limits_{1/2\le h\le 1} \Delta(h)$ and $\Delta_2 = \min\limits_{1/2\le h\le 1} \Delta(h)$, then $\frac{8}{\sqrt{5}}\Delta_1 - 8\Delta_2 =$ **[JEE-Advanced 2013, 4, (–1)]**

Sol.



Tangent at P(h, k) is $\frac{xh}{4}+\frac{ky}{3}=1$

$\Rightarrow \quad R\left(\frac{4}{h}, 0\right)$

$\Delta PQR = k\left(\frac{4}{h}-h\right)$

$= \sqrt{3\left(1-\frac{h^2}{4}\right)}\left(\frac{4}{h}-h\right)$

which is a decreasing function in $\left[\frac{1}{2}, 1\right]$

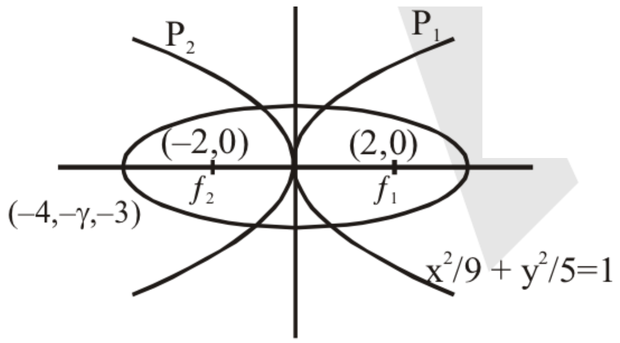
$\Rightarrow \quad \Delta_1 = \sqrt{3\left(1-\frac{1}{16}\right)}\left(8-\frac{1}{2}\right) = \frac{45\sqrt{5}}{8}$

& $\quad \Delta_2 = \sqrt{3\left(1-\frac{1}{4}\right)}(4-1) = \frac{9}{2}$

$\frac{8}{\sqrt{5}}\Delta_1 - 8\Delta_2 = 45-36 = 9$

8. Suppose that the foci of the ellipse $\frac{x^2}{9}+\frac{y^2}{5}=1$ are $(f_1,0)$ and $(f_2,0)$ where $f_1>0$ and $f_2<0$. Let P_1 and P_2 be two parabolas with a common vertex at (0,0) and with foci at $(f_1,0)$ and $(2f_2,0)$, respectively. Let T_1 be a tangent to P_1 which passes through $(2f_2,0)$ and T_2 be a tangent to P_2 which passes through $(f_1,0)$. If m_1 is the slope of T_1 and m_2 is the slope of T_2, then the value of $\left(\frac{1}{m_1^2}+m_2^2\right)$ is

Sol.



$\therefore$ P_1 is $y^2 = 8x$
P_2 is $y^2 = -16x$

$$\therefore\ y = m_1x + \frac{2}{m_1}$$

If passes through (–4,0)

$$\therefore\ -4m_1 + \frac{2}{m_1} = 0$$

$$\therefore\ m_1^2 = \frac{1}{2}$$

equation of tangent to P_2

$$y = m_2x + \frac{(-4)}{m_2}$$

It passes through (2,0) , $2m_2 - \frac{4}{m_2} = 0 \Rightarrow m_2^2 = 2$

$$\therefore\ \frac{1}{m_1^2} + m_2^2 = 4$$

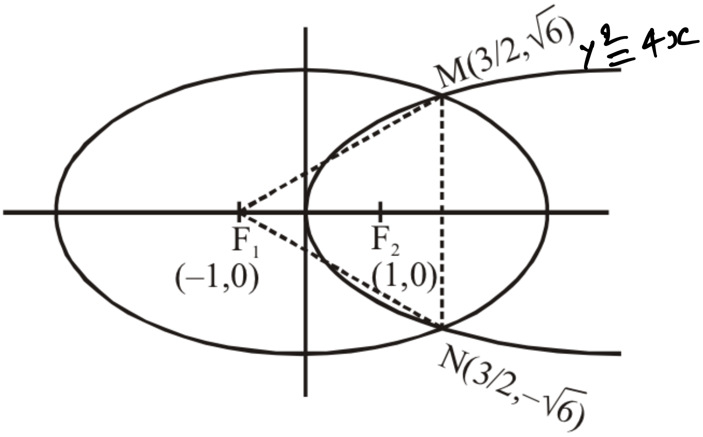
PARAGRAPH : 9 to 10

9. Let $F_1(x_1, 0)$ and $F_2(x_2, 0)$ for $x_1 < 0$ and $x_2 > 0$, be the foci of the ellipse $\frac{x^2}{9}+\frac{y^2}{8}=1$. Suppose a parabola having vertex at the origin and focus at F_2 intersects the ellipse at point M in the first quadrant and at point N in the fourth quadrant.

The orthocentre of the triangle F_1MN is- **[JEE(Advanced)-2016, 4(–2)]**

(A) $\left(-\frac{9}{10},0\right)$ (B) $\left(\frac{2}{3},0\right)$ (C) $\left(\frac{9}{10},0\right)$ (D) $\left(\frac{2}{3},\sqrt{6}\right)$

Sol.



$\frac{x^2}{9}+\frac{4x}{8}=1$

$\Rightarrow 2x^2+9x=18$

$\Rightarrow 2x^2+9x-18=0$

$\Rightarrow (x+6)(2x-3)=0$

$\Rightarrow \boxed{x=3/2}$

$\therefore y^2 = 4\cdot 3/2 \Rightarrow \boxed{y=\pm\sqrt{6}}$

Orthocentre lies on x-axis

Equation of altitude through M : $y-\sqrt{6}=\frac{5}{2\sqrt{6}}\left(x-\frac{3}{2}\right)$

Equation of altitude through F_1 : $y = 0$

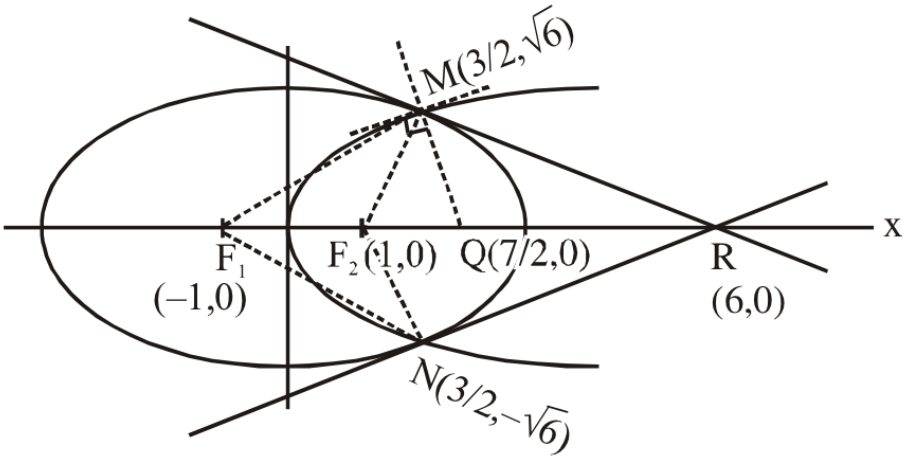
solving, we get orthocentre $\left(-\frac{9}{10},0\right)$

10. If the tangents to the ellipse at M and N meet at R and the normal to the parabola at M meets the x-axis at Q, then the ratio of area of the triangle MQR to area of the quadrilateral MF_1NF_2 is -

[JEE(Advanced)-2016, 3(0)]

(A) 3 : 4 (B) 4 : 5 (C) 5 : 8 (D) 2 : 3

Sol.



Normal to parabola at M : $y-\sqrt{6}=-\frac{\sqrt{6}}{2.1}\left(x-\frac{3}{2}\right)$

Solving it with y = 0, we get $Q \equiv \left(\frac{7}{2}, 0\right)$

Tangent to ellipse at M : $\frac{x.\frac{3}{2}}{9}+\frac{y(\sqrt{6})}{8}=1$

Solving it with y = 0, we get $R \equiv (6, 0)$

$\therefore$ Area of triangle MQR $=\frac{1}{2}.\left(6-\frac{7}{2}\right).\sqrt{6}=\frac{5\sqrt{6}}{4}$

Area of quadrilateral $MF_1NF_2 = 2.\frac{1}{2}.(1-(-1)).\sqrt{6} = 2\sqrt{6}$

Required ratio = 5 : 8

11. Consider two straight lines, each of which is tangent to both the circle $x^2 + y^2 = \frac{1}{2}$ and the parabola $y^2 = 4x$. Let these lines intersect at the point Q. Consider the ellipse whose center is at the origin O(0, 0) and whose semi-major axis is OQ. If the length of the minor axis of this ellipse is $\sqrt{2}$, then the which of the following statement(s) is (are) TRUE ? **[JEE(Advanced)-2018, 4(–2)]**

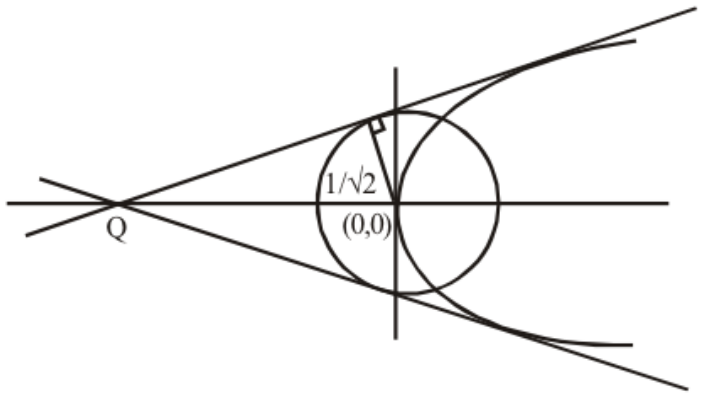
(A) For the ellipse, the eccentricity is $\frac{1}{\sqrt{2}}$ and the length of the latus rectum is 1

(B) For the ellipse, the eccentricity is $\frac{1}{2}$ and the length of the latus rectum is $\frac{1}{2}$

(C) The area of the region bounded by the ellipse between the lines $x = \frac{1}{\sqrt{2}}$ and $x = 1$ is $\frac{1}{4\sqrt{2}}(\pi - 2)$

(D) The area of the region bounded by the ellipse between the lines $x = \frac{1}{\sqrt{2}}$ and $x = 1$ is $\frac{1}{16}(\pi - 2)$

Sol.



Let equation of common tangent is $y = mx + \frac{1}{m}$

$$\therefore \quad \left|\frac{0+0+\frac{1}{m}}{\sqrt{1+m^2}}\right| = \frac{1}{\sqrt{2}} \Rightarrow m^4 + m^2 - 2 = 0 \Rightarrow m = \pm 1$$

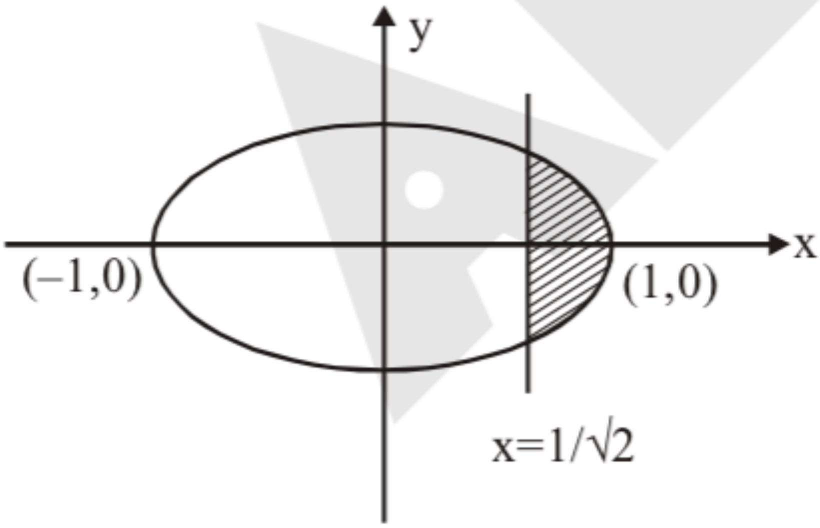
Equation of common tangents are $y = x + 1$ and $y = -x - 1$

point Q is $(-1, 0)$

$\therefore$ Equation of ellipse is $\frac{x^2}{1} + \frac{y^2}{1/2} = 1$

(A) $e = \sqrt{1 - \frac{1}{2}} = \frac{1}{\sqrt{2}}$ and $LR = \frac{2b^2}{a} = 1$

(C)



$$\text{Area} \quad 2.\int_{1/\sqrt{2}}^{1} \frac{1}{\sqrt{2}}.\sqrt{1-x^2}\,dx = \sqrt{2}\left[\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1} x\right]_{1/\sqrt{2}}^{1}$$

$$= \sqrt{2}\left[\frac{\pi}{4} - \left(\frac{1}{4} + \frac{\pi}{8}\right)\right] = \sqrt{2}\left(\frac{\pi}{8} - \frac{1}{4}\right) = \frac{\pi - 2}{4\sqrt{2}}$$

correct answer are (A) and (C)