VECTOR

GENERAL DEFINITIONS :

I. (a) Vector quantities :

Any quantity, such as velocity, momentum, or force, that has both magnitude and direction and for which vector addition is defined and meaningful; is treated as vector quantities.

Note :

Quantities having magnitude and direction but not obeying the vector law of addition will not be treated as vectors.

(b) Scalar quantities :

A quantity, such as mass, length, time, density or energy, that has size or magnitude but does not involve the concept of direction is called scalar quantity.

Directed line segment :

Any given portion of a given straight line where the two end points are distinguished as **Initial** and **Terminal** is called a **Directed Line Segment**.

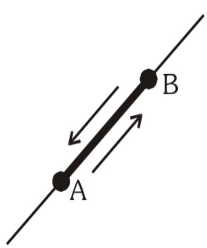
The directed line segment with initial point A and terminal point B is denoted by the symbol $\overrightarrow{AB}$.

The two end points of a directed line segment are not interchangeable and the directed line segments.

$\overrightarrow{AB}$ and $\overrightarrow{BA}$ must be thought of as different.

(a) Vector :

A directed line segment is called vector. Every directed line segment have three essential characteristics.



(i) Length : The length of $\overrightarrow{AB}$ will be denoted by the symbol $|\overrightarrow{AB}|$

Clearly, we have $|\overrightarrow{AB}| = |\overrightarrow{BA}|$

III. Some Special Vectors :

(a) Zero Vector (Null vector) :

A vector of zero magnitude i.e. which has the same initial & terminal point, is called a **Zero Vector**. It is denoted by $\vec{O}$ and its direction is arbitrary.

Note that zero vector has many properties similar to the number zero.

E.g. A boy throwing a ball up and catching it back in his hand, the displacement of the ball is a null vector.

(b) Unit Vector :

A vector of unit magnitude in the direction of vector $\vec{a}$ is called unit vector along $\vec{a}$ and is denoted by $\hat{a}$ symbolically $\boxed{\hat{a} = \frac{\vec{a}}{|\vec{a}|}}$

The concept of unit vector is merely to impart a direction to a physical quantity.

(c) Equal Vectors :

Two vectors are said to be equal if they have the same magnitude, same direction & represent the same physical quantity with same or parallel line of support.

IV. Vectors can be classified as :

(d) Free vectors : Vectors which when transformed into space from one point to another point without affecting their magnitude and direction, can be considered as free vectors i.e. the physical effect produced by them remains unaltered. e.g. displacement, velocity. (Generally **IIT-JEE** deals with free vectors only)

(e) Localised vector : Vectors which when transformed into space from one point to another point without affecting their magnitude and direction, can be considered as localised vectors i.e. the physical effect produced by them are changed e.g. force in case of rotational motion (torque will be changed).

Note that :

(i) Number of distinct unit vectors in space perpendicular to a given plane is 2. (one upward and one downward).

(ii) Number of unit vectors in space parallel to a given plane is infinite.

(iii) Number of distinct unit vectors perpendicular to given line in space. (Infinitely many, think of the line as perpendicular to the xy plane. The unit vector might make any angle θ with the x-axis.)

(iv) Number of distinct unit vectors parallel to a line in space is 2.

(v) Two vectors are equal if they have equal components in an arbitrary direction

(f) Multiplication of vector by scalars :

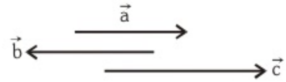
If $\vec{a}$ is a vector & m is a scalar, then $(m\vec{a})$ is a vector parallel to $\vec{a}$ whose modulus is $|m|$ times that of $\vec{a}$. This multiplication is called **Scalar Multiplication**. If $\vec{a}$ & $\vec{b}$ are vectors & m, n are scalars, then :

$$m(\vec{a}) = (\vec{a})m = m\vec{a} \qquad m(n\vec{a}) = n(m\vec{a}) = (mn)\vec{a}$$

$$(m+n)\vec{a} = m\vec{a} + n\vec{a} \qquad m(\vec{a}+\vec{b}) = m\vec{a} + m\vec{b}$$

(g) Collinear Vectors :

Two vectors are said to be collinear if their directed line segments are parallel disregard of their direction. Collinear vectors are also called **Parallel Vectors**. If they have the same direction they are named as like vectors otherwise unlike vectors. ($\vec{a}, \vec{b}, \vec{c}$ are collinear)



Note : Symbolically, two non zero vectors $\vec{a}$ and $\vec{b}$ are collinear if and only if, $\vec{a} = K\vec{b}$, where $K \in R$.

If $K > 0$, like parallel vectors,
$K < 0$, unlike parallel vectors.

Imp **Result :** If $\vec{a}$ and $\vec{b}$ are two non zero non collinear vectors then $x\vec{a} + y\vec{b} = 0 \Rightarrow x = 0$ and $y = 0$.

Proof : Let $y \neq 0$

$$y\vec{b} = -x\vec{a} \Rightarrow \vec{b} = \left(-\frac{x}{y}\right)\vec{a} \Rightarrow \vec{b} = \lambda\vec{a}$$

$\Rightarrow$ $\vec{a}$ and $\vec{b}$ are collinear, which is false.

$\therefore$ $y = 0$

Note : Two non-zero non collinear vectors are also called as **linearly independent vector or base vectors.** If $\vec{a}$ and $\vec{b}$ are collinear then they are known as **linearly dependent vectors**.

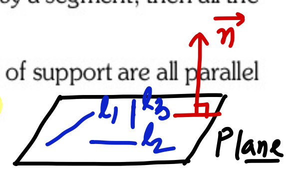
e.g. If $\left(\sin\theta - \frac{1}{2}\right)\vec{a} + \left(\cos\theta - \frac{\sqrt{3}}{2}\right)\vec{b} = 0$ such that $\vec{a}$ & $\vec{b}$ are non colinear vectors then the general solution of θ is

0 & 0

[Ans. $2n\pi + \frac{\pi}{6}$ where $n \in I$]

(h) Plane : A plane is a surface such that any two points on the surface are joining by a segment, then all the point lying on the segment must also lie on the surface.

(i) Coplanar Vectors : A given number of vectors are called coplanar if their line of support are all parallel to the same plane. Note that **"Two Free Vectors Are Always Coplanar"**.

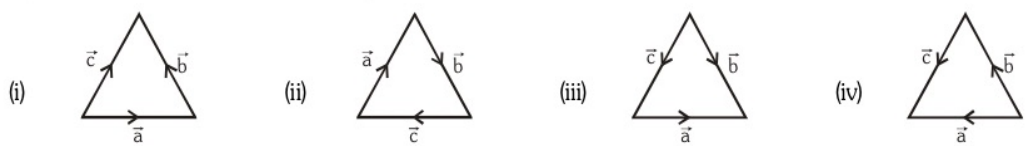


(j) Vector addition :

(i) Triangle law of vectors : If two vectors are represented in magnitude & direction by two sides of a triangle taken in same order then their sum is represented by the third side taken in reverse order.

Example :

E(1) Find the relation between $\vec{a}, \vec{b}$ & $\vec{c}$



[Ans. (i) $\vec{a} + \vec{b} = \vec{c}$ (ii) $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ (iii) $\vec{a} + \vec{c} = \vec{b}$ (iv) $\vec{b} + \vec{c} = \vec{a}$]

(ii) **Parallelogram law of vectors :** If two vectors $\vec{a}$ & $\vec{b}$ represented by $\overrightarrow{OA}$ & $\overrightarrow{OB}$, then their sum $\vec{a}+\vec{b}$ is a vector represented by $\overrightarrow{OC}$, where $\overrightarrow{OC}$ is the diagonal of parallelogram OACB.

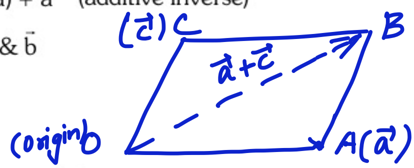
Properties :

$\vec{a}+\vec{b} = \vec{b}+\vec{a}$ (commutative)

$(\vec{a}+\vec{b})+\vec{c} = \vec{a}+(\vec{b}+\vec{c})$ (associativity)

$\vec{a}+\vec{0} = \vec{a} = \vec{0}+\vec{a}$ (additive identity)

$\vec{a}+(-\vec{a}) = \vec{0} = (-\vec{a})+\vec{a}$ (additive inverse)



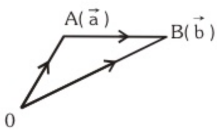
Note : Magnitude of $\vec{a}+\vec{b}$ is not equal to sum of magnitude of $\vec{a}$ & $\vec{b}$

i.e. $\left|\vec{a}+\vec{b}\right| \neq |\vec{a}|+\left|\vec{b}\right|$ (in general)

(k) **Position Vector :**

To specify the position of an object w.r.t. a fixed represence point in 3-D space, position vectors are used. Let O be a fixed origin, then the position vector of a point P is the vector $\vec{OP}$. Hence p.v. of general point P(x, y, z) is the vector extending from the origin to P given by $x\hat{i}+y\hat{j}+z\hat{k}$.

If $\vec{a}$ & $\vec{b}$ are the position vectors of two point A and B w.r.t. origin O then,



$\vec{a}+\overrightarrow{AB} = \vec{b}$

$$\boxed{\overrightarrow{AB} = \vec{b}-\vec{a} = \text{pv of B} - \text{pv of A.}}$$

Note :

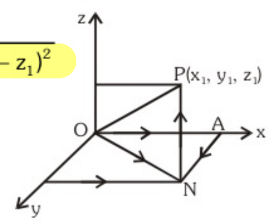
(i) Vector along a line ax + by + c = 0 is given by $\lambda\,(b\hat{i}-a\hat{j})$.

(ii) Equation of any line $\perp$ to ax + by + c = 0 will be of the form bx – ay = k and any vector parallel to this line will be of the form $\lambda\,(a\hat{i}+b\hat{j})$. So $\lambda\,(a\hat{i}+b\hat{j})$ will be a vector perpendicular to line ax + by + c = 0 in xy plane.

V. **Representation of a vector in space in terms of 3 orthonormal triad of unit vectors**

With every point $P(x_1, y_1, z_1)$ in space w.r.t. a fixed origin 'O' we can associate a directed line segment whose initial point is the origin and terminal point is P. Thus $\overrightarrow{OP} = \overrightarrow{OA}+\overrightarrow{AN}+\overrightarrow{NP} = x_1\hat{i}+y_1\hat{j}+z_1\hat{k}$.

Thus if $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ are two points in space



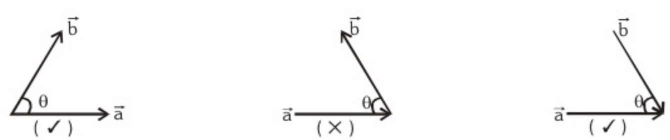
$\overrightarrow{AB} = (x_2-x_1)\hat{i}+(y_2-y_1)\hat{j}+(z_2-z_1)\hat{k}$ & $\left|\overrightarrow{AB}\right| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

Thus any vector $\vec{a}$ in space can be expressed as a linear combination of 3 orthonormal triad of unit vectors as $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$, where

$|\vec{a}|^2 = a_1^2+a_2^2+a_3^2$

ANGLE BETWEEN VECTORS :

Angle between two vectors is angle between head-head or tail-tail. There is only one set of angle between two vectors. angle between $\vec{a}$ & $\vec{b}$ is represented by $\vec{a}\wedge\vec{b} = \theta$.



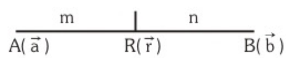
HW

E(1) If the sum of two unit vectors is a unit vector then find the magnitude of their difference and the angle between $\hat{a}$ and $\hat{b}$.

Solⁿ

SECTION FORMULA :

If $\vec{a}$ & $\vec{b}$ are the position vectors of two points A & B then the p.v. of a point R which divides AB internally in the ratio m : n is given by $\boxed{\vec{r} = \dfrac{n\vec{a} + m\vec{b}}{m+n}}$.



Proof : $\dfrac{AR}{RB} = \dfrac{m}{n} \quad \Rightarrow \quad n\overrightarrow{AR} = m\overrightarrow{RB}$

or $\quad n(\vec{r} - \vec{a}) = m(\vec{b} - \vec{r})$

$\therefore \quad \vec{r} = \dfrac{m\vec{b} + n\vec{a}}{m+n}$

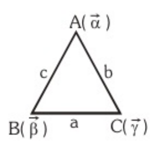
Similarly for external division $\boxed{\vec{r} = \dfrac{m\vec{b} - n\vec{a}}{m-n}}$

Note : p.v. of mid point of AB $= \dfrac{\vec{a} + \vec{b}}{2}$.

Using section formula, we can prove that

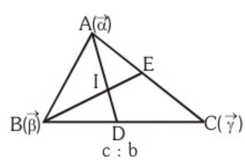
(a) p.v. of the centroid of a triangle ABC $= \dfrac{\vec{\alpha} + \vec{\beta} + \vec{\gamma}}{3}$

(Concurrency of median)



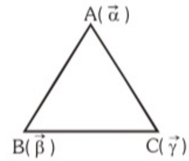
(b) Incentre of the triangle $= \dfrac{a\vec{\alpha} + b\vec{\beta} + c\vec{\gamma}}{a+b+c}$

$\dfrac{AI}{ID} = \dfrac{AB}{BD} = \dfrac{c}{\left(\dfrac{c}{b+c}\right)a} = \dfrac{b+c}{a}$



$\therefore \quad D\left(\dfrac{b\vec{\beta} + c\vec{\gamma}}{b+c}\right)$

Excentres of the Δ are $\frac{-a\vec{\alpha}+b\vec{\beta}+c\vec{\gamma}}{-a+b+c}$; $\frac{a\vec{\alpha}-b\vec{\beta}+c\vec{\gamma}}{a-b+c}$ and $\frac{a\vec{\alpha}+b\vec{\beta}-c\vec{\gamma}}{a+b-c}$



(c) Circumcentre of the $\Delta = \frac{\vec{\alpha}\sin 2A+\vec{\beta}\sin 2B+\vec{\gamma}\sin 2C}{\sum \sin 2A}$

(d) Orthocentre of the $\Delta = \frac{\vec{\alpha}\tan A+\vec{\beta}\tan B+\vec{\gamma}\tan C}{\sum \tan A}$ OR $(\Pi \tan A)$

(Use the fact that distances of orthocentre from the vertices are 2R cosA, 2R cosB, 2R cosC and from the sides are 2R cosB cosC, 2R cosC cosA, 2R cosA cosB)

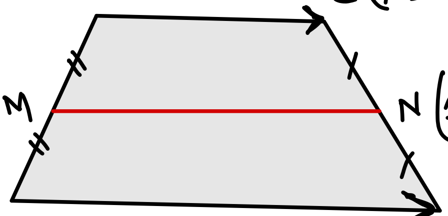
GEOMETRICAL RESULTS WITH VECTORS & PROBLEMS :

(a) Prove that straight line joining the mid points of two non parallel sides of a trapezium is || to the parallel sides and half of their sum.

TPT : (i) MN || AB (or CD)

(ii) $MN = \frac{1}{2}(AB+CD)$

Sol^n



$\vec{AB} = \vec{b}$

Since AB || CD

$\vec{DC} = \lambda \vec{AB}$

pv of C − pv of D $= \lambda\vec{b}$.

pv of C $= (\lambda\vec{b}+\vec{d})$

$\vec{MN}$ = pv of N − pv of M

$= \frac{(\lambda+1)\vec{b}+\vec{d}}{2} - \frac{\vec{d}}{2}$

$\vec{MN} = \left(\frac{\lambda+1}{2}\right)\vec{b} \Rightarrow$ $\vec{MN} = K\,\vec{b} \Rightarrow \vec{MN} = K\,\vec{AB}$

K ∈ scalar

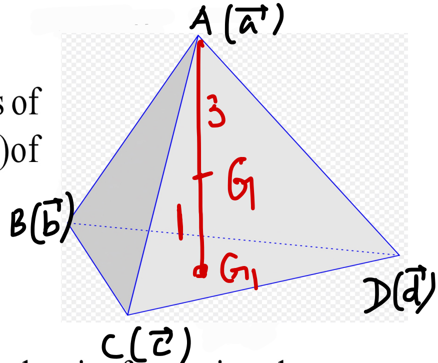
(H.P)

$\vec{MN} = \frac{1}{2}(\lambda\vec{b}+\vec{b})$

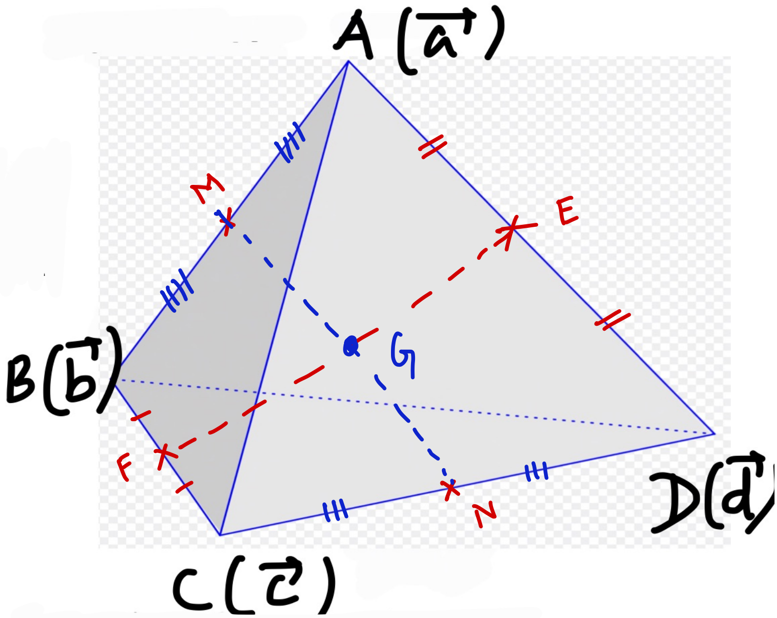
Tetrahedron (a pyramid on a triangular base)

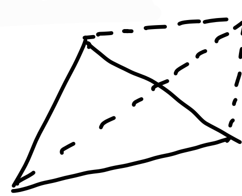
Lines joining the vertices of a tetrahedron to the centroids of the opposite faces are concurrent and this point (P)of concurrency with p.v. $\vec{g} = \frac{\vec{a}+\vec{b}+\vec{c}+\vec{d}}{4}$ is called the centre of the tetrahedron (say G).

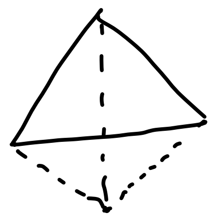
(or Centroid)



2 In a tetrahedron, straight lines joining the mid points of each pair of opposite edges are also concurrent at the centre of the tetrahedron.

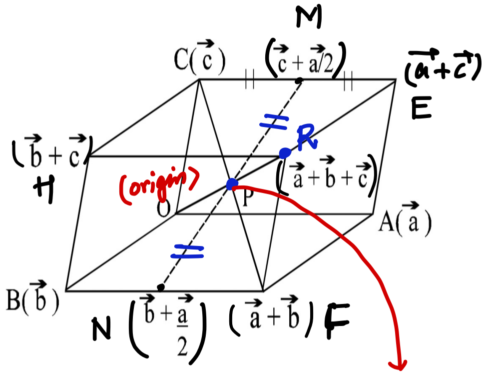






Paralleopiped :-

Four diagonals of any parallelopiped (A prism whose base is a ||gm) and the join of the mid point of each pair of opposite edges are concurrent and are bisected at the point of concurrence. (See the adjacent figure) P is called the centre of the parallopiped with p.v.



pv of $P \equiv \left(\frac{\vec{a}+\vec{b}+\vec{c}}{2}\right)$

$$\frac{\vec{a}+\vec{b}+\vec{c}}{2} \text{ i.e. } \frac{\vec{OA}+\vec{OB}+\vec{OC}}{2}$$

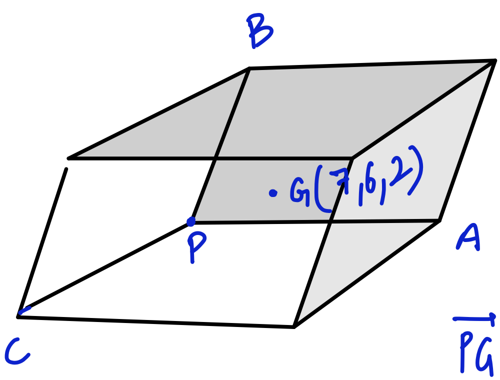
For some general point (other than origin)

$$\frac{\vec{EA}+\vec{EB}+\vec{EC}}{2}$$

Q Centre of the parallelopiped formed by

$\vec{PA}=\hat{i}+2\hat{j}+2\hat{k}$; $\vec{PB}=4\hat{i}-3\hat{j}+\hat{k}$ and $\vec{PC}=3\hat{i}+5\hat{j}-\hat{k}$

is given by the p.v. (7, 6, 2). Then find the p.v. of the point P.



$$\frac{\vec{PA}+\vec{PB}+\vec{PC}}{2} = \vec{PG}$$

$$\frac{8\hat{i}+4\hat{j}+2\hat{k}}{2} = \vec{PG}$$

$$4\hat{i}+2\hat{j}+\hat{k} = \vec{PG}$$

$\vec{PG}$ = pv of G − pv of P

(4, 2, 1) = (7, 6, 2) − pv of P

P (3, 4, 1) Ans.

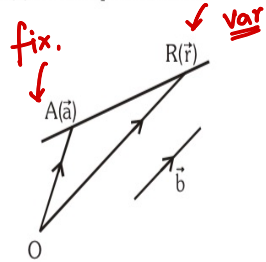
VECTOR EQUATION OF A LINE :

It is possible to express the position vectors of points on given lines and planes in terms of some fixed vectors and variable scalars called parameters, such that

(a) For arbitrary value of the parameter, the resulting position vector represent point on the locus in question, and

(b) Conversely, the position vectors of each of the locus correspond to a definite value(s) of the parameter.

Parametric vector equation of a line :



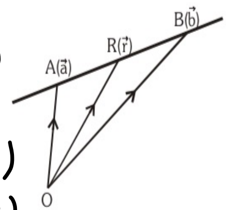
(i) If the line passes through the point $A(\vec{a})$ & is parallel to the vector $\vec{b}$ then its equation is, $\boxed{\vec{r} = \vec{a} + t\vec{b}}$ where t is a parameter.

Rem

Rem

(ii) A line passing through two point $A(\vec{a})$ & $B(\vec{b})$ is given by, $\boxed{\vec{r} = \vec{a} + s(\vec{b} - \vec{a})}$, where s is parameter.

✓ $\vec{r} = \vec{a} + s(\vec{a} - \vec{b})$



These two equation gives the poistion vector of any point on the line and prove to be very useful in vector algebra.

✓ $\vec{r} = \vec{b} + s(\vec{a} - \vec{b})$

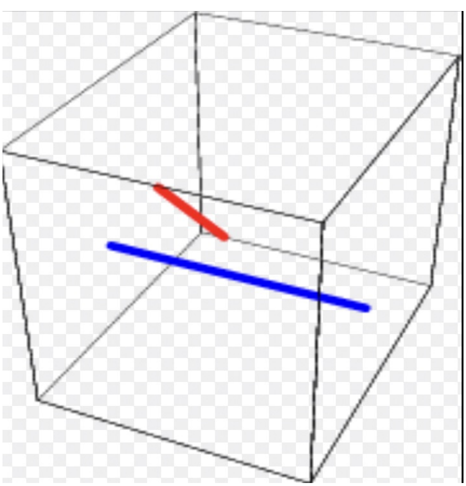
✓ $\vec{r} = \vec{b} + s(\vec{b} - \vec{a})$

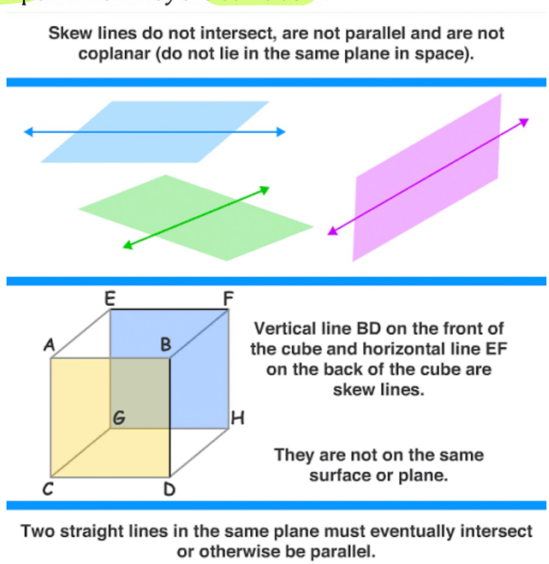
Important Note :

(i) Two lines in a plane are either intersecting or parallel, conversely two intersecting or parallel line must be in the same plane.

(ii) However in space we can have two neither parallel nor intersecting lines. Such non coplanar lines are known as skew lines.

(iii) If two lines are parallel and have a common point then they are coincident.





E(1) Find the p.v. of the point of intersection of the lines (if it exists)

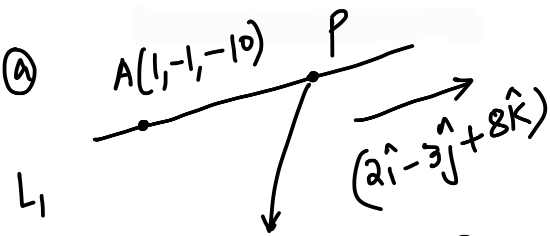
(a) $\vec{r} = \hat{i} - \hat{j} - 10\hat{k} + \lambda(2\hat{i} - 3\hat{j} + 8\hat{k})$
and $\vec{r} = 4\hat{i} - 3\hat{j} - \hat{k} + \mu(\hat{i} - 4\hat{j} + 7\hat{k})$

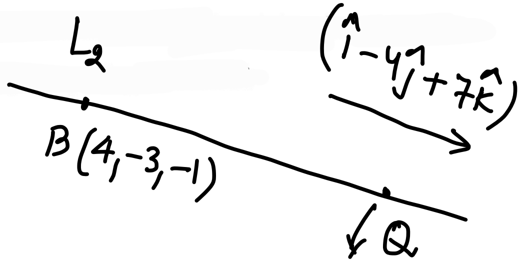
HW (b) $\vec{r} = -3\hat{i} + 6\hat{j} + \lambda(-4\hat{i} + 3\hat{j} + 2\hat{k})$
and $\vec{r} = -2\hat{i} + 7\hat{k} + \mu(-4\hat{i} + \hat{j} + \hat{k})$

(c) $\vec{r} = t(3\hat{i} - \hat{j} + \hat{k})$
and $\vec{r} = 2\hat{i} + s(-6\hat{i} + 2\hat{j} - 2\hat{k})$

HW (d) $\vec{r} = 2\hat{k} + \lambda(3\hat{i} + 2\hat{j} + \hat{k})$
and $\vec{r} = 3\hat{i} + 2\hat{j} + 3\hat{k} + \mu(6\hat{i} + 4\hat{j} + 2\hat{k})$

(a)





General point on L_1 : $\overset{P}{(1+2\lambda,\ -1-3\lambda,\ -10+8\lambda)}$

General point on L_2 : $\overset{Q}{(4+\mu,\ -3-4\mu,\ -1+7\mu)}$

$$\left.\begin{aligned} 1+2\lambda &= 4+\mu \\ -1-3\lambda &= -3-4\mu \end{aligned}\right\} \quad \begin{aligned}\lambda &= 2 \\ \mu &= 1\end{aligned}$$

(coplanar lines)

$-10+8\lambda = -1+7\mu$ (satisfied)

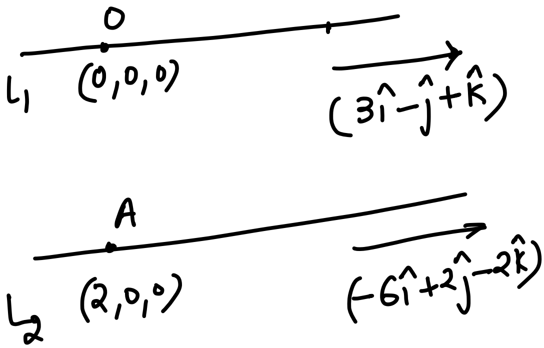
$\therefore$ lines are intersecting at $(5, -7, 6)$ Ans

(c) $\vec{r} = t(3\hat{i} - \hat{j} + \hat{k})$

and $\vec{r} = 2\hat{i} + s(-6\hat{i} + 2\hat{j} - 2\hat{k})$

✓

$\vec{r} = 2\hat{i} + \lambda(3\hat{i} - \hat{j} + \hat{k})$



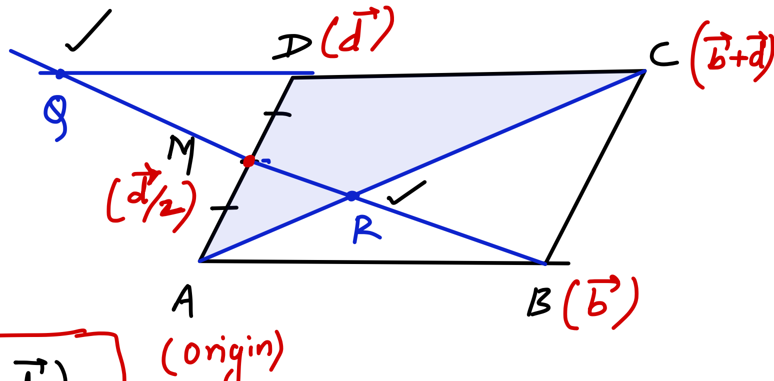
$$\boxed{\vec{OA} = 2\hat{i}}$$

∴ $\vec{OA}$ is not collinear with $(3\hat{i} - \hat{j} + \hat{k})$

∴ parallel but not intersecting.

(L planar lines)

Q Through the middle point M of the side AD of || gm ABCD, a straight line BM is drawn intersecting AC at R and CD produced at Q. use vectors to prove that QR = 2RB.



Vector eqn of line AC:

$$\boxed{\vec{r} = \vec{0} + \lambda(\vec{b}+\vec{d})} \quad —①—$$

Vector eqn of BM: $$\boxed{\vec{r} = \vec{b} + \mu\left(\vec{b} - \frac{\vec{d}}{2}\right)} \quad —②—$$

Solve BM & AC to get pv of R :–

$$\left.\begin{aligned} \lambda &= 1+\mu \\ \lambda &= -\frac{\mu}{2} \end{aligned}\right\} \qquad \begin{aligned} \lambda &= 1/3 \\ \mu &= -2/3 \end{aligned}$$

pv of R $\left(\frac{\vec{b}+\vec{d}}{3}\right)$ **

Vector eqn of CD: $\vec{r} = \vec{d} + t(\vec{b})$ —③—

Solve ② & ③ to get pv of Q :–

$$1+\mu = t$$

$$-\frac{\mu}{2} = 1 \Rightarrow \boxed{\mu = -2}$$

pv of Q : $(-\vec{b} + \vec{d})$

$$\vec{QR} = \text{pv of } R - \text{pv of } Q$$

$$= \frac{\vec{b}+\vec{d}}{3} - (\vec{d} - \vec{b})$$

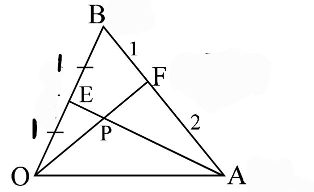
$$\boxed{\vec{QR} = \frac{4\vec{b} - 2\vec{d}}{3} = \frac{2}{3}(2\vec{b} - \vec{d})}$$

$$\vec{RB} = \text{pv of } B - \text{pv of } R$$

$$\boxed{\vec{RB} = \vec{b} - \left(\frac{\vec{b}+\vec{d}}{3}\right) = \frac{1}{3}(2\vec{b} - \vec{d})}$$

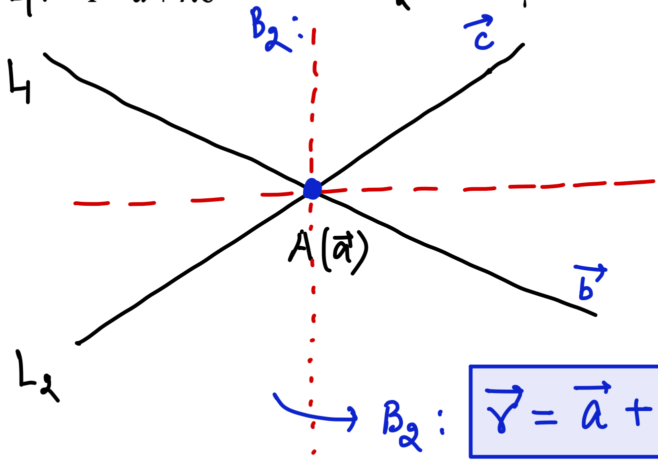
$$\frac{|\vec{QR}|}{|\vec{RB}|} = \frac{\frac{2}{\cancel{3}}\,\cancel{|2\vec{b} - \vec{d}|}}{\frac{1}{\cancel{3}}\,\cancel{|2\vec{b} - \vec{d}|}} = 2$$

Q HW In ΔAOB , E is the mid point of OB and F divides BA in the ratio 1 : 2 use vectors to prove that $\frac{OP}{PF}=\frac{3}{2}$.



Vector equation of the bisectors of the angles between the lines :-

$L_1: \vec{r}=\vec{a}+\lambda\vec{b}$ and $L_2: \vec{r}=\vec{a}+\mu\vec{c}$ are



Rem

B_1 : $\boxed{\vec{r}=\vec{a}+t(\hat{b}+\hat{c})}$

Rem

B_2 : $\boxed{\vec{r}=\vec{a}+k(\hat{b}-\hat{c})}$

TEST OF COLLINEARITY :

Three points A,B,C with position vectors $\vec{a},\vec{b},\vec{c}$ respectively are collinear, if and only if there exist scalars x , y , z not all zero simultaneously such that: $x\vec{a}+y\vec{b}+z\vec{c}=0$, where $x+y+z=0$.

A ($\vec{a}$)
B ($\vec{b}$)
C ($\vec{c}$)

[**Proof : (i) Necessary :**

Let A, B and C collinear $\overrightarrow{AB}=\vec{b}-\vec{a}$; $\overrightarrow{AC}=\vec{c}-\vec{a}$

$\therefore$ A, B, C are collinear $\Rightarrow$ $\overrightarrow{AB}$ and $\overrightarrow{AC}$ are collinear

$\Rightarrow$ $\vec{b}-\vec{a}$ and $\vec{c}-\vec{a}$ are collinear

$\therefore$ $\vec{b}-\vec{a}=\lambda(\vec{c}-\vec{a})$

or $(\lambda-1)\,\vec{a}+\vec{b}-\lambda\vec{c}=0$

$x\vec{a}+y\vec{b}+z\vec{c}=0$ where $x=\lambda-1$, $y=1$, $z=-\lambda$,
obviously, $x+y+z=0$

(ii) Sufficient :

$\therefore$ Again let $x\vec{a}+y\vec{b}+z\vec{c}=0$ where $x+y+z=0$, x, y, z not all zero.

T.P.T. A,B, C are collinear, let $y\neq 0$

now $\vec{b}=-\dfrac{x\vec{a}+z\vec{c}}{y}=\dfrac{x\vec{a}+z\vec{c}}{x+z}$

Hence $B(\vec{b})$ divides the join of $A(\vec{a})$ and $C(\vec{c})$ in the ratio of z and x

$\Rightarrow$ A, B, C are collinear.

Q Find whether the following points are collinear or not

(i) $2\hat{i}+5\hat{j}-4\hat{k}$; $\hat{i}+4\hat{j}-3\hat{k}$; $4\hat{i}+7\hat{j}-6\hat{k}$

(ii) $3\hat{i}-4\hat{j}+3\hat{k}$; $-4\hat{i}+5\hat{j}-6\hat{k}$; $4\hat{i}-7\hat{j}+6\hat{k}$

Note: Collinearly can also be checked by first finding the equation of line through two points and satisfying the third point.

(i) A(2, 5, −4)
B(1, 4, −3)
C(4, 7, −6)

$\overrightarrow{AB}=-\hat{i}-\hat{j}+\hat{k}$

$\overrightarrow{BC}=3\hat{i}+3\hat{j}-3\hat{k}$

$-\dfrac{1}{3}=-\dfrac{1}{3}=-\dfrac{1}{3}$

they are Collinear pts.

$|\overrightarrow{AB}|=\sqrt{3}$; $|\overrightarrow{BC}|=3\sqrt{3}$

$\overrightarrow{AC}=2\hat{i}+2\hat{j}-2\hat{k}$

$|\overrightarrow{AC}|=2\sqrt{3}$; $\boxed{AB+AC=BC}$

Q HW

Vectors $\vec{P}$, $\vec{Q}$, act at 'O' (origin) have a resultant $\vec{R}$. If any transversal cuts their line of action at A, B, C respectively, then show that $\frac{P}{OA}+\frac{Q}{OB}=\frac{R}{OC}$.

Scalar Product (Dot Product) :-

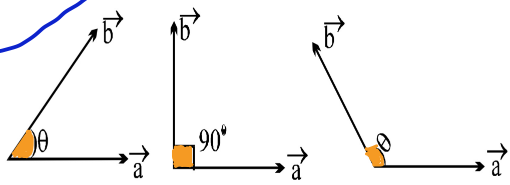
(1) $\vec{a}.\vec{b}=|\vec{a}||\vec{b}|\cos\theta$ for two non zero vectors $\vec{a}\,\&\,\vec{b}$ if

where $\vec{a}\,\&\,\vec{b}$ are not Collinear

(i) $\vec{a}.\vec{b}>0 \quad\Rightarrow\quad \theta$ is acute and $\vec{a}\neq\lambda\vec{b}$

(ii) $\vec{a}.\vec{b}=0 \quad\Rightarrow\quad \theta=\pi/2$

(iii) $\vec{a}.\vec{b}<0 \quad\Rightarrow\quad \theta$ is obtuse and $\vec{a}\neq\lambda\vec{b}$



$\vec{a}\,\&\,\vec{b}$ are not Collinear

① $\vec{a}.\vec{b}>0$

$\Rightarrow \theta\in[0,\pi/2)$

② $\vec{a}.\vec{b}<0$

$\Rightarrow \theta\in(\frac{\pi}{2},\pi]$

③ $\vec{a}.\vec{b}=0$

$\theta=\pi/2$ or $\vec{a}=\vec{0}$ or $\vec{b}=\vec{0}$

* Dot product is commutative i.e. $\vec{a}.\vec{b}=\vec{b}.\vec{a}$

Conventionally $\vec{a}.\vec{a}=|\vec{a}|^2=\vec{a}^2$

$\hat{i}.\hat{i}=\hat{j}.\hat{j}=\hat{k}.\hat{k}=1$; $\hat{i}.\hat{j}=\hat{j}.\hat{k}=\hat{k}.\hat{i}=0$

* Dot product is distributive

$\vec{a}.(\vec{b}+\vec{c})=\vec{a}.\vec{b}+\vec{a}.\vec{c}$

$(\vec{a}+\vec{b})^2=(\vec{a}+\vec{b}).(\vec{a}+\vec{b})$

Rem Simple identities to rembember are ,

(i) $(\vec{a}+\vec{b}).(\vec{a}-\vec{b})=\vec{a}^2-\vec{b}^2$

(ii) $(\vec{a}+\vec{b})^2=\vec{a}^2+2\vec{a}.\vec{b}+\vec{b}^2=(\vec{a}+\vec{b}).(\vec{a}+\vec{b})$

(iii) $(\vec{a}-\vec{b})^2=\vec{a}^2-2\vec{a}.\vec{b}+\vec{b}^2$

(iv) $(\vec{a}+\vec{b})^2=(\vec{a}-\vec{b})^2+4\vec{a}.\vec{b}$

(v) $(\vec{a}+\vec{b}+\vec{c})^2=\vec{a}^2+\vec{b}^2+\vec{c}^2-2\sum(\vec{a}.\vec{b})$

(vi) $\vec{a}.\vec{b}=\frac{1}{4}\left((\vec{a}+\vec{b})^2-(\vec{a}-\vec{b})^2\right)$

Q(1) $|\vec{a}|=11$; $|\vec{b}|=23$ and $|\vec{a}-\vec{b}|=30$, find $|\vec{a}+\vec{b}|$

Q(2) If $\vec{a}+\vec{b}+\vec{c}=0$, $|\vec{a}|=3$; $|\vec{b}|=1$ and $|\vec{c}|=4$. Find $\sum(\vec{a}.\vec{b})$

① $|\vec{a}-\vec{b}|=30 \Rightarrow (\vec{a}-\vec{b})^2=(30)^2 \Rightarrow a^2+b^2-2\vec{a}.\vec{b}=900$

$121+529-900=2\vec{a}.\vec{b}$

$2\vec{a}.\vec{b}=-250$

$(\vec{a}+\vec{b})^2-4\vec{a}.\vec{b}=900.$

$(\vec{a}+\vec{b})^2+500=900$

$(\vec{a}+\vec{b})^2=400 \Rightarrow |\vec{a}+\vec{b}|=20$ Ans

② $(\vec{a}+\vec{b}+\vec{c})^2=0$

$a^2+b^2+c^2+2\underbrace{\sum\vec{a}.\vec{b}}=0$

$9+1+16+2E=0 \Rightarrow E=-13$

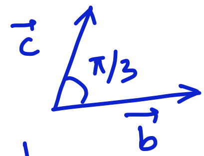
Q If $\vec{a}, \vec{b}, \vec{c}$ are unit vectors such that $\vec{a}$ is perpendicular to plane of $\vec{b}$ and $\vec{c}$ and angle between $\vec{b}$ and $\vec{c}$ is $\frac{\pi}{3}$. Find $|\vec{a}+\vec{b}+\vec{c}| = ?$

Sol^n

$|\vec{a}| = |\vec{b}| = |\vec{c}| = 1$

$\vec{a}\cdot\vec{b} = 0$ & $\vec{a}\cdot\vec{c} = 0$

$\vec{b}\cdot\vec{c} = |\vec{b}||\vec{c}|\cos\pi/3 = 1.1.\frac{1}{2} = \frac{1}{2}$



$$|\vec{a}+\vec{b}+\vec{c}|^2 = a^2+b^2+c^2+2\Sigma\,\vec{a}\cdot\vec{b}$$
$$= 1+1+1+2\left(0+0+\tfrac{1}{2}\right) = 4$$

$$|\vec{a}+\vec{b}+\vec{c}| = 2.$$

* Q Given $a^2+b^2+c^2 = 9$ then find max value of $(a+2b+c)$?

Sol^n

$\vec{V_1} = a\hat{i}+b\hat{j}+c\hat{k}$ $\qquad |\vec{V_1}| = 3.$

$$a+2b+c = \underbrace{(a\hat{i}+b\hat{j}+c\hat{k})}_{\vec{V_1}}\cdot\underbrace{(\hat{i}+2\hat{j}+\hat{k})}_{\vec{V_2}}$$

$\vec{p}\cdot\vec{q} = |\vec{p}||\vec{q}|\cos\theta$

$(\vec{p}\cdot\vec{q})_{max} = |\vec{p}||\vec{q}|$

$(\vec{p}\cdot\vec{q})_{min} = -|\vec{p}||\vec{q}|$

$$(\vec{V_1}\cdot\vec{V_2})_{max} = |\vec{V_1}||\vec{V_2}|$$
$$= (3)(\sqrt{6})$$
$$= 3\sqrt{6}.$$

Q If $a>0$ and A, B, C are variable angles such that $(\sqrt{a^2-4})\tan A + a\tan B + (\sqrt{a^2+4})\tan C = 6a$ then find min value of $\sum \tan^2 A$?

$\frac{\cancel{36}}{\cancel{3}} = 12$

Sol^n $\vec{v}_1 = (\tan A)\hat{i} + (\tan B)\hat{j} + (\tan C)\hat{k}$.

$\vec{v}_2 = (\sqrt{a^2-4})\hat{i} + (a)\hat{j} + (\sqrt{a^2+4})\hat{k}$

$$\vec{v}_1 \cdot \vec{v}_2 = |\vec{v}_1||\vec{v}_2|\cos\theta$$

$$6a = \sqrt{\sum \tan^2 A}\sqrt{a^2\cancel{-4}+a^2+a^2\cancel{+4}}\ \cos\theta.$$

$$\frac{6\cancel{a}}{\sqrt{3}\cancel{a}\cdot \underbrace{\cos\theta}_{\max=1}} = \sqrt{\sum \tan^2 A} \Rightarrow \left(\sqrt{\sum \tan^2 A}\right)_{\min} = \frac{6}{\sqrt{3}}$$

Q If two points P and Q are on curve $y = 2^{x+2}$, such that $\vec{OP}\cdot\hat{i} = -1$ and $\vec{OQ}\cdot\hat{i} = 2$, where $\hat{i}$ is unit vector along x-axis, $|\vec{OQ} - 4\vec{OP}| = ?$

('O' is origin)

Sol^n pv of P $(\alpha\hat{i} + \beta\hat{j})$.

pv of Q $(a\hat{i} + b\hat{j})$.

$\vec{OQ}\cdot\hat{i} = 2$

$\boxed{a=2} \Rightarrow b = 2^{a+2}$

$\boxed{b = 16}$

$\vec{OP} = \alpha\hat{i} + \beta\hat{j}$

$\vec{OP}\cdot\hat{i} = -1$

$\alpha = -1 \Rightarrow \beta = 2^{\alpha+2}$

$\boxed{\begin{matrix}\alpha=-1\\ \beta=2\end{matrix}}$ $\beta = 2^1 = 2$

$$|\vec{OQ} - 4\vec{OP}| = |(2\hat{i}+16\hat{j}) - 4(-\hat{i}+2\hat{j})|$$

$$= |6\hat{i}+8\hat{j}| = 10 \text{ Ans}$$

HW

Q. Let $\vec{r}_0$ be constant vector and $\vec{r}$ is a veriable vector in x-y plane such that both lie on the curve $|\vec{r}|=1$. If a and b are the least and greatest values of $\left(|\vec{r}-\vec{r}_0|+|\vec{r}+\vec{r}_0|\right)$, then (b^2-a^2) is equal to

General Expression for dot product

If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ & $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ then

$\vec{a}.\vec{b}=a_1b_1+a_2b_2+a_3b_3$ & $\cos\theta=\dfrac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}=\dfrac{a_1b_1+a_2b_2+a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}.\sqrt{b_1^2+b_2^2+b_3^2}}$

Projection of vector :–

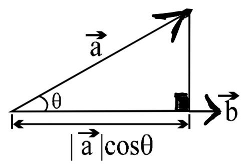
Rem

$\vec{a}.\hat{b}$

$$\boxed{\text{Projection of } \vec{a} \text{ on } \vec{b} = \frac{\vec{a}.\vec{b}}{|\vec{b}|}}$$

. It can be +ve, – ve or zero.

Note that vector component of $\vec{a}$ along $\vec{b}=\left(\dfrac{\vec{a}.\vec{b}}{\vec{b}^2}\right)\vec{b}$ and vector component of $\vec{a}$ perpendicular to $\vec{b}=\vec{a}-\left(\dfrac{\vec{a}.\vec{b}}{\vec{b}^2}\right)\vec{b}$



Q If $\vec{a}=2\hat{i}-3\hat{j}+6\hat{k}$ and $\vec{b}=-2\hat{i}+5\hat{j}-14\hat{k}$. If $\lambda=\dfrac{\text{Projection of } \vec{a} \text{ on } \vec{b}}{\text{Projection of } \vec{b} \text{ on } \vec{a}}$. Find λ.

E(1) Find the p.v. of the point of intersection of the lines (if it exists)

(a) $$\left.\begin{aligned} \vec{r} &= \hat{i}-\hat{j}-10\hat{k}+\lambda(2\hat{i}-3\hat{j}+8\hat{k}) \\ \text{and } \vec{r} &= 4\hat{i}-3\hat{j}-\hat{k}+\mu(\hat{i}-4\hat{j}+7\hat{k}) \end{aligned}\right]$$

HW

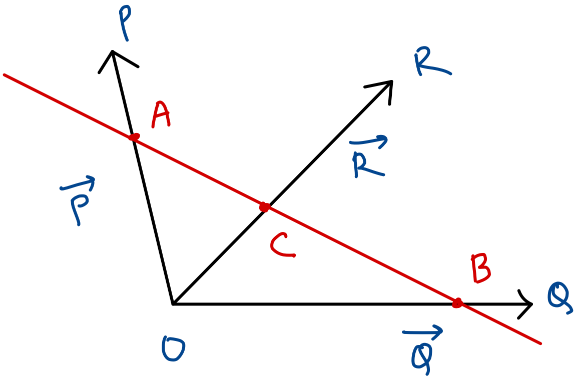
(b) $$\left.\begin{aligned} \vec{r} &= -3\hat{i}+6\hat{j}+\lambda(-4\hat{i}+3\hat{j}+2\hat{k}) \\ \text{and } \vec{r} &= -2\hat{i}+7\hat{k}+\mu(-4\hat{i}+\hat{j}+\hat{k}) \end{aligned}\right] \rightarrow \text{skew lines}$$

(c) $$\left.\begin{aligned} \vec{r} &= t(3\hat{i}-\hat{j}+\hat{k}) \\ \text{and } \vec{r} &= 2\hat{i}+s(-6\hat{i}+2\hat{j}-2\hat{k}) \end{aligned}\right]$$

HW

(d) $$\left.\begin{aligned} \vec{r} &= 2\hat{k}+\lambda(3\hat{i}+2\hat{j}+\hat{k}) \\ \text{and } \vec{r} &= 3\hat{i}+2\hat{j}+3\hat{k}+\mu(6\hat{i}+4\hat{j}+2\hat{k}) \end{aligned}\right] \rightarrow \text{Coincident lines}$$

Q HW Vectors $\vec{P}$, $\vec{Q}$, act at 'O' (origin) have a resultant $\vec{R}$. If any transversal cuts their line of action at A, B, C respectively, then show that $\frac{OP}{OA}+\frac{OQ}{OB}=\frac{OR}{OC}$.



$$\vec{P}+\vec{Q}=\vec{R}.$$

$$\vec{OP}+\vec{OQ}=\vec{OR}$$

$$|\vec{OP}|\,\hat{OP}+|\vec{OQ}|\,\hat{OQ}=|\vec{OR}|\,\hat{OR}$$

$$(OP)\,\hat{OA}+(OQ)\,\hat{OB}-(OR)\,\hat{OC}=0$$

$$(OP)\frac{\vec{OA}}{OA}+(OQ)\frac{\vec{OB}}{OB}-(OR)\frac{\vec{OC}}{OC}=0$$

Since A, B, C are Collinear

hence

$$\frac{OP}{OA}+\frac{OQ}{OB}+\left(\frac{-OR}{OC}\right)=0.$$

(H.P.)

Q Let $\vec{r}_0$ be constant vector and $\vec{r}$ is a veriable vector in x-y plane such that both lie on the curve $|\vec{r}|=1$. If a and b are the least and greatest values of $\left(|\vec{r}-\vec{r}_0|+|\vec{r}+\vec{r}_0|\right)$, then (b^2-a^2) is equal to

Sol^n

$$\vec{r}_0 = (\cos\alpha)\hat{i} + (\sin\alpha)\hat{j}$$

$$\vec{r} = (\cos\theta)\hat{i} + (\sin\theta)\hat{j}$$

$$\underbrace{|\vec{r}-\vec{r}_0| + |\vec{r}+\vec{r}_0|}_{E} = \sqrt{(\cos\alpha-\cos\theta)^2+(\sin\alpha-\sin\theta)^2}$$

$$+\sqrt{(\cos\alpha+\cos\theta)^2+(\sin\alpha+\sin\theta)^2}$$

$$E = \sqrt{2-2\cos(\alpha-\theta)} + \sqrt{2+2\cos(\alpha-\theta)}$$

$$E^2 = 2+2+2\sqrt{4-4\cos^2(\alpha-\theta)}$$

$$E^2_{min} = 4+0 \Rightarrow E_{min} = 2 = a.$$

$$E^2_{max} = 4+2\sqrt{4} \Rightarrow E_{max} = 2\sqrt{2} = b.$$

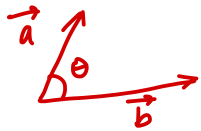
$$\therefore (b^2-a^2) = 8-4 = 4. \text{ Ans}$$

General Expression for dot product

If $\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ & $\vec{b}=b_1\hat{i}+b_2\hat{j}+b_3\hat{k}$ then

$$\vec{a}.\vec{b}=a_1b_1+a_2b_2+a_3b_3 \qquad \& \qquad \cos\theta=\frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}=\frac{a_1b_1+a_2b_2+a_3b_3}{\sqrt{a_1^2+a_2^2+a_3^2}\,.\sqrt{b_1^2+b_2^2+b_3^2}}$$

Projection of vector :— Rem

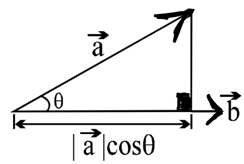


$\theta \in [0,\pi]$

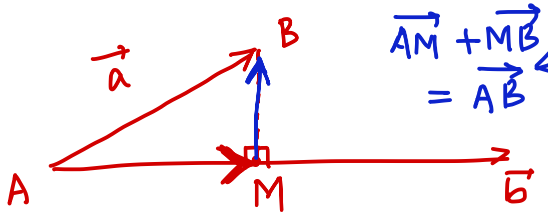
Projection of $\vec{a}$ on $\vec{b}$ $=\dfrac{\vec{a}.\vec{b}}{|\vec{b}|}$. It can be +ve, – ve or zero.

$|\vec{a}|\cos\theta$

Note that vector component of $\vec{a}$ along $\vec{b}=\left(\dfrac{\vec{a}.\vec{b}}{b^2}\right)\vec{b}$ and vector component of $\vec{a}$ perpendicular to $\vec{b}=\vec{a}-\left(\dfrac{\vec{a}.\vec{b}}{b^2}\right)\vec{b}$



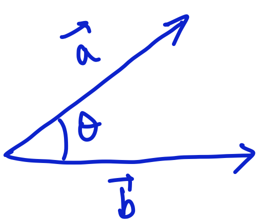
$$\cos\theta=\frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|}$$



$\vec{AM}+\vec{MB}=\vec{AB}$

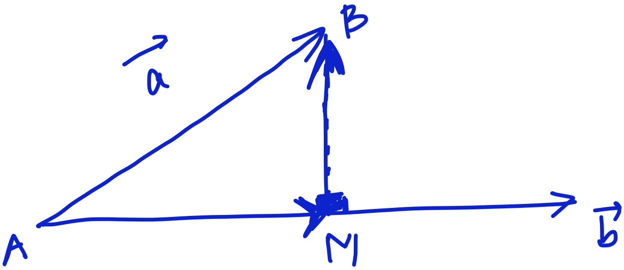
$$\boxed{\vec{AM}=(|\vec{a}|\cos\theta)\,\hat{b}.}$$

Q If $\vec{a}=2\hat{i}-3\hat{j}+6\hat{k}$ and $\vec{b}=-2\hat{i}+5\hat{j}-14\hat{k}$. If $\lambda=\dfrac{\text{Projection of } \vec{a} \text{ on } \vec{b}}{\text{Projection of } \vec{b} \text{ on } \vec{a}}$. Find λ.



$$\lambda=\frac{|\vec{a}|\cos\theta}{|\vec{b}|\cos\theta}=\frac{|\vec{a}|}{|\vec{b}|}$$

Q Express the vector $\vec{a} = 5\hat{i} - 2\hat{j} + 5\hat{k}$ as the sum of two vectors such that one is parallel to $\vec{b} = 3\hat{i} + \hat{k}$ and the other perpendicular to $\vec{b}$.



$$\overrightarrow{AM} = (|\vec{a}|\cos\theta)\hat{b}$$

$$\overrightarrow{AM} + \overrightarrow{MB} = \overrightarrow{AB}.$$

$$\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}||\vec{b}|}$$

General note:

(i) Maximum value of $\vec{a}.\vec{b} = |\vec{a}||\vec{b}|$

(ii) Minimum value of $\vec{a}.\vec{b} = -|\vec{a}||\vec{b}|$

(iii) Any vector $\vec{a}$ can be expressed as $(\vec{a}.\hat{i})\hat{i} + (\vec{a}.\hat{j})\hat{j} + (\vec{a}.\hat{k})\hat{k}$

**

$$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

$$\vec{a}.\hat{i} = a_1$$

$$\vec{a}.\hat{j} = a_2$$

$$\vec{a}.\hat{k} = a_3$$

$$\boxed{\vec{a} = (\vec{a}.\hat{i})\hat{i} + (\vec{a}.\hat{j})\hat{j} + (\vec{a}.\hat{k})\hat{k}}$$

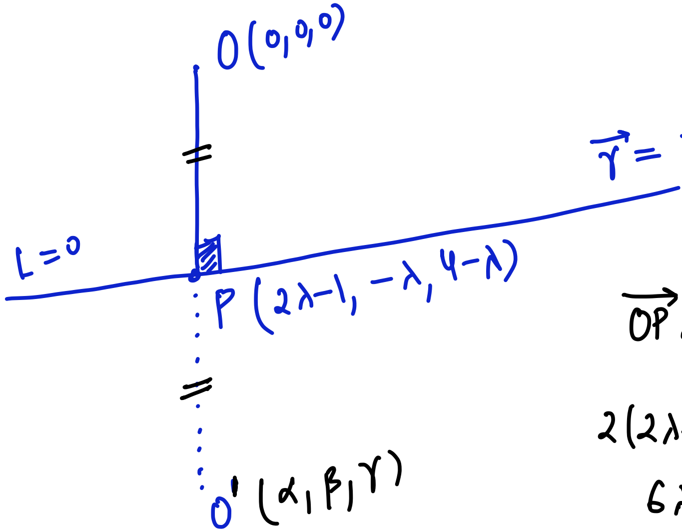
Q1 Find the foot of the perpendicular from the origin on the line $\vec{r} = -\hat{i} + 4\hat{k} + \lambda(2\hat{i} - \hat{j} - \hat{k})$. Also find the p.v. of its image in the line.

Q2 A line passes through a point with p.v. $\hat{i} - 2\hat{j} - \hat{k}$ and is parallel to the vector $\hat{i} - 2\hat{j} + 2\hat{k}$.
HW
Find the distance of a point P (5, 0, – 4) from the line.

Solⁿ 1

$\vec{r} = -\hat{i} + 4\hat{k} + \lambda\underbrace{(2\hat{i} - \hat{j} - \hat{k})}$

direction vector of line



$$\vec{OP} \cdot (2\hat{i} - \hat{j} - \hat{k}) = 0.$$

$$2(2\lambda - 1) + \lambda + \lambda - 4 = 0.$$

$$6\lambda = 6 \Rightarrow \boxed{\lambda = 1}$$

pv of P (1, −1, 3)

image (O') of O in L = 0 is

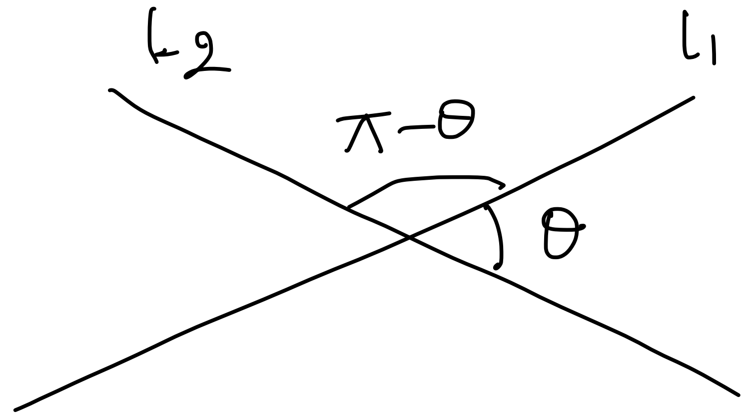
$$\frac{\alpha + 0}{2} = 1 \; ; \; \frac{\beta + 0}{2} = -1 \; ; \; \frac{\gamma + 0}{2} = 3$$

Angle betⁿ 2 lines :-

$L_1: \vec{r} = \vec{a} + \lambda \vec{p}$

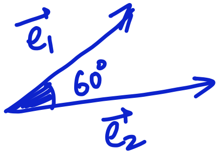
$L_2: \vec{r} = \vec{b} + \mu \vec{q}$

$$\boxed{\vec{p} \cdot \vec{q} = |\vec{p}|\,|\vec{q}| \cos\theta}$$



Q. Two vectors $\vec{e}_1$ and $\vec{e}_2$ with $|\vec{e}_1| = 2$ and $|\vec{e}_2| = 1$ and angle between $\vec{e}_1$ and $\vec{e}_2$ is 60°. The angle between $(2t\vec{e}_1 + 7\vec{e}_2)$ and $(\vec{e}_1 + t\vec{e}_2)$ belongs to the interval (90°, 180°). Find the range of t.

Solⁿ



$|\vec{e}_1| = 2$; $|\vec{e}_2| = 1$

$\vec{e}_1 \cdot \vec{e}_2 = (2)(1) \cdot \cos 60° = 1.$

$\vec{V}_1 = 2t\vec{e}_1 + 7\vec{e}_2$

$\vec{V}_2 = \vec{e}_1 + t\vec{e}_2$

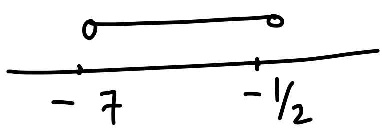
$\vec{V}_1 \cdot \vec{V}_2 < 0$ **

$2t\, e_1^2 + 2t^2\, \vec{e}_1 \cdot \vec{e}_2 + 7\, \vec{e}_1 \cdot \vec{e}_2 + 7t\, e_2^2 < 0$

$2t(4) + 2t^2(1) + 7(1) + 7t(1) < 0.$

$2t^2 + 15t + 7 < 0$

$2t^2 + 14t + t + 7 < 0 \Rightarrow (2t+1)(t+7) < 0.$



$t \in \left(-7, -\frac{1}{2}\right)$ — ① —

Now check when angle betⁿ $\vec{V}_1$ & $\vec{V}_2$ becomes 180°.

$\vec{V}_1$ & $\vec{V}_2$ will be Collinear (Unlike parallel)

$$\frac{2t}{1} = \frac{7}{t} \Rightarrow \quad 2t^2 = 7.$$

$$t^2 = 7/2.$$

$$t = \sqrt{7/2} \; ; \; t = -\sqrt{7/2}$$

$\downarrow$

$**$

finally,

$$t \in \left(-7, -\frac{1}{2}\right) - \left\{-\sqrt{7/2}\right\}$$

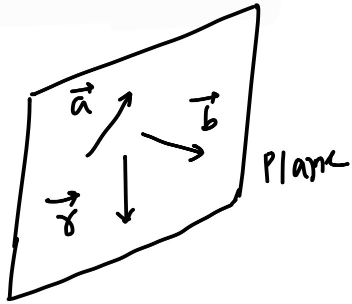
Linear combination

A vector $\vec{r}$ is said to be a linear combination of the vectors $\vec{a}, \vec{b}, \vec{c}$......
if $\exists$ scalars x,y,z, ... such that $\vec{r} = x\vec{a} + y\vec{b} + z\vec{c} + ...$

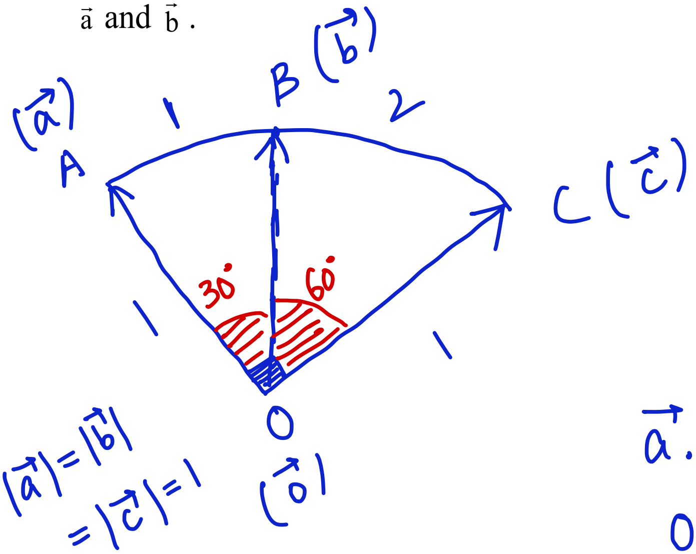
V.imp

Theorem in plane

If $\vec{a}$ and $\vec{b}$ are two non zero non collinear vetors then any vector $\vec{r}$ coplanar with them can be expressed as a linear combination $\vec{r} = x\vec{a} + y\vec{b}$.



(1) Arc AC of the quadrant of a circle with centre as origin and radius unity subtends a right angle at the origin. Point B divides the arc AC in the ratio 1 : 2. Express the vector $\vec{c}$ in terms of $\vec{a}$ and $\vec{b}$.



Theorem in plane :–

$$\vec{c} = x\vec{a} + y\vec{b}. \quad -①-$$

① dot with $\vec{a}$

$$\vec{a}\cdot\vec{c} = x\,\vec{a}^2 + y\,\vec{a}\cdot\vec{b}$$

$$0 = x(1)^2 + y(1)(1)\cdot\frac{\sqrt{3}}{2}$$

$$\boxed{x + \frac{\sqrt{3}}{2}y = 0} \quad -②-$$

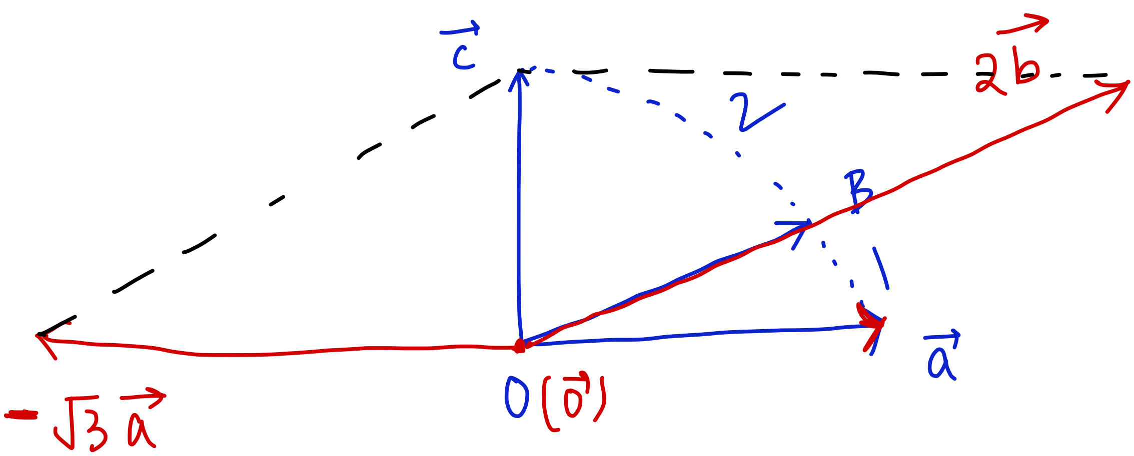
① dot with $\vec{c}$:-

$$\vec{c}\cdot\vec{c} = x\,\cancel{\vec{a}\cdot\vec{c}}^{\,0} + y(\vec{b}\cdot\vec{c})$$

$$1 = y(1)(1).\cos 60^\circ \Rightarrow \boxed{y = 2}$$ put in ②

$x = -\sqrt{3}$

$$\therefore \boxed{\vec{c} = -\sqrt{3}\,\vec{a} + 2\vec{b}}$$



Q Given that $\vec{a}=\hat{i}-\hat{j}$ and $\hat{b}=\hat{i}+2\hat{j}$ are two vectors. Find a unit vector coplanar with $\vec{a}$ and $\vec{b}$ and perpendicular to $\vec{a}$.

Sol

$$\boxed{\vec{c} = x\vec{a} + y\vec{b}} \;;\qquad \left.\begin{array}{l} |\vec{c}| = 1 \\ \vec{c}\cdot\vec{a} = 0 \end{array}\right\} \qquad \begin{array}{l} |\vec{a}| = \sqrt{2}. \\ |\vec{b}| = \sqrt{5}. \end{array}$$

$$\vec{c} = x(\hat{i}-\hat{j}) + y(\hat{i}+2\hat{j})$$

$$\vec{c} = \hat{i}(x+y) + \hat{j}(-x+2y) \quad \text{— ① —}$$

$$|\vec{c}| = \sqrt{(x+y)^2 + (2y-x)^2} = 1 \quad \text{— ② —}$$

$$\cancel{\vec{c}\cdot\vec{a}}^{\,0} = x\,a^2 + y\,\vec{a}\cdot\vec{b}$$

$$0 = x(2) + y(1-2) \quad \text{— ③ —}$$

$$x = \qquad y =$$

Q If $\vec{a}$, $\vec{b}$, $\vec{c}$ are coplanar vectors, prove that $\begin{vmatrix} \vec{a} & \vec{b} & \vec{c} \\ \vec{a}.\vec{a} & \vec{a}.\vec{b} & \vec{a}.\vec{c} \\ \vec{b}.\vec{a} & \vec{b}.\vec{b} & \vec{b}.\vec{c} \end{vmatrix} = 0$

Soln

$$\vec{c} = x\vec{a} + y\vec{b} \quad \text{— ① —}$$

$$\begin{vmatrix} \vec{a} & \vec{b} & x\vec{a}+y\vec{b} \\ \vec{a}.\vec{a} & \vec{a}.\vec{b} & \vec{a}.(x\vec{a}+y\vec{b}) \\ \vec{b}\cdot\vec{a} & \vec{b}.\vec{b} & \vec{b}.(x\vec{a}+y\vec{b}) \end{vmatrix}$$

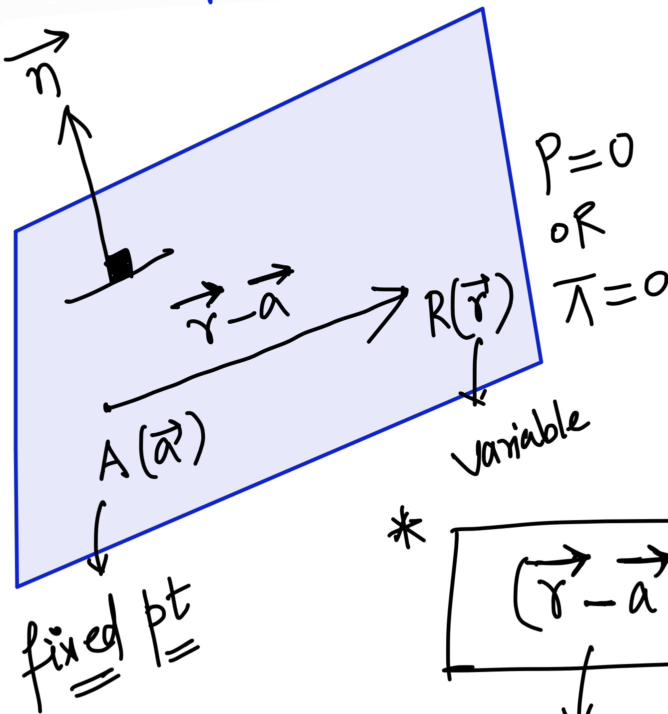
$$\begin{vmatrix} \vec{a} & \vec{b} & x\vec{a} \\ \vec{a}\cdot\vec{a} & \vec{b}\cdot\vec{a} & x\vec{a}^2 \\ \vec{a}\cdot\vec{b} & \vec{b}\cdot\vec{b} & x\,\vec{a}\cdot\vec{b} \end{vmatrix} + \begin{vmatrix} \vec{a} & \vec{b} & y\vec{b} \\ \vec{a}\cdot\vec{a} & \vec{b}\cdot\vec{a} & y\,\vec{a}\cdot\vec{b} \\ \vec{a}\cdot\vec{b} & \vec{b}\cdot\vec{b} & y\,\vec{b}\cdot\vec{b} \end{vmatrix}$$

'x' common

'y' common

$$0 + 0 = 0$$

Equation of plane :-



$\vec{n} \rightarrow$ normal vector of plane

Eqn of plane :-

$$* \quad \boxed{(\vec{r} - \vec{a}) \cdot \vec{n} = 0}$$

$$\vec{r} \cdot \vec{n} = \underbrace{\vec{a} \cdot \vec{n}}_{d}$$

$$\boxed{\vec{r} \cdot \vec{n} = d}$$

$$\boxed{\vec{r} \cdot \hat{n} = K}$$

Angle betⁿ 2 planes :-

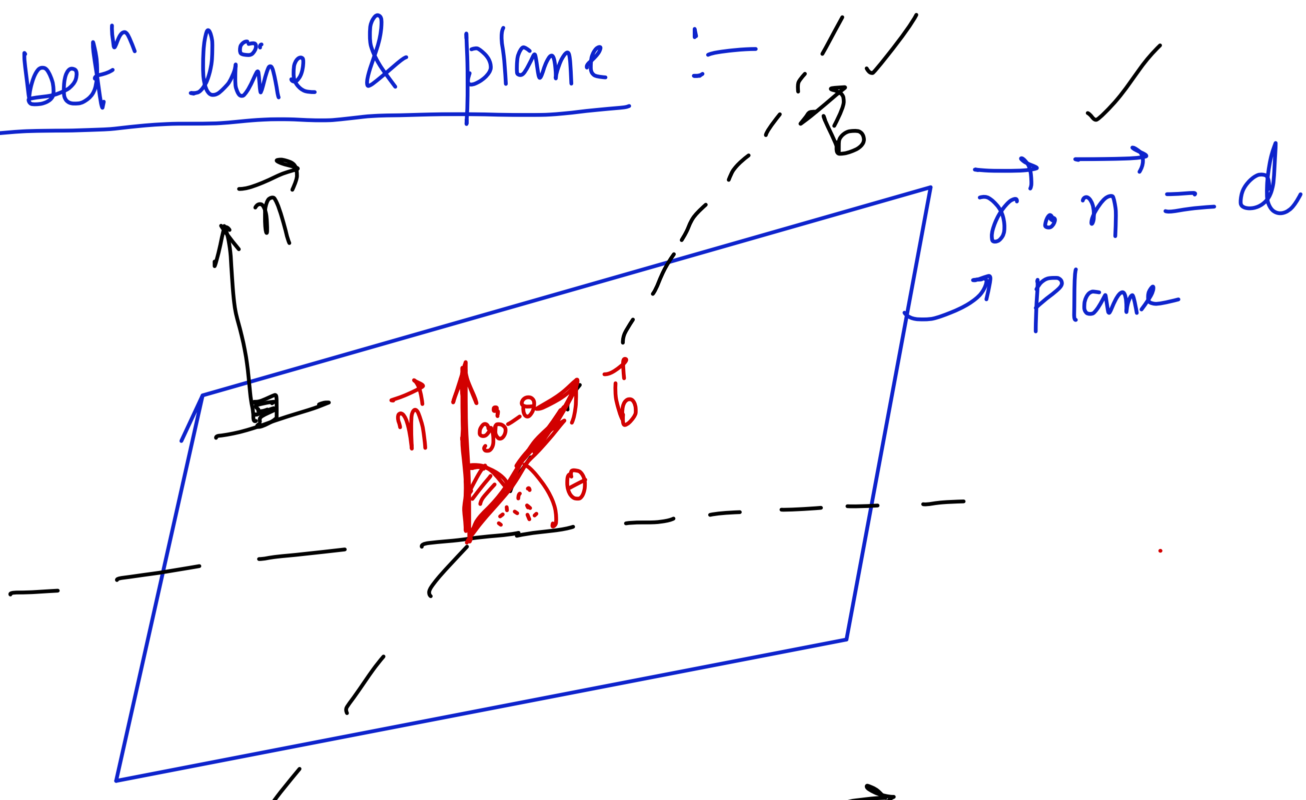
$P_1 : \quad \vec{r} \cdot \vec{n_1} = \lambda$

$P_2 : \quad \vec{r} \cdot \vec{n_2} = K$

Angle between 2 planes is defined as the angle betⁿ their normals.

$$\boxed{\vec{n_1} \cdot \vec{n_2} = |\vec{n_1}|\,|\vec{n_2}| \cos\theta}$$

Angle betⁿ line & plane :-



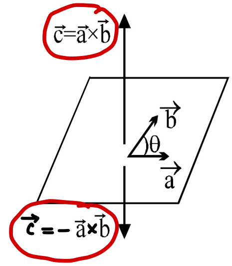
$L: \quad \vec{r} = \vec{a} + \lambda \vec{b}$

Angle betⁿ a line & plane is the complement of the angle between normal vector of plane & direction vector of the line.

$$\boxed{\vec{b} \cdot \vec{n} = |\vec{b}|\,|\vec{n}| \cos(90^\circ - \theta)}$$

Vector product (Cross Product)

(1) $\vec{a}\times\vec{b}=|\vec{a}||\vec{b}|\sin\theta\,\hat{n}$ where $\hat{n}$ is the unit vector perpendicular to the plane containing the vectors $\vec{a}$ and $\vec{b}$ such that $\vec{a}$ and $\vec{b}$ and $\hat{n}$ forms a right handed screw system.



In general $\vec{a}\times\vec{b}\neq\vec{b}\times\vec{a}$

Q Equation of a line which passes through the point with p.v. $\vec{a}$ and perpendicular to the lines $\vec{r}=\vec{b}+\lambda\vec{p}$ and $\vec{r}=\vec{c}+\mu\vec{q}$.

Sol^n $\vec{r}=\vec{a}+t(\vec{p}\times\vec{q})$; $t\in$ scalar

Note :-

(1) Unit vector perpendicular to the plane of $\vec{a}$ & $\vec{b}$ is $\pm\dfrac{\vec{a}\times\vec{b}}{|\vec{a}\times\vec{b}|}$ & a vector of magnitude 'r' perpendicular to the plane of $\vec{a}$ & $\vec{b}$ = $\pm\dfrac{r(\vec{a}\times\vec{b})}{|\vec{a}\times\vec{b}|}$

(2) If θ is the angle between $\vec{a}$ & $\vec{b}$ then $\sin\theta=\pm\dfrac{|\vec{a}\times\vec{b}|}{|\vec{a}||\vec{b}|}$

Langrange's Identity :

$|\vec{a} \times \vec{b}|$ is very frequently needed for which Lagranges identity is used.

i.e. $\left|\vec{a} \times \vec{b}\right|^2 = \vec{a}^2 \vec{b}^2 - (\vec{a}.\vec{b})^2 = \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} \\ \vec{b}.\vec{a} & \vec{b}.\vec{b} \end{vmatrix}$.

Proof: $|\vec{a}\times\vec{b}|^2 = |\vec{a}|^2|\vec{b}|^2\sin^2\theta$

$= |\vec{a}|^2|\vec{b}|^2(1-\cos^2\theta)$

$= a^2b^2 - |\vec{a}|^2|\vec{b}|^2\cos^2\theta$

$= a^2b^2 - (\vec{a}.\vec{b})^2$

Q For any vector $\vec{a}$, $|\vec{a}\times\hat{i}|^2 + |\vec{a}\times\hat{j}|^2 + |\vec{a}\times\hat{k}|^2 = \lambda\vec{a}^2$ then $\lambda = ?$ (2)

Soln

$|\vec{a}\times\hat{i}|^2 = a^2.\hat{i}^2 - (\underbrace{\vec{a}.\hat{i}})^2$

$= a^2 - a_1^2$

$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$

$|\vec{a}| = \sqrt{a_1^2+a_2^2+a_3^2}$

$a^2 = a_1^2+a_2^2+a_3^2$

||ly

$|\vec{a}\times\hat{j}|^2 = a^2 - a_2^2$

$|\vec{a}\times\hat{k}|^2 = a^2 - a_3^2$

$|\vec{a}\times\hat{i}|^2 + |\vec{a}\times\hat{k}|^2 + |\vec{a}\times\hat{j}|^2 = 3a^2 - (\underbrace{a_1^2+a_2^2+a_3^2}_{a^2})$

$= 2a^2$

Note : If $\vec{c}.\vec{a} = \vec{c}.\vec{b} = 0$ where $\vec{a}, \vec{b}$ vector are non-zero, non-parallel vectors, then $\vec{c} = \lambda(\vec{a}\times\vec{b})$.

Properties of cross product :

(i) $\vec{a}\times\vec{b} = \vec{0} \Rightarrow \vec{a} = \lambda\vec{b} (\vec{a} \neq 0;\ \vec{b} \neq 0)$ i.e. $\vec{a}$ and $\vec{b}$ are Collinear/Linearly dependent however if $\vec{a}\times\vec{b} = \vec{0}$ and $\vec{a}.\vec{b} = 0 \Rightarrow \vec{a} = 0$ or $\vec{b} = 0$

$\vec{a}\times\vec{b} = \vec{0}$

$\vec{a} = 0$ OR

$\vec{b} = 0$ OR

$\boxed{\vec{a} = \lambda\vec{b}}$

(ii) $\vec{a}\times\vec{b} \neq \vec{b}\times\vec{a}$ (in general) (not commutative)

(iii) $\vec{a}\times(\vec{b}+\vec{c}) = \vec{a}\times\vec{b}+\vec{a}\times\vec{c}$ (distributive to be proved later using triple product)

(iv) $(\vec{a}\times\vec{b})\times\vec{c} \neq \vec{a}\times(\vec{b}\times\vec{c})$ (in general) (vector product is not associative).

(v) $\hat{i}\times\hat{i} = \hat{j}\times\hat{j} = \hat{k}\times\hat{k} = 0$ and $\hat{i}\times\hat{j} = \hat{k},\ \hat{j}\times\hat{k} = \hat{i},\ \hat{k}\times\hat{i} = \hat{j}$

(vi) $(\vec{a}\times\vec{b}).\vec{a} = 0,\ (\vec{a}\times\vec{b}).\vec{b} = 0$

Expression for $\vec{a}\times\vec{b}$, where $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ & $\vec{b} = b_1\hat{i} + b_2\hat{j} + b_3\hat{k}$ then $\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$

Q1 Find the equation of the line through the point with p.v. $2\hat{i} + 3\hat{j}$ and perpendicular to the vectors $\vec{A} = \hat{i}+2\hat{j}+3\hat{k}$ and $\vec{B} = 3\hat{i}+4\hat{j}+5\hat{k}$.

Q2 If $(2\hat{i}+6\hat{j}+27\hat{k})\times(\hat{i}+\lambda\hat{j}+\mu\hat{k}) = 0$ find λ and μ.

1. $\vec{r} = (2\hat{i}+3\hat{j}) + \lambda(\vec{A}\times\vec{B})$

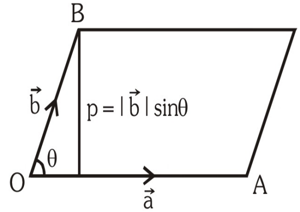
2. $2\hat{i}+6\hat{j}+27\hat{k}$ must be Collinear with $\hat{i}+\lambda\hat{j}+\mu\hat{k}$.

$$\frac{2}{1} = \frac{6}{\lambda} = \frac{27}{\mu} \Rightarrow \left.\begin{matrix}\lambda = 3 \\ \mu = \frac{27}{2}\end{matrix}\right\}$$

Q1 Find the foot of the perpendicular from the origin on the line $\vec{r} = -\hat{i} + 4\hat{k} + \lambda(2\hat{i} - \hat{j} - \hat{k})$.
Also find the p.v. of its image in the line.

Q2 A line passes through a point with p.v. $\hat{i} - 2\hat{j} - \hat{k}$ and is parallel to the vector $\hat{i} - 2\hat{j} + 2\hat{k}$.
HW
Find the distance of a point P (5, 0, – 4) from the line. Ans = 5

Geometrical interpretation : $|\vec{a} \times \vec{b}| = |\vec{a}||\vec{b}|\sin\theta$ denotes the area of parallelogram whose two adjacent sides are the vectors $\vec{a}$ & $\vec{b}$.



Note :

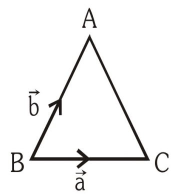
(i) Area of a parallelogram / quad. if diagonal vectors $\vec{d}_1$ & $\vec{d}_2$ are known is given by $\frac{1}{2}\left|\vec{d}_1 \times \vec{d}_2\right|$.

Q If $\vec{d}_1 = 2\hat{i} + 3\hat{j} - 6\hat{k}$ and $\vec{d}_2 = 3\hat{i} - 4\hat{j} - \hat{k}$, then find area of parallelogram.

Vector area of a plane triangle :

(i) Area of the triangle (in terms of sides)

$$= \frac{|\vec{a} \times \vec{b}|}{2} = \frac{1}{2}\left|\overrightarrow{BA} \times \overrightarrow{BC}\right| = \frac{1}{2}\left|\overrightarrow{AB} \times \overrightarrow{AC}\right|$$

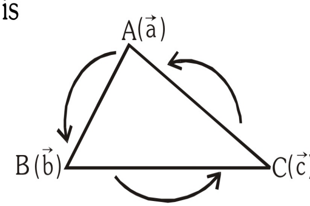


(ii) If $\vec{a}, \vec{b}, \vec{c}$ are the position vectors then the vector area of ΔABC is

$$\vec{\Delta} = \frac{1}{2}[(\vec{c} - \vec{b}) \times (\vec{a} - \vec{b})]$$

Rem

$$\boxed{\vec{\Delta} = \frac{1}{2}[(\vec{a} \times \vec{b}) + (\vec{b} \times \vec{c}) + (\vec{c} \times \vec{a})]}$$



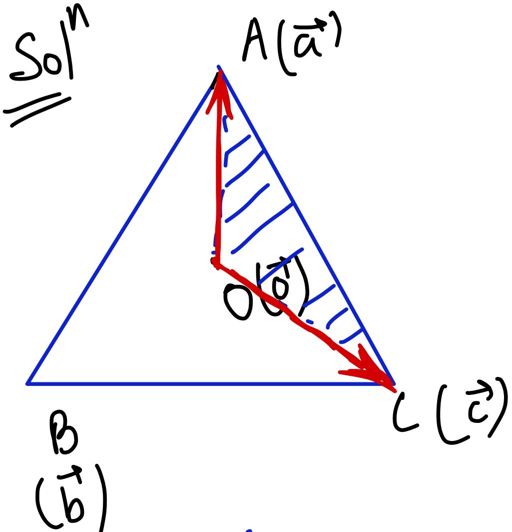
$\overrightarrow{BC} = \vec{c} - \vec{b}$

$\overrightarrow{BA} = \vec{a} - \vec{b}$

$\vec{\Delta} = \frac{1}{2}(\overrightarrow{BC} \times \overrightarrow{BA})$

Q Let 'o' be the interior point of $\triangle ABC$ such that $2\vec{OA} + 5\vec{OB} + 10\vec{OC} = 0$. If the ratio of area of $\triangle ABC$ to area of $\triangle AOC$ is λ (where O is origin) then find λ?

Sol^n



$$\boxed{2\vec{a} + 5\vec{b} + 10\vec{c} = \vec{0}} \quad -①-$$

$$\frac{\text{area}(\triangle ABC)}{\text{area}(\triangle AOC)} = \lambda$$

$$\text{area}(\triangle ABC) = \frac{1}{2}\left|\vec{a}\times\vec{b} + \overbrace{\vec{b}\times\vec{c}} + \vec{c}\times\vec{a}\right|$$

$$\text{area}(\triangle AOC) = \frac{1}{2}\left|\underbrace{\vec{c}\times\vec{a}}\right|.$$

$$2\vec{a} + 5\vec{b} + 10\vec{c} = \vec{0} \quad -①-$$

Cross with $\vec{c}$:-

$$2(\vec{a}\times\vec{c}) + 5(\vec{b}\times\vec{c}) + (0) = 0.$$

$$\vec{a}\times\vec{c} = -\frac{5}{2}(\vec{b}\times\vec{c})$$

$$\vec{c}\times\vec{a} = \frac{5}{2}(\vec{b}\times\vec{c})$$

① Cross with $\vec{a}$:

$$2(\cancel{\vec{a}\times\vec{a}})^{0} + 5(\vec{a}\times\vec{b}) + 10(\vec{a}\times\vec{c}) = \vec{0}$$

$$5(\vec{a}\times\vec{b}) = -10(\vec{a}\times\vec{c})$$

$$\boxed{\vec{a}\times\vec{b} = 2(\vec{c}\times\vec{a}).}$$

$$\frac{\text{area}(\Delta ABC)}{\text{area}(\Delta AOC)} = \frac{\cancel{\frac{1}{2}}\left|2(\vec{c}\times\vec{a}) + \frac{2}{5}(\vec{c}\times\vec{a}) + \vec{c}\times\vec{a}\right|}{\cancel{\frac{1}{2}}\left|\vec{c}\times\vec{a}\right|}$$

$$= \left(2+1+\frac{2}{5}\right)\frac{\cancel{|\vec{c}\times\vec{a}|}}{\cancel{|\vec{c}\times\vec{a}|}}$$

$$\boxed{\lambda = \frac{17}{5}}$$

Note :

(i) If 3 points with position vectors $\vec{a}, \vec{b}$ and $\vec{c}$ are collinear then $\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}=\vec{0}$.

(ii) Unit vector perpendicular to the plane of the ΔABC when $\vec{a}, \vec{b}, \vec{c}$ are the p.v. of its angular point

$\hat{n} = \pm\left(\dfrac{\vec{a}\times\vec{b}+\vec{b}\times\vec{c}+\vec{c}\times\vec{a}}{2\Delta}\right)$, where $\vec{a}, \vec{b}, \vec{c}$ are the position vectors of the angular points of the triangle ABC.

Q For a non zero vector $\vec{a}$, if $\vec{a}.\vec{b} = \vec{a}.\vec{c}$ and $\vec{a}\times\vec{b} = \vec{a}\times\vec{c}$. Prove that $\vec{b}=\vec{c}$.

Sol^n

$\vec{a}.\vec{b} - \vec{a}.\vec{c} = 0$

$\vec{a}.(\vec{b}-\vec{c}) = 0$ — ① —

$\vec{a}\times\vec{b} - \vec{a}\times\vec{c} = \vec{0}$

$\vec{a}\times(\vec{b}-\vec{c}) = \vec{0}$ — ② —

from ① & ② $\quad \vec{a}=\vec{0}$ (✗✗) OR $\vec{b}-\vec{c}=\vec{0}$

$\Downarrow$

$\vec{b}=\vec{c}$

(H.P.)

Q

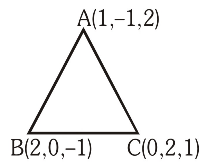
For the given figure find :

(a) A vector of magnitude $\sqrt{6}$ perpendicular to the plane ABC

(b) Area of triangle ABC

(c) Length ot the altitude from A (AB = AC = $\sqrt{11}$)

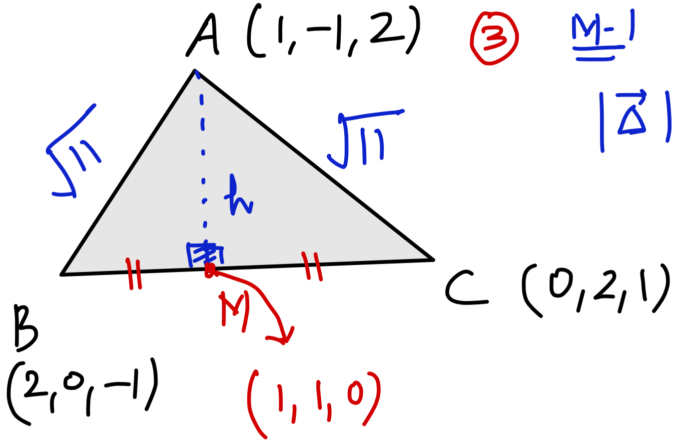
(d) Equation of the plane ABC



Sol^n $\quad \vec{n} = \vec{BC}\times\vec{BA} = (-2\hat{i}+2\hat{j}+2\hat{k})\times(-\hat{i}-\hat{j}+3\hat{k})$

ⓐ $\pm\sqrt{6}\,\hat{n}$ = Req. vector.

ⓑ $\frac{1}{2}|\vec{n}| = \Delta$. OR $\Delta = \frac{1}{2}|\vec{BC}\times\vec{BA}|$



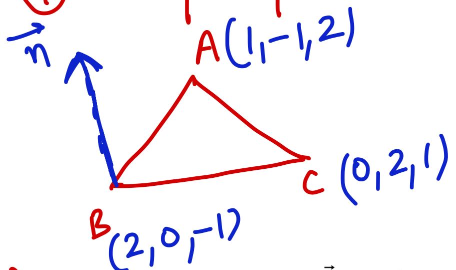
③ M-1

$|\vec{\Delta}| = \frac{1}{2}(\vec{BC})(h)$

M-2

M is mid pt of BC as ΔABC is isosceles Δ

④ Eqn of plane ABC :



$P: (\vec{r} - (\hat{i} - \hat{j} + 2\hat{k})) \cdot \vec{n} = 0$

$\vec{n} = \vec{BC} \times \vec{BA}$.

Q Find the unknown vector $\vec{R}$ satisfying $\vec{R} \times \vec{B} = \vec{C} \times \vec{B}$ and $\vec{R}.\vec{A} = 0$, where $\vec{A} = 2\hat{i} + \hat{k}$; $\vec{B} = \hat{i} + \hat{j} + \hat{k}$ and $\vec{C} = 4\hat{i} - 3\hat{j} + 7\hat{k}$

Soln $\vec{R} \times \vec{B} - \vec{C} \times \vec{B} = \vec{0}$ & $\boxed{\vec{R}.\vec{A} = 0}$

$\Downarrow$

$(\vec{R} - \vec{C}) \times \vec{B} = \vec{0}$ $\Rightarrow$ $\vec{R} - \vec{C} = \vec{0}$ OR $\vec{B} = \vec{0}$ ~~XX~~

OR $\vec{R} - \vec{C}$ is Collinear with $\vec{B}$.

XX

$\vec{R} = 4\hat{i} - 3\hat{j} + 7\hat{k}$.

$\vec{R}.\vec{A} = 8 + 7 \neq 0$. $\therefore$ $\vec{R} - \vec{C} \neq \vec{0}$

$\therefore \boxed{\vec{R} - \vec{C} = \lambda \vec{B}}$; $\lambda \in$ scalar

$$\vec{R} = \vec{C} + \lambda \vec{B}$$

$$\vec{R} = (4\hat{i} - 3\hat{j} + 7\hat{k}) + \lambda(\hat{i} + \hat{j} + \hat{k})$$

$$\vec{A} = 2\hat{i} + \hat{k}$$

$$\boxed{\vec{R} = (4+\lambda)\hat{i} + (\lambda - 3)\hat{j} + (\lambda + 7)\hat{k}}$$

$$\vec{R} \cdot \vec{A} = 0$$

$$2(4+\lambda) + 0 + (\lambda + 7) = 0.$$

$$15 + 3\lambda = 0 \Rightarrow \boxed{\lambda = -5}$$

Q Let $\vec{a}=\hat{i}+4\hat{j}+2\hat{k}$; $\vec{b}=3\hat{i}-2\hat{j}+7\hat{k}$ and $\vec{c}=2\hat{i}-\hat{j}+4\hat{k}$. Find the vector $\vec{d}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$ and satisfy $\vec{c}\cdot\vec{d}=15$.

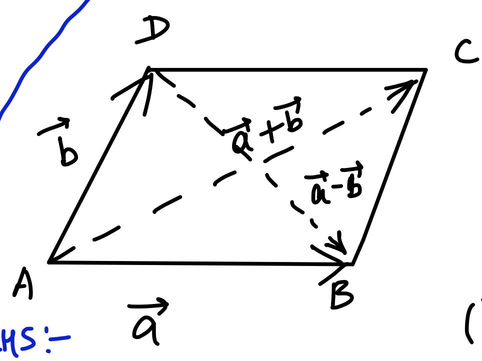
Sol^n

$$\boxed{\vec{d} = \pm\lambda(\vec{a}\times\vec{b})} **$$

$$\vec{d}\cdot\vec{c} = 15$$

Q Prove the identity $(\vec{a}-\vec{b})\times(\vec{a}+\vec{b})=2(\vec{a}\times\vec{b})$ and give its geometrical interpretation.

Sol^n



$$\vec{AC} = \vec{a}+\vec{b}$$
$$\vec{DB} = \vec{a}-\vec{b}$$

$2(\vec{a}\times\vec{b})$ = twice Vector area of ||gm ABCD

LHS:-

$$\cancel{\vec{a}\times\vec{a}} + \vec{a}\times\vec{b} - \vec{b}\times\vec{a} - \cancel{\vec{b}\times\vec{b}}$$

$$2(\vec{a}\times\vec{b}) \quad (\text{H.P})$$

$(\vec{a}-\vec{b})\times(\vec{a}+\vec{b})$ = Vector area of ||gm whose adjacent sides are diagonals of ||gm ABCD.

Q Let $\vec{OA} = \vec{a}$, $\vec{OB} = 10\vec{a} + 2\vec{b}$ and $\vec{OC} = \vec{b}$ where O, A & C are non–collinear points. Let 'p' denote the area of the quadrilateral OABC, and let 'q' denote the area of the parallelogram with OA and OC as adjacent sides. If p = kq. Find k.

HW

SHORTEST DISTANCE BETWEEN 2 SKEW LINES :

Note that

(i) 2 lines in a plane if not || must intersect and 2 lines in a plane if not intersecting must be parallel. Conversely 2 intersecting or parallel lines must be coplanar.

(ii) In space, however we come across situation when two lines neither intersect nor ||. Two such lines in space are known as **skew lines** or **non coplanar lines**. S.D. between two such skew lines is the segment intercepted between the two lines which is perpendicular to both lines.

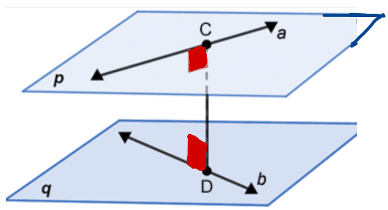
Method 1: Two ways to determine the S.D.

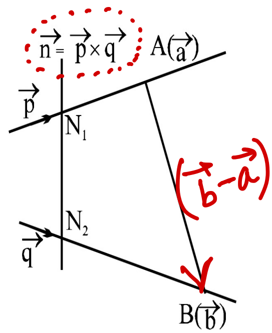
$L_1 : \vec{r} = \vec{a} + \lambda\vec{p}$

$L_2 : \vec{r} = \vec{b} + \mu\vec{q}$

$\vec{n} = \vec{p} \times \vec{q}$

$\overrightarrow{AB} = (\vec{b} - \vec{a})$





$$\text{S.D.} = \left|\text{Projection of } \overrightarrow{AB} \text{ on } \vec{n}\right| = \left|\frac{\overrightarrow{AB}\cdot\vec{n}}{|\vec{n}|}\right| = \left|\frac{(\vec{b}-\vec{a})\cdot(\vec{p}\times\vec{q})}{|\vec{p}\times\vec{q}|}\right|$$

If S.D. = 0 $\Rightarrow$ lines are intersecting and hence coplanar.

Method 2 : p.v. of $N_1 = \vec{a} + \lambda\vec{p}$; p.v. of $N_2 = \vec{b} + \mu\vec{q}$

$\overrightarrow{N_1N_2} = (\vec{b} - \vec{a}) + (\mu\vec{q} - \lambda\vec{p})$

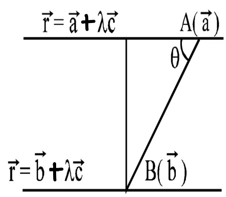
now $\overrightarrow{N_1N_2}\cdot\vec{p} = 0$ and $\overrightarrow{N_1N_2}\cdot\vec{q} = 0$ (two linear equations to get the unique values of λ and μ.)

One p.v's of N_1 and N_2 are known we can also determine the equation to the line of shortest distance and the S.D.

HW

eg $\vec{r} = \hat{i} + 2\hat{j} + \hat{k} + \lambda(\hat{i} - \hat{j} + \hat{k})$; $\vec{r} = 2\hat{i} - \hat{j} - \hat{k} + \mu(2\hat{i} + \hat{j} + 2\hat{k})$. Find SD.

Shortest Distance between two parallel lines



$$\boxed{d = |\vec{a} - \vec{b}| \sin\theta \Rightarrow \left| \frac{(\vec{a} - \vec{b}) \times \vec{c}}{|\vec{c}|} \right|}$$

HW

Q Find the distance between the lines L_1 and L_2 given by

$$\vec{r} = (\hat{i} + 2\hat{j} - 4\hat{k}) + \lambda(2\hat{i} + 3\hat{j} + 6\hat{k})$$

and $\vec{r} = (3\hat{i} + 3\hat{j} - 5\hat{k}) + \mu(2\hat{i} + 3\hat{j} + 6\hat{k})$

PRODUCT OF 3 OR MORE VECTORS :

When 3 vectors are involved with a dot or a cross between them, then 6 different symbols are

✓(i) $(\vec a.\vec b)\vec c$ ~~(ii)~~ $(\vec a.\vec b).\vec c$ ~~(iii)~~ $(\vec a.\vec b)\times\vec c$ ~~(iv)~~ $(\vec a\times\vec b)\vec c$ ✓(v) $\underbrace{(\vec a\times\vec b).\vec c}_{\text{STP} ✓}$ ✓(vi) $\underbrace{(\vec a\times\vec b)\times\vec c}_{\text{VTP} ✓}$

Scalar Triple product (or Box product) :

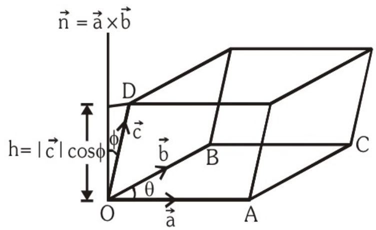
(a) **Definition :** $(\vec a\times\vec b).\vec c = |\vec a|\,|\vec b|\sin\theta\,\hat n.\vec c$

$= |\vec a|\,|\vec b|\,|\vec c|\sin\theta\cos\phi = [\vec a\ \vec b\ \vec c]$

where $\theta = \vec a\ \wedge\ \vec b$; $\phi = \vec n\ \wedge\ \vec c$

but $|\vec a|\,|\vec b|\sin\theta$ = area of $||^{gm}$ OACB and $|\vec c|\cos\phi = h$.

Hence $(\vec a\times\vec b).\vec c$ geometrically describes the volume of the parallelopiped whose three coterminous edges are the vectors $\vec a$, $\vec b$ and $\vec c$.



(b) $(\vec a\times\vec b).\vec c$ is also known as box product and is written $[\vec a\,\vec b\,\vec c]$

(c) If the vector $\vec c$ also line in the plane of $\vec a,\vec b$ then ϕ (angle between $\hat n$ and $\vec c$) is $90°$ and $[\vec a\,\vec b\,\vec c]=0$.

Hence for three vectors, if $[\vec a\,\vec b\,\vec c]=0 \Rightarrow \vec a,\vec b,\vec c$ are coplaner or linearly dependent & conversely.

(d) If $\hat a,\hat b,\hat c$ are unit vectors such that their box product is unity i.e. $[\hat a\ \hat b\ \hat c]=1$

$\Rightarrow$ $\sin\theta\cos\phi = 1$. This is possible only if $\theta = 90°$ and $\phi = 0°$ i.e. $\hat a\ \hat b\ \hat c$ are mutually perpendicular to each other and conversely true e.g. $[\hat i\,\hat j\,\hat k]=1$

General expression for $[\vec a\ \vec b\ \vec c]$: When $\vec a,\vec b,\vec c$ are expressed in terms of $\hat i,\hat j,\hat k$

$$[\vec a\ \vec b\ \vec c] = (\vec a\times\vec b).\vec c = \begin{vmatrix}\hat i & \hat j & \hat k\\ a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\end{vmatrix} . (c_1\hat i+c_2\hat j+c_3\hat k) = \begin{vmatrix}a_1 & a_2 & a_3\\ b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\end{vmatrix}$$

Q If $\vec{u}=2\hat{i}-\hat{j}+\hat{k}$; $\vec{v}=\hat{i}+\hat{j}+\hat{k}$ and $\vec{w}$ is a unit vector then the maximum value of $[\vec{u}\ \vec{v}\ \vec{w}]$ is

Sol^n

$$[\vec{u}\ \vec{v}\ \vec{w}] = (\vec{u}\times\vec{v}).(\vec{w})$$

$$|\vec{u}\times\vec{w}| = \sqrt{14}$$

$$[\vec{u}\ \vec{v}\ \vec{w}]_{max} = (|\vec{u}||\vec{v}|\sin\theta)(1)\underbrace{\cos\phi}_{1}$$

$$= \sqrt{14}\times 1\times 1$$

$$= \sqrt{14}$$ Ans

Properties of STP :

(i) Scalar triple product of three vectors when two of them are collinear / linearly dependent or equal is zero. (two rows identical $\Rightarrow$ determinant is zero).

(ii) If the cyclic order of vector retains then the value of the STP does not change

i.e. $[\vec{a}\,\vec{b}\,\vec{c}] = [\vec{b}\,\vec{c}\,\vec{a}] = [\vec{c}\,\vec{a}\,\vec{b}]$ and if the cyclic order is changed, then the value of STP changes in sign (Prove considering properties of determinants i.e. $[\vec{a}\,\vec{c}\,\vec{b}] = -[\vec{a}\,\vec{b}\,\vec{c}]$

(iii) The position of dot and cross can be interchanged provided the cyclic order of the vectors $\vec{a},\vec{b},\vec{c}$ remains undisturbed.

$$\vec{a}.(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b}).\vec{c}$$

we have $(\vec{a}\times\vec{b}).\vec{c} = (\vec{b}\times\vec{c}).\vec{a} = (\vec{c}\times\vec{a}).\vec{b}$

Also
$$\left.\begin{aligned}(\vec{a}\times\vec{b}).\vec{c} &= \vec{c}.(\vec{a}\times\vec{b})\\ (\vec{b}\times\vec{c}).\vec{a} &= \vec{a}.(\vec{b}\times\vec{c})\\ (\vec{c}\times\vec{a}).\vec{b} &= \vec{b}.(\vec{c}\times\vec{a})\end{aligned}\right]$$ As dot is commutative

(iv) $[\vec{a}+\vec{b}\ \vec{c}\ \vec{d}] = [\vec{a}\,\vec{c}\,\vec{d}] + [\vec{b}\,\vec{c}\,\vec{d}]$

$$[\vec{a}+\vec{b}\quad \vec{c}\quad \vec{d}] = [\vec{a}\ \vec{c}\ \vec{d}] + [\vec{b}\ \vec{c}\ \vec{d}]$$

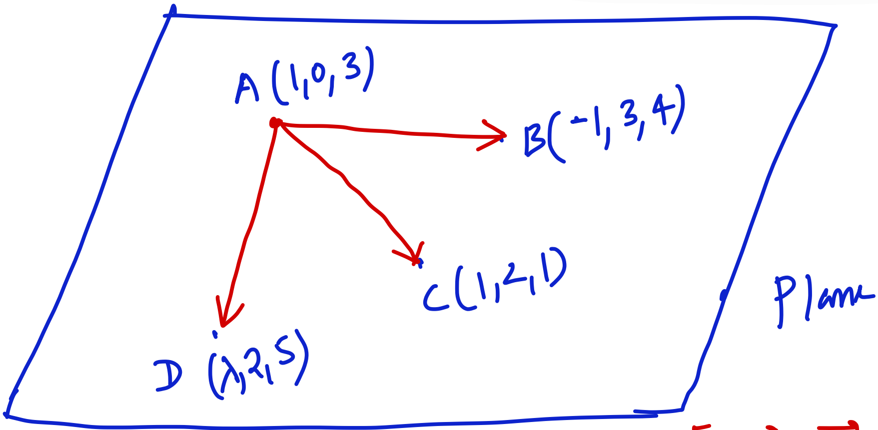
Proof : Let $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and so on

$$\text{Now L.H.S.}=\begin{vmatrix} a_1+b_1 & a_2+b_2 & a_3+b_3\\ c_1 & c_2 & c_3\\ d_1 & d_2 & d_3\end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3\\ c_1 & c_2 & c_3\\ d_1 & d_2 & d_3\end{vmatrix} + \begin{vmatrix} b_1 & b_2 & b_3\\ c_1 & c_2 & c_3\\ d_1 & d_2 & d_3\end{vmatrix} = [\vec{a}\,\vec{c}\,\vec{d}]+[\vec{b}\,\vec{c}\,\vec{d}]$$

(v) For right handed system $[\vec{a}\,\vec{b}\,\vec{c}]>0$ and for left handed system $[\vec{a}\,\vec{b}\,\vec{c}]<0$ — where $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar

Q Find the value of λ for which three points with p.v.'s A(1,0,3) ; B(–1,3,4) ; C(1,2,1) and D (λ, 2,5) are in the same plane.

Sol



$$[\overrightarrow{AB}\ \overrightarrow{AC}\ \overrightarrow{AD}]=0$$

Q Find the value of p for which the vectors $(p+1)\hat{i}-3\hat{j}+p\hat{k}$; $p\hat{i}+(p+1)\hat{j}-3\hat{k}$ and $-3\hat{i}+p\hat{j}+(p+1)\hat{k}$ are linearly dependent/coplanar .

$$\begin{vmatrix} p+1 & -3 & p \\ p & p+1 & -3 \\ -3 & p & p+1 \end{vmatrix}=0.$$

Note :-

$[\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}] = 2[\vec{a}\,\vec{b}\,\vec{c}]$. This identity can be geometrically interpreted as: (Volume of a cuboid whose three coterminous edges are the face diagonals of the cuboid is twice the volume of the cuboid, whose three coterminous edges are the vectors $\vec{a}, \vec{b}, \vec{c}$). This is also conclusive that if $\vec{a}, \vec{b}, \vec{c}$ are coplanar then $\vec{a}+\vec{b}$, $\vec{b}+\vec{c}$, $\vec{c}+\vec{a}$ are also coplanar.

Rem

$$\boxed{[\vec{a}+\vec{b} \quad \vec{b}+\vec{c} \quad \vec{c}+\vec{a}] = 2[\vec{a}\ \vec{b}\ \vec{c}]}$$

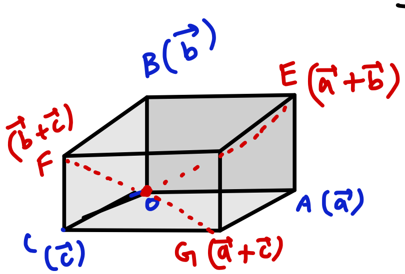
Proof: LHS: $((\vec{a}+\vec{b}) \times (\vec{b}+\vec{c})) \cdot (\vec{c}+\vec{a})$

$$(\vec{a}\times\vec{b} + \vec{a}\times\vec{c} + \cancel{\vec{b}\times\vec{b}}^{\,0} + \vec{b}\times\vec{c}) \cdot (\vec{c}+\vec{a})$$

$$\underbrace{(\vec{a}\times\vec{b})\cdot\vec{c}}_{[\vec{a}\ \vec{b}\ \vec{c}]} + \underbrace{(\vec{a}\times\vec{c})\cdot\vec{c}}_{0} + (\vec{b}\times\vec{c})\cdot\vec{a}$$

$$[\vec{a}\ \vec{b}\ \vec{c}] + \underbrace{[\vec{a}\ \vec{c}\ \vec{c}]}_{\Downarrow\ 0} + [\vec{b}\ \vec{c}\ \vec{a}]$$

$$= 2[\vec{a}\ \vec{b}\ \vec{c}]$$



Rem
* $[\vec{a}-\vec{b}\ \ \vec{b}-\vec{c}\ \ \vec{c}-\vec{a}]$ is always zero.

$\Rightarrow$ $\vec{a}-\vec{b}$, $\vec{b}-\vec{c}$, $\vec{c}-\vec{a}$ are always coplanar whether $\vec{a}, \vec{b}, \vec{c}$ are coplanar or NOT.

Rem

$$\boxed{\vec{a}\times\vec{b} = [\hat{i}\ \vec{a}\ \vec{b}]\hat{i} + [\hat{j}\ \vec{a}\ \vec{b}]\hat{j} + [\hat{k}\ \vec{a}\ \vec{b}]\hat{k}}$$

$$\vec{r} = (\vec{r}\cdot\hat{i})\hat{i} + (\vec{r}\cdot\hat{j})\hat{j} + (\vec{r}\cdot\hat{k})\hat{k}$$

$$\vec{r} = \vec{a}\times\vec{b}$$

(H.P)

Rem

$$[\vec{l}\ \vec{m}\ \vec{n}]\,[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} \vec{l}.\vec{a} & \vec{l}.\vec{b} & \vec{l}.\vec{c} \\ \vec{m}.\vec{a} & \vec{m}.\vec{b} & \vec{m}.\vec{c} \\ \vec{n}.\vec{a} & \vec{n}.\vec{b} & \vec{n}.\vec{c} \end{vmatrix}$$, where $\vec{l}, \vec{m}, \vec{n}$ & $\vec{a}, \vec{b}, \vec{c}$ are non coplanar vectors.

v. imp

$$\boxed{[\vec{a}\ \vec{b}\ \vec{c}]^2 = \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} & \vec{a}.\vec{c} \\ \vec{b}.\vec{a} & \vec{b}.\vec{b} & \vec{b}.\vec{c} \\ \vec{c}.\vec{a} & \vec{c}.\vec{b} & \vec{c}.\vec{c} \end{vmatrix}}$$

$$\begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix} \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Row by Column

Q

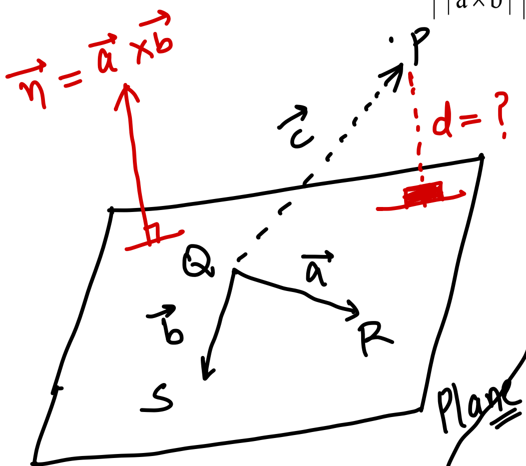
$$[\vec{a}\,\vec{b}\,\vec{c}]\;(\vec{p}\times\vec{q}) = \begin{vmatrix} \vec{p}.\vec{a} & \vec{q}.\vec{a} & \vec{a} \\ \vec{p}.\vec{b} & \vec{q}.\vec{b} & \vec{b} \\ \vec{p}.\vec{c} & \vec{q}.\vec{c} & \vec{c} \end{vmatrix}$$

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \begin{vmatrix} \hat{i} & p_1 & q_1 \\ \hat{j} & p_2 & q_2 \\ \hat{k} & p_3 & q_3 \end{vmatrix}$$

Row by Column

*

Q. Let P be a point not on the plane that passes through Q, R and S. Show that the distance d from P to the plane is $d = \left|\dfrac{[\vec{a}\,\vec{b}\,\vec{c}]}{|\vec{a}\times\vec{b}|}\right|$ where $\vec{a}=\overrightarrow{QR}$; $\vec{b}=\overrightarrow{QS}$ and $\vec{c}=\overrightarrow{QP}$



$$d = \left|\text{projection of } \vec{c} \text{ on } \vec{n}\right|$$

$$d = \left|\vec{c}\cdot\hat{n}\right|$$

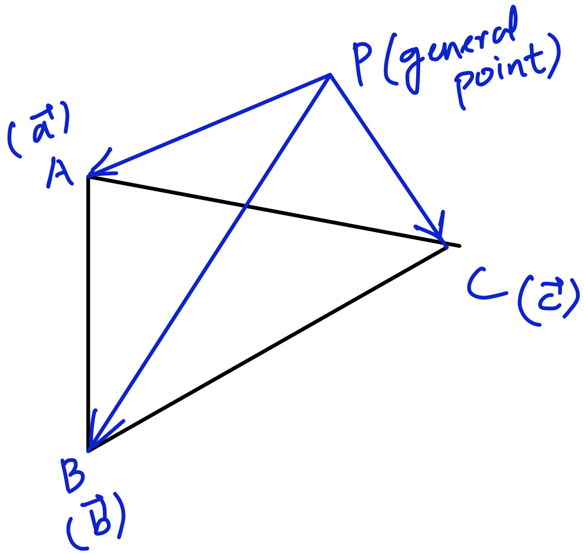
$$d = \left|\frac{\vec{c}\cdot(\vec{a}\times\vec{b})}{|\vec{a}\times\vec{b}|}\right|$$

$$= \left|\frac{[\vec{c}\;\vec{a}\;\vec{b}]}{|\vec{a}\times\vec{b}|}\right| \quad (HP)$$

Rem

v. imp

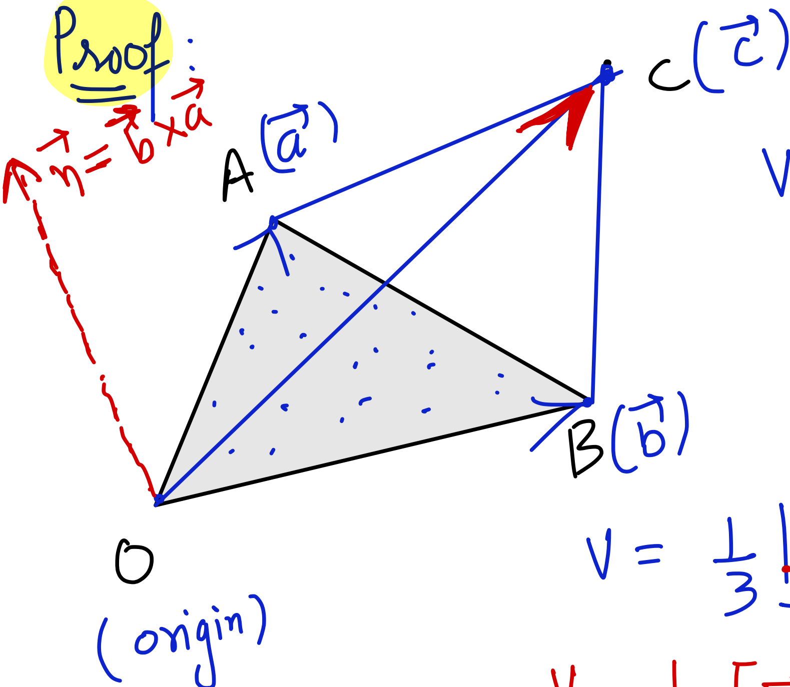
$$\boxed{\text{Volume of a tetrahedron OABC} = \vec{V} = \frac{1}{6}[\vec{a}\,\vec{b}\,\vec{c}] = \frac{1}{6}\left[\underbrace{\vec{OA}\,,\ \vec{OB}\,,\ \vec{OC}}\right] \text{ where O is the origin.}}$$



$$\boxed{V = \frac{1}{6}\left[\vec{PA}\ \vec{PB}\ \vec{PC}\right]}$$

Proof :

$\vec{n} = \vec{b} \times \vec{a}$



Height $= |$proj of $\vec{c}$ on $\vec{n}|$

Vol of tetrahedron

$= \frac{1}{3}$ (Ar of Base) (Height)

$$V = \frac{1}{3}\,\frac{\cancel{|\vec{a}\times\vec{b}|}}{2} \times \frac{\vec{c}\cdot(\vec{a}\times\vec{b})}{\cancel{|\vec{a}\times\vec{b}|}}$$

$$V = \frac{1}{6}\left[\vec{c}\ \vec{a}\ \vec{b}\right]$$

Q Let $\vec{a} = \hat{i} + 4\hat{j} + 2\hat{k}$; $\vec{b} = 3\hat{i} - 2\hat{j} + 7\hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}$. Find the vector $\vec{d}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$ and satisfy $\vec{c} \cdot \vec{d} = 15$.

Sol^n

$$\boxed{\vec{d} = \pm \lambda (\vec{a} \times \vec{b})} \; **$$

$$\vec{d} \cdot \vec{c} = 15$$

$$d = \pm \frac{5}{3} (32\hat{i} - \hat{j} - 14\hat{k})$$

Ans

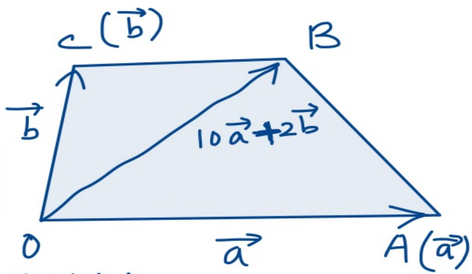
Q HW Let $\vec{OA}=\vec{a}, \vec{OB}=10\vec{a}+2\vec{b}$ and $\vec{OC}=\vec{b}$ where O, A & C are non–collinear points. Let 'p' denote the area of the quadrilateral OABC, and let 'q' denote the area of the parallelogram with OA and OC as adjacent sides. If p = kq. Find k.

$\boxed{K=6}$ Ans.

$$q = |\vec{a}\times\vec{b}|$$

$$p = \frac{1}{2}|\vec{OB}\times\vec{AC}|$$

$$= \frac{1}{2}|(10\vec{a}+2\vec{b})\times(\vec{b}-\vec{a})|$$



$$p = \frac{1}{2}|10\vec{a}\times\vec{b} + 2\vec{a}\times\vec{b}|$$ (origin)

$$p = 6|\vec{a}\times\vec{b}|$$

HW eg $\vec{r}=\hat{i}+2\hat{j}+\hat{k}+\lambda(\hat{i}-\hat{j}+\hat{k})$; $\vec{r}=2\hat{i}-\hat{j}-\hat{k}+\mu(2\hat{i}+\hat{j}+2\hat{k})$. Find SD.

Ans: $\frac{3}{\sqrt{2}}$

HW Q Find the distance between the lines L_1 and L_2 given by

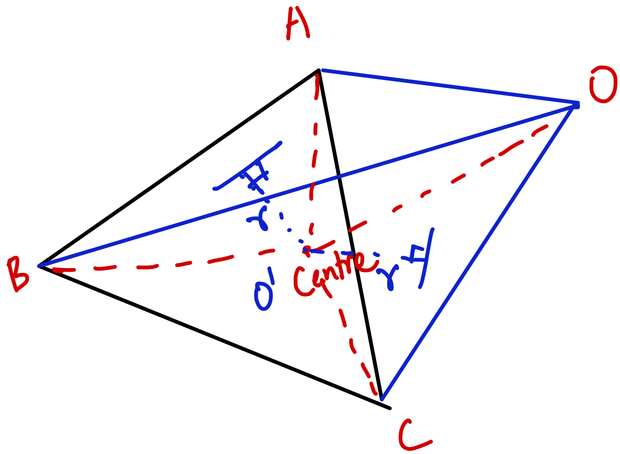
$$\vec{r}=(\hat{i}+2\hat{j}-4\hat{k})+\lambda(2\hat{i}+3\hat{j}+6\hat{k})$$

and $$\vec{r}=(3\hat{i}+3\hat{j}-5\hat{k})+\mu(2\hat{i}+3\hat{j}+6\hat{k})$$

Ans: $\frac{\sqrt{293}}{7}$

Very Important Note : If S_1, S_2, S_3 and S_4 are the areas of the four triangular faces of the tetrahedron with volume V. If r is the radius of the sphere touching the four faces then $V = \frac{1}{3}(S_1 + S_2 + S_3 + S_4)r$

$$\text{Vol} = \frac{1}{3}(\text{ar of base}) \times \text{Height}$$



$$V = V_1 + V_2 + V_3 + V_4$$

$$= \frac{1}{3}(S_1)\cdot r + \frac{1}{3}(S_2)r + \frac{1}{3}(S_3)r + \frac{1}{3}(S_4)r$$

$$\boxed{V = \frac{1}{3}r(S_1 + S_2 + S_3 + S_4)}$$

Note :-

To express scalar triple product of three vectors in terms of any three non coplanar vectors $\vec{l}$, $\vec{m}$ and $\vec{n}$

Let $\vec{a} = a_1\vec{l} + a_2\vec{m} + a_3\vec{n}$; $\vec{b} = b_1\vec{l} + b_2\vec{m} + b_3\vec{n}$; $\vec{c} = c_1\vec{l} + c_2\vec{m} + c_3\vec{n}$

then $$\boxed{[\vec{a}\,\vec{b}\,\vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} [\vec{l}\ \vec{m}\ \vec{n}]}$$

Rem

Q If $\vec{a} = \vec{l} - \vec{m} + \vec{n}$, $\vec{b} = 2\vec{l} + \vec{m} - \vec{n}$, $\vec{c} = \vec{l} + \vec{m} + 2\vec{n}$ and $[\vec{l}\ \vec{m}\ \vec{n}] = 4$ then find $[\vec{a}\ \vec{b}\ \vec{c}] = ?$

Sol^n

$$[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 1 & 1 & 2 \end{vmatrix} [\vec{l}\ \vec{m}\ \vec{n}]$$

Q If the value of $[\vec{a}+2\vec{b}+3\vec{c}\ \ \vec{b}+2\vec{c}+3\vec{a}\ \ \vec{c}+2\vec{a}+3\vec{b}] = k[\vec{a}\ \vec{b}\ \vec{c}]$, then k equals -

(A) 6 (B) 9 (C) 12 (D) 18 ✓

$$\begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{vmatrix} \cancel{[\vec{a}\ \vec{b}\ \vec{c}]} = k\, \cancel{[\vec{a}\ \vec{b}\ \vec{c}]}$$

$k = 18$

Q If $[\bar{a}+2\bar{b}+3\bar{c}\quad 2\bar{a}+3\bar{b}+\bar{c}\quad 3\bar{a}+\bar{b}+2\bar{c}]=-18$, where $\bar{a},\bar{b},\bar{c}$ are 3 non-coplaner vectors, then

$$\begin{vmatrix} \bar{a}.\bar{a} & \bar{a}.\bar{b} & \bar{a}.\bar{c} \\ \bar{b}.\bar{a} & \bar{b}.\bar{b} & \bar{b}.\bar{c} \\ \bar{c}.\bar{a} & \bar{c}.\bar{a} & \bar{c}.\bar{c} \end{vmatrix} \text{ is equal}$$

$$[\vec{a}\ \vec{b}\ \vec{c}]^2$$

$$\boxed{\text{Ans} = 1}$$

$$\begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{vmatrix} [\vec{a}\ \vec{b}\ \vec{c}] = -18$$

Rem

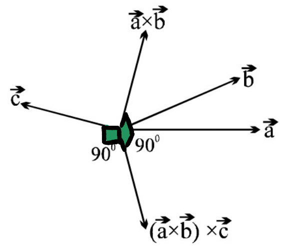
Vector Triple Product

Definition: $(\vec{a}\times\vec{b})\times\vec{c}$ is a vector which is coplanar with $\vec{a}$ and $\vec{b}$ and perpendicular to $\vec{c}$.

Hence $(\vec{a}\times\vec{b})\times\vec{c} = x\vec{a}+y\vec{b}$ ....(1) [linear combination of $\vec{a}$ and $\vec{b}$)

$\vec{c}\cdot(\vec{a}\times\vec{b})\times\vec{c} = x(\vec{a}.\vec{c})+y(\vec{b}.\vec{c})$

$0 = x(\vec{a}.\vec{c})+y(\vec{b}.\vec{c})$ ....(2)



$\therefore \dfrac{x}{\vec{b}.\vec{c}} = -\dfrac{y}{\vec{a}.\vec{c}} = \lambda$ (say)

$\therefore x = \lambda(\vec{b}.\vec{c})$ and $y = -\lambda\ (\vec{a}.\vec{c})$

Substituting the values of x and y in $(\vec{a}\times\vec{b})\times\vec{c} = \lambda(\vec{b}.\vec{c})\vec{a}-\lambda(\vec{a}.\vec{c})\vec{b}$

This is an identity and must be true for all values of $\vec{a},\vec{b},\vec{c}$

Put = $\vec{a}=\hat{i}$; $\vec{b}=\hat{j}$ and $\vec{c}=\hat{i}$

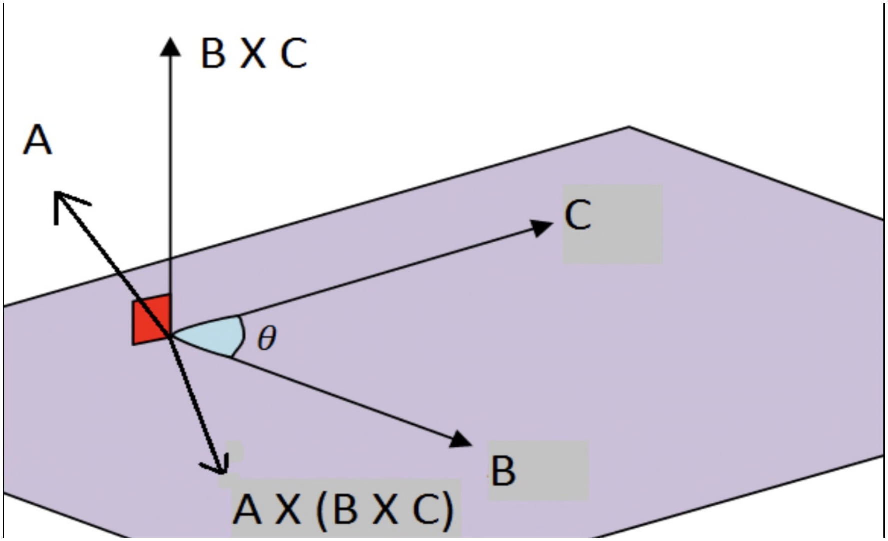
$(\hat{i}\times\hat{j})\times\hat{i} = \lambda(\hat{j}.\hat{i})\hat{i}-\lambda(\hat{i}.\hat{i})\hat{j}$

$\hat{j} = -\lambda\hat{j} \quad\Rightarrow\quad \lambda = -1$

Rem

$$\boxed{(\vec{a}\times\vec{b})\times\vec{c} = (\vec{a}.\vec{c})\vec{b}-(\vec{b}.\vec{c})\vec{a}}$$

durr-durr pass-pass



Note : Unit vector coplanar with $\vec{a}$ and $\vec{b}$ and perpendicular to $\vec{c}$ is $\pm \dfrac{(\vec{a}\times\vec{b})\times\vec{c}}{|(\vec{a}\times\vec{b})\times\vec{c}|}$

|||ly unit vector coplanar with $\vec{a}$ and $\vec{b}$ perpendicular to $\vec{a}$ is $\pm \dfrac{(\vec{a}\times\vec{b})\times\vec{a}}{|(\vec{a}\times\vec{b})\times\vec{a}|}$

Note:-

$$[\vec{a}\times\vec{b} \quad \vec{b}\times\vec{c} \quad \vec{c}\times\vec{a}] = [\vec{a}\,\vec{b}\,\vec{c}]^2 = \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} & \vec{a}.\vec{c} \\ \vec{b}.\vec{a} & \vec{b}.\vec{b} & \vec{b}.\vec{c} \\ \vec{c}.\vec{a} & \vec{c}.\vec{b} & \vec{c}.\vec{c} \end{vmatrix}$$

Note that if $\vec{a},\vec{b},\vec{c}$ are non coplanar vectors then $\vec{a}\times\vec{b}$, $\vec{b}\times\vec{c}$ and $\vec{c}\times\vec{a}$ will also be non coplanar vectors.

Proof

$$[\underbrace{\vec{a}\times\vec{b}}_{\vec{V}_1} \quad \underbrace{\vec{b}\times\vec{c}}_{\vec{V}_2} \quad \underbrace{\vec{c}\times\vec{a}}_{\vec{V}_3}]$$

$$\left(\underbrace{(\vec{a}\times\vec{b})}_{\vec{u}}\times(\vec{b}\times\vec{c})\right).(\vec{c}\times\vec{a}) \qquad \boxed{\vec{u}=\vec{a}\times\vec{b}}$$

$$(\vec{u}\times(\vec{b}\times\vec{c})).(\vec{c}\times\vec{a}) = \left((\vec{u}.\vec{c})\vec{b} - (\overbrace{\cancel{\vec{u}.\vec{b}}}^{0})\vec{c}\right).(\vec{c}\times\vec{a})$$

$$= [\vec{a}\,\vec{b}\,\vec{c}](\underbrace{\vec{b}.(\vec{c}\times\vec{a})}) - 0.$$

Q If $\hat{a}\times(\hat{b}\times\hat{c}) = \frac{1}{2}\hat{b}$ where $\hat{b}$ and $\hat{c}$ are non collinear then find the angle between $\hat{a}$ and $\hat{b}$; between $\hat{a}$ and $\hat{c}$.

Sol^n

$$(\hat{a}.\hat{c})\hat{b} - (\hat{a}.\hat{b})\hat{c} = \frac{1}{2}\hat{b}$$

$$\left(\hat{a}.\hat{c} - \frac{1}{2}\right)\hat{b} = (\hat{a}.\hat{b})\hat{c}$$ Since $\hat{b}$ & $\hat{c}$ are non-collinear vectors

$$\hat{a}.\hat{c} = \frac{1}{2} \quad \text{and} \quad \hat{a}.\hat{b} = 0. \Rightarrow \theta(\hat{a}\wedge\hat{b}) = \frac{\pi}{2}.$$ Ans

$$1.1.\cos\alpha = \frac{1}{2} \Rightarrow \boxed{\alpha = \pi/3}$$ Ans

Q If $\hat{i}\times(\vec{a}\times\hat{i})+\hat{j}\times(\vec{a}\times\hat{j})+\hat{k}\times(\vec{a}\times\hat{k})=\lambda\vec{a}$ then find λ ?

$$\boxed{\lambda=2}$$

$$(\hat{i}\cdot\hat{i})\vec{a}-(\hat{i}\cdot\vec{a})\hat{i}$$
$$(\hat{j}\cdot\hat{j})\vec{a}-(\hat{j}\cdot\vec{a})\hat{j}$$
$$(\hat{k}\cdot\hat{k})\vec{a}-(\hat{k}\cdot\vec{a})\hat{k}$$

$$3\vec{a}-\Big(\underbrace{(\vec{a}\cdot\hat{i})\hat{i}+(\vec{a}\cdot\hat{j})\hat{j}+(\vec{a}\cdot\hat{k})\hat{k}}_{\vec{a}}\Big)=2\vec{a}$$

Q

If $\left.\begin{aligned}\vec{V}_1&=\vec{a}\times(\vec{b}\times\vec{c})\\ \vec{V}_2&=\vec{b}\times(\vec{c}\times\vec{a})\\ \vec{V}_3&=\vec{c}\times(\vec{a}\times\vec{b})\end{aligned}\right]$ then which of the following hold(s) good ?

(1) $\vec{V}_1,\vec{V}_2,\vec{V}_3$ are coplanar

(2) $\vec{V}_1,\vec{V}_2,\vec{V}_3$ can form the sides of a triangle.

(3) $\vec{V}_1+\vec{V}_2+\vec{V}_3$ is a null vectors

(4) $\vec{V}_1,\vec{V}_2,\vec{V}_3$ are linearly dependent

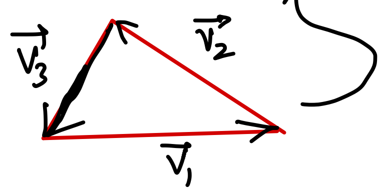
(5) $[\vec{V}_1-\vec{V}_2 \quad \vec{V}_2-\vec{V}_3 \quad \vec{V}_3-\vec{V}_1]=0.$

(6) $[\vec{V}_1+\vec{V}_2 \quad \vec{V}_2+\vec{V}_3 \quad \vec{V}_3+\vec{V}_1]=0.$

(7) $[\vec{V}_1\times\vec{V}_2 \quad \vec{V}_2\times\vec{V}_3 \quad \vec{V}_3\times\vec{V}_1]=0.$

$$\vec{V}_1=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}$$
$$\vec{V}_2=(\vec{b}\cdot\vec{a})\vec{c}-(\vec{b}\cdot\vec{c})\vec{a}$$
$$\vec{V}_3=(\vec{c}\cdot\vec{b})\vec{a}-(\vec{c}\cdot\vec{a})\vec{b}$$

$$\vec{V}_1+\vec{V}_2+\vec{V}_3=\vec{0}$$



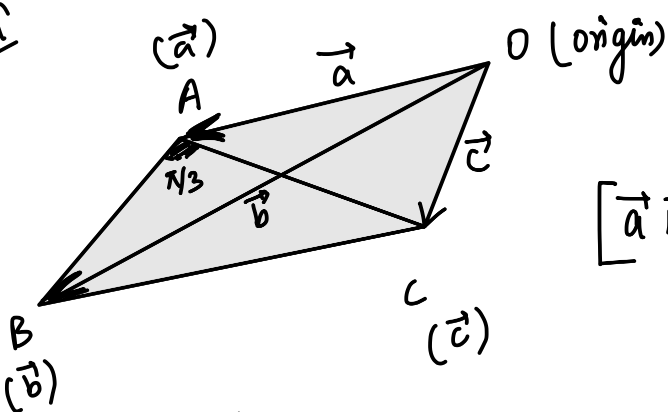
HW

Q If $\vec{a} = \hat{i} + 2\hat{j} + 2\hat{k}$, $\vec{b} = 2\hat{i} - 3\hat{j} + \hat{k}$ and $\vec{c}$ is a vector such that $(\vec{a} \times \vec{b}) \times \vec{c} = \vec{a} \times (\vec{b} \times \vec{c})$ & $|\vec{c}| = 6$. Then the volume of parallelopiped whose co-terminous edges are $\vec{a}, \vec{b}, \vec{b} \times \vec{c}$ will be-

(A) 180 (B) 214 (C) 232 (D) 244

Q** If $\vec{a}, \vec{b}, \vec{c}$ are 3 unit vectors from the vertex of a regular tetrahedron then find value of of absolute value of $[\vec{a}\ \vec{b}\ \vec{c}] = ?$

Sol^n



$$V = \frac{1}{6}[\vec{a}\ \vec{b}\ \vec{c}]$$

$$[\vec{a}\ \vec{b}\ \vec{c}]^2 = \begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c} \\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c} \\ \vec{c}\cdot\vec{a} & \vec{c}\cdot\vec{b} & \vec{c}\cdot\vec{c} \end{vmatrix}$$

$$\vec{a}\cdot\vec{b} = 1.1.\cos\frac{\pi}{3}$$
$$= \frac{1}{2}$$

$$[\vec{a}\ \vec{b}\ \vec{c}]^2 = \underbrace{\begin{vmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 \end{vmatrix}}$$

HW

Q If $\hat{i} \times \left((\vec{a} - \hat{j}) \times \hat{i} \right) + \hat{j} \times \left((\vec{a} - \hat{k}) \times \hat{k} \right) + \hat{k} \times \left((\vec{a} - \hat{i}) \times \hat{k} \right) = 0$ and $\vec{a} = x\hat{i} + y\hat{j} + z\hat{k}$

then value of $8(x^3 - xy + zx) = ?$

Q If three non-zero unequal vectors $\vec{\ell} = a\hat{i} + b\hat{j} + c\hat{k}$, $\vec{m} = b\hat{i} + c\hat{j} + a\hat{k}$ & $\vec{n} = c\hat{i} + a\hat{j} + b\hat{k}$ are such that $\underbrace{\vec{r}.\vec{\ell} = \vec{r}.\vec{m} = \vec{r}.\vec{n} = 0}$ where a, b, c $\in R_0$ & $\vec{r}$ is non zero vector, then value of $\frac{a^2}{bc} + \frac{b^2}{ca} + \frac{c^2}{ab}$?

$R_0 \to \mathbb{R} - \{0\}$

$\vec{l}, \vec{m}$ & $\vec{n}$ must be coplanar.

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0 \Rightarrow a^3 + b^3 + c^3 = 3abc.$$

$$\frac{a^2}{bc} + \frac{b^2}{ca} + \frac{c^2}{ab} = 3$$

Ans

Scalar Product of Four Vector

$$(\vec{a}\times\vec{b}).(\vec{c}\times\vec{d}) = (\vec{a}.\vec{c})(\vec{b}.\vec{d})-(\vec{a}.\vec{d})(\vec{b}.\vec{c}) = \begin{vmatrix} \vec{a}.\vec{c} & \vec{a}.\vec{d} \\ \vec{b}.\vec{c} & \vec{b}.\vec{d} \end{vmatrix}$$

Proof:

$$(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = \vec{u}\cdot(\vec{c}\times\vec{d}) \qquad \vec{u}=\vec{a}\times\vec{b}$$

$$\Downarrow$$

$$(\vec{u}\times\vec{c})\cdot\vec{d} = \left((\vec{a}\times\vec{b})\times\vec{c}\right)\cdot\vec{d}$$

$$= \left((\vec{a}\cdot\vec{c})\,\vec{b} - (\vec{b}\cdot\vec{c})\vec{a}\right)\cdot\vec{d}$$

$$= (\vec{a}\cdot\vec{c})\,(\vec{b}\cdot\vec{d}) - (\vec{b}\cdot\vec{c})(\vec{a}\cdot\vec{d}) \qquad \text{(HP)}$$

Q If $\vec a = \hat i - 2\hat j + 3\hat k$; $\vec b = \hat i + \hat j - 4\hat k$; $\vec c = 4\hat i - 3\hat j + 6\hat k$; $\vec d = 3\hat i - 6\hat j - 5\hat k$ then :

① $\{(\vec a \times \vec b) \times (\vec a \times \vec c)\}.\vec d =$?

② $\vec d.[\vec a \times \{\vec b \times (\vec c \times \vec d)\}] =$?

HW

$$\vec a \cdot \vec d = 3 + 12 - 15 = 0.$$

①

$$\Big(\underbrace{(\vec a \times \vec b)}_{\vec u} \times (\vec a \times \vec c)\Big) \cdot \vec d$$

$$\Big((\vec u \cdot \vec c)\,\vec a - \cancel{(\vec u \cdot \vec a)}^{\,0}\,\vec c\Big) \cdot \vec d$$

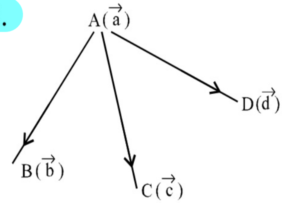
$$\downarrow$$

$$[\vec a\ \vec b\ \vec c]\,\underbrace{(\vec a \cdot \vec d)}_{0} = 0.$$

$\times \qquad 0$

Condition for coplanarity of four points

4 points with pv's $\vec{a},\vec{b},\vec{c},\vec{d}$ are coplanar iff $\exists$ scalars x, y, z and t not all simultaneously zero and satisfying $x\vec{a}+y\vec{b}+z\vec{c}+t\vec{d}=0$ where $x+y+z+t=0$.



Case I : Let the four points A, B, C, D are in the same plane

$\Rightarrow$ the vectors $\vec{b}-\vec{a}, \vec{c}-\vec{a}$ and $\vec{d}-\vec{a}$ are in the same plane.

hence $\vec{d}-\vec{a} = l(\vec{b}-\vec{a})+m(\vec{c}-\vec{a})$

or $\underbrace{(l+m-1)}_{x}\vec{a} - \underbrace{l}_{y}\vec{b} - \underbrace{m}_{z}\vec{c} + \underbrace{1}_{t}\vec{d} = 0 \Rightarrow x\vec{a}+y\vec{b}+z\vec{c}+t\vec{d}=0$ where, $x+y+z+t=0$ and x, y, z, t not all simultaneous zero.

Case II : Let $x\vec{a}+y\vec{b}+z\vec{c}+t\vec{d}=0$ where $x+y+z+t=0$ and not all simultaneously zero

Let $t \neq 0$ $\quad (-y-z-t)\vec{a}+y\vec{b}+z\vec{c}+t\vec{d}=0$ $\quad$ [putting $x=-y-z-t$]

$(\vec{d}-\vec{a})t+y(\vec{b}-\vec{a})+z(\vec{c}-a)=0$

$\Rightarrow$ $\quad \vec{d}-\vec{a}, \vec{b}-\vec{a}$ and $\vec{c}-a$ are coplanar $\Rightarrow$ points A, B, C, D are coplanar

Vector Product of Four Vector

(1) $\vec{V} = (\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d})$

$\vec{u}\times(\vec{c}\times\vec{d}) = [\vec{a}\,\vec{b}\,\vec{d}]\vec{c} - [\vec{a}\,\vec{b}\,\vec{c}]\vec{d}$ ....(1) (where $\vec{u} = \vec{a}\times\vec{b}$)

again $\vec{V} = (\vec{a}\times\vec{b})\times\underbrace{(\vec{c}\times\vec{d})}_{\vec{v}} = (\vec{a}.\vec{v})\vec{b} - (\vec{b}.\vec{v})\vec{a} = [\vec{a}\,\vec{c}\,\vec{d}]\vec{b} - [\vec{b}\,\vec{c}\,\vec{d}]\vec{a}$ ....(2)

from (1) and (2) $[\vec{a}\,\vec{b}\,\vec{d}]\vec{c} - [\vec{a}\,\vec{b}\,\vec{c}]\vec{d} = [\vec{a}\,\vec{c}\,\vec{d}]\vec{b} - [\vec{b}\,\vec{c}\,\vec{d}]\vec{a}$ ....(3)

Note that $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = 0 \Rightarrow$ plane containing the values $\vec{a}$ & $\vec{b}$ and $\vec{c}$ & $\vec{d}$ are parallel.

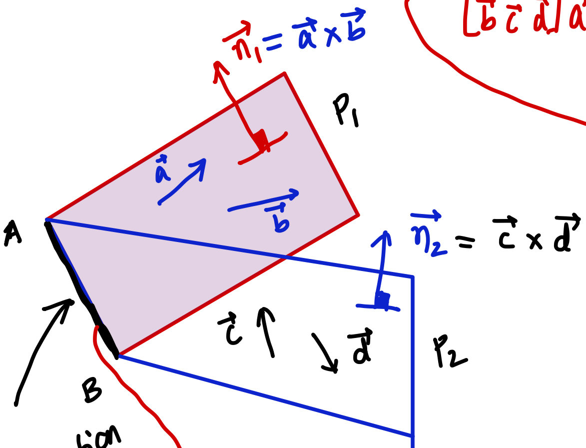
$|||^{ly}$ $(\vec{a}\times\vec{b})\cdot(\vec{c}\times\vec{d}) = 0 \Rightarrow$ the two planes are perpendicular.

(i) equation (3) is suggestive that if $\vec{a},\vec{b},\vec{c},\vec{d}$ are four vectors no 3 three of them are coplanar then each one of them can be expressed as a linear combination of other.

(ii) If $\vec{a},\vec{b},\vec{c},\vec{d}$ are p.v.'s of four points then these four points are in the same plane if

$[\vec{a}\,\vec{b}\,\vec{d}] - [\vec{a}\,\vec{b}\,\vec{c}] = [\vec{a}\,\vec{c}\,\vec{d}] - [\vec{b}\,\vec{c}\,\vec{d}]$

$[\vec{b}\,\vec{c}\,\vec{d}]\vec{a} + [\vec{a}\,\vec{d}\,\vec{c}]\vec{b} + [\vec{a}\,\vec{b}\,\vec{d}]\vec{c} + [\vec{a}\,\vec{c}\,\vec{b}]\vec{d} = \vec{0}$



line of intersection of 2 planes

direction vector of line AB $= \vec{n}_1 \times \vec{n}_2$

Find value of

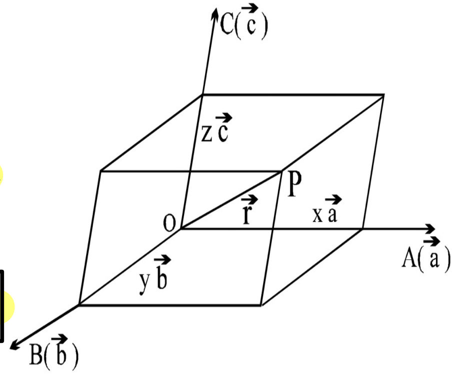
$$\frac{[\vec{a}\,\vec{b}\,\vec{d}] + [\vec{b}\,\vec{c}\,\vec{d}]}{[\vec{a}\,\vec{b}\,\vec{c}] + [\vec{a}\,\vec{c}\,\vec{d}]}$$

Linear combination/linear dependence and independence (Base vectors)

Linear combination

(1) Theorem in plane (already done)

(2) Theorem in space : If $\vec{a}, \vec{b}, \vec{c}$ are 3 non zero non coplanar vectors then any vector $\vec{r}$ can be expressed as a linear combination $\boxed{\vec{r} = x\vec{a} + y\vec{b} + z\vec{c}}$



① Express the non coplanar vectors $\vec{a}, \vec{b}, \vec{c}$ in terms of $\vec{b}\times\vec{c}$, $\vec{c}\times\vec{a}$, $\vec{a}\times\vec{b}$.

HW

② Express $\vec{b}\times\vec{c}$, $\vec{c}\times\vec{a}$, $\vec{a}\times\vec{b}$ in terms of 3 non coplanar vectors $\vec{a}, \vec{b}, \vec{c}$.

Sol^n ① Theorem in space:

$$\vec{a} = x(\vec{b}\times\vec{c}) + y(\vec{c}\times\vec{a}) + z(\vec{a}\times\vec{b}) \quad \text{—①}$$

① dot with $\vec{a}$; ① dot with $\vec{b}$; ① dot with $\vec{c}$

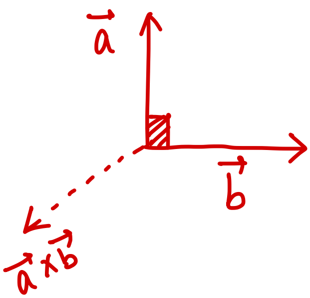
$\vec{a}\cdot\vec{a} = x[\vec{a}\,\vec{b}\,\vec{c}]$ | $\vec{a}\cdot\vec{b} = y[\vec{c}\,\vec{a}\,\vec{b}]$

$x = \dfrac{\vec{a}\cdot\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]}$; | $y = \dfrac{\vec{a}\cdot\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}$ | $z = \dfrac{\vec{a}\cdot\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}$

Q* Given the vector $\vec{a}$ and $\vec{b}$ orthogonal to each other find the vector $\vec{V}$ in terms of $\vec{a}$ and $\vec{b}$ satisfying $\vec{V}.\vec{a}=0$; $\vec{V}.\vec{b}=1$ and $[\vec{V}\ \vec{a}\ \vec{b}]=1$

Sol^n $\vec{a}\cdot\vec{b}=0$

$\vec{V}, \vec{a}, \vec{b}$ are non-coplanar



$$\boxed{\vec{V} = x\vec{a} + y\vec{b} + z(\vec{a}\times\vec{b})} \quad \text{— (1) —}$$

(1) dot with $\vec{a}$

$$\vec{V}\cdot\vec{a} = x\,\vec{a}\cdot\vec{a} + y\,\cancel{\vec{a}\cdot\vec{b}}^{\,0} + z\,\cancel{(\vec{a}\times\vec{b})\cdot\vec{a}}^{\,0}$$

$$\boxed{0 = x}$$

(1) dot with $\vec{b}$

(1) dot with $(\vec{a}\times\vec{b})$.

Real definition of linearly independence

If $\vec{V}_1, \vec{V}_2, \ldots\ldots\ldots \vec{V}_n$ are vectors and $\lambda_1, \lambda_2, \ldots\ldots\ldots \lambda_n$ are scalar and if the linear combination $\lambda_1\vec{V}_1 + \lambda_2\vec{V}_2 + \ldots\ldots + \lambda_n\vec{V}_n = 0$, necessarily implies $\lambda_1 = \lambda_2 = \ldots\ldots\lambda_n = 0$, we say that $\vec{V}_1, \vec{V}_2, \ldots\ldots\ldots \vec{V}_n$ are said to constitutes a linearly independent set of vectors.

Note:

(1) 2 non zero , non collinear vectors are linearly independent.

(2) Three non zero , non coplanar vectors are linearly independent i.e. $[\vec{a}\,\vec{b}\,\vec{c}] \neq 0$.

(3) Four or more vectors in 3D space are always linearly dependent.

Reciprocal system of vectors

(a) If $\vec{a}, \vec{b}, \vec{c}$ and $\vec{a}', \vec{b}', \vec{c}'$ are 2 sets of non coplanar vectors such that $\vec{a}.\vec{a}' = \vec{b}.\vec{b}' = \vec{c}.\vec{c}' = 1$, then $\vec{a}, \vec{b}, \vec{c}$ and $\vec{a}', \vec{b}', \vec{c}'$ are said to be constitute a reciprocal system of vectors.

(b) Reciprocal system of vectors exists only in case of dot product.

(c) It is possible to define $\vec{a}', \vec{b}', \vec{c}'$ in terms of $\vec{a}, \vec{b}, \vec{c}$ as.

Rem //

$$\boxed{\vec{a}' = \frac{\vec{b}\times\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]} \;;\; \vec{b}' = \frac{\vec{c}\times\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]} \;;\; \vec{c}' = \frac{\vec{a}\times\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}} \;;$$

where $\left([\vec{a}\,\vec{b}\,\vec{c}] \neq 0\right)$

Q // Find the set of vector reciprocal to $\underbrace{2\hat{i}+3\hat{j}-\hat{k}}_{\vec{a}}$, $\underbrace{\hat{i}-\hat{j}-2\hat{k}}_{\vec{b}}$ and $\underbrace{-\hat{i}+2\hat{j}+2\hat{k}}_{\vec{c}}$

Ans: $\frac{1}{3}(2\hat{i}+\hat{k})$; $-\frac{1}{3}(8\hat{i}-3\hat{j}+7\hat{k})$

; $\frac{1}{3}(-7\hat{i}+3\hat{j}-5\hat{k})$

Note: (i) $\vec{a}\times\vec{a}'+\vec{b}\times\vec{b}'+\vec{c}\times\vec{c}'=0$ i. e. $\dfrac{\vec{a}\times(\vec{b}\times\vec{c})+\vec{b}\times(\vec{c}\times\vec{a})+\vec{c}\times(\vec{a}\times\vec{b})}{[\vec{a}\,\vec{b}\,\vec{c}]}$

Do yourself

(ii) $(\vec{a}+\vec{b}+\vec{c}).(\vec{a}'+\vec{b}'+\vec{c}')=3$ (as $\vec{a}\cdot\vec{b}'=\vec{a}\cdot\vec{c}'=0$ etc)

(iii) If $[\vec{a}\,\vec{b}\,\vec{c}]=V$ then $[\vec{a}'\,\vec{b}'\,\vec{c}']=\dfrac{1}{V}$ $\Rightarrow$ $[\vec{a}\,\vec{b}\,\vec{c}]\,[\vec{a}'\,\vec{b}'\,\vec{c}']=1$

(iv) $\vec{a}'\times\vec{b}'+\vec{b}'\times\vec{c}'+\vec{c}'\times\vec{a}'=\dfrac{\vec{a}+\vec{b}+\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}$, $[\vec{a}\,\vec{b}\,\vec{c}]\neq 0$

Q If $\vec{a} = \hat{i} + 2\hat{j} + 2\hat{k}$, $\vec{b} = 2\hat{i} - 3\hat{j} + \hat{k}$ and $\vec{c}$ is a vector such that $(\vec{a} \times \vec{b}) \times \vec{c} = \vec{a} \times (\vec{b} \times \vec{c})$ & $|\vec{c}| = 6$. Then the volume of parallelopiped whose co-terminous edges are $\vec{a}, \vec{b}, \vec{b} \times \vec{c}$ will be-

(A) 180 (B) 214 (C) 232 (D) 244

Sol^n

$$(\vec{a} \times \vec{b}) \times \vec{c} = \vec{a} \times (\vec{b} \times \vec{c})$$

$$\cancel{(\vec{a}\cdot\vec{c})\vec{b}} - (\vec{b}\cdot\vec{c})\vec{a} = \cancel{(\vec{a}\cdot\vec{c})\vec{b}} - (\vec{a}\cdot\vec{b})\vec{c}$$

$\Rightarrow$ $\vec{a} = \lambda \vec{c} \Rightarrow \vec{a}$ & $\vec{c}$ are Collinear

$$\therefore \vec{c} = |\vec{c}|\,\hat{a} = 6\cdot\left(\frac{\hat{i} + 2\hat{j} + 2\hat{k}}{3}\right) = 2\hat{i} + 4\hat{j} + 4\hat{k}$$

$$\vec{b} \times \vec{c} = -16\hat{i} - 6\hat{j} + 14\hat{k}$$

Volume of $\|^{piped}$ $= [\vec{a}\ \ \vec{b}\ \ \vec{b}\times\vec{c}] = 244$

Ans

Q If $\hat{i}\times\left((\vec{a}-\hat{j})\times\hat{i}\right)+\hat{j}\times\left((\vec{a}-\hat{k})\times\hat{j}\right)+\hat{k}\times\left((\vec{a}-\hat{i})\times\hat{k}\right)=0$ and $\boxed{\vec{a}=x\hat{i}+y\hat{j}+z\hat{k}}$

then value of $8(x^3-xy+zx)=?$

$x=y=z=\frac{1}{2}$

Sol^n

$$(\hat{i}\cdot\hat{i})(\vec{a}-\hat{j})-(\hat{i}\cdot(\vec{a}-\hat{j}))\hat{i}$$

$$(\hat{j}\cdot\hat{j})(\vec{a}-\hat{k})-(\hat{j}\cdot(\vec{a}-\hat{k}))\hat{j}$$

$$(\hat{k}\cdot\hat{k})(\vec{a}-\hat{i})-(\hat{k}\cdot(\vec{a}-\hat{i}))\hat{k}$$

$$3\vec{a}-(\hat{i}+\hat{j}+\hat{k})-\underbrace{(x\hat{i}+y\hat{j}+z\hat{k})}_{\vec{a}}=0\Rightarrow 2\vec{a}=\hat{i}+\hat{j}+\hat{k}$$

$$\therefore \boxed{\vec{a}=\frac{\hat{i}+\hat{j}+\hat{k}}{2}}$$

Q If $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$; $\vec{b} = \hat{i} + \hat{j} - 4\hat{k}$; $\vec{c} = 4\hat{i} - 3\hat{j} + 6\hat{k}$; $\vec{d} = 3\hat{i} - 6\hat{j} - 5\hat{k}$ then :

① $\{(\vec{a} \times \vec{b}) \times (\vec{a} \times \vec{c})\}.\vec{d} = ?$

HW ② $\vec{d}.[\vec{a} \times \{\vec{b} \times (\vec{c} \times \vec{d})\}] = ?$ -1190 Ans

Isolating an known vectors / Vector Equations :–

Satisfying a given relationship with some known vectors:

There is no general method for solving such equations, however dot or cross with known or unknown vectors or dot with $\vec{a}\times\vec{b}$, generally isolates the unknown vector. Use of linear combination also proves to be advantageous.

(1) Solve for $\vec{x}$

$\vec{x}.\vec{a} = c$ ....(1) where c is a non-zero scalar ; $\vec{a}$ & $\vec{b}$ are non-zero vectors.

and $\vec{a}\times\vec{x} = \vec{b}$ ....(2)

Sol^n

$\vec{x}.\vec{a} = c$ & $\vec{a}\times\vec{x} = \vec{b}$

$\downarrow$ Cross with $\vec{a}$

$$\vec{a}\times(\vec{a}\times\vec{x}) = \vec{a}\times\vec{b}$$

$$(\vec{a}.\vec{x})\,\vec{a} - (\vec{a}.\vec{a})\vec{x} = \vec{a}\times\vec{b}$$

$$c\,\vec{a} - \vec{a}^{\,2}\,\vec{x} = \vec{a}\times\vec{b}$$

$$\boxed{\vec{x} = \frac{c\,\vec{a} - \vec{a}\times\vec{b}}{a^2}}$$

(2) Find the unknown vector $\vec{R}$ satisfying $K\vec{R} + \vec{A}\times\vec{R} = \vec{B}$; $K \neq 0$

(3) Solve the following simultaneous equations for $\vec{x}$ & $\vec{y}$

HW

$\vec{x}+\vec{y} = \vec{a}$ ....(1) ; $\vec{x}\times\vec{y} = \vec{b}$ ....(2) ; $\vec{x}.\vec{a} = 1$ ....(3)

(2) $$K\vec{R} + \vec{A}\times\vec{R} = \vec{B} \quad ; \quad K \neq 0$$

Cross with $\vec{A}$:

$$K(\vec{A}\times\vec{R}) + \vec{A}\times(\vec{A}\times\vec{R}) = \vec{A}\times\vec{B}$$

$$K(\vec{A}\times\vec{R}) + (\vec{A}.\vec{R})\vec{A} - (A^2)\vec{R} = \vec{A}\times\vec{B}$$

dot with $\vec{A}$: $$K\,\vec{A}.\vec{R} + 0 = \vec{A}.\vec{B}$$

$$\boxed{\vec{A}\cdot\vec{R} = \frac{\vec{A}\cdot\vec{B}}{K}}$$

$$K(\vec{B} - K\vec{R}) + \frac{(\vec{A}.\vec{B})}{K}\vec{A} - (A^2)\vec{R} = \vec{A}\times\vec{B}$$

$$\left(\frac{\vec{A}\cdot\vec{B}}{K}\right)\vec{A} + K\vec{B} - \vec{A}\times\vec{B} = (K^2+A^2)\vec{R}$$

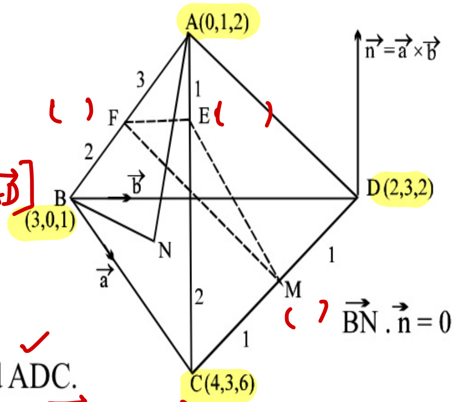
MASTER PROBLEM ON TETRAHEDRON :-

(i) p.v. corresponding to the point of concurrency of the join of the mid points of each pair of opposite edges. $\left(\dfrac{\vec{a}+\vec{b}+\vec{c}+\vec{d}}{4}\right)$

(ii) p.v. of the foot N from the vertex A and the perpendicular distance of A from the face BCD.

(iii) Image of A in plane face BCD.

(iv) Altitude of tetrahedron from the vertex A.



(v) Volume of tetrahedron $V = \frac{1}{6}\left[\vec{AB}\ \vec{AC}\ \vec{AD}\right]$

$\left[\frac{1}{3}\text{ Area of the base} \times h\right]$

$\vec{n}_{ABC} = \vec{AB} \times \vec{AC}$

(vi) Unit vectors normal to the plane face ABC and ADC.

(vii) Acute angle between the planes ABC and ADC. $\dfrac{\vec{n}_{ABC} \cdot \vec{n}_{ADC}}{|\vec{n}_{ABC}|\ |\vec{n}_{ADC}|} = \cos\theta.$

(viii) S.D. between the skew lines AD and BC and the angle between them.

* (ix) A unit vector | | to the plane EFM and perpendicular to the vector $\hat{i}+\hat{j}-\hat{k}$

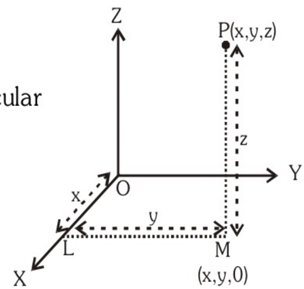
$\vec{n} = (\vec{EF} \times \vec{FM}) \times (\hat{i}+\hat{j}-\hat{k})$

$\pm\hat{n}$

(x) Equation of the plane through ABC. $\left(\vec{r} - (\hat{j}+2\hat{k})\right) \cdot \vec{n}_{ABC} = 0.$

3 - D

COORDINATES OF A POINT IN SPACE :



In two dimensional geometry, magnitude of x & y coordinates are perpendicular distances from y & x axis respectively.

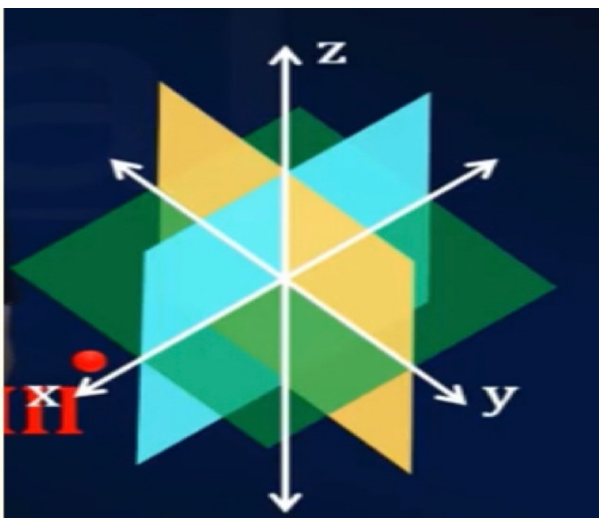
But in case of three dimensional geometry it is understood in different way.

(a) Magnitude of x-coordinate is perpendicular distance from y-z plane, similarly magnitude of y & z coordinates are perpendicular distances from x-z and x-y plane

(b) If a point lies on x-axis then its coordinates are (x, 0, 0) and similarly on y-axis & z-axis the coordinates are (0, y, 0) and (0, 0, z)

Remark : The sign of the coordinates of a point determine the octant in which the point lies. The following table shows the signs of the coordinates in eight octants.

Octants / Coordinates	I	II	III	IV	V	VI	VII	VIII
x	+	−	−	+	+	−	−	+
y	+	+	−	−	+	+	−	−
z	+	+	+	+	−	−	−	−



DISTANCE FORMULA :

The distance between two points A (x_1, y_1, z_1) and B (x_2, y_2, z_2) is given by

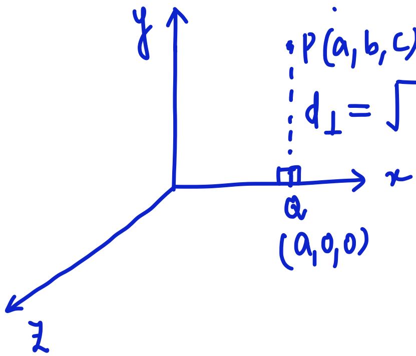
$$AB = \sqrt{[(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2]}$$

SECTION FORMULA :

Let $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ be two points and let $R(x, y, z)$ divide PQ in the ratio $m_1 : m_2$. Then co-ordinates

of $R(x, y, z) = \left(\dfrac{m_1x_2 + m_2x_1}{m_1 + m_2}, \dfrac{m_1y_2 + m_2y_1}{m_1 + m_2}, \dfrac{m_1z_2 + m_2z_1}{m_1 + m_2}\right)$

Q Find the distance of P(a,b,c) from x-axis ?



$d_\perp = \sqrt{0^2 + b^2 + c^2} = \sqrt{b^2 + c^2}$

Q

Find the locus of the point which move such that its distance from x-axis is $\frac{1}{2}$ of its distance from zy-plane.

Sol^n $P(\alpha, \beta, \gamma)$

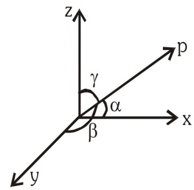
$$\sqrt{\beta^2 + \gamma^2} = \frac{1}{2}|\alpha|$$

$$4(\beta^2 + \gamma^2) = \alpha^2$$

$$\boxed{x^2 - 4y^2 - 4z^2 = 0}$$ Ans

DIRECTION COSINES OF VECTOR :

Let $\vec{a} = a_1 i + a_2 j + a_3 k$ the angles which this vector makes with the +ve directions OX, OY & OZ are called Direction Angles & their cosine are called the direction cosine hence if α, β, γ are the direction angles then the d.c's are



$$\boxed{\cos\alpha = \frac{a_1}{|\vec{a}|}, \cos\beta = \frac{a_2}{|\vec{a}|}, \cos\gamma = \frac{a_3}{|\vec{a}|}}$$

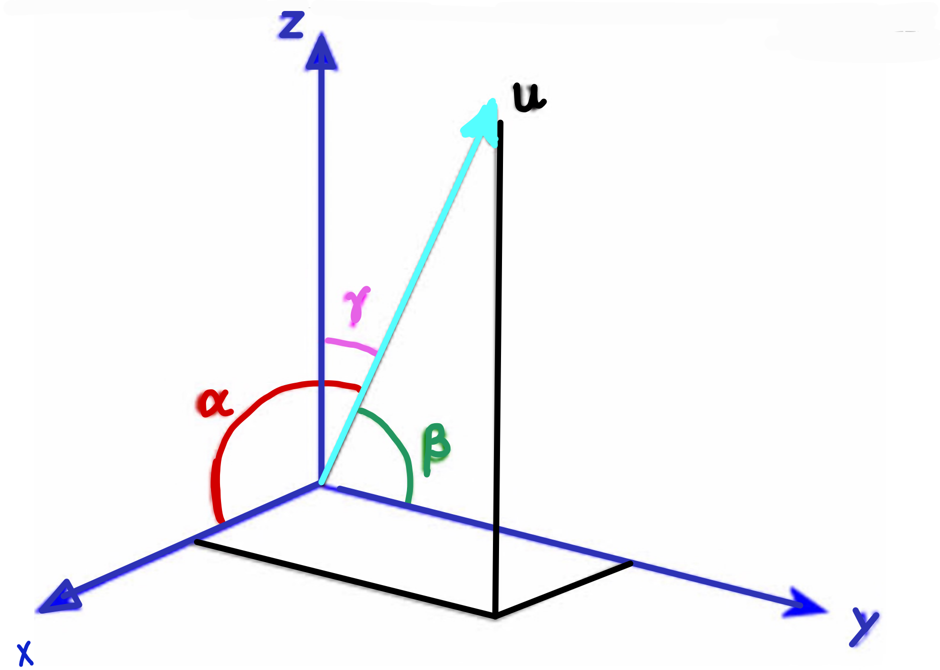
$\cos\alpha$, $\cos\beta$, $\cos\gamma$ are popularly denoted by ℓ, m and n.

Note :

(i) $\ell^2 + m^2 + n^2 = 1 \Rightarrow \cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$

$\Rightarrow \sin^2\alpha + \sin^2\beta + \sin^2\gamma = 2$

(ii) Components of the unit vector denotes the dc's of the vector : $\hat{a} = \ell\hat{i} + m\hat{j} + n\hat{k}$



eg D.C's of the vector $2\hat{j}-2\hat{j}+\hat{k}$ are $\frac{2}{3}, -\frac{2}{3}, \frac{1}{3}$ ✓

eg There exists a vector with direction angles $\alpha = 30°$ and $\beta = 30°$ True or False ?

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

$$\frac{3}{4} + \frac{3}{4} + \cos^2\gamma = 1$$

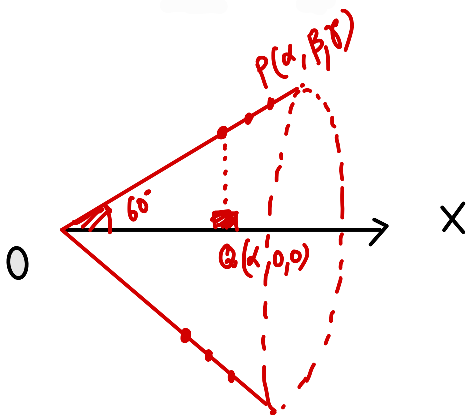
$$\cos^2\gamma = -ve$$

This is not possible

Q (O is Origin)

Find the locus of all points the P for which $\overrightarrow{OP}$ represents a vector whose direction cosine $\cos\alpha = \frac{1}{2}$

Soln



$$PQ = \sqrt{\beta^2+\gamma^2}.$$

$$\tan 60° = \frac{|PQ|}{|OQ|}$$

$$\sqrt{3}\,|\alpha| = \sqrt{\beta^2+\gamma^2}$$

$$3\alpha^2 - \beta^2 - \gamma^2 = 0.$$

$$3x^2 - y^2 - z^2 = 0.$$

Locus of all such points 'P' will be a cone concentric with x-axis.

DIRECTION COSINES OF LINE :

If line makes angles α, β, γ with x, y & z axis respectively then $\pi-\alpha, \pi-\beta$ & $\pi-\gamma$ is another set of angle that line makes with principle axes. Hence if ℓ, m & n are direction cosines of line then $-\ell$, $-m$ & $-n$ are also direction cosines of the same line.

Q Find D.C. of line passing through $\underset{A}{(1,1,1)}$ & $\underset{B}{(3, 2, 3)}$

Sol $\overrightarrow{AB} = 2\hat{i}+\hat{j}+2\hat{k}$

DC's of LINE $\pm\left(\frac{2}{3}, \frac{1}{3}, \frac{2}{3}\right)$

DIRECTION RATIOS OF A LINE :

3 numbers which are proportional to direction cosines of a line are called the direction ratio's of a line. Hence if a, b, c are d.r. of a line, then $\frac{\ell}{a} = \frac{m}{b} = \frac{n}{c}$

RELATIONSHIP BETWEEN DIRECTION COSINE & DIRECTION RATIOS :

If $\frac{\ell}{a} = \frac{m}{b} = \frac{n}{c} = \lambda$ (say) $\Rightarrow$ $\ell = a\lambda;\ m = b\lambda;\ n = c\lambda$

Now $(a^2 + b^2 + c^2)\lambda^2 = 1$ (as $\ell^2 + m^2 + n^2 = 1$)

$\Rightarrow \quad \lambda = \frac{\pm 1}{\sqrt{a^2+b^2+c^2}}$

$\ell = \pm\frac{a}{\sqrt{a^2+b^2+c^2}};\ m = \pm\frac{b}{\sqrt{a^2+b^2+c^2}};\ n = \pm\frac{c}{\sqrt{a^2+b^2+c^2}}$

Note :

(i) Direction ratios of a line joining two points A (x_1, y_1, z_1) and B (x_2, y_2, z_2) are proportional to $x_2 - x_1$; $y_2 - y_1$; $z_2 - z_1$

Since $\overrightarrow{AB} = (x_2 - x_1)\hat{i} + (y_2 - y_1)\hat{j} + (z_2 - z_1)\hat{k}$.

Hence the direction ratios of a vector $\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$ are proportional to the numbers a_1, a_2 and a_3.

(ii) If a line is having direction cosines ℓ, m, n $\Rightarrow$ it is travelling along the vector $\ell\hat{i}+m\hat{j}+n\hat{k}$. and $-\ell\hat{i}-m\hat{j}-n\hat{k}$

Hence angle between two lines with direction cosines ℓ_1, m_1, n_1 and ℓ_2, m_2, n_2 is

$$\cos\theta = \ell_1\ell_2 + m_1m_2 + n_1n_2$$

$$\cos\theta = \frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\times\sqrt{a_2^2+b_2^2+c_2^2}} \text{ (in terms of direction ratios)}$$

(a) If L_1 is perpendicular to L_2 then $\ell_1\ell_2 + m_1m_2 + n_1n_2 = 0$ or $a_1a_2 + b_1b_2 + c_1c_2 = 0$

(b) If L_1 is parallel to L_2 then $\frac{\ell_1}{\ell_2}=\frac{m_1}{m_2}=\frac{n_1}{n_2}$ or $\frac{a_1}{a_2}=\frac{b_1}{b_2}=\frac{c_1}{c_2}$

(iii) Direction cosine of axes :

Since the positive x-axes makes angle 0°, 90°, 90° with axes of x, y and z respectively,

$\therefore$ D.C.'s of x axes are 1, 0, 0.

D.C.'s of y-axis are 0, 1, 0

D.C.'s of z-axis are 0, 0, 1

Q Find the direction cosines of a line perpendicular to two lines whose dr's are 1, 2, 3 and –2, 1, 4.

Sol^n

$\vec{a} = \hat{i}+2\hat{j}+3\hat{k}$

$\vec{b} = -2\hat{i}+\hat{j}+4\hat{k}$

$\vec{v} = \vec{a}\times\vec{b}$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 2 & 3 \\ -2 & 1 & 4 \end{vmatrix}$$

$= \hat{i}(8-3) - \hat{j}(4+6) + \hat{k}(1+4)$

$\vec{a}\times\vec{b} = 5\hat{i} - 10\hat{j} + 5\hat{k}$.

$\vec{v} = 5(\hat{i}-2\hat{j}+\hat{k})$

dc's of line :

$\pm\left(\frac{1}{\sqrt{6}}, \frac{-2}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$

Ans

Q The direction cosines ℓ, m, n of two lines are connected by the relations $\ell + m + n = 0$ and

* $2\ell m + 2\ell n - mn = 0$. Find them and the angles between them.

Sol^n

$$l = -m-n$$

$$2(-m-n)m + 2(-m-n)n - mn = 0.$$

$$-2m^2 - 2mn - 2mn - 2n^2 - mn = 0.$$

$$2m^2 + 2n^2 + 5mn = 0. \Rightarrow 2m^2 + 4mn + mn + 2n^2 = 0$$

$$2m(m+2n) + n(m+2n) = 0.$$

$$(2m+n)(m+2n) = 0.$$

C-I $\quad 2m+n=0$

$$\boxed{\frac{m}{n} = -\frac{1}{2}}$$

$$l+m+n = 0.$$

$$\frac{l}{n} + \frac{m}{n} + 1 = 0.$$

$$\frac{l}{n} = -1 - \frac{m}{n} = -1 + \frac{1}{2}$$

$$\boxed{\frac{l}{n} = -1/2}$$

C-II $\quad m+2n=0.$

$$\boxed{\frac{m}{n} = -2}$$

$$\frac{l}{n} + \frac{m}{n} + 1 = 0.$$

$$\frac{l}{n} - 2 + 1 = 0$$

$$\boxed{\frac{l}{n} = 1}$$

$\therefore$ dr's of 1^{st} line :-

dc's l, m, n

dr's $\frac{l}{n}, \frac{m}{n}, \frac{n}{n} \equiv -\frac{1}{2}, -\frac{1}{2}, 1 \equiv -1, -1, 2$

$$\boxed{\vec{v_1} = -\hat{i} - \hat{j} + 2\hat{k}}$$

dr's of 2^{nd} line :-

$1, -2, 1$

$$\boxed{\vec{v_2} = \hat{i} - 2\hat{j} + \hat{k}}$$

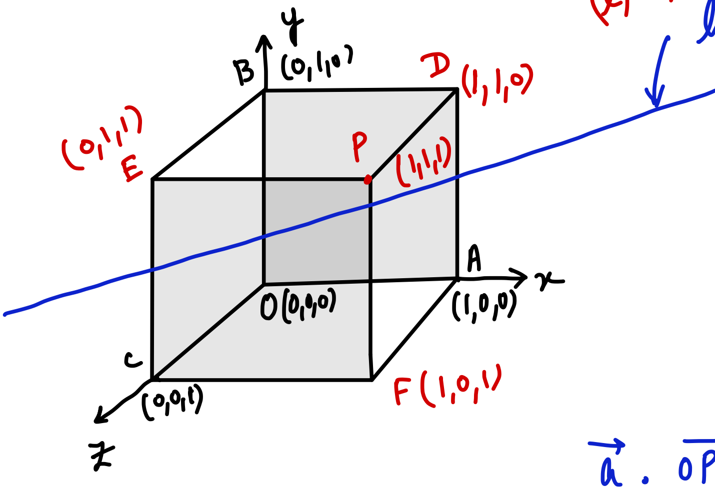
$$\vec{v_1} \cdot \vec{v_2} = |v_1||v_2| \cos\theta .$$

Q HW

A variable line has dc's ℓ, m, n and $\ell + \delta\ell$, $m + \delta m$, $n + \delta n$ in two adjacent positions. If $\delta\theta$ be the angle between the lines in these two positions then prove that $(\delta\theta)^2 = (\delta\ell)^2 + (\delta m)^2 + (\delta n)^2$.

Q A line makes angle $\alpha, \beta, \gamma, \delta$ with four diagonals of a cube. Prove that $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta = \dfrac{4}{3}$.

(l, m, n) line $\quad l\hat{i} + m\hat{j} + n\hat{k} = \vec{a}$



Body diagonals

$\overrightarrow{OP} = \hat{i} + \hat{j} + \hat{k}$

$\overrightarrow{FB} = \hat{i} - \hat{j} + \hat{k}$

$\overrightarrow{AE} = \hat{i} - \hat{j} - \hat{k}$

$\overrightarrow{DC} = \hat{i} + \hat{j} - \hat{k}$.

$\vec{a} \cdot \overrightarrow{OP} = |\vec{a}|\,|\overrightarrow{OP}| \cos\alpha$

$l + m + n = (1)(\sqrt{3}) \cos\alpha$

$l - m + n = \sqrt{3} \cos\beta$

$l - m - n = \sqrt{3} \cos\gamma$

$l + m - n = \sqrt{3} \cos\delta$.

$l^2 + m^2 + n^2 = 1$

$\Vert^{ly}$

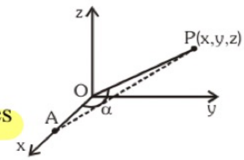
square and add :-

Important point :

Direction cosines of a line have two sets but direction ratios of a line have infinite possible sets.

PROJECTIONS :

(a) Projection of line segment OP on co-ordinate axes :



Let line segment make angle α with x-axis

Thus, the projections of line segment OP on axes are the absolute values of the co-ordinates of P. i.e.

Projection of OP on x-axis = $|x|$

Projection of OP on y-axis = $|y|$

Projection of OP on z-axis = $|z|$

Now, in ΔOAP, angle A is a right angle and OA = x

$$OP = \sqrt{x^2 + y^2 + z^2}$$

$$\therefore \quad \cos\alpha = \frac{x}{\sqrt{x^2+y^2+z^2}} = \frac{x}{|OP|}$$

if $|OP| = r$, then $x = |OP|\cos\alpha = \ell r$

Similarly $y = |OP|\cos\beta = mr$, $z = nr$, where ℓ, m, n are DC's of line

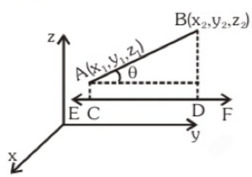
(b) Projection of a line segment AB on coordinate axes :

Projection of the point $A(x_1, y_1, z_1)$ on x-axis is $E(x_1, 0, 0)$. Projection of point $B(x_2, y_2, z_2)$ on x-axis is $F(x_2, 0, 0)$.

Hence projection of AB on x-axis is $EF = |x_2 - x_1|$.

Similarly, projection of AB on y and z-axis are $|y_2 - y_1|$, $|z_2 - z_1|$ respectively.

* **(c) Projection of line segment AB on a line having direction cosines ℓ, m, n :**



Let $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$.

Now projection of AB on EF = CD = AB $\cos\theta$

$$= \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2} \times \frac{|(x_2-x_1)\ell+(y_2-y_1)m+(z_2-z_1)n|}{\sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}}$$

$$= |(x_2-x_1)\ell+(y_2-y_1)m+(z_2-z_1)n|$$

Q Find the length of projection of the line segment joining the points (–1, 0, 3) and (2, 5, 1) on the line whose direction ratios are 6, 2, 3.

Sol^n

The direction cosines ℓ, m, n of the line are given by $\frac{\ell}{6}=\frac{m}{2}=\frac{n}{3}=\frac{\sqrt{\ell^2+m^2+n^2}}{\sqrt{6^2+2^2+3^2}}=\frac{1}{\sqrt{49}}=\frac{1}{7}$

$\therefore \quad \ell=\frac{6}{7},\ m=\frac{2}{7},\ n=\frac{3}{7}$

The required length of projection is given by

$$=|\ell(x_2-x_1)+m(y_2-y_1)+n(z_2-z_1)|=\left|\frac{6}{7}[2-(-1)]+\frac{2}{7}(5-0)+\frac{3}{7}(1-3)\right|$$

$$=\left|\frac{6}{7}\times 3+\frac{2}{7}\times 5+\frac{3}{7}\times -2\right|=\left|\frac{18}{7}+\frac{10}{7}-\frac{6}{7}\right|=\left|\frac{18+10-6}{7}\right|=\frac{22}{7}$$ **Ans.**

Q Find the intercept made by lines

$\ell_1: \quad \vec{r}=\hat{i}+\hat{j}+\hat{k}+\lambda(3\hat{i}-\hat{j})$

$\ell_2: \quad \vec{r}=4\hat{i}-\hat{k}+\mu(2\hat{i}+3\hat{k})$

on a line with dr's (2, 1, 2) ?

PLANES :

DEFINITION OF PLANE :

A plane is a surface such that a line segment joining any two points on the surface lies wholly on it.

A linear equation in three variables of the type $Ax + By + Cz + D = 0$, denotes the general equation of a plane.

Where A, B, C are not simultaneously zero. Dividing by A we get $x + \frac{B}{A}y + \frac{C}{A}z + \frac{D}{A} = 0$. Thus equation of the plane involves only 3 arbitrary constants, hence in order to determine a unique plane 3 independent conditions are needed.

Note :

(i) Equation of y-z plane is $x = 0$.

(ii) Equation of z-x plane is $y = 0$.

(iii) Equation of x-y plane is $z = 0$.

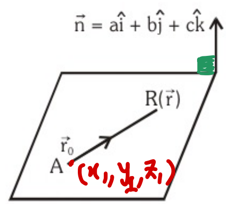
(iv) Equation of the plane parallel to x-y plane at a distance c is $z = c$ or $z = -c$.

(v) Equation of the plane parallel to y-z plane at a distance c is $x = c$ or $x = -c$

(vi) Equation of the plane parallel to z-x plane at a distance c is $y = c$ or $y = -c$.

DIFFERENT FORMS OF THE EQUATIONS OF PLANES

(a) Equation of a plane passing through a fixed point



If $\vec{r}_0$ is p.v. of point on the plane & $\vec{n}$ be vector normal to the plane, then

$\overrightarrow{AR}.\vec{n} = 0$

$(\vec{r} - \vec{r}_0).\vec{n} = 0$ .... (i)

$\vec{r}.\vec{n} = \vec{r}_0.\vec{n}$

Hence $\vec{r}.\vec{n} = d$ is the general equation of a plane in vector form

If $\vec{r} = x\hat{i} + y\hat{i} + z\hat{k}$ and $\vec{r}_0 = x_1\hat{i} + y_1\hat{i} + z_1\hat{k}$

then (i) becomes $a(x - x_1) + b(y - y_1) + c(z - z_1) = 0$

This is the equation of the plane containing the point $(x_1 y_1 z_1)$, where $a\hat{i} + b\hat{j} + c\hat{k}$ is a vector normal to it, where a, b, c are the dr's of a normal to the plane.

eg: P : $2(x+1) - 3(y+2) + 4(3K-1) = 0.$

$2(x+1) - 3(y+2) + 12(K - 1/3) = 0.$

$\vec{n} = 2\hat{i} - 3\hat{j} + 12\hat{k}$;

Plane Parallel to the Coordinate Planes :

(i) Equation of yz plane is $x = 0$.

(ii) Equation of zx plane is $y = 0$.

(iii) Equation of xy plane is $z = 0$.

(iv) Equation of the plane parallel to xy plane at a distance c is $z = c$ or $z = -c$.

(v) Equation of the plane parallel to yz plane at a distance c is $x = c$ or $x = -c$

(vi) Equation of the plane parallel to zx plane at a distance c is $y = c$ or $y = -c$.

Equations of Planes Parallel to the Axes :

If $a = 0$, the plane is parallel to x-axis i.e. equation of the plane parallel to x-axis is $\mathbf{by + cz + d = 0}$.

Similarly, equations of planes parallel to y-axis and parallel to z-axis are $\mathbf{ax + cz + d = 0}$ and $\mathbf{ax + by + d = 0}$, respectively.

Q (a) HW Convert the plane 3x + 4y + z = 9 into vector form.

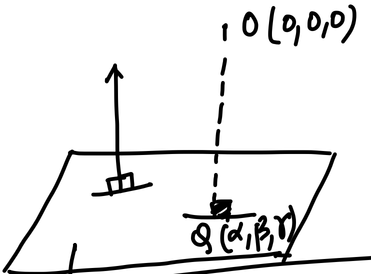
(b) Convert $\vec{r}.(\hat{i}+\hat{j}+\hat{k}) = 3$ into cartesian form

$$(x\hat{i}+y\hat{j}+z\hat{k}).(\hat{i}+\hat{j}+\hat{k}) = 3$$

$$\boxed{x+y+z=3}$$

Q Find the equation of the plane if the feet of normal from origin on a plane is α, β and γ.

$$\vec{n} = \alpha\hat{i}+\beta\hat{j}+\gamma\hat{k}$$



$$\boxed{P:\ \alpha(x-\alpha)+\beta(y-\beta)+\gamma(z-\gamma)=0.}$$

(2) Find the unknown vector $\vec{R}$ satisfying $K\vec{R} + \vec{A}\times\vec{R} = \vec{B}$; $K \neq 0$

HW (3) Solve the following simultaneous equations for $\vec{x}$ & $\vec{y}$

$\vec{x}+\vec{y} = \vec{a}$ ....(1) ; $\vec{x}\times\vec{y} = \vec{b}$ ....(2) ; $\vec{x}.\vec{a} = 1$ ....(3)

3

$$\vec{x} + \vec{y} = \vec{a} \quad \text{— 1 —}$$

dot with $\vec{a}$! $\quad \vec{x}.\vec{a} + \vec{y}.\vec{a} = a^2 \Rightarrow \underbrace{\vec{y}.\vec{a} = a^2 - 1}$.

$$\vec{x}\times\vec{y} = \vec{b}$$

$$\vec{a}\times(\vec{x}\times\vec{y}) = \vec{a}\times\vec{b}$$

$$(\vec{a}.\vec{y})\vec{x} - (\vec{a}.\vec{x})\vec{y} = \vec{a}\times\vec{b}$$

$$(a^2-1)\vec{x} - \vec{y} = \vec{a}\times\vec{b} \quad \text{— 2 —}$$

solve 1 & 2 to get $\vec{x}$ and $\vec{y}$.

Q. A variable line has dc's ℓ, m, n and $\ell + \delta\ell$, $m + \delta m$, $n + \delta n$ in two adjacent positions. If $\delta\theta$ be the angle between the lines in these two positions then prove that $(\delta\theta)^2 = (\delta\ell)^2 + (\delta m)^2 + (\delta n)^2$. ↳ very small.

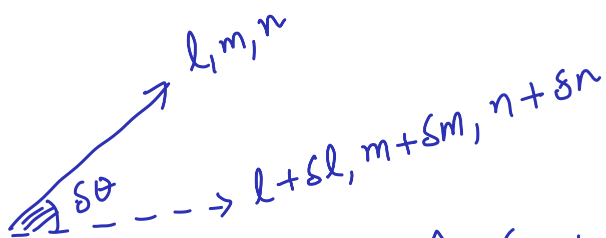
Soln

$$\ell^2 + m^2 + n^2 = 1 \quad \text{— ① —}$$

$$(\ell + \delta\ell)^2 + (m + \delta m)^2 + (n + \delta n)^2 = 1 . \quad \text{— ② —}$$

$$\cancel{\underbrace{(\ell^2 + m^2 + n^2)}_{1}} + (\delta\ell)^2 + (\delta m)^2 + (\delta n)^2 + 2\ell(\delta\ell) + 2m(\delta m) + 2n(\delta n) = \cancel{1}.$$

$$\boxed{(\delta\ell)^2 + (\delta m)^2 + (\delta n)^2 = -2\ell(\delta\ell) - 2m(\delta m) - 2n(\delta n)} \quad \text{— ③ —}$$



$$(\ell\hat{i} + m\hat{j} + n\hat{k}) \cdot ((\ell + \delta\ell)\hat{i} + (m + \delta m)\hat{j} + (n + \delta n)\hat{k})$$
$$= \sqrt{\ell^2 + m^2 + n^2}\sqrt{(\ell + \delta\ell)^2 + (m + \delta m)^2 + (n + \delta n)^2} \cos\delta\theta$$

$$\underbrace{(\ell^2 + m^2 + n^2)}_{1} + \Sigma\,\ell(\delta\ell) = \cos\delta\theta$$

$$1 - \cos\delta\theta = -\Sigma\,\ell(\delta\ell)$$

$$1 - \left(1 - 2\sin^2\frac{\delta\theta}{2}\right) = -\Sigma\,\ell(\delta\ell)$$

$$2\cdot\left(\frac{\delta\theta}{2}\right)^2 = \frac{\Sigma(\delta\ell)^2}{2} \Rightarrow (\delta\theta)^2 = \Sigma(\delta\ell)^2$$

(H.P.)

Q Find the length of projection of the line segment joining the points (–1, 0, 3) and (2, 5, 1) on the line whose direction ratios are 6, 2, 3.

Sol^n

The direction cosines ℓ, m, n of the line are given by $\frac{\ell}{6}=\frac{m}{2}=\frac{n}{3}=\frac{\sqrt{\ell^2+m^2+n^2}}{\sqrt{6^2+2^2+3^2}}=\frac{1}{\sqrt{49}}=\frac{1}{7}$

$\therefore \quad \ell=\frac{6}{7},\ m=\frac{2}{7},\ n=\frac{3}{7}$

The required length of projection is given by

$=|\ell(x_2-x_1)+m(y_2-y_1)+n(z_2-z_1)|=\left|\frac{6}{7}[2-(-1)]+\frac{2}{7}(5-0)+\frac{3}{7}(1-3)\right|$

$=\left|\frac{6}{7}\times 3+\frac{2}{7}\times 5+\frac{3}{7}\times -2\right|=\left|\frac{18}{7}+\frac{10}{7}-\frac{6}{7}\right|=\left|\frac{18+10-6}{7}\right|=\frac{22}{7}$ **Ans.**

Q Find the intercept made by lines

$l_1: \quad \vec{r} = \hat{i}+\hat{j}+\hat{k}+\lambda(3\hat{i}-\hat{j})$

$l_2: \quad \vec{r} = 4\hat{i}-\hat{k}+\mu(2\hat{i}+3\hat{k})$

on a line with dr's (2, 1, 2) ?

Sol^n

General pt on $L_1 : A(1+3\lambda,\ 1-\lambda,\ 1)$

" " " $L_2 : B(4+2\mu,\ 0,\ 3\mu-1)$

$\overrightarrow{AB} = (3+2\mu-3\lambda)\hat{i}-(1-\lambda)\hat{j}+(3\mu-2)\hat{k}$

Now, $\frac{3+2\mu-3\lambda}{2}=\frac{\lambda-1}{1}=\frac{3\mu-2}{2}$

$$3+2\mu-3\lambda = 2\lambda-2 \quad \Big| \quad 2\lambda-2 = 3\mu-2$$

$$2\mu - 5\lambda = -5$$
$$3\mu - 2\lambda = 0 \qquad \mu = \frac{2\lambda}{3}$$

$$2\left(\frac{2\lambda}{3}\right) - 5\lambda = -5$$

$$4\lambda - 15\lambda = -15$$

$$\lambda = \frac{15}{11}; \quad \mu = \frac{\cancel{30}\ 10}{\cancel{33}\ 11}$$

$$\overrightarrow{AB} = \left(3 + \frac{20}{11} - \frac{45}{11}\right)\hat{i} + \left(\frac{15}{11} - 1\right)\hat{j} + \left(\frac{30}{11} - 2\right)\hat{k}$$

$$= \frac{8}{11}\hat{i} + \frac{4}{11}\hat{j} + \frac{8}{11}\hat{k}$$

$$|\overrightarrow{AB}| = \frac{\sqrt{64+16+64}}{11} = \frac{\sqrt{144}}{11} = \frac{12}{11} \text{ Ans}$$

Q (a) HW Convert the plane 3x + 4y + z = 9 into vector form.

(b) Convert $\vec{r}.(\hat{i}+\hat{j}+\hat{k})=3$ into cartesian form

$$\vec{r}.(3\hat{i}+4\hat{j}+\hat{k}) = 9$$

Q Find the equation of the plane if the feet of normal from origin on a plane is α, β and γ.

Done in last lecture.

Q Find the equation of the plane passing through the points A(2, 2, 1) and B(1, –2, 3) and perpendicular to the plane x – 2y + 3z + 4 = 0.

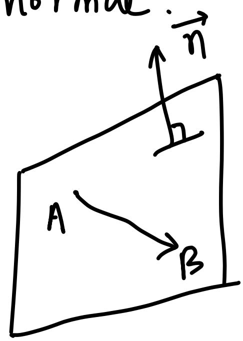
Solⁿ

$$P : a(x-2)+b(y-2)+c(z-1)=0.$$

$a, b, c \to$ dr's of normal.

$$\vec{n} = a\hat{i} + b\hat{j} + c\hat{k}$$

$$\vec{n} = \overrightarrow{AB} \times (\hat{i} - 2\hat{j} + 3\hat{k})$$



Q Find the equation of the plane through the point (2, –3, 1) and || to the plane 3x – 4y + 2z = 5.

Solⁿ $P: 3(x-2)-4(y+3)+2(z-1)=0$

Q Find the equation of the plane through the point (1, 0, –2) and perpendicular to the planes 2x + y – z = 2 and x – y – z = 3.

Solⁿ

$$\vec{n} = (2\hat{i} + \hat{j} - \hat{k}) \times (\hat{i} - \hat{j} - \hat{k})$$

$$= a\hat{i} + b\hat{j} + c\hat{k}$$

$$P: \quad a(x-1) + b(y-0) + c(z+2) = 0.$$

Q Two planes are given by equations x + 2y – 3z = 0 and 2x + y + z + 3 = 0. Find

$\vec{n_1} = \hat{i} + 2\hat{j} - 3\hat{k} \longrightarrow$ dc's $\frac{1}{\sqrt{14}}, \frac{2}{\sqrt{14}}, \frac{-3}{\sqrt{14}}$

(a) DC's of their normals and the acute angle between them.

$$\vec{n_1} \cdot \vec{n_2} = |\vec{n_1}||\vec{n_2}| \cos\theta$$

$$2+2-3 = \sqrt{14}\sqrt{6} \cos\theta \Rightarrow \cos\theta = \frac{1}{\sqrt{84}}$$

$\theta \rightarrow$ acute.

(b) DC's of the their line of intersection. $\vec{n_1} \times \vec{n_2}$ ✓

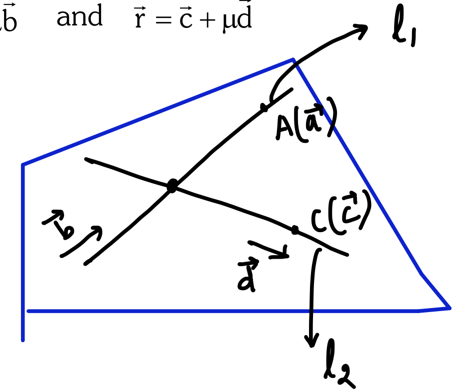
(c) Equation of the plane perpendicular to both of them through the point (2, 2, 1)

$$\vec{n} = \vec{n_1} \times \vec{n_2} = a\hat{i} + b\hat{j} + c\hat{k}$$

$$P: \quad a(x-2) + b(y-2) + c(z-1) = 0$$

Q Find the condition for coplanarity of two lines

$\vec{r} = \vec{a} + \lambda\vec{b}$ and $\vec{r} = \vec{c} + \mu\vec{d}$



$$[\vec{a}-\vec{c} \quad \vec{b} \quad \vec{d}] = 0.$$

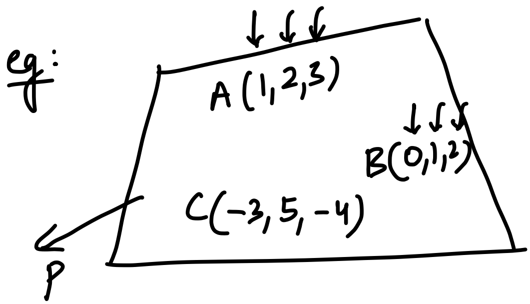
$$[\vec{a}\ \vec{b}\ \vec{d}] - [\vec{c}\ \vec{b}\ \vec{d}] = 0$$

Equation of a Plane through three points :

The equation of the plane through three non-collinear points (x_1, y_1, z_1), (x_2, y_2, z_2) (x_3, y_3, z_3) is

$$\begin{vmatrix} x-x_1 & y-y_1 & z-z_1 \\ x_2-x_1 & y_2-y_1 & z_2-z_1 \\ x_3-x_1 & y_3-y_1 & z_3-z_1 \end{vmatrix} = 0$$

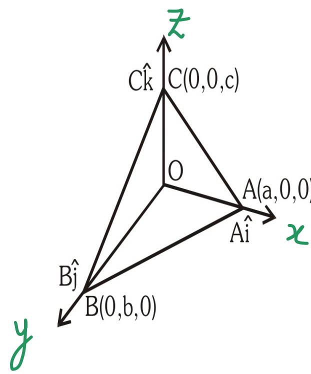
eg:



$$P: \begin{vmatrix} x-1 & y-2 & z-3 \\ -1 & -1 & -1 \\ -4 & 3 & -7 \end{vmatrix} = 0$$

Intercept form of the plane :

$\boxed{\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1}$ Rem where A(a, 0, 0); B(0, b, 0) and C(0, 0, c)



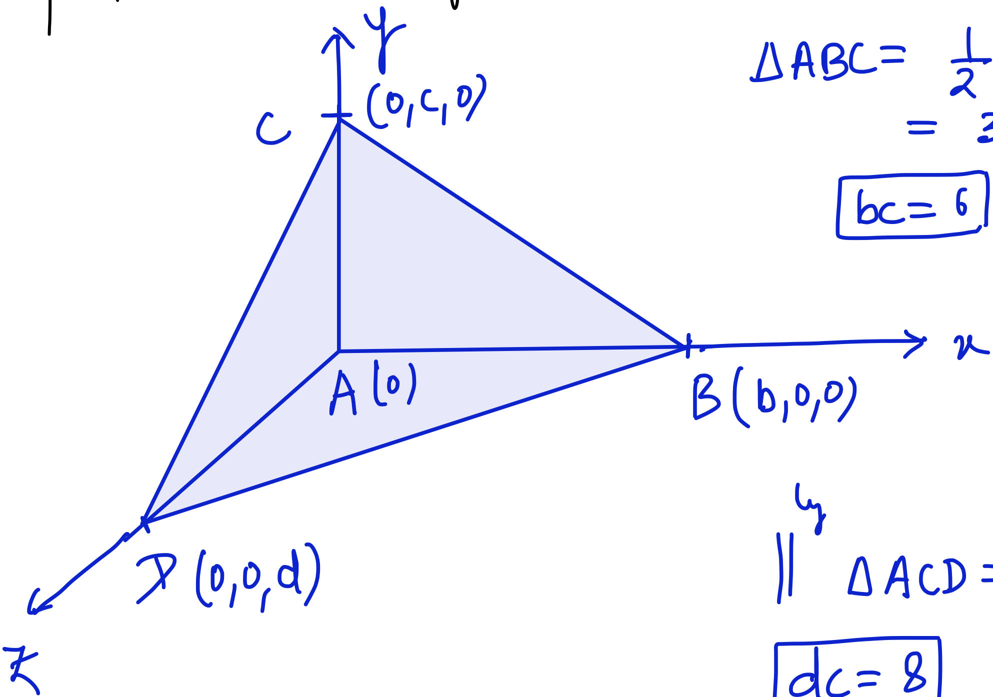
Note : (1) Vector area of $\Delta ABC = \frac{1}{2}\left[(ab(\hat{i}\times\hat{j})+bc(\hat{j}\times\hat{k})+ca(\hat{k}\times\hat{i})\right]$

$$= \frac{1}{2}\left[(bc\,\hat{i}+ca\,\hat{j}+ab\,\hat{k})\right]$$

i.e. $\boxed{\text{Area of the } \Delta ABC = \frac{1}{2}\sqrt{a^2b^2+b^2c^2+c^2a^2}}$ Rem

Q Let ABCD be a tetrahedron such that edges AB, AC, AD are mutually $\perp$. Let the areas of ΔABC, ΔACD, ΔADB are 3, 4, 5 sq. units respectively. then find the area of ΔBCD?

Sol^n



$\Delta ABC = \frac{1}{2} bc$
$= 3$

$\boxed{bc = 6}$

$\Delta ACD = 4$

$\boxed{dc = 8}$

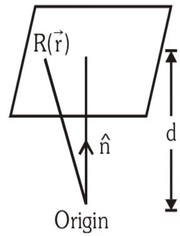
$\Delta ADB = 5$

$\boxed{bd = 10}$

$$\Delta BCD = \frac{1}{2}\sqrt{(bc)^2 + (cd)^2 + (db)^2}$$

$$= \frac{1}{2}\sqrt{8^2 + 6^2 + 10^2}.$$

Normal form of the equation of plane :



Projection of $\vec{r}$ on $\hat{n}$ = d

Rem

$\boxed{\vec{r}.\hat{n} = d}$ ...... (i) $(d > 0)$

$\hat{n} \rightarrow$ unit vector pointing from origin towards plane & perpendicular to it.

$d \rightarrow$ perpendicular distance of the plane from the origin.

$d\hat{n} \rightarrow$ p.v. of foot of perpendicular from origin in plane.

i.e. from (i) $\ell x + my + nz = d$ (cartesian form)

equation (i) helps us to know the distance of the plane from the origin and also the dc's of the normal vector.

Example : Convert given equation into normal form & find D.C. & distance from origin

(1) $\vec{r}.(6\hat{i} - 3\hat{j} - 2\hat{k}) + 1 = 0$

$$\vec{r}\cdot\frac{(6\hat{i}-3\hat{j}-2\hat{k})}{7} + \frac{1}{7} = 0$$

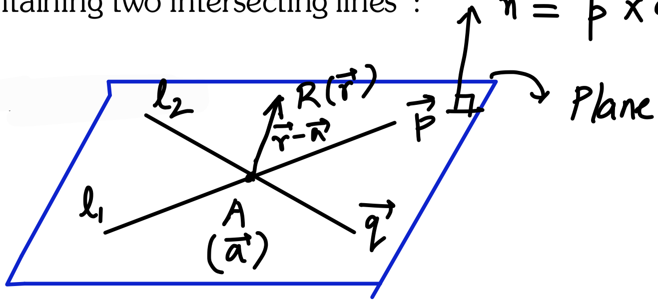
$$\vec{r}\cdot\hat{n} = -\frac{1}{7} \Rightarrow \boxed{\vec{r}\cdot\left(\frac{-6\hat{i}+3\hat{j}+2\hat{k}}{7}\right) = \frac{1}{7}}$$

$$\boxed{\vec{r}\cdot\hat{n} = d}\ ;\ d > 0$$

Q. Find equation of plane containing two intersecting lines :

$$\left.\begin{array}{l}\vec{r}=\vec{a}+\lambda\vec{p}\\ \vec{r}=\vec{a}+\mu\vec{q}\end{array}\right]$$

$\vec{n}=\vec{p}\times\vec{q}$



$$[\vec{r}-\vec{a} \;\; \vec{p} \;\; \vec{q}]=0 \Rightarrow [\vec{r} \;\; \vec{p} \;\; \vec{q}]=[\vec{a} \;\; \vec{p} \;\; \vec{q}]$$

$$\Downarrow$$

$$\vec{r}-\vec{a}=\lambda\vec{p}+\mu\vec{q} \;\; ; \;\; \lambda,\mu\in \text{scalars}$$

Parametric Equation of Plane :—

$$\boxed{\vec{r}=\vec{a}+\lambda\vec{p}+\mu\vec{q}}$$

This denotes a plane containing point $\vec{a}$ & is parallel to two non-collinear vectors $\vec{p}$ & $\vec{q}$

Also known as parametric eqn.

Q Express the equation of a plane $\vec{r} = \hat{i} - 2\hat{j} + \lambda(2\hat{i} + \hat{j} + 3\hat{k}) + \mu(3\hat{i} + 4\hat{j} - \hat{k})$ in

(i) scalar triple product forms

(ii) cartesian form → $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

Sol^n (i)

$$\left[\vec{r} - (\hat{i} - 2\hat{j}) \quad 2\hat{i} + \hat{j} + 3\hat{k} \quad 3\hat{i} + 4\hat{j} - \hat{k}\right] = 0$$

(ii) $\left| \quad \right| = 0.$

Q $\vec{r} = 2\hat{j} + \hat{k} + \lambda(\hat{i} - \hat{j} + \hat{k})$

$\vec{r} = 2\hat{i} + 3\hat{j} + 6\hat{k} + \mu(2\hat{i} + \hat{j} + 5\hat{k})$

Check whether two lines are co-planar or not, if yes find the equation of plane containing them.

$A(2\hat{j} + \hat{k})$; $B(2\hat{i} + 3\hat{j} + 6\hat{k})$

L_1 $\qquad$ L_2

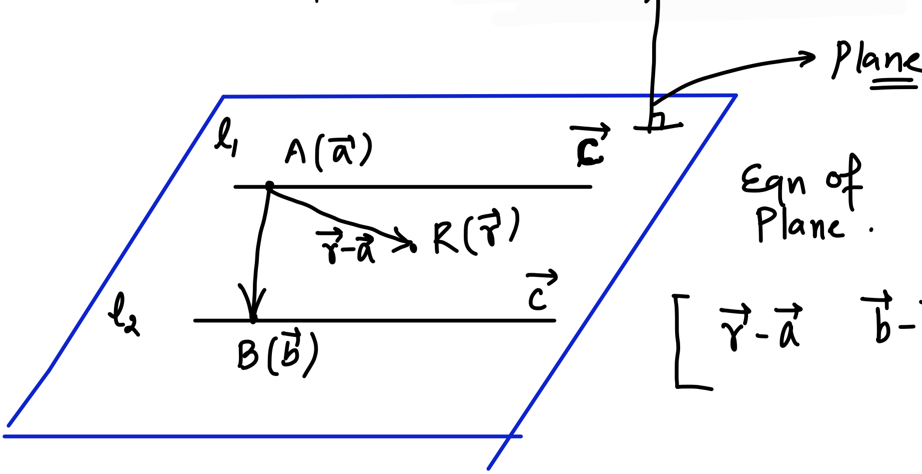
$$\text{STP} = \left[\vec{AB} \quad \hat{i} - \hat{j} + \hat{k} \quad 2\hat{i} + \hat{j} + 5\hat{k}\right]$$

Q. Find equation of plane containing two parallel lines :

$\vec{r} = \vec{a} + \lambda\vec{c}$ and $\vec{r} = \vec{b} + \mu\vec{c}$

Sol^n

$\vec{n} = \overrightarrow{AB} \times \vec{c}$



Eqn of Plane.

$$\left[\vec{r}-\vec{a} \quad \vec{b}-\vec{a} \quad \vec{c}\right] = 0$$

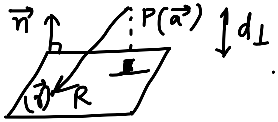
PERPENDICULAR DISTANCE OF A POINT 'P' FROM A PLANE Ax + By + Cz + D = 0 :

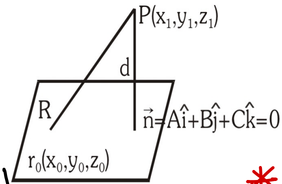
d = | Projection of $\overrightarrow{RP}$ on $\vec{n}$ |

$$d = \left|\frac{\overrightarrow{RP}.\vec{n}}{|\vec{n}|}\right| = \left|\frac{Ax_1 + By_1 + Cz_1 + D}{\sqrt{A^2 + B^2 + C^2}}\right|$$

Rem

$P: \vec{r}.\vec{n} = d$





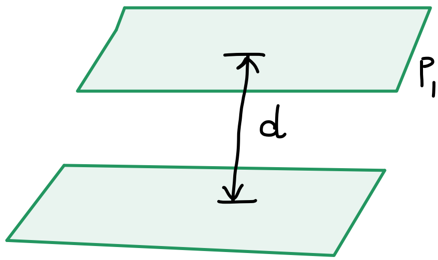
$d_\perp = |\overrightarrow{PR}.\hat{n}|$

$= \dfrac{|(\vec{r}-\vec{a}).\vec{n}|}{|\vec{n}|} = \dfrac{|\vec{a}.\vec{n} - d|}{|\vec{n}|}$ **

Note : Distance between the parallel planes $\left|\dfrac{d_1 - d_2}{\sqrt{a^2 + b^2 + c^2}}\right|$

(Take any point 'P' $(x_1 y_1 z_1)$ on one plane and from P draw perpendicular on the other plane)

** Coeff of x, y, z must be equal in eqn of 2 planes & Constant term must be one same side of eqn of plane.



$P_1: ax+by+cz+d_1 = 0$

$P_2: ax+by+cz+d_2 = 0.$

ANGLE BETWEEN TWO PLANES :

(a) The angle θ, between the two planes $\vec{r}.\vec{n}_1 = q_1$ and $\vec{r}.\vec{n}_2 = q_2$, being equal to the angle between the vectors $\vec{n}_1$ and $\vec{n}_2$ which are normal to the plane, we have $\theta = \cos^{-1}\dfrac{n_1.n_2}{|\vec{n}_1||\vec{n}_2|}$.

(b) Let the plane $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are

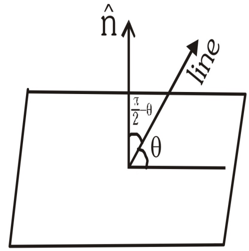
$$\cos\theta = \frac{a_1a_2 + b_1b_2 + c_1c_2}{\sqrt{a_1^2 + b_1^2 + c_1^2}\sqrt{a_2^2 + b_2^2 + c_2^2}}$$

(i) Parallel & distinct if $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2} \neq \dfrac{d_1}{d_2}$

(ii) Perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$

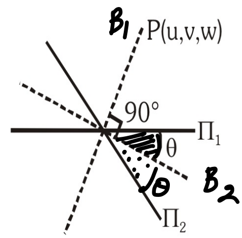
(iii) Identical if $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2} = \dfrac{d_1}{d_2}$

ANGLE BETWEEN A LINE AND A PLANE :



The angle θ, between any line $\vec{r} = \vec{a} + \vec{b}\,\vec{t}$ and any plane $\vec{r}.\vec{n} = q$, being equal to the complement of the angle between the normal vector $\vec{n}$, of the plane and the direction vectors $\vec{b}$ of the line, we have $\theta = \sin^{-1}\left(\dfrac{\vec{n}.\vec{b}}{|\vec{n}|\,|\vec{b}|}\right)$.

EQUATION OF THE BISECTOR PLANES BETWEEN THE PLANES :



$\Pi_1 : a_1x + b_1y + c_1y + d_1 = 0$ and .... (i)

$\Pi_1 : a_2x + b_2y + c_2y + d_2 = 0$ is .... (ii)

$$\boxed{\frac{a_1x+b_1y+c_1z+d_1}{\sqrt{a_1^2+b_1^2+c_1^2}} = \pm\frac{a_2x+b_2y+c_2z+d_2}{\sqrt{a_2^2+b_2^2+c_2^2}}}$$

Acute/Obtuse angle bisectors can be easily isolated by finding Cos θ

where θ is the angle between any one of the two given planes and any one of the two bisector planes.

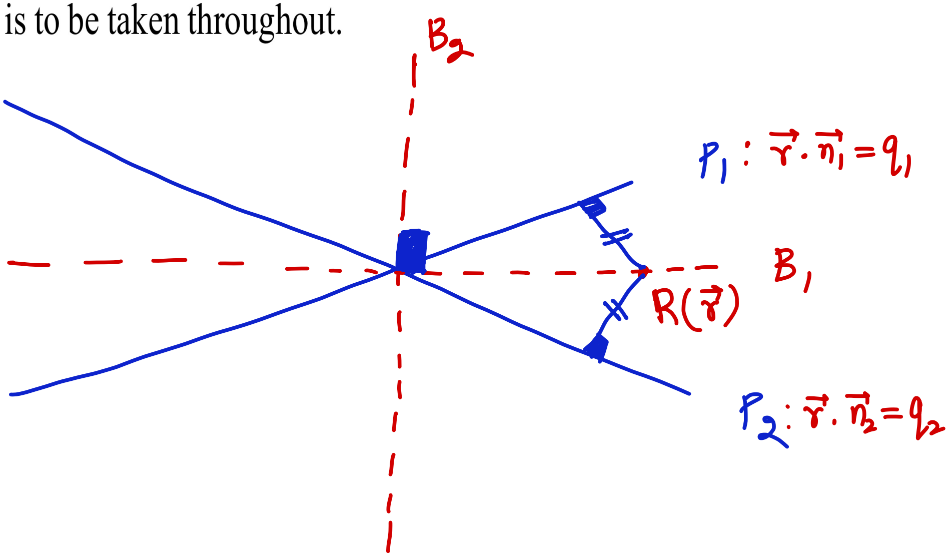
$$\boxed{\text{if } \frac{1}{\sqrt{2}} < |\cos\theta| < 1 \Rightarrow \theta \text{ is acute ; } \quad \text{if } 0 < |\cos\theta| < \frac{1}{\sqrt{2}} \Rightarrow \theta \text{ is obtuse}}$$

Vectorially : (Eqn of bisector planes)

Let $\vec{r}.\vec{n}_1 = q_1$ and $\vec{r}.\vec{n}_2 = q_2$ be the given planes. Perpendicular distance of any point $\vec{r}$ on either bisecting planes from the two given planes being equal, hence

$$\frac{|\vec{r}.\vec{n}_1 - q_1|}{|\vec{n}_1|} = \frac{|\vec{r}.\vec{n}_2 - q_2|}{|\vec{n}_2|} \Rightarrow \frac{\vec{r}.\vec{n}_1 - q_1}{|\vec{n}_1|} = \pm \frac{\vec{r}.\vec{n}_2 - q_2}{|\vec{n}_2|} \text{ or } \vec{r}.\left(\frac{\vec{n}_1}{|\vec{n}_1|} \pm \frac{\vec{n}_2}{|\vec{n}_2|}\right) = \left(\frac{q_1}{|\vec{n}_1|} \pm \frac{q_2}{|\vec{n}_2|}\right)$$

where same sign is to be taken throughout.



Q Find the equation of the plane which is parallel to the plane x + 5y – 4z + 5 = 0 and the sum of whose intercepts on the co-ordinate axes is 19 units. Also find the distance between these planes.

Sol^n

$$P: \ x + 5y - 4z = \lambda \quad -①-$$

$$x_{int} + y_{int} + z_{int} = 19 \Rightarrow \lambda + \frac{\lambda}{5} - \frac{\lambda}{4} = 19.$$

$$\frac{20\lambda + 4\lambda - 5\lambda}{20} = 19$$

$$\boxed{\lambda = 20}$$

$$P: \ x + 5y - 4z = 20.$$
$$P_1: \ x + 5y - 4z + 5 = 0.$$
$$P_1: \ x + 5y - 4z = -5$$

$$\frac{|20 - (-5)|}{\sqrt{1^2 + 5^2 + (-4)^2}} = d_\perp$$

Q Find the equation of the plane parallel to 2x – 6y + 3z = 0 and at a distance of 2 from the point (1, 2, –3).

Sol^n

$$P: \ 2x - 6y + 3z + \lambda = 0.$$

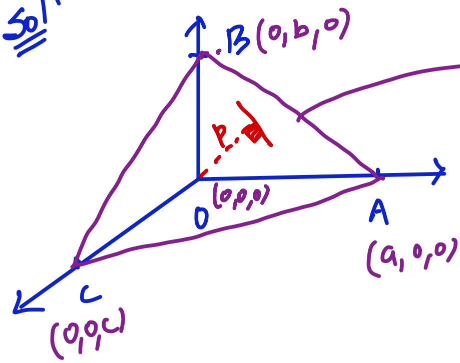
$$\frac{|2(1) - 6(2) + 3(-3) + \lambda|}{\sqrt{2^2 + 6^2 + 3^2}} = 2.$$

$$\lambda \begin{cases} \lambda_1 = \\ \lambda_2 = \end{cases}$$

Q A plane which always remains at a constant distance p from the origin cuts the co-ordinate axes at A, B, C. Find the locus of

(i) Centroid of the plane face ABC (ii) Centre of the tetrahedron OABC

Sol^n



$P: \frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1.$

$$\frac{|0+0+0-1|}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}}=p.$$

$$\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}=\frac{1}{p^2} \quad -①-$$

(i) $G(\alpha,\beta,\gamma)$

$\alpha=\frac{a}{3};\ \beta=\frac{b}{3};\ \gamma=\frac{c}{3}$

$$\frac{1}{(3\alpha)^2}+\frac{1}{(3\beta)^2}+\frac{1}{(3\gamma)^2}=\frac{1}{p^2}$$

$$\boxed{\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}=\frac{9}{p^2}}$$

(ii) $G(\alpha,\beta,\gamma)$

$\alpha=\frac{a}{4};\ \beta=\frac{b}{4};\ \gamma=\frac{c}{4}.$

CONDITION FOR LINE TO LIE COMPLETELY IN PLANE :

Let plane $\vec{r}.\vec{n}=d$ & line is $\vec{r}=\vec{a}+\lambda\vec{b}$.

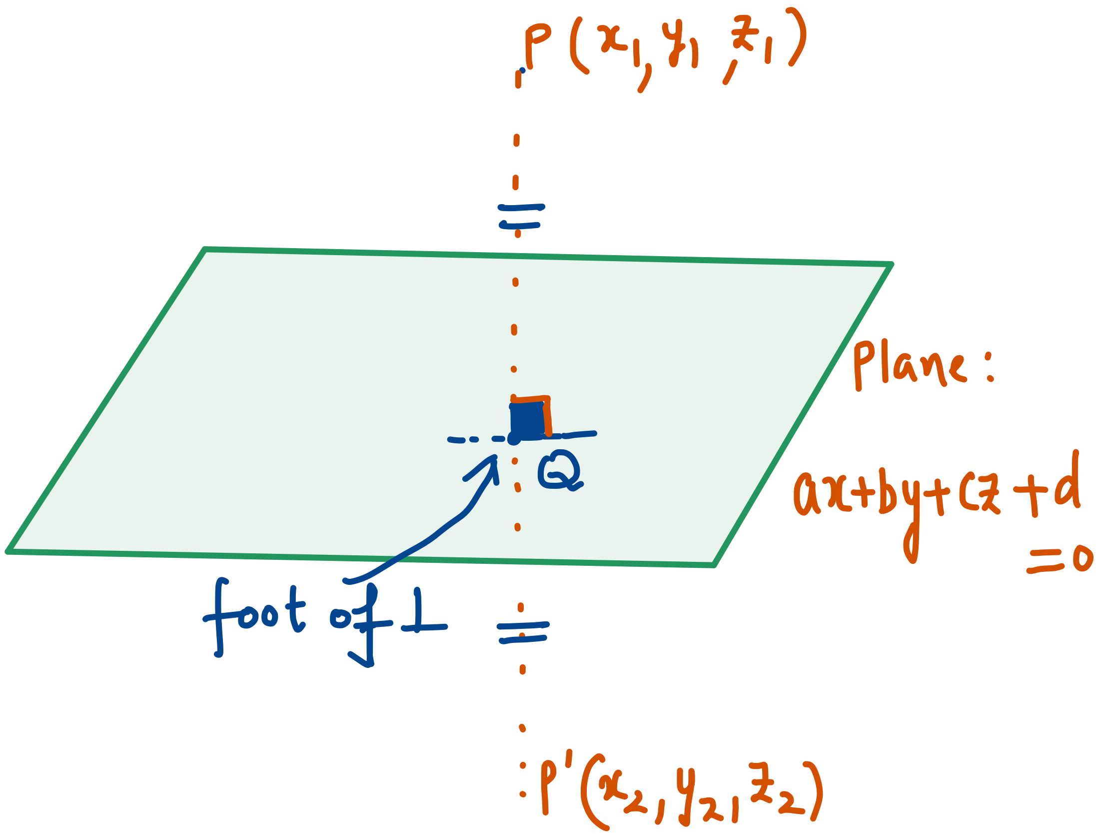
$\vec{n}.\vec{b}=0 \quad \Rightarrow \quad$ line is parallel to plane.

Note :

(i) If a line is parallel to the plane then normal of the plane & the line are perpendicular.

(ii) If a line is perpendicular to the plane then the line & normal are parallel.

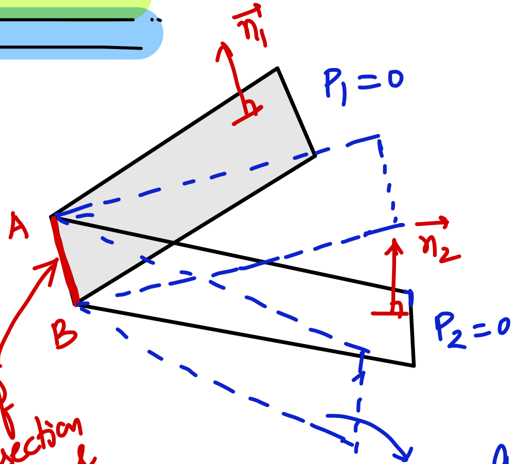
Image of a point in a Plane :-



Rem

$$\frac{x_2-x_1}{a} = \frac{y_2-y_1}{b} = \frac{z_2-z_1}{c} = \frac{-2(ax_1+by_1+cz_1+d)}{a^2+b^2+c^2}$$

Family of Planes :-



line of intersection of $P_1=0$ & $P_2=0$

$\vec{n_1} \times \vec{n_2}$

↓

direction vector of line AB

$aP_1 + bP_2 = 0$; $a, b \in$ scalars

if $a \neq 0$

$$P_1 + \frac{b}{a} P_2 = 0$$

⇓

Rem

$$\boxed{P_1 + \lambda P_2 = 0}$$; $\lambda \in$ scal-ar

family of planes containing line of intersection of $P_1 = 0$ & $P_2 = 0$.

Q To find the equation of the plane coaxial with the planes $\vec{r}.\vec{n}_1 = q_1$ and $\vec{r}.\vec{n}_2 = q_2$ .... (i)

and passing through the point with position vector $\vec{a}$.

Solⁿ
$$P_1 + \lambda P_2 = 0 \Rightarrow (\vec{r}\cdot\vec{n}_1 - q_1) + \lambda(\vec{r}\cdot\vec{n}_2 - q_2) = 0$$

$$\vec{r}\cdot\underbrace{(\vec{n}_1 + \lambda\vec{n}_2)}_{\text{normal vector}} = q_1 + \lambda q_2$$

$\vec{r} = \vec{a}$

get $\lambda = ?$

Q Find the equation of the plane containing the line of intersection of the planes $\vec{r}.\vec{n}_1 = q_1$; $\vec{r}.\vec{n}_2 = q_2$ and is parallel to the line of intersection of the planes $\vec{r}.\vec{n}_3 = q_3$ and $\vec{r}.\vec{n}_4 = q_4$.

Solⁿ
$$P_1 + \lambda P_2 = 0 \Rightarrow (\vec{r}\cdot\vec{n}_1 - q_1) + \lambda(\vec{r}\cdot\vec{n}_2 - q_2) = 0$$

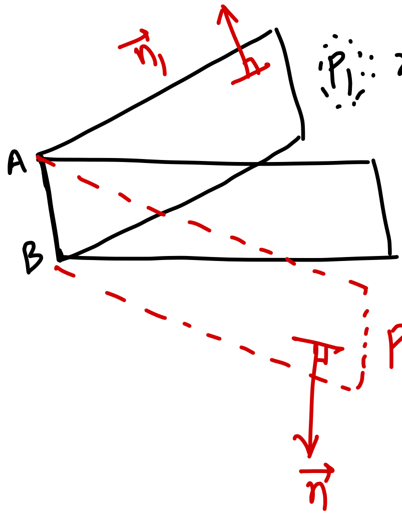
$$\vec{r}\cdot\underbrace{(\vec{n}_1 + \lambda\vec{n}_2)}_{\text{normal vector}} = q_1 + \lambda q_2$$

$$(\vec{n}_1 + \lambda\vec{n}_2)\cdot(\vec{n}_3 \times \vec{n}_4) = 0.$$

$$[\vec{n}_1\ \vec{n}_3\ \vec{n}_4] + \lambda[\vec{n}_2\ \vec{n}_3\ \vec{n}_4] = 0.$$

Q The plane x − y − z = 4 is rotated through 90° about its line of intersection with the plane x + y + 2z = 4. Find its equation in the new position.

Sol^n



$P_1: x - y - z - 4 = 0$

$P_2: x + y + 2z - 4 = 0$

$\boxed{\vec{n}_1 \cdot \vec{n} = 0}$

$P: \boxed{P_1 + \lambda P_2 = 0}$

$$x(1+\lambda) + y(-1+\lambda) + z(-1+2\lambda) - 4 - 4\lambda = 0.$$

$$(1+\lambda)1 - 1(\lambda - 1) - 1(2\lambda - 1) = 0$$

$$\cancel{\lambda} + 1 - \cancel{\lambda} + 1 - 2\lambda + 1 = 0$$

$$\lambda = 3/2$$

STRAIGHT LINES

SYMMETRICAL FORM OF STRAIGHT LINE (CARTESIAN FORM) :

(i) $\frac{x-x_1}{a}=\frac{y-y_1}{b}=\frac{z-z_1}{c}=\lambda$ or $\frac{x-x_1}{\ell}=\frac{y-y_1}{m}=\frac{z-z_1}{n}=\lambda$ (derived from $\vec{r}=\vec{r}_0+\lambda\vec{v}$)

(ii) $\frac{x-x_1}{x_2-x_1}=\frac{y-y_1}{y_2-y_1}=\frac{z-z_1}{z_2-z_1}$ (Two point form)

where a, b, c are the dr's of line or the vector along which the line is travelling.

Note: Equation of x-axis $\frac{x}{1}=\frac{y}{0}=\frac{z}{0}$; y-axis $\frac{x}{0}=\frac{y}{1}=\frac{z}{0}$; z-axis $\frac{x}{0}=\frac{y}{0}=\frac{z}{1}$, 0 in denominator shows that the line is perpendicular to the axis.

eg:- $\frac{x-2}{3}=\frac{y+1}{-2}=\frac{z-2}{0}$ OR $\frac{x-2}{3}=\frac{y+1}{-2}$ & $z=2$ represent a line parallel to xy plane at a distance 2 units.

general point form of line.

$x=\lambda+1$ $\quad z=2\lambda-5$

$y=\frac{3}{2}\lambda-\frac{3}{2}$

$\lambda \in$ scalar

eg:- $\frac{x-1}{1}=\frac{2y+3}{3}=\frac{z+5}{2}$ passes through $1, -\frac{3}{2}$ & -5 with dr's $1, \frac{3}{2}$ & 2.

$\frac{x-1}{1}=\frac{2(y+3/2)}{3}=\frac{z+5}{2}$

L: $\frac{x-1}{1}=\frac{y-(-3/2)}{3/2}=\frac{z-(-5)}{2}$

dr's $(1, \frac{3}{2}, 2)$ or

$(2, 3, 4)$ or

$(\sqrt{2}, \frac{3}{\sqrt{2}}, 2\sqrt{2})$ & so on

L: $\frac{x-2}{2}=\frac{y-0}{3}=\frac{z+3}{4}$

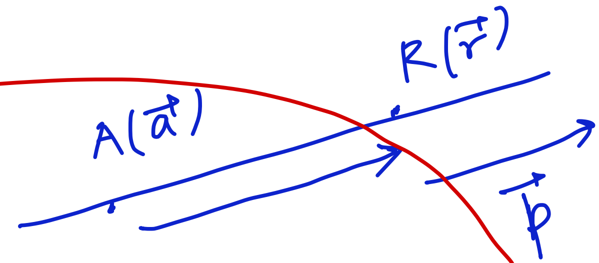
Vector form:

$\vec{r}=(2\hat{i}-3\hat{k})+\mu(2\hat{i}+3\hat{j}+4\hat{k})$

$\boxed{\vec{r}=\vec{a}+\mu\vec{p}}$

Vector eqn of line:

$$\vec{r} = \vec{a} + \lambda \vec{p}.$$



$$\vec{r} - \vec{a} = \lambda \vec{p}$$

$$\vec{a} = a_1\hat{i} + a_2\hat{j} + a_3\hat{k}$$

$$\vec{p} = \alpha\hat{i} + \beta\hat{j} + \gamma\hat{k}$$

$$\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}.$$

$$(x-a_1)\hat{i} + (y-a_2)\hat{j} + (z-a_3)\hat{k} = \lambda(\alpha\hat{i} + \beta\hat{j} + \gamma\hat{k})$$

$$\left.\begin{aligned} x - a_1 &= \lambda\alpha \\ y - a_2 &= \lambda\beta \\ z - a_3 &= \lambda\gamma \end{aligned}\right\} \Rightarrow \boxed{\frac{x-a_1}{\alpha} = \frac{y-a_2}{\beta} = \frac{z-a_3}{\gamma} = \lambda \text{ (say)}}$$

Cartesian form or Symmetric form

Q Convert the equation $3x + 1 = 6y - 2 = 1 - z$ in vector form and find its direction ratios.

$$3\left(x-\left(-\tfrac{1}{3}\right)\right) = 6\left(y-\tfrac{1}{3}\right) = -(z-1)$$

$$\frac{x-\left(-1/3\right)}{1/3} = \frac{y-1/3}{1/6} = \frac{z-1}{-1}$$

dr's $\left(\frac{1}{3}, \frac{1}{6}, -1\right)$ or

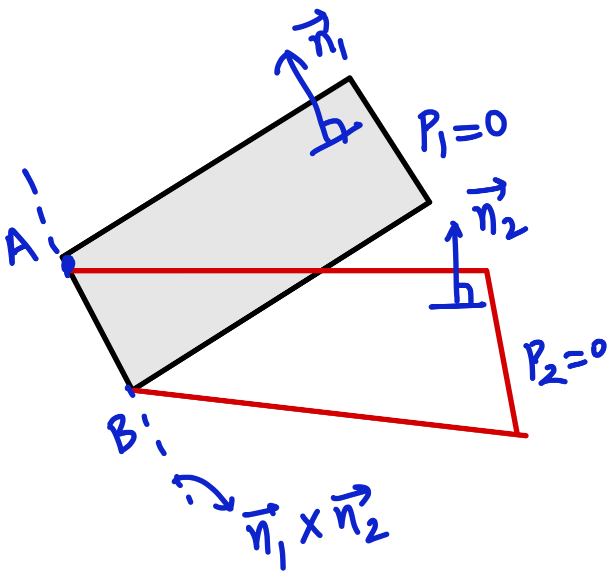
$(2, 1, -6)$ or . . . .

UNSYMMETRICAL FORM OF STRAIGHT LINE :

The equations $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ together represents a line in unsymmetrical form.

$a_1x+b_1y+c_1z+d_1 = a_2x+b_2y+c_2z+d_2=0.$

Note : Vector along the line of intersection of planes $\vec{r}.\vec{n}_1 = q_1$ and $\vec{r}.\vec{n}_2 = q_2$ is $\vec{n}_1 \times \vec{n}_2$.



Q1 Find eqn of line in symmetrical form

$$x+y+z-6=0=2x+y-z-1$$

Sol^n

$$\vec{n}_1 \times \vec{n}_2 = a\hat{i}+b\hat{j}+c\hat{k}.$$

$$\left.\begin{aligned} x+y &= 6 \\ 2x+y &= 1 \end{aligned}\right\} \qquad \begin{aligned} x &= -5 \\ y &= 11 \\ z &= 0 \end{aligned}$$

$$L: \quad \frac{x+5}{a} = \frac{y-11}{b} = \frac{z-0}{c}$$

DEFINITION :

A straight line in space is characterised by the intersection of two planes which are not parallel and, therefore, the equation of a straight line is present as a solution of the system constituted by the equations of the two planes : $a_1 x + b_1 y + c_1 z + d_1 = 0$; $a_2 x + b_2 y + c_2 z + d_2 = 0$

This form is also known as **unsymmetrical form**.

Some particular straight lines :

	Straight lines	Equation
(i)	Through the origin	$y = mx,\ z = nx$
(ii)	x-axis	$y = 0,\ z = 0$ or $\frac{x}{1} = \frac{y}{0} = \frac{z}{0}$
(iii)	y-axis	$x = 0,\ z = 0$ or $\frac{x}{0} = \frac{y}{1} = \frac{z}{0}$
(iv)	z-axis	$x = 0,\ y = 0$ or $\frac{x}{0} = \frac{y}{0} = \frac{z}{1}$
(v)	parallel to x-axis	$y = p,\ z = q$
(vi)	parallel to y-axis	$x = h,\ z = q$
(vii)	parallel to z-axis	$x = h,\ y = p$

EQUATION OF A STRAIGHT LINE IN SYMMETRICAL FORM :

(a) **One point form :** Let $A(x_1, y_1, z_1)$ be a given point on the straight line and ℓ, m, n be the d.c's of the line, then its equation is

$$\boxed{\frac{x - x_1}{\ell} = \frac{y - y_1}{m} = \frac{z - z_1}{n} = r \quad \text{(say)}}$$

It should be noted that $P(x_1 + \ell r, y_1 + mr, z_1 + nr)$ is a general point on this line at a distance r from the point $A(x_1, y_1, z_1)$ i.e. $AP = r$. One should note that for $AP = r$; ℓ, m, n must be d.c.'s not d.r.'s. If a, b, c are direction ratios of the line, then equation of the line is

$$\boxed{\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c} = r} \quad \text{but here } AP \neq r$$

(b) Equation of the line through two points $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ is

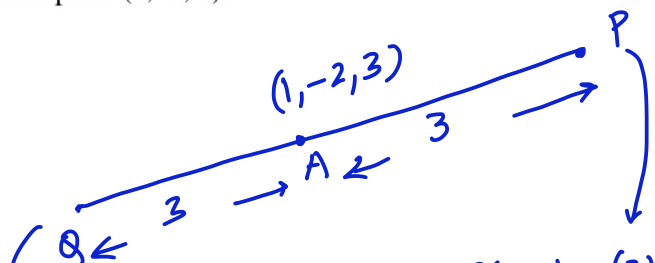
$$\boxed{\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}}$$

Q Find the co-ordinates of those points on the line $\frac{x-1}{2}=\frac{y+2}{3}=\frac{z-3}{6}$ which is at a distance of 3 units from point (1,–2, 3).

dr's: $(2, 3, 6)$

dc's: $\frac{2}{7}, \frac{3}{7}, \frac{6}{7}$

Sol^n



$x = 1 + (-3)\left(\frac{2}{7}\right)$

$y = -2 + (-3)(3/7)$

$z = 3 + (-3)(6/7)$

$x = 1 + (3)\left(\frac{2}{7}\right)$

$y = -2 + (3)\left(\frac{3}{7}\right)$

$z = 3 + 3(6/7)$

Q Find the points in which the line x = 1 + 2t ; y = –1 – t and z = 3t meets the coordinate planes i.e. xy, yz, zx plane P, Q, R, Find the equation of the plane containing the point P, Q, R.

Sol^n

$xy\text{-plane} \rightarrow z = 0 \Rightarrow 3t = 0 \Rightarrow t = 0$

$x = 1 + 2t \Rightarrow x = 1$

$y = -1 - t \Rightarrow y = -1$

$P(1, -1, 0)$

$\|^{ly}$. proceed ursef

Q Show that the straight lines

$$L_1 : 3x + 2y + z - 5 = 0 = x + y - 2z - 3$$

and $L_2 : 8x - 4y - 4z = 0 = 7x + 10y - 8z$ are at right angle.

Sol^n

$$(\vec{n}_1 \times \vec{n}_2) \cdot (\vec{n}_3 \times \vec{n}_4) = 0$$

$$(24 - 8 - 4)(7 + 10 + 16) - (21 + 20 - 8)(8 - 4 + 8)$$

$$(12)(33) - (33)(12) = 0. \qquad \text{(H.P)}$$

Q Find the equation of the line through (1, 4, –2) and parallel to the planes $6x + 2y + 2z + 3 = 0$ & $x + 2y - 6z + 4 = 0$.

Sol^n $L: \quad \dfrac{x-1}{a} = \dfrac{y-4}{b} = \dfrac{z+2}{c}$

$$\vec{n}_1 \times \vec{n}_2 = a\hat{i} + b\hat{j} + c\hat{k}.$$

Q Find the equation of the straight line which passes through the point (2, –1, –1) ; is parallel to the plane $4x + y + z + 2 = 0$ and is perpendicular to the line of intersection of the planes $2x + y = 0 = x - y + z$.

Sol^n

$$L: \quad \frac{x-2}{a} = \frac{y+1}{b} = \frac{z+1}{c}$$

$$a\hat{i} + b\hat{j} + c\hat{k} = (4\hat{i} + \hat{j} + \hat{k}) \times \left((2\hat{i} + \hat{j}) \times (\hat{i} - \hat{j} + \hat{k}) \right)$$

(use VTP)

Q If the lines $\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{z-z_1}{n_1}$ and $\frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2}$ are coplanar/intersect then

Value of $\begin{vmatrix} x_1-x_2 & y_1-y_2 & z_1-z_2 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = ?$ $\quad$ 0

Q Find the equation of the line through the point with p.v. $\vec{c}$ and || to the plane $\vec{r}.\vec{n}=1$ and perpendicular to the line $\vec{r}=\vec{a}+t\vec{b}$ $\quad \vec{r}=\vec{c}+\lambda(\vec{n}\times\vec{b})$ Ans

Q Find the equation of the lines passing through the point with p.v. $\vec{a}$ and parallel to the line of intersection of the planes $\vec{r}.\vec{n}_1=1$ and $\vec{r}.\vec{n}_2=1$.

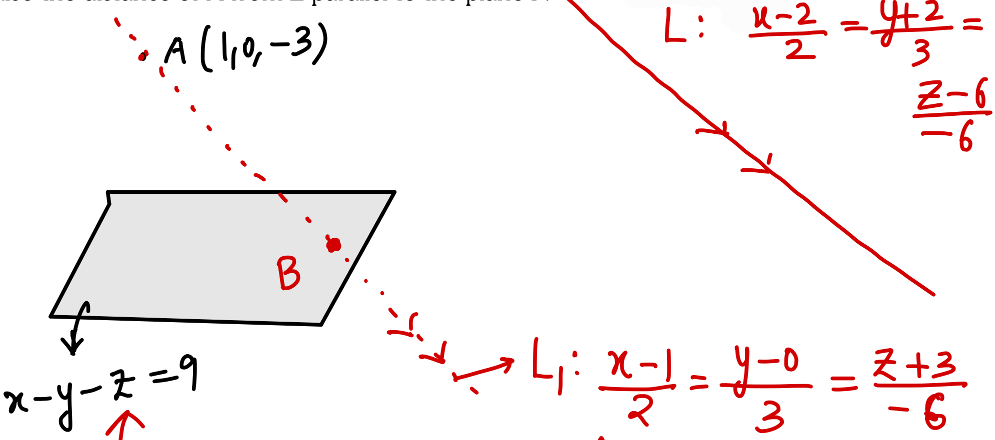
$$\vec{r}=\vec{a}+\lambda(\vec{n}_1\times\vec{n}_2)$$

$\lambda \in$ scalar

E(a) Find the distance of the point A(1, 0, –3) from the plane P : $x - y - z = 9$ measured parallel to the line L : $\frac{x-2}{2} = \frac{y+2}{3} = \frac{z-6}{-6}$.

(b) Find also the distance of A from L parallel to the plane P.

(a)



general pt

$(2\lambda+1,\ 3\lambda,\ -6\lambda-3)$

$(2\lambda+1) - 3\lambda + 6\lambda + 3 = 9$

$5\lambda = 5 \Rightarrow \lambda = 1$

$B(3, 3, -9)$

$AB = \sqrt{2^2 + 3^2 + 6^2} = 7$

(b) Eqn of plane P_1 || to plane $P = 0$ and containing $A(1, 0, -3)$

$P_1 : 1(x-1) - 1(y-0) - 1(z+3) = 0.$

Solve $P_1 = 0$ and Line $L = 0$ to get B

$P_1 : x - y - z - 4 = 0$.

$L : \dfrac{x-2}{2} = \dfrac{y+2}{3} = \dfrac{z-6}{-6}$

general pt : $(2\lambda+2, 3\lambda-2, -6\lambda+6)$

$(2\lambda+2) - (3\lambda-2) - (-6\lambda+6) - 4 = 0$

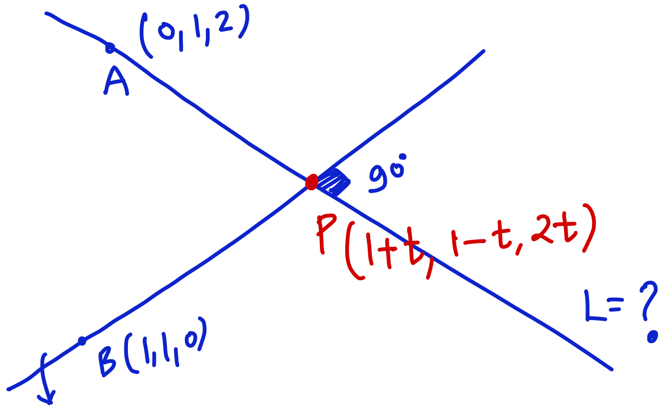
$\lambda =$

$B(\quad)$

$|AB| = ?$

Q Find the parametric equation for the line which passes through the point (0, 1, 2) and is perpendicular to the line x = 1 + t, y = 1 – t and z = 2t and also intersects this line.

Sol^n



$\overrightarrow{AP}\cdot(\hat{i}-\hat{j}+2\hat{k}) = 0.$

$\downarrow$

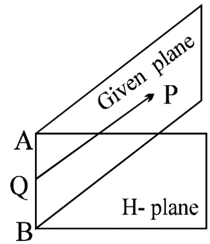
get $t = ?$

pv of P(   )

$L_1: \dfrac{x-1}{1} = \dfrac{y-1}{-1} = \dfrac{z-0}{2}$

Q HW Find the equation of the two lines through the origin which intersect the line $\frac{x-3}{2}=\frac{y-3}{1}=\frac{z}{1}$ at an angle of $\frac{\pi}{3}$. [Ans: $\frac{x}{1}=\frac{y}{2}=\frac{z}{-1}$ or $\frac{x}{-1}=\frac{y}{1}=\frac{z}{-2}$]

Line of Greatest slope in a plane



It is a line in the plane and perpendicular to the line of intersection of the given plane with the horizontal plane. PQ is the line of greatest slope . Its direction cosines can be determined by the facts that

(i) It lies in G-plane

(ii) It is perpendicular to AB, the line of intersection of G and H plane.

Example

Assuming the plane $4x - 3y + 7z = 0$ to be horizontal find the equation of the line of greatest slope through the point (2, 1, 1) in the plane $2x + y - 5z = 0$.

$$L_1: \quad \frac{x-2}{a} = \frac{y-1}{b} = \frac{z-1}{c} \quad \text{— (1) —}$$

$$a\hat{i} + b\hat{j} + c\hat{k} = (2\hat{i}+\hat{j}-5\hat{k}) \times \left((4\hat{i}-3\hat{j}+7\hat{k}) \times (2\hat{i}+\hat{j}-5\hat{k})\right)$$

Q. Determine whether each statement is true or false

(a) Two lines parallel to a third line are parallel. T

(b) Two lines perpendicular to a third line are parallel. F

(c) Two planes parallel to a third plane are parallel. T

(d) Two planes perpendicular to a third plane are parallel. F

(e) Two lines parallel to a plane are parallel. F

(f) Two lines perpendicular to a plane are parallel. T

(g) Two planes parallel to a line are parallel. F

(h) Two planes perpendicular to a line are parallel. T

(i) Two planes either intersect or are parallel. T

(j) Two lines either intersect or are parallel. F

(k) A plane and a line either intersect or are parallel. T

Q If $l^2+m^2+n^2=125$, $a^2+b^2+c^2=5$ and $al+bm+cn=25$, where $a,b,c,l,m,n \in R$, then value of $\frac{lmn}{abc}$ is μ, where sum of digits of μ is

8 Ans

Sol^n

$\vec{v_1} = l\hat{i} + m\hat{j} + n\hat{k}$; $|\vec{v_1}| = \sqrt{125}$

$\vec{v_2} = a\hat{i} + b\hat{j} + c\hat{k}$; $|\vec{v_2}| = \sqrt{5}$

$\vec{v_1} \cdot \vec{v_2} = al + bm + cn = |\vec{v_1}||\vec{v_2}| \cos\theta$

$25 = \sqrt{125}\sqrt{5}\cos\theta$

$\cos\theta = 1 \Rightarrow \boxed{\theta = 0}$

$\therefore$ $\vec{v_1}$ is collinear with $\vec{v_2}$.

$\frac{l}{a} = \frac{m}{b} = \frac{n}{c} = \lambda$ (say) ; $\lambda > 0$

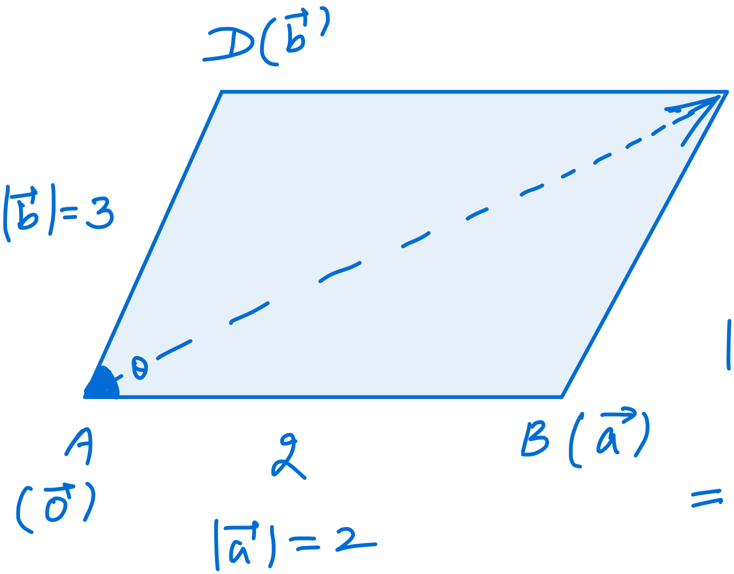
$\boxed{l = a\lambda}$; $\boxed{m = b\lambda}$; $\boxed{n = c\lambda}$

$l^2+m^2+n^2 = 125$

$(a^2+b^2+c^2)\lambda^2 = 125 \Rightarrow \lambda^2 = 25 \Rightarrow \boxed{\lambda = 5}$

$\frac{lmn}{abc} = \left(\frac{l}{a}\right)\left(\frac{m}{b}\right)\left(\frac{n}{c}\right) = \lambda^3 = 125$

Q Given in a parallelogram ABCD, AB = 2, AD = 3 and M, m denotes the maximum and minimum integral value of product $|\vec{AC}||\vec{BD}|$ then $(M-m)$ is ______



$$\vec{AC} = \vec{a}+\vec{b}$$
$$\vec{BD} = \vec{b}-\vec{a}$$

$$|\vec{AC}||\vec{BD}| = |\vec{a}+\vec{b}||\vec{b}-\vec{a}|$$
$$= \left(\sqrt{a^2+b^2+2\vec{a}\cdot\vec{b}}\right)\left(\sqrt{a^2+b^2-2\vec{a}\cdot\vec{b}}\right)$$

**

$$\boxed{0 \le \cos^2\theta < 1}$$

$$= \sqrt{13+12\cos\theta}\sqrt{13-12\cos\theta}$$

$$E = \sqrt{169-144\underbrace{\cos^2\theta}}.$$

$$E \Rightarrow (5, 13]$$

$M = 13$

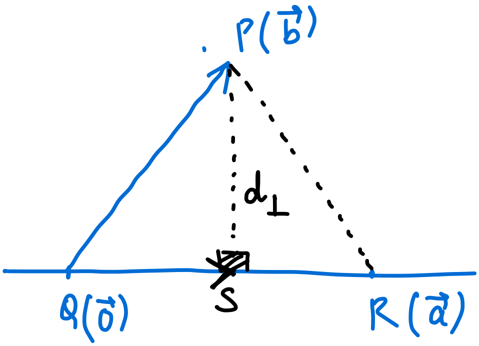
$m = 6$

$(M-m) = 7$ Ans

Q Let P be a point not on the line L that passes through the points Q and R where $\overrightarrow{QR} = \vec{a}$ & $\overrightarrow{QP} = \vec{b}$. The distance d from the point P to the line L is equal to-

(A) $\dfrac{|\vec{a} \times \vec{b}|}{|\vec{b}|}$ (B) $\dfrac{|\vec{a} \times \vec{b}|}{|\vec{a}|}$ (C) $\left|\vec{b} - \left(\dfrac{\vec{a}.\vec{b}}{\vec{a}^2}\right)\vec{a}\right|$ (D) $\sqrt{|\vec{b}|^2 - \left(\dfrac{\vec{b}.\vec{a}}{|\vec{a}|}\right)^2}$

Sol^n



$\Delta PQR = \frac{1}{2}|\vec{a} \times \vec{b}|$

$= \frac{1}{2}|\overrightarrow{QR}| \cdot d_\perp$

$d_\perp = \dfrac{|\vec{a} \times \vec{b}|}{|\vec{a}|}$

$\overrightarrow{QR} = \vec{a}$

$\overrightarrow{QP} = \vec{b}$

$\overrightarrow{QS} + \overrightarrow{SP} = \overrightarrow{QP}$ —①—

$\overrightarrow{QS} = |\text{projection of } \vec{b} \text{ on } \vec{a}|\, \hat{a}$

$\overrightarrow{QS} = (\vec{b} \cdot \hat{a})\, \hat{a}$

$\overrightarrow{SP} = \overrightarrow{QP} - \overrightarrow{QS}$

$= \vec{b} - (\vec{b} \cdot \hat{a})\, \hat{a}$

$d_\perp = |\overrightarrow{SP}| = \left|\vec{b} - \left(\dfrac{\vec{b} \cdot \vec{a}}{|a|^2}\right)\vec{a}\right|$

$PS^2 = PQ^2 - QS^2$

$= b^2 - ((\vec{b} \cdot \hat{a})\hat{a})^2$

Q

The angle θ between two non-zero vectors $\vec{a}$ & $\vec{b}$ satisfies the relation

$\cos\theta = (\vec{a}\times\hat{i}).(\vec{b}\times\hat{i}) + (\vec{a}\times\hat{j}).(\vec{b}\times\hat{j}) + (\vec{a}\times\hat{k}).(\vec{b}\times\hat{k})$,

then the least value of $|\vec{a}|+|\vec{b}|$ is equal to (where $\theta \neq 90°$)

(A) $\frac{1}{2}$ (B) 2 (C) $\sqrt{2}$ (D) 4

Sol^n

$$\cos\theta = (\vec{a}.\vec{b}) - (\vec{a}.\hat{i})(\vec{b}.\hat{i}) + (\vec{a}.\vec{b}) - (\vec{a}.\hat{j})(\vec{b}.\hat{j}) + (\vec{a}.\vec{b}) - (\vec{a}.\hat{k})(\vec{b}.\hat{k})$$

$$\frac{\vec{a}.\vec{b}}{|\vec{a}||\vec{b}|} = 3(\vec{a}.\vec{b}) - (\vec{a}.\vec{b}) \Rightarrow |\vec{a}||\vec{b}| = \frac{1}{2}$$

AM $\geq$ GM

$$\frac{|\vec{a}|+|\vec{b}|}{2} \geq \sqrt{|\vec{a}||\vec{b}|}$$

$$|\vec{a}|+|\vec{b}| \geq 2\cdot\frac{1}{\sqrt{2}}$$

Q In a tetrahedron OABC, the measures of the $\angle BOC$, $\angle COA$ & $\angle AOB$ are α, β & γ respectively, then $(\cos^2\alpha + \cos^2\beta + \cos^2\gamma - 2\cos\alpha\cos\beta\cos\gamma)$ can attain-

(A) $\frac{1}{\sqrt{2}}$ (B) $\frac{\pi}{4}$ (C) 1 (D) 2

Q Let $\vec{a} = 3\hat{i} - 5\hat{k}$, $\vec{b} = 2\hat{i} + 7\hat{j}$ and $\vec{c} = \hat{i} + \hat{j} + \hat{k}$. Consider $\vec{r}$ such that $\vec{r}.\vec{a} = -1$, $\vec{r}.\vec{b} = 6$ and $\vec{r}.\vec{c} = 5$. Then the vector component of $2\hat{i} + 3\hat{j} + 4\hat{k}$ along $\vec{r}$ is $n\left(\frac{\ell\hat{i} + m\hat{k}}{\ell^2 + m^2}\right)$, where ℓ & m are coprimes, then $\ell^2 + m^2 + n^2$ is equal to

Q If $\hat{i}+\hat{j}$ bisects the angle between $\vec{c}$ & $\hat{j}+\hat{k}$, then $\vec{c}.\hat{j}$ is equal to

(A) 0 (B) $\frac{1}{\sqrt{2}}$ (C) $-\frac{1}{\sqrt{2}}$ (D) 1

Q If $x^2+y^2+z^2=1$ where $x, y, z \in \mathbb{R}$ and maximum value of $\underbrace{(2x-y)^2+(3y-2z)^2+(z-3x)^2}$ is λ then $\lambda =$?

Sol^n

$$\vec{v}_1 = x\hat{i}+y\hat{j}+z\hat{k}.$$
$$\vec{v}_2 = \hat{i}+2\hat{j}+3\hat{k}$$

$$\vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x & y & z \\ 1 & 2 & 3 \end{vmatrix} = \vec{v}$$

$$\overbrace{\vec{v}_1 \times \vec{v}_2}^{e} = (2x-y)\hat{k} + (3y-2z)\hat{i} - (3x-z)\hat{j}$$

$$|\vec{v}| = \sqrt{(2x-y)^2+(3y-2z)^2+(3x-z)^2}.$$

$$v^2_{max} = ?$$

$$|\vec{v}|_{max} = |\vec{v}_1||\vec{v}_2| \sin\theta.$$
$$= \sqrt{x^2+y^2+z^2}.$$
$$\sqrt{1^2+2^2+3^2}$$
$$\cdot (1)$$

$$|\vec{v}|_{max} = \sqrt{14}$$
$$|\vec{V}|^2_{max} = \boxed{14. = \lambda}$$ Ans

Q If $\vec{a}, \vec{b}, \vec{c}$ are mutually perpendicular vectors having magnitude 1, 2, 3 respectively then $[\vec{a}+\vec{b}+\vec{c} \quad \vec{b}-\vec{a} \quad \vec{c}] = ?$

Sol^n

$$R_1 \to R_1 + R_2 - R_3$$

$$[2\vec{b} \quad \vec{b}-\vec{a} \quad \vec{c}]$$

$$= \cancelto{0}{[2\vec{b} \quad \vec{b} \quad \vec{c}]} + [2\vec{b} \quad -\vec{a} \quad \vec{c}]$$

$$= 2[\vec{a} \quad \vec{b} \quad \vec{c}] = 2(1)(2)(3)$$

$$= \underline{\underline{12}}$$

Ans

Q. Let $|\vec{a}| = \sqrt{3}, |\vec{b}| = 5, \vec{b}.\vec{c} = 10$ angle between $\vec{b}$ & $\vec{c}$ equal to $\frac{\pi}{3}$. If $\vec{a}$ is perpendicular to $\vec{b} \times \vec{c}$ then find the value of $|\vec{a} \times (\vec{b} \times \vec{c})|$.

Sol^n

$$\vec{b}.\vec{c} = 10 \Rightarrow |\vec{b}||\vec{c}| \cos\frac{\pi}{3} = 10 \Rightarrow |\vec{c}| = 4.$$

$$\theta(\vec{a} \wedge \vec{b} \times \vec{c}) = 90^\circ$$

$$|\vec{a} \times (\vec{b} \times \vec{c})| = |\vec{a}|\,|\vec{b} \times \vec{c}| \sin 90^\circ$$

$$= \sqrt{3}\,(5)(4).\sin\frac{\pi}{3}.\,1$$

$$= \sqrt{3}.\,5.\,\cancel{4}^{2}.\,\frac{\sqrt{3}}{\cancel{2}}$$

$$= 30.$$ Ans

Paragraph

Consider a plane Π whose equation is $x - y = 0$ and two points $A(3,2,4)$ and $B(7,0,-10)$

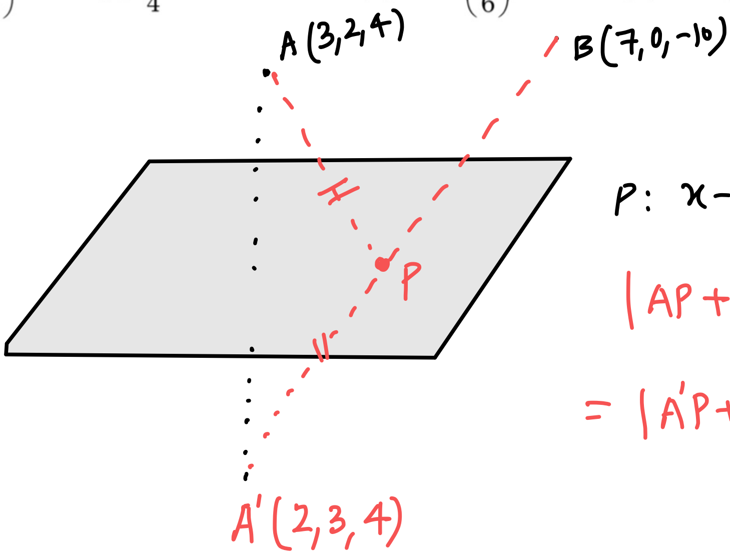
1. Coordinates of point P, which lies on plane Π such that $|PA + PB|$ is minimum, are-

(A) $(21,21,9)$ (B) $\left(\frac{21}{4},\frac{21}{4},\frac{9}{4}\right)$ (C) $\left(\frac{21}{8},\frac{21}{8},\frac{9}{4}\right)$ (D) $\left(\frac{21}{16},\frac{21}{16},\frac{9}{16}\right)$

2. Angle between $\overrightarrow{AB}$ and vector perpendicular to plane Π is-

(A) $\cos^{-1}\left(\frac{1}{2\sqrt{3}}\right)$ (B) $\frac{\pi}{4}$ (C) $\cos^{-1}\left(\frac{1}{6}\right)$ (D) $\cos^{-1}\left(\frac{1}{12}\right)$

Sol^n



P: $x - y = 0$

$|AP + PB|_{min}$

$= |A'P + PB|_{min}$

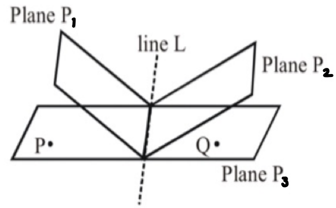
$A'B$: $\dfrac{x-2}{5} = \dfrac{y-3}{-3} = \dfrac{z-4}{-14} = \lambda$

general pt $(5\lambda+2, -3\lambda+3, -14\lambda+4)$

put in plane $x - y = 0$

$\lambda =$

Q In adjacent diagram equation of plane P_1 and P_2 are $x + y - 2z = 1$ and $\alpha x - y + z = 2$ respectively (where $\alpha \in Q$). Coordinates of points P and Q are $(\alpha, 1, 3)$ and $\left(\frac{5}{3}, 1, \alpha\right)$ respectively on the plane P_3. if ℓ, m, n are direction cosines of line L then absolute value of $\frac{\sqrt{35}(\ell + m + n)}{\alpha}$ is



$\frac{9}{2} = \underline{\underline{4.50}}$ Ans

Sol^n $P_3: \quad P_1 + \lambda P_2 = 0.$

$$(x+y-2z-1) + \lambda(\alpha x - y + z - 2) = 0$$

$P(\alpha, 1, 3)$ $\quad$ $Q\left(\frac{5}{3}, 1, \alpha\right)$

$$\boxed{2\alpha^3 + \alpha^2 - 19\alpha + 18 = 0} \Rightarrow \underbrace{\alpha = 2 \in Q}$$

$$dr's = \vec{n_1} \times \vec{n_2} = (\hat{i} + \hat{j} - 2\hat{k}) \times (2\hat{i} - \hat{j} + \hat{k})$$

$$= -\hat{i} - 5\hat{j} - 3\hat{k}$$

$$dc's \equiv \pm\left(\frac{1}{\sqrt{35}}, \frac{5}{\sqrt{35}}, \frac{3}{\sqrt{35}}\right)$$

Q The direction cosines of the projection of the line $\frac{1}{2}(x-1) = -y = z+2$ on the plane $2x + y - 3z = 4$ are-

$\vec{n} = 2\hat{i} + \hat{j} - 3\hat{k}$

(A) $\left(\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$

(B) $\left(\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}\right)$

(C) $\left(\frac{2}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{1}{\sqrt{6}}\right)$

(D) None of these

Sol^n L: $\frac{x-1}{2} = \frac{y}{-1} = \frac{z-(-2)}{1}$

$(2\hat{i} - \hat{j} + \hat{k}) \cdot (2\hat{i} + \hat{j} - 3\hat{k}) = 4 - 1 - 3 = 0.$

$\Rightarrow$ Line L is parallel to plane P.

Q The direction ratios of a normal to the plane passing through (1,0,0), (0,1,0) and making an angle $\frac{\pi}{4}$ with plane x + y = 3 can be-

(A) 0,1,0 (B) $1,1,\sqrt{2}$

(C) 1,0,0 (D) $\sqrt{2},1,1$

Sol^n Let equation of req. plane be

$$P: \quad a(x-1) + b(y-0) + c(z-0) = 0$$

(0,1,0)

$$a(-1) + b + 0 = 0 \Rightarrow \boxed{a = b} \text{ —①—}$$

$$\vec{n} = a\hat{i} + b\hat{j} + c\hat{k}$$

$$\vec{n_1} = \hat{i} + \hat{j}$$

$$\vec{n}\cdot\vec{n_1} = |\vec{n}||\vec{n_1}|\cos 45^\circ$$

$$a + b = \sqrt{a^2+b^2+c^2}\,\cancel{\sqrt{2}}\cdot\frac{1}{\cancel{\sqrt{2}}}$$

$$\cancel{a^2} + \cancel{b^2} + 2ab = \cancel{a^2} + \cancel{b^2} + c^2$$

$$2a^2 = c^2 \Rightarrow c = \sqrt{2}a$$

or

$$c = -\sqrt{2}a$$

$$a : b : c \equiv 1 : 1 : \sqrt{2}$$

$$1 : 1 : -\sqrt{2}$$

Q If for unit vectors $\hat{a}, \hat{b}$ and non-zero $\vec{c}$, $\hat{a} \times \hat{b} + \hat{a} = \vec{c}$ and $\hat{b}.\vec{c} = 0$, then volume of parallelopiped with coterminous edges $\hat{a}, \hat{b}$ and $\vec{c}$ will be (in cu.units)-

(A) 6

(B) 4

(C) 1 ✓

(D) $\frac{1}{2}$

Sol^n

$$V = [\hat{a}\ \hat{b}\ \vec{c}] = (\hat{a} \times \hat{b}).\vec{c}$$

$$\boxed{\hat{b}.\vec{c} = 0} \quad —②—$$

$$\hat{a} \times \hat{b} + \hat{a} = \vec{c} \quad —①—$$

dot with $\vec{c}$

$$\underbrace{(\hat{a} \times \hat{b}).\vec{c}} + \hat{a}.\vec{c} = c^2$$

$$\boxed{V = c^2 - \hat{a}.\vec{c}} \ ***$$

① dot with $\hat{b}$:—

$$\cancel{(\hat{a} \times \hat{b}).\hat{b}}^{\,0} + \hat{a}.\hat{b} = \cancel{\vec{c}.\hat{b}}^{\,0}$$

$$\Rightarrow \boxed{\hat{a}.\hat{b} = 0}$$

$$\theta = 90^\circ$$

① dot with $\hat{a}$:—

$$\cancel{(\hat{a} \times \hat{b}).\hat{a}}^{\,0} + \hat{a}^2 = \hat{a}.\vec{c}$$

$$\boxed{\hat{a}.\vec{c} = 1}$$

$$(\hat{a} \times \hat{b})^2 = (\vec{c} - \hat{a})^2 \Rightarrow \cancel{1} = c^2 + \cancel{1} - 2\hat{a}.\vec{c}$$

$$\boxed{c^2 = 2} *$$

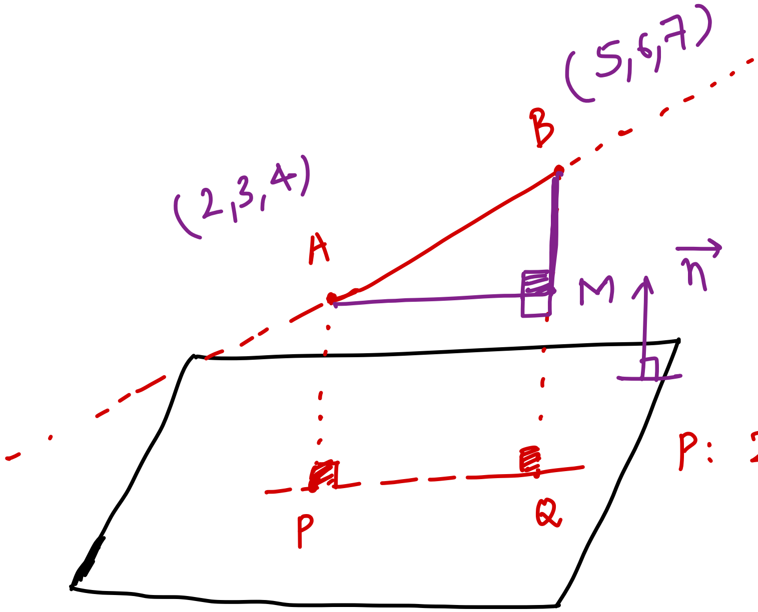
$$V = c^2 - \hat{a}.\vec{c} = 2 - 1 = 1 \text{ Ans}$$

Q The plane $2x-2y+z=3$ is rotated about the line where it cuts the xy plane by an acute angle α. If the new position of plane contains the point (3, 1, 1) then $\cos\alpha$ equal to______ $\frac{7}{9}$

Q Projection of line segment joining (2,3,4) and (5,6,7) on plane $2x + y + z = 1$ is :-

(A) 2 (B) $\sqrt{3}$ (C) 3 (D) $3\sqrt{3}$

Sol^n



$AM = PQ$

$P: 2x + y + z = 1.$

$AB = 3\sqrt{3}$

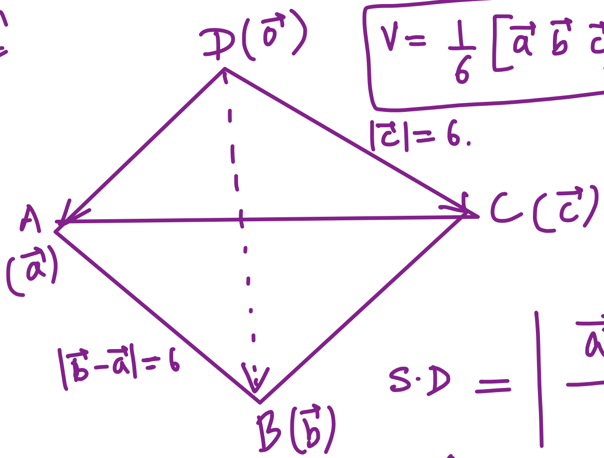
$PQ = AM = \sqrt{AB^2 - BM^2}$

$BM = \left| \text{proj. of } \vec{AB} \text{ on } \vec{n} \right| = 2\sqrt{6}.$

$AM = \sqrt{3}$

Q Let shortest distance between two opposite edges of a tetrahedron is '4 unit' and the length of these opposite edges are same and equal to 6 unit. If angle between these two opposite edges is 30° and volume of tetrahedron is V, the value of $\frac{V}{6}$ is

Sol^n



$$\boxed{V = \frac{1}{6}[\vec{a}\ \vec{b}\ \vec{c}]}$$

AB & CD are opp edges

$|\vec{AB}| = |\vec{CD}| = 6.$

S.D = 4

$$S.D = \left| \frac{\vec{a} \cdot ((\vec{a}-\vec{b}) \times \vec{c})}{|(\vec{a}-\vec{b}) \times \vec{c}|} \right|$$

$$\overset{4}{\widetilde{S.D}} = \left| \frac{[\vec{a}\ \vec{b}\ \vec{c}]}{\underbrace{|(\vec{a}-\vec{b}) \times \vec{c}|}_{18}} \right| \Rightarrow [\vec{a}\ \vec{b}\ \vec{c}] = 72$$

$$|(\vec{a}-\vec{b}) \times \vec{c}| = |\vec{a}-\vec{b}||\vec{c}| \sin 30^\circ$$

$$= 6 \cdot 6 \cdot \frac{1}{2}$$

$$= 18$$

$V = \frac{1}{6}(72)$

$$\boxed{V = 12}$$

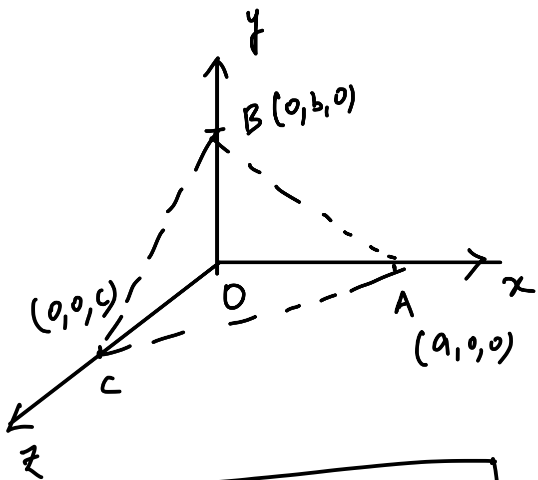
$$\boxed{\frac{V}{6} = 2}$$

Ans

Q A plane passing through (1, 1, 1) cut positive direction of co-ordinate axes at A, B and C, then the volume of tetrahedron OABC (as V) satisfies

(1) $V \le \frac{9}{2}$ (2) $V \ge \frac{9}{2}$ (3) $V = \frac{9}{2}$ (4) None

Sol^n



$a, b, c > 0$

$$P: \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$$

(1,1,1)

$$\boxed{\frac{1}{a} + \frac{1}{b} + \frac{1}{c} = 1} \quad \text{— (1) —}$$

$$\boxed{V = \frac{1}{6} abc}$$

a, b, c

GM ≥ HM

$$(abc)^{1/3} \ge \frac{3}{\frac{1}{a} + \frac{1}{b} + \frac{1}{c}}$$

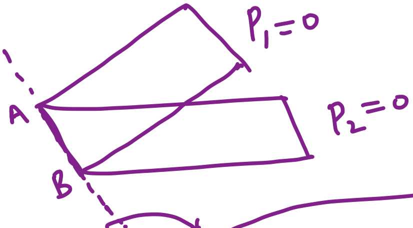
$$(abc)^{1/3} \ge \frac{3}{1} \Rightarrow abc \ge 27$$

$$V = \frac{1}{6} abc \Rightarrow \boxed{V \ge \frac{9}{2}}$$

Q Let $P_1 = x + y + z + 1 = 0$, $P_2 = x - y + 2z + 1 = 0$, $P_3 = 3x + y + 4z + 7 = 0$ be three planes. Find the distance of line of intersection of planes $P_1 = 0$ and $P_2 = 0$ from the plane $P_3 = 0$.

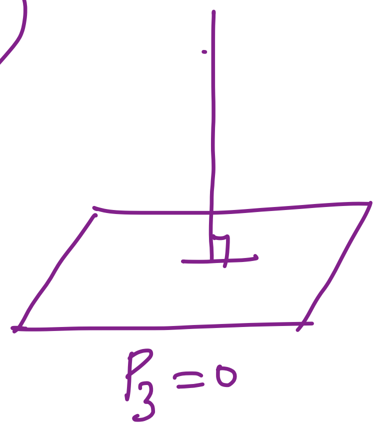
(A) $\frac{2}{\sqrt{26}}$ (B) $\frac{4}{\sqrt{26}}$ ✓ (C) $\sqrt{\frac{1}{26}}$ (D) $\frac{7}{\sqrt{26}}$

Sol^n



L : $\frac{x+1}{3} = \frac{y-0}{-1} = \frac{z-0}{-2}$ → Pt. $(-1,0,0)$

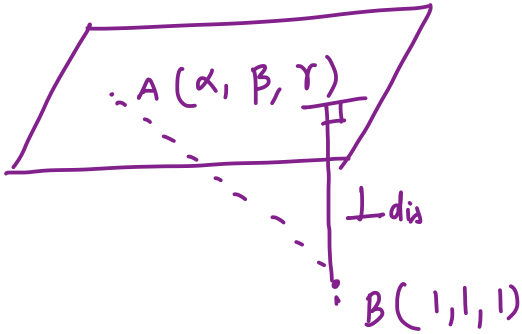
parallel to $P_3=0$.



Q If $a\alpha + b\beta + c\gamma - d = 0, a^2 + b^2 + c^2 = 1$ (where all numbers are real), then the minimum value of $(\alpha-1)^2 + (\beta-1)^2 + (\gamma-1)^2$, is

(A) $2|a+b+c-2d|^2$ (B) $2|a+b+c-d|^2$ ✓(C) $|a+b+c-d|^2$ (D) none of these

Sol^n $(AB)^2_{min} = ?$ $(AB)^2 = (\alpha-1)^2 + (\beta-1)^2 + (\gamma-1)^2$



$P: ax + by + cz - d = 0.$

$A(\alpha, \beta, \gamma)$ lies on $P = 0$.

$$AB_{min} = \frac{|a+b+c-d|}{\sqrt{\underbrace{a^2+b^2+c^2}_{1}}}$$

Paragraph

Let $P_1 : 2x - 3y + 6z + 8 = 0$ and $P_2 : 3x - 2y - 2z + 7 = 0$ be two planes and $A\left(0, \frac{29}{9}, \frac{5}{18}\right)$ lies on both planes. Two points $B(2, 0, -2)$ and C are such that line of intersection of the planes is the internal angle bisector of $\angle A$ of ΔABC and $AB = AC$.

On the basis of above information, answer the following questions :

1. If co-ordinates of C are (α, β, γ), then $(\beta + \gamma)$ is equal to

(A) 6 (B) 4 (C) 2 (D) 0

2. The x-intercept of plane containing the point C and perpendicular to planes P_1 and P_2 is -

(A) 0 (B) $\frac{13}{11}$ (C) $\frac{13}{9}$ (D) $\frac{26}{5}$

Paragraph

A plane p contains the line $L_1: \frac{y}{b}+\frac{z}{c}=1, x=0$ and is parallel to the line $L_2: \frac{x}{a}-\frac{z}{c}=1, y=0$

1. If the shortest distance between L_1 and L_2 is $\frac{1}{4}$ then the value of $\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}$ equals :

(A) 16 (B) 64 (C) 128 (D) 192

2. Distance of image of A(a, 0, 0) in the plane p from $M\left(-\frac{5}{3}, \frac{8}{3}, \frac{11}{3}\right)$. where a = b =c = 1 is equal to :

(A) 1 (B) 2 (C) 3 (D) 4

Q If three points $(2\vec{p}-\vec{q}+3\vec{r})$, $(\vec{p}-2\vec{q}+\alpha\vec{r})$ and $(\beta\vec{p}-5\vec{q})$ (Where $\vec{p}, \vec{q}, \vec{r}$ are non-coplanar vectors) are collinear, then $\frac{1}{(\alpha+\beta)}$ is

(A) 4 (B) 5 (C) 6 (D) 7