RACE # 06

(SPECIAL RACE ON DEFINITE INTEGRATION)

SOLUTION

1. $\int_{0}^{\ell n 2} x\, e^{-x}\, dx$

Put $-x = t \Rightarrow dx = -dt$

$$= \int_{0}^{-\ln 2} t e^{t}\, dt$$

By using Integration By Part

$$= \left[t e^{t} - e^{t} \right]_{0}^{-\ln 2}$$

On putting limits

$$= (-\ln 2) e^{(-\ln 2)} - e^{(-\ln 2)} - 0 + 1$$

$$= -\frac{1}{2}\ln 2 - \frac{1}{2} + 1 = \frac{1}{2} - \frac{1}{2}\ln 2 = \frac{1}{2}(1 - \ln 2)$$

$$= \boxed{\frac{1}{2}\ln\left(e/2\right)}$$

2. Given $f'(x) = \frac{\cos x}{x}$, $f\left(\frac{\pi}{2}\right) = a$, $f\left(\frac{3\pi}{2}\right) = b$. Find the value of the definite integral $\int_{\pi/2}^{3\pi/2} f(x)\,dx$.

Using Integration By Part

$$= \int_{\pi/2}^{3\pi/2} 1.\, f(x)\, dx$$

$$= f(x).x\Big|_{\pi/2}^{3\pi/2} - \int_{\pi/2}^{3\pi/2} f'(x).x\, dx$$

$$= f\left(\frac{3\pi}{2}\right)\cdot\frac{3\pi}{2} - f\left(\frac{\pi}{2}\right)\cdot\frac{\pi}{2} - \int_{\pi/2}^{3\pi/2} \frac{\cos x}{x}.x\, dx$$

$$= b.\left(\frac{3\pi}{2}\right) - a.\left(\frac{\pi}{2}\right) - (\sin x)_{\pi/2}^{3\pi/2}$$

$$= \frac{\pi}{2}(3b-a) - \left(\sin\frac{3\pi}{2} - \sin \pi/2\right)$$

$$= \frac{\pi}{2}(3b-a) - (-1-1)$$

$$= \boxed{2 - \frac{\pi}{2}(a-3b)}$$

3. $\int_{2}^{e} \left(\frac{1}{\ln x} - \frac{1}{\ln^2 x} \right) dx$

Put $\ln x = t \Rightarrow x = e^t$

$\Rightarrow dx = e^t dt$

$$= \int_{\ln 2}^{1} e^t \left(\frac{1}{t} - \frac{1}{t^2} \right) dt$$

$$\left(\because \int e^x (f(x) + f'(x)) dx = e^x f(x) + c \right)$$

$$= \left(e^t \cdot \frac{1}{t} \right)_{\ln 2}^{1}$$

$$= \boxed{e - \frac{2}{\ln 2}}$$

4. $$\int_0^{\pi/4} \frac{\sin 2x}{\sin^4 x + \cos^4 x}\, dx$$

Divide numerator and denominator by $\cos^4 x$

$$= \int_0^{\pi/4} \frac{2 \tan x \sec^2 x}{1 + \tan^4 x}\, dx$$

Put $\tan x = t \Rightarrow \sec^2 x \, dx = dt$

$$= \int_0^1 \frac{2t}{1+t^4}\, dt$$

Put $t^2 = u \Rightarrow 2t \, dt = du$

$$= \int_0^1 \frac{du}{1+u^2}$$

$$= \left(\tan^{-1} u\right)_0^1 = \boxed{\frac{\pi}{4}}$$

5. $$\int_0^{\pi/2} \frac{\cos x \quad dx}{(1+\sin x)\,(2+\sin x)}$$

Put $\sin x = t \Rightarrow \cos x \, dx = dt$

$$= \int_0^1 \frac{dt}{(1+t)(2+t)}$$

$$= \int_0^1 \left(\frac{1}{1+t} - \frac{1}{2+t}\right) dt$$

$$= \ln(1+t) - \ln(2+t)$$

$$= \left[\ln\left(\frac{1+t}{2+t}\right) \right]_0^1$$

$$= \ln\left(\frac{2}{3}\right) - \ln\left(\frac{1}{2}\right) = \boxed{\ln\left(\frac{4}{3}\right)}$$

6. $\displaystyle\int_0^{\pi/4} \frac{\sin^2 x \, . \cos^2 x}{\left(\sin^3 x + \cos^3 x\right)^2} \, dx$

Divide numerator and denominator by $\cos^6 x$

$$= \int_0^{\pi/4} \frac{\tan^2 x \, . \, \sec^2 x}{(1 + \tan^3 x)^2} \, dx$$

Put $\quad 1 + \tan^3 x = t$

$\quad 3 \tan^2 x \sec^2 x \, dx = dt$

$$= \frac{1}{3} \int_1^2 \frac{dt}{t^2}$$

$$= \frac{1}{3} \left[-\frac{1}{t} \right]_1^2$$

$$= -\frac{1}{3} \left[\frac{1}{2} - \frac{1}{1} \right]$$

$$= \boxed{\frac{1}{6}}$$

7. $\int_{2}^{3} \frac{dx}{\sqrt{(x-1)(5-x)}}$

$$= \int_{2}^{3} \frac{dx}{\sqrt{-x^2+6x-5}}$$

$$= \int_{2}^{3} \frac{dx}{\sqrt{2^2-(x-3)^2}}$$

$$= \left[\sin^{-1}\left(\frac{x-3}{2}\right)\right]_{2}^{3}$$

$$= \boxed{\frac{\pi}{6}}$$

8. $\int_{3/2}^{2} \left(\frac{x-1}{3-x}\right)^{1/2} dx$

Put $x = 1.\cos^2\theta + 3.\sin^2\theta$

$dx = 2.\sin 2\theta \, d\theta$

$$= 2\int_{\pi/6}^{\pi/4} \tan\theta . \sin 2\theta \, d\theta$$

$$= 2\int_{\pi/6}^{\pi/4} (2\sin^2\theta) \, d\theta$$

$$= 2\int_{\pi/6}^{\pi/4} (1-\cos 2\theta) \, d\theta$$

$$= \boxed{\frac{\sqrt{3}}{2} - 1 + \frac{\pi}{6}}$$

9. $$\int_0^{\pi/2} \frac{dx}{5+4\sin x}$$

Put $\sin x = \frac{2\tan x/2}{1+\tan^2 x/2}$

$$= \int_0^{\pi/2} \frac{\sec^2 x/2}{5+5\tan^2\frac{x}{2}+8\tan\frac{x}{2}}\,dx$$

Put $\tan\frac{x}{2} = t \Rightarrow \frac{1}{2}\sec^2\frac{x}{2}\,dx = dt$

$$= \int_0^1 \frac{2\,dt}{5+5t^2+8t}$$

$$= \frac{2}{5}\int_0^1 \frac{dt}{\left(t+\frac{8}{10}\right)^2+\left(\frac{6}{10}\right)^2}$$

$$= \left[\frac{2}{3}\tan^{-1}\left(\frac{10t+8}{6}\right)\right]_0^1 = \boxed{\frac{2}{3}\tan^{-1}\left(\frac{1}{3}\right)}$$

10. $\displaystyle\int_2^3 \frac{dx}{(x-1)\sqrt{x^2-2x}}$

Put $x-1 = \frac{1}{t} \Rightarrow dx = -\frac{1}{t^2}\,dt$

$$= \int_1^{1/2} \frac{-\frac{1}{t^2}\,dt}{\frac{1}{t}\sqrt{\left(\frac{1}{t}+1\right)^2 - 2\left(\frac{1}{t}+1\right)}}$$

$$= \int_{1/2}^{1} \frac{dt}{t\sqrt{\left(\frac{1}{t}+1\right)\left(\frac{1}{t}+1-2\right)}}$$

$$= \int_{1/2}^{1} \frac{dt}{t\sqrt{\left(\frac{1}{t}\right)^2-1}}$$

$$= \int_{1/2}^{1} \frac{dt}{\sqrt{1-t^2}} = \left[\sin^{-1} t\right]_{1/2}^{1}$$

$$= \frac{\pi}{2} - \frac{\pi}{6} = \boxed{\frac{\pi}{3}}$$

11. $\displaystyle\int_{0}^{\pi/4} \cos 2x \sqrt{1 - \sin 2x}\, dx$

Put $1 - \sin 2x = t$

$-2 \cos 2x \, dx = dt$

$$= -\frac{1}{2} \int_{1}^{0} \sqrt{t}\, dt \quad = \frac{1}{2} \int_{0}^{1} \sqrt{t}\, dt$$

$$= \boxed{\frac{1}{3}}$$

12. $\int_1^2 \frac{dx}{x(x^4+1)}$

Take x^4 common

$$= \int_1^2 \frac{dx}{x^5(1+x^{-4})}$$

Put $1 + x^{-4} = t$

we get $-4x^{-5}\,dx = dt$

$$= -\int_2^{17/16} \frac{dt}{4t}$$

$$= \frac{1}{4}\int_{17/16}^{2} \frac{dt}{t}$$

$$= \frac{1}{4}\left(\ln t\right)_{17/16}^{2} = \frac{1}{4}\left(\ln 2 - \ln\frac{17}{16}\right)$$

$$= \boxed{\frac{1}{4}\ln\frac{32}{17}}$$

13. **(a)** $\int_{0}^{3\pi/4}\left((1+x)\sin x+(1-x)\cos x\right)dx$ **(b)** $\int_{\pi/2}^{\pi} x^{\sin x}(1+x\cos x\cdot ln\,x+\sin x)\,dx$

(a) $\int_0^{3\pi/4}(\sin x + x\sin x + \cos x - x\cos x)\,dx$

$= \int_0^{3\pi/4}(\sin x + \cos x)\,dx + \int_0^{3\pi/4} x(\sin x - \cos x)\,dx$

Now Apply Integration By Part

$= \int_0^{3\pi/4}(\sin x + \cos x)\,dx + x\cdot(-\cos x - \sin x) - \int_0^{3\pi/4}(-\cos x - \sin x)\,dx$

$= 2\int_0^{3\pi/4}(\sin x + \cos x)\,dx - \left[x(\cos x + \sin x)\right]_0^{3\pi/4}$

$= 2\left[-\cos x + \sin x\right]_0^{3\pi/4} = \boxed{2(\sqrt{2}+1)}$

(b) $\int_{\pi/2}^{\pi} x^{\sin x}(1 + x.\cos x.\ln x + \sin x)\,dx$

$$= \int_{\pi/2}^{\pi} x^{\sin x+1}\left(\frac{1}{x} + \cos x.\ln x + \frac{\sin x}{x}\right)dx$$

Put $x^{\sin x+1} = t$

$$x^{\sin x+1}\left(\frac{1}{x} + \cos x.\ln x + \frac{\sin x}{x}\right)dx = dt$$

$$= \int_{\pi^2/4}^{\pi} dt = [t]_{\pi^2/4}^{\pi} = \boxed{\pi - \frac{\pi^2}{4}}$$

14. Suppose that f, f ' and f '' are continuous on [0, ln 2] and that f (0) = 0, f ' (0) = 3, f (ln 2) = 6, f ' (ln 2) = 4 and $\int_0^{\ln 2} e^{-2x}\cdot f(x)\,dx = 3$. Find the value of $\int_0^{\ln 2} e^{-2x}\cdot f''(x)\,dx$.

Given $\int_0^{\ln 2} e^{-2x} f(x)\,dx = 3$

Apply Integration By Part

$$\left(f(x)\frac{e^{-2x}}{(-2)}\right)_0^{\ln 2} + \frac{1}{2}\int_0^{\ln 2} f'(x)e^{-2x}\,dx = 3$$

$$-\frac{1}{2}\left(f(\ln 2)e^{-2\ln 2} - f(0)e^{-2(0)}\right) + \frac{1}{2}\left(f'(x)\frac{e^{-2x}}{(-2)}\right)_0^{\ln 2} + \frac{1}{4}\int_0^{\ln 2} f''(x)e^{-2x}dx = 3$$

$$-\frac{1}{2}\left(6\times\frac{1}{4} - 0\right) - \frac{1}{4}\left(f'(\ln 2)e^{-2\ln 2} - 3\right) + \frac{1}{4}\int_0^{\ln 2} f''(x)e^{-2x}dx = 3$$

$$-\frac{3}{4} - \frac{1}{4}\left(4\times\frac{1}{4} - 3\right) + \frac{1}{4}\left(\int_0^{\ln 2} f''(x)e^{-2x}dx\right) = 3$$

$$\boxed{\therefore \int_0^{\ln 2} f''(x)e^{-2x}\,dx = 13}$$

15. $$\int_0^1 \frac{dx}{x^2+2x\cos\alpha+1} \quad \text{where } -\pi<\alpha<\pi$$

$$= \int_0^1 \frac{dx}{(x+\cos\alpha)^2+1-\cos^2\alpha}$$

$$= \int_0^1 \frac{dx}{(x+\cos\alpha)^2+(\sin\alpha)^2}$$

$$= \frac{1}{\sin\alpha}\left[\tan^{-1}\left(\frac{x+\cos\alpha}{\sin\alpha}\right)\right]_0^1$$

$$= \frac{1}{\sin\alpha}\left[\tan^{-1}\left(\frac{1+\cos\alpha}{\sin\alpha}\right)-\tan^{-1}\left(\frac{\cos\alpha}{\sin\alpha}\right)\right]$$

$$= \frac{1}{\sin\alpha}\left[\tan^{-1}\left(\frac{2\cos^2\alpha/2}{2\sin\alpha/2\,\cos\alpha/2}\right)-\tan^{-1}(\cot\alpha)\right]$$

$$= \frac{1}{\sin\alpha}\left[\tan^{-1}\left(\cot\frac{\alpha}{2}\right)-\tan^{-1}(\cot\alpha)\right]$$

$$= \frac{1}{\sin\alpha}\left[\tan^{-1}\left(\tan\left(\frac{\pi}{2}-\frac{\alpha}{2}\right)\right)-\tan^{-1}\left(\tan\left(\frac{\pi}{2}-\alpha\right)\right)\right]$$

$$= \frac{1}{\sin\alpha}\left[\frac{\pi}{2}-\frac{\alpha}{2}-\frac{\pi}{2}+\alpha\right]$$

$$\begin{cases} \dfrac{\alpha}{2\sin\alpha} & \text{when } \alpha\neq 0 \\[2ex] \dfrac{1}{2} & \text{when } \alpha=0 \end{cases}$$

16. $$\int_0^1 \frac{1-x^2}{1+x^2+x^4}\,dx$$

$$= \int_0^1 \frac{-x^2\left(1-\frac{1}{x^2}\right)}{x^2\left(x^2+\frac{1}{x^2}+1\right)}\,dx$$

$$= -\int_0^1 \frac{\left(1-\frac{1}{x^2}\right)dx}{\left(x+\frac{1}{x}\right)^2-1}$$

Put $x+\frac{1}{x} = t$

$$\left(1-\frac{1}{x^2}\right)dx = dt$$

$$= -\int_\infty^2 \frac{dt}{t^2-1} = -\frac{1}{2}\ln\left|\frac{t-1}{t+1}\right|$$

$$= \left[\frac{1}{2}\ln\left|\frac{t+1}{t-1}\right|\right]_\infty^2$$

$$= \boxed{\frac{1}{2}\ln(3)}$$

17. $\int_0^{\pi/4} \frac{\sin\theta + \cos\theta}{9 + 16\sin 2\theta} d\theta$

$$= \int_0^{\pi/4} \frac{\sin\theta + \cos\theta}{\frac{9}{16} + 1 - 1 + \sin 2\theta} d\theta$$

$$= \int_0^{\pi/4} \frac{(\sin\theta + \cos\theta)\, d\theta}{\left(\frac{5}{4}\right)^2 - (\cos\theta - \sin\theta)^2}$$

Now put $\cos\theta - \sin\theta = t$

$(\sin\theta + \cos\theta)\, d\theta = -dt$

$$= \int_0^1 \frac{dt}{\left(\frac{5}{4}\right)^2 - (t)^2}$$

$$= \boxed{\frac{1}{20} \ln 3}$$

18. $\int_0^{\pi} \theta \sin^2\theta \cos\theta \, d\theta$

$$= \int_0^{\pi} \theta \cos\theta \left(\frac{1-\cos 2\theta}{2}\right) d\theta$$

$$= \frac{1}{2}\int_0^{\pi} \theta \cos\theta \, d\theta - \frac{1}{2}\int_0^{\pi} \theta \cos\theta \cos 2\theta \, d\theta$$

$$= \frac{1}{2}\int_0^{\pi} \theta \cos\theta \, d\theta - \frac{1}{2}\cdot\frac{1}{2}\int_0^{\pi} \theta(\cos 3\theta + \cos\theta) \, d\theta$$

$$= \frac{1}{2}\int_0^{\pi} \theta \cos\theta \, d\theta - \frac{1}{4}\int_0^{\pi} \theta \cos 3\theta \, d\theta - \frac{1}{4}\int_0^{\pi} \theta \cos\theta \, d\theta$$

$$= \frac{1}{4}\int_0^{\pi} \underset{\mathrm{I}}{\theta} \underset{\mathrm{II}}{\cos\theta} \, d\theta - \frac{1}{4}\int_0^{\pi} \underset{\mathrm{I}}{\theta} \cdot \underset{\mathrm{II}}{\cos 3\theta} \, d\theta$$

Now Apply Integration By Part

$$= \boxed{-\frac{4}{9}}$$

19. $\int_0^{\pi/2} \frac{x + \sin x}{1 + \cos x} \, dx$

$$= \int_0^{\pi/2} \frac{x}{1+\cos x} dx + \int_0^{\pi/2} \frac{\sin x}{1+\cos x} dx$$

$1 + \cos x = t$

$$= \frac{1}{2}\int_0^{\pi/2} \underset{\mathrm{I}}{x} \cdot \underset{\mathrm{II}}{\sec^2 x/2} + \int_1^2 \frac{dt}{t}$$

$$= \frac{1}{2}\left[2x \tan\frac{x}{2} - 2\int \tan\frac{x}{2} \, dx\right] + (\ln t)_1^2$$

$$= \frac{1}{2}\left[2x\tan\frac{x}{2} - 4\ln\sec\frac{x}{2}\right]_0^{\pi/2} + \ln 2$$

$$= \frac{\pi}{2} - \ln 2 + \ln 2 = \boxed{\frac{\pi}{2}}$$

20. $\int_0^{\pi}\left[\cos^2\left(\frac{3\pi}{8}-\frac{x}{4}\right)-\cos^2\left(\frac{11\pi}{8}+\frac{x}{4}\right)\right]dx$

Let $\frac{3\pi}{8}-\frac{x}{4}=A$ & $\frac{11\pi}{8}+\frac{x}{4}=B$

$$=\int_0^{\pi}(\cos^2 A-\cos^2 B)\,dx = \int_0^{\pi}\sin(A+B).\sin(B-A)\,dx$$

$(1-\sin^2 A)-(1-\sin^2 B)$

Using $\sin^2 A-\sin^2 B=\sin(A+B).\sin(A-B)$

$$=\int_0^{\pi}\sin\left(\frac{14\pi}{8}\right)\sin\left(\pi+\frac{x}{2}\right)dx$$

$$=-\sin\left(\frac{14\pi}{8}\right)\int_0^{\pi}\sin\left(\frac{x}{2}\right)dx$$

$$=-\sin\left(\pi+\frac{6\pi}{8}\right).(-2).\left[\cos\left(\frac{x}{2}\right)\right]_0^{\pi}$$

$$=-2.\sin\left(\frac{3\pi}{4}\right)\left[\cos\frac{\pi}{2}-\cos 0\right] = \boxed{\sqrt{2}}$$

21. If $f(\pi)=2$ & $\int_0^{\pi}(f(x)+f''(x))\sin x\;dx=5$, then find $f(0)$

$$\Rightarrow \int_0^{\pi} f(x)\sin x\,dx+\int_0^{\pi} f''(x)\sin x\,dx=5$$

Apply Integration By Part

$$\left(-f(x)\cos x\right)_0^{\pi}+\int_0^{\pi} f'(x)\cos x\,dx+\int_0^{\pi} f''(x)\sin x\,dx=5$$

$$\left(-f(x)\cos x\right)_0^{\pi}+\left(f'(x)\sin x\right)_0^{\pi}-\int_0^{\pi} f''(x)\sin x\,dx+\int_0^{\pi} f''(x)\sin x=5$$

$$-\left[f(\pi)\cos\pi-f(0)\cos 0\right]+\left[f'(\pi)\sin\pi-f'(0).\sin 0\right]=5 \Rightarrow$$

$f(0)=3$

22. $\int_0^1 x\, f''(x)\, dx$, where $f(x) = \cos(\tan^{-1}x)$

By using Integration By Part

$$= x . \int_0^1 f''(x)\, dx - \int_0^1 \left(\frac{d}{dx}(x) \int f''(x)\, dx \right) dx$$

$$= \left(x f'(x) \right)_0^1 - \int_0^1 f'(x)\, dx$$

$$= \left(x f'(x) \right)_0^1 - \left(f(x) \right)_0^1$$

$$= \left(- \frac{x . \sin(\tan^{-1}x)}{1+x^2} \right)_0^1 - \left(\cos(\tan^{-1}x) \right)_0^1$$

$$= \boxed{1 - \frac{3}{2\sqrt{2}}}$$