D.E$^n$

## DEFINITION :

An equation that involves independent (usually 'x') and dependent variables (usually the 1st letter of the physical quantity) and the derivatives of the dependent variables w.r.t. independent variable is called a **Differential equation**.

i.e. $f\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2} \ldots\ldots\right) = 0$, where x is independent & y is dependent variable.

**Note :** A differential equation is said to be **ordinary**, if the differential coefficients have reference to a single independent variable only and it is said to be **Partial** if there are two or more independent variables. We are concerned with ordinary differential equations only.

e.g. $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$ is an ordinary differential equation

$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0; \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = x^2 + y$ are partial differential equation.

Partial derivative

$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$

Slope of a line is 1, no. of lines ≡ infinite

Family of lines having slope 1

$\Rightarrow \boxed{\dfrac{dy}{dx} = 1} \Longleftrightarrow \boxed{y = x + c}$

D. $E^n$ of this family.

Family of lines

where 'c' is an arbitrary constant

$y^2 = 4ax$ Family of parabolas

where 'a' an $\boxed{I \cdot A \cdot C}$

$$2y\frac{dy}{dx} = 4a \Rightarrow 2xy\frac{dy}{dx} = 4ax$$

$$\Rightarrow 2xy\frac{dy}{dx} = y^2 \Rightarrow \boxed{2x\frac{dy}{dx} - y = 0}$$

$x, y, \frac{dy}{dx}$ n/o I. A.C.

D.E^n

## ORDER AND DEGREE OF DIFFERENTIAL EQUATION :

**The order of a differential equation** is the order of the highest differential coefficient occuring in it.

**The degree of a differential equation** which is expressed or can be expressed as a polynomial in the derivatives is the degree of the highest order derivative occuring in it, after it has been expressed in a form free from radicals & fractions so far as derivatives are concerned. Thus the differential equation :

$$f(x,y)\left[\frac{d^m y}{dx^m}\right]^p + \phi(x,y)\left[\frac{d^{m-1}(y)}{dx^{m-1}}\right]^q + \ldots\ldots = 0$$ is of order m & degree p.

If differential equation cannot be written as polynomial in derivatives, then degree is not defined.

**Note :** If the differential equation contains radicals or fractions on derivatives, then in order to define the degree of differential equation minimum power must be raised to convert the differential equation in polynomial form in derivatives.

Degree ≡ Polynomial

e.g. (1) $\sin x\left(\frac{d^3y}{dx^3}\right) + \cos x\left(\frac{dy}{dx}\right)^{100} = 0$ order 3, degree 1

(2) $\sqrt{\frac{dy}{dx}} = \sin x$ order 1, degree 1 $\Rightarrow \left(\frac{dy}{dx}\right) = (\sin x)^2$

Degree = The highest power of the highest order derivative

≡ Polynomial

Degree of a D.E$^n$ is defined only when D.E$^n$ can be expressed as a polynomial in terms of derivative.

$$e^{\left(\frac{dy}{dx}\right)} = 1 + x^2$$

$$\Rightarrow \left(\frac{dy}{dx}\right) = \ln(1+x^2)$$

Degree is defined

E(1) $\dfrac{d^2y}{dx^2} = \left[y + \left(\dfrac{dy}{dx}\right)^6\right]^{1/4}$ — order 2, degree 4

E(2) $\dfrac{dy}{dx} + y = \dfrac{1}{\dfrac{dy}{dx}}$ — order 1, degree 2

E(3) $e^{\frac{d^3y}{dx^3}} - x\dfrac{d^2y}{dx^2} + y = 0$ — order 3, (not applicable)

E(4) $\sin^{-1}\left(\dfrac{dy}{dx}\right) = x + y \Rightarrow \dfrac{dy}{dx} = \sin(x + y)$ — order 1, degree 1

E(5) $\ell n\left(\dfrac{dy}{dx}\right) = ax + by \Rightarrow \dfrac{dy}{dx} = e^{ax+by}$ — order 1, degree 1

where a & b are given numbers

E(6) $y = \ell n\left(1 + \dfrac{dy}{dx} + \dfrac{1}{2!}\left(\dfrac{dy}{dx}\right)^2 + \ldots\ldots\infty\right)$ — order 1, degree 1

E(1) $\left(\dfrac{d^2y}{dx^2}\right)^4 = y + \left(\dfrac{dy}{dx}\right)^6$

E(3) $e^{\frac{d^3y}{dx^3}} - x\dfrac{d^2y}{dx^2} + y = 0$

Not a polynomial

Can't be converted into polynomial

$\Rightarrow$ Degree is not defined

E2 $\frac{dy}{dx} + y = \frac{1}{\frac{dy}{dx}}$

$\Rightarrow \left(\frac{dy}{dx}\right)^2 + y\cdot\left(\frac{dy}{dx}\right) = 1$

Degree = 2, order = 1

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$e^{\frac{dy}{dx}} = 1 + \frac{dy}{dx} + \left(\frac{dy}{dx}\right)^2 \times \frac{1}{2!} \cdots$

E6 $y = \ln\left(1 + \left(\frac{dy}{dx}\right) + \frac{1}{2!}\left(\frac{dy}{dx}\right)^2 + \frac{1}{3!}\left(\frac{dy}{dx}\right)^3 + \cdots \text{ to } \infty\right)$

$= \ln e^{\frac{dy}{dx}} = \frac{dy}{dx} \Rightarrow \frac{dy}{dx} = y$

$\boxed{\text{order} = 1, \text{ degree} = 1}$

**E(7)** $e^{d^2y/dx^2} = \dfrac{dy}{dx}$ < order 2; degree not applicable

## SOLVING DIFFERENTIAL EQUATION :

Finding the unknown function is called **Solving or Integrating** the differential equation. It is a relation between variable involved which satisfies the differential equation. The solution of the differential equation is also called its **Primitive**, because the differential equation can be regarded as a relation derived from it.

e.g. $\dfrac{d^2y}{dx^2} + y = 0$ has a solution y = Acosx + Bsinx because it satisfies the equation.

## FORMATION OF A DIFFERENTIAL EQUATION :

If an equation in independent and dependent variables having some arbitrary constants is given, then a differential equation is obtained as follows :

$f(x, y, c_1, c_2, \dots c_n) = 0$

**(a)** Differentiate the given equation say $f(x, y, c_1) = 0$ w.r.t. the independent variable (say x) as many times as the number of arbitrary constants in it. (Number of independent arbitrary constants decide the order of differential equation).

**(b)** Eliminate the arbitrary constants.

The eliminant is required differential equation.

no. of I·A·C

= order of D·E^n

Family of curves' equation

$y^2 = 4ax$, a is an I·A·C = Parameter

$a = 1$ (given constant), $y^2 = 4x$ one member

$y^2 = 4ax$, $a$ is an I.A.C

Family of curves' equation

$$\boxed{f(x, y, a) = 0}$$

$f(x, y, c_1, c_2, \ldots c_n) = 0$

where $c_1, c_2, \ldots c_n$ are I.A.C.

Family of curves' equation

Now, to form D.E$^n$ | Differentiate given equation $n$ times

$n$ = No. of I.A.C

Now using the above (n+1) equations eliminate I.A.C. & get an eq$^n$ containing $f\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots \frac{d^ny}{dx^n}\right) = 0$

D.E$^n$ of the family

Ex① $y^2 = 4ax$ ← Family

$2y\frac{dy}{dx} = 4a$

Eliminate a

$2xy\frac{dy}{dx} = y^2 \Rightarrow \boxed{2x\frac{dy}{dx} - y = 0}$ D.E$^n$

**Note :**

(i) The arbitrary constants in a differential equation are said to be independent, when it is impossible to deduce an equivalent relation containing fewer arbitrary constants. Thus the two arbitrary constants A, B in the equation $y = Ae^{x+B}$ are not independent since the equation can be written as $y = Ae^{B} . e^{x} = Ce^{x}$. Similarly the relation $y = A\sin x + B\cos(x + C)$ appears to contain three arbitrary constants, but they are really equivalent to two only. Similarly $y = (C_1 + C_2)\cos(x + C_3) - C_4e^{x+C_5}$ has only 3 independent arbitrary constants hence will be of order 3.

(ii) A differential equation represents a family of curves all satisfying some common properties. This can be considered as the geometrical interpretation of the differential equation.

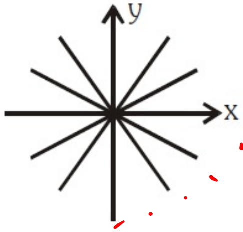
count 1

$C_4 e^{C_5} \times e^{x}$

count 1

**Examples :**

**E(1)** Form the differential equation of family of lines concurrent at the origin



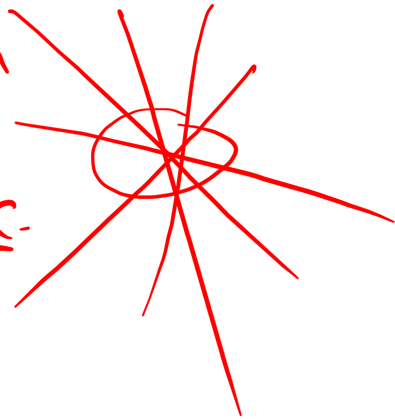

[**Hint :** $(y = mx)$; $\frac{dy}{dx} = m \;\Rightarrow\; y = \frac{dy}{dx}.x \;\Rightarrow\; \boxed{xdy - ydx = 0}$

Note that the order is 1, same as number of constants.]

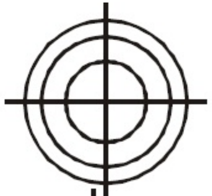
$y = mx$

m is an I.A.C

$\frac{dy}{dx} = m$



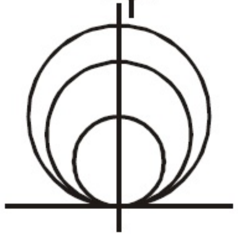
**E(2)** Differential equation of all concentric circles with center at the origin



[**Hint :** $x^2 + y^2 = r^2 \;\Rightarrow\; xdx + ydy = 0$

Note that the order is 1]

**E(3)** Differential equation of all circles touching the x-axis at the origin and centre on y-axis



[**Hint :** $x^2 + (y - a)^2 = a^2 \;\Rightarrow\; x^2 + y^2 - 2ay = 0$ ........ (i)

differentiating, $2x + 2y\frac{dy}{dx} = 2a\frac{dy}{dx}$ or $\boxed{x + y\frac{dy}{dx} = a\frac{dy}{dx}}$

substituting the value of 'a' in (i)

$x^2 + y^2 = R^2$, where R is an I.A.C

$\Rightarrow \boxed{2x + 2y\frac{dy}{dx} = 0}$

dx

**E(4)** Form the differential equation of the system of rectangular hyperbola $xy = c^2$

Where c is an I.A.C

$$\boxed{1\cdot y + x\frac{dy}{dx} = 0}$$

order = 1, degree = 1

order of D.E^n = No. of I.A.C used in the family

## GENERAL AND PARTICULAR SOLUTIONS :

The solution of a differential equation which contains a number of independent arbitrary constants equal to the order of the differential equation is called the **General solution (or complete Integral or complete primitive)**. A solution obtainable from the general solution by giving particular values to the constants is called a **Particular Solution.**

**Note :**

(i) The general solution of a differential equation of $n^{th}$ order contains 'n' & only 'n' independent arbitrary constants.

(ii) In some cases there exist a solution which can not be obtained from the general solution. Such a solution is called singular solution.

## ELEMENTARY TYPES OF FIRST ORDER & FIRST DEGREE DIFFERENTIAL EQUATIONS :

Basic

**Elementary Type of first order and first degree**

- **Variable Separable**
  - **T-1** General
  - **T-2** $\frac{dy}{dx} = f(ax+by+c)$
  - **T-3** Polar Co-ordinates
  - **T-4** Special case
- **Homogeneous**
  - **T-1** General
  - **T-2** Reducible to Homogeneous
- **Linear**
  - **T-1** General
  - **T-2** Reducible to linear
    - By substitution
    - Bernoulli's D.E.
- **Exact**

## VARIABLE SEPARABLE FORM :

If the differential equation can be expressed as; $f(x)dx + g(y)\,dy = 0$ then this is said to be variable separable type.

### Type-1 : General form

A general solution of this is given by $\int f(x)dx + \int g(y)dy = c$; where c is the arbitrary constant and can be chosen in any suitable form, we can replace it by $\tan^{-1}c$, $\ell nc$, $e^c$, etc.

( a & b are not I.A.C)

**Examples :**

a & b are given numbers

**E(1)** $\ell n\dfrac{dy}{dx} = ax + by$ [**Hint :** $\dfrac{dy}{dx} = e^{ax+by} = e^{ax}.e^{by}$ or $\int e^{-by}dy = \int e^{ax}dx$ ]

**E(2)** $x(y^2 + 1)dx + y(x^2 + 1)dy = 0$ [**Ans.** $(x^2 + 1)(y^2 + 1) = c$]

**E(3)** $y'\sin x = y\ell n\, y;\ y|_{x=\frac{\pi}{2}} = e$ find y. [**Ans.** $y = e^{\text{cosec}x - \cot x}$]

$$\frac{dy}{dx} = e^{ax+by} = e^{ax}.e^{by}$$

$$\Rightarrow \int \frac{dy}{e^{by}} = \int e^{ax}dx \Rightarrow \boxed{\frac{e^{-by}}{-b} = \frac{e^{ax}}{a} + C}$$

Solution.

$$\int \frac{x\,dx}{x^2+1} = -\int \frac{y\,dy}{1+y^2} \Rightarrow \boxed{\ln|1+x^2| = -\ln|1+y^2| + C}$$

$$\frac{dy}{dx}\sin x = y\ln y$$

$$\Rightarrow \int \frac{dy}{(\ln y)\, y} = \int \frac{dx}{\sin x}$$

$$\ln|\ln y| = \ln|\operatorname{cosec} x - \cot x| + C$$

$$x = \pi/2,\quad y = e$$

$$0 = 0 + C \Rightarrow \boxed{C = 0}$$

$$|\ln y| = |\operatorname{cosec} x - \cot x|$$

H.W

**D(9)** Form the differential equation of the family of concentric ellipse $ax^2 + by^2 = 1$ with principal axes along the co-ordinates axes. [**Ans.** $x(y'^2 + yy'') = yy'$]

Note that the order is 2.

**E(5)** Find a particular solution of the differential equation $(1 + e^x)\, y\dfrac{dy}{dx} = e^x$. Satisfying the initial condition

$y(0) = 1$ **[Ans.** $y = \sqrt{1 + \ln\left(\dfrac{1+e^x}{2}\right)^2}$ **]**

**E(6)** Find the curve passing through the point (0, –2) such that the slope of the tangent at any of its points is equal to the ordinate of that point increased by a ~~factor of~~ 3. **[Ans.** $y = e^x - 3$**]**

$\dfrac{dy}{dx} = y + 3$

**E(7)** Find the nature of the curve not passing through the origin for which the length of the normal (tangent) at the point P is equal the radius vector of the point P. **[Ans.** circle or rectangular hyperbola**]**

E5

$$\int y\,dy = \int \frac{e^x}{1+e^x}\,dx \Rightarrow \frac{y^2}{2} = \ln|1+e^x| + C$$

$$x = 0,\ y = 1 \Rightarrow \frac{1}{2} = \ln 2 + C \Rightarrow C = \frac{1}{2} - \ln 2$$

$$\frac{y^2}{2} = \ln|1+e^x| + \frac{1}{2} - \ln 2 = \ln\left|\frac{1+e^x}{2}\right| + \frac{1}{2}$$

$$y^2 = 2\ln\left(\frac{1+e^x}{2}\right) + 1 \Rightarrow y = \sqrt{\ln\left(\frac{1+e^x}{2}\right)^2 + 1}$$

$$\frac{dy}{dx} = 3y \Rightarrow \frac{dy}{y} = 3\,dx$$

$$\Rightarrow \ln|y| = 3x + C$$

$$x=0,\ y=-2 \Rightarrow \ln 2 = C$$

$$\Rightarrow \ln|y| = 3x + \ln 2 \Rightarrow \ln\left|\frac{y}{2}\right| = 3x$$

$$\frac{y}{2} = e^{3x}$$

$$y+3 = -e^{3x}$$

$$y+3 = e^{x}\cdot 3$$

$$y =$$

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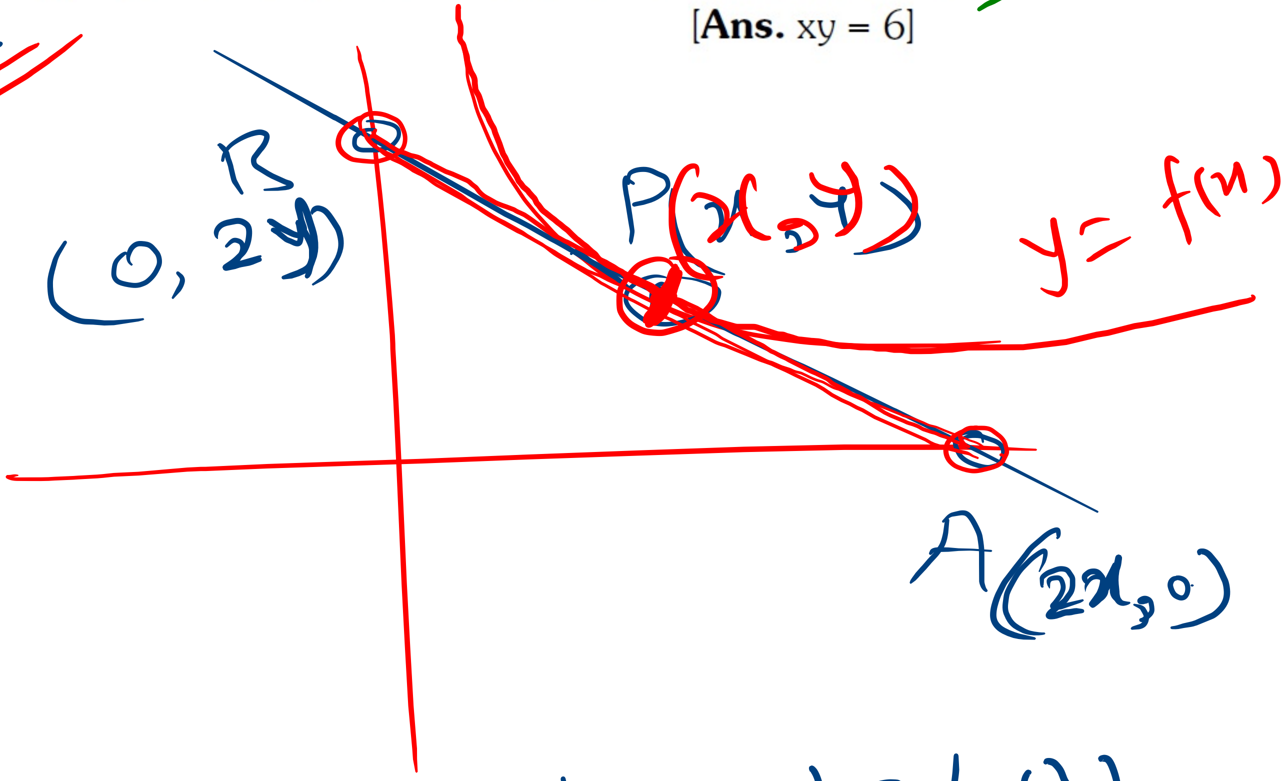
$$\frac{dy}{dx} = y+3 \Rightarrow \frac{dy}{y+3} = dx \Rightarrow \ln|y+3| = x + C$$

$$x=0,\ y=-2 \Rightarrow C=0$$

$$|y+3| = e^{x}$$

$$y = e^{x} - 3$$

**E(8)** Find the curve for which the segment of the tangent contained between the co-ordinates axes is bisected by the point. Curve passes through (2, 3). [**Ans.** xy = 6]




Slope of the tangent

$$= \frac{2y-0}{0-2x}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{y}{x}$$

$$\Rightarrow \frac{dy}{y} = -\frac{dx}{x}$$

$$\Rightarrow \ln|y| = -\ln|x| + \ln\lambda \Rightarrow \ln(xy) = \ln(\lambda)$$

$$\Rightarrow \boxed{xy = c^2}$$

$$xy = \cancel{c^2}$$

**D(11)** Find the foci of the conic passing through (2, 0) and satisfying $y\frac{dy}{dx} + 4x = 0$.

[**Ans.** E : $\frac{x^2}{4} + \frac{y^2}{16} = 1$ ; $e = \frac{13}{12}$; foci $(0, 2\sqrt{3})$, $(0, -2\sqrt{3})$ ]

**D(12)** Find the foci of the conic passing through the point (1, 0) and satisfying the differential equation $(1 + y^2)dx - xydy = 0$. Find also the equation of a circle touching the conic at $(\sqrt{2}, 1)$ and passing through one of its foci.

**D(9)** Form the differential equation of the family of concentric ellipse $ax^2 + by^2 = 1$ with principal axes along the co-ordinates axes.

**[Ans.** $x(y'^2 + yy'') = yy'$]

Note that the order is 2.

I.A.C.

**D(11)** Find the foci of the conic passing through (2, 0) and satisfying $y\frac{dy}{dx} + 4x = 0$.

$(x = 0, y = \pm be)$

**[Ans.** E : $\frac{x^2}{4} + \frac{y^2}{16} = 1$ ; $e = \frac{13}{12}$; foci $(0, 2\sqrt{3}), (0, -2\sqrt{3})$ ]

**D(12)** Find the foci of the conic passing through the point (1, 0) and satisfying the differential equation $(1 + y^2)dx - xydy = 0$. Find also the equation of a circle touching the conic at $(\sqrt{2}, 1)$ and passing through one of its foci.

D(9) D.E^n

$2ax + 2byy' = 0$

$\Rightarrow ax + byy' = 0$

$\Rightarrow \frac{a}{b} = -\frac{y}{x}y'$

$\Rightarrow a + b(y')^2 + byy'' = 0$

$\Rightarrow \frac{a}{b} + (y')^2 + yy'' = 0$

$\Rightarrow \left(-\frac{y}{x}\right)y' + (y')^2 + yy'' = 0$

D(11) $y\,dy = -4x\,dx$ $\Rightarrow \frac{y^2}{2} = -\frac{4x^2}{2} + C$

$\Rightarrow \boxed{2x^2 + \frac{y^2}{2} = C}$

$\Rightarrow$ passing through $(2, 0) \Rightarrow \frac{x^2}{4} + \frac{y^2}{16} = 1$

D(14) Show that if $m_1$ and $m_2$ grams of a radioactive substances are present at time $t_1$ and $t_2$ respectively then its half life is $T = (t_2 - t_1)\dfrac{\ln 2}{\ln(m_1/m_2)}$.

$$(1+y^2)\,dx = xy\,dy \Rightarrow \frac{2\,dx}{x} = \frac{2y\,dy}{1+y^2}$$

$$\Rightarrow \ln x^2 = \ln|1+y^2| + C$$ passing through $(1,0)$

$$\Rightarrow 0 = 0 + c \Rightarrow c = 0$$

$$\Rightarrow x^2 = 1 + y^2 \Rightarrow \boxed{x^2 - y^2 = 1}$$

$e = \sqrt{2}$

$(\sqrt{2}, 1)$ equation of tangent at $(\sqrt{2},1)$

$$\Rightarrow \boxed{x\sqrt{2} - y\cdot 1 = 0}$$

$$\boxed{(x-\sqrt{2})^2 + (y-1)^2 + \lambda(\sqrt{2}x - y) = 0}$$

condition $c^2 = m^2a^2 - b^2$

$$\int_{m_1}^{m_2} \frac{dm}{m} = -\lambda \int_{t_1}^{t_2} dt \Rightarrow \ln\left(\frac{m_2}{m_1}\right) = -\lambda(t_2 - t_1)$$

$$\int_{m}^{\frac{m}{2}} \frac{dm}{m} = -\lambda \int_{0}^{t} dt \Rightarrow \ln\left(\frac{\frac{m}{2}}{m}\right) = -\lambda t$$

$$\Rightarrow -\ln 2 = -\lambda t$$

$$t = \frac{\ln 2}{\lambda}$$

1. If the differential equation of the family of curve given by $y = Ax + Be^{2x}$ where A and B are arbitrary constant is of the form $(1-2x)\frac{d}{dx}\left(\frac{dy}{dx}+ly\right)+k\left(\frac{dy}{dx}+ly\right)=0$ then the ordered pair $(k, l)$ is

(A) $(2, -2)$ (B) $(-2, 2)$ (C) $(2, 2)$ (D) $(-2, -2)$

2. The differential equation for the family of curves $x^2 + y^2 - 2ay = 0$, where a is an arbitrary constant is- **[AIEEE-2004]**

(A) $2(x^2 - y^2)y' = xy$ (B) $2(x^2 + y^2)y' = xy$

(C) $(x^2 - y^2)y' = 2xy$ (D) $(x^2 + y^2)y' = 2xy$

4. The differential representing the family of curves $y^2 = 2c(x + \sqrt{c})$, where $c > 0$, is a parameter, is of order and degree as follows- **[AIEEE-2005]**

(A) order 1, degree 2 (B) order 1, degree 1

(C) order 1, degree 3 (D) order 2, degree 2

5. If $x\frac{dy}{dx} = y(\log y - \log x + 1)$, then the solution of the equation is - **[AIEEE-2005]**

(A) $y\log\left(\frac{x}{y}\right) = cx$ (B) $x\log\left(\frac{y}{x}\right) = cy$ (C) $\log\left(\frac{y}{x}\right) = cx$ (D) $\log\left(\frac{x}{y}\right) = cy$

20. If $y = y(x)$ and $\frac{2+\sin x}{y+1}\left(\frac{dy}{dx}\right) = -\cos x$, $y(0) = 1$, then $y\left(\frac{\pi}{2}\right)$ equals - **[JEE 2004, (Screening) 3M]**

(A) $\frac{1}{3}$ (B) $\frac{2}{3}$ (C) $-\frac{1}{3}$ (D) 1

25. Let f(x) be differentiable on the interval $(0, \infty)$ such that $f(1) = 1$, and $\lim_{t\to x}\frac{t^2 f(x) - x^2 f(t)}{t-x} = 1$, for each $x > 0$. Then f(x) is - **[JEE 2007 (3+3M)]**

(A) $\frac{1}{3x}+\frac{2x^2}{3}$ (B) $\frac{-1}{3x}+\frac{4x^2}{3}$ (C) $\frac{-1}{x}+\frac{2}{x^2}$ (D) $\frac{1}{x}$

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46. Let p(x) be a polynomial such that $p(1) = 0$ and $\frac{d}{dx}(p(x)) > p(x)$ for all $x \geq 1$ show that $p(x) > 0$, for all $x > 1$. **[JEE 2003 (mains), 4M out of 60]**

47. If length of tangent at any point on the curve $y = f(x)$ intercepted between the point and the x-axis is of length 1. Find the equation of the curve. **[JEE 2005 (Mains) 4M out of 60]**

Q. If $y_1, y_2$ are solutions of

$$\frac{dy}{dx} + P(x)\,y = Q(x)$$

T.P.T. $y = y_1 + c\,(y_1 - y_2)$ is the general solution of the given D.E$^n$ where c is an I.A.C.

# Type-2 :

$$\boxed{\frac{dy}{dx} = f(ax + by + c),\ b \neq 0.\ \text{(If } b = 0 \text{ this is directly variable separable)}}$$

To solve this, substitute t = ax + by + c. Then the equation reduces to separable type in the variable t and x which can be solved.

**E(1)** $(x+y)^2 \frac{dy}{dx} = a^2$ where 'a' is a given number

**E(2)** $\sin^{-1}\left(\frac{dy}{dx}\right) = x + y$

E(1) $\frac{dy}{dx} = \frac{a^2}{(x+y)^2}$

$\frac{dy}{dx} = \sin(x+y)$

Put $x+y = t \Rightarrow 1 + \frac{dy}{dx} = \frac{dt}{dx}$

$\frac{dt}{dx} - 1 = \sin t \Rightarrow \frac{dt}{dx} = 1 + \sin t \Rightarrow \int \frac{dt}{1+\sin t} = \int dx$

**E(3)** $\sqrt{1+x+y}\,\dfrac{dy}{dx} = x+y-1$ $\Rightarrow$ $\dfrac{dy}{dx} = \left(\dfrac{x+y-1}{\sqrt{x+y+1}}\right)$

$x+y+1 = t \Rightarrow 1+\dfrac{dy}{dx} = \dfrac{dt}{dx}$ , $x+y-1 = t-2$

$\dfrac{dt}{dx} - 1 = \dfrac{t-2}{\sqrt{t}} \Rightarrow \dfrac{dt}{dx} = \left(\dfrac{t-2}{\sqrt{t}}+1\right) \Rightarrow \int \dfrac{\sqrt{t}\,dt}{t-2+\sqrt{t}} = \int dx$

**E(4)** Find the curve passing through the origin in the form $y = f(x)$ and satisfying the differential equation $\dfrac{dy}{dx} = \cos(10x+8y)$

Put $10x+8y = t \Rightarrow 10+8\dfrac{dy}{dx} = \dfrac{dt}{dx}$

$\dfrac{dy}{dx} = \cos t$

$10+8\cos t = \dfrac{dt}{dx} \Rightarrow \int dx = \int \dfrac{dt}{10+8\cos t}$

$\cos t = \dfrac{1-\tan^2 t/2}{1+\tan^2 t/2} \Rightarrow x = \int \dfrac{\sec^2(t/2)\,dt}{18+2\tan^2 t/2}$

**E(5)** $\frac{dy}{dx} = \frac{2x+3y-1}{4x+6y-5}$ or $\frac{(2x+3y-1)^2}{4x+6y-5}$

$\frac{dy}{dx} = \frac{t}{2t-3}$

Put $2x + 3y - 1 = t$

$2 + 3\frac{dy}{dx} = \frac{dt}{dx}$

$2 + \frac{3\times t}{2t-3} = \frac{dt}{dx}$

$2\frac{dy}{dx} = \frac{4x+6y-2}{4x+6y-5}$

$\Rightarrow 2\frac{dy}{dx} = \frac{t+3}{t}$

$4x + 6y - 5 = t$ ✓

$4 + 6\frac{dy}{dx} = \frac{dt}{dx}$

$4 + 3\times\left(\frac{t+3}{t}\right) = \frac{dt}{dx}$

$\Rightarrow \frac{4t+3t+9}{t} = \frac{dt}{dx}$

$\Rightarrow dx = \int \frac{t}{7t+9}\,dt$

**Type-3 : Use of polar coordinates**

Sometimes transformation to the polar co-ordinates facilitates separation of variables. In this connection it is convenient to remember the following differentials.

**1.** If $x = r\cos\theta$ ; $y = r\sin\theta$, where r and $\theta$ both are variable.

Then $xdx + ydy = rdr$ & $xdy - ydx = r^2d\theta$

**2.** If $x = r\sec\theta$ & $y = r\tan\theta$, then $xdx - ydy = rdr$ and $xdy - ydx = r^2\sec\theta\, d\theta$.

* both r & θ are variable

$dx = (\cos\theta)dr - r(\sin\theta)d\theta$

$dy = (\sin\theta)dr + r\cos\theta\, d\theta$.

$x = r\cos\theta$ ; $y = r\sin\theta$ $\Rightarrow$ $x^2 + y^2 = r^2$ and $\tan\theta = y/x$

Hence $xdx + ydy = rdr$ and $xdy - ydx = x^2\sec^2\theta d\theta$

$xdy - ydx = r^2d\theta$

Similarly $x = r\sec\theta$ and $y = r\tan\theta$ $\Rightarrow$ $x^2 - y^2 = r^2$ and $y/x = \sin\theta$

Hence $xdx - ydy = rdr$ and $xdy - ydx = x^2\cos\theta\, d\theta = r^2\sec\theta\, d\theta$

$\boxed{xdx + ydy = rdr}$

$\boxed{x^2 + y^2 = r^2}$

$2xdx + 2ydy = 2rdr$

**E(1)** $xdx + ydy = x(xdy - ydx)$

**E(2)** $\dfrac{xdx + ydy}{\sqrt{x^2 + y^2}} = \dfrac{ydx - xdy}{x^2}$

$rdr = r\cos\theta\, r^2 d\theta$

$\Rightarrow \dfrac{dr}{r^2} = \cos\theta\, d\theta$

$\Rightarrow -\dfrac{1}{r} = \sin\theta + C$

$\Rightarrow -\dfrac{1}{r} = \dfrac{y}{r} + C$

Ans $\Rightarrow 1 + y + (\sqrt{x^2+y^2})C = 0$

**E(3)** $\dfrac{x + y\dfrac{dy}{dx}}{x\dfrac{dy}{dx} - y} = \sqrt{\dfrac{1 - x^2 - y^2}{x^2 + y^2}}$

**E(4)** $\dfrac{xdx - ydy}{xdy - ydx} = \sqrt{\dfrac{1 + x^2 - y^2}{x^2 - y^2}}$

$(x^2+y^2)$ tells us go for

$x = r\cos\theta$
$y = r\sin\theta$

$(x^2-y^2)$ tells us go for

$x = r\sec\theta$
$y = r\tan\theta$

$$\frac{xdx + ydy}{xdy - ydx} = \sqrt{\frac{1-(x^2+y^2)}{(x^2+y^2)}}$$

Put $x = r\cos\theta$, $y = r\sin\theta$, $x^2+y^2 = r^2$

$xdx + ydy = rdr$, $xdy - ydx = r^2 d\theta$

$$\Rightarrow \frac{\cancel{r}dr}{\cancel{r^2}d\theta} = \frac{\sqrt{1-r^2}}{\cancel{r}} \Rightarrow \int \frac{dr}{\sqrt{1-r^2}} = \int d\theta$$

**Type-4 : Special case**

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2} \quad \text{where } b_1 + a_2 = 0$$

$d(xy) = x \cdot dy + y \cdot dx$

Simply cross multiply & note the perfect differential of xy, Integrate term by term

**E(1)** $\frac{dy}{dx} = \frac{4x - 3y}{3x - 2y}$  **E(2)** $\frac{dy}{dx} = \frac{x - 2y + 5}{2x + 3y - 1}$

$\boxed{b_1 + a_2 = 0}$

$$\frac{dy}{dx} = \frac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}$$

E①

$3x\,dy - 2y\,dy = 4x\,dx - 3y\,dx$

$\Rightarrow 3(x\,dy + y\,dx) = 2y\,dy + 4x\,dx$

$\Rightarrow 3\,d(xy) = d(y^2) + 2\,d(x^2)$

$\boxed{3xy = y^2 + 2x^2 + C}$

$d(y^2) = 2y\,dy$

$d(x^2) = 2x\,dx$

## #II. HOMOGENEOUS EQUATIONS :

### Type 1 : General form

A differential equation of the form $\frac{dy}{dx} = \frac{f(x,y)}{\phi(x,y)}$, where f (x, y) & $\phi$ (x, y) are homogeneous functions of x & y, and of the same degree, is called **HOMOGENEOUS**. This equation may also be reduced to the form $\boxed{\frac{dy}{dx} = g\left(\frac{x}{y}\right) \text{ or } \frac{dy}{dx} = h\left(\frac{y}{x}\right)}$ & is solved by putting y = vx so that the dependent variable y is changed to another variable v, where v is some unknown function. Hence the differential equation is transformed to an equation with variable separable type.

$$\frac{dy}{dx} = f\left(\frac{y}{x}\right) \Rightarrow \text{Homogeneous D.E}^n.$$

$$= f(v)$$

$$y = vx \Rightarrow \frac{dy}{dx} = x\frac{dv}{dx} + v$$

$$f(v) - v = x\frac{dv}{dx}$$

$$\Rightarrow \int \frac{dx}{x} = \int \frac{dv}{f(v) - v}$$

**E(1)** $\frac{dy}{dx} = \frac{x^2+y^2}{xy}$

**E(2)** $\frac{dy}{dx} = \frac{y^2-2xy-x^2}{y^2+2xy-x^2}$

**E(3)** $\left(x\frac{dy}{dx}-y\right)\tan^{-1}\frac{y}{x} = x$ given $y|_{x=1} = 0$

E(1)

$$\frac{dy}{dx} = \left(\frac{x}{y}\right)+\left(\frac{y}{x}\right)$$

$$\frac{dy}{dx} = \frac{1}{v}+v$$

Put $\frac{y}{x} = v$

$y = v\cdot x$

$\frac{dy}{dx} = v\cdot 1 + x\frac{dv}{dx}$

$\frac{1}{v}+\cancel{v} = \cancel{v}+x\frac{dv}{dx}$

$\Rightarrow \frac{1}{v} = x\frac{dv}{dx}$

$$\int\frac{dx}{x} = \int v\,dv$$

H.W.

**E(4)** Find the equation of the curve intersecting with the x-axis at the point x = 1 and possesses the property that the length of the subnormal at any point of the curve is equal to the A.M. of the co-ordinates of this point. [**Ans.** $\frac{1}{2}\ell n\left(\frac{x^2+xy+2y^2}{x^2}\right)-\frac{1}{\sqrt{7}}\tan^{-1}\left(\frac{4y+x}{\sqrt{7}x}\right)=\ell n\frac{c}{x}$]

**E(5)** Find the curve such that the ratio of the subnormal at any point to the sum of its abscissa and ordinates is equal to the ratio of the ordinate of this point to this abscissa. If the curve passes through (1, 0) find all possible equations in the form $y = f(x)$. [**Ans.** $y = x\ell n\,|x|$ or $2xy = 1 - x^2$]

H.W.

**D(8)** Find the curve such that the ordinate of any of its point is the geometrical mean between the abscissa and the sum of the abscissa and subnormal at the point. [**Ans.** $y^2 = \dfrac{x^4 + c}{2x^2}$ or $y^2 + 2x^2 \ln x = cx^2$]

**D(9)** Find the curve such that the angle formed with the x-axis by the tangent to the curve at any of its points, is twice the angle formed by the polar radius of the point of tangency with the x-axis. Interpret the curve.

[**Ans.** $x^2 + y^2 - 2cy = 0$]

Note that the curve is a circle with centre on y-axis and touching the x-axis at the origin.

**D(10)** Find the curve for which the sum of the normal and subnormal is proportional to the abscissa, the proportionality constant being k.

To be discussed in the next lect.

**4.** The solution of the differential equation $ydx + (x + x^2y)dy = 0$ is- **[AIEEE-2004]**

(A) $-\frac{1}{xy} = C$ (B) $-\frac{1}{xy} + \log y = C$ (C) $\frac{1}{xy} + \log y = C$ (D) $\log y = Cx$

**5.** If $x\frac{dy}{dx} = y(\log y - \log x + 1)$, then the solution of the equation is - **[AIEEE-2005]**

(A) $y\log\left(\frac{x}{y}\right) = cx$ (B) $x\log\left(\frac{y}{x}\right) = cy$ (C) $\log\left(\frac{y}{x}\right) = cx$ (D) $\log\left(\frac{x}{y}\right) = cy$

**22.** The solution of primitive integral equation $(x^2 + y^2)dy = xy\,dx$, is $y = y(x)$. If $y(1) = 1$ and $y(x_0) = e$, then $x_0$ is - **[JEE 2005, (Screening) 3M]**

(A) $\sqrt{2(e^2-1)}$ (B) $\sqrt{2(e^2+1)}$ (C) $\sqrt{3}\,e$ (D) $\sqrt{\frac{e^2+1}{2}}$

**25.** Let $f(x)$ be differentiable on the interval $(0, \infty)$ such that $f(1) = 1$, and $\lim_{t\to x}\frac{t^2f(x) - x^2f(t)}{t-x} = 1$, for each $x > 0$. Then $f(x)$ is - **[JEE 2007 (3+3M)]**

(A) $\frac{1}{3x} + \frac{2x^2}{3}$ (B) $\frac{-1}{3x} + \frac{4x^2}{3}$ (C) $\frac{-1}{x} + \frac{2}{x^2}$ (D) $\frac{1}{x}$

**1.** A **2.** C **3.** B **4.** C **5.** C **22.** C **25.** A

35. $\frac{dy}{dx} + \frac{\cos x\,(3\cos y - 7\sin x - 3)}{\sin y\,(3\sin x - 7\cos y + 7)} = 0$

36. Find the curve which passes through the point (2,0) such that the segment of the tangent between the point of tangency & the y-axis has a constant length equal to 2.

37. $xdy + ydx + \frac{xdy - ydx}{x^2 + y^2} = 0$

38. Find the equation of the curve passing through the origin if the middle point of the segment of its normal from any point of the curve to the x - axis lies on the parabola $2y^2 = x$.

39. $3x^2y^2 + \cos(xy) - xy\sin(xy) + \frac{dy}{dx}\{2x^3y - x^2\sin(xy)\} = 0$.

40. Find all the curves possessing the following property; the segment of the tangent between the point of tangency & the x-axis is bisected at the point of intersection with the y-axis.

35. $(\cos y - \sin x - 1)^2(\cos y + \sin x - 1)^5 = c$ 36. $y = \pm\left[\sqrt{4 - x^2} + 2\ln\frac{2 - \sqrt{4 - x^2}}{x}\right]$

37. $xy + \tan^{-1}\frac{y}{x} = c$ 38. $y^2 = 2x + 1 - e^{2x}$ 39. $x(x^2y^2 + \cos xy) = c$ 40. $y^2 = cx$

# Homogeneous D·E$^n$

$$\frac{dy}{dx} = f(x, y)$$

$\frac{dy}{dx} = g\left(\frac{y}{x}\right) \Rightarrow$ It is a homogeneous D·E$^n$

put $\frac{y}{x} = v$

$\Rightarrow y = v \cdot x$

$\Rightarrow \frac{dy}{dx} = 1 \cdot v + x\frac{dv}{dx}$

$g(v) = 1 \cdot v + x\frac{dv}{dx}$

$\Rightarrow g(v) - v = x\frac{dv}{dx}$

$\Rightarrow \int \frac{dx}{x} = \int \frac{dv}{g(v) - v} \Rightarrow$ DO Int$^n$.

Variable - Separable

## Type 2 : Equations Reducible To The Homogeneous Form :

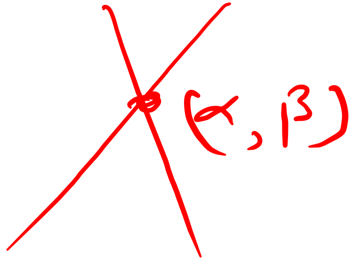
Consider $\boxed{\dfrac{dy}{dx} = \dfrac{a_1x + b_1y + c_1}{a_2x + b_2y + c_2}}$

$\dfrac{a_1}{b_1} = \dfrac{a_2}{b_2}$ | $\dfrac{dy}{dx} = \dfrac{x+y+1}{2x+2y}$ $\Bigg\}$ $\dfrac{dy}{dx} = f(ax+by+c)$

(i) If $a_1b_2 - a_2b_1 = 0$, then a substitution $u = a_1x + b_1y$ transforms the differential equation to an equation with variables separable.

(ii) If $b_1 + a_2 = 0$, then a simple cross multiplication and substituting $d(xy)$ for $x\,dy + y\,dx$ & integrating term by term yields the result easily.

(iii) If $a_1b_2 - a_2b_1 \neq 0$, i.e. $\dfrac{a_1}{b_1} \neq \dfrac{a_2}{b_2}$ and $a_2 + b_1 \neq 0$ Only for this case.




then the substitution $x = u + h$, $y = v + k$ transform this equation to a homogeneous type in the new variables $u$ and $v$ where $h$ and $k$ are arbitrary constants to be chosen so as to make the given equation homogeneous which can be solved by the method as given in Type-1

**E(1)** $\dfrac{dy}{dx} = \dfrac{2x - y + 1}{6x - 5y + 4}$ (Use $x = X + h$, $y = Y + k$) P.O.I $\left(-\dfrac{1}{4}, \dfrac{1}{2}\right)$ | $x = X + \alpha$, $y = Y + \beta$

Not in homogeneous form

Put $x = X - \frac{1}{4}$, $y = Y + \frac{1}{2}$

In Homogeneous.

$x = X - \frac{1}{4}$, $y = Y + \frac{1}{2}$

$$\frac{dy}{dx} = \frac{dY}{dX}$$

$$2x - y + 1 = 2\left(X - \frac{1}{4}\right) - \left(Y + \frac{1}{2}\right) + 1 = 2X - Y - \frac{1}{2} - \frac{1}{2} + 1$$

$$= 2X - Y$$

$$6x - 5y + 4 = 6X - 5Y$$

Given D.E^n becomes: $\frac{dY}{dX} = \frac{2X - Y}{6X - 5Y} = \frac{2 - \frac{Y}{X}}{6 - 5\frac{Y}{X}}$

Put $\frac{Y}{X} = V \Rightarrow \frac{dY}{dX} = V + X\frac{dV}{dX}$

$$V + X\frac{dV}{dX} = \frac{2 - V}{6 - 5V}$$

$$2x - y + 1 = 0$$

$$6x - 5y + 4 = 0$$

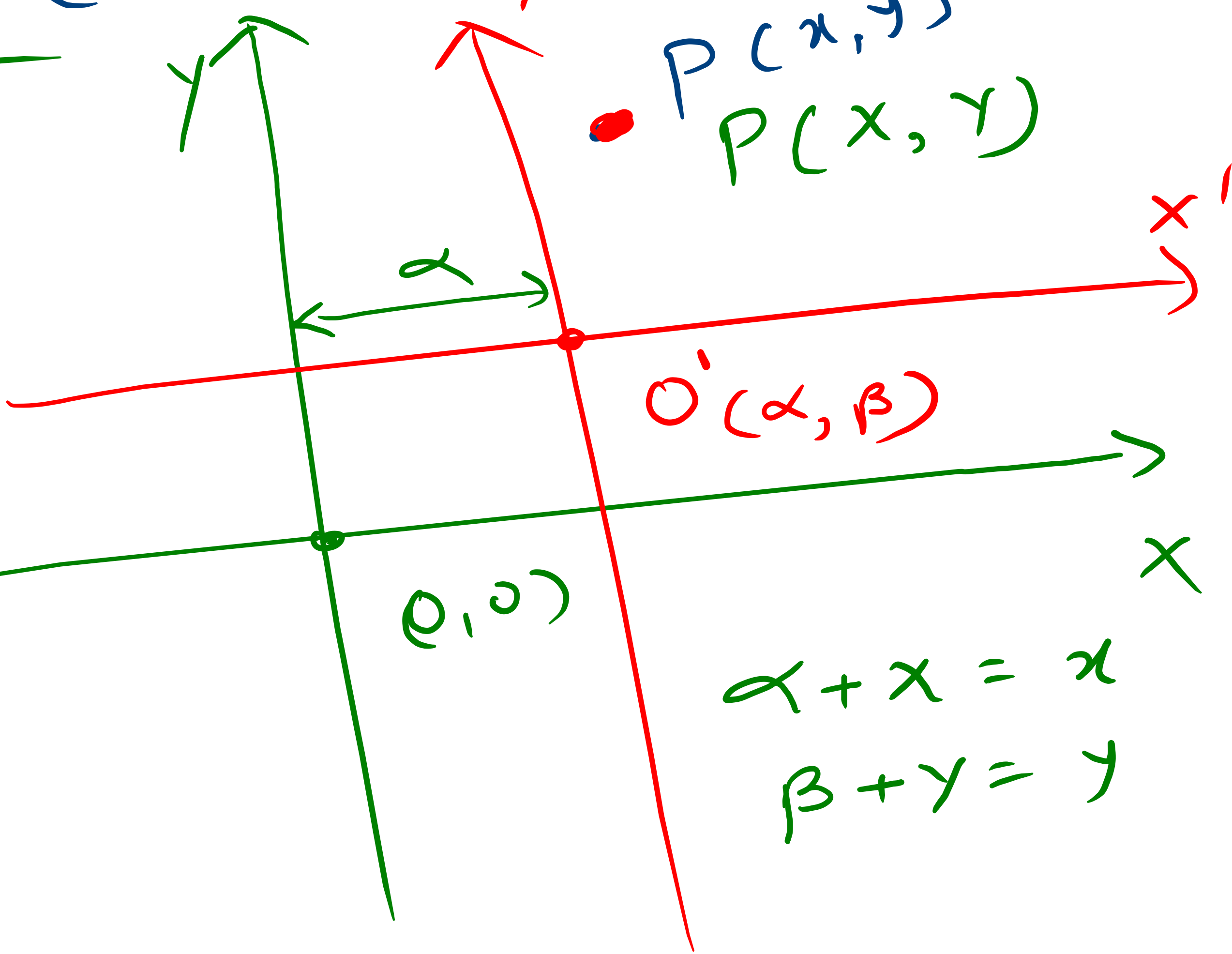
$$P.O.I \equiv (\alpha, \beta)$$

$$\left(-\frac{1}{4}, \frac{1}{2}\right)$$

$$X = x - \alpha$$

$$Y = y - \beta$$

New co-ordinates




$$\alpha + X = x$$

$$\beta + Y = y$$

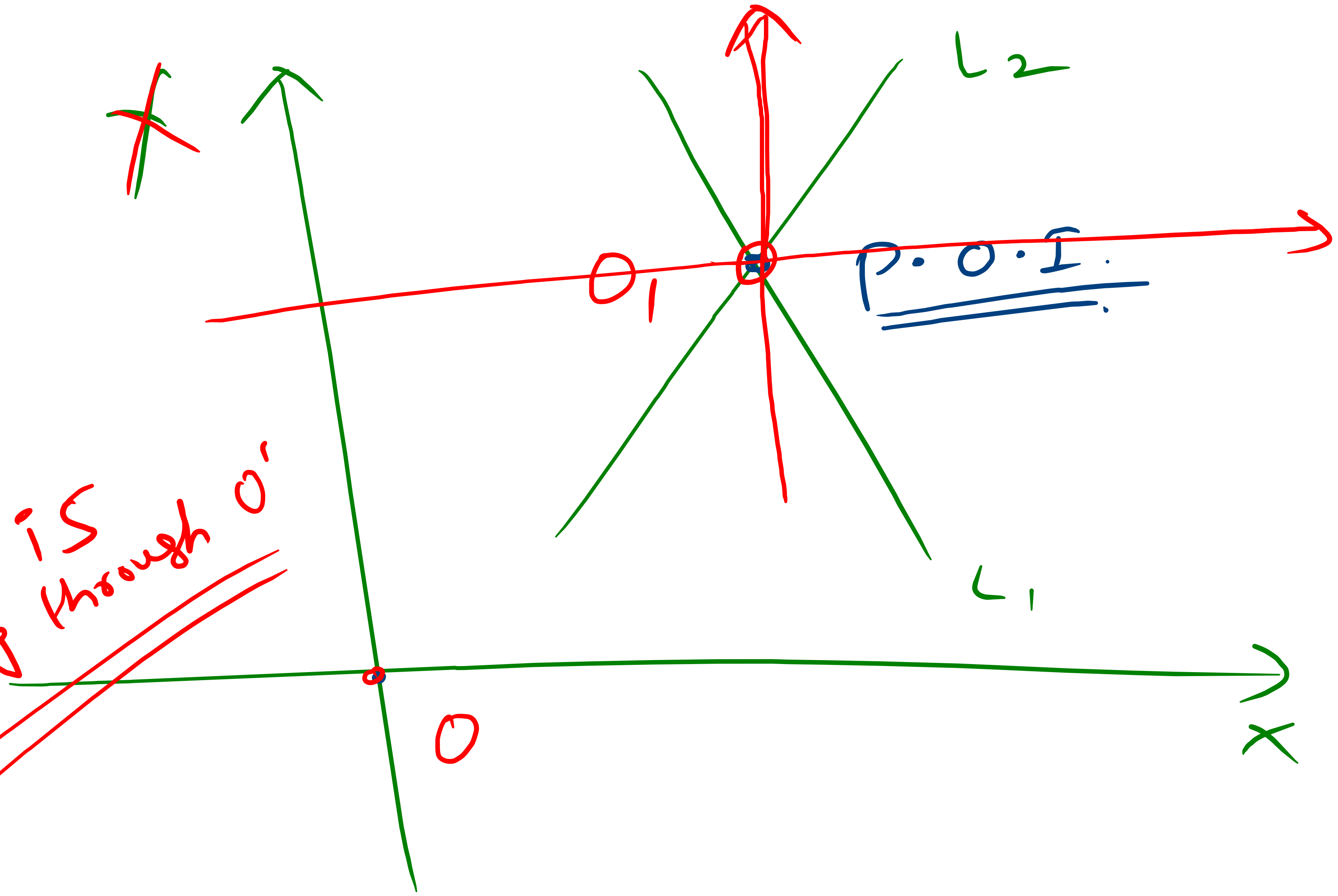
old

$ax+by+c=0$

New

$$\boxed{y=mx}$$

Because line is passing through O'



## III. LINEAR DIFERENTIAL EQUATIONS :

A differential equation is said to be linear if the dependent variable & its differential coefficients occur in the first degree only and are never multiplied together.

The nth order linear differential equation is of the form ;

$a_0(x)\frac{d^n y}{dx^n} + a_1(x)\frac{d^{n-1} y}{dx^{n-1}} + \ldots\ldots + a_n(x).\, y = \phi(x)$, where $a_0(x), a_1(x) \ldots\ldots a_n(x)$ are called the coefficients of the differential equation.

**Note** that a linear differential equation is always of the first degree but every differential equation of the first degree need not be linear. e.g. the differential equation $\frac{d^2 y}{dx^2} + \left(\frac{dy}{dx}\right)^3 + y^2 = 0$ is not linear, though its degree is 1.

$$e^{\int p\,dx} = \underline{I.F.}$$

### Type-1 : Linear Differential Equations Of First Order :

The most general form of a linear differential equations of first order is $\frac{dy}{dx} + Py = Q$, where P & Q are functions of x. (Independent variable)

$$\frac{dy}{dx}.e^{\int Pdx} + Pye^{\int Pdx} = Qe^{\int Pdx} \Rightarrow \frac{dy}{dx}(ye^{\int Pdx}) = Qe^{\int Pdx} \Rightarrow \boxed{ye^{\int Pdx} = \int Q.e^{\int Pdx} + c}$$

$$\frac{dy}{dx} + P.y = Q$$

I.F. =

P & Q function of x only

(i) The factor $e^{\int P\,dx}$, on multiplying by which, the left hand side of the differential equation becomes the differential coefficient of some function of x & y, is called integrating factor of the differential equation popularly abbreviated as I. F.

(ii) It is very important to remember that on multiplying by the integrating factor, the left hand side becomes the derivative of the product of y and the I.F.

(iii) Sometimes a given differential equation becomes linear if we take y as the independent variable and x as the dependent variable. e.g. the equation ; $(x + 2y^3)\frac{dy}{dx} = y$ can be written as

$\frac{dx}{dy} = \frac{x + 2y^3}{y} = \frac{1}{y}x + 2y^2$ which is a linear differential equation.

$$\frac{dx}{dy} + P \cdot x = Q$$

P & Q function of y only.

$$x \times (I.F.) = \int Q \cdot (I.F)\, dy$$

**E(1)** $t(1+t^2)dx = (x + xt^2 - t^2)dt$ ; $x\big|_{t=1} = -\dfrac{\pi}{4}$ **[Ans.** $x = -t\tan^{-1}t$]

**E(2)** $x\ell nx\dfrac{dy}{dx} + y = 2\ell nx$ **[Ans.** $y\ell nx = \ell n^2x + c$]

**E(3)** $x(x^2+1)\dfrac{dy}{dx} = y(1-x^2) + x^2\ell nx$ **[Ans.** $y\left(x + \dfrac{1}{x}\right) = x\ell nx - x + c$]

E(1)

$$t(1+t^2)\,dx = ((1+t^2)x - t^2)\,dt$$

$$\Rightarrow \frac{dx}{dt} = \frac{x}{t} - \frac{t^2}{t(1+t^2)}$$

$$\Rightarrow \boxed{\frac{dx}{dt} + \left(-\frac{1}{t}\right)x = \frac{-t}{(1+t^2)}}$$

$$P = -\frac{1}{t}, \quad Q = \frac{-t}{1+t^2}$$

$$I.F. = e^{\int P\,dt} =$$

$$x(x^2+1)\frac{dy}{dx} = y(1-x^2) + x^2 \ln x$$

$$\Rightarrow \frac{dy}{dx} = y \cdot \frac{1-x^2}{x(x^2+1)} + \frac{x^2 \ln x}{x(x^2+1)}$$

$$\Rightarrow \frac{dy}{dx} + \left(\frac{x^2-1}{(x^2+1)x}\right)y = \frac{x^2 \ln x}{x(x^2+1)}$$

$$P(x) = \frac{x^2-1}{(x^2+1)x}, \quad Q(x) = \frac{x \ln x}{(x^2+1)}$$

$$I.F. = e^{\int P\,dx}$$

$$y \times IF = \int Q \cdot (IF)\,dx + C$$

$$x \ln x \frac{dy}{dx} + y = 2 \ln x$$

$$\frac{dy}{dx} + \left(\frac{1}{x \cdot \ln x}\right) \cdot y = \frac{2 \ln x}{x \cdot \ln x} = \frac{2}{x}$$

$$P = \frac{1}{x \cdot \ln x}, \quad Q = \frac{2}{x}$$

$$y \times I.F. = \int Q\,(I.F.)\,dx + C$$

E(6) $\boxed{\dfrac{dy}{dx}=\dfrac{1}{x\cos y+\sin 2y}}$

Solⁿ. $x \times (I.F.) = \int Q\,(I.F.)\,dy + c$

$\Rightarrow \dfrac{dx}{dy}=x\cos y+\sin 2y$ $\Rightarrow$ $\dfrac{dx}{dy}-(\cos y)x=\sin 2y$

$\dfrac{dx}{dy}+P\cdot x = Q$

P & Q must be function of y only.

$IF = e^{\int P\,dy}$

## Type-2 : Equations Reducible To Linear Form :

### (a) Bernoulli's Equation :

The equation $\frac{dy}{dx} + Py = Q.\, y^n$ where P & Q functions of x, is reducible to the linear form by dividing it by $y^n$ & then substituting $y^{-n+1} = z$. Its solution can be obtained as in normal case.

**E(1)** $\frac{dy}{dx} = xy + x^3y^2$ **[Ans.** $\frac{1}{y} = 2 - x^2 + C$]

1

$\frac{dy}{dx} + (-x)\, y = x^3 . y^2$

$\Rightarrow \frac{1}{y^2}\frac{dy}{dx} + \left(\frac{-1}{y}\right) x = x^3$

put $-\frac{1}{y} = t$

$\Rightarrow \frac{1}{y^2}\frac{dy}{dx} = \frac{dt}{dx}$

$\frac{dt}{dx} + x \cdot t = x^3$

**E(2)** $\frac{dy}{dx} - 2y \tan x + y^2 \tan^4 x = 0$

$\frac{dy}{dx} + (-2\tan x)\, y = (-\tan^4 x)\, y^2$

$$-\frac{1}{y^2}\frac{dy}{dx} + 2\tan x . \frac{1}{y} = \tan^4 x$$

put $\frac{1}{y} = t$ $\therefore -\frac{1}{y^2}\frac{dy}{dx} = \frac{dt}{dx}$

$$\frac{dt}{dx} + 2\tan x . t = \tan^4 x$$

## Change of variable by a suitable substitution :

E(1) $\frac{dy}{dx} = e^{x-y}(e^x - e^y)$ $\Rightarrow$ $e^y \frac{dy}{dx} = e^x(e^x - e^y)$

$\frac{y}{x}$

E(2) $x\frac{dy}{dx} + y\ell ny = xye^x$ $\Rightarrow$ $x \cdot \frac{1}{y}\frac{dy}{dx} + \ln y = x e^x$

$\frac{1}{y}\frac{dy}{dx} + \left(\frac{1}{x}\right)\ln y = e^x$

E(3) $y\sin x\frac{dy}{dx} = \cos x(\sin x - y^2)$

$y \sin x \frac{dy}{dx} = \cos x(\sin x - y^2)$ $\Rightarrow$

$y\frac{dy}{dx} = \cot x(\sin x - y^2)$

$y^2 = t$

$\frac{dt}{dx} = 2\cot x \cdot \sin x - 2t\cot x$

$\Rightarrow \frac{dt}{dx} + 2(\cot x)t = 2\cos x$

$\ln y = t$

$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{dt}{dx}$

$\frac{dt}{dx} + \left(\frac{1}{x}\right)t = e^x$

F. L. D. E

## EXACT DIFFERENTIAL EQUATIONS :

Sometimes just by inspection, a first order degree differential equation can be written in the exact differential form which can be easily integrated.

**Following exact differentials must be remembered :**

(i) $xdy + y\,dx = d(xy)$

(ii) $\dfrac{xdy - ydx}{x^2} = d\left(\dfrac{y}{x}\right)$

(iii) $\dfrac{ydx - xdy}{y^2} = d\left(\dfrac{x}{y}\right)$

(iv) $\dfrac{xdy + ydx}{xy} = d(\ell n\, xy)$

(v) $\dfrac{dx + dy}{x + y} = d(\ell n(x+y))$

(vi) $\dfrac{xdy - ydx}{xy} = d\left(\ell n\dfrac{y}{x}\right)$

(vii) $\dfrac{ydx - xdy}{xy} = d\left(\ell n\dfrac{x}{y}\right)$

(viii) $\dfrac{xdy - ydx}{x^2 + y^2} = d\left(\tan^{-1}\dfrac{y}{x}\right)$

(ix) $\dfrac{ydx - xdy}{x^2 + y^2} = d\left(\tan^{-1}\dfrac{x}{y}\right)$

(x) $\dfrac{xdx + ydy}{x^2 + y^2} = d\left[\ell n\sqrt{x^2 + y^2}\right]$

**E(2)** $\cos\left(\frac{x}{y}\right)(ydx - xdy) = xy^3(xdy + ydx)$ [**Ans.** $\sin\left(\frac{x}{y}\right) = \frac{(xy)^2}{2} + C$ ]

**E(3)** $\left(\frac{1}{x} - \frac{y^2}{(x-y)^2}\right)dx + \left(\frac{x^2}{(x-y)^2} - \frac{1}{y}\right)dy = 0$ [**Ans.** $c = \ell n\frac{x}{y} + \frac{xy}{x-y}$ ]

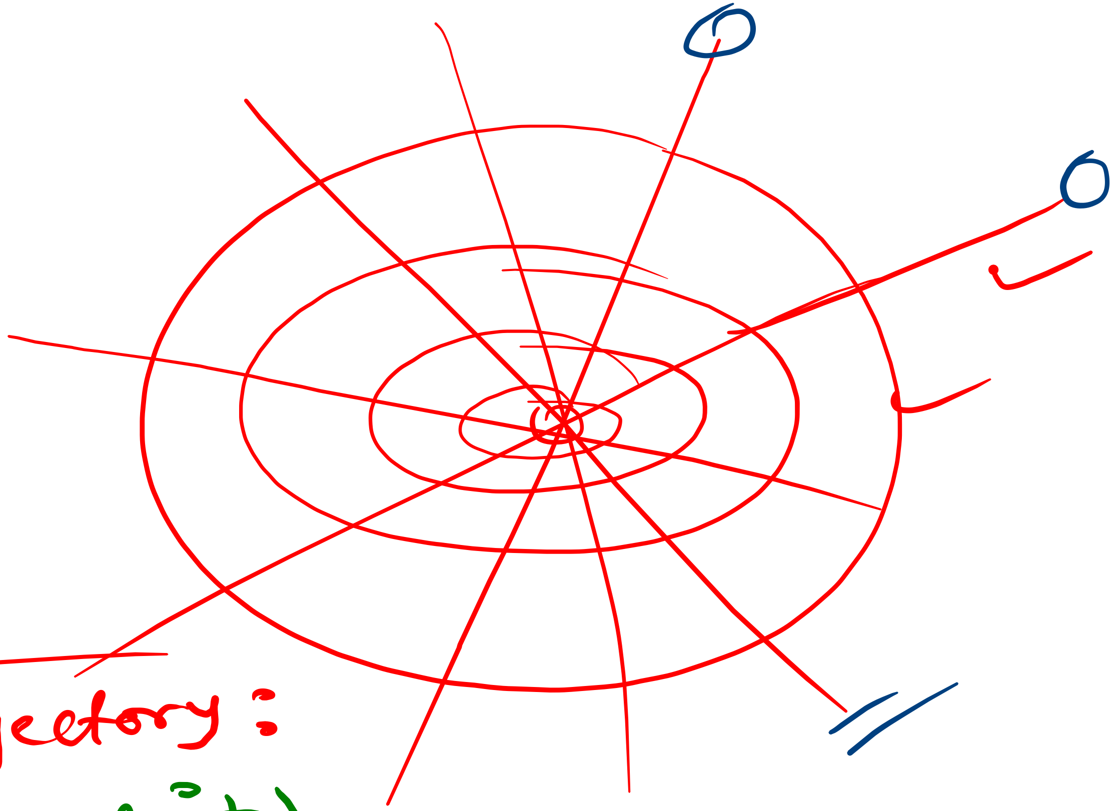
**D(4)** $\left(\frac{1}{y}\sin\frac{x}{y} - \frac{y}{x^2}\cos\frac{y}{x} + 1\right)dx + \left(\frac{1}{x}\cos\frac{y}{x} - \frac{x}{y^2}\sin\frac{x}{y} + \frac{1}{y^2}\right)dy = 0$ [**Ans.** $\sin\frac{y}{x} - \cos\frac{x}{y} + x - \frac{1}{y} = c$ ]

# ORTHOGONAL TRAJECTORY

Family of lines

Family of circles

Intersecting at $90^\circ$



To find orthogonal trajectory:

Given D.E$^n$ (First get it)

Replace $\left(\frac{dy}{dx}\right)$ by $\left(-\frac{dx}{dy}\right)$ $\Rightarrow$ D.E$^n$ solve it

You will get orthogonal trajectory.

$\left(\frac{dy}{dx}\right)$ Given family

Required family $\left(-\frac{dx}{dy}\right)$ Required trajectory

$$\boxed{m_1 m_2 = -1}$$

$x^2+y^2 = R^2$ Family of circles

find equation of a curve orthogonal to this family.

$2x + 2y\frac{dy}{dx} = 0 \Rightarrow x + y\frac{dy}{dx} = 0$

D. E^n of the required family $\Rightarrow x + y\left(-\frac{dx}{dy}\right) = 0$

$\Rightarrow y\frac{dx}{dy} = x \Rightarrow \frac{dx}{x} = \frac{dy}{y}$

## TRAJECTORIES :

A curve which cuts every member of a given family of curves according to a given law is called a Trajectory of the given family.

The trajectory will be called **Orthogonal** if each trajectory cuts every member of given family at right angle.

**Working rule for finding orthogonal trajectory**

**1.** Form the differential equation of family of curves

**2.** Write $-\dfrac{1}{dy/dx}$ for $\dfrac{dy}{dx}$ or $-\dfrac{r^2 d\theta}{dr}$ for $\dfrac{dr}{d\theta}$ if differential equation is in the polar form.

**3.** Solve the new differential equation to get the equation of orthogonal trajectories.

**Note:** A family of curves is self-orthogonal if it is its own orthogonal family.

Q. Find the value of k such that the family of parabolas $y = cx^2 + k$ is the orthogonal trajectory of the family of ellipses $x^2 + 2y^2 - y = c_1$

where $c, c_1$ are I.A.C. k is a given no.

$$x^2 + 2y^2 - y = c_1$$

$$\Rightarrow 2x + 4yy' - y' = 0$$

$$\Rightarrow 2x = (1-4y)\frac{dy}{dx}$$

Replace $\frac{dy}{dx}$ by $\left(-\frac{dx}{dy}\right)$

$$\Rightarrow 2x = (4y-1)\frac{dx}{dy}$$

$$\Rightarrow \frac{dy}{4y-1} = \frac{dx}{2x}$$

$$\Rightarrow \frac{\ln(4y-1)}{4} = \frac{1}{2}\ln x + \ln \lambda$$

$$\Rightarrow \ln(4y-1) = \ln(x^2 \cdot c)$$

$$\Rightarrow 4y - 1 = x^2 \cdot c$$

$$4y = c \cdot x^2 + 1$$

$$y = \left(\frac{c}{4}\right)x^2 + \frac{1}{4}$$

## APPLICATION OF DIFFERENTIAL EQUATIONS :

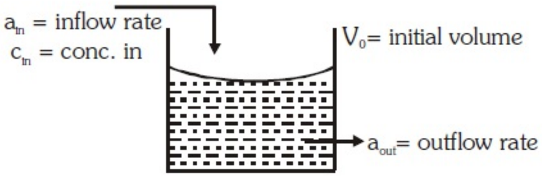
### (a) Mixing Problems

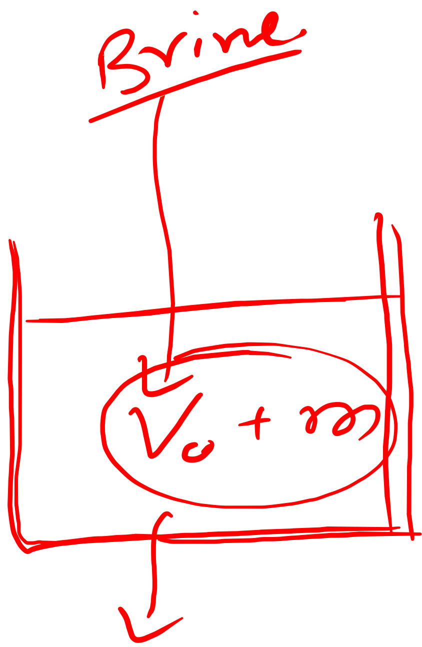
A chemical in a liquid solution with given concentration $c_{in}$ gm/lit. (or dispersed in a gas) runs into a container with a rate of $a_{in}$ lit/min. holding the liquid (or the gas) with, possibly, a specified amount of the chemical dissolved as well. The mixture is kept uniform by stirring and flows out of the container at a known rate ($a_{out}$ litre/min.). In this process it is often important to know the concentration of the chemical in the container at any given time. The differential equation describing the process is based on the formula.

$$\begin{matrix}\text{Rate of change} \\ \text{of amount} \\ \text{in container}\end{matrix} = \begin{pmatrix}\text{rate at which} \\ \text{chemical} \\ \text{arrives}\end{pmatrix} - \begin{pmatrix}\text{rate at which} \\ \text{chemical} \\ \text{departs}\end{pmatrix} \qquad \text{.......(i)}$$

Arrival rate = (conc. in) $\times$ (inflow rate) = $c_{in} \times a_{in}$

If y(t) denotes the amount of substance in the tank at time t & V(t) denotes the amount of mixture in tank at that time



Q. A tank contains 50gm of salt dissolved in 100L of water. Brine that contains $\boxed{2\ gm/L}$ of salt enters the tank at a rate of 5 L/min. The solution is kept thoroughly mixed and it flows out of the tank at the rate of 4 L/min. Find amount of salt m at time t.

$\boxed{5L/min}$

$m_0 = 50gm$

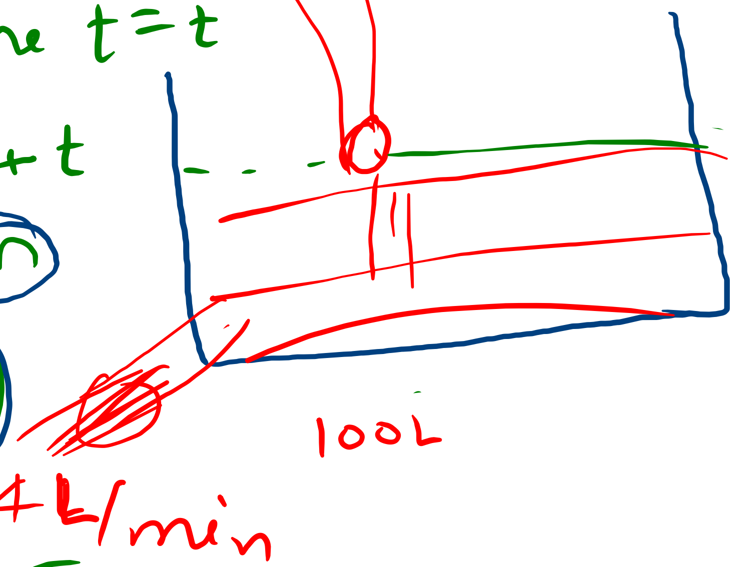
At time $t = t$

$$\boxed{\frac{dm}{dt} = 2\times5 - \frac{4m}{100+t}}$$

$V = 100 + t$

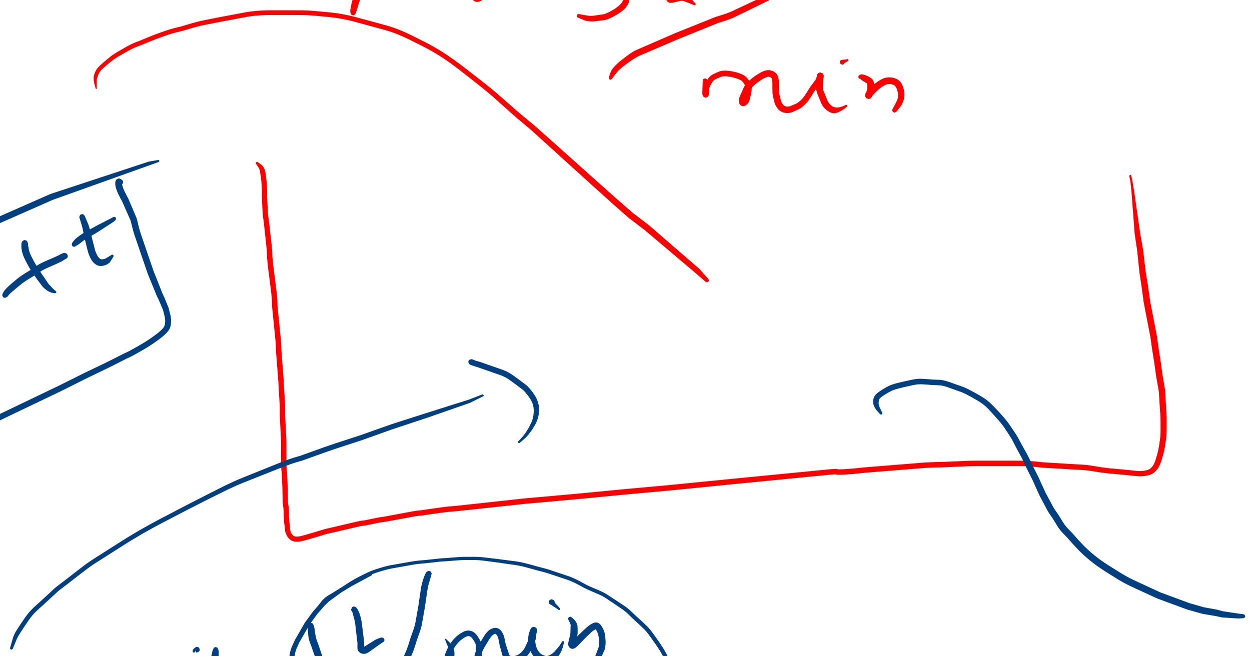
$m$

$\left(\frac{m}{100+t}\right)$




$$\Rightarrow \frac{dm}{dt} + \left(\frac{4}{100+t}\right)m = 10$$

Input 5L/min



At time t

$\boxed{\text{Volume} = 100 + t}$

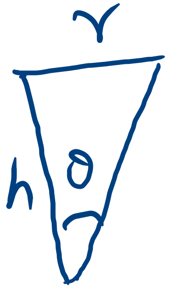
Deposit 1L/min

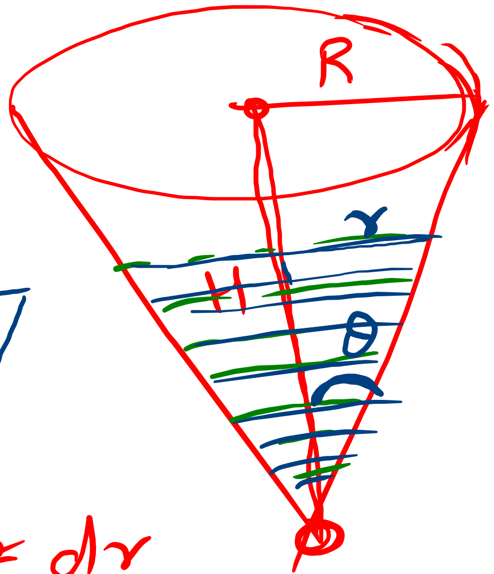
4L/min

In time t added Volume $= t \times 1$

Q. A right circular cone with radius R and height H contains a liquid which evaporates at a rate proportional to its surface area in contact with air (proportionality constant is $K>0$), find the time after which the cone is empty.

$$\boxed{\tan\theta = \frac{R}{H}} \qquad \boxed{\cot\theta = \frac{H}{R}} = \frac{h}{r}$$




$$V = \frac{1}{3}\pi r^2 h,$$

$$\frac{dV}{dt} \propto (\pi r^2)$$

$$\boxed{\frac{dV}{dt} = -K\pi r^2}$$

$$\Rightarrow V = \frac{1}{3}\pi r^2 \times r\frac{H}{R}$$

$$V = \left(\frac{\pi H}{3R}\right) r^3$$

$$\frac{dV}{dt} = \left(\frac{\pi H}{R}\right) r^2 \frac{dr}{dt}$$

$$\Rightarrow -K\cancel{\pi}\cancel{r^2} = \frac{\cancel{\pi} H}{R}\cancel{r^2}\frac{dr}{dt}$$

$$\int_0^t dt = \int_R^0 \left(-\frac{H}{RK}\right) dr \Rightarrow t = \frac{H}{K}$$

Q: A country has a food deficit of 10 %. Its population grows continuously at a rate of 3 % per year. Its annual food production every year is 4 % more than that of the last year. Assuming food consumption per person remains same, P.T. the country will become self-sufficient in food after n years if

$$\boxed{n \geq \frac{\ln 10 - \ln 9}{\ln(1.04) - 0.03}}$$

$t = 0$, $P_0$ $F_0$

$t = n$ $P$ $F$

$\lambda$ per person requirement

$$P_0 \times \lambda \times \frac{90}{100} = F_0 \Rightarrow \boxed{(P_0)\lambda \times 9 = 10 F_0}$$

$$F = (1.04)^n F_0$$

$$\frac{dP}{dt} = P \times \frac{3}{100} \Rightarrow \frac{dP}{P} = \frac{3}{100} dt$$

$$\Rightarrow \ln\left(\frac{P}{P_0}\right) = \frac{3t}{100}$$

$$\Rightarrow P = P_0 e^{\frac{3}{100} \times n}$$

$$P_0 e^{\frac{3}{100} \times n} \times \lambda \leq (1.04)^n F_0$$

continuously grows $\left(\frac{dp}{dt}\right)$

$$\frac{dP}{dt} = \frac{3\times P}{100}$$

$$\Rightarrow \int_{P_0}^{P} \frac{dP}{P} = \frac{3}{100}\int_0^t dt$$

$$\Rightarrow \ln P - \ln P_0 = \frac{3}{100}\times t$$

$$\Rightarrow \ln\left(\frac{P}{P_0}\right) = \frac{3}{100}\times t \Rightarrow \boxed{P = e^{\frac{3}{100}\times t}\times P_0}$$

Compounding

$$F = \left(1+\frac{r}{100}\right)^n P_0$$

$$\boxed{P\times \lambda \le F}$$

**Exponential Growth and Decay :**

In general, if y(t) is the value of quantity y at time t and if the rate of change of y with respect to t is proportional to its value y(t) at that time, then

$$\frac{dy(t)}{dt} = ky(t), \text{ where k is a constant} \qquad ....(i)$$

$$\int \frac{dy(t)}{y(t)} = \int k\,dt$$

Solving, we get $y(t) = Ae^{kt}$

equation (i) is sometimes called the law of natural growth (if $k > 0$) or law of natural decay (if $k < 0$).

$$\frac{dm}{dt} \propto m \Rightarrow \frac{dm}{dt} = -\lambda m, \quad \lambda > 0$$

$$\int_{m_0}^{m} \frac{dm}{m} = -\lambda \int_0^t dt \Rightarrow \ln\left(\frac{m}{m_0}\right) = -\lambda t$$

$$\boxed{m = m_0 e^{-\lambda t}}$$

**Geometrical applications :**

Let $P(x_1, y_1)$ be any point on the curve $y = f(x)$, then slope of the tangent at point P is $\left(\dfrac{dy}{dx}\right)_{(x_1, y_1)}$

(i) The equation of the tangent at P is $y - y_1 = \dfrac{dy}{dx}(x - x_1)$

x-intercept of the tangent $= x_1 - y_1\left(\dfrac{dx}{dy}\right)$

y-intercept of the tangent $= y_1 - x_1\left(\dfrac{dy}{dx}\right)$

(ii) The equation of normal at P is $y - y_1 = -\dfrac{1}{(dy/dx)}(x - x_1)$

x and y-intercepts of normal are ; $x_1 + y_1\dfrac{dy}{dx}$ and $y_1 + x_1\dfrac{dx}{dy}$

(iii) Length of tangent $= PT = |y_1|\sqrt{1 + (dx/dy)^2_{(x_1, y_1)}}$

(iv) Length of normal $= PN = |y_1|\sqrt{1 + (dy/dx)^2_{(x_1, y_1)}}$

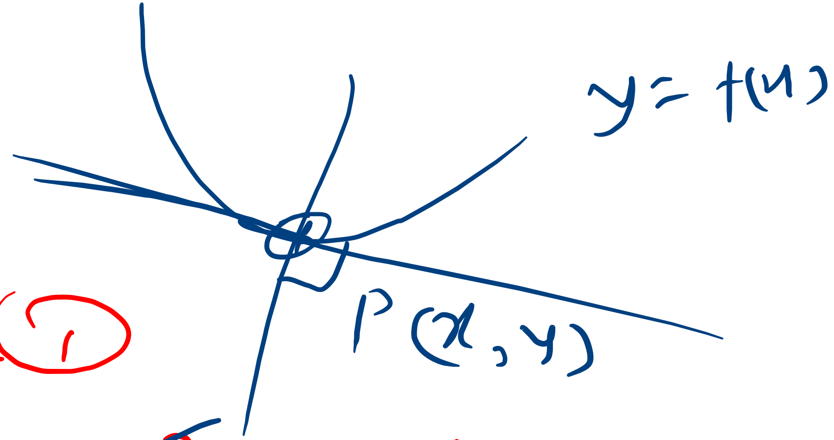
Q. A curve passing through the point (1, 1) has the property that the ⊥r distance of normal at any point P on the curve from (0, 0) is equal to the distance of P from x-axis. Find the curve

equation of normal

$$Y - y = (X - x)\left(\frac{-1}{\frac{dy}{dx}}\right)$$

$$\Rightarrow \left(\frac{dy}{dx}\right)Y - \frac{dy}{dx}y + X - x = 0 \quad \text{— (1)}$$

$$\Rightarrow X + \left(\frac{dy}{dx}\right)Y - \left(x + y\frac{dy}{dx}\right) = 0$$




$$\frac{\left|0 + 0 - \left(x + y\frac{dy}{dx}\right)\right|}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}} = |y|$$

$$x^2 + \cancel{y^2\left(\frac{dy}{dx}\right)^2} + 2xy\frac{dy}{dx} = y^2 + \cancel{y^2\left(\frac{dy}{dx}\right)^2}$$

$$\Rightarrow \boxed{2xy\frac{dy}{dx} + x^2 = y^2}$$


$$PM = \frac{|a\alpha + b\beta + c|}{\sqrt{a^2 + b^2}}$$

Q: Let $u(x)$, $v(x)$ satisfy the D.E$^n$ $\frac{du}{dx} + p(x)\cdot u = f(x)$ and $\frac{dv}{dx} + p(x)v = g(x)$ {where $p(x)$, $f(x)$, $g(x)$ are continuous functions} respectively.

If $\boxed{u(x_1) > v(x_1)}$ for some $x_1$ and $\boxed{f(x) > g(x)}$ for all $\boxed{x > x_1}$. P.T. $y = u(x)$ & $y = v(x)$ will never intersect.

$$\frac{du}{dx} + pu = f$$

$$\frac{dv}{dx} + pv = g$$

$$\Rightarrow \frac{d(u-v)}{dx} + p(u-v) = f-g$$

$$\Rightarrow e^{\int p\,dx}\frac{d(u-v)}{dx} + p(u-v)e^{\int p\,dx} = (f-g)e^{\int p\,dx}$$

$$\Rightarrow \frac{d}{dx}\left((u-v)e^{\int p\,dx}\right) = (f-g)e^{\int p\,dx}$$

$$(u-v)e^{\int p\,dx}$$

$$H(x) = (u-v)e^{\int p\,dx}$$

$H(x_1) = +ve$ no.

$x > x_1$

$H(x) > H(x_1)$

$H(x_1)$ is $+ve$ $\Rightarrow$ $u - v > 0$

Q. Let $f(x)$, $x \geq 0$ be a non-negative continuous function and let $F(x) = \int_0^x f(t)\,dt$, $x \geq 0$, if for some $c > 0$, $\boxed{f(x) \leq c\,F(x)}$ for all $x \geq 0$

T.P.T. $f(x) = 0$ for all $x \geq 0$.

$$F'(x) = f(x) \Rightarrow F'(x) \leq c\,F(x) \Rightarrow F'(x) - c\,F(x) \leq 0$$

$$\Rightarrow e^{-cx} F'(x) - c\,F(x)\,e^{-cx} \leq 0 \Rightarrow \frac{d}{dx}\left(F(x) \times e^{-cx}\right) \leq 0$$

$$H(x) = F(x) \times e^{-cx}, \quad H'(x) \leq 0 \Rightarrow H(x) \leq H(0), \quad \boxed{x \geq 0}$$

$$\Rightarrow F(x) \times e^{-cx} \leq 0 \Rightarrow \boxed{F(x) \leq 0} \Rightarrow F(x) = 0$$

$$\Rightarrow f(x) \leq c\,F(x) = 0 \Rightarrow f(x) \leq 0$$

$$\Rightarrow f(x) = 0$$

**21.** Let y = y(x) be the solution of the differential equation, $\left(x^2+1\right)^2\frac{dy}{dx}+2x\left(x^2+1\right)y=1$ such that $y(0)=0$. If $\sqrt{a}\,y(1)=\frac{\pi}{32}$, then the value of 'a' is : **[JEE(Main)-2019]**

(1) $\frac{1}{2}$ (2) $\frac{1}{16}$ (3) $\frac{1}{4}$ (4) 1

**22.** Consider the differential equation, $y^2dx+\left(x-\frac{1}{y}\right)dy=0$. If value of y is 1 when x = 1, the the value of x for which y = 2, is : **[JEE(Main)-2019]**

(1) $\frac{1}{2}+\frac{1}{\sqrt{e}}$ (2) $\frac{3}{2}-\sqrt{e}$ (3) $\frac{5}{2}+\frac{1}{\sqrt{e}}$ (4) $\frac{3}{2}-\frac{1}{\sqrt{e}}$

**19.** The curve amongst the family of curves, represented by the differential equation, $(x^2 - y^2)dx + 2xy\,dy = 0$ which passes through (1,1) is : **[JEE(Main)-2019]**

(1) A circle with centre on the y-axis

(2) A circle with centre on the x-axis

(3) An ellipse with major axis along the y-axis

(4) A hyperbola with transverse axis along the x-axis

**20.** If y(x) is the solution of the differential equation $\frac{dy}{dx} + \left(\frac{2x+1}{x}\right)y = e^{-2x}$, $x > 0$, where $y(1) = \frac{1}{2}e^{-2}$, then : **[JEE(Main)-2019]**

(1) y(x) is decreasing in (0, 1)

(2) y(x) is decreasing in $\left(\frac{1}{2}, 1\right)$

(3) $y(\log_e 2) = \frac{\log_e 2}{4}$

(4) $y(\log_e 2) = \log_e 4$