Note:- $a, b, c \in \mathbb{R}$

$$a^3+b^3+c^3-3abc = (a+b+c)(a^2+b^2+c^2-ab-bc-ca)$$

$$= \left(\frac{a+b+c}{2}\right)\left((a-b)^2+(b-c)^2+(c-a)^2\right)$$

$$= (a+b+c)\left((a+b+c)^2-3\sum ab\right).$$

$$= (a+b+c)\,\underbrace{(a+b\omega+c\omega^2)}\,\underbrace{(a+b\omega^2+c\omega)}$$

conjugate

Rem

$$|a+b\omega+c\omega^2| = |a+b\omega^2+c\omega| = \sqrt{a^2+b^2+c^2-ab-bc-ca}$$

$$= \sqrt{\frac{(a-b)^2+(b-c)^2+(c-a)^2}{2}}$$

E(1) If ω is imaginary cube root of unity, then find the value of

(a) $(1 + \omega - \omega^2)^7$.

(b) $\sum_{r=0}^{10}(1+\omega^r+\omega^{2r})$

(c) $(1 + 2\omega + 3\omega^2)^{10} + (2 + 3\omega + \omega^2)^{10} + (3 + \omega + 2\omega^2)^{10}$

Sol^n (a) $(\underbrace{1+\omega}_{-\omega^2}-\omega^2)^7 = (-2\omega^2)^7 = -128\,\omega^{14} = -128\,\omega^2$ Ans

(b) $\sum_{r=0}^{10}\left(1^r+\omega^r+\omega^{2r}\right) = 3+3+3+3 = 12$ Ans

$r = 0;\ 1;\ 2;\ 3;\ 6 \cdots;\ 9;\ 10$

(c) $(1+2\omega+3\omega^2)^{10} + (2+3\omega+\omega^2)^{10} + (3+\omega+2\omega^2)^{10}$

$\omega^{10}\left(\frac{1}{\omega}+2+3\omega\right)^{10} + (2+3\omega+\omega^2)^{10} + (\omega^2)^{10}\left(\frac{3}{\omega^2}+\frac{1}{\omega}+2\right)^{10}$

$\frac{1}{\omega} \to \omega^2$

$\left(\frac{3}{\omega^2}+\frac{1}{\omega}+2\right)^{10} \Downarrow (3\omega+\omega^2+2)^{10}$

$(2+3\omega+\omega^2)^{10}\left(\underbrace{\omega^{10}+\omega^{20}+1}_{0}\right) = 0.$

$\Downarrow$

0

E(2) If ω is non-real cube root of unity, then find the value of $\dfrac{1+2\omega+3\omega^2}{2+3\omega+\omega^2}+\dfrac{2+3\omega+\omega^2}{3+\omega+2\omega^2}$.

$$\frac{\omega\left(\cancel{\frac{1}{\omega}+2}+3\omega\right)}{\left(\cancel{2+3\omega}+\omega^2\right)}+\frac{\omega\left(\cancel{\frac{2}{\omega}+3}+\omega\right)}{\cancel{3+\omega}+2\omega^2}$$

2ω Ans

Q Value of $\left(\sqrt{3}-i\right)^{100}+\left(\sqrt{3}+i\right)^{100} \Rightarrow 2^{100}(\underbrace{\omega+\omega^2}_{-1}) = -2^{100}$ Ans

$-1+i\sqrt{3} = 2\omega$

$i(i+\sqrt{3}) = 2\omega \Rightarrow \boxed{\sqrt{3}+i = \frac{2\omega}{i}}$

$$(\sqrt{3}+i)^{100} = \frac{(2\omega)^{100}}{\underbrace{i^{100}}_{1}} = 2^{100}\cdot\underbrace{\omega^{100}}_{\omega}$$

$2\omega^2 = -1-i\sqrt{3}$

$i(2\omega^2) = -i+\sqrt{3} \Rightarrow (\sqrt{3}-i)^{100} = \underbrace{i^{100}}_{1}(2\omega^2)^{100}$

$= 2^{100}\cdot\omega^{200} = 2^{100}\omega^2$

E(5) α & β are roots of $x^2 - x + 1 = 0$, then find value of $\alpha^{2013} + \beta^{2013}$.

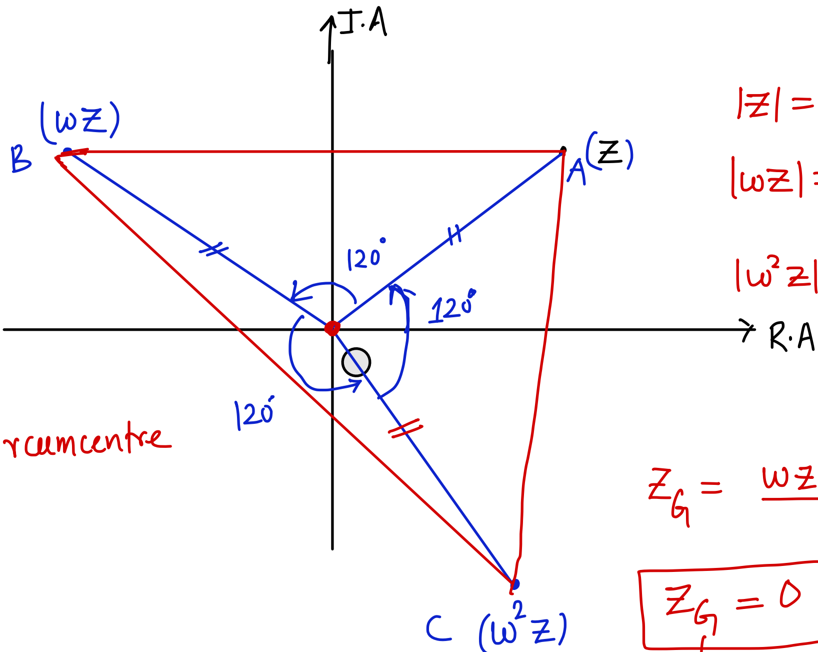
Sol^n

$$x^2 - x + 1 = 0 \quad \begin{cases} -\omega = \alpha \\ -\omega^2 = \beta \end{cases}$$

$$\alpha^{2013} + \beta^{2013} = (-\omega)^{2013} + (-\omega^2)^{2013}$$

$$= -1 - 1 = -2 \text{ Ans}$$

E(6) If ω is non-real cube root of unity, then prove that $z, \omega z, \omega^2 z$ are vertices of equilateral triangle, where $z \neq 0$.



'O' is Circumcentre

$|z| = k$ (let)

$|\omega z| = \underbrace{|\omega|}_{1}|z| = |z|$

$|\omega^2 z| = |z|$

$$z_G = \frac{\omega z + \omega^2 z + z}{3}$$

$$\boxed{z_G = 0}$$

↓ centroid of ΔABC

E(7) Find the solutions of given equations : (i) $z^3 + 27 = 0$ (ii) $z^3 - 27 = 0$ (iii) $4z^2 + 2z + 1 = 0$

(i) $z = (-27)^{\frac{1}{3}} = -3\,(1)^{1/3}$

$-3\,;\ -3\omega\,;\ -3\omega^2$

(ii) $z = (27)^{\frac{1}{3}} = 3\,(1)^{\frac{1}{3}}$

3

3ω

$3\omega^2$.

(iii) $4z^2 + 2z + 1 = 0$

let $2z = t$

$t^2 + t + 1 = 0$

ω

ω^2

$2z = \omega\,;\ \omega^2$

$\therefore z = \frac{\omega}{2}\,;\ \frac{\omega^2}{2}$ Ans

E(8) If α be a complex number satisfying $z^4 + z^3 + 2z^2 + z + 1 = 0$, then find $|\alpha|$.

Sol^n

$$\underbrace{z^4 + z^3 + z^2} + \underbrace{z^2 + z + 1} = 0$$

$$(z^2+1)(z^2+z+1) = 0$$

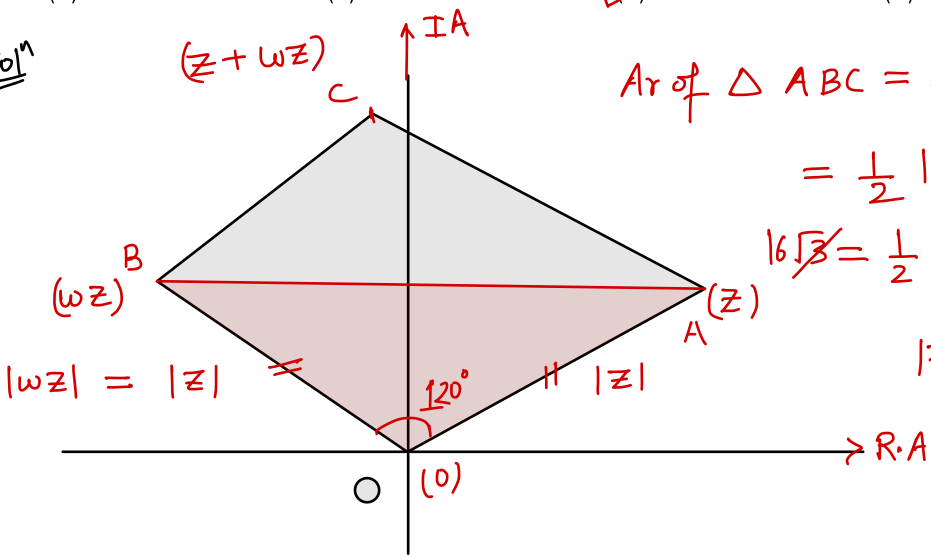
$$z^2 + 1 = 0 \rightarrow i, -i \qquad z^2 + z + 1 = 0 \rightarrow \omega, \omega^2$$

$|\alpha| = 1$ Ans

E(9) If the area of the triangle in the Argand diagram, formed by Z, ωZ and Z + ωZ were ω is the usual complex cube root of unity is $16\sqrt{3}$ square units, then |Z| is -

(A) 16 (B) 4 ✓(C) 8 (D) 3

Sol^n



Ar of Δ ABC = Ar of Δ OAB

$$= \frac{1}{2}|z||z|\sin 120^\circ$$

$$16\cancel{\sqrt{3}} = \frac{1}{2}|z|^2 \cdot \frac{\cancel{\sqrt{3}}}{2}$$

$$|z|^2 = 64$$

$$\therefore |z| = 8$$

Q Find all the complex numbers z satisfying $z^2 + z|z| + |z^2| = 0$.

Sol^n C–I If $|z| = 0$ then $\boxed{z = 0}$ ✓

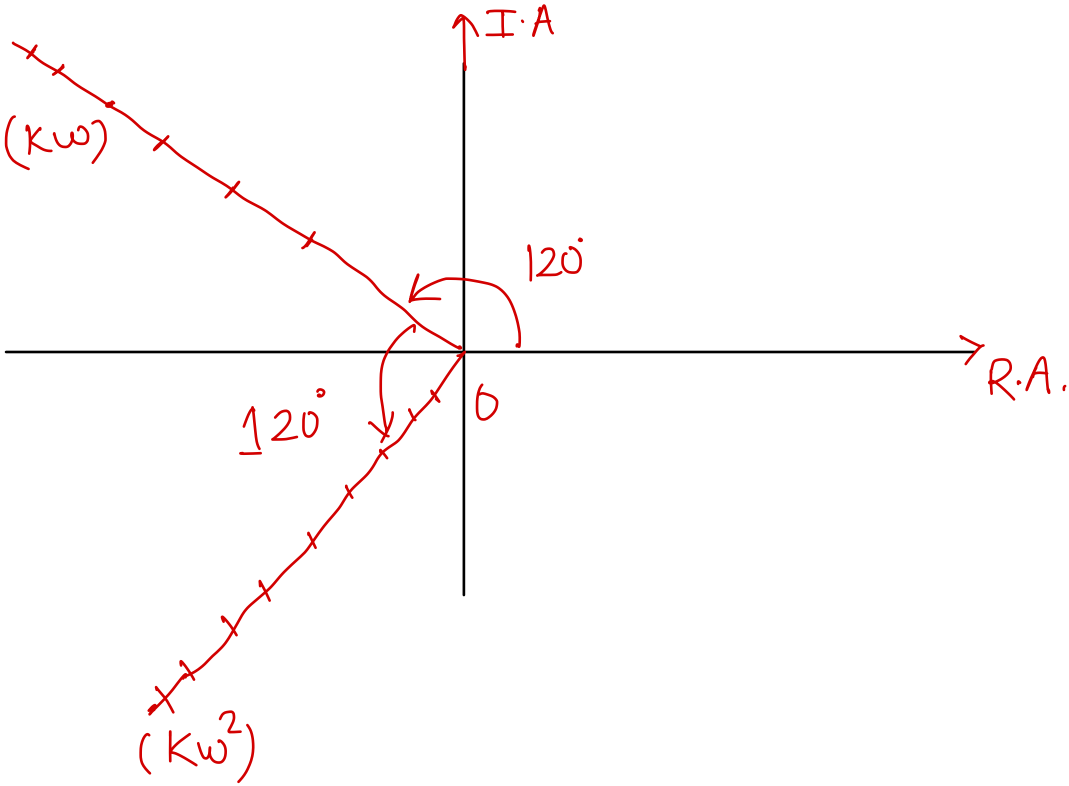
C–II If $|z| \neq 0$

$$\frac{z^2}{|z|^2} + \frac{z}{|z|} + 1 = 0 \qquad \frac{z}{|z|} = t \ (\text{let})$$

$$t^2 + t + 1 = 0 \rightarrow \omega, \ \omega^2$$

$$\frac{z}{|z|} = \omega \ ; \ \omega^2 \Rightarrow z = |z|\omega \ ; \ z = |z|\omega^2$$

where $|z| = k > 0$



HW
Q

Given $A = \begin{bmatrix} 1 & \omega^3 & \omega^2 \\ \omega^3 & 1 & \omega \\ \omega^2 & \omega & 1 \end{bmatrix}$, then (where ω is non real cube root of unity)-

(A) A is a non-singular matrix

(B) A is an orthogonal matrix

(C) A^{-1} is a symmetric matrix

(D) $A^{-1} = \frac{1}{3}\begin{bmatrix} 1-\omega^2 & 0 & \omega-\omega^2 \\ 0 & 1-\omega & \omega^2-\omega \\ \omega-\omega^2 & \omega^2-\omega & 0 \end{bmatrix}$

$$z^3 = 1 \begin{cases} 1 = e^{i0} \\ \omega = e^{i2\pi/3} \\ \omega^2 = e^{i4\pi/3} \end{cases}$$

$$z^3 + 0z^2 + 0z - 1 = 0 \begin{cases} 1 \\ \omega \\ \omega^2 \end{cases}$$

$$\left.\begin{aligned} S.O.R &= 0 \\ P.O.R &= 1 \end{aligned}\right\}$$

$$1^r + \omega^r + (\omega^2)^r \begin{cases} 0 \; ; & r \neq 3\lambda \\ 3 \; ; & r = 3\lambda \end{cases}$$

$\lambda \in I$

$$z^4 = 1 \begin{cases} 1 \\ -1 \\ -i \\ i \end{cases}$$

$$z^4 + 0z^3 + 0z^2 + 0z - 1 = 0$$

$$\begin{aligned} S.O.R &= 0 \\ P.O.R &= -1 \end{aligned}$$

$$1^r + (-1)^r + (i)^r + (-i)^r \begin{cases} 0 \; ; & r \neq 4I \\ 4 \; ; & r = 4I \end{cases}$$

$$\boxed{x^n - a = 0} \rightarrow x^n + 0x^{n-1} + 0x^{n-2} + \cdots - a = 0$$

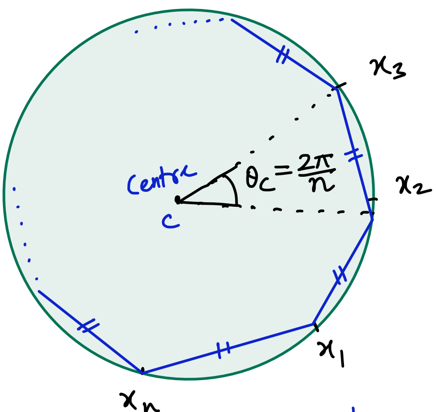
$$x = (a)^{\frac{1}{n}} \rightarrow x_1, x_2, \ldots, x_n$$

$$x^n - a = 0 \rightarrow x_1, x_2, \ldots, x_n$$

$$x_1^n = a \Rightarrow \boxed{x_1^r = (a)^{\frac{r}{n}}}$$

Note:-

1. Total no. of roots $= n$
2. $|x_1| = |x_2| = \cdots = |x_n| = |a^{\frac{1}{n}}|$
3. All roots lie on circle centred at origin.
4. $\sum_{i=1}^{n} x_i = 0$ (S.O.R)
5. $\text{P.O.R} = \prod_{i=1}^{n} x_i \begin{cases} a; & n \in \text{odd} \\ -a; & n \in \text{even} \end{cases}$
6. $x_1^r + x_2^r + \cdots + x_n^r \begin{cases} 0; & r \neq n\lambda,\ \lambda \in I \\ n(a)^{\frac{r}{n}}; & r = n\lambda,\ \lambda \in I \end{cases}$
7. All the roots are in G.P. with common ratio $= e^{i\left(\frac{2\pi}{n}\right)}$

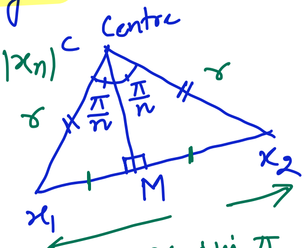


* All sides & all interior $\angle$'s are equal

8. side of n-gon

$r = |x_1| = |x_2| = \cdots = |x_n|$

$r = |a^{\frac{1}{n}}|$



In ΔCMx_1

$$\frac{x_1M}{r} = \sin\left(\frac{\pi}{n}\right) \Rightarrow x_1M = r \cdot \sin\frac{\pi}{n}$$

$$\boxed{x_1x_2 = 2r\sin\frac{\pi}{n}}$$

Perimeter of n-gon $= 2nr\sin\frac{\pi}{n}$

Area of Δx_1x_2C:

$$\Delta = \frac{1}{2} r \cdot r \cdot \sin\left(\frac{2\pi}{n}\right)$$

$$\text{Ar. of n-gon} = \frac{nr^2}{2}\sin\left(\frac{2\pi}{n}\right)$$

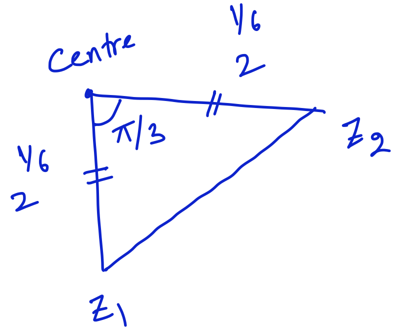
eg: $z = (\sqrt{3}+i)^{\frac{1}{6}} \rightarrow z_1, z_2, \ldots, z_6$

$r = |z| = (2)^{\frac{1}{6}}$

① $\sum z_i = 0$ (S.O.R)

② $P.O.R = \prod_{i=1}^{6} z_i = -(\sqrt{3}+i)$

$z^6 + 0z^5 + 0z^4 + 0z^3 + 0z^2 + 0z - (\sqrt{3}+i) = 0$



Ar. of hexagon $= 6\left(\frac{1}{2}(2)^{1/6}(2)^{1/6} \cdot \sin\frac{\pi}{3}\right)$

eg: $z = (\cos\alpha + i\sin\alpha)^{3/5}$ (where $\alpha \in \mathbb{R}$)

$P.O.R = ?$

$z = \left((\cos\alpha + i\sin\alpha)^3\right)^{\frac{1}{5}} = (\cos 3\alpha + i\sin 3\alpha)^{\frac{1}{5}} \rightarrow x_1, x_2, x_3, x_4, x_5$

① $x_1 x_2 x_3 x_4 x_5 = +(\cos 3\alpha + i\sin 3\alpha)$

② $x_1^4 + x_2^4 + x_3^4 + x_4^4 + x_5^4 = 0.$

HW

JM Q 3, 4, 5, 11, 12, 16 to 19.

O-2 Q3 to 10, 13

JA Q2, 4, 6, 9.