Q Given $A = \begin{bmatrix} 1 & \omega^3 & \omega^2 \\ \omega^3 & 1 & \omega \\ \omega^2 & \omega & 1 \end{bmatrix}$, then (where ω is non real cube root of unity)-

(A) A is a non-singular matrix

(B) A is an orthogonal matrix

(C) A^{-1} is a symmetric matrix

(D) $A^{-1} = \frac{1}{3}\begin{bmatrix} 1-\omega^2 & 0 & \omega-\omega^2 \\ 0 & 1-\omega & \omega^2-\omega \\ \omega-\omega^2 & \omega^2-\omega & 0 \end{bmatrix}$

Ans. (A,C,D)

$|A| = (1 - \omega^2) - \omega^3(\omega^3 - \omega^3) + \omega^2(\omega^4 - \omega^2)$

$\therefore\ |A| = 1 - \omega^2 + \omega^3 - \omega = 2 - (\omega + \omega^2) = 3$

$\therefore$ A is non singular

If A is non-singular symmetric matrix then A^{-1} is also symmetric matrix.

n^{th} Roots Of Unity :

$$z^n = 1$$ → $1, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}$

If $1, \alpha_1, \alpha_2, \alpha_3 \ldots\ldots \alpha_{n-1}$ are the n, n^{th} root of unity then :

(i) They are in G.P. with common ratio $e^{i(2\pi/n)}$ OR $\cos\frac{2\pi}{n} + i\sin\frac{2\pi}{n}$ and lie on standard unit circle on argand plane. n^{th} root of unity represents vertices of n sided polygon.

(ii) $1^p + \alpha_1^p + \alpha_2^p + \ldots\ldots + \alpha_{n-1}^p = 0$ if p is not an integral multiple of n .

$1^p + (\alpha_1)^p + (\alpha_2)^p + \ldots\ldots + (\alpha_{n-1})^p = n$ if p is an integral multiple of n .

* **(iii)** $(1-\alpha_1)(1-\alpha_2) \ldots\ldots (1-\alpha_{n-1}) = n$.

* **(iv)** $(1+\alpha_1)(1+\alpha_2) \ldots\ldots (1+\alpha_{n-1}) = 0$ if n is even and 1 if n is odd.

(v) $1 . \alpha_1 . \alpha_2 . \alpha_3 \ldots\ldots \alpha_{n-1} = 1$ or -1 according as n is odd or even.

* **(vi)** *
$$(w-\alpha_1)(w-\alpha_2) \ldots\ldots (w-\alpha_{n-1}) = \begin{cases} 0 & \text{if } n = 3k \\ 1 & \text{if } n = 3k+1 \\ 1+w & \text{if } n = 3k+2 \end{cases}$$

(vii) $$\underbrace{\sum_{m=0}^{n-1}\left(\cos\frac{2m\pi}{n} + i\sin\frac{2m\pi}{n}\right)}_{0} = 0$$

$$\boxed{\sum_{m=1}^{n-1} \text{Cis}\left(\frac{2m\pi}{n}\right) = -1}$$

$$S.O.R = 1 + \underbrace{\alpha_1 + \alpha_2 + \cdots + \alpha_{n-1}}_{0} = 0$$

$$z = (1)^{\frac{1}{n}} = e^{i\frac{2m\pi}{n}} \quad ; \quad m = 0, 1, 2, \ldots, n-1$$

$m = 0 \Rightarrow z_1 = e^{i0} = 1$ ✓✓

$m = 1 \Rightarrow z_2 = e^{i\frac{2\pi}{n}} = \text{Cis}\left(\frac{2\pi}{n}\right) = \alpha_1 = \alpha$

$m = 2 \Rightarrow z_3 = e^{i\frac{4\pi}{n}} = \text{Cis}\left(\frac{4\pi}{n}\right) = \alpha_2 = \alpha^2$

$\vdots \qquad \vdots \qquad \vdots \qquad \alpha_3 = \alpha^3$

$m = n-1 \Rightarrow z_n = e^{i\frac{2(n-1)\pi}{n}} = \text{Cis}\left(\frac{2(n-1)}{n}\pi\right) = \alpha_{n-1} = \alpha^{n-1}$

* α_1 & α_{n-1} → Conjugate of each other $\Rightarrow \alpha_1 + \alpha_{n-1} = 2\cos\left(\frac{2\pi}{n}\right)$

α_2 & α_{n-2} → " " " "

Rem

$$\boxed{1+x+x^2+\cdots+x^{n-1} = (x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_{n-1})}$$

$Z=-1$

$$\boxed{Z^n-1 = (Z-1)(Z-\alpha_1)(Z-\alpha_2)\cdots\cdots(Z-\alpha_{n-1})}$$

$$\left(\frac{Z^n-1}{Z-1}\right) = (Z-\alpha_1)(Z-\alpha_2)\cdots\cdots(Z-\alpha_{n-1}).$$

put $Z=1$

$$\boxed{1+Z+Z^2+\cdots+Z^{n-1} = (Z-\alpha_1)(Z-\alpha_2)\cdots(Z-\alpha_{n-1})} \quad \text{— (1) —}$$

put $Z=\omega$

$$\frac{\omega^n-1}{\omega-1} = (\omega-\alpha_1)(\omega-\alpha_2)\cdots(\omega-\alpha_{n-1}).$$

$\omega^n \to 1$ if $n=3\lambda$

$\omega^n \to \omega$ if $n=3\lambda+1$ $\quad \underline{\lambda \in I}$

$\omega^n \to \omega^2$ if $n=3\lambda+2$

Rem.

$$\boxed{\sin\frac{\pi}{n}\cdot\sin\frac{2\pi}{n}\sin\frac{3\pi}{n}\cdots\sin\left(\frac{n-1}{n}\right)\pi = \frac{n}{2^{n-1}}}$$

eg: $\sin\frac{\pi}{5}\sin\frac{2\pi}{5}\sin\frac{3\pi}{5}\sin\frac{4\pi}{5} = \frac{5}{2^4} = \frac{5}{16}$.

$z^n = 1$: roots $1, \alpha_1, \alpha_2, \ldots, \alpha_{n-1}$

We know

$$\boxed{(1-\alpha_1)(1-\alpha_2)\cdots(1-\alpha_{n-1}) = n}$$

$$\alpha_1 = e^{i\frac{2\pi}{n}}$$

$$1-\alpha_1 = 1-e^{i\left(\frac{2\pi}{n}\right)} = 1-\left(\cos\frac{2\pi}{n}+i\sin\frac{2\pi}{n}\right) = 2\sin^2\frac{\pi}{n} - \left(2\sin\frac{\pi}{n}\cos\frac{\pi}{n}\right)i$$

$$|1-\alpha_1| = 2\sin\left(\frac{\pi}{n}\right)$$

||ly $|1-\alpha_2| = 2\sin\left(\frac{2\pi}{n}\right)$

$$|1-\alpha_3| = 2\sin\left(\frac{3\pi}{n}\right)$$

$\vdots$

$$|1-\alpha_{n-1}| = 2\sin\left(\frac{n-1}{n}\right)\pi$$

$$|(1-\alpha_1)(1-\alpha_2)\cdots(1-\alpha_{n-1})| = n$$

$$|1-\alpha_1|\,|1-\alpha_2|\cdots|1-\alpha_{n-1}| = n$$

$$\left(2\sin\frac{\pi}{n}\right)\left(2\sin\frac{2\pi}{n}\right)\cdots 2\sin\left(\frac{n-1}{n}\right)\pi = n$$

$z^{13} = 1$

Q ① Evaluate $\sum_{\lambda=0}^{12}\left(\sin\frac{2\lambda\pi}{13} - i\cos\frac{2\lambda\pi}{13}\right)$

$$\frac{1}{i}\underbrace{\sum_{\lambda=0}^{12}\left(\cos\frac{2\lambda\pi}{13} + i\sin\frac{2\lambda\pi}{13}\right)}_{0} = 0.$$

② Evaluate $\sum_{\lambda=1}^{12}\left(\sin\frac{2\lambda\pi}{13} - i\cos\frac{2\lambda\pi}{13}\right) =$

$$\frac{1}{i}\underbrace{\sum_{\lambda=1}^{12}\left(\cos\frac{2\lambda\pi}{13} + i\sin\frac{2\lambda\pi}{13}\right)}_{-1} = \frac{-1}{i} \times \frac{i}{i} = i$$ Ans

③ Evaluate $\sum_{\lambda=1}^{11}\left(\sin\frac{2\lambda\pi}{13} - i\cos\frac{2\lambda\pi}{13}\right)$

$$\frac{1}{i}\sum_{\lambda=1}^{11}\left(\cos\frac{2\lambda\pi}{13} + i\sin\frac{2\lambda\pi}{13}\right)$$

$$\frac{1}{i}\left(\underbrace{\sum_{\lambda=1}^{12} \text{Cis}\,\frac{2\lambda\pi}{13}}_{-1} - \left(\cos\frac{24\pi}{13} + i\sin\frac{24\pi}{13}\right)\right)$$

$$\frac{1}{i}\left(-1 - \text{Cis}\left(\frac{24\pi}{13}\right)\right)$$ Ans

Q $\sum_{K=1}^{12} \cos \frac{2K\pi}{13} = ?$

$z^{13} = 1 \Rightarrow z = (1)^{\frac{1}{13}}$

$$\sum_{m=1}^{12} \left(\underline{\underline{\mathrm{Cis}}} \frac{2m\pi}{13}\right) = -1.$$

$$\left(\cos \frac{2\pi}{13} + i \sin \frac{2\pi}{13}\right)$$
$$+ \left(\cos \frac{4\pi}{13} + i \sin \frac{4\pi}{13}\right)$$
$$+ \left(\cos \frac{6\pi}{13} + i \sin \frac{6\pi}{13}\right) = -1$$
$$\vdots$$
$$+ \left(\cos \frac{24\pi}{13} + i \sin \frac{24\pi}{13}\right)$$

$$\sum_{K=1}^{12} \cos \frac{2K\pi}{13} = -1$$

Q If $1, \alpha, \alpha^2, \alpha^3, \ldots, \alpha^{10}$ are the roots of equation $z^{11} - 1 = 0$ then $\prod_{K=1}^{10} (1+\alpha^K) = ?$

$$(1+\alpha)(1+\alpha^2)(1+\alpha^3)\ldots(1+\alpha^{10})$$

$$(1+\alpha_1)(1+\alpha_2)\ldots\ldots(1+\alpha_{10})$$

Q If 'α' $(\alpha \neq 1)$ be the fifth root of unity then find value of $\log_{\sqrt{3}} \left| \underbrace{1+\alpha+\alpha^2+\alpha^3}_{} - \frac{2}{\alpha} \right| = ?$

Sol^n

$z^5 = 1$: 1, $\alpha_1 = \alpha$, $\alpha_2 = \alpha^2$, $\alpha_3 = \alpha^3$, $\alpha_4 = \alpha^4$

$\boxed{\alpha^5 = 1}$

$|\alpha| = 1$

$$1+\alpha+\alpha^2+\alpha^3+\alpha^4 = 0$$

$$\log_{\sqrt{3}} \left| -\alpha^4 - \frac{2}{\alpha} \right|$$

$$\log_{\sqrt{3}} \left| \frac{-\alpha^5 - 2}{\alpha} \right|$$

$$\log_{\sqrt{3}} \left| -\frac{3}{\alpha} \right|$$

$$\log_{\sqrt{3}} \left(\frac{3}{|\alpha|} \right) = \log_{\sqrt{3}} (3)$$

$|\alpha| \to 1$

$$= 2.$$

Q Let z is a complex number and $\alpha_1, \alpha_2, \ldots\ldots \alpha_{17}$ are 17^{th} roots of unity, then $\dfrac{\left(\sum_{k=1}^{17}|z+\alpha_k|^2\right)-z-\bar{z}}{17\left(|z|^2+1\right)-z-\bar{z}}$ is

(1) Ans

Sol^n

$$z^{17} = 1 \rightarrow \alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_{17}$$

$$|\alpha_i| = 1$$

$$\boxed{\alpha_1+\alpha_2+\ldots+\alpha_{17}=0} \quad \text{—(i)—}$$

$$|z+\alpha_k|^2 = (z+\alpha_k)(\bar{z}+\bar{\alpha}_k)$$

$$= z\bar{z} + z\bar{\alpha}_k + \alpha_k\bar{z} + \alpha_k\bar{\alpha}_k$$

$$= |z|^2 + z\bar{\alpha}_k + \alpha_k\bar{z} + \underbrace{|\alpha_k|^2}_{1}$$

$$\sum_{k=1}^{17}|z+\alpha_k|^2 = \sum_{k=1}^{17}\left(|z|^2 + z\bar{\alpha}_k + \alpha_k\bar{z} + 1\right)$$

$$= 17|z|^2 + z\underbrace{\left(\sum_{k=1}^{17}\bar{\alpha}_k\right)}_{0} + \bar{z}\underbrace{\left(\sum_{k=1}^{17}\alpha_k\right)}_{0} + 17$$

$$= 17\left(|z|^2+1\right)$$

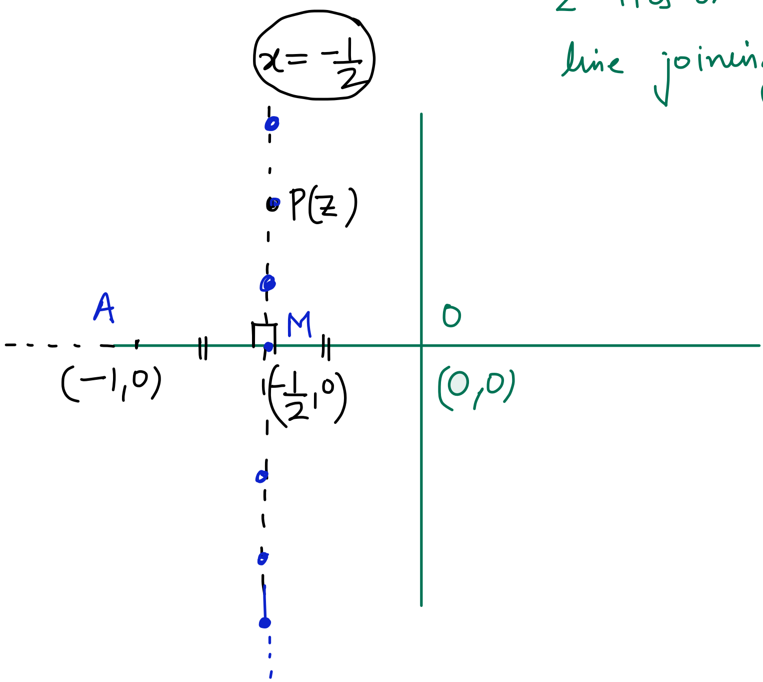
Q Prove that all roots of the equation $\left(\frac{z+1}{z}\right)^n = 1$ are collinear on the complex plane.

Sol^n

$$(z+1)^n = z^n$$

$$|z+1|^n = |z|^n \Rightarrow |z+1| = |z|$$

'z' lies on $\perp$ bisector of line joining $(-1,0)$ & $(0,0)$
A O



Q. If Z_r, $r = 1, 2, 3, \ldots\ldots\ldots 2m$, $m \varepsilon N$ are the roots of the equation

$Z^{2m} + Z^{2m-1} + Z^{2m-2} + \ldots\ldots\ldots\ldots + Z + 1 = 0$ then prove that $\underbrace{\sum_{r=1}^{2m} \frac{1}{Z_r - 1}}_{S} = -m$

Sol^n

$$S = \left(\frac{1}{z_1 - 1}\right) + \frac{1}{z_2 - 1} + \cdots\cdots + \left(\frac{1}{z_{2m} - 1}\right)$$

$$z^{2m} + z^{2m-1} + \cdots\cdots + z + 1 = 0 \Rightarrow \frac{z^{2m+1} - 1}{z - 1} = 0$$

$z\bar{z} = |z|^2 = 1$

$\boxed{z\bar{z} = 1}$

$z = (1)^{\frac{1}{2m+1}}$ → 1 ×, z_1 ✓, z_2 ✓, …, z_{2m} ✓

$|z| = 1$

Conjugate of each other

z_1 & z_{2m} are conjugate of each other $\quad z_{2m} = \frac{1}{z_1}$ ☆

z_2 & z_{2m-1} " " " " $\quad z_{2m-1} = \frac{1}{z_2}$ and so on

$$S = \left(\frac{1}{z_1 - 1} + \frac{1}{z_{2m} - 1}\right) + \left(\frac{1}{z_2 - 1} + \frac{1}{z_{2m-1} - 1}\right) + \cdots\cdots$$

$$= \underbrace{\left(\frac{1}{z_1 - 1} + \frac{1}{\frac{1}{z_1} - 1}\right)}_{-1} + \underbrace{\left(\frac{1}{z_2 - 1} + \frac{1}{\frac{1}{z_2} - 1}\right)}_{-1} + \cdots\cdots$$

$$S = (-1) + (-1) + (-1) + \cdots \text{ m times} = -m$$

M-2

$$z^{2m} + z^{2m-1} + \cdots\cdots + z + 1 = (z - z_1)(z - z_2)\cdots\cdots(z - z_{2m})$$

$$\ln\left(z^{2m} + z^{2m-1} + \cdots + z + 1\right) = \ln(z - z_1) + \ln(z - z_2) + \cdots + \ln(z - z_{2m})$$

differentiate wrt 'z' both sides.

$$\frac{2m \cdot z^{2m-1} + (2m-1)z^{2m-2} + \cdots + 1}{z^{2m} + z^{2m-1} + \cdots + z + 1} = \frac{1}{z - z_1} + \frac{1}{z - z_2} + \cdots + \frac{1}{z - z_{2m}}$$

put $z = 1$ both sides :-

$$\frac{2m + (2m-1) + \cdots + 1}{2m+1} = -\left(\frac{1}{z_1 - 1} + \frac{1}{z_2 - 1} + \cdots + \frac{1}{z_{2m} - 1}\right)$$

$$\frac{(\cancel{2}m)\,\cancel{(2m+1)}}{\cancel{2}\,\cancel{(2m+1)}} = -\left(\qquad\qquad\right)$$

$$-m = \left(\frac{1}{z_1 - 1} + \frac{1}{z_2 - 1} + \cdots + \frac{1}{z_{2m} - 1}\right)$$

Binomial Coefficient :-

$^nC_0, {}^nC_1, {}^nC_2, \ldots, {}^nC_n$ are combinatorial coeff in expansion of $(1+x)^n$.

(i) $^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_n = 2^n$

(ii) $^nC_0 + {}^nC_2 + {}^nC_4 + \cdots = 2^{n-1}$

(iii) $^nC_0 + {}^nC_3 + {}^nC_6 + {}^nC_9 + \cdots = ?$

(iv) $^nC_0 + {}^nC_4 + {}^nC_8 + \cdots = ?$

$$(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \cdots + {}^nC_n x^n \quad \text{—(1)—}$$

put $x = 1$

$$2^n = C_0 + C_1 + C_2 + \cdots + C_n$$

$x^2 = 1 \;\to\; x = 1,\; x = -1$

$$\left.\begin{aligned} 2^n &= C_0 + C_1 + C_2 + \cdots + C_n \\ 0 &= C_0 - C_1 + C_2 + \cdots (-1)^n C_n \end{aligned}\right\}$$

add :

sub :

$$(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \cdots + {}^nC_n x^n \quad \text{— ① —}$$

$$x^3 = 1 \to 1,\ \omega,\ \omega^2$$

$$2^n = C_0 + C_1 + C_2 + C_3 + C_4 + \cdots + C_n$$

$$(1+\omega)^n = C_0 + C_1\omega + C_2\omega^2 + C_3\omega^3 + C_4\omega^4 + \cdots + C_n\omega^n.$$

$$(1+\omega^2)^n = C_0 + C_1\omega^2 + C_2\omega^4 + C_3\omega^6 + C_4\omega^8 + \cdots + C_n\omega^{2n}.$$

Add

$$2^n + (1+\omega)^n + (1+\omega^2)^n = \underline{\underline{3}}\,(\underbrace{C_0 + C_3 + C_6 + \cdots}_{\text{Sum}})$$

$$\frac{1}{3}\left(2^n + \underline{\underline{e^{i\frac{n\pi}{3}}}} + \underline{\underline{e^{-i\frac{n\pi}{3}}}}\right) = C_0 + C_3 + C_6 + \cdots$$

$$\frac{1}{3}\left(2^n + 2\cos\frac{n\pi}{3}\right) = C_0 + C_3 + C_6 + \cdots$$

(iv) $x^4 = 1 \to -1,\ 1,\ i,\ -i$

$$\frac{1}{4}\left(2^n + 0 + \underbrace{(1+i)^n + (1-i)^n}\right) = C_0 + C_4 + C_8 + \cdots$$

$$\frac{1}{4}\left(2^n + 2^{\frac{n}{2}+1}\cos\frac{n\pi}{4}\right) = C_0 + C_4 + C_8 + \cdots$$

Q Find value of

$$\left({}^nC_0 - {}^nC_2 + {}^nC_4 - {}^nC_6 + \cdots\right)^2 + \left({}^nC_1 - {}^nC_3 + {}^nC_5 - \cdots\right)^2$$

Solⁿ

$$(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \cdots + {}^nC_n x^n \quad -(1)-$$

put $x = i$

$$(1+i)^n = {}^nC_0 + {}^nC_1 i - {}^nC_2 - {}^nC_3 i + \cdots$$

$$= \left(\underbrace{C_0 - C_2 + C_4 - C_6 + \cdots}_{\alpha}\right) + i\left(\underbrace{C_1 - C_3 + C_5 - \cdots}_{\beta}\right)$$

$$(1+i)^n = \alpha + i\beta$$

$$|1+i|^n = \sqrt{\alpha^2 + \beta^2}$$

$$\Downarrow$$

$$(\sqrt{2})^n = \sqrt{\alpha^2 + \beta^2} \Rightarrow 2^n = \underbrace{\alpha^2 + \beta^2}_{\text{Ans}}$$

HW :-

Remaining JA