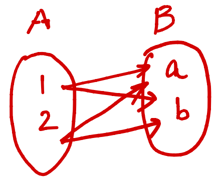
FUNCTIONS & RELATIONS

RELATIONS

1. **CARTESIAN PRODUCT:** Given two non-empty set A & B The Cartesian product $A \times B$ is the set of all ordered pairs (a, b) where $a \in A$ & $b \in B$.

$\therefore \quad A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}$



e.g. $A = \{1, 2\}$, $B = \{a, b\}$

$A \times B = \{(1, a), (1, b), (2, a), (2, b)\}$

Number of elements in set A $\quad n(A) = p$

$n(B) = q$

No. of elements in set A.

$\therefore \quad n(A \times B) = pq$

$(1, a)$ → image of '1'

→ pre-image of 'a'

2. **RELATION:** → $R \subseteq A \times B$.

Every subset of $A \times B$ defined a relation from set A to set B.

If R is a relation from $A \to B$ — 'b' is image of 'a' under R.

$R : \{(a, b) \mid (a, b) \in A \times B \text{ and } a R b\}$ — 'a' is preimage of 'b' under R.

☞ Note that while defining relation there is no constraints. One element may be related with two or more.

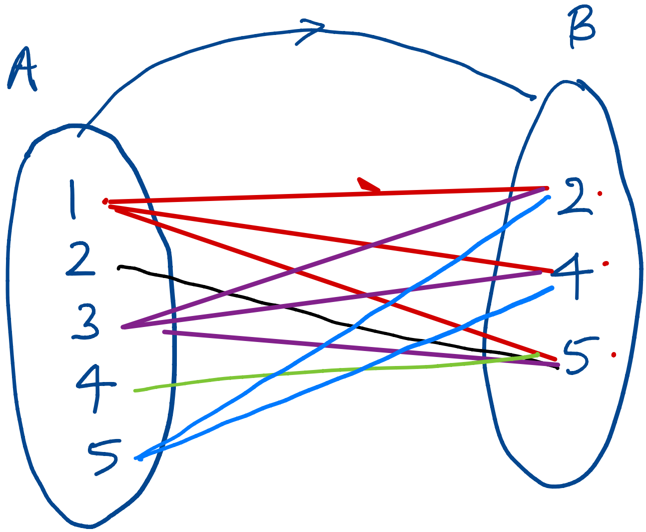
EXAMPLES:

(1) $A = \{1, 2, 3, 4, 5\}$ and $B = \{2, 4, 5\}$

Find relation R

$a R b$ such that a and b are co-prime.

where $a \in A$ and $b \in B$.



R : {(1,2), (1,4), (1,5), (2,5), (3,2), (3,4), (3,5), (4,5), (5,2), (5,4)}

Domain of R : {1, 2, 3, 4, 5}

Range of R : {2, 4, 5}

Number of Relations :-

No. of ways of selecting a subset of $A \times B$ i.e. cartesian product

$$\underbrace{{}^nC_0 + {}^nC_1 + {}^nC_2 + \cdots + {}^nC_n = 2^n}$$

$n(A) = p$

$n(B) = q$

$n(A \times B) = pq$

$$\boxed{{}^{pq}C_0 + {}^{pq}C_1 + \cdots + {}^{pq}C_{pq} = 2^{pq}.}$$

eg: $n(A) = 2$; $n(B) = 3$ then find no. of relations from A to B ?

2^6.

Empty relation: No elements of A is related to any elements of A.

Universal relation: Each element of A is related to every element of A

3. **DOMAIN AND RANGE OF RELATION:**

If $R: A \to B$ (R is a relation defined from set A to set B) then the domain of this relation is

Domain: Set of all the first entries in R

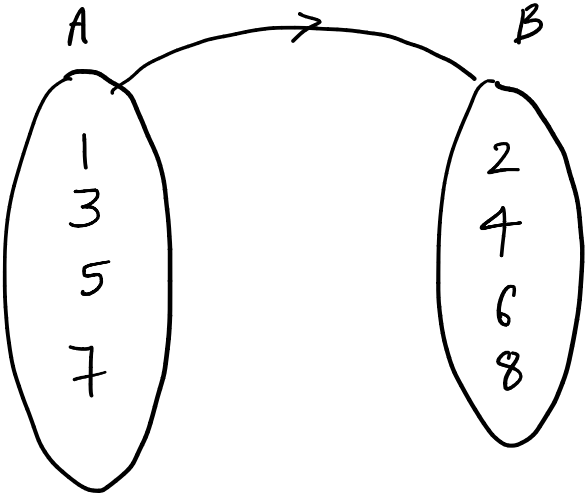
$\{a \mid (a, b) \in R\}$

Range: Set of all the second entries in R

$\{b \mid (a, b) \in R\}$

e.g. If $A = \{1, 3, 5, 7\}$, $B = \{2, 4, 6, 8\}$

Relation is aRb $\Rightarrow$ $a > b$, $a \in A$, $b \in B$



$a > b$.

$R: \{(3,2), (5,2), (5,4), (7,2), (7,4), (7,6)\}$

Domain: $\{3, 5, 7\}$

Range: $\{2, 4, 6\}$

4. **INVERSE RELATION:**

If relation R is defined from A to B then the inverse relation would be defined from B to A, i.e

$R: A \to B \quad \Rightarrow \quad aRb$ where $a \in A$, $b \in B$

$R^{-1}: B \to A \Rightarrow \quad bRa$ where $a \in A$, $b \in B$

Doman of R = Range of R^{-1}

and Range of R = Domain of R^{-1}

$\therefore \quad R^{-1} = \{(b, a) \mid (a, b) \in R\}$

eg A relation R is defined on the set of 1st ten natural numbers.

N is a set of first 10 natural nos. $\therefore N = \{1, 2, 3, ..., 10\}$ & $a, b \in N$

$aRb \Rightarrow a + 2b = 10$

$R: \{(2,4), (4,3), (6,2), (8,1)\}$

$R^{-1}: \{(4,2), (3,4), (2,6), (1,8)\}$

5. **IDENTITY RELATION:**

A relation defined on a set A is said to be an Identity relation if each & every element of A is related to itself & only to itself.

e.g. A relation defined on the set of natural numbers as

$aRb \Rightarrow a = b$ where a & $b \in N$

$R: \{(1,1), (2,2), (3,3), ----- ---\}$

$\underbrace{\text{Range} = \text{Domain}}$

6. CLASSIFICATION OF RELATIONS:

Types of Relations

Reflexive | Symmetric | Transitive

(i) **Reflexive:** A relation defined on a set A is said to be reflexive relation if each & every element of A is related to itself. but not necessarily to itself only.

i.e. if $(a, b) \in R$ then $(a, a) \in R$. However if there is a single ordered pair of $(a, b) \in R$ such $(a, a) \notin R$ then R is not reflexive.

e.g. A relation defined on (set of natural numbers)

$aRb \Rightarrow$ 'a' divides 'b' $a, b \in N$ i.e. if $(a, b) \in R \Rightarrow (a, a) \in R$

$R : \{ (1,1), (2,2), (3,3), \text{--------}$
$(1,2), (1,3), (1,4), \text{- - - -} , (2,4), (2,6), \text{---} \}$

☞ **Note:** Every Identity relation is a reflexive relation but every reflexive relation need not be an Identity.

(ii) **Symmetric:** A relation defined on a set is said to be symmetric if $aRb \Rightarrow bRa$.
If $(a, b) \in R$ then (b, a) must be necessarily there in the same relation.

EXAMPLES:

A relation R defined on the set of lines.

(1) $aRb \Rightarrow a \parallel b$
It is a symmetric relation because if line is | | to 'b' then the line 'b' is | | to 'a'.
where $(a, b) \in L$ {L is a set of | | lines}

(2) $L_1RL_2 \Rightarrow L_1 \perp L_2$ It is a symmetric relation
$L_1, L_2 \in S$ {S is a set of lines}

* (3) $aRb \Rightarrow$ 'a' is brother of 'b' is not a symmetric relation as 'b' may be sister of 'a'.

(4) $aRb \Rightarrow$ 'a' is a cousin of 'b'. This is a symmetric relation.

If R is symmetric

(1) $R = R^{-1}$

(2) Range of R = Domain of R^{-1}

Transitive: A relation on set A is said to be transitive if aRb and bRc implies aRc

i.e. $(a, b) \in R$ and $(b, c) \in R$ then $(a, c) \in R$

and a, b, c need not be distinct.

eg

A relation R defined on a set of natural numbers N.

R: $\{(1, 2), (1, 1)\}$.

eg:- $R: N \to N$; N is set of Natural numbers.

1. $R = \{(x, y) \mid x < y, x \in N, y \in N\}$
2. $R = \{(x, y) \mid x + y = 10, \ x \in N, y \in N\}$
3. $R = \{(x, y) \mid x = y \text{ or } x - y = 1, \ x \in N, y \in N\}$

① $R: \{(1, 2), (1, 3), \ldots$

$(2, 3), \ldots \}$

$\boxed{\overline{R} \cap \overline{S} \cap T}$ Ans

(2) $x+y=10$; $x, y \in N$

$R: \{(1,9), (9,1), (2,8), (8,2)$
$(3,7), (7,3), (6,4), (4,6)$
$(5,5)\}$

$\boxed{\overline{R} \cap S \cap \overline{T}}$ Ans

(3) $x=y$ or $x-y=1$.

$R: \{(1,1), (2,2), (3,3), \ldots$
$(2,1), (3,2), \ldots\}$

$\boxed{R \cap \overline{S} \cap \overline{T}}$ Ans

7. EQUIVALENCE RELATION:

If a relation is Reflexive, Symmetric and Transitive then it is said to be an equivalence relation. All identity relations are equivalence Relat-ion

eg Relation defined on the set of integer (I)

Prove that: xRy $\Rightarrow$ $(x - y)$ is even is an equivalence relation.

[**Proof:**

(a) Reflexive: Let $a \in I$, $a - a = 0 = 2 \times 0 =$ even

$\therefore$ $(a, a) \in R$

Hence this relation is reflexive

(b) Symmetric: Let $a, b \in Z$ and $(a, b) \in R$

Let $a - b = 2m$

$\therefore$ $b - a = -2m = 2 \times (-m) =$ even

$\therefore$ $(b, a) \in R$

Hence this relation is symmetric

(c) Transitive: Let $a, b, c \in z$ $(a, b) \in R$ and $(b, c) \in R$

$\therefore$ $a - b = 2m$ ....(1) $b - c = 2n$ ....(2)

Adding (1) and (2)

$a - b + b - c = 2m + 2n$

$\Rightarrow$ $a - c = 2(m + n) =$ even

$\therefore$ $(a, c) \in R$

Hence this relation is transitive

Since R is reflexive, symmetric and transitive

$\therefore$ R is an equivalence relation.]

Q1) $A = \{1, 2, 3, 4\}$; $R = \{(1, 2), (2, 2), (1, 1), (4, 4), (1, 3), (3, 3), (3, 2)\}$ then

(A) $R \cap S \cap \overline{T}$ (B) $R \cap T \cap \overline{S}$ (C) $S \cap T \cap \overline{R}$ (D) $R \cap S \cap T$

$\boxed{R \cap \overline{S} \cap T}$

Q2) R = {(1, 2), (2, 3)} add minimum number of ordered pairs to make it an equivalence relation.

Sol^n

(1,1), (2,2), (3,3).

(2,1), (3,2), $\underbrace{(3,1), (1,3)}$

$\boxed{\text{Ans} \rightarrow 7}$

Q If R is relation on set of natural numbers N, defined by $\{(x,y);\ 3x+3y=10\}$ then R is

(A) ~~Symmetric~~

(B) ~~Transitive~~

(C) Reflexive

(D) None.

Null set

Q

Let P be relation defined on set of all real numbers such that

$P : \{(a,b) : \sec^2 a - \tan^2 b = 1\}$, then P is

(A) Reflexive ✓

(B) Symmetric ✓

(C) Transitive ✓

(D) None.

(a, b)

(b, c)

(a, c)

Sol^n

$$\sec^2 a - \tan^2 b = 1$$

$$\cancel{1} + \tan^2 a - \tan^2 b = \cancel{1}$$

$$\tan^2 a = \tan^2 b$$

$$\boxed{a = n\pi \pm b} \; ; n \in I$$

$$n = 0$$

$$a = b \quad \text{or} \quad a = -b$$

FUNCTION

Definition-1 : Let A and B be two sets and let there exist a rule or manner or correspondence 'f' which associates to each element of A, a unique element in B. Then f is called a <u>function</u> or <u>mapping</u> from A to B. It is denoted by the symbol

$$f : A \to B \text{ or } A \xrightarrow{f} B$$

which reads 'f' is a function from A to B' or 'f maps A to B,

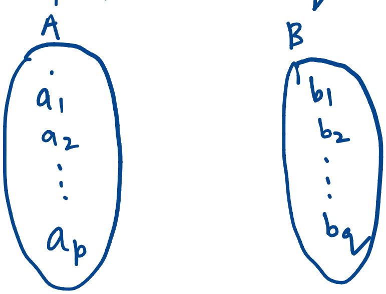
If an element $a \in A$ is associated with an element $b \in B$ then b is called 'the f image of a' or 'image of a under f' or 'the value of the function f at a'. Also a is called the pre-image of b or argument of b under the function f. We write it as

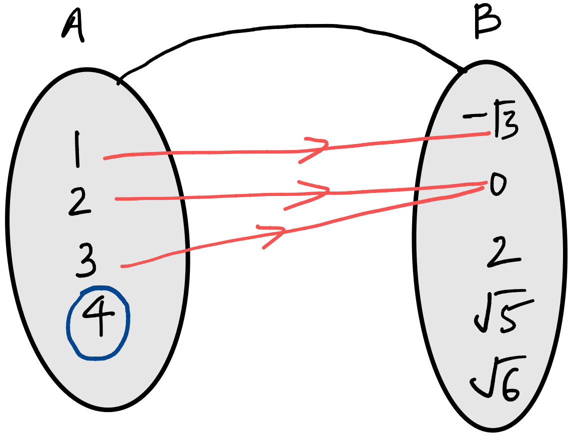
$$b = f(a) \text{ or } f : a \to b \text{ or } f : (a, b)$$

*.

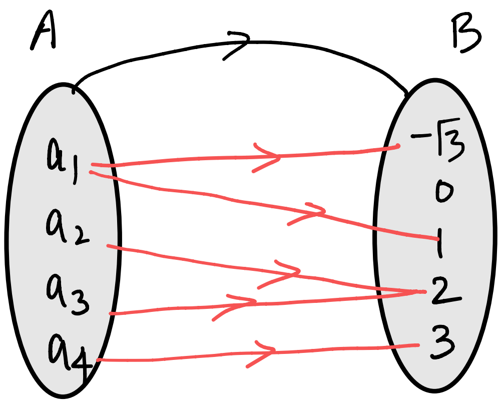
f is subset of $A \times B$. / $f \subseteq R \subseteq A \times B$

If $n(A) = p$ and $n(B) = q$ then no. of functions from A to B is $(q)^p \to (n(B))^{n(A)}$

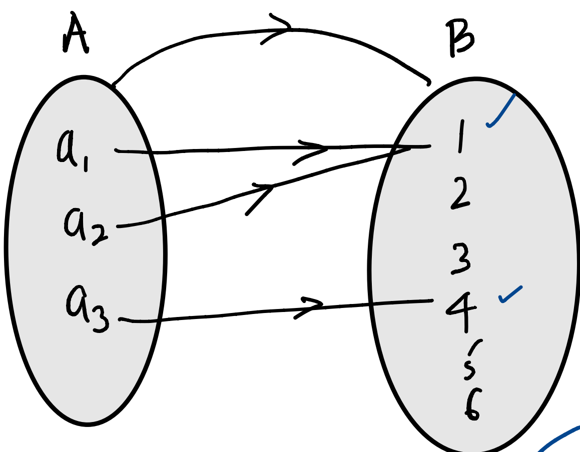




This is Not a function



This is Not a function



This is function

Domain : $\{a_1, a_2, a_3\}$

Range: $\{1, 4\}$

Co-domain : $\{1, 2, 3, 4, 5, 6\}$

Q If $A = \{1, 2, 3, 4\}$, $B = \{1, 4, 9, 16, 25\}$

then which of following relation is/are function

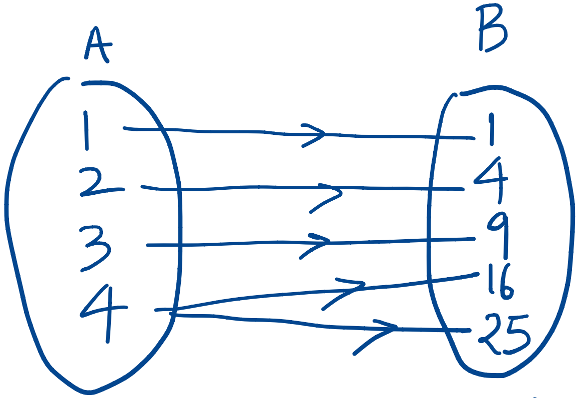
(A) $R_1 : \{(1,1), (2,4), (3,9), (4,16), (4,25)\}$

(B) $R_2 : \{(2,4), (3,9), (4,16)\}$

✓ (C) $R_3 : \{(1,1), (2,4), (3,9), (4,16)\}$

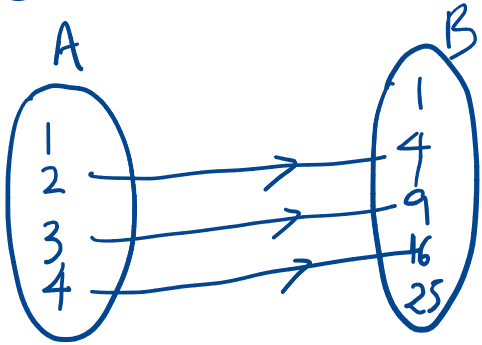
(D) None.

(A)

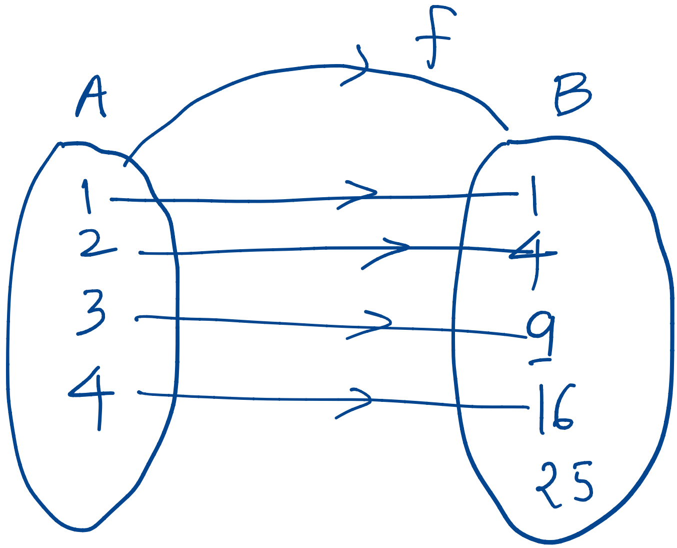


X X

(B)



X X



(C)

Definition-2 : A relation R from a set A to a set B is called a function if

(i) each element of A is associated with some element of B.

(ii) each element of A has unique image in B.

Thus a function 'f' from a set A to a set B is a subset of $A \times B$ in which each 'a' belonging to A appears in one and only one ordered pair belonging to f.

☞ **Note :** Every function is a relation but every relation is not necessarily a function.

DOMAIN, CO–DOMAIN & RANGE OF A FUNCTION :

Let $f: A \rightarrow B$, then the set A is known as the domain of f & the set B is known as co–domain of f . The set of all f images of elements of A is known as the range of f .

Thus : Domain of $f = \{a \mid a \in A, (a, f(a)) \in f\}$

Range of $f = \{f(a) \mid a \in A, f(a) \in B, (a, f(a)) \in f\}$

Range is always subset of Co-domain.

Range depends on domain set.

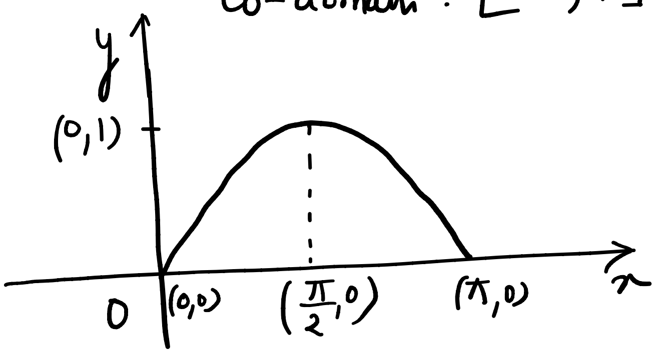
eg: $f : [0, \pi] \rightarrow [-10, 10]$

$f(x) = \sin x$

Co-domain : $[-10, 10]$.

Domain : $[0, \pi]$

Range : $[0, 1]$

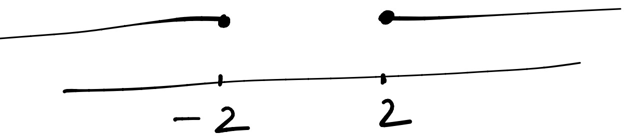


eg: $f(x) = \sqrt{x^2 - 4}$.

Sol^n **Domain:** $x^2 - 4 \geq 0$.

$(x-2)(x+2) \geq 0$

$D_f \in (-\infty, -2] \cup [2, \infty)$



$\boxed{\text{Co-domain : } \mathbb{R}}$

Range: $[0, \infty)$

Definition: $f+g,\ f-g,\ fg,\ f/g$ are defined as

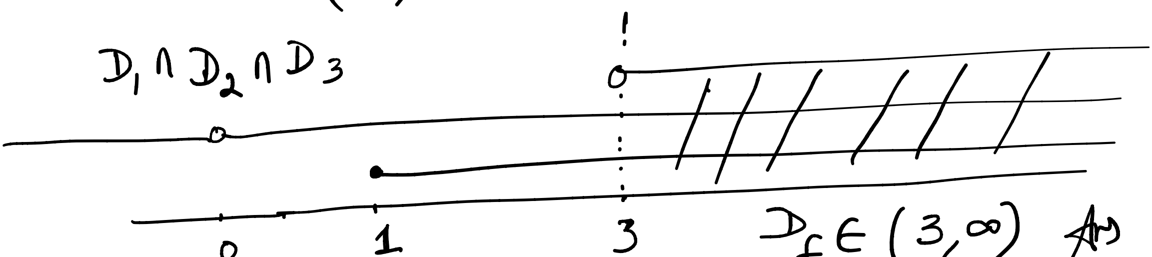
$$(f+g)(x) = f(x) + g(x);\quad \text{Domain } D_1 \cap D_2$$
$$(f-g)(x) = f(x) - g(x);\quad \text{Domain } D_1 \cap D_2$$
$$(f\,g)(x) = f(x)\cdot g(x);\quad \text{Domain } D_1 \cap D_2$$
$$* \quad \left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)};\quad \text{Domain} = \{x \in D_1 \cap D_2 \mid g(x) \neq 0\}$$

eg: $f(x) = \log_2(x-3) + \sqrt{x-1} + \dfrac{1}{x}$

$x-3>0$

$D_1 : (3,\infty)$ $\quad D_2 : [1,\infty)$ $\quad D_3 : R-\{0\}$

$D_1 \cap D_2 \cap D_3$



$D_f \in (3,\infty)$ Ans

3. IMPORTANT TYPES OF FUNCTIONS:

(i) POLYNOMIAL FUNCTION:

If a function f is defined by $f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_{n-1} x + a_n$ where n is a non-negative integer and $a_0, a_1, a_2, \dots, a_n$ are real numbers and $a_0 \neq 0$, then f is called a polynomial function of degree *n*. A polynomial function is always continuous.

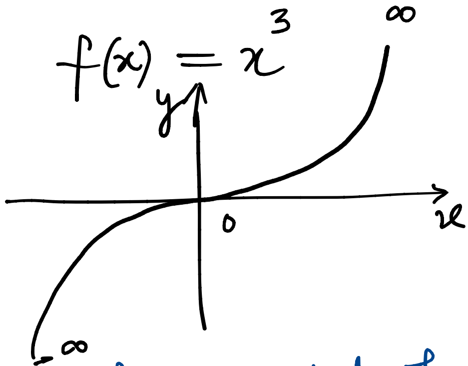
NOTE: **(a)** A polynomial of degree one with no constant term is called an odd linear function . i.e. $f(x) = ax,\ a \neq 0$.
In case $f(x) = 0$ it is a constant function degree is not defined. $\boxed{f(x)=2}$

$f(x) = 2x^0.$

(b) There are two polynomial functions, satisfying the relation ;
$f(x).f(1/x) = f(x) + f(1/x)$ $\forall x \in R-\{0\}$
(i) $f(x) = 1+x^n$ & **(ii)** $f(x) = 1 - x^n$, where n is a positive integer.

(c) A polynomial of degree odd has its range $(-\infty, \infty)$ but a polynomial of degree even has a range which is always subset of R.

eg: $f(x) = x^3$



eg: $f(x) = x^2 + 2x + 3$

Q A polynomial function $f(x)$ satisfies

$$f(x) \cdot f\left(\frac{1}{x}\right) = f(x) + f\left(\frac{1}{x}\right) \quad \forall x \in R - \{0\}.$$

If $f(10) = 1001$ then find $f(3)$?

Sol^n

$$f(x) = 1 + x^n \quad ; \quad f(x) = 1 - x^n \quad \times\times$$

$$f(10) = 1 + 10^n = 1001$$

$$10^n = 10^3$$

$$\boxed{n = 3}$$

$$f(x) = 1 + x^3$$

$$\boxed{f(3) = 28}$$

(ii) ALGEBRAIC FUNCTION:

A function f is called an algebraic function if it can be constructed using algebraic operations such as addition, subtraction, multiplication, division and taking roots, started with polynomials.

eg: $f(x) = (x^3-3)^{1/4} + \dfrac{x^4-2x+1}{x-4}$; $g(x) = x^3+3x+1$

☞ **Note that** all polynomial are algebraic but not the converse. Functions which are not algebraic and known as Transcendental function.

(iii) RATIONAL FUNCTION:

A rational function is a function of the form. $y = f(x) = \dfrac{g(x)}{h(x)}$, where g (x) & h (x) are polynomials & $h(x) \neq 0$. The domain of f (x) is set of reals such that $h(x) \neq 0$.

e.g. $f(x) = \dfrac{2x^4 - x^2 + 1}{x^2 - 4}$; $D \in R - \{\pm 2\}$

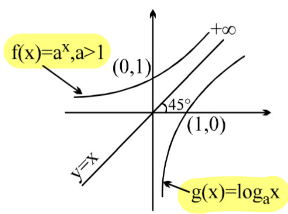
(iv) EXPONENTIAL FUNCTION:

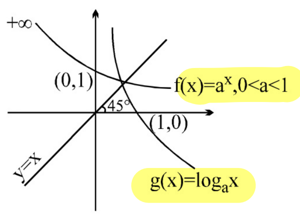
A function $f(x) = a^x = e^{x \ln a}$ $(a > 0,\ a \neq 1,\ x \in R)$ is called an exponential function. $f(x) = a^x$ is called an exponential function because the variable x is the exponent. It should not be confused with power function. $g(x) = x^2$ in which variable x is the base.

For $f(x) = e^x$ domain is R and range is R^+.

For $f(x) = e^{1/x}$ domain is $R - \{0\}$ and range is $R^+ - \{1\}$. i.e. $(0, 1) \cup (1, \infty)$

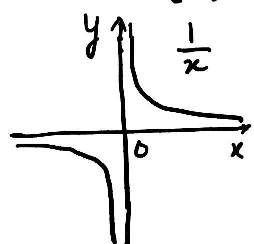
Note that f(x) & g(x) are inverse of each other & their graphs are as shown.





$f(x) = e^{1/x}$

$D: R - \{0\}$



$\frac{1}{x} \in R - \{0\}$

(v) LOGARITHMIC FUNCTION: $y = \log_a x,\ x > 0,\ a > 0,\ a \neq 1$

(vi) ABSOLUTE VALUE FUNCTION/ Modulus function.

A function $y = f(x) = |x|$ is called the absolute value function or Modulus function. It is defined as : $y = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$

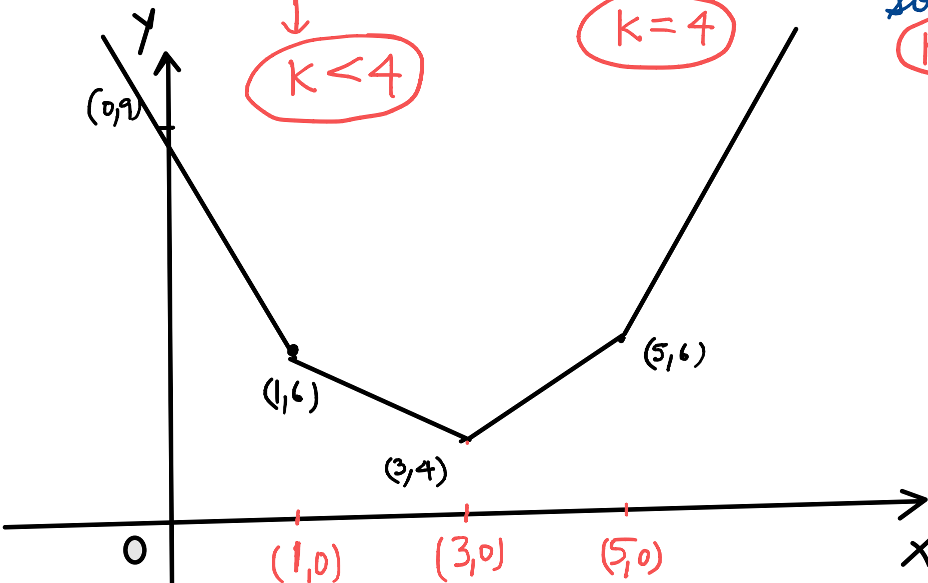
Q If $\underbrace{|x-1| + |x-3| + |x-5|}_{} = K$ then find K for which this equation has :

① No solution → $K < 4$

② One sol. $K = 4$

③ Two distinct solution. $K > 4$

$g(x) = K$



$f(x) = |x-1| + |x-3| + |x-5|$

$f(1) = 6$

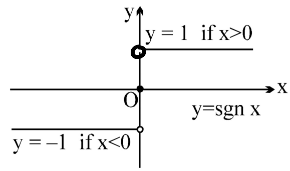
$f(3) = 4$

$f(5) = 6$

$f(0) = 9.$

(vii) SIGNUM FUNCTION:

A function y= f (x) = Sgn (x) is defined as follows :



$$y = f(x) = \begin{bmatrix} 1 & \text{for} & x > 0 \\ 0 & \text{for} & x = 0 \\ -1 & \text{for} & x < 0 \end{bmatrix}$$

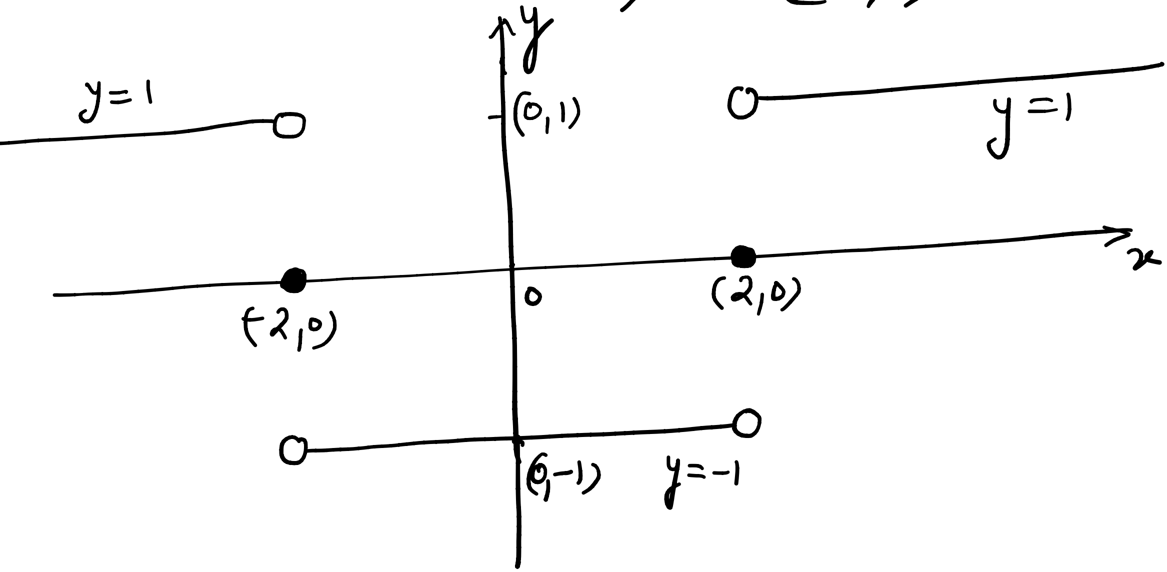
It is also written as Sgn x = |x|/ x;

$x \neq 0$; f (0) = 0

$$f(x) = \begin{cases} \dfrac{|x|}{x} & ; \ x \neq 0 \\ 0 & ; \ x = 0 \end{cases}$$

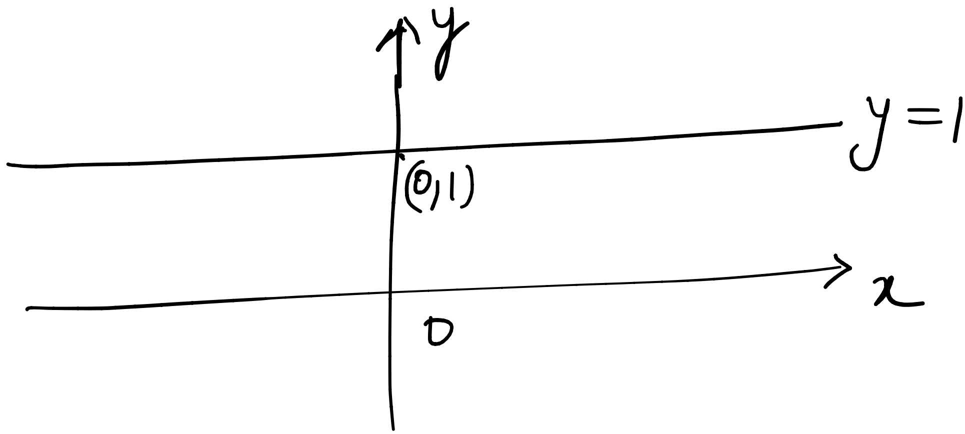
eg: $f(x) = \text{Sgn}(x^2-4)$. $x^2-4 > 0$

$$\text{Sgn}(x^2-4) \begin{cases} 1 & ; \ x \in (-\infty, -2) \cup (2, \infty) \\ 0 & ; \ x = \pm 2 \\ -1 & ; \ x \in (-2, 2) \end{cases}$$



eg: $f(x) = \mathrm{sgn}(e^x)$

$= 1$

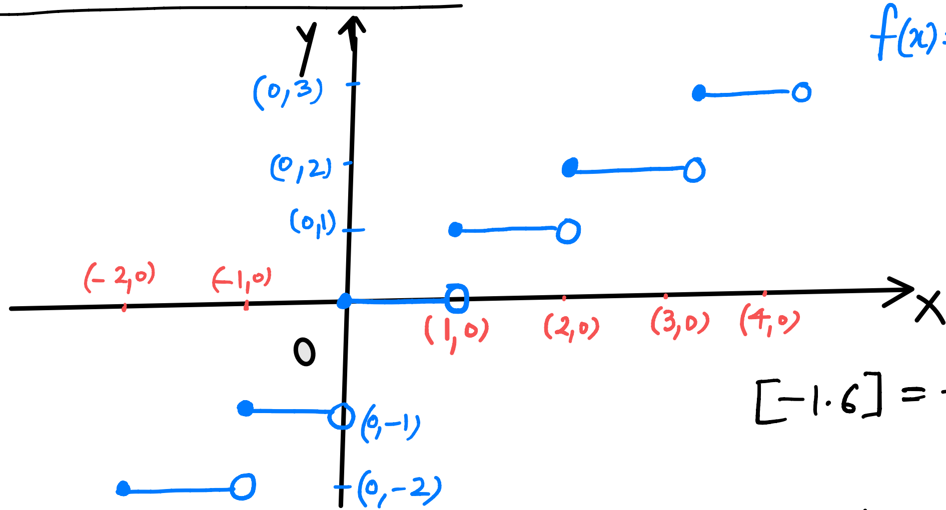


$*\quad f(x) = \mathrm{sgn}(\underbrace{\mathrm{sgn}\,x}) = \mathrm{sgn}(x)$

$f(x) = \mathrm{sgn}(\mathrm{sgn}(\mathrm{sgn}(x))) = \mathrm{sgn}(x)$

(viii) **Greatest Integer Or Step Up Function:** (Gif)

The function $y = f(x) = [x]$ is called the greatest integer function where $[x]$ denotes the greatest integer less than or equal to x .



$[-1.6] = -2$

$[10.6] = 10.$

$[2.6+8] = [2.6]+8 = 10.$

Properties of greatest integer function :

(a) $[x] \leq x < [x]+1$ and

$x-1 < [x] \leq x$, $0 \leq x-[x] < 1$

** **(b)** $[x+m] = [x]+m$ if m is an integer.

(c) $[x]+[y] \leq [x+y] \leq [x]+[y]+1$

* **(d)** $[x]+[-x] = 0$ if x is an integer

$= -1$ otherwise.

(c) ① $x = 2.6$ $\quad [x]=2$

$y = 3.1$ $\quad [y]=3$

$[x+y] = 5$

② $x = 2.6$; $[x]=2$

$y = 3.4$; $[y]=3$

$x+y = 6.$

$[x+y] = 6.$

eg: $x = 2.6$ $\quad [x] = 2$

$-x = -2.6$ $\quad [-x] = -3$

$$[x] + [-x] = -1$$

$x = 5$ $\quad [x] = 5$

$-x = -5$ $\quad [-x] = -5$

$$[x] + [-x] = 0.$$

Note:-

① $$[\underbrace{[x]}_{\text{Integer}}] = [x].$$

$$[[[x]]] = [x].$$

② $$[x + [x + \underbrace{[x]}]] = 3[x].$$

$$[x + \underbrace{[x]} + \underbrace{[x]}]$$

eg:

$$f(x) = \left[\tan x + \left[\sin x + \underbrace{[\cos x]} \right] \right]$$

$$= [\tan x] + [\sin x] + [\cos x].$$

Q1 Find domain of $f(x) = \dfrac{1}{[x]}$?

($[\ ] \rightarrow$ Gif)

Sol^n

$$[x] = 0. \Rightarrow x = ?$$

$\hookrightarrow x \in [0, 1)$

$$D_f \in R - [0, 1).$$ Ans

(ix) FRACTIONAL PART FUNCTION :

It is defined as :

g (x) = {x} = x – [x].

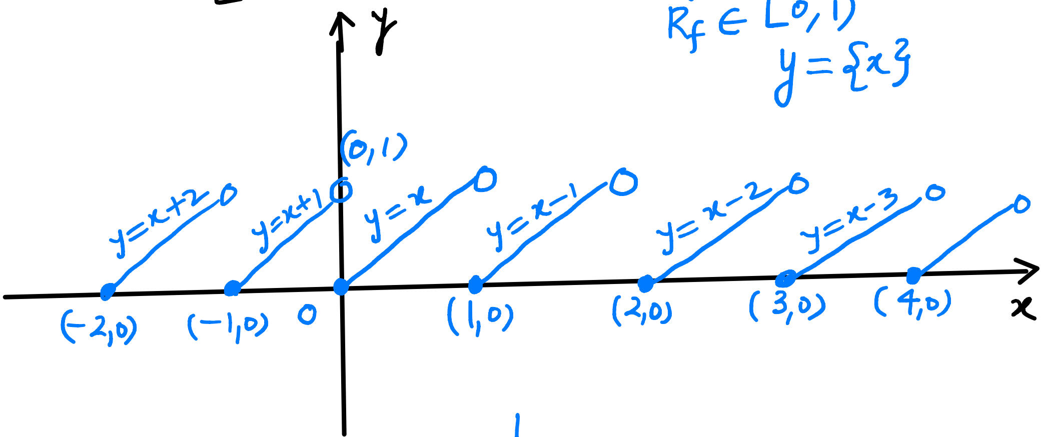
$\{\ \} \rightarrow$ - - - - - - - - $f \in [0, 1)$

$x = [x] + \{x\}$

$D_f \in R.$

$R_f \in [0, 1)$

$y = \{x\}$



$$\{x\} = x - [x] \begin{cases} x+1 & ; \ x \in [-1, 0) \\ x-0 = x & ; \ x \in [0, 1) \\ x-1 & ; \ x \in [1, 2) \\ x-2 & ; \ x \in [2, 3) \\ \vdots & \vdots \end{cases}$$

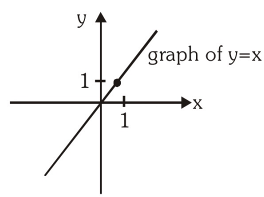
Note: ① $\{x + I\} = \{x\}$

$I \rightarrow$ integer

② $\{x\} + \{-x\} \begin{cases} 0 ; & x \in I \\ 1 ; & x \notin I. \end{cases}$

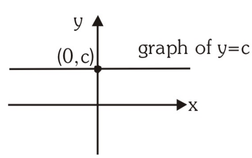
Identity function :

The function $f : A \rightarrow A$ defined by $f(x) = x \ \ \forall \ x \in A$ is called the identity of A and is denoted by I_A. It is easy to observe that identity function defined on R is a bijection.



Constant function :

A function $f : A \rightarrow B$ is said to be a constant function if every element of A has the same f image in B. Thus $f : A \rightarrow B$; $f(x) = c, \ \forall \ x \in A, \ c \in B$ is a constant function. Note that the range of a constant function is a singleton



Real valued function : Function is said to be real valued function if its domain and range are subset of real number.

Note :

(i) The complete set of all positive real numbers is denoted by $\mathbb{R}^{+}$ or $\mathbb{R}_{>0}$

(ii) The complete set of all negative real numbers is denoted by $\mathbb{R}^{-}$ or $\mathbb{R}_{<0}$

(iii) The complete set of all real numbers other then zero is denoted by $\mathbb{R}^{*}$ or $\mathbb{R}_{\neq 0}$ or $\mathbb{R}-\{0\}$ or $\mathbb{R}\setminus\{0\}$

(iv) The complete set of all integers is denoted by $\mathbb{Z}$.

Q $f(x)=\begin{cases} x+1 & x<2 \\ x+3 & x\geq 2 \end{cases}$ & $g(x)=\begin{cases} x^2+2x+7 & x<1 \\ x^2+5x+7 & x\geq 1 \end{cases}$.

Find $f(x)+g(x)$?

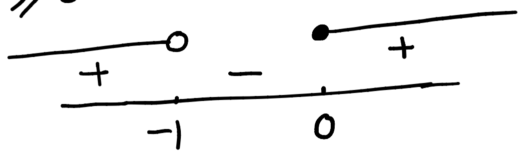
$$f(x)+g(x) \longrightarrow \begin{cases} (x+1)+(x^2+2x+7) \; ; & x<1 \\ (x+1)+(x^2+5x+7) \; ; & 1\leq x<2 \\ (x+3)+(x^2+5x+7) \; ; & x\geq 2 \end{cases}$$

Q Find domain :-

① $f(x) = \sqrt{\dfrac{1-5^x}{7^{-x}-7}}$

$$\frac{1-5^x}{7^{-x}-7} \geq 0 \Rightarrow \left(\frac{1-5^x}{1-7^{x+1}}\right)\underbrace{7^x}_{>0} \geq 0.$$

$$\frac{5^x-1}{7^{x+1}-1} \geq 0.$$



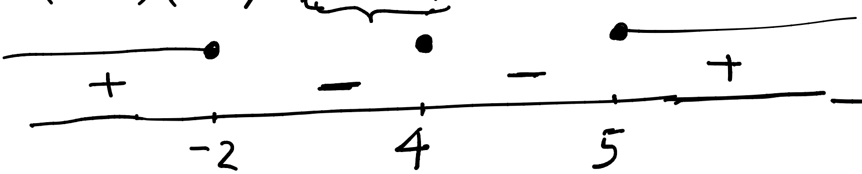
$D_f \in (-\infty, -1) \cup [0, \infty)$

② $f(x) = \sqrt{(x^2-3x-10)\ln^2(x-3)}$

$\underbrace{(x^2-3x-10)}_{0} \cdot \underbrace{\ln^2(x-3)}_{\bullet} \geq 0.$ & $x-3>0$

$\boxed{x>3}$ ✓ —①—

$(x-5)(x+2)\,\underbrace{\ln^2(x-3)}\geq 0$



—②—

① ∩ ② **

$D_f \in [5, \infty) \cup \{4\}$

Note :- For $n \in \mathbb{N}$

$$* \quad [x] + \left[x + \frac{1}{n}\right] + \left[x + \frac{2}{n}\right] + \dots + \left[x + \frac{n-1}{n}\right]$$

(General)

$$= [nx].$$

eg.

$$[x] + \left[x + \frac{1}{3}\right] + \left[x + \frac{2}{3}\right] = [3x]$$

Proof:

$$x = [x] + \{x\} \qquad f \in [0, 1)$$

$$= [x] + f$$

RHS: $[3[x] + 3f]$

$3[x] + [3f]$.

LHS:

$$[[x] + f] + \left[[x] + f + \frac{1}{3}\right] + \left[[x] + f + \frac{2}{3}\right]$$

$$3[x] + [f] + \left[f + \frac{1}{3}\right] + \left[f + \frac{2}{3}\right]$$

C-I :- $f \in \left[0, \frac{1}{3}\right)$

$$3[x] + 0 + 0 + 0 = 3[x].$$

C-I

RHS: $3[x] + 0$

C-II $\quad f \in \left[\frac{1}{3}, \frac{2}{3}\right)$

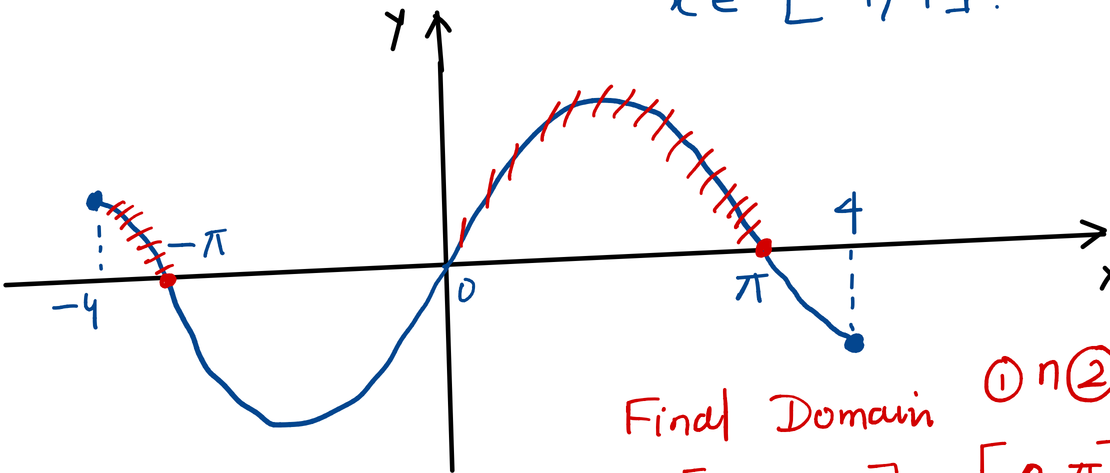
C-III $\quad f \in \left[\frac{2}{3}, 1\right)$

③ $f(x)=\sqrt{\sin x}+\sqrt{16-x^2}$ Find domain ?

$\sin x \geq 0$ — ②—

$16-x^2 \geq 0 \Rightarrow x^2-16 \leq 0.$

$x \in [-4,4].$ —①—



Find Domain ① ∩ ②

$[-4,-\pi] \cup [0,\pi].$

④ $f(x)=\left(\log_{\frac{x-2}{x-3}} 2\right)+\sqrt{9-x^2}$

$\frac{x-2}{x-3}>0$ & $\frac{x-2}{x-3} \neq 1$

— ① —

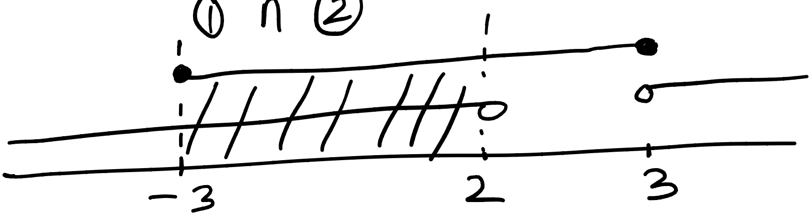
$9-x^2 \geq 0.$

$x^2-9 \leq 0.$

$x \in [-3,3]$

— ② —

① ∩ ②



Domain ⇒ $[-3,2)$

⑤ $f(x) = \sqrt{\log_x(\cos 2\pi x)}$

$\log_x(\cos 2\pi x) \geq 0$; $\cos(2\pi x) > 0$; $x > 0$, $x \neq 1$.

C–I $\quad x \in (1, \infty)$

$\log_x(\cos 2\pi x) \geq 0 \Rightarrow \boxed{\cos 2\pi x \geq 1.}$

$\cos \underbrace{2\pi x}_{\theta} = 1$

$\cancel{2}\cancel{\pi}x = \cancel{2}n\cancel{\pi}$. ; $n \in I$

$\boxed{x = n}$ — ① —

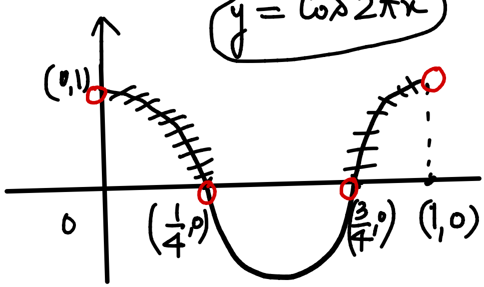
where $n \in N \geq 2$

C–II $\quad 0 < x < 1$

$\log_x(\cos 2\pi x) \geq 0 \Rightarrow \cos(\underbrace{2\pi x}_{\theta}) \leq 1.$

$\therefore \quad \boxed{0 < \cos \underbrace{2\pi x}_{\theta} \leq 1}$

$y = \cos 2\pi x$



$\because x \in (0, 1)$

$\underbrace{2\pi x}_{\theta} \in (0, 2\pi)$

$x \in \left(0, \frac{1}{4}\right) \cup \left(\frac{3}{4}, 1\right)$ — ② —

Finally ① ∪ ②.

⑥ $f(x) = \log_{\underbrace{2\{x\}-3}_{-ve}}(\underbrace{x^2 - 5x + 13}_{N \text{ (always +ve)}})$; $\{\ \} \rightarrow$ fractional part function.

$\{x\} \in [0, 1)$

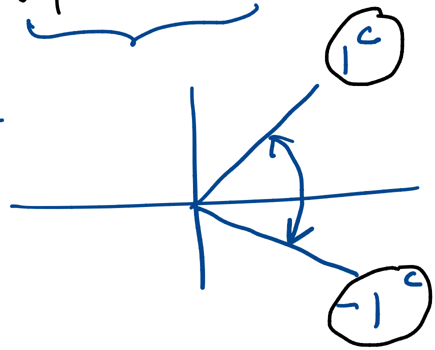
$2\{x\} \in [0, 2)$

$D_f \in \phi.$

⑦ $f(x) = \sqrt{\cos(\underbrace{\sin x}_{\theta})} + \dfrac{\sqrt{\cos x - \frac{1}{2}}}{\sqrt{\underbrace{6 + 35x - 6x^2}}}.$

$\theta \in [-1, 1]$

$\cos\theta > 0.$ $\boxed{D_1 \in R}$ — (1) —



$6 + 35x - 6x^2 > 0.$

$6x^2 - 35x - 6 < 0.$

$6x^2 - 36x + x - 6 < 0.$

$(6x+1)(x-6) < 0. \Rightarrow x \in \left(-\frac{1}{6}, 6\right)$

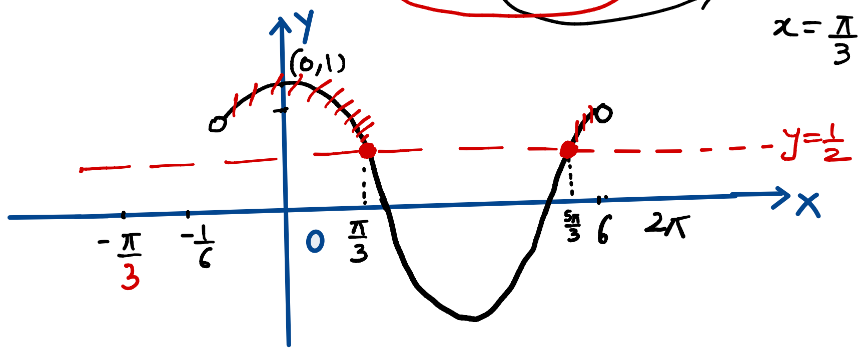
— (2) —

$\cos x - \frac{1}{2} \geq 0 \Rightarrow \cos x \geq \frac{1}{2}$

$\cos x = \frac{1}{2}$

$x = \frac{\pi}{3}; \frac{5\pi}{3}$

$x = -\pi/3$



Domain : $\left(-\frac{1}{6}, \frac{\pi}{3}\right] \cup \left[\frac{5\pi}{3}, 6\right)$ Ans

Q8 $f(x) = \frac{1}{[x]} + \log_{1-\{x\}}\underbrace{(x^2-3x+10)}_{\text{always +ve}} + \frac{1}{\sqrt{2-|x|}} + \frac{1}{\sqrt{\sec(\sin x)}}$

$[\ ] \to$ Gif & $\{\ \} \to$ fractional part function.

$\{x\} \in [0,1)$

$\boxed{D_4 \in R}$

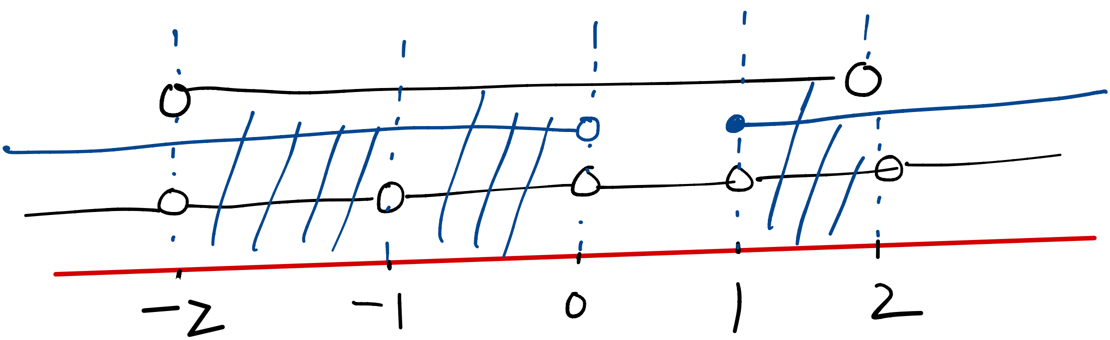
$D_1 \in R - [0,1)$; $D_2 \in R - I$.

$2 - |x| > 0$.

$|x| < 2$

$D_3 : (-2, 2)$

$D_1 \cap D_2 \cap D_3 \cap D_4$



$D_f \in (-2,-1) \cup (-1,0) \cup (1,2)$ Ans

Q9 $f(x) = \dfrac{[x]}{2x-[x]}$; $[\ ] \to$ Gif

Find domain?

$$2x - [x] \neq 0.$$

Solve for $\boxed{2x - [x] = 0}$

S-1 $$x = [x] + \{x\}$$

$$2([x] + \{x\}) - [x] = 0.$$

$$2\{x\} = -[x].$$

$$\{x\} = -\frac{[x]}{2} \quad \text{— ① —}$$

S-2 $$\{x\} \in [0,1)$$

$$\therefore \quad 0 \leq \frac{-[x]}{2} < 1$$

$$0 \leq -[x] < 2$$

$$-2 < [x] \leq 0.$$

$$\therefore [x] = -1 \text{ or } 0.$$

If $[x] = 0$ then $\{x\} = 0$

If $[x] = -1$ then $\{x\} = \frac{1}{2}$

S-3

$$x = [x] + \{x\}$$

$$x = 0 + 0 \Rightarrow x = 0$$

$$x = -1 + \frac{1}{2} \Rightarrow x = -\frac{1}{2}$$

$\therefore$ Domain : $R - \left\{-\frac{1}{2}, 0\right\}$

Q Solve for x :- (Given: $[\ ] \rightarrow$ Gif & $\{\ \} \rightarrow$ ....)

① $[x]^2 = x + 2$

② $5\{x\} = 3[x-1] + x$.

① $\underbrace{[x]^2}_{\text{Integer}} = \underbrace{x}_{\text{shd be integer}} + \underbrace{2}_{\text{Int}}$.

$x^2 - x - 2 = 0$; $x \in I$

$x^2 - 2x + x - 2 = 0$

$(x-2)(x+1) = 0 \Rightarrow x = -1; 2$ Ans

② $5\{x\} = 3[x] - 3 + x$

$5\{x\} = 3[x] - 3 + [x] + \{x\}$

$\underbrace{\{x\}}_{[0,1)} = \dfrac{4[x] - 3}{4}$. —①—

$0 \leq \dfrac{4[x]-3}{4} < 1$

$0 \leq 4[x] - 3 < 4$

$$\frac{3}{4} \leq [x] < \frac{7}{4}.$$

$\therefore [x] = 1$ put in ①

$$\{x\} = \frac{1}{4}$$

$$\therefore x = 1 + \frac{1}{4} = \frac{5}{4} \quad \text{Ans}$$

Q If $\{x\} = \frac{2}{3}$ & $\left[x+\left\{x+\left[x+\{x\}+.......100 \text{ times}\right]\right\}\right] = 5$, then -

~~(A)~~ $x = \frac{14}{3}$ (B) $[x] = 5$ (C) $x = \frac{17}{3}$ ~~(D)~~ $[x] = 4$

(where [.] & {.} denotes greatest integer function & fractional part function respectively)

Sol^n $\{x\} = \frac{2}{3}$

$\{x + I\} = \{x\}$

$$[x + \{x + \text{Int}\}] = 5.$$

$$[x + \{x\}] = 5.$$

$$[[x] + 2\{x\}] = 5.$$

$$[x] + \left[2\cdot\left(\frac{2}{3}\right)\right] = 5$$

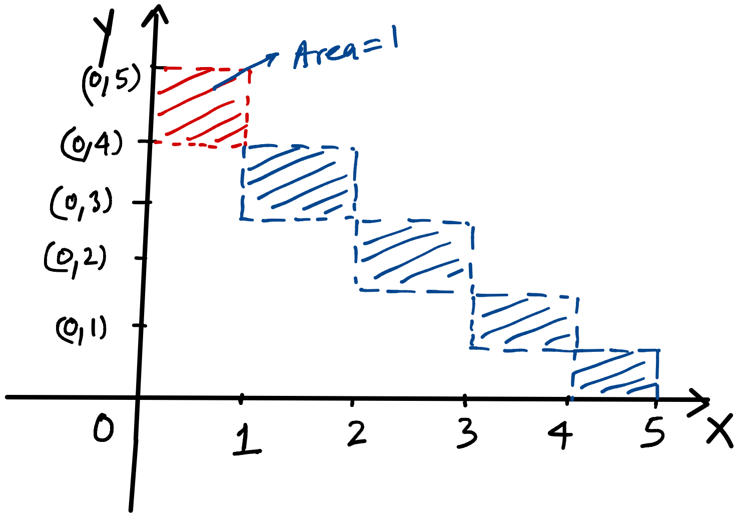
$$[x] = 5 - 1 = 4.$$

$$x = [x] + \{x\} = 4 + \frac{2}{3} = \frac{14}{3}$$

Q. A point P(x, y) moving in 1^{st} quadrant of xy plane such that it always satisfy the relation $[x]+[y]=4$ then find the area of all possible positions of point P?

(Note: [ ] → Gif).

Sol^n



$$[x]+[y]=4$$

$$I_1+I_2=4.$$

$$0+4=4$$

$$[x]=0$$

$$x\in[0,1)$$

$$[y]=4$$

$$y\in[4,5)$$

$$[x]=1$$

$$[y]=3$$

$$x\in[1,2)$$

$$y\in[3,4)$$

Ans → 5

Q

Let n be a positive integer. If the number of integers in the domain of the function $f(x) = ln\left((1-x)(x-n)\right)$ is $2n - 11$, then the value of n is

(A) 8 (B) 9 (C) 10 (D) 11

Sol^n

$(1-x)(x-n) > 0.$

$(x-1)(x-n) < 0$

$x \in (1, n)$

$(1, 5)$

$\cancel{5-1=4}$

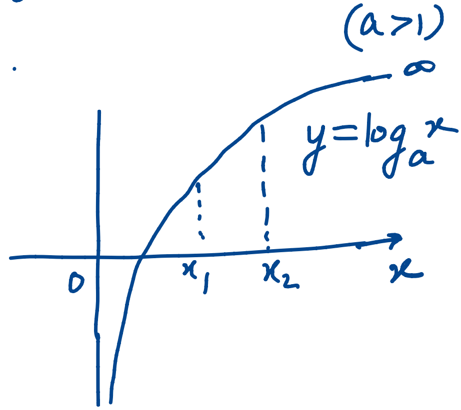
$n-1-1 = 2n-11.$

$n-2 = 2n-11$

$\boxed{n=9}$

Q Find range of following functions :-

(1) $f(x) = \ln(5x^2 - 8x + 4)$.



$5x^2 - 8x + 4 > 0$

$\downarrow$ (always +ve) $\cup$

$\therefore D_f \in R.$

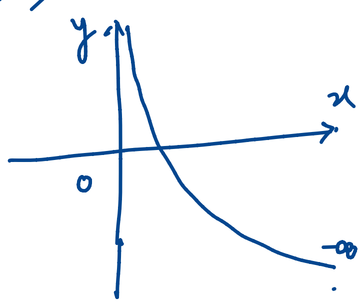
$(5x^2 - 8x + 4) \in \left[-\frac{D}{4a}, \infty\right)$

$D = 8^2 - 4(5)(4)$
$= 64 - 80$
$= -16$

$\therefore \underbrace{5x^2 - 8x + 4}_{M} \in \left[\frac{4}{5}, \infty\right)$

$\log_e M \in \left[\ln\frac{4}{5}, \infty\right)$ Ans

(ii) $g(x) = \log_{\frac{4}{5}}(\underbrace{5x^2 - 8x + 4})$



$5x^2 - 8x + 4 \in \left[\frac{4}{5}, \infty\right)$

$R_g \in (-\infty, 1]$

(2) $f(x) = \log_2\left(2 - \log_{\sqrt{2}}(\underbrace{16\sin^2 x + 1})\right)$

$2 - \log_{\sqrt{2}}(16\sin^2 x + 1) > 0.$

$2 > \log_{\sqrt{2}}(16\sin^2 x + 1)$

$(\sqrt{2})^2 > (16\sin^2 x + 1)$

$16\sin^2 x < 1.$

$\boxed{\sin x \in \left(-\frac{1}{4}, \frac{1}{4}\right)}.$ ✗

$\sin^2 x \in \left[0, \frac{1}{16}\right)$

$16\sin^2 x \in [0, 1)$

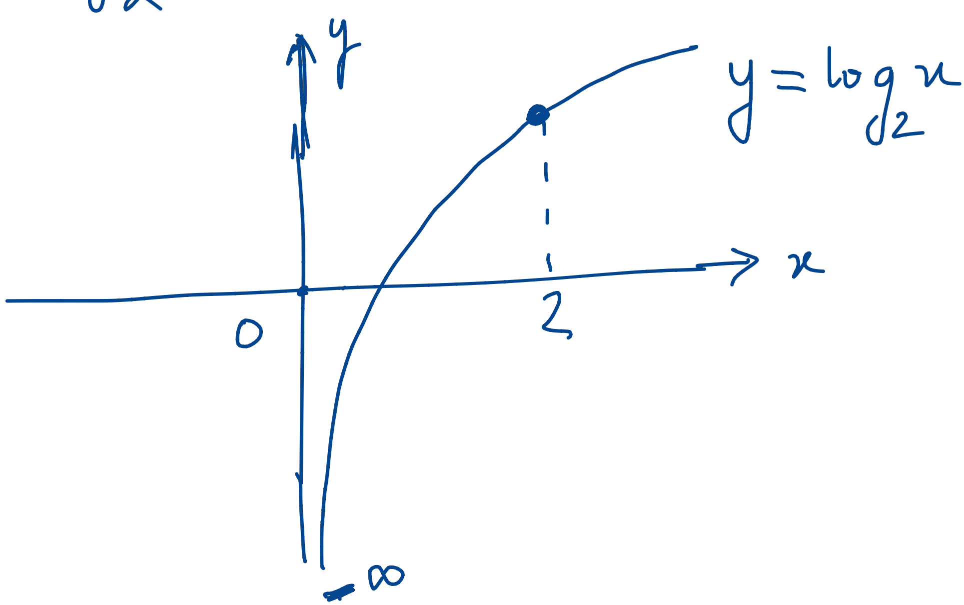
$\boxed{16\sin^2 x + 1 \in [1, 2)}$

$$\log_{\sqrt{2}}(16\underbrace{\sin^2 x}+1) \in [0, 2)$$

$$-\log_{\sqrt{2}}(16\sin^2 x+1) \in (-2, 0]$$

$$\underbrace{2-\log_{\sqrt{2}}(\quad)}_{\text{Masala}=M} \in (0, 2]$$

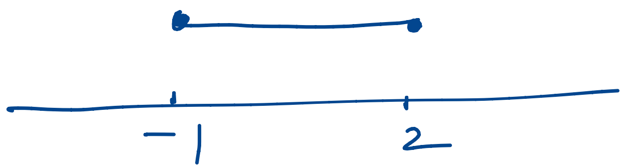
$$\log_2 M \in (-\infty, 1] \;\; \underline{\underline{\text{Ans}}}$$



(3) $f(x) = \sqrt{2-x} + \sqrt{1+x}$

Domain: $2-x \geq 0$ & $1+x \geq 0$

$x-2 \leq 0$ & $x \geq -1$



$D_f \in [-1, 2]$ —①—

$$y = \sqrt{2-x} + \sqrt{1+x} \; ; \; y \geq 0$$

$$y^2 = (2-\cancel{x}) + (1+\cancel{x}) + 2\sqrt{2+2x-x-x^2}$$

$$y^2 = 3 + 2\sqrt{2-x^2+x}$$

$$y^2 = 3 + 2\sqrt{2-(x^2-x)}$$

$$y^2 = 3 + 2\sqrt{2-\left(\left(x-\frac{1}{2}\right)^2-\frac{1}{4}\right)}$$

$$y^2 = 3 + 2\sqrt{\frac{9}{4} - \left(x - \frac{1}{2}\right)^2}$$

$$x \in [-1, 2]$$

$$x - \frac{1}{2} \in \left[-\frac{3}{2}, \frac{3}{2}\right]$$

$$\left(x - \frac{1}{2}\right)^2 \in \left[0, \frac{9}{4}\right].$$

$$-\left(x - \frac{1}{2}\right)^2 \in \left[-\frac{9}{4}, 0\right].$$

$$\frac{9}{4} - \left(x - \frac{1}{2}\right)^2 \in \left[0, 9/4\right]$$

$$2\sqrt{\frac{9}{4} - (\quad)^2} \in [0, 3]$$

$$3 + 2\sqrt{\quad} \in [3, 6].$$

$$y^2 \in [3, 6] \quad (\because y > 0)$$

$$\therefore y \in [\sqrt{3}, \sqrt{6}]$$ Ans

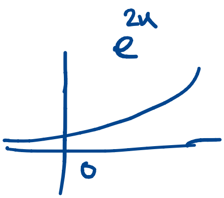
(4) $f(x) = \dfrac{e^{x} - e^{-x}}{e^{x} + e^{-x}}.$

$\boxed{D_f \in R}$

<u>M-1</u>

$$f(x) = \frac{e^{x} + e^{-x} - 2e^{-x}}{e^{x} + e^{-x}}.$$

$$= 1 - 2\left(\frac{e^{-x}}{e^{x} + e^{-x}}\right)$$



$$= 1 - 2\left(\frac{1}{e^{2x} + 1}\right)$$

$$e^{2x} \in (0, \infty)$$

$$e^{2x} + 1 \in (1, \infty)$$

$$\frac{1}{e^{2x} + 1} \in (0, 1)$$

$$\frac{-2}{e^{2x} + 1} \in (-2, 0)$$

$$1 - \frac{2}{e^{2x} + 1} \in (-1, 1)$$

Ans

$\boxed{\text{Note}}$:- $[\ ] \rightarrow$ Gif and $\{\ \} \rightarrow$ fractional part function

$$[\{x\}] = 0$$

$$\{[x]\} = 0$$

$$[\{\text{Masala}\}] = 0$$

$$\{[\text{Masala}]\} = 0.$$

$$[\{x^3 + 3x^2 + 4\}] = 0.$$

$$\{[\sin^3 x + 4x^2 + \ln x]\} = 0.$$

Q If $f(x) = ax^7 + bx^5 + cx^3 + dx + 2$, where a, b, c, d are real constants and $f(-3) = 3$ then find the range of $F(x) = f(3) + \underbrace{3\cos^2x + 4\sin^2x}_{3}$.

Sol^n $f(x) = ax^7 + bx^5 + cx^3 + dx + 2$

$f(-x) = -ax^7 - bx^5 - cx^3 - dx + 2.$

$\boxed{f(x) + f(-x) = 4}$ $\Rightarrow$ $f(3) + f(-3) = 4$

$\boxed{f(3) = 1}$

$F(x) = 1 + 3 + \sin^2x = 4 + \underbrace{\sin^2 x}_{[0,1]}$

$R_F \in [4, 5]$ Ans

Q Which of the following functions have the same range where [x] denotes G.I.F. and {x} denotes fractional part function.

(A) $f(x) = \text{sgn}\left(\sqrt{\log_{[x]}\frac{|x|}{x}}\right)$ (B) $g(x) = \sqrt{ln(\cos(\sin x))}$

(C) $h(x) = \dfrac{\tan(\pi[x^2 - x])}{x^2 - 2x + 3}$ (D) $l(x) = \left[\left\{ln\sqrt{x - [x]}\right\}\right]$

Sol^n $\boxed{h(x) = 0}$ $\boxed{l(x) = 0}$

$f(x) = \text{Sgn}\left(\sqrt{\log_{[x]}(1)}\right) = \text{Sgn}(0) = 0.$

$g(x) = \sqrt{ln(\underbrace{\cos(\sin x)}_{1})}$.

$ln(\cos(\sin x)) \geq 0.$

$\cos(\sin x) \geq 1.$

$\therefore \boxed{\cos(\sin x) = 1}$

$\boxed{g(x) = 0}$

Note :

(i) If a function $f(x)$ is defined on an interval [a, b], then domain of a function $f(g(x))$ can be obtained by solving $a \le g(x) \le b$.

and has range $[2, 10]$

E(1) If a function $f(x)$ is defined on an interval [0, 3], the find the domain of definition & range of following functions :

(a) $h(x) = \dfrac{f(|2x-1|)}{3}$

(b) $K(x) = f(x^2-1) + 5$

(a)

$$h(x) = \frac{1}{3} f(|2x-1|)$$

$$0 \le |2x-1| \le 3$$

$$|2x-1| \le 3$$

$$-3 \le 2x-1 \le 3$$

$$-1 \le x \le 2$$

$$\therefore D_h \in [-1, 2]$$

$$R_h \in \left[\frac{2}{3}, \frac{10}{3}\right].$$

(b)

$$K(x) = f(x^2-1) + 5$$

$$D_k \in [-2,-1] \cup [1,2]$$

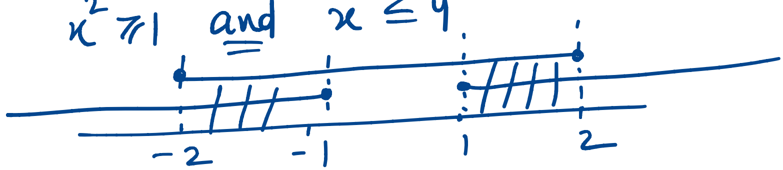
$$R_k \in [7, 15]$$

Ans

$$0 \le x^2-1 \le 3$$

$$1 \le x^2 \le 4$$

$x^2 \geq 1$ and $x^2 \le 4$



EQUAL OR IDENTICAL FUNCTION :

$f(x) = \sin x$
$g(x) = \cos x$ N.I

Two functions f & g are said to be equal if :

(a) The domain of f = the domain of g. and

(b) The range of f = the range of g and

(c) $f(x) = g(x)$, for every x belonging to their domain.

OR

2 functions are said to be identical if their graphs are superimpossible.

① $f(x) = \ln x^2$; $g(x) = 2\ln x$

$D_f \in R - \{0\}$

$D_g \in (0, \infty)$

N·I

② $f(x) = \tan^2 x . \sin^2 x$; $g(x) = \tan^2 x - \sin^2 x$

I

③ $f(x) = \sqrt{\dfrac{1-\cos 2x}{2}}$; $g(x) = \sin x$

$D_g \in R$

D_f : $1 - \cos 2x \geq 0$

$\cos 2x \leq 1$

$x \in R$

N·I

$f(x) = \sqrt{\dfrac{1-(1-2\sin^2 x)}{2}}$

$f(x) = \sqrt{\sin^2 x} = |\sin x|$

④ $f(x) = \sec x; \;\; g(x) = \dfrac{1}{\cos x}$

$D_f \in R - (2n+1)\frac{\pi}{2}$
$n \in I$

$g(x) \in R - (2n+1)\frac{\pi}{2}$
$n \in I$

I

N.I

⑤ $f(x) = \dfrac{1}{1+\dfrac{1}{x}} \;;\;\; g(x) = \dfrac{x}{1+x}$

$D_g \in R - \{-1\}$

$D_f \in R - \{-1, 0\}$

⑥ $f(x) = [\{x\}] \;;\;\; g(x) = \{[x]\}$; ([ ] → Gif
{ } → ......)

I

⑦ $f(x) = \sqrt{x^2 - 1} \;;\; g(x) = \sqrt{x-1} \cdot \sqrt{x+1}$

$x^2 - 1 \geq 0$

$x \in (-\infty, -1] \cup [1, \infty)$

$x - 1 \geq 0$ and
$x + 1 \geq 0$

$D_g \in [1, \infty)$

N.I

Functional Equation :-

f

(1) For $x \in R$, the function $f(x)$ satisfies $2f(x) + f(1-x) = x^2$ then the value of $f(4)$ is equal to

M-1

$$2f(x) + f(1-x) = x^2 \quad \text{— ① —}$$

$$x \to (1-x)$$

$$2f(1-x) + f(\cancel{1}-(\cancel{1}-x)) = (1-x)^2$$

$$\cancel{2f(1-x)} + f(x) = (1-x)^2 \quad \text{— ② —}$$

$$\cancel{2f(1-x)} + 4f(x) = 2x^2$$

$$-3f(x) = (1-x)^2 - 2x^2$$

M-2

$$2f(x) + f(1-x) = x^2.$$

$$\left.\begin{aligned} x &= 4 \\ x &= -3. \end{aligned}\right\}$$

$$2f(4) + f(-3) = 16. \quad \text{— ① —}$$

$$2f(-3) + f(4) = 9. \quad \text{— ② —}$$

Q Let $f(x)$ and $g(x)$ be functions which take integers as arguments. Let $f(x+y)=f(x)+g(y)+8$ for all integer x and y. Let $f(x)=x$ for all negative integers x, and let $g(8)=17$. The value of $f(0)$ is

$$f(x+y) = f(x) + g(y) + 8$$

$$f(-8) = -8.$$

$x = -8$
$y = 8.$

$$f(0) = f(-8) + g(8) + 8$$
$$= -\cancel{8} + 17 + \cancel{8}$$
$$= 17.$$

Q Let $f: R \to R$ be a function such that $\left(f(x^3+1)\right)^{\sqrt{x}} = 5, \ \forall\ x \in (0, \infty)$ then the value of $\left(f\left(\frac{27+y^3}{y^3}\right)\right)^{\sqrt{\frac{27}{y}}}$ for $y \in (0, \infty)$ is equal to

$$x \to \frac{3}{y}$$

$$f\left(\frac{27}{y^3}+1\right)^{\sqrt{\frac{3}{y}}} = 5$$

Sol^n

$$f\left(\frac{27}{y^3}+1\right)^{3\sqrt{\frac{3}{y}}} = ? \quad 125$$

Q If $f(x) = \dfrac{4^x}{4^x + 2}$ then $\underbrace{\sum_{r=1}^{2001} f\left(\dfrac{r}{2002}\right)}_{S} = ?$

$$S = f\left(\frac{1}{2002}\right) + f\left(\frac{2}{2002}\right) + \underset{f\left(\frac{1001}{2002}\right)}{- - - - -} + f\left(\frac{2000}{2002}\right) + f\left(\frac{2001}{2002}\right)$$

$$\underbrace{f(x) + f(1-x)} = \left(\frac{4^x}{4^x + 2} + \frac{4^{1-x}}{4^{1-x} + 2}\right) = 1.$$

$$S = 1000 \times 1 + f\left(\frac{1}{2}\right) = 1000 + \frac{1}{2}$$

$$= 1000.50$$

Ans

Play with Graphs :-

$y = f(x)$

$y = f(-x)$

$y = af(x)$

$y = -f(x)$

$y = f(|x|)$

$y = |f(x)|$

$y = f\left(\frac{x}{a}\right)$

$|y| = f(x)$

$y = f(x) + a$

$y = f(x+a)$

(1) $f(x) = \ln x$

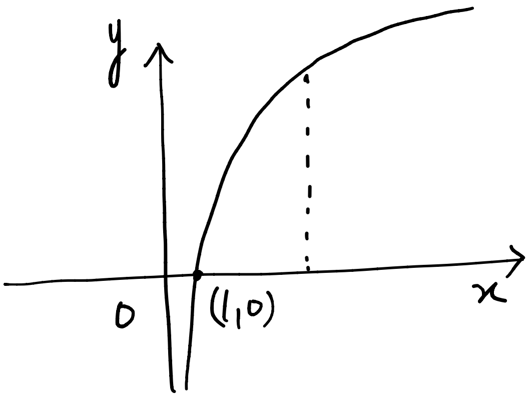
$f(-x)$

$-f(x)$

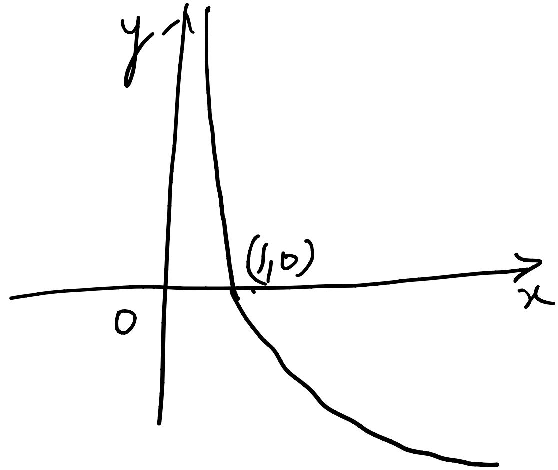
$|f(x)|$

$f(|x|)$

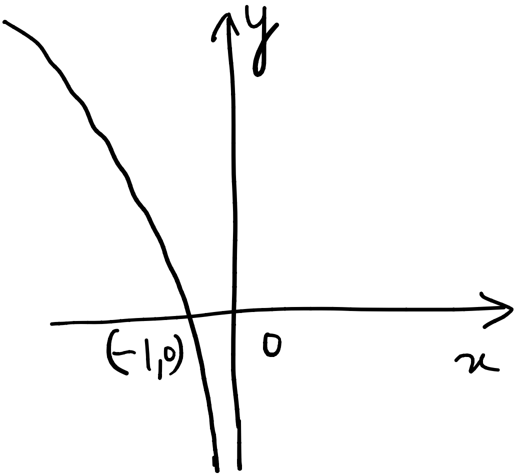
$f(x) = \ln x$



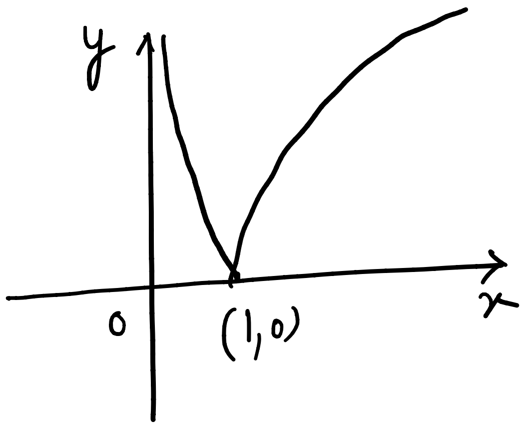
① $-f(x)$



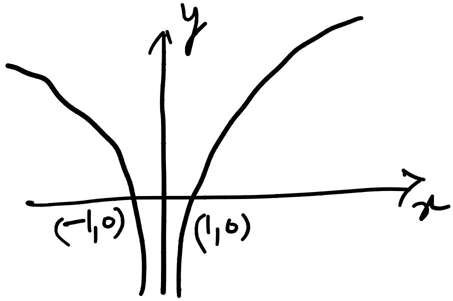
② $f(-x)$



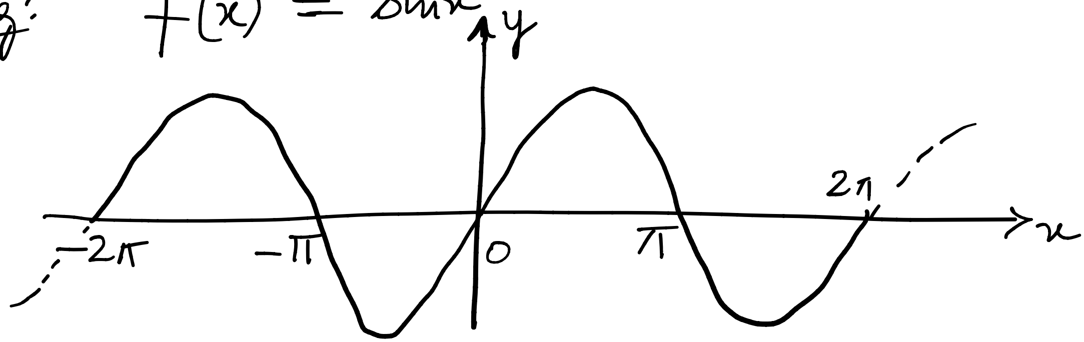
③ $|f(x)|$



④ $f(|x|)$

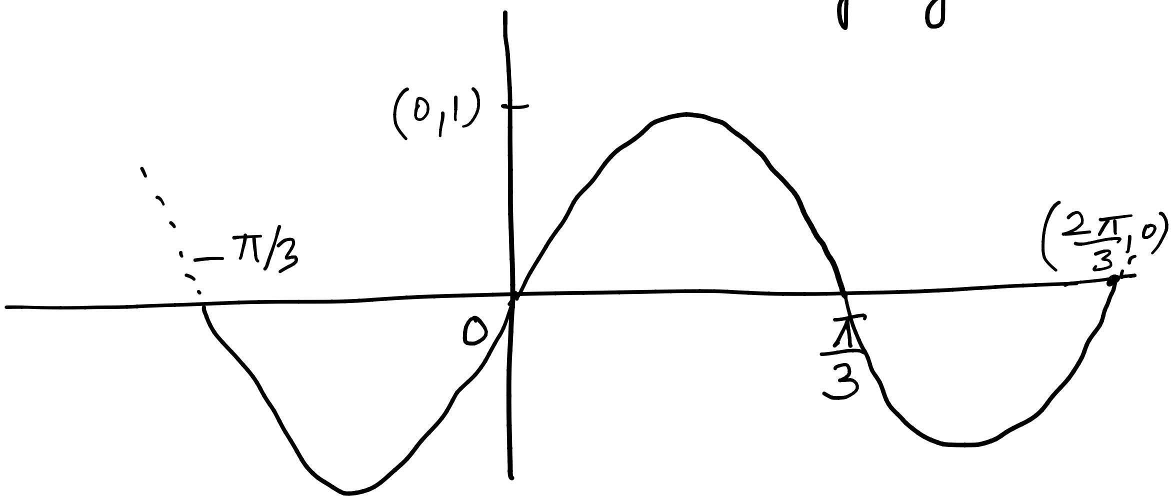


eg: $f(x) = \sin x$

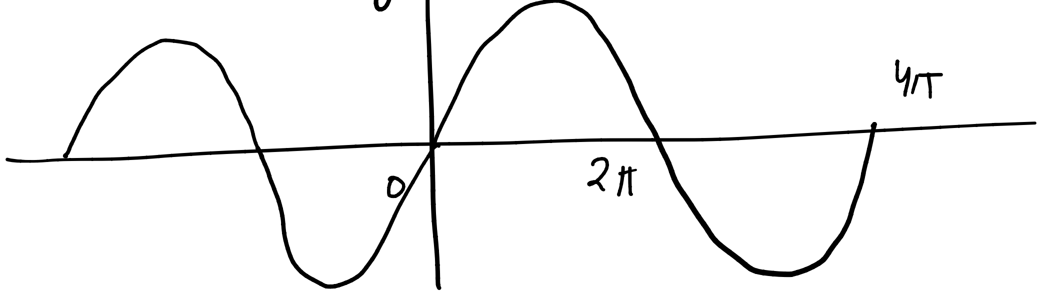


① $g(x) = \sin(3x)$

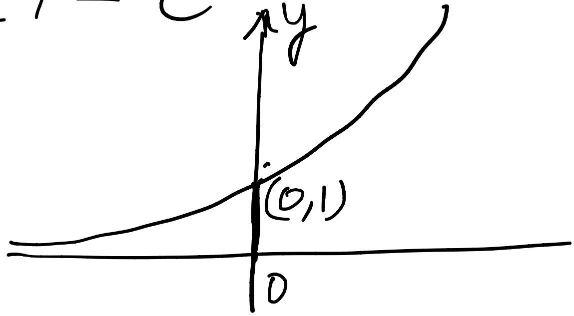
$y = g(x)$



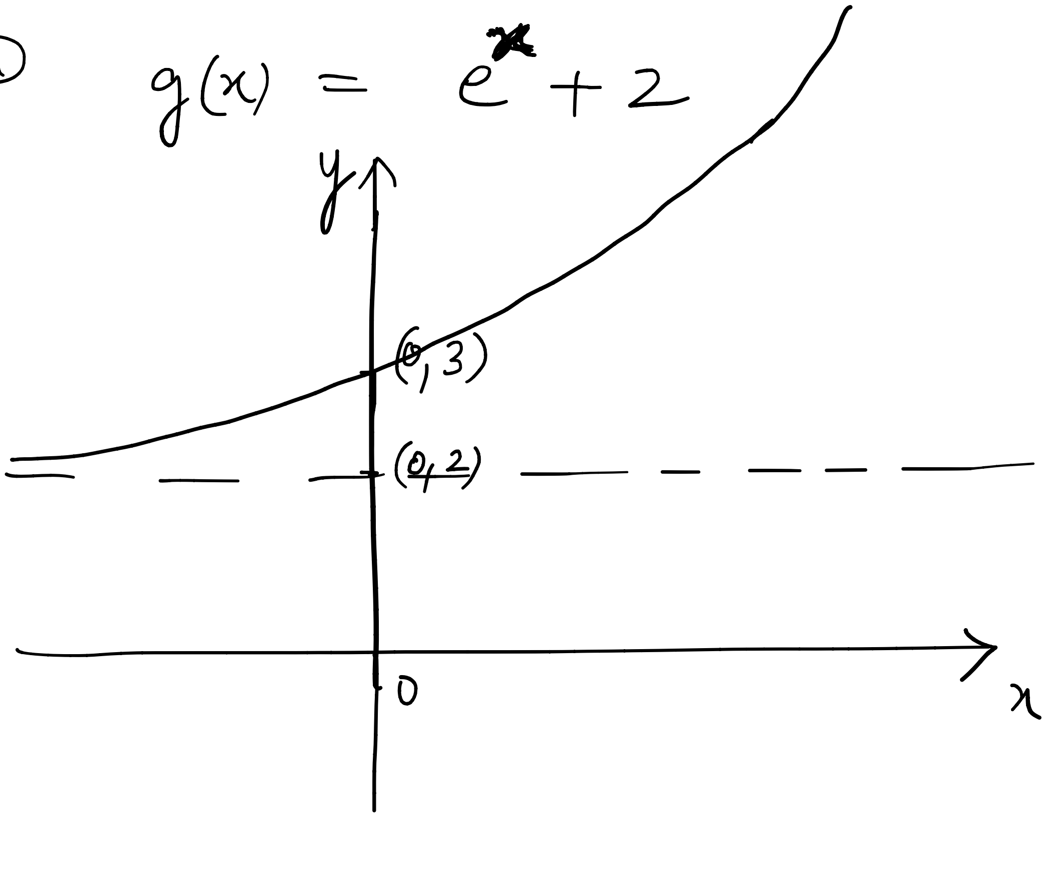
② $h(x) = \sin\left(\frac{x}{2}\right)$



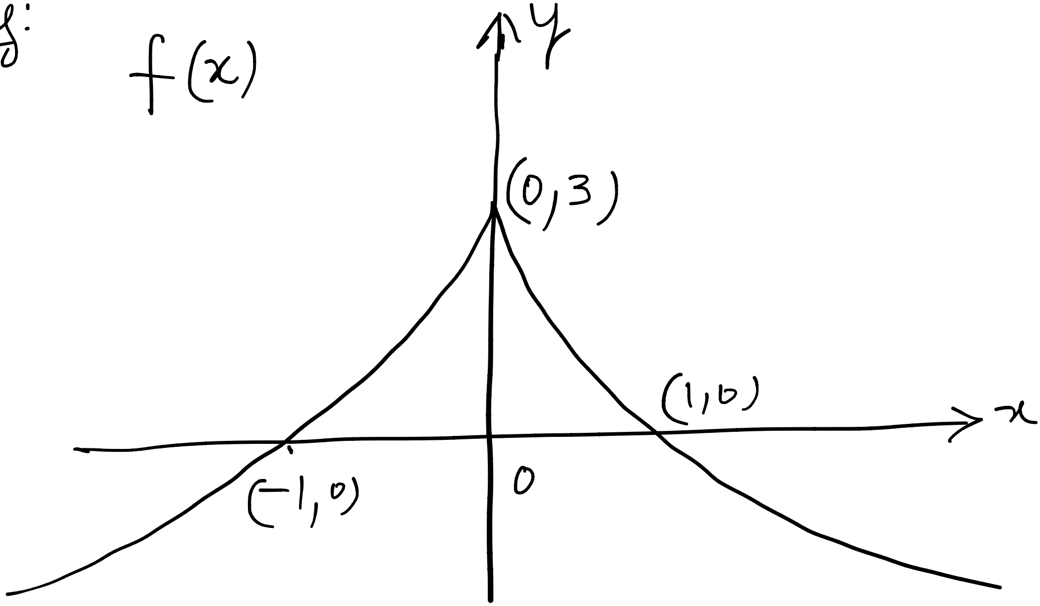
eg: $f(x) = e^x$



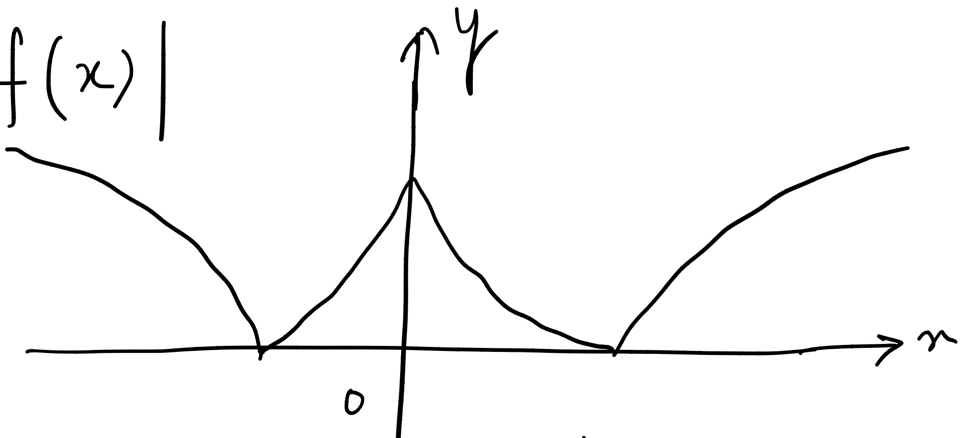
① $g(x) = e^x + 2$



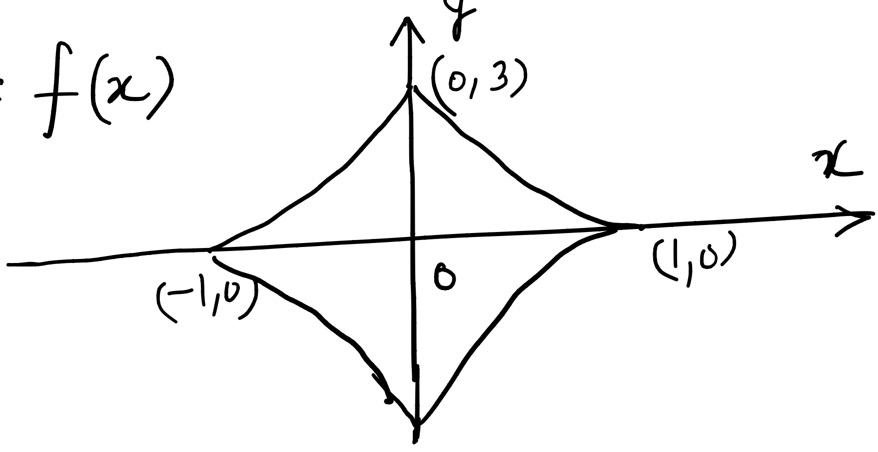
eg: $f(x)$



① $|f(x)|$



② $|y| = f(x)$
*



TRANSFORMATIONS OF THE GRAPH :

(a) **$-f(x)$:** Draw the graph of $f(x)$ in its entire domain, then take its mirror image w.r.t. x-axis.

(b) **$f(-x)$:** Draw the graph of $f(x)$ in its entire domain. Reflect the graph w.r.t. y-axis.

(c) **$|f(x)|$:** Draw the entire graph of $f(x)$ reflect the portion of graph which lies below x-axis w.r.t. x-axis.

(d) **$f(|x|)$:** Draw the graph of $f(x)$ only for $x \geq 0$, then reflect this portion w.r.t. y-axis.

(e) **$f(x) + a$:** Draw the entire graph of $f(x)$, then by keeping axis fix shift the graph along y-axis in upward direction by 'a' units, if $a > 0$ & shift the graph along y-axis in downward direction by 'a' units, if $a < 0$.

OR

By keeping the graph fix shift the co-ordinate axis by 'a' units in downward direction if $a > 0$.

(f) **$f(x + a)$:** Draw the entire graph of $f(x)$, then by keeping coordinate axis fix shift the graph by 'a' units leftward along x-axis if $a > 0$, rightward if $a < 0$.

OR

By keeping graph fixed, shift the coordinate axis by 'a' units rightward if $a > 0$ & leftward if $a < 0$

(g) **$|y| = f(x)$:** Draw the graph of $f(x)$ entirely.

(i) Remove all those branches which are below the x-axis.

(ii) Reflect all those branches w.r.t. x-axis which are above the x-axis.

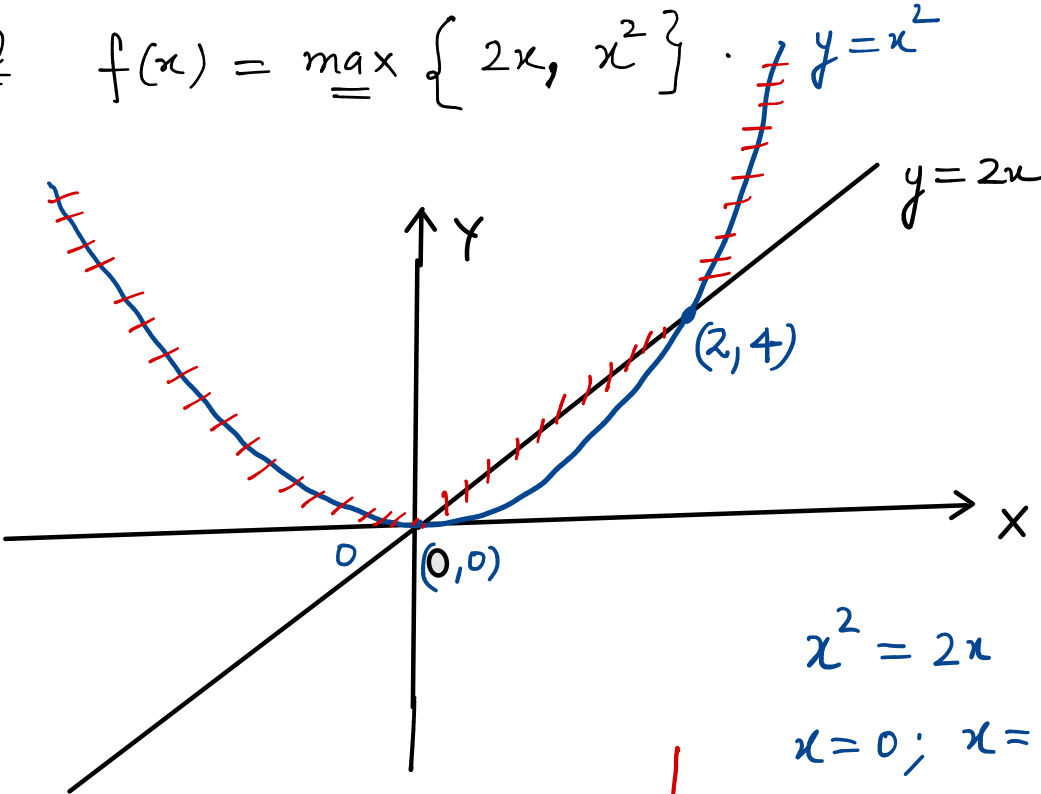
(h) **$af(x)$, $a > 0$:** Draw the entire graph of $f(x)$, then stretch the graph away from the x-axis if $a > 1$, compress the graph towards x-axis if $0 < a < 1$.

(i) **$f(ax)$:** Draw the graph of $f(x)$ entirely.

(i) Compress the graph towards y-axis if $a > 1$.

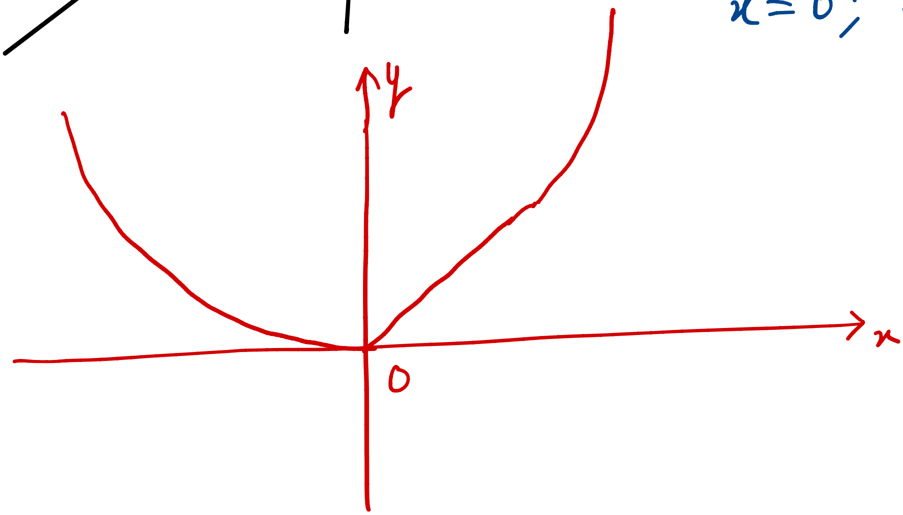
(ii) Stretch the graph away from y-axis if $0 < a < 1$.

Q $f(x) = \max\{2x, x^2\}$.



$x^2 = 2x$

$x = 0$; $x = 2$

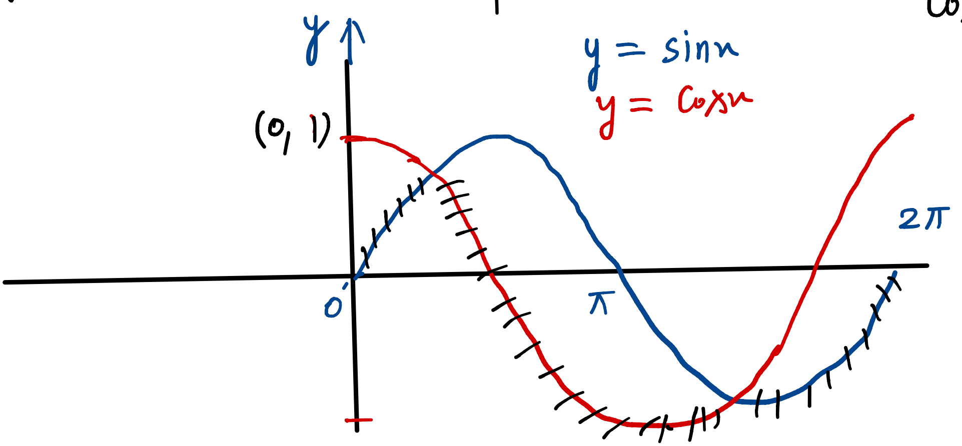


Note:- ☆☆

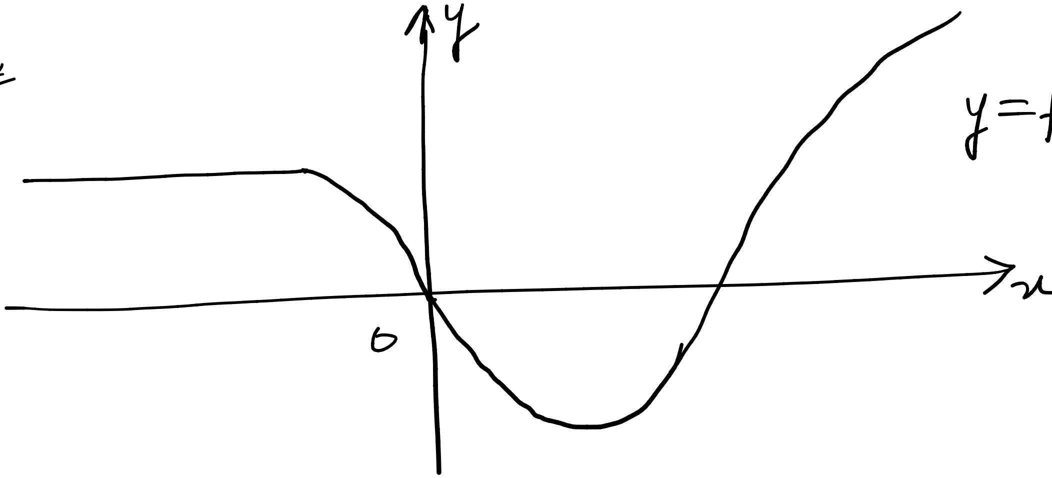
① $f(x) = g(x) + h(x) + |g(x) - h(x)| = 2\max\{g(x), h(x)\}$

② $f(x) = g(x) + h(x) - |g(x) - h(x)| = 2\min\{g(x), h(x)\}$

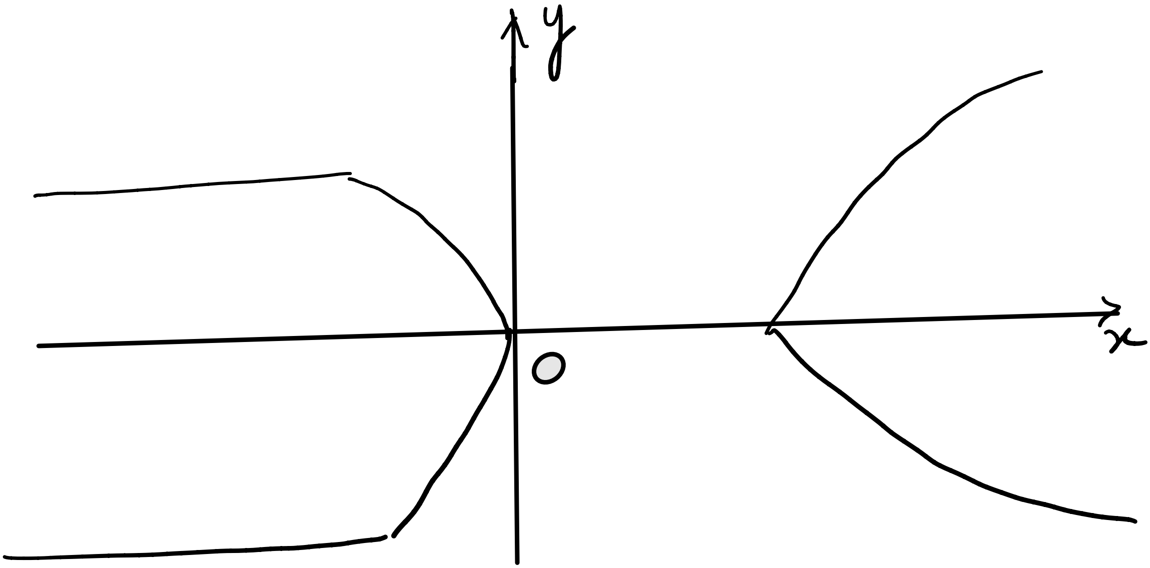
eg: $f(x) = \sin x + \cos x - |\sin x - \cos x| = 2\min\{\sin x, \cos x\}$



Q



$|y| = f(x)$.



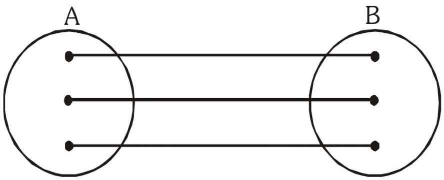
CLASSIFICATION OF FUNCTIONS :

One-One Function (Injective mapping) :

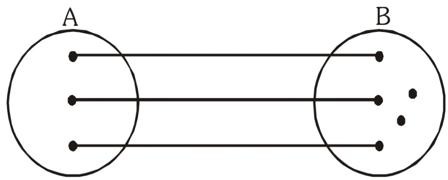
A function $f : A \to B$ is said to be a one-one function or injective mapping if different elements of A have different f images in B.

Thus there exist $x_1, x_2 \in A$ & $f(x_1), f(x_2) \in B$, $\boldsymbol{f(x_1) = f(x_2) \Leftrightarrow x_1 = x_2}$ **or** $\boldsymbol{x_1 \neq x_2 \Leftrightarrow f(x_1) \neq f(x_2)}$.

Diagramatically an injective mapping can be shown as



or

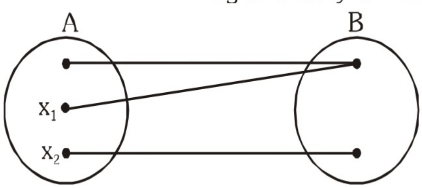


Many-one function (not injective) :

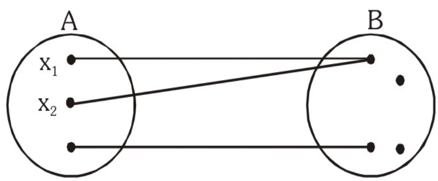
A function $f : A \to B$ is said to be a many one function if two or more elements of A have the same f image in B.

Thus $f : A \to B$ is many one there exist $x_1, x_2 \in A$, $\boldsymbol{f(x_1) = f(x_2)}$ **but** $\boldsymbol{x_1 \neq x_2}$.

Diagramatically a many one mapping can be shown as



or



Note :

(i) If a line parallel to x-axis cuts the graph of the function atmost at one point, then the function is one-one.

(ii) If any line parallel to x-axis cuts the graph of the function atleast at two points, then f is many-one.

(iii) If continuous function $f(x)$ is always increasing or decreasing in whole domain, then $f(x)$ is one-one.

(iv) All linear functions are one-one.

(v) All trigonometric functions in their domain are many one

(vi) All even degree polynomials are many one

(vii) Linear by Linear with no common factor is one-one

$f : A \to B$

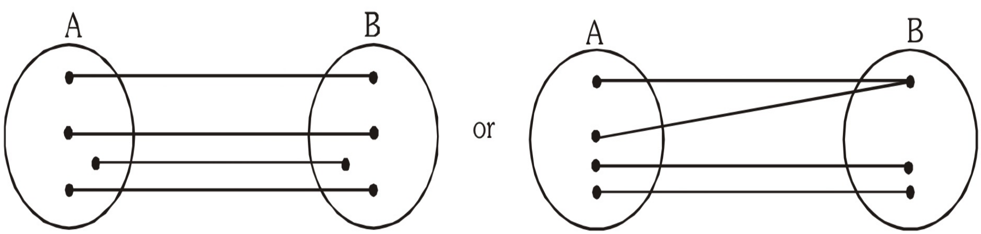
If a function is one-one then it cannot be many-one and vice-versa

No. of one-one functions + No. of many-one functions = Total no. of mappings

Onto function (Surjective mapping) :

If the function $f: A \to B$ is such that each element in B (co-domain) is the f image of atleast one element in A, then we say that f is a function of A 'onto' B. Thus $f: A \to B$ is surjective iff $\forall\ b \in B$, $\exists$ some $a \in A$ such that $f(a) = b$.

Diagramatically surjective mapping can be shown as

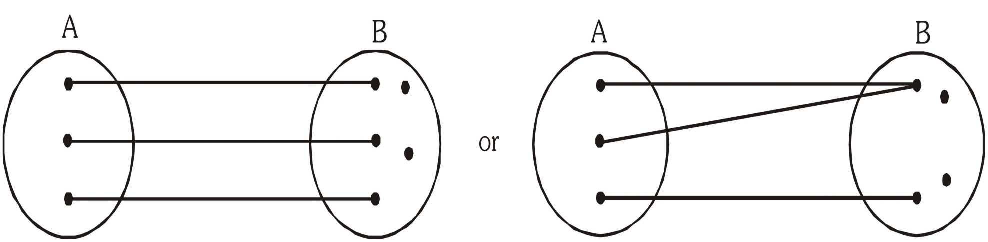


Note that : if range is same as co-domain, then $f(x)$ is onto.

Into function :

If $f: A \to B$ is such that there exists atleast one element in co-domain which is not the image of any element in domain, then $f(x)$ is into.

Diagramatically into function can be shown as



Note :

(i) A polynomial of degree even define from $R \to R$ will always be into

(ii) A polynomial of degree odd defined from $R \to R$ will always be onto.

$f : A \to B$

If a function is onto then it cannot be into and vice-versa.

No. of onto functions + No. of into functions = Total no. of mappings

Any function can be classified in following four ways :

(a) one-one onto (injective & surjective) also known as invertible fns. or Non-singular fns or Bi-uniform fns

(also known as **Bijective** mapping) $(I \cap S)$

(b) one-one into (injective but not surjective) $(I \cap \bar{S})$

(c) many-one onto (surjective but not injective) $(\bar{I} \cap S)$

(d) many-one into (neither surjective nor injective) $(\bar{I} \cap \bar{S})$

Classify the following functions :-

$f : R \to R$

① $f(x) = e^x + e^{-x}$

② $f(x) = |x| \operatorname{Sgn} x$

③ $f(x) = \dfrac{2x^2 - x + 5}{7x^2 + 2x + 10}$

① $f: R \to R$

$f(x) = e^{x} + e^{-x}$

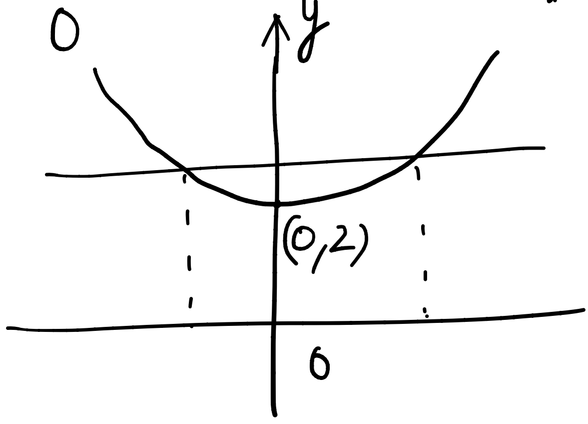
$R_f \in [2, \infty)$

$f'(x) = e^{x} - e^{-x}.$

$= \left(\dfrac{e^{2x} - 1}{e^{x}}\right)$

$\downarrow$ $-$ 0 $+$ $\uparrow$

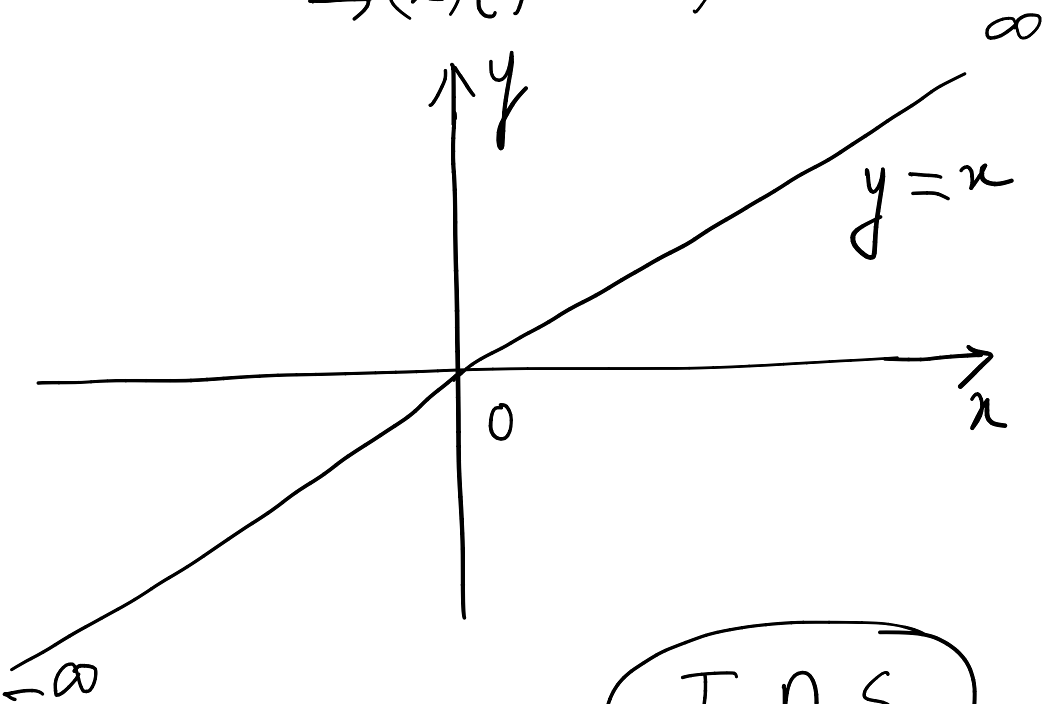
I ∩ S



(2) $f: \underline{\underline{R}} \to \underline{\underline{R}}$

$f(x) = |x| \operatorname{sgn} x$

$$f(x) \begin{cases} (-x)(-1) = x & ; x < 0 \\ 0 & ; x = 0 \\ (x)(1) = x & ; x > 0. \end{cases}$$



INS

Bijection

③ $f: R \to R$

$$f(x) = \frac{\overbrace{2x^2 - x + 5}^{\text{always +ve}}}{\underbrace{7x^2 + 2x + 10}_{\text{always +ve}}}$$

$\therefore$ $f(x) > 0$

✓

M-1

$f'(x)$ ---- ---- ----

M-2

$$\frac{1}{2} = \frac{2x^2 - x + 5}{7x^2 + 2x + 10}$$

$$7x^2 + 2x + \cancel{10} = 4x^2 - 2x + \cancel{10}$$

$3x^2 + 4x = 0$ INS

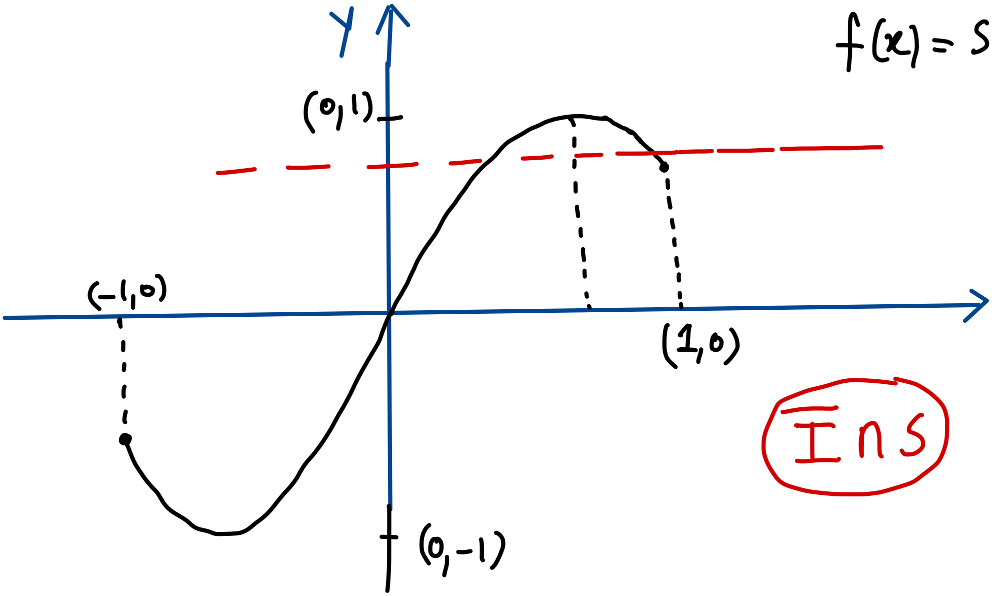
$x = 0$; $x = -\frac{4}{3}$

✓ ✓

④ f: [–1, 1] → [–1, 1]; f(x) = sin2x

$\underbrace{x \in [-1,1]} \Rightarrow 2x \in [-2,2].$

$f(x) = \sin 2x.$

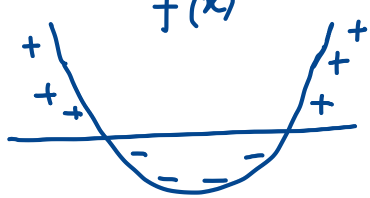


Ins

⑤ $f: R \to R;\ f(x) = x^3 + 3x^2 - 7x + 4$

onto

$f'(x) = \underbrace{3x^2 + 6x - 7}_{D>0}.$



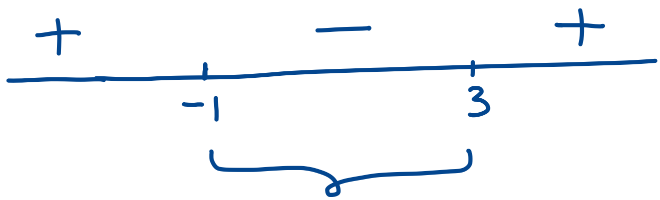
Ins

⑥ $f : [-1, 3] \to [-37, 27]$

$$f(x) = 2x^3 - 6x^2 - 18x + 17.$$

Solⁿ

$$f'(x) = 6x^2 - 12x - 18$$
$$= 6(x^2 - 2x - 3)$$
$$f'(x) = 6(x-3)(x+1)$$



for $x \in (-1, 3)$

$f'(x) < 0. \Rightarrow$ f is one-one fns

(because it is decreasing fns in $(-1, 3)$)

$$G = f(-1) = 27$$

$$L = f(3) = -37.$$

$$R_f \in [-37, 27]$$

Ins

Q

The function $f : [2, \infty) \to Y$ defined by $f(x) = x^2 - 4x + 5$ is both one-one and onto if :

(A) $Y = R$ (B) $Y = [1, \infty)$ (C) $Y = [4, \infty)$ (D) $[5, \infty)$

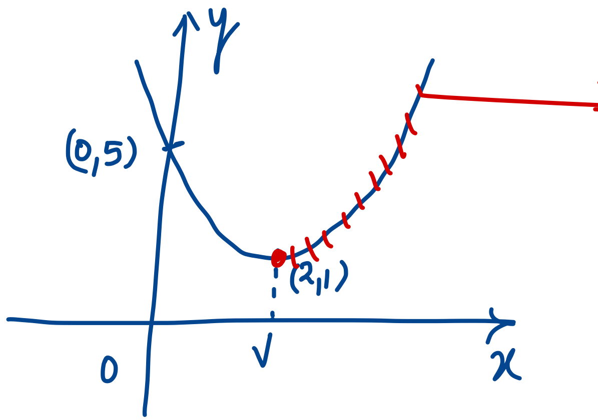
Soln

$$f(x) = x^2 - 4x + 5$$

$$f(x) = \underbrace{(x-2)^2}_{\geq 0} + 1$$

Vertex
$x = 2$
$y = 1$

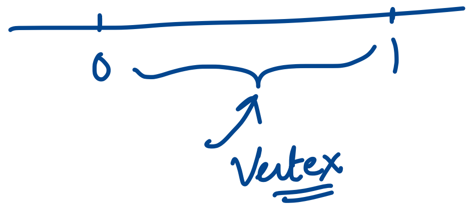
$$R_f \in [1, \infty)$$



Q If the function $f(x) = x^2 + bx + 3$ is not injective for values of x in the interval $0 \le x \le 1$ then b lies in

(A) $(-\infty, \infty)$ (B) $(-2, \infty)$ (C) $(-2, 0)$ (D) $(-\infty, 2)$

Vertex: $x = \frac{-b}{2}$



$$0 < \frac{-b}{2} < 1.$$

$$0 < -b < 2$$

$$\boxed{-2 < b < 0}$$ Ans

Q $f: R \rightarrow R$ is defined as $f(x) = \begin{cases} x^2 + 2mx - 1 & \text{for } x \le 0 \\ mx - 1 & \text{for } x > 0 \end{cases}$.

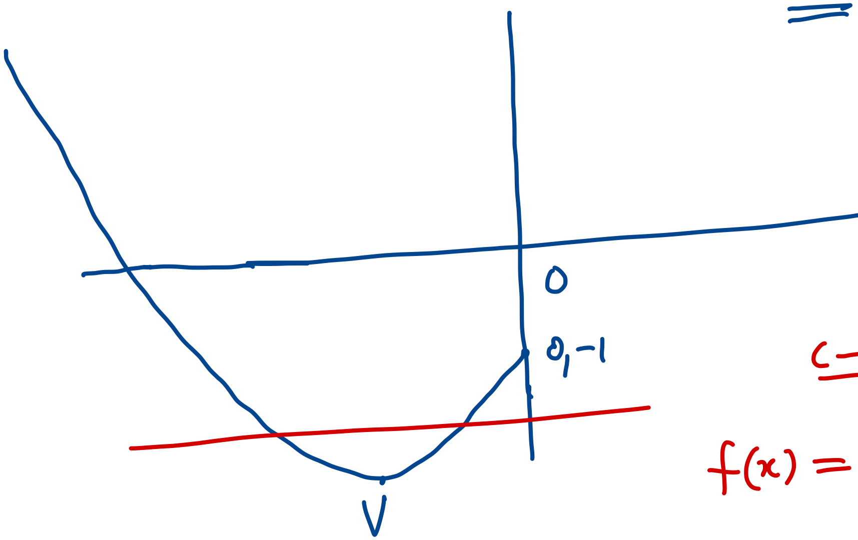
If $f(x)$ is one-one then m must lies in the interval

(A) $(-\infty, 0)$ (B) $(-\infty, 0]$ (C) $(0, \infty)$ (D) $[0, \infty)$

Sol^n

Vertex : $x = \frac{-2m}{2} \Rightarrow \boxed{x = -m}$

C–I $m > 0$ XX



C–II $m = 0$

$f(x) = -1 \,;\, x > 0$

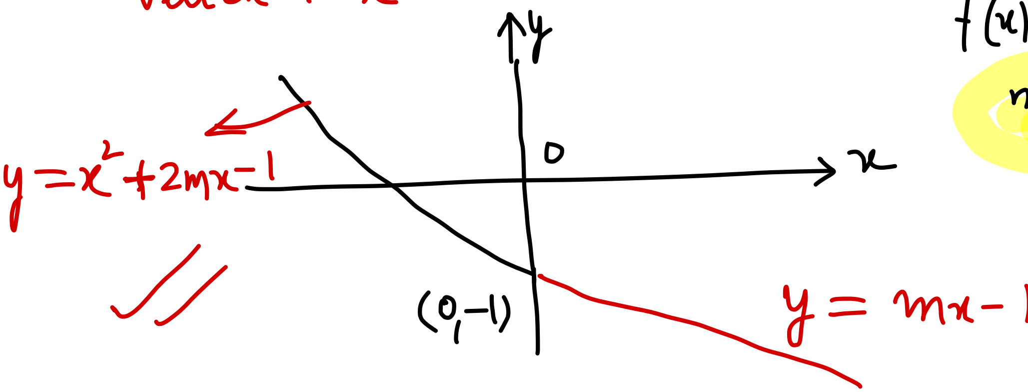
XX

C–III $m < 0$

Vertex : $x = -m \Rightarrow x > 0$

$f(x) = mx - 1$

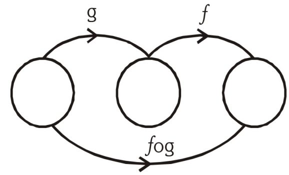
$m < 0$



COMPOSITE OF UNIFORMLY & NON-UNIFORMLY DEFINED FUNCTIONS :

Let $f : A \to B$ & $g : B \to C$ be two functions.

Then the function $gof : A \to C$ defined by $(gof)(x) = g(f(x))\ \forall\ x \in A$ is called the composite of the two functions f & g.



Diagrammatically $\xrightarrow{x} \boxed{f} \xrightarrow{f(x)} \boxed{g} \longrightarrow g(f(x))$.

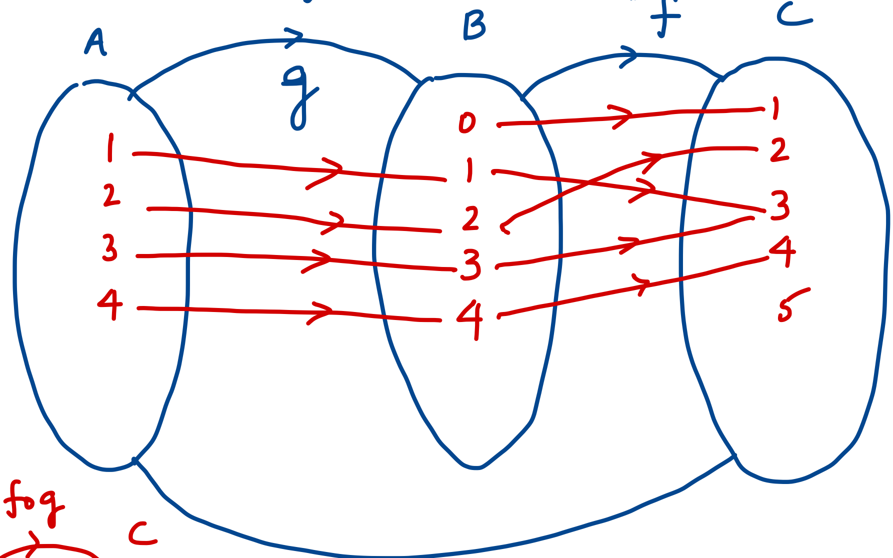
Note :

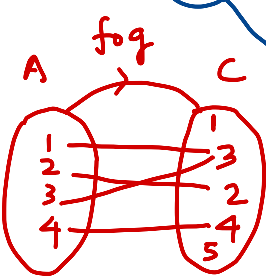
(i) The image of every $x \in A$ under the function gof is the g-image of the f-image of x.

(ii) Domain of gof $= D_{f(x)} \cap \{x : f(x) \in D_{g(x)}\}$

(iii) In general $fog \neq gof$.

* For $fog(x)$ to exist, the range of $g(x)$ must be subset of domain of $f(x)$.





$fog : A \to C$

$fog(1) = 3.$

$fog(2) = 2$

$fog(3) = 3$

$fog(4) = 4$

Q If $f(x) = x^2$ and $g(x) = x - 7$ find $g \circ f$ and $f \circ g$, $f \circ f$, $g \circ g$.

① $f \circ g(x) = f(g(x))$

$= f(x-7)$

$f \circ g(x) = (x-7)^2$; $D_{f \circ g} \in R$

② $g \circ f(x) = g(x^2) = x^2 - 7$. $D_{g \circ f} \in R$

$$\boxed{f \circ g(x) \neq g \circ f(x)}$$

③ $f \circ f(x) = f(f(x)) = f(x^2) = (x^2)^2 = x^4$

④ $g \circ g(x) = g(g(x)) = g(x-7) = (x-7) - 7$

$= (x-14)$.

Q $f : R \to R$ be the function defined by $f(x) = ax^2 - \sqrt{2}$ for some positive a.
If $(fof)(\sqrt{2}) = -\sqrt{2}$ then the value of 'a' is

(A) $\sqrt{2}$ (B) 2 (C) $\frac{1}{2}$ (D*) $\frac{1}{\sqrt{2}}$

Soln

$$f(f(\sqrt{2})) = -\sqrt{2}$$

$$f(\sqrt{2}) = 2a - \sqrt{2}$$

$$f(2a-\sqrt{2}) = -\sqrt{2}$$

$$a(2a-\sqrt{2})^2 - \cancel{\sqrt{2}} = -\cancel{\sqrt{2}}$$

$$a = 0 \quad \text{or} \quad a = \frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}}$$

✗

Q Let $f(x) = \sqrt{x}$; $g(x) = \sqrt{2-x}$, find the domain of ① fog ② gog.

① $fog(x) = f(g(x))$

$$= f(\sqrt{2-x})$$

$$= \sqrt{\sqrt{2-x}}$$

$$2-x \geq 0.$$

$$x-2 \leq 0$$

$$x \leq 2$$

$$D_{fog} : (-\infty, 2]$$

② $gog(x) = g(g(x)) = g(\sqrt{2-x})$

$$g \circ g(x) = \sqrt{2 - \sqrt{2 - x}}$$

$=$

$2 - x \geq 0$ and $\underbrace{\sqrt{2 - \sqrt{2 - x}}} \geq 0$.

$x \leq 2$ — 1 — and

$2 - \sqrt{2 - x} \geq 0$

$2 \geq \sqrt{2 - x}$

$4 \geq 2 - x$

$\boxed{x \geq -2}$ — 2 —

1 $\cap$ 2

$x \in [-2, 2]$ Ans

Q If $f(x) = \log_{100x}\left(\frac{2\log_{10} x + 2}{-x}\right)$ and $g(x) = \{x\}$. If the function (fog)(x) exists then find the range of g(x). (Note : $\{\ \} \rightarrow$ ---------)

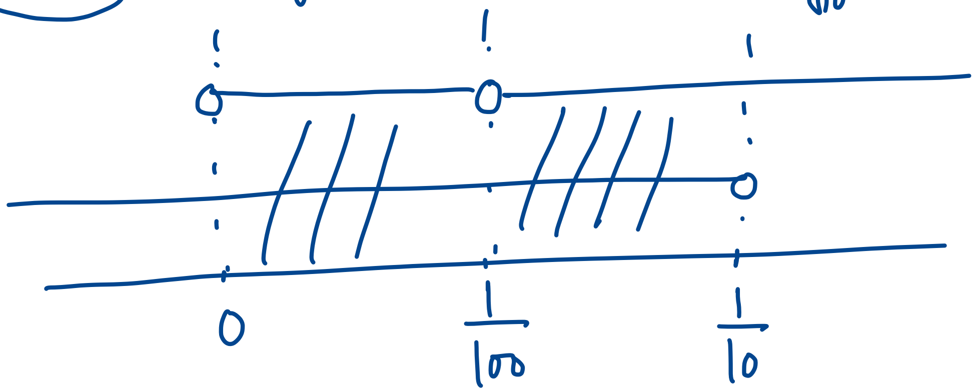
Sol^n Range of ~~$\{x\} \in [0, 1)$~~

For fog(x) to exist, range of g(x) must be subset of domain of f(x).

Domain of f(x):

$x > 0$ & $100x \neq 1$ & $\frac{2\log_{10} x + 2}{-x} > 0$

$\boxed{x < \frac{1}{10}} \Leftarrow \log_{10} x + 1 < 0 \Leftarrow 2\log_{10} x + 2 < 0$



$$D_f \in \left(0, \frac{1}{100}\right) \cup \left(\frac{1}{100}, \frac{1}{10}\right) \quad — 2 —$$

For fog(x) to exist, range of g(x)

$$\left(0, \frac{1}{100}\right) \cup \left(\frac{1}{100}, \frac{1}{10}\right)$$

Q Let $f_1(x) = x$, $f_2(x) = 1 - x$; $f_3(x) = \frac{1}{x}$, $f_4(x) = \frac{1}{1-x}$; $f_5(x) = \frac{x}{x-1}$; $f_6(x) = \frac{x-1}{x}$

Suppose that $f_6(f_m(x)) = f_4(x)$ and $f_n(f_4(x)) = f_3(x)$ then find the value of m & n.

Sol^n

$$f_6(\underbrace{f_m(x)}) = f_4(x)$$

$$\frac{f_m(x) - 1}{f_m(x)} = \frac{1}{1-x} \Rightarrow \boxed{f_m(x) = \frac{x-1}{x}}$$

$$\boxed{m = 6}$$

$$f_n(\underbrace{f_4(x)}) = f_3(x)$$

$$f_n\left(\frac{1}{1-x}\right) = \frac{1}{x}$$

Let $\frac{1}{1-x} = t$

$$\frac{1}{t} = 1 - x$$

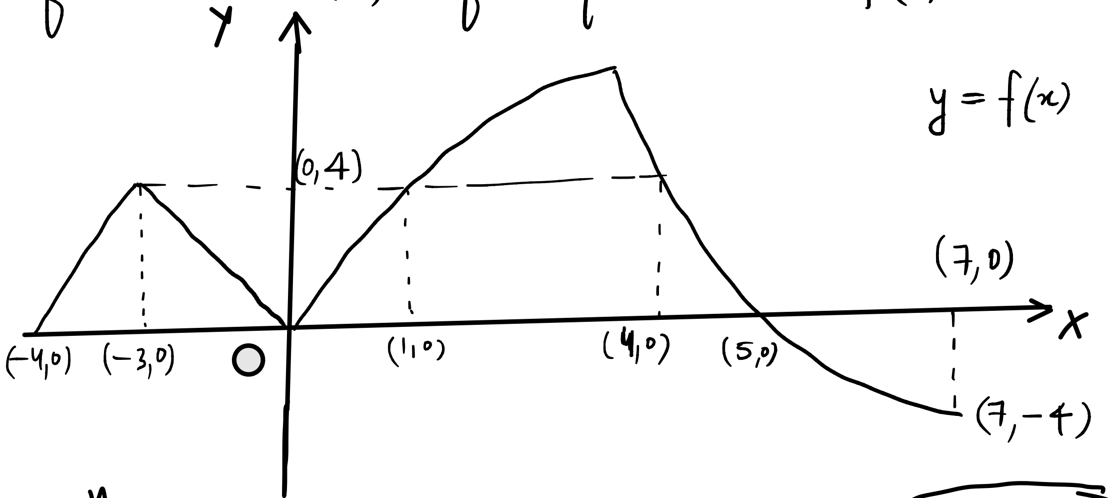
$$x = 1 - \frac{1}{t}$$

$$x = \frac{t-1}{t}$$

$$\boxed{f_n(t) = \frac{t}{t-1}} \Rightarrow n = 5$$

$$\left.\begin{aligned} f(t) &= \sin t \\ f(x) &= \sin x \end{aligned}\right\}$$

Q Given graph of $y = f(x)$ then find no. of solution(s) of equation $f(f(x)) = 4$?



Sol^n

$$f(\underbrace{f(x)}_{t}) = 4. \quad ; \quad t = f(x)$$

$$f(t) = 4 \begin{cases} t = -3 \\ t = 1 \\ t = 4. \end{cases}$$

$f(x) = -3 \longrightarrow 1$ sol^n

$f(x) = 1 \longrightarrow 4$ sol^n

$f(x) = 4. \longrightarrow 3$ sol^n

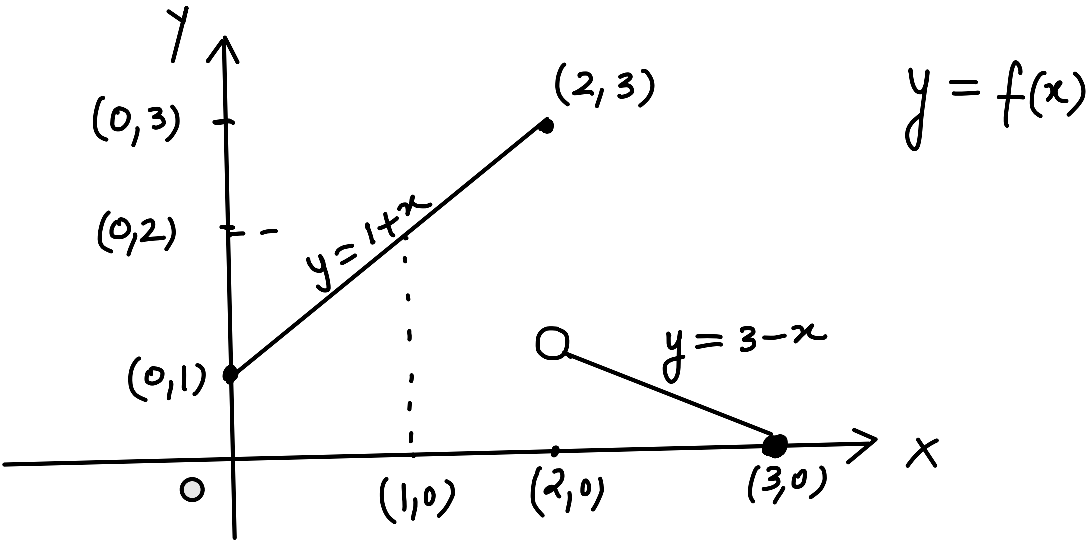
8 sol^n

Examples on composite of nonuniformly defined function :

E(1) $f(x) = \begin{cases} 1+x & \text{if} \quad 0 \le x \le 2 \\ 3-x & \text{if} \quad 2 < x \le 3 \end{cases}$ find $f o f$

Sol^n

$$fof(x) = f(f(x)) \begin{cases} 1+f(x) \ ; & 0 \le f(x) \le 2 \\ 3-f(x) \ ; & 2 < f(x) \le 3. \end{cases}$$

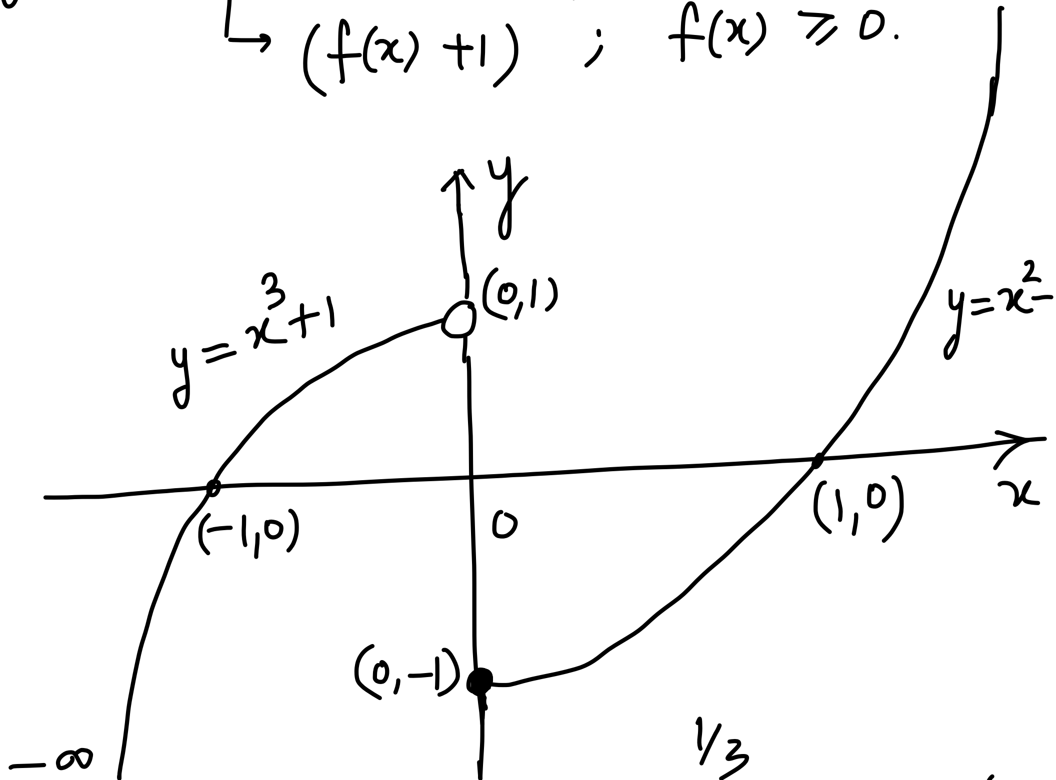


$$f(f(x)) \begin{cases} 1+(1+x) \ ; & x \in [0,1]. \\ 1+(3-x) \ ; & x \in (2,3]. \\ 3-(1+x) \ ; & x \in (1,2]. \end{cases}$$

$$f(f(x)) \rightarrow \begin{cases} 2+x ; & x \in [0,1] \\ 2-x ; & x \in (1,2] \\ 4-x ; & x \in (2,3] \end{cases}$$

② $f(x) = \begin{cases} 1+x^3 & x<0 \\ x^2-1 & x\ge 0 \end{cases}$ and $g(x) = \begin{cases} (x-1)^{1/3} & x<0 \\ (x+1)^{1/2} & x \ge 0 \end{cases}$ find $g(f(x))$

$$g(f(x)) \begin{cases} \to (f(x)-1)^{1/3} \; ; \; f(x) < 0 \\ \to (f(x)+1)^{1/2} \; ; \; f(x) \geq 0. \end{cases}$$



$$g(f(x)) \begin{cases} \to (1+x^3-1)^{1/3} \; ; \; x \in (-\infty, -1) \\ \to (x^2-1-1)^{1/3} \; ; \; x \in [0, 1) \\ \to (x^3+1+1)^{1/2} \; ; \; x \in [-1, 0) \\ \to (x^2-1+1)^{1/2} \; ; \; x \in [1, \infty) \end{cases}$$

(3) Let $f(x) = \begin{cases} x, & x \in Q \\ 1-x, & x \notin Q \end{cases}$ find $fof(x)$

$Q \to$ set of rational nos.

Sol^n

$$fof(x) \begin{cases} \to f(x) \;;\; f(x) \in Q \\ \to 1-f(x) \;;\; f(x) \notin Q \end{cases}$$

$$fof(x) \begin{cases} \to x \;;\; x \in Q \\ \to \cancel{1}-(\cancel{1}-x) \;;\; x \notin Q. \end{cases}$$

$$\boxed{fof(x) = x}$$

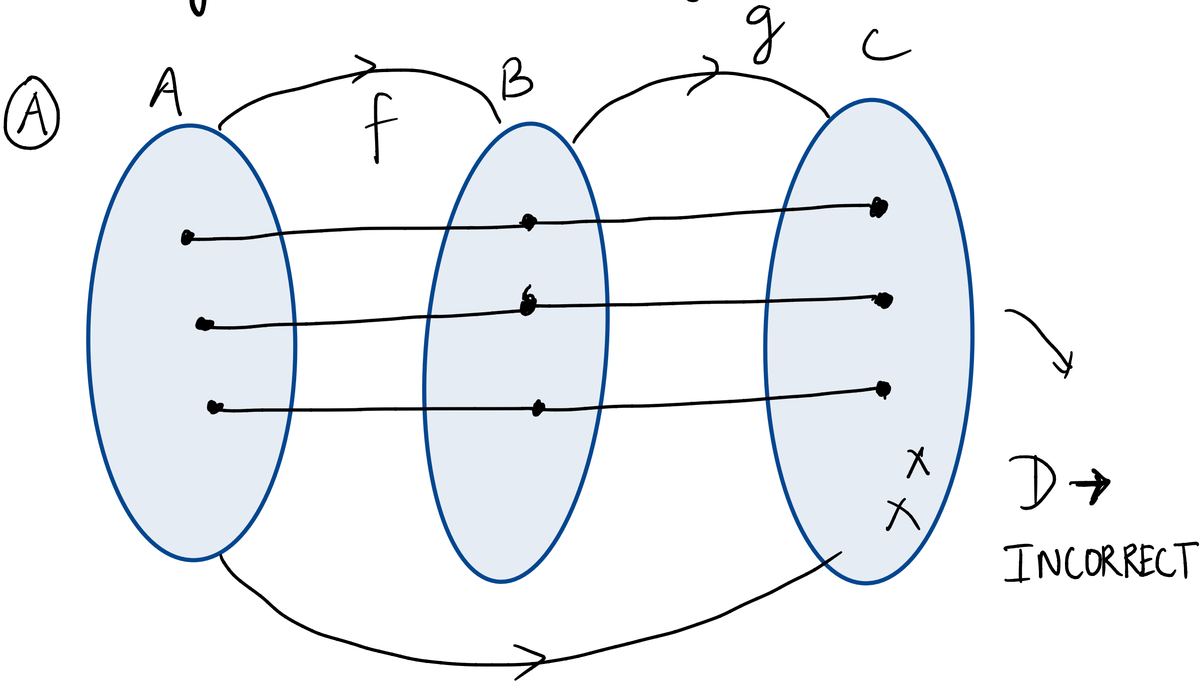
Q Let $f: A \to B$ and $g: B \to C$ be two functions and $gof: A \to C$ is defined. Then which of the following statement(s) is/are INCORRECT?

(A) If gof is into then f must be into.

(B) If f is into and g is onto then gof must be onto function

(C) If gof is one-one then g must be one-one.

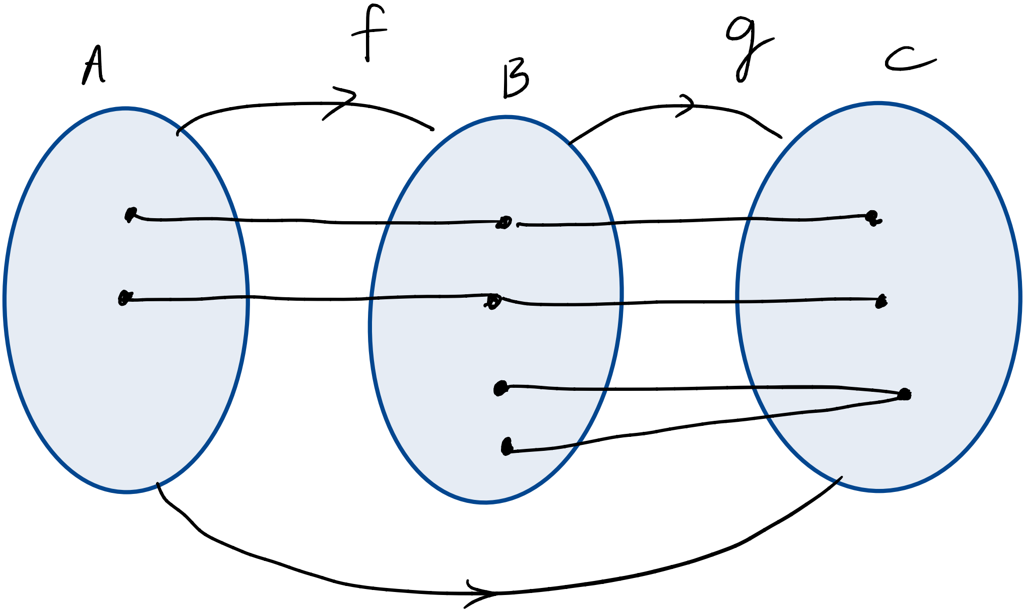
(D) If f is surjective and g is injective then $g \circ f$ must be surjective.



$g \circ f : A \to C$

$\boxed{\text{A is INCORRECT}}$

(B)



$g \circ f : A \to C$

B → INCORRECT

C → INCORRECT

Properties of Composite Functions :

(i) The composite of functions is not commutative i.e. $gof \neq fog$.

(ii) The composite of functions is associative i.e. if f, g, h are three functions such that $fo(goh)$ & $(fog)oh$, are defined, then $fo(goh) = (fog)oh$.

(iii) The composite of two bijections is a bijection i.e. if f & g are two bijections such that gof is defined, then gof is also a bijection.

BOUNDED FUNCTION :

A function is said to be bounded if there exists a finite M such that $|f(x)| \le M, \forall x \in D_f$.

Example :

E(1) Which of the following function(s) is (are) bounded on the intervals as indicated

✓ (a) $f(x) = 2^{\frac{1}{x-1}}$ on $(0, 1)$

✗ (b) $g(x) = \sec^2 x$ on $(0, 2\pi)$

✗ (c) $h(x) = \tan x$ on $\left(\frac{\pi}{3}, \frac{5\pi}{3}\right)$

✓ (d) $K(x) = \tan x \; ; \; \left(0, \frac{\pi}{4}\right)$ $\quad R_K \in (0,1)$

Sol^n

(a) $f(x) = 2^{\frac{1}{x-1}} \; ; \; x \in (0,1)$

$x-1 \in (-1, 0)$

$\frac{1}{x-1} \in (-\infty, -1)$

$2^{\frac{1}{x-1}} \in \left(2^{-\infty}, 2^{-1}\right)$

$R_f \in \left(0, \frac{1}{2}\right)$ $\quad$ Bounded

IMPLICIT & EXPLICIT FORM OF FUNCTION :

A function defined by an equation not solved for the dependent variable is called an **Implicit Function**. For e.g. the equation $x^3 + y^3 = 1$ defines y as an implicit function. If y has been expressed in terms of x alone then it is called an **Explicit Function**.

$$y = f(x)$$

$x \rightarrow$ independent variable

$y \rightarrow$ dependent variable.

eg:

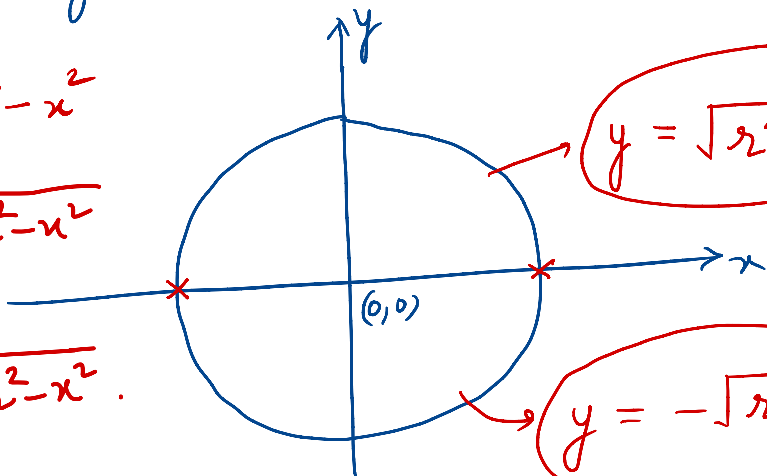
$$x^2 + y^2 = r^2. \longrightarrow \text{Implicit form of function}$$

$$y^2 = r^2 - x^2$$

$$y = \sqrt{r^2 - x^2}$$

or

$$y = -\sqrt{r^2 - x^2}.$$



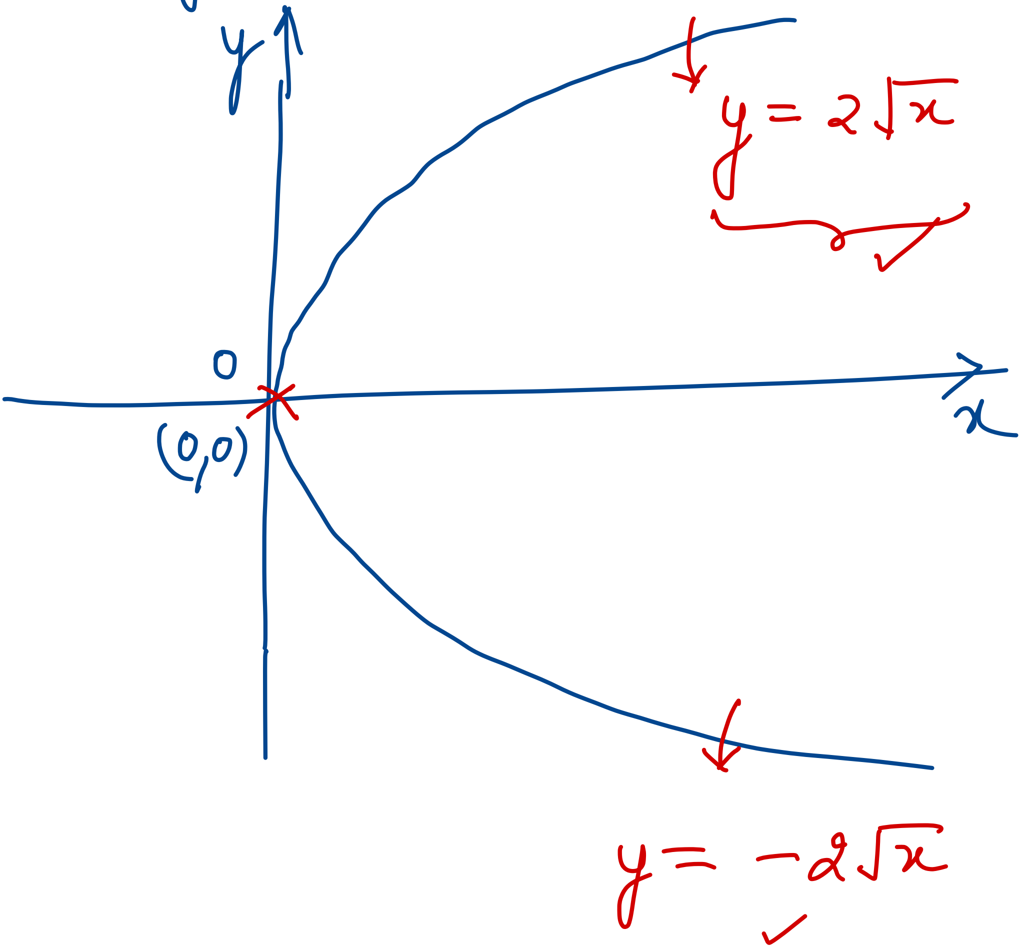
$$y = \sqrt{r^2 - x^2}$$

Explicit

$$y = -\sqrt{r^2 - x^2}$$

eg:

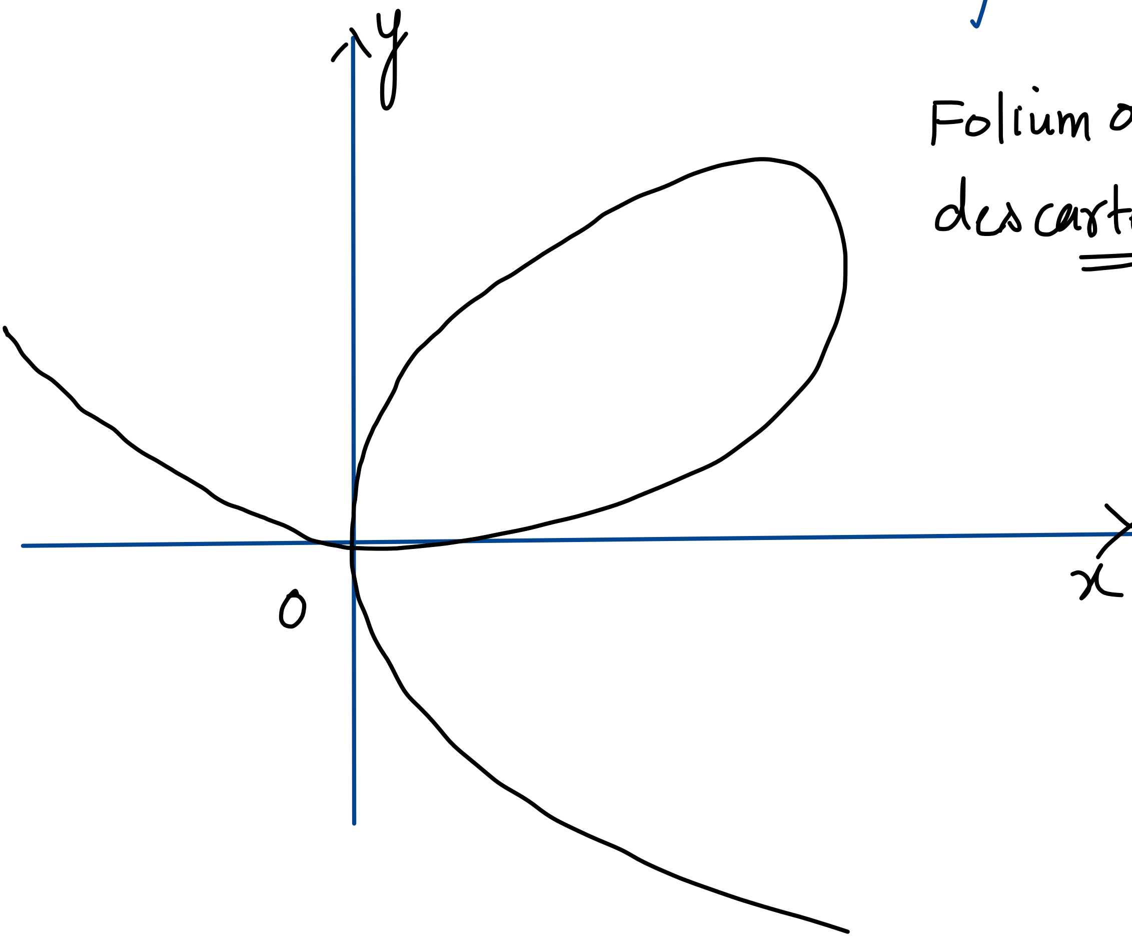
$y^2 = 4x \longrightarrow$ Implicit



$y^2 = 4x$

$$\left.\begin{aligned} y &= 2\sqrt{x} \\ y &= -2\sqrt{x} \end{aligned}\right\} \text{explicit}$$

eg: $x^3 + y^3 = 3xy$ $\longrightarrow$ Implicit form of fns.



Folium of descartes

eg:

$$x\sqrt{1+y} + y\sqrt{1+x} = 0.$$

Implicit form

$y+1 \geq 0$ & $1+x \geq 0$

$$x\sqrt{1+y} = -y\sqrt{1+x}$$

squaring :-

$$x^2(1+y) = y^2(1+x)$$

$$x^2 - y^2 + x^2y - y^2x = 0.$$

$$(x-y)\underbrace{(x+y+xy)} = 0.$$

$y = x$ OR $y = \dfrac{-x}{1+x}$

$$\boxed{x\sqrt{1+y} + y\sqrt{1+x} = 0}$$

$$x\sqrt{1+x} + x\sqrt{1+x} = 0$$

$$x\sqrt{1+x} = 0$$

$x = 0$ OR $x = -1$

$$x\sqrt{1 - \frac{x}{1+x}} + \frac{(-x)}{1+x}\sqrt{1+x}$$

$$x\sqrt{\frac{1}{1+x}} - \frac{x}{\sqrt{1+x}} = 0$$

$$y = \begin{cases} \left(\dfrac{-x}{1+x}\right) & ; \quad x > -1 \\ -1 & ; \quad x = -1 \end{cases}$$

✓

Q The domain of definition of the function, y(x) is given by the equation, $2^x + 2^y = 2$ is -

(A) $0 < x \leq 1$ (B) $0 \leq x \leq 1$

(C) $-\infty < x \leq 0$ (D) $-\infty < x < 1$

Sol^n

$$2^y = 2 - 2^x$$

$$\log_2(2^y) = \log_2(2 - 2^x)$$

$$\boxed{y = \log_2(2 - 2^x)}$$

$$2 - 2^x > 0 \Rightarrow 2 > 2^x$$

$$2^{x-1} < 1$$

$$2^{x-1} < 2^0$$

$$x - 1 < 0 \Rightarrow x < 1$$

Q

If $e^{-|x|} = K$ then find 'K' for which this equation has :

① No solution. $\longrightarrow K \in (-\infty, 0] \cup (1, \infty)$

② One solution. $\longrightarrow K \in \{1\}$

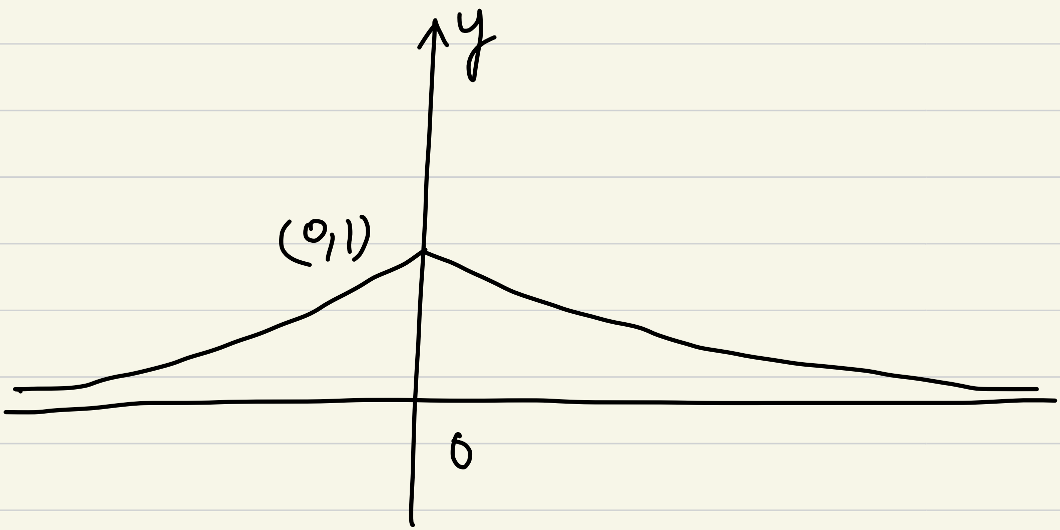
③ Two solution. $\longrightarrow K \in (0,1)$

④ Three solution.

Solⁿ $K \in \phi$

$f(x) = e^{-|x|}$ — $\begin{cases} e^{x} & ; x<0 \\ e^{-x} & ; x \geq 0 \end{cases}$

$\boxed{g(x) = K}$



Q. A bijective function $f(x)$ is defined as :

$f(x) = \dfrac{2x-\alpha}{4-(2\alpha^2-3\alpha)x}$, then find the value of α if function is defined as

(a) $f : R - \left\{\dfrac{4}{2\alpha^2-3\alpha}\right\} \to \underbrace{R - \{2\}}$

(b) $f : R \to R$

Sol^n $f(x) = \dfrac{2x-\alpha}{\underbrace{-(2\alpha^2-3\alpha)x+4}}$

(a)

$$R - \left\{\dfrac{2}{\underbrace{-(2\alpha^2-3\alpha)}}\right\}$$

$$\dfrac{\cancel{2}}{-(2\alpha^2-3\alpha)} = \cancel{2} \Rightarrow 1 = -2\alpha^2+3$$

(b) $2\alpha^2-3\alpha = 0$

$$f(x) = \left(\dfrac{2x-\alpha}{4}\right)$$

$\alpha = 0$

$\alpha = \dfrac{3}{2}$

Q

If functions f (x) and g (x) are defined on $R \to R$ such that

$$f(x)=\begin{cases}0, & x \in \text{rational}\\ x, & x \in \text{irrational}\end{cases},\quad g(x)=\begin{cases}0, & x \in \text{irrational}\\ x, & x \in \text{rational}\end{cases}, \text{ then } (f-g)(x) \text{ is -}$$

(A) one-one and onto ✓

(B) neither one-one nor onto

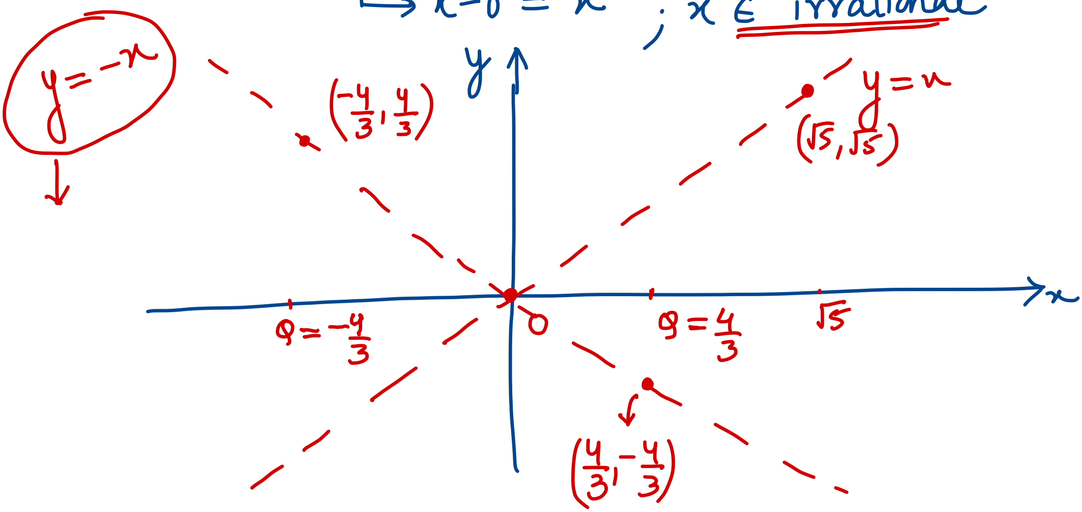
(C) one-one but not onto

(D) onto but not one-one

Sol^n

$(f-g)(x)$ → $0 - x = -x$; $x \in$ rational

→ $x - 0 = x$; $x \in$ irrational



Q

If functions $f(x)$ and $g(x)$ are defined on $R \to R$ such that $f(x) = \begin{cases} x+3 & , \quad x \in \text{rational} \\ 4x & , \quad x \in \text{irrational} \end{cases}$,

$g(x) = \begin{cases} x+\sqrt{5} & , \quad x \in \text{irrational} \\ -x & , \quad x \in \text{rational} \end{cases}$ then $(f-g)(x)$ is -

(A) one-one & onto
(B) neither one-one nor onto
(C) one-one but not onto
(D) onto but not one-one

Sol^n

$$h(x) = (f-g)(x) \begin{cases} \rightarrow (2x+3) & ; x \in \text{rational} \\ \rightarrow (3x-\sqrt{5}) & ; x \in \text{irrational.} \end{cases}$$

$$\underbrace{2x+3}=0 \Rightarrow x = -\frac{3}{2}. \in \text{rational}$$

$$3x-\sqrt{5}=0 \Rightarrow x = \frac{\sqrt{5}}{3} \in \text{irrational.}$$

Many-one

$h(x) = 3x - \sqrt{5}$; $x \in$ irrational.

$$3x - \cancel{\sqrt{5}} = -\cancel{\sqrt{5}} \Rightarrow x = 0$$ which is rational.

ODD & EVEN FUNCTIONS :

Consider a function $f(x)$ such that both x and –x are in its domain then

If $\begin{cases} f(-x) = f(x) & \text{then } f \text{ is said to be an even function} \Rightarrow \\ f(-x) = -f(x) & \text{then } f \text{ is said to be an odd function} \Rightarrow \end{cases}$

$f(x) - f(-x) = 0$

$f(x) + f(-x) = 0$

Note :

(i) $f(x) - f(-x) = 0 \Rightarrow f(x)$ is even & $f(x) + f(-x) = 0 \Rightarrow f(x)$ is odd.

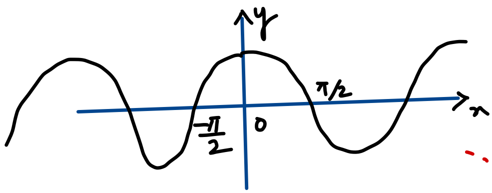
(ii) A function may neither be odd nor even.

(iii) Every non zero constant function is even function, whenever $f(x) = 0$ is even and odd at the same time

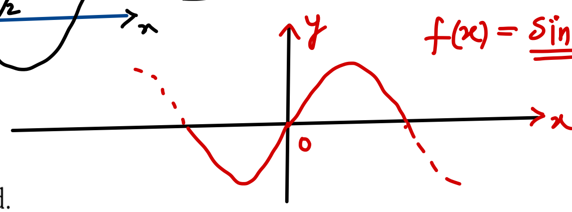
$\boxed{f(x) = 2} \longrightarrow$ Even function

$D_f \in R$

$\boxed{f(x) = \cos x}$



$f(x) = \sin x$



(iv) Inverse of an even function is not defined.

(v) Every even function is symmetric about the y-axis & every odd function is symmetric about the origin.

(vi) Every function which has '–x' in it's domain whenever 'x' is in it's domain, can be expressed as the sum of an even & an odd function .

i.e. $f(x) = \underbrace{\dfrac{f(x)+f(-x)}{2}}_{\text{EVEN}} + \underbrace{\dfrac{f(x)-f(-x)}{2}}_{\text{ODD}}$

EVEN $\downarrow$ $g(x)$ ODD $\downarrow$ $h(x)$

$$g(-x) = \frac{f(-x)+f(x)}{2} = g(x)$$

eg: $f(x) = e^x$

$D_f \in R$

$$f(x) = \underbrace{\left(\frac{e^x + e^{-x}}{2}\right)}_{E} + \underbrace{\left(\frac{e^x - e^{-x}}{2}\right)}_{O}$$

(vii) If $f(x)$ is odd and defined at x = 0, then $f(0) = 0$.

$f(x) + f(-x) = 0$

put $x = 0$

$f(0) + f(0) = 0 \Rightarrow \boxed{f(0) = 0}$

(viii)

$f(x)$	$g(x)$	$f(x) + g(x)$	$f(x) - g(x)$	$f(x) \,.\, g(x)$	$f(x)/g(x)$	$(gof)(x)$	$(fog)(x)$
odd	odd	odd	odd	even	even	odd	odd
even	even	even	even	even	even	even	even
odd	even	neither odd nor even	neither odd nor even	odd	odd	even	even
even	odd	neither odd nor even	neither odd nor even	odd	odd	even	even

1. $f(x) \to$ odd $\qquad f(x) = -f(-x)$

   $g(x) \to$ odd $\qquad g(x) = -g(-x)$.

$h(x) = gof(x) = g(f(x))$

$h(-x) = g(\underbrace{f(-x)}) = g(-\underbrace{f(x)}_{X}) = -g(f(x)) = -h(x)$

$\therefore$ odd fns

Q Check for Odd, Even, $\overline{O} \cap \overline{E}$:

*① $f(x) = \ln\left(x + \sqrt{1+x^2}\right) \rightarrow$ Odd

*② $f(x) = \ln\left(\frac{1+x}{1-x}\right) \rightarrow$ Odd

③ $f(x) = 2x^3 - x + 1$

④ $f(x) = x\left(\frac{2^x+1}{2^x-1}\right)$

$\overbrace{f(x) + f(-x) = 0}^{\text{odd}}$

$\underbrace{f(x) - f(-x) = 0}_{\text{Even}}$

① $f(-x) = \ln\left(-x + \sqrt{1+x^2}\right)$

$f(x) + f(-x) = \ln\left(1 + \cancel{x^2} - \cancel{x^2}\right) = 0.$

③ $f(x) = 2x^3 - x + 1$

$f(-x) = -2x^3 + x + 1.$

$f(x) + f(-x) = 2. \neq 0$

$f(x) - f(-x) = 4x^3 - 2x. \neq 0.$

$\overline{O} \cap \overline{E}$

④ $f(-x) = -x\left(\frac{2^{-x}+1}{2^{-x}-1}\right) = \frac{-x(1+2^x)}{(1-2^x)}$

$= \frac{x(1+2^x)}{2^x-1} = f(x)$ Even

Q

Let $f(x) = ([a]^2 - 5[a] + 4)x^3 - (6\{a\}^2 - 5\{a\} + 1)\,x - (\tan x)\operatorname{sgn} x$, be an even function for all $x \in R$, then sum of all possible values of 'a' is
(where [ ] and { } denote greatest integer function and fractional part functions respectively)

(A) $\frac{17}{6}$ (B) $\frac{53}{6}$ (C) $\frac{31}{3}$ (D) $\frac{35}{3}$ ✓

Sol^n For $f(x)$ to be even.

$[a]^2 - 5[a] + 4 = 0$ and $6\{a\}^2 - 5\{a\} + 1 = 0$

$6\{a\}^2 - 2\{a\} - 3\{a\} + 1 = 0$

$[a] = 1$ or 4

$(2\{a\} - 1)(3\{a\} - 1) = 0$

$\{a\} = \frac{1}{2}$ or $\frac{1}{3}$

$a = [a] + \{a\} = 1 + \frac{1}{2}$ or $1 + \frac{1}{3}$

$= 4 + \frac{1}{2}$ or $4 + \frac{1}{3}$

Sum $= 10 + 1 + \frac{2}{3} = \frac{35}{3}$

Q The smallest natural number k for which $f(x) = \ln\left(x^3 + \sqrt{x^6+1}\right) + \sin 5x + \left[\frac{x^2}{k}\right]$ is an odd function $\forall\ x \in [-2\pi, 2\pi]$, is ([y] denotes largest integer $\le$ y)

(A) 38 (B) 39 (C) 40 (D) 41

$\underbrace{\ln\left(x^3+\sqrt{x^6+1}\right)}_{\text{odd}}$ $\underbrace{\sin 5x}_{\text{odd}}$ $\left[\frac{x^2}{k}\right] \Rightarrow 0.$

Sol^n

$x \in [-2\pi, 2\pi]$

$x^2 \in [0, \underbrace{4\pi^2}_{39.\text{ something}}]$

$\left[\frac{x^2}{k}\right] = 0$ for all $x \in [-2\pi, 2\pi]$

Special Note : If a function $f(x)$ is defined as $f(a+x) = f(a-x)$ then this function is symmetric about line $x = a$

e.g. (a) $f(x) = f(2-x)$ is symmetric about $x = 1$. (b) $f(4+x) = f(6-x)$ is symmetric about $x = 5$.

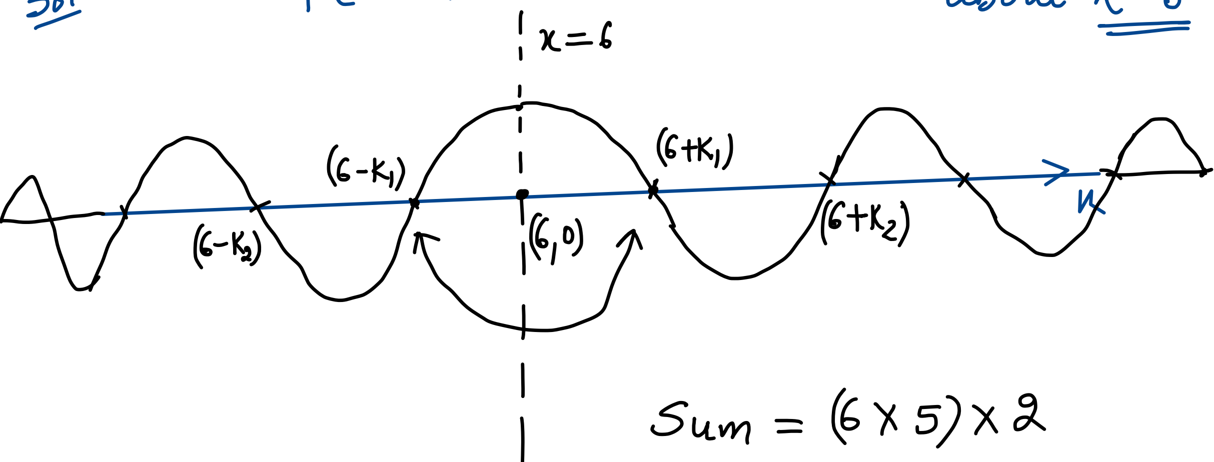
$x \to x+1$

$f(x+1) = f(2-(x+1)) \Rightarrow f(x+1) = f(1-x)$

* If $f(a+x) = f(b-x)$ then $f(x)$ is symmetric about the line $x = \frac{a+b}{2}$.

Q If $y = f(x) = 0$ has 10 distinct real solutions and satisfies the condition $f(4+x) = f(8-x)$ $\forall x \in R$ then find the sum of all the roots.

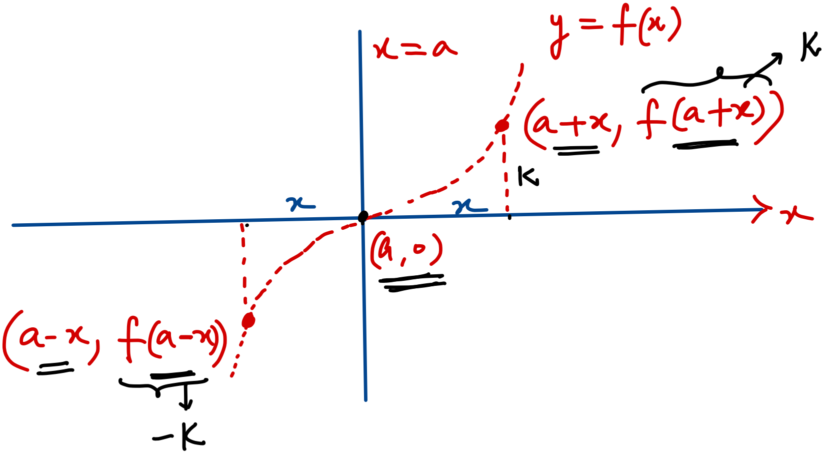
Sol^n $f(4+x) = f(8-x) \Rightarrow f(x)$ is sym. about $x = 6$



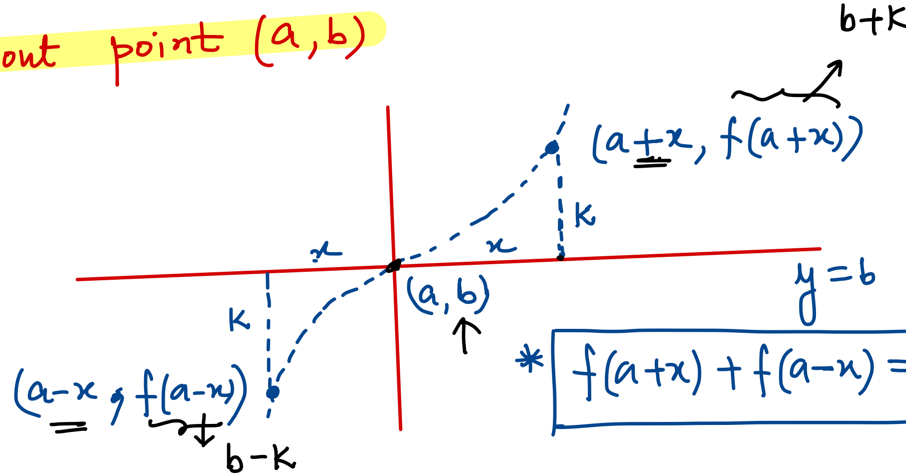
$$\text{Sum} = (6 \times 5) \times 2 = 60.$$

Symmetricity of function about a point :-

If $f(a+x) + f(a-x) = 0$ then function is symmetric about the point $(a, 0)$.



about point (a, b)



$$* \boxed{f(a+x) + f(a-x) = 2b}$$

HOMOGENEOUS FUNCTIONS :

A function is said to be homogeneous with respect to any set of variables when each of its terms is of the same degree with respect to those variables.

For example $5x^2 + 3y^2 - xy$ is homogeneous in x & y. Symbolically if,

$f(tx, ty) = t^n . f(x, y)$ then $f(x, y)$ is homogeneous function of degree n.

1. $$f(x, y) = \frac{x - y\cos x}{y\sin x + x}$$

$x \to tx$, $y \to ty$

$$f(tx, ty) = \frac{tx - ty\cos(tx)}{ty\sin(tx) + tx} = \frac{x - y\cos(tx)}{y\sin(tx) + x}$$

$$\neq t^n f(x, y)$$

Not Homogeneous

2. $f(x,y) =$ $x + y\cos\frac{y}{x}$

$x \to tx$

$y \to ty$

$$f(tx, ty) = tx + ty\cos\left(\frac{\cancel{t}y}{\cancel{t}x}\right)$$

$$= t^1 \cdot f(x, y)$$

Homogeneous of degree $= 1$.

INVERSE OF A FUNCTION :

Let $f : A \to B$ be a one-one & onto function, then their exists a unique function $g : B \to A$ such that $f(x) = y \Leftrightarrow g(y) = x$, $\forall\ x \in A$ & $y \in B$. Then g is said to be inverse of f. Thus $g = f^{-1} : B \to A = \{(f(x), x) \mid (x, f(x)) \in f\}$.

Note :

(i) The inverse of a function exist if & only if it is bijective.

(ii) Domain of f = Range of f^{-1} and Range of f = Domain of f^{-1}

(iii) Inverse of even function does not exist.

(iv) Inverse functions are mirror images of each other in the line $y = x$.

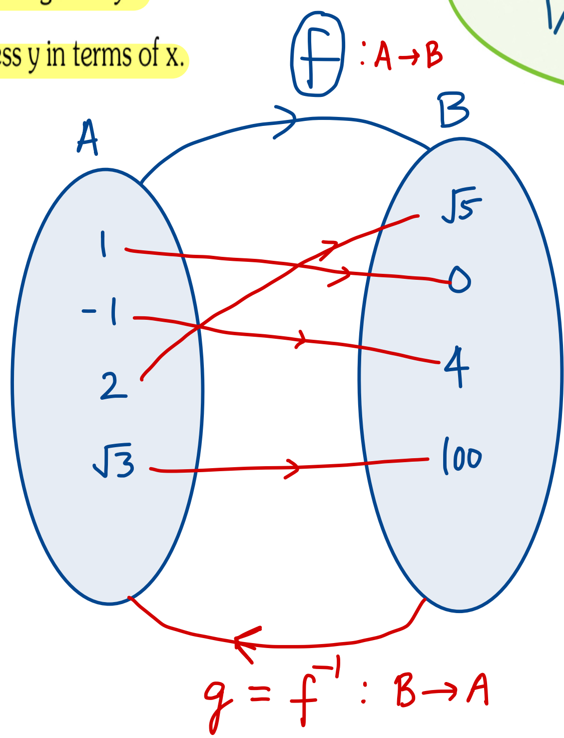
Steps to find inverse :

(i) Write the function as $y = f(x)$.

(ii) Interchange x & y.

(iii) Express y in terms of x.

Inverse of function is Unique



$f(1) = 0$

$f^{-1}(0) = 1.$

$g(0) = 1.$

$f(\sqrt{3}) = 100$

$g(100) = \sqrt{3}$

Q Compute the inverse of the following bijective functions :

E(1) $f(x) = 10^{x+1}$ $\longrightarrow$ $x=0$

$f(0) = 10.$

$$y = 10^{x+1}$$

$$\log_{10} y = (x+1)\log_{10} 10$$

$$\boxed{x = \log_{10} y - 1.}$$

$$\boxed{f^{-1}(x) = \log_{10} x - 1}$$

Ans

$$f^{-1}(10) = \log_{10} 10 - 1$$
$$= 0.$$

E(2) $f(x) = \dfrac{2^x}{1+2^x}$

$$y(1+2^x) = 2^x$$

$$2^x(y-1) = -y.$$

$$2^x = \frac{y}{1-y}$$

$$\log_2 2^x = \log_2\left(\frac{y}{1-y}\right)$$

$$\boxed{f^{-1}(x) = \log_2\left(\frac{x}{1-x}\right)}$$

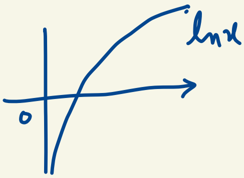
③ $f : [0, \infty) \to [1, \infty)$, $f(x) = \dfrac{e^x + e^{-x}}{2}$ find $f^{-1}(x)$.

$f^{-1} : [1, \infty) \to [0, \infty)$

$$2f(x) = e^x + \frac{1}{e^x}$$

$$2ye^x = e^{2x} + 1. \Rightarrow e^{2x} - e^x(2y) + 1 = 0.$$

$$e^x = \frac{\cancel{2}y \pm \sqrt{\cancel{4}y^2 - \cancel{4}}}{\cancel{2}}$$



$$e^x = y + \sqrt{y^2 - 1} \quad \text{OR} \quad e^x = y - \sqrt{y^2 - 1}.$$

$$\boxed{f^{-1}(x) = \ln\left(x + \sqrt{x^2 - 1}\right)} \quad \text{OR} \quad \boxed{f^{-1}(x) = \ln\left(x - \sqrt{x^2 - 1}\right)}$$

XX

But we know, unique inverse exist.

$$f^{-1}(1) = \ln 1 = 0$$

as $x \to \infty$; $f^{-1}(\infty) \to \infty$

$$f^{-1}(1) = \ln 1 = 0.$$

as $x \to \infty$

$f^{-1}(\infty) \to (-\infty)$

Properties of Inverse functions :-

(i) The inverse of a bijection is unique.

Proof : Let $f : A \to B$ be a bijection. If possible let $g : B \to A$ and $h : B \to A$ be two inverse function of f. Also let $a_1, a_2 \in A$ and $b \in B$ such that $g(b) = a_1$ and $h(b) = a_2$ then

$g(b) = a_1 \quad \Rightarrow \quad f(a_1) = b$

$h(b) = a_2 \quad \Rightarrow \quad f(a_2) = b.$

But since f is one-one, so $f(a_1) = f(a_2) \Rightarrow a_1 = a_2 \Rightarrow g(b) = h(b), \forall\, b \in B$

(ii) Inverse of a bijection is a bijection.

(iii) If f & g are two bijections $f : A \to B$, $g : B \to C$ then the inverse of gof exists and $(gof)^{-1} = f^{-1}og^{-1}$.

In general,

$$(hogof)^{-1} = f^{-1} o g^{-1} o h^{-1}.$$

"Law of Reversal"

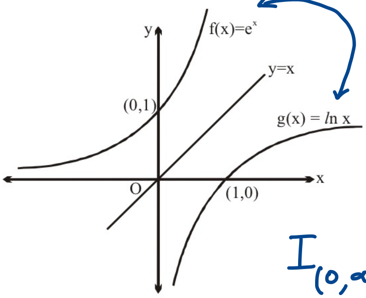
(iv) If $f : A \to B$ is a bijection & $g : B \to A$ is the inverse of f, then $fog = I_B$ and $gof = I_A$, where I_A & I_B are identity functions on the sets A & B respectively.
Note that the graphs of f & g are the mirror images of each other in the line y = x.

$\boxed{(gof)(x) = x \text{ and } (fog)(x) = x}$

gof and fog need not be equal

Note: If f and g are inverse of each other then they are the mirror image of each other in y = x.

mirror images in $y=x$



$f : \mathbb{R} \to (0,\infty)$; $f(x) = e^x$

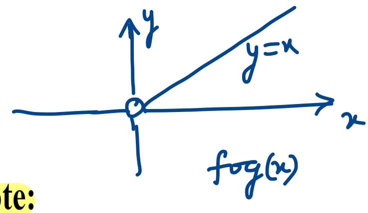
$g : (0,\infty) \to \mathbb{R}$; $g(x) = \ln x$.

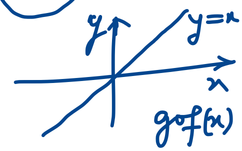
$fog(x) = f(\ln x) = e^{\ln x} = x$

$I_{(0,\infty)}$

$gof(x) = g(e^x) = \ln e^x = x$

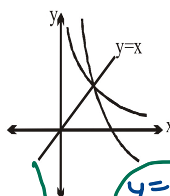
$= I_{\mathbb{R}}$

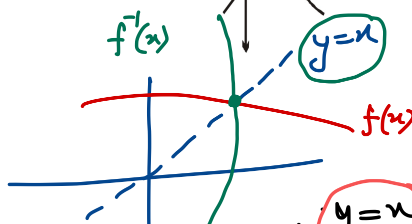




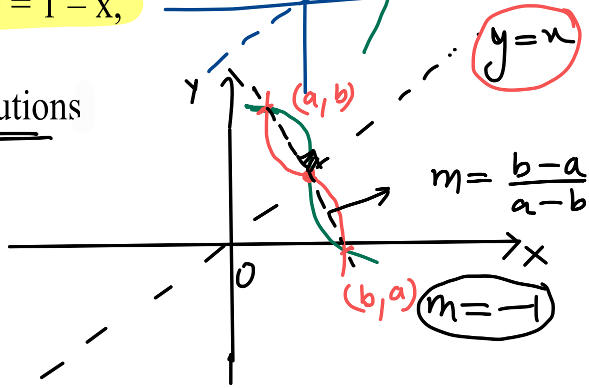
General Note:

$f(x) = f^{-1}(x)$ is either of the form of (α, α)
i.e. (they lie on y = x) or if they are of the form (α, β) (where $\alpha \neq \beta$)
then (β, α) will also be solution of $f(x) = f^{-1}(x)$
i.e. they lie on the line having slope – 1.]





Note that: $f(x) = 1 - x$ and $f^{-1}(x) = 1 - x$,
has the same graph.
$\therefore$ $f(x) = f^{-1}(x)$ has infinite solutions



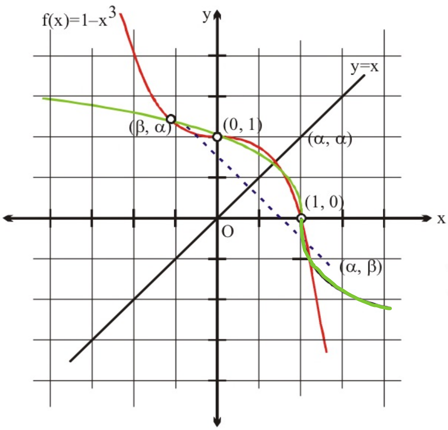
$m = \frac{b-a}{a-b}$

$\boxed{m = -1}$

$f(x) = 1 - x^3$

$g(x) = f^{-1}(x) = (1-x)^{1/3}$

Solution of $f(x) = f^{-1}(x)$ is either of the form (α, α), i.e. on $y = x$ or (α, β) i.e. on the line $(0, 1)$
$m = -1$.

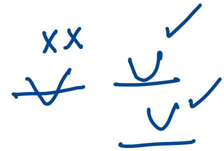


Q If $f : R \to R$, $f(x) = x^3 + (a+2)x^2 + 3ax + 5$ is an invertible mapping find 'a'

Sol^n Since $D_f \in R$ & $f(x)$ is odd-degree poly fun

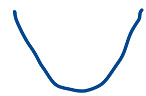
so $R_f \in R$.

For $f(x)$ to be one-one $\Rightarrow f'(x)$



$$f'(x) = 3x^2 + 2(a+2)x + 3a \geq 0$$

(mouth opening upward parabola)



$\therefore \quad D \leq 0.$

$$\cancel{4}(a+2)^2 - \cancel{4}.3.3a \leq 0$$

$$a^2 + 4a + 4 - 9a \leq 0 \Rightarrow a^2 - 5a + 4 \leq 0$$

$$a \in [1, 4] \text{ Ans}$$

Q f & g are 2 bijective functions such that $f : [0, \infty) \to [2, \infty)$, $f(x) = x^2 + 2$

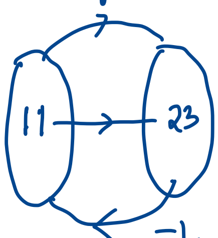
g : $[2, \infty) \to [5, \infty)$, $g(x) = 2x + 1$, find $f^{-1}og^{-1}(23)$

$\boxed{\text{Ans} \to 3}$

Sol^n $f^{-1}og^{-1}(23) = ?$

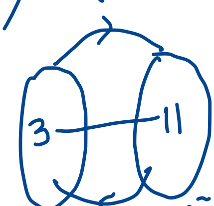
$g^{-1}(23) = ?$

$g(x) = 2x+1$



$g^{-1}(23) = 11$

$f^{-1}(11)$ $\quad f(x) = x^2+2$



$f^{-1}(11) = 3$

M-1 $(gof)^{-1} = f^{-1}og^{-1}$

M-2 Find g^{-1} & f^{-1} then $f^{-1}og^{-1} = ?$

M-3

Q $f(x) = \begin{cases} x & \text{if } x < 1 \\ x^2 & 1 \le x \le 4 \\ 8\sqrt{x} & x > 4 \end{cases}$, find $f^{-1}(x)$.

(i) $f(x) = x \Rightarrow y = x$

$$\boxed{f^{-1}(x) = x}$$

$x \in (-\infty, 1)$

$f(x) = x$

$f(x) \in (-\infty, 1)$

$f : (-\infty, 1) \longrightarrow (-\infty, 1)$

(ii) $f(x) = x^2$; $x \in [1, 4]$

$f(x) \in [1, 16]$

$f : [1, 4] \to [1, 16]$

$f^{-1} : [1, 16] \to [1, 4]$

$y = x^2 \Rightarrow x = \sqrt{y}$ or $x = -\sqrt{y}$

$\underbrace{f^{-1}(x) = \sqrt{x}}_{\checkmark}$ or $\underbrace{f^{-1}(x) = -\sqrt{x}}_{\times\times}$

(iii) $f : (4, \infty) \longrightarrow (16, \infty)$

$f(x) = 8\sqrt{x} \Rightarrow y = 8\sqrt{x} \Rightarrow \sqrt{x} = \frac{y}{8}$

$$\boxed{f^{-1}(x) = \frac{x^2}{64}} \Leftarrow x = \frac{y^2}{8}$$

$$f^{-1}(x) = \begin{cases} x & ; \quad x \in (-\infty, 1) \\ \sqrt{x} & ; \quad x \in [1, 16] \\ \dfrac{x^2}{64} & ; \quad x \in (16, \infty) \end{cases}$$

Q. If $f : R \to R$ such that $f(x)$ is invertible function and $f(x)\, f(y) - f(xy) = x + y \ \forall\, x, y \in R$ and $f(1) > 0$, then find $f(x)$. $f^{-1}(x)$ and also find the number of solution of $f(x) = f^{-1}(x)$.

Sol^n

$$f(x)\, f(y) - f(xy) = x + y \quad \text{— (1) —}$$

put $x = y = 1$.

$$f(1)\, f(1) - f(1) = 2$$

$$f^2(1) - f(1) - 2 = 0.$$

$$f^2(1) - 2f(1) + f(1) - 2 = 0$$

$\boxed{f(1) = 2}$ OR $\boxed{f(1) = -1}$ ✗✗

put $y = 1$.

$$f(x) \cdot \underbrace{f(1)} - f(x) = x + 1.$$

$$\boxed{f(x) = x + 1}$$

$$\boxed{f^{-1}(x) = x - 1}$$

$$f(x) = f^{-1}(x)$$

No Sol^n

$$x + 1 = x - 1$$

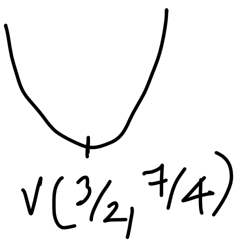
$$1 = -1 \quad \text{(Non-sense)}$$

Q A function $f : \left[\frac{3}{2}, \infty\right) \to \left[\frac{7}{4}, \infty\right)$

$f(x) = x^2 - 3x + 4$. Solve for

$f(x) = f^{-1}(x)$?

Sol^n



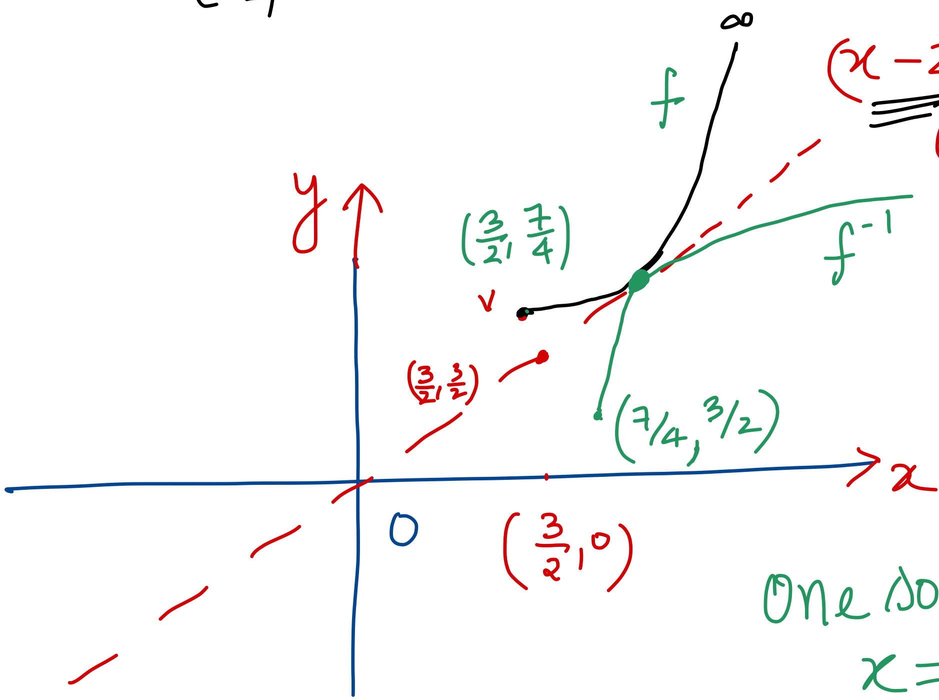
$f(x) = f^{-1}(x)$

$\boxed{f(x) = x}$

$x^2 - 3x + 4 = x$

$(x-2)^2 = 0$

$x = 2$



One sol^n ie

$x = 2$

Q $f(x)$ is bijective function such that $f(x) = f^{-1}(x^3 - x^2 + 2x + 1)$. If $f(\alpha) = \beta$, $f(\beta) = 3$, then find $f^{-1}(\beta)$.

$$\alpha = f^{-1}(\beta).$$

$$f(f(x)) = x^3 - x^2 + 2x + 1.$$

$$x = \alpha$$

$$f(f(\alpha)) = \alpha^3 - \alpha^2 + 2\alpha + 1$$

$$f(\beta) = \alpha^3 - \alpha^2 + 2\alpha + 1$$

$$3 = \alpha^3 - \alpha^2 + 2\alpha + 1$$

$$\boxed{\alpha^3 - \alpha^2 + 2\alpha - 2 = 0}$$

$$\boxed{\alpha = 1}$$

PERIODIC FUNCTION :

A function $f(x)$ is called periodic if there exists a positive number T such that $f(x+T) = f(x) = f(x-T)$, for all values of x within the domain of f. Smallest value of T is called fundamental period of function $f(x)$.

* if x = 0 is in domain

Note :

(i) Odd powers of sinx, cosx, secx, cosecx are periodic with period 2π.

(ii) Non zero integral powers of tanx, cotx are periodic with period π.

(iii) Non zero even powers or modulus of sinx, cosx, secx, cosecx are periodic with period π.

(iv) $f(T) = f(0) = f(-T)$, where 'T' is the period.

(v) if $f(x)$ has a period T then $\boldsymbol{f(ax + b)}$ **has a period** $\mathbf{T}/|\mathbf{a}|$ $(a \neq 0)$.

*

Proof : Let $f(x + T) = f(x)$ and $f[a(x + T') + b] = f(ax + b)$

$f(ax + b + aT') = f(ax + b)$

$f(y + aT') = f(y) = f(y + T) \Rightarrow T = aT' \Rightarrow T' = \dfrac{T}{a}$

Note: $\{x\} \rightarrow \boxed{P = 1}$

eg: $f(x) = \sin^2 4x$

$P = \dfrac{\pi}{4}$

; $g(x) = \{-3x\}$

{ } → fractional part function

$\boxed{P = \dfrac{1}{3}}$

*

(vi) If LCM of (T_1, T_2) exists where T_1 and T_2 are period of $f(x)$ and $g(x)$ respectively, then a period (need not be fundamental) of $f(x) \pm g(x)$, $f(x) \cdot g(x)$, $f(x)/g(x)$ is L.C.M. of (T_1, T_2).

(a) LCM of T_1 & T_2 is defined when T_1 & T_2 is rational.

(b) LCM of $\left\{\dfrac{a}{b}, \dfrac{p}{q}\right\} = \dfrac{\text{LCM of } (a, p)}{\text{HCF of } (b, q)}$

$\left\{\dfrac{3\pi}{1}, \dfrac{5\pi}{2}\right\}$

E(1) $f(x) = \cos\dfrac{2x}{3} - \sin\dfrac{4x}{5}$

$P_1 = \dfrac{2\pi}{(2/3)} = \dfrac{3\pi}{1}$

$P_2 = \dfrac{2\pi}{4/5} = \dfrac{5\pi}{2}$.

$\dfrac{L.C.M(3\pi, 5\pi)}{H.C.F(1, 2)}$

$\boxed{P = \dfrac{15\pi}{1}}$

eg:

$$f(x) = \{x\} + \cos x \quad ; \quad \{\ \} \to \text{fractional part fns.}$$

$$P_1 = \frac{1}{1} \to \text{Rational}$$

$$P_2 = \frac{2\pi}{1} \to \text{irrational}$$

$f(x) \to$ is aperiodic fns.
(Not periodic)

eg: $f(x) = \tan^{31}\left(\frac{2x}{3}\right)$

$$P = \frac{\pi}{(2/3)} = \frac{3\pi}{2}.$$

(2) $f(x) = \tan 3x + \{x\}$; $\{\ \} \to$ fractional part function.

Aperiodic fns

Special Note :

In case if there exists a positive K such that K < LCM of T_1 and T_2 and overall function repeats itself after every K, then fundamental period of the function will be K.

E(3) $f(x) = \sin^4 x + \cos^4 x$ ✓

E(4) $f(x) = \cos(\sin x)$

E(3) $f(x) = \sin^4 x + \cos^4 x$

$P_1 = \frac{\pi}{1}$ $\quad$ $P_2 = \frac{\pi}{1}$

$$P = \frac{L.C.M(\pi, \pi)}{H.C.F(1,1)} = \pi. \longrightarrow \text{This is period.}$$

check for $x = \frac{\pi}{2}$.

$$\boxed{f(x+T) = f(x)} \; ; \; T > 0 \qquad \therefore \boxed{P = \frac{\pi}{2}}$$

$$f\left(x+\frac{\pi}{2}\right) = \sin^4\left(x+\frac{\pi}{2}\right) + \cos^4\left(x+\frac{\pi}{2}\right)$$

$$= \cos^4 x + \sin^4 x = f(x)$$

M-2

$$f(x) = \sin^4 x + \cos^4 x$$

$$f(x) = 1 - \frac{\sin^2 (2x)}{2}$$

$$\boxed{P = \frac{\pi}{2}}$$

eg: $g(x) = \sin^4 x - \cos^4 x$.

$$g(x) = -\cos 2x$$

$$P = \frac{2\pi}{2} = \pi.$$

E(4) $f(x) = \cos(\sin x)$

Period $= 2\pi$.

$g(x) = \cos x$
$h(x) = \sin x$
$f(x) = g(h(x))$

$$\boxed{f(x + \overset{2\pi}{T}) = f(x)}$$

$$f(x+2\pi) = \cos(\sin(2\pi + x)) = \cos(\sin x) = f(x)$$

$$f(x+\pi) = \cos(\sin(\pi + x)) = \cos(-\sin x) = \cos(\sin x) = f(x)$$

$$\therefore \boxed{P = \pi}$$

(vii) Every constant function is always periodic, whose fundamental period is undefined.

e.g. $f(x) = 2$

Find fundamental period of following functions (if exists) :

E(1) $f(x) = \dfrac{x}{x}$

E(2) $f(x) = \dfrac{\sin x}{\sin x}$

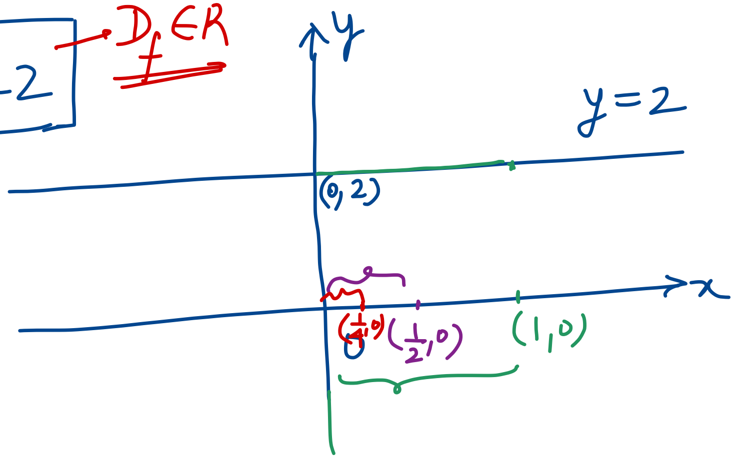
E(3) $f(x) = \tan x \cdot \cot x$.

E(4) $f(x) = \operatorname{sgn}\{x\}$

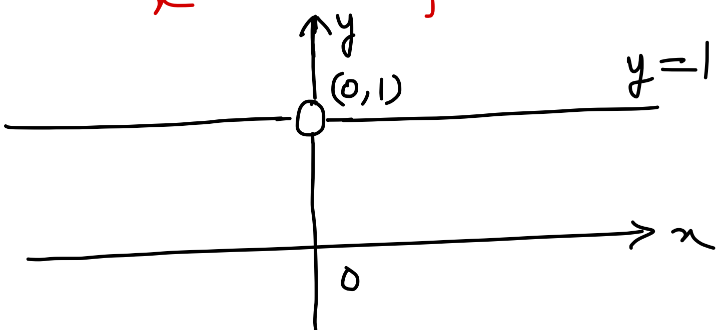
$\{\ \} \rightarrow$ fractional part function.

(viii) Inverse of a periodic function does not exist.

$\boxed{f(x) = 2} \rightarrow D_f \in R$



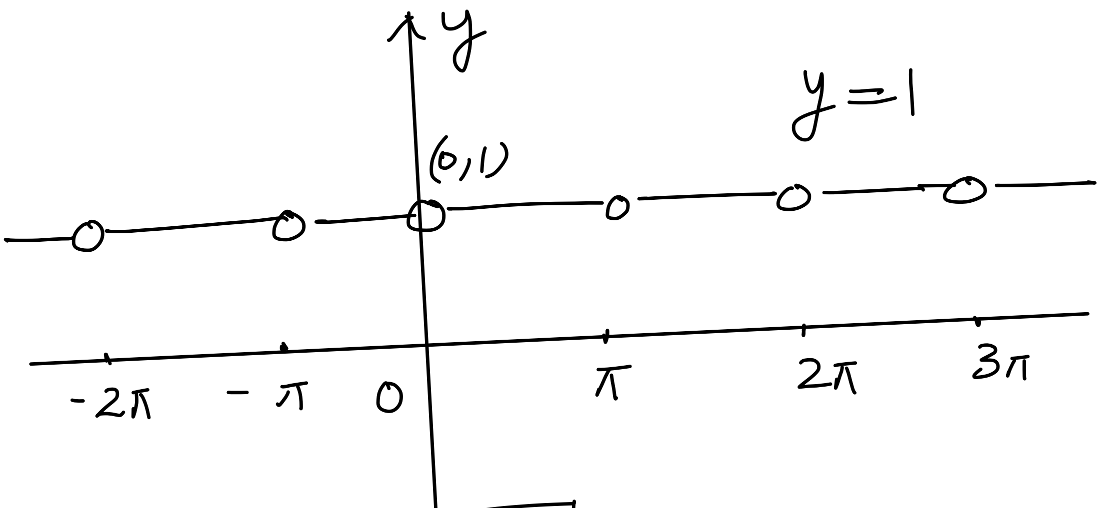
E-1) $f(x) = \dfrac{x}{x}$; $D_f \in R - \{0\}$



(Aperiodic)

E(2)

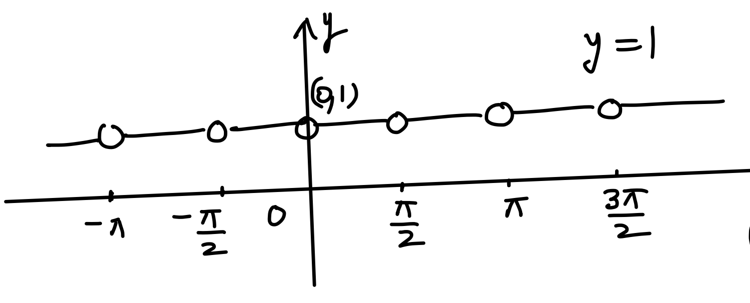
$$f(x) = \frac{\sin x}{\sin x} \quad ; \quad D_f \in R - n\pi$$
$n \in I$



$\boxed{P = \pi}$

E(3) $f(x) = \tan x . \cot x$

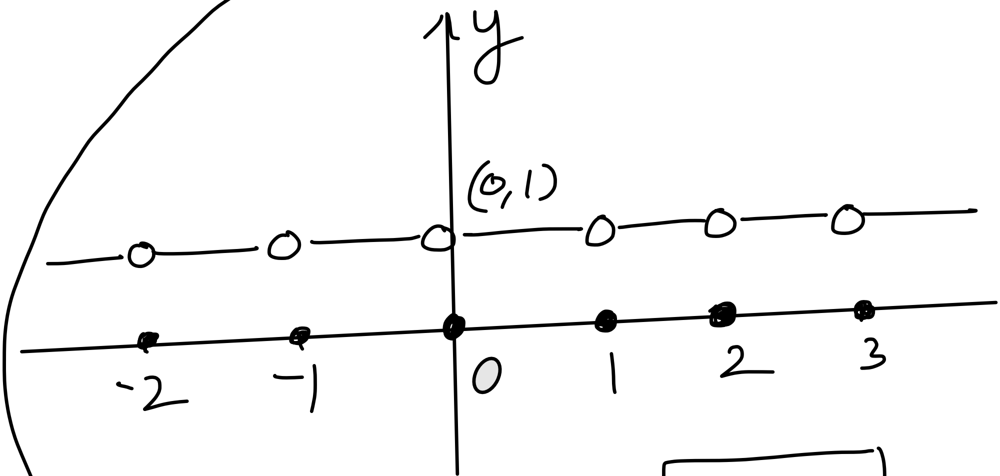
$D_f \in R - \left\{ (2n+1)\frac{\pi}{2} ; \; n\pi \right\} ; n \in I$



$P = \pi/2$

E(4)

$$f(x) = \operatorname{sgn}\{x\} \begin{cases} 0 \;; & x \in I \\ 1 \;; & x \notin I \end{cases}$$



$$\boxed{P = 1}$$

$$f(x) = g(h(x))$$

$$g(x) = \operatorname{sgn} x \;;\; h(x) = \{x\}$$

Q $f(x) = \sin x + \cos ax$ is a periodic function then prove that 'a' must be rational.

$f(x+T) = f(x)$; $\forall x \in R$

$x = 0$ $\quad f(T) = f(0) \Rightarrow \sin T + \cos aT = 1$

$x = -T$ $\quad f(-T) = f(0) \Rightarrow -\sin T + \cos aT = 1.$

Sub :

$2\sin T = 0 \Rightarrow \boxed{T = n\pi}$ $n \in I$

$\cos aT = 1 \Rightarrow \boxed{aT = 2m\pi}$ $m \in I.$

$\frac{a\cancel{\pi}}{\cancel{\pi}} = \frac{2m\cancel{\pi}}{n\cancel{\pi}}$

$\boxed{a = \frac{2m}{n}}$

Rational.

Q Prove that $f(x) = \cos\sqrt{x}$ is Aperiodic ?

Sol^n Let $f(x)$ is periodic with period 'T'

$\boxed{f(x+T) = f(x)}$ $\forall x \in D_f$

$D_f \in [0, \infty)$

put $x = 0$

$\boxed{f(T) = f(0)}$

$\cos\sqrt{T} = 1 \Rightarrow \boxed{\sqrt{T} = 2n\pi}$; $n \in I$

$x = T$

$f(2T) = f(T) = f(0)$

$\cos\sqrt{2T} = 1 \Rightarrow \boxed{\sqrt{2T} = 2m\pi}$; $m \in I$

$$\frac{\sqrt{2\cancel{\pi}}}{\sqrt{\cancel{\pi}}} = \frac{\cancel{2}m\cancel{\pi}}{\cancel{2}n\cancel{\pi}}$$

$$\underbrace{\sqrt{2}}_{\text{irrational}} = \underbrace{\frac{m}{n}}_{\text{Rational}}$$ This is Contraction

Hence, $f(x)$ is A<u>periodic</u>.

Q**

Find the period of real valued function satisfying $f(x+4)+f(x)=f(x+2)+f(x+6)$

Sol^n

Objective

$$\boxed{f(x+T)=f(x)} \; ; \; T>0$$

$$f(\cancel{x+4})+f(x)=f(\cancel{x+2})+f(\cancel{x+6})$$

$x \to x+2$

$$\underline{f(\cancel{x+6})+f(\cancel{x+2})}=\underline{f(\cancel{x+4})+f(x+8)}$$

$$\boxed{f(x)=f(x+8)}$$

$$\boxed{T=8}$$

Q

Find the period of real valued function satisfying $f(x+4) + f(x-4) = f(x)$. ?