Rem *

$$[\vec{l}\ \vec{m}\ \vec{n}]\,[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} \vec{l}.\vec{a} & \vec{l}.\vec{b} & \vec{l}.\vec{c} \\ \vec{m}.\vec{a} & \vec{m}.\vec{b} & \vec{m}.\vec{c} \\ \vec{n}.\vec{a} & \vec{n}.\vec{b} & \vec{n}.\vec{c} \end{vmatrix}$$, where $\vec{l}, \vec{m}, \vec{n}$ & $\vec{a}, \vec{b}, \vec{c}$ are non coplanar vectors.

$$\begin{vmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{vmatrix} \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Rem

$$\boxed{[\vec{a}\ \vec{b}\ \vec{c}]^2 = \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} & \vec{a}.\vec{c} \\ \vec{b}.\vec{a} & \vec{b}.\vec{b} & \vec{b}.\vec{c} \\ \vec{c}.\vec{a} & \vec{c}.\vec{b} & \vec{c}.\vec{c} \end{vmatrix}}$$

Q

$$[\vec{a}\,\vec{b}\,\vec{c}]\ (\vec{p}\times\vec{q}) = \begin{vmatrix} \vec{p}.\vec{a} & \vec{q}.\vec{a} & \vec{a} \\ \vec{p}.\vec{b} & \vec{q}.\vec{b} & \vec{b} \\ \vec{p}.\vec{c} & \vec{q}.\vec{c} & \vec{c} \end{vmatrix}$$

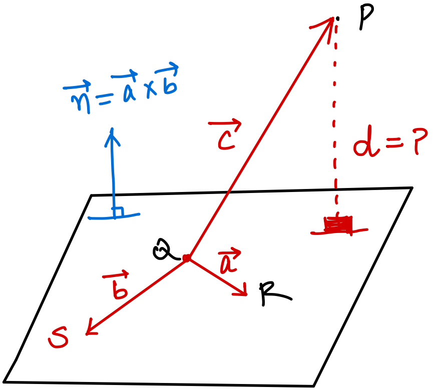
$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} \begin{vmatrix} \hat{i} & p_1 & q_1 \\ \hat{j} & p_2 & q_2 \\ \hat{k} & p_3 & q_3 \end{vmatrix}$$

Row by Column

*

Q. Let P be a point not on the plane that passes through Q, R and S. Show that the distance d from P to the plane is $d = \left| \dfrac{[\vec{a}\,\vec{b}\,\vec{c}]}{|\vec{a}\times\vec{b}|} \right|$ where $\vec{a} = \overrightarrow{QR}$; $\vec{b} = \overrightarrow{QS}$ and $\vec{c} = \overrightarrow{QP}$

Sol^n



$d = |\text{projection of } \vec{c} \text{ on } \vec{n}|$

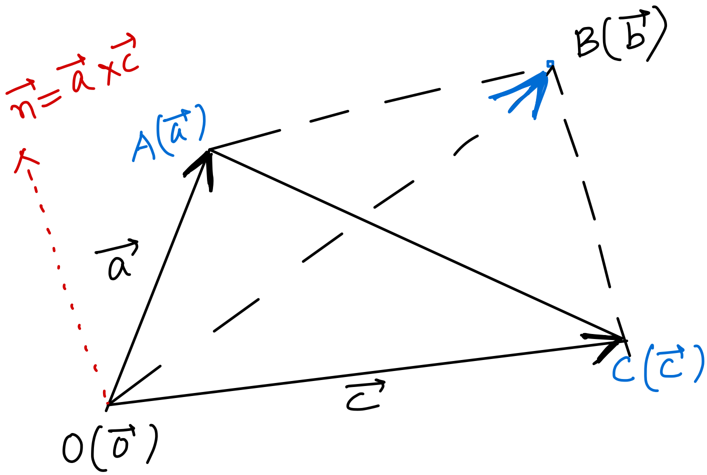
$= \left| \dfrac{\vec{c}\cdot(\vec{a}\times\vec{b})}{|\vec{a}\times\vec{b}|} \right|$

$d = \left| \dfrac{[\vec{a}\,\vec{b}\,\vec{c}]}{|\vec{a}\times\vec{b}|} \right|$ (H.P.)

Rem

Volume of a tetrahedron OABC = $\vec{V} = \frac{1}{6}[\vec{a}\,\vec{b}\,\vec{c}] = \frac{1}{6}\left[\overrightarrow{OA}\,,\,\overrightarrow{OB}\,,\,\overrightarrow{OC}\right]$ where O is the origin.

v. imp



Vol. of tetrahedron

$= \frac{1}{3}$ (Ar of Base) $\times$ Height

$= \frac{1}{3}(\Delta AOC) \times$ Height

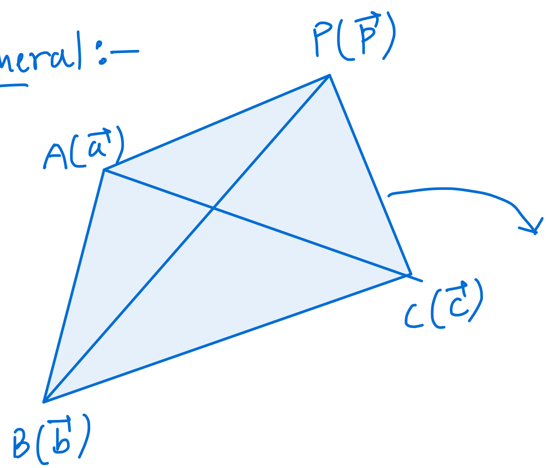
$= \frac{1}{3}\cdot\frac{1}{2}\left|\vec{a} \times \vec{c}\right| \cdot$ (proj. of $\vec{b}$ on $\vec{n}$)

$$V = \frac{1}{6}\,\cancel{\left|\vec{a} \times \vec{c}\right|}\,\left|\frac{\vec{b}\cdot(\vec{a} \times \vec{c})}{\cancel{\left|\vec{a} \times \vec{c}\right|}}\right|$$

$$V = \frac{1}{6}\left|[\vec{a}\ \vec{b}\ \vec{c}]\right|$$

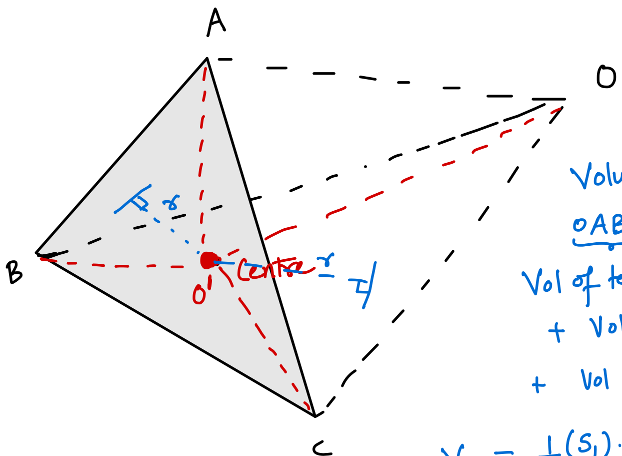
$$V = \frac{1}{6}\left|[\overrightarrow{OA}\ \ \overrightarrow{OB}\ \ \overrightarrow{OC}]\right|$$

General:-



$$\text{Volume} = \frac{1}{6}\left|[\overrightarrow{PA}\ \ \overrightarrow{PB}\ \ \overrightarrow{PC}]\right|$$

Very Important Note : If S_1, S_2, S_3 and S_4 are the areas of the four triangular faces of the tetrahedron with volume V. If r is the radius of the sphere touching the four faces then $V = \frac{1}{3}(S_1 + S_2 + S_3 + S_4)r$



Volume of tetrahedron $\underbrace{OABC}$ =

Vol of tetrahedron O'ABC + Vol O'ABO + Vol^n O'BCO + Vol O'ACO

$V = \frac{1}{3}(S_1) \cdot r + \frac{1}{3}(S_2) \cdot r + \frac{1}{3}(S_3) \cdot r + \frac{1}{3}(S_4) \cdot r.$

Note :-

To express scalar triple product of three vectors in terms of any three non coplanar vectors $\vec{l}$, $\vec{m}$ and $\vec{n}$

Let $\vec{a} = a_1\vec{l} + a_2\vec{m} + a_3\vec{n}$; $\vec{b} = b_1\vec{l} + b_2\vec{m} + b_3\vec{n}$; $\vec{c} = c_1\vec{l} + c_2\vec{m} + c_3\vec{n}$

then $$\boxed{[\vec{a}\,\vec{b}\,\vec{c}] = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} [\vec{l}\ \vec{m}\ \vec{n}]}$$

Rem

Q If $\vec{a} = \vec{l} - \vec{m} + \vec{n}$, $\vec{b} = 2\vec{l} + \vec{m} - \vec{n}$, $\vec{c} = \vec{l} + \vec{m} + 2\vec{n}$

and $[\vec{l}\ \vec{m}\ \vec{n}] = 4$ then find $[\vec{a}\ \vec{b}\ \vec{c}] = ?$

Solⁿ

$$[\vec{a}\ \vec{b}\ \vec{c}] = \begin{vmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ 1 & 1 & 2 \end{vmatrix} [\vec{l}\ \vec{m}\ \vec{n}]$$

*

Q If the value of $\left[\vec{a}+2\vec{b}+3\vec{c}\ \ \vec{b}+2\vec{c}+3\vec{a}\ \ \vec{c}+2\vec{a}+3\vec{b}\right] = k\left[\vec{a}\ \vec{b}\ \vec{c}\right]$, then k equals -

(A) 6 (B) 9 (C) 12 (D) 18

$$[\vec{a}\ \vec{b}\ \vec{c}] \underbrace{\begin{vmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \\ 2 & 3 & 1 \end{vmatrix}}_{} = K\,[\vec{a}\ \vec{b}\ \vec{c}]$$

$$\boxed{K = 18} \rightarrow [D]$$

Q If $[\bar{a}+2\bar{b}+3\bar{c}\ \ 2\bar{a}+3\bar{b}+\bar{c}\ \ 3\bar{a}+\bar{b}+2\bar{c}]=-18$, where $\bar{a},\bar{b},\bar{c}$ are 3 non-coplaner vectors, then

$$\begin{vmatrix} \bar{a}.\bar{a} & \bar{a}.\bar{b} & \bar{a}.\bar{c} \\ \bar{b}.\bar{a} & \bar{b}.\bar{b} & \bar{b}.\bar{c} \\ \bar{c}.\bar{a} & \bar{c}.\bar{a} & \bar{c}.\bar{c} \end{vmatrix}$$ is equal

$$[\vec{a}\ \vec{b}\ \vec{c}] \begin{vmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \\ 3 & 1 & 2 \end{vmatrix} = -18$$

$$[\vec{a}\ \vec{b}\ \vec{c}]^2$$

$$[\vec{a}\ \vec{b}\ \vec{c}] = ?$$

Rem

Vector Triple Product

Definition: $(\vec{a}\times\vec{b})\times\vec{c}$ is a vector which is coplanar with $\vec{a}$ and $\vec{b}$ and perpendicular to $\vec{c}$.

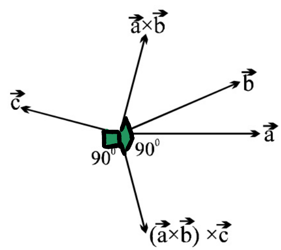
Hence $(\vec{a}\times\vec{b})\times\vec{c} = x\vec{a}+y\vec{b}$ ....(1) [linear combination of $\vec{a}$ and $\vec{b}$)

$\vec{c}\ .(\vec{a}\times\vec{b})\times\vec{c} = x(\vec{a}.\vec{c})+y(\vec{b}.\vec{c})$

$0 = x(\vec{a}.\vec{c})+y(\vec{b}.\vec{c})$ ....(2)

$\therefore \dfrac{x}{\vec{b}.\vec{c}} = -\dfrac{y}{\vec{a}.\vec{c}} = \lambda$ (say)

$\therefore x = \lambda(\vec{b}.\vec{c})$ and $y = -\lambda(\vec{a}.\vec{c})$



Substituting the values of x and y in $(\vec{a}\times\vec{b})\times\vec{c} = \lambda(\vec{b}.\vec{c})\vec{a}-\lambda(\vec{a}.\vec{c})\vec{b}$

This is an identity and must be true for all values of $\vec{a},\vec{b},\vec{c}$

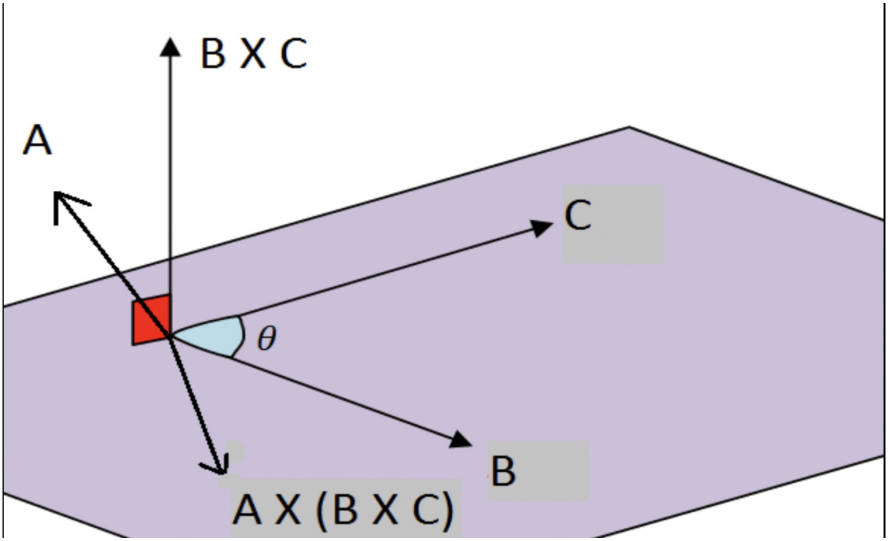
Put = $\vec{a}=\hat{i}$; $\vec{b}=\hat{j}$ and $\vec{c}=\hat{i}$

$(\hat{i}\times\hat{j})\times\hat{i} = \lambda(\hat{j}.\hat{i})\hat{i}-\lambda(\hat{i}.\hat{i})\hat{j}$

$\hat{j}=-\lambda\hat{j} \quad \Rightarrow \quad \lambda=-1$

Rem

$$\boxed{(\vec{a}\times\vec{b})\times\vec{c} = \underbrace{(\vec{a}.\vec{c})\vec{b}}_{\text{durr-durr}} - \underbrace{(\vec{b}.\vec{c})\vec{a}}_{\text{pass-pass}}}$$



Note : Unit vector coplanar with $\vec{a}$ and $\vec{b}$ and perpendicular to $\vec{c}$ is $\pm \dfrac{(\vec{a}\times\vec{b})\times\vec{c}}{|(\vec{a}\times\vec{b})\times\vec{c}|}$

|||ly unit vector coplanar with $\vec{a}$ and $\vec{b}$ perpendicular to $\vec{a}$ is $\pm \dfrac{(\vec{a}\times\vec{b})\times\vec{a}}{|(\vec{a}\times\vec{b})\times\vec{a}|}$

Note:-

$[\vec{a}\times\vec{b} \quad \vec{b}\times\vec{c} \quad \vec{c}\times\vec{a}] = [\vec{a}\,\vec{b}\,\vec{c}]^2 = \begin{vmatrix} \vec{a}.\vec{a} & \vec{a}.\vec{b} & \vec{a}.\vec{c} \\ \vec{b}.\vec{a} & \vec{b}.\vec{b} & \vec{b}.\vec{c} \\ \vec{c}.\vec{a} & \vec{c}.\vec{b} & \vec{c}.\vec{c} \end{vmatrix}$ Note that if $\vec{a},\vec{b},\vec{c}$ are non coplanar vectors then $\vec{a}\times\vec{b}$, $\vec{b}\times\vec{c}$ and $\vec{c}\times\vec{a}$ will also be non coplanar vectors.

Proof:

$$\left(\underbrace{(\vec{a}\times\vec{b})}_{\vec{u}}\times(\vec{b}\times\vec{c})\right)\cdot(\vec{c}\times\vec{a})$$

$\vec{u} = \vec{a}\times\vec{b}$

$$(\vec{u}\times(\vec{b}\times\vec{c}))\cdot(\vec{c}\times\vec{a})$$

$$\left((\vec{u}\cdot\vec{c})\vec{b} - \cancel{(\vec{u}\cdot\vec{b})}^{0}\vec{c}\right)\cdot(\vec{c}\times\vec{a})$$

$$[\vec{a}\,\vec{b}\,\vec{c}]\,\vec{b}\cdot(\vec{c}\times\vec{a}) = [\vec{a}\,\vec{b}\,\vec{c}]\,[\vec{b}\,\vec{c}\,\vec{a}] = [\vec{a}\,\vec{b}\,\vec{c}]^2$$

(H.P.)

Q If $\hat{a}\times(\hat{b}\times\hat{c}) = \frac{1}{2}\hat{b}$ where $\hat{b}$ and $\hat{c}$ are non collinear then find the angle between $\hat{a}$ and $\hat{b}$; between $\hat{a}$ and $\hat{c}$.

Soln

$$\hat{a}\times(\hat{b}\times\hat{c}) = \frac{1}{2}\hat{b}$$

$$(\hat{a}\cdot\hat{c})\hat{b} - (\hat{a}\cdot\hat{b})\hat{c} = \frac{1}{2}\hat{b}$$

$$\underbrace{\left((\hat{a}\cdot\hat{c}) - \frac{1}{2}\right)}_{0}\hat{b} = \underbrace{(\hat{a}\cdot\hat{b})}_{0}\hat{c} \quad —①—$$

Since $\hat{b}$ and $\hat{c}$ are non-collinear vector

$\hat{a}\cdot\hat{b} = 0 \Rightarrow \theta = 90^\circ$

$\hat{a}\cdot\hat{c} = \frac{1}{2} \Rightarrow \phi = 60^\circ$. Ans

Q If $\hat{i}\times(\vec{a}\times\hat{i})+\hat{j}\times(\vec{a}\times\hat{j})+\hat{k}\times(\vec{a}\times\hat{k})=\lambda\vec{a}$ then find λ ?

$\vec{a}=a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$

M-1 put $\vec{a}=\hat{i}$.

M-2

$$\underbrace{(\hat{i}\cdot\hat{i})}_{1}\vec{a}-\underbrace{(\hat{i}\cdot\vec{a})}_{a_1}\hat{i}+\underbrace{(\hat{j}\cdot\hat{j})}_{1}\vec{a}-\underbrace{(\hat{j}\cdot\vec{a})}_{a_2}\hat{j}+\underbrace{(\hat{k}\cdot\hat{k})}_{1}\vec{a}-\underbrace{(\hat{k}\cdot\vec{a})}_{a_3}\hat{k}$$

$$3\vec{a}-(\vec{a})=2\vec{a}$$

$$\therefore \boxed{\lambda=2}$$

Q

If $\left.\begin{array}{l}\vec{V}_1=\vec{a}\times(\vec{b}\times\vec{c})\\ \vec{V}_2=\vec{b}\times(\vec{c}\times\vec{a})\\ \vec{V}_3=\vec{c}\times(\vec{a}\times\vec{b})\end{array}\right]$ then which of the following hold(s) good ?

(1) $\vec{V}_1,\vec{V}_2,\vec{V}_3$ are coplanar

(2) $\vec{V}_1,\vec{V}_2,\vec{V}_3$ can form the sides of a triangle.

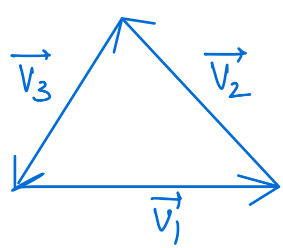
(3) $\vec{V}_1+\vec{V}_2+\vec{V}_3$ is a null vectors

(4) $\vec{V}_1,\vec{V}_2,\vec{V}_3$ are linearly dependent

(5) $[\vec{V}_1-\vec{V}_2 \quad \vec{V}_2-\vec{V}_3 \quad \vec{V}_3-\vec{V}_1]=0.$

(6) $[\vec{V}_1+\vec{V}_2 \quad \vec{V}_2+\vec{V}_3 \quad \vec{V}_3+\vec{V}_1]=0.$

(7) $[\vec{V}_1\times\vec{V}_2 \quad \vec{V}_2\times\vec{V}_3 \quad \vec{V}_3\times\vec{V}_1]=0.$



$\vec{V}_1+\vec{V}_2+\vec{V}_3=\vec{0}$

Sol^n

$$\vec{V}_1=(\vec{a}\cdot\vec{c})\vec{b}-(\vec{a}\cdot\vec{b})\vec{c}$$
$$\vec{V}_2=(\vec{b}\cdot\vec{a})\vec{c}-(\vec{b}\cdot\vec{c})\vec{a}$$
$$\vec{V}_3=(\vec{c}\cdot\vec{b})\vec{a}-(\vec{c}\cdot\vec{a})\vec{b}$$

$$\boxed{\vec{V}_1+\vec{V}_2+\vec{V}_3=\vec{0}}$$

Q If $\vec{a}=\hat{i}+2\hat{j}+2\hat{k}$, $\vec{b}=2\hat{i}-3\hat{j}+\hat{k}$ and $\vec{c}$ is a vector such that $(\vec{a}\times\vec{b})\times\vec{c}=\vec{a}\times(\vec{b}\times\vec{c})$ & $|\vec{c}|=6$. Then the volume of parallelopiped whose co-terminous edges are $\vec{a},\vec{b},\vec{b}\times\vec{c}$ will be-

(A) 180 (B) 214 (C) 232 (D) 244

Solⁿ

$$(\vec{a}\times\vec{b})\times\vec{c}=\vec{a}\times(\vec{b}\times\vec{c})$$

$$\cancel{(\vec{a}\cdot\vec{c})\vec{b}}-(\vec{b}\cdot\vec{c})\vec{a}=\cancel{(\vec{a}\cdot\vec{c})\vec{b}}-(\vec{a}\cdot\vec{b})\vec{c}$$

★★ $\Rightarrow$ $\vec{a}=\lambda\vec{c} \Rightarrow \vec{a}$ & $\vec{c}$ are Collinear

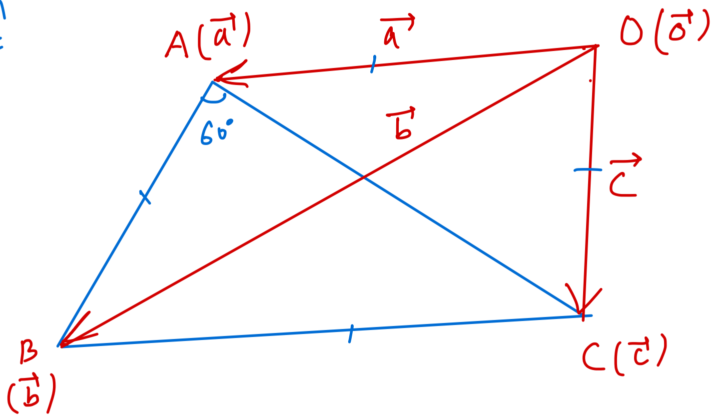
★ $\vec{c}=|\vec{c}|\,\hat{c}=|\vec{c}|\,\hat{a} \Rightarrow \vec{c}=6\cdot\left(\frac{\hat{i}+2\hat{j}+2\hat{k}}{3}\right)$

$$\therefore \boxed{\vec{c}=2\hat{i}+4\hat{j}+4\hat{k}.}$$

$$V=[\vec{a}\ \ \vec{b}\ \ \vec{b}\times\vec{c}]=244$$ Ans $\to$ [D]

Q** If $\vec{a}, \vec{b}, \vec{c}$ are 3 unit vectors from the vertex of a regular tetrahedron then find value of absolute value of $[\vec{a}\ \vec{b}\ \vec{c}] = ?$

Sol^n



$$\text{Volume} = \frac{1}{6}[\vec{OA}\ \vec{OB}\ \vec{OC}]$$

$$V = \frac{1}{6}[\vec{a}\ \vec{b}\ \vec{c}]$$

$$|\vec{a}| = |\vec{b}| = |\vec{c}| = 1$$

$$\vec{a}\cdot\vec{b} = |a|\,|b| \cos 60^\circ$$

$$[\vec{a}\ \vec{b}\ \vec{c}]^2 = \begin{vmatrix} \vec{a}\cdot\vec{a} & \vec{a}\cdot\vec{b} & \vec{a}\cdot\vec{c} \\ \vec{b}\cdot\vec{a} & \vec{b}\cdot\vec{b} & \vec{b}\cdot\vec{c} \\ \vec{c}\cdot\vec{a} & \vec{c}\cdot\vec{b} & \vec{c}\cdot\vec{c} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 1/2 & 1/2 \\ 1/2 & 1 & 1/2 \\ 1/2 & 1/2 & 1 \end{vmatrix}$$

HW

Q If $\hat{i} \times \left((\vec{a} - \hat{j}) \times \hat{i} \right) + \hat{j} \times \left((\vec{a} - \hat{k}) \times \hat{k} \right)$

$+ \hat{k} \times \left((\vec{a} - \hat{i}) \times \hat{k} \right) = 0$ and $\vec{a} = x\hat{i} + y\hat{j} + z\hat{k}$

then value of $8(x^3 - xy + zx) = ?$

Q* If three non-zero unequal vectors $\vec{\ell} = a\hat{i} + b\hat{j} + c\hat{k}$, $\vec{m} = b\hat{i} + c\hat{j} + a\hat{k}$ & $\vec{n} = c\hat{i} + a\hat{j} + b\hat{k}$ are such that $\vec{r}.\vec{\ell} = \vec{r}.\vec{m} = \vec{r}.\vec{n} = 0$ where a, b, c $\in R_0$ & $\vec{r}$ is non zero vector, then value of $\frac{a^2}{bc} + \frac{b^2}{ca} + \frac{c^2}{ab}$?

$R_0 = \mathbb{R} - \{0\}$

Sol^n

** $\vec{l}, \vec{m}, \vec{n}$ are Coplanar Vectors

$$[\vec{l}\ \vec{m}\ \vec{n}] = 0 \Rightarrow \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 0$$

$$3abc - a^3 - b^3 - c^3 = 0$$

$$\boxed{a^3 + b^3 + c^3 = 3abc} \quad \text{— ① —}$$

$$\frac{a^3 + b^3 + c^3}{abc} = \frac{3\cancel{abc}}{\cancel{abc}} = 3 \text{ Ans}$$

HW

0 – 1 33 to 35.

0 – 2 Q1 to 7, 17.