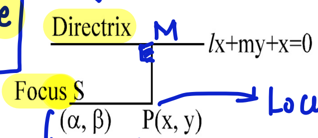
CONIC SECTIONS :

A conic section, or conic is the locus of a point which moves in a plane so that the ratio of its distance from a fixed point to its perpendicular distance from a fixed straight line is a constant i.e. $\boxed{\dfrac{PS}{PM} = \text{constant} = e.}$

$\boxed{\dfrac{PS}{PM} = e}$ eccentricity

fixed line



Locus

fixed point

- The fixed point is called the **FOCUS.**
- The fixed straight line is called the **DIRECTRIX.**
- The constant ratio is called the **ECCENTRICITY** denoted by e.
- The line passing through the focus & perpendicular to the directrix is called the **AXIS.**
- A point of intersection of a conic with its axis is called a **VERTEX.**

GENERAL EQUATION OF A CONIC : (Focal Directrix property)

The general equation of a conic with focus (α, β) & directrix $\ell x + my + n = 0$ is :

$$[(x-\alpha)^2 + (y-\beta)^2] = \frac{e^2(\ell x + my + n)^2}{\ell^2 + m^2}$$

$$\frac{PS}{PM} = e \Rightarrow PS^2 = e^2 PM^2$$

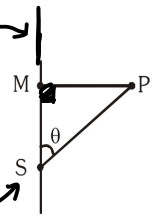
this simplifies to $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

Now two cases arises :

Case (I) : When the focus lies on the directrix. $(\Delta = 0)$

In this case $\Delta \equiv abc + 2fgh - af^2 - bg^2 - ch^2 = 0$ & the general equation of a conic represents a pair of straight lines :

Directrix

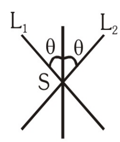


focus

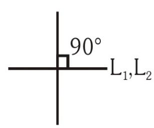
$$e = \frac{PS}{PM} = \operatorname{cosec}\theta$$

If

(i) $e > 1 \Rightarrow$ the lines L_1 and L_2 will be real & distinct intersecting at S.



(ii) $e = 1 \Rightarrow$ the lines will coincident and perpendicular to the given line passing through S.



(iii) $e < 1 \Rightarrow$ the lines will be imaginary (no such real lines are possible)

Case(II) : When the Focus does not lie on directrix. $(\Delta \neq 0)$

a parabola	**an ellipse**	**a hyperbola**	**rectangular hyperbola**
$e = 1 \Rightarrow PS = PM$	$0 < e < 1$	$e > 1$	$e = \sqrt{2}$
$\Delta \neq 0$	$\Delta \neq 0$	$\Delta \neq 0$	$\Delta \neq 0$
$h^2 = ab$	$h^2 < ab$	$h^2 > ab$	$h^2 > ab$
	(if $h = 0$, then $a \neq b$)		$a + b = 0$

$\downarrow$

a circle ($e = 0, \Delta \neq 0, h = 0, a = b \neq 0$)

Note

Hence the exact nature of conic section depends not only on its eccentricity but also on the position of focus relative to directrix.

Q1

Locus of a point which moves such that the ratio of its distance from (1, 2) to its perpendicular distance from $4x - 3y + 2 = 0$ is $\sqrt{3}$

Q2

Identify the locus of P(x,y) satisfying $(x-1)^2 + (y-2)^2 = 3\left(\frac{x+y+3}{\sqrt{2}}\right)^2$ → Hyperbola.

$S(1, 2)$ Focus

Directrix: $x + y + 3 = 0$

$PS^2 = e^2 PM^2$ | $\Delta \neq 0$

$e = \sqrt{3}$

(1) $\frac{PS}{PM} = \underbrace{e = \sqrt{3}}_{e>1}$ & $4(1) - 3(2) + 2 = 0$

$0 = 0 \Rightarrow \boxed{\Delta = 0}$

$\therefore$ Locus will be real & distinct lines.

$PS = \sqrt{3}\, PM \Rightarrow PS^2 = 3PM^2$

$$(h-1)^2 + (k-2)^2 = \frac{3(4h - 3k + 2)^2}{25}$$

THE CENTRE OF CENTRAL CONIC :

The centre of a conic section is a point such that all chords of the conic which pass through it are bisected there.

By partial differentiation method :

Consider general equation of a conic as f(x,y) : $ax^2 + by^2 + 2hxy + 2gx + 2fy + c$. Find $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ equate these to zero. We get

$$2ax + 2hy + 2g = 0 \quad ....(1)$$

$$2by + 2hx + 2f = 0 \quad ....(2)$$

on solving (1) & (2), we get centre of conic (α, β) as $\left(\frac{hf - bg}{ab - h^2}, \frac{gh - af}{ab - h^2}\right)$

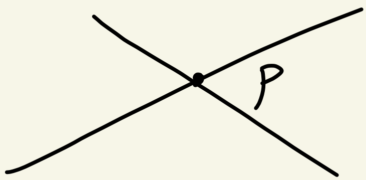
E(1) Find centre of the conic $x^2 + 2y^2 + 3xy - 5x - 7y + 3 = 0$

Sol^n

$$E : x^2 + 2y^2 + 3xy - 5x - 7y + 3$$

$$\frac{\partial E}{\partial x} = 0 \Rightarrow 2x + 0 + 3y - 5 - 0 + 0 = 0 \quad —①—$$

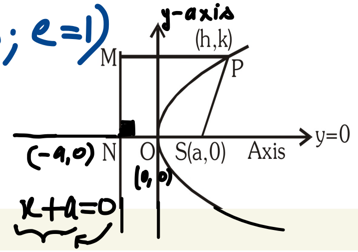
$$\frac{\partial E}{\partial y} = 0 \Rightarrow 0 + 4y + 3x - 0 - 7 + 0 = 0 \quad —②—$$



PARABOLA

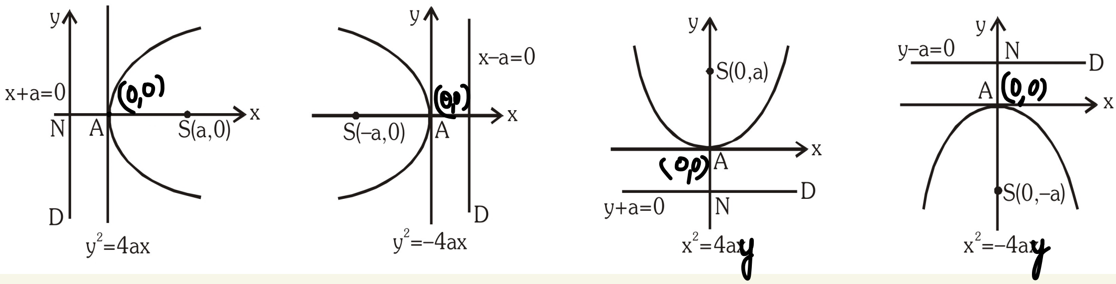
STANDARD EQUATION OF PARABOLA : $(\Delta \neq 0;\ h^2 = ab;\ e = 1)$

Standard equation of parabola is obtained by placing its focus at S(a, 0) and taking its directrix as the line $x + a = 0$ so that origin lies on the curve PS = PM



$$(x - a)^2 + y^2 = (x + a)^2 \quad \Rightarrow \quad y^2 = 4ax \qquad (a > 0)$$

Four Standard Parabolas : $(a > 0)$



General terminology of parabola :

(a) **Axis of parabola :** The line passing through the focus & perpendicular to the directrix is called the axis of parabola.

(b) **Vertex :** A point of intersection of a parabola with axis of parabola is called a Vertex.

(c) **Double ordinate :** A chord of the parabola perpendicular to the axis of the symmetry is called a double ordinate.

(d) **Focal chord :** A chord of the parabola, which passes through the focus is called a Focal chord.

(e) **Latus rectum :** A double ordinate passing through the focus or a focal chord perpendicular to the axis of parabola is called the Latus rectum.

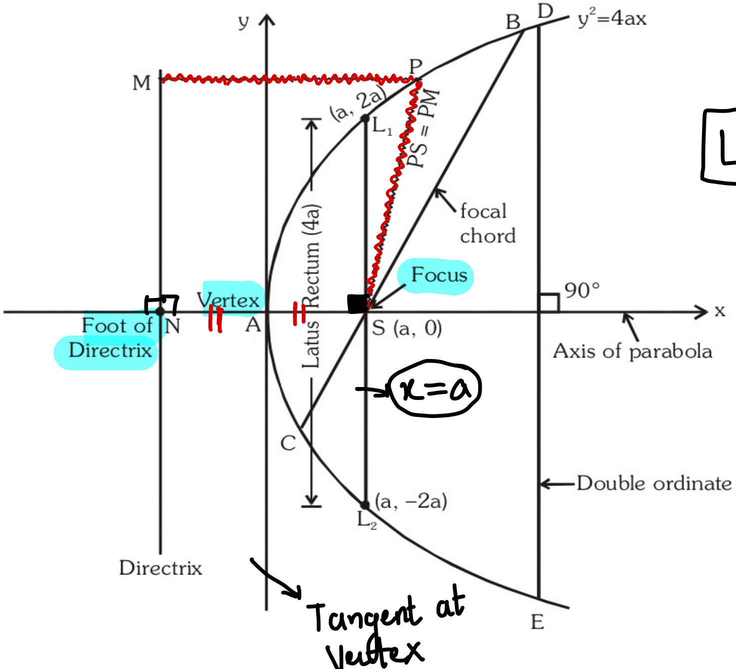
(f) **Focal distance/Focal radii :** The distance of a point P on the parabola from the focus is called the Focal distance of the point and is equal to the distance of point P from the directrix.

(g) **Ends of latus rectum :** Intersection point of latus rectum and parabola is called ends of latus rectum. For parabola $y^2 = 4ax$, ends of the latus rectum are $L_1(a, 2a)$ & $L_2(a, -2a)$.

(h) **Length of Latus rectum :** Distance between ends of latus rectum is called length of latus rectum.

Length of the latus rectum = 4a

Standard parabola $y^2 = 4ax (a > 0)$ at a glance



$\boxed{L \cdot R = 4a}$

$y^2 = \underbrace{8}x$

$L \cdot R = 8.$

$y^2 = -20x$

$LR = 20$

Note that :

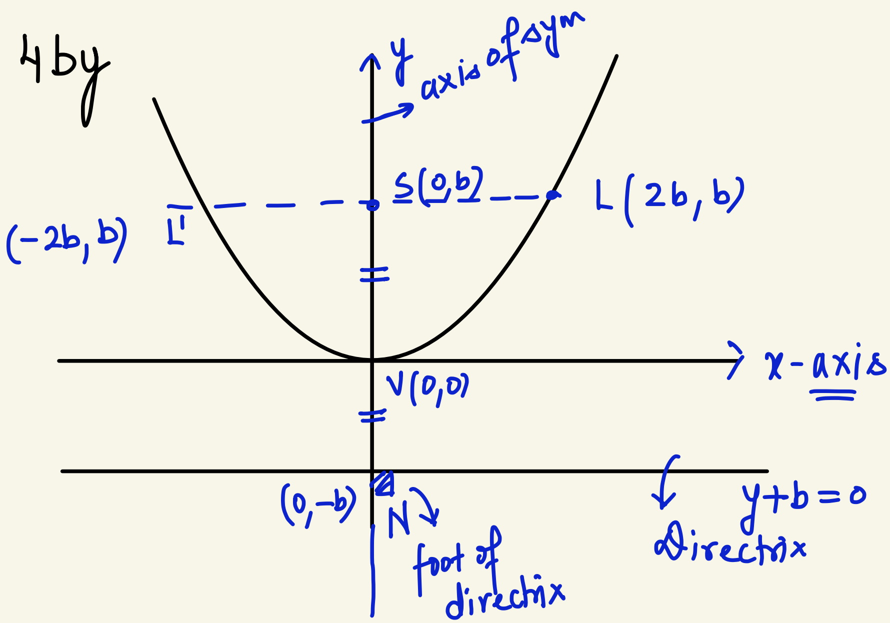
(i) Perpendicular distance from focus to directrix = half the latus rectum.

(ii) Vertex is middle point of the focus & the point of intersection of directrix & axis.

(iii) Point of intersection of axis and directrix is called foot of directrix.

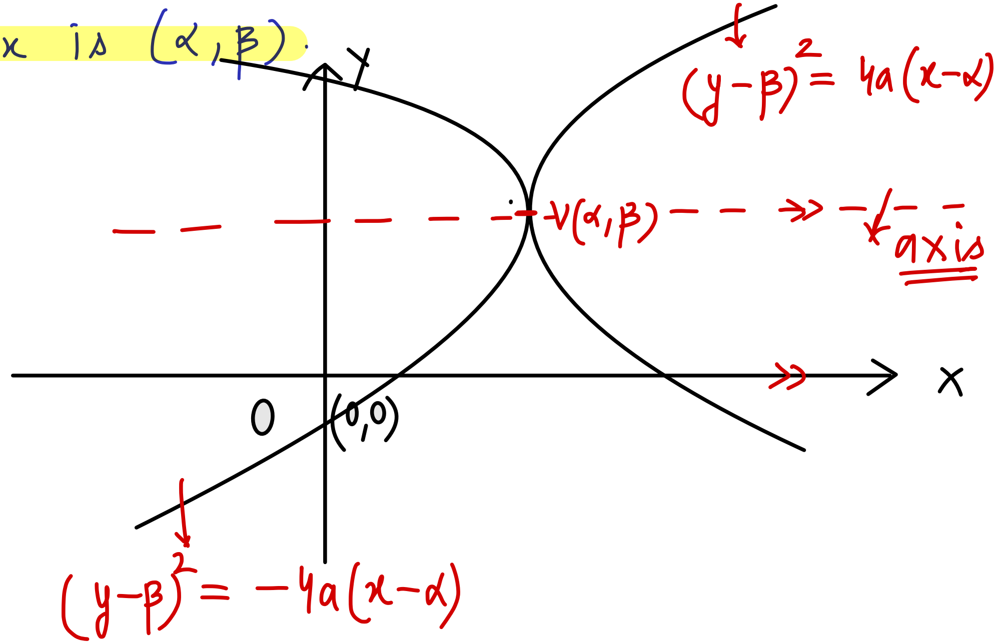
(iv) Two parabolas are said to be equal if they have the same latus rectum.

* $x^2 = 4by$

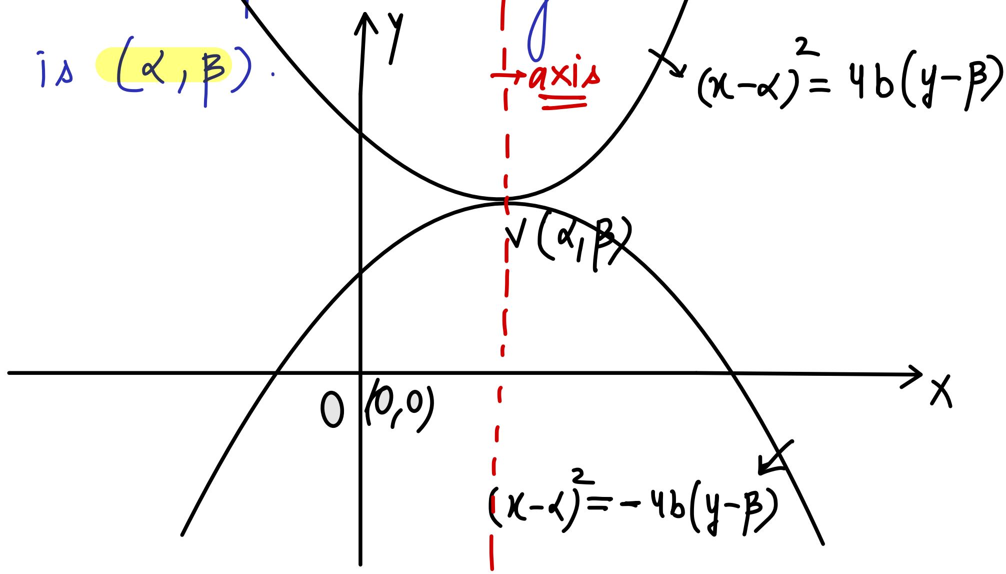


Shifted Parabola :-

(i) If axes is parallel to x-axes and vertex is (α, β).



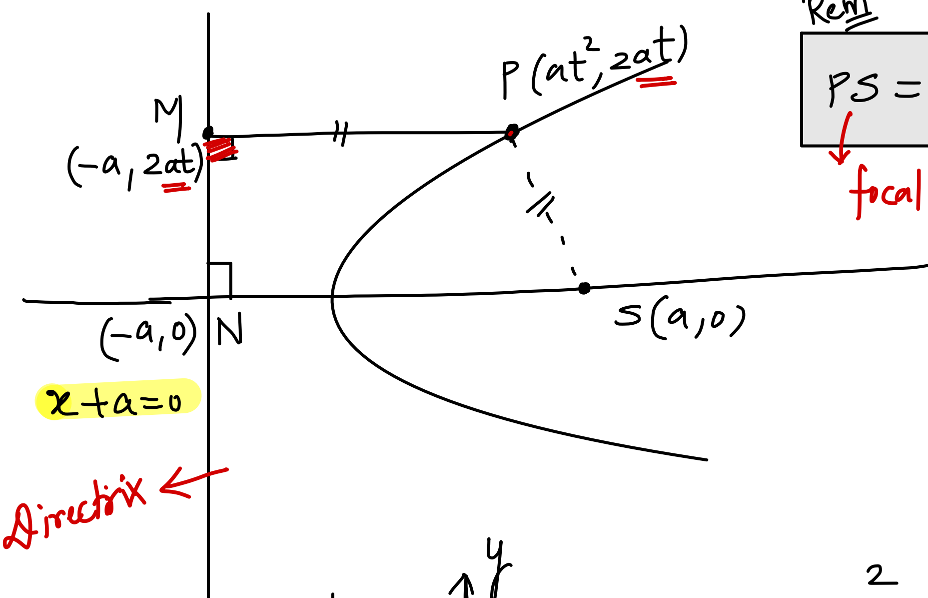
(ii) If axes is parallel to y-axes and vertex is (α, β).



Parametric Representation :-

$\boxed{y^2 = 4ax}$; $x = at^2$ & $y = 2at$; $t \in$ parameter

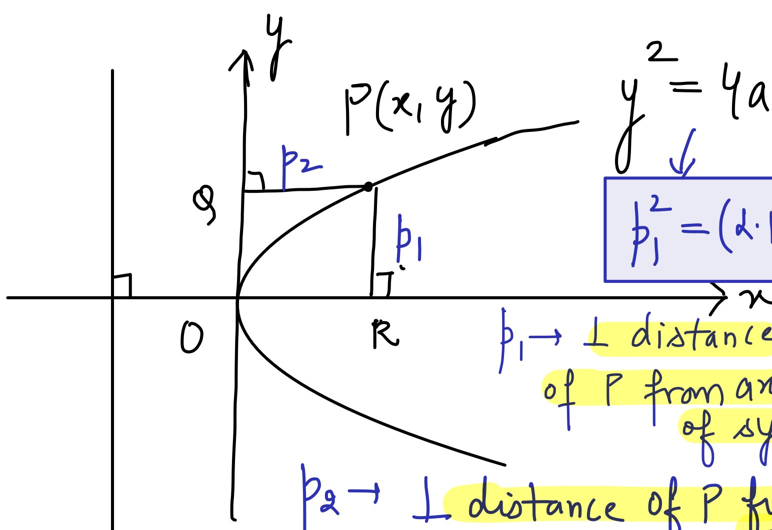
$\boxed{x^2 = 4by}$; $x = 2bt$ & $y = bt^2$; $t \in$ parameter



Rem

$$\boxed{PS = PM = a(1+t^2)}$$

focal distance



$y^2 = 4ax$

Rem

$$\boxed{p_1^2 = (L \cdot R)\, p_2}$$

$p_1 \rightarrow \perp$ distance of P from axes of sym.

$p_2 \rightarrow \perp$ distance of P from tangent at Vertex.

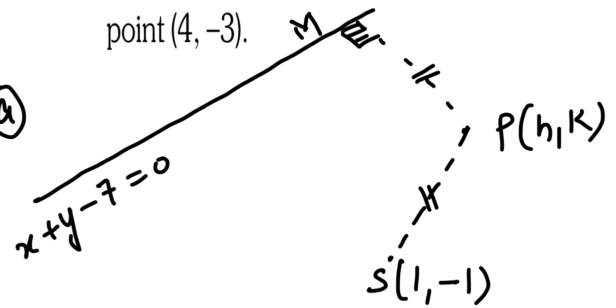
Q Find the equation of the parabola

(a) whose focus is $(1, -1)$ and directrix $x + y - 7 = 0$.

(b) whose vertex is the point $(4, -3)$ and whose latus rectum is 4 and whose axis is parallel to the x-axis.

(c) passing through the point $(-4, -7)$ and whose directrix is parallel to x-axis and whose vertex is the point $(4, -3)$.

(a)



$$\frac{PS}{PM} = e = 1$$

$$PS^2 = PM^2$$

$$(h-1)^2 + (k+1)^2 = \left(\frac{h+k-7}{\sqrt{2}}\right)^2$$

(b) $$(y+3)^2 = \pm 4(x-4)$$

(c) $$(x-4)^2 = 4b(y+3)$$

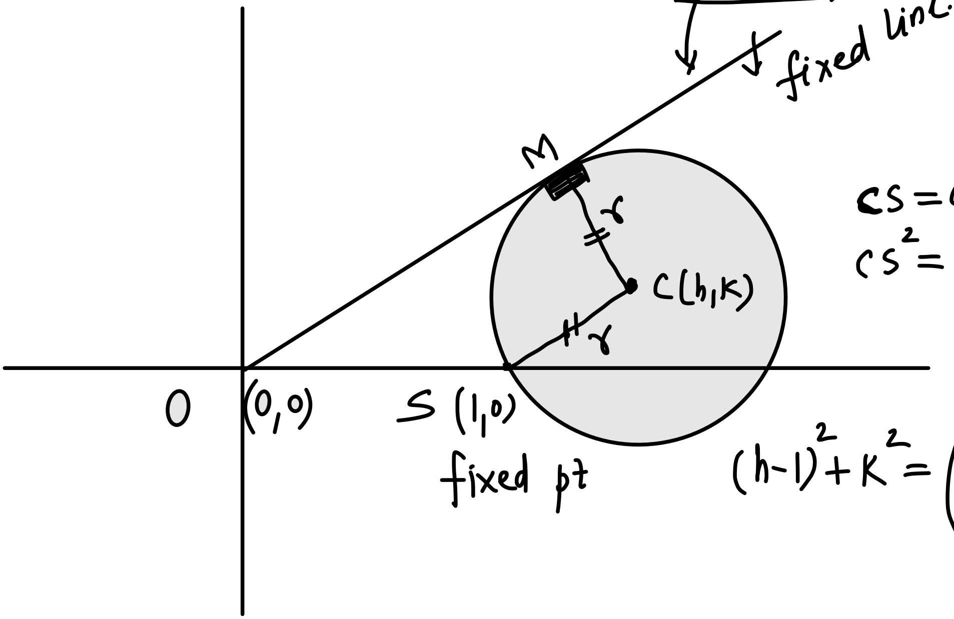
$(-4, -7)$

$$64 = 4b(-4) \Rightarrow \boxed{b = -4}$$

$$\therefore (x-4)^2 = -16(y+3).$$

Q A variable circle always passes through (1, 0) and touches the curve $y = \tan(\tan^{-1}x)$. Find the equation to the locus of its centre.

$$\boxed{y = x}$$

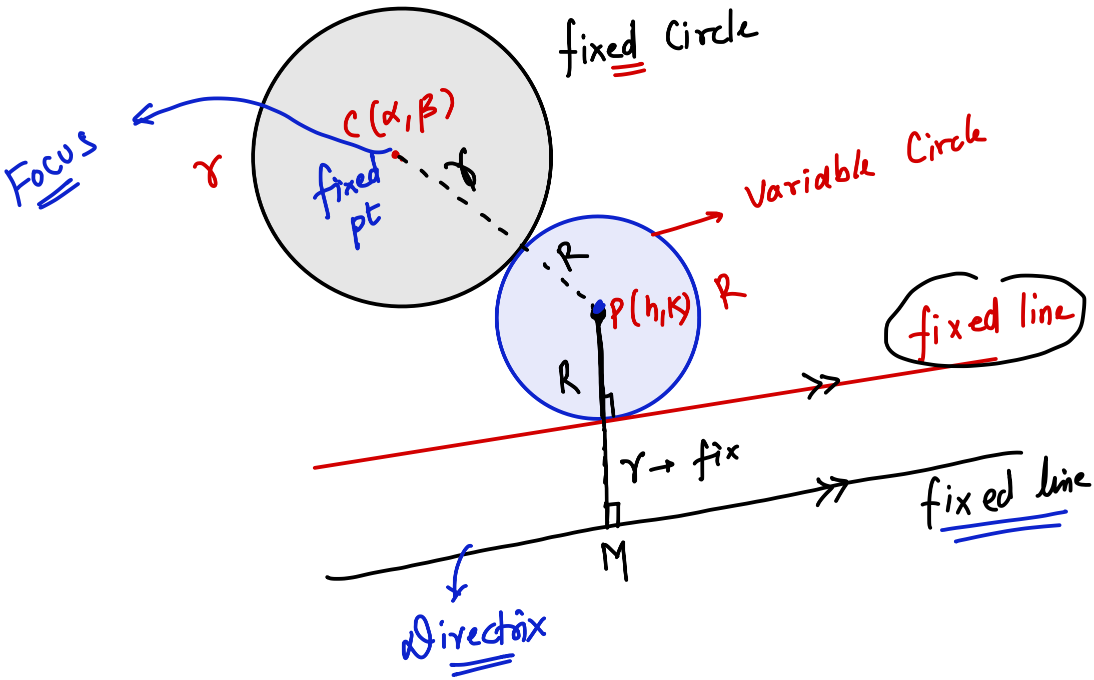


$$CS = CM$$

$$CS^2 = CM^2$$

$$(h-1)^2 + K^2 = \left(\frac{h-K}{\sqrt{2}}\right)^2$$

Q. If a variable circle touches a fixed circle and a fixed line then prove that the locus of the centre of the variable circle is a parabola whose directrix is parallel to a given line at a distance equal to the radius of the given circle.



Q Find everything for the parabola

$$4y^2 + 12x - 20y + 67 = 0$$

Sol^n

$$4y^2 - 20y = -12x - 67$$

$$4(\underbrace{y^2 - 5y}) = -12x - 67$$

$$4\left(\left(y - \frac{5}{2}\right)^2 - \frac{25}{4}\right) = -12x - 67.$$

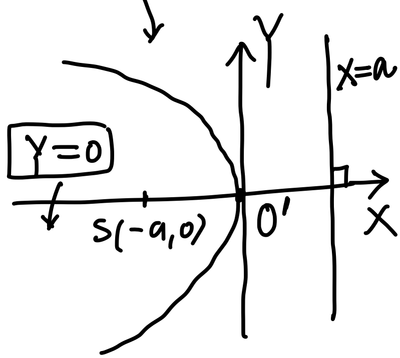
$$4\left(y - \frac{5}{2}\right)^2 = -12x - 42 \Rightarrow \left(y - \frac{5}{2}\right)^2 = -\frac{\cancel{12}^{\,3}}{\cancel{4}}\left(x + \frac{\cancel{42}^{\,7}}{\cancel{12}_{\,2}}\right)$$

$$\left(y - \frac{5}{2}\right)^2 = -3\left(x + \frac{7}{2}\right) \Rightarrow \boxed{Y^2 = -3X} \; ; \; a = \frac{3}{4}$$

$$X = x + 7/2$$
$$Y = y - 5/2$$

$$O(0,0) \longrightarrow O'\left(-\frac{7}{2}, \frac{5}{2}\right)$$

	(X, Y)	(x, y)
① Vertex	$(0,0)$	$\left(-\frac{7}{2}, \frac{5}{2}\right)$
② Focus	$\left(-\frac{3}{4}, 0\right)$	$\left(-\frac{3}{4} - \frac{7}{2}, \frac{5}{2}\right)$
③ Eqn of dir	$X = \frac{3}{4}$	$x + \frac{7}{2} = \frac{3}{4}.$
④ L.R	3	3
⑤ Axis	$Y = 0$	$y - \frac{5}{2} = 0.$
⑥ Tangent at Vertex	$X = 0$	$x + \frac{7}{2} = 0.$



POSITION OF A POINT RELATIVE TO A PARABOLA :

Let P(x, y) be the equation of parabola then to find the position of point P (x_1, y_1)

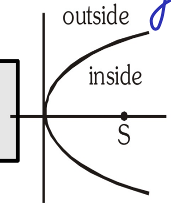
$y^2-4ax=0$

(a) Make the coefficient of x^2 and y^2 non-negative

(b) If $P(x_1, y_1) < 0$ $S_1<0$ $\Rightarrow$ point lies inside the parabola

(c) If $P(x_1, y_1) > 0$ $\Rightarrow$ point lies outside the parabola

$\boxed{S_1 \equiv y_1^2-4ax_1}$



Note : $\Rightarrow S_1>0$

(i) If point lies inside the parabola, then no tangent can be drawn.

(ii) If point lies on the parabola, then one tangent can be drawn.

(iii) If point lies outside the parabola, then two tangents can be drawn.

eg: Discuss position of point (3,4) wrt $y^2=8x$?

Sol^n

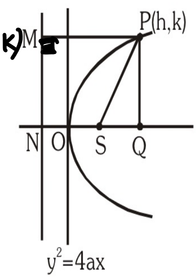
$$y^2-8x=0$$

$$S_1 \equiv 4^2-8(3) = 16-24<0$$

$\therefore$ (3, 4) lies inside parabola.

FOCAL DISTANCE/FOCAL RADII :

Let a point P(h, k) be on the parabola $y^2 = 4ax$, then its focal distance is PS which is equal to PM.



$$PM = ON + OQ = a + h$$

- For parabola $y^2 = 4ax$ or $y^2 = -4ax$, focal distance of any point P(h, k) is **($|h|$ + a)**
- For parabola $x^2 = 4ay$ or $x^2 = -4ay$, focal distance of any point P(h, k) is **($|k|$ + a)**
- For shifted parabola $(y - \beta)^2 = \pm 4a(x - \alpha)$, focal distance any point P(h,k) **($|h-\alpha|$ + a) (a > 0)**
- For shifted parabola $(x - \alpha)^2 = \pm 4a(y - \beta)$, focal distance any point P(h,k) **($|k-\beta|$ + a) (a > 0)**

Note :

(i) Minimum focal distance of any point on the parabola is 'a'.

(ii) If

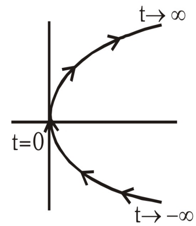
focal distance > a $\Rightarrow$ two such points are possible

focal distance = a $\Rightarrow$ only one point is possible which is vertex of the parabola

focal distance < a $\Rightarrow$ no such points are possible

PARAMETRIC COORDINATES :

	Parabola	Parametric Coordinates
(a)	$y^2 = 4ax$	$(at^2, 2at)$
(b)	$y^2 = -4ax$	$(-at^2, 2at)$
(c)	$x^2 = 4ay$	$(2at, at^2)$
(d)	$x^2 = -4ay$	$(2at, -at^2)$.
(e)	$(y-\beta)^2 = 4a(x-\alpha)$	$(\alpha+at^2, \beta+2at)$
(f)	$(x-\alpha)^2 = 4a(y-\beta)$	$(\alpha+2at, \beta+at^2)$

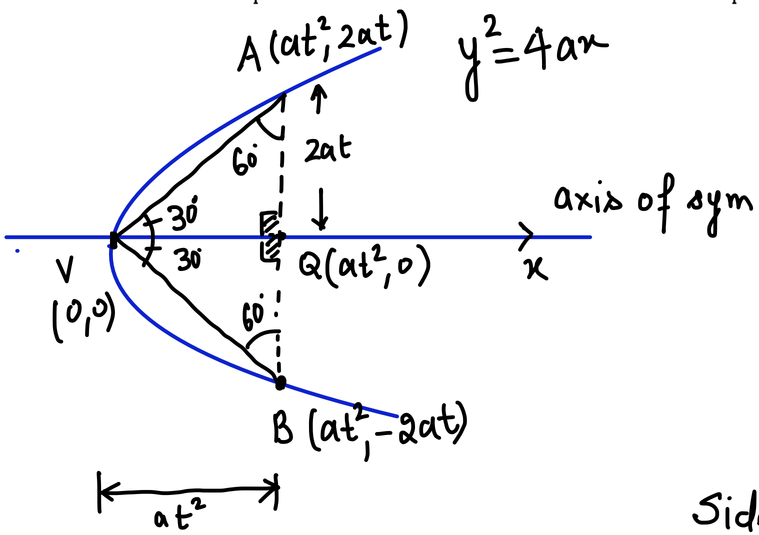


Variation of parameter (t) as point varies on parabola

E(1) Find the side length of an equilateral triangle inscribed in $y^2 = 4ax$, if one of its vertex coincides with the vertex of the parabola.

E(2) Obtain the locus of the point of trisection of double ordinates of the parabola $y^2 = 4ax$.

1



In Δ VQA :

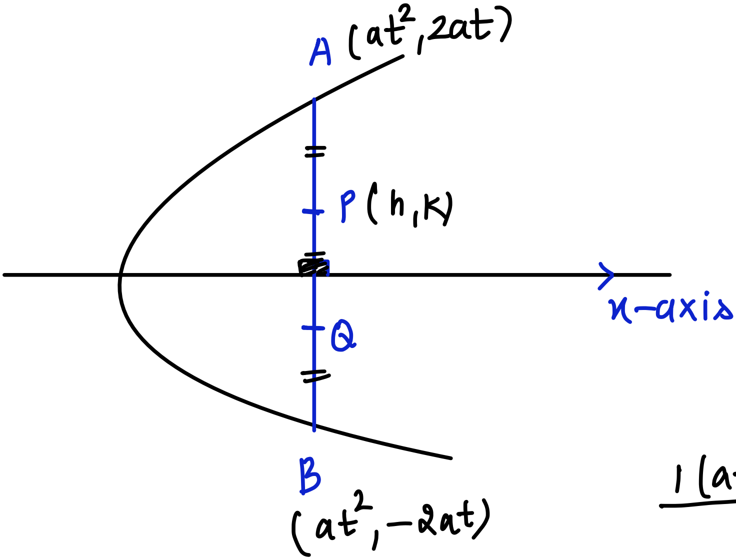
$$\tan 30^\circ = \frac{AQ}{VQ}$$

$$\frac{1}{\sqrt{3}} = \frac{2\cancel{a}\cancel{t}}{\cancel{a}t^{\cancel{2}}}$$

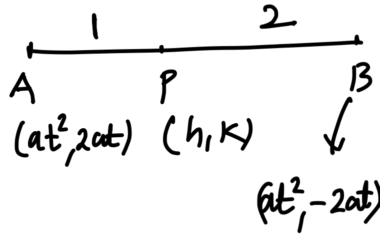
$$\boxed{t = 2\sqrt{3}}$$

Side $= 4at = \underline{\underline{8\sqrt{3}a}}$.

E(2) Obtain the locus of the point of trisection of double ordinates of the parabola $y^2 = 4ax$.



$AP = PQ = BQ.$



$$\frac{1(at^2) + 2at^2}{3} = h$$

$$\boxed{at^2 = h} \quad \text{— (i) —}$$

$$\frac{2at}{3} = K \quad \text{— (ii) —} \Leftarrow \frac{1(-2at) + 2(2at)}{3} = K$$

HW
Q

Identify the conic $\sqrt{\dfrac{x}{3}} + \sqrt{\dfrac{y}{2}} = 1$.

$$\left(\sqrt{\frac{x}{3}}\right)^2 = \left(1 - \sqrt{\frac{y}{2}}\right)^2$$

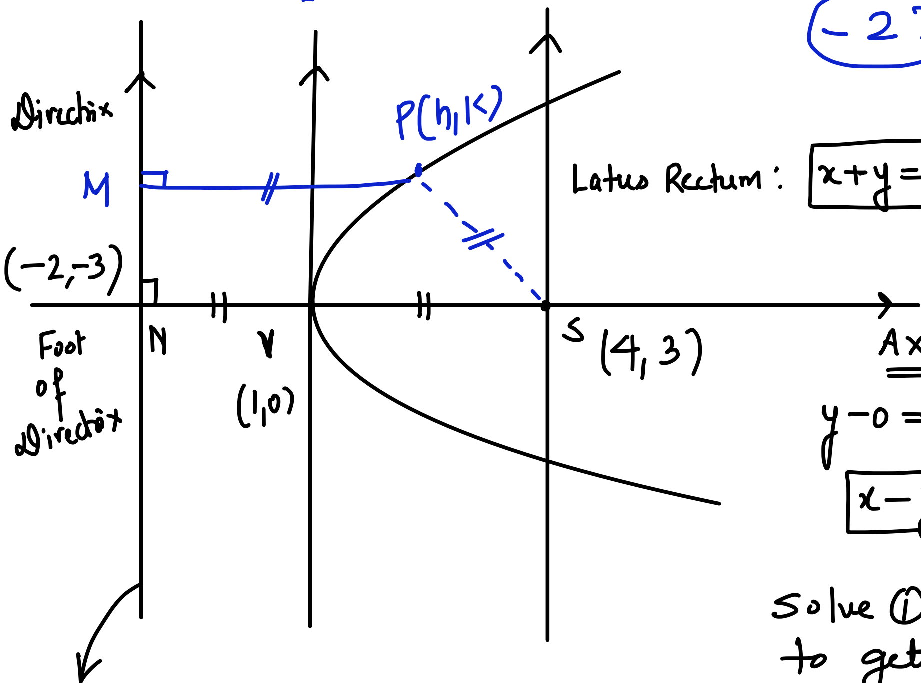
$$\frac{x}{3} = 1 + \frac{y}{2} - 2\sqrt{\frac{y}{2}}$$

$$\left(\frac{x}{3} - 1 - \frac{y}{2}\right)^2 = 4\cdot\left(\frac{y}{2}\right) \longrightarrow 2^{nd} \text{ curve}$$

Q Let (1,0) be the vertex and $x + y - 7 = 0$ be the equation of the latus rectum of a parabola and if the equation of parabola is $x^2 + y^2 - 2xy + px + qy + r = 0$, then $(p + q + r)$ is equal to

Sol^n

(-23) **Ans**



Latus Rectum : $\boxed{x+y=7} \rightarrow m=-1$ —①—

Axis of sym.

$y-0 = 1(x-1)$

$\boxed{x-y-1=0}$ —②—

Solve ① & ② to get S(Focus)

$\boxed{x=4,\ y=3}$

$y+3 = -1(x+2)$

$\boxed{x+y+5=0}$

$$PS^2 = PM^2$$

$$(h-4)^2+(k-3)^2 = \left(\frac{|h+k+5|}{\sqrt{2}}\right)^2$$

$$x^2+y^2-2xy-26x-22y+25=0$$

Q The equation of axis of the parabola $41(x^2 + y^2 + 4x + 4) = (5x - 4y + 1)^2$ is

Sol^n

$$(x^2+4x+4+y^2) = \frac{(5x-4y+1)^2}{41}$$

$$(x+2)^2+(y-0)^2 = \frac{(5x-4y+1)^2}{41}$$

$$\sqrt{(x+2)^2+(y-0)^2} = \frac{|5x-4y+1|}{\sqrt{41}}$$

$$PS = PM$$

Focus: $S(-2,0)$

Directrix: $5x-4y+1=0. \longrightarrow m=\frac{5}{4}$

Axis: $y-0 = -\frac{4}{5}(x+2)$

Alternate definition of parabola :

Consider to perpendicular lines L_1 & L_2

The locus of point P which moves such that the square of its distance from L_1 is proportional to the distance from L_2 is a parabola where proportionality constant is length of latus rectum i.e. $|4a|$ & L_1, L_2 are axis of parabola & tangent at the vertex respectively.

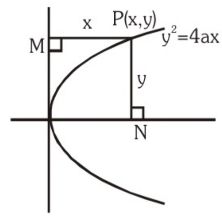
$(PN)^2 \propto PM$

$\Rightarrow \quad (PN)^2 = 4a(PM)$

$\Rightarrow \quad y^2 = 4ax$

$y \rightarrow$ length of perpendicular from P on axis of parabola.

$x \rightarrow$ length of perpendicular from P on tangent at vertex



$y^2 = 4ax$

$4a = \dfrac{y^2}{x}$

Generalised version :

If ax + by + c = 0 is its axis of parabola, bx − ay + d = 0 is its tangent at vertex and 4a is its length of latus rectum, then two parabolas are possible whose equations are :

$$\left(\frac{ax+by+c}{\sqrt{a^2+b^2}}\right)^2 = 4a\left|\frac{bx-ay+d}{\sqrt{a^2+b^2}}\right|$$

No point

Q. Find the points on the parabola $y^2 = 12x$ having focal radii (a) 4 (b) 3 (c) 2

Sol^n

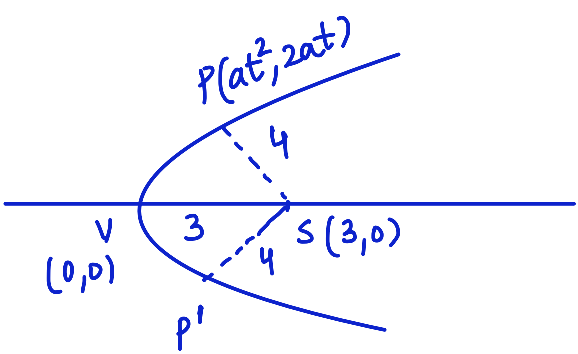
$4R = 12$

$\boxed{a = 3}$

(a) 2 points

(b) One point i.e Vertex (0,0)

(c) No point



$PS = a(1+t^2)$

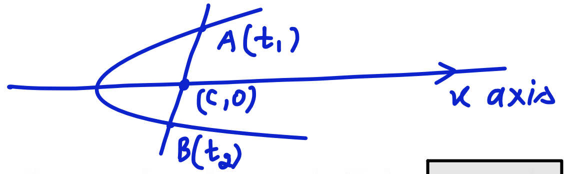
$4 = 3(1+t^2)$

$t^2 = \frac{1}{3} \Rightarrow t = \pm\frac{1}{\sqrt{3}}$

Rem

CHORD JOINING TWO POINTS **(t_1 and t_2) :**

The equation of a chord joining t_1 & t_2 on $y^2 = 4ax$ is $\boxed{2x - (t_1 + t_2)\,y + 2\,at_1 t_2 = 0}$ with $\boxed{\text{slope} = \dfrac{2}{t_1 + t_2}}$



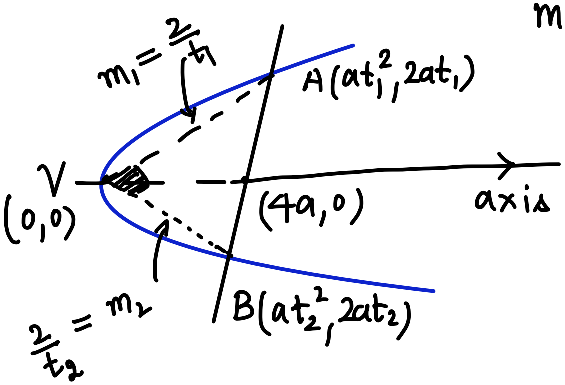
Note :

V.Imp

(1)* If the chord joining t_1 & t_2 pass through a point (c, 0), then $\boxed{t_1 t_2 = -c/a.}$

(2)* If chord joining t_1 & t_2 is a focal chord, then $\boxed{t_1 t_2 = -1.}$

(3)* If chord joining t_1 & t_2 subtends 90° at the vertex, then $\boxed{t_1 t_2 = -4}$ and chords passes through a fixed point (4a, 0).



$$m_1 m_2 = -1 \Rightarrow \frac{2}{t_1} \cdot \frac{2}{t_2} = -1 \Rightarrow \boxed{t_1 t_2 = -4}$$ *

$$Eq_{AB} : 2x - \underbrace{(t_1 + t_2)}_{\lambda} y + 2a\widetilde{t_1 t_2} = 0$$

$$\underbrace{(2x - 8a)}_{L_1} + \lambda \underbrace{y}_{L_2} = 0 \Rightarrow L_1 + \lambda L_2 = 0$$

$L_1 = 0$ & $L_2 = 0$

Length of focal chord :

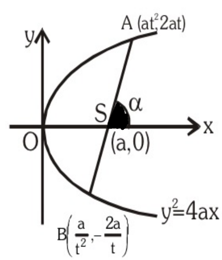
Length of focal chord through t = AB

= AS + SB

$= (a + at^2) + \left(a + \dfrac{a}{t^2}\right)$

Rem

$$= \mathbf{a}\left(\mathbf{t} + \frac{\mathbf{1}}{\mathbf{t}}\right)^2$$ **



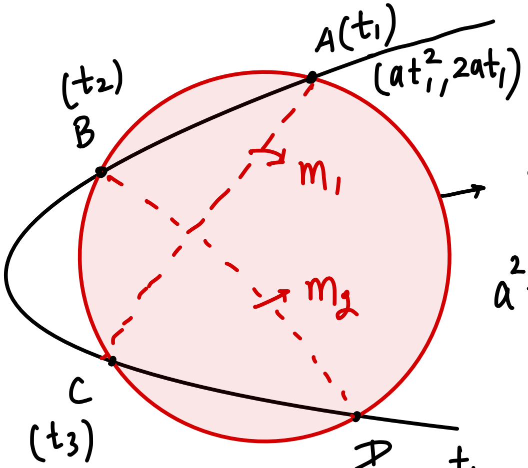
Rem

Note : Length of focal chord inclined with an angle of α with axis of parabola is **$4a\,\text{cosec}^2\alpha$.** *

** The shortest focal chord is Latus Rectum.

Q A circle and a parabola $y^2 = 4ax$ intersect in four points; show that the algebraic sum of the ordinates of the four points is zero. Also show that the line joining one pair of these four points and the line joining the other pair are equally inclined to the axis.

Sol^n



$y^2 = 4ax$ ✓ $\to (at^2, 2at) \leftarrow t_1, t_2, t_3, t_4$

$x^2+y^2+2gx+2fy+c=0$

$a^2t^4+4a^2t^2+2ag\,t^2+4aft+c=0$

$t_1, t_2, t_3, t_4 \to a^2t^4 + 0t^3 + (4a^2+2ag)t^2 + 4aft + c = 0$

$$\boxed{\Sigma t_1 = 0} \quad \text{—(i)—}$$

(i) $2at_1 + 2at_2 + 2at_3 + 2at_4 = 0.$

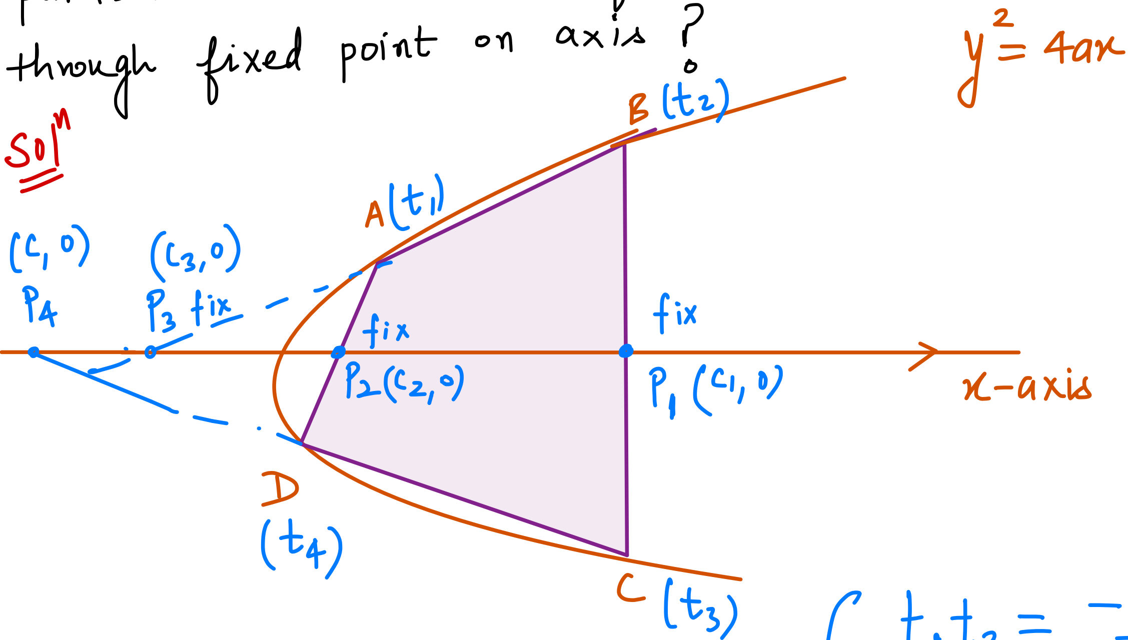
(ii) $m_1 = \dfrac{2}{t_1+t_3}$; $m_2 = \dfrac{2}{t_2+t_4}$

$$m_1+m_2 = \frac{2}{t_1+t_3} + \frac{2}{\underbrace{t_2+t_4}_{-(t_1+t_3)}} = 0.$$

(H.P)

Q A quadrilateral is inscribed in $y^2 = 4ax$ and 3 of its sides passes through fixed point on axis of parabola then show that fourth side also passes through fixed point on axis ?

Sol^n



$$t_2 t_3 = \frac{-c_1}{a}$$

$$t_1 t_4 = -\frac{c_2}{a}$$

$$t_1 t_2 = -\frac{c_3}{a}$$

$$t_3 t_4 = -\frac{c}{a}$$

Multiply

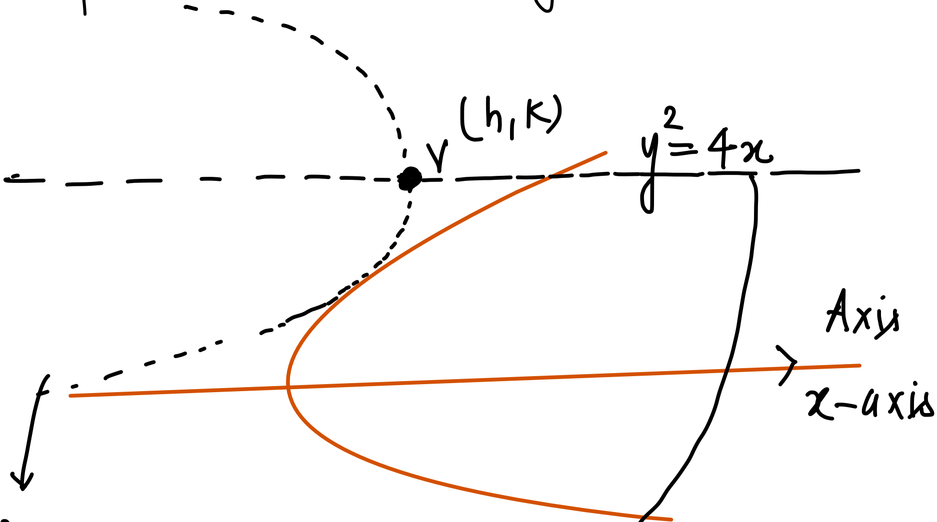
$$(t_1 t_2)(t_3 t_4) = \frac{c_1 c_2}{a^2}$$

$$\left(\frac{-c_3}{\cancel{a}}\right)\left(\frac{-c}{\cancel{a}}\right) = \frac{c_1 c_2}{\cancel{a^2}}$$

$$\Downarrow$$

$$c = \frac{c_1 c_2}{c_3} \rightarrow \text{fix}$$

Q A parabola touches an equal parabola $y^2=4x$. The axes of the parabola are parallel then find the locus of vertex of moving parabola?



$$(y-K)^2=-4(x-h)$$

$$y^2+K^2-2Ky=-4\left(\frac{y^2}{4}-h\right)$$

$$y^2+K^2-2Ky=-y^2+4h$$

$$2y^2-2Ky+K^2-4h=0$$

$$D=0 \Rightarrow \cancel{4}K^2-\cancel{4}\cdot 2\cdot(K^2-4h)=0$$

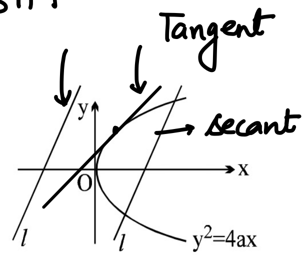
$$\boxed{y^2=8x} \text{ Ans} \Leftarrow -K^2+8h=0$$

Line and Parabola :–

$D = \cancel{4}(cm-2a)^2 - \cancel{4}m^2c^2 = 0$

$\cancel{4}a^2 - \cancel{4}amc = 0$

$\bar{S} \cap \bar{T}$



$y = mx + c \quad \& \quad y^2 = 4ax$

solving them together,

$$(mx + c)^2 - 4ax = 0$$

$$m^2x^2 + 2(cm - 2a)x + c^2 = 0 \qquad ....(1)$$

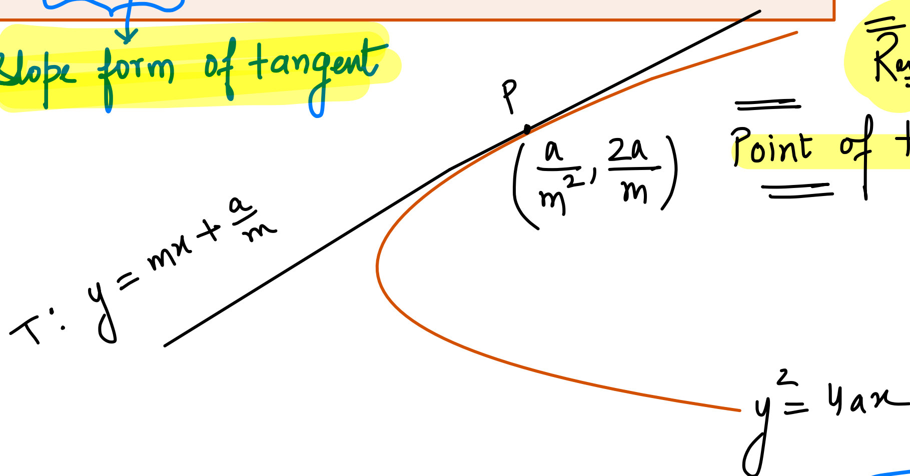
D > 0	D = 0	D < 0
line is a secant	Tangent	$\overline{T} \cap \overline{S}$

$\Downarrow$

Rem

now, $c = \dfrac{a}{m}$ gives the condition of tangency

hence $\underbrace{y = mx + \dfrac{a}{m}}_{\text{Slope form of tangent}}$ is always a tangent to $y^2 = 4ax$ for all $m \neq 0$.



Rem

Point of tangency

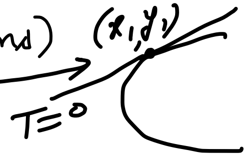
$$(at^2, 2at) \equiv \left(\frac{a}{m^2}, \frac{2a}{m}\right)$$

$$ty = x + at^2 \Rightarrow y = \frac{x}{t} + at.$$

$$y = mx + \frac{a}{m}$$

$\boxed{t \to \frac{1}{m}}$ *

TANGENTS TO THE PARABOLA $y^2 = 4ax$: (in Various forms)

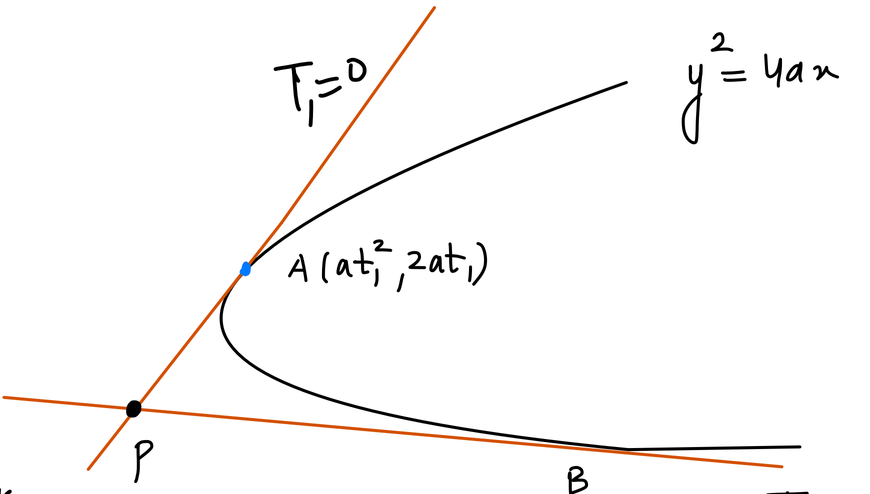


(i) Cartesian form : $yy_1 = 2a(x + x_1)$ at the point (x_1, y_1) ★

(ii) Parametric form : $yt = x + at^2$ at $(at^2, 2at)$ with slope $m = 1/t$.

$x_1 \to at^2$
$y_1 \to 2at$

Note : Point of intersection of two tangents at t_1 & t_2 is $(at_1t_2, a(t_1 + t_2))$. * *



** $\left(at_1t_2,\ a(t_1+t_2)\right)$ $\quad$ $(at_2^2, 2at_2)$ $\quad$ $T_2=0$

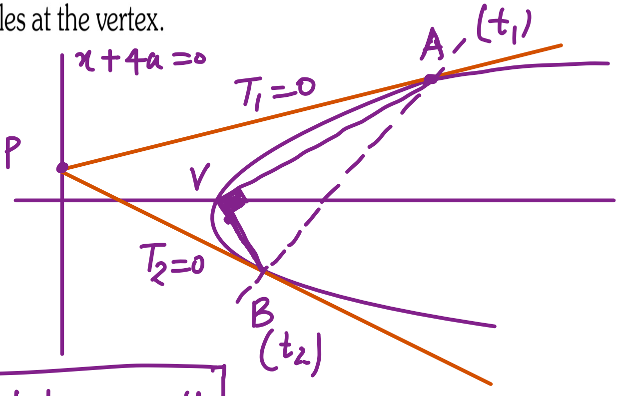
$at_1t_2 \downarrow$ GM of abscissa of A & B

$a(t_1+t_2) \downarrow$ AM of ordinates of A & B.

E(1) From a point on the line $x + 4a = 0$ a pair of tangent are drawn to the parabola $y^2 = 4ax$. Show that their chord of contact subtends a right angles at the vertex.

Sol^n

$(-4a, \alpha)\ P$



$$-4a = \sqrt{at_1^2 \cdot at_2^2}$$

$$-4\cancel{a} = \cancel{a}\, t_1 t_2 \Rightarrow \boxed{t_1 t_2 = -4}$$

(H.P)

E(2) A tangent to the parabola $y^2 = 8x$ makes an angles of $45°$ with the line $y = 3x + 5$. Find its equation and also its point of contact.

Sol^n $y^2 = 8x$; $a = 2$

Let eqn of tangent be

$y = mx + c$; $c = \frac{a}{m}$

$$\boxed{T: \ y = mx + \frac{2}{m}}$$

$$\tan 45° = \left| \frac{m-3}{1+3m} \right|$$

$$\pm 1 = \frac{m-3}{1+3m}$$

$(+)$: $1 + 3m = m - 3 \Rightarrow m = -2$

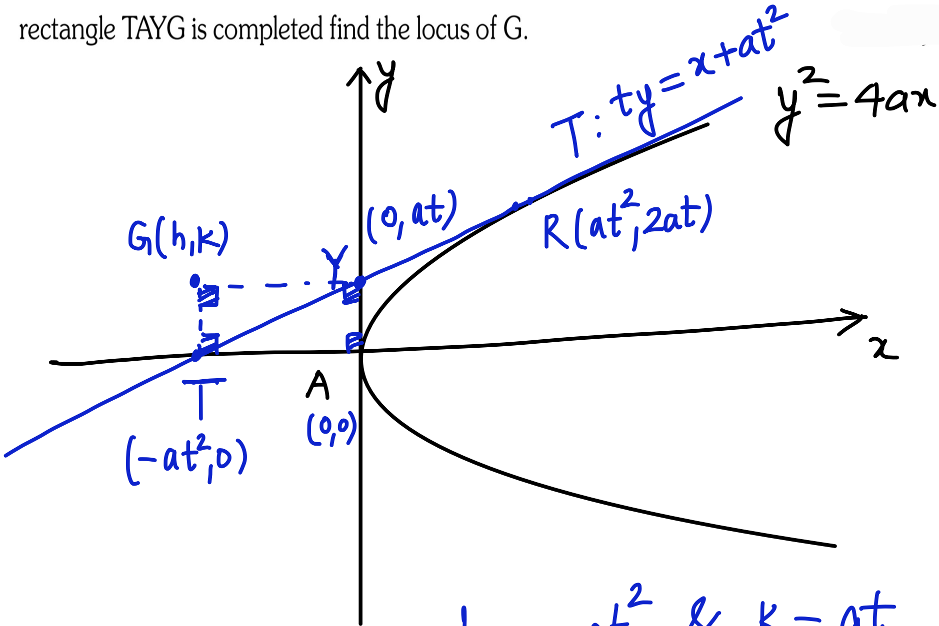
$(-)$: $-1 - 3m = m - 3 \Rightarrow m = \frac{1}{2}$

Pt of Contact

$$\left(\frac{a}{m^2}, \frac{2a}{m}\right)$$

E(3) Let the tangent to the parabola $y^2 = 4ax$ meets the axis in T and the tangent at the vertex A in Y. If the rectangle TAYG is completed find the locus of G.

Sol^n



$$h = -at^2 \quad \& \quad k = at$$

$$h = -a\left(\frac{k}{a}\right)^2$$

$$x = -\frac{y^2}{a} \Rightarrow \boxed{y^2 + ax = 0}$$

Ans

Q* Find the equation of line touching both the parabola's $y^2 = 4x$ and $x^2 = -32y$? Also find their point of contacts on them. and length of common tangents ?

Sol^n Eqn of tangent to $y^2 = 4x$ can be written as $\boxed{y = mx + \frac{1}{m}}$ — ① —

$\downarrow$ $x^2 = -32y$

$$x^2 = -32\left(mx + \frac{1}{m}\right)$$

$$mx^2 + 32m^2x + 32 = 0.$$

$\boxed{D = 0}$ $\quad (32m^2)^2 - 4.m.(32) = 0$

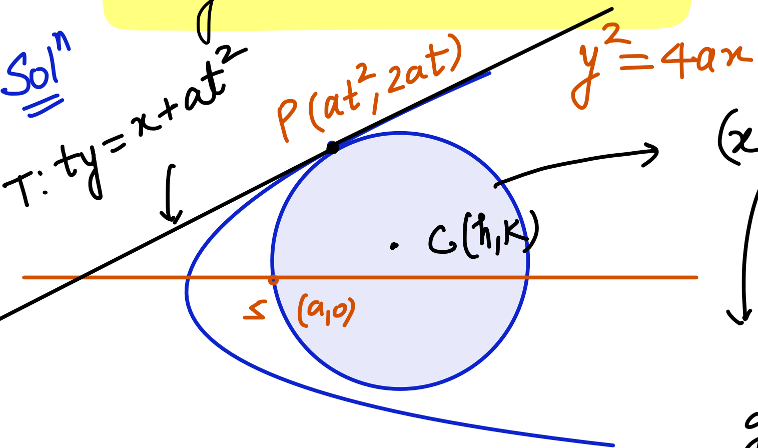
$$32m^4 - 4m = 0$$

$$4m(8m^3 - 1) = 0$$

$m \neq 0$; $\quad m^3 = \frac{1}{8} \Rightarrow m = \frac{1}{2}$

Q P.T equation of circle which pass through focus and touches $y^2 = 4ax$ at $P(at^2, 2at)$ is $x^2 + y^2 - ax(3t^2+1) - ay(3t - t^3) + 3a^2t^2 = 0$. If 't' varies then prove that locus of centre of circle is $27ay^2 = (2x-a)(x-5a)^2$?

Sol^n



$$(x-at^2)^2 + (y-2at)^2 + \lambda(ty - x - at^2) = 0$$

$S(a,0)$

get $\lambda = (\quad)$

$$h = \frac{a(3t^2+1)}{2} \quad -①- \qquad ; \qquad k = \frac{a(3t-t^3)}{2} \quad -②-$$

Alt:

$$(2k)^2 = a^2 \cdot t^2 (3-t^2)^2$$

$$k = \frac{a}{2} t (3 - t^2)$$

$$2k = at\left(3 - \frac{\left(\frac{2h}{a} - 1\right)}{3}\right)$$

$$t = (\quad)$$

Q A pair of tangents are drawn which are equally inclined to a straight line $y = mx + c$ whose inclination to the axis is α, prove that the locus of their point of intersection is the straight line

HW

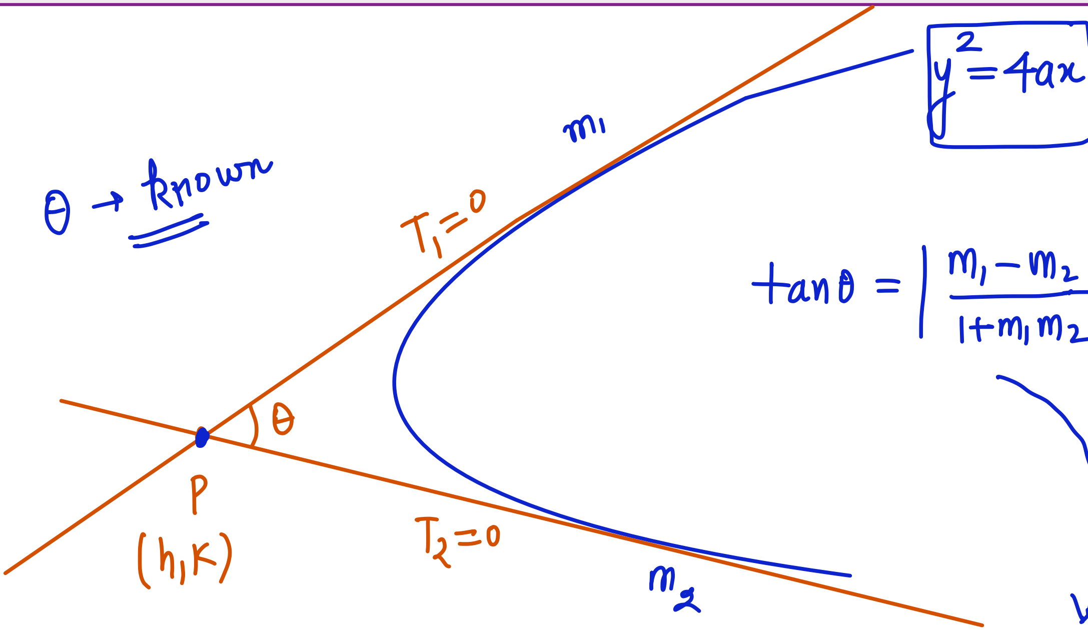
$$y = (x - a) \tan 2\alpha.$$

Q HW Prove that on the axis of any parabola there is a certain point K which has the property that, if a chord PQ of the parabola be drawn through it, then

$$\frac{1}{PK^2} + \frac{1}{QK^2} \text{ is the same for all positions of the chord.}$$

$\star$ To find the locus of all those points from where parabola $y^2 = 4ax$ can be seen at an angle θ.

$\theta \rightarrow$ known



$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$(1+m_1m_2)^2 \tan^2\theta = (m_1 - m_2)^2$$

$$(1+m_1m_2)^2 \tan^2\theta = (m_1 + m_2)^2 - 4m_1m_2$$

$$T: \quad y = mx + \frac{a}{m} \leftarrow (h,k)$$

$$mk = m^2h + a \Rightarrow \boxed{m^2h - mk + a = 0}$$

$m_1 \quad m_2$

$$m_1m_2 = \frac{a}{h} \; ; \quad m_1 + m_2 = \frac{k}{h}$$

Note: If $\theta = \frac{\pi}{2}$ then locus of P will be director Circle of $y^2 = 4ax$

$$m_1 m_2 = -1$$

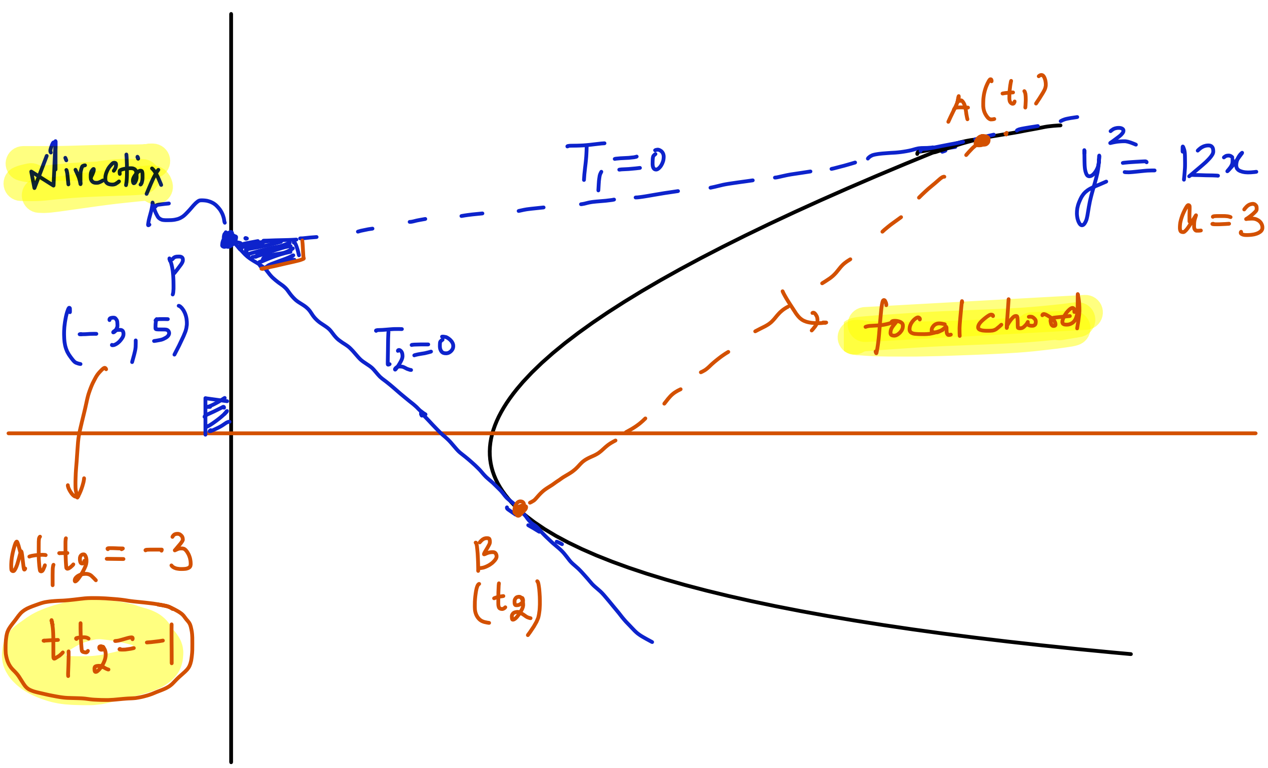
$$\frac{a}{h} = -1 \Rightarrow h + a = 0$$

$$\Downarrow$$

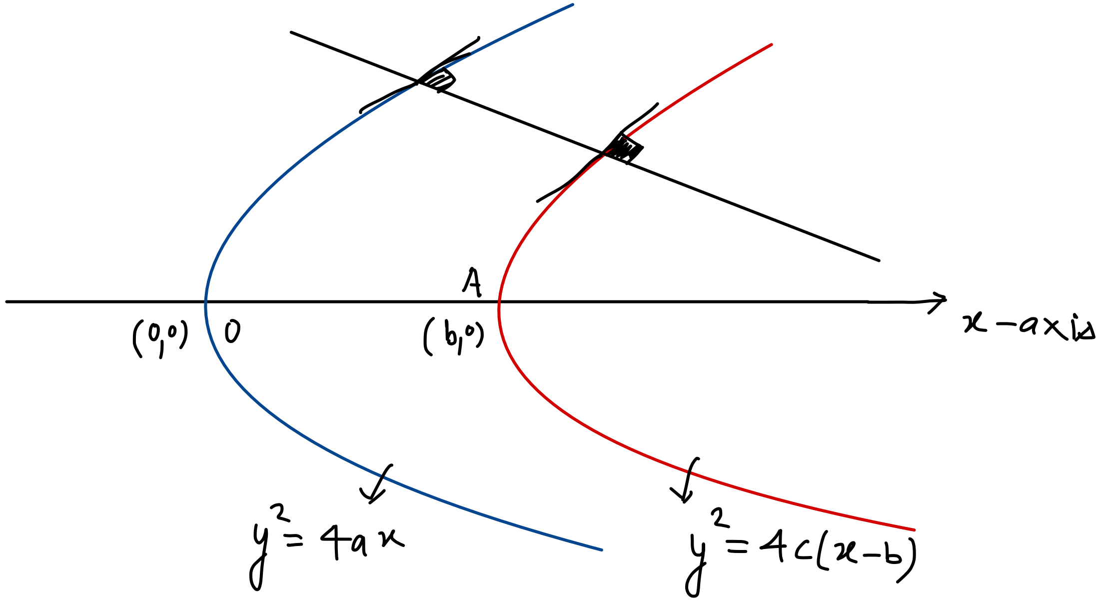
Rem

$$\boxed{x + a = 0}$$

Director Circle of parabola is its own directrix.



Q *Imp* Prove that the two parabolas $y^2 = 4ax$ and $y^2 = 4c(x-b)$ cannot have a common normal, other than the axis, unless $\frac{b}{a-c} > 2$. In other words this gives the condition for the two curves to have a common normal other than x-axis.



Normal to $y^2 = 4ax$ is :-

$$N_1: \; y = mx - 2am - am^3 \quad -①-$$

Normal to $y^2 = 4c(x-b)$ is :-

$$N_2: \; y = m(x-b) - 2cm - cm^3 \quad -②-$$

Both must be same :-

$$-2am - am^3 = -mb - 2cm - cm^3.$$

$$(c-a)m^{\cancel{3}\,2} = (2a-b-2c)\cancel{m} \quad ; \quad m \neq 0$$

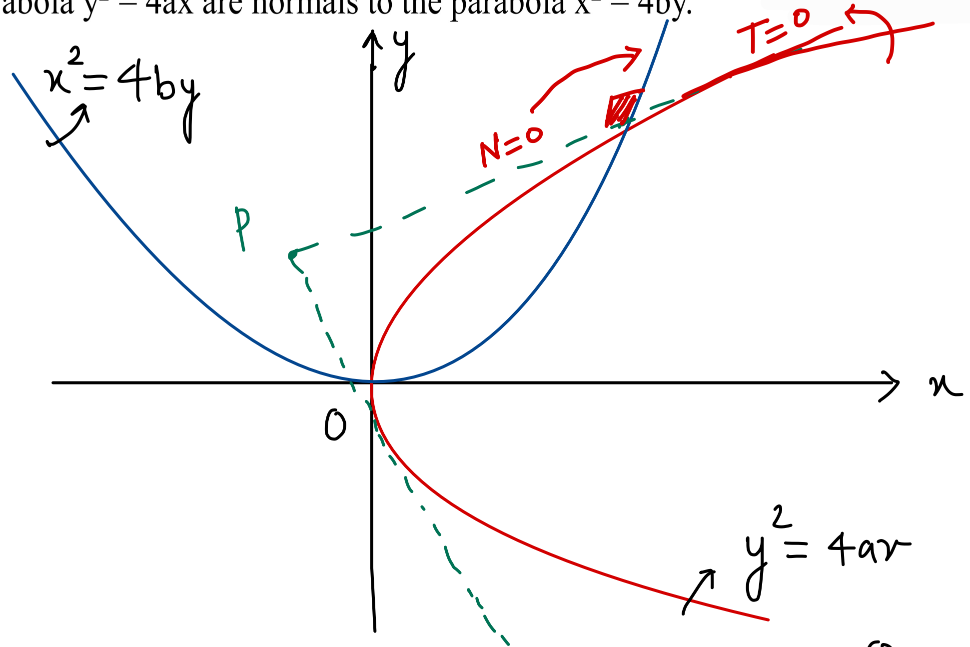
$$m^2 = \frac{2(a-c) - b}{(c-a)} = -2 - \frac{b}{c-a} > 0$$

$$\frac{b}{a-c} > 2$$

Q If $a^2 > 8b^2$, prove that a point can be found such that the two tangents from it to the parabola $y^2 = 4ax$ are normals to the parabola $x^2 = 4by$.

Imp

Sol^n



Tangent to $y^2 = 4ax$ is : $y = mx + \frac{a}{m}$ — ① —

Normal to $x^2 = 4by$ is : $y = mx + 2b + \frac{b}{m^2}$ — ② —

Both are same acc to question :-

$$\frac{a}{m} = 2b + \frac{b}{m^2} \Rightarrow am = 2bm^2 + b$$

$$2bm^2 - am + b = 0 \Rightarrow \boxed{D > 0}$$

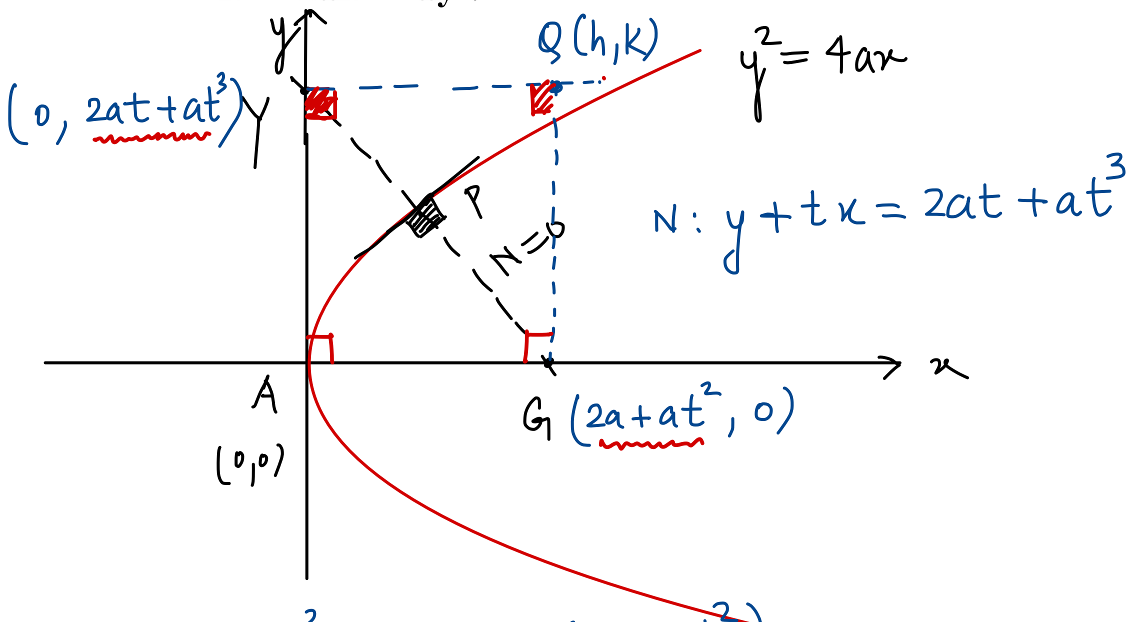
$$a^2 - 4(2b)(b) > 0$$

$$a^2 > 8b^2 \quad \text{(H.P.)}$$

Q HW

The normal at any point P meets the axis in G and the tangent at the vertex in Y, if A be the vertex and the rectangle GAYQ be completed, prove that the equation to the locus of Q is

$$\mathbf{x^3 = 2ax^2 + ay^2}.$$



$h = 2a + at^2$; $K = t(2a + at^2)$

$K = t(h) \Rightarrow \boxed{t = \frac{K}{h}}$

$h = 2a + a \cdot \left(\frac{K^2}{h^2}\right) \Rightarrow h^3 = 2ah^2 + aK^2$

$x^3 = 2ax^2 + ay^2$ (H.P.)

RULES OF TRANSFORMATION :

In the previous articles all the results have been proved for the particular parabola $y^2 = 4ax$ $(a > 0)$.

However all the results with slight transformations are valid for any shifted parabola.

If any equation derived for the parabola $y^2 = 4ax$. $(a > 0)$ is $R(x,y,a,m)$ then result of other parabolas are as follows

$y^2 = 4ax$	$y^2 = -4ax$	$x^2 = 4ay$	$x^2 = -4ay$	$(y-\beta)^2 = 4a(x-\alpha)$	$(x-\alpha)^2 = 4a(y-\beta)$
$R(x,y,a,m)$	$R(x,y-a,m)$	$R\left(y,x,a,\frac{1}{m}\right)$	$R\left(y,x,-a,\frac{1}{m}\right)$	$R(x-\alpha, y-\beta, a,m)$	$R(y-\beta, x-\alpha, a, \frac{1}{m})$

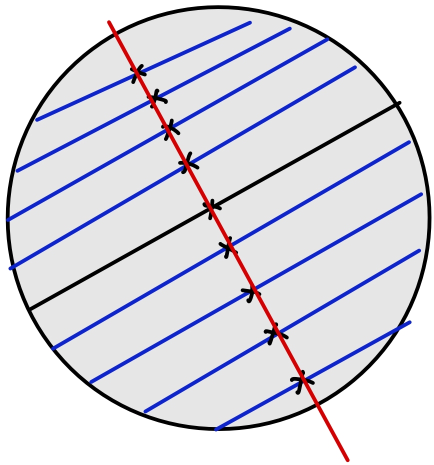
Using the above rule equation of tangent to parabola $x^2 = 4ay$ is $\mathbf{x = \frac{y}{m} + am \Rightarrow y = mx - am^2}$

Similary equation of normal to the parabola $x^2 = 4ay$ is $\mathbf{x = \frac{y}{m} - \frac{2a}{m} - \frac{a}{m^3} \Rightarrow y = mx + 2a + \frac{a}{m^2}}$ and so on.

DIAMETER :

$$y = mx + c$$

m ↓ fix, c ↓ Variable



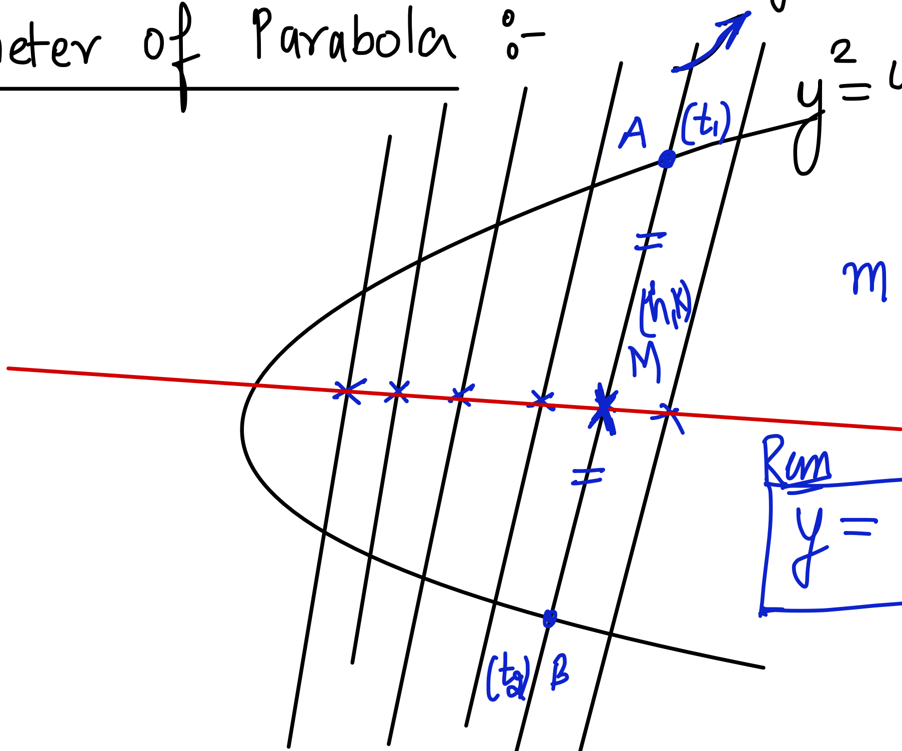
Locus of middle point of the system of parallel chords is known as diameter.

$$y = mx + c$$

m → fix, c → Var.

Diameter of Parabola :-

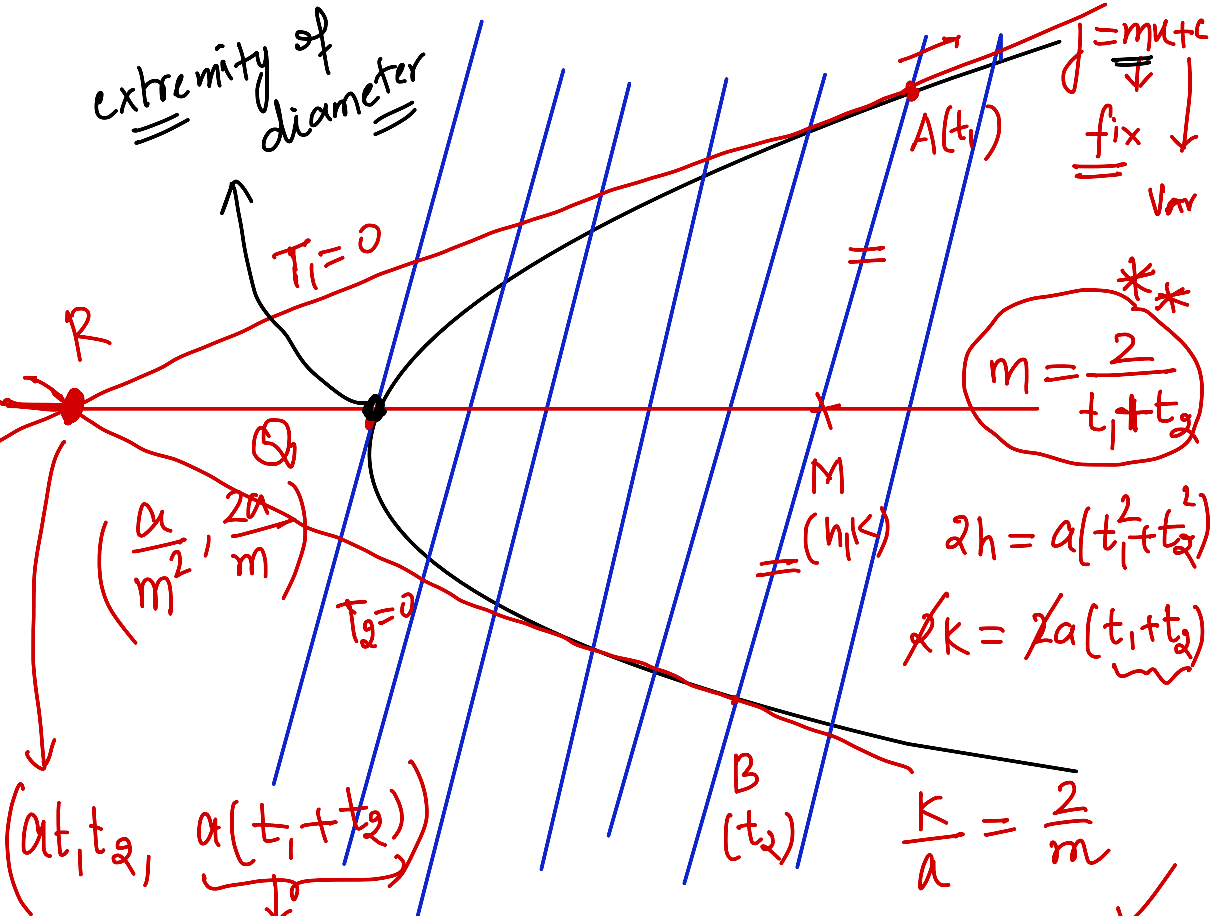
$$y^2 = 4ax$$



$$m = \frac{2a}{k}$$

Rem

$$\boxed{y = \frac{2a}{m}}$$



$= \frac{2a}{m}$

$$\boxed{y = \frac{2a}{m}}$$

|| to axis of parallel

$$\boxed{y = mx + \frac{a}{m}}$$

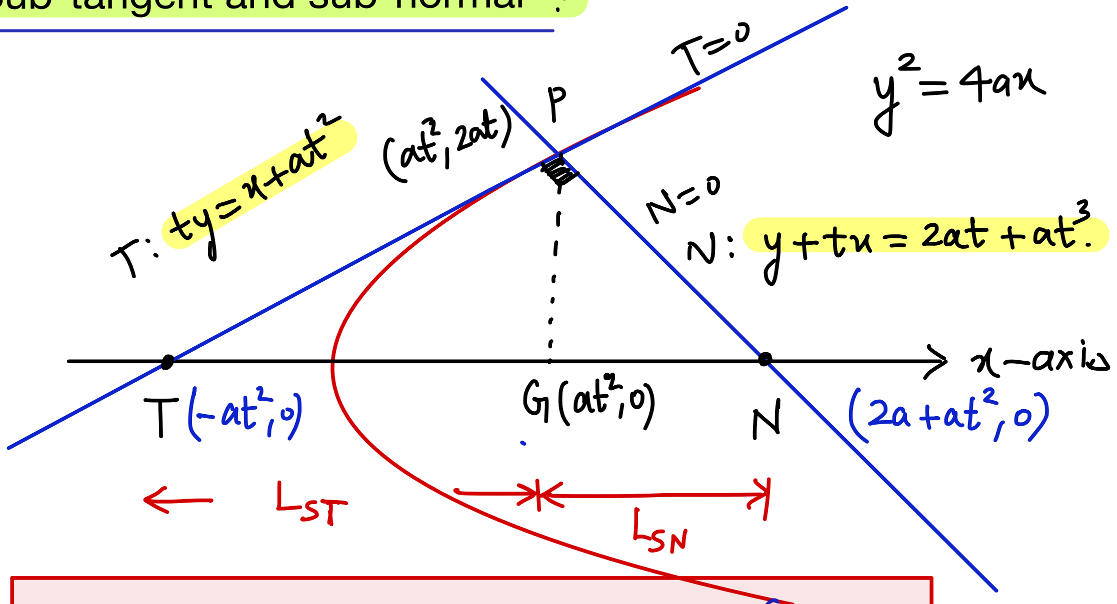
Eqn of tangent at extremity of diameter

$$\boxed{y = \frac{2a}{m}} \rightarrow$$ Eqn of diameter of $y^2 = 4ax$ corresponding to system of || chords $\boxed{y = mx + c}$

The point of intersection of tangent at A & B is $\left(at_1 t_2,\ a(t_1+t_2)\right) \equiv$

$\left(at_1 t_2,\ \frac{2a}{m}\right)$ and hence we can say that tangents at ends of any chord of parabola meets on the diameter which bisects the chord.

Sub-tangent and sub-normal :-



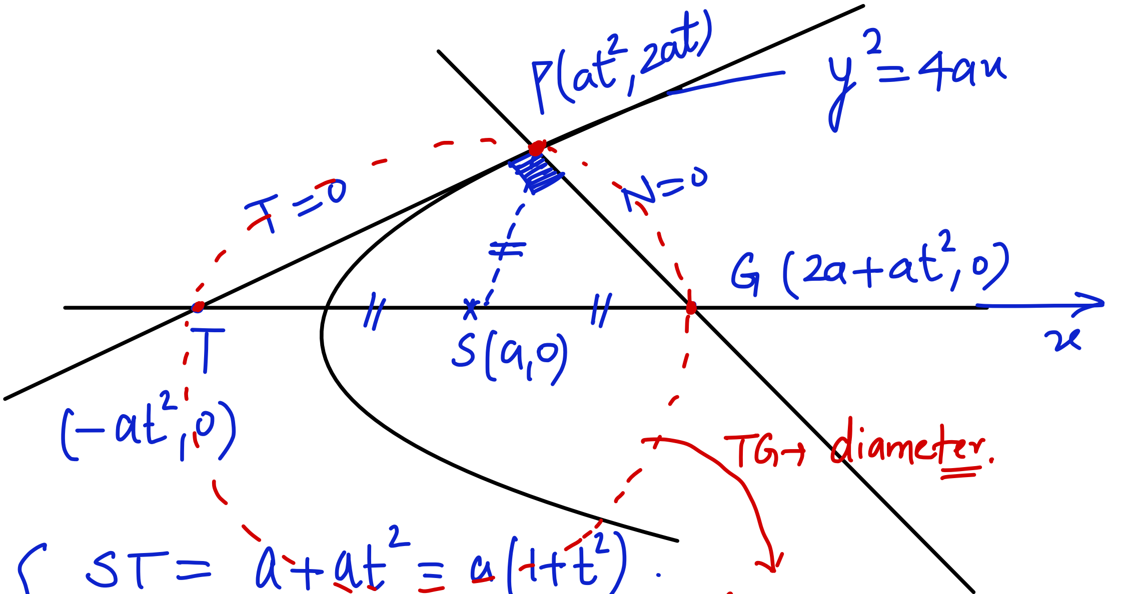
$L_{SN} = 2a$ = Constant = semi (L·R)

$L_{ST} = 2at^2$ = twice of abscissa of point of tangent.

* Highlights to be remembered
(Proof of all won't be given)

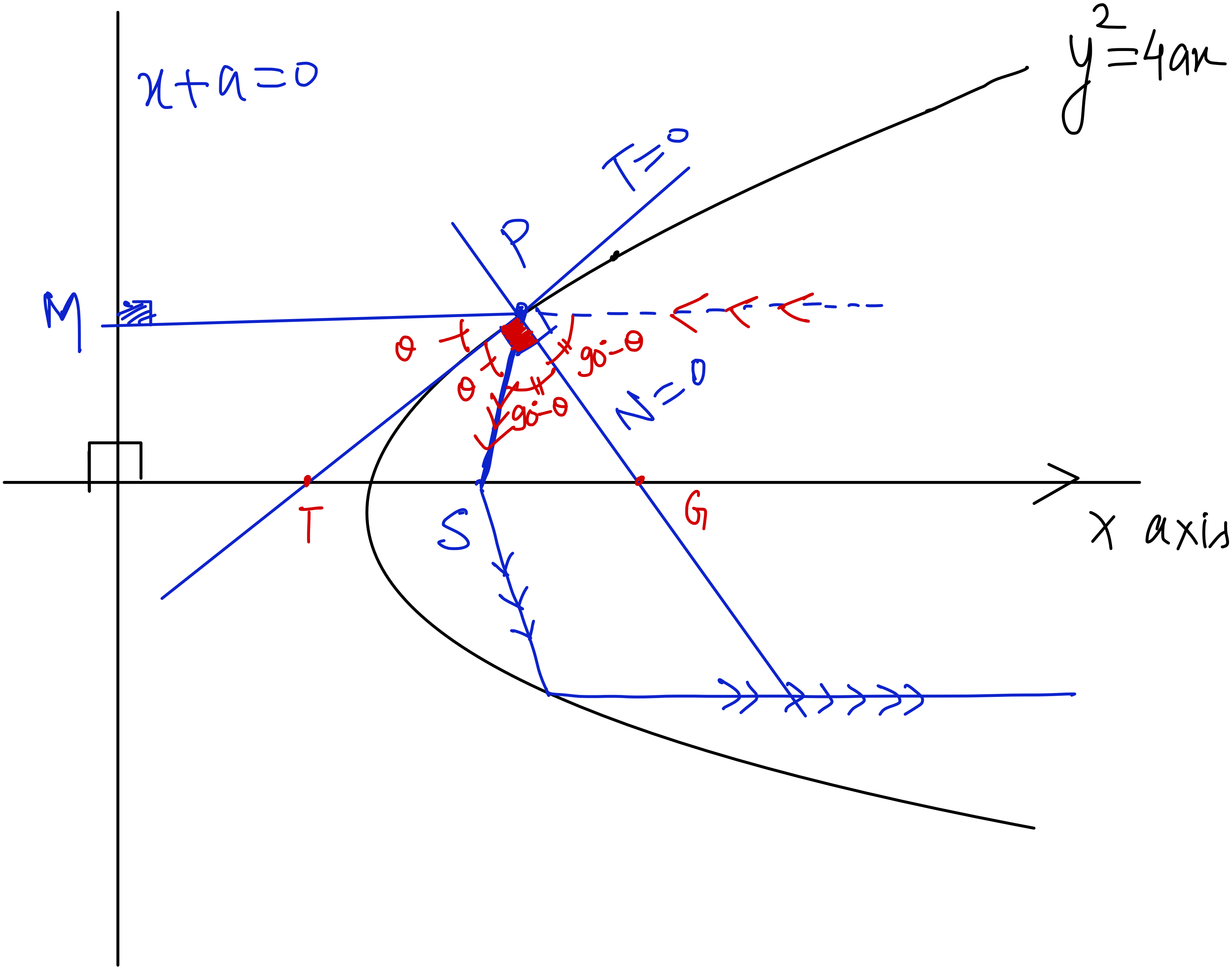
IMPORTANT HIGHLIGHTS :

(1) If the tangent & normal at any point 'P' of the parabola intersect the axis at T & G then ST = SG = SP where 'S' is the focus. In other words the tangent and the normal at a point P on the parabola are the bisectors of the angle between the focal radius SP & the perpendicular from P on the directrix. From this we conclude that all rays emanating from S will become parallel to the axis of the parabola after reflection.



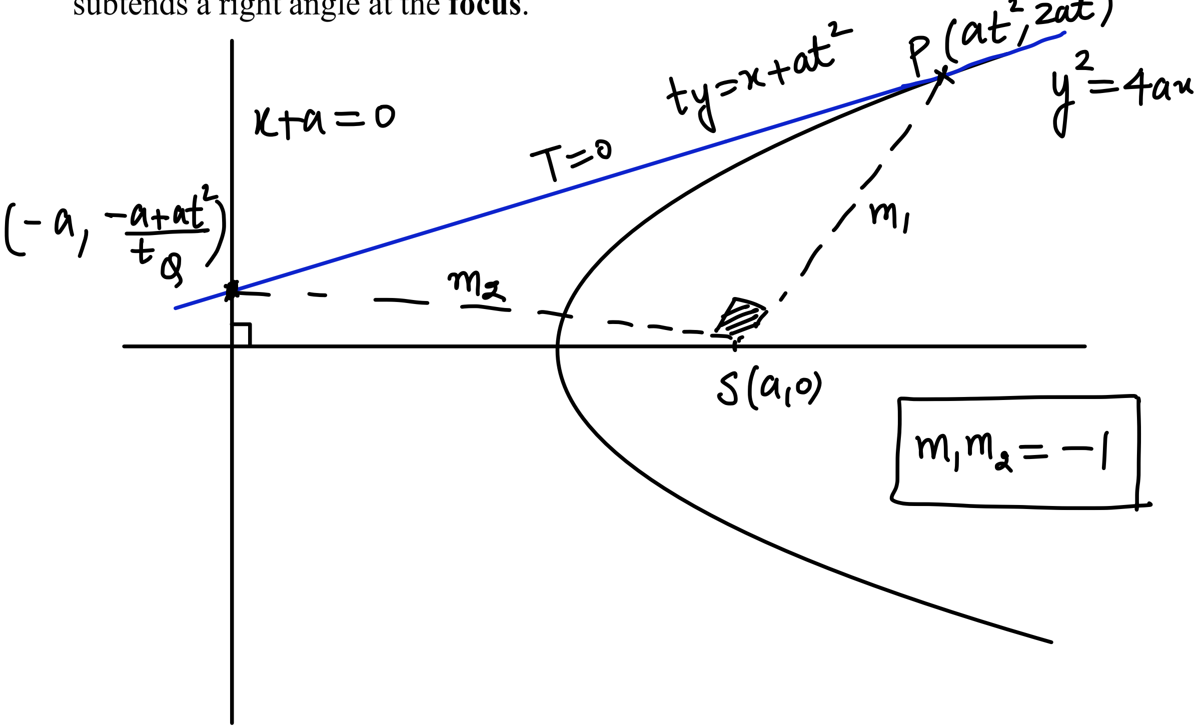
$$\begin{cases} ST = a + at^2 = a(1+t^2). \\ SG = a + at^2. \\ PS = a(1+t^2). \end{cases}$$

Circle Circumscribing Δ TPG is Circle drawn with focus (S) as Centre & radius $= a(1+t^2)$

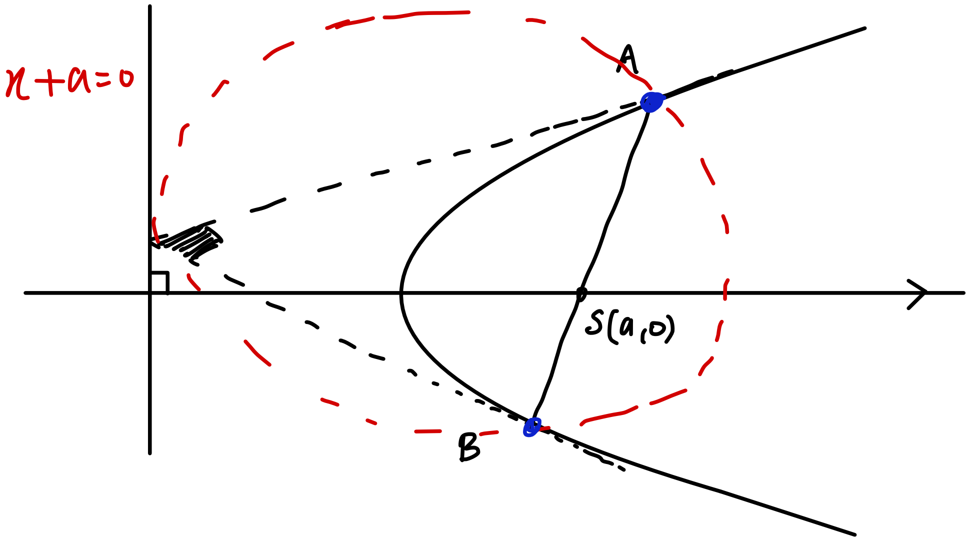


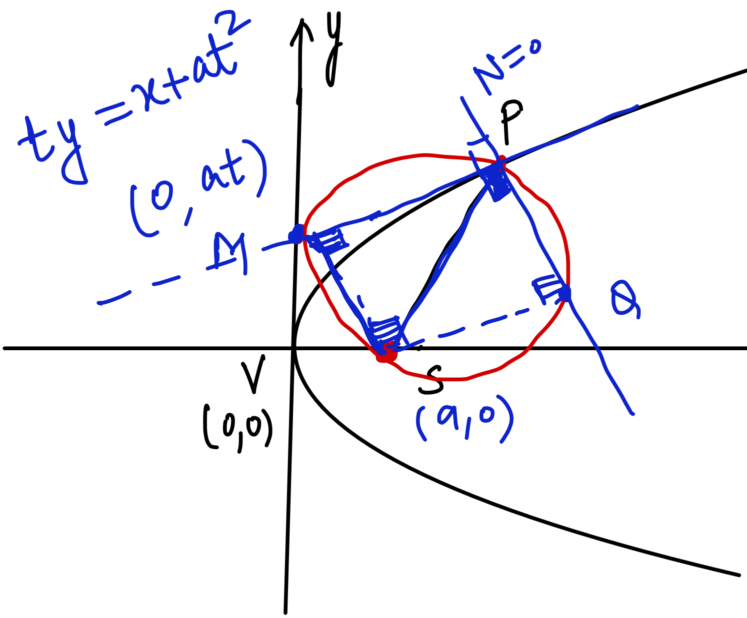
$y^2=4ax$
$x+a=0$
$T=0$
$N=0$
P
M
T
S
G
θ
$90°-\theta$
X axis

② The portion of a tangent to a parabola cut off between the directrix & the curve subtends a right angle at the **focus**.



③ The tangents at the extremities of a focal chord intersect at right angles on the **directrix**, and hence a circle on any focal chord as diameter touches the directrix. Also a circle on any focal radii of a point P $(at^2, 2at)$ as diameter touches the tangent at the vertex and intercepts a chord of length $a\sqrt{1+t^2}$ on a normal at the point P.





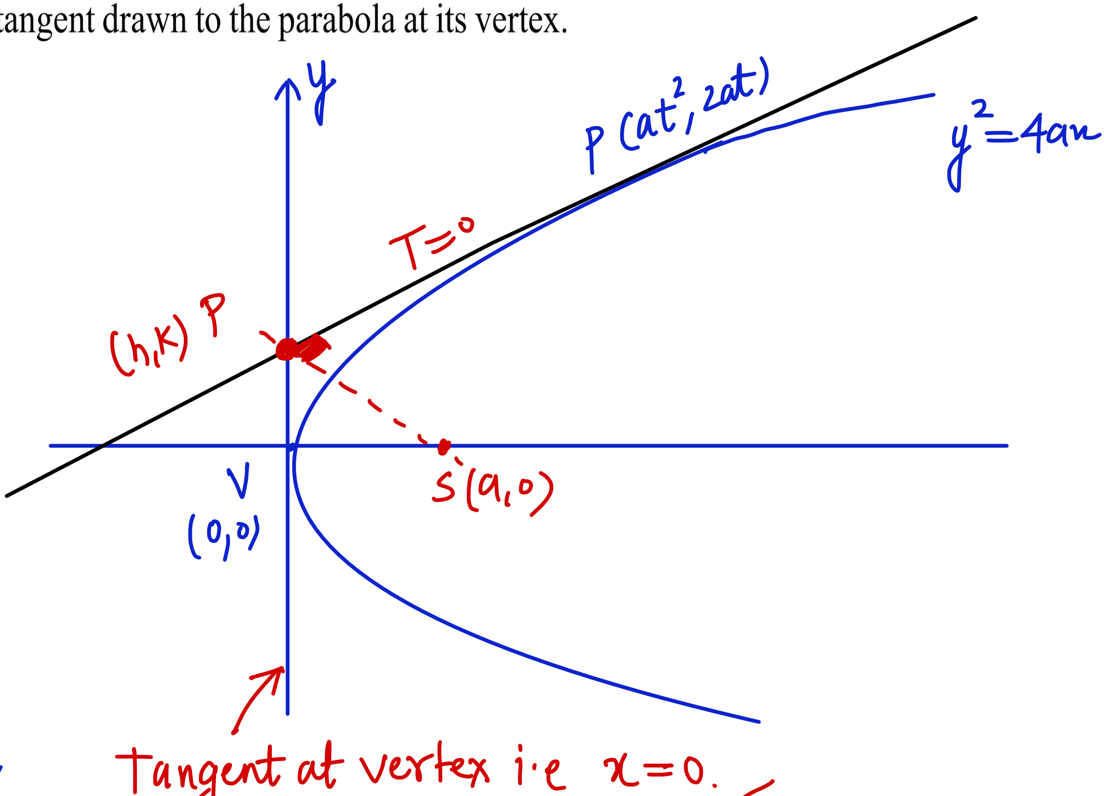
$PQ = a\sqrt{1+t^2}$.

PQSM is Rectangle

$PQ = SM = a\sqrt{1+t^2}$.

④ Any tangent to a parabola & the perpendicular on it from the focus meet on the tangent at the vertex.

i.e. locus of the feet of the perpendicular drawn from focus upon a variable tangent is the tangent drawn to the parabola at its vertex.



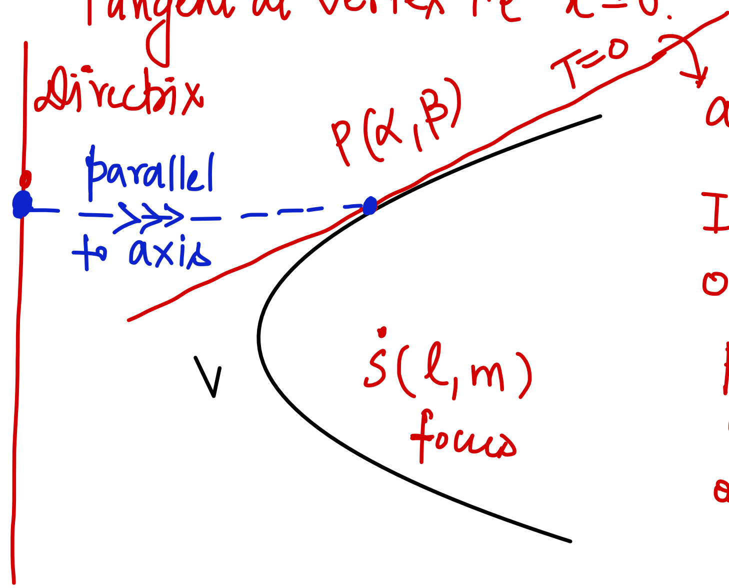


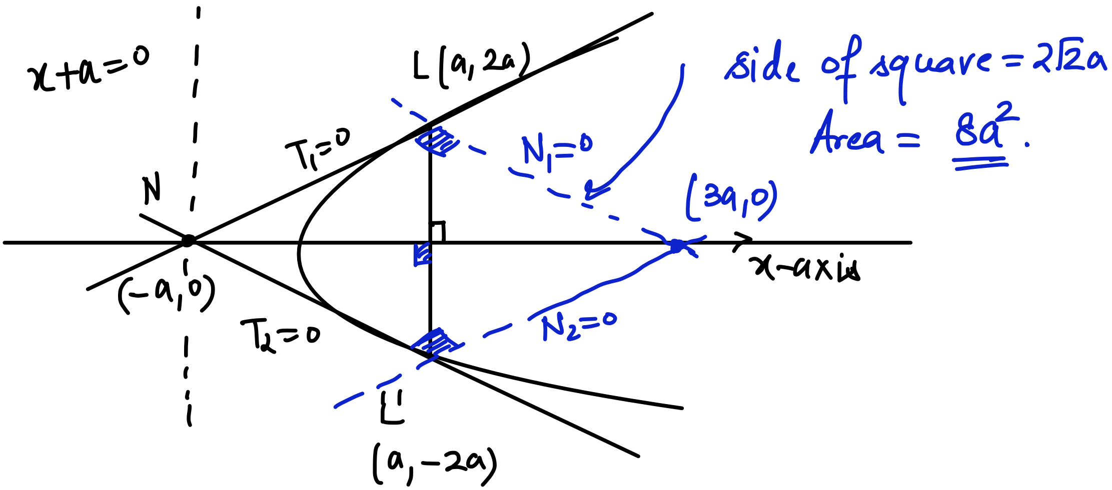
Image of Focus on tangent to parabola lies on directrix of parabola.

(5) Tangents and Normals at the extremities of the latus rectum of a parabola $y^2 = 4ax$ constitute a square, their points of intersection being $(-a, 0)$ & $(3a, 0)$.

Note :

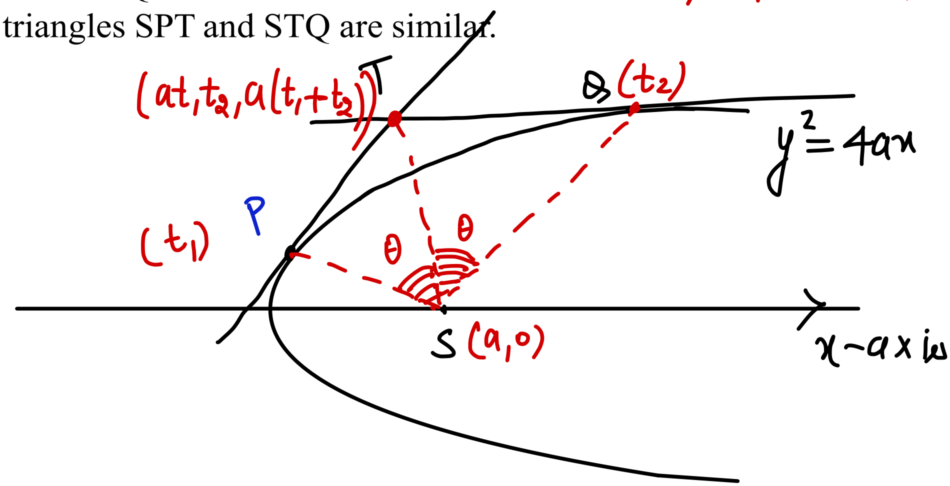
(1) The two tangents at the extremities of focal chord meet on the foot of the directrix.

(2) Figure L_1NL_2G is square of side $2\sqrt{2}a$

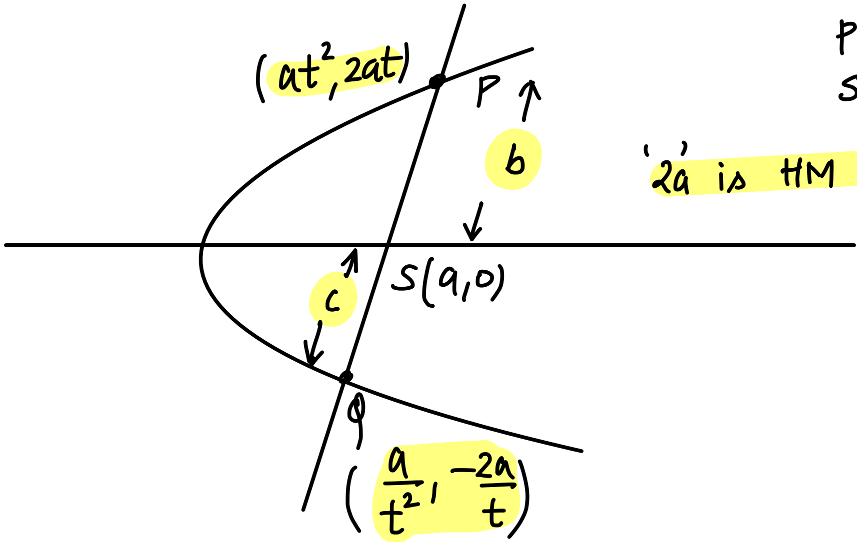


(6) If the tangents at P and Q meet in T, then :

- TP and TQ subtend equal angles at the focus S.
- $ST^2 = SP \cdot SQ$ & → SP, ST, SQ → GP
- The triangles SPT and STQ are similar.

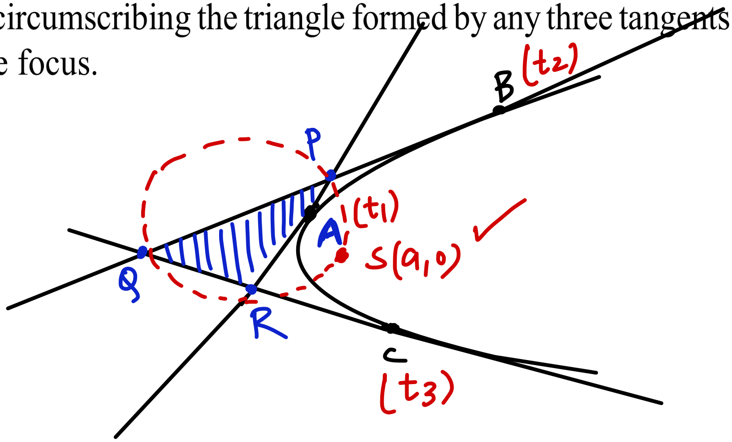


(7) Semi latus rectum of the parabola $y^2 = 4ax$, is the harmonic mean between segments of any focal chord of the parabola is : $\boxed{2a = \dfrac{2bc}{b+c} \textbf{ i.e. } \dfrac{1}{b} + \dfrac{1}{c} = \dfrac{1}{a}.}$



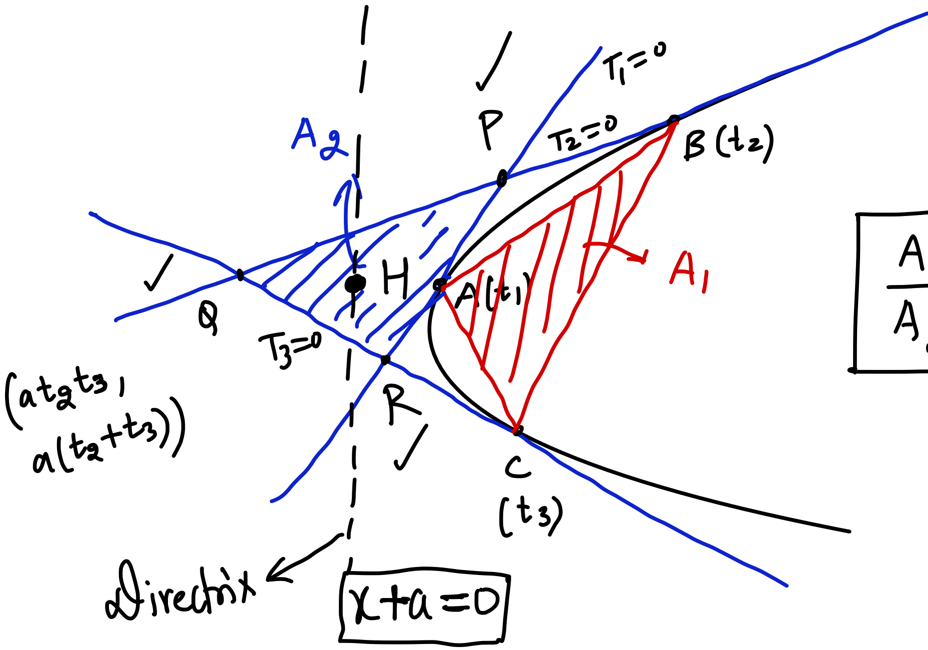
PS = b
SQ = c
'2a' is HM betn 'b' & 'c'.

(8) The circle circumscribing the triangle formed by any three tangents to a parabola passes through the focus.



(9) The orthocentre of any triangle formed by three tangents to a parabola $y^2 = 4ax$ lies on the directrix & has the co-ordinates $\left(-a,\ a(t_1 + t_2 + t_3 + t_1t_2t_3)\right)$.

(10) The area of the triangle formed by three points on a parabola is twice the area of the triangle formed by the tangents at these points.



Rem

$$\boxed{\frac{A_1}{A_2} = 2}$$

(11) Circle circumscribing the Δ formed by 3 Co-normal points will pass through the vertex of parabola and its equation is

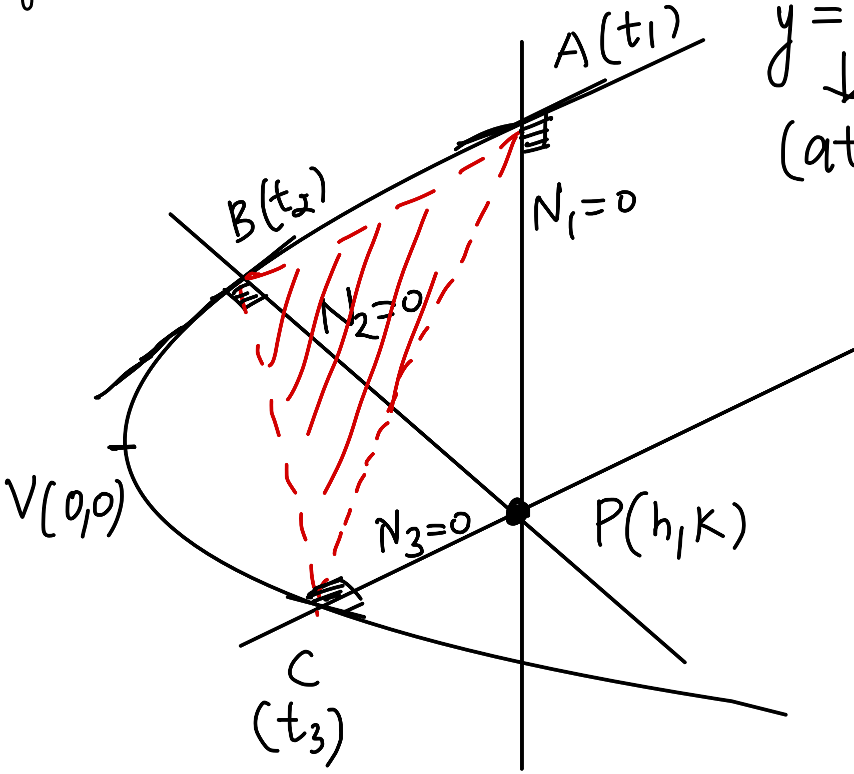
$$2x^2+2y^2-2(h+2a)x-Ky=0$$

where (h,K) is point of intersection of the 3 normals.

$y^2=4ax$

$\downarrow$

$(at^2, 2at)$



Let eqn of $\odot$ be $x^2+y^2+2gx+2fy+c=0$

$x=at^2$ & $y=2at$

$$a^2t^4+4a^2t^2+2gat^2+2(2at)f+c=0$$

$$a^2t^4 + 0t^3 + (4a^2 + 2ag)t^2 + 4aft + c = 0$$

$$\sum t_1 = 0 \Rightarrow \underbrace{t_1 + t_2 + t_3}_{0} + t_4 = 0 \quad \text{--- (1)}$$

N: $y + tx = 2at + at^3 \leftarrow (h, k)$

$$k + th = 2at + at^3$$

$$at^3 + 0t^2 + (2a - h)t - k = 0 \quad \text{--- (2) ---} \quad t_1, t_2, t_3$$

$$\sum t_1 = 0 \Rightarrow \boxed{t_1 + t_2 + t_3 = 0}$$

$$\therefore \boxed{t_4 = 0} \Rightarrow (0, 0) \Rightarrow \boxed{c = 0}$$

$$a^2t^4 + (4a^2 + 2ag)t^2 + 4aft + 0 = 0$$

$$at\left(at^3 + (4a + 2g)t + 4f\right) = 0.$$

$$at^3 + (4a + 2g)t + 4f = 0 \quad \text{--- (3) ---} \quad t_1, t_2, t_3$$

Now compare eqn 2 & 3 to get

'g' & 'f'.

$$\left.\begin{aligned} 2g &= -h-2a \\ 2f &= -\frac{k}{2} \end{aligned}\right\} //$$

Q Let $L_1 : x+y=0$ and $L_2 : x-y=0$ are tangent to parabola whose focus is $S(1,2)$.
If the length of L.R. of the parabola can be expressed as $\frac{m}{\sqrt{n}}$ where m & n are co-prime then find $(m+n)$?

Q If two perpendicular rays from focus of parabolic surface $y^2 = 4x$ are incident at points $A(t_1^2, 2t_1)$ & $B(t_2^2, 2t_2)$ such that $t_1t_2 = -10$, then distance between the reflected rays will be -

(1) 9 (2) 6 (3) 18 (4) not a constant

Q

A parabola whose focus is S(6, 8) is touching the co-ordinate axes, then which of the following statement(s) is/are correct ?

(A) The equation of circle whose diameter is the portion of the tangent at the vertex of the parabola between the co-ordinate axis is $x^2 + y^2 - 6x - 8y = 0$.

(B) If P and Q are the ends of a focal chord of the parabola, then $\frac{1}{SP} + \frac{1}{SQ} = \frac{5}{24}$.

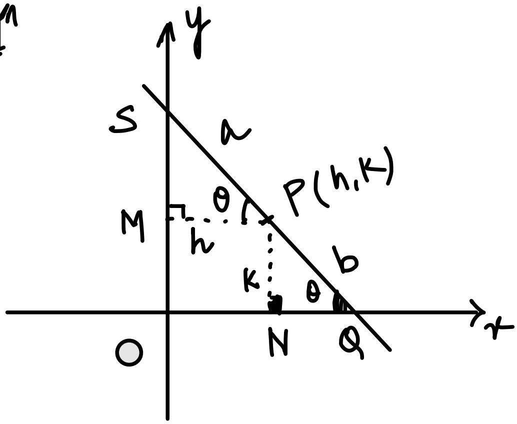
(C) Equation of the latus rectum is $4x + 3y = 48$

(D) If P and Q are the ends of a focal chord of the parabola, then $\frac{1}{SP} + \frac{1}{SQ} = \frac{1}{5}$

Q Tangents are drawn at points (x_i, y_i); $i = 1,2,3$ lying on $y = x^2$ to enclose a triangle of area Δ. If x_1, x_2, x_3 form an increasing AP, where $x_1 = -1$ and $y_3 = 9$, then Δ is

Q A rod of length a + b moves in such a way that both extremities remains on coordinate axes. A point P divides the length in ratio b : a measuring from x-axis. Find locus of P?

Sol^n



$\sin\theta = \frac{k}{b}$

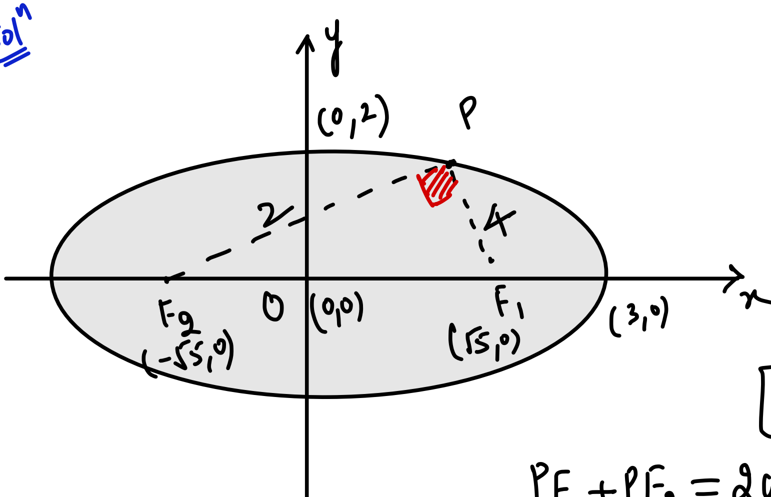
$\cos\theta = \frac{h}{a}$.

$\sin^2\theta + \cos^2\theta = \frac{k^2}{b^2} + \frac{h^2}{a^2}$

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.

Q F_1 and F_2 are the two foci of the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$. Let P be a point on the ellipse such that $|PF_1| = 2|PF_2|$, where F_1 and F_2 are the two foci of the ellipses. The area of ΔPF_1F_2 is

Sol^n



$e = \sqrt{1 - b^2/a^2}$

$e = \sqrt{1 - 4/9} = \frac{\sqrt{5}}{3}$

$F_1(ae, 0) \equiv (\sqrt{5}, 0)$

$F_2(-ae, 0) \equiv (-\sqrt{5}, 0)$

$\boxed{PF_1 = 2PF_2}$

$PF_1 + PF_2 = 2a \Rightarrow 3PF_2 = 2\times 3$

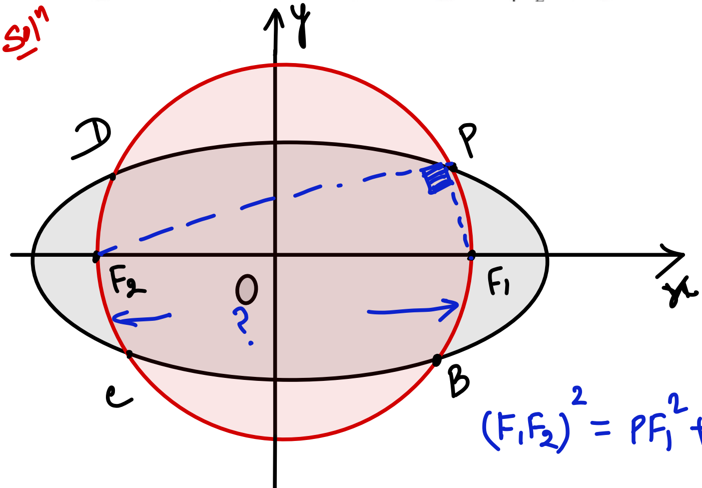
$PF_2 = 2$

$PF_1 = 4$.

$\Delta PF_1F_2 = \frac{1}{2}\times 2\times 4 = 4$ sq. units.

Q A circle has the same centre as an ellipse & passes through the foci F_1 & F_2 of the ellipse, such that the two curves intersect in 4 points. Let 'P' be any one of their point of intersection. If the major axis of the ellipse is 17 & the area of the triangle PF_1F_2 is 30, then the distance between the foci is :

Sol^n



$2a = 17$; $\Delta PF_1F_2 = 30$

$F_1F_2 = ?$

$$* \boxed{PF_1 + PF_2 = 2a = 17} \quad -①-$$

$$\frac{1}{2} \times PF_1(PF_2) = 30$$

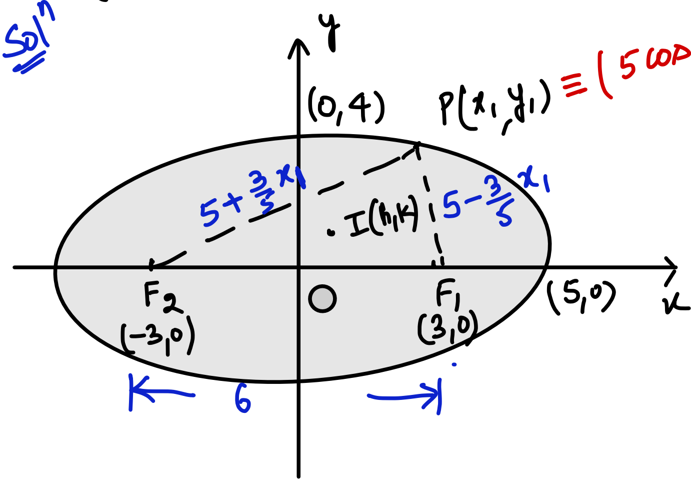
$$\boxed{PF_1 \cdot PF_2 = 60} \quad -②-$$

$$(F_1F_2)^2 = PF_1^2 + PF_2^2$$

*

Q Find the locus of the incentre of ΔPF_1F_2 where F_1 & F_2 are foci of ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ and P is moving on the given ellipse ?

Sol^n



$$e = \sqrt{1 - \frac{16}{25}} = \frac{3}{5}$$

$F_1(3,0)$; $F_2(-3,0)$

$PF_1 = a - ex_1$ *

$PF_2 = a + ex_1$ *

$$h = \frac{6(x_1) + 3\left(5 + \frac{3}{5}x_1\right) - 3\left(5 - \frac{3}{5}x_1\right)}{6 + 5 + 5}$$

$$h = \frac{3}{5}x_1 \Leftarrow h = \frac{\cancel{48}^{3}}{5(\cancel{16})}x_1 \Leftarrow 16h = 6x_1 + \frac{18}{5}x_1$$

$$h = \frac{3}{5}(5\cos\theta)$$

$$h = 3\cos\theta \quad -①-$$

$$K = \frac{6 \cdot y_1}{16} \Rightarrow K = \frac{\cancel{6}^{3}}{\cancel{16}_{\cancel{4}\,2}}(\cancel{4}\sin\theta)$$

$$K = \frac{3}{2}\sin\theta \quad -②-$$

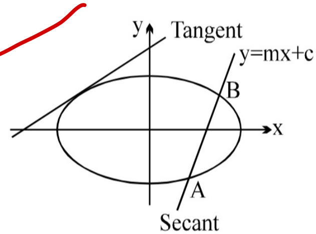
$$\left(\frac{h}{3}\right)^2 + \left(\frac{2K}{3}\right)^2 = \cos^2\theta + \sin^2\theta = 1$$

$$\boxed{\frac{x^2}{9} + \frac{y^2}{9/4} = 1} \rightarrow \text{locus}$$

HW

Q Find everything for ellipse $9x^2+25y^2+18x-100y=116$

Line and an ellipse :



SnT

Line $y = mx + c$ ....(1)

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \qquad ....(2)$$

solving (1) and (2)

$$b^2x^2 + a^2(mx + c)^2 = a^2b^2$$

i.e. $(a^2m^2 + b^2)x^2 + 2a^2cmx + a^2(c^2 - b^2) = 0$

- $D > 0$ Secant
- $D = 0$ Tangent $\Downarrow$ $c^2 = a^2m^2+b^2$
- $D < 0$ Passes outside the ellipse

Rem

Hence $\boxed{y = mx \pm \sqrt{a^2m^2 + b^2} \text{ is always a tangent to the ellipse for all } m \in R.}$

Slope Form

Note that there are two parallel tangents for a given m.

If it passes through (h, k) then

$$k = mh \pm \sqrt{a^2m^2 + b^2}$$

$$(k - mh)^2 = a^2m^2 + b^2$$

$$(h^2 - a^2)m^2 - 2khm + k^2 - b^2 = 0 \begin{smallmatrix} m_1 \\ m_2 \end{smallmatrix} \qquad ....(3)$$

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1m_2}\right|$

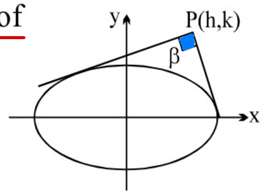
$\tan^2\theta\,(1+m_1m_2)^2 = (m_1+m_2)^2 - 4m_1m_2$

$m_1 + m_2 = \frac{2kh}{h^2 - a^2}$

$m_1m_2 = \frac{k^2 - b^2}{h^2 - a^2}$.

Hence passing through a given point there can be a maximum of two tangents.

Equation (3) can be used to determine the locus of the point of intersection of two tangents enclosing an angle β.



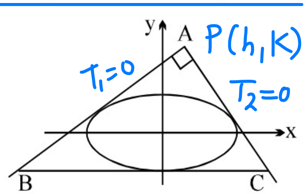
if $\beta = 90°$ then $m_1m_2 = \frac{k^2 - b^2}{h^2 - a^2} = -1$

Rem

i.e. $k^2 - b^2 = a^2 - h^2$; i.e. $\boxed{x^2 + y^2 = a^2 + b^2}$

which is the director circle of the ellipse. Hence director circle of an ellipse is a circle whose centre is the centre of ellipse and whose radius is the length of the line joining the ends of the major and minor axis.

Note that, if a right triangle, right angled at A circumscribes an ellipse then locus of the point A is the director circle of the ellipse.

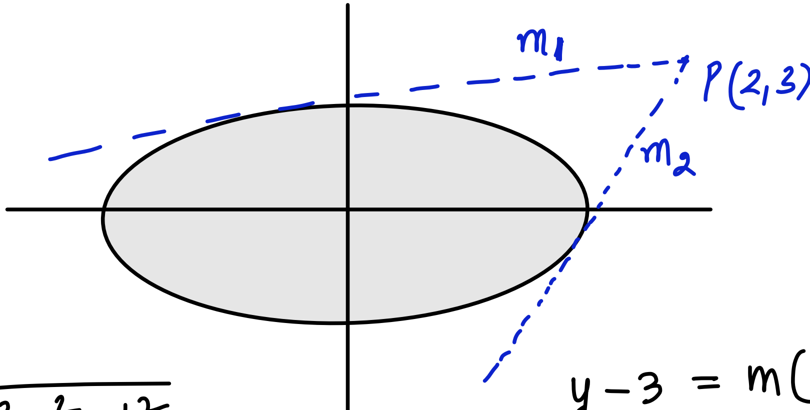


Q Equation of the tangent to an ellipse $9x^2 + 16y^2 = 144$ passing from (2, 3).

Sol^n

$$\frac{x^2}{16} + \frac{y^2}{9} = 1.$$

(2, 3) lies outside ellipse



$$y - 3 = m(x-2)$$

$$T: \boxed{y = mx + \underbrace{(3-2m)}_{c}}$$

$$c = \pm\sqrt{a^2m^2+b^2}$$

$$3-2m = \pm\sqrt{16m^2+9}$$

$$\cancel{9} + 4m^2 - 12m = 16m^2 + \cancel{9}$$

$$12m^2 + 12m = 0 \Rightarrow \underbrace{m=0}; \underbrace{m=-1}$$

$$\left.\begin{aligned} T_1 &: y = 3 \\ T_2 &: y = -x+5 \end{aligned}\right\}$$

Q Tangent to an ellipse makes angles θ_1, θ_2 with major axis. Find the locus of their intersection when $\cot\theta_1 + \cot\theta_2 = k^2$.

+ dirⁿ

$P(\alpha, \beta)$

Solⁿ

$m_1 = \tan\theta_1$

$m_2 = \tan\theta_2$

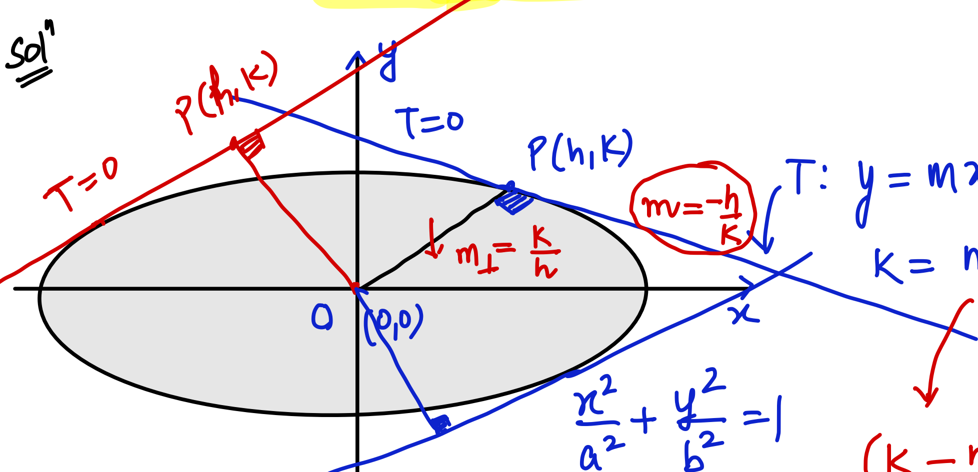
$$\frac{1}{m_1} + \frac{1}{m_2} = k^2 \Rightarrow (m_1 + m_2) = (m_1 m_2)\,k^2$$

$$\left(\frac{2\alpha\beta}{\alpha^2 - a^2}\right) = \left(\frac{\beta^2 - b^2}{\alpha^2 - a^2}\right) k^2$$

$$\frac{2xy}{\cancel{x^2 - a^2}} = \left(\frac{y^2 - b^2}{\cancel{x^2 - a^2}}\right) k^2.$$

Q Locus of the feet of the perpendicular from centre upon a variable tangent to the standard ellipse is $(x^2 + y^2)^2 = a^2x^2 + b^2y^2$.

Solⁿ



$$T: y = mx + \sqrt{a^2m^2 + b^2}$$

$$k = mh + \sqrt{a^2m^2 + b^2}$$

$$(k - mh)^2 = a^2m^2 + b^2$$

$$\left(k + \frac{h^2}{k}\right)^2 = a^2 \cdot \frac{h^2}{k^2} + b^2$$

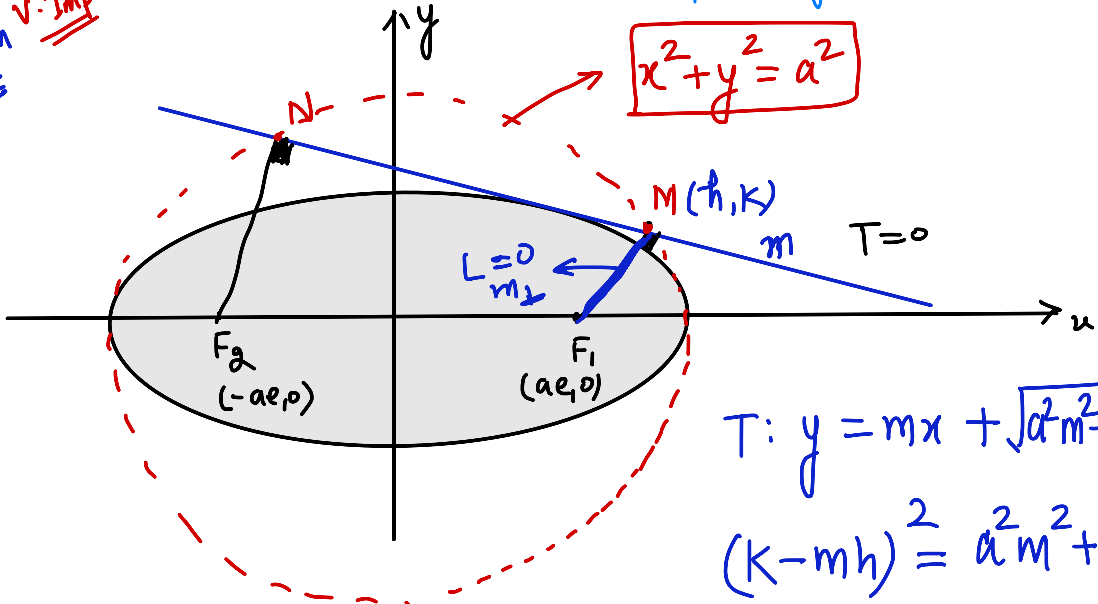
$$= (k^2 + h^2)^2 = a^2h^2 + b^2k^2$$

Q (Rem) Prove that the locus of feet of $\perp$ drawn on variable tangent from either foci is auxillary circle

(Note :- Same Result also holds for hyperbola)

V. Imp

Solⁿ



$T: y = mx + \sqrt{a^2m^2+b^2}$

$(k-mh)^2 = a^2m^2+b^2$ — ① —

$y-0 = m_\perp(x-ae)$

$y = -\frac{1}{m}(x-ae)$

$my + x = ae \Rightarrow (mk+h)^2 = (ae)^2$ — ② —

Add ① & ②.

$k^2+h^2+m^2(h^2+k^2) = a^2m^2+b^2+a^2e^2.$

$$(1+m^2)(h^2+k^2) = a^2m^2 + a^2(1-e^2) + a^2e^2$$

$$\cancel{(1+m^2)}(h^2+k^2) = a^2\cancel{(1+m^2)}$$

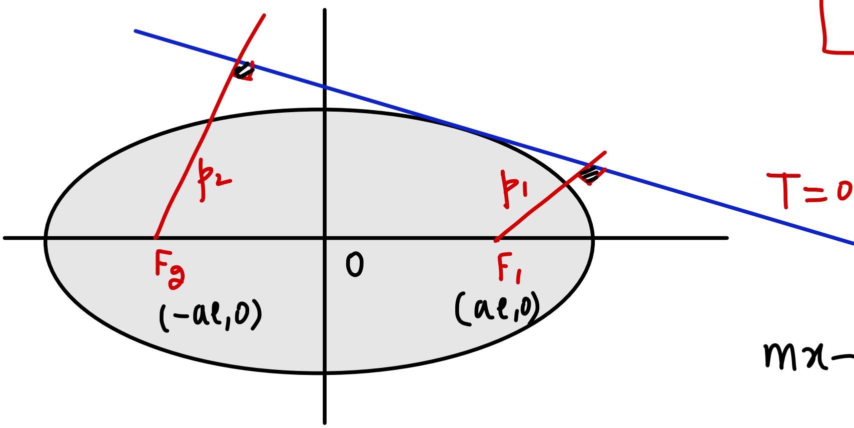
$$\boxed{x^2+y^2 = a^2}$$

(H.P)

Q (Rem, V.Imp) The product of perpendicular drawn from two foci upon a variable tangent is equal to square of semi-minor axes.

(**Note**:- Same Result also holds for hyperbola)

Rem
$$\boxed{p_1 p_2 = b^2}$$



$$y = mx + \sqrt{a^2m^2+b^2}$$

$$mx - y + \sqrt{a^2m^2+b^2} = 0$$

$$p_1 p_2 = \frac{\left|\left(mae + \sqrt{a^2m^2+b^2}\right)\left(-mae + \sqrt{a^2m^2+b^2}\right)\right|}{\sqrt{1+m^2}\,\sqrt{1+m^2}}$$

$$= \frac{\left|a^2m^2 + b^2 - m^2a^2e^2\right|}{(1+m^2)} = \frac{\left|a^2m^2(1-e^2)+b^2\right|}{(1+m^2)}$$

$$= \frac{m^2b^2 + b^2}{(1+m^2)} = b^2. \quad \text{(H.P)}$$

EQUATION OF CHORD OF AN ELLIPSE :

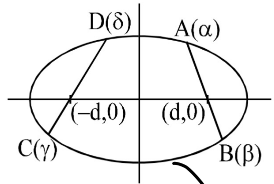
Equation of a chord of an ellipse joining two points P(α) and Q(β) on it is equal to

$$\boxed{\frac{x}{a}\cos\left(\frac{\alpha+\beta}{2}\right)+\frac{y}{b}\sin\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha-\beta}{2}\right)}$$

Rem

If this particular chord passes through (d, 0) then we have

$$\frac{d}{a}\cos\left(\frac{\alpha+\beta}{2}\right)=\cos\left(\frac{\alpha-\beta}{2}\right)\;;\quad \frac{\cos\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)}=\frac{a}{d}$$



$$\frac{\cos\left(\frac{\alpha+\beta}{2}\right)-\cos\left(\frac{\alpha-\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)+\cos\left(\frac{\alpha-\beta}{2}\right)}=\frac{a-d}{a+d}$$

Rem

or $\quad -\dfrac{2\sin\alpha/2\,\sin\beta/2}{2\cos\alpha/2\,\cos\beta/2}=\dfrac{a-d}{a+d}\qquad$ i.e. $\quad \tan\dfrac{\alpha}{2}\tan\dfrac{\beta}{2}=\dfrac{d-a}{d+a}$

if $d = ae$ $\quad$ i.e. PQ is a focal chord then $\tan\dfrac{\alpha}{2}\tan\dfrac{\beta}{2}=\dfrac{e-1}{e+1}$ Rem]

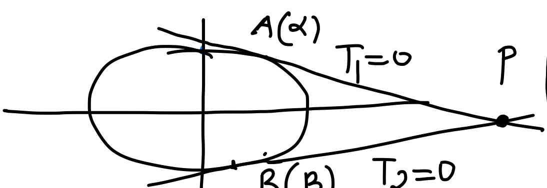
$$\tan\frac{\alpha}{2}\tan\frac{\beta}{2}=\frac{e-1}{e+1}$$

$$\tan\frac{\gamma}{2}\tan\frac{\delta}{2}=\frac{e+1}{e-1}$$

Rem

$$\therefore\quad \boxed{\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}\tan\frac{\delta}{2}=1}$$

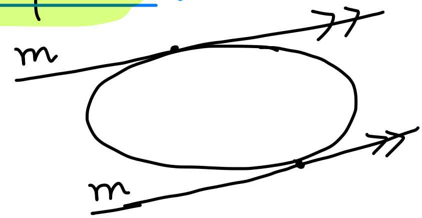
Note :- Similar to what we did in Circle.



$$P\left(\frac{a\cos\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)},\ \frac{b\sin\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)}\right)$$

Equation of tangent in Various forms :-

(1) **Slope Form** :-

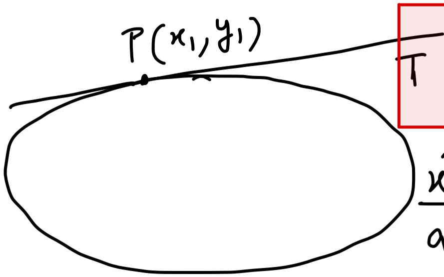


$$\boxed{y = mx \pm \sqrt{a^2m^2+b^2}}$$

For line $y = mx + c$ to be tangent to ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$, $\boxed{c = \pm\sqrt{a^2m^2+b^2}}$

For a given of 'm' we can always draw 2 parallel tangents to ellipse.

(2) **Cartesian Form** :-

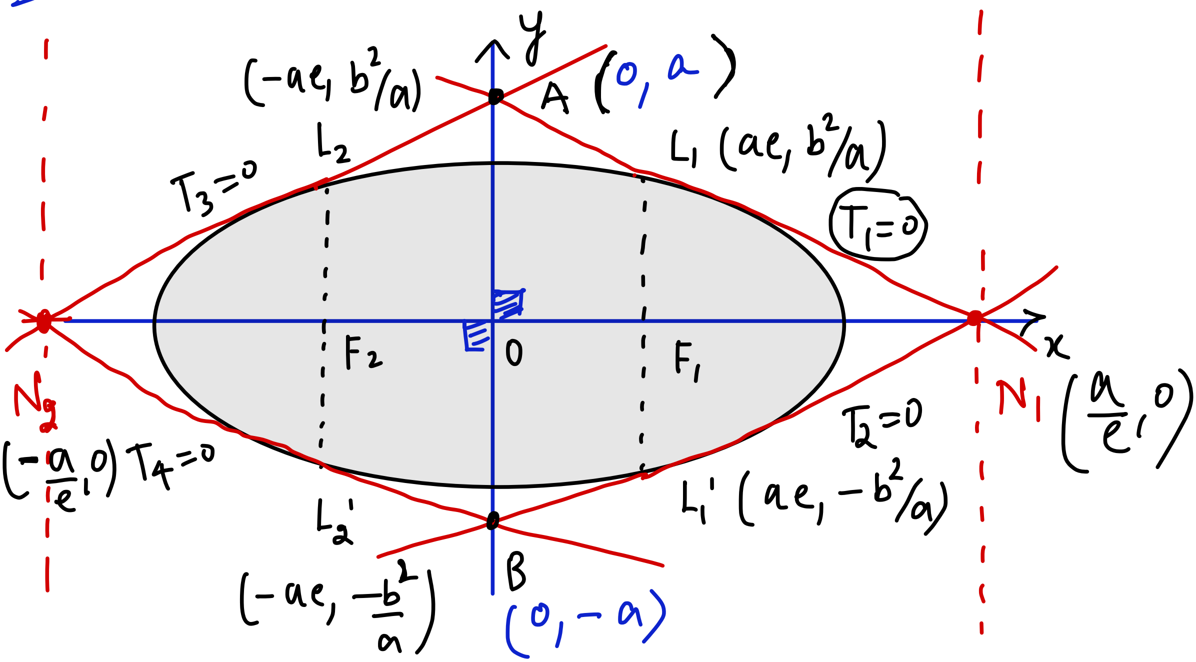


$$\boxed{T : \frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1}$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Note: The tangents at the extremities of the latus rectum of the ellipse intersect at the foot of the corresponding directrix and the figure formed by them is a rhombus of area $\frac{2a^2}{e}$.

Sol^n



$$T_1 : \quad \frac{x(ae)}{a^2} + \frac{y(b^2/a)}{b^2} = 1$$

$$\frac{xe}{a} + \frac{y}{a} = 1 \Rightarrow T_1 : xe + y = a$$

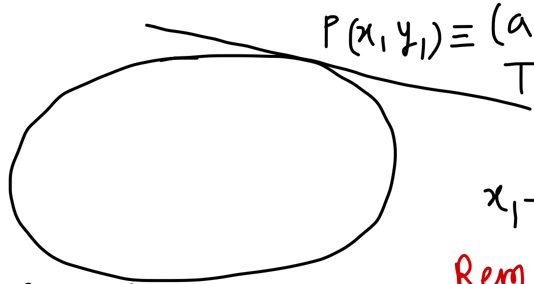
$$T_2 : xe - y = a.$$

$$T_3 : -xe + y = a.$$

$$T_4 : -xe - y = a$$

$$A = \frac{1}{2} d_1 d_2 = \frac{1}{\cancel{2}} (\cancel{2}a)\left(\frac{2a}{e}\right) = \frac{2a^2}{e}.$$

(3) Parametric Form :-



$P(x_1, y_1) \equiv (a\cos\theta, b\sin\theta)$

$$T: \frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$$

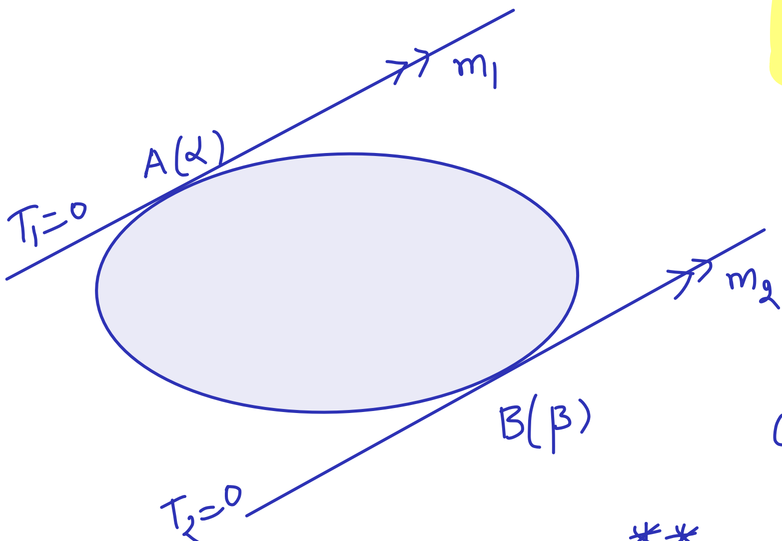
$x_1 \to a\cos\theta \; ; \; y_1 \to b\sin\theta$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

Rem

$$\boxed{T: \frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1}$$

$$m_T = -\frac{b}{a}\cot\theta.$$



$$m_1 = m_2$$

$$-\frac{b}{a}\cot\alpha = -\frac{b}{a}\cot\beta.$$

$$\cot\alpha = \cot\beta.$$

$$\alpha = \pi + \beta.$$

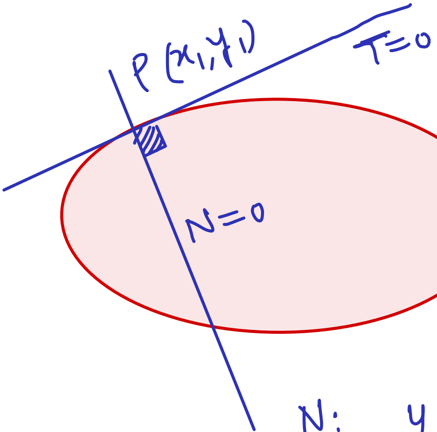
** $\boxed{|\alpha - \beta| = \pi}$

Note :-

The eccentric angles of the point of contact of 2 parallel tangents differ by 'π' & Conversly if eccentric angle of 2 points differ by 'π' then tangents at these points are parallel.

Normal in Various forms :-

① **Cartesian form :** Equation of the normal at (x_1, y_1) is $\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2 = a^2e^2$.



$$T : \frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1 ;$$

$$m_T = \frac{-x_1/a^2}{y_1/b^2} = \frac{-b^2x_1}{a^2y_1}$$

$$N : y - y_1 = m_N (x - x_1)$$

$$\boxed{m_N = \frac{a^2y_1}{b^2x_1}}$$

② **Parametric Form :-**

$x_1 \to a\cos\theta \; ; \; y_1 \to b\sin\theta$

Rem

$$\boxed{N : \frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2e^2 = a^2 - b^2}$$

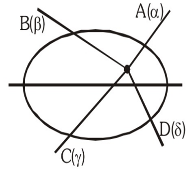
③ **Slope Form :-**

Equation of a normal in terms of its slope 'm' is $y = mx - \frac{(a^2 - b^2)m}{\sqrt{a^2 + b^2m^2}}$.

If the normal passes through (h, k) then biquadratic in m is suggestive that a maximum of four normals can be drawn to an ellipse from a point.

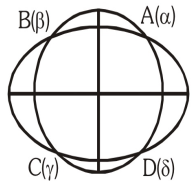
Note : Rem

(i) **Condition of conormal points :**



If A(α), B(β), C(γ) & D(δ) are conormal points then sum of their eccentric angles is odd multiple of π. i.e. $\alpha + \beta + \gamma + \delta = \mathbf{(2n+1)\pi}$.

(ii) **Condition of concyclic points :**



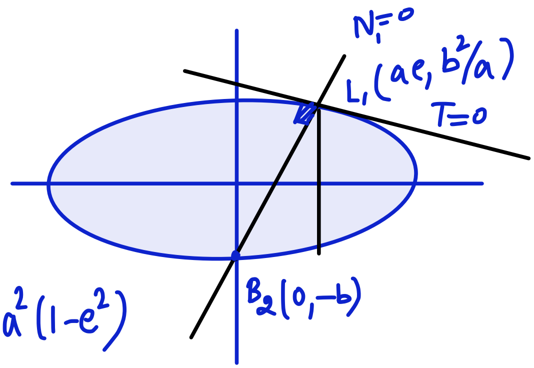
If A(α), B(β), C(γ) & D(δ) are four concyclic points then

sum of their eccentric angles is even multiple of π.

i.e. $\alpha + \beta + \gamma + \delta = \mathbf{2n\pi}$

Q HW Find the equations to the normals at the ends of the latera recta, and prove that each passes through an end of the minor axis if $e^4 + e^2 = 1$.

Sol^n



$b^2 = a^2(1-e^2)$

$y - \frac{b^2}{a} = \frac{a^2}{b^2}\left(\frac{b^2/a}{ae}\right)(x - ae)$

$\uparrow B_2(0,-b)$

$-b - \frac{b^2}{a} = \frac{1}{e}(-ae)$

$b^2 = (a-b)a$

$b^2 = a^2 - ab.$

HW

Q Find the locus of the feet of the perpendiculars from centre on a normal to a standard ellipse.

HW

Q Find everything for ellipse $9x^2+25y^2+18x-100y=116$

Sol^n

$$9(x^2+2x)+25(y^2-4y)=116$$

$$9\left((x+1)^2-1\right)+25\left((y-2)^2-4\right)=116$$

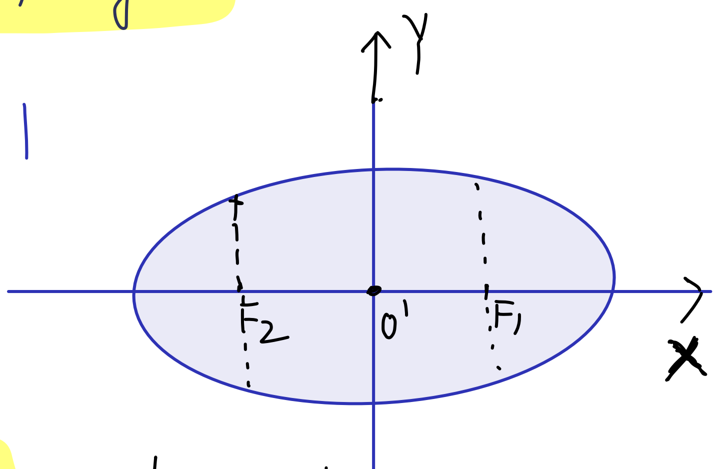
$$9(x+1)^2+25(y-2)^2=225$$

$$\frac{(x+1)^2}{25}+\frac{(y-2)^2}{9}=1$$

$O\ (0,0) \longrightarrow O'(-1,2)$

$$X=x+1 \quad \& \quad Y=y-2$$

$$\frac{X^2}{25}+\frac{Y^2}{9}=1$$



$a=5$; $b=3$

$$e=\sqrt{1-\frac{9}{25}}=\frac{4}{5}$$

$F_1(4,0) \equiv F_1(3,2)$

$F_2(-4,0) \equiv F_2(-5,2)$

$$L\cdot R=\frac{2b^2}{a}=\frac{2\times 9}{5}=\frac{18}{5}$$

Directrix :-

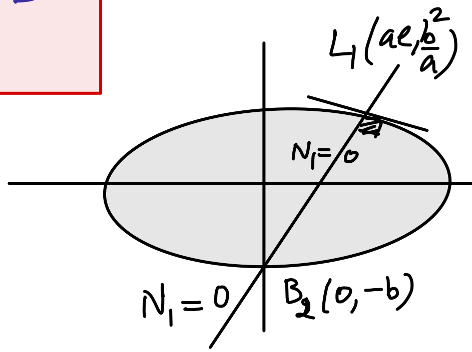
$X=\frac{a}{e}$; $X=-\frac{a}{e}$

$x+1=\frac{5}{4}\times 5$; $x+1=-\frac{25}{4}$

Q Find the equations to the normals at the ends of the latera recta, and prove that each passes through an end of the minor axis if $e^4 + e^2 = 1$.

Rem

N : $$\boxed{\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2e^2 = a^2 - b^2}$$



N_1 : $$\frac{a^2x}{ae} - \frac{b^2y}{b^2/a} = a^2e^2$$

$B_2(0,-b)$

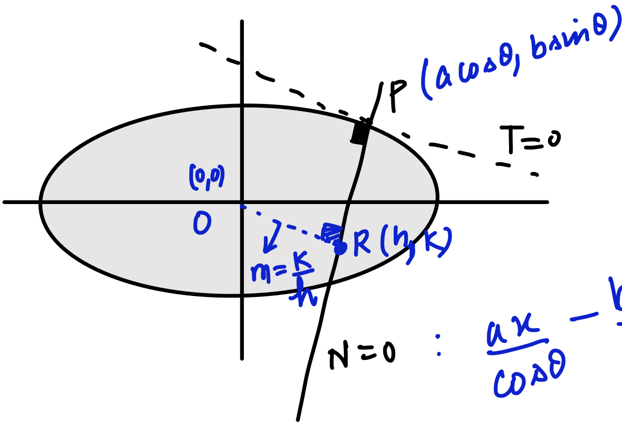
$$\frac{-\cancel{b^2}(-b)\cdot a}{\cancel{b^2}} = a^2e^2$$

$$b = ae^2$$

$$b^2 = a^2e^4 \Rightarrow \cancel{a^2}(1-e^2) = \cancel{a^2}e^4$$

Q Find the locus of the feet of the perpendiculars from centre on a normal to a standard ellipse.

Solⁿ



$N=0$: $$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2e^2 = a^2 - b^2.$$

N: $$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2e^2 \quad \text{—①—}$$

N: $$y - k = -\frac{h}{k}(x-h) \Rightarrow -xh + h^2 = yk - k^2$$

$$xh + yk = h^2 + k^2 \quad \text{—②—}$$

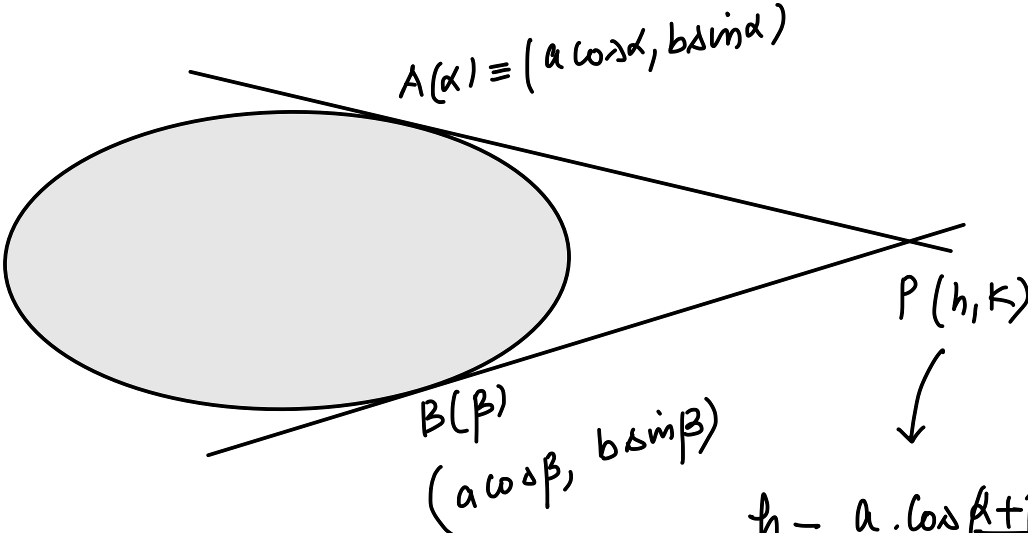
Compare ① & ②

$$\frac{h \cdot \cos\theta}{a} = \frac{k \sin\theta}{-b} = \frac{h^2+k^2}{a^2e^2}$$

$$\left.\begin{aligned} \cos\theta &= \frac{a}{h}\left(\frac{h^2+k^2}{a^2e^2}\right) \\ \sin\theta &= -\frac{b}{k}\left(\frac{h^2+k^2}{a^2e^2}\right) \end{aligned}\right\}$$

Use $\cos^2\theta + \sin^2\theta = 1.$ ✓

Q. Locus of the point of intersection of the pair of tangents to an ellipse if the sum of the ordinates of their point of contact is b.



$$h = \frac{a\cdot\cos\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)} \quad —(2)—$$

$$\cancel{b}\sin\alpha + \cancel{b}\sin\beta = \cancel{b} \quad —(1)—$$

$$k = \frac{b\cdot\sin\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)} \quad —(3)—$$

$$2\sin\left(\frac{\alpha+\beta}{2}\right)\cdot\cos\left(\frac{\alpha-\beta}{2}\right) = 1$$

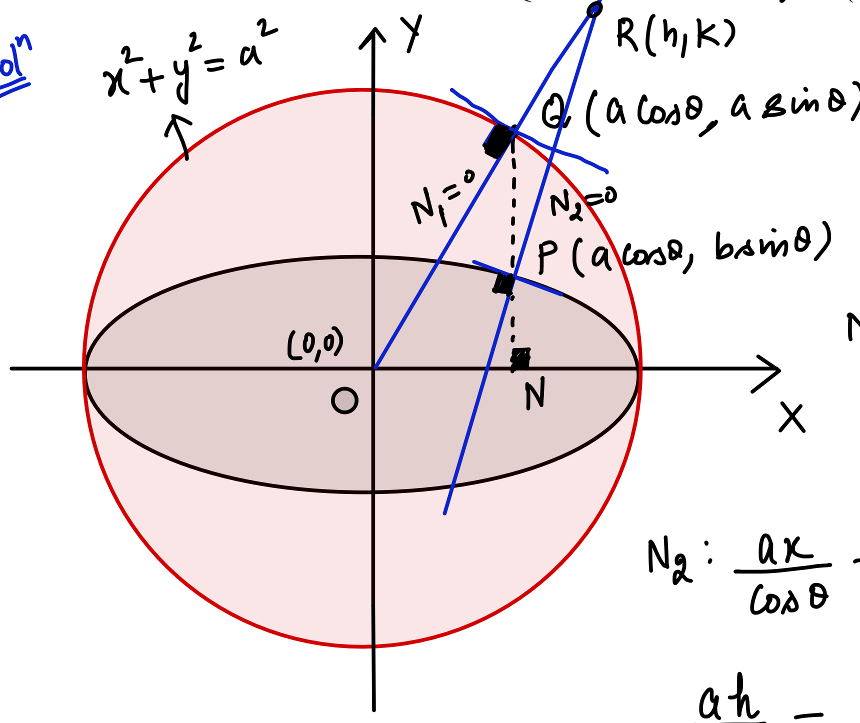
$$2\cdot\frac{k}{b}\cdot\cos^2\left(\frac{\alpha-\beta}{2}\right) = 1$$

$$\left(\frac{h}{a}\right)^2 + \left(\frac{k}{b}\right)^2 = \frac{1}{\cos^2\left(\frac{\alpha-\beta}{2}\right)}$$

$$\cos^2\left(\frac{\alpha-\beta}{2}\right) = \frac{b}{2k}$$

Q Any ordinate NP of an ellipse meets the auxiliary circle in Q ; prove that the locus of the intersection of the normals at P and Q is the circle $x^2 + y^2 = (a + b)^2$.

Sol^n



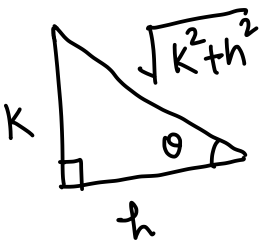
$m_{OQ} = \tan\theta$

$N_1: \; y = (\tan\theta)x$

$k = (\tan\theta).h \quad \text{—(1)—}$

$N_2: \; \dfrac{ax}{\cos\theta} - \dfrac{by}{\sin\theta} = a^2 - b^2$

$\dfrac{ah}{\cos\theta} - \dfrac{bk}{\sin\theta} = a^2 - b^2. \quad \text{—(2)—}$



$\dfrac{a\cancel{h}\sqrt{k^2+h^2}}{\cancel{h}} - \dfrac{b\cancel{k}\sqrt{h^2+k^2}}{\cancel{k}}$

$= (a-b)(a+b)$

$\cancel{(a-b)}\sqrt{k^2+h^2} = \cancel{(a-b)}(a+b)$

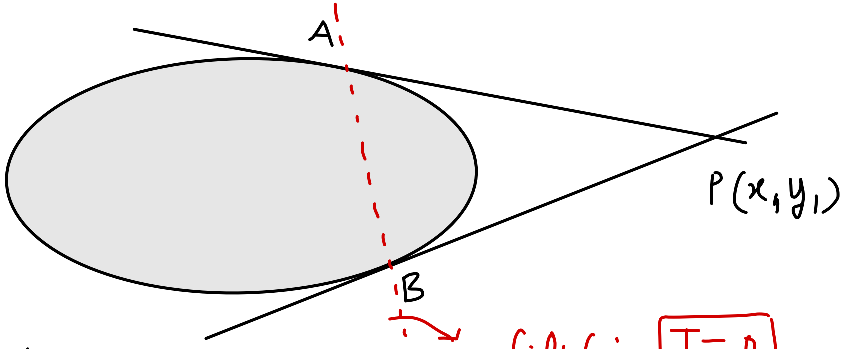
$k^2+h^2 = (a+b)^2$

(H.P)

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$S \equiv \frac{x^2}{a^2}+\frac{y^2}{b^2}-1$$

$$S_1 \equiv \frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}-1$$



C.O.C : $\boxed{T=0}$

$$T \equiv \frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1$$

Chord of contact : $\boxed{\frac{xx_1}{a^2}+\frac{yy_1}{b^2}=1}$ $\boxed{T=0}$

Pair of tangents : $\boxed{SS_1 = T^2}$

Chord with a given middle point : $\boxed{T = S_1}$

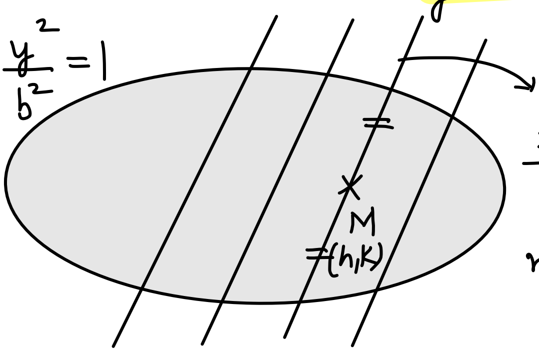
Diameter :-

The locus of the middle points of a system of parallel chords with slope 'm' of an ellipse is a straight line passing through the centre of the ellipse, called its diameter and has the equation $\boxed{y=-\frac{b^2}{a^2m}x.}$ Rem

fix variable

$$y = mx + c$$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$



$$T = S_1$$

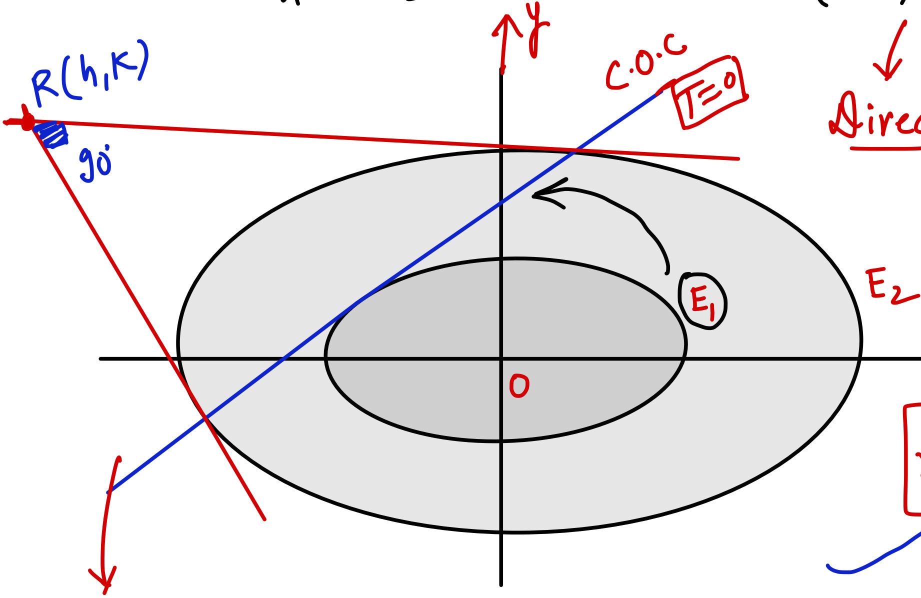
$$\frac{xh}{a^2}+\frac{yk}{b^2}-1=\frac{h^2}{a^2}+\frac{k^2}{b^2}-1$$

$$m = \frac{-h/a^2}{k/b^2} \Rightarrow \boxed{y=\frac{-b^2}{a^2m}x}$$

Q A tangent to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ meets the ellipse $\frac{x^2}{a}+\frac{y^2}{b}=a+b$ in the points P and Q ; prove that the tangents at P and Q are at right angles.

Sol^n

$$E_1: \frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad ; \quad E_2 = \frac{x^2}{a(a+b)}+\frac{y^2}{b(a+b)}=1$$



Director ⊙ of E_2:

$$x^2+y^2 = a(a+b) + b(a+b)$$

$$\boxed{x^2+y^2=(a+b)^2}$$

C.O.C : $T=0$ $\quad \frac{xh}{a(a+b)}+\frac{yk}{b(a+b)}=1$ — (1) —

T: $\quad \frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1.$ — (2) —

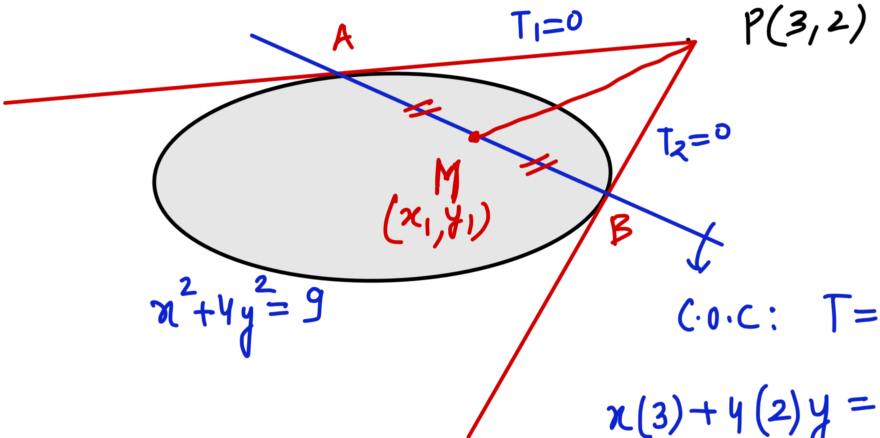
Compare (1) & (2)

$$\frac{\cos\theta\,(\cancel{a})(a+b)}{\cancel{a}\cdot h} = \frac{\sin\theta\,\cancel{b}(a+b)}{\cancel{b}\,k} = 1 \Rightarrow \begin{array}{l}\cos\theta = \frac{h}{a+b}\\ \sin\theta = \frac{k}{a+b}\end{array}$$

$\boxed{c^2+s^2=1}$

Q Tangents are drawn from the point (3, 2) to the ellipse $x^2 + 4y^2 = 9$. Find the equation to their chord of contact and the equation of the straight line joining (3, 2) to the middle point of this chord of contact.

Sol^n



C.O.C: $T=0$

$$x(3)+4(2)y=9$$

Eqn of PM = ?

AB $\hookrightarrow$ $\boxed{3x+8y=9}$ — ① —

Eqn of AB: $T=S_1$

$$xx_1+4yy_1-\cancel{9}=x_1^2+4y_1^2-\cancel{9}$$

$$xx_1+4yy_1=x_1^2+4y_1^2 \quad \text{— ② —}$$

Compare ① & ② to get x_1 & y_1

Q Find the locus of the middle points of chords of an ellipse

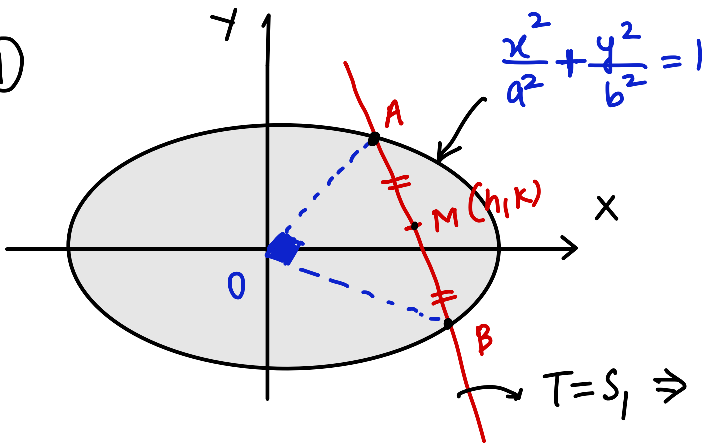
(1) which subtend a right angle at their centre.

HW

(2) the tangent at the ends of which intersect at right angles.

Sol^n

(1)



$T=S_1 \Rightarrow \frac{xh}{a^2}+\frac{yK}{b^2}\cancel{-1}$

$=\frac{h^2}{a^2}+\frac{K^2}{b^2}\cancel{-1}$

$$\frac{xh}{a^2}+\frac{yK}{b^2}=\frac{h^2}{a^2}+\frac{K^2}{b^2}$$

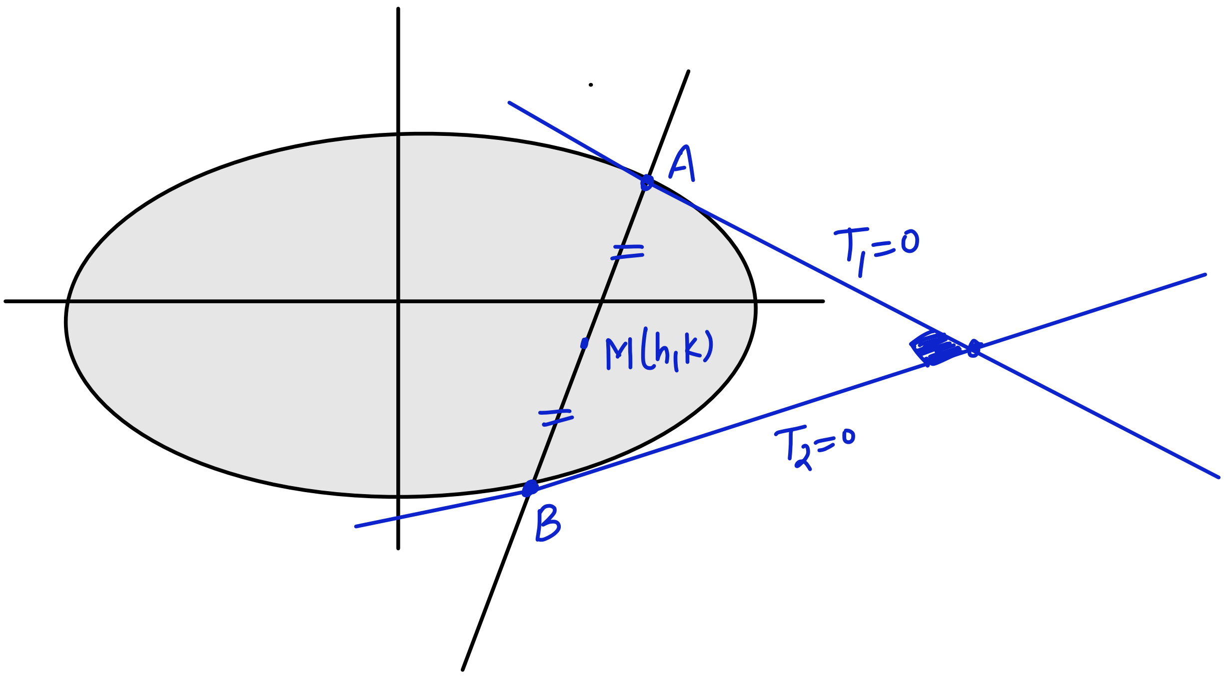
$$\frac{\frac{xh}{a^2}+\frac{yK}{b^2}}{\frac{h^2}{a^2}+\frac{K^2}{b^2}}=1 \quad -(1)-$$

Homogenese ellipse with (1) :-

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\left(\frac{\frac{xh}{a^2}+\frac{yK}{b^2}}{\frac{h^2}{a^2}+\frac{K^2}{b^2}}\right)^2=0$$

Coeff of x^2 + Coeff of $y^2=0$

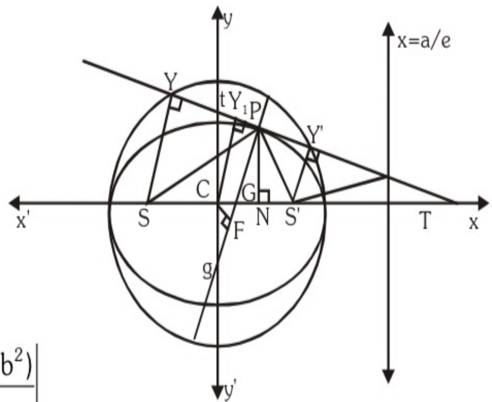
②



Highlights :-

H-1 If P be any point on the ellipse with S & S' as its foci then $\ell(SP) + \ell(S'P) = 2a$.

H-2 The product of the length's of the perpendicular segments from the foci on any tangent to the ellipse is b^2 and the feet of these perpendiculars YY' lie on its auxiliary circle. The tangents at these feet to the auxiliary circle meet on the ordinate of P and that the locus of their point of intersection is a similar ellipse as that of the original one.



[(a) $mx - y + \sqrt{a^2m^2 + b^2} = 0$

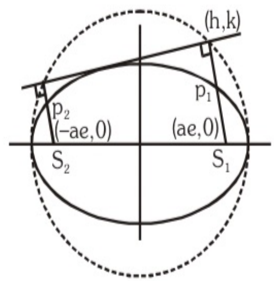
$$p_1p_2 = \left|\frac{mae + \sqrt{a^2m^2 + b^2}}{\sqrt{1+m^2}}\right|$$

$$\left|\frac{-mae + \sqrt{a^2m^2 + b^2}}{\sqrt{1+m^2}}\right|$$

$$= \left|\frac{(a^2m^2 + b^2) - a^2m^2e^2}{1+m^2}\right| = \left|\frac{a^2m^2 + b^2 - m^2(a^2 - b^2)}{1+m^2}\right|$$

$$= \left|\frac{b^2(1+m^2)}{1+m^2}\right| = b^2$$

(b) $y = mx + \sqrt{a^2m^2 + b^2}$



$(k - mh)^2 = a^2m^2 + b^2$ ...... (i)

$$y - 0 = -\frac{1}{m}(x - ae)$$

$$k = -\frac{1}{m}(h - ae)$$

$(km + h)^2 = (ae)^2$ .... (ii)

$h^2 + k^2 + m^2(h^2 + k^2) = a^2m^2 + b^2 + a^2 - b^2$

$(1 + m^2)(h^2 + k^2) = a^2(1 + m^2)$

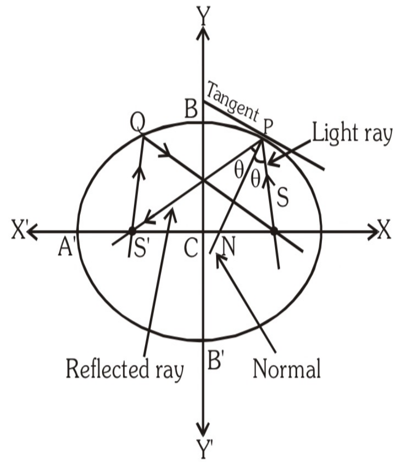
$x^2 + y^2 = a^2$]

H-3 The portion of the tangent to an ellipse between the point of contact & the directrix subtends a right angle at the corresponding focus.

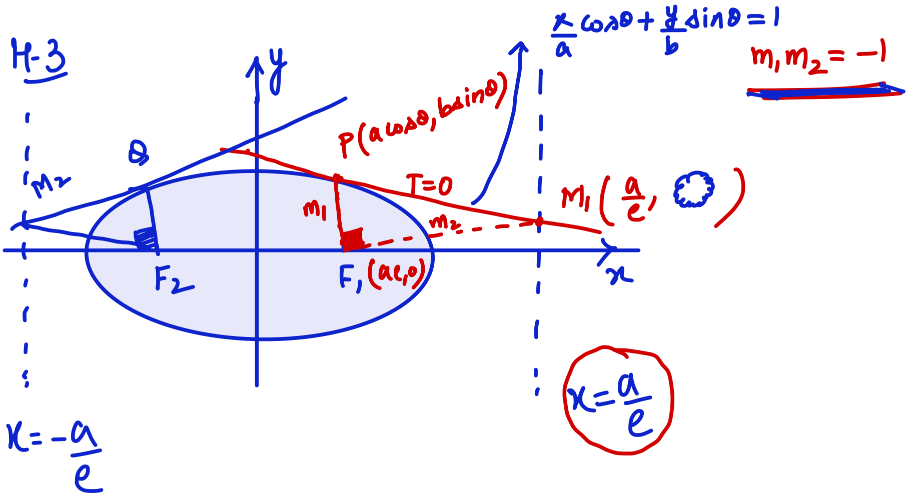
H-4 Harmonic mean of two segments of any focal chord is always constant which is equal to semi latus rectum.

*

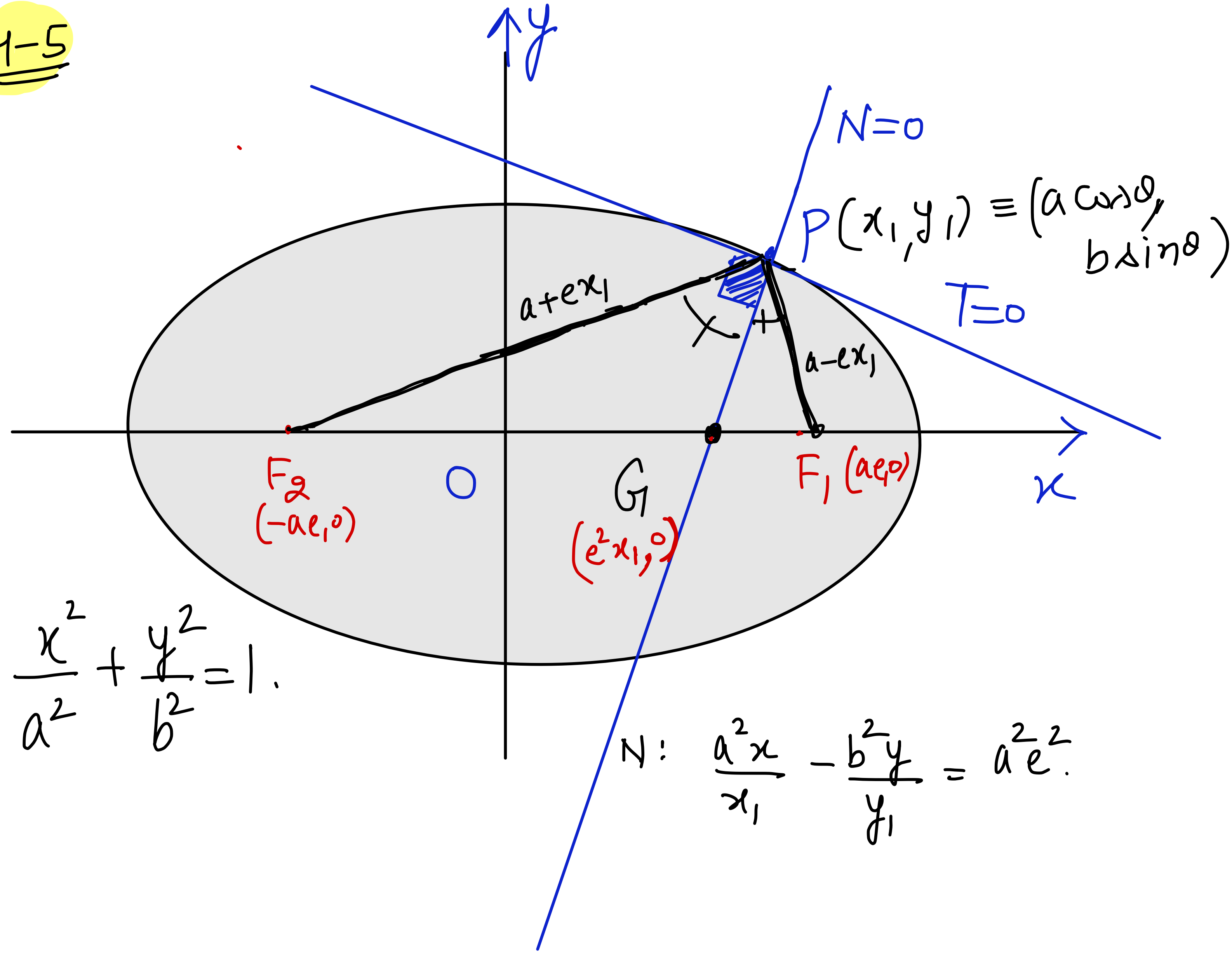
H-5 The tangent & normal at a point P on the ellipse bisect the external & internal angles between the focal distances of P. This refers to the well known reflection property of the ellipse which states that rays from one focus are reflected through other focus & vice-versa. Hence we can deduce that the straight lines joining each focus to the foot of the perpendicular from the other focus upon the tangent at any point P meet on the normal PG and bisects it where G is the point where normal at P meets the major axis.



H-3



H-5



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

$$N: \ \frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2e^2.$$

$$\frac{PF_2}{PF_1}=\frac{a+ex_1}{a-ex_1}$$

$$\frac{F_2G}{F_1G}=\frac{e^2x_1+ae}{ae-e^2x_1}=\frac{\cancel{e}(a+ex_1)}{\cancel{e}(a-ex_1)}$$

$\therefore$ PG is internal $\angle$ bisector of $\underline{|F_2PF_1}$.

H-6 If the normal at any point P on the ellipse with centre C meet the major & minor axes in G & g respectively, & if CF be perpendicular upon this normal, then

(i) PF . PG = b^2 **(ii)** PF . Pg = a^2

(iii) PG . Pg = SP . S′ P **(iv)** CG . CT = $(CS)^2$

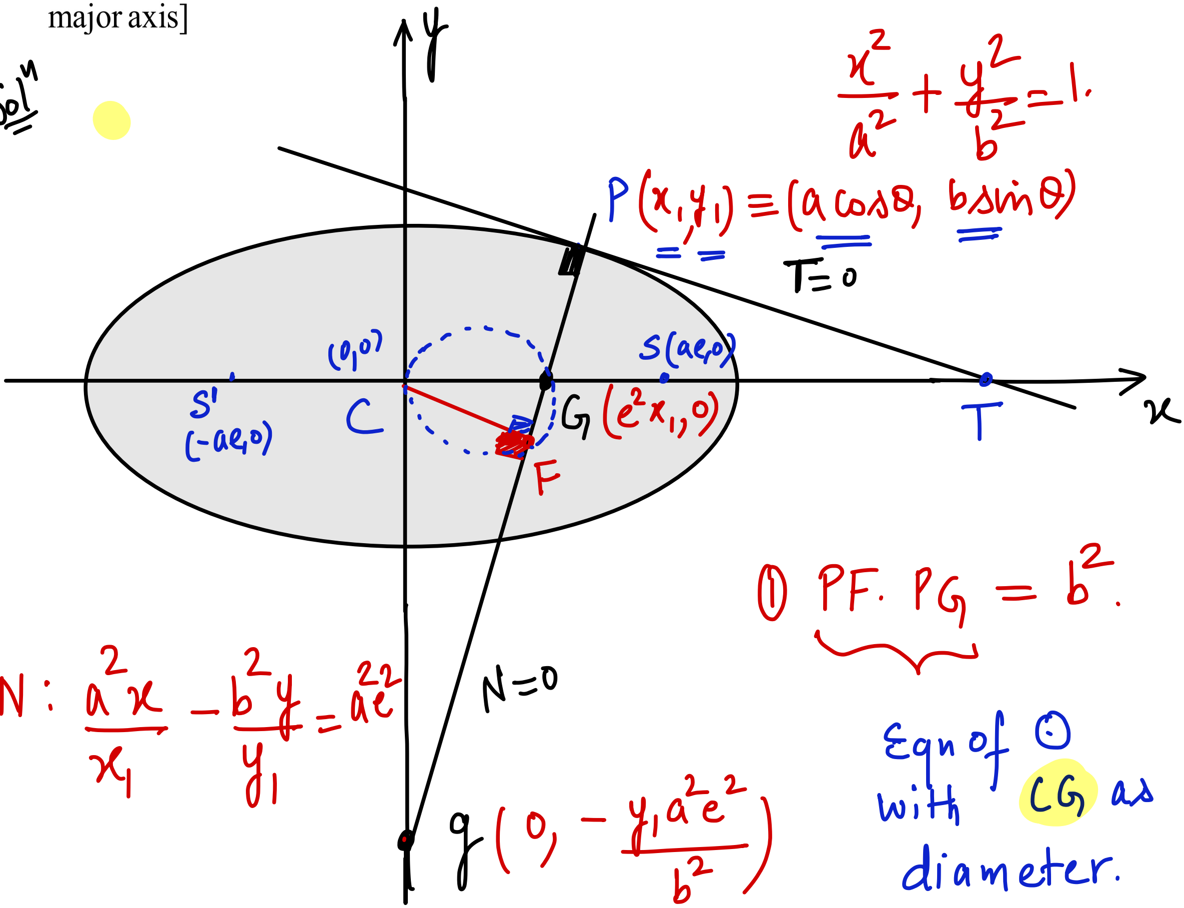
(v) locus of the mid point of Gg is another ellipse having the same eccentricity as that of the original ellipse.

[where S and S′ are the focii of the ellipse and T is the point where tangent at P meet the major axis]

Sol^n

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

$P(x_1,y_1) \equiv (a\cos\theta,\ b\sin\theta)$



$$N:\ \frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2e^2$$

$$g\left(0,\ -\frac{y_1a^2e^2}{b^2}\right)$$

① $\underbrace{PF.\ PG}=b^2.$

Eqn of ⊙ with CG as diameter.

$$(x-0)(x-e^2x_1)+(y-0)(y-0)=0.$$

$$\boxed{x^2+y^2-(e^2x_1)x=0}$$

PF · PG = Power of Pt P wrt above circle

$$PF \cdot PG = x_1^2 + y_1^2 - (e^2 x_1)x_1$$

$$= x_1^2(1-e^2) + y_1^2$$

$$= a^2 \cos^2\theta \, (1-e^2) + b^2 \sin^2\theta \, .$$

$$= b^2(\cos^2\theta + \sin^2\theta) = b^2$$

(H.P)

Remaining parts Do Yourself.

Q
1 Find the locus of the middle points of chords of an ellipse which subtend a right angle at their centre.

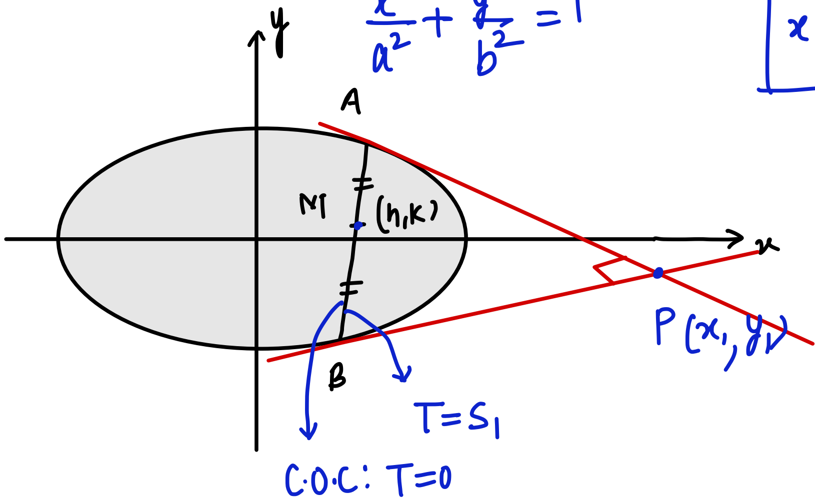
HW

2 the tangent at the ends of which intersect at right angles.

2

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$\boxed{x_1^2+y_1^2=a^2+b^2} \quad -①-$$



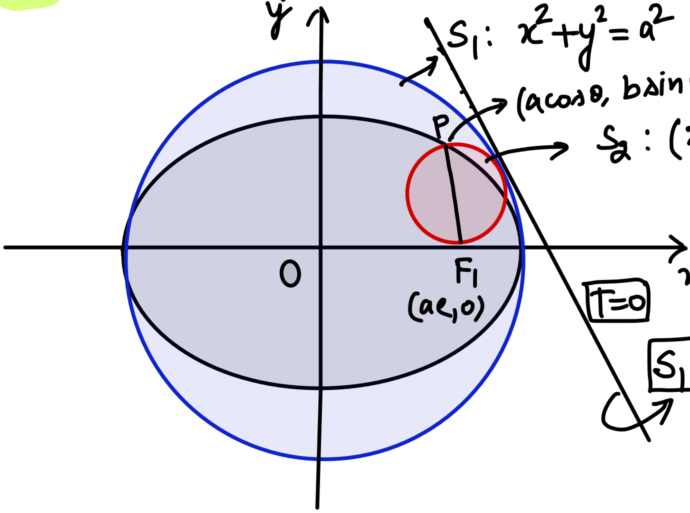
$$T=S_1 \Rightarrow \quad \frac{xh}{a^2}+\frac{yk}{b^2}-\cancel{1}=\frac{h^2}{a^2}+\frac{k^2}{b^2}-\cancel{1}$$

$$\frac{xh}{a^2}+\frac{yk}{b^2}-\left(\frac{h^2}{a^2}+\frac{k^2}{b^2}\right)=0 \quad -②-$$

$$T=0 \Rightarrow \quad \frac{xx_1}{a^2}+\frac{yy_1}{b^2}-1=0 \quad -③-$$

Compare ② & ③ $\Rightarrow$ $x_1=$ ✓ $y_1=$ ✓ put in ①

H-7 The circle on any focal distance as diameter touches the auxiliary circle.



$S_1: x^2 + y^2 = a^2$

$(a\cos\theta, b\sin\theta)$

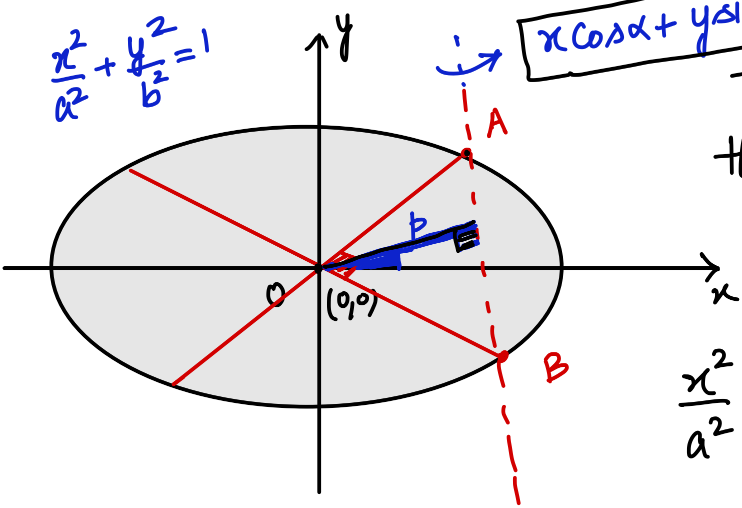
$S_2: (x - ae)(x - a\cos\theta) + (y - 0)(y - b\sin\theta) = 0$

H-8 Perpendiculars from the centre upon all chords which join the ends of any perpendicular diameters of the ellipse are of constant length.

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\boxed{x\cos\alpha + y\sin\alpha = p}$ —(1)—

$p \to$ Constant.



Homogenise ellipse with —(1)—

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \left(\frac{x\cos\alpha + y\sin\alpha}{p}\right)^2 = 0$$

Coeff of x^2 + Coeff of $y^2 = 0$

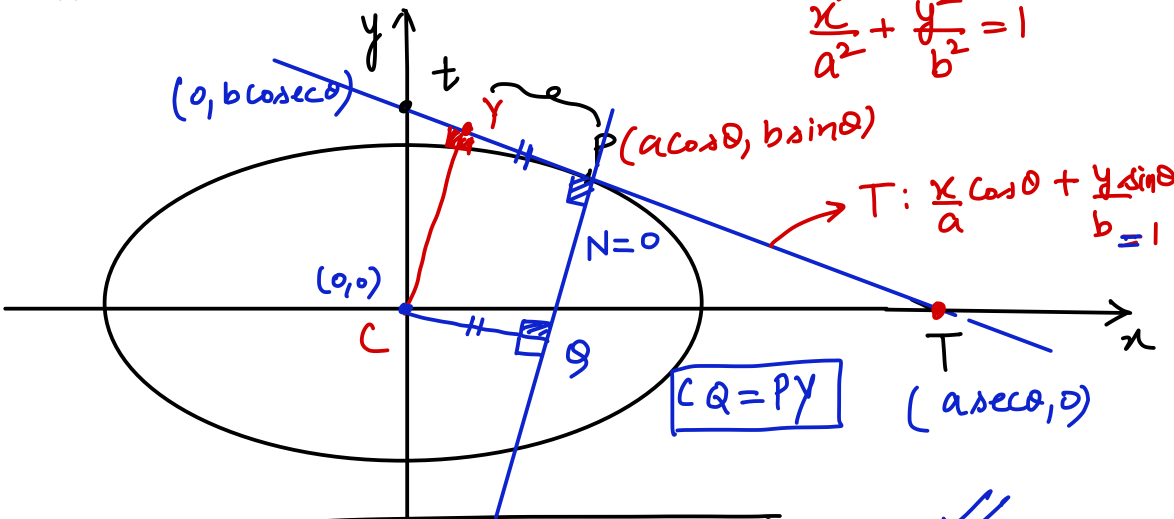
$$\left(\frac{1}{a^2} - \frac{\cos^2\alpha}{p^2}\right) + \left(\frac{1}{b^2} - \frac{\sin^2\alpha}{p^2}\right) = 0 \Rightarrow \boxed{\frac{1}{a^2} + \frac{1}{b^2} = \frac{1}{p^2}}$$

H-9 If the tangent at the point P of a standard ellipse meets the axis in T and t and CY is the perpendicular on it from the centre then, (C → Centre)

** (i) $Tt \cdot PY = a^2 - b^2$ and

(ii) least value of Tt is $a + b$.

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



$$\boxed{Tt = \sqrt{a^2\sec^2\theta + b^2\csc^2\theta}} \quad \text{— ① —}$$

$$Tt = \sqrt{a^2 + b^2 + \underbrace{a^2\tan^2\theta + b^2\cot^2\theta}}$$

(ii) $$\boxed{(Tt)_{min} = \sqrt{a^2 + b^2 + 2ab} = (a + b)}$$

N: $\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$.

$$CQ = \frac{|a^2 - b^2|}{\sqrt{a^2\sec^2\theta + b^2\csc^2\theta}} \quad \text{— ② —}$$

$\Downarrow$

PY

Q

Paragraph

Let the two foci of an ellipse be $(-1, 0)$ and $(3, 4)$ and the foot of perpendicular from the focus $(3, 4)$ upon a tangent to the ellipse be $(4, 6)$.

1. The foot of perpendicular from the focus $(-1, 0)$ upon the same tangent to the ellipse is

(A) $\left(\frac{12}{5}, \frac{34}{5}\right)$ (B) $\left(\frac{7}{3}, \frac{11}{3}\right)$ (C) $\left(2, \frac{17}{4}\right)$ (D) $(-1, 2)$

2. The equation of auxiliary circle of the ellipse is

(A) $x^2 + y^2 - 2x - 4y - 5 = 0$ (B) $x^2 + y^2 - 2x - 4y - 20 = 0$

(C) $x^2 + y^2 + 2x + 4y - 20 = 0$ (D) $x^2 + y^2 + 2x + 4y - 5 = 0$

3. The length of semi-minor axis of the ellipse is

(A) 1 (B) $2\sqrt{2}$ (C) $\sqrt{17}$ (D) $\sqrt{19}$

4. The equations of directrices of the ellipse are

(A) $x - y + 2 = 0,\ x - y - 5 = 0$ (B) $x + y - \frac{21}{2} = 0, x + y + \frac{17}{2} = 0$

(C) $x - y + \frac{3}{2} = 0, x - y - \frac{5}{2} = 0$ (D) $x + y - \frac{31}{2} = 0, x + y + \frac{19}{2} = 0$

Q Consider the parabola $y^2 = 4x$ and the ellipse $2x^2 + y^2 = 6$, intersecting at P and Q.

(a) Prove that the two curves are orthogonal.

(b) Find the area enclosed by the parabola and the common chord of the ellipse and parabola.

(c) If tangent and normal at the point P on the ellipse intersect the x-axis at T and G respectively then find the area of the triangle PTG.

Q Consider two straight lines, each of which is tangent to both the circle $x^2 + y^2 = \frac{1}{2}$ and the parabola $y^2 = 4x$. Let these lines intersect at the point Q. Consider the ellipse whose center is at the origin O(0, 0) and whose semi-major axis is OQ. If the length of the minor axis of this ellipse is $\sqrt{2}$, then the which of the following statement(s) is (are) TRUE ? **[JEE(Advanced)-2018, 4(–2)]**

(A) For the ellipse, the eccentricity is $\frac{1}{\sqrt{2}}$ and the length of the latus rectum is 1

(B) For the ellipse, the eccentricity is $\frac{1}{2}$ and the length of the latus rectum is $\frac{1}{2}$

(C) The area of the region bounded by the ellipse between the lines $x = \frac{1}{\sqrt{2}}$ and $x = 1$ is $\frac{1}{4\sqrt{2}}(\pi - 2)$

(D) The area of the region bounded by the ellipse between the lines $x = \frac{1}{\sqrt{2}}$ and $x = 1$ is $\frac{1}{16}(\pi - 2)$

Hyperbola

$\Delta \neq 0$

General equation : $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ denotes a hyperbola if $h^2 > ab$ and $e > 1$.

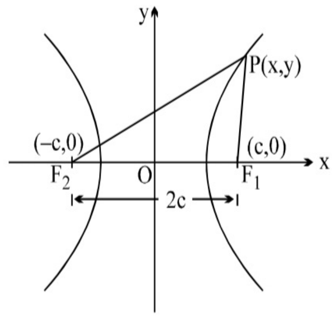
STANDARD EQUATION AND BASIC TERMINOLOGY :

Standard equation of hyperbola is deduced using an important property of hyperbola that the difference of a point moving on it, from two fixed points is constant.

i.e. $|PF_1 - PF_2| = 2a$ $(2a < 2c$ i.e. $c > a)$

i.e. $PF_1 - PF_2 = \pm 2a$

i.e. $\sqrt{(x-c)^2 + y^2} - \sqrt{(x+c)^2 + y^2} = \pm 2a$



simplifying we get $\dfrac{x^2}{a^2} - \dfrac{y^2}{c^2 - a^2} = 1$

or $\boxed{\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1}$ (where $c^2 - a^2 = b^2$) ....(1)

which is the same as the standard equation of ellipse except b^2 has been replaced by $-b^2$. This is a 2nd degree equation in which the powers of *x* and *y* are even, hence the curve is symmetric about both the axes. For each $x > a$ or $x < -a$ there are two values of y symmetrically situated on either side of x-axis and for each value of x lying in $(-a, a)$ the curve fails to exist. Hence the curve denoted by equation (1) consists of two symmetrical branches, each extending to infinity in two directions.

GENERAL TERMINOLOGY OF HYPERBOLA :

(a) **Transverse axis :** Line containing the fixed points S_1 and S_2 (called **Foci**) is called **Transverse Axis** (TA) or Focal Axis (from figure A_1A_2 is transverse axis).

(b) **Conjugate axis :** The perpendicular bisector of Transverse axis is called the **Conjugate Axis** (CA) (from figure B_1B_2 is conjugate axis).

(c) **Principal axes :** Transverse axis and conjugate axis together are called the **Principal Axes.**

(d) **Centre :** The point of intersection of these two axes is called the centre of the hyperbola. (from figure O is the centre).

(e) **Vertices :** Intersection point of hyperbola and transverse axis is known as vertices of the hyperbola (from figure A_1,A_2 are the vertices).

(f) **Length of transverse axis :** Distance between vertices is known as length of transverse axis (from figure length of A_1A_2 is the length of transverse axis)..

(g) **Length of conjugate axis :** Distance between ends of conjugate axis is known as length of conjugate axis (from figure length of B_1B_2 is the length of conjugate axis). .

(h) **Focal chord :** Any chord passing through the foci is known as focal chord.

(i) **Double ordinates :** Any chord perpendicular to the transverse axis is known as double ordinates (from figure AB is the double ordinate).

(j) **Latus rectum :** Double ordinates passing through the foci is known as latus rectum (from figure L_1L_2 is latus rectum).

(k) **Ends of latus rectum :** Intersection point of hyperbola and latus rectum is known as ends of latus rectum (from figureL_1,L_2 are the ends of latus rectum).

(*l*) **Length of latus rectum :** The distance between ends of the latus rectum is known as length of the latus rectum. (from figure distance between L_1,L_2 is the length of latus rectum).

(m) **Foot of directrix :** Intersection point of transverse axis and directrix is known as foot of directrix (from figure N_1,N_2 are the feet of directrix).

(n) **Focal distance/focal radii :** Distance of any point on the hyperbola from the foci is known as focal distance/focal radii. (from figure S_1P , S_2P are the focal distance/focal radii of point P).

ECCENTRICITY :

Defines the curvature of the hyperbola and is mathematically spelled as :

$$e = \frac{\text{distance from centre to focus}}{\text{distance from centre to vertex}} = \frac{c}{a}$$ $\Rightarrow e = \frac{c}{a} \Rightarrow \boxed{c = ae}$

$$e^2 = \frac{c^2}{a^2} = \frac{a^2+b^2}{a^2} = 1 + \frac{b^2}{a^2} = 1 + \left(\frac{CA}{TA}\right)^2$$ $\Rightarrow \boxed{e = \sqrt{1+\left(\frac{C.A}{T.A}\right)^2}}$

$F_1(ae, 0)$

$F_2(-ae, 0)$

$b^2 = a^2(e^2-1)$

<u>Remember that</u> :

(i) $a^2e^2 = a^2 + b^2$

(ii) coordinates of foci : $(\pm ae, 0)$ and

(iii) two hyperbolas are said to be similar if they have the same value of eccentricity.

$\therefore$ equation of hyperbola in terms of eccentricity can be written as $\frac{x^2}{a^2} - \frac{y^2}{a^2(e^2-1)} = 1$

substituting $x = ae$, we get

Latus Rectum $= L_1L_2 = \frac{2b^2}{a} = 2a(e^2-1) = \frac{(CA)^2}{TA}$ and the coordinates of the extremities of latus rectum $\left(\pm ae, \pm\frac{b^2}{a}\right)$

CONJUGATE HYPERBOLA :

Corresponding to every hyperbola there exist a hyperbola such that, the conjugate axis and transverse axis of one is equal to the transverse axis and conjugate axis of other, such hyperbolas are known as conjugate to each other.

Hence for the hyperbola, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ .....(1)

the conjugate hyperbola is, $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ ....(2)

if e_1 and e_2 are the eccentricities of a hyperbola and its conjugate respectively, then

Rem

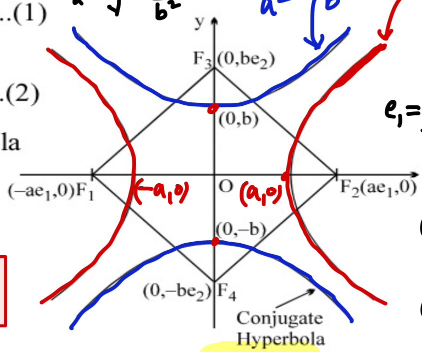
$$\boxed{e_1^{-2} + e_2^{-2} = 1}$$ $\Rightarrow \boxed{\frac{1}{e_1^2} + \frac{1}{e_2^2} = 1}$

Note : The foci of a hyperbola and its conjugate are concylic and form the vertices of a square.

$e_2 = \sqrt{1+\frac{a^2}{b^2}}$

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$



$e_1 = \sqrt{1+\frac{b^2}{a^2}}$

$e_1^2 = \frac{a^2+b^2}{a^2}$

$e_2^2 = \frac{a^2+b^2}{b^2}$

DIRECTRIX : Corresponding to each focus there are two lines $x = \pm\frac{a}{e}$ which are known as directrices and satisfy the focal directrix property of hyperbola.

i.e. $PF_1 = ePM_1 = e\left(x_1 - \frac{a}{e}\right)$

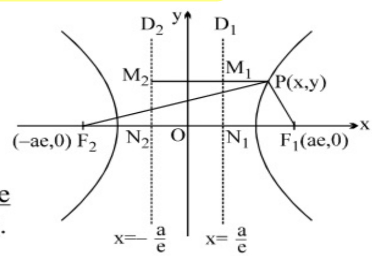
Rem

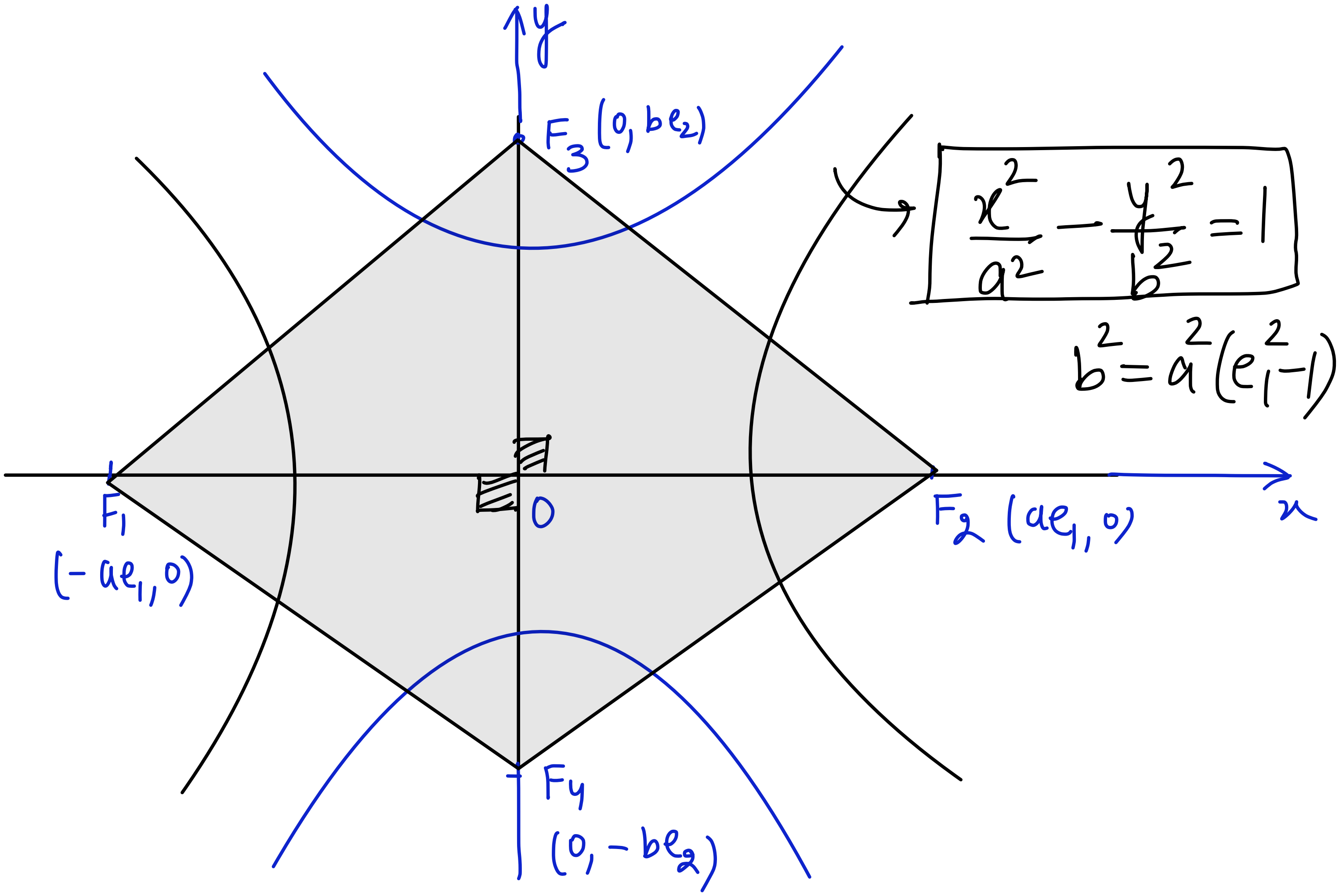
$\boxed{PF_1 = ex_1 - a}$ ....(1)

|||ly $\boxed{PF_2 = ex_1 + a}$ ....(2)

Hence $|PF_1 - PF_2| = 2a$

i.e. <u>the difference of the focal radii of any point on the hyperbola is equal to the length of its transverse axis.</u>





$$b^2 = a^2(e_1^2 - 1)$$

$$\boxed{F_1F_2 = F_3F_4}$$

$$\cancel{2}ae_1 = \cancel{2}be_2$$

$$a^2e_1^2 = b^2e_2^2 \Rightarrow a^2e_1^2 = b^2e_2^2$$

$$\cancel{a^2}e_1^2 = \cancel{a^2}(e_1^2 - 1)e_2^2$$

$$e_1^2 = e_1^2e_2^2 - e_2^2$$

$$e_1^2 + e_2^2 = e_1^2e_2^2 \Rightarrow \boxed{\frac{1}{e_1^2} + \frac{1}{e_2^2} = 1} \ (\text{H.P})$$

TRUE RESULT.

☞ **Note :**

(i) the points of intersection of the directrix with the transverse axis are known as Foot of the directrix (N_1 and N_2).

(ii) l (L.R.) $= 2a(e^2 - 1) = 2e\left(ae - \frac{a}{e}\right)$

l(L.R) = (2e) (distance between the focus and the foot of the corresponding directrix)

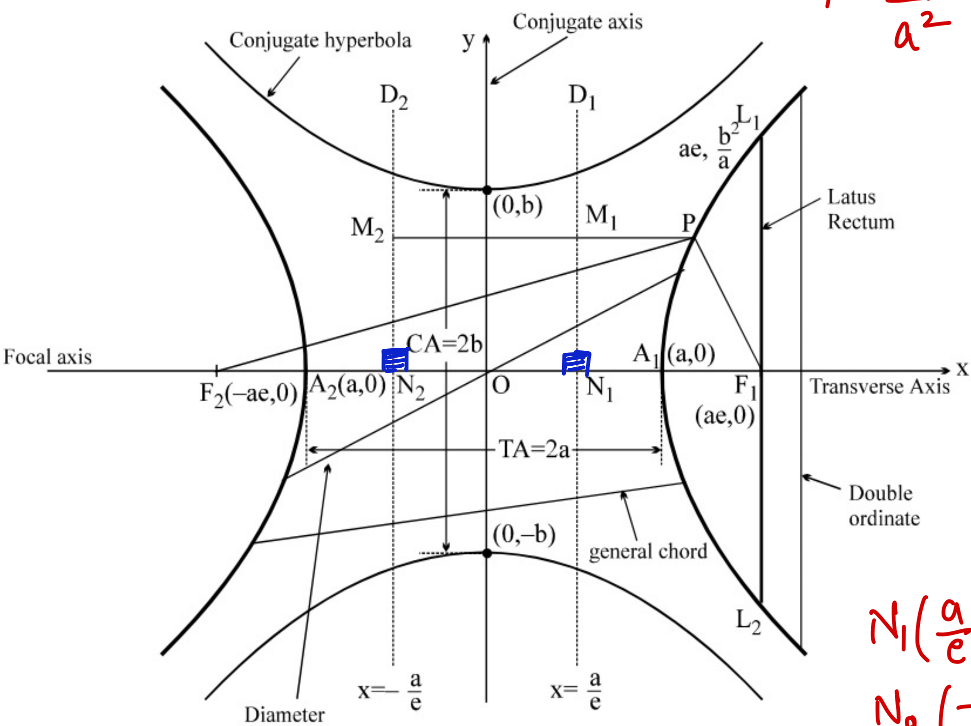
RECTANGULAR HYPERBOLA : If a = b, hyperbola is said to be equilateral or rectangular and has the equation, $x^2 - y^2 = a^2$. Eccentricity for such a hyperbola is $\sqrt{2}$.

l (LR) $= 2a(e^2 - 1) = 2a = l$ (TA)

HYPERBOLA AT A GLANCE

$\boxed{\text{Parametric coordinates } x = a\sec\theta \text{ and } y = b\tan\theta}$

$H: \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$



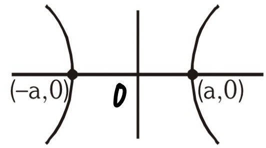
$N_1\left(\frac{a}{e}, 0\right)$

$N_2\left(-\frac{a}{e}, 0\right)$

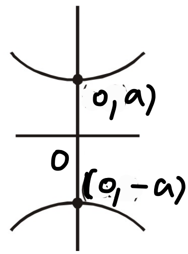
Foot of directrices.

TWO STANDARD HYPERBOLA :

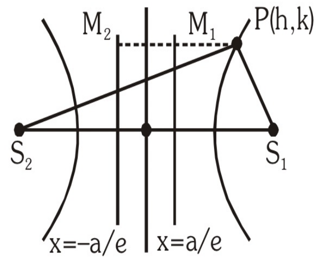
(a) Horizontal Hyperbola : $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$



(b) Vertical Hyperbola : $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$



Focal distance/ Focal radii :



i.e. $PS_1 = ePM_1 = e\left(h - \frac{a}{e}\right)$

$PS_1 = eh - a$ .....(i)

$PS_2 = eh + a$ .....(ii)

Hence $|\mathbf{PS_1 - PS_2}| = \mathbf{2a}$

- For hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ focal distance of any point P(h, k) is **(eh ± a)**
- For hyperbola $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$ focal distance of any point P(h, k) is **(ek ± a)**

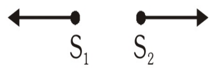
Alternate definition of Hyperbola :

Hyperbola is a locus of point which moves in such a way such that difference of its distances from two fixed points (foci) always remain constant (length of the transverse axis) and this constant must be less than the distance between two fixed point ($2a < 2ae$).

If $|PS_1 - PS_2| = \lambda$

Case I : If $\lambda < S_1S_2 \Rightarrow$ locus of P is a hyperbola.

Case II : If $\lambda = S_1S_2 \Rightarrow$ locus of P is pair of rays.



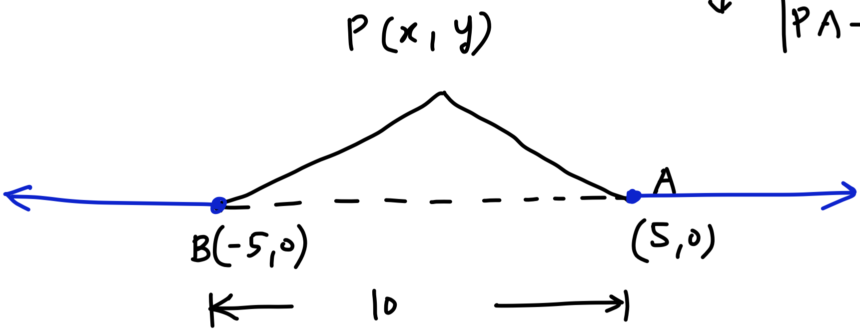
Case III : If $\lambda > S_1S_2 \Rightarrow$ no such point exist.

Q. Find the locus of the point which satisfies $\left|\sqrt{(x-5)^2+y^2} - \sqrt{(x+5)^2+y^2}\right| = 10$.

Soln

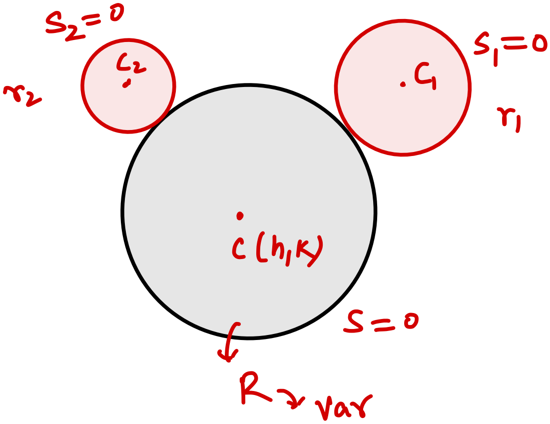
$|PA - PB| = \lambda$

$\boxed{AB = 10}$



The locus of P(x,y) is pair of rays as shown in above fig

Q Show that the locus of the centre of a circle which touches externally two given circles is a hyperbola.



$r_1, r_2 \to$ fix

C_1 & $C_2 \to$ fix

$CC_1 = R + r_1$

$CC_2 = R + r_2$

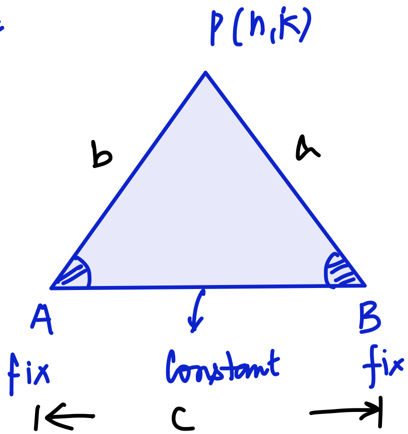
$$|CC_1 - CC_2| = \overbrace{|r_1 - r_2|}^{\lambda} = \text{Constant}$$

$\lambda < C_1C_2$

$\Rightarrow$ locus is hyperbola whose foci are C_1 & C_2 & whose length of TA is $|r_1 - r_2|$.

Q Given the base of a triangle and the ratio of the tangents of half the base angles, prove that the vertex moves on a hyperbola whose foci are the extremities of the base.

Sol^n



$$\boxed{\frac{\tan A/2}{\tan B/2} = K} \text{ (Constant)} \qquad \boxed{K > 0}$$

$$\boxed{\tan\frac{A}{2} = \frac{\Delta}{s(s-a)}}$$

$s \to$ semi-perimeter

$\Delta \to$ area of triangle

$$\frac{\frac{\cancel{\Delta}}{\cancel{s}(s-a)}}{\frac{\cancel{\Delta}}{\cancel{s}(s-b)}} = K$$

$$\boxed{\frac{s-b}{s-a} = K} \Rightarrow \frac{s-b+s-a}{\cancel{s}-b-\cancel{s}+a} = \frac{K+1}{K-1}$$

Componendo & dividendo.

$$\frac{2s-(a+b)}{a-b} = \frac{K+1}{K-1}$$

$2s = a+b+c$

$$\frac{c}{a-b} = \frac{K+1}{K-1}$$

$$\underbrace{a-b}_{\downarrow} = \underbrace{c\left(\frac{K-1}{K+1}\right)}$$

$$\boxed{PB - PA = \lambda}$$

$AB = C$ $\qquad$ $\lambda < C$

$\therefore \lambda < AB$

$\therefore$ Locus is hyp $\qquad$ (H.P)

Q An ellipse and a hyperbola are confocal (have the same focus) and the conjugate axis of hyperbola is equal to the minor axis of the ellipse. If e_1 and e_2 are the eccentricities of ellipse and hyperbola then find value of

$$\frac{1}{e_1^2}+\frac{1}{e_2^2}= ?$$

Solⁿ

$$E : \frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \longrightarrow e_1=\sqrt{1-b^2/a^2}$$

$$e_1^2=\frac{a^2-b^2}{a^2}$$

$$H : \frac{x^2}{A^2}-\frac{y^2}{B^2}=1 \qquad \frac{1}{e_1^2}=\frac{a^2}{a^2-b^2} \quad —(1)—$$

$$e_2=\sqrt{1+\frac{B^2}{A^2}} \Rightarrow \frac{1}{e_2^2}=\frac{A^2}{A^2+B^2} \quad —(2)—$$

Acc to ques $\Rightarrow$ $B=b \Rightarrow B^2=b^2$

$$A^2(e_2^2-1)=a^2(1-e_1^2)$$

$$F_E(ae_1,0), \; F_H(Ae_2,0) \longrightarrow ae_1=\underline{\underline{A}}e_2$$

$$\frac{\cancel{a^2}e_1^2}{e_2^2}(e_2^2-1)=\cancel{a^2}(1-e_1^2)$$

$$e_1^2e_2^2-e_1^2=e_2^2-e_1^2e_2^2$$

$$\boxed{\frac{1}{e_1^2}+\frac{1}{e_2^2}=2} \text{ Ans}$$

Q

Find the equation of hyperbola :

(a) Whose centre is (1, 0); focus is (6, 0) and transverse axis 6. [**Ans.** $\frac{(x-1)^2}{9} - \frac{y^2}{16} = 1$]

HW

~~**(b)**~~ Whose centre is (3, 2), one focus is (5, 2) and one vertex is (4, 2). [**Ans.** $(x-3)^2 - \frac{(y-2)^2}{3} = 1$]

HW

~~**(c)**~~ Whose centre is (–3, 2), one vertex is (– 3, 4) and eccentricity is 5/2. [**Ans.** $\frac{(x+3)^2}{21} - \frac{(y-2)^2}{4} = -1$]

HW

~~**(d)**~~ Whose foci are (4, 2) and (8, 2) and eccentricity is 2. [**Ans.** $\frac{(x-6)^2}{1} - \frac{(y-2)^2}{3} = 1$]

Sol^n (a)

$\boxed{m_{CF_1} = 0}$

$\underbrace{C(1,0) \qquad F_1(6,0)}_{ae=5}$; $TA = 6$

$2a = 6$

$\boxed{a = 3}$

$ae = 5$

$\boxed{e = \frac{5}{3}}$

$b^2 = a^2(e^2 - 1)$

$b^2 = 9\left(\frac{25}{9} - 1\right)$

$\boxed{b^2 = 16}$

$$H: \frac{(x-1)^2}{9} - \frac{(y-0)^2}{16} = 1$$

Q

Paragraph

Let the two foci of an ellipse be $(-1, 0)$ and $(3, 4)$ and the foot of perpendicular from the focus $(3, 4)$ upon a tangent to the ellipse be $(4, 6)$.

1. The foot of perpendicular from the focus $(-1, 0)$ upon the same tangent to the ellipse is

(A) $\left(\frac{12}{5}, \frac{34}{5}\right)$ (B) $\left(\frac{7}{3}, \frac{11}{3}\right)$ (C) $\left(2, \frac{17}{4}\right)$ (D) $(-1, 2)$

2. The equation of auxiliary circle of the ellipse is

(A) $x^2 + y^2 - 2x - 4y - 5 = 0$ (B) $x^2 + y^2 - 2x - 4y - 20 = 0$

(C) $x^2 + y^2 + 2x + 4y - 20 = 0$ (D) $x^2 + y^2 + 2x + 4y - 5 = 0$

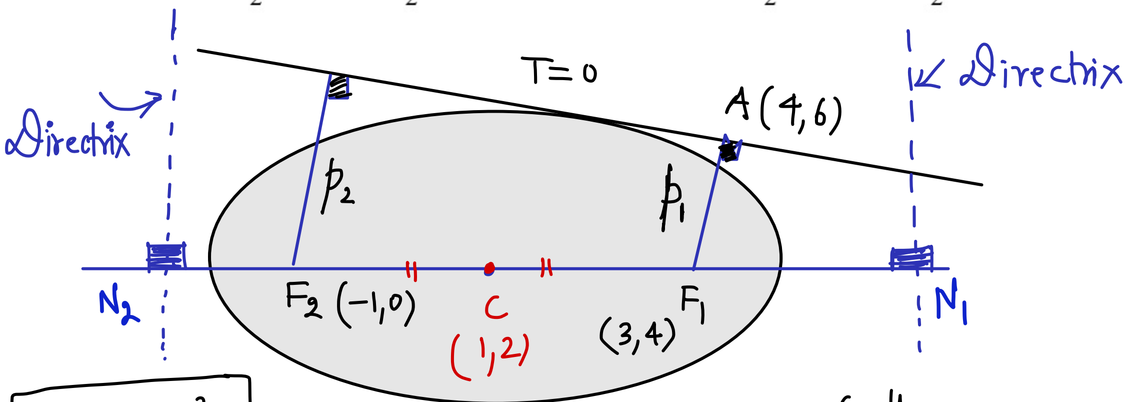
3. The length of semi-minor axis of the ellipse is

(A) 1 (B) $2\sqrt{2}$ (C) $\sqrt{17}$ (D) $\sqrt{19}$

4. The equations of directrices of the ellipse are

(A) $x - y + 2 = 0,\ x - y - 5 = 0$ (B) $x + y - \frac{21}{2} = 0,\ x + y + \frac{17}{2} = 0$

(C) $x - y + \frac{3}{2} = 0,\ x - y - \frac{5}{2} = 0$ (D) $x + y - \frac{31}{2} = 0,\ x + y + \frac{19}{2} = 0$



$$\boxed{p_1 p_2 = b^2}$$

$$\boxed{b^2 = 17}$$

$F_1F_2 = 2ae$

$\sqrt{32} = 2ae.$

$a^2(1-e^2) = 17 \Rightarrow a^2 = 25$

$p_1 = \frac{17}{\sqrt{5}}$

$p_2 = \frac{5}{\sqrt{5}}$

$m_{AF_1} = \frac{6-4}{4-3} = 2$

$\therefore m_T = -\frac{1}{2}$

Eqn of T: $y - 6 = -\frac{1}{2}(x-4)$

T: $\boxed{x + 2y = 16}$

Eqn of auxillary circle $\Rightarrow (x-1)^2+(y-2)^2 = a^2 = 25$

$$x^2+y^2-2x-4y-20=0.$$

$$\frac{a}{e} = \frac{5}{2\sqrt{2}} \times 5 = \frac{25}{2\sqrt{2}} \quad ; \quad m_{F_1F_2} = \frac{4}{4} = 1$$

$$\therefore \; m_{\text{directrix}} = -1$$

Distance betn centre 'c' and foot of directrix $= \dfrac{a}{e}$.

$$\frac{x-1}{\cos 45^\circ} = \frac{y-2}{\sin 45^\circ} = \pm\left(\frac{a}{e}\right)$$

$$\frac{x-1}{1/\sqrt{2}} = \frac{y-2}{1/\sqrt{2}} = \pm \frac{25}{2\sqrt{2}}$$

$$-\frac{25}{4}+1$$

$$x-1 = y-2 = \pm \frac{25}{4}$$

$\oplus$ N_1: $x = \dfrac{29}{4}$; $y = \dfrac{33}{4} \longrightarrow y-\dfrac{33}{4} = -1\left(x-\dfrac{29}{4}\right)$

$\ominus$ N_2: $x = -\dfrac{21}{4}$; $y = \dfrac{-17}{4} \longrightarrow y-\left(\dfrac{-17}{4}\right) = -1\left(x+\dfrac{21}{4}\right)$

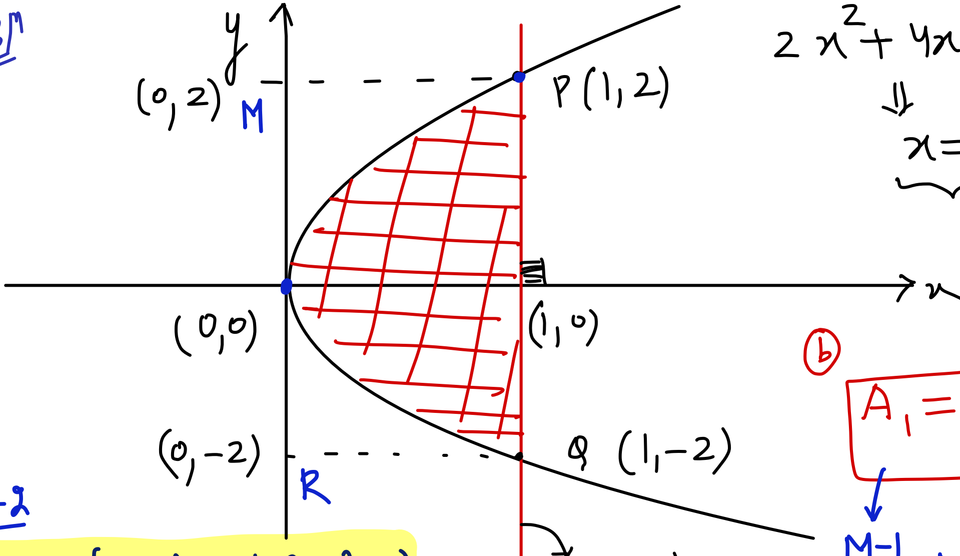
Q Consider the parabola $y^2 = 4x$ and the ellipse $2x^2 + y^2 = 6$, intersecting at P and Q.

AOD (a) Prove that the two curves are orthogonal.

(b) Find the area enclosed by the parabola and the common chord of the ellipse and parabola.

(c) If tangent and normal at the point P on the ellipse intersect the x-axis at T and G respectively then find the area of the triangle PTG.

Solⁿ



$$2x^2 + 4x = 6$$
$$\Downarrow$$
$$\underbrace{x = 1}; \ \cancel{-3}$$

(b) $\boxed{A_1 = \frac{8}{3}}$

M-1

$$A_1 = 2\int_0^1 \sqrt{4x}\,dx$$

M-2

$$A_1 = \frac{2}{3}(\text{Ar of Rect PQRM})$$

$$E: \frac{x^2}{3} + \frac{y^2}{6} = 1.$$

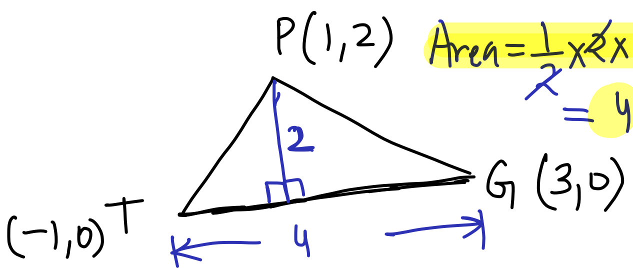
Tangent at P(1,2) : $\frac{x(1)}{3} + \frac{y(\cancel{2})}{\cancel{6}\,3} = 1$

$$\boxed{x + y = 3}$$

(c)

Normal at P(1,2)

$$y - 2 = 1(x - 1)$$
$$\boxed{x - y = -1}$$



$$\text{Area} = \frac{1}{\cancel{2}} \times \cancel{2} \times 4 = 4$$

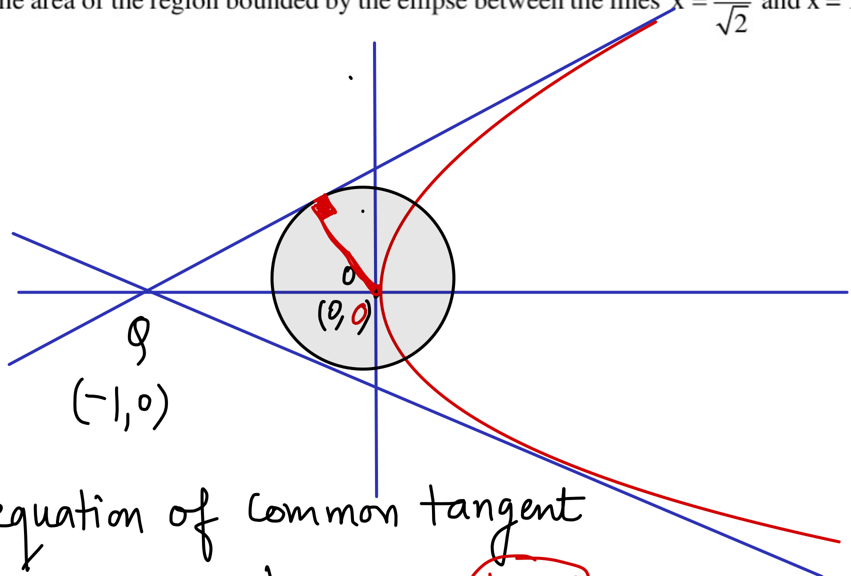
Q Consider two straight lines, each of which is tangent to both the circle $x^2+y^2=\frac{1}{2}$ and the parabola $y^2=4x$. Let these lines intersect at the point Q. Consider the ellipse whose center is at the origin O(0, 0) and whose semi-major axis is OQ. If the length of the minor axis of this ellipse is $\sqrt{2}$, then the which of the following statement(s) is (are) TRUE ? **[JEE(Advanced)-2018, 4(–2)]**

(A) For the ellipse, the eccentricity is $\frac{1}{\sqrt{2}}$ and the length of the latus rectum is 1 ✓

(B) For the ellipse, the eccentricity is $\frac{1}{2}$ and the length of the latus rectum is $\frac{1}{2}$

(C) The area of the region bounded by the ellipse between the lines $x=\frac{1}{\sqrt{2}}$ and $x=1$ is $\frac{1}{4\sqrt{2}}(\pi-2)$ ✓

(D) The area of the region bounded by the ellipse between the lines $x=\frac{1}{\sqrt{2}}$ and $x=1$ is $\frac{1}{16}(\pi-2)$



Let equation of common tangent is $y = mx + \frac{1}{m}$. $p = r$

$$\therefore \frac{\left|0+0+\frac{1}{m}\right|}{\sqrt{m^2+1}} = \frac{1}{\sqrt{2}} \Rightarrow m^4+m^2-2=0$$

$\therefore \boxed{m=\pm 1}$

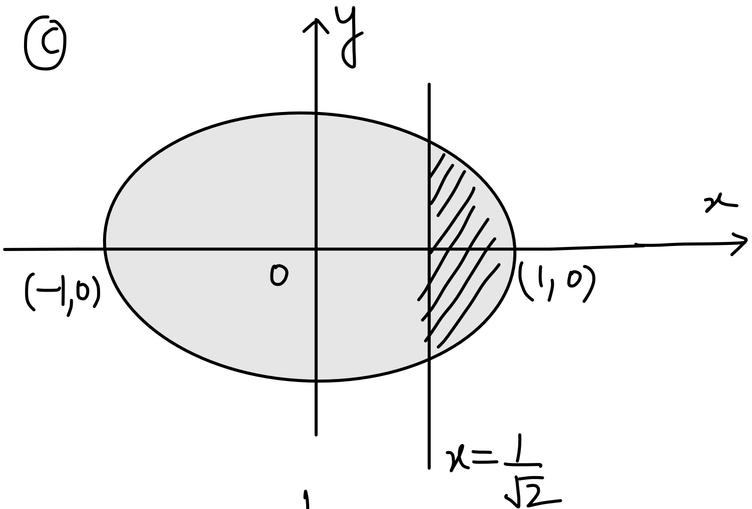
Eqn of common tangents : $\begin{cases} y = x+1 \ \& \\ y=-x-1 \end{cases}$

Q (−1, 0)

Equation of ellipse is $\frac{x^2}{1} + \frac{y^2}{1/2} = 1$.

(A) $e = \sqrt{1 - \frac{1}{2}} = \frac{1}{\sqrt{2}}$; $L.R = \frac{2b^2}{a} = 1$

(C)



$$\text{Area} = 2\int_{\frac{1}{\sqrt{2}}}^{1} \frac{1}{\sqrt{2}}\sqrt{1-x^2}\,dx = \sqrt{2}\int_{1/\sqrt{2}}^{1} \sqrt{1-x^2}\,dx$$

$$= \sqrt{2}\left(\frac{x}{2}\sqrt{1-x^2} + \frac{1}{2}\sin^{-1}\frac{x}{a}\right)\Bigg|_{\frac{1}{\sqrt{2}}}^{1} = \frac{\pi - 2}{4\sqrt{2}}$$

Q Find everything for the hyperbola $9x^2 - 18x - 16y^2 - 64y + 89 = 0$.

[Ans. centre $(1, -2)$; vertices $(1,1)$; $(1, -5)$, foci $(1, 3)$; $(1, -7)$]

Sol^n

$$9(x^2-2x) - 16(y^2+4y) + 89 = 0$$

$$9(x-1)^2 - 9 - 16\left((y+2)^2 - 4\right) + 89 = 0$$

$$9(x-1)^2 - 16(y+2)^2 - 9 + 64 + 89 = 0$$

$$9(x-1)^2 - 16(y+2)^2 = -144$$

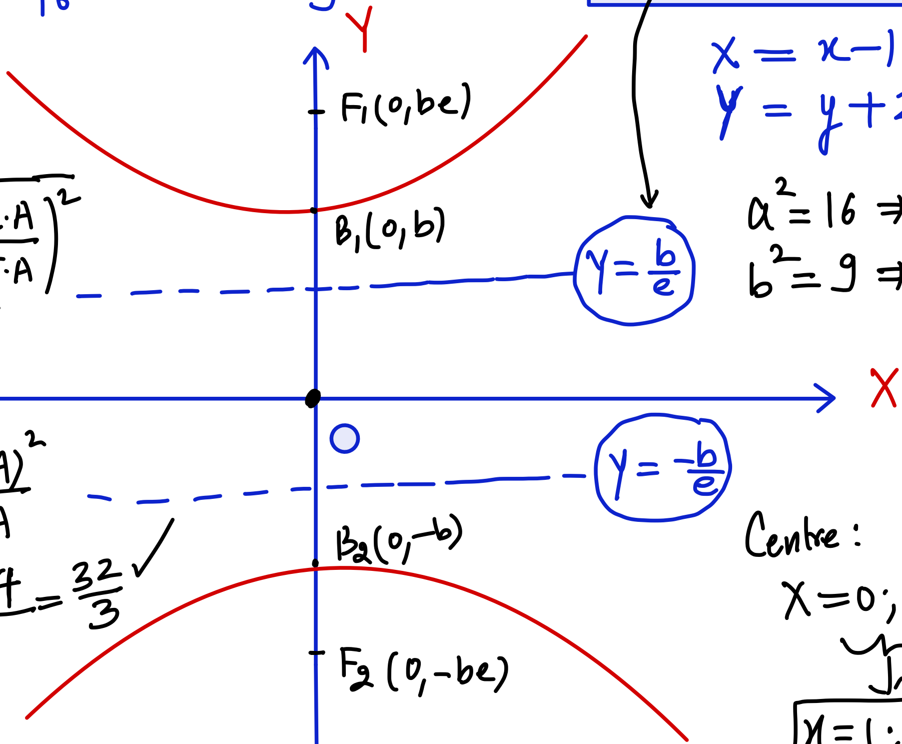
$$\frac{-(x-1)^2}{16} + \frac{(y+2)^2}{9} = 1 \Rightarrow \boxed{\frac{-X^2}{16} + \frac{Y^2}{9} = 1}$$

$X = x-1$

$Y = y+2$

$a^2 = 16 \Rightarrow a = 4$

$b^2 = 9 \Rightarrow b = 3$



$$e = \sqrt{1 + \left(\frac{C.A}{T.A}\right)^2}$$

$e = 5/3$

$$L.R = \frac{(C.A)^2}{T.A} = \frac{64}{6} = \frac{32}{3}$$

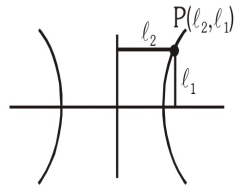
Centre :

$X = 0 ; Y = 0$

$\boxed{x = 1 ; y = -2}$

Q[HW] Find eqn of hyperbola referred to its principal axis as co-ordinate axes, if distance of one of its vertices from foci are 3 & 1.

Remember : Let P be any point on the hyperbola and



$\ell_1 \rightarrow$ Length of perpendicular from P on transverse axis.

$\ell_2 \rightarrow$ Length of perpendicular from P on conjugate axis.

$$\boxed{\frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \Rightarrow \frac{\ell_2^2}{a^2}-\frac{\ell_1^2}{b^2}=1}$$

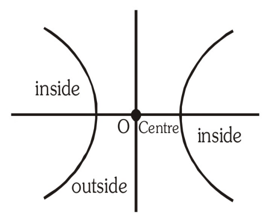
Generalised Version :

If $\ell x + my + n = 0$ is its equation of transverse axis & $mx - \ell y + p = 0$ is its equation of conjugate axis, 2a is the length of transverse axis & 2b is the length of conjugate axis then equation of the ellipse is

$$\frac{\left(\frac{mx-\ell y+p}{\sqrt{\ell^2+m^2}}\right)^2}{a^2}-\frac{\left(\frac{\ell x+my+n}{\sqrt{\ell^2+m^2}}\right)^2}{b^2}=1.$$

POSITION OF POINT W.R.T. HYPERBOLA :

For standard hyperbola :



Let $S(x, y) : \frac{x^2}{a^2}-\frac{y^2}{b^2}-1=0$

its centre is (0, 0) which lies outside the hyperbola and gives negative sign i.e. $S(0, 0) = -1$

The quantity $S_1 \equiv \mathbf{\frac{x_1^2}{a^2}-\frac{y_1^2}{b^2}-1}$ is **positive , zero** or **negative** according as the point (x_1, y_1) lies within, upon or outside the curve.

$S_1 < 0 \Rightarrow$ Pt lies outside hyperbola.

$S_1 > 0 \Rightarrow$ " " inside " .

$S_1 = 0 \Rightarrow$ " " on " .

For general hyperbola :

To find the position of any point w.r.t. any hyperbola $H(x,y)$ we have to take its centre as reference

Step - 1 : Find the centre (α, β) of hyperbola by using the method of partial differentiation.

Step - 2 : If $H(\alpha, \beta) < 0$, then all points lying outside the hyperbola always give negative sign and for all the points lying inside will give positive sign.

If (x_1,y_1) be any point satisfying

$H(x_1,y_1) < 0 \Rightarrow$ Point lies outside the hyperbola $\Rightarrow$ Two tangents are possible

$H(x_1,y_1) = 0 \Rightarrow$ Point lies on the hyperbola $\Rightarrow$ One tangent is possible

$H(x_1,y_1) > 0 \Rightarrow$ Point lies inside the hyperbola $\Rightarrow$ No tangent is possible

E(1) Find the position of the points $(7, -3)$ and $(2, 7)$ relative to the hyperbola $9x^2 - 4y^2 = 36$.

$S: 9x^2 - 4y^2 - 36 = 0.$

$(7,-3)$

$S(7,-3) \equiv 9(7)^2 - 4(-3)^2 - 36 > 0 \rightarrow$ inside hyp.

$S(2,7) \equiv 9\cancel{(2)}^2 - 4(7)^2 - \cancel{36} < 0 \rightarrow$ outside ".

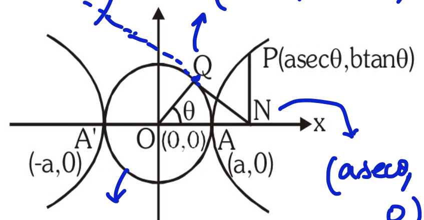
AUXILIARY CIRCLE/ECCENTRIC ANGLE/ PARAMETRIC COORDINATES :

Definition :

Locus of foot of the perpendicular drawn from foci upon any tangent drawn to the hyperbola is known as its auxiliary circle.

T: $x\cos\theta + y\sin\theta = a$

$(a\cos\theta, a\sin\theta)$



$(a\sec\theta, 0)$

$x^2+y^2=a^2.$

Note : Circle described on the transverse axis as diameter is called the auxiliary circle of the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ has the equation $\boxed{x^2 + y^2 = a^2}$ .... (i)

Corresponding points : Perpendicular drawn from the point P on the hyperbola, to the transverse axis meet it at the point N then tangent drawn from the point N to auxiliary circle, touches it at Q then Q is known as corresponding point. From the above figure P & Q are corresponding points of each other.

Eccentric angle of point P : Angle made by radius vector through Q (corresponding point of P) with transverse axis is eccentric angle of point 'P'. $\angle QON = \theta \neq \angle PON$ $\quad 0 \le \theta < 2\pi, \theta \neq \left\{\dfrac{\pi}{2}, \dfrac{3\pi}{2}\right\}$

$P(a\sec\theta, b\tan\theta)$

(i) If $\theta \in (0, \pi/2) \Rightarrow$ P lies on right upper branch of hyp.

(ii) If $\theta \in (\frac{\pi}{2}, \pi) \Rightarrow$ P lies on left lower " " ".

(iii) If $\theta \in (\pi, \frac{3\pi}{2}) \Rightarrow$ P " " " upper " " "

(iv) If $\theta \in (\frac{3\pi}{2}, 2\pi) \Rightarrow$ P " " Right lower " " ".

Q Find the parametric equation of the hyperbola $\frac{(x-5)^2}{36}-\frac{(y+3)^2}{25}=1$.

Sol^n

$$\frac{X^2}{6^2}-\frac{Y^2}{5^2}=1 \quad ; \quad X = 6\sec\theta, \quad Y = 5\tan\theta.$$

$$X = x-5$$
$$Y = y+3$$

$$\boxed{\begin{aligned} x &= 5+6\sec\theta. \\ y &= -3+5\tan\theta. \end{aligned}}$$

CHORD JOINING TWO POINTS WHOSE ECCENTRIC ANGLES ARE α & β :

Chord joining two points with eccentric angles α and β is given by

$$\boxed{\frac{x}{a}\cos\frac{\alpha-\beta}{2}-\frac{y}{b}\sin\frac{\alpha+\beta}{2}=\cos\frac{\alpha+\beta}{2}} \quad \text{.....(i)}$$

Rem

If (i) passes through (d, 0) then $\tan\frac{\alpha}{2}\tan\frac{\beta}{2}=\frac{a-d}{a+d}$

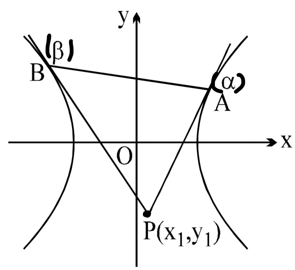
Note :

(i) If $d = ae$ i.e. PQ is a focal chord, then $\tan\frac{\alpha}{2}\tan\frac{\beta}{2}=\frac{1-e}{1+e}$

(ii) If $d = -ae$ i.e. PQ is a focal chord, then $\cot\frac{\alpha}{2}\cot\frac{\beta}{2}=\frac{1-e}{1+e}$

Rem

$$\boxed{x_1 = a\cdot\frac{\cos\frac{\alpha-\beta}{2}}{\cos\frac{\alpha+\beta}{2}} \quad ; \quad y_1 = b\cdot\frac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha+\beta}{2}}}$$



LINE AND A HYPERBOLA :

$y = mx + c$; $\quad b^2x^2 - a^2y^2 = a^2b^2$

$b^2x^2 - a^2(mx + c)^2 = a^2b^2$

$(b^2 - a^2m^2)x^2 - 2a^2mc\, x - a^2(b^2 + c^2) = 0$ ....(1)

$D > 0$	$D = 0$	$D < 0$
Secant	Tangent	$\overline{T} \cap \overline{S}$

condition of tangency gives, $c^2 = a^2m^2 - b^2$

hence, $y = mx \pm \sqrt{a^2m^2 - b^2}$ is a tangent to the standard hyperbola. ....(1)

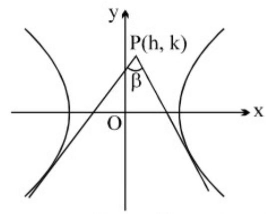
Hence for given m, there can be two parallel tangents to the hyperbola.

If (1), passes through (h, k) then

$(k - mh)^2 = a^2m^2 - b^2$

$\boxed{(h^2 - a^2)m^2 - 2kmh + k^2 + b^2 = 0}$ m_1 m_2 ....(2)

Hence passing through a given point (h, k) there can be a maximum of two tangents



now, $m_1 + m_2 = \dfrac{2kh}{h^2 - a^2}$ ....(3)

$m_1m_2 = \dfrac{k^2 + b^2}{h^2 - a^2}$ ....(4)

Equations (3) and (4) are used to find the locus of the point of intersection of a pair of tangents which enclose an angle β.

Now $\tan^2\beta = \dfrac{(m_1 + m_2)^2 - 4m_1m_2}{(1 + m_1m_2)^2}$

(substituting the values of $m_1 + m_2$ and m_1m_2 to get the locus)

If $\beta = 90°$ then $m_1m_2 = -1$ hence from (4)

$k^2 + b^2 = a^2 - h^2$

$x^2 + y^2 = a^2 - b^2$ which is the equation to the director circle, of the hyperbola.

If l (TA) > l (CA) ; director circle is real with finite radius.

if l (TA) = l (CA) ; director circle is a point circle

if l (TA) < l (CA) ; no real circle

TANGENTS TO A HYPERBOLA :

(a) **Cartesian form :** $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$ is tangent to the hyperbola at (x_1, y_1).

Rem

(b) **Parametric form :** $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$ at the point $(a\sec\theta, b\tan\theta)$ of the hyperbola.

Note :

(i) Slope of tangent at the point $(a\sec\theta, b\tan\theta)$ is $\frac{b}{a}\text{cosec}\theta$.

(ii) The sum of eccentric angles of their point of contact of two parallel tangents is π, chord joinin point of contact always passing through the centre.

$$m_1 = m_2 \Rightarrow \frac{b}{a}\text{cosec}\,\alpha = \frac{b}{a}\text{cosec}\,\beta$$

$$\text{cosec}\,\alpha = \text{cosec}\,\beta$$

$$\sin\alpha = \sin\beta.$$

$$\alpha = \pi - \beta \Rightarrow \boxed{\alpha + \beta = \pi}$$

(C) Slope form :-

Rem

$$\boxed{y = mx \pm \sqrt{a^2m^2 - b^2}}$$

The line $y = mx + c$ is tangent to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

if $c = \pm\sqrt{a^2m^2 - b^2}$ ***

for real tangent, $a^2m^2 - b^2 \geq 0$

$$\Rightarrow \quad m \in \left(-\infty, -\frac{b}{a}\right] \cup \left[\frac{b}{a}, \infty\right)$$

Case (i) : If $m \in \left(-\infty, -\frac{b}{a}\right) \cup \left(\frac{b}{a}, \infty\right)$ two real & distinct tangents can be drawn

Case (ii) : If $m \in \left(-\frac{b}{a}, \frac{b}{a}\right)$ no real tangent is possible

Case (iii) : If $m \in \left\{-\frac{b}{a}, \frac{b}{a}\right\}$ two asymptotes can be drawn

Q The number of possible tangents which can be drawn to the curve $4x^2 - 9y^2 = 36$, which are perpendicular to the straight line $5x + 2y - 10 = 0$ is

(A) zero ✓ (B) 1

(C) 2 (D) 4

$5x + 2y - 10 = 0$

$m = -5/2$

$m_\perp = 2/5$

Slope of tangent

Sol^n

$$\frac{x^2}{9} - \frac{y^2}{4} = 1.$$

$a^2 = 9 \ ; \ b^2 = 4$

$$\underbrace{a^2m^2}_{\text{slope of tangent}} - b^2 = 9 \times \frac{4}{25} - 4 < 0.$$

$\therefore$ No tangent

Q Find equation of common tangent(s) to $y^2 = 8x$ and $3x^2 - y^2 = 3$. Also find length of common tangent ?

Sol^n

$$y^2 = 8x \qquad a = 2 \quad \rightarrow \quad T_1 : y = mx + \frac{2}{m} \quad \text{—}(1)\text{—}$$

$$\frac{x^2}{1} - \frac{y^2}{3} = 1 \quad \rightarrow \quad T_2 : y = mx \pm \sqrt{m^2 - 3}$$

—(2)—

$$\frac{2}{m} = \pm\sqrt{m^2 - 3} \Rightarrow \frac{4}{m^2} = m^2 - 3$$

$$m^4 - 3m^2 - 4 = 0 \Rightarrow m^4 - 4m^2 + m^2 - 4 = 0$$

$$(m^2 - 4)(m^2 + 1) = 0$$

$$\boxed{m = \pm 2}$$

$$\left.\begin{aligned} T_1 &: y = 2x + 1 \\ T_2 &: y = -2x - 1. \end{aligned}\right\}$$

Q Find equation of tangent to $\frac{x^2}{36} - \frac{y^2}{9} = 1$ passing through $(0,4)$?

$a^2 = 36$

$b^2 = 9$

Sol^n

$$y - 4 = m(x - 0)$$

$$\boxed{y = mx + \underbrace{4}_{c}} \quad \text{—(1)—}$$

$$c = \pm\sqrt{a^2m^2 - b^2}$$

$$4 = \pm\sqrt{36m^2 - 9}$$

$$16 + 9 = 36m^2$$

$$\boxed{m = \pm\frac{5}{6}}$$

Q^{HW} Find equation of common tangent(s) to

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$?

Q If one axis of varying central conic (ellipse / hyperbola) be fixed is magnitude and position, prove that the locus of the point of contact of a tangent drawn to it from a fixed points on the other axis is a parabola.

Q Prove that the part of the tangent at any point of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ intercepted between the point of contact and the transverse axis is a harmonic mean between the lengths of the perpendiculars drawn from the foci on the normal at the same point.

NORMAL TO THE HYPERBOLA :

(a) **Cartesian Normal :** $\frac{a^2x}{x_1} + \frac{b^2y}{y_1} = a^2 + b^2 = a^2e^2$ at (x_1, y_1)

$m_N = -\frac{a}{b}\sin\theta$

(b) **Parametric Normal** : $\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2 = a^2e^2$ at $(a\sec\theta, b\tan\theta)$

Note : In general, four normals can be drawn to a hyperbola from any point and if $\alpha, \beta, \gamma, \delta$ the eccentric angles of these four co-normal points, then $\alpha + \beta + \gamma + \delta$ is an odd multiple of π. i.e. $\boldsymbol{\alpha+\beta+\gamma+\delta=(2n+1)\pi}$

(c) **Slope form** : The equation of normal of slope m to the given hyperbola is $y = mx \mp \frac{m(a^2+b^2)}{\sqrt{(a^2-m^2b^2)}}$ foot of normal are $\left(\pm\frac{a^2}{\sqrt{(a^2-m^2b^2)}}, \mp\frac{mb^2}{\sqrt{(a^2-m^2b^2)}}\right)$

Note :-

Equation of normal to $\frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$ at $P(x_1, y_1)$ is

$$\frac{x^2}{(-a^2)} - \frac{y^2}{(-b^2)} = 1$$

$$\boxed{\frac{x^2}{A^2} - \frac{y^2}{B^2} = 1} ; \quad A^2 = -a^2,\ B^2 = -b^2$$

$$N: \frac{A^2x}{x_1} + \frac{B^2y}{y_1} = A^2e^2 = A^2 + B^2.$$

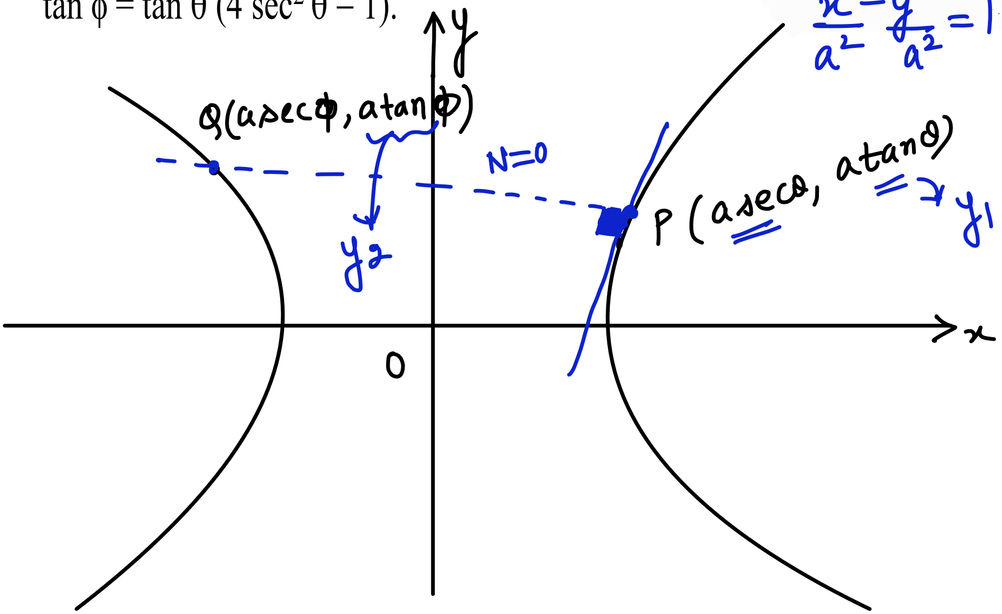
$$-\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = -a^2e^2.$$

Q. If a chord joining the points P (a sec θ, a tan θ) & Q (a sec ϕ, a tan ϕ) on the hyperbola $x^2 - y^2 = a^2$ is a normal to it at P, then show that

$$\tan\phi = \tan\theta\,(4\sec^2\theta - 1).$$

$$\frac{x^2}{a^2} - \frac{y^2}{a^2} = 1$$

Sol^n



$$N: \frac{\cancel{a}x}{\sec\theta} + \frac{\cancel{a}y}{\tan\theta} = 2a^{\cancel{2}} \Rightarrow N: \frac{x}{\sec\theta} + \frac{y}{\tan\theta} = 2a.$$

$$x = \sec\theta\left(2a - \frac{y}{\tan\theta}\right)$$

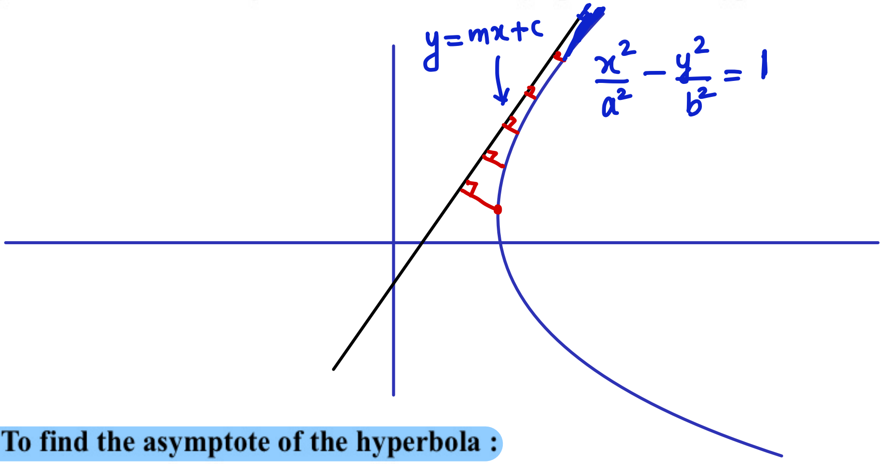
Solve $N=0$ with hyperbola

$$x^2 - y^2 = a^2$$

$$\sec^2\theta\left(2a - \frac{y}{\tan\theta}\right)^2 - y^2 = a^2.$$

ASYMPTOTES :

Definition : If the length of the perpendicular let fall from a point on a hyperbola to a straight line tends to zero as the point on the hyperbola moves to infinity along the hyperbola, then the straight line is called the Asymptote of the Hyperbola.



To find the asymptote of the hyperbola :

Let $y = mx + c$ is the asymptote of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.

Solving these two we get the quadratic as

$$(b^2 - a^2m^2)\, x^2 - 2a^2 mcx - a^2 (b^2 + c^2) = 0 \qquad ....(1)$$

In order that $y = mx + c$ be an asymptote, both roots of equation (1) must approach infinity, the conditions for which are :

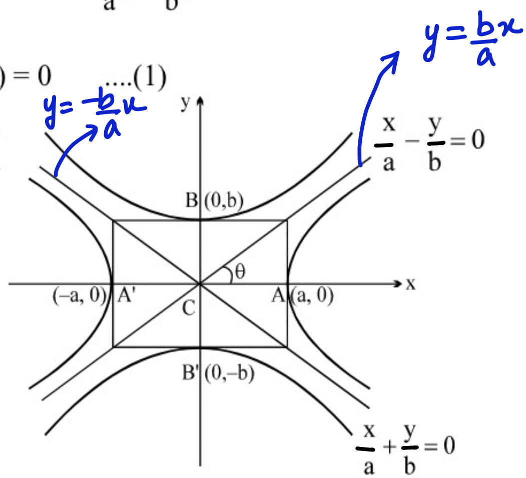
coeff of $x^2 = 0$ & coeff of $x = 0$.

$\Rightarrow \quad b^2 - a^2m^2 = 0$ or $m = \pm\frac{b}{a}$ &

$a^2 mc = 0 \Rightarrow c = 0.$

$\therefore$ equations of asymptote are $\frac{x}{a} + \frac{y}{b} = 0$

and $\frac{x}{a} - \frac{y}{b} = 0$.



combined equation to the asymptotes $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$.

PARTICULAR CASE :

When $b = a$ the asymptotes of the rectangular hyperbola.

$x^2 - y^2 = a^2$ are, $y = \pm x$ which are at right angles.

$x^2 - y^2 = a^2 \rightarrow$ Rectangular hyperbola OR Equilateral hyperbola

$e = \sqrt{2}$

$$\boxed{l(T\cdot A) = l(C\cdot A) = l(L\cdot R) = 2a}$$

$$L\cdot R = \frac{2b^2}{a} = \frac{2a^2}{a} = 2a.$$

$$H : \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$C\cdot A : \frac{x^2}{a^2} - \frac{y^2}{b^2} = 0$$

C.A → Combined equ of asymptotes

$$C\cdot H : \frac{x^2}{a^2} - \frac{y^2}{b^2} = -1$$

Rem

$$\boxed{H - C\cdot A = C\cdot A - C\cdot H}$$

$$\Rightarrow \boxed{C\cdot A = \frac{H + C\cdot H}{2}} \ * *$$

Note :

(i) Equilateral hyperbola $\Leftrightarrow$ rectangular hyperbola.

(ii) If a hyperbola is equilateral then the conjugate hyperbola is also equilateral.

(iii) A hyperbola and its conjugate have the same asymptote.

(iv) The equation of the pair of asymptotes differ the hyperbola & the conjugate hyperbola by the same constant only.

(v) The asymptotes pass through the centre of the hyperbola & the bisectors of the angles between the asymptotes are the axes of the hyperbola.

(vi) The asymptotes of a hyperbola are the diagonals of the rectangle formed by the lines drawn through the extremities of each axis parallel to the other axis.

(vii) Asymptotes are the tangent to the hyperbola from the centre.

(viii) A simple method to find the coordinates of the centre of the hyperbola expressed as a general equation of degree 2 should be remembered as:
Let $f(x, y) = 0$ represents a hyperbola.

Find $\frac{\partial f}{\partial x}$ & $\frac{\partial f}{\partial y}$. Then the point of intersection of $\frac{\partial f}{\partial x} = 0$ & $\frac{\partial f}{\partial y} = 0$

gives the centre of the hyperbola.

Q Find the equation to the hyperbola whose asymptotes are the straight lines $x + 2y + 3 = 0$ and $3x + 4y + 5 = 0$ and which passes through the point $(1, -1)$. Also write the equation to the conjugate hyperbola and the coordinates of its centre.

Sol^n

C.A : $(x+2y+3)(3x+4y+5) = 0$ $(1,-1)$

H : $(x+2y+3)(3x+4y+5) = K$

$(1-2+3)(3-4+5) = K$

$$\boxed{C.A. = \frac{H + C.H}{2}}$$ ✓

$\begin{cases} x+2y+3=0 \\ 3x+4y+5=0 \end{cases}$

Point of int.

$\downarrow$

Centre

Q Find the asymptotes of the hyperbola, $3x^2 - 5xy - 2y^2 + 5x + 11y - 8 = 0$. Also find the equation of the conjugate hyperbola.

Sol^n

C.A : $\underbrace{3x^2 - 5xy - 2y^2 + 5x + 11y + \lambda = 0}$

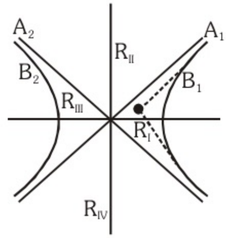
↓ Pair of lines

$\boxed{\Delta = 0}$

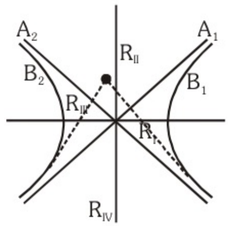
AN IMPORTANT CONCEPT :

Region from where tangents drawn to same/different branches of hyperbola

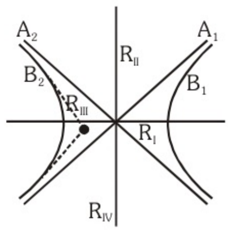
(1) If point lies in region-I (R_I) then both tangents drawn will be on the same branch (B_1) of the hyperbola.



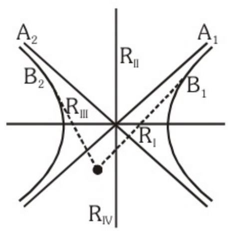
(2) If point lies in region-II (R_{II}) then both tangents drawn will be on the different branches (B_1,B_2) of the hyperbola.



(3) If point lies in region-III (R_{III}) then both tangents drawn will be on the same branch (B_2) of the hyperbola.

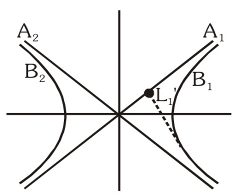


(4) If point lies in region-IV (R_{IV}) then both tangents drawn will be on the different branches (B_1,B_2) of the hyperbola.



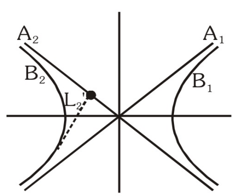
(5)

If point lies on the line A_1 then one of the tangent will be A_1 itself and another will be L_1' on the branch (B_1) of the hyperbola.



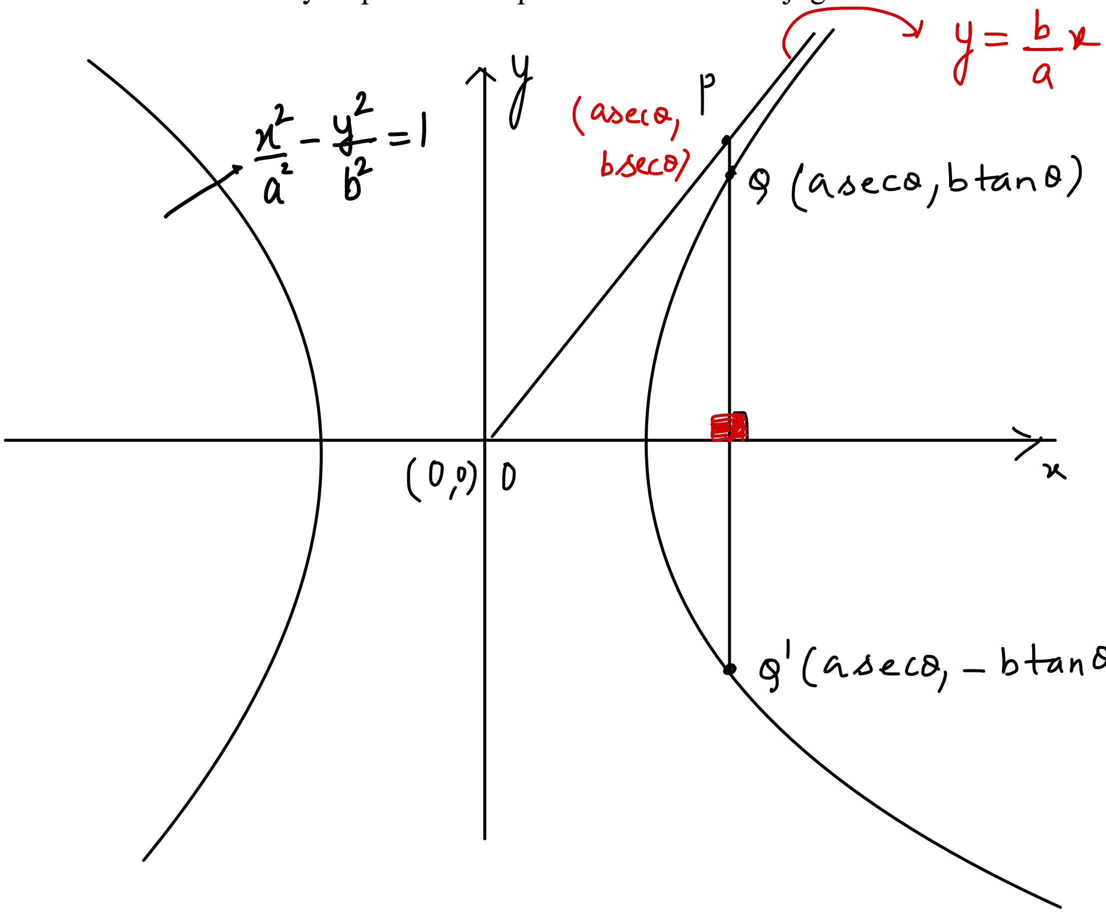
(6)

If point lies on the line A_2 then one of the tangent will be A_2 itself and another will be L_2' on the branch (B_2) of the hyperbola.



HIGHLIGHTS ON ASYMPTOTES :

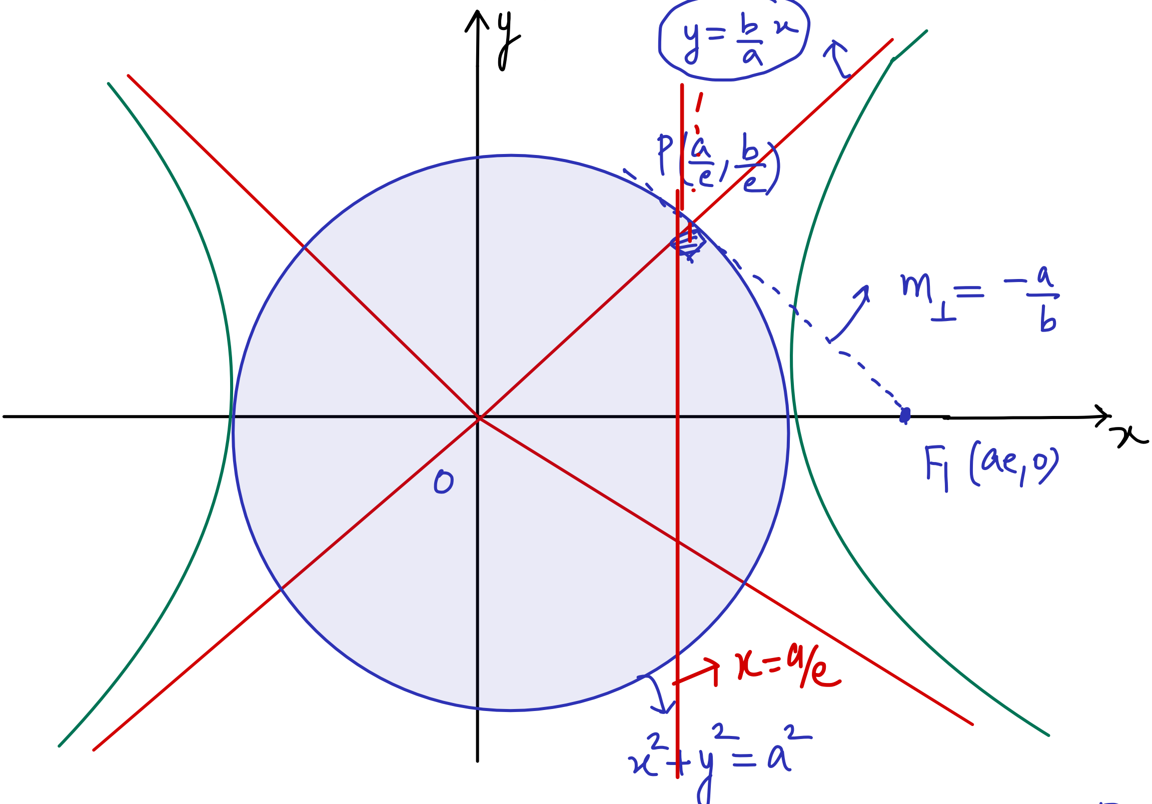
H–1 If from any point on the asymptote a straight line be drawn perpendicular to the transverse axis, the product of the segments of this line, intercepted between the point & the curve is always equal to the square of the semi conjugate axis.



$PQ \cdot PQ' = b(\sec\theta - \tan\theta) \cdot b(\sec\theta + \tan\theta)$

$PQ \cdot PQ' = b^2.$

H–2 Perpendicular from the foci on either asymptote meet it in the same points as the corresponding directrix & the common points of intersection lie on the auxiliary circle.



solve equation of PF_1 with $y=\frac{b}{a}x$ to get P

PF_1 : $y-0=-\frac{a}{b}(x-ae) \Rightarrow P=\left(\frac{a}{e},\frac{b}{e}\right)$

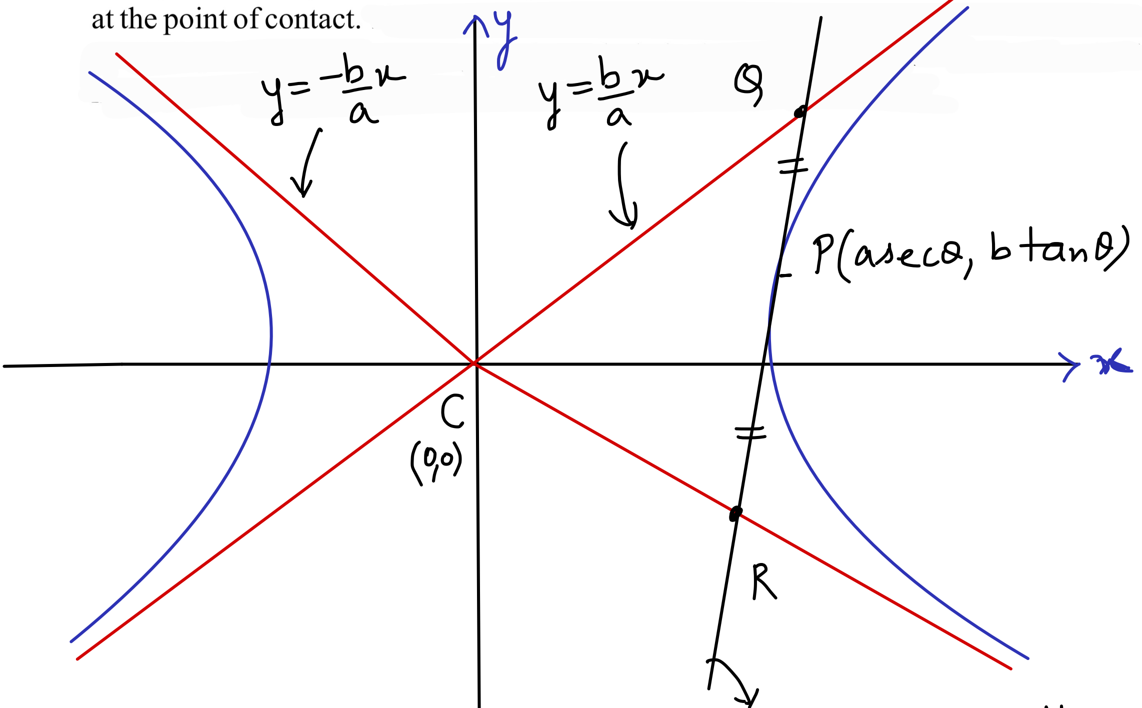
P lies on $x^2+y^2=a^2$

LHS : $\frac{a^2}{e^2}+\frac{b^2}{e^2}=\frac{a^2+a^2(e^2-1)}{e^2}=a^2$

(H.P)

H–3 V. imp

The tangent at any point P on a hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ with centre C, meets the asymptotes in Q and R and cuts off a Δ CQR of constant area equal to ab from the asymptotes & the portion of the tangent intercepted between the asymptote is bisected at the point of contact.



T: $\frac{x}{a}\sec\theta-\frac{y}{b}\tan\theta=1$ — ① —

Solve ① with 2 asymptotes to get Q & R

$Q\left(a(\sec\theta+\tan\theta),\ b(\sec\theta+\tan\theta)\right)$.

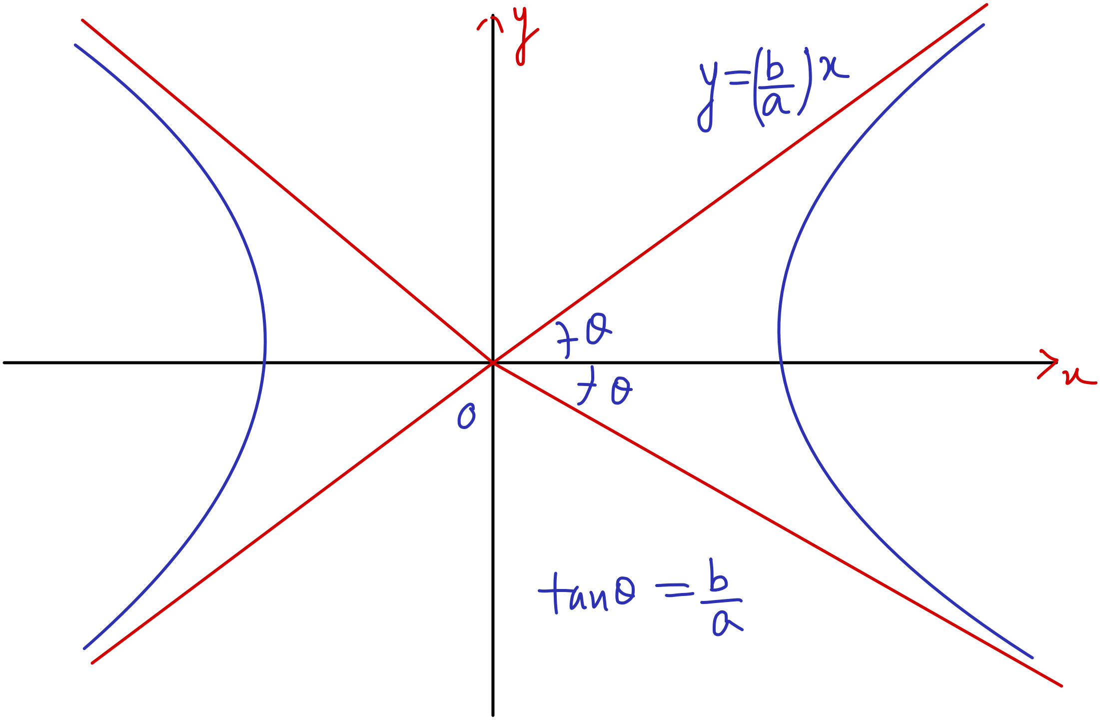
$R\left(a(\sec\theta-\tan\theta),\ -b(\sec\theta-\tan\theta)\right)$.

$\Rightarrow$ 'P' is mid point of QR

$$\Delta=\frac{1}{2}\begin{vmatrix}0 & 0 & 1\\ a(\sec\theta+\tan\theta) & b(\sec\theta+\tan\theta) & 1\\ a(\sec\theta-\tan\theta) & -b(\sec\theta-\tan\theta) & 1\end{vmatrix}=ab$$

$=$ Constant

H–4 If the angle between the asymptote of a hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is 2θ then $e = \sec\theta$.



$$e = \sqrt{1+\frac{b^2}{a^2}} = \sqrt{1+\tan^2\theta} = \sec\theta.$$

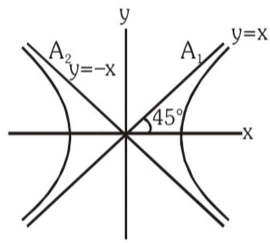
(H.P.)

RECTANGULAR HYPERBOLA :

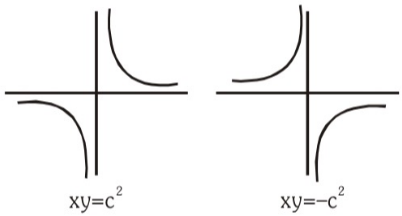
A hyperbola is said to be rectangular hyperbola if its transverse axis is equal to its conjugate axis

i.e. $b = a \Rightarrow e = \sqrt{2}$

In rectangular hyperbola asymptotes are perpendicular to each other then $x^2 - y^2 = a^2$ its asymptotes are $y = \pm x$



Rotating the axes by an angle $-\frac{\pi}{4}$ or $\frac{\pi}{4}$ about the same origin, equation of the rectangular hyperbola $x^2 - y^2 = a^2$ is reduced to $xy = \frac{a^2}{2}$ $(xy = c^2)$ or $xy = -\frac{a^2}{2}$ $(xy = -c^2)$ respectively. In $xy = c^2$ or $xy = -c^2$ asymptotes are coordinates axes.



Note :

(i) Equilateral hyperbola $\Leftrightarrow$ rectangular hyperbola.

(ii) If a hyperbola is equilateral then the conjugate hyperbola is also equilateral.

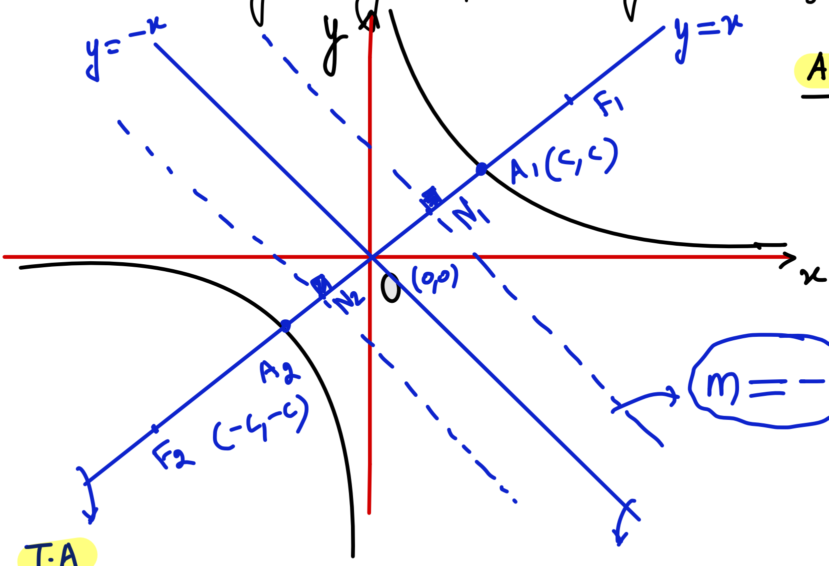
Parametric Coordinates :

If equation is $xy = c^2$ then its parametric representation is **$x = ct$, $y = c/t$, $t \in R - \{0\}$**

Q Find everything for $xy = c^2$?

$e = \sqrt{2}$

Sol^n



Asymptotes :-

x & y - axis

$\left.\begin{array}{l} y = 0 \\ x = 0 \end{array}\right\}$

Vertices

$A_1(c, c)$

$A_2(-c, -c)$

$$\frac{OF_1}{OA_1} = e = \sqrt{2}$$

$$\frac{OA_1}{ON_1} = e = \sqrt{2}$$

$$OF_1 = \sqrt{2} \cdot (\sqrt{2}c)$$

$$\boxed{OF_1 = 2c}$$

$$ON_1 = \frac{OA_1}{\sqrt{2}} = \frac{\cancel{\sqrt{2}}c}{\cancel{\sqrt{2}}}$$

$$\frac{x-0}{\cos 45^\circ} = \frac{y-0}{\sin 45^\circ} = \pm 2c$$

$$\boxed{ON_1 = c}$$

$F_1 : (\sqrt{2}c, \sqrt{2}c)$

$F_2 : (-\sqrt{2}c, -\sqrt{2}c)$

$$\frac{x-0}{\cos 45^\circ} = \frac{y-0}{\sin 45^\circ} = \pm c$$

$$L.R = \frac{2b^2}{a}$$

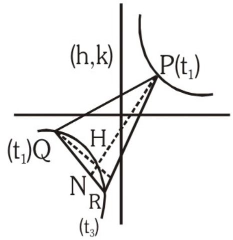
$a = b$

$$L.R = 2a = 2\sqrt{2}c.$$

Foot of Directrix $N_1 : \left(\frac{c}{\sqrt{2}}, \frac{c}{\sqrt{2}}\right)$; $N_2\left(-\frac{c}{\sqrt{2}}, -\frac{c}{\sqrt{2}}\right)$

Q A rectangular hyperbola $xy = c^2$ circumscribing a triangle also passes through the orthocentre of this triangle. If $\left(ct_i, \frac{c}{t_i}\right) i = 1,2,3$ be the angular points P, Q, R then orthocentre is $\left(\frac{-c}{t_1t_2t_3}, -ct_1t_2t_3\right)$.

Sol. slope of $QR = -\frac{1}{t_2t_3}$



$\therefore$ slope of $PN = t_2t_3$

$\therefore$ equation of altitude through P

$$y - \frac{c}{t_1} = t_2t_3(x - ct_1)$$

$$y + c\,t_1\,t_2\,t_3 = \frac{c}{t_1} + xt_2t_3$$

$$y + ct_1t_2t_3 = t_2t_3\left(x + \frac{c}{t_1t_2t_3}\right) \quad .....(i)$$

(i) is suggestive that orthocentre is $\left(\frac{-c}{t_1t_2t_3}, -ct_1t_2t_3\right)$

Q If a circle and the rectangular hyperbola $xy = c^2$ meet in the four points t_1, t_2, t_3 and t_4, then

(a) $t_1t_2t_3t_4 = 1$

(b) The centre of the mean position of the four points bisects the distance between the centres of the two curves.

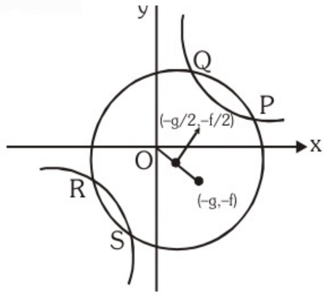
(c) The centre of the circle through the points t_1, t_2 and t_3 is :
$$\left\{\frac{c}{2}\left(t_1 + t_2 + t_3 + \frac{1}{t_1t_2t_3}\right), \frac{c}{2}\left(\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + t_1t_2t_3\right)\right\}$$

(d) If PQRS are the four points of intersection of the circle with rectangular hyperbola then $(OP)^2 + (OQ)^2 + (OR)^2 + (OS)^2 = 4r^2$ where r is the radius of circle.

Sol. (a) Let the equation of the circle be

$x^2 + y^2 + 2gx + 2fy + d = 0$ ......(i)

solving with $xy = c^2$



$$x^2 + \frac{c^4}{x^2} + 2gx + 2f.\frac{c^2}{x} + d = 0$$

$x^4 + 2gx^3 + dx^2 + 2fc^2 x + c^4 = 0$ .....(i)

from (i) $x_1x_2x_3x_4 = c^4$

$c^4 [t_1t_2t_3t_4] = c^4 \quad \Rightarrow \quad t_1t_2t_3t_4 = 1 \Rightarrow$ (a)

(b) again, centre of the mean position of 4 points of intersection $\frac{\sum x_i}{4}, \frac{\sum y_i}{4}$

now from (i)

$x_1 + x_2 + x_3 + x_4 = -2g$ ......(ii) ; hence $\frac{\sum x_i}{4} = -\frac{g}{2}$

using $xy = c^2$

$$y_1 + y_2 + y_3 + y_4 = c^2\left[\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4}\right] = \frac{c^2}{x_1x_2x_3x_4}\sum x_1x_2x_3 = \frac{c^2}{c^4}(-2fc^2) = -2f$$

$$\therefore \frac{\sum y_i}{4} = -\frac{f}{2}; \text{ Hence } \left(\frac{\sum x_i}{4}, \frac{\sum y_i}{4}\right) = \left(-\frac{g}{2}, -\frac{f}{2}\right)$$

(c) centre of the circle through PQR i.e. $(-g, -f)$ is given by

$$\frac{x_1 + x_2 + x_3 + x_4}{2}, \frac{y_1 + y_2 + y_3 + y_4}{2} \text{ (using } t_1t_2t_3t_4 = 1)$$

$$\frac{c}{2}\left[(t_1 + t_2 + t_3) + \frac{1}{t_1t_2t_3}\right], \frac{c}{2}\left[\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + \frac{t_1t_2t_3}{1}\right]$$

(d) $(OP)^2 + (OQ)^2 + (OR)^2 + (OS)^2 = 4r^2$ where $r = g^2 + f^2 = d$

LHS $\left(x_1^2 + x_2^2 + x_3^2 + x_4^2\right) + \left(y_1^2 + y_2^2 + y_3^2 + y_4^2\right)$

$$[(\sum x_1)^2 - 4\sum x_1x_2] + c^4\left[\frac{1}{x_1^2} + \frac{1}{x_2^2} + \frac{1}{x_3^2} + \frac{1}{x_4^2}\right]$$

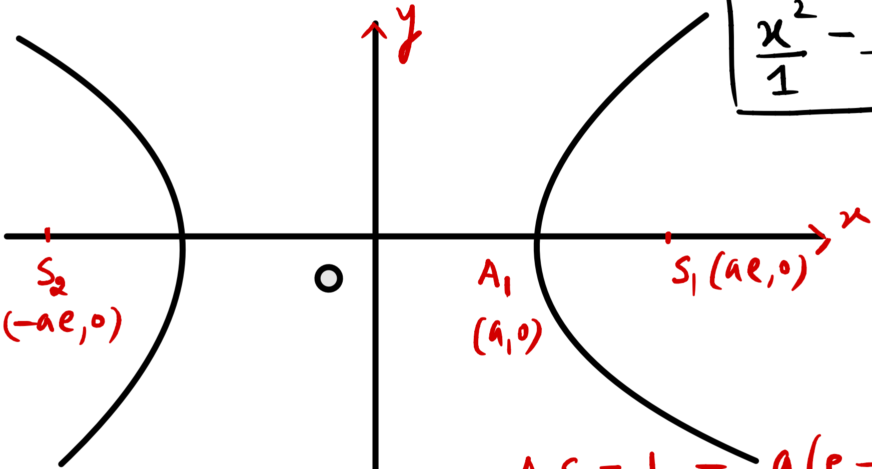
$$(4g^2 - 4d) + c^4\left[\left(\sum \frac{1}{x_1}\right)^2 - 2\sum \frac{1}{x_1x_2}\right]$$

$$(4g^2 - 4d) + c^4\left[\left\{\frac{1}{x_1x_2x_3x_4}\sum x_1x_2x_3\right\}^2 - \frac{4}{x_1x_2x_3x_4}\sum x_1x_2\right]$$

$$(4g^2 - 4d) + c^4\left[\left\{\frac{1}{c^4}(-2fc^2)\right\}^2 - \frac{4d}{c^4}\right] = (4g^2 - 4d) + (4f^2 - 4d) = 4[g^2 + f^2 - d] = 4r^2$$

Q (HW) Find eqn of hyperbola referred to its principal axis as co-ordinate axis, if distance of one of its vertices from foci are 3 & 1.

Sol^n C–I TA : x–axis and C.A : y–axis

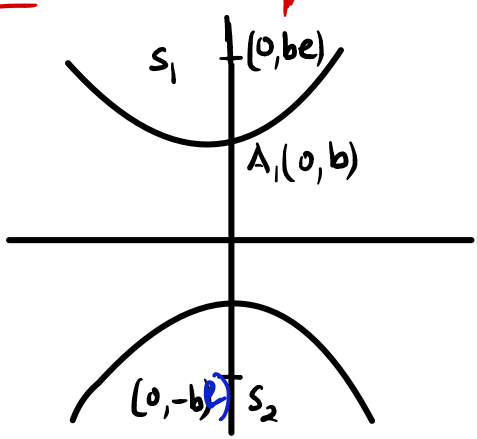


$$\boxed{\frac{x^2}{1}-\frac{y^2}{3}=1}$$

$A_1S_1 = 1 = a(e-1)$
$A_1S_2 = 3 = a(e+1)$
$a=1$, $e=2$

$b^2 = a^2(e^2-1) \Rightarrow b^2 = 3$

C–II TA : y–axis and C.A : x–axis



$A_1S_2 = b+be = 3$
$A_1S_1 = be-b = 1$

$b=1$; $e=2$; $a^2=3$

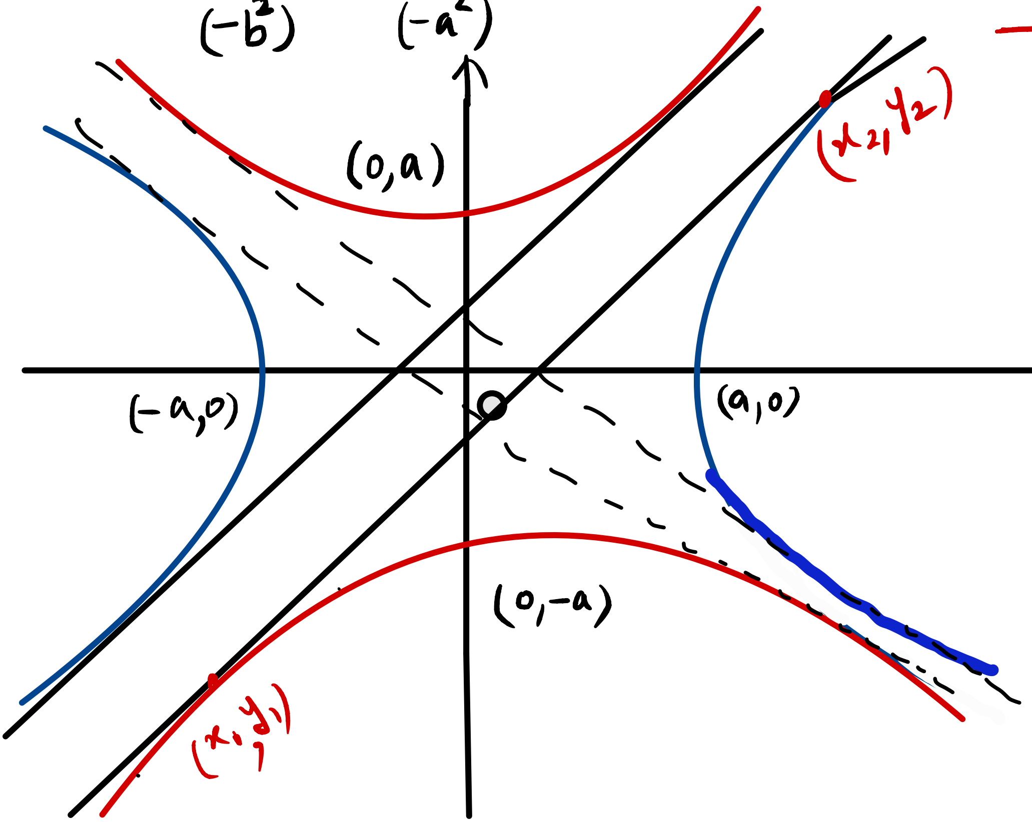
$$\boxed{\frac{x^2}{3}-\frac{y^2}{1}=-1}$$

Q Find equation of common tangent(s) to

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ and $\frac{y^2}{a^2} - \frac{x^2}{b^2} = 1$?

Sol^n H_1: $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \longrightarrow y = mx \pm \sqrt{a^2m^2 - b^2}$ — ① —

H_2: $\frac{x^2}{(-b^2)} - \frac{y^2}{(-a^2)} = 1 \longrightarrow y = mx \pm \sqrt{-b^2m^2 - (-a^2)}$ — ② —

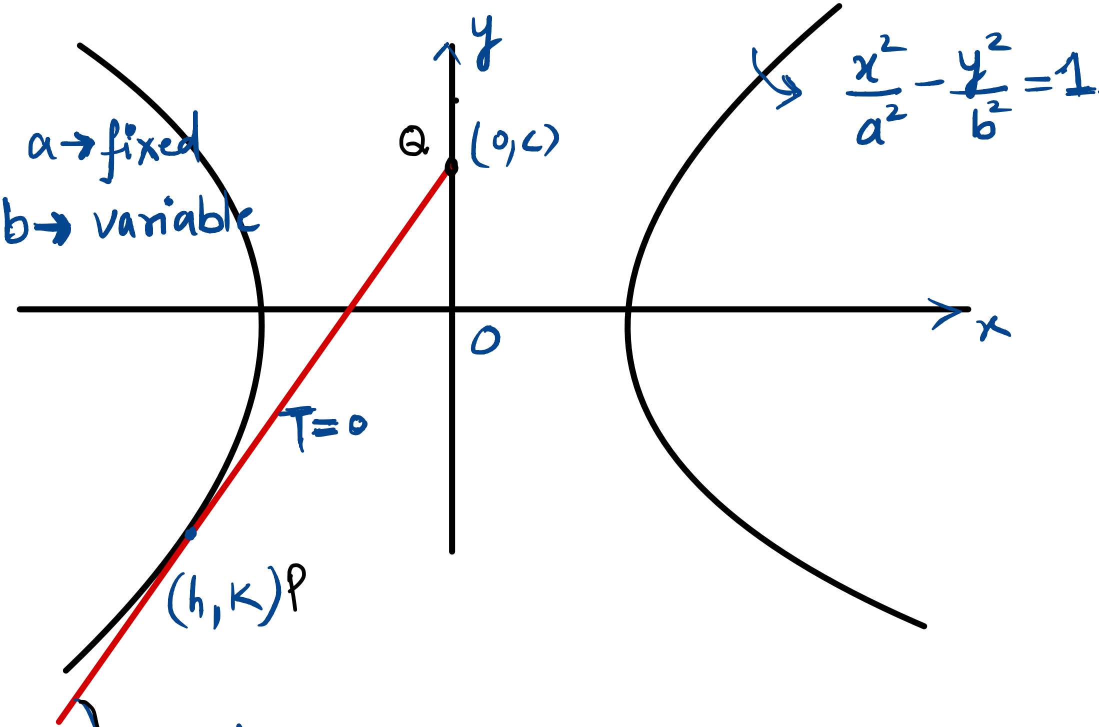


$$a^2m^2 - b^2 = (-b^2)m^2 + a^2$$

$m^2 = 1 \Rightarrow \boxed{m = \pm 1}$

T: $y = \pm x \pm \sqrt{a^2 - b^2}$

Q. If one axis of varying central conic (ellipse / hyperbola) be fixed is magnitude and position, prove that the locus of the point of contact of a tangent drawn to it from a fixed points on the other axis is a parabola.



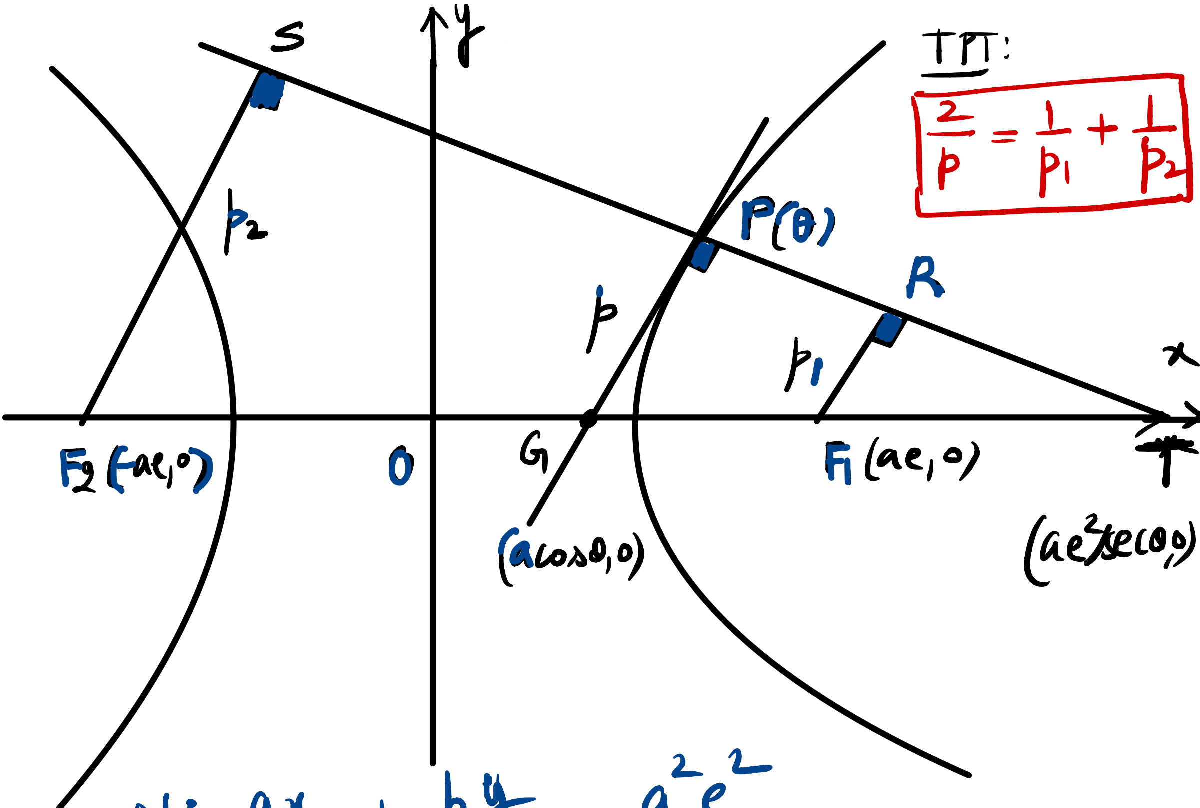
$$\frac{xh}{a^2} - \frac{yK}{b^2} = 1$$

(0,c)

$$\frac{-Kc}{b^2} = 1 \Rightarrow b^2 = -cK$$

$$\boxed{b^2 = -cy}$$

Q Prove that the part of the tangent at any point of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ intercepted between the point of contact and the transverse axis is a harmonic mean between the lengths of the perpendiculars drawn from the foci on the normal at the same point.



TPT:

$$\boxed{\frac{2}{p}=\frac{1}{p_1}+\frac{1}{p_2}}$$

$$N:\ \frac{ax}{\sec\theta}+\frac{by}{\tan\theta}=a^2e^2$$

$$T:\ \frac{x\sec\theta}{a}-\frac{y}{b}\tan\theta=1$$

$\Delta GPT \sim \Delta F_1RT$:

$$\frac{p}{p_1}=\frac{TG}{F_1T}=\frac{ae^2\sec\theta-a\cos\theta}{ae^2\sec\theta-ae}$$

$$\frac{p}{p_1}=\frac{e+\cos\theta}{e}$$

$\Delta F_2ST \sim \Delta GPT$

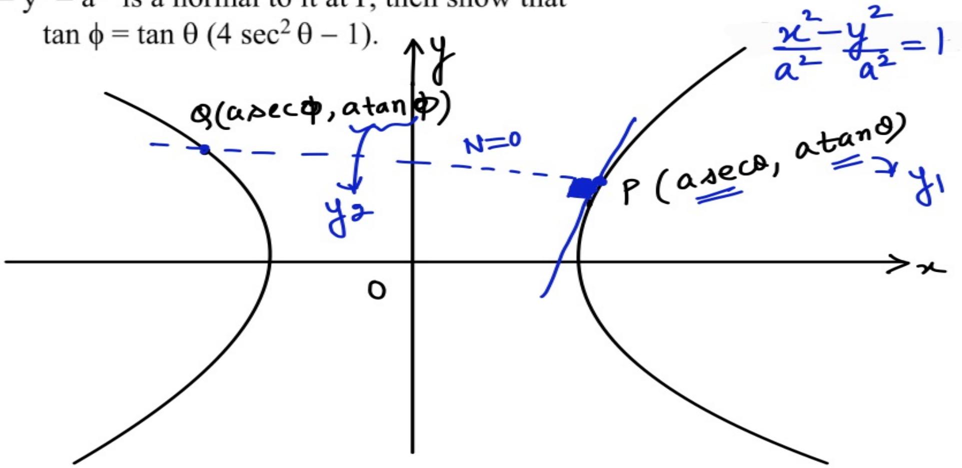
$$\frac{p}{p_2} = \frac{TG}{F_2T} = 1 - \frac{\cos\theta}{e}$$

$$\boxed{\frac{p}{p_1} + \frac{p}{p_2} = 2}$$ (H.P.)

Q. If a chord joining the points P $(a \sec\theta, a\tan\theta)$ & Q $(a\sec\phi, a\tan\phi)$ on the hyperbola $x^2 - y^2 = a^2$ is a normal to it at P, then show that

$$\tan\phi = \tan\theta\,(4\sec^2\theta - 1).$$

Sol^n



$$N: \frac{a x}{\sec\theta} + \frac{a y}{\tan\theta} = 2a^2 \Rightarrow N: \frac{x}{\sec\theta} + \frac{y}{\tan\theta} = 2a.$$

Solve $N = 0$ with hyperbola

$$x^2 - y^2 = a^2$$

$$x = \sec\theta\left(2a - \frac{y}{\tan\theta}\right)$$

$$\sec^2\theta\left(2a - \frac{y}{\tan\theta}\right)^2 - y^2 = a^2.$$

$$\left(\frac{\sec^2\theta}{\tan^2\theta} - 1\right)y^2 - (\quad)\,y + 4a^2\sec^2\theta - a^2 = 0 \;\begin{matrix}\nearrow y_1 \\ \searrow y_2\end{matrix}$$

$$y_1 y_2 = \frac{a^2(4\sec^2\theta - 1)\tan^2\theta}{(\sec^2\theta - \tan^2\theta)}$$

$$(\cancel{a}\tan\phi)(\cancel{a}\cancel{\tan\theta}) = \cancel{a^2}(4\sec^2\theta - 1)\tan^{\cancel{2}}\theta$$

$$\tan\phi = (4\sec^2\theta - 1)\tan\theta. \quad \text{(HP)}$$

Equation of chord of contact :-

$$\boxed{T=0}$$

Equation of chord with given middle point :-

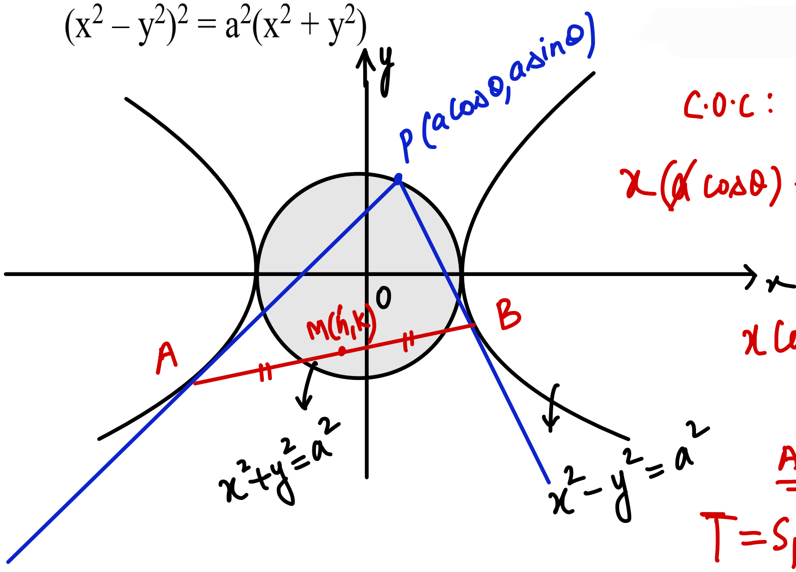
$$\boxed{T=S_1}$$

Equation of pair of tangents :-

$$\boxed{SS_1=T^2}$$

Q From points on the circle $x^2 + y^2 = a^2$ tangents are drawn to the hyperbola $x^2 - y^2 = a^2$; prove that the locus of the middle points of the chords of contact is the curve

$$(x^2 - y^2)^2 = a^2(x^2 + y^2)$$



C.O.C : T = 0

$x(\cancel{a}\cos\theta) - y(\cancel{a}\sin\theta) = a^{\cancel{2}}$

$x\cos\theta - y\sin\theta = a$ —①—

AB :

$T = S_1$

$xh - yk - \cancel{a^2} = h^2 - k^2 - \cancel{a^2}$

$xh - yk = h^2 - k^2$ —②—

Compare ① & ②

$\cos\theta =$

$\sin\theta =$

$\boxed{c^2 + s^2 = 1}$

Q Find the equation to the locus of the middle points of the chords of the hyperbola $2x^2 - 3y^2 = 1$, each of which makes an angle of 45° with the x-axis.

Soln

$$T = S_1$$

$$2xh - 3yk - \cancel{1} = 2h^2 - 3k^2 - \cancel{1}$$

$$m = \frac{-2h}{-3k} = \tan 45^\circ \Rightarrow 2h = 3k$$

$$\boxed{y = \frac{2}{3}x}$$

Q A point P moves such that the chord of contact of the pair of tangents from P on the parabola $y^2 = 4ax$ touches the rectangular hyperbola $x^2 - y^2 = c^2$. Show that the locus of P is the ellipse $\frac{x^2}{c^2} + \frac{y^2}{(2a)^2} = 1$.

HW

Q
HW

A tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ cuts the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at P and Q. Show that the locus of the mid point of PQ is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2=\frac{x^2}{a^2}-\frac{y^2}{b^2}$$

HIGHLIGHTS ON TANGENT AND NORMAL :

H–1 Locus of the feet of the perpendicular drawn from focus of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ upon any tangent is its auxiliary circle i.e. $x^2 + y^2 = a^2$ & the product of the feet of these perpendiculars is $b^2 = (\text{semi C}\cdot\text{A})^2$.

H–2 The portion of the tangent between the point of contact & the directrix subtends a right angle at the corresponding focus.

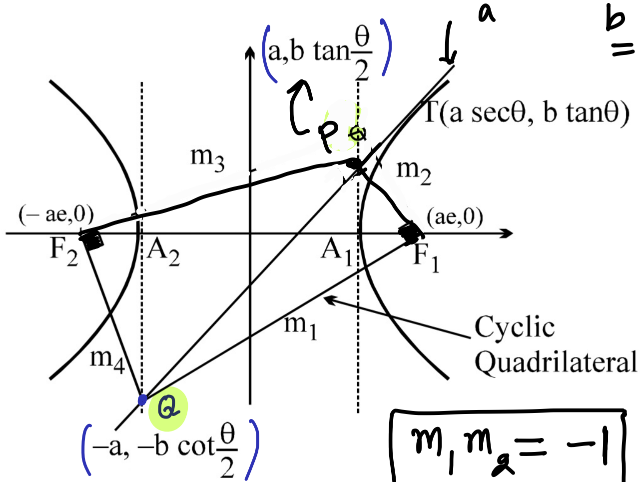
H–3 The foci of the hyperbola and the points P and Q in which any tangent meets the tangents at the vertices are concyclic with PQ as diameter of the circle.

$$T : \frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$$

$$\frac{(\sec\theta - 1)b}{\tan\theta} = y$$

$$y = b\left(\frac{1-\cos\theta}{\sin\theta}\right)$$

$\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1$



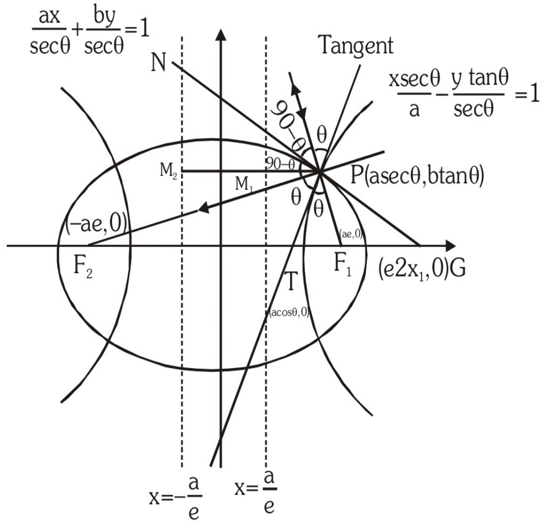
$\boxed{m_1 m_2 = -1}$

$\boxed{m_3 m_4 = -1}$

Family of circles through F_1F_2

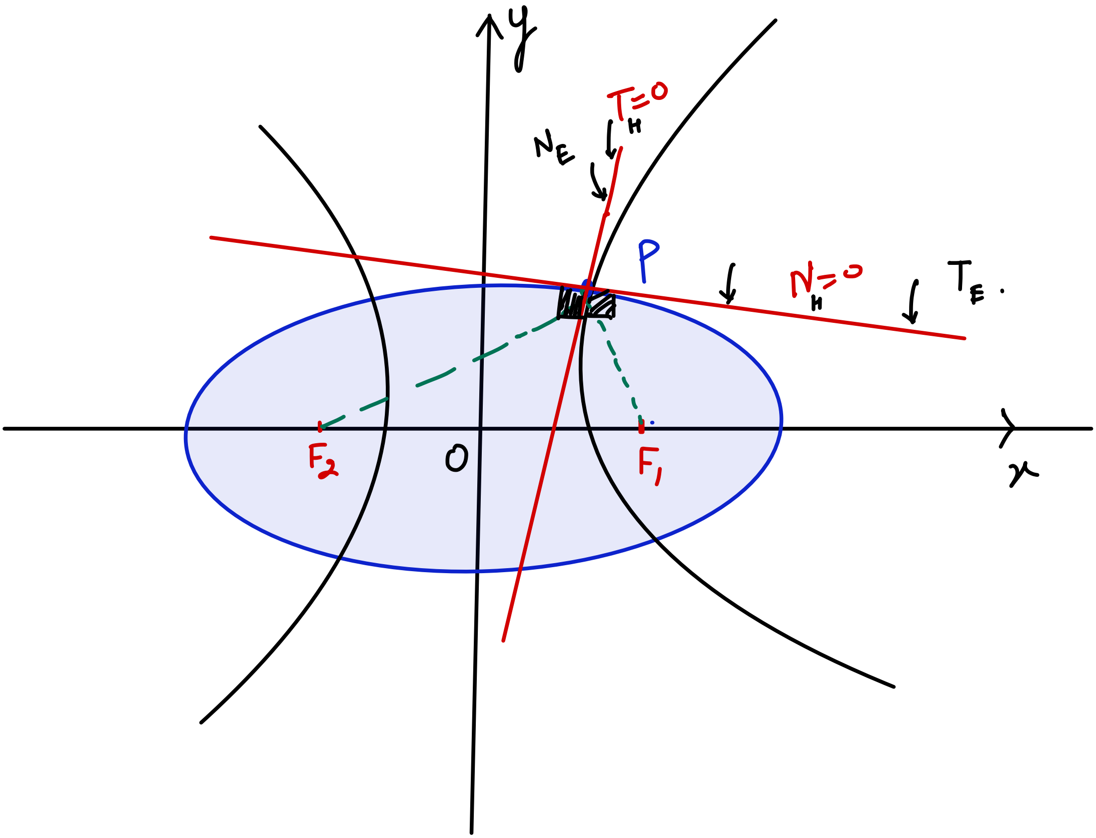
$(x - ae)(x + ae) + y^2 + \lambda y = 0 \quad \Rightarrow \quad$ PQ as diameter]

H-4 **(i)** The tangent and normal at any point of a hyperbola bisect the angle between the focal radii.



(ii) **Reflection property of the hyperbola :** An incoming light ray aimed towards one focus is reflected from the outer surface of the hyperbola towards the other focus.

Note : It follows that if an ellipse and a hyperbola have the same foci, they cut at right angles at any of their common point. (orthogonally)



Q A point P moves such that the chord of contact of the pair of tangents from P on the parabola $y^2 = 4ax$ touches the rectangular hyperbola $x^2 - y^2 = c^2$. Show that the locus of P is the ellipse $\frac{x^2}{c^2} + \frac{y^2}{(2a)^2} = 1$.

Sol^n

$P(h, K)$

C.O.C : $T = 0 \Rightarrow \quad yK = 2a(x+h)$.

$$y = \underbrace{\left(\frac{2a}{K}\right)}_{m} x + \underbrace{\left(\frac{2ah}{K}\right)}_{c} \longrightarrow$$ this is tangent to $\frac{x^2}{c^2} - \frac{y^2}{c^2} = 1$

so **Condition of tangency :-**

$$\frac{2ah}{K} = \pm\sqrt{c^2 \cdot \left(\frac{2a}{K}\right)^2 - c^2}$$

$$\frac{4a^2h^2}{\cancel{K^2}} = \frac{4a^2c^2 - c^2K^2}{\cancel{K^2}}$$

$$4a^2h^2 + c^2K^2 = 4a^2c^2$$

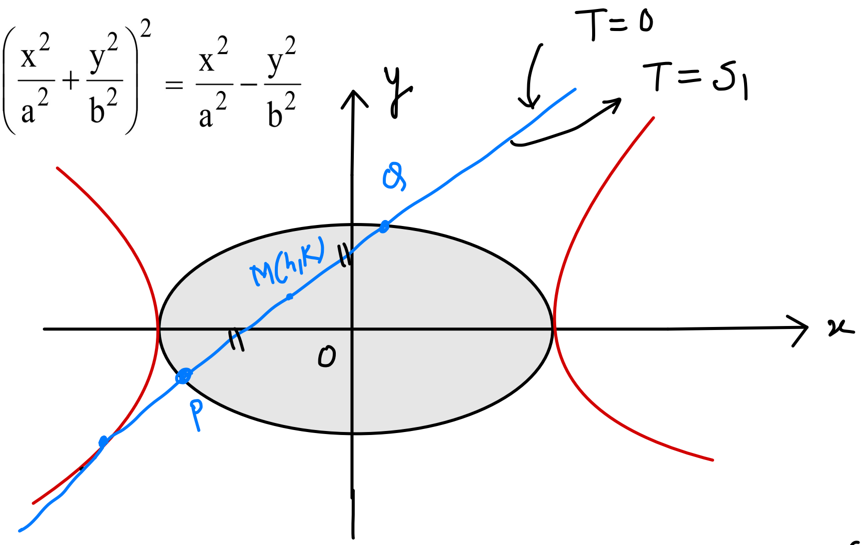
$$\frac{h^2}{c^2} + \frac{K^2}{4a^2} = 1 \Rightarrow \frac{x^2}{c^2} + \frac{y^2}{4a^2} = 1$$

(H.P.)

Q A tangent to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ cuts the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at P and Q. Show that the locus of the mid point of PQ is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2=\frac{x^2}{a^2}-\frac{y^2}{b^2}$$

Sol^n



$$T=0 \Rightarrow \quad \frac{x}{a}\sec\theta-\frac{y}{b}\tan\theta=1 \quad —①—$$

$$T=S_1 \Rightarrow \quad \frac{xh}{a^2}+\frac{yk}{b^2}-\cancel{1}=\frac{h^2}{a^2}+\frac{k^2}{b^2}-\cancel{1} \quad —②—$$

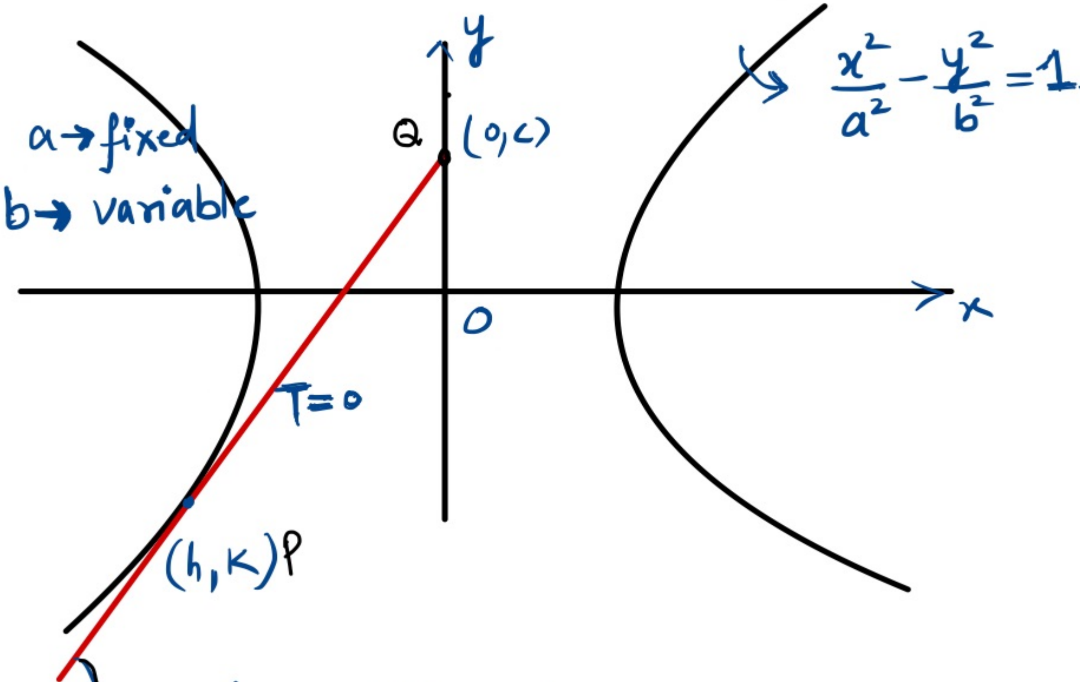
Compare :
$$\frac{\frac{\sec\theta}{a}}{\frac{h}{a^2}}=\frac{-\frac{\tan\theta}{b}}{\frac{k}{b^2}}=\frac{1}{\frac{h^2}{a^2}+\frac{k^2}{b^2}}$$

get $\sec\theta=$

$\tan\theta=$

Use $\sec^2\theta-\tan^2\theta=1$ to get req. locus.

Q. If one axis of varying central conic (ellipse / hyperbola) be fixed is magnitude and position, prove that the locus of the point of contact of a tangent drawn to it from a fixed points on the other axis is a parabola.



$$\frac{xh}{a^2} - \frac{yK}{b^2} = 1$$

(0,c)

$$\frac{-Kc}{b^2} = 1 \Rightarrow b^2 = -cK$$

$$\boxed{b^2 = -cy}$$

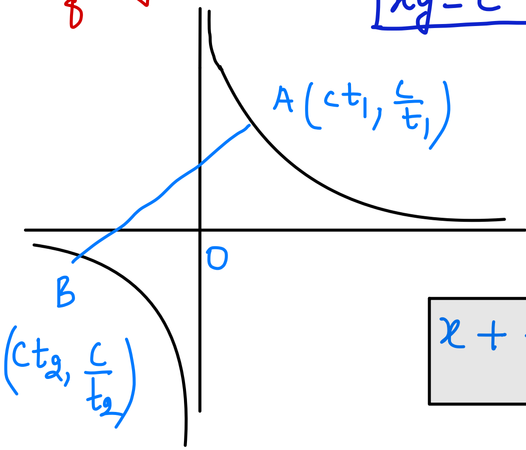
$$\frac{h^2}{a^2} - \frac{K^2}{b^2} = 1$$

$$\frac{h^2}{a^2} - \frac{K^2}{(-cy)} = 1 \Rightarrow y = c\left(1 - \frac{x^2}{a^2}\right)$$

Ans

Eqn of chord :-

$\boxed{xy = c^2}$

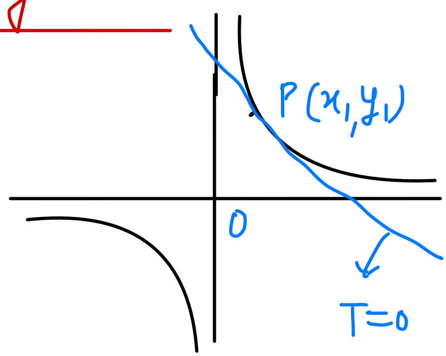


$$m_{AB} = \frac{\frac{c}{t_1} - \frac{c}{t_2}}{ct_1 - ct_2} = -\frac{1}{t_1 t_2}$$

AB : $y - \frac{c}{t_1} = \frac{-1}{t_1 t_2}(x - ct_1)$

$$\boxed{x + t_1 t_2 y = c(t_1 + t_2)}$$

Eqn of tangent at (x_1, y_1) on $xy = c^2$:-



$xy = c^2 \Rightarrow 2xy = 2c^2$

T: $xy_1 + yx_1 = 2c^2$

$xy_1 + yx_1 = 2x_1 y_1$

T : $$\boxed{\frac{x}{x_1} + \frac{y}{y_1} = 2}$$

Eqn of tangent at $\left(ct, \frac{c}{t}\right)$:-

$$\frac{x}{x_1} + \frac{y}{y_1} = 2 \qquad \begin{aligned} x_1 &\to ct \\ y &\to \frac{c}{t} \end{aligned}$$

$$\frac{x}{ct} + \frac{y \cdot t}{c} = 2$$

$$\boxed{\frac{x}{t} + ty = 2c} \rightarrow m_T = -\frac{1}{t^2}$$

Eqn of normal at $\left(ct, \frac{c}{t}\right)$:-

$$N: \; y - \frac{c}{t} = t^2(x - ct)$$

Note :- If normal to $xy = c^2$ at 't_1' is intersecting it again at 't_2' then

$$N: \; y - \frac{c}{t_1} = t_1^2(x - ct_1)$$

$$\left(\frac{c}{t_2} - \frac{c}{t_1}\right) = t_1^2(ct_2 - ct_1)$$

$$\boxed{t_1^3 t_2 = -1} \Rightarrow t_2 = -\frac{1}{t_1^3}$$

If normal passes through general point (h, K) then

$$K - \frac{c}{t} = t^2 (h - ct)$$

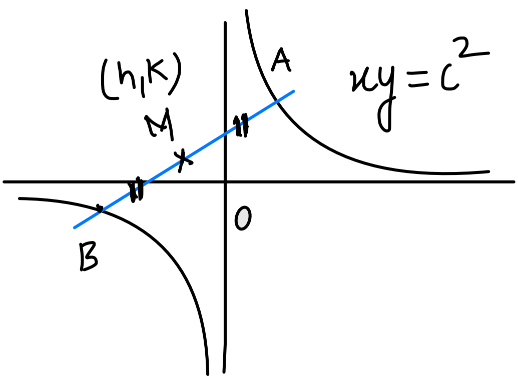
$$Kt - c = t^3 (h - ct)$$

$$ct^4 - t^3 h + Kt - c = 0 \quad \begin{cases} t_1 \\ t_2 \\ t_3 \\ t_4 \end{cases}$$

$$\boxed{t_1 t_2 t_3 t_4 = -1}$$

* If $A(t_1)$; $B(t_2)$; $C(t_3)$; $D(t_4)$ are co-normal point then $t_1 t_2 t_3 t_4 = -1$.

Equation of chord whose middle point is (h, K)



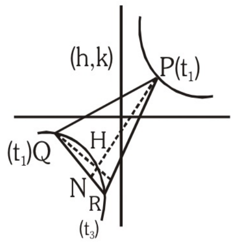
$$T = S_1$$

$$\frac{xK + yK}{2} = hK$$

$$\boxed{\frac{x}{h} + \frac{y}{K} = 2}$$

Q A rectangular hyperbola $xy = c^2$ circumscribing a triangle also passes through the orthocentre of this triangle. If $\left(ct_i, \frac{c}{t_i}\right) i = 1,2,3$ be the angular points P, Q, R then orthocentre is $\left(\frac{-c}{t_1t_2t_3}, -ct_1t_2t_3\right)$.

Sol. slope of $QR = -\frac{1}{t_2t_3}$



$\therefore$ slope of $PN = t_2t_3$

$\therefore$ equation of altitude through P

$$y - \frac{c}{t_1} = t_2t_3(x - ct_1)$$

$$y + c\,t_1\,t_2\,t_3 = \frac{c}{t_1} + xt_2t_3$$

$$y + ct_1t_2t_3 = t_2t_3\left(x + \frac{c}{t_1t_2t_3}\right) \quad .....(i)$$

(i) is suggestive that orthocentre is $\left(\frac{-c}{t_1t_2t_3}, -ct_1t_2t_3\right)$

Q If a circle and the rectangular hyperbola $xy = c^2$ meet in the four points t_1, t_2, t_3 and t_4, then

(a) $t_1t_2t_3t_4 = 1$

(b) The centre of the mean position of the four points bisects the distance between the centres of the two curves.

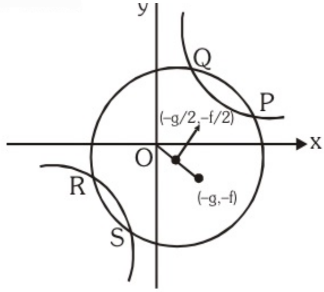
(c) The centre of the circle through the points t_1, t_2 and t_3 is :
$$\left\{\frac{c}{2}\left(t_1 + t_2 + t_3 + \frac{1}{t_1t_2t_3}\right), \frac{c}{2}\left(\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + t_1t_2t_3\right)\right\}$$

(d) If PQRS are the four points of intersection of the circle with rectangular hyperbola then $(OP)^2 + (OQ)^2 + (OR)^2 + (OS)^2 = 4r^2$ where r is the radius of circle.

Sol. (a) Let the equation of the circle be

$x^2 + y^2 + 2gx + 2fy + d = 0$ ......(i)

solving with $xy = c^2$



$$x^2 + \frac{c^4}{x^2} + 2gx + 2f.\frac{c^2}{x} + d = 0$$

$x^4 + 2gx^3 + dx^2 + 2fc^2\,x + c^4 = 0$ .....(i)

from (i) $x_1x_2x_3x_4 = c^4$

$c^4\,[t_1t_2t_3t_4] = c^4 \quad \Rightarrow \quad t_1t_2t_3t_4 = 1 \Rightarrow$ (a)

(b) again, centre of the mean position of 4 points of intersection $\frac{\sum x_i}{4}, \frac{\sum y_i}{4}$

now from (i)

$x_1 + x_2 + x_3 + x_4 = -\,2g$ ......(ii) ; hence $\frac{\sum x_i}{4} = -\frac{g}{2}$

using $xy = c^2$

$$y_1 + y_2 + y_3 + y_4 = c^2\left[\frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4}\right] = \frac{c^2}{x_1x_2x_3x_4}\sum x_1x_2x_3 = \frac{c^2}{c^4}(-2fc^2) = -2f$$

$$\therefore \ \frac{\sum y_i}{4} = -\frac{f}{2};\ \text{Hence}\ \left(\frac{\sum x_i}{4}, \frac{\sum y_i}{4}\right) = \left(-\frac{g}{2}, -\frac{f}{2}\right)$$

(c) centre of the circle through PQR i.e. $(-g, -f)$ is given by

$$\frac{x_1 + x_2 + x_3 + x_4}{2}, \frac{y_1 + y_2 + y_3 + y_4}{2} \quad (\text{using } t_1t_2t_3t_4 = 1)$$

$$\frac{c}{2}\left[(t_1 + t_2 + t_3) + \frac{1}{t_1t_2t_3}\right], \frac{c}{2}\left[\frac{1}{t_1} + \frac{1}{t_2} + \frac{1}{t_3} + \frac{t_1t_2t_3}{1}\right]$$

(d) $(OP)^2 + (OQ)^2 + (OR)^2 + (OS)^2 = 4r^2$ where $r = g^2 + f^2 = d$

LHS $\left(x_1^2 + x_2^2 + x_3^2 + x_4^2\right) + \left(y_1^2 + y_2^2 + y_3^2 + y_4^2\right)$

$$[(\sum x_1)^2 - 4\sum x_1x_2] + c^4\left[\frac{1}{x_1^2} + \frac{1}{x_2^2} + \frac{1}{x_3^2} + \frac{1}{x_4^2}\right]$$

$$(4g^2 - 4d) + c^4\left[\left(\sum \frac{1}{x_1}\right)^2 - 2\sum \frac{1}{x_1x_2}\right]$$

$$(4g^2 - 4d) + c^4\left[\left\{\frac{1}{x_1x_2x_3x_4}\sum x_1x_2x_3\right\}^2 - \frac{4}{x_1x_2x_3x_4}\sum x_1x_2\right]$$

$$(4g^2 - 4d) + c^4\left[\left\{\frac{1}{c^4}(-2fc^2)\right\}^2 - \frac{4d}{c^4}\right] = (4g^2 - 4d) + (4f^2 - 4d) = 4[g^2 + f^2 - d] = 4r^2$$

Note that the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ and the hyperbola $\frac{x^2}{a^2-k^2}-\frac{y^2}{k^2-b^2}=1$ are confocal and therefore orthogonal.

and vice-versa also.

$$C_1: \frac{x^2}{a_1^2}+\frac{y^2}{b_1^2}=1$$

$$C_2: \frac{x^2}{a_2^2}+\frac{y^2}{b_2^2}=1$$

Orthogonal

$$a_1^2-b_1^2=a_2^2-b_2^2$$

$$C_1: \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$C_2: \frac{x^2}{A^2}-\frac{y^2}{B^2}=1$$

$$C_2: \frac{x^2}{A^2}+\frac{y^2}{(-B^2)}=1$$

$$a^2-b^2=A^2-(-B^2)$$