Allen Career Institute

Kota

Differential Equation (Solutions)

Index

Do yourself - 1 :

Find the order and degree of following differential equations

(i) $[1 + (y')^2]^{1/2} = x^2 + y$ **(ii)** $(1 + y')^{1/2} = y''$ **(iii)** $y' = \sin y$

Sol^n (i) $(1+(y')^2)^{1/2} = x^2+y$

$\Rightarrow 1+(y')^2 = (x^2+y)^2$ → order = 1, degree 2

(ii) $(1+y')^{1/2} = y'' \Rightarrow (1+y') = (y'')^2$

$\therefore$ order = 2 ; Degree = 2

(iii) $y' = \sin y$ → order = 1, Degree = 1

Do yourself - 2

Eliminate the arbitrary constants and obtain the differential equation satisfied by it.

(i) $y = 2x + ce^x$ **(ii)** $y = \left(\frac{a}{x^2}\right) + bx$ **(iii)** $y = ae^{2x} + be^{-2x} + c$

Sol^n (i) $y = 2x + ce^x$ ⟶ (I)

$y' = 2 + ce^x$ ⟶ (II)

(II) − (I) : $y' - y = 2 - 2x$ Ans.

(ii) $y = \frac{a}{x^2} + bx$

$\Rightarrow x^2y = a + bx^3 \Rightarrow x^2y' + 2xy = 3bx^2$

$$\Rightarrow y' + \frac{2y}{x} = 3b$$

$$\Rightarrow y'' + \frac{x(2y') - 2y}{x^2} = 0$$

$$\Rightarrow x^2y'' + 2xy' - 2y = 0$$

(iii) $y = ae^{2x} + be^{-2x} + c$

$$y' = 2ae^{2x} - 2be^{-2x} \longrightarrow (1)$$

$$y'' = 4ae^{2x} + 4be^{-2x}$$

$$y''' = 8ae^{2x} - 8be^{-2x}$$

$$y''' = 4(2ae^{2x} - 2be^{-2x})$$

$$\Rightarrow y''' = 4(y') \quad (\text{by } (1))$$

Ans.

Do yourself - 3 :

Solve the following differential equations :

(i) $\frac{2dy}{dx}=\frac{y(x+1)}{x}$ **(ii)** $\sqrt{1+4x^2}dy=y^3xdx$ **(iii)** $(\tan y)\frac{dy}{dx}=\sin(x+y)+\sin(x-y)$

Sol^n

(i) $\frac{2dy}{y}=\left(\frac{x+1}{x}\right)dx \Rightarrow 2\int\frac{dy}{y}=\int 1+\frac{1}{x}\,dx$

$\Rightarrow 2\ln y = x+\ln|x|+C$

$\Rightarrow \ln y^2 = x+\ln|x|+C$

(ii) $\frac{dy}{y^3}=\frac{x\,dx}{\sqrt{1+4x^2}} \Rightarrow \int y^{-3}dy=\frac{1}{8}\int\frac{8x\,dx}{(1+4x^2)^{1/2}}$

$\Rightarrow \frac{y^{-2}}{-2}=\frac{1}{8}\cdot 2\cdot\sqrt{1+4x^2}+C$

$\Rightarrow -\frac{1}{y^2}=\frac{1}{2}\sqrt{1+4x^2}+C$

(iii) $\tan y\frac{dy}{dx}=2\sin x\cos y$

$\Rightarrow \tan y\frac{dy}{dx}=2\sin x\cos y \Rightarrow \int\sec y\tan y\,dy=\int 2\sin x\,dx$

$\Rightarrow \sec y=-2\cos x+C$

Do yourself - 4 :

Solve the following differential equations :

(i) $\frac{dy}{dx} = (y-4x)^2$ **(ii)** $\tan^2(x+y)dx - dy = 0$

Soln (i) $y - 4x = t \Rightarrow \frac{dy}{dx} - 4 = \frac{dt}{dx}$

$\frac{dt}{dx} + 4 = t^2 \Rightarrow \int \frac{dt}{t^2-4} = \int dx = \frac{1}{4}\ln\left(\frac{t-2}{t+2}\right) = x + c$

$\Rightarrow \ln\left(\frac{t-2}{t+2}\right) = e^{4x+c} \Rightarrow \frac{y-4x-2}{y-4x+2} = e^{4x}.K$

(where $K = e^c$)

(ii) $\tan^2(x+y) = dy/dx$ Put $x+y=t$

$\Rightarrow 1 + dy/dx = dt/dx$

$\Rightarrow \tan^2 t = \frac{dt}{dx} - 1 \Rightarrow \frac{dt}{dx} = \sec^2 t$

$\Rightarrow \cos^2 t\, dt = dx \Rightarrow \int (1+\cos 2t)\, dt = \int 2\, dx$

$\Rightarrow t + \frac{\sin 2t}{2} = 2x + C$

$\Rightarrow x + y + \frac{\sin 2t}{2} = 2x + C \Rightarrow \sin 2t = 2(x-y) + 2C$

$\Rightarrow \sin 2(x+y) = 2(x-y) + K$

Do yourself - 5 :

Solve the following differential equations :

(i) $\frac{dy}{dx}=\frac{2x-y+2}{2y-4x+1}$ **(ii)** $\frac{dy}{dx}=\frac{3x-5y}{5x+y+3}$

Sol^n (i) $\frac{dy}{dx} = \frac{2x-y+2}{1-2(2x-y)}$ $2x-y = t$

$\Rightarrow 2 - \frac{dy}{dx} = \frac{dt}{dx}$ $2 - \left(\frac{t+2}{1-2t}\right)$

$\Rightarrow 2 - \frac{dt}{dx} = \frac{t+2}{1-2t} \Rightarrow \frac{dt}{dx} = \frac{-5t}{1-2t}$

$\Rightarrow \frac{2t-1}{t} = 5\,dx \Rightarrow \int 2 - \frac{1}{t}\,dt = 5\int dx$

$\Rightarrow 2t - \ln t = 5x + C$

$\Rightarrow 2(2x-y) - \ln(2x-y) = 5x + C$

$\Rightarrow x + 2y + \ln(2x-y) + C = 0$

(ii) special case. Cross multiply :

$5x\,dy + y\,dy + 3\,dy = 3x\,dx - 5y\,dx$

$\Rightarrow 5(x\,dy + y\,dx) + y\,dy + 3\,dy - 3x\,dx = 0$

$\Rightarrow 5(d(xy)) + y\,dy + 3dy - 3x\,dx = 0$

$\Rightarrow 5xy + \frac{y^2}{2} + 3y - \frac{3x^2}{2} = C$

Do yourself - 6 :

Solve the following differential equations :

(i) $(y - xy^2)dx - (x + x^2y)dy = 0$ **(ii)** $y(1 + 2xy)dx + x(1 - xy)dy = 0$

Sol^n (i) $y(1-xy)\,dx - x(1+xy)\,dy = 0$

put $xy = z \Rightarrow y = \frac{z}{x} \Rightarrow dy = \frac{x\,dz - z\,dx}{x^2}$

$\Rightarrow \frac{z}{x}(1-z)\,dx - x(1+z)\left(\frac{x\,dz - z\,dx}{x^2}\right) = 0$

$\Rightarrow z(1-z)\,dx - (1+z)(x\,dz - z\,dx)$

$\Rightarrow z(1-z)\,dx - x(1+z)\,dz + z(1+z)\,dx = 0$

$\Rightarrow (z - z^2 + z + z^2)\,dx = x(1+z)\,dz$

$\Rightarrow 2z\,dx = x(1+z)\,dz$

$\Rightarrow 2\frac{dx}{x} = \left(\frac{1+z}{z}\right)dz \Rightarrow 2\ln x = \ln z + z + \ln c$

$\Rightarrow 2\ln x = \ln xy + xy + \ln c \Rightarrow \ln\left(\frac{x^2}{cxy}\right) = xy$

$\Rightarrow \frac{x}{cy} = e^{xy} \Rightarrow x = cy\,e^{xy}$ Ans.

(ii) put $xy = z \Rightarrow y = \frac{z}{x} \Rightarrow dy = \frac{x\,dz - z\,dx}{x^2}$

$\frac{z}{x}(1+2z)\,dx + x(1-z)\left(\frac{x\,dz - z\,dx}{x^2}\right) = 0$

$\Rightarrow z(1+2z)\,dx + (1-z)(x\,dz - z\,dx) = 0$

$\Rightarrow (z + 2z^2 - z + z^2)\,dx + (1-z)x\,dz = 0$

$\Rightarrow 3z^2\,dx + (1-z)x\,dz = 0$

$\Rightarrow \frac{dx}{x} = \frac{(z-1)\,dz}{3z^2} \Rightarrow \int\frac{dx}{x} = \frac{1}{3}\cdot\int\left(\frac{1}{z} - \frac{1}{z^2}\right)dz$

$\Rightarrow \ln x = \frac{1}{3}\left(\frac{1}{z} + \ln z\right) + \frac{\ln c}{3} \Rightarrow \ln\frac{x^3}{cz} = \frac{1}{z}$

$\Rightarrow \frac{x^3}{cz} = e^{1/z} \Rightarrow x^2 = cy\,e^{1/xy}$

Do yourself - 7 :

Solve the following differential equations :

(i) $xdx + ydy = xdy - ydx$ **(ii)** $ydx - xdy = xy\,dy - x^2dx$

Soln (i) $x\,dx + y\,dy = x\,dy - y\,dx$

put $x = r\cos\theta$; $y = r\sin\theta$

$\Rightarrow r\,dr = r^2 d\theta \Rightarrow \frac{dr}{r} = d\theta$

$\Rightarrow \ln r = \theta + C \Rightarrow \ln\sqrt{x^2+y^2} = \tan^{-1}\frac{y}{x} + C$

$\Rightarrow \ln(x^2+y^2) = 2\tan^{-1}\frac{y}{x} + C_1$

(ii) $y\,dx - x\,dy = -x(x\,dx - y\,dy)$

$x = r\sec\theta$; $y = r\tan\theta$

$-\cancel{r^2}\cancel{\sec}\theta\,d\theta = -\cancel{r}\cancel{\sec}\theta(\cancel{r}\,dr)$

$\Rightarrow d\theta = dr \Rightarrow \theta = -r + C$

$\Rightarrow \sin^{-1}(y/x) = \sqrt{x^2-y^2} + C$

Do yourself - 8 :

(i) Find the degree of homogeneity of function $f(x,y) = x^3 \ell n\left[\sqrt{x+y}/\sqrt{x-y}\right]$

(ii) Find the degree of homogeneity of function $f(x,y) = ax^{2/3} + hx^{1/3}y^{1/3} + by^{2/3}$

(iii) Determine whether or not each of the following functions is homogeneous.

(a) $f(x,y) = \sqrt{x^2+2xy+3y^2}$ (b) $f(x,y) = x + y\cos\frac{y}{x}$ (c) $f(x,y) = x\sin y + y\sin x$.

(i) $f(\lambda x, \lambda y) = \lambda^3 x^3 \ln\left(\frac{\sqrt{\lambda x+\lambda y}}{\sqrt{\lambda x-\lambda y}}\right)$

$= \lambda^3\left(x^3 \ln\left(\frac{\sqrt{x+y}}{\sqrt{x-y}}\right)\right) = \lambda^3 f(x,y)$

$\therefore$ Degree $=3$

(ii) $f(\lambda x, \lambda y) = a\lambda^{2/3}x^{2/3} + h\lambda^{2/3}x^{1/3}y^{1/3} + b\lambda^{2/3}y^{2/3}$

$= \lambda^{2/3}(ax^{2/3} + hx^{1/3}y^{1/3} + by^{2/3})$

$f(\lambda x, \lambda y) = \lambda^{2/3} f(x,y)$

$\Rightarrow$ Degree $= 2/3$

(iii) (a) $f(\lambda x, \lambda y) = \sqrt{\lambda^2x^2 + 2\lambda^2 xy + 3\lambda^2 y^2}$

$= \lambda\sqrt{x^2 + 2xy + 3y^2}$

$\therefore f(\lambda x, \lambda y) = \lambda f(x,y)$

$\Rightarrow$ Degree $= 1$

(b) $f(\lambda x, \lambda y) = \lambda x + \lambda y \cos\left(\frac{\lambda y}{\lambda x}\right) = \lambda\left(x + y\cos\frac{y}{x}\right)$

$\Rightarrow f(\lambda x, \lambda y) = \lambda f(x,y) \Rightarrow$ Degree $= 1$

(c) $f(\lambda x, \lambda y) = \lambda x \sin \lambda y + \lambda y \sin \lambda x$

$= \lambda(x\sin\lambda y + y\sin\lambda x)$

$\therefore$ f is not homogeneous

Do yourself - 9 :

Solve the following differential equations :

(i) $y' = \dfrac{3x-y}{x+y}$

(ii) $(x - y\ell ny + y\ell nx)\,dx + x(\ell ny - \ell nx)\,dy = 0$

(iii) $(3xy + y^2)dx + (x^2 + xy)dy = 0,\ y(1) = 1$

(i) $\dfrac{dy}{dx} = \dfrac{3x-y}{x+y} \Rightarrow x\,dy + y\,dy = 3x\,dx - y\,dx$

(Homogeneous) $\Rightarrow x\,dy + y\,dx + y\,dy = 3x\,dx$

$\Rightarrow d(xy) + y\,dy = 3x\,dx$

$\Rightarrow xy + y^2/2 = 3x^2/2 + C$

$\Rightarrow 3x^2 - 2xy - y^2 = C_0$

$\Rightarrow (3x+y)(x-y) = C_0$

(ii) $(x - y(\ln y/x))\,dx + x(\ln y/x)\,dy = 0$

$\Rightarrow (x\ln(y/x))\,dy = (y\ln(y/x) - x)\,dx$

$\Rightarrow \dfrac{dy}{dx} = \dfrac{y\ln(y/x) - x}{x\ln y/x} = \dfrac{(y/x)\ln y/x - 1}{\ln(y/x)}$

put $y = vx$

$\Rightarrow v + x\dfrac{dv}{dx} = \dfrac{v\ln v - 1}{\ln v} \Rightarrow x\dfrac{dv}{dx} = \dfrac{-1}{\ln v}$

$\Rightarrow \int \ln v\,dv = \int -\dfrac{dx}{x}$

$\Rightarrow v\ln v - v = -\ln x + C$

$\Rightarrow y\ln(y/x) - y = -x\ln x + Cx$

(iii)

$$\frac{dy}{dx} = -\left(\frac{3xy+y^2}{x^2+xy}\right)$$

$$\Rightarrow \frac{dy}{dx} = -\left(\frac{3\,y/x+(y/x)^2}{1+y/x}\right)$$

$y = vx$

$$\Rightarrow v + \frac{x\,dv}{dx} = -\left(\frac{3v+v^2}{1+v}\right)$$

$$\Rightarrow \frac{x\,dv}{dx} = -\left(\frac{3v+v^2}{1+v} + v\right)$$

$$\Rightarrow \frac{x\,dv}{dx} = -\left(\frac{3v+v^2+v^2+v}{1+v}\right)$$

$$\frac{x\,dv}{dx} = -\left(\frac{2v^2+4v}{1+v}\right)$$

$$\Rightarrow \int \frac{1+v}{v^2+2v}\,dv = -\int 2\frac{dx}{x}$$

$$\Rightarrow \frac{1}{2}\ln(v^2+2v) = -2\ln x + \ln C$$

$$\Rightarrow v^2+2v = \frac{c}{x^4} \Rightarrow \frac{y^2}{x^2} + \frac{2y}{x} = \frac{c^4}{x}$$

$$\Rightarrow x^2(y^2+2xy) = C$$

$\because y(1) = 1 \Rightarrow C = 1$

$\therefore x^2 y(2x+y) = 1$ Ans

Do yourself - 10 :

(i) Solve the differential equation : $\frac{dy}{dx}=\frac{x+2y-5}{2x+y-4}$

Soln

$$\left.\begin{array}{l} x = X+h \\ y = Y+K \end{array}\right\} \quad \text{for } (h,K): \left.\begin{array}{l} h+2K-5=0 \\ 2h+K-4=0 \end{array}\right\} \begin{array}{l} h=1 \\ K=2 \end{array}$$

$$\therefore \frac{dy}{dx}=\frac{x+2y-5}{2x+y-4} \Rightarrow \frac{dY}{dX}=\frac{X+2Y}{2X+Y} \Rightarrow \frac{dY}{dX}=\frac{1+2(Y/X)}{2+(Y/X)}$$

put $Y=VX \Rightarrow V+X\frac{dV}{dX}=\frac{1+2V}{2+V}$

$$\Rightarrow \frac{X\,dV}{dX}=\frac{1-V^2}{2+V} \Rightarrow \frac{(2+V)}{V^2-1}\,dV=-\frac{dX}{X}$$

$$\Rightarrow 2\int\frac{dV}{V^2-1}+\frac{1}{2}\int\frac{2V\,dV}{V^2-1}=-\int\frac{dX}{X}$$

$$\Rightarrow \frac{\cancel{2}}{\cancel{2}}\ln\left(\frac{V-1}{V+1}\right)+\frac{1}{2}\ln(V^2-1)=-\ln X+\ln C$$

$$\Rightarrow \ln\left(\left(\frac{V-1}{V+1}\right)^2\cdot(V^2-1)\right)=\ln\frac{C_1}{X^2}$$

$$\Rightarrow \frac{(V-1)^3}{V+1}=\frac{C_1}{X^2}$$

$$\Rightarrow \frac{(Y-X)^3}{X^2\cdot(Y+X)}=\frac{C_1}{X^2} \Rightarrow (Y-X)^3=C_1(Y+X)$$

$$\Rightarrow (y-2-(x-1))^3=C_1(y-2+x-1) \Rightarrow \boxed{(x-y+1)^3=C_2(x+y-3)}$$

Do yourself - 11 :

Solve the following differential equations :

(i) $\frac{xdy}{dx} = 2y + x^4 + 6x^2 + 2x,\ x \neq 0$ **(ii)** $(x-a)\frac{dy}{dx} + 3y = 12(x-a)^3,\ x > a > 0$

(iii) $y\ \ell n\ y\ dx + (x - \ell n\ y)\ dy = 0$

Solⁿ (i) $\frac{dy}{dx} + \left(-\frac{2}{x}\right)y = x^3 + 6x + 2$

$If = e^{\int -2/x\, dx} = e^{-2\ln x} = 1/x^2$

$\therefore$ Ans: $y(1/x^2) = \int \frac{1}{x^2}(x^3 + 6x + 2)dx$

$\Rightarrow \frac{y}{x^2} = \int x + \frac{6}{x} + 2\cdot x^{-2}\, dx$

$\Rightarrow \frac{y}{x^2} = \frac{x^2}{2} + 6\ln|x| + \frac{2x^{-1}}{-1} + C$

$\Rightarrow \boxed{y = \frac{x^4}{2} + 6x^2\ln|x| - 2x + Cx^2}$

(ii) put $x - a = t$

$\therefore t\frac{dy}{dt} + 3y = 12t^3 \Rightarrow \frac{dy}{dt} + \frac{3}{t}y = 12t^2$

$If = e^{\int \frac{3}{t}dt} \Rightarrow If = e^{3\ln t} = t^3$

$\therefore$ Ans: $y(t^3) = \int 12t^2\cdot t^3\, dt \Rightarrow t^3y = 12\cdot\frac{t^6}{6} + C$

$\Rightarrow t^3y = 2t^6 + C$

$\Rightarrow \boxed{(x-a)^3y = 2(x-a)^6 + C}$ Ans.

(iii) $y \ln y \, dx + (x - \ln y)\, dy = 0$

$$\left(\frac{dx}{dy}\right)(y \ln y) + x - \ln y = 0$$

$$\frac{dx}{dy} + x\left(\frac{1}{y \ln y}\right) = \frac{\ln y}{y \ln y}$$

$$IF = e^{\int \frac{1}{y \ln y} dy} = e^{\ln(\ln y)} = \ln y$$

$$\Rightarrow \text{Ans: } x(\ln y) = \int \frac{\ln y}{y} dy$$

$$\Rightarrow \boxed{x(\ln y) = \frac{(\ln y)^2}{2} + C}$$

Do yourself - 12 :

Solve the following differential equations :

(i) $y' + 3y = e^{3x} y^2$ **(ii)** $xdy - \{y + xy^3 (1 + \ell nx)\}dx = 0$ **(iii)** $\frac{dy}{dx} + y = y^2(\cos x - \sin x)$

(i) $\frac{dy}{dx} + 3y = e^{3x}.y^2$

$\Rightarrow \frac{1}{y^2}\frac{dy}{dx} + \frac{3}{y} = e^{3x}$ put $-\frac{1}{y} = t \Rightarrow \frac{1}{y^2}\frac{dy}{dx} = \frac{dt}{dx}$

$\Rightarrow \frac{dt}{dx} + (-3)t = e^{3x}$

$IF = e^{\int -3\,dx} = e^{-3x} \Rightarrow t(e^{-3x}) = \int e^{-3x}.e^{3x}dx = \int dx$

$\Rightarrow t\,e^{-3x} = x + C$

$\Rightarrow -\frac{e^{-3x}}{y} = x + C \Rightarrow \frac{e^{-3x}}{y} = C_1 - x \Rightarrow \boxed{y = \frac{1}{(C_1 - x)e^{3x}}}$

(ii) $x\frac{dy}{dx} - (y + xy^3(1+\ln x)) = 0$

$\Rightarrow \frac{1}{y^3}\frac{dy}{dx} + \left(\frac{1}{x}\right)\left(-\frac{1}{y^2}\right) = 1 + \ln x$; put $-\frac{1}{y^2} = t \Rightarrow \frac{1}{y^3}\frac{dy}{dx} = \frac{1}{2}.\frac{dt}{dx}$

$\Rightarrow \frac{dt}{dx} + 2\frac{t}{x} = 2(1+\ln x) \Rightarrow IF = e^{\int \frac{2}{x}dx} = x^2$

$\therefore t(x^2) = 2\int x^2(1+\ln x)\,dx = 2\left((1+\ln x)\frac{x^3}{3} - \int \frac{1}{x}.\frac{x^3}{3}dx\right)$

$\Rightarrow -\frac{x^2}{y^2} = 2\left(\frac{x^3}{3}(1+\ln x) - \frac{x^3}{9}\right) + C$

$-\frac{x^2}{y^2} = 2\left(\frac{2x^3}{9} + \frac{x^3}{3}\ln x\right) + C \Rightarrow \boxed{\frac{x^2}{y^2} = -\frac{2}{3}x^3\left(\frac{2}{3} + \ln x\right) + C}$

(iii) $\frac{dy}{dx} + y = y^2(\cos x - \sin x)$

$\Rightarrow \frac{1}{y^2}\frac{dy}{dx} + \frac{1}{y} = \cos x - \sin x$

put $-\frac{1}{y} = t \Rightarrow \frac{1}{y^2}\frac{dy}{dx} = \frac{dt}{dx}$

$\Rightarrow \frac{dt}{dx} + (-1)t = \cos x - \sin x$

$IF = e^{\int -1\,dx} = e^{-x}$

$\Rightarrow t(e^{-x}) = \int e^{-x}(\cos x - \sin x)\,dx$

$-\frac{1}{y}e^{-x} = -e^{-x}(\sin x) + C$

$\Rightarrow \frac{1}{y} = \sin x + C_1 e^{x}$

Do yourself - 13 :

(i) Find the orthogonal trajectories of the following families of curves :

(a) $x + 2y = C$ **(b)** $y = Ce^{-2x}$

Solⁿ (a) $x + 2y = C$

$\Rightarrow 1 + 2\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{-1}{2} \longrightarrow$ DE of curve

$\Rightarrow$ DE of OT : $-\frac{dx}{dy} = \frac{-1}{2}$

$\Rightarrow 2dx = dy \Rightarrow 2x = y + C$

(b) $y = Ce^{-2x} \Rightarrow \frac{dy}{dx} = -2Ce^{-2x}$

$\Rightarrow \frac{dy}{dx} = -2y \longrightarrow$ DE of curve

$\Rightarrow$ DE of OT : $-\frac{dx}{dy} = -2y$

$\Rightarrow 2y\,dy = dx \Rightarrow y^2 = x + C$

Do yourself - 14 :

Solve the following differential equations :

(i) $x\,dx + y\,dy + 4y^3(x^2 + y^2)dy = 0.$ **(ii)** $x\,dy - y\,dx - (1 - x^2)dx = 0.$

Solⁿ (i) $x\,dx + y\,dy + 4y^3(x^2+y^2)dy = 0$

$$\Rightarrow \frac{x\,dx + y\,dy}{x^2+y^2} + 4y^3\,dy = 0$$

$$\Rightarrow \int \frac{1}{2}\left(\frac{2x\,dx + 2y\,dy}{x^2+y^2}\right) + \int 4y^3\,dy = C$$

$$\Rightarrow \frac{1}{2}\ln(x^2+y^2) + y^4 = C$$

(ii) $x\,dy - y\,dx = (1-x^2)\,dx$

$$\int \frac{x\,dy - y\,dx}{x^2} = \int \frac{(1-x^2)}{x^2}\,dx$$

$$\Rightarrow \int d\left(\frac{y}{x}\right) = \int \left(\frac{1}{x^2} - 1\right)dx$$

$$\Rightarrow \frac{y}{x} = -\frac{1}{x} - x + C$$

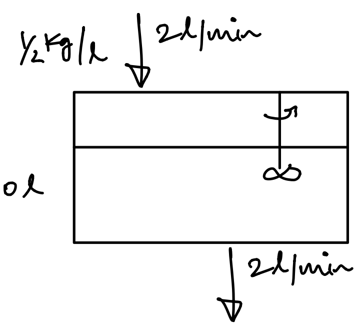
Do yourself - 15 :

(i) A tank initially holds 10 lit. of fresh water. At t = 0, a brine solution containing $\frac{1}{2}$ kg of salt per lit. is poured into the tank at a rate of 2 lit/min. while the well-stirred mixture leaves the tank at the same rate. Find

(a) the amount and

(b) the concentration of salt in the tank at any time t.

<u>Solⁿ</u>



$V_t = 10l$

Let m gm of salt be present in the tank at time t

$\therefore$ Inlet rate $= \frac{1}{2}\,Kg/l \cdot 2l/min = 1\,Kg/min$

Outlet rate $= \frac{m}{10}\,\frac{Kg}{l} \cdot 2\,l/min = \frac{m}{5}\,Kg/min$

$\therefore \frac{dm}{dt} = 1 - \frac{m}{5} \Rightarrow dm = \frac{5-m}{5}dt$

$\Rightarrow \frac{dm}{5-m} = \frac{dt}{5} \Rightarrow \ln(5-m) = -\frac{t}{5} + C$

At $t=0$, $m=0 \Rightarrow C = \ln 5$

$\therefore \ln\left(\frac{5-m}{5}\right) = -\frac{t}{5} \Rightarrow \frac{5-m}{5} = e^{-t/5}$

$\Rightarrow \boxed{m_t = 5 - 5e^{-t/5}\ Kg}$ <u>Am</u>

Concentration $= \frac{m_t}{10}\,Kg/l$ <u>Am</u>.

Do yourself - 16 :

(i) At each point (x,y) of a curve the intercept of the tangent on the y-axis is equal to $2xy^2$. Find the curve.

(ii) Find the equation of the curve for which the normal at any point (x,y) passes through the origin.

Sol^n (i) y intercept of tangent $= y - x\frac{dy}{dx} = 2xy^2$

$\Rightarrow x\frac{dy}{dx} - y = -2xy^2$

$\frac{1}{y^2}\frac{dy}{dx} + \left(\frac{1}{x}\right)\left(\frac{-1}{y}\right) = -2$

put $-1/y = t \Rightarrow (1/y^2)\,dy/dx = dt/dx$

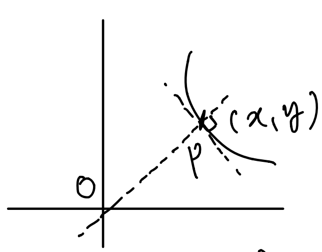
$\Rightarrow \frac{dt}{dx} + (1/x)\,t = -2 \longrightarrow$ LDE

$\therefore$ IF $= e^{\int 1/x\,dx} \Rightarrow x$

$\therefore t(x) = \int -2x\,dx = -x^2 + C$

$\Rightarrow -\frac{x}{y} = -x^2 + C \Rightarrow \frac{x}{y} = x^2 + K$

(ii)



$\underbrace{-\frac{dx}{dy}}_{m_{N\,at\,P}} = \underbrace{y/x}_{\text{Slope of } OP}$

$\Rightarrow -\int x\,dx = \int y\,dy \Rightarrow -\frac{x^2}{2} = \frac{y^2}{2} + C$

$\Rightarrow x^2 + y^2 = K$

Exercise 01

1. Number of values of $m \in N$ for which $y = e^{mx}$ is a solution of the differential equation $D^3y - 3D^2y - 4Dy + 12y = 0$, is

(A) 0 (B) 1 (C) 2 (D) more than 2

Sol.

$$y = e^{mx}$$

$$Dy = y' = m e^{mx}$$

$$D^2y = y'' = m^2 e^{mx}$$

$$D^3y = y''' = m^3 e^{mx}$$

or $m^3 - 3m^2 + 4m + 12 = 0$

or $m = \pm 2, 3$

$m \in N$, $\therefore m = \{2, 3\}$

Ans

2. The value of the constant 'm' and 'c' for which $y = mx + c$ is a solution of the differential equation $D^2y - 3Dy - 4y = -4x$.

(A) is $m = -1; c = 3/4$ (B) is $m = 1; c = -3/4$ (C) no such real m, c (D) is $m = 1; c = 3/4$

Sol. put $y = mx + c$ in

diff. eqn

$-3m - 4(mx + c) = -4x$

$-3m - 4c - 4mx = -4x$

$\therefore \quad m = 1, \quad c = -3/4$

3. Consider the two statements

Statement-1: $y = \sin kt$ satisfies the differential equation $y'' + 9y = 0$.

Statement-2: $y = e^{kt}$ satisfy the differential equation $y'' + y' - 6y = 0$

The value of k for which both the statements are correct is

(A) – 3 (B) 0 (C) 2 (D) 3

Soln

$$y = \sin(kt)$$

$$y' = \cos(kt)(k)$$

$$y'' = -(\sin kt)\, k^2$$

or $y'' + k^2 y = 0 \rightarrow (1)$

again $y = e^{kt}$

$$y' = e^{kt}(k)$$

$$y'' = e^{kt}(k^2)$$

$$y'' + y' - 6y = 0$$

$$k^2 + k - 6 = 0$$

$(k+3)(k-2) = 0$, $k = -3, 2$

✓ $\therefore$ $k = -3$

4. The differential equation corresponding to the family of curves $y = e^x(ax+b)$ is

(A) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} - y = 0$ (B) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$ (C) $\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = 0$ (D) $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} - y = 0$

(i)

$$y = e^x(ax+b)$$

$$\Rightarrow y \cdot e^{-x} = ax + b$$

$$\Rightarrow \frac{dy}{dx} e^{-x} + y(e^{-x})(-1) = a$$

$$\Rightarrow y'' e^{-x} + y'(e^{-x})(-1) - y'e^{-x} + ye^{-x} = 0$$

$$\Rightarrow \frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$$

5. Spherical rain drop evaporates at a rate proportional to its surface area. The differential equation corresponding to the rate of change of the radius of the rain drop if the constant of proportionality is $K > 0$, is

(A) $\frac{dr}{dt} + K = 0$ (B) $\frac{dr}{dt} - K = 0$ (C) $\frac{dr}{dt} = Kr$ (D) none

Sol.

$$V = \frac{4}{3}\pi r^3$$

or $$\frac{dV}{dt} \propto A$$

or $$\frac{dV}{dt} = -KA$$

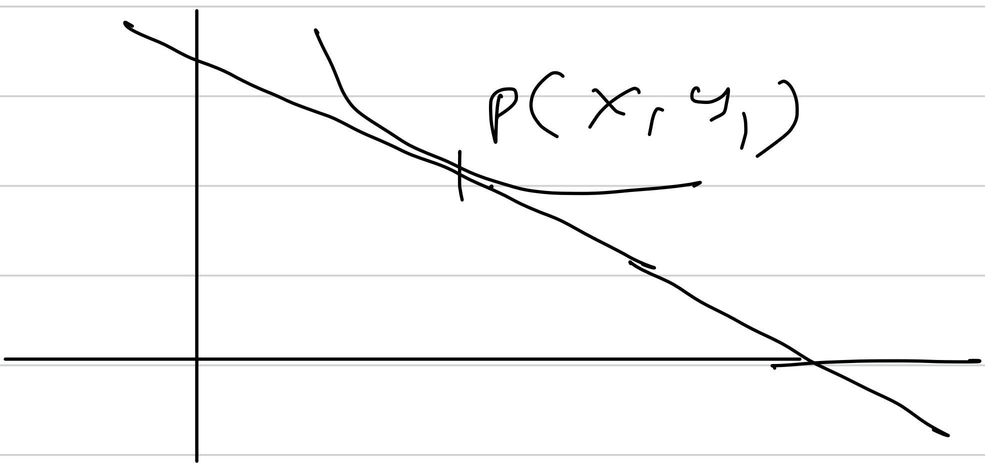
$$4\pi r^2 \frac{dr}{dt} = -KA \qquad (A = 4\pi r^2)$$

or $$\frac{dr}{dt} + K = 0$$

6. The x-intercept of the tangent to a curve is equal to the ordinate of the point of contact. The equation of the curve through the point (1, 1) is

(A) $ye^{\frac{x}{y}} = e$ (B) $xe^{\frac{x}{y}} = e$ (C) $xe^{\frac{y}{x}} = e$ (D) $ye^{\frac{y}{x}} = e$

Sol.



Eq^n of tangent

$$(y - y_1) = m(x - x_1)$$

$$y = 0, \quad x = \left(x_1 - \frac{y_1}{m}\right)$$

$$\therefore \quad x_1 - \frac{y_1}{m} = y_1$$

$$\text{or} \quad \frac{dy}{dx} = \frac{y}{x - y}$$

Put $y = vx$

$$v + x\frac{dv}{dx} = \frac{v}{1-v}$$

$$\int \frac{1-v}{v^2}\,dv = \int \frac{1}{x}\,dx$$

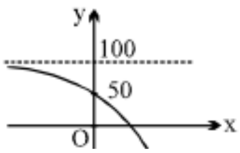
or $\ln(vx) = -\frac{1}{v} - c$

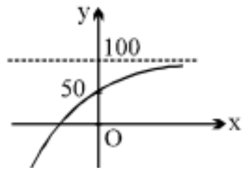
passing through $(1,1)$

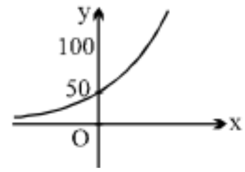
so $c = -1$

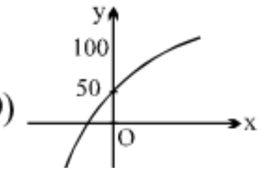
$\therefore \quad y \cdot e^{x/y} = e$

7. Which one of the following curves represents the solution of the initial value problem $Dy = 100 - y$, where $y(0) = 50$

(A) 

(B) 

(C) 

(D) 

Sol.

$$\int \frac{dy}{(100-y)} = \int dx$$

$$-\ln(100-y) = x + C$$

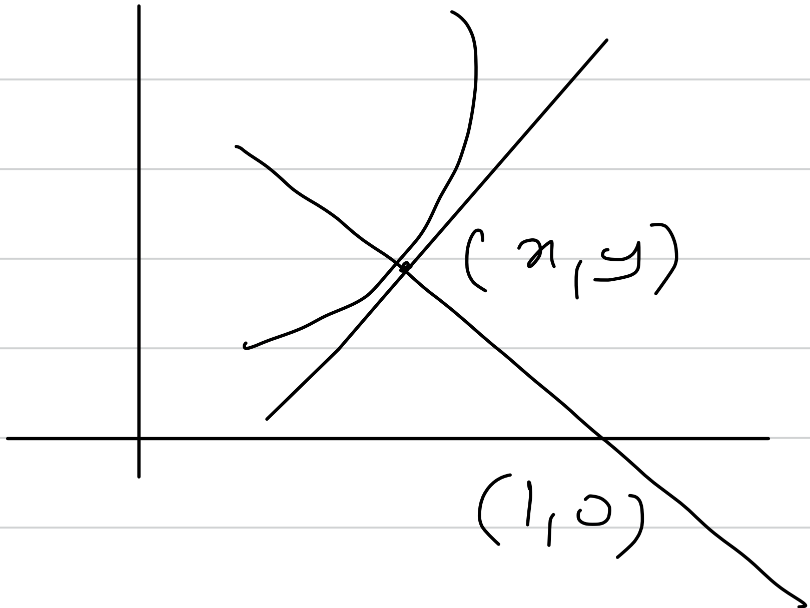
$$x = 0, \quad y = 50, \quad C = \ln 50$$

$$\therefore \quad y = 100 - 50e^{-x}$$

8. A curve C passes through origin and has the property that at each point (x, y) on it the normal line at that point passes through (1, 0). The equation of a common tangent to the curve C and the parabola $y^2 = 4x$ is

(A) $x = 0$ (B) $y = 0$ (C) $y = x + 1$ (D) $x + y + 1 = 0$

Sol.



Slope of Normal $= \frac{y}{(x-1)}$

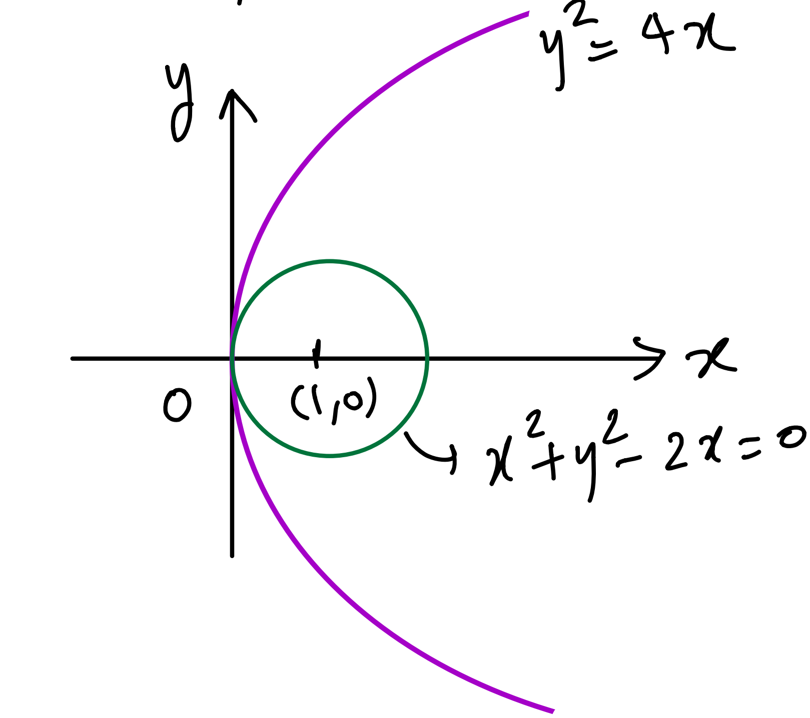
$$\frac{dy}{dx} = \frac{1-x}{y}$$

or $\frac{y^2}{2} = x - \frac{x^2}{2} + c$

which passes through (0, 0), $c = 0$

$\therefore$ $x^2 + y^2 - 2x = 0$

We plot the curves:



Pt. of intⁿ:

$$\begin{cases} y^2 = 4x \\ x^2+y^2-2x=0 \end{cases}$$

$\Rightarrow x^2 + 4x - 2x = 0$

$\Rightarrow x = 0$ or $\underset{\times\times\times}{\underbrace{-2}}$

$\therefore$ both curves touch each other at $(0,0)$

$\therefore$ y axis is common tangent of these curves

Eqⁿ : $x = 0$ <u>Ans</u>

9. A function $y=f(x)$ satisfies $(x+1) \cdot f'(x) - 2(x^2+x) f(x) = \frac{e^{x^2}}{(x+1)}$, $\forall\ x > -1$

If $f(0)=5$, then $f(x)$ is

(A) $\left(\frac{3x+5}{x+1}\right) . e^{x^2}$ (B) $\left(\frac{6x+5}{x+1}\right) . e^{x^2}$ (C) $\left(\frac{6x+5}{(x+1)^2}\right) . e^{x^2}$ (D) $\left(\frac{5-6x}{x+1}\right) . e^{x^2}$

Sol.

$$f'(x) - \frac{2x(x+1)}{(x+1)} f(x) = \frac{e^{x^2}}{(x+1)^2}$$

$$I.F = e^{-\int 2x\, dx}$$

$$= e^{-x^2}$$

$$\therefore f(x) . e^{-x^2} = \int \frac{dx}{(x+1)^2}$$

$$\therefore f(x) . e^{-x^2} = -\frac{1}{(x+1)} + C$$

$x=0$, $f(0)=5$, $C=6$

$$\therefore f(x) = \left(\frac{6x+5}{x+1}\right) e^{x^2}$$

10. If $\int_a^x t\,y(t)dt = x^2 + y(x)$ then y as a function of x is

(A) $y = 2 - (2 + a^2)e^{\frac{x^2-a^2}{2}}$

(B) $y = 1 - (2 + a^2)e^{\frac{x^2-a^2}{2}}$

(C) $y = 2 - (1 + a^2)e^{\frac{x^2-a^2}{2}}$

(D) none

Sol. by diff.

$$x\,y(x) = 2x + y'(x)$$

or $$\frac{dy}{dx} - xy = -2x$$

or $$I.f = e^{\int -x\,dx} = e^{-x^2/2}$$

$$y.e^{-x^2/2} = -\int 2x.e^{-x^2/2}dx$$

or $$y.e^{-x^2/2} = 2e^{-x^2/2} + c$$

If $x = a$, $y(a) = -a^2$

$$\therefore\ c = -(2+a^2)e^{-a^2/2}$$

$$\therefore\ y = 2 - (2+a^2)e^{(x^2-a^2)/2}$$

again

$$x(f' - g') = -(f - g)$$

$$\int \frac{f' - g'}{f - g} = \int + \frac{1}{x}$$

$$\ln(f - g) = \ln x + \ln c$$

$$f - g = cx$$

$$f + g = 4/x$$

$$\therefore \quad f'' - g'' = 0$$

$$f'' + g'' = 8/x^3$$

$$f'' = \frac{4}{x^3} \qquad g'' = \frac{4}{x^3}$$

$$\therefore \; f''(2) \cdot g''(2) = 1/4$$

Exercise 02

EXERCISE (O-2)

One or more than one correct

1. The differential equation, $x\frac{dy}{dx}+\frac{3}{\frac{dy}{dx}}=y^2$:

(A) is of order 1 (B) is of degree 2 (C) is linear (D) is non linear

Solving $x\left(\frac{dy}{dx}\right)^2-y^2\left(\frac{dy}{dx}\right)+3=0$

Is of order one and degree 2

2. A curve $y=f(x)$ has the property that the perpendicular distance of the origin from the normal at any point P of the curve is equal to the distance of the point P from the x-axis. Then the differential equation of the curve

(A) is homogeneous.

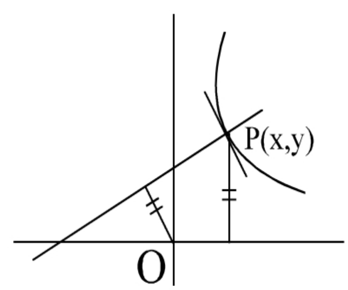
(B) can be converted into linear differential equation with some suitable substitution.

(C) is the family of circles touching the x-axis at the origin.

(D) the family of circles touching the y-axis at the origin.

[Sol. Equation of normal

$$Y - y = -\frac{1}{m}(X - x)$$
$$-my + my = X - x$$
$$X + my - (x + my) = 0$$



perpendicular from $(0, 0) = \left| \frac{x + my}{\sqrt{1+m^2}} \right| = y$

$$x^2 + 2xym = y^2$$

$$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy} \quad \Rightarrow \quad \text{homogeneous} \Rightarrow \quad \textbf{(A)}$$

also $\quad x \cdot 2y \cdot \frac{dy}{dx} - x^2 = y^2 \quad$ put $y^2 = t$; $\quad 2y\frac{dy}{dx} = \frac{dt}{dx}$

$$x \cdot \frac{dt}{dx} + x^2 = t$$

$$\frac{dt}{dx} - \frac{1}{x}t = -x \text{ which is linear differential equation} \quad \Rightarrow B$$

Integration factor $= e^{-\int \frac{1}{x} dx} = e^{-\ln x} = \frac{1}{x}$

$$\Rightarrow \left(\frac{1}{x}\right) \cdot t = -\int 1\, dx = -x + c$$

$$\frac{1}{x} \cdot y^2 = -x + c$$

$$x^2 + y^2 - cx = 0 \Rightarrow D$$

3. The function f(x) satisfying the equation, $f^2(x) + 4f'(x) \cdot f(x) + [f'(x)]^2 = 0$.

(A) $f(x) = c \cdot e^{(2-\sqrt{3})x}$

(B) $f(x) = c \cdot e^{(2+\sqrt{3})x}$

(C) $f(x) = c \cdot e^{(\sqrt{3}-2)x}$

(D) $f(x) = c \cdot e^{-(2+\sqrt{3})x}$

where c is an arbitrary constant.

Soln:– Let $y = f(x)$ & $f'(x) = \frac{dy}{dx}$

$$y^2 + 4\frac{dy}{dx} \cdot y + \left(\frac{dy}{dx}\right)^2 = 0$$

$$\left(\frac{dy}{dx}\right)^2 + (4y)\frac{dy}{dx} + y^2 = 0$$

$$\left(\frac{dy}{dx}\right) = \frac{-4y \pm \sqrt{16y^2 - 4y^2}}{2}$$

$$\frac{dy}{dx} = \frac{-4y \pm 2\sqrt{3}y}{2} = y(-2 \pm \sqrt{3})$$

$$\int \frac{dy}{y} = (-2 \pm \sqrt{3}) \int 1 \cdot dx$$

Integrating we have

$y = f(x) = c(e^{\sqrt{3}-2})x$; $y = f(x) = c \cdot e^{-(2+\sqrt{3})x}$

So

$$\lim_{x \to \frac{\pi}{2}} \left(\frac{\pi}{2} - 0\right) \frac{\sin x}{x} < \int_0^{\pi/2} \frac{(\sin x)}{x} dx < \lim_{x \to \infty} \left(\frac{\pi}{2} - 0\right) \cdot \frac{\sin x}{x}$$

$$1 < \int_0^{\pi/2} \frac{\sin x}{x} dx < \frac{\pi}{2}$$

$\Rightarrow$ A, B, C correct

Also $f(x) = \frac{\sin x}{x}$ is an even function

4. If $y=f(x)$ is solution of the differential equation, $x^2\frac{dy}{dx}.\cos\frac{1}{x}-y\sin\frac{1}{x}=-1$, where $y\to-1$ as $x\to\infty$, then

(A) $f(x)=\sin\frac{1}{x}-\cos\frac{1}{x}$

(B) $\lim\limits_{x\to\infty}\left(-f(x)\right)^x=\frac{1}{e}$

(C) $\lim\limits_{x\to\infty}\left(-f(x)\right)^x=e$

(D) $f'(x)+2xf'(x)+x^2f''(x)=-\frac{2}{x^2}\sin\frac{1}{x}$

$$\frac{dy}{dx}-\frac{y\left(\tan\frac{1}{x}\right)}{x^2}=-\frac{\left(\sec\frac{1}{x}\right)}{x^2}$$

$$I.F.=e^{-\int\frac{\sec\frac{1}{x}}{x^2}dx}=\sec\left(\frac{1}{x}\right)$$

$$y.\left(\sec\frac{1}{x}\right)=-\int\frac{\left(\sec^2\frac{1}{x}\right)}{x^2}dx$$

$$y\sec\left(\frac{1}{x}\right)=\tan\left(\frac{1}{x}\right)+C$$

$$y=\sin\frac{1}{x}+C\cdot\cos\left(\frac{1}{x}\right)$$

$$x\to\infty;\ y\to-1\Rightarrow C=-1$$

So function is

$$y = f(x) = \sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$$

$\Rightarrow$ A is correct

Now $\lim_{x\to\infty} \left(-f(x)\right)^{x} = \lim_{x\to\infty} \left(\cos\frac{1}{x} - \sin\frac{1}{x}\right)^{x}$

Let $x = \frac{1}{t}$ as $x \to \infty$; $t \to 0$

$$= \lim_{t\to 0} (\cos t - \sin t)^{\frac{1}{t}}$$

$$= \lim_{t\to 0} e^{\frac{(\cos t - \sin t - 1)}{t}} = \frac{1}{e}$$

B is correct. For D but $f(x), f'(x)$ & $f''(x)$

5. Consider the differential equation $\frac{dy}{dx} + y\tan x = x\tan x + 1$. Then

(A) The integral curves satisfying the differential equation and given by $y = x + c\cos x$.

(B) The angle at which the integral curves cut the y-axis is $\frac{\pi}{4}$.

(C) Tangents to all the integral curves at their point of intersection with y-axis are parallel.

(D) none of these

Soln:-

$$I.F. = e^{\int \tan x\, dx} = e^{\ln \sec x} = \sec x$$

$$(\sec x)\, y = \int \left(x(\sec x \cdot \tan x) + \sec x\right) dx$$

$$= (x \sec x) + c$$

$$\Rightarrow y = x + c(\cos x)$$

$$\frac{dy}{dx} = (1 - c\sin x)$$

Put $x = 0$ for y axis

$$\Rightarrow \frac{dy}{dx} = 1 \text{ i.e. } \frac{\pi}{4} \text{ angle}$$

Option C is property of Linear Differential Equation we study in class.

Exercise S1

EXERCISE (S-1)

(E-1)

[FORMATION & VARIABLES SEPARABLE]

1. State the order and degree of the following differential equations:

(i) $\left[\frac{d^2x}{dt^2}\right]^3 + \left[\frac{dx}{dt}\right]^4 - xt = 0$

(ii) $\frac{d^2y}{dx^2} = \left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2}$

(i) Order = 2; Degree = 3

(ii) $\left(\frac{d^2y}{dx^2}\right)^2 = \left(1+\left(\frac{dy}{dx}\right)^2\right)^3$

Order = 2, degree = 2

2. (a) Form a differential equation for the family of curves represented by $ax^2 + by^2 = 1$, where a & b are arbitrary constants.

(b) Obtain the differential equation of the family of circles $x^2 + y^2 + 2gx + 2fy + c = 0$; where g , f & c are arbitrary constants.

(c) Obtain the differential equation associated with the primitive,

$y = c_1 e^{3x} + c_2 e^{2x} + c_3 e^x$, where c_1, c_2, c_3 are arbitrary constants.

Soln (a) $ax^2 + by^2 = 1$

$$2ax + 2byy' = 0 \Rightarrow \frac{yy'}{x} = -\frac{2a}{2b}$$

$$\Rightarrow \frac{x(yy'' + (y')^2) - yy' \cdot 1}{x^2} = 0 \Rightarrow xyy'' + x(y')^2 - yy' = 0 \quad \text{Ans}$$

(b) $x^2 + y^2 + 2gx + 2fy + c = 0$

$$2x + 2yy' + 2g + 2fy' = 0$$

$$\Rightarrow x + yy' + g + fy' = 0$$

$$\Rightarrow 1 + yy'' + (y')^2 + fy'' = 0 \Rightarrow \frac{(y')^2 + yy'' + 1}{y''} = -f$$

$$\Rightarrow \frac{y''(2y'y'' + y'y'' + \cancel{yy'''}) - ((y')^2 + \cancel{yy''} + 1)y'''}{(y'')^2} = 0$$

$$\Rightarrow 3y'(y'')^2 - (1 + (y')^2)y''' = 0 \quad \text{Ans}$$

(c) $y = c_1 e^{3x} + c_2 e^{2x} + c_3 e^{x}$ (i)

<u>M1</u>

$y' = 3c_1 e^{3x} + 2c_2 e^{2x} + c_3 e^{x}$ (ii)

$y'' = 9c_1 e^{3x} + 4c_2 e^{2x} + c_3 e^{x}$ (iii)

$y''' = 27c_1 e^{3x} + 8c_2 e^{2x} + c_3 x$ (iv)

(iv) $- 6$ (iii) $+ 11 \cdot$ (ii) $- 6$ (i)

$\Rightarrow y''' - 6y'' + 11y' - 6y = 0$ Ans.

<u>M2</u>

$e^{-x} \cdot y = c_1 e^{2x} + c_2 e^{x} + c_3$

$e^{-x} \cdot y' - e^{-x} y = 2c_1 e^{2x} + c_2 e^{x}$

$e^{-x}(y' - y) = e^{x}(2c_1 e^{x} + c_2)$

$\Rightarrow e^{-2x}(y' - y) = 2c_1 e^{x} + c_2$

$\Rightarrow e^{-2x}(y'' - y') + (y' - y)(-2e^{-2x}) = 2c_1 e^{x}$

$\Rightarrow e^{-3x}(y'' - y') - 2e^{-3x}(y' - y) = 2c_1$

$\Rightarrow e^{-3x}(y'' - y' - 2y' + 2y) = 2c_1$

$\Rightarrow e^{-3x}(y'' - 3y' + 2y) = 2c_1$

$\Rightarrow e^{-3x}(y''' - 3y'' + 2y') - 3e^{-3x}(y'' - 3y' + 2y) = 0$

$\Rightarrow e^{-3x}(y''' - 3y'' + 2y' - 3y'' + 9y' - 6y) = 0$

$\Rightarrow e^{-3x}(y''' - 6y'' + 11y' - 6y) = 0$

$\Rightarrow y''' - 6y'' + 11y' - 6y = 0$ Ans

3. $$\frac{\ell n(\sec x + \tan x)}{\cos x} dx = \frac{\ell n(\sec y + \tan y)}{\cos y} dy$$

let $\ln(\sec x + \tan x) = t$

$$\Rightarrow \frac{\sec x \cdot \tan x + \sec^2 x}{\sec x + \tan x} dx = dt$$

$$\Rightarrow \sec x \, dx = dt.$$

Similarly. $\ln(\sec y + \tan y) = z.$

$$\sec y \cdot dy = dz$$

$$\Rightarrow t \, dt = z \, dz$$

$$\Rightarrow \frac{t^2}{2} = \frac{z^2}{2} + C$$

$$\Rightarrow \ln^2(\sec x + \tan x) = \ln^2(\sec y + \tan y) + C$$

4. $(1 - x^2)(1 - y)\, dx = xy(1 + y)\, dy$

$$\Rightarrow \int \frac{(1-x^2)}{x} dx = \int \frac{y(1+y)}{1-y} . dy$$

$$\Rightarrow \int \frac{1}{x} - x \, dx = \int -y - 2 + \frac{2}{1-y} \, dy$$

$$\Rightarrow \ln x - \frac{x^2}{2} = -\frac{y^2}{2} - 2y - 2\ln(1-y) + C$$

$$\Rightarrow \ln\left(x(1-y)^2\right) = \frac{x^2}{2} - \frac{y^2}{2} - 2y + C$$

5. $\frac{dy}{dx}+\frac{\sqrt{(x^2-1)(y^2-1)}}{xy}=0$

$$\frac{dy}{dx} = -\frac{\sqrt{x^2-1}\cdot\sqrt{y^2-1}}{xy}$$

$$\Rightarrow \int \frac{y}{\sqrt{y^2-1}}\,dy = -\int \frac{\sqrt{x^2-1}}{x}\,dx$$

$$\Rightarrow \sqrt{y^2-1} = -\sqrt{x^2-1} + \sec^{-1}x + C$$

6. $y - x\dfrac{dy}{dx} = a\left(y^2 + \dfrac{dy}{dx}\right)$

$\Rightarrow y - ay^2 = (a+x)\dfrac{dy}{dx}$

$\Rightarrow \int \dfrac{dy}{y(1-ay)} = \int \dfrac{dx}{a+x}$

$\Rightarrow \int\left(\dfrac{1}{y} + \dfrac{a}{1-ay}\right) dy = \ln(a+x) + \ln c$

$\Rightarrow \ln y - a\ln(1-ay) = \ln c(a+x)$

$\Rightarrow y = c(1-ay)(a+x)$

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7. $\dfrac{dy}{dx} = \sin(x+y) + \cos(x+y)$

$x+y = t \Rightarrow 1 + \dfrac{dy}{dx} = \dfrac{dt}{dx}$

$\Rightarrow \dfrac{dt}{dx} - 1 = \sin t + \cos t.$

$\Rightarrow \displaystyle\int \frac{dz}{1+\sin z+\cos z} = \int dx$

$\Rightarrow \displaystyle\int \frac{dz}{1+\frac{2\tan z/2}{1+\tan^2 z/2}+\frac{1-\tan^2 z/2}{1+\tan^2 z/2}} = \int dx$

$\Rightarrow \displaystyle\int \frac{\left(1+\tan^2\frac{z}{2}\right)dz}{1+\cancel{\tan^2\frac{z}{2}}+2\tan\frac{z}{2}+1-\cancel{\tan^2\frac{z}{2}}} = \int dx$

$\Rightarrow \displaystyle\int \frac{\sec^2 z/2 \; dz}{2(1+\tan z/2)} = \int dx$

$\Rightarrow \ln(1+\tan z/2) = x + C$

$\Rightarrow \ln\left(1+\tan\left(\frac{x+y}{2}\right)\right) = x + C$ Ans.

8. $$\frac{dy}{dx}=\frac{x(2\ln x+1)}{\sin y+y\cos y}$$

$$\int(\sin y+y\cos y)\,dy=\int x(2\ln x+1)\,dx$$

$$\Rightarrow \int \sin y\,dy+\int y\cos y\,dy=2\int x\ln x\,dx+\int x\,dx$$

$$\Rightarrow -\cos y+y\sin y-\int\sin y\,dy$$
$$=2\left(\frac{x^2}{2}\ln x-\int\frac{1}{x}\cdot\frac{x^2}{2}\,dx\right)+\frac{x^2}{2}$$

$$\Rightarrow -\cos y+y\sin y+\cos y=x^2\ln x-\int x\,dx+\frac{x^2}{2}+C$$

$$\Rightarrow y\sin y=x^2\ln x+C$$

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9. (a) $\frac{dy}{dx} + \sin\frac{x+y}{2} = \sin\frac{x-y}{2}$

(b) $\sin x . \frac{dy}{dx} = y . \ln y$ if $y = e$, when $x = \frac{\pi}{2}$

$\frac{dy}{dx} = \sin\frac{x-y}{2} - \sin\frac{x+y}{2} = -2\cos\frac{x}{2} . \sin\frac{y}{2}$

$\int \frac{dy}{\sin\frac{y}{2}} = -\int 2\cos\frac{x}{2}\, dx$

$\Rightarrow \frac{\ln\left(\tan\frac{y}{4}\right)}{\frac{1}{2}} = -2\times 2\sin\frac{x}{2} . + C$

$\Rightarrow \ln\tan\frac{y}{4} = -2\sin\frac{x}{2} + C$

(b) $\int \frac{dy}{y\ln y} = \frac{dx}{\sin x} = \int \operatorname{cosec} x . dx$

$\Rightarrow \ln(\ln y) = \ln\left(\tan\frac{x}{2}\right) + \ln C$

$\Rightarrow \ln y = C\left(\tan\frac{x}{2}\right)$ When $x = \frac{\pi}{2}$, $y = e$

$\Rightarrow 1 = C \Rightarrow \ln y = \tan\frac{x}{2}$.

$\Rightarrow y = e^{\tan x/2}$

10. The population P of a town decreases at a rate proportional to the number by which the population exceeds 1000, proportionality constant being $k > 0$. Find

(a) Population at any time t, given initial population of the town being 2500.

(b) If 10 years later the population has fallen to 1900, find the time when the population will be 1500.

(c) Predict about the population of the town in the long run.

$$\frac{dP}{dt} = -k(P-1000)$$

$$\Rightarrow \int \frac{dP}{P-1000} = -\int k\,dt \Rightarrow \ln(P-1000) = -kt + C$$

$$\Rightarrow P = 1000 + Ae^{-kt}$$

at $t=0$, $P=2500$

$$\Rightarrow 2500 = A + 1000 \Rightarrow A = 1500$$

(a). $P = 1500e^{-kt} + 1000$ Ans:

(b) $t = 10$ $P = 1900$

$$\Rightarrow 1900 = 1500e^{-10k} + 1000$$

$$\Rightarrow e^{-10k} = \frac{900}{1500} = \frac{3}{5}$$

$$\Rightarrow k = \frac{1}{10}\ln\frac{5}{3}$$

at $P = 1500$

$\Rightarrow 1500 = 1500\, e^{-kt} + 1000$

$\Rightarrow e^{-kt} = \dfrac{500}{1500} = \dfrac{1}{3}$

$\Rightarrow -kt = \ln\left(\frac{1}{3}\right) = -\ln 3$

$\Rightarrow t = \dfrac{1}{k}\ln 3 = \dfrac{\ln 3}{\frac{1}{10}\ln\left(\frac{5}{3}\right)}$

$= 10\,\dfrac{\ln 3}{\ln 5/3} = 10 \log_{\left(\frac{5}{3}\right)} 3$

(C) $P = 1500\, e^{-kt} + 1000$

When $P = 1000$ then $t \to \infty$.

11. A normal is drawn at a point P(x,y) of a curve. It meets the x-axis at Q. If PQ is of constant length k, then show that the differential equation describing such curves is, $y\frac{dy}{dx} = \pm\sqrt{k^2 - y^2}$. Find the equation of such a curve passing through (0,k).

PQ = length of the normal at P

$$= y\sqrt{1+\left(\frac{dy}{dx}\right)^2} = k \text{ (given)}$$

$$\Rightarrow y^2 + y^2\left(\frac{dy}{dx}\right)^2 = k^2$$

$$\Rightarrow y \cdot \frac{dy}{dx} = \pm\sqrt{k^2 - y^2}$$

$$\Rightarrow \int \frac{y}{\sqrt{k^2-y^2}}\, dy = \pm\int dx$$

$$\Rightarrow -\sqrt{k^2-y^2} = \pm x + C$$ passes through (0, k)

$$\Rightarrow C = 0$$

$$\Rightarrow \pm x = -\sqrt{k^2-y^2} \Rightarrow k^2 - y^2 = x^2$$

$$\Rightarrow x^2 + y^2 = k^2$$

12. Given $y(0) = 2000$ and $\frac{dy}{dx} = 32000 - 20y^2$, then find the value of $\lim_{x\to\infty} y(x)$.

$$\frac{dy}{32000 - 20y^2} = dx$$

$$\Rightarrow \frac{1}{20}\int \frac{dy}{(40)^2 - y^2} = \int dx$$

$$\Rightarrow \frac{1}{20} \cdot \frac{1}{2\cdot 40} \ln\left|\frac{y+40}{y-40}\right| = x + C$$

$$\Rightarrow \frac{y+40}{y-40} = K \cdot e^{1600x}$$

given $y(0) = 2000$

$$\Rightarrow \frac{2040}{1960} = K \Rightarrow K = \frac{51}{49}$$

$$\frac{y+40}{y-40} = \left(\frac{51}{49}\right) e^{1600x}$$

$$\Rightarrow (y+40)e^{-1600x} = (y-40)$$

$$\lim_{x\to\infty} y(x) = 40.$$ Ans.

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13. Let f(x) is a continuous function which takes positive values for $x \geq 0$ and satisfy $\int_0^x f(t)dt = x\sqrt{f(x)}$ with $f(1) = \frac{1}{2}$. Find the value of $f\left(\sqrt{2}+1\right)$.

$$\int_0^x f(t)\,dt = x\sqrt{f(x)}$$

$$\Rightarrow f(x) = \sqrt{f(x)} + \frac{x}{2\sqrt{f(x)}} \cdot f'(x)$$

$$\Rightarrow 2y^{3/2} = 2y + x \cdot \frac{dy}{dx}$$

$$\Rightarrow \left(2y^{3/2} - 2y\right) = x \cdot \frac{dy}{dx}$$

$$\Rightarrow 2\int \frac{dx}{x} = \int \frac{dy}{\sqrt{y}(y-\sqrt{y})}$$

$\sqrt{y} = t$

$\frac{1}{2\sqrt{y}}\,dy = dt$

$$\Rightarrow \ln x = \int \frac{dt}{t^2 - t} = \int \frac{dt}{t(t-1)} = \int \frac{1}{t-1} - \frac{1}{t}\,dt$$

$$= \ln\frac{(\sqrt{y}-1)}{\sqrt{y}} + \ln C$$

$$\Rightarrow x = C\left(\frac{1-\sqrt{y}}{\sqrt{y}}\right)$$

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$$x = \left(\frac{1-\sqrt{y}}{\sqrt{y}}\right) c \qquad f(1) = \frac{1}{2}$$

$$\Rightarrow \quad c = \frac{\sqrt{y}}{1-\sqrt{y}} = \frac{\frac{1}{\sqrt{2}}}{1-\frac{1}{\sqrt{2}}} = \frac{1}{\sqrt{2}-1}$$

$$\frac{\sqrt{y}-1}{\sqrt{y}} \cdot \frac{1}{\sqrt{2}-1} = x$$

$$\frac{1-\sqrt{y}}{\sqrt{y}} = (\sqrt{2}-1)(\sqrt{2}+1) = 1$$

$$\Rightarrow \frac{1}{\sqrt{y}} - 1 = 1 \Rightarrow \sqrt{y} = \frac{1}{2} \Rightarrow y = \frac{1}{4}.$$

$$y-1 = \left(k_2 e^x + (k_2 e - 2k_1 - k_2)x\right)_0^x$$

$$y-1 = k_2 e^x + (k_2 e - 2k_1 - k_2)x - k_2 \quad (4)$$

(1) & (4) are identical so:

$$k_2 e - 2k_1 - k_2 = 0 \qquad 1 - k_2 = -k_1$$

$$\Rightarrow k_2(e-1) = 2k_1 \quad (5) \qquad k_2 = 1 + k_1$$

on solving (3) & (5)

$$(1+k_1)(e-1) = 2k_1$$

$$\Rightarrow e - 1 + k_1 e - k_1 = 2k_1$$

$$\Rightarrow e - 1 = k_1(3-e)$$

$$\Rightarrow k_1 = \frac{e-1}{3-e}, \quad k_2 = \frac{e-1+3-e}{3-e} = \frac{2}{3-e}$$

$$y = k_2 e^x - k_1 = \left(\frac{2}{3-e}\right)e^x - \frac{e-1}{3-e}$$

$$= \frac{1}{3-e}\left(2e^x - e + 1\right)$$

(E-2)
[HOMOGENEOUS]

1. $\dfrac{dy}{dx} = \dfrac{x^2 + xy}{x^2 + y^2}$

Solⁿ. $y = vx \Rightarrow \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{1+v}{1+v^2} \Rightarrow x\frac{dv}{dx} = \frac{1+v-v-v^3}{1+v^2}$$

$$\Rightarrow \frac{(1+v^2)}{(1-v^3)}\,dv = \frac{dx}{x}$$

$$\Rightarrow \left(\frac{1}{1-v^3} + \frac{v^2}{1-v^3}\right)dv = \frac{dx}{x}$$

Integrating.

$$\Rightarrow \int\left(\frac{1/3}{1-v} + \frac{\frac{1}{3}(v+2)}{1+v+v^2} + \frac{v^2}{1-v^3}\right)dv = \int\frac{dx}{x}$$

$$\Rightarrow -\frac{1}{3}\ln|1-v| - \frac{1}{3}\ln|1-v^3| + \frac{1}{6}\int\frac{2v+1}{1+v+v^2}\,dv + \frac{1}{2}\int\frac{dv}{\left(\frac{\sqrt{3}}{2}\right)^2 + \left(v+\frac{1}{2}\right)^2} = \ln x + \ln C$$

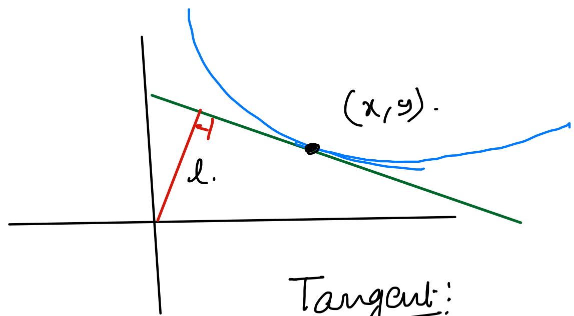
$$\Rightarrow -\frac{1}{3}\left(\ln(1-v)(1-v^3)\right) + \frac{1}{6}\ln(1+v+v^2) + \frac{1}{2}\times\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2v+1}{\sqrt{3}}\right) = \ln x + \ln C$$

$$\Rightarrow \frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x+2y}{\sqrt{3}x}\right) = \ln\left(cx\left(\frac{x-y}{x}\right)^{1/3}\cdot\left(\frac{x^3-y^3}{x^3}\right)^{1/3}\times\left(\frac{x^2}{x^2+xy+y^2}\right)^{1/6}\right)$$

$$\Rightarrow \exp\left(\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x+2y}{\sqrt{3}x}\right)\right) = (x-y)^{2/3}(x^2+xy+y^2)^{1/6}$$

2. The perpendicular from the origin to the tangent at any point on a curve is equal to the abscissa of the point of contact. Find the equation of the curve satisfying the above condition and which passes through (1 , 1).

Solⁿ.



Tangent: $Y - y = \frac{dy}{dx}(X - x)$.

given. $l = x$.

$$\Rightarrow \left| \frac{y - x\frac{dy}{dx}}{\sqrt{1+\left(\frac{dy}{dx}\right)^2}} \right| = |x|.$$

squaring. $y^2 - 2xy\frac{dy}{dx} + x^2\left(\frac{dy}{dx}\right)^2 = x^2 + x^2\left(\frac{dy}{dx}\right)^2.$

$$\Rightarrow \frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$$

$y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}.$

$$\Rightarrow v + x\frac{dv}{dx} = \frac{v^2-1}{2v}.$$

$$\Rightarrow \frac{2v}{1+v^2} = -\frac{dx}{x}.$$

$$\Rightarrow \ln(1+v^2) = -\ln(x) + \ln c.$$

$$\Rightarrow \frac{x^2+y^2}{x} = c.$$

$\because$ It passes through $(1,1) \Rightarrow c = 2$

$\therefore$ curve is. $x^2 + y^2 - 2x = 0.$

3. $(x-y)\,dy = (x+y+1)\,dx$

$$\frac{dy}{dx} = \frac{x+y+1}{x-y} = \frac{(x+1/2)+(y+1/2)}{(x+1/2)-(y+1/2)}$$

$$\left(y+\tfrac{1}{2}\right) = v\,(x+1/2)$$

$$\Rightarrow \frac{dy}{dx} = (x+1/2)\frac{dv}{dx} + v$$

$$\Rightarrow v + (x+1/2)\frac{dv}{dx} = \frac{1+v}{1-v}$$

$$\Rightarrow (x+1/2)\frac{dv}{dx} = \frac{1+v^2}{1-v}$$

$$\Rightarrow \frac{1-v}{1+v^2}\,dv = \frac{dx}{(x+1/2)}$$

Integrating.

$$\Rightarrow \tan^{-1}v - \frac{1}{2}\ln(1+v^2) = \ln(x+1/2) + \ln c$$

$$\Rightarrow \tan^{-1}\left(\frac{2y+1}{2x+1}\right) = \ln\left\{c\sqrt{\left(x+\tfrac{1}{2}\right)^2 + (y+1/2)^2}\right\}$$

4. $\dfrac{dy}{dx} = \dfrac{x+2y-3}{2x+y-3}$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{(x-1)+2(y-1)}{2(x-1)+(y-1)}.$$

$$y-1 = v(x-1).$$

$$\Rightarrow \quad \frac{dy}{dx} = (x-1)\frac{dv}{dx} + v.$$

$$\therefore \quad (x-1)\frac{dv}{dx} + v = \frac{1+2v}{2+v}.$$

$$\Rightarrow \quad (x-1)\frac{dv}{dx} = \frac{1-v^2}{2+v}.$$

$$\Rightarrow \quad \frac{2+v}{1-v^2}\,dv = \frac{dx}{x-1}.$$

Integrating

$$\Rightarrow \quad \ln\left(\frac{1+v}{1-v}\right) - \frac{1}{2}\ln\left(1-v^2\right) = \ln(x-1) + \ln c.$$

$$\Rightarrow \quad \ln\left(\frac{x+y-2}{x-y}\cdot\frac{1}{\sqrt{(x-1)^2-(y-1)^2}}\right) = \ln\left(c(x-1)\right).$$

$$\Rightarrow \quad \frac{(x+y-2)(x-1)}{(x-y)\sqrt{(x-y)(x+y-2)}} = c(x-1).$$

$$\Rightarrow \quad (x+y-2) = c(x-y)^3.$$

5. $$\frac{dy}{dx} = \frac{2(y+2)^2}{(x+y-1)^2}$$

$$y+2 = v(x-3).$$

$$\frac{dy}{dx} = (x-3)\frac{dv}{dx} + v.$$

$$\therefore \quad (x-3)\frac{dv}{dx} + v = \frac{2v^2}{(1+v)^2}.$$

$$(x-3)\frac{dv}{dx} = \frac{2v^2}{(1+v)^2} - v = \frac{2v^2 - v - 2v^2 - v^3}{(1+v)^2}.$$

$$\Rightarrow \quad (x-3)\frac{dv}{dx} = -\frac{(v+v^3)}{(1+v)^2}$$

$$\Rightarrow \quad \frac{(1+v)^2}{v(1+v^2)}\,dv = -\frac{dx}{(x-3)}.$$

$$\Rightarrow \quad \left(\frac{1}{v} + \frac{2}{1+v^2}\right)dv = \frac{-dx}{(x-3)}.$$

$$\Rightarrow \quad \ln v + 2\tan^{-1}(v) = -\ln(x-3) - \ln c.$$

$$\Rightarrow \quad \ln c.(y+2) + 2\tan^{-1}\left(\frac{y+2}{(x-3}\right) =$$

$$\Rightarrow \quad c(y+2) = e^{-2\tan^{-1}\left(\frac{y+2}{x-3}\right)}$$

E(3)

1. $$\frac{dy}{dx}+\frac{x}{1+x^2}y=\frac{1}{2x(1+x^2)}$$

Solⁿ. $$\frac{dy}{dx}+\frac{x}{1+x^2}y=\frac{1}{2x(1+x^2)}.$$

$$IF = e^{\int\frac{x}{1+x^2}dx} = e^{\frac{1}{2}\ln(x^2+1)} = \sqrt{x^2+1}$$

$$y\cdot\sqrt{x^2+1} = \int\frac{dx}{2x\sqrt{x^2+1}} = \frac{1}{2}\ln\left|\tan\left(\frac{1}{2}\tan^{-1}x\right)\right|+c$$

or $$y\sqrt{x^2+1} = \frac{1}{2}\ln\left(\frac{\sqrt{1+x^2}-1}{x}\right)+c.$$

2. $(1 - x^2)\dfrac{dy}{dx} + 2xy = x(1 - x^2)^{1/2}$

$$\frac{dy}{dx} + \left(\frac{2x}{1-x^2}\right) y = \frac{x}{\sqrt{1-x^2}}$$

IF. $\dfrac{1}{1-x^2}$.

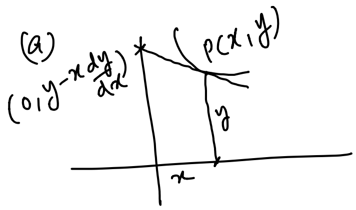
$$\therefore \frac{y}{1-x^2} = \int \frac{x}{(1-x^2)^{3/2}}\,dx = \frac{-1}{\sqrt{1-x^2}} + c$$

$$\Rightarrow y = c(1-x^2) + \sqrt{1-x^2}$$

3. (a) Find the curve such that the area of the trapezium formed by the co-ordinate axes, ordinate of an arbitrary point & the tangent at this point equals half the square of its abscissa.

(b) A curve in the first quadrant is such that the area of the triangle formed in the first quadrant by the x-axis, a tangent to the curve at any of its point P and radius vector of the point P is 2sq. units. If the curve passes through (2,1), find the equation of the curve.

(a)



Ar. of trapezium

$$= \frac{1}{2}\left(y - x\frac{dy}{dx} + y\right)x = \pm\frac{x^2}{2}$$

$$\Rightarrow \quad 2y - x\frac{dy}{dx} = \pm x$$

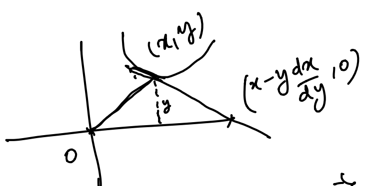
$$\Rightarrow \quad \frac{dy}{dx} - \frac{2y}{x} = \pm 1$$

$$IF = 1/x^2$$

$$\therefore \quad \frac{y}{x^2} = -\int \frac{1}{x^2}\,dx = \pm\frac{1}{x} + c$$

$$\Rightarrow \quad y = cx^2 \pm x$$

(b)



$$\therefore \frac{1}{2}\left(x - y\frac{dx}{dy}\right)x \cdot y = 2$$

$$\Rightarrow \quad xy - y^2\frac{dx}{dy} = 4$$

$$\Rightarrow \quad \frac{dx}{dy} - \frac{x}{y} = -\frac{4}{y^2}$$

$$\frac{x}{y} = -\int \frac{4}{y^3}\,dy \quad \Rightarrow \quad \frac{x}{y} = \frac{2}{y^2} + c$$

$$\Rightarrow \quad xy = 2 + cy^2$$

$x = 2$, $y = 1$ $\Rightarrow$ $c = 0$.

$$\therefore \quad xy = 2$$

4. $x(x-1)\dfrac{dy}{dx}-(x-2)y=x^3(2x-1)$

$$\frac{dy}{dx}-\frac{(x-2)}{x(x-1)}y=\frac{x^3(2x-1)}{x(x-1)}$$

$$\Rightarrow \quad IF = e^{-\int \frac{x-2}{x(x-1)}dx} = e^{-\int\left(\frac{-1}{x-1}+\frac{2}{x}\right)dx} = \frac{x-1}{x^2}$$

$$y\left(\frac{x-1}{x^2}\right)=\int \frac{x^3(2x-1)}{x(x-1)}\times\frac{x-1}{x^2}\,dx$$

$$=\int(2x-1)\,dx = x^2-x+c$$

$$\Rightarrow \quad y(x-1)=x^2\left(x^2-x+c\right)$$

5. $\sin x \dfrac{dy}{dx} + 3y = \cos x$

Sol^n.

$$\frac{dy}{dx} + 3\operatorname{cosec} x\, y = \cot x.$$

$$IF = e^{3\int \operatorname{cosec} x\, dx} = \tan^3 \frac{x}{2}.$$

$$y \tan^3 \frac{x}{2} = \int \cot x . \tan^3 \frac{x}{2} . dx = \int \frac{1-\tan^2 x/2}{2\tan x/2} . \tan^3 x/2\, dx.$$

$$= \frac{1}{2}\int (\tan^2 x/2 - \tan^4 x/2)\, dx =$$

$$= \frac{1}{2}\int (2\tan^2 x/2 - \tan^2 x/2 \sec^2 x/2)\, dx$$

$$= \frac{1}{2}\left(4\tan x/2 - 2x. - \frac{\tan^3 x.}{3}\right) + c$$

$$\Rightarrow \left(y + \frac{1}{3}\right)\tan^3 x/2 = 2\tan x/2 - x + c$$

6. $x(x^2+1)\dfrac{dy}{dx} = y(1-x^2) + x^3 . \ln x$

Soln. $\dfrac{dy}{dx} + y\dfrac{(x^2-1)}{x(x^2+1)} = \dfrac{x^2 \ln x}{x^2+1}$

$$\int \frac{x^2-1}{x(x^2+1)}\,dx = \int\left(\frac{2x}{x^2+1} - \frac{1}{x}\right)dx$$

$$= \ln\left(\frac{x^2+1}{x}\right)$$

$\therefore$ IF $= \left(x+\frac{1}{x}\right) = \dfrac{x^2+1}{x}$

$$\therefore\ y\cdot\left(x+\frac{1}{x}\right) = \int \frac{x^2 \ln x}{x^2+1} \times \frac{x^2+1}{x}\,dx = \int x\ln x\,dx$$

$$= \frac{x^2 \ln x}{2} - \frac{x^2}{4} + c$$

$$\Rightarrow\ 4(x^2+1)y = x^3(2\ln x - 1) + cx$$

7. $x \dfrac{dy}{dx} - y = 2x^2 \operatorname{cosec} 2x$

$$x\frac{dy}{dx} - y = 2x^2 \operatorname{cosec} 2x$$

$$\Rightarrow \frac{dy}{dx} - \frac{y}{x} = 2x \operatorname{cosec} 2x.$$

$$y \cdot \frac{1}{x} = \int 2 \operatorname{cosec} 2x \, dx = \ln \tan x + c.$$

$$\Rightarrow y = cx + x \ln \tan x$$

8. $(1 + y^2)\, dx = (\tan^{-1} y - x)dy$

$$\frac{dx}{dy} + \frac{x}{1+y^2} = \frac{\tan^{-1}y}{1+y^2}$$

$$IF = e^{\tan^{-1}y}$$

$$x.e^{\tan^{-1}y} = \int e^{\tan^{-1}y} \frac{\tan^{-1}y}{1+y^2}\, dy = e^{\tan^{-1}y}(\tan^{-1}y - 1) + c$$

$$\Rightarrow x = ce^{-\tan^{-1}y} + \tan^{-1}y - 1$$

9. Find the differentiable function which satisfies the equation $f(x) = -\int_0^x f(t)\tan t\,dt + \int_0^x \tan(t-x)\,dt$ where $x \in (-\pi/2, \pi/2)$

$$f(x) = -\int_0^x f(t)\tan t\,dt + \int_0^x \tan(t-x)\,dt = -\int_0^x f(t)\tan t\,dt + \ln\sec(t-x)\Big]_0^x$$

$$f'(x) = -f(x)\tan x - \tan x \Rightarrow f'(x) + f(x)\tan x = -\tan x$$

$$IF = \sec x$$

$$\therefore \quad f(x)\cdot\sec x = -\sec x + C$$

$$f(0) = 0 \Rightarrow c = 1 \Rightarrow f(x) = \cos x - 1$$

10. $y - x\,Dy = b(1 + x^2 Dy)$

Soln.

$$y - x\frac{dy}{dx} = b\left(1 + x^2\frac{dy}{dx}\right)$$

$$\Rightarrow (x + bx^2)\frac{dy}{dx} = y - b$$

$$\Rightarrow \frac{dy}{dx} - \frac{y}{x+bx^2} = \frac{-b}{x+bx^2}$$

$$IF = e^{-\int\left(\frac{1}{x} - \frac{b}{bx+1}\right)dx} = \left(\frac{bx+1}{x}\right)$$

$$y\cdot\left(\frac{bx+1}{x}\right) = \int\left(\frac{-b}{x(bx+1)} \times \frac{bx+1}{x}\right)dx$$

$$= \frac{b}{x} + c$$

$$\Rightarrow y(bx+1) = b + cx$$

11. $2\frac{dy}{dx} - y\sec x = y^3 \tan x$

Solⁿ, $-\frac{2}{y^3}\frac{dy}{dx} + \frac{\sec x}{y^2} = -\tan x$

Let $\frac{1}{y^2} = t \Rightarrow -\frac{2}{y^3}\frac{dy}{dx} = \frac{dt}{dx}$

$\Rightarrow \frac{dt}{dx} + t(\sec x) = -\tan x$

$IF = e^{\int \sec x\, dx} = \sec x + \tan x$

$\therefore t\cdot(\sec x + \tan x) = -\int (\sec x + \tan x)\tan x\, dx$

$= -\int (\sec x \tan x + (\sec^2 x - 1))\, dx$

$= -(\sec x + \tan x) + x + c$

$\Rightarrow t = -1 + \frac{(x+c)}{\sec x + \tan x}$

$\Rightarrow \frac{1}{y^2} = -1 + (x+c)\cot\left(\frac{x}{2} + \frac{\pi}{4}\right)$

12. $x^2 y - x^3 \dfrac{dy}{dx} = y^4 \cos x$

Soln.

$$-\frac{1}{y^4}\frac{dy}{dx} + \frac{1}{xy^3} = \frac{\cos x}{x^3}$$

Let $\dfrac{1}{y^3} = z \Rightarrow -\dfrac{1}{y^4}\dfrac{dy}{dx} = \dfrac{1}{3}\dfrac{dz}{dx}.$

$$\Rightarrow \frac{1}{3}\frac{dz}{dx} + \frac{z}{x} = \frac{\cos x}{x^3}.$$

$$\Rightarrow \frac{dz}{dx} + \left(\frac{3}{x}\right) z = \frac{3\cos x}{x^3}.$$

$$IF = e^{\int \frac{3}{x} dx} = x^3.$$

$$\therefore \quad z \cdot x^3 = \int 3\cos x \, dx = 3\sin x + c.$$

$$\Rightarrow \frac{x^3}{y^3} = 3\sin x + c.$$

13. $y(2xy + e^x)\,dx - e^x\,dy = 0$

$$y(2xy + e^{x})\,dx - e^x dy = 0.$$

$$\frac{dy}{dx} = \frac{2xy^2 + ye^x}{e^x} \Rightarrow \frac{dy}{dx} - y = \frac{2xy^2}{e^x}.$$

$$\frac{1}{y^2}\frac{dy}{dx} - \frac{1}{y} = 2xe^{-x}.$$

$$-\frac{1}{y} = z \Rightarrow \frac{1}{y^2}\frac{dy}{dx} = \frac{dz}{dx}.$$

$$\Rightarrow \quad \frac{dz}{dx} + z = 2xe^{-x}.$$

$$IF = e^x.$$

$$z \cdot e^x = \int 2x\,dx \Rightarrow ze^x = x^2 - c.$$

$$\therefore \; -\frac{e^x}{y} = x^2 - c \Rightarrow e^x y^{-1} = c - x^2$$

(E-4)

(GENERAL – CHANGE OF VARIABLE BY A SUITABLE SUBSTITUTION)

1. $(x - y^2)\,dx + 2xy\,dy = 0$

$x\,dx - y^2 dx + 2xy\,dy = 0.$

$\Rightarrow \dfrac{x\,dx}{x^2} + \dfrac{2xy\,dy - y^2 dx}{x^2} = 0.$

$\Rightarrow \dfrac{dx}{x} + d\left(y^2/x\right). = 0$

$\Rightarrow \ln x + y^2/x + \ln c = 0.$ $\Rightarrow$. $y^2 + x \ln cx = 0.$

2. $(x^3 + y^2 + 2)\,dx + 2y\,dy = 0$

Sol^n: $(x^3+y^2+2)\,dx + 2y\,dy = 0 \Rightarrow 2y\frac{dy}{dx} + y^2 = -(x^3+2)$

$y^2 = t \quad \Rightarrow \quad \frac{dt}{dx} + t = -(x^3+2)$

$\therefore \quad t \cdot e^x = -\int (x^3+2)\, e^x\, dx$

$= -e^x(x^3+2) + \int (3x^2) e^x\, dx$

$= -e^x(x^3+2) + 3x^2 e^x - \int 6x e^x\, dx$

$= -e^x(x^3+2) + 3x^2 e^x - 6x e^x + 6e^x + c$

$y^2 e^x = e^x(-x^3+3x^2-6x+4) + c$

$\Rightarrow \quad y^2 = 3x^2 - 6x + 4 - x^3 + ce^{-x}$

3. $x\frac{dy}{dx} + y \ln y = xye^x$

(3) $\frac{1}{y}\frac{dy}{dx} + \frac{\ln y}{x} = e^x$

$\ln y = t \Rightarrow \frac{1}{y}\frac{dy}{dx} = \frac{dt}{dx}$

$\frac{dt}{dx} + \frac{t}{x} = e^x \Rightarrow tx = \int xe^x dx$

$\Rightarrow tx = xe^x - e^x + c \Rightarrow x\ln y = xe^x - e^x + c$

4. $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)\, e^x \sec y$

$$\frac{dy}{dx} - \frac{\tan y}{x+1} = (x+1)e^x \sec y.$$

$$\Rightarrow \quad \cos y \cdot \frac{dy}{dx} - \frac{\sin y}{(x+1)} = (x+1)e^x.$$

$$\sin y = z \Rightarrow \cos y \frac{dy}{dx} = \frac{dz}{dx}.$$

$$\Rightarrow \quad \frac{dz}{dx} - \frac{z}{(x+1)} = (x+1)e^x.$$

$$\text{IF} = e^{-\int \frac{1}{(x+1)} dx} = \frac{1}{(x+1)}$$

$$\therefore \quad \frac{z}{(x+1)} = \int e^x dx = e^x + c.$$

$$\Rightarrow \quad \sin y = (x+1)(e^x + c).$$

5. $\frac{dy}{dx} = \frac{e^{y}}{x^{2}} - \frac{1}{x}$

$$e^{-y}\frac{dy}{dx} + \frac{e^{-y}}{x} = \frac{1}{x^2}$$

$$e^{-y} = t \Rightarrow -e^{-y}\frac{dy}{dx} = \frac{dt}{dx}$$

$$-\frac{dt}{dx} + \frac{t}{x} = \frac{1}{x^2}.$$

$$\Rightarrow \quad \frac{dt}{dx} - \frac{t}{x} = \frac{-1}{x^2}.$$

$$\Rightarrow \quad \frac{t}{x} + c = \frac{1}{2x^2}$$

$$\Rightarrow \quad 2cx^2 + 2xe^{-y} = 1.$$

$$\Rightarrow \quad c\cdot x^2 + 2xe^{-y} = 1.$$

6. $\dfrac{dy}{dx} = \dfrac{y^2 - x}{2y(x+1)}$

$$\frac{dy}{dx} = \frac{y^2 - x}{2y(x+1)}. \Rightarrow 2y\frac{dy}{dx} - \frac{y^2}{(x+1)} = \frac{-x}{(x+1)}.$$

$$\Rightarrow \frac{2y}{(x+1)}\frac{dy}{dx} - \frac{y^2}{(x+1)^2} = \frac{-x}{(x+1)^2}. \Rightarrow \frac{d}{dx}\left(\frac{y^2}{(x+1)}\right) = -\frac{x}{(x+1)^2}$$

$$\Rightarrow \frac{y^2}{(x+1)} = \int -\frac{x}{(x+1)^2}\,dx = -\ln(x+1) - \frac{1}{(x+1)} + \ln c.$$

$$\Rightarrow y^2 = (x+1)\ln\left(\frac{c}{x+1}\right) - 1.$$

7. $\frac{dy}{dx} = e^{x-y}(e^x - e^y)$

Sol^n.

$$e^y \frac{dy}{dx} = e^x \cdot e^x - e^x e^y$$

$$\Rightarrow \quad e^y \frac{dy}{dx} + e^x \cdot e^y = (e^x)^2$$

Let $e^y = z \Rightarrow e^y \frac{dy}{dx} = \frac{dz}{dx}$.

$$\Rightarrow \quad \frac{dz}{dx} + e^x \cdot z = (e^x)^2$$

$$IF = e^{\int e^x dx} = e^{e^x}$$

$$\Rightarrow \quad z \cdot e^{e^x} = \int e^{e^x} \cdot e^x \cdot e^x \, dx$$

$$e^x = t \Rightarrow \int e^t \cdot t \, dt$$

$$= te^t - e^t + c$$

$$z \cdot e^{e^x} = e^x \cdot e^{e^x} - e^{e^x} + c$$

$$\Rightarrow \quad e^y = c \cdot e^{-e^x} + e^x - 1$$

8. $(x^2+y)\frac{dy}{dx}=6x$

$(x^2+y)\frac{dy}{dx}=6x \Rightarrow 2x\frac{dx}{dy}-\frac{x^2}{3}=\frac{y}{3}$

$x^2=t \quad \Rightarrow \quad IF \quad e^{-y/3}$

$x^2 \cdot e^{-y/3}=\int \frac{y}{3}e^{-y/3}dy = -ye^{-y/3}-3e^{-y/3}+$

$\Rightarrow \quad (x^2+y+3)e^{-y/3}=c$

$\Rightarrow \quad ce^{y/3}=(x^2+y+3)$

$\Rightarrow \quad y=3\ln(x^2+y+3)+c$

Exercise S2

1. $\frac{ydx - xdy}{(x-y)^2} = \frac{dx}{2\sqrt{1-x^2}}$, given that y = 2 when x = 1.

$$\frac{(ydx - xdy)}{y^2\left(\frac{x}{y}-1\right)^2} = \frac{dx}{2\sqrt{1-x^2}}$$

$$\int \frac{d(x/y)}{\left(\frac{x}{y}-1\right)^2} = \int \frac{dx}{2\sqrt{1-x^2}}$$

$$\frac{-1}{\left(\frac{x}{y}-1\right)} = \frac{1}{2}\sin^{-1}x + C$$

$$\frac{-y}{x-y} = \frac{1}{2}\sin^{-1}x + C$$

For c: put $x=1$, $y=2$

$$\frac{-2}{-1} = \frac{1}{2}\cdot\frac{\pi}{2} + c \Rightarrow 2 - \frac{\pi}{4} = c$$

$$\therefore \frac{y}{y-x} = \frac{1}{2}\sin^{-1}x + 2 - \frac{\pi}{4}$$

= Ans

2. Find the integral curve of the differential equation, $x\,(1 - x\,\ell n\,y)\,.\dfrac{dy}{dx} + y = 0$ which passes through $\left(1, \dfrac{1}{e}\right)$.

$$x(1 - x\ln y)\frac{dy}{dx} + y = 0$$

$$x - x^2 \ln y + y\frac{dx}{dy} = 0$$

$$x\,dy + y\,dx = x^2 \ln y\, dy$$

$$\int \frac{d(xy)}{x^2y^2} = \int \frac{\ln y}{y^2}\,dy$$

$$\frac{-1}{xy} = \left[\ln y\left(-\frac{1}{y}\right) - \int \frac{1}{y}\left(-\frac{1}{y}\right)dy\right]$$

$$\frac{-1}{xy} = -\frac{1}{y}\ln y - \frac{1}{y} + c$$

$$-1 = -x\ln y - x + cxy$$

since it passes through $(1, 1/e)$:

$$-1 = +1 - 1 + \frac{c}{e} \Rightarrow c = -e$$

$$\therefore \quad \boxed{1 = x\ln y + x + exy}$$

Ans.

$$\pm 1 = \frac{\frac{dy}{dx} - \frac{x}{y}}{1 + \frac{x}{y}\frac{dy}{dx}}$$

$$\pm 1 \pm \frac{x}{y}\frac{dy}{dx} = \frac{dy}{dx} - \frac{x}{y}$$

$$1 + \frac{x}{y}\frac{dy}{dx} = \frac{dy}{dx} - \frac{x}{y}$$

$$1 + \frac{x}{y} = \frac{dy}{dx}\left(1 - \frac{x}{y}\right)$$

$$\frac{dy}{dx} = \frac{y+x}{y-x}$$

$$-1 - \frac{x}{y}\frac{dy}{dx} = \frac{dy}{dx} - \frac{x}{y}$$

$$\frac{x}{y} - 1 = \frac{dy}{dx}\left(1 + \frac{x}{y}\right)$$

$$\frac{dy}{dx} = \frac{x-y}{x+y}$$

Let $y = vx$

$$\frac{dy}{dx} = v + x\frac{dv}{dx}$$

$$v + x\frac{dv}{dx} = \frac{v+1}{v-1}$$

$$x\frac{dv}{dx} = \frac{v+1-v^2+v}{v-1}$$

$$\frac{(v-1)\,dv}{-v^2+2v+1} = \frac{1}{x}\,dx \;,$$

$$-\frac{1}{2}\int \frac{(-2v+2)\,dv}{-v^2+2v+1} = \int \frac{1}{x}\,dx \;,$$

$$-\frac{1}{2}\ln(-v^2+2v+1) = \ln cx$$

$$v + x\frac{dv}{dx} = \frac{1-v}{1+v}$$

$$x\frac{dv}{dx} = \frac{1-v-v-v^2}{1+v}$$

$$-\frac{1}{2}\int \frac{-2(1+v)\,dv}{-v^2-2v+1} = \int \frac{1}{x}\,dx$$

$$-\frac{1}{2}\ln(-v^2-2v+1) = \ln cx$$

$$\ln\left((cx)^2(-v^2-2v+1)\right) = 0$$

$$c^2\left(x^2\,\frac{(-y^2-2xy+x^2)}{x^2}\right) = 1$$

$$\ln(cx)^2 + \ln\left(\frac{-y^2+2xy+x^2}{x^2}\right) = 0$$

$$\ln\left(c^2(-y^2+2xy+x^2)\right) = 0$$

$$\therefore \boxed{c^2(x^2+2xy-y^2) = 1}$$

Ans

$$\boxed{c^2(x^2-2xy-y^2) = 1}$$

Ans

3. Let $f(x, y, c_1) = 0$ and $f(x, y, c_2) = 0$ define two integral curves of a homogeneous first order differential equation. If P_1 and P_2 are respectively the points of intersection of these curves with an arbitrary line, $y = mx$ then prove that the slopes of these two curves at P_1 and P_2 are equal.

Let $P_1 \equiv (x_1 y_1)$, $P_2 \equiv (x_2 y_2)$

differential eqn of both given curve is :

$$\frac{dy}{dx} = \phi(y/x)$$

at P_1 : $\frac{dy}{dx} = \phi\left(\frac{y_1}{x_1}\right) = \phi(m)$ $\{ y_1 = mx_1 , y_2 = mx_2 \}$

at P_2 : $\frac{dy}{dx} = \phi\left(\frac{y_2}{x_2}\right) = \phi(m)$

$\therefore$ $\frac{dy}{dx}$ of both curves at P_1 and P_2 are equal.

4. Let f(x) be a differentiable function and satisfy $f(0) = 2$, $f'(0) = 3$ and $f''(x) = f(x)$. Find

(a) the range of the function f(x)

(b) the value of the function when $x = \ln 2$

(c) the area enclosed by $y = f(x)$ in the 2nd quadrant

$$f''(x) = f(x)$$

$$f''(x)\, f'(x) = f'(x)\, f(x)$$

$$\int f'(x)\, f''(x)\, dx = \int f(x)\, f'(x)\, dx$$

$$\frac{(f'(x))^2}{2} = \frac{(f(x))^2}{2} + c$$

$$(f'(x))^2 = f^2(x) + 2c$$

put $x=0$, $(f'(0))^2 = f^2(0) + 2c \Rightarrow c = \frac{9-4}{2} = \frac{5}{2}$

$\therefore\ f'(x)^2 = f^2(x) + 5$

$\therefore\ f'(x) = \sqrt{f^2(x)+5}$

$$\int \frac{f'(x)}{\sqrt{f^2(x)+5}}\, dx = \int dx$$

$$\ln\left(f(x) + \sqrt{f^2(x)+5}\right) = x + c$$

put $x=0$, $\ln(2+3) = 0 + c \Rightarrow \boxed{c = \ln 5}$

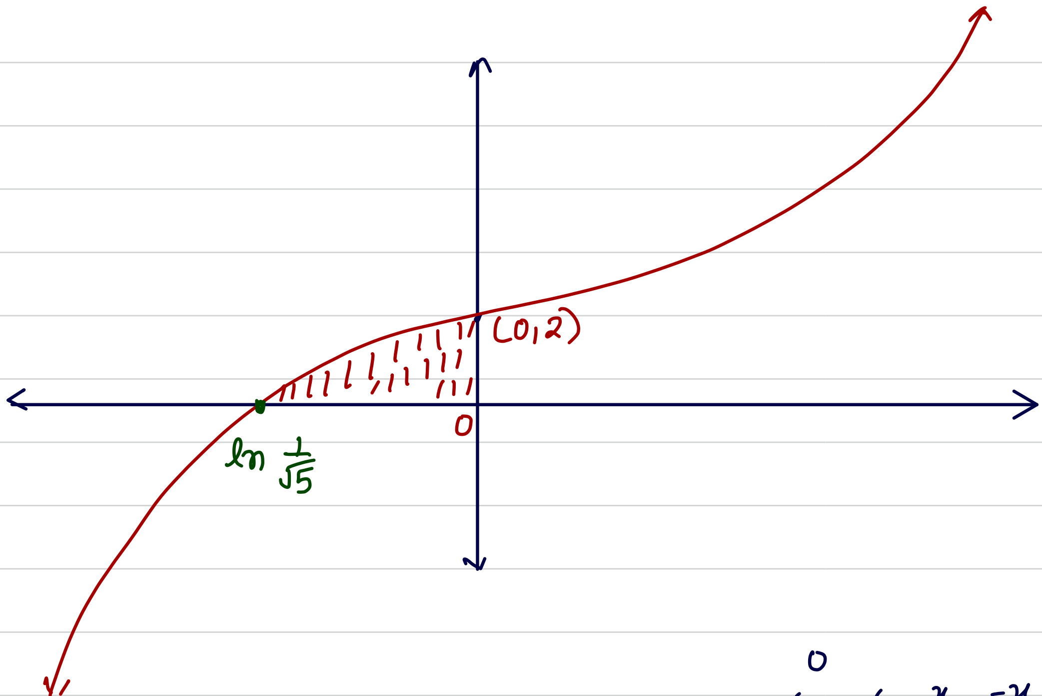
$\therefore\ f(x) + \sqrt{f^2(x)+5} = 5e^{x}$,

(Rationalization)

$\therefore\ f(x) - \sqrt{f^2(x)+5} = -e^{-x}$

$\therefore\ 2f(x) = 5e^{x} - e^{-x} \Rightarrow \boxed{f(x) = \frac{5e^{x} - e^{-x}}{2}}$

$\therefore \quad f(\ln 2) = \dfrac{5e^{\ln 2} - e^{-\ln 2}}{2} \Rightarrow \boxed{f(\ln 2) = \dfrac{19}{4}}$



Area in second quadrant $= \displaystyle\int_{\ln \frac{1}{\sqrt{5}}}^{0} \left(\frac{5e^{x} - e^{-x}}{2}\right) dx$

$= \dfrac{1}{2}\left[5e^{x} + e^{-x}\right]_{\ln \frac{1}{\sqrt{5}}}^{0}$

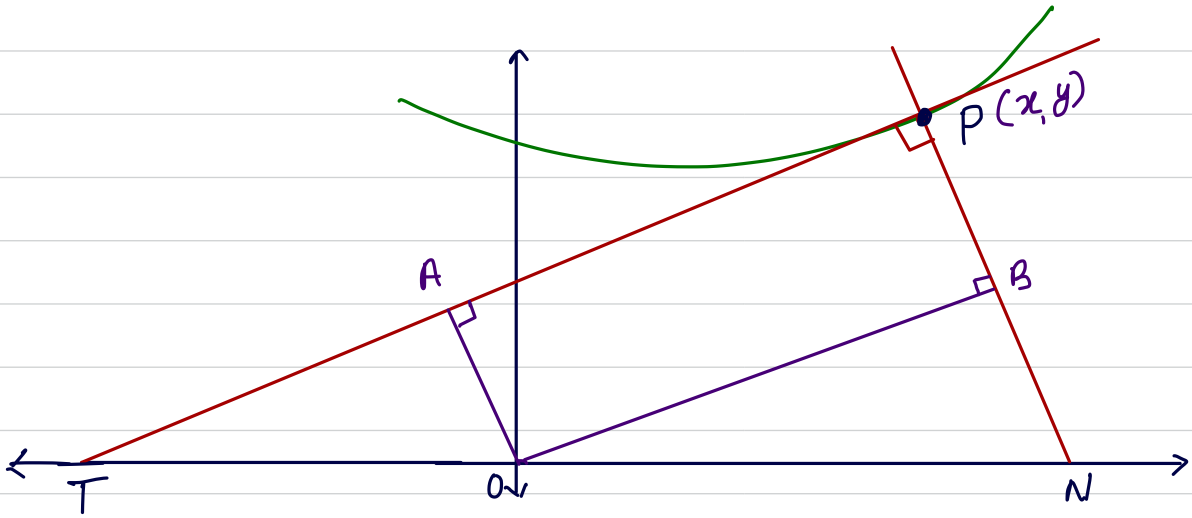
$= \dfrac{1}{2}\left[6 - \left(5 \cdot \dfrac{1}{\sqrt{5}} + \sqrt{5}\right)\right]$

$= \underline{\underline{3 - \sqrt{5}}}$ sq. unit

Ans.

Range of $f(x) \in (-\infty, \infty)$

5. Show that the curve such that the distance between the origin and the tangent at an arbitrary point is equal to the distance between the origin and the normal at the same point, $\sqrt{x^2+y^2} = c\,e^{\pm \tan^{-1}\frac{y}{x}}$.



eqⁿ of tangent at $P(x,y)$:

$$Y - y = \frac{dy}{dx}(X - x)$$

$$X\frac{dy}{dx} - Y - x\frac{dy}{dx} + y = 0$$

Perpendicular distance from Origin :

$$= \left| \frac{y - x\frac{dy}{dx}}{\sqrt{\left(\frac{dy}{dx}\right)^2 + 1}} \right| \quad \text{— (1)}$$

eqⁿ of Normal at $P(x,y)$

$$Y - y = -\frac{dx}{dy}(X - x)$$

$$\therefore \quad -X\frac{dx}{dy} - Y + y + x\frac{dx}{dy} = 0$$

Perpendicular distance from Origin

$$= \left| \frac{y + x\frac{dx}{dy}}{\sqrt{(dx/dy)^2 + 1}} \right| \quad \text{— (2)}$$

from ① and ② :

$$\frac{\left|y - x\dfrac{dy}{dx}\right|}{\sqrt{1+\left(\dfrac{dy}{dx}\right)^2}} = \frac{\left|y + x\dfrac{dx}{dy}\right|}{\sqrt{1+\left(\dfrac{dx}{dy}\right)^2}}$$

$$\frac{\left|y - x\,dy/dx\right|}{\sqrt{1+\left(dy/dx\right)^2}} = \frac{\left|y\,dy/dx + x\right|}{\sqrt{1+\left(dy/dx\right)^2}}$$

$$\therefore \quad y - x\frac{dy}{dx} = \pm\left(y\frac{dy}{dx} + x\right)$$

$$y - x\frac{dy}{dx} = y\frac{dy}{dx} + x$$

$$\frac{dy}{dx} = \frac{y-x}{y+x}$$

Let $y = vx$, $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$$v + x\frac{dv}{dx} = \frac{v-1}{v+1}$$

$$x\frac{dv}{dx} = \frac{v-1-v^2-v}{v+1}$$

$$\frac{v+1}{1+v^2}\,dv = \frac{-1}{x}\,dx$$

$$y - x\frac{dy}{dx} = -y\frac{dy}{dx} - x$$

$$\frac{dy}{dx} = \frac{y+x}{x-y}$$

$$\therefore \quad v + x\frac{dv}{dx} = \frac{v+1}{1-v}$$

$$x\frac{dv}{dx} = \frac{v+1-v+v^2}{1-v}$$

$$\frac{1-v}{1+v^2}\,dv = \frac{1}{x}\,dx$$

$$\frac{1}{2}\ln(1+v^2) + \tan^{-1}v = -\ln x + \ln c$$

$$\ln\left(1+\frac{y^2}{x^2}\right)^{1/2} + \tan^{-1} y/x = -\ln x + \ln c$$

$$\ln\sqrt{x^2+y^2} + \tan^{-1} y/x = \ln c$$

$$\tan^{-1}(y/x) = \ln\left(\frac{c}{\sqrt{x^2+y^2}}\right)$$

$$\boxed{\sqrt{x^2+y^2} = c\cdot e^{-\tan^{-1}(y/x)}}$$

Ans

$$\tan^{-1}v - \frac{1}{2}\ln(1+v^2) = \ln x + \ln c$$

$$\tan^{-1}(y/x) = \ln\left(\sqrt{x^2+y^2}\, c\right)$$

$$\boxed{c\sqrt{x^2+y^2} = e^{\tan^{-1}(y/x)}}$$

Ans

6. Find the curve possessing the property that the intercept, the tangent at any point of a curve cuts off on the y-axis is equal to the square of the abscissa of the point of tangency.

eqⁿ of tangent at Point $P(x,y)$:

$$Y - y = \frac{dy}{dx}(X - x)$$

For y-intercept, put $X = 0$ $\Rightarrow$ $Y = y - x\dfrac{dy}{dx}$

$$\therefore \quad y_{int} = x^2$$

$$\therefore \quad y - x\frac{dy}{dx} = x^2$$

$$\frac{-y\,dx + x\,dy}{x^2} = -dx$$

$$\int d(y/x) = -\int dx$$

$$\frac{y}{x} = -x + c$$

$$y = -x^2 + cx$$

= Ans.

Exercise JM

EXERCISE (JM)

1. Solution of the differential equation cos x dy = y(sin x – y)dx, $0 < x < \frac{\pi}{2}$ is :**[AIEEE-2010]**

(1) sec x = (tan x + c) y
(2) y sec x = tan x + c
(3) y tan x = sec x + c
(4) tan x = (sec x + c) y

Soln:- $\frac{dy}{dx} = y\tan x - y^2\sec x$

$\Rightarrow \frac{1}{y^2}\frac{dy}{dx} = \frac{\tan x}{y} - \sec x$

Let $-\frac{1}{y} = t \Rightarrow \frac{1}{y^2}\frac{dy}{dx} = \frac{dt}{dx}$

$\frac{dt}{dx} + t\cdot\tan x = -\sec x$

Linear DE in $t \Rightarrow IF = e^{\int \tan x\,dx} = \sec x$

$t\cdot\sec x = -\int \sec^2 x\,dx \Rightarrow -\frac{1}{y}\sec x = -\tan x - c$

$\sec x = y\tan x + c\cdot y$

$\sec x = y(\tan x + c)$

option:- (1)

EXERCISE (JM)

2 If $\frac{dy}{dx} = y + 3 > 0$ and $y(0) = 2$, then $y(\ln 2)$ is equal to :- **[AIEEE-2011]**

(1) 13 (2) –2 (3) 7 (4) 5

$$\Rightarrow \int \frac{dy}{y+3} = \int dx \Rightarrow \ln(y+3) = x + C$$

put $x = 0, y = 2$

$$\boxed{\ln 5 = C}$$

$$\Rightarrow \ln(y+3) = x + \ln 5$$

put $x = \ln 2$

$$\ln(y+3) = \ln 2 + \ln 5$$

$$\Rightarrow y + 3 = 10 \Rightarrow \boxed{y = 7}$$

option(3)

3

Let I be the purchase value of an equipment and V(t) be the value after it has been used for t years. The value V(t) depreciates at a rate given by differential equation $\frac{dV(t)}{dt} = -k(T-t)$, where $k > 0$ is a constant and T is the total life in years of the equipment. Then the scrap value V(T) of the equipment is :- **[AIEEE-2011]**

(1) $I - \frac{k(T-t)^2}{2}$ (2) e^{-kT} (3) $T^2 - \frac{I}{k}$ (4) $I - \frac{kT^2}{2}$

Solⁿ

$$\frac{dV(t)}{dt} = -k(T-t)$$

$$\Rightarrow V(t) = \frac{k(T-t)^2}{2} + C$$

at $t=0$, $V(0) = I$

$$\Rightarrow I = \frac{kT^2}{2} + c \Rightarrow c = I - \frac{kT^2}{2}$$

$$\Rightarrow V(t) = \frac{k(T-t)^2}{2} + I - \frac{kT^2}{2}$$

$$\Rightarrow V(t) = \frac{k}{2}(t^2 - 2tT) + I$$

So at $t = T$

$$V(T) = I + \frac{k}{2}(T^2 - 2T^2)$$

$$\boxed{V(T) = I - \frac{k}{2}T^2}$$

→ Scrap value

option (4)

4. The curve that passes through the point (2, 3), and has the property that the segment of any tangent to it lying between the coordinate axes is bisected by the point of contact, is given by : **[AIEEE-2011]**

(1) $\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 2$ (2) $2y - 3x = 0$ (3) $y = \frac{6}{x}$ (4) $x^2 + y^2 = 13$

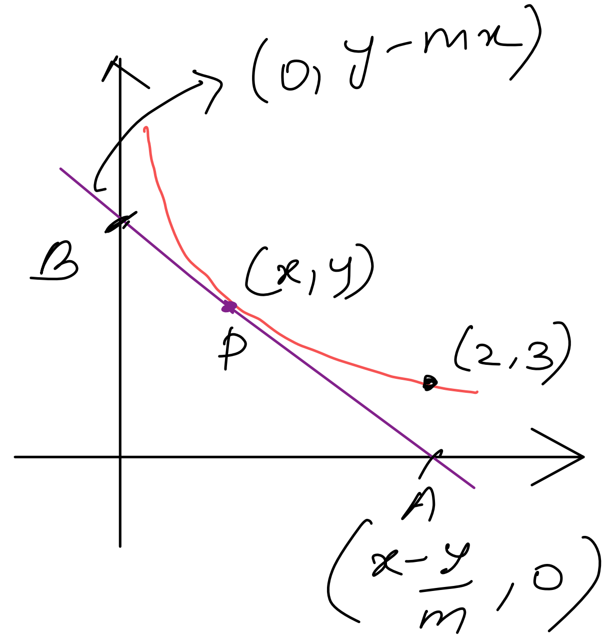
Soln

eqn of tangent

$Y - y = m(X - x)$

For A, $Y = 0$

$\therefore X = x - \frac{y}{m}$



for B, $X = 0$

$Y = y - mx$

Now $x - \frac{y}{m} = 2x \Rightarrow m = -\frac{y}{x}$

$\Rightarrow \frac{dy}{dx} = -y/x \Rightarrow \int \frac{dx}{x} + \int \frac{dy}{y} = \int 0$

$\Rightarrow \ln|x| + \ln|y| = \ln k \Rightarrow xy = \pm k$

it passes through $(2, 3) \Rightarrow \boxed{k = 6}$

Curve $\boxed{xy = 6} \Rightarrow y = 6/x$ Ans,

option (3)

(5) Consider the differential equation $y^2dx + \left(x - \frac{1}{y}\right)dy = 0$. If $y(1) = 1$, then x is given by : **[AIEEE-2011]**

(1) $1 - \frac{1}{y} + \frac{e^{\frac{1}{y}}}{e}$ (2) $4 - \frac{2}{y} - \frac{e^{\frac{1}{y}}}{e}$ (3) $3 - \frac{1}{y} + \frac{e^{\frac{1}{y}}}{e}$ (4) $1 + \frac{1}{y} - \frac{e^{\frac{1}{y}}}{e}$

Sol^n $y^2 dx = -\left(x - \frac{1}{y}\right)dy \Rightarrow \frac{dx}{dy} = \frac{-\left(x - \frac{1}{y}\right)}{y^2}$

$\Rightarrow \frac{dx}{dy} = -\left(\frac{x}{y^2}\right) + \frac{1}{y^3}$

$\Rightarrow \frac{dx}{dy} + \frac{1}{y^2}\cdot x = \frac{1}{y^3}$

$I.F = e^{\int \frac{1}{y^2} dy} = e^{-\frac{1}{y}}$

Solution of D.E.

$x\left(e^{-\frac{1}{y}}\right) = \underbrace{\int \left(e^{-\frac{1}{y}}\right)\left(\frac{1}{y^3}\right)dy}_{} + C$

↓ put $-\frac{1}{y} = t \Rightarrow \frac{1}{y^2}dy = dt$

$I = -\int t e^t \, dt = -\left\{te^t - \int e^t dt\right\}$

$= -te^t + e^t$

$= \frac{1}{y}e^{-1/y} + e^{-1/y}$

So D.E.

$x\left(e^{-1/y}\right) = \left(\frac{1}{y} + 1\right)e^{-1/y} + C$

at $x = 1$, $y = 1$

$(1)e^{-1} = (2)e^{-1} + C \Rightarrow \boxed{C = -e^{-1}}$

$$x\left(e^{-\frac{1}{y}}\right) = \left(\frac{1}{y}+1\right)e^{-1/y} - \frac{1}{e}$$

$$\Rightarrow \quad x = 1 + \frac{1}{y} - \left(\frac{1}{e}\right)\left(e^{1/y}\right)$$

$$\Rightarrow \quad x = 1 + \frac{1}{y} - \frac{e^{1/y}}{e}$$

option (4) Ans

6

The population p(t) at time t of a certain mouse species satisfies the differential equation $\frac{dp(t)}{dt} = 0.5\,p(t) - 450$. If $p(0) = 850$, then the time at which the population becomes zero is :

[AIEEE-2012]

(1) ln18 (2) 2 ln18 (3) ln9 (4) $\frac{1}{2}$ ln18

Soln

$$2\int \frac{dp}{900-p} = \int -dt$$

$$\Rightarrow \quad -2\ln|900-p| = -t + c$$

when $t = 0$, $p = 850 \Rightarrow c = -2\ln 50$

$$2\ln\left|\frac{50}{900-p}\right| = -t \Rightarrow |900-p| = 50e^{t/2}$$

$$\Rightarrow \quad 900 - p = 50e^{t/2} \Rightarrow p = 900 \pm 50^{t/2}$$

Since $P(0) = 850$,

$$\boxed{p = 900 - 50e^{t/2}}$$

{'+' sign's Case Rejected}

Now $p = 0 \Rightarrow 900 - 50e^{t/2}$

$$\Rightarrow \boxed{t = 2\ln 18}$$

option(2)

7 At present a firm is manufacturing 2000 items. It is estimated that the rate of change of production P w.r.t. additional number of workers x is given by $\frac{dP}{dx} = 100 - 12\sqrt{x}$. If the firm employs 25 more workers, then the new level of production of items is : **[JEE (Main)-2013]**

(1) 2500 (2) 3000 (3) 3500 (4) 4500

sol^n

$$\int_{2000}^{P} dP = \int_{0}^{25} (100 - 12\sqrt{x})\,dx$$

$$\Rightarrow P - 2000 = (25)(100) - \frac{12}{3/2}(25)^{3/2}$$

$$\Rightarrow \underline{P = 3500} \text{ Ans.}$$

option (3)

(8) If the surface area of a sphere of radius r is increasing uniformly at the rate 8cm^2/s, then the rate of change of its volume is : **[JEE-Main (On line)-2013]**

(1) proportional to r^2 (2) constant (3) proportional to r (4) proportional to $\sqrt{r}$

\Det.\Eng.p65

Sol^n $A = 4\pi r^2$, given $\frac{dA}{dt} = +8$ cm^2/sec.

$$\Rightarrow \frac{dA}{dt} = (8\pi r)\frac{dr}{dt} \Rightarrow 8 = 8\pi r \frac{dr}{dt}$$

$$\Rightarrow \boxed{\frac{dr}{dt} = \frac{1}{\pi r}}$$

Now $V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$

$$\Rightarrow \frac{dV}{dt} = (4\pi r^2)\left(\frac{1}{\pi r}\right) = \underline{4r}$$

Option (3)

9. Consider the differential equation $\frac{dy}{dx}=\frac{y^3}{2(xy^2-x^2)}$:

Statement 1 : The substitution $z = y^2$ transforms the above equation into a first order homogenous differential equation.

Statement 2 : The solution of this differential equation is $y^2e^{-\frac{y^2}{x}}=C$. **[JEE-Main (On line)-2013]**

(1) Statement 1 is false and statement 2 is true. (2) Both statements are true.

(3) Statement 1 is true and statement 2 is false. (4) Both statements are false.

Sol^n

$$2y\frac{dy}{dx}=\frac{y^4}{xy^2-x^2}$$

put $y^2=t \Rightarrow 2y\frac{dy}{dx}=\frac{dt}{dx}$

$$\frac{dt}{dx}=\frac{t^2}{xt-x^2} \Rightarrow \frac{dx}{dt}=\frac{xt-x^2}{t^2}$$

$$\Rightarrow \frac{dx}{dt}=\frac{x}{t}-\frac{x^2}{t^2} \Rightarrow \boxed{\frac{1}{x^2}\frac{dx}{dt}+\left(\frac{1}{t}\right)\left(\frac{-1}{x}\right)=-\frac{1}{t^2}}$$

put $-\frac{1}{x}=u \Rightarrow \frac{1}{x^2}\frac{dx}{dt}=\frac{du}{dt}$

$$\Rightarrow \frac{du}{dt}+\left(\frac{1}{t}\right)u=-\frac{1}{t^2}, \quad I.F=e^{\int\frac{1}{t}dt}=t$$

Solution of D.E.

$$u(t)=\int(t)\left(-\frac{1}{t^2}\right)dt+\ln C$$

$$u(t)=-\ln t+C \Rightarrow \frac{-y^2}{x}=-\ln(y^2)+\ln C$$

$$\Rightarrow e^{\frac{-y^2}{x}}=\frac{C}{y^2} \Rightarrow \boxed{C=y^2\cdot e^{-y^2/x}}$$

option (2) Ans.

10 If a curve passes through the point $\left(2, \frac{7}{2}\right)$ and has slope $\left(1 - \frac{1}{x^2}\right)$ at any point (x, y) on it, then the ordinate of the point on the curve whose abscissa is – 2 is : **[JEE-Main (On line)-2013]**

(1) $-\frac{5}{2}$ (2) $\frac{5}{2}$ (3) $-\frac{3}{2}$ (4) $\frac{3}{2}$

Sol^n $\frac{dy}{dx} = 1 - \frac{1}{x^2} \Rightarrow \int dy = \int\left(1 - \frac{1}{x^2}\right)dx$

$\Rightarrow y = x + \frac{1}{x} + C$

Curve passes through $\left(2, \frac{7}{2}\right)$

$\Rightarrow \frac{7}{2} = 2 + \frac{1}{2} + C \Rightarrow \boxed{C = 1}$

$\Rightarrow \boxed{y = x + \frac{1}{x} + 1}$

now $x = -2$

$y = -2 - \frac{1}{2} + 1 \Rightarrow \boxed{y = -3/2}$

option (3) Ans.

(11) The equation of the curve passing through the origin and satisfying the differential equation $(1 + x^2)\dfrac{dy}{dx} + 2xy = 4x^2$ is : **[JEE-Main (On line)-2013]**

(1) $(1 + x^2)\,y = x^3$ (2) $3(1 + x^2)\,y = 4x^3$ (3) $3(1+ x^2)\,y = 2x^3$ (4) $(1 + x^2)\,y = 3x^3$

Soln

$$\frac{dy}{dx} + \left(\frac{2x}{1+x^2}\right)y = \frac{4x^2}{1+x^2}$$

$$I.F. = e^{\int \frac{2x}{1+x^2}dx} = e^{\ln(1+x^2)} = 1+x^2$$

Solution of D.E.

$$y(1+x^2) = \int (1+x^2)\left(\frac{4x^2}{1+x^2}\right)dx + C$$

$$\Rightarrow y(1+x^2) = \int 4x^2\,dx + C$$

$$\Rightarrow y(1+x^2) = \frac{4x^3}{3} + C$$

curve passes through origin $\Rightarrow \boxed{C=0}$

$$\Rightarrow y(1+x^2) = \frac{4x^3}{3}$$

$$\Rightarrow \boxed{3(1+x^2)\,y = 4x^3}$$

option (2) Ans.

(12) Let the population of rabbits surviving at a time t be governed by the differential equation $\frac{dp(t)}{dt} = \frac{1}{2}p(t) - 200$. If p(0) = 100, then p(t) equals : **[JEE(Main)-2014]**

(1) $400 - 300\,e^{t/2}$ (2) $300 - 200\,e^{-t/2}$ (3) $600 - 500\,e^{t/2}$ (4) $400 - 300\,e^{-t/2}$

<u>Solⁿ</u>
$$\frac{dp}{dt} = \frac{p-400}{2} \Rightarrow \int \frac{dp}{p-400} = \int \frac{1}{2}dt$$

$$\Rightarrow \ln|p-400| = \frac{1}{2}t + c$$

at $t = 0$, $p = 100 \Rightarrow c = \ln 300$

now, $\ln\left|\frac{p-400}{300}\right| = \frac{1}{2}t \Rightarrow |p-400| = 300 \cdot e^{t/2}$

$$\Rightarrow p - 400 = \pm 300 e^{t/2} \Rightarrow p = 400 \pm 300 \cdot e^{t/2}$$

as at $t = 0$, $p = 100$ (eqⁿ with '+' sign rejected)

$$\Rightarrow p = 400 - 300 e^{t/2}$$

<u>Ans</u> <u>(1)</u>

(13) Let y(x) be the solution of the differential equation $(x \log x)\frac{dy}{dx} + y = 2x \log x$, $(x \geq 1)$.
Then y(e) is equal to : **[JEE(Main)-2015]**

(1) 2 (2) 2e (3) e (4) 0

Solⁿ at $x = 1$ $0 + y = 0 \Rightarrow \boxed{y = 0}$

Now, $(x \log x)\frac{dy}{dx} + y = 2x \log x$

$$\Rightarrow \frac{dy}{dx} + \frac{1}{x \ln x} y = 2$$

$$I.F. = e^{\int \frac{1}{x \ln x} dx} = e^{\ln(\ln x)} = \ln x$$

Solution of D.E.

$$y(\ln x) = \int (\ln x)(2)\,dx + C$$

$$\Rightarrow y(\ln x) = 2(x \ln x - x) + C$$

at $x = 1$, $y = 0$

$$\Rightarrow 0 = 2(-1) + C \Rightarrow \boxed{C = 2}$$

$$\Rightarrow y(\ln x) = 2(x \ln x - x) + 2$$

put $x = e$

$$\boxed{y = 2}$$

option (1) Ans.

14. If a curve $y = f(x)$ passes through the point $(1, -1)$ and satisfies the differential equation, $y(1 + xy)\,dx = x\,dy$, then $f\left(-\frac{1}{2}\right)$ is equal to : **[JEE(Main)-2016]**

(1) $\frac{4}{5}$ (2) $-\frac{2}{5}$ (3) $-\frac{4}{5}$ (4) $\frac{2}{5}$

Solⁿ

$$y(1+xy)\,dx = x\,dy \Rightarrow \frac{dy}{dx} = \frac{y}{x}(1+xy)$$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{x} + y^2$$

$$\Rightarrow \frac{1}{y^2}\frac{dy}{dx} + \left(\frac{-1}{x}\right)\left(\frac{1}{y}\right) = 1$$

put $-\frac{1}{y} = t \Rightarrow \frac{1}{y^2}\frac{dy}{dx} = \frac{dt}{dx}$

$$\Rightarrow \boxed{\frac{dt}{dx} + \frac{t}{x} = 1}$$

$$I.F = e^{\int \frac{1}{x}dx} = |x| = -x \ \forall\ x<0$$

Solution of D.E (for $x<0$)

$$t(-x) = \int (-x)(1)\,dx + C$$

$$\Rightarrow t(-x) = -\frac{x^2}{2} + C \Rightarrow \boxed{\frac{x}{y} = -\frac{x^2}{2} + C}$$

passes through $(1,-1) \Rightarrow -1 = -\frac{1}{2} + C \Rightarrow \boxed{C = -1/2}$

$$\boxed{\frac{x}{y} = -\frac{x^2}{2} - \frac{1}{2}}$$

at $x = -1/2$

$$-\frac{1}{2y} = -\frac{1}{8} - \frac{1}{2} \Rightarrow \boxed{y = 4/5}$$

option (1) Ans

15

If $(2+\sin x)\frac{dy}{dx}+(y+1)\cos x=0$ and y(0) = 1, then $y\left(\frac{\pi}{2}\right)$ is equal to :- **[JEE(Main)-2017]**

(1) $\frac{4}{3}$ (2) $\frac{1}{3}$ (3) $-\frac{2}{3}$ (4) $-\frac{1}{3}$

Sol. $(2 + \sin x)\frac{dy}{dx}+(y+1)\cos x=0$

$$\frac{d}{dx}(2+\sin x)(y+1)=0$$

$$(2 + \sin x)(y + 1) = c$$

$$x = 0,\ y = 1 \Rightarrow c = 4$$

$$y+1=\frac{4}{2+\sin x}$$

$$y\left(\frac{\pi}{2}\right)=\frac{4}{3}-1=\frac{1}{3}$$

option (2)

(16) Let $y = y(x)$ be the solution of the differential equation $\sin x \frac{dy}{dx} + y \cos x = 4x$, $x \in (0, \pi)$. If $y\left(\frac{\pi}{2}\right) = 0$, then $y\left(\frac{\pi}{6}\right)$ is equal to : **[JEE(Main)-2018]**

(1) $\frac{-8}{9\sqrt{3}}\pi^2$ (2) $-\frac{8}{9}\pi^2$ (3) $-\frac{4}{9}\pi^2$ (4) $\frac{4}{9\sqrt{3}}\pi^2$

Soln

$$\sin x \cdot \frac{dy}{dx} + y\cos x = 4x, \quad x \in (0, \pi)$$

$$\Rightarrow \sin x\, dy + y \cos x\, dx = 4x\, dx$$

$$\Rightarrow \int d(y \sin x) = \int 4x\, dx$$

$$\Rightarrow y \sin x = 2x^2 + C$$

given $y\left(\frac{\pi}{2}\right) = 0$

$$\Rightarrow \boxed{C = -\frac{\pi^2}{2}}$$

So, $\boxed{y \sin x = 2x^2 - \frac{\pi^2}{2}}$ is the solution

$\Rightarrow$ put $x = \pi/6$

$$y\left(\frac{1}{2}\right) = \frac{\pi^2}{18} - \frac{\pi^2}{2} \Rightarrow \boxed{y = \frac{-8\pi^2}{9}} \text{ Ans.}$$

option(2)

17. If $y = y(x)$ is the solution of the differential equation, $x\frac{dy}{dx} + 2y = x^2$ satisfying $y(1) = 1$, then $y\left(\frac{1}{2}\right)$ is equal to : **[JEE(Main)-2019]**

(1) $\frac{7}{64}$ (2) $\frac{13}{16}$ (3) $\frac{49}{16}$ (4) $\frac{1}{4}$

Solution:-

$$\frac{dy}{dx} + \frac{2y}{x} = x$$

LDE in y

$$IF = e^{\int \frac{2}{x} dx} = x^2$$

$$y x^2 = \int x^3 \, dx$$

$$y x^2 = \frac{x^4}{4} + C$$

$$y(1) = 1 \Rightarrow C = \frac{3}{4}$$

$$y x^2 = \frac{x^4}{4} + \frac{3}{4}$$

$$y \cdot \frac{1}{4} = \frac{1}{64} + \frac{3}{4} \Rightarrow y\left(\frac{1}{2}\right) = \frac{1}{16} + 3$$

$$y\left(\frac{1}{2}\right) = \frac{49}{16}$$

18 If $\frac{dy}{dx}+\frac{3}{\cos^2 x}y=\frac{1}{\cos^2 x}, x\in\left(\frac{-\pi}{3},\frac{\pi}{3}\right)$ and $y\left(\frac{\pi}{4}\right)=\frac{4}{3}$, then $y\left(-\frac{\pi}{4}\right)$ equals : **[JEE(Main)-2019]**

(1) $\frac{1}{3}+e^6$ (2) $\frac{1}{3}$ (3) $-\frac{4}{3}$ (4) $\frac{1}{3}+e^3$

Sol. $$\frac{dy}{dx}+3\sec^2 x.y=\sec^2 x$$

$$\text{I.F.} = e^{3\int \sec^2 x dx}=e^{3\tan x}$$

$$\text{or } y.e^{3\tan x}=\int \sec^2 x.e^{3\tan x}dx$$

$$\text{or } y.e^{3\tan x}=\frac{1}{3}e^{3\tan x}+C \quad ....(1)$$

Given

$$y\left(\frac{\pi}{4}\right)=\frac{4}{3}$$

$$\therefore \quad \frac{4}{3}.e^3=\frac{1}{3}e^3+C$$

$$\therefore \quad C=e^3$$

Now put $x=-\frac{\pi}{4}$ in equation (1)

$$\therefore \quad y.e^{-3}=\frac{1}{3}e^{-3}+e^3$$

$$\therefore \quad y=\frac{1}{3}+e^6$$

$$\therefore \quad y\left(-\frac{\pi}{4}\right)=\frac{1}{3}+e^6$$

option (1)

19 The curve amongst the family of curves, represented by the differential equation, $(x^2 - y^2)dx + 2xy\, dy = 0$ which passes through (1,1) is : **[JEE(Main)-2019]**

(1) A circle with centre on the y-axis

(2) A circle with centre on the x-axis

(3) An ellipse with major axis along the y-axis

(4) A hyperbola with transverse axis along the x-axis

Sol. $(x^2 - y^2)\, dx + 2xy\, dy = 0$

$$\frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$$

Put $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

Solving we get,

$$\int \frac{2v}{v^2+1}dv = \int -\frac{dx}{x}$$

$$\ln(v^2 + 1) = -\ln x + C$$

$$(y^2 + x^2) = Cx$$

$$1 + 1 = C \Rightarrow C = 2$$

$\boxed{y^2 + x^2 = 2x}$ ⇒ centre of the circle (1,0)

$\therefore$ Option (2)

20. If y(x) is the solution of the differential equation $\frac{dy}{dx} + \left(\frac{2x+1}{x}\right)y = e^{-2x}$, $x > 0$, where $y(1) = \frac{1}{2}e^{-2}$, then : **[JEE(Main)-2019]**

(1) y(x) is decreasing in (0, 1)

(2) y(x) is decreasing in $\left(\frac{1}{2}, 1\right)$

(3) $y(\log_e 2) = \frac{\log_e 2}{4}$

(4) $y(\log_e 2) = \log_e 4$

Solution:- $\frac{dy}{dx} + \left(\frac{2x+1}{x}\right)y = e^{-2x}$

Linear DE in y

IF: $e^{\int\left(2+\frac{1}{x}\right)dx} = e^{2x+\ln x} = e^{2x}\cdot x$

$y\cdot e^{2x}\cdot x = \int e^{-2x}\cdot e^{2x}\cdot x\,dx$

$y\cdot e^{2x}\cdot x = \frac{x^2}{2} + C \Rightarrow \frac{e^{-2}}{2}\cdot e^{2}\cdot 1 = \frac{1}{2} + C \Rightarrow C = 0$

$y = \frac{x}{2}\cdot e^{-2x} \Rightarrow y(\ln 2) = \frac{\ln 2}{2}\cdot e^{-2\ln 2} = 2\ln 2$

$\frac{dy}{dx} = \frac{x}{2}\cdot e^{-2x}\cdot -2 + \frac{e^{-2x}}{2} = \frac{e^{-2x}}{2}(1-2x)$

+ −

0 $\frac{1}{2}$

$\left(0, \frac{1}{2}\right) \to$ inc

$\left(\frac{1}{2}, 1\right) \to$ dec

(2)

(4)

21. Let y = y(x) be the solution of the differential equation, $\left(x^2+1\right)^2 \frac{dy}{dx}+2x\left(x^2+1\right)y=1$ such that y(0) = 0. If $\sqrt{a}\,y(1)=\frac{\pi}{32}$, then the value of 'a' is : **[JEE(Main)-2019]**

(1) $\frac{1}{2}$ (2) $\frac{1}{16}$ (3) $\frac{1}{4}$ (4) 1

<u>Soln:-</u>

$$\frac{dy}{dx}+\frac{2x}{x^2+1}y=\frac{1}{(x^2+1)^2}$$

Linear DE in y

$$IF = e^{\int \frac{2x}{x^2+1}dx} = x^2+1$$

$$y(x^2+1)=\int \frac{1}{x^2+1}dx$$

$$y(x^2+1)=\tan^{-1}x+C \Rightarrow 0=0+C \Rightarrow C=0$$

$$y=\frac{\tan^{-1}x}{x^2+1} \Rightarrow y(1)=\frac{\pi}{8}$$

$$\sqrt{a}\cdot\frac{\pi}{8}=\frac{\pi}{32} \Rightarrow a=\frac{1}{16}$$

option:- (2)

22. Consider the differential equation, $y^2 dx + \left(x - \frac{1}{y}\right) dy = 0$. If value of y is 1 when x = 1, the the value of x for which y = 2, is : **[JEE(Main)-2019]**

(1) $\frac{1}{2} + \frac{1}{\sqrt{e}}$ (2) $\frac{3}{2} - \sqrt{e}$ (3) $\frac{5}{2} + \frac{1}{\sqrt{e}}$ (4) $\frac{3}{2} - \frac{1}{\sqrt{e}}$

Soln:- $y^2 dx = \left(\frac{1}{y} - x\right) dy$

$$\frac{dx}{dy} + \frac{x}{y^2} = \frac{1}{y^3} \Rightarrow \text{Linear DE in } x$$

$$\text{IF}: \quad e^{\int \frac{1}{y^2} dy} = e^{-\frac{1}{y}}$$

$$x \cdot e^{-\frac{1}{y}} = \int \frac{1}{y^3} \cdot e^{-\frac{1}{y}} dy$$

Let $-\frac{1}{y} = t \Rightarrow \frac{1}{y^2} dy = dt$

$$x \cdot e^{-\frac{1}{y}} = -\int \underset{2}{e^t} \cdot \underset{1}{t}\, dt \Rightarrow x \cdot e^{-\frac{1}{y}} = -te^t + \int e^t dt$$

$$x e^{-\frac{1}{y}} = \frac{1}{y} e^{-\frac{1}{y}} + e^{-\frac{1}{y}} + C \Rightarrow x = \frac{1}{y} + 1 + C \cdot e^{1/y}$$

$$y(1) = 1 \Rightarrow 1 = 1 + 1 + C \cdot e \Rightarrow C = -\frac{1}{e}$$

$$x = \frac{1}{y} + 1 - e^{\frac{1}{y} - 1} \Rightarrow x = \frac{1}{2} + 1 - e^{-\frac{1}{2}} = \frac{3}{2} - \frac{1}{\sqrt{e}}$$

option:- (4)

Exercise JA

EXERCISE (JA)

1 Let f be a real-valued differentiable function on R (the set of all real numbers) such that f(1) = 1. If the y-intercept of the tangent at any point P(x,y) on the curve y = f(x) is equal to the cube of the abscissa of P, then the value of f(–3) is equal to

Sol. Ans. 9

Given $y = f(x)$

Tangent at point P(x, y)

$$Y - y = \left(\frac{dy}{dx}\right)_{(x,y)} (X - x)$$

Now y–intercept $\Rightarrow Y = y - x\frac{dy}{dx}$

Given that, $y - x\frac{dy}{dx} = x^3$

$$\Rightarrow \quad \frac{dy}{dx} - \frac{y}{x} = -x^2 \text{ is a linear differential equation}$$

$$\text{with I.F.} = e^{\int -\frac{1}{x}dx} = e^{-\ell nx} = e^{\ell n\left(\frac{1}{x}\right)} = \frac{1}{x}$$

$$\text{Hence, solution is } \frac{y}{x} = \int -x^2.\frac{1}{x}dx + C$$

$$\text{or} \quad \frac{y}{x} = -\frac{x^2}{2} + C$$

Given $f(1) = 1$

Substituing we get, $\quad C = \frac{3}{2}$

$$\text{so} \quad y = -\frac{x^3}{2} + \frac{3}{2}x$$

$$\text{Now} \quad f(-3) = \frac{27}{2} - \frac{9}{2} = 9$$

②

(a) Let $f:[1,\infty)\to[2,\infty)$ be a differentiable function such that $f(1)=2$. If $6\int_1^x f(t)dt = 3x\,f(x) - x^3$

for all $x \geq 1$, then the value of $f(2)$ is [JEE 2011, 4]

NOTE

The question was bonus.

Reason :

If we put $x=1$ in given functional equation,

$$6\int_1^1 f(t)\,dt = 3f(1) - 1$$

$$\Rightarrow f(1) = 1/3$$

but given $f(1) = 2$

both are not possible simultaneously

However, we can correct the question by modifying given rule as:

$$6\int_1^x f(t)\,dt = 3x\,f(x) - x^3 - 5$$

Now we solve the question using this,

Sol:- $\because$ Given

$$6\int_1^x f(t)\cdot dt = 3x\, f(x) - x^3 - 5$$

Differentiating. w.r. to x.

$\Rightarrow$ $6\, f(x) = 3x\, f'(x) + 3f(x) - 3x^2$

$\Rightarrow$ $f(x) = xf'(x) - x^2$

$$\Rightarrow x\frac{dy}{dx} - y = x^2$$

$$\Rightarrow \frac{dy}{dx} - \frac{y}{x} = x$$

$$\left\{ \text{I. F.} = e^{\int -\frac{1}{x}\cdot dx} = \frac{1}{x} \right\}$$

$$\Rightarrow \frac{1}{x}\times y = \int \frac{1}{x}\times x.dx$$

$$\Rightarrow \frac{y}{x} = x + c$$

$\{$ At $x = 1$, $y = 2$,
$\therefore c = 1$ $\}$

$$\Rightarrow \frac{y}{x} = x + 1$$

at $x = 2$

$\boxed{\mathbf{y = 6}}$

(b) Let $y'(x) + y(x)g'(x) = g(x)g'(x)$, $y(0) = 0$, $x \in R$, where $f'(x)$ denotes $\dfrac{df(x)}{dx}$ and $g(x)$ is a given non-constant differentiable function on R with $g(0) = g(2) = 0$. Then the value of $y(2)$ is [JEE 2011, 4]

Sol :-

$\because$ Given

$$\frac{dy}{dx} + g'(x)y = g(x).g'(x)$$

Integrating factor $= e^{\int g'(x)dx} = e^{g(x)}$

$\therefore$ Solution is $y.e^{g(x)} = \int e^{g(x)}.g(x).g'(x)\,dx$

$\Rightarrow\ y\,e^{g(x)} = e^{g(x)}\,g(x) - e^{g(x)} + C$

$\Rightarrow\ y\,e^{g(0)} = e^{g(0)}\,g(0) - e^{g(0)} + C$

$\Rightarrow\ 0 = 0 - e^{\circ} + C$

$\Rightarrow C = 1$

$\therefore$ for $x = 2$,

$y\,e^{g(2)} = e^{g(2)}.g(2) - e^{g(2)} + 1$

$\Rightarrow\ y \times 1 = 0 - 1 + 1$

$\Rightarrow y = 0$

(3) If y(x) satisfies the differential equation y' – ytanx = 2x sec x and y(0) = 0, then [JEE 2012, 4M]

(A) $y\left(\frac{\pi}{4}\right)=\frac{\pi^2}{8\sqrt{2}}$ (B) $y'\left(\frac{\pi}{4}\right)=\frac{\pi^2}{18}$ (C) $y\left(\frac{\pi}{4}\right)=\frac{\pi^2}{9}$ (D) $y'\left(\frac{\pi}{3}\right)=\frac{4\pi}{3}+\frac{2\pi^2}{3\sqrt{3}}$

Sol:-

$$\because \frac{dy}{dx} - y.\tan(x) = 2x\sec(x)$$

$$\Rightarrow \cos(x).\frac{dy}{dx} - y.\sin(x) = 2x$$

$$\Rightarrow \frac{d}{dx}\left(y.\cos(x)\right) = 2x$$

$$\Rightarrow y.\cos(x) = x^2 + C$$

$$\left\{\begin{array}{l}\text{Put } x = y = 0 \\ \text{then } \boxed{C=0}\end{array}\right\}$$

$$\Rightarrow \boxed{f(x) = \frac{x^2}{\cos(x)}}$$

$$\Rightarrow \boxed{f'(x) = \frac{2x\cos(x) - x^2.\sin(x)}{\cos^2(x)}}$$

Now check the option(s).

4. Let $f:\left[\frac{1}{2},1\right]\to R$ (the set of all real numbers) be a positive, non-constant and differentiable function such that $f'(x) < 2f(x)$ and $f\left(\frac{1}{2}\right)=1$. Then the value of $\int_{1/2}^{1} f(x)dx$ lies in the interval

[JEE(Advanced) 2013, 2M]

(A) $(2e - 1, 2e)$ (B) $(e - 1, 2e - 1)$ (C) $\left(\frac{e-1}{2}, e-1\right)$ (D) $\left(0, \frac{e-1}{2}\right)$

Sol:-

$f:\left[\frac{1}{2},1\right]\to R$

$\because f'(x) - 2f(x) < 0$

Multiply both side by e^{-2x}

$e^{-2x}.f'(x) - 2f(x).e^{-2x} < 0$

$\Rightarrow \frac{d}{dx}\left(e^{-2x}.f(x)\right) < 0$

let $g(x) = e^{-2x}.f(x)$

$g(x)$ is decreasing function.

$\because x > \frac{1}{2}$

$\therefore g(x) < g\left(\frac{1}{2}\right)$

$\Rightarrow e^{-2x}.f(x) < \frac{1}{e}$

$\Rightarrow f(x) < \frac{1}{e}e^{2x}$

$\Rightarrow \int_{1/2}^{1} f(x) < \frac{1}{e}\int_{1/2}^{1} e^{2x}.dx$

$\Rightarrow \int_{1/2}^{1} f(x).dx = \frac{1}{e}\left[\frac{e^{2x}}{2}\right]_{\frac{1}{2}}^{1}$

$\Rightarrow \int_{1/2}^{1} f(x)\,dx = \frac{1-e}{2}$

& also $f(x)$ is +ve

$\therefore \boxed{0 < \int_{1/2}^{1} f(x)\,dx < \frac{1-e}{2}}$

Ans.

5. A curve passes through the point $\left(1, \frac{\pi}{6}\right)$. Let the slope of the curve at each point (x, y) be $\frac{y}{x}+\sec\left(\frac{y}{x}\right), x>0$. Then the equation of the curve is [JEE(Advanced) 2013, 2M]

(A) $\sin\left(\frac{y}{x}\right)=\log x+\frac{1}{2}$

(B) $\operatorname{cosec}\left(\frac{y}{x}\right)=\log x+2$

(C) $\sec\left(\frac{2y}{x}\right)=\log x+2$

(D) $\cos\left(\frac{2y}{x}\right)=\log x+\frac{1}{2}$

Sol:-

$\because \frac{y}{x}+\sec\frac{y}{x}; x>0$ is a slope.

$\therefore \frac{dy}{dx}=\frac{y}{x}+\sec\left(\frac{y}{x}\right)$

$\left\{\begin{array}{l}\text{Put: } y=xt \\ \frac{dy}{dx}=t+x.\frac{dt}{dx}\end{array}\right\}$

$\Rightarrow t+x.\frac{dt}{dx}=t+\sec(t)$

$\Rightarrow \int \cos t\, dt=\int\frac{dx}{x}$

$\Rightarrow \sin t=\ln(x)+C$

$\left\{\begin{array}{l}\text{Put } \left[\begin{array}{l}x=1 \\ y=\frac{\pi}{6}\end{array}\right] \therefore t=\frac{y}{x}=\frac{\pi}{6} \\ \text{then, } \boxed{C=\frac{1}{2}}\end{array}\right\}$

$\Rightarrow \boxed{\sin\left(\frac{y}{x}\right)=\ln(x)+\frac{1}{2}}$ Ans.

(6) The function $y = f(x)$ is the solution of the differential equation $\frac{dy}{dx} + \frac{xy}{x^2-1} = \frac{x^4+2x}{\sqrt{1-x^2}}$ in $(-1, 1)$ satisfying $f(0) = 0$. Then $\int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} f(x)dx$ is [JEE(Advanced)-2014, 3(–1)]

(A) $\frac{\pi}{3} - \frac{\sqrt{3}}{2}$ (B) $\frac{\pi}{3} - \frac{\sqrt{3}}{4}$ (C) $\frac{\pi}{6} - \frac{\sqrt{3}}{4}$ (D) $\frac{\pi}{6} - \frac{\sqrt{3}}{2}$

Sol. Ans. (B)

$$\because \frac{dy}{dx} - \frac{xy}{x^2-1} = \frac{x^4+2x}{\sqrt{1-x^2}}$$

$$\Rightarrow \frac{dy}{dx} - \left(\frac{1}{2}\cdot\frac{2x}{1-x^2}\right)y = \frac{x^4+2x}{\sqrt{1-x^2}}$$

$$\left\{ I.F. = e^{-\frac{1}{2}\int \frac{2x}{1-x^2}\cdot dx} = e^{\frac{1}{2}\ln(1-x^2)} = \sqrt{1-x^2} \right\}$$

$$\Rightarrow \int d\left(\sqrt{1-x^2}\cdot y\right) = \int (x^4+2x)\cdot dx$$

$$\Rightarrow \sqrt{1-x^2}\cdot y = \frac{x^5}{5} + x^2 + C$$

$$\left\{ \underline{\text{Put}} \ \begin{matrix} x=0 \\ y=0 \end{matrix} \right] \text{then } \boxed{C=0} \Big\}$$

$$\Rightarrow \sqrt{1-x^2}\cdot y = \frac{x^5}{5} + x^2$$

$$\Rightarrow \boxed{y = \frac{x^5}{5\sqrt{1-x^2}} + \frac{x^2}{\sqrt{1-x^2}}}$$

$$\therefore \int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} f(x) \cdot dx$$

$$= \int_{-\sqrt{3}/2}^{\sqrt{3}/2} \underbrace{\frac{x^5}{5\sqrt{1-x^2}}}_{\text{odd}} + \underbrace{\frac{x^2}{\sqrt{1-x^2}}}_{\text{even}} \cdot dx$$

$$= 2\int_0^{\sqrt{3}/2} \frac{x^2}{\sqrt{1-x^2}} \cdot dx \quad ; \quad \text{Put } x = \sin\theta, \quad dx = \cos\theta \cdot d\theta$$

$$= 2\int_0^{\pi/3} \frac{\sin^2\theta}{\cos\theta} \cdot \cos\theta \cdot d\theta$$

$$= \int_0^{\pi/3} 2\sin^2\theta \cdot d\theta = \int_0^{\pi/3} (1-\cos 2\theta) \cdot d\theta$$

$$= \left[\theta - \frac{\sin 2\theta}{2}\right]_0^{\pi/3}$$

$$= \boxed{\frac{\pi}{3} - \frac{\sqrt{3}}{4}}$$

Ans.

7 Consider the family of all circles whose centers lie on the straight line y = x. If this family of circles is represented by the differential equation $Py'' + Qy' + 1 = 0$, where P,Q are functions of x,y and y' (here $y' = \frac{dy}{dx}, y'' = \frac{d^2y}{dx^2}$), then which of the following statements is (are) true? **[JEE 2015, 4M, –2M]**

(A) $P = y + x$

(B) $P = y - x$

(C) $P + Q = 1 - x + y + y' + (y')^2$

(D) $P - Q = x + y - y' - (y')^2$

Sol. Let Circle

$$x^2 + y^2 - 2ax - 2ay + c = 0$$

{∵ Centre lie on the line y=x}

On differentiation

$$2x + 2yy' - 2a - 2ay' = 0$$

$$\Rightarrow \quad x + yy' - a(1 + y') = 0$$

$$\Rightarrow \quad a = \frac{x + yy'}{1 + y'}$$

again differentiation

$$\frac{(1 + (y')^2 + yy'')(1 + y') - (x + yy')(y'')}{(1 + y')^2} = 0$$

$$\Rightarrow \quad 1 + y'((y')^2 + y' + 1) + y''(y - x) = 0$$

$$\therefore \quad \boxed{\begin{array}{l} P = y - x \\ Q = 1 + y' + (y')^2 \end{array}}$$

Ans.

8 Let $f:(0,\infty)\to\mathbb{R}$ be a differentiable function such that $f'(x)=2-\frac{f(x)}{x}$ for all $x\in(0,\infty)$ and $f(1)\neq 1$. Then **[JEE(Advanced)-2016, 4(–2)]**

(A) $\lim\limits_{x\to 0^+} f'\left(\frac{1}{x}\right)=1$

(B) $\lim\limits_{x\to 0^+} x f\left(\frac{1}{x}\right)=2$

(C) $\lim\limits_{x\to 0^+} x^2 f'(x)=0$

(D) $|f(x)|\le 2$ for all $x\in(0,2)$

Sol. Let $y = f(x)$

Given $\frac{dy}{dx}+\frac{y}{x}=2$ (linear differential equation)

$\left\{\because I.F.=e^{\int\frac{1}{x}\cdot dx}=x\right\}$

$\Rightarrow \int d(x\cdot y)=\int 2x\cdot dx$

$\Rightarrow \therefore yx = x^2 + c$

$\Rightarrow f(x) = x + \frac{c}{x}$; As $f(1)\neq 1 \Rightarrow c\neq 0$

$\Rightarrow f'(x)=1-\frac{c}{x^2}, c\neq 0$

(A) $\lim\limits_{x\to 0^+} f'\left(\frac{1}{x}\right)=\lim\limits_{x\to 0^+}(1-cx^2)=1$

(B) $\lim\limits_{x\to 0^+} x f\left(\frac{1}{x}\right)=\lim\limits_{x\to 0^+} x\left(\frac{1}{x}+cx\right)=\lim\limits_{x\to 0^+}(1+cx^2)=1$

(C) $\lim\limits_{x\to 0^+} x^2 f'(x)=\lim\limits_{x\to 0^+} x^2\left(1-\frac{c}{x^2}\right)=\lim\limits_{x\to 0^+}(x^2-c)=-c$

(D) $f(x)=x+\frac{c}{x}, c\neq 0$

for $c > 0$

$\therefore \lim\limits_{x\to 0^+} f(x)=\infty \Rightarrow$ function is not bounded in (0,2).

9. If y = y(x) satisfies the differential equation $8\sqrt{x}\left(\sqrt{9+\sqrt{x}}\right)dy=\left(\sqrt{4+\sqrt{9+\sqrt{x}}}\right)^{-1}dx,\ x>0$ and $y(0)=\sqrt{7}$, then $y(256)=$ **[JEE(Advanced)-2017, 3(–1)]**

(A) 80 (B) 3 (C) 16 (D) 9

Sol:-

$$\because\ 8\sqrt{x}\left(\sqrt{9+\sqrt{x}}\right)dy=\left(\sqrt{4+\sqrt{9+\sqrt{x}}}\right)^{-1}dx$$

$$\Rightarrow\ dy=\frac{dx}{8\sqrt{x}\sqrt{9+\sqrt{x}}\sqrt{4+\sqrt{9+x}}}$$

$$\Rightarrow\ y=\frac{1}{8}\int\frac{dx}{\sqrt{4+\sqrt{9+x}}.\sqrt{x}.\sqrt{9+\sqrt{x}}}$$

$$\left\{\text{put } \sqrt{9+\sqrt{x}}=t\ \Rightarrow\ \frac{dx}{\sqrt{x}.\sqrt{9+\sqrt{x}}}=4dt\right\}$$

$$\Rightarrow\ \therefore\ y=\frac{4}{8}\int\frac{dt}{\sqrt{4+t}}$$

$$\Rightarrow\ y=\sqrt{4+t}+C$$

$$\Rightarrow\ y(x)=\sqrt{4+\sqrt{9+\sqrt{x}}}+C$$

at x = 0 : $y(0)=\sqrt{7}\ \Rightarrow\ C=0$

$$\therefore\ y(x)=\sqrt{4+\sqrt{9+\sqrt{x}}}$$

$$\Rightarrow\ \boxed{y(256)=3}$$ Ans.

(10) If $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function such that $f'(x) > 2f(x)$ for all $x \in \mathbb{R}$, and $f(0) = 1$, then

[JEE(Advanced)-2017, 3(–2)]

(A) $f(x) > e^{2x}$ in $(0,\infty)$

(B) $f(x)$ is decreasing in $(0,\infty)$

(C) $f(x)$ is increasing in $(0,\infty)$

(D) $f'(x) < e^{2x}$ in $(0,\infty)$

Sol:-

$f'(x) > 2f(x)\ ;\ \forall x \in R$

$\Rightarrow f'(x) - 2f(x) > 0\ ;\ \forall x \in R$

$\therefore e^{-2x}\left(f'(x) - 2f(x)\right) > 0\ ;\ \forall x \in R$

$\Rightarrow \frac{d}{dx}\left(e^{-2x} \cdot f(x)\right) > 0\ ;\ \forall x \in R$

let $g(x) = e^{-2x} \cdot f(x)$

$\left\{\begin{array}{l} \because g'(x) > 0 \\ \therefore g(x) \text{ is } \uparrow \text{ funct}^n \end{array}\right\}$

so $g(x) >$ min.

$\Rightarrow g(x) > g(0)\quad ;\ \underbrace{x \in (0,\infty)}_{\text{through option}}$

$\Rightarrow g(x) > e^0 \underbrace{f(0)}_{\hookrightarrow 1 \text{ (given)}}$

$\Rightarrow g(x) > 1$

$\Rightarrow e^{-2x} \cdot f(x) > 1$

$\Rightarrow \boxed{f(x) > e^{2x}}\ ;\ x \in (0,\infty)$

option A is Correct

$\because$ Given $f'(x) > 2f(x)$

& $f(x) > e^{2x}$

$\therefore f'(x) > 2f(x) > 2 \cdot e^{2x} > 2$ —— (1)

$;\ x \in (0,\infty)$

$\therefore f'(x) \Rightarrow +ve$

so $\boxed{f(x) \text{ is } \uparrow}\ ;\ x \in (0,\infty)$

option B is incorrect

& option C is Correct.

* Through (1) $\Rightarrow \boxed{f'(x) > 2e^{2x}}$

option 'D' is incorrect.

11 Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be two non-constant differentiable functions. If $f'(x) = (e^{(f(x) - g(x))})g'(x)$ for all $x \in \mathbb{R}$, and $f(1) = g(2) = 1$, then which of the following statement(s) is (are) TRUE ?

[JEE(Advanced)-2018, 4(–2)]

(A) $f(2) < 1 - \log_e 2$

(B) $f(2) > 1 - \log_e 2$

(C) $g(1) > 1 - \log_e 2$

(D) $g(1) < 1 - \log_e 2$

Sol. $f'(x) = e^{(f(x) - g(x))}\, g'(x) \ \forall\ x \in \mathbb{R}$

$\Rightarrow\ e^{-f(x)}.\ f'(x) - e^{-g(x)}g'(x) = 0$

$\Rightarrow\ \int\left(e^{-f(x)} f'(x) - e^{-g(x)}.g'(x)\right)dx = C$

$\Rightarrow\ -e^{-f(x)} + e^{-g(x)} = C$

put $x=1$ $\quad -e^{-f(1)} + e^{-g(1)} = c$

put $x=2$ $\quad -e^{-f(2)} + e^{-g(2)} = c$

$\Rightarrow\ -e^{-f(1)} + e^{-g(1)} = -e^{-f(2)} + e^{-g(2)}$

$\Rightarrow\ -\frac{1}{e} + e^{-g(1)} = -e^{-f(2)} + \frac{1}{e}$

$\Rightarrow\ e^{-f(2)} + e^{-g(1)} = \frac{2}{e}$

$\therefore\ e^{-f(2)} < \frac{2}{e}$ and $e^{-g(1)} < \frac{2}{e}$

$\Rightarrow\ -f(2) < \ln 2 - 1$ and $-g(1) < \ln 2 - 1$

$\Rightarrow\ f(2) > 1 - \ln 2$ and $g(1) > 1 - \ln 2$

12 Let $f : (0, \pi) \to \mathbb{R}$ be a twice differentiable function such that

$$\lim_{t \to x} \frac{f(x)\sin t - f(t)\sin x}{t - x} = \sin^2 x \text{ for all } x \in (0, \pi).$$

If $f\left(\frac{\pi}{6}\right) = -\frac{\pi}{12}$, then which of the following statement(s) is (are) TRUE ?

[JEE(Advanced)-2018, 4(–2)]

(A) $f\left(\frac{\pi}{4}\right) = \frac{\pi}{4\sqrt{2}}$

(B) $f(x) < \frac{x^4}{6} - x^2$ for all $x \in (0, \pi)$

(C) There exists $\alpha \in (0, \pi)$ such that $f'(\alpha) = 0$

(D) $f''\left(\frac{\pi}{2}\right) + f\left(\frac{\pi}{2}\right) = 0$

Sol. $$\lim_{t \to x} \frac{f(x)\sin t - f(t)\sin x}{t - x} = \sin^2 x$$

by using L'Hopital

$$\lim_{t \to x} \frac{f(x)\cos t - f'(t)\sin x}{1} = \sin^2 x$$

$\Rightarrow f(x)\cos x - f'(x)\sin x = \sin^2 x$

$$\Rightarrow -\left(\frac{f'(x)\sin x - f(x)\cos x}{\sin^2 x}\right) = 1$$

$$\Rightarrow -d\left(\frac{f(x)}{\sin x}\right) = 1$$

$$\Rightarrow \frac{f(x)}{\sin x} = -x + c$$

Put $x = \frac{\pi}{6}$ & $f\left(\frac{\pi}{6}\right) = -\frac{\pi}{12}$

$\therefore$ $c = 0 \Rightarrow f(x) = -x\sin x$

(A) $f\left(\frac{\pi}{4}\right) = \frac{-\pi}{4} \frac{1}{\sqrt{2}}$

(B) $f(x) = -x \sin x$

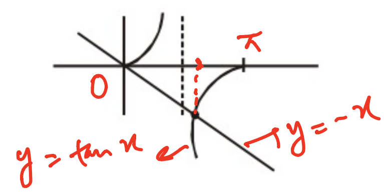
as $\sin x > x - \frac{x^3}{6}$, $-x \sin x < -x^2 + \frac{x^4}{6}$

$\therefore$ $f(x) < -x^2 + \frac{x^4}{6}$ $\forall x \in (0, \pi)$

(C) $f'(x) = -\sin x - x\cos x$

$f'(x) = 0 \Rightarrow \tan x = -x$

$\Rightarrow$ there exist $\alpha \in (0, \pi)$ for which $f'(\alpha) = 0$



(D) $f''(x) = -2\cos x + x\sin x$

$f''\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$, $f\left(\frac{\pi}{2}\right) = -\frac{\pi}{2}$

$$f''\left(\frac{\pi}{2}\right) + f\left(\frac{\pi}{2}\right) = 0$$

(13) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function with $f(0) = 0$. If $y = f(x)$ satisfies the differential equation $\frac{dy}{dx} = (2+5y)(5y-2)$,

then the value of $\lim\limits_{x\to-\infty} f(x)$ is _____ . **[JEE(Advanced)-2018, 3(0)]**

Sol. $$\frac{dy}{dx} = 25y^2 - 4$$

So, $$\frac{dy}{25y^2-4} = dx$$

Integrating, $$\frac{1}{25}\times\frac{1}{2\times\frac{2}{5}}\ln\left|\frac{y-\frac{2}{5}}{y+\frac{2}{5}}\right| = x + c$$

$$\Rightarrow \ln\left|\frac{5y-2}{5y+2}\right| = 20(x+c)$$

Now, c = 0 as f(0) = 0

Hence $$\left|\frac{5y-2}{5y+2}\right| = e^{(20x)}$$

$$\underset{x\to-\infty}{\text{let}}\left|\frac{5f(x)-2}{5f(x)+2}\right| = \underset{x\to-\infty}{\text{let}}\ e^{(20x)}$$

Now, RHS = 0 $\Rightarrow \underset{x\to-\infty}{\text{let}}(5f(x)-2) = 0$

$$\Rightarrow \underset{x\to-\infty}{\text{let}}\ f(x) = \frac{2}{5}$$

(14) Let Γ denote a curve $y = y(x)$ which is in the first quadrant and let the point $(1, 0)$ lie on it. Let the tangent to Γ at a point P intersect the y-axis at Y_P. If PY_P has length 1 for each point P on Γ, then which of the following options is/are correct ? **[JEE(Advanced)-2019, 4(–1)]**

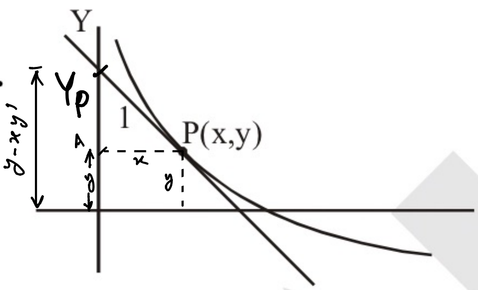
(1) $y = \log_e\left(\frac{1+\sqrt{1-x^2}}{x}\right) - \sqrt{1-x^2}$

(2) $xy' - \sqrt{1-x^2} = 0$

(3) $y = -\log_e\left(\frac{1+\sqrt{1-x^2}}{x}\right) + \sqrt{1-x^2}$

(4) $xy' + \sqrt{1-x^2} = 0$

Sol.



$$Y - y = y'(X - x)$$

So, $Y_P = (0,\ y - xy')$

So, $\underbrace{x^2 + (xy')^2 = 1}_{\text{by } \triangle APY_P} \Rightarrow \frac{dy}{dx} = -\sqrt{\frac{1-x^2}{x^2}}$

[$\frac{dy}{dx}$ can not be positive i.e. $f(x)$ can not be increasing in first quadrant, for $x \in (0, 1)$]

Hence, $\int dy = -\int \frac{\sqrt{1-x^2}}{x}\,dx$

$\Rightarrow y = -\int \frac{\cos^2\theta\, d\theta}{\sin\theta}$; put $x = \sin\theta$

$\Rightarrow y = -\int \text{cosec}\theta\, d\theta + \int \sin\theta\, d\theta$

$\Rightarrow y = \ln(\text{cosec}\theta + \cot\theta) - \cos\theta + C$

$\Rightarrow y = \ell n\left(\frac{1+\sqrt{1-x^2}}{x}\right) - \sqrt{1-x^2} + C$

$\Rightarrow y = \ell n\left(\frac{1+\sqrt{1-x^2}}{x}\right) - \sqrt{1-x^2}$ (as $y(1) = 0$)