Allen Career Institute
Kota

Maxima and Minima
(Solutions)

Index

Do yourself - 1 :

(i) Find local maxima and local minima for the function $f(x) = x^3 - 3x$.

(ii) If function $f(x) = x^3 - 62x^2 + ax + 9$ has local maxima at $x = 1$, then find the value of a.

(i) $f(x) = x^3 - 3x \Rightarrow f'(x) = 3x^2 - 3$

$\Rightarrow f'(x) = 3(x-1)(x+1)$

$f'(x)$: $+$, -1 , $-$, 1 , $+$

$-1 \to M$ $\quad$ $1 \to m$

(ii) $f'(x) = 3x^2 - 124x + a = 0$ at $x = 1$

$\Rightarrow 3 - 124 + a \Rightarrow a = 121$ Ans.

Do yourself - 2 :

(i) Find local maximum value of function $f(x) = \frac{\ell nx}{x}$

(ii) If $f(x) = x^2e^{-2x}$ $(x > 0)$, then find the local maximum value of $f(x)$.

(i) $f'(x) = \dfrac{x\cdot(1/x) - \ln x}{x^2} = \dfrac{1-\ln x}{x^2} = 0$

$\Rightarrow x = e$

$f'(x)$: 0 $+$ e $-$; $e \to M$

$\therefore f(e) = 1/e = f(x)_{max}$

(ii) $f'(x) = 2x e^{-x} - 2x^2\cdot e^{-2x}$

$= -2\cdot e^{-x}\cdot x(x-1)$

$-$ 0 $+$ 1 $-$; m (at 0), $1 \to M$

$\therefore f(x)_{max} = f(1)$

$= 1/e^2$ Ans

Do yourself - 3 :

(i) Identify the point of local maxima/minima in $f(x) = (x-3)^{10}$.

Sol^n $f'(x) = 10(x-3)^9$

$\underset{3}{\overset{- \quad\quad +}{\longleftrightarrow}}$

m

$\therefore f(x)_{min} = f(3) = 0$

No pt. of LM

Do yourself - 4 :

(i) Find the two positive numbers x & y such that their sum is 60 and xy^3 is maximum.

(ii) If from a wire of length 36 metre, a rectangle of greatest area is made, then find its two adjacent sides in metre.

(iii) If $ab = 2a + 3b$ where $a > 0, b > 0$, then find the minimum value of ab.

(iv) Of all closed right circular cylinders of a given volume of 100 cubic centimetres, find the dimensions of cylinder which has minimum surface area.

Sol^n (i) $x + y = 60$; $P = xy^3$

$\Rightarrow P = y^3(60 - y)$

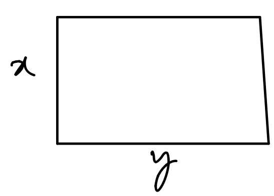
$\frac{dP}{dy} = 3y^2(60 - y) - y^3$

$= y^2(180 - 3y - y) \Rightarrow y = 0$ or $y = 45$

(rej ∵ +ve nos.)

$\therefore x = 15 ; y = 45$ Ans.

(ii)



Perimeter = length of wire

$\Rightarrow 2x + 2y = 36$

$\Rightarrow x + y = 18$

$A = xy = x(18 - x) \Rightarrow A = 18x - x^2$

$\Rightarrow \frac{dA}{dx} = 18 - 2x = 0 \Rightarrow x = 9 ; y = 9$ Ans

(iii) $ab = 2a + 3b \Rightarrow a = \dfrac{3b}{b-2}$

$\therefore \; \varepsilon = ab = \dfrac{3b^2}{b-2}$

$\Rightarrow \dfrac{d\varepsilon}{db} = \dfrac{(b-2)\,6b - 3b^2}{(b-2)^2} = \dfrac{3b(b-4)}{(b-2)^2}$

$\dfrac{d\varepsilon}{db} = 0 \Rightarrow b = 4 \; \Rightarrow a = \dfrac{12}{4-2} = 6$

$\therefore \; \varepsilon = ab = 24$

(iv) $\pi r^2 h = 100 \Rightarrow h = 100/\pi r^2$

$A = 2\pi r^2 + 2\pi r h$

$\Rightarrow A = 2\pi r^2 + \cancel{2\pi}\,\cancel{r} \cdot \dfrac{100}{\cancel{\pi} r \cancel{2}}$

$\Rightarrow A = 2\pi r^2 + \dfrac{200}{r} \Rightarrow \dfrac{dA}{dr} = 4\pi r - \dfrac{200}{r^2}$

$\Rightarrow r^3 = \dfrac{200}{4\pi} \Rightarrow r = \left(\dfrac{50}{\pi}\right)^{1/3}$

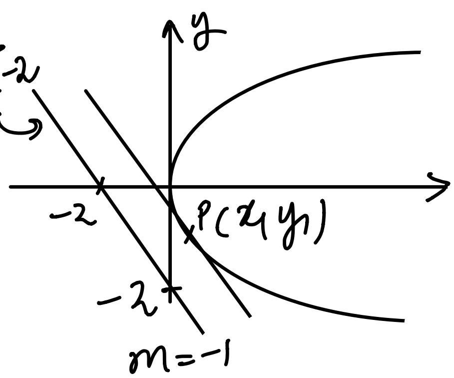
$\therefore \; h = \dfrac{100}{\pi r^2} = \dfrac{100 r}{\pi r^3} = \dfrac{100 r}{\pi \cdot \dfrac{200}{4\pi}} = 2r$

$\therefore \; h = 2\left(\dfrac{50}{\pi}\right)^{1/3}$

Do yourself - 5 :

(i) Find the coordinates of point on the curve $y^2 = 8x$, which is at minimum distance from the line $x + y = -2$.

Sol^n



$y^2 = 8x$

$\Rightarrow 2yy' = 8$

$\Rightarrow y' = \frac{4}{y}$

$\left.\frac{dy}{dx}\right|_{P(x_1, y_1)} = -1$

$\Rightarrow \frac{4}{y_1} = -1 \Rightarrow y_1 = -4$

$\Rightarrow (-4)^2 = 8x_1 \Rightarrow x_1 = 2$

$\therefore P(2, -4)$

Do yourself - 6 :

(i) Find the critical points and stationary point of the function $f(x) = \frac{e^x}{x}$

(ii) Find the point of inflection for the curve $y = x^3 - 6x^2 + 12x + 5$

(iii) For the function $f(x) = \frac{x^4}{12} - \frac{5x^3}{6} + 3x^2 + 7$

(a) Find the interval in which $f''(x) > 0$

(b) Find the interval in which $f''(x) < 0$

(c) Find the points of inflection of f(x).

Solⁿ (i) $f'(x) = \dfrac{x \cdot e^x - e^x \cdot 1}{x^2} = \dfrac{e^x(x-1)}{x}$

$f'(x) = 0 \Rightarrow x = 1$; $f'(x)$: DNE $\Rightarrow x \in \phi$

$\therefore$ $x = 1$ is critical & stationary pt.

(ii) $f(x) = x^3 - 6x^2 + 12x + 5$

$\Rightarrow f'(x) = 3x^2 - 12x + 12 \Rightarrow f'(x) = 3(x-2)^2$

$\Rightarrow f''(x) = 6(x-2)$

$f''(x) = 0 \Rightarrow x = 2$; $f''(x)$: DNE $\Rightarrow x \in \phi$

$\therefore$ Pt. of inflexion : $x = 2$

(iii) $f'(x) = \dfrac{4x^3}{12} - \dfrac{5 \cdot 3x^2}{6} + 6x$

$\Rightarrow f''(x) = x^2 - 5x + 6 = (x-2)(x-3)$

$+ \quad 2 \quad - \quad 3 \quad +$

(a) $f''(x) > 0 \Rightarrow x \in (-\infty, 2) \cup (3, \infty)$

(b) $f''(x) < 0 \Rightarrow x \in (2, 3)$

(c) Pt. of inflexion: $x = 2$; $x = 3$

Exercise 01

1 If $f(x) = |x| + |x - 1| + |x - 2|$, then-

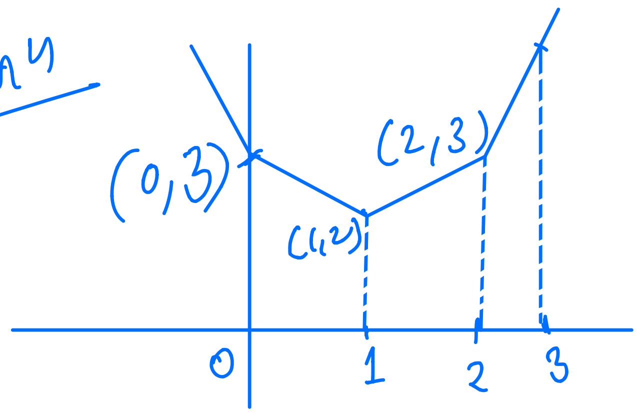
(A) $f(x)$ has minima at $x = 1$

(B) $f(x)$ has maxima at $x = 0$

(C) $f(x)$ has neither maxima nor minima at $x = 3$

(D) None of these

Sol^n



$\therefore$ minima at $x = 1$

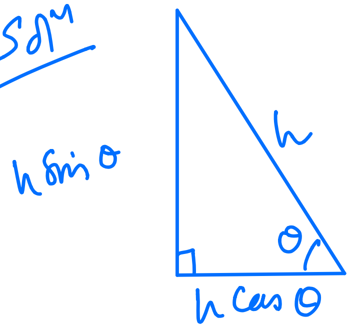
Neither max nor min at $x = 3$

(Ans A, C)

2. The maximum area of a right angled triangle with hypotenuse h is :-

(A) $\frac{h^2}{\sqrt{2}}$ (B) $\frac{h^2}{2}$ (C) $\frac{h^2}{4}$ (D) $\frac{h^2}{2\sqrt{2}}$

Sol^n



$$A = \frac{1}{2}(h\cos\theta)(h\sin\theta)$$

$$\Rightarrow A = \frac{h^2}{4}\sin 2\theta$$

$$A_{max} = h^2/4 \text{ at } \theta = \pi/4$$

(Ans. C)

3. The cost of running a bus from A to B, isRs. $\left(av + \frac{b}{v}\right)$, where v km/h is the average speed of the bus. When the bus travels at 30 km/h, the cost comes out to be Rs. 75 while at 40 km/h, it is Rs. 65. Then the most economical speed (in km/h) of the bus is :

(A) 40 (B) 60 (C) 45 (D) 50

3. $C = aV + \frac{b}{V}$.

$$\left.\begin{aligned} 75 &= 30 \cdot a + \frac{b}{30} \\ 65 &= 40 \cdot a + \frac{b}{40} \end{aligned}\right\} \quad \begin{aligned} a &= 1/2 \\ b &= 1800. \end{aligned}$$

For C_{min} (Most economical cost) : $C' = 0$

$\Rightarrow C' = a - \frac{b}{V^2} = 0 \Rightarrow V^2 = \frac{b}{a} = 3600$

$\Rightarrow V = 60$ Km/h. (Ans B)

4. The greatest value of $x^3 - 18x^2 + 96x$ in the interval (0, 9) is-

(A) 128 (B) 60 (C) 160 (D) 120

Solⁿ $f(x) = x^3 - 18x^2 + 96x$

$\Rightarrow f'(x) = 3x^2 - 36x + 96 = 0 \Rightarrow x = 8, 4$

$\Rightarrow f(4) = 160$

& $f(8) = 128$

$\therefore f(x)_{max} = 180$ (local Max)

5. Difference between the greatest and the least values of the function $f(x) = x(\ln x - 2)$ on $[1, e^2]$ is

(A) 2 (B) e (C) e^2 (D) 1

Sol^n $f(1) = 1(0-2) = -2$

$f(e^2) = e^2(2-2) = 0 \longrightarrow M$

$f'(x) = \ln x - 2 + x \cdot \frac{1}{x} = 0 \Rightarrow \ln x = 1$

$\Rightarrow x = e$

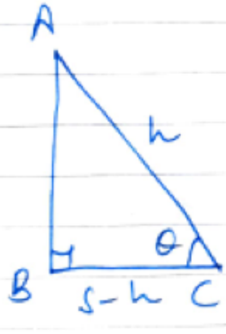
$\therefore f(e) = e(1-2) = -e \longrightarrow m$

$\therefore$ Difference $= e$ (Ans B)

6 - The sum of lengths of the hypotenuse and another side of a right angled triangle is given. The area of the triangle will be maximum if the angle between them is :

(A) $\pi/6$ (B) $\pi/4$ (C) $\pi/3$ (D) $5\pi/12$

6.



Let $AC = h$.

$AC + BC = S$ (given sum)

$\therefore BC = S - h$

Ans: θ, where $\cos\theta = \frac{S-h}{h}$.

$\therefore AB = \sqrt{h^2 - (S-h)^2}$

$\Rightarrow AB = \sqrt{2hS - S^2}$

Area, $A = \frac{1}{2} AB \cdot BC$

$\Rightarrow A = \frac{1}{2}\left(\sqrt{2hS - S^2}\right) \cdot (S - h)$

$\frac{dA}{dh} = 0 \Rightarrow \frac{1}{2}\left[\left(\frac{\cancel{2}\,S}{\cancel{2}\sqrt{2hS - S^2}}\right)(S-h) - \sqrt{2hS - S^2}\right] = 0$

$\Rightarrow S^2 - hS = 2hS - S^2 \Rightarrow 2S^2 = 3hS$

$\Rightarrow S = 3h/2$ or $S = 0$ (reject)

$\therefore \cos\theta = \frac{3h/2 - h}{h} \Rightarrow \cos\theta = 1/2$

(Ans C) $\Rightarrow \theta = \pi/3$ Ans

7. Consider the function $f(x) = x \cos x - \sin x$, then identify the statement which is correct .

(A) f is neither odd nor even

(B) f is monotonic decreasing at $x = 0$

(C) f has a maxima at $x = \pi$

(D) f has a minima at $x = -\pi$

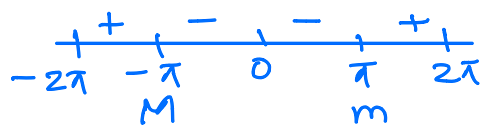
Solⁿ $f'(x) = -x \sin x + \cos x - \cos x$

$\Rightarrow f'(x) = -x \sin x$

$f(x)$ is odd

Maxima at $x = -\pi$.

Minima at $x = \pi$



(Ans. B)

8. If $f(x) = x^3 + ax^2 + bx + c$ is minimum at $x = 3$ and maximum at $x = -1$, then-

(A) $a = -3, b = -9, c = 0$

(B) $a = 3, b = 9, c = 0$

(C) $a = -3, b = -9, c \in R$

(D) none of these

Solⁿ $f'(x) = 3x^2 + 2ax + b$

$$\left.\begin{aligned} f'(3) &= 27 + 6a + b = 0 \\ f'(-1) &= 3 - 2a + b = 0 \end{aligned}\right\} \begin{aligned} a &= -3 \\ b &= -9 \end{aligned} \; ; \; c \in R$$

(Ans C)

9. If $f(x) = \int_{x}^{x^2} (t-1)\,dt$, $1 \le x \le 2$, then global maximum value of $f(x)$ is

(A) 1 (B) 2 (C) 4 (D) 5

Solⁿ.

$$f'(x) = (x^2-1).2x - (x-1)$$

$$= (x-1)(2x(x+1)-1)$$

$$= (x-1)(2x^2+2x-1) \ge 0 \ \forall \ x \in [1,2].$$

$\therefore$ Global maximum $= f(2) = \int_2^4 (t-1)\,dt$

$$= \frac{(t-1)^2}{2}\Bigg]_2^4 = \frac{9}{2} - \frac{1}{2} = 4.$$

10. Range of the function $f(x)=\frac{\ell nx}{\sqrt{x}}$ is

(A) $(-\infty, e)$ (B) $(-\infty, e^2)$ (C) $\left(-\infty, \frac{2}{e}\right]$ (D) $\left(-\infty, \frac{1}{e}\right)$

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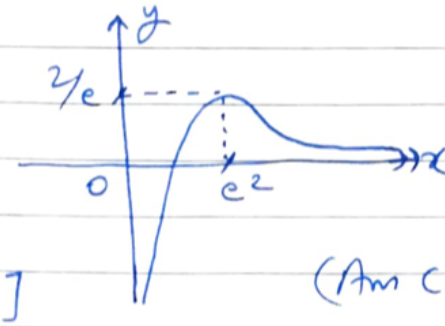
10. $f(x) = \dfrac{\ln x}{\sqrt{x}}$

We plot the graph :



$\therefore$ Range : $(-\infty, 2/e]$

(Ans C).

11. If $ax + \frac{b}{x} \geq c$ for all positive x, where a, b, c > 0, then-

(A) $ab < \frac{c^2}{4}$ (B) $ab \geq \frac{c^2}{4}$ (C) $ab \geq \frac{c}{4}$ (D) None of these

11. $ax + \frac{b}{x} \geqslant c$ for positive x, a, b & $c > 0$.

$\frac{ax + \frac{b}{x}}{2} \geqslant \sqrt{ax \cdot b/x} \Rightarrow ax + \frac{b}{x} \geqslant 2\sqrt{ab}$.

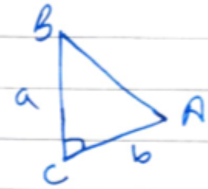
$\therefore \left(ax + \frac{b}{x}\right)_{min} = 2\sqrt{ab} \Rightarrow$ We want $2\sqrt{ab} \geqslant c$

$\Rightarrow ab \geqslant \frac{c^2}{4}$ (Ans B)

12. Two sides of a triangle are to have lengths 'a' cm & 'b' cm. If the triangle is to have the maximum area, then the length of the median from the vertex containing the sides 'a' and 'b' is

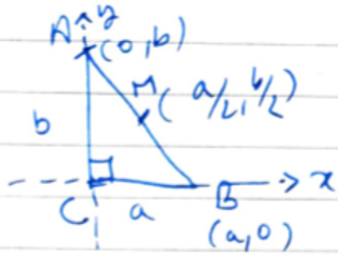
(A) $\frac{1}{2}\sqrt{a^2+b^2}$ (B) $\frac{2a+b}{3}$ (C) $\sqrt{\frac{a^2+b^2}{2}}$ (D) $\frac{a+2b}{3}$

12. $\Delta = \frac{1}{2} ab \sin C$



For Δ_{max} : $\sin C = 1 \Rightarrow C = \pi/2$

$\therefore$



$\therefore CM = \sqrt{(a/2)^2 + (b/2)^2}$

$= \frac{\sqrt{a^2+b^2}}{2}$ (Ans A)

13. $f(x) = 1 + [\cos x]x$, in $0 \le x \le \frac{\pi}{2}$ (where [.] denotes greatest integer function)

(A) has a minimum value 0

(B) has a maximum value 2

(C) is continuous in $\left[0, \frac{\pi}{2}\right]$

(D) is not differentiable at $x = \frac{\pi}{2}$

Sol^n. $f(x) = \begin{cases} 1 & x = 0 \\ 1 & x \in (0, \pi/2) \end{cases}$ $[\cos x] = 0$

$\therefore$ continuous in $[0, \pi/2]$ $\therefore$ (Ans. C)

14. The minimum value of $a\sec x + b\,\mathrm{cosec}\,x$, $0 < a < b$, $0 < x < \pi/2$ is-

(A) $a + b$ (B) $a^{2/3} + b^{2/3}$ (C) $(a^{2/3} + b^{2/3})^{3/2}$ (D) None of these

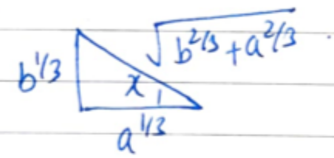
14. $f(x) = a\sec x + b\,\mathrm{cosec}\,x$

$f'(x) = a\sec x\tan x - b\,\mathrm{cosec}\,x\cot x$

$f'(x) = 0 \Rightarrow a\sec x\tan x = b\,\mathrm{cosec}\,x\cot x$

$\Rightarrow \dfrac{a\sin x}{\cos^2 x} = \dfrac{b\cos x}{\sin^2 x} \Rightarrow \tan^3 x = \dfrac{b}{a}.$

$\Rightarrow \tan x = \left(\dfrac{b}{a}\right)^{1/3}$



$f(x)_{min} = \dfrac{a\cdot\sqrt{a^{2/3}+b^{2/3}}}{a^{1/3}} + \dfrac{b\cdot\sqrt{a^{2/3}+b^{2/3}}}{b^{1/3}}.$

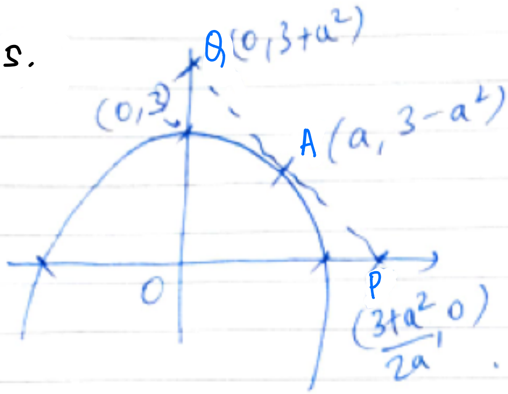
$f(x)_{min} = \left(\sqrt{a^{2/3}+b^{2/3}}\right)\left(a^{2/3}+b^{2/3}\right)$

$f(x)_{max} \to \infty$

$\Rightarrow f(x)_{min} = (a^{2/3}+b^{2/3})^{3/2}$ (Ans C).

15. P is a point on positive x-axis, Q is a point on the positive y-axis and 'O' is the origin. If the line passing through P and Q is tangent to the curve $y = 3 - x^2$ then the minimum area of the triangle OPQ, is

15.



$y = 3 - x^2 \Rightarrow y' = -2x\Big|_{x=a}$

$\Rightarrow m_T = -2a.$

$\therefore$ Tangent at A:

$\therefore y - (3-a^2) = -2a(x-a)$

$\therefore P\left(\frac{3+a^2}{2a}, 0\right); Q(0, 3+a^2)$

$Ar.(\Delta OPQ) = \frac{1}{2}\left(\frac{3+a^2}{2a}\right)(3+a^2) = A.$

$\frac{dA}{da} = 0 \Rightarrow a = 1$ or $a = 3$. (reject)

$\therefore A_{min} = \frac{1}{2}\left(\frac{3+1}{2}\right)(3+1) = 4$ (Ans. B)

16. A minimum value of sinx cos2x is-

(A) 1 (B) –1 (C) $-2/3\sqrt{6}$ (D) None of these

Sol^n $\quad \varepsilon = \sin x \cos 2x = (\sin x)(1 - 2\sin^2 x)$

$$\Rightarrow \varepsilon = \sin x - 2\sin^3 x$$

$$\Rightarrow \frac{d\varepsilon}{dx} = \cos x - 6\sin^2 x \cdot \cos x$$

$$\Rightarrow \frac{d\varepsilon}{dx} = \cos x - 6(1 - \cos^2 x) \cdot \cos x$$

$$\Rightarrow \frac{d\varepsilon}{dx} = (\cos x)(1 - 6 + 6\cos^2 x)$$
$$= -5\cos x + 6\cos^3 x$$

$$\frac{d\varepsilon}{dx} = 0 \Rightarrow \cos x = 0 \text{ or } \cos x = \sqrt{\frac{5}{6}} \text{ or } -\sqrt{\frac{5}{6}}$$

$$\frac{d^2\varepsilon}{dx^2} = 5\sin x - 18\cos^2 x \sin x$$

Minima corresponds to $\cos x = 0$, for which $\frac{d^2\varepsilon}{dx^2}$ is +ve $\Rightarrow x = \pi/2$

$\therefore \varepsilon_{min} = -1$ at $x = \pi/2$ (Ans B)

17. The rate of change of the function $f(x) = 3x^5 - 5x^3 + 5x - 7$ is minimum when

(A) $x = 0$ (B) $x = 1/\sqrt{2}$ (C) $x = -1/\sqrt{2}$ (D) $x = \pm 1/\sqrt{2}$

17. Rate of change of $f(x)$. $f'(x)$

$f'(x) = 15x^4 - 15x^2 + 5$

For Max/min : $f''(x) = 60x^3 - 30x = 0 \Rightarrow x = 0, \pm 1/\sqrt{2}$

$f''(x) = 30x(\sqrt{2}x - 1)(\sqrt{2}x + 1)$

$-$ $+$ $-$ $+$

$-1/\sqrt{2}$ 0 $1/\sqrt{2}$

Min at $\pm 1/\sqrt{2}$ (Ans D).

18. For the function $f(x) = \int_0^x \frac{\sin t}{t} dt$, where $x > 0$,

(A) maximum occurs at $x = n\pi$, n is even

(B) minimum occurs at $x = n\pi$, n is odd

(C) maximum occurs at $x = n\pi$, n is odd

(D) None of these

Solⁿ.

$$f'(x) = \frac{\sin x}{x}.$$

$$f''(x) = \frac{x\cos x - \sin x}{x^2}.$$

$f''((2k+1)\pi) < 0 \Rightarrow$ Max at odd multiple of π.

$f''(2k\pi) > 0 \Rightarrow$ Min. at even multiple of π

19. The set of value(s) of 'a' for which the function $f(x)=\frac{ax^3}{3}+(a+2)x^2+(a-1)x+2$ posses a negative point of inflection.

(A) $(-\infty,-2)\cup(0,\infty)$ (B) $\{-4/5\}$ (C) $(-2, 0)$ (D) empty set

Soln

$$f'(x)=3ax^2+2(a+2)x+a-1$$

$$f''(x)=6ax+2(a+2)=0$$

$$\Rightarrow \quad x=-\frac{(a+2)}{3a}<0$$

$$\Rightarrow \quad \frac{a+2}{a}>0$$

$+$ -2 $-$ 0 $+$

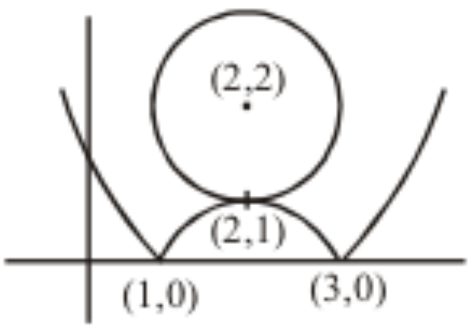
$$\Rightarrow \quad a\in(-\infty,-2)\cup(0,\infty)$$ (Ans A)

Exercise 02

1. If (a, b) be the point on the curve $y = |x^2 - 4x + 3|$ which is nearest to the circle $x^2 + y^2 - 4x - 4y + 7 = 0$, then $(a + b)$ is equal to -

(A) $\frac{7}{4}$ (B) 0 (C) 3 (D) 2

Sol^n



Clearly both the curves touches at (2, 1)

So required point is (2, 1)

(Ans C)

EXERCISE (O-2)

[SINGLE CORRECT CHOICE TYPE]

2. If $f(x)=\begin{cases} e^{x+1}-e^{x} & x\le 0 \\ e^{1-x}-1 & 0<x<1 \\ x+\ell nx & x\ge 1 \end{cases}$, then -

(A) x = 0 is point of local maxima, x = 1 is neither local maxima nor local minima.

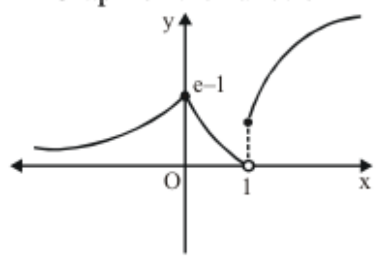
(B) x = 1 is point of local minima, x = 0 is point of local maxima

(C) x = 0 and x = 1 both are points of local maxima

(D) x = 0 and x = 1 both are points of local minima

Ans. (A)

Graph of the function



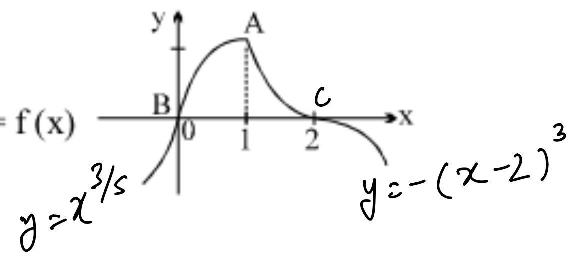
hence x = 0 is point of maxima
and x = 1 is neither maxima nor minima.

3. Let $f(x) = \begin{cases} x^{3/5} & \text{if } x \le 1 \\ -(x-2)^3 & \text{if } x > 1 \end{cases}$, then the number of critical points on the graph of the function is

(A) 1 (B) 2 (C) 3 (D) 4

Sol^n

A, B, C are the 3 critical points of $y = f(x)$



(Ans. C)

4. The set of all values of 'a' for which the function,

$f(x) = (a^2 - 3a + 2)\left(\cos^2\frac{x}{4} - \sin^2\frac{x}{4}\right) + (a-1)x + \sin 1$ does not possess critical points is:

(A) $[1, \infty)$ (B) $(0, 1) \cup (1, 4)$ (C) $(-2, 4)$ (D) $(1, 3) \cup (3, 5)$

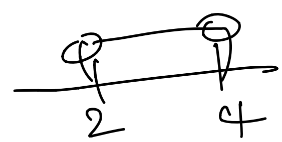
Soln

$f(x) = (a^2-3a+2)\left(\cos\frac{x}{2}\right) + (a-1)x + \sin 1$

$f'(x) = (a^2-3a+2)\left(-\frac{\sin x/2}{2}\right) + a - 1 = 0$

$\Rightarrow \sin(x/2) = \dfrac{2(a-1)}{a^2-3a+2} = \dfrac{2\cancel{(a-1)}}{\cancel{(a-1)}(a-2)}$ $(a \neq 1)$

$\Rightarrow \sin(x/2) = \dfrac{2}{a-2}$



For No critical pts: $\dfrac{2}{a-2} > 1$ or $\dfrac{2}{a-2} < -1$

on solving we get: $a \in (0,2) \cup (2,4) - \{1\}$

Check at $a = 2$: $f(x) = x + \sin 1$

$\therefore$ $a = 2$ accepted

Check at $a = 1$: $f(x) = \sin 1$

Rejected ($\because \infty$ critical pts)

$\therefore$ $a \in (0,1) \cup (1,4)$ (Ans B)

5. Let $f(x) = ax^2 - b|x|$, where a and b are constants. Then at $x = 0$, f(x) has

(A) a maxima whenever $a > 0$, $b > 0$

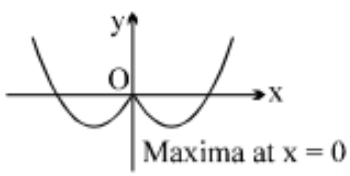
(B) a maxima whenever $a > 0$, $b < 0$

(C) minima whenever $a > 0$, $b > 0$

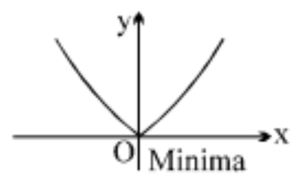
(D) neither a maxima nor minima whenever $a > 0$, $b < 0$

5. A

Sol. for $a > 0$, $b > 0$



for $a > 0$, $b < 0$



6. There are 50 apple trees in an orchard. Each tree produces 800 apples. For each additional tree planted in the orchard, per tree the output drop by 10 times the number of additional trees . Number of trees that should be added to the existing orchard for maximising the output of the trees, is

(A) 5 (B) 10 (C) 15 (D) 20

6. C

Sol. Ans. (C)

Let x tree be added then

$P(x) = (x + 50)(800 - 10x)$

now $P'(x) = 0 \Rightarrow x = 15$

7. Let $f(x)$ be a cubic polynomial such that it has point of inflection at x = 2 and local minima at x = 4, then-

(A) $f(x)$ has local minima at x = 0

(B) $f(x)$ has local maxima at x = 0

(C) $\lim_{x\to\infty} f(x) \to \infty$

(D) $\lim_{x\to\infty} f(x) \to -\infty$

Ans. (B,C)

Sol

$$f''(x) = a(x-2)$$

$$f'(x) = a\left(\frac{x^2}{2} - 2x\right) + b$$

$$f'(4) = 0 \Rightarrow b = 0$$

$$f'(x) = \frac{ax(x-4)}{2}$$

$a > 0$

At x = 0 it has local minima

$$f(x) = a\left(\frac{x^3}{6} - x^2\right) + c$$

$$\lim_{x\to\infty} f(x) \to \infty$$

8. The function $f(x)=\int_{-2015}^{x} t\left(e^{t}-e^{2}\right)\left(e^{t}-1\right)(t+2014)^{2015}(t-2015)^{2016}(t-2016)^{2017}\,dt$ has-

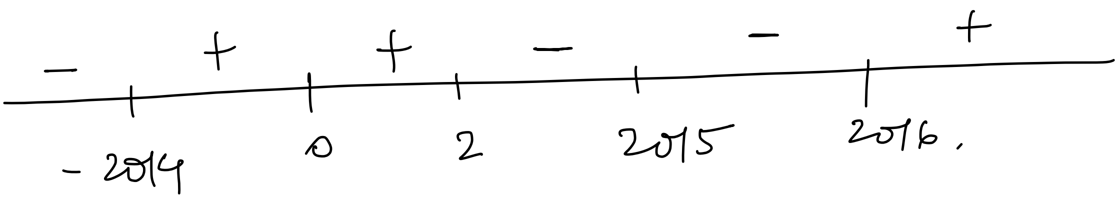
(A) local minima at x = –2014

(B) local minima at x = 2

(C) local maxima at x = 2

(D) local minima at x = 2016

Sol^n. $f'(x) = x\left(e^{x}-e^{2}\right)\left(e^{x}-1\right)(x+2014)^{2015}(x-2015)^{2016}(x-2016)^{2017}$



$\therefore$ Minima at -2014 and 2016.

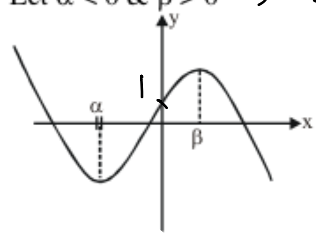
Maxima at 2

9. Let $f(x) = ax^3 + bx^2 + cx + 1$ have extrema at $x = \alpha, \beta$ such that $\alpha\beta < 0$ and $f(\alpha) \cdot f(\beta) < 0$, then which of the following can be true for the equation $f(x) = 0$?

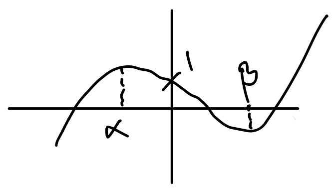
(A) three equal roots
(B) three distinct real roots
(C) one positive root if $f(\alpha) < 0$ and $f(\beta) > 0$
(D) one negative root if $f(\alpha) > 0$ and $f(\beta) < 0$.

Ans. (B,C,D)

Sol. Let $\alpha < 0$ & $\beta > 0$; $f(0) = 1$



$(f(\alpha) < 0, f(\beta) > 0)$
(three distinct real roots with one positive root)

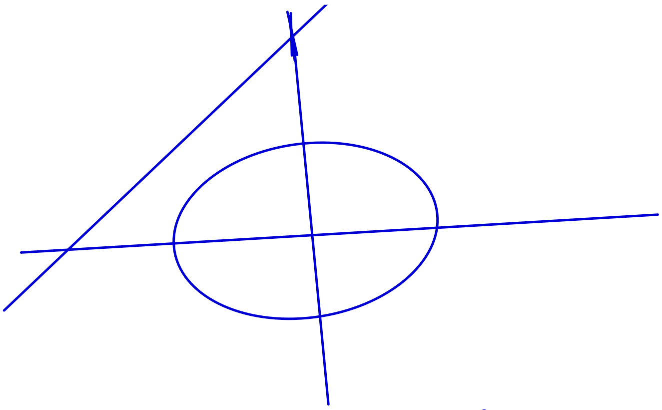


$f(\alpha) > 0, f(\beta) < 0$
three distinct real roots with one negative root.

10. Let A(p,q) and B(h,k) are points on the curve $4x^2 + 9y^2 = 1$, which are nearest and farthest from the line $72 + 8x = 9y$ respectively, then -

(A) $p+q=-\frac{1}{5}$ (B) $p+q=\frac{1}{5}$ (C) $h+k=-\frac{1}{5}$ (D) $h+k=\frac{1}{5}$

Soln.:



Variable pt on $4x^2+9y^2=1$ can be $\left(\frac{\cos\theta}{2}, \frac{\sin\theta}{3}\right)$

distance from $72+8x=9y$

$$D(\theta) = \frac{|4\cos\theta - 3\sin\theta + 72|}{\sqrt{8^2+9^2}}$$

for max. distance $4\cos\theta - 3\sin\theta = 5$

$\Rightarrow \cos\theta = \frac{4}{5}$ & $\sin\theta = -\frac{3}{5}$

$\therefore (h,k) = \left(\frac{2}{5}, -\frac{1}{5}\right)$

$\therefore h+k = \frac{1}{5}$

for minm distance $4\cos\theta - 3\sin\theta = -5$

$\Rightarrow \cos\theta = -\frac{4}{5}$ $\sin\theta = \frac{3}{5}$

$\therefore (p,q) = \left(-\frac{2}{5}, \frac{1}{5}\right)$

$\therefore p+q = -\frac{1}{5}$

11. The function $f(x) = \int_0^x \sqrt{1-t^4}\, dt$ is such that :

(A) it is defined on the interval $[-1, 1]$

(B) it is an strictly increasing function

(C) it is an odd function

(D) the point $(0, 0)$ is the point of inflection

Sol^n. clearly $f(x)$ is defined in $[-1, 1]$.

$f'(x) = \sqrt{1-x^4} > 0. \Rightarrow f(x)$ is $\uparrow$.

Also. $f(-x) = \int_0^{-x} \sqrt{1-t^4}\, dt$

$t = -y \Rightarrow dt = -dy$.

$f(-x) = -\int_0^{x} \sqrt{1-t^4}\, dt = -f(x)$.

$\therefore f(x)$ is odd.

$f''(x) = \dfrac{-4x^3}{2\sqrt{1-x^4}}$; $\overset{+}{}\underset{0}{|}\overset{-}{}$

$\therefore x=0$ is a pt. of inflection.

Exercise S1

1. A cubic f(x) vanishes at $x = -2$ & has relative minimum/maximum at $x = -1$ and $x = 1/3$. If $\int_{-1}^{1} f(x)\,dx = \frac{14}{3}$, find the cubic $f(x)$.

$f'(x) = 0$ at $x = -1$ & $x = 1/3$.

$$\Rightarrow f'(x) = a(x+1)\left(x - 1/3\right).$$

$$= a\left(x^2 + \frac{2x}{3} - \frac{1}{3}\right).$$

$$\Rightarrow f(x) = a\left(\frac{x^3}{3} + \frac{x^2}{3} - \frac{x}{3}\right) + c$$

$$= a'\left(x^3 + x^2 - x\right) + c.$$

$$f(-2) = a'(-8+4+2) + c = 0.$$

$$\Rightarrow \quad c = 2a' \text{ --- } (1)$$

Also: $\int_{-1}^{1} f(x)\,dx = \int_{-1}^{1}\left(a'\left(x^3+x^2-x\right)+c\right)dx = \frac{14}{3}.$

$$\Rightarrow 2\int_{0}^{1}\left(a'(x^2)+c\right)dx = \frac{14}{3}.$$

$$\Rightarrow \frac{a'}{3} + c = 7/3 \text{ --- } (11).$$

from (1) & (11) $\quad a' = 1 \quad c = 2.$

$$\therefore f(x) = x^3 + x^2 - x + 2.$$

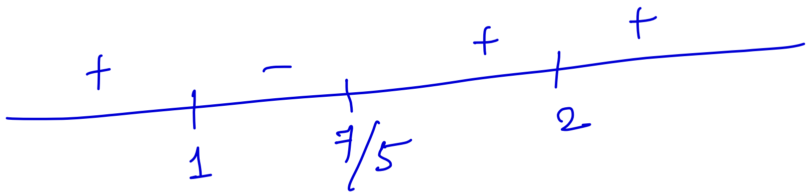
2. Investigate for maxima & minima for the function, $f(x) = \int_1^x [2(t-1)(t-2)^3 + 3(t-1)^2(t-2)^2]dt$

Sol^n.

$$f'(x) = 2(x-1)(x-2)^3 + 3(x-1)^2(x-2)^2$$

$$= (x-1)(x-2)^2[2x-4+3x-3]$$

$$= (x-1)(x-2)^2(5x-7).$$



$\therefore$ Local max at 1. & Local min at 7/5.

3. Find the greatest & least value for the function ;

(a) $y = x + \sin 2x$, $0 \le x \le 2\pi$ (b) $y = 2\cos 2x - \cos 4x$, $0 \le x \le \pi$

3(a). $y = x + \sin 2x \Rightarrow y' = 1 + 2\cos 2x$

$y' = 0 \Rightarrow \cos 2x = -1/2 \Rightarrow x = \frac{\pi}{3}, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}.$

y': $+$ on $(0, \pi/3)$, $-$ on $(\pi/3, \frac{2\pi}{3})$, $+$ on $(\frac{2\pi}{3}, \frac{4\pi}{3})$, $-$ on $(\frac{4\pi}{3}, \frac{5\pi}{3})$, $+$ on $(\frac{5\pi}{3}, 2\pi)$

$0 \quad \pi/3 \quad \frac{2\pi}{3} \quad \frac{4\pi}{3} \quad \frac{5\pi}{3} \quad 2\pi$

M m M m

$f(0) = 0$; $f(2\pi) = 2\pi$.

$f(\pi/3) = \pi/3 + \frac{\sqrt{3}}{2}$; $f(\frac{4\pi}{3}) = \frac{4\pi}{3} + \frac{\sqrt{3}}{2}$

$f(\frac{2\pi}{3}) = \frac{2\pi}{3} - \frac{\sqrt{3}}{2}$; $f(\frac{5\pi}{3}) = \frac{5\pi}{3} - \frac{\sqrt{3}}{2}$

$\therefore$ $f(x)_{max} = 2\pi$; $f(x)_{min} = 0.$

(b) $f(x) = 2\cos 2x - \cos 4x$. , $x \in [0, \pi]$.

$f'(x) = -4\sin 2x + 4\sin 4x$

$= 4(\sin 4x - \sin 2x)$

$f'(x) = 0 \Rightarrow \sin 4x - \sin 2x = 0 \Rightarrow x = 0, \frac{\pi}{2}, \pi, \frac{\pi}{6}, \frac{5\pi}{6}.$

$+$ on $(0, \pi/6)$, $-$ on $(\pi/6, \pi/2)$, $+$ on $(\pi/2, 5\pi/6)$, $-$ on $(5\pi/6, \pi)$

$0 \quad \pi/6 \quad \pi/2 \quad 5\pi/6 \quad \pi$

M m M

$f(0) = 1$; $f(\pi/6) = 2\cos\frac{\pi}{3} - \cos\frac{2\pi}{3} = 2 \cdot \frac{1}{2} - (-1/2) = 3/2$

$f(\pi/2) = 2\cos\pi - \cos 2\pi = 2(-1) - (1) = -3.$

$f(5\pi/6) = 2\cos(\frac{5\pi}{3}) - \cos(\frac{10\pi}{3}) = 2(1/2) - (-1/2) = 3/2$

$f(\pi) = 2\cos 2\pi - \cos 4\pi = 2 - 1 = 1.$

$\therefore$ $f(x)_{max} = 3/2$ at $x = \pi/6$ or $5\pi/6$

$f(x)_{min} = -3$ at $x = \pi/2$.

4. Suppose f(x) is a function satisfying the following conditions :

(i) $f(0) = 2$, $f(1) = 1$ (ii) f has a minimum value at $x = \frac{5}{2}$ and

(iii) for all x, $f'(x) = \begin{vmatrix} 2ax & 2ax-1 & 2ax+b+1 \\ b & b+1 & -1 \\ 2(ax+b) & 2ax+2b+1 & 2ax+b \end{vmatrix}$

Where a, b are some constants. Determine the constants a, b & the function f(x).

Soln.

$f'(x) = 2ax + b.$

$f'(5/2) = 5a + b = 0$ — (I)

$f(x) = ax^2 + bx + c.$

$f(0) = c = 2$ — (II)

$f(1) = a + b + c = 1$ — (III)

$a = \frac{1}{4}$ $b = -\frac{5}{4}$.

$\therefore f(x) = \frac{1}{4}x^2 - \frac{5}{4}x + 2.$

5. Let P(x) be a polynomial of degree 5 having extremum at x = –1, 1 and $\lim_{x\to 0}\left(\frac{P(x)}{x^3}-2\right)=4$. If M and m are the maximum and minimum value of the function y = P'(x) on the set $A = \{x \mid x^2 + 6 \le 5x\}$ then find $\frac{m}{M}$.

(5) $\lim_{x\to 0} \frac{P(x)}{x^3} - 2 = 4 \Rightarrow \lim_{x\to 0} \frac{P(x)}{x^3} = 6$.

Let $P(x) = ax^5 + bx^4 + cx^3 + dx^2 + ex + f$.

$\therefore \lim_{x\to 0} \frac{ax^5+bx^4+cx^3+dx^2+ex+f}{x^3} = 6$

$\Rightarrow d = e = f = 0 \; ; \; c = 6$ (for limit to exist)

$\therefore P(x) = ax^5 + bx^4 + 6x^3$

$P'(-1) = 0 \; ; \; P'(1) = 0 \Rightarrow \left.\begin{matrix} a = -18/5 \\ b = 0 \end{matrix}\right\}$

$\therefore P(x) = -\frac{18}{5}x^5 + 6x^3$.

$P'(x) = -18x^4 + 18x^2 = -18(x^4 - x^2)$

$P''(x) = -18(4x^3 - 2x) = 0$

$\Rightarrow 2x(2x^2-1) = 0 \Rightarrow x = 0 ; \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}$

$A = \{x : x^2 + 6 \le 5x\} \Rightarrow x^2 - 5x + 6 \le 0$

$\Rightarrow x \in [2, 3]$.

$\therefore P'(2) = -18(16-4) = -18\times 12 = M$

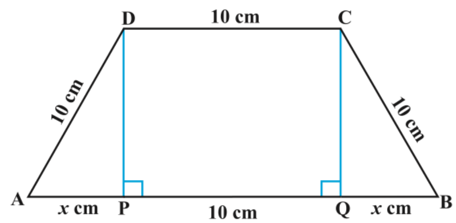
$P'(3) = -18(81-9) = -18\times 72 = m$

$\therefore \frac{m}{M} = \frac{72}{12} = 6$ Ans.

6. The length of three sides of a trapezium are equal, each being 10 cms. Find the maximum area of such a trapezium.

Soln

Let AP = x cm. Note that $\Delta APD \sim \Delta BQC$. Therefore, QB = x cm.



$$A \equiv A(x) = \frac{1}{2}\text{(sum of parallel sides) (height)}$$

$$= \frac{1}{2}(2x+10+10)\left(\sqrt{100-x^2}\right)$$

$$= (x+10)\left(\sqrt{100-x^2}\right)$$

$$A'(x) = (x+10)\frac{(-2x)}{2\sqrt{100-x^2}} + \left(\sqrt{100-x^2}\right)$$

$$= \frac{-2x^2-10x+100}{\sqrt{100-x^2}}$$

$A'(x) = 0$ gives $2x^2 + 10x - 100 = 0$, i.e., $x = 5$ and $x = -10$.

reject

Thus, area of trapezium is maximum at $x = 5$ and the area is given by

$$A(5) = (5+10)\sqrt{100-(5)^2} = 15\sqrt{75} = 75\sqrt{3}\ \text{cm}^2$$

7. Of all the lines tangent to the graph of the curve $y = \frac{6}{x^2+3}$, find the equations of the tangent lines of minimum and maximum slope.

7. $y = \frac{6}{x^2+3} \Rightarrow y' = -6 \cdot \frac{2x}{(x^2+3)^2}$ (slope function)

$$y'' = -12\left[\frac{(x^2+3)^2 - x \cdot 2(x^2+3) \cdot 2x}{(x^2+3)^4}\right] = 0.$$

$$\Rightarrow (x^2+3)(x^2+3-4x^2) = 0 \Rightarrow x^2 = 1$$

$$\Rightarrow x = 1, -1.$$

At $x = 1$: $y = \frac{6}{4} = 3/2$; $y'(1) = \frac{-12(1)}{(1^2+3)^2} = -3/4$;

T: $y - 3/2 = -\frac{3}{4}(x-1)$

$\Rightarrow 3x + 4y - 9 = 0$

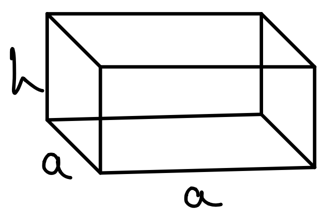
At $x = -1$: $y = \frac{6}{4} = \frac{3}{2}$; $y'(-1) = 3/4$.

T: $y - 3/2 = \frac{3}{4}(x+1)$

$\Rightarrow 3x - 4y + 9 = 0$

8. A closed rectangular box with a square base is to be made to contain 1000 cubic feet. The cost of the material per square foot for the bottom is 15 paise, for the top 25 paise and for the sides 20 paise. The labour charges for making the box are Rs. 3/-. Find the dimensions of the box when the cost is minimum.

Sol^n



$a^2 h = 1000$

$\Rightarrow h = \dfrac{1000}{a^2}$

$C = 15a^2 + 25a^2 + 4ah \cdot 20 + 300$

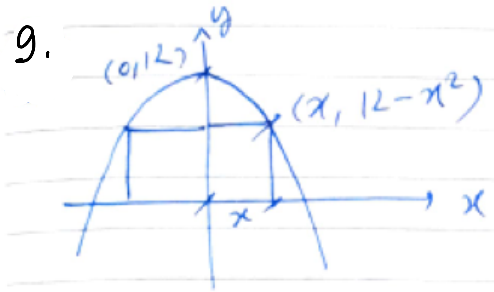
$\Rightarrow C = 40a^2 + 80 \cdot a \cdot \dfrac{1000}{a^2} + 300$

$\dfrac{dC}{da} = 0 = 80a - \dfrac{80000}{a^2} = 0 \Rightarrow a = 10'$

$\therefore h = 10'$

9. Find the area of the largest rectangle with lower base on the x-axis & upper vertices on the curve $y = 12 - x^2$.

9.



$A = 2x(12 - x^2)$

$\Rightarrow A = (12x - x^3).2$.

$\frac{dA}{dx} = 12 - 3x^2 = 0$

$\Rightarrow x = 2$

$\therefore A_{max} = 2(24 - 8) = 16 \cdot 2 = 32$ Ans

10. If $y = \dfrac{ax + b}{(x-1)(x-4)}$ has a turning value at (2, −1) find a & b and show that the turning value is a maximum.

10. $y = \dfrac{ax+b}{(x-1)(x-4)}$ Turning at $x = 2$, $y = -1$.

$\therefore -1 = \dfrac{2a+b}{(1)(-2)} \Rightarrow 2a + b = 2 \longrightarrow (\text{i})$

$y' = 0$ at $x = 2$: $\left. \dfrac{-ax^2 - 2bx + 4a + 5b}{(x^2 - 5x + 4)^2} \right|_{x=2} = 0$

$\Rightarrow -4a - 4b + 4a + 5b = 0 \Rightarrow b = 0. \longrightarrow (\text{ii})$

by (i) & (ii):

$a = 1;\ b = 0.$

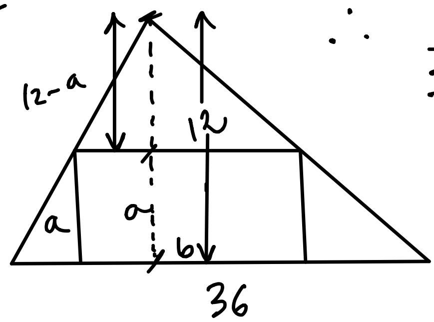
$\therefore y' = \dfrac{-x^2 + 4}{(x^2 - 5x + 4)^2} = \dfrac{-(x-2)(x+2)}{(\quad)^2}$

$+$ | $-$

-2 | 2 Max.

11. What are the dimensions of the rectangular plot of the greatest area which can be laid out within a triangle of base 36 ft. & altitude 12 ft ? Assume that one side of the rectangle lies on the base of the triangle.

Sol^n



$$\therefore \frac{12}{36} = \frac{12-a}{b} \Rightarrow b = 36-3a$$

$$A = ab = a(36-3a)$$

$$\Rightarrow A = 36a - 3a^2$$

$$A' = 36 - 6a = 0 \Rightarrow a = 6'$$

$$\therefore b = 18'$$

12. Let f(x) be a cubic polynomial which has local maximum at x = –1 and f'(x) has a local minimum at x = 1. If f(–1) = 10 and f(3) = –22, then find the distance between its two horizontal tangents.

12. Let $f''(x) = 6a(x-1)$, $\left(a \neq 0 \because \text{cubic}\right)$

$\Rightarrow f'(x) = 6a\left(\frac{x^2}{2} - x\right) + b = 3a(x^2 - 2x) + b.$

$f'(-1) = 0 \Rightarrow 9a + b = 0. \Rightarrow b = -9a.$

$\therefore f'(x) = 3a(x^2 - 2x - 3)$

$f'(x) = 0 \Rightarrow x = -1,\ x = 3.$

$\therefore$ $y = f(-1)$ & $y = f(3)$ are two horizontal tangents.

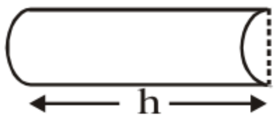
Dist. b/w horizontal tangents $= |f(-1) - f(3)|$

$= (10 - (-22)) = 32.$

Exercise S2

1. A given quantity of metal is to be casted into a half cylinder i.e. with a rectangular base and semicircular ends. Show that in order that total surface area may be minimum , the ratio of the height of the cylinder to the diameter of the semi circular ends is $\pi/(\pi+2)$.

Sol.



$$A = 2rh + \pi r^2 + \pi rh$$

$$V = \frac{\pi r^2 h}{2} \Rightarrow h = \frac{2V}{\pi r^2}$$

To find :

$$\frac{h}{2r} = \frac{1}{2r}\left(\frac{2V}{\pi r^2}\right)$$

$$\Rightarrow \frac{h}{2r} = \frac{V}{\pi r^3}$$

$$A = \frac{2r.2V}{\pi r^2} + \pi r^2 + \frac{\pi r.2V}{\pi r^2}$$

$$\Rightarrow A = \frac{4V}{\pi r} + \pi r^2 + \frac{2V}{r}$$

For A_{min}, $\frac{dA}{dr} = 0$

$$\Rightarrow \frac{dA}{dr} = -\frac{4V}{\pi r^2} + 2\pi r - \frac{2V}{r^2} = 0$$

$$\Rightarrow \quad \pi r = \frac{2V}{\pi r^2} + \frac{V}{r^2} \Rightarrow \pi r = \frac{V}{r^2}\left(\frac{2}{\pi} + 1\right)$$

$$\Rightarrow \quad \frac{V}{\pi r^3} = \frac{\pi}{\pi+2} = \frac{h}{2r}$$

Hence Proved

2. Consider the function $f(x) = \begin{bmatrix} \sqrt{x}\,\ell nx & \text{when} & x > 0 \\ 0 & \text{for} & x = 0 \end{bmatrix}$

(a) Find whether f is continuous at $x = 0$ or not.

(b) Find the absolute minima and absolute maxima (if they exist).

(c) Does $f'(0)$? Find $\lim\limits_{x\to 0} f'(x)$.

(d) Find the inflection points of the graph of $y = f(x)$..

Sol. (a) $\lim\limits_{x\to 0^+} \sqrt{x}\,\ell nx = \lim\limits_{x\to 0^+} \dfrac{\ell nx}{\left(\dfrac{1}{\sqrt{x}}\right)}$

Use L' Hospital Rule

$$\lim_{x\to 0^+} \frac{-x^{3/2}}{2x} = \lim_{x\to 0^+} \frac{-\sqrt{x}}{2} = 0$$

(b) $f'(x) = \sqrt{x} \times \dfrac{1}{x} + \dfrac{\ell nx}{2\sqrt{x}} = \dfrac{2 + \ell nx}{\sqrt{x}}$

$f'(x) = 0$ at $x = e^{-2}$

(c) $f'(0)$ = does not exists

$$\lim_{x\to 0^+} f'(x) = \lim_{x\to 0} \frac{2 + \ell nx}{\sqrt{x}}$$

Use L' Hospital Rule

$$\lim_{x\to 0} \frac{2\sqrt{x}}{x} = \lim_{x\to 0} \frac{1}{\sqrt{x}} \to \infty$$

Does not exist

(d) $f''(x) = \dfrac{\sqrt{x}\left(\dfrac{1}{x}\right) - (2 + \ell nx)\dfrac{1}{2\sqrt{x}}}{x} = \dfrac{-\ell nx}{2x\sqrt{x}}$

$f''(x) = 0 \Rightarrow x = 1$

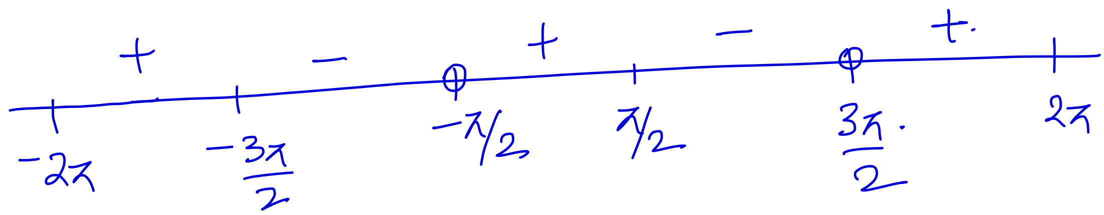
at $x = 1$ f'' change its sign.

Soln.

a) $f(x) = 0 = \ln(1+\sin x)$

$\Rightarrow \sin x = 0 \Rightarrow x = -2\pi, -\pi, 0, \pi, -2\pi.$

c). $f'(x) = \dfrac{\cos x}{1+\sin x} = 0.$



$\therefore$ Maxima at $-\frac{3\pi}{2}, \frac{\pi}{2}, 2\pi.$

Minima at -2π

(b) $f''(x) = \dfrac{(1+\sin x)(-\sin x) - \cos^2 x}{(1+\sin x)^2}$

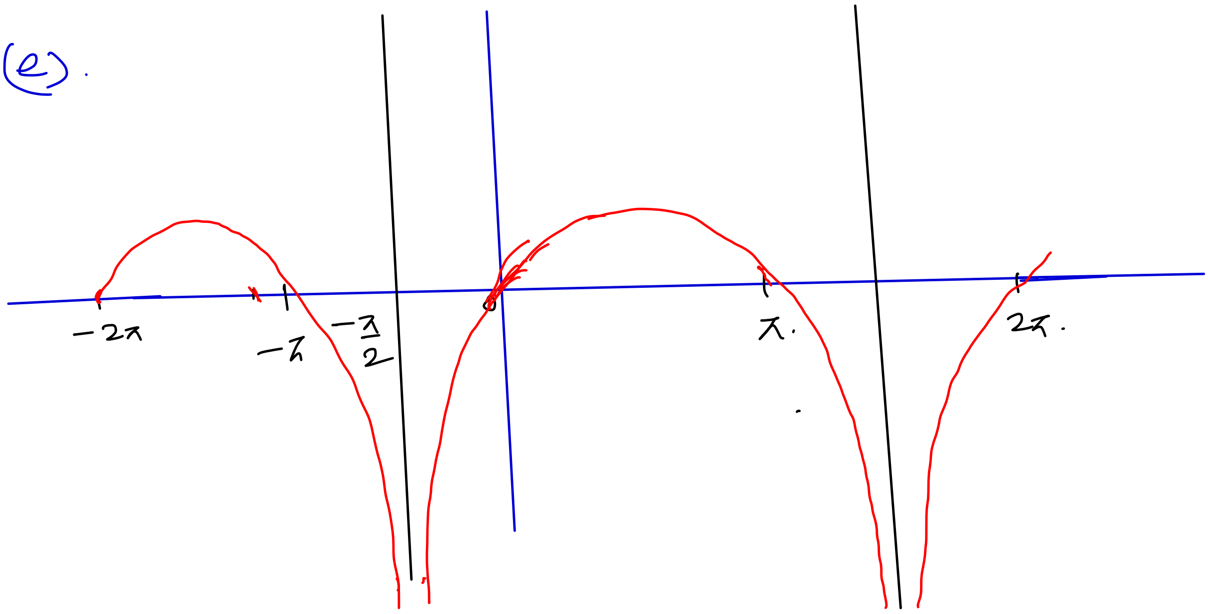
$= \dfrac{-(\sin x + 1)}{(1+\sin x)^2} = \dfrac{-1}{1+\sin x} < 0.$

$\therefore$ No pt of inflection.

(d) $\frac{dy}{dx}$ becomes undefined at $x = -\pi/2$ & $3\pi/2$

$\therefore$ assymptotes are $x = -\pi/2$ & $x = 3\pi/2$

(e).



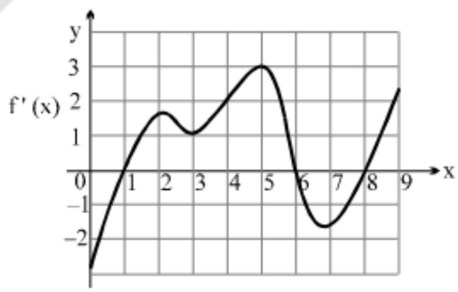
$$I = \int_{-\pi/2}^{\pi/2} \ln(1+\sin x)\,dx.$$

$$I = \int_{-\pi/2}^{\pi/2} \ln(1-\sin x)\,dx$$

$$2I = \int_{-\pi/2}^{\pi/2} \ln(\cos^2 x)\,dx = 2\int_{0}^{\pi/2} 2\ln\cos x\,dx.$$

$$\Rightarrow I = 2\left[-\frac{\pi}{2}\ln 2\right] = -\pi\ln 2.$$

4. The graph of the derivative f' of a continuous function f is shown with $f(0) = 0$



(i) On what intervals f is strictly increasing or strictly decreasing?

(ii) At what values of x does f have a local maximum or minimum?

(iii) On what intervals is $f'' > 0$ or $f'' < 0$

(iv) State the x-coordinate(s) of the point(s) of inflection.

(v) Assuming that $f(0) = 0$, sketch a graph of f.

Solⁿ (i) For ↑ $f'(x) \geq 0$ and for ↓ $f'(x) \leq 0$

$\Rightarrow$ ↑ in $[1,6]$; $[7,8]$; ↓ in $[0,1]$; $[6,8]$

(ii) $f'(x) = 0$ and changes its sign

LM at $x = 0, 6, 9$ { $x = 0$ included as fⁿ. starts and decreases
$x = 9$ included as fⁿ. ends while it was incr. }

Lm at $x = 1, 8$

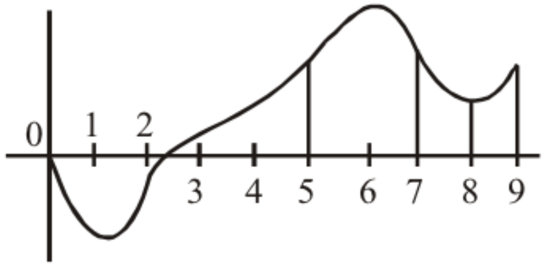
(iii) To find intervals where $f'(x)$ is ↑ or ↓

$f''(x) > 0 \Rightarrow x \in (0,2)$; $(3,5)$; $(7,9)$

$f''(x) < 0 \Rightarrow x \in (2,3)$; $(5,7)$

(iv) Pts. where $f''(x) = 0 \Rightarrow x = 2, 3, 5, 7$

(v)



Estimated graph of $f(x)$

5. Find the set of value of m for the cubic $x^3 - \frac{3}{2}x^2 + \frac{5}{2} = \log_{1/4}(m)$ has 3 distinct solutions.

Sol. $f(x) = x^3 - \frac{3}{2}x^2 + \frac{5}{2} - \log_{1/4}(m)$

$f'(x) = 3x^2 - 3x = 3x(x-1)$

$$\begin{array}{ccccc} + & & - & & + \\ \hline & 0 & & 1 & \end{array}$$

for $f(x) = 0$ to have 3 real & distinct roots

$f(0).f(1) < 0$

$\Rightarrow \left(\frac{5}{2} + \log_4 m\right)\left(2 + \log_4 m\right) < 0$

$\log_4 m$

$$\begin{array}{ccc} & \circ\!\!-\!\!-\!\!-\!\!-\!\!-\!\!\circ & \\ \hline -5/2 & & -2 \end{array}$$

$\therefore -5/2 < \log_4 m < -2$

$\Rightarrow \quad m \in \left(4^{-5/2}, 4^{-2}\right)$

$\Rightarrow \quad m \in \left(\frac{1}{32}, \frac{1}{16}\right)$

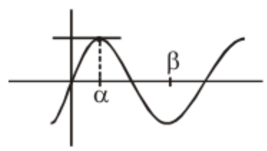
6. The value of 'a' for which $f(x) = x^3 + 3(a-7)x^2 + 3(a^2-9)x - 1$ have a positive point of maximum lies in the interval $(a_1, a_2) \cup (a_3, a_4)$. Find the value of $a_2 + 11a_3 + 70a_4$.

Sol. $f'(x) = 3x^2 + 6(a-7)x + 3(a^2-9)$

$D = 36(a-7)^2 - 4 \times 3 \times 3(a^2-9)$

$= 36(a^2 + 49 - 14a - a^2 + 9)$

$= 36(58 - 14a)$



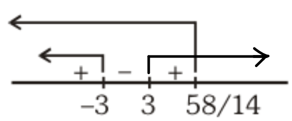
$f'(x) = 0$

at $\quad x = \dfrac{-6(a-7) \pm \sqrt{36(58-14a)}}{6}$

$\alpha = 7 - a - \sqrt{58-14a}$, $\beta = 7 - a + \sqrt{58-14a}$

Now $\alpha > 0$ & $D > 0$

$a^2 - 9 > 0$ & $a < \dfrac{58}{14}$



$a \in (-\infty, -3) \cup \left(3, \dfrac{58}{14}\right)$

So $a_2 = -3$

$a_3 = 3$

$a_4 = \dfrac{58}{14}$

$a_2 + 11a_3 + 70a_4 = 320$

7. The function f (x) defined for all real numbers x has the following properties $f(0) = 0$, $f(2) = 2$ and $f'(x) = k(2x - x^2)e^{-x}$ for some constant $k > 0$. Find

(a) the intervals on which f is strictly increasing and strictly decreasing and any local maximum or minimum values.

(b) the intervals on which the graph f is concave down and concave up.

(c) the function f (x) and plot its graph.

Solⁿ

$$f'(x) = k(2x - x^2)e^{-x}$$

Sign of f'(x) $\quad - \;\; 0 \;\; + \;\; 2 \;\; -$

(a) strictly increasing in [0,2] and strictly decreasing in $(-\infty, 0]$; $[2, \infty)$,

$$f''(x) = k(2x - x^2)(-e^{-x}) + ke^{-x}(2 - 2x)$$

$$= ke^{-x}[2 - 2x - 2x + x^2]$$

$$= ke^{-x}(x^2 - 4x + 2)$$

Sign of f''(x) $\quad + \;\; 2-\sqrt{2} \;\; - \;\; 2+\sqrt{2} \;\; +$

(b) concave up for $(-\infty, 2-\sqrt{2}]$; $[2+\sqrt{2}, \infty)$ and concave down in $[2-\sqrt{2}, 2+\sqrt{2}]$

(c) $\int f'(x)\,dx = \int k\, e^{-x}(x^2 - 2x)\,dx$

$\Rightarrow f(x) = \int k e^{-x}((-x)^2 + (2)(-x))\,dx$

put $-x = t \Rightarrow -dx = dt$

$\Rightarrow \quad f(x) = -k\int e^{t}(t^2 + 2t)\,dt \Rightarrow f(x) = -ke^{t}(t^2) + C$

$\Rightarrow f(x) = ke^{-x} \cdot x^2 + C$

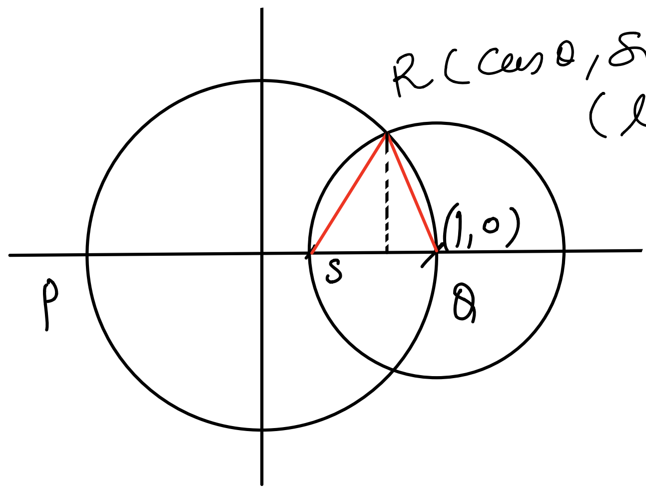
$f(0) = 0 \Rightarrow C = 0$; $f(2) = 2 \Rightarrow 2 = ke^{-2} \cdot 4$

$\Rightarrow k = e^2/2$

$\therefore f(x) = \frac{e^2}{2} \cdot e^{-x} \cdot x^2 = \frac{x^2}{2} \cdot e^{2-x}$

8. The circle $x^2 + y^2 = 1$ cuts the x-axis at P & Q. Another circle with centre at Q and variable radius intersects the first circle at R above the x-axis & the line segment PQ at S. Find the maximum area of the triangle QSR.

Sol^n



$$r = \sqrt{(\cos\theta - 1)^2 + \sin^2\theta}$$

$$\Rightarrow r = \sqrt{2(1-\cos\theta)} = 2\sin\theta/2$$

$$Ar(\Delta QRS) = \frac{1}{2}(\sin\theta)(2\sin\theta/2)$$

$$\Rightarrow \quad A = \sin\theta \cdot \sin\theta/2$$

$$\frac{dA}{d\theta} = \cos\theta \cdot \sin\theta/2 + \frac{\sin\theta \cdot \cos\theta/2}{2} = 0$$

$$\Rightarrow \tan\theta/2 = \sqrt{2}$$

$$\Rightarrow A_{max} = \left(\frac{2\cdot\sqrt{2}}{1+2}\right)\cdot\left(\frac{\sqrt{2}}{\sqrt{3}}\right) = \frac{4}{3\sqrt{3}} \text{ Sq. Units}$$

(Ans)

9. Find all the values of the parameter 'a' for which the function ;

f(x) = 8ax – a sin 6x – 7x – sin 5x increasing & has no critical points for all x ∈ R.

Solⁿ $\therefore f'(x) > 0$

$\Rightarrow f'(x) = 8a - 6a\cos 6x - 7 - 5\cos 5x > 0$

$\Rightarrow 8a - 7 > \underbrace{6a\cos 6x + 5\cos 5x}_{\text{Max. Value } 6a+5}$

At $x = 2k\pi, k \in I$

$\Rightarrow 8a - 7 > 6a + 5$

$\Rightarrow a > 6$ Ans

Exercise JM

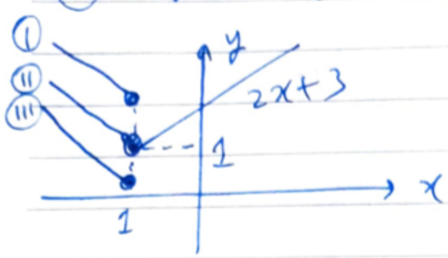
EXERCISE (JM)

1. Let $f : R \rightarrow R$ be defined by $f(x) = \begin{cases} k - 2x, & \text{if } x \leq -1 \\ 2x + 3, & \text{if } x > -1 \end{cases}$ **[AIEEE-2010]**

If f has a local minimum at $x = -1$, then a possible value of k is :

(1) 1 (2) 0 (3) $-\frac{1}{2}$ (4) -1

(1) For local minimum at $x = -1$:



Min only in cond^n. (II) or (III)

$\Rightarrow K - 2(-1) \leq 2(-1) + 3$

$\Rightarrow K + 2 \leq 1$

$\Rightarrow K \leq -1.$

$\therefore$ (Ans. 4).

2. For $x \in \left(0, \frac{5\pi}{2}\right)$, define $f(x) = \int_0^x \sqrt{t}\sin t\, dt$. Then f has :- **[AIEEE-2011]**

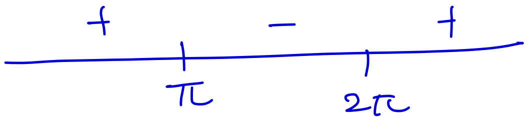
(1) local minimum at π and local maximum at 2π

(2) local maximum at π and local minimum at 2π

(3) local maximum at π and 2π

(4) local minimum at π and 2π

<u>Soln.</u> $f'(x) = \sqrt{x}\sin x = 0 \Rightarrow x = \pi, 2\pi$



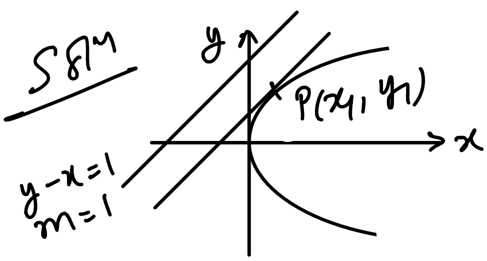
Maxima at $x = \pi$.

Minima at $x = 2\pi$.

3. The shortest distance between line $y - x = 1$ and curve $x = y^2$ is :- **[AIEEE-2011]**

(1) $\frac{8}{3\sqrt{2}}$ (2) $\frac{4}{\sqrt{3}}$ (3) $\frac{\sqrt{3}}{4}$ (4) $\frac{3\sqrt{2}}{8}$

Solⁿ



$y^2 = x \Rightarrow 2yy' = 1 \Rightarrow y' = \frac{1}{2y}$

$\left.\frac{dy}{dx}\right|_P = 1 \Rightarrow \frac{1}{2y_1} = 1 \Rightarrow y_1 = 1/2$

$\therefore x_1 = y_1^2 = 1/4 \Rightarrow P(1/4, 1/2)$

Dist of P from $x - y + 1 = 0$ is $\left|\frac{1/4 - 1/2 + 1}{\sqrt{2}}\right| = \frac{3\sqrt{2}}{8}$

(Ans 4)

4. Let f be a function defined by $f(x)=\begin{cases}\dfrac{\tan x}{x}, & x\neq 0\\ 1, & x=0\end{cases}$

Statement - 1: x = 0 is point of minima of f.

Statement - 2: f′(0) = 0. **[AIEEE-2011]**

(1) Statement-1 is false, statement-2 is true.

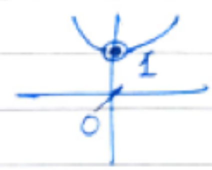
(2) Statement-1 is true, statement-2 is true; Statement-2 is correct explanation for statement-1.

(3) Statement-1 is true, statement-2 is true; Statement-2 is not a correct explanation for statement-1.

(4) Statement-1 is true, statement-2 is false.

4. $f(x)=\begin{cases}\dfrac{\tan x}{x} & x\neq 0.\\ 1 & x=0.\end{cases}$

$\dfrac{\tan x}{x} > 1$ in nbd. of $x=0$.



$\therefore$ $x=0$ is pt. of minima

$f'(0)=\lim\limits_{h\to 0}\dfrac{\dfrac{\tan h}{h}-1}{h} = \lim\limits_{h\to 0}\dfrac{\tan h-h}{h^2}$

$\Rightarrow f'(0)=\dfrac{\sec^2 h-1}{2h} = \lim\limits_{h\to 0}\dfrac{1-\cos^2 h}{2h} = 0.$

$\therefore$ (1) & (2) are true but not related.

$\therefore$ (Ans. 3)

5. A spherical balloon is filled with 4500π cubic meters of helium gas. If a leak in the balloon causes the gas to escape at the rate of 72π cubic meters per minute, then the rate (in meters per minute) at which the radius of the balloon decreases 49 minutes after the leakage began is : **[AIEEE-2012]**

(1) 9/2 (2) 9/7 (3) 7/9 (4) 2/9

5. $V_{t=49} = 4500\pi - 72\pi \times 49 = 972\pi$

$\therefore 972\cancel{\pi} = \frac{4}{3}\cancel{\pi} r^3 \Rightarrow r^3 = 729 \Rightarrow r = 9$

$V = \frac{4}{3}\pi r^3 \Rightarrow \frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt} \Rightarrow -72\cancel{\pi} = 4\cancel{\pi}(9)^2 \frac{dr}{dt}$

$\Rightarrow \frac{dr}{dt} = -\frac{72}{4\times 9\times 9} = -\frac{2}{9}$ m/s ($\because$ -ve decreasing)

(Ans 4)

6. Let a, b ∈ R be such that the function f given by $f(x) = \ln|x| + bx^2 + ax, x \neq 0$ has extreme values at $x = -1$ and $x = 2$.

Statement–1 : f has local maximum at $x = -1$ and at $x = 2$.

Statement–2 : $a = \frac{1}{2}$ and $b = \frac{-1}{4}$. **[AIEEE-2012]**

(1) Statement–1 is true, Statement–2 is false.

(2) Statement–1 is false, Statement–2 is true.

(3) Statement–1 is true, Statement–2 is true ; Statement–2 is a correct explanation for Statement–1.

(4) Statement–1 is true, Statement–2 is true ; Statement–2 is not a correct explanation for Statement–1.

Soln $f'(x) = \frac{1}{x} + 2bx + a$

$f'(-1) = 0 \Rightarrow -1 - 2b + a = 0 \Rightarrow \boxed{a - 2b = 1}$

$f'(2) = 0 \Rightarrow 1/2 + b + a = 0 \Rightarrow \boxed{a + b = -1/2}$

Solving these : $a = 1/2$; $b = -1/4$

$f''(x) = \frac{-1}{x^2} + 2b \Rightarrow f''(x) = \frac{-1}{x^2} - \frac{1}{2}$

$\therefore f''(x) < 0 \;\forall x$ in domain $\Rightarrow$ maxima at $x = -1$; $x = 2$

$\therefore$ 1 is true and 2 is true and is an explanation ($\because$ value of b is used for establishing st. 1)

(Ans 3)

7. The real number k for which the equation $2x^3 + 3x + k = 0$ has two distinct real roots in [0, 1]

[JEE-MAIN 2013]

(1) lies between 1 and 2.

(2) lies between 2 and 3.

(3) lies between –1 and 0

(4) does not exist

Sol^n Let $f(x) = 2x^3 + 3x + k$

$f'(x) = 6x^2 + 3 \Rightarrow f'(x) > 0 \ \forall x \in R$

$\therefore$ $f(x)$ is always increasing

$\therefore$ Graph of $f(x)$ will cross x axis exactly once $\Rightarrow$ No value of K for two roots.

(Ans 4)

8. If $x = -1$ and $x = 2$ are extreme points of $f(x) = \alpha \log|x| + \beta x^2 + x$ then : **[JEE-MAIN 2014]**

(1) $\alpha = -6,\ \beta = \frac{1}{2}$ (2) $\alpha = -6,\ \beta = -\frac{1}{2}$

(3) $\alpha = 2,\ \beta = -\frac{1}{2}$ (4) $\alpha = 2,\ \beta = \frac{1}{2}$

8. $f(x) = \alpha \ln|x| + \beta x^2 + x.$

Derivative of $\ln|x|$ is $1/x$

$f'(x) = \frac{\alpha}{x} + 2\beta x + 1$

$f'(-1) = 0 \Rightarrow -\alpha + 2\beta + 1 = 0.$

$f'(2) = 0 \Rightarrow \frac{\alpha}{2} + 4\beta + 1 = 0$

$\left.\right\} \alpha = 2,\ \beta = -1/2$ (Ans 3)

9. Let $f(x)$ be a polynomial of degree four having extreme values at $x = 1$ and $x = 2$. If $\lim_{x\to 0}\left[1+\frac{f(x)}{x^2}\right] = 3$, then $f(2)$ is equal to : **[JEE-MAIN 2015]**

(1) 0 (2) 4 (3) –8 (4) –4

9. $\lim_{x\to 0} 1 + \frac{f(x)}{x^2} = 3 \Rightarrow \lim_{x\to 0} \frac{f(x)}{x^2} = 2.$

Let $f(x) = ax^4 + bx^3 + cx^2 + dx + e.$

$\therefore \lim_{x\to 0} \frac{ax^4+bx^3+cx^2+dx+e}{x^2} = 2.$

$\Rightarrow$ $c = 2;\ d = 0;\ e = 0$ (for limit to exist)

$\therefore f(x) = ax^4 + bx^3 + 2x^2$

$f'(x) = 4ax^3 + 3bx^2 + 4x.$

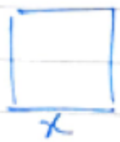
$$\left.\begin{array}{l} f'(1) = 0 \Rightarrow 4a+3b+4=0 \\ f'(2)=0 \Rightarrow 32a+12b+8=0 \end{array}\right\} \begin{array}{l} a = 1/2 \\ b = -2. \end{array}$$

$\therefore f(x) = \frac{x^4}{2} - 2x^3 + 2x^2.$

$\Rightarrow f(2) = 8 - 16 + 8 = 0.$ (Ans 1).

10. A wire of length 2 units is cut into two parts which are bent respectively to form a square of side = x units and a circle of radius = r units. If the sum of the areas of the square and the circle so formed is minimum, then : **[JEE-MAIN 2016]**

(1) $2x = r$ (2) $2x = (\pi + 4)r$ (3) $(4 - \pi)x = \pi r$ (4) $x = 2r$

10. 

$4x + 2\pi r = 2 \Rightarrow r = \frac{1-2x}{\pi}$

$A = x^2 + \pi r^2 = x^2 + \frac{(1-2x)^2}{\pi}$

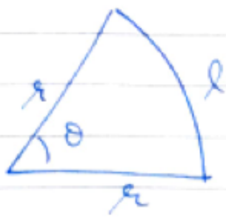
$\frac{dA}{dx} = 0 \Rightarrow x = \frac{2}{\pi+4} \Rightarrow r = \frac{1}{\pi+4} \Rightarrow \boxed{x = 2r}$

Ans. 4

11. Twenty meters of wire is available for fencing off a flower-bed in the form of a circular sector. Then the maximum area (in sq. m) of the flower -bed, is :- **[JEE-MAIN 2017]**

(1) 30 (2) 12.5 (3) 10 (4) 25

11.



$l = r\theta$

$20 = 2r + r\theta.$

$\Rightarrow \theta = \frac{20-2r}{r}.$

$A = \frac{1}{2}r^2\theta \Rightarrow A = \frac{1}{2}r^2\left(\frac{20-2r}{r}\right)$

$\Rightarrow A = 10r - r^2.$

$\frac{dA}{dr} = 0 \Rightarrow r = 5 \Rightarrow A_{max} = 25$ (Ans 4)

12. Let $f(x) = x^2 + \frac{1}{x^2}$ and $g(x) = x - \frac{1}{x}$, $x \in R - \{-1, 0, 1\}$. If $h(x) = \frac{f(x)}{g(x)}$, then the local minimum value of $h(x)$ is : **[JEE-MAIN 2018]**

(1) -3 (2) $-2\sqrt{2}$ (3) $2\sqrt{2}$ (4) 3

12. $h(x) = \frac{f(x)}{g(x)} = \frac{x^2 + 1/x^2}{x - 1/x}$.

$h(x) = \frac{(x - 1/x)^2 + 2}{x - 1/x} \Rightarrow h(x) = x - \frac{1}{x} + \frac{2}{(x - \frac{1}{x})}$

When $x - 1/x < 0$: $x - \frac{1}{x} + \frac{2}{(x - 1/x)} \leq -2\sqrt{2}$

$\therefore$ $-2\sqrt{2}$ will be local max.

When $x - 1/x > 0$: $x - 1/x + \frac{2}{x - 1/x} \geq 2\sqrt{2}$.

$\therefore$ $2\sqrt{2}$ is local Minimum.

(Ans 3)

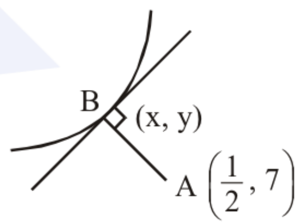
13. A helicopter is flying along the curve given by $y - x^{3/2} = 7$, $(x \geq 0)$. A soldier positioned at the point $\left(\frac{1}{2}, 7\right)$ wants to shoot down the helicopter when it is nearest to him. Then this nearest distance is : **[JEE-MAIN 2019]**

(1) $\frac{1}{2}$ (2) $\frac{1}{3}\sqrt{\frac{7}{3}}$ (3) $\frac{1}{6}\sqrt{\frac{7}{3}}$ (4) $\frac{\sqrt{5}}{6}$

Soln

$y - x^{3/2} = 7 \ (x \geq 0)$

$$\frac{dy}{dx} = \frac{3}{2}x^{1/2}$$



$$\left(\frac{3}{2}\sqrt{x}\right)\left(\frac{7-y}{\frac{1}{2}-x}\right) = -1$$

$$\left(\frac{3}{2}\sqrt{x}\right)\left(\frac{-x^{3/2}}{\frac{1}{2}-x}\right) = -1$$

$$\frac{3}{2}\cdot x^2 = \frac{1}{2} - x$$

$3x^2 = 1 - 2x$

$3x^2 + 2x - 1 = 0$

$3x^2 + 3x - x - 1 = 0$

$(x + 1)(3x - 1) = 0$

$\therefore \quad x = -1$ (rejected)

$$x = \frac{1}{3}$$

$$y = 7 + x^{3/2} = 7 + \left(\frac{1}{3}\right)^{3/2}$$

$$\ell_{AB} = \sqrt{\left(\frac{1}{2} - \frac{1}{3}\right)^2 + \left(\frac{1}{3}\right)^3} = \sqrt{\frac{1}{36} + \frac{1}{27}}$$

$$= \sqrt{\frac{3+4}{9 \times 12}}$$

$$= \sqrt{\frac{7}{108}} = \frac{1}{6}\sqrt{\frac{7}{3}}$$

Option (3)

Exercise JA

EXERCISE (JA)

1. **(a)** Let f, g and h be real-valued functions defined on the interval [0, 1] by $f(x) = e^{x^2} + e^{-x^2}$, $g(x) = xe^{x^2} + e^{-x^2}$ and $h(x) = x^2e^{x^2} + e^{-x^2}$. If a, b and c denote respectively, the absolute maximum of f, g and h on [0, 1], then

(A) a = b and c ≠ b (B) a = c and a ≠ b (C) a ≠ b and c ≠ b (D) a = b = c

(b) Let f be a function defined *on* ***R*** (the set of all real numbers) such that

$f'(x) = 2010\ (x–2009)\ (x–2010)^2\ (x–2011)^3\ (x – 2012)^4$, for all $x \in R$.

If g is a function defined on R with values in the interval $(0,\infty)$ such that

$f(x) = \ell n\ (g(x))$, for all $x \in R$,

then the number of points in R at which g has a local maximum is **[JEE 2010, 3 + 3]**

Sol^n (a)

$$f(x) = e^{x^2} + e^{-x^2} \Rightarrow f'(x) = 2x\left(e^{x^2} - e^{-x^2}\right) \geq 0 \ \forall\ x \in [0, 1]$$

Clearly for $0 \leq x \leq 1$: $1 \geq x \geq x^2$

$f(x) \geq g(x) \geq h(x)$

$\because f(1) = g(1) = h(1) = e + \dfrac{1}{e}$ and f(1) is the greatest

$\therefore a = b = c = e + \dfrac{1}{e} \Rightarrow a = b = c.$ (Ans D)

(b)

$f(x) = \ln\{g(x)\}$
$g(x) = e^{f(x)}$
$g'(x) = e^{f(x)} . f'(x)$
$g'(x) = 0 \Rightarrow f'(x) = 0$ as $e^{f(x)} \neq 0$
$\Rightarrow 2010(x – 2009)(x – 2010)^2(x – 2011)^3(x – 2012)^4 = 0$
so there is only one point of local maxima.

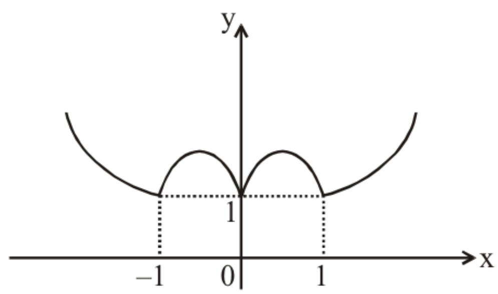
1

2. Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = |x| + |x^2 - 1|$. The total number of points at which f attains either a local maximum or a local minimum is **[JEE 2012, 4M]**

Ans. 5

Sol. $f(x) = |x| + |(x + 1)\ (x - 1)|$

$$\Rightarrow \quad f(x) = \begin{cases} x^2 - x - 1 & x < -1 \\ -x^2 - x + 1 & -1 \le x < 0 \\ -x^2 + x + 1 & 0 \le x < 1 \\ x^2 + x - 1 & x \ge 1 \end{cases}$$



$\therefore$ f has 5 points where it attains either a local maximum or local minimum.

3. Let p(x) be a real polynomial of least degree which has a local maximum at x = 1 and a local minimum at x = 3. If p(1) = 6 and p(3) = 2, then p'(0) is **[JEE 2012, 4M]**

Sol. Let $P'(x) = k(x - 1)(x - 3)$

$= k(x^2 - 4x + 3)$

$\Rightarrow P(x) = k\left(\frac{x^3}{3} - 2x^2 + 3x\right) + c$

$\because P(1) = 6$

$\Rightarrow \frac{4k}{3} + c = 6$ ....(1)

$P(3) = 2$

$\Rightarrow c = 2$ ....(2)

by (i) and (ii)

$k = 3$

$\therefore P'(x) = 3(x - 1)(x - 3)$

$\Rightarrow P'(0) = 9$

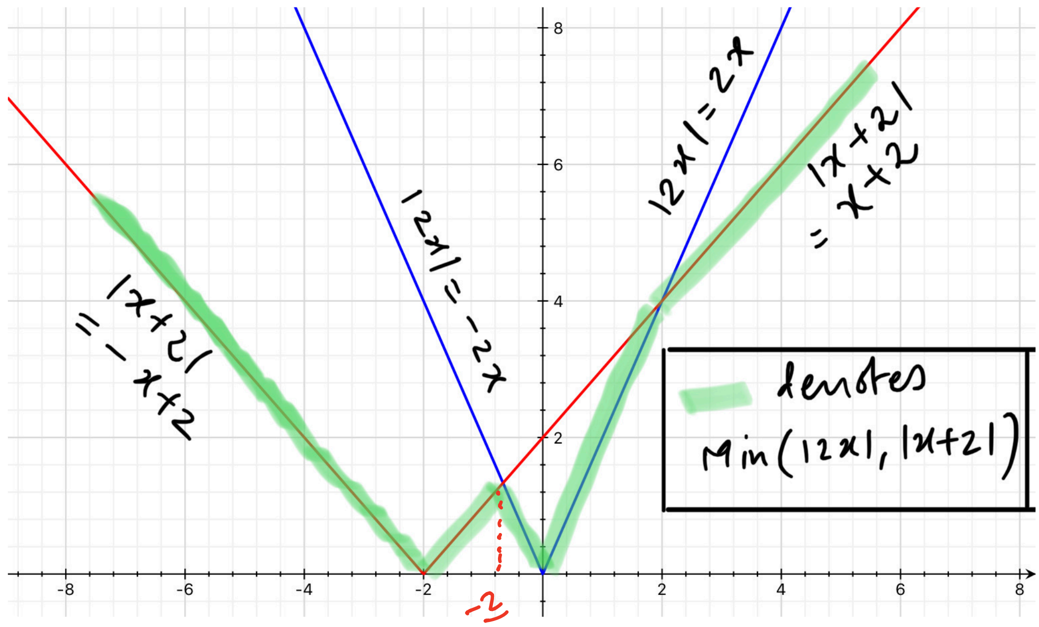
4. The function $f(x) = 2|x| + |x + 2| - ||x + 2| - 2|x||$ has a local minimum or a local maximum at x = **[JEE 2013, 3M,–1M]**

(A) -2 (B) $\frac{-2}{3}$ (C) 2 (D) $\frac{2}{3}$

Sol. **(A), (B)**

As, $$\frac{f(x)+g(x)-|f(x)-g(x)|}{2} = \text{Min }(f(x), g(x))$$

$$\Rightarrow \frac{2|x|+|x+2|-||x+2|-2|x||}{2} = \text{Min }(|2x|, |x+2|)$$



According to the figure shown, points of local minima/maxima are $x = -2, \frac{-2}{3}, 0$.

5. The least value of $\alpha \in \mathbb{R}$ for which $4\alpha x^2 + \frac{1}{x} \geq 1$, for all x > 0, is -

[JEE(Advanced)-2016, 3(–1)]

(A) $\frac{1}{64}$ (B) $\frac{1}{32}$ (C) $\frac{1}{27}$ (D) $\frac{1}{25}$

Ans. (C)

Sol. $f(x) = 4\alpha x^2 + \frac{1}{x}; x > 0$

$f'(x) = 8\alpha x - \frac{1}{x^2}$

$= \frac{8\alpha x^3 - 1}{x^2}$

f(x) attains its minimum at $x = \left(\frac{1}{8\alpha}\right)^{1/3}$

$f\left(\left(\frac{1}{8\alpha}\right)^{1/3}\right) = 1$

$\Rightarrow 4\alpha\left(\frac{1}{8\alpha}\right)^{2/3} + (8\alpha)^{1/3} = 1$

$\Rightarrow 3\alpha^{1/3} = 1 \Rightarrow \alpha = \frac{1}{27}$

6. Let $f : \mathbb{R} \to (0, \infty)$ and $g : \mathbb{R} \to \mathbb{R}$ be twice differentiable function such that f'' and g'' ar continuous functions on $\mathbb{R}$. Suppose $f'(2) = g(2) = 0$, $f''(2) \neq 0$ and $g'(2) \neq 0$. If $\lim\limits_{x\to 2} \dfrac{f(x)g(x)}{f'(x)g'(x)} = 1$, then

[JEE(Advanced)-2016, 4(–2)]

(A) f has a local minimum at $x = 2$

(B) f has a local maximum at $x = 2$

(C) $f''(2) > f(2)$

(D) $f(x) - f''(x) = 0$ for at least one $x \in \mathbb{R}$

Sol^n.

Using L'Hôpital's Rule

$$\lim_{x\to 2} \frac{f'(x)g(x) + f(x)g'(x)}{f''(x)g'(x) + f'(x)g''(x)} = 1$$

$$\Rightarrow \quad \frac{f(2)g'(2)}{f''(2)g'(2)} = 1 \quad \Rightarrow \quad f''(2) = f(2) > 0$$

option (D) is right and option (C) is wrong

also $f'(2) = 0$ and $f''(2) > 0 \quad \therefore \quad x = 2$ is local minima.

(Ans A, D)

7. If $f(x) = \begin{vmatrix} \cos(2x) & \cos(2x) & \sin(2x) \\ -\cos x & \cos x & -\sin x \\ \sin x & \sin x & \cos x \end{vmatrix}$, then **[JEE(Advanced)-2017]**

(A) $f'(x) = 0$ at exactly three points in $(-\pi, \pi)$

(B) $f(x)$ attains its maximum at $x = 0$

(C) $f(x)$ attains its minimum at $x = 0$

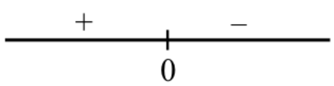
(D) $f'(x) = 0$ at more than three points in $(-\pi, \pi)$

Ans. (B,D)

Sol. Expansion of determinant

$f(x) = \cos 2x + \cos 4x$

$f'(x) = -2\sin 2x - 4\sin 4x = -2\sin 2x(1 + 4\cos 2x)$



$\therefore$ maxima at $x = 0$

$f'(x) = 0 \Rightarrow x = \frac{n\pi}{2}, \cos 2x = -\frac{1}{4}$

$\Rightarrow$ more than two solutions

8. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = (x-1)(x-2)(x-5)$. Define $F(x) = \int_0^x f(t)\,dt$, $x > 0$. Then which of the following options is/are correct ? **[JEE(Advanced)-2019, 4(–1)]**

(1) F has a local minimum at x = 1

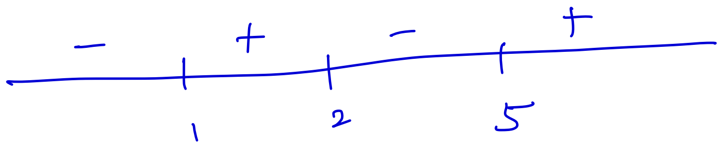
(2) F has a local maximum at x = 2

(3) $F(x) \neq 0$ for all $x \in (0, 5)$

(4) F has two local maxima and one local minimum in $(0, \infty)$

Soln.

$F'(x) = f(x) = (x-1)(x-2)(x-5)$



Minima at $x = 1$. & $x = 5$

Maxima at $x = 2$.

$F(x) < 0 \;\forall\; x \in (0, 5)$.

9. Let $f(x)=\frac{\sin \pi x}{x^2}, x>0$

Let $x_1 < x_2 < x_3 < \ldots < x_n < \ldots$ be all the points of local maximum of f and $y_1 < y_2 < y_3 < \ldots < y_n < \ldots$ be all the points of local minimum of f.

Then which of the following options is/are correct ? **[JEE(Advanced)-2019, 4(–1)]**

(1) $|x_n - y_n| > 1$ for every n

(2) $x_1 < y_1$

(3) $x_n \in \left(2n, 2n+\frac{1}{2}\right)$ for every n

(4) $x_{n+1} - x_n > 2$ for every n

Ans. (1,3,4)

$$f(x)=\frac{\sin \pi x}{x^2}$$

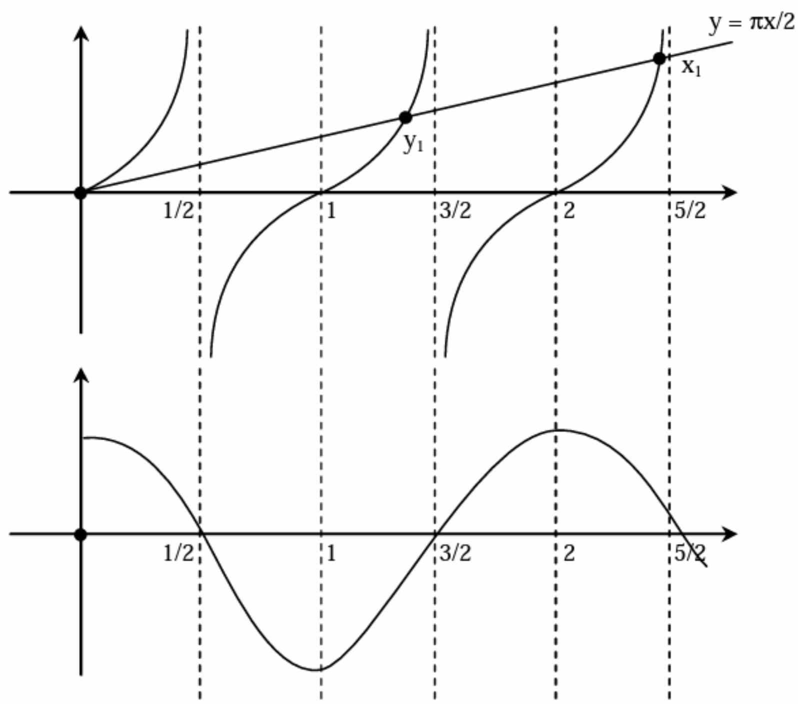
$$\Rightarrow \quad f'(x)=\frac{\pi x^2 \cos \pi x - 2x \sin \pi x}{x^4}$$

$$=\frac{2\cos \pi x\left(\frac{\pi x}{2}-\tan \pi x\right)}{x^3}$$

$$f'(x)=0 \Rightarrow \cos\pi x = 0 \text{ or } \frac{\pi x}{2}=\tan\pi x$$

$$\Rightarrow \quad \pi x = (2n+1)\frac{\pi}{2} \text{ or } \frac{\pi x}{2}=\tan \pi x$$

$$x=\frac{(2n+1)}{2}, n \in I$$



from graph, we can see that $\forall\ x=\frac{2n+1}{2}$

$\Rightarrow$ f′(x) doesn't change sign so these points are neither local maxima nor local minimum.

Similarly, $\forall\ x : \frac{\pi x}{2}=\tan \pi x$

Where $y_n \in (2n-1, 2n-½)\ \forall\ n = 1, 2, 3, \ldots$

and $x_n \in (2n, 2n+½)\ \forall\ n = 1, 2, 3, \ldots$

$x_{n+1} - y_{n+1} > 1$ and $y_{n+1} - x_n > 1 \Rightarrow x_{n+1} - x_n > 2$.