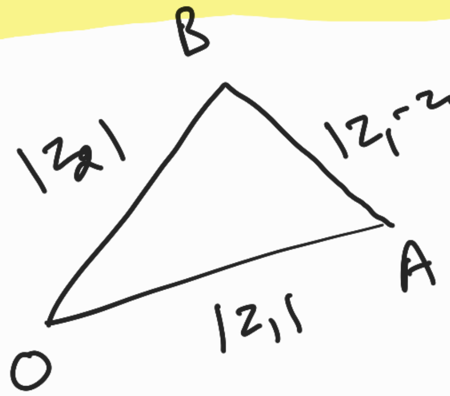
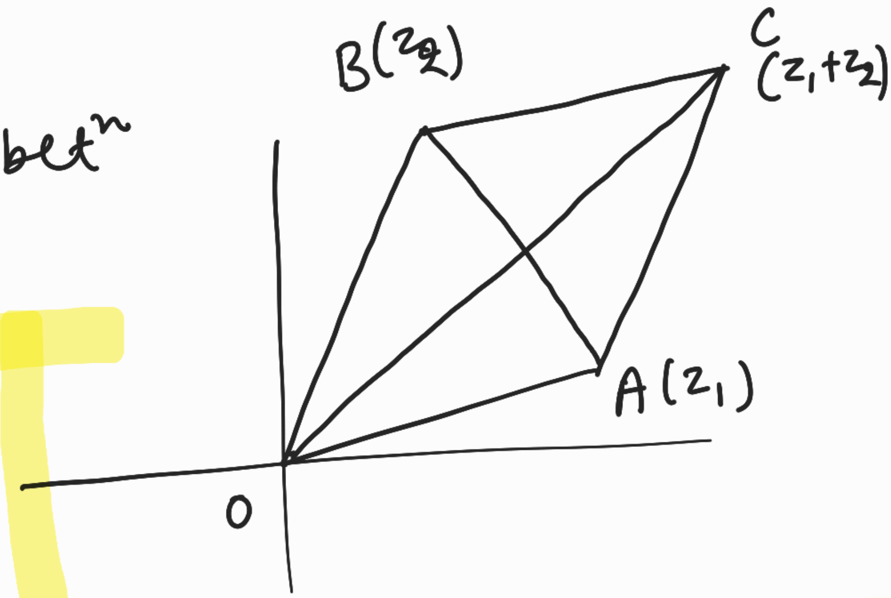
02/04/22

Triangular inequalities

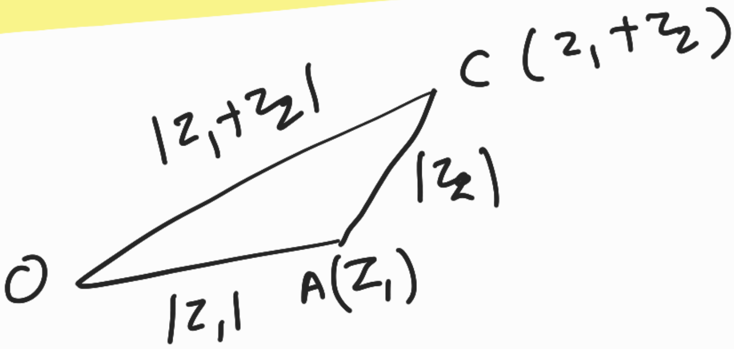
$$\big||z_1| - |z_2|\big| \leq |z_1 \pm z_2| \leq |z_1| + |z_2|$$

$|z_2 - z_1| =$ distance betn z_1 & z_2





$$\big||z_1| - |z_2|\big| < |z_1 - z_2| < |z_1| + |z_2|$$



$$\big||z_1| - |z_2|\big| < |z_1 + z_2| < |z_1| + |z_2|$$

Ex:- find minimum and Maximum value of $|z|$ if $\left|z - \frac{4}{z}\right| = 2$

$\underline{\text{Sol}^n}$

$$\left|z - \frac{4}{z}\right| = 2$$

$$\boxed{\big||z_1| - |z_2|\big| \le |z_1 - z_2| \le |z_1| + |z_2|}$$

$$\left||z| - \left|\frac{4}{z}\right|\right| \le \left|z - \frac{4}{z}\right| \le |z| + \left|\frac{4}{z}\right|$$

$$\underbrace{\left||z| - \frac{4}{|z|}\right| \le 2}\ \underbrace{\le |z| + \frac{4}{|z|}}$$

$$\Downarrow$$

$$-2 \le |z| - \frac{4}{|z|} \le 2$$

$$2|z| \le |z|^2 + 4$$

$$0 \le |z|^2 - 2|z| + 4$$

$$0 \le (|z| - 1)^2 + 3$$

Always True

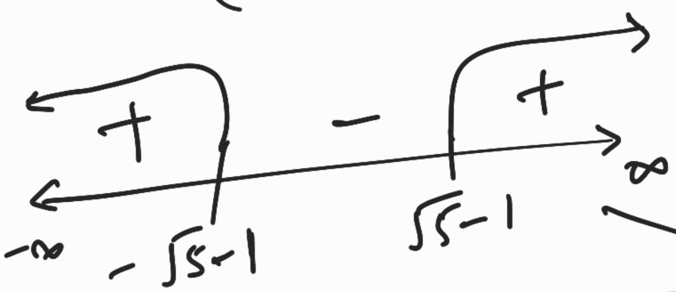
$$\underbrace{-2|z| \le |z|^2 - 4}\ \&\ \underbrace{|z|^2 - 4 \le 2|z|}$$

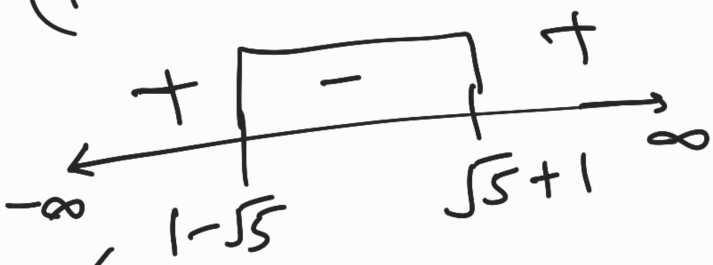
$$|z|^2 - 2|z| - 4 \le 0$$

$$0 \le |z|^2 + 2|z| - 4$$

$$0 \le (|z| + 1)^2 - 5$$

$$(|z| - 1)^2 - 5 \le 0$$





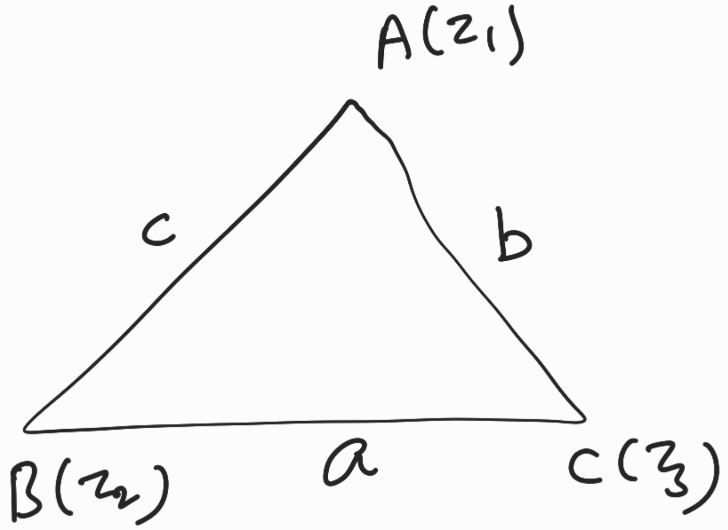
$$\cap$$

$$\boxed{\sqrt{5} - 1 \le |z| \le \sqrt{5} + 1}$$

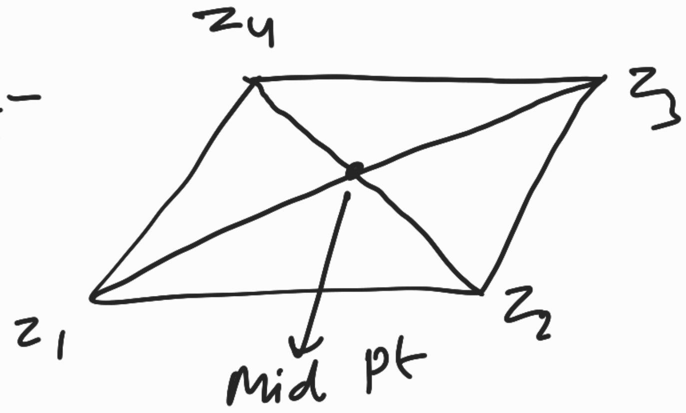
Geometry in Complex plane

centroid $= \dfrac{z_1+z_2+z_3}{3}$

incentre $= \dfrac{az_1+bz_2+cz_3}{a+b+c}$



Ex :-

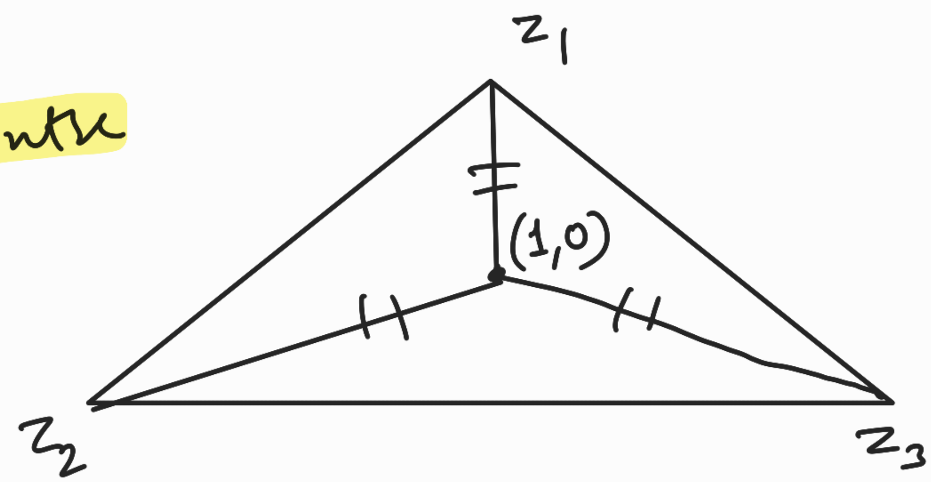


If z_1, z_2, z_3, z_4 are vertices of llgram. then p.T. $z_1+z_3 = z_2+z_4$

$$\frac{z_1+z_3}{2} = \frac{z_2+z_4}{2}$$

Exn If z_1, z_2, z_3 are vertices of Δ such that $|z_1-1| = |z_2-1| = |z_3-1|$ & $z_1+z_2+z_3=3$ then p.T. Δ is Equilateral Δ.

$z=1=$ Circumcentre

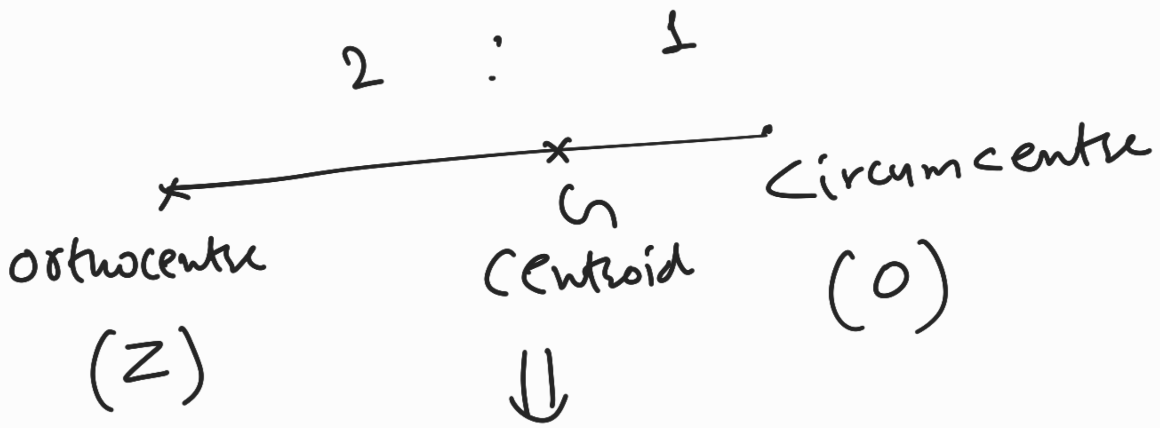


$$\text{Centroid} = \frac{z_1+z_2+z_3}{3} = 1$$

centroid, circumcentre are coincide

$\Rightarrow \Delta$ is equilateral

Ex:- If origin is circum-centre and z is orthocentre of a Δ with vertices z_1, z_2, z_3 then P.T. $z = z_1+z_2+z_3$



$\Downarrow$

$$\frac{z_1+z_2+z_3}{3} = \frac{z+0}{3}$$

Some locus problem

Ex:- find locus of z

$$z(1+i)+\bar{z}(1-i)+3=0$$

Sol^n put $\left.\begin{array}{l} z=x+iy \\ \bar{z}=x-iy \end{array}\right\}$ $\begin{array}{l} z+\bar{z}=2x \\ z-\bar{z}=2iy \end{array}$

$(z+\bar{z})+i(z-\bar{z})+3=0$

$2x+i(2iy)+3=0$

$2x-2y+3=0$

Ex:- find curve $|z-1+3i|=5$

$z=x+iy$ $\quad |(x-1)+i(y+3)|=5$

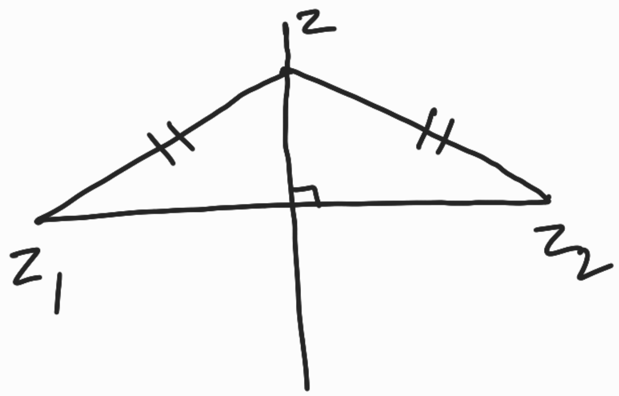
$\sqrt{(x-1)^2+(y+3)^2}=5$

$(x-1)^2+(y+3)^2=25$

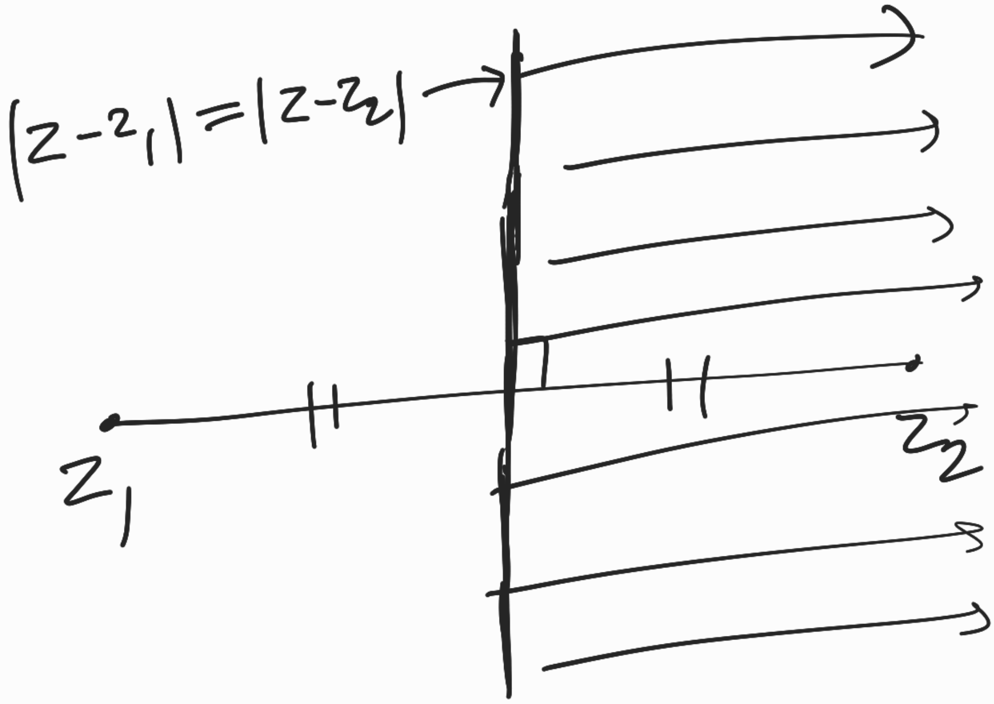
circle $\Rightarrow$ centre $(1,-3)$

radius $=5$

① $|z-z_1|=|z-z_2| \Rightarrow$ locus of z is ⊥ar bisector of line joining z_1 & z_2



(2) $|z-z_1| \geq |z-z_2|$ draw region



(3) Eqn of S.L.

$$\bar{\alpha} z + \alpha \bar{z} + \beta = 0$$

$\alpha = $ C.N. $\qquad \beta = $ real

proof

$2D \Rightarrow ax + by + c = 0 \qquad \boxed{a, b, c \in R}$

$$\left.\begin{array}{l} z = x+iy \\ \bar{z} = x-iy \end{array}\right\} \Rightarrow x = \frac{z+\bar{z}}{2}, \; y = \frac{z-\bar{z}}{2i}$$

$$a\left(\frac{z+\bar{z}}{2}\right) + b\left(\frac{z-\bar{z}}{2i}\right) + c = 0$$

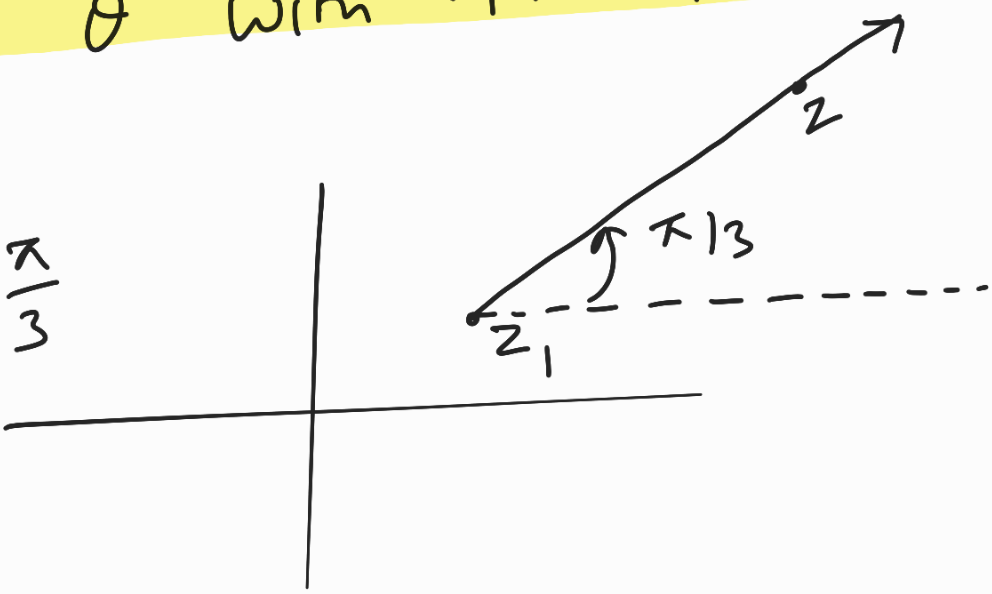
$$\underbrace{\left(\frac{a}{2}-\frac{ib}{2}\right)}_{\bar{\alpha}} z+\underbrace{\left(\frac{a}{2}+\frac{ib}{2}\right)}_{\alpha} \bar{z}+c=0$$

$$\bar{\alpha} z+\alpha \bar{z}+c=0$$

(4) Eqn of ray

If $\text{amp}(z-z_1)=\theta$ then locus of z is a ray from point z_1 which makes an angle θ with $+x$ axis

$\theta=+\frac{\pi}{3}$



Ex:- find locus of z If

$$\arg(z-1-i)=\frac{\pi}{4}$$

Sol^n $\arg\left(z-(1+i)\right) = \frac{\pi}{4}$

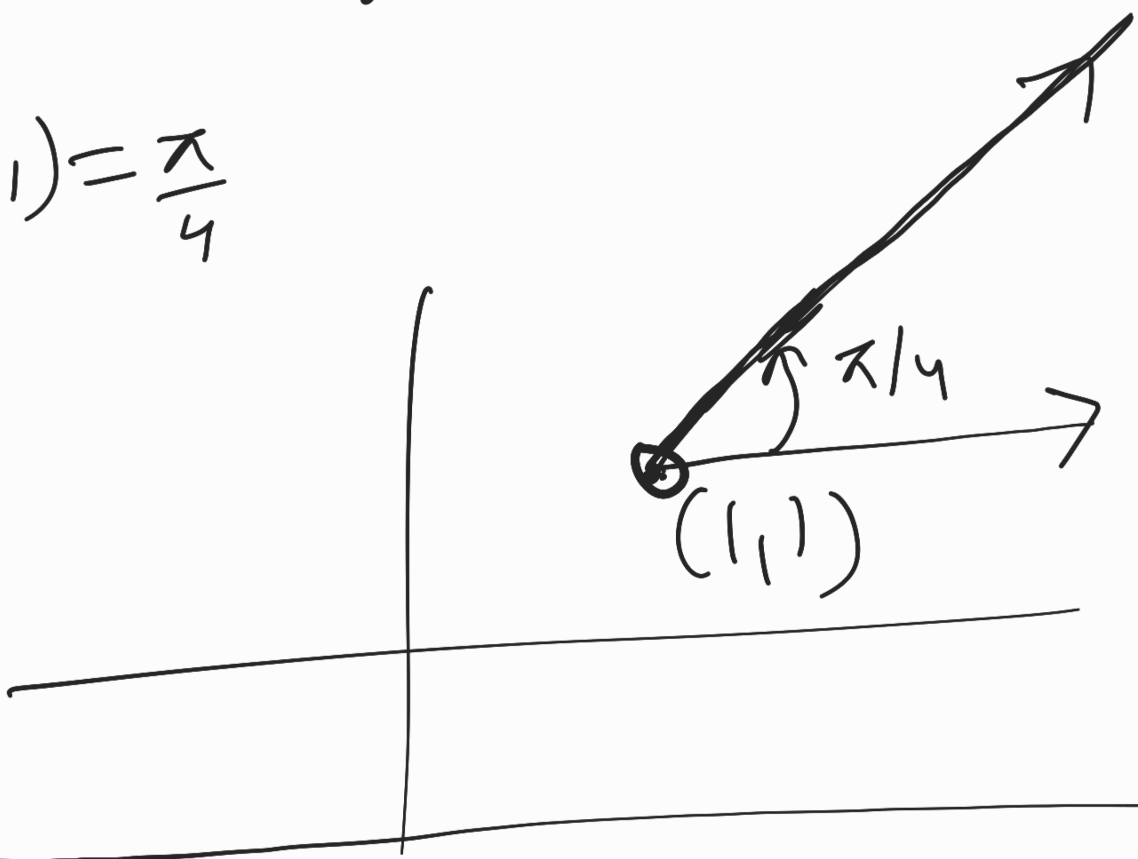
$z = x+iy$, $\arg\left((x-1)+i(y-1)\right) = \frac{\pi}{4}$

$$\tan^{-1}\left(\frac{y-1}{x-1}\right) = \pi/4$$

$$\frac{y-1}{x-1} = 1 \Rightarrow y = x$$

$\arg(z-z_1) = \frac{\pi}{4}$

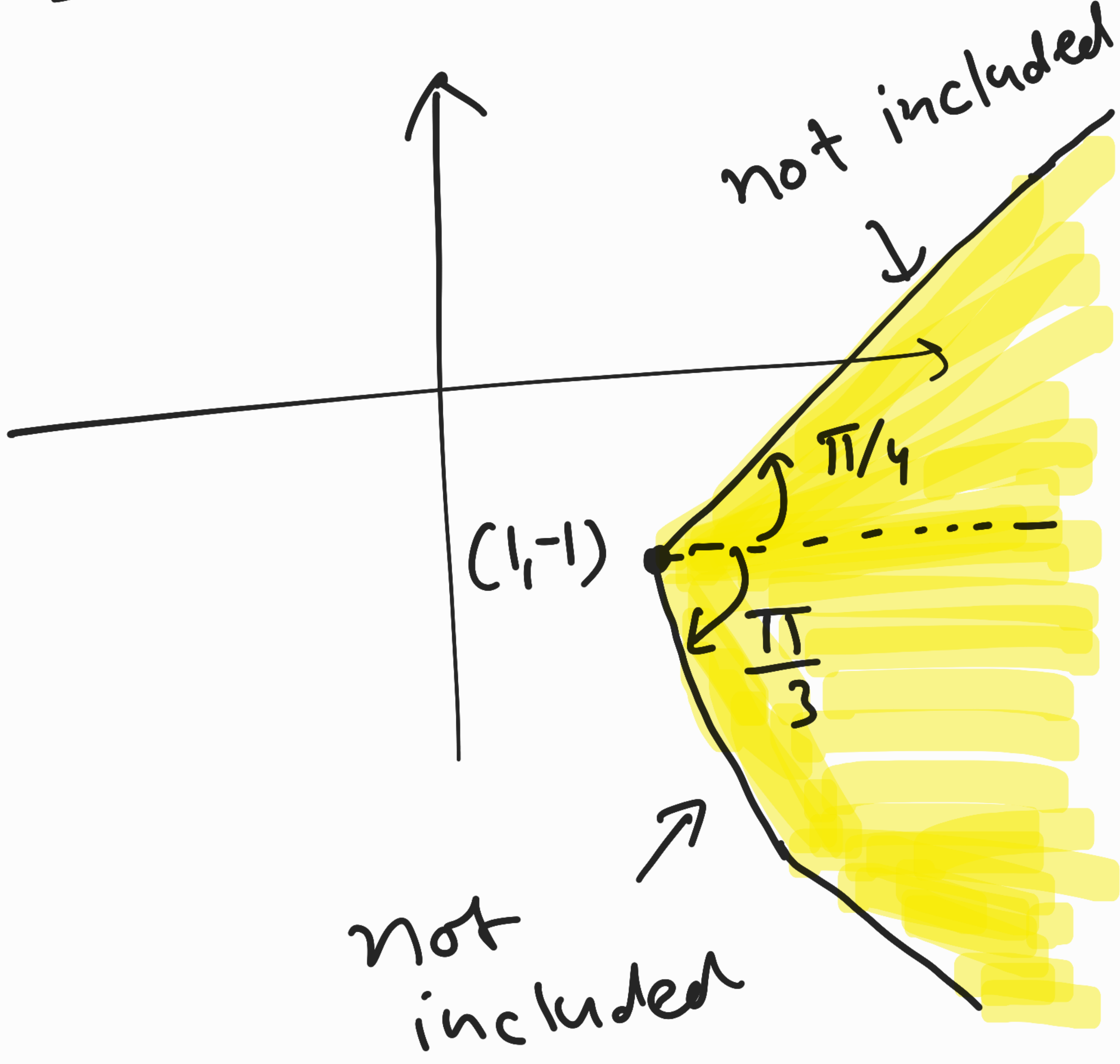
$z_1 = (1,1)$



Ex:- $-\frac{\pi}{3} < \arg(z-1+i) < \frac{\pi}{4}$

Draw region

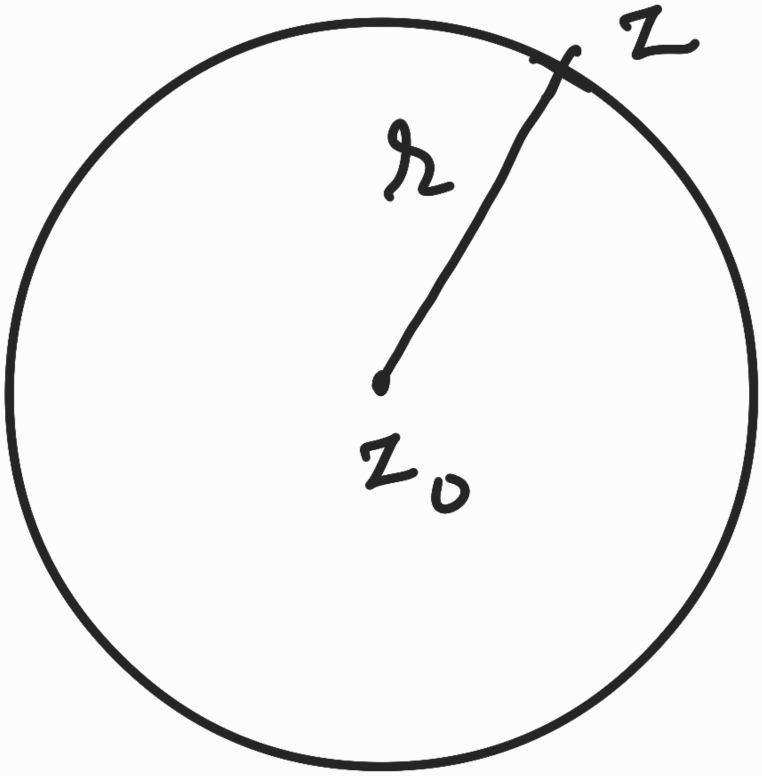
$$-\frac{\pi}{3} < \arg\left(z-(1-i)\right) < \frac{\pi}{4}$$



5 Cirle

z_0 = fixed point

r = constant



$$* \quad |z - z_0| = r \quad *$$

z_0 = centre

r = radius

Ex:- $|z+1+\sqrt{3}i| = 5$

$$\Big|z - \underbrace{(-1-\sqrt{3}i)}_{\text{centre } (-1,-\sqrt{3})}\Big| = \underbrace{5}_{\text{radius}}$$

general form of circle

$|z - z_0| = r \longrightarrow$ Circle

$$|z - z_0|^2 = r^2$$

$$(z - z_0)(\overline{z} - \overline{z_0}) = r^2$$

$$z\bar{z} - z\bar{z}_0 - z_0\bar{z} + z_0\bar{z}_0 - r^2 = 0$$

$$\boxed{z\bar{z} - z\bar{z}_0 - z_0\bar{z} + \underbrace{|z_0|^2 - r^2}= 0}$$

$$\Downarrow$$

$$\underbrace{z\bar{z} + \bar{\alpha}z + \alpha\bar{z} + \beta = 0}_{\text{general form}}$$

$$\alpha = -z_0 \qquad \beta = |z_0|^2 - r^2$$

Ex:- If $|z| = 1$ and $w = \dfrac{z-1}{z+1}$ $(z \neq -1)$ then find locus of w (w is C.N.).

Sol^n $\Rightarrow$ $w = \dfrac{z-1}{z+1}$

$$wz + w = z - 1$$

$$w+1 = z - zw$$

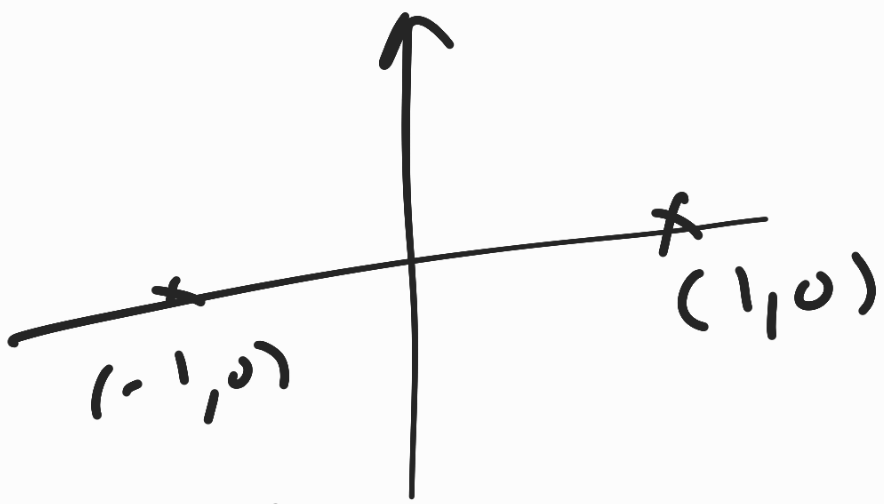
$$\frac{w+1}{1-w} = z$$

$$\left|\frac{w+1}{1-w}\right| = |z| = 1$$

$$|1+w| = |1-w|$$

$$|w-(-1)| = |w-1|$$

$w = x+iy$



$x = 0$

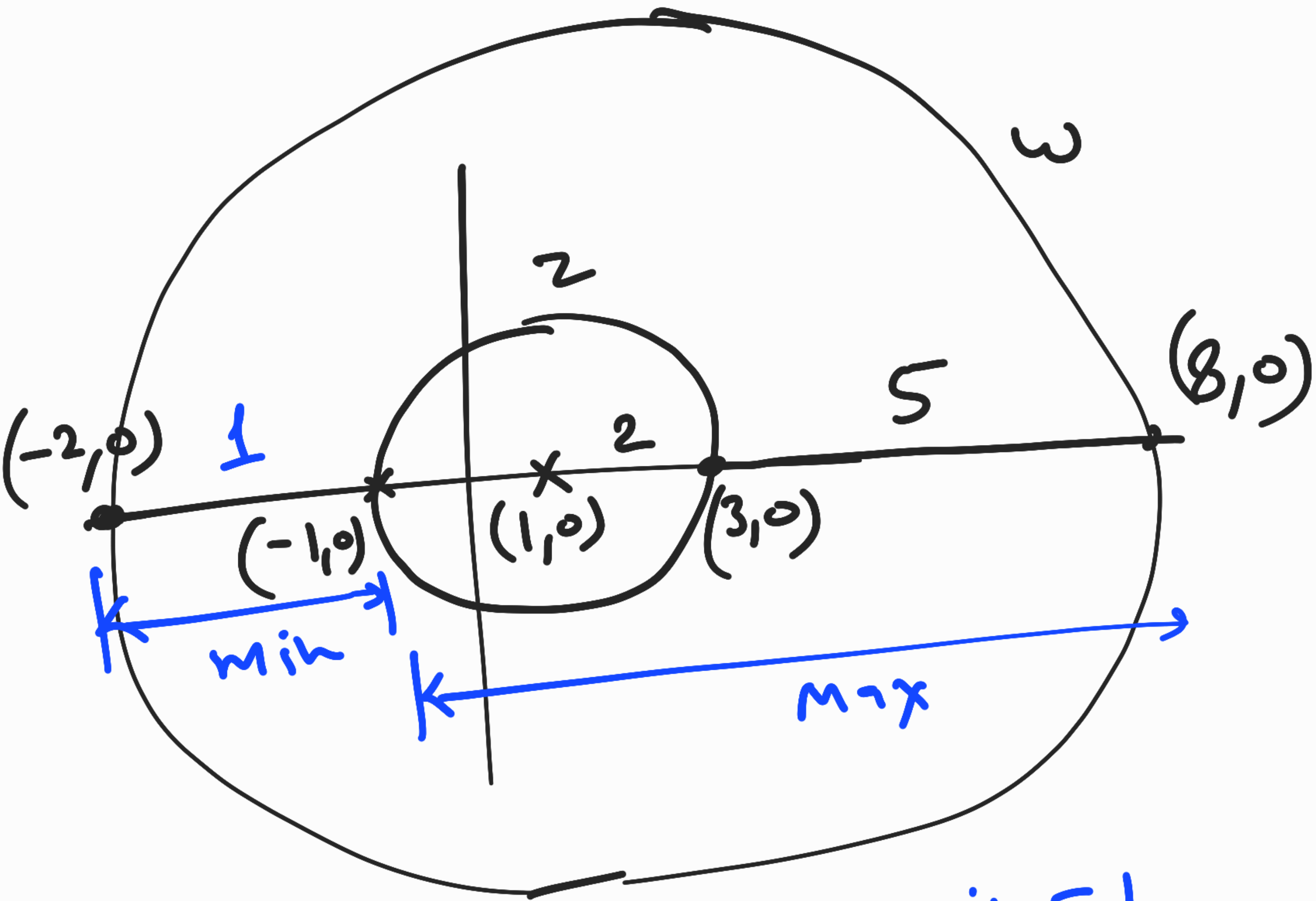
Ex:- If $|z-1| = 2$

and $|w-3| = 5$

find min and max

Value of $|z-w| = ?$

& $|2z - 3w| = ?$



$|z-w|$ min $= 1$, max $= 9$

$|2z - 3w| = |z_1 - w_1|$

Let $2z = z_1$, $3w = w_1$

$|z-1| = 2$

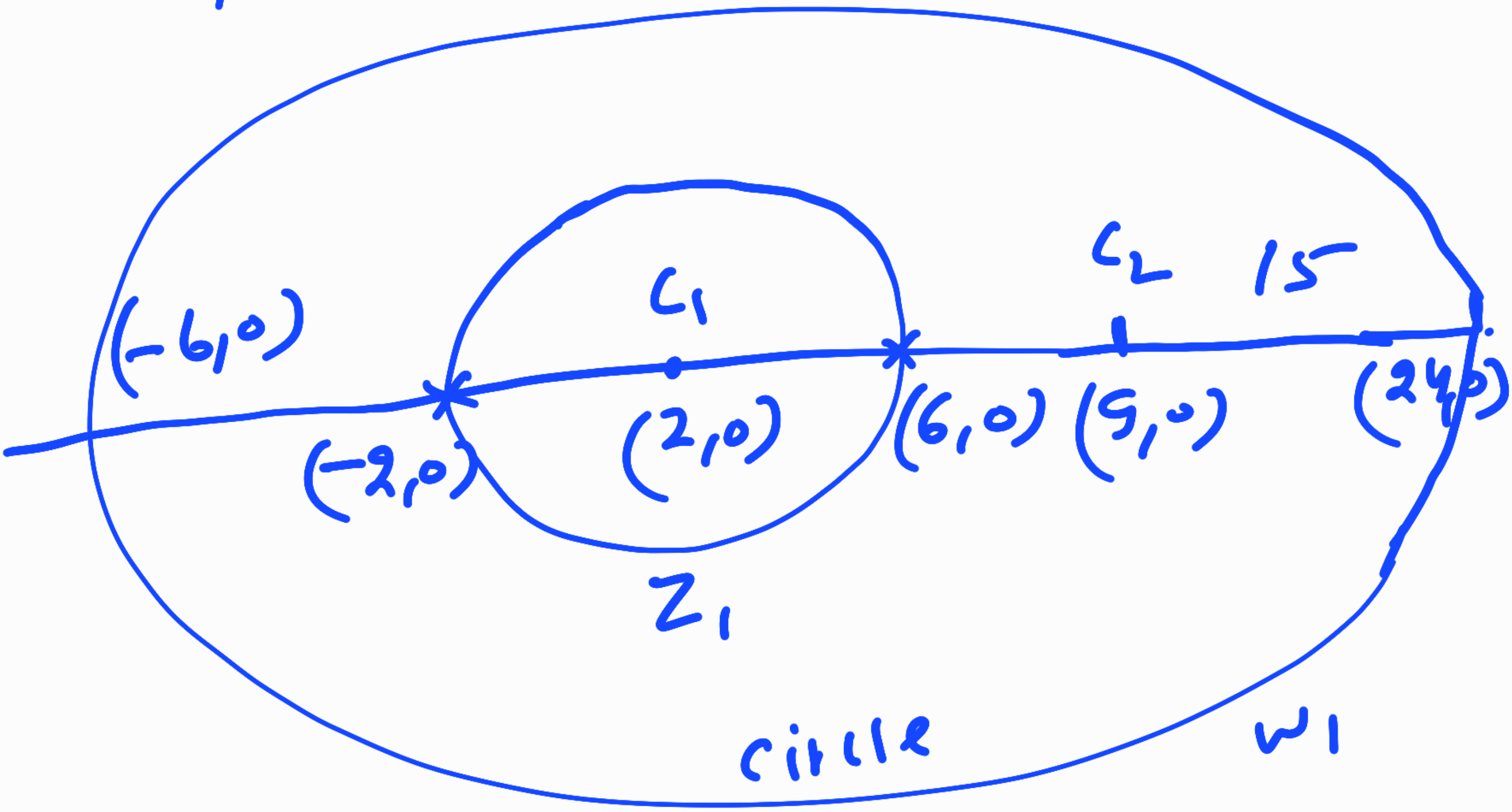
$|2z-2| = 4$

$|z_1 - 2| = 4$

$|w-3| = 5$

$|3w-9| = 15$

$|w_1 - 9| = 15$

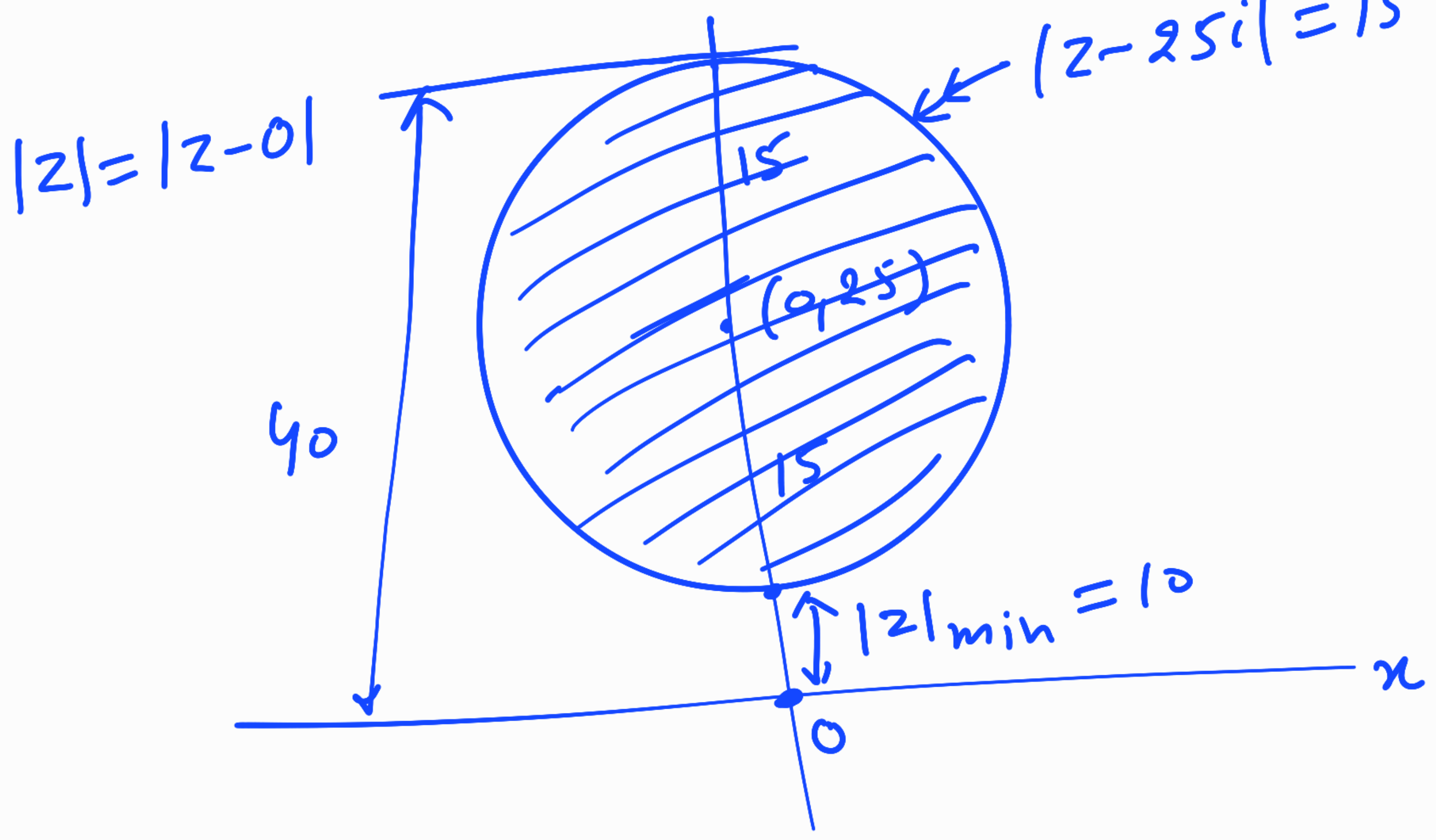


$|z_1 - w_1|$ — Min $= 4$, Max $= 26$

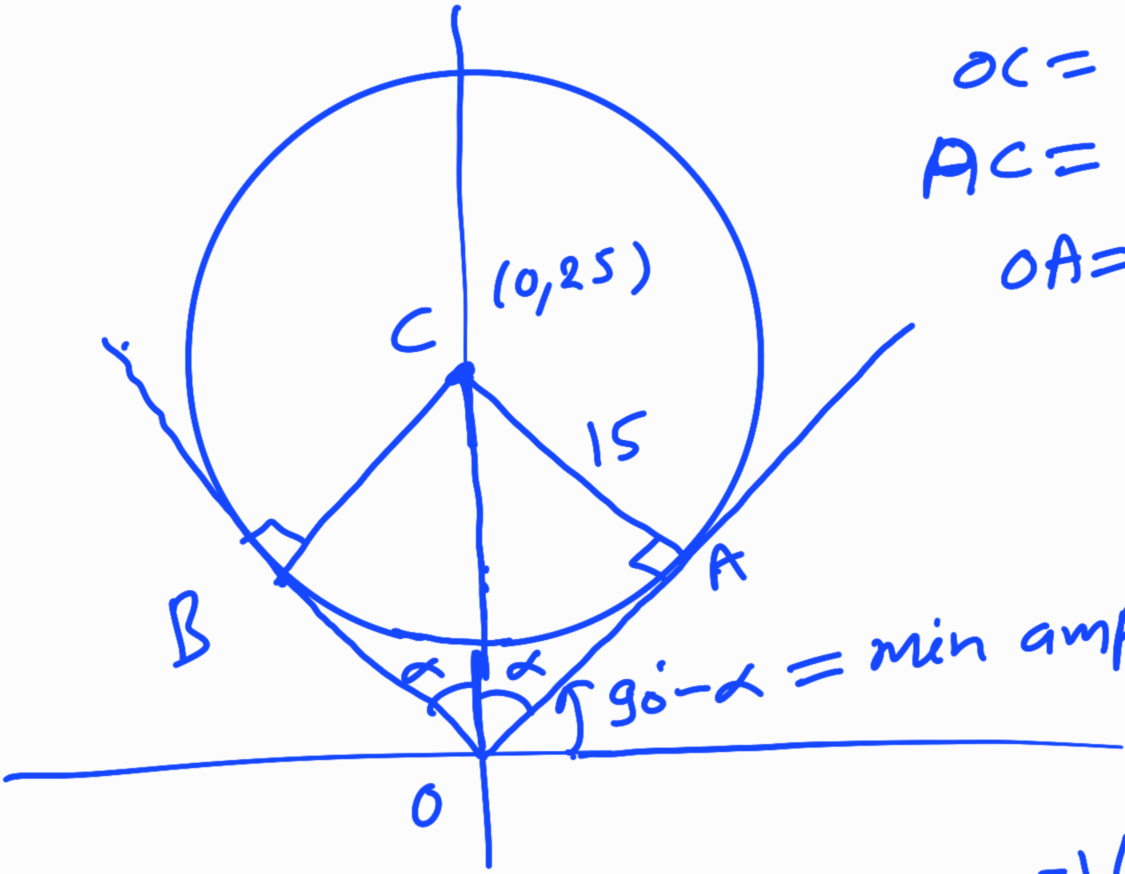
Ex:- If $|z-25i| \le 15$ then find max and min value of $|z|$ and $amp(z)$

$$|z-25i| \le 15$$

centre $(0,25)$ $\quad r=15$



$$|z|_{max} = 40$$



$OC = 25$

$AC = 15 = BC$

$OA = 20 = OB$

$$\tan\alpha = \frac{15}{20} \Rightarrow \alpha = \tan^{-1}(3/4)$$

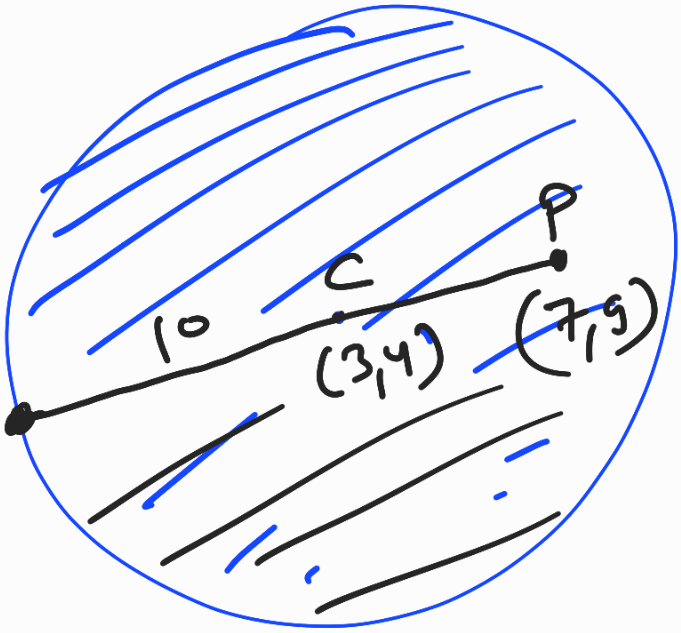
$$\text{Min } amp(z) = \frac{\pi}{2} - \tan^{-1}\left(\frac{3}{4}\right)$$

$$\text{Max } amp(z) = \frac{\pi}{2} + \tan^{-1} 3/4$$

Ex:- If $|z-3-4i| \le 10$ then find min and Max value of $|z-7-9i|$

$|z-(7+9i)|$

$|z-3-4i| \le 10$



$$|z-7-9i|_{min} = 0$$

$$|z-7-9i|_{Max} = 10+\sqrt{41}$$

$$|z-3-4i| \le 10$$

$$|7+9i-3-4i| \le 10$$

$$|4+5i| \le 10$$

$$\sqrt{16+25} \le 10 \text{ True}$$

(7, 9) lies inside circle

H.w. = elementary exercise