Allen Career Institute

Kota

CIRCLE
(Solutions)

Index

Do-yourself I

(1) find the centre and radius of the circle $2x^2 + 2y^2 = 3x - 5y + 7$

Soln

$$2x^2 + 2y^2 - 3x + 5y - 7 = 0$$

$$x^2 + y^2 - \frac{3}{2}x + \frac{5}{2}y - 7/2 = 0$$

Compare with $x^2 + y^2 + 2gx + 2fy + c = 0$

Centre $(-g, -f)$ radius $= \sqrt{g^2 + f^2 - c}$

$2g = -3/2$	$2f = 5/2$	$r = \sqrt{\left(\frac{3}{4}\right)^2 + \left(\frac{-5}{4}\right)^2 - \left(-7/2\right)}$
$-g = +3/4$	$-f = \frac{-5}{4}$	$r = \frac{3\sqrt{10}}{4}$

Centre : $\left(\frac{3}{4}, \frac{-5}{4}\right)$, radius $= \frac{3\sqrt{10}}{4}$

(2) find the equation of the circle whose centre is the point of intersection of the lines

$2x - 3y + 4 = 0$

$3x + 4y - 5 = 0$

and passes through the origin.

Soln

$3(2x - 3y + 4) = 0$

$2(3x + 4y - 5) = 0$

$-17y + 22 = 0$

$y = \frac{22}{17}$ $\qquad x = \frac{3\left(\frac{22}{17}\right) - 4}{2}$

$x \Rightarrow \frac{-1}{17}$

Centre: $\left(\frac{-1}{17}, \frac{22}{17}\right)$

radius = distance b/w origin & centre

$= \sqrt{\left(\frac{22}{17}\right)^2 + \left(\frac{-1}{17}\right)^2} = \frac{1}{17}\sqrt{485}$

equation of circle

$(x-h)^2 + (y-k)^2 = r^2$

$\left(x - \left(-\frac{1}{17}\right)\right)^2 + \left(y - \frac{22}{17}\right)^2 = \left(\frac{\sqrt{485}}{17}\right)^2$

$$\boxed{17(x^2+y^2) + \frac{2x}{17} - 44y = 0}$$

(3) find the parametric form of the equation of the circle $x^2+y^2+px+py=0$

Soln

$$x^2+y^2+px+py=0$$

$$x^2+px+\left(\frac{p}{2}\right)^2-\left(\frac{p}{2}\right)^2+y^2+py+\left(\frac{p}{2}\right)^2-\left(\frac{p}{2}\right)^2=0$$

$$\left(x+\frac{p}{2}\right)^2+\left(y+\frac{p}{2}\right)^2=\frac{p^2}{2}$$

So $r=\frac{p}{\sqrt{2}}$

Paramatric form

$$x+\frac{p}{2}=\frac{p}{\sqrt{2}}\cos\theta$$

$$x=\frac{p}{2}\left(-1+\sqrt{2}\cos\theta\right)$$

Similarly

$$y+\frac{p}{2}=\frac{p}{\sqrt{2}}\sin\theta$$

$$y=\frac{p}{2}\left(-1+\sqrt{2}\sin\theta\right)$$

(4) find the equation of circle the end points of whose diametre are the centres of the circles

$C_1: x^2+y^2+16x-14y=1$

&

$C_2: x^2+y^2-4x+10y=2$

Sol^n

Centre of C_1 : $(-8, 7)$

Centre of C_2 : $(2, -5)$

Desired circle have end point of diameter as C_1 & C_2

Diametric form of circle

$$(x-x_1)(x-x_2)+(y-y_1)(y-y_2)=0$$

$$\Rightarrow (x-(-8))(x-2)+(y-7)(y-(-5))=0$$

$$\Rightarrow \boxed{x^2+y^2+6x-2y-51=0}$$

Do-yourself II

(1) find the position of point $(1,2)$ & $(6,0)$ W.r.t the circle $x^2+y^2-4x+2y-11=0$

Sol^n

$S: \quad x^2+y^2-4x+2y-11=0$

$S_{(1,2)}: \quad (1)^2+(2)^2-4(1)+2(2)-11$

$\Rightarrow \quad 5-11 = -6 \quad (<0)$

Point $(1,2)$ lies inside circle

$S_{(6,0)}: \quad (6)^2+(0)^2-4(6)+2(0)-11$

$\Rightarrow \quad 36-24-11 = 1 \quad (>0)$

Point $(6,0)$ lies outside circle.

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(2) find the greatest & least ~~value~~ distance of a point $P(7,3)$ from circle $x^2+y^2-8x-6y+16=0$

Also find power of point P w.r.t circle.

Solⁿ

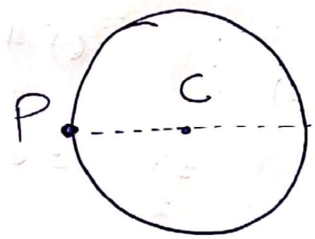
$S : x^2+y^2-8x-6y+16=0$

$S_P : (7)^2+(3)^2-8(7)-6(3)+16$

$-7-9+16 = 0$

Point P lies on circle.

$\boxed{\text{Power} = 0}$



$\boxed{\text{least distance} = 0}$

greatest distance $= 2 \times$ radius

radius $= \sqrt{g^2+f^2-c} = 3$

$\boxed{\text{greatest dist.} = 2\times3 = 6}$

Do-yourself III

(1) find the equation of tangent to the circle $x^2+y^2-2ax=0$ at the point $(a(1+\cos\alpha), a\sin\alpha)$

Sol^n

Tangent at point

$$\boxed{T=0}$$

$$T: \quad xx_1 + yy_1 - 2a\frac{(x+x_1)}{2} = 0$$

(x_1, y_1) : $(a(1+\cos\alpha), a\sin\alpha)$

$$x(\cancel{a}(1+\cos\alpha)) + y(\cancel{a}\sin\alpha) - \cancel{a}(x + a(1+\cos\alpha)) = 0$$

$$\cancel{x} + x\cos\alpha + y\sin\alpha - \cancel{x} - a(1+\cos\alpha) = 0$$

$$\boxed{x\cos\alpha + y\sin\alpha = a(1+\cos\alpha)}$$

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(2) find the equation of tangents to the circle $x^2+y^2-6x+4y-12=0$ which are parallel to the line $4x-3y+6=0$

Sol^n

Tangent Parallel to line $4x-3y+6=0$ will be $\boxed{4x-3y+\lambda=0}$

Tangency condition $\boxed{P=r}$

Perpendicular dist from centre.

$S:\ x^2+y^2-6x+4y-12=0$

Centre : $(3,-2)$

radius : $\sqrt{9+4-(-12)} = 5$

So

$$5 = \frac{|4(3)-3(-2)+\lambda|}{\sqrt{4^2+3^2}}$$

$$25 = |18+\lambda|$$

$$\lambda = 7,\ -43$$

lines will be

$$4x-3y+7=0\ ,\quad 4x-3y-43=0$$

(3) find the equation of the tangents to the Circle $x^2+y^2=4$ which are perpendicular to the line $12x-5y+9=0$ also find point of Contact.

Sol^n

Tangent is perpendicular to given line

So eq. of tangent will be

$$5x+12y+\lambda=0$$

Tangency Cond^n $\boxed{P=r}$

Centre $(0,0)$

radius : (2)

$$2=\frac{|\lambda|}{\sqrt{5^2+12^2}}$$

$$\lambda=\pm 26$$

Tangents $5x+12y=\pm 26$

$$m=\frac{-5}{12}$$

Contact points $\left(\mp\frac{am}{\sqrt{1+m^2}},\ \pm\frac{a}{\sqrt{1+m^2}}\right)$

$\boxed{a = \text{radius} = 2}$

$$\left(\mp\frac{2\left(\frac{-5}{12}\right)}{\sqrt{1+\left(\frac{5}{12}\right)^2}},\ \pm\frac{2}{\sqrt{1+\left(\frac{5}{12}\right)^2}}\right)=\left(\mp\frac{10}{13},\mp\frac{24}{13}\right)$$

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(4) find the Value of 'c' if the line $y = c$ is a tangent to the circle $x^2+y^2-2x+2y-2=0$ at the point $(1,1)$

Soln

$y = c$ (Passing through $(1,1)$)

So

$y = 1$

Solvieng Circle $x^2+y^2-2x+2y-2$ & line $y=1$

Simultaneouslyly.

$$\Rightarrow \quad x^2+(1)^2-2x+2(1)-2=0$$

$$\Rightarrow \quad x^2-2x+1 = 0$$

$D=0$ (only one point common)

$x = 1$

Tangent exist & it will be

$y=1$

$$\boxed{c=1}$$

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Do-your self IV

(1) find the equation of Normal to the circle $x^2+y^2=2x$, which is parallel to the line $x+2y=3$

Sol^n eq. of Normal will be

$$x+2y=\lambda$$

★ Normal to circle always passes through centre.

Circle : $x^2-2x+y^2=0$

Centre : $(1,0)$

So Normal $(1,0)$

$$1+2(0)=\lambda$$

$$\lambda=1$$

eq. of Normal will be

$$\boxed{x+2y=1}$$

Do-yourself - 5

1. Find the eqⁿ of the chord of contact of the point (1,2) with respect to the circle $x^2+y^2+2x+3y+1=0$

Solⁿ:- Eqⁿ of chord contact is $T=0$

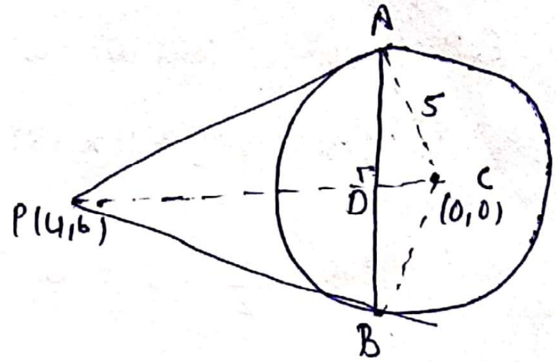
So. $1\cdot x + 2\cdot y + (x+1) + \frac{3}{2}(y+2) + 1 = 0$

$2x + \frac{7y}{2} + 5 = 0$

$4x + 7y + 10 = 0$

(ii) Tangents are drawn from the point $P(4,6)$ to the circle $x^2+y^2=25$. Find the area of the triangle formed by them & their chord of contact.

Solⁿ:-



eqⁿ of chord of contact AB is

$4x+6y=25$

length of CD $= \dfrac{|0+0-25|}{\sqrt{16+36}}$

$= \dfrac{25}{2\sqrt{13}}$

length of AD $= \sqrt{25-\dfrac{625}{52}}$

$= 5\sqrt{1-\dfrac{25}{52}}$

AD $= \dfrac{15\sqrt{3}}{2\sqrt{13}}$

AB $= \dfrac{15\sqrt{3}}{\sqrt{13}}$

leng of PD $= \dfrac{|16+36-25|}{2\sqrt{13}}$

$= \dfrac{27}{2\sqrt{13}}$

Area of Δ PAB $= \dfrac{1}{2} \times AB \times PD$

$= \dfrac{1}{2} \times \dfrac{15\sqrt{3}}{\sqrt{13}} \times \dfrac{27}{2\sqrt{13}}$

$= \dfrac{405\sqrt{3}}{52}$

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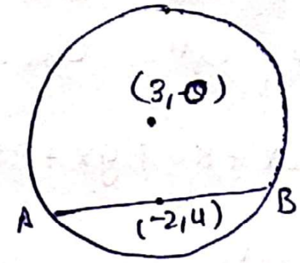
Do yourself - 6

(i) Find the eqn of the chord of $x^2+y^2 - 6x + 10y - 9 = 0$ which is bisected at $(-2, 4)$.

Soln:- $T = S_1$



$$-2x + 4y - 3(x-2) + 0 + \cancel{10} - 9$$
$$= 4 + 16 + 12 + \cancel{10} - 9$$

$$-5x + 4y = 26$$
$$5x - 4y + 26 = 0$$

(ii) Find the locus of mid point of chord of $x^2+y^2+2gx+2fy+c=0$ that passes through origin.

Soln:- Let the mid point of chord is (h,k)

so eqn of chord with given middle point is $T = S_1$

$$hx + ky + gx + gh + fy + fk + \cancel{c} = h^2 + k^2 + 2gh + 2fk + \cancel{c}$$

$$(h+g)x + (k+f)y - h^2 - k^2 - gh - fy = 0$$

it passes through origin so.

$$(h+g)0 + (k+f)0 - h^2 - k^2 - gh - fy = 0$$

$$h^2 + k^2 + gh + fy = 0$$

Replacing (h,k) by (x,y)

$$\boxed{x^2 + y^2 + gx + fy = 0}$$

Do-yourself-7

(i) Find the equation of the director circle of the circle $(x-h)^2+(y-k)^2 = a^2$

Sol^n.

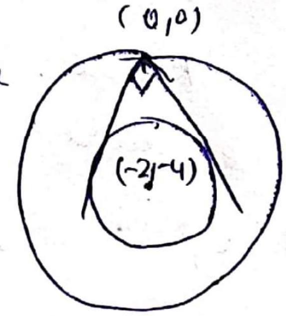
Radius of director circle is $\sqrt{2}a$

Centre of director circle is (h,k)

So eq^n $\Rightarrow$ $(x-h)^2+(y-k)^2 = 2a^2$

(ii) If the angle b/w the tangents drawn to $x^2+y^2+4x+8y+c=0$ from $(0,0)$ is $\pi/2$, then find value of c

Sol^n: If tangents from $(0,0)$ made angle $\frac{\pi}{2}$ So. there is a director circle which passes through $(0,0)$.



So $S_1 \equiv x^2+y^2+4x+8y+c=0$

Center $= (-2,-4)$

Radius $= \sqrt{20-c}$

So eq^n of director circle is =

$(x+2)^2+(y+4)^2 = 2(20-c)$

and it passes through $(0,0)$ so.

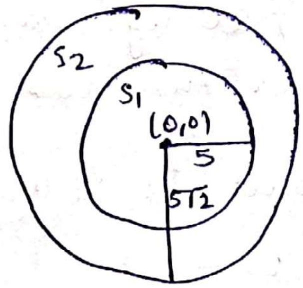
$4+16 = 40-2c$

$2c = 20$

$\boxed{c = 10}$

(iii) If two tangents are drawn from a point on the circle $x^2+y^2=50$ to the circle $x^2+y^2=25$ then find the angle b/w the tangents.

Solⁿ:-



$\because$ Circle S_1 & S_2 has same center and Radius of circle S_2 is $\sqrt{2}$ times circle S_1

So S_2 is the director circle of S_1

$\therefore$ Angle b/w tangents is $\pi/2$

Do yourself - 8

(i) Prove that the polar of a given point with respect to any one of circles $x^2+y^2-2kx+c^2=0$ where k is variable, always passes through a fixed point, whatever the value of k

Sol^n:- Let the given point is (a,b) so. eq^n of Polar w.r.t. (a,b) is -

$$T=0$$

$$ax+by-kx-ka+c^2=0$$

$$ax+by+c^2-k(x+a)=0$$

It represent family of st. line which always passes through intersection of the lines $ax+by+c^2=0$ & $x+a=0$ so it does not depend on the value of k.

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(ii) Find the eqn of the circle passing through the points of intersection of the circle $x^2+y^2-6x+2y+4=0$ and $x^2+y^2+2x-4y-6=0$ and with its centre on the line $y=x$.

Soln. eqn of circle is $S_1+dS_2=0$

$$x^2+y^2-6x+2y+4+d(x^2+y^2+2x-4y-6)=0$$

$$x^2(1+d)+y^2(1+d)+2x(d-3)+2y[1-2d]+4-6d=0$$

Centre $\equiv$ $(3-d,\ 2d-1)$

Centre is passes through the line $y=x$

i.e. $3-d=2d-1$

$4=3d$

$d=4/3$

$$x^2(1+4/3)+y^2(1+4/3)+2x(4/3-3)+2y(1-8/3)+4-6\times 4/3=0$$

$$\boxed{x^2+y^2-\frac{10}{7}x-\frac{10}{7}y-\frac{12}{7}=0}$$

(iii) Find the equation of circle through the points of intersection of the circles $x^2+y^2+2x+3y-7=0$ and $x^2+y^2+3x-2y-1=0$ and passing through the point $(1,2)$.

Sol^n:-

$$S_1+dS_2=0$$

$\Rightarrow$ $x^2+y^2+2x+3y-7+d(x^2+y^2+3x-2y-1)=0$

$\Rightarrow$ $\cancel{x^2(1+d)}$ and it passes through $(1,2)$ so.

$1+4+2+6-7+d(1+4+3-4-1)=0$

$6+3d=0$

$d=-2$

$\Rightarrow$ $x^2+y^2+2x+3y-7-2x^2-2y^2-6x+4y+2=0$

$\Rightarrow$ $-x^2-y^2-4x+7y-5=0$

$\Rightarrow$ $x^2+y^2+4x-7y+5=0$

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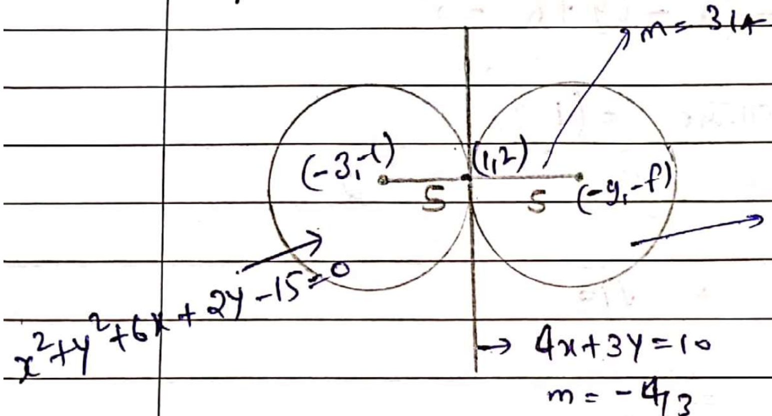
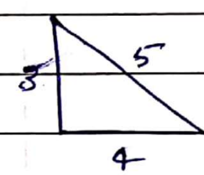
Do your self (9)

(i) Two Circles with radius 5 touches at the point (1,2). If the equation of common tangent is $4x+3y=10$ and one of the circle is $x^2+y^2+6x+2y-15=0$. Find the equation of other circle.





eq.^{n} of other circle $= \dfrac{x-x_1}{\cos\theta} = \dfrac{y-y_1}{\sin\theta} = \pm r$

$$\frac{x-1}{4/5} = \frac{y-2}{3/5} = \pm 5$$

$x = (\pm 4+1)$ & $y = (\pm 3+2)$

$\therefore$ (5,5) (-3,-1)

$\therefore$ eq.^{n} of circle $(x-5)^2+(y-5)^2=25$

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(ii) Find the Number of common tangents of the circles $x^2+y^2=1$ & $x^2+y^2-2x-6y+6=0$

let $S_1 = x^2+y^2=1 \longrightarrow$ centre $=(0,0)$

radius $= 1$

let $S_2 = x^2+y^2-2x-6y+6=0$

Centre $=(1,3)$

radius $= \sqrt{g^2+f^2-c} = 2$

$C_1C_2 = \sqrt{9+1} = \sqrt{10}$

$r_1+r_2 = 1+2 = 3$

$\therefore \quad C_1C_2 > r_1+r_2$

i.e. there are $\boxed{4}$ common tangent.

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Do your Self (10)

(P) Find the angle of intersection of two circles

$S = x^2+y^2-4x+6y+11=0$

$S' = x^2+y^2-2x+8y+13=0$

$S = x^2+y^2-4x+6y+11=0$ — centre $(2,-3)$, radius $= \sqrt{2}$

$S' = x^2+y^2-2x+8y+13=0$ — centre $(1,-4)$, radius $= 2$

$$\cos\theta = \left|\frac{r_1^2+r_2^2-d^2}{2r_1r_2}\right|$$

$$\cos\theta = \left|\frac{\cancel{2}+4-\cancel{2}}{2\times 2\sqrt{2}}\right|$$

$$\cos\theta = \left|\frac{\cancel{4}}{\cancel{4}\sqrt{2}}\right|$$

$$\theta = 45^\circ, \boxed{135^\circ}$$

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(ii) Find the equation of the radical axis of the circle $x^2+y^2-3x-4y+5=0$ & $3x^2+3y^2-7x-8y+11=0$

∵ We know that eqn of radical axis is

$$\boxed{S_1 - S_2 = 0}$$

$$\therefore \cancel{x^2}+\cancel{y^2}-3x-4y+5-\cancel{x^2}-\cancel{y^2}+\frac{7}{3}x+\frac{8}{3}y-\frac{11}{3}=0$$

$$x\left[-3+\frac{7}{3}\right]+y\left[-4+\frac{8}{3}\right]+5-\frac{11}{3}=0$$

$$-\frac{2}{3}x-\frac{4}{3}y+\frac{4}{3}=0$$

$$2x+4y-4=0$$

$$\boxed{x+2y=2}$$

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(iii) Find the radical centre of three circles described on the three sides $4x - 7y + 10 = 0$, $x + y - 5 = 0$ & $7x + 4y - 15 = 0$ of a triangle as diameters.

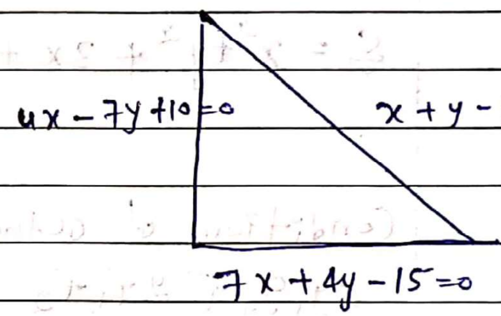
$S_1 = 4x - 7y + 10 = 0 \Rightarrow m_1 = +4/7$

$S_2 = x + y - 5 = 0 \Rightarrow m_2 = -1$

$S_3 = 7x + 4y - 15 = 0 \Rightarrow m_3 = -7/4$

$m_1 m_3 = -1$

$\therefore S_1 \perp S_3$



$$\therefore \begin{array}{r} 16x - \cancel{28y} + 40 = 0 \\ 49x + \cancel{28y} - 105 = 0 \\ \hline 65x = 65 \end{array} \Rightarrow \boxed{x = 1}$$

$$\boxed{y = 2}$$

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Do your self (11)

(i) When the circles $x^2+y^2+4x+6y+3=0$ & $2(x^2+y^2)+6x+4y+c=0$ intersect orthogonally then find the value of $c = ?$

$S_1 = x^2+y^2+4x+6y+3=0$ — centre $(-2,-3)$, radius $= \sqrt{10}$

$S_2 = x^2+y^2+3x+2y+\frac{c}{2}=0$ — centre $(-3/2, -1)$, radius $= \frac{1}{2}\sqrt{13-2c}$

Condition of orthogonally

$2g_1g_2 + 2f_1f_2 = c_1+c_2$

$\cancel{2}(-2)\left(\frac{-3}{\cancel{2}}\right) + 2(-3)(-1) = 3+\frac{c}{2}$

$12 = \frac{c}{2}+3$

$\frac{c}{2} = 9$

$\boxed{c=18}$

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(ii) Write the condition so that circles $x^2+y^2+2ax+c=0$ & $x^2+y^2+2by+c=0$ touch externally.

$S_1 = x^2+y^2+2ax+c=0$ — centre $(-a,0)$, radius $\sqrt{a^2-c}$

$S_2 = x^2+y^2+2by+c=0$ — centre $(0,-b)$, radius $\sqrt{b^2-c}$

When two circles touches externally

$$C_1C_2 = r_1+r_2$$

$$\sqrt{b^2+a^2} = \sqrt{a^2-c}+\sqrt{b^2-c}$$

$$\left(\sqrt{a^2+b^2}\right)^2 = \left[\sqrt{a^2-c}+\sqrt{b^2-c}\right]^2$$

$$\cancel{a^2}+\cancel{b^2} = \cancel{a^2}-c+\cancel{b^2}-c+2\sqrt{(a^2-c)(b^2-c)}$$

$$\cancel{2}c = \cancel{2}\sqrt{(a^2-c)(b^2-c)}$$

$$\cancel{c^2} = a^2b^2-a^2c-b^2c+\cancel{c^2}$$

$$a^2b^2 = a^2c+b^2c$$

$$1 = \frac{c}{b^2}+\frac{c}{a^2}$$

$$\boxed{\frac{1}{c} = \frac{1}{a^2}+\frac{1}{b^2}}$$

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EXERCISE (O-1)

[SINGLE CORRECT]

1. Centres of the three circles $x^2 + y^2 - 4x - 6y - 14 = 0$, $x^2 + y^2 + 2x + 4y - 5 = 0$ and $x^2 + y^2 - 10x - 16y + 7 = 0$

(A) are the vertices of a right triangle

(B) the vertices of an isosceles triangle which is not regular

(C) vertices of a regular triangle

(D) are collinear

Solution: Centre of first circle is $A(2, 3)$

" " second " " $B(-1, -2)$

" " third " " $C(5, 8)$

Now, $AB = \sqrt{3^2 + 5^2} = \sqrt{34}$

$BC = \sqrt{6^2 + 10^2} = \sqrt{136} = 2\sqrt{34}$

$AC = \sqrt{3^2 + 5^2} = \sqrt{34}$

As $AB + AC = BC$. So, points A, B, C are collinear

option (D)

2. $y - 1 = m_1(x - 3)$ and $y - 3 = m_2(x - 1)$ are two family of straight lines, at right angled to each other. The locus of their point of intersection is

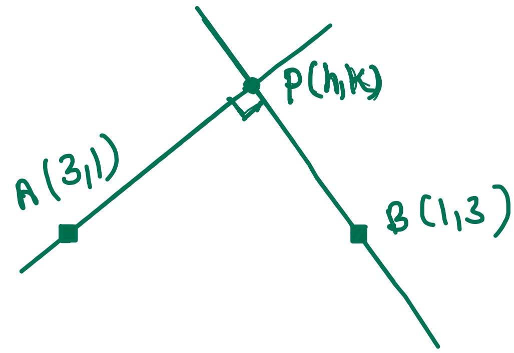
(A) $x^2 + y^2 - 2x - 6y + 10 = 0$ (B) $x^2 + y^2 - 4x - 4y + 6 = 0$

(C) $x^2 + y^2 - 2x - 6y + 6 = 0$ (D) $x^2 + y^2 - 4x - 4y - 6 = 0$

Solution: First family passes through (3,1)
and second " " " (1,3)

Let point of intersection is $P(h,k)$

So, $m_{AP} \cdot m_{BP} = -1$ (For locus)



$$\Rightarrow \left(\frac{k-1}{h-3}\right) \cdot \left(\frac{k-3}{h-1}\right) = -1$$

$$\Rightarrow (k-1)(k-3) = -(h-3)(h-1)$$

$$\Rightarrow k^2 - 4k + 3 = -(h^2 - 4h + 3) \Rightarrow h^2 + k^2 - 4h - 4k + 6 = 0$$

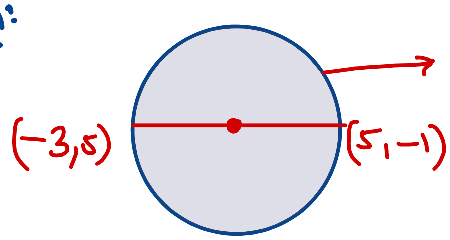
$$\Rightarrow x^2 + y^2 - 4x - 4y + 6 = 0$$ option (B)

---

3. Suppose that the equation of the circle having (–3, 5) and (5, –1) as end points of a diameter is $(x - a)^2 + (y - b)^2 = r^2$. Then $a + b + r$, $(r > 0)$ is

(A) 8 (B) 9 (C) 10 (D) 11

Solution:



$S_1: (x+3)(x-5) + (y-5)(y+1) = 0$

$\Rightarrow S_1: x^2 + y^2 - 2x - 4y - 20 = 0$

$\Rightarrow$ centre $(1, 2) \equiv (a, b)$

radius $r = \sqrt{1^2 + 2^2 - (-20)} = \sqrt{25} = 5$

$\boxed{\therefore \quad a + b + r = 1 + 2 + 5 = 8}$ option (A)

OR, centre $(a, b) = \left(\frac{-3+5}{2}, \frac{5+(-1)}{2}\right) \equiv (1, 2)$

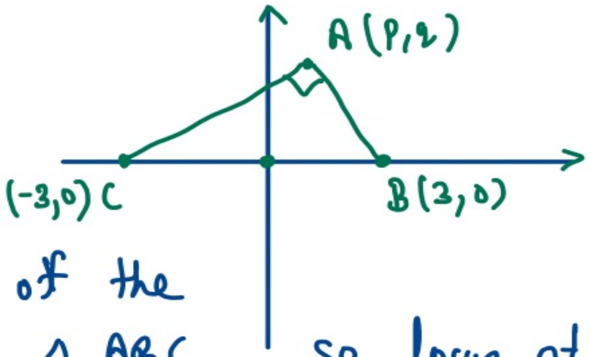
$$r = \frac{1}{2}\sqrt{(5+3)^2 + (5+1)^2} = 5$$

4. B and C are fixed points having co-ordinates (3, 0) and (–3, 0) respectively. If the vertical angle BAC is 90°, then the locus of the centroid of the ΔABC has the equation :

(A) $x^2 + y^2 = 1$ (B) $x^2 + y^2 = 2$ (C) $9(x^2 + y^2) = 1$ (D) $9(x^2 + y^2) = 4$

Solution:

Let $A(p, q)$



$\angle BAC = 90°$

$\Rightarrow$ BC is diameter of the circumcircle of Δ ABC. so locus of 'A' is

$(p+3)(p-3) + (q-0)(q-0) = 0 \Rightarrow p^2 + q^2 = 9$

Now, let centroid of Δ ABC is $G(h, k)$

$\therefore h = \frac{p}{3}$ and $k = \frac{q}{3}$

as $p^2 + q^2 = 9 \Rightarrow h^2 + k^2 = 1 \Rightarrow x^2 + y^2 = 1$ option(A)

5. The equation of the image of the circle $x^2 + y^2 + 16x - 24y + 183 = 0$ by the line mirror $4x + 7y + 13 = 0$ is

(A) $x^2 + y^2 + 32x - 4y + 235 = 0$

(B) $x^2 + y^2 + 32x + 4y - 235 = 0$

(C) $x^2 + y^2 + 32x - 4y - 235 = 0$

(D) $x^2 + y^2 + 32x + 4y + 235 = 0$

Solution:

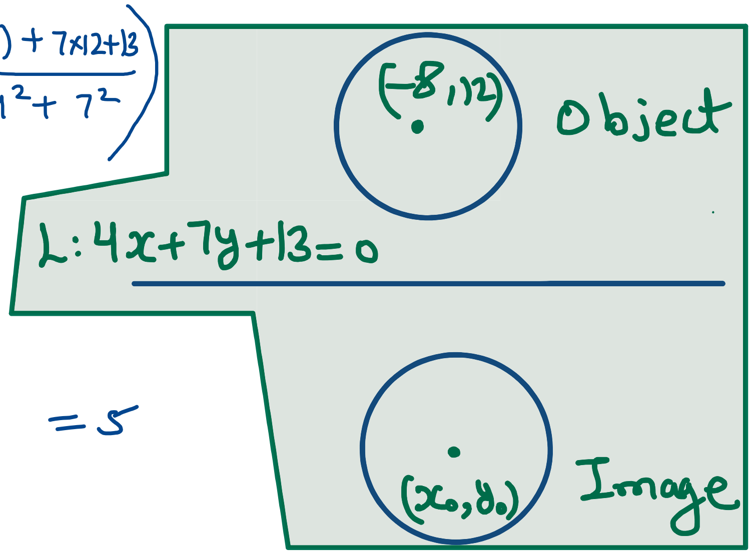
Image of a circle in a line is a circle with same radius and centre as the reflection of centre of the given circle.

Let image is (x_0, y_0) of $(-8, 12)$ in line $L: 4x + 7y + 13 = 0$

So, $\dfrac{x_0 - (-8)}{4} = \dfrac{y_0 - 12}{7} = (-2)\left(\dfrac{4\times(-8) + 7\times12 + 13}{4^2 + 7^2}\right)$

$\Rightarrow \dfrac{x_0 + 8}{4} = \dfrac{y_0 - 12}{7} = -2$

$\Rightarrow x_0 = -16$ & $y_0 = -2$



and radius, $r = \sqrt{(-8)^2 + (12)^2 - 183} = 5$

$\therefore$ Equation of the circle is

$(x+16)^2 + (y+2)^2 = 5^2$

$\Rightarrow x^2 + y^2 + 32x + 4y + 235 = 0$ option (D)

6

In the xy plane, the segment with end points (3, 8) and (–5, 2) is the diameter of the circle. The point (k, 10) lies on the circle for

(A) no value of k

(B) exactly one integral k

(C) exactly one non integral k

(D) two real values of k

Solution:

Equation of circle is $S: (x-3)(x+5)+(y-8)(y-2)=0$

$\Rightarrow S: x^2+2x-15+y^2-10y+16=0$

Put $x=K$ and $y=10$ we get

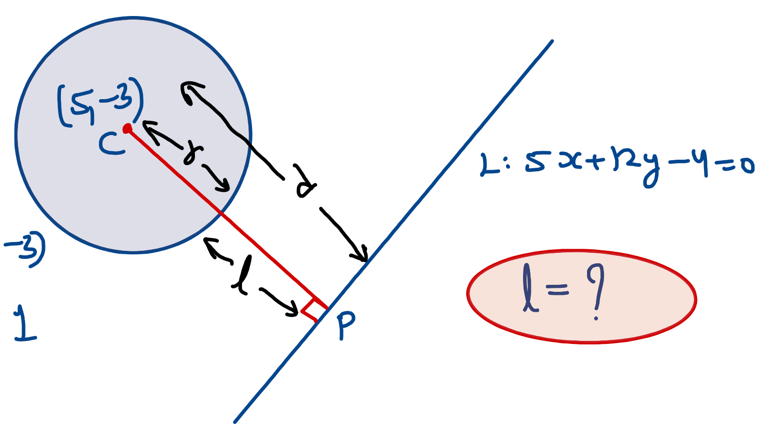
$K^2+2K+1=0 \Rightarrow K=-1$ option (B)

7

The smallest distance between the circle $(x-5)^2 + (y+3)^2 = 1$ and the line $5x + 12y - 4 = 0$, is

(A) 1/13 (B) 2/13 (C) 3/15 (D) 4/15

Solution:



Centre $\equiv (5,-3)$

radius, $r = 1$

$$d = \left|\frac{5(5)+12(-3)-4}{\sqrt{5^2+12^2}}\right| = \left|\frac{25-36-4}{13}\right| = \frac{15}{13}$$

$l = |d - r|$ = minimum distance b/w line & circle

$$\Rightarrow l = \left|\frac{15}{13} - 1\right| = \frac{2}{13}$$

Option (B)

8

Consider the points P(2, 1); Q(0, 0); R(4, –3) and the circle $S : x^2 + y^2 - 5x + 2y - 5 = 0$

(A) exactly one point lies outside S

(B) exactly two points lie outside S

(C) all the three points lie outside S

(D) none of the point lies outside S

Solution: $S_P = 2^2 + 1^2 - 5\cdot 2 + 2\cdot 1 - 5 < 0$ ('P' is inside)

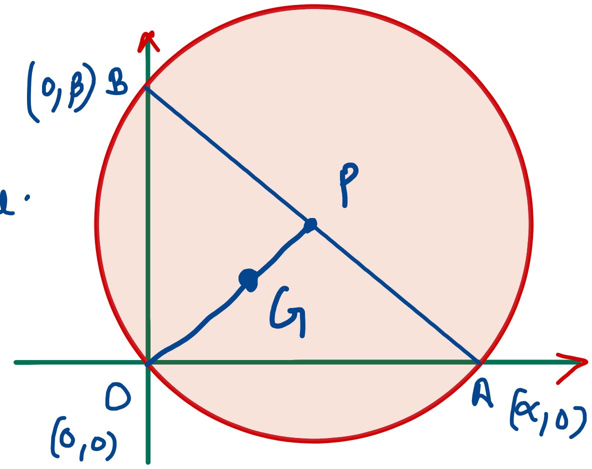
$S_Q = 0^2 + 0^2 - 5\cdot 0 + 2\cdot 0 - 5 < 0$ ('Q' is inside)

$S_R = 4^2 + (-3)^2 - 5(4) + 2(-3) - 5 < 0$ ('R' is inside)

Option (D)

9. If a circle of constant radius 3k passes through the origin 'O' and meets co-ordinate axes at A and B then the locus of the centroid of the triangle OAB is -

(A) $x^2 + y^2 = (2k)^2$ (B) $x^2 + y^2 = (3k)^2$ (C) $x^2 + y^2 = (4k)^2$ (D) $x^2 + y^2 = (6k)^2$



Solution:

Let $A(\alpha, 0)$ & $B(0, \beta)$

$\angle BOA = 90^\circ \Rightarrow$ AB is diameter of the circle.

Let centroid is $G(p, q)$

So, $p = \frac{\alpha + 0 + 0}{3}$ & $q = \frac{0 + \beta + 0}{3}$

$\Rightarrow \alpha = 3p$ & $\beta = 3q$

Note: $AB = 2 \times \text{radius} = 6K$

$\Rightarrow \sqrt{\alpha^2 + \beta^2} = 6K \Rightarrow (3p)^2 + (3q)^2 = 36K^2$

$\Rightarrow p^2 + q^2 = 4K^2 \Rightarrow \boxed{x^2 + y^2 = 4K^2}$ option (A)

M-II OP is median of ΔOAB from 'O'.

where 'P' is the centre of the circle.

So, $\frac{OG}{GP} = \frac{2}{1}$ ($G \equiv$ centroid)

$\Rightarrow OG = \frac{2}{3}(OP) \Rightarrow OG = 2K$

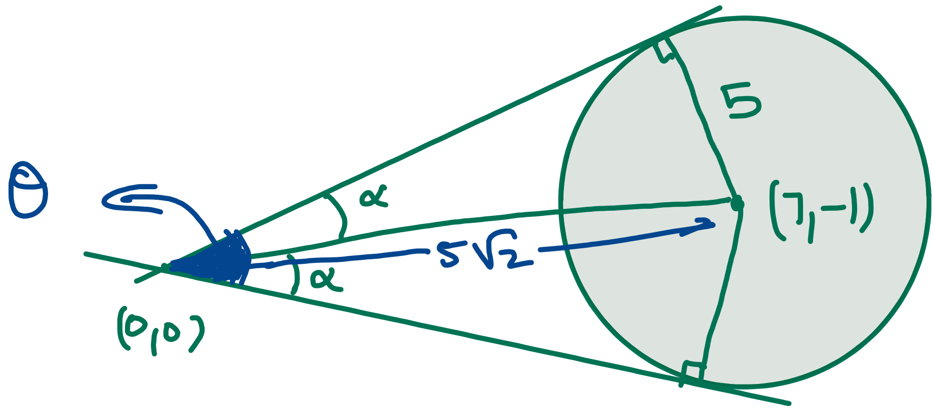
So, $\sqrt{p^2 + q^2} = 2k \Rightarrow p^2 + q^2 = (2k)^2$

$\Rightarrow x^2 + y^2 = (2k)^2$

10 The angle between the two tangents from the origin to the circle $(x-7)^2 + (y+1)^2 = 25$ equals

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{3}$ (C) $\frac{\pi}{2}$ (D) $\frac{\pi}{4}$

Solution:



$\theta = 2\alpha;$

$\sin\alpha = \frac{5}{5\sqrt{2}} = \frac{1}{\sqrt{2}} \Rightarrow \alpha = \frac{\pi}{4} \Rightarrow \theta = \frac{\pi}{2}$ option(C)

---

11 Tangents are drawn from (4, 4) to the circle $x^2 + y^2 - 2x - 2y - 7 = 0$ to meet the circle at A and B. The length of the chord AB is

(A) $2\sqrt{3}$ (B) $3\sqrt{2}$ (C) $2\sqrt{6}$ (D) $6\sqrt{2}$

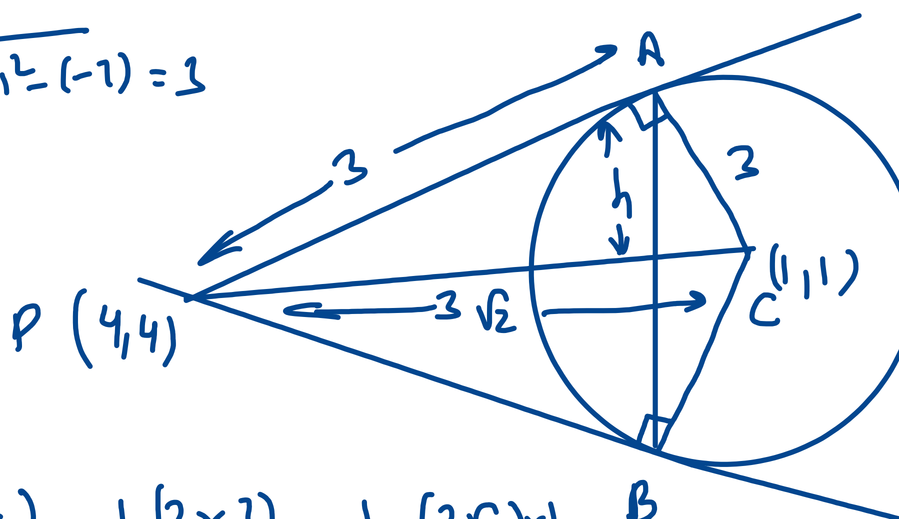
Solution:

$r = \sqrt{1^2+1^2-(-7)} = 3$

$AB = 2h$

$PC = 3\sqrt{2}$

$AP = 3$



$ar(\triangle APC) = \frac{1}{2}(3\times 3) = \frac{1}{2}(3\sqrt{2})\times h$

$\Rightarrow h = \frac{3}{\sqrt{2}}$

$\therefore A = 2h = 3\sqrt{2}$ option (B)

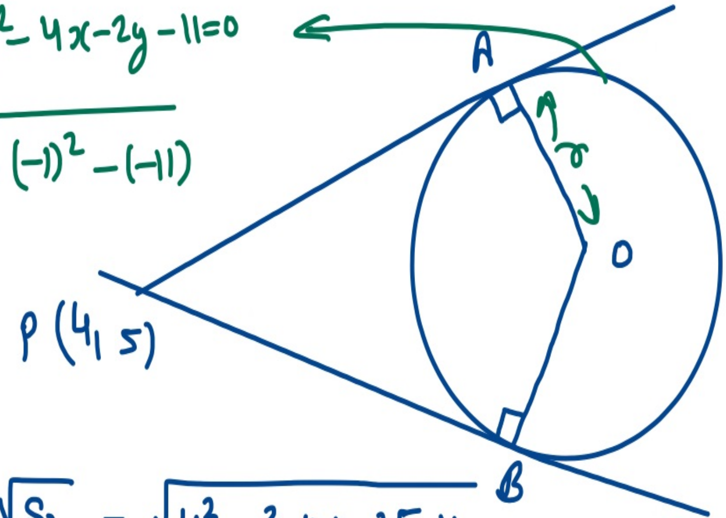
12. The area of the quadrilateral formed by the tangents from the point (4, 5) to the circle $x^2 + y^2 - 4x - 2y - 11 = 0$ with the pair of radii through the points of contact of the tangents is :

(A) 4 sq. units (B) 8 sq. units (C) 6 sq. units (D) none

Solution: $S : x^2 + y^2 - 4x - 2y - 11 = 0$

$r = \sqrt{(-2)^2 + (-1)^2 - (-11)}$

$\boxed{r = 4}$



$PA = PB = \sqrt{S_1} = \sqrt{4^2 + 5^2 - 4\cdot4 - 2\cdot5 - 11}$

$\Rightarrow PA = PB = 2$

$\therefore \ ar(PAOB) = 2 ar(\triangle PAO) = 2\left(\frac{1}{2} PA \times OA\right) = 8$ Option (B)

13. Combined equation to the pair of tangents drawn from the origin to the circle $x^2 + y^2 + 4x + 6y + 9 = 0$ is

(A) $3(x^2 + y^2) = (x + 2y)^2$
(B) $2(x^2 + y^2) = (3x + y)^2$
(C) $9(x^2 + y^2) = (2x + 3y)^2$
(D) $x^2 + y^2 = (2x + 3y)^2$

Solution: $\boxed{SS_1 = T^2}$ is formula for pair of tangents.

where, $S = x^2 + y^2 + 4x + 6y + 9$

$S_1 = 9$

$T = 0\cdot x + 0\cdot y + 4\left(\frac{x+0}{2}\right) + 6\left(\frac{y+0}{2}\right) + 9 = 2x + 3y + 9$

So, $(x^2 + y^2 + 4x + 6y + 9)(9) = (2x + 3y + 9)^2$

$\Rightarrow 9(x^2 + y^2) + 18(2x + 3y) + 81 = (2x+3y)^2 + 18(2x+3y) + 81$

$\Rightarrow 9(x^2 + y^2) = (2x + 3y)^2$ option (C)

14 From (3, 4) chords are drawn to the circle $x^2 + y^2 - 4x = 0$. The locus of the mid points of the chords is :

(A) $x^2 + y^2 - 5x - 4y + 6 = 0$ (B) $x^2 + y^2 + 5x - 4y + 6 = 0$

(C) $x^2 + y^2 - 5x + 4y + 6 = 0$ (D) $x^2 + y^2 - 5x - 4y - 6 = 0$

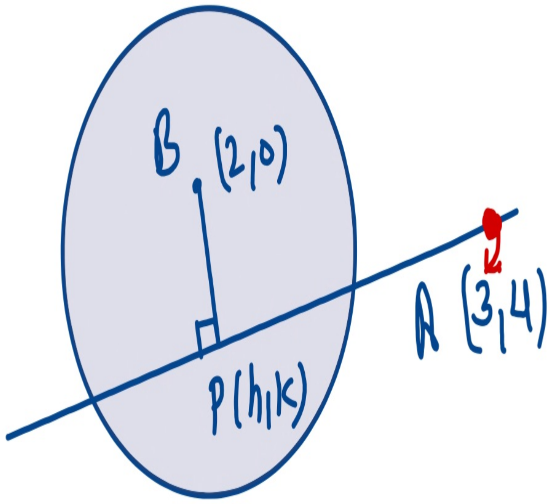
Solution:

$$m_{AP} \cdot m_{BP} = -1$$

$$\Rightarrow \left(\frac{k-4}{h-3}\right)\left(\frac{k-0}{h-2}\right) = -1$$

$$\Rightarrow x^2 + y^2 - 5x - 4y + 6 = 0$$

Option(A)

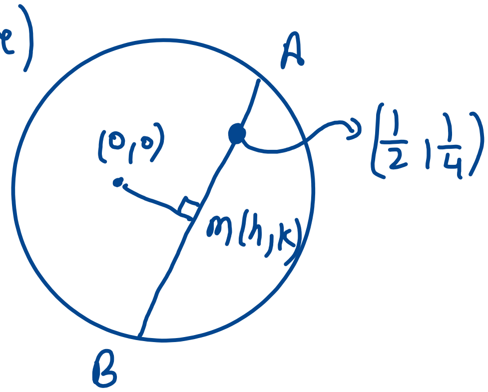


As chord AB is of the form $L_1 + \lambda L_2 = 0$, it means AB will pass through intersection point of $L_1 = 0$ and $L_2 = 0$.
i.e., $(4y-1=0$ & $x-2y=0)$

That intersection point is $\left(\frac{1}{2}, \frac{1}{4}\right)$.

$\left[\text{See } 4y-1=0 \Rightarrow y=\frac{1}{4}; \text{ use } x=2y=\frac{1}{2}\right]$

So, chord AB is always passing through a fixed point. (See figure)



locus of $m(h,k)$ is given as

$$\left(\frac{k-0}{h-0}\right)\left(\frac{k-\frac{1}{4}}{h-\frac{1}{2}}\right) = -1$$

$$\Rightarrow x^2+y^2-\frac{x}{2}-\frac{y}{4}=0 \Rightarrow 4(x^2+y^2)=2x+y$$

Option (C)

15 The locus of the center of the circles such that the point (2, 3) is the mid point of the chord $5x + 2y = 16$ is

(A) $2x - 5y + 11 = 0$ (B) $2x + 5y - 11 = 0$ (C) $2x + 5y + 11 = 0$ (D) none

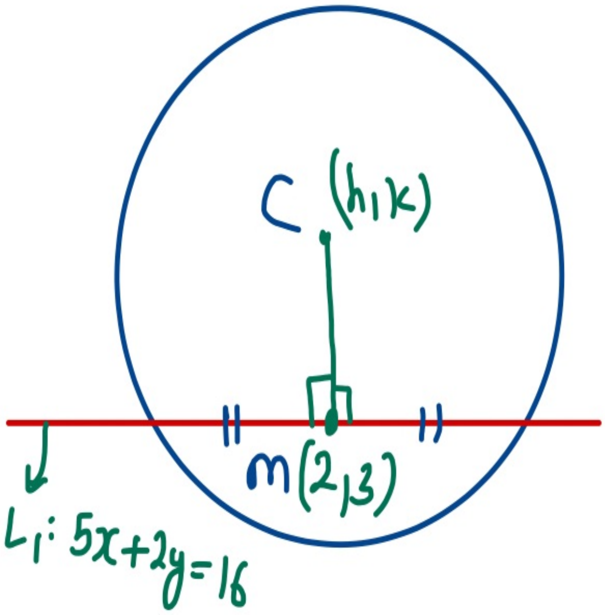
Solution:

$$m_{CM} \cdot m_{L_1} = -1$$

$$\Rightarrow \left(\frac{k-3}{h-2}\right)\left(\frac{-5}{2}\right) = -1$$

$$\Rightarrow 5k - 15 = 2h - 4$$

$$\Rightarrow 2x - 5y + 11 = 0$$ option(A)

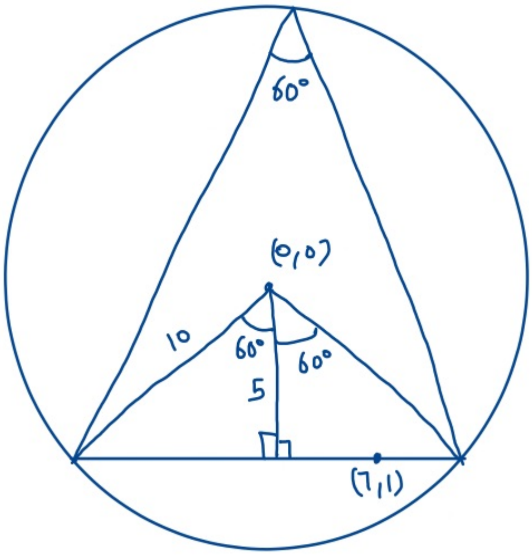


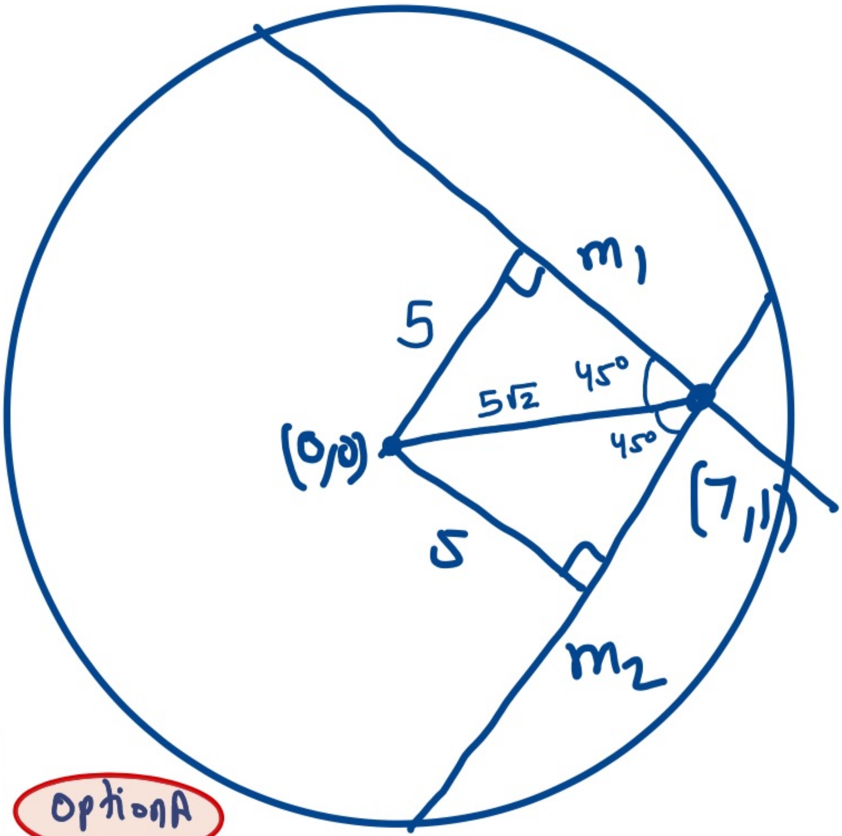
16

Chord AB of the circle $x^2 + y^2 = 100$ passes through the point (7, 1) and subtends an angle of 60° at the circumference of the circle. If m_1 and m_2 are the slopes of two such chords then the value of m_1m_2, is

(A) –1 (B) 1 (C) 7/12 (D) –3

Solution:





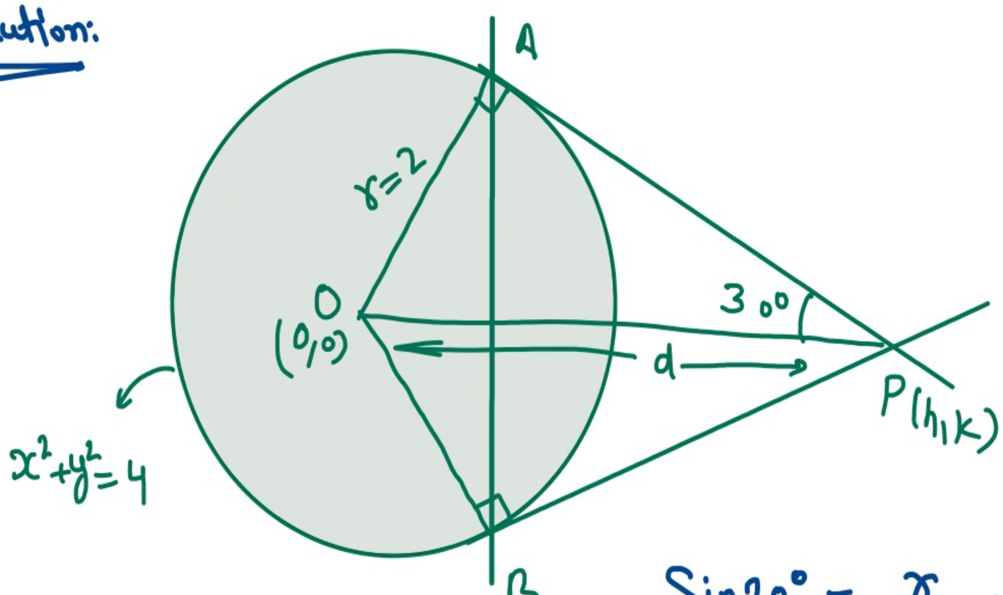
Clearly, $\boxed{m_1 \cdot m_2 = -1}$ Option A

17

Tangents PA and PB are drawn to the circle $x^2 + y^2 = 4$, then the locus of the point P if the triangle PAB is equilateral, is equal to-

(A) $x^2 + y^2 = 16$ (B) $x^2 + y^2 = 8$ (C) $x^2 + y^2 = 64$ (D) $x^2 + y^2 = 32$

Solution:



$$\sin 30^\circ = \frac{r}{d} \Rightarrow d = 4$$

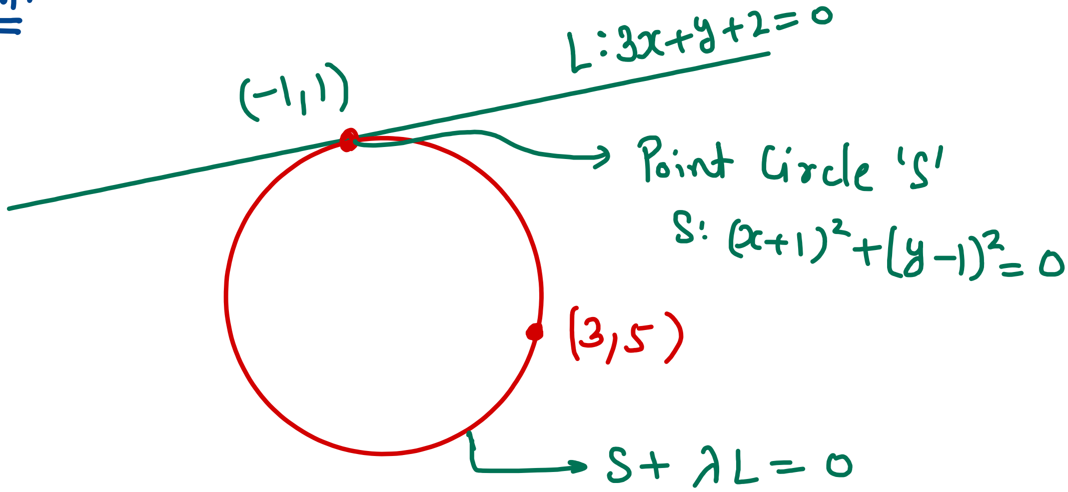
$$\Rightarrow |OP| = 4 \Rightarrow h^2 + k^2 = 16$$

option (A)

18. Sum of the abscissa and ordinate of the centre of the circle touching the line $3x + y + 2 = 0$ at the point $(-1,1)$ and passing through the point $(3,5)$ is-

(A) 2 (B) 3 (C) 4 (D) 5

Solution:



From family of circles the equation of required circle is given by $\boxed{S+\lambda L=0}$

$$S_1: (x+1)^2+(y-1)^2+\lambda(3x+y+2)=0$$

which passes through $(3,5)$

so, $(3+1)^2+(5-1)^2+\lambda(3\cdot 3+5+2)=0$

$$\Rightarrow \lambda=-2$$

$$\therefore S_1: (x+1)^2+(y-1)^2-2(3x+y+2)=0$$

$$\Rightarrow S_1: x^2+y^2-4x-4y-2=0$$

centre $\equiv (2,2)$ so, answer $=4$ option(C)

19. If L_1 and L_2 are the length of the tangent from (0, 5) to the circles $x^2 + y^2 + 2x - 4 = 0$ and $x^2 + y^2 - 2x - y + 1 = 0$ then

(A) $L_1 = 2L_2$ (B) $L_2 = 2L_1$ (C) $L_1 = L_2$ (D) $L_1^2 = L_2$

Solution: Length of tangent from Ext. pt. $= \sqrt{S_1}$

$$L_1 = \sqrt{0^2 + 25 + 2(0) - 4} = \sqrt{21}$$

$$L_2 = \sqrt{0 + 25 - 2(0) - 5 + 1} = \sqrt{21}$$

$\therefore$ $L_1 = L_2$ (Ans C)

20

Let C_1 and C_2 are circles defined by $x^2 + y^2 - 20x + 64 = 0$ and $x^2 + y^2 + 30x + 144 = 0$. The length of the shortest line segment PQ that is tangent to C_1 at P and to C_2 at Q is -

(A) 15 (B) 18 (C) 20 (D) 24

Solution: Shortest common tangent is transverse common tangent.

$$L_{TCT} = \sqrt{(C_1C_2)^2 - (r_1+r_2)^2}$$

$$= \sqrt{25^2 - (6+9)^2} = 20$$

option C

C_1C_2 = distance b/w centres of the circles
= 25

$r_1 = \sqrt{10^2 - 64} = 6$; $r_2 = \sqrt{15^2 - 144} = 9$

21. Two congruent circles with centres at (2,3) and (5,6) which intersect at right angles has radius equal to-

(A) $2\sqrt{2}$ (B) 3 (C) 4 (D) none

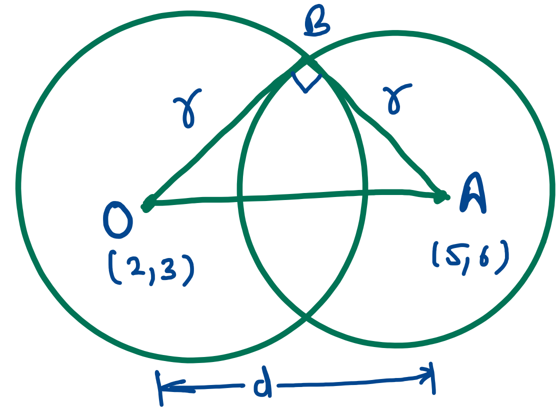
Solution:

Condition of orthogonality is $r_1^2 + r_2^2 = d^2$

$$\Rightarrow r^2 + r^2 = \left(\sqrt{(5-2)^2 + (6-3)^2}\right)^2$$

$$\Rightarrow 2r^2 = 9 + 9$$

$$\Rightarrow r = 3 \quad \text{option (B)}$$



22. The equation of a circle which touches the line x + y = 5 at N(–2,7) and cuts the circle $x^2 + y^2 + 4x - 6y + 9 = 0$ orthogonally, is -

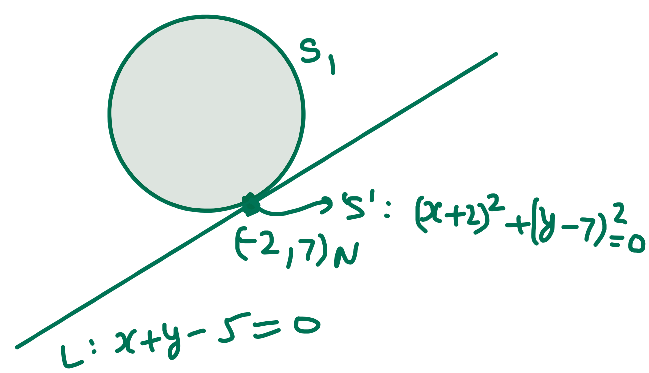
(A) $x^2 + y^2 + 7x - 11y + 38 = 0$ (B) $x^2 + y^2 = 53$

(C) $x^2 + y^2 + x - y - 44 = 0$ (D) $x^2 + y^2 - x + y - 62 = 0$

Solution:

$$S_1 : S + \lambda L = 0$$

$$\Rightarrow S_1 : (x+2)^2 + (y-7)^2 + \lambda(x+y-5) = 0$$



$$\Rightarrow S_1 : x^2 + y^2 + (\lambda+4)x + (\lambda-14)y + (53 - 5\lambda) = 0$$

which is orthogonal to $S_2 : x^2 + y^2 + 4x - 6y + 9 = 0$

Condition of orthogonality is $\boxed{2g_1g_2 + 2f_1f_2 = c_1 + c_2}$

$$\therefore 2\left(\frac{\lambda+4}{2}\right)(2) + 2\left(\frac{\lambda-14}{2}\right)(-3) = (53 - 5\lambda) + 9$$

$$\Rightarrow 2\lambda + 8 - 3\lambda + 42 = 62 - 5\lambda \Rightarrow 4\lambda = 12$$

$$\Rightarrow \lambda = 3$$

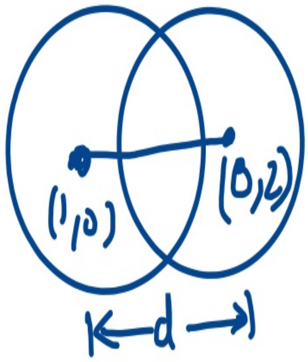
$$\therefore S_1 : x^2 + y^2 + 7x - 11y + 38 = 0 \quad \text{option (A)}$$

23 The angle at which the circle $(x-1)^2 + y^2 = 10$ and $x^2 + (y-2)^2 = 5$ intersect is -

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{4}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{2}$

Solution:

$$|\cos\theta| = \left|\frac{r_1^2 + r_2^2 - d^2}{2r_1 r_2}\right|$$



$$= \frac{10+5-5}{2\times\sqrt{10}\times\sqrt{5}} = \frac{1}{\sqrt{2}}$$

$$\Rightarrow \cos\theta = \pm\frac{1}{\sqrt{2}}$$

option (B)

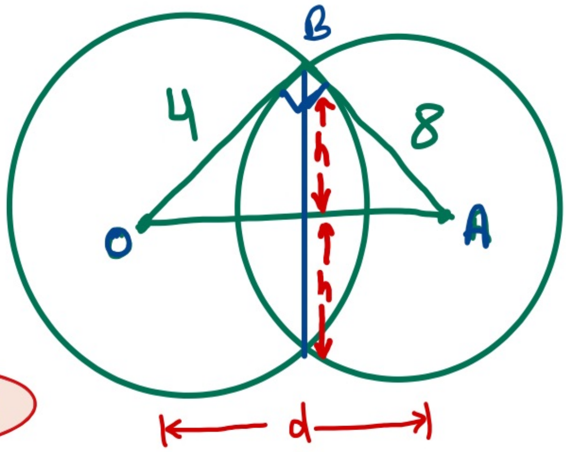
24

Two circles whose radii are equal to 4 and 8 intersect at right angles. The length of their common chord is-

(A) $\frac{16}{\sqrt{5}}$ (B) 8 (C) $4\sqrt{6}$ (D) $\frac{8\sqrt{5}}{5}$

Solution: Tangents drawn at the common points of the orthogonal circles are diameter of each other.

$$d = \sqrt{4^2+8^2} = 4\sqrt{5}$$



$$ar(\Delta OAB) = \frac{1}{2}\times 4\times 8 = \frac{1}{2}\times d\times h$$

$$\Rightarrow 32 = 4\sqrt{5}\times h \Rightarrow h = \frac{8}{\sqrt{5}}$$

$$\therefore \text{Common Chord} = 2h = \frac{16}{\sqrt{5}}$$

option (A)

25

The points (x_1, y_1), (x_2, y_2), (x_1, y_2) and (x_2, y_1) are always

(A) collinear

(B) concyclic

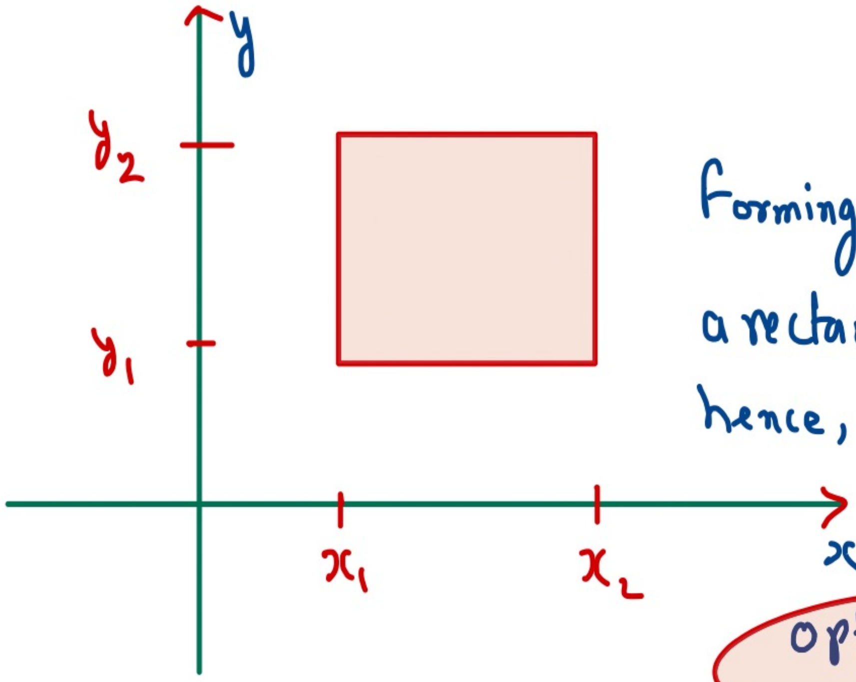
(C) vertices of a square

(D) vertices of a rhombus

Solution:

Let $x_1 < x_2$

& $y_1 < y_2$



Forming a rectangle hence, concyclic

Option (B)

EXERCISE

O-2

Paragraph for question Nos. 1 to 4

Consider the circle $S : x^2 + y^2 - 4x - 1 = 0$ and the line $L : y = 3x - 1$. If the line L cuts the circle at A & B.

1. Length of the chord AB equal -

(A) $2\sqrt{5}$ (B) $\sqrt{5}$ (C) $5\sqrt{2}$ (D) $\sqrt{10}$

2. The angle subtended by the chord AB in the minor arc of S is-

(A) $\frac{3\pi}{4}$ (B) $\frac{5\pi}{6}$ (C) $\frac{2\pi}{3}$ (D) $\frac{\pi}{4}$

3. Acute angle between the line L and the circle S is -

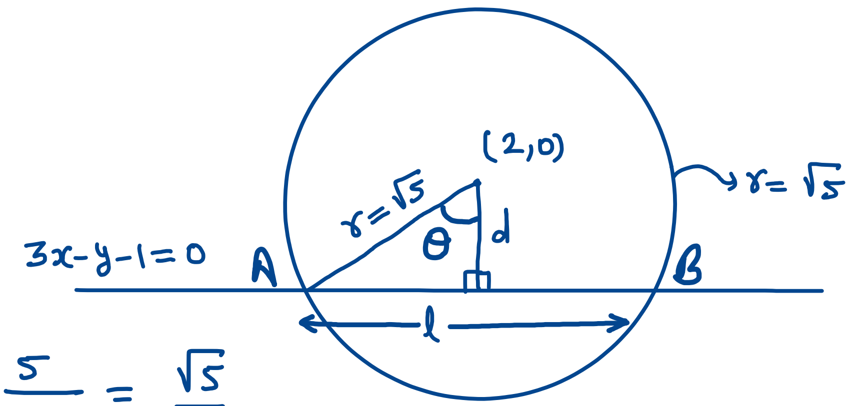
(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{3}$ (C) $\frac{\pi}{4}$ (D) $\frac{\pi}{6}$

4. If the equation of the circle on AB as diameter is of the form $x^2 + y^2 + ax + by + c = 0$ then the magnitude of the vector $\vec{V} = a\hat{i} + b\hat{j} + c\hat{k}$ has the value equal to-

(A) $\sqrt{8}$ (B) $\sqrt{6}$ (C) $\sqrt{9}$ (D) $\sqrt{10}$

Solution: Paragraph (1 to 4)

(1)



$$d = \left|\frac{3\cdot 2 - 0 - 1}{\sqrt{3^2+1^2}}\right| = \frac{5}{\sqrt{10}} = \frac{\sqrt{5}}{\sqrt{2}}$$

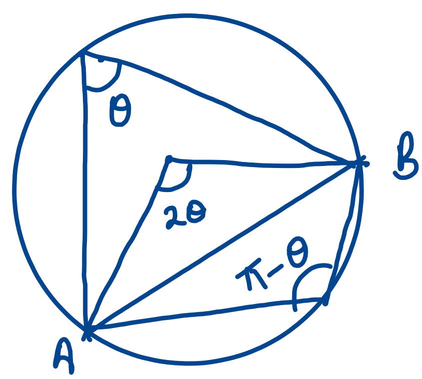
$$l = 2\sqrt{r^2 - d^2} = 2\sqrt{5 - \frac{5}{2}} = \sqrt{10}$$ option (D)

---

(2) $\cos\theta = \frac{d}{r} = \frac{1}{\sqrt{2}}$

$\Rightarrow \theta = \frac{\pi}{4}$

$\therefore \pi - \theta = \frac{3\pi}{4}$ option (A)



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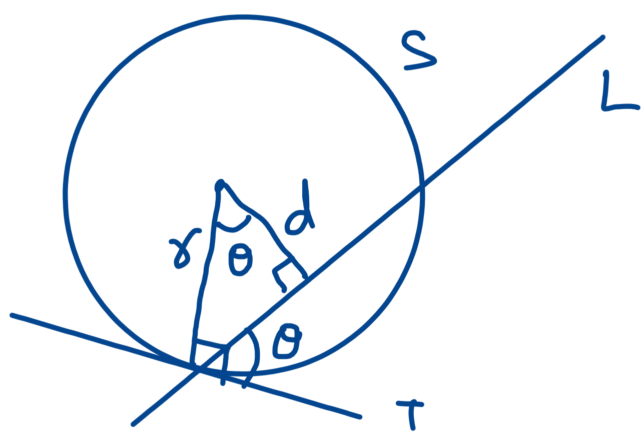
(3) Angle b/w line 'L' & circle 'S' is angle b/w tangent 'T' & line 'L' which is 'θ'

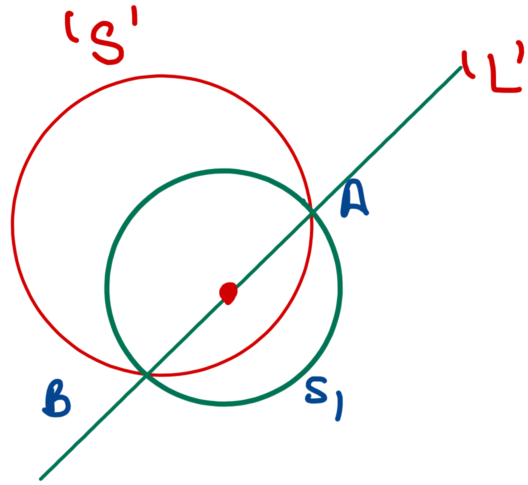


$\cos\theta = \frac{d}{r} = \frac{1}{\sqrt{2}} \Rightarrow \theta = \frac{\pi}{4}$ option (C)

(4)

$S: x^2+y^2-4x-1=0$

$L: 3x-y-1=0$



$S_1: S+\lambda L=0$

$\Rightarrow S_1: x^2+y^2-4x-1+\lambda(3x-y-1)=0$

S_1 has 'AB' as diameter so centre of S_1 lie's on line 'L'.

$\therefore S_1: x^2+y^2+(3\lambda-4)x-\lambda y-(\lambda+1)=0$

Centre is $\left(2-\frac{3\lambda}{2}, \frac{\lambda}{2}\right)$

$\downarrow\ 3x-y-1=0$

$3\left(2-\frac{3\lambda}{2}\right)-\frac{\lambda}{2}-1=0$

$\Rightarrow 6-\frac{9\lambda+\lambda}{2}-1=0 \Rightarrow \boxed{\lambda=1}$

$\therefore$ Circle S_1 is given by

$\Rightarrow S_1: x^2+y^2-x-y-2=0$

$|\vec{v}|=\sqrt{a^2+b^2+c^2}=\sqrt{1+1+4}=\sqrt{6}$ option (B)

[MULTIPLE CHOICE]

(5) Which of the following lines have the intercepts of equal lengths on the circle, $x^2 + y^2 - 2x + 4y = 0$?

(A) $3x - y = 0$ (B) $x + 3y = 0$ (C) $x + 3y + 10 = 0$ (D) $3x - y - 10 = 0$

Solution: Lines which cut equal intercepts on the circle are equidistant from the centres of the circle.

Here centre is $(1, -2)$

(A) $$d_1 = \left|\frac{3(1) - (-2)}{\sqrt{3^2 + (-1)^2}}\right| = \frac{5}{\sqrt{10}}$$

(B) $$d_2 = \left|\frac{1 + 3(-2)}{\sqrt{1^2 + 3^2}}\right| = \frac{5}{\sqrt{10}}$$

(C) $$d_3 = \frac{|1 + 3(-2) + 10|}{\sqrt{1^2 + 3^2}} = \frac{5}{\sqrt{10}}$$

(D) $$d_4 = \left|\frac{3(1) - (-2) - 10}{\sqrt{3^2 + 1^2}}\right| = \frac{5}{\sqrt{10}}$$

As $d_1 = d_2 = d_3 = d_4$ so, all the options are correct.

$\boxed{A, B, C, D}$

16) $\frac{x-x_1}{\cos\theta} = \frac{y-y_1}{\sin\theta} = r$, represents :

(A) equation of a straight line, if θ is constant and r is variable

(B) equation of a circle, if r is constant and θ is a variable

(C) a straight line passing through a fixed point and having a known slope

(D) a circle with a known centre and a given radius.

Solution: If 'θ' is constant then

$$\boxed{\tan\theta = \frac{y-y_1}{x-x_1} = \text{Constant}}$$

A straight line passing through a fixed point (x_1, y_1) and have a constant slope.

If 'θ' is variable and 'r' is constant

then $x - x_1 = r\cos\theta$; $y - y_1 = r\sin\theta$

$\Rightarrow (x-x_1)^2 + (y-y_1)^2 = r^2$

Circle with a given centre (x_1, y_1) & radius 'r'.

A, B, C, D

(7) Tangents PA and PB are drawn to the circle $S \equiv x^2 + y^2 - 2y - 3 = 0$ from the point P(3,4). Which of the following alternative(s) is/are correct ?

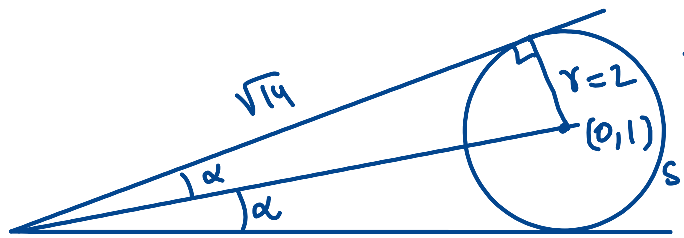
(A) The power of point P(3,4) with respect to circle $S = 0$ is 14.

(B) The angle between tangents from P(3,4) to the circle $S = 0$ is $\frac{\pi}{3}$

(C) The equation of circumcircle of ΔPAB is $x^2 + y^2 - 3x - 5y + 4 = 0$

(D) The area of quadrilateral PACB is $3\sqrt{7}$ square units where C is the centre of circle $S = 0$.

Solution:



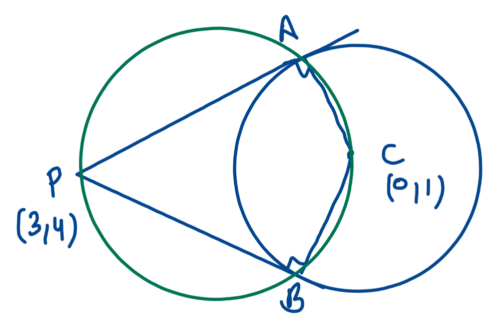
$r = \sqrt{1^2+3} = 2$

$S: x^2+y^2-2y-3=0$

(A) Power of point $P(3,4) = S_P = 3^2+4^2-2\cdot4-3 = 14$

(B) angle b/w the tangents $= 2\alpha = 2\tan^{-1}\frac{2}{\sqrt{14}} \neq \frac{\pi}{3}$

(C) Circle passing through P, A, B will also pass through C as P, A, B, C are concyclic.



So, 'PC' will be diameter of circum circle of ΔPAB

So, equation is $(x-3)(x-0)+(y-4)(y-1)=0$

$\Rightarrow x^2+y^2-3x-5y+4=0$

(D) $ar(PACB) = 2\times\left(\frac{1}{2}\times 2\times\sqrt{14}\right) = 2\sqrt{14}$

A, C

(8) Which of the following is/are True ?

The circles $x^2 + y^2 - 6x - 6y + 9 = 0$ and $x^2 + y^2 + 6x + 6y + 9 = 0$ are such that -

(A) they do not intersect

(B) they touch each other

(C) their exterior common tangents are parallel.

(D) their interior common tangents are perpendicular.

Solution:

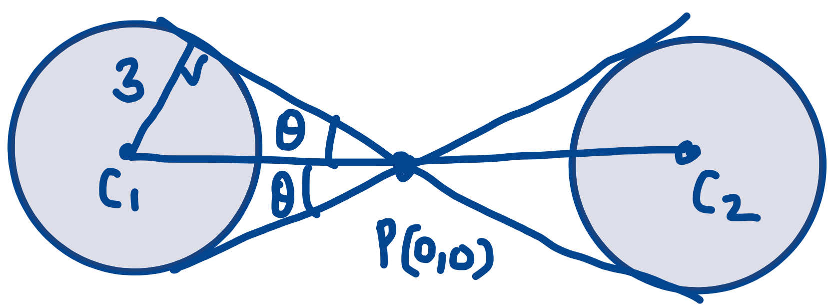
$S_1: x^2+y^2-6x-6y+9=0;$

$S_2: x^2+y^2+6x+6y+9=0$

	S_1	S_2
Centre	$C_1\ (3,3)$	$C_2\ (-3,-3)$
Radius	$r_1 = 3$	$r_2 = 3$

$C_1C_2 = 6\sqrt{2}$; $r_1 + r_2 = 6$; $r_1 - r_2 = 0$

$C_1C_2 > r_1 + r_2 \Rightarrow$ Circles are away



A, C, D

$PC_1 = 3\sqrt{2} \Rightarrow \theta = 45^\circ$

So, interior common tangents are $\perp$r.

Note: Intersection point of interior common tangent divides the line joing centre of the circles internally in the ratio of radii

(9) Consider the circles $S_1 : x^2 + y^2 = 4$ and $S_2 : x^2 + y^2 - 2x - 4y + 4 = 0$ which of the following statements are correct ?

(A) Number of common tangents to these circles is 2.

(B) If the power of a variable point P w.r.t. these two circles is same then P moves on the line $x + 2y - 4 = 0$

(C) Sum of the y-intercepts of both the circles is 6.

(D) The circles S_1 and S_2 are orthogonal.

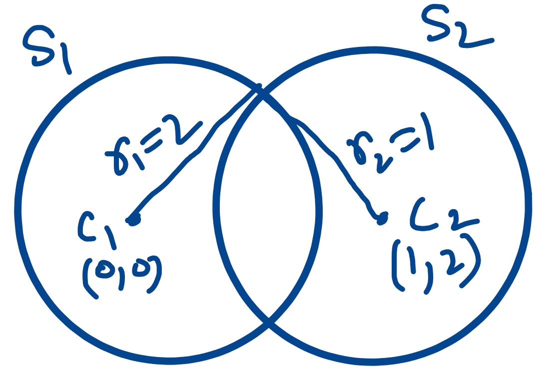
Solution: Draw both circles.

Centre of S_1 is C_1 $(0,0)$

Radius " S_1 is $r_1 = 2$

Centre of S_2 is C_2 $(1,2)$

Radius of S_2 is $r_2 = 1$



$C_1C_2 = \sqrt{5} < r_1 + r_2$ and $\sqrt{5} > |r_1 - r_2|$

i.e., $|r_1 - r_2| < C_1C_2 < r_1 + r_2$

$\Rightarrow$ Circle intersect each other at two distinct points.

(A) is true

(B) Radical Axis $S_1 - S_2 = 0$

$$\Rightarrow (x^2+y^2-4) - (x^2+y^2-2x-4y+4) = 0$$

$$\Rightarrow 2x + 4y - 8 = 0 \Rightarrow x + 2y - 4 = 0$$

(C) $2\sqrt{f^2 - c} = 4$ (for S_1)

$2\sqrt{f^2 - c} = 2\sqrt{4-4} = 0$ (for S_2)

A, B, D

(D) Note: $r_1^2 + r_2^2 = (C_1C_2)^2$ So orthogonal

10. Two circles $x^2 + y^2 + px + py - 7 = 0$ and $x^2 + y^2 - 10x + 2py + 1 = 0$ intersect each other orthogonally then the value of p is -

(A) 1 (B) 2 (C) 3 (D) 5

Solution: Condition of orthogonality is

$$\boxed{2g_1g_2 + 2f_1f_2 = c_1 + c_2}$$

Here, $g_1 = \frac{p}{2}$; $f_1 = \frac{p}{2}$; $c_1 = -7$

$g_2 = -5$; $f_2 = p$; $c_2 = 1$

$\therefore \quad 2\left(\frac{p}{2}\right)(-5) + 2\left(\frac{p}{2}\right)(p) = (-7) + 1$

$\Rightarrow \quad -5p + p^2 = -6 \Rightarrow p^2 - 5p + 6 = 0$

$\Rightarrow \quad p = 2 \text{ or } 3$ B, C

EXERCISE

S-1

(1) Find the equation to the circles which pass through the points :

(i) (0, 0), (a, 0) and (0, b) **(ii)** (1, 2), (3, –4) and (5, –6)

(iii) (1, 1), (2, –1) and (3, 2)

Sol :- i.) let Circle

$$x^2+y^2+2gx+2fy+c=0$$

* pass through $(0,0) \Rightarrow \boxed{c=0}$

* pass through $(a,0) \Rightarrow a^2+2ag+c=0$

$$\Rightarrow a^2+2ag=0$$

$$\Rightarrow \boxed{g=-\frac{a}{2}}$$

* pass through $(0,b) \Rightarrow b^2+2fb+c=0$

$$\Rightarrow b^2+2fb=0$$

$$\Rightarrow \boxed{f=\frac{-b}{2}}$$

$\therefore$ eq^n of circle

$$x^2+y^2+2\left(\frac{-a}{2}\right)x+2\left(\frac{-b}{2}\right)y+0=0$$

$$\Rightarrow \boxed{x^2+y^2-ax-by=0}$$

Ans.

ii.) let Circle

$$x^2+y^2+2gx+2fy+c=0$$

* pass through $(1,2) \Rightarrow 5+2g+4f+c=0$ —— (1)
* pass through $(3,-4) \Rightarrow 25+6g-8f+c=0$ —— (2)
* pass through $(5,-6) \Rightarrow 61+10g-12f+c=0$ —— (3)

solve (1), (2) & (3)

$g = -11$

$f = -2$

$c = 25$

$\therefore$ eq^n of circle

$$x^2+y^2+2(-11)x+2(-2)y+25=0$$

$$\Rightarrow \boxed{x^2+y^2-22x-4y+25=0}$$ Ans.

iii.) let Circle

$$x^2+y^2+2gx+2fy+c=0$$

* pass through $(1,1) \Rightarrow 2+2g+2f+c=0$ —— ①
* pass through $(2,-1) \Rightarrow 5+4g-2f+c=0$ —— ②
* pass through $(3,2) \Rightarrow 13+6g+4f+c=0$ —— ③

solve ①, ② & ③

$g = -\frac{5}{2}$

$f = -\frac{1}{2}$

$c = 4$

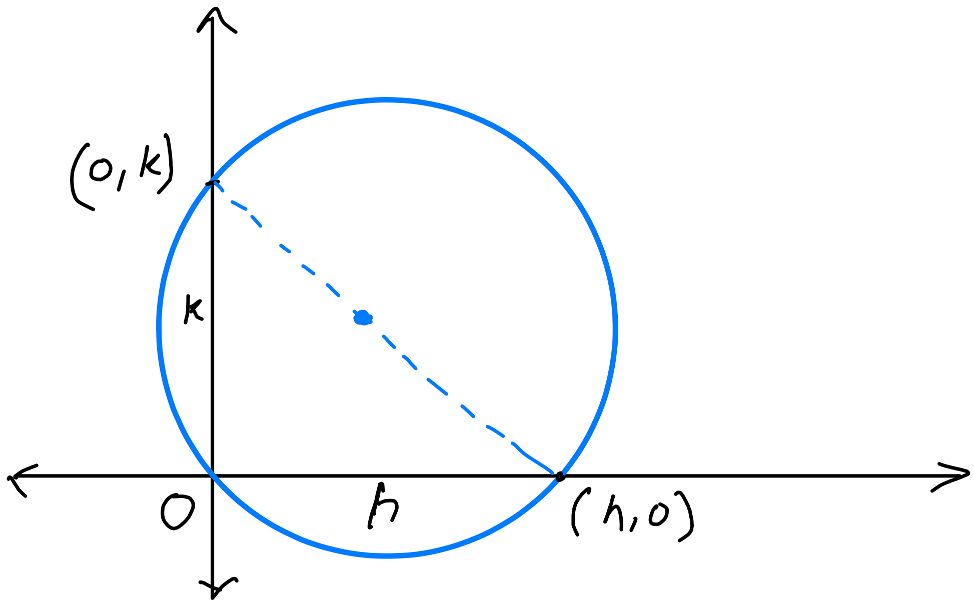
$\therefore$ eqn of circle

$$\Rightarrow x^2+y^2+2\left(-\frac{5}{2}\right)x+2\left(-\frac{1}{2}\right)y+4=0$$

$$\Rightarrow \boxed{x^2+y^2-5x-y+4=0}$$

(2) Find the equation to the circle which goes through the origin and cuts off intercepts equal to h and k from the positive parts of the axes.

Sol:-



$\therefore$ eq^n of circle (diametric form)

$$(x-0)(x-h)+(y-0)(y-k)=0$$

$$\Rightarrow \boxed{x^2+y^2-hx-ky=0}$$ Ans.

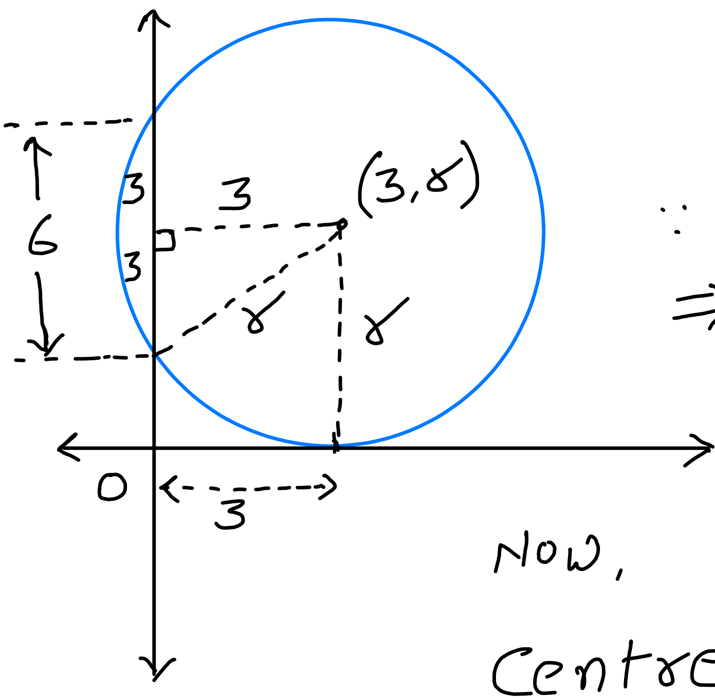
(3) Find the equation to the circle which touches the axis of :

(a) x at a distance +3 from the origin and intercepts a distance 6 on the axis of y.

(b) x, pass through the point (1, 1) and have line x + y = 3 as diameter.

Sol:-

a.)



$\because \; r^2 = 3^2 + 3^2$

$\Rightarrow \boxed{r = 3\sqrt{2}}$

Now,

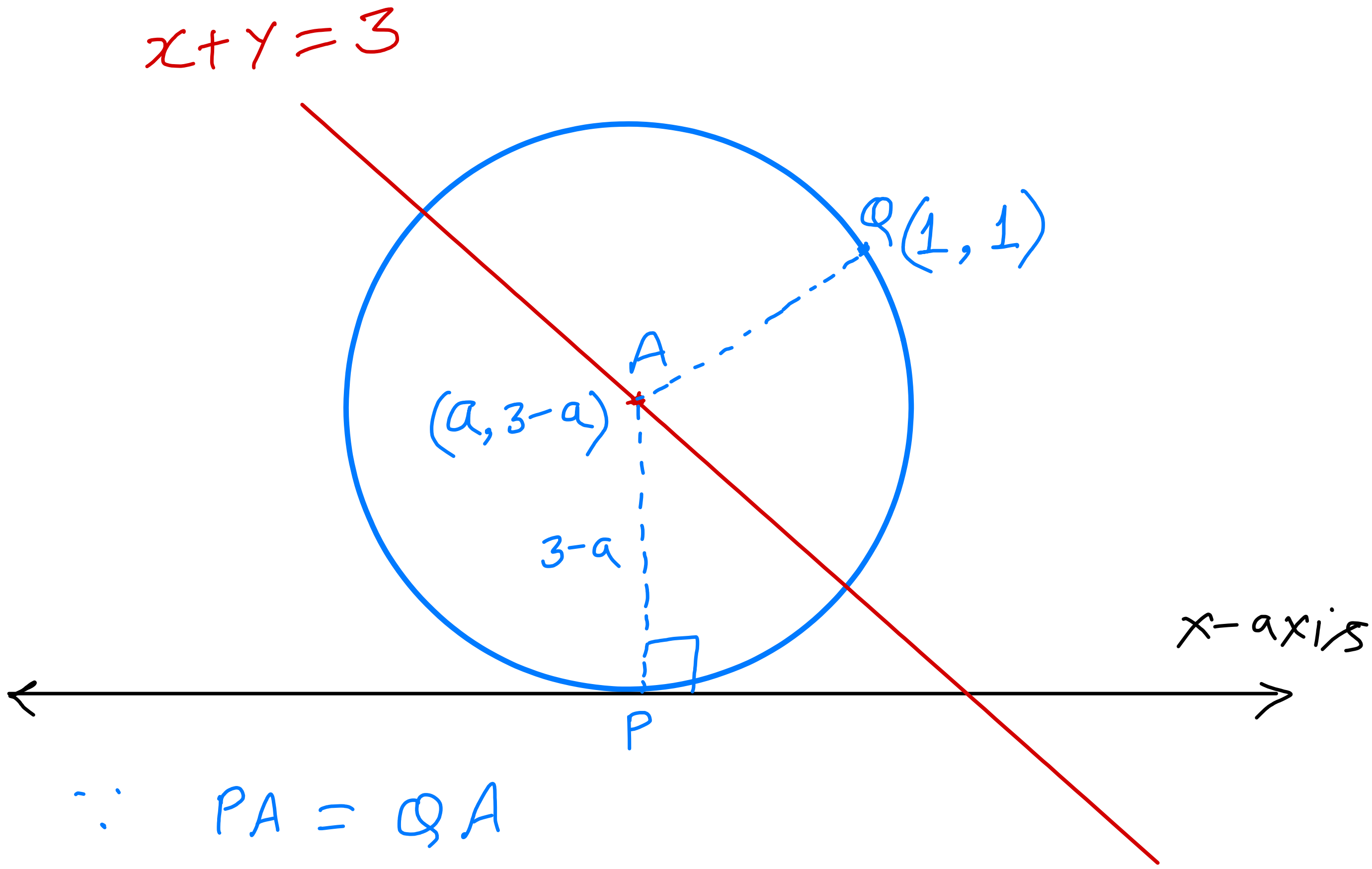
Centre : $(3, \pm 3\sqrt{2})$

& radius : $3\sqrt{2}$

$\therefore$ eq^n $\Rightarrow (x-3)^2 + (y \pm 3\sqrt{2})^2 = (3\sqrt{2})^2$

$\Rightarrow \boxed{x^2 + y^2 - 6x \pm 6\sqrt{2}\,y + 9 = 0}$ Ans.

b.)



$\because \quad PA = QA$

$\Rightarrow \quad 3-a = \sqrt{(a-1)^2 + (3-a-1)^2}$

$\Rightarrow \quad 9 + a^2 - 6a = a^2 - 2a + 1 + 4 - 4a + a^2$

$\Rightarrow \quad a^2 = 4 \quad \Rightarrow \quad \boxed{a = \pm 2}$

$\therefore$ Centre : $(2, 1)$

radius : 1

eqn $\Rightarrow (x-2)^2 + (y-1)^2 = 1$

$\Rightarrow \boxed{x^2 + y^2 - 4x - 2y + 4 = 0}$

Ans.

or

Centre : $(-2, 5)$

radius : 5

eqn $\Rightarrow (x+2)^2 + (y-5)^2 = 5^2$

$\Rightarrow \boxed{x^2 + y^2 + 4x - 10y + 4 = 0}$

Ans.

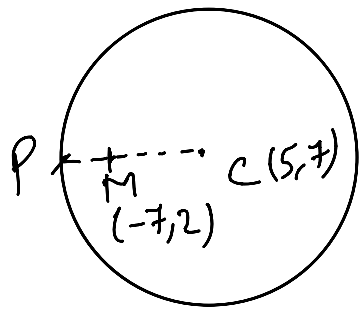
(4) (a) Find the shortest distance from the point M(–7, 2) to the circle $x^2 + y^2 - 10x - 14y - 151 = 0$.

Sol^n $S \equiv x^2+y^2-10x-14y-151$

center $(5, 7)$; $R = \sqrt{25+49+151} = 15$

wrt $M(-7, 2)$: $S_1 = 49+4+70-28-151 = -56$

$\Rightarrow$ M lies inside the circle



$\therefore d_{min} = MP = R - MC$

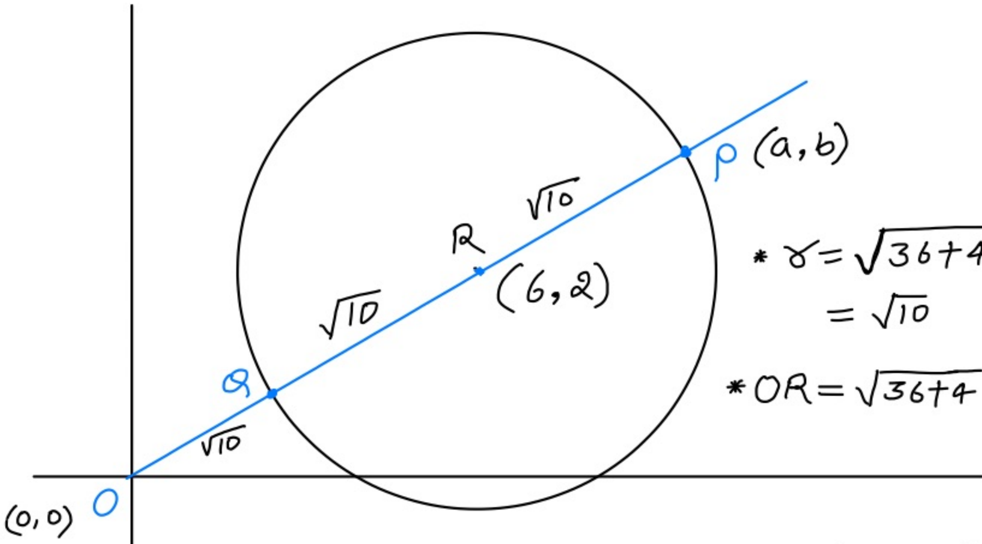
$= 15 - \sqrt{12^2+5^2}$

$= 15 - 13$

$\Rightarrow d_{min} = 2$ Ans.

(b) Find the co-ordinate of the point on the circle $x^2 + y^2 - 12x - 4y + 30 = 0$, which is farthest from the origin.

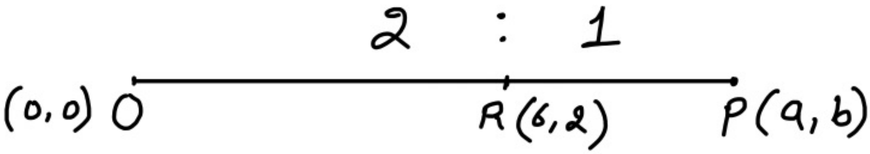
Sol:-



$* \; r = \sqrt{36+4-30}$
$= \sqrt{10}$

$* \; OR = \sqrt{36+4} = 2\sqrt{10}$

Farthest Point on the Circle is "P."



$\therefore \quad \dfrac{2a+0}{3} = 6 \quad \& \quad \dfrac{2b+0}{3} = 2$

$a = 9 \quad \Big| \quad b = 3$

So, $P(a, b) \equiv (9, 3)$ Ans.

(5) If the points $(\lambda, -\lambda)$ lies inside the circle $x^2 + y^2 - 4x + 2y - 8 = 0$, then find the range of λ.

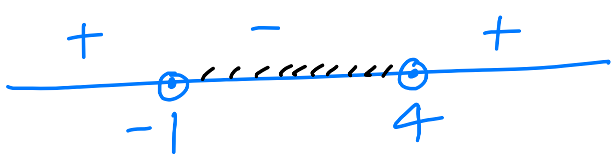
Sol:- If Point $P(\lambda, -\lambda)$ lie inside the circle $x^2 + y^2 - 4x + 2y - 8 = 0$

$\therefore$ Put $P(\lambda, -\lambda)$ in the eqn of circle < 0

$$\Rightarrow \lambda^2 + \lambda^2 - 4\lambda - 2\lambda - 8 < 0$$

$$\Rightarrow \lambda^2 - 3\lambda - 4 < 0$$

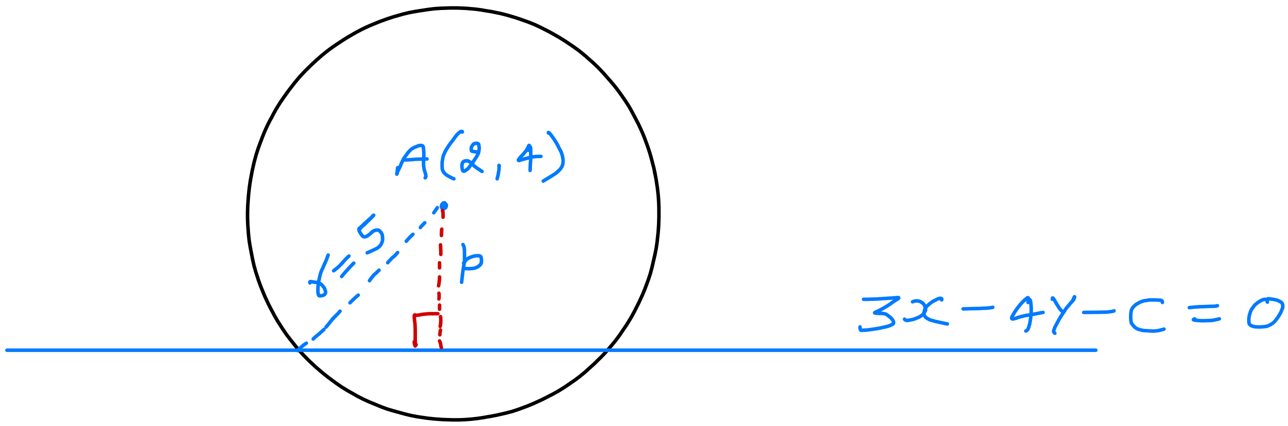
$$\Rightarrow (\lambda - 4)(\lambda + 1) < 0$$



$$\therefore \boxed{\lambda \in (-1, 4)}$$

(6) Show that the line $3x - 4y - c = 0$ will meet the circle having centre at (2, 4) and the radius 5 in real and distinct points if $-35 < c < 15$.

Sol:-



$$\therefore \quad p < r$$

$$\Rightarrow \frac{|6-16-c|}{\sqrt{9+16}} < 5$$

$$\Rightarrow |c+10| < 25$$

$$\Rightarrow -25 < c+10 < 25$$

$$\Rightarrow \boxed{-35 < c < 15}$$

7 Find the equation of the tangent to the circle

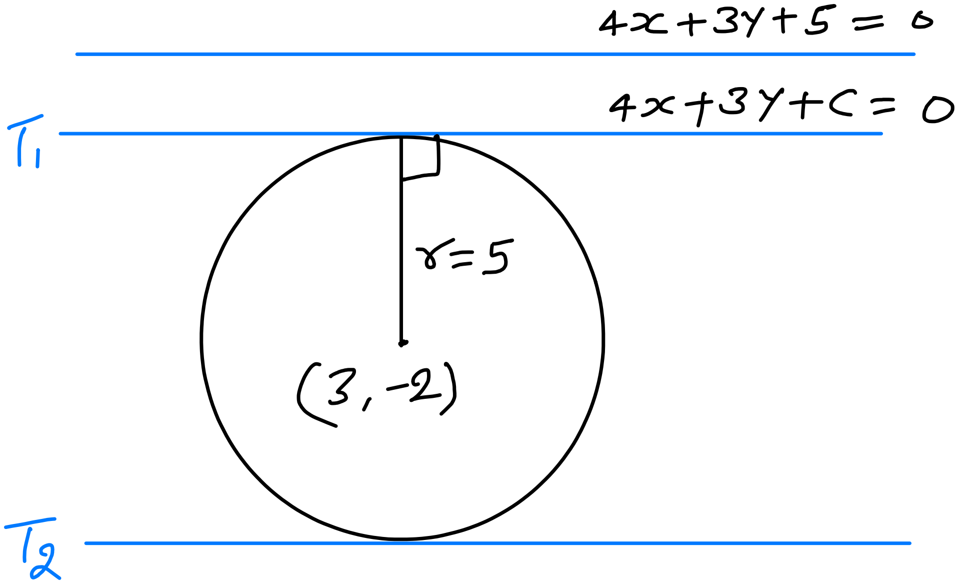
(a) $x^2 + y^2 - 6x + 4y = 12$, which are parallel to the straight line $4x + 3y + 5 = 0$.

(b) $x^2 + y^2 - 22x - 4y + 25 = 0$, which are perpendicular to the straight line $5x + 12y + 9 = 0$

(c) $x^2 + y^2 = 25$, which are inclined at 30° to the axis of x.

Sol:-

a.)



apply, $p = r$

$\Rightarrow$ $\perp^r$ drop from $(3,-2)$ to the line $4x+3y+C=0$ $= 5$

$$\Rightarrow \frac{|12-6+C|}{\sqrt{16+9}} = 5$$

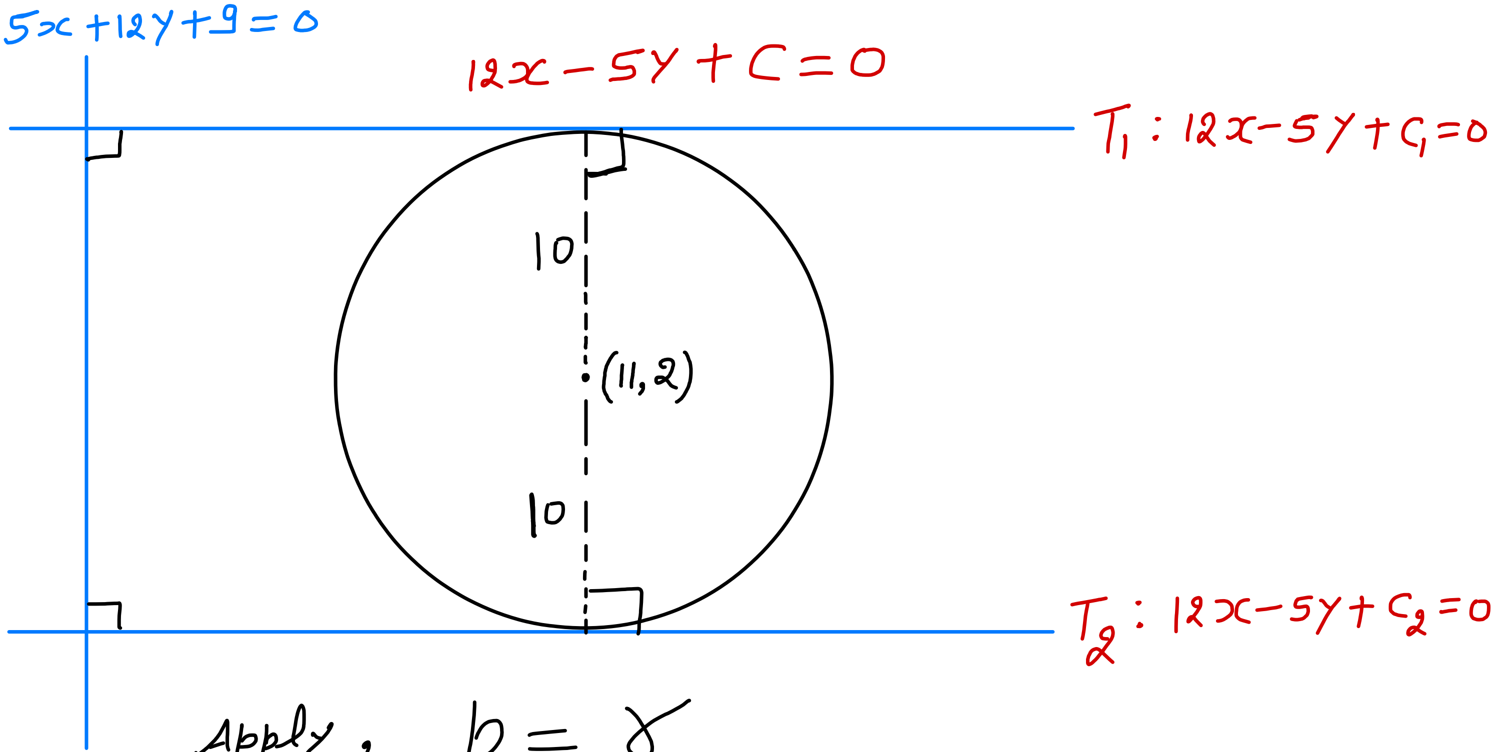
$$\Rightarrow c+6 = \pm 25$$

$$\Rightarrow c = 19 \text{ or } -31$$

$\therefore$ T_1 : $\boxed{4x+3y+19=0}$ Ans.

T_2 : $\boxed{4x+3y-31=0}$ Ans.

b.)



Apply, $p = r$

$$\Rightarrow \frac{\perp^r \text{ drop from } (11,2)}{\text{to line } 12x-5y+C=0} = 10$$

$$\Rightarrow \frac{|132-10+C|}{\sqrt{144+25}} = 10$$

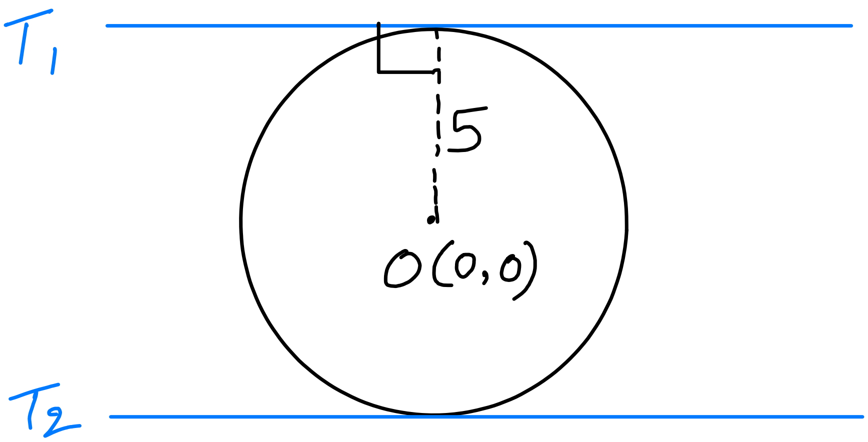
$$\Rightarrow |C+122| = 130$$

$$\Rightarrow C = 8 \ \underline{\text{or}} \ -252$$

$\therefore$ Tangents, $T_1 : \boxed{12x-5y+8=0}$ Ans.

& $T_2 : \boxed{12x-5y-252=0}$ Ans.

c.)



$m = \frac{1}{\sqrt{3}}$ $\{\because$ inclination $= 30^\circ$

Tangent T: $y = mx + c$

$$\Rightarrow y = \frac{1}{\sqrt{3}}x + c$$

$$\Rightarrow \boxed{x - \sqrt{3}y + \sqrt{3}c = 0} \quad \text{—— (1)}$$

$\therefore$ apply $p = r$

$$\Rightarrow \frac{|0 - 0 + \sqrt{3}c|}{\sqrt{1+3}} = 5$$

$$\Rightarrow c = \pm \frac{10}{\sqrt{3}} \quad \text{Put in (1)}$$

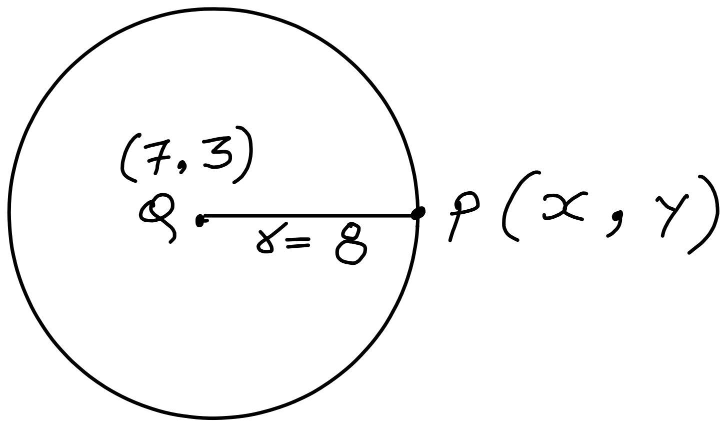
$\therefore$ Tangents, T_1 : $\boxed{x - \sqrt{3}y + 10 = 0}$ Ans

&

T_2 : $\boxed{x - \sqrt{3}y - 10 = 0}$ Ans.

(8) Given that $x^2 + y^2 = 14x + 6y + 6$, find the largest possible value of the expression $E = 3x + 4y$.

Sol:- $\because$ Circle : $x^2 + y^2 - 14x - 6y - 6 = 0$



$\therefore$ Parametric Co-ordinate

$$P(x, y) \equiv (7 + 8\cos\theta,\ 3 + 8\sin\theta)$$

Now, $E = 3x + 4y$

$$= 3(7 + 8\cos\theta) + 4(3 + 8\sin\theta)$$

$$= 33 + 8(\underbrace{3\cos\theta + 4\sin\theta}_{\Downarrow \ -5 \text{ to } +5})$$

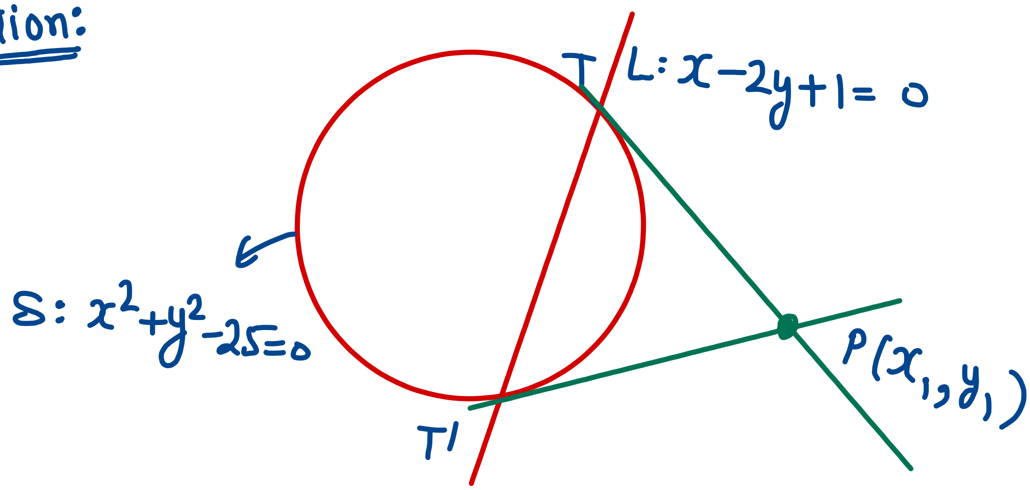
$$\therefore E_{Max.} = 33 + 8(5)$$

$$= \boxed{73}$$

Ans.

(9) The straight line $x - 2y + 1 = 0$ intersects the circle $x^2 + y^2 = 25$ in points T and T', find the co-ordinates of a point of intersection of tangents drawn at T and T' to the circle.

Solution:



TT' is Chord of Contact of point $P(x_1, y_1)$

$$\therefore \; TT': xx_1 + yy_1 - 25 = 0$$

Comparing with $L: x - 2y + 1 = 0$

$$\boxed{\frac{x_1}{1} = \frac{y_1}{-2} = \frac{-25}{1}}$$

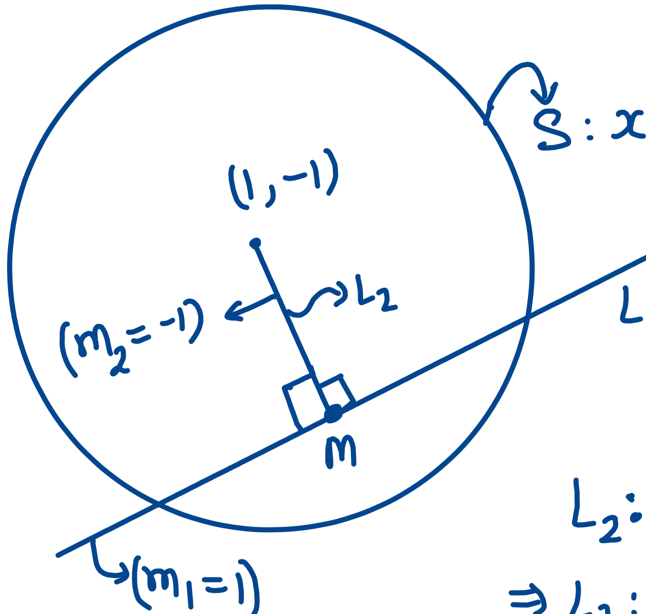
$$\Rightarrow \quad x_1 = -25; \quad y_1 = 50$$

So, $\boxed{P(-25, 50)}$

(10) Find the co-ordinates of the middle point of the chord which the circle $x^2 + y^2 - 2x + 2y - 2 = 0$ cuts off on the line $y = x - 1$.

Find also the equation of the locus of the middle point of all chords of the circle which are parallel to the line $y = x - 1$.

Solution:



$S: x^2+y^2-2x+2y-2=0$

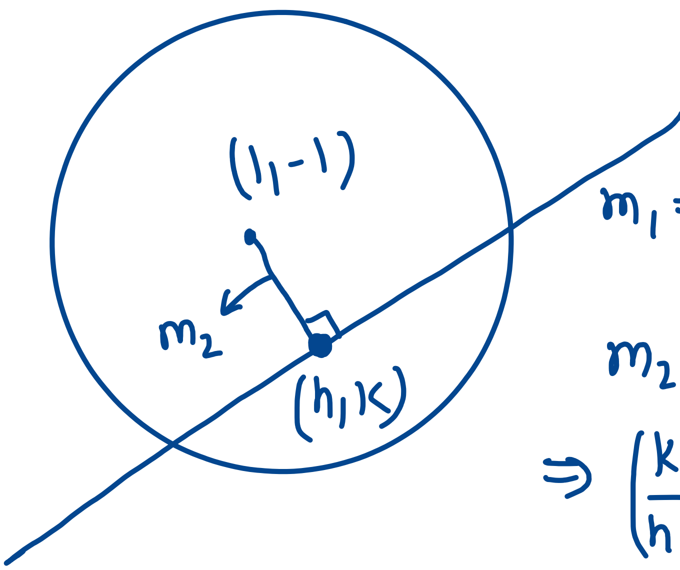
$L_1: x-y-1=0$

$L_2: (y+1) = (-1)(x-1)$

$\Rightarrow L_2: x+y=0$

Solving with $L_1: x-y-1=0$

$\boxed{\text{we get } M\left(\frac{1}{2}, -\frac{1}{2}\right)}$



$m_1 = 1$

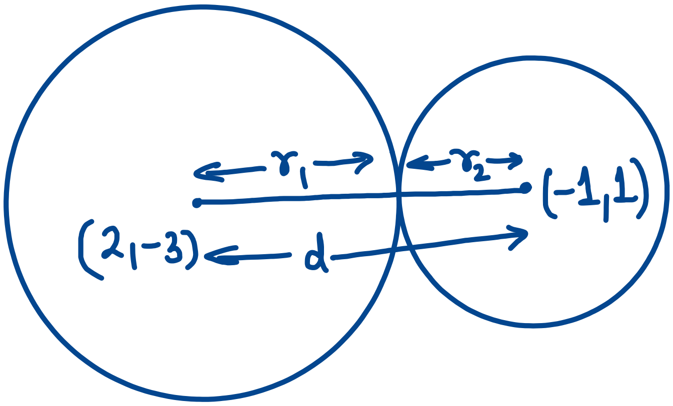
$m_2 m_1 = -1$

$\Rightarrow \left(\frac{k+1}{h-1}\right)(1) = -1$

$\Rightarrow \boxed{x+y=0}$

(11) A circle S = 0 is drawn with its centre at (–1, 1) so as to touch the circle $x^2 + y^2 - 4x + 6y - 3 = 0$ externally. Find the intercept made by the circle S = 0 on the coordinate axes.

Solution:



$$d = \sqrt{(2-(-1))^2 + (-3-1)^2} = \sqrt{3^2+4^2} = 5$$

$$r_1 = \sqrt{(-2)^2 + 3^2 - (-3)} = \sqrt{4+9+3} = 4$$

$$r_2 = d - r_1 = 1$$

$\therefore$ Equation required circle is

$$(x+1)^2 + (y-1)^2 = (1)^2$$

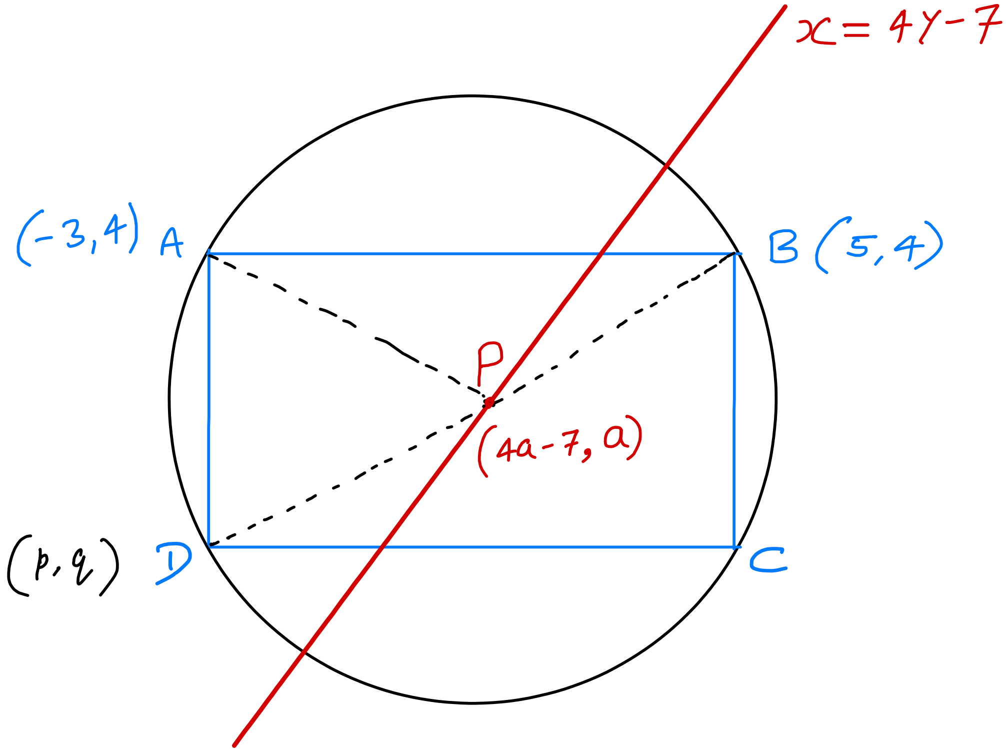
$$S: \boxed{\Rightarrow x^2+y^2+2x-2y+1=0}$$

Intercept on x-axis $= 2\sqrt{g^2-c} =$ zero

Intercept on y-axis $= 2\sqrt{f^2-c} =$ zero

(12) One of the diameters of the circle circumscribing the rectangle ABCD is $4y = x + 7$. If A & B are the points $(-3, 4)$ & $(5,4)$ respectively, then find the area of the rectangle.

Sol:-



$$PB = PA$$

$$\Rightarrow \sqrt{(4a-12)^2+(a-4)^2} = \sqrt{(4a-4)^2+(a-4)^2}$$

$$\Rightarrow \boxed{a = 2} \quad \therefore P(1,2)$$

$\because$ P(1,2) is mid Point B(5,4) & D(p,q)

$$\therefore \frac{p+5}{2} = 1 \;\&\; \frac{q+4}{2} = 2 \Rightarrow (p,q) \equiv (-3,0)$$

Now, Area of Rectangle ABCD

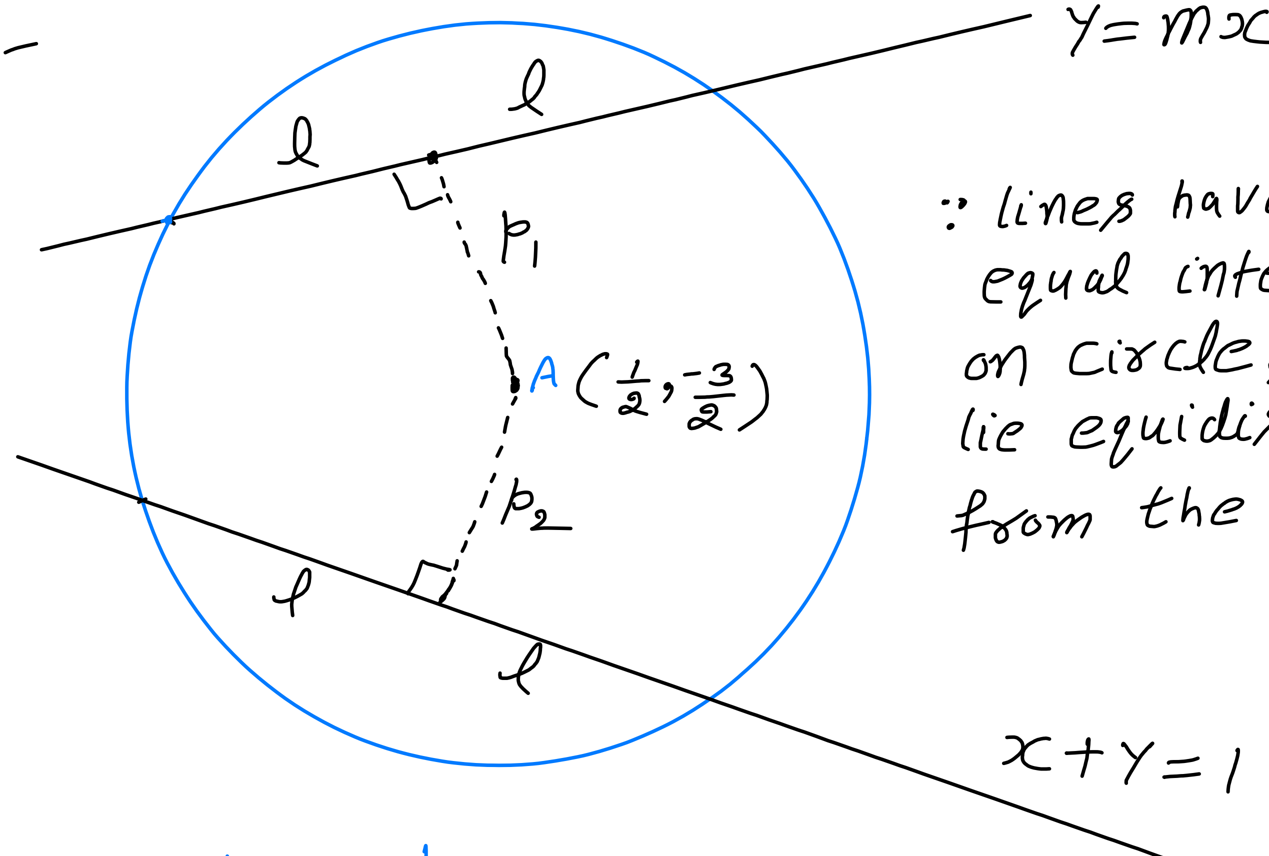
$$= AB \times AC$$

$$= \sqrt{64+0} \times \sqrt{0+16} = 8 \times 4$$

$$= \boxed{32} \text{ sq. unit}$$ Ans.

(13) Let L_1 be a straight line through the origin and L_2 be the straight line $x + y = 1$. If the intercepts made by the circle $x^2 + y^2 - x + 3y = 0$ on L_1 & L_2 are equal, then find the equation(s) which represent L_1.

Sol:-



∵ lines having equal intercept on circle, always lie equidistant from the origin.

$\therefore \quad p_1 = p_2$

$\Rightarrow$ ⊥r drop from $A\left(\frac{1}{2}, \frac{-3}{2}\right)$ to line $mx - y = 0$ = ⊥r drop from $A\left(\frac{1}{2}, \frac{-3}{2}\right)$ to the line $x + y - 1 = 0$

$$\Rightarrow \frac{\left|\frac{m}{2} + \frac{3}{2}\right|}{\sqrt{m^2+1}} = \frac{\left|\frac{1}{2} - \frac{3}{2} - 1\right|}{\sqrt{1+1}} \Rightarrow \boxed{m = 1 \text{ or } \frac{-1}{7}}$$

* if $m = 1$, then line

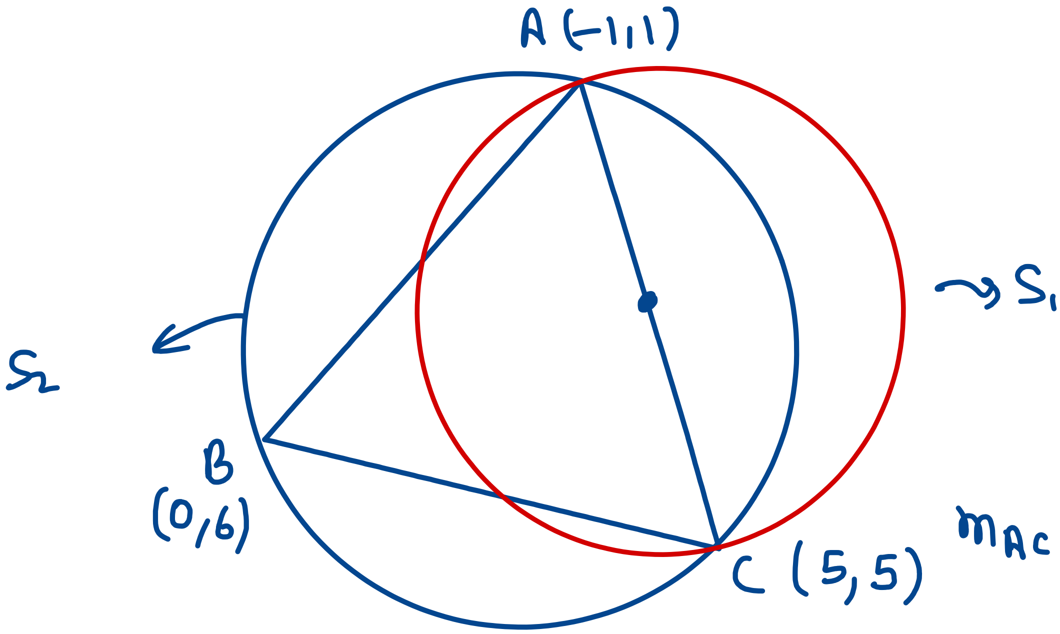
$\boxed{y = x}$ Ans.

* if $m = \frac{-1}{7}$, then line

$\boxed{x + 7y = 0}$ Ans.

(14) A circle passes through the points (–1, 1), (0, 6) and (5, 5). Find the points on the circle the tangents at which are parallel to the straight line joining origin to the centre.

Solution:



$m_{AC} = \dfrac{5-1}{5-(-5)} = \dfrac{2}{3}$

Equation AC: $L: 2x - 3y = 2(-1) - 3(1)$

$\Rightarrow L: 2x - 3y + 5 = 0$

Equation of S_1 (with AC as diameter)

$\Rightarrow S_1: (x+1)(x-5) + (y-1)(y-5) = 0$

$\Rightarrow S_1: x^2 + y^2 - 4x - 6y = 0$

Equation of S_2 : $S_1 + \lambda L = 0$

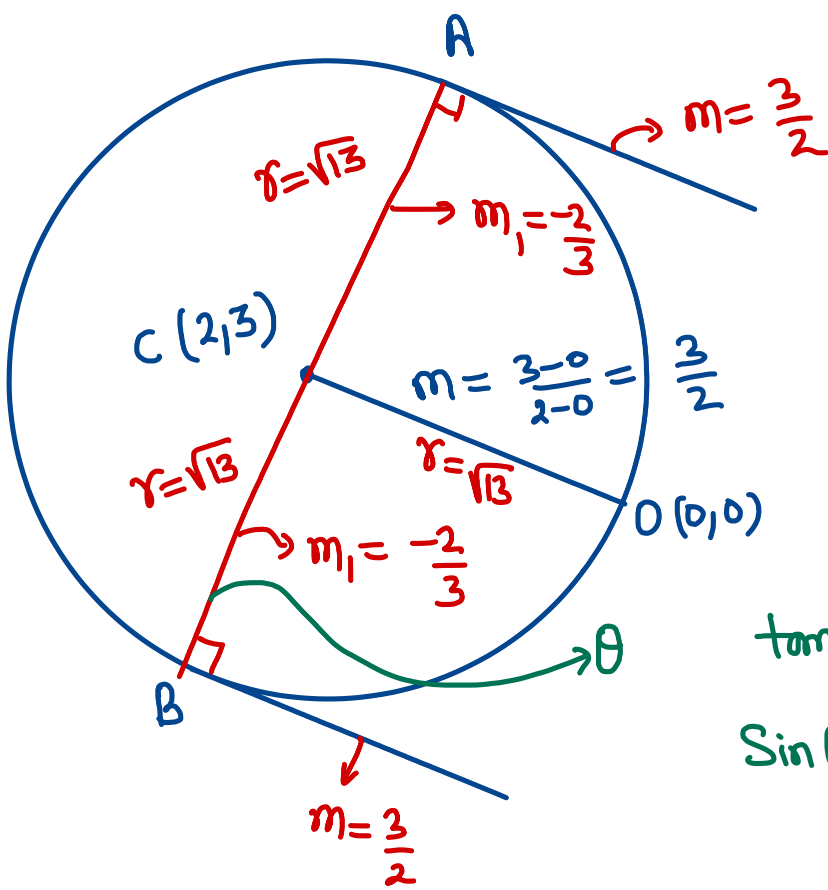
$\Rightarrow S_2: x^2 + y^2 - 4x - 6y + \lambda(2x - 3y + 5) = 0$

will pass through B(0,6)

$0^2 + 6^2 - 4\times 0 - 6^2 + \lambda(2\times 0 - 3\times 6 + 5) = 0$

$\Rightarrow \lambda = 0$

$$\boxed{\therefore\ S_2: x^2 + y^2 - 4x - 6y = 0}$$



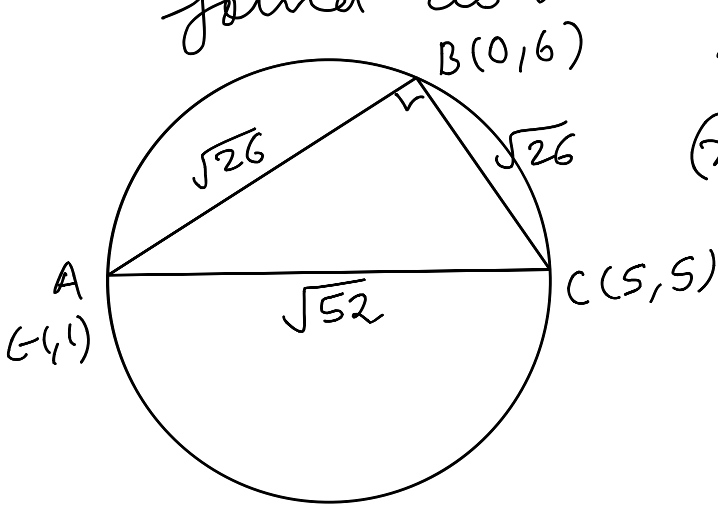
$\tan\theta = -\frac{2}{3}$

$\sin\theta = \frac{2}{\sqrt{13}}$; $\cos\theta = \frac{-3}{\sqrt{13}}$

$A \equiv (2 + r\cos\theta,\ 3 + r\sin\theta) \equiv \boxed{(-1, 5)}$

$B \equiv (2 - r\cos\theta,\ 3 - r\sin\theta) \equiv \boxed{(5, 1)}$

Note: Eq^n. of circle can also be found as:



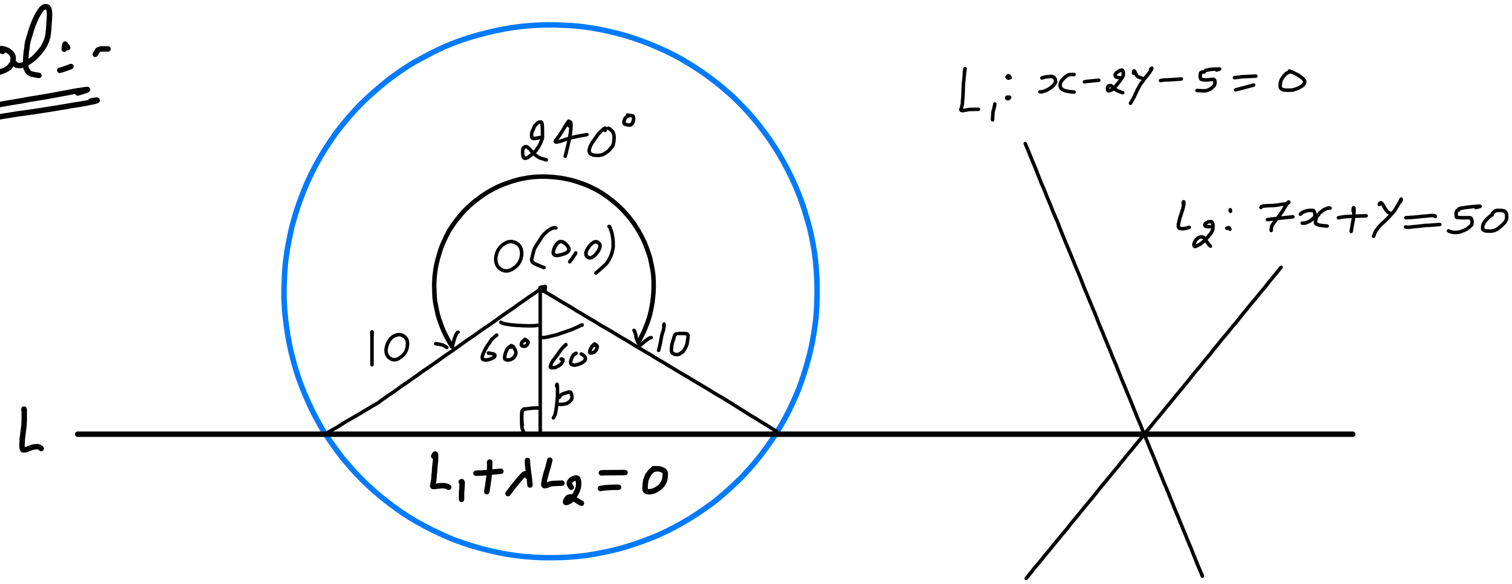
$\therefore$ Eq^n. of circle:

$(x+1)(x-5) + (y-5)(y-1) = 0$

$= x^2 + y^2 - 4x - 6y = 0$

(15) Find the equations of straight lines which pass through the intersection of the lines $x - 2y - 5 = 0$, $7x + y = 50$ & divide the circumference of the circle $x^2 + y^2 = 100$ into two arcs whose lengths are in the ratio $2:1$.

Sol:-



Line L: $L_1 + \lambda L_2 = 0$

$$\Rightarrow (x - 2y - 5) + \lambda(7x + y - 50) = 0$$

$$\Rightarrow (1 + 7\lambda)x + (\lambda - 2)y - (5 + 50\lambda) = 0 \quad \text{——— (1)}$$

* $p = \perp^r$ drop from $O(0,0)$ to line L

$$\Rightarrow p = \frac{|5 + 50\lambda|}{\sqrt{(1 + 7\lambda)^2 + (\lambda - 2)^2}}$$

* $\because \cos 60^\circ = \frac{p}{10} \Rightarrow p = 5$

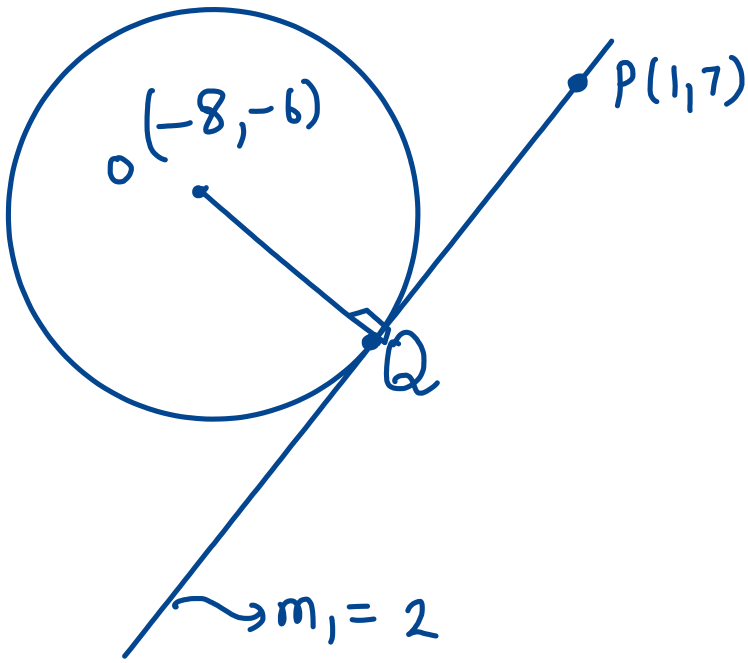
$$\Rightarrow \frac{|5 + 50\lambda|}{\sqrt{(1 + 7\lambda)^2 + (\lambda - 2)^2}} = 5 \Rightarrow \underbrace{\lambda = \frac{-2}{5} \text{ or } \frac{1}{5}}_{\text{Put in (1), then}}$$

$$\Rightarrow 4x - 3y - 25 = 0 \ \textbf{OR} \ 3x + 4y - 25 = 0$$

Ans.

(16) A line with gradient 2 is passing through the point P(1, 7) and touches the circle $x^2 + y^2 + 16x + 12y + c = 0$ at the point Q. If (a, b) are the coordinates of the point Q, then find the value of $(7a + 7b + c)$.

Solution:



$$PQ: (y-7) = 2(x-1) \Rightarrow PQ: 2x - y + 5 = 0 \;\cdots\; (1)$$

$$OQ = r = \sqrt{g^2 + f^2 - c} = \left| \frac{2(-8) - (-6) + 5}{\sqrt{2^2 + 1^2}} \right|$$

$$\Rightarrow \sqrt{8^2 + 6^2 - c} = \sqrt{5} \Rightarrow \boxed{c = 95}$$

Equation OQ: $x + 2y = (-8) + 2(-6)$

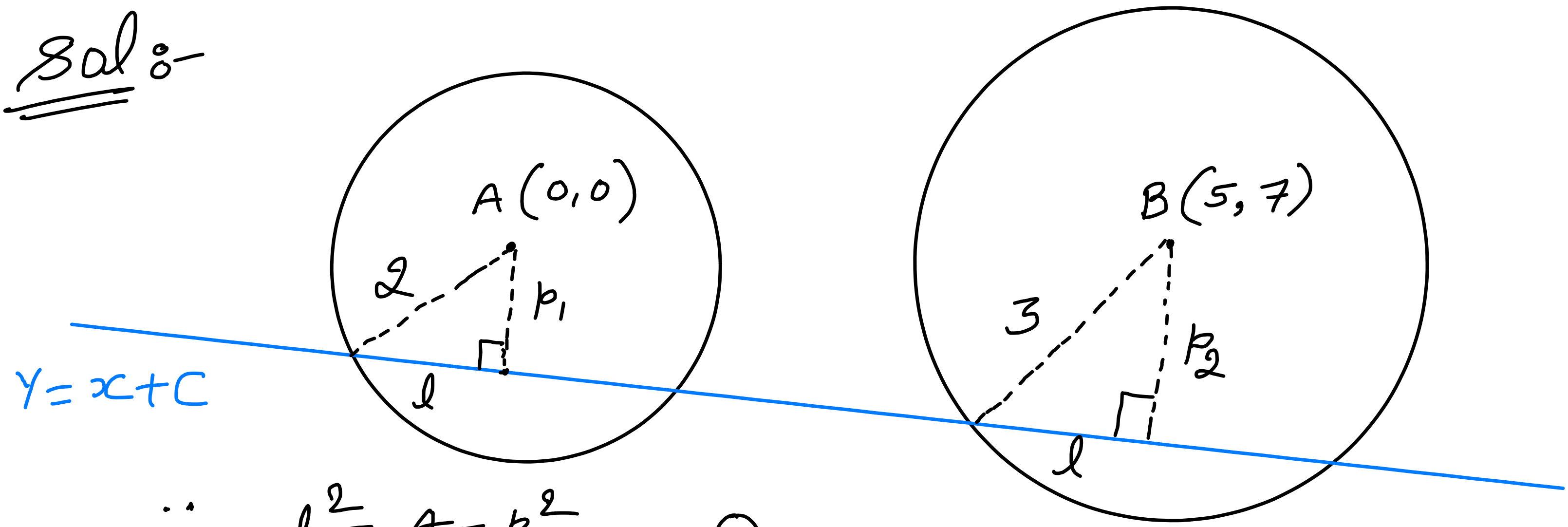
$$\Rightarrow x + 2y + 20 = 0 \;\cdots\; (2)$$

Solving (1) & (2) we get $Q \equiv (-6, -7) \equiv (a, b)$

So, $7a + 7b + c = -42 - 49 + 95 = \boxed{4}$

(17) Find the equation of a line with gradient 1 such that the two circles $x^2 + y^2 = 4$ and $x^2 + y^2 - 10x - 14y + 65 = 0$ intercept equal length on it.

Sol :-



$\because \quad l^2 = 4 - p_1^2$ —— (1)

$l^2 = 9 - p_2^2$ —— (2)

from (1) & (2)

$$4 - p_1^2 = 9 - p_2^2$$

$$\Rightarrow \quad p_2^2 - p_1^2 = 5$$

$$\Rightarrow \left(\frac{|5-7+c|}{\sqrt{1+1}}\right)^2 - \left(\frac{|0-0+c|}{\sqrt{1+1}}\right)^2 = 5$$

$$\Rightarrow \quad \boxed{C = -\frac{3}{2}}$$

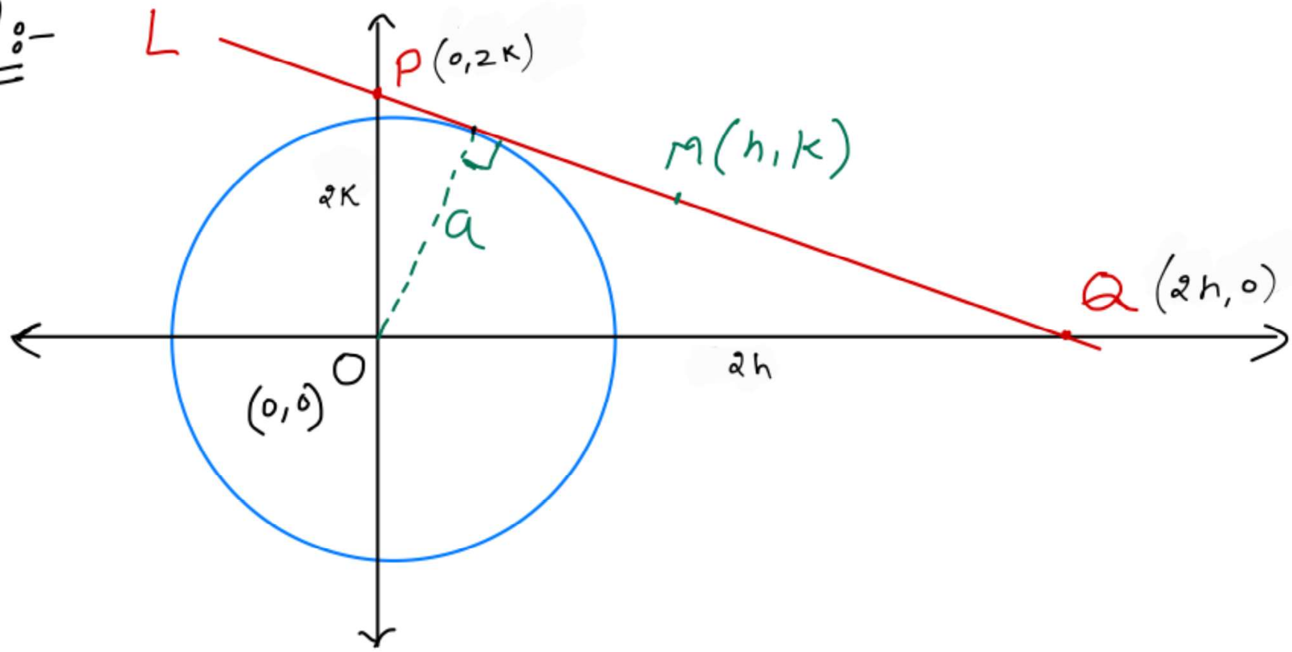
$\therefore$ eq^n of line $\Rightarrow$ $Y = x - \frac{3}{2}$

$$\Rightarrow \boxed{2x - 2Y - 3 = 0}$$

Ans.

(18) Find the locus of the middle points of portions of the tangents to the circle $x^2 + y^2 = a^2$ terminated by the coordinate axes.

Sol:-



line $L: \dfrac{x}{2h} + \dfrac{y}{2K} = 1$

$$\Rightarrow \boxed{xK + yh = 2hK}$$

$\because$ line is Tangent to the Circle

$\therefore$ apply $p = r$

$$\Rightarrow \frac{|0+0-2hK|}{\sqrt{K^2+h^2}} = a$$

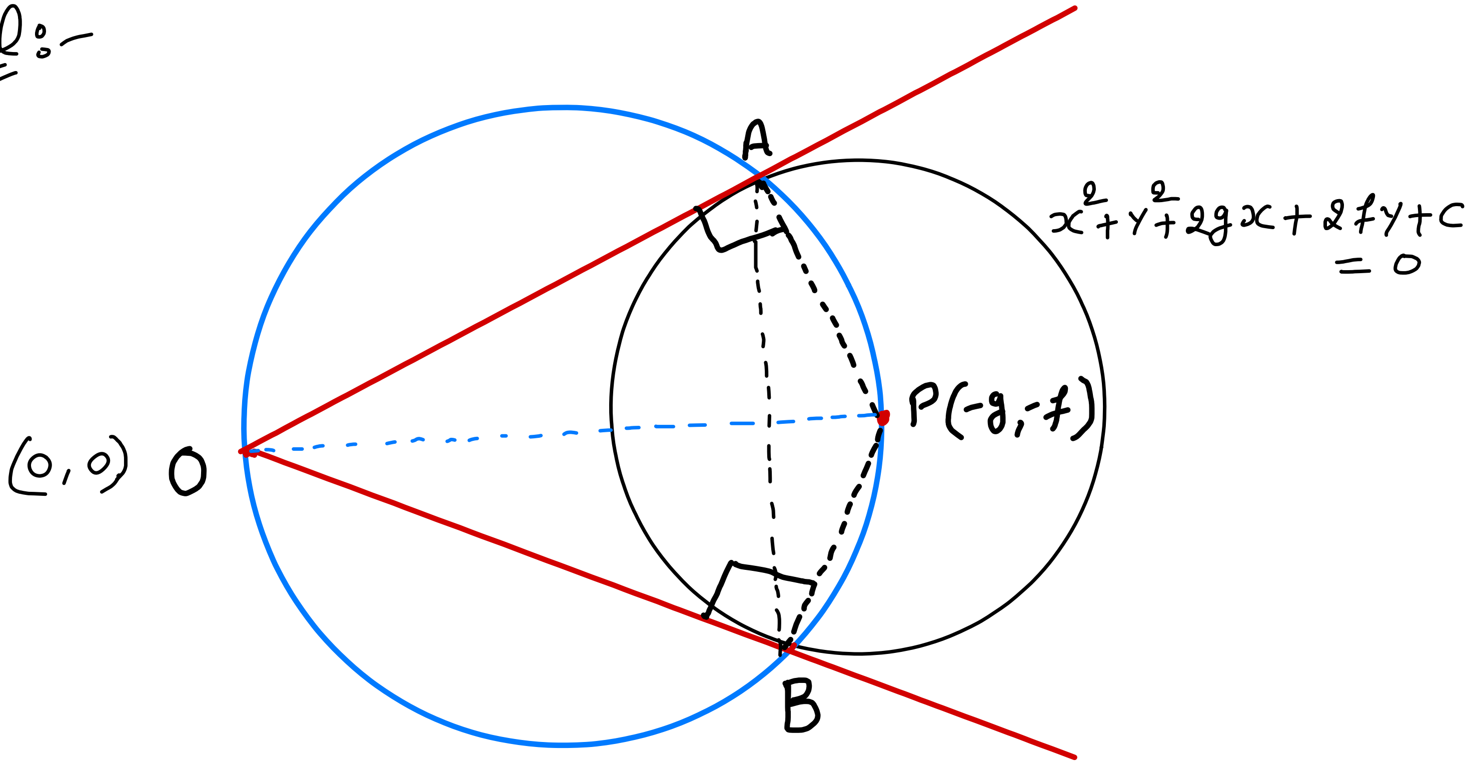
$$\Rightarrow 4h^2K^2 = a^2(h^2+K^2)$$

$$\Rightarrow \boxed{a^2(x^2+y^2) = 4x^2y^2}$$

Ans.

(19) Tangents OP and OQ are drawn from the origin O to the circle $x^2 + y^2 + 2gx + 2fy + c = 0$. Find the equation of the circumcircle of the triangle OPQ.

Sol:-



Circumcircle of ΔOAB & Circumcircle of quadrilateral OAPB are same.
And Quadrilateral OAPB is cyclic quadrilateral.

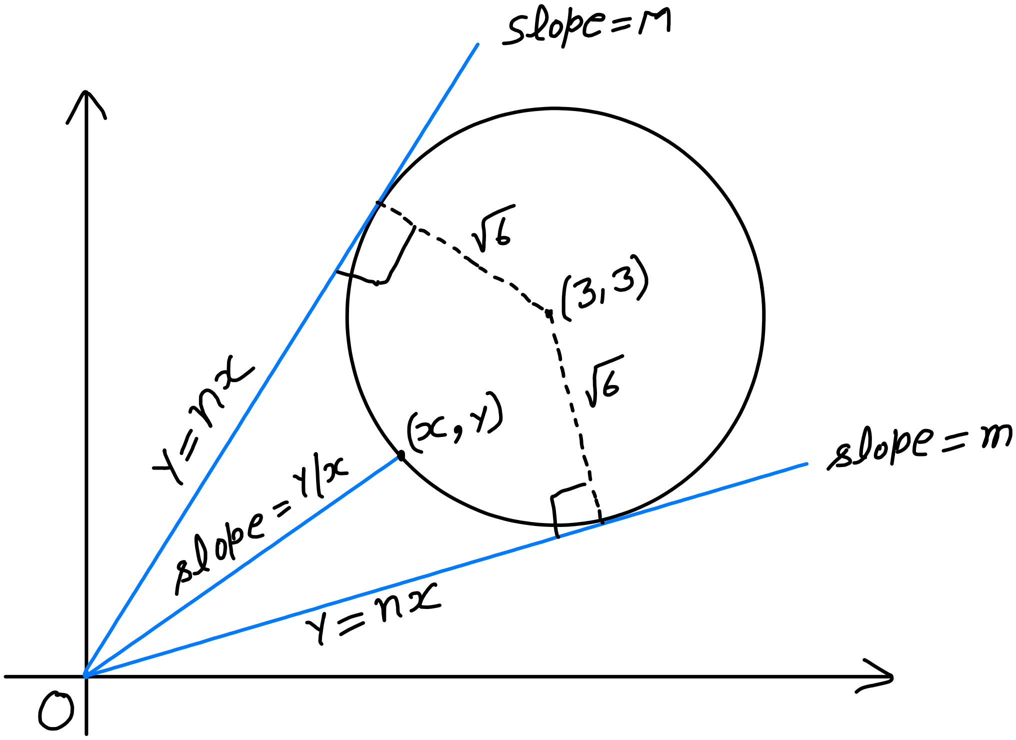
$\therefore$ OP is diameter of required circle

$$(x-0)(x+g)+(y-0)(y+f)=0$$

$$\Rightarrow \boxed{x^2+y^2+gx+fy=0}$$

20. If M and m are the maximum and minimum values of $\frac{y}{x}$ for pair of real number (x,y) which satisfy the equation $(x-3)^2 + (y-3)^2 = 6$, then find the value of (M + m).

Sol:-



$\because$ $y = nx$ is Tangent to the circle

$\therefore$ apply, $p = r$

$$\Rightarrow \frac{|3n-3|}{\sqrt{n^2+1}} = \sqrt{6}$$; n have two values m & M.

$$\Rightarrow 9(n-1)^2 = 6(n^2+1)$$

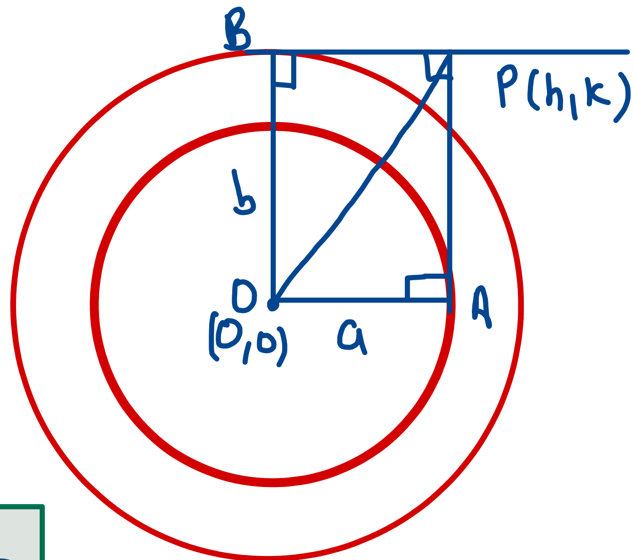
$$\Rightarrow 3n^2 - 18n + 3 = 0$$

$$\Rightarrow n^2 - 6n + 1 = 0 \;<\; \begin{matrix} m \\ M \end{matrix}$$

$\therefore$ $\boxed{m + M = 6}$ Ans.

(21) Tangents are drawn to the concentric circles $x^2 + y^2 = a^2$ and $x^2 + y^2 = b^2$ at right angle to one another. Show that the locus of their point of intersection is a 3rd concentric circle. Find its radius.

Solution:



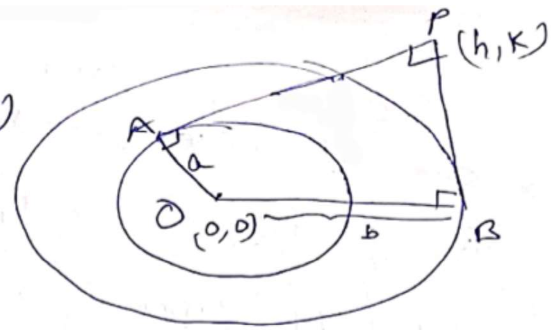
$$OP = \sqrt{a^2+b^2}$$

$$\Rightarrow \sqrt{h^2+k^2} = \sqrt{a^2+b^2}$$

$$\Rightarrow h^2+k^2 = a^2+b^2$$

$$\boxed{\Rightarrow x^2+y^2 = a^2+b^2}$$

Method 2



Let $A(a\cos\theta, a\sin\theta)$

eq^n of AP

$T=0$

$x\cos\theta + y\sin\theta = a$

eq^n of PB ($\perp$ to AP)

$x\sin\theta - y\cos\theta = \lambda$

$OB = b$

$$\left|\frac{0-\lambda}{\sqrt{\sin^2\theta+\cos^2\theta}}\right| = b$$

$\lambda = \pm b$

eq^n of PB

$x\sin\theta - y\cos\theta = \pm b$

PA & PB passes through $P(h,k)$

$h\cos\theta + k\sin\theta = a$ —— (1)

$h\sin\theta - k\cos\theta = \pm b$ —— (2)

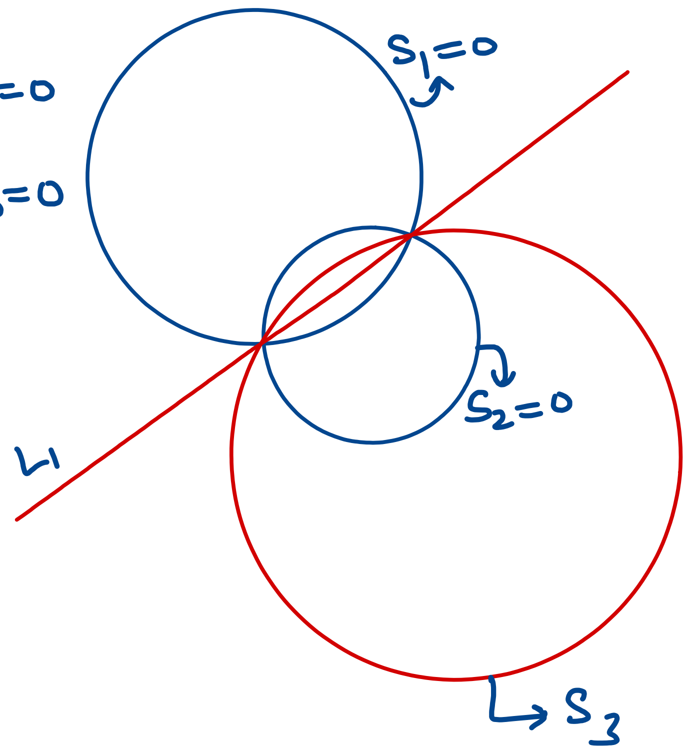
$(1)^2 + (2)^2$

$h^2+k^2 = a^2+b^2$

$x^2+y^2 = a^2+b^2$

(22) Find the equation of the circle passing through the point of intersection of the circles $x^2 + y^2 - 6x + 2y + 4 = 0$, $x^2 + y^2 + 2x - 4y - 6 = 0$ and with its centre on the line $y = x$.

Solution:



$S_1: x^2+y^2-6x+2y+4=0$

$S_2: x^2+y^2+2x-4y-6=0$

$L_1: S_1-S_2=0$

$\Rightarrow L_1: -8x+6y+10=0$

$\Rightarrow \boxed{L_1: 4x-3y-5=0}$

$S_3: S_1+\lambda L=0$

$\Rightarrow S_3: x^2+y^2-6x+2y+4+\lambda(4x-3y-5)=0$

$\Rightarrow S_3: x^2+y^2+(4\lambda-6)x+(2-3\lambda)y+(4-5\lambda)=0$

Centre $\left(3-2\lambda, \frac{3\lambda-2}{2}\right)$ line on $y=x$

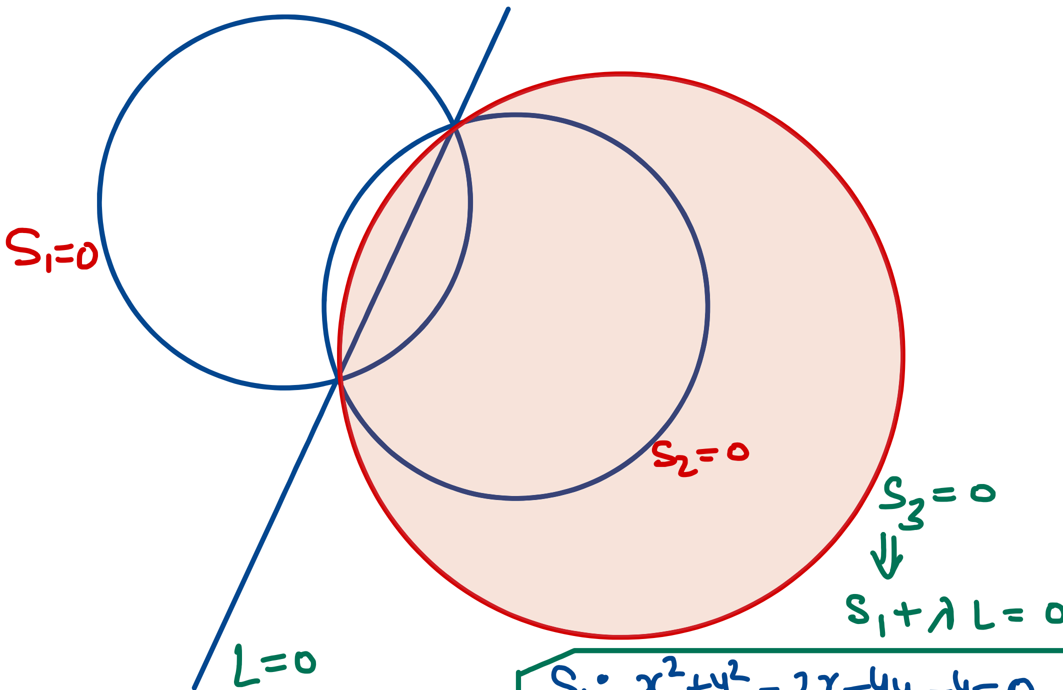
So, $\frac{3\lambda-2}{2} = 3-2\lambda \Rightarrow 3\lambda-2=6-4\lambda \Rightarrow \boxed{\lambda=\frac{8}{7}}$

So, $S_3: x^2+y^2+\left(4\times\frac{8}{7}-6\right)x+\left(2-3\times\frac{8}{7}\right)y+\left(4-5\times\frac{8}{7}\right)=0$

$\Rightarrow \boxed{S_3: 7x^2+7y^2-10x-10y-12=0}$

(23) Find the equation of the circle passing through the points of intersection of the circles $x^2 + y^2 - 2x - 4y - 4 = 0$ and $x^2 + y^2 - 10x - 12y + 40 = 0$ and whose radius is 4.

Solution:



$L: S_1 - S_2 = 0$

$S_1: x^2+y^2-2x-4y-4=0$

$S_2: x^2+y^2-10x-12y+40=0$

$L: S_1 - S_2 = 0 \Rightarrow 8x+8y-44=0$

$\Rightarrow L: 2x+2y-11=0$

So, $S_3: S_1+\lambda L=0$

$\Rightarrow S_3: (x^2+y^2-2x-4y-4)+\lambda(2x+2y-11)=0$

$\Rightarrow S_2: x^2+y^2+(2\lambda-2)x+(2\lambda-4)y-(11\lambda+4)=0$

$r=\sqrt{(\lambda-1)^2+(\lambda-2)^2+11\lambda+4} = 4$

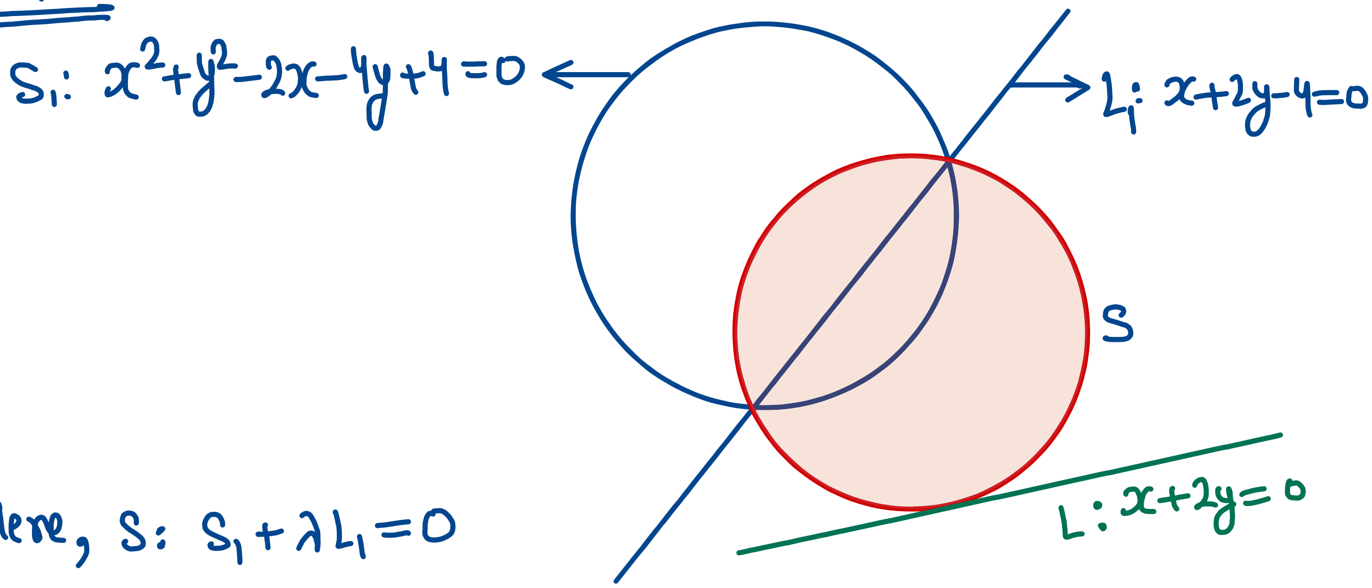
$\Rightarrow 2\lambda^2+5\lambda-7=0 \Rightarrow \lambda=1;\ \lambda=-\frac{7}{2}$

Put $\lambda=1$ we get $\boxed{x^2+y^2-2y-15=0}$

Put $\lambda=-\frac{7}{2}$ we get $\boxed{2x^2+2y^2-18x-22y+69=0}$

(24) Find the equation of the circle through points of intersection of the circle $x^2 + y^2 - 2x - 4y + 4 = 0$ and the line $x + 2y = 4$ which touches the line $x + 2y = 0$.

Solution:



Here, $S: S_1 + \lambda L_1 = 0$

$$\Rightarrow S: x^2+y^2-2x-4y+4+\lambda(x+2y-4)=0 \quad \ldots \boxed{1}$$

'S' is tangent to L. Use $x=-2y$ (Put in $\boxed{1}$)

So, $$(-2y)^2+y^2-2(-2y)-4y+4+\lambda(-2y+2y-4)=0$$

$$\Rightarrow 5y^2+4-4\lambda=0$$

Condition of tangency, $D=0$

$$\Rightarrow 0^2-4(5)(4-4\lambda)=0 \Rightarrow \boxed{\lambda=1}$$

$$\therefore S: x^2+y^2-2x-4y+4+x+2y-4=0$$

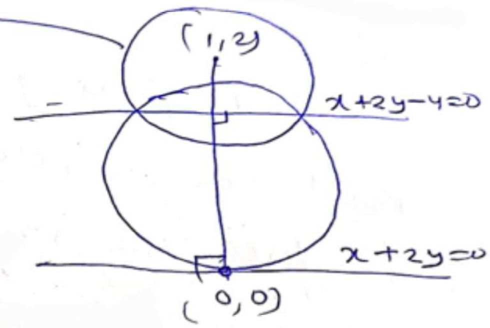
$$\boxed{\Rightarrow S: x^2+y^2-x-2y=0}$$

M2

$x^2+y^2-2x-4y+4=0$

foot of $\perp$ from $(1,2)$ on

$x+2y=0$ is



$$\frac{x-1}{1}=\frac{y-2}{2}=-\frac{(1+4)}{1+4}=-1$$

$x=0,\ y=0$

eq^n of circle is given by $S+\lambda L=0$

$x^2+y^2-2x-4y+4+\lambda(x+2y-4)=0$ Passes through origin

$\boxed{\lambda=1}$

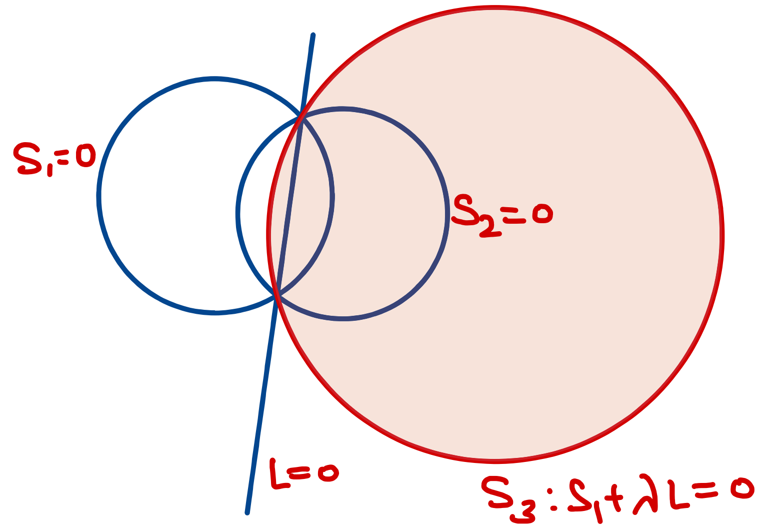
$x^2+y^2-x-2y=0$

(25) Find the equations of the circles which pass through the common points of the following pair of circles.

(a) $x^2 + y^2 + 2x + 3y - 7 = 0$ and $x^2 + y^2 + 3x - 2y - 1 = 0$ through the point (1,2)

(b) $x^2 + y^2 + 4x - 6y - 12 = 0$ and $x^2 + y^2 - 5x + 17y = 19$ and having its centre on $x + y = 0$.

Solution:



(a) $S_1: x^2+y^2+2x+3y-7=0$

and $S_2: x^2+y^2+3x-2y-1=0$

$L: S_1-S_2=0 \Rightarrow L: x-5y+6=0$

$\therefore S_3: S_1+\lambda L=0 \Rightarrow S_3: x^2+y^2+2x+3y-7+\lambda(x-5y+6)=0$

will pass through (1,2)

So, $1^2+2^2+2\cdot1+3\cdot2-7+\lambda(1-5\cdot2+6)=0$

$\Rightarrow 6-3\lambda=0 \Rightarrow \boxed{\lambda=2}$

$\therefore S_3: x^2+y^2+2x+3y-7+2(x-5y+6)=0$

$$\boxed{\Rightarrow S_3: x^2+y^2+4x-7y+5=0}$$

(b) $S_1: x^2+y^2+4x-6y-12=0$

and $S_2: x^2+y^2-5x+17y-19=0$

$L: S_1-S_2=0 \Rightarrow L: 9x-23y+7=0$

$S_3: (x^2+y^2+4x-6y-12)+\lambda(9x-23y+7)=0 \quad \text{(1)}$

$\Rightarrow S_3: x^2+y^2+(9\lambda+4)x-(6+23\lambda)y+(7\lambda-12)=0$

Centre $= \left(-\frac{(9\lambda+4)}{2}, +\frac{(6+23\lambda)}{2}\right)$

will lie on $x+y=0$

$\Rightarrow -\left(\frac{9\lambda+4}{2}\right)+\left(\frac{6+23\lambda}{2}\right)=0$

$\Rightarrow -9\lambda-4+6+23\lambda=0 \Rightarrow \lambda=-\frac{1}{7}$

Put in (1)

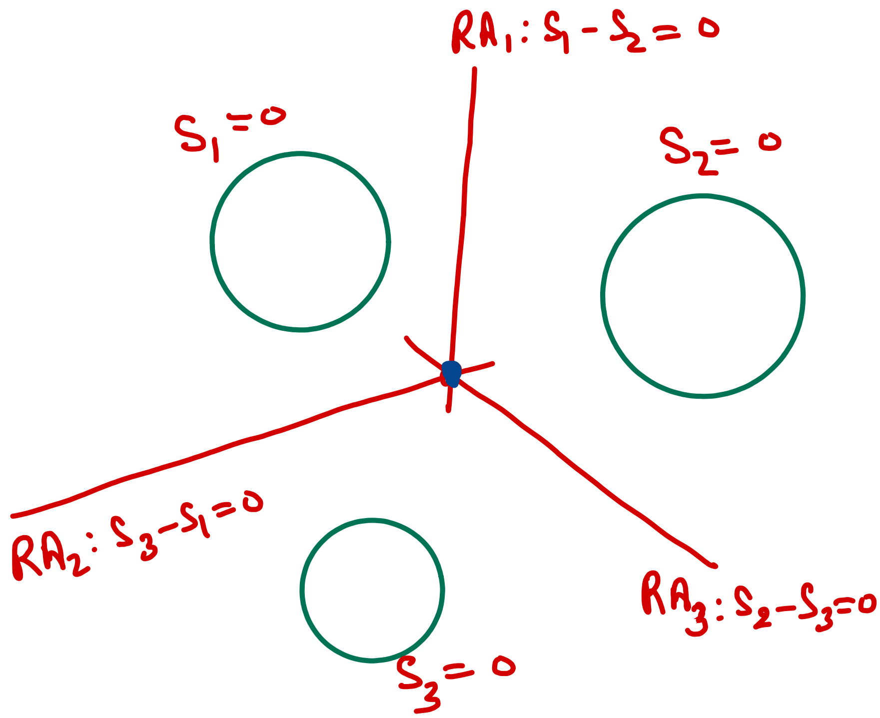
$\therefore S_3: \boxed{7(x^2+y^2)+19x-19y-91=0}$

Ans.

(26) Find the radical centre of the following set of circles

$x^2 + y^2 - 3x - 6y + 14 = 0$; $x^2 + y^2 - x - 4y + 8 = 0$; $x^2 + y^2 + 2x - 6y + 9 = 0$

Solution:



$S_1: x^2+y^2-3x-6y+14=0$

$S_2: x^2+y^2-x-4y+8=0$

$S_3: x^2+y^2+2x-6y+9=0$

$\therefore RA_1: -2x-2y+6=0 \Rightarrow x+y=3$

$RA_2: -3x+2y-1=0 \Rightarrow 3x-2y=-1$

Solving, both we get radical centre as $(1, 2)$

(27) Find the equation to the circle orthogonal to the two circles
$x^2 + y^2 - 4x - 6y + 11 = 0$; $x^2 + y^2 - 10x - 4y + 21 = 0$ and has $2x + 3y = 7$ as diameter.

Solution: The orthogonal to two circles have its centre on radical axis of the two circles.

$S_1: x^2+y^2-4x-6y+11=0$
and $S_2: x^2+y^2-10x-4y+21=0$

Radical axis: $S_1 - S_2 = 0 \Rightarrow 6x - 2y - 10 = 0$
$\Rightarrow \boxed{3x - y - 5 = 0}$

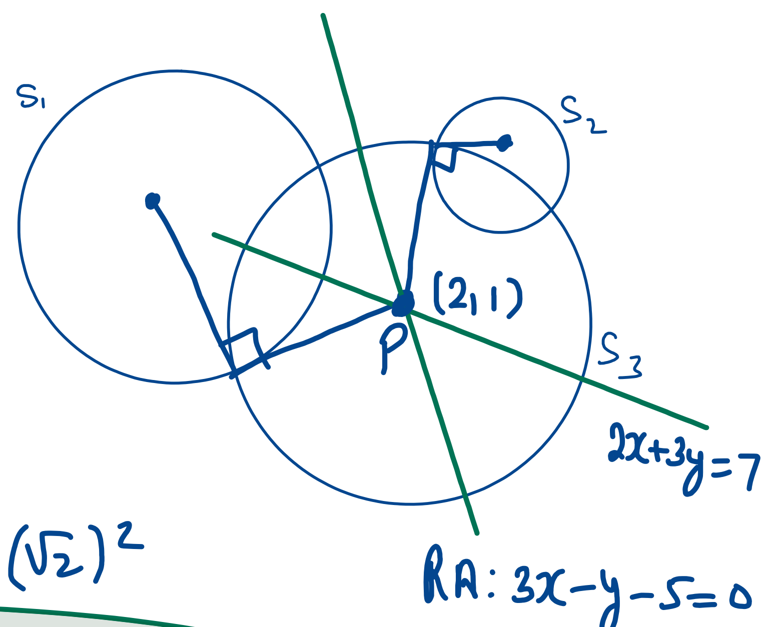
$\therefore$ Two diameters are $3x-y-5=0$ and $2x+3y=7$

Solving both we get $x=2;\ y=1$

$\therefore$ centre is $P(2,1)$

radius $= \sqrt{S_{1P}}$

$R = \sqrt{2^2+1^2-4\cdot 2-6\cdot 1+11}$
$= \sqrt{2}$



$\therefore S_3: (x-2)^2+(y-1)^2 = (\sqrt{2})^2$

$\Rightarrow x^2+y^2-4x-2y+3=0$

(28) Find the equation to the circle, cutting orthogonally each of the following circles :

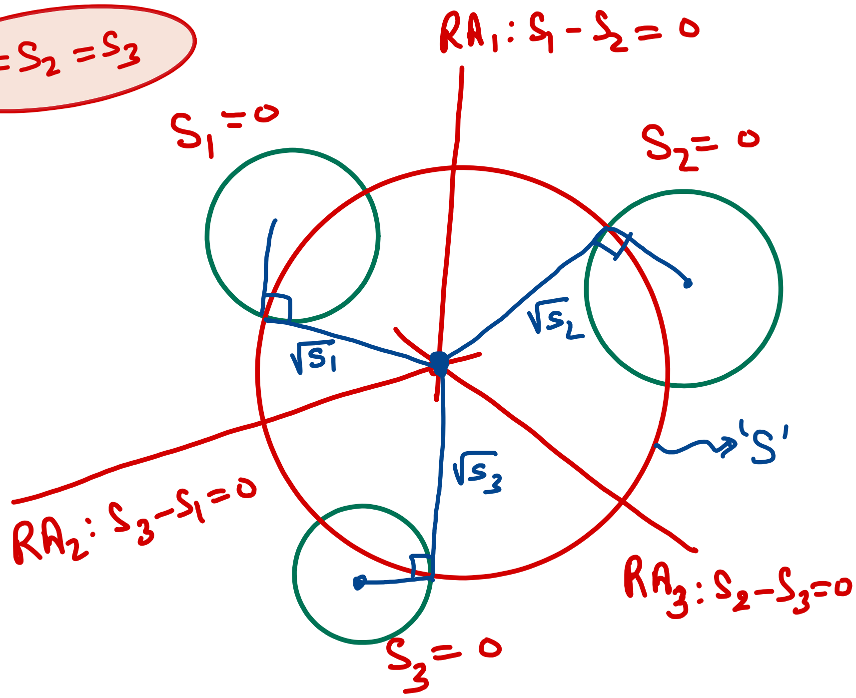
$x^2 + y^2 - 2x + 3y - 7 = 0$; $x^2 + y^2 + 5x - 5y + 9 = 0$; $x^2 + y^2 + 7x - 9y + 29 = 0$.

Solution: S_1: $x^2 + y^2 - 2x + 3y - 7 = 0$;

S_2: $x^2 + y^2 + 5x - 5y + 9 = 0$; and S_3: $x^2 + y^2 + 7x - 9y + 29 = 0$.

Circle cutting three given circles orthogonaly has its centre as radical center of the three circle.

$S_1 = S_2 = S_3$



$RA_1: S_1 - S_2 = 0 \Rightarrow RA_1: -7x + 8y - 16 = 0 \Rightarrow 7x - 8y + 16 = 0$

$RA_2: S_3 - S_1 = 0 \Rightarrow RA_2: 9x - 12y + 36 = 0 \Rightarrow 3x - 4x + 12 = 0$

Solving RA_1 and RA_2 we get centre $= (8, 9)$

radius $= \sqrt{S_1} = \sqrt{8^2 + 9^2 - 2\times8 + 3\times9 - 7} = \sqrt{149}$

$\therefore$ Equation of circle is $(x-8)^2 + (y-9)^2 = (\sqrt{149})^2$

$$\boxed{\Rightarrow x^2 + y^2 - 16x - 18y - 4 = 0}$$

(29) Find the equation of the circle through the points of intersection of circles $x^2 + y^2 - 4x - 6y - 12 = 0$ and $x^2 + y^2 + 6x + 4y - 12 = 0$ & cutting the circle $x^2 + y^2 - 2x - 4 = 0$ orthogonally.

Solution:

$S_1: x^2+y^2-4x-6y-12=0$

$S_2: x^2+y^2+6x+4y-12=0$

$S_3: x^2+y^2-2x-4=0$

$L: S_1-S_2=0 \Rightarrow L: -10x-10y=0$

$\Rightarrow \boxed{L: x+y=0}$

Circle 'S' passing through intersection of S_1 and S_2

$\therefore S: S_1+\lambda L=0$

$\Rightarrow S: x^2+y^2-4x-6y-12+\lambda(x+y)=0$

$\boxed{\Rightarrow S: x^2+y^2+(\lambda-4)x+(\lambda-6)y-12=0}$

is orthogonal to 'S_3'.

Use $\boxed{2g_1g_2+2f_1f_2=C_1+C_2}$

$2\left(\frac{\lambda-4}{2}\right)(-1)+2\left(\frac{\lambda-6}{2}\right)(0)=(-12)+(-4)$

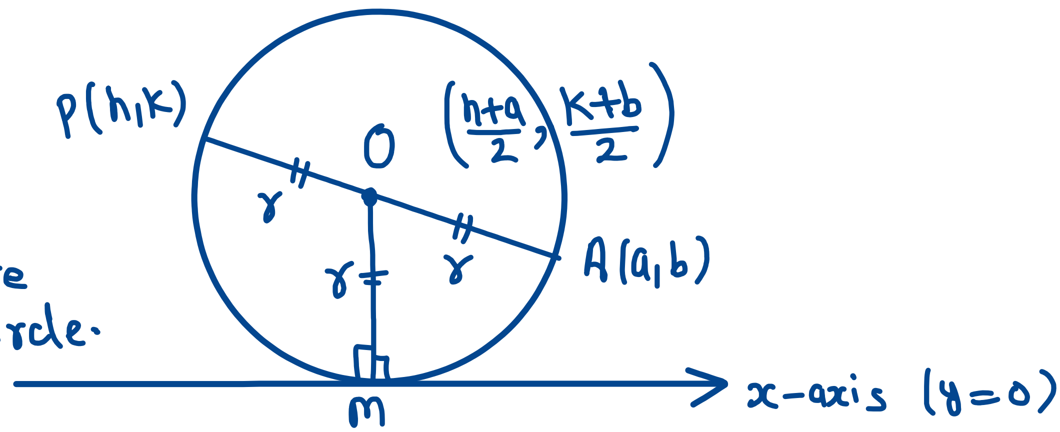
$\Rightarrow (\lambda-4)(-1)=-16 \Rightarrow \lambda=20$

$\therefore$ circle 'S' is given as

$\boxed{x^2+y^2+16x+14y-12=0}$

(30) A variable circle passes through the point A (a, b) & touches the x-axis. Show that the locus of the other end of the diameter through A is $(x - a)^2 = 4by$.

Solution:



'O' is centre of the circle.

$$|OM| = r = \left|\frac{k+b}{2}\right| \Rightarrow 2r = |k+b| = AP$$

$$\Rightarrow |k+b| = \sqrt{(h-a)^2 + (k-b)^2}$$

$$\Rightarrow (k+b)^2 = (h-a)^2 + (k-b)^2$$

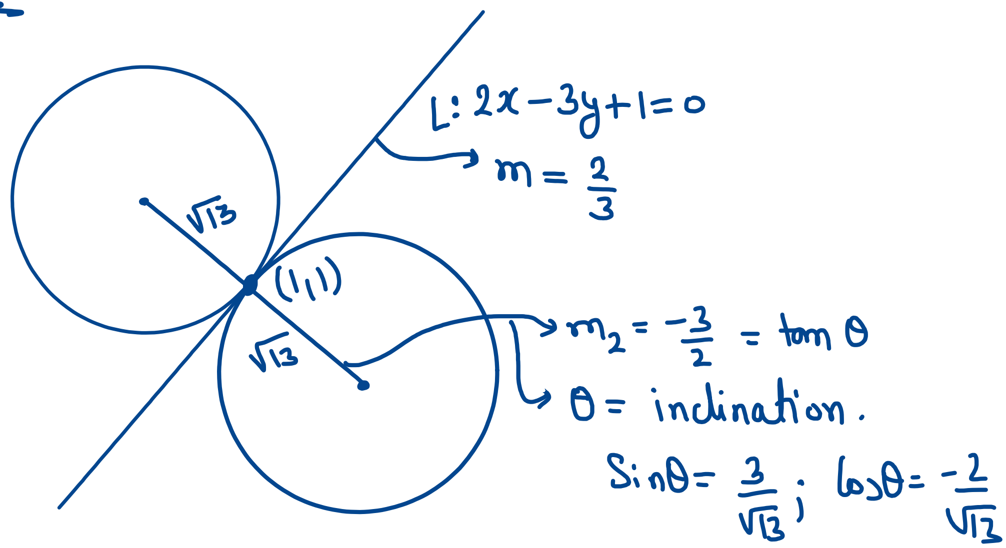
$$\Rightarrow k^2 + 2bk + b^2 = (h-a)^2 + k^2 - 2bk + b^2$$

$$\Rightarrow 4bk = (h-a)^2 \Rightarrow \boxed{(x-a)^2 = 4by}$$

[Hence Proved]

(31) The line $2x - 3y + 1 = 0$ is tangent to a circle $S = 0$ at $(1, 1)$. If the radius of the circle is $\sqrt{13}$. Find the equation of the circle S.

Solution:



centre of circle $(1+\sqrt{13}\cos\theta, 1+\sqrt{13}\sin\theta)$

or $(1-\sqrt{13}\cos\theta, 1-\sqrt{13}\sin\theta)$

Two possibilities for the circle which are

$(-1,4)$ or $(3,-2)$; radius is $\sqrt{13}$

$\therefore$ Circles are

$$(x+1)^2+(y-4)^2 = 13 \Rightarrow \boxed{x^2+y^2+2x-8y+4=0}$$

or $$(x-3)^2+(y+2)^2 = 13 \Rightarrow \boxed{x^2+y^2-6x+4y=0}$$

M2

eq^n of circle can be given as

$$(x-1)^2+(y-1)^2+\lambda(2x-3y+1)=0$$

$$x^2+y^2-2x-2y+2+\lambda(2x-3y+1)=0$$

$$x^2+y^2+x(2\lambda-2)+y(-3\lambda-2)+\lambda+2=0$$

$$r=\sqrt{13}$$

$$(\lambda-1)^2+\left(\frac{3\lambda+2}{2}\right)^2-\lambda-2=13$$

$$4(\lambda^2+1-2\lambda)+9\lambda^2+4+12\lambda-4\lambda=60$$

$$13\lambda^2-52=0$$

$$\lambda^2=4$$

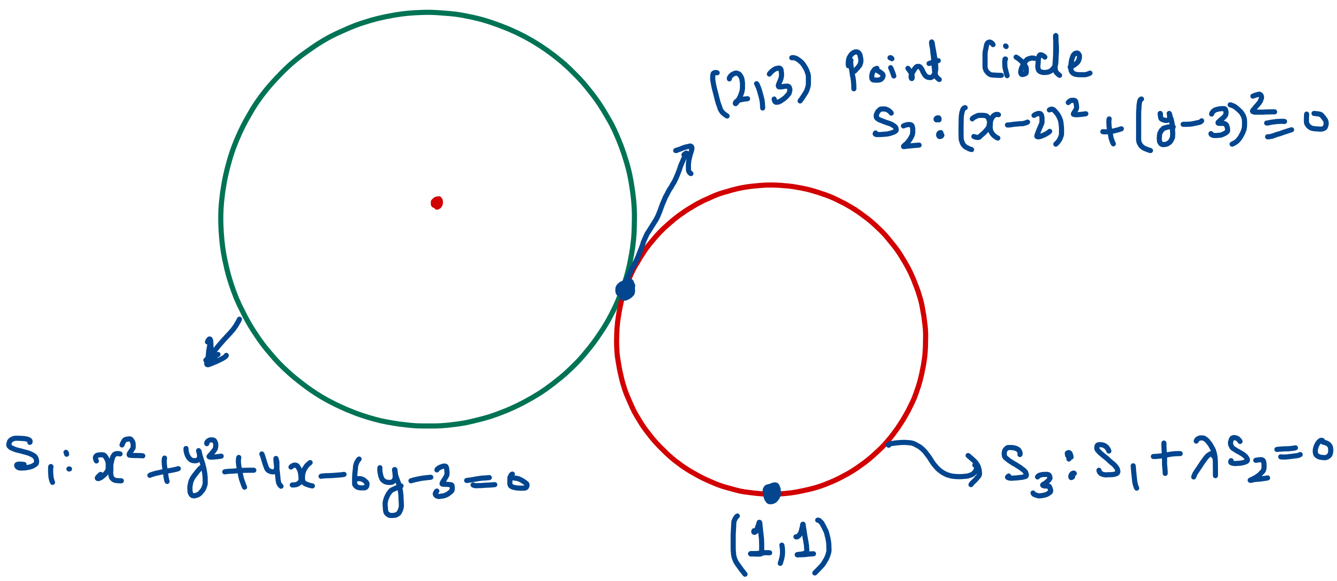
$$\lambda=\pm 2$$

$$x^2+y^2+2x-8y+4=0 \quad [\text{for } \lambda=2]$$

$$x^2+y^2-6x+4y=0 \quad [\text{for } \lambda=-2]$$

(32) Find the equation of the circle which passes through the point (1, 1) & which touches the circle $x^2 + y^2 + 4x - 6y - 3 = 0$ at the point (2, 3) on it.

Solution:



$$S_3 : S_1 + \lambda S_2 = 0 \Rightarrow (x^2+y^2+4x-6y-3) + \lambda\left((x-2)^2+(y-3)^2\right) = 0$$

passes through (1,1)

So, $(1+1+4-6-3) + \lambda(1+4) = 0$

$$\Rightarrow \quad \lambda = \frac{3}{5}$$

Put $\lambda = \frac{3}{5}$ in S_3

$$(x^2+y^2+4x-6y-3) + \frac{3}{5}(x^2+y^2-4x-6y+13) = 0$$

$$\Rightarrow 8x^2 + 8y^2 + 8x - 48y + 24 = 0$$

$$\boxed{\Rightarrow \; x^2+y^2+x-6y+3=0}$$

M2

eq of tangent at (2,3)

$x^2+y^2+4x-6y-3=0$

$T=0$

$2x+3y+2(x+2)-3(y+3)-3=0$

$4x-8=0$

$x-2=0$

eq^n of circle can be given as

$(x-2)^2+(y-3)^2+\lambda(x-2)=0$

passes through (1,1)

$1+4+\lambda(-1)=0$

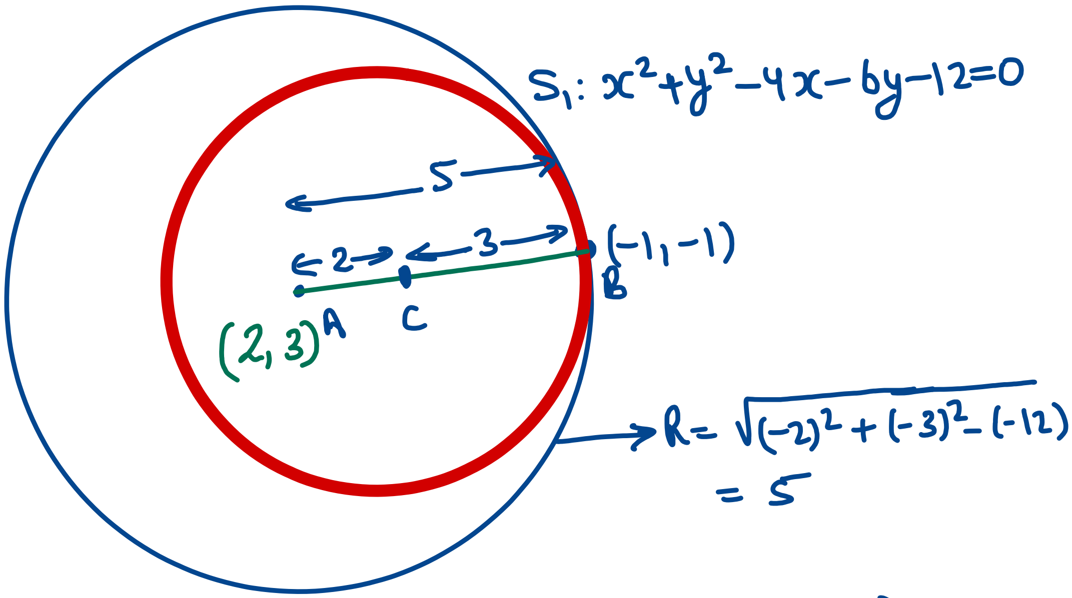
$\lambda=5$

$x^2+y^2-4x-6y+4+9+5x-10=0$

$x^2+y^2+x-6y+3=0$

(33) Find the equation of the circle whose radius is 3 and which touches the circle $x^2 + y^2 - 4x - 6y - 12 = 0$ internally at the point $(-1, -1)$.

Solution:



Centre, $C \equiv \left(\dfrac{2\times(-1)+3\times(2)}{2+3}, \dfrac{2\times(-1)+3\times(3)}{2+3} \right)$

$\equiv \left(\dfrac{4}{5}, \dfrac{7}{5} \right)$

Equation of the circle is:

$$\left(x-\frac{4}{5}\right)^2+\left(y-\frac{7}{5}\right)^2=3^2$$

$$\begin{array}{r} 225 \\ -\ 65 \\ \hline 160 \end{array}$$

$$\Rightarrow x^2+y^2-\frac{8}{5}x-\frac{14}{5}y+\frac{16}{25}+\frac{49}{25}-9=0$$

$$\Rightarrow 25x^2+25y^2-40x-70y-160=0$$

$$\boxed{\Rightarrow 5x^2+5y^2-8x-14y-32=0}$$

(34) The centre of the circle S = 0 lie on the line $2x - 2y + 9 = 0$ & S = 0 cuts orthogonally the circle $x^2 + y^2 = 4$. Show that circle S = 0 passes through two fixed points & find their coordinates.

Solution: Let $S=0$ is $x^2+y^2+2gx+2fy+c=0$

centre : $(-g,-f)$

$2x-2y+9=0$

$-2g+2f+9=0 \Rightarrow \boxed{2g=2f+9}$

and 'S' is orthogonal to $x^2+y^2-4=0$

$\therefore \; 2g_1g_2+2f_1f_2=c_1+c_2$

$\Rightarrow \; 2\cdot g\cdot 0+2f\cdot 0=c+(-4) \Rightarrow \boxed{c=4}$

$\therefore$ S: $x^2+y^2+(2f+9)x+2fy+4=0$

$\Rightarrow$ S: $(x^2+y^2+9x+4)+2f(x+y)=0$

Of the form $S_1+\lambda L_1=0$ which is passing through intersection of circle $x^2+y^2+9x+4=0$ and $x+y=0$

Solving, Put $y=-x$; $x^2+x^2+9x+4=0$

$\Rightarrow \; 2x^2+8x+x+4=0$

$\Rightarrow \; x=-\frac{1}{2};\; x=-4$

$\therefore$ Points are $\left(-\frac{1}{2},\frac{1}{2}\right)$ and $(-4,4)$

EXERCISE

5-2

(1) Consider a family of circles passing through two fixed points A(3, 7) & B(6, 5). The chords in which the circle $x^2 + y^2 - 4x - 6y - 3 = 0$ cuts the members of the family are concurrent at a point. Find the coordinates of this point.

Sol:- $A(3,7)$, $B(6,5)$

Equation of AB is

$L \equiv y - 7 = \frac{7-5}{3-6}(x-3)$

$\Rightarrow y - 7 = \frac{-2}{3}(x-3)$

$\Rightarrow 3y - 21 = -2x + 6$

$\Rightarrow 2x + 3y - 27 = 0$

$L: 2x + 3y - 27 = 0$

Equation of circle with AB as diameter is

$S: x^2 + y^2 - 9x - 12y + 53 = 0$

The family of circles through A and B is $S + \lambda L = 0$

$\Rightarrow x^2 + y^2 - 9x - 12y + 53 + \lambda(2x + 3y - 27) = 0$ —— (1)

Circle $x^2 + y^2 - 4x - 6y - 3 = 0$ —— (2) cuts the family (1) in concurrent chords.

The equation of common chord is

(1) − (2) = 0

$\Rightarrow -5x - 6y + 56 + \lambda(2x + 3y - 27) = 0$

The above equation represents concurrent chords, which pass through the point of intersection of

$5x + 6y - 56 = 0$ and

$2x + 3y - 27 = 0$

Solving these two gives

$x = 2, y = \frac{23}{3}$

Hence the chords are concurrent at the point $\left(2, \frac{23}{3}\right)$.

---

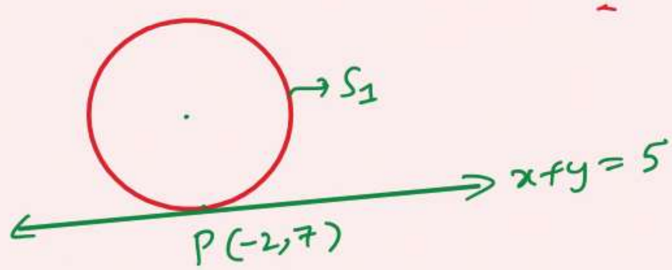
(2) Find the equation of a circle which touches the line x + y = 5 at the point (–2, 7) and cuts the circle $x^2 + y^2 + 4x - 6y + 9 = 0$ orthogonally.

11. Sol:- $S_1 = 0$ touches $x+y=5$ at $P(-2,7)$.

Equation of S_1 is $S+\lambda L = 0$

where $S=0$ is point circle at P

and $L=0$ is tangent



$\Rightarrow (x+2)^2 + (y-7)^2 + \lambda(x+y-5) = 0$

$\Rightarrow x^2+y^2+x(4+\lambda)+y(\lambda-14)+53-5\lambda = 0$

This cuts $x^2+y^2+4x-6y+9=0$ orthogonally.

$\Rightarrow 2g_1g_2 + 2f_1f_2 = c_1+c_2$

$\Rightarrow (4+\lambda)(2) + (\lambda-14)(-3) = 53-5\lambda+9$

$\Rightarrow 4\lambda = 12 \Rightarrow \lambda = 3$

$\therefore$ Required circle is

$\boxed{x^2+y^2+7x-11y+38=0}$

3. A circle is drawn with its centre on the line x + y = 2 to touch the line 4x – 3y + 4 = 0 and pass through the point (0, 1). Find its equation.

3 sol: Let centre be $C = (K, 2-K)$

Circle touches the line

$L: 4x - 3y + 4 = 0$ and passes through $P(0,1)$

$\Rightarrow$ $CP = \perp$ar distance from C to L

= radius

$$\Rightarrow \sqrt{K^2 + (K-1)^2} = \frac{|4K - 3(2-K) + 4|}{5}$$

$$\Rightarrow [K^2 + (K-1)^2]25 = (7K-2)^2$$

$$\Rightarrow 50K^2 - 50K + 25 = 49K^2 - 28K + 4$$

$$\Rightarrow K^2 - 22K + 21 = 0$$

$$-21 \quad -1$$

$$\Rightarrow K = 1, 21$$

If $K = 1$ then radius = 1

If $K = 21$ then radius $= \sqrt{21^2 + 20^2}$

$\therefore$ The equation of circle is

$$(x-1)^2 + (y-1)^2 = 1^2 \text{ (OR) } (x-21)^2 + (y+19)^2 = 21^2 + 20^2$$

$$\Rightarrow x^2 + y^2 - 2x - 2y + 1 = 0$$

(OR)

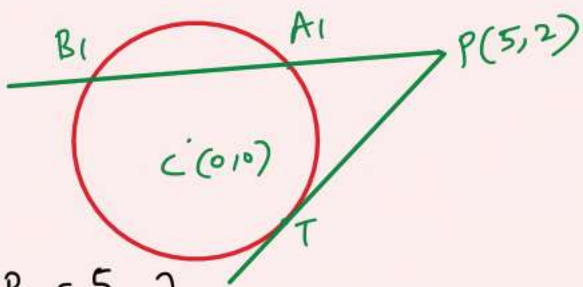
$$x^2 + y^2 - 42x + 38y - 39 = 0$$

(4) Through a given point P(5, 2), secants are drawn to cut the circle $x^2 + y^2 = 25$ at points $A_1(B_1)$, $A_2(B_2)$, $A_3(B_3)$, $A_4(B_4)$ and $A_5(B_5)$ such that $PA_1 + PB_1 = 5$, $PA_2 + PB_2 = 6$, $PA_3 + PB_3 = 7$, $PA_4 + PB_4 = 8$ and $PA_5 + PB_5 = 9$. Find the value of $\sum_{i=1}^{5} PA_i^2 + \sum_{i=1}^{5} PB_i^2$.

[**Note :** $A_r(B_r)$ denotes that the line passing through P(5, 2) meets the circle $x^2 + y^2 = 25$ at two points A_r and B_r.]

Sol :- $P(5,2)$

$S: x^2+y^2=25$



$$\left.\begin{aligned} PA_1+PB_1 &= 5 \\ PA_2+PB_2 &= 6 \\ PA_3+PB_3 &= 7 \\ PA_4+PB_4 &= 8 \\ PA_5+PB_5 &= 9 \end{aligned}\right\} \text{(Given)}$$

$PA_i \cdot PB_i = PT^2 = S_1 = 5^2+2^2-25 = 4$

for $i = 1, 2, 3, 4, 5$

Now $\sum_{i=1}^{5} PA_i^2 + \sum_{i=1}^{5} PB_i^2$

$= (PA_1^2+PB_1^2) + \cdots + (PA_5^2+PB_5^2)$

$= [(PA_1+PB_1)^2 - 2PA_1PB_1] + \cdots [(PA_5+PB_5)^2 - 2PA_5PB_5]$

$= [5^2-2(4)] + [6^2-2(4)] + \cdots + [9^2-2(4)]$

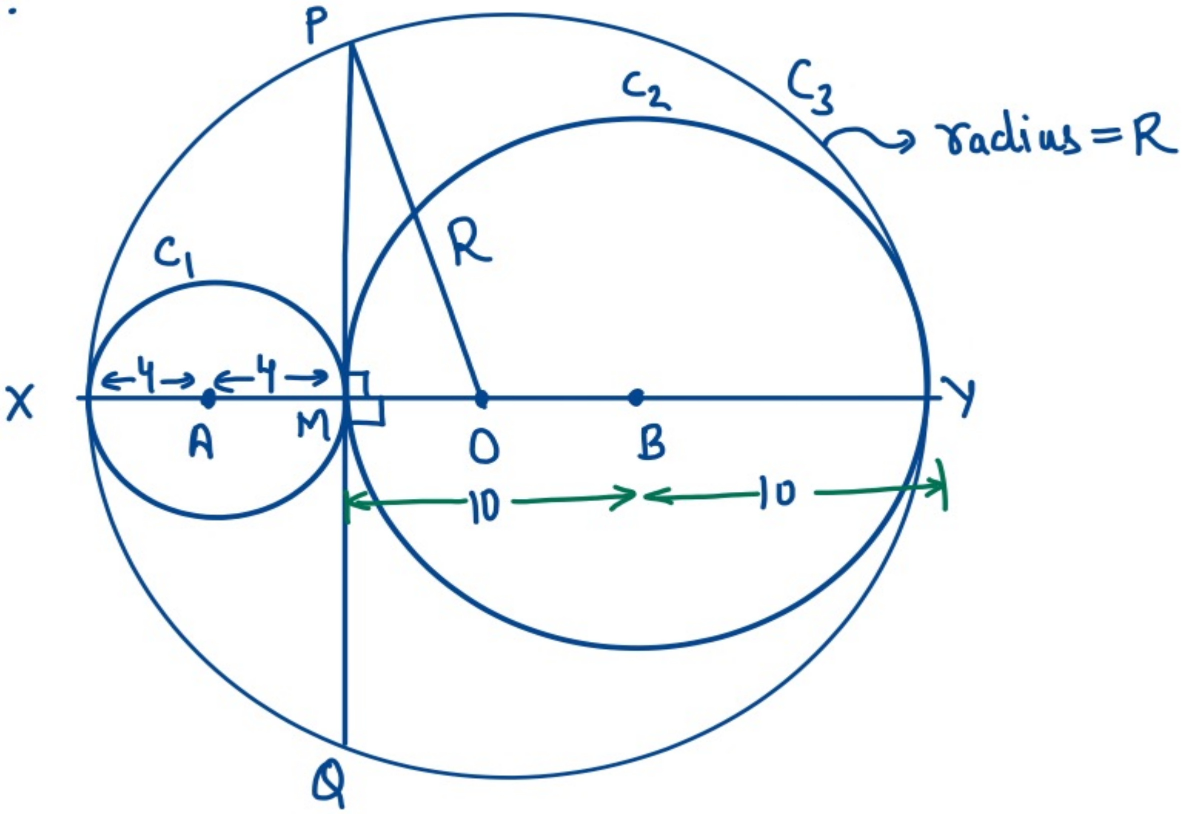
$= 5^2+6^2+7^2+8^2+9^2 - 8\times 5$

$= 215$

5

Circles C_1 and C_2 are externally tangent and they are both internally tangent to the circle C_3. The radii of C_1 and C_2 are 4 and 10, respectively and the centres of the three circles are collinear. A chord of C_3 is also a common internal tangent of C_1 and C_2. Given that the length of the chord is $\frac{m\sqrt{n}}{p}$ where m, n and p are positive integers, m and p are relatively prime and n is not divisible by the square of any prime, find the value of $(m+n+p)$.

Solution:



$$XY = \text{diameter of } C_3 = 2(XA) + 2(BY) = 18$$

$$\Rightarrow 2(OX) = 28 \Rightarrow OX = 14 = R = OP$$

$$OM = OX - XM = 14 - 8 = 6$$

$$PM = \sqrt{OP^2 - OM^2} = \sqrt{14^2 - 6^2} = 4\sqrt{10}$$

$$\Rightarrow PQ = 2PM = \frac{8\sqrt{10}}{1} = \frac{m\sqrt{n}}{p}$$

$$m + n + p = 8 + 10 + 1 = \boxed{19}$$

(6) A circle with center in the first quadrant is tangent to $y = x + 10$, $y = x - 6$, and the y-axis. Let (h, k) be the center of the circle. If the value of $(h + k) = a + b\sqrt{a}$ where $\sqrt{a}$ is a surd, find the value of $a + b$.

Sol:- Circle $S=0$ is tangent to

$y=x+10$, $y=x-6$ and Y axis.

Centre $C=(h,k)$ in first Quadrant

$$\Rightarrow \frac{|h-k+10|}{\sqrt{2}} = \frac{|h-k-6|}{\sqrt{2}} = |h|$$

(⊥ar distance from centre to tangent = radius)

$$\Rightarrow (h-k+10)^2 = 2h^2 \quad \text{— (1)}$$

$$(h-k-6)^2 = 2h^2 \quad \text{— (2)}$$

$$(1)-(2) \Rightarrow 32(h-k)+64=0$$

$$\Rightarrow h-k=-2$$

From (1), $(-2+10)^2 = 2h^2$

$$\Rightarrow h^2 = 32 \Rightarrow h = 4\sqrt{2}$$

$$\Rightarrow k = 2+4\sqrt{2}$$

$$\Rightarrow h+k = 2+8\sqrt{2} = a+b\sqrt{a}$$

$$\Rightarrow \boxed{a+b = 2+8 = 10}$$

EXERCISE

JM

EXERCISE (JM)

1. For a regular polygon, let r and R be the radii of the inscribed and the circumscribed circles. A false statement among the following is :- **[AIEEE-2010]**

(1) There is a regular polygon with $\frac{r}{R}=\frac{1}{2}$ (2) There is a regular polygon with $\frac{r}{R}=\frac{1}{\sqrt{2}}$

(3) There is a regular polygon with $\frac{r}{R}=\frac{2}{3}$ (4) There is a regular polygon with $\frac{r}{R}=\frac{\sqrt{3}}{2}$

sol:-

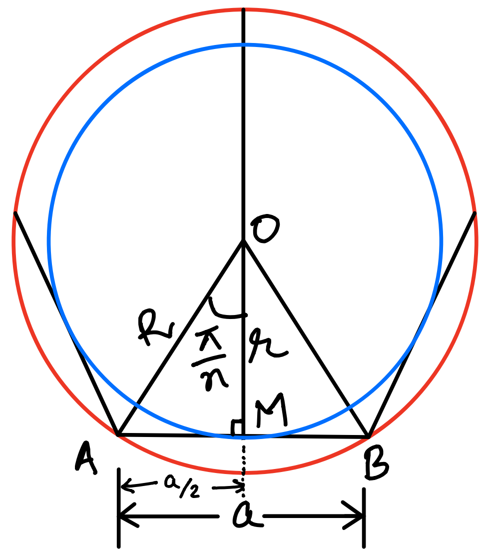
$r = \frac{a}{2}\cot\frac{\pi}{n}$

'a' is side of polygon.

$R = \frac{a}{2}\operatorname{cosec}\frac{\pi}{n}$

$\frac{r}{R} = \frac{\cot\frac{\pi}{n}}{\operatorname{cosec}\frac{\pi}{n}} = \cos\frac{\pi}{n}$

$\cos\frac{\pi}{n} \neq \frac{2}{3}$ for any $n \in N$.



2. The circle $x^2 + y^2 = 4x + 8y + 5$ intersects the line $3x - 4y = m$ at two distinct points if :-

[AIEEE-2010]

(1) $-85 < m < -35$ (2) $-35 < m < 15$ (3) $15 < m < 65$ (4) $35 < m < 85$

Solution:

Circle $x^2 + y^2 - 4x - 8y - 5 = 0$

Centre = (2, 4), Radius = $\sqrt{4+16+5} = 5$

If circle is intersecting line $3x - 4y = m$

at two distinct points.

$\Rightarrow$ length of perpendicular from centre < radius

$\Rightarrow \frac{|6-16-m|}{5} < 5$

$\Rightarrow |10 + m| < 25$

$\Rightarrow -25 < m + 10 < 25$

$\Rightarrow -35 < m < 15.$

(option (2))

3. The two circles $x^2 + y^2 = ax$ and $x^2 + y^2 = c^2$ $(c > 0)$ touch each other if :- **[AIEEE-2011]**

(1) $a = 2c$ (2) $|a| = 2c$ (3) $2|a| = c$ (4) $|a| = c$

Solution:

$x^2 + y^2 = ax$ ....(1)

$\Rightarrow$ centre $c_1\left(-\frac{a}{2}, 0\right)$ and radius $r_1 = \left|\frac{a}{2}\right|$

$x^2 + y^2 = c^2$ ....(2)

$\Rightarrow$ centre c_2 (0, 0) and radius $r_2 = c$

both touch each other iff

$|c_1c_2| = r_1 \pm r_2$

$$\frac{a^2}{4} = \left(\pm\frac{a}{2} \pm c\right)^2 \quad \Rightarrow \frac{a^2}{4} = \frac{a^2}{4} \pm |a|\,c + c^2 \quad \Rightarrow |a| = c$$

(option (4))

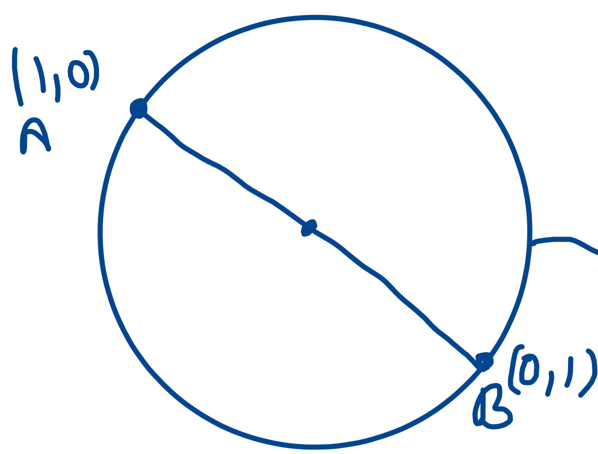
---

4. The equation of the circle passing through the points (1, 0) and (0, 1) and having the smallest radius is - **[AIEEE-2011]**

(1) $x^2 + y^2 + x + y - 2 = 0$ (2) $x^2 + y^2 - 2x - 2y + 1 = 0$

(3) $x^2 + y^2 - x - y = 0$ (4) $x^2 + y^2 + 2x + 2y - 7 = 0$

Solution:



Smallest circle is circle whose diameter is AB.

$(x-1)(x-0)+(y-0)(y-1)=0$

$\Rightarrow x^2+y^2-x-y=0$

option (3)

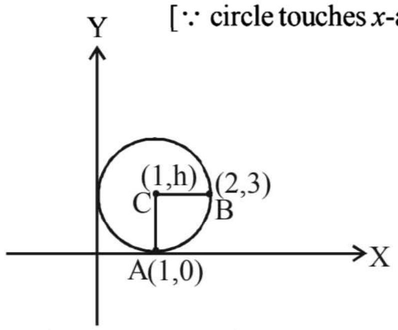
5. The length of the diameter of the circle which touches the x-axis at the point (1, 0) and passes through the point (2, 3) is : **[AIEEE-2012]**

(1) 5/3 (2) 10/3 (3) 3/5 (4) 6/5

Sotion:

Let centre of the circle be (1,h)

[$\because$ circle touches x-axis at (1,0)]



Let the circle passes through the point B (2,3)

$\therefore$ $CA = CB$ (radius)

$\Rightarrow$ $CA^2 = CB^2$

$\Rightarrow$ $(1-1)^2 + (h-0)^2 = (1-2)^2 + (h-3)^2$

$\Rightarrow$ $h^2 = 1 + h^2 + 9 - 6h \Rightarrow h = \frac{10}{6} = \frac{5}{3}$

Thus, diameter is $2h = \frac{10}{3}$. option (2)

---

6. The circle passing through (1, – 2) and touching the axis of x at (3, 0) also passes through the point : **[JEE (Main)-2013]**

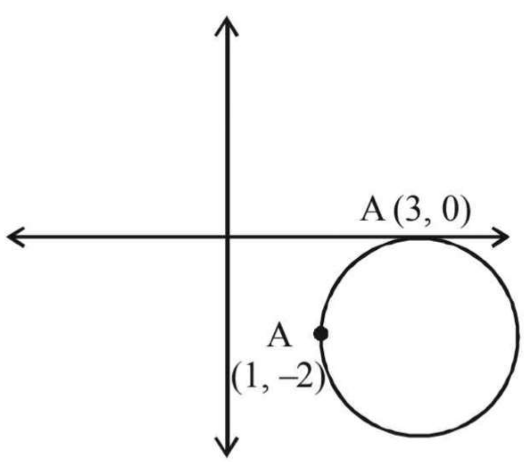
(1) (–5, 2) (2) (2, –5) (3) (5, –2) (4) (–2, 5)

Solution:

Since circle touches x-axis at (3, 0)

$\therefore$ The equation of circle be

$(x-3)^2 + (y-0)^2 + \lambda y = 0$



As it passes through (1, –2)

$\therefore$ Put $x = 1, y = -2$

$\Rightarrow$ $(1-3)^2 + (-2)^2 + \lambda(-2) = 0 \Rightarrow \lambda = 4$

$\therefore$ equation of circle is $(x-3)^2 + y^2 - 8 = 0$

Now, from the options (5, –2) satisfies equation of circle.

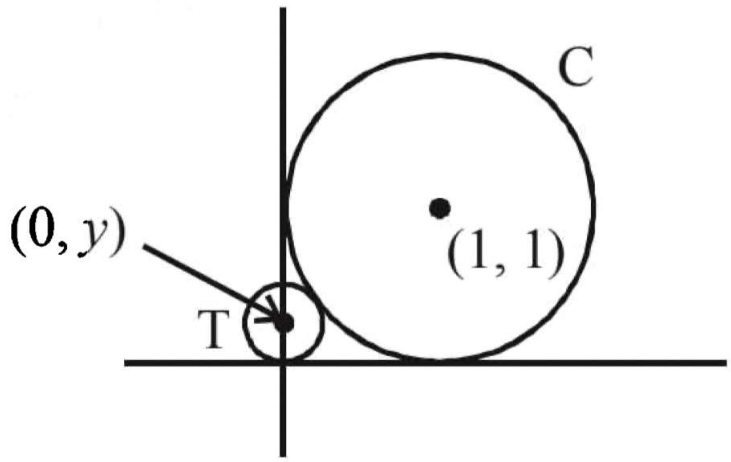
Option (3)

(7) Let C be the circle with centre at (1, 1) and radius = 1. If T is the circle centred at (0, y), passing through origin and touching the circle C externally, then the radius of T is equal to :

[JEE(Main)-2014]

(1) $\frac{\sqrt{3}}{\sqrt{2}}$ (2) $\frac{\sqrt{3}}{2}$ (3) $\frac{1}{2}$ (4) $\frac{1}{4}$

Solution:



Equation of circle $C \equiv (x-1)^2 + (y-1)^2 = 1$

Radius of T = $|y|$
T touches C externally
therefore,
Distance between the centres = sum of their radii

$\Rightarrow \quad \sqrt{(0-1)^2 + (y-1)^2} = 1 + |y|$
$\Rightarrow \quad (0-1)^2 + (y-1)^2 = (1+|y|)^2$
$\Rightarrow \quad 1 + y^2 + 1 - 2y = 1 + y^2 + 2|y|$
$2|y| = 1 - 2y$

If $y > 0$ then $2y = 1 - 2y \Rightarrow y = \frac{1}{4}$

If $y < 0$ then $-2y = 1 - 2y \Rightarrow 0 = 1$
(not possible)

$\therefore \quad y = \frac{1}{4}$

option(4)

(8) The number of common tangents to the circle
$x^2 + y^2 - 4x - 6y - 12 = 0$ and $x^2 + y^2 + 6x + 18y + 26 = 0$, is : **[JEE(Main)-2015]**

(1) 3 (2) 4 (3) 1 (4) 2

Solution:

$x^2 + y^2 - 4x - 6y - 12 = 0$ ...(i)

Centre, $C_1 = (2, 3)$ and Radius, $r_1 = 5$ units

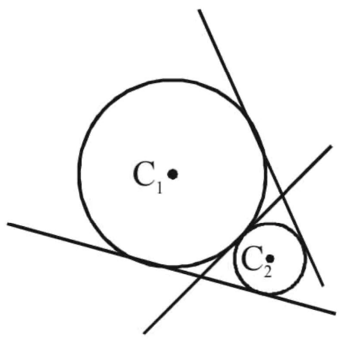
$x^2 + y^2 + 6x + 18y + 26 = 0$ ...(ii)

Centre, $C_2 = (-3, -9)$ and Radius, $r_2 = 8$ units

$C_1C_2 = \sqrt{(2+3)^2 + (3+9)^2} = 13$ units

$r_1 + r_2 = 5 + 8 = 13$

$\therefore C_1C_2 = r_1 + r_2$

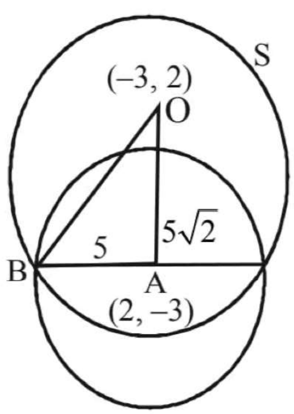


Therefore there are three common tangents. option (1)

(9) If one of the diameters of the circle, given by the equation, $x^2 + y^2 - 4x + 6y - 12 = 0$, is a chord of a circle S, whose centre is at $(-3, 2)$, then the radius of S is :- **[JEE(Main)-2016]**

(1) 10 (2) $5\sqrt{2}$ (3) $5\sqrt{3}$ (4) 5

Solution:



Centre of S : O (–3, 2) centre of given circle A(2, –3)

$\Rightarrow OA = 5\sqrt{2}$

Also AB = 5 ($\because$ AB = r of the given circle)

$\Rightarrow$ Using pythagoras theorem in ΔOAB

$r = 5\sqrt{3}$

option (3)

(10) The centres of those circles which touch the circle, $x^2 + y^2 - 8x - 8y - 4 = 0$, externally and also touch the x-axis, lie on :- **[JEE(Main)-2016]**

(1) A parabola
(2) A circle
(3) An ellipse which is not a circle
(4) A hyperbola

Solution: ∵ $x^2 + y^2 - 8x - 8y - 4 = 0$

Centre (4, 4)

Radius $= \sqrt{4^2 + 4^2 + 4} = 6$

Let centre of the circle is (h, k)

$\sqrt{(h-4)^2 + (k-4)^2} = (6 + k)$

$(h-4)^2 + (k-4)^2 = (6+k)^2$

$h^2 - 8h + 16 + k^2 - 8k + 16 = 36 + k^2 + 12k$

$h^2 - 8h - 20k - 4 = 0$

$x^2 - 8x - 20y - 4 = 0$

Which is an equation of parabola

Option(1)

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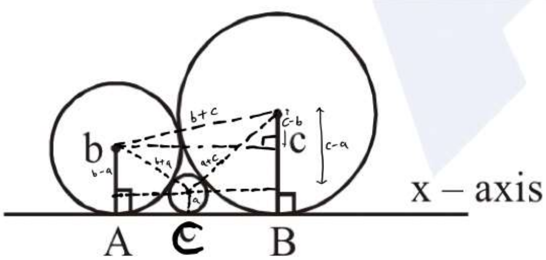
(11) Three circles of radii a, b, c(a < b < c) touch each other externally. If they have x-axis as a common tangent, then : **[JEE(Main)-2019]**

(1) $\frac{1}{\sqrt{a}} = \frac{1}{\sqrt{b}} + \frac{1}{\sqrt{c}}$
(2) a, b, c are in A.P.
(3) $\sqrt{a}, \sqrt{b}, \sqrt{c}$ are in A.P.
(4) $\frac{1}{\sqrt{b}} = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{c}}$

Sol.



AB = AC + CB

$\Rightarrow \sqrt{(b+c)^2 - (b-c)^2}$

$= \sqrt{(b+a)^2 - (b-a)^2} + \sqrt{(a+c)^2 - (a-c)^2}$

$\Rightarrow \sqrt{bc} = \sqrt{ab} + \sqrt{ac}$

$\Rightarrow \frac{1}{\sqrt{a}} = \frac{1}{\sqrt{c}} + \frac{1}{\sqrt{b}}$

Option (1)

(12) If a circle C passing through the point (4,0) touches the circle $x^2 + y^2 + 4x - 6y = 12$ externally at the point (1, –1), then the radius of C is : **[JEE(Main)-2019]**

(1) $\sqrt{57}$ (2) 4 (3) $2\sqrt{5}$ (4) 5

Sol. $x^2 + y^2 + 4x - 6y - 12 = 0$

Equation of tangent at (1, –1)

$x - y + 2(x + 1) - 3(y - 1) - 12 = 0$

$3x - 4y - 7 = 0$

$\therefore$ Equation of circle is

$(x^2 + y^2 + 4x - 6y - 12) + \lambda(3x - 4y - 7) = 0$

It passes through (4, 0) :

$(16 + 16 - 12) + \lambda(12 - 7) = 0$

$\Rightarrow 20 + \lambda(5) = 0$

$\Rightarrow \lambda = -4$

$\therefore (x^2 + y^2 + 4x - 6y - 12) - 4(3x - 4y - 7) = 0$

or $x^2 + y^2 - 8x + 10y + 16 = 0$

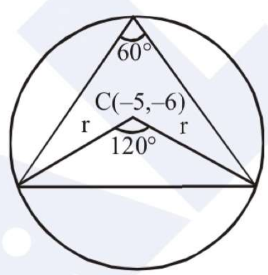
Radius $= \sqrt{16 + 25 - 16} = 5$

---

(13) If the area of an equilateral triangle inscribed in the circle, $x^2 + y^2 + 10x + 12y + c = 0$ is $27\sqrt{3}$ sq. units then c is equal to : **[JEE(Main)-2019]**

(1) 20 (2) 25 (3) 13 (4) –25

Sol. $3\left(\frac{1}{2}r^2.\sin 120°\right) = 27\sqrt{3}$



$\frac{r^2}{2}\frac{\sqrt{3}}{2} = \frac{27\sqrt{3}}{3}$

$r^2 = \frac{108}{3} = 36$

Radius $= \sqrt{25 + 36 - C} = \sqrt{36}$

$\boxed{C = 25}$

$\therefore$ Option (2)

(14) A square is inscribed in the circle $x^2 + y^2 - 6x + 8y - 103 = 0$ with its sides parallel to the corrdinate axes. Then the distance of the vertex of this square which is nearest to the origin is :- **[JEE(Main)-2019]**

(1) 13 (2) $\sqrt{137}$ (3) 6 (4) $\sqrt{41}$

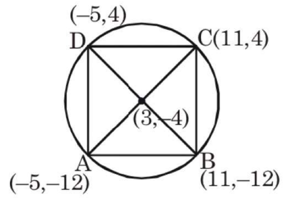
Sol. $R = \sqrt{9+16+103} = 8\sqrt{2}$

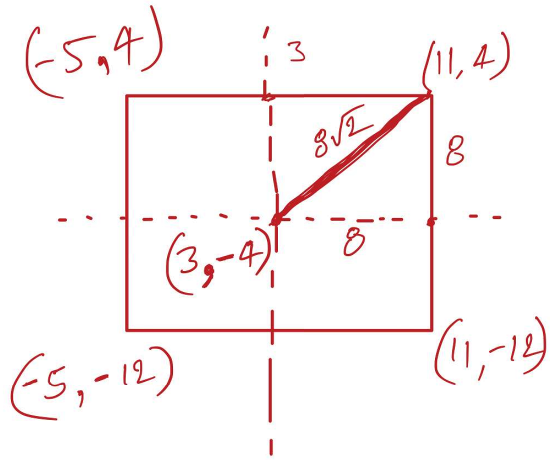
OA = 13

$OB = \sqrt{265}$

$OC = \sqrt{137}$

$OD = \sqrt{41}$





(15) A circle cuts a chord of length 4a on the x-axis and passes through a point on the y-axis, distant 2b from the origin. Then the locus of the centre of this circle, is :- **[JEE(Main)-2019]**

(1) A hyperbola (2) A parabola (3) A straight line (4) An ellipse

Sol. Let equation of circle is

$x^2 + y^2 + 2fx + 2fy + e = 0$, it passes through $(0, 2b)$

$\Rightarrow 0 + 4b^2 + 2g \times 0 + 4f + c = 0$

$\Rightarrow 4b^2 + 4f + c = 0$ ...(i)

$2\sqrt{g^2 - c} = 4a$ ...(ii)

$g^2 - c = 4a^2 \Rightarrow c = \left(g^2 - 4a^2\right)$

Putting in equation (1)

$\Rightarrow 4b^2 + 4f + g^2 - 4a^2 = 0$ $\{$put $g = -x$, $f = -y\}$

$\Rightarrow x^2 + 4y + 4(b^2 - a^2) = 0$, it represent a parabola.

(16) If a variable line, $3x + 4y - \lambda = 0$ is such that the two circles $x^2 + y^2 - 2x - 2y + 1 = 0$ and $x^2 + y^2 - 18x - 2y + 78 = 0$ are on its opposite sides, then the set of all values of λ is the interval :-

[JEE(Main)-2019]

(1) [12, 21] (2) (2, 17) (3) (23, 31) (4) [13, 23]

Sol. Centre of circles are opposite side of line

$(3 + 4 - \lambda)(27 + 4 - \lambda) < 0$

$(\lambda - 7)(\lambda - 31) < 0$

$\lambda \in (7, 31)$ —— (1)

distance from S_1

$\left|\frac{3+4-\lambda}{5}\right| \geq 1 \Rightarrow \lambda \in (-\infty, 2] \cup [(12, \infty]$ —— (2)

distance from S_2

$\left|\frac{27+4-\lambda}{5}\right| \geq 2 \Rightarrow \lambda \in (-\infty, 21] \cup [41, \infty)$ —— (3)

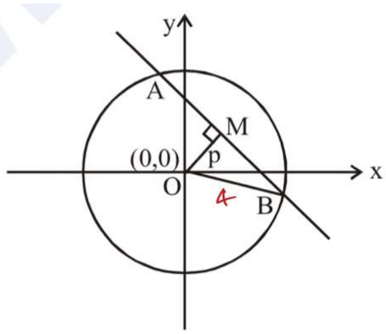
so $\lambda \in [12, 21]$ $\{(1) \cap (2) \cap (3)\}$

---

(17) The sum of the squares of the lengths of the chords intercepted on the circle, $x^2 + y^2 = 16$, by the lines, $x + y = n$, $n \in N$, where N is the set of all natural numbers, is : **[JEE(Main)-2019]**

(1) 320 (2) 160 (3) 105 (4) 210

Sol.



$(AB)^2 = \ell^2 < \boxed{64}$ ⇓ Diameter

$p = \frac{n}{\sqrt{2}}$, but $\frac{n}{\sqrt{2}} < 4 \Rightarrow n = 1, 2, 3, 4, 5.$

Length of chord AB $= 2\sqrt{16 - \frac{n^2}{2}}$

$= \sqrt{64 - 2n^2} = \ell(\text{say})$

For $n = 1, \ell^2 = 62$

$n = 2, \ell^2 = 56$

$n = 3, \ell^2 = 46$

$n = 4, \ell^2 = 32$

$n = 5, \ell^2 = 14$

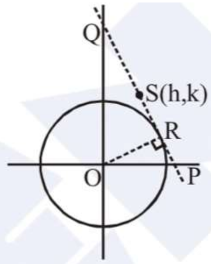
$\therefore$ Required sum $= 62 + 56 + 46 + 32 + 14 = 210$

(18) If a tangent to the circle $x^2 + y^2 = 1$ intersects the coordinate axes at distinct points P and Q, then the locus of the mid-point of PQ is **[JEE(Main)-2019]**

(1) $x^2 + y^2 - 2xy = 0$ (2) $x^2 + y^2 - 16x^2y^2 = 0$

(3) $x^2 + y^2 - 4x^2y^2 = 0$ (4) $x^2 + y^2 - 2x^2y^2 = 0$

Sol.



Let the mid point be S(h,k)

$\therefore$ P(2h,0) and Q(0,2k)

equation of PQ : $\frac{x}{2h} + \frac{y}{2k} = 1$

$\because$ PQ is tangent to circle at R(say)

$\therefore$ OR = 1 $\Rightarrow \left|\frac{-1}{\sqrt{\left(\frac{1}{2h}\right)^2 + \left(\frac{1}{2k}\right)^2}}\right| = 1$

$\Rightarrow \frac{1}{4h^2} + \frac{1}{4k^2} = 1$

$\Rightarrow x^2 + y^2 - 4x^2y^2 = 0$

Aliter :

tangent to circle

$x\cos\theta + y\sin\theta = 1$

P : $(\sec\theta, 0)$

Q : $(0, \operatorname{cosec}\theta)$

$2h = \sec\theta \Rightarrow \cos\theta = \frac{1}{2h}$ & $\sin\theta = \frac{1}{2k}$

(19) The common tangent to the circles $x^2 + y^2 = 4$ and $x^2 + y^2 + 6x + 8y - 24 = 0$ also passes through the point :- **[JEE(Main)-2019]**

(1) (–4, 6) (2) (6, –2) (3) (–6, 4) (4) (4, –2)

Sol. Circle touches internally

$C_1(0, 0)$; $r_1 = 2$

$C_2 : (-3, -4)$; $r_2 = 7$

$C_1C_2 = |r_1 - r_2|$

$S_1 - S_2 = 0 \Rightarrow$ eqn. of common tangent

$6x + 8y - 20 = 0$

$3x + 4y = 10$

(6, –2) satisfy it

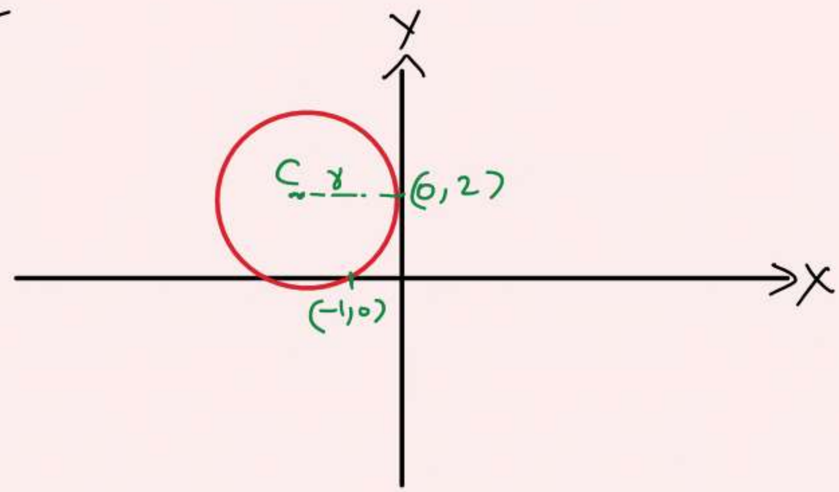
EXERCISE

JA

(1) The circle passing through the point (–1,0) and touching the y-axis at (0,2) also passes through the point - **[JEE 2011, 3M, –1M]**

(A) $\left(-\frac{3}{2},0\right)$ (B) $\left(-\frac{5}{2},2\right)$ (C) $\left(-\frac{3}{2},\frac{5}{2}\right)$ (D) (–4,0)

2 Sol:-



Centre lies on $y=2$ and also the perpendicular bisector of $A(-1,0)$ and $B(0,2)$

Slope of $AB = 2$

Midpoint of $AB = \left(\frac{-1}{2}, 1\right)$

$\Rightarrow$ Perpendicular bisector of AB is $y-1 = \frac{-1}{2}\left(x+\frac{1}{2}\right)$

$\Rightarrow 2y-2 = -x-\frac{1}{2}$

$\Rightarrow x+2y-\frac{3}{2} = 0$ —(1)

C lies on (1) and $y=2$

$\therefore C = \left(\frac{-5}{2}, 2\right)$

$\Rightarrow$ Radius $= \frac{5}{2}$

Out of the given options, we need to find the point through which circle passes.

A) $\left(-\frac{3}{2}, 0\right) = P$

$$CP = \sqrt{1^2 + 2^2} = \sqrt{5} \neq \text{radius}$$

B) $\left(-\frac{5}{2}, 2\right)$ is the centre itself.

C) $\left(-\frac{3}{2}, \frac{5}{2}\right) = P$

$$CP = \sqrt{1^2 + \left(\frac{1}{2}\right)^2} = \frac{\sqrt{5}}{2} \neq \text{radius}$$

D) $(-4, 0) = P$

$$CP = \sqrt{\left(\frac{3}{2}\right)^2 + 2^2} = \sqrt{\frac{25}{4}} = \frac{5}{2} = \text{radius}$$

$\therefore$ $(-4, 0)$ lies on the circle

Altier :- Eqⁿ of Circle touching line $x=0$ at $(0,2)$ is

$$S_1 + \lambda L = 0$$

where S_1 = point circle = $x^2+(y-2)^2=0$

& $L = x = 0$

$\therefore$ reqd. circle $\equiv x^2+(y-2)^2+\lambda(x)=0$

it passes th. $(-1,0)$

$\Rightarrow \quad (-1)^2+(0-2)^2+\lambda(-1)=0 \Rightarrow \boxed{\lambda=5}$

$\therefore$ reqd. circle $= x^2+(y-2)^2+5x=0$

$\Rightarrow \quad x^2+y^2+5x-4y+4=0$

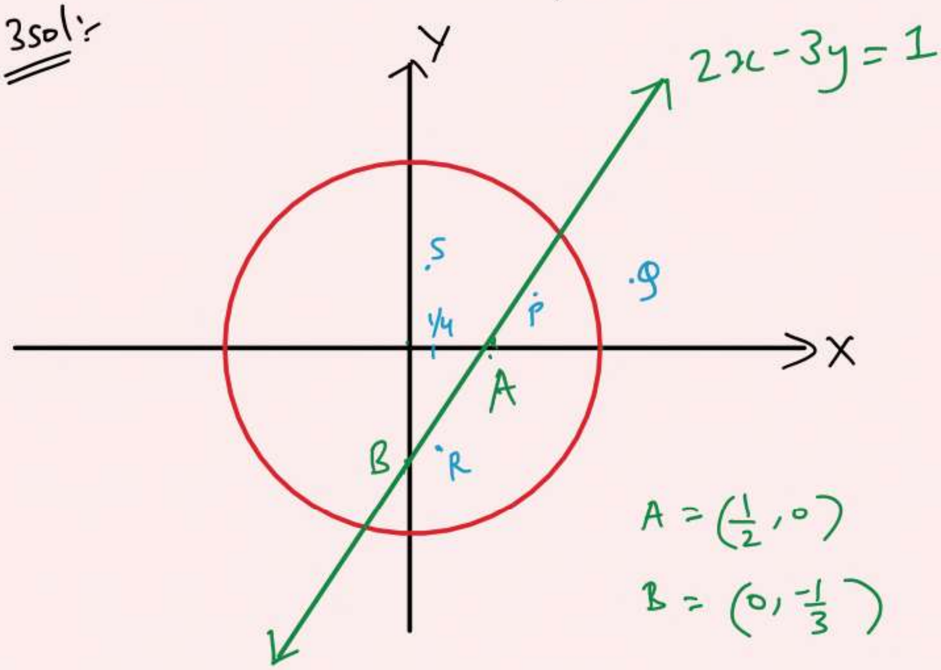
Now chk options

(2) The straight line $2x - 3y = 1$ divides the circular region $x^2 + y^2 \le 6$ into two parts. If

$$S = \left\{\left(2, \frac{3}{4}\right), \left(\frac{5}{2}, \frac{3}{4}\right), \left(\frac{1}{4}, -\frac{1}{4}\right), \left(\frac{1}{8}, \frac{1}{4}\right)\right\},$$

then the number of point(s) in S lying inside the smaller part is **[JEE 2011, 4M]**

Sol:-



$x^2 + y^2 \le 6$, Line is L: $2x - 3y = 1$

$\Rightarrow y = \frac{2x-1}{3}$

(i) $\left(2, \frac{3}{4}\right) = P$

at $x = 2$, Corresponding point on the line whose x coordinate is 2 is $y = \frac{2(2)-1}{3} = 1 > \frac{3}{4}$

$(2, 1)$

$\therefore \left(2, \frac{3}{4}\right)$ lies in the smaller region.

(ii) $\left(\frac{5}{2}, \frac{3}{4}\right) = Q$

Here $\frac{5}{2} > \sqrt{6}$

$\Rightarrow Q$ is outside the circle

(iii) $\left(\frac{1}{4}, \frac{-1}{4}\right) = R$

at $x = \frac{1}{4}$, $y = \dfrac{2\left(\frac{1}{4}\right) - 1}{3} = \frac{-1}{6} < \frac{-1}{4}$

$\therefore \left(\frac{1}{4}, \frac{-1}{4}\right)$ lies inside smaller region

(iv) $\left(\frac{1}{8}, \frac{1}{4}\right) = S$

at $x = \frac{1}{8}$, $y = \dfrac{2\left(\frac{1}{8}\right) - 1}{3} = -\frac{1}{4} < \frac{1}{4}$

$\therefore \left(\frac{1}{8}, \frac{1}{4}\right)$ lies outside smaller region.

Hence number of points in smaller region is 2

(P and R)

(3) The locus of the mid-point of the chord of contact of tangents drawn from points lying on the straight line $4x - 5y = 20$ to the circle $x^2 + y^2 = 9$ is- **[JEE 2012, 3M, –1M]**

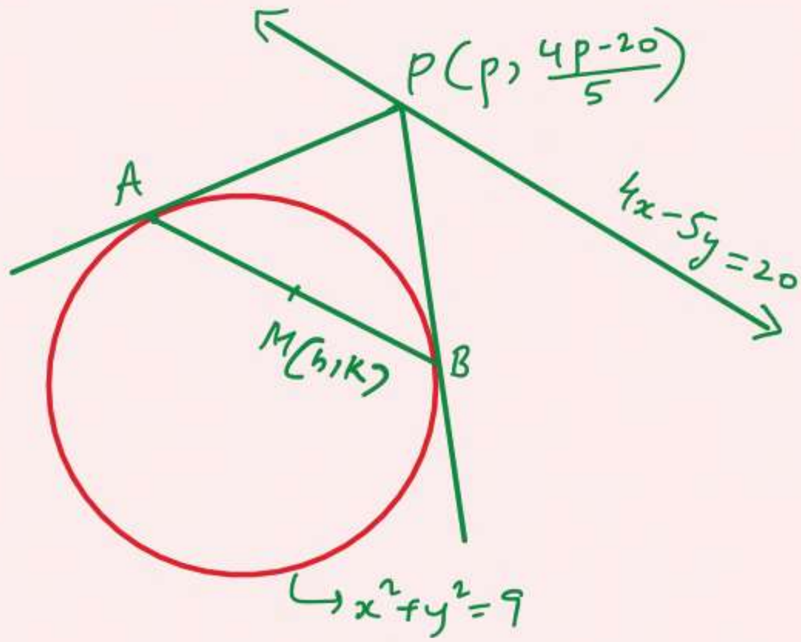
(A) $20(x^2 + y^2) - 36x + 45y = 0$

(B) $20(x^2 + y^2) + 36x - 45y = 0$

(C) $36(x^2 + y^2) - 20x + 45y = 0$

(D) $36(x^2 + y^2) + 20x - 45y = 0$

Sol:-



Let arbitary point on $4x-5y=20$ is $P\left(p, \frac{4p-20}{5}\right)$

Chord of contact of P is $T=0$

$$\Rightarrow x(p) + y\left(\frac{4p-20}{5}\right) - 9 = 0 \quad \text{— (1)}$$

Equation of AB with $M(h,k)$ as middle point is $T = S_1$

$$\Rightarrow x(h) + y(k) - \cancel{9} = h^2 + k^2 - \cancel{9}$$

$$\Rightarrow x(h) + y(k) - (h^2 + k^2) = 0 \quad \text{— (2)}$$

1, 2 represents the same equation

i.e AB

$$\Longrightarrow \frac{h}{p} = \frac{k}{\frac{4p-20}{5}} = \frac{h^2+k^2}{9}$$

$$p = \frac{9h}{h^2+k^2}$$

Eliminating p, we get

$$9k = \frac{(h^2+k^2)}{5}(4p-20)$$

$$\Longrightarrow 9k = \frac{h^2+k^2}{5}\left(\frac{36h}{h^2+k^2} - 20\right)$$

$$\Longrightarrow 45k = 36h - 20(h^2+k^2)$$

$\therefore$ Locus is $\boxed{20(x^2+y^2) - 36x + 45y = 0}$

Paragraph for Question 4 and 5

A tangent PT is drawn to the circle $x^2 + y^2 = 4$ at the point $P(\sqrt{3}, 1)$. A straight line L, perpendicular to PT is a tangent to the circle $(x-3)^2 + y^2 = 1$.

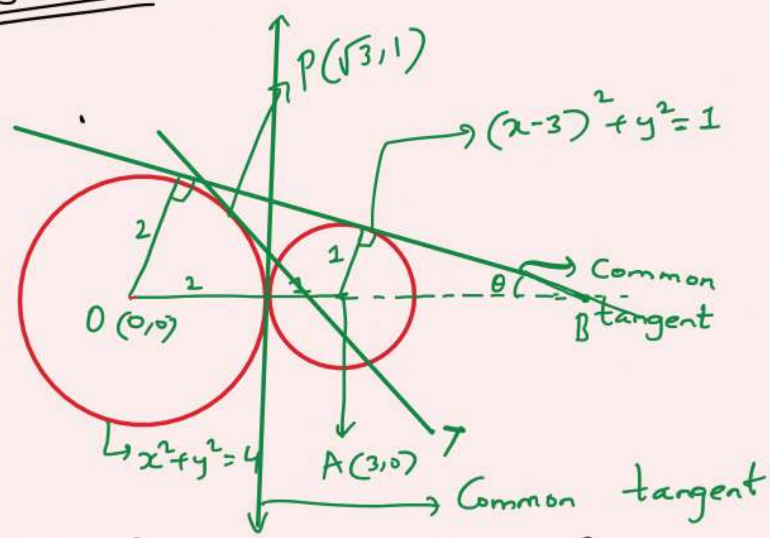
(4) A common tangent of the two circles is **[JEE 2012, 3M, –1M]**

(A) $x = 4$ (B) $y = 2$ (C) $x + \sqrt{3}y = 4$ (D) $x + 2\sqrt{2}y = 6$

(5) A possible equation of L is **[JEE 2012, 3M, –1M]**

(A) $x - \sqrt{3}y = 1$ (B) $x + \sqrt{3}y = 1$ (C) $x - \sqrt{3}y = -1$ (D) $x + \sqrt{3}y = 5$

Paragraph 5, 6



5 sol:

From figure, $\sin\theta = \frac{1}{AB} = \frac{2}{OB}$

$\Rightarrow \frac{1}{AB} = \frac{2}{OA + AB}$

$\Rightarrow \frac{1}{AB} = \frac{2}{3 + AB}$

$\Rightarrow AB = 3$

$\Rightarrow \sin\theta = \frac{1}{3} \Rightarrow \tan\theta = \pm\frac{1}{2\sqrt{2}}$

$OA = AB = 3 \Rightarrow A = \frac{O + B}{2}$

$\Rightarrow B = 2A - O$

$B = (6, 0)$

$\therefore$ Slope common tangent $= \pm\frac{1}{2\sqrt{2}}$

it passes through $B(6,0)$

$\Rightarrow$ Equation of common tangent is

$$y = \pm\frac{1}{2\sqrt{2}}(x-6)$$

$$\Rightarrow \boxed{x \pm 2\sqrt{2}y = 6}$$

Also another common tangent is

|| to Y axis through $(2,0)$

$$\Rightarrow \boxed{x=2}$$

6 sol:- L is ⊥ar to tangent at P

Slope of $OP = \frac{1}{\sqrt{3}}$

$\Rightarrow$ Slope of $L = \frac{1}{\sqrt{3}}$

$\Rightarrow$ Let the equation of L be

$L: x-\sqrt{3}y = c$

L is tangent to $(x-3)^2+y^2=1$

$\Rightarrow$ ⊥ar distance from $(3,0)$ to $L = 1$

$$\Rightarrow \frac{|3-c|}{\sqrt{1^2+(\sqrt{3})^2}} = 1$$

$\Rightarrow |3-c| = 2 \Rightarrow c = 1, 5$

$\therefore$ Possible equation of L is

$$\boxed{x-\sqrt{3}y = 1}$$

(6) Circle(s) touching x-axis at a distance 3 from the origin and having an intercept of length $2\sqrt{7}$ on y-axis is (are) **[JEE(Advanced) 2013, 3, (–1)]**

(A) $x^2 + y^2 - 6x + 8y + 9 = 0$ (B) $x^2 + y^2 - 6x + 7y + 9 = 0$

(C) $x^2 + y^2 - 6x - 8y + 9 = 0$ (D) $x^2 + y^2 - 6x - 7y + 9 = 0$

Sol: Circle touches X axis at 3 units from origin.

$\Rightarrow$ Let centre be $(3, \pm r)$ where r is the radius of circle.

$\Rightarrow$ Equation of circle is

$(x-3)^2 + (y \pm r)^2 = r^2$

$\Rightarrow x^2+y^2 - 6x \pm 2ry + 9 = 0$

Y intercept $= 2\sqrt{7}$

$\Rightarrow 2\sqrt{f^2-c} = 2\sqrt{7}$

$\Rightarrow r^2 - 9 = 7$

$\Rightarrow r = 4$

$\Rightarrow$ $\boxed{\begin{array}{c} x^2+y^2-6x+8y+9=0 \\ \text{(OR)} \\ x^2+y^2-6x-8y+9=0 \end{array}}$

(7) A circle S passes through the point (0, 1) and is orthogonal to the circles $(x-1)^2 + y^2 = 16$ and $x^2 + y^2 = 1$. Then :- **[JEE(Advanced)-2014, 3]**

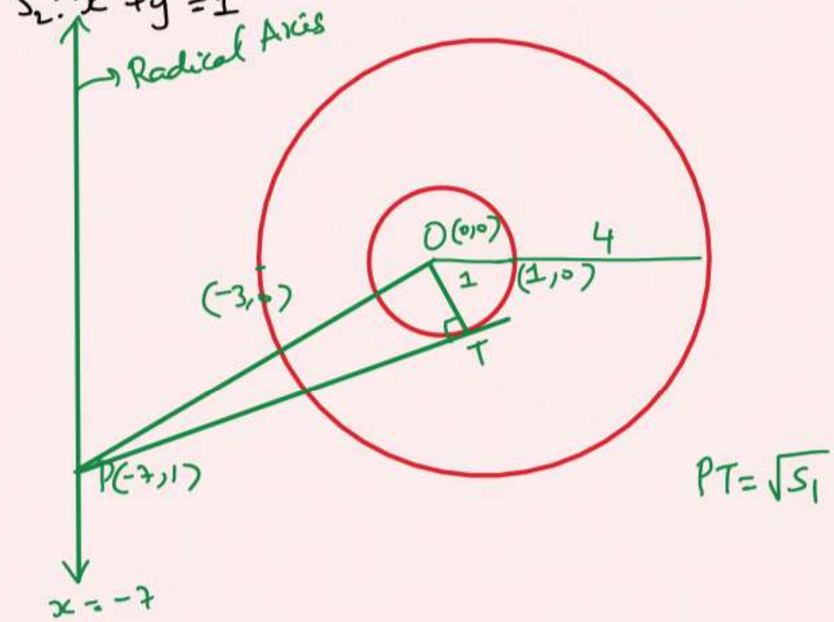
(1) radius of S is 8

(B) radius of S is 7

(3) centre of S is (–7, 1)

(D) centre is S is (–8, 1)

Sol:- S passes through (0,1) and is orthogonal to $S_1 : (x-1)^2 + y^2 = 16$ and $S_2 : x^2 + y^2 = 1$



Radical axis of S_1 and S_2 is

$S_1 - S_2 = 0$

$\Rightarrow -2x - 14 = 0 \Rightarrow x = -7$

$\because$ Required circle is orthogonal to S_1 and S_2, its centre lies on radical axis of S_1 and S_2 which is $x = -7$

Let centre be $(-7, k)$

$\Rightarrow$ Radius, $r = \sqrt{(-7)^2 + (k-1)^2}$

[$\because$ o/c passes through (0,1)]

Also $d^2 = r_1^2 + r_2^2$ (Orthogonality condition)

$\Rightarrow PO^2 = r^2 + 1^2$

$\Rightarrow PT^2 + OT^2 = r^2 + 1^2$

$\Rightarrow S_1 + \cancel{1^2} = r^2 + \cancel{1^2}$

$\Rightarrow \cancel{(-7)^2} + K^2 - 1 = \cancel{(-7)^2} + (K-1)^2$

$\Rightarrow K^2 - 1 = K^2 - 2K + 1$

$\Rightarrow K = 1$

$\Rightarrow \boxed{\text{Centre} = (-7, 1)}$

Radius, $r = \sqrt{(-7)^2 + (1-1)^2}$

$$\boxed{r = 7}$$

Altier :- let the required circle be

$$x^2+y^2+2gx+2fy+c=0 \quad \text{———— (1)}$$

$\Rightarrow$ It passes th. $(0,1)$ so

$$0^2+1^2+2g\cdot(0)+2f(1)+c=0$$

$$\Rightarrow 1+2f+c=0 \quad \text{———— (2)}$$

$\Rightarrow$ Also (1) is orthogonal to $x^2+y^2=1$ &

$x^2+y^2-2x-15=0$, so

$$2(g)0+2f(0)=c-1 \Rightarrow c=1 \quad \text{———— (3)}$$

& $$2(g)(-1)+2f(0)=c-15 \Rightarrow 2g+c=15 \quad \text{———— (4)}$$

Solving (2), (3) & (4) we get $c=1$, $g=7$, $f=-1$

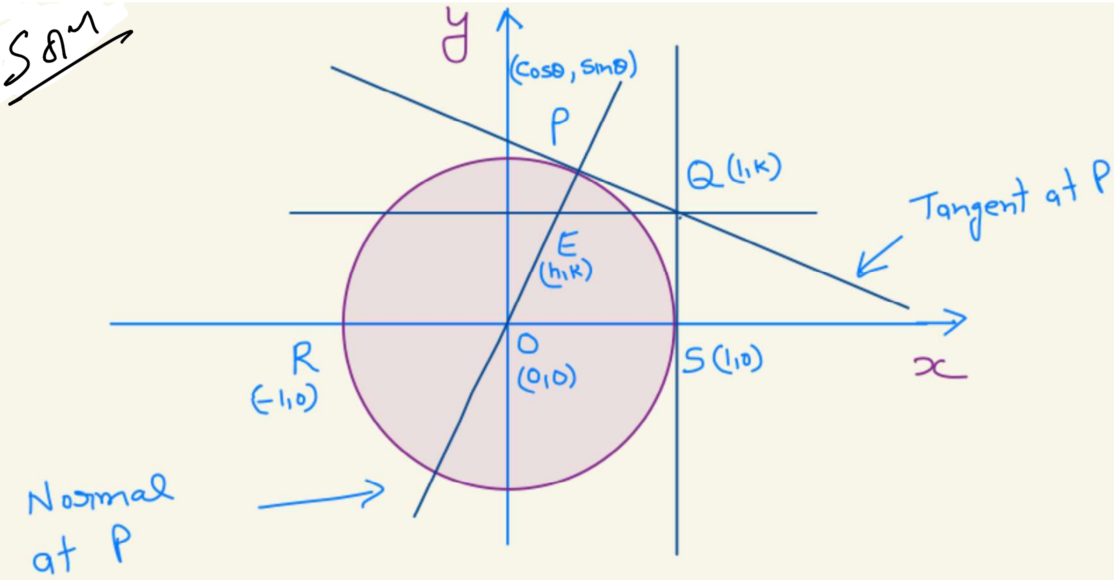
$\therefore$ reqd circle is

$$x^2+y^2+14x-2y+1=0$$

(8) Let RS be the diameter of the circle $x^2 + y^2 = 1$, where S is the point (1,0). Let P be a variable point (other than R and S) on the circle and tangents to the circle at S and P meet at the point Q. The normal to the circle at P intersects a line drawn through Q parallel to RS at point E. then the locus of E passes through the point(s)- **[JEE(Advanced)-2016, 4(–2)]**

(A) $\left(\frac{1}{3}, \frac{1}{\sqrt{3}}\right)$ (B) $\left(\frac{1}{4}, \frac{1}{2}\right)$ (C) $\left(\frac{1}{3}, -\frac{1}{\sqrt{3}}\right)$ (D) $\left(\frac{1}{4}, -\frac{1}{2}\right)$

Sol



Let $E = (h, K)$

$EQ \parallel RS \Rightarrow Q = (1, K)$

Let $P = (\cos\theta, \sin\theta)$

Tangent at P is $x(\cos\theta) + y(\sin\theta) = 1$

It passes through $Q(1, K)$

$\Rightarrow \cos\theta + K\sin\theta = 1$ — (1)

Also P, E, O are collinear

$\Rightarrow \dfrac{\sin\theta - K}{\cos\theta - h} = \dfrac{K}{h}$ — (2)

To get locus, we need to eliminate

θ from (1), (2)

(1) $\Rightarrow \cos\theta = 1 - k\sin\theta$

(2) $\Rightarrow h(\sin\theta - k) = k(\cos\theta - h)$

$\Rightarrow h\sin\theta - \cancel{hk} = k(1 - k\sin\theta) - \cancel{hk}$

$\Rightarrow \sin\theta(h + k^2) = k$

$\Rightarrow \sin\theta = \dfrac{k}{h+k^2}$

$\Rightarrow \cos\theta = 1 - k\sin\theta = 1 - \dfrac{k^2}{h+k^2}$

$\cos\theta = \dfrac{h}{h+k^2}$

$\sin^2\theta + \cos^2\theta = 1$

$\Rightarrow \dfrac{k^2 + h^2}{(h+k^2)^2} = 1$

$\Rightarrow k^2 + \cancel{h^2} = \cancel{h^2} + k^4 + 2hk^2$

$\Rightarrow$ Locus is $k^2 + 2h = 1$

$\Rightarrow \boxed{y^2 + 2x = 1}$

Clearly the locus passes through

$\left(\frac{1}{3}, \frac{1}{\sqrt{3}}\right)$ and $\left(\frac{1}{3}, -\frac{1}{\sqrt{3}}\right)$ among

the given options

(9) For how many values of p, the circle $x^2 + y^2 + 2x + 4y - p = 0$ and the coordinate axes have exactly three common points ? **[JEE(Advanced)-2017, 3]**

Sol:- Circle $x^2+y^2+2x+4y-p=0$

Circle and coordinate axes should have exactly 3 common points.

X intercept $= 2\sqrt{g^2-c}$

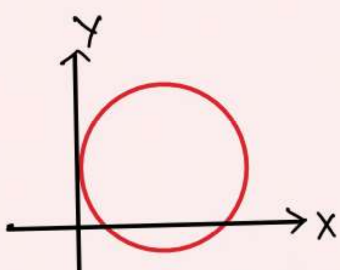
$= 2\sqrt{1+p} > 0$ if $p > -1$

Y intercept $= 2\sqrt{f^2-c}$

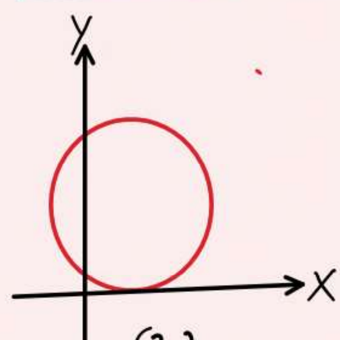
$= 2\sqrt{2^2+p}$

$= 2\sqrt{4+p} > 0$ if $p > -4$

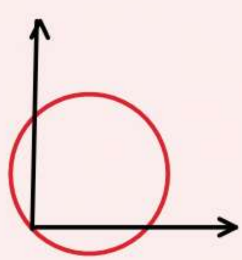
Cases for exactly 3 points common



(1)



(2)



(3)

For case 1, Y intercept = 0 $\Rightarrow p = -4$

But if $p = -4$, X intercept $\nexists$ $\times$ Rejected

$\therefore$ Only 1 point in common

i.e Case 1 is not possible

For case 2, X intercept = 0 $\Rightarrow \boxed{p = -1}$

If $p = -1$, Y intercept > 0

$\therefore$ 3 points in common

For case 3, $\boxed{p = 0}$ (To pass through origin)

If $p = 0$, Both X and Y intercepts are positive

$\therefore$ 3 points in common

Hence $p = -1, p = 0$ are the two possible values of p.

Paragraph "X"

(10) Let S be the circle in the xy-plane defined by the equation $x^2 + y^2 = 4$.

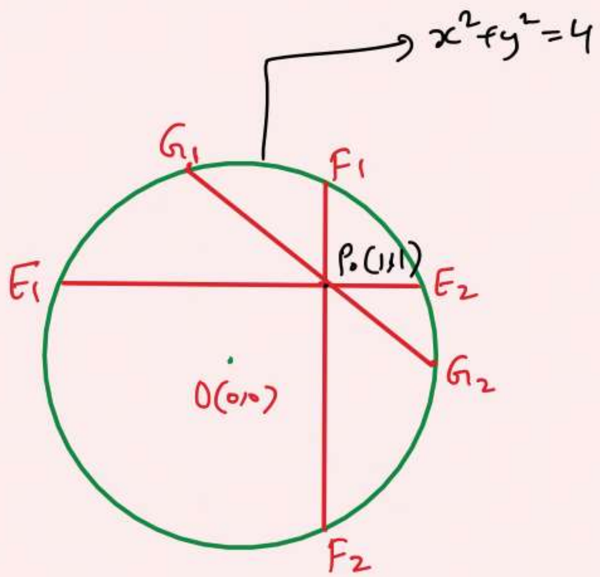
(*There are two question based on Paragraph "X", the question given below is one of them*)

Let E_1E_2 and F_1F_2 be the chord of S passing through the point $P_0(1, 1)$ and parallel to the x-axis and the y-axis, respectively. Let G_1G_2 be the chord of S passing through P_0 and having slop -1. Let the tangents to S at E_1 and E_2 meet at E_3, the tangents of S at F_1 and F_2 meet at F_3, and the tangents to S at G_1 and G_2 meet at G_3. Then, the points E_3, F_3 and G_3 lie on the curve **[JEE(Advanced)-2018, 3(–1)]**

(A) $x + y = 4$ (B) $(x - 4)^2 + (y - 4)^2 = 16$

(C) $(x - 4)(y - 4) = 4$ (D) $xy = 4$

Paragraph 11, 12



The information given in the question is illustrated in the figure.

Also given, tangents at E_1, E_2 meet at E_3; at G_1, G_2 meet at G_3; at F_1, F_2 meet at F_3.

The points E_3, F_3, G_3 lie on the polar of the point $(1,1)$ which is $T = 0$

$\Rightarrow \boxed{x + y = 4}$

ALITER:-

Let $Q(h,k)$ be a point outside the circle, the chord of contact of Q passes through $P_0(1,1)$.

Chord of contact is $T=0$

$\Rightarrow$ $x(h)+y(k)=4$ - passes through $(1,1)$

$\Rightarrow$ $h+k=4$

$\Rightarrow$ Locus of Q is $x+y=4$

$\Rightarrow$ E_3, F_3, G_3 lies on $\boxed{x+y=4}$

Paragraph "X"

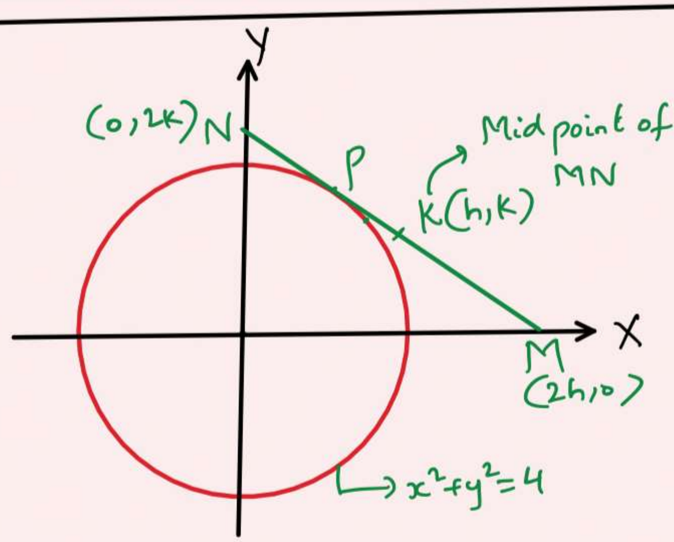
(11) Let S be the circle in the xy-plane defined by the equation $x^2 + y^2 = 4$.

(*There are two question based on Paragraph "X", the question given below is one of them*)

Let P be a point on the circle S with both coordinates being positive. Let the tangent to S at P intersect the coordinate axes at the points M and N. Then, the mid-point of the line segment MN must lie on the curve - **[JEE(Advanced)-2018, 3(–1)]**

(A) $(x+y)^2 = 3xy$ (B) $x^{2/3} + y^{2/3} = 2^{4/3}$ (C) $x^2 + y^2 = 2xy$ (D) $x^2 + y^2 = x^2y^2$

12 sol:-



Let $K(h,k)$ be the mid point of MN

$\Rightarrow$ $M = (2h, 0)$, $N = (0, 2k)$

$\Rightarrow$ Equation of MN is

$$\frac{x}{2h} + \frac{y}{2k} = 1$$

This is tangent to circle.

$$\Rightarrow \frac{|0+0-1|}{\sqrt{\left(\frac{1}{2h}\right)^2+\left(\frac{1}{2k}\right)^2}} = 2$$

$$\Rightarrow \frac{1}{4h^2} + \frac{1}{4k^2} = \frac{1}{4}$$

$$\Rightarrow h^2+k^2 = h^2k^2$$

$\Rightarrow$ Locus of midpoint of MN

is $\boxed{x^2+y^2 = x^2y^2}$

(12) Let T be the line passing through the points P(–2, 7) and Q(2, –5). Let F_1 be the set of all pairs of circles (S_1, S_2) such that T is tangents to S_1 at P and tangent to S_2 at Q, and also such that S_1 and S_2 touch each other at a point, say, M. Let E_1 be the set representing the locus of M as the pair (S_1, S_2) varies in F_1. Let the set of all straight line segments joining a pair of distinct points of E_1 and passing through the point R(1, 1) be F_2. Let E_2 be the set of the mid-points of the line segments in the set F_2. Then, which of the following statement(s) is (are) TRUE ? **[JEE(Advanced)-2018, 4(–2)]**

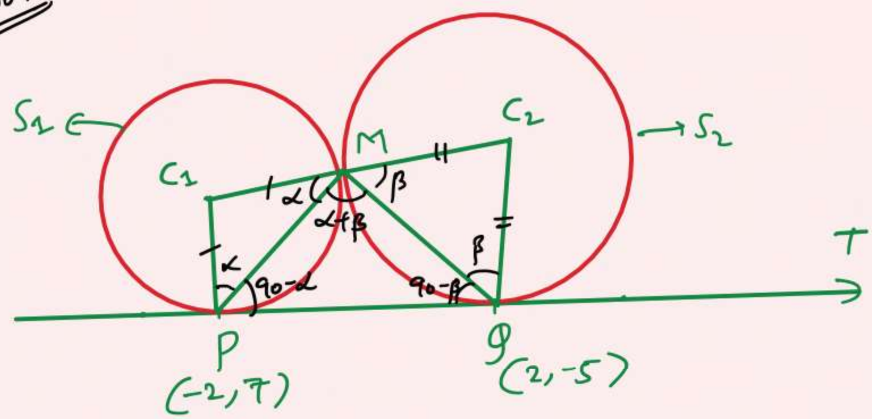
(A) The point (–2, 7) lies in E_1

(B) The point $\left(\frac{4}{5},\frac{7}{5}\right)$ does **NOT** lie in E_2

(C) The point $\left(\frac{1}{2},1\right)$ lies in E_2

(D) The point $\left(0,\frac{3}{2}\right)$ does **NOT** lie in E_1

13 Sol :-



Circle S_1, S_2 are tangents at T.

We need to find locus of M.

Let $\angle C_1PM = \alpha$, $\angle C_2QM = \beta$

$\Rightarrow \angle C_1MP = \alpha$, $\angle C_2MQ = \beta$

Also $\angle MPQ = 90^\circ - \alpha$, $\angle MQP = 90^\circ - \beta$

$\Rightarrow \angle PMQ = \alpha + \beta$

$\because$ C_1, M, C_2 are collinear,

$2(\alpha+\beta) = 180^\circ \Rightarrow \alpha+\beta = 90^\circ$

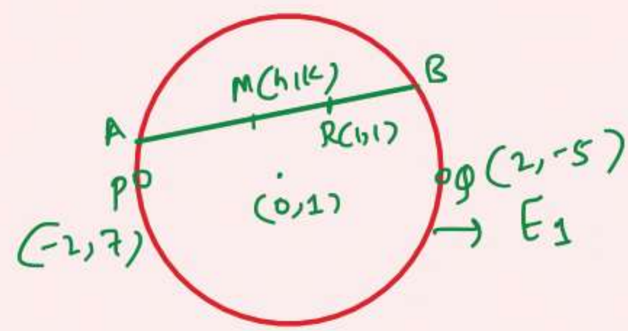
$\Rightarrow$ M is such that $\angle PMQ = 90°$

also M cannot be the points P or Q.

So locus of M is $\bigcirc^{le}$ with PQ as diameter excluding P and Q

$E_1: (x-2)(x+2) + (y-7)(y+5) = 0$

$\Rightarrow x^2+y^2-2y-39=0$ except P, Q



Let middle point of chords through R(1,1) be M(h,k)

$\Rightarrow$ AB: $T = S_1$

$\Rightarrow$ AB: $x(h)+y(k)-(y+k)-\cancel{39}$
$= h^2+k^2-2k-\cancel{39}$
— passes through (1,1)

$\Rightarrow h+\cancel{k}-1-\cancel{k} = h^2+k^2-2k$

$\Rightarrow$ Locus of M is

$E_2: x^2+y^2-x-2y+1=0$ but

in this equation we need to delete two points. They are the midpoints of the chords joining PR and RQ.

A) $(-2,7)$ lies in E_1

False as P, Q are excluded in E_1

B) $\left(\frac{4}{5},\frac{7}{5}\right)$ does not lies in E_2

First let us check the equation of E_2

$$\left(\frac{4}{5}\right)^2+\left(\frac{7}{5}\right)^2-\left(\frac{4}{5}\right)-2\left(\frac{7}{5}\right)+1$$

$$=0$$

Now let us check whether $\left(\frac{4}{5},\frac{7}{5}\right)$ is mid point of chord joining one of PR or QR

i) Check if $P(-2,7)$, $\underset{\uparrow K}{\left(\frac{4}{5},\frac{7}{5}\right)}$, $R(1,1)$ are collinear

$$m_{PK}=\frac{7-\frac{7}{5}}{-2-\frac{4}{5}}=\frac{28}{-14}=-2$$

$$m_{PR}=\frac{6}{-3}=-2$$

$\therefore \left(\frac{4}{5},\frac{7}{5}\right)$ is mid point of chord joining P, R

Hence it is not in E_2

So true statement

C) $\left(\frac{1}{2}, 1\right)$ lies in E_2

Checking with eqn of E_2

$\left(\frac{1}{2}\right)^2 + 1^2 - \left(\frac{1}{2}\right) - 2(1) + 1 \neq 0$

$\therefore$ $\left(\frac{1}{2}, 1\right)$ does not lies in E_2

False statement

D) $\left(0, \frac{3}{2}\right)$ does not lie in E_1

Checking with eqn of E_1

$0^2 + \left(\frac{3}{2}\right)^2 - 2\left(\frac{3}{2}\right) - 39 \neq 0$

$\therefore$ $\left(0, \frac{3}{2}\right)$ does not lie in E_1

True statement

$\therefore$ B, D are correct

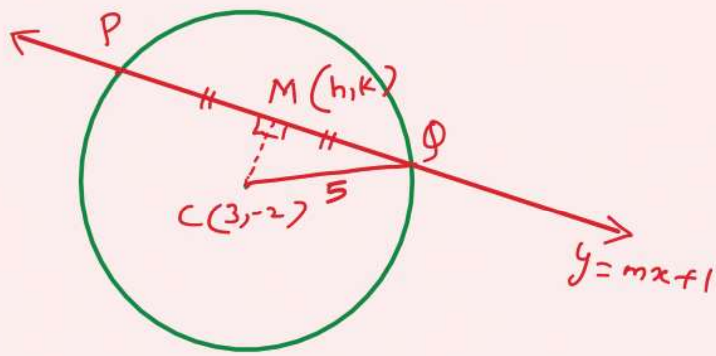
(13) A line $y = mx + 1$ intersects the circle $(x-3)^2 + (y+2)^2 = 25$ at the points P and Q. If the midpoint of the line segment PQ has x-coordinate $-\frac{3}{5}$, then which one of the following options is correct ?

[JEE(Advanced)-2019, 3(–1)]

(1) $6 \le m < 8$ (2) $2 \le m < 4$ (3) $4 \le m < 6$ (4) $-3 \le m < -1$

14

Sol :- $S: (x-3)^2 + (y+2)^2 = 25$



M = midpoint

$h = -\frac{3}{5} \Rightarrow k = -\frac{3}{5}m + 1$

Slope of $MC = \dfrac{-\frac{3}{5}m + 1 + 2}{-\frac{3}{5} - 3}$

$= \dfrac{-3m + 15}{-18}$

$m_{MC} \cdot m_{PQ} = -1$

$\Rightarrow \dfrac{-3m + 15}{-18} \cdot m = -1$

$\Rightarrow -3m^2 + 15m = 18$

$\Rightarrow m^2 - 5m + 6 = 0$

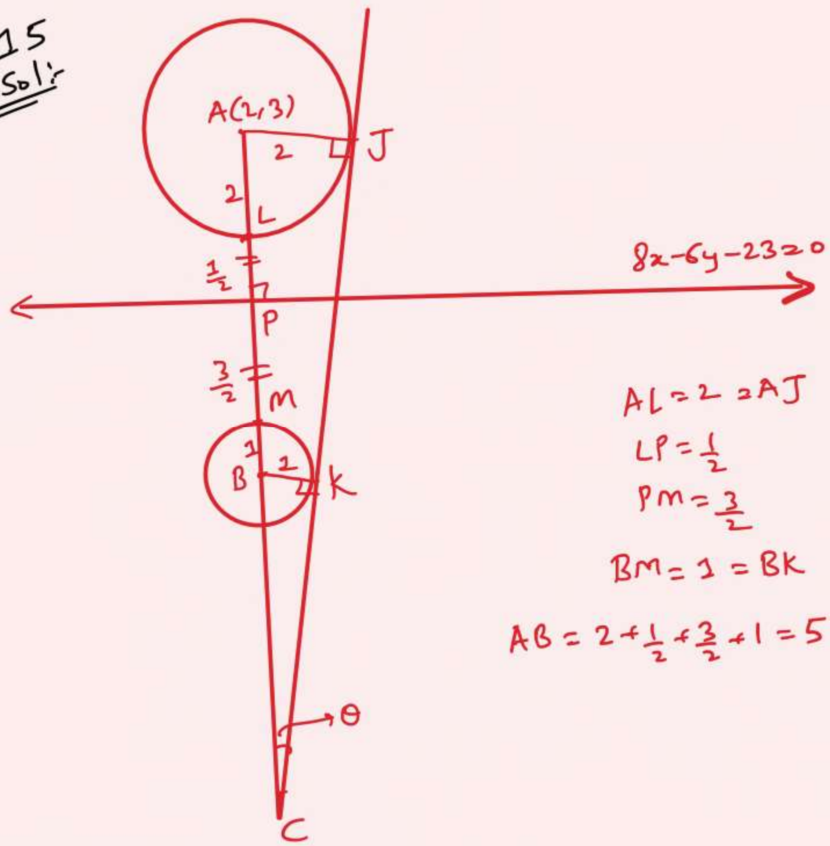
$\Rightarrow \boxed{m = 2, 3}$

$\Rightarrow \boxed{2 \le m \le 4}$

(14) Let the point B be the reflection of the point A(2, 3) with respect to the line $8x - 6y - 23 = 0$. Let Γ_A and Γ_B be circles of radii 2 and 1 with centres A and B respectively. Let T be a common tangent to the circles Γ_A and Γ_B such that both the circles are on the same side of T. If C is the point of intersection of T and the line passing through A and B, then the length of the line segment AC is _____ **[JEE(Advanced)-2019, 3(0)]**

15

Sol:



$AL = 2 = AJ$

$LP = \frac{1}{2}$

$PM = \frac{3}{2}$

$BM = 1 = BK$

$AB = 2 + \frac{1}{2} + \frac{3}{2} + 1 = 5$

$$AP = \frac{|16 - 18 - 23|}{10} = \frac{5}{2}$$

From the figure, ΔBKC is similar to ΔAJC

$$\Rightarrow \frac{BK}{BC} = \frac{AJ}{AC}$$

$$\Rightarrow \frac{1}{BC} = \frac{2}{5 + BC}$$

$$\Rightarrow BC = 5$$

$$\Rightarrow AC = AB + BC = 5 + 5$$

$$\boxed{AC = 10}$$

Answer the following by appropriately matching the lists based on the information given in the paragraph

Let the circles $C_1 : x^2 + y^2 = 9$ and $C_2 : (x-3)^2 + (y-4)^2 = 16$, intersect at the points X and Y. Suppose that another circle $C_3 : (x-h)^2 + (y-k)^2 = r^2$ satisfies the following conditions :

(i) centre of C_3 is collinear with the centres of C_1 and C_2

(ii) C_1 and C_2 both lie inside C_3, and

(iii) C_3 touches C_1 at M and C_2 at N.

Let the line through X and Y intersect C_3 at Z and W, and let a common tangent of C_1 and C_3 be a tangent to the parabola $x^2 = 8\alpha y$.

There are some expression given in the List-I whose values are given in List-II below :

	List-I		**List-II**
(I)	$2h + k$	(P)	6
(II)	$\dfrac{\text{Length of ZW}}{\text{Length of XY}}$	(Q)	$\sqrt{6}$
(III)	$\dfrac{\text{Area of triangle MZN}}{\text{Area of triangle ZMW}}$	(R)	$\dfrac{5}{4}$
(IV)	α	(S)	$\dfrac{21}{5}$
		(T)	$2\sqrt{6}$
		(U)	$\dfrac{10}{3}$

(15) Which of the following is the only INCORRECT combination ? **[JEE(Advanced)-2019, 3(–1)]**

(1) (IV), (S) (2) (IV), (U) (3) (III), (R) (4) (I), (P)

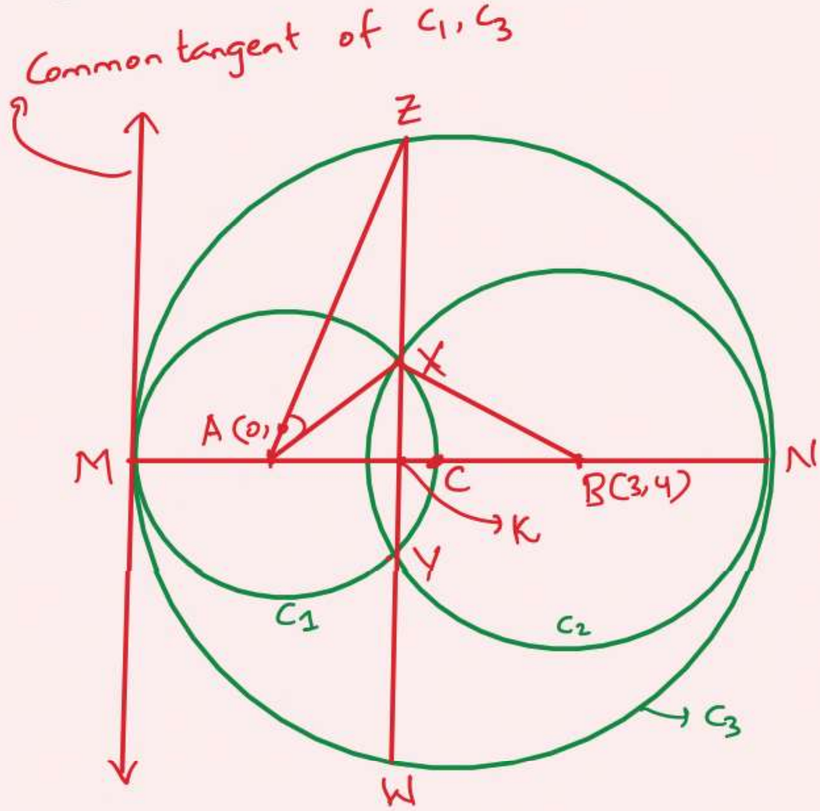
(16) Which of the following is the only CORRECT combination ? **[JEE(Advanced)-2019, 3(–1)]**

(1) (II), (T) (2) (I), (S) (3) (I), (U) (4) (II), (Q)

Paragraph 16 and 17

$C_1: x^2+y^2=9$, $C_2: (x-3)^2+(y-4)^2=16$

$C_3: (x-h)^2+(y-k)^2=r^2$



Radius of $C_1 = r_1 = AM = 3$

Radius of $C_2 = r_2 = BN = 4$

$AB = 5$

$\Rightarrow MN = 3+5+4 = 12$

$\Rightarrow$ Radius of $C_3 = 6$

C = centre of $C_3 = (h, k)$

$AC = 3, CB = 2$

$$\Rightarrow \quad C = \frac{3B+2A}{5}$$

$$= \frac{(9,12)+(0,0)}{5}$$

$$C = \left(\frac{9}{5}, \frac{12}{5}\right) = (h, k)$$

$$\Rightarrow \quad 2h+k = 2\left(\frac{9}{5}\right)+\frac{12}{5}$$

(i) $\boxed{2h+k = 6}$

Let intersection of XY and AB be K

Let length $XK = a$

$$\Rightarrow \quad AK = \sqrt{9-a^2}$$

and $BK = \sqrt{16-a^2}$

$AK + BK = 5$

$$\Rightarrow \quad \sqrt{9-a^2} + \sqrt{16-a^2} = 5$$

$$\Rightarrow \quad 9-\cancel{a^2} = 25+16-\cancel{a^2} - 10\sqrt{16-a^2}$$

$$\Rightarrow \quad 16 = 5\sqrt{16-a^2}$$

$$\Rightarrow \quad \frac{256}{25} = 16-a^2$$

$$\Rightarrow \quad a^2 = 16-\frac{256}{25} = \frac{144}{25}$$

$$\Rightarrow \quad a = \frac{12}{5} = XK$$

$$\Rightarrow \boxed{XY = 2a = \frac{24}{5}}$$

$$AK = \sqrt{9-a^2} = \sqrt{9-\frac{144}{25}} = \frac{9}{5}$$

$$\Rightarrow \quad CK = AC - AK = 3-\frac{9}{5} = \frac{6}{5}$$

$$\Rightarrow \quad ZK = \sqrt{CZ^2-CK^2}$$

$$= \sqrt{6^2-\left(\frac{6}{5}\right)^2}$$

$$ZK = \frac{6\sqrt{24}}{5} = \frac{12\sqrt{6}}{5}$$

$$\Rightarrow \boxed{ZW = \frac{24\sqrt{6}}{5}}$$

(ii) $\dfrac{ZW}{XY} = \dfrac{24\sqrt{6}}{5\left(\frac{24}{5}\right)} = \sqrt{6}$

(iii) $\dfrac{\text{Area of } \triangle MZN}{\text{Area of } \triangle ZMW} = \dfrac{\frac{1}{2}\cdot MN\cdot ZK}{\frac{1}{2}\cdot ZW\cdot MK}$

$$= \frac{12\cdot \cancel{ZK}}{2(\cancel{ZK})\cdot\left(3+\frac{9}{5}\right)}$$

$$= \frac{6}{\frac{24}{5}}$$

$$= \frac{5}{4}$$

(iv) To find M,

$$M = 2A - C$$
$$= -C$$
$$M = \left(\frac{-9}{5}, \frac{-12}{5}\right)$$

$$m_{MC} = \frac{\frac{12}{5}}{\frac{9}{5}} = \frac{4}{3}$$

$\Longrightarrow$ slope of Common tangent (CT)

$$= \frac{-1}{m_{MC}} = \frac{-3}{4}$$

$\Longrightarrow$ CT: $y + \frac{12}{5} = \frac{-3}{4}\left(x + \frac{9}{5}\right)$

$\Rightarrow 4(5y+12) = -3(5x+9)$

$\Rightarrow 15x + 20y + 75 = 0$

$\Longrightarrow 3x + 4y + 15 = 0$

This is tangent to $x^2 = 8\alpha y$

Solving $x^2 = 8\alpha\left(\frac{-15-3x}{4}\right)$

$\Rightarrow x^2 = -30\alpha - 6\alpha x$

$\Rightarrow x^2 + 6\alpha x + 30\alpha = 0$

$D = 0$

$\Rightarrow 36\alpha^{\cancel{2}} = 4(30)\cancel{\alpha}$

$\Rightarrow \alpha = \frac{\cancel{4}^{2} \cdot \cancel{30}^{5}}{\cancel{36}_{\cancel{6}\,3}}$

$$\boxed{\alpha = \frac{10}{3}}$$