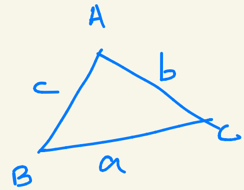
Compound Angles

Tri — 3
Gon — sides
Metron — measurement



* $a+b > c$.
* $|a-b| < c$.

Common identities:—

① $\sin^2\theta + \cos^2\theta = 1$.

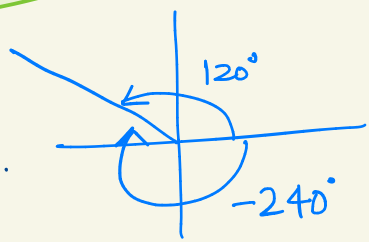
② $\sec^2\theta - \tan^2\theta = 1$.

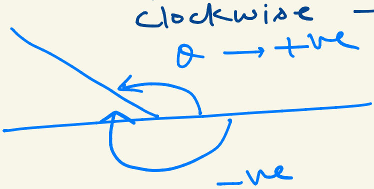
③ $\csc^2\theta - \cot^2\theta = 1$.

$$\boxed{\sec\theta - \tan\theta = \frac{1}{\sec\theta + \tan\theta}}.$$

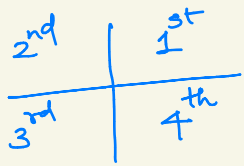
Sign Convention

Anticlockwise → +ve
clockwise → −ve.
θ → +ve



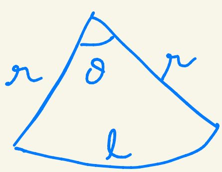


Quadrants



radians

$$\boxed{\pi^c = 180^\circ \Rightarrow 1^c = \frac{180}{\pi} \simeq 57.3^\circ}$$



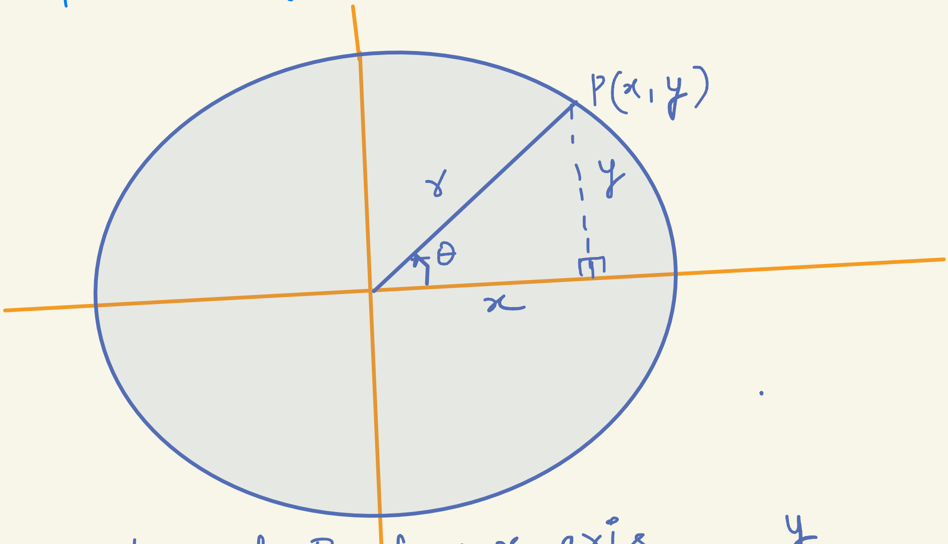
$$\boxed{\theta = \frac{\text{arc}}{\text{Rad}}}$$

length of arc

& Area of sector $= \frac{1}{2} r^2 \theta$.

$\theta \rightarrow$ Radians.

Real definition of sine & Cosine



$$\sin\theta = \frac{\perp \text{dis of } P \text{ from } x\text{-axis}}{\text{Radius}} = \frac{y}{r}$$

$$\cos\theta = \frac{\perp \text{dis of } P \text{ from } y\text{-axis}}{\text{Radius}} = \frac{x}{r}.$$

Reduction formulae :-

1. $\sin(90^\circ - \theta) = \cos\theta.$
2. $\cos(90^\circ - \theta) = \sin\theta.$
3. $\tan(90^\circ - \theta) = \cot\theta.$
4. $\sec(90^\circ - \theta) = \operatorname{cosec}\theta.$
5. $\operatorname{cosec}(90^\circ - \theta) = \sec\theta.$
6. $\cot(90^\circ - \theta) = \tan\theta.$

$\sin(90^\circ + \theta) = \cos\theta$

$\cos(90^\circ + \theta) = -\sin\theta$

$\tan(90^\circ + \theta) = -\cot\theta$

$\vdots \quad \vdots$

$\pi - \theta$

$\sin(\pi - \theta) = \sin\theta$

$\cos(\pi - \theta) = -\cos\theta$

$\tan(\pi - \theta) = -\tan\theta$

$\pi + \theta$

$\sin(\pi + \theta) = -\sin\theta$

$\cos(\pi + \theta) = -\cos\theta$

$\tan(\pi + \theta) = \tan\theta.$

$270^\circ - \theta$

$\sin\left(\frac{3\pi}{2} - \theta\right) = -\cos\theta$

$\cos\left(\frac{3\pi}{2} - \theta\right) = -\sin\theta$

$\tan\left(\frac{3\pi}{2} - \theta\right) = \cot\theta$

$\frac{3\pi}{2} + \theta$

$\sin\left(\frac{3\pi}{2} + \theta\right) = -\cos\theta$

$\cos\left(\frac{3\pi}{2} + \theta\right) = \sin\theta$

$\tan\left(\frac{3\pi}{2} + \theta\right) = -\cot\theta$

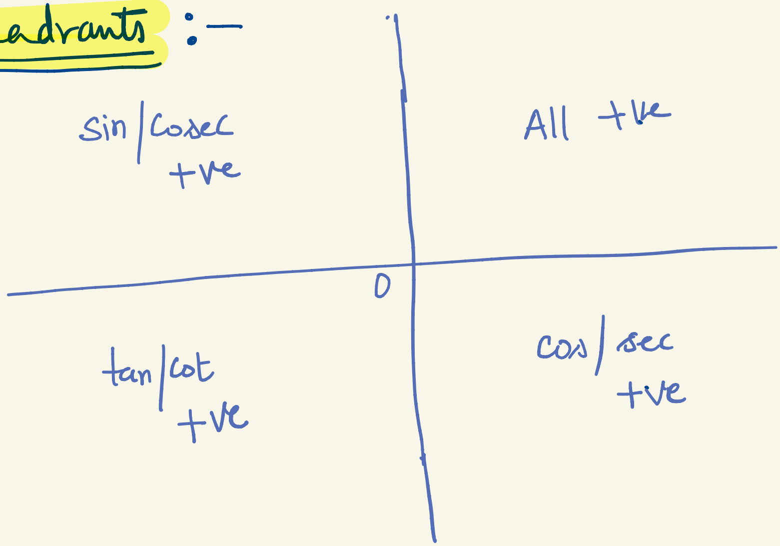
$2\pi - \theta$

$\sin(2\pi - \theta) = \sin(-\theta) = -\sin\theta$

$\cos(2\pi - \theta) = \cos(-\theta) = \cos\theta$

$\tan(2\pi - \theta) = \tan(-\theta) = -\tan\theta.$

4 Quadrants :-



* Remember

$\sin\theta \in [-1, 1]$ $\quad$ $\csc\theta \in (-\infty, -1] \cup [1, \infty)$

$\sin^2\theta \in [1, \infty)$ $\quad$ $\csc^2\theta \in [1, \infty)$

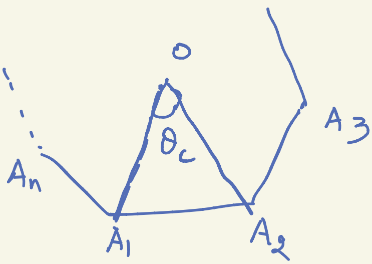
$\tan\theta \in R$

$\tan^2\theta \in [0, \infty)$

Regular Polygon

Central angle of polygon.

① $\boxed{\theta_c = \frac{2\pi}{n}}$ $\quad$ $n \to$ no. of sides



② Interior angle $= \frac{(n-2)\pi}{n}$

③ Sum of all interior angles $= (n-2)\pi$

Pentagon → 5 sides

Hexagon → 6 "

Octagon → 8 "

Heptagon → 7 "

Decagon → 10 "

Dodecagon → 12 sides

Quindecagon → 15 " .

① $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

② $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

③ $2 \sin A \cos B = \sin(A+B) + \sin(A-B)$

④ $2 \cos A \sin B = \sin(A+B) - \sin(A-B)$.

⑤ $2 \cos A \cos B = \cos(A+B) + \cos(A-B)$

⑥ $2 \sin A \sin B = \cos(A-B) - \cos(A+B)$.

⑦ $\sin(A+B) \cdot \sin(A-B) = \sin^2 A - \sin^2 B$

$= \cos^2 B - \cos^2 A$.

⑧ $\cos(A+B) \cdot \cos(A-B) = \cos^2 A - \sin^2 B$.

$= \cos^2 B - \sin^2 A$.

⑨ $\sin C + \sin D = 2 \sin\left(\frac{C+D}{2}\right) \cos\left(\frac{C-D}{2}\right)$

$\sin C - \sin D = 2 \sin\left(\frac{C-D}{2}\right) \cos\left(\frac{C+D}{2}\right)$

⑩ $\cos C + \cos D = 2 \cos\left(\frac{C+D}{2}\right) \cos\left(\frac{C-D}{2}\right)$

* $\cos C - \cos D = 2 \sin\left(\frac{C+D}{2}\right) \sin\left(\frac{D-C}{2}\right)$

(Note carefully)

$$\sin\frac{\pi}{12} = \cos\frac{5\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}} = \frac{\sqrt{6}-\sqrt{2}}{4} = \frac{1}{\sqrt{6}+\sqrt{2}}.$$

$$\sin\frac{5\pi}{12} = \cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}} = \frac{\sqrt{6}+\sqrt{2}}{4} = \frac{1}{\sqrt{6}-\sqrt{2}}.$$

$$\tan\frac{\pi}{12} = \cot\frac{5\pi}{12} = 2-\sqrt{3}.$$

$$\tan\frac{5\pi}{12} = \cot\frac{\pi}{12} = 2+\sqrt{3}.$$

———— x ———— x ———— x ————

$$\tan(A+B) = \frac{\tan A + \tan B}{1-\tan A\tan B}.$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1+\tan A\tan B}.$$

$$\tan\left(\frac{\pi}{4}+\theta\right) = \frac{1+\tan\theta}{1-\tan\theta}$$

$$\tan\left(\frac{\pi}{4}-\theta\right) = \frac{1-\tan\theta}{1+\tan\theta}$$

Remember vice-versa also.

$$\cot(A+B) = \frac{\cot A\cot B - 1}{\cot B + \cot A}$$

$$* \cot(A-B) = \frac{\cot A\cot B + 1}{\cot B - \cot A}$$

(Note carefully)

Note:

If $A+B=\frac{\pi}{4}$ then :

① $(1+\tan A)(1+\tan B)=2.$

② $(1-\cot A)(1-\cot B)=2.$

* (Rem this very useful)

$$\sin 2\theta = 2\sin\theta\cos\theta.$$

$$= \frac{2\tan\theta}{1+\tan^2\theta}$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$= 2\cos^2\theta - 1$$

$$= 1-2\sin^2\theta$$

$$= \frac{1-\tan^2\theta}{1+\tan^2\theta}.$$

** Imp forms from above :

① $\sin^2\theta = \frac{1-\cos 2\theta}{2}$

② $\cos^2\theta = \frac{1+\cos 2\theta}{2}$

③ $\tan^2\theta = \frac{1-\cos 2\theta}{1+\cos 2\theta}$

$$\tan\frac{\pi}{8} = \tan 22.5^\circ = \cot\frac{3\pi}{8} = \cot 67.5^\circ = \sqrt{2}-1$$

$$\cot\frac{\pi}{8} = \cot 22.5^\circ = \tan\frac{3\pi}{8} = \tan 67.5^\circ = \sqrt{2}+1$$

$$\left.\begin{aligned} \sin 3\theta &= 3\sin\theta - 4\sin^3\theta \\ \cos 3\theta &= 4\cos^3\theta - 3\cos\theta. \\ \tan 3\theta &= \frac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}. \end{aligned}\right\}$$

$$\left.\begin{aligned} \sin 5\theta &= 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta. \\ \cos 5\theta &= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta. \end{aligned}\right\}$$

$\bigstar$

$$1+\sin 2\theta = (\sin\theta + \cos\theta)^2$$

$$1-\sin 2\theta = (\sin\theta - \cos\theta)^2$$

$$\sqrt{1+\sin\theta} = \sqrt{\left(\sin\frac{\theta}{2} + \cos\frac{\theta}{2}\right)^2}$$

Note carefully $$= \left|\sin\frac{\theta}{2} + \cos\frac{\theta}{2}\right|$$

* If $A+B=C$ then

$$\boxed{\tan A \tan B \tan C = \tan C - \tan B - \tan A}$$

$$\tan 7.5^\circ = \tan \frac{\pi}{24} = (\sqrt{3}-\sqrt{2})(\sqrt{2}-1)$$
$$= \sqrt{6}-\sqrt{4}-\sqrt{3}+\sqrt{2}$$

$$\cot 7.5^\circ = \cot \frac{\pi}{24} = (\sqrt{3}+\sqrt{2})(\sqrt{2}+1)$$
$$= \sqrt{6}+\sqrt{4}+\sqrt{3}+\sqrt{2}.$$

$$\cos \frac{\pi}{8} = \cos 22.5^\circ = \frac{\sqrt{2+\sqrt{2}}}{2}.$$

$$\sin \frac{\pi}{8} = \sin 22.5^\circ = \frac{\sqrt{2-\sqrt{2}}}{\sqrt{2}}.$$

Imp

$$\sin\theta \ \sin(60^\circ-\theta) \ \sin(60^\circ+\theta) = \frac{\sin 3\theta}{4}$$

$$\cos\theta \ \cos(60^\circ-\theta) \ \cos(60^\circ+\theta) = \frac{\cos 3\theta}{4}.$$

* $\tan\theta \ \tan(60^\circ-\theta) \ \tan(60^\circ+\theta) = \tan 3\theta.$
* $\cot\theta \ \cot(60^\circ-\theta) \ \cot(60^\circ+\theta) = \cot 3\theta.$

$$\cos\theta \cdot \cos 2\theta \cos 2^2\theta \ldots\ldots \cos 2^{n-1}\theta = \frac{\sin 2^n\theta}{2^n \sin\theta}.$$

* (a) $\cot A - \tan A = 2\cot 2A$ (b) $\cot A + \tan A = 2\operatorname{cosec} 2A$

(a) $\sin^2\theta + \sin^2(60^\circ + \theta) + \sin^2(60^\circ - \theta) = \dfrac{3}{2}$

(b) $\cos^2\theta + \cos^2(60^\circ + \theta) + \cos^2(60^\circ - \theta) = \dfrac{3}{2}$

(c) $\tan\theta + \tan(60^\circ + \theta) + \tan(120^\circ + \theta) = 3\tan 3\theta$

Imp

① $\sin^4\theta + \cos^4\theta = 1 - \dfrac{\sin^2 2\theta}{2}$

② $\sin^6\theta + \cos^6\theta = 1 - \dfrac{3}{4}\sin^2 2\theta.$

③ $\cos^4\theta - \sin^4\theta = \cos 2\theta.$

very useful

$$\sin\frac{\pi}{10} = \sin 18^\circ = \cos 72^\circ = \cos\frac{2\pi}{5} = \frac{\sqrt{5}-1}{4}$$

$$\cos\frac{\pi}{5} = \cos 36^\circ = \sin 54^\circ = \sin\frac{3\pi}{10} = \frac{\sqrt{5}+1}{4}$$

TRIGONOMETRIC IDENTITIES IN A TRIANGLE (CONDITIONAL IDENTITIES) :

If A, B, C are the angles of a triangle i.e. $A + B + C = \pi$, then

(i) $\sin(A + B) = \sin(\pi - C) = \sin C$

(ii) $\cos(A + B) = \cos(\pi - C) = -\cos C$

(iii) $\tan(A + B) = \tan(\pi - C) = -\tan C$

(iv) $\sin(2A + 2B) = \sin(2\pi - 2C) = -\sin 2C$

(v) $\cos(2A + 2B) = \cos(2\pi - 2C) = \cos 2C$

(vi) $\tan(2A + 2B) = \tan(2\pi - 2C) = -\tan 2C$

(vii) $\sin\left(\frac{A+B}{2}\right) = \sin\frac{\pi - C}{2} = \cos\frac{C}{2}$

(viii) $\cos\left(\frac{A+B}{2}\right) = \sin\frac{C}{2}$

(ix) $\tan\left(\frac{A+B}{2}\right) = \cot\frac{C}{2}$

(x) If A, B, C are the angles of a triangle, then $\sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C$

[**Sol.** LHS $= 2\sin(A + B)\cos(A - B) + 2\sin C\cos C = 2\sin C[\cos(A - B) + \cos(\pi - A - B)]$

$= 2\sin C[\cos(A - B) - \cos(A + B)] = 4\sin A\sin B\sin C$]

(xi) $\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A\cos B\cos C$ (where $A + B + C = \pi$)

[**Sol.** $2\cos(A + B)\cos(A - B) + 2\cos^2 C - 1 = 2\cos(\pi - C)\cos(A - B) + 2\cos^2 C - 1$]

$= 2\cos C[\cos C - \cos(A - B)] - 1 = 2\cos C[\cos(\pi - \overline{A+B}) - \cos(A - B)] - 1$

$= 2\cos C[-\cos(A + B) - \cos(A - B)] - 1 = -4\cos A\cos B\cos C - 1$]

(xii) $\Sigma\tan A = \Pi\tan A$

(xiii) $\sum\cot A\cot B = 1$

(xiv) $\sum\tan\frac{A}{2}\tan\frac{B}{2} = 1$

(xv) $\sum\cot\frac{A}{2} = \Pi\cot\frac{A}{2}$

[**Sol.** (i) $\tan(A + B) = \tan(\pi - C) = -\tan C \Rightarrow \frac{\tan A + \tan B}{1 - \tan A\tan B} = -\tan C$]

Note Carefully :-

(a) If $\tan A + \tan B + \tan C = \tan A \tan B \tan C$, then $A + B + C = n\pi, n \in I$

(b) If $\tan A \tan B + \tan B \tan C + \tan C \tan A = 1$, then $A + B + C = (2n + 1)\frac{\pi}{2}, n \in I$

MAXIMUM/MINIMUM VALUES OF TRIGONOMETRIC EXPRESSIONS :

(1) a cos θ + b sin θ will always lie in the interval $[-\sqrt{a^2+b^2}, \sqrt{a^2+b^2}]$ i.e. the maximum and minimum value are $\sqrt{a^2+b^2}$, $-\sqrt{a^2+b^2}$ respectively.

(2) Minimum value of $a^2\tan^2\theta + b^2\cot^2\theta$ is 2ab where a, b > 0

(3) If A, B, C are the angles of a triangle then

(a) $\sin A + \sin B + \sin C \le \frac{3\sqrt{3}}{2}$

(b) $1 < \cos A + \cos B + \cos C \le \frac{3}{2}$

(c) $\tan A + \tan B + \tan C \ge 3\sqrt{3}$ (for acute angled triangle)

(d) $\cot A + \cot B + \cot C \ge \sqrt{3}$

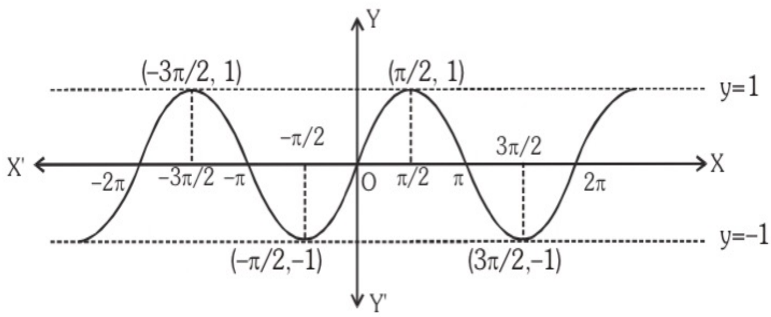
Note : Equality holds for equilateral triangle.

(e) $\cot^2 A + \cot^2 B + \cot^2 C \geq 1$

(f) $\cos A \cos B \cos C \le \frac{1}{8}$.

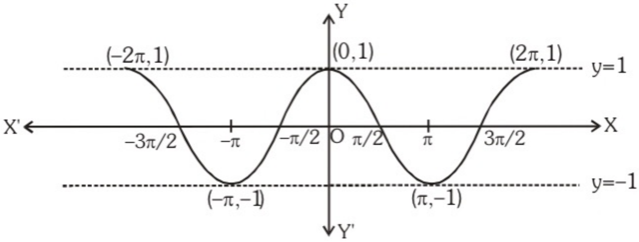
(a) Graphs of trigonometric ratios :

(1) y = sinx, x∈R, y∈ [–1, 1]



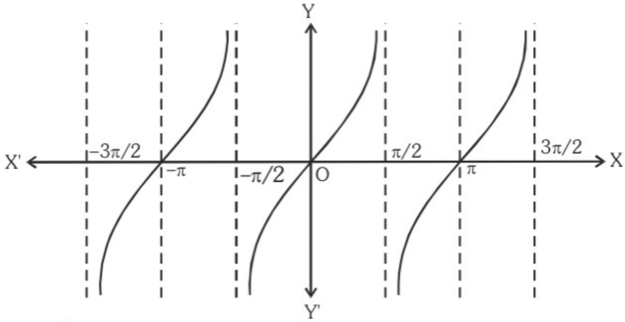
Periodic with period 2π

(2) $y = \cos x$



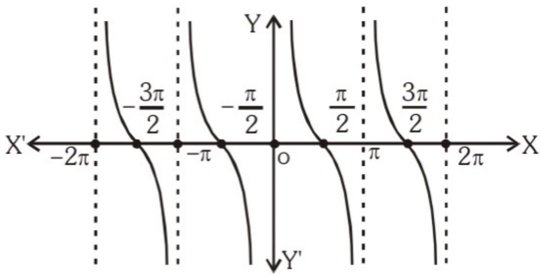
Periodic with period 2π

(3) $y = \tan x,\ x \neq \frac{\pi}{2}, \frac{3\pi}{2} \ldots\ldots \neq (2n-1)\frac{\pi}{2}\ : n \in I$

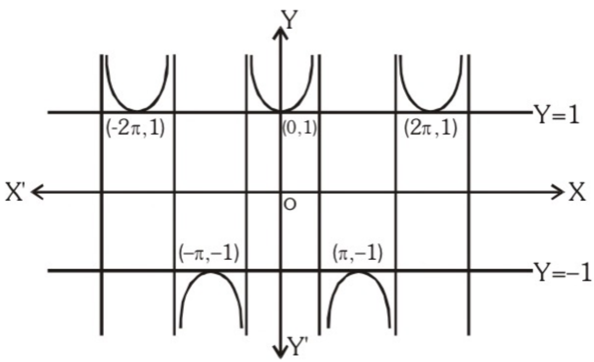


Periodic with period π

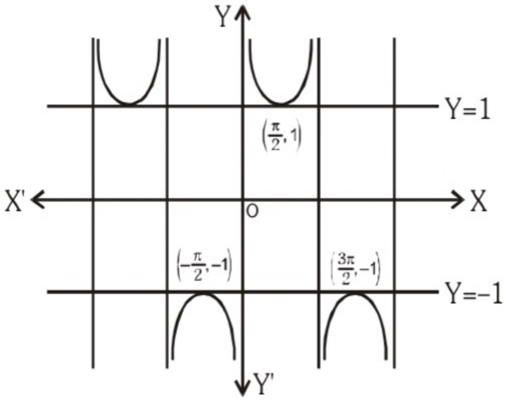
(4) **y = cotx**



(5) **y = secx**



(6) $\mathbf{y = cosecx}$



(b) $-1 \le \sin\theta \le 1$

(c) $-1 \le \cos\theta \le 1$

(d) $\sin 0 = 0;\quad \sin\pi = 0;\ \sin 2\pi = 0$

$\sin(n\pi) = 0,\ n \in I$

i.e. sine of integral multiple of $\pi = 0$

Also $\tan(n\pi) = 0$ and $\sin\frac{\pi}{2} = \sin\frac{5\pi}{2} = \sin\left(2n\pi + \frac{\pi}{2}\right) = 1$ and $\sin\frac{3\pi}{2} = \sin\frac{7\pi}{2} = \sin\left(2n\pi - \frac{\pi}{2}\right) = -1$

(e) $\cos\frac{\pi}{2} = 0;\ \cos\frac{3\pi}{2} = 0;\ \cos\frac{5\pi}{2} = 0$

$\cos(2n-1)\frac{\pi}{2} = 0,\ n \in I$

i.e. cosine of odd integral multiple of $\frac{\pi}{2}$ is zero.

Also $\cot(2n-1)\frac{\pi}{2} = 0 : n \in I$

(f) $\cos 0 = 1;\quad \cos 2\pi = 1;\ \cos 4\pi = 1$

$\cos 2m\pi = 1,\ m \in I$

i.e. cos of even multiple of $\pi = 1$

(g) $\cos\pi = -1;\ \cos 3\pi = -1;\ \cos 5\pi = -1$

$\cos(2m-1)\pi$ = cos of odd multiple of $\pi = -1$

(h) $\sin(2n\pi + \theta) = \sin\theta,\ n \in I$

$\cos(2n\pi + \theta) = \cos\theta,\ n \in I$

$\tan(n\pi + \theta) = \tan\theta,\ n \in I$