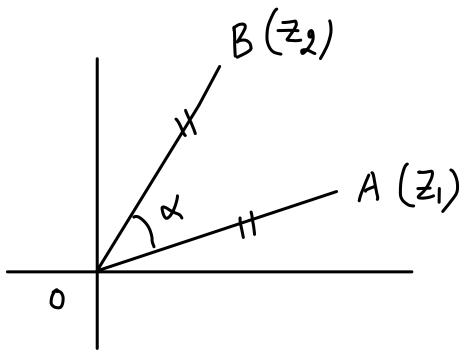
Q HW Let z_1 and z_2 be roots of the equation $z^2 + pz + q = 0$, where the coefficients p and q may be complex numbers. Let A and B represent z_1 and z_2 in the complex plane. If $\angle AOB = \alpha \neq 0$ and $OA = OB$, where O is the origin, prove that $p^2 = 4q \cos^2\alpha/2$.

Sol^n

$$z_1 + z_2 = -p \quad \& \quad z_1 z_2 = q$$



$$z_2 = z_1 e^{i\alpha} \quad -①-$$

$$1 + \frac{z_2}{z_1} = e^{i\alpha} + 1$$

$$\frac{z_2 + z_1}{z_1} = 1 + e^{i\alpha}$$

$$\frac{-p}{z_1} = 1 + \cos\alpha + i\sin\alpha$$

$$\frac{-p}{z_1} = 2\cos^2\frac{\alpha}{2} + 2i\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}$$

$$\frac{-p}{z_1} = 2\cos\frac{\alpha}{2}\, e^{i\frac{\alpha}{2}}$$

$$\frac{p^2}{z_1^2} = 2^2\cos^2\frac{\alpha}{2} \cdot e^{i\alpha}$$

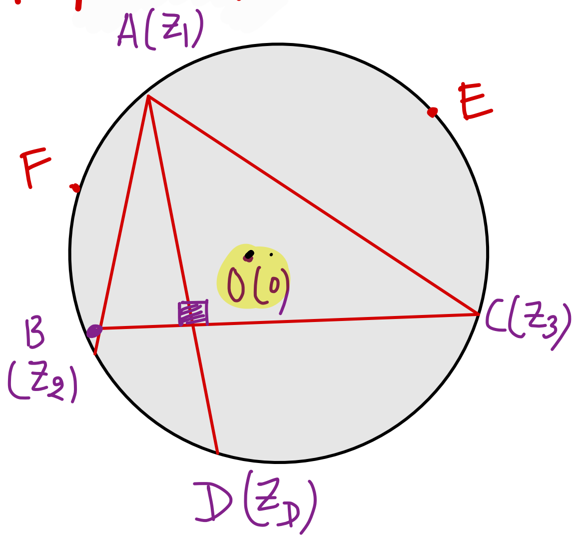
from ①

$$p^2 = 4\cos^2\frac{\alpha}{2} \cdot \cancel{z_1^2} \cdot \left(\frac{z_2}{\cancel{z_1}}\right)$$

$$p^2 = 4\cos^2\frac{\alpha}{2} \cdot (q) \qquad (H.P)$$

Q (HW) A, B, C are the points representing the complex numbers z_1, z_2 z_3 respectively and the circumcentre of the triangle ABC lies at the origin. If the altitudes of the triangle through the opposite vertices meets the circumcircle at D, E, F respectively. Find the complex numbers corresponding to the D, E, F in terms of z_1, z_2, z_3.

$$|z_1|^2 = |z_2|^2 = |z_3|^2 = r^2.$$



Sol^n

$$z_1 \bar{z}_1 = r^2$$

$$\bar{z}_1 = \frac{r^2}{z_1}$$

$$\omega_{AD} + \omega_{BC} = 0.$$

$$\frac{z_D - z_1}{\bar{z}_D - \bar{z}_1} + \frac{z_2 - z_3}{\bar{z}_2 - \bar{z}_3} = 0.$$

$$\frac{z_D - z_1}{\frac{r^2}{z_D} - \frac{r^2}{z_1}} + \frac{z_2 - z_3}{\frac{r^2}{z_2} - \frac{r^2}{z_3}} = 0$$

$$\left(\frac{z_D - z_1}{z_1 - z_D}\right) \cdot z_1 z_D + \left(\frac{z_2 - z_3}{z_3 - z_2}\right) z_2 z_3 = 0.$$

$$\boxed{z_D = -\frac{z_2 z_3}{z_1}} \; ; \quad z_E = -\frac{z_1 z_3}{z_2}.$$

$$z_F = -\frac{z_1 z_2}{z_3}.$$

Alt :- Use Rotation.

DEMOIVRE'S THEOREM : **

Statement :

If $z = (\cos\theta + i\sin\theta)^n$, then

(i) $(\cos n\theta + i\sin n\theta)$ is the value of z, if $n \in \mathbb{Z}$ integer

(ii) $(\cos n\theta + i\sin n\theta)$ is one of the value of z, if $n = p/q$, where p & $q \in \mathbb{Z}$, $q \neq 0$ and are coprime.

$z = (\cos\theta + i\sin\theta)^5 = (\cos 5\theta + i\sin 5\theta)$ is only value of z

$z = (\cos\theta + i\sin\theta)^{\frac{1}{5}} = \left(\cos\frac{\theta}{5} + i\sin\frac{\theta}{5}\right)$ is one of the value of z.

$$x^3 = 1 \Rightarrow x = (1)^{\frac{1}{3}} = (\cos 0 + i\sin 0)^{\frac{1}{3}} = (\cos 2m\pi + i\sin 2m\pi)^{\frac{1}{3}}$$

$$= \left(\cos\frac{2m\pi}{3} + i\sin\frac{2m\pi}{3}\right)$$

$$m = 0, 1, 2$$

4 basic steps to determine the roots of a complex number are :

(a) Write the complex number whose roots are to be determined in polar form/ exponential form.

(b) Add $2m\pi$ to the argument

(c) Apply DMT

(d) Put $m = 0, 1, 2, 3, \ldots(n-1)$ to get all then n ; n^{th} roots.

eg: $\underbrace{(\alpha)^{1/3} \rightarrow 3 \text{ Roots}}$ $\underbrace{(a)^{3/5} = (a^3)^{\frac{1}{5}} \rightarrow 5 \text{ Roots}}$

Q

(i) $z = \left(1 + i\sqrt{3}\right)^{1/4}$ (ii) $z = (-8)^{2/3}$

(i) $z = (1 + i\sqrt{3})^{1/4} = 2^{1/4}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)^{\frac{1}{4}}$.

$$z = (2)^{1/4}\left(\cos\left(\frac{2m\pi + \pi/3}{4}\right) + i\sin\left(\frac{2m\pi + \pi/3}{4}\right)\right).$$

$m = 0, 1, 2, 3$

if $m = 0$ then $z_1 = (2)^{1/4}\,\text{cis}\left(\frac{\pi}{12}\right) = (2)^{1/4} e^{i(\pi/12)}$

if $m = 1$ " $z_2 = (2)^{1/4}\,\text{cis}\left(\frac{7\pi}{12}\right) = (2)^{1/4} e^{i(7\pi/12)}$

if $m = 2$ " $z_3 = (2)^{1/4}\,\text{cis}\left(\frac{13\pi}{12}\right) = (2)^{1/4} e^{i\left(\frac{13\pi}{12}\right)}$

if $m = 3$ " $z_4 = (2)^{1/4}\,\text{cis}\left(\frac{19\pi}{12}\right) = (2)^{1/4} e^{i\left(\frac{19\pi}{12}\right)}$

(2) $z = (-8)^{2/3} = (64)^{1/3} = 4\,\underline{(1)^{1/3}}$

$= 4\left(\cos\frac{2m\pi}{3} + i\sin\frac{2m\pi}{3}\right)$

$m = 0, 1, 2$

Q $$2\sqrt{2}z^4 = \left(\sqrt{3}-1\right)+i\left(\sqrt{3}+1\right)$$

$$z = \left(\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right) + i\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)\right)^{1/4}$$

$$z = \left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right)^{1/4} \quad ; \quad |z| = 1$$

$$z = 1.\left(\cos\left(\frac{2m\pi + \frac{5\pi}{12}}{4}\right) + i\sin\left(\frac{\frac{5\pi}{12} + 2m\pi}{4}\right)\right)$$

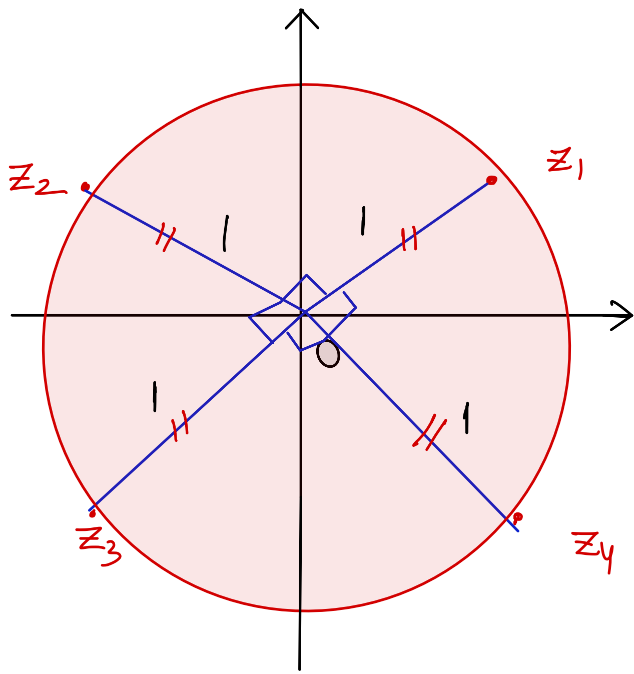
$m = 0, 1, 2, 3.$

$$z_1 = e^{i\,5\pi/48}$$

$$z_2 = e^{i\,29\pi/48}$$

$$z_3 = e^{i\,53\pi/48}$$

$$z_4 = e^{i\,77\pi/48}$$



Q Find the roots of the equation $z^5 + z^4 + z^3 + z^2 + z + 1 = 0$ having the least positive argument.

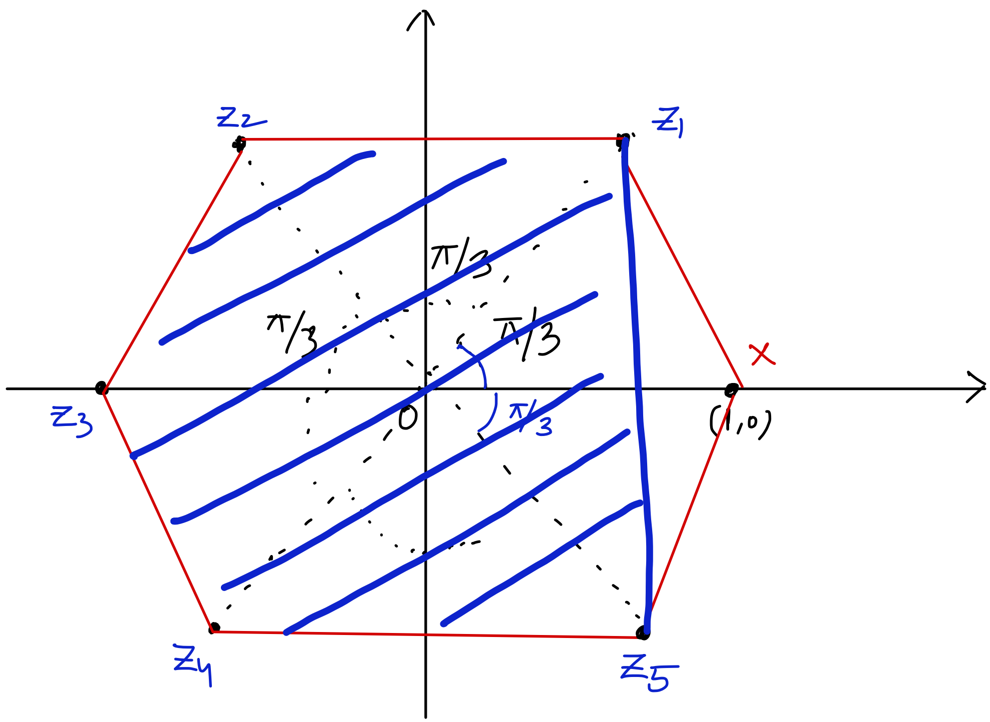
Sol^n

$$1 \cdot \left(\frac{z^6 - 1}{z - 1} \right) = 0 \Rightarrow \quad z = (1)^{1/6} \; ; \; z \neq 1$$

$$z = \left(\cos \frac{2m\pi}{6} + i \sin \frac{2m\pi}{6} \right) ; \quad m = \cancel{0}, 1, 2, 3, 4, 5$$

$$z_{\text{least Positive arg}} = \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right)$$
$(m = 1)$

(ii)



Q Find the number of roots of the equation $z^{10} - z^5 - 992 = 0$ with real part –ve.

Sol^n

$$z^5 = t$$

$$t^2 - t - 992 = 0 \Rightarrow (t-32)(t+31) = 0.$$

$$(t-32)(t+31) = 0$$

$$(z^5 - 32)(z^5 + 31) = 0.$$

$$z = (32)^{1/5}$$

$$z = (2)\left(\cos\frac{2m\pi}{5} + i\sin\frac{2m\pi}{5}\right)$$

$m = 0, 1, 2, 3, 4$
✗ ✗ ✓ ✓ ✗

2 values of z with –ve real part.

$$z = (-31)^{1/5}$$

$$z = (31)^{1/5}(-1)^{1/5}$$

$$= (31)^{1/5}(\cos\pi + i\sin\pi)^{\frac{1}{5}}$$

$$= (31)^{1/5}\left(\cos\left(\frac{2m\pi+\pi}{5}\right) + i\sin\left(\frac{2m\pi+\pi}{5}\right)\right)$$

$m = 0, 1, 2, 3, 4$
✗ ✓ ✓ ✓ ✗

3 values of z with –ve real part

Hence, total 5 complex no. with –ve real part.

Q Solve the following equations : (i) $\bar{z} = iz^2$ (ii) $z^5 = \bar{z}$ (iii) $z^4 = |z|$

Rem: $\boxed{z^n = \bar{z}} \rightarrow (n+2)$ solutions

$$z^n = \bar{z} \Rightarrow |z^n| = |\bar{z}| \Rightarrow |z|^n = |z|$$

$$|z|\left(|z|^{n-1} - 1\right) = 0$$

$$\therefore \boxed{|z| = 0} \text{ OR } \boxed{|z| = 1}$$

$$\boxed{z = 0} \rightarrow 1 \text{ sol}^n$$

when $|z| = 1$ then

$$z^n = \bar{z}$$

$$z^{n+1} = z\bar{z} = |z|^2$$

$$z^{n+1} = 1 \Rightarrow z = (1)^{\frac{1}{n+1}} \rightarrow (n+1) \text{ sol}^n$$

$$z = \left(\cos\frac{2m\pi}{n+1} + i\sin\frac{2m\pi}{n+1}\right)$$

$m = 0, 1, 2, \ldots, n$

$(n+2)$ solutions

(iii) $z^4 = |z|$

$|z|^4 = ||z|| = |z|$

$|z|(|z|^3 - 1) = 0$

$|z| = 0$ OR $|z| = 1$

$\boxed{z = 0}$ $\quad z^4 = 1 \Rightarrow$ 4 solutions

1 sol^n $\quad 1, -1, i, -i$

5 sol^n

(i) $\bar{z} = i z^2$

$|\bar{z}| = |iz^2| = |i|\,|z^2|$

$|z| = |z|^2$

$|z|(|z| - 1) = 0$

$|z| = 0$ OR $|z| = 1 \Rightarrow$

$\boxed{z = 0 + 0i}$

$\bar{z} = i z^2$

$z\bar{z} = i z^3$

$|z|^2 = i z^3 \Rightarrow z^3 = \frac{1}{i} \times \frac{i}{i}$

$\boxed{z^3 = -i} \rightarrow$ 3 sol^n

alt :-

$\bar{z} = i z^2$

$\arg(\bar{z}) = \arg(i) + \arg(z^2) + 2K\pi$.

$-\arg z = \frac{\pi}{2} + 2\arg(z) + 2K\pi$.

$3\arg z = -\frac{\pi}{2} - 2K\pi. \Rightarrow \underbrace{\arg z}_{(-\pi, \pi]} = -\frac{\pi}{6} - \frac{2K\pi}{3}$

Cube roots of unity : Imp

$$z^3 = 1 \Rightarrow z = (1)^{\frac{1}{3}} = \left(\cos\frac{2m\pi}{3} + i\sin\frac{2m\pi}{3}\right); \quad m = 0; 1; 2$$

$m = 0 \Rightarrow z_1 = \cos 0 + i\sin 0 = \text{cis}\, 0 = e^{i0}$

$m = 1 \Rightarrow z_2 = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} = \text{cis}\,\frac{2\pi}{3} = e^{i2\pi/3} \Rightarrow \omega = \frac{-1+i\sqrt{3}}{2}$

$m = 2 \Rightarrow z_3 = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} = \text{cis}\,\frac{4\pi}{3} = e^{i4\pi/3} \Rightarrow \omega^2 = \frac{-1-i\sqrt{3}}{2}$

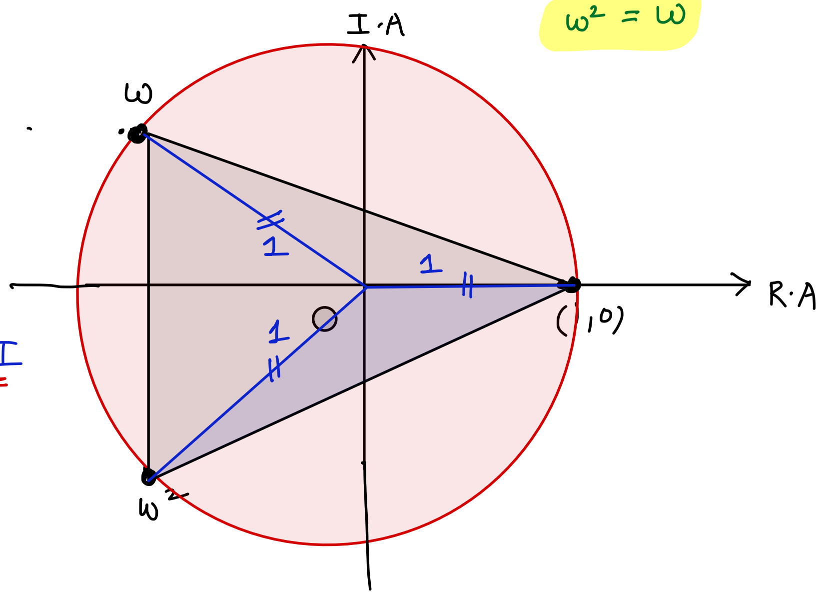
$$\boxed{|\omega| = |\omega^2| = 1}$$

$\overline{\omega} = \omega^2$

$\overline{\omega^2} = \omega$

$$\boxed{\begin{aligned} 2\omega &= -1+i\sqrt{3} \\ 2\omega^2 &= -1-i\sqrt{3} \end{aligned}}$$

$z^3 = 1 \begin{cases} 1 \\ \omega \\ \omega^2 \end{cases}$



$\omega^3 = 1$; $\omega^6 = 1$

$\Rightarrow \boxed{\omega^{3n} = 1}$; $n \in I$ Rem

$$\boxed{\begin{aligned} \omega^{3n+1} &= \omega \\ \omega^{3n+2} &= \omega^2 \end{aligned}}$$

Rem

$\boxed{1 + \omega + \omega^2 = 0} \Rightarrow \boxed{\omega + \omega^2 = -1}$

** $1^r + \omega^r + (\omega^2)^r \rightarrow$
- 0 ; r is not integral multiple of '3' i.e $r \neq 3\lambda$; $\lambda \in I$.
- 3 ; $r = 3\lambda$; $\lambda \in I$

eg: $r = 4$

$1^4 + \omega^4 + (\omega^2)^4 = 1 + \omega + \omega^2 = 0$

$1+w=-w^2$

Note : $1+w^2=-w$

(i) $\omega = \overline{\omega^2}$, $\omega^{3n} = 1$, $\omega^{3n+1} = \omega$, $\omega^{3n+2} = \omega^2$, $\frac{1}{\omega} = \omega^2$, $\frac{1}{\omega^2} = \omega$, $1.\omega.\omega^2 = 1$

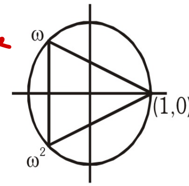
(ii) If w is one of the imaginary cube roots of unity then $1 + w + w^2 = 0$.

In general $1 + w^r + w^{2r} = 0$; when $r \neq 3\lambda$

$1 + w^r + w^{2r} = 3$; when $r = 3\lambda$

(iii) $|1| = |\omega| = |\omega^2|$

(iv) The three cube roots of unity when plotted on the argand plane constitute the vertices of an equilateral triangle. and they lie on a Unit Circle centered at origin.



(v) The following factorisation should be remembered :

(a, b, c ∈ R & ω is the cube root of unity)

Rem

$a^3 - b^3 = (a - b)(a - \omega b)(a - \omega^2 b)$; $x^2 + x + 1 = (x - \omega)(x - \omega^2)$;

$a^3 + b^3 = (a + b)(a + \omega b)(a + \omega^2 b)$; $x^2+1 = (x-i)(x+i)$

$a^3 + b^3 + c^3 - 3abc = (a + b + c)(a + \omega b + \omega^2 c)(a + \omega^2 b + \omega c)$

$x^2-x+1 = (x+w)(x+w^2)$

$a^3-b^3 = (a-b)(a^2+ab+b^2) = (a-b)\, b^2\left(\underbrace{\left(\frac{a}{b}\right)}_{t}^2 + \left(\frac{a}{b}\right)+1\right)$

$= (a-b)\, b^2 \left(\frac{a}{b}-w\right)\left(\frac{a}{b}-w^2\right)$

$= (a-b)(a-bw)(a-bw^2)$.

$x^3-1=0 \Rightarrow (x-1)(x^2+x+1)=0$

$x=1$ $x=w$ $w^2=x$

$x^2+x+1 = (x-w)(x-w^2)$.

$x^3+1=0 \Rightarrow (x+1)(x^2-x+1)$

$x=-1$ $x=-w$; $x=-w^2$

$x = (-1)^{\frac{1}{3}} = (-1)(1)^{1/3}$

$-1; -w^2; -w$ $1, w, w^2$

$$z^3 + 27 = 0 \Rightarrow \cancel{z =} \ (-27)^{\frac{1}{3}} \begin{cases} \to -3 \\ \to -3w \\ \to -3w^2. \end{cases}$$

$$z = (-3)\,(1)^{1/3}$$

$$1, w, w^2$$

$$z = (-8)^{2/3} = (64)^{\frac{1}{3}} = 4\,(1)^{1/3} \begin{cases} 4 \\ 4w \\ 4w^2 \end{cases}$$

No HW

(Do Revision)