HYPERBOLA

Complete Solutions Of DYS & Exercises

DO YOURSEF

Do yourself - 1 :

(i) Find the eccentricity of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ which passes through (4, 0) & $(3\sqrt{2}, 2)$

(ii) Find the equation to the hyperbola, whose eccentricity is $\frac{5}{4}$, focus is (a, 0) and whose directrix is $4x - 3y = a$.

(iii) In the hyperbola $4x^2 - 9y^2 = 36$, find length of the axes, the co-ordinates of the foci, the eccentricity, and the latus rectum.

(iv) Find the equation to the hyperbola, the distance between whose foci is 16 and whose eccentricity is $\sqrt{2}$.

Q1

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

(4, 0) $\quad (3\sqrt{2}, 2)$

$$\frac{16}{a^2}-0=1 \qquad \frac{18}{16}-\frac{4}{b^2}=1$$

$$a^2=16$$

$$\frac{4}{b^2}=\frac{18-16}{16}$$

$$b^2=32$$

$$e^2=1+\frac{b^2}{a^2}=1+\frac{32}{16}=3$$

$$e=\sqrt{3}$$

Q2 $P(x,y)$

$$\frac{PS}{PM} = e$$

$$(x-a)^2 + y^2 = \frac{25}{16}\left(\frac{(4x-3y-9)}{25}\right)^2$$

$$7y^2 + 24xy - 24ax$$
$$-6ay + 15a^2 = 0$$

Q3 $$\frac{x^2}{9} - \frac{y^2}{4} = 1$$

$a=3 \qquad b=2$

$$e = 1 + \frac{b^2}{a^2} = 1 + \frac{4}{9} \qquad e = \frac{\sqrt{13}}{3}$$

Taxis $= 2a = 6$

Caxis $: 2b = 4$

$f_1, f_2 \; (\pm ae, 0) = (\pm\sqrt{13}, 0)$

$$LLR = \frac{2b^2}{a} = \frac{8}{3}$$

Q4 $2ae = 16$ $e = \sqrt{2}$

$a = 4\sqrt{2}$

$e^2 = 1 + \frac{b^2}{a^2} = 2$

$b^2 = a^2$

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ (assuming standard hyperbola

$x^2 - y^2 = 32$

Do yourself - 2 :

(i) Find eccentricity of conjugate hyperbola of hyperbola $4x^2 - 16y^2 = 64$, also find area of quadrilateral formed by foci of hyperbola & its conjugate hyperbola

$$\frac{x^2}{16} - \frac{y^2}{4} = 1$$

$$e^2 = 1 + \frac{4}{16} = \frac{20}{16}$$

$$e = \frac{2\sqrt{5}}{4}$$

$$f_1, f_2 = (\pm ae, 0)$$

$$= (\pm 2\sqrt{5}, 0)$$

conjugate hyperbola

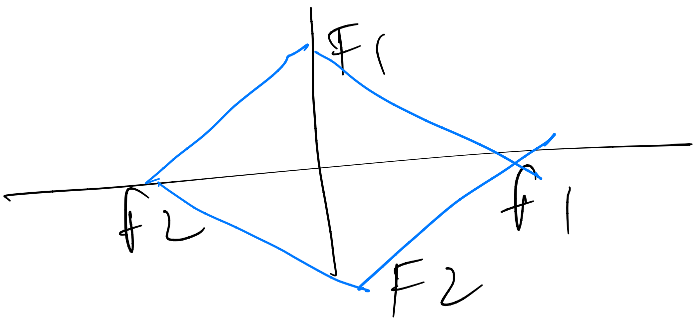
$$\frac{y^2}{4} - \frac{x^2}{16} = 1$$

$$E^2 = 1 + \frac{16}{4} = \frac{20}{4}$$

$$E = \frac{2\sqrt{5}}{2}$$

$$F_1, F_2 = (0, \pm BE)$$

$$= (0, \pm 2\sqrt{5})$$



$$A = \frac{1}{2} d_1 d_2 \sin 90 = \frac{1}{2} 4\sqrt{5} \times 4\sqrt{5}$$

$$= 8 \times 5 = 40$$

Do yourself - 3 :

(i) Find the condition for the line $\ell x + my + n = 0$ to touch the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

(ii) If the line y = 5x +1 touch the hyperbola $\frac{x^2}{4} - \frac{y^2}{b^2} = 1$ $\{b > 4\}$, then -

(A) $b^2 = \frac{1}{5}$ (B) $b^2 = 99$ (C) $b^2 = 4$ (D) $b^2 = 100$

(i) $lx + my + n = 0$ $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

POI

$b^2x^2 - a^2\left(\frac{-lx-n}{m}\right)^2 = a^2b^2$

$(m^2b^2 - a^2l^2)x^2 - 2a^2lnx - a^2n^2 - m^2a^2b^2 = 0$

for tangent

$D = 0$

$4a^4l^2n^2 - 4(m^2b^2 - a^2l^2)(-a^2n^2 - m^2a^2b^2) = 0$

$\cancel{a^4l^2n^2} + (a^2m^2b^2n^2 + a^2b^4m^4 - \cancel{a^4l^2n^2} - a^4l^2m^2b^2) = 0$

$n^2 + b^2m^2 - a^2l^2 = 0$

$n^2 = a^2l^2 - b^2m^2$

(11) $y = 5x+1$ $\quad \frac{x^2}{4} - \frac{y^2}{b^2} = 1$

|| PoI

$$b^2x^2 - 4(5x+1)^2 = 4b^2$$

$$(b^2-100)x^2 - 40x - 4 - 4b^2 = 0$$

tangent

$$D = 0$$

$$40\times 40 - 4.(b^2-100)(-4-4b^2) = 0$$

$$100 + (b^2-100)(1+b^2) = 0$$

$$100 + b^2 + b^4 - 100 - 100b^2 = 0$$

$$b^4 - 99b^2 = 0$$

$b^2 = 0 \qquad b^2 = 99$

$\times \qquad$ Ans

Do yourself - 4 :

(i) Find the equation of the tangent to the hyperbola $4x^2 - 9y^2 = 1$, which is parallel to the line $4y = 5x + 7$.

(ii) Find the equation of the tangent to the hyperbola $16x^2 - 9y^2 = 144$ at $\left(5, \frac{16}{3}\right)$.

(iii) Find the common tangent to the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ and an ellipse $\frac{x^2}{4} + \frac{y^2}{3} = 1$.

$$\frac{x^2}{1/4} - \frac{y^2}{1/9} = 1$$

tangent $y = mx \pm \sqrt{\frac{1}{4}m^2 - \frac{1}{9}}$

$\parallel \; m = 5/4$

$$y = \frac{5x}{4} \pm \sqrt{\frac{25}{64} - \frac{1}{9}}$$

$$y = \frac{5x}{4} \pm \frac{\sqrt{161}}{8 \times 3}$$

$$24y = 30x \pm \sqrt{161}$$

(ii) tangent $T=0$

$$16xx_1 - 9yy_1 = 144$$

$$16x5 - 9y\frac{16}{3} = 144$$

$$5x - 3y = 9$$

(iii) $\frac{x^2}{16} - \frac{y^2}{9} = 1$ $\qquad$ $\frac{x^2}{4} + \frac{y^2}{3} = 1$

$y = mx \pm \sqrt{16m^2-9}$ $\qquad$ $y = mx \pm \sqrt{4m^2+3}$

common tangent

compare

$$\frac{1}{1} = \frac{m}{m} = \frac{\pm\sqrt{16m^2-9}}{\pm\sqrt{4m^2+3}}$$

$$4m^2+3 = 16m^2-9$$

$$12m^2 = 12$$

$$m^2 = 1$$

$$m = \pm 1$$

$\boxed{y = \pm x \pm \sqrt{7}}$ Ans

Do yourself - 5 :

(i) Find the equation of normal to the hyperbola $\frac{x^2}{25}-\frac{y^2}{16}=1$ at (5, 0).

(ii) Find the equation of normal to the hyperbola $\frac{x^2}{16}-\frac{y^2}{9}=1$ at the point $\left(6, \frac{3}{2}\sqrt{5}\right)$.

(iii) Find the condition for the line $\ell x + my + n = 0$ is normal to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

(i) normal $\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2+b^2$

$$\frac{25x}{5}+\frac{b^2y}{0}=a^2+b^2$$

$$\boxed{y=0}$$ Ans

(ii)

$$\frac{a^2x}{x_1}+\frac{b^2y}{y_1}=a^2+b^2$$

$$\frac{16x}{6}+\frac{9y}{3/2\sqrt{5}}=16+9$$

$$8\sqrt{5}x+18y=75\sqrt{5}$$

(iii) normal at $P(\theta)$

$$\frac{a^2x}{a\sec\theta}+\frac{b^2y}{b\tan\theta}=a^2+b^2$$

$$ax\cos\theta + by\sin\theta = a^2+b^2$$

$$lx+my+n=0$$

compare

$$\frac{a\cos\theta}{l} = \frac{b\cot\theta}{+m} = \frac{a^2+b^2}{-n}$$

$$\sec\theta = -\frac{an}{l(a^2+b^2)}$$

$$\tan\theta = -\frac{bn}{m(a^2+b^2)}$$

$$\sec^2\theta - \tan^2\theta = 1$$

$$\frac{a^2}{l^2} - \frac{b^2}{m^2} = \frac{(a^2+b^2)^2}{n^2}$$

Do yourself - 6 :

(i) Find the equation to the chords of the hyperbola $x^2 - y^2 = 9$ which is bisected at (5, –3)

(ii) If m_1 and m_2 are the slopes of the tangents to the hyperbola $\frac{x^2}{25} - \frac{y^2}{16} = 1$ which pass through the point (6, 2), then find the value of $11m_1m_2$ and $11(m_1 + m_2)$.

(iii) Find the locus of the mid points of the chords of the circle $x^2 + y^2 = 16$ which are tangents to the hyperbola $9x^2 - 16y^2 = 144$.

(iv) The asymptotes of a hyperbola are parallel to lines $2x + 3y = 0$ and $3x + 2y = 0$. The hyperbola has its centre at (1, 2) and it passes through (5, 3). Find its equation.

(i) chord with middle Point (5, -3)

$$T = S_1$$

$$5x - (-3)y \cancel{-} 9 = 25 - 9 - 9$$

$$5x + 3y = 16$$

(ii) $y = mx \pm \sqrt{25m^2 - 16}$ tangent

$$\parallel (6, 2)$$

$$(2 - 6m)^2 = 25m^2 - 16$$

$$11m^2 - 24m + 20 = 0$$

$$m_1 m_2 = \frac{20}{11} \qquad m_1 + m_2 = \frac{24}{11}$$

$$11 m_1 m_2 = 20 \qquad 11(m_1 + m_2) = 24$$

(iii) $x^2+y^2=16$

Chord with middle Point (h,k)

$T=S_1$

$hx+ky-16=h^2+k^2-16$

$hx+ky=h^2+k^2 \quad \text{—(i)}$

tangent of hyperbola $9x^2-16y^2=144$

at $P(\theta)$

$$\frac{x^2}{16}-\frac{y^2}{9}=1$$

$$\frac{x\cdot 4\sec\theta}{16}-\frac{y\cdot 3\tan\theta}{9}=1$$

$$\frac{x\sec\theta}{4}-\frac{y\tan\theta}{3}=1 \quad \text{—(ii)}$$

$$\frac{\sec\theta}{4h}=\frac{-\tan\theta}{3k}=\frac{1}{h^2+k^2}$$

$$\sec\theta=\frac{4h}{h^2+k^2} \qquad \tan\theta=\frac{-3k}{h^2+k^2}$$

$\sec^2\theta - \tan^2\theta = 1$

$16h^2 - 9k^2 = (h^2+k^2)^2$

$$\boxed{16x^2 - 9y^2 = (x^2+y^2)^2}$$

(iv) assymptotes will be

$2x+3y = d_1$ $\qquad$ $3x+2y = d_2$

|| (1,2) they always pass through center || (1,2)

$d_1 = 8$ $\qquad$ $d_2 = 7$

$2x+3y = 8$ $\qquad$ $3x+2y = 7$

pair of assymptotes

$(2x+3y-8)(3x+2y-7) = 0$

hyperbola and pair of assymptotes only differ by a constant

$$(2x+3y-8)(3x+2y-7)=M$$

$$\parallel (5,3)$$

$$11 \times 14 = M$$

$$M = 154$$

hyperbola $\quad (2x+3y-8)(3x+2y-7)$

$$=154$$

Do yourself - 7 :

(i) If equation $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a rectangular hyperbola then write required conditions.

(ii) Find the equation of tangent at the point (1, 2) to the rectangular hyperbola xy = 2.

(iii) Prove that the locus of point, tangents from where to hyperbola $x^2 - y^2 = a^2$ inclined at an angle α & β with x-axis such that $\tan\alpha \tan\beta = 2$ is also a hyperbola. Find the eccentricity of this hyperbola.

(i) $\Delta = abc + 2fgh - af^2 - bg^2 - ch^2 \neq 0$

$h^2 - ab > 0$

$a + b = 0$

(ii) tangent is $T = 0$

$$\frac{xy_1 + yx_1}{2} - 2 = 0$$

$$\frac{x2 + y}{2} - 2 = 0$$

$$2x + y = 4$$

(iii) assume Point (h, k)

tangent $y = mx \pm \sqrt{a^2m^2 - a^2}$

$$\parallel (h, k)$$

$$(k - mh)^2 = a^2 (m^2 - 1)$$

$$m^2 (h^2 - a^2) - 2mkh + k^2 + a^2 = 0$$

$$m_1 m_2 = \tan\alpha \tan\beta = \frac{2hk}{h^2 - a^2} = 2$$

$$h^2 - a^2 = hk$$

$$x^2 - xy = a^2$$

Pair of assymtotes (only diff by a constant)

$$x^2 - xy = c$$

angle btw assymptotes

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{a+b}$$

$$= \frac{2\sqrt{\frac{1}{4} - 0}}{1 + 0}$$

$$= 1$$

$$\theta = \pi/4$$

eccentricity of hyperbola is $= \sec \theta/2$

$= \sec \frac{\pi}{6}$

$= \sqrt{3}$

0-1

1. Consider the hyperbola $9x^2 - 16y^2 + 72x - 32y - 16 = 0$. Find the following:

(a) centre (b) eccentricity (c) focii (d) equation of directrix

(e) length of the latus rectum (f) equation of auxilary circle

(g) equation of director circle

Solⁿ 1

$$9x^2 + 72x - 16y^2 - 32y = 16$$

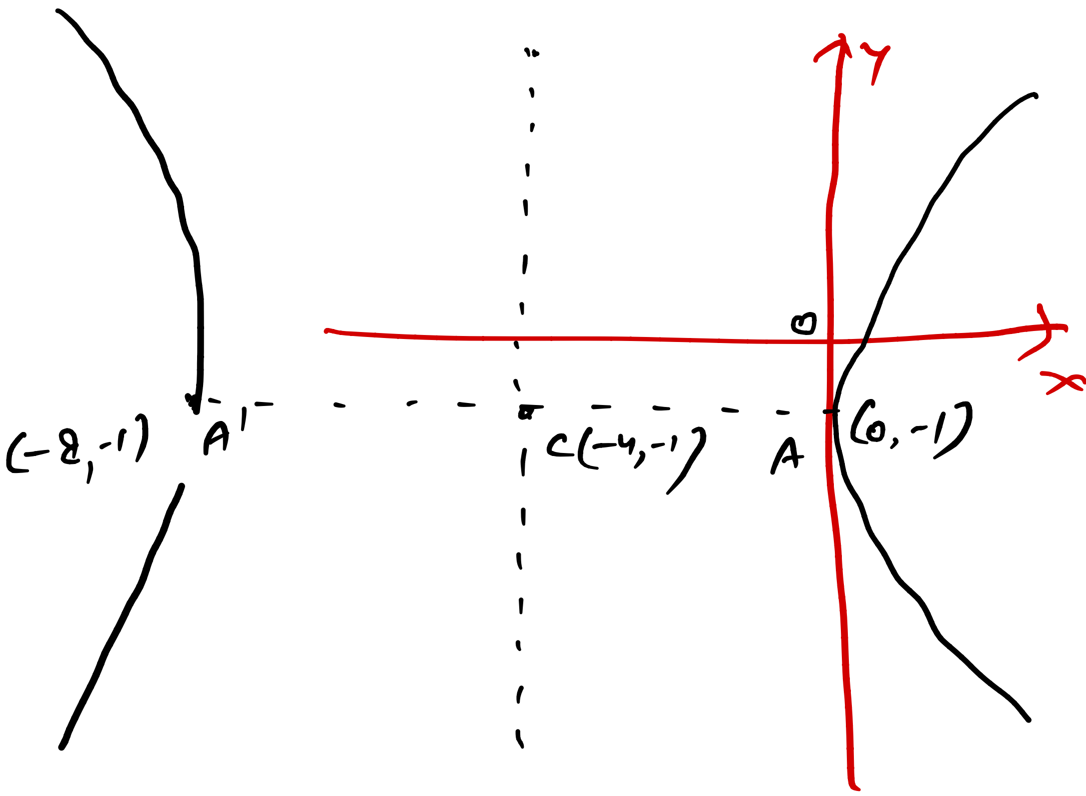
$$9(x^2 + 2.4.x + 16) - 16(y^2 + 2y + 1) = 144$$

$$9(x+4)^2 - 16(y+1)^2 = 144$$

ie $$\frac{(x+4)^2}{16} - \frac{(y+1)^2}{9} = 1$$

(a) $x + 4 = 0$ & $y + 1 = 0$ gives centre

$\Rightarrow$ centre $C(-4, -1)$



(b) $a = 4, \quad b = 3$

$$e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{9}{16}} = \frac{5}{4}$$

(c) $S(-4 \pm ae, -1) \equiv S(-4 \pm 5, -1)$

$\therefore S_1(-9, -1)$ & $S_2(1, -1)$

(d) equation of directrix

$$x = -4 \pm \frac{a}{e},$$

$$x = -4 \pm \frac{16}{5}$$

$\therefore x = -\frac{36}{5}$ & $x = -\frac{4}{5}$

(e) $l(L.R) = \frac{2b^2}{a} = \frac{2.9}{4} = \frac{9}{2}$

(f) Auxiliary circle is $(x+4)^2 + (y+1)^2 = 4^2$

(g) Director circle is $(x+4)^2 + (y+1)^2 = a^2 - b^2$

ie $(x+4)^2 + (y+1)^2 = 7$ Ans

2 Eccentricity of the hyperbola conjugate to the hyperbola $\frac{x^2}{4}-\frac{y^2}{12}=1$ is

(A) $\frac{2}{\sqrt{3}}$ (B) 2 (C) $\sqrt{3}$ (D) $\frac{4}{3}$

Solⁿ 3

$H: \frac{x^2}{4}-\frac{y^2}{12}=1$

$a=2,\ b=\sqrt{12},$

$e_1=\sqrt{1+\frac{b^2}{a^2}}=2$

Let e_2 is eccentricity of its conjugate hyperbola

& we know that

$$\frac{1}{e_1^2}+\frac{1}{e_2^2}=1$$

$\therefore\ e_2=\frac{2}{\sqrt{3}}$ Ans (A)

3 The foci of a hyperbola coincide with the foci of the ellipse $\frac{x^2}{25}+\frac{y^2}{9}=1$. Then the equation of the hyperbola with eccentricity 2 is

(A) $\frac{x^2}{12}-\frac{y^2}{4}=1$ (B) $\frac{x^2}{4}-\frac{y^2}{12}=1$ (C) $3x^2-y^2+12=0$ (D) $9x^2-25y^2-225=0$

Solⁿ

$E: \frac{x^2}{25}+\frac{y^2}{9}=1$

its eccentricity $e_1=\sqrt{1-\frac{b^2}{a^2}}=\frac{4}{5}$

$\therefore$ its foci $S(\pm ae, 0) \equiv (\pm 4, 0)$

Let $H: \frac{x^2}{A^2}-\frac{y^2}{B^2}=1$

its eccentricity $e_2=2$

& its focus $(\pm Ae_2, 0)$

According to question $Ae_2=4 \Rightarrow A=2$

& $B^2=A^2(e_2^2-1)$
$=12$

$\therefore$ equation of Hyperbola is

$\frac{x^2}{4}-\frac{y^2}{12}=1$ Ans

4 If the eccentricity of the hyperbola $x^2 - y^2 \sec^2\alpha = 5$ is $\sqrt{3}$ times the eccentricity of the ellipse $x^2 \sec^2\alpha + y^2 = 25$, then a value of α is :

(A) $\pi/6$ (B) $\pi/4$ (C) $\pi/3$ (D) $\pi/2$

Sol^n 5

H: $\dfrac{x^2}{(\sqrt{5})^2} - \dfrac{y^2}{(\sqrt{5}\cos\alpha)^2} = 1$

& $e_1 = \sqrt{1 + \dfrac{b^2}{a^2}} = \sqrt{1 + \cos^2\alpha}$

E: $\dfrac{x^2}{(5\cos\alpha)^2} + \dfrac{y^2}{(5)^2} = 1$

Here $5\cos\alpha < 5$ (ie $a < b$)

$\therefore$ $5^2\cos^2\alpha = 5^2(1 - e_2^2)$

$e_2 = \sin\alpha$

According to question, $e_1 = \sqrt{3}\, e_2$

ie $1 + \cos^2\alpha = 3\sin^2\alpha$

ie $\sin^2\alpha + 2\cos^2\alpha = 3\sin^2\alpha$

$\tan^2\alpha = 1$

$\therefore$ $\alpha = \pi/4$ Ans B

5 The foci of the ellipse $\frac{x^2}{16}+\frac{y^2}{b^2}=1$ and the hyperbola $\frac{x^2}{144}-\frac{y^2}{81}=\frac{1}{25}$ coincide. Then the value of b^2 is-

(A) 5 (B) 7 (C) 9 (D) 4

Sol^n

H: $\frac{x^2}{\left(\frac{12}{5}\right)^2}-\frac{y^2}{\left(\frac{9}{5}\right)^2}=1$, $e_1=\frac{15}{12}$

$\therefore$ foci $(\pm ae, 0) \equiv (\pm 3, 0)$

E: $\frac{x^2}{16}+\frac{y^2}{b^2}=1$

$a=4$,

& $S(ae, 0)$

when $ae=3$

$\therefore$ $e=3/4$ $(\because a=4)$

& $e=\sqrt{1-\frac{b^2}{a^2}}$

$\therefore$ $\frac{9}{16}=1-\frac{b^2}{a^2} \Rightarrow \frac{b^2}{a^2}=\frac{7}{16}$

$b^2=7$ Ans

6 The graph of the equation $x + y = x^3 + y^3$ is the union of -

(A) line and an ellipse (B) line and a parabola (C) line and hyperbola (D) line and a point

Sol^n

$$x^3 + y^3 = x + y$$

$$(x+y)(x^2 + y^2 - xy) - (x+y) = 0$$

$$(x+y).[x^2 + y^2 - xy - 1] = 0$$

$\Rightarrow$ either $x + y = 0$ (gives line)

or

$$x^2 + y^2 - xy - 1 = 0$$

Compare it with $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

$\therefore$ $a = 1$, $b = 1$, $h = -\frac{1}{2}$, $g = f = 0$, $c = -1$

$$\Delta = abc + 2fgh - af^2 - bg^2 - ch^2$$

$$= -1 + 0 - 0 - 0 + \frac{1}{4} \neq 0$$

& $h^2 < ab$

$\Rightarrow$ represent ellipse.

$\therefore$ Ans (A)

7. The focal length of the hyperbola $x^2 - 3y^2 - 4x - 6y - 11 = 0$, is-

(A) 4 (B) 6 (C) 8 (D) 10

Sol^n 8

$$x^2 - 4x - 3(y^2 + 2y) = 11$$

$$(x^2 - 4x + 4) - 3(y^2 + 2y + 1) = 11 + 4 - 3$$

$$(x-2)^2 - 3(y+1)^2 = 12$$

$$\frac{(x-2)^2}{12} - \frac{(y+1)^2}{4} = 1$$

$$e = \sqrt{1 + \frac{4}{12}} = \frac{2}{\sqrt{3}}$$

$\therefore$ $S(2 \pm ae, -1)$

ie $S_1(6, -1)$ & $S_2(-2, 1)$

focal length of hyperbola
= distance between foci
= 8 Ans

8. The equation $\frac{x^2}{29-p}+\frac{y^2}{4-p}=1$ $(p \neq 4, 29)$ represents -

(A) an ellipse if p is any constant greater than 4

(B) a hyperbola if p is any constant between 4 and 29.

(C) a rectangular hyperbola if p is any constant greater than 29.

(D) no real curve is p is less than 29.

Sol 9

for equation to represent an ellipse.

$29-p>0$ & $4-p>0$

$\Rightarrow \boxed{p<4}$ for ellipse

for hyperbola

$29-p>0$ & $4-p<0$

$\Rightarrow p<29$ & $p>4$

$\Rightarrow \boxed{4<p<29}$ for hyperbola

9 If $\frac{x^2}{\cos^2\alpha} - \frac{y^2}{\sin^2\alpha} = 1$ represents family of hyperbolas where 'α' varies then -

(A) distance between the foci is constant

(B) distance between the two directrices is constant

(C) distance between the vertices is constant

(D) distances between focus and the corresponding directrix is constant

Solⁿ 10

$$e = \sqrt{1 + \frac{\sin^2\alpha}{\cos^2\alpha}} = \sec\alpha$$

$\therefore$ $S(\pm ae, 0) \equiv (\pm 1, 0)$

$\boxed{\text{Here } a = \cos\alpha, \; b = \sin\alpha}$

$\therefore$ Distance between foci $= 2ae = 2$

Distance between directrix

$= 2\frac{a}{e} = 2\cos^2\alpha$

Distance between vertices $= 2\cos\alpha$ & $2\sin\alpha$

Distance between focus & corresponding directrix

$= ae - \frac{a}{e}$

$= \cos\alpha(\sec\alpha - \cos\alpha)$

$= \sin^2\alpha$

Ans A

10 The locus of the point of intersection of the lines $\sqrt{3}x - y - 4\sqrt{3}\,t = 0$ & $\sqrt{3}tx + ty - 4\sqrt{3} = 0$ (where t is a parameter) is a hyperbola whose eccentricity is

(A) $\sqrt{3}$ (B) 2 (C) $\frac{2}{\sqrt{3}}$ (D) $\frac{4}{3}$

Sol (1) from equation (1), $t = \frac{\sqrt{3}x - y}{4\sqrt{3}}$

from equation (2), $t = \frac{4\sqrt{3}}{\sqrt{3}x + y}$

To get locus of point of intersection, remove parameter t, we set

$$\frac{\sqrt{3}x - y}{4\sqrt{3}} = \frac{4\sqrt{3}}{\sqrt{3}x + y}$$

$$\Rightarrow \quad 3x^2 - y^2 = 48$$

$$\frac{x^2}{16} - \frac{y^2}{48} = 1$$

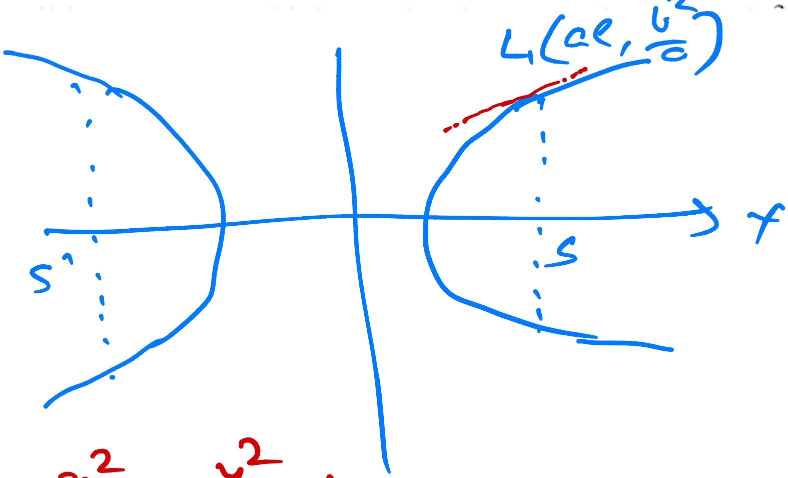
$$\therefore\ e = \sqrt{1 + \frac{b^2}{a^2}} = \sqrt{1 + \frac{48}{16}} = 2$$

Ans

11 The magnitude of the gradient of the tangent at an extremity of latera recta of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is equal to (where e is the eccentricity of the hyperbola)

(A) be (B) e (C) ab (D) ae

Sol^n 13



$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$$

differentiate wrt x,

$$\frac{2x}{a^2}-\frac{2yy'}{b^2}=0$$

$$\Rightarrow \quad y' = \frac{xb^2}{ya^2}$$

$$\therefore \quad (y')_{(ae,\ \frac{b^2}{a})} = \frac{ae\,b^2}{\frac{b^2}{a}\cdot a^2} = e$$

Ans B

12 The number of possible tangents which can be drawn to the curve $4x^2 - 9y^2 = 36$, which are perpendicular to the straight line $5x + 2y - 10 = 0$ is :

(A) zero (B) 1 (C) 2 (D) 4

Sol^n 14

$$\frac{x^2}{9} - \frac{y^2}{4} = 1$$

given line is

$2y = -5x + 10$

its slope $m_1 = -\frac{5}{2}$

Now any tangent to the hyperbola is

$$y = mx \pm \sqrt{a^2m^2 - b^2}$$

where $m \cdot m_1 = -1$

$$y = \frac{2}{5}x \pm \sqrt{9\cdot\left(\frac{2}{5}\right)^2 - 4}$$

$$y = \frac{2}{5}x \pm \sqrt{-\frac{64}{25}}$$

(here $m_1 \cdot m = -1$)

So no real tangent possible. Ans

13 Locus of the point of intersection of the tangents at the points with eccentric angles ϕ and $\frac{\pi}{2}-\phi$ on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is :

(A) $x = a$ (B) $y = b$ (C) $x = ab$ (D) $y = ab$

Soln we know that point of intersection of tangents at θ_1 & θ_2 is

$$x = a\frac{\cos\left(\frac{\theta_1-\theta_2}{2}\right)}{\cos\left(\frac{\theta_1+\theta_2}{2}\right)} \quad \& \quad y = b\tan\left(\frac{\theta_1+\theta_2}{2}\right)$$

So here $\theta_1 = \phi$, $\theta_2 = \frac{\pi}{2}-\phi$

$$\therefore \frac{\theta_1-\theta_2}{2} = \phi - \frac{\pi}{4}$$

$$\& \quad \frac{\theta_1+\theta_2}{2} = \frac{\pi}{4}$$

$$\therefore y = b$$

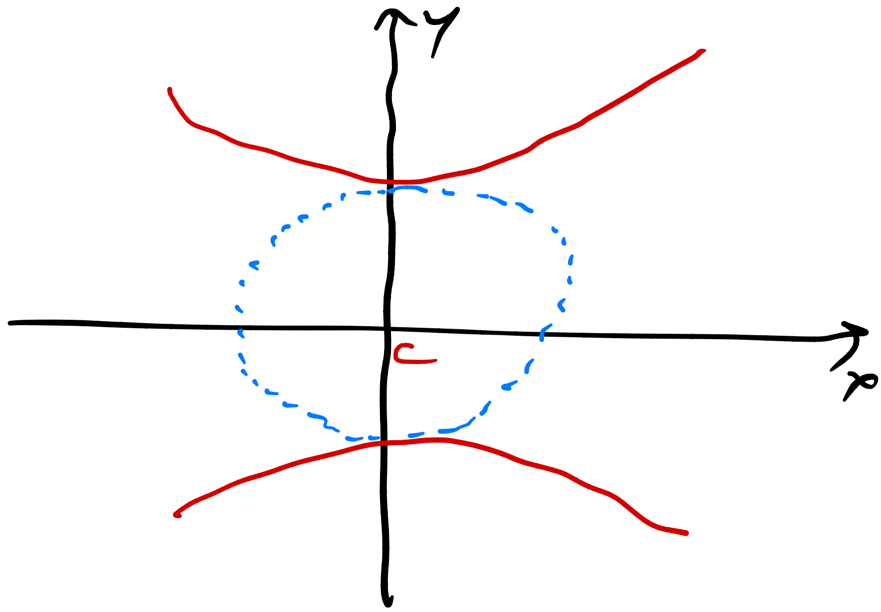
Ans B

14 Locus of the feet of the perpendiculars drawn from either foci on a variable tangent to the hyperbola $16y^2 - 9x^2 = 1$ is

(A) $x^2 + y^2 = 9$ (B) $x^2 + y^2 = 1/9$ (C) $x^2 + y^2 = 7/144$ (D) $x^2 + y^2 = 1/16$

Sol 16

$$\frac{x^2}{1/9} - \frac{y^2}{1/16} = -1$$



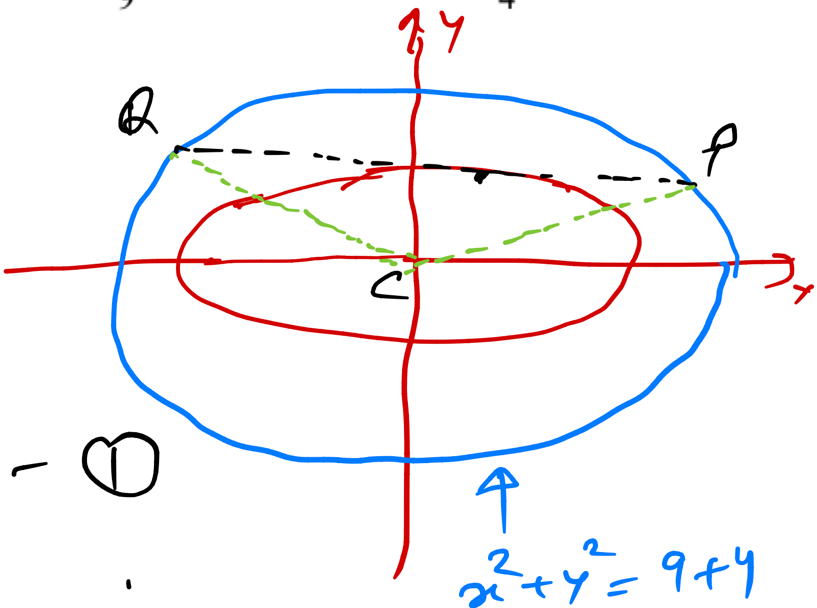
Locus of feet of perpendicular from foci on variable tangent always meet at its corresponding Auxiliary circle

Auxiliary circle of Hyperbola is

$$x^2 + y^2 = \frac{1}{16}$$ Ans

15 A tangent to the ellipse $\frac{x^2}{9}+\frac{y^2}{4}=1$ with centre C meets its director circle at P and Q. Then the product of the slopes of CP and CQ, is -

(A) $\frac{9}{4}$ (B) $\frac{-4}{9}$ (C) $\frac{2}{9}$ (D) $-\frac{1}{4}$



Solⁿ

variable tangent to the ellipse is

$y = mx + \sqrt{9m^2+4}$ — (1)

or $\frac{y-mx}{\sqrt{9m^2+4}} = 1$ — (2)

Homogenise director circle of ellipse with equation 1, we get

$x^2+y^2-13\cdot\left(\frac{y-mx}{\sqrt{9m^2+4}}\right)^2=0$

$x^2\left(1-\frac{13m^2}{9m^2+4}\right)+y^2\left(1-\frac{13}{9m^2+4}\right)+\frac{26m}{\sqrt{9m^2+4}}xy=0$

This represent pair of straight line representing CP & CQ.

Put $y = Mx$, we get

$$m^2\left(1 - \frac{13}{9m^2+4}\right) + \frac{26mM}{\sqrt{9m^2+4}} + \left(1 - \frac{13m^2}{9m^2+4}\right) = 0$$

it is quadratic in M, which has two roots M_1 (slope of CP) & M_2 (slope of CQ)

$$\therefore \quad M_1 M_2 = \frac{1 - \frac{13m^2}{9m^2+4}}{1 - \frac{13}{9m^2+4}}$$

$$= -\frac{4}{9}$$

16 In which of the following cases maximum number of normals can be drawn from a point P lying in the same plane

(A) circle (B) parabola (C) ellipse (D) hyperbola

Sol^n (a)

(A) In case of circle, we can draw infinite

(B) In case of parabola, we can draw atmost 3 normals

(C & D) In case of Ellipse & hyperbola, we can draw at most 4 real normals.

$\therefore$ Ans A

ie $\frac{\cos\theta}{5} = \frac{2h}{17}$ & $\frac{\sin\theta}{5} = 2k$

$\because \cos^2\theta + \sin^2\theta = 1$

$\therefore \frac{4h^2}{17^2} + \frac{2k^2}{1} = \frac{1}{25}$

$\therefore$ Locus of $T(h,k)$ is

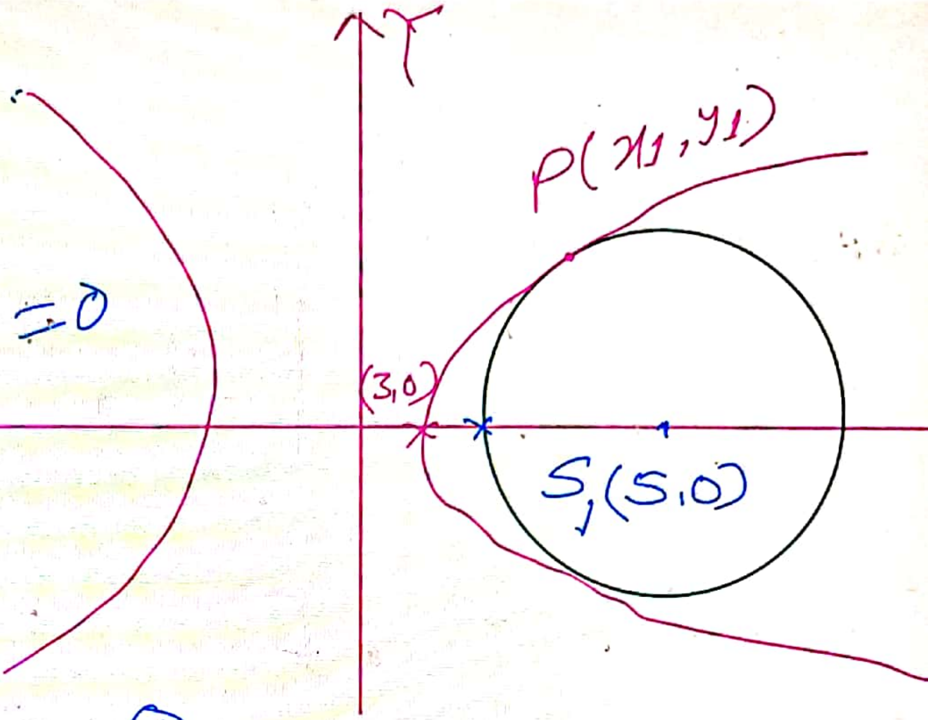
$$\frac{4x^2}{17^2} + \frac{2y^2}{1} = \frac{1}{25}$$

ie an ellipse.

17 With one focus of the hyperbola $\frac{x^2}{9} - \frac{y^2}{16} = 1$ as the centre , a circle is drawn which is tangent to the hyperbola with no part of the circle being outside the hyperbola. The radius of the circle is

(A) less than 2 (B) 2 (C) $\frac{11}{3}$ (D) none

$$e = \sqrt{1 + \frac{16}{9}} = \frac{5}{3}$$



$$\frac{2x}{9} - \frac{2y}{16}\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{x}{9} = \frac{y}{16}\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{16x}{9y} \quad —(1)$$

$$\text{slope of } PS_1 = \frac{y_1 - 0}{x_1 - 5} \quad —(2)$$

$$(1)\cdot(2) = -1 \Rightarrow \frac{16x}{9\cancel{y}}\cdot\frac{\cancel{y}}{x-5} = -1$$

$$\Rightarrow 16xy = 9y(5-x)$$

$$\Rightarrow y(16x - 45 + 9x) = 0$$

$$\Rightarrow y(25x - 45) = 0$$

$$\Rightarrow y = 0, \qquad x = \frac{9}{5}.$$

since, $x = \frac{9}{5}$ doesnot satisfying the equation of Hyperbola. so, $y = 0$

$x = 3$

$P(3, 0)$

radius = 2. Ans ⇒ option B

[MULTIPLE OBJECTIVE TYPE]

18 Let p and q be non-zero real numbers. Then the equation $(px^2 + qy^2 + r)(4x^2 + 4y^2 - 8x - 4) = 0$ represents

(A) two straight lines and a circle, when $r = 0$ and p, q are of the opposite sign.

(B) two circles, when $p = q$ and r is of sign opposite to that of p.

(C) a hyperbola and a circle, when p and q are of opposite sign and $r \neq 0$.

(D) a circle and an ellipse, when p and q are unequal but of same sign and r is of sign opposite to that of p.

$4(x^2+y^2-2x-1)=0$ Always represents a circle with centre (1,0) and $r=\sqrt{2}$

$px^2+qy^2+r=0$

(A) This represents pair of st. line if $r=0$ and p & q are opposite sign.

(B) If $p=q$ and r is opposite to p then circle

(C) If p & q are opposite sign & $r \neq 0$ then Hyperbola

(D) ellipse when p & q are unequal and same sign & r is opposite

19 If θ is eliminated from the equations $a \sec\theta - x\tan\theta = y$ and $b\sec\theta + y\tan\theta = x$ (a and b are constant), then the eliminant denotes the equation of

(A) the director circle of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

(B) auxiliary circle of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

(C) Director circle of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

(D) Director circle of the circle $x^2 + y^2 = \frac{a^2+b^2}{2}$.

$a\sec\theta - x\tan\theta = y \quad — (1)$

$b\sec\theta + y\tan\theta = x \quad — (2)$

$(1)\cdot y + (2)\cdot x \Rightarrow ay\sec\theta + bx\sec\theta = x^2+y^2$

$\sec\theta = \frac{x^2+y^2}{ay+bx} \quad — (3)$

$(1)\, b - (2)\cdot a \Rightarrow$

$ab\sec\theta - bx\tan\theta = by$

$ab\sec\theta + ay\tan\theta = ax$

$(-bx-ay)\tan\theta = by-ax$

$\tan\theta = \frac{ax-by}{bx+ay} \quad — (4)$

$\sec^2\theta - \tan^2\theta = 1$

$\Rightarrow (x^2+y^2)^2 - (ax-by)^2 = (bx+ay)^2$

$\Rightarrow x^4+y^4+2x^2y^2 - a^2x^2 - b^2y^2 + 2abxy = b^2x^2+a^2y^2 + 2abxy$

$(x^2+y^2)^2 = (a^2+b^2)(x^2+y^2)$

$x^2+y^2 = a^2+b^2 \Rightarrow$ (C) & (D)

20

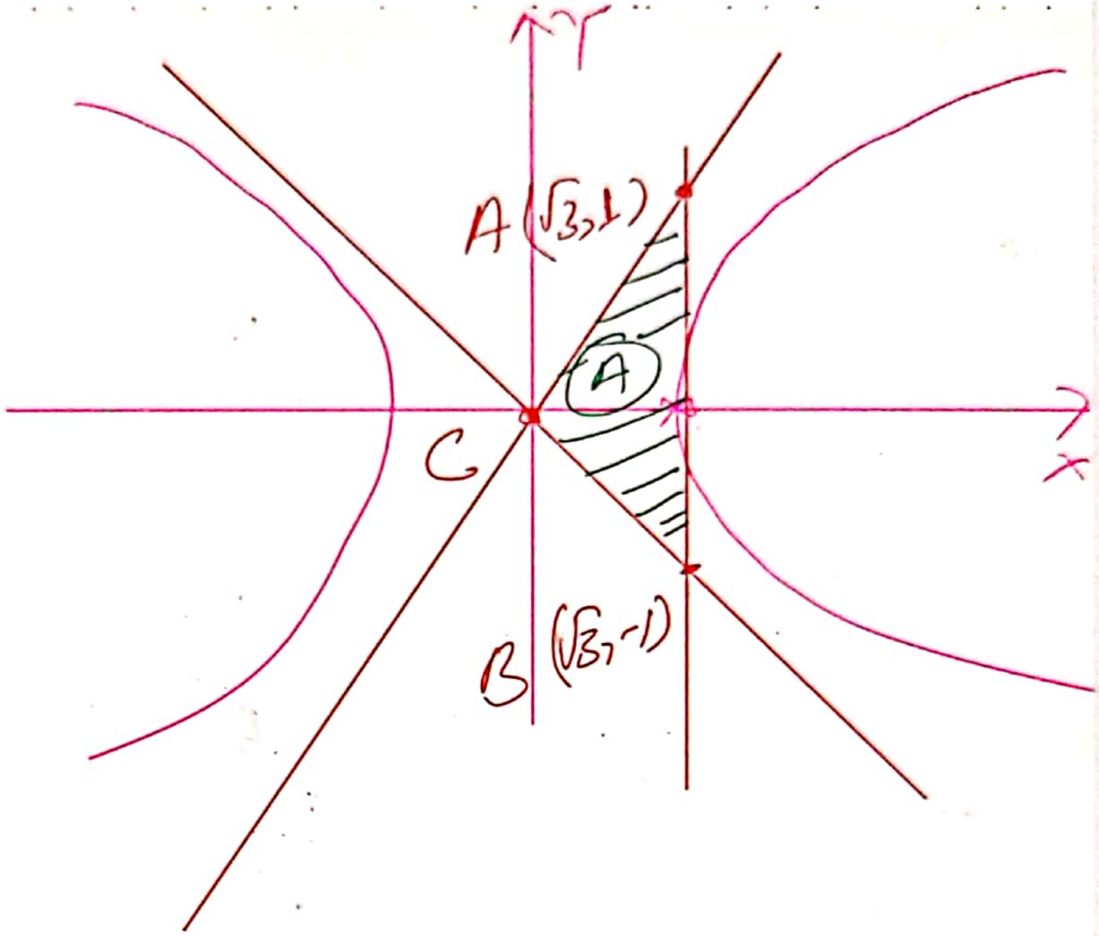
The tangent to the hyperbola, $x^2 - 3y^2 = 3$ at the point $\left(\sqrt{3}, 0\right)$ when associated with two asymptotes constitutes -

(A) isosceles triangle which is not equilateral

(B) an equilateral triangle

(C) a triangles whose area is $\sqrt{3}$ sq. units

(D) a right isosceles triangle.

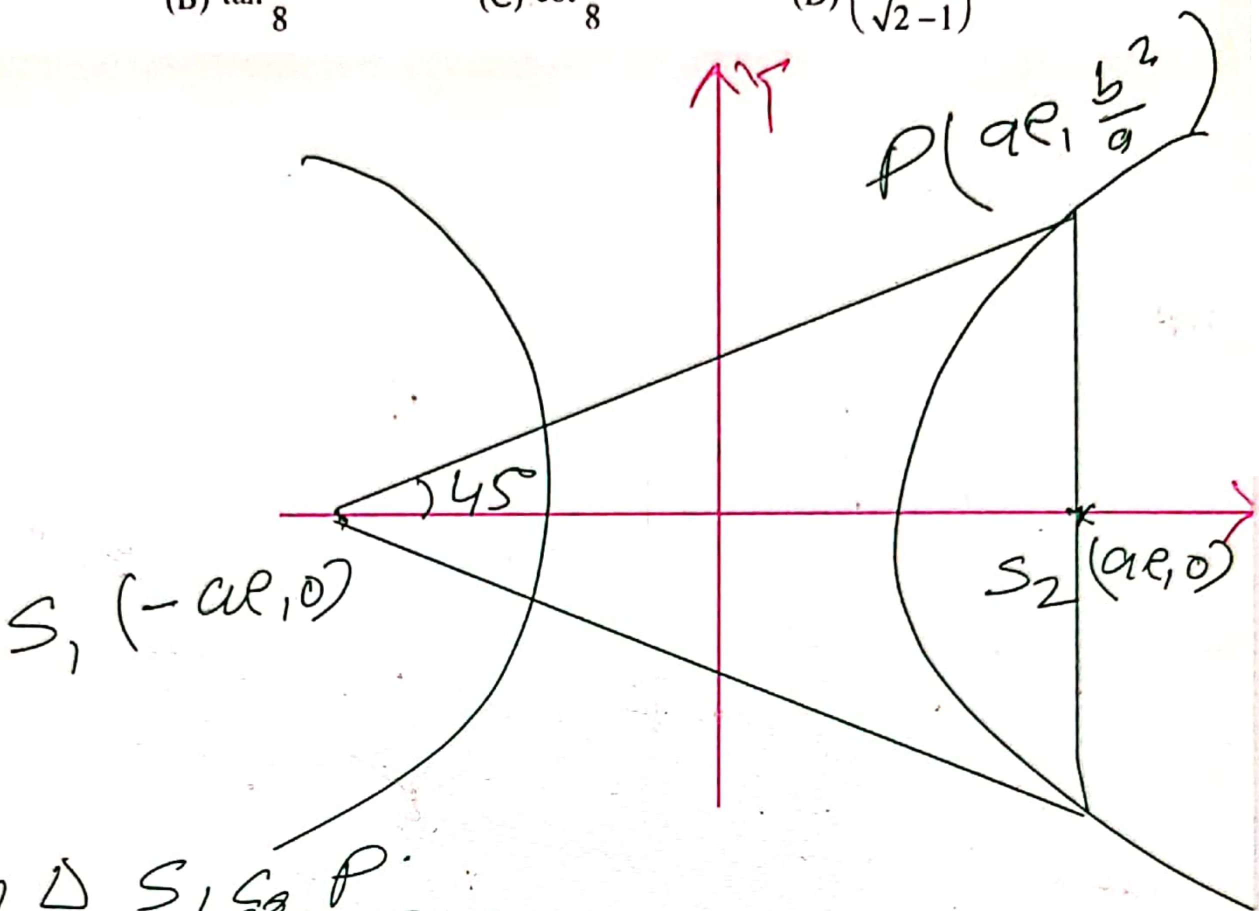


Area of $(\triangle ACB) = \frac{1}{2} \times 2 \times \sqrt{3} = \sqrt{3}$.

B & C

21 If latus rectum of a hyperbola subtends a right angle at other focus of hyperbola, then eccentricity is equal to-

(A) $1-\sqrt{2}$ (B) $\tan\frac{\pi}{8}$ (C) $\cot\frac{\pi}{8}$ (D) $\left(\frac{1}{\sqrt{2}-1}\right)$



In ΔS_1S_2P:

$\tan 45^\circ = \dfrac{\frac{b^2}{a}}{2ae} \Rightarrow 2ae = \dfrac{b^2}{a}$

$\Rightarrow 2e = \dfrac{b^2}{a^2}$ And $e^2 = 1 + \dfrac{b^2}{a^2}$

$= 1 \cdot e^2$

$\Rightarrow e^2 + 2e - 1 = 0$

$e = \dfrac{-2 \pm \sqrt{4+4}}{2} = -1 + \sqrt{2}$

$\Rightarrow$ C & D

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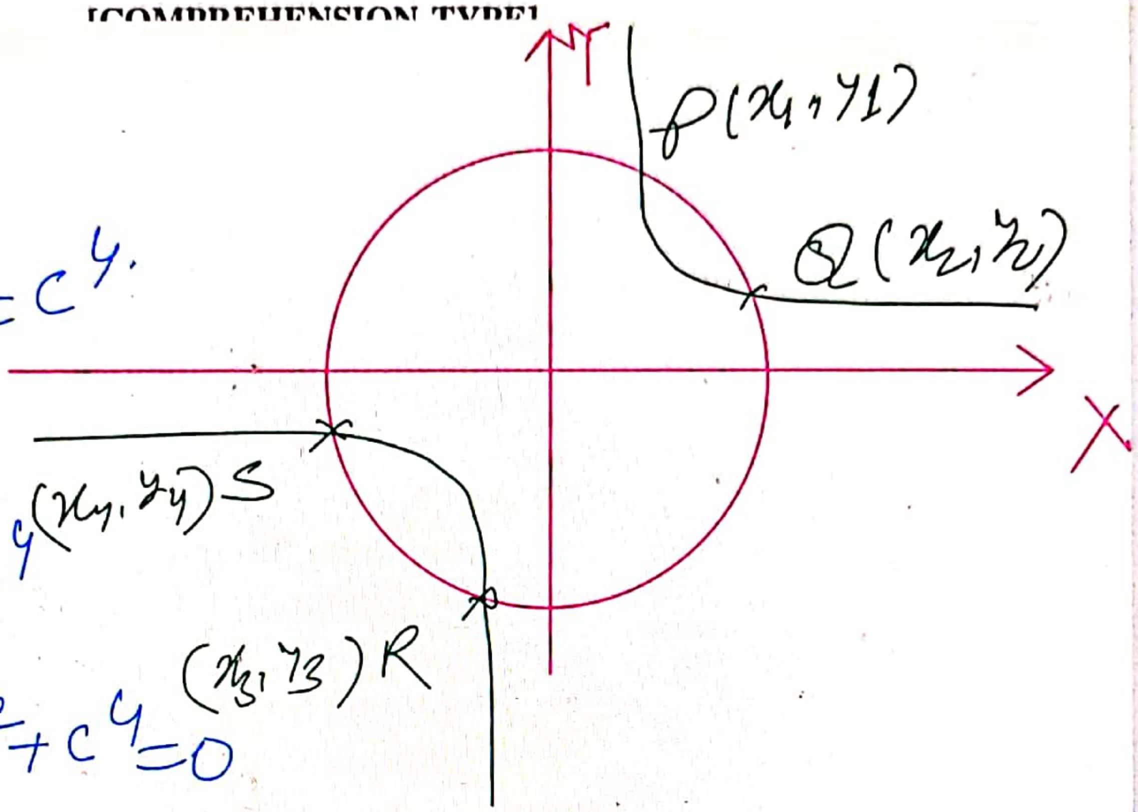
22 If the circle $x^2 + y^2 = a^2$ intersects the hyperbola $xy = c^2$ in four points $P(x_1,y_1)$, $Q(x_2,y_2)$, $R(x_3,y_3)$, $S(x_4,y_4)$, then -

(A) $x_1 + x_2 + x_3 + x_4 = 0$ (B) $y_1 + y_2 + y_3 + y_4 = 0$

(C) $x_1x_2x_3x_4 = c^4$ (D) $y_1y_2y_3y_4 = c^4$

[COMPREHENSION TYPE]



$x^2y^2 = c^4$

$\Rightarrow x^2(a^2 - x^2) = c^4.$

$\Rightarrow a^2x^2 - x^4 = c^4$

$\Rightarrow x^4 - a^2x^2 + c^4 = 0$

$x_1 + x_2 + x_3 + x_4 = 0$

$x_1x_2x_3x_4 = c^4$

similarly for y.

[COMPREHENSION TYPE]

Paragraph for question nos. 31 to 33

The graph of the conic $x^2-(y-1)^2=1$ has one tangent line with positive slope that passes through the origin. the point of tangency being (a, b). Then

23 The value of $\sin^{-1}\left(\frac{a}{b}\right)$ is

(A) $\frac{5\pi}{12}$ (B) $\frac{\pi}{6}$ (C) $\frac{\pi}{3}$ (D) $\frac{\pi}{4}$

24 Length of the latus rectum of the conic is

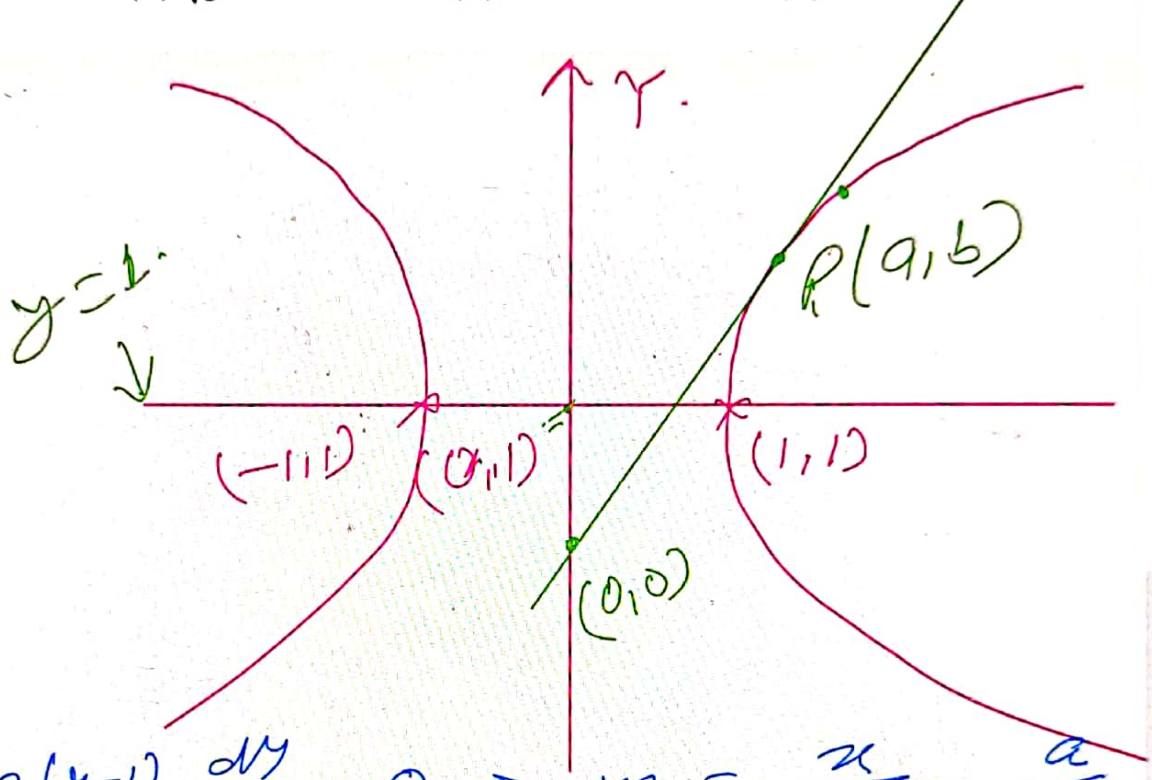
(A) 1 (B) $\sqrt{2}$ (C) 2 (D) none

25 Eccentricity of the conic is

(A) $\frac{4}{3}$ (B) $\sqrt{3}$ (C) 2 (D) none

$a=\sqrt{2}$

$b=2$



$2x-2(y-1)\cdot\frac{dy}{dx}=0 \Rightarrow m=\frac{x}{y-1}=\frac{a}{b-1}$

Slope of line OP $=\frac{b}{a}$

$\Rightarrow \frac{b}{a}=\frac{a}{b-1} \Rightarrow a^2=b^2-b$ — (1)

$a^2=1+(b-1)^2$ — — — (2)

from (1) & (2) $\Rightarrow \cancel{b^2}-b=1+\cancel{b^2}-2b+1 \Rightarrow b=2$

$\therefore a = \sqrt{2}$

$\sin^{-1}\left(\frac{a}{b}\right) = \sin^{-1}\left(\frac{\sqrt{2}}{2}\right) \Rightarrow \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$.

24.) length of LR $= \frac{2b^2}{a} = 2 \Rightarrow$ (C)

25.) $e = \sqrt{1+1} = \sqrt{2} \Rightarrow$ (D)

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0-2

EXERCISE (O-2)

[STRAIGHT OBJECTIVE TYPE]

1. Let F_1, F_2 are the foci of the hyperbola $\frac{x^2}{16} - \frac{y^2}{9} = 1$ and F_3, F_4 are the foci of its conjugate hyperbola. If e_H and e_C are their eccentricities respectively then the statement which holds true is

(A) Their equations of the asymptotes are different.

(B) $e_H > e_C$

(C) Area of the quadrilateral formed by their foci is 50 sq. units.

(D) Their auxiliary circles will have the same equation.

Soln: Ans C

$e_H = 5/4$; $e_C = 5/3$

$$\text{area} = \frac{d_1 d_2}{2} = \frac{100}{2} = 50$$

$A_C: x^2 + y^2 = 16$; $A_H = x^2 + y^2 = 9$; Area of the quadraliteral $= 2(a^2 + b^2)$]

2 AB is a double ordinate of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ such that ΔAOB (where 'O' is the origin) is an equilateral triangle, then the eccentricity e of the hyperbola satisfies

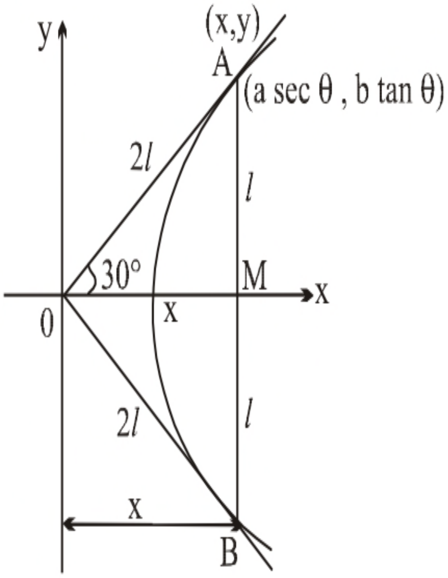
(A) $e>\sqrt{3}$ (B) $1<e<\frac{2}{\sqrt{3}}$ (C) $e=\frac{2}{\sqrt{3}}$ (D) $e>\frac{2}{\sqrt{3}}$

Ans D

[Sol. $\tan 30=\frac{b\tan\theta}{a\sec\theta}$

$\frac{1}{\sqrt{3}}=\frac{b}{a}\sin\theta \Rightarrow \operatorname{cosec}^2\theta=\frac{3b^2}{a^2}>1 \Rightarrow \frac{b^2}{a^2}>\frac{1}{3}$

$e^2-1>\frac{1}{3} \Rightarrow e^2>\frac{4}{3} \Rightarrow e>\frac{2}{\sqrt{3}}$



Alternatively: $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ where $y=l$

$\frac{x^2}{a^2}=1+\frac{l^2}{b^2} \Rightarrow x^2=(b^2+l^2)\frac{a^2}{b^2}$ ....(1)

now $x^2+l^2=4l^2 \Rightarrow x^2=3l^2$ ....(2)

from (1) and (2) $\frac{a^2(b^2+l^2)}{b^2}=3l^2 \Rightarrow a^2b^2+a^2l^2=3b^2l^2$

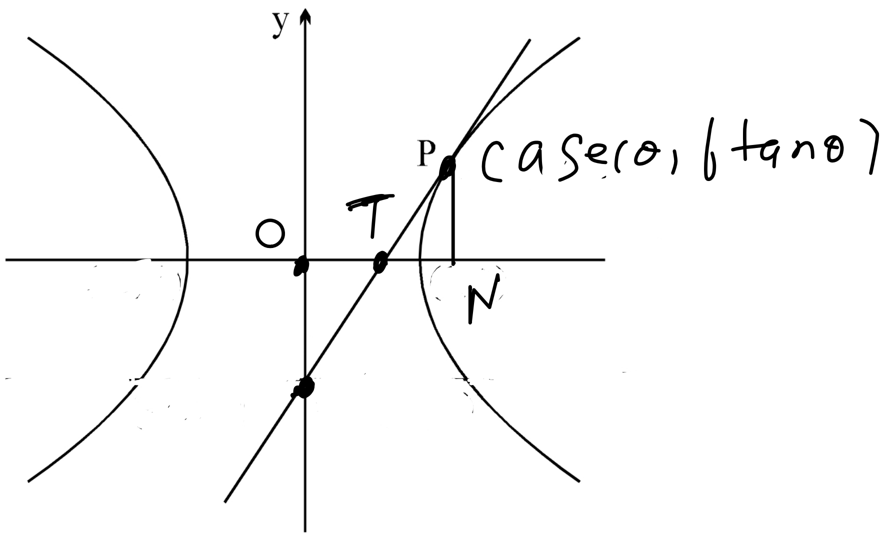
$l^2(3b^2-a^2)=a^2b^2$

$l^2=\frac{a^2b^2}{3b^2-a^2}>0 \Rightarrow 3b^2-a^2>0 \Rightarrow \frac{b^2}{a^2}>\frac{1}{3};\ 1+\frac{b^2}{a^2}>\frac{4}{3} \Rightarrow e^2>\frac{4}{3} \Rightarrow e>\frac{2}{\sqrt{3}}$

Note: $\frac{b}{a}>\frac{1}{\sqrt{3}} \Rightarrow 1+\frac{b^2}{a^2}>\frac{4}{3} \Rightarrow e^2>\frac{4}{3} \Rightarrow e>\frac{2}{\sqrt{3}}$]

3 P is a point on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, N is the foot of the perpendicular from P on the transverse axis. The tangent to the hyperbola at P meets the transverse axis at T . If O is the centre of the hyperbola, then OT.ON is equal to :

(A) e^2 (B) a^2 (C) b^2 (D) b^2/a^2



Eq^n of tangent is

$$\frac{x}{a}\sec\theta - \frac{y}{b}\tan\theta = 1$$

Put $y=0$

$\Rightarrow OT = x = a\cos\theta$

$ON = a\sec\theta$

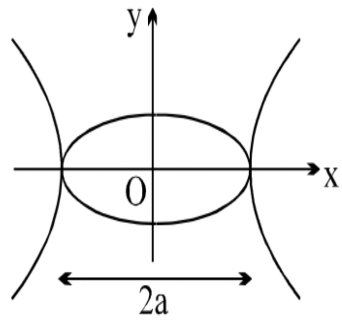
$OT \cdot ON = a^2$

4 Let the major axis of a standard ellipse equals the transverse axis of a standard hyperbola and their director circles have radius equal to 2R and R respectively. If e_1 and e_2 are the eccentricities of the ellipse and hyperbola then the correct relation is

(A) $4e_1^2 - e_2^2 = 6$ (B) $e_1^2 - 4e_2^2 = 2$ (C) $4e_2^2 - e_1^2 = 6$ (D) $2e_1^2 - e_2^2 = 4$

Ans C

[Sol. $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ....(1); $\frac{x^2}{a^2} - \frac{y^2}{b_1^2} = 1$ ....(2)



$R = \sqrt{a^2 - b_1^2}$

$2R = \sqrt{a^2 + b^2}$

$\therefore \quad 2\sqrt{a^2 - b_1^2} = \sqrt{a^2 + b^2}$ $\left[e_1^2 = 1 - \frac{b^2}{a^2};\ e_2^2 = 1 + \frac{b_1^2}{a^2}\right]$

$4(a^2 - b_1^2) = a^2 + b^2$

$4\left(1 - \frac{b_1^2}{a^2}\right) = 1 + \frac{b^2}{a^2}$

$4[(1 - (e_2^2 - 1)] = 1 + 1 - e_1^2$

$8 - 4e_2^2 = 2 - e_1^2$

$4e_2^2 - e_1^2 = 6$ Ans.]

[MULTIPLE OBJECTIVE TYPE]

5 Which of the following equations in parametric form can represent a hyperbolic profile, where 't' is a parameter.

(A) $x = \frac{a}{2}\left(t + \frac{1}{t}\right)$ & $y = \frac{b}{2}\left(t - \frac{1}{t}\right)$

(B) $\frac{tx}{a} - \frac{y}{b} + t = 0$ & $\frac{x}{a} + \frac{ty}{b} - 1 = 0$

(C) $x = e^{t} + e^{-t}$ & $y = e^{t} - e^{-t}$

(D) $x^2 - 6 = 2\cos t$ & $y^2 + 2 = 4\cos^2\frac{t}{2}$

(A) $x = \frac{a}{2}\left(t + \frac{1}{t}\right)$

$$\frac{2x}{a} = t + \frac{1}{t} \quad \cdots\cdots \text{I}$$

$$\frac{2y}{b} = t - \frac{1}{t} \quad \cdots \text{II}$$

Add

$$\frac{2x}{a} + \frac{2y}{b} = 2t$$

$$\frac{x}{a} + \frac{y}{b} = t$$

Put in I (or II)

$$2\frac{x}{a} = \frac{x}{a} + \frac{y}{b} + \frac{1}{\frac{x}{a} + \frac{y}{b}}$$

$$2\left(\frac{x}{a}\right)\left(\frac{x}{a}+\frac{y}{b}\right)=\left(\frac{x}{a}+\frac{y}{b}\right)^2+1$$

$$2\frac{x^2}{a^2}+\frac{2xy}{ab}=\frac{x^2}{a^2}+\frac{y^2}{b^2}+1$$

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{2xy}{ab}=1$$

In above $D=\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} \neq 0$

& $H^2 > AB$ so hyberbola

(B) Similarly in II eleminate t it is not hyberbola

(c) $x = e^{t} + e^{-t}$

$x^2 = e^{2t} + e^{-2t} + 2$ ... I

Similarly

$y^2 = e^{2t} + e^{-2t} - 2$ ... II

I − II

$x^2 - y^2 = 4$

Hyperbola

(D) $x^2 - 6 = 2\cos t = 2(2\cos^2\frac{t}{2} - 1)$

$x^2 - 6 = 4\cos^2\frac{t}{2} - 2$ ... I

$y^2 + 2 = 4\cos^2\frac{t}{2}$ ... II

I − II

$x^2 - y^2 - 8 = -2$

$x^2 - y^2 = 6$ Hyperbola

6 let $A(-1,0)$ and $B(2,0)$ be two points on the x-axis. A point M is moving in xy-plane (other than x-axis) in such a way that $\angle MBA = 2\angle MAB$, Then the point M moves along a conic whose

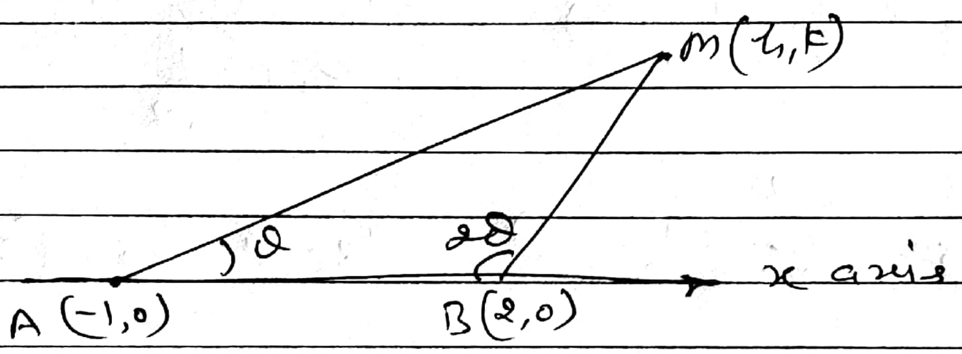
(A) Co-ordinate of vertices are $(\pm 3, 0)$

(B) length of latus rectum equals 6.

(C) Eccentricity equals 2

D equation of directrices are $x = \pm 1/2$

Solution:-



here $\tan 2\theta = \dfrac{-k}{h-2}$ and $\tan\theta = \dfrac{k}{h+1}$ — (2)

$\Rightarrow \dfrac{2\tan\theta}{1-\tan^2\theta} = \dfrac{-k}{h-2}$ — (1)

solving (1) + (2) $\dfrac{2\left(\frac{k}{h+1}\right)}{1-\left(\frac{k}{h+1}\right)^2} = \dfrac{-k}{h-2} \Rightarrow \dfrac{x^2}{1} - \dfrac{y^2}{3} = 1$

is a hyperbola

$\Rightarrow a = 1,\ b = \sqrt{3},\ e = \sqrt{1 + \dfrac{b^2}{a^2}} = 2$

Foci $\equiv (\pm 2, 0)$

equation of Directrices $\Rightarrow x = \pm a/e = \pm 1/2$

length of L.R. $= \dfrac{2b^2}{a} = 6$

7 Hyperbola $\frac{x^2}{a^2} - \frac{y^2}{3} = 1$ of eccentricity e is confocal with the ellipse $\frac{x^2}{8} + \frac{y^2}{4} = 1$. Let A, B, C, D are points of intersection of hyperbola & ellipse, then

(A) $e = 5/2$ (B) $e = 2$ (C) A, B, C, D are concyclic points (D) Number of common tangent is 2

<u>Solⁿ:-</u>

$$\frac{x^2}{8} + \frac{y^2}{4} = 1 \Rightarrow a_1 = 2\sqrt{2},\ b_1 = 2$$

$$e_1 = \frac{1}{\sqrt{2}}$$

$$\frac{x^2}{a^2} - \frac{y^2}{3} = 1 \Rightarrow a_2 = a,\ b_2 = \sqrt{3},\ e_2 = \sqrt{1 + \frac{3}{a^2}}$$

both are confocal so $a_1 e_1 = a_2 e_2$

$$\Rightarrow \underline{\underline{a = 1}}$$

$$\Rightarrow e_2 = 2,$$

It is obvious that A, B, C, D are the vertices of a rectangle so concyclic.

$y = mx + \sqrt{a^2m^2 + b^2}$ tangent to ellipse

& $y = mx + \sqrt{a^2m^2 - b^2}$ " " Hyperbola

$$\Rightarrow 8m^2 + 4 = m^2 - 3$$

$$\Rightarrow 7m^2 = -7 \Rightarrow m^2 = -1 \text{ not possible}$$

So no common tangent can be drawn.

8 If the ellipse $4x^2+9y^2+12x+12y+5=0$ is confocal with a hyperbola having same principal axes, then

(A) Angle between normals at their each point of intersection is 90°.

(B) Centre of the ellipse is $\left(-\frac{3}{2}, -\frac{2}{3}\right)$

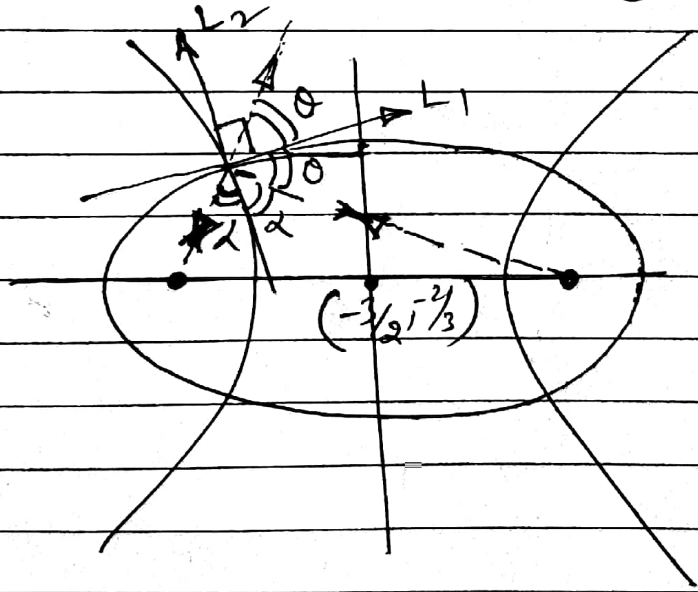
(C) Distance between foci of the hyperbola is $\frac{2\sqrt{10}}{3}$

(D) Ellipse and hyperbola has same length of latus rectum.

Sol^n:-

$$4x^2+9y^2+12x+12y+5=0$$

$$\Rightarrow \frac{\left(x+\frac{3}{2}\right)^2}{2}+\frac{\left(y+\frac{2}{3}\right)^2}{8/9}=1$$



at the point of intersection normals will be perpendicular because of reflection property of ellipse & Hyperbola which is shown in figure

Centre of the ellipse is $\left(-\frac{3}{2}, -\frac{2}{3}\right)$

Distance between foci $= 2ae = 2\sqrt{2}\cdot\sqrt{1-\frac{4}{9}}$

$$= \frac{2\sqrt{10}}{3}$$

9 If the eccentricity of the ellipse $\frac{x^2}{(\log a)^2} + \frac{y^2}{(\log b)^2} = 1$ $(a > b > 0, a, b \neq 1)$ is $\frac{1}{\sqrt{2}}$

and e be the eccentricity of the hyperbola $\frac{x^2}{(\log_b a)^2} - \frac{y^2}{1} = 1$, then e^2 is greater than

(A) $3/2$ (B) $1/2$ (C) $2/3$ (D) $5/4$

Solⁿ:-

$$\frac{1}{\sqrt{2}} = \sqrt{1 - \frac{(\log b)^2}{(\log a)^2}} = \sqrt{1 - (\log_a b)^2}$$

$$\Rightarrow \frac{1}{2} = 1 - (\log_a b)^2 \Rightarrow (\log_a b)^2 = \frac{1}{2} \quad \text{--- (1)}$$

and $e^2 = 1 + \frac{1}{(\log_b a)^2} = 1 + (\log_a b)^2 = 3/2$

S-1

1. Find the equation to the hyperbola whose directrix is $2x+y=1$, focus $(1, 1)$ & eccentricity $\sqrt{3}$. Find also the length of its latus rectum.

Sol.

Equation $(x-1)^2+(y-1)^2=3\dfrac{(2x+y-1)^2}{5}$

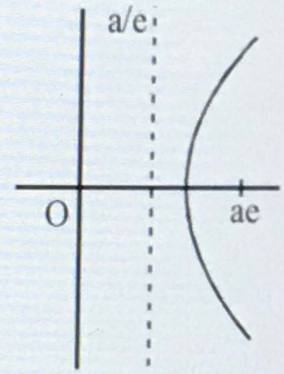
L.R. $=2e$ (distance from focus to this foot of the corresponding directrix)

[$\perp$ distance from $(1, 1)$ on $2x+y=1$]

$$=2e\left(ae-\frac{a}{e}\right)$$

$$=2\sqrt{3}\left[\frac{2+1-1}{\sqrt{5}}\right]=\frac{4\sqrt{3}}{\sqrt{5}}=\frac{4\sqrt{15}}{5}$$

$$=\sqrt{\frac{48}{5}}$$ **Ans.]**



2. The hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ passes through the point of intersection of the lines, $7x + 13y - 87 = 0$ and $5x - 8y + 7 = 0$ & the latus rectum is $32\sqrt{2}/5$. Find 'a' & 'b'.

Sol.

$$\frac{2b^2}{a} = \frac{32\sqrt{2}}{5} \quad \text{......(1)}$$

[New typed]

$$\frac{25}{a^2} - \frac{16}{b^2} = 1 \quad \text{......(2)}$$

solve (1) and (2)

$$7x + 13y - 87 = 0$$
$$5x - 8y + 7 = 0$$

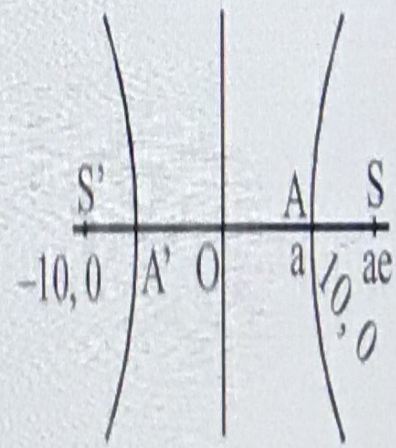
Solving $x = 5$ and $y = 4$]

3. For the hyperbola $\frac{x^2}{100} - \frac{y^2}{25} = 1$, prove that

(i) eccentricity $= \sqrt{5}/2$

(ii) SA. S′A = 25, where S & S′ are the foci & A is the vertex.

Sol:



$$e^2 = 1 + \frac{b^2}{a^2} = 1 + \frac{25}{100} = \frac{5}{4}$$

$$e = \frac{\sqrt{5}}{2}$$

$$SA \cdot S'A = (ae - a)(ae + a) = a^2e^2 - a^2$$

$$= a^2 + b^2 - a^2 = 25 \textbf{ Ans.}]$$

4. Find the centre, the foci, the directrices, the length of the latus rectum, the length & the equations of the axes of the hyperbola $16x^2 - 9y^2 + 32x + 36y - 164 = 0$.

$$\frac{(x+1)^2}{9} - \frac{(y-2)^2}{16} = 1 \qquad e = 5/3$$

① centre $= (-1, 2)$

② foci : $x + 1 = \pm(3)5/3$

$x + 1 = \pm 5 \quad , \quad y =$

$$\left.\begin{matrix} x = +4 \; , \; y = 2 \\ x = -6 \; , \; y = 2 \end{matrix}\right\} \text{foci}$$

③ Directrix

$x - 1 = \pm(3)/e$

$x - 1 = \pm \dfrac{9}{5}$

$x = \pm \dfrac{9}{5} + 1 = \dfrac{14}{5}$ or $-\dfrac{4}{5}$

5. Find the equation of the tangent to the hyperbola $x^2 - 4y^2 = 36$ which is perpendicular to the line $x - y + 4 = 0$.

[Sol. Any line perpendicular to $x - y + 4 = 0$ is

$$x + y = 1 \qquad .....(1)$$

If (1) is tangent to $y = -x + \lambda$

$$\frac{x^2}{36} - \frac{y^2}{9} = 1 \text{ then}$$

$$\lambda^2 = 36(-1)^2 - (9)$$

$$\lambda^2 = 27 \Rightarrow \lambda = \lambda = \pm 3\sqrt{3}$$

$\therefore \quad x + y = \pm 3\sqrt{3}$]

6. Tangents are drawn to the hyperbola $3x^2 - 2y^2 = 25$ from the point (0, 5/2). Find their equations.

Let Eqn of tangent is

$$y = mx + \sqrt{a^2m^2 - b^2}$$

$$\frac{x^2}{25/3} - \frac{y^2}{25/2} = 1$$

$$\therefore \quad y = mx + \sqrt{\frac{25}{3}m^2 - \frac{25}{2}}$$

or by passing through $(0, 5/2)$

$$\left(\frac{5}{2}\right)^2 = \frac{25m^2}{3} - \frac{25}{2}$$

$$\frac{2\cancel{5}}{4} + \frac{\cancel{25}}{2} = \cancel{25}\frac{m^2}{3}$$

$$\frac{9}{4} = m^2 \Rightarrow \quad m = \pm \frac{3}{2}$$

$\therefore \quad 3x + 2y - 5 = 0 \quad \& \quad 3x - 2y + 5 = 0$

7. A conic C satisfies the differential equation, $(1 + y^2)dx - xy\,dy = 0$ and passes through the point (1,0). An ellipse E which is confocal with C having its eccentricity equal to $\sqrt{2/3}$.

(a) Find the length of the latus rectum of the conic C

(b) Find the equation of the ellipse E.

(c) Find the locus of the point of intersection of the perpendicular tangents to the ellipse E.

[Sol. $(1 + y^2)dx = xy\,dy$

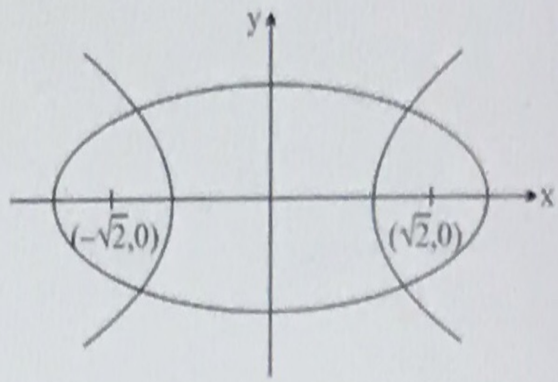
$$\int \frac{dx}{x} = \int \frac{y\,dy}{1+y^2}$$

$2\,ln\,x = ln(1 + y^2) + C$

given $x = 1,\ y = 0 \quad \Rightarrow \quad C = 0$

hence equation of C is $x^2 - y^2 = 1$

which is a rectangular hyperbola with eccentricity $e = \sqrt{2}$.



(i) length of the latus rectum of rectangular hyperbola $= 2a = 2$ **Ans.**

(ii) Now for ellipse,

$$ae = \sqrt{2} \Rightarrow a^2e^2 = 2 \Rightarrow a^2 \cdot \frac{2}{3} = 2 \Rightarrow a^2 = 3$$

and $\quad b^2 = a^2(1 - e^2) = 3\left(1 - \frac{2}{3}\right) = 1.$

Hence equation of ellipse is $\frac{x^2}{3} + \frac{y^2}{1} = 1$ **Ans.**

(iii) Locus of the point of intersection of the perpendicular tangents is the director circle of the ellipse equation is $x^2 + y^2 = 4$.]

8. A hyperbola has one focus at the origin and its eccentricity $= \sqrt{2}$ and one of its directrix is $x + y + 1 = 0$. Find the equation to its asymptotes.

Sol.

$$\sqrt{x^2+y^2} = \sqrt{2}\left(\frac{x+y+1}{\sqrt{2}}\right)$$

$$\therefore \quad x^2+y^2=(x+y+1)^2$$

$$2xy+2x+2y+1=0$$

Let the combined equation of the asymptotes is

$$2xy+2x+2y+c=0$$

put $D=0$ to get $c=2$

hence combined equation of the asymptotes are

$$xy+x+y+1=0$$

$$(x+1)(y+1)=0 \quad \Rightarrow \quad x+1=0 \text{ and } y+1=0 \quad]$$

9. If the lines $x + y + 1 = 0$ and $2x - y + 2 = 0$ are the asymptotes of a hyperbola. If the line $x - 2 = 0$ touches the hyperbola then the equation of the hyperbola is $4(x + y + 1)(2x - y + 2) = \lambda$. Find the value of λ.

[Sol. equation is $(x+y+1)(2x-y+2)=c$

solving it with $x=2$, we get

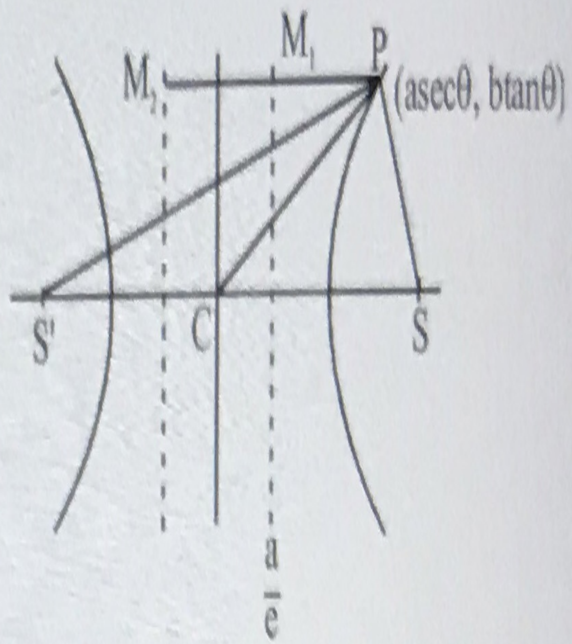
$$y^2 - 3y + (c - 18) = 0$$

put $D=0 \Rightarrow c = \dfrac{81}{4}$]

10. If C is the centre of a hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, S, S′ its foci and P a point on it.

Prove that $SP. S'P = CP^2 - a^2 + b^2$.

[Sol.



$SP \cdot S'P$

$= e\left(a\sec\theta - \frac{a}{e}\right) \times e\left(a\sec\theta - \frac{a}{e}\right)$

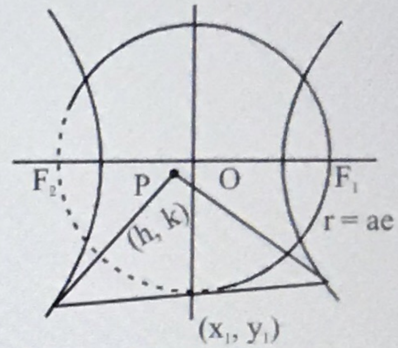
$= a^2 \cdot e^2 \sec^2\theta - a^2$

$= (a^2 + b^2)\sec^2\theta - a^2$

$= a^2\tan^2\theta + b^2\sec^2\theta$

$= a^2(\sec^2\theta - 1) + b^2(1 + \tan^2\theta) = (CP)^2 - a^2 + b^2$ **Ans.**]

11. Chords of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ are tangents to the circle drawn on the line joining the foci as diameter. Find the locus of the point of intersection of tangents at the extremities of the chords.



[Sol. $x^2 + y^2 = a^2e^2$

$x^2 + y^2 = (a^2 + b^2)$ ......(1)

C.O.C of P(h, k)

$$\frac{hx}{a^2}-\frac{ky}{b^2}=1 \qquad \text{......(2)}$$

This must be the same as that of the tangent at (x_1, y_1) to the circle

$$xx_1 + yy_1 = (a^2 + b^2)$$

$$\frac{x_1 .a^2}{h}=\frac{-y_1b^2}{k}=\frac{(a^2+b^2)}{1}$$

$$x_1=\frac{h(a^2+b^2)}{a^2}; \qquad y_1=\frac{-k(a^2+b^2)}{b^2}$$

Hence locus of P is

$$\frac{h^2}{a^4}(a^2+b^2)^2+\frac{k^2(a^2+b^2)^2}{b^4}=(a^2+b^2)$$

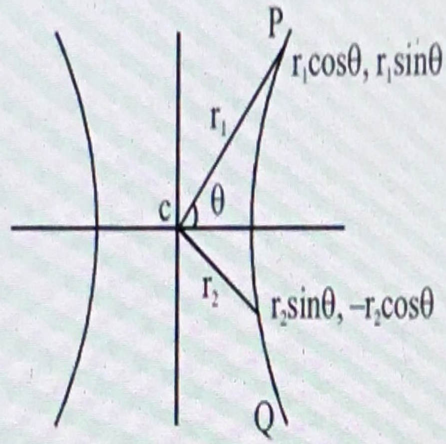
$$\frac{x^2}{\left(\frac{a^4}{a^2+b^2}\right)}+\frac{y^2}{\left(\frac{b^4}{a^2+b^2}\right)}=1$$

$$\frac{x^2}{a^4}+\frac{y^2}{b^4}=\frac{1}{a^2+b^2}$$

$$\frac{x^2}{a^4}+\frac{y^4}{b^4}=\frac{1}{(a^2+b^2)} \quad]$$

12. If two points P & Q on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ whose centre is C be such that CP is perpendicular to CQ & a < b, then prove that $\frac{1}{CP^2}+\frac{1}{CQ^2}=\frac{1}{a^2}-\frac{1}{b^2}$.

[Sol. $\frac{1}{CP^2}+\frac{1}{CQ^2}=\frac{1}{r_1}+\frac{1}{r_2^2}$



Now $\frac{x-0}{\cos\theta}=\frac{y-0}{\sin\theta}=r_1$

$x=r_1\cos\theta$

$y=r_1\sin\theta$

$$\frac{r_1^2\cos^2\theta}{a^2}-\frac{r_1^2\sin^2\theta}{b^2}=1 \Rightarrow \frac{\cos^2\theta}{a^2}-\frac{\sin^2\theta}{b^2}=\frac{1}{r_1^2}$$

$$\frac{r_2^2\sin^2\theta}{a^2}-\frac{r_2^2\cos^2\theta}{b^2}=1 \Rightarrow \frac{\sin^2\theta}{a^2}-\frac{\cos^2\theta}{b^2}=\frac{1}{r_2^2}$$

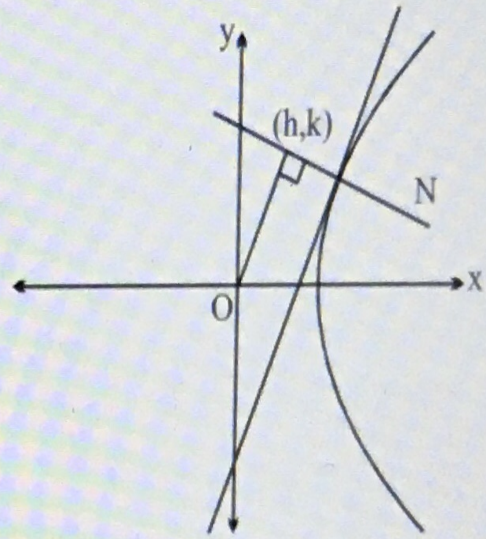
$\therefore \quad \frac{1}{r_1}+\frac{1}{r_2^2}=\frac{1}{a^2}-\frac{1}{b^2}$ Hence proved]

13. Locus of the feet of the perpendicular from centre of the hyperbola $x^2 - 4y^2 = 4$ upon a variable normal to it has the equation, $(x^2 + y^2)^2 (4y^2 - x^2) = \lambda x^2y^2$, find λ.

[Sol. Equation of N is

$hx + ky = h^2 + k^2$

Compair it with the normal are Q



$$\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2$$

i.e., $$\frac{2x}{\sec\theta} + \frac{y}{\tan\theta} = 5$$ $\boxed{\begin{array}{c}\frac{x^2}{4} - \frac{y^2}{1} = 1 \\ a = 2 ; b = 1\end{array}}$

$$\frac{h \cdot \sec\theta}{2} = k\tan\theta = \frac{h^2 + k^2}{5}$$

$$\sec\theta = \frac{2(h^2 + k^2)}{5h}$$

$$\tan\theta = \frac{h^2 + k^2}{5k}$$

$$\therefore \quad \frac{4}{25}\frac{(h^2 + k^2)^2}{25h^2} - \frac{(h^2 + k^2)^2}{25k^2} = 1$$

$$4\frac{(h^2 + k^2)^2}{20}\left(\frac{4}{h^2} - \frac{1}{k^2}\right) = 1 \Rightarrow k = 25 \Rightarrow A.]$$

14. Let P (a sec θ, b tan θ) and Q (a sec φ, b tan φ), where $\theta + \phi = \frac{\pi}{2}$, be two points on the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If (h, k) is the point of intersection of the normals at P & Q, then find k.

[Sol. $P(a \sec\theta, b\tan\theta)$ $\qquad \theta + \phi = \frac{\pi}{2}$

$Q(a \sec\phi, b\tan\phi) = Q(a\,\text{cosec}\,\phi, b\cot\phi)$

Normal at P $\qquad \frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2$

Normal at Q $\qquad \frac{ax}{\cos ec\theta} + \frac{by}{\cot\theta} = a^2 + b^2$

$$ax\cos\theta + by\cot\theta = a^2 + b^2 \qquad \text{......(1)}$$

$$ax\sin\theta + by\tan\theta = a^2 + b^2 \qquad \text{......(2)}$$

multiplying (1) by sin θ and (2) by cos θ and subtract to get the answer.

$$D = \begin{vmatrix} a\cos\theta & b\cot\theta \\ a\sin\theta & b\tan\theta \end{vmatrix} = ab\sin\theta - ab\cos\theta = ab(\sin\theta - \cos\theta)$$

$$D_1 = \begin{vmatrix} a^2+b^2 & b\cos\theta \\ a^2+b^2 & b\tan\theta \end{vmatrix} = (a^2+b^2)b(\tan\theta - \cot\theta)$$

$$D_2 = \begin{vmatrix} a\cos\theta & a^2+b^2 \\ a\sin\theta & a^2+b^2 \end{vmatrix}$$

$$x = \frac{(a^2+b^2)\,b\,(\tan\theta - \cos\theta)}{ab\,(\sin\theta - \cos\theta)}$$

$$y = k = \frac{(a^2+b^2)\,a\,(\cos\theta - \sin\theta)}{ab\,(\sin\theta - \cos\theta)}$$

$$= -\frac{a^2+b^2}{b}$$ **Ans.]**

S-2

EXERCISE (S-2)

1. Tangent and normal are drawn at the upper end (x_1,y_1) of the latus rectum P with $x_1>0$ and $y_1>0$, of the hyperbola $\frac{x^2}{4}-\frac{y^2}{12}=1$, intersecting the transverse axis at T and G respectively. Find the area of the triangle PTG.

$$e = \sqrt{1+\frac{12}{4}} = 2$$

$$a = 2, \quad b = \sqrt{12}$$

$$x_1 = ae = 4$$

$$y_1 = \frac{b^2}{a} = \frac{12}{2} = 6$$

$\Rightarrow$ P(4, 6)

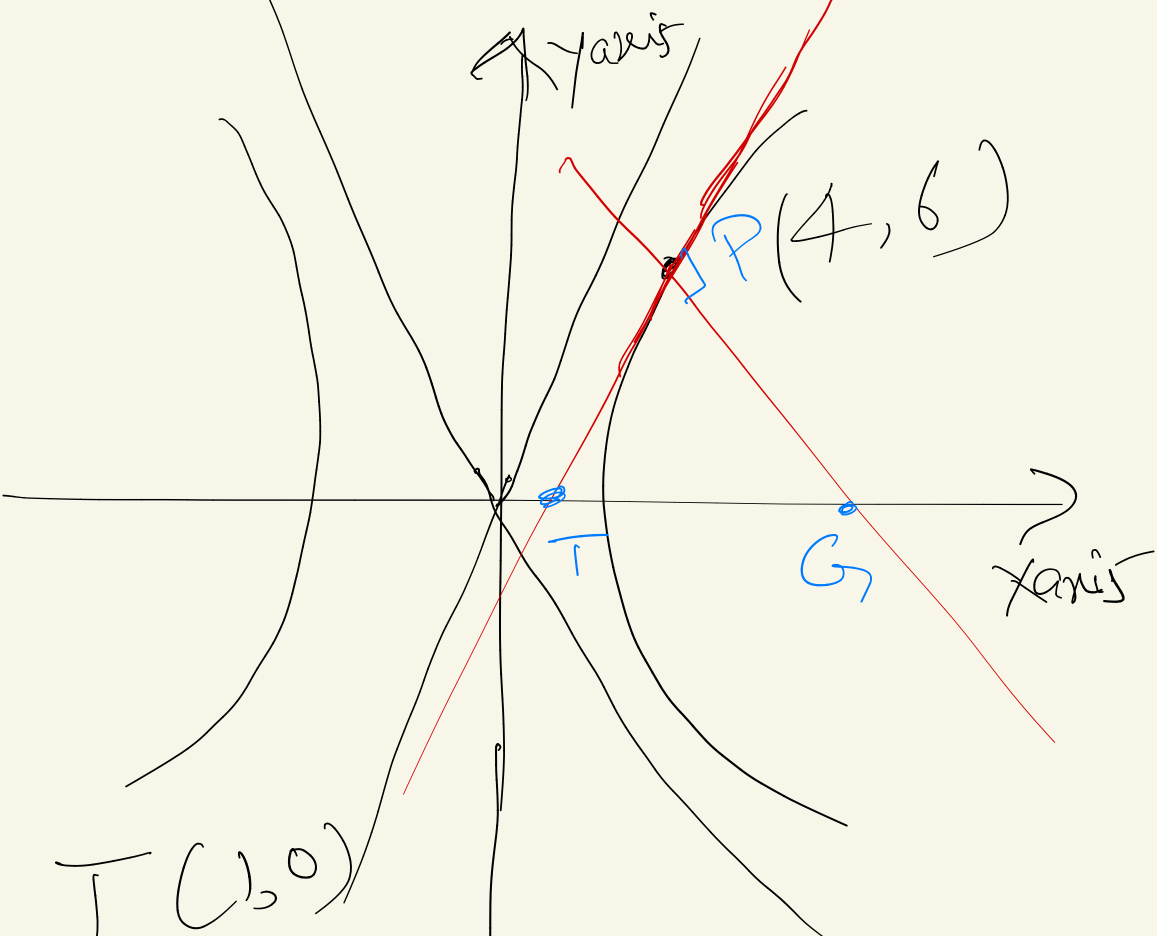
equation of tangent at P

$$\Rightarrow \frac{x4}{4} - \frac{y \times 6}{12} = 1$$

$$\Rightarrow 2x - y - 2 = 0$$

equation of normal at P

$$x + 2y - 16 = 0$$



T(1, 0)
G(16, 0)

Area of $\triangle PTG$

$= \frac{1}{2} \times 15 \times 6 = 45$

2. Find the equations of the tangents to the hyperbola $x^2 - 9y^2 = 9$ that are drawn from (3, 2). Find the area of the triangle that these tangents form with their chord of contact.

x^2 y^2

Let m be the slope of the tangent drawn from $(3, 2)$.

$\Rightarrow y - 2 = m(x - 3)$

$\Rightarrow y = mx + 2 - 3m$

using condition for tangency

$$c^2 = m^2a^2 - b^2$$

$\Rightarrow (2 - 3m)^2 = m^2 9 - 1$

$\Rightarrow 4 + 9m^2 - 12m = 9m^2 - 1$

$\Rightarrow$ Either m is not defined

or $m = \frac{5}{12}$

$\Rightarrow$ equation of tangents are—

$\boxed{x = 3}$ $\boxed{12y = 5x + 9}$

Point of contact of $x = 3$ is $(3, 0)$

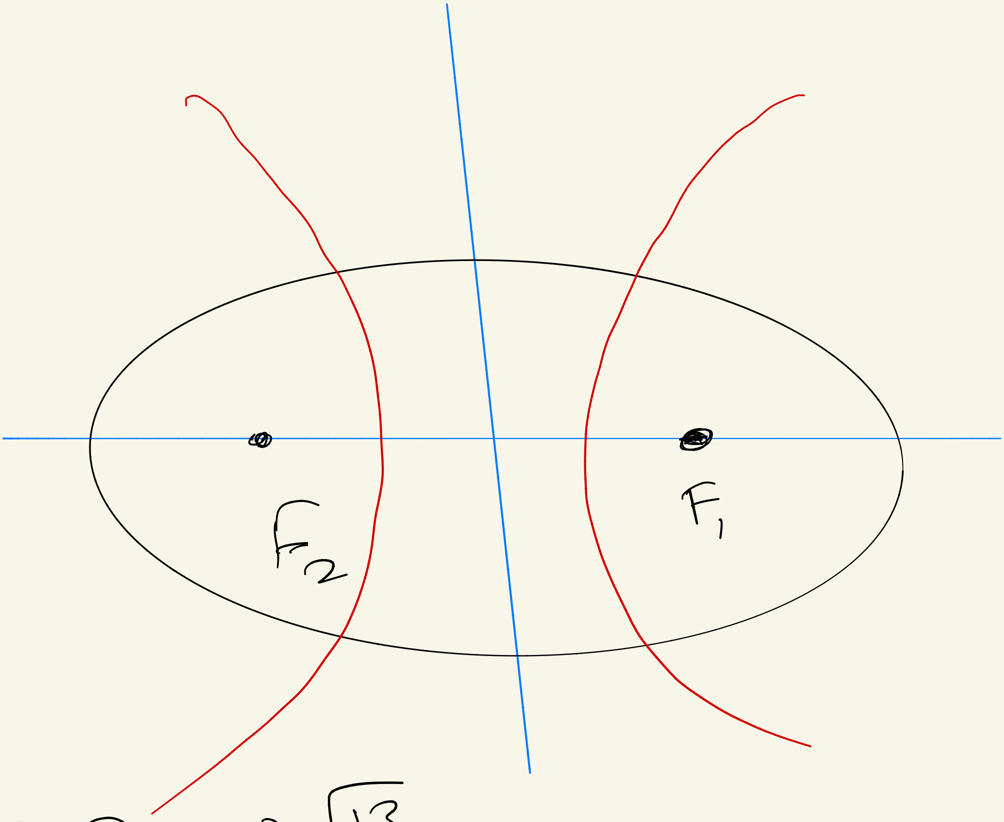
Point of contact of $y = \frac{5}{12}x + \frac{3}{4}$ is $\left(-5, -\frac{4}{3}\right)$

Area of Δ formed by tangents & chord of contact

$$= \frac{1}{2} \left| \begin{vmatrix} 3 & 2 & 1 \\ -5 & -4/3 & 1 \\ 3 & 0 & 1 \end{vmatrix} \right|$$

$$= \frac{1}{2} \times 16 = 8$$

3 An ellipse and a hyperbola have their principal axes along the coordinate axes and have a common foci separated by a distance $2\sqrt{13}$, the difference of their focal semi axes is equal to 4. If the ratio of their eccentricities is 3/7. Find the equation of these curves.



$F_1F_2 = 2\sqrt{13}$

equation of ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \quad a > b$$

Equation of hyperbola

$$\Rightarrow \frac{x^2}{A^2} - \frac{y^2}{B^2} = 1$$

$$a - A = 4$$

$$\frac{e}{E} = \frac{3}{7} \Rightarrow \frac{ae}{AE} = \frac{3a}{7A} \quad \cdots (1)$$

Also, we have $F_1F_2 = F_1F_2$

$$\Rightarrow 2ae = 2AE \quad \cdots (2)$$

using (1) & (2)

$$\Rightarrow 3a = 7A$$

$$\Rightarrow 3a - 3A = 4A$$

$$\Rightarrow 12 = 4A \Rightarrow A = 3, \; a = 7$$

$$\Rightarrow b^2 = a^2 - a^2e^2 \quad \& \quad B^2 = A^2E^2 - A^2$$

$$= 49 - 13 \qquad\qquad = 13 - 9$$

$$\Rightarrow b = 6 \qquad\qquad \Rightarrow B = 2$$

$\Rightarrow$ Equation of ellipse

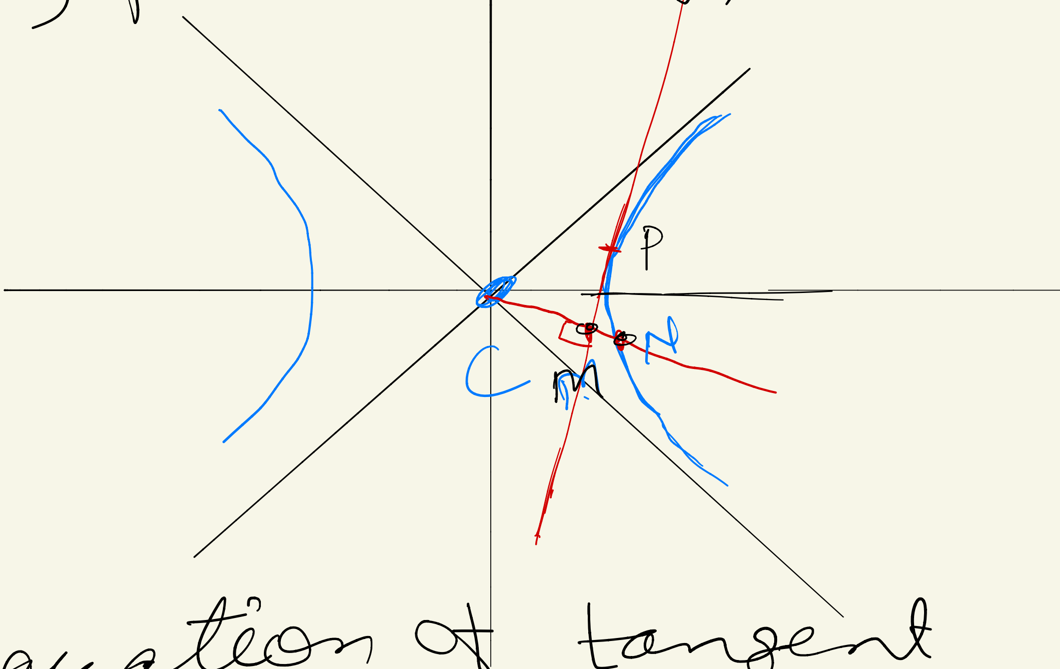
$$\frac{x^2}{49} + \frac{y^2}{36} = 1$$

& Equation of hyperbola

$$\Rightarrow \frac{x^2}{9} - \frac{y^2}{4} = 1$$

4 From the centre C of the hyperbola $x^2 - y^2 = 9$, CM is drawn perpendicular to the tangent at any point of the curve, meeting the tangent at M and the curve at N. Find the value of the product (CM)(CN).

Let $P(3\sec\theta, 3\tan\theta)$ be any point on the hyperbola



Equation of tangent at $P \Rightarrow x\sec\theta - y\tan\theta = 3$

equation of CM $\Rightarrow y = (-\sin\theta)x$

$\Rightarrow N(\pm 3\sec\theta, \mp 3\tan\theta)$

Now $(CM)^2(CN)^2 = \dfrac{9}{(\sec^2\theta + \tan^2\theta)} \times 9(\sec^2\theta + \tan^2\theta)$

$\Rightarrow (CM)(CN) = 9$

5 Tangents are drawn from the point (α, β) to the hyperbola $3x^2 - 2y^2 = 6$ and are inclined at angles θ and ϕ to the x-axis. If $\tan\theta . \tan\phi = 2$, prove that $\beta^2 = 2\alpha^2 - 7$.

equation of tangent passing through (α, β)

$\Rightarrow \; y - \beta = m(x - \alpha)$

$\Rightarrow \; y = mx + (\beta - m\alpha)$

where $\; c^2 = m^2a^2 - b^2$

$\Rightarrow \; (\beta - m\alpha)^2 = m^2 \times 2 - 3$

$\Rightarrow \; \beta^2 + \alpha^2 m^2 - 2\alpha\beta m = 2m^2 - 3$

$\Rightarrow \; (\alpha^2 - 2)m^2 - 2\alpha\beta m + \beta^2 - 3 = 0$

which is quadratic in m

$\Rightarrow \; m_1 m_2 = \dfrac{\beta^2 - 3}{\alpha^2 - 2}$

$\Rightarrow \; 2 = \dfrac{\beta^2 - 3}{\alpha^2 - 2} \Rightarrow \boxed{\beta^2 = 2\alpha^2 + 7}$

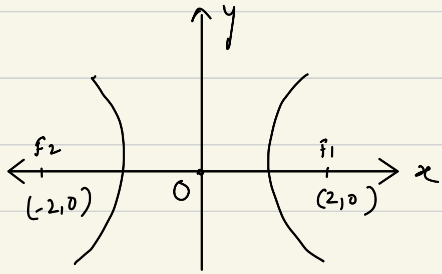
JM

EXERCISE (JM)

1. The equation of the hyperbola whose foci are (–2,0) and (2, 0) and eccentricity is 2 is given by : **[AIEEE-2011]**

(1) $-3x^2 + y^2 = 3$ (2) $x^2 - 3y^2 = 3$ (3) $3x^2 - y^2 = 3$ (4) $-x^2 + 3y^2 = 3$

sol^n $ae = 2$ and $e = 2$



$\Rightarrow \boxed{a = 1}$

NOW $b^2 = a^2(e^2 - 1) \Rightarrow b^2 = 3$

req. eq^n of hyperbola is

$$\frac{x^2}{1} - \frac{y^2}{3} = 1 \Rightarrow \boxed{3x^2 - y^2 = 3}$$

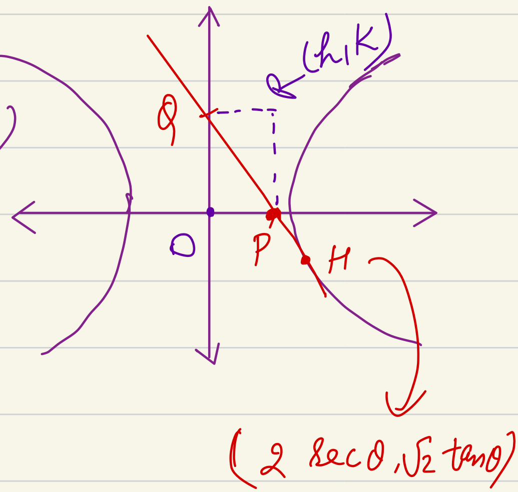
Ans $\rightarrow$ option (3)

2. A tangent to the hyperbola $\frac{x^2}{4} - \frac{y^2}{2} = 1$ meets x-axis at P and y-axis at Q. Lines PR and QR are drawn such that OPRQ is a rectangle (where O is the origin). Then R lies on : **[JEE-Main (On line)-2013]**

(1) $\frac{2}{x^2} - \frac{4}{y^2} = 1$ (2) $\frac{4}{x^2} - \frac{2}{y^2} = 1$ (3) $\frac{4}{x^2} + \frac{2}{y^2} = 1$ (4) $\frac{2}{x^2} + \frac{4}{y^2} = 1$

sol^n

eq^n of tangent at point $H(2\sec\theta, \sqrt{2}\tan\theta)$



$$\frac{x\sec\theta}{2} - \frac{y\tan\theta}{\sqrt{2}} = 1$$

$\Rightarrow P(2\cos\theta, 0)$

and $Q(0, -\sqrt{2}\cot\theta)$

So coordinates of $R \equiv (2\cos\theta, -\sqrt{2}\cot\theta)$

$\Rightarrow \quad h = 2\cos\theta, \quad k = -\sqrt{2}\cot\theta$

$\Rightarrow \quad \sec^2\theta = \frac{4}{h^2}, \quad \tan^2\theta = \frac{2}{k^2}$

$\Rightarrow \quad \frac{4}{h^2} - \frac{2}{k^2} = 1$

$\Rightarrow$ locus of R is $\boxed{\frac{4}{x^2} - \frac{2}{y^2} = 1}$

Ans:- option (2)

3. A common tangent to the conics $x^2 = 6y$ and $2x^2 - 4y^2 = 9$ is : **[JEE-Main (On line)-2013]**

(1) $x + y = \frac{9}{2}$ (2) $x + y = 1$ (3) $x - y = \frac{3}{2}$ (4) $x - y = 1$

Sol^n eq^n of tangent to the parabola $x^2 = 6y$ is

$x = \frac{1}{m}y + \left(\frac{3}{2}\right)m \Rightarrow y = mx - \frac{3}{2}m^2$ —— (1)

eq^n of tangent to the hyperbola $\frac{x^2}{\frac{9}{2}} - \frac{y^2}{\frac{9}{4}} = 1$ is

$y = mx \pm \sqrt{\frac{9}{2}m^2 - \frac{9}{4}}$ —— (2)

eq (1) and eq (2) represent same line

$$\Rightarrow \sqrt{\frac{9}{2}m^2 - \frac{9}{4}} = \frac{3}{2}m^2$$

$$\Rightarrow \frac{9}{2}m^2 - \frac{9}{4} = \frac{9}{4}m^4$$

$$\Rightarrow m^4 - 2m^2 + 1 = 0 \Rightarrow (m^2 - 1)^2 = 0$$

$$\Rightarrow \boxed{m = \pm 1}$$

from eq (1),

$$y = x - \frac{3}{2} \Rightarrow \boxed{x - y = \frac{3}{2}} \rightarrow \text{option (3)}$$

or $y = -x - \frac{3}{2} \Rightarrow x + y = -\frac{3}{2}$

Ans → option (3)

4. The eccentricity of the hyperbola whose length of the latus rectum is equal to 8 and the length of its conjugate axis is equal to half of the distance between its foci, is : **[JEE (Main) 2016]**

(1) $\sqrt{3}$ (2) $\frac{4}{3}$ (3) $\frac{4}{\sqrt{3}}$ (4) $\frac{2}{\sqrt{3}}$

Ans. (4)

Sol. Given

$$\frac{2b^2}{a}=8 \quad \text{.... (1)}$$

$$2b = ae \quad \text{.... (2)}$$

we know

$$b^2 = a^2(e^2 - 1) \quad \text{.... (3)}$$

substitute $\frac{b}{a} = \frac{e}{2}$ from (2) in (3)

$$\Rightarrow \frac{e^2}{4} = e^2 - 1$$

$$\Rightarrow 4 = 3e^2$$

$$\Rightarrow e = \frac{2}{\sqrt{3}}$$

5. A hyperbola passes through the point $P(\sqrt{2},\sqrt{3})$ and has foci at (± 2, 0). Then the tangent to this hyperbola at P also passes through the point : **[JEE (Main) 2017]**

(1) $(-\sqrt{2},-\sqrt{3})$ (2) $(3\sqrt{2},2\sqrt{3})$ (3) $(2\sqrt{2},3\sqrt{3})$ (4) $(\sqrt{3},\sqrt{2})$

Ans → option (3)

Sol. Equation of hyperbola is $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$

foci is (±2, 0) hence $ae = 2, \Rightarrow a^2e^2 = 4$

$b^2 = a^2(e^2 - 1)$

$\therefore\ a^2 + b^2 = 4$ ...(1)

Hyperbola passes through $(\sqrt{2},\sqrt{3})$

$\therefore\ \frac{2}{a^2}-\frac{3}{b^2}=1$ ...(2)

On solving (1) and (2)

$a^2 = 8$ (is rejected) and $a^2 = 1$ and $b^2 = 3$

$\therefore\ \frac{x^2}{1}-\frac{y^2}{3}=1$

Equation of tangent is $\frac{\sqrt{2}x}{1}-\frac{\sqrt{3}y}{3}=1$

Hence $(2\sqrt{2},3\sqrt{3})$ satisfy it.

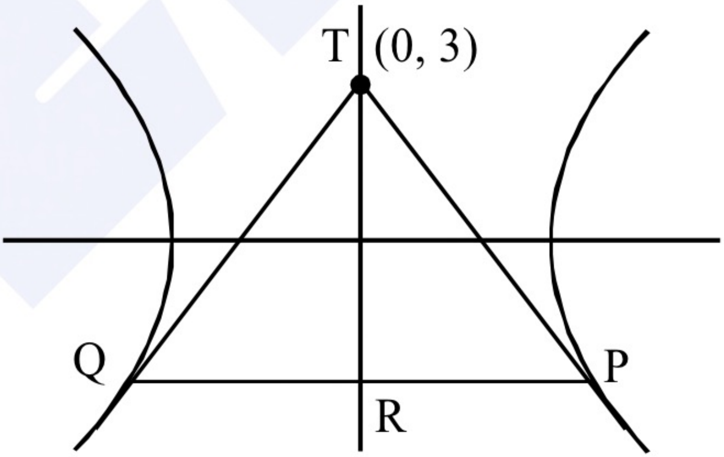
6. Tangents are drawn to the hyperbola $4x^2 - y^2 = 36$ at the point P and Q. If these tangents intersect at the point T(0, 3) then the area (in sq. units) of ΔPTQ is - **[JEE (Main) 2018]**

(1) $54\sqrt{3}$ (2) $60\sqrt{3}$ (3) $36\sqrt{5}$ (4) $45\sqrt{5}$

Ans. (4)

Sol. Equation PQ : chord of contact T = 0

$y = -12$



$$\frac{x^2}{9} - \frac{y^2}{36} = 1$$

Area : $\frac{1}{2}$ PQ · TR

TR = 3 + 12 = 15, Point P $\left(3\sqrt{5}, -12\right)$

$\Rightarrow$ PQ = $6\sqrt{5}$

Area of $\Delta PTQ = \frac{1}{2} \cdot 15 \cdot 6\sqrt{5} = 45\sqrt{5}$ sq. units

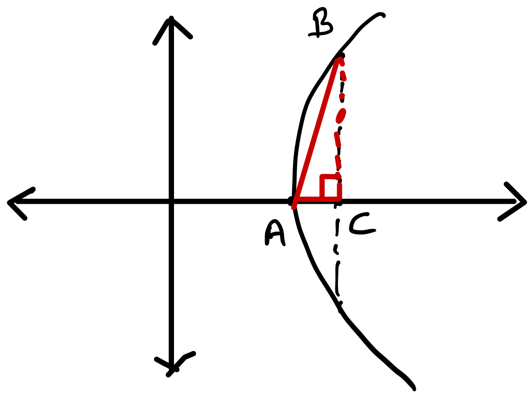
J-A

EXERCISE (JA)

1. Consider a branch of the hyperbola, $x^2 - 2y^2 - 2\sqrt{2}\,x - 4\sqrt{2}\,y - 6 = 0$ with vertex at the point A. Let B be one of the end points of its latus rectum. If C is the focus of the hyperbola nearest to the point A, then the area of the triangle ABC is **[JEE 2008, 3]**

(A) $1-\sqrt{\frac{2}{3}}$ (B) $\sqrt{\frac{3}{2}}-1$ (C) $1+\sqrt{\frac{2}{3}}$ (D) $\sqrt{\frac{3}{2}}+1$

Sol. :



Hyperbola is $\dfrac{(x-\sqrt{2})^2}{4} - \dfrac{(y+\sqrt{2})^2}{2} = 1$

$a = 2,\ b = \sqrt{2}$

$e = \sqrt{\dfrac{3}{2}}$

$$\text{Area} = \frac{1}{2}a(e-1)\times\frac{b^2}{a} = \frac{1}{2}\frac{(\sqrt{3}-\sqrt{2})\times 2}{\sqrt{2}} = \frac{(\sqrt{3}-\sqrt{2})}{\sqrt{2}}$$

$$\Rightarrow \text{Area} = \left(\sqrt{\frac{3}{2}}-1\right).$$

2. The locus of the orthocentre of the triangle formed by the lines $(1+p)x - py + p(1+p) = 0$, $(1+q)x - qy + q(1+q) = 0$ and $y = 0$, where $p \neq q$, is **[JEE 2009, 3]**

(A) a hyperbola (B) a parabola (C) an ellipse (D) a straight line

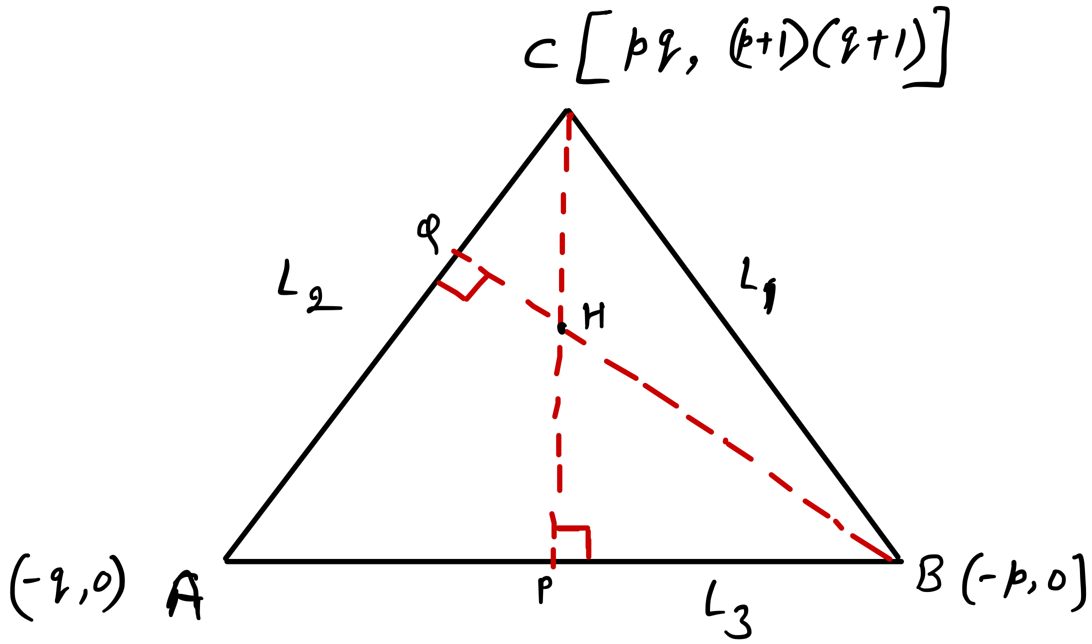
Sol:

$L_1 : (1+p)x - py + p(1+p) = 0$
$L_2 : (1+q)x - qy + q(1+q) = 0$
$L_3 : y = 0$

Intersection point of y = 0 with first line is B(–p, 0)
Intersection point of y = 0 with second line is A(–q, 0)
Intersection point of the two lines is C(pq, (p + 1)(q + 1))
Altitude from C to AB is x = pq

Altitude from B to AC is $y = -\dfrac{q}{1+q}(x+p)$

Solving these two we get x = pq and y = – pq
$\therefore$ locus of orthocentre is x + y = 0.



CP : $\boxed{x = pq}$ — ① & BQ : $\boxed{y = \dfrac{-q}{1+q}(x+p)}$ — ②

solve ① & ②

$\left.\begin{aligned} x &= pq \\ y &= -pq \end{aligned}\right\}$ $\boxed{x + y = 0}$

3. An ellipse intersects the hyperbola $2x^2 - 2y^2 = 1$ orthogonally. The eccentricity of the ellipse is reciprocal of that of the hyperbola. If the axes of the ellipse are along the coordinate axes, then **[JEE 2009, 4]**

(A) Equation of ellipse is $x^2 + 2y^2 = 2$

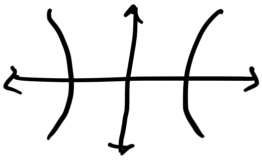
(B) The foci of ellipse are $(\pm 1, 0)$

(C) Equation of ellipse is $x^2 + 2y^2 = 4$

(D) The foci of ellipse are $(\pm \sqrt{2}, 0)$

Sol:-

$H: \boxed{\dfrac{x^2}{\frac{1}{2}} - \dfrac{y^2}{\frac{1}{2}} = 1}$



$a = b = \dfrac{1}{\sqrt{2}}$

$e = \sqrt{2}$

Ellipse and hyperbola will be confocal

$\Rightarrow (\pm ae, 0) \equiv (\pm 1, 0)$ (focus of hyperbola)

$\Rightarrow \left(\pm a \times \dfrac{1}{\sqrt{2}},\ 0\right) \equiv (\pm 1, 0)$ (focus of ellipse)

So for ellipse

$\Rightarrow a = \sqrt{2}$ and $e = \dfrac{1}{\sqrt{2}}$

$\Rightarrow b^2 = a^2(1 - e^2) \Rightarrow b^2 = 1$

$\therefore$ Equation of ellipse $\dfrac{x^2}{2} + \dfrac{y^2}{1} = 1$.

Comprehension : 4 & 5 **[JEE 2010, 3+3]**

The circle $x^2 + y^2 - 8x = 0$ and hyperbola $\frac{x^2}{9} - \frac{y^2}{4} = 1$ intersect at the points A and B.

4 Equation of a common tangent with positive slope to the circle as well as to the hyperbola is -

(A) $2x - \sqrt{5}y - 20 = 0$ (B) $2x - \sqrt{5}y + 4 = 0$

(C) $3x - 4y + 8 = 0$ (D) $4x - 3y + 4 = 0$

Sol. Let equation of tangent to ~~ellipse~~ Hyperbala

$$\frac{\sec\theta}{3}x - \frac{\tan\theta}{2}y = 1$$

$2\sec\theta\, x - 3\tan\theta\, y = 6$

It is also tangent to circle $x^2 + y^2 - 8x = 0$

$$\Rightarrow \frac{|8\sec\theta - 6|}{\sqrt{4\sec^2\theta + 9\tan^2\theta}} = 4$$

$(8\sec\theta - 6)^2 = 16(13\sec^2\theta - 9)$

$\Rightarrow 12\sec^2\theta + 8\sec\theta - 15 = 0$

$\Rightarrow \sec\theta = \frac{5}{6}$ and $-\frac{3}{2}$ but $\sec \neq \frac{5}{6}$

$\Rightarrow \sec\theta = -\frac{3}{2}$ $\Rightarrow \tan\theta = \frac{\sqrt{5}}{2}$ $\therefore$ slope is positive

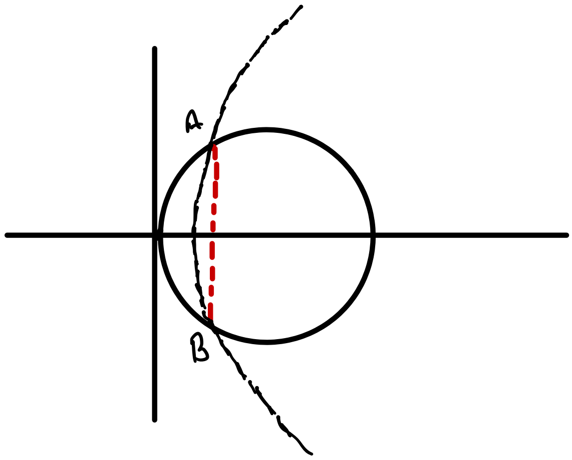
Equation of tangent $= 2x - \sqrt{5}y + 4 = 0$

5 Equation of the circle with AB as its diameter is -

(A) $x^2 + y^2 - 12x + 24 = 0$ (B) $x^2 + y^2 + 12x + 24 = 0$

(C) $x^2 + y^2 + 24x - 12 = 0$ (D) $x^2 + y^2 - 24x - 12 = 0$

Sol. $x^2 + y^2 - 8x = 0$



$\frac{x^2}{9} - \frac{y^2}{4} = 1 \Rightarrow 4x^2 - 9y^2 = 36$

$\Rightarrow 4x^2 - 9(8x - x^2) = 36$

$13x^2 - 72x - 36 = 0$

$13x^2 - 78x + 6x - 36 = 0$

$(13x + 6)(x - 6) = 0$

$\Rightarrow x = -\frac{6}{13}$ and $x = 6$

But $x > 0 \Rightarrow x = 6$

$\Rightarrow A(6, \sqrt{2})$ and $B(6, -\sqrt{2})$

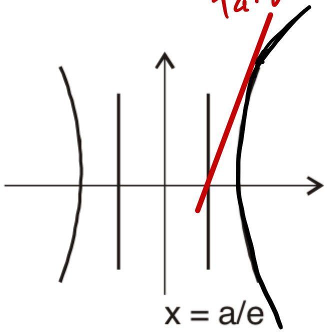
$\Rightarrow$ Equation of circle with AB as a diameter $x^2 + y^2 - 12x + 24 = 0$

6 The line $2x + y = 1$ is tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. If this line passes through the point of intersection of the nearest directrix and the x-axis, then the eccentricity of the hyperbola is

[JEE 2010, 3]

Tangent : $y = mx + \sqrt{a^2m^2 - b^2}$

Sol.



$$y = -2x + 1$$

ek minus sign remove karna hai

$$0 = - -\frac{2a}{e} + 1$$

$$\Rightarrow \frac{a}{e} = \frac{1}{2}$$

$$e = 2a$$

$$c^2 = a^2 m^2 - b^2$$

$$\Rightarrow 1 = 4a^2 - b^2$$

$$\Rightarrow 1 + b^2 - 4a^2 = 0$$

$$e^2 = 1 + \frac{b^2}{a^2}$$

$$= 1 + \frac{(4a^2 - 1)}{a^2}$$

$$e^2 = 1 + 4 - \frac{1}{a^2}$$

$$e^2 = 5 - \frac{4}{e^2}$$

$$\Rightarrow e^4 - 5e^2 + 4 = 0$$

$$\Rightarrow (e^2 - 1)(e^2 - 4) = 0$$

$$e^2 - 1 \neq 0 \qquad e = 2$$

7 Let the eccentricity of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ be reciprocal to that of the ellipse $x^2 + 4y^2 = 4$. If the hyperbola passes through a focus of the ellipse, then - **[JEE 2011, 4]**

(A) the equation of the hyperbola is $\frac{x^2}{3}-\frac{y^2}{2}=1$

(B) a focus of the hyperbola is (2,0)

(C) the eccentricity of the hyperbola is $\sqrt{\frac{5}{3}}$

(D) the equation of the hyperbola is $x^2-3y^2=3$

Sol :-

Ellipse is $\frac{x^2}{2^2}+\frac{y^2}{1^2}=1$

$1^2 = 2^2(1-e^2) \Rightarrow e = \frac{\sqrt{3}}{2}$

$\therefore$ eccentricity of the hyperbola is $\frac{2}{\sqrt{3}} \Rightarrow b^2 = a^2\left(\frac{4}{3}-1\right) \Rightarrow 3b^2 = a^2$

Foci of the ellipse are $\left(\sqrt{3},\ 0\right)$ and $\left(-\sqrt{3},\ 0\right)$.

Hyperbola passes through $\left(\sqrt{3},\ 0\right)$

$\frac{3}{a^2} = 1 \Rightarrow a^2 = 3$ and $b^2 = 1$

$\therefore$ Equation of hyperbola is $x^2 - 3y^2 = 3$

Focus of hyperbola is $(ae, 0) \equiv \left(\sqrt{3}\times\frac{2}{\sqrt{3}},\ 0\right) \equiv (2, 0)$.

8 Let P(6, 3) be a point on the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. If the normal at the point P intersects the x-axis at (9,0), then the eccentricity of the hyperbola is - **[JEE 2011, 3]**

(A) $\sqrt{\frac{5}{2}}$ (B) $\sqrt{\frac{3}{2}}$ (C) $\sqrt{2}$ (D) $\sqrt{3}$

Sol:-

$\because \frac{x^2}{a^2}-\frac{y^2}{b^2}=1 \Rightarrow \frac{2x}{a^2}-\frac{2y}{b^2}\cdot\frac{dy}{dx}=0$

$\Rightarrow m_T=\frac{x_1 b^2}{y_1 a^2}=\frac{6b^2}{3a^2}$

$\therefore m_N=\frac{-a^2}{2b^2}$

{$\because$ Normal pass through (6,3) & (9,0)}

$\Rightarrow \frac{0-3}{9-6}=\frac{-a^2}{2b^2} \Rightarrow \frac{a^2}{b^2}=2$

$\because e^2=1+\frac{b^2}{a^2} \Rightarrow e^2=1+\frac{1}{2} \Rightarrow \boxed{e=\sqrt{\frac{3}{2}}}$

9 Tangents are drawn to the hyperbola $\frac{x^2}{9}-\frac{y^2}{4}=1$, parallel to the straight line $2x - y = 1$. The points of contact of the tangents on the hyperbola are **[JEE 2012, 4M]**

(A) $\left(\frac{9}{2\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

(B) $\left(-\frac{9}{2\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

(C) $\left(3\sqrt{3}, -2\sqrt{2}\right)$

(D) $\left(-3\sqrt{3}, 2\sqrt{2}\right)$

Sol. **(A, B)**

Slope of tangent = 2

The tangents are $y = 2x \pm \sqrt{9\times 4-4}$

i.e., $2x - y = \pm\, 4\sqrt{2}$

$\Rightarrow \frac{x}{2\sqrt{2}}-\frac{y}{4\sqrt{2}}=1$ and $-\frac{x}{2\sqrt{2}}+\frac{y}{4\sqrt{2}}=1$

Comparing it with $\frac{xx_1}{9}-\frac{yy_1}{4}=1$

We get point of contact as $\left(\frac{9}{2\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{9}{2\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$

Alternate:

Equation of tangent at P (θ) is $\left(\frac{\sec\theta}{3}\right)x-\left(\frac{\tan\theta}{2}\right)y=1$

$\Rightarrow$ Slope $= \frac{2\sec\theta}{3\tan\theta}=2$

$\Rightarrow \sin\theta = \frac{1}{3}$

$\Rightarrow$ points are $\left(\frac{9}{2\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{9}{2\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$.

10 Consider the hyperbola H : $x^2 - y^2 = 1$ and a circle S with center $N(x_2, 0)$. Suppose that H and S touch each other at a point $P(x_1, y_1)$ with $x_1 > 1$ and $y_1 > 0$. The common tangent to H and S at P intersects the x-axis at point M. If (l, m) is the centroid of the triangle ΔPMN, then the correct expression(s) is(are) **[JEE 2015, 4M, –0M]**

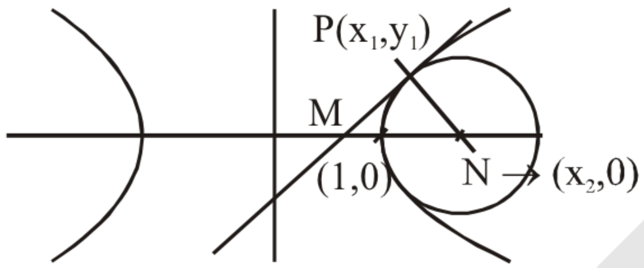
(A) $\frac{dl}{dx_1} = 1 - \frac{1}{3x_1^2}$ for $x_1 > 1$

(B) $\frac{dm}{dx_1} = \frac{x_1}{3\left(\sqrt{x_1^2 - 1}\right)}$ for $x_1 > 1$

(C) $\frac{dl}{dx_1} = 1 + \frac{1}{3x_1^2}$ for $x_1 > 1$

(D) $\frac{dm}{dy_1} = \frac{1}{3}$ for $y_1 > 0$

Sol. Given H : $x^2 - y^2 = 1$



Now, equation of family of circle touching hyperbola at (x_1, y_1) is

$(x - x_1)^2 + (y - y_1)^2 + \lambda(xx_1 - yy_1 - 1) = 0$

Now, it centre is $(x_2, 0)$

$\therefore \quad 2y_1 + \lambda y_1 = 0 \quad \Rightarrow \quad \lambda = -2$

$$\therefore \quad x_2 = \frac{2x_1 - \lambda x_1}{2} = \frac{4x_1}{2} = 2x_1$$

$$\therefore \quad P \equiv (x_1, \sqrt{x_1^2 - 1})$$

$$N \equiv (2x_1, 0)$$

$$\& \quad M \equiv \left(\frac{1}{x_1}, 0\right)$$

$$\therefore \quad \ell = \frac{3x_1 + \frac{1}{x_1}}{3} = x_1 + \frac{1}{3x_1} \quad \Rightarrow \quad \frac{d\ell}{dx_1} = 1 - \frac{1}{3x_1^2} \qquad x_1 > 1$$

$$m = \frac{\sqrt{x_1^2 - 1}}{3} \quad \Rightarrow \quad \frac{dm}{dx_1} = \frac{x_1}{3\sqrt{x_1^2 - 1}}$$

$$\text{Also } m = \frac{y_1}{3} \quad \Rightarrow \quad \frac{dm}{dy_1} = \frac{1}{3} \qquad y_1 > 0$$

$\therefore \quad$ (A), (B) & (D)

11 If $2x - y + 1 = 0$ is tangent to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{16} = 1$, then which of the following CANNOT be sides of a right angled triangle ? **[JEE(Advanced)-2017, 4(–2)]**

(A) 2a, 4, 1 (B) 2a, 8, 1 (C) a, 4, 1 (D) a, 4, 2

Sol. The line y = mx + c is tangent to hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$, if $c^2 = a^2m^2 - b^2$

$\therefore (1)^2 = 4a^2 - 16 \Rightarrow a^2 = \frac{17}{4}$

$\Rightarrow a = \frac{\sqrt{17}}{2}$

For option (A), sides are $\sqrt{17}, 4, 1$ ($\Rightarrow$ Right angled triangle)

For option (B), sides are $\sqrt{17}, 8, 1$ ($\Rightarrow$ Triangle is not possible)

For option (C), sides are $\frac{\sqrt{17}}{2}, 4, 1$ ($\Rightarrow$ Triangle is not possible)

For option (D), sides are $\frac{\sqrt{17}}{2}, 4, 2$ ($\Rightarrow$ Triangle exist but not right angled)

12 Let H : $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$, where a > b > 0, be a hyperbola in the xy-plane whose conjugate axis LM subtends an angle of 60° at one of its vertices N. Let the area of the triangle LMN be $4\sqrt{3}$.

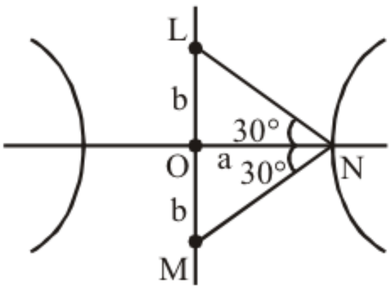
LIST-I	LIST-II
P. The length of the conjugate axis of H is	**1.** 8
Q. The eccentricity of H is	**2.** $\frac{4}{\sqrt{3}}$
R. The distance between the foci of H is	**3.** $\frac{2}{\sqrt{3}}$
S. The length of the latus rectum of H is	**4.** 4

The correct option is :

(A) **P → 4; Q → 2, R → 1; S → 3**
(B) **P → 4; Q → 3; R → 1; S → 2**
(C) **P → 4; Q → 1, R → 3; S → 2**
(D) **P → 3; Q → 4; R → 2; S → 1**

[JEE(Advanced)-2018, 3(–1)]

Sol. :



$$\tan 30° = \frac{b}{a}$$

$$\Rightarrow \quad a = b\sqrt{3}$$

Now area of $\Delta LMN = \frac{1}{2}.2b.b\sqrt{3}$

$$4\sqrt{3} = \sqrt{3}b^2$$

$$\Rightarrow \quad b = 2 \quad \& \quad a = 2\sqrt{3}$$

$$\Rightarrow \quad e = \sqrt{1+\frac{b^2}{a^2}} = \frac{2}{\sqrt{3}}$$

P. Length of conjugate axis = 2b = 4

So P → 4

Q. Eccentricity $e = \frac{2}{\sqrt{3}}$

So Q → 3

R. Distance between foci = 2ae

$$= 2(2\sqrt{3})\left(\frac{2}{\sqrt{3}}\right) = 8$$

So R → 1

S. Length of latus rectum $= \frac{2b^2}{a} = \frac{2(2)^2}{2\sqrt{3}} = \frac{4}{\sqrt{3}}$

So S → 2