Allen Career Institute

Kota

Methods of Differentiation

(Solutions)

Index

METHODS OF DIFFERENTIATION

Do yourself -1 :

(i) Differentiate each of following functions by first principle:

(a) $f(x) = \ell nx$ (b) $f(x) = \dfrac{1}{x}$

Sol^n (a) $f'(x) = \lim\limits_{h\to 0} \dfrac{\ln(x+h) - \ln x}{h}$

$= \lim\limits_{h\to 0} \dfrac{\ln\left(1+\frac{h}{x}\right)}{h}$

$= \lim\limits_{h\to 0} \left(\dfrac{\frac{h}{x} - \frac{h^2}{2x^2} + - \cdots}{h}\right) = \boxed{\dfrac{1}{x}}$

(b) $f(x) = \dfrac{1}{x}$

$\Rightarrow f'(x) = \lim\limits_{h\to 0} \dfrac{\frac{1}{x+h} - \frac{1}{x}}{h}$

$= \lim\limits_{h\to 0} \dfrac{-h}{x(x+h)h}$

$= \lim\limits_{h\to 0} \dfrac{-1}{x(x+h)} = \boxed{-\dfrac{1}{x^2}}$

Do yourself -2 :

(i) Find $\frac{dy}{dx}$ if -

(a) $y = (x + 1)(x + 2)(x + 3)$ (b) $y = e^{5x} \tan(x^2 + 2)$

Soln ⓐ $y = \underset{\mathrm{I}}{(x+1)}\ \underbrace{(x+2)(x+3)}_{\mathrm{II}}$ using Product Rule

$$\Rightarrow y' = (x+1)\frac{d}{dx}\{\underset{\mathrm{I}}{(x+2)}\underset{\mathrm{II}}{(x+3)}\} + (x+2)(x+3)\frac{d}{dx}(x+1)$$

$$y' = (x+1)\{(x+2)\cdot 1 + (x+3)\cdot 1\} + (x+2)(x+3)$$

$$= 2x^2 + 5x + 2x + 5 + x^2 + 5x + 6$$

$$\boxed{y' = 3x^2 + 12x + 11}$$

Soln ⓑ $y = \underset{\mathrm{I}}{e^{5x}}\ \underset{\mathrm{II}}{\tan(x^2+2)}$

$$\Rightarrow y' = e^{5x}\frac{d}{dx}\left(\tan(x^2+2)\right) + \tan(x^2+2)\frac{d}{dx}e^{5x}$$

$$= e^{5x}\left(\sec^2(x^2+2)\cdot 2x\right) + 5e^{5x}\tan(x^2+2)$$

$$\boxed{y' = 2x\,e^{5x}\sec^2(x^2+2) + 5e^{5x}\tan(x^2+2)}$$

Do yourself -3 :

(i) Find $\frac{dy}{dx}$ if $y = x^x$ **(ii)** Find $\frac{dy}{dx}$ if $y = e^x.e^{x^2}.e^{x^3}.e^{x^4}$

Solⁿ (i) $y = x^x$

$\Rightarrow \log y = \log x^x = x \log x$

Differentiating w.r.t x

$\Rightarrow \frac{1}{y} y' = x \times \frac{1}{x} + \log x . 1$

$$\boxed{y' = y(1+\log x) = x^x(1+\log x)}$$

Solⁿ (ii) $y = e^x e^{x^2} . e^{x^3} . e^{x^4}$

$y = e^{(x+x^2+x^3+x^4)}$ ——— (1)

$\Rightarrow y' = e^{(x+x^2+x^3+x^4)} . \frac{d}{dx}(x+x^2+x^3+x^4)$

$$\boxed{y' = y(1+2x+3x^2+4x^3)}$$

Do yourself-4 :

(i) Find $\frac{dy}{dx}$ at $t=\frac{\pi}{4}$ if $y=\cos^4 t$ & $x=\sin^4 t$.

(ii) Find the slope of the tangent at a point P(t) on the curve $x = at^2$, $y=2at$.

(iii) Differentiate $x^{\ell nx}$ with respect to ℓnx.

Solⁿ (i) $y = \cos^4 t$, $x = \sin^4 t$

$\Rightarrow \frac{dy}{dt} = 4\cos^3 t(-\sin t)$, $\frac{dx}{dt} = 4\sin^3 t(\cos t)$

$\Rightarrow \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-4\cos^3 t \sin t}{4\sin^3 t \cos t} = -\cot^2 t$

$$\boxed{\left.\frac{dy}{dx}\right)_{t=\frac{\pi}{4}} = -\cot^2\frac{\pi}{4} = -1}$$

Solⁿ (ii) $x = at^2 \Rightarrow \frac{dx}{dt} = 2at$

$y = 2at \Rightarrow \frac{dy}{dt} = 2a$

$$\Rightarrow \boxed{\text{slope of tangent} = \frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2a}{2at} = \frac{1}{t}}$$

Solⁿ (iii) let $y = x^{\ln x}$ & $u = \ln x$ we have to find $\frac{dy}{du}$

$\frac{du}{dx} = \frac{1}{x}$ —(2)

$\Rightarrow \log y = (\ln x)(\ln x)$

$\Rightarrow \frac{1}{y}y' = 2(\ln x)\cdot\left(\frac{1}{x}\right)$

$\Rightarrow \frac{dy}{dx} = 2x^{\ln x}\cdot(\ln x)\left(\frac{1}{x}\right)$ —(1)

$\Rightarrow \frac{dy}{du} = \frac{dy/dx}{du/dx}$

$$\boxed{\frac{dy}{du} = 2x^{\ln x}(\ln x)}$$

Do yourself -5 :

(i) Find $\frac{dy}{dx}$, if $x + y = \sin(x - y)$

(ii) If $x^2 + xe^y + y = 0$, find y', also find the value of y' at point (0,0).

Sol^n (i)

$$x + y = \sin(x-y)$$

Diff. w.r.t x

$$1 + y' = \cos(x-y)\left[1 - y'\right]$$

$$\Rightarrow \left(1 + \cos(x-y)\right) y' = \cos(x-y) - 1$$

$$\boxed{y' = \frac{\cos(x-y) - 1}{\cos(x+y) + 1}}$$

Sol^n (ii)

$$x^2 + x e^y + y = 0$$

Diff. w.r.t x

$$2x + x\left(\frac{d}{dx} e^y\right) + e^y \cdot \frac{d}{dx}(x) + y' = 0$$

$$2x + x e^y \cdot y' + e^x + y' = 0$$

$$(x e^y + 1) y' = -(2x + e^x)$$

$$\boxed{y' = \frac{-(e^x + 2x)}{(x e^x + 1)}} \quad \& \quad y'\Big)_{(0,0)} = \frac{-(e^0 + 0)}{(0+1)}$$

$$\Rightarrow \boxed{y'_{\substack{at \\ x=0 \\ y=0}} = -1}$$

Do yourself : 6

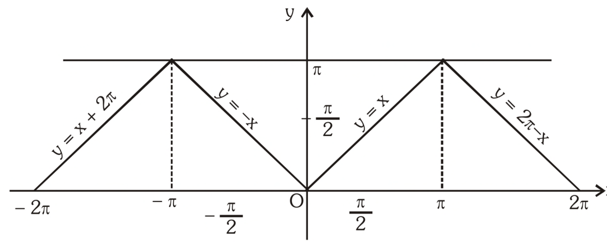
(i) If $y = \cos^{-1}(4x^3 - 3x)$, then find :

(a) $f'\left(-\frac{\sqrt{3}}{2}\right)$, (b) $f'(0)$, (c) $f'\left(\frac{\sqrt{3}}{2}\right)$.

Soln $y = \cos^{-1}(4x^3 - 3x)$

Put $x = \cos\theta \Rightarrow \theta = \cos^{-1}x$

$\because \cos^{-1}(\cos x) = \begin{cases} -x & , -\pi \le x \le 0 \\ x & , 0 < x \le \pi \end{cases}$



$y = \cos^{-1}\cos 3\theta = \begin{cases} 3\cos^{-1}x - 2\pi & \text{if } -1 \le x < -\frac{1}{2} \\ 2\pi - 3\cos^{-1}x & \text{if } -\frac{1}{2} \le x \le \frac{1}{2} \\ 3\cos^{-1}x & \text{if } \frac{1}{2} < x \le 1 \end{cases}$

(a) $f(x) = 3\cos^{-1}x - 2\pi$ for $x = -\frac{\sqrt{3}}{2}$

$f'(x) = \frac{-3}{\sqrt{1-x^2}} \Rightarrow \boxed{f'\left(-\frac{\sqrt{3}}{2}\right) = -6}$

(b) for $x = 0$

$f(x) = 2\pi - 3\cos^{-1}x \Rightarrow f'(x) = \frac{3}{\sqrt{1-x^2}}$

$\Rightarrow \boxed{f'(0) = 3}$

(c) for $x = \frac{\sqrt{3}}{2}$, $f(x) = 3\cos^{-1}x$

$\Rightarrow f'(x) = \frac{-3}{\sqrt{1-x^2}} \Rightarrow \boxed{f'\left(\frac{\sqrt{3}}{2}\right) = -6}$

Do yourself -7 :

(i) If g is inverse of f and f (x) = 2x + sinx; then g'(x) equals:

(A) $-\frac{3}{x^2}+\frac{1}{\sqrt{1-x^2}}$ (B) $2+\sin^{-1}x$ (C) $2+\cos g(x)$ (D) $\frac{1}{2+\cos(g(x))}$

Soln (i)

$f(x) = 2x + \sin x$

$\Rightarrow f'(x) = 2 + \cos x$ —— (1)

$\because$ g is inverse of f

$\Rightarrow f(g(x)) = x$

$f'(g(x))\, g'(x) = 1$

$$\boxed{g'(x) = \frac{1}{f'(g(x))} = \frac{1}{2+\cos(g(x))}}$$

$\because f'(x) = 2 + \cos x$

$\Rightarrow f'(g(x)) = 2 + \cos(g(x))$

Do yourself : 8

(i) If $y = xe^{x^2}$ then find y''.

(ii) Find y'' at $x = \pi/4$, if $y = x \tan x$.

(iii) Prove that the function $y = e^x \sin x$ satisfies the relationship $y'' - 2y' + 2y = 0$.

Soln (I) $y = xe^{x^2} \Rightarrow y' = x(e^{x^2} \cdot 2x) + e^{x^2}$

$$\Rightarrow y' = (2x^2+1)e^{x^2} \quad \text{—— (1)}$$

$$\Rightarrow y'' = (2x^2+1)e^{x^2}(2x) + 4(xe^{x^2})$$

$$y'' = y'(2x) + 4y \qquad \text{Using Eqn (1)}$$

$$\boxed{y'' = 2xy' + 4y}$$

(II) $y = x\tan x \Rightarrow y' = x\sec^2 x + \tan x$

$$\Rightarrow y'' = x(2\sec x)(\sec x \tan x) + \sec^2 x + \sec^2 x$$

$$y'' = 2(x\tan x + 1)\sec^2 x$$

$$\boxed{y''\big)_{x=\pi/4} = 4\left(\frac{\pi}{4}+1\right) = \pi + 4}$$

(III) $y = e^x \sin x \Rightarrow y' = e^x\cos x + e^x \sin x$ —— (1)

$$y' = e^x\cos x + y \Rightarrow y'' = -e^x\sin x + e^x\cos x + y'$$

$$\Rightarrow y'' = -2e^x\sin x + \underbrace{e^x\cos x + e^x\sin x}_{} + y'$$

$$y'' = -2y + y' + y'$$

$$\Rightarrow \boxed{y'' - 2y' + 2y = 0}$$

Do yourself : 9

(i) If $f(x)=\begin{vmatrix} e^x & x^2 \\ \ell nx & \sin x \end{vmatrix}$, then find $f'(1)$. **(ii)** If $f(x)=\begin{vmatrix} 2x & x^2 & x^3 \\ x^2+2x & 1 & 3x+1 \\ 2x & 1-3x^2 & 5x \end{vmatrix}$, then find $f'(1)$.

Soln (1)

$$f'(x)=\begin{vmatrix} \frac{d}{dx}(e^x) & \frac{d}{dx}(x^2) \\ \ln x & \sin x \end{vmatrix}+\begin{vmatrix} e^x & x^2 \\ \frac{d}{dx}(\ln x) & \frac{d}{dx}(\sin x) \end{vmatrix}$$

$$f'(x)=\begin{vmatrix} e^x & 2x \\ \ln x & \sin x \end{vmatrix}+\begin{vmatrix} e^x & x^2 \\ \frac{1}{x} & \cos x \end{vmatrix}$$

$$f'(1)=\begin{vmatrix} e & 2 \\ 0 & \sin 1 \end{vmatrix}+\begin{vmatrix} e & 1 \\ 1 & \cos 1 \end{vmatrix}$$

$$\boxed{f'(1)=e\sin 1+e\cos 1-1=e(\sin 1+\cos 1)-1}$$

Soln (II) $f(x)=\begin{vmatrix} 2x & x^2 & x^3 \\ x^2+2x & 1 & 3x+1 \\ 2x & 1-3x^2 & 5x \end{vmatrix}$

$$f'(x)=\begin{vmatrix} 2 & 2x & 3x^2 \\ x^2+2x & 1 & 3x+1 \\ 2x & 1-3x^2 & 5x \end{vmatrix}+\begin{vmatrix} 2x & x^2 & x^3 \\ 2x+2 & 0 & 3 \\ 2x & 1-3x^2 & 5x \end{vmatrix}+\begin{vmatrix} 2x & x^2 & x^3 \\ x^2+2x & 1 & 3x+1 \\ 2 & -6x & 5 \end{vmatrix}$$

$$f'(1)=\begin{vmatrix} 2 & 2 & 3 \\ 3 & 1 & 4 \\ 2 & -2 & 5 \end{vmatrix}+\begin{vmatrix} 2 & 1 & 1 \\ 4 & 0 & 3 \\ 2 & -2 & 5 \end{vmatrix}+\begin{vmatrix} 2 & 1 & 1 \\ 3 & 1 & 4 \\ 2 & -6 & 5 \end{vmatrix}$$

$$\boxed{f'(1)=-12-10+31=9}$$

or First expand det after that diff.

Do yourself : 10

(i) Using L'Hôpital's rule find (a) $\lim\limits_{x\to 0}\frac{\tan x - x}{x^3}$ (b) $\lim\limits_{x\to 0}\frac{e^x - x - 1}{x^2}$

(ii) Using L'Hôpital's rule verify that : (a) $\lim\limits_{x\to 0}\frac{\sin x - \tan x}{x^3} = -\frac{1}{2}$ (b) $\lim\limits_{x\to 0}\frac{\ell n(1+x)}{x} = 1$

Sol (1) (a) $\lim\limits_{x\to 0}\frac{\tan x - x}{x^3}$ using L' Hopital Rule

$= \lim\limits_{x\to 0}\frac{\sec^2 x - 1}{3x^2} = \lim\limits_{x\to 0}\left(\frac{\tan x}{x}\right)^2 \cdot \frac{1}{3}$

$= \frac{1}{3}$ Ans

Soln 1 (b) $\lim\limits_{x\to 0}\frac{e^x - x - 1}{x^2} = \lim\limits_{x\to 0}\frac{e^x - 1}{2x}$

$= \lim\limits_{x\to 0}\frac{e^x}{2} = \frac{1}{2}$ Ans

(ii) Using L'Hôpital's rule verify that : (a) $\lim_{x \to 0} \frac{\sin x - \tan x}{x^3} = -\frac{1}{2}$ (b) $\lim_{x \to 0} \frac{\ell n(1+x)}{x} = 1$

Soln (a) $\lim_{x \to 0} \frac{\sin x - \tan x}{x^3}$

$$= \lim_{x \to 0} \frac{\frac{d}{dx}(\sin x - \tan x)}{\frac{d}{dx}(x^3)} = \lim_{x \to 0} \frac{\cos x - \sec^2 x}{3x^2}$$

$$= \lim_{x \to 0} \frac{\sin x - 2\sec x \sec x \tan x}{6x}$$

$$= \lim_{x \to 0} \left(\frac{\sin x}{x}\right)\left(\frac{1 - 3\sec^3 x}{6}\right)$$

$$= 1 \cdot \left(-\frac{1}{2}\right) = -\frac{1}{2} \text{ Ans}$$

(b) $\lim_{x \to 0} \frac{\ln(1+x)}{x}$

$$= \lim_{x \to 0} \frac{\frac{d}{dx}(\ln(1+x))}{\frac{d}{dx}(x)} = \lim_{x \to 0} \frac{1}{(1+x) \cdot 1}$$

$$= \underline{\underline{1}}$$

EXERCISE (O-1)

1. If $y=\dfrac{1}{1+x^{\beta-\alpha}+x^{\gamma-\alpha}}+\dfrac{1}{1+x^{\alpha-\beta}+x^{\gamma-\beta}}+\dfrac{1}{1+x^{\alpha-\gamma}+x^{\beta-\gamma}}$, then $\dfrac{dy}{dx}$ is equal to-

(A) 0 (B) 1

(C) $(\alpha+\beta+\gamma)x^{\alpha+\beta+\gamma-1}$ (D) $\alpha\beta\gamma$

Sol

$$y=\frac{1}{1+\frac{x^{\beta}}{x^{\alpha}}+\frac{x^{\gamma}}{x^{\alpha}}}+\frac{1}{1+\frac{x^{\alpha}}{x^{\beta}}+\frac{x^{\gamma}}{x^{\beta}}}+\frac{1}{1+\frac{x^{\beta}}{x^{\gamma}}+\frac{x^{\alpha}}{x^{\gamma}}}$$

$$\Rightarrow y=\frac{x^{\alpha}}{x^{\alpha}+x^{\beta}+x^{\gamma}}+\frac{x^{\beta}}{x^{\alpha}+x^{\beta}+x^{\gamma}}+\frac{x^{\gamma}}{x^{\alpha}+x^{\beta}+x^{\gamma}}$$

$$\Rightarrow y=\frac{x^{\alpha}+x^{\beta}+x^{\gamma}}{x^{\alpha}+x^{\beta}+x^{\gamma}}\quad\Rightarrow\quad y=1$$

$$\Rightarrow \frac{dy}{dx}=0 \qquad (\text{Ans } A)$$

2. If f(x) = |cosx|, then $f'\left(\frac{3\pi}{4}\right)$ is equal to -

(A) $-\frac{1}{\sqrt{2}}$ (B) $\frac{1}{\sqrt{2}}$ (C) 1 (D) –1

Solⁿ

$f(x) = |\cos x|$

For $x \in (\pi/2, \pi)$: $f(x) = -\cos x$

$\Rightarrow f'(x) = -(-\sin x)$

$\Rightarrow f'(x) = \sin x$

$\therefore f'(3\pi/4) = \sin\frac{3\pi}{4} = \frac{1}{\sqrt{2}}$

(Ans B)

3. $\frac{d}{dx}(e^x \sin\sqrt{3}\,x)$ equals-

(A) $e^x \sin(\sqrt{3}\,x + \pi/3)$

(B) $2e^x \sin(\sqrt{3}\,x + \pi/3)$

(C) $\frac{1}{2} e^x \sin(\sqrt{3}\,x + \pi/3)$

(D) $\frac{1}{2} e^x \sin(\sqrt{3}\,x - \pi/3)$

Solⁿ

$$\frac{d}{dx}(e^x \sin\sqrt{3}\,x) = e^x \cdot \sqrt{3}\cos\sqrt{3}x + e^x \sin\sqrt{3}x$$

$$= 2e^x\left(\frac{\sqrt{3}}{2}\cos\sqrt{3}x + \frac{1}{2}\sin\sqrt{3}x\right)$$

$$= 2e^x\left(\sin\frac{\pi}{3}\cos\sqrt{3}x + \cos\frac{\pi}{3}\sin\sqrt{3}x\right)$$

$$= 2e^x\left(\sin\left(\sqrt{3}x + \frac{\pi}{3}\right)\right)$$

(Ans B)

4. $\frac{d}{dx}(\ell n \sin\sqrt{x})$ is equal to-

(A) $\frac{\tan\sqrt{x}}{2\sqrt{x}}$ (B) $\frac{\cot\sqrt{x}}{\sqrt{x}}$ (C) $\frac{\cot\sqrt{x}}{2x}$ (D) $\frac{\cot\sqrt{x}}{2\sqrt{x}}$

Sol^n

$$\frac{d}{dx}(\ell n \sin\sqrt{x}) = \frac{\cos(\sqrt{x})}{\sin\sqrt{x}} \cdot \frac{1}{2\sqrt{x}}$$

$$= \frac{\cot(\sqrt{x})}{2\sqrt{x}} \quad \text{(AnsD)}$$

5. If $y=\sqrt{\frac{1-x}{1+x}}$, then $\frac{dy}{dx}$ equals-

(A) $\frac{y}{1-x^2}$ (B) $\frac{y}{x^2-1}$ (C) $\frac{y}{1+x^2}$ (D) $\frac{y}{y^2-1}$

Solⁿ

$$y=\left(\frac{1-x}{1+x}\right)^{1/2}$$

$$\ln y=\frac{1}{2}\ln\left(\frac{1-x}{1+x}\right)$$

$$\ln y=\frac{1}{2}\left(\ln(1-x)-\ln(1+x)\right)$$

$$\Rightarrow \frac{y'}{y}=\frac{1}{2}\left(\frac{-1}{1-x}-\frac{1}{1+x}\right)$$

$$\Rightarrow \frac{y'}{y}=\frac{-1}{2}\left(\frac{2}{1-x^2}\right)$$

$$\Rightarrow y'=\frac{y}{x^2-1}$$ (Ans B)

6. If $y = \ell n\left\{\frac{x+\sqrt{(a^2+x^2)}}{a}\right\}$, then the value of $\frac{dy}{dx}$ is-

(A) $\sqrt{a^2-x^2}$ (B) $a\sqrt{a^2+x^2}$ (C) $\frac{1}{\sqrt{a^2+x^2}}$ (D) $x\sqrt{a^2+x^2}$

Sol^n

$$y = \ln\left(x+\sqrt{a^2+x^2}\right) - \ln a$$

$$\Rightarrow y' = \frac{1+\frac{\cancel{2}x}{\cancel{2}\sqrt{a^2+x^2}}}{x+\sqrt{a^2+x^2}} - 0$$

$$\Rightarrow y' = \frac{\cancel{(x+\sqrt{a^2+x^2})}}{\cancel{(x+\sqrt{a^2+x^2})}} \cdot \frac{1}{\sqrt{a^2+x^2}}$$

$$\Rightarrow y' = \frac{1}{\sqrt{a^2+x^2}}$$ (Ans C)

7. If $(\cos x)^y = (\sin y)^x$, then $\frac{dy}{dx}$ equals-

(A) $\dfrac{\log \sin y - y \tan x}{\log \cos x + x \cot y}$

(B) $\dfrac{\log \sin y + y \tan x}{\log \cos x - x \cot y}$

(C) $\dfrac{\log \sin y + y \tan x}{\log \cos x + x \cot y}$

(D) $\dfrac{\log \sin y + y \tan x}{\log \cos y - y \cot x}$

Sol^n

$$(\cos x)^y = (\sin y)^x$$

Taking log on both sides:

$$y \ln(\cos x) = x \ln(\sin y)$$

$$\frac{y(-\sin x)}{\cos x} + y' \ln(\cos x) = \frac{x(\cos y)}{\sin y} \cdot y' + \ln(\sin y)$$

$$\Rightarrow y' = \frac{\ln \sin y + y \tan x}{\ln \cos x - x \cot y}$$ (Ans B)

8 If $2^x + 2^y = 2^{x+y}$, then $\frac{dy}{dx}$ is equal to-

(A) $\frac{2^x+2^y}{2^x-2^y}$ (B) $\frac{2^x+2^y}{1+2^{x+y}}$ (C) $2^{x-y}\left(\frac{2^y-1}{1-2^x}\right)$ (D) $\frac{2^x+y-2^x}{2^y}$

Solⁿ

$$2^x + 2^y = 2^{x+y}$$

$$2^x \cancel{\ln 2} + (2^y \cancel{\ln 2})y' = 2^{x+y}(\cancel{\ln 2})(1+y')$$

$$\Rightarrow 2^x + 2^y y' = 2^{x+y}(1+y')$$

$$\Rightarrow y'(2^y - 2^{x+y}) = 2^{x+y} - 2^x$$

$$\Rightarrow y' = \frac{2^{x+y} - 2^x}{2^y - 2^{x+y}}$$

$$\Rightarrow y' = \frac{2^x(2^y-1)}{2^y(1-2^x)}$$

$$\Rightarrow y' = 2^{x-y}\left(\frac{2^y-1}{1-2^x}\right)$$ (Ans)

9) If $x = a(t - \sin t)$, $y = a(1 + \cos t)$, then $\frac{dy}{dx}$ equals-

(A) $-\tan\frac{t}{2}$ (B) $\cot\frac{t}{2}$ (C) $-\cot\frac{t}{2}$ (D) $\tan\frac{t}{2}$

Sol^n

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{a(0 - \sin t)}{a(1 - \cos t)}$$

$$\Rightarrow \frac{dy}{dx} = -\frac{2\sin t/2 \cos t/2}{2\sin^2 t/2}$$

$$\Rightarrow \frac{dy}{dx} = -\cot t/2 \quad (\text{Ans } C)$$

(10) The differential coefficient of $\sec^{-1}\left(\dfrac{1}{2x^2-1}\right)$ w.r.t. $\sqrt{1-x^2}$ is-

(A) $1/x^2$ (B) $2/x^3$ (C) $x/2$ (D) $2/x$

Sol^n

$$\sec^{-1}\left(\frac{1}{2x^2-1}\right) = \cos^{-1}(2x^2-1)$$

$$\therefore \frac{d\left(\cos^{-1}(2x^2-1)\right)}{d\left(\sqrt{1-x^2}\right)} = \frac{\dfrac{d\left(\cos^{-1}(2x^2-1)\right)}{dx}}{\dfrac{d\left(\sqrt{1-x^2}\right)}{dx}}$$

$$= \frac{\dfrac{\cancel{-}1}{\sqrt{1-(2x^2-1)^2}}\cdot(4\cancel{x})}{\dfrac{\cancel{-}\cancel{2}\cancel{x}}{\cancel{2}\sqrt{1-x^2}}} = \frac{\dfrac{4}{2(\sqrt{1-x^2})x}}{\dfrac{1}{\sqrt{1-x^2}}}$$

$$= 2/x \text{ (Ans D)}$$

11 If $x^3 - y^3 + 3xy^2 - 3x^2y + 1 = 0$, then at (0, 1) $\frac{dy}{dx}$ equals-

(A) 1 (B) –1 (C) 2 (D) 0

Soln

$$\underbrace{x^3 - y^3 + 3xy^2 - 3x^2y}_{} + 1 = 0,$$

$$(x-y)^3 + 1 = 0$$

Differentiate wrt x:

$$3(x-y)^2(1-y') + 0 = 0$$

$x = 0,\ y = 1$

$$\Rightarrow 3(1-y') = 0 \Rightarrow y' = 1$$

(Ans A)

(12) $\frac{d}{d\theta}\left(\tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)\right)$ equals, if $-\pi < \theta < \pi$ -

(A) 1/2 (B) 1 (C) $\sec\theta$ (D) $\text{cosec}\theta$

Solⁿ

$$\frac{d}{d\theta}\left(\tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)\right) = \frac{d}{d\theta}\left(\tan^{-1}\left(\frac{2\sin^2\theta/2}{2\sin\theta/2\cos\theta/2}\right)\right)$$

$$= \frac{d}{d\theta}\left(\tan^{-1}\left(\tan\theta/2\right)\right)$$

$\because \theta \in (-\pi, \pi) \Rightarrow \theta/2 \in (-\pi/2, \pi/2)$

$\Rightarrow \tan^{-1}(\tan\theta/2) = \theta/2$

$$\therefore \frac{d}{d\theta}\left(\tan^{-1}\left(\tan\theta/2\right)\right) = \frac{d}{d\theta}(\theta/2)$$

$$= 1/2 \quad \text{(Ans A)}$$

(13) If $y = \tan^{-1}(\cot x) + \cot^{-1}(\tan x)$, then $\frac{dy}{dx}$ is equal to-

(A) 1 (B) 0 (C) –1 (D) –2

Sol^n $y = \tan^{-1}(\cot x) + \cot^{-1}(\tan x)$

$$\Rightarrow y = \frac{\pi}{2} - \underbrace{\cot^{-1}\cot x}_{} + \frac{\pi}{2} - \underbrace{\tan^{-1}\tan x}_{}$$

$$\Rightarrow y = \pi - \left(\left[x + \alpha\pi\right] + \left[x + \beta\pi\right]\right)$$

(where α, β depends on interval of x)

$$\Rightarrow y' = -2 \quad \text{(Ans D)}$$

(14) If g is the inverse of f and $f'(x) = \frac{1}{1+x^3}$ then $g'(x)$ is equal to-

(A) $1 + [g(x)]^3$ (B) $\frac{-1}{2(1+x^2)}$ (C) $\frac{1}{2(1+x^2)}$ (D) $\frac{1}{1+[g(x)]^3}$

Solⁿ

$\because$ g is inverse of f :

$g(f(x)) = x$ and $f(g(x)) = x$

$\Rightarrow g'(f(x)) \cdot f'(x) = 1$

$\Rightarrow g'(f(x)) = \dfrac{1}{f'(x)}$

$\Rightarrow g'(f(x)) = \dfrac{1}{\frac{1}{1+x^3}} = 1 + x^3$

Replace $x \to g(x)$

$g'(f(g(x))) = 1 + (g(x))^3$

(Ans A)

15. If $x^2 + y^2 = 1$, then-

(A) $yy'' - 2(y')^2 + 1 = 0$

(B) $yy'' + (y')^2 + 1 = 0$

(C) $yy'' + (y')^2 - 1 = 0$

(D) $yy'' + 2(y')^2 + 1 = 0$

Solⁿ $x^2 + y^2 = 1$

$\Rightarrow 2x + 2yy' = 0$

$\Rightarrow x + yy' = 0$

$\Rightarrow 1 + (y')^2 + yy'' = 0$

(Ans B)

16

Let f be a function defined for all $x \in R$. If f is differentiable and $f(x^3) = x^5$ for all $x \in R$ ($x \neq 0$), then the value of $f'(27)$ is-

(A) 15 (B) 45 (C) 0 (D) 35

Sol^n

$$f(x^3) = x^5$$

$$\Rightarrow 3x^2 \cdot f'(x^3) = 5x^4$$

$$\Rightarrow f'(x^3) = \frac{5x^2}{3}$$

Put $x = 3$:

$$f'(27) = \frac{5(3)^2}{3} = 15 \text{ (Ans A)}$$

EXERCISE (O-2)

1. Let u(x) and v(x) are differentiable functions such that $\frac{u(x)}{v(x)} = 7$. If $\frac{u'(x)}{v'(x)} = p$ and $\left(\frac{u(x)}{v(x)}\right)' = q$, then $\frac{p+q}{p-q}$ has the value equal to -

(A) 1 (B) 0 (C) 7 (D) –7

(1)

$$\frac{u(x)}{v(x)} = 7$$

$$\left(\frac{u(x)}{v(x)}\right)' = 0 = q$$

$$\frac{p+q}{p-q} = \frac{p+0}{p-0} = 1$$

2. Let f (x) be a polynomial function of second degree. If $f(1) = f(-1)$ and a, b, c are in A.P., then f '(a), f '(b) and f '(c) are in

(A) G.P. (B) H.P. (C) A.G.P. (D) A.P.

---

(2) Let $f(x) = Ax^2 + Bx + C$

$f(1) = f(-1)$

$A + B + c = A - B + c$

$\boxed{B = 0}$

$f(x) = Ax^2 + C$

$f'(x) = 2Ax$

a, b, c are in A.P.

$2Aa, 2Ab, 2Ac$ are in AP

$f'(a), f'(b), f'(c)$ are in AP

3. Suppose the function $f(x) - f(2x)$ has the derivative 5 at $x = 1$ and derivative 7 at $x = 2$. The derivative of the function $f(x) - f(4x)$ at $x = 1$, has the value equal to

•(A) 19 (B) 9 (C) 17 (D) 14

3

$$\left(f(x) - f(2x)\right)' = f'(x) - 2f'(2x)\Big|_{x=1} = 5$$

$$f'(1) - 2f'(2) = 5 \quad \text{——— } (1)$$

at $x = 2 \Rightarrow f'(2) - 2f'(4) = 7 \quad \text{——— } (2)$

$$\left(f(x) - f(4x)\right)' = f'(x) - 4f'(4x)$$

at $\underline{x = 1} \Rightarrow f'(1) - 4f'(4)$

$eq^n (1) + 2 \times eq^n (2)$

$$\boxed{f'(1) - 4f'(4) = 19}$$

4. Suppose that $f(0)=0$ and $f'(0)=2$, and let $g(x)=f\left(-x+f\left(f(x)\right)\right)$. The value of $g'(0)$ is equal to

(A) 0 (B) 1 • (C) 6 (D) 8

$$g(x) = f\left(-x + f\left(f(x)\right)\right)$$

$$g'(x) = f'\left(-x + f\left(f(x)\right)\right)\left(-1 + f'\left(f(x)\right)\cdot f'(x)\right)$$

$$g'(0) = f'\left(f\left(f(0)\right)\right)\left(-1 + f'\left(f(0)\right)\cdot f'(0)\right)$$

$$= f'(0)\left(-1 + f'(0)\cdot f'(0)\right)$$

$$= 2\left(-1 + 2^2\right)$$

$$\boxed{g'(0) = 6}$$

5 If f is differentiable in (0, 6) & $f'(4) = 5$, then $\underset{x\to 2}{\text{Limit}} \frac{f(4) - f(x^2)}{2 - x} =$

(A) 5 (B) 5/4 (C) 10 (D) 20

$$\lim_{x\to 2} \frac{f(4) - f(x^2)}{2 - x} \quad \left(\frac{0}{0} \text{ form}\right)$$

Apply L'hopital Rule

$$\lim_{x\to 2} \frac{0 - f'(x^2)\cdot 2x}{0 - 1}$$

$$= 4 f'(4)$$

$$\boxed{= 20}$$

6 If $y = x + e^x$ then $\frac{d^2x}{dy^2}$ is :

(A) e^x (B) $-\frac{e^x}{(1+e^x)^3}$ (C) $-\frac{e^x}{(1+e^x)^2}$ (D) $\frac{-1}{(1+e^x)^3}$

$$y = x + e^x$$

$$\frac{dy}{dx} = 1 + e^x$$

$$\frac{dx}{dy} = \frac{1}{1+e^x}$$

$$\frac{d^2x}{dy^2} = -\frac{1}{(1+e^x)^2} \cdot e^x \cdot \frac{dx}{dy}$$

$$\boxed{\frac{d^2x}{dy^2} = \frac{-e^x}{(1+e^x)^3}}$$

7. If $\frac{d^2x}{dy^2}\left(\frac{dy}{dx}\right)^3+\frac{d^2y}{dx^2}=K$ then the value of K is equal to

(A) 1 (B) –1 (C) 2 (D) 0

we know that

$$\frac{d^2x}{dy^2} = -\frac{\frac{d^2y}{dx^2}}{\left(\frac{dy}{dx}\right)^3}$$

$$\frac{d^2x}{dy^2}\left(\frac{dy}{dx}\right)^3+\frac{d^2y}{dx^2}=0$$

$$\boxed{K=0}$$

8. Let $f(x)$ be differentiable at $x = h$, then $\lim_{x \to h} \frac{(x+h)f(x) - 2hf(h)}{x-h}$ is equal to -

(A) $f(h) + 2hf'(h)$ (B) $2f(h) + hf'(h)$ (C) $hf(h) + 2f'(h)$ (D) $hf(h) - 2f'(h)$

$$\lim_{x \to h} \frac{(x+h)f(x) - 2hf(h)}{x-h} \quad \left(\frac{0}{0} \text{ form}\right)$$

Apply L'hopital Rule

$$\lim_{x \to h} \frac{(x+h)f'(x) + f(x) - 0}{1-0}$$

$$\boxed{= 2hf'(h) + f(h)}$$

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(9) Let $f(x) = x + \sin x$. Suppose g denotes the inverse function of f. The value of $g'\left(\frac{\pi}{4} + \frac{1}{\sqrt{2}}\right)$ has the value equal to :-

$f(x) = x + \sin x$

$x + \sin x = \frac{\pi}{4} + \frac{1}{\sqrt{2}}$

$g'(y) = \frac{1}{f'(x)}$

$\therefore\ x = \pi/4$

$= \frac{1}{1+\cos y}$

$= \frac{1}{1+\cos(\pi/4)} \Rightarrow \frac{\sqrt{2}}{\sqrt{2}+1} \times \frac{\sqrt{2}-1}{\sqrt{2}-1}$

$\boxed{2-\sqrt{2}}$

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(10) If $f(x) = (2x-3\pi)^5 + \frac{4}{3}x + \cos x$ and g is the inverse function of f, then find Right answer :-

$$f(x) = (2x-3\pi)^5 + \frac{4}{3}x + \cos x = 2\pi$$

Put $x = 3\pi/2$

$f(x) = 2\pi$

$$g'(f(x)) = \frac{1}{5(2x-3\pi)^4 \cdot 2 + \frac{4}{3} - \sin x}$$

$$= \frac{1}{5\left(2\cdot\frac{3\pi}{2} - 3\pi\right)^4 \cdot 2 + \frac{4}{3} - \sin\left(\frac{3\pi}{2}\right)}$$

$$= \frac{1}{\frac{4}{3}+1} \Rightarrow \frac{3}{7}$$

$$g' f(x) = \frac{1}{f'(x)}$$

$$g'' f(x)\, f'(x) = -\frac{1}{f'(x)^2} \cdot f''(x)$$

$$g'' f(x) = \frac{-1}{(f'(x))^3} \cdot f''(x)$$

$$g''(2\pi) = \frac{-1}{\left(\frac{4}{3}+1\right)^3}\left[20\cdot 2\cdot (2x-3\pi)^3 \cdot 2 - \cos\left(\frac{3\pi}{2}\right)\right]$$

$$= 0$$

$\therefore$ $\boxed{B}$ & $\boxed{D}$

(11) If $f(x) = x \cdot |x|$ then its derivative is :-

$$f(x) = \begin{cases} x^2 , & x > 0 \\ 0 , & x = 0 \\ -x^2 , & x < 0 \end{cases}$$

$$f'(x) = \begin{cases} 2x , & x > 0 \\ 0 , & x = 0 \\ -2x , & x < 0 \end{cases}$$

$$f'\,0^+ = \lim_{h \to 0} \frac{h^2}{h} = 0$$

$$f'\,0^- = \lim_{h \to 0} \frac{(-h)^2}{-h} = 0$$

$f'(0) = 0$

$\therefore\ f'(x) = 2|x|$

$= 2x\,\text{Sgn}(x)$

$\therefore\ \boxed{C}\ \&\ \boxed{D}$

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(12) $\lim_{x\to\pi} \dfrac{x^\pi - \pi^x}{x^x - \pi^\pi}$ is equal to :-

$\lim_{x\to\pi} \dfrac{\pi x^{\pi-1} - \pi^x \ln\pi}{x^x(1+\ln x)}$

$\dfrac{\pi\cdot\pi^{\pi-1} - \pi^\pi \ln\pi}{\pi^\pi(1+\ln\pi)}$

$\dfrac{\cancel{\pi^\pi}[1-\ln\pi]}{\cancel{\pi^\pi}[1+\ln\pi]} \Rightarrow \dfrac{\ln(e/\pi)}{\ln(e\pi)}$

$\Rightarrow \log_{e\pi}(e/\pi)$

(C) $\tan[\cot^{-1}(\ln x) - \cot^{-1}(1)]$

$\tan\left[\tan^{-1}\left(\frac{1}{\ln\pi}\right) - \tan^{-1}(1)\right]$

$\tan\left[\tan^{-1} \dfrac{\frac{1}{\ln\pi} - 1}{1+\frac{1}{\ln\pi}}\right]$

$\dfrac{1-\ln\pi}{1+\ln\pi} \rightarrow \log_{e\pi}(e/\pi)$

(D) $\tan[\tan^{-1}(1) - \tan^{-1}(\ln\pi)]$

$\dfrac{1-\ln\pi}{1+\ln\pi}$

$\log_{e\pi}(e/\pi)$

$\therefore$ $\boxed{A}$ $\boxed{C}$ $\boxed{D}$

S-1

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① Let f, g, h are differentiable functions. if $f(0)=1$, $g(0)=2$, $h(0)=3$ and the derivatives of their pair wise products at $x=0$ are $(fg)'(0)=6$, $(gh)'(0)=4$, $(hf)'(0)=5$ then compute the value of $(fgh)'(0)$:-

By Property

$$(fgh)'(0) = \frac{(fg)'(0)\,h(0) + (fh)'(0)\,g(0) + (gh)'(0)\,f'(0)}{2}$$

$$= \frac{6\times3 + 5\times2 + 4\times1}{2} = \boxed{16}$$

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(2) (A) If $y = (\cos x)^{\ln x} + (\ln x)^x$ find $\frac{dy}{dx}$.

$$\frac{dy}{dx} = \ln x\,(\cos x)^{\ln x - 1}(-\sin x) + (\cos x)^{\ln x} \cdot \frac{\ln \cos x}{x} +$$

$$\cancel{x}(\ln x)^{x-1} \cdot \frac{1}{\cancel{x}} + (\ln x)^x \ln \ln x$$

$$= (\cos x)^{\ln x}\left[-\ln x \tan x + \frac{\ln \cos x}{x}\right] +$$

$$(\ln x)^x \left[\frac{1}{\ln x} + \ln \ln x\right]$$

(B) if $y = e^{x^{e^x}} + e^{x^{x^e}} + x^{e^{e^x}}$ find $\frac{dy}{dx}$.

let $y_1 = e^{x^{e^x}}$ $\qquad \ln y_1 = x^{e^x}$

$$\ln \ln y_1 = e^x \cdot \ln x$$

$$\frac{1}{\ln y_1} \cdot \frac{1}{y_1} \cdot y' = e^x \ln x + \frac{e^x}{x}$$

$$y' = y_1 \cdot \ln y_1 (x \ln x + 1) \frac{e^x}{x} \quad \text{———} \; (1)$$

let $y_2 = e^{x^{x^e}}$ $\qquad \ln y_2 = x^{x^e}$

$$\ln \ln y_2 = x^e \cdot \ln x$$

$$\frac{1}{\ln y_2} \cdot \frac{1}{y_2} \cdot y_2' = e \cdot x^{e-1} \ln x + \frac{x^e}{x}$$

$$y_2' = y_2 \cdot \ln y_2 \,(e \ln x + 1)\, x^{e-1} \quad \text{———} \; (2)$$

let $y_3 = x^{e^{e^x}}$ $\qquad \ln y_3 = e^{e^x} \cdot \ln x$

$$\ln \ln y_3 = e^x + \ln \ln x$$

$$\frac{1}{\ln y_3} \cdot \frac{1}{y_3} \, y_3' = e^x + \frac{1}{\ln x} \cdot \frac{1}{x} \quad \text{———} \; (3)$$

Add (1) + (2) + (3)

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(3) If $y = \frac{x^2}{2} + \frac{1}{2} x\sqrt{x^2+1} + \ln\sqrt{x+\sqrt{x^2+1}}$ Prove that $2y = xy' + \ln y'$ where ' denotes the derivatives

$$y = \frac{x^2}{2} + \frac{1}{2} x\sqrt{x^2+1} + \ln\left(x+\sqrt{x^2+1}\right)^{\frac{1}{2}}$$

$$y = \frac{x^2}{2} + \frac{1}{2} x\sqrt{x^2+1} + \frac{1}{2}\ln\left[x+\sqrt{x^2+1}\right]$$

$$y' = x + \frac{\sqrt{x^2+1}}{2} + \frac{x}{2}\cdot\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+1}} + \frac{1}{2}\,\frac{1+\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+1}}}{\left(x+\sqrt{x^2+1}\right)}$$

$$= x + \frac{\sqrt{x^2+1}}{2} + \frac{x^2}{2\sqrt{x^2+1}} + \frac{1}{2\sqrt{x^2+1}}$$

$$= x + \frac{\sqrt{x^2+1}}{2} + \frac{x^2+1}{2\sqrt{x^2+1}}$$

$$= x + \frac{\sqrt{x^2+1}}{2} + \frac{\sqrt{x^2+1}}{2}$$

$$y' = x + \sqrt{x^2+1}$$

$$\therefore\ x y' = x^2 + x\sqrt{x^2+1}$$

$$\ln y' = \ln\left[x+\sqrt{x^2+1}\right]$$

$$\boxed{2y = xy' + \ln y'}$$

(4) if $y = \ln\left(x^{e^{x} \cdot a^{y}}\right)^{y^{x}}$ find $\frac{dy}{dx}$.

$$= y^{x} \cdot \ln\left(x^{e^{x} \cdot a^{y}}\right)$$

$$y = y^{x} \cdot e^{x} \cdot a^{y} \ln x$$

$$\ln y = x \ln y + x + y \ln a + \ln \ln x \qquad [\text{taking log}]$$

$$\frac{1}{y} \cdot y' = \ln y + \frac{x}{y} \cdot y' + 1 + y' \ln a + \frac{1}{\ln x} \cdot \frac{1}{x}$$

$$y'\left[\frac{1}{y} - \frac{x}{y} - \ln a\right] = \ln y + 1 + \frac{1}{x \ln x}$$

$$y' = \frac{(x \ln x \cdot \ln y + x \ln x + 1)}{(x \ln x)} \cdot \frac{y}{(1 - x - y \ln a)}$$

(5) If $x = \operatorname{cosec}\theta - \sin\theta$ & $y = \operatorname{cosec}^n\theta - \sin^n\theta$ then show that $(x^2+4)\left(\frac{dy}{dx}\right)^2 - n^2(y^2+4) = 0$

Sol^n

$$y = \operatorname{cosec}^n\theta - \sin^n\theta$$

$$\frac{dy}{d\theta} = -n\operatorname{cosec}^{n-1}\theta.(\operatorname{cosec}\theta\cot\theta) - n\sin^{n-1}\theta.\cos\theta$$

$$\frac{dx}{d\theta} = -\operatorname{cosec}\theta\cot\theta - \cos\theta$$

$$\frac{dy}{dx} = \frac{n\operatorname{cosec}^n\theta\cot\theta + \sin^{n-1}\theta\cos\theta}{\operatorname{cosec}\theta\cot\theta + \cos\theta}$$

$$\frac{dy}{dx} = \frac{n\operatorname{cosec}^n\theta\frac{\cos\theta}{\sin\theta} + \sin^{n-1}\theta\cos\theta}{\operatorname{cosec}\theta.\frac{\cos\theta}{\sin\theta} + \cos\theta}$$

$$\frac{dy}{dx} = \frac{n\cancel{\cos\theta}(\operatorname{cosec}^n\theta + \sin^n\theta)}{\cancel{\cos\theta}\cancel{\sin\theta}\left(\frac{\operatorname{cosec}\theta + \sin\theta}{\cancel{\sin\theta}}\right)}$$

$$\frac{dy}{dx} = \frac{n(\operatorname{cosec}^n\theta + \sin^n\theta)}{\operatorname{cosec}\theta + \sin\theta}$$

$$\rightarrow x^2 + 4 = (\operatorname{cosec}\theta - \sin\theta)^2 + 4$$

$$= (\operatorname{cosec}\theta + \sin\theta)^2$$

$$(x^2+4)\left(\frac{dy}{dx}\right)^2 = \cancel{(\operatorname{cosec}\theta + \sin\theta)^2} \cdot \frac{n^2\left((\operatorname{cosec}^n\theta + \sin^n\theta)\right)^2}{\cancel{(\operatorname{cosec}\theta + \sin\theta)^2}}$$

$$(x^2+4)\left(\frac{dy}{dx}\right)^2 = n^2(\operatorname{cosec}^n\theta + \sin^n\theta)^2 \quad \text{—— (i)}$$

$$n^2(y^2+4) = n^2\left[(\operatorname{cosec}^n\theta - \sin^n\theta)^2 + 4\right]$$

$$n^2(y^2+4) = n^2(\operatorname{cosec}^n\theta + \sin^n\theta)^2 \quad \text{—— (ii)}$$

By (i) & (ii)

$$\Rightarrow \boxed{(x^2+4)\left(\frac{dy}{dx}\right)^2 - n^2(y^2+4) = 0}$$

⑥ Differentiate

$$\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}} \quad \text{w.r.t } \sqrt{1-x^4}$$

Soln:

$$y=\frac{\sqrt{1+x^2}+\sqrt{1-x^2}}{\sqrt{1+x^2}-\sqrt{1-x^2}}, \qquad z=\sqrt{1-x^4}$$

$$y=\frac{\left(\sqrt{1+x^2}+\sqrt{1-x^2}\right)^2}{1+x^2-1+x^2} \qquad \text{(Rationalize)}$$

$$y=\frac{1+\sqrt{1-x^4}}{x^2}$$

$$\frac{dy}{dz}=\frac{\dfrac{\dfrac{-4x^3}{2\sqrt{1-x^4}}\cdot x^2-\left(1+\sqrt{1-x^4}\right)2x}{x^4}}{\dfrac{-4x^3}{2\sqrt{1-x^4}}}$$

$$=\frac{-2x^5-2x\left(1+\sqrt{1-x^4}\right)\left(\sqrt{1-x^4}\right)}{x^4\left(-2x^3\right)}$$

$$\Rightarrow \frac{x^5+x\left(\sqrt{1-x^4}+1-x^4\right)}{x^7}$$

$$\Rightarrow \frac{x^4+\sqrt{1-x^4}+1-x^4}{x^6}=\frac{1+\sqrt{1-x^4}}{x^6}$$

7. If $\sqrt{1-x^6} + \sqrt{1-y^6} = a^3(x^3-y^3)$

Prove that $\dfrac{dy}{dx} = \dfrac{x^2}{y^2}\sqrt{\dfrac{1-y^6}{1-x^6}}$

Sol^n:

$$\sqrt{1-x^6} + \sqrt{1-y^6} = a^3(x^3-y^3)$$

Diff w.r.t x

$$\frac{1 \cdot -\cancel{6}x^5}{\cancel{2}\sqrt{1-x^6}} + \frac{1 \cdot -6y^5y'}{2\sqrt{1-y^6}} = a^3(\cancel{3}x^2 - \cancel{3}y^2y')$$

$$\frac{-x^5}{\sqrt{1-x^6}} - \frac{y^5y'}{\sqrt{1-y^6}} = \frac{\sqrt{1-x^6}+\sqrt{1-y^6}}{x^3-y^3}(x^2-y^2y')$$

putting value of a^3

$$y'\left[\frac{-y^5}{\sqrt{1-y^6}} + \frac{\sqrt{1-x^6}+\sqrt{1-y^6}}{x^3-y^3}\,y^2\right]$$

$$= \frac{\sqrt{1-x^6}+\sqrt{1-y^6}}{x^3-y^3}\cdot x^2 + \frac{x^5}{\sqrt{1-x^6}}$$

$$y'y^2\left(\frac{-x^3y^3+\cancel{y^6}+\sqrt{1-x^6}\sqrt{1-y^6}+1-\cancel{y^6}}{\cancel{(x^3-y^3)}\sqrt{1-y^6}}\right)$$

$$= x^2\left[\frac{1-\cancel{x^6}+\sqrt{1-x^6}\sqrt{1-y^6}+\cancel{x^6}-y^3x^3}{\cancel{(x^3-y^3)}\sqrt{1-x^6}}\right]$$

$$y' y^2 \left[\frac{1 - x^3 y^3 + \cancel{\sqrt{1-x^6}\sqrt{1-y^6}}}{\sqrt{1-y^6}} \right] = x^2 \left[\frac{1 - x^3 y^3 + \cancel{\sqrt{1-x^6}\sqrt{1-y^6}}}{\sqrt{1-x^6}} \right]$$

$$y' = \frac{x^2}{y^2} \sqrt{\frac{1-y^6}{1-x^6}}$$

$$\boxed{\frac{dy}{dx} = \frac{x^2}{y^2} \frac{\sqrt{1-y^6}}{\sqrt{1-x^6}}}$$

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(8) Suppose $f(x) = \tan(\sin^{-1}(2x))$

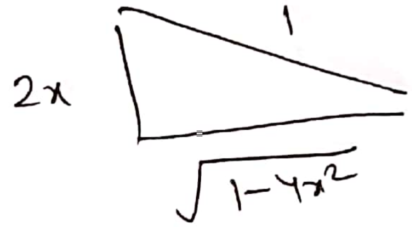
(a) find the Domain & Range of f

(b) Express $f(x)$ as an algebric function of x

(c) find $f'\left(\frac{1}{4}\right)$

Solⁿ:

$$f(x) = \tan(\sin^{-1}(2x))$$



(a) Domain

$$-1 \leq 2x \leq 1$$

$$x \in \left(\frac{-1}{2}, \frac{1}{2}\right)$$

Range → R

(b) $$f(x) = \frac{2x}{\sqrt{1-4x^2}}$$

(c)

$$f'(x) = \frac{2\sqrt{1-4x^2} - \dfrac{2x.(-8x)}{2\sqrt{1-4x^2}}}{1-4x^2}$$

$$f'\left(\frac{1}{4}\right) = \frac{2\sqrt{1-4\times\frac{1}{16}} - \dfrac{2\times\frac{1}{4}\times -8\times\frac{1}{4}}{2\sqrt{1-\frac{1}{4}}}}{1-\frac{1}{4}}$$

$$f'\left(\frac{1}{4}\right) = \frac{\dfrac{2\sqrt{3}}{2} + \dfrac{1}{\frac{2\sqrt{3}}{2}}}{3/4} \Rightarrow \frac{16\sqrt{3}}{9}$$

(9) If $y = \tan^{-1}\left(\frac{1}{x^2+x+1}\right) + \tan^{-1}\left(\frac{1}{(x^2+3x+3)}\right)$

$+ \tan^{-1}\left(\frac{1}{x^2+5x+7}\right) + \tan^{-1}\left(\frac{1}{x^2+7x+13}\right) ---$

to n terms

find $\frac{dy}{dx}$, expressing your Answer in 2 terms

Sol^n:

$$y = \tan^{-1}\left(\frac{1}{1+x(x+1)}\right) + \tan^{-1}\left(\frac{1}{1+(x+1)(x+2)}\right)$$

$$+ \tan^{-1}\left(\frac{1}{1+(x+2)(x+3)}\right) ----$$

$$y = \tan^{-1}\left(\frac{(x+1)-(x)}{1+x(x+1)}\right) + \tan^{-1}\left(\frac{(x+2)-(x+1)}{1+(x+1)(x+2)}\right)$$

$$+ \tan^{-1}\left(\frac{(x+3)-(x+2)}{1+(x+2)(x+3)}\right) -------$$

$$y = \left(\cancel{\tan^{-1}(x+1)} - \tan^{-1}x\right) + \left(\cancel{\tan^{-1}(x+2)} - \cancel{\tan^{-1}(x+1)}\right)$$

$$+ \left(\tan^{-1}(x+3) - \cancel{\tan^{-1}(x+2)}\right) ----$$

$$y = \tan^{-1}(x+n) - \tan^{-1}x$$

$$\frac{dy}{dx} = \frac{1}{1+(x+n)^2} - \frac{1}{1+x^2}$$

(10) If $y = \tan^{-1}\frac{u}{\sqrt{1-u^2}}$ & $x = \sec^{-1}\left(\frac{1}{2u^2-1}\right)$

$u \in \left(0, \frac{1}{\sqrt{2}}\right) \cup \left(\frac{1}{\sqrt{2}}, 1\right)$, Prove that $2\left(\frac{dy}{dx}\right)+1=0$

Soln:

$y = \tan^{-1}\left(\frac{u}{\sqrt{1-u^2}}\right)$, $x = \cos^{-1}(2u^2-1)$

Put $u = \cos\theta$, $\theta \in \left(0, \frac{\pi}{4}\right) \cup \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

$y = \tan^{-1}\left(\frac{\cos\theta}{\sin\theta}\right) = \tan^{-1}(\cot\theta)$

$y = \frac{\pi}{2} - \cot^{-1}\cot\theta = \frac{\pi}{2} - \cos^{-1}(u)$

$x = \cos^{-1}(2\cos^2\theta - 1)$

$x = \cos^{-1}(\cos 2\theta)$ $\qquad 2\theta \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)$

$x = 2\theta$

$x = 2\cos^{-1}u$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{-\left(\frac{-1}{\cancel{\sqrt{1-u^2}}}\right)}{2\,\frac{-1}{\cancel{\sqrt{1-u^2}}}}$$

$$\frac{dy}{dx} = -\frac{1}{2}$$

$$\boxed{2\left(\frac{dy}{dx}\right) + 1 = 0}$$

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11 (a) If $y = y(x)$ and it follows the relation $e^{xy} + y\cos x = 2$ then find (i) $y'(0)$ & (ii) $y''(0)$

(b) A twice differentiable function $f(x)$ is defined for all real numbers and satisfy the following Condition $f(0) = 2$, $f'(0) = 5$ & $f''(0) = 3$

The function $g(x)$ is defined by $g(x) = e^{ax} + f(x)$ $\forall\ x \in R$, where a is any constant.

If $g'(0) + g''(0) = 0$ Find the value(s) of 'a'

Sol^n: (a) $e^{xy} + y\cos x = 2$ | at $x = 0$

$e^0 + y \times 1 = 2$

$y = 1$

Diff w.r.t x

$$e^{xy}(y + xy') + y'\cos x - \sin y = 0$$

$$y' = \frac{\sin x . y - e^{xy} . y}{\cos x + x e^{xy}}$$

(i) $$y'(0) = \frac{0 . y - e^0 . (1)}{\cos 0 + 0} = -1$$

(ii) $$y(e^{xy} - \sin x) + y'(\cos x + x e^{xy}) = 0$$

Diff. w.r.to x

$$y'(e^{xy} - \sin x) + y(e^{xy}(y + xy') - \cos x)$$
$$+ y''(\cos x + x e^{xy}) + y'(-\sin x + e^{xy} + x(e^{xy}(y + xy'))$$

$x=0, \ y=1$

$$(-1)(e^0-0)+1\left(e^0(1+0)-1\right)+y''(1+0)$$
$$+(-1)(0+e^0+0)=0$$

$$-1+1\cdot(1-1)+y''(1)-1=0$$

$$\boxed{y''(0)=2}$$

at $\underline{\underline{x=0}}$

(b) $$g(x)=e^{ax}+f(x)$$

$$g'(x)=a e^{ax}+f'(x)$$

$$g'(0)=a e^0+f'(0)=a+(-5)$$
$$=a-5$$

$$g''(x)=a^2 e^{ax}+f''(x)$$

$$g''(0)=a^2+f''(0)=a^2+3$$

$$g'(0)+g''(0)=0$$

$$a-5+a^2+3=0$$

$$a^2+a-2=0$$

$$\boxed{a=-2,\ 1}$$

(12) If $x = 2\cos t - \cos 2t$ & $y = 2\sin t - \sin 2t$ find the value of $\left(\frac{d^2y}{dx^2}\right)$ when $t = \left(\frac{\pi}{2}\right)$

Soln:

$$x = 2\cos t - \cos 2t \quad , \quad y = 2\sin t - \sin 2t$$

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{2\cos t - 2\cos 2t}{-2\sin t + 2\sin 2t}$$

$$= \frac{\cos t - \cos 2t}{-\sin t + \sin 2t}$$

$$= \frac{2\sin\left(\frac{3t}{2}\right)\cancel{\sin\left(\frac{t}{2}\right)}}{2\cos\left(\frac{3t}{2}\right)\cancel{\sin\left(\frac{t}{2}\right)}}$$

$$\frac{dy}{dx} = \tan\left(\frac{3t}{2}\right)$$

$$\frac{d^2y}{dx^2} = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx}$$

$$= \frac{3}{2}\sec^2\frac{3t}{2} \times \left(\frac{1}{-2\sin t + 2\sin 2t}\right)$$

$$\left.\frac{d^2y}{dx^2}\right|_{t=\pi/2} = \frac{3}{2}\,\frac{2}{(-2+0)} \Rightarrow -3/2$$

EXERCISE (S-2)

1. Find a polynomial function f(x) such that f(2x) = f '(x) f "(x).

Sol :- Let degree of $f(x)$ be n

$\Rightarrow$ Degree of $f(2x) = n$

Degree of $f'(x) = n-1$

Degree of $f''(x) = n-2$

Since $f(2x) = f'(x) f''(x)$

$\Rightarrow$ Degree of LHS $= n =$ Degree of RHS $= n-1+n-2 = 2n-3$

$\Rightarrow n = 2n-3 \Rightarrow \boxed{n=3}$

Let $f(x) = ax^3+bx^2+cx+d \quad (a \neq 0)$

$f(2x) = 8ax^3+4bx^2+2cx+d$

$f'(x) = 3ax^2+2bx+c$

$f''(x) = 6ax+2b$

$8ax^3+4bx^2+2cx+d = (3ax^2+2bx+c)(6ax+2b)$

Equating the coefficients of x^3, x^2, x and constant, we get

$$8a = 18a^2 \implies a = 4/9 \quad [\because a \neq 0]$$

$$4b = 6ab + 12ab = 18ab \implies a = 2/9$$
or $b = 0$

$$\implies b = 0 \quad [\because a = 4/9]$$

$$2c = 4b^2 + 6ac \implies 2c = 6ac$$

$$\implies a = \frac{1}{3} \quad \text{or } c = 0$$

$$\implies c = 0 \quad [\because a = 4/9]$$

$$d = 2bc = 0$$

$$\implies f(x) = \frac{4}{9}x^3$$

2. If $2x = y^{\frac{1}{5}} + y^{-\frac{1}{5}}$ then $(x^2 - 1)\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} = ky$, then find the value of 'k'.

Sol:-

$$y^{1/5} + y^{-1/5} = 2x \quad \text{—— (1)}$$

$$\Rightarrow \left[\frac{1}{5} y^{\frac{1}{5}-1} - \frac{1}{5} y^{-\frac{1}{5}-1}\right] y' = 2$$

$$\Rightarrow \frac{1}{5} y^{-1} \left[y^{\frac{1}{5}} - y^{-\frac{1}{5}} \right] = 2$$

$$\Rightarrow y' (y^{1/5} - y^{-1/5}) = 10y$$

$$\Rightarrow y'^2 (y^{1/5} - y^{-1/5})^2 = 100y^2$$

$$\Rightarrow y'^2 \left[(y^{1/5} + y^{-1/5})^2 - 4 \right] = 100y^2$$

$$\Rightarrow 4y'^2 [x^2 - 1] = 100y^2$$

$$\Rightarrow y'^2 (x^2 - 1) = 25y^2$$

$$\Rightarrow y'^2 . 2x + (x^2 - 1) 2y' . y'' = 50y . y'$$

$$\Rightarrow (x^2 - 1) y'' + x y' = 25y$$

$$\Rightarrow K = 25$$

3. Let y = x sin kx. Find the possible value of k for which the differential equation $\frac{d^2y}{dx^2} + y = 2k\cos kx$ holds true for all $x \in R$.

Sol :-

$$y = x\sin kx$$

$$\Rightarrow y' = kx\cos kx + \sin kx$$

$$\Rightarrow y'' = k\cos kx - k^2 x\sin kx + k\cos kx$$

$$y'' = 2k\cos kx - k^2 x\sin kx$$

$$y'' + y = 2k\cos kx \quad \forall x \in R$$

$$\Rightarrow \cancel{2k\cos kx} - k^2 x\sin kx + x\sin kx = \cancel{2k\cos kx}$$

$$\Rightarrow x\sin kx\,(k^2 - 1) = 0 \quad \forall x \in R$$

$$\Rightarrow k^2 - 1 = 0 \quad \text{or} \quad \sin kx = 0 \ \forall x \in R$$

$$\Rightarrow k = \pm 1 \quad \text{or} \quad k = 0$$

$$\Rightarrow k = -1, 0, 1$$

4 The function f: R → R satisfies $f(x^2) \cdot f''(x) = f'(x) \cdot f'(x^2)$ for all real x. Given that $f(1) = 1$ and $f'''(1) = 8$, compute the value of $f'(1) + f''(1)$.

Sol:

$$f(x^2) \cdot f''(x) = f'(x) \cdot f'(x^2) \quad \forall\, x \in R \quad \text{— (1)}$$

$$\Rightarrow f(x^2) \cdot f'''(x) + f''(x) \cdot f'(x^2) \cdot 2x$$

$$= f'(x)\, f''(x^2) \cdot 2x + f'(x^2) \cdot f''(x)$$

Put $x = 1$

$$\Rightarrow f(1) \cdot f'''(1) + \cancel{f''(1) \cdot f'(1) \cdot 2}$$

$$= \cancel{f'(1) \cdot f''(1) \cdot 2} + f'(1) \cdot f''(1)$$

$$\Rightarrow 1 \cdot 8 = f'(1) \cdot f''(1)$$

$$\Rightarrow f'(1) \cdot f''(1) = 8 \quad \text{— (2)}$$

Put $x = 1$ in (1)

$$\Rightarrow f(1) \cdot f''(1) = f'(1) \cdot f'(1)$$

$$\Rightarrow f''(1) = [f'(1)]^2 \quad \text{— (3)}$$

From (2) and (3),

$$[f'(1)]^3 = 8 \Rightarrow f'(1) = 2$$

$$\Rightarrow f''(1) = 4$$

$$\Rightarrow f'(1) + f''(1) = 6$$

(5) Let $f(x) = \frac{\sin x}{x}$ if $x \neq 0$ and $f(0) = 1$. Define the function $f'(x)$ for all x and find $f''(0)$ if it exist.

Sol :-

$$f(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$$

For $x \neq 0$, $f'(x) = \frac{x \cdot \cos x - \sin x}{x^2}$.

For $x = 0$, $f'(x) = \underset{h \to 0}{Lt} \frac{f(h) - f(0)}{h}$

$$= \underset{h \to 0}{Lt} \frac{\frac{\sin h}{h} - 1}{h}$$

$$= \underset{h \to 0}{Lt} \frac{\sin h - h}{h^2}$$

$$= \underset{h \to 0}{Lt} \frac{\cancel{h} - \frac{h^3}{3!} + \dots - \cancel{h}}{h^2}$$

$$f'(0) = 0$$

$$\Rightarrow f'(x) = \begin{cases} \frac{x \cos x - \sin x}{x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Now, $f''(0) = \underset{h \to 0}{Lt} \dfrac{f'(h) - f'(0)}{h}$

$$= \underset{h \to 0}{Lt} \frac{h \cos h - \sin h}{h^2 \cdot h}$$

$$= \underset{h \to 0}{Lt} \frac{h\left[1 - \frac{h^2}{2!} + \frac{h^4}{4!}\right] - \left[h - \frac{h^3}{3!} + \cdots\right]}{h^3}$$

$$= \underset{h \to 0}{Lt} \frac{\cancel{h} - \frac{h^3}{2!} - \cancel{h} + \frac{h^3}{3!}}{h^3}$$

$$= \frac{1}{6} - \frac{1}{2}$$

$$f''(0) = \frac{-1}{3}$$

⑥ If $y = \tan^{-1}\frac{x}{1+\sqrt{1-x^2}} + \sin\left(2\tan^{-1}\sqrt{\frac{1-x}{1+x}}\right)$, then find $\frac{dy}{dx}$ for $x \in (-1, 1)$.

Sol: Let $f(x) = \tan^{-1}\frac{x}{1+\sqrt{1-x^2}}$

Put $\sin^{-1}x = \theta$ i.e $x = \sin\theta$

$\because x \in (-1, 1)$, $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$\Rightarrow f(x) = \tan^{-1}\frac{\sin\theta}{1+\cos\theta}$

$\Rightarrow f(x) = \tan^{-1}\frac{\cancel{2}\sin\frac{\theta}{2}\cancel{\cos\frac{\theta}{2}}}{\cancel{2}\cos^{\cancel{2}}\frac{\theta}{2}}$

$= \tan^{-1}\left(\tan\frac{\theta}{2}\right)$

$= \frac{\theta}{2}$

$f(x) = \frac{1}{2}\sin^{-1}x$

Let $g(x) = \sin\left(2\tan^{-1}\sqrt{\frac{1-x}{1+x}}\right)$

Let $\theta = \cos^{-1}x$ i.e $x = \cos\theta$

$\because x \in (-1, 1)$, $\theta \in (0, \pi)$

$\Rightarrow g(x) = \sin\left(2\tan^{-1}\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}\right)$

$= \sin\left(2\tan^{-1}\sqrt{\frac{2\sin^2\theta/2}{2\cos^2\theta/2}}\right)$

$$= \sin\left(2\tan^{-1}\left|\tan\frac{\theta}{2}\right|\right)$$

$$= \sin\left(2\tan^{-1}\left(\tan\frac{\theta}{2}\right)\right)$$

$$\left[\because \frac{\theta}{2} \in \left(0, \frac{\pi}{2}\right), \tan\frac{\theta}{2} > 0\right]$$

$$= \sin\left(2\frac{\theta}{2}\right)$$

$$= \sin\theta$$

$$g(x) = \sqrt{1-x^2} \quad \left[\because \sin\theta > 0 \text{ for } \theta \in (0, \pi)\right]$$

$$\Rightarrow y = f(x) + g(x)$$

$$\Rightarrow y = \frac{1}{2}\sin^{-1}x + \sqrt{1-x^2}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{2} \cdot \frac{1}{\sqrt{1-x^2}} + \frac{1}{\cancel{2}\sqrt{1-x^2}} \cdot -\cancel{2}x$$

$$\frac{dy}{dx} = \frac{1-2x}{2\sqrt{1-x^2}}$$

7. If $f: R \to R$ is a function such that $f(x) = x^3 + x^2 f'(1) + x f''(2) + f'''(3)$ for all $x \in R$, then prove that $f(2) = f(1) - f(0)$.

Sol:- $f(x) = x^3 + x^2 f'(1) + x f''(2) + f'''(3)$ — (1)
$\forall x \in R$

$\Rightarrow f'(x) = 3x^2 + 2f'(1)x + f''(2) \ \forall x \in R$ — (2)

$\Rightarrow f''(x) = 6x + 2f'(1) \ \forall x \in R$ — (3)

$\Rightarrow f'''(x) = 6 \ \forall x \in R$ — (4)

$\Rightarrow f'''(3) = 6$

Put $x = 2$ in (3) $\Rightarrow f''(2) = 12 + 2f'(1)$

Put $x = 1$ in (2) $\Rightarrow f'(1) = 3 + 2f'(1) + f''(2)$

$\Rightarrow f'(1) = 3 + 2f'(1) + 12 + 2f'(1)$

$\Rightarrow 3f'(1) = -15$

$\Rightarrow f'(1) = -5$

$\Rightarrow f''(2) = 12 + 2f'(1) = 12 + 2(-5)$

$f''(2) = 2$

$\Rightarrow f(x) = x^3 - 5x^2 + 2x + 6$

$$f(2) = 8 - 20 + 4 + 6 = -2$$

$$f(1) = 1 - 5 + 2 + 6 = 4$$

$$f(0) = 6$$

$$\Rightarrow \quad f(2) = f(1) - f(0)$$

(8) let $g(x)$ be a polynomial, of degree one & $f(x)$ be defined by $f(x) = \begin{cases} g(x), & x \le 0 \\ \left(\frac{1+x}{2+x}\right)^{1/x}, & x > 0 \end{cases}$

Find the continous function $f(x)$ satisfying

$$f'(1) = f(-1)$$

Soln:

$$g(x) = ax + b$$

$$f(x) = \begin{cases} ax+b & x \le 0 \\ \left(\frac{1+x}{2+x}\right)^{1/x} & x > 0 \end{cases}$$

$$f(0^+) = f(0) = f(0)$$

$$\left(\frac{1}{2}\right)^{\infty} = b = b$$

$$\boxed{b = 0}$$

in vicinity of '1' $\quad f(x) = \left(\frac{1+x}{2+x}\right)^{1/x}$ say y

$$y = \left(\frac{1+x}{2+x}\right)^{1/x}$$

$$\ln y = \frac{1}{x}\left(\ln(1+x) - \ln(2+x)\right)$$

$$\frac{1}{y}\, y' = \frac{\frac{1}{1+x}\cdot\frac{1}{x} + \frac{\ln(1+x)}{x^2}}{x^2} - \frac{\frac{1}{(2+x)}\frac{1}{x} + \frac{\ln(2+x)}{x^2}}{x^2}$$

at 1

$$y = \left(\frac{1+1}{2+1}\right)^{1} = \frac{2}{3}$$

$$y' = \frac{2}{3}\left(\frac{\frac{1}{2}\times\frac{1}{1} + \frac{\ln 2}{1}}{1} - \frac{\frac{1}{3}\cdot\frac{1}{1} + \ln 3}{1}\right)$$

$$y' = \frac{2}{3}\left(\frac{1}{6} - \ln\left(\frac{2}{3}\right)\right)$$

$$f(-1) = -a + b \qquad (b=0)$$

$$= -a$$

$$y'(1) = f(-1) \Rightarrow a = -\frac{2}{3}\left(\frac{1}{6} + \ln \frac{3}{2}\right)$$

$$f(x) = \begin{cases} -\frac{2}{3}\left(\frac{1}{6} + \ln\frac{3}{2}\right)x, & x \le 0 \\ \left(\frac{1+x}{2+x}\right)^{1/x}, & x > 0 \end{cases}$$

9

(a) let $f(x) = x^2 - 4x - 3$, $x > 2$ and let g be the inverse of f. find the value of g' where $f(x) = 2$.

(b) let $f: R \to R$ be defined as $f(x) = x^3 + 3x^2 + 6x - 5 + 4e^{2x}$

and $g(x) = f^{-1}(x)$, then find $g'(-1)$

(c) Suppose f^{-1} is the inverse function of diff. function f and let $G(x) = \dfrac{1}{f^{-1}(x)}$

If $f(3) = 2$ & $f'(3) = \frac{1}{9}$ find $G'(2)$

Solⁿ:

(a)

$$g(f(x)) = x$$

$$g'(f(x)) \cdot f'(x) = 1$$

$$g'(f(x)) = \frac{1}{f'(x)}$$

$$g'(2) = \frac{1}{2x-4}$$

$$= \frac{1}{2\times5-4} = \frac{1}{6}$$

$f(x) = 2$

$x^2 - 4x - 3 = 2$

$x^2 - 4x - 5 = 0$

$(x-5)(x+1) = 0$

$x = 5$, -1 (x) $(x > 2)$

(b) $f(x) = x^3 + 3x^2 + 6x - 5 + 4e^{2x}$

$$g'(f(x)) = g'(-1) = \frac{1}{f'(x)\big|_{x=0}}$$

$$\Rightarrow \frac{1}{3x^2 + 6x + 6 + 8e^{2x}}$$

$$\Rightarrow \frac{1}{0 + 0 + 6 + 8} = \frac{1}{14}$$

(C) $G(x) = \dfrac{1}{f^{-1}(x)}$

$$G'(x) = \frac{-1}{(f^{-1}(x))^2} (f^{-1}(x))'$$

$$= \frac{-1}{(f^{-1}(2))^2} 9$$

$$= \frac{-1}{3^2} \times 9$$

$$= -1$$

$$(f^{-1}(x))' = \frac{1}{f'(x)}$$

$$= \frac{1}{f'(3)}$$

$$= \frac{1}{1/9}$$

$$= 9$$

$f(3) = 2$

$f^{-1}(2) = 3$

1. $\lim_{x\to 0}\left[\frac{1}{x\sin^{-1}x} - \frac{1-x^2}{x^2}\right]$

Soln.:

$$\lim_{x\to 0}\frac{x-(1-x^2)\sin^{-1}x}{x^2\sin^{-1}x}$$

$$\Rightarrow \lim_{x\to 0}\frac{x-\sin^{-1}x+x^2\sin^{-1}x}{x^2\left(\frac{\sin^{-1}x}{x}\right)\times x}$$

$$\Rightarrow \lim_{x\to 0}\frac{x-\sin^{-1}x+x^2\sin^{-1}x}{x^3}$$

Applying

$$\Rightarrow \lim_{x\to 0}\frac{1-\frac{1}{\sqrt{1-x^2}}+\frac{x^2}{\sqrt{1-x^2}}+2x\sin^{-1}x}{3x^2}$$

$$\Rightarrow \lim_{x\to 0}\frac{1-\frac{(1-x^2)}{\sqrt{1-x^2}}+2x\sin^{-1}x}{3x^2}$$

$$\Rightarrow \lim_{x\to 0}\frac{1-\sqrt{1-x^2}+2x\sin^{-1}x}{3x^2}$$

$$\Rightarrow \lim_{x\to 0}\frac{-x+\frac{1}{2}\times\frac{1}{\sqrt{1-x^2}}\times(-2x)+2\sin^{-1}x+\frac{2x}{\sqrt{1-x^2}}}{6x}$$

$$\Rightarrow \lim_{x\to 0}\frac{\frac{3x}{\sqrt{1-x^2}}+2\sin^{-1}x}{6x}$$

$$\Rightarrow \lim_{x\to 0}\frac{\frac{3}{\sqrt{1-x^2}}+\frac{3x}{(1-x^2)^{3/2}}\times(-\frac{1}{2})\times(-2x)+\frac{2}{\sqrt{1-x^2}}}{6}$$

$$\Rightarrow \frac{5}{6}$$

2. $\lim_{x\to 0} \dfrac{x+\ln(\sqrt{x^2+1}-x)}{x^3}$

$\Rightarrow \lim_{x\to 0} \dfrac{1+\dfrac{1}{(\sqrt{x^2+1}-x)} \times \left(\dfrac{2x}{2\sqrt{x^2+1}}-1\right]}{3x^2}$

$\Rightarrow \lim_{x\to 0} \dfrac{1+\dfrac{-1}{\sqrt{x^2+1}}}{3x^2}$

$\Rightarrow \lim_{x\to 0} \dfrac{-1 \times \dfrac{+1}{\cancel{2} \times (x^2+1)^{3/2}} \times \cancel{2}x}{6x}$

$\Rightarrow \lim_{x\to 0} \dfrac{1}{6(x^2+1)^{3/2}}$

$\Rightarrow \dfrac{1}{6}$

3. $\lim_{x \to 0} \left(\frac{1}{x^2} - \frac{1}{\sin^2 x} \right)$

$\Rightarrow \lim_{x \to 0} \frac{\sin^2 x - x^2}{x^2 \left(\frac{\sin^2 x}{x^2}\right) \times x^2}$

$\Rightarrow \lim_{x \to 0} \frac{\sin^2 x - x^2}{x^4}$

Applying L' Hospital's Rule

$\Rightarrow \lim_{x \to 0} \frac{2 \sin x \cos x - 2x}{4x^3}$

$\Rightarrow \lim_{x \to 0} \frac{\sin 2x - 2x}{4x^3}$

$\Rightarrow \lim_{x \to 0} \frac{2 \cos 2x - 2}{12 x^2}$

$\Rightarrow \lim_{x \to 0} \frac{-4 \sin 2x}{24 x}$

$\Rightarrow \lim_{x \to 0} \frac{-8 \cos 2x}{24}$

$\Rightarrow \frac{-8}{24}$

$\Rightarrow -\frac{1}{3}$

4. $\lim_{x \to 0} \dfrac{1+\sin x - \cos x + \ln(1-x)}{x \tan^2 x}$

Sol^n

$\lim_{x \to 0} \dfrac{1+\sin x - \cos x + \ln(1-x)}{x \times \left(\dfrac{\tan^2 x}{x^2}\right) \times x^2}$

$\Rightarrow \lim_{x \to 0} \dfrac{1+\sin x - \cos x + \ln(1-x)}{x^3}$

$\Rightarrow \lim_{x \to 0} \dfrac{\cos x + \sin x + \dfrac{-1}{(1-x)}}{3x^2}$

$\Rightarrow \lim_{x \to 0} \dfrac{-\sin x + \cos x - \dfrac{1}{(1-x)^2}}{6x}$

$\Rightarrow \lim_{x \to 0} \dfrac{-\cos x - \sin x - \dfrac{2}{(1-x)^3}}{6}$

$\Rightarrow \dfrac{-1-0-2}{6}$

$\Rightarrow -\dfrac{3}{6} = -\dfrac{1}{2}$

5. $\lim_{x\to 0} \dfrac{x^{6000} - (\sin x)^{6000}}{x^2 (\sin x)^{6000}}$

Solⁿ. $\lim_{x\to 0} \dfrac{x^{6000} - (\sin x)^{6000}}{x^2 \left(\dfrac{(\sin x)^{6000}}{x^{6000}}\right) \times x^{6000}}$

$\Rightarrow \lim_{x\to 0} \dfrac{x^{6000} - (\sin x)^{6000}}{x^{6002}}$

$\Rightarrow \lim_{x\to 0} \dfrac{1 - \left(\dfrac{\sin x}{x}\right)^{6000}}{x^2}$

$\Rightarrow$ Applying L' Hospital Rule

$\Rightarrow \lim_{x\to 0} \dfrac{0 - 6000 \times \left(\dfrac{\sin x}{x}\right)^{5999} \times \dfrac{x\cos x - \sin x}{x^2}}{2x}$

$\Rightarrow \lim_{x\to 0} -3000 \times \dfrac{x\cos x - \sin x}{x^3}$

$\Rightarrow \lim_{x\to 0} -3000 \times \dfrac{[\cancel{\cos x} - x\sin x - \cancel{\cos x}]}{3x^2}$

$\Rightarrow \lim_{x\to 0} \dfrac{3000}{3} \times \dfrac{\cancel{x}\sin x}{x^{\cancel{2}}}$

$\Rightarrow$ 1000

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(6) Given at real valued function $f(x)$ as follows.

$f(x) = \frac{x^2+2\cos x-2}{x^4}$ for $x<0$; $f(0) = \frac{1}{12}$ & $f(x) = \frac{\sin x - \ln(e^x \cos x)}{6x^2}$ for $x>0$. Test the continuity & differentiability of $f(x)$ at $x=0$

<u>Soln</u>:- For continuity:

L.H.L. $\Rightarrow \lim_{x\to 0^-} \frac{x^2+2\cos x-2}{x^4}$

$$\Rightarrow \lim_{x\to 0^-} \frac{2x - 2\sin x}{4x^3}$$

$$\Rightarrow \lim_{x\to 0^-} \frac{2 - 2\cos x}{12x^2}$$

$$\Rightarrow \lim_{x\to 0^-} \frac{+2\sin x}{24x}$$

$$\Rightarrow \lim_{x\to 0^-} \frac{2\cos x}{24}$$

$$\Rightarrow \frac{2}{24} = \frac{1}{12}$$

<u>R.H.L.</u> $\Rightarrow \lim_{x\to 0^+} \frac{\sin x - \ln(e^x \cos x)}{6x^2}$

$$\Rightarrow \lim_{x\to 0^+} \frac{\cos x - \frac{1}{e^x \cos x} \times (e^x \cos x + e^x(-\sin x))}{12x}$$

$$\Rightarrow \lim_{x\to 0^+} \frac{\cos x - \frac{(\cos x - \sin x)}{\cos x}}{12x}$$

$$\Rightarrow \lim_{x\to 0^+} \frac{\cos x - 1 + \tan x}{12x}$$

$$\Rightarrow \frac{-\sin x - 0 + \sec^2 x}{12}$$

$$\Rightarrow \frac{1}{12}$$

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$\because$ R.H.L = L.H.L = $f(0) = \frac{1}{12}$ So $f(x)$ is Continuous

For Differentiability :-

L.H.D. :- $\lim_{x \to 0^-} \dfrac{\dfrac{x^2+2\cos x-2}{x^4} - \dfrac{1}{12}}{x}$

$$\Rightarrow \lim_{h \to 0} \dfrac{\dfrac{(-h)^2 + 2\cos(-h)-2}{h^4} - \dfrac{1}{12}}{-h}$$

$$\Rightarrow \lim_{h \to 0} \frac{12(h^2+2\cos h-2) - h^4}{-12h^5}$$

$\Rightarrow$ Applying L'Hospital Rule :-

$$\Rightarrow \lim_{h \to 0} \frac{12[2h - 2\sin h] - 4h^3}{-60h^4}$$

$$\Rightarrow \lim_{h \to 0} \frac{12[2-2\cos h] - 12h^2}{-60 \times 4 \times h^3}$$

$$\Rightarrow \lim_{h \to 0} \frac{0 + 2\sin h - 2h}{-60h^2}$$

$$\Rightarrow \lim_{h \to 0} \frac{2\cos h - 2}{-120h}$$

$$\Rightarrow \lim_{h \to 0} \frac{-2\sin h}{-120} = 0$$

For R.H.D. $\lim_{h \to 0} \dfrac{\dfrac{\sinh - \ln(e^h \cosh)}{6h^2} - \dfrac{1}{12}}{h}$

$$\Rightarrow \lim_{h \to 0} \frac{2(\sinh - \ln(e^h \cosh)) - h^2}{12h^3}$$

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Applying L'Hospital Rule:-

$$\lim_{h\to 0} \frac{2\left(\cosh - \frac{1}{e^h \cosh}\left[e^h \cosh - e^h \sinh\right]\right) - 2h}{36h^2}$$

$$\Rightarrow \lim_{h\to 0} \frac{2\left(\cosh - 1 + \tanh\right] - 2h}{36h^2}$$

$$\Rightarrow \lim_{h\to 0} \frac{2\left[-\sinh - 0 + \sec^2 h\right] - 2}{\cancel{7}2h}$$

$$\Rightarrow \lim_{h\to 0} \frac{2\left[-\cosh + 2\sec h \cdot \sec h \tanh\right] - 0}{72}$$

$$= \frac{-2}{72} = -\frac{1}{36}$$

R.H.D $\neq$ L.H.D. So Not Differentiable

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(7) Coloumn I contains function Defined on R and Coloumn II contains their properties.

(A) $\lim_{n\to\infty}\left[\dfrac{1+\tan\frac{\pi}{2n}}{1+\sin\frac{\pi}{3n}}\right]^{n}$

$\Rightarrow e^{\lim_{n\to\infty}\left[\frac{1+\tan\frac{\pi}{2n}}{1+\sin\frac{\pi}{3n}}-1\right]\times n}$

$\Rightarrow e^{\lim_{n\to\infty}\left[\frac{\tan\frac{\pi}{2n}-\sin\frac{\pi}{3n}}{1+\sin\frac{\pi}{3n}}\right]\times n}$

let $n=\frac{1}{t}$

$\Rightarrow e^{\lim_{t\to0}\frac{\left[\tan\frac{\pi}{2}t-\sin\frac{\pi}{3}t\right]}{\left[1+\sin\frac{\pi}{3}t\right]t}}$

Applying L'Hospital Rule:-

$\Rightarrow e^{\lim_{t\to0}\frac{\left[\sec^2\frac{\pi}{2}t\cdot\frac{\pi}{2}-\cos\frac{\pi}{3}t\cdot\frac{\pi}{3}\right]}{\left(1+\sin\frac{\pi}{3}t\right)+t\cos\frac{\pi}{3}t\cdot\pi/3}}$

$\Rightarrow e^{\lim_{t\to0}\frac{\pi}{2}-\frac{\pi}{3}}$

$\Rightarrow e^{\pi/6}$

(B) $\lim\limits_{x\to 0^+} \dfrac{1}{(1+\operatorname{cosec}x)^{\frac{1}{\ln\sin x}}}$

Sol^n. $l = \lim\limits_{x\to 0^+} (1+\operatorname{cosec}x)^{-\frac{1}{\ln\sin x}}$

$$\ln l = \lim_{x\to 0^+} -\frac{\ln(1+\operatorname{cosec}x)}{\ln\sin x}$$

Applying L'Hospital Rule

$$\Rightarrow \ln l = \lim_{x\to 0^+} \frac{+\frac{1}{(1+\operatorname{cosec}x)} \times \cot x \operatorname{cosec}x}{\frac{1}{\sin x}\times\cos x}$$

$$= \lim_{x\to 0^+} \frac{\operatorname{cosec}x}{1+\operatorname{cosec}x}$$

$$\Rightarrow \lim_{x\to 0^+} \frac{1}{\frac{1}{\operatorname{cosec}x}+1}$$

$\ln l \Rightarrow 1$

$\boxed{l = e}$

c. $l = \lim_{x \to 0} \left(\frac{2}{\pi} \cos^{-1} x \right)^{1/x}$

$$l = e^{\lim_{x \to 0} \frac{\frac{2}{\pi}\cos^{-1}x - 1}{x}}$$

$$= e^{\lim_{x \to 0} \frac{\frac{-2}{\pi\sqrt{1-x^2}}}{1}}$$

$$= e^{\lim_{x \to 0} \frac{-2}{\pi\sqrt{1-x^2}}}$$

$$\boxed{l = e^{-2/\pi}}$$

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EXERCISE (JM)

1. Let $f : (-1, 1) \to R$ be a differentiable function with $f(0) = -1$ and $f'(0) = 1$. Let $g(x) = [f(2f(x) + 2)]^2$. Then $g'(0)$:- **[AIEEE-2010]**

(1) 4 (2) –4 (3) 0 (4) –2

Sol :· $\because \; g(x) = \left[f(2f(x)+2)\right]^2$

$$\Rightarrow g'(x) = 2\left[f(2f(x)+2)\right] \cdot f'(2f(x)+2) \cdot \left[2f'(x)\right]$$

Put $x = 0$

$$\Rightarrow g'(0) = 2\left[f(2f(0)+2)\right] f'(2f(0)+2)\left[2f'(0)\right]$$

$$= 2\left[f(-2+2)\right] f'(-2+2)\left[2\times 1\right]$$

$$= 2f(0) \cdot f'(0) \cdot 2$$

$$= 2(-1) \cdot 1 \cdot 2$$

$$= \boxed{-4}$$

Ans.

2. $\dfrac{d^2x}{dy^2}$ equals :- **[AIEEE-2011]**

(1) $\left(\dfrac{d^2y}{dx^2}\right)\left(\dfrac{dy}{dx}\right)^{-2}$ (2) $-\left(\dfrac{d^2y}{dx^2}\right)\left(\dfrac{dy}{dx}\right)^{-3}$ (3) $\left(\dfrac{d^2y}{dx^2}\right)^{-1}$ (4) $-\left(\dfrac{d^2y}{dx^2}\right)^{-1}\left(\dfrac{dy}{dx}\right)^{-3}$

Sol:- $\because \quad \dfrac{dx}{dy} = \dfrac{1}{\left(\dfrac{dy}{dx}\right)}$

$$\Rightarrow \frac{d}{dy}\left(\frac{dx}{dy}\right) = \frac{d}{dy}\left(\frac{1}{\left(\frac{dy}{dx}\right)}\right)$$

$$\Rightarrow \frac{d^2x}{dy^2} = \frac{d}{dx}\left(\frac{1}{\left(\frac{dy}{dx}\right)}\right)\cdot\frac{dx}{dy}$$

$$\Rightarrow \frac{d^2x}{dy^2} = \left[-\frac{1}{\left(\frac{dy}{dx}\right)^2}\cdot\frac{d^2y}{dx^2}\right]\cdot\frac{dx}{dy}$$

$$\Rightarrow \frac{d^2x}{dy^2} = \left[-\frac{\left(\frac{d^2y}{dx^2}\right)}{\left(\frac{dy}{dx}\right)^2}\right]\cdot\frac{1}{\left(\frac{dy}{dx}\right)}$$

$$\Rightarrow \boxed{\frac{d^2x}{dy^2} = -\left(\frac{d^2y}{dx^2}\right)\cdot\left(\frac{dy}{dx}\right)^{-3}}$$

Ans.

3. If $y = \sec(\tan^{-1}x)$, then $\frac{dy}{dx}$ at $x = 1$ is equal to : **[JEE(Main)-2013]**

(1) $\frac{1}{\sqrt{2}}$ (2) $\frac{1}{2}$ (3) 1 (4) $\sqrt{2}$

Sol :- $y' = \sec(\tan^{-1}(x)) \cdot \tan(\tan^{-1}x) \cdot \frac{1}{1+x^2}$

Put $\boxed{x=1}$

$\Rightarrow y'(1) = \sec\left(\frac{\pi}{4}\right) \cdot \tan\left(\frac{\pi}{4}\right) \cdot \frac{1}{2}$

$= \sqrt{2} \cdot 1 \cdot \frac{1}{2}$

$= \boxed{\frac{1}{\sqrt{2}}}$ Ans.

4. If g is the inverse of a function f and $f'(x) = \frac{1}{1+x^5}$, then g'(x) is equal to :

[JEE(Main)-2014]

(1) $1 + x^5$ (2) $5x^4$ (3) $\frac{1}{1+\{g(x)\}^5}$ (4) $1+\{g(x)\}^5$

Sol: $\because g'(f(x)) = \frac{1}{f'(x)}$

$\Rightarrow g'(f(x)) = 1 + x^5$

$\Rightarrow g'(t) = 1 + (g(t))^5$

Put $\left\{ \begin{array}{l} t = f(x) \\ f^{-1}(t) = x \\ g(t) = x \end{array} \right\}$

$\therefore \boxed{g'(x) = 1 + (g(x))^5}$ Ans.

5. If for $x \in \left(0, \frac{1}{4}\right)$, the derivative of $\tan^{-1}\left(\frac{6x\sqrt{x}}{1-9x^3}\right)$ is $\sqrt{x} \cdot g(x)$, then g(x) equals :-

[JEE(Main)-2017]

(1) $\frac{3}{1+9x^3}$ (2) $\frac{9}{1+9x^3}$ (3) $\frac{3x\sqrt{x}}{1-9x^3}$ (4) $\frac{3x}{1-9x^3}$

Sol :- $y = \tan^{-1}\left(\frac{2 \cdot 3x^{3/2}}{1-(3x^{3/2})^2}\right)$

$\Rightarrow \boxed{y = 2\tan^{-1}(3x^{3/2})}$

$\Rightarrow \frac{dy}{dx} = \frac{\cancel{2}}{1+9x^3} \cdot 3 \cdot \frac{3}{\cancel{2}} \cdot \sqrt{x}$

$\Rightarrow \sqrt{x} \cdot g(x) = \frac{9\sqrt{x}}{1+9x^3} \Rightarrow \boxed{g(x) = \frac{9}{1+9x^3}}$ 2.

6. If $x = 3\tan t$ and $y = 3\sec t$, then the value of $\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{4}$, is: **[JEE(Main)-2019]**

(1) $\frac{3}{2\sqrt{2}}$ (2) $\frac{1}{3\sqrt{2}}$ (3) $\frac{1}{6}$ (4) $\frac{1}{6\sqrt{2}}$

Sol. $$\frac{dx}{dt} = 3\sec^2 t$$

$$\frac{dy}{dt} = 3\sec t \tan t$$

$$\frac{dy}{dx} = \frac{\tan t}{\sec t} = \sin t$$

$$\frac{d^2y}{dx^2} = \cos t \frac{dt}{dx}$$

$$= \frac{\cos t}{3\sec^2 t} = \frac{\cos^3 t}{3} = \frac{1}{3.2\sqrt{2}} = \boxed{\frac{1}{6\sqrt{2}}}$$

7. Let $f(x) = \begin{cases} -1, & -2 \le x < 0 \\ x^2 - 1, & 0 \le x \le 2 \end{cases}$ and $g(x) = |f(x)| + f(|x|)$. Then, in the interval $(-2, 2)$, g is :-

[JEE(Main)-2019]

(1) differentiable at all points
(2) not differentiable at two points
(3) not continuous
(4) not differentiable at one point

Sol. $$|f(x)| = \begin{cases} 1 & , & -2 \le x < 0 \\ 1 - x^2 & , & 0 \le x < 1 \\ x^2 - 1 & , & 1 \le x \le 2 \end{cases}$$

and $f(|x|) = x^2 - 1,\ x \in [-2, 2]$

Hence $$g(x) = \begin{cases} x^2 & , & x \in [-2, 0) \\ 0 & , & x \in [0, 1) \\ 2(x^2 - 1) & , & x \in [1, 2] \end{cases}$$

$\boxed{\text{It is not differentiable at } x = 1}$ Ans.

8. For $x > 1$, if $(2x)^{2y} = 4e^{2x-2y}$, then $\left(1+\log_e 2x\right)^2 \frac{dy}{dx}$ is equal to : **[JEE(Main)-2019]**

(1) $\log_e 2x$

(2) $\frac{x\log_e 2x + \log_e 2}{x}$

(3) $x\log_e 2x$

(4) $\frac{x\log_e 2x - \log_e 2}{x}$

Sol. $(2x)^{2y} = 4e^{2x-2y}$

$2y\ell n2x = \ell n4 + 2x - 2y$

$$y = \frac{x+\ell n2}{1+\ell n2x}$$

$$y' = \frac{(1+\ell n2x)-(x+\ell n2)\frac{1}{x}}{(1+\ell n2x)^2}$$

$$\boxed{y'(1+\ell n2x)^2 = \left[\frac{x\ell n2x-\ell n2}{x}\right]}$$

Ans

9. If $f(1) = 1$, $f'(1) = 3$, then the derivative of $f(f(f(x))) + (f(x))^2$ at $x = 1$ is : **[JEE(Main)-2019]**

(1) 12 (2) 33 (3) 9 (4) 15

Sol. $y = f(f(f(x))) + (f(x))^2$

$$\frac{dy}{dx} = f'(f(f(x)))f'(f(x))f'(x) + 2f(x)f'(x)$$

$$= f'(1)f'(1)f'(1) + 2f(1)f'(1)$$

$$= 3 \times 5 \times 3 + 2 \times 1 \times 3$$

$$= 27 + 6$$

$$= \boxed{33}$$ Ans.

10. If $e^y + xy = e$, the ordered pair $\left(\frac{dy}{dx}, \frac{d^2y}{dx^2}\right)$ at x = 0 is equal to : **[JEE(Main)-2019]**

(1) $\left(-\frac{1}{e}, \frac{1}{e^2}\right)$ (2) $\left(\frac{1}{e}, \frac{1}{e^2}\right)$ (3) $\left(\frac{1}{e}, -\frac{1}{e^2}\right)$ (4) $\left(-\frac{1}{e}, -\frac{1}{e^2}\right)$

Sol. $e^y + xy = e$

differentiate w.r.t. x

$$e^y\frac{dy}{dx} + x\frac{dy}{dx} + y = 0 \quad \text{--- (1)}$$

$$\frac{dy}{dx}(x + e^y) = -y \,, \boxed{\left.\frac{dy}{dx}\right|_{(0,\,1)} = -\frac{1}{e}}$$ Ans.

again differentiate w.r.t. x in (1)

$$e^y \cdot \frac{d^2y}{dx^2} + \frac{dy}{dx} \cdot e^y \cdot \frac{dy}{dx} + x \cdot \frac{d^2y}{dx^2} + \frac{dy}{dx} + \frac{dy}{dx} = 0$$

$$(x + e^y)\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 \cdot e^y + 2\frac{dy}{dx} = 0$$

$$e\frac{d^2y}{dx^2} + \frac{1}{e^2}e + 2\left(-\frac{1}{e}\right) = 0$$

$$\therefore \boxed{\frac{d^2y}{dx^2} = \frac{1}{e^2}}$$ Ans.

11. The derivative of $\tan^{-1}\left(\frac{\sin x-\cos x}{\sin x+\cos x}\right)$, with respect to $\frac{x}{2}$, where $\left(x\in\left(0,\frac{\pi}{2}\right)\right)$ is : **[JEE(Main)-2019]**

(1) $\frac{1}{2}$ (2) $\frac{2}{3}$ (3) 1 (4) 2

Sol. $f(x)=\tan^{-1}\left(\frac{\sin x-\cos x}{\sin x+\cos x}\right)$

$$=\tan^{-1}\left(\frac{\tan x-1}{\tan x+1}\right)=\tan^{-1}\left(\tan\left(x-\frac{\pi}{4}\right)\right)$$

$$\because\ x-\frac{\pi}{4}\in\left(-\frac{\pi}{4},\frac{\pi}{4}\right)$$

$$\therefore\ f(x)=x-\frac{\pi}{4}$$

$\Rightarrow$ its derivative w.r.t. $\frac{x}{2}$ is $\frac{1}{1/2}=\boxed{2}$ Ans.

EXERCISE (JA)

(1) If the function $f(x) = x^3 + e^{\frac{x}{2}}$ and $g(x) = f^{-1}(x)$, then the value of $g'(1)$ is **[JEE 2009, 4]**

Sol:- $\because g(x) = f^{-1}(x)$

$\therefore g(f(x)) = x$

$\Rightarrow g'(f(x)) \cdot f'(x) = 1$

$\Rightarrow g'(f(x)) = \dfrac{1}{f'(x)}$

$\Rightarrow g'\left(x^3 + e^{\frac{x}{2}}\right) = \dfrac{1}{3x^2 + \frac{1}{2} \cdot e^{x/2}}$

Put $x = 0$

$\Rightarrow \boxed{g'(1) = 2}$ Ans.

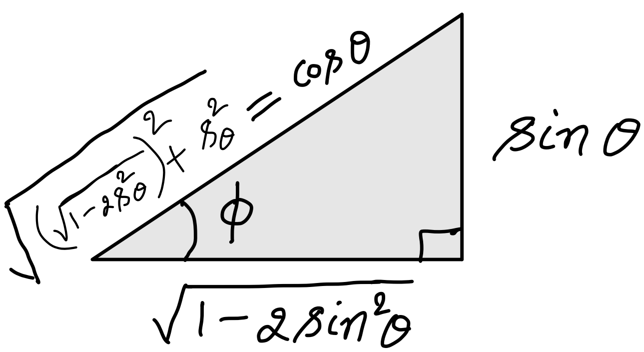
② Let $f(\theta) = \sin\left(\tan^{-1}\left(\dfrac{\sin\theta}{\sqrt{\cos 2\theta}}\right)\right)$, where $-\dfrac{\pi}{4} < \theta < \dfrac{\pi}{4}$. Then the value of $\dfrac{d}{d(\tan\theta)}(f(\theta))$ is

[JEE 2011, 4]

Sol:- $\because f(\theta) = \sin\left(\tan^{-1}\left(\dfrac{\sin\theta}{\sqrt{\cos 2\theta}}\right)\right)$

let $\tan^{-1}\left(\dfrac{\sin\theta}{\sqrt{\cos 2\theta}}\right) = \phi$

$$\Rightarrow \frac{\sin\theta}{\sqrt{1-2\sin^2\theta}} = \tan\phi$$



$$\Rightarrow f(\theta) = \sin(\phi) = \frac{\sin\theta}{\cos\theta} \quad \{\text{from } \Delta\}$$

$$\Rightarrow \boxed{f(\theta) = \tan\theta}$$

find $\Rightarrow \dfrac{d}{d(\tan\theta)} f(\theta) = \dfrac{d(\tan\theta)}{d(\tan\theta)} = \boxed{1}$

Ans.

(3) The slope of the tangent to the curve $(y - x^5)^2 = x(1 + x^2)^2$ at the point (1,3) is 8

[JEE(Advanced)-2014, 3]

Sol:- $\because (y - x^5)^2 = x(1+x^2)^2$

Differenciate w.r.t. (x)

$$\Rightarrow 2(y-x^5)\cdot[y' - 5x^4] = 1\cdot(1+x^2)^2 + x\cdot 2(1+x^2)\cdot[2x]$$

Now, Put $x = 1$, $y = 3$

$$\Rightarrow 2(3-1)\cdot[y'-5] = 1\cdot(1+1)^2 + 1\cdot 2(1+1)\cdot[2]$$

$$\Rightarrow 4\cdot(y'-5) = 4 + 8$$

$$\Rightarrow y' - 5 = 3$$

$$\Rightarrow y' = 8$$

$$\Rightarrow \boxed{\left.\frac{dy}{dx}\right)_{(1,3)} = 8}$$

Ans.

(4) Let $f:\mathbb{R}\to\mathbb{R}$, $g:\mathbb{R}\to\mathbb{R}$ and $h:\mathbb{R}\to\mathbb{R}$ be differentiable functions such that $f(x)=x^3+3x+2$, $g(f(x))=x$ and $h(g(g(x)))=x$ for all $x\in\mathbb{R}$. Then- **[JEE(Advanced)-2016, 4(–2)]**

(A) $g'(2)=\frac{1}{15}$ (B) $h'(1)=666$ ✓ (C) $h(0)=16$ ✓ (D) $h(g(3))=36$

Sol:- Given

$\because\ g(f(x))=x$

$\Rightarrow g'(f(x)).f'(x)=1$

$\Rightarrow g'(f(x))=\frac{1}{f'(x)}$

$\Rightarrow g'(x^3+3x+2)=\frac{1}{3x^2+3}$

Put $x=0$

$\Rightarrow \boxed{g'(2)=\frac{1}{3}}$ Option A

$*$ $h(g(g(x)))=x$; given

$x \xrightarrow{\text{replace}} f(x)$

$\Rightarrow h(g(g(f(x))))=f(x)$; given $g\,f(x)=x$

$\Rightarrow \boxed{h(g(x))=f(x)} \xrightarrow[x=3]{\text{Put}} h(g(3))=f(3)=3^3+3\times3+2$

$\Rightarrow \boxed{h(g(3))=38}$ option D

$x \xrightarrow{\text{replace}} f(x)$

$\Rightarrow h(\underbrace{g(f(x))}_{\hookrightarrow x})=f(f(x))$

$\Rightarrow \boxed{h(x)=f(f(x))} \xrightarrow[x=0]{\text{Put}} h(0)=f(\underbrace{f(0)}_{\hookrightarrow 2})=f(2)=16$

$\Rightarrow \boxed{h(0)=16}$ option C

$\Rightarrow h'(x)=f'(f(x)).f'(x)$

$\Rightarrow h'(x)=f'(x^3+3x+2).(3x^2+3)$

Put $x=1$

$\Rightarrow h'(1)=f'(6).\ 6$

$\Rightarrow h'(1)=111\times6 \Rightarrow \boxed{h'(1)=666}$ Option B

$\because\ f(x)=x^3+3x+2$

$\Rightarrow f'(x)=3x^2+3$

$\Rightarrow f'(6)=111$