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Do your self ①

① If (α, β) are roots of the equation $6x^2 + 11x + 3 = 0$ then

(A) Both $\cos^{-1}\alpha$ & $\cos^{-1}\beta$ are real

(B) Both $\operatorname{cosec}^{-1}\alpha$ & $\operatorname{cosec}^{-1}\beta$ are real

(C) Both $\cot^{-1}\alpha$ & $\cot^{-1}\beta$ are real

(D) ~~Both~~ N.O.D.

$$6x^2 + 11x + 3 = 0$$
$$6x^2 + 9x + 2x + 3 = 0$$
$$3x(2x+3) + 1(2x+3) = 0$$
$$(2x+3)(3x+1) = 0$$

let
$x = -3/2 = \alpha$
$x = -1/3 = \beta$

(A) ✗ $\cos^{-1}(-3/2)$ Not possible bcz of domain is $[-1,1]$

(B) ✗ $\operatorname{cosec}^{-1}(-1/3)$ Not possible bcz of $x \in R - (-1,1)$

(C) ✓ $\cot^{-1}\alpha$, $\cot^{-1}\beta$

$\cot^{-1}(-3/2)$, $\cot^{-1}(-1/3)$ $\in R$

$$\boxed{C}$$

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(2) if $\sin^{-1}x + \sin^{-1}y = \pi$ and $x = ky$ then find the value of $39^{2k} + 5^{k}$

put $x = \sin\frac{\pi}{2}$ & $y = \sin\frac{\pi}{2}$

$\therefore \sin^{-1}\sin\frac{\pi}{2} + \sin^{-1}\sin\frac{\pi}{2} = \pi$ $\therefore$ Satisfy

and $\sin\frac{\pi}{2} = k\sin\frac{\pi}{2}$

$\therefore k = 1$

$\therefore 39^{2(1)} + 5^{(1)} \Rightarrow \boxed{1526}$

M-II

$\sin^{-1}\left[x\sqrt{-y^2+1} + y\sqrt{-x^2+1}\right] = \pi$

$x\sqrt{-y^2+1} + y\sqrt{-x^2+1} = \sin\pi$

$x\sqrt{-y^2+1} + y\sqrt{-x^2+1} = 0$

$x\sqrt{-y^2+1} = -y\sqrt{-x^2+1}$

$x^2(1-y^2) = y^2(1-x^2)$

$x^2 - \cancel{x^2y^2} = y^2 - \cancel{x^2y^2}$

$x^2 = y^2$

$\because x = ky$ $\therefore k^2\cancel{y^2} = \cancel{y^2} \Rightarrow k = \pm 1$

+ve $39^{(2)} + 5^{(1)} \Rightarrow \boxed{1526}$

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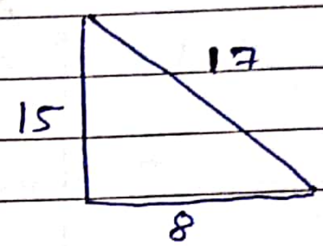
Do your self ②

Evaluate the following :-

(i) $\tan\left[\cos^{-1}\left(\frac{8}{17}\right)\right]$ —— ①



$\cos^{-1}\left(\frac{8}{17}\right) = \theta$ $\therefore \cos\theta = \frac{8}{17}$

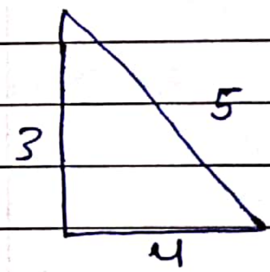
~~$\tan[\tan^{-1} \therefore \tan\theta$~~

from ① $\tan\theta$

$\therefore \boxed{\frac{15}{8}}$

(ii) $\sin\left[\frac{1}{2}\cos^{-1}\left(\frac{4}{5}\right)\right]$ —— ①

$\cos^{-1}\left(\frac{4}{5}\right) = \theta \Rightarrow \cos\theta = \frac{4}{5}$



$\because \cos 2\theta = 1 - 2\sin^2\theta$

$\therefore \cos\theta = 1 - 2\sin^2\theta/2$

$\therefore \sin\frac{\theta}{2} = \frac{1}{\sqrt{10}}$

in $0 < \theta < \pi/2$

$\sin\theta/2 > 0$

from ① $\sin\frac{\theta}{2} = \boxed{\frac{1}{\sqrt{10}}}$

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(iii) $\sin\left[\frac{\pi}{3} - \sin^{-1}\left(-\frac{1}{2}\right)\right]$ —— (1) in $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\sin^{-1}\sin\theta = \theta$

∴ $\sin\left[\frac{\pi}{3} - (-\pi/6)\right]$

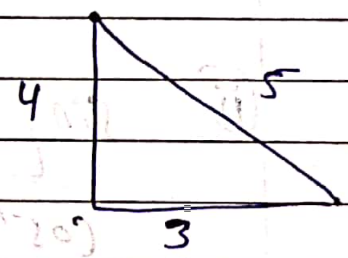
$\sin\left[\frac{\pi}{3} + \frac{\pi}{6}\right]$

$\sin\left(\frac{\cancel{3}\pi}{\cancel{6}_2}\right) \Rightarrow \boxed{1}$

(iv) $\sin\left[\cos^{-1}\frac{3}{5}\right]$

$\cos^{-1}\frac{3}{5} = \theta$

$\cos\theta = \frac{3}{5}$



$\sin\theta$

$= \boxed{\frac{4}{5}}$

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Do your self (3)

Evaluating the following

(i) $\cos^{-1}\left(\cos \frac{13\pi}{6}\right)$ in $\theta \in [0,\pi]$

$\cos^{-1}\left[\cos\left(2\pi+\frac{\pi}{6}\right)\right]$ $\cos^{-1}\cos\theta=\theta$

$\cos^{-1}\left[\cos\left(\frac{\pi}{6}\right)\right]$

$\boxed{\frac{\pi}{6}}$

(ii) $\tan^{-1}\left[\tan\left(\frac{7\pi}{6}\right)\right]$ in $\theta \in (-\pi/2, \pi/2)$

$\tan^{-1}\left[\tan\left(\pi+\frac{\pi}{6}\right)\right]$ $\tan^{-1}\tan\theta=\theta$

$\tan^{-1}\tan\left(\frac{\pi}{6}\right)$

$\boxed{\frac{\pi}{6}}$

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iii $\sin^{-1}\left[\sin\left(\frac{5\pi}{6}\right)\right]$

$\theta \in [-\pi/2, \pi/2]$

$\sin^{-1}\left[\sin\left(\pi - \frac{\pi}{6}\right)\right]$ $\sin^{-1}\sin\theta = \theta$

$\sin^{-1}\sin\left(\frac{\pi}{6}\right)$

$\boxed{\left(\frac{\pi}{6}\right)}$

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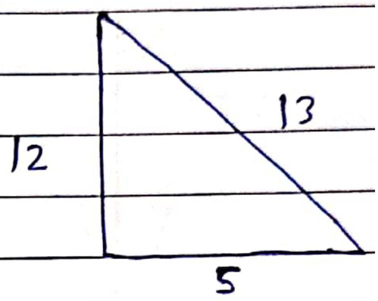
Do your self (4)

(1) prove the following :-

(A) $\cos^{-1}\left(\frac{5}{13}\right) = \tan^{-1}\left(\frac{12}{5}\right)$

L.H.S. $\cos^{-1}\left(\frac{5}{13}\right) = \theta$



$\cos\theta = \frac{5}{13}$

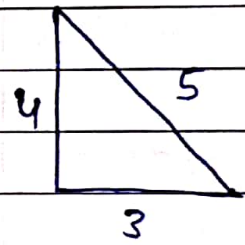
$\therefore \tan^{-1}\left(\frac{12}{5}\right)$ R.H.S.

(B) $\sin^{-1}\left(\frac{-4}{5}\right) = \tan^{-1}\left(\frac{-4}{3}\right) = \cos^{-1}\left(\frac{-3}{5}\right) \times (-\pi)$

$\sin^{-1}\left(\frac{-4}{5}\right) = -\sin^{-1}\left(\frac{4}{5}\right) = \theta$

I

Ist part



$-\tan^{-1}\left(\frac{4}{3}\right)$

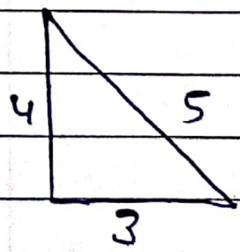
$\tan^{-1}\left(-4/3\right)$ IInd Part

IIIrd part

$\cos^{-1}\left(-\frac{3}{5}\right)(-\pi)$

$\cancel{\pi} - \cos^{-1}(3/5) - \cancel{\pi}$

$-\cos^{-1}(3/5) = \theta$



$-\tan^{-1}(4/3)$

$\tan^{-1}(-4/3)$ IInd part

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(ii) Find the value of $\sin\left[\tan^{-1}a + \tan^{-1}\frac{1}{a}\right]$; $a \neq 0$

$\sin\left[\tan^{-1}a + \tan^{-1}\frac{1}{a}\right]$ when $x > 0$

$\sin\left[\tan^{-1}a + \cot^{-1}a\right]$

$\because \tan^{-1}\theta + \cot^{-1}\theta = \pi/2$

$\sin\left(\frac{\pi}{2}\right) \Rightarrow \boxed{1}$

when $x < 0$

$\sin\left[\tan^{-1}a - \pi + \cot^{-1}a\right]$

$\sin\left[\frac{\pi}{2} - \pi\right]$

$\sin\left(-\frac{\pi}{2}\right) \Rightarrow \boxed{-1}$

$\therefore \begin{cases} 1, & \text{when } a > 0 \\ -1, & \text{when } a < 0 \end{cases}$

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Do your self (5)

Prove the following :-

(i) $\sin^{-1}\left(\frac{3}{5}\right) + \sin^{-1}\left(\frac{8}{17}\right) = \cos^{-1}\left(\frac{36}{85}\right)$

L.H.S. $\sin^{-1}\left(\frac{3}{5}\right) + \sin^{-1}\left(\frac{8}{17}\right)$

$\because \sin^{-1}x + \sin^{-1}y = \sin^{-1}\left[x\sqrt{1-y^2} + y\sqrt{1-x^2}\right]$

$$= \sin^{-1}\left[\frac{3}{5}\sqrt{1-\left(\frac{8}{17}\right)^2} + \frac{8}{17}\sqrt{1-\left(3/5\right)^2}\right]$$

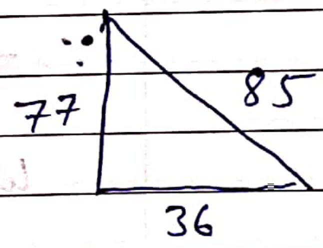
$$= \sin^{-1}\left[\frac{3}{5}\sqrt{\frac{225}{289}} + \frac{8}{17}\sqrt{\frac{16}{25}}\right]$$

$$= \sin^{-1}\left[\frac{3}{5}\times\frac{15}{17} + \frac{8}{17}\times\frac{4}{5}\right]$$

$$= \sin^{-1}\left[\frac{32+45}{85}\right]$$

$$= \sin^{-1}\left(\frac{77}{85}\right)$$

$\because$



$\therefore \sin^{-1}\left(\frac{3}{5}\right) + \sin^{-1}\left(\frac{8}{17}\right) = \cos^{-1}\left(\frac{36}{85}\right)$.

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(ii) $\tan^{-1}\left(\frac{3}{4}\right) + \tan^{-1}\left(\frac{3}{5}\right) - \tan^{-1}\left(\frac{8}{19}\right) = \frac{\pi}{4}$

$\because \tan^{-1}(a) + \tan^{-1}(b) = \tan^{-1}\left[\frac{a+b}{1-ab}\right]$; $ab<1$

where $a = \frac{3}{4}$, $b = \frac{3}{5}$ $\quad \frac{3}{4} \times \frac{3}{5}$

$\therefore \quad = \tan^{-1}\left[\frac{\frac{3}{4} + \frac{3}{5}}{1 - \frac{3}{4} \times \frac{3}{5}}\right]$ $\quad \frac{15}{20} < 1$

$= \tan^{-1}\left[\frac{15+12}{20} \times \frac{20}{20-9}\right]$

$\tan^{-1}\left[\frac{27}{\cancel{20}} \times \frac{\cancel{20}}{11}\right]$

$\therefore \tan^{-1}\left(\frac{27}{11}\right) - \tan^{-1}\left(\frac{8}{19}\right)$

$\because \tan^{-1}(a) - \tan^{-1}(b) = \tan^{-1}\left[\frac{a-b}{1+ab}\right]$; $ab > -1$

where $a = \frac{27}{11}$ & $b = \frac{8}{19}$ $\Rightarrow \frac{27}{11} \times \frac{8}{19} > -1$

$\therefore \tan^{-1}\left[\frac{\frac{27}{11} - \frac{8}{19}}{1 + \frac{27}{11} \times \frac{8}{19}}\right]$

$\tan^{-1}\left(\frac{425}{425}\right)$

$\tan^{-1}(1) \Rightarrow \boxed{\frac{\pi}{4}}$

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(iii) $\tan^{-1}\left(\frac{2}{11}\right) + \tan^{-1}\left(\frac{7}{24}\right) = \tan^{-1}\left(\frac{1}{2}\right)$

$\because \tan^{-1}(a) + \tan^{-1}(b) = \tan^{-1}\left[\frac{a+b}{1-ab}\right]$

L.H.S. $\therefore \tan^{-1}\left[\frac{\frac{2}{11}+\frac{7}{24}}{1-\frac{2}{11}\times\frac{7}{24}}\right]$

$\tan^{-1}\left[\frac{\frac{48+77}{(24)(11)}}{1-\frac{14}{(11)(24)}}\right]$

$\tan^{-1}\left[\frac{125}{\cancel{(11)(24)}} \times \frac{\cancel{(11)(24)}}{250}\right]$

$\tan^{-1}\left(\frac{1}{2}\right)$. R.H.S.

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Do your self (6)

Prove the following Results :-

(1) $2\tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{8}\right) = \tan^{-1}\left(\frac{4}{7}\right)$

$\because \; 2\tan^{-1}x = \tan^{-1}\frac{2x}{1-x^2}$

$\therefore$ L.H.S. $2\tan^{-1}\left(\frac{1}{5}\right) + \tan^{-1}\left(\frac{1}{8}\right)$

$\tan^{-1}\left[\frac{\frac{2}{5}}{1-\left(\frac{1}{5}\right)^2}\right] + \tan^{-1}\left(\frac{1}{8}\right)$

$\tan^{-1}\left[\frac{2}{5} \times \frac{25}{24}\right] \tan^{-1}\left(\frac{1}{8}\right)$

$\tan^{-1}\left(\frac{5}{12}\right) + \tan^{-1}\left(\frac{1}{8}\right)$

$\tan^{-1}\left[\frac{\frac{5}{12}+\frac{1}{8}}{1-\frac{5}{12}\times\frac{1}{8}}\right]$

$\tan^{-1}\left[\frac{40+12}{\cancel{(12)(8)}} \times \frac{\cancel{(12)(8)}}{96-5}\right]$

$\tan^{-1}\left(\frac{52}{91}\right) \Rightarrow \tan^{-1}\left(\frac{4}{7}\right)$ R.H.S.

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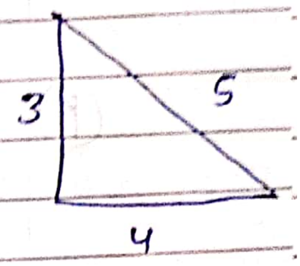
(ii) $2\sin^{-1}\left(\frac{3}{5}\right) - \tan^{-1}\left(\frac{17}{31}\right) = \frac{\pi}{4}$

$\sin^{-1}\left(\frac{3}{5}\right) = \theta \Rightarrow \sin\theta = \frac{3}{5}$



$\therefore 2\tan^{-1}\left(\frac{3}{4}\right) - \tan^{-1}\left(\frac{17}{31}\right) = \frac{\pi}{4}$

$\because 2\tan^{-1} x = \tan^{-1} \frac{2x}{1-x^2}$

$\therefore$ L.H.S. $\tan^{-1}\left[\frac{\cancel{2} \times \frac{3}{\cancel{4}\,2}}{1-\left(\frac{3}{4}\right)^2}\right] - \tan^{-1}\left(\frac{17}{31}\right)$

$\tan^{-1}\left[\frac{3}{\cancel{2}} \times \frac{\cancel{16}\,8}{7}\right] - \tan^{-1}\left(\frac{17}{31}\right)$

$\tan^{-1}\left(\frac{24}{7}\right) - \tan^{-1}\left(\frac{17}{31}\right)$

$\tan^{-1}\left[\frac{\frac{24}{7} - \frac{17}{31}}{1 + \frac{24}{7} \times \frac{17}{31}}\right]$

$\tan^{-1}\left[\frac{(744-119)}{\cancel{(31)(7)}} \times \frac{\cancel{(31)(7)}}{217+408}\right]$

$\tan^{-1}\left[\frac{\cancel{625}}{\cancel{625}}\right]$

$\tan^{-1}(1)$

$\tan^{-1}(\tan \pi/4)$

$\therefore \boxed{\frac{\pi}{4}}$ R.H.S.

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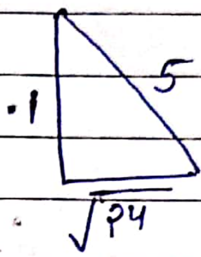
Do your self (7)

Solve the following equation for x :-

(i) $\sin\left[\sin^{-1}\left(\frac{1}{5}\right)+\cos^{-1}x\right]=1$

$\sin^{-1}\left(\frac{1}{5}\right)=\theta \Rightarrow \frac{1}{5}=\sin\theta$

$\therefore \frac{\sqrt{24}}{5}=\cos\theta$



$\cos^{-1}x=\phi \Rightarrow x=\cos\phi$

$\sqrt{1-x^2}=\sin\phi$

$\therefore \sin[\theta+\phi]=1$

$\sin\theta\cos\phi+\cos\theta\sin\phi=1$

$\frac{1}{5}\cdot x+\frac{\sqrt{24}}{5}\cdot\sqrt{1-x^2}=1$

$\left[\frac{\sqrt{24}\cdot\sqrt{1-x^2}}{5}\right]^2=\left(1-\frac{x}{5}\right)^2$

$\frac{(24)(1-x^2)}{25}=1+\frac{x^2}{25}-\frac{2}{5}x$

$\frac{24}{25}-\frac{24}{25}x^2=1+\frac{x^2}{25}-\frac{2}{5}x$

$x^2-\frac{2}{5}x+\frac{1}{25}=0$

$\left(x-\frac{1}{5}\right)^2=0$

$$\boxed{x=\frac{1}{5}}$$

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(ii)

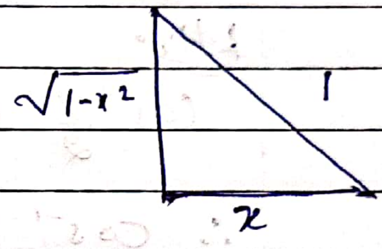
$$\cos^{-1}x + \sin^{-1}\frac{x}{2} = \frac{\pi}{6}$$

$$\sin^{-1}\frac{x}{2} = \frac{\pi}{6} - \cos^{-1}x$$

$$\sin\left(\sin^{-1}\frac{x}{2}\right) = \sin\left[\frac{\pi}{6} - \cos^{-1}x\right]$$

$$\frac{x}{2} = \sin\frac{\pi}{6}\cos(\cos^{-1}x) - \cos\frac{\pi}{6}\sin(\cos^{-1}x)$$

$$\cancel{\frac{x}{2}} = \cancel{\frac{1}{2}x} - \frac{\sqrt{3}}{2}\sqrt{1-x^2}$$



$$0 = -\frac{\sqrt{3}}{2}\sqrt{1-x^2}$$

$$\sqrt{1-x^2} = 0$$

$$x^2 = 1 \Rightarrow x = \pm 1$$

+1 satisfy the given equation

$$\therefore \boxed{x = 1}$$

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Do your self (8)

(1) Solve the inequality $\tan^{-1}x > \cot^{-1}x$

$$\tan^{-1}x > \frac{\pi}{2} - \tan^{-1}x$$

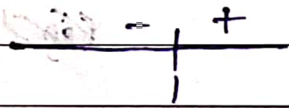
$$2\tan^{-1}x > \frac{\pi}{2}$$

$$\tan^{-1}x > \frac{\pi}{4}$$

$$\tan^{-1}x > \frac{\pi}{4}$$

$$x > \tan\left(\frac{\pi}{4}\right)$$

$$x > 1$$



$$\boxed{x \in (1, \infty)}$$

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(ii) Complete solution set of inequation

$(\cos^{-1}x)^2 - (\sin^{-1}x)^2 > 0$ is = ?

$(\cos^{-1}x)^2 - (\sin^{-1}x)^2 > 0$

$(\cos^{-1}x + \sin^{-1}x) \cdot (\cos^{-1}x - \sin^{-1}x) > 0$

$\downarrow$

$\frac{\pi}{2} \cdot (\cos^{-1}x - \sin^{-1}x) > 0$

$\cos^{-1}x - \sin^{-1}x > 0$

$\cos^{-1}x > \sin^{-1}x$

$\frac{\pi}{2} - \sin^{-1}x > \sin^{-1}x$

$\frac{\pi}{2} > 2\sin^{-1}x \Rightarrow \sin^{-1}x < \frac{\pi}{4}$

$x < \sin\frac{\pi}{4} \Rightarrow x < \frac{1}{\sqrt{2}}$

$- \quad + $

$1/\sqrt{2}$

But $\sin\theta \in [-1,1]$

$\therefore \boxed{x \in [-1, 1/\sqrt{2})}$

(B)

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Do your self (9)

Evaluate :- $\sum_{r=1}^{\infty} \tan^{-1}\left[\frac{2}{1+(2r+1)(2r-1)}\right]$

$$\sum_{r=1}^{\infty} \tan^{-1}\left[\frac{(2r+1)-(2r-1)}{1+(2r+1)(2r-1)}\right]$$

$$\sum_{r=1}^{\infty} \cancel{\tan^{-1}}\left[\tan^{-1}(2r+1) - \tan^{-1}(2r-1)\right]$$

at $r=1$ $\quad \cancel{\tan^{-1}(3)} - \tan^{-1}(1)$

$r=2$ $\quad \cancel{\tan^{-1}(5)} - \cancel{\tan^{-1}(3)}$

$r=3$ $\quad \cancel{\tan^{-1}(7)} - \cancel{\tan^{-1}(5)}$

$\vdots \qquad \vdots \qquad \vdots$

$r=n$ $\quad \tan^{-1}(2n+1) - \cancel{\tan^{-1}(2n-1)}$

$\therefore \quad \tan^{-1}(2n+1) - \tan^{-1}(1) \qquad n\to\infty$

$\therefore \quad \frac{\pi}{2} - \frac{\pi}{4}$

$$\boxed{\frac{\pi}{4}}$$

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EXERCISE (O-1)

1. The domain of the function $\sin^{-1}\left(\log_2\left(\frac{x}{3}\right)\right)$ is-

(A) $\left[\frac{1}{2},3\right]$ (B) $\left[\frac{1}{2},3\right]$ (C) $\left[\frac{3}{2},6\right]$ (D) $\left[\frac{1}{2},2\right]$

Solution. $-1 \le \log_2\left(x/3\right) \le 1 \Rightarrow 3/2 \le x \le 6$ —— ①

Also, $x/3 > 0 \Rightarrow x > 0$ —— ②

from ① & ② : $x \in \left[3/2, 6\right]$ —— C

2. Domain of the function $f(x) = \log_e \cos^{-1}\{\sqrt{x}\}$ is, where $\{.\}$ represents fractional part function -

(A) $x \in \mathbb{R}$ (B) $x \in [0, \infty)$ (C) $x \in (0, \infty)$ (D) $x \in \mathbb{R} - \{x \mid x \in I\}$

Solution.

$\cos^{-1}\{\sqrt{x}\} > 0$ and $-1 \leq \{\sqrt{x}\} \leq 1$ and $x \geq 0$

It is true It is true

$x \in [0, \infty)$ — (B)

3. The value of $\tan^2(\sec^{-1}3) + \cot^2(\text{cosec}^{-1}4)$ is -

(A) 9 (B) 16 (C) 25 (D) 23

Solution.

Let $\sec^{-1}3 = \alpha$ and $\text{cosec}^{-1}4 = \beta$

$\sec\alpha = \frac{3}{1}$ and $\text{cosec}\,\beta = \frac{4}{1}$

$\therefore \tan^2\alpha + \cot^2\beta = \sec^2\alpha - 1 + \text{cosec}^2\beta - 1$

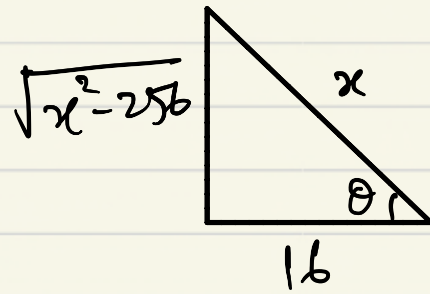
$\Rightarrow 9 - 1 + 16 - 1 \Rightarrow 23$ — (D).

64. If $x > 0$ $\cos^{-1}\left(\frac{12}{x}\right) = \frac{\pi}{2} - \cos^{-1}\left(\frac{16}{x}\right)$ then x is -

(A) 12 (B) 16 (C) 20 (D) 320

Solution.

$$\frac{12}{x} = \sin\left(\cos^{-1}\frac{16}{x}\right)$$



$$\frac{12}{x} = \sin\left(\sin^{-1}\frac{\sqrt{x^2-256}}{x}\right)$$

$$\frac{12}{x} = \frac{\sqrt{x^2-256}}{x}$$

$$144 = x^2 - 256$$

$$x^2 = 400$$

$$x = 20 \quad \{\because x > 0\} \text{ — } C$$

5. If $\cos^{-1}(2x^2 - 1) = 2\pi - 2\cos^{-1}x$, then -

(A) $x \in [-1, 0]$ (B) $x \in [0, 1]$ (C) $x \in \left[0, \frac{1}{\sqrt{2}}\right]$ (D) $x \in \left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$

sol: Put $x = \cos\theta$ ie $\theta = \cos^{-1}x$

$$\therefore \quad \cos^{-1}(2\cos^2\theta - 1) = 2\pi - 2\theta$$

$$\underbrace{\cos^{-1}(\cos 2\theta) = 2\pi - 2\theta}_{\Downarrow \text{ Possible when}}$$

$$0 \le 2\pi - 2\theta \le \pi$$

$$\Rightarrow 0 - 2\pi \le -2\theta \le \pi - 2\pi$$

$$\Rightarrow \pi \ge \theta \ge \pi/2$$

$$\Rightarrow \cos\pi \le \cos\theta \le \cos \pi/2$$

$$\Rightarrow -1 \le x \le 0$$

$$\therefore x \in [-1, 0]$$ Ans (A)

6. Number of integral ordered pairs (a,b) for which $\sin^{-1}(1+b+b^2+\ldots\infty)+\cos^{-1}\left(a-\frac{a^2}{3}+\frac{a^3}{9}-\ldots\infty\right)=\frac{\pi}{2}$ is-

(A) 0 (B) 4 (C) 9 (D) Infinitely many

Solution. Let $x = 1+b+b^2+\text{-----}\infty = \frac{1}{1-b}$

$y = a-\frac{a^2}{3}+\frac{a^3}{9}-\text{-----}\infty = \frac{a}{1+a/3}$

Also $\frac{1}{1-b} = \frac{3a}{a+3}$

$\frac{2a-3}{3a} = b$ — (1)

$|a|<3$ and $|b|<1$

Possible integral values

$\therefore\ a \in \{-2,-1,0,1,2\}$ and $b \in \{0\}$

But for $b=0$ $a = 3/2 \notin$ Integer (from (1))

$\therefore$ No Integral Pair — (A)

7. The value of $\sin^{-1}\left\{\cot\left(\sin^{-1}\sqrt{\frac{2-\sqrt{3}}{4}}+\cos^{-1}\frac{\sqrt{12}}{4}+\sec^{-1}\sqrt{2}\right)\right\}$ is-

(A) 5 (B) 6 (C) 0 (D) 10

Sol:-

$$= \sin^{-1}\left(\cot\left(\sin^{-1}\sqrt{\frac{4-2\sqrt{3}}{8}} + \cos^{-1}\left(\frac{\sqrt{3}}{2}\right) + \sec^{-1}(\sqrt{2})\right)\right)$$

$$= \sin^{-1}\left(\cot\left(\sin^{-1}\left(\sqrt{\frac{(\sqrt{3}-1)^2}{8}}\right) + \frac{\pi}{6} + \frac{\pi}{4}\right)\right)$$

$$= \sin^{-1}\left(\cot\left(\sin^{-1}\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right) + \frac{\pi}{6} + \frac{\pi}{4}\right)\right)$$

$$= \sin^{-1}\left(\cot\left(\frac{\pi}{12} + \frac{\pi}{6} + \frac{\pi}{4}\right)\right)$$

$$= \sin^{-1}\left(\cot\frac{\pi}{2}\right)$$

$$= \sin^{-1}(0)$$

$= 0$ option C Ans.

8. Consider the function $f(x) = e^x$ and $g(x) = \sin^{-1}x$, then which of the following is/are necessarily true.

(A) Domain of $g \circ f$ = Domain of f

(B) Range of $g \circ f \subset$ Range of g

(C) Domain of $g \circ f$ is $(-\infty, 0]$

(D) Range of $g \circ f$ is $\left[-\frac{\pi}{2}, 0\right)$

Solution.

$$gof \Rightarrow g(f(x)) = \sin^{-1}(e^x)$$

(A). Domain of gof :-

$$-1 \le e^x \le 1 \Rightarrow -\infty < x \le 0$$

But Domain of f : $x \in R$

(B). Range of gof :-

$$\because e^x \in (0, 1] \ \forall \ x \in (-\infty, 0]$$

$\Rightarrow \sin^{-1}(e^x) \in (0, \pi/2]$, which is the subset of $\underbrace{\text{range of g(x)}}_{\Updownarrow}$

$g(x) \in [-\pi/2, \pi/2]$

(C). See Option (A).

(D). See Option (B). — BC

9. Let $f(x) = \sin^{-1}(\tan x) + \cos^{-1}(\cot x)$ then

(A) $f(x) = \frac{\pi}{2}$ wherever defined

(B) domain of $f(x)$ is $x = n\pi \pm \frac{\pi}{4}$, $n \in I$

(C) period of $f(x)$ is $\frac{\pi}{2}$

(D) $f(x)$ is many one function

Solution. $f(x)$ is defined when

$-1 \le \tan x \le 1$ and $-1 \le \cot x \le 1$

$\Downarrow$

$\Rightarrow \tan x \in (-\infty, -1] \cup [1, \infty)$

$\Downarrow$

$\therefore \tan x = -1, 1$ $\qquad \therefore f(x) = \pi/2$

$\Rightarrow x = n\pi \pm \pi/4$, $n \in I$

$\therefore$ for $n \in I$ $f(x)$ is periodic with period $= \pi/2$

Also, for diff. values of x, we have diff values of x, for which $f(x) = \pi/2$

Hence it is many one function $\longrightarrow$ ABCD

10. Let $f(x)=e^{x^3-x^2+x}$ be an invertible function such that $f^{-1}=g$, then -

(A) $g(e)=0$

(B) Domain of 'g' is $\mathbb{R}^+$

(C) Range of 'g' is $\mathbb{R}$

(D) $f(g(e))=e$

Solution. $\because g=f^{-1}$

Range of g is the domain of f & domain of g is the range of f.

$\therefore f(x)=e^{x(x^2-x+1)}$ → always >0

For $x\geq 0$; $f(x)\geq 1$ and $x<0$; $0<f(x)<1$

$\therefore f(x)\in(0,\infty)\ \forall x\in R$

$\therefore$ Domain of $g(x)$ = Range of $f(x)$ = $(0,\infty)$ = R^+

$\therefore$ Range of $g(x)$ = Domain of $f(x)$ = $\in R$

Now,

$$f^{-1}(x)=g(x)$$

$$f^{-1}(f(x))=g(f(x)) \Rightarrow g(f(x))=x \quad \text{①}$$

Now, at $x=1 \Rightarrow f(1)=e^{1-1+1}=e^1$

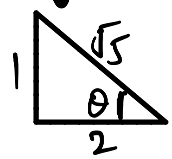
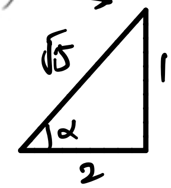
$\therefore$ Put $x=1$ in eq ①

$\Rightarrow g(f(1))=1 \Rightarrow g(e)=1$

option ⓓ: $f(g(e))=f(1)=e$

$\therefore$ Ans: BCD

11. Value of $3\tan^{-1}\left(\frac{1}{3}\right)+\tan^{-1}\left(\frac{1}{2}\right)+\sin^{-1}\left(\frac{1}{\sqrt{5}}\right)+\cos^{-1}\left(\frac{2}{\sqrt{5}}\right)$ is greater than

(A) $\frac{\pi}{2}$ (B) $\frac{2\pi}{3}$  (C) $\frac{3\pi}{4}$  (D) $\frac{5\pi}{6}$

Solution.

$$3\tan^{-1}(1/3)+\tan^{-1}(1/2)+\tan^{-1}(1/2)+\tan^{-1}\left(\frac{1}{2}\right)$$

$$3\left[\tan^{-1}(1/3)+\tan^{-1}(1/2)\right]$$

$$3\tan^{-1}\left(\frac{1/3+1/2}{1-1/3\cdot 1/2}\right)=3\tan^{-1}(1)$$

$$=\frac{3\pi}{4} \text{ — } AB$$

12. Which of the following is/are correct ?

(A) $\cot^{-1}(x) = \tan^{-1}\left(\frac{1}{x}\right) \forall\ x \in R - \{0\}$

(B) If $f : R \to R$ such that $f(x) = \mathrm{sgn}(e^x)$ then $f(x)$ is an into function.

(C) If $f : R^+ \to R$ such that $f(x) = \sin x + x$ then $f(x)$ is an odd function.

(D) If $f : R \to R$ such that $f(x) = \frac{e^x}{e^{[x]}}$ then $f(x)$ is a periodic function .

(where [.] represents greatest integer function)

Solution.

(A). $\cot^{-1} x = \begin{cases} \tan^{-1}\frac{1}{x} & ; \quad x > 0 \\ \pi + \tan^{-1}\frac{1}{x} & ; \quad x < 0 \end{cases}$

(B). $f(x) = \mathrm{Sgn}\,(e^x)$

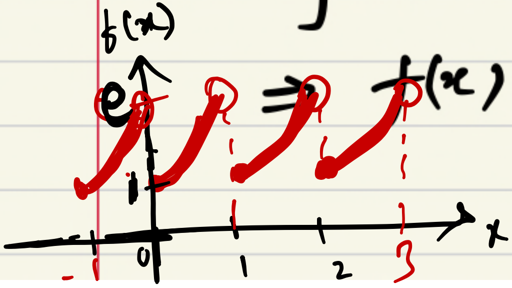
$\because\ e^x > 0\ \forall x \in R$

$\Rightarrow f(x) = 1$,hence Into.

(C). For odd function, $f(x) + f(-x) = 0$

But $x \in R^+$ (R^- is not in domain $\therefore f(-x)$ is not defined)

(D). $f(x) = e^{x-[x]} \Rightarrow e^{\{x\}}$



$\Rightarrow f(x) \in [1, e)\ \forall x \in R$, hence periodic.

($\because\ \{x\} \in [0, 1)$)

— BD

0-2

EXERCISE (O-2)

Straight Objective Type

1. The range of the function $f(x) = \sin^{-1}(\log_2(-x^2 + 2x + 3))$ is -

(A) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (B) $\left[-\frac{\pi}{2}, 0\right]$ (C) $\left[0, \frac{\pi}{2}\right]$ (D) $[-1, 1]$

Sol:-

$\because$ Range of $ax^2+bx+c=0$ when $a<0$ is $\left(-\infty, -\frac{D}{4a}\right]$

$$-\infty < -x^2+2x+3 \leq 4$$

log function is defined only when

$$0 < -x^2+2x+3 \leq 4$$

$$-\infty < \log_2(-x^2+2x+3) \leq 2$$

$\because$ $\sin^{-1} M$ is defined only when $-1 \leq M \leq 1$

So $$-1 \leq \log_2(-x^2+2x+3) \leq 1$$

$$-\frac{\pi}{2} \leq \sin^{-1}\left(\log_2(-x^2+2x+3)\right) \leq \frac{\pi}{2}$$

Range of function $f(x)$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

option (A) Ans.

2. Range of $f(x) = \cot^{-1}(\log_e(1 - x^2))$ is -

(A) $(0,\pi)$ (B) $\left(0,\frac{\pi}{2}\right)$ (C) $\left[\frac{\pi}{2},\pi\right)$ (D) $\left(0,\frac{\pi}{2}\right]$

Sol:-

$\because\ 0 \le x^2 < \infty$

$\Rightarrow -\infty < -x^2 \le 0$

$\Rightarrow -\infty < 1 - x^2 \le 1$

$\Rightarrow 0 < 1 - x^2 \le 1$

$\Rightarrow -\infty < \log_e(1-x^2) \le 0$

$$\frac{\pi}{2} \le \cot^{-1}(\log_e(1-x^2)) < \pi$$

So Range of function $f(x)$ is $\left[\frac{\pi}{2}, \pi\right)$

option C Ans.

3. $\lim_{n\to\infty}\sum_{r=1}^{n}\tan^{-1}\frac{2r+1}{r^4+2r^3+r^2+1}$ is equal to -

(A) $\frac{\pi}{4}$ (B) $\frac{3\pi}{4}$ (C) $\frac{\pi}{2}$ (D) $-\frac{\pi}{8}$

Sol^n:

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\tan^{-1}\left(\frac{2r+1}{r^2(r^2+2r+1)+1}\right)$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\tan^{-1}\left(\frac{(r^2+2r+1)-r^2}{r^2(r+1)^2+1}\right)$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\left(\tan^{-1}(r+1)^2-\tan^{-1}(r^2)\right)$$

$$= \lim_{n\to\infty}\Big[\left(\cancel{\tan^{-1}(2^2)}-\tan^{-1}(1^2)\right)+\left(\cancel{\tan^{-1}(3^2)}-\cancel{\tan^{-1}(2^2)}\right)+\left(\cancel{\tan^{-1}(4^2)}-\cancel{\tan^{-1}(3^2)}\right)+\dots+\left(\tan^{-1}((n+1)^2)-\cancel{\tan^{-1}(n^2)}\right)\Big]$$

$$= \lim_{n\to\infty}\left[\tan^{-1}((n+1)^2)-\tan^{-1}(1)\right]$$

$$= \tan^{-1}(\infty)-\pi/4 = \frac{\pi}{2}-\frac{\pi}{4} = \frac{\pi}{4}$$

Ans (A)

4. Number of solution(s) of the equation $\cos^{-1}\sqrt{x}-\sin^{-1}\sqrt{x-1}+\cos^{-1}\sqrt{1-x}-\sin^{-1}\frac{1}{\sqrt{x}}=\frac{\pi}{2}$ is -

(A) 0 (B) 1 (C) 2 (D) 4

Sol:- Domain $(x>0 \cap x-1\geq 0 \cap 1-x\geq 0)$

So Domain is only $x=1$

Then we will check the solution only at that point.

$$= \cos^{-1}\sqrt{1}-\sin^{-1}\sqrt{1-1}+\cos^{-1}\sqrt{1-1}-\sin^{-1}\frac{1}{\sqrt{1}}$$

$$= 0 - 0 + \frac{\pi}{2} - \frac{\pi}{2}$$

$$= 0$$

But this point does't satisfy the equation.

So number of solution of this equation is 0.

Option (A) Ans.

5. $\sum_{r=0}^{\infty} \tan^{-1}\left(\frac{r((r+1)!)}{(r+1)+((r+1)!)^2}\right)$ is equal to -

(A) $\frac{\pi}{2}$ (B) $\frac{\pi}{4}$ (C) $\cot^{-1}3$ (D) $\tan^{-1}2$

Sol:-

$$= \sum_{r=0}^{\infty} \tan^{-1}\left(\frac{(r+1-1).(r+1)!}{(r+1)+((r+1)!)^2}\right)$$

$$= \sum_{r=0}^{\infty} \tan^{-1}\left(\frac{(r+1).(r+1)! - 1.(r+1)!}{(r+1)+((r+1)!)^2}\right)$$

Divide N^r and D^r by $(r+1)$

$$= \sum_{r=0}^{\infty} \tan^{-1}\left(\frac{(r+1)! - r!}{1 + r!\,(r+1)!}\right)$$

$$= \sum_{r=0}^{\infty} \left(\tan^{-1}(r+1)! - \tan^{-1} r!\right)$$

Now expand this & we get

$$= \lim_{r\to\infty} \left(\tan^{-1}(r+1)! - \tan^{-1}(0!)\right)$$

$$= \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

option (B) Ans.

Multiple Correct Answer Type

6. Let $f(x)=\sqrt{\cos^{-1}\sqrt{1-x^2}-\sin^{-1}x}$ then which of the following statement/s is/are correct -

(A) Domain of $f(x)$ is $[-1,1]$

(B) Domain of $f(x)$ is $[0,1]$

(C) Range of $f(x)$ is $\{0\}$

(D) Range of $f(x)$ is $\left[0,\sqrt{\pi}\right]$

Sol:-

For $\boxed{0 \le x \le 1}$

$\cos^{-1}\sqrt{1-x^2} = \sin^{-1}x$

$f(x) = \sqrt{\sin^{-1}x - \sin^{-1}x}$

$f(x) = 0$

For $\boxed{-1 \le x \le 0}$

$\cos^{-1}\sqrt{1-x^2} = -\sin^{-1}x$

$f(x) = \sqrt{-\sin^{-1}x - \sin^{-1}x}$

$f(x) = \sqrt{-2\sin^{-1}x}$

So Domain of $f(x)$ is $[-1, 1]$

Since $-2\sin^{-1}x$ is a decreasing function

So we get maximum value at $x = -1$

$f(x) = \sqrt{-2\sin^{-1}(-1)} = \sqrt{(-2)(-\pi/2)}$

$= \sqrt{\pi}$

And minimum value of $f(x) = 0$

So Range of $f(x)$ is $[0, \sqrt{\pi}]$

Option (A) & (D) Ans

7. If $\alpha = 2\tan^{-1}(\sqrt{3-2\sqrt{2}}) + \sin^{-1}\left(\frac{1}{\sqrt{6}-\sqrt{2}}\right)$, $\beta = \cot^{-1}\left(\sqrt{3}-2\right) + \frac{1}{8}\sec^{-1}(-2)$ & $\gamma = \tan^{-1}\frac{1}{\sqrt{2}} + \cos^{-1}\frac{1}{\sqrt{3}}$, then

(A) $\alpha = \beta$ (B) $\alpha + \beta = 3\gamma$ (C) $4(\beta - \gamma) = \alpha$ (D) $\beta = \gamma$

Sol.

$$\alpha = 2\tan^{-1}\sqrt{3-2\sqrt{2}} + \sin^{-1}\left(\frac{1}{\sqrt{6}-\sqrt{2}}\right)$$

$$= 2\tan^{-1}\sqrt{(\sqrt{2}-1)^2} + \sin^{-1}\left(\frac{\sqrt{6}+\sqrt{2}}{4}\right)$$

$$= 2\tan^{-1}(\sqrt{2}-1) + \sin^{-1}\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)$$

$$= 2\times\frac{\pi}{8} + \frac{5\pi}{12} = 2\pi/3$$

$$\beta = \cot^{-1}(-(2-\sqrt{3})) + \frac{1}{8}\cdot\left(\pi - \sec^{-1}(2)\right)$$

$$= \pi - \frac{5\pi}{12} + \frac{\pi}{8} - \frac{\pi}{24} = 2\pi/3$$

$$\gamma = \tan^{-1}\left(\frac{1}{\sqrt{2}}\right) + \tan^{-1}\sqrt{2}$$

$$= \tan^{-1}\left(\frac{\frac{1}{\sqrt{2}}+\sqrt{2}}{1-\sqrt{2}\cdot\frac{1}{\sqrt{2}}}\right) = \pi/2$$

Now check the options.

8. If α is only real root of the equation $x^3 + (\cos 1)x^2 + (\sin 1)x + 1 = 0$, then $\left(\tan^{-1}\alpha + \tan^{-1}\frac{1}{\alpha}\right)$ **cannot** be equal to-

(A) 0 (B) $\frac{\pi}{2}$ (C) $-\frac{\pi}{2}$ (D) π

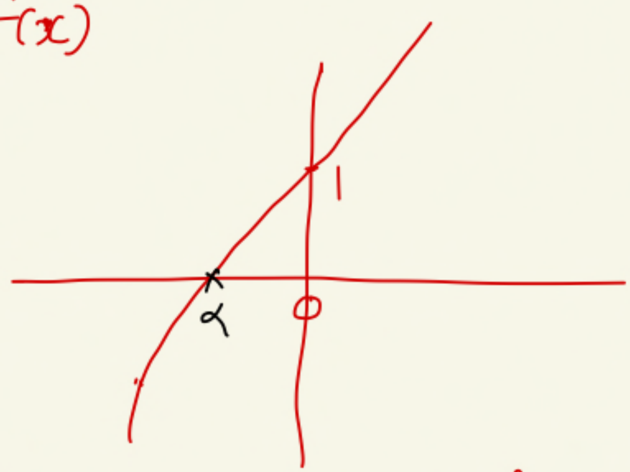
Sol:-

$f(x) = x^3 + (\cos 1)x^2 + (\sin 1)x + 1$

Graph of $f(x)$

$\because f(0) = 1$

$f(-\infty) = -\infty$



It means α is negative real root $(\alpha < 0)$

$\Rightarrow \tan^{-1}\alpha + \tan^{-1}\frac{1}{\alpha}$

$\left\{ \tan^{-1}\left(\frac{\alpha + 1/\alpha}{1 - \alpha \cdot \frac{1}{\alpha}}\right) = \tan^{-1}\left(\frac{\alpha + 1/\alpha}{0}\right) = -\frac{\pi}{2} \right.$

$\Rightarrow (-ve) + (-ve)$

$\Rightarrow (-ve)$

It means $\left(\tan^{-1}\alpha + \tan^{-1}\frac{1}{\alpha}\right)$ cannot be equal to

options (A), (B) & (D) are correct Ans.

S-1

1\. (a) Find the following :

(i) $\tan\left[\cos^{-1}\frac{1}{2}+\tan^{-1}\left(\frac{-1}{\sqrt{3}}\right)\right]$ (ii) $\cos^{-1}\left(\cos\frac{7\pi}{6}\right)$

(iii) $\cos\left(\tan^{-1}\frac{3}{4}\right)$ (iv) $\tan\left(\sin^{-1}\frac{3}{5}+\cot^{-1}\frac{3}{2}\right)$

1 (a) (1) $\tan\left(\cos^{-1}\frac{1}{2}-\tan^{-1}\frac{1}{\sqrt{3}}\right) = \tan\left(\frac{\pi}{3}-\frac{\pi}{6}\right) = \frac{1}{\sqrt{3}}$

---

(2) $\cos^{-1}\cos\frac{7\pi}{6} = 2\pi - \frac{7\pi}{6} = \frac{5\pi}{6}$

---

(3) $\cos\left(\tan^{-1}\frac{3}{4}\right)$, let $\tan^{-1}\frac{3}{4} = \theta \in [0, \pi/2)$

$\tan\theta = \frac{3}{4} \Rightarrow \cos\theta = \frac{4}{5}$

---

(4) $\tan\left(\sin^{-1}\frac{3}{5} + \cot^{-1}\frac{3}{2}\right)$

let $\sin^{-1}\frac{3}{5} = A$, $\cot^{-1}\frac{3}{2} = B$

$\sin A = \frac{3}{5}$ & $\cot B = \frac{3}{2}$

$\Rightarrow \tan A = \frac{3}{4}$ & $\tan B = \frac{2}{3}$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \tan B} = \frac{\frac{3}{4}+\frac{2}{3}}{1-\frac{2}{4}}$$

$$= \frac{17}{6}$$

(b) Find the following :

(i) $\sin\left[\dfrac{\pi}{2} - \sin^{-1}\left(\dfrac{-\sqrt{3}}{2}\right)\right]$

(ii) $\cos\left[\cos^{-1}\left(\dfrac{-\sqrt{3}}{2}\right) + \dfrac{\pi}{6}\right]$

(iii) $\tan^{-1}\left(\tan\dfrac{3\pi}{4}\right)$

(b) (i) $\text{Sin}\left(\frac{\pi}{2} + \text{Sin}^{-1}\frac{\sqrt{3}}{2}\right) = \text{Sin}\left(\frac{\pi}{2} + \frac{\pi}{3}\right) = \text{Cos}\frac{\pi}{3} = \frac{1}{2}$

(ii) $\text{Cos}\left(\text{Cos}^{-1}\left(-\frac{\sqrt{3}}{2}\right) + \frac{\pi}{6}\right) = \text{Cos}\left(\frac{5\pi}{6} + \frac{\pi}{6}\right) = -1$

(iii) $\tan^{-1}\tan\frac{3\pi}{4} = \frac{3\pi}{4} - \pi = -\pi/4$

2. Find the domain of definition the following functions.

(Read the symbols [*] and { * } as greatest integers and fractional part functions respectively)

(i) $f(x) = \arccos\dfrac{2x}{1+x}$

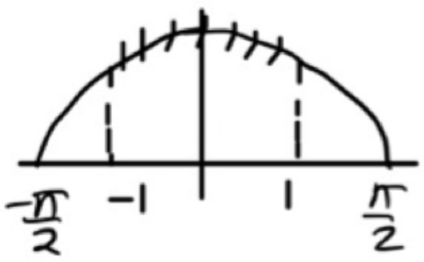
(i) $f(x) = \cos^{-1}\dfrac{2x}{1+x}$, $\dfrac{2x}{1+x} \in [-1,1]$

$\dfrac{1+x}{2x} \in (-\infty,-1] \cup [1,\infty) \Rightarrow \dfrac{1}{x}+1 \in (-\infty,-2] \cup [2,\infty)$

$\dfrac{1}{x} \in (-\infty,-3] \cup [1,\infty) \Rightarrow \boxed{x \in [-\tfrac{1}{3}, 1]}$

(ii) $f(x) = \sqrt{\cos(\sin x)} + \sin^{-1}\dfrac{1+x^2}{2x}$

sol^n (ii) $f(x) = \sqrt{\cos(\sin x)} + \sin^{-1}\left(\dfrac{1+x^2}{2x}\right)$



$\cos(\sin x) \geq 0 \Rightarrow x \in R$

$\sin^{-1}\left(\dfrac{1+x^2}{2x}\right)$: $\dfrac{1+x^2}{2x} \in [-1,1]$ $\Rightarrow x = \{-1, 1\}$

Now $y = \frac{1}{2}\left(x + \frac{1}{x}\right) \in (-\infty,-1] \cup [1,\infty)$ Both

$\dfrac{1+x^2}{2x} = 1, -1 \Rightarrow \boxed{x = 1, -1}$

(iii) $f(x) = \sin^{-1}\left(\dfrac{x-3}{2}\right) - \log_{10}(4-x)$

sol^n: $\sin^{-1}\left(\dfrac{x-3}{2}\right)$: $\dfrac{x-3}{2} \in [-1,1] \Rightarrow x \in [1,5]$ (1)

$\log_{10}(4-x)$: $4-x>0 \Rightarrow x \in (-\infty, 4)$ (2)

From (1) & (2)

$\boxed{x \in [1,4)}$

2. Find the domain of definition the following functions.

(Read the symbols [*] and { * } as greatest integers and fractional part functions respectively)

(iv) $f(x) = \sin^{-1}(2x + x^2)$

Sol^n: $f(x) = \sin^{-1}(x^2+2x)$

$x^2+2x \in [-1,1] \Rightarrow (x+1)^2-1 \in [-1,1]$

$(x+1)^2 \in [0,2] \Rightarrow (x+1)^2 \le 2$

$|x+1| \le \sqrt{2} \Rightarrow x+1 \in [-\sqrt{2}, \sqrt{2}]$

$x \in [-\sqrt{2}-1, \sqrt{2}-1]$

2. Find the domain of definition the following functions.

(Read the symbols [*] and { * } as greatest integers and fractional part functions respectively)

(v) $f(x)=\sqrt{3-x}+\cos^{-1}\left(\dfrac{3-2x}{5}\right)+\log_6(2|x|-3)+\sin^{-1}(\log_2 x)$

Sol:

$\sqrt{3-x}$: $\quad 3-x \geq 0 \Rightarrow x \leq 3 \Rightarrow x \in (-\infty, 3]$ —(1)

$\cos^{-1}\left(\dfrac{3-2x}{5}\right)$: $\quad \dfrac{3-2x}{5} \in [-1,1]$

$\dfrac{2x-3}{5} \in [-1,1] \Rightarrow 2x-3 \in [-5,5]$

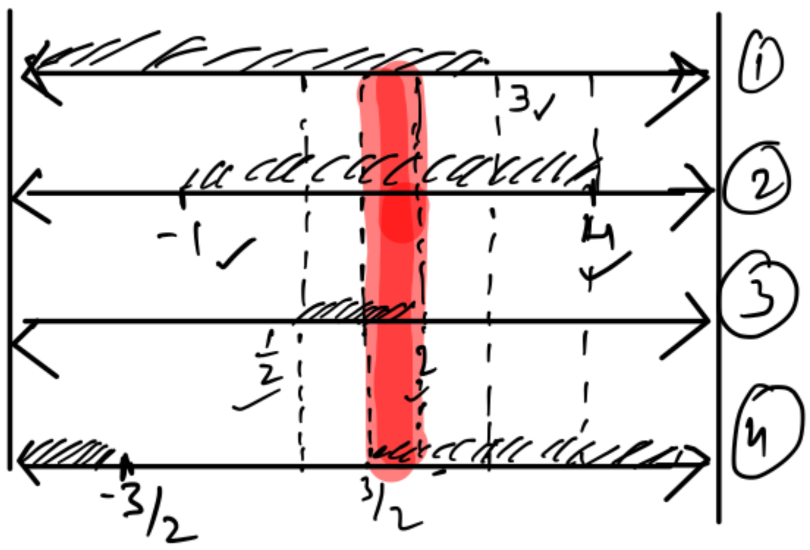
$2x \in [-2,8] \Rightarrow x \in [-1,4]$ —(2)

$\log_6(2|x|-3)$: $\quad 2|x|-3>0 \Rightarrow |x|>3/2$

$x \in \left(-\infty,-\frac{3}{2}\right) \cup \left(\frac{3}{2},\infty\right)$ —(3)

$\sin^{-1}(\log_2 x)$: $\quad \log_2 x \in [-1,1] \Rightarrow x \in \left[\frac{1}{2},2\right]$ —(4)

(1) ∩ (2) ∩ (3) ∩ (4)



$\boxed{x \in \left(\frac{3}{2},2\right]}$

3. Identify the pair(s) of functions which are identical. Also plot the graphs in each case.

(a) $y=\tan(\cos^{-1}x);\ y=\dfrac{\sqrt{1-x^2}}{x}$

(b) $y=\tan(\cot^{-1}x);\ y=\dfrac{1}{x}$

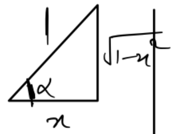
(c) $y=\sin(\arctan x);\ y=\dfrac{x}{\sqrt{1+x^2}}$

(d) $y=\cos(\arctan x);\ y=\sin(\operatorname{arccot} x)$

a) $y=\tan(\cos^{-1}x)$

$=\tan\left(\tan^{-1}\left(\dfrac{\sqrt{1-x^2}}{x}\right)\right)$

$=\dfrac{\sqrt{1-x^2}}{x}$



Domain: $x\in[-1,1]-\{0\}$

$y=\dfrac{\sqrt{1-x^2}}{x}$

$1-x^2\geq 0$ and $x\neq 0$

$x^2-1\leq 0$

$x\in[-1,1]-\{0\}$

∴ Both are identical

(b) $y=\tan(\cot^{-1}x),$

$=\tan(\tan^{-1}(1/x))$

$=1/x$

D: $x\neq 0$

$y=\dfrac{1}{x}$

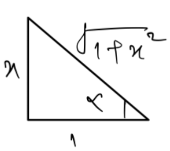
$x\neq 0$

∴ Both are identical.

(c) $y=\sin(\tan^{-1}x)$

Let $\alpha=\tan^{-1}x$

$\tan\alpha=x$



∴ $y=\sin\alpha=\dfrac{x}{\sqrt{1+x^2}}$

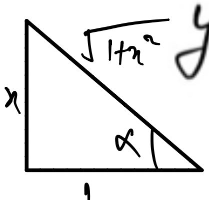
Domain: $x\in R$

$y=\dfrac{x}{\sqrt{1+x^2}}$

Domain: $x\in R$

∴ Both are identical

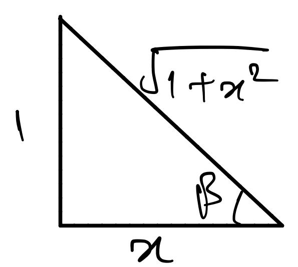
(D) $y=\cos(\tan^{-1}x)$, $y=\sin(\cot^{-1}x)$



$y=\dfrac{1}{\sqrt{x^2+1}}$

D_1: $x\in R$

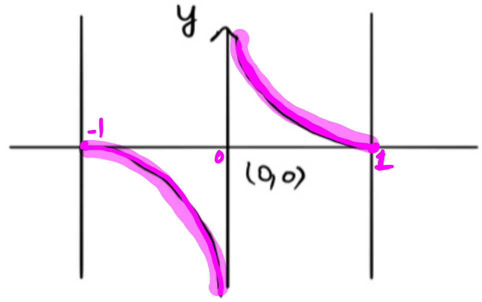
$y=\dfrac{1}{\sqrt{x^2+1}}$



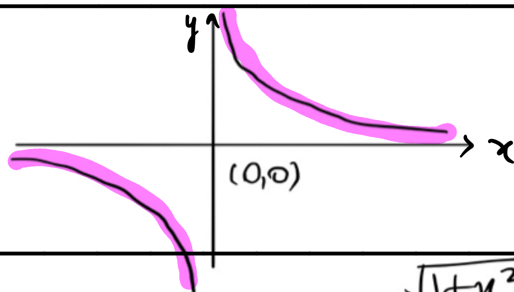
D_2: $x\in R$

Both are identical

(A) $y = \frac{\sqrt{1-x^2}}{x}$

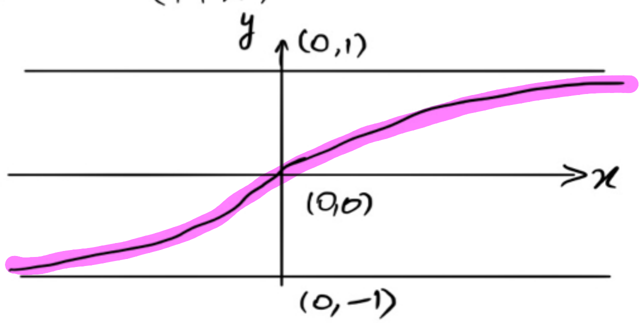


(B) $y = \frac{1}{x}$



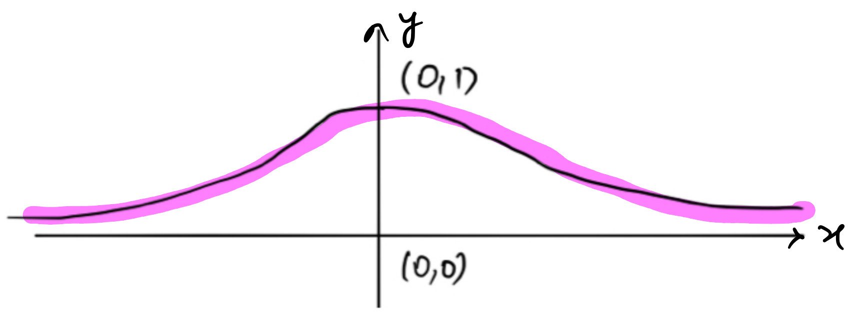
(C) $y = \frac{x}{\sqrt{1+x^2}}$, $\frac{dy}{dx} = \frac{\sqrt{1+x^2} - \frac{x^2}{\sqrt{1+x^2}}}{(1+x^2)}$

$y' = \frac{1}{(1+x^2)^{3/2}}$



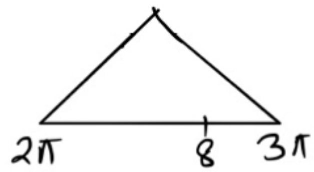
(D) $y = \frac{1}{\sqrt{x^2+1}}$

$y = \frac{-1 \times 2x}{2(x^2+1)^{3/2}} = \frac{-x}{(x^2+1)^{3/2}}$

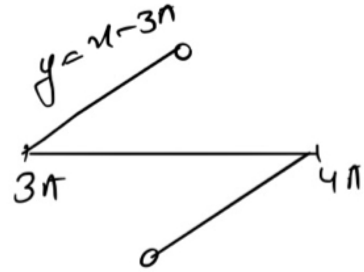


4. Let $y = \sin^{-1}(\sin 8) - \tan^{-1}(\tan 10) + \cos^{-1}(\cos 12) - \sec^{-1}(\sec 9) + \cot^{-1}(\cot 6) - \text{cosec}^{-1}(\text{cosec } 7)$. If y simplifies to $a\pi + b$, then find $(a - b)$.

$\sin^{-1}\sin 8$, $\quad 8 \in (2\pi, 3\pi)$



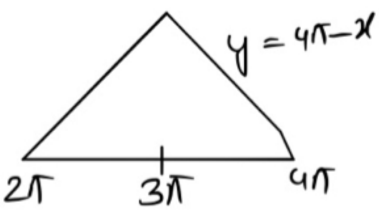
$\sin^{-1}\sin 8 = \boxed{3\pi - 8}$



$\tan^{-1}\tan 10$, $\quad 10 \in (3\pi, 4\pi)$

$\tan^{-1}\tan 10 = \boxed{10 - 3\pi}$

$\cos^{-1}\cos 12$, $\quad 12 \in (3\pi, 4\pi)$



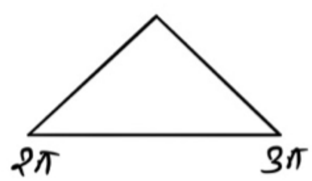
$\cos^{-1}\cos 12 = \boxed{4\pi - 12}$

$\sec^{-1}\sec 9$, $\quad 9 \in (2\pi, 3\pi)$

$= \boxed{9 - 2\pi}$

$\cot^{-1}\cot 6$, $\quad 6 \in (\pi, 2\pi)$

$= \boxed{6 - \pi}$



$\text{Cosec}^{-1}\text{Cosec } 7$, $\quad 7 \in (2\pi, 3\pi)$

$= \boxed{7 - 2\pi}$

$$y = (3\pi - 8) - (10 - 3\pi) + (4\pi - 12) - (9 - 2\pi) + (6 - \pi) - (7 - 2\pi)$$

$$y = 13\pi - 40 = a\pi + b$$

$a = 13$, $b = -40$, $\quad \boxed{a - b = 53}$

5. If α and β are the roots of the equation $x^2 + 5x - 49 = 0$, then find the value of $\cot(\cot^{-1}\alpha + \cot^{-1}\beta)$.

$\hookrightarrow \left.\begin{matrix}\alpha+\beta = -5 \\ \alpha\beta = -49\end{matrix}\right\}$

Solⁿ: Let $\cot^{-1}\alpha = t_1$ and $\cot^{-1}\beta = t_2$

$\cot t_1 = \alpha$ $\quad\quad$ $\cos t_2 = \beta$

Now $\cot(t_1+t_2) = \dfrac{\cot t_1 \cot t_2 - 1}{\cot t_1 + \cot t_2} = \dfrac{\alpha\beta - 1}{\alpha+\beta}$

$= \dfrac{-49-1}{-5} = 10$ Ans

6. If $a > b > c > 0$, then find the value of : $\cot^{-1}\left(\frac{1+ab}{a-b}\right)+\cot^{-1}\left(\frac{1+bc}{b-c}\right)+\cot^{-1}\left(\frac{1+ca}{c-a}\right)$.

$a > b > c > 0$

$$\cot^{-1}\left(\frac{1+ab}{a-b}\right)+\cot^{-1}\left(\frac{1+bc}{b-c}\right)+\pi-\cot^{-1}\left(\frac{1+ac}{a-c}\right)$$

$$\tan^{-1}\left(\frac{a-b}{1+ab}\right)+\tan^{-1}\left(\frac{b-c}{1+bc}\right)+\pi-\tan^{-1}\left(\frac{a-c}{1+ac}\right)$$

$$(\tan^{-1}a-\tan^{-1}b)+(\tan^{-1}b-\tan^{-1}c)+\pi-(\tan^{-1}a-\tan^{-1}c)$$

$$=\boxed{\pi}$$

7. Find all values of k for which there is a triangle whose angles have measure $\tan^{-1}\left(\frac{1}{2}\right), \tan^{-1}\left(\frac{1}{2}+k\right)$ and $\tan^{-1}\left(\frac{1}{2}+2k\right)$.

$A+B+C=\pi = \tan(A+B+C)=0$

$S_1 = S_3 \Rightarrow \frac{1}{2}+\left(\frac{1}{2}+K\right)+\left(\frac{1}{2}+2K\right)=\frac{1}{2}\left(\frac{1}{2}+K\right)\left(\frac{1}{2}+2k\right)$

$\frac{3}{2}+3K=\frac{1}{2}\left(2K^2+\frac{3K}{2}+\frac{1}{4}\right) \Rightarrow 8K^2+6K+1=12+24K$

$8K^2-18K-11=0 \Rightarrow \boxed{K=11/4}, -\frac{1}{2}$ (Not possible)

at $K=-1/2$

Angle $=\tan^{-1}\left(\frac{1}{2}+K\right)=\tan^{-1}(0)$
$=0$

$\therefore$ Ans: $K=11/4$

8. Find the simplest value of

(a) $f(x) = \text{arc}\cos x + \text{arc}\cos\left(\frac{x}{2} + \frac{1}{2}\sqrt{3-3x^2}\right)$, $x \in \left(\frac{1}{2}, 1\right)$

(b) $f(x) = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$, $x \in \mathbb{R} - \{0\}$

(a) $f(x) = \cos^{-1}x + \cos^{-1}\left(\frac{x}{2} + \sqrt{1-x^2} \cdot \frac{\sqrt{3}}{2}\right)$

put $\cos^{-1}x = \theta \in (0, \pi/3)$

$f(x) = \theta + \cos^{-1}\cos(\theta - \pi/3)$, $\theta - \pi/3 \in \left(\frac{-\pi}{3}, 0\right)$

$f(x) = \theta + \frac{\pi}{3} - \theta = \pi/3$

Let $x = \tan\theta$

(b) $\tan^{-1}x = \theta \in (-\pi/2, \pi/2)$ $f(x) = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$

$f(x) = \tan^{-1}\left(\frac{\sec\theta - 1}{\tan\theta}\right) = \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right)$

$f(x) = \tan^{-1}\tan\frac{\theta}{2} = \frac{\tan^{-1}x}{2}$

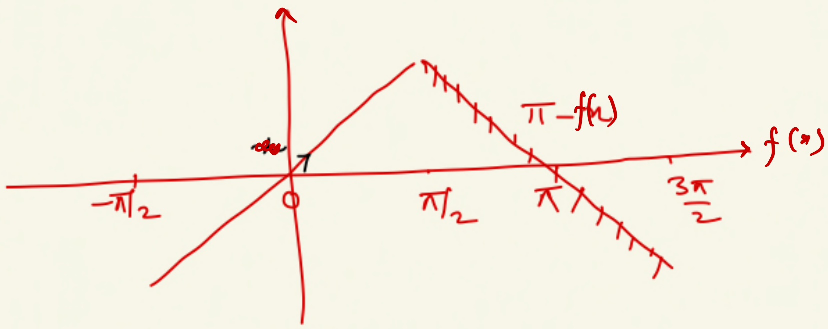
9. Least integral value of x for which inequality $\sin^{-1}\left(\sin\left(\underbrace{\frac{2e^x+3}{e^x+1}}_{f(x)}\right)\right) > \pi - \frac{5}{2}$ holds, is

Sol:

$$f(x) = \frac{2e^x+3}{e^x+1} = 2 + \frac{1}{\underbrace{e^x+1}_{>1}}$$

$$2 < f(x) < 3 \qquad \left\{ \because 0 < \frac{1}{e^x+1} < 1 \right\}$$

Since $\sin^{-1}(\sin f(x)) = \pi - f(x)$ when $f(x) \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$



So $\sin^{-1}\left(\sin\left(\frac{2e^x+3}{e^x+1}\right)\right) = \pi - \frac{2e^x+3}{e^x+1}$

$\therefore$ $\sin^{-1}\left(\sin\left(\frac{2e^x+3}{e^x+1}\right)\right) > \pi - \frac{5}{2}$

$$\pi - \frac{2e^x+3}{e^x+1} > \pi - \frac{5}{2}$$

$$\frac{2e^x+3}{e^x+1} < \frac{5}{2}$$

$$4e^x + 6 < 5e^x + 5$$

$$e^x > 1$$

$$x > 0$$

So least integral value of x is 1.

Ans

10. Number of integral solutions of the equation $2\sin^{-1}\sqrt{x^2-x+1}+\cos^{-1}\sqrt{x^2-x}=\frac{3\pi}{2}$ is

Sol^n:

$0 \le \sqrt{x^2-x+1} \le 1$ and $0 \le \sqrt{x^2-x} \le 1$ — (2)

$\Rightarrow 0 \le x^2-x+1 \le 1$

$\Rightarrow -1 \le x^2-x \le 0$ — (1)

from (1) and (2)

$x^2-x=0 \Rightarrow x(x-1)=0$

$\Rightarrow x=0, 1$

Both values of x satisfy the equation.

So number of integral solutions of equation is 2 Ans

$$= \sin^{-1}\left(-\sin\frac{\pi}{3}\right) + \cos^{-1}\left(-\cos\frac{\pi}{3}\right) + (10 - 3\pi)$$

$$= -\frac{\pi}{3} + \pi - \frac{\pi}{3} + 10 - 3\pi$$

$$= -\frac{8\pi}{3} + 10$$

$$= a\pi + b$$

So $a = -\frac{8}{3}$, $b = 10$

$$\therefore \left|\frac{3ab}{80}\right| = \left|\frac{3 \cdot (-8/3) \cdot 10}{80}\right| = 1$$

Ans.

(18)

If $\sin^{-1}\sin\left(\frac{10\pi}{3}\right)+\cos^{-1}\cos\frac{22\pi}{3}+\tan^{-1}\tan 10 = a\pi + b$, then $\left|\frac{3ab}{80}\right|$ is equal to

$$= \sin^{-1}\sin\left(3\pi+\frac{\pi}{3}\right) + \cos^{-1}\cos\left(7\pi+\frac{\pi}{3}\right) + \tan^{-1}(\tan 10)$$

$$= \sin^{-1}\left(-\sin\frac{\pi}{3}\right) + \cos^{-1}\left(-\cos\frac{\pi}{3}\right) + (10-3\pi)$$

$$= -\frac{\pi}{3} + \pi - \frac{\pi}{3} + 10 - 3\pi$$

$$= -\frac{8\pi}{3} + 10$$

$$= a\pi + b$$

So $a = -\frac{8}{3}$, $b = 10$

$$\left|\frac{3\cdot(-8/3)\cdot 10}{80}\right| = 1$$

Ans.

12. Solve the following :

(a) $\sin^{-1} x + \sin^{-1} 2x = \frac{\pi}{3}$

Sol: let $\sin^{-1} x = \alpha$ and $\sin^{-1} 2x = \beta$

$\sin \alpha = x$ $\qquad$ $\sin \beta = 2x$

$\therefore \quad \alpha + \beta = \pi/3$

$\Rightarrow \quad \alpha = \frac{\pi}{3} - \beta$

$\Rightarrow \quad \sin(\alpha) = \sin\left(\frac{\pi}{3} - \beta\right)$

$\Rightarrow \quad \sin \alpha = \sin \frac{\pi}{3} \cdot \cos \beta - \cos \frac{\pi}{3} \sin \beta$

$\Rightarrow \quad x = \frac{\sqrt{3}}{2} \sqrt{1-4x^2} - \frac{1}{2}(2x)$

$\Rightarrow \quad 2x = \sqrt{3}\sqrt{1-4x^2} - 2x$

$\Rightarrow \quad 4x = \sqrt{3}\sqrt{1-4x^2}$ .... (1)

Squaring

$\Rightarrow \quad 16x^2 = 3(1-4x^2) \Rightarrow 28x^2 = 3$

$\Rightarrow \quad x^2 = 3/28$

$\Rightarrow \quad x = \frac{1}{2}\frac{\sqrt{3}}{\sqrt{7}}$, $\underbrace{-\frac{1}{2}\frac{\sqrt{3}}{\sqrt{7}}}$

Ans

Rejected because in eqn (1) x is +ve.

12. **Solve the following :**

(b) $\tan^{-1}\frac{1}{1+2x}+\tan^{-1}\frac{1}{1+4x}=\tan^{-1}\frac{2}{x^2}$

Sol^n: $\tan^{-1}\left[\dfrac{\frac{1}{1+2x}+\frac{1}{1+4x}}{1-\left(\frac{1}{1+2x}\right)\left(\frac{1}{1+4x}\right)}\right]=\tan^{-1}\left(\frac{2}{x^2}\right)$

$$\Rightarrow \frac{(1+4x)+(1+2x)}{(1+2x)(1+4x)-1}=\frac{2}{x^2}$$

$$\Rightarrow \frac{6x+2}{8x^2+6x}=\frac{2}{x^2}$$

$$\Rightarrow (3x+1)x^2 = 8x^2+6x$$

$$\Rightarrow 3x^3+x^2 = 8x^2+6x$$

$$\Rightarrow 3x^3-7x^2-6x=0$$

$$\Rightarrow x(3x^2-7x-6)=0$$

$$\Rightarrow x(3x^2-9x+2x-6)=0$$

$$\Rightarrow x[3x(x-3)+2(x-3)]=0$$

$$\Rightarrow x(x-3)(3x+2)=0$$

$$\Rightarrow x = 0, 3, -2/3$$

$x=0$: Not in domain

$-2/3$: Not satisfying given eq^n

Ans 3

12. Solve the following :

(c) $\tan^{-1}(x-1)+\tan^{-1}(x)+\tan^{-1}(x+1)=\tan^{-1}(3x)$

Sol^n: $\tan^{-1}(x-1)+\tan^{-1}(x+1)=\tan^{-1}(3x)-\tan^{-1}(x)$

$$\Rightarrow \tan^{-1}\left(\frac{(x-1)+(x+1)}{1-(x-1)(x+1)}\right)=\tan^{-1}\left(\frac{3x-\cancel{x}}{1+(3x)(x)}\right)$$

$$\Rightarrow \frac{2x}{1-x^2+1}=\frac{2x}{1+3x^2}$$

$$\Rightarrow 2x\left(\frac{1}{2-x^2}-\frac{1}{1+3x^2}\right)=0$$

$$\Rightarrow x=0 \quad \text{or} \quad 2-x^2=1+3x^2$$

$$4x^2=1$$

$$x=\pm 1/2$$

Ans $\boxed{x\in\{-1/2,0,1/2\}}$

12. Solve the following :

(a) $\sin^{-1}x+\sin^{-1}y=\frac{2\pi}{3}$ & $\cos^{-1}x-\cos^{-1}y=\frac{\pi}{3}$

Solⁿ : $\sin^{-1}x+\sin^{-1}y=\frac{2\pi}{3}$ —— (1)

$\cos^{-1}x-\cos^{-1}y=\frac{\pi}{3}$

$\Rightarrow \left(\frac{\pi}{2}-\sin^{-1}x\right)-\left(\frac{\pi}{2}-\sin^{-1}y\right)=\pi/3$

$\Rightarrow \sin^{-1}y-\sin^{-1}x=\pi/3$ —— (2)

Add (1) & (2)

$\Rightarrow 2\sin^{-1}y=\pi \Rightarrow \sin^{-1}y=\pi/2 \Rightarrow \boxed{y=1}$

Put in (1) we get $\Rightarrow \sin^{-1}x+\frac{\pi}{2}=\frac{2\pi}{3}$

$\Rightarrow \sin^{-1}x=\frac{\pi}{6} \Rightarrow x=\sin\pi/6=1/2$

$\Rightarrow \boxed{x=1/2}$

Ans

13. Find the sum of the series :

(a) $\cot^{-1}7 + \cot^{-1}13 + \cot^{-1}21 + \cot^{-1}31 + \ldots\ldots$ to n terms.

Sol^n:

15 (a)

$$S_n = 7 + 13 + 21 + 31 + \cdots + T_n$$

$$S_n = \quad 7 + 13 + 21 + \cdots\cdots + T_n$$

$$T_n = 7 + (6 + 8 + 10 + \cdots + (n-1) \text{ terms})$$

$$T_n = 3 + (4 + 6 + 8 + \cdots + n \text{ terms})$$

$$T_n = 3 + \frac{n}{2}\left(8 + (n-1)\times 2\right)$$

$$T_n = 3 + n(4 + n - 1) = n^2 + 3n + 3$$

$$t_n = \cot^{-1}(n^2+3n+3) = \tan^{-1}\left(\frac{1}{n^2+3n+3}\right)$$

$$t_n = \tan^{-1}\left(\frac{(n+2)-(n+1)}{1+(n+1)(n+2)}\right)$$

$$t_n = \tan^{-1}(n+2) - \tan^{-1}(n+1)$$

$$S_n = \sum_{n=1}^{n} t_n = \sum_{n=1}^{n}\left[\tan^{-1}(n+2) - \tan^{-1}(n+1)\right]$$

$$S_n = \tan^{-1}(n+2) - \tan^{-1}2$$

$$S_n = \tan^{-1}\left(\frac{n}{1+2n+4}\right) = \tan^{-1}\left(\frac{n}{2n+5}\right)$$

13. Find the sum of the series :

(b) $\tan^{-1}\frac{1}{3} + \tan^{-1}\frac{2}{9} + \ldots\ldots + \tan^{-1}\frac{2^{n-1}}{1+2^{2n-1}} + \ldots\ldots \infty$

sol^n:

$$T_n = \tan^{-1}\left(\frac{2^{n-1}}{1+2^{2n-1}}\right)$$

$$T_n = \tan^{-1}\left(\frac{2^n - 2^{n-1}}{1+2^n \cdot 2^{n-1}}\right)$$

$$T_n = \tan^{-1}2^n - \tan^{-1}2^{n-1}$$

$$S_n = \sum_{n=1}^{n} T_n = \sum_{n=1}^{n}\left(\tan^{-1}2^n - \tan^{-1}2^{n-1}\right)$$

$$= \left(\cancel{\tan^{-1}2^1} - \tan^{-1}1\right)$$
$$+ \left(\cancel{\tan^{-1}2^2} - \cancel{\tan^{-1}2^1}\right)$$
$$+ \left(\cancel{\tan^{-1}2^3} - \cancel{\tan^{-1}2^2}\right)$$
$$\vdots \qquad\qquad \vdots$$
$$+ \left(\tan^{-1}2^n - \cancel{\tan^{-1}2^{n-1}}\right)$$

$$= \tan^{-1}(2^n) - \tan^{-1}(1)$$

$$S_\infty = \tan^{-1}(\infty) - \pi/4 = \pi/2 - \pi/4 = \pi/4$$

13. Find the sum of the series :

(c) $\tan^{-1}\frac{1}{x^2+x+1}+\tan^{-1}\frac{1}{x^2+3x+3}+\tan^{-1}\frac{1}{x^2+5x+7}+\tan^{-1}\frac{1}{x^2+7x+13}$ to n terms.

sol^n: $= \tan^{-1}\left(\frac{(x+1)-x}{1+x(x+1)}\right) + \tan^{-1}\left(\frac{(x+2)-(x+1)}{1+(x+2)(x+1)}\right)$

$+ \tan^{-1}\left(\frac{(x+3)-(x+2)}{1+(x+3)(x+2)}\right) + \cdots\cdots$

$$= \left(\tan^{-1}(x+1) - \tan^{-1}x\right)$$
$$+ \left(\tan^{-1}(x+2) - \tan^{-1}(x+1)\right)$$
$$+ \left(\tan^{-1}(x+3) - \tan^{-1}(x+2)\right)$$
$$\vdots \qquad \vdots$$
$$+ \tan^{-1}(x+n) - \tan^{-1}(x+(n-1))$$

$$= \tan^{-1}(x+n) - \tan^{-1}x$$

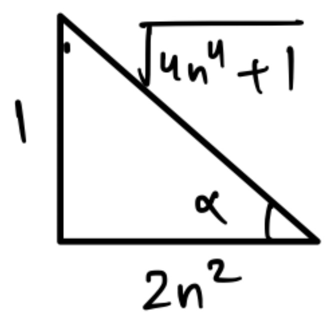
Ans

13. Find the sum of the series :

(d) $\sin^{-1}\frac{1}{\sqrt{5}}+\sin^{-1}\frac{1}{\sqrt{65}}+\sin^{-1}\frac{1}{\sqrt{325}}+\ldots\ldots+\sin^{-1}\frac{1}{\sqrt{4n^4+1}}+\ldots\ldots\infty$ terms.

Sol^n

(d) $T_n = \sin^{-1}\left(\frac{1}{\sqrt{4n^4+1}}\right) = \tan^{-1}\left(\frac{1}{2n^2}\right)$



$T_n = \tan^{-1}\left(\frac{2}{1+4n^2-1}\right) = \tan^{-1}\left(\frac{(2n+1)-(2n-1)}{1+(4n^2-1)}\right)$

$T_n = \tan^{-1}(2n+1) - \tan^{-1}(2n-1)$

$\therefore S_n = \sum_{n=1}^{n} T_n = \left(\tan^{-1}(3) - \tan^{-1}(1)\right)$
$+ \left(\tan^{-1}(5) - \tan^{-1}(3)\right)$
$+ \left(\tan^{-1}(7) - \tan^{-1}(5)\right)$
$\vdots$
$+ \tan^{-1}(2n+1) - \tan^{-1}(2n-1)$

$S_n = \tan^{-1}(2n-1) - \tan^{-1}(1)$

$S_\infty = \tan^{-1}(\infty) - \tan^{-1}(1)$
$= \pi/2 - \pi/4 = \pi/4$ Ans

S-2

EXERCISE (S-2)

1. If $\alpha = 2 \text{ arc } \tan\left(\frac{1+x}{1-x}\right)$ & $\beta = \text{arc } \sin\left(\frac{1-x^2}{1+x^2}\right)$ for $0 < x < 1$, then prove that $\alpha + \beta = \pi$, what the value of $\alpha + \beta$ will be if $x > 1$.

Soln

Soln

Case I : $0 < x < 1$

Put $x = \tan\theta$, $\theta \in \left(0, \frac{\pi}{4}\right)$

So, $\alpha = 2\tan^{-1}\left(\tan\left(\frac{\pi}{4}+\theta\right)\right)$

$\Rightarrow \alpha = 2\left(\frac{\pi}{4}+\theta\right)$ as $\frac{\pi}{4}+\theta \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

$\Rightarrow \alpha = \frac{\pi}{2} + 2\theta$ ...... (i)

Also, $\beta = \sin^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \frac{\pi}{2} - \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$

$\Rightarrow \beta = \frac{\pi}{2} - \cos^{-1}(\cos 2\theta) = \frac{\pi}{2} - 2\theta$ ...... (ii)

(as $2\theta \in \left(0, \frac{\pi}{2}\right)$)

Hence, $\boxed{\alpha + \beta = \pi}$

Case II : $x > 1$

Again we let $x = \tan\theta$; $\theta \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right)$

So, $\alpha = 2\tan^{-1}\left(\tan\left(\frac{\pi}{4}+\theta\right)\right) = 2\left(\frac{\pi}{4}+\theta-\pi\right)$

$\Rightarrow \alpha = 2\theta - \frac{3\pi}{2}$ ...... (iii)

Also, $\beta = \frac{\pi}{2} - \cos^{-1}(\cos 2\theta) = \frac{\pi}{2} - 2\theta$ ...... (iv)

[as $2\theta \in \left(\frac{\pi}{2}, \pi\right)$]

Hence, $\boxed{\alpha + \beta = -\pi}$ Ans

2. Solve the following :

(a) $\tan^{-1}\frac{x-1}{x+1}+\tan^{-1}\frac{2x-1}{2x+1}=\tan^{-1}\frac{23}{36}$

(b) $2\tan^{-1}x=\cos^{-1}\frac{1-a^2}{1+a^2}-\cos^{-1}\frac{1-b^2}{1+b^2}\ (a>0,\ b>0)$

Sol^n (a.) Taking tangent both sides :—

$$\frac{\frac{x-1}{x+1}+\frac{2x-1}{2x+1}}{1-\left(\frac{x-1}{x+1}\right)\left(\frac{2x-1}{2x+1}\right)}=\frac{23}{36}$$

$$\Rightarrow \frac{(2x^2+x-2x-1+2x^2+2x-x-1)}{(2x^2+2x+x+1-2x^2+x+2x-1)}=\frac{23}{36}$$

$$\Rightarrow \frac{4x^2-2}{6x}=\frac{23}{36} \Rightarrow 24x^2-23x-12=0$$

$$\Rightarrow 24x^2-32x+9x-12=0$$

$$\Rightarrow 8x(3x-4)+3(3x-4)=0$$

$$\Rightarrow x=\frac{4}{3},\ \frac{-3}{8}$$

But x cannot be $\frac{-3}{8}$ as both $\frac{x-1}{x+1}$ as well as $\frac{2x-1}{2x+1}$ are $-$ve for $x=\frac{-3}{8}$ and so LHS <0 & RHS >0.

Checking for $x=4/3$

$$\text{LHS}=\tan^{-1}\left(\frac{1}{7}\right)+\tan^{-1}\left(\frac{5}{11}\right)=\tan^{-1}\left(\frac{46}{72}\right)=\text{RHS}$$

Hence, $\boxed{x=\frac{4}{3}}$ Ans

2

(b) $2\tan^{-1}x = \cos^{-1}\frac{1-a^2}{1+a^2} - \cos^{-1}\frac{1-b^2}{1+b^2}$ $(a>0,\ b>0)$

Solⁿ:

(b) Put $a = \tan\alpha$ & $b = \tan\beta$;

So $\alpha, \beta \in \left(0, \frac{\pi}{2}\right)$ as $a, b > 0$

Now, Equation reduces to :—

$2\tan^{-1}x = \cos^{-1}(\cos 2\alpha) - \cos^{-1}(\cos 2\beta)$

$\Rightarrow 2\tan^{-1}x = 2\alpha - 2\beta$ [as $2\alpha, 2\beta \in (0, \pi)$]

$\Rightarrow \tan^{-1}x = \tan^{-1}a - \tan^{-1}b = \tan^{-1}\left(\frac{a-b}{1+ab}\right)$

[as $a, b > 0$]

$\Rightarrow \boxed{x = \frac{a-b}{1+ab}}$ [as $\tan^{-1}x$ is injective]

Ans

3. Find all the positive integral solutions of, $\tan^{-1}x + \cos^{-1}\frac{y}{\sqrt{1+y^2}} = \sin^{-1}\frac{3}{\sqrt{10}}$.

Sol^n Put $y = \tan\theta$, $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

So, $\cos^{-1}\frac{y}{\sqrt{1+y^2}} = \cos^{-1}\left(\frac{\tan\theta}{|\sec\theta|}\right) = \cos^{-1}\left(\frac{\tan\theta}{\sec\theta}\right)$ [as $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$]

$= \cos^{-1}(\sin\theta) = \frac{\pi}{2} - \sin^{-1}(\sin\theta) = \frac{\pi}{2} - \theta$

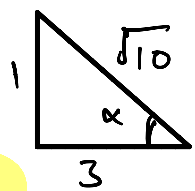
$= \frac{\pi}{2} - \tan^{-1}y$

Now, given equation simplifies to :-

$$\frac{\pi}{2} + \tan^{-1}x - \tan^{-1}y = \sin^{-1}\frac{3}{\sqrt{10}}$$

$$\Rightarrow \tan^{-1}\left(\frac{x-y}{1+xy}\right) = -\cos^{-1}\frac{3}{\sqrt{10}} \quad [\text{as } x, y > 0]$$

$$\Rightarrow \tan^{-1}\left(\frac{x-y}{1+xy}\right) = \tan^{-1}\left(\frac{-1}{3}\right)$$



$$\Rightarrow \frac{x-y}{1+xy} = \frac{-1}{3} \Rightarrow y + 3 = \frac{10}{3-x}$$

on solving

So, $3 - x = 1$ or 2 (as $y + 3 \geq 4$)

$$\Rightarrow \boxed{(x, y) = (2, 7) \text{ or } (1, 2)}$$ Ans

4. Prove that the equation, $(\sin^{-1}x)^3 + (\cos^{-1}x)^3 = \alpha\pi^3$ has no roots for $\alpha < \frac{1}{32}$ and $\alpha > \frac{7}{8}$.

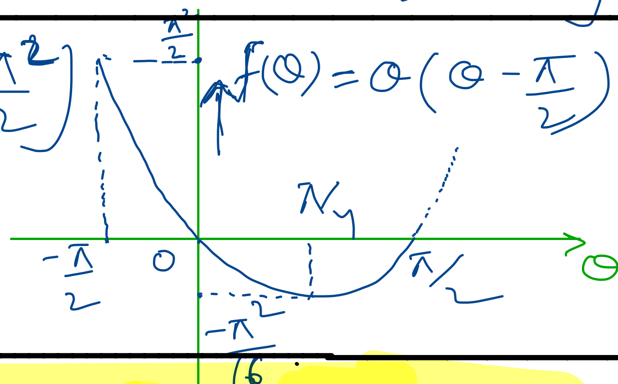
Let's find range of $y = (\sin^{-1}x)^3 + (\cos^{-1}x)^3$

$\Rightarrow y = (\sin^{-1}x + \cos^{-1}x)\left[(\sin^{-1}x + \cos^{-1}x)^2 - 3\sin^{-1}x\cos^{-1}x\right]$

$\Rightarrow y = \frac{\pi}{2}\left[\frac{\pi^2}{4} - 3\theta\left(\frac{\pi}{2} - \theta\right)\right]$; $\theta = \sin^{-1}x$

$\Rightarrow y = \frac{\pi}{2}\left[\frac{\pi^2}{4} + 3\theta\left(\theta - \frac{\pi}{2}\right)\right]$; $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Since, $f(\theta) \in \left[-\frac{\pi^2}{16}, \frac{\pi^2}{2}\right]$



$\Rightarrow y = \frac{\pi}{2}\left[\frac{\pi^2}{4} + 3f(\theta)\right] \in \left[\frac{\pi^3}{32}, \frac{7\pi^3}{8}\right]$

Thus, $y = \alpha\pi^3$

$\Rightarrow \alpha\pi^3 \in \left[\frac{\pi^3}{32}, \frac{7\pi^3}{8}\right]$

$\Rightarrow \alpha \in \left[\frac{1}{32}, \frac{7}{8}\right]$

Hence, neither α can be less than $\frac{1}{32}$ nor it can be greater than $\frac{7}{8}$

5. Solve the following inequalities :

(a) arc $\cot^2 x - 5$ arc $\cot x + 6 > 0$ (b) arc $\sin x >$ arc $\cos x$ (c) $\tan^2(\text{arc}\sin x) > 1$

Sol^n

(a) $\theta^2 - 5\theta + 6 > 0$; $\theta = \cot^{-1} x \in (0, \pi)$

$\Rightarrow (\theta - 2)(\theta - 3) > 0$

$\Rightarrow \theta \in (-\infty, 2) \cup (3, \infty)$

$\Rightarrow \theta \in (0, 2) \cup (3, \pi)$ [as $\theta \in (0, \pi)$]

$\Rightarrow \cot\theta = \boxed{x \in (-\infty, \cot 3) \cup (\cot 2, \infty)}$ Ans

---

(b) $\sin^{-1} x > \cos^{-1} x \Rightarrow \sin^{-1} x > \frac{\pi}{2} - \sin^{-1} x$

$\Rightarrow \sin^{-1} x > \left(\frac{\pi}{4} = \sin^{-1}\frac{1}{\sqrt{2}}\right)$

$\Rightarrow \boxed{x \in \left(\frac{1}{\sqrt{2}}, 1\right]}$ Ans

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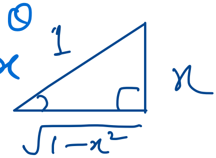
(c) $\left(\tan(\sin^{-1} x)\right)^2 > 1$ let $\sin^{-1} x = \theta$, $\sin\theta = x$



$\Rightarrow (\tan\theta)^2 > 1$

$\Rightarrow \left(\frac{x}{\sqrt{1-x^2}}\right)^2 > 1$; $x \in (-1, 1)$

$\Rightarrow x^2 > 1 - x^2$

$\Rightarrow x^2 > \frac{1}{2}$

$\Rightarrow \boxed{x \in \left(-1, \frac{-1}{\sqrt{2}}\right) \cup \left(\frac{1}{\sqrt{2}}, 1\right)}$ Ans

6. If the sum $\sum_{n=1}^{10}\sum_{m=1}^{10}\tan^{-1}\left(\frac{m}{n}\right) = k\pi$, find the value of k.

Sol^n $\frac{m}{n} \in \left\{ \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{10}, \frac{2}{1}, \frac{2}{2}, \frac{2}{3}, \ldots, \frac{2}{10}, \ldots, \frac{10}{1}, \frac{10}{2}, \frac{10}{3}, \ldots, \frac{10}{10} \right\}$

Pairs $\left(\frac{p}{q}, \frac{q}{p}\right)$

So, these 100 elements include 1 repeated ten times and 45 pairs of $\left(\frac{p}{q}, \frac{q}{p}\right)$; $p \neq q$.

Also, $\tan^{-1}\left(\frac{p}{q}\right) + \tan^{-1}\left(\frac{q}{p}\right) = \tan^{-1}\frac{p}{q} + \cot^{-1}\frac{p}{q} = \frac{\pi}{2}$

Hence, LHS $= 10\left(\tan^{-1} 1\right) + 45 \times \left(\frac{\pi}{2}\right) = K\pi$ (given)

$\Rightarrow \frac{5\pi}{2} + \frac{45\pi}{2} = K\pi$

$\Rightarrow \boxed{K = 25}$ Ans

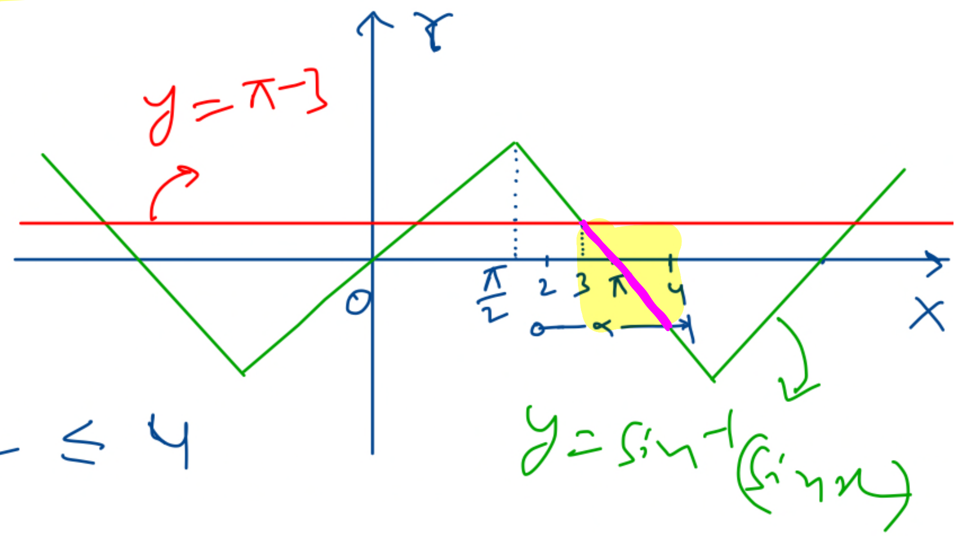
7. Solve for x : $\sin^{-1}\left(\sin\left(\frac{2x^2+4}{1+x^2}\right)\right) < \pi - 3$.

Sol^n Say $\alpha = \frac{2x^2+4}{1+x^2} = 2 + \frac{2}{1+x^2} \in (2, 4]$

Given, $\sin^{-1}(\sin\alpha) < \pi - 3$

So, from graph we find that

$\alpha \in (3, 4]$



$\Rightarrow 3 < 2 + \frac{2}{1+x^2} \leq 4$

$\Rightarrow 1 < \frac{2}{1+x^2} \leq 2$

$\Rightarrow \frac{1}{2} < \frac{1}{1+x^2} \leq 1 \Rightarrow 2 > 1+x^2 \geq 1$

$\Rightarrow 1 > x^2 \geq 0 \Rightarrow x \in (-1, 1)$

Ans

8. Find the set of values of 'a' for which the equation $2\cos^{-1}x = a + a^2(\cos^{-1}x)^{-1}$ posses a solution.

Sol^n Let $\cos^{-1}x = \theta \in [0, \pi]$

so, $2\theta = a + \dfrac{a^2}{\theta}$; $\theta \neq 0$ (i.e. now $\theta \in (0, \pi]$)

$\Rightarrow \quad 2\theta^2 - a\theta - a^2 = 0$

$\Rightarrow \quad (\theta - a)(2\theta + a) = 0$

$\Rightarrow \quad \theta = a$ or $\dfrac{-a}{2}$

$\Rightarrow$ Either $a \in (0, \pi]$ or $\dfrac{-a}{2} \in (0, \pi]$

$\Rightarrow$ Either $a \in (0, \pi]$ or $a \in [-2\pi, 0)$

$\Rightarrow \quad \boxed{a \in [-2\pi, \pi] - \{0\}}$

Ans

JM

1. Let $\tan^{-1} y = \tan^{-1} x + \tan^{-1}\left(\frac{2x}{1-x^2}\right)$, where $|x| < \frac{1}{\sqrt{3}}$. Then a value of y is : **[JEE (Main)-2015]**

(1) $\frac{3x-x^3}{1+3x^2}$ (2) $\frac{3x+x^3}{1+3x^2}$ (3) $\frac{3x-x^3}{1-3x^2}$ (4) $\frac{3x+x^3}{1-3x^2}$

Soln :-

$$\tan^{-1} y = \tan^{-1} x + \tan^{-1}\left(\frac{2x}{1-x^2}\right)$$

when $|x| < \frac{1}{\sqrt{3}}$

$$\Rightarrow \tan^{-1} y = \tan^{-1}\left(\frac{x + \frac{2x}{1-x^2}}{1 - x\cdot\left(\frac{2x}{1-x^2}\right)}\right)$$

$$\tan^{-1} y = \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$$

$$\Rightarrow y = \frac{3x-x^3}{1-3x^2}$$

2. If $\cos^{-1}\left(\frac{2}{3x}\right)+\cos^{-1}\left(\frac{3}{4x}\right)=\frac{\pi}{2}\left(x>\frac{3}{4}\right)$, then x is equal to : **[JEE(Main)-2019]**

(1) $\frac{\sqrt{145}}{12}$ (2) $\frac{\sqrt{145}}{10}$ (3) $\frac{\sqrt{146}}{12}$ (4) $\frac{\sqrt{145}}{11}$

Soln! — $\cos^{-1}\left(\frac{2}{3x}\right)+\cos^{-1}\left(\frac{3}{4x}\right)=\frac{\pi}{2}$

$$\Rightarrow \cos^{-1}\left(\frac{2}{3x}\times\frac{3}{4x}-\sqrt{1-\frac{4}{9x^2}}\sqrt{1-\frac{9}{16x^2}}\right)=\frac{\pi}{2}$$

$$\frac{1}{2x^2}-\frac{\sqrt{9x^2-4}\sqrt{16x^2-9}}{9x^2\times 16x^2}=\cos\frac{\pi}{2}=0$$

$$\sqrt{9x^2-4}\sqrt{16x^2-9}=6$$

Squaring & Solving

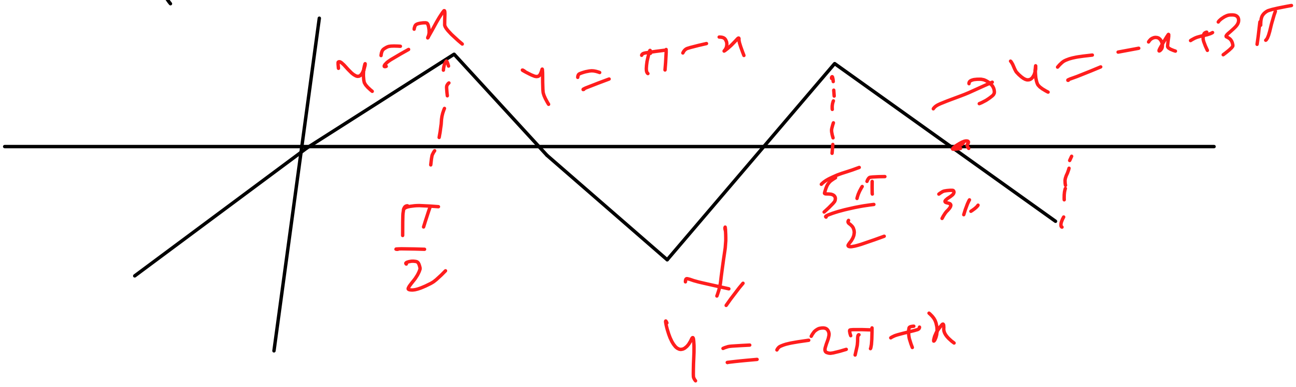
$$x=\pm\frac{\sqrt{145}}{2} \quad \text{as } x>3$$

So $x=\frac{\sqrt{145}}{2}$

3. If $x = \sin^{-1}(\sin 10)$ and $y = \cos^{-1}(\cos 10)$, then $y - x$ is equal to: **[JEE(Main)-2019]**

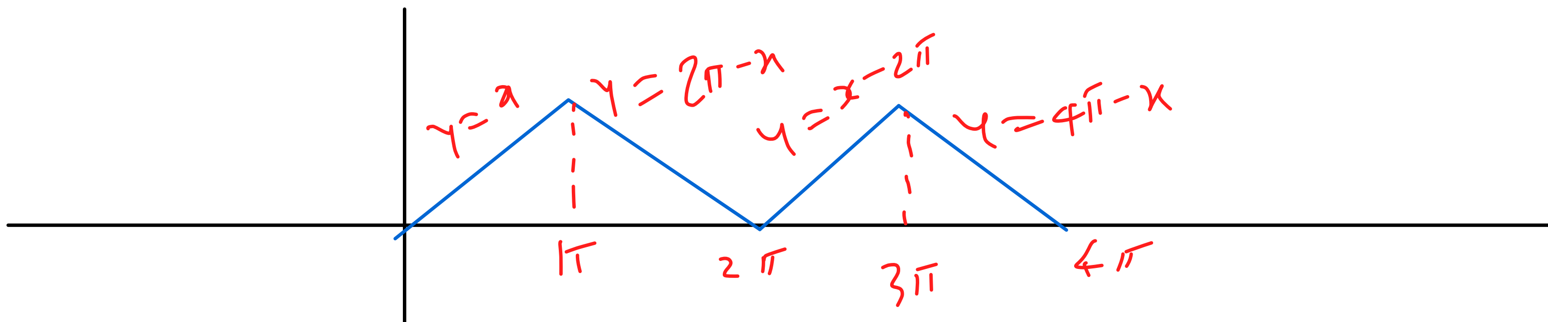
(1) π (2) 7π (3) 0 (4) 10

$$y = \sin^{-1}\sin 10 = -10 + 3\pi$$



Similarly

$$y = \cos^{-1}\cos 10 = -10 + 4\pi$$



$$\boxed{y - x = \pi}$$

4. The value of $\cot\left(\sum_{n=1}^{19}\cot^{-1}\left(1+\sum_{p=1}^{n}2p\right)\right)$ is : **[JEE(Main)-2019]**

(1) $\frac{22}{23}$ (2) $\frac{23}{22}$ (3) $\frac{21}{19}$ (4) $\frac{19}{21}$

$$\sum_{p=1}^{n} 2p = n(n+1)$$

$$\cot^{-1}(1+(n)(n+1)) = \tan^{-1}\frac{1}{1+n(n+1)}$$

$$= \tan^{-1}\left(\frac{n+1-n}{1+n(n+1)}\right) = \tan^{-1}(n+1) - \tan^{-1}n$$

$$\sum_{n=1}^{19}\cot^{-1}\left(1+\sum_{p=1}^{n}2p\right) = \sum_{n=1}^{19}\left(\tan^{-1}(n+1) - \tan^{-1}n\right)$$

$$= (\cancel{\tan^{-1}2} - \tan^{-1}1) + (\cancel{\tan^{-1}3} - \cancel{\tan^{-1}2}) + (\tan^{-1}4 - \cancel{\tan^{-1}3})$$

$$+ \dots (\tan^{-1}20 - \tan^{-1}19)$$

$\Rightarrow$ sum is $= \tan^{-1}20 - \tan^{-1}1$

Now $\cot\sum_{n=1}^{19}\cot^{-1}\left(1+\sum_{p=1}^{n}(2p)\right) = \cot(\tan^{-1}20 - \tan^{-1}1)$

$$\cot\left(\tan^{-1}20 - \tan^{-1}1\right) = \frac{1 + 20 \times 1}{20 - 1} = \frac{21}{19}$$

5. All x satisfying the inequality $(\cot^{-1} x)^2 - 7(\cot^{-1} x) + 10 > 0$, lie in the interval :-

[JEE(Main)-2019]

(1) $(-\infty, \cot 5) \cup (\cot 4, \cot 2)$

(2) $(\cot 5, \cot 4)$

(3) $(\cot 2, \infty)$

(4) $(-\infty, \cot 5) \cup (\cot 2, \infty)$

Soln:- $(\cot^{-1}x - 2)\underbrace{(\cot^{-1}x - 5)}_{-ve} > 0$

$(\cot^{-1}x - 2) < 0$

$\cot^{-1}x < 2$

$x > \cot 2$

$\Rightarrow (\cot 2, \infty)$

6. Considering only the principal values of inverse functions, the set $A=\left\{x\geq 0:\tan^{-1}(2x)+\tan^{-1}(3x)=\frac{\pi}{4}\right\}$

[JEE(Main)-2019]

(1) is an empty set

(2) Contains more than two elements

(3) Contains two elements

(4) is a singleton

Soln:– $\tan^{-1}2x+\tan^{-1}3x=\frac{\pi}{4}$; $x\geq 0$

$$\tan\left(\tan^{-1}2x+\tan^{-1}3x\right)=\tan\pi/4$$

$$\frac{2x+3x}{1+6x^2}=1$$

$$6x^2+5x-1=0$$

$x=\frac{1}{6}$; $x\neq -1$ as $x>0$

7. If $\alpha = \cos^{-1}\left(\frac{3}{5}\right), \beta = \tan^{-1}\left(\frac{1}{3}\right)$, where $0 < \alpha, \beta < \frac{\pi}{2}$, then $\alpha - \beta$ is equal to : **[JEE(Main)-2019]**

(1) $\sin^{-1}\left(\frac{9}{5\sqrt{10}}\right)$ (2) $\tan^{-1}\left(\frac{9}{14}\right)$ (3) $\cos^{-1}\left(\frac{9}{5\sqrt{10}}\right)$ (4) $\tan^{-1}\left(\frac{9}{5\sqrt{10}}\right)$

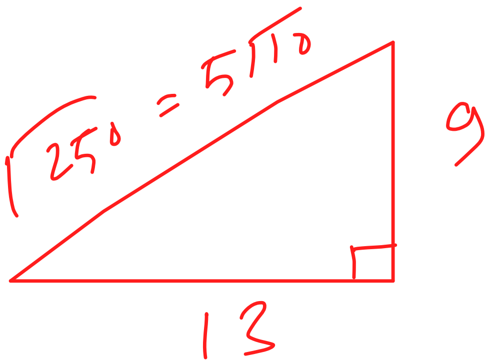
$$\alpha = \cos^{-1}\left(\frac{3}{5}\right) = \tan^{-1}\left(\frac{4}{3}\right)$$

$$\beta = \tan^{-1}\frac{1}{3} \qquad \left(\alpha, \beta \in \left(0, \frac{\pi}{2}\right)\right)$$

$$\alpha - \beta = \tan^{-1}\frac{4}{3} - \tan^{-1}\frac{1}{3} = \tan^{-1}\left(\frac{\frac{4}{3} - \frac{1}{3}}{1 + \frac{4}{3} \times \frac{1}{3}}\right)$$

$$\alpha - \beta = \tan^{-1}\left(\frac{1}{1 + \frac{4}{9}}\right) = \tan^{-1}\left(\frac{9}{13}\right)$$

$$\alpha - \beta = \tan^{-1}\left(\frac{9}{13}\right) = \sin^{-1}\left(\frac{9}{5\sqrt{10}}\right)$$



8. If $\cos^{-1}x - \cos^{-1}\frac{y}{2} = \alpha$, where $-1 \le x \le 1$, $-2 \le y \le 2$, $x \le \frac{y}{2}$, then for all x, y, $4x^2 - 4xy\cos\alpha + y^2$ is equal to **[JEE(Main)-2019]**

(1) $4\sin^2\alpha - 2x^2y^2$ (2) $4\cos^2\alpha + 2x^2y^2$ (3) $4\sin^2\alpha$ (4) $2\sin^2\alpha$

Soln:- $\cos^{-1}x - \cos^{-1}\frac{y}{2} = \alpha$

$$\cos^{-1}\left(\frac{xy}{2} + \sqrt{(1-x^2)\left(1-\frac{y^2}{4}\right)}\right) = \alpha$$

$$\cos^{-1}\left(\frac{xy + \sqrt{4-y^2-4x^2+x^2y^2}}{2}\right) = \alpha$$

$$\Rightarrow xy + \sqrt{4-y^2-4x^2+x^2y^2} = 2\cos\alpha$$

$$\sqrt{4-y^2-4x^2+x^2y^2} = 2\cos\alpha - xy$$

Squaring

$$4-y^2-4x^2+\cancel{x^2y^2} = 4\cos^2\alpha + \cancel{x^2y^2} - 4xy\cos\alpha$$

$$4x^2+y^2-4xy\cos\alpha = 4\sin^2\alpha$$

9. The value of $\sin^{-1}\left(\frac{12}{13}\right)-\sin^{-1}\left(\frac{3}{5}\right)$ is equal to : **[JEE(Main)-2019]**

(1) $\pi-\sin^{-1}\left(\frac{63}{65}\right)$ (2) $\pi-\cos^{-1}\left(\frac{33}{65}\right)$ (3) $\frac{\pi}{2}-\sin^{-1}\left(\frac{56}{65}\right)$ (4) $\frac{\pi}{2}-\cos^{-1}\left(\frac{9}{65}\right)$

$$\sin^{-1}\frac{12}{13}-\sin^{-1}\frac{3}{5}=\sin^{-1}\left(\frac{12}{13}\sqrt{1-\frac{9}{25}}-\frac{3}{5}\sqrt{1-\frac{144}{25}}\right)$$

$$=\sin^{-1}\left(\frac{33}{65}\right)$$

$$=\cos^{-1}\left(\frac{56}{65}\right)$$

$$=\frac{\pi}{2}-\sin^{-1}\left(\frac{56}{65}\right)$$

J·A

1. If $0 < x < 1$, then $\sqrt{1+x^2}\,[\{x\cos(\cot^{-1}x) + \sin(\cot^{-1}x)\}^2 - 1]^{1/2} =$ **[JEE 2008, 3]**

(A) $\frac{x}{\sqrt{1+x^2}}$ (B) x (C) $x\sqrt{1+x^2}$ (D) $\sqrt{1+x^2}$

$$\sqrt{1+x^2}\left[\left\{x\cos(\cot^{-1}x) + \sin(\cot^{-1}x)\right\}^2 - 1\right]^{1/2}$$

$$\sqrt{1+x^2}\left[\left\{x\cdot\frac{x}{\sqrt{1+x^2}} + \frac{1}{\sqrt{1+x^2}}\right\}^2 - 1\right]^{1/2}$$

$$\sqrt{1+x^2}\left[1+x^2-1\right]^{1/2}$$

$$|x|\sqrt{1+x^2}$$

2. The value of $\cot\left(\sum_{n=1}^{23}\cot^{-1}\left(1+\sum_{k=1}^{n}2k\right)\right)$ is **[JEE(Advanced) 2013, 2]**

(A) $\frac{23}{25}$ (B) $\frac{25}{23}$ (C) $\frac{23}{24}$ (D) $\frac{24}{23}$

$$\sum_{n=1}^{23} \cot^{-1}(1+n(n+1))$$

$$\sum_{n=1}^{23} \tan^{-1}\left(\frac{(n+1)-n}{1+n(n+1)}\right)$$

$$\sum_{n=1}^{23} \left(\tan^{-1}(n+1) - \tan^{-1}n\right)$$

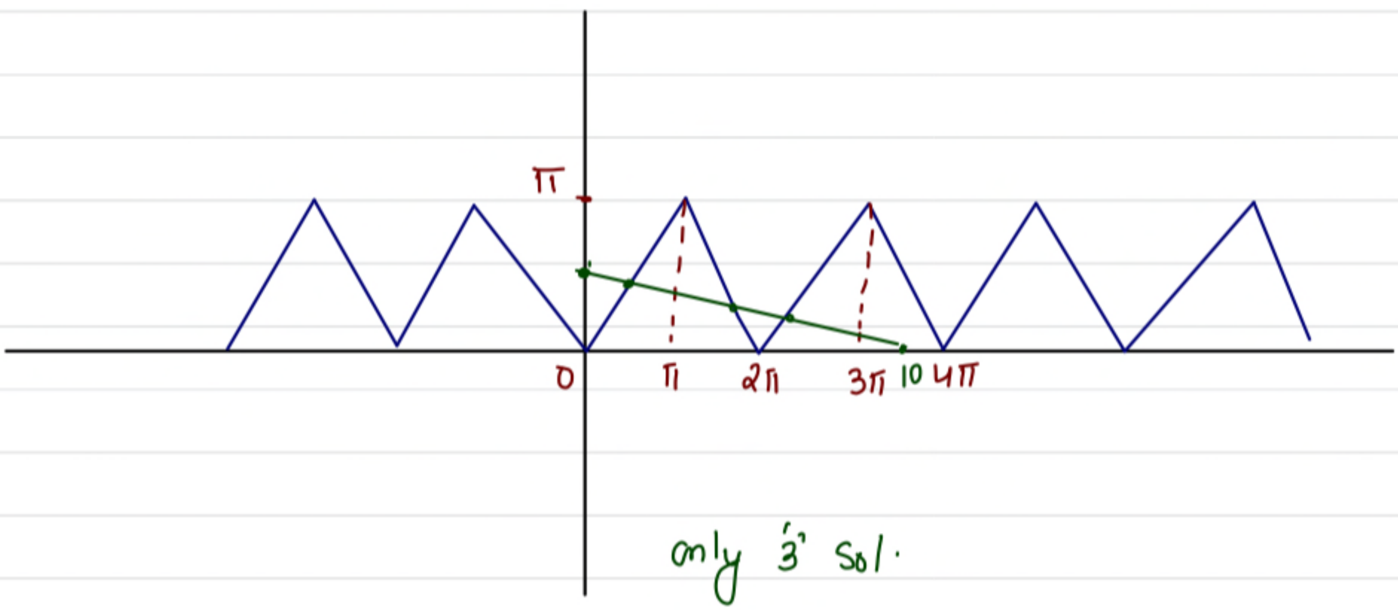
$$\left[(\tan^{-1}2 - \tan^{-1}1) + (\tan^{-1}3 - \tan^{-1}2) + \cdots + (\tan^{-1}24 - \tan^{-1}23)\right]$$

$$\tan^{-1}24 - \tan^{-1}1$$

$$\therefore \cot\left(\tan^{-1}\frac{23}{25}\right) = \frac{25}{23}$$

3. Let $f : [0,4\pi] \to [0,\pi]$ be defined by $f(x) = \cos^{-1}(\cos x)$. The number of points $x \in [0,4\pi]$ satisfying the equation $f(x) = \frac{10-x}{10}$ is **[JEE(Advanced)-2014, 3]**

$f : [0,4\pi] \to [0,\pi]$, $f(x) = \cos^{-1}(\cos x)$, $f(x) = \frac{10-x}{10}$



only '3' sol.

4. If $\alpha = 3\sin^{-1}\left(\frac{6}{11}\right)$ and $\beta = 3\cos^{-1}\left(\frac{4}{9}\right)$ where the inverse trigonometric functions take only the principal values, then the correct option(s) is(are) **[JEE(Advanced)-2015, 4]**

(A) $\cos\beta > 0$ (B) $\sin\beta < 0$ (C) $\cos(\alpha+\beta) > 0$ (D) $\cos\alpha < 0$

$\sin\frac{\alpha}{3} = \frac{6}{11}$, $\cos\left(\frac{\beta}{3}\right) = \frac{4}{9}$

$\sin\alpha = 3\sin\frac{\alpha}{3} - 4\sin^3\frac{\alpha}{3}$, $\cos\beta = 4\cos^3\frac{\beta}{3} - 3\cos\frac{\beta}{3}$

$\sin\alpha = 3\cdot\frac{6}{11} - 4\cdot\frac{216}{(11)^3}$, $\cos\beta = 4\cdot\frac{64}{729} - \frac{4}{3}$

$\sin\beta < 0$, $\cos(\alpha+\beta) > 0$ $\cos\alpha < 0$

5. The number of real solutions of the equation

$$\sin^{-1}\left(\sum_{i=1}^{\infty}x^{i+1}-x\sum_{i=1}^{\infty}\left(\frac{x}{2}\right)^{i}\right)=\frac{\pi}{2}-\cos^{-1}\left(\sum_{i=1}^{\infty}\left(-\frac{x}{2}\right)^{i}-\sum_{i=1}^{\infty}(-x)^{i}\right)$$ lying in the interval $\left(-\frac{1}{2},\frac{1}{2}\right)$ is____

(Here, the inverse trigonometric functions $\sin^{-1}x$ and $\cos^{-1}x$ assume value in $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ and $[0, \pi]$, respectively.) **[JEE(Advanced)-2018]**

$$x\sum_{i=1}^{\infty}x^{i}-x\sum_{i=1}^{\infty}\left(\frac{x}{2}\right)^{i}=\sum_{i=1}^{\infty}\left(-\frac{x}{2}\right)^{i}-\sum_{i=1}^{\infty}(-x)^{i}$$

$$x\left(\frac{x}{1-x}\right)-x\left(\frac{x/2}{1-x/2}\right)=\frac{-x/2}{1+x/2}-\frac{(-x)}{1+x}$$

$x=0,$

$$\frac{x}{1-x}-\frac{x}{2-x}=\frac{-1}{2+x}+\frac{1}{1+x}$$

$$\frac{x(2-x-1+x)}{(1-x)(2-x)}=\frac{-1-x+2+x}{(2+x)(1+x)}$$

$$\frac{x}{(1-x)(2-x)}=\frac{1}{(2+x)(1+x)}$$

$\Rightarrow \quad x=0, \quad x^3+2x^2+5x-2=0$

it has one sol. in $0<x<\frac{1}{2}$

$\therefore$ 2 solution

6. Let $E_1 = \left\{x \in \mathbb{R} : x \neq 1 \text{ and } \frac{x}{x-1} > 0\right\}$ and $E_2 = \left\{x \in E_1 : \sin^{-1}\left(\log_e\left(\frac{x}{x-1}\right)\right) \text{ is a real number}\right\}$.

$\left(\text{Here, the inverse trigonometric function } \sin^{-1}x \text{ assumes values in } \left[-\frac{\pi}{2}, \frac{\pi}{2}\right].\right)$

Let $f : E_1 \to \mathbb{R}$ be the function defined by $f(x) = \log_e\left(\frac{x}{x-1}\right)$

and $g : E_2 \to \mathbb{R}$ be the function defined by $g(x) = \sin^{-1}\left(\log_e\left(\frac{x}{x-1}\right)\right)$.

LIST-I	LIST-II
P. The range of f is	**1.** $\left(-\infty, \frac{1}{1-e}\right] \cup \left[\frac{e}{e-1}, \infty\right)$
Q. The range of g contains	**2.** (0, 1)
R. The domain of f contains	**3.** $\left[-\frac{1}{2}, \frac{1}{2}\right]$
S. The domain of g is	**4.** $(-\infty, 0) \cup (0, \infty)$
	5. $\left(-\infty, \frac{e}{e-1}\right]$
	6. $(-\infty, 0) \cup \left(\frac{1}{2}, \frac{e}{e-1}\right]$

The correct option is : **[JEE(Advanced)-2018]**

(A) **P → 4; Q → 2; R → 1; S → 1**
(B) **P → 3; Q → 3; R → 6; S → 5**
(C) **P → 4; Q → 2; R → 1; S → 6**
(D) **P → 4; Q → 3; R → 6; S → 5**

$f(x) = \ln\left(\frac{x}{x-1}\right)$

Domain of $f(x) \Rightarrow x \in (-\infty, 0) \cup (1, \infty)$

Range of $f(x)$:

$e^y = \frac{x}{x-1} \Rightarrow xe^y - e^y = x$

$\Rightarrow x = \frac{e^y}{e^y - 1}$

$\therefore \frac{e^y}{e^y-1} < 0 \; , \; \frac{e^y}{e^y-1} > 1$

$y < 0 \; , \; y > 0$

$y \in R - \{0\}$

$g(x) = \sin^{-1} \ln\left(\frac{x}{x-1}\right)$

for domain : $-1 \leq \ln\left(\frac{x}{x-1}\right) \leq 1$

$e^{-1} \leq \frac{x}{x-1} \leq e \Rightarrow x \in \left(-\infty, \frac{1}{1-e}\right) \cup \left(\frac{e}{e-1}, \infty\right)$

Range: $y \in [-1, 1] - \{0\}$

7. The value of $\sec^{-1}\left(\frac{1}{4}\sum_{k=0}^{10}\sec\left(\frac{7\pi}{12}+\frac{k\pi}{2}\right)\sec\left(\frac{7\pi}{12}+\frac{(k+1)\pi}{2}\right)\right)$ in the interval $\left[-\frac{\pi}{4},\frac{3\pi}{4}\right]$ equals

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$$\sum_{k=0}^{10} \frac{1}{\cos\left(\frac{7\pi}{12}+\frac{k\pi}{2}\right)\cdot\cos\left(\frac{7\pi}{12}+\frac{(k+1)\pi}{2}\right)}$$

$$\sum_{k=0}^{10} \frac{\sin\left(\left(\frac{7\pi}{12}+(k+1)\frac{\pi}{2}\right)-\left(\frac{7\pi}{12}+\frac{k\pi}{2}\right)\right)}{\cos\left(\frac{7\pi}{12}+\frac{k\pi}{2}\right)\cdot\cos\left(\frac{7\pi}{12}+\frac{(k+1)\pi}{2}\right)}$$

$$\sum_{k=0}^{10} \left(\tan\left(\frac{7\pi}{12}+(k+1)\frac{\pi}{2}\right)-\tan\left(\frac{7\pi}{12}+k\frac{\pi}{2}\right)\right)$$

$$\left[\left(\tan\left(\frac{7\pi}{12}+\frac{\pi}{2}\right)-\tan\left(\frac{7\pi}{12}\right)\right)+\left(\tan\left(\frac{7\pi}{12}+2\cdot\frac{\pi}{2}\right)-\tan\left(\frac{7\pi}{12}+\frac{\pi}{2}\right)\right)+\cdots\right]$$

$$\Rightarrow \tan\left(\frac{7\pi}{12}+\frac{11\pi}{2}\right)-\tan\frac{7\pi}{12}$$

$$\frac{\sin\frac{11\pi}{2}}{\cos\left(\frac{7\pi}{12}+\frac{11\pi}{2}\right)\cdot\cos\left(\frac{7\pi}{12}\right)}$$

$$\therefore \sec^{-1}\left(\frac{1}{4}\,\frac{(-1)}{\cos\frac{\pi}{12}\cdot\cos\frac{7\pi}{12}}\right)$$

$$= 0.00$$