Q If $\lim_{x \to \frac{1}{2}} \frac{f(x)}{2x-1}$ exists then $\lim_{x \to \frac{1}{2}} f(x) = ?$

as $x \to \frac{1}{2}$; $D^r \to 0$.

$\therefore$ For limit to exist as $x \to \frac{1}{2}$ $N^r \to 0$.

$\therefore \lim_{x \to \frac{1}{2}} f(x) = 0$.

Q If $\lim_{x \to 0} \frac{\sqrt{ax+b}-2}{x} = 1$ find a and b.

Sol^n

as $x \to 0$; $D^r \to 0$

$\therefore N^r \to 0. \Rightarrow \sqrt{b} - 2 = 0$

$\therefore \boxed{b = 4}$

$$\lim_{x \to 0} \frac{\sqrt{ax+4} - 2}{x} = 1.$$

$$\lim_{x \to 0} \frac{(a\cancel{x} + \cancel{4}) - \cancel{4}}{\cancel{x}\,(\underbrace{\sqrt{ax+4} + 2})} = 1 \Rightarrow \frac{a}{4} = 1$$

$\therefore \boxed{a = 4}$

Q $\lim\limits_{n\to\infty} \dfrac{(3(n+1))!}{(n+1)^3(3n)!}$ equals $\left(\dfrac{\infty}{\infty}\right)$ $\quad \underline{|n} = n! = 1.2.3\ldots n$

$$\lim_{n\to\infty} \frac{(3n+3)!}{(n+1)^3\,(3n)!} = \lim_{n\to\infty} \frac{(3n+3)(3n+2)(3n+1)\,\cancel{(3n)!}}{(n+1)^3 \cdot \cancel{(3n)!}}$$

$$\lim_{n\to\infty} \frac{\cancel{n^3}\left(3+\frac{3}{n}\right)\left(3+\frac{2}{n}\right)\left(3+\frac{1}{n}\right)}{\cancel{n^3}\left(1+\frac{1}{n}\right)^3} = 27.$$

$\frac{4}{3}-\frac{3}{2} = \frac{8-9}{6} = -\frac{1}{6}$

Q $\lim\limits_{n\to\infty} \dfrac{\sqrt{n^3-2n^2+1}+\sqrt[3]{n^4+1}}{\sqrt[4]{n^6+6n^5+2}-\sqrt[5]{n^7+3n^3+1}}$

$$\lim_{n\to\infty} \frac{n^{3/2}\left(1-\frac{2}{n}+\frac{1}{n^3}\right)^{\frac{1}{2}} + n^{4/3}\left(1+\frac{1}{n^4}\right)^{\frac{1}{3}}}{n^{3/2}\left(1+\frac{6}{n}+\frac{2}{n^6}\right)^{\frac{1}{4}} - n^{7/5}\left(1+\frac{3}{n^4}+\frac{1}{n^7}\right)^{\frac{1}{5}}}.$$

$$\lim_{n\to\infty} \frac{\cancel{n^{3/2}}\left(\underbrace{\left(1-\frac{2}{n}+\frac{1}{n^3}\right)}^{\frac{1}{2}} + \overbrace{\frac{1}{n^{1/6}}\left(1+\frac{1}{n^4}\right)^{\frac{1}{3}}}^{0}\right)}{\cancel{n^{3/2}}\left(\underbrace{\left(1+\frac{6}{n}+\frac{2}{n^6}\right)}^{1/4} - \underbrace{\frac{1}{n^{\varepsilon}}\left(1+\frac{3}{n^4}+\frac{1}{n^7}\right)^{\frac{1}{5}}}_{0}\right)}$$

1 Ans

Q $\displaystyle \operatorname*{Limit}_{x \to \underbrace{-\infty}_{}} \frac{\sqrt{x^2+1}}{3x-6}$

$-x$

$$\lim_{x \to (-\infty)} \frac{|x| \sqrt{1+\frac{1}{x^2}}}{x\left(3-\frac{6}{x}\right)} = -\frac{1}{3}.$$

Q Let $S_n = 1 + 2 + 3 + \ldots\ldots + n$ and $P_n = \prod_{n=2}^{n} \frac{S_n}{S_n - 1}$

where $n \in N$ $(n \geq 2)$. Find $\lim_{n \to \infty} P_n$.

Sol^n

$$S_n = \frac{n(n+1)}{2} \qquad S_n - 1 = \frac{n^2+n}{2} - 1.$$

$$= \frac{n^2+n-2}{2} = \frac{(n+2)(n-1)}{2}$$

$$P_n = \prod_{n=2}^{n} \frac{n(n+1)}{(n+2)(n-1)} = \prod_{n=2}^{n} \underbrace{\left(\frac{n}{n-1}\right)} \underbrace{\left(\frac{n+1}{n+2}\right)}$$

$$\left(\frac{\cancel{2}}{1} \cdot \frac{3}{\cancel{2}} \cdot \frac{\cancel{4}}{3} \cdots \frac{n}{\cancel{n-1}}\right)\left(\frac{3}{\cancel{4}} \cdot \frac{\cancel{4}}{\cancel{5}} \cdots \frac{\cancel{n+1}}{n+2}\right)$$

$$P_n = \frac{3n}{n+2}$$

$$\lim_{n \to \infty} \frac{\cancel{n}(3)}{\cancel{n}\left(1+\frac{2}{n}\right)} = 3. \text{ Ans}$$

Rem :

① $(x^n + a^n) = (x+a)(x^{n-1} - x^{n-2}.a + \cdots\cdots + a^{n-1})$

where n is odd natural number.

② $(x^n - a^n) = (x-a)(x^{n-1} + x^{n-2}a + x^{n-3}a^2 + \cdots + a^{n-1})$

$n \in N$

$$\boxed{\lim_{x\to a}\left(\frac{x^n - a^n}{x-a}\right) = n\,a^{n-1}}$$

Rem

$n \in Q$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \frac{n(n-1)(n-2)}{3!}x^3 + \cdots\cdots\infty$$

$n \in Q$

(x is small compared to 1)

Proof: put $x = a+h$.

$$\lim_{h\to 0}\left(\frac{(a+h)^n - a^n}{(\cancel{a}+h) - \cancel{a}}\right) = \lim_{h\to 0}\frac{a^n\left(\left(1+\frac{h}{a}\right)^n - 1\right)}{h}$$

$$= \lim_{h\to 0}\frac{a^n\left(\cancel{1} + n\cdot\frac{h}{a} + \frac{n(n-1)}{2!}\left(\frac{h}{a}\right)^2 + \cdots - \cancel{1}\right)}{h}$$

$$a^n\cdot\frac{n}{a} = n.a^{n-1}$$

eg: If $\lim_{x \to a} \left(\frac{x^5 - a^5}{x - a} \right) = 80$. then find 'a' ?

$$5 \cdot a^4 = 80 \Rightarrow a^4 = 16 \Rightarrow \boxed{a = \pm 2}$$

* $\lim_{x \to 1} \left(\frac{x^m - 1}{x^n - 1} \right) = \lim_{x \to 1} \frac{\left(\frac{x^m - 1^m}{x - 1} \right)}{\left(\frac{x^n - 1^n}{x - 1} \right)} = \frac{m}{n}.$

** $\lim_{x \to 1} \left(\frac{x^{\frac{1}{11}} - x^{\frac{1}{5}}}{x^{\frac{1}{7}} - x^{\frac{1}{3}}} \right) = \lim_{x \to 1} \frac{\frac{(x^{\frac{1}{11}} - 1)}{x-1} - \frac{(x^{\frac{1}{5}} - 1)}{x-1}}{\frac{(x^{\frac{1}{7}} - 1)}{x-1} - \frac{(x^{\frac{1}{3}} - 1)}{x-1}}$

$$= \frac{\frac{1}{11} - \frac{1}{5}}{\frac{1}{7} - \frac{1}{3}}.$$

Q $\lim_{x\to 0} \dfrac{(\cos x)^{1/3} - (\cos x)^{1/2}}{\sin^2 x}$ equals

Let $\boxed{\cos x = t^6}$

$$\lim_{t\to 1} \left(\frac{t^2 - t^3}{1 - t^{12}}\right) = \lim_{t\to 1} \frac{t^2(t-1)}{(t^{12}-1)}$$

$$= \lim_{t\to 1} \frac{t^2}{\left(\frac{t^{12}-1}{t-1}\right)} = \frac{1}{12}.$$

**Q $\lim_{x\to\infty}\left(\sqrt[3]{x^3+3x^2} - \sqrt{x^2-2x}\right)$ $\quad(\infty - \infty)$

$$\lim_{x\to\infty} \left(x\left(1+\frac{3}{x}\right)^{\frac{1}{3}} - \underbrace{|x|}_{(x)}\left(1+\left(\frac{-2}{x}\right)\right)^{\frac{1}{2}}\right)$$

$$\lim_{x\to\infty} x \left(\underbrace{\left(1+\frac{3}{x}\right)^{\frac{1}{3}}} - \underbrace{\left(1+\left(\frac{-2}{x}\right)\right)^{\frac{1}{2}}}\right)$$

$$\lim_{x\to\infty} x\left(\left(\cancel{1}+\frac{1}{\cancel{3}}\left(\frac{\cancel{3}}{x}\right)+\underbrace{\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\left(\frac{3}{x}\right)^2}+\cdots\right)-\left(\cancel{1}+\frac{1}{\cancel{2}}\left(\frac{-\cancel{2}}{x}\right)+\cdots\right)\right)$$

$$\lim_{x\to\infty} x^1\left(\frac{2}{x}+\underbrace{\frac{(\,)}{x^2}+\frac{(\,)}{x^3}+\cdots\cdots}\right)$$

2. AM

Q $$\lim_{x\to\infty}\left(((x+a)(x+b)(x+c))^{1/3}-x\right) \qquad (\infty-\infty)$$

Sol^n $$\lim_{x\to\infty}\left(\left(x^3+x^2(a+b+c)+x(ab+bc+ca)+abc\right)^{\frac{1}{3}}-x\right)$$

$$\lim_{x\to\infty} x\left(\left(1+\left(\frac{a+b+c}{x}+\frac{ab+bc+ca}{x^2}+\frac{abc}{x^3}\right)\right)^{\frac{1}{3}}-1\right)$$

$$\lim_{x\to\infty} x\left(\cancel{1}+\frac{1}{3}\left(\frac{a+b+c}{x}+\frac{ab+bc+ca}{x^2}+\frac{abc}{x^3}\right)-\cancel{1}\right)$$

$$\left(\frac{a+b+c}{3}\right) = \text{AM of } a, b, c$$

Note: $$\lim_{x\to\infty}\left(\left((x+1)(x+2)(x+3)\ldots(x+100)\right)^{\frac{1}{100}}-x\right)$$

$$\left(\frac{1+2+3+\cdots+100}{100}\right)$$

M-2 Q $\lim_{x\to\infty}\left(\underbrace{((x+a)(x+b)(x+c))^{1/3}}_{A} - \underbrace{x}_{B}\right)$

$$\boxed{A^3 - B^3 = (A-B)(A^2+AB+B^2)}$$

$$A-B = \frac{A^3-B^3}{A^2+AB+B^2}$$

$x^3 + x^2(a+b+c) + x(ab+bc+ca) + abc$

$$\lim_{x\to\infty}\left(\frac{(x+a)(x+b)(x+c) - x^3}{\underbrace{((x+a)(x+b)(x+c))^{2/3}} + \underbrace{((x+a)(x+b)(x+c))^{1/3}\cdot x} + \underbrace{x^2}}\right)$$

Q

If $\lim_{x\to\infty} x^p\left(\sqrt[3]{x+1}+\sqrt[3]{x-1}-2\sqrt[3]{x}\right)$ exists and is equal to L, where L is non-zero real number then which of the following is/are correct options ?

(A) $L=\frac{2}{9}$ (B) $p=\frac{5}{3}$ (C) $L=\frac{5}{3}$ (D) $L=-\frac{2}{9}$

Sol^n

$$\lim_{\underbrace{x\to\infty}} x^{p+\frac{1}{3}}\left(\left(1+\frac{1}{x}\right)^{\frac{1}{3}}+\left(1-\frac{1}{x}\right)^{\frac{1}{3}}-2\right)$$

$$\lim_{x\to\infty} x^{p+\frac{1}{3}}\left(\cancel{1}+\underbrace{\cancel{\frac{1}{3}\left(\frac{1}{x}\right)}}+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\left(\frac{1}{x}\right)^2+\cdots\right.$$
$$\left.+\cancel{1}+\underbrace{\cancel{\frac{1}{3}\left(\frac{-1}{x}\right)}}+\frac{\frac{1}{3}\left(\frac{1}{3}-1\right)}{2!}\left(\frac{-1}{x}\right)^2+\cdots-\cancel{2}\right)$$

$$\lim_{x\to\infty} x^{p+\frac{1}{3}}\left(\underbrace{\frac{-2}{9x^2}+\cdots\cdots}\right)$$

$$\lim_{x\to\infty} \underbrace{x^{\left(p+\frac{1}{3}-2\right)}}\left(-\frac{2}{9}+\underbrace{\frac{\omega}{x^2}+\frac{\omega}{x^4}+\cdots}\right)$$

$$\boxed{p+\frac{1}{3}-2=0.} \qquad \boxed{L=-\frac{2}{9}}$$

Q If $\lim_{x\to 0}\frac{\sqrt{a^2+x^2}-\sqrt{b^2-x^2}+x^n}{x^2}=2$ a,b > 0 & n ∈ ℕ, then find possible values of a,b and n

$$\lim_{x\to 0}\frac{a\left(1+\frac{x^2}{a^2}\right)^{\frac{1}{2}}-b\left(1-\frac{x^2}{b^2}\right)^{\frac{1}{2}}+x^n}{x^2}=2.$$

$$\lim_{x\to 0}\frac{a\left(1+\frac{1}{2}\cdot\frac{x^2}{a^2}+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(\frac{x^2}{a^2}\right)^2+\cdots\right)-b\left(1+\frac{1}{2}\left(-\frac{x^2}{b^2}\right)+\frac{\frac{1}{2}\left(\frac{1}{2}-1\right)}{2!}\left(-\frac{x^2}{b^2}\right)^2+\cdots\right)+x^n}{x^2}$$

$$\lim_{x\to 0}\frac{\overbrace{(a-b)}^{0}+\overbrace{x^2\left(\frac{1}{2a}+\frac{1}{2b}\right)+x^4(\quad)\cdots+x^n}^{2}}{x^2}=2$$

$\therefore \boxed{a=b}$

$$\lim_{x\to 0}\left(\left(\frac{1}{2a}+\frac{1}{2b}\right)+x^2(\quad)+\cdots+x^{\overbrace{n-2}}\right)=2$$

$$\lim_{x\to 0}\left(\left(\frac{1}{2a}+\frac{1}{2b}\right)+\underbrace{x^{2}\{\} + \cdots}_{0} + x^{n-2}\right) = 2.$$

C–I If $n < 2$ then Limit dne.

C–II If $n = 2$ then $\frac{1}{2a} + \frac{1}{2b} + 1 = 2$ where $a = b$

$$\frac{1}{2a} + \frac{1}{2a} = 1 \Rightarrow \frac{1}{a} = 1 \Rightarrow \boxed{a = b = 1}$$

C-III If $n > 2$ then $\frac{1}{2a} + \frac{1}{2b} = 2.$ where $a = b.$

$$\frac{1}{2a} + \frac{1}{2a} = 2. \Rightarrow \boxed{a = b = \frac{1}{2}}$$

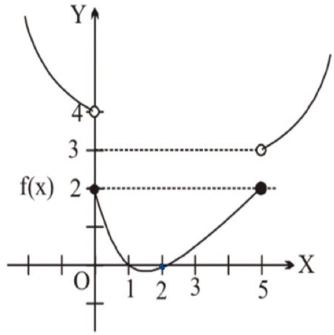
Q. Refer to the graph of $y=f(x)$ and $g(x)=(x-2)^2, x<2$

$= 7-x, x\ge 2$

then which of the following limits are non existent.

(a) $\lim\limits_{x\to 2} f(g(x))$ (b) $\lim\limits_{x\to 0} g(f(x))$ (c) $\lim\limits_{x\to 5} g(f(x))$

(d) $f(2)$ (e) $\lim\limits_{x\to 5} f(x)$ (f) $\lim\limits_{x\to 2} g(x)$



Sol^n (d) $f(2)=0$

(e) $\lim\limits_{x\to 5} f(x) \begin{cases} \xrightarrow{\text{LHL}} 2 \\ \xrightarrow{\text{RHL}} 3. \end{cases}$

Limit dne

(f) $\lim\limits_{x\to 2} g(x) \begin{cases} \xrightarrow{\text{LHL}} \lim\limits_{x\to 2^-} (x-2)^2 = 0. \\ \xrightarrow{\text{RHL}} \lim\limits_{x\to 2^+} (7-x) = 5 \end{cases}$

Limit dne

**

(a) $\lim\limits_{x\to 2} f(g(x)) \begin{cases} \xrightarrow{\text{LHL}} \lim\limits_{x\to 2^-} f(\overbrace{g(2^-)}^{0^+}) = 2. \\ \xrightarrow{\text{RHL}} \lim\limits_{x\to 2^+} f(\underbrace{g(2^+)}_{5^-}) = 2. \end{cases}$

$\Downarrow$

2

(b) $\lim\limits_{x\to 0} g(f(x)) \begin{cases} \xrightarrow{\text{LHL}} \lim\limits_{x\to 0^-} g(\underbrace{f(0^-)}_{4^+}) = 3. \\ \xrightarrow{\text{RHL}} \lim\limits_{x\to 0^+} g(\underbrace{f(0^+)}_{2^-}) = 0 \end{cases}$

$\Downarrow$

dne

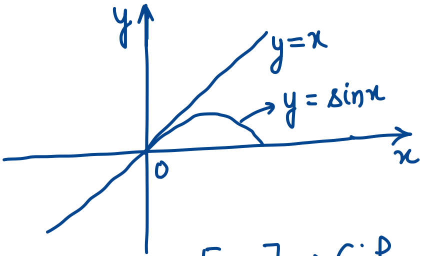
Limit of trigonometric functions :-

$$\boxed{\lim_{x \to 0} \frac{\sin x}{x} = 1}$$

it approaches '1' from LHS

i.e $0.\underline{9}\,\underline{9}\,\underline{9}\ldots\ldots\underline{9}$

($x \to$ radians)



* for $x > 0$

$x > \sin x$

* for $x < 0$

$\sin x > x$

[ ] → Gif

* $\lim_{x \to 0} \left[\frac{\sin x}{x}\right] = 0$; $\lim_{x \to 0} \left[\frac{x}{\sin x}\right] = 1$

Rem

$$\lim_{x \to 0} \frac{\sin x}{x} = \lim_{x \to 0} \frac{x}{\sin x} = \lim_{x \to 0} x \operatorname{cosec} x = \lim_{x \to 0} \left(\frac{\sin^{-1} x}{x}\right)$$

$$= \lim_{x \to 0} \left(\frac{x}{\sin^{-1} x}\right) = 1$$

Proof: $\lim_{x \to 0} \frac{\sin^{-1} x}{x} = 1.$

Let $\sin^{-1} x = \theta \Rightarrow x = \sin\theta$

$$\lim_{\theta \to 0} \left(\frac{\theta}{\sin\theta}\right) = 1.$$

$$\lim_{x\to 0}\left[\frac{2\sin^{-1}x}{x}\right] = 2.$$

$$\lim_{x\to 0}\left[\frac{4x}{\sin^{-1}x}\right] = 3.$$

$$\boxed{\lim_{x\to 0}\frac{\tan x}{x} = 1}$$

it approaches '1' from RHS

i.e 1.0000.....01

Rem

$x \to$ radians

[ ] → Gif

$$\boxed{\begin{array}{l} \text{for } x>0 \Rightarrow \tan x > x \\ \text{for } x<0 \Rightarrow \tan x < x \end{array}}$$

$$\lim_{x\to 0}\left[\frac{5x}{\tan x}\right] = 4.$$

For $x > 0$

$$\left[\lim_{x\to 0}\frac{5x}{\tan x}\right] = 5.$$

** $\underbrace{\tan x > x > \sin x}$

Rem

$$\boxed{\lim_{x\to 0}\frac{\tan x}{x} = \lim_{x\to 0}\frac{x}{\tan x} = \lim_{x\to 0} x.\cot x = \lim_{x\to 0}\frac{\tan^{-1}x}{x} = \lim_{x\to 0}\frac{x}{\tan^{-1}x} = 1}$$

$$\lim_{x\to 0}\left[\frac{7\tan^{-1}x}{x}\right] = 6.$$

as $x \to 1$

$M \to 0.$

eg: $$\lim_{x\to 1}\frac{\sin(x-1)}{(x^2-1)} = \lim_{x\to 1}\underbrace{\left(\frac{\sin M}{M}\right)}_{1}\cdot\underbrace{\left(\frac{1}{x+1}\right)}_{1/2} = \frac{1}{2}.$$

eg: $\lim_{x\to 2} \frac{\tan(x-2)}{(x-2)} = \lim_{x\to 2} \frac{\tan M}{M} = 1.$

eg: $\lim_{x\to \pi} \left(\frac{\tan x}{x}\right) = \frac{0}{\pi} = 0.$

eg: $\lim_{x\to(-1)} \frac{\sin(x^3+1)}{(x+1)} = \lim_{x\to(-1)} \underbrace{\left(\frac{\sin(x^3+1)}{(x^3+1)}\right)}_{1} \cdot \left(\frac{x^3+1}{x+1}\right)$

$= \lim_{x\to(-1)} \left(\quad\right) \frac{\cancel{(x+1)}(x^2-x+1)}{\cancel{(x+1)}}$

$(1)(3) = 3$

Rem

Note: $\boxed{\lim_{x\to 0} \left(\frac{1-\cos x}{x^2}\right) = \frac{1}{2}}$

$\lim_{x\to 0} \left(\frac{1-(1-2\sin^2\frac{x}{2})}{x^2}\right) = \lim_{x\to 0} \frac{2.\sin^2\frac{x}{2}}{x^2}$

$= 2.\lim_{x\to 0} \underbrace{\left(\frac{\sin x/2}{x/2}\right)^2}\cdot\frac{\cancel{x^2}/4}{\cancel{x^2}}$

$= 2\times\frac{1}{4} = \frac{1}{2}.$

Q

Let $f(x)$ = xcosx, then $\lim_{x\to 0}\left(\left[\frac{f(x)}{\sin x}\right]+\left[\frac{f(2x)}{\sin 2x}\right]+\left[\frac{f(3x)}{\sin 3x}\right]+\ldots\ldots+\left[\frac{f(2015x)}{\sin(2015x)}\right]\right)$ is

where [ ] → Gif ?

Sol^n

$$\lim_{x\to 0}\left(\left[\frac{x}{\tan x}\right]+\left[\frac{2x}{\tan 2x}\right]+\left[\frac{3x}{\tan 3x}\right]+\cdots+\left[\frac{2015x}{\tan 2015x}\right]\right)$$

$$\downarrow \qquad \downarrow \qquad \downarrow \qquad \downarrow$$

$$0 \qquad 0 \qquad 0 \qquad 0$$

0.

Q Let $f(x)=\left[\frac{\sin x}{x}\right]+\left[\frac{2\sin 2x}{x}\right]+\left[\frac{3\sin 3x}{x}\right]+$

$- - - - + \left[\frac{10 \sin 10x}{x}\right]$ then $\lim_{x\to 0} f(x) = ?$

[ ] → Gif.

Sol^n

$$\lim_{x\to 0}\left(\left[\frac{\sin x}{x}\right]+\left[2^2\cdot\left(\frac{\sin 2x}{2x}\right)\right]+\left[\frac{3^2\sin 3x}{3x}\right]+\cdots+\left[\frac{10^2\sin 10x}{10x}\right]\right)$$

$$(1^2-1)+(2^2-1)+(3^2-1)+\cdots+(10^2-1)$$

① $\lim_{x\to 0}\left(\frac{1-\cos 5x}{3x^2}\right) = \lim_{x\to 0}\left(\frac{1-\cos 5x}{(5x)^2}\right)\cdot\frac{(5x)^2}{3x^2}$

$$\downarrow$$

$$\left(\frac{1}{2}\right)\times\frac{25}{3} = \frac{25}{6}.$$

② $\lim_{x\to 0}\frac{1}{x}\sqrt{\frac{1-\cos 2x}{2}} = \lim_{x\to 0}\frac{1}{x}\sqrt{\frac{1-(1-2\sin^2 x)}{2}}$

$$\lim_{x\to 0}\frac{1}{x}|\sin x|. \quad \xrightarrow{LHL} (-1), \quad \xrightarrow{RHL} (1)$$

Limit dne.

③ $\lim_{x\to 1}(1-x)\tan\frac{\pi x}{2}$

LHL: put $x = 1-h$

$$\lim_{h\to 0}\left(1-(1-h)\right)\tan\left(\frac{\pi}{2}(1-h)\right)$$

$$\lim_{h\to 0}(h)\cot\left(\frac{\pi h}{2}\right)$$

$$\lim_{h\to 0}\frac{h}{\tan\frac{\pi h}{2}}\cdot\frac{\pi h/2}{\pi h/2} = \frac{2}{\pi}$$

RHL: put $x = 1+h$

$$\lim_{h\to 0} \left(1-(1+h)\right) \tan\left(\frac{\pi}{2}(1+h)\right)$$

$$\lim_{h\to 0} (-h)\left(-\cot\frac{\pi h}{2}\right) = \frac{2}{\pi}.$$

$$\text{Limit} = \frac{2}{\pi}.$$

4) $$\lim_{x\to 0} \frac{1-\cos(\overbrace{1-\cos x}^{M})}{\sin^4 x} = \qquad \left(\frac{0}{0} \text{ form}\right)$$

$$\lim_{x\to 0} \left(\frac{1-\cos M}{M^2}\right) \cdot \frac{(1-\cos x)^2}{\left(\frac{\sin^4 x}{x^4}\right)\cdot x^4}$$

$$\lim_{x\to 0} \underbrace{\left(\quad\right)} \left(\frac{1-\cos x}{x^2}\right)^2 \cdot \frac{1}{\left(\frac{\sin x}{x}\right)^4}.$$

$$\left(\frac{1}{2}\right) \cdot \left(\frac{1}{2}\right)^2 \cdot \quad 1 = \frac{1}{8}.$$

Q (a) $\lim_{x\to\pi/2} \frac{\tan x}{\tan 3x}$; (b) $\lim_{x\to\pi/2} \frac{\tan x}{\tan 5x}$; (c) $\lim_{x\to\frac{\pi}{2}} \left(\frac{\tan x}{\tan 4x}\right)$

(a) $\lim_{x\to\frac{\pi}{2}} \left(\frac{\tan x}{\tan 3x}\right) \xrightarrow[x=\frac{\pi}{2}-h]{LHL} \lim_{h\to 0} \frac{\tan\left(\frac{\pi}{2}-h\right)}{\tan\left(\frac{3\pi}{2}-3h\right)}$

$$= \lim_{h\to 0} \frac{\cot h}{\cot 3h}$$

$$= \lim_{h\to 0} \frac{\tan 3h}{\tan h}$$

$$= 3.$$

RHL: $x = \frac{\pi}{2}+h$

$\downarrow$

3

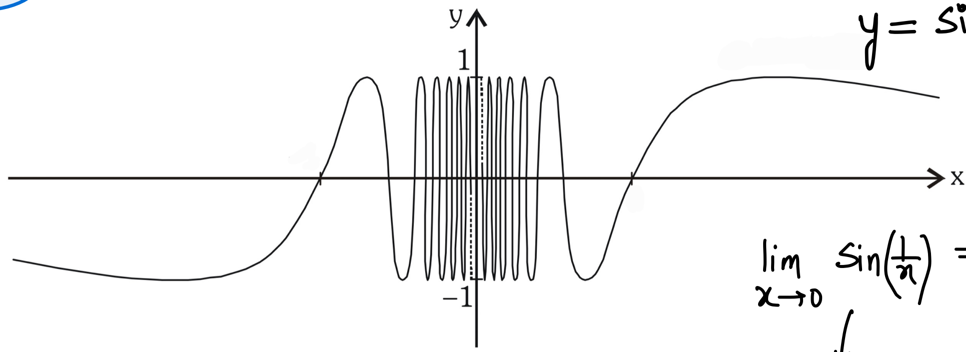
(b) 5

(c) $\lim_{x\to\frac{\pi}{2}} \left(\frac{\tan x}{\tan 4x}\right) \xrightarrow[x=\frac{\pi}{2}-h]{LHL} \lim_{h\to 0} \frac{\tan\left(\frac{\pi}{2}-h\right)}{\tan\left(2\pi-4h\right)}$

$$\lim_{h\to 0} \frac{-1}{\tan h \cdot \tan 4h}.$$

(dne)

Note :-



$y = \sin\left(\frac{1}{x}\right)$

$\lim\limits_{x\to 0} \sin\left(\frac{1}{x}\right) = \underline{\underline{dne}}$

$\downarrow$

oscillates from -1 to 1.

$\lim\limits_{x\to 0} \cos\left(\frac{1}{x}\right) = \underline{\underline{dne}}$

eg:
$$\lim_{x\to 0} \underbrace{x^3}_{\downarrow\, 0} \cdot \underbrace{\sin\left(\frac{1}{x}\right)}_{[-1,1]} = 0.$$

$0 \times [-1,1]$

Q
$$\lim_{x\to 0} \frac{x^2 \cos\frac{1}{x}}{\sin x}$$

$$\lim_{x\to 0} \underbrace{\left(\frac{x}{\sin x}\right)}_{1} \cdot \underbrace{x}_{0} \cdot \underbrace{\cos\left(\frac{1}{x}\right)}_{[-1,1]} = 0.$$

$1 \times 0 \times [-1,1]$

**

Q If $\lim\limits_{x\to 0} \dfrac{\cos\left(\dfrac{\pi}{2\cos x}\right)}{\sin(\sin x^2)} = l$, then find the value of $\{l\}$ $\left(\frac{0}{0}\right)$

$\{\ \} \to$ fractional part fns.

Soln

$$\lim_{x\to 0} \frac{\sin\left(\overbrace{\frac{\pi}{2} - \frac{\pi}{2\cos x}}^{M_1}\right)}{\underbrace{\left(\frac{\sin M}{M}\right)}\cdot\underbrace{\left(\frac{\sin x^2}{x^2}\right)}\cdot\underbrace{x^2}}$$

$$\lim_{x\to 0} \left(\frac{\sin M_1}{M_1}\right) \frac{\frac{\pi}{2}\left(\frac{\cos x - 1}{\cos x}\right)}{(\ \ )(\ \ )\ x^2} = \lim_{x\to 0} \frac{(\ )}{(\ )(\ )} \frac{\frac{\pi}{2}\left(\frac{\cos x-1}{x^2}\right)}{(\cos x)}$$

$$l = \left(\frac{\pi}{2}\right)\left(\frac{-1}{2}\right) = \frac{-\pi}{4}.$$

$$\{l\} = l - [l] = \frac{-\pi}{4} - \underbrace{\left[\frac{-\pi}{4}\right]}_{-1} = 1 - \frac{\pi}{4} \text{ Ans}$$

Q $\lim\limits_{x\to 0} x^2 \cos\underbrace{(\pi \underbrace{\sec^2 x}_{1})}_{\pi} \operatorname{cosec}(\pi \sec^2 x)$ $(0\times\infty)$

$$\lim_{x\to 0} \frac{x^2\left(\cos\left(\pi\sec^2 x\right)\right)}{\sin(\underbrace{\pi\sec^2 x}_{\pi})} \quad \left(\frac{0}{0}\right)$$

$$\lim_{x\to 0} \frac{x^2 \overbrace{\cos(\pi\sec^2 x)}^{-1}}{\frac{\sin(\pi - \pi\sec^2 x)}{(\pi-\pi\sec^2 x)} \cdot \pi(1-\sec^2 x)} = \lim_{x\to 0} \frac{x^2\cdot(\overset{-1}{\ }) \cos^2 x}{(\ \ )\,\pi(\cos^2 x - 1)}$$

$$= \lim_{x\to 0} \left(-\frac{1}{\pi}\right) \frac{x^2(-1)\cos^2 x}{(\sin^2 x)(1\ )}$$

Ans $\dfrac{1}{\pi}$

Q ** $\lim_{x \to 0} \left(\frac{\cos^{-1}(1-x)}{\sqrt{x}} \right)$

Sol^n (One sided limit)

M-1 Let $\cos^{-1}(1-x) = \theta$

$1 - x = \cos\theta$

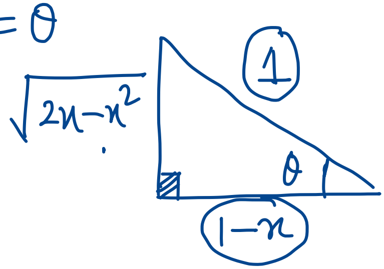
$x = 1 - \cos\theta.$

$$\lim_{\theta \to 0^+} \left(\frac{\theta}{\sqrt{1-\cos\theta}} \right) = \lim_{\theta \to 0^+} \frac{\theta}{\sqrt{2 \cdot \sin^2 \frac{\theta}{2}}}$$

$$** = \lim_{\theta \to 0^+} \frac{\cancel{\theta}}{\sqrt{2} \left(\frac{\sin \frac{\theta}{2}}{\frac{\theta}{2}} \right) \cdot \frac{\cancel{\theta}}{2}}$$

$$\frac{2}{\sqrt{2}} = \sqrt{2}.$$

M-2 $\cos^{-1}(1-x) = \theta$



$$\lim_{x \to 0^+} \left(\frac{\sin^{-1}\sqrt{2x-x^2}}{\sqrt{x}} \right) = \lim_{x \to 0^+} \underbrace{\left(\frac{\sin^{-1} M}{M} \right)}_{1} \cdot \sqrt{\frac{2x-x^2}{x}}$$

$1 \times \sqrt{2} = \sqrt{2}$ Ans

Q $\lim_{x\to 0} \dfrac{\sqrt{(1-\cos x)+\sqrt{(1-\cos x)+\sqrt{(1-\cos x)+\cdots\cdots\infty}}}-1}{x^2}$

$E = \sqrt{(1-\cos x) + E}$; $E \geq 0$

$E^2 - E - (1-\cos x) = 0 \Rightarrow E = \dfrac{1 \pm \sqrt{1+4(1-\cos x)}}{2}$

$E = \dfrac{1+\sqrt{5-4\cos x}}{2}$ or $E = \dfrac{1-\sqrt{5-4\cos x}}{2} \Rightarrow -ve$

XX (reject)

$$\lim_{x\to 0} \frac{\left(\dfrac{1+\sqrt{5-4\cos x}}{2} - 1\right)}{x^2} = \lim_{x\to 0} \frac{\sqrt{5-4\cos x}-1}{2x^2}$$

$$\lim_{x\to 0} \frac{(5-4\cos x-1)}{2x^2\left(\sqrt{5-4\cos x}+1\right)} = \lim_{x\to 0} \frac{\overset{2}{\cancel{4}}(1-\cos x)}{\cancel{2}x^2(\ \cdot\ )} = \frac{1}{2}$$

$\frac{1}{\cancel{2}}$

2

Ans

LIMIT OF EXPONENTIAL FUNCTIONS :

Note :-

$$\lim_{x\to\infty} a^x = \begin{cases} 0, & 0<a<1 \\ 1, & a=1 \\ \infty, & a>1 \end{cases}$$

$$a^x = e^{x\ln a}$$

Rem

(a) $a^x = 1 + \dfrac{x\,\ell na}{1!} + \dfrac{x^2\ell n^2 a}{2!} + \dfrac{x^3\ell n^3 a}{3!} + \ldots, a>0, a\neq 1, x\in\mathbb{R}$

* **(b)** $e^x = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \ldots, x\in\mathbb{R}$

$$\boxed{\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots}$$

(c) $\ell n(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \dfrac{x^4}{4} + \ldots$ for $-1 < x \le 1$

$$e^{2x} = 1 + \frac{2x}{1!} + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \text{------}$$

Rem

① $\displaystyle\lim_{x\to 0}\frac{a^x-1}{x} = \ln a \quad (a>0)$; ② $\displaystyle\lim_{x\to 0}\frac{e^x-1}{x} = 1$ and $\displaystyle\lim_{x\to\infty} x\left(e^{1/x}-1\right) = 1$

$x = \frac{1}{t}$

Rem

③ $\displaystyle\lim_{x\to 0}(1+x)^{\frac{1}{x}} = e = \lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x$

$x = \frac{1}{t}$

Rem

④ $\displaystyle\lim_{x\to 0}\frac{\ln(1+x)}{x} = 1$

① $\displaystyle \lim_{x\to 0} \frac{e^{\tan x} - e^{x}}{\tan x - x} = \lim_{x\to 0} \frac{e^{x}\left(e^{\tan x - x} - 1\right)}{(\tan x - x)}$

$= e^{0} \times 1 = 1.$

② $\displaystyle \lim_{x\to 0} \left(\frac{e^{x^2} - \cos x}{x^2}\right) = \lim_{x\to 0}\left(\frac{e^{x^2}-1}{x^2} + \frac{1-\cos x}{x^2}\right)$

$= 1 + \frac{1}{2} = \frac{3}{2}.$

(3) $\displaystyle \lim_{x\to 0} \left(\frac{e^{4x} - 1}{\tan x}\right) = \lim_{x\to 0} \underbrace{\left(\frac{e^{4x}-1}{4x}\right)}\cdot \frac{4\cancel{x}}{\left(\frac{\tan x}{x}\right)\cdot \cancel{x}} = 4.$

(4) $\displaystyle \lim_{x\to\infty} \frac{e^{1/x^2} - 1}{2\arctan x^2 - \pi} = \lim_{x\to\infty} \frac{e^{\left(\frac{1}{x^2}\right)} - 1}{2\underbrace{\left(\tan^{-1} x^2 - \frac{\pi}{2}\right)}_{-\cot^{-1} x^2}}.$

$\displaystyle = \lim_{x\to\infty} \frac{e^{\frac{1}{x^2}} - 1}{-2\cot^{-1} x^2}$

$x = \frac{1}{t}$

$\displaystyle \lim_{t\to 0^+} \left(\frac{e^{t^2}-1}{t^2}\right)\cdot \frac{t^2}{-2\cot^{-1}\left(\frac{1}{t^2}\right)}$

$\displaystyle \lim_{t\to 0^+} (\ )\left(\frac{-1}{2}\right)\underbrace{\left(\frac{t^2}{\tan^{-1} t^2}\right)}$

$1 \times \left(-\frac{1}{2}\right) \times 1 = -\frac{1}{2}$ Ans

5) $\lim_{h\to 0}\left(\frac{a^{x+h}+a^{x-h}-2a^{x}}{h^2}\right)$; $a>0$

$$a^{x}\cdot\lim_{h\to 0}\left(\frac{a^{h}+\frac{1}{a^{h}}-2}{h^2}\right)=a^{x}\cdot\lim_{h\to 0}\left(\frac{a^{2h}-2a^{h}+1}{h^2\cdot a^{h}}\right)$$

$$a^{x}\lim_{h\to 0}\underbrace{\left(\frac{a^{h}-1}{h}\right)^2}\cdot\frac{1}{a^{h}}.$$

$$a^{x}\cdot(\ln a)^2\cdot(1).$$

$$\underline{\underline{\text{limit}}}=\left(a^{x}\ln^2 a\right)$$

6) $\lim_{x\to a}\left(\frac{a^{x}-a^{a}}{x-a}\right)$; $\underline{\underline{a>0}}$ $\left(\frac{0}{0}\text{ form}\right)$

$$a^{a}\lim_{x\to a}\left(\frac{a^{\overbrace{x-a}^{M}}-1}{\underbrace{x-a}_{M}}\right)=a^{a}\cdot\ln a.=\ln(a)^{a^{a}}$$

Ans

Q $$\lim_{n\to\infty} \frac{1}{n}\left(\underbrace{1+e^{\frac{1}{n}}+e^{\frac{2}{n}}+e^{\frac{3}{n}}+\cdots+e^{\frac{n-1}{n}}}\right).$$

$$\lim_{n\to\infty} \frac{1}{n}\,\frac{\left(1\cdot\ 1-\left(e^{\frac{1}{\cancel{n}}}\right)^{\cancel{n}}\right)}{\left(1-e^{1/n}\right)}$$

$$n=\frac{1}{t}$$

$$\lim_{n\to\infty} \frac{1}{n}\left(\frac{1-e}{1-e^{1/n}}\right)$$

$$\lim_{t\to 0} \frac{(1-e)}{\left(\frac{1-e^t}{t}\right)} = \left(\frac{1-e}{-1}\right)$$

$$=(e-1)$$

Ans

7) $\lim\limits_{x\to 0} \dfrac{\ln(1+x)}{3^x-1} = \lim\limits_{x\to 0} \left(\dfrac{\ln(1+x)}{x}\right) \dfrac{\cancel{x}}{\left(\dfrac{3^x-1}{x}\right)\cdot \cancel{x}}$

$= 1 \times \dfrac{1}{\ln 3} = \log_3 e.$

8) $\lim\limits_{x\to 0} (1+\underbrace{2x}_{})^{5/x}$ → 1^∞ form

$\lim\limits_{x\to 0} \left((1+2x)^{\frac{1}{2x}}\right)^{2\cancel{x}\cdot\frac{5}{\cancel{x}}} = e^{10}.$

$(1+2x)^{\frac{1}{2x}} \to e$

9) $\lim\limits_{x\to 1} \underbrace{(1-x)}_{} \underbrace{\log_x 2}_{}$ $\quad 0\times\infty.$

$(1-x)\dfrac{\ln 2}{\ln x}$

<u>RHL:</u>
put $x=(1+h)$

$\lim\limits_{h\to 0} (\cancel{1}-(\cancel{1}+h))\cdot\dfrac{\ln 2}{\ln(1+h)} = \lim\limits_{h\to 0} -\dfrac{\ln 2}{\dfrac{\ln(1+h)}{(h)}} = -\ln 2$

<u>LHL:</u> put $x=1-h$

$\lim\limits_{h\to 0} (\cancel{1}-(\cancel{1}-h))\cdot\dfrac{\ln 2}{(-h)\times\dfrac{\ln(1-h)}{-h}}$

$= -\ln 2.$

$\boxed{\text{Limit} = -\ln 2}$

10) $\lim\limits_{\underbrace{x\to e}_{}} \dfrac{\ln x - 1}{x-e}$ $\quad 1 \to \ln e$ $\quad \left(\dfrac{0}{0} \text{ form}\right)$

put $x=e+h$

$\lim\limits_{h\to 0} \dfrac{\ln(e+h) - \ln e}{(\cancel{e}+h-\cancel{e})} = \lim\limits_{h\to 0} \dfrac{\ln\left(1+\frac{h}{e}\right)}{\frac{h}{e}} \cdot \dfrac{1}{e} = \dfrac{1}{e}$

<u>M-2</u> $\quad \underline{\ln x} = t \Rightarrow x = e^t.$

$\lim\limits_{t\to 1} \left(\dfrac{t-1}{e^t-e}\right) = \lim\limits_{t\to 1} \dfrac{1}{e}\left(\dfrac{t-1}{e^{t-1}-1}\right)$
$= 1/e.$

11) * $\lim_{x\to a}\left(\dfrac{x^x-a^x}{x-a}\right)$ $\left(\frac{0}{0}\text{ form}\right)$

Note
$x^x = e^{x\ln x}$
$a^x = e^{x\ln a}$.

$$\lim_{x\to a}\left(\frac{e^{x\ln x}-e^{x\ln a}}{x-a}\right) = \lim_{x\to a}\frac{e^{x\ln a}\left(e^{\overbrace{x\ln x - x\ln a}^{M}}-1\right)}{(x-a)}$$

$$= \lim_{x\to a} a^x\left(\frac{e^M-1}{M}\right)\cdot\left(\frac{x\ln x - x\ln a}{x-a}\right)$$

put $x=(a+h)$

$$\lim_{h\to 0}\left(a^{a+h}\right)\left(\quad\right)\left(\frac{(a+h)\ln(a+h)-(a+h)\ln a}{\cancel{a}+h-\cancel{a}}\right)$$

$$\lim_{h\to 0}\left(a^{a+h}\right)\left(\quad\right)\left(\frac{a\left(\ln\underbrace{(a+h)-\ln a}\right)+h\left(\ln(a+h)-\ln a\right)}{h}\right)$$

$$\lim_{h\to 0}\left(a^{a+h}\right)\left(\quad\right)\left(\frac{\cancel{a}\cdot\ln\left(1+\frac{h}{a}\right)}{\frac{h}{a}\cdot\cancel{a}}+\ln\left(1+\frac{h}{a}\right)\right)$$

$$a^a\cdot 1\cdot\left(1+\ln 1\right) = a^a.$$

Note :

If a function $f(x)$ has continuous derivatives up to $(n + 1)^{th}$ order, then this function can be expanded in the following fashion (the series is called MacLuarin Series

$$f(x) = f(0) + \frac{f'(0)}{1!}x + \frac{f''(0)}{2!}x^2 + \ldots\ldots + \frac{f'''(0)}{3!}x^3 + \ldots\ldots$$

$$f(x) = \sin x = 0 + x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots$$

$$\ldots\ldots$$

$$f(x) = \sin x$$

$$f'(x) = \underbrace{\cos x}$$

$$f''(x) = -\sin x$$

$$f'''(x) = -\cos x \checkmark$$

$$f''''(x) = \sin x$$

$$\vdots$$

To be Remembered :-

(a) $a^x = 1 + \frac{x\,\ell na}{1!} + \frac{x^2\ell n^2 a}{2!} + \frac{x^3\ell n^3 a}{3!} + \ldots, a > 0, a \neq 1, x \in \mathbb{R}$

(b) $e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots, x \in \mathbb{R}$

(c) $\ell n\left(1+x\right) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \text{ for } -1 < x \leq 1$

(d) $\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots, x \in \mathbb{R}$

(e) $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots, x \in \mathbb{R}$

(f) $\tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \ldots, x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

(g) $\tan^{-1} x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \ldots, x \in [-1, 1]$

(h) $\sin^{-1} x = x + \frac{1^2}{3!}x^3 + \frac{1^2.3^2}{5!}x^5 + \frac{1^2.3^2.5^2}{7!}x^7 + \ldots, x \in (-1, 1)$

(i) $\sec^{-1} x = 1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \frac{61x^6}{6!} + \ldots, |x| > 1$

(j) $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \ldots, n \in \mathbb{R}, x \in (-1, 1)$

Special Limits :—

Rem

(1) $\lim\limits_{x\to 0}\left(\dfrac{x-\sin x}{x^3}\right)=\dfrac{1}{6}$

(2) $\lim\limits_{x\to 0}\left(\dfrac{\tan x-x}{x^3}\right)=\dfrac{1}{3}$

(3) $\lim\limits_{x\to 0}\left(\dfrac{e^x-x-1}{x^2}\right)=\dfrac{1}{2}$

(4) $\lim\limits_{x\to 0}\left(\dfrac{e^x-e^{-x}-2x}{x^3}\right)=\dfrac{1}{3}$.

(5) $\lim\limits_{x\to 0}\left(\dfrac{x-\ln(1+x)}{x^2}\right)=\dfrac{1}{2}$.

$\dfrac{1}{2}\cdot\dfrac{1}{2^6}=\dfrac{1}{2^7}$

① $\lim\limits_{x\to 0}\dfrac{e^{x^3}-1-x^3}{\sin^6 2x} = \lim\limits_{x\to 0}\underbrace{\left(\dfrac{e^{x^3}-x^3-1}{x^6}\right)}\cdot\dfrac{x^6}{\underbrace{\left(\dfrac{\sin 2x}{2x}\right)^6}\cdot(2x)^6}$

② $\lim\limits_{x\to\infty}\left(x-x^2\ \ell n\left(1+\dfrac{1}{x}\right)\right)$

$x=\dfrac{1}{t}$ $\qquad \lim\limits_{t\to 0}\left(\dfrac{1}{t}-\dfrac{1}{t^2}\ln(1+t)\right)=\lim\limits_{t\to 0}\left(\dfrac{t-\ln(1+t)}{t^2}\right)$

$1/2$

* (3) $\lim\limits_{x\to 0} \dfrac{(1+x)^{1/x} - e}{x}$

$$(1+x)^{\frac{1}{x}} = e^{\frac{1}{x}\ln(1+x)}$$

$$\lim_{x\to 0} \left(\frac{e^{\frac{1}{x}\ln(1+x)} - e}{x} \right) = e \cdot \lim_{x\to 0} \left(\frac{e^{\overbrace{\frac{\ln(1+x)}{x} - 1}^{M}} - 1}{x} \right)$$

$$e \lim_{x\to 0} \left(\frac{e^{M} - 1}{M} \right) \cdot \left(\frac{\frac{\ln(1+x)}{x} - 1}{x} \right)$$

$$e \lim_{x\to 0} \left(\quad \right) \left(\frac{\ln(1+x) - x}{x^2} \right)$$

$$e \times -\frac{1}{2} = \frac{-e}{2} \quad \text{Ans}$$

4) $\lim\limits_{x\to 0} \dfrac{\tan^2 x - x^2}{x^2 \underbrace{\tan^2 x}}$ $= \lim\limits_{x\to 0} \dfrac{(\tan x - x)}{x^3} \dfrac{(\tan x + x)}{x}$

$$= \left(\frac{1}{3}\right)(1+1) = \frac{2}{3}.$$

5) $$\lim_{x\to 0}\left(\frac{e^{x}-e^{-x}-2x}{\sin x - x}\right)$$

$$\lim_{x\to 0}\frac{\left(\frac{e^{x}-e^{-x}-2x}{x^{3}}\right)}{\left(\frac{\sin x - x}{x^{3}}\right)} = \frac{1/3}{-1/6} = -2$$

Q If $\lim_{x\to 0} \dfrac{10^x - 2^x - 5^x + 1}{\ln(\sec x)}$ is equal to $(\ln k)(\ln w)$, find least value of $(k + w)$.

(where $k, w \in N$)

Soln

$$\lim_{x\to 0} \frac{\dfrac{(2^x-1)}{x} \cdot \dfrac{(5^x-1)}{x} \cdot x^2}{-\ln \cos x} = \lim_{x\to 0} \frac{(\ )(\ )\ x^2}{-\ln(1+\underbrace{\cos x - 1}_{})}$$

$$-\lim_{x\to 0} \frac{(\ )(\ )\ \cancel{x^2}}{\underbrace{\left(\dfrac{\ln(1+\cos x-1)}{(\cos x-1)}\right)}_{} \cdot \dfrac{(\cos x-1)}{x^2} \cdot \cancel{x^2}}$$

$$\frac{-(\ln 2)(\ln 5)}{-1/2} = 2(\ln 2)(\ln 5) = (\ln 2^2)(\ln 5)$$

$$= (\ln 4)(\ln 5)$$

$$\therefore (k+w)_{least} = 9$$

Q If $p < 0 < q$ and $p, q \in I$ such that

$$\lim_{n\to\infty}\lim_{x\to 0}\frac{(1+x)^{\frac{1}{x}}+(1+2x)^{\frac{1}{2x}}+(1+3x)^{\frac{1}{3x}}+\dots\dots+(1+nx)^{\frac{1}{nx}}-\overset{(e+e+e+\dots+e)}{ne}}{n^2x}$$ has the value equal to $\left(\frac{p}{q}\right)e$

then find the least value of $(p+q)$. [Ans. 3]

Sol^n

$$\lim_{n\to\infty}\lim_{x\to 0}\frac{\left(e^{\frac{1}{x}\ln(1+x)}-e\right)+\left(e^{\frac{1}{2x}\ln(1+2x)}-e\right)+\dots+\left(e^{\frac{\ln(1+nx)}{nx}}-e\right)}{n^2x}$$

$$e\lim_{n\to\infty}\lim_{x\to 0}\frac{\left(e^{\frac{\ln(1+x)}{x}-1}-1\right)+\left(e^{\frac{\ln(1+2x)}{2x}-1}-1\right)+\dots+\left(e^{\frac{\ln(1+nx)}{nx}-1}-1\right)}{n^2x}$$

$$e\lim_{n\to\infty}\lim_{x\to 0}\frac{\left[\underbrace{\left(\frac{e^{M_1}-1}{M_1}\right)}_{1}\cdot\underbrace{\left(\frac{\ln(1+x)-x}{x^2}\right)}_{-1/2}+2\underbrace{\left(\frac{e^{M_2}-1}{M_2}\right)}_{1}\cdot\underbrace{\left(\frac{\ln(1+2x)-2x}{2^2x^2}\right)}_{}+\dots+n\left(\frac{e^{M_n}-1}{M_n}\right)\left(\frac{\ln(1+nx)-nx}{n^2x^2}\right)\right]}{n^2}$$

$$-\frac{e}{2}\lim_{n\to\infty}\left(\frac{1+2+\dots+n}{n^2}\right)=-\frac{e}{2}\lim_{n\to\infty}\frac{n(n+1)}{2n^2}=-\frac{e}{4}=\left(\frac{p}{q}\right)e$$

$(p+q)_{least}=3$ Ans. $p=-1;\ q=4$

Q. If $\lim\limits_{x \to 0} \dfrac{A\cos x + Bx\sin x - 5}{x^4}$ exists & finite. Find A & B and also the limit.

Sol^n

$$\lim_{x \to 0} \frac{A\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + Bx\left(x - \frac{x^3}{3!} + \cdots\right) - 5}{x^4}$$

$$\lim_{x \to 0} \frac{(A-5) + \left(-\frac{A}{2} + B\right)x^2 + \left(\frac{A}{24} - \frac{B}{6}\right)x^4 + \cdots\cdots}{x^4}$$

$A - 5 = 0$ and $-\frac{A}{2} + B = 0$

$$L = \frac{A}{24} - \frac{B}{6}.$$

Q Find the values of a, b & c so that $\lim\limits_{x \to 0} \dfrac{ae^{x} - b\cos x + ce^{-x}}{x.\sin x} = 2$

Sol^n

$$\lim_{x \to 0} \frac{a\left(1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) - b\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \cdots\right) + c\left(1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} \cdots\right)}{x\left(x - \frac{x^3}{3!} + \cdots\right)} = 2$$

$$\lim_{x \to 0} \frac{\underbrace{(a-b+c)} + \underbrace{(a-c)}x + \left(\frac{a}{2} + \frac{b}{2} + \frac{c}{2}\right)x^2 + \cdots}{x^2\left(1 - \frac{x^2}{6} + \cdots\right)} = 2.$$

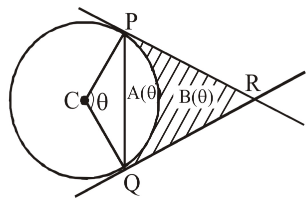
$\underbrace{a - b + c = 0}$ and $\underbrace{a - c = 0}$.

$$\frac{a + b + c}{2} = 2.$$

Q Let $f(x)=\dfrac{4+\sin 2x+A\sin x+B\cos x}{x^2}$. If $\lim\limits_{x\to 0} f(x)$ exists and finite find A and B and the limit.

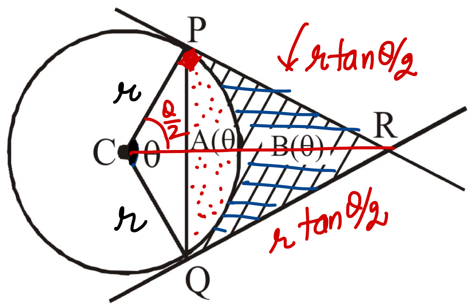
$$\underbrace{[\text{Ans}: A=-2,\ B=-4 \text{ and } l=2]}$$

Q An arc PQ of a circle subtends a central angle θ as shown. Let A(θ) be the area between the chord PQ and the arc PQ. Let B(θ) be the area between the tangent lines PR and QR and the arc PQ.



Find $\lim\limits_{\theta\to 0}\dfrac{A(\theta)}{B(\theta)}$

Sol^n

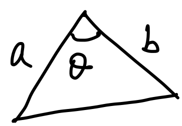


Area of sector CQPC $= \frac{1}{2}r^2\theta$.

Area of Δ CPQ $= \frac{1}{2} r.r.\sin\theta$.

$$A(\theta) = \frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin\theta$$

$$\boxed{A(\theta) = \frac{1}{2}r^2(\theta-\sin\theta).}$$



$\Delta = \frac{1}{2}ab\sin\theta.$

Area of quad CQRP $= \cancel{2}\cdot\frac{1}{\cancel{2}}\cdot\left(r\tan\frac{\theta}{2}\cdot r\right)$

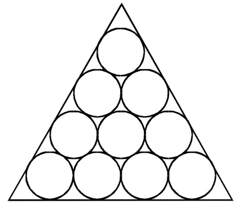
$$= r^2\tan\frac{\theta}{2}.$$

$$B(\theta) = r^2\tan\frac{\theta}{2} - \frac{1}{2}r^2\theta = \boxed{r^2\left(\tan\frac{\theta}{2}-\frac{\theta}{2}\right) = B(\theta)}$$

$$\lim_{\theta\to 0}\frac{A(\theta)}{B(\theta)} = \lim_{\theta\to 0}\left(\frac{1}{2}\right)\frac{(\theta-\sin\theta)}{\theta^3}\cdot\frac{\cancel{\theta^3}}{\dfrac{\tan\theta/2-\theta/2}{(\theta/2)^3}\cdot\left(\frac{\cancel{\theta}}{2}\right)^3} = \frac{1}{2}\cdot\frac{1}{6}\cdot\frac{8}{1/3}$$

$$= 2 \text{ Ans}$$

Q. Suppose that circle of equal diameter are packed tightly in n rows inside an equilateral triangle. (The figure illustrates the case n=4.) If A is the area of the triangle and A_n is the total area occupied by the circles in n rows then $\lim\limits_{n\to\infty} \frac{A_n}{A}$ equals

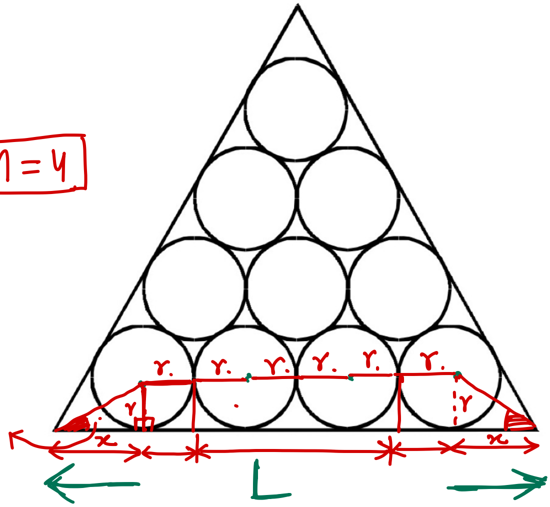


(A) $\frac{\pi}{\sqrt{3}}$ (B) $\frac{\pi\sqrt{3}}{6}$ (C) $\frac{\pi}{2\sqrt{3}}$ (D) $\frac{\pi}{6}$

Sol^n

$$A = \frac{\sqrt{3}}{4}L^2 \quad -(1)-$$

$\boxed{n=4}$



For $n = n$

total no. of circle $= 1+2+\cdots+n$

$= \frac{n(n+1)}{2}$

30°

$\frac{\cancel{r}}{x} = \tan 30°$

$\boxed{\sqrt{3}r = x}$

$L = x + 6r + x$

$L = 2x + 2(2r) + r + r.$

$(n-2)2r + r + r + 2x = L$

$(n-2)2r + 2r + 2\sqrt{3}r = L$

$$r = \frac{L}{(n-1)2+2\sqrt{3}} \Rightarrow A_n = \pi \cdot \left(\frac{L}{(n-1)2+2\sqrt{3}}\right)^2 \times \frac{n(n+1)}{2}$$

$$\lim_{n\to\infty}\frac{A_n}{A} = \lim_{n\to\infty} \frac{\pi \cancel{L^2}\, \frac{n(n+1)}{2}}{((n-1)2+2\sqrt{3})^2}\left(\frac{4}{\sqrt{3}\cancel{L^2}}\right) = \frac{2\pi}{4\sqrt{3}}$$

$$= \frac{\pi}{2\sqrt{3}}.$$

Generalised form of 1^∞ :-

* If $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) \to \infty$.

then

$$\boxed{\lim_{x\to a} (f(x))^{g(x)} = e^{\lim_{x\to a} g(x)(f(x)-1)}}.$$

Rem

Proof:

$$\lim_{x\to a} \left(1 + \underbrace{f(x)-1}_{x}\right)^{\frac{1}{f(x)-1}\cdot(f(x)-1)\cdot g(x)} = \lim_{x\to a} e^{g(x)(f(x)-1)}$$

(H.P)

① $\lim_{n\to\infty} \left(5^{\frac{1}{n}} + 3^{\frac{1}{n}} - 1\right)^n$ $\to$ (1^∞ form) $\to$ $e^{L} = e^{\ln 15} = 15$ Ans

$$L = \lim_{n\to\infty} n\left(5^{\frac{1}{n}} + 3^{\frac{1}{n}} - 1 - 1\right) = \lim_{n\to\infty} \frac{\left(5^{\frac{1}{n}}-1\right)}{1/n} + \frac{\left(3^{\frac{1}{n}}-1\right)}{1/n}$$

$$L = \ln 5 + \ln 3$$

② $$\lim_{x\to 0}\left(\frac{5}{2+\sqrt{9+x}}\right)^{\operatorname{cosec} x} = e^{L} = e^{-1/30} \quad \text{Ans}$$

$$L=\lim_{x\to 0} \operatorname{cosec} x\left(\frac{5}{2+\sqrt{9+x}}-1\right)$$

$$L=\lim_{x\to 0} \frac{5-2-\sqrt{9+x}}{\left(2+\sqrt{9+x}\right)\sin x} = \lim_{x\to 0} \frac{3-\sqrt{9+x}}{\left(2+\sqrt{9+x}\right)\sin x}$$

$$L=\lim_{x\to 0} \frac{\cancel{9}-(\cancel{9}+x)}{\left(2+\sqrt{9+x}\right)\left(3+\sqrt{9+x}\right)\left(\frac{\sin x}{x}\right)\cdot(x)} \Rightarrow L=\frac{-1}{30}.$$

③ $$\lim_{x\to 0}(\cos mx)^{\frac{n}{x^2}},\ m, n \in N = e^{L} = e^{-\frac{m^2 n}{2}} \quad \text{Ans}$$

$$L=\lim_{x\to 0} \frac{n}{x^2}\left(\frac{\cos mx-1}{(mx)^2}\right)\cdot(mx)^2$$

$$L=-\frac{1}{2}\cdot n\cdot m^2$$

4) $$\lim_{x\to\infty}\left(\frac{x^2+2x-1}{2x^2-3x-2}\right)^{\frac{2x+1}{x-1}}$$

$$\lim_{x\to\infty}\frac{\cancel{x}\left(2+\frac{1}{x}\right)}{\cancel{x}\left(1-\frac{1}{x}\right)}$$

$$\boxed{\text{Power} = 2}$$

$$\text{Base} = \lim_{x\to\infty}\frac{\cancel{x^2}\left(1+\frac{2}{x}-\frac{1}{x^2}\right)}{\cancel{x^2}\left(2-\frac{3}{x}-\frac{2}{x^2}\right)}$$

$$\text{Base} = \frac{1}{2}$$

$$\left(\frac{1}{2}\right)^2 = \frac{1}{4}$$

Ans

5) $$\lim_{x\to\infty}\left(\frac{x+2}{2x-1}\right)^{x^2} = 0$$

Ans

Power $\longrightarrow \infty$.

$$\text{Base} = \lim_{x\to\infty}\frac{\cancel{x}\left(1+\frac{2}{x}\right)}{\cancel{x}\left(2-\frac{1}{x}\right)}$$

$$\text{Base} = \frac{1}{2}$$

$$\left(\frac{1}{2}\right)^\infty = 0$$

6) $$\lim_{x\to\infty}\left(\frac{1^{1/x}+2^{1/x}+3^{1/x}+\dots\dots\dots+n^{1/x}}{n}\right)^{nx} \quad n\in N$$

$$= e^{L} = e^{\ln(n!)} = \underline{\underline{n!}}$$ Ans

$$L=\lim_{x\to\infty} nx\left(\frac{1^{\frac{1}{x}}+2^{\frac{1}{x}}+\dots+n^{\frac{1}{x}}}{n}-1\right)$$

$$x=\frac{1}{t}$$

$$L=\lim_{t\to 0}\frac{\cancel{n}}{t}\left(\frac{1^{t}+2^{t}+\dots+n^{t}-n}{\cancel{n}}\right)$$

$1+1+1+\dots+1$

$$L=\lim_{t\to 0}\left(\frac{1^{t}-1}{t}\right)+\left(\frac{2^{t}-1}{t}\right)+\dots+\left(\frac{n^{t}-1}{t}\right)$$

$$L=\ln 1+\ln 2+\dots+\ln n=\ln(1\cdot 2\cdot 3\dots n)=\ln(n!)$$

7) $$\lim_{n\to\infty}\left(\frac{a-1+\sqrt[n]{b}}{a}\right)^{n} \quad a>0,\ b>0,\ n\in N\geq 2$$

$$\sqrt[n]{b}=(b)^{\frac{1}{n}}$$

e^{L}

$$L=\lim_{n\to\infty} n\left(\frac{a-1+(b)^{\frac{1}{n}}}{a}-1\right)=\lim_{n\to\infty} n\left(\frac{\cancel{a}-1+(b)^{1/n}-\cancel{a}}{a}\right)$$

$$L=\lim_{n\to\infty}\left(\frac{b^{\frac{1}{n}}-1}{\frac{1}{n}}\right)\cdot\frac{1}{a}=\frac{1}{a}(\ln b)$$

8)

$$\lim_{x\to 0}\left(\sin^2\frac{\pi}{2-ax}\right)^{\sec^2\frac{\pi}{2-bx}} = e^{L} = e^{-a^2/b^2}$$

Ans

$$L = \lim_{x\to 0} \sec^2\left(\frac{\pi}{2-bx}\right)\left(\sin^2\left(\frac{\pi}{2-ax}\right) - 1\right)$$

$$L = \lim_{x\to 0} -\left(\frac{\cos\left(\frac{\pi}{2-ax}\right)}{\cos\left(\frac{\pi}{2-bx}\right)}\right)^2$$

$$L = -\lim_{x\to 0}\left(\frac{\frac{\sin\left(\frac{\pi}{2}-\frac{\pi}{2-ax}\right)}{\left(\frac{\pi}{2}-\frac{\pi}{2-ax}\right)}\left(\frac{\pi}{2}-\frac{\pi}{2-ax}\right)}{\frac{\sin\left(\frac{\pi}{2}-\frac{\pi}{2-bx}\right)}{\left(\frac{\pi}{2}-\frac{\pi}{2-bx}\right)}\left(\frac{\pi}{2}-\frac{\pi}{2-bx}\right)}\right)^2$$

$$L = -\lim_{x\to 0}\left(\frac{(\;)}{(\;)}\;\frac{\frac{\cancel{\pi}(\cancel{2}-ax-\cancel{2})}{\cancel{2}(2-ax)}}{\frac{\cancel{\pi}(\cancel{2}-bx-\cancel{2})}{\cancel{2}(2-bx)}}\right)^2 \Rightarrow L = \frac{-a^2}{b^2}$$

Q $\lim\limits_{x\to\infty} x^2 \sin\left(\ln\sqrt{\cos\frac{\pi}{x}}\right)$ $(0 \times \infty)$

$$\lim_{x\to\infty} x^2 \sin\left(\overbrace{\frac{1}{2}\ln\left(\cos\frac{\pi}{x}\right)}^{M}\right)$$

$$\lim_{x\to\infty} x^2 \left(\frac{\sin M}{M}\right)\cdot \frac{1}{2}\cdot \ln\left(\cos\frac{\pi}{x}\right)$$

$$\lim_{x\to\infty} x^2 \left(\ \right) \frac{1}{2} \frac{\ln\left(1+\overbrace{\cos\frac{\pi}{x}-1}\right)}{\left(\cos\frac{\pi}{x}-1\right)} \cdot \left(\cos\frac{\pi}{x}-1\right)$$

$$\frac{1}{2}\lim_{x\to\infty} \left(\ \right)\left(\ \right)\left(\frac{\cos\left(\frac{\pi}{x}\right)-1}{\left(\frac{\pi}{x}\right)^2}\right)\cdot\left(\frac{\pi}{\cancel{x}}\right)^2\cdot \cancel{x^2}$$

$$\frac{1}{2}(1)(1)\left(-\frac{1}{2}\right)\left(\pi^2\right) \Longrightarrow -\frac{\pi^2}{4} \text{ Ans}$$

Q Let $\ell = \lim_{x\to\infty} \frac{a^x - a^{-x}}{a^x + a^{-x}} (a > 0)$, then

(A) $\ell = 1\ \forall\ a > 0$ (B) $\ell = -1\ \forall\ a \in (0, 1)$ (C) $\ell = 0$, if $a = 1$ (D) $\ell = 1\ \forall\ a > 1$

Sol^n

C-I $a \in (0,1)$

as $x \to \infty$; $a^x \to 0$.

as $x \to \infty$; $a^{-x} \to \infty$.

$$\ell = \lim_{x\to\infty} \frac{\cancel{a^{-x}}\left(\frac{a^x}{a^{-x}} - 1\right)}{\cancel{a^{-x}}\left(\frac{a^x}{a^{-x}} + 1\right)}$$

$$\boxed{\ell = -1}$$

C-II $a = 1$

as $x \to \infty$; $a^x = 1$

$a^{-x} = 1$.

$$\ell = \frac{1-1}{1+1} = 0.$$

C-III $a > 1$

as $x \to \infty$; $a^x \to \infty$

$a^{-x} \to 0$.

$$\lim_{x\to\infty} \frac{\cancel{a^x}\left(1 - \frac{a^{-x}}{a^x}\right)}{\cancel{a^x}\left(1 + \frac{a^{-x}}{a^x}\right)}$$

$$\boxed{\ell = 1}$$

Imp LIMITS OF FUNCTIONS HAVING BUILT IN LIMIT WITH THEM :

Q1 If $f(x)=\lim_{n\to\infty}\frac{\tan \pi x^2+(x+1)^n \sin x}{x^2+(x+1)^n}$ then $\lim_{x\to 0} f(x)$?

Sol^n

$$f(x) \to \begin{cases} \frac{\tan \pi x^2}{x^2} & ; x \in (0^-, 0) \\ 0 & ; x = 0. \\ \sin x & ; x \in (0, 0^+) \end{cases}$$

$$\lim_{x\to 0} f(x) \begin{cases} \xrightarrow{LHL} \lim_{x\to 0} \pi\left(\frac{\tan \pi x^2}{\pi x^2}\right) = \pi \\ \xrightarrow{RHL} \lim_{x\to 0} (\sin x) = 0 \end{cases}$$

Limit dne

Q If $f(x) = \underbrace{\lim_{n\to\infty}} \dfrac{\cos\pi x - x^{2n}\sin(x-1)}{1+x^{2n+1}-x^{2n}}$, find $\lim_{x\to 1} f(x)$?

Sol^n

$$f(x) \longrightarrow \begin{cases} \cos\pi x & ; x \in (1^-, 1) \\ -1 & ; x = 1 \\ -\dfrac{\sin(x-1)}{(x-1)} & ; x \in (1, 1^+) \end{cases}$$

$$\lim_{n\to\infty} \frac{\cancel{x^{2n}}\left(\dfrac{\cos\pi x}{x^{2n}} - \sin(x-1)\right)}{\cancel{x^{2n}}\left(\dfrac{1}{x^{2n}} + x - 1\right)}$$

$$LHL = \lim_{x\to 1}(\cos\pi x) = -1$$

$$RHL = \lim_{x\to 1} -\frac{\sin(x-1)}{(x-1)} = -1$$

$$\boxed{\text{Limit} = -1}$$

Q

Let $f(x) = \lim_{n\to\infty} \dfrac{\cos\pi x - e^{-nx}\sin(x-1)}{1+e^{-nx}\cdot(x-1)}$, then

(A) $f(1) = -1$ (B) $f(0^+) = 1$ (C) $f(0^-) = -\sin 1$.

(D) $f(1) = 1$.

Sol^n

if $x > 0$ then $\lim_{n\to\infty} e^{-nx} \longrightarrow 0$. *

if $x < 0$ then $\lim_{n\to\infty} e^{-nx} \longrightarrow \infty$.

$$\boxed{f(1) = -1}$$

$$f(x) = \begin{cases} -\dfrac{\sin(x-1)}{(x-1)} \; ; & x\in(0^-,0) \\ \cos(\pi x) \; ; & x\in(0,0^+) \end{cases}$$

Q The value of $\sqrt{\frac{1}{2}}\cdot\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\cdot\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}\cdot$ ....∞ is

Soln

$$\frac{1}{\sqrt{2}} = \cos\theta \;; \quad \theta = \frac{\pi}{4} \quad *$$

$$\cos\theta \cdot \sqrt{\frac{1}{2}+\frac{1}{2}\cos\theta}\;\sqrt{\quad} \cdots\cdots \infty$$

$$\cos\theta \cdot \sqrt{\cancel{\frac{1}{2}}\left(\cancel{1}+\cancel{2}\cos^2\frac{\theta}{2}-\cancel{1}\right)}\;\sqrt{\quad} \cdots\cdots \infty$$

$$\cos\frac{\theta}{2^0}\cdot\cos\frac{\theta}{2}\cdot\cos\frac{\theta}{2^2}\cdot\cos\frac{\theta}{2^3}\cdots\cos\left(\frac{\theta}{2^{n-1}}\right)\cdot$$

$$\lim_{n\to\infty}\frac{\sin 2\theta\cdot}{\underbrace{2^n}\dfrac{\sin\left(\frac{\theta}{2^{n-1}}\right)}{\left(\frac{\theta}{2^{n-1}}\right)}\cdot\underbrace{\left(\frac{\theta}{2^{n-1}}\right)}} = \frac{\sin 2\theta}{2\theta}$$

$$= \frac{1}{2\cdot\left(\frac{\pi}{4}\right)}$$

$$= \frac{2}{\pi} \text{ Ans}$$

Q If $\lim_{n \to \infty} \dfrac{1}{\left(\frac{3}{\pi}\tan^{-1} 2x\right)^{2n} + 5} = 0$ then find range of x ?

Sol^n

$$\left|\frac{3}{\pi}\tan^{-1} 2x\right| > 1.$$

$$|\tan^{-1} 2x| > \frac{\pi}{3}.$$

$$2x > \tan\frac{\pi}{3} \quad \text{OR} \quad 2x < -\tan\frac{\pi}{3}$$

$$2x > \sqrt{3} \quad \text{OR} \quad 2x < -\sqrt{3}$$

$$x \in \left(-\infty, -\frac{\sqrt{3}}{2}\right) \cup \left(\frac{\sqrt{3}}{2}, \infty\right)$$ Ans

Q

If $\ell = \lim_{n\to\infty} \sum_{r=0}^{n} \frac{2^r}{5^{2^r}+1}$, then 24ℓ ,is

Sol^n

$2^r = t$ ✓

$2t = 2^{r+1}$

$$\sum_{r=0}^{n} \left(\frac{t}{5^t+1}\right)\cdot\left(\frac{5^t-1}{5^t-1}\right)$$

$$\sum_{r=0}^{n} \left(\frac{t(5^t+1-2)}{(5^t+1)(5^t-1)}\right)$$

$$\sum_{r=0}^{n} \left(\frac{t}{5^t-1} - \frac{2t}{(5^{2t}-1)}\right)$$

$$\sum_{r=0}^{n} \left(\frac{2^r}{5^{2^r}-1} - \frac{2^{r+1}}{5^{2^{r+1}}-1}\right)$$

$$\lim_{n\to\infty} \left(\frac{1}{4} - \frac{2^{n+1}}{5^{2^{n+1}}-1}\right) = \frac{1}{4} \text{ Ans}$$

If $\lim_{x\to 0} \frac{(\tan x)^n - x^n}{x^m} = L$, where m,n $\in$ N, then -

(A) If $m = n + 2$ then $L = \frac{n}{3}$

(B) If $m < n + 2$ then $L = 0$

(C) If $m > n + 2$ then L does not exist

(D) If $m > n + 2$ then $L = 0$

Sol^n

$$(a^n - b^n) = (a-b)(a^{n-1} + a^{n-2}\cdot b + \cdots + b^{n-1})$$

$$\lim_{x\to 0} \left(\frac{(\tan x - x)}{x^3}\right)\frac{\left((\tan x)^{n-1} + x\cdot(\tan x)^{n-2} + \cdots + x^{n-1}\right)}{x^{m-3}}$$

$$\lim_{x\to 0} \underbrace{\left(\frac{\tan x - x}{x^3}\right)}\underbrace{\left(x^{\overbrace{n-m+2}}\right)}\underbrace{\left(\left(\frac{\tan x}{x}\right)^{n-1} + \left(\frac{\tan x}{x}\right)^{n-2} + \cdots + \frac{x^{n-1}}{x^{n-1}}\right)}$$

Sandwich theorem :----

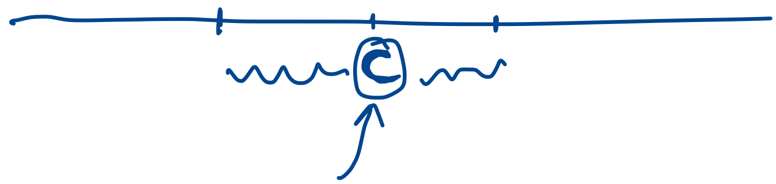
$\leq \quad <$

$< \quad \leq$

$$\lim_{x \to c} f(x)$$

$$h(x) \leq f(x) \leq g(x)$$

$< \quad <$



$$\lim_{x \to c} h(x) = L$$

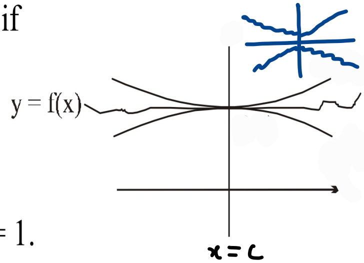
$$\lim_{x \to c} g(x) = L$$

Sandwich / Squeeze play theorem :-

Statement: If f, g and h are 3 functions such that $g(x) \le f(x) \le h(x)$ for all x in some interval containing the point x = c, and if

$$\underset{x\to c}{\text{Limit}}\ g(x) = \underset{x\to c}{\text{Limit}}\ h(x) = L$$

then $\underset{x\to c}{\text{Limit}}\ f(x) = L$



From the figure note that $\underset{x\to 0}{\text{Limit}}\ f(x) = 1$.

Note: (i) the quantity c may be a finite number, $+\infty$ or $-\infty$. Similarly L may be finite number, $+\infty$ or $-\infty$.

Q

Evaluate $\underset{x\to -\infty}{\text{Lim}} \dfrac{5x^2 - \sin 3x}{x^2+10}$ using sandwich theorem?

$$-1 \le -\sin 3x \le 1$$

$$\frac{5x^2-1}{x^2+10} \le \frac{5x^2-\sin 3x}{x^2+10} \le \frac{5x^2+1}{x^2+10}.$$

* $g(x) \le f(x) \le h(x)$

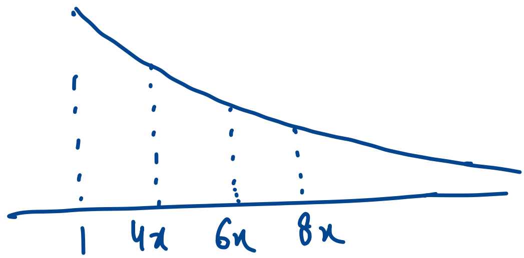
* $\lim\limits_{x\to(-\infty)} \left(\dfrac{5x^2-1}{x^2+10}\right) = 5$; $\lim\limits_{x\to(-\infty)} \left(\dfrac{5x^2+1}{x^2+10}\right) = 5$

Q Let $f:(1,\infty)\rightarrow(0,\infty)$ be a continuous decreasing function with $\underbrace{\lim\limits_{x\to\infty}\frac{f(4x)}{f(8x)}=1}$.

Then $\lim\limits_{x\to\infty}\frac{f(6x)}{f(8x)}$ is equal to

(A) $\frac{4}{8}$ (B) $\frac{4}{6}$ (C) $\frac{6}{8}$ (D) 1 ✓

$$f(8x)\le f(6x)\le f(4x)$$



$$\frac{f(8x)}{f(8x)}\le\frac{f(6x)}{f(8x)}\le\frac{f(4x)}{f(8x)}$$

$$\underbrace{1}\le\frac{f(6x)}{f(8x)}\le\underbrace{\frac{f(4x)}{f(8x)}}$$

Q The value of the limit $\lim\limits_{x\to 0}\frac{x}{a}\left[\frac{b}{x}\right]$ $(a\ne 0)$ (where [ ] denotes the greatest integer function) is equal to

(A) a (B) b (C) b/a ✓ (D) 1 – (b/a)

Solⁿ We know,

$$x-1<[x]\le x$$

$$\frac{b}{x}-1<\left[\frac{b}{x}\right]\le\frac{b}{x}$$

$$\frac{x}{a}\left(\frac{b}{x}-1\right)<\frac{x}{a}\left[\frac{b}{x}\right]\le\frac{\cancel{x}}{a}\cdot\frac{b}{\cancel{x}}$$

$$\left(\frac{b}{a}-\frac{x}{a}\right)<f(x)\le\frac{b}{a}$$

Q $\lim_{n\to\infty} \underbrace{\left(\frac{n}{n^2+1} + \frac{n}{n^2+2} + \frac{n}{n^2+3} + \cdots + \frac{n}{n^2+n} \right)}_{f(n)} = 1$

Ans

Greatest term $= \dfrac{n}{n^2+1}$

Least term $= \dfrac{n}{n^2+n}$

$$\begin{cases} \dfrac{n}{n^2+1} = \dfrac{n}{n^2+1} \checkmark \\ \dfrac{n}{n^2+1} > \dfrac{n}{n^2+2} \checkmark \\ \dfrac{n}{n^2+1} > \dfrac{n}{n^2+3} \checkmark \\ \vdots \qquad \vdots \\ \dfrac{n}{n^2+1} > \dfrac{n}{n^2+n} \checkmark \end{cases}$$

$$\frac{n^2}{n^2+1} > f(n)$$

$$\frac{n}{n^2+n} < \frac{n}{n^2+1}$$

$$\frac{n}{n^2+n} < \frac{n}{n^2+2}$$

$$\vdots$$

$$\frac{n}{n^2+n} = \frac{n}{n^2+n}$$

$$\frac{n^2}{n^2+n} < f(n)$$

$$\boxed{\frac{n^2}{n^2+n} < f(n) < \frac{n^2}{n^2+1}}$$

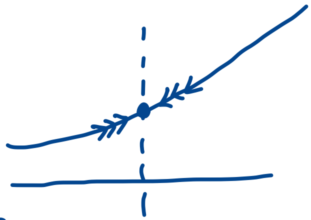
Q $\lim_{n\to\infty}\left(\frac{1}{1+n^2}+\frac{2}{2+n^2}+\frac{3}{3+n^2}+\cdots+\frac{n}{n+n^2}\right)$

Continuity of function

FORMULATIVE DEFINITION OF CONTINUITY :

A function $f(x)$ is said to be continuous at $x = a$, if $\lim\limits_{x \to a} f(x)$ exists and is equal to $f(a)$. Symbolically $f(x)$ is continuous at $x = a$ if $\lim\limits_{h \to 0} f(a-h) = \lim\limits_{h \to 0} f(a+h) = f(a) =$ finite quantity.

i.e. $\text{LHL}\big|_{x=a} = \text{RHL}\big|_{x=a} =$ value of $f(x)\big|_{x=a} =$ finite quantity.

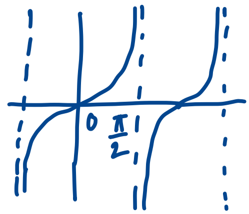


Note:

We talk about continuity of functions in $x = a$ their respective domain.

eg: $f(x) = \tan x$.

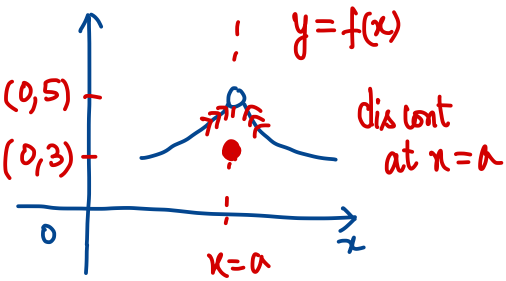
Continuous fns in its domain.

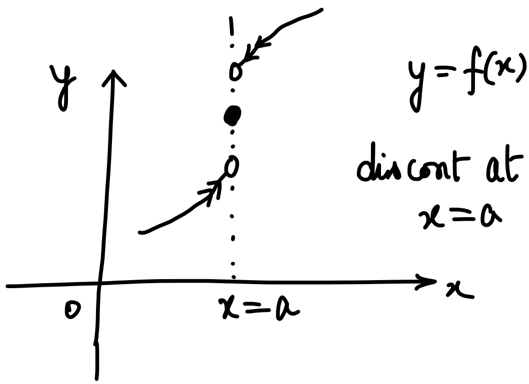


** Point functions are treated as continuous functions.

eg: $f(x) = \sqrt{x} + \sqrt{-x}$; $D_f \in \{0\}$

Continuous fns.





CONTINUITY IN AN INTERVAL :

(a) A function $f(x)$ is said to be continuous in (a, b) if $f(x)$ is continuous at each & every point $\in (a, b)$.

(b) A function $f(x)$ is said to be continuous in a closed interval [a, b] if :

(i) $f(x)$ is continuous in the open interval (a, b) &

(ii) $f(x)$ is right continuous at 'a' i.e. $\lim_{x\to a^+} f(x) = f(a) =$ finite quantity.

(iii) $f(x)$ is left continuous at 'b' i.e. $\lim_{x\to b^-} f(x) = f(b) =$ finite quantity.

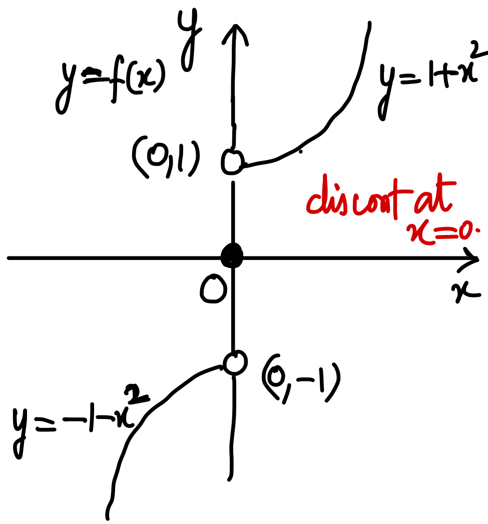
Note :-

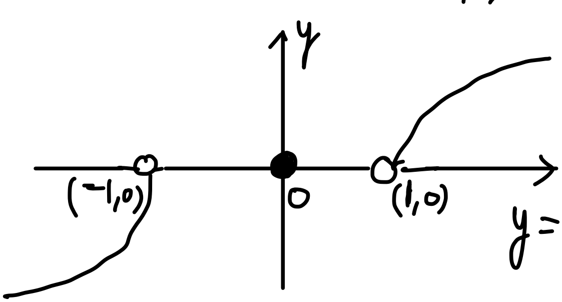
① All polynomial fns, rational fns, trigonometric functions, logarithmic fns & exponential fns are always continuous and differentiable in their respective domain.

② Inverse of a discontinuous fn (at $x=a$) can be Continuous. (at $x=a$)

eg:

$$f(x) = \begin{cases} 1+x^2 \ ; & x>0 \\ 0 \ ; & x=0 \\ -(1+x^2) \ ; & x<0. \end{cases}$$





$y = f^{-1}(x)$ → Cont at $x=0$.

Q $f(x) = \begin{cases} (\cos x)^{\cot^2 x} & x \neq 0 \\ e^{-1/2} \quad \text{if} & x = 0 \end{cases}$ find whether the f (x) is continuous at x = 0 or not.

Soln

$f(0) = e^{-1/2}$

$\lim_{x \to 0} (\cos x)^{\cot^2 x} = e^{\lim_{x \to 0} \cot^2 x \left(\frac{\cos x - 1}{x^2}\right) \cdot x^2}$

$= e^{-1/2}$

Yes f(x) is Cont at $x = 0$

Q Let $f(x) = \begin{cases} (1+|\sin x|)^{\frac{a}{|\sin x|}} & \text{for } -\frac{\pi}{6} < x < 0 \\ b & \text{for } x = 0 \\ e^{\frac{\tan 2x}{\tan 3x}} & \text{for } 0 < x < \frac{\pi}{6} \end{cases}$ Find 'a' and 'b' if f is continuous at x =0.

Sol^n

$$LHL = RHL = f(0)$$

$$LHL = \lim_{x\to 0^-} (1+|\sin x|)^{\frac{a}{|\sin x|}} = e^{\lim_{x\to 0^-} \frac{a}{|\sin x|}(\cancel{1}+|\sin x|-\cancel{1})}$$

$$= e^{a}$$

$$RHL = \lim_{x\to 0^+} e^{\frac{\tan 2x}{\tan 3x}} = e^{\frac{2}{3}}.$$

$$e^{a} = e^{2/3} = b.$$

$$a = \frac{2}{3}\ ;\ b = e^{2/3}.$$

Q $f(x) = \begin{cases} 3 - \left[\cot^{-1}\dfrac{2x^3 - 3}{x^2}\right] & \text{if } x > 0 \\ \{x^2\}\cos e^{\frac{1}{x}} & \text{if } x < 0 \end{cases}$ where $[x]$ and $\{x\}$ denotes greatest integer & fractional part. Can $f(x)$ be made continuous.

Soln

$$RHL = \lim_{x \to 0^+} 3 - \left[\cot^{-1}\left(\underbrace{2x - \frac{3}{x^2}}_{(-\infty)}\right)\right]$$

$$(3 - [\pi]) = 3 - 3 = 0.$$

$$LHL = \lim_{x \to 0^-} \{x^2\}\cos e^{\left(\frac{1}{x}\right)}$$

as $x \to 0^-$

$\frac{1}{x} \to -\infty$

$e^{\frac{1}{x}} \to 0.$

$$\lim_{x \to 0^-} \left(x^2 - [x^2]\right) \underbrace{\cos\left(e^{\left(\frac{1}{x}\right)}\right)}_{1} = 0.$$

$$(0 - 0) \times 1$$

Q If $f(x) = \begin{cases} \dfrac{a+\sin x}{|x-2|} & \text{if } x<0 \\ b+5x & \text{if } 0 \le x < 1 \\ e^{cx^2} & \text{if } x \ge 1 \end{cases}$ (where a, b, c $\in \mathbb{R}$) is continuous for all x $\in \mathbb{R}$, then find

$(a - 3b + e^c)$.

Soln

$$\lim_{x\to 0^-} f(x) = \underbrace{f(0) = \lim_{x\to 0^+} f(x)}$$

$$\lim_{x\to 0^-} \frac{a+\sin x}{|x-2|} = b \Rightarrow \frac{a}{2} = b \Rightarrow \boxed{a = 2b}.$$

$$\lim_{x\to 1^-} f(x) = \underbrace{f(1) = \lim_{x\to 1^+} f(x)}$$

$$\boxed{b+5 = e^c}$$

$$a - 3b + e^c = (2b-3b) + b + 5 = 5$$

Ans

Q Let $f(x) = \begin{cases} 5 & , \quad x \le 2 \\ \alpha x & , \quad 2 < x < 4 \\ 3x + \beta & , \quad x \ge 4 \end{cases}$ and $g(x) = \begin{cases} 3x + 2 , & x < 1 \\ x + 1 , & x \ge 1 \end{cases}$. If $f(g(x))$ is continuous at $x = 1$, then find α and β.

Solⁿ

$$f(g(1)) = \lim_{x \to 1^-} f(g(x)) = \lim_{x \to 1^+} f(g(x))$$

$$f(g(1)) = f(2) = 5.$$

$$\lim_{x \to 1^-} f(g(x)) = f(5^-) = 15 + \beta.$$

$$\lim_{x \to 1^+} f(g(x)) = f(2^+) = 2\alpha$$

$$5 = 15 + \beta = 2\alpha$$

$$\beta = -10 ; \quad \alpha = \frac{5}{2}$$

Q If $f(x) = \cos\left(x \cos\frac{1}{x}\right)$ and $g(x) = \frac{\ln(\sec^2 x)}{x \sin x}$

are both continuous at $x=0$ then

(A) $f(0) = g(0)$ (B) $g(0) = 2f(0)$ (C) $f(0) - 2g(0) = 1$

(D) $f^2(0) - g(0) = 0$.

Solⁿ

$$f(0) = \lim_{x\to 0} f(x) = \lim_{x\to 0} \cos\left(x \cos\frac{1}{x}\right) = 1.$$

$$g(0) = \lim_{x\to 0} \frac{\ln(1+\tan^2 x)}{x^2 \cdot \frac{\sin x}{x} \cdot \tan^2 x} \cdot \tan^2 x = 1.$$

Q

If $f(x)$ is continuous $\forall\, x \in \mathbb{R}$ satisfying $x^2 + x(1 - f(x)) + 2f(x) - 6 = 0$, then find $f(2)$.

Sol^n

$$x^2 + x - 6 - xf(x) + 2f(x) = 0$$

$$f(x) = \frac{x^2 + x - 6}{x - 2}$$

$$f(2) = \lim_{x \to 2} \left(\frac{x^2 + x - 6}{x - 2} \right)$$

$$= \lim_{x \to 2} \frac{\cancel{(x-2)}(x+3)}{\cancel{(x-2)}} = 5 \quad \text{Ans}$$

$$f(x) = \begin{cases} \dfrac{x^2 + x - 6}{x - 2} & ; \; x \neq 2 \\ 5 & ; \; x = 2 \end{cases}$$

Q1 Let $f(x)$ be a real valued continuous function such that $f\left(e^x\right)=\dfrac{4}{f(1+|x|)}$, where $f(x)>0\ \forall\ x\in\mathbb{R}$, then find $\lim\limits_{x\to 1} f(x)$

Q2 Let f(x) be a continuous function defined for $1\le x\le 3$. If f(x) takes rational values for all real x and f(2) = 10, then f(1.5) = 10 **[JEE 1997]**

Q3 Let $f:\mathbb{R}\to\mathbb{R}$ is continuous function satisfying $f(x)=f(2x)\ \forall\ x\in\mathbb{R}$ and $f(1)=2$, then find $f(2020)$.

① $\lim\limits_{x\to 1} f(x) = f(1)$.

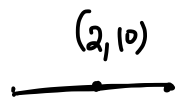
$$f(e^x) = \frac{4}{f(1+|x|)}$$

put $x=0$

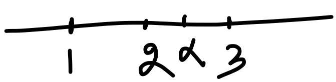
$$f(1) = \frac{4}{f(1)} \Rightarrow f^2(1) = 4$$

$$\boxed{f(1) = 2}$$

②



f(x) is constant fns



③ $f: R \to R$ continuous fns

$f(x) = f(2x)$

$x \to \frac{x}{2}$ $\quad \cancel{f\left(\frac{x}{2}\right)} = f(x)$

$x \to \frac{x}{2}$ $\quad f\left(\frac{x}{2^2}\right) = \cancel{f\left(\frac{x}{2}\right)}$

$\vdots$

$f\left(\frac{x}{2^n}\right) = \cancel{f\left(\frac{x}{2^{n-1}}\right)}$

Multiply :- $f\left(\frac{x}{2^n}\right) = f(x)$

$$\lim_{n\to\infty} f\left(\frac{x}{2^n}\right) = \lim_{n\to\infty} f(x) \Rightarrow f(x) = f(0)$$

$$f(x) = f(0) = \underline{\underline{\text{Constant}}}$$

$\therefore$ $f(x)$ is Constant fn.

$f(2) = 2$; $f(2020) = 2$ Ans

Q Let $f(x) = \dfrac{\sqrt{x^2+kx+1}}{x^2-k}$. The interval(s) of all possible values of k for which f is continuous for every $x \in R$, is

Sol^n

$x^2+Kx+1 \geq 0 \quad \forall x \in R$

$D \leq 0.$

$K^2-4 \leq 0$

$K \in [-2, 2]$ —①—

and $x^2-K \neq 0$

$x^2 \neq K$

$[0, \infty)$

$\boxed{K<0}$

—②—

① ∩ ②

$K \in [-2, 0)$ Ans

Q $\lim_{n\to\infty} \underbrace{\left(\frac{1}{1+n^2} + \frac{2}{2+n^2} + \frac{3}{3+n^2} + \cdots + \frac{n}{n+n^2}\right)}_{f(n)}$

$\lim_{n\to\infty} \left(\frac{1}{1+n^2} + \frac{1}{1+\frac{n^2}{2}} + \frac{1}{1+\frac{n^2}{3}} + \cdots + \frac{1}{1+\frac{n^2}{n}}\right)$

$G.T = \frac{1}{1+\frac{n^2}{n}}$; $L.T = \frac{1}{1+n^2}$.

$$\frac{1}{1+n^2} = \frac{1}{1+n^2}$$
$$\frac{2}{2+n^2} < \frac{2}{1+n^2}$$
$$\frac{3}{3+n^2} < \frac{3}{1+n^2}$$
$$\vdots$$
$$\frac{n}{n+n^2} < \frac{n}{1+n^2}$$

Add

$$f(n) < \frac{1+2+\cdots+n}{1+n^2}$$

$$\frac{1}{1+n^2} > \frac{1}{n+n^2}$$
$$\frac{2}{2+n^2} > \frac{2}{n+n^2}$$
$$\vdots$$
$$\frac{n}{n+n^2} = \frac{n}{n+n^2}$$

$$f(n) > \frac{1+2+\cdots+n}{n+n^2}$$

$$\underbrace{\frac{n(n+1)}{2(n+n^2)}}_{g(n)} < f(n) < \underbrace{\frac{n(n+1)}{2(1+n^2)}}_{h(n)}$$

$$\lim_{n\to\infty} \frac{\cancel{n^2}\left(1+\frac{1}{n}\right)}{2\cancel{n^2}\left(\frac{1}{n}+1\right)} = \frac{1}{2}$$

$$\therefore \lim_{n\to\infty} f(n) = \frac{1}{2}.$$

Ans

$$\lim_{n\to\infty} \frac{\cancel{n^2}\left(1+\frac{1}{n}\right)}{2\cancel{n^2}\left(\frac{1}{n^2}+1\right)} = \frac{1}{2}.$$

Q Let $f(x) = \lim_{n\to\infty} \frac{x^{2n-1} + ax^2 + bx}{x^{2n} + 1}$. If $f(x)$ is continuous for all $x \in R$ then find the values of a and b. $(n \in N)$

Sol^n

$$f(x) \rightarrow \begin{cases} ax^2 + bx & ; x \in (-1, 1) \checkmark \\ \frac{1}{x} & ; x \in (-\infty, -1) \cup (1, \infty) \checkmark \\ \frac{1+a+b}{2} & ; x = 1 \\ \frac{-1+a-b}{2} & ; x = -1 \end{cases}$$

$$\frac{\cancel{x^{2n-1}}\left(1 + \frac{a}{x^{\infty}} + \frac{b}{x^{\infty}}\right)}{x\,\cancel{x^{2n}}\left(1 + \frac{1}{x^{2n}}\right)}$$

Cont at $x = 1$

$$f(1) = \lim_{x\to 1} f(x)$$

$$\frac{1+a+b}{2} = 1$$

$$\boxed{1+a+b = 2} \quad -①-$$

$$\boxed{a+b = 1}$$

Cont at $x = -1$

$$\frac{-1+a-b}{2} = -1$$

$$-1+a-b = -2.$$

$$\boxed{a-b = -1} \quad -②-$$

$a = 0$ Ans

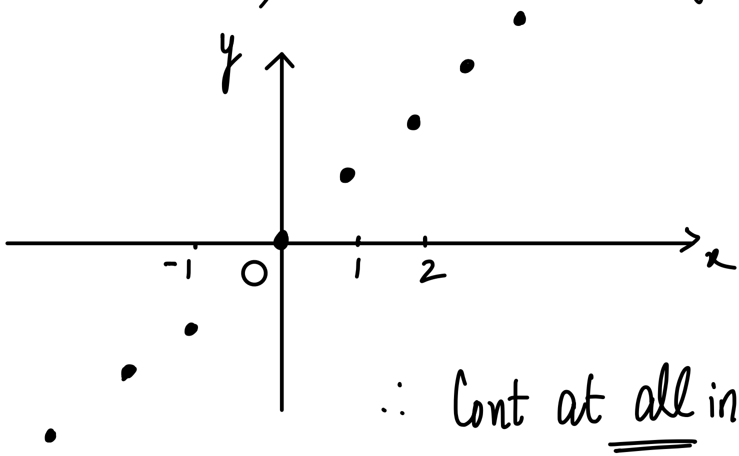
$b = 1$

(1) $f(x) = \sin^{-1}\sqrt{x} + \sec^{-1}\sqrt{x}$

Domain: $\{1\}$ → Point fns

$\therefore$ Continuous at $x = \underline{\underline{1}}$ only.

(2) $f : \mathbb{Z} \to \mathbb{Z},\ f(x) = x$; $\mathbb{Z}$ is set of integers.



$\therefore$ Cont at <u>all</u> integers.

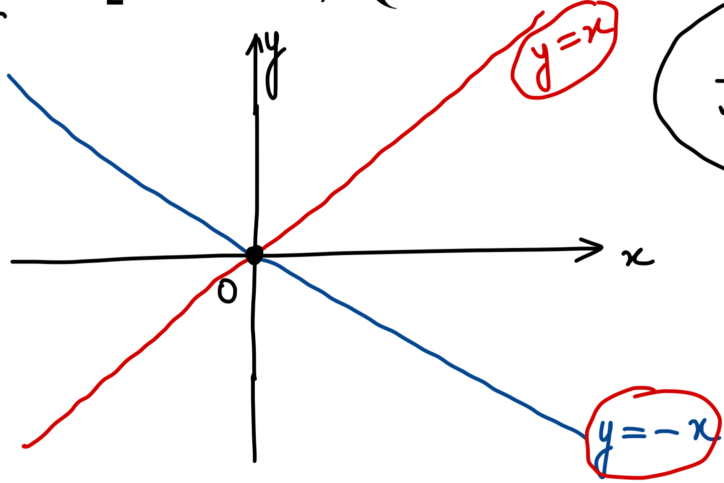
Q Discuss continuity of $f(x)$ where

$$f(x) = \begin{cases} x & \text{if } x \in Q \checkmark \\ -x & \text{if } x \notin Q \checkmark \end{cases}$$

$Q \rightarrow$ set of rational nos.

$$f(x) - \begin{cases} 1; & x \in Q \\ 0; & x \notin Q. \end{cases}$$

Cont at $x=0$ ONLY



Only at $\underline{x=0}$ we should check for continuity.

$\boxed{f(0) = 0}$ ✓

$$\lim_{x\to0^+} f(x) \xrightarrow{Q} \lim_{x\to0^+} (x) = 0$$

$$\lim_{x\to0^+} f(x) \xrightarrow{Q^c} \lim_{x\to0^+} (-x) = 0$$

$\therefore$ $\boxed{RHL = 0}$ ✓

$$\lim_{x\to0^-} f(x) \xrightarrow{Q} \lim_{x\to0^-} (x) = 0$$

$$\lim_{x\to0^-} f(x) \xrightarrow{Q^c} \lim_{x\to0^-} (-x) = 0$$

$\boxed{LHL = 0}$ ✓

Q Discuss continuity of $f(x) = \begin{cases} x^2 & \text{if } x \in Q \\ 1 & \text{if } x \notin Q \end{cases}$.

Sol^n $x^2 = 1 \Rightarrow \boxed{x = \pm 1}$

Continuous at $x = \pm 1$ only.

Q Let $f(x) = \begin{cases} \text{sgn}\, x, & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$, then find number of points, where $f(x)$ is continuous (where sgn(.) denotes signum function).

Sol^n $\text{sgn}\, x = 0 \Rightarrow \boxed{x = 0}$ **

Check !!!

discont everywhere.

Q Let $f(x)$ is an odd function, continuous in $\mathbb{R}$. If $f(1) = 5$ and $f(3) = -5$, $f(6) = 7$, then find minimum number of zeroes of $f(x) = 0$.

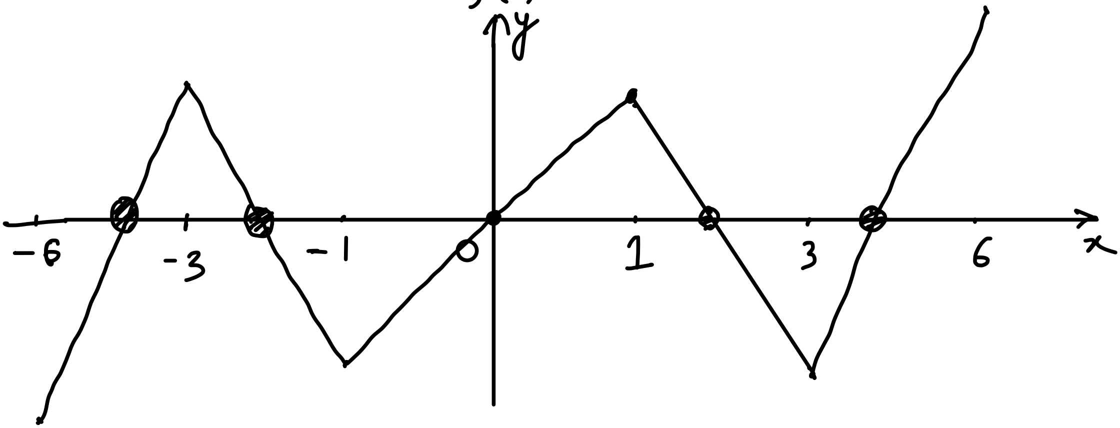
Sol^n $\boxed{f(0) = 0}$ *

$f(1) = 5 \Rightarrow f(-1) = -5$

$f(3) = -5 \Rightarrow f(-3) = 5$

$f(6) = 7 \Rightarrow f(-6) = -7.$

5 Ans



Q Find the number of points of discontinuity of:

([ ] → Gif.)

(i) $f(x) = [[x]] - [x+2]$

Sol^n

$$f(x) = \cancel{[x]} - \cancel{[x]} - 2$$

$$f(x) = -2 \Rightarrow \text{Constant fn.}$$

$\therefore$ '0' points of discontinuity.

(ii) $f(x) = [3x+1]$ in $[1, 3]$?

$\underline{\underline{\text{Sol}^n}}$ $f(x) = \underbrace{[3x]}_{} + 1$; $[1, 3]$

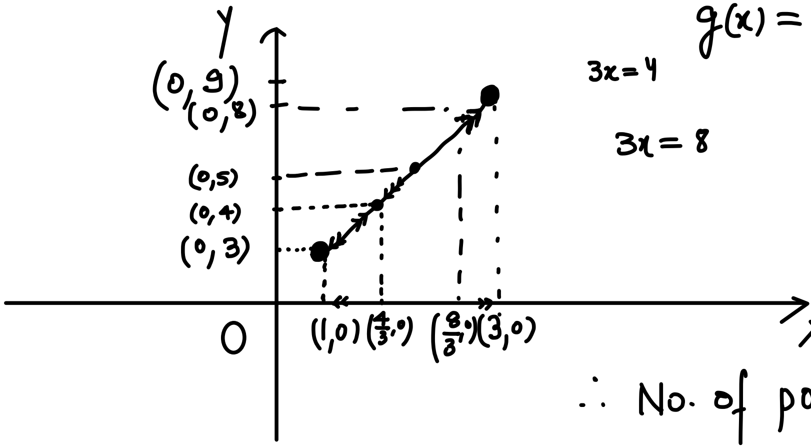
$g(x) = 3x$; $x \in [1, 3]$

$3x \in [3, 9]$

$\underbrace{4, 5, 6, 7, 8}$

$3x = 4$

$3x = 8$



$\therefore$ No. of points of discont $= \underline{\underline{6}}$

(iii) $f(x) = [2x^3 - 5]$ in $[1, 2)$?

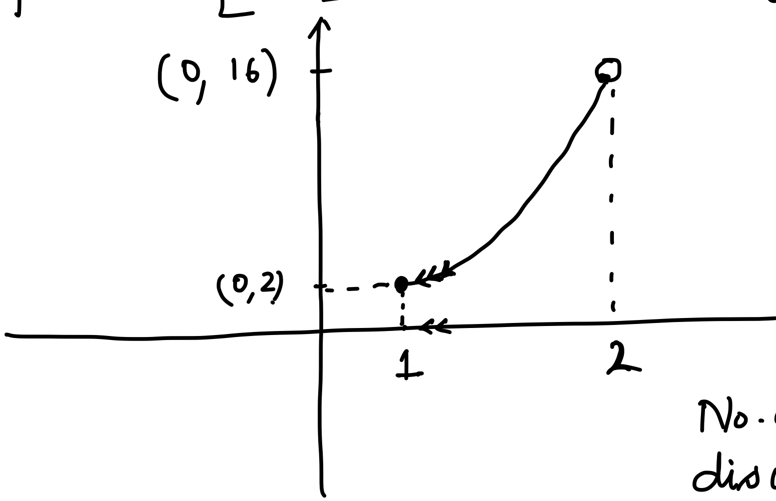
$f(x) = [2x^3] - 5$

$g(x) = 2x^3$.

in $[1, 2)$

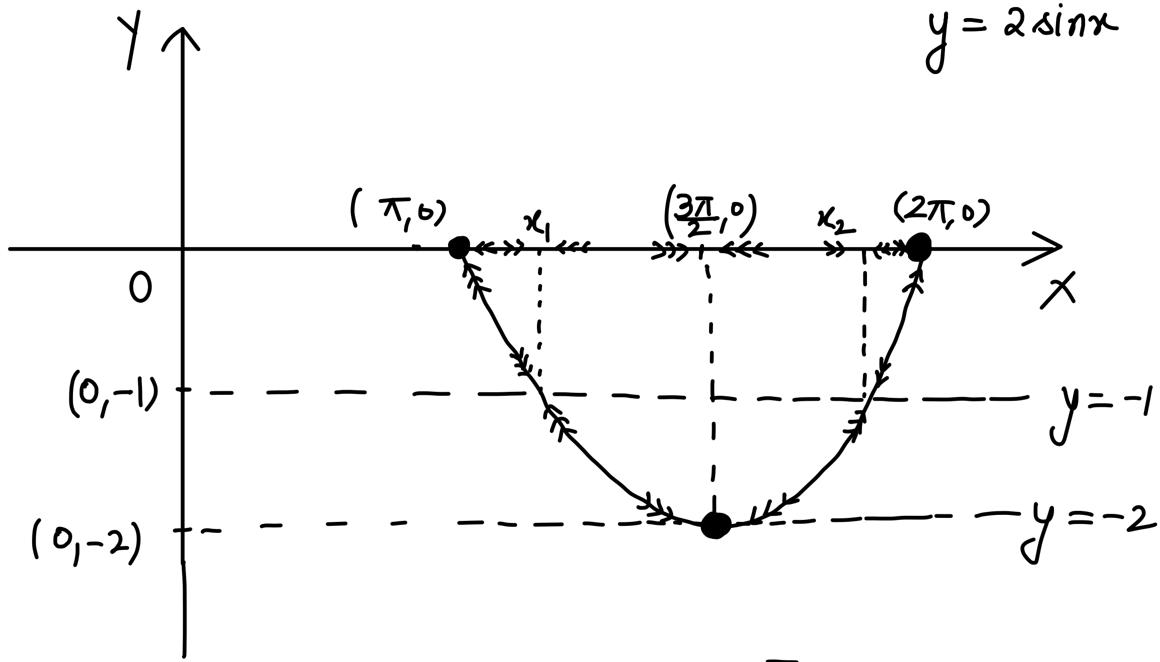
$x \in [1, 2)$

$2x^3 \in [2, 16)$



No. of points of discont $= \underline{\underline{13.}}$

(iv) $f(x) = [2\sin x]$ in $[\pi, 2\pi]$?



disCont at $\underline{\underline{4}}$ points in $[\pi, 2\pi]$.

Sum of 'x' $= \pi + x_1 + x_2 + 2\pi = 6\pi$.

$x_1 \downarrow (\pi + \theta)$ $\quad$ $x_2 \downarrow (2\pi - \theta)$

(v) $f(x) = [x] + \{-x\} + x$ in $[-2, 2]$?

$\{\ \} \to$ fractional part fns & $[\ ] \to$ Gif.

Sol^n

$$f(x) = [x] + (\cancel{-x} - [-x]) + \cancel{x} \qquad -x \in (1, 2)$$

$$f(x) = [x] - [-x]$$

$$f(x) \to \begin{cases} -4 & ;\ x = -2 \\ (-2) - (1) = -3 & ;\ x \in (-2, -1) \\ \cdot & ;\ x = -1 \\ \cdot & ;\ x \in (-1, 0) \\ \cdot & ;\ x = 0 \\ \cdot & ;\ x \in (0, 1) \\ \cdot & ;\ x = 1 \\ \cdot & ;\ x \in (1, 2) \\ & ;\ x = 2 \end{cases}$$

Discont at $\underline{x = -2;\ -1,\ 0;\ 1;\ 2}$ Ans

Q Discuss continuity : [ ] → Gif

(i) $f(x) = \left[x\left[\frac{1}{x}\right]\right]$ at $x = \frac{1}{2}$.

(ii) $f(x) = \left[\frac{x}{[x]}\right]$ at $x = 2$.

Sol^n (i)

$$f\left(\frac{1}{2}\right) = \left[\frac{1}{2}\underbrace{\left[\frac{1}{1/2}\right]}\right] = \left[\frac{1}{\cancel{2}}\cdot\cancel{2}\right]$$

$= 1.$

RHL : $\lim\limits_{h\to 0}\left[\left(\frac{1}{2}+h\right)\left[\frac{1}{\left(\frac{1}{2}+h\right)}\right]\right]$

$$\lim_{h\to 0}\left[\left(\frac{1}{2}+h\right)\underbrace{\left[\frac{2}{2h+1}\right]}_{1}\right] = 0.$$

Discont at $x = \frac{1}{2}$.

(ii) $f(2) = \left[\frac{2}{[2]}\right] = 1.$

RHL $= \lim\limits_{h\to 0}\left[\frac{2+h}{[2+h]}\right] = \lim\limits_{h\to 0}\left[\frac{2+h}{2}\right] = 1$

LHL = 1.

Theorems on Continuity

(1) If $f(x)$ & $g(x)$ are two functions that are continuous at $x = c$ then the function defined by :

$F_1(x) = f(x) \pm g(x)$; $F_2(x) = K f(x)$, where K is any real number ; $F_3(x) = f(x).g(x)$ are also continuous at $x = c$.

Further, if $g(c)$ is not zero, then $F_4(x) = \frac{f(x)}{g(x)}$ is also continuous at $x = c$.

(2) If $f(x)$ is continuous at $x = a$ & $g(x)$ is discontinuous at $x = a$, then $f(x) \pm g(x)$ must be discontinuous at $x = a$ .However nothing directly can be said about product and division function. So always check continuity in such cases.

(3) If f(x) and g(x) are both discontinuous at x=a then nothing directly can be said about their sum, difference, product or division function , so check continuity always in such cases.

eg: $$f(x) = \left(\frac{x^4 - 2x + 1}{x^2 - 4}\right) + \sin(4x+1) + \ln(x^2+1).$$

$D_f \in R - \{\pm 2\}$

f(x) is continuous everywhere in its domain.

eg: $f(x) = [x]$; $g(x) = \{x\}$; $[\ ] \to$ Gif, $\{\ \} \to \ldots\ldots$

$f(x)$: discont. $\forall x \in I$

$g(x)$: discont $\forall x \in I$

$h(x) = [x] + \{x\} = x$ cont $\forall x \in R$

eg: $f(x) = \text{sgn}(x^2-4) + \frac{x^2+7x-4}{(x-2)}$.

$D_f \in R - \{2\}$

$\text{sgn}(M) = 0 \Rightarrow M = 0$

$x^2 - 4 = 0$

$\cancel{x=2}$ OR $\boxed{x=-2}$

$f(x) = g(x) + h(x)$

$g(x) \downarrow$ discont at $x=-2$

$h(x) \longrightarrow$ Cont at $x=-2$.

$\therefore$ using theorem, $f(x)$ will be discont at $x=-2$ only in its domain.

④ ** If $g(x)$ defined at $x=a$ is discontinuous at $x=a$ with finite LHL & RHL and $f(x)$ is continuous at $x=a$ such that $f(a)=0$ then the product function $y=f(x)\cdot g(x)$ is cont. at $x=a$.

eg: $f(x) = \underbrace{[x]}_{g(x)} \underbrace{\cos\left(2x-1\right)\frac{\pi}{2}}_{h(x)}$; $[\ ] \to$ Gif

$f(x)$ is cont $\forall x \in R$.

eg: $f(x) = \text{sgn}\, x \cdot \sin x$

$\hookrightarrow$ Cont $\forall x \in R$

eg: $f(x) = \{x\} \underbrace{\sin \pi x}_{0}$; $\{\ \} \to$ ....

$\hookrightarrow$ Cont $\forall x \in R$.

Q Discuss number of points of discontinuity of

$f(x) = [5x] + \{3x\}$ in $[0, 5]$?

(where $[\ ] \to$ Gif & $\{\ \} \to$ fractional part fns)

$\boxed{\text{Ans} \to 30}$

Sol^n $x \in [0, 5]$ $3x \in \underbrace{[0, 15]}$.

$5x \in \underbrace{[0, 25]}$

C-I $x \in [0, 1]$

$5x \in [0, 5] \Rightarrow 5x = 0, 1, 2, 3, 4, 5$

$3x \in [0, 3]. \Rightarrow 3x = 0, 1, 2, 3.$

[0, 1]

[1, 2]

[2, 3] ⋰

[3, 4]

[4, 5]

$x = 0, \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, 1$

$x = 0, \frac{1}{3}, \frac{2}{3}, 1.$

Check at $x=0$ & $x=1$.

$f(x) = [5x] + \{3x\}$

$f(0) = 0 + 0 = 0.$

$RHL = \lim_{h \to 0} \left([5(0+h)] + 3(0+h) - [3(0+h)]. \right)$

$0 + 0 - 0 = 0.$

$f(x) = [5x] + \{3x\}$

$f(1) = 5 + 0 = 5$

$LHL = \lim_{h \to 0} [5(1-h)]$

$+ \underbrace{3(1-h)} - [3(1-h)]$

$4 + 3 - 2 = 5$

Q For every integer n, let a_n and b_n be real numbers. Let function $f : \mathbb{R} \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} a_n + \sin \pi x, & \text{for } x \in [2n, 2n+1] \\ b_n + \cos \pi x, & \text{for } x \in (2n-1, 2n) \end{cases}, \text{ for all integers n.}$$

$n \to n+1$

If f is continuous, then which of the following holds(s) for all n ? **[JEE 2012, 4M]**

(A) $a_{n-1} - b_{n-1} = 0$ (B) $a_n - b_n = 1$ (C) $a_n - b_{n+1} = 1$ (D) $a_{n-1} - b_n = -1$

Sol^n Cont at $x = 2n$.

$$f(2n) = \lim_{x \to 2n^-} f(x) \Rightarrow a_n + \sin \pi(2n) = b_n + \cos \pi(2n)$$

$$a_n + 0 = b_n + 1.$$

$$\boxed{a_n - b_n = 1}$$

$n \in I$

$n \to n+1$
$n \to n-1$
$n \to 2n$
$n \to n+4$
$\vdots$

Cont at $x = 2n+1$.

$$f(2n+1) = \lim_{x \to (2n+1)^+} f(x)$$

$f(x)$ — $a_n + \sin \pi x$, $x \in [2n, 2n+1]$; $b_{n+1} + \cos \pi x$, $x \in (2n+1, 2n+2)$

$$a_n + \sin \pi(2n+1) = b_{n+1} + \cos \pi(2n+1).$$

$$a_n + 0 = b_{n+1} + (-1) \Rightarrow \boxed{a_n - b_{n+1} = -1}$$

$n \to n-1$

TYPES OF DISCONTINUITIES :

T.O.D

- **Type-1 (Removable discontinuity)**
- **Type-2 (Non removable discontinuity)**

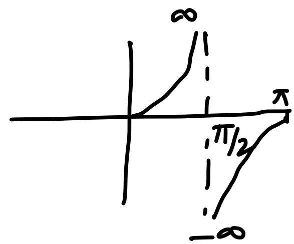
Type-1 (Removable discontinuity)

Here $\lim_{x\to a} f(x)$ necessarily exists.
Therefore it is possible to redefine the function in such a manner that $\lim_{x\to a} f(x) = f(a)$ and thus making the function continuous.

Type-2 (Non removable discontinuity)

Here $\lim_{x\to a} f(x)$ does not exists, therefore it is not possible to redefine the function in any manner to make it continuous.

eg: $\lim_{x\to \frac{\pi}{2}} \left(2^{\tan x}\right)$ $\xrightarrow{LHL} \infty$, $\xrightarrow{RHL} 2^{-\infty} = 0.$



Non-Removable. discont.

eg: $\lim_{x\to 0} \left(1 + \underbrace{\frac{1}{3}\sin(\underbrace{\ln|x|})}_{\left[-\frac{1}{3}, \frac{1}{3}\right]}\right)$

as $x \to 0^+$
$\ln x \to (-\infty)$

Non-Removable

oscillate from $\left[\frac{2}{3}, \frac{4}{3}\right]$.

eg: $\lim_{x\to 2} \left([x] + [-x]\right)$; $[\ ] \to$ Gif.

$\to$ LHL = RHL = -1. $f(2) = 0$

Removable.

CONTINUITY OF COMPOSITE FUNCTIONS :

If $f(x)$ is continuous at $x = a$ & $g(x)$ is continuous at $x = f(a)$, then the composite $g[f(x)]$ is continuous at $x = a$.

Q

$$f(x)=\begin{cases}1+x & x\in[0,2]\\ 3-x & x\in(2,3]\end{cases} \quad \& \quad g(x)=\begin{cases}1-x & x\in[0,1]\\ 3-x & x\in(1,3]\end{cases}$$

Discuss continuity of $g(f(x))$

Sol^n M-1 Construct $g(f(x))$ as we did in function.

M-2 $f(x)$ is dis cont at $x = 2$.

$g(x)$ is dis cont at $x = 1$.

For $g(f(x))$ check continuity at $x = 2$, doubtful points.

$f(x) = 1 \Rightarrow x = ??$

$1+x = 1 \Rightarrow \boxed{x=0}$ ✓

$3-x = 1 \Rightarrow \boxed{x=2}$ XX

Check cont at $x=0$ & $x=2$

$y = g(f(x)) \Rightarrow g(f(0)) = g(1) = 0.$ ✓

RHL at $x=0 \Rightarrow \lim\limits_{x\to 0^+} g(f(x)) = \lim\limits_{x\to 0} g((1+x))$

$= 2.$ ✓

$\therefore$ $\boxed{\text{discont at } x=0}$

Check at $x=2$:

$g(f(2)) = g(3) = 0.$ ✓

RHL:

$\lim_{x \to 2^+} g(f(x)) = \lim_{h \to 0} g(1-h) = 0.$ ✓

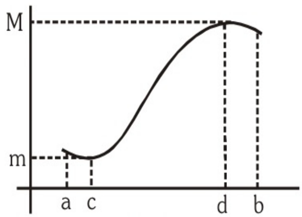
$||^{ly}$ $LHL = 0$ ✓

$\therefore$ $g(f(x))$ is cont at $\underline{\underline{x=2}}$

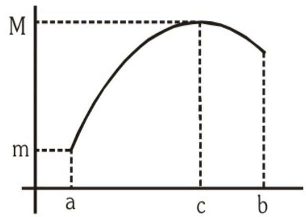
Q If $f(x) = [x]$ and $g(x) = e^{\{x\}}$, then discuss the continuity of $f o g(x)$ and $g o f(x)$ in $\mathbb{R}$.

[ ] → Gif & { } → - - - - - - -

Extreme value Theorem : If f is continuous on [a, b] then f takes on, a least value m and a greatest value M on this interval.



Minimum value 'm' occurs at x = c and maximum value M occurs at x = d, c, d $\in$ (a, b)



Minimum value 'm' occurs at the end point x = a and maximum value M occurs at x=c, c $\in$ (a,b)

PROPERTIES OF CONTINUOUS FUNCTION

(a) If a function f (x) is continuous on a closed interval [a, b], then it is bounded

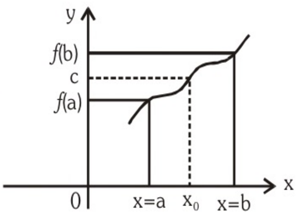
(b) A continuous function whose domain is some closed interval must have its range also in closed interval.

V.imp

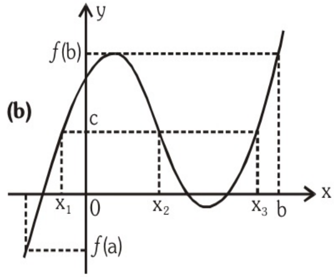
Intermediate value theorem (IMVT) : (IVT)

If f is continuous on [a, b], then for every value c between f(a) and f(b) (including both), there is at least one number x_0 in [a, b] for which $f(x_0) = c$.

Explain by following graphs :



(a)



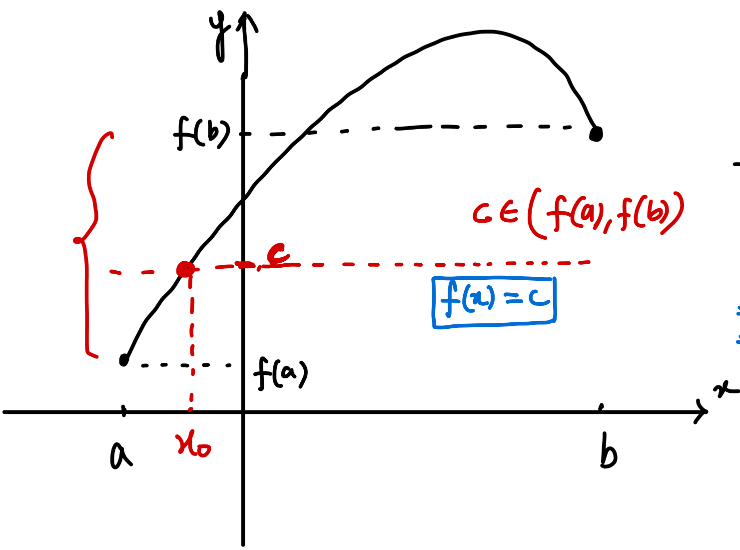
(b)

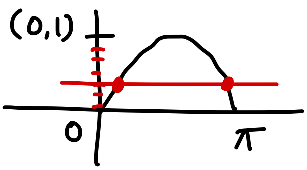
NOTE :

(i) If $f(x)$ is continuous in [a,b], then for every m,n $\in \mathbb{R}^+$ there exist atleast one corresponding c $\in$ [a, b] such that $f(c) = \dfrac{m f(a) + n f(b)}{m + n}$.

(ii) A function f which is continuous in [a,b] possesses the following property

If f(a) & f(b) posses opposite signs, then there exists atleast one root of the equation f(x) = 0 in the open interval (a,b).





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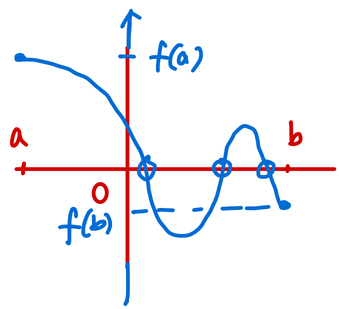


Q Suppose that f (x) is continuous in [0, 1] and f (0) = 0, f (1) = 0. Prove that f(c)=1 – 2c²
for some c ∈ (0, 1)

Solⁿ $f(c) - 1 + 2c^2 = 0.$

$\boxed{g(x) = \underbrace{f(x)}\underbrace{-1+2x^2}} \longrightarrow$ Cont in $[0,1]$

$g(0) = f(0) - 1 + 0 = -1$

$g(1) = f(1) - 1 + 2 = 1$

$g(0)\cdot g(1) < 0 \Rightarrow$ from IVT thr exist atleast one $x \in (0,1)$ s.t $\boxed{g(x) = 0}$

Q Let f : [0, 2] → R be continuous and f (0) = f (2). Prove that there exists x_1 and x_2 in (0, 2) such that $x_2 - x_1 = 1$ and $f(x_2) = f(x_1)$

Solⁿ $f : [0,2] \to R$ $\boxed{f(0) = f(2) = K}$ (say)

$f(x_2) = f(x_1) \Rightarrow f(x_1+1) = f(x_1)$

$f(x_1+1) - f(x_1) = 0 \Rightarrow$

$g(x) = f(x+1) - f(x)$

$g(0) = f(1) - f(0)$

$g(1) = f(2) - f(1)$

$g(0)\cdot g(1) = (\underbrace{f(1)-K})(\underbrace{K-f(1)})$

Q Prove that there is a number x such that $x^{1995} + \frac{1}{1+\sin^2 x} = 2011$

Sol^n

$$f(x) = x^{1995} + \frac{1}{1+\sin^2 x} - 2011.$$

$f(x)$ is cont $\forall x \in R$.

$$f(0) = 1 - 2011 = -ve.$$

$$f(2) = 2^{1995} + \frac{1}{1+\sin^2 2} - 2011 = +ve$$

$$f(0) \cdot f(2) < 0 \Rightarrow \cdots\cdots\cdots\cdots$$

Q

$f(x)$ is continuous function on [1, 3] and $f(1) = 2$, $f(3) = -2$, then which of the following not necessarely hold good ?

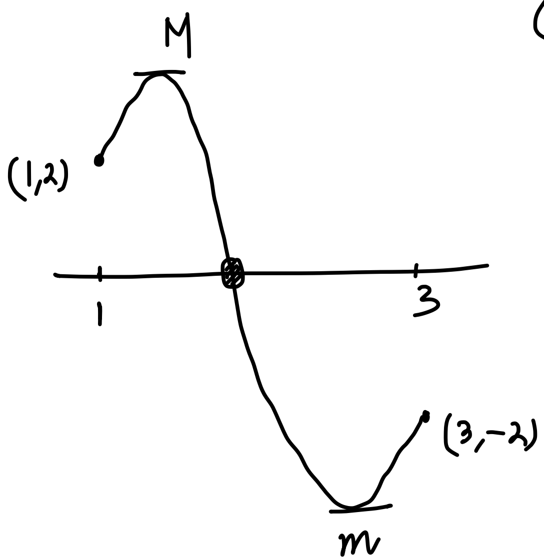
~~(A)~~ $f(2) \geq 0$

(B) $x f(x) = 0$ has a root on (1, 3)

~~(C)~~ $-2 \leq f(x) \leq 2 \ \forall \ x \in [1, 3]$

(D) $f(x) - x = 0$ has a root on (1, 3)

Sol^n



(B) $g(x) = x f(x)$

$g(1) = 1 f(1) = +ve$

$g(3) = 3 f(3) = -ve$

$g(1)\, g(3) < 0.$

(D) $g(x) = f(x) - x$

$g(1) = f(1) - 1$

$= 2 - 1 = 1$

$g(3) = f(3) - 3$

$= -2 - 3 = -5$

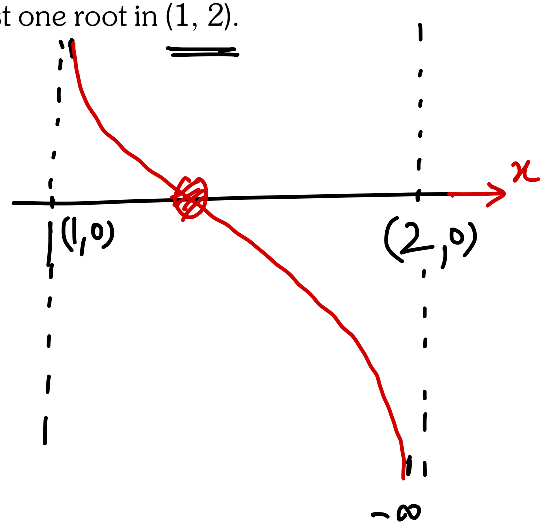
$g(1) \cdot g(3) < 0 \Rightarrow \ldots$

Q. Prove that equation $\frac{1}{x-1}+\frac{2}{x-2}=0$ has atleast one root in (1, 2).

Sol^n $f(x)=\frac{1}{x-1}+\frac{2}{x-2}$

$\lim_{x\to 1^+} f(x)\to \infty$

$\lim_{x\to 2^-} f(x)\to -\infty$



Differentiability of function

(Two fold meaning of derivability)

Geometrical meaning of derivative

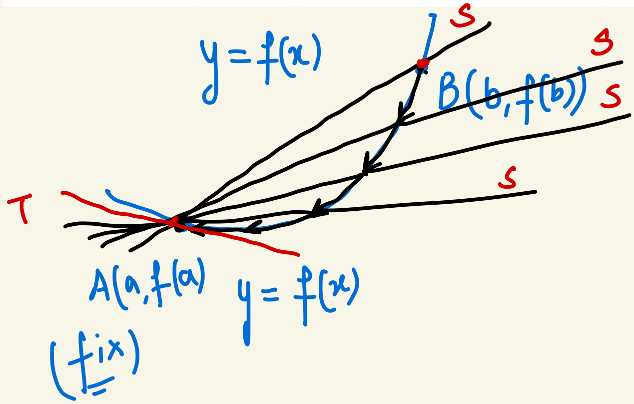
Slope of the tangent drawn to the curve at x = a if it exists

Physical meaning of derivative

(functions which are differentiable)

Instantaneous rate of change of function

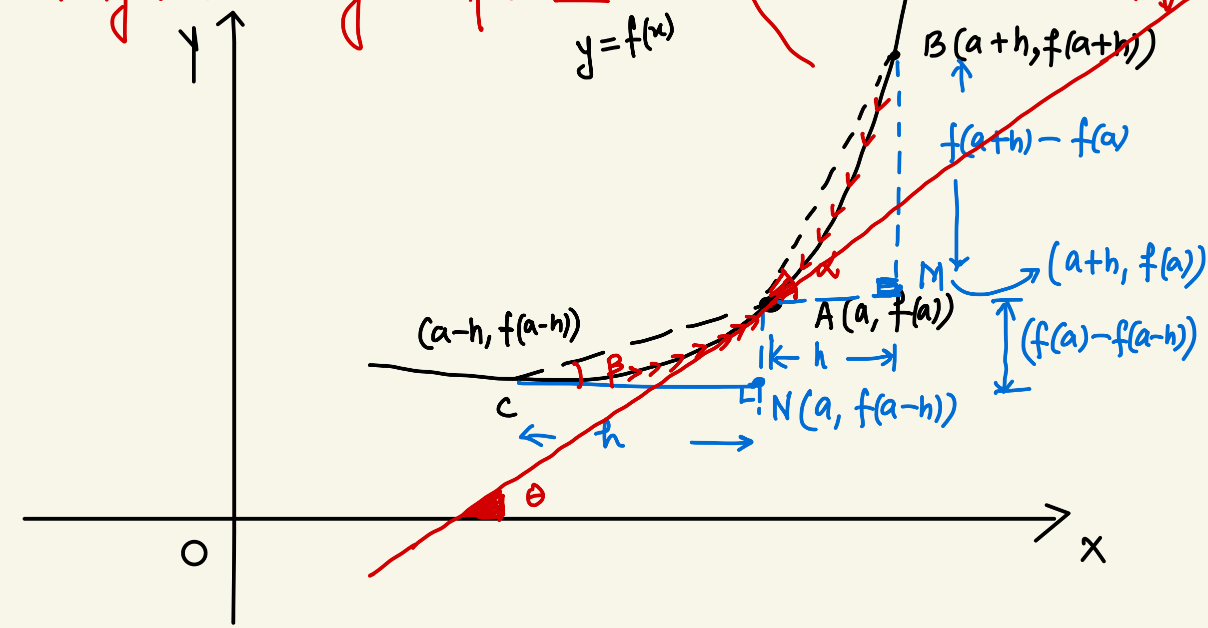
Note : "Tangent at a point 'A' is the the limiting case of secant through A."



$$\tan\alpha = \frac{f(a+h) - f(a)}{h}$$

$$\tan\beta = \frac{f(a) - f(a-h)}{h}$$

Tangent is limiting case of secant.



$$\boxed{\lim_{h\to 0}\left(\frac{f(a+h)-f(a)}{h}\right) = RHD = Rf'(a) = f'_+(a)}$$
(Right hand derivative)

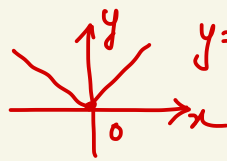
$$\boxed{\lim_{h\to 0}\left(\frac{f(a-h)-f(a)}{-h}\right) = LHD = Lf'(a) = f'_-(a)}$$

Rem

$\therefore$ For a fns $f(x)$ to be derivable at $x = a$

LHD = RHD = a finite value.

$$\boxed{LHD = RHD = f'(a)}$$



$y = |x|$

Note: If f(x) is derivable for every point of its domain of definition, then it is continuous in that domain .

The Converse of the above result is not true :

For a function f :

(1) Differentiability $\Rightarrow$ Continuity ; (2) Continuity $\not\Rightarrow$ derivability ;

(3) Non derivibality $\not\Rightarrow$ discontinuous (4) But discontinuity $\Rightarrow$ Non derivability

(i) If the function y = f(x) is differentiable at x = a, then a unique non-vertical tangent can be drawn to the curve y = f(x) at the point P(a, f(a)) & f'(a) represent the slope of the tangent at the point.

(ii) * All polynomial, rational, trigonometric, logarithmic and exponential function are continuous and differentiable in their domains.

Q If $f(x) = [x]$ and $g(x) = e^{\{x\}}$, then discuss the continuity of $fog(x)$ and $gof(x)$ in $\mathbb{R}$.

$[\ ] \to$ Gif & $\{\ \} \to$ ------

Sol^n

$$gof(x) = g(f(x)) = g([x]) = e^{\{[x]\}} = e^0 = 1$$

↳ Constant fns.

$\therefore$ $gof(x)$ is Cont $\forall x \in R$.

$$fog(x) = f\left(e^{\{x\}}\right) = \left[e^{\{x\}}\right]$$

$\{x\} \in [0, 1)$

$e^{\{x\}} \in [1, e)$

$\left(e^{\{x\}}\right)_{int} = 1, 2.$

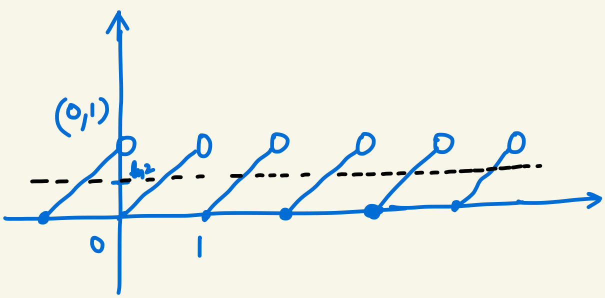
$e^{\{x\}} = 1$; $e^{\{x\}} = 2$

$\{x\} = \ln 1$; $\{x\} = \ln 2$

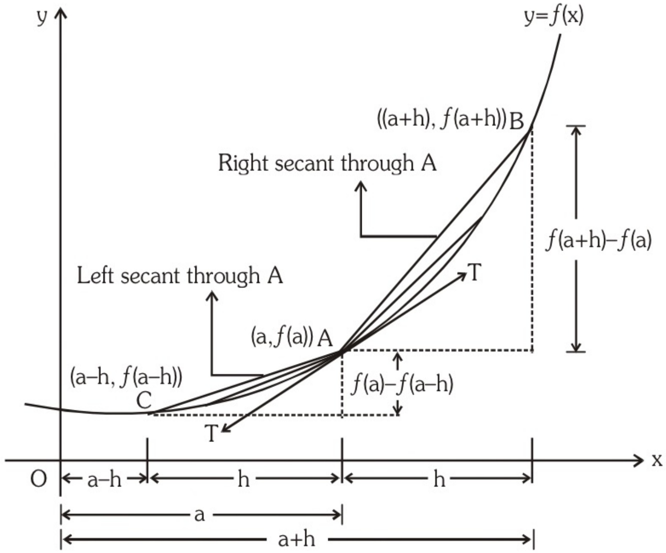
$\{x\} = 0$ | $\{x\} = \underline{0.693}$

$\boxed{x \in I}$ | $\boxed{x = I + \ln 2}$

$\{I + x\} = \{x\}$



Union



Tangent to the curve $y = f(x)$ at a point $A(a, f(a))$ is a limiting case of secant through A

$\Rightarrow$ Slope of tangent at point A (if it exists) = Limiting value of slope of secant through A (if it exist)
i.e. limiting value of slope of secant AB, where B on the right side of A and tends to A
= limiting value of slope of secant AC, where C on the left side of A and tends to A.

$\Rightarrow$ Derivative at $x = a$ is limiting value of slope of secant through A

Formally, let a secant through $A(a, f(a))$ intersects the curve at $B(a + h, f(a + h))$ on the right and another secant through A intersects the curve at $C(a-h, f(a-h))$ on the left.

$$\Rightarrow m_{AB} = \frac{f(a+h)-f(a)}{h},\ h > 0 \quad \text{and} \quad m_{AC} = \frac{f(a)-f(a-h)}{h},\ h > 0$$

$$\Rightarrow \quad \lim_{B\to A} m_{AB} = \lim_{C\to A} m_{AC} = \text{slope of tangent at A. i.e. derivative at } x = a.$$

Now as, $B \to A$ & $C \to A \Rightarrow h \to 0$

$$\Rightarrow \quad \lim_{B\to A} m_{AB} = \lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h} = \text{Slope of tangent at A through right side}$$

$$\text{and} \quad \lim_{C\to A} m_{AC} = \lim_{h\to 0^+} \frac{f(a-h)-f(a)}{-h} = \text{Slope of tangent at A through left side}$$

$$\Rightarrow \quad \text{derivative of } f(x) \text{ at } x = a \text{ exist, if } \lim_{h\to 0^+} \frac{f(a-h)-f(a)}{-h} = \lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h} = \text{Finite quantity}$$

(a) Right hand derivative :

The right hand derivative of f(x) at $x = a$ denoted by $Rf'(a)$ or $f'_+(a)$ is defined as :

$$Rf'(a) = \lim_{h\to 0^+} \frac{f(a+h)-f(a)}{h}, \text{ provided the limit exists \& is finite. } (h > 0)$$

(b) Left hand derivative :

The left hand derivative of f(x) at $x = a$ denoted by $Lf'(a)$ or $f'_-(a)$ is defined as :

$$Lf'(a) = \lim_{h\to 0^+} \frac{f(a-h)-f(a)}{-h}, \text{ provided the limit exists \& is finite. } (h > 0)$$

Hence f(x) is said to be **derivable or differentiable at x = a**,

if $\boxed{\mathbf{f'_+(a) = f'_-(a) = \text{Unique finite quantity}}}$, then derivate of $f(x)$ at $x = a$ is denoted by $f'(a)$,

where $\mathbf{f'(a) = f'_-(a) = f'_+(a)}$ at $x = a$.

Note :-

Let $f'_+(a) = p$ and $f'_-(a) = q$ then

① If $p = q =$ finite then $f(x)$ is derivable at $x = a$ and hence continuous at $x = a$.

② If $p \neq q$ but both are finite then $f(x)$ is non-derivable at $x = a$, however $f(x)$ is still continuous at $x = a$. Geometrically this implies a sharp corner at $x = a$.

③ If p or q or both fails to exist then $f(x)$ is non-derivable at $x = a$, however continuity needs to be checked seperately.

Q Discuss derivability :-

(1) $f(x) = |\ln x|$ at $x=1$.

(2) $f(x) = \ln^2 x$ at $x=1$. ✓

(3) $f(x) = e^{-|x|}$ at $x=0$. ✓

Sol^n $f(x) = |\ln x|$ at $x=1$.

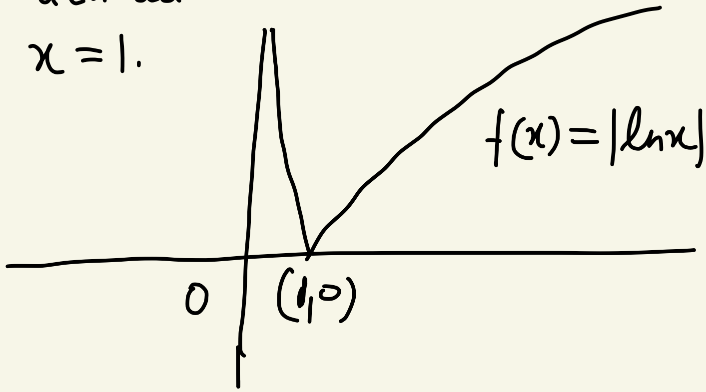
$$RHD = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h} = \lim_{h\to 0} \frac{|\ln(1+h)| - |\cancel{\ln 1}|}{h}$$

$$= \lim_{h\to 0} \left(\frac{\ln(1+h)}{h}\right) = 1.$$

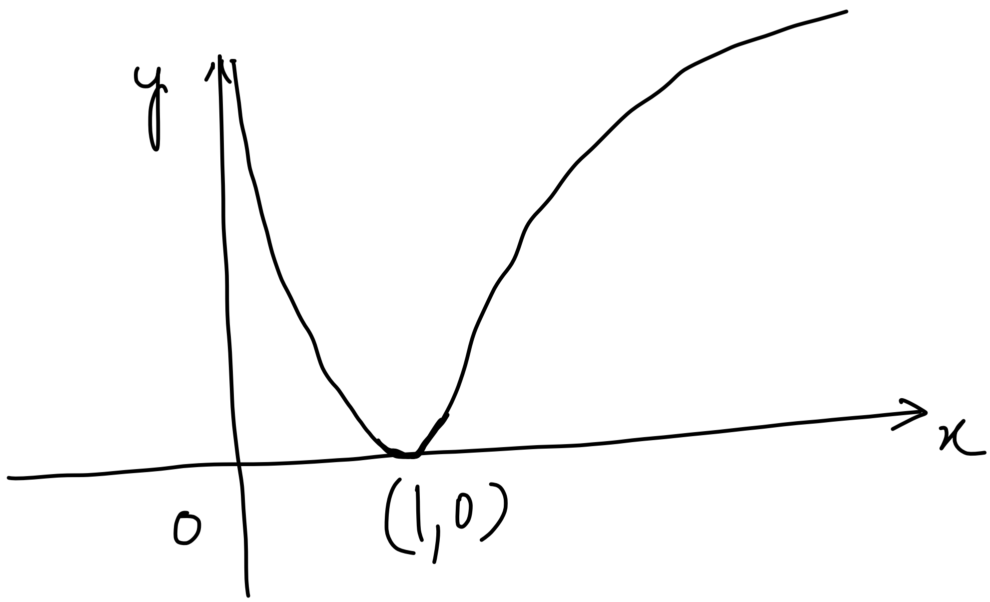
$$LHD = \lim_{h\to 0} \left(\frac{f(1-h) - f(1)}{-h}\right) = \lim_{h\to 0} \frac{|\ln(1-h)| - |\cancel{\ln 1}|}{-h}$$

$$= \lim_{h\to 0} -\frac{\ln(1-h)}{-h} = -1$$

$LHD \neq RHD \Rightarrow$ Non-derivable at $x=1$.



(ii) $y = \ln^2 x$



Do yourself.

(iii) Do yourself.

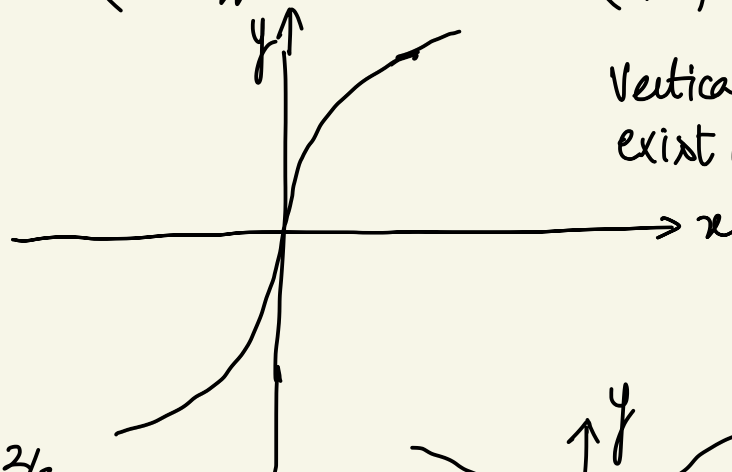
Vertical tangent : If y = f(x) is continuous at x = a and $\lim_{x\to a}|f'(x)|$ approaches to ∞, then y = f(x) has a vertical tangent at x = a. If a function has vertical tangent at x = a then it is non differentiable at x = a.

eg: $f(x) = x^{1/3}$ at $x=0$.

Continuous at $x=0$.

$$LHD = \lim_{h\to 0}\left(\frac{(0-h)^{1/3} - 0^{1/3}}{-h}\right) = \lim_{h\to 0} \frac{-(h)^{1/3}}{-(h)} = \infty$$

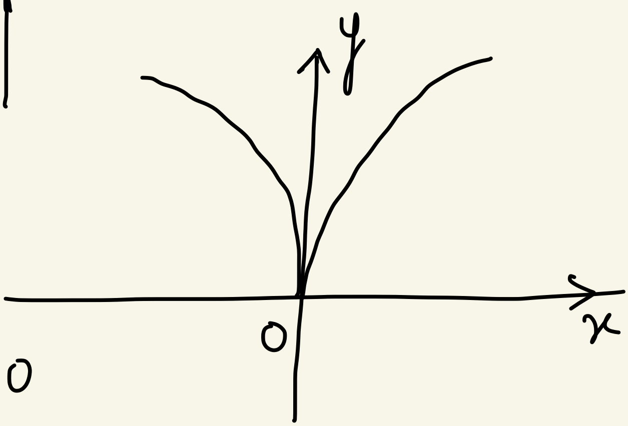
$$RHD = \lim_{h\to 0}\left(\frac{(0+h)^{1/3} - (0)^{1/3}}{h}\right) = \lim_{h\to 0}\left(\frac{h^{1/3}}{h}\right) = \infty.$$



Vertical tangent exist at $x=0$.

eg: $f(x) = x^{2/3}$; at $x=0$

Do yourself



V.T exist at $x=0$

DERIVABILITY OVER AN INTERVAL :

(a) f(x) is said to be differentiable over an open interval (a, b) if it is differentiable at each & every point of the open interval (a, b).

(b) f(x) is said to be differentiable over the closed interval [a, b] if :

(i) f(x) is differentiable in (a, b) &

(ii) for the points a and b, $f'_+(a)$ & $f'_-(b)$ exist.

Theorem on derivability :

(1) If f (x) & g(x) are two functions that are derivable at x = c then the function defined by :

$F_1(x) = f(x) \pm g(x)$; $F_2(x) = K\,f(x)$, where K is any real number ; $F_3(x) = f(x).g(x)$ are also derivable at x = c.

Further, if g(c) is not zero, then $F_4(x) = \dfrac{f(x)}{g(x)}$ is also derivable at x = c.

(2) If f(x) is derivable at x = a & g(x) is non-derivable at x = a, then f (x) ± g(x) must be non-derivable at x = a .However nothing directly can be said about product and division function. So always check derivability in such cases.

(3) If f(x) and g(x) are both non-derivable at x=a then nothing directly can be said about their sum, difference, product or division function , so check derivability always in such cases.

Note :

If f is continuous from right and RHL of f '(x) at x = a exist, then it is equal to RHD of f(x) at x = a

i.e. $f'_+(a) = \lim\limits_{x\to a^+} f'(x) = f'(a^+)$

Similarly if f is continuous from left and LHL of f '(x) at x = a exist, then it is equal to LHD of f(x) at x = a. i.e. $f'_-(a) = \lim\limits_{x\to a^-} f'(x) = f'(a^-)$

Hence if f(x) is continuous at x = a and $\lim\limits_{x\to a} f'(x)$ exist, then it is equal to f '(a).

Note : If f is continuous at x = a but $\lim\limits_{x\to a} f'(x)$ fails to exist, then nothing can said about differentiability of function at x = a. To determine it we have to use first principle.

Q If $f(x)=\begin{cases} ax+b & \text{for } x\le -1 \\ ax^3+x+2b & x>-1 \end{cases}$ is differentiable for all $x\in\mathbb{R}$ find 'a' & 'b'.

Sol^n Cont at $x=-1$

$$-\cancel{a}+b=-\cancel{a}-1+2b.$$

$$\boxed{b=1}$$

der at $x=-1$.

$$a=3a(-1)^2+1.$$

$$a=3a+1 \Rightarrow 2a=-1 \Rightarrow \boxed{a=\frac{-1}{2}}$$

Q

$f(x)=\begin{cases} \sin x & ; \text{if } x<\pi \\ mx+n & ; \text{if } x\ge\pi \end{cases}$ where m and n are constants. Determine m and n such that f is derivable on set of real numbers.

Cont at $x=\pi$.

$$\sin\pi = m\pi+n \Rightarrow \boxed{0=m\pi+n} \quad —①—$$

der at $x=\pi$.

$$\cos(\pi)=m \Rightarrow \boxed{m=-1}$$

Q

$$f(x) = \begin{cases} \sqrt{4x^2 - 12x + 9}.\{x\} & \text{for } 1 \le x \le 2 \\ \cos\left(\frac{\pi}{2}(|x| - \{x\}\right) & -1 \le x < 1 \end{cases}$$

check the differentiability in [–1, 2]

Sol^n

$\{x\} = x - [x]$

$$f(x) - \begin{cases} |2x-3|\ \{x\} & ;\ 1 \le x \le 2 \\ \cos\left(\frac{\pi}{2}(|x| - \{x\})\right) & ;\ -1 \le x < 1. \end{cases}$$

$$f(x) - \begin{cases} 0 & ;\ x = 1 \ \&\ x = 2 \\ (3-2x)\{x\} & ;\ 1 \le x < \frac{3}{2} \\ (2x-3)\{x\} & ;\ \frac{3}{2} \le x < 2 \\ \cos\left(\frac{\pi}{2}(-x - \{x\})\right) & ;\ -1 \le x < 0 \\ \cos\left(\frac{\pi}{2}(x - \{x\}\right) & ;\ 0 \le x < 1 \end{cases}$$

$$f(x) - \begin{cases} 0 & ;\ x = 1 \ \&\ x = 2 \\ (3-2x)(x-1) & ;\ x \in (1, 3/2) \\ (2x-3)\ (x-1) & ;\ x \in [\frac{3}{2}, 2) \\ \cos\left(\frac{\pi}{2}(-x - (x+1)\right) & ;\ x \in [-1, 0) \\ \cos 0 & ;\ x \in [0, 1) \end{cases}$$

$$f(x) = \begin{cases} \cos\left(\frac{\pi}{2}(-2x-1)\right) & ; \quad -1 \le x < 0 \\ 1 & ; \quad 0 \le x < 1 \\ 0 & ; \quad x = 1 \\ (3-2x)(x-1) & ; \quad x \in (1, 3/2) \\ (2x-3)(x-1) & ; \quad x \in \left[\frac{3}{2}, 2\right) \\ 0 & ; \quad x = 2 \end{cases}$$

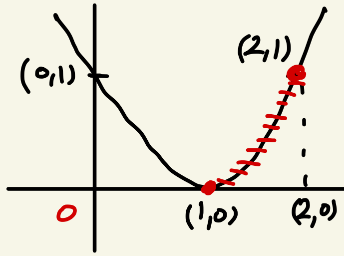
discont at $x = 0 ; 1 ; 2$.

Non-der at $\underbrace{x = 0 ; 1 ; 2 ; \frac{3}{2}}_{\text{Ans}}$.

Q $f(x) = \begin{cases} |1-4x^2| & 0 \le x < 1 \\ [x^2-2x] & 1 \le x \le 2 \end{cases}$. Check the differentiability in $[0, 2]$, where [ ] denotes greatest integer function.

Soln

$$f(x) \rightarrow \begin{cases} 1-4x^2 \; ; & 0 \le x < 1/2. \\ 4x^2-1 \; ; & \frac{1}{2} \le x < 1. \\ [\underbrace{(x-1)^2}_{0}]-1 \; ; & \underbrace{1 \le x \le 2}_{0}. \end{cases}$$



$$f(x) \rightarrow \begin{cases} 1-4x^2 \; ; & 0 \le x < \frac{1}{2} \\ 4x^2-1 \; ; & \frac{1}{2} \le x < 1. \\ 0-1 \; ; & 1 \le x < 2 \\ 1-1 \; ; & x = 2. \end{cases}$$

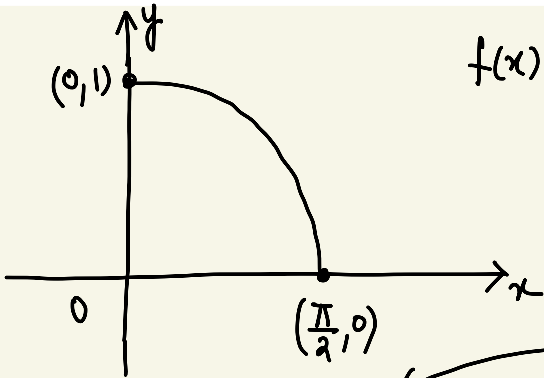
discontinuous at $x = 1; 2.$

Non-der at $x = 1; 2; \frac{1}{2}$. Ans

Q

$f(x) = \cos x$ and $g(x) = \begin{cases} \min(f(t)/0 \le t \le x) & \text{for } 0 \le x \le \frac{\pi}{2} \\ \frac{\pi}{2} - x & \text{for } \frac{\pi}{2} < x \le 3 \end{cases}$ Check continuity and ~~derivable~~ derivability in [0, 3] of g(x).

Sol^n



$f(x) = \cos x$

$$g(x) \begin{cases} \cos x & ; \ 0 \le x < \pi/2 \\ 0 & ; \ x = \pi/2 \\ \frac{\pi}{2} - x & ; \ \frac{\pi}{2} < x \le 3. \end{cases}$$

✓ Cont in $[0, 3]$

✓ der in $[0, 3]$.

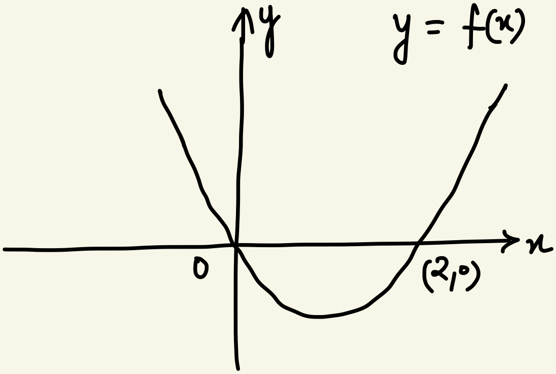
$LHD\Big|_{x=\frac{\pi}{2}} = -1$

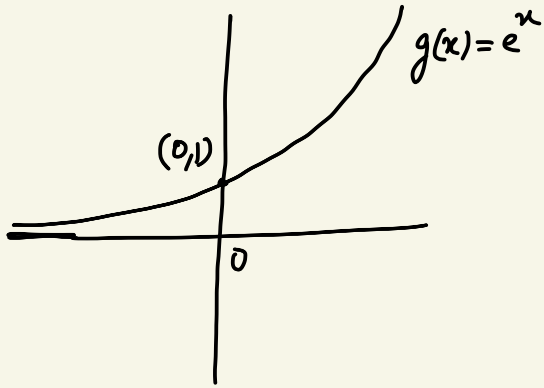
$RHD\Big|_{x=\pi/2} = -1$

Q

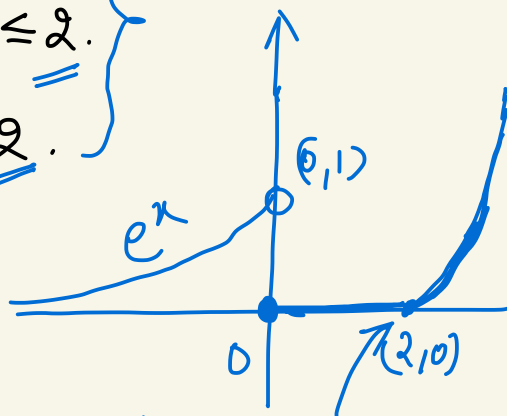
Let $f(x) = x^2 - 2x$, $g(x) = e^x$ and $H(x) = \begin{cases} \max(g(t), -\infty < t \le x) & , \; x < 0 \\ \max(f(t), 0 \le t \le x) & , \; x \ge 0 \end{cases}$.

If the number of points in $\mathbb{R}$ where H(x) is discontinuous is ℓ and the number of points where it is not differentiable is m, then find ℓ and m.





$$H(x) \begin{cases} e^x & ; \; x < 0. \\ 0 & ; \; 0 \le x \le 2. \\ x^2 - 2x & ; \; x > 2. \end{cases}$$



dis cont at $x = 0$.

Non-der at $x = 0; 2$.

Q

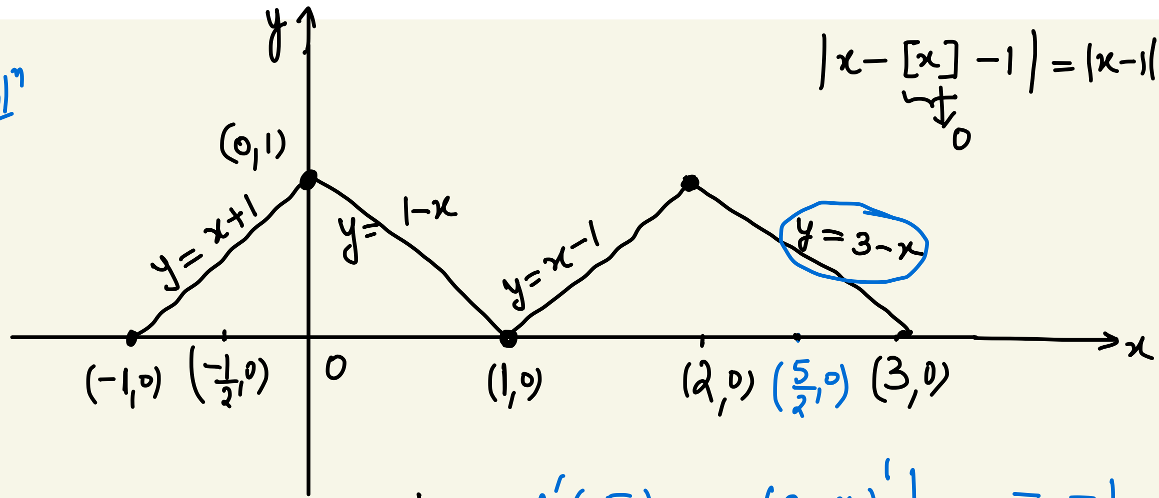
Let $f : \mathbb{R} \to \mathbb{R}$, defined by $f(x) = \begin{cases} |x-[x]| & \text{if } [x] \text{ is odd} \\ |x-[x+1]| & \text{if } [x] \text{ is even} \end{cases}$

(where [.] denotes greatest integer function), then find the value of

(i) $f'\left(\frac{5}{2}\right)$ (ii) Number of points where $f(x)$ is non-derivable for $x \in \left[-\frac{1}{2}, \frac{5}{2}\right]$

Sol^n

$|x-\underbrace{[x]}_{0}-1| = |x-1|$



(i) $f'\left(\frac{5}{2}\right) = (2-x)'\Big|_{x=\frac{5}{2}} = -1$

(ii) Non der at 3 points in $\left[-\frac{1}{2}, \frac{5}{2}\right]$

Ans

Q

Let f be differentiable at $x = a$ and $f'(a) = 4$ let $f(a) \neq 0$. Evaluate $\lim\limits_{n\to\infty} \left\{\dfrac{f(a+1/n)}{f(a)}\right\}^n$.

Sol^n

$$e^{L} \quad \text{where } L = \lim_{n\to\infty} n\left(\frac{f(a+\frac{1}{n})}{f(a)} - 1\right) \qquad (1^\infty \text{ form})$$

$$L = \lim_{n\to\infty} \underbrace{\left(\frac{f(a+\frac{1}{n}) - f(a)}{\frac{1}{n}}\right)}_{Rf'(a) = f'(a)} \cdot \frac{1}{f(a)}$$

$$L = \frac{f'(a)}{f(a)}$$

Note :-

Let $f(x)$ be a function whose derivative is $f'(x)$ then $\lim\limits_{y\to x} \dfrac{f(x) - f(y)}{x - y} = f'(x)$

Q

If $f(x)$ is differentiable at $x = a$ and $f'(a) = \dfrac{1}{4}$. Find $\lim\limits_{h\to 0} \dfrac{f(a+2h^2) - f(a-2h^2)}{h^2}$

Sol^n

$$\lim_{h\to 0}\left(\frac{f(a+2h^2) - f(a)}{h^2} + \frac{f(a) - f(a-2h^2)}{h^2}\right)$$

$$\lim_{h\to 0}\left(\underbrace{\frac{f(a+2h^2) - f(a)}{2h^2}}_{Rf'(a)} \cdot 2\right) + 2\left(\underbrace{\frac{f(a-2h^2) - f(a)}{-2h^2}}_{Lf'(a)}\right)$$

$$2f'(a) + 2f'(a) = 4f'(a) = 4 \times \frac{1}{4} = 1.$$

Q If f is derivable function and

$$\lim_{h \to 0} \frac{f((a+h)^2) - f(a^2)}{h} = 5$$ then find the

value of $f'(a^2) = ?$

Soln

$$\underbrace{\lim_{h \to 0} \left(\frac{f(a^2 + h^2 + 2ah) - f(a^2)}{(h^2 + 2ah)} \right)} \cdot \underbrace{(h+2a)} = 5$$

$$f'(a^2) \cdot 2a = 5$$

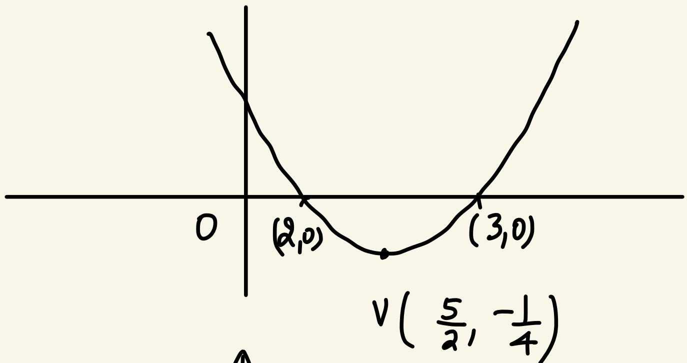
$$\boxed{f'(a^2) = \frac{5}{2a}}$$ Ans

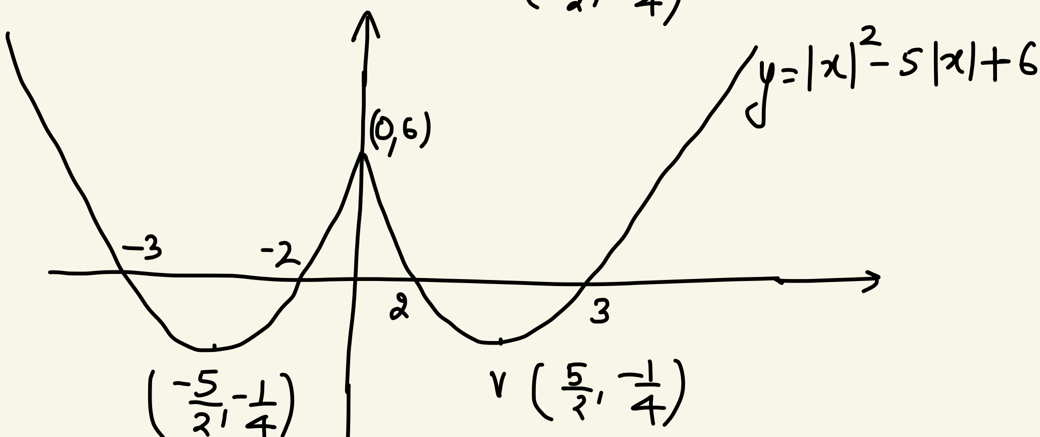
Q Find number of points where $f(x) = |x^2 - 5|x| + 6|$ is non-derivable ?

Sol^n

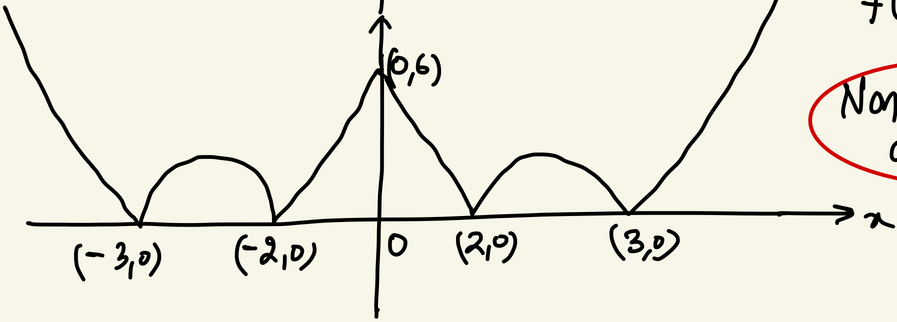
$$x^2 - 5x + 6 \xrightarrow{x \to |x|} x^2 - 5|x| + 6 \xrightarrow{|\ |} f(x)$$

$$g(x) = x^2 - 5x + 6$$





f(x)



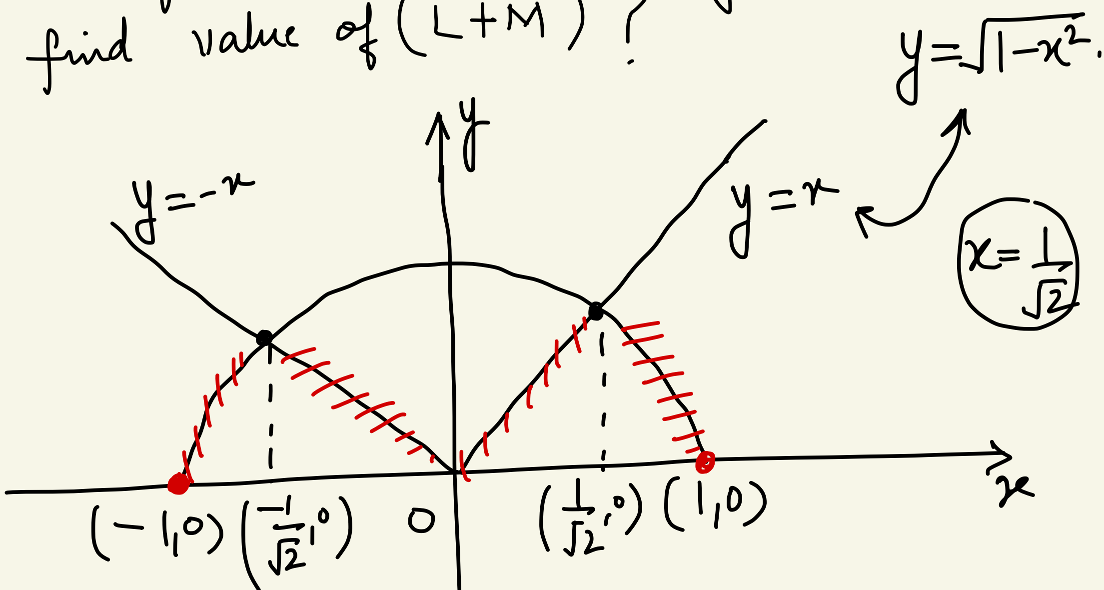
Non-der at 5 points

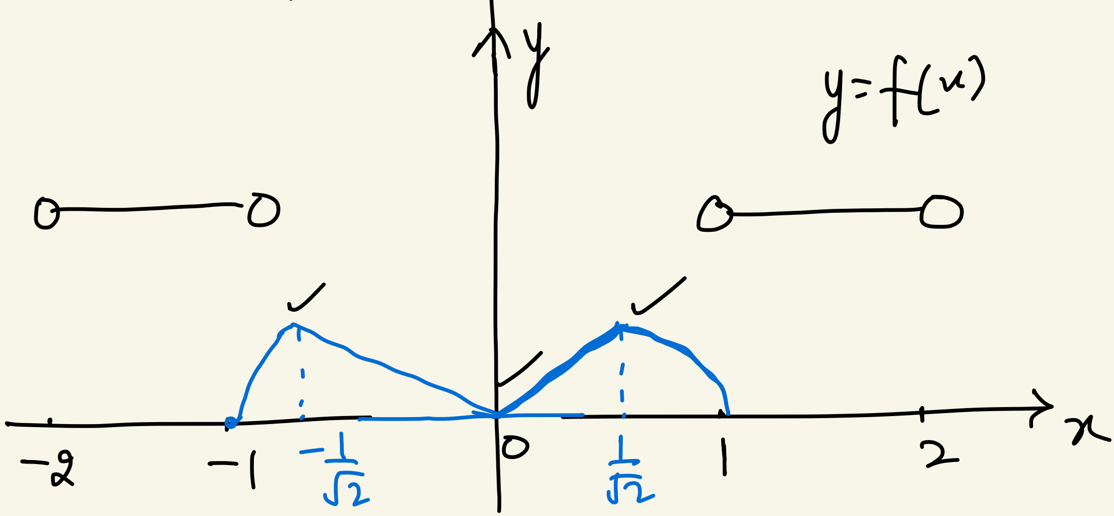
Q Consider $f(x) \to \begin{cases} [|x|] \; ; \; 1 < |x| < 2 \\ \min\left(|x|, \sqrt{1-x^2}\right) ; \; -1 \le x \le 1 \end{cases}$

([ ] → Gif)

If 'L' be no. of points of non-derivability and 'M' be no. of points of discontinuity in $(-2, 2)$ then find value of $(L+M)$?

Sol^n





discont at 2 points $x = \pm 1$

$\therefore \boxed{M = 2}$

Non-der at $-1; 1; -\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0$

$\boxed{L = 5}$

$\therefore \boxed{L + M = 7}$ Ans

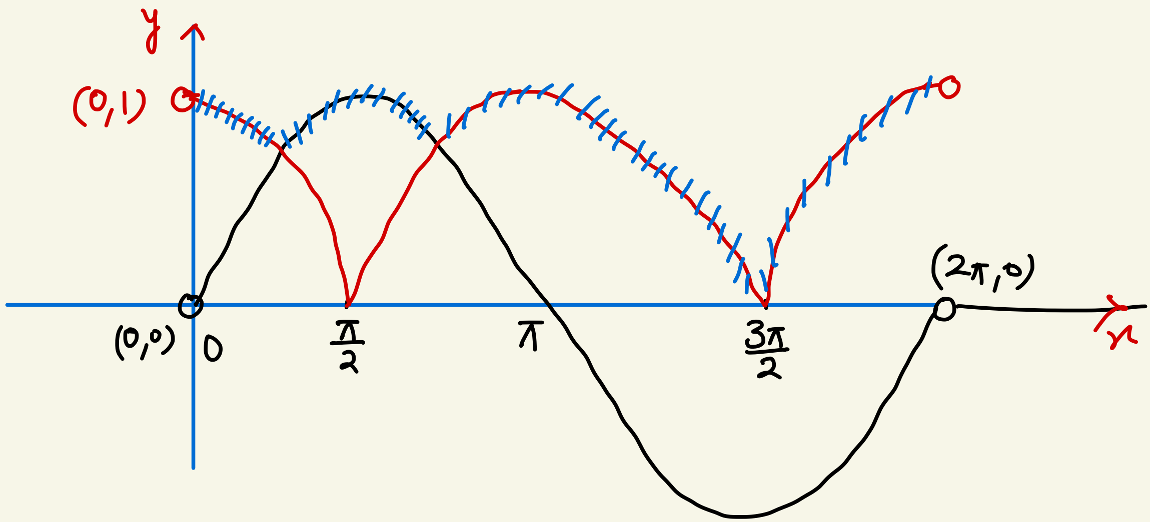
Q. Let $f(x) = \sin x + |\cos x| + |\sin x - |\cos x||$; $x \in (0, 2\pi)$ then find no. of points where $f(x)$ is non-derivable ?

Q/ Let $f(x) = \sin x + |\cos x| + \left|\sin x - |\cos x|\right|$; $x \in (0, 2\pi)$ then find no. of points where $f(x)$ is non-derivable?

Sol^n

$$f(x) = h(x) + g(x) + |h(x) - g(x)| = 2\max(h(x), g(x))$$

$$f(x) = 2\max(\sin x, |\cos x|).$$



$\therefore$ Non-der at 3 points

$x = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{3\pi}{2}$

Q

If $f(x) = \begin{cases} (\ell n(1+x))^m . \sin\left(\dfrac{1}{\ell n(1+x)}\right) & x > 0 \\ 0 & x = 0 \end{cases}$, then find the set of value of m for which

(a) $f(x)$ is continuous but not differentiable.

(b) $f(x)$ is continuous and differentiable.

(c) $f(x)$ is discontinuous

Sol^n (a) $f(x)$ is cont at $x=0$

$$f(0) = \lim_{x \to 0^+} f(x) \Rightarrow 0 = \lim_{x \to 0^+} \underbrace{(\ln(1+x))^m}_{0} \cdot \underbrace{\sin\left(\frac{1}{\ln(1+x)}\right)}_{\text{oscillatory}}$$

$$0 \times [-1,1]$$

$$\therefore \boxed{m > 0} \; — (1) —$$

$f(x)$ is Non-der at $x=0$.

$$f'(0) = \lim_{h \to 0}\left(\frac{f(0+h) - \cancel{f(0)}}{h}\right)$$

$$= \lim_{h \to 0} \frac{(\ln(1+h))^m \cdot \sin\left(\frac{1}{\ln(1+h)}\right)}{h}$$

$$= \lim_{h \to 0} \underbrace{\left(\frac{\ln(1+h)}{h}\right)^m}_{} \cdot \underbrace{h^{m-1}}_{0} \cdot \underbrace{\sin(\quad)}_{0}$$

$$\boxed{m - 1 \leq 0} \; — (2) —$$

From ① ∩ ②

$$m \in (0, 1] \quad \text{Ans}$$

(b) $m - 1 > 0 \Rightarrow m \in (1, \infty)$.

(c) $m \leq 0. \Rightarrow m \in (-\infty, 0]$.

Note :-

(1)

Derivative of continuous function need not be a continuous function i.e. if $f'(x)$ exists everywhere in its domain then $f'(x)$ need not to be necessarily continuous everywhere.

★ e.g. $f(x) = \begin{cases} x^2 \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$. Here $f(x)$ is differentiable $\forall x \in \mathbb{R}$, but $f'(x)$ is not continuous at $x = 0$.

$$f(0) = 0 \;;\; \lim_{x\to 0} f(x) = \lim_{x\to 0}\left(x^2 \sin\frac{1}{x}\right) = 0.$$

$f(x)$ is continuous at $\underline{x=0}$

$$f'(x) \to \begin{cases} 2x\sin\left(\frac{1}{x}\right) + \cancel{x^2}\cdot\cos\left(\frac{1}{x}\right)\left(\frac{-1}{\cancel{x^2}}\right) \;; & x \neq 0 \\ 0 \;; & x = 0 \end{cases}$$

$$f'(x) \to \begin{cases} 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right) \;; & x \neq 0 \\ 0 \;; & x = 0. \end{cases}$$

Not continuous at $\underline{x=0}$

** (2) If $f(x)$ is derivable at $x=a$ such that $f(a)=0$ and $g(x)$ is continuous at $x=a$ then the product function $f(x)\cdot g(x)$ will be derivable at $x=a$.

eg: $f(x) = \underbrace{(x^2-1)}_{\downarrow}\ \underbrace{|x-1|}_{\downarrow}$.

Soln $f(x)$ is der $\forall\ x \in \underline{R}$

OR

M-2 proper method

$f'_+(1) =$

$f'_-(1) =$

2 $f(x) = (x-1)(x+2)|(x+2)(x-3)| + \sin|x+2| + \cos|x| - \text{sgn}|x|$ is non derivable at

(where sgn(.) denotes signum function)

(A) 1 (B) –2 ✓ (C) 3 ✓ (D) 0 ✓

Sol^n

$$f(x) = h(x) + g(x) + w(x) + m(x)$$

$h(x) = (x-1)(x+2)\,|(x+2)(x-3)| \longrightarrow$ Non-der at $x=3$

$g(x) = \sin|x+2| \longrightarrow$ Non-der at $x=-2$

$w(x) = \cos x \longrightarrow$ der everywhere

$m(x) = -\text{sgn}|x| \longrightarrow$ Non-der at $x=0$.

(3)

Number of points of non-differentiability of the function $f(x) = x(x-1)|e^x - 1|\,|x-3|^3$ on $x \in (-\infty, \infty)$ is equal to [**Ans.** 0]

Sol^n

$$f(x) = x(x-1)\,|e^x - 1|\,|x-3|^3 \quad **$$

$|e^x-1| \to x=0 \qquad |x-3|^3 \to x=3$

$|x-3|^3 \to (x-3)^2|x-3|$.

$\therefore$ $f(x)$ is der at $x=0$ ($\because$ ....)

Check at $x=3$ only.

$f'_+(3) = 0 \qquad f'_-(3) = 0$

Q

Let $f(x) = |\sin x||(x-1)||(x-\pi)||(x-2\pi)|\left|x-\frac{\pi}{2}\right|$ (where $\frac{9}{10} \le x \le 2\pi$) and $g(x) = \ell n(x-1) + |x-2|$ $+ |x+3|$ (where $x > 1$). Let number of points where $f(x)$ and $g(x)$ are non differentiable are respectively λ_1, λ_2, then $\lambda_1 + \lambda_2$ is

$f(x) \rightarrow$ doubtful points

$x = n\pi$

$x = \pi, 2\pi$ ✓

$\left.\begin{array}{l} x = 1 \\ x = \pi/2 \end{array}\right\}$ ✓

$\therefore f(x)$ is der at $x = \pi$ & $x = 2\pi$.

$f(x)$ is Non der at $x = 1$ & $x = \frac{\pi}{2}$

$\therefore \boxed{\lambda_1 = 2}$

$g(x) = \ln(x-1) + |x-2| + \underbrace{|x+3|}_{+ve}$

$x > 1$

Non-der at $x = 2$ only.

$\boxed{\lambda_2 = 1}$

DETERMINATION OF FUNCTION SATISFYING THE GIVEN FUNCTIONAL EQUATION :

(derivable)

(a) Write down the expression for $f'(x)$ as $f'(x) = \frac{f(x+h) - f(x)}{h}$

(b) Manipulate $f(x + h) - f(x)$ in such a way that the given functional rule is applicable. Now apply the functional rule and simplify the RHS to get $f'(x)$ as a function of x along with constant if any.

(c) Integrate $f'(x)$ to get $f(x)$ as a function of x and a constant of integration. In some cases a Differential Equation is formed which can be solved to get $f(x)$.

(d) Apply the boundary value conditions to determine the value of this constant.

Q1 Suppose f is a derivable function that satisfies the equation

$$f(x + y) = f(x) + f(y) + x^2y + xy^2$$

for all real numbers x and y. Suppose that $\lim_{x\to 0} \frac{f(x)}{x} = 1$, find $f(x)$

Sol^n

$$f'(x) = \lim_{h\to 0}\left(\frac{f(x+h) - f(x)}{h}\right)$$

M-1

$x = y = 0$

$\boxed{f(0) = 0}$

$$= \lim_{h\to 0}\left(\frac{\cancel{f(x)} + f(h) + x^2h + xh^2 - \cancel{f(x)}}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{f(h)}{h} + \frac{x^2\cancel{h}}{\cancel{h}} + \frac{xh^{\cancel{2}}}{\cancel{h}}\right)$$

$$\boxed{f'(x) = 1 + x^2} \xrightarrow[\text{wrt } 'x']{\text{integrate}} \boxed{f(x) = x + \frac{x^3}{3} + C}$$

$x = 0$

$f(0) = 0 + 0 + C$

$\boxed{0 = C}$

M-2

$$\boxed{\lim_{x\to 0} \frac{f(x)}{x} = 1 \quad \& \quad f(0) = 0} \to$$

$$f'(0) = \lim_{h\to 0}\left(\frac{f(0+h) - f(0)}{h}\right)$$

$$\boxed{f'(0) = 1}$$

$$f(x+y) = f(x) + f(y) + x^2 y + xy^2.$$

(partial differentiation method)

**

diff wrt 'x' :-

$$f'(x+y) = f'(x) + 0 + 2xy + y^2.$$

put $x=0$

$$f'(y) = f'(0) + y^2$$

$$f'(y) = 1 + y^2 \xrightarrow[\text{wrt 'y'}]{\text{integrate}} \boxed{f(y) = y + \frac{y^3}{3} + C}$$

Q2 Let 'f' be derivable function satisfying $f\left(\frac{x}{y}\right) = f(x) - f(y)$ for all $x, y > 0$ and if $f'(1) = 1$ then find $f(x)$?

$x \to x+h$
$y \to x$

$x = y = 1$

$\boxed{f(1) = 0}$

Sol^n M-1 $f'(x) = \lim_{h \to 0} \left(\frac{f(x+h) - f(x)}{h} \right)$

$= \lim_{h \to 0} \left(\frac{f\left(\frac{x+h}{x}\right)}{h} \right) = \lim_{h \to 0} \frac{f\left(1+\frac{h}{x}\right) - 0}{h}$

$f'(x) = \lim_{h \to 0} \left(\frac{f\left(1+\frac{h}{x}\right) - f(1)}{\frac{h}{x} \cdot x} \right)$

$f'(1)$

$\boxed{f'(x) = \frac{1}{x}}$ $\xrightarrow[\text{wrt } 'x']{\text{integrate}}$ $f(x) = \ln|x| + c$

$\boxed{f(x) = \ln x + C}$

$f(1) = 0 + C$

$\boxed{C = 0}$

M-2 $\quad f\left(\frac{x}{y}\right) = f(x) - f(y) \quad ; \quad f(1) = 0$

diff wrt 'y' :-

$$f'\left(\frac{x}{y}\right) \cdot x\left(-\frac{1}{y^2}\right) = 0 - f'(y)$$

put $y = 1$.

$$x\, f'(x) \cdot (-1) = -f'(1)$$

$$f'(x) = \frac{f'(1)}{x}.$$

$$\boxed{f'(x) = \frac{1}{x}}$$

$\longrightarrow$ integrate

Q

Let $f:\mathbb{R}\to(-\pi,\pi)$ be differentiable function such that $f(x)+f(y)=f\left(\frac{x+y}{1-xy}\right), \forall x, y \in \mathbb{R}$ and $xy \neq 1$.

If $f(1)=\frac{\pi}{2}$ and $\lim_{x\to 0}\frac{f(x)}{x}=2$, find $f(x)$.

$x=y=0$

$f(0)+f(0)=f(0) \Rightarrow \boxed{f(0)=0}$

$y\to(-x)$

$f(x)+f(-x)=f(0)$

$\boxed{f(x)+f(-x)=0}$

$f(x)\to$ odd fn

$x\to x+h$

$y\to -x$

Sol^n

$$f'(x)=\lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right)$$

$$f'(x)=\lim_{h\to 0}\left(\frac{f(x+h)+f(-x)}{h}\right)$$

$$=\lim_{h\to 0}\frac{f\left(\frac{\cancel{x}+h-\cancel{x}}{1+(x+h).x}\right)}{h}$$

$$f'(x)=\lim_{h\to 0}\frac{f\left(\frac{h}{1+x^2+xh}\right)}{h}$$

$\boxed{C=0}$

$$f'(x)=\lim_{h\to 0}\frac{f\left(\frac{h}{1+x^2+xh}\right)}{\left(\frac{h}{1+x^2+xh}\right)(x^2+xh+1)}=\frac{2}{x^2+1}$$

Integrate wrt 'x' :

$$\boxed{f(x)=2\tan^{-1}x+C}$$

$$f(1)=2\cdot(\pi/4)+C$$

Q

If $f\left(\frac{2x+3y}{5}\right)=\frac{2f(x)+3f(y)}{5}\ \forall x, y \in \mathbb{R}$, $f(0)=1$, $f'(1)=1$, then find $f(x)$

Q

A function $f:\mathbb{R}-\{0\}\to\mathbb{R}-\{0\}$ is defined as $f\left(\frac{x}{y}\right)=\frac{f(x)}{f(y)}$ and $f'(1)=3$ then find $f(x)$.

L'HÔPITAL'S RULE : (L'H RULE)

Indeterminate forms of the type $\frac{0}{0}, \frac{\infty}{\infty}$. If the function f(x) and g(x) are differentiable in certain neighbourhood of the point a, except, may be, at the point a itself, and $g'(x) \neq 0$, and if

$\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$ or $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = \infty$,

then $\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} = \lim_{x\to a} \frac{f''(x)}{g''(x)}$ and so on.

provided the limit $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists (L' Hôpital's rule). The point 'a' may be either finite or improper $+\infty$ or $-\infty$.

Q1 $\lim_{x\to 0} \left(\frac{x\cos x - \ln(1+x)}{x^2}\right)$ $\left(\frac{0}{0} \text{ form}\right)$

Sol^n M-1 Use expansion

M-2 Apply LH Rule.

$$\lim_{x\to 0} \frac{x(-\sin x) + \cos x - \left(\frac{1}{1+x}\right)}{2x} \quad \left(\frac{0}{0} \text{ form}\right)$$

Apply LH Rule

$$\lim_{x\to 0} \frac{-(x\cos x + \sin x) - \sin x + \left(\frac{1}{1+x}\right)^2}{2} = \frac{1}{2}$$

② $$\lim_{x\to 0} \frac{1-\cos x\cdot\cos 2x\cdot\cos 3x}{x^2}$$ $\left(\frac{0}{0}\right)$ $(fgh)' = f'gh + fg'h + fgh'$

LH Rule :

$$\lim_{x\to 0} \frac{0 - \left((-\sin x)\cos 2x\cos 3x + (\cos x)(-2\sin 2x)\cos 3x + (\cos x)(\cos 2x)(-3\sin 3x)\right)}{2x}$$

$$\lim_{x\to 0} \frac{1}{2}\left(\underbrace{\left(\frac{\sin x}{x}\right)}_{1}\underbrace{\cos 2x}_{1}\underbrace{\cos 3x}_{1} + 2^2\frac{\sin 2x}{2x}\cos x\cos 3x + 3^2\frac{\sin 3x}{3x}\cos x\cos 2x\right)$$

$$\frac{1}{2}\left(1^2 + 2^2 + 3^2\right)$$

Note:

$$\lim_{x\to 0}\left(\frac{1-\cos(ax)\cos(bx)\cos(cx)\cos(dx)}{x^2}\right)$$

$$\frac{1}{2}\left(a^2+b^2+c^2+d^2\right).$$ Aoy

3) $$\lim_{x\to 0} \frac{\log_{\sec\frac{x}{2}}(\cos x)}{\log_{\sec x}\left(\cos(x/2)\right)}$$

$$\lim_{x\to 0} \frac{\left(\dfrac{\ln \cos x}{\ln \sec\frac{x}{2}}\right)}{\left(\dfrac{\ln \cos x/2}{\ln \sec x}\right)} = \lim_{x\to 0}\left(\frac{\ln \cos x}{\ln \cos\frac{x}{2}}\right)^2 \quad \left(\frac{0}{0}\text{ form}\right)$$

$$= \Bigg(\underbrace{\lim_{x\to 0} \frac{\ln \cos x}{\ln \cos\frac{x}{2}}}_{l}\Bigg)^2 = l^2$$

$$l = \lim_{x\to 0}\left(\frac{\ln \cos x}{\ln \cos\frac{x}{2}}\right)$$

LH Rule

$$l = \lim_{x\to 0}\left(\frac{\frac{1}{\cos x}\cdot(-\sin x)}{\frac{1}{\cos\frac{x}{2}}\left(-\sin\frac{x}{2}\right)\cdot\frac{1}{2}}\right) \Rightarrow l = \lim_{x\to 0}\left(2\,\frac{\tan x}{\tan\frac{x}{2}}\right)$$

$$\boxed{l = 4}$$

$$4^2 = 16 \text{ Ans}$$

** 4) $$\lim_{x\to 1} \frac{x^x - x}{x - 1 - \ln x}$$ $\left(\frac{0}{0}\text{ form}\right)$

Rem $\boxed{(x^x)' = x^x(1+\ln x).}$

$$x^x = e^{x\ln x}$$

$$\left(e^{x\ln x}\right)' = e^{x\ln x}\left(\cancel{x}\cdot\frac{1}{\cancel{x}} + 1\cdot\ln x\right)$$

Apply LH Rule :

$$\lim_{x\to 1} \frac{x^x(1+\ln x) - 1}{1 - 0 - \left(\frac{1}{x}\right)} \quad \left(\frac{0}{0}\text{ form}\right)$$

Apply LH Rule.

$$\lim_{x\to 1} \frac{x^x\left(\frac{1}{x}\right) + x^x(1+\ln x)\cdot(1+\ln x)}{0 + \left(\frac{1}{x^2}\right)} = \frac{1+1}{1} = 2$$

5) HW

$$\lim_{x\to 4} \frac{\sqrt{2x+1}+\sqrt{x-3}-4}{\sqrt{(3x+4)}+\sqrt{5x+5}-9}$$

$\left(\frac{0}{0} \text{ form}\right)$ $(\sqrt{x})' = \frac{1}{2\sqrt{x}}$

Apply LH Rule.

$$\lim_{x\to 4} \left(\frac{\frac{2}{2\sqrt{2x+1}} + \frac{1}{2\sqrt{x-3}}}{\frac{3}{2\sqrt{3x+4}} + \frac{5}{2\sqrt{5x+5}}} \right) =$$

**

6)

$$\lim_{x\to 0^+} (\operatorname{cosec} x)^{\frac{1}{\ln x}}$$

(∞^0)

as $x \to 0^+$

$\ln x \to -\infty$

$\frac{1}{\ln x} \to 0^-$

$\operatorname{cosec} x \to \infty$

$$\ln l = \lim_{x\to 0^+} \frac{1}{\ln x} \cdot \ln(\operatorname{cosec} x)$$

$$\ln l = \lim_{x\to 0^+} \left(\frac{\ln \operatorname{cosec} x}{\ln x} \right)$$

$\left(\frac{\infty}{\infty} \text{ form}\right)$

Apply LH Rule

$$\ln l = \lim_{x \to 0^+} \left(\frac{\frac{1}{\cancel{\operatorname{cosec} x}} \left(- \cancel{\operatorname{cosec} x} \cot x \right)}{\frac{1}{x}} \right)$$

$$\ln l = \lim_{x \to 0^+} \left(- \frac{x}{\tan x} \right) = -1$$

$$\ln l = -1$$

$$\boxed{l = e^{-1}} \text{ Ans}$$

⑦ $$\lim_{x\to 0}\left(\frac{1}{x}\right)^{\tan x}$$

∞^0

$$\ln l = \lim_{x\to 0^+} \tan x \ln\left(\frac{1}{x}\right)$$

$$\ln l = \lim_{x\to 0^+} \left(-\frac{\ln x}{\cot x}\right) \quad \left(\frac{\infty}{\infty}\right)$$

<u>LH Rule</u>

$$\ln l = \lim_{x\to 0^+} \frac{(-1/x)}{(-\operatorname{cosec}^2 x)}$$

$$\ln l = \lim_{x\to 0^+} \left(\frac{\sin^2 x}{x}\right) = 0.$$

$$\ln l = 0 \Rightarrow \boxed{l = e^0 = 1}$$

⑧ $$\lim_{x\to 0} (x)^x$$

9) $\lim_{x \to 0^+} \left(x^x + (\tan x)^{\operatorname{cosec} x} + (\operatorname{cosec} x)^{\tan x} \right)$

** 10) If 'f' is derivable function with $f'(2) = 3$ then find the value of $\lim_{x \to 2} \left(\frac{f(x) - f(2)}{\sqrt{x} - \sqrt{2}} \right)$?

Sol^n LH Rule

$$\lim_{x \to 2} \frac{f'(x) - 0}{\left(\frac{1}{2\sqrt{x}} \right)} = \lim_{x \to 2} \left(2\sqrt{x}\, f'(x) \right)$$

$$= 2\sqrt{2} \cdot (3)$$

$$= 6\sqrt{2}$$

Note :-

$$f(x) \begin{cases} x^2 \sin\left(\frac{1}{x}\right) & ; x \neq 0 \\ 0 & ; x = 0 \end{cases}$$

LHD : $f'_-(0) = \lim_{h \to 0} \left(\frac{f(0-h) - \cancel{f(0)}^{0}}{-h} \right) = \lim_{h \to 0} \frac{h^2 \sin\left(-\frac{1}{h}\right)}{(-h)}$

$= 0.$

RHD : $f'_+(0) = \lim_{h \to 0} \frac{h^2 \sin\left(\frac{1}{h}\right)}{h} = 0.$

$f(x)$ is derivable at $x = 0$.

$$f'(x) \begin{cases} 2x \sin\left(\frac{1}{x}\right) + \cancel{x^2} \cdot \cos\left(\frac{1}{x}\right)\left(\frac{-1}{\cancel{x^2}}\right) & ; x \neq 0 \\ 0 & ; x = 0. \end{cases}$$

Q1

If $f\left(\frac{2x+3y}{5}\right)=\frac{2f(x)+3f(y)}{5}\ \forall x,y\in\mathbb{R}$, $f(0)=1$, $f'(1)=1$, then find $f(x)$

Q2

A function $f:\mathbb{R}-\{0\}\to\mathbb{R}-\{0\}$ is defined as $f\left(\frac{x}{y}\right)=\frac{f(x)}{f(y)}$ and $f'(1)=3$ then find $f(x)$.

1 Sol^n

$$f'\left(\frac{2x+3y}{5}\right)\cdot\frac{\cancel{2}}{\cancel{5}}=\frac{\cancel{2}f'(x)}{\cancel{5}}$$

$$f'\left(\frac{3y}{5}\right)=1. \qquad \frac{3y}{5}=t$$

$$f'(t)=1 \Rightarrow f(t)=t+c.$$

$$f(0)=0+c \Rightarrow c=1.$$

$$\boxed{f(x)=x+1}\ ✓$$

2) A function $f:\mathbb{R}-\{0\}\to\mathbb{R}-\{0\}$ is defined as $f\left(\frac{x}{y}\right)=\frac{f(x)}{f(y)}$ and $f'(1)=3$ then find $f(x)$. **[Ans.** x^3]

Solⁿ

$x=y=1$

$f(1)=\frac{f(1)}{f(1)}$

$\Downarrow$

$\boxed{f(1)=1}$

$$f'(x)=\lim_{h\to0}\left(\frac{f(x+h)-f(x)}{h}\right)$$

$$f'(x)=\lim_{h\to0} f(x)\left(\frac{f\left(\frac{x+h}{x}\right)-1}{h}\right)$$

$$\frac{f'(x)}{f(x)}=\lim_{h\to0}\frac{f\left(1+\frac{h}{x}\right)-f(1)}{\frac{h}{x}\cdot x}$$

$$\frac{f'(x)}{f(x)}=\frac{f'(1)}{x}=\frac{3}{x}$$

$$\ln|f(x)|=3\ln|x|+c.$$

$$\ln|f(x)|-\ln|x|^3=c\Rightarrow\ln\left(\frac{|f(x)|}{|x|^3}\right)=c\Rightarrow f(x)=\pm\lambda x^3$$

$$f(1)=\pm\lambda\Rightarrow\lambda=\pm1$$

$\therefore f(x)=x^3$ OR $f(x)=-x^3$

But $f(1)=1$

$\therefore \boxed{f(x)=x^3}$ Ans

⑧ $\lim_{x\to 0} (x)^x = \lim_{x\to 0^+} (x)^x$

$$\ln l = \lim_{x\to 0^+} x.(\ln x) \quad ; \qquad (0\times\infty)$$

$$\ln l = \lim_{x\to 0^+} \frac{(\ln x)}{(1/x)} \qquad \left(\frac{\infty}{\infty}\right)$$

LH Rule:

$$\ln l = \lim_{x\to 0^+} \frac{\left(\frac{1}{x}\right)}{\left(-\frac{1}{x^2}\right)} = 0. \Rightarrow \boxed{l = e^0 = 1}$$

Ans

9) $\lim_{x\to 0^+} \left(x^x + \underbrace{(\tan x)^{\operatorname{cosec} x}}_{0^\infty} + \underbrace{(\operatorname{cosec} x)^{\tan x}}_{\infty^0} \right) = 2$ Ans

$$(1) + 0 + \quad l \to 1$$

$$\ln l = \lim_{x\to 0^+} (\tan x) \ln(\operatorname{cosec} x) \qquad (0\times\infty)$$

$$\ln l = \lim_{x\to 0^+} \left(\frac{\ln \operatorname{cosec} x}{\cot x}\right) \qquad \left(\frac{\infty}{\infty}\right)$$

LH Rule $\lim_{x\to 0^+} \dfrac{\frac{1}{\cancel{\operatorname{cosec} x}} \times (-\cancel{\operatorname{cosec} x} \cot x)}{-\operatorname{cosec}^2 x} = 0$

Method of differentiation (MOD)

DERIVATIVE BY FIRST PRINCIPLE :

$x \to x+\Delta x$
$y \to y+\Delta y$

Let $y = f(x)$; $y + \Delta y = f(x + \Delta x)$

$$\therefore \quad \boxed{\frac{\Delta y}{\Delta x} = \frac{f(x+\Delta x)-f(x)}{\Delta x}} \text{(average rate of change of function)}$$

$$\therefore \quad \boxed{\frac{dy}{dx} = \lim_{\Delta x \to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0}\frac{f(x+\Delta x)-f(x)}{\Delta x}} \quad \ldots\ldots \text{(i)}$$

(i) denotes the instantaneous rate of change of function and gives slope of tangent at any point on curve $y = f(x)$.
Finding the value of the limit given by (i) in respect of variety of functions is called finding the derivative by first principle/by delta method/by ab-initio method/by fundamental definition of calculus.

Note then if $y = f(x)$ then the symbols $\frac{dy}{dx} = Dy = f'(x) = y_1$ or y' have the same meaning.

2^{nd} order derivative $\frac{d^2y}{dx^2} = D^2y = f''(x) = y_2$ or y''.

Q

Find derivative of $y = \cos(x^2)$ using first principle.

$$\frac{dy}{dx} = \lim_{h\to 0}\frac{\cos(x+h)^2 - \cos x^2}{h} = \lim_{h\to 0}\frac{2\sin\left(x^2+hx+\frac{h^2}{2}\right)\sin\left(-hx-\frac{h^2}{2}\right)}{h} = -2\sin x^2 . \lim_{h\to 0}\frac{\sin\left(h\left(x+\frac{h}{2}\right)\right)}{h}$$

$= -2x\sin(x^2)$

Note : If y = f(x) is a derivable function $\lim_{\delta x \to 0} \frac{\delta x}{\delta y} \cdot \frac{\delta y}{\delta x} = 1 \Rightarrow \frac{dy}{dx} \cdot \frac{dx}{dy} = 1$

(None of individual limits tends towards zero)

Q

Find D($\sin^{-1}$ x) using first principle :

$y = \sin^{-1}x \qquad -\frac{\pi}{2} \le y \le \frac{\pi}{2}$

$x = \sin y \qquad \Rightarrow \qquad x + \Delta x = \sin(y + \Delta y)$

$$\frac{\Delta x}{\Delta y} = \frac{\sin(y+\Delta y) - \sin y}{\Delta y} \Rightarrow \frac{\Delta y}{\Delta x} = \frac{1}{\frac{\sin(y+\Delta y)-\sin y}{\Delta y}} \Rightarrow \lim_{\Delta x \to 0}\frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0}\frac{1}{\frac{\sin(y+\Delta y)-\sin y}{\Delta y}}$$

$\frac{dy}{dx} = \frac{1}{\cos y} \qquad y \ne \pm\frac{\pi}{2}$

$\frac{dy}{dx} = \frac{1}{\sqrt{1-\sin^2 y}} = \frac{1}{\sqrt{1-x^2}} \qquad x \ne \pm 1$

$* \frac{d}{dx}(\log_a x) = \frac{d}{dx}\left(\frac{\ln x}{\ln a}\right)$

$= \frac{1}{\ln a} \cdot \left(\frac{1}{x}\right) = \frac{1}{x}\log_a e$

DERIVATIVE OF STANDARD FUNCTIONS :

	f(x)	f'(x)		f(x)	f'(x)
(i)	x^n	nx^{n-1}	(ii)	e^x	e^x
(iii)	a^x	$a^x \ell na$, $a > 0$	(iv)	ℓnx	$1/x$
(v)	$\log_a x$	$(1/x)\log_a e$, $a > 0$, $a \ne 1$	(vi)	sinx	cosx
(vii)*	cosx	– sinx	(viii)	tanx	$\sec^2 x$
(ix)	secx	secx tanx	(x) *	cosecx	– cosecx . cotx
(xi)*	cotx	$-\text{cosec}^2 x$	(xii)	constant	0

Note that the derivative value of all co-trigonometric functions begin with a –ve sign.

Derivative inverse trigonometric function

	f(x)	f'(x)		f(x)	f'(x)
(i)	$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$, $-1 < x < 1$	(ii)*	$\cos^{-1} x$	$-\frac{1}{\sqrt{1-x^2}}$, $-1 < x < 1$
(iii)	$\tan^{-1} x$	$\frac{1}{1+x^2}$, $x \in R$	(iv)	$\sec^{-1} x$	$\frac{1}{\lvert x \rvert\sqrt{x^2-1}}$, $\lvert x \rvert > 1$
(v)*	$\text{cosec}^{-1} x$	$-\frac{1}{\lvert x \rvert\sqrt{x^2-1}}$, $\lvert x \rvert > 1$	(vi)*	$\cot^{-1} x$	$-\frac{1}{1+x^2}$, $x \in R$

$$\frac{d}{dx}\left(\ln(\underbrace{3x+2}_{X})\right) = \frac{1}{(3x+2)}\cdot(3) = \frac{3}{3x+2}.$$

$$\frac{d}{dx}\left(\sin(\underbrace{4x-7}_{X})\right) = 4\cdot\cos(4x-7).$$

$$\frac{d}{dx}\left(\sin\left(\underbrace{\ln\left(x^3\right)}_{X}\right)\right) = \cos\left(\ln\left(x^3\right)\right)\cdot\frac{1}{(x^3)}\cdot(3x^2)$$

(1) $D\left(\cos^4\frac{x}{2} - \sin^4\frac{x}{2}\right) = D(\cos x) = -\sin x.$

(2) $D\left(\underbrace{\tan^{-1}}\, x + \underbrace{\cot^{-1}}\, x\right) = D(\pi/2) = 0.$

Q

If $y = \{x\}^{[x]}$, $x \notin I$, then $y'\left(\frac{e}{2}\right)$ is equal to (where $\{.\}$ denotes fractional part function & $[.]$ denotes greatest integer function)

Sol^n

$$y = (x - [x])^{[x]}$$

$$y' = [x]\,(x - [x])^{[x]-1}$$

$$y'\left(\frac{e}{2}\right) = \left[\frac{e}{2}\right]\left(\frac{e}{2} - \left[\frac{e}{2}\right]\right)^{\left[\frac{e}{2}\right]-1}$$

$$= 1.\left(\frac{e}{2} - 1\right)^0 = 1 \text{ Ans}$$

$y = x^n$

$y' = nx^{n-1}.$

$\left[\frac{e}{2}\right] = 1.$

SUPPLEMENTARY THEOREMS/RESULT :

(a) Sum/difference rule : $D(f(x) \pm g(x)) = f'(x) \pm g'(x)$

E(1) If $y = e^x + 3\ln x - 4\tan^{-1}x + \underbrace{\sin 3x + 4\sin^3 x}$, then $\frac{dy}{dx}$ is ....

Sol^n

$$\frac{dy}{dx} = e^x + \frac{3}{x} - \frac{4}{1+x^2} + 3\cos 3x + 12\sin^2 x . \cos x$$

(b) Product rule : $D(f(x) . g(x)) = f(x) . g'(x) + g(x) . f'(x)$

Note : If 3 functions are involved then remember

Rem ① $D(\underbrace{f(x) . g(x) . h(x)}) = f(x) . g(x) . h'(x) + g(x) . h(x) . f'(x) + h(x) . f(x) . g'(x)$

② $$\boxed{D(fgh) = \frac{(fg)'h + (gh)'f + (fh)'g}{2}}$$

E(1) Let $F(x) = f(x) \,.\, g(x) \,.\, h(x)$. If for some $x = x_0$, $F'(x_0) = 21\, F(x_0)$; $f'(x_0) = 4f(x_0)$; $g'(x_0) = -7g(x_0)$ and $h'(x_0) = k\, h(x_0)$, then find k. **[JEE 97]**

E(2) If $f(x) = (1 + x)(3 + x^2)^{1/2}(9 + x^3)^{1/3}$, then $f'(-1)$ is equal to -

(A) 0 (B) $2\sqrt{2}$ (C) 4 (D) 6

① $\boxed{F(x) = f(x)\, g(x)\, h(x)}$ $F'(x_0) = 21 F(x_0)$

$$F'(x_0) = f'(x_0)\, g(x_0)\, h(x_0) + f(x_0)\, g'(x_0)\, h(x_0) + f(x_0)\, g(x_0)\, h'(x_0)$$

$$21\, \cancel{F(x_0)} = \cancel{f(x_0)\, g(x_0)\, h(x_0)}\, (4 - 7 + K).$$

$$21 + 3 = K \Rightarrow \boxed{K = 24} \text{ Ans}$$

② $f(x) = \underbrace{(1+x)}_{f}\, \underbrace{(3+x^2)^{1/2}}\, \underbrace{(9+x^3)^{1/3}}$; $f'(-1) = ?$

$$f'(x) = (1+x)(3+x^2)^{1/2}\left((9+x^3)^{1/3}\right)'$$
$$+ (1+x)(9+x^3)^{1/3}\left((3+x^2)^{1/2}\right)'$$
$$+ 1.\,(3+x^2)^{1/2}(9+x^3)^{1/3}$$

$$f'(-1) = \sqrt{4}\,.\,(8)^{1/3} = 2 \times 2 = 4.$$

Q If $f(x) = 1 + x + x^2 + x^3 + \cdots + x^{100}$

then find $f'(1)$?

Sol^n

$$f'(x) = 0 + 1 + 2x + 3x^2 + \cdots + 100x^{99}$$

$$f'(1) = 1 + 2 + 3 + \cdots + 100 = 5050.$$

(c) **Quotient rule :** $y = \dfrac{f(x)}{g(x)}$

Note :- Think twice before applying Quotient Rule.

$$D\left(\frac{f(x)}{g(x)}\right) = \frac{g(x).f'(x) - f(x).g'(x)}{g^2(x)}$$, to be remembered as $$D\left(\frac{N^r}{D^r}\right) = \frac{D^r\frac{d}{dx}(N^r) - N^r\frac{d}{dx}(D^r)}{(D^r)^2}$$

① If $y = \dfrac{x^4 - 3x^2 + 1}{x^2 + \sqrt{5}x + 1}$ then find y' ?

Solⁿ

$$y = \frac{(x^4 + 2x^2 + 1) - 5x^2}{x^2 + \sqrt{5}x + 1} \Rightarrow y = \frac{(x^2+1)^2 - (\sqrt{5}x)^2}{x^2 + \sqrt{5}x + 1}$$

$$y = x^2 + 1 - \sqrt{5}x$$

$$y' = 2x - \sqrt{5} \quad \text{Ans}$$

②

If $y = \dfrac{\sec x + \tan x - 1}{\tan x - \sec x + 1}$, find $\left.\dfrac{dy}{dx}\right]_{x=\pi/4}$

Solⁿ

$$y = \frac{(\sec x + \tan x) - (\sec^2 x - \tan^2 x)}{\tan x - \sec x + 1}$$

$$y = \frac{(\sec x + \tan x)\cancel{(\qquad)}}{\cancel{(\qquad)}}$$

$$y' = (\sec x \tan x + \sec^2 x)\Big|_{x=\pi/4}$$

③ If $y = \dfrac{\tan^{-1}x - \cot^{-1}x}{\tan^{-1}x + \cot^{-1}x}$, find $\left.\dfrac{dy}{dx}\right]_{x=-1}$

Solⁿ $y = \dfrac{\tan^{-1}x - \left(\frac{\pi}{2} - \tan^{-1}x\right)}{\pi/2}$

$y = \frac{2}{\pi}\left(2\tan^{-1}x - \frac{\pi}{2}\right)$

$y' = \frac{2}{\pi}\left(\frac{2}{1+x^2}\right)\Big|_{x=-1} = \frac{4}{\pi(2)} = \frac{2}{\pi}$ Ans

4) If $y = \cos x \cos 2x \cos 4x \cos 8x \cos 16x$

then $\left.\frac{dy}{dx}\right|_{x=\frac{\pi}{4}}$?

Solⁿ $y = \dfrac{\sin 32x}{2^5 \sin x}$

$\Rightarrow \frac{dy}{dx} = \frac{1}{32}\left(\frac{32 \sin x . \cos 32x - \sin 32x . \cos x}{\sin^2 x}\right)\Big|_{x=\pi/4}$

$= \frac{1}{\cancel{32}}\left(\frac{\cancel{32}.\left(\frac{1}{\sqrt{2}}\right)(1)}{(1/2)}\right) = \frac{2}{\sqrt{2}} = \sqrt{2}$ Ans

Chain Rule :-

Imp

If $y = f(u)$; $u = g(v)$; $v = w(x)$ then

$$\frac{dy}{dx} = ?$$

$$\frac{dy}{dx} = \left(\frac{dy}{du}\right)\left(\frac{du}{dx}\right) = \left(\frac{dy}{du}\right)\left(\frac{du}{dv}\right)\cdot\left(\frac{dv}{dx}\right)$$

$$\frac{d}{dx}(\sin x) = \cos x$$

$x \to$ radians

eg: $\frac{d}{dx}(\sin x^\circ)$

$$\frac{d}{dx}\left(\sin\frac{\pi x}{180}\right) = \frac{\pi}{180}\cdot\cos\left(\frac{\pi x}{180}\right)$$

Note :-

(1) $\frac{d}{dx}\left(fog(x)\right) = \frac{d}{dx}\left(f(\underset{\sim}{g(x)})\right) = f'(g(x)) \cdot g'(x)$

(2) $\frac{d}{dx}\left(\frac{1}{f(x)}\right) = \frac{-1}{f^2(x)} \cdot f'(x).$

(3) $\frac{d}{dx}\left(\sqrt{f(x)}\right) = \frac{1}{2\sqrt{f(x)}} \cdot f'(x).$

(4) $\frac{d}{dx}\left((f(x))^n\right) = n(f(x))^{n-1} \cdot f'(x).$

Q1 Find $\frac{dy}{dx}$ if :

(1) $y = \cos(\underbrace{\ln x^3})$?

$$\frac{dy}{dx} = -\sin(\ln x^3) \cdot \left(\frac{1}{\cancel{x^3}}\right)(3\cancel{x^2}) \cdot$$

$$= \frac{-3\sin(\ln x^3)}{x}$$

(2) $$y = (x^3 - x^2 + 1)^{10}.$$

$$\frac{dy}{dx} = 10\,(x^3 - x^2 + 1)^9 \cdot (3x^2 - 2x).$$

(3) $$y = \sin^2(x^5 + x^2 - 4x) = \Big(\underbrace{\sin(x^5+x^2-4x)}_{X}\Big)^2$$

$$\frac{dy}{dx} = \underbrace{2 \cdot \sin(x^5 + x^2 - 4x) \cdot \cos(x^5 + x^2 - 4x)} \cdot (5x^4 + 2x - 4).$$

Q

Suppose that f is a differentiable function such that f (2) = 1 and f ' (2) = 3 and let $g(x) = f(x\,f(x))$. Find g ' (2).

Sol^n

$$g(x) = f(\underbrace{x \cdot f(x)})$$

$$g'(x) = f'(x\ f(x)) \cdot (x\ f'(x) + f(x))$$

$$g'(2) = f'(2 \cdot f(2)) \cdot (2 f'(2) + f(2))$$

$$= f'(2) \cdot (2 \times 3 + 1)$$

$$= 3 \times (7) = 21$$

Q

Let h (x) = f(g(sin 4x)), find $h'\left(\frac{\pi}{24}\right)$. If $g\left(\frac{1}{2}\right) = 2$; $f'(2) = 4$, $g'\left(\frac{1}{2}\right) = \sqrt{3}$.

Sol^n

$$h'(x) = f'(g(\sin 4x)) \cdot g'(\sin 4x) \cdot (\cos 4x) \cdot 4.$$

$$h'\left(\frac{\pi}{24}\right) = f'\left(g\left(\sin\frac{\pi}{6}\right)\right) g'\left(\frac{1}{2}\right) \cdot \frac{\sqrt{3}}{2} \cdot 4$$

$$= f'(2) \cdot \sqrt{3} \cdot \frac{\sqrt{3}}{\cancel{2}} \cdot \cancel{4}^{\,2}$$

$$= 4 \cdot 3 \cdot 2 = 24 \text{ Ans}$$

Q If $y = \log_2 \log_3 x$ then find $y'(3)$?

$$\boxed{\frac{d}{dx}\left(\log_a x\right) = \left(\frac{1}{x \ln a}\right)}$$

Sol

$y = \log_2 X$; $X = \underbrace{\log_3 x}$

$$\frac{dy}{dx} = \frac{1}{\log_3 x} \cdot \frac{1}{\ln 2} \cdot \left(\frac{1}{\ln 3}\right)\left(\frac{1}{x}\right)$$

$$\left.\frac{dy}{dx}\right|_{x=3} = \frac{1}{\ln 2 (\ln 3)} \cdot \left(\frac{1}{3}\right) = \frac{1}{(\ln 3)(\ln 8)}$$

$$= \frac{1}{(\ln 27)(\ln 2)}.$$

Logarithmic Differentiation :

Type: 1

$$y = f(x) \cdot g(x)\, h(x)\, w(x) \cdot \phi(x) \cdots\cdots$$

$$\ln y = \ln f(x) + \ln g(x) + \ln h(x) + \cdots\cdots$$

$$\frac{1}{y} \cdot \frac{dy}{dx} = \left(\frac{f'(x)}{f(x)} + \frac{g'(x)}{g(x)} + \cdots\cdots \right)$$

Type: 2

$$y = \frac{f(x) \cdot g(x) \cdot h(x) \cdots\cdots}{l(x) \cdot m(x) \cdot n(x) \cdots\cdots}$$

* **Type: 3**

$$y = (f(x))^{g(x)}$$

M-1

$$\ln y = g(x) \cdot \ln f(x)$$

$$\frac{1}{y}\left(\frac{dy}{dx}\right) = \left(g(x) \cdot \left(\frac{f'(x)}{f(x)}\right) + g'(x) \cdot \ln f(x) \right)$$

** M-2

$$y = \underbrace{e^{g(x) \ln f(x)}}$$

$$\frac{dy}{dx} = (f(x))^{g(x)} \left(g(x) \cdot \frac{f'(x)}{f(x)} + g'(x) \ln f(x) \right)$$

1) If $f(x) = (x + 1)(x + 2)(x + 3) \ldots\ldots (x + n)$ then $f'(0)$ is

$$\ln f(x) = \ln(x+1) + \ln(x+2) + \cdots + \ln(x+n)$$

diff wrt 'x' :-

$$\frac{f'(x)}{f(x)} = \left(\frac{1}{x+1} + \frac{1}{x+2} + \cdots + \frac{1}{x+n} \right)$$

$$f'(x) = f(x) \left(\quad \right)$$

$$f'(0) = f(0) \left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} \right)$$

$$f(0) = 1.2.3.4 \ldots n = \underline{|n} = n!$$

2) If $f(x)=\prod_{n=1}^{100}(x-n)^{n(101-n)}$ then find $\dfrac{f(101)}{f'(101)}$

$$\ln f(x) = \sum_{n=1}^{100} n(101-n)\ln(x-n).$$

$$\ln f(x) = 1.100\ln(x-1) + 2.99\ln(x-2) + \cdots + 100.1.\ln(x-100)$$

diff wrt 'x':

$$\frac{f'(x)}{f(x)} = \left(\frac{1.100}{x-1} + \frac{2.99}{x-2} + \cdots + \frac{100.1}{x-100}\right)$$

put $x=101$

$$\frac{f'(101)}{f(101)} = \left(\frac{1.\cancel{100}}{\cancel{100}} + \frac{2.\cancel{99}}{\cancel{99}} + \cdots + \frac{100.1}{1}\right)$$

$$= 5050.$$

$$\frac{f(101)}{f'(101)} = \frac{1}{5050}$$

HW

(2) $y = \tan^{-1}\left(\dfrac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right)$ find $\dfrac{dy}{dx} = ?$

put $x = \cos\theta.$; $\theta \in [0, \pi]$; $\dfrac{\theta}{2} \in [0, \pi/2]$

$$y = \tan^{-1}\left(\frac{\sqrt{1+\cos\theta}-\sqrt{1-\cos\theta}}{\sqrt{1+\cos\theta}+\sqrt{1-\cos\theta}}\right)$$

$$y = \tan^{-1}\left(\frac{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}+\sin\frac{\theta}{2}}\right)$$

$$y = \tan^{-1}\left(\frac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}}\right) = \tan^{-1}\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$$

$$y = \frac{\pi}{4}-\frac{\theta}{2} \Rightarrow y = \frac{\pi}{4}-\frac{1}{2}\cos^{-1}x$$

$$\frac{dy}{dx} = \frac{1}{2}\left(\frac{1}{\sqrt{1-x^2}}\right)$$ Ans

Q $x = a\sqrt{\cos 2t}\ \cos t$ and $y = a\sqrt{\cos 2t}\ \sin t$ then, find $\left.\frac{dy}{dx}\right|_{t=\pi/6}$.

$$\frac{dx}{dt} = \frac{-a\cos t(\sin 2t).2}{2\sqrt{\cos 2t}} - a\sqrt{\cos 2t}\ \sin t$$

$$\Rightarrow \frac{dx}{dt} = -\frac{a\sin 3t}{\sqrt{\cos 2t}}$$

$$\frac{dy}{dt} = \frac{-a\sin t(\sin 2t).2}{2\sqrt{\cos 2t}} + a\sqrt{\cos 2t}.\cos t$$

$$\frac{dy}{dt} = \frac{a\cos 3t}{\sqrt{\cos 2t}}$$

$$\therefore \frac{dy}{dx} = -\cot(3t)\Big|_{t=\pi/6} = 0. \text{ Ans}$$

HW

4) If $y = (\sin x)^{\ln x}\,\mathrm{cosec}(e^x(a + bx))$ and $a + b = \frac{\pi}{2e}$, then the value of $\frac{dy}{dx}$ at $x = 1$ is -

(A) $(\sin 1)\ell n\sin(1)$ (B) 0 (C) $\ell n\sin(1)$ ✓ (D) indeterminate

$$\ln y = \ln x.\ln\sin x + \ln(\mathrm{cosec}(e^x(a+bx)))$$

$$\frac{y'}{y} = \frac{\ln\sin x}{x} + \cot x.\ln x - \frac{\mathrm{cosec}(e^x(a+bx))\cot(e^x(a+bx))}{\mathrm{cosec}(e^x(a+bx))} \times \left(e^x(b) + e^x(a+bx)\right)$$

$$\left.\frac{dy}{dx}\right|_{x=1} = \ln\sin 1 \text{ Ans}$$

Q If $x = 2\cos t - \cos 2t$; $y = 2\sin t - \sin 2t$, find $\frac{d^2y}{dx^2}$ at $t = \frac{\pi}{2}$.

Sol^n

$$\frac{dx}{dt} = -2\sin t + 2\sin 2t. ✓$$

$$\frac{dy}{dt} = 2\cos t - 2\cos 2t.$$

$$\frac{dy}{dx} = \frac{\cos t - \cos 2t}{\sin 2t - \sin t} = \frac{\cancel{2}\sin\frac{3t}{2}\cdot\cancel{\sin\frac{t}{2}}}{\cancel{2}\cancel{\sin\frac{t}{2}}\cos\frac{3t}{2}} = \tan\frac{3t}{2}$$

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\tan\frac{3t}{2}\right) = \frac{d}{dt}\left(\tan\frac{3t}{2}\right)\cdot\frac{dt}{dx}$$

$$\left.\frac{d^2y}{dx^2}\right|_{t=\frac{\pi}{2}} = \left.\left(\frac{3}{2}\sec^2\left(\frac{3t}{2}\right)\cdot\frac{1}{2(\sin 2t - \sin t)}\right)\right|_{t=\frac{\pi}{2}}$$

$$= \frac{3}{4}(2)\cdot\frac{1}{(0-1)} = -\frac{3}{2} \text{ Ans}$$

Q

Starting with $\frac{dx}{dy} = \frac{1}{\frac{dy}{dx}}$. Prove that $\frac{d^2x}{dy^2} = -\frac{\frac{d^2y}{dx^2}}{\left(\frac{dy}{dx}\right)^3}$

Rem

$$\frac{d^2y}{dx^2} = \frac{-\frac{d^2x}{dy^2}}{\left(\frac{dx}{dy}\right)^3}$$

Sol^n

$$\frac{dx}{dy} = \frac{1}{dy/dx}$$

$$\frac{d}{dy}\left(\frac{dx}{dy}\right) = \frac{d}{dy}\left(\frac{1}{dy/dx}\right) = \underbrace{\frac{d}{dx}\left(\frac{1}{dy/dx}\right)} \cdot \frac{dx}{dy}.$$

$$\frac{d^2x}{dy^2} = -\frac{1}{(dy/dx)^2} \cdot \left(\frac{d^2y}{dx^2}\right) \cdot \frac{1}{(dy/dx)}$$

$$= \frac{-\frac{d^2y}{dx^2}}{\left(\frac{dy}{dx}\right)^3}.$$

(H.P.)

Q Let g(x) = ln f(x + 1), where f(x) is a twice differentiable positive function on $(0, \infty)$ such that $f(x+2) - f(x+1) = x f(x+1)$. Determine the value of $g''(2n+1) - g''(1)$.

Sol^n

$$g(x) = \ln f(x+1) \quad ; \quad f(x+2) - f(x+1) = x f(x+1)$$

$$\downarrow x \to x+1 \qquad\qquad f(x+2) = (x+1) f(x+1)$$

$$g(x+1) = \ln\left(f(x+2)\right)$$

$$g(x+1) = \ln(x+1) + \underbrace{\ln f(x+1)}_{g(x)}$$

$$g(x+1) - g(x) = \ln(x+1).$$

$$g'(x+1) - g'(x) = \frac{1}{x+1}$$

$$g''(x+1) - g''(x) = \frac{-1}{(x+1)^2}.$$

$x = 1$

$x = 2$

$\vdots$

$x = 2n$

$$\sum \left\{ \begin{aligned} g''(\cancel{2}) - g''(1) &= -\frac{1}{2^2} \\ g''(\cancel{3}) - \cancel{g''(2)} &= -\frac{1}{3^2} \\ &\vdots \\ g''(2n+1) - \cancel{g''(2n)} &= -\frac{1}{(2n+1)^2} \end{aligned} \right.$$

Q Let f be twice differentiable function such that $f''(x) = -f(x)$ and $f'(x) = g(x)$. If $h(x) = f^2(x) + g^2(x)$ then find $h(0)$ if $h(5) = 11$?

Sol^n

$$h(x) = f^2(x) + g^2(x)$$

$$g(x) = f'(x)$$
$$g'(x) = f''(x)$$
$$\underline{f(x) = -f''(x)}$$

$$h'(x) = 2f(x)f'(x) + 2g(x)g'(x)$$

$$= 2f(x)f'(x) + 2f'(x)f''(x)$$

$$= -2f''(x)f'(x) + 2f'(x)f''(x)$$

$$\boxed{h'(x) = 0} \Rightarrow h(x) = \text{constant fn}$$

$$\therefore h(5) = h(0) = 11.$$

Ans

Q

If f is twice differentiable such that $f''(x) = -f(x), f'(x) = g(x)$

$$h'(x) = [f(x)]^2 + [g(x)]^2 \text{ and}$$

$$h(0) = 2, h(1) = 4$$

then the equation $y = h(x)$ represents :

(A) a curve of degree 2

(B) a curve passing through the origin

(C) a straight line with slope 2

(D) a straight line with y intercept equal to -2.

Sol^n

$$h'(x) = f^2(x) + g^2(x)$$

$$h''(x) = 2f(x)f'(x) + 2g(x)g'(x)$$

$$h''(x) = 0 \Rightarrow h'(x) = \text{Constant}$$

$$h'(x) = \lambda.$$

Integrate wrt 'x' :-

$$h(x) = \lambda x + C.$$

$$h(0) = C = 2$$

$$h(1) = \lambda + C = 4$$

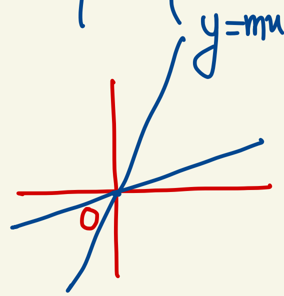
$$\lambda = 2.$$

$$\therefore \boxed{h(x) = 2x + 2}$$

Note : A homogeneous equation of degree n represents '*n*' straight lines passing through the origin, hence

$$\boxed{\frac{dy}{dx} = \frac{y}{x}} \quad \text{and} \quad \boxed{\frac{d^2y}{dx^2} = 0}$$

$ax^2 + 2hxy + by^2 = 0$ Homogeneous eqn of degree-2.



diff wrt 'x'

$$\cancel{2}ax + \cancel{2}h(xy' + y) + \cancel{2}by \cdot y' = 0.$$

$$ax + hy = -(hx + by) \cdot y'$$

$$y' = -\frac{(ax+hy)}{(hx+by)} = \frac{y}{x} = \text{Constant.}$$

$$ax^2 + hxy + hxy + by^2 = 0$$

$$x(ax+hy) + y(hx+by) = 0$$

$$\boxed{-\frac{(ax+hy)}{(hx+by)} = \frac{y}{x}}$$

1) If $x^3 + 3x^2y - 6xy^2 + 2y^3 = 0$, then $\left.\frac{d^2y}{dx^2}\right]_{(1,1)} = 0.$

2) Given the curve $\left(\frac{x^2 - y^2}{xy}\right)^3 = \frac{512}{27}$, then find the value of $\left(2\frac{dy}{dx} - 5\frac{d^2y}{dx^2}\right)_{(3,1)}$ $= 2\cdot\left(\frac{1}{3}\right) - 0.$

$= \frac{2}{3}.$

3) If $x^m \cdot y^n = (x+y)^{m+n}$ then find $\frac{dy}{dx} = ?$

$\frac{y}{x}$ Ans

Rem

① The derivative of an even differentiable function is an odd function and the derivative of an odd differentiable function is an even function. but not vice-versa necessarily.

② If $(x - r)$ is a factor of the polynomial $f(x) = a_n x^n + a_{n-1} x^{n-1} + a_{n-2} x^{n-2} + \dots + a_0$ repeated m times where $1 \le m \le n$ then r is a root of the equation $f'(x) = 0$ repeated $(m - 1)$ times.

$x = r$

① $f(x) = \sin x \longrightarrow$ Odd fns

$g(x) = f'(x) = \cos x \longrightarrow$ even fns.

$g'(x) = -\sin x \longrightarrow$ Odd fns.

$f'(x) = \cos x$

$\int f'(x)\,dx = \int \cos x\,dx \Longrightarrow \boxed{f(x) = \sin x + C}$

Constant of integration

② $f(x) = (x-1)^2 (2x+1).$

$f'(x) = (x-1)^2 (2) + (2x+1).2(x-1)$

$= 2(x-1)(x - \cancel{1} + 2x + \cancel{1})$

$= 6x\underbrace{(x-1)}$

Q If $f(x) = (x-1)^4 (x-2)^3 (x-3)^2$ then the value of $f'''(1) + f''(2) + f'(3)$ is

$$\underset{0}{f'''(1)} + \underset{0}{f''(2)} + \underset{0}{f'(3)}$$

Sol^n 0.

DERIVATIVE OF FUNCTIONS EXPRESSED IN THE DETERMINANT FORM:

Let $F(x) = \begin{vmatrix} f & g & h \\ u & v & w \\ l & m & n \end{vmatrix}$ where all functions are differentiable then

Row-wise

$$F'(x) = \begin{vmatrix} f' & g' & h' \\ u & v & w \\ l & m & n \end{vmatrix} + \begin{vmatrix} f & g & h \\ u' & v' & w' \\ l & m & n \end{vmatrix} + \begin{vmatrix} f & g & h \\ u & v & w \\ l' & m' & n' \end{vmatrix}$$

Similarly we can do Column-wise.

Q $f(x) = \begin{vmatrix} e^x & x^2+x+1 & \sin x \\ x^2 & \cos x & 1 \\ x^3+x & \tan x & 3x^2+1 \end{vmatrix}$, find $f'(0)$

$\boxed{f'(0) = 1}$ Ans

Sol^n Column-wise.

$$f'(x) = \begin{vmatrix} e^x & x^2+x+1 & \sin x \\ 2x & \cos x & 1 \\ 3x^2+1 & \tan x & 3x^2+1 \end{vmatrix} + \begin{vmatrix} (\ ) & 2x+1 & (\ ) \\ (\ ) & -\sin x & (\ ) \\ (\ ) & \sec^2 x & (\ ) \end{vmatrix}$$

$$+ \begin{vmatrix} (\ ) & (\ ) & \cos x \\ (\ ) & (\ ) & 0 \\ (\ ) & (\ ) & 6x \end{vmatrix}$$

Q

If f, g and h are differentiable functions of x & $D=\begin{vmatrix} f & g & h \\ (xf)' & (xg)' & (xh)' \\ (x^2f)'' & (x^2g)'' & (x^2h)'' \end{vmatrix}$ prove that

$$D'=\begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^3f'')' & (x^3g'')' & (x^3h'')' \end{vmatrix}$$

Sol^n

$$D=\begin{vmatrix} f \\ (xf)' \\ (x^2f)'' \end{vmatrix}$$

$(xf)' = xf' + f$

$(x^2f)'' = (2xf + x^2f')'$

$= 2f + 2xf' + 2xf' + x^2f''$

$(x^2f)'' = (2f + 4xf' + x^2f'')$

$$D = \begin{vmatrix} f & g & h \\ xf' + f & xg' + g & xh' + h \\ 2f + 4xf' + x^2f'' & 2g + 4xg' + x^2g'' & 2h + 4xh' + x^2h'' \end{vmatrix}$$

$R_2 \to R_2 - R_1$ and $R_3 \to R_3 - 2R_1$

$$D = \begin{vmatrix} f & g & h \\ xf' & xg' & xh' \\ 4xf' + x^2f'' & 4xg' + x^2g'' & 4xh' + x^2h'' \end{vmatrix}$$

$R_3 \to R_3 - 4R_2$

$$D = \begin{vmatrix} f & g & h \\ xf' & xg' & xh' \\ x^2 f'' & x^2 g'' & x^2 h'' \end{vmatrix}$$

$$D = \begin{vmatrix} f & g & h \\ f' & g' & h' \\ x^3 f'' & x^3 g'' & x^3 h'' \end{vmatrix}$$

$$D' = \cancelto{0}{\begin{vmatrix} f' & g' & h' \\ f' & g' & h' \\ (\ ) & (\ ) & (\ ) \end{vmatrix}} + \cancelto{0}{\begin{vmatrix} f & g & h \\ f'' & g'' & h'' \\ x^3 f'' & x^3 g'' & x^3 h'' \end{vmatrix}} + \begin{vmatrix} f & g & h \\ f' & g' & h' \\ (x^3 f'')' & - & - \end{vmatrix}$$

(H.P.)

Q

If α be a repeated root of a quadratic equation $f(x) = 0$ & $A(x)$, $B(x)$, $C(x)$ be the polynomials of degree 3, 4 & 5 respectively, then show that $\begin{vmatrix} A(x) & B(x) & C(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$ is divisible by $f(x)$, where dash denotes the derivative.

Sol^n

$$f(x) = \lambda (x-\alpha)^2$$

$A(\alpha) = \underline{\text{Constant}}$

$A'(\alpha) = \text{Constant}$

$$D(x) = \begin{vmatrix} A(x) & B(x) & C(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix}$$

$D(x) \rightarrow$ poly fns.

Row-wise differentiate :-

$$D'(x) = \begin{vmatrix} A'(x) & B'(x) & C'(x) \\ A(\alpha) & B(\alpha) & C(\alpha) \\ A'(\alpha) & B'(\alpha) & C'(\alpha) \end{vmatrix} + 0 + 0.$$

$\boxed{D'(\alpha) = 0}$ & $\boxed{D(\alpha) = 0}$

$$D(x) = (x-\alpha)^2 \times \underbrace{g(x)}_{\text{poly of atmost '3' degree.}}$$

Q

If $f(x)=\begin{vmatrix}\cos(x+x^2) & \sin(x+x^2) & -\cos(x+x^2)\\ \sin(x-x^2) & \cos(x-x^2) & \sin(x-x^2)\\ \sin 2x & 0 & \sin 2x^2\end{vmatrix}$ then find $f'(x)$.

$$x+x^2 = A \qquad 2x = A+B.$$
$$x-x^2 = B. \qquad 2x^2 = A-B$$

$$f(x) = \begin{vmatrix}\cos A & \sin A & -\cos A\\ \sin B & \cos B & \sin B\\ \sin(A+B) & 0 & \sin(A-B)\end{vmatrix}$$

$$f(x) = \sin 2A = \sin(2x+2x^2)$$

$$f'(x) = \cos(2x+2x^2)(2+4x) \quad \text{Ans}$$

Q

If $\cos\frac{x}{2}.\cos\frac{x}{2^2}.\cos\frac{x}{2^3}.........\cos\frac{x}{2^n} = \frac{\sin x}{2^n \sin(x/2^n)}$, then prove that $\sum_{r=1}^{n}\frac{1}{2^r}\tan\frac{x}{2^r} = \frac{1}{2^n}\cot\frac{x}{2^n} - \cot x$

Sol^n Take ln both sides :-

$$\ln\cos\frac{x}{2} + \ln\cos\left(\frac{x}{2^2}\right) + \dots + \ln\cos\left(\frac{x}{2^n}\right) = \ln\sin x - \ln 2^n - \ln\sin\left(\frac{x}{2^n}\right)$$

diff wrt 'x' :-

$$-\frac{1}{2}.\tan\frac{x}{2} - \frac{1}{2^2}\tan\frac{x}{2^2} - \dots - \frac{1}{2^n}.\tan\frac{x}{2^n} = \cot x - \frac{1}{2^n}\cot\left(\frac{x}{2^n}\right)$$

$$\sum_{r=1}^{n}\frac{1}{2^r}\tan\left(\frac{x}{2^r}\right) = \frac{1}{2^n}\cot\frac{x}{2^n} - \cot x$$

(H.P)

8

Let $f : \mathbb{R} \to \mathbb{R}$, $g : \mathbb{R} \to \mathbb{R}$ and $h : \mathbb{R} \to \mathbb{R}$ be differentiable functions such that $f(x) = x^3 + 3x + 2$, $g(f(x)) = x$ and $h(g(g(x))) = x$ for all $x \in \mathbb{R}$. Then- **[JEE(Advanced)-2016, 4(–2)]**

(A) $g'(2) = \frac{1}{15}$ (B) $h'(1) = 666$ (C) $h(0) = 16$ (D) $h(g(3)) = 36$

$f(x) = x^3 + 3x + 2$; $g(f(x)) = x$; $h(g(g(x))) = x$

$f'(x) = 3x^2 + 3$

$g'(2) = \frac{1}{f'(0)} = \frac{1}{3}$.

$x \to f(x)$

$h(g(\underbrace{g(f(x))}_{x})) = f(x)$

$f(3) = 3^3 + 3 \times 3 + 2$

$= 27 + 9 + 2$

$= 38.$

$h(g(x)) = f(x)$

put $x = 3$

$h(g(3)) = f(3)$

$x \to f(x)$

$h(\underbrace{g(f(x))}_{x}) = f(f(x))$

$\boxed{h(x) = f(f(x))}$

$h(0) = f(f(0))$

$= f(2)$

$= 8 + 6 + 2$

$= 16$

$h'(x) = f'(f(x)) \cdot f'(x)$

$h'(1) = f'(f(1)) \cdot f'(1)$

Q Let y = f(x) is a positive function which satisfies equation $\sqrt{y^2+2x}+\sqrt{y^2-2x}=2x^2$, then $\frac{dy}{dx}$ is equal to-

(A) $\frac{2x^3-x^{-3}}{y}$ (B) $\frac{2x^4-x^{-2}}{y}$ (C) $\frac{2x^4-x^{-2}}{\sqrt{x^6+1}}$ (D) $\frac{2x^3-x^{-3}}{\sqrt{1+x^6}}$

Q Let $y = f(x)$ is a positive function which satisfies equation $\sqrt{y^2 + 2x} + \sqrt{y^2 - 2x} = 2x^2$, then $\frac{dy}{dx}$ is equal to-

(A) $\frac{2x^3 - x^{-3}}{y}$ (B) $\frac{2x^4 - x^{-2}}{y}$ (C) $\frac{2x^4 - x^{-2}}{\sqrt{x^6 + 1}}$ (D) $\frac{2x^3 - x^{-3}}{\sqrt{1 + x^6}}$

Q.

If $f\left(\frac{2x+3y}{5}\right)=\frac{2f(x)+3f(y)}{5}\ \forall x,y\in\mathbb{R}$, $f(0)=1$, $f'(1)=1$, then find $f(x)$

Sol^n

$$f'\left(\frac{2x+3y}{5}\right)\cdot\frac{\cancel{2}}{\cancel{5}}=\frac{\cancel{2}}{\cancel{5}}f'(x)$$

put $x=1$

$$f'\left(\frac{3y+2}{5}\right)=1$$

$$f'(t)=1$$

$$f(t)=t+c\xrightarrow{\text{put } t=0}f(0)=c\Rightarrow\boxed{c=1}$$

$$f(t)=t+1\Rightarrow\boxed{f(x)=x+1}$$ Ans

Q In a sequence $\{a_n\}$, $a_1 = 2$, $a_{n+1} = 1 - \frac{1}{a_n}$ for $n \geq 1$. Let P_n be the product of its first n terms, then the value of P_{2017} is

(A) $\frac{-1}{2}$ (B) 2 (C) $\frac{1}{2}$ (D) 1

Sol^n $a_1 = 2$; $a_{n+1} = 1 - \frac{1}{a_n}$

$a_2 = 1 - \frac{1}{a_1} = 1 - \frac{1}{2} = \frac{1}{2}$

$a_3 = 1 - \frac{1}{a_2} = 1 - \frac{1}{1/2} = -1$

$a_1 = a_4 = 1 - \frac{1}{a_3} = 1 - (-1) = 2$

$a_2 = a_5 = a_8 = \cdots = 1/2$

$a_1 = a_4 = a_7 = \cdots = 2$

$a_3 = a_6 = a_9 = \cdots = -1$

$P_{2017} = (P_3)^{672} \cdot P_1$

$= 2.$

Q. Let S_n represent sum to first n terms of a G.P. whose first term is 1 and common ratio is 3 and T_n represent the n^{th} term of another G.P. whose first term is 5 and common ratio is 5, then $\sum_{n=1}^{\infty} \frac{S_{n+1}}{T_{n+2}}$ is equal to

Ans 0.08 or 0.09

$$S_n = \frac{3^n - 1}{2} \Rightarrow S_{n+1} = \frac{3^{n+1} - 1}{2}$$

$$T_n = 5 \cdot 5^{n-1} = 5^n \Rightarrow T_{n+2} = 5^{n+2}$$

$$\sum_{n=1}^{\infty} \frac{S_{n+1}}{T_{n+2}} = \sum_{n=1}^{\infty} \frac{\frac{3^{n+1} - 1}{2}}{5^{n+2}} = \sum_{n=1}^{\infty} \frac{3 \cdot 3^n - 1}{2 \cdot 5^2 \cdot 5^n}$$

$$= \frac{3}{50}\Big(\underbrace{\sum_{n=1}^{\infty} \left(\frac{3}{5}\right)^n}_{GP}\Big) - \frac{1}{50}\Big(\underbrace{\sum_{n=1}^{\infty} \frac{1}{5^n}}_{GP}\Big)$$

$$= \frac{17}{200}.$$

Q

Number of integral solution of equation $\tan\left(\frac{\pi}{2}[x]\right) = 0$ in $x \in \left[-\frac{9\pi}{2}, \frac{11\pi}{2}\right]$ (where [.] represents greatest integer function) is -

(A) 20 (B) 15 (C) 16 (D) 29

$$\tan\left(\frac{\pi}{2}[x]\right) = 0$$

$$\tan(n\pi) = 0$$

$$n \in I$$

$[x]$ = Even integer

$$x \in \left[-\frac{9\pi}{2}, \frac{11\pi}{2}\right]$$

$$[-14.\omega, 17.\omega]$$

Q If x,y,z are three real number such that x + y + z = 4 and $x^2 + y^2 + z^2 = 6$, then-

(A) $x \in \left[\frac{2}{3}, 2\right]$ (B) $y \in \left[\frac{2}{3}, 2\right]$ (C) $z \in (2,3)$ (D) $y \in (2,3)$

$z = 4 - x - y$

$x, y, z \in R$

$x^2 + y^2 + (4 - x - y)^2 = 6.$

$x^2 + (y-4)x + (y^2 + 5 - 4y) = 0.$

Since $x \in R$ $\therefore$ $D \geq 0.$

$y \in \left[\frac{2}{3}, 2\right]$

$\| ^{ly}$ $x \in \left[\frac{2}{3}, 2\right]$

$z \in \left[\frac{2}{3}, 2\right].$

Q

1 Consider an equation $(x-a)(x-b)(x-c)+(x-1)(x^2+x+1)=0$ whose roots are α,β,γ.
The roots of equation $2(x-\alpha)(x-\beta)(x-\gamma)+1-x^3=0$ are-

(A) a,b,c (B) $\frac{1}{a},\frac{1}{b},\frac{1}{c}$ (C) –a, – b, –c (D) $-\frac{1}{a},-\frac{1}{b},-\frac{1}{c}$

2 Sum of roots of the equation $2(x-\alpha)(x-\beta)(x-\gamma)-(x-a)(x-b)(x-c)=0$, is-

(A) –1 (B) 1 (C) 0 (D) –2

Sol^n

$$(x-a)(x-b)(x-c)+\underbrace{(x-1)(x^2+x+1)}_{(x^3-1)}=2(x-\alpha)(x-\beta)(x-\gamma)$$

$$2(x-\alpha)(x-\beta)(x-\gamma)+1-x^3=(x-a)(x-b)(x-c)$$

$$2(x-\alpha)(x-\beta)(x-\gamma)-(x-a)(x-b)(x-c)=x^3-1$$

$$x^3+0x^2+0x-1=0$$

$$\boxed{S.O.R=0}$$

Q

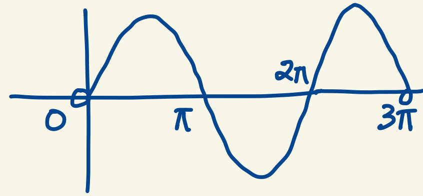
$f(x)=\prod_{k=1}^{\infty}\left(\frac{1+2\cos\left(\frac{2x}{3^k}\right)}{3}\right)$, then number of points where $[x f(x)]+|x\ f(x)|+(x-1)|x^2-3x+2|$ is non-differentiable in $x \in (0, 3\pi)$ is equal to (where [·] denotes greatest integer function)

$[\sin x] + |\sin x| + (x-1)\,|(x-1)(x-2)|$

5 Ans

Sol^n

$\frac{x}{3^k}=\theta$.



$$f(x)=\prod_{k=1}^{\infty}\left(\frac{1+2\cos 2\theta}{3}\right)$$

$$=\prod_{k=1}^{\infty}\left(\frac{1+2(1-2\sin^2\theta)}{3}\right)$$

$$=\prod_{k=1}^{\infty}\left(\frac{3-4\sin^2\theta}{3}\right)=\prod_{k=1}^{\infty}\left(\frac{3\sin\theta-4\sin^3\theta}{3\sin\theta}\right)$$

$$=\prod_{k=1}^{\infty}\left(\frac{\sin 3\theta}{3\sin\theta}\right)=\lim_{K\to\infty}\prod_{k=1}^{K}\left(\frac{\sin\left(\frac{x}{3^{k-1}}\right)}{3\sin\left(\frac{x}{3^k}\right)}\right)$$

$$f(x)=\lim_{K\to\infty}\frac{1}{3^K}\left(\frac{\sin x}{\cancel{\sin\frac{x}{3}}}\cdot\frac{\cancel{\sin\frac{x}{3}}}{\cancel{\sin\frac{x}{3^2}}}\cdots\frac{\cancel{\sin\frac{x}{3^{K-1}}}}{\sin\frac{x}{3^K}}\right)$$

$$f(x)=\lim_{K\to\infty}\frac{1}{\cancel{3^K}}\;\frac{\sin x}{\frac{\sin\left(\frac{x}{3^K}\right)}{\left(\frac{x}{3^K}\right)}\cdot\left(\frac{x}{\cancel{3^K}}\right)}=\frac{\sin x}{x}$$

$$\boxed{x f(x)=\sin x}$$

Q Let $f(x)$ and $g(x)$ be derivable functions such that $f'(2) = 5$ and $g'(3) = 4$, then if $L = \lim_{h\to 0} \frac{f(2+h)-f(2)}{h^2+5h}$ and $M = \lim_{h\to 0} \frac{g(3+h^2)-g(3)}{2h^2}$, then the value of $(L^2 + M^2)$ is

5

$$L = \lim_{h\to 0}\left(\frac{f(2+h)-f(2)}{h(h+5)}\right) = \frac{f'(2)}{5} = \frac{5}{5} = 1$$

$$M = \lim_{h\to 0} \frac{g(3+h^2)-g(3)}{2h^2} = \frac{4}{2} = 2$$

Q Let $f(x) = [x-1]x^2 + (a^2+ab+2)\,\mathrm{sgn}(x-2) + (a+b)\{x\}\ \forall\ x \in R$. If $\lim\limits_{x\to 2} f(x)$ exists for some real value(s) of a, then the smallest positive integral value of b is

([.], {.} greatest integer function and fractional part function & sgn(x) denotes signum function of x respectively)

$\boxed{\text{Ans} \to 4}$ ✓

Sol^n

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x)$$

$$\{x\} = x - \underbrace{[x]}$$

$$-(a^2+ab+2) + (a+b)(x-1) = x^2 + (a^2+ab+2) + (a+b)(x-2)$$

$$-(a^2+ab+2) + (a+b) = 4 + (a^2+ab+2) + 0.$$

$\underbrace{\text{get QE in 'a'}}$; $\boxed{D \geqslant 0}$

Q

Let $f(x)$ be a differentiable function everywhere and satisfies relation

$f(x + y) + f(x) + f(y) = 2f(x - y) + 6xy - 4y \ \forall \ x, y \in R$ and $f'(0) = -1$, then

(A) $f(x)$ is an even function

(B) $f''(x)$ is an even function

(C) $f'(1) = 1$

(D) $f(4)$, $f'(4)$ and $f''(4)$ are in A.P.

Sol^n

Diff wrt 'y'

$x = y = 0$

$$f'(x+y) + f'(y) = 2f'(x-y)\cdot(-1) + 6x - 4.$$

put $y = 0$

$$f'(x) + f'(0) = -2f'(x) + 6x - 4$$

$$3f'(x) = 6x - 3. \Rightarrow f'(x) = 2x - 1$$

Integrate

$$\boxed{f(x) = x^2 - x + c}$$

$$\boxed{c = 0}$$

Q Let $\lim_{x\to\infty}\sqrt{x^4+Ax^3+Ax^2+x-1}-\sqrt{Cx^4+2x^3+7x^2-4x+1}=L$, then

(A) $C\neq 1 \Rightarrow$ L does not exist

(B) $C=1,\ A\neq 2 \Rightarrow$ L does not exist

(C) $C=1,\ A=2 \Rightarrow L=-\frac{5}{2}$

(D) $A=C \Rightarrow$ L does not exist

Soln

$$L=\lim_{x\to\infty}\left(\frac{x^4(1-c)+x^3(A-2)+x^2(A-7)+5x-2}{\sqrt{x^4+Ax^3+Ax^2+x-1}+\sqrt{Cx^4+2x^3+7x^2-4x+1}}\right)$$

$$L=\lim_{x\to\infty}\frac{x^4(1-c)+x^3(A-2)+x^2(A-7)+5x-2}{x^2\left(\sqrt{1+\frac{A}{x}+\frac{A}{x^2}+\frac{1}{x^3}-\frac{1}{x^4}}+\sqrt{C+\frac{2}{x}+\frac{7}{x^2}-\frac{4}{x^3}+\frac{1}{x^4}}\right)}$$

$C=1$ & $A=2$

$$L=\frac{A-7}{1+\sqrt{C}}=\frac{2-7}{2}=\frac{-5}{2}$$

Q Number of solutions of the equation $\underbrace{[y+[y]]}$ = 2cosx is

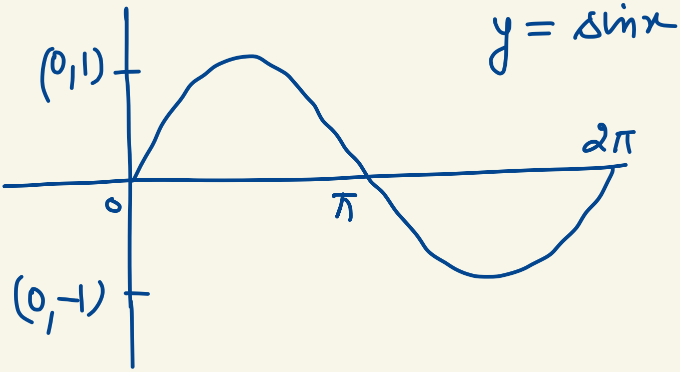
(where $y = \frac{1}{3}\left[\sin x + \left[\sin x + [\sin x]\right]\right]$ and [.] is greatest integer function)-

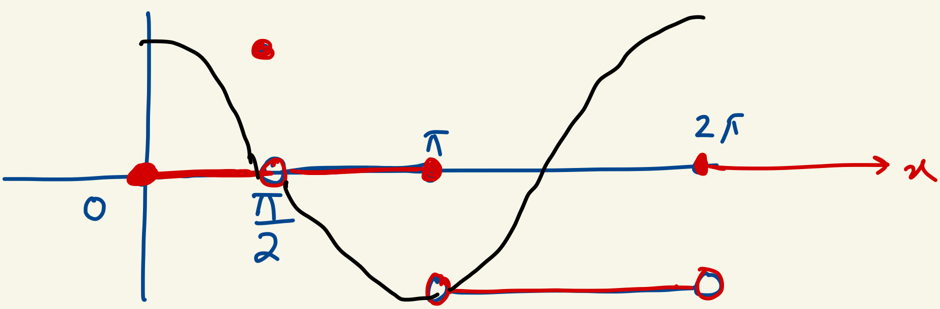
(A) 0 (B) 1 (C) 2 (D) infinite

Sol^n

$y = [\sin x] \quad \cancel{2}[y] = \cancel{2}\cos x$

$\underbrace{[\sin x]}_{\underbrace{-1,\ 0,\ 1}} = \cos x$





Q Let $f(x)$ be a bijective function such that $f(x)=\frac{\alpha x}{x+1}$, $x \neq -1$, where $fofof(\beta)=1$ and $f^{-1}(2)=1$, then which of the following option(s) is/are correct ?

(A) $\alpha = 4$

(B) $f(3)=3$

(C) $\alpha+\frac{1}{\beta}=47$

(D) if $|f(x)|<1$, then $x \in \left(-\frac{1}{17}, \frac{1}{15}\right)$

$f(1)=2$

$$f(x)=\frac{\alpha x}{x+1} \xrightarrow{x=1} f(1)=\frac{\alpha}{2} \Rightarrow \boxed{\alpha=4}$$

$$\boxed{f(x)=\frac{4x}{x+1}} \longrightarrow f(3)=\frac{12}{4}=3$$

$$y=\frac{4x}{x+1}$$

$$fo\underbrace{fof(\beta)}=1 \Rightarrow fof(\beta)=f^{-1}(1)$$

$$fof(\beta)=\frac{1}{3}$$

$$f(\beta)=f^{-1}\left(\frac{1}{3}\right)$$

$$f(\beta)=\frac{1}{11}$$

$$\beta=f^{-1}\left(\frac{1}{11}\right)=\frac{1}{43}$$

$$\frac{1}{\beta}=43$$

Q. Consider $f : N \to N$ where $f(n) = \frac{4}{\pi}\sum_{i=1}^{n}\sum_{j=1}^{n}\cot^{-1}\left(\frac{i}{j}\right)$.

1. Function f will be -

(A) one one onto

(B) one one into

(C) many one onto

(D) many one into

2. Value of $f(5) + f(10)$ is -

(A) 75

(B) 100

(C) 125

(D) 150

Sol^n

$$f(n) = \frac{4}{\pi}\sum_{i=1}^{n}\sum_{j=1}^{n}\cot^{-1}\left(\frac{i}{j}\right)$$

$$= \frac{4}{\pi}\sum_{i=1}^{n}\underbrace{\left(\cot^{-1}(i) + \cot^{-1}\left(\frac{i}{2}\right) + \cot^{-1}\left(\frac{i}{3}\right) + \cdots + \cot^{-1}\left(\frac{i}{n}\right)\right)}_{S}$$

$$S = \cot^{-1}1 + \cot^{-1}2 + \cot^{-1}3 + \cdots + \cot^{-1}n +$$

$$\cot^{-1}\frac{1}{2} + \cot^{-1}\frac{2}{2} + \cot^{-1}\frac{3}{2} + \cdots + \cot^{-1}\frac{n}{2}.$$

$$\cot^{-1}\left(\frac{1}{3}\right) + \cot^{-1}\left(\frac{2}{3}\right) + \cot^{-1}\left(\frac{3}{3}\right) + \cdots + \cot^{-1}\left(\frac{n}{3}\right)$$

$$\vdots$$

$$\cot^{-1}\left(\frac{1}{n}\right) + \cot^{-1}\left(\frac{2}{n}\right) + \cdots\cdots + \cot^{-1}\left(\frac{n}{n}\right)$$

Total terms $= n^2$; Diagonal $= n$.

$$S = n \cdot \frac{\pi}{4} + \left(\frac{n^2 - n}{2}\right) \cdot \frac{\pi}{2} = \frac{n^2 \pi}{4}$$

$$f(n) = \frac{4}{\pi} S = \frac{\cancel{4}}{\cancel{\pi}} \cdot \frac{n^2 \cancel{\pi}}{\cancel{4}}$$

$$\boxed{f(n) = n^2}$$

$$f: N \to N$$

$$f(n) = n^2.$$

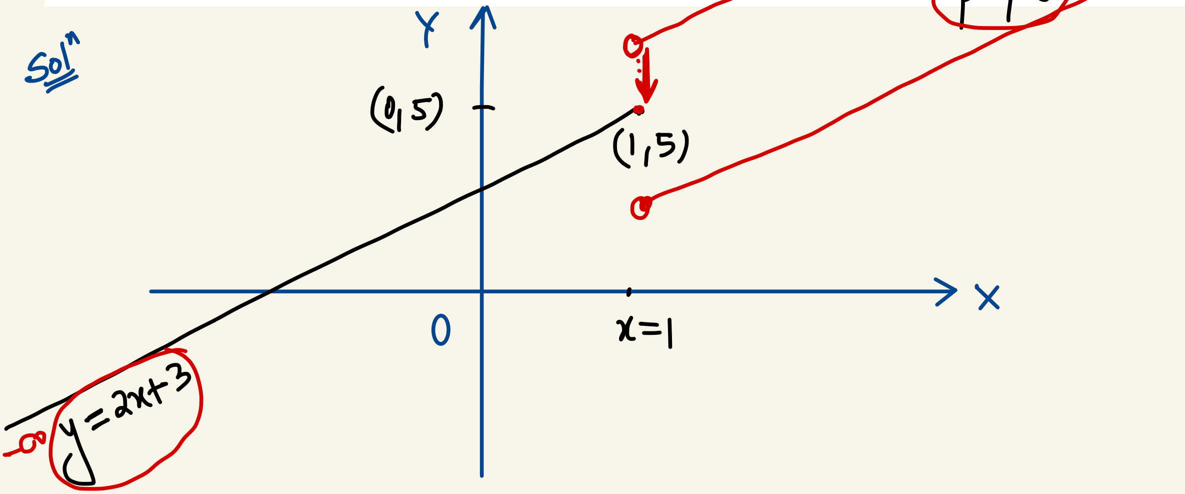
$$f(5) = 25$$

$$f(10) = 100$$

$$R_f \in \{1, 2^2, 3^2, \ldots\}$$

Q Let $f: R \to R$ be defined $f(x) = \begin{cases} 2x+3 & x \le 1 \\ p^2x+1 & x > 1 \end{cases}$ where $f(x)$ is onto function. If p_1 and p_2 are minimum and maximum value of p respectively, then $\left|\frac{p_2+1}{p_1}\right|$ is equal to $\left|\frac{3}{-2}\right| = 1.50$ Ans

Sol^n



$p^2+1 \le 5$

$p^2 \le 4$

$p \in [-2, 2] - \{0\}$

$p_2 = 2$

$p_1 = -2$

Q If $e^{f(x)} = \ln x$ and $g(x)$ is inverse of $f(x)$ then $g'(x)$ is equal to (A) $e^x + x$ (B) e^{e^x}

(C) $e^{e^{e^x}} e^{e^x} e^x$ (D) $e^{e^x + x}$ ✓

Solⁿ

$$e^{f(x)} = \ln x \Rightarrow x = e^{e^y}$$

$$f^{-1}(x) = g(x) = e^{e^x}$$

$$g'(x) = e^{e^x} \cdot e^x = e^{e^x + x}$$

Q If $y = x + e^x$ then $\left.\frac{d^2x}{dy^2}\right|_{x=\ln 2} = ?$

Solⁿ

$$\frac{d^2x}{dy^2} = \frac{-\frac{d^2y}{dx^2}}{\left(\frac{dy}{dx}\right)^3} = \frac{-2}{(3)^3} = \frac{-2}{27}$$

Ans

$$\frac{dy}{dx} = 1 + e^x \Big|_{x=\ln 2} = 1 + e^{\ln 2} = 3$$

$$\frac{d^2y}{dx^2} = e^x \Big|_{x=\ln 2} = 2$$