Allen Career Institute
Kota

Definite Integration
Sheet Solutions
(For Leader)

Do Yourself

Do yourself -1 :

(i) $\int_0^3 |x^2 - x - 2|\, dx$

Solution: $|x^2-x-2| = |(x+1)(x-2)| = \begin{cases} (x+1)(x-2) & ; \quad x \in (-\infty,-1] \cup [2,\infty) \\ -(x+1)(x-2) & ; \quad x \in [-1,2] \end{cases}$

$$\therefore \quad I = \int_0^3 |x^2-x-2|\,dx = \int_0^2 |x^2-x-2|\,dx + \int_2^3 |x^2-x-2|\,dx$$

$$= \int_0^2 (-x^2+x+2)\,dx + \int_2^3 (x^2-x-2)\,dx$$

$$= -\frac{x^3}{3} + \frac{x^2}{2} + 2x\Big|_0^2 + \frac{x^3}{3} - \frac{x^2}{2} - 2x\Big|_2^3$$

$$= \left[\left(-\frac{8}{3} + \frac{4}{2} + 4\right) - 0\right] + \left[\left(\frac{27}{3} - \frac{9}{2} - 6\right) - \left(\frac{8}{3} - \frac{4}{2} - 4\right)\right]$$

$$= 3 - \frac{9}{2} - \frac{16}{3} + \frac{8}{2} + 8 = 15 - \frac{9}{2} - \frac{16}{3}$$

$$= \frac{90-27-32}{6} = \boxed{\frac{31}{6}}$$

(ii) $\int_0^4 \{x\}dx$, where $\{.\}$ denotes fractional part of x.

Solution:

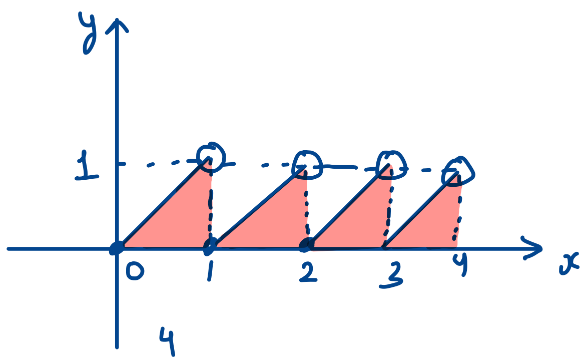
M-I

$$\{x\} = \begin{cases} x; & 0 \le x < 1 \\ x-1; & 1 \le x < 2 \\ x-2; & 2 \le x < 3 \\ x-3; & 3 \le x < 4 \end{cases}$$

$$I = \int_0^4 \{x\}\,dx = \int_0^1 x\,dx + \int_1^2 (x-1)\,dx + \int_2^3 (x-2)\,dx + \int_3^4 (x-3)\,dx$$

$$= \left.\frac{x^2}{2}\right|_0^1 + \left.\frac{(x-1)^2}{2}\right|_1^2 + \left.\frac{(x-2)^2}{2}\right|_2^3 + \left.\frac{(x-3)^2}{2}\right|_3^4 = 4\times\frac{1}{2} = \boxed{2}$$

M-II

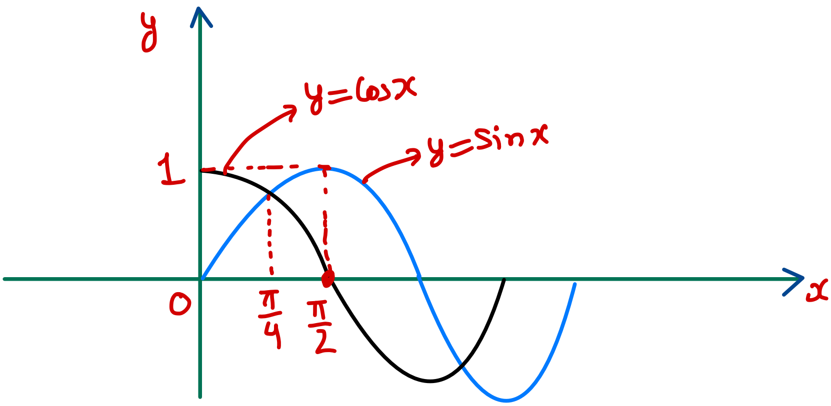


$$I = \int_0^4 \{x\}dx = \text{Algebric sum of area shaded}$$

$$= 4\times\left(\frac{1}{2}\times1\times1\right) = 4\times\frac{1}{2} = 2$$

(iii) $\int_0^{\pi/2} |\sin x - \cos x| dx$

Solution:



$$I = \int_0^{\pi/2} |\sin x - \cos x| dx$$

$$= \int_0^{\pi/4} (\cos x - \sin x) dx + \int_{\pi/4}^{\pi/2} (\sin x - \cos x) dx$$

$$= (\sin x + \cos x)\Big|_0^{\pi/4} + (-\cos x - \sin x)\Big|_{\pi/4}^{\pi/2}$$

$$= \left[\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - (0+1)\right] + \left[(-0-1) - \left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)\right]$$

$$= \left[\sqrt{2} - 1\right] + \left[-1 + \sqrt{2}\right] = 2\sqrt{2} - 2 = \boxed{2(\sqrt{2}-1)}$$

(iv) If $f(x)=\begin{cases}2 & 0\le x\le 1\\ x+[x] & 1\le x<3\end{cases}$, where [.] denotes the greatest integer function. Evaluate $\int_0^2 f(x)dx$

Solution:

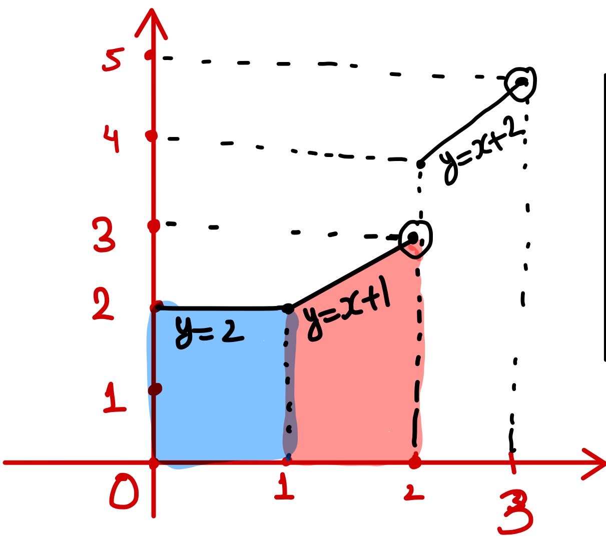
$$I=\int_0^2 f(x)dx = \int_0^1 f(x)dx+\int_1^2 f(x)dx$$

$$= \int_0^1 2\,dx + \int_1^2 (x+[x])dx$$

$$= \int_0^1 2\,dx + \int_1^2 (x+1)\,dx$$

$$= 2x\Big|_0^1 + \frac{(x+1)^2}{2}\Big|_1^2 = 2(1-0)+\left[\frac{3^2}{2}-\frac{2^2}{2}\right]$$

$$= 2+\frac{9}{2}-2 = \boxed{\frac{9}{2}}$$



Observe the shaded Area

also $= 2\times1+\frac{1}{2}(2+3)\times1$

$= 2+\frac{5}{2}=\frac{9}{2}$

Do yourself -2 :

(i) $\int_{-\pi/2}^{\pi/2} (x^2 \sin^3 x + \cos x)dx$

Solution: Observe, $f(x) = x^2 \sin^3 x + \cos x$

So, be $f(-x) = -x^2 \sin^3 x + \cos x$

See that $f(x) + f(-x) = 2\cos x$

$$\boxed{\therefore \int_{-a}^{a} f(x)\,dx = \int_{0}^{a} (f(x) + f(-x))\,dx}$$

$$\Rightarrow \int_{-\pi/2}^{\pi/2} f(x)\,dx = \int_{0}^{\pi/2} 2\cos x\,dx = 2\sin x\Big|_{0}^{\pi/2} = \boxed{2}$$

Proof: Let $I = \int_{-a}^{a} f(x)dx$

replace, $x = -t$; $I = \int_{a}^{-a} f(-t)\,(d(-t)) = \int_{-a}^{a} f(-t)\,dt = \int_{-a}^{a} f(-x)dx$

$$\Rightarrow I + I = \int_{-a}^{a} (f(x) + f(-x))\,dx = 2\int_{0}^{a} (f(x) + f(-x))\,dx \Rightarrow \boxed{2I = 2\int_{0}^{a} (f(x) + f(-x))dx}$$

[As $g(x) = f(x) + f(-x)$ is an even function]

$\boxed{\text{OR}}$ $I = \underbrace{\int_{-\pi/2}^{\pi/2} x^2 \sin^3 x\,dx}_{\text{Odd}} + \underbrace{\int_{-\pi/2}^{\pi/2} \cos x\,dx}_{\text{Even}} = \underbrace{0}_{\text{zero}} + \underbrace{2\int_{0}^{\pi/2} \cos x\,dx}_{\text{twice}}$

$\boxed{= 2}$

(ii) $\displaystyle\int_{-\pi/2}^{\pi/2} \ell n\left[2\left(\frac{4-\sin\theta}{4+\sin\theta}\right)\right]d\theta$

Solution:

$$I = \int_{-\pi/2}^{\pi/2} \ln\left[2\left(\frac{4-\sin\theta}{4+\sin\theta}\right)\right]dx$$

$$\boxed{\because \int_{-a}^{a} f(x)\,dx = \int_{0}^{a} \big(f(x)+f(-x)\big)\,dx}$$

$$\therefore I = \int_{0}^{\pi/2} \left\{\ln\left[2\left(\frac{4-\sin\theta}{4+\sin\theta}\right)\right] + \ln\left[2\left(\frac{4+\sin\theta}{4-\sin\theta}\right)\right]\right\} d\theta$$

$$= \int_{0}^{\pi/2} \ln\left\{\left[2\left(\frac{4-\sin\theta}{4+\sin\theta}\right)\right]\left[2\left(\frac{4+\sin\theta}{4-\sin\theta}\right)\right]\right\} d\theta$$

$$= \int_{0}^{\pi/2} \ln 4\, d\theta = \frac{\pi}{2}\ln 4 = \boxed{\pi \ln 2}$$

Note: Any, $\displaystyle f(x) = \underbrace{\frac{f(x)+f(-x)}{2}}_{\text{Even Extension}} + \underbrace{\frac{f(x)-f(-x)}{2}}_{\text{Odd Extension}}$

$$\boxed{\therefore \int_{-a}^{a} f(x)dx = \int_{-a}^{a}\left(\frac{f(x)+f(-x)}{2}\right)dx + 0 = 2\int_{0}^{a}\left(\frac{f(x)+f(-x)}{2}\right)dx}$$

Do yourself - 3 :

(i) $\displaystyle\int_1^5 \frac{\sqrt{x}}{\sqrt{6-x}+\sqrt{x}}dx$ (ii) $\displaystyle\int_{\pi/6}^{\pi/3} \frac{dx}{1+\tan^5 x}$

$$\boxed{\text{King Property: } \int_a^b f(x)dx = \int_a^b f(a+b-x)\,dx}$$

Solution: (i) Let $I = \displaystyle\int_1^5 \frac{\sqrt{x}}{\sqrt{6-x}+\sqrt{x}}dx$

By King,

$$I = \int_1^5 \frac{\sqrt{6-x}}{\sqrt{6-(6-x)}+\sqrt{6-x}}dx$$

$$= \int_1^5 \frac{\sqrt{6-x}}{\sqrt{x}+\sqrt{6-x}}dx$$

Now,

$$I+I = \int_1^5 \frac{\sqrt{x}}{\sqrt{6-x}+\sqrt{x}}dx + \int_1^5 \frac{\sqrt{6-x}}{\sqrt{x}+\sqrt{6-x}}dx$$

$$\Rightarrow 2I = \int_1^5 \frac{\sqrt{x}+\sqrt{6-x}}{\sqrt{6-x}+\sqrt{x}}dx = \int_1^5 1\,dx$$

$$\Rightarrow 2I = 4 \Rightarrow \boxed{I=2}$$

(ii) Let $I = \displaystyle\int_{\pi/6}^{\pi/3} \frac{dx}{1+\tan^5 x}$

$$\Rightarrow I = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\frac{\sin^5 x}{\cos^5 x}}$$

$$\Rightarrow I = \int_{\pi/6}^{\pi/3} \frac{\cos^5 x}{\cos^5 x+\sin^5 x}dx$$

By King,

$$I = \int_{\pi/6}^{\pi/3} \frac{\cos^5\left(\frac{\pi}{3}+\frac{\pi}{6}-x\right)}{\cos^5\left(\frac{\pi}{3}+\frac{\pi}{6}-x\right)+\sin^5\left(\frac{\pi}{3}+\frac{\pi}{6}-x\right)}dx$$

$$I = \int_{\pi/6}^{\pi/3} \frac{\sin^5 x}{\sin^5 x+\cos^5 x}dx$$

Now, $I+I = \displaystyle\int_{\pi/6}^{\pi/3} 1\,dx = 1\left(\frac{\pi}{3}-\frac{\pi}{6}\right)$

$$\Rightarrow 2I = \frac{\pi}{6} \Rightarrow \boxed{I=\frac{\pi}{12}}$$

Do yourself -4 :

(i) $\displaystyle\int_{-\sqrt{3}}^{\sqrt{3}} \frac{dx}{(1+e^x)(1+x^2)}$

Solution: Using, $\boxed{\because \displaystyle\int_{-a}^{a} f(x)\,dx = \int_{0}^{a} \left(f(x) + f(-x)\right) dx}$ [Refer Solutions of DYS #2]

$$I = \int_0^{\sqrt{3}} \left(\frac{1}{(1+e^x)(1+x^2)} + \frac{1}{(1+e^{-x})(1+x^2)} \right) dx$$

$$= \int_0^{\sqrt{3}} \left(\frac{1}{(1+e^x)(1+x^2)} + \frac{e^x}{(e^x+1)(1+x^2)} \right) dx$$

$$= \int_0^{\sqrt{3}} \left(\frac{1+e^x}{(1+e^x)(1+x^2)} \right) dx = \int_0^{\sqrt{3}} \frac{1}{1+x^2}\,dx$$

$$= \tan^{-1}x \Big|_0^{\sqrt{3}} = \tan^{-1}\sqrt{3} - \tan^{-1}0 = \frac{\pi}{3} - 0$$

$$\boxed{I = \boxed{\frac{\pi}{3}}}$$

Note: Any, $f(x) = \underbrace{\frac{f(x)+f(-x)}{2}}_{\text{Even Extension}} + \underbrace{\frac{f(x)-f(-x)}{2}}_{\text{Odd Extension}}$

$$\boxed{\therefore \int_{-a}^{a} f(x)dx = \int_{-a}^{a} \left(\frac{f(x)+f(-x)}{2}\right) dx + 0 = 2\int_0^a \left(\frac{f(x)+f(-x)}{2}\right) dx}$$

(ii) $\int_0^{\pi/2} \ell n\left(\sin^2 x \cos x\right) dx$

<u>Solution:</u> Let $I = \int_0^{\pi/2} \ln(\sin^2 x \cdot \cos x)\,dx$

Apply king, $I = \int_0^{\pi/2} \ln(\cos^2 x \cdot \sin x)\,dx$

& Add,

$$I + I = 2I = \int_0^{\pi/2} \left(\ln(\sin^2 x \cdot \cos x) + \ln(\cos^2 x \cdot \sin x)\right)dx$$

$$\Rightarrow 2I = \int_0^{\pi/2} \left(\ln(\sin^3 x \cdot \cos^3 x)\right)dx = \int_0^{\pi/2} (3\ln\sin x + 3\ln\cos x)\,dx$$

$$\Rightarrow 2I = 3\left(-\frac{\pi}{2}\ln 2\right) + 3\left(-\frac{\pi}{2}\ln 2\right)$$

$$\Rightarrow \boxed{I = -\frac{3\pi}{2}\ln 2}$$

Remember:

$$\int_0^{\pi/2} \log(\sin x)dx = \int_0^{\pi/2} \log(\cos x)dx = -\frac{\pi}{2}\log 2$$

King Property: $\int_a^b f(x)dx = \int_a^b f(a+b-x)\,dx$

(iii) $\displaystyle\int_0^{\pi/2} \frac{\sin x - \cos x}{1+\sin x \cos x}\,dx$

Solution: Let $\displaystyle I = \int_0^{\pi/2} \frac{\sin x - \cos x}{1+\sin x \cdot \cos x}\,dx$

Apply King: $\displaystyle I = \int_0^{\pi/2} \frac{\sin\left(\frac{\pi}{2}-x\right) - \cos\left(\frac{\pi}{2}-x\right)}{1+\sin\left(\frac{\pi}{2}-x\right)\cdot\cos\left(\frac{\pi}{2}-x\right)}\,dx$

$\displaystyle \Rightarrow I = \int_0^{\pi/2} \frac{\cos x - \sin x}{1+\cos x \cdot \sin x}\,dx$

& add, $\displaystyle 2I = \int_0^{\pi/2} \left(\frac{\sin x - \cos x}{1+\sin x \cdot \cos x} + \frac{\cos x - \sin x}{1+\cos x \cdot \sin x}\right)dx$

$\Rightarrow 2I = 0 \Rightarrow \boxed{I = 0}$

King Property: $\displaystyle \int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

(iv) $\int_{-\pi/2}^{\pi/2} \sqrt{\cos x - \cos^3 x}\, dx$

Solution:

$$I = \int_{-\pi/2}^{\pi/2} \sqrt{\cos x(1-\cos^2 x)}\; dx$$

$$= 2\int_{0}^{\pi/2} \sqrt{\cos x \cdot \sin^2 x}\; dx \qquad \{\text{Even Function}\}$$

$$= 2\int_{0}^{\pi/2} \sqrt{\cos x}\cdot |\sin x|\, dx$$

$$I = 2\int_{0}^{\pi/2} \sqrt{\cos x}\cdot \sin x\, dx$$

Let $\cos x = t \Rightarrow -\sin x \cdot dx = dt$

x	0	$\pi/2$
t	1	0

$$\therefore\; I = 2\int_{1}^{0} \sqrt{t}\,(-dt) = 2\int_{0}^{1} \sqrt{t}\, dt = 2\left.\frac{\left(t^{3/2}\right)}{3/2}\right|_{0}^{1}$$

$$\boxed{I = \frac{4}{3}}$$

Do yourself -5 :

Evaluate :

(i) $\int_{-1.5}^{10} \{2x\}dx$, where {.} denotes fractional part of x.

Solution: Period of $\{2x\}$ is $\frac{1}{2}$ or 0.5

$$\therefore \quad I = \int_{(-3)\times 0.5}^{20\times 0.5} \{2x\}dx = (20-(-3))\int_0^{1/2} \{2x\}dx$$

$$= 23\int_0^{1/2} (2x)dx = 23\times x^2\Big|_0^{1/2} = \boxed{\frac{23}{4}}$$

Note: If 'T' is fundamental period of a function f(x)

then $\boxed{\int_{n_1T}^{n_2T} f(x)\,dx = (n_2-n_1)\int_0^T f(x)dx}$

also, $$\int_{n_1T+a}^{n_2T+b} f(x)\,dx = \int_{n_1T+a}^{n_2T+a} f(x)dx + \int_a^b f(x)dx$$

$$= (n_2-n_1)\int_0^T f(x)dx + \int_a^b f(x)dx$$

Think this result to get derived by your own.

(ii) $\displaystyle\int_{20\pi+\frac{\pi}{6}}^{20\pi+\frac{\pi}{3}} (\sin x + \cos x)dx$

Solution: $\displaystyle I = \int_{20\pi+\frac{\pi}{6}}^{20\pi+\frac{\pi}{3}} (\sin x + \cos x)\, dx$

Let $f(x) = \sin x + \cos x \Rightarrow \boxed{f(x+\frac{\pi}{2}) = f(x)} \rightarrow T = \frac{\pi}{2}$

$$\therefore\ I = \int_{40\frac{\pi}{2}+\frac{\pi}{6}}^{40\frac{\pi}{2}+\frac{\pi}{3}} (f(x))dx = (40-40)\int_0^{\pi/2} f(x)dx + \int_{\frac{\pi}{6}}^{\pi/3} f(x)\, dx$$

$$= 0 + \int_{\pi/6}^{\pi/3} (\sin x + \cos x)\, dx$$

$$= 0 + (-\cos x + \sin x)\Big|_{\pi/6}^{\pi/3} = \left(\frac{-1}{2} + \frac{\sqrt{3}}{2}\right) - \left(\frac{-\sqrt{3}}{2} + \frac{1}{2}\right)$$

$$\Rightarrow \boxed{I = \sqrt{3} - 1}$$

Use, $\displaystyle\int_{n_1T+a}^{n_2T+b} f(x)\, dx = \int_{n_1T+a}^{n_2T+a} f(x)dx + \int_a^b f(x)dx$

$$= (n_2 - n_1)\int_0^T f(x)dx + \int_a^b f(x)dx$$

$\boxed{\text{Proofs:}}$ We have $\int_0^{nT} f(x)\,dx = n\int_0^{T} f(x)\,dx$

where $f(x+T) = f(x)$

Now:

$$\boxed{\int_a^{nT+a} f(x)\,dx = n\int_0^{T} f(x)\,dx}$$

Proof: LHS $= \int_a^{0} f(x)dx + \int_0^{nT} f(x)dx + \int_{nT}^{nT+a} f(x)\,dx$

$= \int_a^{0} f(x)\,dx + \int_0^{nT} f(x)dx + \int_0^{a} f(nT+t)dt$ $\quad \{f(nT+t) = f(t)\}$

[Put $x = nT+t$]

$= \int_0^{nT} f(x)dx = n\int_0^{T} f(x)dx$

And, $\int_{n_1T}^{n_2T} f(x)\,dx = \int_{n_1T}^{0} f(x)dx + \int_0^{n_2T} f(x)\,dx$

$= -\int_0^{n_1T} f(x)dx + \int_0^{n_2T} f(x)\,dx$

$= -n_1\int_0^{T} f(x)dx + n_2\int_0^{T} f(x)\,dx$

$$\boxed{\int_{n_1T}^{n_2T} f(x)\,dx = (n_2-n_1)\int_0^{T} f(x)\,dx}$$

$$\boxed{I = \int_a^{a+T} f(x)\,dx = \int_0^{T} f(x)\,dx}$$

$$\int_{n_1T+a}^{n_2T+a} f(x)\,dx = \int_{n_1T+a}^{a} f(x)\,dx + \int_{a}^{n_2T+a} f(x)\,dx$$

$$= -\int_{a}^{n_1T+a} f(x)\,dx + \int_{a}^{n_2T+a} f(x)\,dx$$

$$= -\int_{0}^{n_1T} f(x)\,dx + \int_{0}^{n_2T} f(x)\,dx$$

$$= (n_2-n_1)\int_{0}^{T} f(x)\,dx.$$

$$I = \int_{n_1T+a}^{n_2T+b} f(x)\,dx = \int_{n_1T+a}^{n_2T+a} f(x)\,dx + \int_{n_2T+a}^{n_2T+b} f(x)\,dx$$

$$\boxed{\int_{n_1T+a}^{n_2T+b} f(x)\,dx = (n_2-n_1)\int_{0}^{T} f(x)\,dx + \int_{a}^{b} f(x)\,dx}$$

Do yourself - 6 :

(i) If $f(x)=\int_{1/x}^{\sqrt{x}} \sin t\,dt$, then find $f'(1)$.

(ii) $\int_{\pi/3}^{x} \sqrt{3-\sin^2 t}\,dt + \int_{0}^{y} \cos t\,dt = 0$, then evaluate $\frac{dy}{dx}$.

Solution: (i) $f(x) = \int_{1/x}^{\sqrt{x}} \sin t \, dt$

$$\Rightarrow f'(x) = (\sin\sqrt{x})\cdot\frac{d}{dx}(\sqrt{x}) - \sin\left(\frac{1}{x}\right)\cdot\frac{d}{dx}\left(\frac{1}{x}\right)$$

$$= \frac{\sin\sqrt{x}}{2\sqrt{x}} + \frac{\sin\left(\frac{1}{x}\right)}{x^2}$$

$$\therefore f'(1) = \frac{\sin 1}{2} + \frac{\sin 1}{1} = \boxed{\frac{3}{2}\sin 1}$$

(ii) $\int_{\frac{\pi}{3}}^{x} \sqrt{3-\sin^2 t}\,dt + \int_0^y \cos t \cdot dt = 0$

$$\Rightarrow \frac{d}{dx}\left(\int_{\frac{\pi}{3}}^{x} \sqrt{3-\sin^2 t}\,dt + \int_0^y \cos t\cdot dt\right) = 0$$

$$\Rightarrow \frac{d}{dx}\left(\int_{\pi/3}^{x} \sqrt{3-\sin^2 t}\,dt\right) + \frac{d}{dy}\left(\int_0^y \cos t\,dt\right)\cdot\frac{dy}{dx} = 0$$

$$\Rightarrow \sqrt{3-\sin^2 x}\cdot\frac{d}{dx}(x) - \sqrt{3-\sin^2\frac{\pi}{3}}\cdot\frac{d}{dx}\left(\frac{\pi}{3}\right) + \left(\cos y\cdot 1 - \cos 0\cdot\frac{d}{dy}(0)\right)\frac{dy}{dx} = 0$$

$$\Rightarrow \sqrt{3-\sin^2 x}\cdot 1 - 0 + \cos y\cdot\frac{dy}{dx} = 0$$

$$\boxed{\Rightarrow \frac{dy}{dx} = \frac{-\sqrt{3-\sin^2 x}}{\cos y}}$$

Do yourself - 7 :

Evaluate :

(i) $\lim_{n\to\infty}\left[\frac{1}{n+2.1}+\frac{1}{n+2.2}+\frac{1}{n+2.3}\ldots\ldots.\frac{1}{3n}\right]$ **(ii)** $\lim_{n\to\infty}\sum_{r=0}^{n-1}\frac{1}{\sqrt{n^2-r^2}}$

Steps for : $\lim_{n\to\infty}\sum_{r=1}^{n} f(r,n)$

(faster)

(Short-cut)

Replace, $r \to nx$; $1 \to n.dx$ & $\lim_{n\to\infty}\sum \to \int_a^b$

Solution: (i) $\lim_{n\to\infty}\sum_{r=1}^{n}\left(\frac{1}{n+2\cdot r}\right)$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{(n+2r)}\cdot 1 = \int_0^1 \frac{1}{(n+2nx)}\cdot ndx = \int_0^1\frac{dx}{(1+2x)}$$

$$= \frac{1}{2}.\ln|1+2x|\Big|_0^1 = \frac{1}{2}[\ln 3-\ln 0] = \boxed{\frac{1}{2}\ln 3}$$

(ii) $$\lim_{n\to\infty}\sum_{r=0}^{n-1}\frac{1}{\sqrt{n^2-r^2}} = \lim_{n\to\infty}\sum_{r=0}^{n-1}\frac{1}{\sqrt{n^2-r^2}}\cdot 1$$

$$= \int_0^1\frac{1}{\sqrt{n^2-n^2x^2}}\cdot ndx = \int_0^1\frac{dx}{\sqrt{1-x^2}}$$

$$= \sin^{-1}x\Big|_0^1 = \boxed{\frac{\pi}{2}}$$

Do yourself - 8 :

(i) Prove that $4 \le \int_1^3 \sqrt{3+x^2}\,dx \le 4\sqrt{3}$

Solution: In $x \in [1,3]$; $f(x) = \sqrt{3+x^2}$ is increasing.

$$f'(x) = \frac{x}{\sqrt{3+x^2}} > 0 \text{ for } x \in [1,3]$$

$\therefore \quad f(1) \le f(x) \le f(3)$

$$\Rightarrow \int_1^3 f(1)\,dx \le \int_1^3 f(x)\,dx \le \int_1^3 f(3)\,dx$$

$$\Rightarrow \int_1^3 \sqrt{3+1^2}\,dx \le \int_1^3 \sqrt{3+x^2}\,dx \le \int_1^3 \sqrt{3+3^2}\,dx$$

$$\Rightarrow \int_1^3 2\,dx \le \int_1^3 \sqrt{3+x^2}\,dx \le \int_1^3 2\sqrt{3}\,dx$$

$$\Rightarrow \boxed{4 \le \int_1^3 \sqrt{3+x^2}\,dx \le 4\sqrt{3}}$$

[Hence Proved]

(ii) Prove that $\frac{\pi}{4} \le \int_0^{2\pi} \frac{dx}{5+3\sin x} \le \pi$.

Solution: For $x \in [0, 2\pi]$; $2 \le 5+3\sin x \le 8$

$$\therefore \frac{1}{8} \le \frac{1}{5+3\sin x} \le \frac{1}{2}$$

$$\Rightarrow \int_0^{2\pi} \frac{1}{8}\,dx \le \int_0^{2\pi} \frac{1}{5+3\sin x} \cdot dx \le \int_0^{2\pi} \frac{1}{2}\,dx$$

$$\Rightarrow \frac{2\pi}{8} \le \int_0^{2\pi} \frac{1}{5+3\sin x} \cdot dx \le \frac{2\pi}{2}$$

$$\Rightarrow \boxed{\frac{\pi}{4} \le \int_0^{2\pi} \frac{1}{5+3\sin x} \cdot dx \le \pi}$$

(iii) Show that $\frac{3}{5}\left(2^{1/3}-1\right) \le \int_0^1 \frac{x^4}{(1+x^6)^{2/3}}\,dx \le 1$

Solution: $f(x) = \frac{x^4}{(1+x^6)^{2/3}}$ [Please Go through complete Solution carefully]

Analysing $f(x)$ for $x \in [0,1]$,

$$f(0)=0; \quad f(1) = \frac{1}{2^{2/3}} = 2^{-2/3}$$

for $x \neq 0$, $f(x) = \frac{x^4}{\left(x^6\left(\frac{1}{x^6}+1\right)\right)^{2/3}} = \frac{x^4}{(x^6)^{2/3}\left(\frac{1}{x^6}+1\right)^{2/3}} = \frac{1}{(x^{-6}+1)^{2/3}}$

In $x \in (0,1]$; $\quad x^{-6} \downarrow \Rightarrow (x^{-6}+1) \downarrow$

$$\Rightarrow \frac{1}{x^{-6}+1} \uparrow \Rightarrow f(x) \uparrow$$

$\Rightarrow \quad f(x) \uparrow$ for $x \in [0,1]$

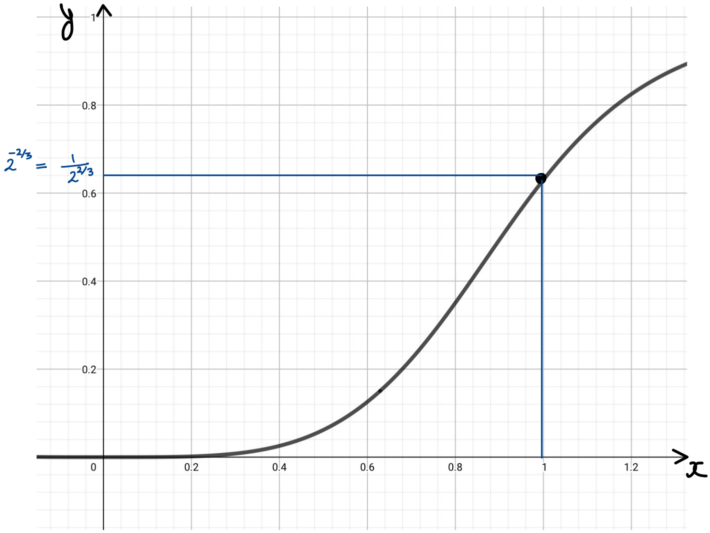
$f(0) \le f(x) \le f(1) \Rightarrow \quad 0 \le f(x) \le \frac{1}{2^{2/3}}$ (But this analysis is not helping much for required result)

Also, $\quad x^6 \le (x^6+1) \le 2 \quad$ for $0 \le x < 1$

$$\Rightarrow (x^6)^{2/3} \le (x^6+1)^{2/3} \le 2^{2/3}$$

$$\Rightarrow x^4 \le (x^6+1)^{2/3} \le 2^{2/3}$$

$$\Rightarrow \frac{1}{2^{2/3}} \le \frac{1}{(x^6+1)^{2/3}} \le \frac{1}{x^4} \Rightarrow \frac{x^4}{2^{2/3}} \le \frac{x^4}{(x^6+1)^{2/3}} \le 1$$



$$y = \frac{x^4}{(1+x^6)^{2/3}}$$

One side is matched with $\boxed{\frac{x^4}{(1+x^6)^{2/3}} \leq 1}$

and for other side to get '5' in denominator, think

$$x^6 \leq x^5 \Rightarrow \quad 1+x^6 \leq 1+x^5$$

$$\Rightarrow \quad (1+x^6)^{2/3} \leq (1+x^5)^{2/3}$$

$$\Rightarrow \quad \frac{1}{(1+x^5)^{2/3}} \leq \frac{1}{(1+x^6)^{2/3}} \Rightarrow \frac{x^4}{(1+x^5)^{2/3}} \leq \frac{x^4}{(1+x^6)^{2/3}}$$

Hence, $$\int_0^1 \frac{x^4}{(1+x^5)^{2/3}}\,dx \leq \int_0^1 \frac{x^4}{(1+x^6)^{2/3}}\,dx \leq \int_0^1 1\,dx$$

$$\Rightarrow \frac{1}{5}\int_0^1 \frac{5x^4}{(1+x^5)^{2/3}}\,dx \leq \int_0^1 \frac{x^4}{(1+x^6)^{2/3}}\,dx \leq 1 \Rightarrow \frac{1}{5}\left.\frac{(1+x^5)^{1/3}}{1/3}\right|_0^1 \leq I \leq 1$$

$$\Rightarrow \boxed{\frac{3}{5}(2^{1/3}-1) \leq I \leq 1}$$

[Hence Proved]

EXERCISE (O-1)

EXERCISE (O-1)

[STRAIGHT OBJECTIVE TYPE]

1. If $g(x) = \int_0^x \cos^4 t\, dt$, then $g(x+\pi)$ equals

(A) $g(x) + g(\pi)$ (B) $g(x) - g(\pi)$ (C) $g(x)\,g(\pi)$ (D) $[g(x)/g(\pi)]$

Solution – $g(x+\pi) = \int_0^{x+\pi} \cos^4 t\, dt$

$= \int_0^{x} \cos^4 t\, dt + \int_x^{x+\pi} \cos^4 t\, dt$

$= g(x) + \int_0^{\pi} \cos^4 t\, dt$ ($\because$ period of $\cos^4 t$ is π)

$= g(x) + g(\pi)$

Scanned with CamScanner

2. Variable x and y are related by equation $x = \int_0^y \frac{dt}{\sqrt{1+t^2}}$. The value of $\frac{d^2y}{dx^2}$ is equal to

(A) $\frac{y}{\sqrt{1+y^2}}$ (B) y (C) $\frac{2y}{\sqrt{1+y^2}}$ (D) 4y

Solution - $\frac{dx}{dy} = \frac{1}{\sqrt{1+y^2}}$

$$\Rightarrow \frac{dy}{dx} = \sqrt{1+y^2}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dy}\left(\frac{dy}{dx}\right)\cdot\frac{dy}{dx}$$

$$= \frac{\cancel{2}y}{\cancel{2}\cancel{\sqrt{1+y^2}}}\cdot\cancel{\sqrt{1+y^2}}$$

$$= y \qquad (\text{Ans } B)$$

3

If $\int_0^x f(t)\,dt = x + \int_x^1 t^2 \cdot f(t)\,dt + \frac{\pi}{4} - 1$, then the value of the integral $\int_{-1}^{1} f(x)\,dx$ is equal to

(A) 0 (B) $\pi/4$ (C) $\pi/2$ (D) π

Solution - differentiate both the sides w.r.t. 'x'

$f(x) = 1 + 0 - x^2 f(x)$

$\Rightarrow f(x) = \frac{1}{1+x^2} \Rightarrow \int_{-1}^{1} \frac{1}{1+x^2}\,dx = 2\left[\tan^{-1}x\right]_0^1 = \frac{\pi}{2}$

4

If $I = \int_0^{\pi/2} \ell n(\sin x)dx$ then $\int_{-\pi/4}^{\pi/4} \ell n(\sin x + \cos x)dx$

(A) $\frac{I}{2}$ (B) $\frac{I}{4}$ (C) $\frac{I}{\sqrt{2}}$ (D) I

Solution -

$$I_1 = \int_{-\pi/4}^{\pi/4} \ln(\sin x + \cos x)\, dx$$

$$= \int_{-\pi/4}^{\pi/4} \ln(-\sin x + \cos x)\, dx \quad \text{(king prop.)}$$

$$2I_1 = \int_{-\pi/4}^{\pi/4} \ln(\cos^2 x - \sin^2 x)\, dx = \int_{-\pi/4}^{\pi/4} \ln(\cos 2x)\, dx$$

$$= 2\int_0^{\pi/4} \ln(\cos 2x)\, dx \quad \text{(even function)}$$

put $2x = t$

$$\Rightarrow \quad I_1 = \frac{1}{2}\int_0^{\pi/2} \ln(\cos t)\, dt = \frac{1}{2}\int_0^{\pi/2} \ln(\sin t)\, dt \quad \text{(king)}$$

$$\Rightarrow \quad I_1 = \frac{I}{2}$$

Scanned with CamScanner

⑤ The value of the definite integral $\int_{0}^{\pi/2} \sin x \sin 2x \sin 3x \, dx$ is equal to :

(A) $\frac{1}{3}$ (B) $-\frac{2}{3}$ (C) $-\frac{1}{3}$ (D) $\frac{1}{6}$

Solution - $I = \int_0^{\frac{\pi}{2}} \sin x \sin 2x \sin 3x \, dx$

$= -\int_0^{\frac{\pi}{2}} \cos x \sin 2x \cos 3x \, dx$ (king prop.)

$\Rightarrow 2I = \int_0^{\pi/2} \sin 2x (\sin 3x \sin x - \cos 3x \cos x) \, dx$

$= -\int_0^{\pi/2} \sin 2x \cos 4x \, dx$

$= -\frac{1}{2} \int_0^{\pi/2} [\sin 6x - \sin 2x] \, dx$

$= \frac{1}{2} \left[\frac{\cos 6x}{6} - \frac{\cos 2x}{2}\right]_0^{\pi/2}$

$= \frac{1}{2} \left[\left(-\frac{1}{6} + \frac{1}{2}\right) - \left(\frac{1}{6} - \frac{1}{2}\right)\right]$

$= \frac{1}{3} \Rightarrow I = \frac{1}{6}$

6

Value of the definite integral $\int_{-1/2}^{1/2} (\sin^{-1}(3x-4x^3) - \cos^{-1}(4x^3-3x))\,dx$

(A) 0 (B) $-\frac{\pi}{2}$ (C) $\frac{7\pi}{2}$ (D) $\frac{\pi}{2}$

Solution -

$$I = \int_{-\frac{1}{2}}^{\frac{1}{2}} [\sin^{-1}(3x-4x^3) - (\pi - \cos^{-1}(3x-4x^3))]\,dx$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} [\sin^{-1}(3x-4x^3) + \cos^{-1}(3x-4x^3)]\,dx - \pi\int_{-\frac{1}{2}}^{\frac{1}{2}} dx$$

$$= \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{\pi}{2}\,dx - \pi\int_{-\frac{1}{2}}^{\frac{1}{2}} dx$$

$$= -\frac{\pi}{2}$$

7. The value of the definite integral $\int_{1}^{e}\left((x+1)e^{x}.\ell nx\right)dx$ is -

(A) e (B) e^{e+1} (C) $e^{e}(e-1)$ (D) $e^{e}(e-1)+e$

Solution–

$$\int_{1}^{e}(x+1)e^{x}\ln x = \int_{1}^{e}(x\ln x+\ln x)e^{x}dx$$

$$= \int_{1}^{e}[\underbrace{x\ln x}_{f(x)} + \underbrace{(1+\ln x)}_{f'(x)}]e^{x}dx - \int_{1}^{e}e^{x}dx$$

$$= \left[(x\ln x)e^{x}\right]_{1}^{e} - \left(e^{x}\right]_{1}^{e} = e.e^{e} - e^{e} + e$$

8

$\lim\limits_{n\to\infty} \dfrac{1}{\sqrt{n}\sqrt{n+1}} + \dfrac{1}{\sqrt{n}\sqrt{n+2}} + ... + \dfrac{1}{\sqrt{n}\sqrt{4n}}$ is equal to -

(A) 2 (B) 4 (C) $2\left(\sqrt{2}-1\right)$ (D) $2\sqrt{2}-1$

Sol^n

$$\Rightarrow \lim_{n\to\infty} \sum_{r=1}^{3n} \frac{1}{\sqrt{n}\sqrt{n+r}}$$

$$\Rightarrow \lim_{n\to\infty} \sum_{r=1}^{3n} \frac{1}{n\cdot\sqrt{1+\frac{r}{n}}}$$

$$\Rightarrow \int_0^3 \frac{1}{\sqrt{1+x}}\,dx$$

$$\Rightarrow \left(2\sqrt{1+x}\right)_0^3$$

$$\Rightarrow 4-2=2$$

A

9

The value of the definite integral $\int_{19}^{37}\left(\{x\}^2+3(\sin 2\pi x)\right)dx$ where { x } denotes the fractional part function.

(A) 0 (B) 6 (C) 9 (D) can't be determined

Solution-

∵ $\{x\}$ and $\sin 2\pi x$ both have period equals to

∴ $I = (37-19)\int_0^1 \left(\{x\}^2 + 3\sin 2\pi x\right)dx$

$= 18\left(\int_0^1 x^2\,dx + 3\int_0^1 \sin 2\pi x\,dx\right)$

$= 18\left(\frac{1}{3} + 3\cdot 0\right) = 6$

(10)

$$\lim_{x\to 0} \frac{\int_0^x \sin t^2 dt}{x(1-\cos x)} \text{ equals -}$$

(A) $\frac{1}{3}$ (B) 2 (C) $\frac{1}{2}$ (D) $\frac{2}{3}$

Solution -

$$= \lim_{x\to 0} \frac{\int_0^x \sin t^2 \, dt}{x\left(\frac{1-\cos x}{x^2}\right) \cdot x^2} = 2\lim_{x\to 0} \frac{\int_0^x \sin t^2 \, dt}{x^3}$$

$$= \lim_{x\to 0} 2\, \frac{\sin x^2}{3x^2} \quad \text{(Applying L'ô pital)}$$

$$= \frac{2}{3}$$

Scanned with CamScanner

(11) If g (x) is the inverse of f (x) and f (x) has domain $x \in [1, 5]$, where $f(1) = 2$ and $f(5) = 10$ then the values of $\int_1^5 f(x)\,dx + \int_2^{10} g(y)\,dy$ equals -

(A) 48 (B) 64 (C) 71 (D) 52

Solution - $\because \int_a^b f(x)\,dx + \int_{f(a)}^{f(b)} f^{-1}(x)\,dx = bf(b) - af(a)$.

$\therefore \int_1^5 f(x)\,dx + \int_{f(1)}^{f(5)} f^{-1}(x)\,dx = 5 \cdot f(5) - 1 \cdot f(1) = 48$

12

The value of the definite integral $\int_2^4 \left(x(3-x)(4+x)(6-x)(10-x)+\sin x\right)dx$ equals-

(A) $\cos 2+\cos 4$ (B) $\cos 2-\cos 4$ (C) $\sin 2+\sin 4$ (D) $\sin 2-\sin 4$

Solution - $I = \int_2^4 [x(3-x)(4+x)(6-x)(10-x)]\,dx + \int_2^4 \sin x\,dx$

$\Rightarrow I = -\int_2^4 \underbrace{(6-x)(3-x)(10-x)\,x\,(4+x)}_{\text{(king prop.)}} + [-\cos x]_2^4$

$\Rightarrow 2I = 2[-\cos 4+\cos 2] \Rightarrow I = \cos 2-\cos 4$

13

The true solution set of the inequality, $\sqrt{5x-6-x^2}+\left(\frac{\pi}{2}\int_0^x dz\right) > x\int_0^{\pi}\sin^2 x dx$ is :

(A) R (B) (1,6) (C) (–6,1) (D) (2,3)

Solution - $\sqrt{5x-6-x^2}+\frac{\pi}{2}.x > \cancel{x}.2.\int_0^{\pi/2}\sin^2 x\, dx$ (Queen prop.)

$\Rightarrow \sqrt{5x-6-x^2}+\frac{\pi}{2}x > x.\cancel{2}.\left(\frac{1}{\cancel{2}}.\frac{\pi}{2}\right)$ (wallis theorem)

$\Rightarrow \sqrt{5x-6-x^2} > 0$ which is true in domain except $x=2,3$

$\Rightarrow 5x-6-x^2 \geq 0$ (domain)

$\Rightarrow (x-2)(x-3) \leq 0$ but $x \neq 2,3$

$\Rightarrow x \in (2,3)$.

14

$$\int_{\frac{1}{2}}^{3\frac{1}{2}} \left\{\frac{1}{2}(|x-3|+|1-x|-4)\right\} dx \text{ equals-}$$

(A) $-\frac{3}{2}$ (B) $\frac{9}{8}$ (C) $\frac{1}{4}$ (D) $\frac{3}{2}$

Where {.} denotes the fraction part function.

Solution put $x-2=t$

$$I = \int_{-3/2}^{3/2} \left\{\frac{|t-1|+|t+1|}{2} - 2\right\} dt$$

$$= \int_{-3/2}^{3/2} \left\{\underbrace{\frac{|t-1|+|t+1|}{2}}_{\text{even function}}\right\} dt \qquad (\because \{x+n\}=\{x\},\ n\in I)$$

$$= 2\int_{0}^{3/2} \left\{\frac{|t-1|+t+1}{2}\right\} dt$$

$$= 2\left[\int_{0}^{1}\left\{\frac{1-\cancel{t}+\cancel{t}+1}{2}\right\}dt + \int_{1}^{3/2}\left\{\frac{t-1+t+1}{2}\right\}dt\right]$$

$$= 2\left[0 + \int_{1}^{3/2}\{t\}\,dt\right]$$

$$= 2\int_{1}^{3/2}(t-1)\,dt = \left[(t-1)^2\right]_{1}^{3/2}$$

$$= \frac{1}{4}$$

15

Suppose that F(x) is an antiderivative of $f(x) = \frac{\sin x}{x}, x > 0$ then $\int_1^3 \frac{\sin 2x}{x}$ can be expressed as -

(A) $F(6) - F(2)$ (B) $\frac{1}{2}(F(6) - F(2))$ (C) $\frac{1}{2}(F(3) - F(1))$ (D) $2(F(6) - F(2))$

Solution – $\int_1^3 \frac{\sin 2x}{2x} \cdot 2dx$ put $2x = t$

$= \int_2^6 \frac{\sin t}{t} dt = [F(t)]_2^6 = F(6) - F(2).$

Scanned with CamScanner

16

$$\int_0^{\pi/2} \frac{\cos\theta - \sin\theta}{(1+\cos\theta)(1+\sin\theta)} d\theta \text{ equals -}$$

(A) $\cos^{-1}\left(\frac{1}{\sqrt{2}}\right)$ (B) $\cos^{-1}(0)$ (C) $\cos^{-1}(1)$ (D) $\cos^{1}(-1)$

Solution -

$$I = \int_0^{\pi/2} \frac{\sin\theta - \cos\theta}{(1+\sin\theta)(1+\cos\theta)} d\theta \quad \text{(king prop.)}$$

$$= -I$$

$$\Rightarrow I = 0$$

Scanned with CamScanner

17 Let $f(x)$ be a continuous function on [0,4] satisfying $f(x)\,f(4-x)=1$.

The value of the definite integral $\int_0^4 \frac{1}{1+f(x)}dx$ equals-

(A) 0 (B) 1 (C) 2 (D) 4

Solution-

$$I = \int_0^4 \frac{dx}{1+f(4-x)} \quad \text{(using prop.)}$$

$$= \int_0^4 \frac{dx}{1+\frac{1}{f(x)}} = \int_0^4 \frac{f(x)\,dx}{1+f(x)}$$

$$\Rightarrow 2I = \int_0^4 1 \cdot dx = 4 \Rightarrow I = 2$$

(18)

If $g(x) = \int_1^x e^{t^2} dt$ then the value of $\int_3^{x^3} e^{t^2} dt$ equals

(A) $g(x^3) - g(3)$ (B) $g(x^3) + g(3)$ (C) $g(x^3) - 3$ (D) $g(x^3) - 3g(x)$

if $g(x) = \int_1^x e^{t^2} dt$ then the value of $\int_3^{x^3} e^{t^2} dt$ equals.

Solution :- $g(x^3) = \int_1^{x^3} e^{t^2} dt$

$= \int_1^3 e^{t^2} dt + \int_3^{x^3} e^{t^2} dt$

$= g(3) + \int_3^{x^3} e^{t^2} dt$

$\Rightarrow \quad g(x^3) - g(3) = \int_3^{x^3} e^{t^2} dt.$

EXERCISE (O-2)

EXERCISE (O-2)

[STRAIGHT OBJECTIVE TYPE]

1

15. The absolute value of $\frac{\int_0^{\pi/2}(x\cos x+1)e^{\sin x}dx}{\int_0^{\pi/2}(x\sin x-1)e^{\cos x}dx}$ is equal to -

(A) e (B) πe (C) e/2 (D) π/e

Sol^n

$$E = \frac{\int_0^{\pi/2} e^{\sin x} + x\left(e^{\sin x}\cdot \cos x\right)dx}{\int_0^{\pi/2} e^{\cos x} + x\left(e^{\cos x}(-\sin x)\right)dx}$$

$$\Rightarrow E = \frac{x\cdot e^{\sin x}\Big|_0^{\pi/2}}{x\cdot e^{\cos x}\Big|_0^{\pi/2}}$$

$$\Rightarrow E = \frac{\pi/2\cdot e - 0}{\pi/2\cdot e^0 - 0} = e \ (\text{Ans } A)$$

② Suppose f is continuous and satisfies $f(x)+f(-x)=x^2$ then the integral $\int_{-1}^{1} f(x)dx$ has the value equal to

(A) $\frac{2}{3}$ (B) $\frac{1}{3}$ (C) $\frac{4}{3}$ (D) zero

Soln

$I = \int_{-1}^{1} f(x)\,dx$

King

$I = \int_{-1}^{1} f(-x)\,dx$

Add

$\therefore\ 2I = \int_{-1}^{1} f(x)+f(-x)\,dx \Rightarrow 2I = \int_{-1}^{1} x^2\,dx$

$\Rightarrow 2I = \left.\frac{x^3}{3}\right|_{-1}^{1} \Rightarrow 2I = \frac{1}{3} - \left(-\frac{1}{3}\right)$

$\Rightarrow I = 1/3$ (Ans B)

3. If the value of the integral $\int_1^2 e^{x^2} dx$ is α, then the value of $\int_e^{e^4} \sqrt{\ell nx}\, dx$ is -

(A) $e^4 - e - \alpha$ (B) $2e^4 - e - \alpha$ (C) $2(e^4 - e) - \alpha$ (D) $2e^4 - 1 - \alpha$

Soln

We know that

$$\boxed{\int_a^b f(x)\,dx + \int_{f(a)}^{f(b)} f^{-1}(x)\,dx = bf(b) - af(a)}$$

If $f(x) = e^{x^2}$ then $f^{-1}(x) = \sqrt{\ln x}$

$$\therefore \underbrace{\int_1^2 e^{x^2} dx}_{\alpha} + \int_e^{e^4} \sqrt{\ln x}\, dx = 2e^4 - e$$

$$\therefore \int_e^{e^4} \sqrt{\ln x}\, dx = 2e^4 - e - \alpha$$ (Ans. B)

4

18. The value of $\lim_{n\to\infty} \sum_{r=1}^{r=4n} \frac{\sqrt{n}}{\sqrt{r}\left(3\sqrt{r}+4\sqrt{n}\right)^2}$ is equal to

(A) $\frac{1}{35}$ (B) $\frac{1}{14}$ ~~(C)~~ $\frac{1}{10}$ (D) $\frac{1}{5}$

Sol^n

$$l = \lim_{n\to\infty} \sum_{r=1}^{4n} \frac{\sqrt{n}}{\sqrt{r}\left(n\left(3\cdot\sqrt{\frac{r}{n}}+4\right)^2\right.}$$

$$L = \lim_{n\to\infty} \sum_{r=1}^{4n} \frac{1}{n\cdot\left(\sqrt{\frac{r}{n}}\left(3\cdot\sqrt{\frac{r}{n}}+4\right)^2\right)}$$

$$L = \int_0^4 \frac{dx}{\sqrt{x}\,(3\sqrt{x}+4)^2} \qquad 3\sqrt{x}+4 = t, \quad \frac{3}{2\sqrt{x}}\,dx = dt$$

$$\Rightarrow L = \frac{2}{3}\cdot\int_4^{10} \frac{dt}{t^2} \Rightarrow L = \frac{2}{3}\left(-\frac{1}{t}\right)\Bigg|_4^{10}$$

$$\Rightarrow L = \frac{2}{3}\left[\frac{1}{4}-\frac{1}{10}\right] \Rightarrow L = \frac{2}{3}\cdot\frac{6}{4\times 10}.$$

$$\Rightarrow L = \frac{1}{10} \quad (\text{Ans } C)$$

5 The value of $\int_{\pi}^{2\pi}[2\cos x]dx$ where [.] represents the greatest integer function, is -

(A) $-\frac{5\pi}{6}$ (B) $-\frac{\pi}{2}$ (C) $-\pi$ (D) none

Sol^n $I = \int_{\pi}^{2\pi} [2\cos x]\, dx$

King $x \to 3\pi - x$

$I = \int_{\pi}^{2\pi} [-2\cos x]\, dx$

$\therefore\ 2I = \int_{\pi}^{2\pi} [2\cos x] + [-2\cos x]\, dx$

$\Rightarrow 2I = \int_{\pi}^{2\pi} (-1)\, dx \Rightarrow I = -\pi/2$ Ans

$\left(\because\ [x] + [-x] = \begin{cases} 0 & x \in \text{Integer} \\ -1 & x \notin \text{Integer} \end{cases}\right.$

(6)

14. $\lim_{k\to 0}\frac{1}{k}\int_0^k (1+\sin 2x)^{\frac{1}{x}}\,dx$ -

(A) 2 (B) 1 (C) e^2 (D) non existent

Sol^n

$$\lim_{K\to 0}\frac{\int_0^K (1+\sin 2x)^{1/x}\,dx}{K}$$

$$= \lim_{K\to 0}\frac{(1+\sin 2K)^{1/K}}{1}$$

$\therefore$ Ans: e^L where $L = \lim_{K\to 0}\frac{1}{K}(1+\sin 2K - 1)$

$$\Rightarrow L = \lim_{K\to 0}\frac{\sin 2K}{K} = 2$$

$\therefore$ Ans: e^2 (Ans C)

Paragraph for Question Nos. 7 to 9

Let the function f satisfies

$$f(x)\cdot f'(-x) = f(-x)\cdot f'(x) \text{ for all x and } f(0) = 3.$$

7. The value of $f(x)\cdot f(-x)$ for all x, is

(A) 4 (B) 9 (C) 12 (D) 16

8. $\int_{-51}^{51} \frac{dx}{3+f(x)}$ has the value equal to

(A) 17 (B) 34 (C) 102 (D) 0

9. Number of roots of $f(x) = 0$ in $[-2, 2]$ is

(A) 0 (B) 1 (C) 2 (D) 4

Sol^n $f(x)\cdot f'(-x) = f(-x)\cdot f'(x)$

$$\Rightarrow \frac{f'(-x)}{f(-x)} = \frac{f'(x)}{f(x)}$$

$$\Rightarrow \int \frac{f'(-x)}{f(-x)}\,dx = \int \frac{f'(x)}{f(x)}\,dx$$

$$\Rightarrow -\ln f(-x) + C = f(x)$$

$$\Rightarrow \ln f(x)\cdot f(-x) = C.$$

$$\Rightarrow f(x)\cdot f(-x) = C_1$$

At $x = 0$: $C_1 = f(0)\cdot f(0) \Rightarrow C_1 = 9$

$$\therefore f(x)\cdot f(-x) = 9$$

7 $f(x)\cdot f(-x) = 9$ (Ans. B)

8 $I = \int_{-51}^{51} \frac{dx}{3+f(x)} \longrightarrow$ 1

King (

$$I = \int_{-51}^{51} \frac{dx}{3+f(-x)} \Rightarrow I = \int_{-51}^{51} \frac{dx}{3+\frac{9}{f(x)}}$$

$$\Rightarrow I = \int_{-51}^{51} \frac{f(x)}{3f(x)+9}\,dx$$

$$\Rightarrow I = \frac{1}{3}\int_{-51}^{51} \frac{f(x)}{f(x)+3}\,dx$$

$$\Rightarrow I = \frac{1}{3}\int_{-51}^{51} \frac{f(x)+3-3}{f(x)+3}\,dx$$

$$\Rightarrow I = \frac{1}{3}\left(\int_{-51}^{51} dx - 3\cdot\int_{-51}^{51} \frac{dx}{f(x)+3}\right)$$

$$\Rightarrow I = \frac{1}{3}(51-(-51)) - I$$

$$\Rightarrow 2I = \frac{102}{3} \Rightarrow I = 17 \text{ (Ans A)}$$

9 $\because f(x)\cdot f(-x) = 9 \;\forall x$

$\therefore f(x)$ can never be zero for any x as this product is non zero.

$\therefore$ No root for $f(x)=0$ (Ans A)

[MULTIPLE OBJECTIVE TYPE]

10 $\int_0^{\infty} \frac{x}{(1+x)(1+x^2)} dx =$

(A) $\frac{\pi}{4}$

(B) $\frac{\pi}{2}$

(C) is same as $\int_0^{\infty} \frac{dx}{(1+x)(1+x^2)}$

(D) cannot be evaluated

Soln $I = \int_0^{\infty} \frac{x}{(1+x)(1+x^2)} dx \longrightarrow (I)$

Put $x = 1/t \Rightarrow dx = -1/t^2 \, dt$

$\therefore I = \int_{\infty}^{0} \frac{(1/t)}{(1+1/t)(1+1/t^2)} \left(-\frac{1}{t^2}\right) dt$

$\Rightarrow I = -\int_{\infty}^{0} \frac{dt}{(1+t)(1+t^2)} \Rightarrow I = \int_0^{\infty} \frac{dt}{(1+t)(1+t^2)}$

$\Rightarrow I = \int_0^{\infty} \frac{dx}{(1+x)(1+x^2)} \longrightarrow (II)$

(I) + (II) : $2I = \int_0^{\infty} \frac{dx}{1+x^2} = \tan^{-1}x \Big|_0^{\infty}$

$\Rightarrow 2I = \frac{\pi}{2} - 0 \Rightarrow I = \pi/4$

$\therefore$ (Ans : A, C)

11

Let $u = \int_0^\infty \frac{dx}{x^4+7x^2+1}$ & $v = \int_0^\infty \frac{x^2dx}{x^4+7x^2+1}$ then -

(A) $v > u$ (B) $6v = \pi$ (C) $3u + 2v = 5\pi/6$ (D) $u + v = \pi/3$

Solⁿ $V = \int_0^\infty \frac{x^2\,dx}{x^4+7x^2+1}$

$x = 1/t \Rightarrow dx = -1/t^2\,dt$

$\Rightarrow V = \int_\infty^0 \frac{(1/t)^2(-1/t^2)\,dt}{(1/t)^4+7(1/t)^2+1}$

$\Rightarrow V = \int_0^\infty \frac{dt}{t^4+7t^2+1} = U \Rightarrow \boxed{V = U}$

$\therefore V + U = \int_0^\infty \frac{x^2+1}{x^4+7x^2+1}\,dx$

$\Rightarrow V + U = \int_0^\infty \frac{1+\frac{1}{x}}{x^2+\frac{1}{x^2}+7}\,dx$

$\Rightarrow V + U = \int_0^\infty \frac{\left(1+\frac{1}{x^2}\right)dx}{\left(x-\frac{1}{x}\right)^2+9}$

put $x - \frac{1}{x} = t \Rightarrow \left(1+\frac{1}{x^2}\right)dx = dt$

(11) Continues...

$$\Rightarrow \quad V + \mho = \int_{-\infty}^{\infty} \frac{dt}{t^2+9}$$

$$\Rightarrow \quad V + \mho = \frac{1}{3} \tan^{-1}\left(\frac{t}{3}\right) \Big|_{-\infty}^{\infty}$$

$$\Rightarrow \quad V + \mho = \frac{1}{3}\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) \Rightarrow V + \mho = \frac{\pi}{3}$$

$$\because \quad V = \mho \quad \Rightarrow \quad \therefore \boxed{V = \mho = \pi/6}$$

12. Which of the following statement(s) is/are **TRUE ?**

(A) $\int_0^1 \ell n x dx = -1$

(B) $\lim_{n\to\infty} \frac{1}{n} \ell n\left(\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\ldots\ldots\left(1+\frac{n}{n}\right)\right) = 1 + 2\ell n 2.$

(C) Let f be a continuous and non-negative function defined on [a.b].

If $\int_a^b f(x)dx = 0$ then $f(x) = 0 \;\forall\; x \in [a,b]$

(D) Let f be a continuous function defined on [a ,b] such that $\int_a^b f(x)dx = 0$, then there exists atleast one $c \in (a,b)$ for which $f(c) = 0$.

23. (A) $\int_0^1 \ln x\, dx = -1$

$(x\ln x - x)_0^1$

$[0 - 1 - 0]$

$= -1$

(B) $\lim_{n\to\infty} \frac{1}{n} \ln\left[(1+1/n)(1+2/n)(1+3/n) \text{---} (1+n/n)\right]$

$\lim_{n\to\infty} \frac{1}{n}\left[\ln(1+1/n) + \ln(1+2/n) + \ln(1+3/n) + \text{---} + \ln(1+n/n)\right]$

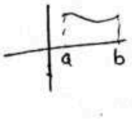
$\lim_{n\to\infty} 1/n \sum_{r=1}^{r=n} \ln[1+r/n]$

$\int_0^1 \ln(1+x) = \left[(1+x)(\ln(1+x) - 1)\right]_0^1$

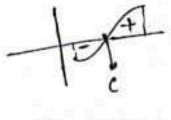
$2(\ln 2 - 1) - 1(0-1)$

$2\ln 2 - 1$

(C) If f(x) is a non-negative continuous f^n then if area ~~must be zero~~ of f(x) is zero than f(x) is also zero



(D)



for $f(c) = 0$

13 Let $f(x)=\begin{cases} x+1, & 0\le x\le 1 \\ 2x^2-6x+6, & 1<x\le 2 \end{cases}$ and $g(t)=\int_{t-1}^{t} f(x)dx$ for $t\in[1,2]$

Which of the following hold(s) good ?

(A) $f(x)$ is continuous and differentiable in [0,2]

(B) g'(t) vanishes for t = 3/2 and 2

(C) g(t) is maximum at t = 3/2

(D) g(t) is minimum at t = 1

24. $f(x)=\begin{cases} x+1 & 0\le x\le 1 \\ 2x^2-6x+6 & 1<x\le 2 \end{cases}$; $g(t)=\int_{t-1}^{t} f(x)dx$ for $t\in[1,2]$

$\Rightarrow$ $f(1^-) = 2 = f(1)$

$f(1^+) = 2-6+6 = 2$

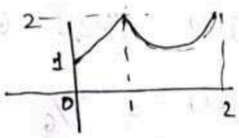
R.H.L = L.H.L = f(1)

So continuous

$\Rightarrow$ $f'(x)=\begin{cases} 1 & 0\le x<1 \\ 4x-6 & 1<x\le 2 \end{cases}$

$\left.\begin{array}{l} f'(1^+) = -2 \\ f'(1^-) = 1 \end{array}\right\} \rightarrow$ So non differentiable

$\Rightarrow$ $g(t)=\int_{t-1}^{t} f(x)dx$



$g(t) = \int_{(t-1)}^{1} f(x)dx + \int_{1}^{t} f(x)dx$

$g'(t) = f(t) - f(t-1) = 0$

$g'(3/2) = f(3/2) - f(1/2)$

$= 2(3/2)^2 - 6(3/2)+6-(1/2+1)$

$= 9/2 - 9+6-3/2$

$= 0$

$f(t) = f(t-1)$

$2t^2-6t+6 = (t-1)+1$

$2t^2-6t+6 = t$

$2t^2-7t+6=0$

$t = 2, 3/2$

13 Continues...

$g'(2) = f(2) - f(1)$

$= 2 - 2 = 0$

$g'(t)$ Vanishes for $t = 3/2 , 2$

$g''(t) = f'(t) - f'(t-1)$

$g''(3/2) = (4(3/2) - 6) - (1)$

$= 6 - 6 - 1 = -1$

$g(t)$ is max^m at $3/2$

$\Rightarrow$ $g(t) = \int_{t-1}^{t} f(x)\,dx$

$g(1) = \int_0^1 f(x) = \int_0^1 (x+1)\,dx$

$= \left(\frac{x^2}{2} + x\right)_0^1 = 1/2 + 1 = 3/2$

$g(2) = \int_1^2 (2x^2 - 6x + 6)\,dx$

$g(2) = \left[\frac{2x^3}{3} - \frac{6x^2}{2} + 6x\right]_1^2$

$g(2) = 10/6$

So min^m at $t = 1$

14. Which of the following is/are true ?

(A) $\int_{a}^{\pi-a} x f(\sin x)dx = \frac{\pi}{2}\int_{a}^{\pi-a} f(\sin x)dx$

(B) $\int_{0}^{n\pi} f(\cos^2 x)dx = n\int_{0}^{\pi} f(\cos^2 x)dx$

(C) $\int_{-a}^{a} f(x^2)dx = 2\int_{0}^{a} f(x^2)dx$

(D) $\int_{0}^{b-c} f(x+c)dx = \int_{c}^{b} f(x)dx$

Solution:

(A) $$\int_{a}^{\pi-a} x f(\sin x)\, dx = I$$

$\boxed{\text{By King,}}$ $$I = \int_{a}^{\pi-a} (\pi - x) f(\sin(\pi - x))\, dx$$

$$= \int_{a}^{\pi-a} (\pi - x) f(\sin x)\, dx$$

add, $$I + I = \int_{a}^{\pi-a} \pi f(\sin x)\, dx = 2I$$

$\boxed{\text{(A) is Correct}}$

(B) Period of $\cos^2 x = \pi$

$$\therefore \int_{0}^{n\pi} f(\cos^2 x)\, dx = n\int_{0}^{\pi} f(\cos^2 x)\, dx$$

$\boxed{\text{By Jack}}$

$\boxed{\text{(B) is Correct}}$

(14) Continues....

(C) $I = \int_{-a}^{a} f(x^2)\,dx = 2\int_{0}^{a} f(x^2)\,dx$

as $f(x^2)$ is an even function

$\boxed{\text{(C) is Correct}}$

(D) $\int_{0}^{b-c} f(x+c)\,dx = \int_{c}^{b} f(t)\,dt = \int_{c}^{b} f(x)\,dx$

Let $x+c=t$ $\left\{\begin{array}{l}\text{when } x=0;\ t=c \\ \text{when } x=b-c;\ t=b\end{array}\right\}$

& $dx=dt$

$\boxed{\text{(D) is Correct}}$

$\boxed{\text{Answer: A, B, C, D}}$

15. Which of the following definite integral reduces to $\frac{\pi}{2}$?

(A) $\int_0^{\pi} \frac{dx}{1+(\sin x)^{\cos x}}$

(B) $\int_0^{\pi/2} \frac{dx}{1+(\tan x)^5}$

(C) $\int_0^{\infty} \frac{x^2+1}{x^4-x^2+1} dx$

(D) $\int_0^{\pi/2} (\ell n(\sec x))\left(e^{\ell n(\ell n 2)}\right)^{-1} dx$

Solution: (A) $I = \int_0^{\pi} \frac{dx}{1+(\sin x)^{\cos x}}$

By king, $I = \int_0^{\pi} \frac{dx}{1+(\sin(\pi-x))^{\cos(\pi-x)}} = \int_0^{\pi} \frac{dx}{1+(\sin x)^{-\cos x}}$

$= \int_0^{\pi} \frac{(\sin x)^{\cos x}}{(\sin x)^{\cos x}+1} dx$

$\therefore I+I = \int_0^{\pi} \left(\frac{1}{1+(\sin x)^{\cos x}} + \frac{(\sin x)^{\cos x}}{(\sin x)^{\cos x}+1}\right) dx$

$\Rightarrow 2I = \int_0^{\pi} 1\,dx = \pi \Rightarrow \boxed{I = \frac{\pi}{2}}$ A ✓

(B) $I = \int_0^{\pi/2} \frac{dx}{1+(\tan x)^5} = \int_0^{\pi/2} \frac{(\cos x)^5}{(\cos x)^5+(\sin x)^5} dx$

By king, $I = \int_0^{\pi/2} \frac{(\sin x)^5}{(\sin x)^5+(\cos x)^5} dx$

$I+I = \int_0^{\pi/2} \left(\frac{(\cos x)^5}{(\cos x)^5+(\sin x)^5} + \frac{(\sin x)^5}{(\sin x)^5+(\cos x)^5}\right) dx = \int_0^{\pi/2} 1 dx$

$\Rightarrow 2I = \frac{\pi}{2} \Rightarrow I = \frac{\pi}{4}$ B ✗

(C) $I = \int_0^\infty \frac{x^2+1}{x^4-x^2+1}\,dx = \int_0^\infty \frac{1+\frac{1}{x^2}}{x^2-1+\frac{1}{x^2}}\,dx$

$= \int_{-\infty}^{\infty} \frac{dt}{t^2+2-1} = \int_{-\infty}^{\infty} \frac{dt}{t^2+1} = \tan^{-1} t \Big|_{-\infty}^{\infty}$

$= \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \boxed{\pi}$ $\boxed{C}$ (X)

Let $x - \frac{1}{x} = t \Rightarrow \left(1+\frac{1}{x^2}\right)dx = dt$

$\Rightarrow x^2 + \frac{1}{x^2} = t^2 + 2$

(D) $I = \int_0^{\pi/2} \left(\ln(\sec x)\right)\left(e^{\ln(\ln 2)}\right)^{-1} dx$

$= \int_0^{\pi/2} \ln\left((\cos x)^{-1}\right) (\ln 2)^{-1}\, dx$

$= \frac{1}{\ln 2} \int_0^{\pi/2} \left(-\ln(\cos x)\right) dx$

$= \frac{1}{\ln 2} (-1) \left(-\frac{\pi}{2}\ln 2\right) = \boxed{\frac{\pi}{2}}$ (D) ✓

$\boxed{\text{Answer: (A) \& (D)}}$

16. Let $A = \int_1^e \log^2(x)\,dx$, then -

(A) $A > e-1$ (B) $A < e-1$ (C) $A > \frac{e-1}{2}$ (D) $A < \log^2 2 + (e-2)$

Solution:

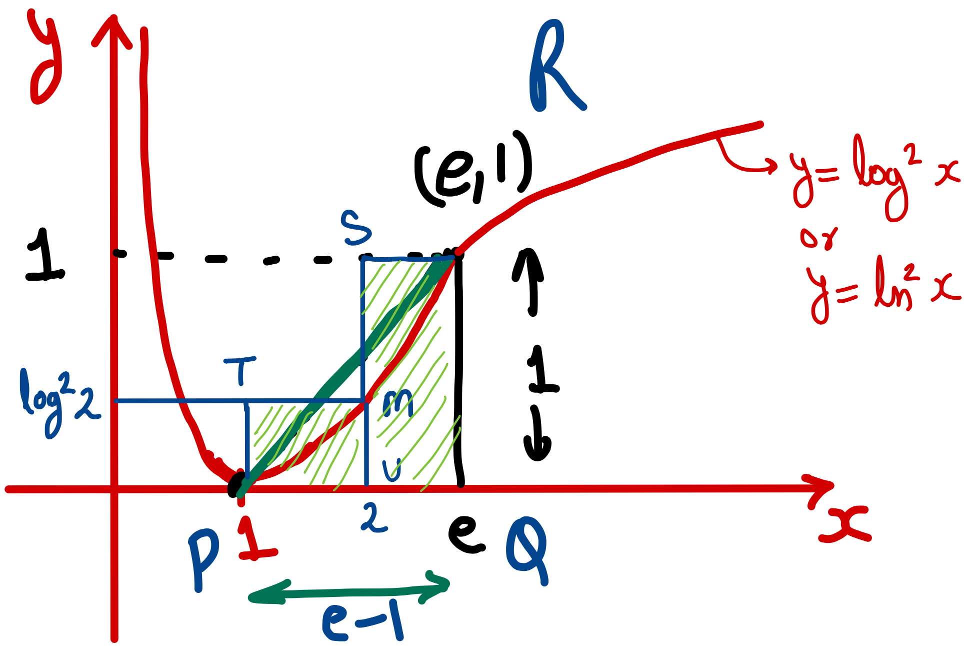
$y = f(x) = \log^2 x$

$f'(x) = \frac{2\log x}{x} > 0;\ x \in (1,e)$

$f(1) = 0;\quad f(e) = 1$

$f''(x) = 2\left[\frac{x\cdot\frac{1}{x} - \log x\cdot 1}{x^2}\right] = \frac{2[1-\log x]}{x^2}$ → ⊕ $(1,e)$, ⊖ (e,∞)

Minima at $x=1$ & point of inflection at $x=e$



16 Continues...

As, $\log^2 x < 1$ for $x \in (1, e)$

$\int_1^e \log^2 x\,dx < \int_1^e 1\,dx \Rightarrow \boxed{A < e-1} \longrightarrow$ $\boxed{\text{(B) is correct}}$

(A) is incorrect

also ; $A <$ ar$(\Delta PQR) \Rightarrow A < \frac{1}{2} \times (e-1) \times 1$

$\Rightarrow A < \frac{e-1}{2}$ (C) is incorrect

Also, $A <$ ar $(PUMT) +$ ar $(UQRS)$

$\Rightarrow \boxed{A < (\log^2 2)(2-1) + 1(e-2)} \longrightarrow$ $\boxed{\text{D is Correct}}$

$\boxed{\text{Answer : B, D}}$

17. The value of $\int_{0}^{\pi}\left(\sqrt[2015]{\cos x}+\sqrt[2015]{\sin x}+\sqrt[2015]{\tan x}\right)dx$ is equal to-

(A) 0

(B) $\int_{1/2}^{2}\frac{\ell nx}{1+x^{2}}dx$

(C) $2\int_{0}^{\pi/2}(\sin x)^{\frac{1}{2015}}dx$

(D) $2\int_{0}^{\pi/2}(\cos x)^{\frac{1}{2015}}dx$

Solution: Let $I=\int_{0}^{\pi}\left(\sqrt[2015]{\cos x}+\sqrt[2015]{\sin x}+\sqrt[2015]{\tan x}\right)dx$

By King,

$$I=\int_{0}^{\pi}\left(\sqrt[2015]{-\cos x}+\sqrt[2015]{\sin x}+\sqrt[2015]{-\tan x}\right)dx$$

$$\Rightarrow I=\int_{0}^{\pi}\left(-\sqrt[2015]{\cos x}+\sqrt[2015]{\sin x}-\sqrt[2015]{\tan x}\right)dx$$

$$\therefore I+I=\int_{0}^{\pi}2\sqrt[2015]{\sin x}\,dx$$

$$\Rightarrow 2I=2\int_{0}^{\pi}(\sin x)^{\frac{1}{2015}}dx \Rightarrow \boxed{I=\int_{0}^{\pi}(\sin x)^{\frac{1}{2015}}dx}$$

By Queen,

$$I=2\int_{0}^{\pi/2}(\sin x)^{\frac{1}{2015}}dx=\underbrace{2\int_{0}^{\pi/2}(\cos x)^{\frac{1}{2015}}dx}_{\text{(By King)}}$$

$$\boxed{\text{Answer: C, D}}$$

18. $650\int_0^2 x(2-x)^{24}\,dx$ is divisible by -

(A) 2^{25} (B) 2^{26} (C) 2^{27} (D) 2^{28}

Solution:

$$I = 650\int_0^2 x(2-x)^{24}\,dx = 650 I_1$$

$0 < x < 2 \Rightarrow$ Let $x = 2\sin^2\theta$; $\theta \in (0, \frac{\pi}{2})$

$$\text{Let } I_1 = \int_0^{\pi/2} (2\sin^2\theta)(2-2\sin^2\theta)^{24} \cdot 4\cdot\sin\theta\cos\theta\,d\theta$$

$$= \int_0^{\pi/2} 2^{27}\sin^3\theta\cdot\cos^{49}\theta\,d\theta$$

$$= \int_0^{\pi/2} 2^{27}\sin^2\theta\cdot\cos^{49}\theta\sin\theta\,d\theta$$

$$= \int_0^{\pi/2} 2^{27}(1-\cos^2\theta)\cos^{49}\theta\sin\theta\,d\theta$$

$$= 2^{27}\int_1^0 (1-t^2)\,t^{49}(-dt)$$

$$= 2^{27}\int_0^1 (t^{49}-t^{51})\,dt$$

$$= 2^{27}\left(\frac{1}{50}-\frac{1}{52}\right) = 2^{26}\left(\frac{1}{25}-\frac{1}{26}\right)$$

$$I_1 = 2^{26}\times\frac{1}{25\times 26} = 2^{25}\times\frac{1}{5\times 65}$$

$$\text{So, } I = 650\times 2^{25}\times\frac{1}{5\times 65} = \boxed{2^{26}} \rightarrow (B)$$

2^{26} is divisible by 2^{25} & 2^{26} $\longrightarrow$ $\boxed{A, B}$ Answers

19. Let $f(x) = 2014\tan^{2015}x + 2014\tan^{2013}x - 2010\tan^{2011}x - 2010\tan^{2009}x$ for all $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then the correct expression(s) is(are)

(A) $\int_0^{\pi/4} x f(x)dx = \frac{1}{2011.2013}$

(B) $\int_0^{\pi/4} f(x)dx = 0$

(C) $\int_0^{\pi/4} x f(x)dx = \frac{2}{2011.2013}$

(D) $\int_0^{\pi/4} f(x)dx = 1$

Solution: Observe,

$$f(x) = 2014\tan^{2013}x\,(\tan^2 x + 1) - 2010\tan^{2009}x\,(1 + \tan^2 x)$$

$$f(x) = \left(2014\tan^{2013}x - 2010\tan^{2009}x\right)\sec^2 x$$

$$\int_0^{\pi/4} f(x)\,dx = \int_0^1 \left(2014\,t^{2013} - 2010\,t^{2009}\right)dt$$

$$= t^{2014} - t^{2010}\Big|_0^1 = 0 \longrightarrow B$$

$$I = \int_0^{\pi/4} x f(x)\,dx = \left\{x \int f(x)dx - \int\left(1 \int f(x)\,dx\right)dx\right\}_0^{\pi/4}$$

$$= \left[x\left((\tan x)^{2014} - (\tan x)^{2010}\right) - \int\left((\tan x)^{2014} - (\tan x)^{2010}\right)dx\right]_0^{\pi/4}$$

$$= \int_0^{\pi/4}\left((\tan x)^{2010} - (\tan x)^{2014}\right)dx = \int_0^{\pi/4}(\tan x)^{2010}(1 - \tan^4 x)\,dx$$

$$I = \int_0^{\pi/4}(\tan x)^{2010}(1 - \tan^2 x)\sec^2 x\,dx = \int_0^1\left(t^{2010} - t^{2012}\right)dt$$

$$= \frac{1}{2011} - \frac{1}{2013} = \boxed{\frac{2}{2011\cdot 2013}} \longrightarrow C$$

Answer: B, C

EXERCISE (S-1)

EXERCISE (S-1)

1

PAGE NO. :
DATE : / /

Evaluate : (i) $\int_0^1 e^{\ln \tan^{-1}x} \cdot \sin^{-1}(\cos x) \cdot dx$

(ii) $\int_{1/3}^{3} \frac{\sin^{-1}\frac{x}{\sqrt{1+x^2}}}{x} dx$

Solⁿ (i) $\int_0^1 \tan^{-1}x \; \sin^{-1}(\cos x)\, dx$

$\int_0^1 \tan^{-1}x \left[\frac{\pi}{2} - \cos^{-1}\cos x\right] dx$

$\int_0^1 \tan^{-1}x \left(\frac{\pi}{2} - x\right) dx$ $\left\{\because \cos^{-1}\cos x = x \quad 0 < x < 1\right\}$

$\frac{\pi}{2}\int_0^1 \underset{II}{1}\cdot \underset{I}{\tan^{-1}x}\, dx - \int_0^1 \underset{II}{x}\, \underset{I}{\tan^{-1}x}\, dx$

$\frac{\pi}{2}\left[\left(x\tan^{-1}x\right)_0^1 - \int_0^1 \frac{1}{1+x^2}\times x\, dx\right] - \left[\left(\frac{x^2}{2}\tan^{-1}x\right)_0^1 - \int_0^1 \frac{x^2}{2(1+x^2)}dx\right]$

$\frac{\pi}{2}\left[\frac{\pi}{4} - \frac{1}{2}\ln(1+x^2)\Big|_0^1\right] - \left[\frac{\pi}{8} - \frac{1}{2}\left(\int_0^1 1 - \frac{1}{1+x^2}dx\right)\right]$

$\frac{\pi}{2}\left[\frac{\pi}{4} - \frac{1}{2}\ln 2\right] - \left[\frac{\pi}{8} - \frac{1}{2}\left[x - \tan^{-1}x\right]_0^1\right.$

$\frac{\pi^2}{8} - \frac{\pi\ln 2}{4} - \frac{\pi}{8} + \frac{1}{2}\left(1 - \frac{\pi}{4}\right)$

$\frac{\pi^2}{8} - \frac{\pi\ln 2}{4} - \frac{\pi}{8} + \frac{1}{2} - \frac{\pi}{8}$

$\frac{\pi^2}{8} - \frac{\pi}{4}(1+\ln 2) + \frac{1}{2}$

① Continues.—

PAGE NO.:
DATE: / /

put $x = \frac{1}{t}$ $\Rightarrow$ $dx = -\frac{1}{t^2}\,dt$

$$I = \int_{3}^{1/3} \sin^{-1}\frac{1}{\sqrt{1+t^2}}\, t\cdot\left(-\frac{1}{t^2}\right)dt$$

$$I = \int_{1/3}^{3} \frac{\sin^{-1}\frac{1}{\sqrt{1+t^2}}}{t}\,dt$$

$t = \tan\theta$
$dt = \sec^2\theta\, d\theta$

$$I = \int_{\tan^{-1}1/3}^{\tan^{-1}3} \frac{\sin^{-1}\cos\theta}{\tan\theta}\sec^2\theta\, d\theta$$

$$I = \int_{\cot^{-1}3}^{\tan^{-1}3} \frac{\left(\frac{\pi}{2} - \cos^{-1}\cos\theta\right)\frac{1}{\cos^2\theta}}{\frac{\sin\theta}{\cos\theta}}\,d\theta$$

$$I = \int_{\cot^{-1}3}^{\tan^{-1}3} \left(\frac{\pi}{2} - \theta\right)\frac{1}{\sin\theta\cos\theta}\,d\theta.$$

$$I \stackrel{K}{=} \int_{\cot^{-1}3}^{\tan^{-1}3} \left[\frac{\pi}{2} - \left(\frac{\pi}{2} - \theta\right)\right]\cdot\frac{1}{\sin\left(\frac{\pi}{2}-\theta\right)\cos\left(\frac{\pi}{2}-\theta\right)}\,d\theta$$

$$I = \int_{\cot^{-1}3}^{\tan^{-1}3} \theta\,\frac{1}{\sin\theta\cos\theta}\,d\theta$$

$$2I = \int_{\cot^{-1}3}^{\tan^{-1}3} \frac{\pi}{2\sin\theta\cos\theta}\,d\theta$$

$$2I = \int_{\cot^{-1}3}^{\tan^{-1}3} \frac{\pi\sec^2\theta}{2\tan\theta}\,d\theta \qquad \tan\theta = t$$

$$2I = \int_{1/3}^{3} \frac{\pi\,dt}{2t} \Rightarrow \frac{\pi}{2}\left(\ln t\right)_{1/3}^{3}$$

$$I = \frac{\pi}{4}\left(\ln 3 - \ln\frac{1}{3}\right)$$

$$\frac{\pi}{4}\,2\ln 3 \Rightarrow \boxed{\frac{\pi\ln 3}{2}}$$

PAGE NO. :
DATE : / /

② Evaluate : $\int_0^1 \frac{\sin^{-1}\sqrt{x}}{\sqrt{x(1-x)}}\,dx$

Sol^n Put $x = t^2$ $\quad dx = 2t\,dt$

$$I = \int_0^1 \frac{\sin^{-1}t \cdot 2t\,dt}{\sqrt{t^2(1-t^2)}}$$

$$I = \int_0^1 \frac{\sin^{-1}t \; 2\cancel{t}}{\cancel{t}\sqrt{1-t^2}}\,dt$$

$$\frac{I}{2} = \int_0^1 \underset{I}{\sin^{-1}t} \; \underset{II}{\frac{1}{\sqrt{1-t^2}}}\,dt$$

$$\frac{I}{2} = \left[\sin^{-1}t \;\; \sin^{-1}t\right]_0^1 - \int_0^1 \frac{1}{\sqrt{1-t^2}} \sin^{-1}t \; dt$$

$$\frac{I}{2} = \left(\sin^{-1}t\right)^2 - \frac{I}{2}$$

$$\therefore \boxed{I = \frac{\pi^2}{4}}$$

Scanned with CamScanner

3

Evaluate :- $\displaystyle\int_0^{\pi/2} \frac{1+2\cos x}{(2+\cos x)^2}\, dx$

Sol^n $\displaystyle\int_0^{\pi/2} \frac{1+2\cos x}{(2+\cos x)^2} \times \frac{\operatorname{cosec}^2 x}{\operatorname{cosec}^2 x}\, dx$

$\displaystyle\int_0^{\pi/2} \frac{\operatorname{cosec}^2 x + 2\cot x \operatorname{cosec} x}{(2\operatorname{cosec} x + \cot x)^2}\, dx$

put $\cot x + 2\operatorname{cosec} x = t$

$-\operatorname{cosec}^2 x - 2\cot x \operatorname{cosec} x\, dx = dt$

$\therefore \displaystyle\int_\infty^2 (-)\frac{1}{t^2}\, dt$

$(-)\left[-\frac{1}{t}\right]_\infty^2$

$-\left[\frac{-1}{2} - \left(\frac{-1}{\infty}\right)\right]$

$\boxed{\frac{1}{2}}$

PAGE NO.:
DATE: / /

4

Evaluate: $\int_0^{\pi/2} e^x \left\{ \cos(\sin x)\cos^2\frac{x}{2} + \sin(\sin x)\sin^2\frac{x}{2} \right\} dx$

Solⁿ $I = \int_0^{\pi/2} e^x \left\{ \cos(\sin x)\left(\frac{1+\cos x}{2}\right) + \sin(\sin x)\left(\frac{1-\cos x}{2}\right) \right\} dx$

$\int_0^{\pi/2} \frac{e^x}{2} \left[\cos\sin x + \sin\sin x + (\cos\sin x - \sin\sin x)\cos x \right.$

let $f(x) = \cos\sin x + \sin\sin x$

$f'(x) = -\sin\sin x \cdot \cos x + \cos\sin x \cdot \cos x$

$\frac{1}{2}\int_0^{\pi/2} e^x \left[f(x) + f'(x) \right] dx$

$\frac{1}{2}\left[e^x (\cos\sin x + \sin\sin x)_0^{\pi/2} \right]$

$\frac{1}{2}\left[\left(e^{\pi/2} \cos\sin\frac{\pi}{2} + \sin\sin\frac{\pi}{2}\right) - \left(e^0 \cos\sin 0 + \sin\sin 0\right) \right]$

$\frac{1}{2}\left[e^{\pi/2} (\cos 1 + \sin 1) - (\cos 0 + \sin 0) \right]$

$\boxed{\frac{1}{2}\left[e^{\pi/2} (\cos 1 + \sin 1) - 1 \right]}$

Scanned with CamScanner

⑤ Evaluate : $\int_1^e \left[\{1+x\}e^x + \{1-x\}e^{-x}\right] \underset{I}{\ln x}\, dx$

Sol^n $\left[\ln x\left(xe^x + xe^{-x}\right)\right]_1^e - \int_1^e \underset{II}{\frac{1}{x}}\left[xe^x + xe^{-x}\right] dx$

$\ln e\left(e\cdot e^e + e\cdot e^{-e}\right) - \int_1^e \left(e^x + e^{-x}\right) dx$

$e\cdot e^e + e\cdot e^{-e} - \left(e^x - e^{-x}\right)_1^e$

$e^{1+e} + e^{1-e} - e^e + e^{-e} + e - e^{-1}$

6

PAGE NO.:

DATE: / /

If $P = \int_0^\infty \frac{x^2}{1+x^4} dx$ $\phi = \int_0^\infty \frac{x\,dx}{1+x^4}$

$R = \int_0^\infty \frac{dx}{1+x^4}$ then prove that

(A) $\phi = \frac{\pi}{4}$ (B) $P = R$ (C) $P - \sqrt{2}\phi + R = \frac{\pi}{2\sqrt{2}}$

Solⁿ. Replace $x = \frac{1}{t} \Rightarrow dx = \frac{-1}{t^2} dt$ in P

$P = \int_\infty^0 \frac{1/t^2}{1+\frac{1}{t^4}} \cdot -1/t^2\, dt$

$P = \int_\infty^0 \frac{-1}{t^4+1} dt \Rightarrow \int_0^\infty \frac{1}{1+x^4} dx = R$

$\therefore \boxed{P = R}$

Put $x^2 = t \Rightarrow 2x\,dx = dt$ in ϕ

$\phi = \int_0^\infty \frac{1}{2(1+t^2)} dt \Rightarrow \frac{1}{2}\left(\tan^{-1}t\right)_0^\infty$

$\frac{1}{2}\left(\tan^{-1}\infty - 0\right)$.

$\therefore \boxed{\phi = \frac{\pi}{4}}$

Now $P+R = \int_0^\infty \frac{x^2}{1+x^4} dx + \int_0^\infty \frac{dx}{1+x^4}$

$\Rightarrow \int_0^\infty \frac{x^2+1}{x^4+1} dx$

$\Rightarrow \int_0^\infty \frac{1+1/x^2}{x^2+1/x^2} dx \Rightarrow \int_0^\infty \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+2} dx$

put $x - \frac{1}{x} = t \Rightarrow 1 + \frac{1}{x^2}\, dx = dt$

$\therefore P+R = \int_{-\infty}^{\infty} \frac{dt}{t^2+2} \Rightarrow \frac{1}{\sqrt{2}}\left(\tan^{-1}\frac{t}{\sqrt{2}}\right)_{-\infty}^{\infty}$

6 Continues...

DATE : / /

$$\frac{1}{\sqrt{2}}\left[\tan^{-1}\infty - \tan^{-1}(-\infty)\right]$$

$$\frac{1}{\sqrt{2}}\left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] \Rightarrow \frac{\pi}{\sqrt{2}}$$

Now $P + R - \sqrt{2}Q = \frac{\pi}{\sqrt{2}} - \sqrt{2}\left(\frac{\pi}{4}\right)$

$$\frac{\pi}{\sqrt{2}} - \frac{\pi}{2\sqrt{2}} \Rightarrow \boxed{\frac{\pi}{2\sqrt{2}}}$$

(7) $\displaystyle\int_1^2 \frac{(x^2-1)\,dx}{x^3\cdot\sqrt{2x^4-2x^2+1}} = \frac{U}{V}$ Where U & V are in their lowest form. find $(1000)\frac{U}{V}$

Sol^n Put $x = 1/t$ $\quad dx = \frac{-1}{t^2}\,dt$

$$\int_1^{1/2} \frac{\left(\frac{1}{t^2}-1\right)\left(\frac{-1}{t^2}\right)dt}{\left(\frac{1}{t^3}\right)\sqrt{\frac{2}{t^4}-\frac{2}{t^2}+1}} \Rightarrow \int_1^{1/2} \frac{\frac{(1-t^2)(-1)}{t^4}}{\frac{1}{t^3}\frac{\sqrt{t^4-2t^2+2}}{t^2}}\,dt$$

$$\int_1^{1/2} \frac{(t^2-1)\,t}{\sqrt{t^4-2t^2+2}}\,dt$$

Put $t^4-2t^2+2 = x^2$

$4t^3-4t\,dt = 2x\,dx$

$(t^2-1)\,t = \frac{1}{2}x\,dx$

$$\therefore \int_1^{5/4} \frac{\cancel{x}\,dx}{2\cancel{x}}$$

$\frac{1}{2}\left[\frac{5}{4}-1\right]$

$\frac{1}{2}\times\frac{1}{4} = \frac{1}{8}$

$\therefore \frac{U}{V} = \frac{1}{8}$

$\therefore \frac{1000\,U}{V} = \frac{1000}{8} = \boxed{125}$

8

PAGE NO. :

DATE : / /

Evaluate : $\int_0^{\pi/2} \sqrt{\frac{1-\sin 2x}{1+\sin 2x}}\, dx$

$$I = 2\int_0^{\pi/4} \sqrt{\frac{1-\sin 2x}{1+\sin 2x}}\, dx$$

$$= 2\int_0^{\pi/4} \sqrt{\frac{(\cos x - \sin x)^2}{(\cos x + \sin x)^2}}\, dx$$

$$= 2\int_0^{\pi/4} \frac{|\cos x - \sin x|}{|\cos x + \sin x|}\, dx$$

$$= 2\int_0^{\pi/4} \frac{\cos x - \sin x}{\cos x + \sin x}\, dx$$

$$= 2 \ln(\sin x + \cos x)\Big|_0^{\pi/4}$$

$$= 2 \ln \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}\right) - 1$$

$$= 2 \ln(\sqrt{2})$$

$$\boxed{\ln 2}$$

Scanned with CamScanner

9

PAGE NO.:
DATE: / /

If a_1, a_2, a_3 are the three values of a which satisfy the eq.ⁿ $\int_0^{\pi/2} (\sin x + a\cos x)^3 dx - \frac{4a}{\pi-2}\int_0^{\pi/2} x\cos x\, dx = 2$ then find the value of $1000(a_1^2 + a_2^2 + a_3^2)$.

Solⁿ: $\int_0^{\pi/2} \sin^3 x + a^3\cos^3 x + 3a\sin^2 x\cos x + 3a^2 \sin x\cos^2 x\, dx - \frac{4a}{\pi-2}\int_0^{\pi/2} \underset{I}{x}\cdot \underset{II}{\cos x}\, dx = 2$

$\because \sin 3x = 3\sin x - 4\sin^3 x$ & $\cos 3x = 4\cos^3 x - 3\cos x$

$$\int_0^{\pi/2} \frac{3\sin x - \sin 3x}{4} + a^3\frac{(\cos 3x + 3\cos x)}{4} + 3a\sin^2 x\cos x + 3a^2\sin x\cos^2 x\, dx - \frac{4a}{\pi-2}\left[(x\sin x)_0^{\pi/2} - \int_0^{\pi/2}\sin x\, dx\right] = 2$$

$$\left[-\frac{3\cos x}{4} + \frac{\cos 3x}{12} + \frac{a^3\sin 3x}{12} + \frac{3a^3\sin x}{4} + a\sin^3 x - a^2\cos^3 x\right]_0^{\pi/2} - \frac{4a}{\pi-2}\left[\frac{\pi}{2} + (\cos x)_0^{\pi/2}\right] = 2$$

$$\left[0 + 0 - \frac{a^3}{12} + \frac{3a^3}{4} + a\right] - \left[\frac{-3}{4} + \frac{1}{12} - a^2\right] - \frac{4a}{\pi-2}\left(\frac{\pi}{2} - 1\right) = 2$$

$$\frac{8a^3}{12} + a + \frac{8}{12} + a^2 - 2a = 2$$

$$\frac{2a^3}{3} + a^2 - a + \frac{2}{3} - 2 = 0$$

$$2a^3 + 3a^2 - 3a - 4 = 0$$

$a_1 + a_2 + a_3 = -3/2$ & $a_1a_2 + a_2a_3 + a_3a_1 = -3/2$

$a_1^2 + a_2^2 + a_3^2 = (-3/2)^2 - 2(-3/2) = 21/4$

$\therefore 1000 \times \frac{21}{4} \Rightarrow 250 \times 21 \Rightarrow \boxed{5250}$

(10)

PAGE NO.:
DATE: / /

Evaluate: $\displaystyle\int_{-\sqrt{2}}^{\sqrt{2}} \frac{2x^7+3x^6-10x^5-7x^3-12x^2+x+1}{x^2+2}\,dx$

Solⁿ

$$I \overset{K}{=} \int_{-\sqrt{2}}^{\sqrt{2}} \frac{-2x^7+3x^6+10x^5+7x^3-12x^2-x+1}{x^2+2}\,dx$$

$$2I = \int_{-\sqrt{2}}^{\sqrt{2}} \frac{2\left[3x^6-12x^2+1\right]}{x^2+2}\,dx$$

$$2I \overset{P}{=} 2\cdot 2\int_0^{\sqrt{2}} \frac{3x^6-12x^2}{x^2+2} + \frac{1}{x^2+2}\,dx$$

$$I = 2\int_0^{\sqrt{2}} \frac{3x^2\cdot(x^4-4)}{x^2+2} + \frac{1}{x^2+2}\,dx$$

$$I = 2\int_0^{\sqrt{2}} 3x^2(x^2-2) + \frac{1}{x^2+2}\,dx$$

$$= 2\int_0^{\sqrt{2}} 3x^4-6x^2+\frac{1}{x^2+2}\,dx$$

$$= 2\left[\frac{3x^5}{5} - 2x^3 + \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x}{\sqrt{2}}\right)\right]_0^{\sqrt{2}}$$

$$= 2\left(\frac{3\cdot 4\sqrt{2}}{5} - 2\cdot 2\sqrt{2} - \frac{1}{\sqrt{2}}\tan^{-1}(1)\right)$$

$$2\left[\frac{12\sqrt{2}}{5} - 4\sqrt{2} + \frac{\pi}{4\sqrt{2}}\right]$$

$$\frac{24\sqrt{2}}{5} - 8\sqrt{2} + \frac{\pi}{2\sqrt{2}}$$

$$\boxed{\frac{\pi}{2\sqrt{2}} - \frac{16\sqrt{2}}{5}}$$

(11)

PAGE NO. :
DATE : / /

Evaluate : $\int_{-2}^{2} \frac{x^2 - x}{\sqrt{x^2+4}} dx$

Sol^n $I = \int_{-2}^{2} \frac{x^2 + x}{\sqrt{x^2+4}} dx$

$2I = \int_{-2}^{2} \frac{2x^2}{\sqrt{x^2+4}} dx$

$I = \int_{-2}^{2} \frac{x^2}{\sqrt{x^2+4}} dx \Rightarrow 2\int_{0}^{2} \frac{x^2}{\sqrt{x^2+4}} dx$

$I = 2\int_{0}^{2} \frac{x^2+4-4}{\sqrt{x^2+4}} dx$

$I = 2\int_{0}^{2} \sqrt{x^2+4} - \frac{4}{\sqrt{x^2+4}} dx$

$= 2\left[\frac{x\sqrt{x^2+4}}{2} + \frac{4}{2}\ln(x+\sqrt{x^2+4}) - 4\ln(x+\sqrt{x^2+4})\right]_0^2$

$= 2\left[\frac{x\sqrt{x^2+4}}{2} - 2\ln(x+\sqrt{x^2+4})\right]_0^2$

$2\left[\frac{2\sqrt{4+4}}{2} - 2\ln(2+\sqrt{4+4}) - \{0 - 2\ln(0+2)\}\right]$

$2\left[2\sqrt{2} - 2\ln(2+2\sqrt{2}) + 2\ln 2\right]$

$4\sqrt{2} - 4\ln\left(\frac{2+2\sqrt{2}}{2}\right)$

$\boxed{4\sqrt{2} - 4\ln(1+\sqrt{2})}$

(12)

PAGE NO.:
DATE: / /

Evaluate : $\int_0^{\pi/4} \frac{x\,dx}{\cos x\,(\cos x + \sin x)}$

$I = \int_0^{\pi/4} \frac{x\,dx}{\cos^2 x + \cos x \sin x} \Rightarrow \int_0^{\pi/4} \frac{x\,dx}{\frac{1+\cos 2x}{2} + \frac{\sin 2x}{2}}$

$I = \int_0^{\pi/4} \frac{2x\,dx}{1+\cos 2x + \sin 2x}$

$I = \int_0^{\pi/4} \frac{2\left(\frac{\pi}{4} - x\right)}{1+\sin 2x + \cos 2x}\,dx$

$2I = \int_0^{\pi/4} \frac{\pi/2\,dx}{1+\sin 2x + \cos 2x} \Rightarrow \frac{\pi}{4}\int_0^{\pi/4} \frac{dx}{1+\frac{2\tan x}{1+\tan^2 x} + \frac{1-\tan^2 x}{1+\tan^2 x}}$

$I = \frac{\pi}{4}\int_0^{\pi/4} \frac{\sec^2 x\,dx}{1+\cancel{\tan^2 x} + 2\tan x + 1 - \cancel{\tan^2 x}}$

$\frac{\pi}{8}\int_0^{\pi/4} \frac{\sec^2 x}{1+\tan x}\,dx$

$\tan x = t$

$\sec^2 x\,dx = dt$

$\frac{\pi}{8} \ln(1+\tan x)\Big|_0^{\pi/4}$

$\boxed{\frac{\pi}{8}\ln 2}$

Scanned with CamScanner

13. Evaluate : $\displaystyle\int_1^{\frac{1+\sqrt{5}}{2}} \frac{x^2+1}{x^4-x^2+1} \ln\left(1+x-\frac{1}{x}\right)dx$

Solution:

$$I = \int_1^{\alpha} \frac{x^2\left(1+\frac{1}{x^2}\right)}{x^2\left(x^2-1+\frac{1}{x^2}\right)} \ln\left(1+x-\frac{1}{x}\right)dx$$

where $\alpha = \frac{1+\sqrt{5}}{2} \Rightarrow 2\alpha = 1+\sqrt{5} \Rightarrow 2\alpha-1=\sqrt{5}$

$\Rightarrow 4\alpha^2-4\alpha+1=5 \Rightarrow \boxed{\alpha^2-\alpha-1=0} \Rightarrow \boxed{\alpha^2=\alpha+1}$

Let $x-\frac{1}{x}=t \Rightarrow \left(1+\frac{1}{x^2}\right)dx = dt$ and $x^2+\frac{1}{x^2}=t^2+2$

at $x=1$; $t=0$

at $x=\alpha$; $t = \alpha-\frac{1}{\alpha} = \frac{\alpha^2-1}{\alpha} = \frac{(\alpha+1)-1}{\alpha} = 1$

$$\therefore\ I = \int_0^1 \frac{\ln(1+t)}{(t^2+2-1)}dt = \int_0^1 \frac{\ln(1+t)}{1+t^2}dt$$

Let $t=\tan\theta$

$$I = \int_0^{\pi/4} \ln(1+\tan\theta)d\theta$$

By king, $I = \displaystyle\int_0^{\pi/4} \ln\left(1+\tan\left(\frac{\pi}{4}-\theta\right)\right)d\theta$

$$\therefore\ I+I = \int_0^{\pi/4}\left[\ln(1+\tan\theta)+\ln\left(1+\tan\left(\frac{\pi}{4}-\theta\right)\right)\right]d\theta$$

(13) Continues...

$$\Rightarrow 2I = \int_0^{\pi/4} \ln\left[\left(1+\tan\theta\right)\left(1+\tan\left(\frac{\pi}{4}-\theta\right)\right)\right] d\theta$$

$$= \int_0^{\pi/4} \ln 2 \, d\theta = \frac{\pi}{4}\ln 2$$

$$\Rightarrow \boxed{I = \frac{\pi}{8}\ln 2}$$

[Note: $\left(1+\tan\theta\right)\left(1+\tan\left(\frac{\pi}{4}-\theta\right)\right) = \left(1+\tan\theta\right)\left(1+\dfrac{\tan\frac{\pi}{4}-\tan\theta}{1+\tan\frac{\pi}{4}\cdot\tan\theta}\right)$

$$= \left(1+\tan\theta\right)\left(1+\frac{1-\tan\theta}{1+\tan\theta}\right) = \left(1+\tan\theta\right) + \left(1+\tan\theta\right)\left(\frac{1-\tan\theta}{1+\tan\theta}\right)$$

$$= 1+\tan\theta + 1-\tan\theta = 2$$]

(14)

PAGE NO.:
DATE: / /

Evaluate: $\int_0^{\pi/2} \frac{a\sin x + b\cos x}{\sin\left(\frac{\pi}{4}+x\right)}\,dx$

Solⁿ

$$I = \int_0^{\pi/2} \frac{a\sin x + b\cos x}{\frac{1}{\sqrt{2}}\cos x + \frac{1}{\sqrt{2}}\sin x}\,dx$$

$$I = \sqrt{2}\int_0^{\pi/2} \frac{a\sin x + b\cos x}{\sin x + \cos x}\,dx$$

$$I \overset{K}{=} \sqrt{2}\int_0^{\pi/2} \frac{a\cos x + b\sin x}{\cos x + \sin x}\,dx$$

$$2I = \sqrt{2}\int_0^{\pi/2} \frac{a\sin x + a\cos x + b\cos x + b\sin x}{\sin x + \cos x}\,dx$$

$$2I = \sqrt{2}\,(a+b)\left(\frac{\pi}{2}\right)$$

$$\boxed{I = (a+b)\frac{\pi}{2\sqrt{2}}}$$

Scanned with CamScanner

(15)

PAGE NO.:
DATE: / /

Evaluate : $\int_0^{\pi} \frac{(2x+3)\sin x}{(1+\cos^2 x)} dx$

Sol^n $I = \int_0^{\pi} \frac{(2\pi - 2x+3)\sin(\pi - x)}{1+\cos^2(\pi - x)} dx$

$I = \int_0^{\pi} \frac{(2\pi - 2x+3)\sin x}{1+\cos^2 x} dx$

$2I = \int_0^{\pi} \frac{(2\pi+6)\sin x}{1+\cos^2 x} dx$

Put $\cos x = t$
$-\sin x\, dx = dt$

$I = (\pi+3)\, 2 \int_0^{\pi/2} \frac{\sin x}{1+\cos^2 x} dx$

$I = 2(\pi+3) \int_1^{0} \frac{-dt}{1+t^2}$

$= 2(\pi+3)\left(\tan^{-1} t\right)_0^1$

$2(\pi+3)(\pi/4)$

$\boxed{\frac{(\pi+3)\pi}{2}}$

16. Evaluate : $\int_0^3 \sqrt{\frac{x}{3-x}}\,dx$

Solution:

Domain: $0 \le x < 3$

$\therefore$ Let $\theta = \sin^{-1}\sqrt{\frac{x}{3}}$ $\{\text{idea } x = 3\sin^2\theta\}$

i.e., $x = 3\sin^2\theta$ $\{\theta \in [0, \frac{\pi}{2})\}$

$dx = 6\sin\theta\cos\theta\,d\theta$

$$\therefore \quad I = \int_0^{\pi/2} \sqrt{\frac{3\sin^2\theta}{3-3\sin^2\theta}} \cdot 6\sin\theta\cos\theta\,d\theta$$

$$= \int_0^{\pi/2} \sqrt{\frac{\sin^2\theta}{\cos^2\theta}} \cdot 6\sin\theta\cos\theta\,d\theta$$

$$I = 6\int_0^{\pi/2} \sin^2\theta\,d\theta$$

By king, $I = 6\int_0^{\pi/2} \cos^2\theta\,d\theta$

$$2I = 6\int_0^{\pi/2} 1\,d\theta$$

$$\Rightarrow \boxed{I = \frac{3\pi}{2}}$$

PAGE NO.:

DATE: / /

(17)

Evaluate : $\int_0^{\frac{\ln 3}{2}} \frac{e^x+1}{e^{2x}+1}\,dx$

Sol^n put $e^x = t \Rightarrow e^x\,dx = dt$

$$\int_1^{\sqrt{3}} \frac{t+1}{t^2+1}\,\frac{dt}{t}$$

$$\int_1^{\sqrt{3}} \frac{\cancel{t}}{(t^2+1)\cancel{t}}\,dt + \int_1^{\sqrt{3}} \frac{1}{(t^2+1)\,t}\,dt$$

$$(\tan^{-1}t)_1^{\sqrt{3}} + \int_1^{\sqrt{3}} \frac{t}{t^2(t^2+1)}\,dt \qquad \text{Put } t^2 = p$$

$$\left(\frac{\pi}{3}-\frac{\pi}{4}\right) + \frac{1}{2}\int_1^3 \frac{dp}{p(p+1)}$$

$$\frac{\pi}{12} + \frac{1}{2}\int_1^3 \left(\frac{1}{p}-\frac{1}{p+1}\right)dp$$

$$\frac{\pi}{12} + \frac{1}{2}\left(\ln p - \ln(p+1)\right)_1^3$$

$$\frac{\pi}{12} + \frac{1}{2}\left[\ln 3 - \ln 4 - \ln 1 + \ln 2\right]$$

$$\boxed{\frac{1}{2}\left[\frac{\pi}{6} + \ln 3 - \ln 2\right]}$$

(18)

PAGE NO.:
DATE: / /

Let $I = \int_0^1 \frac{2+3x+4x^2}{2\sqrt{1+x+x^2}}\,dx$ Find I^2.

Sol^n $I = \int_0^1 \frac{2x+3x^2+4x^3}{2x\sqrt{1+x+x^2}}\,dx$

$x^2+x^3+x^4 = t^2$

$I = \int_0^1 \frac{2x+3x^2+4x^3}{2\sqrt{x^2+x^3+x^4}}\,dx$

$(2x+3x^2+4x^3)\,dx = 2t\,dt$

$\therefore$ (1) $= \int_0^{\sqrt{3}} \frac{\cancel{2t}\,dt}{\cancel{2t}}$ $\Rightarrow$ $I = \sqrt{3}$

$\therefore$ $\boxed{I^2 = 3}$

(19) $\displaystyle\int_{-1}^{1} \frac{(2x^{332} + x^{998} + 4x^{1668} \sin x^{691})}{1+x^{666}}\, dx$

Sol^n

$$I = \int_{-1}^{1} \frac{(2x^{332} + x^{998} + \underbrace{4x^{1668} \sin x^{691}}_{\text{odd } f^n})}{1+x^{666}}\, dx$$

$$I = \int_{-1}^{1} \frac{2x^{332} + x^{998}}{1+x^{666}}\, dx$$

$$I = 2\int_{0}^{1} \frac{2x^{332} + x^{998}}{1+x^{666}}\, dx$$

$$I = 2\int_{0}^{1} \frac{x^{332} + x^{332}(1+x^{666})}{1+x^{666}}\, dx$$

$$I = 2\left[\int_{0}^{1} \frac{x^{332}}{1+x^{666}}\, dx + \int_{0}^{1} x^{332}\, dx\right]$$

$x^{333} = t$

$333\, x^{332}\, dx = dt$

$$I = 2\left[\frac{1}{333}\left(\int_{0}^{1} \frac{1}{1+t^2}\, dt\right) + \frac{1}{333}(1-0)\right]$$

(13) Continues...

$$I = 2\left\{\frac{1}{333}\left[\tan^{-1}t\right]_0^1 + \frac{1}{333}\right\}$$

$$I = \frac{2}{333}\left(\frac{\pi}{4}\right) + \frac{2}{333}$$

$$\boxed{I = \frac{\pi+4}{666}}$$

Scanned with CamScanner

20 9 let $g(x) = x^c . e^{2x}$, $f(x) = \int_0^x e^{2t}(3t^2+1)^{1/2}$

for a certain value of c the limit of

$\frac{f'(x)}{g'(x)}$ as $x \to \infty$ is non zero finite.

Determine 'c' & limit

Solⁿ

$$\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = \lim_{x\to\infty} \frac{e^{2x}(3x^2+1)^{1/2}}{cx^{c-1}e^{2x} + x^c e^{2x}.2}$$

$$\lim_{x\to\infty} \frac{(3x^2+1)^{1/2}}{x^c\left(\frac{c}{x}+2\right)}$$

$$x = \frac{1}{t}$$

$$\lim_{t\to 0^+} \frac{\left(\frac{3}{t^2}+1\right)^{1/2}}{\frac{1}{t^c}(ct+2)}$$

$$\lim_{t\to 0^+} \frac{(t^2+3)^{1/2}\ t^{c-1}}{(ct+2)}$$

for a finite non zero value C must be equal to (1)

limit $\Rightarrow \frac{\sqrt{3}}{2}$

find a, b $(a>0)$

(20) (b) such that

$$\lim_{x \to 0} \frac{\int_0^x \frac{t^2\,dt}{\sqrt{a+t}}}{bx - \sin x} = 1$$

Sol^n

L/H

$$\lim_{x \to 0} \frac{\frac{x^2}{\sqrt{a+x}}}{b - \cos x} = 1$$

$$\lim_{x \to 0} \frac{x^2}{\sqrt{a+x}\,(b-\cos x)} = 1$$

$$\boxed{b=1}$$

$$\lim_{x \to 0} \frac{x^2}{(1-\cos x)\sqrt{a+x}} = 1$$

$$\lim_{x \to 0} \frac{1}{\sqrt{a+x}\,(1/2)} = 1$$

$$\frac{2}{\sqrt{a}} = 1$$

$$a = 4, \quad b = 1$$

Scanned with CamScanner

(21) $$\lim_{x\to\infty} \frac{d}{dx} \int_{2\sin\left(\frac{1}{x}\right)}^{3\sqrt{x}} \frac{3t^4+1}{(t-3)(t^2+3)}\, dt$$

Sol^n

$$\lim_{x\to\infty} \frac{3(3\sqrt{x})^4+1}{(3\sqrt{x}-3)((3\sqrt{x})^2+3)}\left(\frac{3}{2\sqrt{x}}\right) - \frac{3\left(2\sin\left(\frac{1}{x}\right)\right)^4+1}{\left(2\sin\left(\frac{1}{x}\right)-3\right)\left(\left(2\sin\frac{1}{x}\right)^2+3\right)} \cdot \left(2\cos\frac{1}{x}\left(\frac{-1}{x^2}\right)\right)$$

$$\lim_{x\to\infty} \frac{243x^2+1}{3(\sqrt{x}-1)(3x+1)} \frac{1}{2\sqrt{x}} + \frac{\left(48\sin^4\left(\frac{1}{x}\right)+1\right) 2\cos\left(\frac{1}{x}\right)\cdot\frac{1}{x^2}}{\left(2\sin\left(\frac{1}{x}\right)-3\right)\left(4\sin^2\frac{1}{x}+3\right)}$$

$$x = \frac{1}{t}$$

$$\lim_{t\to 0^+} \frac{\frac{243}{t^2}+1}{3\left(\frac{1}{\sqrt{t}}-1\right)\left(\frac{3}{t}+1\right)} \frac{\sqrt{t}}{2} + \frac{(48\sin^4 t+1)(2\cos t\cdot t^2)}{(2\sin t-3)(4\sin^2 t+3)}$$

$$\lim_{t\to 0} \frac{243+t^2}{2\times 3(1-\sqrt{t})(3+t)} + \frac{(48\sin^4 t+1)(2t^2)\cos t}{(2\sin t-3)(4\sin^2 t+3)}$$

$$\frac{243}{2\times 3\times 3} + 0 = \frac{27}{2}$$

$$\Rightarrow 13.5$$

22 a

$$\lim_{n\to\infty}\left[\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\cdots\cdots\left(1+\frac{n^2}{n^2}\right)\right]^{1/n}$$

Sol^n

$$l=\lim_{n\to\infty}\left[\left(1+\frac{1}{n^2}\right)\left(1+\frac{2^2}{n^2}\right)\cdots\left(1+\frac{n^2}{n^2}\right)\right]^{1/n}$$

$$\ln(l)=\lim_{n\to\infty}\frac{1}{n}\left[\ln\left(1+\frac{1}{n^2}\right)+\ln\left(1+\frac{2^2}{n^2}\right)\cdots\cdots\ln\left(1+\frac{n^2}{n^2}\right)\right]$$

$$\ln(l)=\lim_{n\to\infty}\sum_{r=1}^{n}\ln\left(1+\frac{r^2}{n^2}\right)\frac{1}{n}$$

$$\ln(l)=\int_0^1 \underset{II}{1}\cdot\underset{I}{\ln(1+x^2)}\,dx$$

$$\ln(l)=\left(x\ln(1+x^2)\right)_0^1-\int_0^1 x\,\frac{2x}{1+x^2}\,dx$$

$$=\ln 2-0-2\int_0^1\frac{x^2+1-1}{x^2+1}\,dx$$

$$\Rightarrow \ln 2-2\left[x-\tan^{-1}x\right]_0^1$$

$$\ln l=\ln 2-2+\frac{\pi}{2}$$

$$l=e^{\ln 2+\pi/2-2}$$

$$\boxed{l=2e^{\frac{1}{2}(\pi-4)}}$$

Scanned with CamScanner

22 b $\lim\limits_{n\to\infty} \frac{1}{n}\left[\frac{1}{n+1} + \frac{2}{n+2} \cdots\cdots \frac{3n}{4n}\right]$

Sol^n

$$\lim_{n\to\infty} \sum_{r=1}^{3n} \frac{1}{n}\left(\frac{r}{n+r}\right)$$

$$\lim_{n\to\infty} \sum_{r=1}^{3n} \frac{r/n}{\left(1+r/n\right)}\left(\frac{1}{n}\right)$$

$$\Rightarrow \int_0^3 \frac{x}{1+x}\,dx$$

$$\Rightarrow \int_0^3 \frac{x+1-1}{x+1}\,dx$$

$$\Rightarrow \int_0^3 1\,dx - \int_0^3 \frac{1}{1+x}\,dx$$

$$\Rightarrow [x]_0^3 - \left(\ln(x+1)\right)_0^3$$

$$\Rightarrow 3 - \ln 4$$

Scanned with CamScanner

23 Find a positive real valued continously differentiable function f on the real line such that for all x

$$f^2(x) = \int_0^x \left((f(t))^2 + (f'(t))^2 \right) dt + e^2$$

Solⁿ

put $x = 0$

$$f^2(0) = 0 + e^2$$

$$f(0) = +e, \ \cancel{-e}$$

(reject) → positive real valued fⁿ

Diff. equation

$$2 f(x) f'(x) = (f(x))^2 + (f'(x))^2 + 0$$

$$(f(x) - f'(x))^2 = 0$$

$$f(x) = f'(x)$$

$$\frac{f'(x)}{f(x)} = 1$$

$$\ln f(x) = x + C$$

$$f(x) = A e^x$$

$$f(0) = A = e$$

$$\boxed{f(x) = e^{x+1}}$$

(24) let $f(x)$ be a function defined on R Suchthat

$f'(x) = f'(3-x) \quad \forall \quad x \in [0,3]$ with

$f(0) = -32$ & $f(3) = 46$

$$\int_0^3 f(x)\,dx = ?$$

Solⁿ

$$f'(x) = f'(3-x)$$

$$f(x) = \frac{f(3-x)}{-1} + C$$

$$f(x) + f(3-x) = C$$

Put $x = 0$

$$f(0) + f(3) = C$$

$$-32 + 46 = C$$

$$C = 14$$

So $\quad f(x) + f(3-x) = 14 \quad \text{——(i)}$

$$I = \int_0^3 f(x)\,dx$$

$$I \overset{k}{=} \int_0^3 f(3-x)\,dx$$

$$2I = \int_0^3 \left(f(x) + f(3-x)\right)dx = \int_0^3 14\,dx$$

$$I = \frac{(14)\times(3-0)}{2} = 21$$

(25) (a)

$$\left[\frac{\pi}{6}\right] < \int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} < \left[\frac{\pi\sqrt{2}}{8}\right]$$

Sol^n

$$\int_0^1 \frac{dx}{\sqrt{4-x^2}} < \int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} < \int_0^1 \frac{dx}{\sqrt{4-2x^2}}$$

$$\left(\sin^{-1}\left(\frac{x}{2}\right)\right)_0^1 < \int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} < \left(\frac{1}{\sqrt{2}}\sin^{-1}\frac{x}{\sqrt{2}}\right)_0^1$$

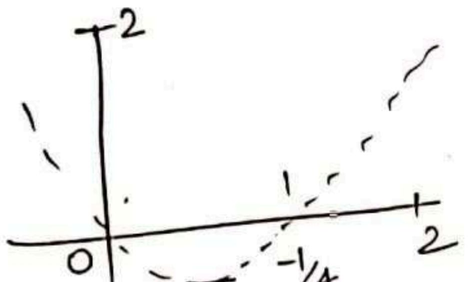
$$\sin^{-1}\frac{1}{2} < \quad '' \quad < \frac{1}{\sqrt{2}}\sin^{-1}\frac{1}{\sqrt{2}}$$

$$\frac{\pi}{6} < \int_0^1 \frac{dx}{\sqrt{4-x^2-x^3}} < \frac{\pi}{4\sqrt{2}}$$

(b)

$$2e^{-1/4} < \int_0^2 e^{x^2-x} < 2e^2$$

Sol^n



$f(x) = x^2 - x$

$$\int_0^2 e^{-1/4} < \int_0^2 e^{x^2-x} < \int_0^2 e^2$$

$$2e^{-1/4} < \int_0^2 e^{x^2-x} < 2e^2$$

EXERCISE (S-2)

EXERCISE (S-2)

1

Evaluate $\int_0^1 x\,(\tan^{-1}x)^2\,dx$

Sol :

$$I = (\tan^{-1}x)^2 \frac{x^2}{2}\Big|_0^1 - \int_0^1 (2\tan^{-1}x)\frac{1}{1+x^2}\cdot\frac{x^2}{2}\,dx$$

$$= \frac{1}{2}\left(\frac{\pi}{4}\right)^2 - \int_0^1 \tan^{-1}x\left(1-\frac{1}{1+x^2}\right)dx$$

$$= \frac{\pi^2}{32} - \int_0^1 \tan^{-1}x\,dx + \int_0^1 \frac{\tan^{-1}x}{1+x^2}\,dx$$

$$I = \frac{\pi^2}{32} - \left[\tan^{-1}x\cdot x\Big|_0^1 - \int_0^1 \frac{1}{1+x^2}\cdot x\,dx\right] + I_1$$

$$I = \frac{\pi^2}{32} - \left[\frac{\pi}{4} - \frac{1}{2}\ln(1+x^2)\Big|_0^1\right] + I_1$$

$$I_1 = \int_0^1 \frac{\tan^{-1}x}{1+x^2}\,dx = \tan^{-1}x\cdot(\tan^{-1}x)\Big|_0^1 - \int_0^1 \frac{1}{1+x^2}\tan^{-1}x$$

$$\Rightarrow I_1 = (\tan^{-1}x)^2\Big|_0^1 - I_1$$

$$\Rightarrow I_1 = \cancel{\frac{(\tan^{-1}x)^2}{2}}\ \frac{1}{2}(\tan^{-1}x)^2\Big|_0^1$$

$$\Rightarrow I = \frac{\pi^2}{32} - \left[\frac{\pi}{4} - \frac{1}{2}\ln 2\right] + \frac{(\tan^{-1}x)^2}{2}\Big|_0^1$$

$$= \frac{\pi^2}{32} - \frac{\pi}{4} + \frac{1}{2}\ln 2 + \frac{1}{2}\left(\frac{\pi^2}{16}\right)$$

$$= \frac{\pi^2}{16} - \frac{\pi}{4} + \frac{1}{2}\ln 2$$

$$I = \frac{\pi}{4}\left(\frac{\pi}{4}-1\right) + \frac{1}{2}\ln 2$$

2

Evaluate $\int_0^1 x^5 \sqrt{\frac{1+x^2}{1-x^2}}\, dx$

Sol^n: Put $x^2 = \cos\theta \Rightarrow 2x\,dx = (-)\sin\theta\, d\theta$

$$I = \int_{\pi/2}^{0} (x^2)^2 \sqrt{\frac{1+\cos\theta}{1-\cos\theta}}\,.\,x\,dx$$

$$= \int_{\pi/2}^{0} \cos^2\theta \cot\left(\frac{\theta}{2}\right) \cdot \frac{(-)\sin\theta\, d\theta}{2}$$

$$= \frac{1}{2}\int_0^{\pi/2} \cos^2\theta \cdot \frac{\cos\frac{\theta}{2}}{\cancel{\sin\frac{\theta}{2}}} \cdot 2\cancel{\sin\frac{\theta}{2}} \cos\frac{\theta}{2}\, d\theta$$

$$= \int_0^{\pi/2} \cos^2\theta \left(\cos^2\frac{\theta}{2}\right) d\theta$$

$$= \int_0^{\pi/2} \cos^2\theta \frac{(1+\cos\theta)}{2}\, d\theta$$

$$= \frac{1}{2}\left[\int_0^{\pi/2} \cos^2\theta\, d\theta + \int_0^{\pi/2} \cos^3\theta\, d\theta\right]$$

$$= \frac{1}{2}\left[\int_0^{\pi/2} \left(\frac{1+\cos 2\theta}{2}\right) d\theta + \int_0^{\pi/2} \cos\theta\,(1-\sin^2\theta)\, d\theta\right]$$

$$= \frac{1}{2}\left[\left(\frac{1}{2}\right)\left(\frac{\pi}{2}\right) + \sin\theta\Big|_0^{\pi/2} - \int_0^{\pi/2} \cos\theta \sin^2\theta\, d\theta\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{4} + 1 - \frac{\sin^3\theta}{3}\Big|_0^{\pi/2}\right]$$

$$= \frac{1}{2}\left[\frac{\pi}{4} + 1 - \frac{1}{3}\right]$$

$$= \frac{\pi}{8} + \frac{1}{3}$$

$$I = \frac{3\pi+8}{24}$$

③

Let $A = \int_{3/4}^{4/3} \frac{2x^2+x+1}{x^3+x^2+x+1}\,dx$, then find the value of e^A.

Sol:- $A = \int_{3/4}^{4/3} \frac{2x^2+x+1}{(x^2+1)(x+1)}\,dx$

$$\frac{2x^2+x+1}{(x^2+1)(x+1)} = \frac{A}{x+1} + \frac{Bx+C}{x^2+1}$$

$$\frac{2x^2+x+1}{(x^2+1)(x+1)} = \frac{(A+B)x^2+(B+C)x+A+C}{(x+1)(x^2+1)}$$

$\Rightarrow A+B=2,\ B+C=1,\ A+C=1$

$\Rightarrow A=B=1,\ C=0$

$$\Rightarrow A = \int_{3/4}^{4/3} \frac{1}{x+1}\,dx + \int_{3/4}^{4/3} \frac{x}{x^2+1}\,dx$$

$$= \ln(x+1)\Big|_{3/4}^{4/3} + \frac{1}{2}\ln(x^2+1)\Big|_{3/4}^{4/3}$$

$$= \ln\left(\frac{7/3}{7/4}\right) + \frac{1}{2}\ln\left(\frac{25/9}{25/16}\right)$$

$$= \ln\frac{4}{3} + \ln\left(\frac{16}{9}\right)^{1/2}$$

$$= 2\ln\left(\frac{4}{3}\right)$$

$$A = \ln\left(\frac{16}{9}\right)$$

$$\Rightarrow e^A = \frac{16}{9}$$

4

Evaluate $\int_0^1 \frac{2-x^2}{(1+x)\sqrt{1-x^2}}\,dx$

Sol:- $I = \int_0^1 \frac{1}{(1+x)\sqrt{1-x^2}}\,dx + \int_0^1 \frac{1-x^2}{(1+x)\sqrt{1-x^2}}\,dx$

$I = (I_1) + (I_2)$

$I_1 = \int_0^1 \frac{dx}{(1+x)\sqrt{1-x^2}}$ Put $x = \cos\theta$ $\Rightarrow dx = (-)\sin\theta\, d\theta$

$= \int_{\pi/2}^{0} \frac{(-)\cancel{\sin\theta}\, d\theta}{(1+\cos\theta)\cancel{\sin\theta}}$

$= \int_0^{\pi/2} \frac{d\theta}{(1+\cos\theta)}$

$= \int_0^{\pi/2} \frac{(1-\cos\theta)}{\sin^2\theta}\,d\theta$

$= \int_0^{\pi/2} (\operatorname{cosec}^2\theta - \operatorname{cosec}\theta\cot\theta)\,d\theta$

$= -\cot\theta \Big|_0^{\pi/2} + \operatorname{cosec}\theta \Big|_0^{\pi/2}$

$= (-)0 + \underset{\theta\to 0}{Lt} \cot\theta + 1 - \underset{\theta\to 0}{Lt} \operatorname{cosec}\theta$

$I_1 = 1 + \underset{\theta\to 0}{Lt} \frac{\cos\theta - 1}{\sin\theta} = 1+0 = 1$

$I_2 = \int_0^1 \frac{(1-x)\cancel{(1+x)}}{\cancel{(1+x)}\sqrt{(1-x)(1+x)}} = \int_0^1 \sqrt{\frac{1-x}{1+x}}\,dx$

Put $x = \cos\theta$

$= \int_{\pi/2}^{0} \tan\frac{\theta}{2}\cdot(-)\sin\theta\, d\theta$

$I_2 = \int_0^{\pi/2} (1-\cos\theta)\,d\theta = \frac{\pi}{2} - 1$

$\Rightarrow I = I_1 + I_2 = \frac{\pi}{2}$

(5) Evaluate $\int_0^1 \frac{1-x}{1+x} \frac{dx}{\sqrt{x+x^2+x^3}}$

Sol:- $I = \int_0^1 \frac{(1-x)(1+x)}{(1+x)^2\sqrt{x+x^2+x^3}}\,dx$

$= \int_0^1 \frac{1-x^2}{(x^2+2x+1)\sqrt{x+x^2+x^3}}\,dx$

$= \int_0^1 \frac{\frac{1}{x^2}-1}{\left(x+2+\frac{1}{x}\right)\sqrt{\frac{1}{x}+1+x}}\,dx$

Put $x+\frac{1}{x} = t \Rightarrow \left(1-\frac{1}{x^2}\right)dx = dt$

$I = \int_{\infty}^{2} \frac{-dt}{(2+t)\sqrt{1+t}} = \int_2^{\infty} \frac{dt}{(2+t)\sqrt{1+t}}$

Put $1+t = y^2$
$\Rightarrow dt = 2y\,dy$

$I = \int_{\sqrt{3}}^{\infty} \frac{2y\,dy}{(y^2+1)\cancel{y}}$

$= 2\tan^{-1}y\Big|_{\sqrt{3}}^{\infty}$

$= 2\left[\frac{\pi}{2}-\frac{\pi}{3}\right]$

$I = \frac{\pi}{3}$

⑥ Evaluate $\displaystyle\int_0^{\sqrt{3}} \sin^{-1} \frac{2x}{1+x^2}\, dx$

Sol:- Put $x = \tan\theta$

$$\Rightarrow I = \int_0^{\pi/3} \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right)\sec^2\theta\, d\theta$$

$$= \int_0^{\pi/3} \sin^{-1}(\sin 2\theta)\sec^2\theta\, d\theta$$

$$= \int_0^{\pi/4} 2\theta \sec^2\theta\, d\theta + \int_{\pi/4}^{\pi/3} (\pi - 2\theta)\sec^2\theta\, d\theta$$

$$= 2\left[\theta\tan\theta\Big|_0^{\pi/4} - \int_0^{\pi/4} \tan\theta\, d\theta\right] + \pi\tan\theta\Big|_{\pi/4}^{\pi/3}$$

$$- 2\left[\theta\tan\theta\Big|_{\pi/4}^{\pi/3} - \int_{\pi/4}^{\pi/3}\tan\theta\, d\theta\right]$$

$$= 2\left[\frac{\pi}{4} - \ln|\sec\theta|\Big|_0^{\pi/4}\right] + \pi\left[\sqrt{3}-1\right]$$

$$- 2\left[\frac{\pi}{3}\sqrt{3} - \frac{\pi}{4} - \ln|\sec\theta|\Big|_{\pi/4}^{\pi/3}\right]$$

$$= 2\left[\frac{\pi}{4} - \ln\sqrt{2}\right] + \pi\sqrt{3} - \pi - \frac{2\pi}{3}\sqrt{3} + \frac{\pi}{2} + 2\ln\left(\frac{\cancel{2}}{\cancel{\sqrt{2}}}\right)$$

$$2\ln\left(\frac{2}{\sqrt{2}}\right)$$

$$= \cancel{\frac{\pi}{2}} - \cancel{2\ln\sqrt{2}} + \pi\sqrt{3} - \cancel{\pi} - \frac{2\pi}{3}\sqrt{3} + \cancel{\frac{\pi}{2}} + \cancel{2\ln\sqrt{2}}$$

$$= \pi\sqrt{3} - \frac{2\pi}{3}\sqrt{3}$$

$$I = \frac{\pi}{3}\sqrt{3}$$

⑦ Evaluate: $\int_1^{16} \tan^{-1}\sqrt{\sqrt{x}-1}\, dx$

Sol: Put $x = \sec^4\theta \Rightarrow dx = 4\sec^3\theta\, \sec\theta\tan\theta\, d\theta$

$$I = \int_0^{\pi/3} \tan^{-1}\sqrt{\sec^2\theta - 1}\,.\, 4\sec^3\theta\,.\sec\theta\tan\theta\, d\theta$$

$$= \int_0^{\pi/3} 4\theta\sec^4\theta\tan\theta\, d\theta$$

$$I = 4\int_0^{\pi/3} \theta\sec^4\theta\tan\theta\, d\theta$$

$$\int \sec^4\theta\tan\theta\, d\theta = \int t^3 dt \qquad \text{where } t = \sec\theta$$

$$= \frac{\sec^4\theta}{4}$$

$$\Rightarrow I = 4\cdot\left[\theta\frac{\sec^4\theta}{4}\Big|_0^{\pi/3} - \int_0^{\pi/3}(1)\frac{\sec^4\theta}{4}\, d\theta\right]$$

$$= \frac{\pi}{3}.(16) - \int_0^{\pi/3}\sec^4\theta\, d\theta$$

$$= \frac{16\pi}{3} - \left[\int_0^{\pi/3}(\sec^2\theta + \sec^2\theta\tan^2\theta)\, d\theta\right]$$

$$= \frac{16\pi}{3} - \tan\theta\Big|_0^{\pi/3} - \frac{\tan^3\theta}{3}\Big|_0^{\pi/3}$$

$$= \frac{16\pi}{3} - \sqrt{3} - \frac{3\sqrt{3}}{3}$$

$$I = \frac{16\pi}{3} - 2\sqrt{3}$$

8

A curve C_1 is defined by : $\frac{dy}{dx} = e^x \cos x$ for $x \in [0, 2\pi]$ and passes through the origin. Prove that the roots of the function $y = 0$ (other than zero) occurs in the ranges $\frac{\pi}{2} < x < \pi$ and $\frac{3\pi}{2} < x < 2\pi$.

$$\frac{dy}{dx} = e^x \cos x .$$

$$y = \int \underset{\mathrm{II}}{e^x} \underset{\mathrm{I}}{\cos x}\, dx$$

$$y = \cos x\, e^x - \int -\sin x\, e^x\, dx$$

$$y = \cos x\, e^x + \left[\sin x\, e^x - \int \cos x\, e^x\, dx\right)$$

$$2y = (\cos x + \sin x) e^x + c$$

∵ curve passes through origin

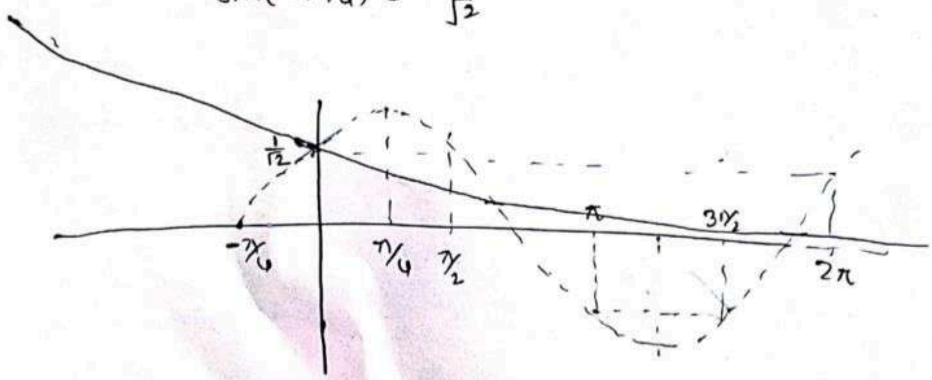
$$0 = (1) + c$$

$$c = -1$$

$$y = \frac{1}{2}\left[(\cos x + \sin x) e^x - 1\right] = 0$$

$$(\cos x + \sin x) e^x = 1$$

$$\sin(x + \pi/4)\, e = \frac{1}{\sqrt{2}} e^{-x}$$



So one root is lie b/w $\pi/2 < x < \pi$

& other root is lie b/w $3\pi/2 < x < 2\pi$

9

Let $F(x)=\int_{-1}^{x}\sqrt{4+t^2}\,dt$ and $G(x)=\int_{x}^{1}\sqrt{4+t^2}\,dt$ then compute the value of $(FG)'(0)$ where dash denotes the derivative.

$$F(x) = \int_{-1}^{x}\sqrt{4+t^2}\,dt \qquad G(x) = \int_{x}^{1}\sqrt{4+t^2}\,dt$$

$$F(0) = \int_{-1}^{0}\sqrt{4+t^2}\,dt$$

let $t = -p$

$dt = -dp$

$$F(0) = \int_{1}^{0}\sqrt{4+p^2}\,(-dp)$$

$$= -\int_{1}^{0}\sqrt{4+p^2}\,dp$$

$$F(0) = \int_{0}^{1}\sqrt{4+p^2}\,dp$$

$$F(0) = G(0) = \int_{0}^{1}\sqrt{4+p^2}\,dp$$

$$F'(x) = \sqrt{4+x^2} \qquad G'(x) = -\sqrt{4+x^2}$$

$$F'(0) = 2 \qquad G'(0) = -2$$

$$(FG)'(0) = F'(0)\,G(0) + F(0)\cdot G'(0)$$

$$= 2(G(0)) + F(0)(-2)$$

$$= 2\left[-F(0) + G(0)\right]$$

$$(FG)'(0) = 0$$

10 (a) $\lim_{n\to\infty}\left[\frac{n!}{n^n}\right]^{1/n}$

30. (a) $y = \lim_{n\to\infty}\left[\frac{n!}{n^n}\right]^{1/n}$

$\ln y = \lim_{n\to\infty}\frac{1}{n}\ln\left[\frac{n!}{n^n}\right]$

$\ln y = \lim_{n\to\infty}\frac{1}{n}\ln\left[\frac{1}{n}\cdot\frac{2}{n}\cdot\frac{3}{n}\cdot\text{-----}\cdot\frac{n}{n}\right]$

$\ln y = \lim_{n\to\infty}\frac{1}{n}\left[\ln 1/n + \ln 2/n + \ln 3/n + -- + \ln n/n\right)$

$\ln y = \lim_{n\to\infty}\sum_{n=1}^{n} 1/n \cdot \ln r/n$

$\ln y = \int_0^1 \ln x \, . \, dx$

$\ln y = [x\ln x - x]_0^1$

$\ln y = [0-1-[0.0]$

$y = \frac{1}{e}$

10 (b) Let $P_n = \sqrt[n]{\dfrac{(3n)!}{(2n)!}}$ (n= 1,2,3.......), then find $\lim\limits_{n\to\infty} \dfrac{P_n}{n}$.

(b) $P_n = \left(\dfrac{3n!}{2n!}\right)^{1/n}$

$P_n = \left[\left((2n+1)(2n+2)(2n+3)\ldots(2n+n)\right)\right]^{1/n}$

$I = \lim\limits_{n\to\infty} \dfrac{P_n}{n}$

$I = \lim\limits_{n\to\infty} \dfrac{\left[(2n+1)(2n+2)\ldots(2n+n)\right]^{1/n}}{n}$

$I = \lim\limits_{n\to\infty} \left[\dfrac{(2n+1)(2n+2)\ldots(2n+n)}{n^n}\right]^{1/n}$

$\ln I = \lim\limits_{n\to\infty} \dfrac{1}{n} \ln\left[\dfrac{(2n+1)}{n} \cdot \dfrac{(2n+2)}{n} \ldots \dfrac{2n+n}{n}\right]$

$\ln I = \lim\limits_{n\to\infty} \dfrac{1}{n}\left[\ln(2+1/n) + \ln[2+2/n] + \ldots + \ln[2+n/n]\right]$

$\ln I = \lim\limits_{n\to\infty} \dfrac{1}{n} \sum\limits_{n=1}^{n} \ln[2+r/n]$

$\ln I = \int_0^1 \ln(2+x)$

$\ln I = \left[(x+2)\ln(x+2) - (x+2)\right]_0^1$

$\ln I = \left[3\ln 3 - 3 - 2\ln 2 + 2\right]$

$\ln I = \left[\ln 27/4 - 1\right]$

$\ln I = \ln \dfrac{27}{4e}$

$I = \dfrac{27}{4e}$

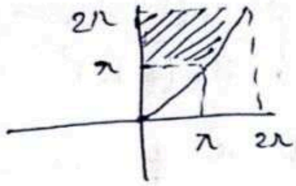
11

If $f(x) = x + \sin x$ and I denotes the value of integral $\int_{\pi}^{2\pi} \left(f^{-1}(x) + \sin x\right) dx$ then the value of $\left[\frac{2I}{3}\right]$

(where [.] denotes greatest integer function)

$f(x) = x + \sin x$

$f'(x) = 1 + \cos x \geq 0$



$I = \int_{\pi}^{2\pi} (f^{-1}(x) + \sin x)\, dx$

$I = \int_{\pi}^{2\pi} f^{-1}(x)\, dx + \int_{\pi}^{2\pi} \sin x\, dx$

$I = [(2\pi \times 2\pi) - (\pi \times \pi)] - \int_{\pi}^{2\pi} f(x) dx + [-\cos x]_{\pi}^{2\pi}$

$I = 3\pi^2 - \int_{\pi}^{2\pi} (x + \sin x)\, dx + [-\cos x]_{\pi}^{2\pi}$

$I = 3\pi^2 - \frac{1}{2}[(2\pi)^2 - (\pi)^2] - \cancel{[-\cos x]_{\pi}^{2\pi}} + \cancel{[-\cos x]_{\pi}^{2\pi}}$

$I = 3\pi^2 - \frac{3\pi^2}{2}$

$I = \frac{3\pi^2}{2}$

$\left[\frac{2I}{3}\right] = [\pi^2]$

$= 9$

(12) Prove the inequalities :

(a) $\frac{1}{3} < \int_0^1 x^{(\sin x + \cos x)^2} dx < \frac{1}{2}$ (b) $\frac{1}{2} \le \int_0^2 \frac{dx}{2+x^2} \le \frac{5}{6}$

31. (a) $\frac{1}{3} < \int_0^1 x^{(\sin x + \cos x)^2} dx < \frac{1}{2}$

$\Rightarrow 0 < x < 1$

$\Rightarrow 0 < 2x < 2$

$\Rightarrow 0 < \sin 2x < \sin 2$

$\Rightarrow 1 < 1 + \sin 2x < 1 + \sin 2 < 2$

$\therefore 0 < \sin 2 < 1$

$1 < 1 + \sin 2 < 2$

$\Rightarrow 1 < 1 + \sin 2x < 2$

$\Rightarrow 1 < 1 + \sin 2x < 2$

$\Rightarrow x^2 < x^{1+\sin 2x} < x^1$

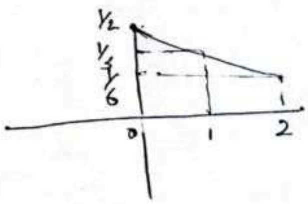
$\Rightarrow \int_0^1 x^2 dx < \int_0^1 x^{(\sin x + \cos x)^2} dx < \int_0^1 x^1 dx$

$\Rightarrow \left[\frac{x^3}{3}\right]_0^1 < \int_0^1 x^{(\sin x + \cos x)^2} dx < \left[\frac{x^2}{2}\right]_0^1$

$\frac{1}{3} < \int_0^1 x^{(\sin x + \cos x)^2} dx < \frac{1}{2}$

(b) $\frac{1}{2} \le \int_0^2 \frac{dx}{2+x^2} \le \frac{5}{6}$

$\Rightarrow 1 \times \frac{1}{3} + 1 \times \frac{1}{6} \le \int_0^2 \frac{dx}{2+x^2} < 1 \times \frac{1}{2} + 1 \times \frac{1}{3}$



$\frac{1}{2} < \int_0^2 \frac{dx}{2+x^2} < \frac{5}{6}$

(13)

Evaluate $I = \int_0^{2\pi} \frac{dx}{2+\sin 2x}$

Sol^n $I = \int_0^{2\pi} \frac{dx}{2-\sin 2x}$ [King property]

$$I + I = \int_0^{2\pi} \left(\frac{1}{2+\sin 2x} + \frac{1}{2 - \sin 2x} \right) dx$$

$$\Rightarrow 2I = \int_0^{2\pi} \frac{4}{4-\sin^2 2x} dx$$

$$\Rightarrow I = 2\int_0^{2\pi} \frac{1}{4-\sin^2 2x} dx$$

$$= 2(2)\int_0^{\pi} \frac{1}{4-\sin^2 2x} dx \quad \text{[Queen property]}$$

$$= 2(2)(2)\int_0^{\pi/2} \frac{1}{4-\sin^2 2x} dx \quad \text{[Queen again]}$$

$$I = 16\int_0^{\pi/4} \frac{1}{4-\sin^2 2x} dx \quad \text{[Queen again]}$$

$$= 16\int_0^{\pi/4} \frac{dx}{3+\cos^2 2x}$$

$$= 16\int_0^{\pi/4} \frac{\sec^2 2x \, dx}{3\sec^2 2x + 1}$$

$\tan 2x = t$

$\Rightarrow 2\sec^2 2x \, dx = dt$

$$= 8\int_0^{\infty} \frac{dt}{4+3t^2}$$

$$= \frac{8}{3}\int_0^{\infty} \frac{dt}{t^2 + \left(\frac{2}{\sqrt{3}}\right)^2}$$

$$= \frac{8}{3} \cdot \frac{1}{\frac{2}{\sqrt{3}}} \left[\tan^{-1}\left(\frac{t}{2/\sqrt{3}} \right) \right]_0^{\infty}$$

$$I = \frac{4}{\sqrt{3}}\left(\frac{\pi}{2}\right) = \frac{2\pi}{\sqrt{3}}$$

(14)

Let $I = \int_0^{\pi/2} \frac{\cos x + 4}{3\sin x + 4\cos x + 25}\,dx$ and

$J = \int_0^{\pi/2} \frac{\sin x + 3}{3\sin x + 4\cos x + 25}\,dx$. If $25I = a\pi + b\ln\frac{c}{d}$

where a, b, c and $d \in N$ and $\frac{c}{d}$ is not a perfect square of a rational then find the value of $a+b+c+d$

Sol: $4I + 3J = \int_0^{\pi/2} \frac{4\cos x + 3\sin x + 25}{3\sin x + 4\cos x + 25}\,dx = \frac{\pi}{2}$ —— (1)

$3I - 4J = \int_0^{\pi/2} \frac{3\cos x - 4\sin x}{3\sin x + 4\cos x + 25}\,dx$

$= \ln(3\sin x + 4\cos x + 25)\Big|_0^{\pi/2}$

$3I - 4J = \ln\left(\frac{28}{29}\right)$ —— (2)

From (1), (2) Solving for I, we get

$25I = 2\pi + 3\ln\left(\frac{28}{29}\right)$

$a = 2,\ b = 3,\ c = 28,\ d = 29$

$\Rightarrow a + b + c + d = 2 + 3 + 28 + 29$

$a + b + c + d = 62$

Scanned by CamScanner

EXERCISE (JM)

EXERCISE (JM)

1. Let p(x) be a function defined on R such that p'(x) = p'(1 – x), for all x ∈ [0, 1], p(0) = 1 and p(1) = 41. Then $\int_0^1 p(x)\,dx$ equals :- **[AIEEE-2010]**

(1) $\sqrt{41}$ (2) 21 (3) 41 (4) 42

Solution:

$$p'(x) = p'(1-x)$$

$$\Rightarrow \int p'(x)\,dx = \int p'(1-x)\,dx$$

$$\Rightarrow p(x) = -p(1-x) + C$$

$$\Rightarrow p(x) + p(1-x) = C$$

Put $x = 0$; $p(0) + p(1) = c \Rightarrow 1 + 41 = c$

$$\Rightarrow \boxed{c = 42}$$

Hence, $\boxed{p(x) + p(1-x) = 42}$

Let $I = \int_0^1 p(x)\,dx$

also $I = \int_0^1 p(1-x)\,dx$ [By king]

$$\Rightarrow 2I = \int_0^1 [p(x) + p(1-x)]\,dx = \int_0^1 42\,dx$$

$$\Rightarrow 2I = 42 \Rightarrow \boxed{I = 21} \rightarrow \text{Option (2)}$$

② The value of $\int_0^1 \frac{8\log(1+x)}{1+x^2}dx$ is :- 2011

(1) $\frac{\pi}{2}\log 2$ (2) $\log 2$ (3) $\pi \log 2$ (4) $\frac{\pi}{8}\log 2$

Sol $I = 8\int_0^1 \frac{\log(1+x)}{1+x^2}.dx$

Put $x = \tan\theta$, $dx = \sec^2\theta.d\theta$

$\Rightarrow I = 8\int_0^{\pi/4} \frac{\log(1+\tan\theta).\cancel{\sec^2\theta}.d\theta}{\cancel{(1+\tan^2\theta)}}$

$\Rightarrow I = 8\int_0^{\pi/4} \log(1+\tan\theta)\, d\theta$ ——— ①

King

$\Rightarrow I = 8\int_0^{\pi/4} \log\left(1+\frac{1-t_\theta}{1+t_\theta}\right) d\theta$

$\Rightarrow I = 8\int_0^{\pi/4} \log\left(\frac{2}{1+t_\theta}\right) d\theta$ ——— ②

① + ②

$2I = 8\int_0^{\pi/4} \log(2).d\theta$

$\Rightarrow 2I = 8.\log(2).\frac{\pi}{4}$

$\Rightarrow \boxed{I = \pi.\log(2)}$ 3.

(3) Let [.] denote the greatest integet function then the value of $\int_0^{1.5} x[x^2]dx$ is :- **[AIEEE-2011**

(1) $\frac{5}{4}$ (2) 0 (3) $\frac{3}{2}$ (4) $\frac{3}{4}$

Sol)

$$\int_0^{3/2} x.[x^2]\,dx$$

Put

$x^2 = t$

$2x\,dx = dt$

$$= \frac{1}{2}\int_0^{9/4} [t]\,dt$$

$$= \frac{1}{2}\left[\int_0^1 0.dt + \int_1^2 1.dt + \int_2^{9/4} 2.dt\right]$$

$$= \frac{1}{2}\left[0 + [x]_1^2 + 2[x]_2^{9/4}\right]$$

$$= \frac{1}{2}\left[(2-1) + 2\left(\frac{9}{4}-2\right)\right]$$

$$= \frac{1}{2}\left(1+\frac{1}{\cancel{4}\,2}\right) = \boxed{\frac{3}{4}}\ \text{Ans.}$$

4 If $g(x) = \int_0^x \cos 4t \, dt$, then $g(x+\pi)$ equals : **[AIEEE-2012]**

(1) $g(x) \cdot g(\pi)$ (2) $\dfrac{g(x)}{g(\pi)}$ (3) $g(x) + g(\pi)$ (4) $g(x) - g(\pi)$

Sol $g(x) = \int_0^x \cos(4t) \cdot dt$

lebnitz

$\Rightarrow g'(x) = \cos(4x)$

$\Rightarrow \int g'(x)\, dx = \int \cos(4x) \cdot dx$

$\Rightarrow g(x) = \dfrac{\sin(4x)}{4} + C$ $\left\{ \begin{array}{l} \text{Put } x=0 \\ \text{then } C=0 \end{array} \right\}$

$\Rightarrow \boxed{g(x) = \dfrac{\sin(4x)}{4}}$

$\Rightarrow g(\pi) = \dfrac{\sin 4\pi}{4} = 0$

find $\Rightarrow g(\pi + x)$

$= \dfrac{\sin(4(\pi+x))}{4}$

$= \dfrac{\sin(4\pi + 4x)}{4}$

$= \dfrac{\sin(4x)}{4}$

$= \boxed{g(x)} \pm 0$

$= \boxed{g(x) \pm g(\pi)}$ 1.

5 **Statement-I :** The value of the integral $\int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}}$ is equal to $\frac{\pi}{6}$. **[JEE-MAIN-2013]**

Statement-II : $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx$.

(1) Statement-I is true, Statement-II is true; Statement-II is a **correct** explanation for Statement-I.

(2) Statement-I is true, Statement-II is true; Statement-II is **not** a correct explanation for Statement-I.

(3) Statement-I is true, Statement-II is false.

(4) Statement-I is false, Statement-II is true.

Sol. (4)

$$I = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan x}} \quad \text{—} \ 1$$

Apply

$\left\{ \int_a^b f(x)dx = \int_a^b f(a+b-x)dx \text{ it is property} \right\}$

King

$$I = \int_{\pi/6}^{\pi/3} \frac{dx}{1+\sqrt{\tan\left(\frac{\pi}{2}-x\right)}}$$

$$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\tan x}\, dx}{1+\sqrt{\tan x}}$$

$$I = \int_{\pi/6}^{\pi/3} \frac{\sqrt{\tan x}\, dx}{1+\sqrt{\tan x}} \quad \text{—} \ 2$$

1 + 2

$$\Rightarrow \quad 2I = \int_{\pi/6}^{\pi/3} dx$$

$$\Rightarrow \quad I = \frac{1}{2}\left[\frac{\pi}{3} - \frac{\pi}{6}\right] = \frac{\pi}{12},$$

statement -1 is false

6

The integral $\int_0^{\pi}\sqrt{1+4\sin^2\frac{x}{2}-4\sin\frac{x}{2}}\,dx$ equals : **[JEE-MAIN-2014]**

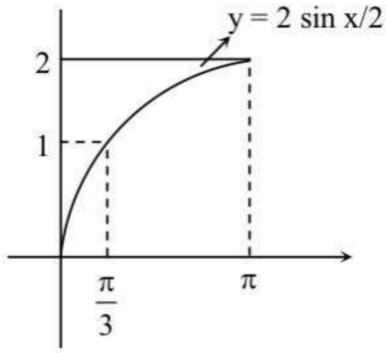
(1) $\pi - 4$ (2) $\frac{2\pi}{3}-4-4\sqrt{3}$ (3) $4\sqrt{3}-4$ (4) $4\sqrt{3}-4-\frac{\pi}{3}$

Solⁿ

$$I=\int_0^{\pi}\sqrt{1+4\sin^2\frac{x}{2}-4\sin\frac{x}{2}}\;dx$$

$$=\int_0^{\pi}\left|1-2\sin\frac{x}{2}\right|dx$$

$$=\int_0^{\pi/3}\left(1-2\sin\frac{x}{2}\right)dx+\int_{\pi/3}^{\pi}\left(2\sin\frac{x}{2}-1\right)dx$$



$$=\left(x+4\cos\frac{x}{2}\right)\Big|_0^{\pi/3}+\left(-4\cos\frac{x}{2}-x\right)\Big|_{\pi/3}^{\pi}$$

$$=-\frac{\pi}{3}+8\cdot\frac{\sqrt{3}}{2}-4$$

$$=4\sqrt{3}-4-\frac{\pi}{3}$$

7 The integral $\int_2^4 \frac{\log x^2}{\log x^2 + \log(36-12x+x^2)} dx$ is equal to : **[JEE-MAIN-2015]**

(1) 1 (2) 6 (3) 2 (4) 4

Sol:-

$$I = \int_2^4 \frac{\log(x)^2}{\log(x)^2 + \log(36-12x+x^2)} \cdot dx \quad \text{--- } (1)$$

King

$$I = \int_2^4 \frac{\log(6-x)^2}{\log(6-x)^2 + \log\left[36-12(6-x)+(6-x)^2\right]} \cdot dx$$

$$\Rightarrow \quad I = \int_2^4 \frac{\log(6-x)^2}{\log(36-12x+x^2) + \log(x)^2} \cdot dx \quad (2)$$

(1) + (2)

$$2I = 2\int_0^4 1 \cdot dx$$

$$\Rightarrow \quad I = \boxed{4} \quad \text{Ans.}$$

8. $\lim_{n\to\infty}\left(\frac{(n+1)(n+2)....3n}{n^{2n}}\right)^{1/n}$ is equal to :- **[JEE-MAIN-2016]**

(1) 3 log3 – 2 (2) $\frac{18}{e^4}$ (3) $\frac{27}{e^2}$ (4) $\frac{9}{e^2}$

Solution:

Let $L = \lim_{n\to\infty}\left(\frac{(n+1)(n+2)\ldots(n+2n)}{n^{2n}}\right)^{1/n}$

$$= \lim_{n\to\infty}\left(\left(\frac{n+1}{n}\right)\left(\frac{n+2}{n}\right)\ldots\left(\frac{n+2n}{n}\right)\right)^{1/n}$$

$$= \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\ldots\left(1+\frac{2n}{n}\right)\right]^{1/n}$$

$$L = \lim_{n\to\infty}\left(\prod_{r=1}^{2n}\left(1+\frac{r}{n}\right)\right)^{1/n}$$

$$\Rightarrow \log_e L = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{2n}\log_e\left(1+\frac{r}{n}\right) = \boxed{\int_0^2 \ln(1+x)\,dx}$$

$$\Rightarrow \log_e L = \left\{(1+x)\ln(1+x) - (1+x)\right\}\Big|_0^2$$

$$= (3\ln 3 - 3) - (0 - 1)$$

$$= 3\ln 3 - 2 = \ln 27 - \ln e^2 = \ln\left(\frac{27}{e^2}\right)$$

$$\Rightarrow \boxed{L = \frac{27}{e^2}} \longrightarrow \text{Option (3)}$$

9. The integral $\int_{\frac{\pi}{4}}^{\frac{3\pi}{4}} \frac{dx}{1+\cos x}$ is equal to :- **[JEE-MAIN-2017]**

(1) –1 (2) –2 (3) 2 (4) 4

Sol

$$I = \int_{\pi/4}^{3\pi/4} \frac{dx}{1+\cos(x)} \quad \text{—— (1)}$$

King

$$I = \int_{\pi/4}^{3\pi/4} \frac{dx}{1-\cos(x)} \quad \text{—— (2)}$$

(1) + (2)

$$2I = \int_{\pi/4}^{3\pi/4} \frac{2}{\sin^2(x)} \cdot dx$$

$$\Rightarrow I = \int_{\pi/4}^{3\pi/4} \operatorname{cosec}^2(x) \cdot dx$$

$$\Rightarrow I = \left[\cot(x)\right]_{\pi/4}^{3\pi/4}$$

$$\Rightarrow I = 2 \quad 1.$$

10. The value of $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\sin^2 x}{1+2^x} dx$ is : **[JEE-MAIN-2018]**

(1) $\frac{\pi}{2}$ (2) 4π (3) $\frac{\pi}{4}$ (4) $\frac{\pi}{8}$

Sol

$$I = \int_{-\pi/2}^{\pi/2} \frac{\sin^2 x}{1+2^x} dx \qquad ..... (i)$$

using property $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$

$$I = \int_{-\pi/2}^{\pi/2} \frac{\sin^2 x}{1+2^{-x}} dx \qquad ..... (ii)$$

adding (i) and (ii)

$$2I = \int_{-\pi/2}^{\pi/2} \sin^2 x\,dx$$

$$\Rightarrow 2I = 2.\int_0^{\pi/2} \sin^2 x\,dx$$

$$\Rightarrow 2I = 2 \times \frac{\pi}{4} \Rightarrow I = \frac{\pi}{4}$$

11. If $\int_0^{\frac{\pi}{3}} \frac{\tan\theta}{\sqrt{2k\sec\theta}} d\theta = 1 - \frac{1}{\sqrt{2}}, (k > 0)$, then the value of k is : **[JEE (Main)-2019]**

(1) 2 (2) $\frac{1}{2}$ (3) 4 (4) 1

Solution:

$$\int_0^{\pi/3} \frac{\tan\theta}{\sqrt{2k\sec\theta}} d\theta = 1 - \frac{1}{\sqrt{2}}$$

$$\Rightarrow \frac{1}{\sqrt{2}\sqrt{k}} \int_0^{\pi/3} \frac{\tan\theta}{\sqrt{\sec\theta}} d\theta = \frac{(\sqrt{2}-1)}{\sqrt{2}}$$

$$\Rightarrow \int_0^{\pi/3} \frac{\tan\theta}{\sqrt{\sec\theta}} d\theta = (\sqrt{2}-1)\sqrt{k}$$

$$\Rightarrow \int_0^{\pi/3} \frac{\sec\theta \tan\theta}{(\sec\theta)^{3/2}} d\theta = (\sqrt{2}-1)\sqrt{k}$$

$$\Rightarrow \left. \frac{(\sec\theta)^{-1/2}}{-1/2} \right|_0^{\pi/3} = (\sqrt{2}-1)\sqrt{k}$$

$$\Rightarrow \left. 2(\cos\theta)^{1/2} \right|_{\pi/3}^{0} = (\sqrt{2}-1)\sqrt{k}$$

$$\Rightarrow 2\left[1 - \frac{1}{\sqrt{2}}\right] = (\sqrt{2}-1)\sqrt{k} \Rightarrow \sqrt{2}(\sqrt{2}-1) = (\sqrt{2}-1)\sqrt{k}$$

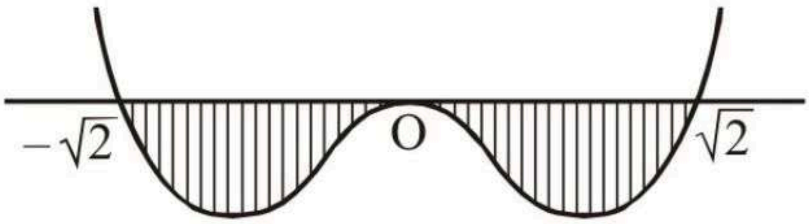
$$\Rightarrow \boxed{k=2} \longrightarrow (1) \text{ Option}$$

12

Let $I=\int_a^b (x^4-2x^2)dx$. If I is minimum then the ordered pair (a, b) is : **[JEE-MAIN 2019]**

(1) $(-\sqrt{2},0)$ (2) $(-\sqrt{2},\sqrt{2})$ (3) $(0,\sqrt{2})$ (4) $(\sqrt{2},-\sqrt{2})$

Sol. Let $f(x) = x^2(x^2 - 2)$



As long as $f(x)$ lie below the x-axis, definite integral will remain negative,

so correct value of (a, b) is $(-\sqrt{2},\sqrt{2})$ for minimum of I

13

The value of $\int_{-\pi/2}^{\pi/2} \frac{dx}{[x]+[\sin x]+4}$, where [t] denotes the greatest integer less than or equal to t, is :

[JEE-MAIN 2019]

(1) $\frac{1}{12}(7\pi+5)$ (2) $\frac{3}{10}(4\pi-3)$ (3) $\frac{1}{12}(7\pi-5)$ (4) $\frac{3}{20}(4\pi-3)$

Sol. $I = \int_{\frac{-\pi}{2}}^{\frac{\pi}{2}} \frac{dx}{[x]+[\sin x]+4}$

$$I = \int_{\frac{-\pi}{2}}^{-1} \frac{dx}{-2-1+4} + \int_{-1}^{0} \frac{dx}{-1-1+4}$$

$$+\int_{0}^{1} \frac{dx}{0+0+4} + \int_{1}^{\frac{\pi}{2}} \frac{dx}{1+0+4}$$

$$I = \int_{\frac{-\pi}{2}}^{-1} \frac{dx}{1} + \int_{-1}^{0} \frac{dx}{2} + \int_{0}^{1} \frac{dx}{4} + \int_{1}^{\frac{\pi}{2}} \frac{dx}{5}$$

$$= \left(-1+\frac{\pi}{2}\right) + \frac{1}{2}(0+1) + \frac{1}{4} + \frac{1}{5}\left(\frac{\pi}{2}-1\right)$$

$$= -1+\frac{1}{2}+\frac{1}{4}-\frac{1}{5}+\frac{\pi}{2}+\frac{\pi}{10}$$

$$= \frac{-20+10+5-4}{20} + \frac{6\pi}{10}$$

$$= \frac{-9}{20} + \frac{3\pi}{5} = \frac{3}{20}(4\pi-3)$$

14

If $\int_0^x f(t)dt = x^2 + \int_x^1 t^2 f(t)dt$, then f'(1/2) is : **[JEE-MAIN 2019]**

(1) $\frac{6}{25}$ (2) $\frac{24}{25}$ (3) $\frac{18}{25}$ (4) $\frac{4}{5}$

Sol. $\int_0^x f(t)dt = x^2 + \int_x^1 t^2 f(t)dt$ $\qquad f'\left(\frac{1}{2}\right) = ?$

Differentiate w.r.t. 'x'

$f(x) = 2x + 0 - x^2 f(x)$

$$f(x) = \frac{2x}{1+x^2} \Rightarrow f'(x) = \frac{(1+x^2)2 - 2x(2x)}{(1+x^2)^2}$$

$$f'(x) = \frac{2x^2 - 4x^2 + 2}{(1+x^2)^2}$$

$$f'\left(\frac{1}{2}\right) = \frac{2-2\left(\frac{1}{4}\right)}{\left(1+\frac{1}{4}\right)^2} = \frac{\left(\frac{3}{2}\right)}{\frac{25}{16}} = \frac{48}{50} = \frac{24}{25}$$

15. The value of the integral $\int_{-2}^{2} \frac{\sin^2 x}{\left[\frac{x}{\pi}\right]+\frac{1}{2}} dx$ (where [x] denotes the greatest integer less than or equal to x) is : **[JEE (Main)-2019]**

(1) 4 (2) 4 – sin4 (3) sin 4 (4) 0

Solution:

$$I = \int_{-2}^{2} \frac{\sin^2 x}{\left[\frac{x}{\pi}\right]+\frac{1}{2}} dx$$

Apply King,

$$I = \int_{-2}^{2} \frac{\sin^2 (-x)}{\left[\frac{-x}{\pi}\right]+\frac{1}{2}} dx$$

$[x] + [-x] = -1 \ ; \ x \notin I$

$$\boxed{\frac{1}{2}+\left[-\frac{x}{\pi}\right] = -1-\left[\frac{x}{\pi}\right]+\frac{1}{2} = -\left(\left[\frac{x}{\pi}\right]+\frac{1}{2}\right)}$$

$$\Rightarrow I = \int_{-2}^{2} \frac{\sin^2 x}{-\left(\left[\frac{x}{\pi}\right]+\frac{1}{2}\right)} dx = -I$$

$$\Rightarrow I + I = 0 \Rightarrow \boxed{I = 0} \longrightarrow \text{(option (4))}$$

16. $\lim_{n\to\infty}\left(\frac{(n+1)^{1/3}}{n^{4/3}}+\frac{(n+2)^{1/3}}{n^{4/3}}+\ldots\ldots+\frac{(2n)^{1/3}}{n^{4/3}}\right)$ is equal to : **[JEE (Main)-2019]**

(1) $\frac{4}{3}(2)^{4/3}$ (2) $\frac{3}{4}(2)^{4/3}-\frac{4}{3}$ (3) $\frac{3}{4}(2)^{4/3}-\frac{3}{4}$ (4) $\frac{4}{3}(2)^{3/4}$

Solution:

limit $L = \lim_{n\to\infty} \sum_{r=1}^{n} \frac{(n+r)^{1/3}}{n^{4/3}}$

Conversion, $\Sigma \to \int$; $r = nx$; $1 = n\,dx$

$$L = \int_0^1 \frac{(n+nx)^{1/3}}{n^{4/3}} \cdot n\,dx$$

$$= \int_0^1 (1+x)^{1/3}\,dx = \left.\frac{(1+x)^{4/3}}{4/3}\right|_0^1$$

$$= \boxed{\frac{3}{4}\left[2^{4/3}-1\right]} \longrightarrow \text{Option (3)}$$

17. The integral $\int_{\pi/6}^{\pi/3} \sec^{2/3} x \operatorname{cosec}^{4/3} x \, dx$ equal to: **[JEE (Main)-2019]**

(1) $3^{7/6} - 3^{5/6}$ (2) $3^{5/3} - 3^{1/3}$ (3) $3^{4/3} - 3^{1/3}$ (4) $3^{5/6} - 3^{2/3}$

Solution:

$$I = \int_{\pi/6}^{\pi/3} \operatorname{Sec}^{2/3} x \cdot \operatorname{Cosec}^{4/3} x \, dx$$

[Note: $\frac{2}{3} + \frac{4}{3} = 2 \Rightarrow$ mutiply & divide by $\sec^2 x$]

$$I = \int_{\pi/6}^{\pi/3} \frac{1}{\operatorname{Cos}^{2/3} x \cdot \operatorname{Sin}^{4/3} x} \cdot \frac{\operatorname{Sec}^2 x}{\operatorname{Sec}^2 x} \, dx$$

$$= \int_{\pi/6}^{\pi/3} \frac{\operatorname{Sec}^2 x}{\tan^{4/3} dx} \, dx = \int_{1/\sqrt{3}}^{\sqrt{3}} \frac{1}{t^{4/3}} dx$$

$$= \left. \frac{t^{-1/3}}{-1/3} \right|_{1/\sqrt{3}}^{\sqrt{3}} = -3\left((\sqrt{3})^{-1/3} - \left(\frac{1}{\sqrt{3}}\right)^{-1/3} \right)$$

$$= 3^{1+\frac{1}{6}} - 3^{1-\frac{1}{6}} = \boxed{3^{7/6} - 3^{5/6}}$$

Option (1)

EXERCISE (JA)

EXERCISE (JA)

1. The value of $\lim\limits_{x\to 0}\frac{1}{x^3}\int_0^x \frac{t\ell n(1+t)}{t^4+4}dt$ is **[JEE 2010, 3 (–1)]**

(A) 0 (B) $\frac{1}{12}$ (C) $\frac{1}{24}$ (D) $\frac{1}{64}$

Solution:

$$L = \lim_{x\to 0}\frac{\int_0^x \frac{t\ln(1+t)}{t^4+4}dt}{x^3} \qquad \left(\frac{\to 0}{\to 0}\text{ form}\right)$$

By L-Hospital's Rule,

$$L = \lim_{x\to 0}\frac{\frac{x\ln(1+x)}{x^2+4}}{3x^2} \qquad [\text{By Leibnitz's Rule}]$$

$$= \lim_{x\to 0}\frac{\ln(1+x)}{3x}\cdot\frac{1}{x^2+4} = \frac{1}{3}\times\frac{1}{4} = \boxed{\frac{1}{12}}$$

B

2. The value(s) of $\int_0^1 \frac{x^4(1-x)^4}{1+x^2}dx$ is (are) **[JEE 2010, 3]**

(A) $\frac{22}{7}-\pi$ (B) $\frac{2}{105}$ (C) 0 (D) $\frac{71}{15}-\frac{3\pi}{2}$

Solution:

$$I = \int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx$$

Degree of Numerator > Degree of Denominator

$$x^4(1-x)^4 = (x-x^2)^4 = x^4 - 4x^3\cdot x^2 + 6x^2x^4 - 4x\cdot x^6 + x^8$$

$$= x^4 - 4x^5 + 6x^6 - 4x^7 + x^8$$

$$x^2+1\,\big)\,x^8 - 4x^7 + 6x^6 - 4x^5 + x^4\,\big(\,x^6 - 4x^5 + 5x^4 - 4x^2 + 4$$

$$\begin{array}{l}
\underline{\quad x^8 \quad + x^6} \\
\quad (-) \quad\ (-) \\
\quad -4x^7 + 5x^6 - 4x^5 + x^4 \\
\quad -4x^7 \qquad\ \ -4x^5 \\
\quad\ \underline{(+) \qquad\quad\ \ (+)} \\
\qquad\qquad 5x^6 + x^4 \\
\qquad\qquad \underline{5x^6 + 5x^4} \\
\qquad\qquad (-) \quad\ (-) \\
\qquad\qquad\qquad -4x^4 \\
\qquad\qquad\qquad -4x^4 - 4x^2 \\
\qquad\qquad\qquad \underline{\ \ (+) \qquad (+)} \\
\qquad\qquad\qquad\qquad 4x^2 \\
\qquad\qquad\qquad\qquad \underline{4x^2 + 4} \\
\qquad\qquad\qquad\qquad (-) \quad (-) \\
\qquad\qquad\qquad\qquad \underline{\ \ -4\ \ }
\end{array}$$

(2) Continues...

$$\therefore\ I = \int_0^1 \left((x^6 - 4x^5 + 5x^4 - 4x^2 + 4) - \frac{4}{1+x^2} \right) dx$$

$$= \frac{1}{7} - \frac{2}{3} + 1 - \frac{4}{3} + 4 - 4\tan^{-1}(1)$$

$$= \frac{3 - 14 + 21 - 28 + 84}{21} - 4 \times \frac{\pi}{4}$$

$$= \frac{66}{21} - \pi = \boxed{\frac{22}{7} - \pi}$$

(A)

3. Let f be a real-valued function defined on the interval $(-1,1)$ such that $e^{-x}f(x) = 2+\int_0^x \sqrt{t^4+1}\,dt$, for all $x\in(-1,1)$, and let f^{-1} be the inverse function of f. Then $(f^{-1})'(2)$ is equal to-

[JEE2010, 5 (–2)]

(A) 1 (B) $\frac{1}{3}$ (C) $\frac{1}{2}$ (D) $\frac{1}{e}$

Solution:

$$e^{-x}.f(x) = 2+\int_0^x \sqrt{t^4+1}\,dt \quad \text{...}(1)$$

Put $x=0$, $\quad e^{-0}.f(0) = 2+\int_0^0 \sqrt{t^4+1}\,dt \Rightarrow \boxed{f(0)=2}$

Different (1) both sides w.r.t. x,

$$e^{-x}f'(x) - e^{-x}f(x) = \sqrt{x^4+1}$$

Put $x=0$; $\quad e^{-0}f'(0) - e^{-0}f(0) = 1$

$$\Rightarrow f'(0) = 1 + f(0) = \boxed{3 = f'(0)}$$

Now, let $g(x) = f^{-1}(x)$

Thus, $(f^{-1})'(2) = \left.\frac{d}{dx}g(x)\right|_{x=2} = g'(2)$

We know that, $g(f(x)) = x \Rightarrow g'(f(x)).f'(x) = 1$

$$\Rightarrow g'(f(x)) = \frac{1}{f'(x)}$$

Put $x=0 \Rightarrow g'(f(0)) = \frac{1}{f'(0)} \Rightarrow \boxed{g'(2) = \frac{1}{3}} \rightarrow$ (B)

4 The value of $\int_{\sqrt{\ln 2}}^{\sqrt{\ln 3}} \frac{x \sin x^2}{\sin x^2 + \sin(\ln 6 - x^2)} dx$ is [JEE 2011, 3 (–1)]

(A) $\frac{1}{4}\ln\frac{3}{2}$ (B) $\frac{1}{2}\ln\frac{3}{2}$ (C) $\ln\frac{3}{2}$ (D) $\frac{1}{6}\ln\frac{3}{2}$

Sol:-

$$I = \int_{\sqrt{\ln 2}}^{\sqrt{\ln 3}} \frac{x \cdot \sin(x^2) \quad dx}{\sin(x^2) + \sin(\ln 6 - x^2)} \quad — (1)$$

Put $\left\{ \begin{array}{l} x^2 = t \\ x\,dx = \frac{dt}{2} \end{array} \right\}$

$$I = \frac{1}{2}\int_{\ln 2}^{\ln 3} \frac{\sin(t) \cdot dt}{\sin(t) + \sin(\ln 6 - t)} \quad — (1)$$

King

$$I = \frac{1}{2}\int_{\ln 2}^{\ln 3} \frac{\sin(\ln 6 - t) \cdot dt}{\sin(\ln 6 - t) + \sin(t)} \quad — (2)$$

(1) + (2)

$$2I = \frac{1}{2}\int_{\ln 2}^{\ln 3} 1 \cdot dt$$

$$2I = \frac{1}{2}\ln\left(\frac{3}{2}\right)$$

$$\Rightarrow \boxed{I = \frac{1}{4}\ln\left(\frac{3}{2}\right)}$$ Ans.

5 The value of the integral $\int_{-\pi/2}^{\pi/2}\left(x^2+\ln\frac{\pi+x}{\pi-x}\right)\cos x\,dx$ is **[JEE 2012, 3, (–1)]**

(A) 0 (B) $\frac{\pi^2}{2}-4$ (C) $\frac{\pi^2}{2}+4$ (D) $\frac{\pi^2}{2}$

Ans. (B)

Sol. $$\int_{-\pi/2}^{\pi/2} x^2\cos x\,dx + \int_{-\pi/2}^{\pi/2}\ell n\left(\frac{\pi+x}{\pi-x}\right)\cos x\,dx$$

$$= \int_{-\pi/2}^{\pi/2} x^2\cos x\,dx = 2\int_0^{\pi/2} \underset{\text{I}}{x^2}\,\underset{\text{II}}{\cos x}\,dx$$

$$= 2\left((x^2\sin x)_0^{\pi/2} - 2\int_0^{\pi/2} x\sin x\,dx\right)$$

$$= 2\left(\frac{\pi^2}{4} - 2\left(-(x\cos x)_0^{\pi/2} + \int_0^{\pi/2}\cos x\,dx\right)\right)$$

$$= 2\left(\frac{\pi^2}{4} - 2\int_0^{\pi/2}\cos x\,dx\right)$$

$$= 2\left(\frac{\pi^2}{4} - 2\right) = \frac{\pi^2}{2} - 4$$

6

For $a \in R$ (the set of all real numbers), $a \neq -1$.

$$\lim_{n\to\infty} \frac{(1^a + 2^a + ... + n^a)}{(n+1)^{a-1}[(na+1)+(na+2)+......+(na+n)]} = \frac{1}{60}$$

Then a =

(A) 5 (B) 7 (C) $\frac{-15}{2}$ (D) $\frac{-17}{2}$

Ans. (B)

Sol. $$L = \lim_{n\to\infty} \frac{1^a + 2^a + ... + n^a}{(n+1)^{a-1}\left[\underbrace{na+na+na+.......+na}_{n\,times}+1+2+3+.....+n\right]} = \lim_{n\to\infty} \frac{\sum_{r=1}^{n} r^a}{(n+1)^{a-1}\left[n^2a + \frac{n(n+1)}{2}\right]}$$

$$= \lim_{n\to\infty} \frac{\left(\frac{1}{n}\sum_{r=1}^{n}\frac{r^a}{n^a}\right)n^{a+1}}{(n+1)^{a-1}\left[n^2a + \frac{n(n+1)}{2}\right]} = \lim_{n\to\infty} \frac{\left(\frac{1}{n}\sum_{r=1}^{n}\frac{r^a}{n^a}\right)}{\left(\frac{n+1}{n}\right)^{a-1}\left[\frac{n^2a + \frac{n(n+1)}{2}}{n^2}\right]}$$

$$= \frac{\int_0^1 x^a dx}{\left(a+\frac{1}{2}\right)} = \frac{1}{60} \Rightarrow \frac{2}{(a+1)(2a+1)} = \frac{1}{60} \quad \Rightarrow 2a^2 + 3a - 119 = 0 \quad \Rightarrow a = 7 \text{ \& } -\frac{17}{2}$$

$a = -\frac{17}{2}$ will be rejected as $\int_0^1 x^{-\frac{17}{2}} dx$ is not defined.

7 Let $f : [a, b] \to [1, \infty)$ be a continuous function and let $g : R \to R$ be defined as

$$g(x) = \begin{cases} 0 & \text{if } x < a \\ \int_a^x f(t)\,dt & \text{if } a \le x \le b \\ \int_a^b f(t)\,dt & \text{if } x > b \end{cases}$$

Then

(A) g(x) is continuous but not differentiable at a

(B) g(x) is differentiable on R

(C) g(x) is continuous but not differentiable at b

(D) g(x) is continuous and differentiable at either a or b but not both.

Ans. (A,C)

Given that $f : [a, b] \to [1, \infty)$

$$g(x) = \begin{cases} 0 & x < a \\ \int_a^x f(t)\,dt, & a \le x \le b \\ \int_a^b f(t)\,dt, & x > b \end{cases}$$

Now $g(a^-) = 0 = g(a^+) = g(a)$

$$g(b^-) = g(b^+) = g(b) = \int_a^b f(t)\,dt$$

$\Rightarrow$ g is continuous $\forall\ x \in R$

Now $g'(x) = \begin{cases} 0 & : x < a \\ f(x) & : a < x < b \\ 0 & : x > b \end{cases}$

$g'(a^-) = 0$ but $g'(a^+) = f(a) \ge 1$

$\Rightarrow$ g is non differentiable at x = a

and $g'(b^+) = 0$ but $g'(b^-) = f(b) \ge 1$

$\Rightarrow$ g is non differentiable at x = b

8 The value of $\int_0^1 4x^3\left\{\frac{d^2}{dx^2}\left(1-x^2\right)^5\right\}dx$ is **[JEE(Advanced)-2014, 3]**

Sol:

using integration by part

$$\int_0^1 4x^3\left(\left(1-x^2\right)^5\right)'' dx$$

$$=4x^3\left(\left(1-x^2\right)^5\right)'\Big]_0^1-\int_0^1 12x^2\left(\left(1-x^2\right)^5\right)' dx$$

using integration by part

$$=-12\left[x^2\left(\left(1-x^2\right)^5\right)\Big|_0^1-\int_0^1 2x\left(1-x^2\right)^5 dx\right]$$

$$=12.2.\int_0^1 x(1-x^2)^5 dx$$

Let $1-x^2=t \Rightarrow xdx=-\frac{dt}{2}$

$$=24\int_1^0 t^5\left(-\frac{dt}{2}\right)$$

$$=12\int_0^1 t^5 dt=2$$

9 Let $f : [0, 2] \to \mathbb{R}$ be a function which is continuous on $[0, 2]$ and is differentiable on $(0, 2)$ with $f(0) = 1$. Let $F(x) = \int_0^{x^2} f(\sqrt{t})dt$ for $x \in [0, 2]$. If $F'(x) = f'(x)$ for all $x \in (0, 2)$, then $F(2)$ equals -

(A) $e^2 - 1$ (B) $e^4 - 1$ (C) $e - 1$ (D) e^4

[JEE(Advanced)-2014, 3(–1)]

Sol. Ans. (B)

$$f(x) = \int_0^{x^2} f(\sqrt{t})dt$$

$$f'(x) = 2x\ f(x) \qquad \because x \in [0, 2]$$

$$\Rightarrow f'(x) = 2x\ f(x) \qquad \because F'(x) = f'(x)$$

$$\Rightarrow \frac{f'(x)}{f(x)} = 2x \Rightarrow \ell n(f(x)) = x^2 + c$$

$$c = 0 \quad (\because f(0) = 1)$$

$$\Rightarrow f(x) = e^{x^2}$$

$$f(x) = \int_0^{x^2} f(\sqrt{t})dt = \int_0^{x^2} e^t.dt$$

$$\therefore f(2) = \int_0^{x^2} e^t\,dt = e^4 - 1.$$

Paragraph For Questions 10 and 11

Given that for each $a \in (0,1)$, $\lim\limits_{h\to 0^+} \int_h^{1-h} t^{-a}(1-t)^{a-1}\,dt$ exists. Let this limit be g(a). In addition, it is given that the function g(a) is differentiable on (0,1).

10 The value of $g\left(\frac{1}{2}\right)$ is - **[JEE(Advanced)-2014, 3(–1)]**

(A) π (B) 2π (C) $\frac{\pi}{2}$ (D) $\frac{\pi}{4}$

Sol. Ans. (A)

$$g\left(\frac{1}{2}\right) = \lim_{h\to 0^+}\int_h^{1-h}\frac{dt}{\sqrt{t(1-t)}}$$

$$= \lim_{h\to 0^+}\int_h^{1-h}\frac{dt}{\sqrt{\frac{1}{4}-(t-\frac{1}{2})^2}}$$

$$= \sin^{-1}\left(\frac{t-\frac{1}{2}}{\frac{1}{2}}\right)\Bigg|_0^1 = \sin^{-1}(1)-\sin^{-1}(-1) = \pi$$

11 The value of $g'\left(\frac{1}{2}\right)$ is-

(A) $\frac{\pi}{2}$ (B) π (C) $-\frac{\pi}{2}$ (D) 0

Sol. Ans. (D)

Given,

$$g(a) = \lim_{h\to 0^+}\int_h^{1-h} t^{-a}(1-t)^{a-1}dt$$

$$g^1(a) = \lim_{h\to 0^+}\int_h^{1-h} t^{-a}(1-t)^{a-1}(-\ln t + \ln(1-t))dt$$

$$g'\left(\frac{1}{2}\right) = \lim_{h\to 0^+}\int_h^{1-h}\frac{\ell n\left(\frac{1-t}{t}\right)dt}{\sqrt{t(1-t)}} \quad ....(1)$$

$$g^1\left(\frac{1}{2}\right)\lim_{h\to 0^+}\int_h^{1-h}\frac{\ln\left(\frac{1-(1-t)}{1-t}\right)}{\sqrt{(1-t)t}}dt \quad ....(ii) \quad \left(\text{Apply } \int_a^b f(x)dx = \int_a^b f(a+b-x)dx\right)$$

$$\Rightarrow 2g^1\left(\frac{1}{2}\right) = \lim_{h\to 0^+}\int_h^{1-h} 0\,dt \qquad \Rightarrow g^1\left(\frac{1}{2}\right) = 0$$

(12) Let $f : \mathbb{R} \to \mathbb{R}$ be a function defined by $f(x) = \begin{cases} [x] & , \quad x \le 2 \\ 0 & , \quad x > 2 \end{cases}$,

where [x] is the greatest integer less than or equal to x. If $I = \int_{-1}^{2} \frac{x f(x^2)}{2 + f(x+1)} dx$, then the value of (4I – 1) is **[JEE 2015, 4M, –0M]**

Sol. Given $f(x) = \begin{bmatrix} [x] & x \le 2 \\ 0 & x > 2 \end{bmatrix}$

where [x] denotes greatest integer function.

Now $I = \int_{-1}^{2} \frac{x f(x^2)}{2 + f(x+1)} dx$

$$I = \int_{-1}^{0} \frac{x f(x^2)}{2 + f(x+1)} dx + \int_{0}^{1} \frac{x f(x^2)}{2 + f(x+1)} dx + \int_{1}^{\sqrt{2}} \frac{x f(x^2)}{2 + f(x+1)} dx + \int_{\sqrt{2}}^{\sqrt{3}} \frac{x f(x^2)}{2 + f(x+1)} dx + \int_{\sqrt{3}}^{2} \frac{x f(x^2)}{2 + f(x+1)} dx$$

$\therefore \quad I = I_1 + I_2 + I_3 + I_4 + I_5$

Clearly I_1, I_2, I_4 & I_5 are zero using definition of $f(x)$

$$\therefore \quad I = I_3 = \int_{1}^{\sqrt{2}} \frac{x f(x^2)}{2 + f(x+1)} dx = \int_{1}^{\sqrt{2}} \frac{x.1}{2+0} dx = \left.\frac{x^2}{4}\right|_{1}^{\sqrt{2}} = \frac{2}{4} - \frac{1}{4} = \frac{1}{4}$$

$\therefore \quad 4I - 1 = 0$

13 If $\alpha = \int_0^1 \left(e^{9x+3\tan^{-1}x}\right)\left(\frac{12+9x^2}{1+x^2}\right)dx$, where $\tan^{-1}x$ takes only principal values, then the value of $\left(\log_e|1+\alpha| - \frac{3\pi}{4}\right)$ is **[JEE 2015, 4M, –0M]**

Ans. 9

Sol. $$\alpha = \int_0^1 e^{\left(9x+3\tan^{-1}x\right)}\left(9+\frac{3}{1+x^2}\right)dx$$

$$\alpha = \left(e^{9x+3\tan^{-1}x}\right)_0^1$$

$$= e^{9+\frac{3\pi}{4}} - 1$$

$$\log|\alpha + 1| = 9 + \frac{3\pi}{4} \Rightarrow \text{Ans.} = 9$$

14 Let $f:\mathbb{R}\to\mathbb{R}$ be a continuous odd function, which vanishes exactly at one point and $f(1)=\frac{1}{2}$. Suppose that $F(x)=\int_{-1}^{x} f(t)\,dt$ for all $x\in[-1,2]$ and $G(x)=\int_{-1}^{x} t\left|f(f(t))\right|dt$ for all $x\in[-1,2]$. If $\lim_{x\to 1}\frac{F(x)}{G(x)}=\frac{1}{14}$, then the value of $f\left(\frac{1}{2}\right)$ is **[JEE 2015, 4M, –0M]**

Sol. $$\lim_{x\to 1}\frac{F(x)}{G(x)}=\lim_{x\to 1}\frac{F'(x)}{G'(x)}=\lim_{x\to 1}\frac{f(x)}{x\left|f(f(x))\right|}=\frac{1}{14}$$

$$\Rightarrow \frac{\frac{1}{2}}{\left|f\left(\frac{1}{2}\right)\right|}=\frac{1}{14}\Rightarrow\left|f\left(\frac{1}{2}\right)\right|=7$$

$$f\left(\frac{1}{2}\right)=7$$

$f\left(\frac{1}{2}\right)\neq -7$ as $f(x)$ vanishes exactly at one point.

15 Let $f(x) = 7\tan^8 x + 7\tan^6 x - 3\tan^4 x - 3\tan^2 x$ for all $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$. Then the correct expression(s) is(are) **[JEE 2015, 4M, –0M]**

(A) $\int_0^{\pi/4} x f(x)dx = \frac{1}{12}$

(B) $\int_0^{\pi/4} f(x)dx = 0$

(C) $\int_0^{\pi/4} x f(x)dx = \frac{1}{6}$

(D) $\int_0^{\pi/4} f(x)dx = 1$

Ans. (A,B)

Sol. Given $f(x) = \left(7\tan^6 x - 3\tan^2 x\right)\sec^2 x$

$$\therefore \int_0^{\pi/4} \underset{\mathrm{I}}{x} \underbrace{\left(7\tan^6 x - 3\tan^2 x\right)\sec^2 x}_{\mathrm{II}} dx$$

Using I.B.P.

$$= x.\left(\tan^7 x - \tan^3 x\right)_0^{\pi/4} - \int_0^{\pi/4} \left(\tan^7 x - \tan^3 x\right)dx$$

dx

$$= -\int_0^{\pi/4} \tan^3 x\left(\tan^2 x - 1\right)\sec^2 x dx$$

Put tanx = t

$$= \int_0^1 \left(t^3 - t^5\right)dt = \frac{1}{4} - \frac{1}{6} = \frac{3-2}{12} = \frac{1}{12}$$

Also, $\int_0^{\pi/4} \left(7\tan^6 x - 3\tan^2 x\right)\sec^2 x dx = \tan^7 x - \tan^3 x\Big|_0^{\pi/4} = 0$

16 The value of $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{x^2 \cos x}{1+e^x} dx$ is equal to **[JEE(Advanced)-2016, 3(–1)]**

(A) $\frac{\pi^2}{4} - 2$ (B) $\frac{\pi^2}{4} + 2$ (C) $\pi^2 - e^{\frac{\pi}{2}}$ (D) $\pi^2 + e^{\frac{\pi}{2}}$

Ans. (A)

Sol. Let $I = \int_{-\pi/2}^{\pi/2} \frac{x^2 \cos x}{1+e^x} dx = \int_0^{\pi/2} \left(\frac{1}{1+e^x} + \frac{1}{1+e^{-x}}\right) x^2 \cos x \, dx$

$$= \int_0^{\pi/2} \underset{\text{(I)}}{x^2} \, \underset{\text{(II)}}{\cos x} \, dx = \left(x^2 \sin x\right)_0^{\pi/2} - 2\int_0^{\pi/2} \underset{\text{(I)}}{x} . \underset{\text{(II)}}{\sin x} \, dx$$

$$= \frac{\pi^2}{4} - 2\left[-\left(x.\cos x\right)_0^{\pi/2} + \int_0^{\pi/2} 1.\cos x\right] = \frac{\pi^2}{4} - 2[0 + 1] = \left(\frac{\pi^2}{4} - 2\right)$$

17 Let $f : R \to R$ be a differentiable function such that $f(0) = 0$, $f\left(\frac{\pi}{2}\right) = 3$ and $f'(0) = 1$. If

$$g(x) = \int_x^{\frac{\pi}{2}} \left[f'(t)\operatorname{cosec} t - \cot t \operatorname{cosec} t\, f(t)\right] dt$$

for $x \in \left(0, \frac{\pi}{2}\right]$, then $\lim_{x \to 0} g(x) =$ **[JEE(Advanced)-2017, 3]**

Sol. $g(x) = \int_x^{\pi/2} (f'(t)\operatorname{cosec} t - f(t)\operatorname{cosec} t \cot t)dt$

$$= \int_x^{\pi/2} (f(t)\operatorname{cosec} t)' dt \quad = \left[f(t) \cdot \operatorname{cosec}(t)\right]_x^{\pi/2}$$

$$= f\left(\frac{\pi}{2}\right)\operatorname{cosec}\left(\frac{\pi}{2}\right) - \frac{f(x)}{\sin x} = 3 - \frac{f(x)}{\sin x}$$

$\therefore \quad \lim_{x\to 0} g(x) = 3 - \lim_{x\to 0}\frac{f(x)}{\sin x}$; as $f'(0) = 1$

$= 3 - \lim_{x\to 0} \frac{f'(x)}{\cos(x)}$ L'pital

$\Rightarrow \quad \lim_{x\to 0} g(x) = 3 - 1 = 2$

18 If $g(x) = \int_{\sin x}^{\sin(2x)} \sin^{-1}(t)\, dt$, then **[JEE(Advanced)-2017, 4]**

(A) $g'\left(\frac{\pi}{2}\right) = -2\pi$ (B) $g'\left(-\frac{\pi}{2}\right) = 2\pi$ (C) $g'\left(\frac{\pi}{2}\right) = 2\pi$ (D) $g'\left(-\frac{\pi}{2}\right) = -2\pi$

Sol. $g(x) = \int\limits_{\sin x}^{\sin 2x} \sin^{-1} t\, dt \Rightarrow g'(x) = 2\sin^{-1}(\sin 2x) \times \cos 2x - \sin^{-1}(\sin x)\cos x$

$\Rightarrow g'\left(\frac{\pi}{2}\right) = 0 \ \& \ g'\left(-\frac{\pi}{2}\right) = 0$

No option matches the result

$\Rightarrow$ BONUS

(19) For each positive integer n, let

$$y_n = \frac{1}{n}\left[(n+1)(n+2)...(n+n)\right]^{1/n}$$

For $x \in \mathbb{R}$, let [x] be the greatest integer less than or equal to x. If $\lim\limits_{n\to\infty} y_n = L$, then the value of [L] is ______ **[JEE(Advanced)-2018, 3(0)]**

Sol. $$y_n = \left\{\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right).....\left(1+\frac{n}{n}\right)\right\}^{\frac{1}{n}}$$

$$y_n = \prod_{r=1}^{n}\left(1+\frac{r}{n}\right)^{1/n}$$

$$\log y_n = \frac{1}{n}\sum_{r=1}^{n}\ell n\left(1+\frac{r}{n}\right)$$

$$\Rightarrow \lim_{n\to\infty}\log y_n = \lim_{x\to\infty}\sum_{r=1}^{n}\frac{1}{n}\ell n\left(1+\frac{r}{n}\right)$$

$$\Rightarrow \log L = \int_0^1 \ell n(1+x)dx$$

$$\Rightarrow \log L = \log\frac{4}{e}$$

$$\Rightarrow L = \frac{4}{e}$$

$$\Rightarrow [L] = 1$$

20 The value of the integral

$$\int_0^{\frac{1}{2}} \frac{1+\sqrt{3}}{\left((x+1)^2(1-x)^6\right)^{\frac{1}{4}}}dx$$

is ______ . **[JEE(Advanced)-2018, 3(0)]**

Sol. $$\int_0^{\frac{1}{2}} \frac{\left(1+\sqrt{3}\right)dx}{\left[(1+x)^2(1-x)^6\right]^{1/4}}$$

$$\int_0^{\frac{1}{2}} \frac{\left(1+\sqrt{3}\right)dx}{(1+x)^2\left[\dfrac{(1-x)^6}{(1+x)^6}\right]^{1/4}}$$

Put $\dfrac{1-x}{1+x} = t \Rightarrow \dfrac{-2dx}{(1+x)^2} = dt$

$$I = \int_1^{1/3} \frac{\left(1+\sqrt{3}\right)dt}{-2t^{6/4}} = \frac{-(1+\sqrt{3})}{2} \times \left|\frac{-2}{\sqrt{t}}\right|_1^{1/3} = \left(1+\sqrt{3}\right)\left(\sqrt{3}-1\right) = 2$$

21 If $I=\frac{2}{\pi}\int_{-\pi/4}^{\pi/4}\frac{dx}{(1+e^{\sin x})(2-\cos 2x)}$ then $27I^2$ equals _____ **[JEE(Advanced)-2019, 3(0)]**

Sol. $$2I=\frac{2}{\pi}\int_{-\pi/4}^{\pi/4}\left[\frac{1}{(1+e^{\sin x})(2-\cos 2x)}+\frac{1}{(1+e^{-\sin x})(2-\cos 2x)}\right]dx \text{ (using King's Rule)}$$

$$\Rightarrow \quad I=\frac{1}{\pi}\int_{-\pi/4}^{\pi/4}\frac{dx}{2-\cos 2x}$$

$$\Rightarrow \quad I=\frac{2}{\pi}\int_{0}^{\pi/4}\frac{dx}{2-\cos 2x}=\frac{2}{\pi}\int_{0}^{\pi/4}\frac{\sec^2 dx dx}{1+3\tan^2 x}$$

$$=\frac{2}{\sqrt{3}\pi}\left[\tan^{-1}\left(\sqrt{3}\tan x\right)\right]_0^{\pi/4}=\frac{2}{3\sqrt{3}}$$

$$\Rightarrow \quad 27I^2=27\times\frac{4}{27}=4$$

22 For $a \in \mathbb{R}$, $|a| > 1$, let $\lim_{n\to\infty}\left(\dfrac{1+\sqrt[3]{2}+.....+\sqrt[3]{n}}{n^{7/3}\left(\dfrac{1}{(an+1)^2}+\dfrac{1}{(an+2)^2}+.......+\dfrac{1}{(an+n)^2}\right)}\right)=54$. Then the possible value(s) of a is/are : **[JEE(Advanced)-2019, 4(–1)]**

(1) 8 (2) –9 (3) –6 (4) 7

Sol. $$\lim_{n\to\infty}\frac{n^{1/3}\left(\sum_{r=1}^{n}\left(\frac{r}{n}\right)^{1/3}\right)}{n^{7/3}\left(\sum_{r=1}^{n}\frac{1}{(an+r)^2}\right)}=54 \Rightarrow \lim_{n\to\infty}\left(\frac{\frac{1}{n}\sum_{r=1}^{n}\left(\frac{r}{n}\right)^{1/3}}{\frac{1}{n}\sum_{r=1}^{n}\frac{1}{(a+r/n)^2}}\right)=54 \Rightarrow \frac{\int_0^1 x^{1/3}dx}{\int_0^1\frac{1}{(a+x)^2}dx}=54 \Rightarrow \frac{\frac{3}{4}}{\frac{1}{a(a+1)}}=54$$

$$\Rightarrow \quad a(a+1)=72 \quad \Rightarrow \quad a^2+a-72=0 \quad \Rightarrow \quad a=-9,\ 8$$

23 The value of the integral $\int_0^{\pi/2} \frac{3\sqrt{\cos\theta}}{\left(\sqrt{\cos\theta}+\sqrt{\sin\theta}\right)^5} d\theta$ equals **[JEE(Advanced)-2019, 3(0)]**

Sol. $$I = \int_0^{\pi/2} \frac{3\sqrt{\cos\theta}}{\left(\sqrt{\cos\theta}+\sqrt{\sin\theta}\right)^5} d\theta$$

$$= \int_0^{\pi/2} \frac{3\sqrt{\sin\theta}}{\left(\sqrt{\cos\theta}+\sqrt{\sin\theta}\right)} d\theta$$

$$2I = \int_0^{\pi/2} \frac{3d\theta}{\left(\sqrt{\cos\theta}+\sqrt{\sin\theta}\right)^4}$$

$$= 3\int_0^{\pi/2} \frac{\sec^2\theta d\theta}{\left(1+\sqrt{\tan\theta}\right)^4}$$

Let $1+\sqrt{\tan\theta} = t$

$$\frac{\sec^2\theta}{2\sqrt{\tan\theta}} d\theta = dt$$

$$\sec^2\theta d\theta = 2(t-1)dt$$

$$= 3\int_1^{\infty} \frac{2(t-1)dt}{t^4}$$

$$= 6\int_1^{\infty} \left(t^{-3}-t^{-4}\right)dt$$

$$2I = 6\left(\frac{t^{-2}}{-2}-\frac{t^{-3}}{-3}\right)_1^{\infty} = 6\left[0-0-\left\{-\frac{1}{2}+\frac{1}{3}\right\}\right]$$

I = 0.50