Allen Career Institute

Kota

Matrix (Solutions)
(for Leader)

Do yourself -1 :

1 Find 2×3 matrix $[a_{ij}]_{2\times 3}$, where $a_{ij} = i + 2j$

2. Find the minimum number of zeroes in a triangular matrix of order 4.

3. Find minimum number of zeros in a diagonal matrix of order 6.

4. If $\begin{bmatrix} 2x+y & 2 & x-2y \\ a-b & 2a+b & -3 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 4 \\ 4 & -1 & -3 \end{bmatrix}$, then find the values of x,y,a and b.

(1) $a_{ij} = \hat{i} + 2j$

$a_{11} = 1+2 = 3$, $a_{12} = 1+4 = 5$

$a_{13} = 1+6 = 7$, $a_{21} = 2+2 = 4$

$a_{22} = 2+4 = 6$, $a_{23} = 2+6 = 8$

Required matrix is

$$\begin{bmatrix} 3 & 5 & 7 \\ 4 & 6 & 8 \end{bmatrix}$$

(2) minimum no. of zeros

$$= \frac{n(n-1)}{2}$$

$$= \frac{4\times 3}{2} = 6$$

(3) minimum no. of zeros

$= n(n-1)$

$= 6 \times 5 = 30$

DYS#1

4. If $\begin{bmatrix} 2x+y & 2 & x-2y \\ a-b & 2a+b & -3 \end{bmatrix} = \begin{bmatrix} 3 & 2 & 4 \\ 4 & -1 & -3 \end{bmatrix}$, then find the values of x,y,a and b.

$\Rightarrow 2x + y = 3$ ---- (1)

$x - 2y = 4$ ---- (2)

$a - b = 4$ ---- (3)

$2a + b = -1$ ---- (4)

using (1) & (2) $x = 2, y = -1$

using (3) & (4) $a = 1, b = -3$

Ans (C

Do yourself-2 :

1. If $A=\begin{bmatrix}2 & 3 & 9\\ 8 & -2 & 5\end{bmatrix}$ and $B=\begin{bmatrix}-5 & -7 & 2\\ 6 & 4 & 8\end{bmatrix}$, then find a matrix C such that $A-B+C=\mathbf{O}$ and also find the order of the matrix C.

2. If $A=\begin{bmatrix}8 & 9\\ 7/2 & 8\\ 1 & -1\end{bmatrix}$, then find the additive inverse of A and show that additive inverse of additive inverse will be the matrix itself.

(1) $A-B+C=0$

$\Rightarrow C=B-A$

$=\begin{bmatrix}-5-2 & -7-3 & 2-9\\ 6-8 & 4+2 & 8-5\end{bmatrix}$

$=\begin{bmatrix}-7 & -10 & -7\\ -2 & 6 & 3\end{bmatrix}$

order of matrix C = 2×3

2. If $A = \begin{bmatrix} 8 & 9 \\ 7/2 & 8 \\ 1 & -1 \end{bmatrix}$, then find the additive inverse of A and show that additive inverse of additive inverse will be the matrix itself.

D4S#2

$A + B = O$

then B is called additive inverse

Additive inverse of A

$= -A = \begin{bmatrix} -8 & -9 \\ -\frac{7}{2} & -8 \\ -1 & 1 \end{bmatrix}$

Additive inverse of $(-A)$

$= -(-A) = A$

Do yourself - 3 :

1. If $A=\begin{bmatrix}2 & 9\\ -4 & 5\end{bmatrix}$, $B=\begin{bmatrix}1 & 5\\ 6 & 8\end{bmatrix}$ and $C=\begin{bmatrix}9 & -7\\ -2 & 4\end{bmatrix}$, then show that $A(B+C) = AB + AC$.

2. If $A=\begin{bmatrix}2 & -1\\ -1 & 2\end{bmatrix}$ and $B=\begin{bmatrix}1 & 4\\ -1 & 1\end{bmatrix}$, then prove that $(A-B)^2 \neq A^2 - 2AB + B^2$.

3. Find the value of x : $2\begin{bmatrix}3 & 1 & -2\\ -1 & -3 & 4\end{bmatrix} + x\begin{bmatrix}1 & 0 & -1\\ 3 & 4 & 5\end{bmatrix} = \begin{bmatrix}4 & 2 & -2\\ -8 & -14 & -2\end{bmatrix}$

(i) $B+C = \begin{bmatrix}10 & -2\\ 4 & 12\end{bmatrix}$

$$A(B+C) = \begin{bmatrix}2 & 9\\ -4 & 5\end{bmatrix}\begin{bmatrix}10 & -2\\ 4 & 12\end{bmatrix}$$

$$= \begin{bmatrix}56 & 104\\ -20 & 68\end{bmatrix} \text{ ---- } (1)$$

$$AB = \begin{bmatrix}2 & 9\\ -4 & 5\end{bmatrix}\begin{bmatrix}1 & 5\\ 6 & 8\end{bmatrix} = \begin{bmatrix}56 & 82\\ 26 & 20\end{bmatrix}$$

$$AC = \begin{bmatrix}2 & 9\\ -4 & 5\end{bmatrix}\begin{bmatrix}9 & -7\\ -2 & 4\end{bmatrix} = \begin{bmatrix}0 & 22\\ -46 & 48\end{bmatrix}$$

$$AB+AC = \begin{bmatrix}56 & 104\\ -20 & 68\end{bmatrix} = A(B+C)$$

$\{$From (1)$\}$

2. If $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 4 \\ -1 & 1 \end{bmatrix}$, then prove that $(A - B)^2 \neq A^2 - 2AB + B^2$.

D4S#3

$$A - B = \begin{bmatrix} 1 & -5 \\ 0 & 1 \end{bmatrix}$$

$$(A-B)^2 = (A-B)(A-B)$$

$$= \begin{bmatrix} 1 & -5 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -5 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & -10 \\ 0 & 1 \end{bmatrix} \quad \ldots\text{—}\ 1$$

$$A^2 = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}$$

$$AB = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 7 \\ -3 & -2 \end{bmatrix}$$

$$\Rightarrow -2AB = \begin{bmatrix} -6 & -14 \\ 6 & 4 \end{bmatrix}$$

$$B^2 = \begin{bmatrix} 1 & 4 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 1 & 4 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} -3 & 8 \\ -2 & -3 \end{bmatrix}$$

DYS#3

Now

$A^2 - 2AB + B^2$

$= \begin{bmatrix} -4 & -10 \\ 0 & 6 \end{bmatrix}$

$\neq (A-B)^2$ {from ①}

3. Find the value of x : $2\begin{bmatrix} 3 & 1 & -2 \\ -1 & -3 & 4 \end{bmatrix} + x\begin{bmatrix} 1 & 0 & -1 \\ 3 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 4 & 2 & -2 \\ -8 & -14 & -2 \end{bmatrix}$

equating a_{11} from both the sides

$\Rightarrow 6 + x = 4 \Rightarrow \boxed{x = -2}$

Do yourself -4 :

1 If $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$, then prove that $A^n = \begin{bmatrix} \cos n\alpha & \sin n\alpha \\ -\sin n\alpha & \cos n\alpha \end{bmatrix}$ where n is positive integer.

2. If $A = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$, where $i = \sqrt{-1}$ and $x \in N$, then A^{4x} equals -

(A) $\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$ (B) $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ (C) $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (D) $\begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$

(i) Suppose

$$P(n): A^n = \begin{bmatrix} \cos n\alpha & \sin n\alpha \\ -\sin n\alpha & \cos n\alpha \end{bmatrix}$$

When $n = 1$

$$A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \quad \text{(given)}$$

$\Rightarrow$ P(1) is true ...—— (1)

Suppose P(K) is true for some $K \in N$

DYS#4

$$\Rightarrow A^K = \begin{bmatrix} \cos K\alpha & \sin K\alpha \\ -\sin K\alpha & \cos K\alpha \end{bmatrix}$$

$$\Rightarrow A^K A = \begin{bmatrix} \cos K\alpha & \sin K\alpha \\ -\sin K\alpha & \cos K\alpha \end{bmatrix} \times \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos(K+1)\alpha & \sin(K+1)\alpha \\ -\sin(K+1)\alpha & \cos(K+1)\alpha \end{bmatrix}$$

$$\Rightarrow A^{K+1} = \begin{bmatrix} \cos(K+1)\alpha & \sin(K+1)\alpha \\ -\sin(K+1)\alpha & \cos(K+1)\alpha \end{bmatrix}$$

$\Rightarrow$ $P(K+1)$ is true when $P(K)$ is true ---- (2)

From ① & ② and DYS#4
using P.M.I

$$A^n = \begin{bmatrix} \cos n\alpha & \sin n\alpha \\ -\sin n\alpha & \cos n\alpha \end{bmatrix}$$

for all $n \in N$.

(ii) $A = \begin{bmatrix} i & 0 \\ 0 & i \end{bmatrix}$

$$A^4 = \begin{bmatrix} i^4 & 0 \\ 0 & i^4 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$\Rightarrow A^{4x} = I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

for all $x \in N$.

Ans. (C)

Do yourself - 5 :

1 The matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ -1 & -2 & -3 \end{bmatrix}$ is

(A) idempotent matrix (B) involutary matrix

(C) nilpotent matrix (D) periodic matrix

2. If $A = \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix}$ and A^2 is the identity matrix, then find the value of x

(1) $A^2 = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ -1 & -2 & -3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \\ -1 & -2 & -3 \end{bmatrix}$

$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\Rightarrow$ A is a nilpotent matrix

Ans. C

(2) $A^2 = \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Equate $a_{21} \Rightarrow x = 0$

$a_{11} \Rightarrow x^2 + 1 = 1 \Rightarrow x^2 = 0$

Ans. $x = 0$

Do yourself - 6 :

1. If $A = \begin{bmatrix} 4 & 2 & -5 \\ 1 & 0 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 6 & -7 & 0 \\ -1 & 2 & 5 \\ 1 & 0 & 3 \end{bmatrix}$, then show that $(AB)^T = B^T.A^T$.

2. If $A = \begin{bmatrix} 2 & 3 & -4 \\ -1 & 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 \\ 3 & 4 \\ -5 & -6 \end{bmatrix}$, then find $A + B^T$.

3. If $A = \begin{bmatrix} 9 & -3 & 6 \\ 8 & ½ & 7 \\ -1 & 0 & 0 \end{bmatrix}$, then, show that $(A^T)^T = A$.

4. Show that the matrix $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$ is an orthogonal matrix.

(1) $AB = \begin{bmatrix} 4 & 2 & -5 \\ 1 & 0 & 3 \end{bmatrix} \begin{bmatrix} 6 & -7 & 0 \\ -1 & 2 & 5 \\ 1 & 0 & 3 \end{bmatrix}$

$$= \begin{bmatrix} 17 & -24 & -5 \\ 9 & -7 & 9 \end{bmatrix}$$

$$(AB)^T = \begin{bmatrix} 17 & 9 \\ -24 & -7 \\ -5 & 9 \end{bmatrix} \quad \text{---} \quad (1)$$

DYS#6

$R.H.S. = B^T A^T$

$$= \begin{bmatrix} 6 & -1 & 1 \\ -7 & 2 & 0 \\ 0 & 5 & 3 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 2 & 0 \\ -5 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 17 & 9 \\ -24 & -7 \\ -5 & 9 \end{bmatrix} = L.H.S.$$

$\{\text{from } ①\}$

$$\Rightarrow (AB)^T = B^T A^T$$

Verified.

DYS#6

2. If $A = \begin{bmatrix} 2 & 3 & -4 \\ -1 & 2 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 2 \\ 3 & 4 \\ -5 & -6 \end{bmatrix}$, then find $A + B^T$.

$$A + B^T = \begin{bmatrix} 2 & 3 & -4 \\ -1 & 2 & 3 \end{bmatrix} + \begin{bmatrix} -1 & 3 & -5 \\ 2 & 4 & -6 \end{bmatrix}$$

$$= \begin{bmatrix} 2-1 & 3+3 & -4-5 \\ -1+2 & 2+4 & 3-6 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 6 & -9 \\ 1 & 6 & -3 \end{bmatrix}$$

DYS#6

3. If $A = \begin{bmatrix} 9 & -3 & 6 \\ 8 & \frac{1}{2} & 7 \\ -1 & 0 & 0 \end{bmatrix}$, then, show that $(A^T)^T = A$.

$$A^T = \begin{bmatrix} 9 & -3 & 6 \\ 8 & \frac{1}{2} & 7 \\ -1 & 0 & 0 \end{bmatrix}^T = \begin{bmatrix} 9 & 8 & -1 \\ -3 & \frac{1}{2} & 0 \\ 6 & 7 & 0 \end{bmatrix}$$

Now $(A^T)^T = \begin{bmatrix} 9 & 8 & -1 \\ -3 & \frac{1}{2} & 0 \\ 6 & 7 & 0 \end{bmatrix}^T$

$$= \begin{bmatrix} 9 & -3 & 6 \\ 8 & \frac{1}{2} & 7 \\ -1 & 0 & 0 \end{bmatrix} = A$$

Verified.

DYS #6

4. Show that the matrix $A = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$ is an orthogonal matrix.

$$AA^T = \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$$A^TA = \begin{bmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\alpha & \sin\alpha \\ -\sin\alpha & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

$\Rightarrow AA^T = A^TA = I$

$\Rightarrow$ A is an orthogonal matrix

Do yourself - 7 :

1. If $A = \begin{bmatrix} -2 & -1 & 1 \\ -1 & 7 & 4 \\ 1 & -x & -3 \end{bmatrix}$ be symmetric matrix then find the value of x.

2. Express matrix $A = \begin{bmatrix} 2 & 5 & 7 \\ 9 & -7 & 2 \\ 1 & -1 & 0 \end{bmatrix}$ as a sum of a symmetric and a skew symmetric matrix.

(i) $A = A^T$

$$\Rightarrow \begin{bmatrix} -2 & -1 & 1 \\ -1 & 7 & 4 \\ 1 & -x & -3 \end{bmatrix} = \begin{bmatrix} -2 & -1 & 1 \\ -1 & 7 & -x \\ 1 & 4 & -3 \end{bmatrix}$$

equate a_{32}

$$\Rightarrow -x = 4 \Rightarrow x = -4$$

(2) $A = \begin{bmatrix} 2 & 5 & 7 \\ 9 & -7 & 2 \\ 1 & -1 & 0 \end{bmatrix}$

$$\Rightarrow A^T = \begin{bmatrix} 2 & 9 & 1 \\ 5 & -7 & -1 \\ 7 & 2 & 0 \end{bmatrix}$$

DYS#7

$$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$$

$$= \begin{bmatrix} 2 & 7 & 4 \\ 7 & -7 & \frac{1}{2} \\ 4 & \frac{1}{2} & 0 \end{bmatrix} + \begin{bmatrix} 0 & -2 & 3 \\ 2 & 0 & \frac{3}{2} \\ -3 & \frac{-3}{2} & 0 \end{bmatrix}$$

Symmetric matrix + Skew-Symmetric matrix

Do yourself - 8 :

1. For any 2×2 matrix, if $A(\text{Adj}A) = \begin{bmatrix} 15 & 0 \\ 0 & 15 \end{bmatrix}$, then $|A|$ is equal -

   (A) 20 (B) 625 (C) 15 (D) 0

2. Which of the following is/are incorrect ?

   (A) Adjoint of a symmetric matrix is skew symmetric matrix.

   (B) Adjoint of a diagonal matrix is a diagonal matrix.

   (C) $A(\text{Adj}A) = (\text{Adj}A)A = |A|I$

   (D) Adjoint of a unit matrix is a diagonal matrix

3. If A be a square matrix of the order 5 and B = Adj(A) then find Adj (5A).

4. If A be a square matrix of order 4 and $|A| = 3$ then find adj(adjA).

Solⁿ ① $A(\text{Adj}A) = 15\,I$

$\Rightarrow |A|\,I = 15\,I \Rightarrow |A| = 15$ Ans (C)

② (A) $A^T = A$ (given)

According to statement : $(\text{Adj}(A))^T = -\text{Adj}A$

LHS $(\text{Adj}A)^T = \text{Adj}(A^T)$ (property)

$= \text{Adj}A$

$\therefore$ Statement is wrong

(B) eg $A = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \Rightarrow \text{Adj}A = \begin{bmatrix} d_2d_3 & 0 & 0 \\ 0 & d_1d_3 & 0 \\ 0 & 0 & d_1d_2 \end{bmatrix}$

So adjoint of a diagonal matrix will also be a diagonal matrix

(C) Property of Adjoint Matrix

(D) $\text{Adj}(I) = I$ $\therefore$ (Ans A)

3. If A be a square matrix of the order 5 and B = Adj(A) then find Adj (5A).

DYS #8

$A = [a_{ij}]_{5\times 5}$

We know that

$$Adj(\lambda A) = \lambda^{n-1}(Adj A)$$

$$\Rightarrow Adj(5A) = 5^{5-1}(Adj A)$$

$$= 625 B$$

4. If A be a square matrix of order 4 and |A| = 3 then find adj(adjA).

We know that

$$adj(adj A) = |A|^{n-2} A$$

$$= 3^{4-2} A = 9A$$

Do yourself -9 :

1. If 'A' is a square matrix such that $A^2 = I$ then A^{-1} is equal to -

(A) $A + I$ (B) A (C) 0 (D) $2A$

2. If 'A' is an orthogonal matrix, then A^{-1} equals -

(A) A (B) A^T (C) A^2 (D) none of these

3. If $A = \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}$, then $(A^{-1})^3$ is equal to -

(A) $\frac{1}{27}\begin{bmatrix} 1 & -26 \\ 0 & 27 \end{bmatrix}$ (B) $\frac{1}{27}\begin{bmatrix} 1 & 26 \\ 0 & 27 \end{bmatrix}$ (C) $\frac{1}{27}\begin{bmatrix} 1 & -26 \\ 0 & -27 \end{bmatrix}$ (D) $\frac{1}{27}\begin{bmatrix} -1 & -26 \\ 0 & -27 \end{bmatrix}$

(1) $A^2 = I \Rightarrow |A| = \pm 1$

$\Rightarrow A^{-1}$ exists

Now $A^2 = I$

$\Rightarrow A^{-1} A A = A^{-1} I$

$\Rightarrow I A = A^{-1}$

$\Rightarrow A = A^{-1}$

Ans. (B)

(2) A is an orthogonal matrix

$\Rightarrow A A^T = A^T A = I$

$\Rightarrow A^{-1} = A^T$

Ans. (B)

DYS#9

3) $A = \begin{bmatrix} 3 & 2 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow A^3 = \begin{bmatrix} 27 & 26 \\ 0 & 1 \end{bmatrix}$

$\mathrm{Adj}(A^3) = \begin{bmatrix} 1 & -26 \\ 0 & 27 \end{bmatrix}$

$|A^3| = 27$

Now $(A^{-1})^3 = (A^3)^{-1}$

$= \dfrac{1}{|A^3|}(\mathrm{Adj}(A^3))$

$= \dfrac{1}{27}\begin{bmatrix} 1 & -26 \\ 0 & 27 \end{bmatrix}$

Ans. A

Do yourself -10 :

1. Determine the characteristic roots of the matrix A. Hence find the trace and determinant value of A.

Where $A = \begin{bmatrix} 8 & -6 & 2 \\ -6 & 7 & -4 \\ 2 & -4 & 3 \end{bmatrix}$ and also prove that $A^3 - 18A^2 + 45A = \mathbf{O}$.

Solⁿ

$$|A - xI| = 0 \Rightarrow \begin{vmatrix} 8-x & -6 & 2 \\ -6 & 7-x & -4 \\ 2 & -4 & 3-x \end{vmatrix} = 0$$

$$\Rightarrow (8-x)\left((7-x)(3-x) - 16\right) + 6\left(-6(3-x) + 8\right) + 2\left(24 - 2(7-x)\right) = 0$$

$$\Rightarrow (8-x)(x^2 - 10x + 5) + 6(6x - 10) + 2(2x + 10) = 0$$

$$\Rightarrow 8x^2 - 80x + 40 - x^3 + 10x^2 - 5x + 36x - 60 + 4x + 20 = 0$$

$$\Rightarrow -x^3 + 18x^2 - 45x = 0$$

$$\Rightarrow x^3 - 18x^2 + 45x = 0 \longrightarrow \text{charactenstic Equation of A}$$

$\therefore tr(A) = $ Sum of roots $= 18$

$|A| =$ Product of roots $= 0$

Also, as A will satisfy its charactenstic equation $\Rightarrow A^3 - 18A^2 + 45A = 0$

Do yourself -11 :

1. The system of equations x + 2y – 3z = 1, x – y + 4z = 0, 2x + y + z = 1 has -

(A) only two solutions (B) only one solution

(C) no solution (D) infinitely many solutions

2. The system of equations x + y + z = 8, x – y + 2z = 6, 3x + 5y – 7z = 14 has-

(A) Unique solution (B) infinite number of solutions

(C) no solution (D) none of these

(i) $x+2y-3z=1$ -- (1)

$x-y+4z=0$ -- (2)

$2x+y+z=1$ -- (3)

(3) – (1)

$\Rightarrow x-y+4z=0$

which is (2)

$\Rightarrow$ System has infinitely many Solution.

(2) $x+y+z=8$ --- (1)

$x-y+2z=6$ --- (2)

$3x+5y-7z=14$ --- (3)

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & -1 & 2 \\ 3 & 5 & -7 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 1 & -2 & 1 \\ 3 & 2 & -10 \end{vmatrix} = 18 \neq 0$$

$\Rightarrow$ unique solution.

Exercise 01

EXERCISE (O-1)

1 Let $A+2B=\begin{bmatrix}1 & 2 & 0\\ 6 & -3 & 3\\ -5 & 3 & 1\end{bmatrix}$ and $2A-B=\begin{bmatrix}2 & -1 & 5\\ 2 & -1 & 6\\ 0 & 1 & 2\end{bmatrix}$, then Tr (A) – Tr (B) has the value equal to

(A) 0 (B) 1 (C) 2 (D) none

Soln :-

$$A+2B=\begin{bmatrix}1 & 2 & 0\\ 6 & -3 & 3\\ -5 & 3 & 1\end{bmatrix} \text{ ——— (1)}$$

$$2A-B=\begin{bmatrix}2 & -1 & 5\\ 2 & -1 & 6\\ 0 & 1 & 2\end{bmatrix} \text{ ——— (2)}$$

Solving eqn (1) & (2)

$$5A=\begin{bmatrix}5 & 0 & 10\\ 10 & -5 & 15\\ -5 & 5 & 5\end{bmatrix} \Rightarrow A=\begin{bmatrix}1 & 0 & 2\\ 2 & -1 & 3\\ -1 & 1 & 1\end{bmatrix}$$

$$\& \quad B=\begin{bmatrix}0 & 1 & -1\\ 2 & -1 & 0\\ -2 & 1 & 0\end{bmatrix}$$

$$\Rightarrow \text{Tr}(A)-\text{Tr}(B)$$

$$\Rightarrow 1-(-1)=2$$

EXERCISE (O-1)

2. If $\begin{bmatrix} x & 3x-y \\ zx+z & 3y-w \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 4 & 7 \end{bmatrix}$, then

(A) $x = 3$, $y = 7$, $z = 1$, $w = 14$

(B) $x = 3$, $y = -5$, $x = -1$, $w = -4$

(C) $x = 3$, $y = 6$, $z = 2$, $w = 7$

(D) None of these

Solth:- Comparing

$x = 3$

$3x - y = 2 \Rightarrow y = 7$

$z(x+1) = 4 \Rightarrow z = 1$

$3y - w = 7 \Rightarrow w = 14$

3. The matrix $A^2 + 4A - 5I$, where I is identity matrix and $A = \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}$ equals :[JEE-MAIN Online 2013]

(A) $32\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ (B) $4\begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix}$ (C) $4\begin{bmatrix} 0 & -1 \\ 2 & 2 \end{bmatrix}$ (D) $32\begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix}$

Solⁿ

$$A = \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}$$

$$\Rightarrow A^2 = \begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 4 & -3 \end{bmatrix} = \begin{bmatrix} 9 & -4 \\ -8 & 17 \end{bmatrix}$$

$$4A = \begin{bmatrix} 4 & 8 \\ 16 & -12 \end{bmatrix} \; ; \; -5I = \begin{bmatrix} -5 & 0 \\ 0 & -5 \end{bmatrix}$$

$$\therefore \; A^2 + 4A - 5I = \begin{bmatrix} 9+4-5 & -4+8+0 \\ -8+16+0 & 17-12-5 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 4 \\ 8 & 0 \end{bmatrix} = 4\begin{bmatrix} 2 & 1 \\ 2 & 0 \end{bmatrix}$$

(Ans B)

4 If $M = \begin{bmatrix} 0 & 2 \\ 5 & 0 \end{bmatrix}$ and $N = \begin{bmatrix} 0 & 5 \\ 2 & 0 \end{bmatrix}$, then M^{2011} is -

(A) $10^{1005}M$ (B) $10^{1005}N$ (C) $10^{2010}M$ (D) $10^{2011}M$

Solⁿ:-

$$M^2 = \begin{bmatrix} 0 & 2 \\ 5 & 0 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 5 & 0 \end{bmatrix} = \begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix}$$

$$M^3 = \begin{bmatrix} 0 & 2 \\ 5 & 0 \end{bmatrix}\begin{bmatrix} 10 & 0 \\ 0 & 10 \end{bmatrix} = \begin{bmatrix} 0 & 20 \\ 50 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2\times 10 \\ 5\times 10 & 0 \end{bmatrix}$$

$$M^4 = \begin{bmatrix} 0 & 20 \\ 50 & 0 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 5 & 0 \end{bmatrix} = \begin{bmatrix} 100 & 0 \\ 0 & 100 \end{bmatrix} = \begin{bmatrix} 10^2 & 0 \\ 0 & 10^2 \end{bmatrix}$$

$$M^5 = \begin{bmatrix} 100 & 0 \\ 0 & 100 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 5 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 200 \\ 500 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 2\times 10^2 \\ 5\times 10^2 & 0 \end{bmatrix}$$

$$\Rightarrow \therefore M^{2011} = \begin{bmatrix} 0 & 2\times 10^{1005} \\ 5\times 10^{1005} & 0 \end{bmatrix}$$

$$= 10^{1005} \cdot M$$

5 If $A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$ and $A^2 - kA - I_2 = 0$, then value of k is-

(A) 4 (B) 2 (C) 1 (D) –4

Sol^n

$$A^2 = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} = \begin{bmatrix} 5 & 8 \\ 8 & 13 \end{bmatrix}$$

$$-KA = \begin{bmatrix} -K & -2K \\ -2K & -3K \end{bmatrix}$$

$$-I_2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$$

$$\Rightarrow A^2 - KA - I_2 = \begin{bmatrix} 5-K-1 & 8-2k \\ 8-2K & 13-3k-1 \end{bmatrix} = 0$$

$\Rightarrow$ $K = 4$ only

(Ans. A)

6 Let three matrices are $A = \begin{bmatrix} 2 & 1 \\ 4 & 1 \end{bmatrix}$; $B = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix}$ and $C = \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}$, then

$$t_r(A) + t_r\left(\frac{ABC}{2}\right) + t_r\left(\frac{A(BC)^2}{4}\right) + t_r\left(\frac{A(BC)^3}{8}\right) + \ldots\ldots + \infty \text{ is equal to-}$$

(A) 6 (B) 9 (C) 12 (D) none

Solⁿ :-

$$BC = \begin{bmatrix} 3 & 4 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 3 & -4 \\ -2 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 9-8 & -12+12 \\ 6-6 & -8+9 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

Now

$$tr(A) + tr\left(\frac{A \cdot I}{2}\right) + tr\left(\frac{A \cdot (I)^2}{4}\right) + \text{---} \infty$$

$$\Rightarrow 3 + \frac{3}{2} + \frac{3}{4} + \text{-----} \infty$$

$$\Rightarrow \frac{3}{1-1/2} = 6 \ \underline{\underline{An}}$$

7 For a matrix $A = \begin{bmatrix} 1 & 2r-1 \\ 0 & 1 \end{bmatrix}$, the value of $\prod_{r=1}^{50} \begin{bmatrix} 1 & 2r-1 \\ 0 & 1 \end{bmatrix}$ is equal to -

(A) $\begin{bmatrix} 1 & 100 \\ 0 & 1 \end{bmatrix}$ (B) $\begin{bmatrix} 1 & 4950 \\ 0 & 1 \end{bmatrix}$ (C) $\begin{bmatrix} 1 & 5050 \\ 0 & 1 \end{bmatrix}$ (D) $\begin{bmatrix} 1 & 2500 \\ 0 & 1 \end{bmatrix}$

Soln:- $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix} \cdots\cdots \begin{bmatrix} 1 & 99 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 3+1 \\ 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix} \cdots\cdots \begin{bmatrix} 1 & 99 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 5+3+1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 7 \\ 0 & 1 \end{bmatrix} \cdots\cdots \begin{bmatrix} 1 & 99 \\ 0 & 1 \end{bmatrix}$

observing the pattern, finally we get

$\begin{bmatrix} 1 & 99+97+\cdots\cdots+5+3+1 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 1 & 2500 \\ 0 & 1 \end{bmatrix}$ Ans

8 A and B are two given matrices such that the order of A is 3×4 , if A′ B and BA′ are both defined then

(A) order of B′ is 3×4
(B) order of B′A is 4×4
(C) order of B′A is 3×3
(D) B′A is undefined

Soln:- Given order of A is 3×4

let order of B is $m \times n$

Now if $A'B$ is defined then

no. of coloumn of A' = no. of rows of B

i.e A' will be of 4×3 order, so

$3 = m$ ——— (1)

& Similarly BA' is defined, so

$n = 4$ ——— (2)

∴ order of B is 3×4

order of B' is 4×3

order of $B'A$ is 4×4

9 If the product of n matrices $\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 3\\0 & 1\end{bmatrix}\ldots\ldots\begin{bmatrix}1 & n\\0 & 1\end{bmatrix}$ is equal to the matrix $\begin{bmatrix}1 & 378\\0 & 1\end{bmatrix}$ then the value of n is equal to -

(A) 26 (B) 27 (C) 377 (D) 378

Solt^n:- $\begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 2\\0 & 1\end{bmatrix} = \begin{bmatrix}1 & 2+1\\0 & 1\end{bmatrix}$

Now $\begin{bmatrix}1 & 2+1\\0 & 1\end{bmatrix}\begin{bmatrix}1 & 3\\0 & 1\end{bmatrix} = \begin{bmatrix}1 & 3+2+1\\0 & 1\end{bmatrix}$

So observing the pattern, multiplying all the matrices we get

$$\Rightarrow \begin{bmatrix}1 & 1+2+3+\ldots+n\\0 & 1\end{bmatrix} = \begin{bmatrix}1 & 378\\0 & 1\end{bmatrix}$$

$$\Rightarrow \frac{n(n+1)}{2} = 378$$

$$\Rightarrow n(n+1) = 27\times 28$$

$$\Rightarrow n = 27 \quad \underline{\text{Ans}}$$

10 Consider a matrix $A(\theta) = \begin{bmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{bmatrix}$ then

(A) $A(\theta)$ is symmetric

(B) $A(\theta)$ is skew symmetric

(C) $A^{-1}(\theta) = A(\pi - \theta)$

(D) $A^2(\theta) = A\left(\frac{\pi}{2} - 2\theta\right)$

Solⁿ:- $A(\theta)$ is neither symm. nor skew symm.

Now $A^2 = \begin{bmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{bmatrix} \times \begin{bmatrix} \sin\theta & \cos\theta \\ -\cos\theta & \sin\theta \end{bmatrix}$

$= \begin{bmatrix} -\cos 2\theta & \sin 2\theta \\ -\sin 2\theta & -\cos 2\theta \end{bmatrix} \neq A\left(\frac{\pi}{2} - 2\theta\right)$

Now $A^{-1} = \frac{\operatorname{adj} A}{|A|}$

$\Rightarrow |A| = \sin^2\theta + \cos^2\theta = 1$

$A^{-1}(\theta) = \begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix} = A(\pi - \theta)$

11 If p, q, r are 3 real number satisfying the matrix equation, $[p\ q\ r]\begin{bmatrix}3 & 4 & 1\\ 3 & 2 & 3\\ 2 & 0 & 2\end{bmatrix} = [3\ 0\ 1]$, then

2p + q – r equals :- [JEE-MAIN Online 2013]

(A) –1 (B) 4 (C) –3 (D) 2

Sol^n:- Given $PA = B$, where $P = [p\ q\ r]$, $A = \begin{bmatrix}3 & 4 & 1\\ 3 & 2 & 3\\ 2 & 0 & 2\end{bmatrix}$, $B = [3\ 0\ 1]$

$\Rightarrow P = B \cdot A^{-1}$

Now $A^{-1} = \dfrac{\text{adj}A}{|A|}$

$$A^{-1} = \frac{1}{8}\begin{bmatrix}4 & 0 & -4\\ -8 & 4 & 8\\ 10 & -6 & -6\end{bmatrix}^T \Rightarrow \begin{bmatrix}\frac{1}{2} & -1 & 5/4\\ 0 & \frac{1}{2} & -3/4\\ -\frac{1}{2} & 1 & -3/4\end{bmatrix}$$

$\therefore$ from $P = B \cdot A^{-1}$

$$[p\ q\ r] = [3\ 0\ 1]\begin{bmatrix}1/2 & -1 & 5/4\\ 0 & 1/2 & -3/4\\ -1/2 & 1 & -3/4\end{bmatrix}$$

$\Rightarrow [p\ q\ r] = [1\ -2\ 3]$

$\therefore p = 1, q = -2, r = 3$

So $2p + q - r = -3$

EXERCISE (O-1)

12 If A, B and C are $n \times n$ matrices and det(A) = 2, det(B) = 3 and det(C) = 5, then the value of the det(A^2BC^{-1}) is equal to

(A) $\frac{6}{5}$ (B) $\frac{12}{5}$ (C) $\frac{18}{5}$ (D) $\frac{24}{5}$

Solⁿ :- Given $|A| = 2$, $|B| = 3$ and $|C| = 5$

$\therefore \quad |A^2BC^{-1}| = |A^2|\,|B|\,|C^{-1}|$

$= |A|^2 \cdot |B| \cdot \frac{1}{|C|}$

$= (2)^2 (3) \cdot \frac{1}{5}$

$= \frac{12}{5}$ Ans

13. Which of the following is an orthogonal matrix -

(A) $\begin{bmatrix} 6/7 & 2/7 & -3/7 \\ 2/7 & 3/7 & 6/7 \\ 3/7 & -6/7 & 2/7 \end{bmatrix}$

(B) $\begin{bmatrix} 6/7 & 2/7 & 3/7 \\ 2/7 & -3/7 & 6/7 \\ 3/7 & 6/7 & -2/7 \end{bmatrix}$

(C) $\begin{bmatrix} -6/7 & -2/7 & -3/7 \\ 2/7 & 3/7 & 6/7 \\ -3/7 & 6/7 & 2/7 \end{bmatrix}$

(D) $\begin{bmatrix} 6/7 & -2/7 & 3/7 \\ 2/7 & 2/7 & -3/7 \\ -6/7 & 2/7 & 3/7 \end{bmatrix}$

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Solⁿ :- for a matrix to be orthogonal $\boxed{AA^T = I}$

or $\sum_{i=1}^{3} a_i^2 = \sum b_i^2 = \sum c_i^2 = 1$ and $\sum_{i=1}^{3} a_i b_i = \sum b_i c_i = \sum c_i a_i = 0$

So by checking options

$\sum_{i=1}^{3} a_i^2 = \sum b_i^2 = \sum c_i^2 = 1$ in all

but $\sum_{i=1}^{3} a_i b_i = \sum b_i c_i = \sum c_i a_i = 0$ in option (A) only.

$\therefore$ (A) is orthogonal matrix

14 Matrix A = $\begin{bmatrix} x & 3 & 2 \\ 1 & y & 4 \\ 2 & 2 & z \end{bmatrix}$, if xyz = 60 and 8x + 4y + 3z = 20, then A (adj A) is equal to -

(A) $\begin{bmatrix} 64 & 0 & 0 \\ 0 & 64 & 0 \\ 0 & 0 & 64 \end{bmatrix}$ (B) $\begin{bmatrix} 88 & 0 & 0 \\ 0 & 88 & 0 \\ 0 & 0 & 88 \end{bmatrix}$ • (C) $\begin{bmatrix} 68 & 0 & 0 \\ 0 & 68 & 0 \\ 0 & 0 & 68 \end{bmatrix}$ (D) $\begin{bmatrix} 34 & 0 & 0 \\ 0 & 34 & 0 \\ 0 & 0 & 34 \end{bmatrix}$

Solt^n:- We know that,

$$A(\operatorname{adj} A) = |A| \cdot I_n$$

$$\therefore \quad |A| = x(yz-8) - 3(z-8) + 2(2-2y)$$

$$= xyz - 8x - 3z - 4y + 28$$

$$= 60 - (20) + 28 = 68$$

$$\therefore \quad A \cdot (\operatorname{adj} A) = 68 \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 68 & 0 & 0 \\ 0 & 68 & 0 \\ 0 & 0 & 68 \end{bmatrix}$$

15 The matrix $\begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$ is a

(A) non-singular • (B) Idempotent (C) Nipotent (D) Orthogonal

Solt^n:- let given matrix

$$A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$$

Now

$|A| = 0$ so it is singular

$$\Rightarrow A^2 = A \cdot A = \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & -2 & -4 \\ -1 & 3 & 4 \\ 1 & -2 & -3 \end{bmatrix} = A$$

Since $A^2 = A$, So matrix is Idempotent

16 If A and B are symmetric matrices, then ABA is -

(A) symmetric matrix (B) skew symmetric matrix

(C) diagonal matrix (D) scalar matrix

Soln

Given

$$\left.\begin{aligned} A^T &= A \\ B^T &= B \end{aligned}\right\}$$ As 'A' and 'B' are symmetric matrices —— (1)

$\Rightarrow$ $(ABA)^T = ((AB)A)^T$

$\Rightarrow A^T \cdot (AB)^T$

$\Rightarrow A^T \cdot B^T \cdot A^T$

$\Rightarrow ABA$ from eqn (1)

$\therefore$ ABA is symmetric matrix

17 Number of real values of λ for which the matrix $A = \begin{bmatrix} \lambda-1 & \lambda & \lambda+1 \\ 2 & -1 & 3 \\ \lambda+3 & \lambda-2 & \lambda+7 \end{bmatrix}$ has no inverse

(A) 0 (B) 1 (C) 2 • (D) infinite

Soln:- for no inverse, $\det(A) = 0$

ie: $\begin{vmatrix} \lambda-1 & \lambda & \lambda+1 \\ 2 & -1 & 3 \\ \lambda+3 & \lambda-2 & \lambda+7 \end{vmatrix} = 0$

$\Rightarrow$ $R_3 \to R_3 - R_1$

$\begin{vmatrix} \lambda-1 & \lambda & \lambda+1 \\ 2 & -1 & 3 \\ 4 & -2 & 6 \end{vmatrix} = 0$

$\Rightarrow (\lambda-1)(-6+6) - \lambda(12-12) + (\lambda+1)(-4+4) = 0$

$\Rightarrow$ $0 = 0$ ie this true of all

$\lambda \in R$

18 If $A = [a_{ij}]_{2\times 2}$ where $a_{ij} = \begin{cases} i+j & i \neq j \\ i^2 - 2j & i = j \end{cases}$, then A^{-1} is equal to -

✓(A) $\frac{1}{9}\begin{bmatrix} 0 & 3 \\ 3 & 1 \end{bmatrix}$ (B) $\frac{1}{9}\begin{bmatrix} 0 & -3 \\ 3 & -1 \end{bmatrix}$ (C) $\frac{1}{9}\begin{bmatrix} 0 & -3 \\ -3 & -1 \end{bmatrix}$ (D) $\frac{1}{3}\begin{bmatrix} 0 & 3 \\ 3 & 1 \end{bmatrix}$

Sol^n :- By given condition,

$$A = \begin{bmatrix} -1 & 3 \\ 3 & 0 \end{bmatrix}$$

Now $\quad \operatorname{adj} A = \begin{bmatrix} 0 & -3 \\ -3 & -1 \end{bmatrix}$

$$|A| = -9$$

$$\therefore \quad A^{-1} = \frac{\operatorname{adj} A}{|A|} = -\frac{1}{9}\begin{bmatrix} 0 & -3 \\ -3 & -1 \end{bmatrix}$$

or

$$\frac{1}{9}\begin{bmatrix} 0 & 3 \\ 3 & 1 \end{bmatrix}$$

Exercise 02

EXERCISE (O-2)

1. Let A, other than I or –I, be a 2×2 real matrix such that $A^2 = I$, I being the unit matrix. Let Tr(A) be the sum of diagonal elements of A. [JEE-MAIN Online 2013]

Statement–1 : Tr (A) = 0

Statement–2 : det (A) = –1

(A) Statement-1 and Statement-2 are true and Statement-2 is not the correct explanation for Statement-1

(B) Statement-1 and Statement-2 are true and Statement-2 is a correct explanation for Statement-1.

(C) Statement-1 is true and Statement-2 is false.

(D) Statement-1 is false and Statement-2 is true.

given $A^2 = I \Rightarrow |A|^2 = 1 \Rightarrow |A| = 1, -1$

so, $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

$A^2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

so, $tr(A) = 0$ and $det(A) = -1$.

so, statement 1 and 2 both are true but not the correct explanation

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②

Let $S = \left\{ \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} : a_{ij} \in \{0, 1, 2\}, a_{11} = a_{22} \right\}$. Then the number of non-singular matrices in the set S is :

[JEE-MAIN Online 2013]

(A) 24 (B) 10 (C) 20 (D) 27

$|S| = a_{11} \cdot a_{22} - a_{21} \cdot a_{12}$ (given $a_{11} = a_{22}$)

$= (a_{11})^2 - a_{21} \cdot a_{12}$

For non singular matrics $|S| \neq 0$

$\Rightarrow (a_{11})^2 - a_{21} \cdot a_{12} \neq 0$

I) $a_{11} = 0$ then ${}^2C_1 \times {}^2C_1 = 4$ cases

II) $a_{11} = 1$, then $(0,0), (0,1), (0,2)$
$(1,0), (2,0), (1,2), (2,1), (2,2)$ 8 cases

III) $a_{11} = 2$, then $(0,0), (0,1), (0,2), (1,0), (2,0)$
$(1,2), (2,1), (1,1)$ 8 cases

total $4+8+8 = 20$ Ans.

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3

Let $A=\begin{bmatrix}2&2&1\\2&5&2\\1&2&2\end{bmatrix}$ and $B=\begin{bmatrix}-x&-y&z\\0&y&2z\\x&-y&z\end{bmatrix}$ where $x,y,z \in R$. If $B^TAB=\begin{bmatrix}8&0&0\\0&27&0\\0&0&42\end{bmatrix}$ then the number of ordered triplet (x,y,z) is-

(A) 2 (B) 6 (C) 8 (D) 9

$$AB = \begin{pmatrix}2&2&1\\2&5&2\\1&2&2\end{pmatrix}\begin{pmatrix}-x&-y&z\\0&y&2z\\x&-y&z\end{pmatrix}$$

$$= \begin{pmatrix}-2x+x & -2y+2y-y & 2z+4z+z\\ -2x+2x & -2y+5y-2y & 2z+10z+2z\\ -x+2x & -y+2y-2y & z+4z+2z\end{pmatrix}$$

$$= \begin{pmatrix}-x & -y & 7z\\ 0 & y & 14z\\ x & -y & 7z\end{pmatrix}$$

$$B^TAB = \begin{pmatrix}-x&0&x\\-y&y&-y\\z&2z&z\end{pmatrix}\begin{pmatrix}-x&-y&7z\\0&y&14z\\x&-y&7z\end{pmatrix}$$

$$= \begin{pmatrix}x^2+x^2 & xy-xy & -7xz+7xz\\ xy-xy & y^2+y^2+y^2 & -7yz+14yz-7yz\\ -xz+xz & -yz+2yz-yz & 7z^2+28z^2+7z^2\end{pmatrix}$$

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③ Continues...

$$B^TAB = \begin{pmatrix} 2x^2 & 0 & 0 \\ 0 & 3y^2 & 0 \\ 0 & 0 & 42z^2 \end{pmatrix}$$

given

$$= \begin{pmatrix} 8 & 0 & 0 \\ 0 & 27 & 0 \\ 0 & 0 & 42 \end{pmatrix}$$

On Comparing: $x = \pm 2$, $y = \pm 3$, $z = \pm 1$

So, total ordered triplet $= 8$
$= 2 \times 2 \times 2$
$= 8$ //

(x, y, z)

z: 2 options (± 1)

y: 2 options (± 3)

x: 2 options (± 2)

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4

Let $A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{bmatrix}$ and $10B = \begin{bmatrix} 4 & 2 & 2 \\ -5 & 0 & \alpha \\ 1 & -2 & 3 \end{bmatrix}$. If B is the inverse of matrix A, then α is -

(A) –2 (B) –1 (C) 2 (D) 5

$$10BA = \begin{pmatrix} 4 & 2 & 2 \\ -5 & 0 & \alpha \\ 1 & -2 & 3 \end{pmatrix} \begin{pmatrix} 1 & -1 & 1 \\ 2 & 1 & -3 \\ 1 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 4+4+2 & -4+2+2 & 4-6+2 \\ -5+\alpha & 5+\alpha & -5+\alpha \\ 1-4+3 & -1-2+3 & 1+6+3 \end{pmatrix}$$

$$= \begin{pmatrix} 10 & 0 & 0 \\ \alpha-5 & \alpha+5 & \alpha-5 \\ 0 & 0 & 10 \end{pmatrix}$$

$\Rightarrow \alpha = 5 \Rightarrow$ option (D)

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[ONE OR MORE THAN ONE ARE CORRECT]

5

Let $\det(\operatorname{adj}(\operatorname{adj}A)) = 14^4$ where $A = \begin{bmatrix} x & 2 & -1 \\ -1 & 1 & 2 \\ 2 & -1 & 1 \end{bmatrix}$, $x \neq -\frac{11}{3}$, then

(A) $x = 1$ (B) $\det(2A) = 112$ (C) $x = 2$ (D) $\det(2A) = 256$

$$|\operatorname{Adj}(\operatorname{Adj}A)| = |A|^{(n-1)^2} = (14)^4 \quad \text{—} (1)$$

$$A = \begin{pmatrix} x & 2 & -1 \\ -1 & 1 & 2 \\ 2 & -1 & 1 \end{pmatrix}$$

$$|A| = x(1+2) - 2(-1-4) - 1(1-2)$$

$$= 3x + 10 + 1 = 3x + 11$$

from (1)

$$(3x+11)^{(n-1)^2} = (14)^4$$

$$\Rightarrow 3x + 11 = 14 \Rightarrow x = 1.$$

and $|A| = 14$

So, $|2A| = 2^3 \cdot 14 = 112$

option (A) and (B)

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6

Let $A^{-1} = \begin{bmatrix} 1 & -1 & 2 \\ -1 & 2 & 1 \\ 2 & 1 & -1 \end{bmatrix}$, then -

(A) $7|A| = \frac{1}{2}$

(B) $|\text{adj } A| = \frac{1}{196}$

(C) $\text{trace}(\text{adj}A) = -\frac{1}{7}$

(D) Matrix A is a symmetric matrix

$|A^{-1}| = 1(-2-1) + 1(1-2) + 2(-1-4)$

$= -3 - 1 - 10 = -14.$

$\frac{1}{|A|} = -14 \Rightarrow 7|A| = -\frac{1}{2}$

Also $A^{-1} = \frac{Adj(A)}{|A|}$

$\therefore Adj\, A = A^{-1}|A|$

$|Adj\, A| = |A^{-1}|A|| = |A|^3 \cdot |A^{-1}|$

$= \left(-\frac{1}{14}\right)^3 \cdot (-14) = \frac{1}{196}$

(B) option

$tr(adj\, A) = tr(|A| \cdot A^{-1})$

$= -\frac{1}{14}(2) = -\frac{1}{7} \Rightarrow$ (C) option

(D) is also true, as A^{-1} is symmetric

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7

If $A = \begin{bmatrix} 3 & 0 & 0 \\ 2 & 2 & 0 \\ 4 & 5 & 3 \end{bmatrix}$, then which of the following is(are) true ?

(trace of A denotes sum of principal diagonal elements of A)

(A) A is invertible

(B) trace(adj(adj(A))) = 144

(C) trace(adj(adj(A))) = 8

(D) |adj A| is less than 400

$|A| = 3(6-0) = 18 \neq 0$

so, A is invertible ⇒ option (A)

$adj(adj\,A) = |A|^{n-2} \cdot A$

$= 18^{3-2} \begin{pmatrix} 3 & 0 & 0 \\ 2 & 2 & 0 \\ 4 & 5 & 3 \end{pmatrix}$

$= 18 \begin{pmatrix} 3 & 0 & 0 \\ 2 & 2 & 0 \\ 4 & 5 & 3 \end{pmatrix}$

$\therefore tr(adj(adj\,A)) = 18(3+2+3) = 144$

⇒ option (B)

$|Adj\,A| = |A^{-1}|A|| = 18^3 \cdot |A^{-1}| = \frac{18^3}{18} = 324$

⇒ option (D)

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8. Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & -1 \\ 3 & 0 & k \end{bmatrix}$ and $f(x) = x^3 - 2x^2 - \alpha x + \beta = 0$. If A satisfies $f(x) = 0$, then-

(A) $k = 1, \alpha = 14$ (B) $\alpha = 14, \beta = 22$ (C) $k = -1, \beta = 22$ (D) $\alpha = -14, \beta = -22$

Sol: Characteristic equation $\Rightarrow |A - \lambda I| = 0$

$$\Rightarrow \begin{vmatrix} 1-\lambda & 2 & 3 \\ 2 & 2-\lambda & -1 \\ 3 & 0 & k-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1-\lambda)(2-\lambda)(k-\lambda) - 2(2k - 2\lambda) + 3(-2 - 6 + 3\lambda) = 0$$

$$\Rightarrow (\lambda^2 - 3\lambda + 2)(k - \lambda) - 4k + 4\lambda - 24 + 9\lambda = 0$$

$$\Rightarrow \lambda^2 k - 3\lambda k + 2k - \lambda^3 + 3\lambda^2 - 2\lambda - 4k - 24 + 13\lambda = 0$$

$$\Rightarrow -\lambda^3 + (3+k)\lambda^2 + (-3k+11)\lambda - 2k - 24 = 0$$

$$\therefore f(x) = x^3 - (k+3)x^2 + (3k-11)\lambda + 2k + 24 = 0$$

↓ given

$$x^3 - 2x^2 - \alpha x + \beta$$

On comparing.

$k + 3 = 2$

$\Rightarrow k = -1$

$3k - 11 = -\alpha$

$\therefore -\alpha = 3(-1) - 11$

$\alpha = 14$

$\beta = 2k + 24$

$\beta = 2(-1) + 24$

$\beta = 22$

Ans $\boxed{B, C}$

9

If A and B are 3×3 matrices and $|A| \neq 0$, then which of the following are true?

(A) $|AB| = 0 \Rightarrow |B| = 0$

(B) $|AB| = 0 \Rightarrow B = 0$

(C) $|A^{-1}| = |A|^{-1}$

(D) $|A + A| = 2|A|$

$|AB| = 0 \Rightarrow |A|.|B| = 0 \Rightarrow |A| = 0$ OR $|B| = 0$

$\Rightarrow$ option (A)

$|A^{-1}| = |A|^{-1} \Rightarrow$ option (C)

Option D: $|A+A| = |2A| = 2^3 |A|$

$(\because |kA| = k^n |A|)$

$\therefore$ Ans A, C

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10

If D_1 and D_2 are two 3×3 diagonal matrices where none of the diagonal element is zero, then -

(A) D_1D_2 is a diagonal matrix

(B) $D_1D_2 = D_2D_1$

(C) $D_1^2 + D_2^2$ is a diagonal matrix

(D) none of these

Since. D_1 & D_2 are diagonal matrix.

So, $D_1 \cdot D_2$ is also a diagonal matrix.

$D_1 \cdot D_2 = D_2 \cdot D_1$

$D_1^2 + D_2^2 =$ a diagonal matrix

So, (A), (B) & (C) options are correct.

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11

Let $A = a_{ij}$ be a matrix of order 3 where $a_{ij} = \begin{bmatrix} x & \text{if} \quad i=j, \quad x \in R \\ 1 & \text{if} \quad |i-j|=1 \\ 0 & \text{otherwise} \end{bmatrix}$, then which of the following hold(s) good ?

(A) for $x = 2$, A is a diagonal matrix.

(B) A is a symmetric matrix

(C) for $x = 2$, det A has the value equal to 6

(D) Let $f(x) = \det A$, then the function $f(x)$ has both the maxima and minima.

$$A = \begin{pmatrix} x & 1 & 0 \\ 1 & x & 1 \\ 0 & 1 & x \end{pmatrix}$$

$\det(A) = x(x^2-1) - 1(x)$

$f(x) = x^3 - x - x = x^3 - 2x$

$f(2) = 8 - 4 = 4$

$f'(x) = 3x^2 - 2 = 0 \Rightarrow x = \pm\sqrt{\frac{2}{3}}$

$\therefore f(x)$ has both maxima & minima

$\Rightarrow$ option (B) & (D)

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12 If A & B are square matrices of order 2 such that $A + adj(B^T) = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ & $A^T - adj(B) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$,

then-

(A) B is symmetric matrix

(B) $A^n = A \ \forall \ n \in N$

(C) $|A + A^2 + A^3 + A^4 + A^5| = 0$

(D) $|B + B^2 + B^3 + B^4 + B^5| = 0$

$A + (Adj\ B^T) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$, $A^T - Adj\ B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ — (2)

$\Rightarrow A^T + Adj\ B = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ — (1)

(1) + (2)

so, $2A^T = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 2 \end{pmatrix}$

$\therefore A^T = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \Rightarrow A = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

also $A + Adj(B^T) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

$\therefore Adj(B^T) = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} - \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\therefore B = I$

so, B is symmetric $\Rightarrow$ option (A)

and $|A| = 0 \Rightarrow$ option (C)

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13

If $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ & $A^n = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, (where $n \geq 2$ & $n \in N$), then -

(A) $a = d$

(B) $b = c$

(C) $b = a + 1$ if n is odd

(D) $b = a - 1$ if n is even

$$A^2 = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 1+4 & 2+2 \\ 2+2 & 4+1 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} = \begin{pmatrix} 13 & 14 \\ 14 & 13 \end{pmatrix}$$

So, all options are correct.

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14

If A and B are two orthogonal matrices of order 3, then -

(A) A and B both will be invertible matrices (B) matrix ABA will also be orthogonal

(C) matrix A^2B^2 will also be orthogonal (D) maximum value of $\det\left(\frac{A}{2}\mathrm{adj}(2B)\right)$ is 8.

$A \cdot A^T = I$ and $B \cdot B^T = I$.

$\Rightarrow A^T = A^{-1}$ and $B^T = B^{-1}$

$\Rightarrow$ option (A)

$(ABA)^T = A^T B^T A^T$

$(ABA)\cdot(ABA)^T = AB\overbrace{A \cdot A^T}^{I} B^T A^T$

$= A\underbrace{B\ B^T}_{I} A^T = \underbrace{AA^T}_{I} = I.$

So, ABA is orthogonal $\Rightarrow$ option (B)

$(A^2B^2)\cdot(A^2B^2)^T = A^2 B^2 \cdot (B^2)^T \cdot (A^2)^T$

$= A^2 \cdot (A^T)^2 = I.$

$\Rightarrow$ option (C)

$\det\left(\frac{A}{2} \cdot adj(2B)\right) = \left|\frac{A}{2}\right| \cdot 2^6 \ |B| = \frac{1}{2^3}|A| \cdot 2^6|B|$

Max of $8|A|\cdot|B| = 8(1)(1) = 8$

$(\because |A|_{max} = 1 = |B|_{max}$ as both are orthogonal$)$

Ans: A, B, C, D

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15 If A & B are two non singular matrices of order 3×3 such that $A^T + B = I$ & $BA^T = -B$, then which is/are always true (where X^T denotes transpose of X and I denotes unit matrix)-

(A) $|B| = 2$ (B) $|B| = 8$ (C) $|A| = -1$ (D) $|A| = 1$

(where $|X|$ denotes determinant value of X)

$BA^T = -B \Rightarrow |B| \cdot |A^T| = -|B|$

$\Rightarrow |B|(|A^T| + 1) = 0$

$\Rightarrow |B| = 0$, OR $|A^T| = -1 = |A|$

But B is non singular so, $|A| = -1$

given $B \cdot A^T = -B$

$\Rightarrow B(I - B) + B = 0$

$\Rightarrow B(2I - B) = 0$

$\Rightarrow B = 2I \Rightarrow |B| = |2I| = 2^3|I| = 2^3(1)$

$\Rightarrow |B| = 8 \Rightarrow$ (B) & (C) options

Paragraph for Question 16 to 17

Consider the system AX = B, where $A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 1 & 1 \\ 3 & 2 & 2 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, B = \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$.

16. Sum of elements of (adjA) B is-

(A) –1 (B) 2 (C) –2 (D) –4

17. Value of $tr(XB^T)$ is (where tr(A) denotes trace of matrix A)-

(A) 0 (B) 1 (C) 2 (D) 3

$C_{11} = 2-2 = 0$ $\quad C_{21} = -4$ $\quad C_{31} = 2$

$C_{12} = -(4-3) = -1$ $\quad C_{22} = 5$ $\quad C_{32} = -3$

$C_{13} = 4-3 = 1$ $\quad C_{23} = 1$ $\quad C_{33} = -1$

$$\text{Adj } A = \begin{pmatrix} 0 & -4 & 2 \\ -1 & 5 & -3 \\ 1 & 1 & -1 \end{pmatrix}$$

$$(\text{Adj } A)\, B = \begin{pmatrix} 0 & -4 & 2 \\ -1 & 5 & -3 \\ 1 & 1 & -1 \end{pmatrix} \begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$$

$$= \begin{pmatrix} -8+6 \\ +1+10-9 \\ -1+2-3 \end{pmatrix} = \begin{pmatrix} -2 \\ 2 \\ -2 \end{pmatrix}$$

Sum of elements = -2. $\Rightarrow$ option (C)

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Paragraph for Question 16 to 17 Continues...

$$Ax = B \Rightarrow X = A^{-1}B$$

$$|A| = 1(2-2) - 1(4-3) - 1(4-3)$$

$$= -1 - 1 = -2$$

$$A^{-1} = \frac{Adj(A)}{|A|} = -\frac{1}{2}\begin{pmatrix} 0 & -4 & 2 \\ -1 & 5 & -3 \\ 1 & 1 & -1 \end{pmatrix}$$

$$\therefore A^{-1}B = -\frac{1}{2}\begin{pmatrix} 0 & -4 & 2 \\ -1 & 5 & -3 \\ 1 & 1 & -1 \end{pmatrix}\begin{pmatrix} -1 \\ 2 \\ 3 \end{pmatrix}$$

$$= -\frac{1}{2}\begin{pmatrix} -8+6 \\ 1+10-9 \\ -1+2-3 \end{pmatrix} = -\frac{1}{2}\begin{pmatrix} -2 \\ 2 \\ -2 \end{pmatrix}$$

$$X = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$XB^T = \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}\begin{pmatrix} -1 & 2 & 3 \end{pmatrix}$$

$$= \begin{pmatrix} -1 & 2 & 3 \\ 1 & -2 & -3 \\ -1 & 2 & 3 \end{pmatrix}$$

$$\therefore tr(XB^T) = 0 \Rightarrow \text{option } (A)$$

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Exercise S1

1 Find the number of 2×2 matrix satisfying following conditions :

(i) a_{ij} is 1 or -1 ; (ii) $a_{11}a_{21} + a_{12}a_{22} = 0$

Sol. $\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}_{2\times 2} \Rightarrow a_{11}a_{21} + a_{12}a_{22} = 0$

$\therefore \quad 1 + (-1) = 0$

or $(-1) + (1) = 0$

$a_{11}a_{21} = 1 \Rightarrow$ possible ways $= 2$

$a_{12}a_{22} = -1 \Rightarrow$ possible ways $= 2$ 4

So total No. of ways are $= 4 \times 2$

$= 8$ ✓

(2) Find the value of x and y that satisfy the equations

$$\begin{bmatrix} 3 & -2 \\ 3 & 0 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} y & y \\ x & x \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3y & 3y \\ 10 & 10 \end{bmatrix}$$

Soln

$$\begin{bmatrix} 3y-2x & 3y-2x \\ 3y & 3y \\ 2y+4x & 2y+4x \end{bmatrix} = \begin{bmatrix} 3 & 3 \\ 3y & 3y \\ 10 & 10 \end{bmatrix}$$

On comparing

$$\left.\begin{aligned} 3y-2x &= 3 \\ 2y+4x &= 10 \end{aligned}\right\} \Rightarrow x = 3/2 \,,\ y = 2$$

3

If $A = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 2 & 1 \\ 2 & 0 & 3 \end{bmatrix}$, then show that the matrix A is a root of the polynomial $f(x) = x^3 - 6x^2 + 7x + 2$.

Sol. for characteristic eqⁿ

$\Rightarrow |A - xI| = 0$

$$\Rightarrow \begin{vmatrix} 1-x & 0 & 2 \\ 0 & 2-x & 1 \\ 2 & 0 & 3-x \end{vmatrix} = 0$$

$\Rightarrow x^3 - 6x^2 + 7x + 2 = 0$

4

If the matrices $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

(a, b, c, d not all simultaneously zero) commute, find the value of $\frac{d-b}{a+c-b}$. Also show that the

matrix which commutes with A is of the form $\begin{bmatrix} \alpha-\beta & 2\beta/3 \\ \beta & \alpha \end{bmatrix}$

[Sol : Since given matrices commute $\Rightarrow AB = BA$

$$\therefore \quad \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$\begin{bmatrix} a+2c & b+2d \\ 3a+4c & 3b+4d \end{bmatrix} = \begin{bmatrix} a+3b & 2a+4b \\ c+3d & 2c+4d \end{bmatrix}$$

$$\therefore \quad a + 2c = a + 3b \quad \Rightarrow \quad b = \frac{2c}{3} \qquad ....(1)$$

$$b + 2d = 2a + 4b \quad \Rightarrow \quad 2a + 3b = 2d \qquad ....(2)$$

$$3a + 4c = c + 3d \quad \Rightarrow \quad a + c = d \qquad ....(3)$$

$$3b + 4d = 2c + 4d \quad \Rightarrow \quad b = \frac{2c}{3} \qquad ....(4)$$

$$\therefore \quad \frac{d-b}{a+c-b} = \frac{d-b}{d-b} = 1 \qquad (\text{as } a + c = d)$$

Now Let $c = \beta$ (say) and $b = \frac{2\beta}{3}$ $\Rightarrow 3b = 2\beta$

from (2), $2a + 2\beta = 2d \Rightarrow a + \beta = d$; from (3) $a + \beta = d$

if $d = \alpha$ then $a = \alpha - \beta$

Hence the required matrices is of the form $\begin{bmatrix} \alpha-\beta & 2\beta/3 \\ \beta & \alpha \end{bmatrix}$]

5 If $\begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$ is an idempotent matrix. Find the value of f(a), where $f(x) = x - x^2$, when $bc = 1/4$.

Hence otherwise evaluate a.

[Sol. Idempotent matrix $A^2 = A$

$$A^2 = \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}\begin{bmatrix} a & b \\ c & 1-a \end{bmatrix} = \begin{bmatrix} a^2+bc & ab+b-ab \\ ca+c-ac & bc+(1-a)^2 \end{bmatrix}$$

$$\begin{bmatrix} a^2+bc & b \\ c & bc+(1-a)^2 \end{bmatrix} = \begin{bmatrix} a & b \\ c & 1-a \end{bmatrix}$$

$a^2 + bc = a \quad \text{.......} (1)$

$a^2 + 1/4 = a$

$4a^2 - 4a + 1 = 0$

$2a = 1$

$a = 1/2$

$bc + (1 - a)^2 = 1 - a$

$a - a^2 + (1 - a)^2 = 1 - a$

$a - a^2 + 1 + a^2 - 2a = 1 - a$

$1 - a = 1 - a$ which is true

$$f(a) = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$$]

6 If the matrix A is involutary, show that $\frac{1}{2}(I + A)$ and $\frac{1}{2}(I - A)$ are idempotent and $\frac{1}{2}(I + A).\ \frac{1}{2}(I - A) = \mathbf{O}$.

[Sol. $A^2 = I$

$\frac{1}{2}(I + A) = C$

$C^2 = \frac{1}{4}(I + A)^2 = \frac{1}{4}(1 + A^2 + 2IA) = \frac{1}{4}(I + A^2 + 2A) = \frac{1}{4}(I + I + 2A) = \frac{1}{2}\ (I + A) = C$

hence $C^2 = C \quad \Rightarrow \quad C = \frac{1}{2}\ (I + A)$ is involutory

|||ly $\frac{1}{2}(I - A) = O$ can be proved.]

7 Show that the matrix $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}$ can be decomposed as a sum of a unit and a nilpotent matrix.

Hence evaluate the matrix $\begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix}^{2007}$.

[Sol. $A = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$ ....(1)

$= I + B$ (say)

now $B^2 = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}\begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

hence B is nilpotent with order of nilpotency 2. So $B^k = \mathbf{O}$ for $k \geq 2$

Now from (1) using binomial expansion, we get

$A^{2007} = (I + B)^{2007} = I + 2007\,B$ (remaining terms would be null matrices)

$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 4014 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 4014 & 1 \end{bmatrix}$ Ans.]

(4) $A = \begin{pmatrix} 3 & a & -1 \\ 2 & 5 & c \\ b & 8 & 2 \end{pmatrix}$ is Symmetric and $B = \begin{pmatrix} d & 3 & a \\ b-a & e & -2b-c \\ -2 & 6 & -f \end{pmatrix}$ is Skew Symmetric, then find AB.

Is AB a symmetric, Skew Symmetric or neither of them. Justify your answer.

Sol. for symmetric $a_{ij} = a_{ji}$

for skew symm. $a_{ij} = -a_{ji}$

& diagonal elements $= 0$

So in A, $a = 2$, $b = -1$, $c = 8$

and in B; $d = e = f = 0$

$b - a = -3$, $a = 2$ $2b + c = 6$

$\therefore A = \begin{bmatrix} 3 & 2 & -1 \\ 2 & 5 & 8 \\ -1 & 8 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 3 & 2 \\ -3 & 0 & -6 \\ -2 & 6 & 0 \end{bmatrix}$

$AB = \begin{bmatrix} -4 & 3 & -6 \\ -31 & 54 & -26 \\ -28 & 9 & -50 \end{bmatrix}$ neither symm. nor skew symm.

9. Express the matrix $\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & -6 \\ -1 & 0 & 4 \end{bmatrix}$ as a sum of a lower triangular matrix & an upper triangular matrix with zero in its leading diagonal. Also express the matrix as a sum of a symmetric and a skew symmetric matrix.

Sol.

$$\begin{bmatrix} 1 & 2 & 5 \\ 2 & 3 & -6 \\ -1 & 0 & 4 \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 2 & 3 & 0 \\ -1 & 0 & 4 \end{bmatrix}}_{\text{Lower tr.}} + \underbrace{\begin{bmatrix} 0 & 2 & 5 \\ 0 & 0 & -6 \\ 0 & 0 & 0 \end{bmatrix}}_{\text{upper tr.}}$$

$$\therefore \quad A = \underbrace{\frac{1}{2}(A + A^T)}_{\text{symm.}} + \underbrace{\frac{1}{2}(A - A^T)}_{\text{skew sym.}}$$

$$\text{symm.} = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 3 & -3 \\ 2 & -3 & 4 \end{bmatrix}, \quad \text{skew sym.} = \begin{bmatrix} 0 & 0 & 3 \\ 0 & 0 & -3 \\ -3 & 3 & 0 \end{bmatrix}$$

10 (a) A is a square matrix of order n.

ℓ = maximum number of distinct entries if A is a triangular matrix

m = maximum number of distinct entries if A is a diagonal matrix

p = minimum number of zeroes if A is a triangular matrix.

If $\ell + 5 = p + 2m$, find the order of the matrix.

(b) Let A be the set of all 3×3 skew symmetric matrices whose entries are either –1, 0 or 1. If there are exactly three 0's, three 1's and three (–1)'s, then find the number of such matrices.

[Sol.$_{13/05}$ (a) $l = \dfrac{n(n+1)}{2} + 1$, $m = n + 1$, $p = \dfrac{n(n-1)}{2}$

$\therefore \quad \dfrac{n(n+1)}{2} + 6 = \dfrac{n(n-1)}{2} + 2(n+1)$

$\Rightarrow \quad n^2 + n + 12 = n^2 - n + 4n + 4$

$\Rightarrow \quad 8 = 2n \quad \Rightarrow n = 4$

$\therefore$ order of matrix = 4

(b) In a skew symmetric matrix, diagonal elements are zero. Also $a_{ij} + a_{ji} = 0$

$$\begin{bmatrix} 0 & \times & \times \\ \times & 0 & \times \\ \times & \times & 0 \end{bmatrix}$$

Hence number of matrices = $2 \times 2 \times 2 = 8$ **Ans.**]

11 If A is an idempotent non-zero matrix and I is an identity matrix of the same order, find the value of n, n $\in$N, such that $(A + I)^n = I + 127 A$.

Sol. By Binomial expansion

$$(A+I)^n = A^n + {}^nC_1 A^{n-1}.I + {}^nC_2 A^{n-2}.I$$

$$+ \ldots + I^n$$

and we know $A^2 = A^3 = A^4 \ldots A^n = A$

$$\therefore (A+I)^n = A({}^nC_0 + {}^nC_1 + \cdots {}^nC_{n-1}) + I$$

$$= (2^n - 1)A + I = 127A + I$$

$$2^n - 1 = 127$$

$$2^n = 128$$

$$n = 7$$

12 Let $A = \begin{bmatrix} 2 & 0 & 7 \\ 0 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -x & 14x & 7x \\ 0 & 1 & 0 \\ x & -4x & -2x \end{bmatrix}$ are two matrices such that $AB = (AB)^{-1}$ and $AB \neq I$ (where I is an identity matrix of order 3×3).

Find the value of Tr. $\left(AB + (AB)^2 + (AB)^3 + ... + (AB)^{100}\right)$,

where Tr. (A) denotes the trace of matrix A.

[Sol. $AB = (AB)^{-1} \Rightarrow (AB)^2 = I$

$$AB = \begin{bmatrix} 2 & 0 & 7 \\ 0 & 1 & 0 \\ 1 & -2 & 1 \end{bmatrix} \begin{bmatrix} -x & 14x & 7x \\ 0 & 1 & 0 \\ x & -4x & -2x \end{bmatrix} = \begin{bmatrix} 5x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 10x-2 & 5x \end{bmatrix}$$

$$(AB)^2 = \begin{bmatrix} 5x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 10x-2 & 5x \end{bmatrix} \begin{bmatrix} 5x & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 10x-2 & 5x \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\Rightarrow \quad \begin{bmatrix} 25x^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 10x-2+50x^2-10x & 25x^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\Rightarrow \quad 25x^2 = 1 \Rightarrow x = \pm \frac{1}{5}$

$\quad 50x^2 - 2 = 0 \Rightarrow x = \pm \frac{1}{5}$

at $\quad x = \frac{1}{5}, AB = I$ but $AB \neq I$

$\therefore \quad x = -\frac{1}{5}$

$$AB = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & -1 \end{bmatrix}$$

$\because \quad (AB)^2 = I \Rightarrow (AB)^3 = AB$

$\quad (AB)^4 = (AB)^2 = I$

$\therefore \quad (AB)^{2n} = I$ and $(AB)^{2n-1} = AB, n \in N$

$\therefore \quad AB + (AB)^2 + (AB)^3 + ... + (AB)^{100}$

$\quad = AB + I + AB + I + ... + I$

$\quad = 50\,(AB) + 50\,I$

Tr. $(AB + (AB)^2 + ... + (AB)^{100})$ = Tr. $(50(AB) + 50I)$

= Tr. $(50\,(AB))$ + Tr. $(50\,I)$

= 50 Tr.(AB) + 50 Tr. (I)

$= 50\,(-1+1-1) + 50\,(1+1+1)$

$= -50 + 150 = 100$. **Ans.**]

13. If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then prove that value of f and g satisfying the matrix equation $A^2 + fA + gI = \mathbf{O}$ are equal to $- t_r(A)$ and determinant of A respectively. Given a, b, c, d are non zero reals and $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$; $O = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$.

Solution:

Every square matrix satisfies its characteristic equation.

$$|A - \lambda I| = 0 \Rightarrow \begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (a-\lambda)(d-\lambda) - bc = 0$$

$$\Rightarrow \lambda^2 - (a+d)\lambda + ad - bc = 0$$

$$\therefore A^2 - (a+d)A + (ad-bc) = 0$$

Hence, $f = -(a+d) = -tr(A)$

$g = ad - bc = |A|$

14 Let $A=\begin{bmatrix}3x^2\\1\\6x\end{bmatrix}$, $B=[a\ b\ c]$ and $C=\begin{bmatrix}(x+2)^2 & 5x^2 & 2x\\5x^2 & 2x & (x+2)^2\\2x & (x+2)^2 & 5x^2\end{bmatrix}$ be three given matrices, where a, b, c and $x \in R$. Given that tr(AB) = tr(C) $\forall\, x \in R$, where tr(A) denotes trace of A. Find the value of (a + b + c)

[Sol. We have $AB=\begin{bmatrix}3ax^2 & 3bx^2 & 3cx^2\\a & b & c\\6ax & 6bx & 6cx\end{bmatrix}$

Now $\text{tr}\cdot(AB)=\text{tr}\cdot(C) \Rightarrow 3ax^2+b+6cx=(x+2)^2+2x+5x^2 \quad \forall\, x \in R$ (Identity)

$\Rightarrow 3ax^2+6cx+b=6x^2+6x+4$

Hence $a=2, c=1, b=4 \qquad \Rightarrow \qquad a+b+c=7$ **Ans.**]

Exercise S2

1

For the matrix $A = \begin{bmatrix} 4 & -4 & 5 \\ -2 & 3 & -3 \\ 3 & -3 & 4 \end{bmatrix}$ find A^{-2}.

Solⁿ.

$$A^{-1} = \begin{bmatrix} 3 & 1 & -3 \\ -1 & 1 & 2 \\ -3 & 0 & 4 \end{bmatrix}.$$

$$(A^{-1})^2 = A^{-2} = \begin{bmatrix} 17 & 4 & -19 \\ -10 & 0 & 13 \\ -21 & -3 & 25 \end{bmatrix}$$

2

(a) Given $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 4 & 1 \\ 2 & 3 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 \\ 3 & 4 \end{bmatrix}$. Find P such that $BPA = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

Soln. (a) $BPA = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$.

$P = B^{-1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} A^{-1}$

$|A| = 1(4-3) - 1(2-2) + 1(6-8)$

$\boxed{|A| = -1}$

where $B^{-1} = \begin{bmatrix} -4 & 3 \\ 3 & -2 \end{bmatrix}$, $A^{-1} = \begin{bmatrix} -1 & -2 & 3 \\ 0 & 1 & -1 \\ 2 & 1 & -2 \end{bmatrix}$

$\left(\because A^{-1} = \dfrac{\operatorname{adj} A}{|A|} \right)$

$\therefore P = B^{-1} \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} A^{-1}$

$= \begin{bmatrix} -4 & 3 \\ 3 & -2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} -1 & -2 & 3 \\ 0 & 1 & -1 \\ 2 & 1 & -2 \end{bmatrix}$

$= \begin{bmatrix} -4 & 3 & -4 \\ 3 & -2 & 3 \end{bmatrix} \begin{bmatrix} -1 & -2 & 3 \\ 0 & 1 & -1 \\ 2 & 1 & -2 \end{bmatrix}$

$= \begin{bmatrix} (4+0-8) & (8+3-4) & (-12-3+8) \\ (-3+0+6) & (-6-2+3) & (9+2-6) \end{bmatrix}$

$\Rightarrow P = \begin{bmatrix} -4 & 7 & -7 \\ 3 & -5 & 5 \end{bmatrix}$

//Ans

2

(b) Find the matrix A satisfying the matrix equation, $\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix} . A . \begin{bmatrix} 3 & 2 \\ 5 & -3 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 3 & -1 \end{bmatrix}$.

Sol:

$$A = \left(\begin{bmatrix} 2 & 1 \\ 3 & 2 \end{bmatrix}\right)^{-1} \begin{bmatrix} 2 & 4 \\ 3 & -1 \end{bmatrix} \left(\begin{bmatrix} 3 & 2 \\ 5 & -3 \end{bmatrix}\right)^{-1}.$$

$$= \begin{bmatrix} 2 & -1 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 4 \\ 3 & -1 \end{bmatrix} \left\{ \frac{1}{-19} \begin{bmatrix} -3 & -2 \\ -5 & 3 \end{bmatrix} \right\}$$

$$= -\frac{1}{19} \begin{bmatrix} (4-3) & (8+1) \\ (-6+6) & (-12-2) \end{bmatrix} \begin{bmatrix} -3 & -2 \\ -5 & 3 \end{bmatrix}$$

$$= -\frac{1}{19} \begin{bmatrix} 1 & 9 \\ 0 & -14 \end{bmatrix} \begin{bmatrix} -3 & -2 \\ -5 & 3 \end{bmatrix}$$

$$= -\frac{1}{19} \begin{bmatrix} (-3-45) & (-2+27) \\ (0+70) & (0-42) \end{bmatrix}$$

$$= -\frac{1}{19} \begin{bmatrix} -48 & 25 \\ 70 & -42 \end{bmatrix}$$

$$\Rightarrow A = \frac{1}{19} \begin{bmatrix} 48 & -25 \\ -70 & 42 \end{bmatrix}$$

// Ans

3

Let A be a 3×3 matrix such that $a_{11} = a_{33} = 2$ and all the other $a_{ij} = 1$. Let $A^{-1} = xA^2 + yA + zI$, then find the value of $(x + y + z)$ where I is a unit matrix of order 3.

Solⁿ.

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

Characteristic equation of A.

$$|A - \lambda I| = 0 = \begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0.$$

$$\Rightarrow 1 - 5\lambda + 5\lambda^2 - \lambda^3 = 0.$$

$\because$ Every matrix satisfies its characteristic equation.

$$\Rightarrow \quad I - 5A + 5A^2 - A^3 = 0.$$

$$I = 5A - 5A^2 + A^3.$$

$\Rightarrow$ Multiply both sides by A^{-1}

$$\Rightarrow \quad A^{-1} = 5I - 5A + A^2.$$

Comparing we have $x = 1$, $y = -5$, $z = 5$.

$\therefore \quad x + y + z = 1.$

4

Given that $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 2 & 3 \\ 1 & -1 & 3 \end{bmatrix}$, $C = \begin{bmatrix} 2 & 1 & 1 \\ 2 & 2 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, $D = \begin{bmatrix} 10 \\ 13 \\ 9 \end{bmatrix}$ and that Cb = D.

Solve the matrix equation Ax = b.

Solⁿ.

$$Ax = b.$$

$$\Rightarrow x = A^{-1}b.$$

Also. $Cb = D.$

$$\Rightarrow b = C^{-1}D.$$

$$\therefore x = A^{-1}C^{-1}D. = (CA)^{-1}D. = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

Where.

$$CA = \begin{bmatrix} 5 & 5 & 10 \\ 7 & 7 & 13 \\ 4 & 3 & 8 \end{bmatrix}; (CA)^{-1} = \begin{bmatrix} -17/5 & 2 & 1 \\ 4/5 & 0 & -1. \\ 7/5 & -1 & 0 \end{bmatrix}$$

5

Let $A = \begin{bmatrix} 1 & \frac{3}{2} \\ 1 & 2 \end{bmatrix}$, $B = \begin{bmatrix} 4 & -3 \\ -2 & 2 \end{bmatrix}$ and $C_r = \begin{bmatrix} r.3^r & 2^r \\ 0 & (r-1)3^r \end{bmatrix}$ be 3 given matrices.

Compute the value of $\sum_{r=1}^{50} \text{tr.}\left((AB)^r C_r\right)$. (where tr.(A) denotes trace of matrix A)

Sol^n.

$$AB = \begin{bmatrix} 1 & 3/2 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} 4 & -3 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I.$$

$$\sum_{r=1}^{50} tr\left((AB)^r Cr\right) = \sum_{r=1}^{50} tr(Cr).$$

$$= \sum_{r=1}^{50}\left(r.3^r + (r-1)3^r\right)$$

$$S = \sum_{r=1}^{50} (2r-1).\,3^r = 3\left[(49)(3^{50})+1\right].$$

$$S = 1(3)^1 + 3(3)^2 + 5(3)^3 + \cdots + 99(3)^{50}$$

$$3S = \quad 1(3)^2 + 3(3)^3 + \cdots\cdots + 99(3)^{51}$$

$$-2S = 3 + \underbrace{2(3)^2 + 2(3)^3 + \cdots\cdots}_{49 \text{ terms GP}} - 99(3)^{51}$$

$$\Rightarrow -2S = 3 + 2(3)^2\left(\frac{3^{49}-1}{3-1}\right) - 99(3)^{51}$$

$$\Rightarrow -2S = 3 + 3^{51} - 9 - 99(3)^{51} = -98(3)^{51} - 6$$

$$\Rightarrow S = 49(3)^{51} + 3 = 3\left(49(3)^{50}+1\right) \text{ Ans}$$

6 Given $A = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$; $B = \begin{bmatrix} 9 & 3 \\ 3 & 1 \end{bmatrix}$. I is a unit matrix of order 2. Find all possible matrix X in the following cases.

(a) $AX = A$ (b) $XA = I$ (c) $XB = \mathbf{O}$ but $BX \neq \mathbf{O}$.

Soln. Let $X = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$.

(i) $AX = A \Rightarrow \begin{bmatrix} 2a+c & 2b+d \\ 2a+c & 2b+d \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}$.

$\Rightarrow 2a+c = 2$ and $2b+d = 1$.

$\Rightarrow c = 2-2a$ and $d = 1-2b$.

$\therefore X = \begin{bmatrix} a & b \\ 2-2a & 1-2b \end{bmatrix}$. Ans

(ii) $XA = I \Rightarrow \begin{bmatrix} 2a+2b & a+b \\ 2c+2d & c+d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

$\Rightarrow 2a+2b = 1$ and $a+b = 0$

Not possible, Hence X does not exist. Ans

(iii) $XB = 0$ but $BX \neq 0$.

$XB = \begin{bmatrix} 9a+3b & 3a+b \\ 9c+3d & 3c+d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow 3a+b = 0 \Rightarrow b = -3a$

and $3c+d = 0 \Rightarrow d = -3c$.

$BX = \begin{bmatrix} 9a+3c & 9b+3d \\ 3a+c & 3b+d \end{bmatrix} \neq 0$

$\Rightarrow 3a+c \neq 0$ and $3b+d \neq 0$

$\therefore X = \begin{bmatrix} a & -3a \\ c & -3c \end{bmatrix}$ Ans

7

Find the product of two matrices A & B, where $A = \begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix}$ & $B = \begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}$ and use it to

solve the following system of linear equations,

$x + y + 2z = 1; 3x + 2y + z = 7; 2x + y + 3z = 2.$

Solⁿ.

$$AB = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

$$x + y + 2z = 1.$$
$$3x + 2y + z = 7.$$
$$2x + y + 3z = 2.$$

$$\Rightarrow \begin{bmatrix} 1 & 1 & 2 \\ 3 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix}.$$

$$\Rightarrow BX = C.$$

$$\Rightarrow X = B^{-1}C.$$

$$AC = \begin{bmatrix} -5 & 1 & 3 \\ 7 & 1 & -5 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 7 \\ 2 \end{bmatrix} = \begin{bmatrix} 8 \\ 4 \\ -4 \end{bmatrix}$$

Also. $AB = 4I.$

$$\Rightarrow \left(\frac{A}{4}\right) = B^{-1}.$$

$$\therefore X = \frac{1}{4} AC = \frac{1}{4} \begin{bmatrix} 8 \\ 4 \\ -4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$$

$$\Rightarrow x = 2 \quad y = 1, \quad z = -1.$$

8

Determine the values of a and b for which the system $\begin{bmatrix} 3 & -2 & 1 \\ 5 & -8 & 9 \\ 2 & 1 & a \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b \\ 3 \\ -1 \end{bmatrix}$

(a) has a unique solution ; (b) has no solution and (c) has infinitely many solutions

Sol^n.

$|A| = -14(a+3)$.

$$\text{adj}A = \begin{bmatrix} -8a-9 & 2a+1 & -10 \\ 18-5a & 3a-2 & -22 \\ 21 & -7 & -14 \end{bmatrix}$$

$$(\text{adj}A)B = \begin{bmatrix} -8ab-9b+6a+13 \\ 18b-5ab+9a+16 \\ 21b-7 \end{bmatrix}$$

(a) For unique sol^n $a \neq -3$.

(b) For no solution $a = -3$ and $b \neq 1/3$.

(c) For infinite sol^n. $a = -3$ and $b = 1/3$.

9

If $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$; $B = \begin{bmatrix} 3 & 1 \\ 1 & 0 \end{bmatrix}$; $C = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$ and $X = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}$, then solve the following matrix equation.

(a) $AX = B - I$ (b) $(B - I)X = IC$ (c) $CX = A$

Soln.

(a) $AX = (B-I) \Rightarrow X = A^{-1}(B-I). = -\frac{1}{2}\begin{bmatrix} 4 & -2 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & -1 \end{bmatrix}$

$= \begin{bmatrix} -3 & -3 \\ \frac{5}{2} & 2 \end{bmatrix}.$

(b) $X = (B-I)^{-1}C. = -\frac{1}{3}\begin{bmatrix} -1 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ -1 & -2 \end{bmatrix}$

(c) $\because |C| = 0 \Rightarrow$ No solution [as C^{-1} does not exist].

10 $A_{3\times 3}$ is a matrix such that $|A|=a$, $B = (\text{adj } A)$ such that $|B|= b$. Find the value of $(ab^2 + a^2b + 1)S$ where $\frac{1}{2}S = \frac{a}{b} + \frac{a^2}{b^3} + \frac{a^3}{b^5} + \ldots\ldots$ up to ∞, and $a = 3$.

Solⁿ.

$$S = 2\left[\frac{a/b}{1 - a/b^2}\right] = \frac{2ab}{b^2 - a}.$$

Also. $B = \text{adj } A \Rightarrow |B| = |A|^2 \Rightarrow b = a^2 = 9.$

$$\therefore (ab^2 + a^2b + 1)S = (ab^2 + a^2b + 1)\left(\frac{2ab}{b^2 - a}\right).$$

$$= (3\times 81 + 9\times 9 + 1)\left(\frac{2\times 3\times 9}{81 - 3}\right) = 225.$$

11 If A and B are square matrices of order 3, where $|A| = -2$ and $|B| = 1$, then find $\left|\left(A^{-1}\right)\operatorname{adj}\left(B^{-1}\right)\operatorname{adj}\left(2A^{-1}\right)\right|$.

Soln.

$$\left|\left(A^{-1}\right)\operatorname{adj}\left(B^{-1}\right)\operatorname{adj}\left(2A^{-1}\right)\right|.$$

$$= |A^{-1}|\,|\operatorname{adj}(B^{-1})|\,|\operatorname{adj}(2A^{-1})|.$$

$$= |A^{-1}|\,|B^{-1}|^2\,|2A^{-1}|^2.$$

$$= \frac{1}{|A|}\cdot\frac{1}{|B|^2}\cdot\frac{2^6}{|A|^2}.$$

$$= \frac{1}{-2}\times\frac{1}{1}\times\frac{2^6}{4} = -8.$$

Exercise JM

1 Let $A = \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{pmatrix}$. If u_1 and u_2 are column matrices such that $Au_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ and $Au_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$, then $u_1 + u_2$ is equal to : **[AIEEE-2012]**

(1) $\begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix}$ (2) $\begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}$ (3) $\begin{pmatrix} -1 \\ 1 \\ -1 \end{pmatrix}$ (4) $\begin{pmatrix} -1 \\ -1 \\ 0 \end{pmatrix}$

Solⁿ Let $U_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \Rightarrow AU_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$

$$\Rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \Rightarrow \begin{bmatrix} x \\ 2x+y \\ 3x+2y+z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

$\Rightarrow x = 1$; $2x+2y=0$; $3x+2y+z=0$

$\Rightarrow y = -2$; $\Rightarrow 3(1)+2(-2)+z=0$

$\Rightarrow z = 1$

$\therefore U_1 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$

Similarly we get $U_2 = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$

$\therefore U_1 + U_2 = \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$ (Ans. 1).

2. 7. If $P = \begin{bmatrix} 1 & \alpha & 3 \\ 1 & 3 & 3 \\ 2 & 4 & 4 \end{bmatrix}$ is the adjoint of a 3×3 matrix A and $|A| = 4$, then α is equal to

[JEE(Main) - 2013]

(1) 4 (2) 11 (3) 5 (4) 0

Sol^n

$|P| = 2\alpha - 6$

For 3x3 matrix $|adj(A)| = |A|^2$

$\Rightarrow \quad 2\alpha - 6 = 16 \Rightarrow \boxed{\alpha = 11}$

Ans (2)

3

If A is an 3 × 3 non-singular matrix such that AA' = A'A and B = A^{-1} A', the BB' equals :

[JEE(Main) - 2014]

(1) I + B (2) I (3) B^{-1} (4) $(B^{-1})'$

Sol. $B = A^{-1}\ A^T$

$B^T = (A^{-1}\ A^T)^T = A\ (A^{-1})^T$

$\therefore\ BB^T = A^{-1}\ A^T\ A\ (A^{-1})^T$

$= A^{-1}\ AA^T\ (A^T)^{-1}\ (\because\ AA^T = A^TA)$

$= I$

Ans (2)

4 If $A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & 2 & b \end{bmatrix}$ is a matrix satisfying the equation $AA^T = 9I$, where I is 3×3 identity matrix, then the ordered pair (a, b) is equal to : **[JEE(Main)-2015]**

(1) (2, 1) (2) (–2, –1) (3) (2, –1) (4) (–2, 1)

Ans. (2)

Sol. $\left(\frac{A}{3}\right)\left(\frac{A}{3}\right)^T = I \Rightarrow \frac{A}{3}$ is orthogonal matrix

$$\frac{A}{3} = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{a}{3} & \frac{2}{3} & \frac{b}{3} \end{bmatrix}$$

$\because$ Three rows are corresponding to three mutually perpendicular unit vectors

$\Rightarrow \frac{a^2+4+b^2}{9} = 1$, a+2b+4 = 0 & 2a–2b+2 =0

$\Rightarrow$ a = –2 & b = –1

If $A = \begin{bmatrix} 5a & -b \\ 3 & 2 \end{bmatrix}$ and A adj A = A A^T, then 5a + b is equal to : **[JEE(Main)-2016]**

(1) 13 (2) –1 (3) 5 (4) 4

Ans. (3)

Sol. $A = \begin{bmatrix} 5a & -b \\ 3 & 2 \end{bmatrix}$ and $A^T = \begin{bmatrix} 5a & 3 \\ -b & 2 \end{bmatrix}$

$$AA^T = \begin{bmatrix} 25a^2+b^2 & 15a-2b \\ 15a-2b & 13 \end{bmatrix}$$

Now, A adj A = $|A|I_2 = \begin{bmatrix} 10a+3b & 0 \\ 0 & 10a+3b \end{bmatrix}$

Given AA^T = A. adj A

$15a - 2b = 0$ ...(1)

$10a + 3b = 13$ ...(2)

Solving we get

$5a = 2$ and $b = 3$

$\therefore \quad 5a + b = 5$

6

If $A=\begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix}$, then adj $(3A^2 + 12A)$ is equal to :- **[JEE(Main)-2017]**

(1) $\begin{bmatrix} 72 & -63 \\ -84 & 51 \end{bmatrix}$ (2) $\begin{bmatrix} 72 & -84 \\ -63 & 51 \end{bmatrix}$ (3) $\begin{bmatrix} 51 & 63 \\ 84 & 72 \end{bmatrix}$ (4) $\begin{bmatrix} 51 & 84 \\ 63 & 72 \end{bmatrix}$

Ans. (3)

Sol. Given $A = \begin{bmatrix} 2 & -3 \\ -4 & 1 \end{bmatrix}$

$$3A^2 = 3\begin{bmatrix} 16 & -9 \\ -12 & 13 \end{bmatrix} = \begin{bmatrix} 48 & -27 \\ -36 & 39 \end{bmatrix}$$

$$12A = \begin{bmatrix} 24 & -36 \\ -48 & 12 \end{bmatrix}$$

$$\therefore \quad 3A^2 + 12A = \begin{bmatrix} 72 & -63 \\ -84 & 51 \end{bmatrix}$$

$$\text{adj}\,(3A^2 + 12A) = \begin{bmatrix} 51 & 63 \\ 84 & 72 \end{bmatrix}$$

7 Let $A=\begin{pmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{pmatrix}, (\alpha\in R)$ such that $A^{32}=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix}$. Then a value of α is

[JEE(Main) Apr-2019]

(1) $\frac{\pi}{16}$ (2) 0 (3) $\frac{\pi}{32}$ (4) $\frac{\pi}{64}$

Sol. $A=\begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}$

$$A^2=\begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}\begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}$$

$$=\begin{bmatrix}\cos 2\alpha & -\sin 2\alpha\\ \sin 2\alpha & \cos 2\alpha\end{bmatrix}$$

$$A^3=\begin{bmatrix}\cos 2\alpha & -\sin 2\alpha\\ \sin 2\alpha & \cos 2\alpha\end{bmatrix}\begin{bmatrix}\cos\alpha & -\sin\alpha\\ \sin\alpha & \cos\alpha\end{bmatrix}$$

$$=\begin{bmatrix}\cos 3\alpha & -\sin 3\alpha\\ \sin 3\alpha & \cos 3\alpha\end{bmatrix}$$

Similarly

$$A^{32}=\begin{bmatrix}\cos 32\alpha & -\sin 32\alpha\\ \sin 32\alpha & \cos 32\alpha\end{bmatrix}=\begin{bmatrix}0 & -1\\ 1 & 0\end{bmatrix}$$

$$\Rightarrow \quad \cos 32\alpha = 0 \ \&\ \sin 32\alpha = 1$$

$$\Rightarrow \quad 32\alpha=(4n+1)\frac{\pi}{2}, n\in I$$

$$\alpha=(4n+1)\frac{\pi}{64}, n\in I$$

$$\alpha=\frac{\pi}{64} \text{ for } n=0$$

Ans (4)

8. If $A=\begin{bmatrix} e^t & e^{-t}\cos t & e^{-t}\sin t \\ e^t & -e^{-t}\cos t-e^{-t}\sin t & -e^{-t}\sin t+e^{-t}\cos t \\ e^t & 2e^{-t}\sin t & -2e^{-t}\cos t \end{bmatrix}$. Then A is - **[JEE(Main)-2019]**

(1) Invertible only if $t=\frac{\pi}{2}$

(2) not invertible for any tεR

(3) invertible for all tεR

(4) invertible only if t=π

Solution:

$$|A| = e^{-t}\begin{vmatrix} 1 & \cos t & \sin t \\ 1 & -\cos t-\sin t & -\sin t+\cos t \\ 1 & 2\sin t & -2\cos t \end{vmatrix}$$

$R_1 \to R_1 - R_2$ & $R_2 \to R_2 - R_3$

$$|A| = e^{-t}\begin{vmatrix} 0 & \sin t+2\cos t & 2\sin t-\cos t \\ 0 & -\cos t-3\sin t & -\sin t+3\cos t \\ 1 & 2\sin t & -2\cos t \end{vmatrix}$$

$$\Rightarrow |A| = e^{-t}\left((S+2C)(3C-S)+(2S-C)(C+3S)\right)$$

$$= e^{-t}\left(3CS+6C^2-S^2-2CS+2SC+6S^2-C^2-3CS\right)$$

$$= 5e^{-t} \neq 0$$

9. Let $A = \begin{bmatrix} 2 & b & 1 \\ b & b^2+1 & b \\ 1 & b & 2 \end{bmatrix}$ where $b > 0$. Then the minimum value of $\frac{\det(A)}{b}$ is : **[JEE(Main)-2019]**

(1) $\sqrt{3}$ (2) $-\sqrt{3}$ (3) $-2\sqrt{3}$ (4) $2\sqrt{3}$

Solution:

$$\frac{|A|}{b} = \frac{2(2b^2+2-b^2) - b(2b-b) + 1(b^2-b^2-1)}{b}$$

$$= \frac{2(b^2+2) - b^2 - 1}{b}$$

$$= \frac{b^2+3}{b} = b + \frac{3}{b}$$

By $AM \geq GM$ $(b>0)$

$$b + \frac{3}{b} \geq 2\sqrt{b \cdot \frac{3}{b}}$$

$$\Rightarrow b + \frac{3}{b} \geq 2\sqrt{3}$$

10. Let $A = \begin{pmatrix} 0 & 2q & r \\ p & q & -r \\ p & -q & r \end{pmatrix}$. It $AA^T = I_3$, then $|p|$ is : **[JEE(Main)-2019]**

(1) $\frac{1}{\sqrt{2}}$ (2) $\frac{1}{\sqrt{5}}$ (3) $\frac{1}{\sqrt{6}}$ (4) $\frac{1}{\sqrt{3}}$

Solution: $AA^T = I_3$

$\Rightarrow$ A is orthogonal matrix

$\therefore \; 0^2 + p^2 + p^2 = 1 \Rightarrow p^2 = \frac{1}{2} \Rightarrow |p| = \frac{1}{\sqrt{2}}$

11 Let A and B be two invertible matrices of order 3×3. If $\det(ABA^T) = 8$ and $\det(AB^{-1}) = 8$, then $\det(BA^{-1}B^T)$ is equal to :- **[JEE(Main) Jan-2019]**

(1) 16 (2) $\frac{1}{16}$ (3) $\frac{1}{4}$ (4) 1

Ans. (2)

Sol. $|A|^2.|B| = 8$ and $\dfrac{|A|}{|B|} = 8 \Rightarrow |A| = 4$ and $|B| = \dfrac{1}{2}$

$\therefore \det(BA^{-1}.B^T) = \dfrac{1}{4} \times \dfrac{1}{4} = \dfrac{1}{16}$

12 Let $P = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 9 & 3 & 1 \end{bmatrix}$ and $Q = [q_{ij}]$ be two 3×3 matrices such that $Q - P^5 = I_3$. Then $\frac{q_{21} + q_{31}}{q_{32}}$ is equal to :

[JEE(Main) Jan-2019]

(1) 15 (2) 9 (3) 135 (4) 10

Soln

Ans. (4)

Sol. $P = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 9 & 3 & 1 \end{bmatrix}$

$$P^2 = \begin{bmatrix} 1 & 0 & 0 \\ 3+3 & 1 & 0 \\ 9+9+9 & 3+3 & 1 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} 1 & 0 & 0 \\ 3+3+3 & 1 & 0 \\ 6.9 & 3+3+3 & 1 \end{bmatrix}$$

$$P^n = \begin{bmatrix} 1 & 0 & 0 \\ 3n & 1 & 0 \\ \frac{n(n+1)}{2}3^2 & 3n & 1 \end{bmatrix}$$ (by observing pattern)

$$P^5 = \begin{bmatrix} 1 & 0 & 0 \\ 5.3 & 1 & 0 \\ 15.9 & 5.3 & 1 \end{bmatrix}$$

$Q = P^5 + I_3$

$$Q = \begin{bmatrix} 2 & 0 & 0 \\ 15 & 2 & 0 \\ 135 & 15 & 2 \end{bmatrix}$$

$$\Rightarrow \frac{q_{21} + q_{31}}{q_{32}} = \frac{15 + 135}{15}$$

$$\boxed{= 10}$$

Exercise JA

1 The number of 3×3 matrices A whose entries are either 0 or 1 and for which the system $A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ has exactly two distinct solutions, is

(A) 0 (B) $2^9 - 1$ (C) 168 (D) 2

Sol:-

$$A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Three planes can not meet at two distinct points.

$\therefore$ Exactly 2 solutions are not possible

$\therefore$ No. of matrices $= 0$

2. Let k be a positive real number and let

$$A = \begin{bmatrix} 2k-1 & 2\sqrt{k} & 2\sqrt{k} \\ 2\sqrt{k} & 1 & -2k \\ -2\sqrt{k} & 2k & -1 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 2k-1 & \sqrt{k} \\ 1-2k & 0 & 2\sqrt{k} \\ -\sqrt{k} & -2\sqrt{k} & 0 \end{bmatrix}.$$

If det (adj A) + det(adj B) = 10^6, then [k] is equal to

[**Note :** adj M denotes the adjoint of a square matrix M and [k] denotes the largest integer less than or equal to k].

Sol:-

$$\because A = \begin{bmatrix} 2K-1 & 2\sqrt{K} & 2\sqrt{K} \\ 2\sqrt{K} & 1 & -2K \\ -2\sqrt{K} & 2K & -1 \end{bmatrix} \Rightarrow |A| = (2K+1)^3$$

$$\& \; B = \begin{bmatrix} 0 & 2K-1 & \sqrt{K} \\ 1-2K & 0 & 2\sqrt{K} \\ -\sqrt{K} & -2\sqrt{K} & 0 \end{bmatrix} \Rightarrow |B| = 0$$

(skew sym. matrix of odd order)

$$\because \det(\operatorname{adj}(A)) = |A|^{n-1}$$

$$\det(\operatorname{adj}(A)) = |A|^{3-1} = |A|^2$$

$$\because \det(\operatorname{adj}(B)) = |B|^{3-1} = |B|^2$$

$\because$ Given

$$\det(\operatorname{adj}(A)) + \det(\operatorname{adj}(B)) = 10^6$$

$$|A|^2 + |B|^2 = 10^6$$

$$\Rightarrow ((2K+1)^3)^2 + 0 = 10^6$$

$$\Rightarrow 2K+1 = 10$$

$$\Rightarrow K = 9/2 = 4.5$$

$$\therefore [K] = 4$$

3

Let $\omega \neq 1$ be a cube root of unity and S be the set of all non-singular matrices of the form $\begin{bmatrix} 1 & a & b \\ \omega & 1 & c \\ \omega^2 & \omega & 1 \end{bmatrix}$,

where each of a,b and c is either ω or ω^2. Then the number of distinct matrices in the set S is-

(A) 2 (B) 6 (C) 4 (D) 8

[JEE 2011, 3, (–1)]

Solve this question after complex numbers

Sol:-

$\because \omega^3 = 1$

$\because 1 + \omega + \omega^2 = 0$

let

$A = \begin{bmatrix} 1 & a & b \\ \omega & 1 & c \\ \omega^2 & \omega & 1 \end{bmatrix}$ & Given 'A' is Non-singular so $|A| \neq 0$

then $|A| = 1 - \omega(a+c) + ac\,\omega^2$

* if $a = c = \omega$; then $|A| = 1 - 2\omega^2 + \omega^4$

$\Rightarrow |A| = 1 - 2\omega^2 + \omega \neq 0$

* In all other cases

$\boxed{\begin{matrix} a=\omega \\ c=\omega^2 \end{matrix} \Big/ \begin{matrix} a=\omega^2 \\ c=\omega \end{matrix} \Big/ \begin{matrix} a=\omega^2 \\ c=\omega^2 \end{matrix}}$

↳ the $|A| = 0$ so $A \rightarrow$ singular matrix

but given $A \rightarrow$ Non-singular

$\therefore$ possible Triplets of

$(a,b,c) = \begin{cases} \rightarrow (\omega, \omega, \omega) \\ \rightarrow (\omega, \omega^2, \omega) \end{cases} \Bigg\}$ 2 A

4

Let M be a 3 × 3 matrix satisfying

$$M\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}-1\\2\\3\end{bmatrix}, M\begin{bmatrix}1\\-1\\0\end{bmatrix}=\begin{bmatrix}1\\1\\-1\end{bmatrix}, \text{ and } M\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}0\\0\\12\end{bmatrix}.$$

Then the sum of the diagonal entries of M is

Sol. **(9)**

$$\text{Let } M=\begin{bmatrix}a & b & c\\ d & e & f\\ g & h & i\end{bmatrix}$$

$$M\begin{bmatrix}0\\1\\0\end{bmatrix}=\begin{bmatrix}-1\\2\\3\end{bmatrix}\Rightarrow b=-1,\ e=2,\quad h=3$$

$$M\begin{bmatrix}1\\-1\\0\end{bmatrix}=\begin{bmatrix}1\\1\\-1\end{bmatrix}\Rightarrow a=0,\ d=3,\quad g=2$$

$$M\begin{bmatrix}1\\1\\1\end{bmatrix}=\begin{bmatrix}0\\0\\12\end{bmatrix}\Rightarrow g+h+i=12\Rightarrow i=7$$

$\therefore$ Sum of diagonal elements = 9.

5 Let $P = [a_{ij}]$ be a 3×3 matrix and let $Q = [b_{ij}]$, where $b_{ij} = 2^{i+j}a_{ij}$ for $1 \le i, j \le 3$. If the determinant of P is 2, then the determinant of the matrix Q is - **[JEE 2012, 3M, –1M]**

(A) 2^{10} (B) 2^{11} (C) 2^{12} (D) 2^{13}

Sol:

$\because Q = [b_{ij}]$; $b_{ij} = 2^{i+j} a_{ij}$

$\because P = [a_{ij}]$ & Given $|P| = 2$

so $$|Q| = \begin{vmatrix} 2^2 a_{11} & 2^3 a_{12} & 2^4 a_{13} \\ 2^3 \cdot a_{21} & 2^4 a_{22} & 2^5 a_{23} \\ 2^4 \cdot a_{31} & 2^5 a_{32} & 2^6 a_{33} \end{vmatrix}$$

$$= 2^2 \cdot 2^3 \cdot 2^4 \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ 2 \cdot a_{21} & 2 \cdot a_{22} & 2 \cdot a_{23} \\ 2^2 \cdot a_{31} & 2^2 a_{32} & 2^2 \cdot a_{33} \end{vmatrix}$$

$$|Q| = 2^{12} \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

$$|Q| = 2^{12} \cdot |P| = 2^{12} \cdot (2) = 2^{13} \text{ Ans.}$$

6 If P is a 3 × 3 matrix such that $P^T = 2P + I$, where P^T is the transpose of P and I is the 3 × 3 identity matrix, then there exists a column matrix $X = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ such that **[JEE 2012, 3M, –1M]**

(A) $PX = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ (B) $PX = X$ (C) $PX = 2X$ (D) $PX = -X$

Sol:

$\because P^T = 2P + I$

$\Rightarrow (P^T)^T = (2P + I)^T$

$\Rightarrow P = 2P^T + I^T$

$\Rightarrow P = 2P^T + I$

$\Rightarrow P = 2(2P + I) + I$

$\Rightarrow P = 4P + 3I$

$\Rightarrow 3(P + I) = 0$

$\Rightarrow P + I = 0$

$\Rightarrow PX + X = 0$

$\Rightarrow PX = -X$ 1.

7 If the adjoint of a 3×3 matrix P is $\begin{bmatrix} 1 & 4 & 4 \\ 2 & 1 & 7 \\ 1 & 1 & 3 \end{bmatrix}$, then the possible value(s) of the determinant of P is (are) - **[JEE 2012, 4M]**

(A) –2 (B) –1 (C) 1 (D) 2

Sol :-

$\because \ Adj(P) = \begin{bmatrix} 1 & 4 & 4 \\ 2 & 1 & 7 \\ 1 & 1 & 3 \end{bmatrix}$

$\Rightarrow \because \ |Adj(P)| = |P|^{3-1}$

$\Rightarrow \begin{vmatrix} 1 & 4 & 4 \\ 2 & 1 & 7 \\ 1 & 1 & 3 \end{vmatrix} = |P|^2$

$\Rightarrow |P|^2 = 4$

$\Rightarrow |P| = \pm 2$ Ans. (A, D)

8 For 3 × 3 matrices M and N, which of the following statement(s) is (are) **NOT** correct ?

(A) N^TMN is symmetric or skew symmetric, according as M is symmetric or skew symmetric

(B) MN – NM is skew symmetric for all symmetric matrices M and N

(C) MN is symmetric for all symmetric matrices M and N

(D) (adj M) (adj N) = adj (M N) for all invertible matrices M and N

[JEE-Advanced 2013, 4, (–1)]

Sol:-

A.) let

$P = N^TMN$

$\Rightarrow P^T = (N^TMN)^T$

$\Rightarrow P^T = N^TM^TN$

if $M^T = M$ || if $M^T = -M$

$\Rightarrow P^T = N^TMN$

$\Rightarrow \underbrace{P^T = P}$

'P' is symmetric

$\Rightarrow P^T = -N^TMN$

$\Rightarrow \underbrace{P^T = -P}$

'P' is skew symmetric

B.) let $Q = MN - NM$

$\Rightarrow Q^T = (MN)^T - (NM)^T$

$\Rightarrow Q^T = N^TM^T - M^TN^T$

$\left\{ \begin{array}{l} \because M^T = M \\ \because N^T = N \end{array} \text{ Given} \right\}$

$\Rightarrow Q^T = NM - MN$

$\Rightarrow Q^T = -Q$

$\therefore Q \rightarrow$ is skew symmetric

C.) let $R = MN$

$\Rightarrow R^T = (MN)^T$

$\Rightarrow R^T = N^TM^T$

$\left\{ \because \text{Given } \begin{array}{l} N^T = N \\ M^T = M \end{array} \right\}$

$\Rightarrow R^T = NM$

so $R^T \neq R$ so R is Not symmetric

D.)

$\because \boxed{adj(MN) = adj(N) . adj(M)}$

but Given

$\boxed{adj(MN) = adj(M).adj(N)}$

so it is incorrect.

(Ans C, D)

9 Let M be a 2 × 2 symmetric matrix with integer entries. Then M is invertible if

(A) the first column of M is the transpose of the second row of M

(B) the second row of M is the transpose of the first column of M

(C) M is a diagonal matrix with nonzero entries in the main diagonal

(D) the product of entries in the main diagonal of M is not the square of an integer

[JEE(Advanced)-2014, 3]

Sol:-

Let $M=\begin{bmatrix} a & b \\ b & c \end{bmatrix}$; $a, b, c \in I-\{0\}$

A. $\because$ Given that $\begin{bmatrix} a \\ b \end{bmatrix}=\begin{bmatrix} b \\ c \end{bmatrix}$

$\Rightarrow a=b=c=\alpha$ (let

then $M=\begin{bmatrix} \alpha & \alpha \\ \alpha & \alpha \end{bmatrix}$

$\Rightarrow |M|=0 \Rightarrow$ Non-invertible

B. Given that $[b, c]=[a \;\; b]$

$\Rightarrow a=b=c=\alpha$ (let)

$\therefore |M|=0 \Rightarrow$ Non-invertible

C. As given $M=\begin{bmatrix} a & 0 \\ 0 & c \end{bmatrix}$

$\therefore |M|=ac \neq 0 \Rightarrow M$ is invertible

D. $M=\begin{bmatrix} a & b \\ b & c \end{bmatrix} \Rightarrow |M|=ac-b^2 \neq 0$

$\because$ Given ac is not equal to sq. of integer.

$\therefore M$ is invertible

10 Let M and N be two 3 × 3 matrices such that MN = NM. Further, if $M \neq N^2$ and $M^2 = N^4$, then

(A) determinant of $(M^2 + MN^2)$ is 0

(B) there is a 3 × 3 non-zero matrix U such that $(M^2 + MN^2)U$ is zero matrix

(C) determinant of $(M^2 + MN^2) \geq 1$

(D) for a 3 × 3 matrix U, if $(M^2 + MN^2)U$ equals the zero matrix then U is the zero matrix

$M^2 - N^4 = 0$

$(M - N^2)(M + N^2)$

[JEE(Advanced)-2014, 3]

Sol. Ans. (A,B)

(A) $(M - N^2)(M + N^2) = \mathbf{O}$ ...(1) ($\therefore MN^2 = N^2M$)

$\Rightarrow |M - N^2|\,|M + N^2| = 0$

Case I : If $|M + N^2| = 0$

$\therefore |M^2 + MN^2| = 0$

Case II : If $|M + N^2| \neq 0 \Rightarrow M + N^2$ is invertible

from (1)

$(M - N^2)(M + N^2)(M + N^2)^{-1} = \mathbf{O} \Rightarrow M - N^2 = \mathbf{O}$ which is wrong

(B) $(M + N^2)(M - N^2) = \mathbf{O}$

pre-multiply by M

$\Rightarrow (M^2 + MN^2)(M - N^2) = \mathbf{O}$ ...(2)

Let $M - N^2 = U$

$\Rightarrow$ from equation (2) there exist same non zero 'U'

$(M^2 + MN^2)U = \mathbf{O}$

11 Let X and Y be two arbitrary, 3 × 3, non-zero, skew-symmetric matrices and Z be an arbitrary 3 × 3, non-zero, symmetric matrix. Then which of the following matrices is (are) skew symmetric ? **[JEE(Advanced)-2015, 4M, –2M]**

(A) $Y^3Z^4 - Z^4Y^3$ (B) $X^{44} + Y^{44}$ (C) $X^4Z^3 - Z^3X^4$ (D) $X^{23} + Y^{23}$

Ans. (C,D)

Sol. $x^T = -x,\ y^T = -y,\ z^T = z$

(A) Let $P = y^3z^4 - z^4y^3$

$P^T = (y^3z^4)^T - (z^4y^3)^T = (z^4)^T(y^3)^T - (y^3)^T(z^4)^T = (z^T)^4(y^T)^3 - (y^T)^3(z^T)^4$

$= -z^4y^3 + y^3z^4 = P \Rightarrow$ symmetric

(B) Let $P = x^{44} + y^{44}$

$P^T = (x^{44})^T + (y^{44})^T = (x^T)^{44} + (y^T)^{44} = (-x)^{44} + (-y)^{44} = P$

$\therefore$ P is symm.

(C) Let $P = x^4z^3 - z^3x^4$

$P^T = (z^3)^T(x^4)^T - (x^4)^T(z^3)^T$

$= z^3x^4 - x^4z^3 = -P \Rightarrow$ skew symmetric

(D) Let $P = x^{23} + y^{23}$

$P^T = -x^{23} - y^{23} = -P \Rightarrow$ skew symmetric

(Ans. C, D)

12

Let $P=\begin{bmatrix}3 & -1 & -2\\ 2 & 0 & \alpha\\ 3 & -5 & 0\end{bmatrix}$, where $\alpha \in \mathbb{R}$, Suppose $Q=[q_{ij}]$ is a matrix such that $PQ = kI$, where $k \in \mathbb{R}$,

$\rightarrow |P| = 12\alpha + 20$

$k \neq 0$ and I is the identity matrix of order 3. If $q_{23} = -\frac{k}{8}$ and $\det(Q) = \frac{k^2}{2}$, then-

(A) $\alpha = 0,\ k = 8$

✓(B) $4\alpha - k + 8 = 0$

✓(C) $\det(P\operatorname{adj}(Q)) = 2^9$

(D) $\det(Q\operatorname{adj}(P)) = 2^{13}$

[JEE(Advanced)-2016, 4(–2)]

Sol. $PQ = kI \Rightarrow |PQ| = |kI| \Rightarrow |P||Q| = k^3|I|$

$\Rightarrow |P|.|Q| = k^3 \Rightarrow |P|\left(\frac{k^2}{2}\right) = k^3$

$\Rightarrow |P| = 2k \neq 0 \Rightarrow$ P is an invertible matrix

$\because PQ = kI$

$\therefore Q = kP^{-1}I = k.\frac{1}{|P|}.\operatorname{adj}(P).I = k.\frac{1}{2k}.\operatorname{adj}(P)$

$\therefore Q = \frac{\operatorname{adj} P}{2} \Longrightarrow Q = \frac{1}{2}\begin{bmatrix}5\alpha & 10 & -\alpha\\ 3\alpha & 6 & -(3\alpha+4)\\ -10 & 12 & 3\end{bmatrix}$

$\because q_{23} = -\frac{k}{8}$

$\therefore \frac{-(3\alpha+4)}{2} = -\frac{k}{8}$ —— (1)

$\therefore |P| = 2k \Rightarrow k = 10 + 6\alpha$ —— (2)

solve (1) & (2) $\alpha = -1$, $k = 4$

Now check option

Put value of k in (i).. we get $\alpha = -1$

$\therefore 4\alpha - k + 8 = 0$

& $\det(P(\operatorname{adj}.Q)) = |P|\ |\operatorname{adj}.Q| = 2k.\left(\frac{k^2}{2}\right)^2 = \frac{k^5}{2} = 2^9$

(Ans B, C)

13 Let $P = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 16 & 4 & 1 \end{bmatrix}$ and I be the identity matrix of order 3. If $Q = [q_{ij}]$ is a matrix such that $P^{50} - Q = I$, then $\frac{q_{31}+q_{32}}{q_{21}}$ equals **[JEE(Advanced)-2016, 3(–1)]**

(A) 52 (B) 103 (C) 201 (D) 205

Ans. (B)

Sol. $P = \begin{bmatrix} 1 & 0 & 0 \\ 4 & 1 & 0 \\ 16 & 4 & 1 \end{bmatrix} \Rightarrow P^2 = \begin{bmatrix} 1 & 0 & 0 \\ 8 & 1 & 0 \\ 16+32 & \underset{4\times 2}{8} & 1 \end{bmatrix}$

so, $P^3 = \begin{bmatrix} 1 & 0 & 0 \\ 12 & 1 & 0 \\ 16+32+48 & \underset{4\times 3}{12} & 1 \end{bmatrix}$ (from the symmetry)

$a \longrightarrow 16+32+48 \ldots 50 \text{ terms}$

$\Rightarrow a = 16(1+2+3+\ldots 50 \text{ terms})$

$= \frac{16(50(50+1))}{2}$

$b \longrightarrow 4 \times 50 = 200$

$P^{50} = \begin{bmatrix} 1 & 0 & 0 \\ 200 & 1 & 0 \\ \underbrace{\frac{16.50.51}{2}}_{a} & \underbrace{200}_{b} & 1 \end{bmatrix}$

by solving.

$Q = \begin{bmatrix} 0 & 0 & 0 \\ 200 & 0 & 0 \\ \frac{16.50.51}{2} & 200 & 0 \end{bmatrix}$

As, $\boxed{P^{50} - Q = I} \Rightarrow q_{31} = \frac{16.50.51}{2}$

$q_{32} = 200$ and $q_{21} = 200$

$\therefore \frac{q_{31}+q_{32}}{q_{21}} = \frac{16.50.51}{2.200} + 1 = 102 + 1 = 103$

(Ans B)

14 Which of the following is(are) NOT the square of a 3 × 3 matrix with real entries ?

[JEE(Advanced)-2017, 4(–2)]

(A) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ ✓ (B) $\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ ✓ (C) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (D) $\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$

Sol:- check the determinant value of all options. Matrix which is the square of another matrix has +ve determinant value.

A.) $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{vmatrix} = -1$

B.) $\begin{vmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix} = -1$

C.) $\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{vmatrix} = 1 \rightarrow$ we have $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3^2$

D.) $\begin{vmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{vmatrix} = 1$

We have:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{vmatrix}^2$$

15 How many 3×3 matrices M with entries from $\{0,1,2\}$ are there, for which the sum of the diagonal entries of M^TM is 5 ? **[JEE(Advanced)-2017, 3(–1)]**

(A) 198 (B) 126 (C) 135 (D) 162

Sol. Let $M = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$

$\therefore$ $tr(M^TM) = a^2 + b^2 + c^2 + d^2 + e^2 + f^2 + g^2 + h^2 + i^2 = 5$, where entries are $\{0,1,2\}$

Only two cases are possible.

(I) five entries 1 and other four zero

$\therefore$ ${}^9C_5 \times 1 = 126$

(II) One entry is 2, one entry is 1 and others are 0.

$\therefore$ ${}^9C_2 \times 2! = 72$

Total = 126 + 72 = 198.

$$M^TM = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

$$tr(M^TM) = (a^2+d^2+g^2) + (b^2+e^2+h^2) + (c^2+f^2+i^2)$$

16 For a real number α, if the system

$$\begin{bmatrix} 1 & \alpha & \alpha^2 \\ \alpha & 1 & \alpha \\ \alpha^2 & \alpha & 1 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

of linear equations, has infinitely many solutions, then $1 + \alpha + \alpha^2 =$ **[JEE(Advanced)-2017, 3]**

determinant

Sol. $\Delta = 0 \Rightarrow \quad 1(1-\alpha^2) - \alpha(\alpha - \alpha^3) + \alpha^2(\alpha^2 - \alpha^2) = 0$

$$(1-\alpha^2) - \alpha^2 + \alpha^4 = 0$$

$$(\alpha^2 - 1)^2 = 0 \Rightarrow \alpha = \pm 1$$

but at $\alpha = 1$ No solution (so rejected)

at $\alpha = -1$ all three equation become

$x - y + z = 1$ (coincident planes)

$\therefore \quad 1 + \alpha + \alpha^2 = 1$

17

Let S be the set of all column matrices $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ such that $b_1, b_2, b_3 \in \mathbb{R}$ and the system of equations (in real variables)

$$\left.\begin{aligned} -x + 2y + 5z &= b_1 \\ 2x - 4y + 3z &= b_2 \\ x - 2y + 2z &= b_3 \end{aligned}\right\} \Rightarrow \begin{bmatrix} -1 & 2 & 5 \\ 2 & -4 & 3 \\ 1 & -2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

has at least one solution. Then, which of the following system(s) (in real variables) has (have) at least one solution of each $\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \in S$? **[JEE(Advanced)-2018, 4(–2)]**

(A) $x + 2y + 3z = b_1$, $4y + 5z = b_2$ and $x + 2y + 6z = b_3$

(B) $x + y + 3z = b_1$, $5x + 2y + 6z = b_2$ and $-2x - y - 3z = b_3$

(C) $-x + 2y - 5z = b_1$, $2x - 4y + 10z = b_2$ and $x - 2y + 5z = b_3$

(D) $x + 2y + 5z = b_1$, $2x + 3z = b_2$ and $x + 4y - 5z = b_3$

Ans. (A,C,D)

Sol. We find D = 0 & since no pair of planes are parallel, so there are infinite number of solutions.

Let $\alpha P_1 + \lambda P_2 = P_3$

$\Rightarrow \quad P_1 + 7P_2 = 13P_3$

$\Rightarrow \quad b_1 + 7b_2 = 13b_3$

(A) $D \neq 0 \Rightarrow$ unique solution for any b_1, b_2, b_3

(B) $D = 0$ but $P_1 + 7P_2 \neq 13P_3$

(C) $D = 0$ Also $b_2 = -2b_1$, $b_3 = -b_1$

Satisfies $b_1 + 7b_2 = 13b_3$ (Actually all three planes are co-incident)

(D) $D \neq 0$

18 Let P be a matrix of order 3×3 such that all the entries in P are from the set $\{-1, 0, 1\}$. Then, the maximum possible value of the determinant of P is ______ . **[JEE(Advanced)-2018, 3(0)]**

Sol. $$\Delta = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = \underbrace{(a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2)}_{x\ \le 3} - \underbrace{(a_3b_2c_1 + a_2b_1c_3 + a_1b_3c_2)}_{y\ \ge -3}$$

Now if $x \le 3$ and $y \ge -3$

the Δ can be maximum 6

But it is not possible

as $x = 3 \Rightarrow$ each term of $x = 1$

and $y = 3 \Rightarrow$ each term of $y = -1$

$a_1b_2c_3 = a_2b_3c_1 = a_3b_1c_2 = 1$

Multiply:

$a_1a_2a_3b_1b_2b_3c_1c_2c_3 = 1 \longrightarrow$ (I)

$a_3b_2c_1 = a_2b_1c_3 = a_1b_3c_2 = -1$

Multiply:

$a_1a_2a_3b_1b_2b_3c_1c_2c_3 = -1 \longrightarrow$ (II)

$\therefore$ by (I) & (II):

$$\Rightarrow \prod_{i=1}^{3} a_ib_ic_i = 1 \text{ and } \prod_{i=1}^{3} a_ib_ic_i = -1$$

which is contradiction

so now next possibility is 4

which is obtained as $$\begin{vmatrix} 1 & 1 & 1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{vmatrix} = 1(1+1) - 1(-1-1) + 1(1-1) = 4$$

19. Let $M=\begin{bmatrix}\sin^4\theta & -1-\sin^2\theta\\ 1+\cos^2\theta & \cos^4\theta\end{bmatrix}=\alpha I+\beta M^{-1}$,

where $\alpha = \alpha(\theta)$ and $\beta = \beta(\theta)$ are real number, and I is the 2×2 identity matrix. If

α^* is the minimum of the set $\{\alpha(\theta) : \theta \in [0, 2\pi)\}$ and

β^* is the minimum of the set $\{\beta(\theta) : \theta \in [0, 2\pi)\}$,

then the value of $\alpha^* + \beta^*$ is **[JEE(Advanced)-2019, 3(–1)]**

(1) $-\frac{37}{16}$ (2) $-\frac{29}{16}$ (3) $-\frac{31}{16}$ (4) $-\frac{17}{16}$

Ans. (2)

Sol. Given $M = \alpha I + \beta M^{-1}$

$\Rightarrow M^2 - \alpha M - \beta I = O$

By putting values of M and M^2, we get

$$\alpha(\theta) = 1 - 2\sin^2\theta\cos^2\theta = 1-\frac{\sin^2 2\theta}{2} \geq \frac{1}{2}$$

Also, $\beta(\theta) = -(\sin^4\theta\cos^4\theta + (1+\cos^2\theta)(1+\sin^2\theta))$

$= -(\sin^4\theta\cos^4\theta + 1 + \cos^2\theta + \sin^2\theta + \sin^2\theta\cos^2\theta)$

$$= -(t^2 + t + 2),\ t = \frac{\sin^2 2\theta}{4} \in \left[0, \frac{1}{4}\right]$$

$$\Rightarrow \quad \beta(\theta) \geq -\frac{37}{16}$$

20. Let $M=\begin{bmatrix}0 & 1 & a\\ 1 & 2 & 3\\ 3 & b & 1\end{bmatrix}$ and $\operatorname{adj}M=\begin{bmatrix}-1 & 1 & -1\\ 8 & -6 & 2\\ -5 & 3 & -1\end{bmatrix}$ where a and b are real numbers. Which of the following options is/are correct ? **[JEE(Advanced)-2019, 4(–1)]**

(1) $a + b = 3$

(2) $\det(\operatorname{adj}M^2) = 81$

(3) $(\operatorname{adj}M)^{-1} + \operatorname{adj}M^{-1} = -M$

(4) If $M\begin{bmatrix}\alpha\\ \beta\\ \gamma\end{bmatrix}=\begin{bmatrix}1\\ 2\\ 3\end{bmatrix}$, then $\alpha - \beta + \gamma = 3$

Ans. (1,3,4)

Sol. $(\operatorname{adj}M)_{11} = 2 - 3b = -1 \Rightarrow b = 1$

Also, $(\operatorname{adj}M)_{22} = -3a = -6 \Rightarrow a = 2$

Now, $\det M=\begin{vmatrix}0 & 1 & 2\\ 1 & 2 & 3\\ 3 & 1 & 1\end{vmatrix}=-2$

$\Rightarrow \det(\operatorname{adj}M^2) = (\det M^2)^2$

$= (\det M)^4 = 16$

Also $M^{-1}=\dfrac{\operatorname{adj}M}{\det M}$

$\Rightarrow \operatorname{adj}M = -2M^{-1}$

$\Rightarrow (\operatorname{adj}M)^{-1} = \dfrac{-1}{2}M$

And, $\operatorname{adj}(M^{-1}) = (M^{-1})^{-1}\det(M^{-1})$

$=\dfrac{1}{\det M}M=\dfrac{-M}{2}$

Hence, $(\operatorname{adj}M)^{-1} + \operatorname{adj}(M^{-1}) = -M$

Further, $MX = b$

$\Rightarrow X = M^{-1}b = \dfrac{-\operatorname{adj}M}{2}b$

$$=\frac{-1}{2}\begin{bmatrix}-1 & 1 & -1\\ 8 & -6 & 2\\ -5 & 3 & -1\end{bmatrix}\begin{bmatrix}1\\ 2\\ 3\end{bmatrix}$$

$$=\frac{-1}{2}\begin{bmatrix}-2\\ 2\\ -2\end{bmatrix}=\begin{bmatrix}1\\ -1\\ 1\end{bmatrix}$$

$\Rightarrow (\alpha, \beta, \gamma) = (1, -1, 1)$

ALLEN

21. Let $P_1 = I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $P_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, $P_3 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $P_4 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$, $P_5 = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$,

$P_6 = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ and $X = \sum_{k=1}^{6} P_K \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \\ 3 & 2 & 1 \end{bmatrix} P_K^T$

where P_K^T denotes the transpose of the matrix P_K. Then which of the following options is/are correct?

[JEE(Advanced)-2019, 4(–1)]

(1) $X - 30I$ is an invertible matrix

(2) The sum of diagonal entries of X is 18

(3) If $X \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, then $\alpha = 30$

(4) X is a symmetric matrix

Ans. (2,3,4)

Sol. Let $Q = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 0 & 2 \\ 3 & 2 & 1 \end{bmatrix}$

$$X = \sum_{k=1}^{6} \left(P_k Q P_K^T\right)$$

$$X^T = \sum_{k=1}^{6} \left(P_k Q P_K^T\right)^T = X.$$

X is symmetric

Let $R = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$$XR = \sum_{k=1}^{6} P_K Q P_K^T R. \quad [\because P_K{}^T R = R]$$

$$= \sum_{K=1}^{6} P_K QR. = \left(\sum_{K=1}^{6} P_K\right) QR$$

$$\sum_{K=1}^{6} P_K = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} \qquad QR = \begin{bmatrix} 6 \\ 3 \\ 6 \end{bmatrix}$$

$$\Rightarrow XR = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} 6 \\ 3 \\ 6 \end{bmatrix} = \begin{bmatrix} 30 \\ 30 \\ 30 \end{bmatrix} = 30R$$

$\Rightarrow \alpha = 30.$

$$\text{Trace } X = \text{Trace}\left(\sum_{K=1}^{6} P_K Q P_K^T\right)$$

$$= \sum_{K=1}^{6} \text{Trace}\left(P_K Q P_K^T\right) = 6\left(\text{Trace } Q\right) = 18$$

$$X \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = 30 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

$$\Rightarrow \left(X - 30I\right) \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = O \Rightarrow |X - 30I| = 0$$

$\Rightarrow X - 30I$ is non-invertible

ALLEN

22. Let $x \in \mathbb{R}$ and let $P = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix}$, $Q = \begin{bmatrix} 2 & x & x \\ 0 & 4 & 0 \\ x & x & 6 \end{bmatrix}$ and $R = PQP^{-1}$.

Then which of the following options is/are correct ? **[JEE(Advanced)-2019, 4(–1)]**

(1) For x = 1, there exists a unit vector $\alpha\hat{i}+\beta\hat{j}+\gamma\hat{k}$ for which $R\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

(2) There exists a real number x such that PQ = QP

(3) $\det R = \det \begin{bmatrix} 2 & x & x \\ 0 & 4 & 0 \\ x & x & 5 \end{bmatrix} + 8$, for all $x \in \mathbb{R}$

(4) For x = 0, if $R\begin{bmatrix} 1 \\ a \\ b \end{bmatrix} = 6\begin{bmatrix} 1 \\ a \\ b \end{bmatrix}$, then a + b = 5

Ans. (3,4)

Sol. $\det(R) = \det(PQP^{-1}) = (\det P)(\det Q)\left(\dfrac{1}{\det P}\right)$

$= \det Q$

$= 48 - 4x^2$

Option-1 :

for x = 1 det (R) = 44 ≠ 0

$\therefore$ for equation $R\begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

We will have trivial solution

$\alpha = \beta = \gamma = 0$

Option-2 :

PQ = QP

$PQP^{-1} = Q$

R = Q

No value of x.

Option-3 :

$\det \begin{bmatrix} 2 & x & x \\ 0 & 4 & 0 \\ x & x & 5 \end{bmatrix} + 8$

$= (40 - 4x^2) + 8 = 48 - 4x^2 = \det R \ \forall x \in R$

Option-4 :

$R = \begin{bmatrix} 2 & 1 & 2/3 \\ 0 & 4 & 4/3 \\ 0 & 0 & 6 \end{bmatrix}$

$(R - 6I)\begin{bmatrix} 1 \\ a \\ b \end{bmatrix} = O$

$\Rightarrow -4 + a + \dfrac{2b}{3} = 0$

$-2a + \dfrac{4b}{3} = 0$

$\Rightarrow a = 2 \qquad b = 3$

a + b = 5