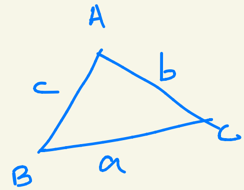
Compound Angles

Tri — 3
Gon — sides
Metron — measurement



* $a+b > c$.
* $|a-b| < c$.

Common identities:—

① $\sin^2\theta + \cos^2\theta = 1$.

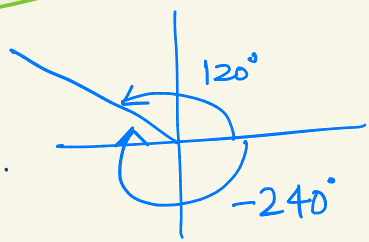
② $\sec^2\theta - \tan^2\theta = 1$.

③ $\operatorname{cosec}^2\theta - \cot^2\theta = 1$.

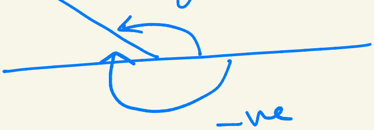
$$\sec\theta - \tan\theta = \frac{1}{\sec\theta + \tan\theta}.$$

Sign Convention

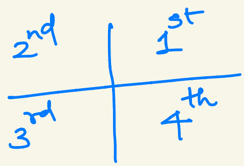
Anticlockwise → +ve
clockwise → −ve.



$\theta \rightarrow$ +ve

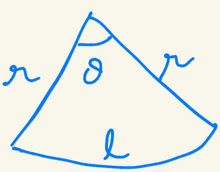


Quadrants



radians

$$\boxed{\pi^c = 180^\circ \Rightarrow 1^c = \frac{180}{\pi} \simeq 57.3^\circ}$$

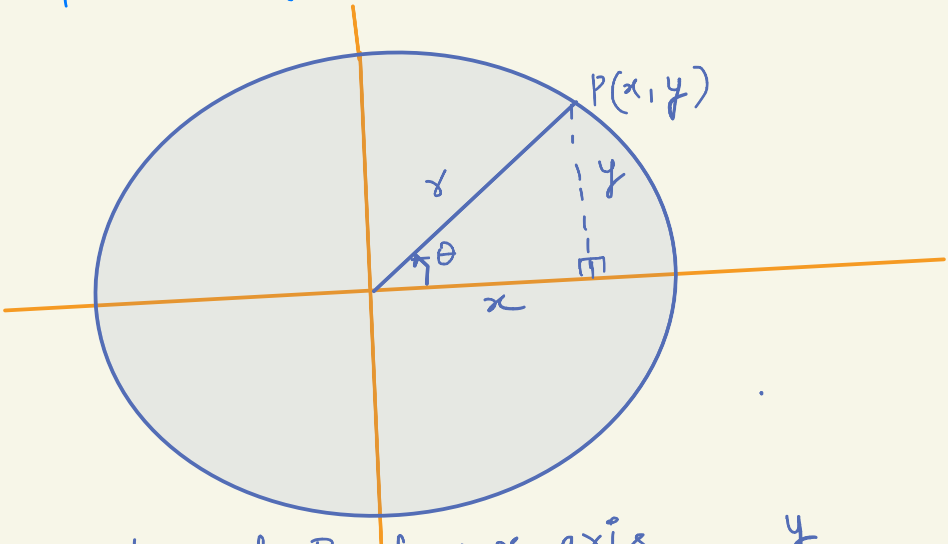


$\boxed{\theta = \dfrac{\text{arc}}{\text{Rad}}}$ & Area of sector $= \frac{1}{2} r^2 \theta$.

length of arc

$\theta \rightarrow$ Radians.

Real definition of sine & Cosine



$$\sin\theta = \frac{\perp \text{dis of } P \text{ from } x\text{-axis}}{\text{Radius}} = \frac{y}{r}$$

$$\cos\theta = \frac{\perp \text{dis of } P \text{ from } y\text{-axis}}{\text{Radius}} = \frac{x}{r}.$$

Reduction formulae :–

1. $\sin(90^\circ - \theta) = \cos\theta.$
2. $\cos(90^\circ - \theta) = \sin\theta.$
3. $\tan(90^\circ - \theta) = \cot\theta.$
4. $\sec(90^\circ - \theta) = \operatorname{cosec}\theta.$
5. $\operatorname{cosec}(90^\circ - \theta) = \sec\theta.$
6. $\cot(90^\circ - \theta) = \tan\theta.$

$\sin(90^\circ + \theta) = \cos\theta$

$\cos(90^\circ + \theta) = -\sin\theta$

$\tan(90^\circ + \theta) = -\cot\theta$

⋮ ⋮

$\pi - \theta$

$\sin(\pi - \theta) = \sin\theta$

$\cos(\pi - \theta) = -\cos\theta$

$\tan(\pi - \theta) = -\tan\theta$

$\pi + \theta$

$\sin(\pi + \theta) = -\sin\theta$

$\cos(\pi + \theta) = -\cos\theta$

$\tan(\pi + \theta) = \tan\theta.$

$270^\circ - \theta$

$\sin\left(\frac{3\pi}{2} - \theta\right) = -\cos\theta$

$\cos\left(\frac{3\pi}{2} - \theta\right) = -\sin\theta$

$\tan\left(\frac{3\pi}{2} - \theta\right) = \cot\theta$

$\frac{3\pi}{2} + \theta$

$\sin\left(\frac{3\pi}{2} + \theta\right) = -\cos\theta$

$\cos\left(\frac{3\pi}{2} + \theta\right) = \sin\theta$

$\tan\left(\frac{3\pi}{2} + \theta\right) = -\cot\theta$

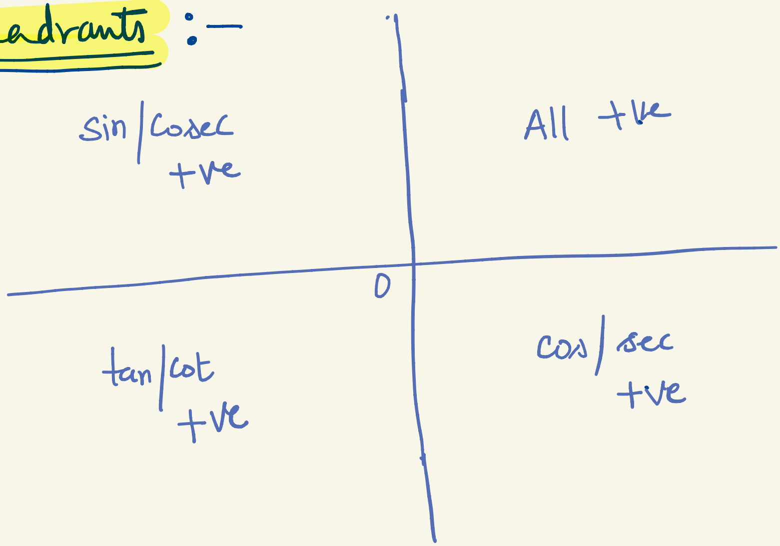
$2\pi - \theta$

$\sin(2\pi - \theta) = \sin(-\theta) = -\sin\theta$

$\cos(2\pi - \theta) = \cos(-\theta) = \cos\theta$

$\tan(2\pi - \theta) = \tan(-\theta) = -\tan\theta.$

4 Quadrants :–



* Remember

$\sin\theta \in [-1, 1]$

$\sin^2\theta \in [1, \infty)$

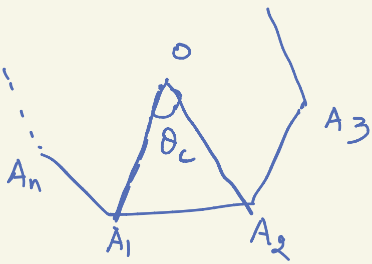
$\tan\theta \in R$

$\tan^2\theta \in [0, \infty)$

$\operatorname{cosec}\theta \in (-\infty, -1] \cup [1, \infty)$

$\operatorname{cosec}^2\theta \in [1, \infty)$

Regular Polygon



① $\boxed{\theta_c = \dfrac{2\pi}{n}}$ → Central angle of polygon.

$n \to$ no. of sides

② Interior angle $= \dfrac{(n-2)\pi}{n}$

③ Sum of all interior angles $= (n-2)\pi$

Pentagon → 5 sides
Hexagon → 6 "
Octagon → 8 "
Heptagon → 7 "
Decagon → 10 "

Dodecagon → 12 sides
Quindecagon → 15 "

① $\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$

② $\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$

③ $2 \sin A \cos B = \sin(A+B) + \sin(A-B)$

④ $2 \cos A \sin B = \sin(A+B) - \sin(A-B).$

⑤ $2 \cos A \cos B = \cos(A+B) + \cos(A-B)$

⑥ $2 \sin A \sin B = \cos(A-B) - \cos(A+B).$

⑦ $\sin(A+B) \cdot \sin(A-B) = \sin^2 A - \sin^2 B$

$= \cos^2 B - \cos^2 A.$

⑧ $\cos(A+B) \cdot \cos(A-B) = \cos^2 A - \sin^2 B.$

$= \cos^2 B - \sin^2 A.$

⑨ $\sin C + \sin D = 2 \sin\left(\frac{C+D}{2}\right) \cos\left(\frac{C-D}{2}\right)$

$\sin C - \sin D = 2 \sin\left(\frac{C-D}{2}\right) \cos\left(\frac{C+D}{2}\right)$

⑩ $\cos C + \cos D = 2 \cos\left(\frac{C+D}{2}\right) \cos\left(\frac{C-D}{2}\right)$

* $\cos C - \cos D = 2 \sin\left(\frac{C+D}{2}\right) \sin\left(\frac{D-C}{2}\right)$

(Note carefully)

$$\sin\frac{\pi}{12} = \cos\frac{5\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}} = \frac{\sqrt{6}-\sqrt{2}}{4} = \frac{1}{\sqrt{6}+\sqrt{2}}.$$

$$\sin\frac{5\pi}{12} = \cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}} = \frac{\sqrt{6}+\sqrt{2}}{4} = \frac{1}{\sqrt{6}-\sqrt{2}}.$$

$$\tan\frac{\pi}{12} = \cot\frac{5\pi}{12} = 2-\sqrt{3}.$$

$$\tan\frac{5\pi}{12} = \cot\frac{\pi}{12} = 2+\sqrt{3}.$$

———— x ———— x ———— x ————

$$\tan(A+B) = \frac{\tan A + \tan B}{1-\tan A\tan B}.$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1+\tan A\tan B}.$$

$$\tan\left(\frac{\pi}{4}+\theta\right) = \frac{1+\tan\theta}{1-\tan\theta}$$

$$\tan\left(\frac{\pi}{4}-\theta\right) = \frac{1-\tan\theta}{1+\tan\theta}$$

Remember vice-versa also.

$$\cot(A+B) = \frac{\cot A\cot B - 1}{\cot B + \cot A}$$

$$* \cot(A-B) = \frac{\cot A\cot B + 1}{\cot B - \cot A}$$

(Note carefully)

Note:

If $A + B = \frac{\pi}{4}$ then :

① $(1+\tan A)(1+\tan B) = 2$.

② $(1-\cot A)(1-\cot B) = 2$.

* (Rem this very useful)

$$\sin 2\theta = 2\sin\theta\cos\theta.$$

$$= \frac{2\tan\theta}{1+\tan^2\theta}$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$= 2\cos^2\theta - 1$$

$$= 1 - 2\sin^2\theta$$

$$= \frac{1-\tan^2\theta}{1+\tan^2\theta}.$$

☆☆

Imp forms from above :

① $\sin^2\theta = \frac{1-\cos 2\theta}{2}$

② $\cos^2\theta = \frac{1+\cos 2\theta}{2}$

③ $\tan^2\theta = \frac{1-\cos 2\theta}{1+\cos 2\theta}$

$$\tan\frac{\pi}{8} = \tan 22.5^\circ = \cot\frac{3\pi}{8} = \cot 67.5^\circ = \sqrt{2}-1$$

$$\cot\frac{\pi}{8} = \cot 22.5^\circ = \tan\frac{3\pi}{8} = \tan 67.5^\circ = \sqrt{2}+1$$

$$\left.\begin{aligned} \sin 3\theta &= 3\sin\theta - 4\sin^3\theta \\ \cos 3\theta &= 4\cos^3\theta - 3\cos\theta. \\ \tan 3\theta &= \frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}. \end{aligned}\right\}$$

$$\left.\begin{aligned} \sin 5\theta &= 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta. \\ \cos 5\theta &= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta. \end{aligned}\right\}$$

★

$$1+\sin 2\theta = (\sin\theta+\cos\theta)^2$$

$$1-\sin 2\theta = (\sin\theta-\cos\theta)^2$$

$$\sqrt{1+\sin\theta} = \sqrt{\left(\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right)^2}$$

Note carefully $\quad = \left|\sin\frac{\theta}{2}+\cos\frac{\theta}{2}\right|$

* If $A+B=C$ then

$$\boxed{\tan A \tan B \tan C = \tan C - \tan B - \tan A}$$

$$\tan 7.5^\circ = \tan \frac{\pi}{24} = (\sqrt{3}-\sqrt{2})(\sqrt{2}-1)$$
$$= \sqrt{6}-\sqrt{4}-\sqrt{3}+\sqrt{2}$$

$$\cot 7.5^\circ = \cot \frac{\pi}{24} = (\sqrt{3}+\sqrt{2})(\sqrt{2}+1)$$
$$= \sqrt{6}+\sqrt{4}+\sqrt{3}+\sqrt{2}.$$

$$\cos \frac{\pi}{8} = \cos 22.5^\circ = \frac{\sqrt{2+\sqrt{2}}}{2}.$$

$$\sin \frac{\pi}{8} = \sin 22.5^\circ = \frac{\sqrt{2-\sqrt{2}}}{\sqrt{2}}.$$

Imp

$$\sin\theta \, \sin(60^\circ-\theta) \, \sin(60^\circ+\theta) = \frac{\sin 3\theta}{4}$$

$$\cos\theta \, \cos(60^\circ-\theta) \, \cos(60^\circ+\theta) = \frac{\cos 3\theta}{4}.$$

* $\tan\theta \, \tan(60^\circ-\theta) \, \tan(60^\circ+\theta) = \tan 3\theta.$
* $\cot\theta \, \cot(60^\circ-\theta) \, \cot(60^\circ+\theta) = \cot 3\theta.$

$$\cos\theta \cdot \cos 2\theta \cos 2^2\theta \ldots\ldots \cos 2^{n-1}\theta = \frac{\sin 2^n\theta}{2^n \sin\theta}.$$

* (a) $\cot A - \tan A = 2\cot 2A$ (b) $\cot A + \tan A = 2\operatorname{cosec} 2A$

(a) $\sin^2\theta + \sin^2(60° + \theta) + \sin^2(60° - \theta) = \dfrac{3}{2}$

(b) $\cos^2\theta + \cos^2(60° + \theta) + \cos^2(60° - \theta) = \dfrac{3}{2}$

(c) $\tan\theta + \tan(60° + \theta) + \tan(120° + \theta) = 3\tan 3\theta$

Imp

① $\sin^4\theta + \cos^4\theta = 1 - \dfrac{\sin^2 2\theta}{2}$

② $\sin^6\theta + \cos^6\theta = 1 - \dfrac{3}{4}\sin^2 2\theta.$

③ $\cos^4\theta - \sin^4\theta = \cos 2\theta.$

very useful

$$\sin\frac{\pi}{10} = \sin 18° = \cos 72° = \cos\frac{2\pi}{5} = \frac{\sqrt{5}-1}{4}$$

$$\cos\frac{\pi}{5} = \cos 36° = \sin 54° = \sin\frac{3\pi}{10} = \frac{\sqrt{5}+1}{4}$$

TRIGONOMETRIC IDENTITIES IN A TRIANGLE (CONDITIONAL IDENTITIES) :

If A, B, C are the angles of a triangle i.e. $A + B + C = \pi$, then

(i) $\sin(A + B) = \sin(\pi - C) = \sin C$

(ii) $\cos(A + B) = \cos(\pi - C) = -\cos C$

(iii) $\tan(A + B) = \tan(\pi - C) = -\tan C$

(iv) $\sin(2A + 2B) = \sin(2\pi - 2C) = -\sin 2C$

(v) $\cos(2A + 2B) = \cos(2\pi - 2C) = \cos 2C$

(vi) $\tan(2A + 2B) = \tan(2\pi - 2C) = -\tan 2C$

(vii) $\sin\left(\frac{A+B}{2}\right) = \sin\frac{\pi - C}{2} = \cos\frac{C}{2}$

(viii) $\cos\left(\frac{A+B}{2}\right) = \sin\frac{C}{2}$

(ix) $\tan\left(\frac{A+B}{2}\right) = \cot\frac{C}{2}$

(x) If A, B, C are the angles of a triangle, then $\sin 2A + \sin 2B + \sin 2C = 4\sin A \sin B \sin C$

[**Sol.** LHS $= 2\sin(A + B)\cos(A - B) + 2\sin C\cos C = 2\sin C[\cos(A - B) + \cos(\pi - A - B)]$

$= 2\sin C[\cos(A - B) - \cos(A + B)] = 4\sin A\sin B\sin C$]

(xi) $\cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A\cos B\cos C$ (where $A + B + C = \pi$)

[**Sol.** $2\cos(A + B)\cos(A - B) + 2\cos^2 C - 1 = 2\cos(\pi - C)\cos(A - B) + 2\cos^2 C - 1$]

$= 2\cos C[\cos C - \cos(A - B)] - 1 = 2\cos C\,[\cos(\pi - \overline{A+B}) - \cos(A - B)] - 1$

$= 2\cos C\,[-\cos(A + B) - \cos(A - B)] - 1 = -4\cos A\cos B\cos C - 1$]

(xii) $\Sigma\tan A = \Pi\tan A$

(xiii) $\sum\cot A\cot B = 1$

(xiv) $\sum\tan\frac{A}{2}\tan\frac{B}{2} = 1$

(xv) $\sum\cot\frac{A}{2} = \Pi\cot\frac{A}{2}$

[**Sol.** (i) $\tan(A + B) = \tan(\pi - C) = -\tan C \Rightarrow \frac{\tan A + \tan B}{1 - \tan A\tan B} = -\tan C$]

Note Carefully :-

(a) If $\tan A + \tan B + \tan C = \tan A\tan B\tan C$, then $A + B + C = n\pi, n \in I$

(b) If $\tan A\tan B + \tan B\tan C + \tan C\tan A = 1$, then $A + B + C = (2n + 1)\frac{\pi}{2}, n \in I$

MAXIMUM/MINIMUM VALUES OF TRIGONOMETRIC EXPRESSIONS :

(1) $a\cos\theta + b\sin\theta$ will always lie in the interval $[-\sqrt{a^2+b^2}, \sqrt{a^2+b^2}]$ i.e. the maximum and minimum value are $\sqrt{a^2+b^2}$, $-\sqrt{a^2+b^2}$ respectively.

(2) Minimum value of $a^2\tan^2\theta + b^2\cot^2\theta$ is $2ab$ where $a, b > 0$

(3) If A, B, C are the angles of a triangle then

(a) $\sin A + \sin B + \sin C \le \frac{3\sqrt{3}}{2}$

(b) $1 < \cos A + \cos B + \cos C \le \frac{3}{2}$

(c) $\tan A + \tan B + \tan C \ge 3\sqrt{3}$ (for acute angled triangle)

(d) $\cot A + \cot B + \cot C \ge \sqrt{3}$

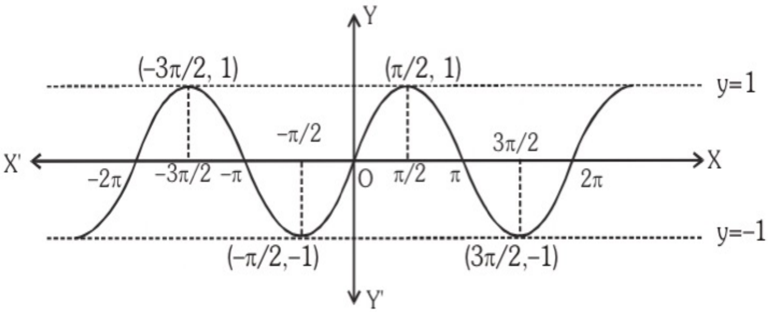
Note : Equality holds for equilateral triangle.

(e) $\cot^2 A + \cot^2 B + \cot^2 C \geq 1$

(f) $\cos A \cos B \cos C \le \frac{1}{8}$.

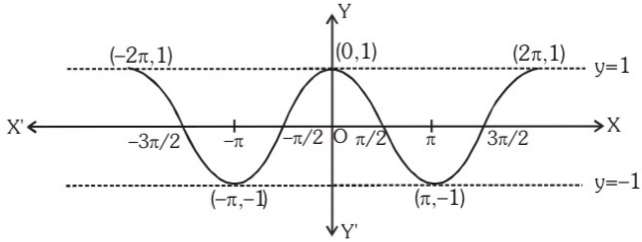
(a) Graphs of trigonometric ratios :

(1) $y = \sin x$, $x \in R$, $y \in [-1, 1]$



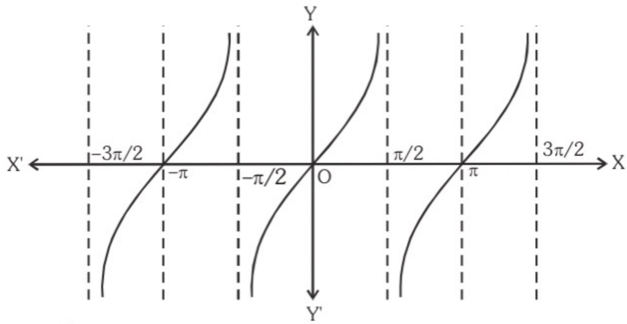
Periodic with period 2π

(2) $y = \cos x$



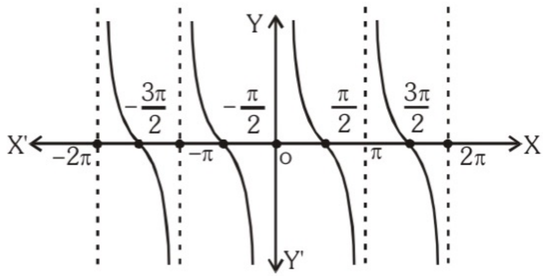
Periodic with period 2π

(3) $y = \tan x,\ x \neq \frac{\pi}{2}, \frac{3\pi}{2} \ldots\ldots \neq (2n-1)\frac{\pi}{2}\ :\ n \in I$

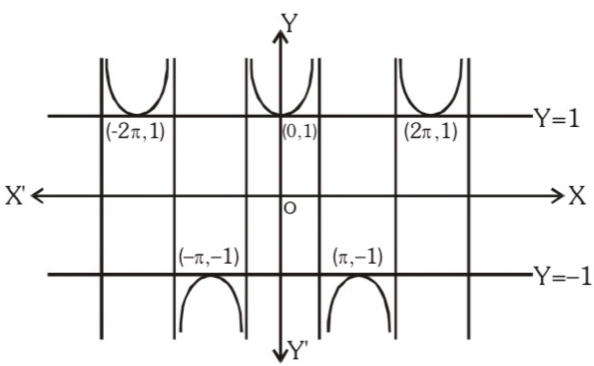


Periodic with period π

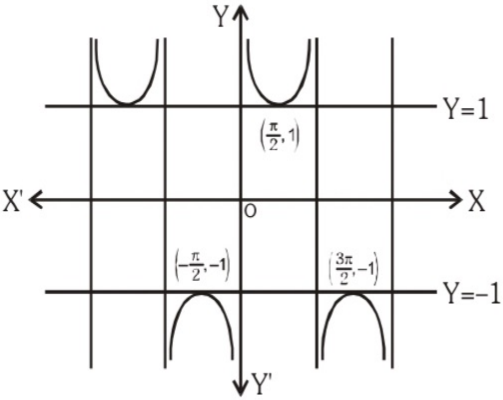
(4) **y = cotx**



(5) **y = secx**



(6) **y = cosecx**



(b) $-1 \leq \sin\theta \leq 1$

(c) $-1 \leq \cos\theta \leq 1$

(d) $\sin 0 = 0$; $\sin\pi = 0$; $\sin 2\pi = 0$

$\sin(n\pi) = 0,\ n \in I$

i.e. sine of integral multiple of $\pi = 0$

Also $\tan(n\pi) = 0$ and $\sin\frac{\pi}{2} = \sin\frac{5\pi}{2} = \sin\left(2n\pi + \frac{\pi}{2}\right) = 1$ and $\sin\frac{3\pi}{2} = \sin\frac{7\pi}{2} = \sin\left(2n\pi - \frac{\pi}{2}\right) = -1$

(e) $\cos\frac{\pi}{2} = 0$; $\cos\frac{3\pi}{2} = 0$; $\cos\frac{5\pi}{2} = 0$

$\cos(2n-1)\frac{\pi}{2} = 0,\ n \in I$

i.e. cosine of odd integral multiple of $\frac{\pi}{2}$ is zero.

Also $\cot(2n-1)\frac{\pi}{2} = 0 : n \in I$

(f) $\cos 0 = 1$; $\cos 2\pi = 1$; $\cos 4\pi = 1$

$\cos 2m\pi = 1,\ m \in I$

i.e. cos of even multiple of $\pi = 1$

(g) $\cos\pi = -1$; $\cos 3\pi = -1$; $\cos 5\pi = -1$

$\cos(2m-1)\pi$ = cos of odd multiple of $\pi = -1$

(h) $\sin(2n\pi + \theta) = \sin\theta,\ n \in I$

$\cos(2n\pi + \theta) = \cos\theta,\ n \in I$

$\tan(n\pi + \theta) = \tan\theta,\ n \in I$

Common log :

$\downarrow$ Base = 10

$\log_{10} N$.

Natural log

Base = e $\rightarrow$ napier's Constant

e $\rightarrow$ irrational quantity whose value lies in (2.7, 2.8)

$\pi \rightarrow$ irrational no.

$\pi \simeq \frac{22}{7}$

$\pi \simeq 3.14$

Q Comment on $\log_3 \pi + \log_\pi 3$?

Sol^n If $\log_3 \pi = x$ then $\log_\pi 3 = \frac{1}{x}$.

$x > 0$; $x \neq 1$

$$x + \frac{1}{x} > 2$$

Ans

Note:

$$\left.\begin{aligned} \log_2 8 &= 3 \quad (+ve) \\ \log_{(\frac{1}{2})}\left(\frac{1}{16}\right) &= 4 \quad (+ve) \\ \log_{(\frac{1}{2})}(32) &= -5 \quad (-ve) \end{aligned}\right\}$$

Note: whenever the number & base are on the same side of unity then value of log will be +ve but if they are on opposite side of unity then value of log will be -ve

$$\text{antilog}_{\frac{1}{100}}\left(-\frac{1}{2}\right) = \left(\frac{1}{100}\right)^{-1/2}$$

$$= (10)^{-2\times -\frac{1}{2}} = 10.$$

Note:

Rem

$$\boxed{\log_{(a^b)}(m^p) = \frac{p}{b}\left(\log_a m\right)}$$

$$\frac{\log_c m^p}{\log_c a^b} = \frac{p}{b}\frac{(\log_c m)}{(\log_c a)}$$

$$= \frac{p}{b}(\log_a m) \quad (H.P)$$

Q1 $A = \log_{11}\left(11^{\log_{11} 1331}\right)$; $B = \log_{385} 5 + \log_{385} 7 + \log_{385} 11$

$C = \log_4\left(\log_2\left(\log_5 625\right)\right)$; $D = 10^{\log_{100} 16}$ then find value of

$(A + B \div C - D)$?

Sol^n

$1331 = 11^3$

$A = \log_{11}(1331) = \log_{11}(11)^3 \Rightarrow A = 3$

$B = \log_{385}(5 \times 7 \times 11) = \log_{385}(385) = 1$

$C = \log_4\left(\log_2\left(\underbrace{\log_5 625}_{4}\right)\right) = \log_4\left(\underbrace{\log_2(4)}_{2}\right) = \log_4 2 = \log_{2^2} 2 = \frac{1}{2}$

$D = 10^{\log_{(10^2)}(4^2)} = 10^{\log_{10} 4} \Rightarrow \boxed{D = 4}$

$$\boxed{\begin{aligned} &A + B \div C - D \\ &3 + \frac{1}{1/2} - 4 \\ &5 - 4 = 1 \text{ Ans} \end{aligned}}$$

Q2 If $\log_2 3 \cdot \log_3 4 \cdot \log_4 5 \cdots \log_n (n+1) = 10$, find $n = ?$

$$\log_2 (n+1) = 10 \Rightarrow n+1 = 2^{10} = 1024$$

$$n = 1023 \text{ Ans}$$

Q3 Simplify $\cancel{7^{\log_3 5}} + \cancel{3^{\log_5 7}} - \cancel{5^{\log_3 7}} - \cancel{7^{\log_5 3}} = 0.$

$$a^{\log_c b} = b^{\log_c a}$$

Q Find value of $\frac{1}{\log_{3} 2} + \frac{2}{\log_{9} 4} - \frac{3}{\log_{27} 8}$

Sol^n

$$\log_2 3 + 2\log_4 9 - 3\log_8 27$$

$$\log_2 3 + 2 \cdot \frac{\cancel{2}}{\cancel{2}} \cdot \log_2 3 - 3 \cdot \frac{\cancel{3}}{\cancel{3}} \cdot \log_2 3$$

$$3\log_2 3 - 3\log_2 3 = 0.$$

Q. If $N = \dfrac{\log_3 12}{\log_{36} 3} - \dfrac{\log_3 4}{\log_{108} 3}$ then N is

(A) Prime Number.

(B) Rational No.

(C) Irrational No.

(D) Natural No.

Solⁿ

$$N = \log_3 (2^2 \cdot 3) \cdot \log_3 (2^2 \cdot 3^2) - \log_3 (2^2) \cdot \log_3 (3^3 \cdot 2^2)$$

$$= (2 \log_3 2 + 1)(2 \log_3 2 + 2) - (2 \log_3 2)(3 + 2 \log_3 2).$$

$\log_3 2 = t$.

$$= (2t+1)(2t+2) - 2t(3+2t)$$

$$= \cancel{4t^2} + \cancel{6t} + 2 - \cancel{6t} - \cancel{4t^2} = 2$$

Q If $\log_6 15 = \alpha$ and $\log_{12} 18 = \beta$ then find $\log_{25} 24$ in terms of α and β ?

Sol^n ✱ Base ✱

$$\frac{\log_3(15)}{\log_3 6} = \alpha \Rightarrow \alpha = \frac{1+\log_3 5}{1+\log_3 2} \quad -①-$$

get $\log_3 5 = \,?$ ✓

$$\frac{\log_3(18)}{\log_3(12)} = \beta \Rightarrow \frac{2+\log_3 2}{2\log_3 2+1} = \beta \Rightarrow 2+\log_3 2 = 2\beta.\log_3 2 + \beta.$$

$$\frac{2-\beta}{2\beta-1} = \log_3 2 \quad ✱$$

✓

$$\frac{\log_3 24}{\log_3(25)} = \frac{1+3\log_3 2}{2\log_3 5}$$

Q which is greater ?

$a = \log_3 5$ and $b = \log_{17} 25$.

$$\frac{a}{b} = \frac{\log_3 5}{2\log_{17} 5} = \frac{1}{2}\left(\log_3 5 \cdot \log_5 17\right)$$

$$\frac{a}{b} = \frac{1}{2}\left(\underbrace{\log_3 17}\right) > 1 \longrightarrow \underline{\underline{a > b}} \text{ Ans}$$

$$9 < 17 < 27.$$

$$\log_3 9 < \log_3 17 < \log_3 27$$

$$\boxed{2 < \log_3 17 < 3}$$

① $a - b$

$\oplus$: $a > b$

$\ominus$: $a < b$

0: $a = b$.

② $\left(\frac{a}{b}\right)$

$> 1 \Rightarrow a > b$

$< 1 \Rightarrow a < b$

$= 1 \Rightarrow a = b$.

Log equations :-

Q1 $x^2 + 7^{\log_7 x} - 2 = 0.$

$x^2 + x - 2 = 0$

$x^2 + 2x - x - 2 = 0$

$(x+2)(x-1) = 0$

$x = 1$; $x = -2$ ✗✗

✓

CHECK !!!

Q2 $\dfrac{\log_2(9-2^x)}{(x-3)} = 1.$

$\log_2(9-2^x) = x-3.$

$(9-2^x) = 2^{x-3}$

$9 - 2^x = \dfrac{2^x}{8} \Rightarrow 72 - 8.(2^x) = 2^x$

$72 = 9.(2^x)$

$2^x = 8 \Rightarrow x = 3$ ✗✗

$x \in \phi.$ Ans

Q3 $\log_4\left(\underbrace{2\log_3\left(1+\log_2\left(1+3\log_3 x\right)\right)}_{N_1}\right)=\frac{1}{2}$

$\log_4 N_1 = \frac{1}{2} \Rightarrow N_1 = 4^{1/2} = 2$

$\cancel{2}\log_3\left(\underbrace{1+\log_2\left(1+3\log_3 x\right)}_{N_2}\right) = \cancel{2}$

$\log_3 (N_2) = 1 \Rightarrow \boxed{N_2 = 3}$

$1+\log_2\left(1+3\log_3 x\right) = 3$

$\log_2\left(1+3\log_3 x\right) = 2$

$1+3\log_3 x = 4 \Rightarrow \log_3 x = 1 \Rightarrow x = 3$

Check!!!

Q

$$5^{1+\log_4 x} + 5^{\log_{1/4} x - 1} = \frac{26}{5}.$$

$$5^1 \cdot 5^{\log_4 x} + 5^{-\log_4 (x)} \cdot 5^{-1} = \frac{26}{5}.$$

$$5 \cdot 5^{\log_4 x} + \frac{1}{5} \cdot \frac{1}{5^{\log_4 x}} = \frac{26}{5}.$$

Let $5^{\log_4 x} = t$; $t > 0$ **

$$5t + \frac{1}{5t} = \frac{26}{5}$$

$$25t^2 + 1 = 26t.$$

$$25t^2 - 26t + 1 = 0$$

$$25t^2 - 25t - t + 1 = 0$$

$$(25t - 1)(t - 1) = 0$$

$$t = 1; \quad t = \frac{1}{25}$$

$$5^{\log_4 x} = 5^0 \text{ or } 5^{-2}$$

$$\log_4 x = 0 \text{ or } -2$$

$$x = 4^0 \text{ or } 4^{-2}$$

$x = 1$ or $x = \frac{1}{16}$

Ans check   Ans

Q

$$(x+1)^{\log_{10}(x+1)} = 100(x+1).$$

Take log both sides to the base '10'

$$\log_{10}(x+1)^{\log_{10}(x+1)} = \log_{10}(100\,(x+1))$$

$$\log_{10}(x+1).\log_{10}(x+1) = \log_{10}100 + \log_{10}(x+1)$$

$$\log_{10}(x+1).\log_{10}(x+1)$$

Let $\log_{10}(x+1) = t$

$$t^2 - t - 2 = 0.$$

$$t^2 - 2t + t - 2 = 0$$

$$t = 2 \text{ or } t = -1$$

$$\log_{10}(x+1) = 2 \text{ or } -1$$

$$x+1 = 10^2 \text{ or } 10^{-1}$$

$$x+1 = 100 \text{ or } \frac{1}{10}$$

$x = 99$ or $x = -0.9$

check

Note :-

$\sqrt[n]{a} \rightarrow (a)^{\frac{1}{n}}$; $n \in R - \{0\}$

* $n \in N \geq 2$

$\sqrt[5]{2}$ ✓✓

$\cancel{\sqrt[3.7]{2}}$

Q $\log_5\left(5^{\frac{1}{x}} + 125\right) = \underbrace{\log_5 6} + 1 + \frac{1}{2x}$

$\log_5\left(\frac{5^{\frac{1}{x}} + 125}{6}\right) = 1 + \frac{1}{2x}$

$\frac{5^{1/x} + 125}{6} = 5^{1+\frac{1}{2x}}$

$5^{\frac{1}{x}} + 125 = 6\left(5^1 \cdot 5^{\frac{1}{2x}}\right)$

$5^{\frac{1}{x}} + 125 = 30 \cdot 5^{\frac{1}{2x}}$

Let $5^{\frac{1}{2x}} = t$; $t > 0$ *

$5^{\frac{1}{x}} = t^2$

$t^2 - 30t + 125 = 0 \Rightarrow t = 5$ or 25

$5^{\frac{1}{2x}} = 5$ or $5^2 \Rightarrow x = \frac{1}{2}$ or $\frac{1}{4}$ ✓ Ans

Q $\log_5\left(\sqrt[x]{5} + 125\right) = \log_5 6 + 1 + \frac{1}{2x}$.

calculation
same as previous question

$x = \frac{1}{2}$ or $x = \frac{1}{4}$

X X

$x \in \phi$ Ans

Q $\log_{10}^2(100x) + \log_{10}^2(10x) = 14 + \log_{10}\left(\frac{1}{x}\right)$

$\cancel{\log_{10}^2(100x) = 2\log_{10}(100x)}$

$\log_{10}^2(100x) = \left(\log_{10} 100x\right)^2$

$\left(\log_{10}(100 \cdot x)\right)^2 + \left(\log_{10}(10 \cdot x)\right)^2 = 14 - \log_{10} x$

$(2 + \log_{10} x)^2 + (1 + \log_{10} x)^2 = 14 - \log_{10} x$

let $\log_{10} x = t$

$x = 10 \,;\; 10^{-9/2}$ +ve

Ans

Note:

① $x = \sqrt{7\sqrt{7\sqrt{7\sqrt{7 \cdots \infty}}}}$; $x > 0$

$x = \sqrt{7x}$

$x^2 = 7x \Rightarrow x(x-7) = 0$

$x = 0$; $x = 7$

② $x = \sqrt{7 + \sqrt{7 + \sqrt{7 + \cdots \infty}}}$

$x = \sqrt{7 + x}$; $x > 0$

Q. If $x = \log_2\left(\sqrt{56+\sqrt{56+\sqrt{56+\cdots\infty}}}\right)$ then find x ?

Sol^n

Let $y = \sqrt{56+\sqrt{56+\sqrt{56+\cdots\infty}}}$; $y>0$.

$y = \sqrt{56+y} \Rightarrow y^2 - y - 56 = 0$.

$y^2 - 8y + 7y - 56 = 0$

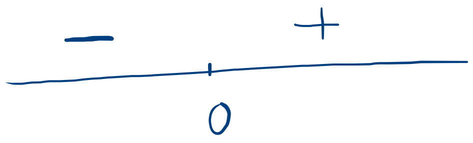
$(y-8)(y+7) = 0$

$y = 8$; $\cancel{y = -7}$

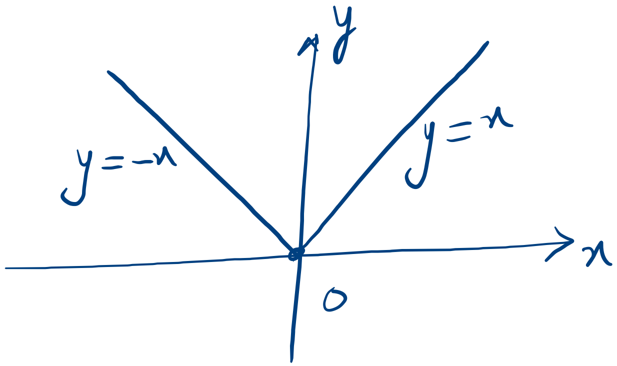
$x = \log_2(8) = 3$ Ans

Absolute Value function / Modulus function

$$f(x) = |x| \longrightarrow \begin{cases} -x & ; x < 0 \\ x & ; x \geq 0 \end{cases}$$



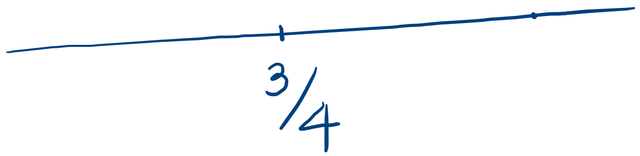
$R_f \in [0, \infty)$.

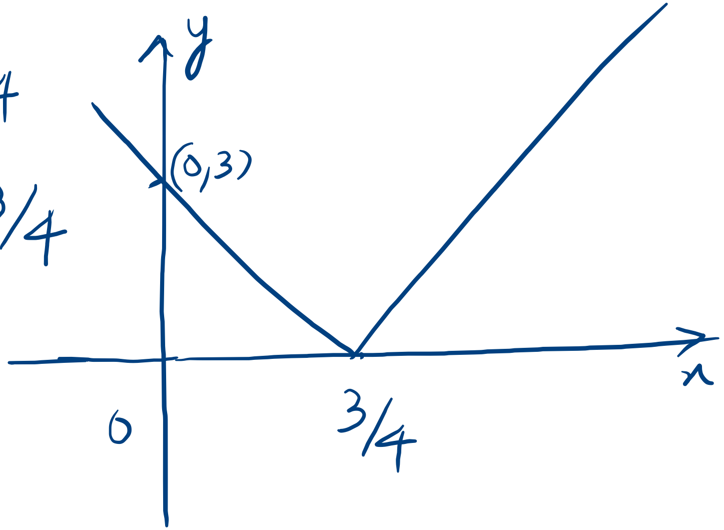


$f(x) = |x|$.

$$|x| = 5 \Rightarrow \boxed{x = \pm 5}$$

$$f(x) = |\underbrace{3 - 4x}| \longrightarrow \begin{cases} 3 - 4x & ; x \leq 3/4 \\ -(3 - 4x) & ; x > 3/4 \end{cases}$$





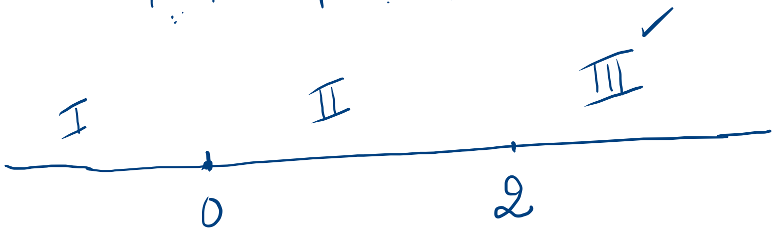
* $\sqrt{x^2-2x+1} = \sqrt{(x-1)^2} = |x-1|$. *

$\sqrt{(\text{Masala})^2} = |\text{Masala}|$.

$$|x| \longrightarrow \begin{cases} -x\ ; & x<0 \\ x\ ; & x \geq 0 \end{cases}$$

$$|x-2| \longrightarrow \begin{cases} -(x-2)\ ; & x<2 \\ (x-2)\ ; & x \geq 2 \end{cases}$$

① $|x| - |x-2| = 2$.



C–I $\quad \underline{x<0}$ ✓

$-\cancel{x} + \cancel{x} - 2 = 2$

$-2 = 2$

Non-sense

C–II $\quad 0 \leq x < 2$

$x + x - 2 = 2$.

$2x = 4$

$x = 2$ ✗✗

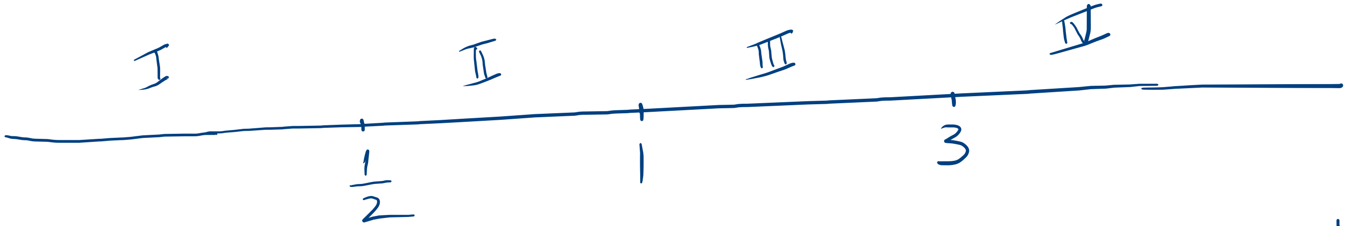
C–III $\quad x \geq 2$

$x - (x-2) = 2$

$2 = 2$ Always True

$\therefore \quad x \in [2, \infty)$ Ans

(2) $|x-1| + |2x-1| + |6-2x| = 5$



C – I

$x < \frac{1}{2}$

C – II

$\frac{1}{2} \le x < 1$

C – III

$1 \le x < 3$

C – IV

$x \geq 3$

Q3 $\quad |x-5| > |\underbrace{x^2-5x+9}_{\substack{a>0\ \&\ D<0 \\ \text{(always +ve)}}}|$.

$$|x-5| > x^2-5x+9$$

I $\qquad$ II

$\qquad\quad$ 5

C-I $\quad \underline{\underline{x<5}}$

$$-x+5 > x^2-5x+9$$

$$x^2-4x+4 < 0$$

$$(x-2)^2 < 0$$

$$x \in \phi \quad \text{--- (1) ---}$$

C-II $\quad \underline{\underline{x \geq 5}}$

$$x-5 > x^2-5x+9$$

$$x^2-6x+14 < 0 \;\Rightarrow\; \underline{\underline{x \in \phi}} \quad \text{--- (2) ---}$$

$a>0$ & $D<0$

always +ve

finally $\underline{\underline{x \in \phi}}$ Ans

Note:

$$|x| \le 2 \Rightarrow \boxed{-2 \le x \le 2}$$

$a > 0$

$$|x| \le a \Rightarrow -a \le x \le a.$$

$$|x| \geq 2 \Rightarrow x \in (-\infty, -2] \cup [2, \infty)$$

$$|x| \geq a \Rightarrow x \in (-\infty, -a] \cup [a, \infty)$$

$a > 0$

Q

$$\left| \frac{x^2 - 3x - 1}{x^2 + x + 1} \right| < 3$$

$$-3 < \frac{x^2 - 3x - 1}{\underbrace{x^2 + x + 1}_{(>0 \text{ always})}} < 3. \Rightarrow \underbrace{-3x^2 - 3x - 3 < x^2 - 3x - 1}_{1} \overbrace{< 3x^2 + 3x + 3}^{2}$$

$1 \cap 2$

Q $\log_4(x^2-1) - \log_4(x^2-2x+1) = \log_4\sqrt{16+x^2-8x}$.

$$\log_4\left(\frac{x^2-1}{(x-1)^2}\right) = \log_4\sqrt{(4-x)^2} \quad **$$

$$\log_4\left(\quad\right) = \log_4 |4-x|.$$

$$\frac{(x^2-1)}{(x-1)^2} = |4-x| \Rightarrow (x^2-1) = (x-1)^2|4-x|.$$

$$(x-1)\left((x+1)-(x-1)|4-x|\right) = 0$$

$x-1=0 \Rightarrow x=1$ X X

$$(x+1)-(x-1)\,|\underbrace{4-x}|=0$$

I | II

4

C-I $\quad x<4$

$$(x+1)-(x-1)(4-x)=0.$$

$$x+1-(4x-x^2-4+x)=0.$$

$$\cancel{x}+1-4x+x^2+4-\cancel{x}=0$$

$$x^2-4x+5=0.$$

$D<0.$ No sol^n

C-II $\quad x\geq 4$

$$(x+1)+(x-1)(4-x)=0.$$

$$x+1+(4x-x^2-4+x)=0$$

$$x^2-6x+3=0.$$

$$x=\frac{6\pm\sqrt{36-12}}{2}$$

$$x=\frac{6\pm 2\sqrt{6}}{2}$$

$x=3+\sqrt{6}$ or $x=3-\sqrt{6}$

$$x\in\{3+\sqrt{6}\}\rightarrow \text{Ans}$$

Properties of Modulus :-

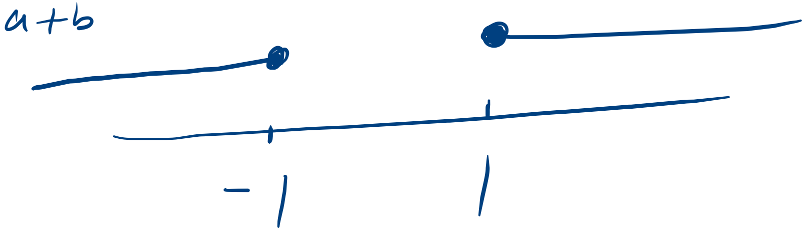
① $|a| + |b| = |a+b|$ only if $ab \geq 0$.

② $|a| + |b| = |a-b|$ only if $ab \leq 0$.

③ $\big||a| - |b|\big| = |a-b|$ only if $ab \geq 0$

④ $|a+b+c| = |a|+|b|+|c|$ only if $\left.\begin{array}{l} ab \geq 0 \\ bc \geq 0 \\ ca \geq 0 \end{array}\right\}$

Q.1 $|\underbrace{x+1}_{a}| + |\underbrace{x-1}_{b}| = |\underbrace{2x}_{a+b}|$.



$(x+1)(x-1) \geq 0$

$x \in (-\infty, -1] \cup [1, \infty)$ Ans

Q $\underbrace{|x+1|}_{a}+\underbrace{|x-1|}_{b}=\underset{\underset{a-b.}{\downarrow}}{2}$

$(x+1)(x-1)\le 0 \Rightarrow x\in[-1,1].$

Q Find the smallest integral value of 'a' such that

$|x+a-3|+|x-2a|=|2x-a-3|$ is true $\forall x\in R$?

Sol^n $\underbrace{|x+a-3|}_{A}+\underbrace{|x-2a|}_{B}=|\underbrace{2x-a-3}_{A+B}|.$

$(x+a-3)(x-2a)\ge 0 \quad \forall x\in R$

$x^2-(a+3)x+6a-2a^2\ge 0 \quad \forall x\in R.$

$\boxed{D\le 0}$

$(a+3)^2-4(6a-2a^2)\le 0$

$(a-1)^2\le 0.$

$\downarrow$

$a=1$ Ans

Q Find value of $\log_5\left(\underbrace{\sqrt{7-\sqrt{48}}+\sqrt{5-\sqrt{24}}+\sqrt{3-\sqrt{8}}}_{\text{Masala}}\right)$. ?

Solⁿ $\sqrt{3-\sqrt{8}} = \sqrt{3-2\sqrt{2}} = \sqrt{(\sqrt{2})^2+(1)^2-2\cdot1\cdot\sqrt{2}} = \sqrt{(\sqrt{2}-1)^2} = |\sqrt{2}-1| = \sqrt{2}-1.$ ✓

$\sqrt{5-\sqrt{24}} = \sqrt{5-2\sqrt{6}} = \sqrt{(\sqrt{3}-\sqrt{2})^2} = |\sqrt{3}-\sqrt{2}| = \sqrt{3}-\sqrt{2}$ ✓

$\sqrt{6} \downarrow \sqrt{2}\cdot\sqrt{3}$

$\sqrt{7-\sqrt{48}} = \sqrt{7-4\sqrt{3}} = \sqrt{7-2\cdot2\cdot\sqrt{3}} = \sqrt{(2-\sqrt{3})^2} = |2-\sqrt{3}| = (2-\sqrt{3})$ ✓

Masala $= \cancel{\sqrt{2}}-1+\cancel{\sqrt{3}}-\cancel{\sqrt{2}}+2-\cancel{\sqrt{3}} = 1$

$\log_5(1) = 0.$

Note :-

$$\boxed{a^x = a^y}$$

① $x = y$

② $a = 1$

③ $a = -1$

but x & y must be both odd or both even.

④ $a = 0$

(but make sure $x, y > 0$)

Q $|x-2|^{10x^2-1} = |x-2|^{3x}$

① $10x^2 - 1 = 3x$

$10x^2 - 3x - 1 = 0$

$10x^2 - 5x + 2x - 1 = 0$

$(2x-1)(5x+1) = 0.$

$x = \frac{1}{2}$ or $x = -\frac{1}{5}$

② $|x-2| = 1.$

$x - 2 = \pm 1$

$x = 3$ or $x = 1$

③ $|x-2| = -1$

$\times$

④ $|x-2| = 0 \Rightarrow x = 2$

power.

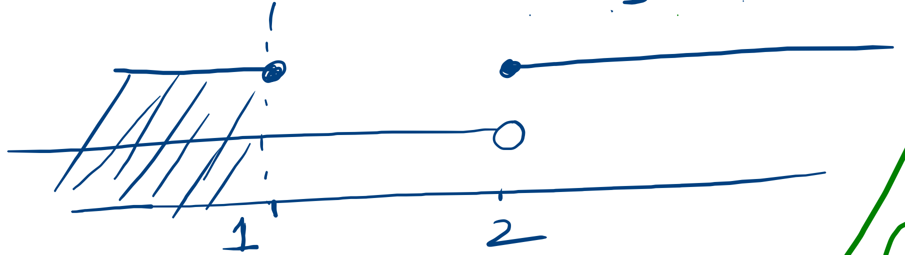
$x \in \left\{-\frac{1}{5}, \frac{1}{2}, 1, 2, 3\right\}$

Q. Solve for x : $\sqrt{x^2-3x+2} > \underbrace{x-2}$

Sol^n

$x^2-3x+2 \geq 0 \Rightarrow (x-1)(x-2) \geq 0$

$x \in (-\infty, 1] \cup [2, \infty)$ — ①



C-I

$x-2 < 0 \Rightarrow \boxed{x < 2}$ — ②

RHS − ve
LHS + ve

LHS > RHS

② ∩ ①

$x \in (-\infty, 1]$ — Ⓐ —

C-II

$\boxed{x-2 \geq 0} \Rightarrow \boxed{x \geq 2}$

$\sqrt{x^2-3x+2} > x-2$

$\cancel{x^2} - 3x+2 > \cancel{x^2}+4-4x \Rightarrow x > 2$

∩

$x \in (2, \infty)$ — ③ —

① ∩ ③

$x \in (2, \infty)$. — Ⓑ —

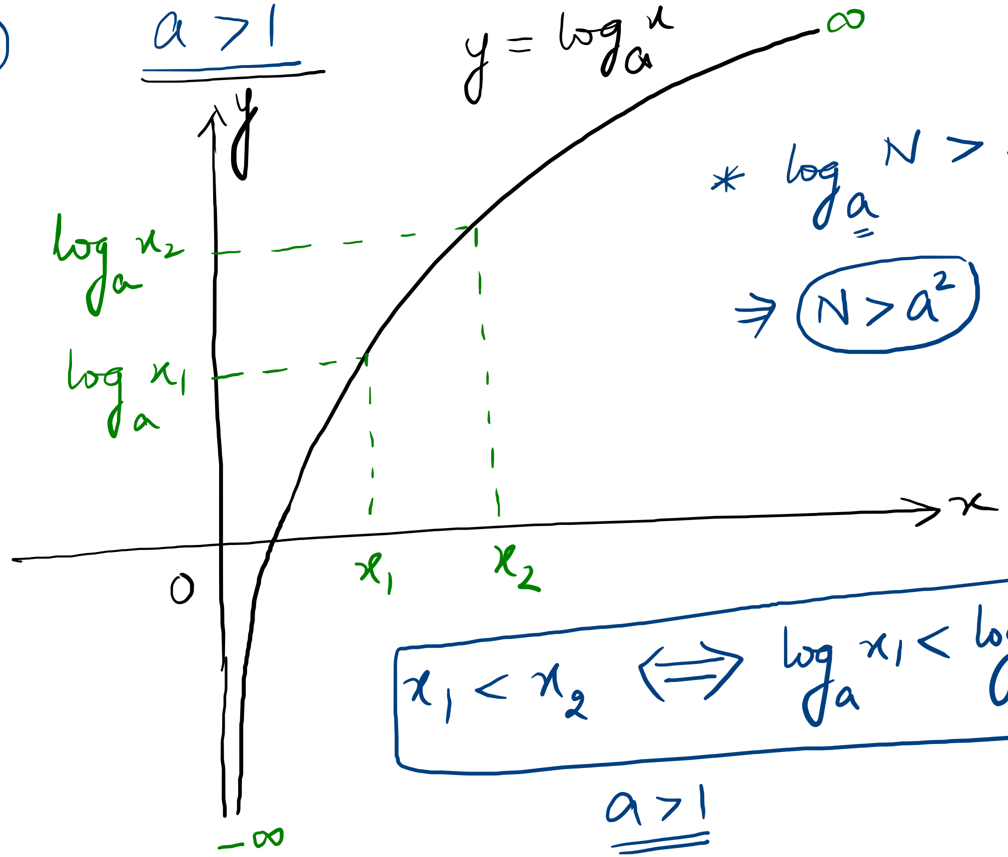
Ⓐ ∪ Ⓑ $\Rightarrow x \in (-\infty, 1] \cup (2, \infty)$

Ans

Log inequality :-

$f(x) = \log_a x$; $\quad x > 0$, $\quad a > 0 \; \& \; a \neq 1.$

(I) $\underline{a > 1}$



* $\log_a N > 2$

$\Rightarrow N > a^2$

$$\boxed{x_1 < x_2 \iff \log_a x_1 < \log_a x_2}$$

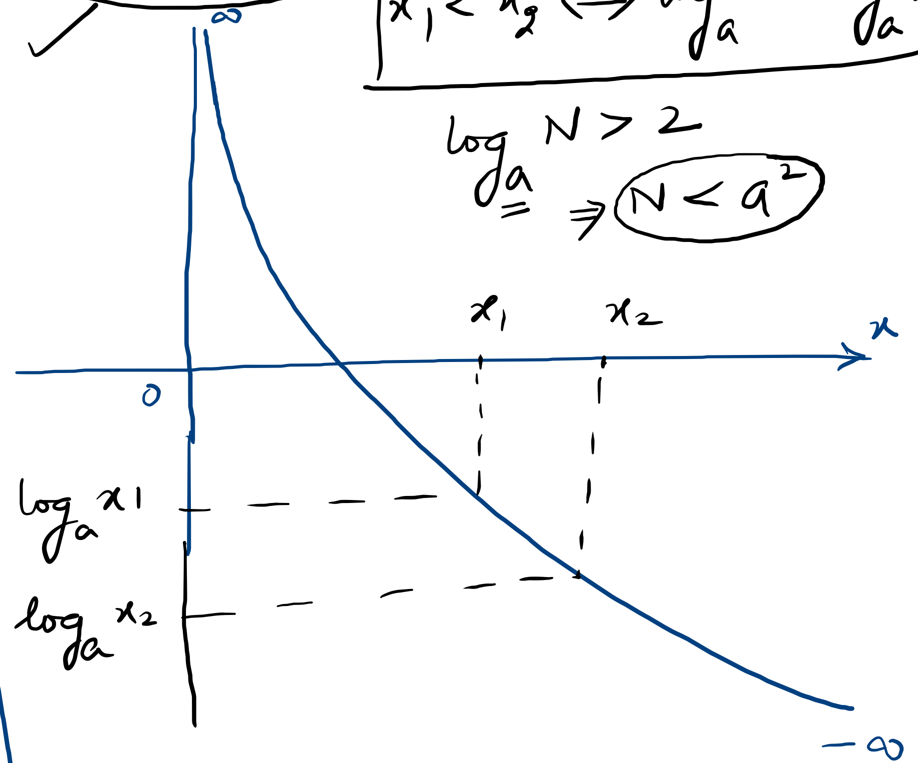
$\underline{\underline{a > 1}}$

(II) $0 < a < 1$ ✓

$$\boxed{x_1 < x_2 \iff \log_a x_1 > \log_a x_2}$$

$\log_a N > 2$

$\Rightarrow N < a^2$



Q1 $\log_7\left(\frac{2x-6}{2x-1}\right) > 0$.

$*$ $\frac{2x-6}{2x-1} > 0$ — ① ✓

$\frac{2x-6}{2x-1} > 1$ — ② ✓

① ∩ ②

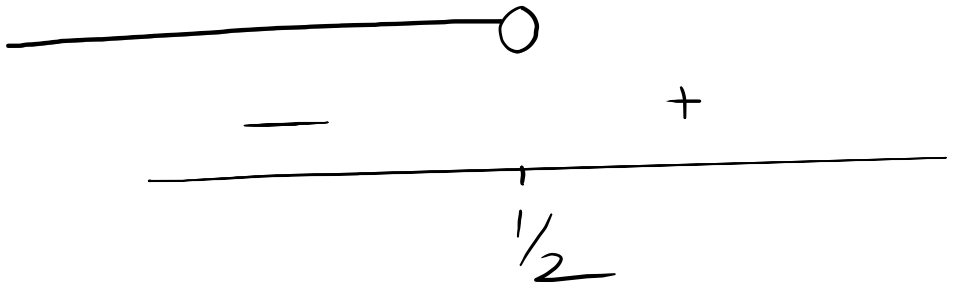
$\frac{2x-6}{2x-1} > 1$.

$\frac{2x-6}{2x-1} - 1 > 0$.

$\frac{\cancel{2x}-6-\cancel{2x}+1}{2x-1} > 0$.

$\frac{-5}{2x-1} > 0$.

$\frac{1}{2x-1} < 0$.



$x \in \left(-\infty, \frac{1}{2}\right)$ Ans

Q2 $\log_{0.2}(\underbrace{x^2-x-2}) > \log_{0.2}(\underbrace{-x^2+2x+3})$.

$x^2-x-2>0$ — ① —

$-x^2+2x+3>0$ — ② —

$\underbrace{x^2-x-2} < \underbrace{-x^2+2x+3}$ — ③ —

$0 < x^2-x-2 < -x^2+2x+3$

$\left.\begin{array}{l} x^2-x-2>0 \\ -x^2+2x+3 > x^2-x-2 \end{array}\right\} \cap$ $\quad x \in \left(2, \frac{5}{2}\right)$ Ans

Q3 $\log_{(2x+3)} x^2 < 1$.

S-1 $\quad x^2 > 0 \Rightarrow x \in R - \{0\}$

$\underbrace{2x+3 > 0}_{x > -3/2}$ & $\underbrace{2x+3 \neq 1}_{x \neq -1} \Rightarrow x \in \left(-\frac{3}{2}, -1\right) \cup (-1, \infty)$.

$\cap$

$$\boxed{x \in \left(-\frac{3}{2}, -1\right) \cup (-1, 0) \cup (0, \infty)}$$ — 1 —

S-2

Base > 1

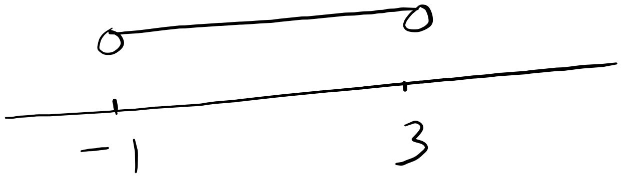
C-I $\quad 2x+3 > 1$.

$\boxed{x > -1}$ — 2 —

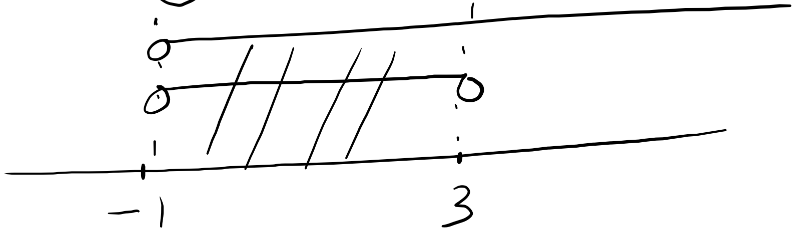
$\log_{(2x+3)} x^2 < 1 \Rightarrow x^2 < 2x+3$

$x^2 - 2x - 3 < 0 \Rightarrow x^2 - 3x + x - 3 < 0$.

$(x-3)(x+1) < 0$ — (3) —



(2) $\cap$ (3)



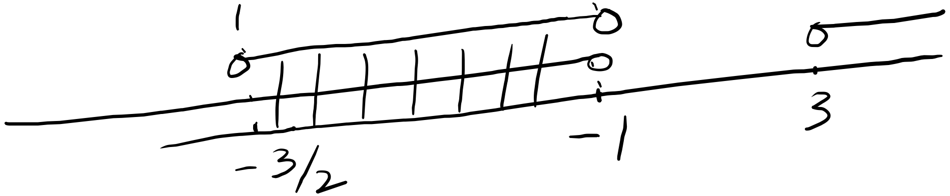
$\boxed{x \in (-1,3)}$ — (A) —

C – II

$0 < \text{Base} < 1$

$0 < 2x+3 < 1$

$-\frac{3}{2} < x < -1$ — (4) —



$\cap$ $\boxed{x \in \left(-\frac{3}{2}, -1\right)}$ — (B) —

$\log_{2x+3}(x^2) < 1 \Rightarrow$

$x^2 > 2x+3$

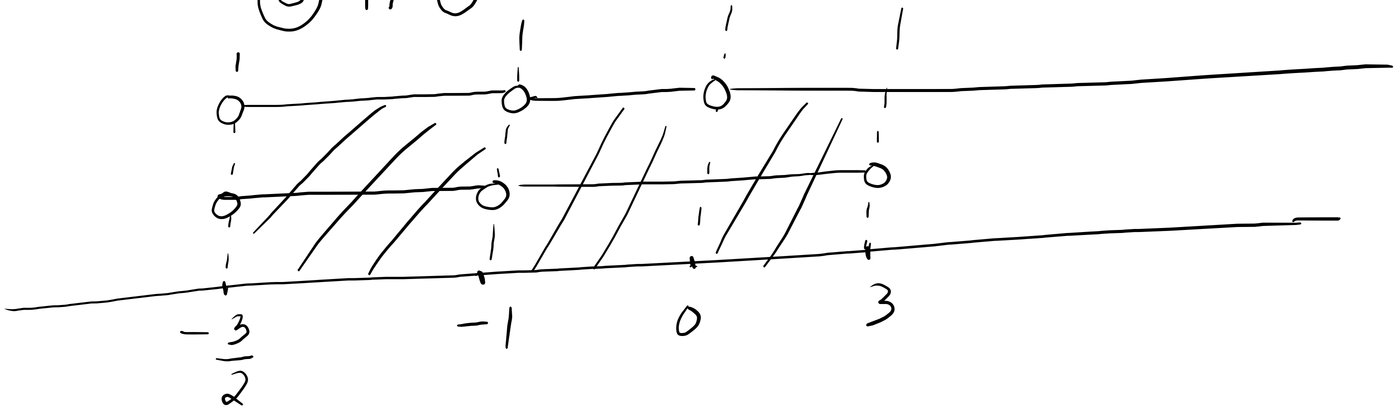
$(x-3)(x+1) > 0$. — (5) —

(A) $\cup$ (B)

$x \in \left(-\frac{3}{2}, -1\right) \cup (-1, 3)$ — ⑥ —

S-3

⑥ ∩ ①



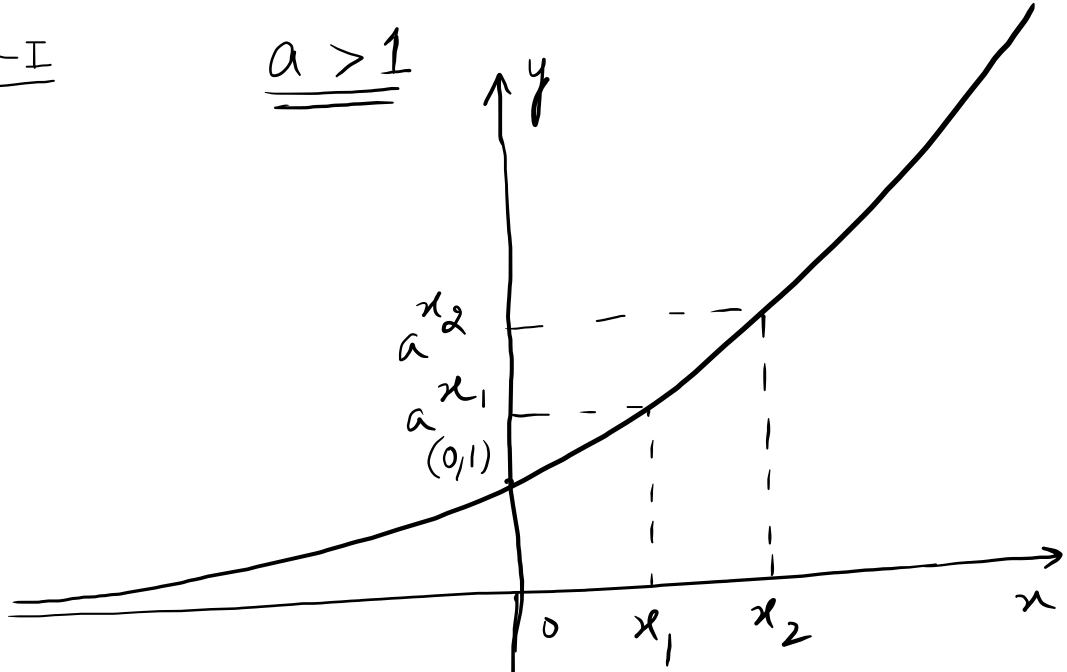
Finally, $x \in \left(-\frac{3}{2}, -1\right) \cup (-1, 0) \cup (0, 3)$

Ans

Exponential inequality :-

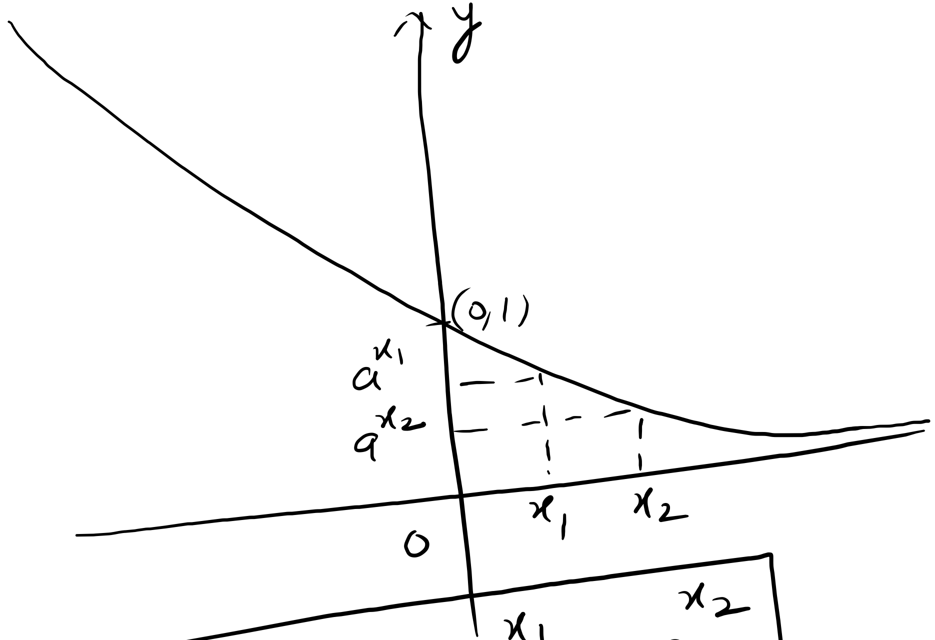
$f(x) = a^{x}$; $a \in (0,1) \cup (1, \infty)$.

C − I $\quad$ $a > 1$



$$\boxed{x_1 < x_2 \iff a^{x_1} < a^{x_2}}$$

C − II $\quad$ $a \in (0,1)$.



$$\boxed{x_1 < x_2 \iff a^{x_1} > a^{x_2}}$$

Q Solve for x :-

$$2^{x+2} > \left(\frac{1}{4}\right)^{\frac{1}{x}}.$$

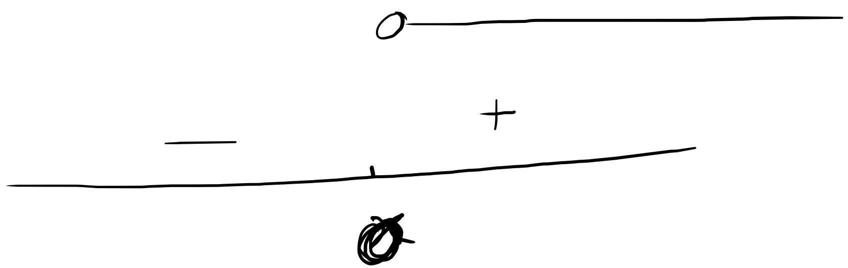
$$(2)^{x+2} > (2)^{-\frac{2}{x}}.$$

$$x+2 > -\frac{2}{x}.$$

$$x+2+\frac{2}{x} > 0.$$

always +ve

$$\frac{\overbrace{x^2+2x+2}}{x} > 0$$



$x \in (0, \infty)$. Ans

Note

$N = 2.65 = \underbrace{2}_{\text{integral part}} + \underbrace{0.65}_{\text{fractional part} = f \in [0,1)}$

$N = 5 = \underbrace{5}_{I} + \underbrace{0.0}_{f=0}$

$N = -5.6 = -5 + \cancel{(-0.6)}$

$= \underbrace{-6}_{I} + \underbrace{0.4}_{f}$

$\log_e N = \ln N$

$e^{\ln N} = N$ ✓

$\ln 2 = 0.693$

$\ln 10 = 2.303$

$\log_{10} 2 = 0.3010$

$\log_{10} 3 = 0.4771$

Characteristic & Mantissa :-

$\log_{10} N = 2.8$

I = characteristic = 2.

f = Mantissa = 0.8.

$\log_5 N = -8.6$

$= -9 + 0.4.$

$= \bar{9} + 0.4$

Note

$\log_{10}(10) = 1.$

$\log_{10}(100) = 2.$

$\log_{10}(1000) = 3.$

Say $\log_{10} N = 1.5$

Char = 1

$\boxed{\text{(char + 1) = No. of digits in N before decimal}}$

$\log_{10} N = 2.2$

char = 2

$(2+1) = 3$ = No of digits in N before decimal

$\log_{10}(0.001) = -3$

$\log_{10} N = -2.7$

$= -3 + 0.3$

$\log_{10}(0.01) = -2$

char $= -3.$

$|-3+1| = 2.$

Rem

$$\boxed{|\text{Char} + 1| = \text{No. of zeroes just after decimal before a non-zero digit starts}}$$

$72.\underbrace{000}32000207$

Q Find no. of digits before decimal in $N = (2.5)^{200}$?

Sol^n

$$\log_{10} N = 200 . \log_{10}(2.5)$$

$$= 200 . \log_{10}\left(\frac{\cancel{25}}{\cancel{10}}\right)$$

$$= 200 . \log_{10}\left(\frac{5}{2}\right)$$

$$= 200\left(\log_{10} 5 - \log_{10} 2\right)$$

$\log_{10} 2 = 0.3010.$

$\log_{10} 3 = 0.4771$

$\log_{10} 7 = 0.8451.$

$$\log_{10} 5 = \log_{10}\left(\frac{10}{2}\right)$$

$$= 1 - \log_{10} 2$$

$$= (1 - 0.3010)$$

$$\log_{10} N = 79.6$$

$$= \quad \text{Char} = 79$$

$79 + 1 = 80$ digits in N before decimal.

Q. Let $\log_3 N = \alpha_1 + \beta_1$; $\log_5 N = \alpha_2 + \beta_2$; $\log_7 N = \alpha_3 + \beta_3$

where $\alpha_1, \alpha_2, \alpha_3$ are integer & $\beta_1, \beta_2, \beta_3 \in [0, 1)$. Find

① No. of integral values of N if $\alpha_1 = 4$ & $\alpha_2 = 2$ ✓ 44

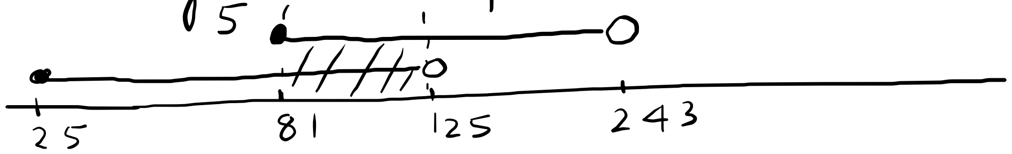
② largest integral value of N if $\alpha_1 = 5$, $\alpha_2 = 3$ & $\alpha_3 = 2$.

Sol^n ① $\alpha_1 = 4$ & $\alpha_2 = 2$ $\quad 4 + \underbrace{\beta_1}$ $\quad \beta_1 \in [0, 1)$

$$\log_3 N = 4 + \beta_1 \Rightarrow N = 3$$

$$3^4 \le N < 3^5 \quad —①—$$

$$\log_5 N = 2 + \beta_2 \Rightarrow N = 5^{2+\beta_2} \Rightarrow 5^2 \le N < 5^3 \quad —②—$$



$\leftarrow$ ① $\cap$ ② $\Rightarrow N \in [81, 125)$

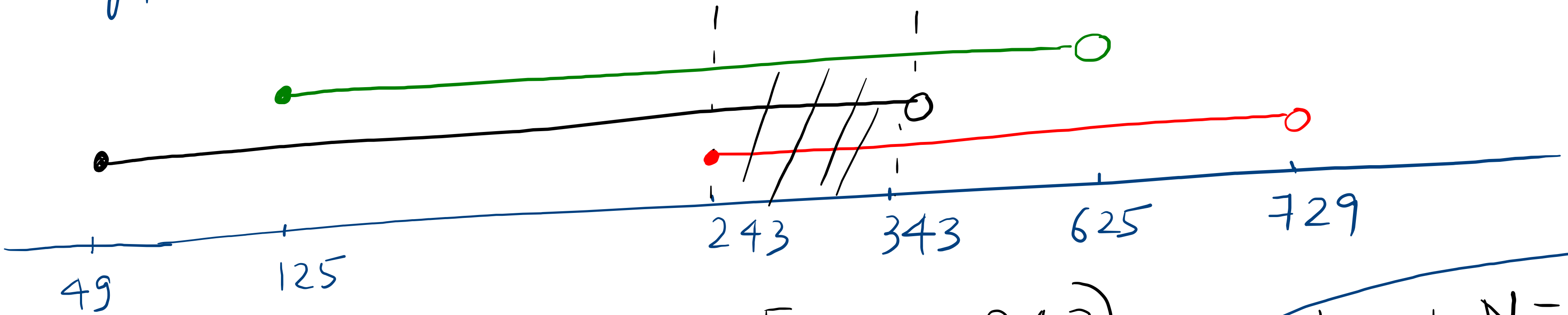
(2) $\alpha_1 = 5$; $\alpha_2 = 3$ & $\alpha_3 = 2$

$\log_3 N = 5 + \beta_1 \Rightarrow N \in [3^5, 3^6)$ — (1) —

$\log_5 N = 3 + \beta_2 \Rightarrow N \in [5^3, 5^4)$ — (2) —

$\log_7 N = 2 + \beta_3 \Rightarrow N \in [7^2, 7^3)$ — (3) —

$\cap$

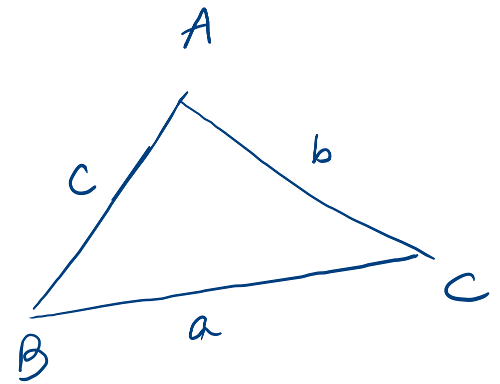


$N \in [243, 343)$

largest $N = 342$ Ans

Compound Angles

Tri — 3
gon — sides
metron — measurement



① $\sin^2\theta + \cos^2\theta = 1$

② $\sec^2\theta - \tan^2\theta = 1 \Rightarrow (\sec\theta - \tan\theta)(\sec\theta + \tan\theta) = 1$

$\sec\theta + \tan\theta = t.$

$\sec\theta - \tan\theta = \frac{1}{t}$

$2\sec\theta = t + \frac{1}{t}.$

③ $\operatorname{cosec}^2\theta - \cot^2\theta = 1$

$$\left.\begin{aligned} \operatorname{cosec}\theta + \cot\theta &= t \\ \operatorname{cosec}\theta - \cot\theta &= \frac{1}{t} \end{aligned}\right\}$$

$a + b > c \Rightarrow |c - b| < a$

$b + c > a$

$c + a > b$

(4) $\dfrac{\sin^4\theta}{\cos^2\theta} = \tan^2\theta \sin^2\theta = \tan^2\theta - \sin^2\theta$ ✓

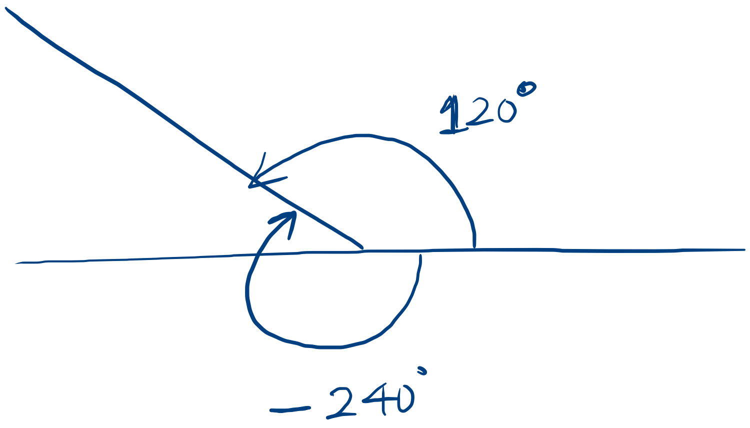
$$\underset{=}{\tan^2\theta}\,(1 - \cos^2\theta) = \tan^2\theta - \sin^2\theta.$$

(5) $\dfrac{\cos^4\theta}{\sin^2\theta} = \cot^2\theta \cdot \cos^2\theta = \cot^2\theta - \cos^2\theta.$

(6) $\sec^2\theta \ \operatorname{cosec}^2\theta = \sec^2\theta + \operatorname{cosec}^2\theta$

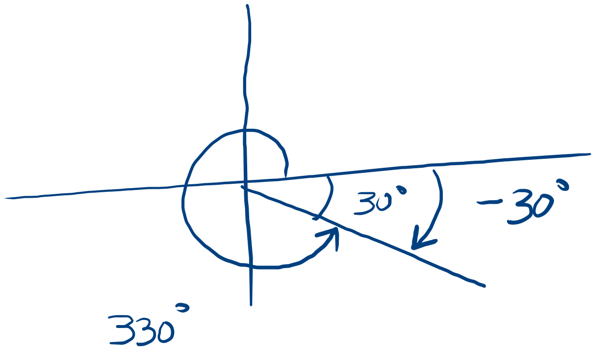
$$\frac{1}{\cos^2\theta} + \frac{1}{\sin^2\theta} = \frac{\sin^2\theta + \cos^2\theta}{\cos^2\theta \sin^2\theta} = \frac{1}{\cos^2\theta \sin^2\theta}.$$

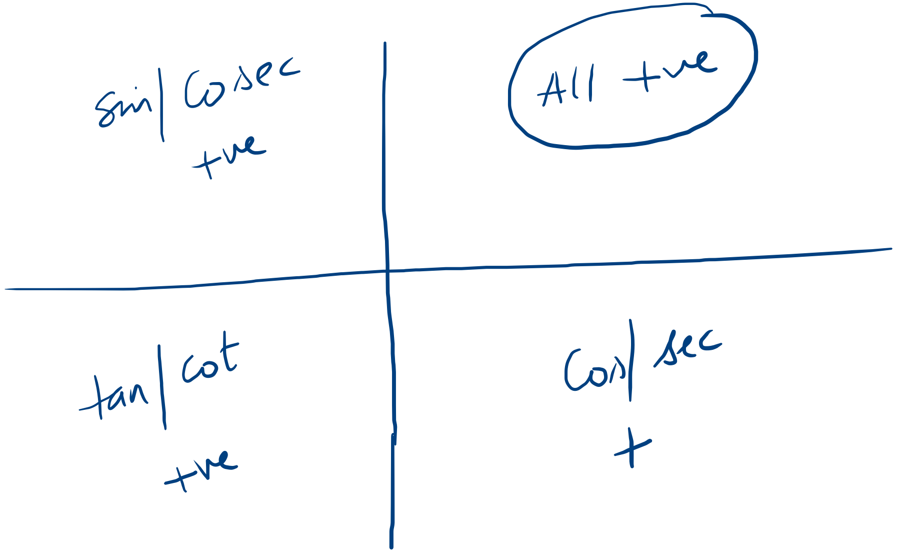
Sign Convention

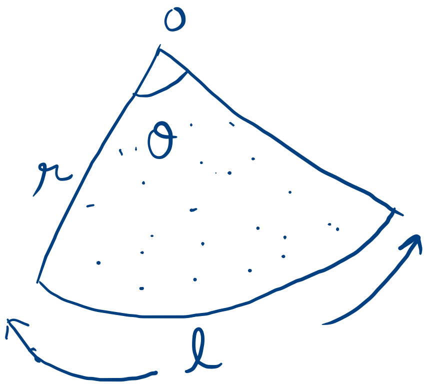


anticlockwise → +ve

clockwise → −ve







$$\boxed{\theta = \frac{arc}{Rad} = \frac{l}{r}}$$

$\theta \in$ Radians

$$\boxed{\text{Area of sector} = \frac{1}{2} r^2 \theta}$$

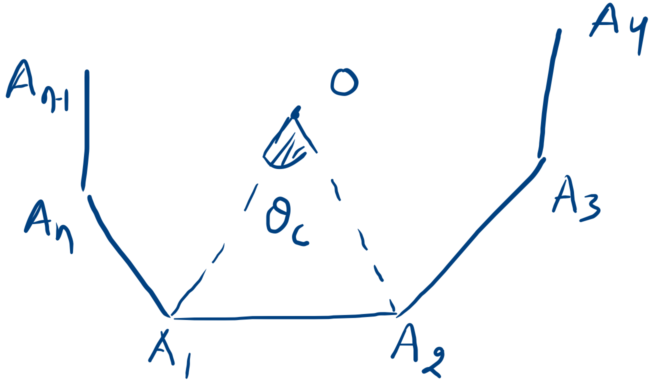
$\pi^c = 180^\circ$

$1^c = \frac{180^\circ}{\pi} \simeq 57.3^\circ$

$\sin x \Rightarrow x \in$ Radians

Regular Polygon :-

All sides are equal



θ_c = Central angle

$$\boxed{\theta_c = \frac{2\pi}{n}}$$

$n \rightarrow$ No. of sides

$$\boxed{\text{Interior } \angle = \left(\frac{n-2}{n}\right)\pi}$$

$\alpha + \beta = 90^\circ$ Complementary angles.

$\alpha + \beta = 180^\circ$ supplementary angles

Reduction :-

$\sin(\pi - \theta) = \sin\theta$

$\cos(\pi - \theta) = -\cos\theta$.

$\vdots$

$$\boxed{\cos\theta + \cos(\pi - \theta) = 0}$$

$\sin(\pi + \theta) = -\sin\theta$

$\tan(\pi + \theta) = \tan\theta$

$\vdots$

$$\boxed{\sin\theta + \sin(\pi + \theta) = 0}$$

and so on

(1) $\sum_{r=1}^{89} \log_{10}(\tan r^\circ) = ?$

$\log_{10} \tan 1^\circ + \log_{10} \tan 2^\circ + \cdots + \log_{10} \tan 89^\circ$

$\log_{10}\left(\cancel{\tan 1^\circ}\ \cancel{\tan 2^\circ}\ \cancel{\tan 3^\circ} \cdots \underbrace{\tan 45^\circ}_{1} \cdots \underbrace{\cancel{\tan 89^\circ}}_{\cot 1^\circ}\right)$

$\log_{10}(1) = 0.$

Q2 $\tan\cancel{\frac{\pi}{11}} + \tan\cancel{\frac{2\pi}{11}} + \tan\cancel{\frac{4\pi}{11}} + \tan\cancel{\frac{7\pi}{11}} + \tan\cancel{\frac{9\pi}{11}} + \underbrace{\tan\cancel{\frac{10\pi}{11}}}_{\tan\left(\pi - \frac{\pi}{11}\right)} = 0$

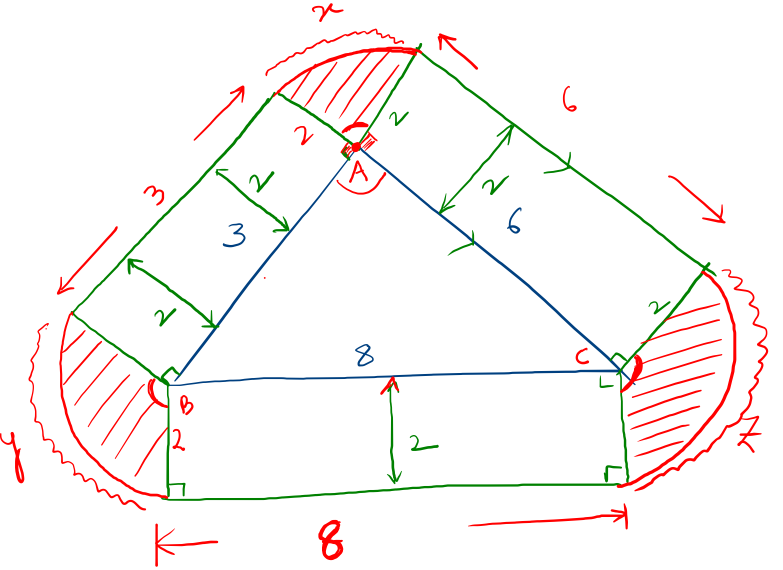
$\frac{\pi}{11} + \frac{10\pi}{11} = \pi$

$-\tan\frac{\pi}{11}$

$$\boxed{\tan\theta + \tan(\pi - \theta) = 0}$$

Q3 Consider a Δ with sides 3, 6, 8 resp.
Now if a man runs around the Δ in such a way that he is always at a distance of 2 units from the sides of Δ then how much distance will he cover in one round?

Sol^n

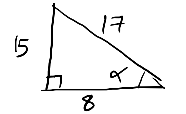


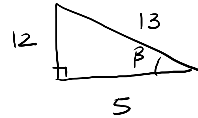
$$(3+6+8)+(\underbrace{x+y+z})$$

$$\boxed{17+4\pi}$$ Ans

$$x+y+z = 2\pi(2)$$

Q If $\sin\alpha = \frac{15}{17}$; $\cos\beta = -\frac{5}{13}$ then find $\cos(\alpha-\beta)$?





$\sin\alpha = \frac{15}{17} > 0$

$\alpha \in$ 1st Quad or 2nd Quad.

$\cos\beta = -\frac{5}{13} < 0$

$\beta \in$ 2nd Quad / 3rd Quad.

α	β
1st	2nd
1st	3rd
2nd	2nd
2nd	3rd

$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta.$$

C–I $\quad \alpha \in 1^{st}$ & $\beta \in 2^{nd}$

$$\cos(\alpha-\beta) = \left(\frac{8}{17}\right)\left(-\frac{5}{13}\right) + \left(\frac{15}{17}\right)\left(\frac{12}{13}\right).$$

C–II $\quad \alpha \in 1^{st}$ & $\beta \in 3^{rd}$

$$\cos(\alpha-\beta) = \left(\frac{8}{17}\right)\left(-\frac{5}{13}\right) + \left(\frac{15}{17}\right)\left(\frac{-12}{13}\right).$$

and so on

$\sin C - \sin D$

$\sin C + \sin D$.

Q If $\alpha = \frac{\pi}{19}$ then find value of

$$\frac{\sin 23\alpha - \sin 3\alpha}{\sin 16\alpha + \sin 4\alpha} \; ?$$

$$\frac{\cancel{2 \sin 10\alpha} \cdot \cos 13\alpha}{\cancel{2 \sin 10\alpha} \; \cos 6\alpha}$$

$$\frac{\cos(\pi - 6\alpha)}{\cos 6\alpha} = -1 \text{ Ans}$$

$\alpha = \frac{\pi}{19}$ ✓✓

$19\alpha = \pi$

$\underbrace{2\alpha + 17\alpha}= \pi$

Q Find value of $\dfrac{\sin 8\theta \cos\theta - \sin 6\theta \cdot \cos 3\theta}{\cos 2\theta \cos\theta - \sin 3\theta \cdot \sin 4\theta}$ for $\theta = 7.5^\circ$

$$\boxed{\theta = 7.5^\circ}$$

$$\left.\begin{array}{l} 2\sin A\cos B \\ 2\sin A\sin B \\ 2\cos A\cos B \end{array}\right\}$$

$$\frac{2\sin 8\theta\cos\theta - 2\sin 6\theta\cos 3\theta}{2\cos 2\theta\cos\theta - 2\sin 3\theta\sin 4\theta}.$$

$$\frac{(\cancel{\sin 9\theta} + \sin 7\theta) - (\cancel{\sin 9\theta} + \sin 3\theta)}{(\cos 3\theta + \cancel{\cos\theta}) - (\cancel{\cos\theta} - \cos 7\theta)}$$

$$\frac{\sin 7\theta - \sin 3\theta}{\cos 3\theta + \cos 7\theta} = \frac{\cancel{2}\sin 2\theta \cdot \cancel{\cos 5\theta}}{\cancel{2}\cancel{\cos 5\theta}\cdot\cos 2\theta}$$

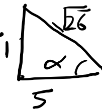
$$= \tan 2\theta = \tan 15^\circ$$

$$= 2 - \sqrt{3}$$

Ans//

Q If $\tan\alpha = \frac{1}{5}$ where $\alpha = 12\beta$, $\alpha \in (0, \pi/2)$ then find value of $(\operatorname{cosec} 4\beta - 5\sec 4\beta)$?

$\tan\alpha = \frac{1}{5}$ *



$\alpha = 12\beta$ *

$$\left(\frac{1}{\sin 4\beta} - \frac{\cot\alpha}{\cos 4\beta}\right) = \frac{1}{\sin 4\beta} - \frac{\cos\alpha}{\sin\alpha \cos 4\beta}$$

$$\frac{\sin\alpha \cos 4\beta - \cos\alpha \sin 4\beta}{\sin\alpha \sin 4\beta \cos 4\beta} = \frac{2\,\cancel{\sin(\alpha - 4\beta)}}{\sin\alpha \cdot \cancel{\sin 8\beta}}$$

$$= 2\operatorname{cosec}\alpha.$$

$$= 2 \cdot \frac{\sqrt{26}}{1} = 2\sqrt{26}.$$

Ans

Q

$$\left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1+\cos\frac{5\pi}{8}\right)\left(1+\cos\frac{7\pi}{8}\right)$$

$$\frac{\pi}{8}+\frac{7\pi}{8}=\pi.$$

$$\cos(\pi-\theta)=-\cos\theta$$

$$\left(1-\cos\frac{3\pi}{8}\right)\left(1-\cos\frac{\pi}{8}\right)$$

$$\frac{\pi}{8}+\frac{3\pi}{8}=\frac{\pi}{2}$$

$$\left(1-\cos^2\frac{\pi}{8}\right)\left(1-\cos^2\frac{3\pi}{8}\right)$$

$$\sin^2\frac{\pi}{8}\cdot\sin^2\frac{3\pi}{8}=\left(\sin\frac{\pi}{8}\cos\frac{\pi}{8}\right)^2$$

$$=\left(\frac{\sin\pi/4}{2}\right)^2=\frac{1}{8}\ \text{Ans}$$

Q If $\cos(\alpha+\beta)\cos(\gamma+\pi/3) = \cos(\alpha-\beta)\cos(\gamma-\pi/3)$ then find value of $(\tan\alpha\tan\beta\cot\gamma)$?

$$\frac{\cos(\alpha+\beta)}{\cos(\alpha-\beta)} = \frac{\cos(\gamma-\pi/3)}{\cos(\gamma+\pi/3)}$$

Componendo & dividendo

$$\frac{\cos(\alpha+\beta)+\cos(\alpha-\beta)}{\cos(\alpha+\beta)-\cos(\alpha-\beta)} = \frac{\cos(\gamma-\pi/3)+\cos(\gamma+\pi/3)}{\cos(\gamma-\pi/3)-\cos(\gamma+\pi/3)}$$

$$\frac{\cancel{2}\cos\alpha.\cos\beta}{\cancel{2}\sin\alpha.\sin(-\beta)} = \frac{\cancel{2}\cos\gamma.\cos.\pi/3}{\cancel{2}\sin\gamma.\sin.\pi/3}$$

$$-\cot\alpha\cot\beta = \cot\gamma.\frac{1}{\sqrt{3}} \Rightarrow \boxed{\tan\alpha\tan\beta\cot\gamma = -\sqrt{3}}$$

Ans

Q $\sin^2(127.5^\circ) - \sin^2(112.5^\circ)$

$\sin^2 A - \sin^2 B$.

$\sin(240^\circ)\sin(15^\circ)$

$\left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)$

$73^\circ + 47^\circ = 120^\circ$

26°

$55^\circ + 65^\circ = 120^\circ$

10°

Q $\cos^2 73^\circ + \cos^2 47^\circ + \cos 73^\circ \cos 47^\circ$.

$\cos^2 A + \cos^2 B = ?$ × ×

$\cos^2 A - \sin^2 B$ ✓✓

$\cos^2 73^\circ + 1 - \sin^2 47^\circ + \frac{1}{2}\left(2\cos 73^\circ \cos 47^\circ\right)$

$\cos 120^\circ \cdot \cos 26^\circ + 1 + \frac{1}{2}\left(\cos 120^\circ + \cos 26^\circ\right)$

$-\cancel{\frac{\cos 26^\circ}{2}} + 1 - \frac{1}{4} + \cancel{\frac{\cos 26^\circ}{2}} = \frac{3}{4}$ Ans

Q If $(\tan 46^\circ - \tan 1^\circ - 1)(\tan 47^\circ - \tan 2^\circ - 1)(\tan 48^\circ - \tan 3^\circ - 1)$

$\ldots\ldots\ (\tan 89^\circ - \tan 44^\circ - 1) = 2^{(n-121)}$

then find n ?

$$\boxed{A - B = 45^\circ}$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}$$

$$1 + \tan A \tan B = \tan A - \tan B$$

$$\boxed{(\tan A - \tan B - 1) = \tan A \tan B}$$

LHS: $\cancel{(\tan 46^\circ \tan 1^\circ)}\ \cancel{(\tan 47^\circ \tan 2^\circ)} \ldots\ldots \cancel{(\tan 89^\circ \tan 44^\circ)}$

LHS $= 1$

$$2^{n-121} = 1 = 2^0$$

$$\boxed{n = 121}$$ Ans

Q $\cot 16^\circ \cot 44^\circ + \cot 44^\circ \cot 76^\circ - \cot 76^\circ \cot 16^\circ$

$$\cot(\underbrace{16^\circ + 44^\circ}) = \frac{\cot 16^\circ \cot 44^\circ - 1}{\cot 16^\circ + \cot 44^\circ}$$

$$\cot(76^\circ - 16^\circ) = \frac{\cot 76^\circ \cot 16^\circ + 1}{\cot 16^\circ - \cot 76^\circ}$$

*

$$\cot 60^\circ(\cot 16^\circ + \cot 44^\circ) + 1$$

$$+ \cot 120^\circ(\cot 44^\circ + \cot 76^\circ) + 1$$

$$-(\cot 60^\circ(\cot 16^\circ - \cot 76^\circ) - 1).$$

$$3 + \frac{1}{\sqrt{3}}\left(\cancel{\cot 16^\circ} + \cancel{\cot 44^\circ} - \cancel{\cot 44^\circ} - \cancel{\cot 76^\circ} - \cancel{\cot 16^\circ} + \cancel{\cot 76^\circ}\right)$$

3 Ans//

*
$$\cot(A+B) = \frac{\cot A \cot B - 1}{\cot B + \cot A}.$$

***
$$\cot(A-B) = \frac{\cot A \cot B + 1}{\cot B - \cot A}.$$

**

If $A+B=\pi/4$ then

$$(1+\tan A)(1+\tan B)=2$$
$$(\cot A-1)(\cot B-1)=2$$
$$(1-\cot A)(1-\cot B)=2.$$

Q $(1-\tan 1^\circ)(1-\tan 2^\circ)(1-\tan 3^\circ)(1+\tan 46^\circ)(1+\tan 47^\circ)(1+\tan 48^\circ)=?$

$$\underbrace{(1+\tan(-1^\circ))(1+\tan 46^\circ)}_{2}\ \overbrace{(1+\tan(-2^\circ))(1+\tan 47^\circ)}^{2}$$
$$\underbrace{(1+\tan(-3^\circ))(1+\tan 48^\circ)}_{2}$$

8 Ans

Q $(1-\cot 2^\circ)(1-\cot 9^\circ)(1-\cot 43^\circ)(1-\cot 36^\circ)=?$

$(1-\cot 2^\circ)(1-\cot 43^\circ) \to 2$

$(1-\cot 9^\circ)(1-\cot 36^\circ) \to 2$

4 Ans

** //

$$\boxed{\cot\theta - \tan\theta = 2\cot 2\theta}$$

Q $\tan\frac{\pi}{16} + \tan\frac{5\pi}{16} + \tan\frac{9\pi}{16} + \tan\frac{13\pi}{16}$.

$$\frac{9\pi}{16} - \frac{\pi}{16} = \frac{8\pi}{16} = \frac{\pi}{2}$$

$$\frac{13\pi}{16} - \frac{5\pi}{16} = \frac{\pi}{2}.$$

$$\tan\frac{\pi}{16} + \tan\frac{5\pi}{16} + \tan\left(\frac{\pi}{2} + \frac{\pi}{16}\right) + \tan\left(\frac{\pi}{2} + \frac{5\pi}{16}\right)$$

$$\tan\frac{\pi}{16} + \tan\frac{5\pi}{16} - \cot\frac{\pi}{16} - \cot\frac{5\pi}{16}.$$

$$5\frac{\pi}{8} - \frac{\pi}{8} = \frac{\pi}{2}$$

$$-2\cot\left(\frac{\pi}{8}\right) - 2\cot\left(\frac{5\pi}{8}\right)$$

$$-2\left(\cot\frac{\pi}{8} + \cot\left(\frac{\pi}{2} + \frac{\pi}{8}\right)\right) = -2\left(\underbrace{\cot\frac{\pi}{8} - \tan\frac{\pi}{8}}\right)$$

$$= -2\left(2.\cot\frac{\pi}{4}\right)$$

$$= -4 \text{ Ans//}$$

Note :-

If $A + B = C$

$\tan A \tan B \tan C$

$= \tan C - \tan B - \tan A$

Q. The product $\tan(\ln x) \cdot \tan\left(\ln \frac{x}{2}\right) \tan(\ln 2)$ whenever defined is equal to

(A) $\tan(\ln x) + \tan\left(\ln \frac{x}{2}\right) + \tan(\ln 2)$

(B) $\tan(\ln x) + \tan\left(\ln \frac{x}{2}\right) - \tan(\ln 2)$

(C) $\tan(\ln x) - \tan\left(\ln \frac{x}{2}\right) - \tan(\ln 2)$

(D) None.

$A = \ln x$

$B = \ln \frac{x}{2}$

$C = \ln 2$

$\boxed{A = B + C}$

$\tan A \tan B \tan C$

$= \tan A - \tan B - \tan C$.

Q

$$\frac{\sin\frac{2\pi}{7}+\sin\frac{4\pi}{7}-\sin\frac{6\pi}{7}}{\sin\frac{\pi}{7}\cdot\sin\frac{3\pi}{7}\cdot\sin\frac{2\pi}{7}}.$$

Sol^n

$$\frac{2\sin\frac{3\pi}{7}\cos\frac{\pi}{7}-2\sin\frac{3\pi}{7}\cos\frac{3\pi}{7}}{\sin\frac{\pi}{7}\sin\frac{2\pi}{7}\sin\frac{3\pi}{7}}$$

$$\frac{2\cancel{\sin\frac{3\pi}{7}}\left(\cos\frac{\pi}{7}-\cos\frac{3\pi}{7}\right)}{\sin\pi/7\,\sin\frac{2\pi}{7}\,\cancel{\sin\frac{3\pi}{7}}}=\frac{2.\left(2\cancel{\sin\frac{2\pi}{7}}\,\cancel{\sin\frac{\pi}{7}}\right)}{\cancel{\sin\pi/7}\,\cancel{\sin 2\pi/7}}$$

$$=4$$ Ans

Q If $\theta = 2021\frac{\pi}{5}$ then $\sin^2(4\theta) + \cos^2\theta = ?$

$$\boxed{5\theta = 2021\pi}$$

$$4\theta = 2021\pi - \theta$$

$$\boxed{\sin^2\theta = \frac{1 - \cos 2\theta}{2}}$$

$$\boxed{\cos^2\theta = \frac{1 + \cos 2\theta}{2}}$$

$$\sin^2(2021\pi - \theta) + \cos^2\theta$$

$$\sin^2\theta + \cos^2\theta = 1 \quad \text{Ans}$$

$$\frac{1-\cos 2\theta}{1+\cos 2\theta} = \tan^2\theta$$

Q If $\dfrac{3 - 4\cos 2A + \cos 4A}{3 + 4\cos 2A + \cos 4A} = K$

then find no. of solution(s) of equation $K = 1$ in $(0, 3\pi)$?

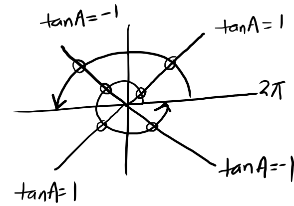
$$\frac{3 - 4\cos 2A + (2\cos^2 2A - 1)}{3 + 4\cos 2A + (2\cos^2 2A - 1)} = \frac{\cancel{2}(\cos^2 2A - 2\cos 2A + 1)}{\cancel{2}(\cos^2 2A + 2\cos 2A + 1)}$$

$$= \left(\frac{1-\cos 2A}{1+\cos 2A}\right)^2 = (\tan^2 A)^2 = \tan^4 A = K.$$

$$\tan^4 A = 1 \Rightarrow \boxed{\tan A = \pm 1}$$

6 sol^n in $(0, 3\pi)$

Ans



Q The expression defined by

$A = \cos^4 x + K\cos^2 2x + \sin^4 x$, where K is constant. If expression is constant for all values of x then K = ?

$$\cos^4 x + \sin^4 x = 1 - \frac{\sin^2 2x}{2}$$

$$A = \left(1 - \frac{1}{2}\sin^2 2x\right) + K\left(1 - \sin^2 2x\right)$$

$$A = (1+K) - \sin^2 2x\left(\frac{1}{2} + K\right) \to 0$$

For A to be constant $\frac{1}{2} + K = 0 \Rightarrow K = -\frac{1}{2}$ Ans

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta.$$

$$\boxed{\sin^3\theta = \frac{3\sin\theta - \sin 3\theta}{4}} \quad *$$

Q Simplify $\sin^3 10^\circ + \sin^3 50^\circ - \sin^3 70^\circ$?

$$\left(\frac{3\sin 10^\circ - \sin 30^\circ}{4}\right) + \left(\frac{3\sin 50^\circ - \sin 150^\circ}{4}\right) - \left(\frac{3\sin 70^\circ - \sin 210^\circ}{4}\right)$$

$$\frac{3}{4}\left(\underbrace{\sin 10^\circ + \sin 50^\circ} - \sin 70^\circ\right) - \frac{1}{4}\left(\sin 30^\circ + \sin 150^\circ - \sin 210^\circ\right)$$

$$\frac{3}{4}\left(\underbrace{\cancel{2}\cancel{\sin 30^\circ}\cdot\cos 20^\circ - \cos 20^\circ}_{0}\right) - \frac{1}{4}\left(\frac{1}{2} + \frac{1}{2} - \left(\frac{-1}{2}\right)\right)$$

$$0 - \frac{1}{4}\left(\frac{3}{2}\right) = -\frac{3}{8} \quad \text{Ans}$$

Q If $f(\theta) = 16\cos^3 2\theta - 32\sin^3\theta - 12\cos 2\theta + 24\sin\theta$

and $f\left(\frac{\pi}{30}\right) = 3\sqrt{a} - b$ where $a, b \in N$

find $(a+b)$? 6 Ans

$$f(\theta) = \left(16\cos^3 2\theta - 12\cos 2\theta\right) + \left(24\sin\theta - 32\sin^3\theta\right)$$

$$= 4\left(4\cos^3 2\theta - 3\cos 2\theta\right) + 8\left(3\sin\theta - 4\sin^3\theta\right)$$

$$f(\theta) = 4\cos 6\theta + 8\sin 3\theta.$$

$$f\left(\frac{\pi}{30}\right) = 4\cos\left(\frac{6\pi}{30}\right) + 8\sin\left(\frac{3\pi}{30}\right)$$

$$= 4\cos 36^\circ + 8\sin 18^\circ$$

$$= 4\left(\frac{\sqrt{5}+1}{4}\right) + 8\left(\frac{\sqrt{5}-1}{4}\right).$$

Q If $\theta = \pi/16$ then find value of

① $\sin^4\theta + \sin^4 3\theta + \sin^4 5\theta + \sin^4 7\theta$.

② $\sin^6\theta + \sin^6 3\theta + \sin^6 5\theta + \sin^6 7\theta$. DYS

$$\boxed{16\theta = \pi}$$

$$\boxed{8\theta = \pi/2}$$

$$7\theta = \frac{\pi}{2} - \theta.$$

$$\sin^4\theta + \cos^4\theta = 1 - \frac{1}{2}\sin^2 2\theta.$$

① $\sin^4\theta + \sin^4 3\theta + \sin^4 5\theta + \sin^4 7\theta$

$$(\sin^4\theta + \cos^4\theta) + (\sin^4 3\theta + \cos^4 3\theta)$$

$$\left(1 - \frac{1}{2}\sin^2 2\theta\right) + \left(1 - \frac{\sin^2 6\theta}{2}\right)$$

$$2 - \frac{1}{2}\Big(\sin^2 2\theta + \underset{\downarrow\ \cos^2 2\theta}{\sin^2 6\theta}\Big) = 2 - \frac{1}{2} = \frac{3}{2} \text{ Ans}$$

Q Given $\begin{cases} \sin\theta + \sin\phi = a \\ \cos\theta + \cos\phi = b \end{cases}$ then

find ① $\tan\left(\frac{\theta-\phi}{2}\right)$ ② $\tan(\theta+\phi)$.

Square & add 2 eqn

$$\sin^2\theta + \sin^2\phi + 2\sin\theta\sin\phi + \cos^2\theta + \cos^2\phi + 2\cos\theta\cos\phi = a^2+b^2$$

$$2 + 2\left(\cos(\theta-\phi)\right) = a^2+b^2$$

$$\cos(\theta-\phi) = \frac{a^2+b^2-2}{2}$$

$$\boxed{\frac{1-\cos(\theta-\phi)}{1+\cos(\theta+\phi)} = \tan^2\left(\frac{\theta-\phi}{2}\right)}$$

$$\sin\theta + \sin\phi = a$$

$$2\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2} = a$$

$$\cos\theta + \cos\phi = b.$$

$$2\cos\frac{\theta+\phi}{2}\cdot\cos\frac{\theta-\phi}{2} = b$$

divide.

$$\boxed{\tan\left(\frac{\theta+\phi}{2}\right) = \frac{a}{b}}$$

$$\tan(\theta+\phi) = \frac{2\tan\left(\frac{\theta+\phi}{2}\right)}{1-\tan^2\left(\frac{\theta+\phi}{2}\right)}$$

Q Find value of $\sqrt{3}\,\text{cosec}\,20^\circ - \sec 20^\circ$

$$\frac{\tan 60^\circ}{\sin 20^\circ} - \frac{1}{\cos 20^\circ} = \frac{\sin 60^\circ}{\cos 60^\circ \sin 20^\circ} - \frac{1}{\cos 20^\circ}$$

$$= \frac{\sin 60^\circ \cos 20^\circ - \cos 60^\circ \sin 20^\circ}{\frac{1}{2} \underbrace{\sin 20^\circ \cos 20^\circ}} = \frac{\overset{2}{\cancel{\sin 40^\circ}}}{\frac{1}{2}\left(\cancel{\sin 20 \cos 20^\circ}\right)}$$

$$= 4 \text{ Ans}$$

Note :-

$$\sqrt{(\sin\theta + \cos\theta)^2} = \sqrt{1 + \sin 2\theta}$$

$$* \quad |\sin\theta + \cos\theta| = \sqrt{1 + \sin 2\theta}$$

$$|\sin\theta - \cos\theta| = \sqrt{1 - \sin 2\theta}$$

Q $\sin\left(67\frac{1}{2}\right)^\circ + \cos\left(67\frac{1}{2}\right)^\circ$ is equal to

(A) $\frac{1}{2}\sqrt{4+2\sqrt{2}}$ (B) $\frac{1}{2}\sqrt{4-2\sqrt{2}}$ (C) $\frac{1}{4}\sqrt{4-2\sqrt{2}}$ (D) None

$\theta \in 1^{st}$ Quad.

$$\sin\theta + \cos\theta = \sqrt{1 + \sin 135^\circ}$$

$$= \sqrt{1 + \frac{1}{\sqrt{2}}}$$

Product Series :-

① $\underbrace{\sin 20^\circ \sin 40^\circ \underline{\underline{\sin 60^\circ}} \sin 80^\circ}$ =

$$\frac{1}{4}\sin(3\times 20^\circ)\ \sin 60^\circ = \frac{1}{4}\cdot\left(\frac{\sqrt{3}}{2}\right)^2$$

$$\sin\theta\ \sin(60^\circ-\theta)\ \sin(60^\circ+\theta) = \frac{1}{4}\sin 3\theta.$$

$$\tan\theta\ \tan(60^\circ-\theta)\tan(60^\circ+\theta) = \tan 3\theta.$$

② $\underbrace{\tan 16^\circ \tan 42^\circ \tan 44^\circ \tan 76^\circ}$ =

$$\tan 42^\circ \tan 48^\circ = 1$$

Rem **Note that:** If continued product of cosine series is given such that each angle is double of previous angle, not necessarily the last one then multiply and divide the series by 2 Sine of smallest angle.

Q $\cos 12^\circ \cos 24^\circ \cos 48^\circ \cos 84^\circ$

$$\frac{2\sin 12^\circ \cos 12^\circ \cos 24^\circ \cos 48^\circ \cos 84^\circ}{2\sin 12^\circ} = \frac{2\sin 96^\circ \cos 84^\circ}{2(8\sin 12^\circ)}$$

$$= \frac{\cancel{\sin 180^\circ}^{\,0} + \sin 12^\circ}{16\sin 12^\circ} = \frac{1}{16}$$

Q $\cos\frac{\pi}{65}\cos\frac{2\pi}{65}\cos\frac{8\pi}{65}\cos\frac{16\pi}{65}\cos\frac{33\pi}{65}\cos\frac{61\pi}{65}$

$\left(-\cos\frac{32\pi}{65}\right)\left(-\cos\frac{4\pi}{65}\right)$

$$\begin{cases}\frac{33\pi}{65} = \pi - \frac{32\pi}{65} \\ \frac{61\pi}{65} = \pi - \frac{4\pi}{65}\end{cases}$$

$$\left(\cos\frac{\pi}{65}\cos\frac{2\pi}{65}\cos\frac{4\pi}{65}\cos\frac{8\pi}{65}\cos\frac{16\pi}{65}\cos\frac{32\pi}{65}\right) = \frac{1}{64}$$ Ans

Q $\cos\frac{\pi}{7}\cos\frac{2\pi}{7}\cos\frac{3\pi}{7}$

$$\frac{2\sin\pi/7\cos\pi/7\cos\frac{2\pi}{7}\cos\frac{3\pi}{7}}{2\sin\pi/7} = \frac{2\sin\frac{4\pi}{7}\cos\frac{3\pi}{7}}{2\times4\sin\pi/7} = \frac{\sin\overset{0}{\cancel{\pi}} + \sin\pi/7}{8\sin\pi/7}$$

$$= \frac{1}{8}.$$

Q $\prod_{r=1}^{7}\sin(2r-1)\frac{\pi}{14}$

$$P = \underbrace{\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}}\ \underbrace{\sin\frac{7\pi}{14}}_{1}\ \underbrace{\sin\frac{9\pi}{14}\sin\frac{11\pi}{14}\sin\frac{13\pi}{14}}_{\sin\frac{5\pi}{14}\cdot\sin\frac{3\pi}{14}\cdot\sin\pi/14}$$

$$P = \left(\sin\frac{\pi}{14}\sin\frac{3\pi}{14}\sin\frac{5\pi}{14}\right)^2.$$

$$P = \left(\underbrace{\cos\frac{3\pi}{7}\cos\frac{2\pi}{7}\cos\pi/7}\right)^2 = \frac{1}{64}$$ Ans

**
$$\sin\left(\frac{5\pi}{14}\right) = \cos\left(\frac{\pi}{2} - \frac{5\pi}{14}\right)$$
$$= \cos\left(\frac{7\pi - 5\pi}{14}\right)$$
$$= \cos\left(\frac{\pi}{7}\right)$$

Q $\sin\frac{\pi}{16}\ \sin\frac{3\pi}{16}\ \sin\frac{5\pi}{16}\ \sin\frac{7\pi}{16}$

$$\frac{\pi}{16}+\frac{7\pi}{16}=\frac{\pi}{2}$$

$$\frac{\pi}{8}+\frac{3\pi}{8}=\frac{\pi}{2}$$

$\sin\frac{\pi}{16}\ \sin\frac{3\pi}{16}\ \cos\frac{3\pi}{16}\ \cos\pi/16$

$$\frac{1}{4}\left(\sin\frac{\pi}{8}\cdot\sin\frac{3\pi}{8}\right)=\frac{1}{4}\left(\sin\frac{\pi}{8}\cos\frac{\pi}{8}\right)=\frac{1}{8}\left(\sin\frac{\pi}{4}\right)=\frac{1}{8\sqrt{2}}\ \text{Ans}$$

* Q If $\sin 1^\circ \sin 3^\circ \sin 5^\circ \ldots \sin 89^\circ = \frac{1}{2^n}$ then find $n = ?$

$\cos 2^\circ$ $\cos 1^\circ$

$$\boxed{n = 44.50}$$

$$\frac{\sin 1^\circ\ \sin 2^\circ\ \sin 3^\circ\ \sin 4^\circ \ldots \sin 45^\circ \ldots \sin 88^\circ\ \sin 89^\circ}{\sin 2^\circ\ \sin 4^\circ\ \sin 6^\circ \ldots \sin 88^\circ}$$

$$\frac{2(\sin 1^\circ \cos 1^\circ)\ 2(\sin 2^\circ \cos 2^\circ)\ldots 2(\sin 44^\circ \cos 44^\circ)\cdot \sin 45^\circ}{2^{44}(\sin 2^\circ\ \sin 4^\circ \ldots \sin 88^\circ)}=\frac{1}{\sqrt{2}}\cdot\frac{1}{2^{44}}$$

$$=\frac{1}{2^{44.5}}$$

Q If $f(n,\theta) = \tan\frac{\theta}{2}(1+\sec\theta)(1+\sec 2\theta)(1+\sec 2^2\theta)$

$\dots\dots\dots\dots(1+\sec 2^n\theta)$; $n \in N$

find $f\left(2, \frac{\pi}{16}\right)$?

$$f(n,\theta) = \underbrace{\tan\frac{\theta}{2}(1+\sec\theta)}_{T_1}(1+\sec 2\theta)(1+\sec 2^2\theta)\dots(1+\sec 2^n\theta)$$

$$T_1 = \frac{\sin\theta/2}{\cos\theta/2}\left(\frac{1+\cos\theta}{\cos\theta}\right) = \frac{\sin\theta/2}{\cancel{\cos\theta/2}}\left(\frac{\cancel{1}+2\cos^2\frac{\theta}{2}-\cancel{1}}{\cos\theta}\right)$$

$$\boxed{T_1 = \tan\theta}$$

$$= \underbrace{\tan\theta\,(1+\sec 2\theta)}_{\tan 2\theta}(1+\sec 2^2\theta)\dots(1+\sec 2^n\theta)$$

$$\boxed{f(n,\theta) = \tan 2^n\theta}$$

$$f\left(2, \frac{\pi}{16}\right) = \tan\left(2^2\cdot\frac{\pi}{16}\right) = 1 \text{ Ans}$$

Q $\cos^2 48° - \sin^2 12°$

$\cos^2 A - \sin^2 B = ?$

$$\cos 60° . \cos 36° = \left(\frac{1}{2}\right)\left(\frac{\sqrt{5}+1}{4}\right).$$

Q $4\cos\left(\frac{\pi}{10}\right) - 3\sec\left(\frac{\pi}{10}\right) - 2\tan\left(\frac{\pi}{10}\right).$

$2\cos^2\theta = 1 + \cos 2\theta.$

$$4\cos 18° - \frac{3}{\cos 18°} - \frac{2\sin 18°}{\cos 18°}$$

$$\frac{4\cos^2 18° - 3 - 2\sin 18°}{\cos 18°} = \frac{2(1+\cos 36°) - 3 - 2\sin 18°}{\cos 18°}$$

$$\frac{2\left(1 + \frac{\sqrt{5}+1}{4}\right) - 3 - 2\left(\frac{\sqrt{5}-1}{4}\right)}{\cos 18°} = 0.$$

Conditional Identities :-

** If $A+B+C=\pi$ then

$\sum \sin 2A = 4 \sin A \sin B \sin C$.

Q $$\frac{\sin 50^\circ + \sin 100^\circ + \sin 210^\circ}{\underbrace{\sin 25^\circ}_{A} \ \underbrace{\sin 50^\circ}_{B} \ \underbrace{\sin 105^\circ}_{C}}$$

$A+B+C=180^\circ$

$$\frac{4 \cancel{\sin 25^\circ \sin 50^\circ \sin 105^\circ}}{\cancel{(\qquad)}} = 4$$

Q If A, B, C are angles of Δ such that $\sum \sin^2 A = 2$ then comment upon nature of Δ ?

Sol^n

$\sin^2 A + \sin^2 B + \sin^2 C = 2$

$\cancel{2} + 2\cos A \cos B \cos C = \cancel{2}$

$\underbrace{\cos A}\ \underbrace{\cos B}\ \underbrace{\cos C} = 0. \Rightarrow \cos A = 0$ OR $\cos B = 0$ OR $\cos C = 0$

$\downarrow$

one of the angle is 90°.

(Right angle Δ)

Q If $\alpha+\beta+\gamma=\pi$ then find value of $\left(\frac{\cos\alpha}{\sin\beta\sin\gamma}\right)+\frac{\cos\beta}{\sin\alpha\sin\gamma}+\frac{\cos\gamma}{\sin\alpha\sin\beta}$

Sol^n

M-1

$$\frac{2\sin\alpha\cos\alpha+2\sin\beta\cos\beta+2\sin\gamma\cos\gamma}{2\sin\alpha\sin\beta\sin\gamma}$$

$$\frac{\sin2\alpha+\sin2\beta+\sin2\gamma}{2\sin\alpha\sin\beta\sin\gamma}=\frac{4\left(\prod\sin\alpha\right)}{2\left(\prod\sin\alpha\right)}=2.$$

M-2

$\alpha+\beta+\gamma=\pi.$

$$\frac{\cos\alpha}{\sin\beta\sin\gamma}=\frac{\cos(\pi-(\beta+\gamma))}{\sin\beta\sin\gamma}=\frac{-\cos(\beta+\gamma)}{\sin\beta\sin\gamma}=\frac{\sin\beta\sin\gamma-\cos\beta\cos\gamma}{\sin\beta\sin\gamma}.$$

$$=(1-\cot\beta\cot\gamma)$$

$$(1-\cot\beta\cot\gamma)+(1-\cot\alpha\cot\gamma)+(1-\cot\alpha\cot\beta)$$

$$3-(\underbrace{\Sigma\cot\alpha\cot\beta}_{1})=3-1=2.$$

Note: If $(A+B+C) = \pi$ then ① $\sum \tan A = \Pi(\tan A)$. ② $\sum \cot A \cot B = 1$

$*$ If $\sum \tan A = \Pi \tan A$ then $\boxed{A+B+C = n\pi}$; $n \in I$

Note: If $A+B+C = \frac{\pi}{2}$ then

① $\sum \cot A = \Pi \cot A$

② $\sum \tan A \tan B = 1$

$$\tan A + \tan B + \tan C = \tan A \tan B \tan C$$

$$\tan A + \tan B = \tan C\,(\underbrace{\tan A \tan B - 1})$$

$$\frac{\tan A + \tan B}{1 - \tan A \tan B} = -\tan C$$

$$\tan(A+B) = \tan(n\pi - C) \; ; \; n \in I$$

$$\boxed{A + B = n\pi - C}$$

* $\tan(A+B+C) = \dfrac{\tan A + \tan B + \tan C - \tan A \tan B \tan C}{1 - (\tan A \tan B + \tan B \tan C + \tan C \tan A)}$ ✓

$$= \frac{\sum \tan A - \prod \tan A}{1 - \sum \tan A \tan B} = \frac{s_1 - s_3}{1 - s_2}$$ ✓

s_1 : sum of the tangent of angles taken one at a time.

s_2 : " " " " " " " two " " ".

s_3 : " " " " " " " three " " ".

$$\tan(A+B+C+D+\cdots) = \frac{s_1 - s_3 + s_5 - s_7 + \cdots}{1 - s_2 + s_4 - s_6 + \cdots}$$

$$\tan(A+B+C+D) = \frac{s_1 - s_3}{1 - s_2 + s_4}.$$

Q If A, B, C are the angles of ΔABC and N denotes the number of possible integral value(s) of expression $(\sin A)^{-2} + (\sin B)^{-2} + (\sin C)^{-2}$ in $[2, 20]$ * then find N?

If $A+B+C=\pi$ then $\sum \cot^2 A \geq 1$.

Sol^n

$$E = \operatorname{cosec}^2 A + \operatorname{cosec}^2 B + \operatorname{cosec}^2 C$$

$$= 1+\cot^2 A + 1+\cot^2 B + 1+\cot^2 C$$

$$= 3 + \underbrace{\sum \cot^2 A}_{\geq 1}$$

$$\boxed{E \geq 4}$$

$$\boxed{N = 17}$$ Ans

Range of Trigonometric functions

① $\cos^4\theta - \sin^4\theta = \cos 2\theta$

$y = \cos x$

② $x \neq (2n+1)\frac{\pi}{2}$ *

$n \in I$

$y = 4\,\frac{\sin x}{\cancel{\cos x}} \cdot \cancel{\cos x}$

$\boxed{y = 4\sin x}$

$(-1, 1)$

1. Find range of y, where $y = \cos^4\frac{x}{2} - \sin^4\frac{x}{2}$. $R \in [-1,1]$ Ans

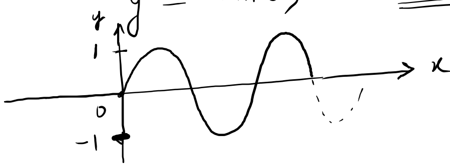
2. Find range of $y = 4\tan x \cdot \cos x$ $R \in (-4,4)$ Ans

3. Let $y = \sin\sqrt{x}$. Find the range of y $(x \geq 0)$.

4. $y = \cos(2\sin x)$, Find range of y.

③ $y = \sin\underbrace{\sqrt{x}}_{\theta}$; $\boxed{x \geq 0}$ *

$y = \sin\theta$; $\theta \in [0, \infty)$



$R \in [-1,1]$ Ans

Graph of Sine, Cos, tan, cot, sec, cosec.

$\sin x \in [-1,1]$.

$\sin^2 x \in [0,1]$

$\tan x \in R$

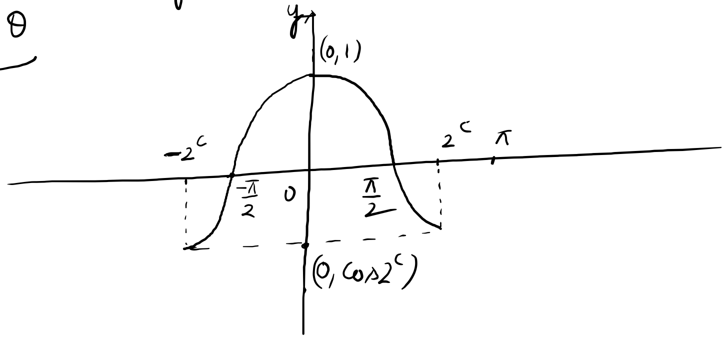
$\tan^2 x \in [0,\infty)$

and so on.

④ $y = \cos(\underbrace{2\sin x}_{\theta})$ $\theta = 2\underbrace{\sin x}_{\downarrow}$ $[-1, 1]$.

$y = \cos\theta$; $\theta \in [-2^c, 2^c]$

$\underbrace{\cos(-\theta) = \cos\theta}$ ✓✓



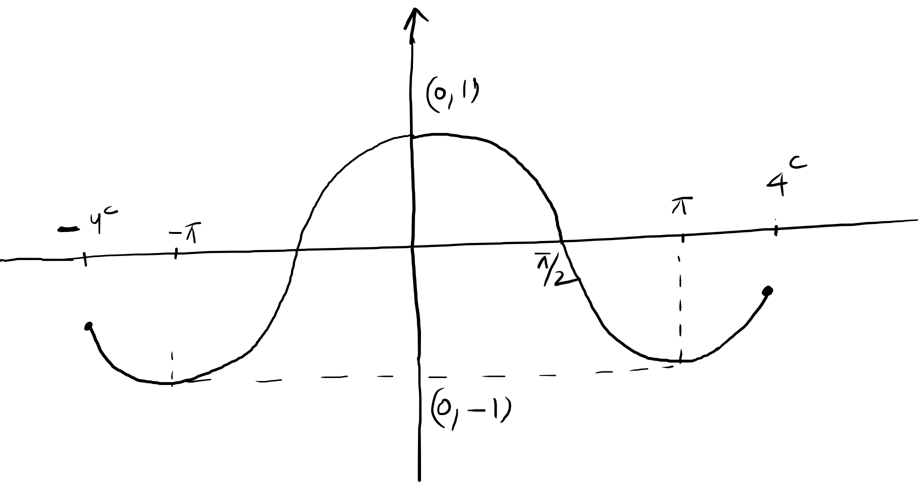
$[\cos 2^c, \underline{1}]$

$\cancel{[-\cos 2^c, 1]}$

⑤ $y = \cos(\underbrace{4\sin x}_{\theta})$.

$\theta = 4\sin x$

$\theta \in [-4^c, 4^c]$



$R \in [-1, 1]$ Ans

Special case:

(1) When argument of sine and cosine are same

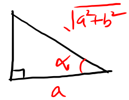
General form $y = a \sin x + b \cos x + c$. Find min. and max. value of y.

$$y \in \left[-\sqrt{a^2+b^2}+c,\ \sqrt{a^2+b^2}+c\right]$$ Ans

$$a \sin\theta + b \cos\theta$$

$$\sqrt{a^2+b^2}\left(\underbrace{\frac{a}{\sqrt{a^2+b^2}}}_{\cos\alpha}\sin\theta + \underbrace{\frac{b}{\sqrt{a^2+b^2}}}_{\sin\alpha}\cos\theta\right)$$

$$\sqrt{a^2+b^2}\ \underbrace{\sin(\theta+\alpha)}_{[-1,1]}$$



$$a \sin\theta + b\cos\theta \to \left[-\sqrt{a^2+b^2},\ \sqrt{a^2+b^2}\right]$$

(a) $y = \dfrac{17 + 5\sin x + 12\cos x}{17 - 5\sin x - 12\cos x}$. Find min. and max. value of y.

$5 \sin x + 12 \cos x \in [-13, 13]$

$$y = \frac{17 + (5\sin x + 12\cos x)}{17 - (5\sin x + 12\cos x)}$$

$$y_{max} = \frac{17+13}{17-13} = \frac{15}{2}$$

$$y_{min} = \frac{17-13}{17-(-13)} = \frac{2}{15}$$

(b) $y = \log_2\left(\dfrac{3\sin x - 4\cos x + 15}{10}\right)$. Find range / min. and max. value of y.

Masala

$3 \sin x - 4\cos x \in [-5, 5]$

$3 \sin x - 4\cos x + 15 \in [10, 20]$

$\dfrac{3 \sin x - 4\cos x + 15}{10} \in [1, 2]$

$y \in [0, 1]$ Ans

$y = 3\cos\underbrace{\left(x + \frac{\pi}{3}\right)} + 5\cos x + 3$. Find min. and max. value of y.

$y = 3\left(\cos x \cos\frac{\pi}{3} - \sin x \sin\frac{\pi}{3}\right) + 5\cos x + 3.$

$y = 3\left(\frac{\cos x}{2} - \sin x \cdot \frac{\sqrt{3}}{2}\right) + 5\cos x + 3$

$y = \underbrace{\left(\frac{13}{2}\cos x - \frac{3\sqrt{3}}{2}\sin x\right)} + 3$

Note

$x \in [-2, 1]$

$x^2 \in [0, 4]$

Type-II: Argument of sine and cosine are different or a quadratic in sine / cos is given then we make a perfect square in sine / cosine and interpret.

(1) $y = \cos^2 x - 4\cos x + 13$. Find min. and max value of y.

$y = \underbrace{(\cos x - 2)^2} + 9$

$\cos x \in [-1, 1].$

$\cos x - 2 \in [-3, -1]$

$(\cos x - 2)^2 \in [1, 9]$

$y_{min} = 1 + 9 = 10.$

$y_{max} = 9 + 9 = 18.$

Type-III: Making use of reciprocal relationship between tan and cot, sin/cosec and cos/sec.

EXAMPLES

Ex.1 $y = a^2\tan^2 x + b^2\cot^2 x$ (a, b > 0). find y_{min} $\forall$ $x \in R$.

$y_{min} = 2ab.$

Ex.2 $y = a^2 \sec^2\theta + b^2 \mathrm{cosec}^2\theta$, find y_{min} (a, b > 0).

(1)

$y = a^2\tan^2 x + b^2\cot^2 x$; $a, b > 0$

AM $\geq$ GM

$$\frac{a^2\tan^2 x + b^2\cot^2 x}{2} \geq \sqrt{a^2 b^2}$$

$$a^2\tan^2 x + b^2\cot^2 x \geq 2|ab|$$

$$\geq 2ab.$$

$$a^2\tan^2 x = b^2\cot^2 x$$

$$\tan^4 x = \left(\frac{b^2}{a^2}\right) > 0.$$

✓✓

M-2

$$y = a^2\tan^2 x + b^2\cot^2 x$$

$$y = (\underbrace{a\tan x - b\cot x}_{\geq 0})^2 + 2ab.$$

$a\tan x = b\cot x$

$\tan^2 x = b/a > 0.$ ✓

$$\boxed{y_{min} = 2ab} *$$

2) $y = \sin^2 x - 20\cos x + 1.$

$$y = 1 - \cos^2 x - 20\cos x + 1$$

$$y = 2 - \cos^2 x - 20\cos x$$

$$y = 2 - (\underbrace{\cos^2 x + 20\cos x})$$

$$= 2 - ((\cos x + 10)^2 - 100)$$

$$y = 102 - (\cos x + 10)^2$$

$\cos x \in [-1, 1]$

$\cos x + 10 \in [9, 11]$

$(\cos x + 10)^2 \in [81, 121]$

$y_{min} = 102 - 121$

$= -19$

$y_{max} = 102 - 81$

$= 21.$

(2) $y = a^2 \sec^2\theta + b^2 \operatorname{cosec}^2\theta.$

$$= a^2(1+\tan^2\theta) + b^2(1+\cot^2\theta)$$

$$= a^2 + b^2 + (\underbrace{a^2\tan^2\theta + b^2\cot^2\theta}_{\geq 2ab})$$

$$y_{min} = a^2 + b^2 + 2ab$$

$$= (a+b)^2.$$

$*$ ① $y = 4\sin^2x + \text{cosec}^2x$, find y_{min}. $= 4$

② $y = 8\sec^2x + 18\cos^2x$, find y_{min}. $= ?$ $\boxed{y_{min} = 24}$

① $y = 4\sin^2x + \text{cosec}^2x$

M-1

AM $\geq$ GM

$$\frac{4\sin^2x + \text{cosec}^2x}{2} \geq \sqrt{4}$$

$$4\sin^2x + \text{cosec}^2x \geq 4$$

$$4\sin^2x = \text{cosec}^2x$$

$$\sin^4x = \frac{1}{4}$$

M-2

$$y = 4\sin^2x + \text{cosec}^2x$$

$$= \underbrace{(2\sin x - \text{cosec}\,x)^2}_{\geq 0} + 4 \Rightarrow \boxed{y_{min} = 4}$$

$$2\sin x = \text{cosec}\,x \Rightarrow \sin^2x = \frac{1}{2}$$

$*$ ③ $y = \sin^2x + 4\,\text{cosec}^2x$ then $y_{min} = ?$

AM $\geq$ GM

X X

$$\frac{\sin^2x + 4\,\text{cosec}^2x}{2} \geq \sqrt{4}$$

$$\sin^2x + 4\,\text{cosec}^2x \geq 4.$$

$$\sin^2x = 4\,\text{cosec}^2x$$

$$\sin^4x = 4 \quad \text{X X X}$$

$$y = (\sin^2x + \text{cosec}^2x) + 3\,\text{cosec}^2x$$

$$y = \underbrace{(\sin x - \text{cosec}\,x)^2}_{\geq 0} + 2 + \underbrace{3\,\text{cosec}^2x}_{\geq 1}$$

$$(\sin x - \text{cosec}\,x)^2 \geq 0.$$

$$\sin x = \text{cosec}\,x$$

$$\boxed{\sin^2x = 1}$$

$$\boxed{y_{min} = 5}$$ Ans

Miscellaneous :-

① $y = \frac{\sin 3x}{\sin x}$?

$\boxed{\sin x \neq 0} \rightarrow \sin x \in [-1,1] - \{0\}$

$\sin^2 x \in (0, 1]$

$$y = \frac{3\sin x - 4\sin^3 x}{\sin x} \Rightarrow y = 3 - 4\sin^2 x$$

$-4\sin^2 x \in [-4, 0)$

$3 - 4\sin^2 x \in [-1, 3)$

$y \in [-1, 3)$ Ans

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$\left.\begin{array}{l} x = a\cos\theta \\ y = b\sin\theta \end{array}\right\}$

$x^2 + y^2 = r^2$

$\left.\begin{array}{l} x = r\cos\theta \\ y = r\sin\theta \end{array}\right\}$

② If $a^2 + b^2 = 4$ and $x^2 + y^2 = 9$ then find range of $(ax + by)$?

$x = 3\cos\phi$

$y = 3\sin\phi$.

Sol^n $a = 2\cos\theta$; $b = 2\sin\theta$

$$(ax + by) = 6(\cos\theta\cos\phi + \sin\theta\sin\phi)$$
$$= 6\underbrace{\cos(\theta - \phi)}_{[-1,1]}$$

$[-6, 6]$ Ans

③ If $4\sin x\cos y + 2\sin x + 2\cos y + 1 = 0$ where $x, y \in [0, 2\pi]$, find the largest possible value of $(x+y)$?

Sol^n

$$4\sin x\cos y + 2\sin x + 2\cos y + 1 = 0$$

$$(2\sin x + 1)(2\cos y + 1) = 0.$$

$2\sin x + 1 = 0$ $\boxed{OR}$ $2\cos y + 1 = 0.$

$\sin x = -\frac{1}{2}$

$$\boxed{x = \frac{7\pi}{6}, \frac{11\pi}{6}}$$

$\cos y = -\frac{1}{2}$

$y = \frac{2\pi}{3}, \frac{4\pi}{3}$

$$(x+y)_{largest} = \left(\frac{11\pi}{6} + \frac{4\pi}{3}\right) \times \times.$$

$$(x+y)_{largest} = \left(\frac{11\pi}{6}\right) + (2\pi)$$

Q If x and y are real nos such that $x^2+2xy-y^2=6$ then find the minimum value of $(x^2+y^2)^2$?

$(x^2+y^2)^2_{min} = ?$

$r^4_{min} = ?$

Sol^n

$$\left.\begin{aligned} x &= r\cos\theta \\ y &= r\sin\theta \end{aligned}\right\}$$

$$x^2+2xy-y^2=6$$

$$r^2\cos^2\theta + 2r^2\underbrace{\sin\theta\cos\theta} - r^2\sin^2\theta = 6.$$

$$r^2(\cos 2\theta + \sin 2\theta) = 6.$$

$$r^2 = \frac{6}{\underbrace{\cos 2\theta + \sin 2\theta}} \Rightarrow r^2_{min} = \frac{6}{\sqrt{2}} \Rightarrow r^4_{min} = \frac{36}{2}$$

$$= 18.$$

Ans//

SUMMATION OF TRIGONOMETRIC SERIES

(1) To find the sum of the series of sine / cosine of n angles which are in A.P. i.e. successive arguments of sine or cosine have the same difference.

1st angle = α

C.d of angles = β.

Multiply by $2\sin\frac{\beta}{2}$.

Rem $S = \sin(\alpha) + \sin(\alpha+\beta) + \sin(\alpha+2\beta) + \dots + \sin(\alpha+(n-1)\beta)$

$$= \left(\frac{\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}\right)\sin\left(\frac{1^{st}.\angle + last.\angle}{2}\right)$$

Rem $S = \cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+2\beta) + \dots + \cos(\alpha+(n-1)\beta)$

$$= \left(\frac{\sin\left(\frac{n\beta}{2}\right)}{\sin\left(\frac{\beta}{2}\right)}\right)\cos\left(\frac{1^{st}\angle + last\ \angle}{2}\right)$$

Q $S = \cos\frac{\pi}{11} + \cos\frac{3\pi}{11} + \cos\frac{5\pi}{11} + \cos\frac{7\pi}{11} + \cos\frac{9\pi}{11}$. Find the sum of series.

$\alpha = \frac{\pi}{11}$; $\beta = \frac{2\pi}{11}$; $n = 5$.

$$S = \frac{\sin\left(5 \cdot \frac{\pi}{11}\right)}{\sin\left(\frac{\pi}{11}\right)} \cos\left(\frac{\frac{\pi}{11} + \frac{9\pi}{11}}{2}\right)$$

$$\frac{\pi}{11} + \frac{10\pi}{11} = \pi$$

$$S = \frac{2\sin\frac{5\pi}{11} \cdot \cos\frac{5\pi}{11}}{2\sin\frac{\pi}{11}} = \frac{\cancel{\sin\left(\frac{10\pi}{11}\right)}}{2\cancel{\sin\left(\frac{\pi}{11}\right)}} = \frac{1}{2} \text{ Ans}$$

$=$

Q Given $f_n(\alpha) = \dfrac{\sin\alpha + \sin 3\alpha + \sin 5\alpha + ... + \sin(2n-1)\alpha}{\cos\alpha + \cos 3\alpha + \cos 5\alpha + ... + \cos(2n-1)\alpha}$. Then match the entries of **column-I** with the values given in **column-II**.

Column-I		Column-II	
(A)	$f_3\left(\frac{\pi}{12}\right)$	(P)	1
(B)	$f_4\left(\frac{\pi}{32}\right)$	(Q)	$\sqrt{2}+1$
(C)	$f_5\left(\frac{\pi}{12}\right)$	(R)	$\sqrt{2}-1$
(D)	$f_5\left(\frac{\pi}{60}\right)$	(S)	$2+\sqrt{3}$
		(T)	$2-\sqrt{3}$

$$f_n(\alpha) = \tan\left(\frac{\alpha + (2n-1)\alpha}{2}\right) = \tan(n\alpha)$$

(A) $$f_3\left(\frac{\pi}{12}\right) = \tan\left(3\cdot\frac{\pi}{12}\right) = \tan\frac{\pi}{4} = 1.$$

Note that: Some series may be brought under this rule

(1) $S = \sin^2\theta + \sin^2 2\theta + \sin^2 3\theta + \ldots\ldots + \sin^2 n\theta$ / $\cos^2\theta + \cos^2 2\theta + \cos^2 3\theta + \ldots\ldots$

It can be written as

$$S = \frac{1}{2}[1 - \cos 2\theta + 1 - \cos 4\theta + 1 - \cos 6\theta + \ldots + 1 - \cos 2n\theta]$$

$$= \frac{1}{2}[n - \underbrace{(\cos 2\theta + \cos 4\theta + \cos 6\theta + \ldots + \cos 2n\theta)}_{\text{formula.}}]$$

(Change to double angle)

$$\sin^2\theta = \frac{1 - \cos 2\theta}{2}$$

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2}$$

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$$

$$\sin^3\theta = \left(\frac{3\sin\theta - \sin 3\theta}{4}\right)$$

(2) $S = \sin^3\theta + \sin^3 2\theta + \sin^3 3\theta + \cdots + \sin^3 n\theta.$

$$S = \left(\frac{3\sin\theta - \sin 3\theta}{4}\right) + \left(\frac{3\sin 2\theta - \sin 6\theta}{4}\right) + \left(\quad\right) + \cdots$$

$$S = \frac{3}{4}\left(\underbrace{\sin\theta + \sin 2\theta + \sin 3\theta + \cdots}_{n \text{ terms}}\right) - \frac{1}{4}\left(\underbrace{\sin 3\theta + \sin 6\theta + \sin 9\theta + \cdots}_{n\text{-terms}}\right).$$

Type-II of summation of sine/cosine series

Splitting the n^{th} term as difference of 2 terms.

(1) $\operatorname{cosec} x + \operatorname{cosec} 2x + \operatorname{cosec} 4x + \ldots\ldots + \operatorname{cosec} 2^n x$

(2) $\sin x \sec 3x + \sin 3x \sec 9x + \sin 9x \sec 27x \ldots$ upto n terms.

(3) Compute the value of the sum $\sum_{r=1}^{n} \left(\frac{\tan 2^{r-1}}{\cos 2^r}\right)$

(1) $S = \underbrace{\operatorname{cosec} x}_{T_1} + \underbrace{\operatorname{cosec} 2x}_{T_2} + \underbrace{\operatorname{cosec} 2^2 x}_{T_3} + \cdots + \underbrace{\operatorname{cosec} 2^n x}_{T_{n+1}}$

$$T_1 = \frac{1}{\sin x \sin \frac{x}{2}} \times \sin(x/2) = \frac{\sin\left(x - \frac{x}{2}\right)}{\sin x \sin \frac{x}{2}} = \frac{\sin x \cos \frac{x}{2} - \cos x \sin \frac{x}{2}}{\sin x \sin \frac{x}{2}}$$

$$\left(x - \frac{x}{2}\right) = \frac{x}{2}$$

$$T_1 = \cot \frac{x}{2} - \cancel{\cot x}$$

$$T_2 = \cancel{\cot x} - \cancel{\cot 2x}$$

$$\vdots$$

$$T_{n+1} = \cancel{\cot 2^{n-1} x} - \cot 2^n x$$

$$\boxed{Sum = \cot \frac{x}{2} - \cot 2^n x}$$

Ans

(2) $S = \underbrace{\sin x \sec 3x}_{T_1} + \underbrace{\sin 3x \sec 9x}_{T_2} + \sin 9x \sec 27x + \cdots$ upto n terms.

$$T_1 = \frac{2\sin x \cos x}{2\cos 3x \cos x} = \frac{\sin 2x}{2\cos 3x \cos x} = \frac{\sin(3x - x)}{2\cos 3x \cos x}$$

$3x - x = 2x$

$$T_1 = \frac{1}{2}\left(\tan 3x - \tan x\right).$$

$$T_2 = \frac{1}{2}\left(\tan 3^2 x - \tan 3x\right).$$

$\vdots$

③ $$S = \sum_{r=1}^{n} \left(\frac{\tan(2^{r-1})}{\cos(2^{r})} \right) = \sum_{r=1}^{n} \frac{\sin(2^{r-1}x)}{\underbrace{\cos 2^{r}x}\ \underbrace{\cos 2^{r-1}x}}$$ //

$$x(2^{r} - 2^{r-1}) = 2^{r-1}(2^{1} - 2^{0})\,x$$
$$= 2^{r-1}x$$

$$\sum_{r=1}^{n} \frac{\sin(2^{r} - 2^{r-1})x}{\cos 2^{r}x \cdot \cos 2^{r-1}x}$$

$$S = \sum_{r=1}^{n} \left(\tan 2^{r}x - \tan 2^{r-1}x \right).$$

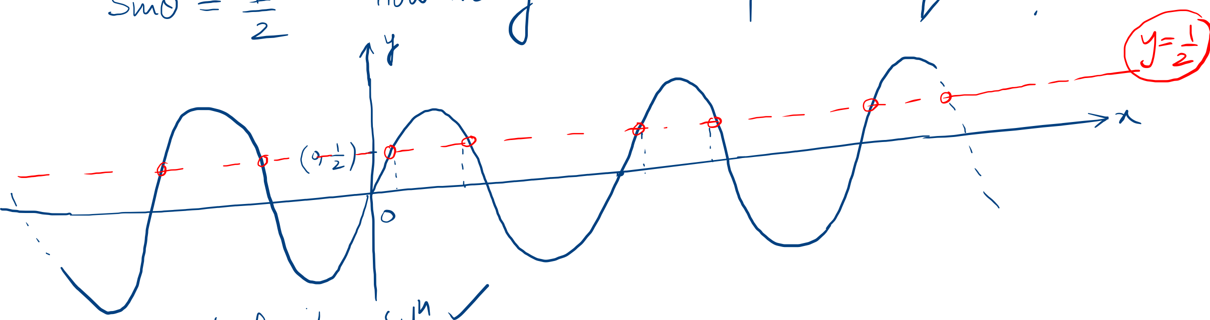
**

$$\sum \operatorname{Cosec}(\ )\operatorname{Cosec}(\ )$$

$$\sum \frac{1}{\sin(\ )\sin(\ )}$$

Trigonometric Equations

$\sin\theta = \frac{1}{2}$ How many solutions of this equation ?



infinite Sol^n ✓

TE

Principal Sol^n → $[0, 2\pi)$

General Sol^n ✓

General solutions :-

① $\sin\theta = \sin\phi$; $\phi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Rem* $\boxed{\theta = n\pi + (-1)^n \phi}$; $n \in I$

$n \in$ Even $\Rightarrow n = 2m$; $m \in I$

$\boxed{\theta = 2m\pi + \phi}$

$n \in$ odd $\Rightarrow n = 2m+1$; $m \in I$

$\boxed{\theta = (2m+1)\pi - \phi}$

Proof :- $\sin\theta - \sin\phi = 0$.

$$2\sin\left(\frac{\theta - \phi}{2}\right)\cos\left(\frac{\theta+\phi}{2}\right) = 0.$$

$\sin\left(\frac{\theta-\phi}{2}\right) = 0$ OR $\cos\left(\frac{\theta+\phi}{2}\right) = 0$

$\frac{\theta - \phi}{2} = n\pi$; $n \in I$

$\boxed{\theta = 2n\pi + \phi}$; $n \in I$ —①—

$\frac{\theta+\phi}{\cancel{2}} = (2n+1)\frac{\pi}{\cancel{2}}$; $n \in I$

$\boxed{\theta = (2n+1)\pi - \phi}$; $n \in I$ —②—

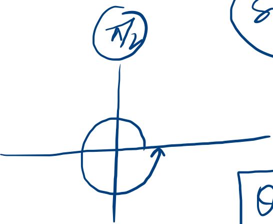
① ∪ ② final answer.

eg: $\sin\theta = \frac{1}{2} \Rightarrow \sin\theta = \sin(\pi/6)$

$$\boxed{\theta = n\pi + (-1)^n \frac{\pi}{6}} \; ; n \in I.$$

$\sin\theta = 1 \Rightarrow \sin\theta = \sin(\pi/2)$

✓ $$\boxed{\theta = n\pi + (-1)^n \pi/2} \; ; n \in I$$



$$\boxed{\theta = 2n\pi + \frac{\pi}{2}} \; ; n \in I$$ ✓

✓ $n = 0 \; ; \; \theta = \pi/2$

✓ $n = 1 \; ; \; \theta = 5\pi/2$

$\vdots$

$n = 0 \; ; \; \theta = \pi/2$ ✓

$n = 1 \; ; \; \theta = \pi/2$ ✓

$n = 2 \; ; \; \theta = 5\pi/2$

$\vdots$

(2) $\cos\theta = \cos\phi$; $\phi \in [0, \pi]$

Rem $\boxed{\theta = 2n\pi \pm \phi}$; $n \in I$

eg: $\cos\theta = \frac{-1}{2} \Rightarrow \cos\theta = \cos\frac{2\pi}{3}$

$\theta = 2n\pi \pm \frac{2\pi}{3}$; $n \in I$

$\cos\theta = -1 \Rightarrow \theta = (2n+1)\pi$; $n \in I$

(3) $\tan\theta = \tan\phi$; $\phi \in \left(-\pi/2, \pi/2\right)$

Rem $\boxed{\theta = n\pi + \phi}$; $n \in I$

$\sin^2\theta = \sin^2\phi$ ✓

$\cos^2\theta = \cos^2\phi$ ✓

$\tan^2\theta = \tan^2\phi$ ✓

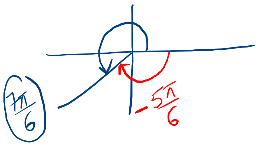
$\boxed{\theta = n\pi \pm \phi}$

$n \in I$

EXAMPLES :

Ex.1 What is the most general values of which satisfy both the equation.

$$\sin\theta = -\frac{1}{2} \quad \text{and} \quad \tan\theta = \frac{1}{\sqrt{3}}$$



$\sin\theta < 0$ and $\tan\theta > 0 \Rightarrow \theta \in 3^{rd}$ Quad.

$\boxed{\theta = 2n\pi + \frac{7\pi}{6}}$; $n \in I$ Ans

$\boxed{\theta = 2n\pi - \frac{5\pi}{6}}$; $n \in I$ Ans

Q2

Principal solution of equation : $\sin^2 x + 2\tan^2 x + \frac{4}{\sqrt{3}}\tan x - \sin x + \frac{11}{12} = 0.$

$$(\sin^2 x - \sin x) + 2\left(\tan^2 x + \frac{2}{\sqrt{3}}\tan x\right) + \frac{11}{12} = 0.$$

$$\left(\sin x - \frac{1}{2}\right)^2 - \frac{1}{4} + 2\left(\left(\tan x + \frac{1}{\sqrt{3}}\right)^2 - \frac{1}{3}\right) + \frac{11}{12} = 0.$$

$$\left(\sin x - \frac{1}{2}\right)^2 + 2\left(\tan x + \frac{1}{\sqrt{3}}\right)^2 \cancel{-\frac{1}{4}} \cancel{-\frac{2}{3}} \cancel{+\frac{11}{12}} = 0.$$

$\alpha^2 + \beta^2 = 0 \Rightarrow \alpha = \beta = 0$ simultaneously

$\sin x = \frac{1}{2}$ and $\tan x = -\frac{1}{\sqrt{3}}$

$x = \frac{5\pi}{6}$ Ans

Find general solution Q1 $(2\sin x - \cos x)(1+\cos x) = \sin^2 x$.

Q2 $2\cos x \cdot \cos 2x = \cos x$

Q3 $\cot x - \cos x = 1 - \cot x \cdot \cos x$

1 $(2\sin x - \cos x)(1+\cos x) = \sin^2 x$

$= (1-\cos^2 x) = (1-\cos x)(1+\cos x)$

$(1+\cos x)(2\sin x - \cancel{\cos x} - 1 + \cancel{\cos x}) = 0$

$\cos x = -1$ OR $\sin x = \frac{1}{2}$

$\Downarrow$

$\boxed{x = (2n+1)\pi}$; $n \in I$ — 1 —

$\boxed{x = m\pi + (-1)^m \frac{\pi}{6}}$; $m \in I$ — 2 —

1 $\cup$ 2

(2) $2\cos x \cos 2x = \cos x$

$\cos x(\underbrace{2\cos 2x - 1}) = 0.$

$\cos x = 0$ OR $\cos 2x = \frac{1}{2}$

$\downarrow$

$\boxed{x = (2n+1)\frac{\pi}{2}}$ —(1)— $n \in I$

OR

$2x = 2n\pi \pm \pi/3$

$\boxed{x = n\pi \pm \pi/6}$ —(2)— $n \in I$

(1) $\cup$ (2)

(3) $\cot x - \cos x = 1 - \cot x \cos x$

V. imp

$\boxed{x \neq n\pi}$

$n \in I$

$\cot x - \cos x - 1 + \cot x \cos x = 0$.

$\cot x(1 + \cos x) - 1(\cos x + 1) = 0$.

$(\cot x - 1)(1 + \cos x) = 0$.

$\cot x = 1$ OR $\boxed{\cos x = -1}$ Rejected

$\tan x = 1$

$\boxed{x = n\pi + \frac{\pi}{4}}$; $n \in I$

Q Find the general solution of the trigonometric equation
$\cos 4x + 6 = 7\cos 2x$ and also find the sum of all the solution in $[0, 314]$

$[0, 314]$ ✓

$\pi > 3.14$

Sol^n

$2\cos^2 2x - 1 + 6 = 7\cos 2x$

$2\cos^2 2x - 7\cos 2x + 5 = 0.$

$2\cos^2 2x - 2\cos 2x - 5\cos 2x + 5 = 0.$

$(2\cos 2x - 5)(\cos 2x - 1) = 0.$

$\cos 2x = \frac{5}{2}$ XXXX OR $\cos 2x = 1 = \cos 0.$

$\cancel{2}x = \cancel{2}n\pi \; ; \; n \in I$

$\boxed{x = n\pi} \; ; \; n \in I.$

$n = 0 \; ; \; x = 0$

$n = 1 \; ; \; x = \pi$

$n = 2 \; ; \; x = 2\pi$

$\vdots$

$n = 99 \; ; \; x = 99\pi$

$n = 100 \; ; \; x = 100\pi.$ XX

Sum $= \pi + 2\pi + \cdots + 99\pi$

$= \pi(4950).$ Ans

Solving trigonometric equations by introducing an Auxilliary argument.

Equation of the form of $\boxed{a\cos\theta + b\sin\theta = c.}$ ★★

For Real soln ★★

$a\cos\theta + b\sin\theta = \text{Ⓒ}$

$\underbrace{a\cos\theta + b\sin\theta}_{[-\sqrt{a^2+b^2},\ \sqrt{a^2+b^2}]}$

$\boxed{|c| \le \sqrt{a^2+b^2}}$ ★

$\sin\theta = 1.5$?

No soln

$|x| \le 2$

$-2 \le x \le 2$

Note : (1) First check for real solution

(2) Avoid squaring.

Q1 $\sqrt{3}\cos x + \sin x = 2$

Q2 $\sin x + \cos x = 1.5$ → $x \in \phi$

Q3 $(\sec x - 1) = (\sqrt{2} - 1)\tan x.$

$\sqrt{3}\cos x + \sin x \in [-2, 2]$

① $\sqrt{3}\cos x + 1\sin x = 2$

$\frac{\sqrt{3}}{2}\cos x + \frac{1}{2}\sin x = \frac{2}{2} = 1$

$\cos\frac{\pi}{6}\cos x + \sin\frac{\pi}{6}\sin x = 1$

$\cos(x - \pi/6) = 1 \Rightarrow x - \frac{\pi}{6} = 2n\pi.$

$\boxed{x = 2n\pi + \pi/6}$; $n \in I$

③ $(\sec x - 1) = (\sqrt{2}-1)\tan x$

$\left\{ \begin{array}{l} x \neq (2n+1)\frac{\pi}{2} \\ n \in I \end{array} \right.$

$\left(\frac{1-\cos x}{\cos x}\right) = \left(\tan\frac{\pi}{8}\right)\frac{\sin x}{\cos x}$

$\cos x\,(1-\cos x) = \tan\frac{\pi}{8}\sin x \cos x$

$\underbrace{\cos x}_{\neq 0}\left((1-\cos x) - \tan\frac{\pi}{8}\sin x\right) = 0.$

$(1-\cos x) = \tan\frac{\pi}{8}\sin x$

$\cancel{1}-(\cancel{1}-\cancel{2}\sin^2\frac{x}{2}) = \tan\frac{\pi}{8}\left(\cancel{2}\sin\frac{x}{2}\cos\frac{x}{2}\right)$

$\sin\frac{x}{2}\left(\underbrace{\sin\frac{x}{2} - \tan\frac{\pi}{8}\cos\frac{x}{2}}\right) = 0.$

$\frac{x}{2} = n\pi.$

$\boxed{x = 2n\pi}$; $n \in I$ — ① —

$\tan\frac{x}{2} = \tan\frac{\pi}{8}$

$\frac{x}{2} = n\pi + \frac{\pi}{8}.$

$\boxed{x = 2n\pi + \frac{\pi}{4}}$

$n \in I$ — ② —

finally

① ∪ ②

Q $1 + \sin^3 x + \cos^3 x = \frac{3}{2} \sin 2x.$

If $a^3 + b^3 + c^3 = 3abc$ then

$a + b + c = 0$ OR

$a = b = c$.

$1^3 + \sin^3 x + \cos^3 x = \frac{3}{\cancel{2}} \cdot \cancel{2} \sin x \cos x$

$a = 1$

$b = \sin x$

$c = \cos x$

$a + b + c = 0 \Rightarrow \sin x + \cos x = -1$ → proceed yourself

OR

$a = b = c \Rightarrow \boxed{1 = \sin x = \cos x}$

XX Not possible

★ Q For what value of K the equation

$K\cos x - 3\sin x = K+1$ has real solution?

$$|K+1| \le \sqrt{K^2+9}$$

$$\cancel{K^2}+2K+1 \le \cancel{K^2}+9$$

$$\boxed{K \le 4} \quad \text{Ans}$$

Solving equations by change of variable or by substitution method.

Q $\sin^4 2x + \cos^4 2x = \sin 2x \cdot \cos 2x.$

$$1 - \frac{1}{2}\sin^2 4x = \frac{1}{2}\sin 4x$$

$$2 - \sin^2 4x = \sin 4x$$

$$\sin^2 4x + \sin 4x - 2 = 0$$

$$\sin^2 4x + 2\sin 4x - \sin 4x - 2 = 0$$

$$(\sin 4x - 1)(\sin 4x + 2) = 0.$$

$\sin 4x = 1$ OR $\sin 4x = -2$ ✗✗

$$4x = 2n\pi + \frac{\pi}{2}$$

$$\boxed{x = \frac{n\pi}{2} + \frac{\pi}{8}}\ ;\ n \in I$$

Ans

Solving equations with the use of the boundness of the function sin x / cos x etc

Q1 $\sin^3 x - \cos^3 x = 1 + \sin x \cdot \cos x$

Q2 $\cos x + \cos 2x + \cos 3x = 3.$

$$\begin{cases} \sin x \in [-1, 1] \\ \sin^2 x \in [0, 1] \\ \operatorname{cosec}^2 x \in [1, \infty) \\ \tan^2 x \in [0, \infty). \end{cases}$$

(1) $(\sin x - \cos x)(\underbrace{\sin^2 x + \cos^2 x}_{1} + \sin x \cos x)$

$= 1 + \sin x \cos x$

$\underbrace{(1 + \sin x \cos x)}\ \underbrace{(\sin x - \cos x - 1)} = 0.$

$\sin x \cos x = -1$

$\sin 2x = -2$

X X

$\dfrac{\sin x}{\sqrt{2}} - \dfrac{\cos x}{\sqrt{2}} = \dfrac{1}{\sqrt{2}}$ (proceed now)

② $\underbrace{\cos x} + \underbrace{\cos 2x} + \underbrace{\cos 3x} = 3.$ $\qquad (\cos\theta)_{max} = 1$

$\cos x = \cos 2x = \cos 3x = 1$ (simultaneously)

$\cos x = 1$	$\cos 2x = 1$	$\cos 3x = 1$
$x = 2n\pi \; ; n \in I$	$2x = 2n\pi$	$3x = 2n\pi$
	$\boxed{x = n\pi} \; ; n \in I$	$\boxed{x = \frac{2n\pi}{3}} \rightarrow n \in I$
$\downarrow$	$\downarrow$	$\downarrow$
$---, -2\pi, 0, 2\pi, 4\pi, 6\pi, -----$	$---, -2\pi, -\pi, 0, \pi, 2\pi, -----$	$---, \frac{-4\pi}{3}, \frac{-2\pi}{3}, 0, \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{6\pi}{3}, ---$

Common sol^n

finally; $\boxed{x = 2n\pi}$; $n \in I$

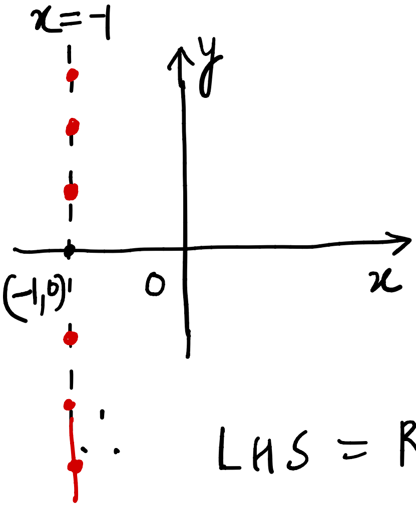
Q Solve for x and y, $1 - 2x - x^2 = \tan^2(x+y) + \cot^2(x+y)$.

RHS : $\underbrace{t + \frac{1}{t}}_{\geq 2}$; $t = \tan^2(x+y)$

$t > 0$

LHS: $1 - 2x - x^2 = 1 - (x^2 + 2x)$

$= 1 - ((x+1)^2 - 1)$



$$\text{LHS} = \underbrace{2 - \underbrace{(x+1)^2}_{\geq 0}}_{\leq 2}$$

$\therefore$ LHS = RHS = 2

$\tan^2(x+y) = 1$ and $(x+1) = 0$

$\boxed{x = -1}$ ✓

$x + y = n\pi \pm \frac{\pi}{4}$.

$\boxed{y = n\pi \pm \frac{\pi}{4} + 1}$; $n \in I$

Q Number of ordered pair satisfying the inequality, $\frac{1}{2^{\sin^2 x}}\sqrt{y^2-2y+2} \le 2.$

$$\underbrace{2^{\underbrace{\operatorname{cosec}^2 x}_{\geq 1}}}_{\geq 2} \underbrace{\sqrt{\underbrace{(y-1)^2}_{\geq 0}+1}}_{\geq 1} \le 2$$

LHS ≥ 2.

$\therefore$ LHS $=$ RHS $= 2$

$\operatorname{cosec}^2 x = 1$ and $y - 1 = 0. \Rightarrow y = 1$

$\sin^2 x = 1$

$\boxed{x = n\pi \pm \frac{\pi}{2}}$; $n \in I$

Q $\sin x\left(\cos\frac{x}{4}-2\sin x\right)+\left(1+\sin\frac{x}{4}-2\cos x\right)\cos x=0$

$$\left(\sin x\cos\frac{x}{4}+\sin\frac{x}{4}\cos x\right)-2\sin^2 x-2\cos^2 x+\cos x=0.$$

$$\sin\frac{5x}{4}+\cos x=2.$$

$$\sin\frac{5x}{4}=1 \quad \underline{\underline{\text{and}}} \quad \cos x=1.$$

$$\frac{5x}{4}=2n\pi+\frac{\pi}{2};\ n\in I \quad \& \quad x=2n\pi;\ n\in I$$

$$x=\frac{8n\pi}{5}+\frac{2\pi}{5};\ \underline{\underline{n\in I}} \quad \& \quad x=2n\pi;\ \underline{\underline{n\in I}}$$

$x \rightarrow$ $\ldots, -\frac{6\pi}{5}, \frac{2\pi}{5}, \frac{10\pi}{5}, \frac{18\pi}{5}, \frac{26\pi}{5}, \frac{34\pi}{5}, \frac{42\pi}{5}, \frac{50\pi}{5}, \ldots$

$x \rightarrow$ $\ldots, -2\pi, 0, 2\pi, 4\pi, 6\pi, 8\pi, 10\pi, 12\pi, \ldots$

$\ldots, -6\pi, 2\pi, 10\pi, 18\pi, \ldots$ AP

$\underline{\underline{c.d=8\pi}}$

① $x = -6\pi + (n-1)8\pi \quad ; \; n \in I$

$\boxed{x = 8n\pi - 14\pi} \quad ; \; n \in I.$

② $x = 2\pi + (n-1)8\pi \; ; \; n \in I$

$\boxed{x = 8n\pi - 6\pi} \quad ; \; n \in I$

$n \to n-1$

Note: Solving equations by Transforming a sum of Trigonometric functions into a product.

Solving equations by transforming a product of trigonometric functions into a sum.

Q General solution of the trigonometric equation, $\sin x + \sin 5x = \sin 2x + \sin 4x$ is

(A) $\frac{n\pi}{3}$ (B) $\frac{2n\pi}{3}$ (C) $2n\pi$ (D) $n\pi$

Sol^n

$$\cancel{2}\sin 3x \cdot \cos 2x = \cancel{2}\sin 3x \cos x$$

$$\sin 3x (\cos 2x - \cos x) = 0.$$

$\sin 3x = 0$ OR $\cos 2x = \cos x$

$3x = n\pi$

$\boxed{x = \frac{n\pi}{3}}$; $n \in I$

$2x = 2n\pi \pm x$

(+) $\boxed{x = 2n\pi}$; $n \in I$

(−) $\boxed{x = \frac{2n\pi}{3}}$; $n \in I$

$n = 6n$

$n = 2n$

$\cdots, -\frac{\pi}{3}, 0, \frac{\pi}{3}, \frac{2\pi}{3}, \frac{3\pi}{3}, \frac{4\pi}{3}, \frac{5\pi}{3}, \frac{6\pi}{3}, \cdots$

Q Number of solutions of the trigonometric equation in $[0, \pi]$

$$\sin 3\theta = 4 \sin \theta \cdot \sin 2\theta \cdot \sin 4\theta$$

(A) 4 (B) 6 (C) 8 (D) 10

Sol^n

$$3 \sin\theta - 4 \sin^3\theta = 4 \sin\theta \sin 2\theta \sin 4\theta$$

$$\sin\theta \left(\underbrace{3 - 4\sin^2\theta - 4 \sin 2\theta \sin 4\theta}\right) = 0.$$

$\Downarrow$

$\sin\theta = 0$

$\boxed{\theta = n\pi}$

$n \in I$

$n = 0 ;\ \theta = 0$

$n = 1 ;\ \theta = \pi$

2 solutions

$$3 - 4\sin^2\theta - 2(\cos 2\theta - \cos 6\theta) = 0.$$

$$3 - 2(1 - \cancel{\cos 2\theta})$$

$$-2 \cancel{\cos 2\theta} + 2\cos 6\theta = 0.$$

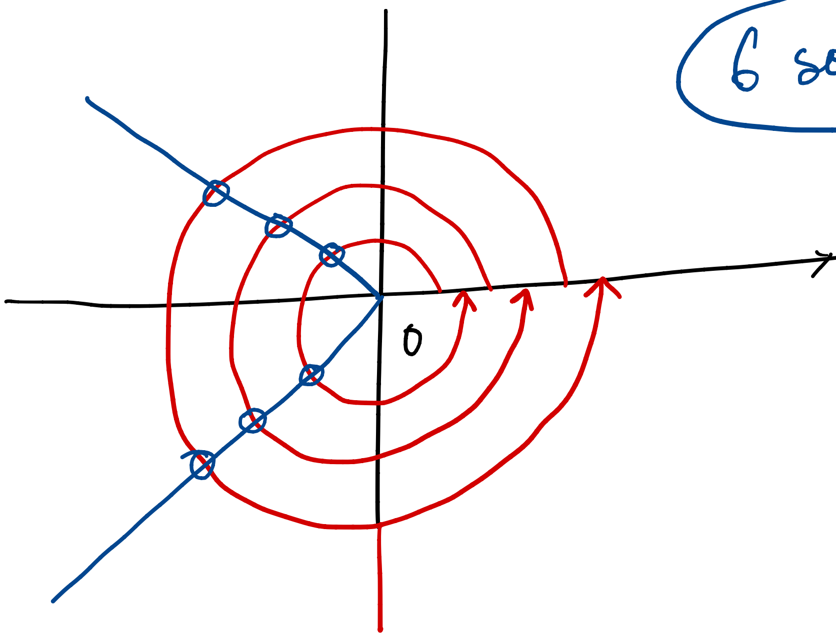
$$\cos 6\theta = -\frac{1}{2}$$

$$6\theta = 2n\pi \pm \frac{2\pi}{3}$$

$$\boxed{\theta = \frac{n\pi}{3} \pm \frac{\pi}{9}} ;\ n \in I$$

$\cos 6\theta = -\frac{1}{2}$; $\theta \in [0, \pi]$

$\underline{6\theta} \in [0, 6\pi]$



6 solutions

$\therefore$ Total 8 $\underline{\text{solutions}}$

Q $\cos^2 x + \cos^2 2x + \cos^2 3x + \cos^2 4x = 2.$

$$\cancel{1} - \sin^2 x + \cancel{1} - \sin^2 2x + \cos^2 3x + \cos^2 4x = \cancel{2}$$

$$\underbrace{(\cos^2 4x - \sin^2 2x)} + \underbrace{(\cos^2 3x - \sin^2 x)} = 0.$$

$$\cos 6x \cdot \cos 2x + \cos 4x \cos 2x = 0.$$

$$\underbrace{\cos 2x} \, (\underbrace{\cos 6x + \cos 4x}) = 0.$$

$\cos 2x = 0$ OR $2\cos 5x \cos x = 0.$

$\Downarrow$

$2x = (2n+1)\frac{\pi}{2}$

$x = (2n+1)\frac{\pi}{4}$ ①

$n \in I$

OR

$\cos x = 0$

$x = (2m+1)\frac{\pi}{2}$

$m \in I$ ②

OR

$\cos 5x = 0$

$5x = (2p+1)\frac{\pi}{2}$

$x = (2p+1)\frac{\pi}{10}$

$p \in I$ ③

① ∪ ② ∪ ③

Q $\text{cosec } x - \text{cosec } 2x = \text{cosec } 4x.$

$n \in I$

$$\frac{1}{\sin x} - \frac{1}{\sin 2x} = \frac{1}{\sin 4x};\quad \begin{cases} x \neq n\pi \\ 2x \neq n\pi \\ 4x \neq n\pi \end{cases}$$

$$\frac{\sin 2x - \sin x}{\sin 2x \sin x} = \frac{1}{\sin 4x}$$

$$2\sin 4x \sin 2x - 2\sin x \sin 4x$$
$$= 2\sin 2x \sin x$$

$$(\cos 2x - \cos 6x) - (\cancel{\cos 3x} - \cos 5x)$$
$$= \cos x - \cancel{\cos 3x}$$

$$(\cos 6x + \cos x) = \cos 5x + \cos 2x$$

$$\cancel{2}\cos\frac{7x}{2}\cos\frac{5x}{2} = \cancel{2}\cos\frac{7x}{2}\cos\frac{3x}{2}$$

$$\underbrace{\cos\frac{7x}{2}}\ \underbrace{\left(\cos\frac{5x}{2}-\cos\frac{3x}{2}\right)}=0.$$

$$\cos\frac{7x}{2}=0$$

OR

$$\cos\frac{5x}{2}=\cos\frac{3x}{2}$$

$$\downarrow$$

$$\frac{7x}{\cancel{2}}=(2n+1)\frac{\pi}{\cancel{2}}$$

$$\boxed{x=(2n+1)\frac{\pi}{7}}$$

$n\in I$

$$\frac{5x}{2}=2n\pi\pm\frac{3x}{2}.$$

$\oplus$ $x=2n\pi$; $n\in I$ XX

$\ominus$ $4x=2n\pi$

$$\boxed{x=\frac{n\pi}{2}}\ ;\ n\in I$$

XX

where

$n\neq\ldots,-4,3,10,17,----$ → $3+(m-1)7$

* OR $x=(2n+1)\dfrac{\pi}{7}$ where $n\neq 7m-4$

$n\in I$ $\qquad m\in I$

Q Find the number of solution of the equation in [0, 2π]

$$\tan(\underbrace{5\pi\cos\alpha}_{\theta}) = \cot(\underbrace{5\pi\sin\alpha}_{\phi})$$

$$\tan(5\pi\cos\alpha) = \tan\left(\frac{\pi}{2} - 5\pi\sin\alpha\right).$$

$$5\cancel{\pi}\cos\alpha = n\cancel{\pi} + \frac{\cancel{\pi}}{2} - 5\cancel{\pi}\sin\alpha.$$

$$5(\underbrace{\sin\alpha + \cos\alpha}) = \frac{(2n+1)}{2}$$

$$\frac{\sin\alpha}{\sqrt{2}} + \frac{\cos\alpha}{\sqrt{2}} = \frac{(2n+1)}{10\sqrt{2}}$$

$$\underbrace{\cos(\alpha - \pi/4)}_{[-1,\,1]} = \underbrace{\frac{(2n+1)}{10\sqrt{2}}}_{[-1,\,1]} \; ; \; n \in I$$

$n \searrow$ $\quad 10\sqrt{2} = 14.14$

$$\boxed{-7, \ldots, -2, -1, 0, 1, 2, 3, 4, 5, 6}$$

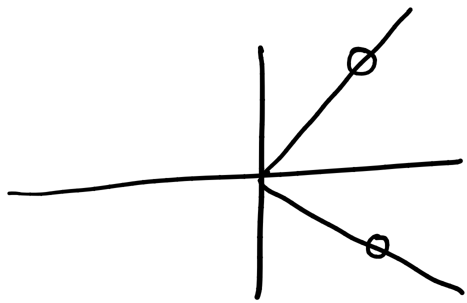
14 values of 'n'.

for $\underline{\underline{n=0}}$

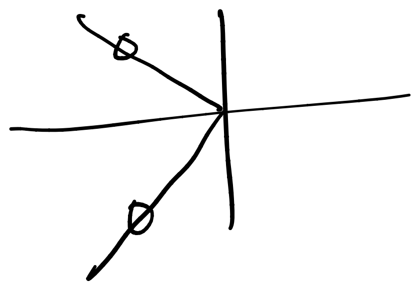
$$\cos\left(\alpha-\frac{\pi}{4}\right)=\left(\frac{1}{10\sqrt{2}}\right)$$

lies $[-1,1]$

for $n=-2$



$$\cos\left(\alpha-\frac{\pi}{4}\right)=\frac{-3}{10\sqrt{2}}$$



Total no. of solutions = $\underline{\underline{28}}$.

Solution of trigonometric equation of the form of $f(x) = \sqrt{\phi(x)}$.

Q $\sqrt{1-\cos x} = \sin x.$

squaring both sides.

$$1-\cos x = \sin^2 x = 1-\cos^2 x$$

$$(1-\cos x)\left(\cancel{1}-(\cancel{1}+\cos x)\right) = 0.$$

$1-\cos x = 0$ OR $\cos x = 0.$

$\cos x = 1$ * OR $\cos x = 0.$

$\boxed{x = 0}$ ✓ Check !!! ... $x = \frac{\pi}{2}$ ✓ ; $\frac{3\pi}{2}$ ✗ ✗ ✗

Check !!! ... Check !!! ...

$\boxed{x = 2n\pi}$ $n \in I$ Union $\boxed{x = 2n\pi + \frac{\pi}{2}}$ $n \in I$

Q $2\sin\left(3x+\frac{\pi}{4}\right) = \sqrt{1+8\sin 2x\cos^2 2x}$

$$2\left(\sin 3x \cdot \cos\frac{\pi}{4} + \cos 3x \sin\frac{\pi}{4}\right) = \sqrt{\quad}$$

$$\sqrt{2}\left(\underbrace{\sin 3x + \cos 3x}\right) = \sqrt{1+8\sin 2x\cos^2 2x}$$

squaring:

$$2(1+\sin 6x) = 1+4\cos 2x(2\sin 2x\cos 2x)$$

$$2+2\sin 6x = 1+4\cos 2x\sin 4x$$

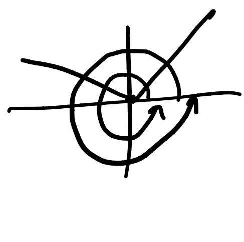
$$2+\cancel{2\sin 6x} = 1+2(\cancel{\sin 6x}+\sin 2x)$$

$$\boxed{\sin 2x = \frac{1}{2}}$$

$x\in[0,2\pi)$

$2x\in[0,4\pi)$

$$2x = \frac{\pi}{6}, \frac{5\pi}{6}, 2\pi+\frac{\pi}{6}, 2\pi+\frac{5\pi}{6}$$



$$2x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}$$

$$x = \underbrace{\frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}}_{\text{Check!!!}}$$

✓ ✗ ✗ ✓

acceptable

$$x = 2n\pi + \frac{\pi}{12}; n \in I$$

$$x = 2n\pi + \frac{17\pi}{12}; n \in I$$

union

Ans

Q Find general solution of TE

$$3^{\sin 2x + 2\cos^2 x} + 3^{1 - \sin 2x + 2\sin^2 x} = 28.$$

Sol^n

$$3^{\sin 2x + 2\cos^2 x} + 3^{1 - \sin 2x + 2(1-\cos^2 x)} = 28.$$

$$3^{\sin 2x + 2\cos^2 x} + 3^{3 - (\sin 2x + 2\cos^2 x)} = 28.$$

$$\boxed{3^{\sin 2x + 2\cos^2 x} = t}$$

$$t + \frac{27}{t} = 28. \Rightarrow t^2 - 28t + 27 = 0$$

$$t = 1 \text{ or } 27$$

$$3^{\sin 2x + 2\cos^2 x} = 3^0 \text{ or } 3^3.$$

$$\underbrace{\sin 2x} + \underbrace{2\cos^2 x} = 0 \quad \text{OR} \quad \sin 2x + 2\cos^2 x = 3$$

$$\text{OR} \quad \underbrace{\sin 2x + \cos 2x = 2}_{\text{Not possible}}.$$

$$\sin 2x + \cos 2x = -1$$

$$\frac{\sin 2x}{\sqrt{2}} + \frac{\cos 2x}{\sqrt{2}} = \frac{-1}{\sqrt{2}}$$

$$\cos(2x - \pi/4) = \cos\frac{3\pi}{4}.$$

Q2 If the equation $x^2+4+3\sin(ax+b)-2x=0$ has atleast one real solution where $a,b\in[0,2\pi]$ then one of the possible value of $(a+b)$ is

(A) $\frac{7\pi}{2}$ (B) $\frac{5\pi}{2}$ (C) $\frac{9\pi}{2}$ (D) $\frac{5\pi}{4}$.

Sol^n

$$(x^2-2x+1)+3+3\sin(ax+b)=0.$$

$$\underbrace{(x-1)^2+3}_{\geq 3} = \underbrace{-3\sin(ax+b)}_{\leq 3}$$

$\therefore$ LHS = RHS = 3

$x-1=0$ & $\sin(ax+b)=-1$

$x=1$

$\sin(a+b)=-1$

$$\boxed{a+b = 2n\pi-\pi/2} \; ; n\in I$$

Q Solve for x & y :-

(1) $\cos x \cos y = \frac{3}{4}$

$\sin x \sin y = \frac{1}{4}$.

Sol^n Add: $\cos(x-y) = 1$.

$\boxed{x-y = 2n\pi}$; $n \in I$. —(1)—

Sub:

$\cos(x+y) = \frac{1}{2}$

$\boxed{x+y = 2m\pi \pm \frac{\pi}{3}}$ —(2)—

$m \in I$

Add (1) & (2)

$2x = 2n\pi + 2m\pi \pm \pi/3$.

$\boxed{x = n\pi + m\pi \pm \pi/6}$

$n, m \in I$

(2) $$\underbrace{x+y=\frac{2\pi}{3}}_{=} \quad \text{and} \quad \frac{\sin x}{\sin y}=2.$$

$$\sin x = 2\sin\left(\frac{2\pi}{3}-x\right)$$

$$\sin x = 2\left(\sin\frac{2\pi}{3}\cos x - \cos\frac{2\pi}{3}\sin x\right)$$

$$\cancel{\sin x} = 2\left(\frac{\sqrt{3}}{2}\cos x + \frac{1}{2}\cancel{\sin x}\right)$$

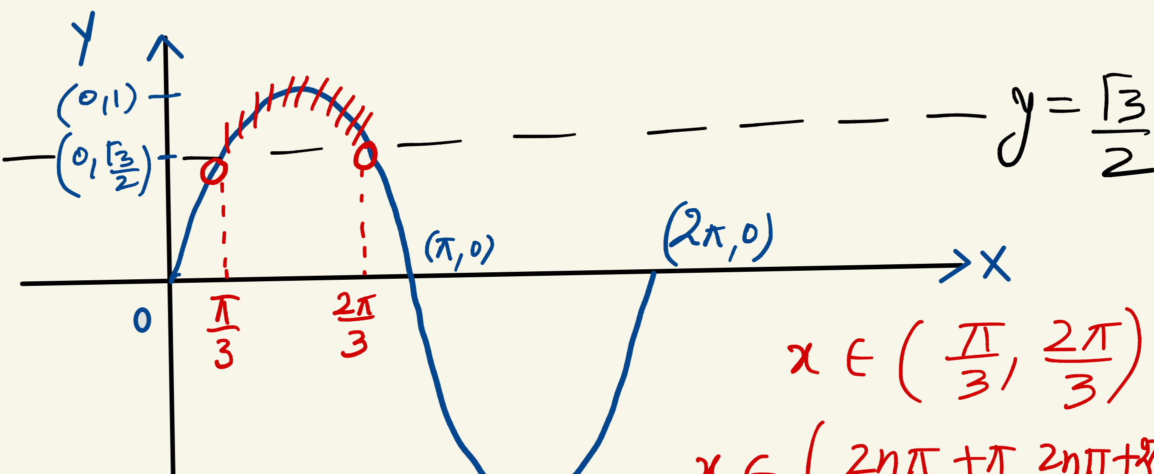
$$\cos x = 0 \Rightarrow \boxed{x=(2n+1)\frac{\pi}{2}};\ n\in I$$

Trigonometric Inequalities

$\left.\begin{array}{l}\sin x\\ \cos x\\ \sec x\\ \operatorname{cosec} x\end{array}\right\}$ Period $= 2\pi$

$\left.\begin{array}{l}\tan x\\ \cot x\end{array}\right\}$ Period $= \pi$.

Q1 $\sin x > \frac{\sqrt{3}}{2}$



$x \in \left(\frac{\pi}{3}, \frac{2\pi}{3}\right)$

$x \in \left(2n\pi + \frac{\pi}{3}, 2n\pi + \frac{2\pi}{3}\right)$

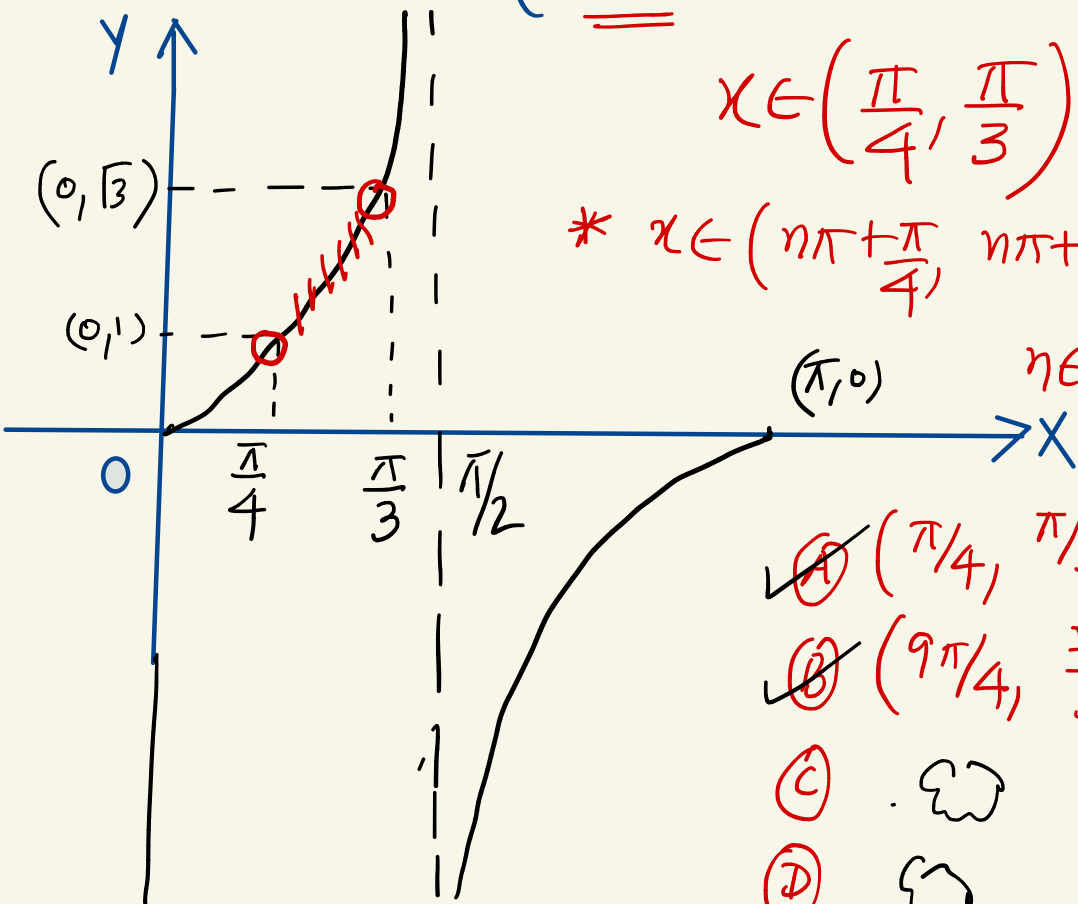
$n \in I$.

② $\tan^2 x - (\sqrt{3}+1)\tan x + \sqrt{3} < 0.$

$\tan^2 x - \sqrt{3}\tan x - \tan x + \sqrt{3} < 0.$

$(\tan x - \sqrt{3})(\tan x - 1) < 0.$

$\tan x \in (1, \underline{\underline{\sqrt{3}}}).$



$x \in \left(\frac{\pi}{4}, \frac{\pi}{3}\right)$

* $x \in \left(n\pi + \frac{\pi}{4}, n\pi + \frac{\pi}{3}\right)$ $n \in I.$

✓ Ⓐ $(\pi/4, \pi/3)$

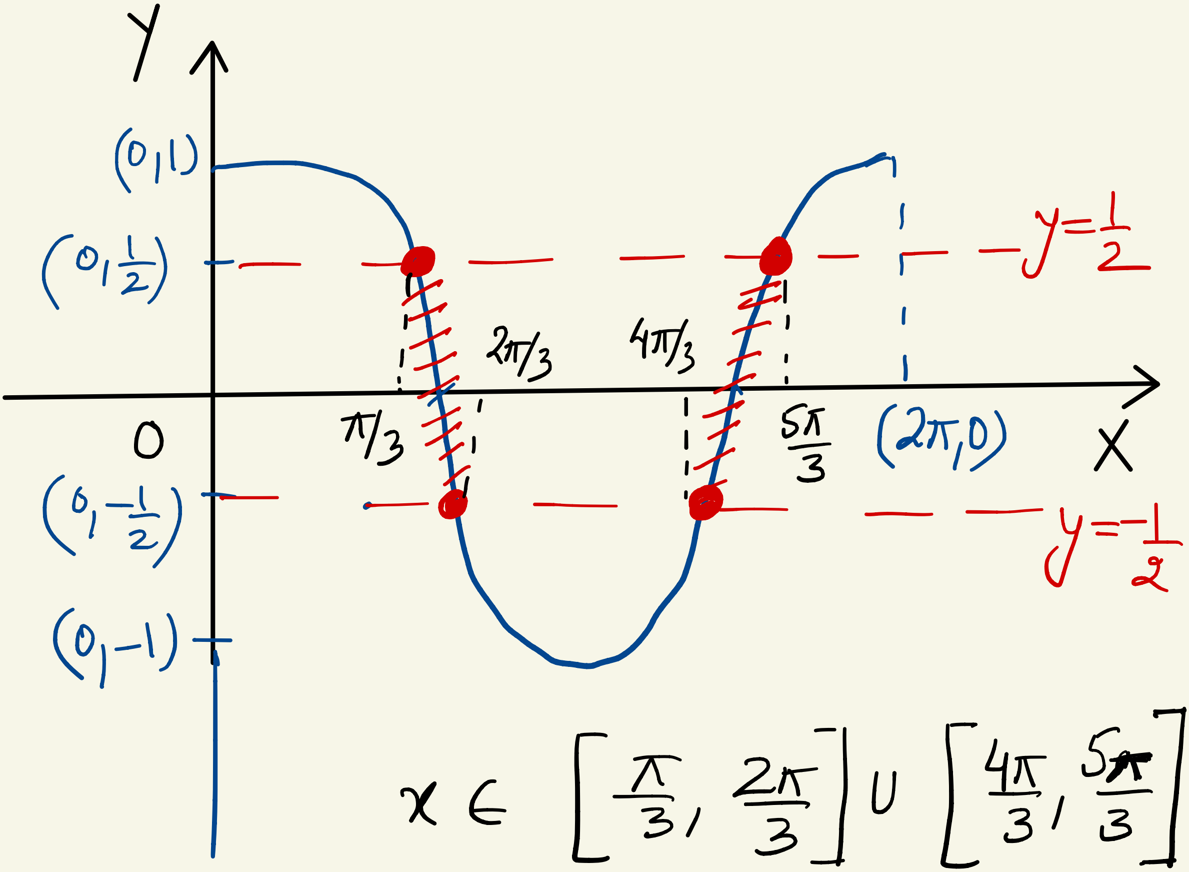
✓ Ⓑ $(9\pi/4, \frac{7\pi}{3})$

Ⓒ

Ⓓ

Q3 $|\cos x| \leq \frac{1}{2}$

$$-\frac{1}{2} \leq \cos x \leq \frac{1}{2}$$



$$x \in \left[\frac{\pi}{3}, \frac{2\pi}{3}\right] \cup \left[\frac{4\pi}{3}, \frac{5\pi}{3}\right]$$

$$x \in \left[2n\pi + \frac{\pi}{3}, 2n\pi + \frac{2\pi}{3}\right] \cup \left[2n\pi + \frac{4\pi}{3}, 2n\pi + \frac{5\pi}{3}\right]$$

$n \in I$

Q.

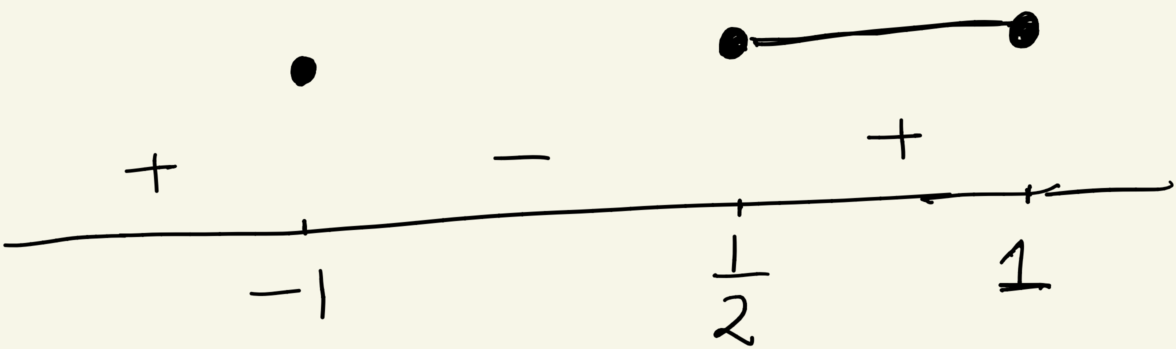
$$\sin x \geq \cos 2x.$$

$$\sin x \geq 1 - 2\sin^2 x$$

$$2\sin^2 x + \sin x - 1 \geq 0.$$

$$2\sin^2 x + 2\sin x - \sin x - 1 \geq 0.$$

$$(2\sin x - 1)(\sin x + 1) \geq 0.$$

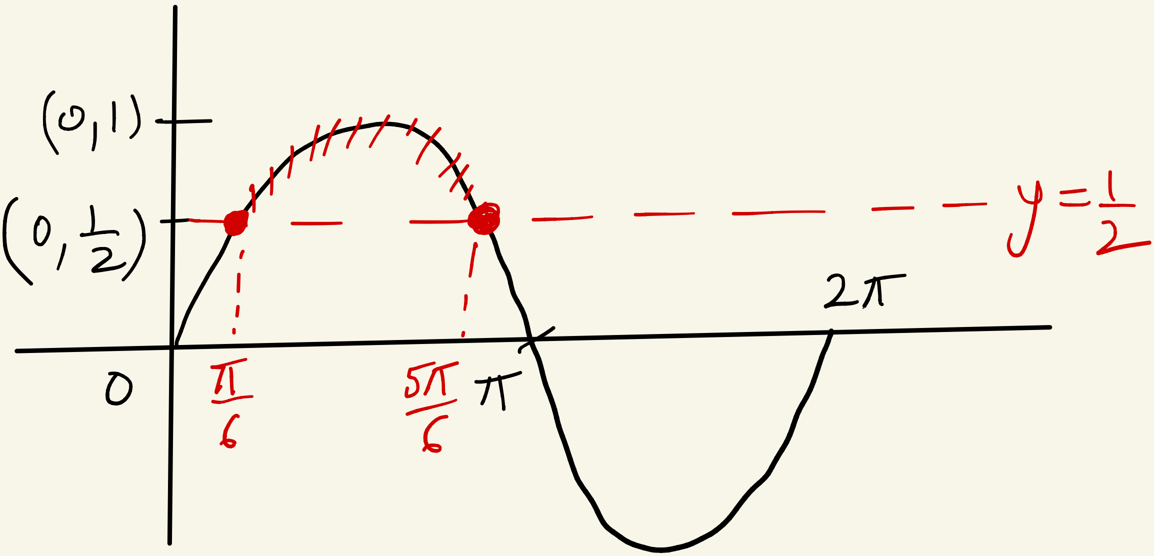


$\sin x = -1$ and $\frac{1}{2} \leq \sin x \leq 1$

$$\boxed{x = 2n\pi - \frac{\pi}{2}}$$

$n \in I$ — (1) —

$$\frac{1}{2} \le \sin x \le 1$$



$$x \in \left[2n\pi + \frac{\pi}{6},\ 2n\pi + \frac{5\pi}{6}\right]$$

$n \in I$ — 2 —

1 ∪ 2

Q

$$\sqrt{5-2\sin x} \geq \underbrace{6\sin x - 1}_{}.$$

$$5 - 2\sin x \geq 0.$$

$$\sin x \leq \frac{5}{2}$$ always true

$$x \in R$$

C–I $\quad 6\sin x - 1 < 0.$

$$\boxed{\sin x < \frac{1}{6}}$$ *

LHS ≥ 0 but RHS < 0

LHS $\geq$ RHS

TRUE

$$\therefore \boxed{\sin x < \frac{1}{6}} \quad \text{—(1)—}$$

C-II

$$6\sin x - 1 \geq 0$$

$$\boxed{\sin x \geq \frac{1}{6}} \; ** $$

squaring :-

$$5 - 2\sin x \geq 36\sin^2 x - 12\sin x + 1.$$

$$36\sin^2 x - 10\sin x - 4 \leq 0.$$

$$18\sin^2 x - 5\sin x - 2 \leq 0.$$

$$18\sin^2 x - 9\sin x + 4\sin x - 2 \leq 0$$

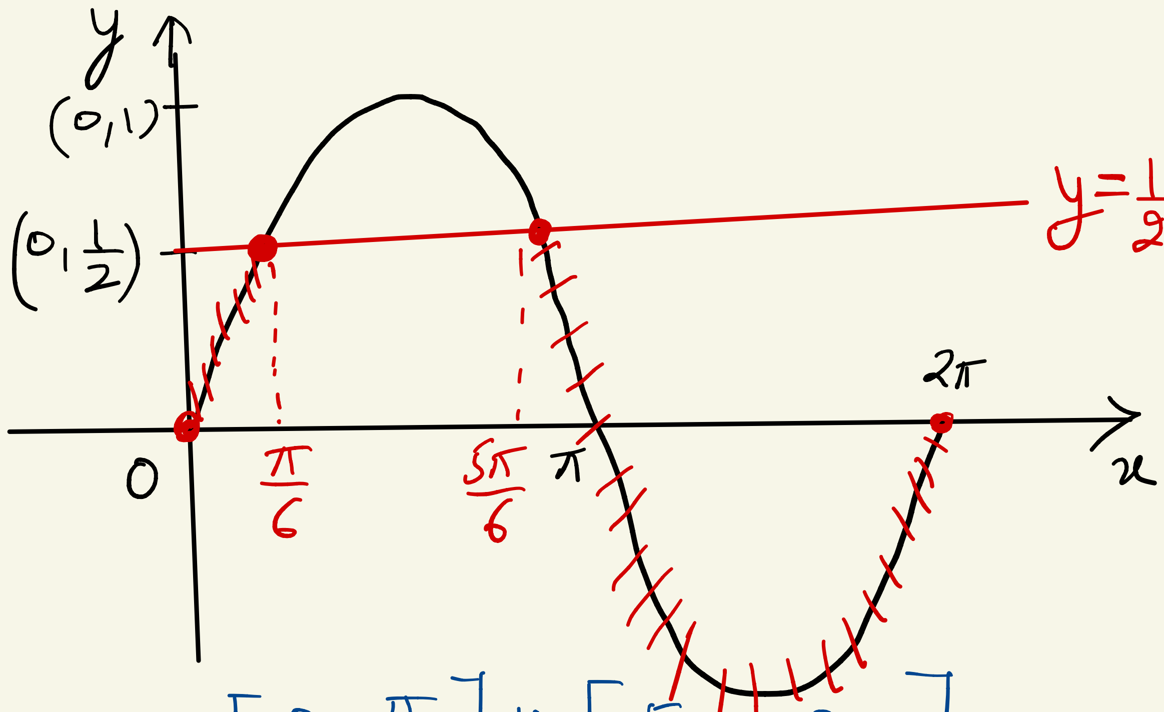
$$\underbrace{(9\sin x + 2)}_{+ve}\underbrace{(2\sin x - 1)} \leq 0.$$

$$\sin x \leq \frac{1}{2}$$

$\sin x \in \left[\frac{1}{6}, \frac{1}{2}\right]$ — (2) —

(1) $\cup$ (2)

$$\boxed{\sin x \le \frac{1}{2}}$$



$x \in \left[0, \frac{\pi}{6}\right] \cup \left[\frac{5\pi}{6}, 2\pi\right]$

OR

$x \in \left[\frac{5\pi}{6}, \frac{13\pi}{6}\right]$.

Q If $x, y \in [0, 2\pi]$ and $\sin^4 x = 1 + \cos^6 y$ then find no. of possible ordered pairs (x, y) which satisfy the equation?

Sol^n

$$\underbrace{\sin^4 x}_{\leq 1} = \underbrace{1 + \cos^6 y}_{\geq 1}$$

$\boxed{4}$ Ans

$\sin^4 x = 1$ and $\cos^6 y = 0$

$\cos y = 0$

$\sin x = \pm 1$

$\boxed{x = \frac{\pi}{2}, \frac{3\pi}{2}}$

$\boxed{y = \frac{\pi}{2}, \frac{3\pi}{2}}$

Q. $\tan x + \tan\left(x+\frac{\pi}{3}\right) + \tan\left(x+\frac{2\pi}{3}\right) = 3.$

$$\tan x + \left(\frac{\tan x + \sqrt{3}}{1-\sqrt{3}\tan x}\right) + \left(\frac{\tan x - \sqrt{3}}{1+(\sqrt{3})\tan x}\right)$$

$$= 3.$$

$\tan x = t$

$$\cancel{3}\left(\frac{3t - t^3}{1-3t^2}\right) = \cancel{3}.$$

$$\tan 3x = 1.$$

$$3x = n\pi + \frac{\pi}{4}$$

$$\boxed{x = \frac{n\pi}{3} + \frac{\pi}{12}}\ ; n \in I.$$