VECTOR

GENERAL DEFINITIONS :

I. (a) Vector quantities :

Any quantity, such as velocity, momentum, or force, that has both magnitude and direction and for which vector addition is defined and meaningful; is treated as vector quantities.

Note :

Quantities having magnitude and direction but not obeying the vector law of addition will not be treated as vectors.

(b) Scalar quantities :

A quantity, such as mass, length, time, density or energy, that has size or magnitude but does not involve the concept of direction is called scalar quantity.

Directed line segment :

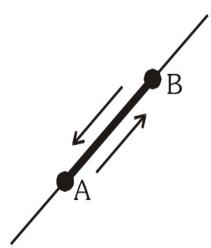
Any given portion of a given straight line where the two end points are distinguished as **Initial** and **Terminal** is called a **Directed Line Segment**.

The directed line segment with initial point A and terminal point B is denoted by the symbol $\overrightarrow{AB}$.
The two end points of a directed line segment are not interchangeable and the directed line segments.

$\overrightarrow{AB}$ and $\overrightarrow{BA}$ must be thought of as different.

(a) Vector :

A directed line segment is called vector. Every directed line segment have three essential characteristics.



(i) Length : The length of $\overrightarrow{AB}$ will be denoted by the symbol $|\overrightarrow{AB}|$

Clearly, we have $|\overrightarrow{AB}| = |\overrightarrow{BA}|$

III. Some Special Vectors :

(a) Zero Vector (Null vector) :

A vector of zero magnitude i.e. which has the same initial & terminal point, is called a **Zero Vector**. It is denoted by $\vec{O}$ and its direction is arbitrary.

Note that zero vector has many properties similar to the number zero.

E.g. A boy throwing a ball up and catching it back in his hand, the displacement of the ball is a null vector.

(b) Unit Vector :

A vector of unit magnitude in the direction of vector $\vec{a}$ is called unit vector along $\vec{a}$ and is denoted by $\hat{a}$

symbolically $\boxed{\hat{a} = \frac{\vec{a}}{|\vec{a}|}}$

The concept of unit vector is merely to impart a direction to a physical quantity.

(c) Equal Vectors :

Two vectors are said to be equal if they have the same magnitude, same direction & represent the same physical quantity with same or parallel line of support.

IV. Vectors can be classified as :

(d) Free vectors : Vectors which when transformed into space from one point to another point without affecting their magnitude and direction, can be considered as free vectors i.e. the physical effect produced by them remains unaltered. e.g. displacement, velocity. (Generally **IIT-JEE** deals with free vectors only)

(e) Localised vector : Vectors which when transformed into space from one point to another point without affecting their magnitude and direction, can be considered as localised vectors i.e. the physical effect produced by them are changed e.g. force in case of rotational motion (torque will be changed).

Note that :

(i) Number of distinct unit vectors in space perpendicular to a given plane is 2. (one upward and one downward).

(ii) Number of unit vectors in space parallel to a given plane is infinite.

(iii) Number of distinct unit vectors perpendicular to given line in space. (Infinitely many, think of the line as perpendicular to the xy plane. The unit vector might make any angle θ with the x-axis.)

(iv) Number of distinct unit vectors parallel to a line in space is 2.

(v) Two vectors are equal if they have equal components in an arbitrary direction

(f) Multiplication of vector by scalars :

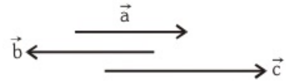
If $\vec{a}$ is a vector & m is a scalar, then $(m\vec{a})$ is a vector parallel to $\vec{a}$ whose modulus is $|m|$ times that of $\vec{a}$. This multiplication is called **Scalar Multiplication**. If $\vec{a}$ & $\vec{b}$ are vectors & m, n are scalars, then :

$$m(\vec{a}) = (\vec{a})m = m\vec{a} \qquad m(n\vec{a}) = n(m\vec{a}) = (mn)\vec{a}$$

$$(m+n)\vec{a} = m\vec{a} + n\vec{a} \qquad m(\vec{a}+\vec{b}) = m\vec{a} + m\vec{b}$$

(g) Collinear Vectors :

Two vectors are said to be collinear if their directed line segments are parallel disregard of their direction. Collinear vectors are also called **Parallel Vectors**. If they have the same direction they are named as like vectors otherwise unlike vectors. ($\vec{a}, \vec{b}, \vec{c}$ are collinear)



Note : Symbolically, two non zero vectors $\vec{a}$ and $\vec{b}$ are collinear if and only if, $\vec{a} = K\vec{b}$, where $K \in R$.

If $K > 0$, like parallel vectors,
$K < 0$, unlike parallel vectors.

Result : $\boxed{\text{If } \vec{a} \text{ and } \vec{b} \text{ are two non zero non collinear vectors then } x\vec{a} + y\vec{b} = 0 \Rightarrow x = 0 \text{ and } y = 0.}$

Proof : Let $y \neq 0$

$$y\vec{b} = -x\vec{a} \Rightarrow \vec{b} = \left(-\frac{x}{y}\right)\vec{a} \Rightarrow \vec{b} = \lambda\vec{a}$$

$\Rightarrow$ $\vec{a}$ and $\vec{b}$ are collinear, which is false.

$\therefore$ $y = 0$

Note : Two non-zero non collinear vectors are also called as **linearly independent vector or base vectors.** If $\vec{a}$ and $\vec{b}$ are collinear then they are known as **linearly dependent vectors**.

e.g. If $\left(\sin\theta - \frac{1}{2}\right)\vec{a} + \left(\cos\theta - \frac{\sqrt{3}}{2}\right)\vec{b} = 0$ such that $\vec{a}$ & $\vec{b}$ are non zero & non colinear vectors then the general solution of θ is [**Ans.** $2n\pi + \frac{\pi}{6}$ where $n \in I$]

(h) Plane : A plane is a surface such that any two points on the surface are joining by a segment, then all the point lying on the segment must also lie on the surface.

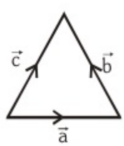
(i) Coplanar Vectors : A given number of vectors are called coplanar if their line of support are all parallel to the same plane. Note that **"Two Free Vectors Are Always Coplanar"**.

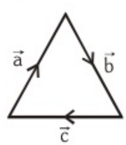
(j) Vector addition :

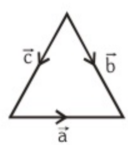
(i) Triangle law of vectors : If two vectors are represented in magnitude & direction by two sides of a triangle taken in same order then their sum is represented by the third side taken in reverse order.

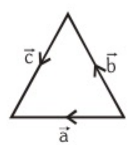
Example :

E(1) Find the relation between $\vec{a}, \vec{b}$ & $\vec{c}$

(i)


(ii)


(iii)


(iv)


[**Ans.** (i) $\vec{a} + \vec{b} = \vec{c}$ (ii) $\vec{a} + \vec{b} + \vec{c} = \vec{0}$ (iii) $\vec{a} + \vec{c} = \vec{b}$ (iv) $\vec{b} + \vec{c} = \vec{a}$]

(ii) Parallelogram law of vectors : If two vectors $\vec{a}$ & $\vec{b}$ represented by $\overrightarrow{OA}$ & $\overrightarrow{OB}$, then their sum $\vec{a}+\vec{b}$ is a vector represented by $\overrightarrow{OC}$, where $\overrightarrow{OC}$ is the diagonal of parallelogram OACB.

Properties :

$\vec{a}+\vec{b} = \vec{b}+\vec{a}$ (commutative) $\qquad$ $(\vec{a}+\vec{b})+\vec{c} = \vec{a}+(\vec{b}+\vec{c})$ (associativity)

$\vec{a}+\vec{0} = \vec{a} = \vec{0}+\vec{a}$ (additive identity) $\qquad$ $\vec{a}+(-\vec{a}) = \vec{0} = (-\vec{a})+\vec{a}$ (additive inverse)

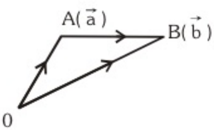
Note : Magnitude of $\vec{a}+\vec{b}$ is not equal to sum of magnitude of $\vec{a}$ & $\vec{b}$

i.e. $|\vec{\mathbf{a}}+\vec{\mathbf{b}}| \neq |\vec{\mathbf{a}}|+|\vec{\mathbf{b}}|$ (in general)

(k) Position Vector :

To specify the position of an object w.r.t. a fixed represence point in 3-D space, position vectors are used. Let O be a fixed origin, then the position vector of a point P is the vector $\overrightarrow{OP}$. Hence p.v. of general point P(x, y, z) is the vector extending from the origin to P given by $x\hat{i}+y\hat{j}+z\hat{k}$.

If $\vec{a}$ & $\vec{b}$ are the position vectors of two point A and B w.r.t. origin O then,



$\vec{a}+\overrightarrow{AB} = \vec{b}$

$\boxed{\overrightarrow{\mathbf{AB}} = \vec{\mathbf{b}}-\vec{\mathbf{a}} = \textbf{pv of B} - \textbf{pv of A.}}$

Note :

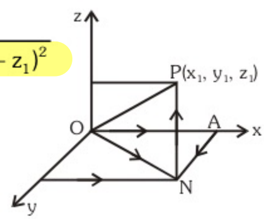
(i) Vector along a line ax + by + c = 0 is given by $\lambda(b\hat{i}-a\hat{j})$.

(ii) Equation of any line $\perp$ to ax + by + c = 0 will be of the form bx – ay = k and any vector parallel to this line will be of the form $\lambda(a\hat{i}+b\hat{j})$. So $\lambda(a\hat{i}+b\hat{j})$ will be a vector perpendicular to line ax + by + c = 0 in xy plane.

V. Representation of a vector in space in terms of 3 orthonormal triad of unit vectors

With every point $P(x_1, y_1, z_1)$ in space w.r.t. a fixed origin 'O' we can associate a directed line segment whose initial point is the origin and terminal point is P. Thus $\overrightarrow{OP} = \overrightarrow{OA}+\overrightarrow{AN}+\overrightarrow{NP} = x_1\hat{i}+y_1\hat{j}+z_1\hat{k}$.

Thus if $A(x_1, y_1, z_1)$ and $B(x_2, y_2, z_2)$ are two points in space



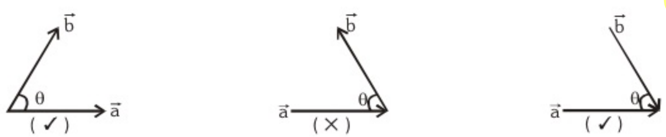
$\overrightarrow{AB} = (x_2-x_1)\hat{i}+(y_2-y_1)\hat{j}+(z_2-z_1)\hat{k}$ & $|\overrightarrow{AB}| = \sqrt{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2}$

Thus any vector $\vec{a}$ in space can be expressed as a linear combination of 3 orthonormal triad of unit vectors as $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$, where

$|\vec{a}|^2 = a_1^2+a_2^2+a_3^2$

ANGLE BETWEEN VECTORS :

Angle between two vectors is angle between head-head or tail-tail. There is only one set of angle between two vectors. angle between $\vec{a}$ & $\vec{b}$ is represented by $\vec{a} \wedge \vec{b} = \theta$. $\theta \in [0, \pi]$

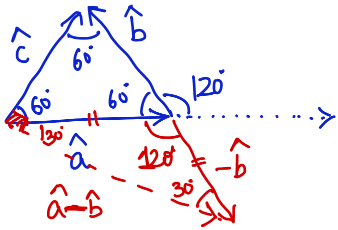


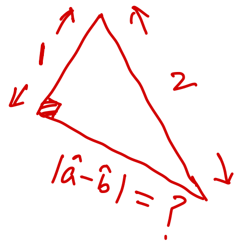
E(1) If the sum of two unit vectors is a unit vector then find the magnitude of their difference and the angle between $\hat{a}$ and $\hat{b}$.

120°

Sol^n

$|\hat{a}-\hat{b}| = ?$



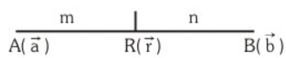


$\sqrt{3}$

SECTION FORMULA :

If $\vec{a}$ & $\vec{b}$ are the position vectors of two points A & B then the p.v. of a point R which divides AB internally in the ratio m : n is given by $\boxed{\vec{r} = \dfrac{n\vec{a}+m\vec{b}}{m+n}}$.

Proof : $\dfrac{AR}{RB} = \dfrac{m}{n} \quad \Rightarrow \quad n\overrightarrow{AR} = m\overrightarrow{RB}$



or $\quad n(\vec{r}-\vec{a}) = m(\vec{b}-\vec{r})$

$\therefore \quad \vec{r} = \dfrac{m\vec{b}+n\vec{a}}{m+n}$

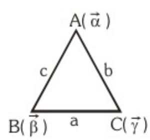
Similarly for external division $\boxed{\vec{r} = \dfrac{m\vec{b}-n\vec{a}}{m-n}}$

Note : p.v. of mid point of AB $= \dfrac{\vec{a}+\vec{b}}{2}$.

Using section formula, we can prove that

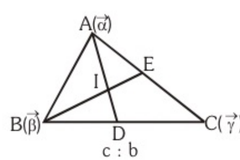
(a) p.v. of the centroid of a triangle ABC $= \dfrac{\vec{\alpha}+\vec{\beta}+\vec{\gamma}}{3}$

(Concurrency of median)



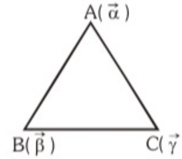
(b) Incentre of the triangle $= \dfrac{a\vec{\alpha}+b\vec{\beta}+c\vec{\gamma}}{a+b+c}$

$$\frac{AI}{ID} = \frac{AB}{BD} = \frac{c}{\left(\frac{c}{b+c}\right)a} = \frac{b+c}{a}$$



$\therefore \quad D\left(\dfrac{b\vec{\beta}+c\vec{\gamma}}{b+c}\right)$

Excentres of the Δ are $\dfrac{-a\vec{\alpha}+b\vec{\beta}+c\vec{\gamma}}{-a+b+c}$; $\dfrac{a\vec{\alpha}-b\vec{\beta}+c\vec{\gamma}}{a-b+c}$ and $\dfrac{a\vec{\alpha}+b\vec{\beta}-c\vec{\gamma}}{a+b-c}$



(c) Circumcentre of the $\Delta = \dfrac{\vec{\alpha}\sin 2A+\vec{\beta}\sin 2B+\vec{\gamma}\sin 2C}{\sum \sin 2A \text{ or } 4\left(\prod \sin A\right)}$

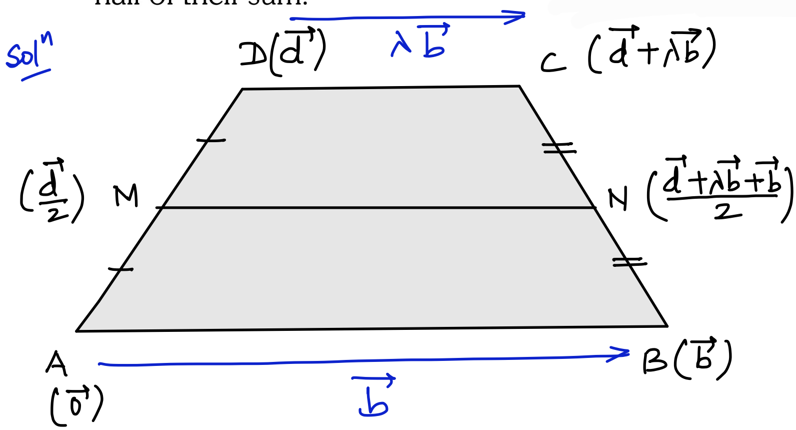
(d) Orthocentre of the $\Delta = \dfrac{\vec{\alpha}\tan A+\vec{\beta}\tan B+\vec{\gamma}\tan C}{\sum \tan A \text{ or } \left(\prod \tan A\right)}$

(Use the fact that distances of orthocentre from the vertices are 2R cosA, 2R cosB, 2R cosC and from the sides are 2R cosB cosC, 2R cosC cosA, 2R cosA cosB)

GEOMETRICAL RESULTS WITH VECTORS & PROBLEMS :

(a) Prove that straight line joining the mid points of two non parallel sides of a trapezium is || to the parallel sides and half of their sum.

Sol^n



$\overrightarrow{AB} = \vec{b}-\vec{0} = \vec{b}$

$\overrightarrow{AD} = \vec{d}$

$\overrightarrow{AB} \parallel \overrightarrow{DC}$

$\overrightarrow{DC} = \lambda\overrightarrow{AB}$

$\overrightarrow{DC} = \lambda\vec{b}$

pv. of C − pv of D $= \lambda\vec{b}$

pv of C = pv of D $+ \lambda\vec{b}$

$= \vec{d}+\lambda\vec{b}$

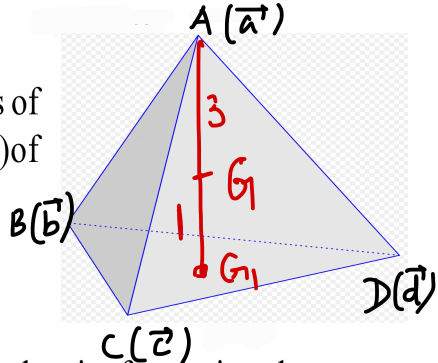
$\overrightarrow{MN}$ = pv of N − pv of M

$$\overrightarrow{MN} = \frac{\vec{d}+\lambda\vec{b}+\vec{b}}{2} - \frac{\vec{d}}{2} = \frac{(\lambda+1)}{2}\vec{b} \checkmark$$

$$\boxed{\overrightarrow{MN} = K\vec{b} = K\overrightarrow{AB}}$$ $K \in$ scalar

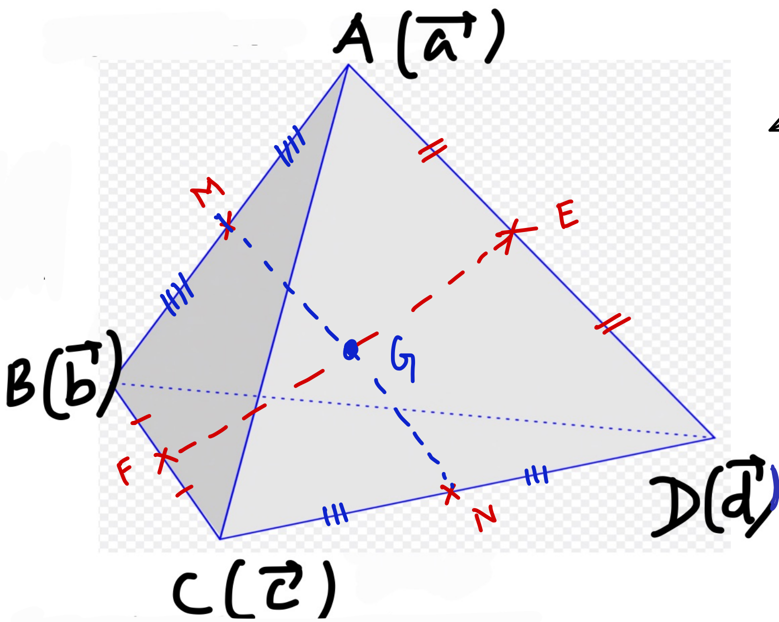
$$\overrightarrow{MN} = \frac{\lambda\vec{b}+\vec{b}}{2} = \frac{\overrightarrow{DC}+\overrightarrow{AB}}{2}$$

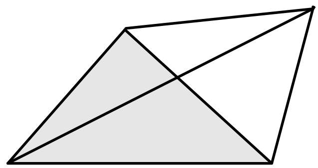
Tetrahedron (a pyramid on a triangular base)

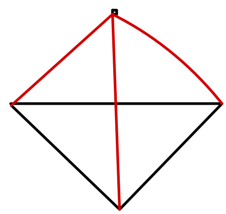


Lines joining the vertices of a tetrahedron to the centroids of the opposite faces are concurrent and this point (P)of concurrency with p.v. $\vec{g}=\dfrac{\vec{a}+\vec{b}+\vec{c}+\vec{d}}{4}$ is called the centre of the tetrahedron (say G).
(or centroid)

2 In a tetrahedron, straight lines joining the mid points of each pair of opposite edges are also concurrent at the centre of the tetrahedron.

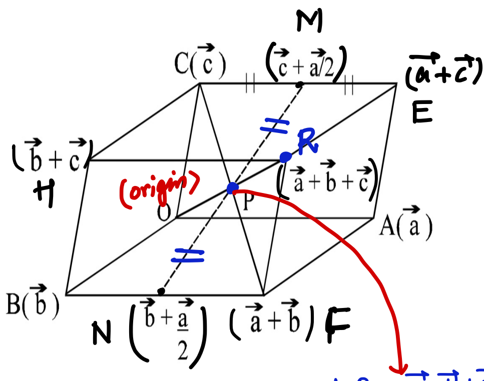






Paralleopiped :-

Four diagonals of any parallelopiped (A prism whose base is a ||gm) and the join of the mid point of each pair of opposite edges are concurrent and are bisected at the point of concurrence.
(See the adjacent figure) P is called the centre of the parallopiped with p.v.



p.v of $P = \dfrac{\vec{a}+\vec{b}+\vec{c}}{2}$

$$\frac{\vec{a}+\vec{b}+\vec{c}}{2} \text{ i.e. } \frac{\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}}{2}$$

For some general point (other than origin)

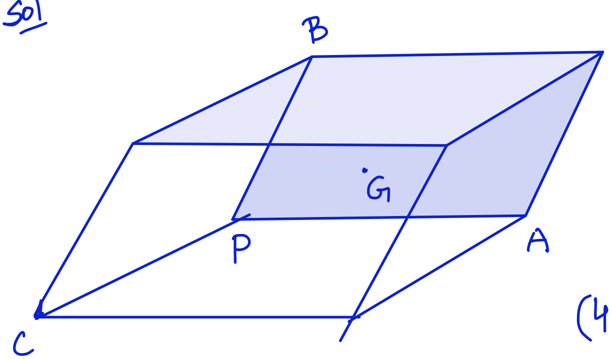
$$\left(\frac{\overrightarrow{EA}+\overrightarrow{EB}+\overrightarrow{EC}}{2}\right)$$

Q Centre of the parallelopiped formed by

$\overrightarrow{PA}=\hat{i}+2\hat{j}+2\hat{k}$; $\overrightarrow{PB}=4\hat{i}-3\hat{j}+\hat{k}$ and $\overrightarrow{PC}=3\hat{i}+5\hat{j}-\hat{k}$

is given by the p.v. (7, 6, 2). Then find the p.v. of the point P.

Sol



$$\frac{\overrightarrow{PA}+\overrightarrow{PB}+\overrightarrow{PC}}{2} = \overrightarrow{PG}$$

$$4\hat{i}+2\hat{j}+\hat{k} = \overrightarrow{PG}$$

$$\overrightarrow{PG} = \text{pv of } G - \text{pv of } P.$$

$$(4,2,1) = (7,6,2) - \text{pv of } P$$

$$\therefore \text{Pv. of } P = (3,4,1) \rightarrow 3\hat{i}+4\hat{j}+\hat{k}.$$

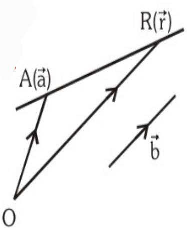
VECTOR EQUATION OF A LINE :

It is possible to express the position vectors of points on given lines and planes in terms of some fixed vectors and variable scalars called parameters, such that

(a) For arbitrary value of the parameter, the resulting position vector represent point on the locus in question, and

(b) Conversely, the position vectors of each of the locus correspond to a definite value(s) of the parameter.

Parametric vector equation of a line :



(i) If the line passes through the point $A(\vec{a})$ & is parallel to the vector $\vec{b}$ then its

Rem

equation is, $\boxed{\vec{r} = \vec{a} + t\vec{b}}$ where t is a parameter.

Rem

(ii) A line passing through two point $A(\vec{a})$ & $B(\vec{b})$ is given by, $\boxed{\vec{r} = \vec{a} + s(\vec{b} - \vec{a})}$,

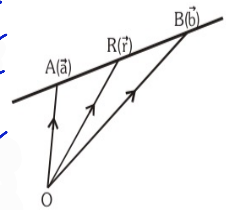
$\vec{r} = \vec{b} + \lambda(\vec{b} - \vec{a})$

$\vec{r} = \vec{b} + \lambda(\vec{a} - \vec{b})$

where s is parameter.

These two equation gives the position vector of any point on the line and prove to be very useful in vector algebra.

$\vec{r} = \vec{a} + \lambda(\vec{a} - \vec{b})$

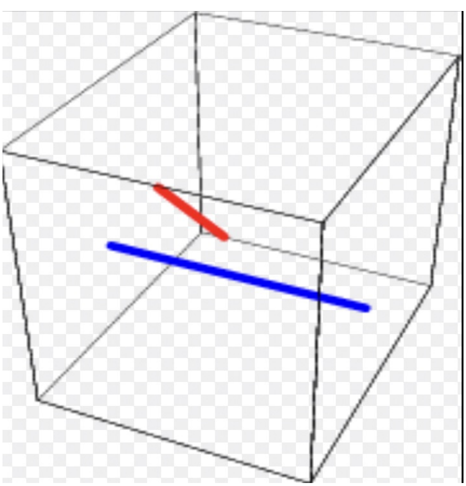


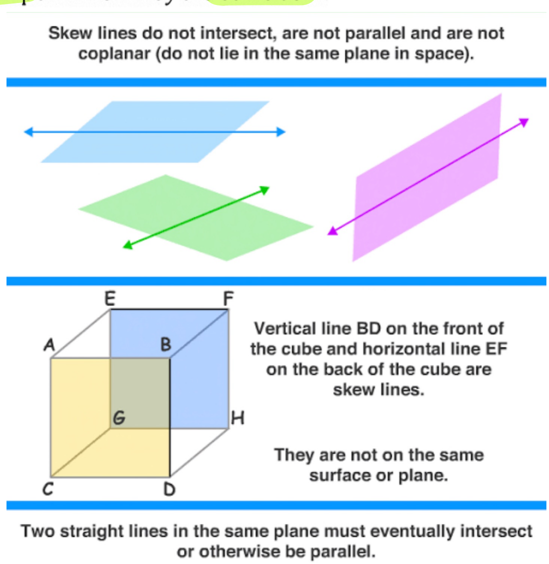
Important Note :

(i) Two lines in a plane are either intersecting or parallel, conversely two intersecting or parallel line must be in the same plane.

(ii) However in space we can have two neither parallel nor intersecting lines. Such non coplanar lines are known as skew lines.

(iii) If two lines are parallel and have a common point then they are coincident.





E(1) Find the p.v. of the point of intersection of the lines (if it exists)

(a) $\vec{r} = \hat{i} - \hat{j} - 10\hat{k} + \lambda(2\hat{i} - 3\hat{j} + 8\hat{k})$
and $\vec{r} = 4\hat{i} - 3\hat{j} - \hat{k} + \mu(\hat{i} - 4\hat{j} + 7\hat{k})$

$\frac{2}{1} \neq \frac{3}{4} \neq \frac{8}{7}$

HW (b) $\vec{r} = -3\hat{i} + 6\hat{j} + \lambda(-4\hat{i} + 3\hat{j} + 2\hat{k})$
and $\vec{r} = -2\hat{i} + 7\hat{k} + \mu(-4\hat{i} + \hat{j} + \hat{k})$ $\rightarrow$ skew lines

(c) $\vec{r} = t(3\hat{i} - \hat{j} + \hat{k})$
and $\vec{r} = 2\hat{i} + s(-6\hat{i} + 2\hat{j} - 2\hat{k})$ $\rightarrow$ $\frac{3}{-6} = \frac{-1}{2} = \frac{1}{-2}$

$\parallel$ lines.

HW (d) $\vec{r} = 2\hat{k} + \lambda(3\hat{i} + 2\hat{j} + \hat{k})$
and $\vec{r} = 3\hat{i} + 2\hat{j} + 3\hat{k} + \mu(6\hat{i} + 4\hat{j} + 2\hat{k})$ $\rightarrow$ Coincident lines

(a) General point of L_1 : $(1+2\lambda, -1-3\lambda, -10+8\lambda)$
" " " L_2 : $(4+\mu, -3-4\mu, -1+7\mu)$

L1 $(1,-1,-10)$ $\quad 2\hat{i} - 3\hat{j} + 8\hat{k}$

L2 $(4,-3,-1)$ $\quad \hat{i} - 4\hat{j} + 7\hat{k}$

$$\left.\begin{aligned} 1+2\lambda &= 4+\mu \\ -1-3\lambda &= -3-4\mu \end{aligned}\right\} \quad \lambda = 2 \checkmark, \ \mu = 1 \checkmark$$

$$-10+8\lambda = -1+7\mu.$$

satisfies

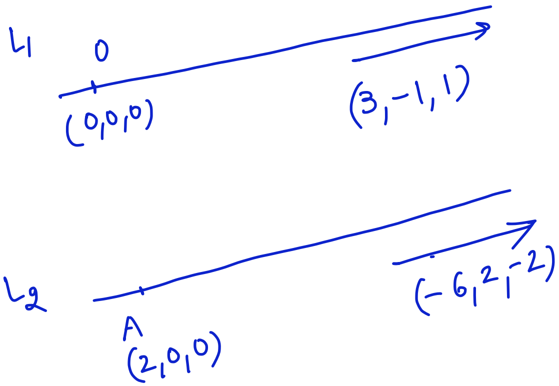
$\therefore$ lines are intersecting at $(5, -7, 6)$

(c) $\vec{r} = t(3\hat{i} - \hat{j} + \hat{k})$

and $\vec{r} = 2\hat{i} + s(-6\hat{i} + 2\hat{j} - 2\hat{k})$

$\rightarrow \dfrac{3}{-6} = \dfrac{-1}{2} = \dfrac{1}{-2}$

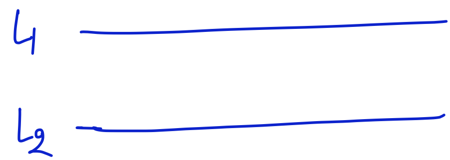
$\parallel$ lines.



$\overrightarrow{OA} = 2\hat{i}$

$\overrightarrow{OA}$ is not Collinear with $(3\hat{i} - \hat{j} + \hat{k})$

Hence they are distinct parallel lines. i.e. not Coincident



HW

JA Hyperbola