Differentiability- (L)

Complete Solutions Of DYS & Exercises

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MON TUE WED THU FRI SAT SUN

Subject

Do your self (1)

(1) A function is defined as follows:-

$$f(x) = \begin{cases} x^3 ; & x^2 < 1 \\ x ; & x^2 \geq 1 \end{cases}$$ discuss the continuity and differentiability at $x=1$

$$f(x) = \begin{cases} x ; & x \leq -1 \\ x^3 ; & -1 < x < 1 \\ x ; & x \geq 1 \end{cases}$$

Continuity at $x=1$

$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} x \Rightarrow 1$

$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} x^3 \Rightarrow (1)^3 \Rightarrow 1$

$\lim_{x\to 1} f(x) = \lim_{x\to 1} x \Rightarrow 1.$

Differentiability at $x=1$

$f'(1^-) = \lim_{h\to 0} \frac{(1-h)^3 - 1}{-h} \Rightarrow \lim_{h\to 0} \frac{\cancel{1} - h^3 - 3h + 3h^2 - \cancel{1}}{-h} \Rightarrow 3$

$f'(1^+) = \lim_{h\to 0} \frac{(1+h) - 1}{h} \Rightarrow \lim_{h\to 0} \frac{\cancel{h}}{\cancel{h}} \Rightarrow 1$

$\therefore$ $\boxed{\text{Continuous at } x=1 \text{ but not differentiable}}$

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(11) If $f(x) = \begin{cases} ax^3+b & ; \ 0 \le x \le 1 \\ 2\cos \pi x + \tan^{-1} x & ; \ 1 < x \le 2 \end{cases}$

be the differentiable function in $[0,2]$, then find a & b.

$\because \lim_{x\to 1^-} f'(x) = \lim_{x\to 1^+} f'(x)$

$\therefore \lim_{x\to 1^-} 3ax^2 = \lim_{x\to 1^+} -2\sin \pi x \cdot \pi + \frac{1}{1+x^2}$

$\sin \pi x \to 0$

$3a = \frac{1}{1+1} \Rightarrow 3a = \frac{1}{2}$

$\boxed{a = \frac{1}{6}}$

Now $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x)$

$\lim_{x\to 1^-} ax^3+b = \lim_{x\to 1^+} 2\cos \pi x + \tan^{-1} x$

$a+b = -2 + \frac{\pi}{4}$

$a = \frac{1}{6} \Rightarrow \frac{1}{6} + b = -2 + \frac{\pi}{4}$

$\boxed{b = \frac{\pi}{4} - \frac{13}{6}}$

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Do yourself (2)

(1) Let $f(x) = (x-1)\cdot|x-1|$.
Discuss the continuity and Differentiability of $f(x)$ at $x=1$.

$$\overset{-}{\quad}\;|\;\overset{+}{\quad}$$
$$1$$

$$f(x) = \begin{cases} x \geq 1 & (x-1)^2 \\ x < 1 & -(x-1)^2 \end{cases}$$

for $f(x)$ continuous

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^-} f(x) = f(1)$$

$$\lim_{x\to 1^+} (x-1)^2 = \lim_{x\to 1^-} -(x-1)^2 = 0$$

$$0 = 0 = 0 \qquad \therefore \text{ Continuous}$$

for $f(x)$ Differentiable at $x=1$

$$f'(1^+) = \lim_{h\to 0} \frac{(1+h-1)^2 - 0}{h} \Rightarrow \lim_{h\to 0} \frac{h^2}{h} \Rightarrow \lim_{h\to 0} h = 0$$

$$f'(1^-) = \lim_{h\to 0} \frac{-(1-h-1)^2 - 0}{-h} \Rightarrow \lim_{h\to 0} \frac{-h^2}{-h} \Rightarrow \lim_{h\to 0} h = 0$$

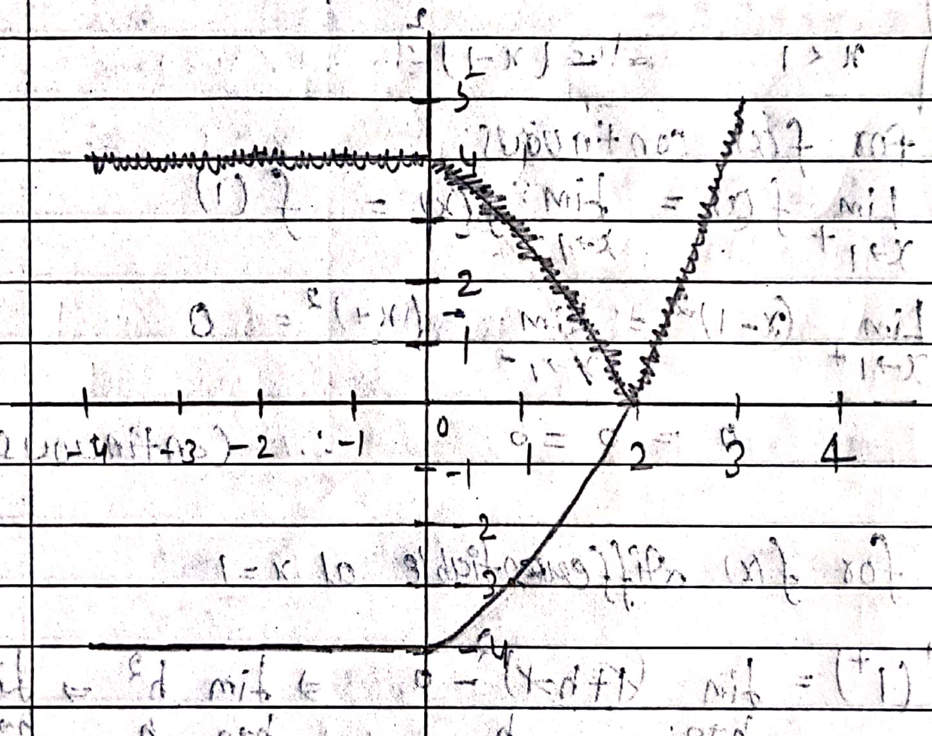
L.H.D. = R.H.D.

$\therefore$ Differentiable at $x=1$

Do your self ③

1. Let $f(x) = \begin{cases} -4, & -4 < x < 0 \\ x^2 - 4, & 0 \le x < 4 \end{cases}$

Discuss the continuity and differentiability of $g(x) = |f(x)|$



$\therefore$ $g(x)$ is continuous everywhere but not differentiable at $x = 2$.

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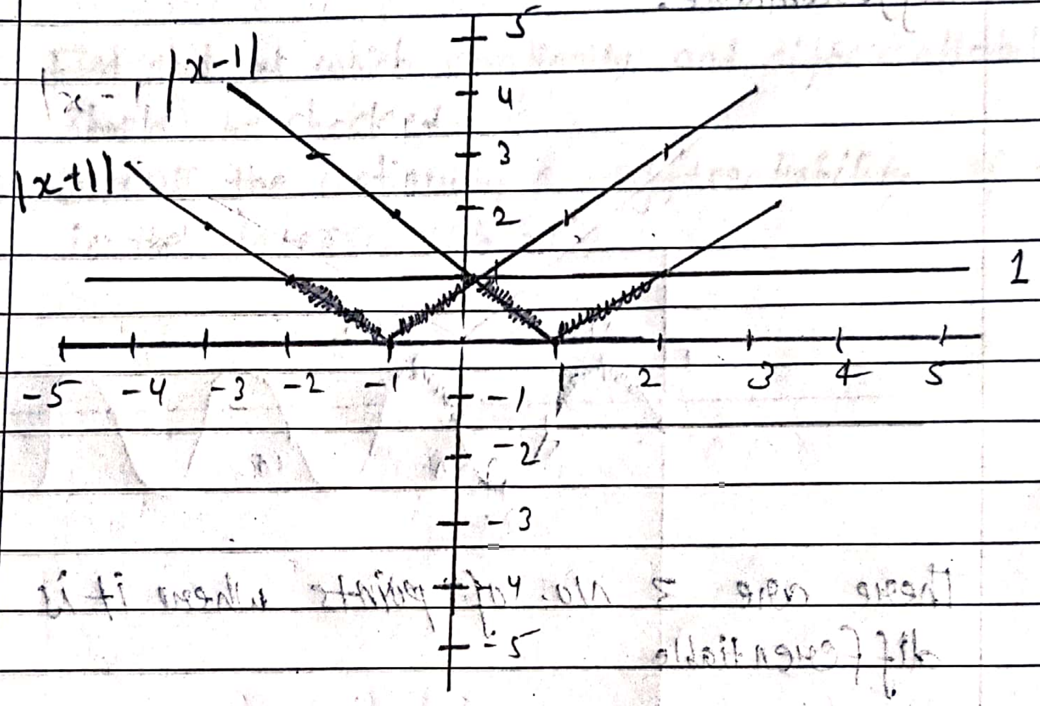
(ii) Let $f(x) = \min\{|x-1|, |x+1|, 1\}$. Find the number of points where it is not differentiable.



There are 5 points where it is not differentiable.

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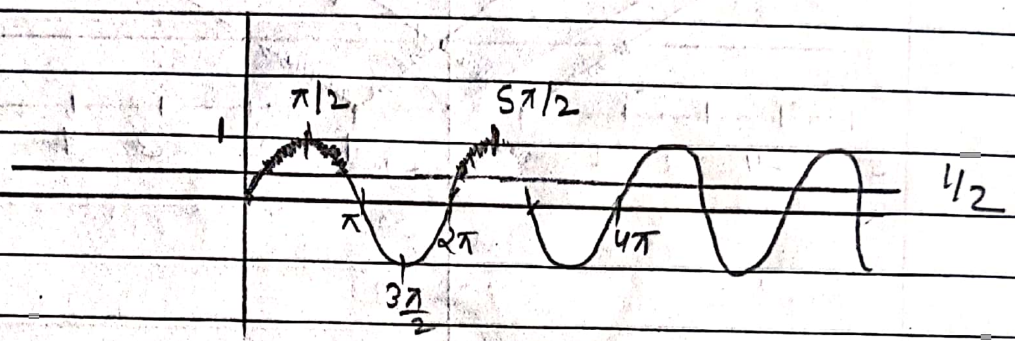
Do your self (4)

(i) Let $f(x) = \max\{\sin x, \frac{1}{2}\}$, where $0 \le x \le \frac{5\pi}{2}$. find the Number of points where it is not differentiable.



There are 3 No. of points where it is not differentiable

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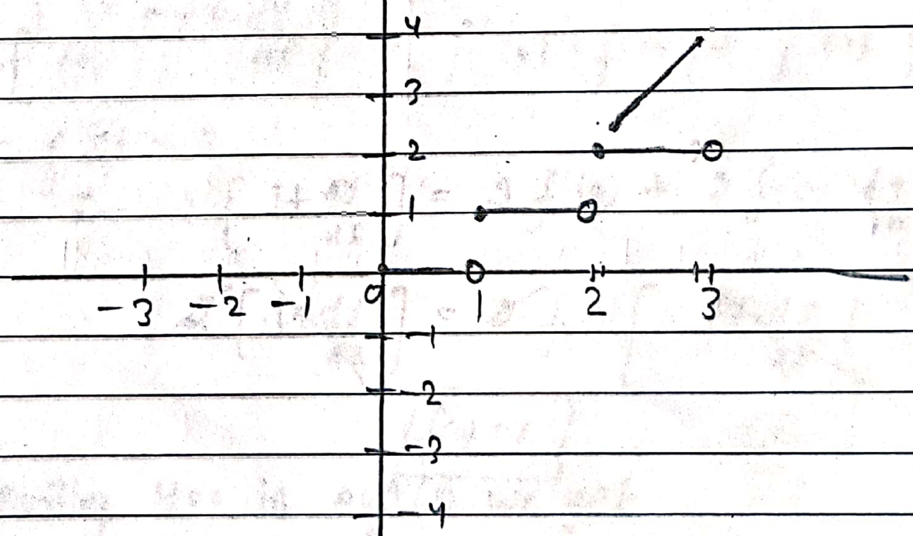
(ii) Let $f(x) = \begin{cases} [x] & ; 0 < x \leq 2 \\ 2x-2 & ; 2 < x < 3 \end{cases}$ where $[\cdot]$ denotes G.I.F.

(A) Find that at which continuity and differentiability should be checked.

(B) Discuss the continuity & differentiability of $f(x)$ in the interval $(0,3)$.



Accordingly to graph we should be checked continuity & Differentiability at 1 & 2.

$\because$ G.I.f. is Not continuous at integer

$\therefore$ Not continuous at $x = 1 \& 2$

also Not Differentiable at $x = 1 \& 2$.

Do yourself - 5 :

(i) A function $f : \mathbb{R} \to \mathbb{R}$ satisfies the equation $f(x + y) = f(x).f(y)$ for all $x, y \in \mathbb{R}$, $f(x) \neq 0$. Suppose that the function is differentiable everywhere and $f'(0) = 2$. Prove that $f'(x) = 2f(x)$.

Sol. $f(x+y) = f(x).f(y)$

put $x = 0,\ y = 0$

$f(0) = (f(0))^2 \Rightarrow f(0) = 0$ or $f(0) = 1$

(Given $f(x) \neq 0$)

So $f(0) = 1$

Now partial differtiate w.r.t x

$\left(\dfrac{\partial f}{\partial x},\ y = \text{constant} \right)$

$f'(x+y) = f'(x)\, f(y)$

put $x = 0$

$f'(y) = f'(0)\, f(y)$

or $f'(y) = 2 f(y)$ (Given $f'(0) = 2$)

or $\displaystyle\int \frac{f'(y)}{f(y)} = \int 2$

or $\ln f(y) = 2y + c$,

or $c = 0$, as $f(0) = 1$

$\Rightarrow \quad f(y) = e^{2y}$ or $f(x) = e^{2x}$

Now $f'(x) = e^{2x}(2)$

$= 2f(x)$

Hence proved

EXERCISE (O-1)

EXERCISE (O-1)

[SINGLE CORRECT CHOICE TYPE]

1. Let $f(x) = [\tan^2 x]$, (where [.] denotes greatest integer function). Then -

(A) $\lim_{x\to 0} f(x)$ does not exist

(B) $f(x)$ is continuous at x = 0.

(C) $f(x)$ is not differentiable at x = 0

(D) $f'(0) = 1$

Solⁿ:

$f(x) = [\tan^2 x]$

$f(x) = 0$ in $x \in (0-\delta, 0+\delta)$
where $\delta \to 0$

$\Rightarrow$ f(x) is continous as well

as differentiable at $x = 0$

2. Let f be differentiable at $x = 0$ and $f'(0) = 1$. Then $\lim_{h\to 0}\frac{f(h)-f(-2h)}{h} =$

(A) 3 (B) 2 (C) 1 (D) –1

Solⁿ:

$$\lim_{h\to 0}\frac{f(h)-f(-2h)}{h} = ?$$

$$= \lim_{h\to 0}\frac{f(h)-f(0) - f(-2h)+f(0)}{h}$$

$$= \lim_{h\to 0}\left(\frac{f(h)-f(0)}{h} - \frac{f(-2h)-f(0)}{h}\right)$$

$$= \lim_{h\to 0}\left(\frac{f(h)-f(0)}{h}\right) + 2\lim_{h\to 0}\left(\frac{f(-2h)-f(0)}{-2h}\right)$$

$$= f'(0) + 2f'(0) = 1+2(1) = 3$$

Ans

3. Let $g(x) = \begin{cases} 3x^2 - 4\sqrt{x} + 1 & \text{for } x < 1 \\ ax + b & \text{for } x \geq 1 \end{cases}$.

If g(x) is continuous and differentiable at x = 1 then -

(A) a = b = 4

(B) a = b = – 4

(C) a = 4 and b = – 4

(D) a = – 4 and b = 4

Sol^n: $\because$ g(x) is continuous at $x = 1$

$$g(1^-) = g(1^+) = g(1)$$

$$\Rightarrow 3 - 4 + 1 = a + b \Rightarrow \boxed{a + b = 0} \quad \text{—— (1)}$$

Now, g(x) is differentiable

$$g'(1^-) = g'(1^+)$$

$$\lim_{x \to 1^-} \left(6x - \frac{4}{2\sqrt{x}}\right) = \lim_{x \to 1^+} a$$

$$6 - 2 = a \Rightarrow \boxed{a = 4}$$

Put in (1)

$$b = -a = -4$$

$$\boxed{b = -4}$$

4. If $f(x)\ f(y) + 2 = f(x) + f(y) + f(xy)$ and $f(1) = 2$, $f'(1) = 2$ then sgn $f(x)$ is equal to (where sgn denotes signum function) -

(A) 0 (B) 1 (C) –1 (D) 4

sol: Replace x by $1/x$

$\Rightarrow f(x)\cdot f\left(\frac{1}{x}\right) + \cancel{2} = f(x) + f(1/x) + \cancel{f(1)}$

{as $f(1) = 2$}

$\Rightarrow \boxed{f(x)\cdot f(1/x) = f(x) + f(1/x)}$
①

$\therefore f(x) = 1 \pm x^n$

put $x = 1 \Rightarrow f(1) = 1 \pm (1)^n$

$2 = 1 \pm 1$

$\therefore$ take +ve sign for ②.

$\therefore f(x) = 1 + x^n$

$f'(x) = 0 + n(x)^{n-1}$

$f'(1) = n(1)^{n-1}$

$\boxed{2 = n}$

$\therefore \boxed{f(x) = 1 + x^2}$

$\therefore \text{sgn}(f(x)) = \text{sgn}(1+x^2) = \text{sgn}(+ve)$

$= 1$

5. The function $g(x)=\begin{cases} x+b, & x<0 \\ \cos x, & x\geq 0 \end{cases}$ can be made differentiable at x = 0 -

(A) if b is equal to zero
(B) if b is not equal to zero
(C) if b takes any real value
(D) for no value of b

sol:

For above to be differentiable

L.H.D = R.H.D. at $x=0$

$\Rightarrow 1 = -\sin x$

which does not hold at $x=0$

So function is not differentiable

at $x=0$

6. Which one of the following functions is continuous everywhere in its domain but has atleast one point where it is not differentiable ?

(A) $f(x) = x^{1/3}$ (B) $f(x) = \frac{|x|}{x}$ (C) $f(x) = e^{-x}$ (D) $f(x) = \tan x$

Soln:- $f(x) = (x)^{\frac{1}{3}}$ is not differentiable at $x = 0$ because of vertical tangent

All other given functions are differentiable in their domain

7. If the right hand derivative of $f(x) = [x] \tan \pi x$ at $x = 7$ is $k\pi$, then k is equal to ([y] denotes greatest integer $\leq$ y)

(A) 6 (B) 7 (C) –7 (D) 49

Soln: For R.H.D. at $x = 7$

$f(x) = 7 \tan \pi x$; as derivative exist

$$f'(7^+) = \lim_{h \to 0} \frac{f(7+h) - f(7)}{h}$$

$$= \lim_{h \to 0} \frac{7 \tan \pi h}{h} = 7\pi$$

$$\Rightarrow k = 7$$

8. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous onto function satisfying $f(x) + f(-x) = 0$, $\forall x \in \mathbb{R}$. If $f(-3) = 2$ and $f(5) = 4$ in $[-5,5]$, then the equation $f(x) = 0$ has -

(A) exactly three real roots

(B) exactly two real roots

(C) atleast five real roots

(D) atleast three real roots

Soln:- $f(x) + f(-x) = 0$ --- I

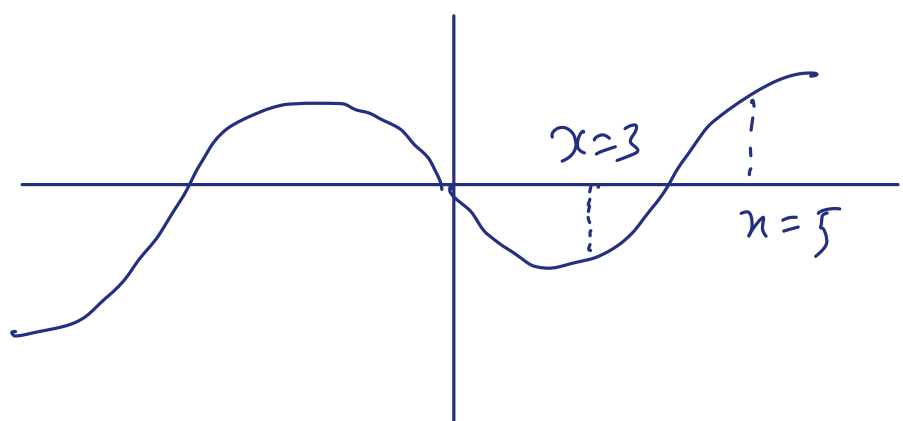
$\Rightarrow$ $f(x)$ is odd function

Put $x = 0$ in I^{st}

$\Rightarrow f(0) = 0$

Given $f(-3) = 2 \Rightarrow f(3) = -2$

$f(5) = 4 \Rightarrow f(-5) = -4$



$\Rightarrow$ Eqn has at least 3 real roots

9. If $2x + 3|y| = 4y$, then y as a function of x i.e. $y = f(x)$, is -

(A) discontinuous at one point

(B) non differentiable at one point

(C) discontinuous & non differentiable at same point

(D) continuous & differentiable everywhere

Soln:- (I) Let $y \geq 0$

$$2x + 3y = 4y$$

$$y = 2x \Rightarrow x \geq 0$$

(II) Let $y < 0$

$$2x - 3y = 4y$$

$$7y = 2x$$

$$y = \frac{2x}{7} \Rightarrow x < 0$$

$$y = \begin{cases} 2x & ; x \geq 0 \\ \frac{2x}{7} & ; x < 0 \end{cases}$$

$\Rightarrow$ Not Differentiable at $x = 0$

10. If $f(x) = (x^5 + 1)\,|x^2 - 4x - 5| + \sin|x| + \cos(|x - 1|)$, then $f(x)$ is not differentiable at -

(A) 2 points (B) 3 points (C) 4 points (D) zero points

Soln:-

$$f(x) = (x^5+1)\,|(x+1)(x-5)| + \sin|x| + \cos(x-1)$$

So $f(x)$ is not differentiable

at $x = 5$ & $x = 0$

11. Let $\mathbb{R}$ be the set of real numbers and $f : \mathbb{R} \to \mathbb{R}$, be a differentiable function such that $|f(x) - f(y)| \le |x - y|^3 \ \forall \ x,y \in \mathbb{R}$. If $f(10) = 100$, then the value of $f(20)$ is equal to -

(A) 0 (B) 10 (C) 20 (D) 100

Soln:- Replace x by $x+h$ & y by x where $h \to 0$

$$\lim_{h\to 0} |f(x+h) - f(x)| \le |h|^3$$

$$\lim_{h\to 0} \left| \frac{f(x+h) - f(x)}{h} \right| \le |h^2|$$

$$\Rightarrow |f'(x)| \le 0$$

$$\Rightarrow f'(x) = 0 \quad ; \quad f(x) = \text{constant}$$

$$\Rightarrow f(x) = 100 \quad (\because f(10) = 100)$$

So $f(20) = 100$

12. For what triplets of real numbers (a, b, c) with $a \neq 0$ the function

$$f(x) = \begin{cases} x & , \ x \le 1 \\ ax^2 + bx + c & , \ \text{otherwise} \end{cases}$$ is differentiable for all real x ?

(A) $\{(a, 1-2a, a) \mid a \in \mathbb{R}, a \neq 0\}$

(B) $\{(a, 1-2a, c) \mid a, c \in \mathbb{R}, a \neq 0\}$

(C) $\{(a, b, c) \mid a, b, c \in \mathbb{R}, a + b + c = 1\}$

(D) $\{(a, 1-2a, 0) \mid a \in \mathbb{R}, a \neq 0\}$

Sol^n: $\because$ $f(x)$ is Continuous $\forall x \in R$

at $x=1$

$$f(1^-) = f(1) = f(1^+)$$

$$\Rightarrow \lim_{x \to 1^-} (x) = 1 = \lim_{x \to 1^+} (ax^2 + bx + c)$$

$$\Rightarrow 1 = 1 = a + b + c$$

$$\Rightarrow \boxed{a + b + c = 1} \text{ ——— } (1)$$

Now $f(x)$ is differentiable $\forall x \in R$.

$$f'(x) = \begin{cases} 1 & ; x < 1 \\ 2ax + b & ; x > 1 \end{cases}$$

$\therefore$ at $x = 1 \rightarrow$ LHD = RHD

$$f'(1^-) = f'(1^+) \Rightarrow 1 = 2a(1) + b$$

$$\Rightarrow \boxed{2a + b = 1} \longrightarrow \text{Put in eq } (1)$$

we get $\Rightarrow$ $a + (1 - 2a) + c = 1 \Rightarrow \boxed{c = a}$

$\therefore$ $(a, b, c) \equiv (a, 1-2a, a)$; $a \in R$, $a \neq 0$

13. Number of points of non-differentiability of the function

$g(x) = [x^2]\{\cos^2 4x\} + \{x^2\}[\cos^2 4x] + x^2 \sin^2 4x + [x^2][\cos^2 4x] + \{x^2\}\{\cos^2 4x\}$ in $(-50, 50)$ where $[x]$ and $\{x\}$ denotes the greatest integer function and fractional part function of x respectively, is equal to :-

(A) 98 (B) 99 (C) 100 (D) 0

Soln :

$$g(x) = \{\cos^2 4x\}\left([x^2] + \{x^2\}\right) + [\cos^2 4x]\left([x^2] + \{x^2\}\right) + x^2 \sin^2 4x$$

(Using $x = [x] + \{x\}$)

$$\Rightarrow g(x) = x^2\{\cos^2 4x\} + x^2[\cos^2 4x] + x^2 \sin^2 4x$$

$$\Rightarrow g(x) = x^2\left(\left(\underbrace{\{\cos^2 4x\} + [\cos^2 4x]}\right) + \sin^2 4x\right)$$

$$\Rightarrow g(x) = x^2\left(\cos^2 4x + \sin^2 4x\right)$$

$$\Rightarrow g(x) = x^2$$

g(x) is always differentiable

14. Let $f(x) = [n + p \sin x]$, $x \in (0, \pi)$, $n \in \mathbb{Z}$ and p is a prime number. The number of points where f(x) is not differentiable is :-

(A) $p - 1$ (B) $p + 1$ (C) $2p + 1$ (D) $2p - 1$

Here [x] denotes greatest integer function.

Soln:- $f(x) = [n + p\sin x]$

$$f(x) = n + [p \sin x]$$

Above function is not differentiable if

Sin x takes values $\Rightarrow \underbrace{\frac{1}{p}, \frac{2}{p}, \frac{3}{p}, \ldots \frac{p-1}{p}}_{\text{occur twice in } (0,\pi)}, \frac{p}{p}$ at $x = \frac{\pi}{2}$

So total values $= 2(p-1)+1 = 2p-1$

15. The function $f(x) = (x^2 - 1)\,|\,x^2 - 3x + 2\,| + \cos(|x|)$ is NOT differentiable at :

(A) –1 (B) 0 (C) 1 (D) 2

Soln:- $f(x) = (x^2-1)\,|(x-1)(x-2)| + \cos x$

Clearly not differentiable at $x=2$

EXERCISE (O-2)

1. The function $f(x)=\begin{cases}|x-3|, x\geq 1\\ \left(\frac{x^2}{4}\right)-\left(\frac{3x}{2}\right)+\left(\frac{13}{4}\right), x<1\end{cases}$ is -

(A) continuous at x = 1

(B) differentiable at x = 1

(C) continuous at x = 3

(D) differentiable at x = 3

Solⁿ :

$$f(x)=\begin{cases}\frac{x^2}{4}-\frac{3x}{2}+\frac{13}{4} & ; \quad x<1\\ 3-x & ; \quad 1\leq x\leq 3\\ x-3 & ; \quad x\geq 3\end{cases}$$

check continuity $f(1)=f(1^+)=f(1^-)=2$

$f(3)=f(3^+)=f(3^-)=0$, Cont at $x=1,3$

$$f'(x)=\begin{cases}\frac{x}{2}-\frac{3}{2} & ; \quad x<1\\ -1 & ; \quad 1<x<3\\ 1 & ; \quad x>3\end{cases}$$

$f'(1^+)=-1$ = RHD

$f'(1^-)=-1$ = LHD

$f'(3^+)=1$ (RHD)

$f'(3^-)=-1$ (LHD)

Differentiable at $x=1$

Non-diff at $x=3$

(ABC)

2. Select the correct statements -

(A) The function f defined by $f(x)=\begin{bmatrix} 2x^2+3 & \text{for} & x\le 1 \\ 3x+2 & \text{for} & x>1 \end{bmatrix}$ is neither differentiable nor continuous at x = 1.

(B) The function $f(x) = x^2|x|$ is twice differentiable at x = 0.

(C) If f is continuous at x = 5 and $f(5) = 2$ then $\lim_{x\to 2} f(4x^2-11)$ exists

(D) If $\lim_{x\to a}(f(x)+g(x))=2$ and $\lim_{x\to a}(f(x)-g(x))=1$ then $\lim_{x\to a} f(x).g(x)$ need not exist.

Sol^n:

(A) $f'(x) = \begin{cases} 4x & ; x \le 1 \\ 3 & ; x>1 \end{cases}$

$f'(1^+)=3$

$f'(1^-)=4$

$\downarrow$

ND at $x=1$

but it is continuous at $x=1$

(B) $f(x) = \begin{cases} x^3 & ; x\ge 0 \\ -x^3 & ; x\le 0 \end{cases}$

$f'(x) = \begin{cases} 3x^2 & ; x\ge 0 \\ -3x^2 & ; x\le 0 \end{cases}$ $\quad f'(0^+)=0$, $f'(0^-)=0$

$f''(x) = \begin{cases} 6x & ; x\ge 0 \\ -6x & ; x\le 0 \end{cases}$ $\quad f''(0^+)=0$, $f''(0^-)=0$

Twice differentiable at $x=0$

(C) $\lim_{x\to 2} f(4x^2-11) = f(5) = 2$

$4x^2-11$ → This will become either 5^+ or 5^-

$f(5) = f(5^+) = f(5^-)$

(BC)

(i)

$$\lim_{x\to a} f(x) + \lim_{x\to a} g(x) = 2$$

$$\lim_{x\to a} f(x) - \lim_{x\to a} g(x) = 1$$

$$\lim_{x\to a} f(x) = \frac{3}{2}, \quad \lim_{x\to a} g(x) = 1/2$$

$$\lim_{x\to a} f(x)\cdot g(x) = \frac{3}{4}$$

(In this case limit exist)

3. If $f(x) = \text{sgn}(x^5)$, then which of the following is/are **false** (where sgn denotes signum function)

(A) $f'_+(0) = 1$

(B) $f'_-(0) = -1$

(C) f is continuous but not differentiable at $x = 0$

(D) f is discontinuous at $x = 0$

$$f(x) = \begin{cases} 1 & ; \ x > 0 \\ 0 & ; \ x = 0 \\ -1 & ; \ x < 0 \end{cases}, \quad \text{DC at } x = 0$$

$$f'_+(0) = \lim_{h \to 0^+} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0^+} \frac{1-0}{h}$$

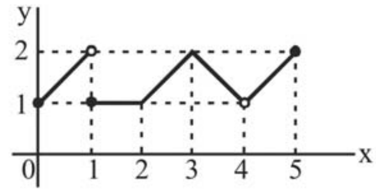
$f'_+(0) = \infty$

$\therefore$ $f(x)$ is Non-diff at $x = 0$

$f'_-(0) = \infty$

(ABC)

4. Graph of $f(x)$ is shown in adjacent figure, then



(A) $f(x)$ has non removable discontinuity at two points

(B) $f(x)$ is non differentiable at three points in its domain

(C) $\lim\limits_{x \to 1} f(f(x)) = 1$

(D) Number of points of discontinuity = number of points of non-differentiability

$\underline{\underline{S}}$ $f(x)$ is non- differentiable at 3 points

$(x = 1, 2, 3)$

$f(x)$ is discont. at $x = 1$

Now

$RHL = f(f(1^+)) = f(1) = 1$

$LHL = f(f(1^-)) = f(2^-) = 1$

$\Big] \lim\limits_{x \to 1} f(f(x)) = 1$

Non - removable discontinuity at $x = 1$ $\begin{pmatrix} RHL = 1 \\ LHL = 2 \end{pmatrix}$

(BC)

5. Let S denotes the set of all points where $\sqrt[5]{x^2|x|^3} - \sqrt[3]{x^2|x|} - 1$ is not differentiable then S is a subset of -

(A) $\{0,1\}$ (B) $\{0,1,-1\}$ (C) $\{0,1\}$ (D) $\{0\}$

⑥ $f(x) = \sqrt[5]{|x|^5} - \sqrt[3]{|x|^3} - 1 = |x| - |x| - 1 = -1$

$f(x)$ is always differentiable $\forall\ x \in R$

(ABCD)

6. Which of the following statements is/are correct ?

(A) There exist a function $f:[0,1]\to\mathbb{R}$ which is discontinuous at every point in [0,1] & $|f(x)|$ is continuous at every point in [0,1]

(B) Let $F(x)=f(x).\,g(x)$. If $f(x)$ is differentiable at $x=a$, $f(a)=0$ and $g(x)$ is continuous at $x=a$ then $F(x)$ is always differentiable at $x=a$.

(C) If $Rf'(a)=2$ & $Lf'(a)=3$, then $f(x)$ is non-differentiable at $x=a$ but will be always continuous at $x=a$

(D) If $f(a)$ and $f(b)$ possess opposite signs then there must exist at least one solution of the equation $f(x)=0$ in (a,b) provided f is continuous on [a,b]

7) A $f(x)=\begin{cases}1 & ; x\in Q\\ -1 & ; x\notin Q\end{cases}$

$|f(x)|=\begin{cases}1 & x\in Q\\ 1 & x\notin Q\end{cases}$ True

B $F(x)=f(x)\cdot g(x)$

$f'(a^+)=f'(a^-)=d$, $f(a)=0$

$g(a^+)=g(a^-)=g(a)=d_1$

$F'(x)=f'(x)\,g(x)+f(x)\,g'(x)$

$F'(a^+)=d\,d_1=F'(a^-)$ Hence $F(x)$ is differentiable at $x=a$

C If $f'_+(a)=2$ and $f'_-(a)=3$

If RHD and LHD are finite then function will be definately cont at $x=a$

D Statement of IMVT (Hence True) (ABCD)

7. Let $f : \mathbb{R} \to \mathbb{R}$ be a function. Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = |f(x)|$ for all x. Then which of the following is/are not always true-

(A) If f is continuous then g is also continuous (B) If f is one-one then g is also one-one

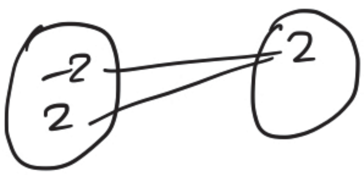
(C) If f is onto then g is also onto (D) If f is differentiable then g is also differentiable

S $\quad g(x) = |f(x)| \qquad g : R \to R$

(C) $g(x)$ cannot take negative values so into

$f(x) = x, \quad g(x) = |x|$ (assume)

(B) $f(x) = x, \quad g(x) = |x|$ is many-one



(A) If $f(x)$ is cont at $x = a$

$f(a^+) = f(a^-) = f(a) = d$

$g(x) = |f(x)|$

$g(a^+) = g(a^-) = g(a) = |d|$

It is also continuous

(D) $f(x) = x, \quad g(x) = |x|$ is not differentiable (BCD)

8.

[MATCH THE COLUMN]

Column - I	Column - II
(A) If f(x) is derivable at $x = 3$ & $f'(3) = 2$, then $\underset{h\to 0}{\text{Limit}}\dfrac{f(3+h^2)-f(3-h^2)}{2h^2}$ equals	(P) 0
(B) Let f(x) be a function satisfying the condition $f(-x) = f(x)$ for all real x. If $f'(0)$ exists, then its value is equal to	(Q) 1
(C) For the function $f(x) = \begin{cases} \dfrac{x}{1+e^{1/x}}, x \neq 0 \\ 0,\ x = 0 \end{cases}$ the derivative from the left $Lf'(0)$ equals	(R) 2
(D) The number of points at which the function $f(x) = \max.\ \{a - x, a + x, b\}$, $-\infty < x < \infty$, $0 < a < b$ cannot be differentiable is	(S) 3

0 < a < b cannot be differentiable is

(A) $$\lim_{h\to 0}\left[\frac{f(3+h^2)-f(3-h^2)}{(3+h^2)-(3-h^2)}\right] = f'(3) = 2$$

(B) $$f(x) = f(-x)$$

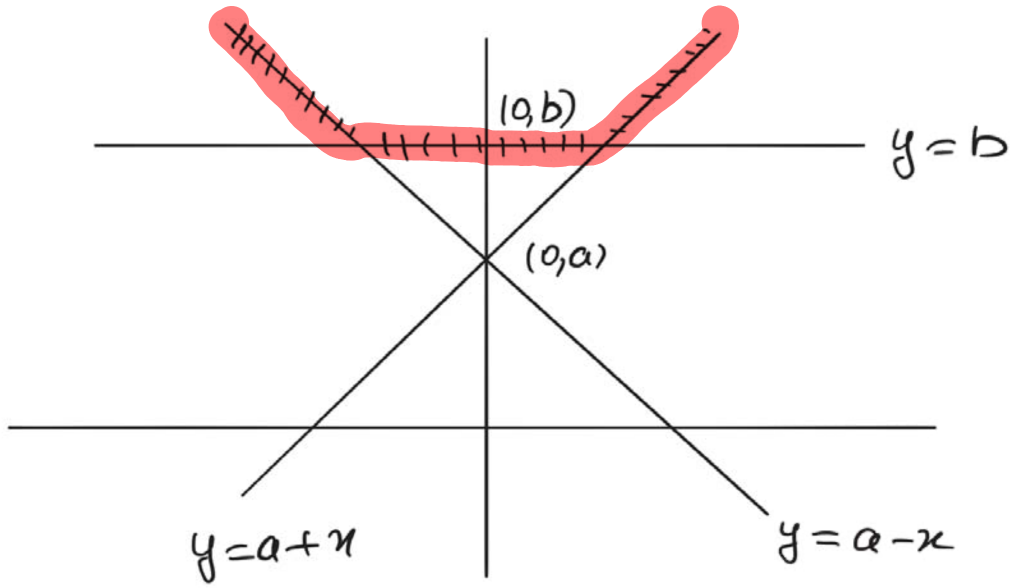
$$f'(x) = -f'(-x) \qquad \text{put } x = 0$$

$$f'(0) = -f'(0) \Rightarrow f'(0) = 0$$

(C) $$f'_-(0) = \lim_{h\to 0^+}\frac{f(0-h)-f(0)}{-h}$$

$$f'_-(0) = \lim_{h\to 0^+}\left(\frac{-h}{1+e^{-1/h}}\right)\left(-\frac{1}{h}\right) = \lim_{h\to 0^+}\frac{e^{1/h}}{e^{1/h}+1}$$

$$= 1$$



clearly f(x) is ND at two points

EXERCISE (S-1)

1. Discuss the continuity & differentiability of the function $f(x) = \sin x + \sin|x|, x \in \mathbb{R}$. Draw a rough sketch of the graph of f(x).

Sol^n. $\because$ $x=0$ is the only point of doubt as differentiability is concerned.

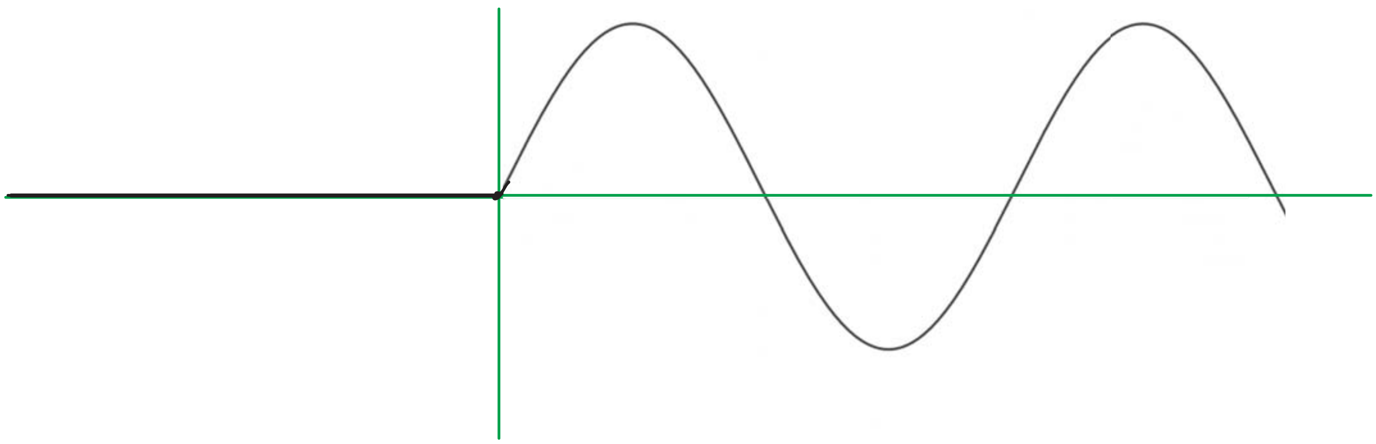
$\therefore$ LHD (at $x=0$) $= \lim\limits_{h\to 0} \dfrac{f(0-h)-f(0)}{-h} = \lim\limits_{h\to 0} \dfrac{-\sin h+\sin h}{-h} = 0.$

RHD (at $x=0$) $= \lim\limits_{h\to 0} \dfrac{f(0+h)-f(0)}{h} = \lim\limits_{h\to 0} \dfrac{2\sin h}{h} = 2.$

$\because$ LHD and RHD are both finite $\Rightarrow$ f(x) is continous at $x=0$.

Also $\because$ LHD $\neq$ RHD $\Rightarrow$ f(x) is not differentiable at $x=0$.

$$f(x) = \sin x+\sin|x| = \begin{cases} 0 & x<0. \\ 2\sin x. & x\geq 0. \end{cases}$$



Clearly from the graph f(x) is continous at $x=0$.
non differentiable at $x=0$.

2. Examine the continuity and differentiability of $f(x) = |x| + |x-1| + |x-2|\ x \in \mathbb{R}$. Also draw the graph of $f(x)$.

Sol^n.

$$f(x) = \begin{cases} -3x+3. & x<0. \\ -x+3. & 0 \le x < 1 \\ x+1 & 1 \le x < 2. \\ 3x-3. & 2 \le x. \end{cases}$$

$f'_-(0) = -3$; $f'_+(0) = -1$

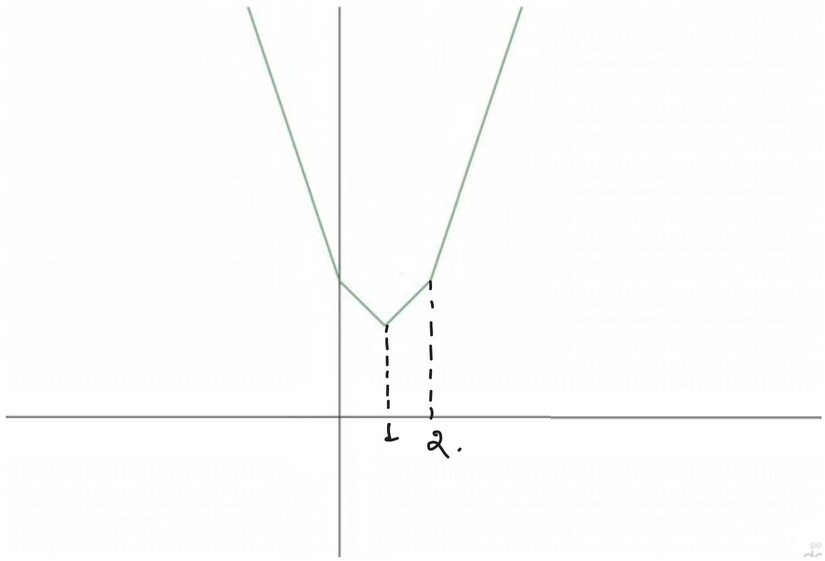
$\therefore$ continous but not differentiable at $x=0$

$f'_-(1) = -1.$; $f'_+(1) = 1$

$\therefore$ continous but not differentiable at $x=1.$

$f'_-(2) = 1$; $f'_+(2) = 3.$

$\therefore$ continous but not differentiable at $x=2.$



3. A function f is defined as follows : $f(x) = \begin{cases} 1 & \text{for } -\infty < x < 0 \\ 1+|\sin x| & \text{for } 0 \le x < \frac{\pi}{2} \\ 2+\left(x-\frac{\pi}{2}\right)^2 & \text{for } \frac{\pi}{2} \le x < +\infty \end{cases}$

Discuss the continuity & differentiability at x = 0 & x = π/2.

Sol^n.

$f'_-(0) = 0.$

$f'_+(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{1+\sin h - 1}{h} = 1.$

$\therefore$ continous but not differentiable at $x = 0$.

$f'_-(\pi/2) = \lim_{h \to 0} \frac{f(\pi/2 - h) - f(\pi/2)}{-h} = \lim_{h \to 0} \frac{1+\cos h - 2}{-h} = 0.$

$f'_+(\pi/2) = \lim_{h \to 0} \frac{f(\pi/2 + h) - f(\pi/2)}{h} = \lim_{h \to 0} \frac{2+h^2 - 2}{h} = 0.$

$\Rightarrow$ continous and differentiable at $x = \pi/2$.

4. Let f(0) = 0 and f'(0) = 1. For a positive integer k, show that

$$\lim_{x\to 0}\frac{1}{x}\left(f(x)+f\left(\frac{x}{2}\right)+........+f\left(\frac{x}{k}\right)\right)=1+\frac{1}{2}+\frac{1}{3}+......+\frac{1}{k}$$

Sol^n.

$$\lim_{x\to 0}\left(\frac{f(x)-f(0)}{x}+\frac{f(x/2)-f(0)}{2\times\left(\frac{x}{2}\right)}+---+\frac{f(x/k)-f(0)}{k\cdot(x/k)}\right)$$

$$\left\{\because \lim_{x\to a}\frac{f(x)-f(a)}{x-a}=f'(a)\right\}$$

$$= f'(0)+\frac{1}{2}f'(0)+---+\frac{1}{k}f'(0)$$

$$= 1+\frac{1}{2}+----+\frac{1}{k}$$

5. If $f(x) = \begin{cases} ax^2 - b & \text{if } |x| < 1 \\ -\dfrac{1}{|x|} & \text{if } |x| \geq 1 \end{cases}$ is derivable at x = 1. Find the values of a & b.

Sol^n.

$$f'_-(1) = \lim_{h \to 0} \frac{f(1-h) - f(1)}{-h}$$

$$= \lim_{h \to 0} \frac{a(1-h)^2 - b + 1}{-h}$$

$$= \lim_{h \to 0} \frac{(a-b+1) - 2ah + ah^2}{-h}$$

for limit to exist $a - b + 1 = 0$ — (i)

Also $f'_-(1) = 2a$.

$$f'_+(1) = \lim_{h \to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h \to 0} \frac{-\frac{1}{1+h} + 1}{h} = 1.$$

$\therefore$ for differentiability at $x = 1$.

$2a = 1 \Rightarrow a = 1/2$ — (ii)

from (i) and (ii).

$a = \frac{1}{2}$ $\quad b = \frac{3}{2}$

6. Let $g(x) = \begin{cases} a\sqrt{x+2}, & 0 < x < 2 \\ bx+2, & 2 \le x < 5 \end{cases}$. If g(x) is derivable on (0, 5), then find (2a + b).

Sol^n.

$$f'_-(2) = \lim_{h \to 0} \frac{f(2-h) - f(2)}{-h}$$

$$= \lim_{h \to 0} \frac{a\sqrt{4-h} - (2b+2)}{-h}$$

$$= \lim_{h \to 0} \frac{2a\left(1-\frac{h}{4}\right)^{1/2} - (2b+2)}{-h}$$

using binomial approximation.

$$= \lim_{h \to 0} \frac{2a\left(1-\frac{h}{8}\right) - (2b+2)}{-h}$$

$$= \lim_{h \to 0} \frac{2(a-b-1) - \frac{ah}{4}}{-h}$$

for limit to exist. $a - b - 1 = 0$ — (I).

Also. $f'_-(2) = \frac{a}{4}$.

$f'_+(2) = b$.

for differentiability at $x = 2$.

$$f'_-(2) = f'_+(2) \Rightarrow \frac{a}{4} = b \quad \text{— (II)}.$$

from (I) and (II). $b = \frac{1}{3}$ and $a = \frac{4}{3}$.

$\therefore \quad 2a + b = 3.$

EXERCISE (S-2)

1. Discuss the continuity & the derivability in [0, 2] of $f(x) = \begin{cases} |2x-3|[x] & \text{for } x \geq 1 \\ \sin\frac{\pi x}{2} & \text{for } x < 1 \end{cases}$

where [.] denotes the greatest integer function

$$f(x) = \begin{cases} |2x-3|[x] & \forall\, x \geq 1. \\ \sin\frac{\pi x}{2}, & \forall\, x < 1. \end{cases}$$

at $x = 1$

$$L.H.L. = \lim_{x \to 1^-} \sin\left(\frac{\pi x}{2}\right) = 1$$

$$R.H.L. = \lim_{x \to 1^+} |2x-3|[x] = 1.$$

$\Rightarrow$ Continuous at $x = 1$.

at $x = 3/2$

$L.H.L. = 0 = R.H.L. \Rightarrow$ Continuous.

at $x = 2$

$$L.H.L. = \lim_{x \to 2^-} |2x-3|[x] = 1.$$

$$R.H.L. = \lim_{x \to 2^+} |2x-3|[x] = 1 \cdot 2 = 2.$$

discontinuous at $x = 2$.

For differentiability:

at $x = 1$

$L.H.D. = \left(\cos\frac{\pi x}{2} \cdot \frac{\pi}{2}\right)_{x=1} = 0$

$R.H.D = \left(\frac{d}{dx}(3-2x)\right)_{x=1} = -2.$

Not differentiable at $x = 1$.

at $x = \frac{3}{2}$.

$$f(x) = \begin{cases} 2x-3, & x \in \left[\frac{3}{2}, 2\right) \\ 3-2x & x \in \left(1, \frac{3}{2}\right) \end{cases}$$

$L.H.D. = -2$ & $R.H.D = 2$.

So, non differentiable at $x = \frac{3}{2}$.

at $x = 2$. $f(x)$ is non differentiable

Finally. $f(x)$ is continious at $x = 1, \frac{3}{2}$
discontinuous at $x = 2$.
$f(x)$ is non diff. at $x = 1, \frac{3}{2}$ and 2.

2. Examine the function, $f(x) = x \cdot \frac{a^{1/x} - a^{-1/x}}{a^{1/x} + a^{-1/x}}$, $x \neq 0$ $(a > 0)$ and $f(0) = 0$ for continuity and existence of the derivative at the origin.

$$f(x) = x\left(\frac{a^{1/x} - a^{-1/x}}{a^{1/x} + a^{-1/x}}\right), \quad x \neq 0$$

I) $a \in (0,1)$

$$f(x) = x\left(\frac{a^{2/x} - 1}{a^{2/x} + 1}\right), \quad x \neq 0$$

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} x \cdot \left(\frac{a^{2/x} - 1}{a^{2/x} + 1}\right) = 0$$

$f(0) = 0$ given.

So $f(x)$ is continuous at $x = 0$

II) $a = 1$.

$$f(x) = x \cdot 0 = 0, \quad x \neq 0$$

$f(0) = 0 \Rightarrow f(x)$ is continuous at $x = 0$

II) $a \in (1, \infty)$

$$f(x) = x \cdot \left(\frac{1 - a^{-2/x}}{1 + a^{-2/x}} \right), \quad x \neq 0$$

$$\lim_{x \to 0} f(x) = 0 \quad \& \quad f(0) = 0$$

$\Rightarrow f(x)$ is continuous at $x = 0$.

For differentiability

$$f(x) = x \cdot \left(\frac{a^{2/x} - 1}{a^{2/x} + 1} \right)$$

$$f'(x) = \frac{a^{2/x} - 1}{a^{2/x} + 1} + x \frac{d}{dx} \left(\frac{a^{2/x} - 1}{a^{2/x} + 1} \right)$$

I) $a \in (0, 1)$

$$Lf'(0) = 1. \quad R.f'(0) = -1$$

so, non differentiable at $x = 0$

II) $a=1$

$f(x) = 0$ Constant function

So, Continuous & differentiable

III) $a \in (1, \infty)$

$$f'(x) = \frac{a^{1/x}-1}{a^{1/x}+1} + x \cdot \frac{d}{dx}\left(\frac{a^{1/x}-1}{a^{1/x}+1}\right)$$

$Lf'(0) = -1$

$R.f'(0) = 1$

So, $f(x)$ is continuous but not differentiable

3. If $\lim_{x\to 0}\dfrac{1-\cos\left(1-\cos\dfrac{x}{2}\right)}{2^m x^n}$ is equal to the left hand derivative of $e^{-|x|}$ at $x = 0$, then find the value of $(n - 10m)$

$$\lim_{x\to 0}\frac{1-\cos\left(2\sin^2\frac{x}{4}\right)}{2^m \cdot x^n} = 1$$

$f(x) = e^{-|x|}$

$Lf'(0) = 1.$

$$\lim_{x\to 0}\left(\frac{1-\cos\left(2\sin^2\frac{x}{4}\right)}{\left(2\sin^2\frac{x}{4}\right)^2}\right)\cdot\frac{4\sin^4\frac{x}{4}}{2^m\, x^n} = 1$$

$$\Rightarrow \lim_{x\to 0}\left(\frac{1-\cos\left(2\sin^2\frac{x}{4}\right)}{\left(2\sin^2\frac{x}{4}\right)^2}\right)\left(\frac{\sin\frac{x}{4}}{\frac{x}{4}}\right)^4\cdot\frac{x^4\cdot 4}{4^4\cdot 2^m\, x^n} = 1$$

$$\Rightarrow \left(\frac{1}{2}\right)\cdot\frac{x^{4-n}}{4^3\cdot 2^m} = 1 \quad\Rightarrow 4 = n.$$

and $m + 7 = 0 \Rightarrow m = -7$

$n - 10m = 4 + 70 = 74.$

4. If f is a differentiable function such that $f\left(\frac{x+y}{3}\right)=\frac{f(x)+f(y)+f(0)}{3}$, $\forall$ x,y $\in \mathbb{R}$ and $f'(0) = 2$, find $f(x)$

put $y = 2x$ then

$$f(x) = \frac{f(x) + f(2x) + f(0)}{3}$$

$$\Rightarrow 2f(x) = f(2x) + f(0)$$

$$\Rightarrow f(x) - f(0) = f(2x) - f(x)$$

$$\Rightarrow \lim_{x\to 0} \frac{f(x) - f(0)}{x} = \lim_{x\to 0} \frac{f(2x) - f(x)}{x}$$

$$\Rightarrow f'(0) = f'(x) = 2.$$

$\Rightarrow f(x) = 2x + C$ where C is Constant.

5. If $\lim_{x\to 0}\frac{f(3-\sin x)-f(3+x)}{x}=8$, then $|f'(3)|$ is

Sol^n:

Since, its $\frac{0}{0}$ form; so, use L'Hospital's Rule.

$$\lim_{x\to 0} f'(3-\sin x)(-\cos x)-f'(3+x)=8$$

$$\Rightarrow -f'(3)-f'(3)=8$$

$$\Rightarrow |f'(3)|=4.$$

6. Let $f(x)$ be a differentiable function such that $2f(x+y) + f(x-y) = 3f(x) + 3f(y) + 2xy$ $\forall$ x, y $\in \mathbb{R}$ & $f'(0) = 0$, then $f(10) + f'(10)$ is equal to

Solⁿ:

$$2f(x+y) + f(x-y) = 3f(x) + 3f(y) + 2xy$$

diff. w.r.t. 'y' we get:

$$2f'(x+y) - f'(x-y) = 3f'(y) + 2x$$

Also $f(0) = 0$ so put $y = 0$.

$$2f'(x) - f'(x) = 3f'(0) + 2x$$

$$\Rightarrow f'(x) = 2x \Rightarrow f(x) = x^2 + \lambda$$

$$\Rightarrow f(0) = 0 \Rightarrow \lambda = 0$$

$$\Rightarrow f(x) = x^2$$

$$\therefore f(10) = 100 \quad f'(10) = 20$$

$$\therefore f(10) + f'(10) = 120$$

EXERCISE (JM)

1. For $x \in R$, $f(x) = |\log 2 - \sin x|$ and $g(x) = f(f(x))$, then : **[JEE(Main)-2016]**

(1) g is differentiable at $x = 0$ and $g'(0) = -\sin(\log 2)$

(2) g is not differentiable at $x = 0$

~~(3)~~ $g'(0) = \cos(\log 2)$

(4) $g'(0) = -\cos(\log 2)$

Sol^n) $f(x) = |\log 2 - \sin x|$

f(x) is continuous but not differentiable at some x where $\sin x = \log 2$ —— (i)

$g(x) = f(f(x)) = |\log 2 - \sin(f(x))|$

g(x) is continuous but not differentiable where

$\sin f(x) = \log 2$ —— (ii)

Now According to options

at $x = 0 \Rightarrow f(0) = \log 2$

as $\sin x < x$

$\sin(\log 2) < \log 2$ which

means at $x=0$ (ii) does not hold

So $g(x)$ is diff^nble at $x=0$

At $x=0$ $\quad f(x) = \log 2 - \sin x$

$g(x) = \log 2 - \sin(f(x))$

$g'(x) = -\cos(f(x)) \cdot f'(x)$

$g'(0) = -\cos(f(0)) \cdot (-\cos 0)$

$g'(0) = \cos(\log 2)$

2. Let S = {t ∈ R : $f(x) = |x - \pi| \cdot (e^{|x|} - 1) \sin|x|$ is not differentiable at t}. Then the set S is equal to: **[JEE(Main)-2018]**

(1) $\{0\}$ (2) $\{\pi\}$ (3) $\{0, \pi\}$ (4) ϕ (an empty set)

Sol^n)

$$f(x) = |x - \pi|(e^{|x|} - 1)\sin|x|$$

$\Rightarrow |x - \pi|$ is not differentiable at $x = \pi$ but at $x = \pi$, $\sin|x| = 0$

$\Rightarrow (e^{|x|} - 1)$ is not differentiable at $x = 0$ but $(\sin|x|)_{x=0} = 0$

$\Rightarrow \sin|x|$ is not differentiable at $x = 0$ but $(e^{|x|} - 1)_{x=0} = 0$

So $f(x)$ is differentiable $\forall$ $x \in R$ $\Rightarrow S = \phi$

3. Let f be a differentiable function from R to R such that $|f(x)-f(y)| \le 2|x-y|^{\frac{3}{2}}$, for all x, y $\in$ R.

If $f(0) = 1$ then $\int_0^1 f^2(x)dx$ is equal to **[JEE(Main)-2019]**

(1) 0 (2) $\frac{1}{2}$ (3) 2 (4) 1

Sol^n)

$$\left|\frac{f(x)-f(y)}{x-y}\right| \le 2|x-y|^{1/2}$$

$$\lim_{y\to x}\left|\frac{f(x)-f(y)}{x-y}\right| \le \lim_{y\to x} 2|x-y|^{\frac{1}{2}}$$

$$|f'(x)| \le 0 \Rightarrow f'(x)=0$$

$\Rightarrow f(x)$ is constant function

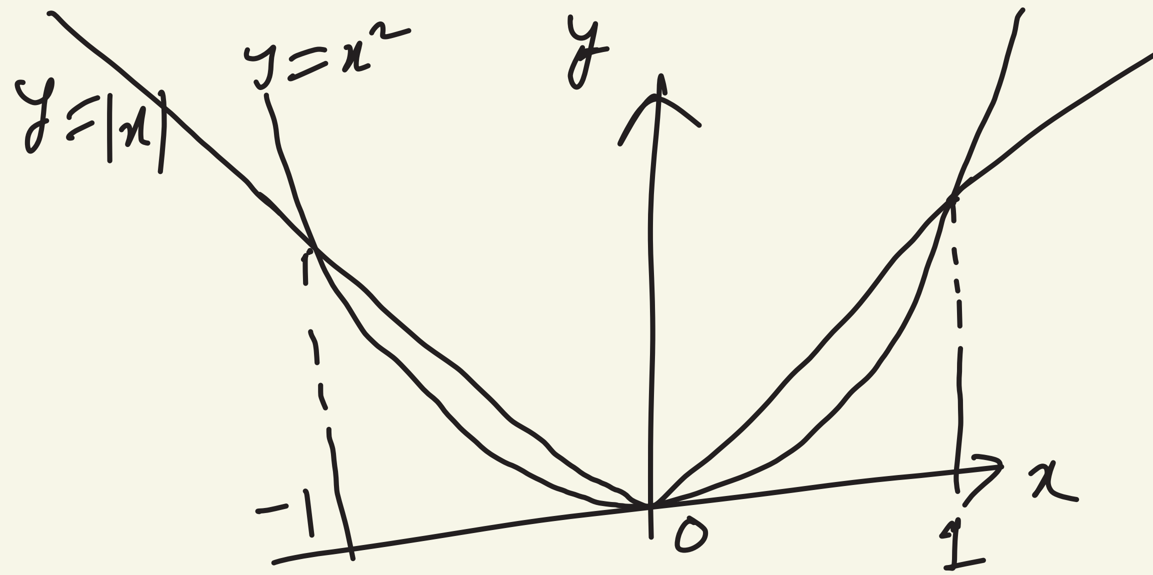
given $f(0)=1 \Rightarrow \boxed{f(x)=1}\ \forall x\in R$

$$\int_0^1 f^2(x)\,dx = \int_0^1 dx = 1$$

4. Let $f(x)=\begin{cases}\max\{|x|,x^2\}, & |x|\le 2\\ 8-2|x|, & 2<|x|\le 4\end{cases}$. Let S be the set of points in the interval (–4,4) at which f is not differentiable. Then S : **[JEE(Main)-2019]**

(1) is an empty set
(2) equals {–2, –1, 1, 2}
(3) equals {–2, –1, 0, 1, 2}
(4) equals {–2, 2}

Sol^n



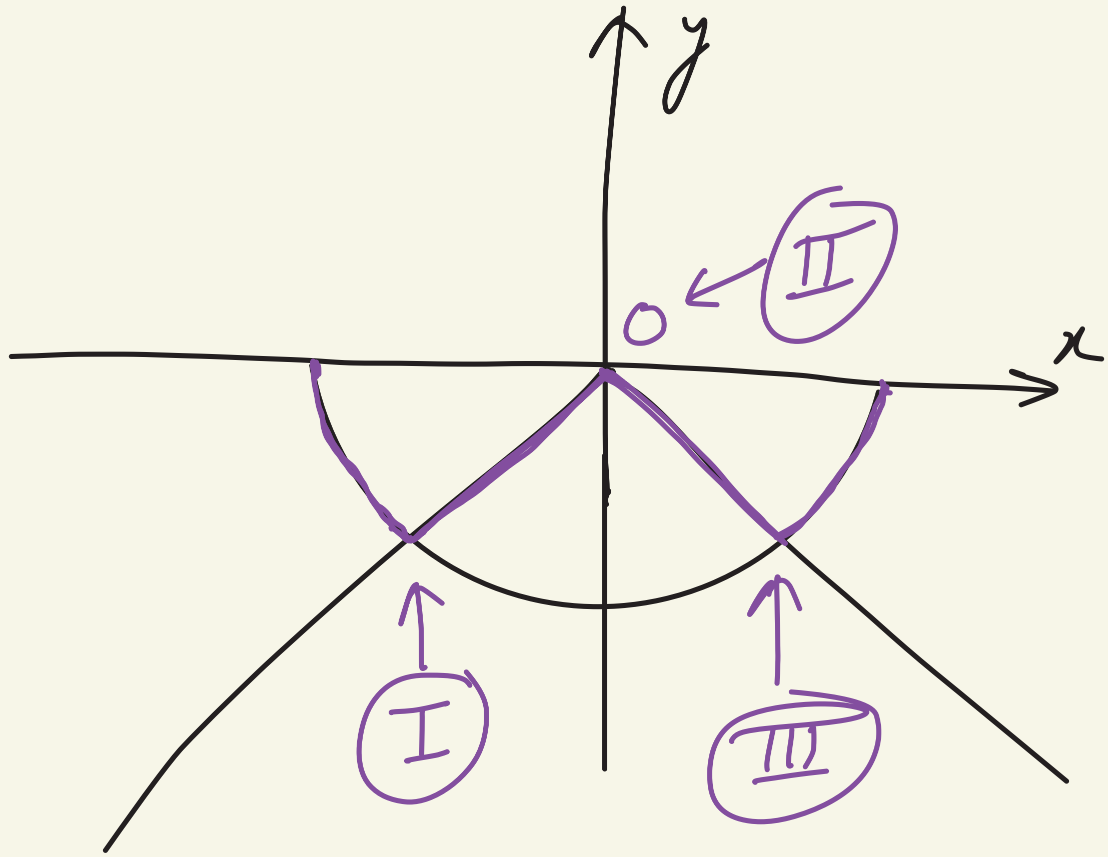
$$f(x)=\begin{cases}8+2x & -4\le x<-2\\ x^2 & -2\le x\le -1\\ |x| & -1<x\le 1\\ x^2 & 1<x\le 2\\ 8-2x & 2<x\le 4\end{cases}$$

$f(x)$ is not diff^n at $x=0,1,-1,2,-2$

5. Let $f : (-1,1) \to R$ be a function defined by $f(x) = \max\left\{-|x|, -\sqrt{1-x^2}\right\}$. If K be the set of all points at which f is not differentiable, then K has exactly : **[JEE(Main)-2019]**

~~(1) Three~~ elements (2) One element (3) Five elements (4) Two elements

Soln)



f(x) is not differentiable at three points

6. Let K be the set of all real values of x where the function $f(x) = \sin|x| - |x| + 2(x - \pi)\cos|x|$ is not differentiable. Then the set K is equal to :- **[JEE(Main)-2019]**

(1) $\{\pi\}$ (2) $\{0\}$ (3) ϕ (an empty set) (4) $\{0, \pi\}$

Sol^n

Let $g(x) = \sin|x| - |x| = \begin{cases} \sin x - x, & x \geq 0 \\ -\sin x + x, & x < 0 \end{cases}$

Check differentiability at $x = 0$

$g'(x) = \begin{cases} \cos x - 1, & x \geq 0 \\ -\cos x + 1, & x < 0 \end{cases}$

at $x = 0$ $g'(0^+) = g'(0^-) = 0 \Rightarrow g'(0) = 0$

hence $g(x)$ is diff^{able} $\forall x \in R$

Also $2(x - \pi)\cos|x| = 2(x - \pi)\cos x$

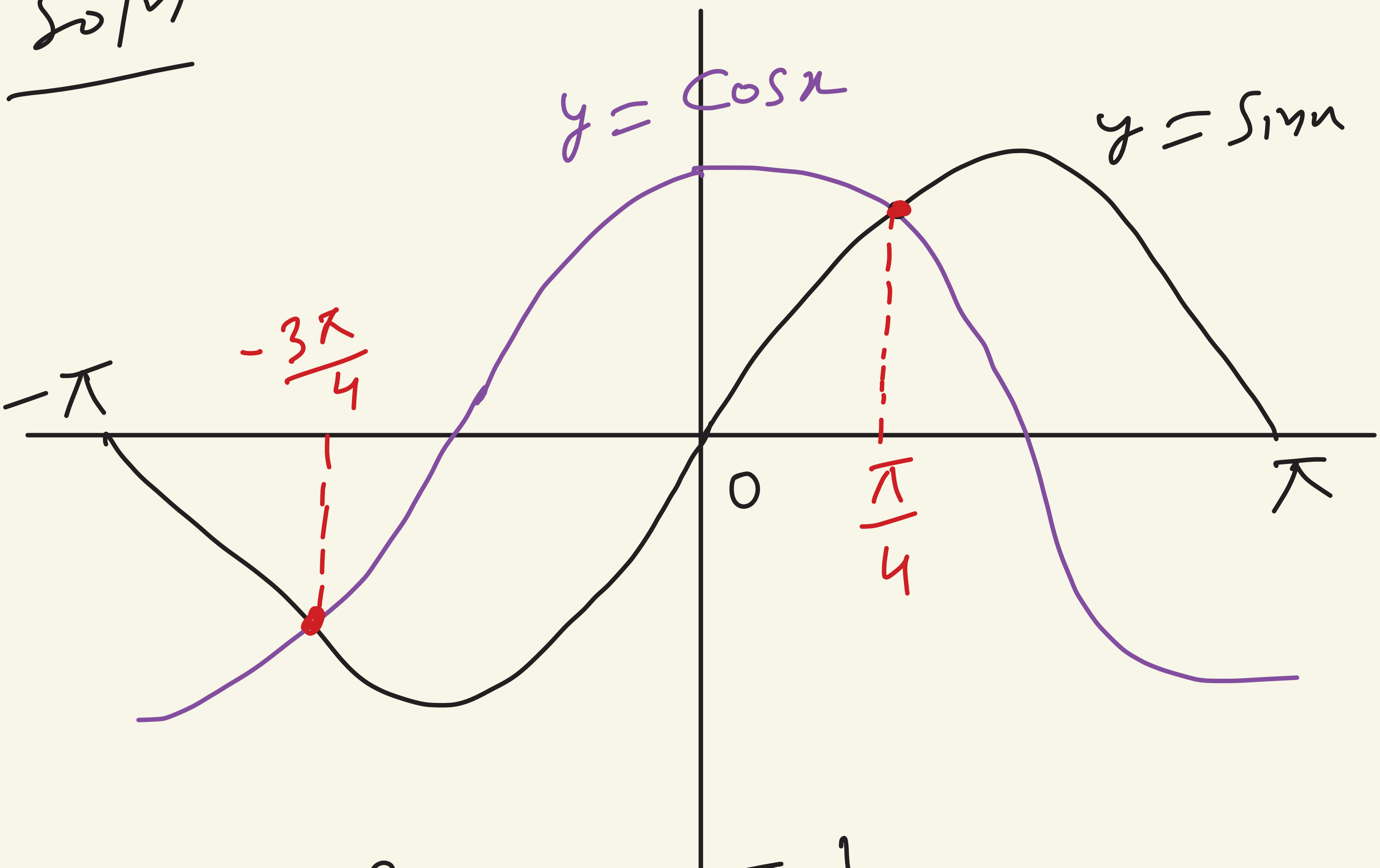
is always diff^{able} $\Rightarrow f(x)$ is always

diff^{able} $\Rightarrow K = \phi$

7. Let S be the set of all points in $(-\pi,\pi)$ at which the function, $f(x) = \min\{\sin x, \cos x\}$ is not differentiable. Then S is a subset of which of the following? **[JEE(Main)-2019]**

~~(1)~~ $\left\{-\frac{3\pi}{4}, -\frac{\pi}{4}, \frac{3\pi}{4}, \frac{\pi}{4}\right\}$ (2) $\left\{-\frac{3\pi}{4}, -\frac{\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{4}\right\}$ (3) $\left\{-\frac{\pi}{2}, -\frac{\pi}{4}, \frac{\pi}{4}, \frac{\pi}{2}\right\}$ (4) $\left\{-\frac{\pi}{4}, 0, \frac{\pi}{4}\right\}$

Soln



$$S = \left\{-\frac{3\pi}{4}, \frac{\pi}{4}\right\}$$

8. Let $f(x) = 15 - |x - 10|$; $x \in R$. Then the set of all values of x, at which the function, $g(x) = f(f(x))$ is not differentiable, is : **[JEE(Main)-2019]**

(1) {5,10,15,20} (2) {10,15} ~~(3)~~ {5,10,15} (4) {10}

Solⁿ

$$f(x) = 15 - |x-10| = \begin{cases} 25-x, & x \geq 10 \\ x+5 & x < 10 \end{cases}$$

$f(x)$ is not diff^{nble} at $x=10$

$$g(x) = f(f(x)) = 15 - |f(x) - 10|$$

$g(x)$ is not diff^{nble} where

$f(x) = 10$ and $x = 10$

$$15 - |x-10| = 10 \Rightarrow |x-10| = 5$$

$$x = 5, 15$$

$g(x)$ is not diff^{nble} at $x = 5, 10, 15$

9. Let $f : R \to R$ be differentiable at $c \in R$ and $f(c) = 0$. If $g(x) = |f(x)|$, then at $x = c$, g is : **[JEE(Main)-2019]**

~~(1)~~ differentiable if $f'(c) = 0$
(2) not differentiable
(3) differentiable if $f'(c) \neq 0$
(4) not differentiable if $f'(c) = 0$

Sol^n)

$$g(x) = |f(x)| = \begin{cases} f(x) & f(x) \geq 0 \\ -f(x) & f(x) < 0 \end{cases}$$

$$g'(x) = \begin{cases} f'(x), & f(x) \geq 0 \\ -f'(x), & f(x) < 0 \end{cases}$$

now if $g(x)$ is diff^{nble} then

$$f'(x) = -f'(x) \Rightarrow f'(x) = 0$$

it means if At $x = c$ $g(x)$ is diff^{nble} then $f'(c) = 0$ (must be)

$\Rightarrow$ option (i) is Correct

EXERCISE (JA)

1. Let $f : R \to R$ be a function such that

$f(x + y) = f(x) + f(y), \forall\ x, y \in R.$

If $f(x)$ is differentiable at $x = 0$, then

(A) $f(x)$ is differentiable only in a finite interval containing zero

(B) $f(x)$ is continuous $\forall\ x \in R$

(C) $f'(x)$ is constant $\forall\ x \in R$

(D) $f(x)$ is differentiable except at finitely many points **[JEE 2011, 4M]**

Solution:

If $f(x)$ is differentiable at $x=0$

$\Rightarrow$ $f(x)$ is continuous at $x=0$

$\Rightarrow \lim\limits_{h\to 0} f(h) = f(0) = 0$ — (I)

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h\to 0} \frac{f(h)}{h}$$

exists finitely. ($x, y = 0$, $f(0) = 2f(0)$ $\Rightarrow f(0) = 0$)

- For continuity

Limit at x

$$\lim_{h\to 0} f(x+h) = \lim_{h\to 0}\left(f(x) + f(h)\right)$$

$$= f(x) + \lim_{h\to 0} f(h) = f(x) = \text{output at } x$$

$\lim\limits_{h\to 0} f(x+h) = f(x)$ for any $x \in R$

$\Rightarrow$ $f(x)$ is continuous $\forall\ x \in R$

- $$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{f(x) + f(h) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{f(h)}{h} = f'(0)$$

$\Rightarrow$ $f'(x)$ is constant $\forall\ x \in R$

$\boxed{1 - B, C}$

2. If $f(x)=\begin{cases}-x-\dfrac{\pi}{2}, & x\le -\dfrac{\pi}{2}\\ -\cos x, & -\dfrac{\pi}{2}<x\le 0\\ x-1, & 0<x\le 1\\ \ell n x, & x>1\end{cases}$ then - **[JEE 2011, 4M]**

(A) $f(x)$ is continuous at $x=-\dfrac{\pi}{2}$

(B) $f(x)$ is not differentiable at $x = 0$

(C) $f(x)$ is differentiable at $x = 1$

(D) $f(x)$ is differentiable at $x=-\dfrac{3}{2}$

Solution:

$$\lim_{x\to -\frac{\pi}{2}^-} f(x) = \lim_{x\to -\frac{\pi}{2}^-}\left(-x-\frac{\pi}{2}\right) = 0 = f\left(-\frac{\pi}{2}\right)$$

$$\lim_{x\to -\frac{\pi}{2}^+} f(x) = \lim_{x\to -\frac{\pi}{2}^+}(-\cos x) = 0$$

$$\lim_{x\to -\frac{\pi}{2}^-} f(x) = \lim_{x\to -\frac{\pi}{2}^+} f(x) = f\left(-\frac{\pi}{2}\right)$$

$\Rightarrow$ $f(x)$ is continuous at $x=-\pi/2$

At $x=0$

$$L.H.L = \lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(-\cos x) = -1 = f(0)$$

$$R.H.L = \lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}(x-1) = -1$$

$L.H.L = R.H.L = f(0) \Rightarrow f(x)$ is continuous at $x=0$.

$$f'(x) = \begin{cases}\sin x, & x<0\\ 1, & x>0\end{cases},$$

$L.H.D$ at 0; $f'(0^-)=0$

$R.H.D$ at 0, $f'(0^+)=1$

$\Rightarrow$ $f(x)$ is non-differentiable at 0.

At $x=1$

$L.H.L = \lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (x-1) = 0 = f(1)$

$R.H.L = \lim_{x \to 1^+} f(x) = \lim_{x \to 1^+} \ln x = 0$

$L.H.L = R.H.L = f(1)$

$\Rightarrow$ $f(x)$ is continuous at $x=1$.

$$f(x) = \begin{cases} x-1, & x \le 1 \\ \ln x, & x > 1 \end{cases}$$

$$\Rightarrow f'(x) = \begin{cases} 1, & x < 1 \\ \frac{1}{x}, & x > 1 \end{cases}$$

$L.H.D = f'(1^-) = 1$

$R.H.D = f'(1^+) = 1$

$L.H.D = R.H.D \Rightarrow$ $f(x)$ is differentiable at $x=1$

- for $x \in [-\frac{\pi}{2}, 0]$, $f(x) = -\cos x$

  $f(x)$ is differentiable for $x \in (-\frac{\pi}{2}, 0)$

  So $f(x)$ is differentiable at $x = -3/2$.

$\boxed{2- A, B, C, D}$.

3. Let $f(x)=\begin{cases} x^2\left|\cos\dfrac{\pi}{x}\right| & , \quad x\neq 0 \\ 0 & , \quad x=0 \end{cases}, x\in\mathbb{R}$, then f is - **[JEE 2012, 3M, –1M]**

(A) differentiable both at x = 0 and at x = 2

(B) differentiable at x = 0 but not differentiable at x = 2

(C) not differentiable at x = 0 but differentiable at x = 2

(D) differentiable neither at x = 0 nor at x = 2

Solution.

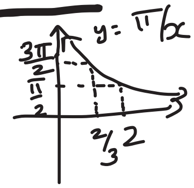
$$\lim_{x\to 0} f(x) = \lim_{x\to 0} x^2\left|\cos\frac{\pi}{x}\right| = 0 = f(0)$$

$\Rightarrow f(x)$ is continuous at $x=0$.

$$f'(0) = \lim_{h\to 0}\frac{f(h)-f(0)}{h} = \lim_{h\to 0}\frac{h^2\left|\cos \pi/h\right|}{h}$$

$$= \lim_{h\to 0} h\left|\cos \pi/h\right| = 0$$

$\Rightarrow$ $f(x)$ is differentiable at $x=0$.



$$f(x) = \begin{cases} -x^2\cos \pi/x , & \frac{2}{3}<x\leq 2 \\ x^2\cos\pi/x & 2<x \end{cases}$$

$$f'(x) = \begin{cases} -2x\cos\pi/x - x^2\sin\frac{\pi}{x}\times\left(\frac{\pi}{x^2}\right), & x\in\left(\frac{2}{3},2\right) \\ 2x\cos\pi/x + x^2\sin\pi/x\times(\pi/x^2), & x\in(2,\infty) \end{cases}$$

$$= \begin{cases} -2x\cos\pi/x - \pi\sin\pi/x , & x\in(2/3,2) \\ 2x\cos\pi/x + \pi\sin\pi/x, & x\in(2,\infty) \end{cases}$$

L.H.D at $x=2$, $f'(2^-) = -4\cos\frac{\pi}{2} - \pi\sin\pi/2 = -\pi$

R.H.D at $x=2$, $f'(2^+) = 4\cos\pi/2 + \pi\sin\pi/2 = \pi$

L.H.D $\neq$ R.H.D $\Rightarrow$ $f(x)$ is not differentiable at $x=2$

$\boxed{3 - B}$

4. Let $f_1 : \mathbb{R} \to \mathbb{R}$, $f_2 : [0, \infty) \to \mathbb{R}$, $f_3 : \mathbb{R} \to \mathbb{R}$ and $f_4 : \mathbb{R} \to [0, \infty)$ be defined by

$$f_1(x) = \begin{cases} |x| & \text{if} \quad x < 0, \\ e^x & \text{if} \quad x \ge 0; \end{cases}$$

$$f_2(x) = x^2;$$

$$f_3(x) = \begin{cases} \sin x & \text{if} \quad x < 0, \\ x & \text{if} \quad x \ge 0 \end{cases}$$

$$\text{and } f_4(x) = \begin{cases} f_2(f_1(x)) & \text{if} \quad x < 0, \\ f_2(f_1(x)) - 1 & \text{if} \quad x \ge 0. \end{cases}$$

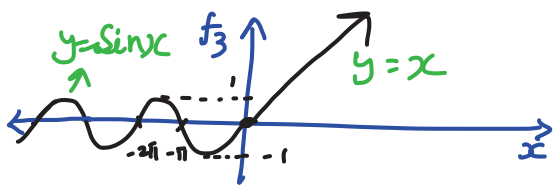
	List-I		**List-II**
P.	f_4 is	1.	onto but not one-one
Q.	f_3 is	2.	neither continuous nor one-one
R.	$f_2 o f_1$ is	3.	differentiable but not one-one
S.	f_2 is	4.	continuous and one-one

Codes :

	P	Q	R	S
(A)	3	1	4	2
(B)	1	3	4	2
(C)	3	1	2	4
(D)	1	3	2	4

[JEE(Advanced)-2014, 3(–1)]

$$f_3(x) = \begin{cases} \sin x, & x < 0 \\ x, & x \ge 0, \end{cases}$$



$$f_3'(x) = \begin{cases} \cos x, & x < 0 \\ 1, & x \ge 0, \end{cases} \Rightarrow \begin{array}{l} \text{L.H.D at } 0 = f'(0^-) = 1 \\ \text{R.H.D at } 0 = f'(0^+) = 1 \end{array}$$

$\Rightarrow$ $f_3(x)$ is differentiable but not one-one.

$\boxed{Q \to 3}$ — (1)

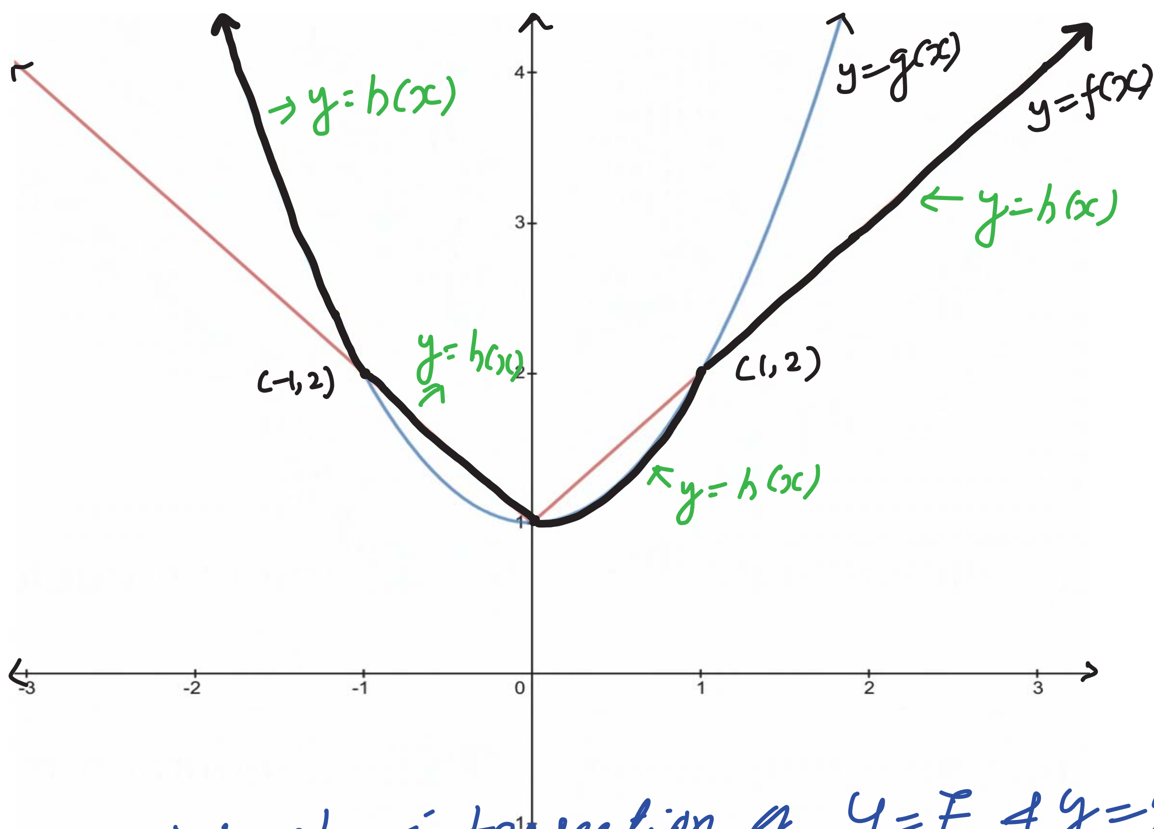
Both B & D options have $Q \to 3$

In both B & D $P \to 1$, but R & S have different choices

5. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be respectively given by $f(x) = |x| + 1$ and $g(x) = x^2 + 1$. Define

$$h : \mathbb{R} \to \mathbb{R} \text{ by } h(x) = \begin{cases} \max\{f(x), g(x)\} & \text{if } x \le 0, \\ \min\{f(x), g(x)\} & \text{if } x > 0. \end{cases}$$

The number of points at which h(x) is not differentiable is **[JEE(Advanced)-2014, 3]**



For point of intersection of $y=f$ & $y=g$

$|x|+1 = x^2+1 \Rightarrow x^2 = |x| \Rightarrow x = 0, \pm 1.$

$$h(x) = \begin{cases} x^2+1, & x \in (-\infty,-1] \cup [0,1] \\ |x|+1, & x \in (-1,0) \cup (1,\infty) \end{cases} = \begin{cases} x^2+1, & x \in (-\infty,-1] \cup [0,1] \\ -x+1, & x \in (-1,0) \\ x+1, & x \in (1,\infty) \end{cases}$$

$$h'(x) = \begin{cases} 2x, & x \in (-\infty,-1) \cup (0,1) \\ -1, & x \in (-1,0) \\ 1, & x \in (1,\infty) \end{cases}$$

At $x=-1$,
L.H.D $= h'(-1^-) = -2$
R.H.D $= h'(-1^+) = -1$

At $x=0$
L.H.D $= h'(0^-) = -1$
R.H.D $= h'(0^+) = 0$

At $x=1$
L.H.D $= h'(1^-) = 2$
R.H.D $= h'(1^+) = 1$

The number of points at which $h(x)$ is not differentiable is $\boxed{3}$.

6. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function with $f(0) = 1$ and satisfying the equation

$$f(x + y) = f(x)f'(y) + f'(x)f(y) \text{ for all } x, y \in \mathbb{R}.$$

Then, then value of $\log_e(f(4))$ is _____ . **[JEE(Advanced)-2018]**

Soln:

$f(0) = 1,$

$f(x+y) = f(x)f'(y) + f'(x)f(y) \cdot x, y \in R$

for $x = y = 0$

$f(0) = 2f'(0) \cdot f(0) \Rightarrow f'(0) = \frac{1}{2}$

let $y = 0$.

$f(x) = f(x) \cdot f'(0) + f'(x) \cdot f(0)$

$\Rightarrow f'(x) = \frac{f(x)}{2}$

$\Rightarrow \int \frac{df(x)}{f(x)} = \int \frac{dx}{2}$

$\Rightarrow \ln|f(x)| = \frac{x}{2} + c$

$\Rightarrow f(x) = e^{x/2 + c}$

putting $x = 0$, $f(0) = e^c \Rightarrow c = 0$

$\Rightarrow f(x) = e^{x/2}$

$\ln f(4) = \ln e^{4/2} = \boxed{2}$.

7. Let $f_1 : \mathbb{R} \to \mathbb{R}$, $f_2 : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$, $f_3 : \left(-1, e^{\frac{\pi}{2}} - 2\right) \to \mathbb{R}$ and $f_4 : \mathbb{R} \to \mathbb{R}$ be functions defined by

(i) $f_1(x) = \sin\left(\sqrt{1 - e^{-x^2}}\right)$

(ii) $f_2(x) = \begin{cases} \dfrac{|\sin x|}{\tan^{-1} x} & \text{if } x \neq 0 \\ 1 & \text{if } x = 0 \end{cases}$, where the inverse trigonometric function $\tan^{-1}x$ assumes values in $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$,

(iii) $f_3(x) = [\sin(\log_e(x + 2))]$, where for $t \in \mathbb{R}$, $[t]$ denotes the greatest integer less than or equal to t,

(iv) $f_4(x) = \begin{cases} x^2 \sin\left(\dfrac{1}{x}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

	List-I		**List-II**
P.	The function f_1 is	**1.**	**NOT** continuous at $x = 0$
Q.	The function f_2 is	**2.**	continuous at $x = 0$ and **NOT** differentiable at $x = 0$
R.	The function f_3 is	**3.**	differentiable at $x = 0$ and its derivative is **NOT** continuous at $x = 0$
S.	The function f_4 is	**4.**	differentiable at $x = 0$ and its derivative is continuous at $x = 0$

The correct option is :

(A) **P → 2; Q → 3, R → 1; S → 4**

(B) **P → 4; Q → 1; R → 2; S → 3**

(C) **P → 4; Q → 2, R → 1; S → 3**

(D) **P → 2; Q → 1; R → 4; S → 3** **[JEE(Advanced)-2018]**

Solution:

At $x = 0$

$-x^2$ is continuous for $x \in \mathbb{R}$

$\Rightarrow \sqrt{1 - e^{-x^2}}$ is continuous for $x \in \mathbb{R}$

$\Rightarrow f_1(x) = \sin\sqrt{1 - e^{-x^2}}$ is continuous for $x \in \mathbb{R}$

$$f_1'(0) = \lim_{h \to 0} \frac{f(h) - f(0)}{h} = \lim_{h \to 0} \frac{\sin\sqrt{\frac{e^{h^2} - 1}{e^{h^2}}} - 0}{h}$$

$$= \lim_{h \to 0} \underbrace{\frac{\sin\sqrt{\frac{e^{h^2}-1}{e^{h^2} \times h^2} \times h^2}}{\sqrt{\frac{e^{h^2}-1}{h^2 \times e^{h^2}} \times h^2}}}_{\to 1} \underbrace{\sqrt{\frac{e^{h^2}-1}{h^2 \times e^{h^2}}}}_{\to 1} \times \frac{|h|}{h}$$

$R \cdot H \cdot D = f_1'(0^+) = 1$ and $L \cdot H \cdot D = f_1'(0^-) = -1$

at $x = 0$, $f_1(x)$ is continuous & not differentiable.

$\boxed{P-2}$

$$f_2(x) = \begin{cases} \dfrac{|\sin x|}{\tan^{-1} x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$$

At $x = 0$

$$L \cdot H \cdot L = \lim_{h \to 0^-} \frac{\left(\frac{|\sin h|}{|h|}\right)}{\left(\frac{\tan^{-1} h}{h}\right)} \times \frac{|h|}{h} = -1$$

$$R \cdot H \cdot L = \lim_{h \to 0^+} \frac{\left(\frac{|\sin h|}{|h|}\right)}{\left(\frac{\tan^{-1} h}{h}\right)} \times \frac{|h|}{h} = 1$$

$L \cdot H \cdot L \neq R \cdot H \cdot L \Rightarrow f_2(x)$ is not continuous at $x = 0$.

$\boxed{Q-1}$

$$f_3(x) = [\sin \ln(x+2)]$$

$$0 < \ln(x+2) < \pi/2 \Rightarrow x \in (-1, e^{\pi/2} - 2)$$

$$0 < \sin \ln(x+2) < 1 \Rightarrow [\sin \ln(x+2)] = 0$$

$\Rightarrow f_3(x) = 0, \quad \forall x \in (-1, e^{\pi/2} - 2)$, $e^{\pi/2} - 2 > 0$ (Since $f_3(x)$ is constant for this interval)

$\Rightarrow f(0) = f'(0) = 0$

$f_3(x)$ is differentiable at $x = 0$, and its derivative continuous at $x = 0$.

$\boxed{R-4}$

In option D, P-2, Q-1, R-4.

$\boxed{9-D}$ Ans.

We can check for $f_4(x)$

$$f_4'(0) = \lim_{h \to 0} \frac{h^2 \sin(\frac{1}{h}) - 0}{h} = 0$$

$\Rightarrow f_4(x)$ is differentiable at $x = 0$.

but $f_4'(x) = 2x \sin(\frac{1}{x}) + x^2 \cos(\frac{1}{x}) \times (-\frac{1}{x^2}), \quad x \neq 0$

$= 2x \sin \frac{1}{x} - \cos \frac{1}{x}, \quad x \neq 0$

$\lim_{x \to 0} f_4'(x)$ does not exist, so $f_4'(x)$ is not continuous at 0