Q The first term of an AP is 5 and last term is 45 and sum of this AP is 400 then find no. of terms & the common diff ?

Sol^n 1st term = $a = 5$

last term = $l = 45$

$$S_n = \frac{n}{2}(a+l) \Rightarrow 400 = \frac{n}{2}(5+45) \Rightarrow \underline{n=16}$$

$$l = a+(n-1)d$$

$$45 = 5+(15).d \Rightarrow d = \frac{8}{3}.$$

Q Find sum of series :

$$S = \underbrace{100^2 - 99^2} + \underbrace{98^2 - 97^2} + \cdots + 2^2 - 1^2.$$

Sol^n

$$S = \underbrace{(100-99)}_{1}(100+99) + \underbrace{(98-97)}_{1}(98+97) + \cdots + \underbrace{(2-1)}_{1}(2+1).$$

$$S = (100+99) + (98+97) + \cdots + (2+1)$$

$$S = 5050.$$

Q The sum of n terms of two AP's are in ratio $7n+1 : 4n+27$, then find the ratio of their 11^{th} term?

Sol^n

$$\frac{S_n}{S_n'} = \frac{\cancel{\frac{n}{2}}\left(2a+(n-1)d\right)}{\cancel{\frac{n}{2}}\left(2a'+(n-1)d'\right)} = \frac{7n+1}{4n+27}.$$

$$= \frac{2a+(n-1)d}{2a'+(n-1)d'} = \frac{7n+1}{4n+27}$$

$$= \frac{a+\frac{(n-1)}{2}d}{a'+\left(\frac{n-1}{2}\right)d'} = \frac{7n+1}{4n+27}$$

$$\frac{n-1}{2} = 10 \Rightarrow n=21$$

$$= \frac{T_{11}}{T_{11}'} = \frac{\cancel{148}}{\cancel{111}} = \frac{4}{3}.$$

$$\frac{T_{11}}{T_{11}'} = ?$$

$$\frac{T_{11}}{T_{11}'} = \frac{a+10d}{a'+10d'} = ?$$

Q Find sum of all integers between 1 to 100 which are divisible by 2 or 3.

Soln

2, 3, 4, 5, 6, 7, ... , 96, 97, 98, 99

S_2 : 2, 4, 6, 8, 10, 12, , 98. $\xrightarrow{n=49}$ div by '2'. ✓

S_3 : 3, 6, 9, 12, - - -, 99 $\xrightarrow{n=33}$ div by '3'. ✓

S_6 : 6, 12, - - -, 96 $\xrightarrow{n=16}$ " " '6'. ✓

$S_2 = \frac{49}{2}(2+98)$; $S_3 = \frac{33}{2}(3+99)$; $S_6 = \frac{16}{2}(6+96)$

$S_2 + S_3 - S_6 = 3317$ Ans

Properties of A.P :–

(1) 3 Nos in A.P : $\underbrace{a-d}_{1^{st}\text{ term}}, a, a+d. \longrightarrow c.d = d$

* 4 Nos in A.P : $\overbrace{a-3d}, a-d, a+d, a+3d. \longrightarrow c.d = 2d.$

5 Nos in A.P : $\overbrace{a-2d}, a-d, a, a+d, a+2d$

(2) 1, 3, 5, 7, 9, 11, 13. $\longrightarrow$ A.P. with cd = 2

If we pick the term of an A.P in a particular interval, then picked sequence is also an AP with common diff interval times the original common difference.

3) If $a_1, a_2, a_3, \ldots\ldots a_n$ and $b_1, b_2, b_3, \ldots, b_n$ are two AP's then

$a_1 \pm b_1,\ a_2 \pm b_2,\ a_3 \pm b_3, \ldots\ldots\ a_n \pm b_n$ are also in A.P.

but $a_1 b_1, a_2 b_2, a_3 b_3, \ldots, a_n b_n$ and $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \ldots, \frac{a_n}{b_n}$

<u>may</u> or <u>may</u> not be in A.P.

AP_1 : $1, 3, 5, 7, 9, 11 \longrightarrow d_1 = 2$

AP_2 : $5, 10, 15, 20, 25, 30. \longrightarrow d_2 = 5$

AP_3 : $6, 13, 20, 27, 34, 41. \longrightarrow$ A.P. $\longrightarrow d_3 = 7$

$5, 30, 75, 140, \ldots\ldots$ <u>Not in A.P.</u>

4) AP_1: 1, 3, 5, 7, 9, 11. $\longrightarrow$ $d_1 = 2$ ✓

add '3' AP_2 : 4, 6, 8, 10, 12, 14 $\longrightarrow$ $d_2 = 2$. ✓

AP_1:
mult by $\underbrace{'3'}$ ✓

AP_3 : 3, 9, 15, 21, 27, 33 $\longrightarrow$ $d_3 = 6$

* If each term of an AP is increased or decreased by the same number then resulting sequence is also an A P having same common diff.

* If each term of an AP is multiplied or divided by same non-zero number 'K' then resulting sequence is also an A.P whose common diff is 'kd' or '$\frac{d}{K}$' respectively. ($d \rightarrow$ common diff of original AP).

5)

$2, \ 5^{*}, \ 8, \ 11, \ 14, \ 17, \ 20 \longrightarrow AP.$

T_k

$r \qquad r$

$$\frac{5+17}{2} = 11.$$

$$\boxed{T_k = \frac{T_{k+r} + T_{k-r}}{2}}$$

Q The sum of 1^{st} 3 terms of an AP is 27 and sum of their square is 293 find S_n ?

Sol^n Let 3 nos in AP : $\underbrace{a-d}_{1^{st}\text{ term}}, a, a+d$.

$a-\cancel{d}+a+a+\cancel{d}=27 \Rightarrow \boxed{a=9}$

$(a-d)^2+a^2+(a+d)^2 = 293. \Rightarrow 3a^2+2d^2=293 \Rightarrow d^2=25$

$\boxed{d=\pm 5}$

C-I If $d=5$.

1^{st} term $= a-d = 4$

$S_n = \frac{n}{2}\left(2(4)+(n-1)5\right)$

$\boxed{S_n = \frac{n}{2}(5n+3)}$.

C-II If $d=-5$.

1^{st} term $= a+d = 14$.

$S_n = \frac{n}{2}\left(2\times 14+(n-1)(-5)\right)$

$\boxed{S_n = \frac{n}{2}(33-5n)}$

Q Let $a_1, a_2, a_3, ---, a_{29}$ be an AP such that $a_1 + a_3 + a_{27} + a_{29} = 48$ then find sum of all terms of this AP ?

Sol^n $a_1, a_2, ---, a_{29} \longrightarrow$ A P.

$$a_1 + a_3 + a_{27} + a_{29} = 48 \Rightarrow (a_1 + a_{29}) + (a_1 + a_{29}) = 48.$$

$$\cancel{2}(a_1 + a_{29}) = \cancel{48}.\ 24$$

$$S_{29} = \frac{29}{2}(a_1 + a_{29}) = \frac{29}{2} \times 24 = \underline{\underline{348}}$$

Q If a, b, c are in AP then prove that :

✓a) $b+c,\ c+a,\ a+b$ are also in AP.

✓b) $(b+c)^2-a^2,\ (c+a)^2-b^2;\ (a+b)^2-c^2$ are also in A.P.

Sol^n $a, b, c \longrightarrow$ A.P.

(a) $-a, -b, -c \longrightarrow$ A.P.

add $(a+b+c)$:

$b+c,\ a+c,\ a+b \longrightarrow$ A.P.

(b) $a, b, c \longrightarrow$ A.P.

$-2a, -2b, -2c \rightarrow$ A.P.

add $(a+b+c)$:

$\underbrace{b+c}-\underbrace{a},\ a+c-b,\ a+b-c \longrightarrow$ A.P.

multiply by $(a+b+c)$:

$(b+c)^2-a^2,\ (c+a)^2-b^2,\ (a+b)^2-c^2$

Arithmetic Mean (AM) :-

when three nos are in A.P then middle one is called AM of the other two.

i.e if a, b, c are in AP then 'b' is AM of 'a' & 'c'

$$\boxed{b = \frac{a+c}{2}}$$

AM of 3 nos $a, b, c = \dfrac{a+b+c}{3}$

AM of 'n' nos $x_1, x_2, \ldots, x_n = \dfrac{x_1 + x_2 + \cdots + x_n}{n}$.

How to insert 'n' AM's bet 'a' & 'b' :-

$$\underset{1^{st}\text{ term}}{a},\ \underbrace{A_1, A_2, A_3, \ldots\ldots, A_n}_{n \text{ AM's}},\ b \rightarrow A.P$$

$$b \rightarrow T_{n+2}$$

$$b = a + (\underbrace{n+2} - 1)\cdot d \Rightarrow \boxed{d = \frac{b-a}{n+1}}$$

Rem

$$A_1 = a + d$$
$$A_2 = a + 2d$$
$$\vdots \qquad \vdots$$
$$A_n = a + nd$$

$$A_1 + A_2 + \cdots + A_n = na + d\,(\underbrace{1+2+\cdots+n})$$

$$\sum_{i=1}^{n} A_i = na + \left(\frac{b-a}{\cancel{n+1}}\right)\frac{n\cancel{(n+1)}}{2}$$

$$\boxed{\sum_{i=1}^{n} A_i = n\underbrace{\left(\frac{a+b}{2}\right)}}$$

Rem

= n times single AM bet^n a & b.

Q1 If 101 means are inserted between 1 and 99 then find sum of these means ?

Sol^n

$$1, \underbrace{A_1, A_2, \ldots\ldots, A_{101}}_{101}, 99 \longrightarrow A.P.$$

$$\sum_{i=1}^{101} A_i = 101\left(\frac{1+99}{2}\right) = 5050.$$

Q2 If 'p' AM's are inserted between 5 and 41 so that $\frac{A_3}{A_{p-1}} = \frac{2}{5}$ then find 'p' ?

Sol^n

$$d = \frac{41-5}{p+1} \Rightarrow d = \frac{36}{p+1}$$

$$A_3 = 5 + 3d$$

$$A_{p-1} = 5 + (p-1)d$$

$$\frac{5 + \frac{108}{p+1}}{5 + (p-1)\cdot\frac{36}{p+1}} = \frac{2}{5}$$

$$p = 11$$ Ans

Q A number sequence $a_1, a_2, a_3, \ldots, a_n$ is such that

$a_1 = 0$; $|a_2| = |a_1+1|$; $|a_3| = |a_2+1|$; $\ldots\ldots$, $|a_n| = |a_{n-1}+1|$.

Prove that AM of $a_1, a_2, \ldots, a_n$ is not less than $-\frac{1}{2}$?

Sol^n

$$AM = \frac{a_1+a_2+\cdots+a_n}{n}.$$

$$\to \cancel{a_1^2} = 0$$

$$\cancel{a_2^2} = \cancel{a_1^2} + 2a_1 + 1.$$

$$\cancel{a_3^2} = \cancel{a_2^2} + 2a_2 + 1.$$

$$\cancel{a_n^2} = \cancel{a_{n-1}^2} + 2a_{n-1} + 1.$$

add

$$a_{n+1}^2 = \cancel{a_n^2} + 2a_n + 1$$

$$a_{n+1}^2 = 2(a_1+a_2+\cdots+a_n) + n.$$

$$\frac{a_{n+1}^2 - n}{2n} = \frac{a_1+a_2+\cdots+a_n}{n}$$

$$\boxed{\frac{a_{n+1}^2}{2n} - \frac{1}{2} = AM.}$$

$$\underbrace{\underbrace{\frac{a_{n+1}^2}{2n}}_{\geq 0} - \frac{1}{2}}_{\geq -\frac{1}{2}}.$$

Q Given $a_1, a_2, \ldots, a_n$ are in AP then Prove that

$$\frac{1}{a_1 a_n} + \frac{1}{a_2 a_{n-1}} + \cdots + \frac{1}{a_n a_1} = \frac{2}{\underbrace{a_1 + a_n}}\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right)$$

Sol^n

$$\left(\frac{a_1 + a_n}{a_1 a_n}\right) + \frac{\overbrace{a_1 + a_n}^{a_2 + a_{n-1}}}{\underbrace{a_2}\,\underbrace{a_{n-1}}} + \frac{\overbrace{a_1 + a_n}^{a_3 + a_{n-2}}}{\underbrace{a_3}\,\underbrace{a_{n-2}}} + \cdots + \frac{a_1 + a_n}{a_n a_1} = 2\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right)$$

LHS:

$$\left(\frac{1}{a_n} + \frac{1}{a_1}\right) + \left(\frac{1}{a_{n-1}} + \frac{1}{a_2}\right) + \left(\frac{1}{a_{n-2}} + \frac{1}{a_3}\right) + \cdots + \left(\frac{1}{a_1} + \frac{1}{a_n}\right)$$

$$2\left(\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}\right) = RHS$$

(H.P)

Geometrical Progression (GP) :-

Standard appearance :

$$\underset{\substack{\downarrow \\ T_1 = 1^{st}\text{ term}}}{a}, \quad \underset{\substack{\downarrow \\ T_2}}{ar}, \quad \underset{\substack{\downarrow \\ T_3}}{ar^2}, \quad ar^3, \; - - - - - - - , \; \underbrace{ar^{n-1}}$$

Rem

$$\boxed{T_n = ar^{n-1}}$$

$a \rightarrow 1^{st}$ term

$r \rightarrow$ Common ratio

$n \rightarrow$ no. of terms.

Q1 In a GP if $T_3 = 2$ and $T_6 = \frac{-1}{4}$ then $T_{10} = ?$

Sol^n

$$T_3 = ar^2 = 2$$

$$T_6 = ar^5 = -1/4$$

$$\frac{\cancel{a}r^2}{\cancel{a}r^5} = \frac{2}{-1/4} = -8$$

$$r^3 = -1/8 \Rightarrow r = -1/2$$

$$a = 8$$

$$T_{10} = ar^9$$

$$= -1/64.$$

Q2 If p^{th}, q^{th}, r^{th} terms of G.P are x, y, z respectively then prove that $x^{q-r} \cdot y^{r-p} \cdot z^{p-q} = 1$?

Sol^n

$$T_p = AR^{p-1} = x \quad ; \quad T_q = AR^{q-1} = y \quad ; \quad T_r = AR^{r-1} = z.$$

$$x^{q-r} = \frac{x^q}{x^r}$$

$$\frac{x^q}{x^r} \cdot \frac{y^r}{y^p} \cdot \frac{z^p}{z^q} = \left(\frac{x}{z}\right)^q \left(\frac{y}{x}\right)^r \left(\frac{z}{y}\right)^p$$

$$\frac{x}{z} = \frac{AR^{p-1}}{AR^{r-1}} = R^{p-r}$$

$$= \left(R^{p-r}\right)^q \left(R^{q-p}\right)^r \cdot \left(R^{r-q}\right)^p$$

$$= R^{\cancel{pq}-\cancel{qr}+\cancel{qr}-\cancel{pr}+\cancel{pr}-\cancel{pq}}$$

$$= R^0 = 1 \quad (\text{H.P})$$

Sum to n-terms of GP :-

$$S_n = a + \cancel{ar} + \cancel{ar^2} + \cdots + \cancel{ar^{n-1}}$$

$$r S_n = \cancel{ar} + \cancel{ar^2} + \cdots + \cancel{ar^{n-1}} + ar^n$$

$$(1-r) S_n = a - ar^n \Rightarrow \boxed{S_n = a\left(\frac{r^n - 1}{r-1}\right) = a\left(\frac{1-r^n}{1-r}\right)}$$

Rem

it is not valid for $r=1$.

$a \rightarrow 1^{st}$ term

$r \rightarrow$ Common ratio

$n \rightarrow$ no. of terms.

* If $r=1$ then

$$\boxed{S_n = a + a + \cdots + a = na}$$

Sum to infinite term of GP :-

valid for $r \in (-1,1) - \{0\}$

Rem

$$\boxed{S_\infty = \frac{a}{1-r}}$$

Q1 The sum of an infinite no. of terms of a G.P is 15 and sum of their squares is 45 then find the series ?

Soln

Let G.P : $a, ar, ar^2, ar^3, \ldots \infty$ $\quad r \in (-1,1)-\{0\}$

$$\frac{a}{1-r} = 15 \quad \text{--- (1) ---} \rightarrow \quad \frac{a^2}{(1-r)^2} = (15)^2 \quad \text{--- (3) ---}$$

GP_1 : $a^2, (ar)^2, (ar^2)^2, \ldots \infty$

$$\frac{a^2}{1-r^2} = 45 \quad \text{--- (2) ---}$$

Divide (3) & (2)

$$\frac{\dfrac{\cancel{a^2}}{(1-r)^2}}{\dfrac{\cancel{a^2}}{1-r^2}} = \frac{\overset{5}{\cancel{225}}}{\cancel{45}}$$

$$\frac{1-r^2}{(1-r)^2} = 5 \Rightarrow \frac{1+r}{1-r} = 5$$

$$\begin{cases} r = 2/3 \text{ put in (1)} \\ a = 5 \end{cases}$$

Q Find sum of series :

① $S = 9 + 99 + 999 + \cdots + \underbrace{999\ldots9}_{n\text{ — times.}}$

$S = (10-1) + (10^2-1) + (10^3-1) + \cdots + (10^n-1)$

$S = (10 + 10^2 + 10^3 + \cdots + 10^n) - n$

$S = \dfrac{10 \cdot (10^n - 1)}{(10-1)} - n.$

② $S = 7 + 77 + 777 + \cdots + \underbrace{777\ldots7}_{n\text{ times}}$

$S = \dfrac{7}{9}\left(9 + 99 + 999 + \cdots + \underbrace{99\ldots9}_{n\text{ times}}\right)$

(3)

$$S = 0.5 + 0.55 + 0.555 + \cdots + 0.\underbrace{555\ldots5}_{n \text{ times}}$$

$$S = \frac{5}{10} + \frac{55}{10^2} + \frac{555}{10^3} + \cdots + \frac{555\ldots5}{10^n}.$$

$$S = \frac{5}{9}\left(\frac{9}{10} + \frac{99}{10^2} + \frac{999}{10^3} + \cdots + \frac{99\ldots9}{10^n} \right)$$

$$= \frac{5}{9}\left(\frac{10-1}{10} + \frac{10^2-1}{10^2} + \frac{10^3-1}{10^3} + \cdots + \frac{10^n-1}{10^n} \right)$$

$$= \frac{5}{9}\left(\underbrace{(1+1+\cdots+1)} - \underbrace{\left(\frac{1}{10} + \frac{1}{10^2} + \cdots + \frac{1}{10^n}\right)}_{GP} \right)$$

$$= \frac{5}{9}\left(n - \quad \right)$$

Q $S = \frac{3}{19} + \frac{33}{19^2} + \frac{333}{19^3} + \frac{3333}{19^4} + \cdots \infty.$

$$S = \frac{3}{9}\left(\frac{9}{19} + \frac{99}{19^2} + \frac{999}{19^3} + \cdots \infty\right)$$

$$= \frac{1}{3}\left(\frac{10-1}{19} + \frac{10^2-1}{19^2} + \frac{10^3-1}{19^3} + \cdots \infty\right)$$

$$= \frac{1}{3}\left(\left(\frac{10}{19} + \left(\frac{10}{19}\right)^2 + \cdots \infty\right) - \left(\frac{1}{19} + \frac{1}{19^2} + \cdots \infty\right)\right)$$

$$= \frac{1}{3}\left(\frac{10/19}{1-10/19} - \frac{1/19}{1-1/19}\right) = \frac{19}{54} \text{ Ans}$$

Note : eg: $\underbrace{111}_{3\text{-digit}} = 1 + 10 + 10^2 = \dfrac{1.(10^3-1)}{(10-1)}$ ✓✓

$$\underbrace{111\ldots 1}_{n\text{-times}} = \frac{10^n - 1}{9}$$

Q Prove that $\underbrace{444\ldots 4}_{8\text{ times}} - 8888 = (6666)^2$.

Sol^n

$$44\ldots 4 = 4(11\ldots 1) = \frac{4(10^8-1)}{9}$$

$$8888 = \frac{8(10^4-1)}{9}.$$

$$\underbrace{44\ldots 4}_{8\text{times}} - 8888 = 4\left(\frac{10^8-1}{9}\right) - \frac{8(10^4-1)}{9} = \frac{4}{9}\left((10^8-1) - 2(10^4-1)\right)$$

$$\frac{4}{9}\left(\underbrace{10^8 - 2.10^4 + 1}\right) = \frac{4}{9}(10^4-1)^2$$

$$= 9^{\cancel{2}} \times \frac{4}{\cancel{9}}\left(\frac{10^4-1}{9}\right)^2$$

$$36(1111)^2$$

$$(6666)^2$$

(H.P)

Note : $x = 0.\overline{423} \longrightarrow 0.423\,423\,423\,423 \ldots \infty$

$x = 0.\underbrace{423}\,\underbrace{423}\,423 \ldots \infty$

sub $1000x = 423.\,423\,423\,423 \ldots \infty$

$999x = 423 \Rightarrow x = \dfrac{423}{999} = \dfrac{47}{111}.$

Q The 1^{st} term of an infinite G.P. is 'x' and its sum is 5 then find range of x ?

Sol^n $S_\infty = \dfrac{a}{1-r} \Rightarrow 5 = \dfrac{x}{1-r} \Rightarrow 1 - r = \dfrac{x}{5}$

$r = 1 - \dfrac{x}{5}$; $r \in (-1, 1) - \{0\}$

$-1 < 1 - \dfrac{x}{5} < 1$ but $1 - \dfrac{x}{5} \neq 0.$

$-2 < -\dfrac{x}{5} < 0$ but $x \neq 5 \Rightarrow x \in (0, 10) - \{5\}$ Ans

Q. Evaluate $\displaystyle\sum_{r=1}^{n}\underbrace{\sum_{s=1}^{n}\underbrace{\delta_{rs}}\,2^{r}\cdot 3^{s}}$ where $^{*}\delta_{rs} \to \begin{cases} 1 \; ; \text{if } r=s \\ \underline{0} \; ; \text{if } r\neq s. \end{cases}$

Sol^n $\displaystyle\sum_{r=1}^{n} \underline{2^{r}} \left(\underline{\delta_{r1}}\, 3^{1} + \delta_{r2}\, 3^{2} + \cdots + \delta_{rn}\cdot 3^{n}\right).$

$$\left(\sum_{r=1}^{n}\underbrace{2^{r}\cdot 3^{1}\cdot \delta_{r1}}\right) + \left(\sum_{r=1}^{n}\cdot\underbrace{2^{r}\cdot 3^{2}\cdot \delta_{r2}}\right) + \cdots + \left(\sum_{r=1}^{n}\underbrace{2^{r}\cdot 3^{n}\cdot \delta_{rn}}\right)$$

$$\left(\underbrace{2^{1}\cdot 3^{1}}\cdot\underbrace{\delta_{11}}_{1} + 2^{2}\cdot 3^{1}\cdot\cancel{\delta_{21}}^{\,0} + \cdots \cancel{}^{\,0} + 2^{n}\cdot 3^{1}\cdot\cancel{\delta_{n1}}^{\,0}\right) + \left(\cdots + 2^{2}\cdot 3^{2}\cdot\delta_{22} + \cdots\right) + \cdots + \left(\quad\right)$$

$$6 + 6^{2} + 6^{3} + \cdots + 6^{n}.$$

Q. If the p^{th}, q^{th}, r^{th}, s^{th} terms of an AP are in G.P then $p-q$, $q-r$, $r-s$ are in (A) AP ~~(B)~~ GP (C) HP (D) None

Sol^n

$$T_p = a+(p-1)d. \checkmark$$

$$T_q = a+(q-1)d. \checkmark$$

$$T_r = a+(r-1)d. \checkmark$$

$$T_s = a+(s-1)d. \checkmark$$

$T_p, T_q, T_r, T_s \longrightarrow$ G.P.

$$\frac{T_q}{T_p} = \frac{T_r}{T_q} = \frac{T_s}{T_r} \Rightarrow \frac{T_r - T_q}{T_q - T_p} = \frac{T_s - T_r}{T_r - T_q}$$

$$T_r - T_q = (r-q)d.$$

$$T_q - T_p = (q-p)d.$$

$$T_s - T_r = (s-r)d.$$

$$\frac{(r-q)\cancel{d}}{(q-p)\cancel{d}} = \frac{(s-r)\cancel{d}}{(r-q)\cancel{d}}$$

Note:

$$\frac{a}{b} = \frac{c}{d} = \frac{a-c}{b-d} = \frac{a+c}{b+d} \ast\ast$$

$$\frac{p}{r} = \frac{t}{x} = \frac{p-t}{r-x} = \frac{p+t}{r+x} \checkmark\checkmark$$

Properties of GP :-

① 3 Nos in GP: $\frac{a}{r}, a, ar$ → $C \cdot r = r$

1st term

4 Nos in GP: $\frac{a}{r^3}, \frac{a}{r}, ar, ar^3.$ → $C \cdot r = r^2.$

5 Nos in GP: $\frac{a}{r^2}, \frac{a}{r}, a, ar, ar^2$

② If a, b, c $(\underbrace{a, b, c}_{} > 0)$ are in GP then $\log a, \log b, \log c$ are in A.P.

Proof: $b^2 = ac \Rightarrow$ Take log $\log b^2 = \log(ac)$

$= 2\log b = \log a + \log c.$

If $x_1, x_2, x_3, \ldots, x_n$ are in GP (where $x_i > 0$) then $\log x_1, \log x_2, \ldots \log x_n$ are in A.P.

③ GP_1 : 2, 4, 8, 16, 32, 64 $\quad r_1 = 2$

add '1' : 3, 5, 9, 17, 33, 65 $\quad$ ✗✗ Not GP.

multiply '3' :

GP_2 : 6, 12, 24, 48, 96, ... $\quad$ GP ✓ $\quad r_2 = 2$

If each term of GP be multiplied or divided by same non-zero quantity then resulting sequence is also G.P.

④ 1, 3, 9, 27, 81, 243.

243

243

243

In GP product of k^{th} term from beginning & k^{th} term from end is constant and is equal to product of 1^{st} & last term.

$$\boxed{T_k \, T_{n-k+1} = a.l}$$

$a \rightarrow 1^{st}$ term.

$l \rightarrow$ last term.

(5) $1, 3, 9, \overset{T_K}{27}, 81, 243$

GP shd be +ve ✓✓

$\sqrt{243 \times 3} = 27.$

eg: $2, -4, 8, -16, 32, -64$

$T_K = \sqrt{T_{K-r} \cdot T_{K+r}}$

$\sqrt{8 \times 32} \neq -16.$

(6) GP_1: $2, 4, 8, 16, 32 \longrightarrow c \cdot r_1 = 2$

GP_2: $2^3, 4^3, 8^3, 16^3, 32^3 \longrightarrow$ ✓✓ $c \cdot r_2 = 2^3.$

If each term of GP be raised to same power then resulting sequence is also G.P.

(7) GP_1 : $5, 10, 20, 40, 80 \longrightarrow r_1 = 2$

GP_2 : $3, 9, 27, 81, 243 \longrightarrow r_2 = 3.$

GP_3: $5\times 3, \quad 10\times 9, \quad 20\times 27, \quad 40\times 81, \quad 80\times 243 \longrightarrow$ ✓✓

If $a_1, a_2, \dots a_n$ & $b_1, b_2, \dots b_n$ are 2 GP's

then $\frac{a_1}{b_1}, \frac{a_2}{b_2}, \frac{a_3}{b_3}, \dots \frac{a_n}{b_n} \longrightarrow GP$ ✓✓

$a_1b_1, \quad a_2b_2, \quad a_3b_3, \dots, a_nb_n \longrightarrow GP.$

Q The sum of 1st 3 Consecutive terms of GP is 19 and their product is 216 then find S_n and S_∞ (if it exist).

Sol^n

$\frac{a}{r}$, a, ar.

$\frac{a}{r} \cdot a \cdot ar = 216 \Rightarrow a^3 = 216$

$\Rightarrow \boxed{a = 6}$

$\frac{a}{r} + a + ar = 19 \Rightarrow r = \frac{2}{3}$ or $\frac{3}{2}$

C–I : If $r = \frac{2}{3}$

1st term : $\frac{a}{r} = \frac{6}{2/3} = 9$.

$$S_n = \frac{9\left(1 - \left(\frac{2}{3}\right)^n\right)}{\left(1 - \frac{2}{3}\right)} = 27\left(1 - \left(\frac{2}{3}\right)^n\right)$$

$\boxed{S_\infty = 27}$

C–II If $r = \frac{3}{2}$

1st term : $\frac{a}{r} = \frac{6}{3/2} = 4$

$$S_n = \frac{4\left(\left(\frac{3}{2}\right)^n - 1\right)}{\left(\frac{3}{2} - 1\right)} \Rightarrow S_n = 8\left(\left(\frac{3}{2}\right)^n - 1\right)$$

$S_\infty = \text{dne}$

Q2 If $a_1, a_2, a_3 \ldots$ in GP such that $a_1 + a_2 + a_3 = 13$ and $a_1^2 + a_2^2 + a_3^2 = 91$. Find 1st term and common ratio?

Sol^n

$a_1, a_2, a_3 \longrightarrow$ GP

$\frac{a}{r}, a, ar \longrightarrow$ GP

1st term $= \frac{a}{r}$; common ratio $= r$

1st term $= \frac{a}{r} \Rightarrow$ ① 1 ② 9

$$\left.\begin{aligned} \frac{a}{r} + a + ar &= 13 \\ \left(\frac{a}{r}\right)^2 + a^2 + (ar)^2 &= 91 \end{aligned}\right\} \longrightarrow \begin{aligned} a &= 3 \\ \underbrace{r = 3;\ 1/3}& \end{aligned}$$

9, 3, 1.

OR

1, 3, 9

Geometrical Mean (GM)

If a, b, c are positive nos in GP then 'b' is called GM of 'a' & 'c'. i.e $b^2 = ac \Rightarrow b = (ac)^{1/2} \Rightarrow b = \sqrt{ac}$.

GM of 3 nos $x_1, x_2, x_3 = (x_1 x_2 x_3)^{1/3}$

GM of 'n' nos $x_1, x_2, \ldots, x_n = (x_1 x_2 \cdots x_n)^{1/n}$.

How to insert 'n' GM's between a & b :-

$$a, \underbrace{G_1, G_2, \dots, G_n}_{n \text{ GM's}}, b \longrightarrow \text{GP.}$$

$b \to T_{n+2}$

$$b = a \cdot (R)^{n+2-1} \Rightarrow R = \left(\frac{b}{a}\right)^{\frac{1}{n+1}}$$

Rem

$$G_1 = a R$$
$$G_2 = a R^2$$
$$\vdots$$
$$G_n = a R^n$$

$$G_1 G_2 \dots G_n = a^n \cdot R^{1+2+\dots+n}$$

$$= a^n \cdot \left(\frac{b}{a}\right)^{\frac{1}{\cancel{n+1}} \cdot \frac{n\cancel{(n+1)}}{2}} = a^n \cdot \frac{b^{n/2}}{a^{n/2}}$$

Rem

$$\boxed{\prod_{i=1}^{n} G_i = \left(\sqrt{ab}\right)^n}$$

Product of 'n' GM's between a & b is equal to n^{th} power of single GM bet a & b.

Q Find the product of 4 GM's inserted between 5 and 160?

Sol^n

$$5 \quad \underbrace{G_1 \quad G_2 \quad G_3 \quad G_4} \quad 160$$

$$r = \left(\frac{160}{5}\right)^{\frac{1}{5}}$$

$$\prod_{i=1}^{4} G_i = \left(\sqrt{5 \times 160}\right)^4 \checkmark$$

$$r = (32)^{\frac{1}{5}} = 2$$

$$G_1 = 5(2) \checkmark$$

$$G_2 = 5(2)^2 \checkmark$$

Q If sum of 2 positive nos a, b where $a > b$ is 'n' times their GM then show that $\dfrac{a}{b} = \dfrac{n+\sqrt{n^2-4}}{n-\sqrt{n^2-4}}$?

Sol

<u>Given</u> :

$a+b = n\sqrt{ab}$. $\Rightarrow \dfrac{a+b}{n} = \sqrt{ab} = K$ (say).

$a+b = Kn \rightarrow$ S.O.R

$ab = K^2 \rightarrow$ P.O.R

QE = 0 $\rightarrow a$, $\rightarrow b$ $\quad a > b$ *

$x^2 - (Kn)x + K^2 = 0$

Roots : $\dfrac{Kn \pm \sqrt{(Kn)^2 - 4K^2}}{2}$

$a = \dfrac{Kn + K\sqrt{n^2-4}}{2}$

$b = \dfrac{Kn - K\sqrt{n^2-4}}{2}$

Q If a, b, c are in GP and x, y are respectively AM between a, b and b, c then prove that $\frac{1}{x}+\frac{1}{y}=\frac{2}{b}$.

Sol^n $b^2 = ac$ — ① —

$$x = \frac{a+b}{2} \Rightarrow \frac{1}{x} = \frac{2}{a+b}$$

$$y = \frac{b+c}{2} \Rightarrow \frac{1}{y} = \frac{2}{b+c}$$

$$\frac{1}{x}+\frac{1}{y} = \frac{2}{a+b}+\frac{2}{b+c} = \frac{2(b+c+a+b)}{ab+ac+b^2+bc}$$

$$= \frac{2(2b+a+c)}{ab+bc+2b^2} = \frac{2\cancel{(2b+a+c)}}{b\cancel{(a+c+2b)}}$$

(H.P)

Harmonic Progression (H.P.)

A non-zero sequence is said to be in H.P if the reciprocal of its terms are in A.P. i.e if $a_1, a_2, a_3, \dots, a_n$ are in H.P then

$$\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \dots, \frac{1}{a_n} \text{ are in A.P.}$$

1. No term of H.P can be zero.
2. There is no general formula for finding sum to n-terms of H.P.

Std appearance of H.P :

$$\frac{1}{a}, \frac{1}{a+d}, \frac{1}{a+2d}, \dots, \frac{1}{a+(n-1)d}.$$

* If a, b, c are in H.P then

$\frac{1}{a}, \frac{1}{b}, \frac{1}{c} \longrightarrow$ A.P

$$\boxed{\frac{2}{b} = \frac{1}{a} + \frac{1}{c}}$$ Rem $\Rightarrow$ $$\boxed{b = \frac{2ac}{a+c}}$$ Rem $\Rightarrow$ $$b = \frac{2}{\frac{1}{a}+\frac{1}{c}}$$

$$\frac{1}{b} - \frac{1}{a} = \frac{1}{c} - \frac{1}{b}$$

$$\frac{a-b}{a\cancel{b}} = \frac{b-c}{\cancel{b}c} \Rightarrow \boxed{\frac{c}{a} = \frac{b-c}{a-b}}$$

$$ac - bc = ab - ac$$

$$2ac = b(a+c) \Rightarrow b = \frac{2ac}{a+c}$$

Q. If 3^{rd}, 6^{th} and last term of a H.P are $\frac{1}{3}, \frac{1}{5}, \frac{3}{203}$ find no. of terms & also 10^{th} term of H.P

Sol^n

$$a+2d = 3 \quad \text{—①—}$$
$$a+5d = 5. \quad \text{—②—}$$

$$d = \frac{2}{3}$$
$$a = \frac{5}{3}$$

$$a+(n-1)d = \frac{203}{3} \quad \text{—③—}$$

$$\frac{5}{\cancel{3}}+(n-1)\cdot\frac{2}{\cancel{3}} = \frac{203}{\cancel{3}}$$

$$\boxed{n=100} \text{ Ans}$$

10^{th} term of AP $= a+9d = \frac{5}{3}+9\cdot\frac{2}{3} = \frac{23}{3}$

$$\therefore \boxed{10^{th} \text{ term of H.P} = \frac{3}{23}}$$

Q2 If a, b, c are in H.P then find the value of $\frac{b+a}{b-a} + \frac{b+c}{b-c}$?

Soln

$$b = \frac{2ac}{a+c}$$

$$\frac{b}{a} = \frac{2c}{a+c} \qquad \frac{b}{c} = \frac{2a}{a+c}$$

$$\frac{\frac{b}{a}+1}{\frac{b}{a}-1} + \frac{\frac{b}{c}+1}{\frac{b}{c}-1}$$

$$\frac{\frac{2c}{a+c}+1}{\frac{2c}{a+c}-1} + \frac{\frac{2a}{a+c}+1}{\frac{2a}{a+c}-1}$$

$$= 2 \text{ Ans}$$

Q3 If $a_1, a_2, a_3, \dots, a_n$ are in H.P then prove that

$$\underbrace{a_1 a_2} + a_2 a_3 + a_3 a_4 + \dots + a_{n-1} a_n = \underbrace{(n-1) a_1 a_n}.$$

✓// **

$$\frac{1}{a_n} = \frac{1}{a_1} + (n-1)d.$$

$$\frac{1}{a_n} - \frac{1}{a_1} = (n-1)d$$

$$\frac{a_1 - a_n}{a_1 a_n} = (n-1)d.$$

$$\frac{a_1 - a_n}{d} = (n-1)a_1 a_n$$

Sol^n

$$\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, \dots, \frac{1}{a_n} \longrightarrow AP.$$

$$\frac{1}{a_2} - \frac{1}{a_1} = d. \Rightarrow \frac{a_1 - a_2}{a_1 a_2} = d \Rightarrow$$

$$a_1 a_2 = \frac{a_1 - a_2}{d}.$$

$$a_2 a_3 = \frac{a_2 - a_3}{d}.$$

$$\vdots$$

$$a_{n-1} a_n = \frac{a_{n-1} - a_n}{d}.$$

LHS:

$$a_1 a_2 + a_2 a_3 + \dots + a_{n-1} a_n = \frac{(a_1 - \cancel{a_2}) + (\cancel{a_2} - \cancel{a_3}) + (\cancel{a_3} - \cancel{a_4}) + \dots + (\cancel{a_{n-1}} - a_n)}{d}$$

$$= \frac{(a_1 - a_n)}{d} = (n-1) a_1 a_n.$$

Harmonical Mean (HM)

If a, b, c are in H.P then 'b' is called HM betⁿ a & c.

i.e $\boxed{b = \frac{2ac}{a+c}} \Rightarrow \boxed{b = \frac{2}{\frac{1}{a} + \frac{1}{c}}}$

* HM bet 3 nos $x, y, z = \frac{3}{\frac{1}{x} + \frac{1}{y} + \frac{1}{z}}$.

* HM bet 'n' nos $a_1, a_2, \ldots, a_n = \frac{n}{\frac{1}{a_1} + \frac{1}{a_2} + \cdots + \frac{1}{a_n}}$.

Q If α, β, γ are the roots of cubic $x^3 - 4x^2 + 7x - 8 = 0$ then find HM betn the roots ?

Soln

$$x^3 - 4x^2 + 7x - 8 = 0 \quad \begin{matrix} \alpha \\ \beta \\ \gamma \end{matrix}$$

$$\Sigma\alpha = 4$$

$$\Sigma\alpha\beta = 7$$

$$\alpha\beta\gamma = 8.$$

$$\text{HM of } \alpha, \beta, \gamma = \frac{3}{\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}}$$

$$= \frac{3\alpha\beta\gamma}{\alpha\beta+\beta\gamma+\gamma\alpha}$$

$$= \frac{3(8)}{7} = \frac{24}{7}$$

Ans

How to insert 'n' HM's betn 'a' & 'b' :-

$$a, H_1, H_2, H_3, \ldots, H_n, b \longrightarrow H.P.$$

$$\frac{1}{a}, \frac{1}{H_1}, \frac{1}{H_2}, \frac{1}{H_3}, \ldots, \frac{1}{H_n}, \underbrace{\frac{1}{b}}_{T_{n+2}} \longrightarrow \underline{\underline{A.P}}$$

$$\frac{1}{b} = \frac{1}{a} + (n+2-1)d \Rightarrow \boxed{d = \frac{\frac{1}{b} - \frac{1}{a}}{n+1}}$$ Rem

$$\frac{1}{H_1} = \frac{1}{a} + d.$$

$$\frac{1}{H_2} = \frac{1}{a} + 2d. \quad \ldots \quad \frac{1}{H_n} = \frac{1}{a} + nd.$$

$$\frac{1}{H_1}+\frac{1}{H_2}+\cdots+\frac{1}{H_n}=\frac{n}{a}+d\,(1+2+3+\cdots+n)$$

$$=\frac{n}{a}+\frac{\frac{1}{b}-\frac{1}{a}}{\cancel{(n+1)}}\cdot\frac{n\,\cancel{(n+1)}}{2}$$

$$=\frac{n}{a}+\frac{n}{2b}-\frac{n}{2a}$$

$$\sum_{i=1}^{n}\frac{1}{H_i}=\frac{n}{2a}+\frac{n}{2b}$$

$$=\frac{n}{2}\left(\frac{a+b}{ab}\right)=n\left(\frac{a+b}{2ab}\right)$$

$$\boxed{\sum_{i=1}^{n}\frac{1}{H_i}=\frac{n}{\text{HM bet}^n\ \text{'}a\text{' \& '}b\text{'}}}$$

Rem

Q Between 1 and $\frac{1}{100}$ if 100 HM's are inserted then find $\sum_{i=1}^{100} \frac{1}{H_i} = ?$

Sol^n

$$\sum_{i=1}^{n} \frac{1}{H_i} = \frac{100}{\frac{2(1)\left(\frac{1}{100}\right)}{\left(1+\frac{1}{100}\right)}} = 5050 \text{ Ans}$$

Q If $9(25x^2+y^2)+25(z^2-3xz)=15y(3x+z)$ then $\underbrace{y, z, x}$ are in

~~(A)~~ AP (B) G.P (C) H.P (D) None.

Sol^n

$$225x^2+9y^2+25z^2-75xz-45xy-15yz=0.$$

$$(15x)^2+(3y)^2+(5z)^2-(15x)(5z)-(15x)(3y)-(5z)(3y)=0.$$

$$\downarrow$$

** $A^2+B^2+C^2-AB-BC-CA=0 \Rightarrow \boxed{A=B=C}$

$15x = 3y = 5z = 15k$ (say)

$x = k$; $y = 5k$; $z = 3k$.

$y, z, x \longrightarrow$ AP

** **Q** If $a_1, a_2, a_3, \ldots, a_{10}$ are in AP and $H_1, H_2, \ldots, H_{10}$ are in H.P. and $a_1 = H_1 = 2$ & $a_{10} = H_{10} = 3$ then find value of $a_4 H_7$?

Sol^n

$a_1 = 2 = H_1$

$a_{10} = 3 \Rightarrow a_{10} = a_1 + 9d$

$3 = 2 + 9d$

$\boxed{d = 1/9}$

$a_4 = a_1 + 3d$

$= 2 + 3\left(\frac{1}{9}\right)$

$\boxed{a_4 = 2 + \frac{1}{3} = \frac{7}{3}}$

$H_1 = 2 \quad \& \quad H_{10} = 3$

$\frac{1}{H_1}, \frac{1}{H_2}, \ldots, \frac{1}{H_{10}} \rightarrow AP$

$\frac{1}{H_{10}} = \frac{1}{H_1} + 9d' \Rightarrow \frac{1}{3} = \frac{1}{2} + 9d'$

$\frac{1}{3} - \frac{1}{2} = 9d' \Rightarrow d' = \frac{-1}{54}$

$\frac{1}{H_7} = \frac{1}{H_1} + 6d'$

$= \frac{1}{2} + 6\left(-\frac{1}{54}\right)$

$\frac{1}{H_7} = \frac{1}{2} - \frac{1}{9} = \frac{9-2}{18} = \frac{7}{18}$

$\therefore H_7 = \frac{18}{7}$; $a_4 = \frac{7}{3}$

$\boxed{a_4 H_7 = 6}$

Note: If n AM's & n HM's are inserted betⁿ 2 nos 'a' & 'b' then product of K^{th} AM from beginning & K^{th} HM from end is always constant equal to $\underline{ab}$.

Q Let $A_1, A_2, A_3, ---, A_{11}$ be 11 AM's ✓
$H_1, H_2, H_3, ---, H_{11}$ be 11 HM's ✓
$G_1, G_2, G_3, ---, G_{11}$ be 11 GM's ✓
between two numbers 1 and 9 then find value of $\prod_{K=1}^{5} \left(A_K \, G_{12-2K} \, H_{12-K}\right)$

$\underline{\underline{\text{sol}^n}}$

$$\prod_{k=1}^{5}\left(A_k\, G_{12-2k}\, H_{12-k}\right) = (A_1 G_{10} H_{11})(A_2 G_8 H_{10})(A_3 G_6 H_9)(A_4 G_4 H_8)(A_5 G_2 H_7).$$

$$A_1 H_{11} = A_2 H_{10} = A_3 H_9 = A_4 H_8 = A_5 H_7 = 9.$$

$$9^5\left(G_{10} G_8 G_6 G_4 G_2\right)$$

$$G_2 \quad G_4 \quad G_6 \quad G_8 \quad G_{10}.$$

$$\sqrt{G_4 G_8} = G_6$$

$$\sqrt{G_2 G_{10}} = G_6 \Rightarrow G_6 = \sqrt{9} = 3$$

$$9^5 (G_6)^5 = 9^5 \cdot 3^5$$

$$= 3^{10} \cdot 3^5 = 3^{15} \quad \underline{\underline{\text{Ans}}}$$

Arithmetic - Geometric Progression (AGP) :-

Std appearance

$$a, \quad (a+d)r, \quad (a+2d)r^2, \quad \dots\dots \quad \underbrace{(a+(n-1)d)\,r^{n-1}}_{T_n}$$

Q1 If $|x| < 1$ then compute the sum :

① $S = 1 + 2x + 3x^2 + 4x^3 + \dots\dots \infty$

Sol^n

$$S = 1 + 2x + 3x^2 + 4x^3 + \dots\dots \infty.$$

$$xS = \quad x + 2x^2 + 3x^3 + \dots\dots \infty$$

$$(1-x)S = \underbrace{1 + x + x^2 + x^3 + \dots\dots \infty}_{GP} \Rightarrow (1-x)S = \frac{1}{(1-x)}$$

$$\boxed{S_\infty = \frac{1}{(1-x)^2}} \text{ Ans}$$

(2) $S = 1 + 3x + 6x^2 + 10x^3 + \cdots \infty$

$\underbrace{1+3x}_{2},\ \underbrace{3x+6x^2}_{3},\ \underbrace{6x^2+10x^3}_{4}, \ldots \longrightarrow$ A P.

$$
\begin{aligned}
S &= 1 + 3x + 6x^2 + 10x^3 + \cdots \infty \\
-\ xS &= \phantom{1 +{}} x + 3x^2 + 6x^3 + \cdots \infty \\
\hline
(1-x)S &= 1 + 2x + 3x^2 + 4x^3 + \cdots \infty \longrightarrow \text{AGP} \\
-\ x(1-x)S &= \phantom{1 +{}} x + 2x^2 + 3x^3 + \cdots \infty \\
\hline
(1-x)^2 S &= 1 + x + x^2 + x^3 + \cdots \infty
\end{aligned}
$$

$$\boxed{S = \frac{1}{(1-x)^3}}$$ Ans

Q Find sum to n-terms :

$$S = 1 + \frac{4}{5} + \frac{7}{5^2} + \frac{10}{5^3} + \cdots \text{ n terms.}$$

Sol^n

AP : $1, 4, 7, 10, \ldots \underbrace{1+(n-1)\cdot 3}$

GP : $1, \frac{1}{5}, \frac{1}{5^2}, \ldots, \frac{1}{5^{n-1}}$

n-terms

$n \to n-1$

$$\underbrace{S = 1 + \frac{4}{5} + \frac{7}{5^2} + \frac{10}{5^3} + \cdots + \frac{3n-5}{5^{n-2}}} + \frac{3n-2}{5^{n-1}}$$

$$\frac{S}{5} = \frac{1}{5} + \frac{4}{5^2} + \frac{7}{5^3} + \cdots + \frac{3n-5}{5^{n-1}} + \frac{3n-2}{5^n}$$

n-1 terms

$$\frac{4}{5} S = \underbrace{1 + \left(\frac{3}{5} + \frac{3}{5^2} + \frac{3}{5^3} + \cdots + \frac{3}{5^{n-1}}\right)} - \underbrace{\left(\frac{3n-2}{5^n}\right)}$$

$$\frac{4}{5}S = 1 + \frac{\frac{3}{5}\left(1-\left(\frac{1}{5}\right)^{n-1}\right)}{1-\frac{1}{5}} - \frac{(3n-2)}{5^n}.$$

$$S = \frac{35}{16} - \left(\frac{12n+7}{16 \cdot 5^{n-1}}\right) \quad \text{Ans}$$

AM - GM - HM Inequality

If a, b are 2 positive nos then $AM \geq GM$ where equality holds only when $a = b$.

$$AM = \frac{a+b}{2} \quad ; \quad GM = \sqrt{ab}.$$

$$AM - GM = \frac{a+b}{2} - \sqrt{ab} = \frac{a+b-2\sqrt{ab}}{2} = \frac{(\sqrt{a}-\sqrt{b})^2}{2} \geq 0.$$

$$AM - GM \geq 0 \Rightarrow \boxed{AM \geq GM}$$

$\frac{1}{a}, \frac{1}{b}$

$AM \geq GM$

$$\frac{\frac{1}{a}+\frac{1}{b}}{2} \geq \sqrt{\frac{1}{ab}}.$$

$$\sqrt{ab} \geq \frac{2}{\frac{1}{a}+\frac{1}{b}} \Rightarrow \boxed{GM \geq HM}$$

where equality holds when nos are equal

finally $\boxed{AM \geq GM \geq HM}$

where equality holds when nos are equal.

RMS (Root mean square)

RMS of a & b $= \sqrt{\dfrac{a^2+b^2}{2}}$

RMS of a, b, c $= \sqrt{\dfrac{a^2+b^2+c^2}{3}}$

$$\boxed{RMS \geq AM \geq GM \geq HM}$$ Rem

where equality holds when nos are all equal

For 'n' positive numbers, $RMS \geq AM \geq GM \geq HM$

where equality holds when all numbers are equal.

Relation between AM, GM & HM :-

If a & b are 2 positive nos

$$AM = A = \frac{a+b}{2}$$

$$GM = G = \sqrt{ab}$$

$$HM = H = \frac{2ab}{a+b}$$

$$\boxed{G^2 = AH}$$

Rem

* This relation is valid for 2 nos only. For this relation to be valid for more than 2 nos. those nos must be in G.P.

Q AM between 2 Nos exceeds GM by 10 and exceed HM by 16. then find those 2 nos?

Sol^n $\quad AM - GM = A - G = 10. \Rightarrow G = A - 10$

$A - H = 16. \Rightarrow H = A - 16.$

We know, $\quad G^2 = AH$

$$(A-10)^2 = A(A-16)$$

$$\cancel{A^2} + 100 - 20A = \cancel{A^2} - 16A$$

$$\boxed{A = 25} \Rightarrow \frac{a+b}{2} = 25 \Rightarrow a+b = 50$$

$$ab = 225$$

$G = 15$

$\sqrt{ab} = 15$

45 & 5

Q2 If $\underbrace{x, y, z > 0}$ then find the least value of $\dfrac{(x+y)(y+z)(z+x)}{xyz}$?

Sol^n Consider x & y ; AM $\geq$ GM

$$\boxed{\frac{x+y}{2} \geq \sqrt{xy}} \Rightarrow x+y \geq 2\sqrt{xy} \quad \text{—①—}$$

$$y+z \geq 2\sqrt{yz} \quad \text{—②—}$$

$$z+x \geq 2\sqrt{zx} \quad \text{—③—}$$

||ly

Multiply :

$$(x+y)(y+z)(z+x) \geq 8\sqrt{(xyz)^2}$$

$$(x+y)(y+z)(z+x) \geq 8\,|\underbrace{xyz}_{>0}|.$$

$$\frac{(x+y)(y+z)(z+x)}{xyz} \geq 8 \quad \text{Ans}$$

Q2 If l, m, n are 3 positive roots of $x^3 - ax^2 + bx - 48 = 0$ then find minimum value of $\left(\frac{1}{l} + \frac{2}{m} + \frac{3}{n}\right)$?

Sol^n $x^3 - ax^2 + bx - 48 = 0$ $\begin{cases} l \\ m \\ n \end{cases}$ $\Sigma l = a$; $\Sigma lm = b$; $lmn = 48$

$AM \geq GM$

Consider $\frac{1}{l}, \frac{2}{m}, \frac{3}{n}$

$$\frac{\frac{1}{l} + \frac{2}{m} + \frac{3}{n}}{3} \geq \left(\frac{1}{l} \cdot \frac{2}{m} \cdot \frac{3}{n}\right)^{1/3}$$

$$\frac{1}{l} + \frac{2}{m} + \frac{3}{n} \geq 3\left(\frac{\cancel{6}}{\cancel{48}_{8}}\right)^{1/3} \Rightarrow \boxed{\frac{1}{l} + \frac{2}{m} + \frac{3}{n} \geq \frac{3}{2}}$$

Ans

Q3 If $x, y, z > 0$ then find least value of $(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$?

Sol^n AM $\geq$ HM

$$\frac{(x+y+z)}{3} \geq \frac{3}{\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)} \Rightarrow (x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \geq 9.$$

Ans

Q4 If $a+b+c=3$ & $a, b, c > 0$ then prove that $a^2 b^3 c^2 \leq \frac{3^{10} \cdot 2^4}{7^7}$?

Sol^n

$a \to a/2,\ a/2$

$b \to b/3,\ b/3,\ b/3$

$c \to c/2,\ c/2$

<u>Consider</u> $\frac{a}{2}, \frac{a}{2}, \frac{b}{3}, \frac{b}{3}, \frac{b}{3}, \frac{c}{2}, \frac{c}{2}$..

AM $\geq$ GM

$$\frac{\frac{a}{2}+\frac{a}{2}+\frac{b}{3}+\frac{b}{3}+\frac{b}{3}+\frac{c}{2}+\frac{c}{2}}{7} \geq \left(\frac{a^2 \cdot b^3 \cdot c^2}{2^2 \cdot 3^3 \cdot 2^2}\right)^{\frac{1}{7}}$$

$\left\{ \frac{a}{2} = \frac{b}{3} = \frac{c}{2} = K \right.$ (say) ✓

$a = 2K$; $b = 3K$; $c = 2K$.

$a+b+c = 3 \Rightarrow 7K = 3 \Rightarrow K = 3/7 \in R$

$$\frac{a+b+c}{7} \geq \left(\frac{a^2 b^3 c^2}{2^4 \cdot 3^3}\right)^{1/7}$$

$$\left(\frac{3}{7}\right)^7 \geq \frac{a^2 b^3 c^2}{2^4 \cdot 3^3} \Rightarrow a^2 b^3 c^2 \leq \frac{3^{10} \cdot 2^4}{7^7}. \quad \underline{\underline{(H.P)}}$$

Q If $P(x) = ax^3 - 6ax^2 + bx - 8a$; $a \neq 0$ and $\underbrace{P(x) = 0}$ has $\underline{\underline{\text{positive}}}$ roots then find value of $\frac{b}{2a}$?

$\alpha, \beta, \gamma > 0$.

$\underline{\underline{\text{Sol}^n}}$

$$ax^3 - 6ax^2 + bx - 8a = 0 \quad \alpha, \beta, \gamma$$

$$\underbrace{\Sigma\alpha = 6}; \quad \boxed{\Sigma\alpha\beta = \frac{b}{a}}; \quad \underbrace{\alpha\beta\gamma = 8}$$

$$\boxed{\alpha = \beta = \gamma = 2}$$

$$\Uparrow$$

Consider $\alpha, \beta, \gamma \Rightarrow$ AM $\geq$ GM

$$\frac{\alpha+\beta+\gamma}{3} \geq (\alpha\beta\gamma)^{1/3} \Rightarrow \frac{6}{3} = (8)^{1/3} \Rightarrow AM = GM$$

$$\Sigma \alpha\beta = \frac{b}{a}$$

$$4+4+4 = \frac{b}{a} \Rightarrow \frac{b}{a} = 12$$

$$\boxed{\frac{b}{2a} = 6}$$ Ans

$\alpha \quad \beta \quad \delta \quad \gamma$

Q If the equation $x^4 + ax^3 + bx^2 - 32x + 16 = 0$ has all positive roots then which of the following hold(s) good?

(A) $a = 8$

(B) S.O.R $= 8$

(C) $b = 8$

(D) $b = 24$.

Sol^n

$$x^4 + ax^3 + bx^2 - 32x + 16 = 0 \quad \alpha, \beta, \gamma, \delta$$

$$\Sigma\alpha = -a \; ; \quad \Sigma\alpha\beta = b \; ; \quad \Sigma\alpha\beta\gamma = 32 \; ; \quad \alpha\beta\gamma\delta = 16.$$

$$b = \alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta = 24$$

Consider $\alpha, \beta, \gamma, \delta$

$$GM = (\alpha\beta\gamma\delta)^{1/4} = (16)^{1/4} = 2$$

$$HM = \frac{4}{\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}+\frac{1}{\delta}} = \frac{4\alpha\beta\gamma\delta}{(\beta\gamma\delta+\alpha\gamma\delta+\alpha\beta\delta+\alpha\beta\gamma)}$$

$$= \frac{4\times 16}{32} = 2$$

$$GM = HM \Rightarrow \alpha=\beta=\gamma=\delta=2$$

Q. If a, b, c are three distinct positive reals in H.P. then prove that $a^n + c^n > 2b^n$?

Sol^n a, b, c are in H.P $\Rightarrow$ $\boxed{\text{'b' is HM bet}^n\ a\ \&\ c}$ *

Consider a^n & c^n $\Rightarrow$ AM > GM

$$\frac{a^n + c^n}{2} > \sqrt{a^n c^n} \Rightarrow a^n + c^n > 2(ac)^{n/2} \quad \text{— (1) —}$$

Consider 'a' & 'c'

GM > HM

$$\sqrt{ac} > b \quad ** \Rightarrow (ac)^{\frac{1}{2}} > b. \Rightarrow (ac)^{\frac{n}{2}} > b^n \quad \text{— (2) —}$$

from (1) & (2) $\Rightarrow$ $\boxed{a^n + c^n > 2b^n}$

(A) $a^{15} + c^{15} > 2b^{15}$ ✓

(B) $a^{15} + c^{15} < 2b^{15}$ ✗

(C) $a^{25} + c^{25} > 2b^{25}$ ✓

(D) $a^{25} + c^{25} < 2b^{25}$ ✗

Q If the equations $x^{14} - Kx^4 + x^2 + 2 = 0$ has no real solution then find the greatest possible integral value of K?

Sol^n

$$x^{14} + x^2 + 2 = Kx^4$$

$$x^{10} + \frac{1}{x^2} + \frac{2}{x^4} = K.$$

Consider $x^{10}, \frac{1}{x^2}, \frac{1}{x^4}, \frac{1}{x^4}$

AM $\geq$ GM.

$$\frac{x^{10} + \frac{1}{x^2} + \frac{1}{x^4} + \frac{1}{x^4}}{4} \geq (1.)^{\frac{1}{4}}$$

$$\frac{K}{4} \geq 1 \Rightarrow \boxed{K \geq 4}$$

For No real sol^n. $\Rightarrow K < 4$.

greatest +ve int value of $K = 3$

Ans

$x = \sin\theta$

$x \in (-\infty, -1) \cup (1, \infty)$ for no sol^n

Miscellaneous Sequences :-

Note :- $\displaystyle\sum_{i=1}^{n} i = 1+2+3+\cdots+n.$

Summation

$\displaystyle\prod_{r=1}^{n} (2r) = (2)(4)(6)\ldots(2n).$

Product

Note:

① $\displaystyle\sum_{r=1}^{n} (a_r \pm b_r) = \sum_{r=1}^{n} a_r \pm \sum_{r=1}^{n} b_r$

② $\displaystyle\sum_{r=1}^{n} K\,a_r = K \sum_{r=1}^{n} a_r$

$K \in$ Constant

③ $\displaystyle\sum_{r=1}^{n} K = K \sum_{r=1}^{n} 1 = Kn.$

$K \in$ Constant

S_n = sum to n-terms.

$$S_n = T_1 + T_2 + T_3 + \cdots + T_n$$

$$\boxed{S_n = \sum_{r=1}^{n} T_r = \sum T_n}$$ Rem

Note

① $$\sum_{n=1}^{n} n = 1+2+3+\cdots+n = \frac{n(n+1)}{2}$$

② $$\sum_{n=1}^{n} n^2 = 1^2+2^2+\cdots+n^2 = \frac{n(n+1)(2n+1)}{6}.$$

③ $$\sum_{n=1}^{n} n^3 = 1^3+2^3+\cdots+n^3 = \left(\frac{n(n+1)}{2}\right)^2 = \left(\sum_{n=1}^{n} n\right)^2.$$

Q1 Find S_n if $T_n = n^2 - n + 1$. ?

Sol^n

$$S_n = \sum T_n = \sum n^2 - \sum n + \sum 1$$

$$= \frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2} + n.$$

Q2 $$\underbrace{\sum_{i=1}^{n} \sum_{j=1}^{i} \sum_{k=1}^{j} 1}$$

$$\underbrace{\sum_{i=1}^{n} \sum_{j=1}^{i} (j)} = \sum_{i=1}^{n} \left(1+2+3+\cdots+i \right) = \sum_{i=1}^{n} \left(\frac{i(i+1)}{2} \right)$$

$$= \frac{1}{2} \left(\sum_{i=1}^{n} i^2 + \sum_{i=1}^{n} i \right)$$

$$= \frac{1}{2} \left(\frac{n(n+1)(2n+1)}{6} + \frac{n(n+1)}{2} \right).$$

Q Find the value(s) of positive integer 'n' for which the QE

$$\sum_{K=1}^{n} (x+K-1)(x+K) = 10n$$ has solutions α and $\alpha+1$ for some α?

Sol^n

$$\sum_{K=1}^{n} \left(x^2 + (2K-1)x + (K^2-K) \right) = 10n.$$

$$x^2 \sum_{K=1}^{n} 1 + x \sum_{K=1}^{n} (2K-1) + \left(\sum_{K=1}^{n} K^2 - \sum_{K=1}^{n} K \right) = 10n.$$

$$\cancel{n}x^2 + x(n^{\cancel{2}}) + \frac{\cancel{n}(n+1)(2n+1)}{6} - \frac{\cancel{n}(n+1)}{2} = 10\cancel{n}.$$

$(\because n \in I^+)$

$$3x^2 + 3nx + n^2 - 31 = 0 \quad \begin{cases} \alpha + 1 \ \checkmark \\ \alpha \ \checkmark \end{cases}$$

Diff of roots $= 1 = \dfrac{\sqrt{D}}{|a|}$

$$1 = \frac{\sqrt{9n^2 - 4.(3)(n^2 - 31)}}{3}$$

$$(3)^2 = 9n^2 - 12(n^2 - 31)$$

$$n^2 = 121 \Rightarrow \boxed{n = 11} \text{ or } n = -11 \ \times$$

Q $S = 1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + 3 \cdot 4 \cdot 5 + 4 \cdot 5 \cdot 6 + \cdots$ upto n-terms.

$T_n = n(n+1)(n+2)$.

$S_n = \sum T_n = \sum n(n^2 + 3n + 2)$

$= \sum n^3 + 3\sum n^2 + 2\sum n$.

T-2 Here $T_1, T_2, T_3, \ldots$ are the terms of sequence in which

$\underbrace{T_2 - T_1}_{x}, \underbrace{T_3 - T_2}_{y}, \underbrace{T_4 - T_3}_{z}, \ldots$ are in A.P or G.P.

$\underbrace{x \quad y}_{y-x}, \underbrace{\quad z}_{z-y}, \ldots \longrightarrow$ AP or GP.

Q1 $S = 6 + 13 + 22 + 33 + \ldots$ upto n - terms.

$7, \quad 9, \quad 11, \ldots$ AP.

$$S_n = 6 + 13 + 22 + 33 + \ldots + T_{n-1} + T_n$$

$$S_n = \quad 6 + 13 + 22 + \ldots + T_{n-1} + T_n$$

$$T_n = 6 + \underbrace{7 + 9 + 11 + \ldots + (T_n - T_{n-1})}_{(n-1)\text{ terms}}$$

$$T_n = 6 + \left(\frac{n-1}{\cancel{4}}\right)\left(\cancel{2}\times 7 + (n-1-1)\cdot\cancel{2}\right)$$

$$T_n = 6 + (n-1)(n+5)$$

$$= 6 + n^2 + 5n - n - 5$$

$$\boxed{T_n = n^2 + 4n + 1}$$

$$S_n = \sum T_n = \sum n^2 + 4\sum n + \sum 1$$

Am

Q2 $S = 5 + 7 + 13 + 31 + 85 + \cdots$ upto n-terms.

$\underbrace{5+7}_{2}, \underbrace{7+13}_{6}, \underbrace{13+31}_{18}, \underbrace{31+85}_{54}, \ldots \rightarrow$ GP.

$$S = 5 + 7 + 13 + 31 + 85 + \cdots + T_{n-1} + T_n$$
$$S = \quad 5 + 7 + 13 + 31 + \cdots + T_{n-1} + T_n$$

$$T_n = 5 + (\underbrace{2 + 6 + 18 + 54 + \cdots + T_n - T_{n-1}}_{(n-1)\text{ terms.}})$$

$$= 5 + \cancel{2}\cdot\frac{(3^{n-1} - 1)}{\cancel{(3-1)}} \Rightarrow T_n = 3^{n-1} + 4$$

$$\Sigma T_n = S_n = \Sigma 3^{n-1} + \Sigma 4$$

$$= (\underbrace{3^0 + 3^1 + \cdots + 3^{n-1}}_{\text{GP.}}) + 4n.$$

T-3 In this type of series each term is composed of r factors in AP the first factor of several terms in same AP.

$$S = 1\cdot 2\cdot 3\cdot 4 + 2\cdot 3\cdot 4\cdot 5 + 3\cdot 4\cdot 5\cdot 6 + \cdots\cdots \text{ upto } n\text{-terms.}$$

$$\boxed{T_n = n(n+1)(n+2)(n+3)}$$

$n \to n+1$

$$T_n = \frac{1}{5} n(n+1)(n+2)(n+3)\Big((n+4) - (n-1)\Big)$$

$$T_n = \frac{1}{5}\Big(n(n+1)(n+2)(n+3)(n+4) - (n-1)n(n+1)(n+2)(n+3) \Big)$$

$$\left\{\begin{aligned}
T_1 &= \frac{1}{5}\left(\cancel{1\cdot 2\cdot 3\cdot 4\cdot 5} - 0 \right)\\
T_2 &= \frac{1}{5}\left(\cancel{2\cdot 3\cdot 4\cdot 5\cdot 6} - \cancel{1\cdot 2\cdot 3\cdot 4\cdot 5} \right)\\
T_3 &= \frac{1}{5}\left(\cancel{3\cdot 4\cdot 5\cdot 6\cdot 7} - \cancel{2\cdot 3\cdot 4\cdot 5\cdot 6} \right)\\
&\;\;\vdots\\
T_n &= \frac{1}{5}\left(n(n+1)(n+2)(n+3)(n+4) - \cancel{(n-1)\cdot n(n+1)(n+2)(n+3)} \right)
\end{aligned}\right.$$

$$S_n = \frac{1}{5} n(n+1)(n+2)(n+3)(n+4).$$

T-4 V. Imp

In this series each term is composed of the reciprocal of the product of 'r' factors in AP, the first factor of several terms being in same A.P.

Q1 $$S = \frac{1}{1.2.3.4} + \frac{1}{2.3.4.5} + \frac{1}{3.4.5.6} + \cdots + \frac{1}{n(n+1)(n+2)(n+3)}$$

$$T_n = \frac{1}{n(n+1)(n+2)(n+3)} = \frac{1}{3}\left(\frac{(n+3) - n}{n(n+1)(n+2)(n+3)}\right)$$

$n \to n+1$

$$T_n = \frac{1}{3}\left(\frac{1}{n(n+1)(n+2)} - \frac{1}{(n+1)(n+2)(n+3)}\right)$$

$$
\begin{aligned}
T_1 &= \frac{1}{3}\left(\frac{1}{1.2.3} - \cancel{\frac{1}{2.3.4}}\right) \\
T_2 &= \frac{1}{3}\left(\cancel{\frac{1}{2.3.4}} - \cancel{\frac{1}{3.4.5}}\right) \\
&\vdots \\
T_n &= \frac{1}{3}\left(\cancel{\frac{1}{n(n+1)(n+2)}} - \frac{1}{(n+1)(n+2)(n+3)}\right)
\end{aligned}
$$

$$
S_n = \frac{1}{3}\left(\frac{1}{6} - \frac{1}{(n+1)(n+2)(n+3)}\right)
$$

as $n \to \infty$

$$
\boxed{S_\infty = \frac{1}{18}}
$$

Q. Find the absolute term in QE

$$\sum_{K=1}^{n}\left(x-\frac{1}{K+1}\right)\left(x-\frac{1}{K}\right) \quad \text{as } n \to \infty.$$

Sol^n

$$\text{Absolute term} = \sum_{K=1}^{n}\left(\frac{1}{K(K+1)}\right) = \sum_{K=1}^{n}\left(\frac{K+1-K}{K(K+1)}\right)$$

$$\text{Absolute term} = \sum_{K=1}^{n}\left(\frac{1}{K}-\frac{1}{K+1}\right)$$

$$\cancel{\frac{1}{1}} - \cancel{\frac{1}{2}}$$
$$\cancel{\frac{1}{2}} - \cancel{\frac{1}{3}}$$
$$\vdots$$
$$\cancel{\frac{1}{n}} - \frac{1}{n+1}$$

$$= \left(1-\frac{1}{n+1}\right)$$

$$n \to \infty$$

$$\text{Absolute term} = 1$$

Q3 Find S_n if $t_n = \dfrac{n}{n^4+n^2+1}$?

Note :-

** $n^4+n^2+1 = (n^4+2n^2+1) - n^2$

$= (n^2+1)^2 - (n)^2$

$= (n^2+1+n)\,(n^2+1-n)$

Solⁿ

$$t_n = \frac{n}{(n^2+n+1)(n^2-n+1)}$$

$2n$

$$= \frac{1}{2}\left(\frac{(n^2+1+n) - (n^2-n+1)}{(n^2+n+1)\,(n^2-n+1)}\right)$$

$$t_n = \frac{1}{2}\left(\frac{1}{n^2-n+1} - \frac{1}{n^2+n+1}\right)$$

$$\boxed{S_n = \frac{1}{2}\left(1 - \frac{1}{n^2+n+1}\right)}$$

$\frac{1}{2}\left(\frac{1}{1} - \cancel{\frac{1}{3}}\right)$

$\frac{1}{2}\left(\cancel{\frac{1}{3}} - \cancel{\frac{1}{7}}\right)$

$\frac{1}{2}\left(\cancel{\frac{1}{7}} - \frac{1}{13}\right)$

$\vdots$

$\frac{1}{2}\left(\cancel{\frac{1}{(\;)}} - \frac{1}{(\;)}\right)$

Q4 $S = \frac{3}{1.2.4} + \frac{4}{2.3.5} + \frac{5}{3.4.6} + \cdots$ upto n- terms.

Sol^n

$$T_n = \frac{(n+2)}{n(n+1)(n+3)} \Rightarrow T_n = \frac{(n+2)}{n(n+1)(n+2)(n+3)}$$

(n^2+4n+4)

$$T_n = \frac{(n^2+4n+3)+1}{n(n+1)(n+2)(n+3)}$$

$$T_n = \frac{\cancel{(n+1)}\cancel{(n+3)}}{n\cancel{(n+1)}(n+2)\cancel{(n+3)}} + \frac{1}{n(n+1)(n+2)(n+3)}$$

(already done)

$$T_n = \left\{\frac{1}{n(n+2)}\right\} + \left\{\frac{1}{n(n+1)(n+2)(n+3)}\right\} \Rightarrow \boxed{T_n = t_n + t_n'}$$

$$\Sigma T_n = \Sigma t_n + \Sigma t_n'$$

$$t_n = \frac{1}{n(n+2)} = \frac{(n+2)-n}{n(n+2)}$$

$$t_n = \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+2}\right)$$

$$t_1 = \frac{1}{2}\left(\frac{1}{1} - \cancel{\frac{1}{3}}\right)$$

$$t_2 = \frac{1}{2}\left(\frac{1}{2} - \frac{1}{4}\right)$$

$$t_3 = \frac{1}{2}\left(\cancel{\frac{1}{3}} - \frac{1}{5}\right)$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$t_{n-1} = \frac{1}{2}\left(\frac{1}{n-1} - \frac{1}{n+1}\right)$$

$$t_n = \frac{1}{2}\left(\frac{1}{n} - \frac{1}{n+2}\right)$$

$$\Sigma t_n = \frac{1}{2}\left(1+\frac{1}{2} - \left(\underbrace{\frac{1}{n+1} + \frac{1}{n+2}}\right)\right)$$

Q If $\sum_{K=1}^{\infty} \frac{1}{(K+2)\sqrt{K} + K\sqrt{K+2}} = \frac{\sqrt{a}+\sqrt{b}}{\sqrt{c}}$ where $a, b, c \in N$ and $a, b, c \in [1, 15]$ then find $(a+b+c)$?

$\boxed{a+b+c=11}$

Sol^n

$$\sum_{K=1}^{\infty} \frac{1}{\sqrt{K}\sqrt{K+2}\left(\sqrt{K+2}+\sqrt{K}\right)} \frac{\left(\sqrt{K+2}-\sqrt{K}\right)}{\left(\sqrt{K+2}-\sqrt{K}\right)}$$

$$\sum_{K=1}^{\infty} \frac{\sqrt{K+2}-\sqrt{K}}{2\left(\sqrt{K}\sqrt{K+2}\right)} = \sum_{K=1}^{\infty} \frac{1}{2}\left(\frac{1}{\sqrt{K}} - \frac{1}{\sqrt{K+2}}\right)$$

$$T_1 = \frac{1}{2}\left(\frac{1}{\sqrt{1}} - \cancel{\frac{1}{\sqrt{3}}}\right)$$

$$= \frac{1}{2}\left(\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{4}}\right)$$

$$= \frac{1}{2}\left(\cancel{\frac{1}{\sqrt{3}}} - \cancel{\frac{1}{\sqrt{5}}}\right)$$

$\vdots \quad \vdots \quad \vdots$

$$S = \frac{1}{2}\left(\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{K+1}} - \frac{1}{\sqrt{K+2}}\right)$$

$$S_{\infty} = \frac{1}{2}\left(\frac{\sqrt{2}+1}{\sqrt{2}}\right) = \frac{\sqrt{2}+1}{\sqrt{8}}$$

Q. If sum of 1st n terms of A.P is $S_n = 3n^2 - 2n \quad \forall n \in N$ then find value of

$$\sum_{n=1}^{\infty} \frac{21}{(S_n S_{n+2} + S_{n-1} S_{n+1}) - (S_n S_{n+1} + S_{n-1} S_{n+2})} \ ?$$

Sol^n

$$\sum_{n=1}^{\infty} \frac{21}{S_n(\underbrace{S_{n+2} - S_{n+1}}_{T_{n+2}}) - S_{n-1}(\underbrace{-S_{n+1} + S_{n+2}}_{T_{n+2}})}$$

$$\sum_{n=1}^{\infty} \frac{21}{T_{n+2}(\underbrace{S_n - S_{n-1}}_{T_n})} = \sum_{n=1}^{\infty} \frac{21}{T_n \cdot T_{n+2}}$$

$S_n = 3n^2 - 2n$

$T_n = S_n - S_{n-1}$

$\boxed{T_n = 6n - 5}$

$T_{n+2} = 6(n+2) - 5$

$= 6n + 7.$

$$\sum_{n=1}^{\infty} \frac{21}{(6n-5)(6n+7)}$$

$$\frac{\cancel{21}}{\cancel{12}} \sum_{n=1}^{\infty} \left(\frac{(6n+7)-(6n-5)}{(6n-5)(6n+7)} \right)$$

$$\frac{7}{4} \sum_{n=1}^{\infty} \left(\frac{1}{6n-5} - \frac{1}{6n+7} \right)$$

$$\boxed{\text{Ans} = 2}$$

Q Let $S_n = \sum_{K=1}^{4n} (-1)^{\frac{K(K+1)}{2}} \frac{K(K+1)}{2}$ then S_n can be

(A) 480
~~(B)~~ 2112
~~(C)~~ 1860
~~(D)~~ 840.

Sol^n

$$S_n = \frac{1}{2}\left(\sum_{K=1}^{4n} (-1)^{\frac{K(K+1)}{2}} \cdot K^2 + \sum_{K=1}^{4n} (-1)^{\frac{K(K+1)}{2}} \cdot K\right)$$

$$S_n = \frac{1}{2}\left[\left(\underbrace{-1^2 - 2^2 + 3^2 + 4^2 - 5^2 - 6^2 + 7^2 + 8^2 - \cdots}_{4n \text{ terms}}\right) + \left(\underbrace{-1 - 2 + 3 + 4 - 5 - 6 + 7 \cdots}_{4n \text{ terms}}\right)\right]$$

$$= \frac{1}{\cancel{2}}\left[\cancel{2}\left(1 + 2 + 3 + 4 + \cdots + 4n\right) + \cancel{2}(2n)\right]$$

$$= \frac{2\,\cancel{4}n(4n+1)}{\cancel{2}} + 2n$$

$$S_n = 2n(4n+2)$$

$$\boxed{S_n = 4n(2n+1)}$$

$n = 10$
$n = 15$
$n = 16$

Logarithms

Every positive real number N can be expressed in exponential form

as $\boxed{N = a^x}$

$N > 0$;

$a \rightarrow$ Base

$a \in (0, \infty) - \{1\}$

$x \rightarrow$ exponent or power.

$$\boxed{N = a^x \Longleftrightarrow \log_a N = x}$$

where $N > 0$.
$a > 0$ & $a \neq 1$.

Rem

Note : $\log_1 N$ ✗✗ Non-sense.

$\log_1 1$ ✗✗ Non-sense.

$N = a^x \iff \log_a N = x$

eg: $\log_2 16 = ?$

$\log_2 16 = x$

$16 = 2^x \Rightarrow 2^4 = 2^x$

$\therefore \boxed{x = 4}$

eg: $\log_2 1 = ?$

$\log_2 1 = x \Rightarrow 2^x = 1 = 2^0$

$\therefore \boxed{x = 0}$

Note:-

① $\boxed{a^{\log_a N} = N}$ **Rem**

$2^{\log_2 5} = 5.$

Proof:

$N = a^x.$

$\log_a N = x$

$a^{\log_a N} = a^x = N.$ **(M.P)**

② $\log_N N = 1$; $N > 0$ & $N \neq 1.$

③ $\log_{\frac{1}{N}}(N) = -1$; $N > 0$ & $N \neq 1.$

④ $\log_a 1 = 0$

$a > 0$ & $a \neq 1$

eg: $\log_{(2-\sqrt{3})}(2+\sqrt{3}) = -1$

$(2+\sqrt{3})(2-\sqrt{3}) = 1$

$$\left.\begin{aligned} 2+\sqrt{3} &= N \\ 2-\sqrt{3} &= \frac{1}{N} \end{aligned}\right\}$$

eg: $\log_{10}(0.\bar{9}) = \,?$

$\log_{10}(1) = 0.$

Note:

$$\begin{aligned} x &= 0.999\ldots9 \\ 10x &= 9.999\ldots9 \\ \hline 9x &= 9 \Rightarrow x = 1 \end{aligned}$$

Note: The Number 'N' is called antilog of 'x' to base 'a'

$N = a^x \implies \log_a N = x$

$$\boxed{\text{antilog}_a\, x = N}$$

$\rightarrow a^x = N$

eg: $\text{antilog}_2 5 = ?$

$$(2)^5 = 32.$$

Properties of log :-

① $\log_a(m) + \log_a(n) = \log_a(mn)$; $m > 0$; $n > 0$; $a > 0$ & $a \neq 1$

* $\log_a(mn) = \log_a|m| + \log_a|n|$

② $\log_a m - \log_a n = \log_a\left(\frac{m}{n}\right)$; $m > 0$; $n > 0$
$a > 0$ & $a \neq 1.$

③ $\log_a (m)^p = p \cdot \log_a(|m|)$

$**$

eg: ① $\log_2(x^2-1) = 3$

$x^2 - 1 = 2^3$

$x^2 = 9$

$x = 3$ or $x = -3$

Check !!!

② $\log_2(x-1) + \log_2(x+1) = 3.$

$\log_2(x^2-1) = 3.$

$x^2 - 1 = 8$

$x = 3$ or $x = -3$ XX

Check !!!

Base changing theorem :-

Rem $$\boxed{\log_b a = \frac{\log_c a}{\log_c b}}$$

$(c > 0 ;\ c \neq 1)$

$$* \quad \boxed{\log_b a = \frac{1}{\log_a b}}$$

$$\left.\begin{aligned} &\log_b a \cdot \log_a b = 1 \\ &\frac{\cancel{\log_c a}}{\cancel{\log_c b}} \cdot \frac{\cancel{\log_c b}}{\cancel{\log_c a}} = 1. \end{aligned}\right\}$$

eg: $\log_b a \cdot \log_c b \cdot \log_d c = \log_d a$.

$$\frac{\log a}{\cancel{\log b}} \cdot \frac{\cancel{\log b}}{\cancel{\log c}} \cdot \frac{\cancel{\log c}}{\log d} = \log_d a.$$

Note: $\log_e 2 = \ln 2 = \log_e 2$

$* \ \log_{10} x - 1 \rightarrow (\log_{10} x) - 1$

Different $\log_{10}(x-1)$

Note :-

Rem ① $a^{\log_c b} = b^{\log_c a}$

Proof : $a^{\log_c b} = a^{(\log_a b \cdot \log_c a)}$

$= \left(a^{\log_a b}\right)^{\log_c a} = (b)^{\log_c a}$ (H.P)

★ ② $\boxed{a^{\sqrt{\log_a b}} = b^{\sqrt{\log_b a}}}$

$a^{\log_a b} = b$ *

$x = \log_a b$

Proof : $a^{\sqrt{\log_a b}} = a^{\sqrt{x} \cdot \frac{\sqrt{x}}{\sqrt{x}}} = \left(a^{x}\right)^{\frac{1}{\sqrt{x}}} = \left(a^{\log_a b}\right)^{\frac{1}{\sqrt{\log_a b}}} = (b)^{\sqrt{\log_b a}}$ HP