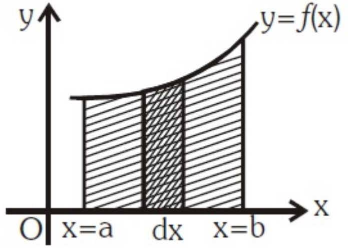
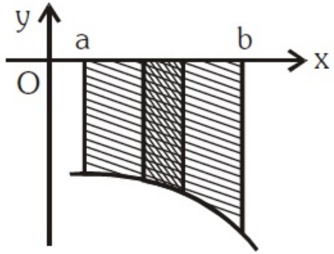
1. **AREA UNDER THE CURVES (given by cartesian equation) :**

(a) Area bounded by the curve, the x-axis and the ordinate at x = a and x = b is given by $A = \int_a^b y\,dx$, where $y = f(x)$ lies above the x-axis and b > a.

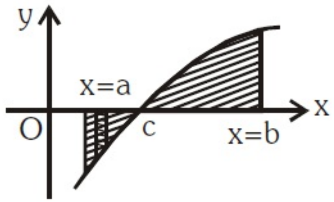
Here vertical strip of thickness dx is considered at distance x.



(b) If $y = f(x)$ lies completely below the x-axis then $A = \left|\int_a^b y\,dx\right|$

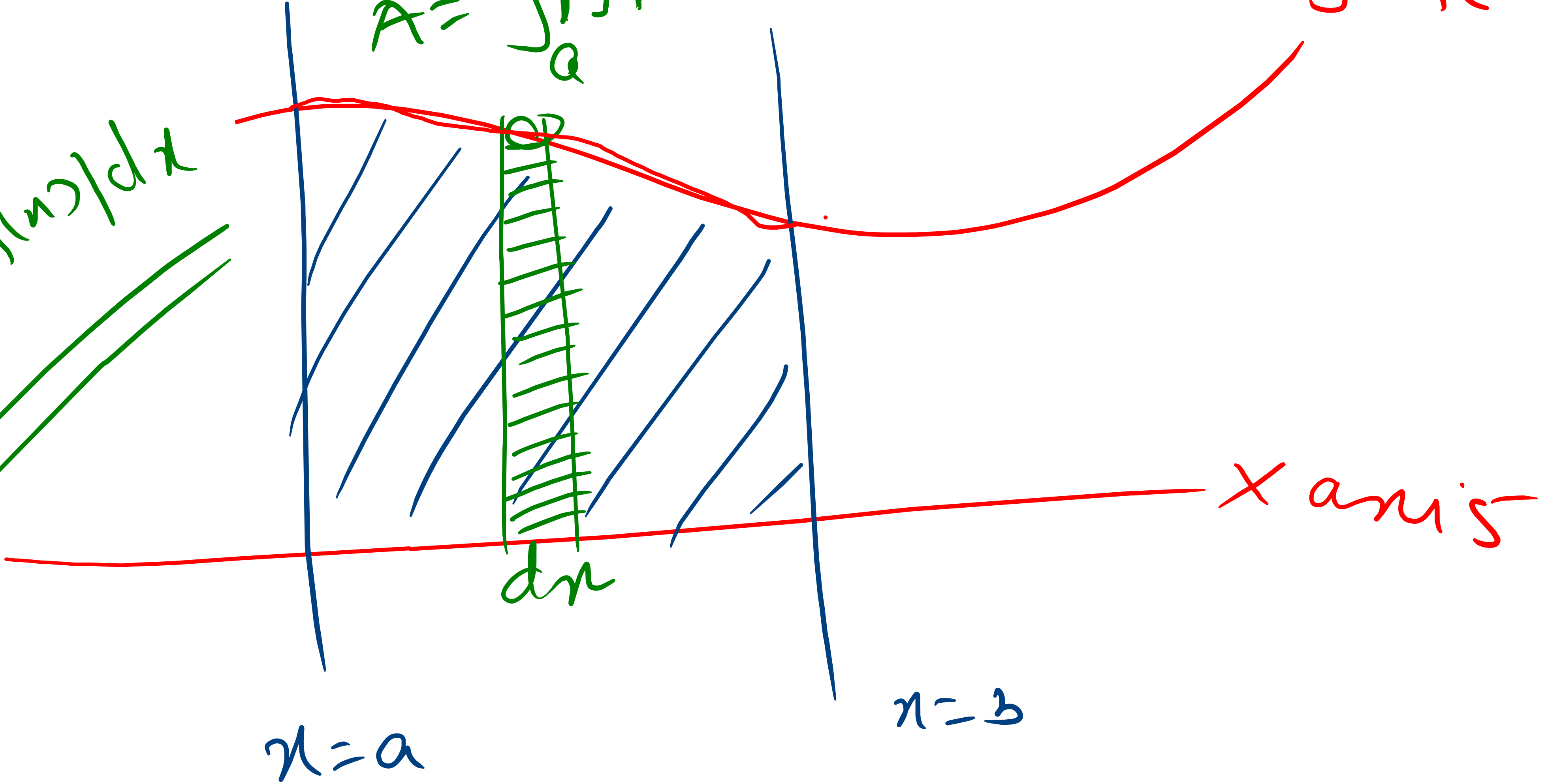


(c) If curve crosses the x-axis at x = c, then $A = \left|\int_a^c y\,dx\right| + \int_c^b y\,dx$



Area
- using x axis
- using y-axis
- B/w 2 (or more than 2) curves

$$A = \int_a^b |y|\,dx = \int_a^b |f(x)|\,dx$$



Q. Let f(x) be a non-negative continuous function such that the area bounded by the curve y= f(x), x-axis and the ordinates $x = \frac{\pi}{4}$ and $x = \beta > \frac{\pi}{4}$ is $\left(\beta \sin\beta + \frac{\pi}{4}\cos\beta + \sqrt{2}\beta\right)$. Then $f\left(\frac{\pi}{2}\right)$ is- **[AIEEE-2005]**

(A) $\left(\frac{\pi}{4} + \sqrt{2} - 1\right)$ (B) $\left(\frac{\pi}{4} - \sqrt{2} + 1\right)$ (C) $\left(1 - \frac{\pi}{4} - \sqrt{2}\right)$ (D) $\left(1 - \frac{\pi}{4} + \sqrt{2}\right)$

$$\int_{\pi/4}^{\beta} |f(x)|\, dx = \beta \sin\beta + \frac{\pi}{4}\cos\beta + \sqrt{2}\beta$$

Differentiate w.r.t. β

$$\Rightarrow \quad f(\beta) = 1\cdot \sin\beta + \beta\cos\beta + \frac{\pi}{4}(-\sin\beta) + \sqrt{2}$$

$$f\left(\frac{\pi}{2}\right) = 1 + 0 - \frac{\pi}{4} + \sqrt{2}$$

Consider the functions defined implicitly by the equation $y^3 - 3y + x = 0$ on various intervals in the real line.

For Q 1 to 3

If $x \in (-\infty, -2) \cup (2, \infty)$, the equation implicitly defines a unique real valued differentiable function $y = f(x)$.

If $x \in (-2, 2)$, the equation implicitly defines a unique real valued differentiable function $y = g(x)$ satisfying $g(0) = 0$.

Q.1 If $f(-10\sqrt{2}) = 2\sqrt{2}$, then $f''(-10\sqrt{2}) =$ **[JEE 2008, 4]**

(A) $\frac{4\sqrt{2}}{7^3 3^2}$ (B) $-\frac{4\sqrt{2}}{7^3 3^2}$ (C) $\frac{4\sqrt{2}}{7^3 3}$ (D) $-\frac{4\sqrt{2}}{7^3 3}$

Q.2 The area of the region bounded by the curve $y = f(x)$, the x-axis, and the lines $x = a$ and $x = b$, where $-\infty < a < b < -2$, is **[JEE 2008, 4]**

(A) $\int_a^b \frac{x}{3\left((f(x))^2 - 1\right)} dx + b f(b) - a f(a)$

(B) $-\int_a^b \frac{x}{3\left((f(x))^2 - 1\right)} dx + b f(b) - a f(a)$

(C) $\int_a^b \frac{x}{3\left((f(x))^2 - 1\right)} dx - b f(b) + a f(a)$

(D) $-\int_a^b \frac{x}{3\left((f(x))^2 - 1\right)} dx - b f(b) + a f(a)$

Q.3 $\int_{-1}^{1} g'(x)\, dx =$ **[JEE 2008, 4]**

(A) $2g(-1)$ (B) 0 (C) $-2g(1)$ (D) $2g(1)$

Q4

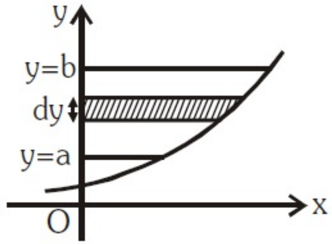
Column-I	**Column-II**
(A) The cosine of the angle between the curves $y = 3^{x-1} \ln x$ and $y = x^x - 1$ at their point of intersection on the line $y = 0$, is	(P) 0
(B) The area bounded by the curves $x = -4y^2$ and $(x-1) = -5y^2$ is	(Q) 1
(C) The value of the integral $\int_0^{\pi/2} (\sin x)^{\cos x}(\cos x \cot x - \ln(\sin x)^{\sin x})\,dx$, is	(R) $\frac{4}{3}$
(D) A continuous function $f : [1, 6] \to [0, \infty)$ is such that $f'(x) = \frac{2}{x + f(x)}$ and $f(1) = 0$, then the maximum value of f cannot exceed	(S) $2 \ln 6$

[JEE 2006, 6]

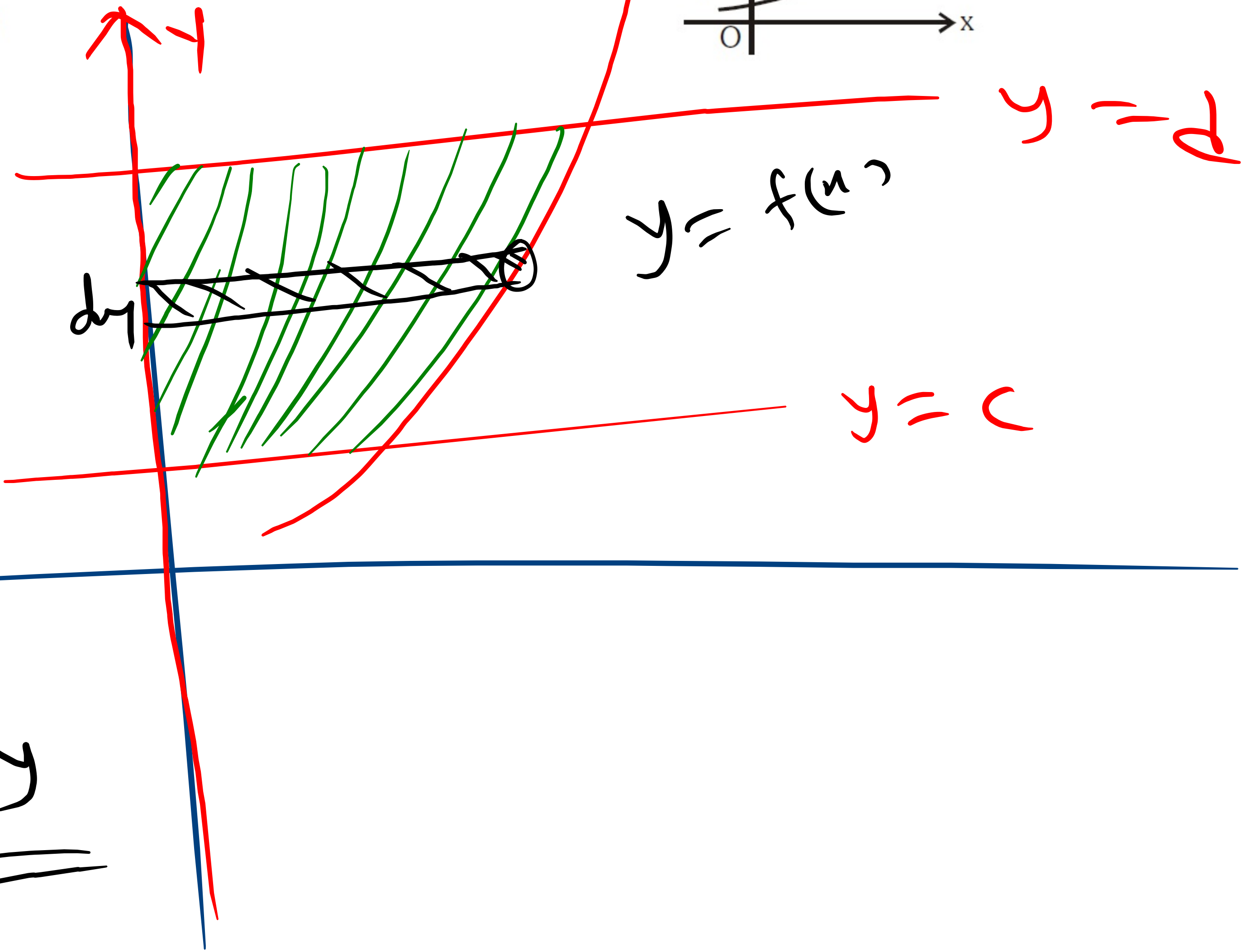
(d) Sometimes integration w.r.t. y is very useful i.e. (horizontal strip) :

Area bounded by the curve, y-axis and the two abscissa at

$y = a$ & $y = b$ is written as $A = \int_a^b x\,dy$



$$A = \int_c^d |x|\,dy$$



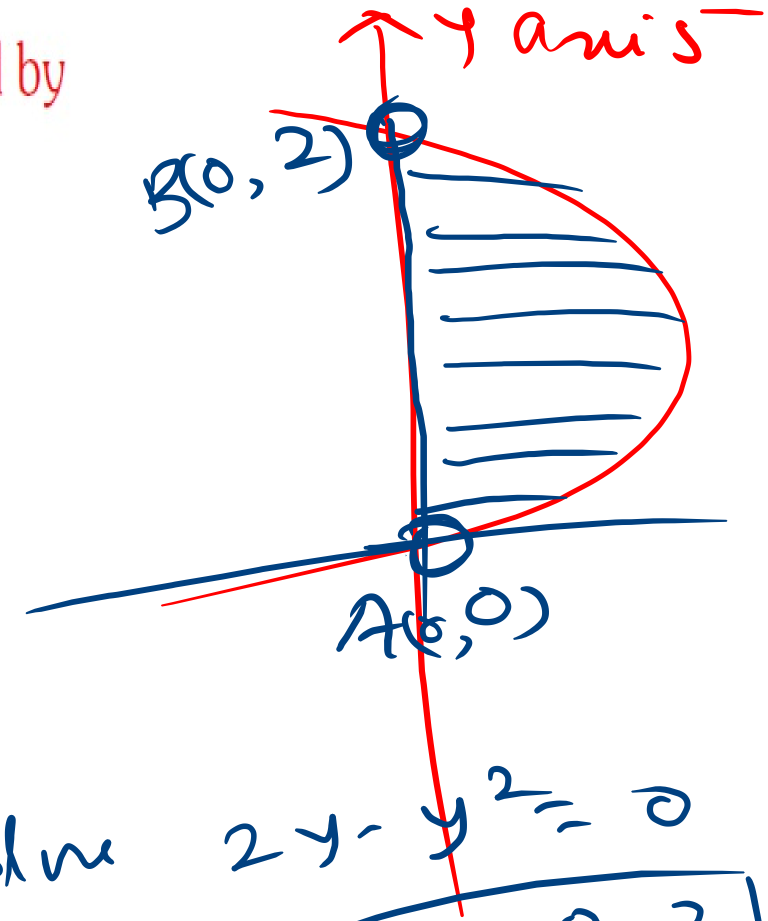
Convert/express

x in terms of y

Q: Area bounded by x = 2y – y² and the y-axis or area bounded by y = 2x – x² and x-axis

$$\int_0^2 |x|\, dy$$

$$= \int_0^2 |2y-y^2|\, dy$$



Solve $2y-y^2=0$

$\Rightarrow \boxed{y=0,2}$

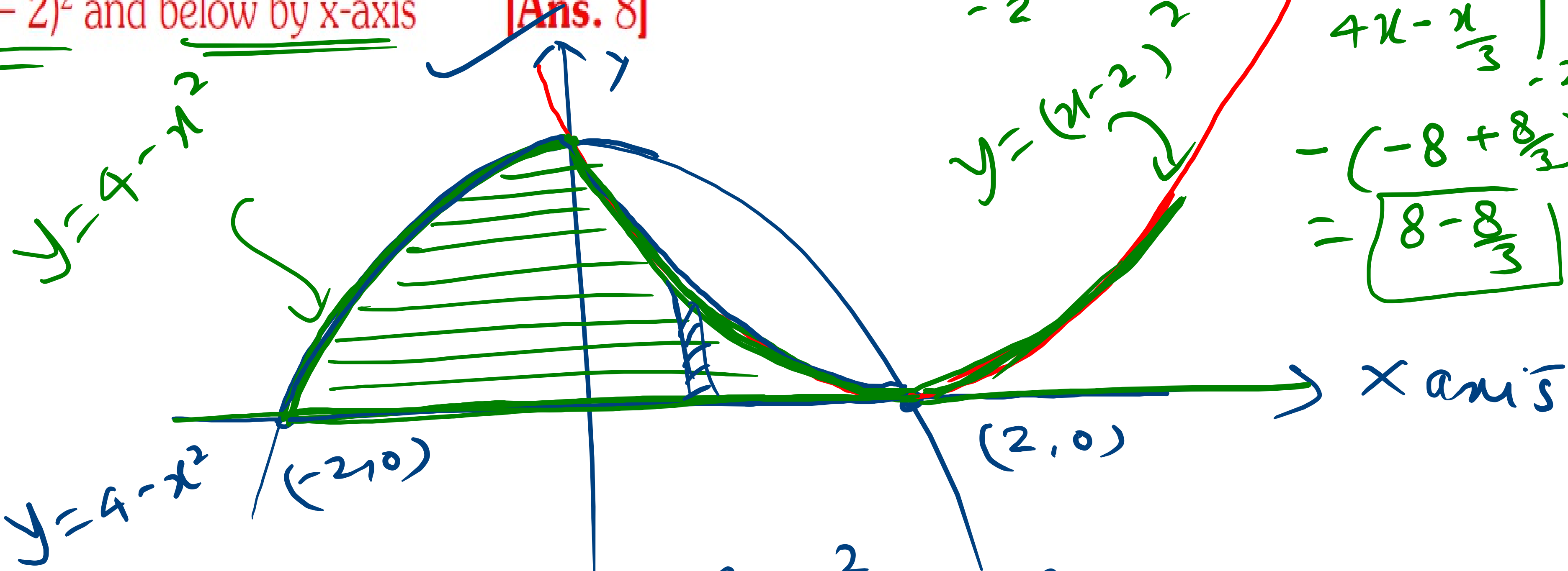
$$\boxed{|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}}$$

Q. Area of region that is bounded left by $y = 4 - x^2$, right by $y = (x-2)^2$ and below by x-axis [Ans. 8]

$$A_1 = \int_{-2}^{0} (4 - x^2)\,dx = 4x - \frac{x^3}{3}\Big|_{-2}^{0}$$

$$-\left(-8 + \frac{8}{3}\right)$$

$$= \boxed{8 - \frac{8}{3}}$$

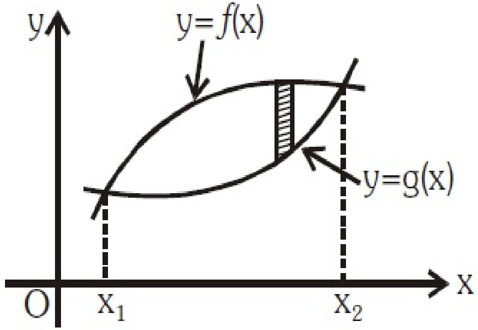


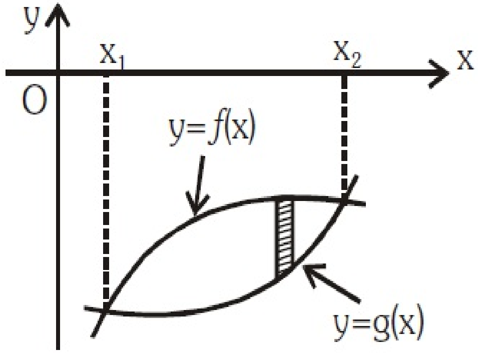
$$A_2 = \int_0^2 (x-2)^2\,dx = \frac{(x-2)^3}{3}\Big|_0^2 = +\frac{8}{3}$$

$$A = A_1 + A_2 = 8 - \cancel{\frac{8}{3}} + \cancel{\frac{8}{3}}$$

2. AREA ENCLOSED BETWEEN TWO CURVES :

(a) Area bounded by two curves $y = f(x)$ & $y = g(x)$ such that $f(x) > g(x)$ is



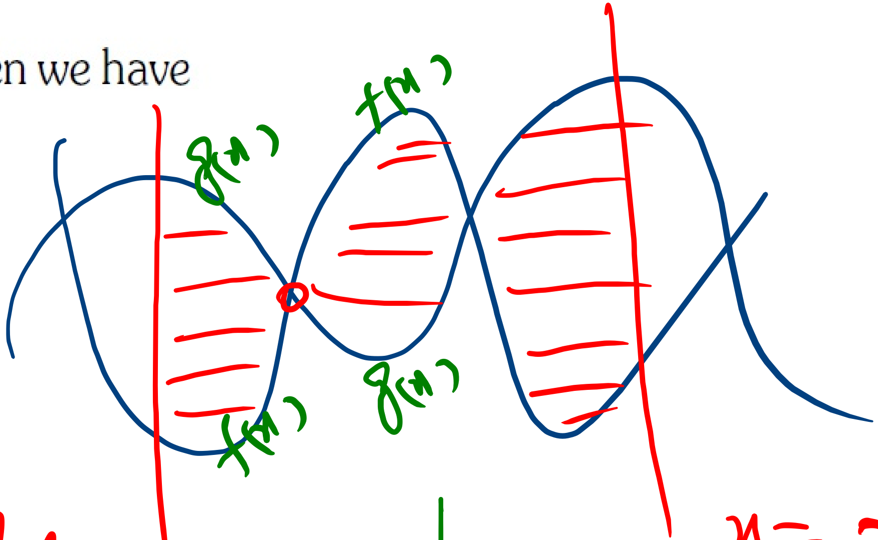


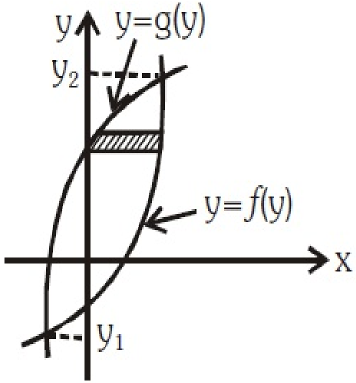
$$A = \int_{x_1}^{x_2} [f(x) - g(x)]dx$$

(b) In case horizontal strip is taken we have

$$A = \int_{y_1}^{y_2} (x_2 - x_1)dy$$

$$A = \int_{y_1}^{y_2} [f(y) - g(y)]dy$$





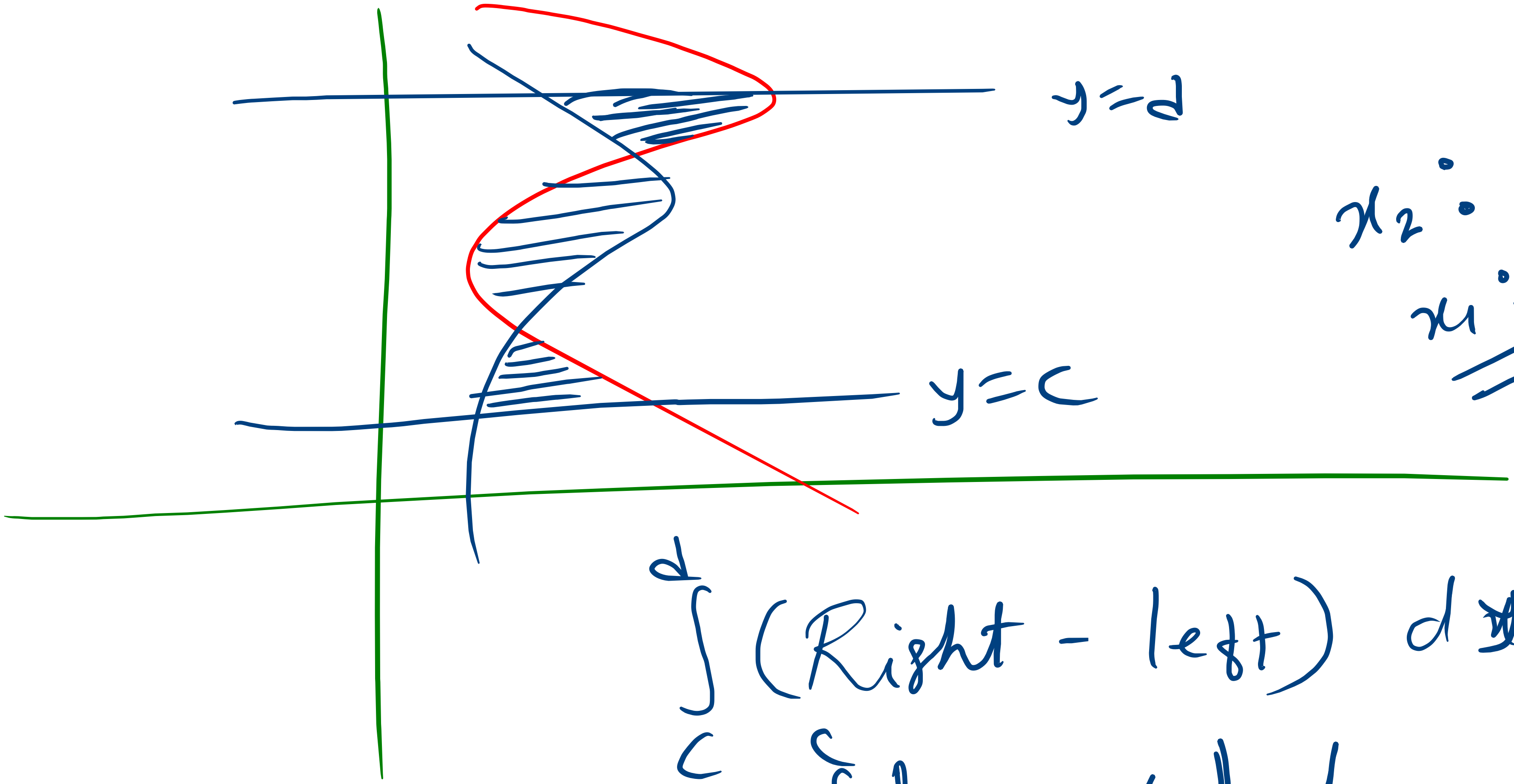
$$\int_{x_1}^{x_2} (\text{upper} - \text{lower})\,dx$$

Vertical strip

$x = x_1$

$x = x_2$

$$\int_a^b |f(x) - g(x)|\,dx$$



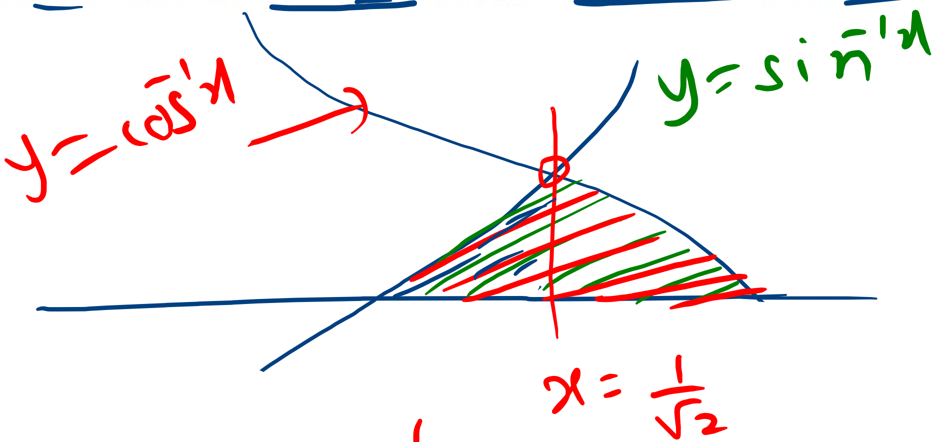
$x_2 : y = f(x)$

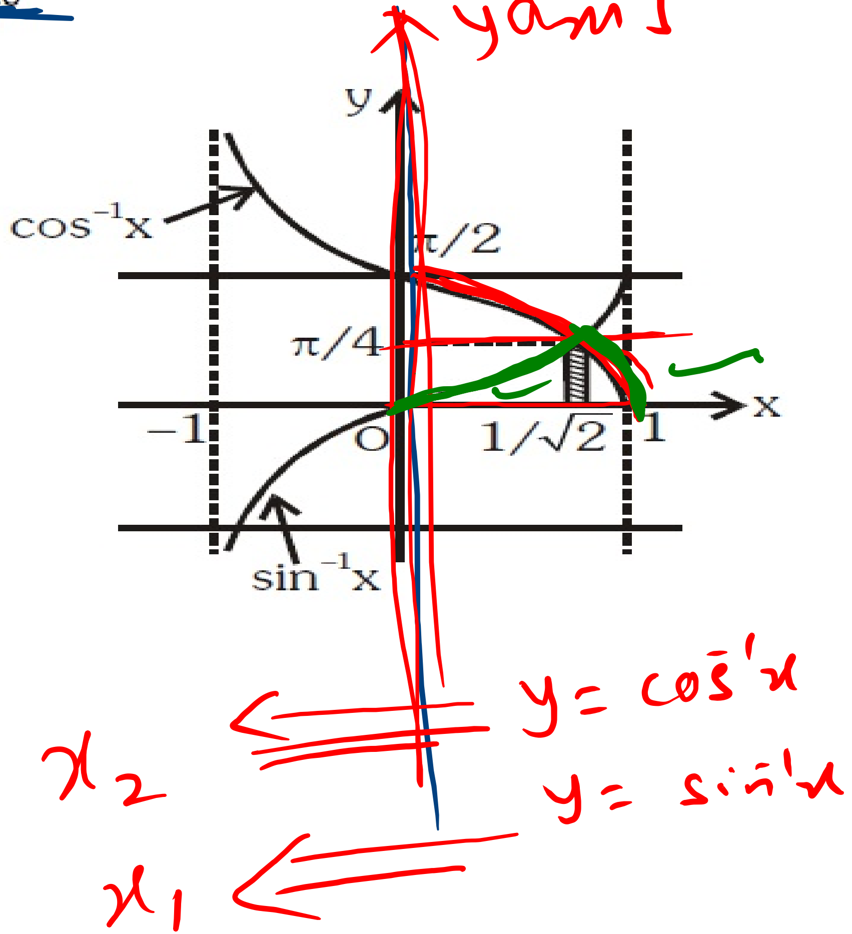
$x_1 : y = g(x)$

$$\int_c^d (\text{Right} - \text{left})\, dy$$

$$\int_c^c |x_2 - x_1|\, dy$$

Q: Find the area bounded by $y = \sin^{-1}x$; $y = \cos^{-1}x$ and the x-axis

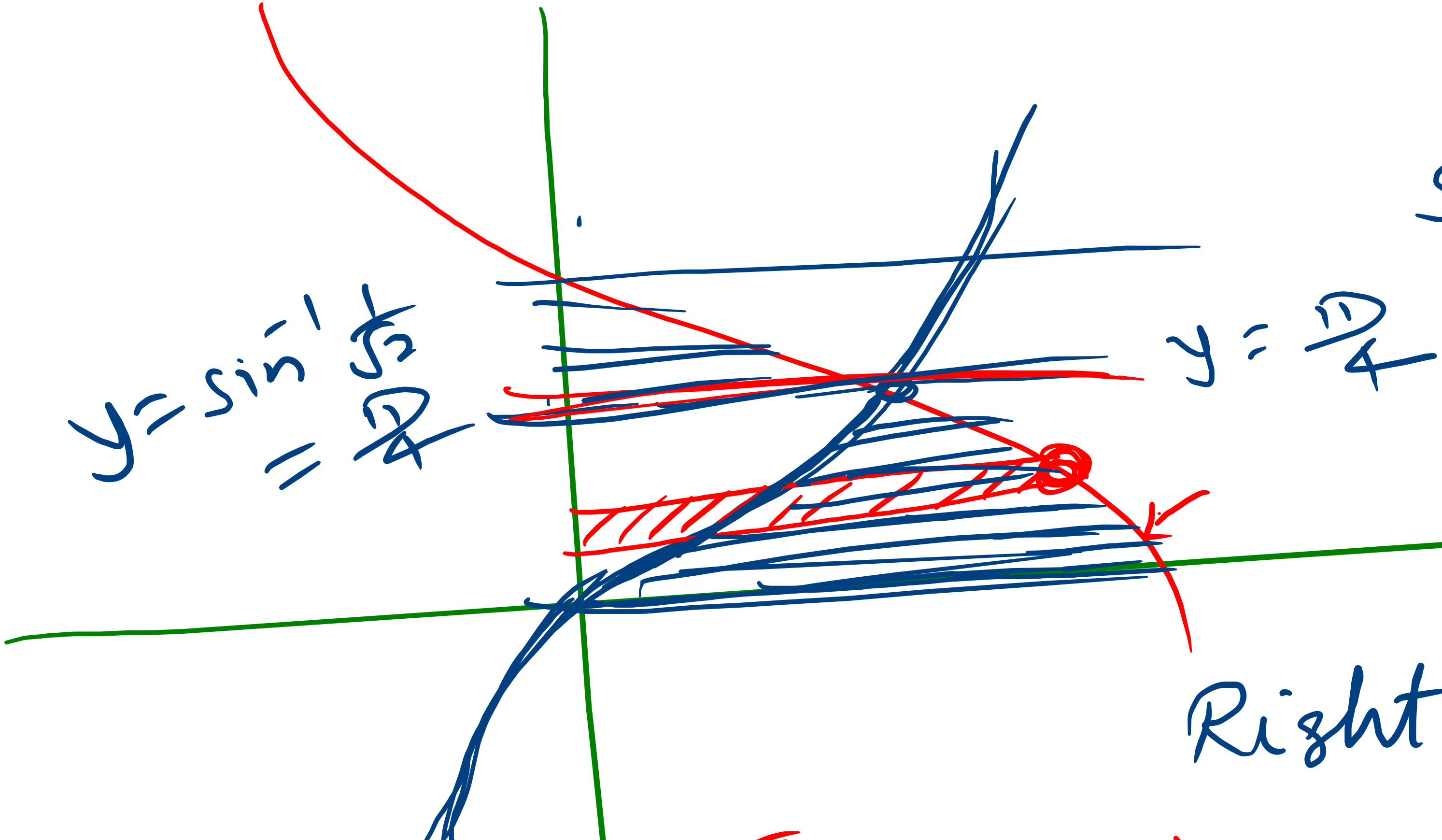




$$\int_0^{1/\sqrt{2}} \sin^{-1}x\,dx + \int_{1/\sqrt{2}}^{1} \cos^{-1}x\,dx$$

$$\int_0^{\pi/4} (\cos y - \sin y)\,dy =$$

$x_2 - x_1$



$\sin^{-1}x = \cos^{-1}x$

$\Rightarrow x = \frac{1}{\sqrt{2}}$

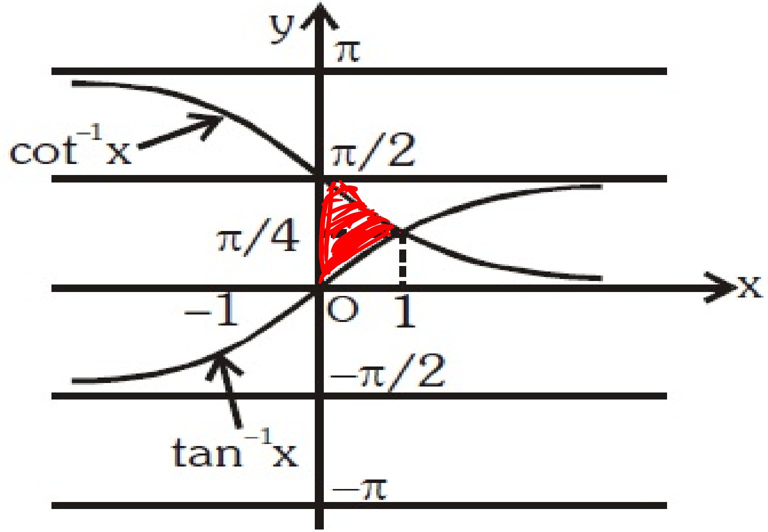
Right ≡ Red

Left ≡ Blue.

$$\int_0^{\pi/4} (\cos y - \sin y)\, dy$$

Q: Compute the area enclosed between

$y = \tan^{-1}x$; $y = \cot^{-1}x$ and y-axis.

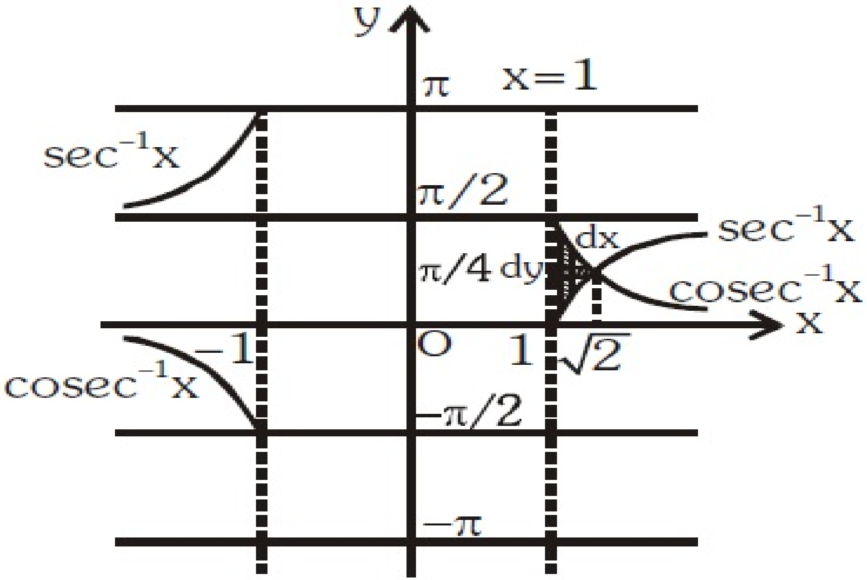


$$\int_0^{\pi/4} \tan y \, dy + \int_{\pi/4}^{\pi/2} \cot y \, dy$$

$$\int_0^1 (\text{upper} - \text{lower})\, dx = \int_0^1 (\cot^{-1}x - \tan^{-1}x)\, dx$$

$y = \sec^{-1}x$; $y = \text{cosec}^{-1}x$ and line $x - 1 = 0$

H.W.



Q. Area enclosed by the curves $(y - \sin^{-1}x)^2 = x - x^2$.

two curves

$$y - \sin^{-1}x = \pm\sqrt{x - x^2} \Rightarrow y = \sin^{-1}x + \sqrt{x - x^2} \quad \ldots (1)$$

$$y = \sin^{-1}x - \sqrt{x - x^2} \quad \ldots (2)$$

P.O.I.S of (1) & (2)

$\Rightarrow A(0,0)$, $B\left(1, \frac{\pi}{2}\right)$

$$\int_0^1 (\text{upper} - \text{lower})\,dx = \int_0^1 \left(\cancel{\sin^{-1}x} + \sqrt{x - x^2} \cancel{- \sin^{-1}x} + \sqrt{x - x^2}\right)dx$$

$$= 2\int_0^1 \sqrt{x - x^2}\,dx$$

$$= \frac{\pi}{4}$$

$$\int_{x_1}^{x_2} |f(x) - g(x)|\,dx$$

$$\int \sqrt{a^2-x^2}\,dx = \frac{x\sqrt{a^2-x^2}}{2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right)$$

$$\int_0^1 \sqrt{x-x^2}\,dx = \int_0^1 \sqrt{\frac{1}{4}-\left(x-\frac{1}{2}\right)^2}$$

$$= \left.\frac{\left(x-\frac{1}{2}\right)\sqrt{\frac{1}{4}-\left(x-\frac{1}{2}\right)^2}}{2} + \frac{1}{4\times 2}\sin^{-1}\left(\frac{x-\frac{1}{2}}{\frac{1}{2}}\right)\right|_0^1$$

$$= \frac{1}{8}\times\frac{\pi}{2} + \frac{1}{8}\times\frac{\pi}{2} = 2\times\frac{\pi}{16} = \frac{\pi}{8}$$

Q. The area of the region between the curves $y = \sqrt{\dfrac{1+\sin x}{\cos x}}$ and $y = \sqrt{\dfrac{1-\sin x}{\cos x}}$ bounded by the lines $x = 0$ and $x = \dfrac{\pi}{4}$ is **[JEE 2008, 3]**

(A) $\displaystyle\int_0^{\sqrt{2}-1} \frac{t}{(1+t^2)\sqrt{1-t^2}}\,dt$

(B) $\displaystyle\int_0^{\sqrt{2}-1} \frac{4t}{(1+t^2)\sqrt{1-t^2}}\,dt$

(C) $\displaystyle\int_0^{\sqrt{2}+1} \frac{4t}{(1+t^2)\sqrt{1-t^2}}\,dt$

(D) $\displaystyle\int_0^{\sqrt{2}+1} \frac{t}{(1+t^2)\sqrt{1-t^2}}\,dt$

$$\int_0^{\pi/4} \frac{\sqrt{1+\sin x} - \sqrt{1-\sin x}}{\sqrt{\cos x}}\,dx$$

Q: Area enclosed between the curves $y = ex.\ell nx$ and $y = \dfrac{\ell nx}{ex}$

AREA UNDER VARIOUS CASES :

Where the curve sketching is very significant :

$$\lim_{x \to 0^+} ex \cdot \ln x = 0$$

$$\frac{dy}{dx} = e\left(1 \cdot \ln x + \frac{x}{x}\right)$$

$-ve \quad \frac{1}{e} \quad +ve$

0

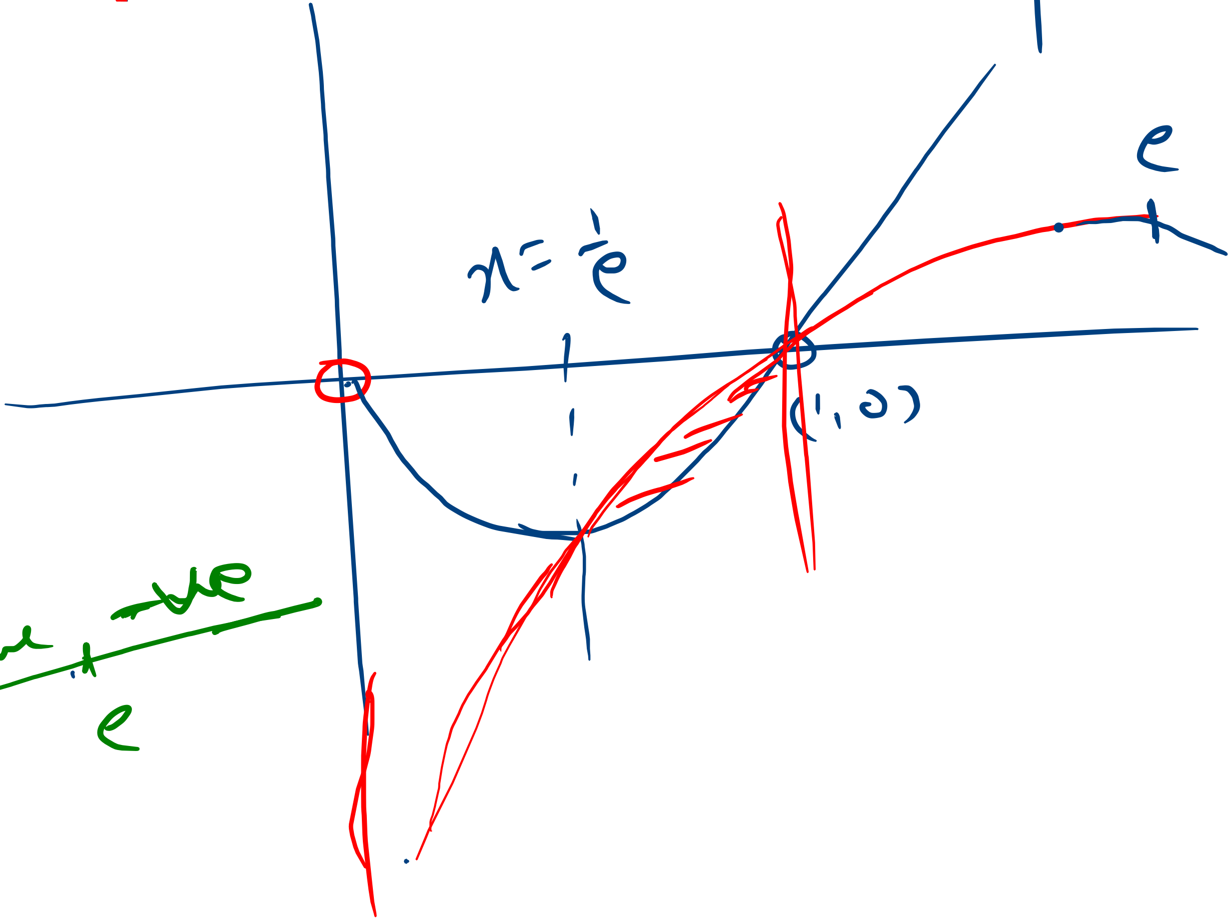
$$\lim_{x \to 0^+} \frac{\ln x}{ex} = -\infty$$

$$\frac{\frac{1}{x} \times x - \ln x}{x^2}$$

$+ve \quad e \quad -ve$

0

$$y' = \frac{1}{e}$$



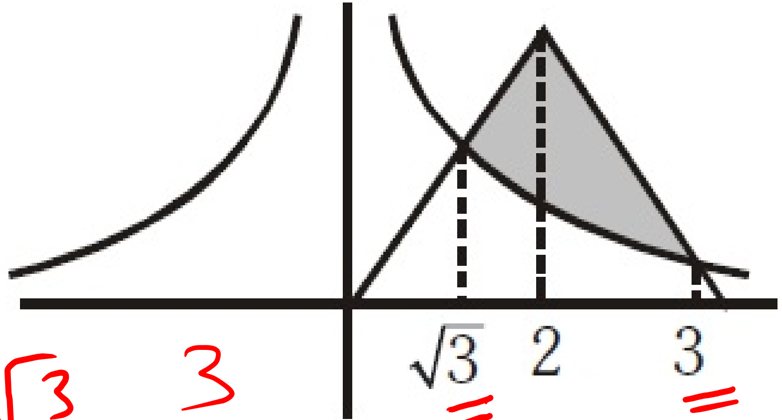
Between 2 curves

$x = a$, $x = b$, $y = f(x)$, $y = g(x)$

$f(x) = g(x)$ has only 2 solutions

$$\text{Required area} = \left| \int_a^b \left(f(x) - g(x)\right) dx \right|$$

Area of the closed figure bounded by the curves $y = 2 - |2 - x|$ and $y = \frac{3}{|x|}$ [**Ans.** $\frac{4 - 3\ln 3}{2}$]



Step I: Get P.O.I.

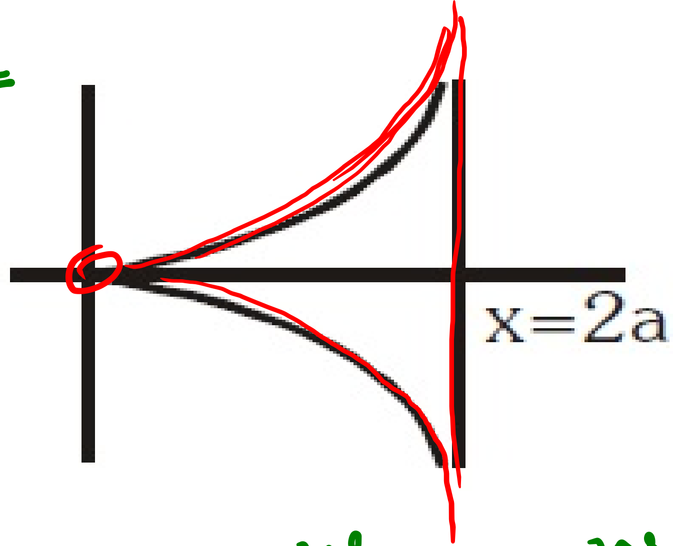
$$\frac{3}{|x|} = 2 - |2 - x| \Rightarrow x = \sqrt{3},\ 3$$

$$A = \int_{\sqrt{3}}^{3} (\text{upper} - \text{lower})\,dx$$

$$\int_{\sqrt{3}}^{3} \left| \quad \right| dx$$

$$\left| \int_{\sqrt{3}}^{3} \left(\frac{3}{|x|} - 2 + |2 - x| \right) dx \right| =$$

Q. Area bounded by the curve $y^2(2a - x) = x^3$ and its asymptote. , $a > 0$



$$y^2(2a-x) = x^3$$

$$(2a-x)\cdot x^3 \ge 0$$

-ve 0 +ve 2a -ve

$$2\int_0^{2a} x^{3/2}\frac{dx}{\sqrt{2a-x}}$$

$$y = \pm\frac{x^{3/2}}{\sqrt{2a-x}}$$

$$2a - x \to 0 \Rightarrow \boxed{x = 2a}$$

Q. Let $p(x)$ be a polynomial function of degree 3. If $p(x)+2$ is divisible by $(x-2)^2$ and $p(x)-2$ is div. by $(x+2)^2$, then find the area bounded by $y=p(x)$, x axis, $x=0$ & $x=1$.

Solⁿ.

$$\boxed{p(x)+2=(x-2)^2 g_1(x)} \Rightarrow p'(x) = 2(x-2)g_1(x)+(x-2)^2 g_1'(x)$$

$$\Rightarrow \boxed{p'(2)=0} \qquad \boxed{p'(-2)=0}$$

$$p'(x) = 3a(x^2-4)$$

$$\boxed{p(x) = ax^3-12ax+c}$$

$$\left.\begin{aligned} p(2) &= -2 \\ p(-2) &= 2 \end{aligned}\right\}$$

$$-2 = 8a-24a+c$$

$$2 = -8a+24a+c$$

$$\boxed{c=0}$$

$$\Rightarrow 4 = 32a \Rightarrow \boxed{a=\frac{1}{8}}$$

$$p(x) = \frac{1}{8}x^3 - \frac{12}{8}x$$

48. Find the area bounded by the curves $x^2 = y$, $x^2 = -y$ and $y^2 = 4x - 3$ **[JEE 2005 (Mains), 4]**.

49. Let f(x) be a quadratic polynomial and a, b, c be distinct real numbers such that

$$\begin{bmatrix} 4a^2 & 4a & 1 \\ 4b^2 & 4b & 1 \\ 4c^2 & 4c & 1 \end{bmatrix}\begin{bmatrix} f(-1) \\ f(1) \\ f(2) \end{bmatrix} = \begin{bmatrix} 3a^2+3a \\ 3b^2+3b \\ 3c^2+3c \end{bmatrix}.$$

Let V be the point of maximum of the curve $y = f(x)$. If A and B are the points on this curve such that the curve meets the positive x-axis at A and the chord AB subtends a right angle at V, then find the area enclosed by the curve and the chord AB. **[JEE 2005 (Mains), 6]**

Extra Q^n

45. Find the area of the region bounded by the curves $y = x^2$, $y = |2 - x^2|$ and $y = 2$, which lies to the right of the line $x = 1$. **[JEE 2002, (Mains) 5M out of 60]**

48. $\frac{1}{3}$ sq. units

49. $\frac{125}{3}$ sq. units

45. $\left(\frac{20}{3} - 4\sqrt{2}\right)$ sq. units

43. $f(x) = x^3 - x^2$

$$p(x) + 2 \equiv (x-2)^2 g_1(x)$$

$\forall x \in$ Complex.

Put $x = 2$, $\quad p(2) + 2 = 0$

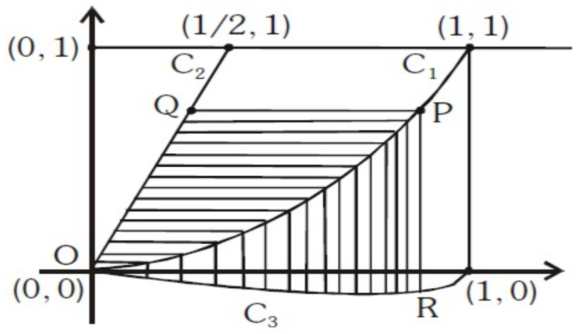
$\Rightarrow p(2) = -2$

diff. w.r.t. x

$$p'(x) = 2(x-2)\,g_1(x) + (x-2)^2 g_1'(x)$$

$$p'(2) = 0 + 0 \Rightarrow \Rightarrow p'(2) = 0$$

43. Let C_1 & C_2 be the graphs of the functions $y = x^2$ & $y = 2x$, $0 \le x \le 1$ respectively. Let C_3 be the graph of a function $y = f(x)$, $0 \le x \le 1$, $f(0) = 0$. For a point P on C_1, let the lines through P, parallel to the axes, meet C_2 & C_3 at Q & R respectively (see figure). If for every position of P (on C_1), the areas of the shaded regions OPQ & ORP are equal, determine the function f (x). **[JEE 98, 8M]**

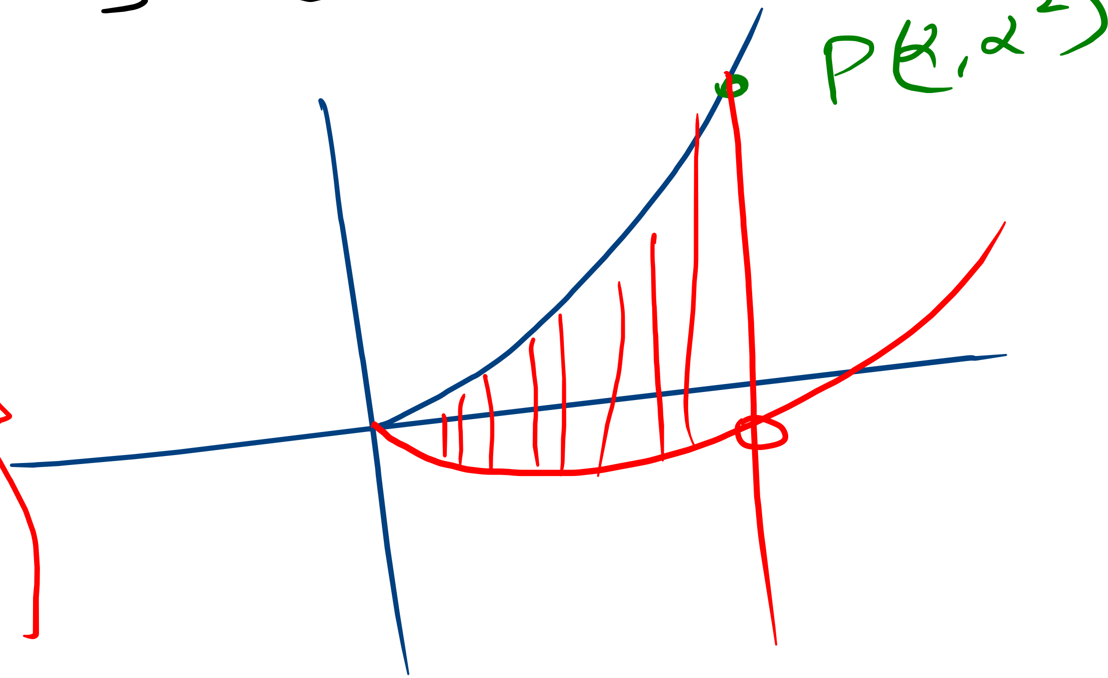


$C_1 : y = x^2$, $C_2 : y = 2x$, $C_3 : y = f(x)$

$$\int_0^{\alpha^2} \left(\sqrt{y} - \frac{y}{2}\right) dy = \int_0^{\alpha} (x^2 - f(x))\, dx$$

Differentiate w.r.t α

$\Rightarrow$



passage for Q 1, 2

Consider the polynomial (2010)
$f(x) = 1 + 2x + 3x^2 + 4x^3$.

Q1. Let s be the sum of all distinct real roots of $f(x)$ and let $t = |s|$. The real numbers lies in the interval

(a) $\left(-\frac{1}{4}, 0\right)$ (b) $\left(-11, -\frac{3}{4}\right)$

✓ (c) $\left(-\frac{3}{4}, -\frac{1}{2}\right)$ (d) $\left(0, \frac{1}{4}\right)$

Q2. The area bounded by the curve $y = f(x)$ and the lines $x = 0$, $y = 0$ and $x = t$, lies in the interval

✓ (a) $\left(\frac{3}{4}, 3\right)$ (b) $\left(\frac{21}{64}, \frac{11}{16}\right)$

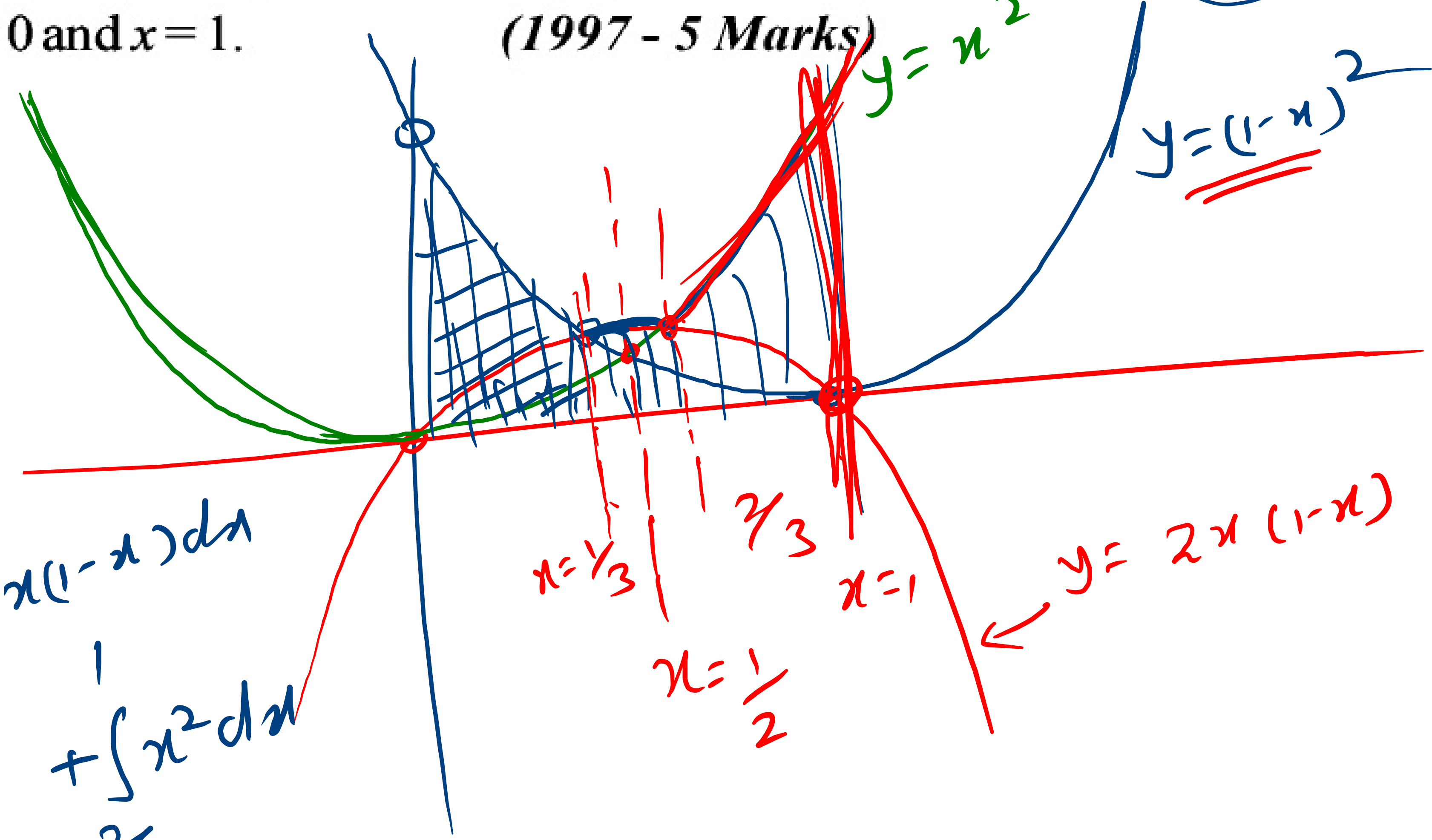
(c) $(9, 10)$ (d) $\left(0, \frac{21}{64}\right)$

Q. Let $f(x) = \text{Maximum}\ \{x^2, (1-x)^2, 2x(1-x)\}$, where $0 \le x \le 1$. Determine the area of the region bounded by the curves $y = f(x)$, x-axis, $x = 0$ and $x = 1$. ***(1997 - 5 Marks)***

$\frac{17}{27}$

P.O.I.

$x = \frac{1}{3}, \frac{2}{3}, \quad x = 1$



$$\int_0^{1/3} (1-x)^2 dx + \int_{1/3}^{2/3} 2x(1-x) dx + \int_{2/3}^{1} x^2 dx$$

Q Let $f(x)$ be a continuous function given by

$$\frac{257}{192}$$

$$f(x) = \begin{cases} 2x, & |x| \le 1 \\ x^2 + ax + b, & |x| > 1 \end{cases}$$

(1999 - 10 Marks)

Find the area of the region in the third quadrant bounded by the curves $x = -2y^2$ and $y = f(x)$ lying on the left of the line $8x + 1 = 0$.

$$f(x) = \begin{cases} x^2 + ax + b, & -\infty < x < -1 \Rightarrow \\ 2x, & -1 \le x \le 1 \\ x^2 + ax + b, & 1 < x \end{cases}$$

$f(-1) = 1 - a + b$

$f(-1^+) = -2$

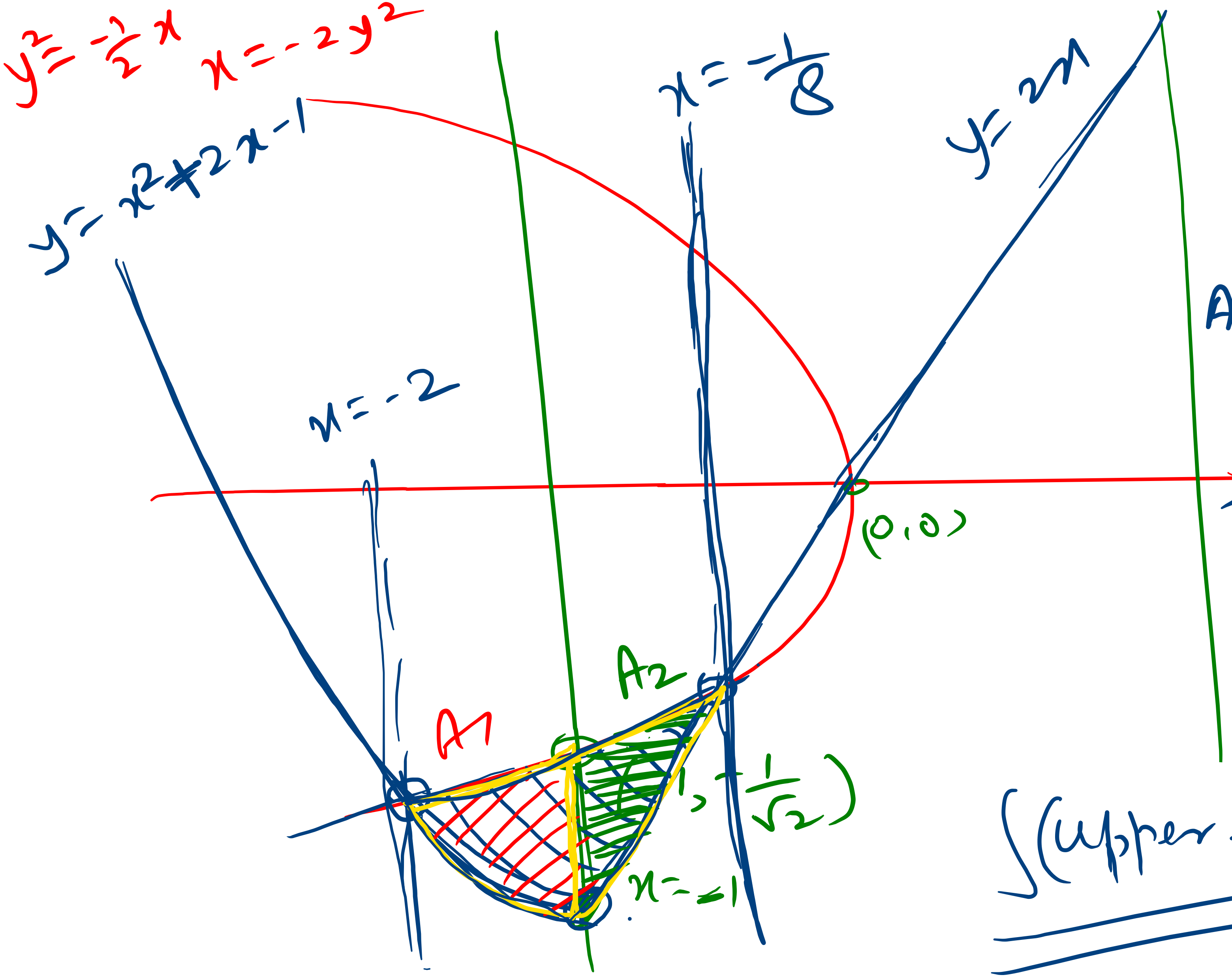
$1 - a + b = -2$

$\boxed{a - b = 3}$

$f(1^-) = 2 = 1 + a + b \Rightarrow \boxed{a + b = 1}$

$\boxed{a = 2,\ b = -1}$

$$f(x) = \begin{cases} x^2 + 2x - 1, & -\infty < x < -1 \\ 2x, & -1 \le x \le 1 \\ x^2 + 2x - 1, & 1 < x \end{cases}$$



$$y = x^2 + 2x - 1$$

$$A_1 = \int_{-2}^{-1} \left(\sqrt{\frac{-x}{2}} - x^2 - 2x + 1\right) dx$$

$$A_2 = \int_{-1}^{-\frac{1}{8}} \left(\sqrt{-x/2} - 2x\right) dx$$

$$=$$

$$\int (\text{upper} - \text{lower})\, dx > 0$$

$f(x)$ is a differentiable function and $g(x)$ is a double differentiable function such that $|f(x)| \leq 1$ and $f'(x) = g(x)$. If $f^2(0) + g^2(0) = 9$. Prove that there exists some $c \in (-3, 3)$ such that $g(c).g''(c) < 0$. ***(2005 - 6 Marks)***

Tough

Q. Let $b \neq 0$ and for $j = 0, 1, 2, \ldots, n$, let S_j be the area of the region bounded by the y–axis and the curve $xe^{ay} = \sin by$, $\boxed{\frac{j\pi}{b} \leq y \leq \frac{(j+1)\pi}{b}}$. Show that $S_0, S_1, S_2, \ldots, S_n$ are in geometric progression. Also, find their sum for $a = -1$ and $b = \pi$. *(2001 - 5 Marks)*

$$\frac{S_{j+1}}{S_j} = \text{Const.}$$

$$x = e^{-ay} \sin by$$

$$S_j = \int_{\frac{j\pi}{b}}^{(j+1)\frac{\pi}{b}} |x|\, dy$$

$$\Rightarrow S_j = \left| \int_{\frac{j\pi}{b}}^{(j+1)\frac{\pi}{b}} e^{-ay} \sin(by)\, dy \right|$$

$$S_{j+1} = \int_{\frac{(j+1)\pi}{b}}^{(j+2)\frac{\pi}{b}} e^{-ay} \sin(by)\, dy$$

$$\int_a^b f(x)\,dx = \int_{a-c}^{b-c} f(x+c)\,dx$$

$$S_{j+1} = \left| \int_{(j+1)\frac{\pi}{b}-\frac{\pi}{b}}^{(j+2)\frac{\pi}{b}-\frac{\pi}{b}} e^{-a\left(y+\frac{\pi}{b}\right)} \sin\left(b\left(y+\frac{\pi}{b}\right)\right) dy \right| = \left| K \int_{\frac{j\pi}{b}}^{(j+1)\frac{\pi}{b}} e^{-ay} \sin(by)\,dy \right|$$

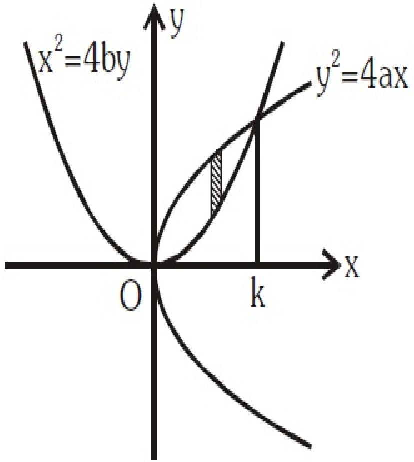
$$e^{-ay} \times e^{-\frac{\pi a}{b}}$$

$$\boxed{S_{j+1} = e^{-\frac{\pi a}{b}} S_j}$$

$$K = e^{-\frac{\pi a}{b}}$$

STANDARD AREAS (to be remembered) :

(a) Area bounded by parabolas $y^2 = 4ax$; $x^2 = 4by$, $a > 0$; $b > 0$



$$A = \int_0^k \left(2\sqrt{a}\sqrt{x} - \frac{x^2}{4b}\right) dx \text{, where } k = 4a^{1/3}b^{2/3}$$

$$A = \frac{16ab}{3}$$

$y^2 = 4x$

$x^2 = 4y$

$A(0,0), B(4,4)$

$$\int_0^4 \left(2\sqrt{x} - \frac{x^2}{4}\right) dx =$$

$y^2 = 4x$, $x^2 = 4y$

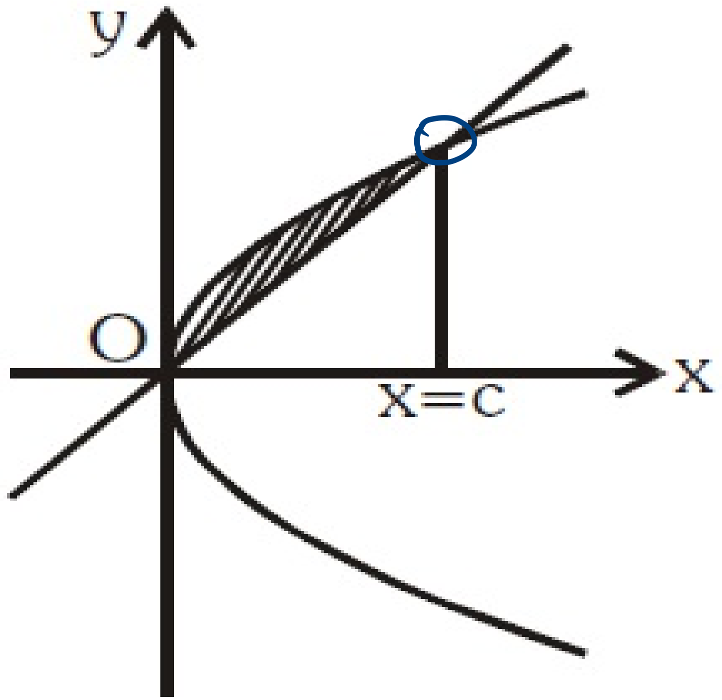
$\left(\frac{x^2}{4}\right)^2 = 4x \Rightarrow x^4 = 64x^3$

$\Rightarrow x = 0, 4$

Area bounded by $y^2 = 4ax$ and $y = mx$

$$A = \int_0^c (2\sqrt{a}\sqrt{x} - mx)dx\text{, where } c = \frac{4a}{m^2}$$

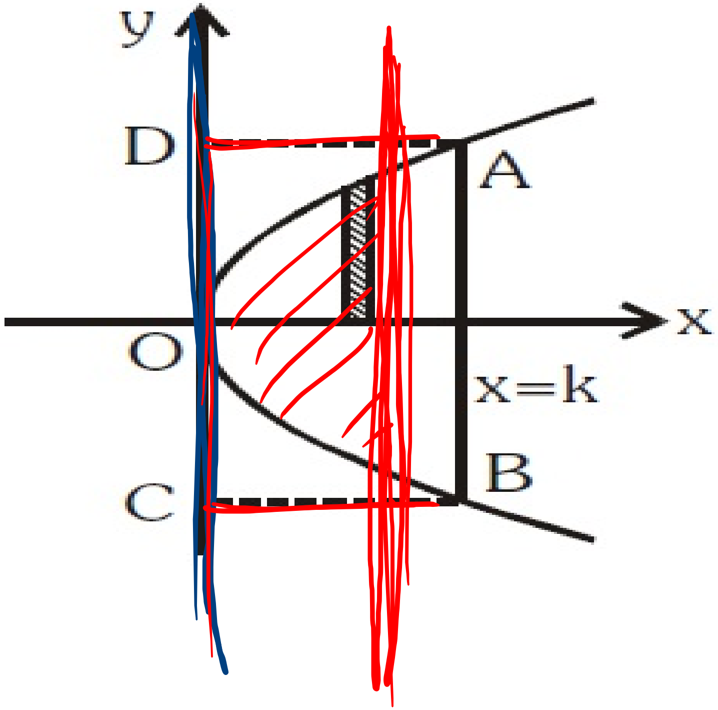
$$A = \frac{8a^2}{3m^3}$$



Note that for curve $x^2 = 4ay$ and line $y = mx$, area enclosed will be $A = \frac{8a^2m^3}{3}$

Area enclosed by $y^2 = 4ax$ and its double ordinate at $x = k$

(chord perpendicular to the axis of symmetry)



$$A = 2.\int_0^k \left(2\sqrt{ax}\right)dx = \frac{8}{3}\sqrt{a}.k^{3/2}$$

This simplifies to

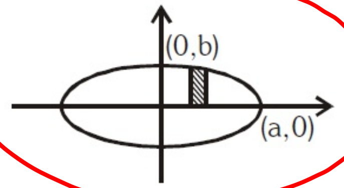
Area of AOB = $\frac{2}{3}$(area □ABCD)

Find area enclosed by $y = 2x - x^2$ & $y + 3 = 0$;

Whole area of ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$A = 4\int_0^a \left(b\sqrt{1 - \frac{x^2}{a^2}}\right)dx = \pi ab$$

πab



Problems based on inequalities :

E(1) Find the area of common region of $3|x| + 4|y| \leq 25$ and $x^2 + y^2 \geq 25$ **[Ans.** $\frac{625}{6} - 25\pi$]

E(2) Find the area of common region of $y \geq x\ell nx$ & $y \leq 2x - 2x^2$ **[Ans.** $\frac{7}{12}$]

E(3) Find the area enclosed by $2 \leq |x - y| + |x + y| \leq 4$ **[Ans. 12]**

E(4) Find the area enclosed by $y(x^2 + y^2 - 1) \leq 0$ and $\frac{x^2}{25} + \frac{y^2}{16} \leq 1$ **[Ans. 10π]**

III. Shifting of origin :

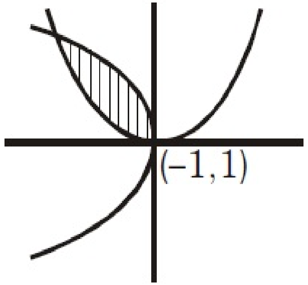
Next lectr.

Since area remains invariant even if the coordinate axes are shifted, hence shifting of origin in many cases prove to be very convenient in computing the areas.

Examples :

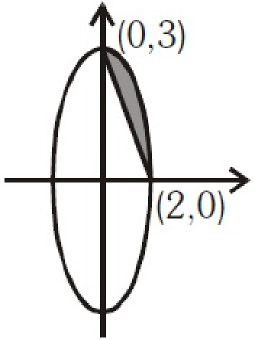
E(1) (a) Area enclosed between the parabolas $y^2 - 2y + 4x + 5 = 0$ and

$x^2 + 2x - y + 2 = 0$.



(b) Smaller area enclosed between the ellipse $9x^2 + 4y^2 - 36x + 8y + 4 = 0$

and the line $3x + 2y = 10$ in the first quadrant.



[**Ans.** (a) 4/3, (b) $3\pi/2 - 3$]

Area bounded by the circle $x^2 + y^2 + 4x + 6y + 4 = 0$ & pair of tangents drawn to it from (3, –3).

[**Ans.** $12 - 9\tan^{-1}(4/3)$]

Inverse.

$f(x) = x^3 + 2x^2 + 2x + 1$ and $g(x)$ is the inverse of it. Then compute the area bounded by $g(x)$, x-axis and the ordinate at $x = -3$ and $x = 6$. [**Ans.** 101/12]

Area enclosed by inverse of a function :

Let $f(x) = x^3 + 3x + 2$ and $g(x)$ is the inverse of it. Find the area bounded by $g(x)$, the x-axis and the ordinate at $x = -2$ and $x = 6$. [**Ans.** 9/2]

Area enclosed by inverse of a function :

$g(x) = f^{-1}(x)$

Q. Let $f(x) = x^3 + 3x + 2$ and $g(x)$ is the inverse of it. Find the area bounded by $g(x)$, the x-axis and the ordinate at $x = -2$ and $x = 6$. [**Ans.** 9/2]

$y = f^{-1}(x)$
$x = -2$
$x = 6$
x axis

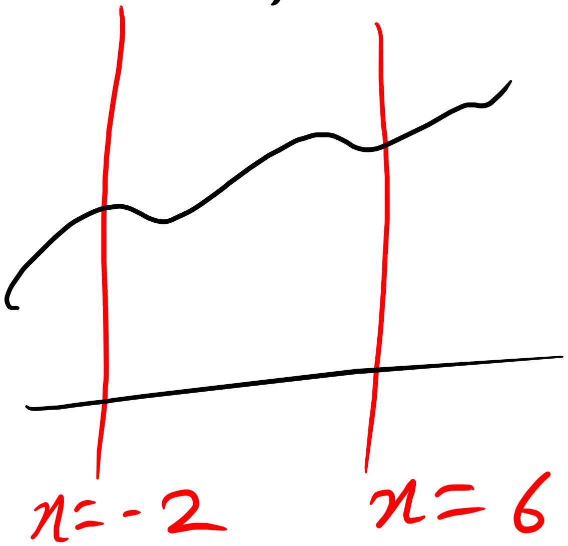
* $f^{-1}(x)$ is not given, $f(x)$ is given

$$\text{Area} = \int_{-2}^{6} \left| f^{-1}(x) \right| dx$$

$f^{-1}(x) = t$

$\Rightarrow f(t) = x$

$\Rightarrow dx = f'(t)\,dt$



$$= \int_{-1}^{1} |t|\, f'(t)\,dt$$

$$= \int_{-1}^{0} (-t)(3t^2+3)\,dt + \int_{0}^{1} t\,(3t^2+3)\,dt$$

$$\left. \frac{3t^4}{4} + \frac{3t^2}{2} \right|_0^1$$

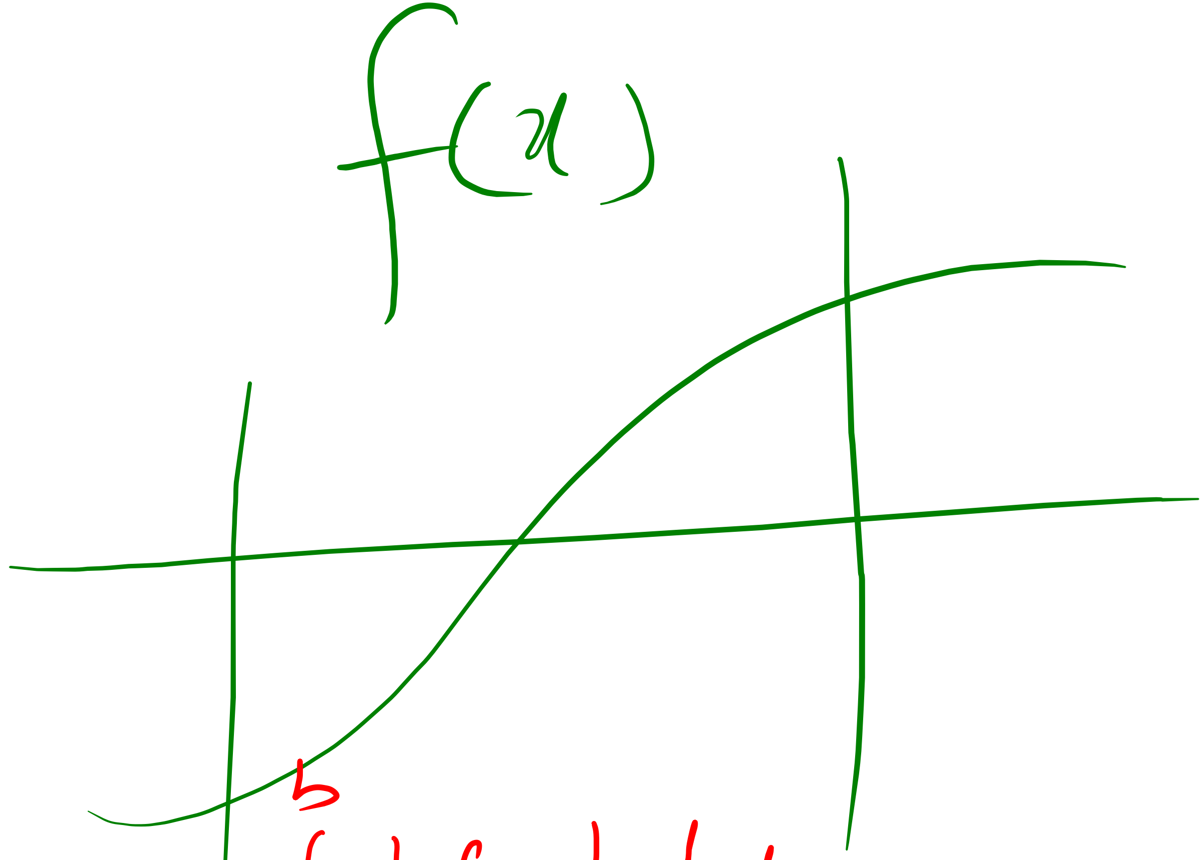
when $x = -2$

$\Rightarrow t^3 + 3t + 2 = -2$

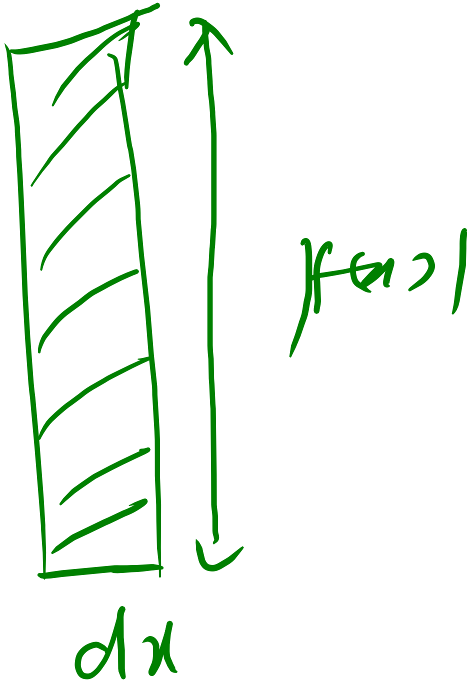
$\Rightarrow t = -1$

when $x = 6$

$\Rightarrow t = 1$



$$= \int_a^b |f(x)|\,dx$$



$$dA = |f(x)|\,dx$$

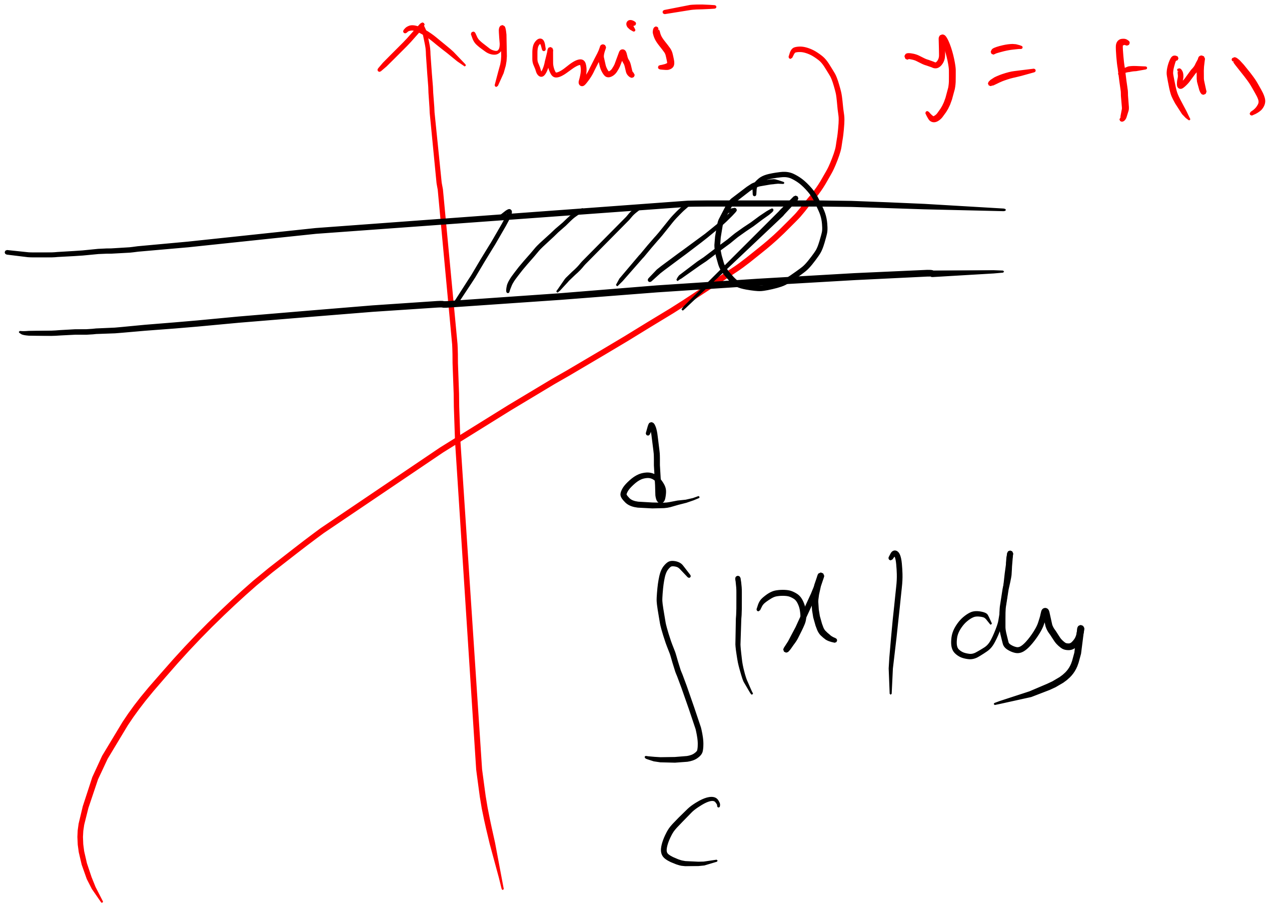
$$A = \int_a^b |f(x)|\,dx$$

Convert

Express

x in terms

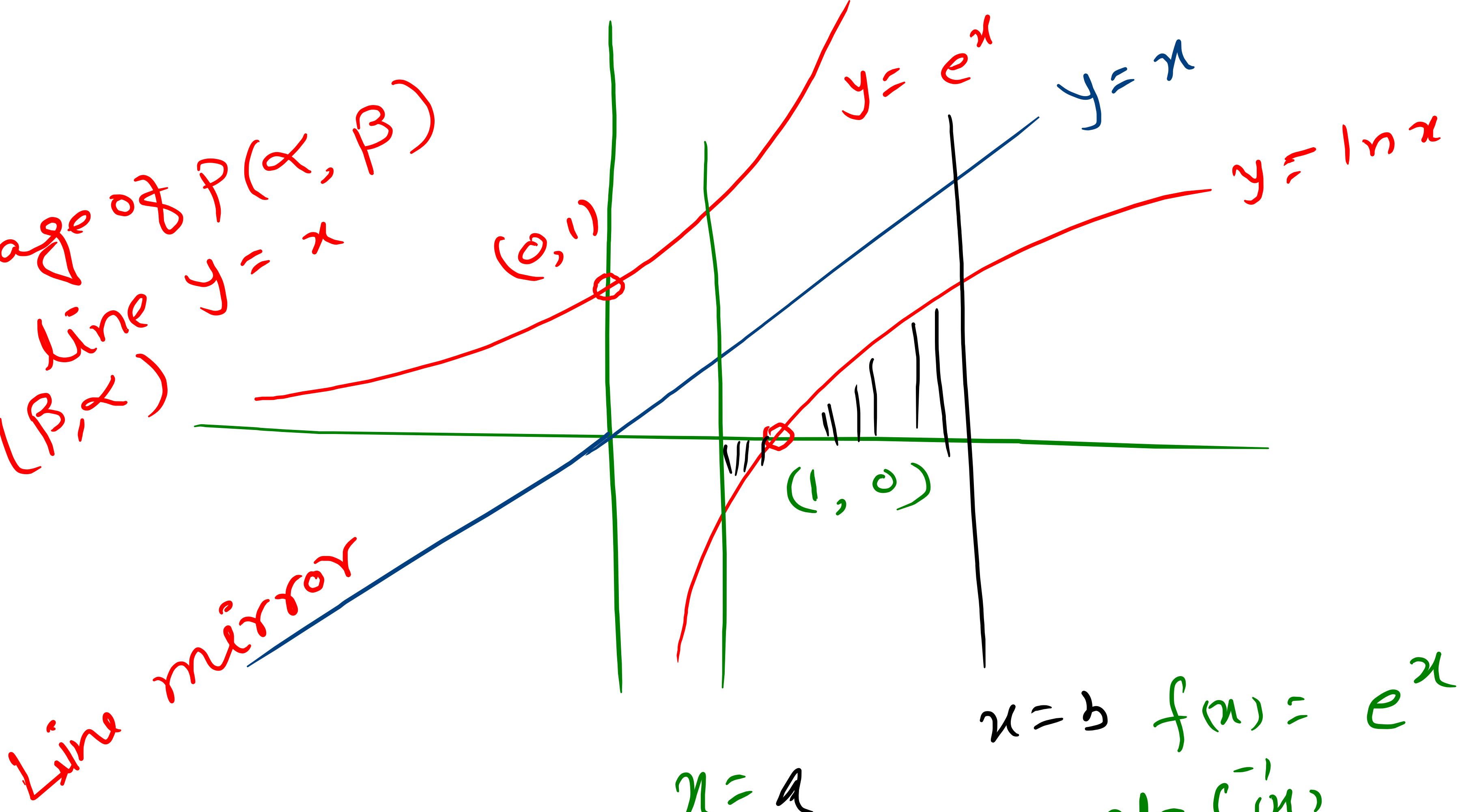
of y



$$\int_c^d |x|\, dy$$

$$\left.\begin{array}{l} y = \ln x \\ y = e^x \end{array}\right\}$$

Mirror image of $P(\alpha, \beta)$ w.r.to line $y = x$ is $Q(\beta, \alpha)$



$f(x) = e^x$

$y = f^{-1}(x)$

Q. $f(x) = x^3 + 3x + 4$, $\int_{-1}^{1} f(x)\,dx + \int_{0}^{4} f^{-1}(x)\,dx$

$$\int_{0}^{4} f^{-1}(x)\,dx$$

Put $f^{-1}(x) = t \Rightarrow x = f(t)$

$dx = f'(t)\,dt$

$$= \int_{-1}^{0} t \cdot f'(t)\,dt = \int_{-1}^{0} (3t^2+3)\,t\,dt$$

$=$

I.B.P.

when $x = 0$,

$t^3 + 3t + 4 = 0$

$\Rightarrow \boxed{t = -1}$

when $x = 4$

$t^3 + 3t + 4 = 4$

$t = 0$

Q. $f(x) = x^3 + 2x^2 + 2x + 1$ and g(x) is the inverse of it. Then compute the area bounded by g(x), x-axis and the ordinate at x = −3 and x = 6. [**Ans.** 101/12]

$$\int_{-3}^{6} |g(x)|\, dx =$$

$$= \int_{-2}^{1} |t|(3t^2+4t+2)\, dt =$$

$g(x) = f^{-1}(x)$

put $g(x) = t$

$\Rightarrow x = f(t)$

$dx = f'(t)\, dt$

when $x = -3$

$\Rightarrow t^3 + 2t^2 + 2t + 1 = -3$

$\Rightarrow \boxed{t = -2}$

$\boxed{t = 1}$

Shifting of origin :

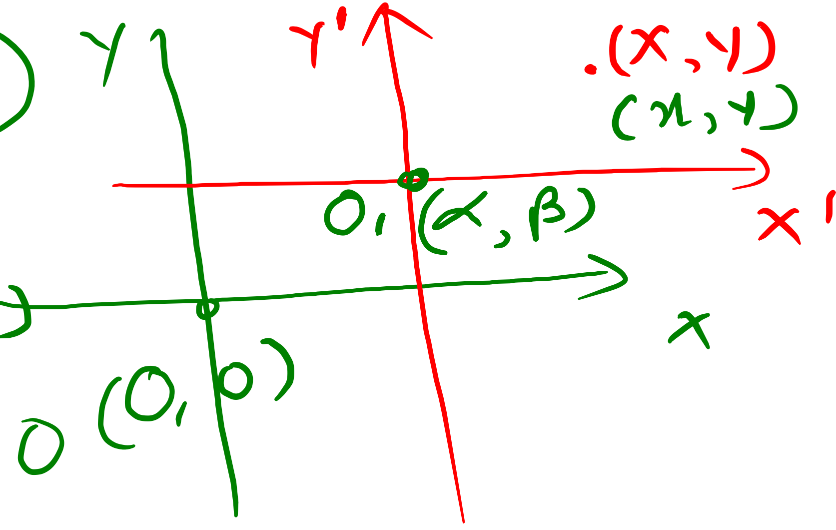
(a) Area enclosed between the parabolas $y^2 - 2y + 4x + 5 = 0$ and $x^2 + 2x - y + 2 = 0$.

$(x+1)^2 = (y-1)$

$(y-1)^2 = -4(x+1)$

origin at $(-1,1)$

$\frac{4}{3}$



Ex: $(y-1)^2 = 4(x-1)$

Shift origin at $(1, 1)$

Now find equation in new co-ordinate system.

$$\boxed{\begin{aligned} X + \alpha &= x \\ Y + \beta &= y \end{aligned}}$$

$x = X+1, \quad y = Y+1$

$(Y+1-1)^2 = 4(X+1-1) \Rightarrow \boxed{Y^2 = 4X}$

$\frac{16ab}{3}$

Q. Smaller area enclosed between the ellipse $9x^2 + 4y^2 - 36x + 8y + 4 = 0$ and the line $3x + 2y = 10$ in the first quadrant.

$9(x^2 - 4x) + 4(y^2 + 2y) + 4 = 0$

$\Rightarrow 9(x-2)^2 + 4(y+1)^2 = 36$

$\Rightarrow \dfrac{(x-2)^2}{4} + \dfrac{(y+1)^2}{9} = 1$

origin to $\rightarrow (2, -1)$

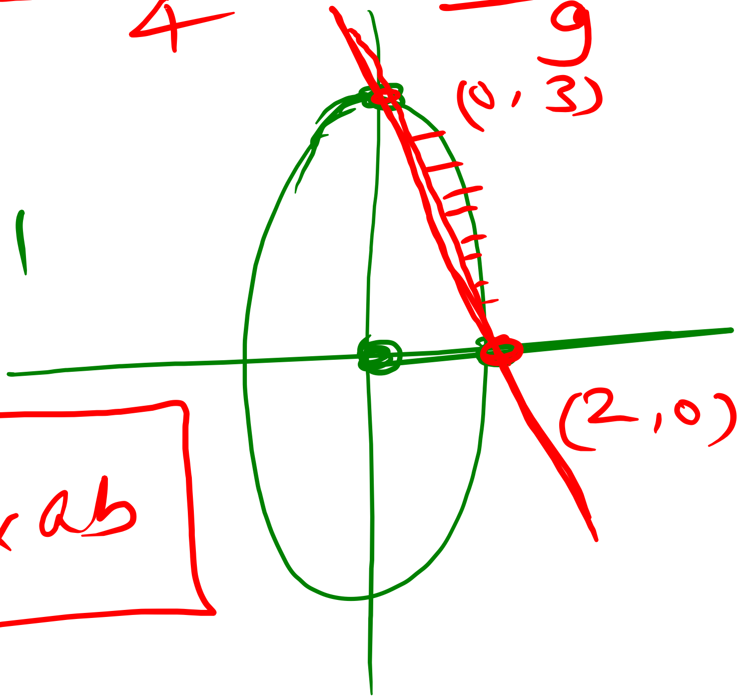
$x = X + 2, \quad y = Y - 1$

$3X + 6 + 2Y - 2 = 10$

$3X + 2Y = 6 \Rightarrow \dfrac{X}{2} + \dfrac{Y}{3} = 1$

$\dfrac{X^2}{4} + \dfrac{Y^2}{9}$

$$\boxed{A = \frac{\pi ab}{4} - \frac{1}{2} \times ab}$$



Q. Find the area enclosed by the parabola $(y-2)^2 = x - 1$ and the tangent to it at (2, 3) and x-axis.

[**Ans.** $A = \int_{-2}^{1} [Y^2 - (2Y - 1]dY = 9$]

$H = \omega$

$(y-2)^2 = x-1$

$(1, 2)$

$2(y-2)y' = 1 \Rightarrow y' = \frac{1}{2}$

$x = X + 1$

$y = Y + 2$

$\boxed{Y^2 = X}$

$y - 3 = (x-2)\frac{1}{2}$

$\Rightarrow 2y - 6 = x - 2$

$\boxed{x - 2y + 4 = 0}$

$X + 1 - 2Y - 4 + 4 = 0$

$\boxed{X - 2Y + 1 = 0}$

$X = x - 1 = 1$

$Y = 3 - 2 = 1$

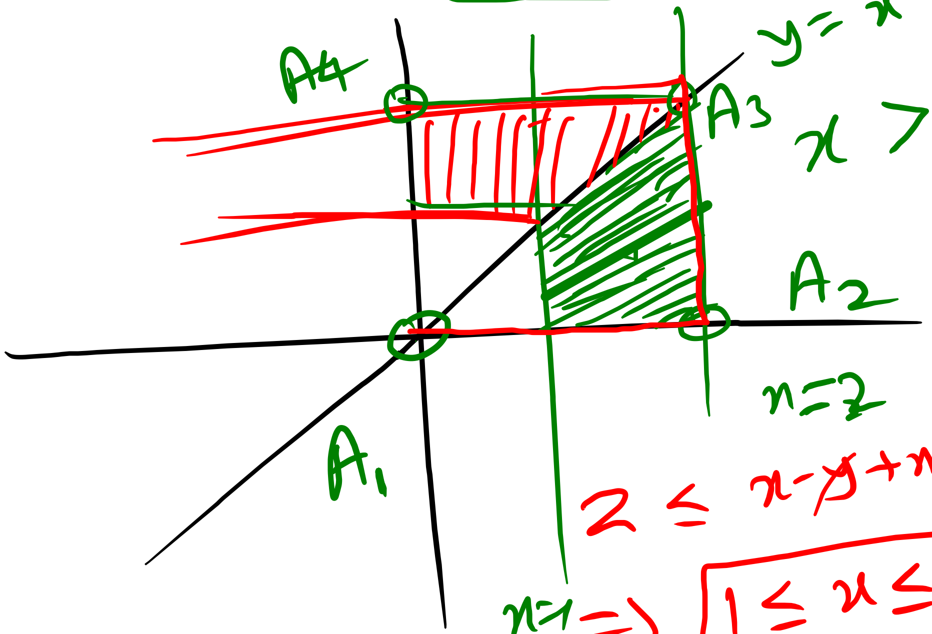
$(1, 1)$

Xaxis $\boxed{y = 0}$

$\boxed{Y = -2}$

Problems based on inequalities :

Q: Find the area enclosed by $2 \le |x-y| + |x+y| \le 4$



$2 \le x - y + x + y \le 4$

$x = 1 \Rightarrow \boxed{1 \le x \le 2}$

$4 \times (4 - 1) = 12$

Symmetric about

X-axis: $y \to -y$, No change in eqn.

Y-axis: $x \to -x$, No change in eqn

$y = x$ line

$x \to y$, $y \to x$: $(x, y) \to (y, x)$

No change in equation

Find the area enclosed by $y(x^2 + y^2 - 1) \leq 0$ and $\dfrac{x^2}{25} + \dfrac{y^2}{16} \leq 1$ **[Ans. 10π]**

H.W.

Find the area of common region of $y \geq x\ell nx$ & $y \leq 2x - 2x^2$ [**Ans.** $\frac{7}{12}$]

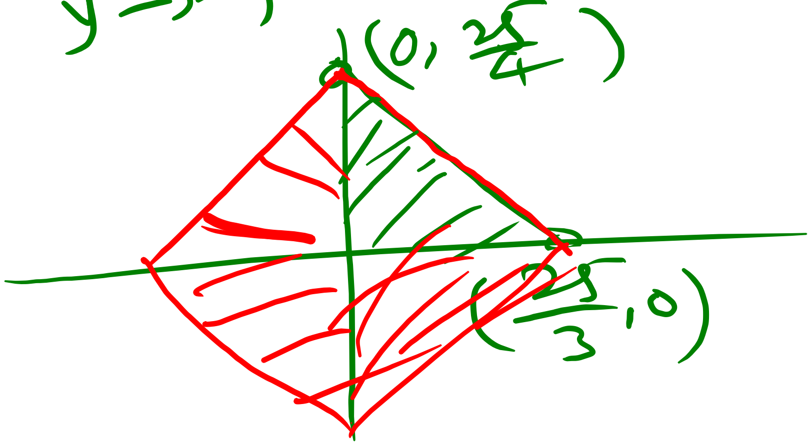
H.W.

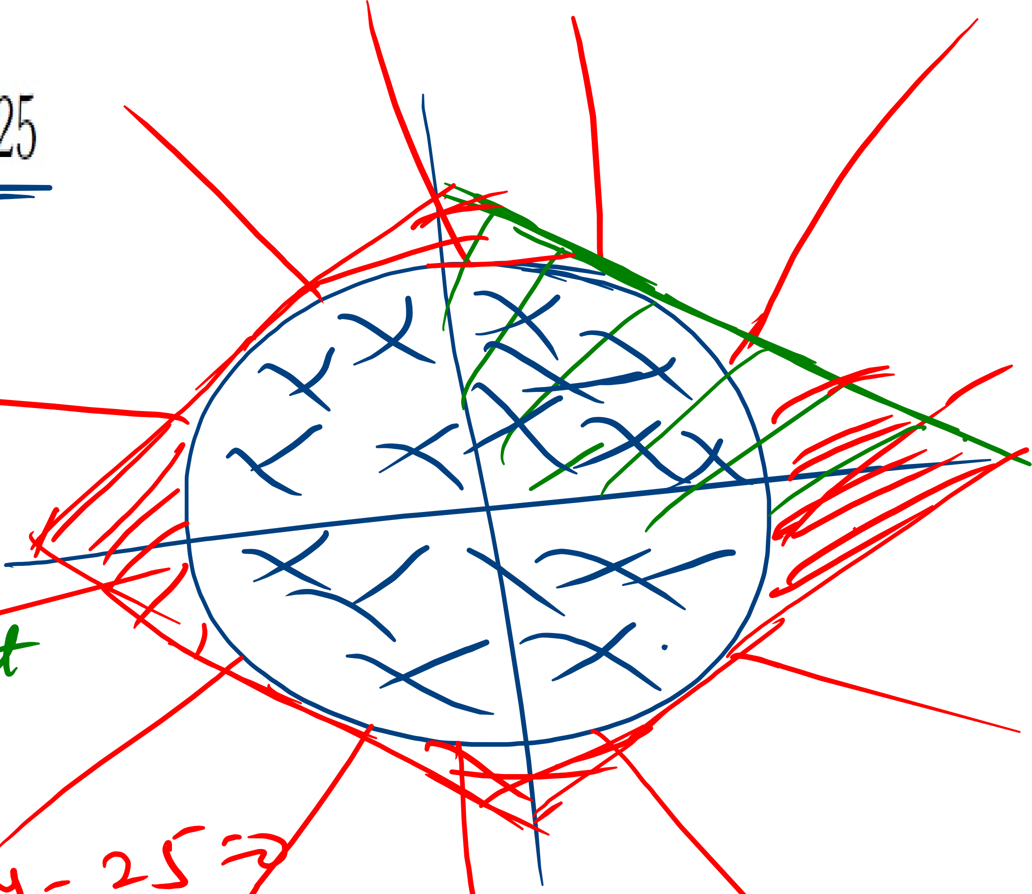
Q. Find the area of common region of $3|x|+4|y|\le 25$ and $x^2+y^2\ge 25$

$\boxed{3|x|+4|y|\le 25}$

$x \to -x$ Symmetric about Y axis

$y \to -y \Rightarrow$ Symmetric about X axis





$3x+4y-25 \Rightarrow$

$\frac{25}{5}=5$

Area $= 4\times\left(\frac{1}{2}\times\frac{625}{12}-\frac{25\pi}{4}\right)$

$\boxed{\text{Parametric form}}$

$x = g(t), \quad y = h(t)$

$$\left.\begin{aligned} x &= \cos\theta \\ y &= \sin\theta \end{aligned}\right\}$$

$$\int_{x_1}^{x_2} |y|\,dx \quad , \quad \int_{y_1}^{y_2} |x|\,dy$$

$$\text{Area} = \int_{x_1}^{x_2} |y|\,dx = \int_{t_1}^{t_2} |h(t)| \left(\frac{dx}{dt}\right) dt = \int_{t_1}^{t_2} |h(t)|\, g'(t)\, dt$$

For $\boxed{x_1 < x < x_2}$

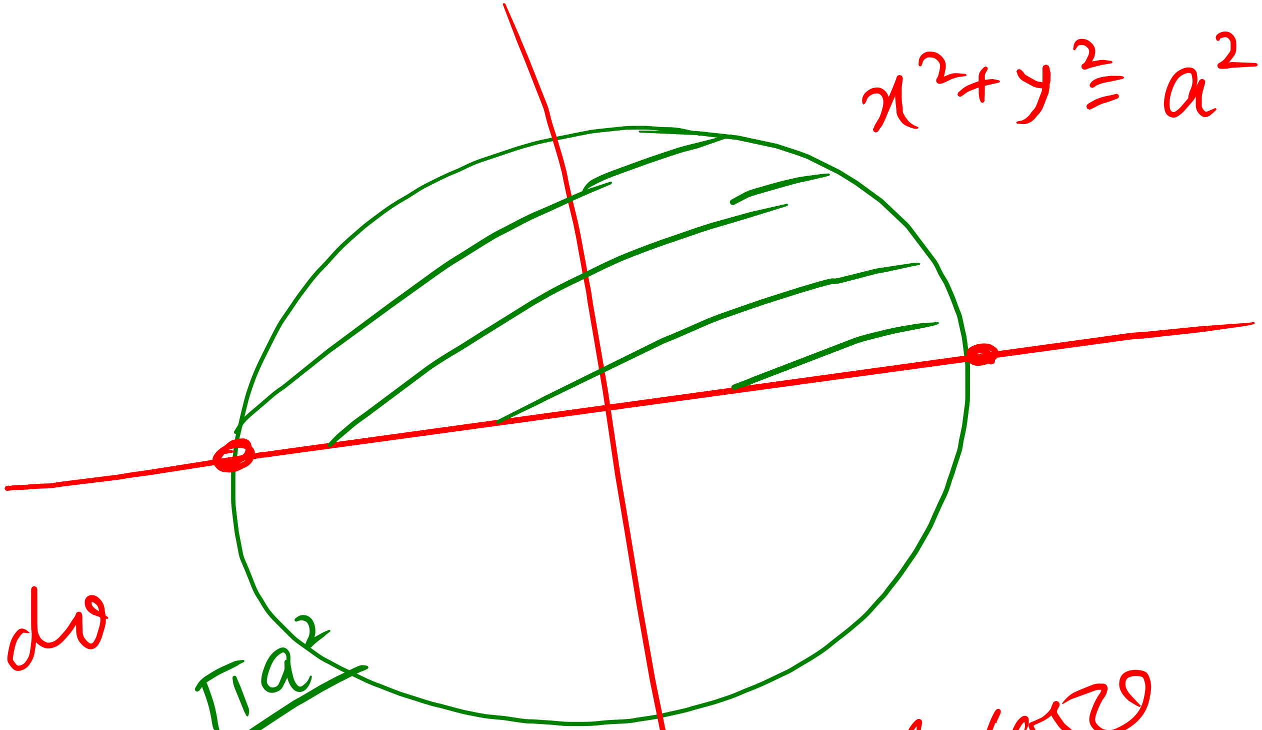
$\equiv \boxed{t_1 < t < t_2}$

$$\int_{-a}^{a} |f(x)|\,dx$$

$$= \int_{\pi}^{0} |a\sin\theta| \left(\frac{dx}{d\theta}\right) d\theta$$

$$= \int_{\pi}^{0} |a\sin\theta|(-a\sin\theta)\,d\theta$$

$$= -a^2 \int_{\pi}^{0} \sin^2\theta\,d\theta = \frac{\pi a^2}{2}$$



$x = a\cos\theta$

$y = a\sin\theta$

$-a = a\cos\theta$

$\Rightarrow \theta = \pi$

$a = a\cos\theta$

$\theta = 0$

DETERMINATION OF UNKNOWN PARAMETERS :

Find the value of c for which the area of the figure bounded by the curves $y = \frac{4}{x^2}$; $x = 1$ and $y = c$ is equal to $\frac{9}{4}$.

[**Ans.** $c = \frac{1}{4}, c = \frac{49}{4}$]

Find the area of the figure bounded by the parabola $y = ax^2 + 12x - 14$ and the straight line $y = 9x - 32$ if the tangent drawn to the parabola at the point $x = 3$ is known to make an angle $\pi - \tan^{-1}6$ with the x-axis. [**Ans.** $A = \int_{-2}^{3} [-3x^2 + 12x - 14] - (9x - 32)]dx$]

E(3) For what value of 'a' is the area of the figure bounded by the curves, $y = \frac{1}{x}$, $y = \frac{1}{2x-1}$, $x = 2$ & $x = a$ is equal to $\ln\frac{4}{\sqrt{5}}$?

[**Ans.** $a = 8, \frac{2}{5}\left(6-\sqrt{21}\right)$]

E(4) If the area bounded by $y = x^2 + 2x - 3$ and the line $y = kx + 1$ is least. Find k and also the least area.

[**Ans.** $k = 2, A_{min} = \frac{32}{3}$]

Find the area of the region bounded by the curves $y = x^2$, $y = |2 - x^2|$ and $y = 2$, which lies to the right of the line $x = 1$.

(2002 - 5 Marks)

CONCEPT OF VARIABLE AREA (greatest and least value) :

If $y = f(x)$ is a monotonic function in (a, b) then the area bounded by the ordinates at $x = a$, $x = b$, $y = f(x)$ and $y = f(c)$, [where $c \in (a, b)$] is minimum when $c = \frac{a+b}{2}$.

If the area bounded by $f(x) = \frac{x^3}{3} - x^2 + a$ and the straight lines x = 0; x = 2 and the x-axis is minimum then find the value of 'a'. [**Ans.** 2/3]

DETERMINATION OF UNKNOWN PARAMETERS :

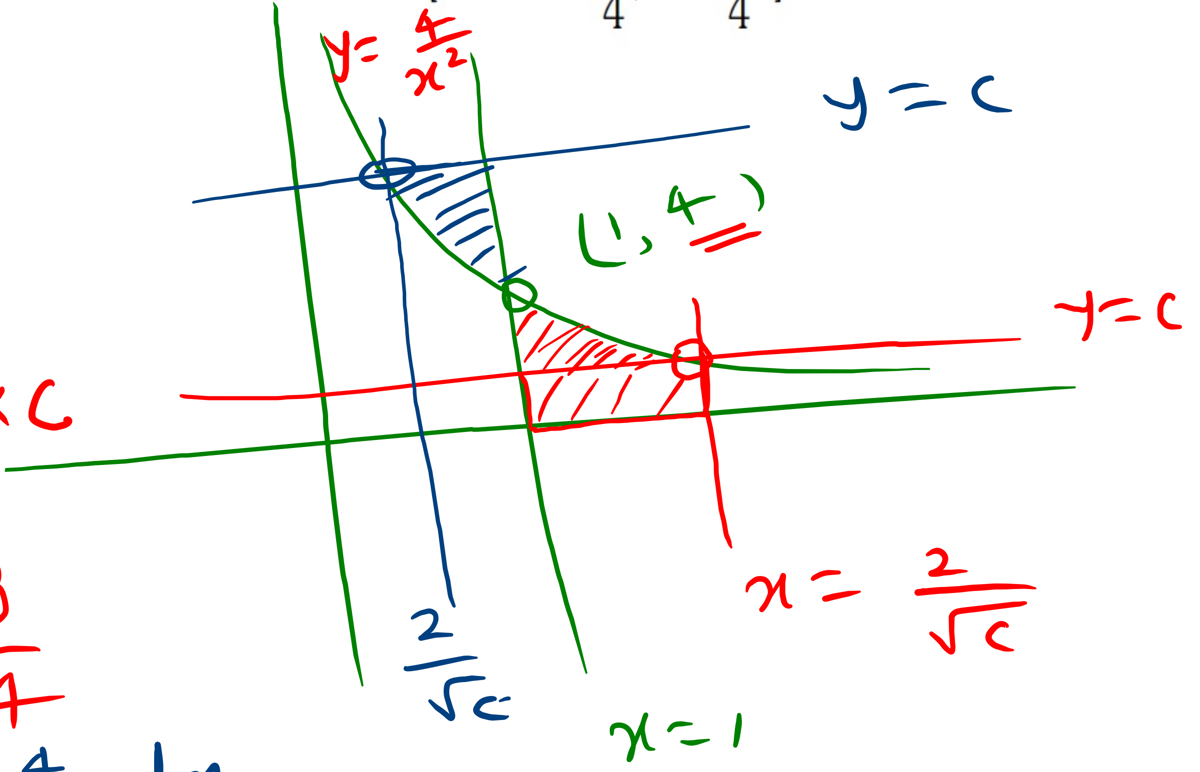
E(1) Find the value of c for which the area of the figure bounded by the curves $y = \frac{4}{x^2}$; $x = 1$ and $y = c$ is equal to $\frac{9}{4}$.

[**Ans.** $c = \frac{1}{4}, c = \frac{49}{4}$]

$c = \frac{4}{x^2}$

$x = \frac{2}{\sqrt{c}}$

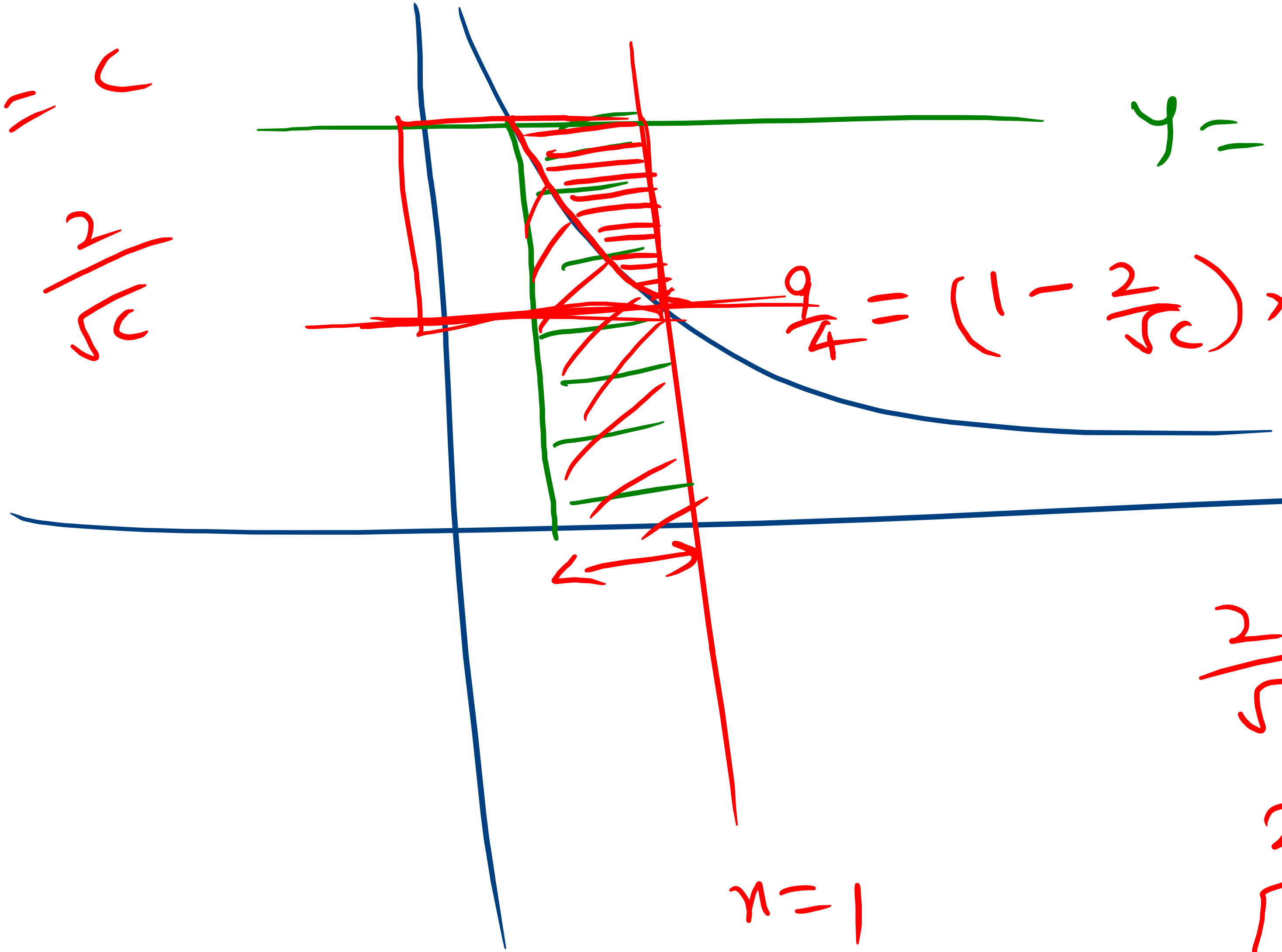
$$\int_1^{\frac{2}{\sqrt{c}}} \frac{4}{x^2}\,dx - \left(\frac{2}{\sqrt{c}} - 1\right) \times c = \frac{9}{4}$$



$$\text{Area} = \left(1 - \frac{2}{\sqrt{c}}\right) \times c - \int_{\frac{2}{\sqrt{c}}}^{1} \frac{4}{x^2}\,dx$$

$$\frac{4}{x^2} = c$$

$$x = \frac{2}{\sqrt{c}}$$



$$\frac{9}{4} = \left(1 - \frac{2}{\sqrt{c}}\right) \times c - \int_{\frac{2}{\sqrt{c}}}^{1} \frac{4}{x^2}\,dx$$

$$\frac{2}{\sqrt{c}} < 1$$

$$2 < \sqrt{c}$$

$$\boxed{4 < c}$$

Q **E(2)** Find the area of the figure bounded by the parabola $y = ax^2 + 12x - 14$ and the straight line $y = 9x - 32$ if the tangent drawn to the parabola at the point $x = 3$ is known to make an angle $\pi - \tan^{-1}6$ with the x-axis. [**Ans.** $A = \int_{-2}^{3} [-3x^2 + 12x - 14] - (9x - 32)]dx$]

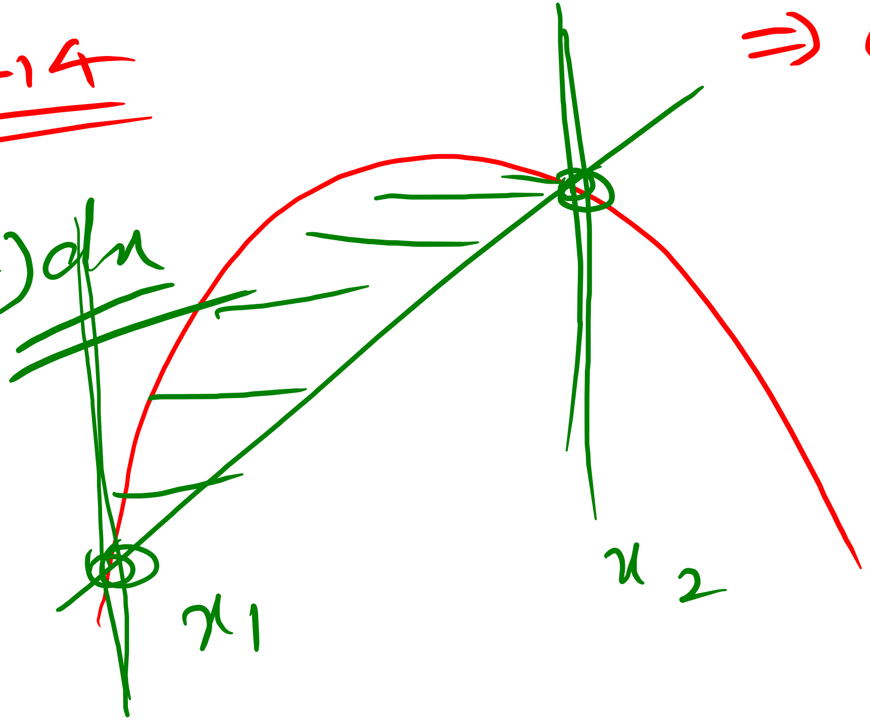
$\tan\theta = \tan(\pi - \tan^{-1}6) = -6$

$\frac{dy}{dx} = 2ax + 12 \Rightarrow -6 = +6a + 12 \Rightarrow -18 = +6a$

$\Rightarrow \boxed{a = -3}$

$y = -3x^2 + 12x - 14$

$\int_{x_1}^{x_2} (-3x^2 + 12x - 14 - 9x + 32)\,dx$



E(3) For what value of 'a' is the area of the figure bounded by the curves, $y = \frac{1}{x}$, $y = \frac{1}{2x-1}$, $x = 2$ & $x = a$ is equal to $\ln\frac{4}{\sqrt{5}}$?

[**Ans.** $a = 8, \frac{2}{5}(6-\sqrt{21})$]

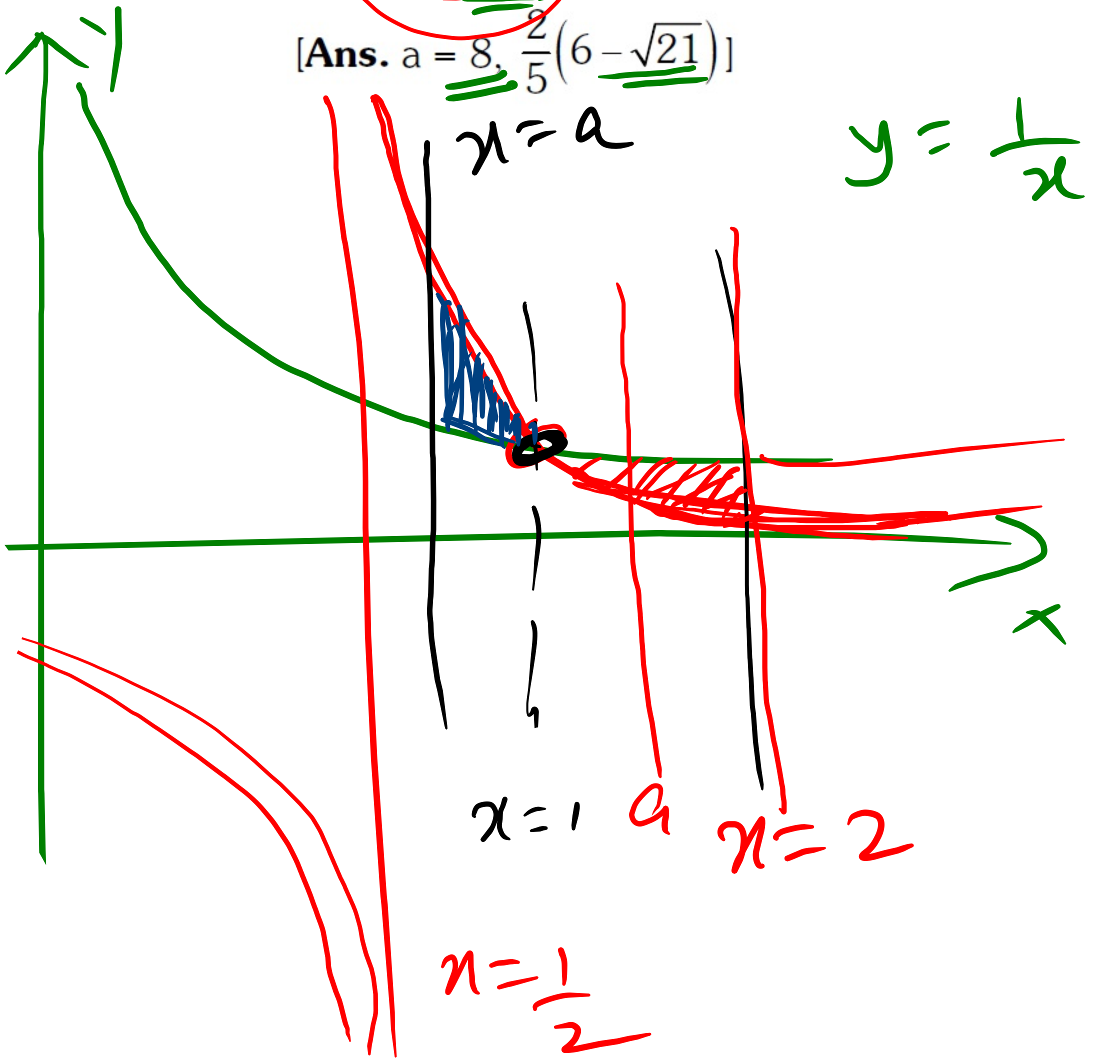
$x = 2x - 1 \Rightarrow \boxed{x = 1}$

$a < 1$

$\int_1^2 \left(\frac{1}{x} - \frac{1}{2x-1}\right)dx < \ln\frac{4}{\sqrt{5}}$

Case I: $a < 1$

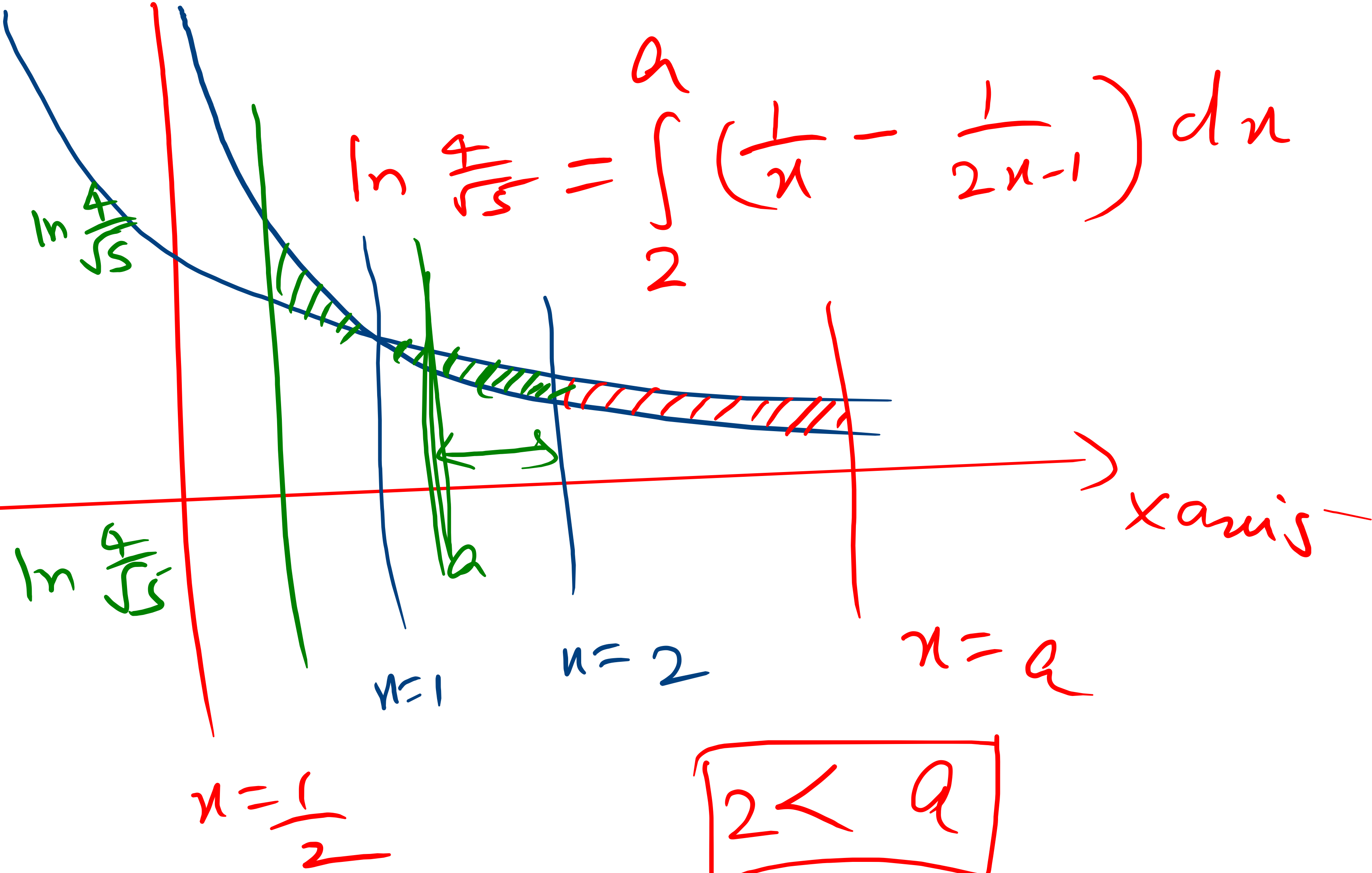
Case II: $a > 2$



$$\int_1^2 (\quad)\,dx > \ln\frac{4}{\sqrt{5}}$$

$$\int_1^2 (\quad)\,dx < \ln\frac{4}{\sqrt{5}}$$

$$\ln\frac{4}{\sqrt{5}} = \int_2^a \left(\frac{1}{x} - \frac{1}{2x-1}\right)dx$$



$$\boxed{2 < a}$$

Q: **E(4)** If the area bounded by $y = x^2 + 2x - 3$ and the line $y = kx + 1$ is least. Find k and also the least area.

How? How?

$y = kx + 1$
$(0, 1)$

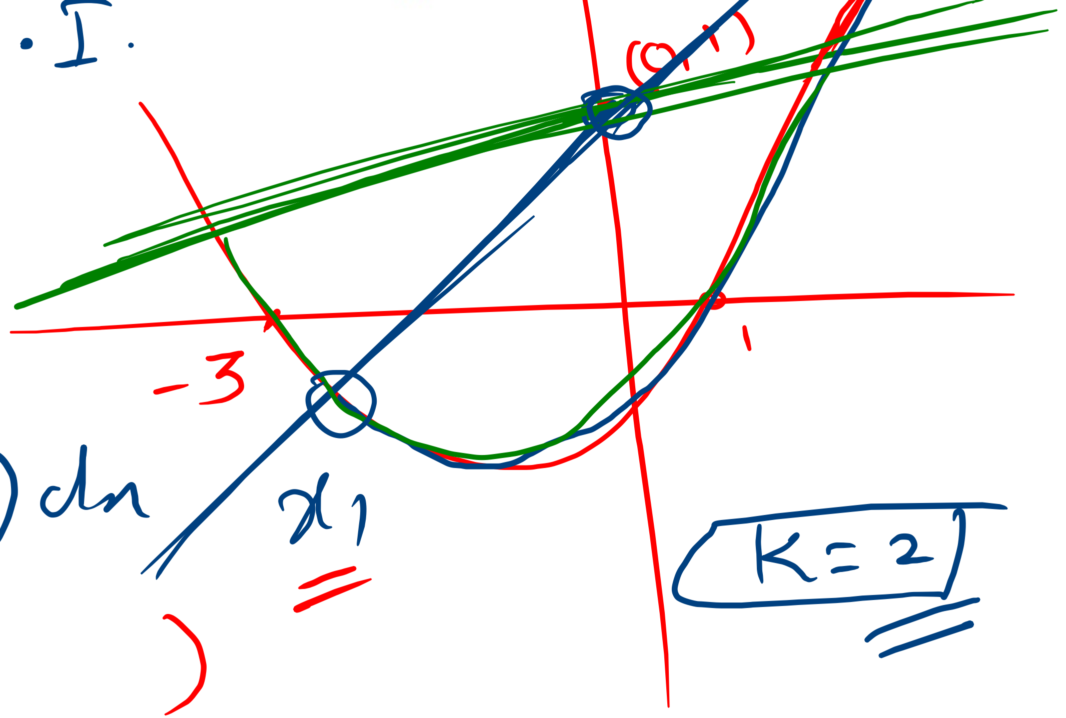
[**Ans.** $k = 2$, $A_{min} = \frac{32}{3}$]

For what value of k is the area of the figure bounded by the curves $y = x^2 - 3$ and $y = kx + 2$ is the least. Determine the least area. [**Ans.** $k = 0$, $a = \frac{20\sqrt{5}}{3}$]

Area Variable | P.O.I.

$y = kx + 1$

$$A(k) = \int_{x_1}^{x_2} (kx + 1 - x^2 - 2x + 3)\,dx = (x_1 - x_2)(\quad)$$



$$kx+1 = x^2+2x-3$$

$$x^2+(2-k)x-4=0 \Rightarrow \begin{cases} x_1+x_2 = k-2 \\ x_1x_2=-4 \end{cases}$$

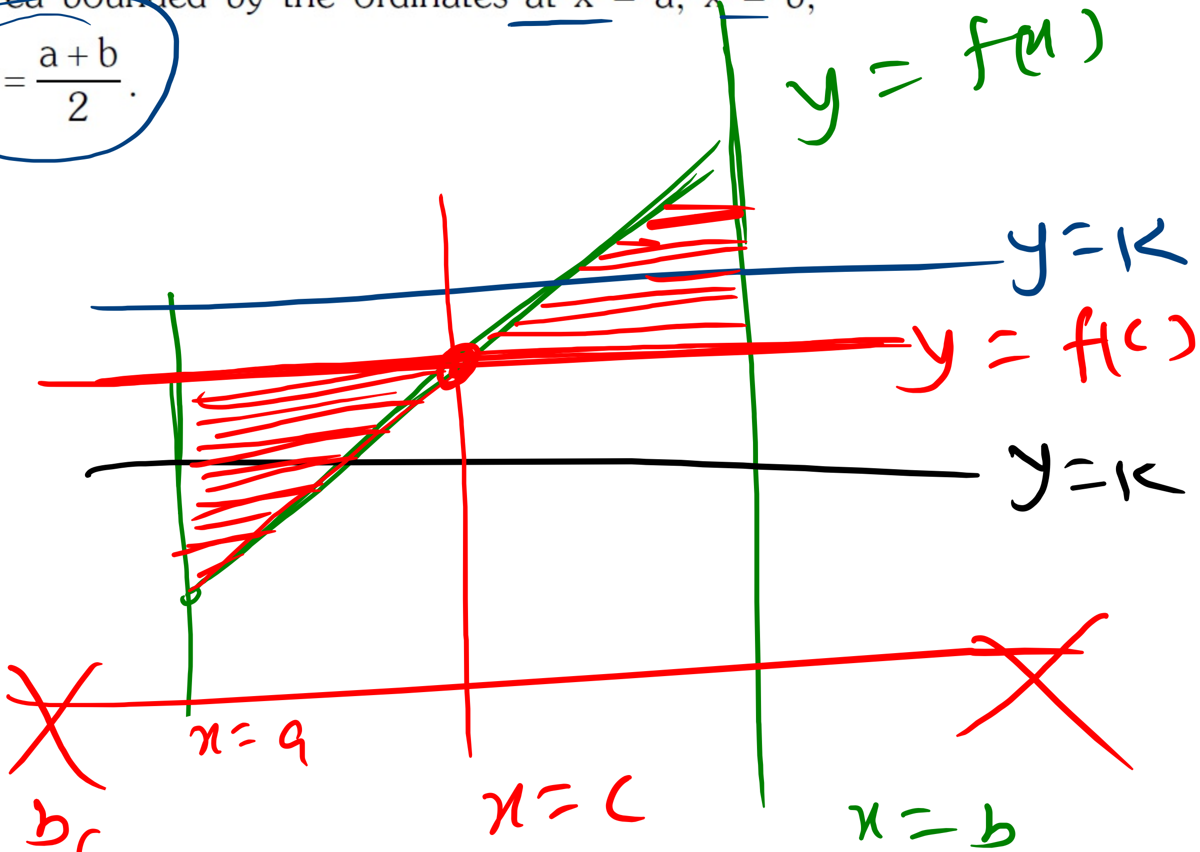
$$(x_1-x_2)^2 = (x_1+x_2)^2 - 4x_1x_2 = (k-2)^2+16$$

$$\int_{x_1}^{x_2} (\quad) \, dx = A(k)$$

$$\boxed{y=f(c) \text{ lim}} \qquad \frac{dA}{dk}=0 \Rightarrow \boxed{c=\frac{a+b}{2}}$$

CONCEPT OF VARIABLE AREA (greatest and least value) :

If $y = f(x)$ is a monotonic function in (a, b) then the area bounded by the ordinates at $x = a$, $x = b$, $y = f(x)$ and $y = f(c)$, [where $c \in (a, b)$] is minimum when $c = \frac{a+b}{2}$.

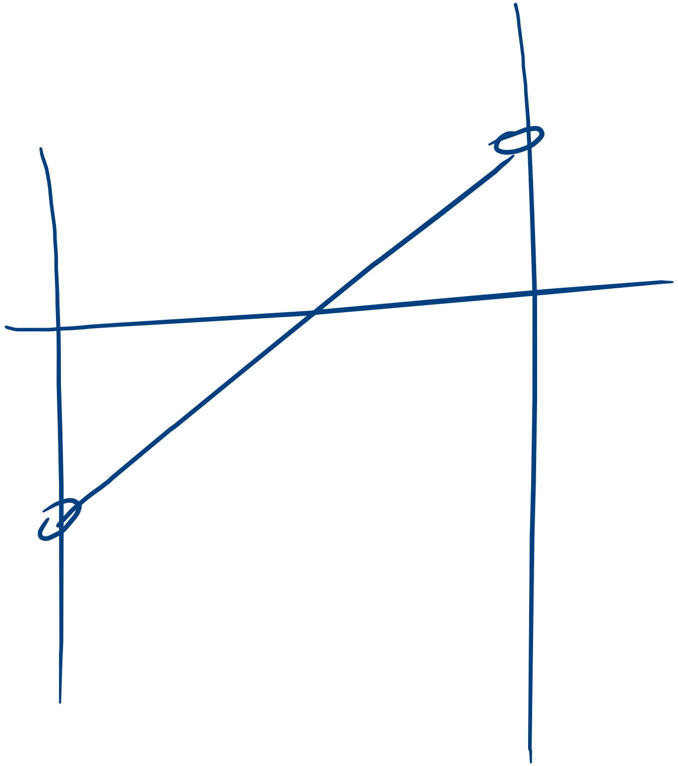


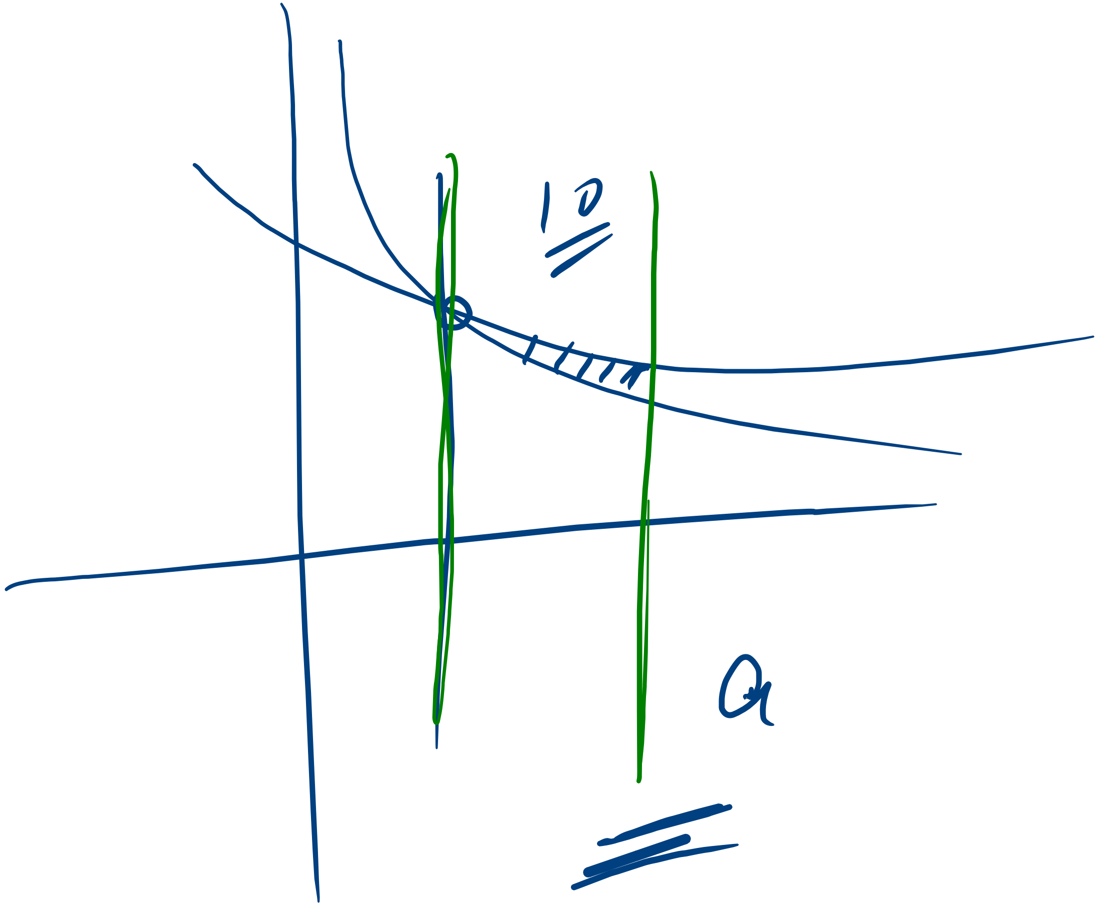
Always minm at

$$x = \frac{a+b}{2} = c$$

$$y = f(c) = K$$

$$A = \int_a^c (f(c) - f(x))\,dx + \int_c^b (f(x) - f(c))\,dx \Rightarrow \frac{dA}{dc} = 0$$




1D

Q. Find the value of the parameter 'a' for which the area of the figure bounded by the abscissa axis, the graph of the function $y = x^3 + 3x^2 + x + a$ and the straight lines which are parallel to the axis of ordinates and cut the abscissa axis at the point of extremum of the function, is the least. [**Ans.** $a = -1$]

$$\frac{dy}{dx} = 3x^2 + 6x + 1 = 0 \quad \begin{cases} \alpha \\ \beta \end{cases}$$

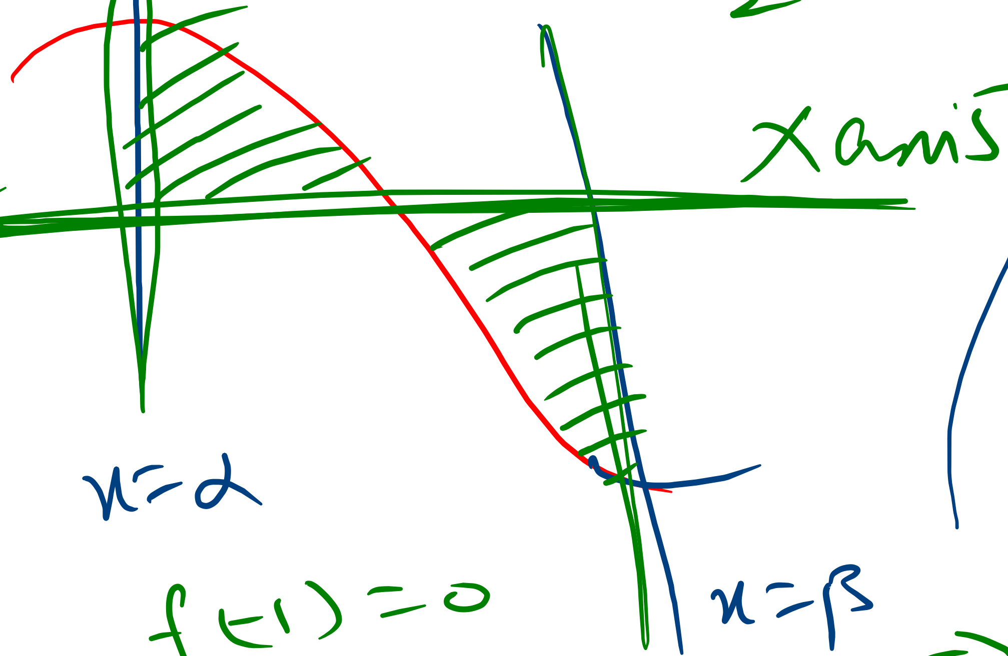
$\alpha < \beta$

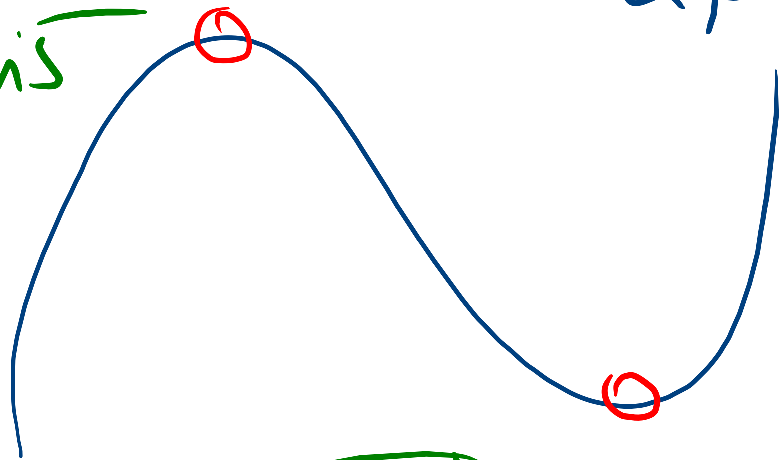
$\alpha + \beta = -2$

$\alpha\beta = \frac{1}{3}$

$C = \frac{\alpha+\beta}{2} = -1$

$y = f(x) = 0$





$c = \frac{\alpha+\beta}{2}$

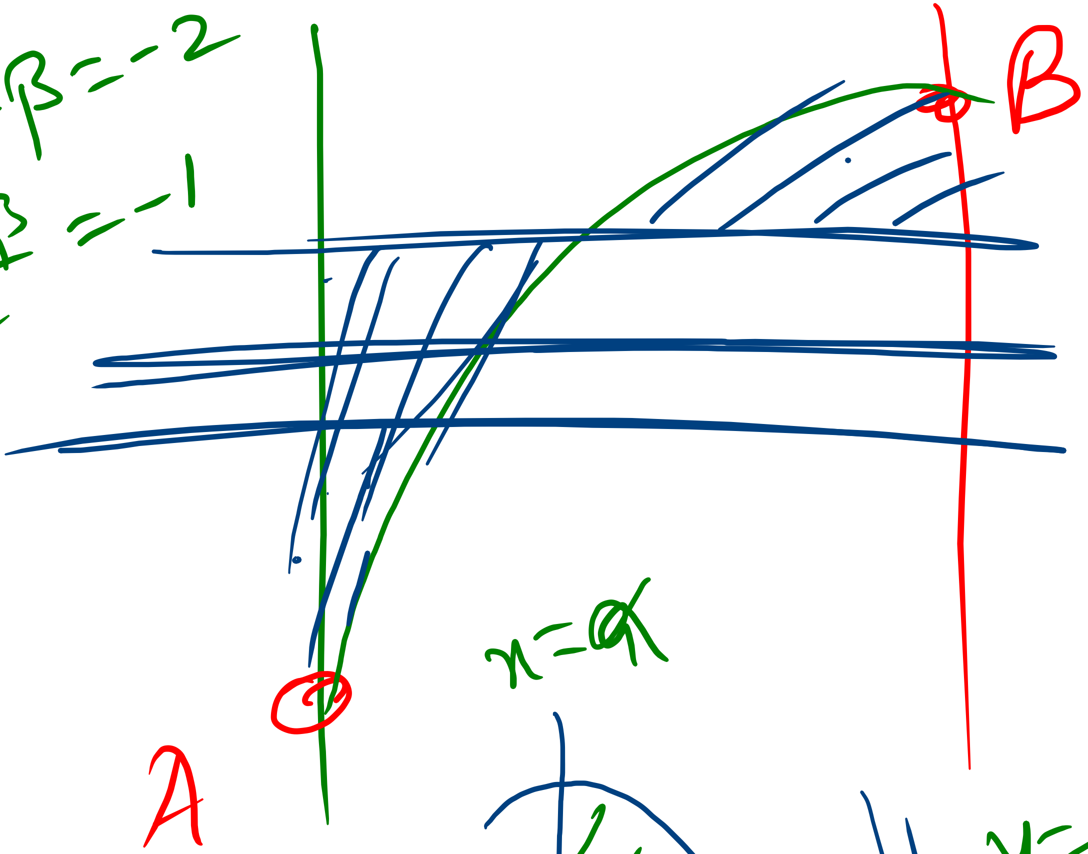
$c = -1$

$f(-1) = 0$

$\Rightarrow -1 + 3 + (-1) + a = 0 \Rightarrow \boxed{a = -1}$

$\alpha+\beta=-2$

$\frac{\alpha+\beta}{2}=-1$



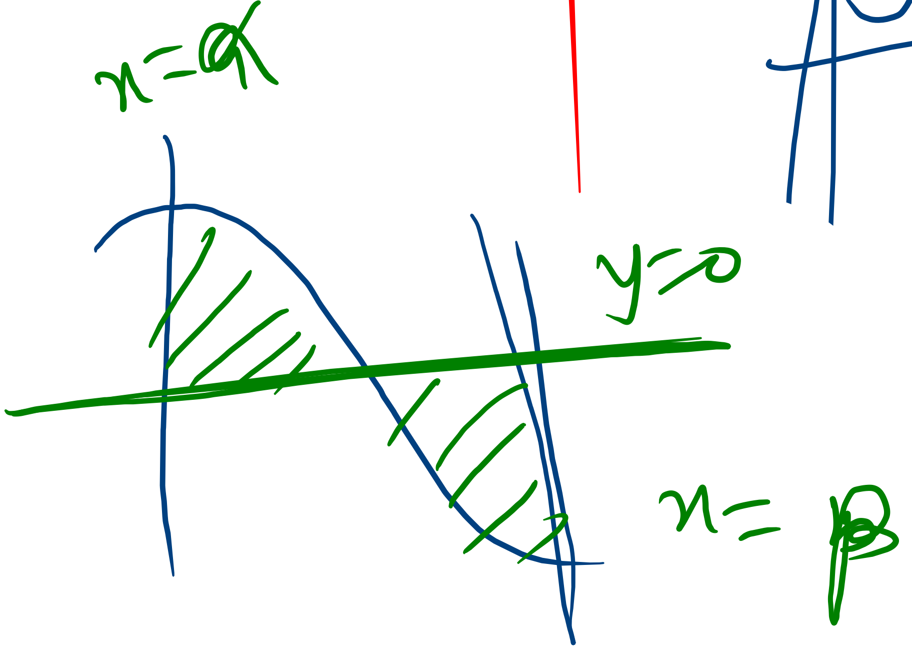
Monotonic

line को

$y=f(c)$

होना होगा।

जहाँ $c=\frac{a+b}{2}$



$f(c)=0$

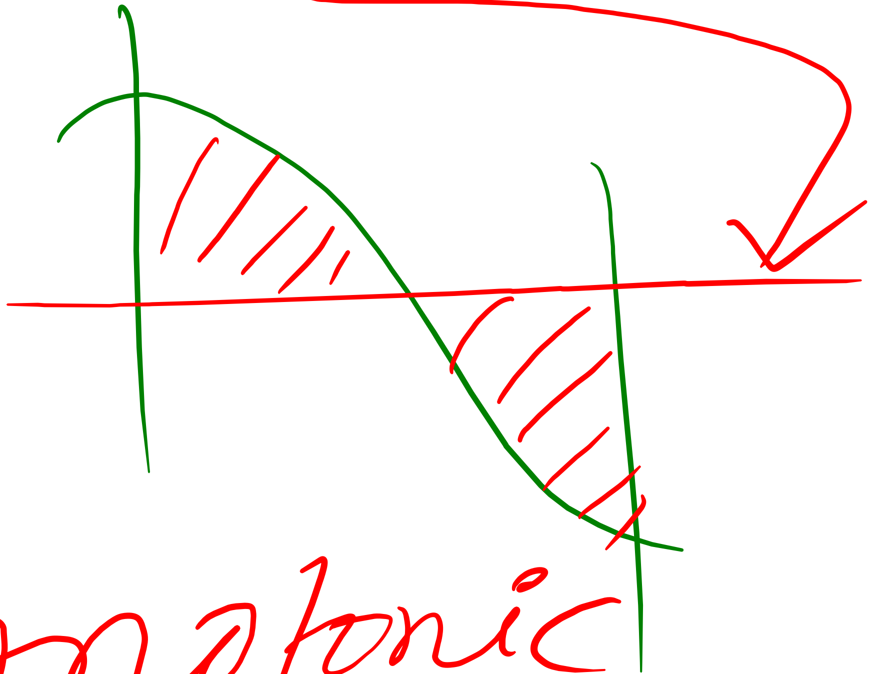
$c=\frac{\alpha+\beta}{2}$

$\boxed{c=-1}$

$f(-1)=0$

$a=-1$

Q. If the area bounded by $f(x) = \frac{x^3}{3} - x^2 + a$ and the straight lines $x = 0$; $x = 2$ and the x-axis is minimum then find the value of 'a'. [**Ans.** 2/3]



$a = \frac{2}{3}$

0

$\parallel$

$y = f(c)$

$c = \frac{0+2}{2} = 1$

$f(c) = \frac{1}{3} - 1 + a$

Monotonic

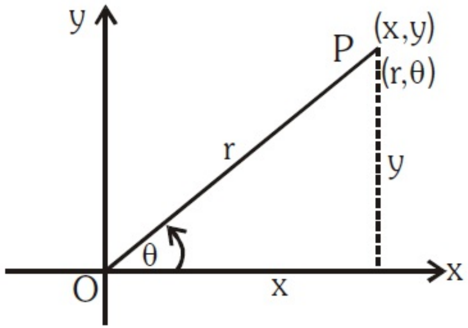
CO-ORDINATE SYSTEM :

Rectangular Cartesian co-ordinates system :

x-coordinate $\rightarrow$ abscissa

y-coordinate $\rightarrow$ ordinate

Polar system :



Note :

1. Radius vector is positive if measured from O along line bounding θ and negative if measured in opposite direction.

2. Polar coordinates of a point are not unique e.g. $\left(-3,\frac{\pi}{6}\right) \equiv \left(3,\frac{7\pi}{6}\right) \equiv \left(3,\frac{-5\pi}{6}\right)$

Find the area bounded by $\begin{cases} y = x^2 \text{ and} \\ y = \dfrac{2}{1+x^2} \end{cases}$

$$x^2 = \frac{2}{1+x^2} \Rightarrow x = 1, -1$$

$$\int_{-1}^{1} \left| \frac{2}{1+x^2} - x^2 \right| dx$$