Integration

Indefinite Integration | Definite Integration

Integration

Reverse process of differentiation (Converse) | Process of summation

Indefinite Integration :-

$$(x^2+1)' = 2x \checkmark$$

$$(x^2+5)' = 2x$$

$$(x^2+\sqrt{6})' = 2x$$

$$\int 2x \, dx = \underbrace{x^2 + C}.$$

ANTIDERIVATIVE :

A function $\phi(x)$ is called antiderivative of f(x) on the interval [a, b] if at all points of interval $\phi'(x) = f(x)$

{It is reverse process of differentiation. In differentiation we use to calculate derivative of given function and here we calculate the function whose derivative is given}

For Ex. $(\sin x)' = \cos x \Rightarrow$ antiderivative of $\cos x = \sin x$

antiderivative of $x^2 = \dfrac{x^3}{3}$ as $\left(\dfrac{x^3}{3}\right)' = x^2$

But antiderivative of x^2 can be $\dfrac{x^3}{3}+5$ as $\left(\dfrac{x^3}{3}+5\right)' = x^2$

But antiderivative of x^2 can be $\dfrac{x^3}{3}+8$ as $\left(\dfrac{x^3}{3}+8\right)' = x^2$

So antiderivative of x^2 is not a unique function. It can be any member of family of function $\dfrac{x^3}{3}+C$ as $\left(\dfrac{x^3}{3}+C\right)' = x^2$

Theorem : If $\phi_1(x)$ & $\phi_2(x)$ are two antiderivatives of a function f(x) on [a, b] then difference between them is constant.

Proof : Let $\phi_1(x)$ & $\phi_2(x)$ are two antiderivatives of a function f(x)

$$\Rightarrow \phi_1'(x) = f(x) = \phi_2'(x) \qquad \Rightarrow \phi_1'(x) - \phi_2'(x) = 0 \quad \Rightarrow \quad \phi_1(x) - \phi_2(x) = \text{const.}$$

Indefinite integral :

If $\phi(x)$ is antiderivative of f(x) then $\phi(x) + C$ is the indefinite integral of f(x) and is denoted by

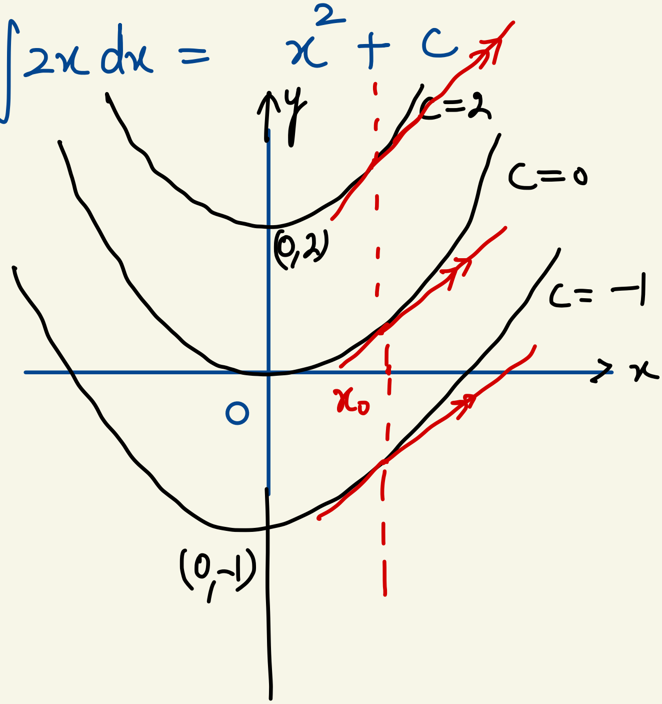
$$\int \underbrace{f(x)}_{\text{Integrand}}\underbrace{dx}_{\text{wrt 'x'}} = \underbrace{\phi(x)}_{\text{Primitive / anti-derivative / integral.}} + C \leftarrow \text{Constant of integration}$$

$\int$: Sign of integration

Geometrical interpretation of indefinite integration :-

$\int f(x)\,dx$ geometrically denotes family of curves all satisfying some common property such that the slope of tangent drawn to these curves at some $x = x_0$ is same.

eg: $\int 2x\,dx = x^2 + c$



PROPERTIES OF INTEGRATION :

1. $\int (f_1(x) \pm f_2(x))dx = \int f_1(x)dx \pm \int f_2(x)dx$

2. $\int af(x)dx = a\int f(x)dx$; $a \in$ Constant

***3.** If $\int f(x)dx = \phi(x) + C \Rightarrow \int f(ax+b)dx = \frac{1}{a}\phi(ax+b) + C$

Proof : $\phi'(x) = f(x)$

$\phi'(ax+b) = f(ax+b)$

$\frac{d}{dx}\phi(ax+b) = af(ax+b)$

eg: $\int \sin(\underbrace{2x+3}_{\text{Linear poly}})dx$

$= -\frac{\cos(2x+3)}{2} + C$

4. $\frac{d}{dx}\left(\int f(x)dx\right) = f(x)$

5. $\int \left(\frac{d}{dx}f(x)\right)dx = f(x) + C$

Integral form (LOVING)

1. $$\int (ax+b)^n\,dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C$$

$n \neq -1,\ n \in R$

eg: $$\int (2x+1)^{3/4}\,dx = \frac{(2x+1)^{\frac{3}{4}+1}}{2\left(\frac{3}{4}+1\right)} + C$$

2. $$\int \frac{dx}{ax+b} = \frac{\ln|ax+b|}{a} + C$$

eg: $$\int \frac{dx}{-5x+1} = \frac{\ln|-5x+1|}{-5} + C.$$

3. (a) $$\int a^{px+q}\,dx = \frac{1}{p}\frac{a^{px+q}}{\ln a} + C \quad (a > 0)$$

(b) $$\int e^{ax+b}\,dx = \frac{e^{ax+b}}{a} + C$$

eg: $$\int 2^{5x+7}\,dx = \frac{1}{5}\left(\frac{2^{5x+7}}{\ln 2}\right) + C.$$

4. (a) $$\int \sin(ax+b)\,dx = -\frac{1}{a}\cos(ax+b) + C\ ;$$

(b) $$\int \cos(ax+b)\,dx = \frac{1}{a}\sin(ax+b) + C$$

5. (a) $$\int \sec^2(ax+b)\,dx = \frac{1}{a}\tan(ax+b) + C$$

(b) $$\int \operatorname{cosec}^2(ax+b)\,dx = -\frac{1}{a}\cot(ax+b) + C$$

6. (a) $$\int \sec(ax+b).\tan(ax+b)\,dx$$

$$= \frac{1}{a}\sec(ax+b) + C$$

(b) $$\int \operatorname{cosec}(ax+b).\cot(ax+b)\,dx$$

$$= -\frac{1}{a}\operatorname{cosec}(ax+b) + C$$

where

$C \rightarrow$ Constant of Integration

7. $$\int \frac{dx}{1+x^2};\ \int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$$

$$\int \frac{dx}{\sqrt{1-x^2}};\ \int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} + C$$

$$\int \frac{dx}{x\sqrt{x^2-1}};\ \int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\sec^{-1}\frac{x}{a} + C$$

eg: $$\int \frac{dx}{4x^2+1} = \frac{1}{4}\int \frac{dx}{x^2+\left(\frac{1}{4}\right)}$$

$$= \frac{1}{4}\left(\frac{1}{(1/2)}\tan^{-1}\left(\frac{x}{1/2}\right)\right) + C$$

$$-\frac{1}{a}\cot^{-1}\frac{x}{a} + C = \frac{1}{a}\left(\frac{\pi}{2} - \cot^{-1}\frac{x}{a}\right) + C$$

Q1 $\int \frac{1}{2\sqrt{x}}\,dx = \frac{1}{2}\int x^{-1/2}\,dx$

$$\frac{1}{2}\left(\frac{x^{-1/2+1}}{-\frac{1}{2}+1}\right) + C = \sqrt{x} + C.$$

2) $\int 2^{\ln x}\,dx = \int x^{\overbrace{\ln 2}^{e}}\,dx$

$$= \frac{x^{\ln 2+1}}{\ln 2+1} + C.$$

$$= \frac{x^{\ln(2e)}}{\ln(2e)} + C$$

3) $\int \frac{dx}{3-2x} = \frac{\ln|3-2x|}{-2} + C.$

4) $\int \frac{x}{x^2+2x+1}\,dx = \int \frac{x}{(x+1)^2}\,dx$

$$= \int \frac{x+1-1}{(x+1)^2}\,dx$$

$$= \int \frac{1}{x+1}\,dx - \int (x+1)^{-2}\,dx$$

$$= \ln|x+1| - \frac{(x+1)^{-2+1}}{(-2+1)} + C.$$

5) $\int \frac{2x+3}{x^2+3x-10}\,dx$

$$\int \frac{2x+3}{(x+5)(x-2)}\,dx = \int \frac{(x+5)+(x-2)}{(x+5)\,(x-2)}\,dx$$

$$\int \frac{dx}{x-2} + \int \frac{dx}{x+5}$$

$$\ln|x-2| + \ln|x+5| + C.$$

6) $$\int \frac{8x+13}{\sqrt{4x+7}}\,dx = \int \frac{2(4x+7) - 1}{\sqrt{4x+7}}\,dx$$

$$\int 2\,(4x+7)^{1/2}\,dx - \int (4x+7)^{-1/2}\,dx$$

$$2\,\frac{(4x+7)^{3/2}}{4\cdot\left(\frac{3}{2}\right)} - \frac{(4x+7)^{1/2}}{4\left(\frac{1}{2}\right)} + C.$$

7) $$\int \frac{\sqrt{x^4+x^{-4}+2}}{x^3}\,dx = \int \frac{\sqrt{\left(x^2+x^{-2}\right)^2}}{x^3}\,dx$$

$$\int \frac{x^2+x^{-2}}{x^3}\,dx = \int \frac{1}{x}\,dx + \int x^{-5}\,dx$$

8) $\int \sqrt{1+\sin 2x}\, dx$

$$\int |\underbrace{\sin x + \cos x}| \, dx$$

$$= \operatorname{sgn}(\sin x + \cos x)\left(-\cos x + \sin x\right) + C$$

9) $$\int a^{mx} \cdot b^{nx} \, dx = \int (a^{m} \cdot b^{n})^{x} \, dx$$

$$= \frac{(a^{m} \cdot b^{n})^{x}}{\ln(a^{m} \cdot b^{n})} + C.$$

10) $$\int \frac{2^{x+1} - 5^{x-1}}{10^{x}} \, dx = \int \left(2\,(0.2)^{x} - \frac{1}{5}\,(0.5)^{x}\right) dx$$

$$2\,\frac{(0.2)^{x}}{\ln(0.2)} - \frac{1}{5}\,\frac{(0.5)^{x}}{\ln(0.5)} + C$$

11) $\int \frac{e^{3x}+e^{5x}}{e^{x}+e^{-x}}\,dx$

$$\int \frac{e^{3x}\cancel{(1+e^{2x})}\cdot e^{x}}{\cancel{(e^{2x}+1)}}\,dx = \int e^{4x}\,dx$$

$$= \frac{e^{4x}}{4} + C.$$

Rem

Note:-

$\int \sin^2 x\,dx \;\Big|\; \int \cos^2 x\,dx$ ✓

$\int \sin^3 x\,dx \;\Big|\; \int \cos^3 x\,dx$ ✓

$\int \sin^4 x\,dx \;\Big|\; \int \cos^4 x\,dx$ ✓

(1) $\int \sin^2 x\,dx = \frac{1}{2}\int (1-\cos 2x)\,dx$

*

$$= \frac{1}{2}\left(x - \frac{\sin 2x}{2}\right) + C.$$

$$* \int \sin^3 x \, dx = \int \frac{3\sin x - \sin 3x}{4} \, dx$$

$$\boxed{\sin 3x = 3\sin x - 4\sin^3 x}$$

$$* \int \sin^4 x \, dx = \frac{1}{4} \int (2\sin^2 x)^2 \, dx$$

$$\frac{1}{4} \int (1 - \cos 2x)^2 \, dx$$

$$\frac{1}{4} \int (1 + \cos^2 2x - 2\cos 2x) \, dx$$

Q)* $\displaystyle\int \frac{\cos 5x + \cos 4x}{\underbrace{2\cos 3x - 1}} \, dx$

$$\int \frac{2\cos\frac{9x}{2}\cos\frac{x}{2}}{(2\cos 3x - 1)} \, dx = \int \frac{2\cos\frac{x}{2}\cos\left(3\cdot\frac{3x}{2}\right)}{(\qquad)} \, dx$$

$$\int \frac{2\cos\frac{x}{2}\left(4\cos^3\frac{3x}{2} - 3\cos\frac{3x}{2}\right)}{(\qquad)} \, dx$$

$$\int 2\cos\frac{x}{2}\cos\frac{3x}{2} \, \frac{\cancel{\left(4\cos^2\frac{3x}{2} - 3\right)}}{\cancel{\left(2\left(2\cos^2\frac{3x}{2} - 1\right) - 1\right)}} \, dx$$

$$\int (\cos 2x + \cos x) \, dx$$

Rem ★★

$\int \tan^2 x \, dx$ | $\int \cot^2 x \, dx$

Rem

$$\boxed{\begin{aligned} \int \sec^2 x \, dx &= \tan x + C. \\ \int \operatorname{cosec}^2 x \, dx &= -\cot x + C \end{aligned}}$$

$\int \tan^2 x \, dx = \int (\sec^2 x - 1) \, dx = \tan x - x + C$

Rem

$$\boxed{\int \tan^2 x \, dx = (\tan x - x) + C}$$

Rem

$$\boxed{\int \cot^2 x \, dx = (-\cot x - x) + C.}$$

$$* \int \frac{\sin^4 x}{\cos^2 x}\,dx \quad \Big| \quad \int \underbrace{\tan^2 x \cdot \sin^2 x}\,dx$$

$$* \int \frac{\cos^4 x}{\sin^2 x}\,dx \quad \Big| \quad \int \cot^2 x \cdot \cos^2 x\,dx$$

$$* \int \underbrace{\sec^2 x \cdot \csc^2 x}\,dx$$

$$\tan^2 x \cdot \sin^2 x = \tan^2 x - \sin^2 x$$

$$\int \frac{\sin^4 x}{\cos^2 x}\,dx = \int (\tan^2 x - \sin^2 x)\,dx$$

$$\cot^2 x \cdot \cos^2 x = \cot^2 x - \cos^2 x$$

$$\sec^2 x \, \csc^2 x = \underbrace{\sec^2 x} + \underbrace{\csc^2 x}$$

Q) (i) $\int \frac{dx}{4+x^2}$

$\frac{1}{2}\underbrace{\tan^{-1}\left(\frac{x}{2}\right)} + C$

(ii) $\int \frac{dx}{9x^2+16}$.

$\frac{1}{9}\int \frac{dx}{x^2+\left(\frac{4}{3}\right)^2}$

$\frac{1}{9}\left(\frac{1}{\left(\frac{4}{3}\right)}\tan^{-1}\left(\frac{x}{4/3}\right)\right)$

$+C$

Q) (i) $\int \frac{dx}{\sqrt{2-x^2}}$

$\sin^{-1}\left(\frac{x}{\sqrt{2}}\right) + C$

(ii) $\int \frac{dx}{\sqrt{9-4x^2}}$

$\frac{1}{2}\int \frac{dx}{\sqrt{\left(\frac{3}{2}\right)^2 - x^2}}$

$\frac{1}{2}\sin^{-1}\left(\frac{x}{3/2}\right) + C$

Q) $\int \frac{dx}{x\sqrt{x^2-9}}$

$\frac{1}{3}\sec^{-1}\left(\frac{x}{3}\right) + C.$

Q) $\int \frac{x^2}{1+x^2}\, dx$

$$\int \frac{x^2+1-1}{1+x^2}\, dx = \int dx - \int \frac{dx}{1+x^2}.$$

Q) $\int \frac{x^4}{1+x^2}\, dx$

$$\int \frac{x^4-1 \quad +1}{1+x^2}\, dx$$

$$\int (x^2-1)\, dx + \int \frac{dx}{1+x^2}.$$

Q $\displaystyle\int \frac{x^2+3}{x^6(x^2+1)}\,dx$

$$\int \frac{x^2+1 \;+\; 2}{x^6\,(x^2+1)}\,dx \;=\; \int x^{-6}\,dx + 2\int \frac{dx}{x^6(x^2+1)}$$

$$\frac{x^{-5}}{-5} + 2\int \frac{\overbrace{(x^6+1)} - x^6}{x^6\,(x^2+1)}\,dx$$

$$-\frac{1}{5}x^{-5} + 2\left[\int \frac{(x^2)^3+1}{x^6\,(x^2+1)}\,dx - \int \frac{dx}{x^2+1}\right]$$

$$-\frac{1}{5}x^{-5} + 2\left[\underbrace{\int \frac{x^4-x^2+1}{x^6}\,dx} - \tan^{-1}x\right] + C$$

Q $$\int \frac{dx}{\underbrace{(2x-7)}\sqrt{\underbrace{(x-3)(x-4)}}}.$$

$$\int \frac{dx}{(2x-7)\sqrt{(4x^2-28x+49)-1}}$$

$$2\int \frac{dx}{(2x-7)\sqrt{(2x-7)^2-1}} = \cancel{2}.\frac{\sec^{-1}(2x-7)}{\cancel{2}} + c$$

$$* \int \frac{dx}{x\sqrt{x^2-a^2}}$$

$(2x-7)^2 =$

$4x^2+49-28x$

$(x-3)(x-4)$

$(x^2-7x+12.)$

Q A function g defined for all positive real numbers, satisfies $g'(x^2)=x^3$ for all $x>0$ and $g(1)=1$. Compute $g(4)$.

$\boxed{g'(x^2) = x^3}$. ; $x>0$; $\boxed{g(1) = 1.}$

$x^2 = t \Rightarrow x = \sqrt{t}$

$g'(t) = t^{3/2}$.

$\int g'(t)\,dt = \int t^{3/2}\,dt$

$g(t) = \dfrac{2t^{5/2}}{5} + C \Rightarrow \boxed{g(t) = \dfrac{2}{5}t^{5/2} + \dfrac{3}{5}}$

$g(1) = \dfrac{2}{5} + C$

$1 - \dfrac{2}{5} = C \Rightarrow \boxed{C = \dfrac{3}{5}}$

Techniques of Integration

- Substitution
- By part (product rule) of integration
- Partial (fraction)
- Kuturputur & Misc.

INTEGRATION BY SUBSTITUTION :

$$I = \int f(x)\,dx \quad \text{and let } x = \phi(z)$$

$$\frac{dI}{dx} = f(x) \;;\; \frac{dx}{dz} = \phi'(z)$$

$$\Rightarrow \quad \frac{dI}{dz} = \frac{dI}{dx} \cdot \frac{dx}{dz} = f(x) \cdot \phi'(z) \quad \text{or} \quad \frac{dI}{dz} = f(\phi(z)) \cdot \phi'(z)$$

$$\text{Hence } I = \int f(\phi(z)) \cdot \phi'(z)\,dz \qquad \text{.............(i)}$$

Rem

LOVING INTEGRANDS

Formulae :

- $\int \tan x = -\ln|\cos x| + C = \ln|\sec x| + C$
- $\int \cot x = \ln|\sin x| + C$
- $\int \sec x\,dx = \ln|\sec x + \tan x| + C$ or $\ln\left|\tan\left(\frac{\pi}{4} + \frac{x}{2}\right)\right| + C$
- $\int \operatorname{cosec} x\,dx = \ln|\operatorname{cosec} x - \cot x| + C = \ln\left|\tan\frac{x}{2}\right| + C$

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = \int -\frac{dt}{t} = -\ln|t| + c$$

$$\cos x = t \Rightarrow -\sin x\,dx = dt$$

$$\int \sec x \, dx = \int \frac{\sec x(\sec x + \tan x)}{\sec x + \tan x} dx = \int \frac{dt}{t}$$

$$\sec x + \tan x = t$$

$$(\sec x \tan x + \sec^2 x) \, dx = dt$$

Q $\displaystyle\int \frac{d(x^2+1)}{\sqrt{x^2+2}}$

$x^2+1 = t.$

M-1 $\displaystyle\int \frac{dt}{\sqrt{t+1}} = \int (t+1)^{-1/2}\,dt = \frac{(t+1)^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C$

$$= \frac{(x^2+2)^{1/2}}{1/2} + C.$$

M-2 $\displaystyle\int \frac{2x\,dx}{\sqrt{x^2+2}}$

$\sqrt{x^2+2} = t.$

$x^2+2 = t^2$

$2x\,dx = 2t\,dt$

$\displaystyle\int \frac{2\cancel{t}}{\cancel{t}}\,dt = 2t + C$

$$= 2\sqrt{x^2+2} + C.$$

Q $\int \frac{\tan(\ln x)}{x} dx$

$\ln x = t \Rightarrow \frac{1}{x} dx = dt$

$\int \tan t \cdot dt = \ln|\sec t| + c.$

Q $\int \frac{\cos x}{\cos(x-a)} dx$

$x - a = t$

$dx = dt$

$\int \frac{\cos(a+t)}{\cos t} \cdot dt$

$\int \frac{\cos a \cos t - \sin a \sin t}{\cos t} dt$

$\int \cos a \, dt - \sin a \int \tan t \, dt$

$t \cdot \cos a - \sin a \ln|\sec t| + c.$

Q $\int \frac{x\cos x}{(x\sin x+\cos x)^2}\,dx$

$x\sin x+\cos x=t.$

$(x\cos x+\cancel{\sin x}-\cancel{\sin x})dx$
$=dt$

$\int \frac{dt}{t^2}.$

Q $\int \frac{\ln(\ln x)}{x\ln x}\,dx$

$\ln(\ln x)=t$

$\frac{1}{(\ln x)}\cdot\frac{1}{(x)}\,dx=dt$

$\int t\,dt.$

Q $\displaystyle\int \frac{dx}{\sin(x-1)\cos(x-2)}$

$$\frac{1}{\cos 1}\int \frac{\cos\left[(x-1)-(x-2)\right]}{\sin(x-1)\cdot\cos(x-2)}\,dx = \frac{1}{\cos 1}\left[\int \cot(x-1)\,dx + \int \tan(x-2)\,dx\right]$$

$$\frac{1}{\cos 1}\left(\ln|\sin(x-1)| + \ln|\sec(x-2)|\right) + C.$$

Q $\displaystyle\int \sec^2 x\sqrt{5+\tan x}\,dx$

$\sqrt{5+\tan x} = t \Rightarrow \quad 5+\tan x = t^2$

$\sec^2 x\,dx = 2t\,dt$

$$\int t\cdot 2t\cdot dt$$

Q $\displaystyle\int \frac{x^5}{1+x^{12}}\,dx$

$x^6 = t \Rightarrow 6x^5\,dx = dt$

$$\frac{1}{6}\int \frac{dt}{1+t^2} = \frac{1}{6}\tan^{-1}(t) + C$$

Q $\displaystyle\int \sec x \ln(\sec x + \tan x)\,dx$

$\ln(\sec x + \tan x) = t$

$\sec x\,dx = dt$

$$\int t \cdot dt$$

$$\frac{t^2}{2} + C.$$

Q $\int \frac{x^2 \tan^{-1} x^3}{1+x^6} dx$

$\tan^{-1} x^3 = t$

$\frac{1}{1+x^6} \cdot 3x^2 \cdot dx = dt$

$\frac{x^2}{1+x^6} dx = \frac{1}{3} dt$

$\frac{1}{3} \int t \cdot dt$

Q $\int \frac{\sqrt{\tan x}}{\sin 2x} dx$

$\int \frac{\sqrt{\tan x}}{2 \tan x} \sec^2 x \, dx$

$\sqrt{\tan x} = t$

$\tan x = t^2$

$\sec^2 x \, dx = 2t dt$

$\frac{1}{2} \int \frac{\sec^2 x}{\sqrt{\tan x}} dx$

$\frac{1}{\cancel{2}} \int \frac{\cancel{2t}}{\cancel{t}} dt$

Q $\displaystyle\int \frac{\ln^2(\sec x)}{\cot x}\, dx$

$\ln \sec x = t$

$\dfrac{1}{\cancel{\sec x}} \cdot \cancel{\sec x} \cdot \tan x \, dx = dt$

$\displaystyle\int t^2 \cdot dt$

Q $\displaystyle\int \frac{\tan\sqrt{x}\,\sec^2\sqrt{x}}{\sqrt{x}}\, dx$

$\tan\sqrt{x} = t$

$\sec^2\sqrt{x} \cdot \dfrac{1}{2\sqrt{x}}\, dx = dt.$

$\displaystyle 2\int t \cdot dt$

Q $\displaystyle\int \frac{\sec^4 x}{\sqrt{\tan x}}\,dx$

$\sqrt{\tan x} = t$

$\tan x = t^2$

$\sec^2 x\,dx = 2t\,dt$

$$\int \frac{(1+\tan^2 x)\ \sec^2 x}{\sqrt{\tan x}}\,dt = \int \frac{(1+t^4)\cdot 2\cancel{t}}{\cancel{t}}\,dt$$

Q $\displaystyle\int \frac{\ln^2\left(\frac{x}{x+1}\right)}{x(x+1)}\,dx$

$\ln\left(\frac{x}{x+1}\right) = t \Rightarrow \dfrac{1}{\left(\frac{x}{\cancel{x+1}}\right)} \dfrac{(x+1)\cdot 1 - x}{(x+1)^{\cancel{2}}}\,dx = dt$

$\displaystyle\int t^2 \cdot dt$

$\dfrac{1}{x(x+1)}\,dx = dt$

Q $\displaystyle\int \frac{x+e^{x}(\sin x+\cos x)+\sin x\cos x}{(x^{2}+2e^{x}\sin x-\cos^{2}x)^{2}}dx$

$x^2+2e^x\sin x-\cos^2 x = t$

$(2x+2(e^x\sin x+e^x\cos x)+2\cos x\sin x)\,dx = dt$

$\frac{1}{2}\int \frac{dt}{t^2} = \frac{1}{2}\left(-\frac{1}{t}\right)+c$

Q $\int x \cdot \sqrt{\dfrac{2\sin(x^2+1) - \sin 2(x^2+1)}{2\sin(x^2+1) + \sin 2(x^2+1)}}\, dx$

$$\int x \sqrt{\frac{\cancel{2\sin(x^2+1)}\left(1-\cos(x^2+1)\right)}{\cancel{2\sin(x^2+1)}\left(1+\cos(x^2+1)\right)}}\, dx$$

$$\int x \sqrt{\tan^2\left(\frac{x^2+1}{2}\right)}\, dx = \int x \cdot \tan\left(\frac{x^2+1}{2}\right) \cdot dx = \int \tan t\, dt.$$

*

$$\frac{x^2+1}{2} = t$$

$$\cancel{2}x\,dx = \cancel{2}dt$$

Q ** $\int \frac{dx}{e^x+1}$

$$\int \frac{e^{-x}}{1+e^{-x}}\,dx = -\int \frac{dt}{t}$$

$$1+e^{-x}=t$$

$$-e^{-x}\,dx=dt$$

$$\int \frac{e^x}{e^{2x}+e^x}\,dx.$$

$$e^x=t.$$

$$e^x\,dx=dt$$

$$\int \frac{dt}{t(t+1)}$$

$$\int \frac{(t+1)-t}{t(t+1)}\,dt$$

$$\int \frac{dt}{t}-\int \frac{dt}{t+1}$$

Q $\int \frac{e^x-1}{e^x+1}\,dx$

$$\int \frac{e^x+1-2}{e^x+1}\,dx$$

$$\int dx-2\int \frac{dx}{e^x+1}.$$

✓✓

Q $\displaystyle\int \frac{e^x(1+x)}{\sin^2(xe^x)}\,dx$

$xe^x = t$

$(e^x + x.e^x)dx = dt$

$$\int \frac{dt}{\sin^2 t} = \int \operatorname{cosec}^2 t \, dt = -\cot(t) + c$$

Q
$$\int \frac{x\,dx}{(1+x^2)+\sqrt{(1+x^2)^3}}$$

$\sqrt{1+x^2} = t \Rightarrow 1+x^2 = t^2$

$\cancel{2}x\,dx = \cancel{2}t\,dt$

$$\int \frac{t}{t^2+t^3} \cdot dt = \int \frac{dt}{t+t^2} = \int \frac{(t+1)-t}{t(t+1)}\,dt$$

Standard Substitution :

For terms of the form $x^2 + a^2$ or $\sqrt{x^2 + a^2}$, put $x = a\tan\theta$ or $a\cot\theta$.

For terms of the form $x^2 - a^2$ or $\sqrt{x^2 - a^2}$, put $x = a\sec\theta$ or $a\,\text{cosec}\,\theta$.

For terms of the form $a^2 - x^2$ or $\sqrt{a^2 - x^2}$, put $x = a\sin\theta$ or $a\cos\theta$.

$$\sqrt{\frac{a-x}{a+x}} \; ; \quad x = a\cos 2\theta$$

Remember :-

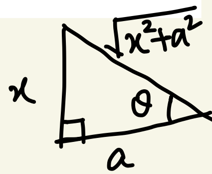
$$* \int \frac{dx}{\sqrt{\text{Quad}}} \quad ✓✓$$

- $\displaystyle\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + c$
- $\displaystyle\int \frac{dx}{\sqrt{x^2 + a^2}} = \ell n\left|x + \sqrt{x^2 + a^2}\right| + c$
- $\displaystyle\int \frac{dx}{\sqrt{x^2 - a^2}} = \ell n\left|x + \sqrt{x^2 - a^2}\right| + c$
- $\displaystyle\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + c$

$$I = \int \frac{dx}{\sqrt{x^2 + a^2}}$$

put $x = a\tan\theta \longrightarrow$

$dx = a\sec^2\theta\, d\theta$



$$\int \frac{\cancel{a}\sec^{\cancel{2}}\theta}{\cancel{a}\,\cancel{\sec\theta}} \cdot d\theta = \ln|\sec\theta + \tan\theta| + c$$

$$= \ln\left|\frac{\sqrt{x^2+a^2} + x}{a}\right| + c$$

$$= \ln\left|\sqrt{x^2+a^2} + x\right| \underbrace{- \ln|a| + c}_{C}$$

Rem:-

$$\boxed{\begin{aligned}\int \frac{dx}{x^2-a^2} &= \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C \\ \int \frac{dx}{a^2-x^2} &= \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| + C.\end{aligned}}$$

* $\int \frac{dx}{\text{Quad}}$ ✓✓

$$\int \frac{dx}{(x-a)(x+a)} = \frac{1}{2a}\int \frac{(x+a)-(x-a)}{(x-a)(x+a)}\,dx$$

$$\frac{1}{2a}\left[\int \frac{dx}{x-a} - \int \frac{dx}{x+a}\right]$$

$$\frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + C$$

Q $\displaystyle\int \frac{e^x}{4+e^{2x}}\,dx$

$e^x = t \Rightarrow e^x\,dx = dt$

$$\int \frac{dt}{4+t^2} = \frac{1}{2}\tan^{-1}\left(\frac{t}{2}\right) + C.$$

Q $\displaystyle\int \frac{dx}{a^2e^x + b^2e^{-x}}$

$$\int \frac{e^x}{a^2e^{2x}+b^2}\,dx$$

$e^x = t$

$e^x\,dx = dt$

$$\int \frac{dt}{a^2t^2+b^2} = \frac{1}{a^2}\int \frac{dt}{t^2+\left(\frac{b}{a}\right)^2}$$

$$\frac{1}{a^{\cancel{2}}}\cdot\left(\frac{1}{(b/\cancel{a})}\tan^{-1}\left(\frac{t}{b/a}\right)\right)+C$$

Q $\int \frac{e^x}{\sqrt{5-4e^x-e^{2x}}}\,dx$

$e^x = t \Rightarrow e^x\,dx = dt$

$$\int \frac{dt}{\sqrt{5-4t-t^2}} = \int \frac{dt}{\sqrt{5-(t^2+4t+4-4)}}$$

$$\int \frac{dt}{\sqrt{9-(t+2)^2}} = \sin^{-1}\left(\frac{t+2}{3}\right) + C.$$

Q $\int \frac{x}{\sqrt{(x^2-a^2)(b^2-x^2)}}\,dx$ where $b^2 > a^2$ ✓ * *

$x^2 - a^2 = t^2 \Rightarrow x^2 = a^2 + t^2$

$\cancel{2}x\,dx = \cancel{2}t\,dt$

$$\int \frac{\cancel{t}}{\cancel{t}\sqrt{b^2-a^2-t^2}}\,dt = \int \frac{dt}{\sqrt{K^2-t^2}}$$

where $\boxed{K^2 = b^2 - a^2}$

$$= \sin^{-1}\left(\frac{t}{K}\right) + C$$

$$= \sin^{-1}\left(\frac{\sqrt{x^2-a^2}}{\sqrt{b^2-a^2}}\right) + C.$$

Q $\displaystyle\int \frac{dx}{\sqrt{4x^2+1}}$

$$\frac{1}{2}\int \frac{dx}{\sqrt{x^2+\frac{1}{4}}}$$

$$\frac{1}{2}\ln\left|x+\sqrt{x^2+\frac{1}{4}}\right|+C.$$

Q $\displaystyle\int \frac{dx}{\sqrt{9x^2-4}}$

$$\frac{1}{3}\int \frac{dx}{\sqrt{x^2-\left(\frac{2}{3}\right)^2}}$$

*

$$\int \frac{dx}{\sqrt{x^2-a^2}} \checkmark$$

Q $\displaystyle\int \frac{dx}{25+16x^2} = \frac{1}{16}\int \frac{dx}{x^2+\left(\frac{5}{4}\right)^2}$.

Q $\displaystyle\int \frac{dx}{\sqrt{9-(\underbrace{2x-3})^2}} = \frac{1}{2}\sin^{-1}\left(\frac{2x-3}{3}\right) + C$

* *

Q $\displaystyle\int \frac{dx}{\sqrt{(\underbrace{4x-1})^2+25}} = \frac{1}{4}\ln\left|(4x-1)+\sqrt{(4x-1)^2+25}\right| + C$

Q $\displaystyle\int \frac{dx}{(2x+1)(x-5)}$

M-1

$$= \frac{2}{11}\int \frac{(2x+1)-(2x-10)}{(2x+1)\,(2x-10)}\,dx$$

M-2

$$\int \frac{dx}{(2x^2-9x-5)} = \frac{1}{2}\int \frac{dx}{\left(x^2-\frac{9}{2}x\right)-\frac{5}{2}}$$

$$\frac{1}{2}\int \frac{dx}{\left(x-\frac{9}{\cancel{4}}\right)^2-\frac{81}{16}-\frac{5}{2}}.$$

Q $\int \sqrt{\frac{x}{a-x}}\, dx$

$\boxed{x = a\sin^2\theta}$. $\Rightarrow$ $dx = 2a\sin\theta\cos\theta\, d\theta$

$$\int \sqrt{\frac{\cancel{a}\sin^2\theta}{\cancel{a}\cancel{\cos^2}\theta}} \cdot 2a\sin\theta \cdot \cancel{\cos\theta}\, d\theta$$

$$a\int 2\sin^2\theta\, d\theta = a\int (1-\cos 2\theta)\, d\theta.$$

$$= a\left(\theta - \frac{\sin 2\theta}{2}\right) + C.$$

Q $\displaystyle\int \frac{\tan x}{\sqrt{a+b\tan^2 x}}\,dx \;;\; b>a>0.$

M-1

$\sqrt{a+b\tan^2 x} = t \Rightarrow a+b\tan^2 x = t^2 \Rightarrow \tan^2 x = \frac{t^2-a}{b}$

$\cancel{2}b\tan x\cdot\sec^2 x\,dx = \cancel{2}t\,dt$

$\tan x\,dx = \frac{t\,dt}{b\left(1+\frac{t^2-a}{b}\right)}$

$\tan x\,dx = \frac{t}{\underbrace{b-a}+t^2}\,dt$

$\displaystyle\int \frac{\cancel{t}\,dt}{((b-a)+t^2)\cdot\cancel{t}}$

$\boxed{b-a=K^2}$

$\displaystyle\int \frac{dt}{K^2+t^2} = \frac{1}{K}\cdot\tan^{-1}\left(\frac{t}{K}\right)+C.$

M-2

$\displaystyle\int \frac{\sin x}{\sqrt{a\cos^2 x + b\sin^2 x}}\,dx$

$\sin^2 x \to (1-\cos^2 x)$

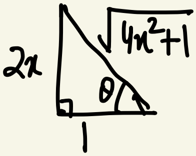
$\displaystyle\int \frac{\sin x}{\sqrt{b+(a-b)\cos^2 x}}\,dx$

$\cos x = t$

$-\sin x\,dx = dt$

Q

$$\int \frac{dx}{(x^2+4)\sqrt{4x^2+1}}$$



$$\boxed{2x = \tan\theta}$$

$$2\,dx = \sec^2\theta\, d\theta$$

$$\frac{1}{2}\int \frac{\sec^{\cancel{2}}\theta}{\left(\frac{\tan^2\theta}{4}+4\right)\cancel{\sec\theta}}\,d\theta = 2\int \frac{\sec\theta}{\tan^2\theta+16}\,d\theta$$

$$= 2\int \frac{\cos\theta}{\sin^2\theta+16\cos^2\theta}\,d\theta$$

$$= 2\int \frac{\cos\theta}{\sin^2\theta+16(1-\sin^2\theta)}\,d\theta$$

$$\boxed{\sin\theta = z}$$

$$\cos\theta\, d\theta = dz$$

$$2\int \frac{dz}{16-15z^2} = \frac{2}{15}\int \frac{dz}{\left(\frac{16}{15}-z^2\right)}$$

$$\frac{\cancel{2}}{15}\cdot\frac{1}{\cancel{2}\left(\frac{4}{\sqrt{5}}\right)}\ln\left|\frac{\frac{4}{\sqrt{15}}+z}{\frac{4}{\sqrt{15}}-z}\right| + C$$

$$\frac{1}{4\sqrt{15}}\ln\left|\frac{\frac{4}{\sqrt{15}}+\frac{2x}{\sqrt{4x^2+1}}}{\frac{4}{\sqrt{15}}-\frac{2x}{\sqrt{4x^2+1}}}\right| + C$$

Q

HW

$$\int \cos\left(2\cot^{-1}\sqrt{\frac{1-x}{1+x}}\right)dx$$

Integration by parts: (Product Rule of Integration)

$$\frac{d}{dx}[f(x).g(x)] = f(x).g'(x) + g(x).f'(x)$$

$$\therefore \quad \int \underset{\text{I}}{f(x)}.\underset{\text{II}}{g'(x)}dx = f(x).g(x) - \int g(x).f'(x)dx$$

Rem

$$\text{Let } I = \int \underset{\text{I}}{f(x)}.\underset{\text{II}}{g(x)}\,dx$$

$$= 1^{st} \text{ function} \times \text{integral of 2nd} - \int (\text{diff. coeff. of } I^{st}) \times (\text{integral of } 2^{nd})\,dx$$

ILATE RULE

Inverse ✓

Log

Algebraic

Trig ✓

exponential

$$* \quad \int \underset{\text{II}}{1}.\underset{\text{I}}{f(x)}\,dx$$

$$* \quad \int f^2(x)\,dx$$

$$\int \underset{\text{I}}{f(x)} \cdot \underset{\text{II}}{f(x)}\,dx$$

Rem :-

$$\boxed{\int \ln x \, dx = x\ln x - x + C}$$

$$\int \underset{\mathrm{II}}{1} \cdot \underset{\mathrm{I}}{\ln x} \, dx = (\ln x)(x) - \int \frac{1}{\cancel{x}} \cdot \cancel{x} \cdot dx$$

$$x \ln x - x + C.$$

$$\int \ln^2 x \, dx = \int (\underset{\mathrm{I}}{\ln x} \cdot \underset{\mathrm{II}}{\ln x}) \, dx$$

$$(\ln x)(x\ln x - x) - \int \frac{1}{x} \cdot (x\ln x - x) \, dx$$

$$\ln x (x \ln x - x) - \int (\ln x - 1) \, dx$$

Rem

$$\int \sec^3 x\,dx \Big/ \int \mathrm{cosec}^3 x\,dx$$

$$I = \int \sec^3 x\,dx$$

$$I = \int \underset{\text{II}}{\sec^2 x}\cdot \underset{\text{I}}{\sec x}\,dx$$

$$I = (\sec x)(\tan x) - \int (\sec x\tan x)(\tan x)\,dx$$

$$I = (\sec x)(\tan x) - \int \sec x(\sec^2 x - 1)\,dx$$

$$I = (\sec x)(\tan x) - I + \int \sec x\,dx$$

$$2I = (\sec x)(\tan x) + \ln|\sec x + \tan x| + C$$

$$I = \frac{1}{2}(\quad)(\quad) + \frac{1}{2}\ln|\qquad| + C.$$

Q $\int \underset{\text{II}}{x} \, \underset{\text{I}}{\tan^{-1} x} \, dx = (\tan^{-1} x) \cdot \frac{x^2}{2} - \int \frac{1}{1+x^2} \cdot \frac{x^2}{2} \cdot dx$

$= (\tan^{-1} x) \frac{x^2}{2} - \frac{1}{2} \int \frac{x^2+1-1}{1+x^2} dx$

Q $\int \underbrace{\operatorname{cosec}^2 x}_{\text{II}} \, \underbrace{\ln(\sec x)}_{\text{I}} \, dx$

$\ln(\sec x) \cdot (-\cot x) - \int \frac{\cancel{\sec x} \tan x}{\cancel{\sec x}} (-\cot x) \, dx$

$\ln(\cos x) \cot x + \int dx$

Q $\int \frac{\sin^{-1}x}{(1-x^2)^{3/2}}\,dx$

*

$\sin^{-1}x = \theta.$

$x = \sin\theta.$

$dx = \cos\theta\, d\theta$

$$\int \frac{\theta \cdot \cos\theta}{(\cos^2\theta)^{3/2}} \cdot d\theta$$

$$\int \underset{\mathrm{I}}{\theta} \cdot \underset{\mathrm{II}}{\sec^2\theta}\; d\theta.$$

Q $\int \cos^{-1}\left(\frac{1}{x}\right) dx \;;\; (x>0)$

$\sec^{-1}x = \theta$

$x = \sec\theta.$

$dx = \sec\theta \tan\theta\, d\theta$

$$\int \sec^{-1}(x)\; dx$$

$$\int \underset{\mathrm{I}}{\theta} \cdot \underbrace{(\sec\theta \tan\theta)}_{\mathrm{II}}\, d\theta$$

HW

Q. $\int \cos\left(2\cot^{-1}\sqrt{\frac{1-x}{1+x}}\right)dx$

put $\boxed{x = -\cos 2\theta}$ $\Rightarrow$ $dx = 2\sin 2\theta\, d\theta.$

$\int \cos\left(2\cot^{-1}\sqrt{\frac{1+\cos 2\theta}{1-\cos 2\theta}}\right) \cdot 2\sin 2\theta\, d\theta.$

$\int \cos\left(2\cot^{-1}\cot\theta\right) \cdot 2\sin 2\theta\, d\theta.$

$\int 2\cos 2\theta \sin 2\theta\, d\theta = \int \sin 4\theta\, d\theta$

$= -\frac{\cos 4\theta}{4} + C.$

$= -\left(\frac{2\cos^2 2\theta - 1}{4}\right) + C$

$= \frac{1-2x^2}{4} + C.$

$= -\frac{x^2}{2} + C$

Q $\int e^x (x+1) \ln(xe^x)\,dx$

$xe^x = t \Rightarrow (xe^x + e^x)\,dx = dt$

$\int \ln t \, dt = (t \ln t - t) + c$

Q $\int \underbrace{x^2}_{\mathrm{I}} \underbrace{e^{3x}}_{\mathrm{II}} dx$ $\quad\quad$ $\int e^{ax} \cdot x^n dx$

*

$n \in N$

M-1

$$x^2 \cdot \frac{e^{3x}}{3} - \int (2x) \cdot \frac{e^{3x}}{3} dx$$

$$x^2 \frac{e^{3x}}{3} - \frac{2}{3} \int \underset{\mathrm{I}}{x}\ \underset{\mathrm{II}}{e^{3x}} dx = x^2 \frac{e^{3x}}{3} - \frac{2}{3}\left(x \cdot \frac{e^{3x}}{3} - \int \frac{e^{3x}}{3} dx \right)$$

$$= x^2 \frac{e^{3x}}{3} - \frac{2}{9} x \cdot e^{3x} + \frac{2}{9} \int e^{3x} dx$$

$$= x^2 \frac{e^{3x}}{3} - \frac{2}{9} x e^{3x} + \frac{2}{27} e^{3x} + C$$

$$= e^{3x}\left(\frac{x^2}{3} - \frac{2}{9} x + \frac{2}{27} \right) + C \checkmark$$

M-2

$$\int x^2 e^{3x} dx = \underbrace{e^{3x}\left(P x^2 + Q x + R \right)}_{} + C$$

Const of integration

diff wrt 'x' :-

$$x^2 \cancel{e^{3x}} = \cancel{e^{3x}} \left(2Px + Q \right) + 3\cancel{e^{3x}} \left(Px^2 + Qx + R \right)$$

$1 = 3P \Rightarrow P = 1/3$

$0 = 2P + 3Q \Rightarrow Q = -\frac{2P}{3} = -\frac{2}{9}$ $\quad | \quad$ $0 = Q + 3R$

Q $\int x^3 \ln^2 x \, dx$

$\ln x = t \Rightarrow x = e^t$

$dx = e^t \, dt$

$\int e^{3t} \cdot t^2 \cdot e^t \cdot dt = \int e^{4t} \cdot t^2 \, dt$

✓

Rem

$$\int \underbrace{e^{ax}}_{\text{II}} \underbrace{\cos bx}_{\text{I}}\, dx \;/\; \int e^{ax} \sin bx\, dx$$

$\int e^{2x} \cos 3x\, dx \Rightarrow$ M-1 Apply By parts

M-2 $\int e^{2x} \cos 3x\, dx = e^{2x}\left(A \cos 3x + B \sin 3x\right) + C$

diff wrt 'x' :-

$$\cancel{e^{2x}} \cos 3x = \cancel{e^{2x}}\left(-3A \sin 3x + 3B \cos 3x\right) + 2\cancel{e^{2x}}\left(A \cos 3x + B \sin 3x\right)$$

$$\left.\begin{aligned} 1 &= 3B + 2A. \\ 0 &= -3A + 2B. \end{aligned}\right\} \quad \begin{aligned} A &= \\ B &= \end{aligned}$$

Q $\displaystyle\int \frac{\cos^{-1}x}{x^3}\,dx$

$\cos^{-1}x = \theta$

$x = \cos\theta$

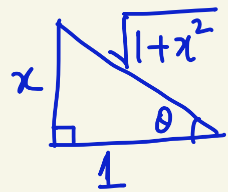
$dx = -\sin\theta\,d\theta$

$$\int \frac{\theta}{\cos^3\theta}\cdot(-\sin\theta)\,d\theta$$

$$-\int \underbrace{\theta}_{\mathrm{I}}\ (\underbrace{\tan\theta\cdot\sec^2\theta}_{\mathrm{II}})\,d\theta$$

Q $\displaystyle\int \frac{\tan^{-1}x}{x^4}\,dx$

$\tan^{-1}x = \theta$



$x = \tan\theta$

$dx = \sec^2\theta\,d\theta$

$$\int \frac{\theta}{\tan^4\theta}\cdot\sec^2\theta\cdot d\theta$$

$$\int \underbrace{\theta}_{\mathrm{I}}\cdot\underbrace{\left(\frac{\sec^2\theta}{\tan^4\theta}\right)}_{\mathrm{II}}\,d\theta$$

* $\displaystyle\int \cot^3\theta\,d\theta$

$$\int \underbrace{\cot\theta}_{\mathrm{I}}\cdot\underbrace{\cot^2\theta}_{\mathrm{II}}\cdot d\theta$$

Q $\int \ln\left(x+\sqrt{x^2+a^2}\right) dx$

$$\int \underset{\text{II}}{1} \cdot \underbrace{\ln\left(x+\sqrt{x^2+a^2}\right)}_{\text{I}} dx = \ln\left(x+\sqrt{x^2+a^2}\right) \cdot x - \int \frac{\left(1+\frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+a^2}}\right)}{\left(x+\sqrt{x^2+a^2}\right)} \cdot x \cdot dx$$

$$x \cdot \ln\left(x+\sqrt{x^2+a^2}\right) - \int \frac{x}{\sqrt{x^2+a^2}} dx$$

Q $\int \frac{x}{(1+\cos x)} dx$

$$\int \frac{x}{2} \cdot \sec^2 \frac{x}{2} \, dx$$

$\frac{x}{2} = t$

$dx = 2dt$

$$2\int \underset{\text{I}}{t} \cdot \underset{\text{II}}{\sec^2 t} \cdot dt$$

Q $\displaystyle\int \frac{x}{1+\sin x}\,dx$

$$\int \frac{x}{(1+\sin x)}\,\frac{(1-\sin x)}{(1-\sin x)}\,dx = \int \frac{x\,(1-\sin x)}{\cos^2 x}\,dx$$

$$\int \underset{\substack{\downarrow\\ \mathrm{I}}}{x}\cdot \underset{\substack{\downarrow\\ \mathrm{II}}}{\sec^2 x}\,dx - \int \underset{\substack{\downarrow\\ \mathrm{I}}}{x}\cdot \underbrace{(\tan x \sec x)}_{\mathrm{II}}\,dx$$

Q $\displaystyle\int \sin^{-1}\sqrt{\frac{x}{a+x}}\,dx$ ★★

$$\boxed{x = a\tan^2\theta.}$$

$$dx = a\cdot(2\tan\theta\,\sec^2\theta)\,d\theta$$

$$\int \sin^{-1}\sqrt{\frac{\cancel{a}\tan^2\theta}{\cancel{a}+\cancel{a}\tan^2\theta}}\;\cdot\; 2a\tan\theta\,\sec^2\theta.\,d\theta.$$

$$2a\int \big(\underbrace{\sin^{-1}\sin\theta}_{\theta}\big)\,(\tan\theta\,\sec^2\theta)\,d\theta. = 2a\int \underset{\mathrm{I}}{\theta}\cdot \underset{\mathrm{II}}{(\tan\theta\,\sec^2\theta)}\,d\theta$$

Rem

① $\int \sqrt{a^2 - x^2}\,dx = \frac{x}{2}\sqrt{a^2 - x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + C$

② $\int \sqrt{x^2 + a^2}\,dx = \frac{x}{2}\sqrt{x^2 + a^2} + \frac{a^2}{2}\ell n\left(x + \sqrt{x^2 + a^2}\right) + C$

③ $\int \sqrt{x^2 - a^2}\,dx = \frac{x}{2}\sqrt{x^2 - a^2} - \frac{a^2}{2}\ell n\left(x + \sqrt{x^2 - a^2}\right) + C$

$\int \sqrt{\text{Quad.}}\,dx$

② $I = \int \underset{\mathrm{II}}{1} \cdot \underset{\mathrm{I}}{\sqrt{x^2 + a^2}}\,dx \Rightarrow I = \sqrt{x^2+a^2} \cdot x - \int \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+a^2}} \cdot x \cdot dx$

$$I = x\sqrt{x^2+a^2} - \int \frac{x^2+a^2-a^2}{\sqrt{x^2+a^2}}\,dx$$

$$I = x\sqrt{x^2+a^2} - I + a^2\int \frac{dx}{\sqrt{x^2+a^2}}.$$

$$2I = x\sqrt{x^2+a^2} + a^2 \cdot \ln\left|x + \sqrt{x^2+a^2}\right| + C.$$

* $\displaystyle\int \frac{\text{Linear poly}}{\sqrt{\text{Quad poly}}}$ / $\displaystyle\int \frac{\text{linear poly}}{\text{Quad poly}}$ / $\displaystyle\int \text{Linear poly}\sqrt{\text{Quad. poly}}$

Rem

$$\boxed{\text{Linear poly} = A\frac{d}{dx}(\text{Quad poly}) + B}$$

Q $\displaystyle\int \frac{2x-1}{\sqrt{3x^2-4x+5}}\,dx =$

$$\boxed{2x-1 = A\cdot(6x-4) + B}$$

$2 = 6A \Rightarrow A = 1/3$

$-1 = -4A+B \Rightarrow B = 4A-1 = \frac{4}{3}-1 = \frac{1}{3}.$

$$\int \frac{\frac{1}{3}(6x-4)}{\sqrt{3x^2-4x+5}}\,dx + \frac{1}{3}\int\frac{dx}{\sqrt{3x^2-4x+5}}.$$

$3x^2-4x+5 = t^2$

eg:

$$\int (2x-1)\sqrt{3x^2-4x+5}\;dx$$

$$\underbrace{(2x-1)} = A\,(6x-4) + B \Rightarrow \quad A = 1/3, \quad B = 1/3$$

$$\int \left(\frac{1}{3}\underbrace{(6x-4)} + \frac{1}{3}\right)\sqrt{\underbrace{3x^2-4x+5}}\;dx$$

$$\frac{1}{3}\int \left(\sqrt{\underbrace{3x^2-4x+5}}\right)(6x-4)\,dx + \frac{1}{3}\int \sqrt{3x^2-4x+5}\;dx$$

Two Classic Integrands : V. Imp

(a) $\int e^x(f(x)+f'(x))dx = e^x f(x)+c$ & (b) $\int (f(x)+xf'(x))\,dx = x f(x)+c$ **

"Kahi yeh Woh toh Nahi"

Proof: (a) $\int \underset{\mathrm{II}}{e^x}\, \underset{\mathrm{I}}{f(x)}\, dx + \underbrace{\int e^x f'(x)\, dx}_{\text{hold}}$

$f(x) \cdot e^x - \cancel{\int f'(x) \cdot e^x dx} + \cancel{\int e^x f'(x) dx} + C$

(b) $\int \underset{\mathrm{II}}{1} \cdot \underset{\mathrm{I}}{f(x)}\, dx + \int x f'(x)\, dx$ hold

$f(x) \cdot x - \cancel{\int f'(x) \cdot x\, dx} + \cancel{\int x f'(x) dx} + C$

Dhokha

* $\int e^x \left(f(x) - f''(x) \right) dx = e^x\left(f(x) - f'(x)\right) + c$

$\int e^x \left(f(x) + f'(x) - f'(x) - f''(x) \right) dx = \int e^x \left(f(x) + f'(x)\right) dx - \int e^x \left(f'(x) + f''(x)\right) dx$

Q $\int e^x \left(\underbrace{\ln(\sec x + \tan x)}_{f} + \underbrace{\sec x}_{f'} \right) dx$

$e^x \left(\ln(\sec x + \tan x)\right) + C.$

Q $\int \frac{e^x (x^2+1)}{(x+1)^2} dx = \int \frac{e^x (x^2 - 1 + 2)}{(x+1)^2} dx$

$\int e^x \left(\frac{x-1}{x+1} + \frac{2}{(x+1)^2} \right) dx = e^x \cdot \left(\frac{x-1}{x+1}\right) + C.$

$f = \frac{x-1}{x+1} \Rightarrow f' = \frac{(x+1)\cdot 1 - (x-1)\cdot 1}{(x+1)^2}$

$= \frac{2}{(x+1)^2}$

Q $\displaystyle\int e^x \frac{x^2}{(x+2)^2}\,dx$

$$\int e^x \left(\frac{x^2-4+4}{(x+2)^2}\right) dx$$

$$\int e^x \left(\underbrace{\frac{x-2}{x+2}}_{f} + \underbrace{\frac{4}{(x+2)^2}}_{f'}\right) dx$$

Q $\displaystyle\int \Big(\overbrace{\sin(\ln x)}^{f} + \overbrace{\cos(\ln x)}^{xf'}\Big)\,dx$

M-1 $\ln x = t \Rightarrow x = e^t \Rightarrow dx = e^t\,dt$

$$\int e^t(\underbrace{\sin t}_{f} + \underbrace{\cos t}_{f'})\,dt = e^t \sin t + c = e^{\ln x}\cdot\sin(\ln x) + c$$

$e^{\ln x} \rightarrow x$

M-2 $f(x) = \sin(\ln x)$

$f'(x) = \cos(\ln x)\cdot\frac{1}{x} \Rightarrow \boxed{x f'(x) = \cos(\ln x)}$

Q $\displaystyle\int \frac{x+\sin x}{1+\cos x}\,dx$

$$\int \frac{x+2\sin\frac{x}{2}\cos\frac{x}{2}}{2\cos^2\frac{x}{2}}\,dx$$

$\frac{x}{2}=t$ **

$dx = 2dt$

$$\int\left(\frac{x}{2}\sec^2\frac{x}{2}+\tan\frac{x}{2}\right)dx = 2\int\left(\underbrace{t\cdot\sec^2 t}_{t f'}+\underbrace{\tan t}_{f}\right)dt$$

Q $\displaystyle\int e^x\left(\frac{1+x+x^3}{(1+x^2)^{3/2}}\right)dx$

$$\int e^x\left(\underbrace{\frac{1}{(1+x^2)^{3/2}}}_{f'}+\underbrace{\frac{x}{(1+x^2)^{1/2}}}_{f}\right)dx$$

$f(x)=\dfrac{x}{(1+x^2)^{1/2}}$

$$\underline{Q} \int e^{2x} \frac{(\sin 4x - 2)}{(1-\cos 4x)}\, dx$$

$2x = t \Rightarrow 2\,dx = dt$

$$\frac{1}{2}\int e^{t} \cdot \left(\frac{\sin 2t - 2}{1-\cos 2t}\right) dt$$

$$\frac{1}{2}\int e^{t} \left(\frac{\cancel{2}\sin t\cos t - \cancel{2}}{\cancel{2}\sin^2 t}\right) dt \cdot$$

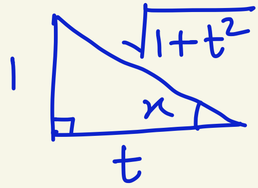
$$\frac{1}{2}\int e^{t} \Big(\underbrace{\cot t}_{f} + \underbrace{(-\operatorname{cosec}^2 t)}_{f'}\Big)\, dt$$

Q $\int e^{\cot x}\left(\cos x - \operatorname{cosec} x\right) dx$

M-1

$\cot x = t$ ✓

$-\operatorname{cosec}^2 x \, dx = dt$



$$\int e^t \left(\frac{t}{\sqrt{1+t^2}} - \sqrt{1+t^2} \right) \frac{(-)dt}{(1+t^2)}$$

$$\int e^t \left(\underbrace{\frac{1}{\sqrt{1+t^2}}}_{f} - \underbrace{\frac{t}{(1+t^2)^{3/2}}}_{f'} \right) dt.$$

$$f = \frac{1}{\sqrt{1+t^2}}$$

$$e^t \cdot \frac{1}{\sqrt{1+t^2}} + C.$$

$$e^{\cot x} \cdot \sin x + C.$$

M-2

$$\int e^{\cot x}\left(\cos x - \operatorname{cosec} x\right) dx.$$

$$\int \underset{\mathrm{I}}{e^{\cot x}} \cdot \underset{\mathrm{II}}{\cos x}\, dx - \int e^{\cot x} \cdot \underset{\text{Hold}}{\operatorname{cosec} x}\, dx$$

$$e^{\cot x} \cdot (\sin x) - \cancel{\int e^{\cot x} \cdot \left(-\operatorname{cosec}^2 x\right) \cdot (\sin x)\, dx}$$

$$- \cancel{\int e^{\cot x} \operatorname{cosec} x\, dx} + C$$

Q $\int e^{\tan^{-1}x}\left(\frac{1+x+x^2}{1+x^2}\right)dx$

$\tan^{-1}x = t \Rightarrow \frac{1}{1+x^2}\,dx = dt$

$\int e^t\left(1 + \tan t + \tan^2 t\right)dt$

$\int e^t\left(\underset{f}{\tan t} + \underset{f'}{\sec^2 t}\right)dt$

Q

HW

$$\int e^{\tan x}\left(x\sec^2 x+\sin 2x\right)dx$$

Q $\int \left(\ln(\ln x) + \frac{1}{\ln^2 x} \right) dx$

$\ln x = t \Rightarrow x = e^t \Rightarrow dx = e^t dt$

$\int e^t \left(\ln t + \frac{1}{t^2} \right) dt = \int e^t \left(\ln t + \frac{1}{t} - \frac{1}{t} + \frac{1}{t^2} \right) dt$

$\int e^t \left(\ln t + \frac{1}{t} \right) dt - \int e^t \left(\frac{1}{t} - \frac{1}{t^2} \right) dt$

$e^t \ln t - e^t \left(\frac{1}{t} \right) + C$

Q $\int \frac{\sqrt{1+\sin 2x}}{(1+\cos 2x)\, e^{-x}} \, dx \; ; \; x \in 1^{st}$ Quad

$\int \frac{e^x (\sin x + \cos x)}{2\cos^2 x} \, dx$

$\frac{1}{2} \int e^x \left(\underbrace{\tan x \sec x}_{f'} + \underbrace{\sec x}_{f} \right) dx$

Q $$\int\left(\frac{\ln x-1}{\ln^2 x+1}\right)^2 dx$$

$$\int \frac{(\ln^2 x+1)-2\ln x}{(\ln^2 x+1)^2}\,dx$$

$$\int\left(\underbrace{\frac{1}{\ln^2 x+1}}_{f}-\underbrace{\frac{2\ln x}{(1+\ln^2 x)^2}}_{x f'}\right)dx$$

$$f(x)=\frac{1}{\ln^2 x+1}$$

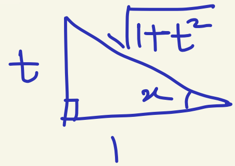
$$f'(x)=\frac{-1}{(\ln^2 x+1)^2}\cdot 2\ln x\cdot\left(\frac{1}{x}\right)$$

$$x f'(x)=\frac{-2\ln x}{(\ln^2 x+1)^2}$$

HW

$$\int e^{\tan x}\left(x\sec^2 x + \sin 2x\right)dx$$

$$\tan x = t \Rightarrow \sec^2 x\,dx = dt$$



$$\int e^{\tan x}\left(x\sec^2 x + \frac{2\tan x}{1+\tan^2 x}\right).dx$$

$$\int e^{t}\left((\tan^{-1}t)(1+t^2) + \frac{2t}{1+t^2}\right)\frac{dt}{1+t^2}$$

$$\int e^{t}\left(\tan^{-1}t + \frac{2t}{(1+t^2)^2}\right)dt$$

$$\int e^{t}\left(\tan^{-1}t + \frac{1}{1+t^2} - \frac{1}{1+t^2} + \frac{2t}{(1+t^2)^2}\right)dt$$

$$\int e^{t}\left(\tan^{-1}t + \frac{1}{1+t^2}\right)dt - \int e^{t}\left(\frac{1}{1+t^2} - \frac{2t}{(1+t^2)^2}\right)dt$$

$$e^{t}\left(\tan^{-1}t - \frac{1}{1+t^2}\right) + C \Rightarrow \boxed{e^{\tan x}\left(x - \cos^2 x\right) + C}$$

Ans.

Q $\int e^x \frac{(x^2+5x+7)}{(x^2+6x+9)} dx$

$$\int e^x \left(\frac{(x^2+6x+9)-x-2}{x^2+6x+9} \right) dx$$

$$\int e^x dx - \int \frac{e^x (x+2)}{(x+3)^2} dx$$

$$e^x - \int e^x \left(\frac{x+3 \quad -1}{(x+3)^2} \right) dx$$

$$e^x - \int e^x \left(\underbrace{\frac{1}{x+3}}_{f} + \underbrace{\left(\frac{-1}{(x+3)^2} \right)}_{f'} \right) dx$$

$$e^x - e^x \cdot \left(\frac{1}{x+3} \right) + C.$$

Rem

$$\boxed{\int e^{g(x)}\left(f(x)\,g'(x)+f'(x)\right)dx = e^{g(x)}f(x)+C.}$$

Proof: $\int \underbrace{e^{g(x)}\cdot g'(x)}_{\text{II}}\cdot \underbrace{f(x)}_{\text{I}}\,dx + \int e^{g(x)} f'(x)\,dx$

HOLD.

$$f(x)\cdot e^{g(x)} - \cancel{\int f'(x)\cdot e^{g(x)}dx} + \cancel{\int e^{g(x)} f'(x)\,dx} + C$$

Q $\int e^{\sin x}\left(\overbrace{\sin x}+\overbrace{\sec^2 x}\right)dx = e^{\sin x}\cdot \tan x + C.$

$g(x) = \sin x$

$g'(x) = \cos x$

$\sin x = g'(x)\,f(x)$

$f(x) = \tan x$

$f'(x) = \sec^2 x$

Q $\int e^{x^2}\left(\underbrace{2}_{} - \frac{1}{x^2}\right) dx$

$g(x) = x^2$

$g'(x) = 2x$

$g'(x)\, f(x) = 2$ ✓

$f(x) = \frac{1}{x}$

$f'(x) = \frac{-1}{x^2}$

$\downarrow$

$e^{x^2}\left(\frac{1}{x}\right) + C.$

Q $\int e^{(x\sin x + \cos x)}\left(\underbrace{\cos^2 x}_{} - \left(\frac{x\sin x + \cos x}{x^2}\right)\right) dx$

$g(x) = x\sin x + \cos x$

$g'(x) = x\cos x + \cancel{\sin x} - \cancel{\sin x} = x\cos x$

$f(x) = \frac{\cos x}{x}$

$f'(x) = \frac{x(-\sin x) - \cos x}{x^2}$

$\downarrow$

$e^{(x\sin x + \cos x)} \cdot \left(\frac{\cos x}{x}\right)$

$+ C.$

INTEGRALS OF TRIGONOMETRIC FUNCTION :

Type-I : **Substition for trigonometric functions of the form** $\int \sin^n x . \cos^m x \, dx$

$n, m \in N$

(i) m & n both odd :-

$$\int \sin^{75} x \cos^3 x \, dx = \int t^{75} . (1-t^2) . dt .$$

$$\sin x = t \Rightarrow \cos x \, dx = dt$$

(ii) One is odd other is even.

$$\int \sin^4 x \cos^3 x \, dx$$

$$\sin x = t \quad \Rightarrow \cos x \, dx = dt$$

$$\int t^4 (1-t^2) . dt .$$

(ii) Both even :-

$$\int \sin^4 x \cos^6 x \, dx$$

$$\int (\sin x \cos x)^4 \cdot \underbrace{\cos^2 x}\, dx$$

$$\frac{1}{2^5}\int \sin^4 2x\,(1+\cos 2x)\,dx = \frac{1}{32}\left[\int \sin^4 2x\,dx + \int \sin^4 2x \cos 2x\,dx\right]$$

** Note:- $\int \sin^p x \cos^q x \, dx$

when $(p+q)$ is negative even integer

$\tan x = t$. **

Q1 $$\int \frac{dx}{\sqrt{\cos^3 x \sin^5 x}}$$

$$\int \cos^{-3/2} x \cdot \sin^{-5/2} x \cdot dx$$

$$-\frac{3}{2}-\frac{5}{2} = -4.$$

$$\int (\tan x)^{-5/2} \cos^{-\frac{3}{2}-\frac{5}{2}} x \, dx = \int (\tan x)^{-5/2} (1+\tan^2 x) \sec^2 x \, dx$$

$\tan x = t$

Q $\int \sqrt{\cot x}\, \sec^4 x \, dx$

$$\int \frac{(\cos x)^{1/2}}{(\sin x)^{1/2}} \cdot (\cos^{-4} x)\, dx = \int (\sin x)^{-1/2} \cdot (\cos x)^{-7/2}\, dx$$

$\checkmark$

$\boxed{\text{Type: 2}}$

* (i) $\displaystyle\int \frac{a\sin x+b}{(a+b\sin x)^2}\,dx$

N^r & $D^r \rightarrow$ multiply by $\sec^2 x$

(ii) * $\displaystyle\int \frac{a\cos x+b}{(a+b\cos x)^2}\,dx$

N^r & $D^r \rightarrow$ multiply by $\operatorname{cosec}^2 x$.

eg:

$$\int \frac{2\sin x+3}{(2+3\sin x)^2}\,dx$$

$$\int \frac{2\tan x\sec x+3\sec^2 x}{(2\sec x+3\tan x)^2}\,dx$$

$$2\sec x+3\tan x=t.$$

$$(2\sec x\tan x+3\sec^2 x)\,dx=dt.$$

Type-3

$$\int \frac{dx}{a+b\sin^2 x} \Big/ \int \frac{dx}{a+b\cos^2 x} \Big/ \int \frac{dx}{a\sin^2 x+b\cos^2 x+c\sin x\cos x}$$

Method $\rightarrow$ Multiply & divide by $\sec^2 x$

eg: $\displaystyle\int \frac{dx}{2+3\cos^2 x}$

$$\int \frac{\sec^2 x}{2\sec^2 x+3}\,dx = \int \frac{\sec^2 x}{2(1+\tan^2 x)+3}\,dx$$

$\tan x = t$

proceed

$\boxed{\text{Type - 4}}$

$$\int \frac{dx}{a+b\sin x} \Bigg/ \int \frac{dx}{a+b\cos x} \Bigg/\!\!\Bigg/ \int \frac{dx}{a\sin x+b\cos x+c} \quad ✓$$

Method → Convert sine | Cosine into half of the tangents.

eg: $\displaystyle\int \frac{dx}{3\sin x+4\cos x}.$

$$\int \frac{dx}{3\cdot\dfrac{2\tan x/2}{1+\tan^2 x/2}+\dfrac{4\left(1-\tan^2 x/2\right)}{\left(1+\tan^2 x/2\right)}}$$

$$\int \frac{\sec^2\frac{x}{2}}{6\tan\frac{x}{2}+4-4\tan^2\frac{x}{2}}\,dx$$

$$\tan\frac{x}{2}=t \Rightarrow \frac{1}{2}\sec^2\frac{x}{2}\,dx=dt.$$

Q $\displaystyle\int \frac{dx}{\cos x(4\cos x+5)}$

$$\frac{1}{5}\int \frac{(4\cos x+5)-4\cos x}{\cos x\,(4\cos x+5)}\,dx$$

$$\frac{1}{5}\left(\int \sec x\,dx - 4\int \frac{dx}{4\cos x+5}\right)$$

Q $\displaystyle\int \frac{\sin 2x}{(\sin^4 x+\cos^4 x)}\,dx$

M-I $\displaystyle\int \frac{\sin 2x}{1-\frac{1}{2}\underbrace{\sin^2 2x}}\,dx = \int \frac{\sin 2x}{1-\frac{1}{2}(1-\cos^2 2x)}\,dx$

$\cos 2x = t$

$-2\sin 2x\,dx = dt.$

$$-\frac{1}{2}\int \frac{dt}{\frac{1}{2}+\frac{1}{2}t^2} = -\int \frac{dt}{(t^2+1)} = -\tan^{-1}(t)+c$$

$$= -\tan^{-1}(\cos 2x) + c$$

M-2

$$\int \frac{\sin 2x \cdot \sec^4 x}{\tan^4 x+1}\,dx$$

$$\int \frac{\frac{2\tan x}{\cancel{1+\tan^2 x}} \cdot \cancel{\sec^2 x} \cdot \sec^2 x\,dx}{\tan^4 x+1}$$

$$\int \frac{2\tan x \cdot \sec^2 x}{\tan^4 x+1}\,dx$$

$$\tan^2 x = t \Rightarrow 2\tan x \sec^2 x\,dx = dt$$

$$\int \frac{dt}{t^2+1} = \tan^{-1} t + c$$

$$= \underbrace{\tan^{-1}(\tan^2 x) + C}$$

Note:

$$\int \frac{dx}{1+\tan x} \;\Big|\; \int \frac{dx}{1+\cot x} \;\Big|\; \int \frac{\tan x}{1+\tan x}\,dx \;\Big/\; \int \frac{\cot x}{1+\cot x}\,dx$$

$$\int \frac{dx}{1+\tan x} = \int \frac{\cos x}{\sin x+\cos x}\,dx$$

$$\frac{1}{2}\int \frac{(\cos x+\sin x)+(\cos x-\sin x)}{(\sin x+\cos x)}\,dx$$

$$\frac{1}{2}\int dx \quad + \quad \frac{1}{2}\int \frac{\cos x-\sin x}{\sin x+\cos x}\,dx$$

Type: 5

$$\int \frac{a\sin x + b\cos x + c}{p\sin x + q\cos x + r}\,dx \quad \Bigg| \quad \int \frac{ae^{x} + be^{-x} + c}{pe^{x} + qe^{-x} + r}\,dx$$

Method →

$$\boxed{N^r = P(D^r) + Q\,\frac{d}{dx}(D^r) + R}$$

Q $$\int \frac{6 + 3\sin x + 14\cos x}{3 + 4\sin x + 5\cos x}\,dx$$

$$\underbrace{6 + 3\sin x + 14\cos x}_{} = P(3 + 4\sin x + 5\cos x) + Q(4\cos x - 5\sin x) + R$$

$$\left.\begin{aligned} 6 &= 3P + R \\ 3 &= 4P - 5Q \\ 14 &= 5P + 4Q. \end{aligned}\right\}$$

P =
Q =
R =

$$P\int \frac{\cancel{3 + 4\sin x + 5\cos x}}{\cancel{3 + 4\sin x + 5\cos x}}\,dx + Q\int \frac{4\cos x - 5\sin x}{\underbrace{3 + 4\sin x + 5\cos x}_{}}\,dx + R\int \frac{dx}{3 + 4\sin x + 5\cos x}$$

Type-6

NOTE If denominator contains $a + \sin 2x$ or $\sqrt{b + \sin 2x}$ then generate $\cos x + \sin x$ or $\cos x - \sin x$ in the numerator. However if $\cos x + \sin x$ or $\cos x - \sin x$ appears in denominator and $\sin 2x$ in numerator, then manipulate it as $\sin 2x = (\sin x + \cos x)^2 - 1$ or $1 - (\sin x - \cos x)^2$.

Q $\displaystyle\int \frac{\cos x + \sin x}{\sqrt{10 + \sin 2x}}\, dx$

?? integral of $(\cos x + \sin x)$

$(\sin x - \cos x) = t.$

$$\int \frac{\cos x + \sin x}{\sqrt{11 - (\sin x - \cos x)^2}}\, dx$$

$$\int \frac{dt}{\sqrt{11 - t^2}} = \sin^{-1}\left(\frac{t}{\sqrt{11}}\right) + C.$$

Q $\displaystyle\int (\cos x + \sin x)\sqrt{10 - \sin 2x}\, dx$

$$\int (\cos x + \sin x)\sqrt{9 + (\sin x - \cos x)^2}\, dx$$

$\sin x - \cos x = t.$

$$\int \sqrt{3^2 + t^2}\, dt.$$

Q $\displaystyle\int \frac{\cos x}{\sqrt{10+\underbrace{\sin 2x}}}\,dx$

$$\frac{1}{2}\int\left(\frac{(\cos x+\sin x)}{\sqrt{10+\sin 2x}}+\frac{(\cos x-\sin x)}{\sqrt{10+\sin 2x}}\right)dx$$

Q $\displaystyle\int\left(\sqrt{\tan x}\pm\sqrt{\cot x}\right)dx$

$$\int\left(\sqrt{\frac{\sin x}{\cos x}}+\sqrt{\frac{\cos x}{\sin x}}\right)dx$$

$$\sqrt{2}\int\frac{\sin x+\cos x}{\sqrt{\sin 2x}}\,dx$$

✓

Q $\int \frac{dx}{\operatorname{cosec}x - \sec x}$

$$\frac{1}{2}\int \frac{2\sin x\cos x}{\cos x - \sin x}\,dx = \frac{1}{2}\int \frac{\sin 2x}{(\cos x - \sin x)}\,dx$$

$$\frac{1}{2}\int \frac{1-(\cos x - \sin x)^2}{(\cos x - \sin x)}\,dx$$

$$\frac{1}{2}\int \frac{dx}{\cos x - \sin x} - \frac{1}{2}\int (\cos x - \sin x)\,dx$$

Integration of Rational function

$\dfrac{f(x)}{g(x)}$ where $f(x)$ & $g(x)$ are polynomial functions

I Degree of $N^r \geq$ Degree of D^r

eg $\displaystyle\int \frac{x^4 - 5x^2 + 7}{x^2+4}\,dx$

II Degree of $N^r <$ Degree of D^r

Partial fraction (Certain Rules)

Dividend = Divisor X Quotient + Remainder

$$
\begin{array}{r|l}
 & x^2 - 9 \\
\hline
(x^2+4) & x^4 - 5x^2 + 7 \\
 & \underline{x^4 + 4x^2} \\
 & -9x^2 + 7 \\
 & \underline{-9x^2 - 36} \\
 & (43)
\end{array}
$$

$$\int \frac{(x^2+4)(x^2-9) + 43}{(x^2+4)}\,dx = \int (x^2-9)\,dx + 43\int \frac{dx}{(x^2+4)}$$

Rules :- (when degree of $N^r <$ degree of D^r)

S. No.	Form of the rational function		Form of the partial fraction
1.	$\dfrac{px^2+qx+r}{(x-a)(x-b)(x-c)}$	Non-Repeated Linear factors	$\dfrac{A}{x-a}+\dfrac{B}{x-b}+\dfrac{C}{x-c}$
2.	$\dfrac{px^2+qx+r}{(x-a)^2(x-b)}$	Repeated Linear factors	$\dfrac{A}{x-a}+\dfrac{B}{(x-a)^2}+\dfrac{C}{x-b}$
3.	$\dfrac{px^2+qx+r}{(x-a)(x^2+bx+c)}$		$\dfrac{A}{x-a}+\dfrac{Bx+C}{x^2+bx+c}$

where x^2+bx+c cannot be factorised further

4. $$\frac{px^2+qx+r}{(x-a)(lx^2+mx+n)^2} \rightarrow \frac{A}{x-a}+\frac{Bx+C}{lx^2+mx+n}+\frac{Dx+E}{(lx^2+mx+n)^2}$$

Q1 $$\int \frac{x^2+x+1}{\underbrace{x^3-6x^2+11x-6}} \, dx$$

$$\int \frac{x^2+x+1}{(x-1)(x-2)(x-3)} \, dx$$

$$\frac{x^2+x+1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1}+\frac{B}{x-2}+\frac{C}{x-3}.$$

$$\frac{x^2+x+1}{(x-1)(x-2)(x-3)} = \frac{A(x-2)(x-3)+B(x-1)(x-3)+C(x-1)(x-2)}{(x-1)(x-2)(x-3)}$$

$$x^2+x+1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$$

$x=1 \Rightarrow 3 = A(-1)(-2)$

$A = 3/2$

$x=2 \Rightarrow B =$

$x=3 \Rightarrow C =$

Q2 $\int \frac{x}{(x-1)(x^2+4)}\,dx$

$$\frac{x}{(x-1)(x^2+4)} = \frac{A}{x-1} + \frac{Bx+C}{x^2+4}$$

$$x = A(x^2+4) + (x-1)(Bx+C)$$

$A = 1/5$

$B =$

$C =$

Q3 $\displaystyle\int \frac{x^3}{x^4+3x^2+2}\,dx$

$$\int \frac{x^3}{(x^2+1)(x^2+2)}\,dx$$

$x^2+1=t.$

$2x\,dx=dt.$

$$\frac{1}{2}\int \frac{(t-1)}{t(t+1)}\cdot dt = \frac{1}{2}\int \frac{dt}{t+1} - \frac{1}{2}\int \frac{dt}{t(t+1)}$$

Rem

Q4 $\displaystyle\int \frac{dx}{x^3+1} \;/\; \int \frac{dx}{x^3-1}$

$$\int \frac{dx}{(x+1)(x^2-x+1)}$$

$$\int \frac{x^2-(x^2-1)}{(x^3+1)}\,dx = \int \frac{x^2}{x^3+1}\,dx - \int \frac{(x-1)}{x^2-x+1}\,dx$$

Q5 $\displaystyle\int \frac{1}{\sin x\,(3+2\cos x)}\,dx$

$\cos x = t$

$\sin x\,dx = -dt$

$$\int \frac{\sin x}{(1-\cos^2 x)(3+2\cos x)}\,dx$$

$$\int \frac{-\,dt}{(1-t)(1+t)(3+2t)}$$

Q Suppose $f(x)$ is a quadratic function such that $f(0)=1$ and $f(-1)=4$. If $\int \frac{f(x)\,dx}{x^2(x+1)^2}$ is a rational function, find the value of $f(10)$.

$$\int \frac{f(x)}{x^2(x+1)^2}\,dx \longrightarrow \text{Rational function}$$

$$\int \frac{f(x)}{x^2(x+1)^2} = \int \frac{A}{x} + \int \frac{B}{x^2} + \int \frac{C}{x+1} + \int \frac{D}{(x+1)^2}$$

$$\boxed{A=0 \ ; \ C=0}$$

$$\frac{f(x)}{x^2(x+1)^2} = \frac{B}{x^2} + \frac{D}{(x+1)^2}$$

$$\boxed{f(x) = B(x+1)^2 + Dx^2}$$

$f(0) = B = 1$

$f(-1) = D = 4$

$f(x) = (x+1)^2 + 4x^2$

$f(10) = 11^2 + 4(10)^2$

$= 121 + 400$

$= 521$ Ans

Miscellaneous :-

Rem

$$\int \frac{x^2 \pm 1}{x^4 + kx^2 + 1}\, dx$$

** Coeff of x^4 and Constant term (in D^r) is same i.e '1'

Approach :- Divide N^r & D^r by x^2.

Q1 $\displaystyle\int \frac{x^2+1}{x^4+7x^2+1}\, dx$

$$\int \frac{\left(1+\frac{1}{x^2}\right)}{x^2+7+\frac{1}{x^2}}\, dx$$

integral $\left(x-\frac{1}{x}\right)$

$x-\frac{1}{x}=t$

$$\int \frac{1+\frac{1}{x^2}}{9+\left(x-\frac{1}{x}\right)^2}\, dx \qquad = \int \frac{dt}{9+t^2}.$$

Q

$$\int \frac{x^2+3}{x^4+8x^2+9}\,dx$$

$x = (9)^{1/4} t$

$dx = \sqrt{3}\,dt$

$$\int \frac{(3t^2+3)}{9t^4+8\times 3.t^2+9} \cdot (\sqrt{3})\,dt.$$

$$\frac{\cancel{3}\sqrt{3}}{\cancel{9}_{3}} \int \frac{t^2+1}{\left(t^4+\frac{8}{3}t^2+1\right)}\,dt$$

$$\frac{1}{\sqrt{3}} \int \frac{\left(1+1/t^2\right)}{\left(t^2+\frac{8}{3}+\frac{1}{t^2}\right)}\,dt$$

Q $\int \frac{dx}{\underbrace{x^4+1}_{K=0}}$ | $\int \frac{x^2}{\underbrace{x^4+1}_{K=0}} dx$ ✓

$$\int \frac{dx}{x^4+1} = \frac{1}{2} \int \frac{(1+x^2)+(1-x^2)}{(x^4+1)} dx$$

$$\frac{1}{2} \int \frac{x^2+1}{x^4+1} dx - \frac{1}{2} \int \frac{x^2-1}{x^4+1} dx$$

$$\frac{1}{2} \int \frac{\left(1+\frac{1}{x^2}\right)}{\left(x^2+\frac{1}{x^2}\right)} dx - \frac{1}{2} \int \frac{\left(1-\frac{1}{x^2}\right)}{x^2+\frac{1}{x^2}} dx$$

Q $\displaystyle\int \frac{(x+1)^2}{x^4+x^2+1}\,dx$

$\displaystyle\int \frac{x^2+1}{x^4+x^2+1}\,dx + \int \frac{2x}{x^4+x^2+1}\,dx$

$x^2 = t$

Q $\displaystyle\int \frac{x^{17}}{1+x^{24}}\,dx$

$x^6 = t \Rightarrow 6x^5\,dx = dt$

$\displaystyle\frac{1}{6}\int \frac{t^2}{1+t^4}\,dt$

Rem $\int \sqrt{\tan x}\, dx \;\Big|\; \int \sqrt{\cot x}\, dx$

$\tan x = t^2$

$\sec^2 x\, dx = 2t\, dt$

$dx = \frac{2t}{1+t^4} dt$

$$\int t \cdot \frac{2t}{1+t^4} dt = \int \frac{2t^2}{1+t^4} dt$$

Q ★ $\int \frac{dx}{\sin^4 x + \cos^4 x}$

$$\int \frac{\sec^4 x}{\tan^4 x + 1} dx$$

$\tan x = t$

$\sec^2 x \, dx = dt$

$$\int \frac{(1+\tan^2 x) \cdot \sec^2 x \cdot dx}{(1+\tan^4 x)}$$

$$\int \frac{1+t^2}{1+t^4} dt \quad \checkmark$$

KUTUR-PUTUR :-

$$\int \frac{dx}{x(x^n+1)} \, ; \; n \in N \rightarrow \int \frac{x^{-(n+1)}\,dx}{(\underbrace{1+x^{-n}})}$$

eg: $$\int \frac{dx}{x(x^5+1)}$$

M-1

$$\int \frac{x^{-6}}{1+x^{-5}}\,dx$$

$1+x^{-5}=t$

$-5x^{-6}\cdot dx = dt$

$$-\frac{1}{5}\int \frac{dt}{t}$$

M-2 S.O.S (save of soul) "Ijjat bachao" substitution

$$x = \frac{1}{t} \Rightarrow dx = -\frac{1}{t^2}dt$$

$$\int \frac{-1/t^2}{\frac{1}{t}\left(\frac{1}{t^n}+1\right)}\,dt = -\int \frac{t^{n-1}}{(1+t^n)}\,dt$$

Q $\int \frac{x^7}{(1-x^2)^5} dx$

$$\int \frac{x^7 \cdot x^{-10}}{(x^{-2}-1)^5} dx = \int \frac{x^{-3}}{(x^{-2}-1)^5} dx$$

$x^{-2}-1 = t.$

M-2 $\Rightarrow x = \frac{1}{t}$

Q $\int \frac{x}{(1-x^4)^{3/2}} dx$

$$\int \frac{x^{-6} \cdot x}{(x^{-4}-1)^{3/2}} dx = \int \frac{x^{-5}}{(x^{-4}-1)^{3/2}} dx$$

M-2 $x = \frac{1}{t}$

Q $\displaystyle\int \frac{dx}{x^2\left(x+\sqrt{x^2+1}\right)}$

M-1 $\displaystyle\int \frac{x^{-3}}{1+\sqrt{1+x^{-2}}}\,dx$ $\qquad \sqrt{1+x^{-2}} = t$

$1+x^{-2} = t^2.$

M-2 $x = \frac{1}{t} \Rightarrow dx = -\frac{1}{t^2}dt$

$$\int \frac{-\cancel{1/t^2}}{\cancel{\frac{1}{t^2}}\left(\frac{1}{t}+\sqrt{\frac{1}{t^2}+1}\right)}dt = -\int \frac{t}{1+\sqrt{t^2+1}}dt$$

M-3 $x = \tan\theta$.

M-4 $x + \sqrt{x^2+1} = t$.

M-5 Rationalise

Q $\int \frac{x-1}{x^2\sqrt{2x^2-2x+1}}\,dx$

Sol^n $\int \frac{(x-1)}{x^3\sqrt{2-2x^{-1}+x^{-2}}}\,dx$

$\int \frac{x^{-2}-x^{-3}}{\sqrt{2-2x^{-1}+x^{-2}}}\,dx$

$2-2x^{-1}+x^{-2}=t^2$

$(\cancel{2}x^{-2}-\cancel{2}x^{-3})\,dx=\cancel{2}t\,dt$

Q $\displaystyle\int \frac{x^4-1}{x^2\sqrt{x^4+x^2+1}}\,dx$

$$\int \frac{x^4-1}{x^4\sqrt{1+x^{-2}+x^{-4}}}\,dx$$

$$\int \frac{1-x^{-4}}{\sqrt{1+x^{-2}+x^{-4}}}\,dx.$$

$$\int \frac{(x^4-1)}{x^3\sqrt{x^2+1+x^{-2}}}\,dx$$

$$\int \frac{x-x^{-3}}{\sqrt{x^2+1+x^{-2}}}\,dx$$

$$\sqrt{x^2+1+x^{-2}}=t$$

$$x^2+1+x^{-2}=t^2$$

Q

$$\int \frac{x\cos\alpha+1}{(x^2+2x\cos\alpha+1)^{3/2}}\,dx$$

$$\int \frac{x\cos\alpha+1}{x^3\left(1+2x^{-1}\cos\alpha+x^{-2}\right)^{3/2}}\,dx$$

$$\int \frac{x^{-2}\cos\alpha+x^{-3}}{\left(1+2x^{-1}\cos\alpha+x^{-2}\right)^{3/2}}\,dx$$

$$1+2x^{-1}\cos\alpha+x^{-2}=t^2$$

$$\left(-\cancel{2}x^{-2}\cos\alpha-\cancel{2}x^{-3}\right)dx=\cancel{2}t\,dt.$$

Q $\displaystyle\int \frac{(ax^2-b)}{x\sqrt{c^2x^2-(ax^2+b)^2}}\,dx$ $a, b, c \in$ real constants

$$\int \frac{ax^2-b}{x\sqrt{\underset{=}{c^2x^2}-\underset{=}{x^2}\left(ax+\frac{b}{x}\right)^2}}\,dx$$

$$\int \frac{a-bx^{-2}}{\sqrt{c^2-\left(ax+\frac{b}{x}\right)^2}}\,dx$$

$ax+\frac{b}{x} = t$

$\left(a-\frac{b}{x^2}\right)dx = dt$

$$\int \frac{dt}{\sqrt{c^2-t^2}} = \sin^{-1}\left(\frac{t}{c}\right)+C.$$

Q ① $\int \dfrac{dx}{\left(\underbrace{(x+3)}^{2021}(x-1)^{2019}\right)^{\frac{1}{2020}}}$ / ② $\int \dfrac{dx}{\sqrt[3]{x^{5/2}(x+1)^{7/2}}}$

① $x+3=t \Rightarrow dx = c$

$$\int \frac{dt}{\left(t^{2021}\cdot(t-3-1)^{2019}\right)^{\frac{1}{2020}}}$$

$$\int \frac{dt}{\left(\underbrace{t^{2021}\cdot t^{2019}}\left(1-\frac{4}{t}\right)^{2019}\right)^{\frac{1}{2020}}}$$

$$\int \frac{t^{-2}}{\left(1-\frac{4}{t}\right)^{\frac{2019}{2020}}}\,dt$$

$1-\frac{4}{t} = z^{2020}$.

INTEGRATION OF IRRATIONAL ALGEBRAIC FUNCTION :

T-1

$$\int \frac{dx}{(ax+b)\sqrt{px+q}}$$

put $px+q = t^2$

eg: $$\int \frac{dx}{(2x+1)\sqrt{4x+3}}.$$

$$\boxed{t = \sqrt{4x+3}}$$

$$4x+3 = t^2 \Rightarrow 2x = \frac{t^2-3}{2}$$

$$4dx = 2t\,dt \Rightarrow dx = \frac{t}{2}dt$$

$$\int \frac{(\cancel{t}/2)}{\left(\frac{t^2-3}{2}+1\right)\cdot \cancel{t}}\cdot dt = \int \frac{dt}{t^2-1}.$$

$$= \frac{1}{2}\ln\left|\frac{t-1}{t+1}\right| + C.$$

Type-2 : $\displaystyle\int \frac{dx}{(ax+b)\sqrt{px^2+qx+r}}$; Put $ax+b=\dfrac{1}{t}$

eg: $$\int \frac{dx}{(x+1)\sqrt{1+x-x^2}}$$

put $x+1=\dfrac{1}{t} \Rightarrow dx = -\dfrac{1}{t^2}dt$

$$\int \frac{-1/t^2}{\frac{1}{t}\sqrt{\frac{1}{t}-\left(\frac{1}{t}-1\right)^2}} \cdot dt$$

$$-\int \frac{1/t}{\sqrt{\frac{1}{t}-\left(\frac{1}{t^2}+1-\frac{2}{t}\right)}} dt = -\int \frac{1/t}{\sqrt{\frac{3}{t}-\frac{1}{t^2}-1}} dt$$

$$-\int \frac{\cancel{1/t}}{\cancel{1/t}\sqrt{3t-1-t^2}} dt$$

Type-3 : $\int \frac{dx}{(ax^2+bx+c)\sqrt{px+q}}$; Put $px+q=t^2$

Q $\int \frac{dx}{(x^2+5x+2)\sqrt{x-2}}$

$x-2=t^2$

$dx = 2t\,dt$

$$\int \frac{2\cancel{t}}{\left((t^2+2)^2+5(t^2+2)+2\right)(\cancel{t})}\,dt$$

$$2\int \frac{dt}{t^4+9t^2+16}.$$

✓✓

Type-4 : $\int \frac{dx}{(ax^2+b)\sqrt{px^2+q}}$ then put $x = \frac{1}{t}$ or the trigonometric substitutions are also helpful.

(already done)

Q $\int \frac{dx}{(x^2+4)\sqrt{4x^2+1}}$

HW

Type-5 Imp

(i) $\int \sqrt{\dfrac{x-\alpha}{\beta-x}}\,dx$ / $\int \dfrac{dx}{\sqrt{(x-\alpha)(\beta-x)}}$ / $\int \dfrac{dx}{(x-\alpha)\sqrt{(x-\alpha)(\beta-x)}}$

(where $\beta > \alpha$)

put $\boxed{x = \alpha\cos^2\theta + \beta\sin^2\theta.}$

$$x-\alpha = (\beta-\alpha)\sin^2\theta.$$

$$\beta-x = (\beta-\alpha)\cos^2\theta.$$

$$dx = (\beta-\alpha)\sin 2\theta\, d\theta.$$

eg: $\int \sqrt{\dfrac{x-3}{7-x}}\,dx$ $\qquad \alpha = 3$, $\beta = 7.$

$$x = 3\cos^2\theta + 7\sin^2\theta.$$

$$x-3 = 4\sin^2\theta. \qquad dx = 4\sin 2\theta \cdot d\theta.$$

$$7-x = 4\cos^2\theta$$

$$\int \frac{\sin\theta}{\cancel{\cos\theta}} \cdot 4\cdot(2\sin\theta\cdot\cancel{\cos\theta})\cdot d\theta$$

$$4\int (1-\cos 2\theta)\,d\theta$$

(ii) $\displaystyle\int \sqrt{\frac{x-\alpha}{x-\beta}}\,dx \;\Big|\; \int \frac{dx}{(x-\alpha)\sqrt{(x-\alpha)(x-\beta)}}$

put $x = \alpha \sec^2\theta - \beta \tan^2\theta.$

$dx = 2(\alpha-\beta) \sec^2\theta . \tan\theta \, d\theta$

$x-\alpha = (\alpha-\beta) . \tan^2\theta .$

$x-\beta = (\alpha-\beta) \sec^2\theta .$

$Q \to$ Quad poly.

Type : $\int \frac{Q_1(x)}{(Q_2(x))^2}\,dx$

$$\boxed{Q_1(x) = \alpha\, Q_2(x) + \beta \frac{d}{dx}(Q_2(x)) + \gamma}$$

eg $\int \frac{5x^2-12}{(x^2-6x+13)^2}\,dx$

$5x^2-12 = \alpha.(x^2-6x+13) + \beta.(2x-6)+\gamma.$

$5 = \alpha.$

$0 = -6\alpha+2\beta \Rightarrow \beta = 3\alpha \Rightarrow \beta = 15$

$-12 = 13\alpha - 6\beta + \gamma.$

$-12 = 65 - 90 + \gamma$

$-12+25 \Rightarrow \boxed{\gamma = 13}$

$$\underbrace{\int \frac{5\,\cancel{(x^2-6x+13)}}{(x^2-6x+13)^{\cancel{2}}}\,dx}_{I_1} + \underbrace{\int \frac{15\,(2x-6)\,dx}{(x^2-6x+13)^2}}_{I_2} + \underbrace{\int \frac{13\,dx}{(x^2-6x+13)^2}}_{I_3}$$

// ✓//

$$I_3 = 13 \int \frac{dx}{(x^2-6x+13)^2}$$

$$13 \int \frac{dx}{((x-3)^2 + 2^2)^2}$$

$x - 3 = 2\tan\theta.$
$dx = 2\sec^2\theta \, d\theta.$

$$13 \int \frac{2\sec^2\theta}{(2^2 \sec^2\theta)^2} \, d\theta. = \frac{13}{8} \int \cos^2\theta \cdot d\theta.$$

Forcing integrand by Parts :-

1) $\int \frac{dx}{(x^4-1)^2}$

$x^4-1=t$

$4x^3 \cdot dx = dt$

$$\int \underbrace{\left(\frac{4x^3}{(x^4-1)^2}\right)}_{\text{II}} \cdot \underbrace{\left(\frac{1}{4x^3}\right)}_{\text{I}} dx = \frac{1}{4x^3}\left(\frac{-1}{x^4-1}\right) - \int \frac{1}{4}\frac{(-3)}{x^4}\left(\frac{-1}{x^4-1}\right) dx$$

$$\frac{-1}{4x^3(x^4-1)} - \frac{3}{4}\int \frac{dx}{x^4(x^4-1)}$$

$$\frac{-1}{4x^3(x^4-1)} - \frac{3}{4}\left(\int \frac{x^4-(x^4-1)}{x^4(x^4-1)} dx\right)$$

$$\frac{-1}{4x^3(x^4-1)} - \frac{3}{4}\left(\int \frac{dx}{(x^2-1)(x^2+1)} - \int x^{-4} dx\right)$$

$$\frac{-1}{4x^3(x^4-1)} - \frac{3}{4}\left(\frac{1}{2}\int \frac{(x^2+1)-(x^2-1)}{(x^2-1)(x^2+1)} dx - \int x^{-4} dx\right)$$

2) ① $\int \frac{x^2}{(x\sin x+\cos x)^2}\,dx$ | ② $\int \frac{x^2}{(x\cos x-\sin x)^2}\,dx$

M-1 ✓

M-2 ✓

$x=\cot\theta$

①
$$x\sin x+\cos x=t$$
$$(x\cos x+\cancel{\sin x}-\cancel{\sin x})\,dx=dt$$

$$\int \underbrace{\frac{x\cos x}{(x\sin x+\cos x)^2}}_{\text{II}}\cdot\underbrace{\left(\frac{x}{\cos x}\right)}_{\text{I}}\,dx$$

$$\left(\frac{x}{\cos x}\right)\left(\frac{-1}{x\sin x+\cos x}\right)-\int \frac{\cancel{(\cos x+x\sin x)}}{\cos^2 x}\cdot\left(\frac{-1}{\cancel{x\sin x+\cos x}}\right)dx$$

M-2 $\quad x=\tan\theta$

$\star$

Q $\displaystyle\int \sin \underbrace{51x}\,.\,\sin^{49} x \, dx$

$$\int \sin(\underbrace{50x} + \underbrace{x}) \sin^{49} x \, dx$$

$$\int (\sin 50x \cos x + \cos 50x \sin x) \sin^{49} x \, dx$$

$$\int \underbrace{\sin 50x.}_{I} \quad \underbrace{\cos x.\sin^{49} x}_{II} \, dx \quad + \int \cos 50x \sin^{50} x \, dx$$

Hold

$$(\sin 50x) \frac{(\sin x)^{50}}{50} - \int \cancel{50}.(\cos 50x).\frac{(\sin x)^{50}}{\cancel{50}}$$

$$+ \int \cos 50.\sin^{50} x \, dx + C$$

Reduction formulae :-

$$I_n = \int \sin^n x \, dx \;\Big|\; \int \cos^n x \, dx$$

$$I_n = \int \underbrace{\sin^{n-1} x}_{\mathrm{I}} \cdot \underbrace{\sin x}_{\mathrm{II}} \, dx$$

$$I_n = (\sin x)^{n-1}(-\cos x) - \int (n-1)\sin^{n-2} x \cdot \cos x \cdot (-\cos x) \, dx$$

$$I_n = -(\sin x)^{n-1} \cdot \cos x + (n-1)\int \sin^{n-2} x \cdot (1-\sin^2 x) \, dx$$

$$I_n = -(\sin x)^{n-1} \cos x + (n-1)\left[I_{n-2} - I_n\right]$$

$$I_n = \frac{(-\sin x)^{n-1} \cos x}{n} + \left(\frac{n-1}{n}\right) I_{n-2}.$$

2) $I_n = \int \tan^n x \, dx \;\Big|\; \int \cot^n x \, dx$

$$I_n = \int \tan^{n-2} x \cdot \tan^2 x \cdot dx$$

$$I_n = \int (\tan x)^{n-2} (\sec^2 x - 1) \, dx$$

$$I_n + I_{n-2} = \int (\tan x)^{n-2} \sec^2 x \, dx$$

$$\boxed{I_n + I_{n-2} = \frac{(\tan x)^{n-1}}{n-1}} + C$$

eg: If $I_n = \int_0^{\pi/4} (\tan x)^n \, dx$ then

Find $I_3 + I_5$?

$$\left. \frac{(\tan x)^4}{4} \right|_0^{\pi/4} = \frac{1}{4}.$$

3) $\int \sec^n x \, dx$ / $\int \operatorname{cosec}^n x \, dx$

$$\downarrow$$

$$I_n = \int \underset{\text{I}}{\sec^{n-2} x} \cdot \underset{\text{II}}{\sec^2 x} \, dx$$

proceed.

Q Value of $\int \frac{\sec x\,(2+\sec x)}{(1+2\sec x)^2}\,dx = \frac{a\sin x}{b+\cos x}+C$

where C is constant of integration, a & b are co-prime then find value of $(a+b)$?

Sol^n differentiate wrt 'x' :-

$$\frac{\sec x\,(2+\sec x)}{(1+2\sec x)^2} = \frac{(b+\cos x)(a\cos x)-(a\sin x)(-\sin x)}{(b+\cos x)^2}$$

$$\frac{2\cos x+1}{(\cos x+2)^2} = \frac{ab\cos x + \underbrace{a\cos^2 x + a\sin^2 x}_{a}}{(b+\cos x)^2}$$

$\therefore \boxed{b=2}$; $ab=2$

$\boxed{a=1}$

Q

$$\int (x^{12}+x^{8}+x^{4})(2x^{8}+3x^{4}+6)^{\frac{1}{4}}\,dx$$

$$= a\,(2x^{12}+3x^{8}+6x^{4})^{5/4}+C.$$

then find 'a' ?

Sol^n

$$\int (x^{11}+x^{7}+x^{3})(2x^{12}+3x^{8}+6x^{4})^{\frac{1}{4}}\,dx$$

$$2x^{12}+3x^{8}+6x^{4}=t^{4}$$

$$(24x^{11}+24x^{7}+24x^{3})\,dx = 4t^{3}\,dt$$

$$a=\frac{1}{30}$$ Ans

Q $$\int \frac{x^2(x^6+x^5-1)dx}{(2x^6+3x^5+2)^2}$$

$$\int \frac{x^2(x^6+x^5-1)}{x^6(2x^3+3x^2+2x^{-3})^2}\,dx$$

$$\int \frac{x^2+x-x^{-4}}{(2x^3+3x^2+2x^{-3})^2}\,dx = \frac{-(2x^3+3x^2+2x^{-3})^{-1}}{6} + C.$$

$$2x^3+3x^2+2x^{-3}=t$$

Q *

$$\int \frac{t-1}{(t+1)\sqrt{t+t^2+t^3}}\,dt$$

$$1+t+\frac{1}{t} = z^2$$

$$\left(1-\frac{1}{t^2}\right)dt = 2z\,dz$$

$$\int \frac{(t-1)}{(t+1)\,t\,\sqrt{\frac{1}{t}+1+t}}\,dt$$

$$\int \frac{(t^2-1)}{\underbrace{(t^2+2t+1)\cdot t}\sqrt{t+1+\frac{1}{t}}}\,dt$$

$$\int \frac{\left(1-\frac{1}{t^2}\right)}{\left(t+2+\frac{1}{t}\right)\sqrt{t+\frac{1}{t}+1}}\,dt = \int \frac{2\cancel{z}}{(z^2+1)\cdot\cancel{z}}\,dz$$

$$= 2\tan^{-1}(z) + C$$

Q $\int (\sin x)^{\cos x - 1}(\cos^2 x - (\sin^2 x)\ell n \sin x)dx$ is equal to -

(A) $(\sin x)^{\cos x + 1} + C$

(B) $(\cos x)^{\sin x} + C$

(C) $\cos x.(\sin x)^{\cos x} + C$

(D) $(\sin x)^{\cos x} + C$ ✓

Sol^n

$$\frac{d}{dx}\left(\underbrace{(\sin x)^{\cos x}}\right) = (\sin x)^{\cos x - 1} \cdot \left(\cos^2 x - (\sin^2 x)(\ln \sin x)\right)$$

Q $\int e^{x^3}(x^8+2x^5+x^2)\,dx$

$x^3 = t \Rightarrow 3x^2\,dx = dt$

$\frac{1}{3}\int e^t(t^2+2t+1)\,dt = \frac{1}{3}\int e^t(\underbrace{t^2+1}_{f}+\underbrace{2t}_{f'})\,dt$

$\frac{1}{3}e^t(t^2+1)+C$

$\operatorname{cosec}^2\theta = \sec^2(90°-\theta)$

Q $\int e^x\left(\frac{2\tan x}{1+\tan x}+\cot^2\left(x+\frac{\pi}{4}\right)\right)dx$

$\int e^x\left(\frac{2\tan x}{1+\tan x}+\operatorname{cosec}^2\left(x+\frac{\pi}{4}\right)-1\right)dx$

$\int e^x\left(\frac{\tan x-1}{1+\tan x}+\underbrace{\operatorname{cosec}^2\left(x+\frac{\pi}{4}\right)}\right)dx$

$\int e^x\left(\underbrace{\tan\left(x-\frac{\pi}{4}\right)}_{f}+\underbrace{\sec^2\left(x-\frac{\pi}{4}\right)}_{f'}\right)dx = e^x\tan\left(x-\frac{\pi}{4}\right)+C$

Q $\int (2 - x\tan x)\sqrt{\cos x}\, dx$

$$\int \Big(\underbrace{2\sqrt{\cos x}}_{f} - \underbrace{x\sqrt{\cos x}\tan x}_{x f'}\Big)\, dx = x\left(2\sqrt{\cos x}\right) + C.$$

$$f(x) = 2\sqrt{\cos x}$$

$$f'(x) = \frac{\cancel{2}}{\cancel{2}\sqrt{\cos x}} \cdot (-\sin x)$$

$$f'(x) = -\frac{\sin x}{\sqrt{\cos x}} \cdot \frac{\sqrt{\cos x}}{\sqrt{\cos x}} = -\tan x \cdot \sqrt{\cos x}$$

Q If $\int \frac{dx}{\sqrt{x}+\sqrt[3]{x}} = a\sqrt{x}+b.\sqrt[3]{x}+c.\sqrt[6]{x}$
$+d.\ln(\sqrt[6]{x}+1)+e,$

e being constant of integration then find value of $(20a+b+c+d)$? 37

Sol^n $x=t^6. \Rightarrow dx = 6t^5\,dt$

$$\int \frac{6t^5}{t^3+t^2}.\,dt = 6\int \frac{t^3+1-1}{t+1}\,dt$$

$$2x^{1/2} - 3x^{1/3} + 6x^{1/6} - 6\ln(1+x^{1/6}) + e$$

$a=2$; $b=-3$; $c=6$; $d=-6$

Q

$$\int \frac{1}{x}\sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}\,dx$$

$$x = \cos^2 2\theta \Rightarrow dx = 2.\cos 2\theta.(-\sin 2\theta).2.d\theta.$$

$$dx = -2^2 \sin 2\theta.\cos 2\theta.\,d\theta.$$

$$\int \frac{-4}{\cos^{\cancel{2}} 2\theta}\sqrt{\frac{1-\cos 2\theta}{1+\cos 2\theta}}.\ \sin 2\theta.\ \cancel{\cos} 2\theta.\ d\theta$$

$$-4\int \frac{\sin\theta}{\cos 2\theta.\cancel{\cos\theta}}.(2\sin\theta.\cancel{\cos\theta})\ d\theta.$$

$$-4\int \frac{2\sin^2\theta}{\cos 2\theta}d\theta = -4\int \frac{1-\cos 2\theta}{\cos 2\theta}d\theta$$

$$-4\left(\int \sec 2\theta\, d\theta - \int d\theta\right)$$

$$-4\,\frac{\ln|\sec 2\theta + \tan 2\theta|}{2} + 4\theta + C.$$

Q $\displaystyle\int e^{\cos^{-1}x}\left(\frac{x+1+\sqrt{1-x^2}}{(x+1)^2\sqrt{1-x^2}}\right)dx$

$\boxed{g(x) = \cos^{-1}x} \Rightarrow \boxed{g'(x) = \dfrac{-1}{\sqrt{1-x^2}}.}$

$f(x) = \dfrac{-1}{x+1}$

$$\int e^{\cos^{-1}x}\left(\frac{1}{(x+1)\sqrt{1-x^2}} + \frac{1}{(1+x)^2}\right)dx$$

$$e^{\cos^{-1}x}\cdot\left(\frac{-1}{x+1}\right) + C$$

Ans

Paragraph :-

Let a, b, c, d be 4 distinct roots of the equation $x^4 - 4x + 3 = x(x^3 - f'(1)x^2 + f''(1)x - 4) + f(1)$ and $f(x)$ is a monic polynomial of degree 3.

1. The value of $\lim_{x\to1}\frac{(x-1)^3 - \sin^3(x-1)}{(f(x)-3)^{5/3}}$ is -

(A) $\frac{1}{2}$ ✓ (B) 2 (C) $\frac{1}{6}$ (D) $\frac{2}{3}$

$x-1=t$

$\lim_{t\to0}\left(\frac{t^3-\sin^3 t}{t^5}\right) = \lim_{t\to0}\frac{(t-\sin t)}{t^3}\left(\frac{t^2+\sin^2 t+t\sin t}{t^2}\right)$

$= \frac{1}{6}\times3 = \frac{1}{2}.$

2. $\int\frac{dx}{\frac{f(x)-3}{x-1}+4} = \frac{1}{2}g(x)+C$, where C is constant of integration and $g(3)=\frac{\pi}{4}$, then the value of g(5) + g(7) is -

(A) $\frac{\pi}{2}$ (B) π (C) $\frac{5\pi}{4}$ (D) $\frac{3\pi}{4}$ ✓

$\underbrace{f'(1)}x^3 + \underbrace{f''(1)}x^2 + \underbrace{(f(1)-3)} = 0$ — roots a, b, c, d

It is an identity.

$f'(1) = f''(1) = 0$ & $f(1)-3=0 \Rightarrow \boxed{f(1)=3}$

$f(x) = (x-1)^3+3$

$g(x) = \tan^{-1}\left(\frac{x-1}{2}\right)$

$f''(x) = a(x-1).$

$f'(x) = \frac{a(x-1)^2}{2} + C. \Rightarrow f'(1) = \boxed{C=0}$

$f'(x) = \frac{a(x-1)^2}{2} \longrightarrow f(x) = \frac{a(x-1)^3}{6} + \lambda$

Q Let $I = \int \frac{e^x}{e^{4x}+e^{2x}+1}dx$, $J = \int \frac{e^{-x}}{e^{-4x}+e^{-2x}+1}dx$. Then, for an arbitrary constant c, the value of J – I equals **[JEE 2008, 3 (–1)]**

(A) $\frac{1}{2}\ell n\left(\frac{e^{4x}-e^{2x}+1}{e^{4x}+e^{2x}+1}\right)+c$

(B) $\frac{1}{2}\ell n\left(\frac{e^{2x}+e^{x}+1}{e^{2x}-e^{x}+1}\right)+c$

(C) $\frac{1}{2}\ell n\left(\frac{e^{2x}-e^{x}+1}{e^{2x}+e^{x}+1}\right)+c$

(D) $\frac{1}{2}\ell n\left(\frac{e^{4x}+e^{2x}+1}{e^{4x}-e^{2x}+1}\right)+c$

$$I = \int \frac{e^x}{e^{4x}+e^{2x}+1}dx \; ; \quad J = \int \frac{e^{3x}}{1+e^{2x}+e^{4x}}dx$$

$$J - I = \int \frac{e^{3x}-e^{x}}{e^{4x}+e^{2x}+1}dx$$

$$= \int \frac{e^{x}\left(e^{2x}-1\right)}{e^{4x}+e^{2x}+1}dx$$

$e^x = t$

$e^x\,dx = dt$

$$= \int \frac{(t^2-1)\cdot}{(t^4+t^2+1)}dt$$

Q The integral $\int \frac{\sec^2 x}{(\sec x + \tan x)^{9/2}} dx$ equals (for some arbitrary constant K) **[JEE 2012, 3M, –1M]**

(A) $-\frac{1}{(\sec x + \tan x)^{11/2}}\left\{\frac{1}{11} - \frac{1}{7}(\sec x + \tan x)^2\right\} + K$

(B) $\frac{1}{(\sec x + \tan x)^{11/2}}\left\{\frac{1}{11} - \frac{1}{7}(\sec x + \tan x)^2\right\} + K$

(C) $-\frac{1}{(\sec x + \tan x)^{11/2}}\left\{\frac{1}{11} + \frac{1}{7}(\sec x + \tan x)^2\right\} + K$

(D) $\frac{1}{(\sec x + \tan x)^{11/2}}\left\{\frac{1}{11} + \frac{1}{7}(\sec x + \tan x)^2\right\} + K$

Sol^n

$$\sec x + \tan x = t$$
$$\sec x - \tan x = \frac{1}{t}.$$

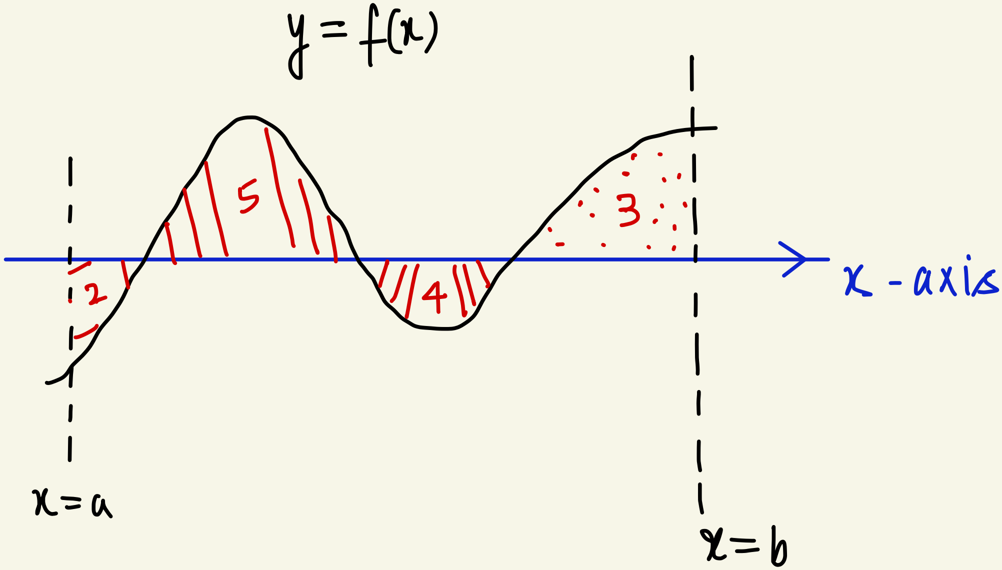
$$\sec x = \frac{1}{2}\left(t + \frac{1}{t}\right)$$
$$\tan x = \frac{1}{2}\left(t - \frac{1}{t}\right)$$

$$(\sec x \tan x + \sec^2 x)\, dx = dt$$
$$\sec x\,(t) \cdot dx = dt$$
$$dx = \frac{dt}{t \cdot \frac{1}{2}\left(t + \frac{1}{t}\right)}$$

Definite Integration

A definite integral is denoted by $\int_a^b f(x)dx$ which represents the algebraic area bounded by the curve $y = f(x)$, the ordinates $x = a$, $x = b$ and the x axis.

This is also geometrical interpretation of D.I.



$$\int_a^b f(x)\,dx = -2 + 5 - 4 + 3 = 8 - 6 = 2.$$

** $$\int_a^b f(x) \leq \left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f(x)|\,dx$$

* If $f(x)$ is continuous in $[a,b]$ and $\int f(x)\,dx = g(x)+c$ then $\int_a^b f(x)\,dx = g(b)-g(a)$

$= g(x)\Big|_a^b$

eg: $\int_0^1 (x+1)\,dx = \frac{(x+1)^2}{2}\Big|_0^1 = \frac{2^2-1^2}{2} = \frac{3}{2}.$

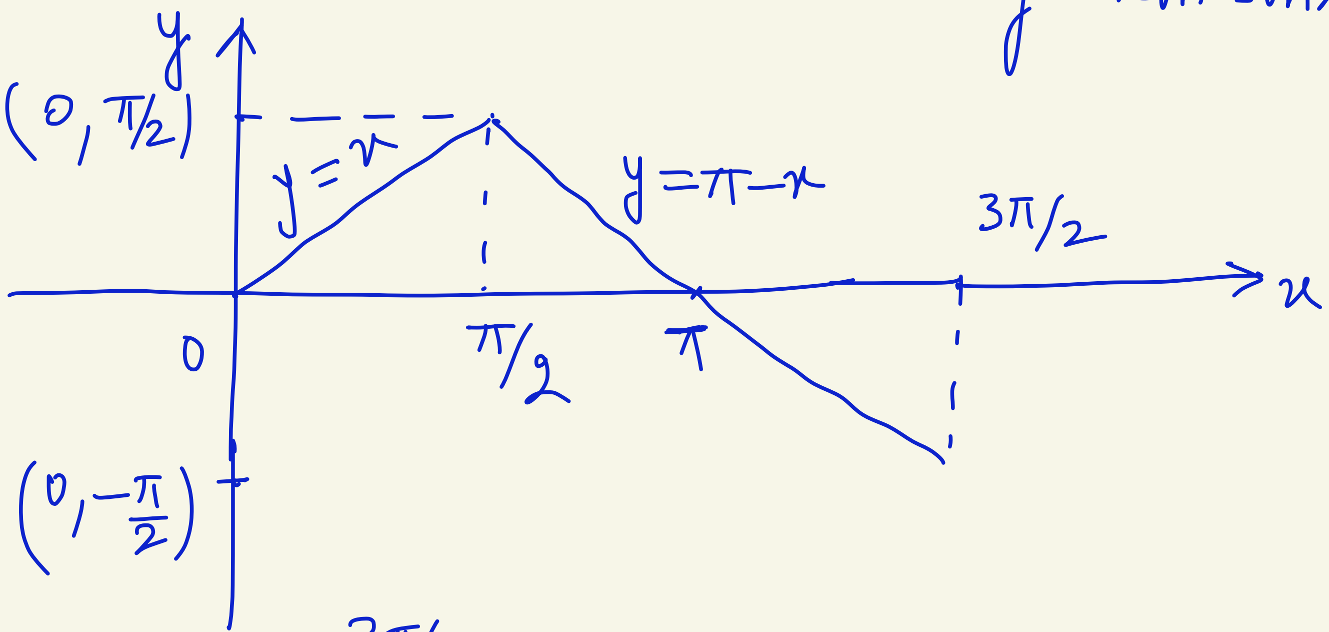
Note:- If $y=f(x)$ is continuous in $[a,b]$ and $f(x)>0$ in $[a,b]$ then $\int_a^b f(x)\,dx > 0$ (where $a<b$)

Rem

$$\int_a^b f(x)\,dx \leq \left| \int_a^b f(x)\,dx \right| \leq \int_a^b |f(x)|\,dx$$

Q Evaluate $\displaystyle\int_0^{3\pi/2} \sin^{-1}(\sin x)\,dx$?

$y = \sin^{-1}\sin x$



$$\int_0^{\pi/2} x \cdot dx + \int_{\pi/2}^{3\pi/2} (\pi - x)\,dx = \frac{\pi^2}{8}.$$

Q

Evaluate $\displaystyle\int_0^3 f(x)\,dx$ where $f(x) = \begin{cases} \{x\} ; & 0 \le x < 1 \\ [x+2] ; & 1 \le x < 2 \\ 6-2x ; & 2 \le x < 3 \end{cases}$

$\{\ \} \to \ldots$

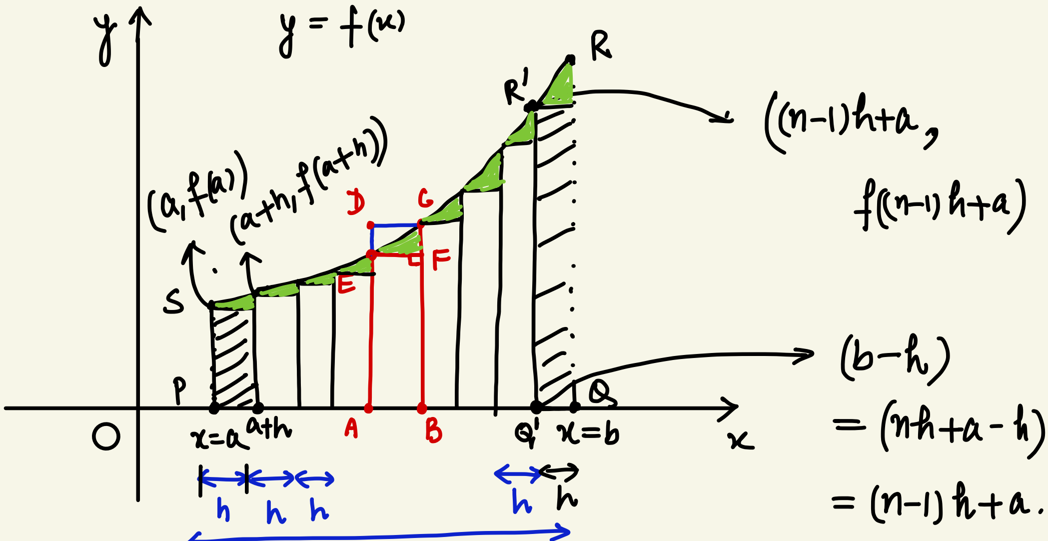
$[\ ] \to$ Gif.

$$\int_0^1 x \cdot dx + \int_1^2 3 \cdot dx + \int_2^3 (6-2x)\,dx$$

$$= \frac{9}{2} = 4.5$$

DEFINITE INTEGRAL AS THE LIMIT OF SUM :

If $f(x) > 0$ for all $x \in [a,b]$ then the definite integral $\int_a^b f(x)dx$ is numerically equal to the area bounded by the curve $y = f(x)$, the ordinates $x = a$, $x = b$ and the x-axis.

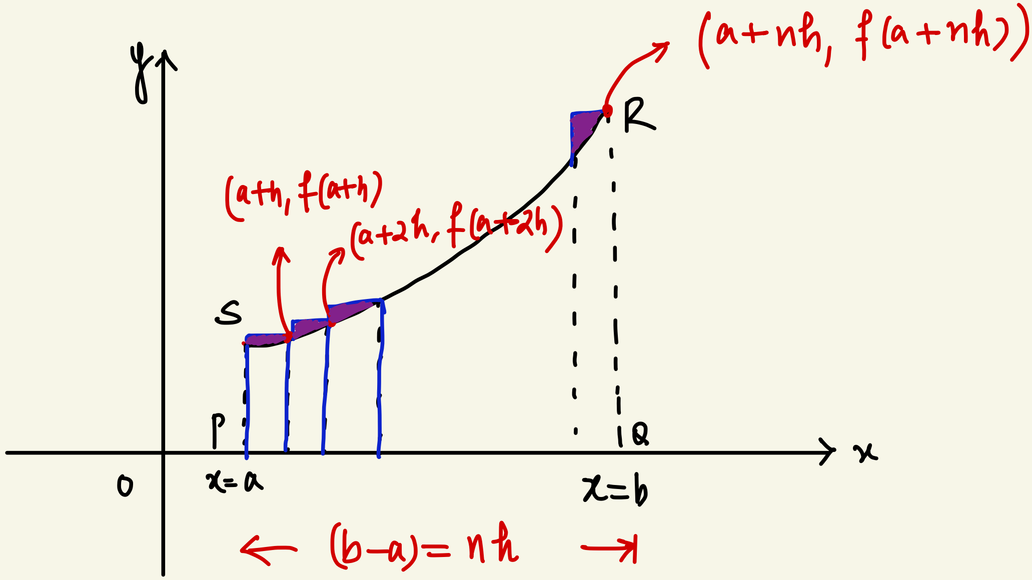


$$\boxed{nh = b - a}$$

Ar of Rect ABFE < Area of region ABCEA < Ar of Rect ABCD

$$\boxed{h\left(f(a)+f(a+h)+\cdots+f(a+(n-1)h)\right) < \underbrace{\text{Ar of region PQRSP}}_{A}}$$

$$\underbrace{\sum_{r=0}^{n-1} h\, f(a+rh)}_{s} < A.$$



$$\underbrace{\text{Area of region } PQRSP}_{A} < h\left(f(a+h) + f(a+2h) + \cdots \cdots + f(a+nh) \right)$$

$$A < \underbrace{\sum_{r=1}^{n} h\, f(a+rh)}_{S}$$

$$\boxed{s < A < S}$$

$$\lim_{\substack{n\to\infty \\ h\to 0}} s = A. \quad \& \quad \lim_{\substack{n\to\infty \\ h\to 0}} S = A.$$

$$\boxed{\lim_{\substack{n\to\infty \\ h\to 0}} h\sum_{n=0}^{n-1} f(a+nh) = A. = \int_a^b f(x)\,dx.}$$

$$b-a = nh. \Rightarrow \frac{b-a}{h} = n.$$

The above expression is known as the definition of definite integral as the limit of sum.

Evaluating a definite integral by evaluating the limit of a sum is called evaluating definite integral by first principle or by ab-initio method.

Note :-

(1)

(a) $\displaystyle\int_0^{\pi/2} \sin x\,dx = \int_0^{\pi/2} \cos x\,dx = 1$ (b) $\displaystyle\int_0^{\pi/2} \sin^2 x\,dx = \int_0^{\pi/2} \cos^2 x\,dx = \frac{\pi}{4}$

(c) $\displaystyle\int_0^{\pi/2} \sin^3 x\,dx = \int_0^{\pi/2} \cos^3 x\,dx = \frac{2}{3}$ (d) $\displaystyle\int_0^{\pi/2} \sin^4 x\,dx = \int_0^{\pi/2} \cos^4 x\,dx = \frac{3\pi}{16}$

(2) $$\int_a^b f(x)\cdot d(g(x)) = \int_{g^{-1}(a)}^{g^{-1}(b)} f(x)\cdot g'(x)\,dx.$$

$g(x) = a$

$x = g^{-1}(a).$

eg $$\int_{-1}^{1} x^2\, d(\ln x) = \int_{e^{-1}}^{e} x^{\cancel{2}}\cdot\frac{1}{\cancel{x}}\cdot dx$$

$\ln x = -1$

$x = e^{-1}$

$$= \left.\frac{x^2}{2}\right|_{1/e}^{e}$$

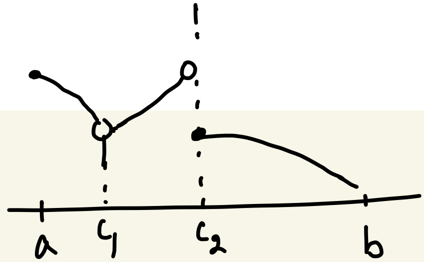
$\ln x = 1$

$x = e.$

$$= \frac{e^2 - 1/e^2}{2}$$ Ans

3) $\int_a^b \frac{d}{dx}(f(x))\,dx = [f(x)]_a^b$ if f(x) is continuous in (a, b) however if f(x) is discontinuous in (a, b) at $x = c \in (a, b)$

$\int_a^b \frac{d}{dx}(f(x)) = [f(x)]_a^{c^-} + [f(x)]_{c^+}^b$



eg $\int_{-1}^{1} \frac{d}{dx}\left(\cot^{-1}\frac{1}{x}\right) \cdot dx$

$$\cot^{-1}\left(\frac{1}{x}\right)\Big|_{-1}^{0^-} + \cot^{-1}\left(\frac{1}{x}\right)\Big|_{0^+}^{1}$$

$$\left(\cot^{-1}(-\infty) - \cot^{-1}(-1)\right) + \left(\cot^{-1}1 - \cot^{-1}(\infty)\right)$$

$$\left(\pi - \frac{3\pi}{4}\right) + \left(\frac{\pi}{4} - 0\right) = \frac{\pi}{2} \text{ Ans}$$

4) $$\lim_{n\to\infty}\left(\int_a^b f_n(x)\,dx\right) = \int_a^b\left(\lim_{n\to\infty} f_n(x)\right)dx\,;$$

eg $$\lim_{n\to\infty}\int_0^{10}\frac{e^x}{1+x^n}\,dx$$

$$\int_0^1 \frac{e^x}{1}\,dx + \cancel{\int_1^{10} 0 \cdot dx} = e^x\Big|_0^1 = (e-1)$$

$$\boxed{\lim_{x\to 0}\frac{a^x-1}{x} = \ln a}$$

eg: $$\lim_{n\to 0}\int_0^1 \frac{x^n-1}{n}\,dx$$

$$\int_0^1 \ln x \cdot dx = (x\ln x - x)\Big|_{0^+}^1$$

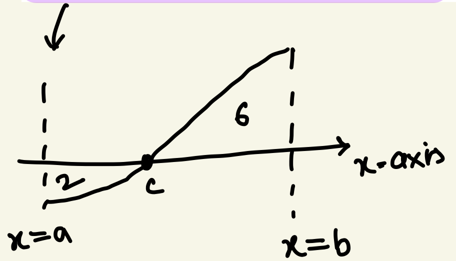
$$= (1\cdot\cancel{\ln 1}^{\,0} - 1) - \lim_{x\to 0^+}(x\ln x - \cancel{x}^{\,0})$$

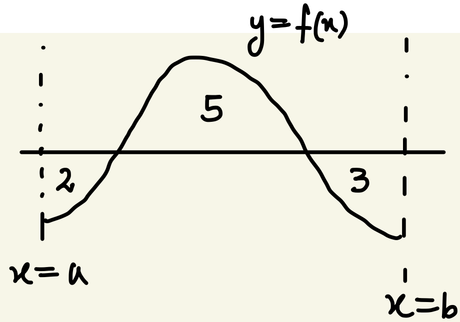
$$= (-1) - \lim_{x\to 0^+}\left(\frac{\ln x}{1/x}\right)$$

LH Rule $$= (-1) - \cancel{\lim_{x\to 0^+}\left(\frac{1/x}{-1/x^2}\right)}^{\,0} = -1.$$

5) If $\int_a^b f(x)\,dx = 0$, then the equation $f(x)=0$ has atleast one root in (a, b) provided f is continuous in (a, b).

Note that the converse is not true.





eg If $2a+3b+6c=0$ then prove that quadratic equation $ax^2+bx+c=0$ must have a root in $(0,1)$?

Sol^n

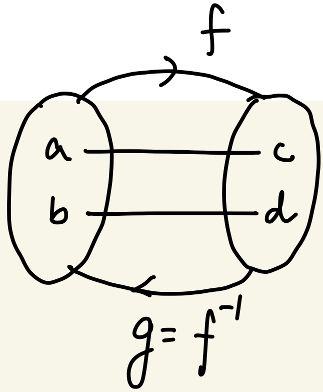
$$\int_0^1 (ax^2+bx+c)\,dx = \left(\frac{ax^3}{3}+\frac{bx^2}{2}+cx\right)\Big|_0^1$$

$$= \left(\frac{a}{3}+\frac{b}{2}+c\right)\Big|_0^1$$

$$= \frac{2a+3b+6c}{6} = 0.$$

6) If $g(x)$ is the inverse of $f(x)$ and $f(x)$ has domain $x \in [a, b]$ where $f(a) = c$ and $f(b) = d$ then the value of

$$\int_a^b f(x)\,dx + \int_c^d g(y)\,dy = (bd - ac)$$



$$y = f(x)$$
$$dy = f'(x)\,dx$$

$$f^{-1}(c) = a$$
$$f^{-1}(d) = b$$

$$\int_a^b f(x)\,dx + \int_{f^{-1}(c)=a}^{f^{-1}(d)=b} \underbrace{g(f(x))}_{x} \cdot f'(x)\,dx$$

$$\int_a^b \left(f(x) + x f'(x)\right) dx = x f(x)\Big|_a^b = b \cdot f(b) - a \cdot f(a)$$
$$= bd - ac. \text{ (H.P)}$$

Q1 If f is one-one function such that $f(2) = 3$; $f(5) = 7$ and $\int_2^5 f(x)\,dx = 17$ then

$$\int_3^7 f^{-1}(x)\,dx = ?$$

$$f^{-1}(3) = 2$$
$$f^{-1}(7) = 5$$
$$\Downarrow$$
$$f(5) = 7$$

Sol^n

$$\int_2^5 f(x)\,dx + \int_3^7 f^{-1}(x)\,dx = 35 - 6$$
$$= 29$$

Q Let $f:[0,1]\to[e, e^{\sqrt{e}}]$ then

$$I=\int_0^1 \underbrace{e^{\sqrt{e^x}}}_{f^{-1}}\,dx+2\int_e^{e^{\sqrt{e}}} \ln(\ln x)\,dx = ?$$

$$1\times e^{\sqrt{e}}-0\times e$$

$$= e^{\sqrt{e}}. \quad \text{Ans}$$

$$f(x)=2\ln(\ln x)$$

$$y=2\ln(\ln x) \Rightarrow \frac{y}{2}=\ln(\ln x)$$

$$e^{y/2}=\ln x$$

$$x=e^{e^{y/2}}=e^{\sqrt{e^y}}$$

$$f^{-1}(x)=e^{\sqrt{e^x}}.$$

Q 8.
$$\int_{3}^{8} \frac{\sin\sqrt{x+1}}{\sqrt{x+1}}\,dx$$

$$\underbrace{\sqrt{x+1}}\ = t$$

$$x+1 = t^2$$

$$dx = 2t\,dt$$

$$\int_{2}^{3} \frac{\sin t}{\cancel{t}} \cdot 2\cancel{t} \cdot dt$$

$$2\left(-\cos t\right)\Big|_{2}^{3} = -2\left(\underbrace{\cos 3} - \underbrace{\cos 2}\right)$$

Q
$$\int_{2}^{5} \sqrt{\frac{x-2}{5-x}}\,dx$$

$$x = 2\cos^2\theta + 5\sin^2\theta.$$

$$x-2 = 3\sin^2\theta.$$

$$5-x = 3\cos^2\theta.$$

$$dx = 6\sin\theta\cos\theta\,d\theta.$$

$$\int_{0}^{\pi/2} \frac{\sin\theta}{\cancel{\cos\theta}} \cdot 6\sin\theta\cancel{\cos\theta}\,d\theta = 6\int_{0}^{\pi/2} \sin^2\theta\,d\theta$$

$$= 6\cdot\left(\frac{\pi}{4}\right)$$

Q. $\displaystyle\int_{\pi/4}^{\pi/2}\left(\sqrt{\frac{\sin x}{x}}+\sqrt{\frac{x}{\sin x}}\cos x\right)dx$

$\displaystyle\int_{\pi/4}^{\pi/2}\left(\frac{\sin x+x\cos x}{\sqrt{x\sin x}}\right)dx$

$\sqrt{x\sin x}=t.$

$x\sin x=t^2$

$(x\cos x+\sin x)\,dx=2t\,dt$

$\displaystyle\int_{\sqrt{\frac{\pi}{4\sqrt{2}}}}^{\sqrt{\frac{\pi}{2}}}\frac{2\cancel{t}}{\cancel{t}}\,dt. \quad =(2t)\Big|_{\sqrt{\frac{\pi}{4\sqrt{2}}}}^{\sqrt{\frac{\pi}{2}}}$

$\underline{Q}$ $\displaystyle\int_0^{\pi} \frac{dx}{1+\cos^2 x}$.

$$\int_0^{\pi} \frac{\sec^2 x}{\sec^2 x + 1}\, dx = \int_0^{\pi} \frac{\sec^2 x}{2+\tan^2 x}\, dx = \int_0^{0} \frac{dt}{2+t^2}$$

$\Downarrow 0$

$\tan x = t$

$\sec^2 x\, dx = dt$

$$\int_0^{\frac{\pi}{2}^-} \frac{\sec^2 x}{2+\tan^2 x}\, dx + \int_{\frac{\pi}{2}^+}^{\pi} \frac{\sec^2 x}{2+\tan^2 x}\, dx$$

$\tan x = t$

$$\int_0^{\infty} \frac{dt}{2+t^2} + \int_{-\infty}^{0} \frac{dt}{2+t^2} = \frac{1}{\sqrt{2}} \tan^{-1} \frac{t}{\sqrt{2}} \Big|_0^{\infty} + \frac{1}{\sqrt{2}} \tan^{-1} \frac{t}{\sqrt{2}} \Big|_{-\infty}^{0}$$

$$= \frac{1}{\sqrt{2}} \left(\frac{\pi}{2} - 0 \right) + \frac{1}{\sqrt{2}} \left(0 - \left(-\pi/2\right) \right)$$

$$= \frac{\pi}{\sqrt{2}}.$$

$\underline{\text{M-2}}$

$$\int_0^{\pi} \frac{\operatorname{cosec}^2 x}{\operatorname{cosec}^2 x + \cot^2 x}\, dx$$

$\cot x = t$

Q $\displaystyle\int_0^{\pi/2} \frac{1+2\sin x}{(2+\sin x)^2}\,dx$

$$\int_0^{\pi/2} \frac{\sec^2 x + 2\tan x \sec x}{(2\sec x + \tan x)^2}\,dx$$

$2\sec x + \tan x = t$

$(2\sec x \tan x + \sec^2 x)\,dx = dt$

$$\int_2^{\infty} \frac{dt}{t^2} = -\frac{1}{t}\Big|_2^{\infty} = -\left(\frac{1}{\infty} - \frac{1}{2}\right)$$

$$= \frac{1}{2}.$$

Q $\int_0^{2\pi}[\cos x(1+x)+(1-x)\sin x]\,dx$

$$\int_0^{2\pi}\left((\underbrace{\cos x+\sin x}_{f}) + x(\underbrace{\cos x-\sin x}_{f'})\right)dx$$

$$x\cdot(\cos x+\sin x)\Big|_0^{2\pi} = 2\pi(1)-0.$$
$$= 2\pi.$$

Q $\int_0^1 x\;\ell n(1+2x)\;dx$

$x \downarrow$ II, $\ell n(1+2x) \downarrow$ I

$$\left(\ln(1+2x)\cdot\frac{x^2}{2}\right)\Big|_0^1 - \int_0^1 \frac{\cancel{2}}{1+2x}\cdot\frac{x^2}{\cancel{2}}\cdot dx$$

$$\ln 3.\left(\frac{1^2}{2}\right) - 0. \quad -\frac{1}{4}\int_0^1 \frac{4x^2-1+1}{2x+1}dx$$

$$\frac{\ln 3}{2} - \frac{1}{4}\int_0^1(2x-1)\,dx - \frac{1}{4}\int_0^1\frac{dx}{2x+1}$$

Q

Let $I = \int_0^{\pi/2} \frac{\cos x}{\underbrace{a\cos x + b\sin x}} dx$ and $J = \int_0^{\pi/2} \frac{\sin x}{\underbrace{a\cos x + b\sin x}} dx$, where $a > 0$ and $b > 0$.

Compute the values of I and J.

$$aI + bJ = \int_0^{\pi/2} \left(\frac{\cancel{a\cos x + b\sin x}}{\cancel{a\cos x + b\sin x}} \right) dx$$

$$aI + bJ = \frac{\pi}{2} \quad — ① —$$

$$bI - aJ = \int_0^{\pi/2} \frac{(b\cos x - a\sin x)}{(a\cos x + b\sin x)} dx$$

$$bI - aJ = \ln |a\cos x + b\sin x| \Big|_0^{\pi/2}$$

$$bI - aJ = \quad — ② —$$

Q $$\int_{1/2}^{2}\left(1+x-\frac{1}{x}\right)e^{x+\frac{1}{x}}\,dx$$

$$\int_{1/2}^{2}\left(e^{x+\frac{1}{x}}+x\left(1-\frac{1}{x^2}\right)\cdot e^{x+\frac{1}{x}}\right)\cdot dx$$

$$\left(x\cdot e^{x+\frac{1}{x}}\right)\Big|_{1/2}^{2}=2e^{5/2}-\frac{1}{2}e^{5/2}$$

$$=e^{5/2}\left(\frac{3}{2}\right)\text{ Ans}$$

HW

Q $$\lim_{n\to\infty}\int_0^1\left(\sum_{K=0}^{n}\frac{x^{K+2}\,2^{K}}{K!}\right)dx$$

Q $\lim\limits_{n\to\infty} \int_0^1 \left(\sum_{k=0}^{n} \frac{x^{k+2}\, 2^{k}}{k!} \right) dx$

$$\int_0^1 x^2 \underbrace{\sum_{k=0}^{\infty} \frac{(2x)^k}{k!}} \cdot \, dx$$

$$\boxed{e^{2x} = 1 + \frac{2x}{1!} + \frac{(2x)^2}{2!} + \ldots\ldots}$$

$$\int_0^1 x^2 \cdot e^{2x} \cdot dx = \int_0^1 \underset{\mathrm{II}}{\underset{\downarrow}{e^{2x}}} \cdot \underset{\mathrm{I}}{\underset{\downarrow}{x^2}} \, dx$$

Q.

Let $f(x)$ be a function satisfying $f(x)=f\left(\frac{100}{x}\right)\ \forall x>0$. If $\int_1^{10}\frac{f(x)}{x}dx=5$, then the value of $\int_1^{100}\frac{f(x)}{x}dx$ is equal to 10

Sol^n

$$\int_1^{100}\frac{f(x)}{x}dx=\underbrace{\int_1^{10}\frac{f(x)}{x}dx}_{5}+\underbrace{\int_{10}^{100}\frac{f(x)}{x}dx}_{I}=5+I$$

$$I=\int_{10}^{100}\frac{f(x)}{x}dx=\int_{10}^{100}\frac{f\left(\frac{100}{x}\right)}{x}dx$$

$\frac{100}{x}=t$

$-\frac{100}{x^2}dx=dt$

$$=\int_{10}^{1}\frac{t\,f(t)}{100}\cdot\frac{(-1)}{100}\left(\frac{100}{t}\right)^2\cdot dt.$$

$$I=-\int_{10}^{1}\frac{f(t)}{t}dt=\int_1^{10}\frac{f(t)\,dt}{t}$$

$$I=5$$

Q. If $I_1 = \int_0^{\pi/2} (\sin x)^{\sqrt{2}-1} dx$, $I_2 = \int_0^{\pi/2} (\sin x)^{\sqrt{2}+1} dx$

then $\frac{I_1}{I_2} = ?$

Sol^n

$$I_2 = \int_0^{\pi/2} \underset{\text{I}}{(\sin x)^{\sqrt{2}}} \cdot \underset{\text{II}}{(\sin x)}\, dx$$

$$I_2 = \cancel{\left((\sin x)^{\sqrt{2}} \cdot (-\cos x)\right)}\Big|_0^{\pi/2} - \int_0^{\pi/2} \sqrt{2}\,(\sin x)^{\sqrt{2}-1} \cdot \cos x\,(-\cos x)\,dx$$

0

$$I_2 = \sqrt{2} \int_0^{\pi/2} (\sin x)^{\sqrt{2}-1} \cdot \underbrace{\cos^2 x}_{(1-\sin^2 x)}\, dx$$

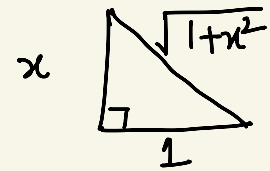
$$I_2 = \sqrt{2}\left(\int_0^{\pi/2} (\sin x)^{\sqrt{2}-1} \cdot dx - \int_0^{\pi/2} (\sin x)^{\sqrt{2}-1+2} \cdot dx\right)$$

$$I_2 = \sqrt{2}\,(I_1 - I_2) \Rightarrow \boxed{(\sqrt{2}+1)\, I_2 = \sqrt{2}\, I_1}$$

Q $\displaystyle\int_{1/3}^{3} \frac{\sin^{-1}\left(\frac{x}{\sqrt{1+x^2}}\right)}{x}\, dx$

Note :- When limits are reciprocal of each other then generally $x = \frac{1}{t}$ is useful.

$$I = \int_{1/3}^{3} \frac{\tan^{-1} x}{x}\, dx \quad \text{—①—}$$



$x = \frac{1}{t} \Rightarrow dx = -\frac{1}{t^2}\, dt$

$$I = \int_{3}^{1/3} \frac{\tan^{-1}(1/t)}{1/t} \cdot \left(-\frac{1}{t^2}\right) dt$$

$$I = -\int_{3}^{1/3} \frac{\cot^{-1}(t)}{t}\, dt. \Rightarrow I = \int_{1/3}^{3} \frac{\cot^{-1} t}{t}\, dt \quad \text{—②—}$$

Add ① & ②

$$\underbrace{2I} = \int_{1/3}^{3} \frac{\pi/2}{t}\, dt \Rightarrow I = \frac{1}{2} \cdot \frac{\pi}{2} \cdot (\ln t)\Big|_{1/3}^{3}$$

Q $\displaystyle\int_{-1}^{-1/\sqrt{3}} \cot^{-1}x \, dx$

$\boxed{\cot^{-1}x = \theta} \Rightarrow x = \cot\theta.$

$dx = -\operatorname{cosec}^2\theta \, d\theta.$

$$\int_{3\pi/4}^{2\pi/3} \underset{\mathrm{I}}{\theta} \cdot \left(-\underset{\mathrm{II}}{\operatorname{cosec}^2\theta}\right) d\theta = -\left(\left(\theta\cdot(-\cot\theta)\right)\Big|_{\frac{3\pi}{4}}^{\frac{2\pi}{3}} + \int_{\frac{3\pi}{4}}^{\frac{2\pi}{3}} \cot\theta \cdot d\theta\right)$$

$$= \frac{2\pi}{3}\left(-\frac{1}{\sqrt{3}}\right) - \frac{3\pi}{4}(-1) - \left(\ln\sin\theta\right)\Big|_{\frac{3\pi}{4}}^{\frac{2\pi}{3}}$$

Walli's Theorem :-

where $m, n \in W$

$$\int_0^{\pi/2} \sin^n x . \cos^m x \, dx = \frac{[(n-1)(n-3)(n-5)....1 \text{ or } 2][(m-1)(m-3)....1 \text{ or } 2]}{(m+n)(m+n-2)(m+n-4)....1 \text{ or } 2} \cdot K$$

$$\text{Where } K = \begin{cases} \dfrac{\pi}{2} & \text{if both m and n are even } (m, n \in W) \\ 1 & \text{otherwise} \end{cases}$$

eg:

$$\int_0^{\pi/2} \sin^6 x \, dx = \frac{(5.\cancel{3}.1)}{\underset{2}{\cancel{6}}.4.2.} \frac{\pi}{2} = \frac{5\pi}{32}.$$

$$\int_0^{\pi/2} \sin^7 x . \cos^{10} x \, dx = \frac{(6.4.2)(9.7.5.3.1)}{(17.15.13. \ldots 1)} . 1$$

PROPERTIES OF DEFINITE INTEGRAL :

P–1 $\int_a^b f(x)dx = \int_a^b f(t)\,dt$ (change of variable does not change value of integral)

P–2 $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$

Q1. If $\dfrac{d}{dx}f(x) = \dfrac{e^{\sin x}}{x}$, $x > 0$ & $\int_1^4 \dfrac{2e^{\sin x^2}}{x}dx = f(k) - f(1)$, then find possible values of k.

$$f(x) = \int \frac{e^{\sin x}}{x}\,dx \quad -(1)-$$

$$\int_1^4 \frac{2e^{\sin x^2}}{x^2}\cdot x\,dx = \boxed{f(k) - f(1)}$$

$$x^2 = t \Rightarrow 2x\,dx = dt$$

$$\int_1^{16} \frac{\cancel{2}\cdot e^{\sin t}}{t}\cdot\frac{dt}{\cancel{2}} = \boxed{f(16) - f(1)}$$

$K = 16$ Ans

2. $\int_0^1 \frac{\tan^{-1} x}{x} dx = \lambda \int_0^{\pi/2} \frac{\theta}{\sin\theta} d\theta$, then find λ.

Solⁿ LHS: $\tan^{-1} x = z$.

$x = \tan z \Rightarrow dx = \sec^2 z \, dz$.

$$\int_0^{\pi/4} \frac{z \cdot \sec^2 z}{\tan z} dz = \int_0^{\pi/4} \frac{2z}{2\sin z \cos z} dz$$

$$\int_0^{\pi/4} \frac{2z}{\sin 2z} dz$$

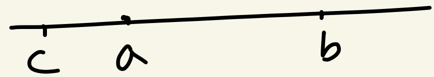
$2z = \alpha$

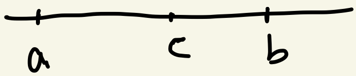
$2dz = d\alpha$

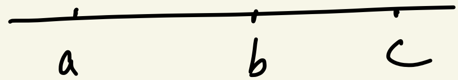
$$\frac{1}{2}\int_0^{\pi/2} \frac{\alpha}{\sin\alpha} d\alpha$$

$\therefore \boxed{\lambda = \frac{1}{2}}$

P–3 $\int_a^b f(x) = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$







[ ] → Gif

1 $\int_1^3 [x]dx = ?$

2. $\int_0^5 |x-2|\,dx = ?$

3. $\int_0^{\pi} \sqrt{\frac{1+\cos 2x}{2}}\,dx = ?$

①

$$\int_1^2 1.\,dx + \int_2^3 2.\,dx$$

②

$$\int_0^2 (2-x)dx + \int_2^5 (x-2)\,dx$$

③

$$\int_0^{\pi} \sqrt{\frac{\cancel{1}+\cancel{2}\cos^2 x-\cancel{1}}{\cancel{2}}}\,dx = \int_0^{\pi} |\cos x|\,dx$$

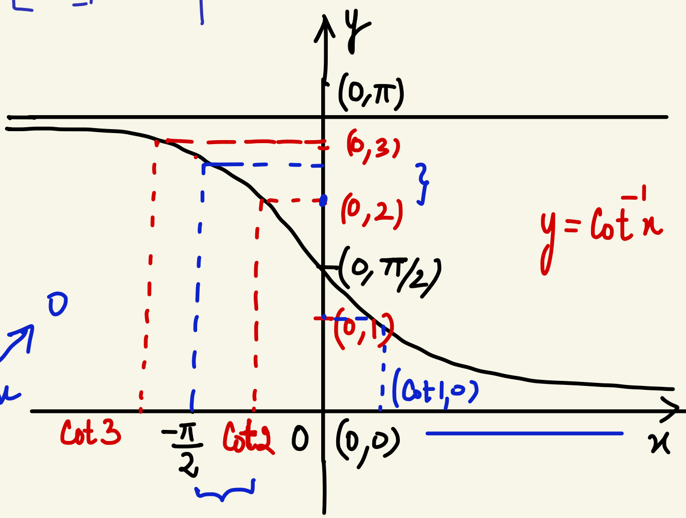
$$= \int_0^{\pi/2} \cos x + \int_{\pi/2}^{\pi} (-\cos x)dx$$

Q $\displaystyle\int_0^{\pi/4} \left[\sin x + \left[\tan x + \left[\sec x + [\cos x]\right]\right]\right] dx$; [ ] → Gif

$$\int_0^{\pi/4} \Big(\underbrace{[\sin x]}_{0} + \underbrace{[\tan x]}_{0} + \underbrace{[\sec x]}_{1} + \underbrace{[\cos x]}_{0} \Big)\, dx$$

$$\int_0^{\pi/4} 1 . dx = \frac{\pi}{4}.$$

Q $\displaystyle\int_{-\pi/2}^{\infty} [\cot^{-1} x]\, dx$; [ ] → Gif



$$\int_{-\pi/2}^{\cot 2} 2 . dx + \int_{\cot 2}^{\cot 1} 1 . dx + \cancel{\int_{\cot 1}^{\infty} 0 . dx} \to 0$$

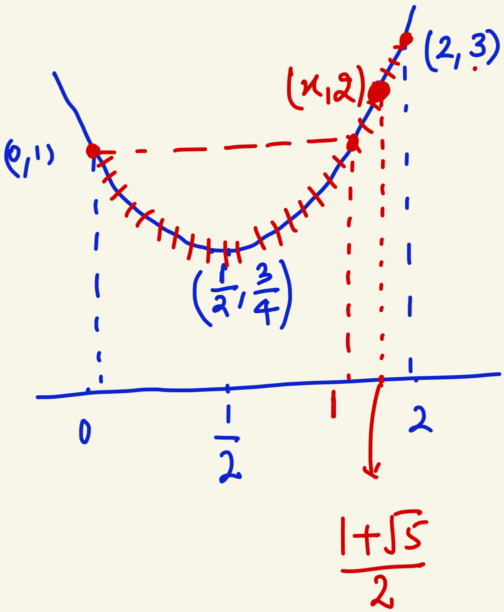
$$2(x)\Big|_{-\pi/2}^{\cot 2} + (x)\Big|_{\cot 2}^{\cot 1}$$

Q $\int_0^2 [x^2 - x + 1]\, dx$; $[\ ] \rightarrow$ Gif

$y = x^2 - x + 1$

Vertex: $x = \frac{1}{2}$

$y = \frac{5}{4} - \frac{1}{2} = \frac{3}{4}$.



$x^2 - x + 1 = 1$

$x^2 - x = 0$

$x = 0$; $x = 1$

$x^2 - x + 1 = 2$

$x^2 - x - 1 = 0$

$x = \frac{1 \pm \sqrt{5}}{2}$

$x = \frac{1+\sqrt{5}}{2}$

$$\cancelto{0}{\int_0^1 0 \cdot dx} + \int_1^{\frac{1+\sqrt{5}}{2}} 1 \cdot dx + \int_{\frac{1+\sqrt{5}}{2}}^{2} 2 \cdot dx.$$

Q $\displaystyle\int_0^{2\pi} \sqrt{1-\sin x}\, dx$

$\sqrt{1-\sin 2\theta} = |\sin\theta - \cos\theta|$

$\displaystyle\int_0^{2\pi} \left|\sin\frac{x}{2} - \cos\frac{x}{2}\right| dx$

$\frac{x}{2} = t \Rightarrow x = 2t$

$dx = 2dt$

$\displaystyle 2\int_0^{\pi} |\sin t - \cos t| .\, dt = 2\int_0^{\pi/4} (\cos t - \sin t)\, dt$

$\displaystyle + 2\int_{\pi/4}^{\pi} (\sin t - \cos t)\, dt.$

Q

$$\int_0^2 |(1-x)\ln x|\,dx$$

$$= \int_0^2 (x-1)\ln x \cdot dx$$

$$\int_0^2 x \cdot \ln x\, dx - \int_0^2 \ln x\, dx$$

P-4 $\int_{-a}^{a} f(x)dx = \int_{0}^{a} (f(x) + f(-x))\ dx = \begin{cases} 0 & \text{if } f(x) \text{ is odd} \\ 2\int_{0}^{a} f(x)dx & \text{if } f(x) \text{ is even} \end{cases}$

eg $\int_{-\pi/2}^{\pi/2} \cos x\, dx$

$2\int_{0}^{\pi/2} \cos x\, dx = 2(1)$
$= 2.$

eg $\int_{-\pi/3}^{\pi/3} \underbrace{\tan x}_{\text{odd fns}}\, dx = 0.$

Q $\int_{-5}^{6} x^3 dx = \cancel{\int_{-5}^{5} x^3 dx}^{\,0} + \int_{5}^{6} x^3 dx = \frac{x^4}{4}\Big|_{5}^{6}.$

Q $\int_{-1/2}^{1/2} \overbrace{\underbrace{\sec x}_{\text{Even}}\ \underbrace{\ln\left(\frac{1-x}{1+x}\right)}_{\text{Odd}}}^{\text{Odd}}\ dx = 0.$

Q $$\int_{-\pi/4}^{\pi/4} \frac{\overbrace{x^7 - 3x^5 + 3x^3 - x}^{} + 1}{\cos^2 x}\, dx$$

$$2\int_0^{\pi/4} \sec^2 x\, dx = 2(\tan x)\Big|_0^{\pi/4}$$
$$= 2.$$

Q $$\int_{-1}^{3} \left(\tan^{-1}\left(\frac{x}{x^2+1}\right) + \tan^{-1}\left(\frac{x^2+1}{x}\right)\right) dx$$

$$\underbrace{\int_{-1}^{1} \left(\tan^{-1}(\quad) + \tan^{-1}(\quad)\right) dx}_{\text{odd}} \xrightarrow{} 0 \quad + \int_1^3 \left(\tan^{-1}\left(\frac{x}{x^2+1}\right) + \cot^{-1}\left(\frac{x}{x^2+1}\right)\right) dx$$

$$\int_1^3 \left(\frac{\pi}{2}\right) dx = \frac{\pi}{2}(3-1).$$
$$= \pi.$$

Q

$$\int_{-1}^{11} \left((x-1)(x-3)(x-5)(x-7)(x-9)\right) dx$$

$x-5 = t \Rightarrow dx = dt$

$x = 5+t$

$$\int_{-6}^{6} (4+t)(2+t).\, t\, (t-2)(t-4).\ dt$$

$$\int_{-6}^{6} (\underbrace{t^2-4}_{E})(\underbrace{t^2-16}_{E})\underbrace{t}_{O}.\ dt \;=\; 0.$$

E . E . O

Q.

Evaluate : $\int_{-2}^{2}(\underbrace{x^3 f(x)}_{0}+\underbrace{x.f''(x)}_{0}+\underbrace{2}_{E})dx$, where f(x) is an even differentiable function.

$$\int_{-2}^{2} 2\,dx = 2.2.\int_{0}^{2} dx = 4(2-0) = 8.$$

KING.

P-5 $\int_a^b f(x)\,dx = \int_a^b f(a+b-x)\,dx$ **or** $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$

$$x = a+b-t \Rightarrow dx = -dt$$

$$-\int_b^a f(a+b-t).dt = \int_a^b f(a+b-t)\,dt.$$

$$I = \int_a^b f(x)\,dx \quad -①-$$

$$I = \int_a^b f(a+b-x)\,dx \quad -②-$$

Add

$$I = \frac{1}{2}\int_a^b (f(x) + f(a+b-x))\,dx$$

Q $I=\int_{\pi/6}^{\pi/3} \sin 2x \, ln(\tan x) \, dx$

KING : $\pi/3$

$$I = \int_{\pi/6}^{\pi/3} \sin 2\left(\frac{\pi}{2}-x\right) \cdot \ln\left(\tan\left(\frac{\pi}{2}-x\right)\right) dx$$

$$I = \int_{\pi/6}^{\pi/3} \sin 2x \cdot \ln \cot x \, dx$$

Add : $$2I = \int_{\pi/6}^{\pi/3} \sin 2x \cdot \underbrace{\ln\left(\cancel{\tan x} \cdot \cancel{\cot x}\right)}_{0} dx = 0.$$

Q $\int_{50}^{100} \frac{\ell n \, x}{\ell n \, x + \ell n(150-x)} dx$

Directly :

$$I = \frac{1}{2} \int_{50}^{100} \left(\frac{\ln x}{\ln x + \ln(150-x)} + \frac{\ln(150-x)}{\ln(150-x) + \ln x} \right) dx$$

$$= \frac{1}{2}(50) = 25.$$

HW

Q.

$$\int_{\pi/8}^{3\pi/8} \ell n\left(\frac{4+3\sin x}{4+3\cos x}\right) dx$$

HW

Q.

$$\int_{0}^{\pi/2} \frac{\sin^3 x}{\sin x + \cos x} dx$$

HW Q

$$\int_0^{\pi} \frac{dx}{1 + 2^{\tan x}}$$

Q

$$\int_2^3 \frac{x^2\, dx}{2x^2 - 10x + 25}$$

$$I = \int_2^3 \frac{x^2}{x^2 + (x-5)^2}\, dx$$

directly :

$$I = \frac{1}{2} \int_2^3 \left(\frac{x^2}{x^2 + (x-5)^2} + \frac{(5-x)^2}{(5-x)^2 + x^2} \right) dx$$

$$= \frac{1}{2}(3-2) = \frac{1}{2} \text{ Ans}$$

Q $$I = \int_0^{\infty} \frac{\ln x \, dx}{ax^2 + bx + a}$$

Rem $= 0.$

$$\int_0^{\infty} \frac{\ln x}{3x^2+4x+3} dx = 0.$$

$$I = \int_0^{\infty} \frac{\ln x}{ax^2+bx+a} dx \quad \text{— (1) —}$$

$$x = \frac{1}{t} \Rightarrow dx = -\frac{1}{t^2} dt$$

$$I = \int_{\infty}^{0} \frac{\ln\left(\frac{1}{t}\right)}{\frac{a}{t^2} + \frac{b}{t} + a} \cdot \left(\frac{-1}{t^2}\right) dt$$

$$I = \int_{\infty}^{0} \frac{+\ln t}{(a+bt+at^2)} \cdot dt$$

$$I = -I \Rightarrow \boxed{I = 0}.$$

$$\boxed{\int_0^{\pi/4} \ln(1+\tan x)\,dx = \frac{\pi}{8}\ln 2}$$

Rem

$$I = \int_0^{\pi/4} \ln(1+\tan x)\,dx \quad \text{— (1) —}$$

KING:

$$I = \int_0^{\pi/4} \ln\left(1+\tan\left(\frac{\pi}{4}-x\right)\right) dx$$

$$I = \int_0^{\pi/4} \ln\left(1+\frac{1-\tan x}{1+\tan x}\right) dx$$

$$I = \int_0^{\pi/4} \ln\left(\frac{2}{1+\tan x}\right) dx \quad \text{— (2) —}$$

$$2I = \int_0^{\pi/4} \ln 2 \cdot dx \Rightarrow I = \frac{1}{2}(\ln 2)\left(\frac{\pi}{4}\right)$$

Q $\int_0^1 \frac{ln(1+x)}{1+x^2}dx$

$x = \tan\theta$
$dx = \sec^2\theta \, d\theta$

$$\int_0^{\pi/4} \frac{\ln(1+\tan\theta)}{\cancel{\sec^2\theta}} \cdot \cancel{\sec^2\theta} \, d\theta$$

Q $I = \int_{\pi/2}^{3\pi/2} [2\sin x]dx$

$[\ ] \rightarrow$ Gif

$[x] + [-x] \begin{cases} \rightarrow 0; x \in I \\ \rightarrow -1; x \notin I \end{cases}$

KING

$$I = \int_{\pi/2}^{3\pi/2} [-2\sin x] \, dx$$

$$2I = \int_{\pi/2}^{3\pi/2} \Big(\underbrace{[2\sin x] + [-2\sin x]}_{-1} \Big) dx$$

$$I = \frac{1}{2}(-1) \cdot \left(\frac{3\pi}{2} - \frac{\pi}{2} \right)$$

Q If $\int_0^{\pi/2} \frac{x \sin x \cos x}{\sin^4 x + \cos^4 x} dx = \frac{\pi^2}{K}$ then find K ?

$$I = \int_0^{\pi/2} \frac{x \sin x \cos x}{\sin^4 x + \cos^4 x} dx \quad \text{—(1)—}$$

KING:

$$I = \int_0^{\pi/2} \frac{\left(\frac{\pi}{2} - x\right) \cos x \sin x}{\cos^4 x + \sin^4 x} dx \quad \text{—(2)—}$$

Add:

$$2I = \frac{\pi}{2} \int_0^{\pi/2} \frac{\sin x \cos x}{\sin^4 x + \cos^4 x} \cdot dx$$

$$= \frac{\pi}{2} \int_0^{\pi/2} \frac{\tan x . \sec^2 x .}{\tan^4 x + 1} dx$$

$\tan^2 x = t$.

$2 \tan x \sec^2 x dx = dt$

$$I = \frac{\pi}{4}\left(\frac{1}{2}\right) \int_0^{\infty} \frac{dt}{t^2+1} \Rightarrow I = \frac{\pi}{8}\left(\tan^{-1} t\right)\Big|_0^{\infty}$$

$$= \frac{\pi^2}{16}$$

$\therefore \boxed{K = 16}$

$$\int_0^1 \cot^{-1}(1 - x + x^2)\,dx$$

Q. $$\int_{\pi/8}^{3\pi/8} \ell n\left(\frac{4+3\sin x}{4+3\cos x}\right) dx$$

Ans $\rightarrow 0$

Q. $$\int_{0}^{\pi/2} \frac{\sin^3 x}{\sin x+\cos x} dx$$

Ans $\rightarrow \dfrac{\pi-1}{4}$.

Q $\displaystyle\int_0^{\pi} \frac{dx}{1+2^{\tan x}}$

Ans → $\frac{\pi}{2}$

Q $\int_0^1 \cot^{-1}(1-x+x^2)\,dx$

Ans → $\frac{\pi}{2} - \ln 2.$

$$\int_0^1 \tan^{-1}\left(\frac{x-(x-1)}{1+x(x-1)}\right) dx$$

$$\int_0^1 \left(\tan^{-1}x - \tan^{-1}(x-1)\right) dx$$

$$\int_0^1 \tan^{-1}x\,dx - \int_0^1 \tan^{-1}(x-1)\,dx$$

↓ King

$$\int_0^1 \tan^{-1}x\,dx - \int_0^1 \tan^{-1}(\cancel{1}-x-\cancel{1})\,dx$$

$$2\int_0^1 \tan^{-1}x\,dx$$

By parts.

P-6 $\int_0^{2a} f(x)dx = \int_0^{a} f(x)dx + \int_0^{a} f(2a-x)dx = \begin{cases} 0 & \text{if } f(2a-x) = -f(x) \\ 2\int_0^{a} f(x)\, dx & \text{if } f(2a-x) = f(x) \end{cases}$

QUEEN

$$\int_0^{2a} f(x)\,dx = \int_0^{a} f(x)\,dx + \int_a^{2a} f(x)\,dx$$

$x = 2a - t$

$dx = -dt$

$$= \int_0^{a} f(x)\,dx - \int_a^{0} f(2a-t)\,dt$$

eg

$$\int_0^{2\pi} \underbrace{\sin^4 x}\,dx$$

Queen

$$2\int_0^{\pi} \sin^4 x\,dx \xrightarrow{\text{Queen}} 4\int_0^{\pi/2} \sin^4 x\,dx = 4\left(\frac{3\pi}{16}\right)$$

Q $\displaystyle\int_0^{\pi} \frac{\sin x}{\sin 4x} = 0.$

$$\int_0^{\pi} \frac{\sin 8x}{\sin x}\,dx = 0.$$

Rem

$$\int_0^{\pi/2} \ln \sin x\,dx = \int_0^{\pi/2} \ln \cos x\,dx = \int_0^{\pi/2} \ln \sin 2x\,dx = \frac{-\pi}{2}\ln 2.$$

Rem

$$\int_0^{\pi/2} \ln \tan x\,dx = \int_0^{\pi/2} \ln \cot x\,dx = 0$$

$$I = \int_0^{\pi/2} \ln \sin x \, dx \quad \text{—①—}$$

KING

$$I = \int_0^{\pi/2} \ln \cos x \, dx \quad \text{—②—}$$

Add ① & ②

$$2I = \int_0^{\pi/2} \ln\left(\frac{2\sin x \cos x}{2}\right) dx$$

$$2I = \underbrace{\int_0^{\pi/2} \ln \sin 2x \, dx}_{I_1} - \int_0^{\pi/2} \ln 2 \, dx.$$

$$\boxed{2I = I_1 - \ln 2 \cdot \left(\frac{\pi}{2}\right)} \quad \text{—(iii)—}$$

$$I_1 = \int_0^{\pi/2} \ln \sin 2x \, dx$$

$2x = t$

$dx = \frac{1}{2} dt$

$$I_1 = \frac{1}{2} \int_0^{\pi} \ln(\sin t) \cdot dt$$

↓ Queen

$$I_1 = \frac{\cancel{2}}{\cancel{2}} \int_0^{\pi/2} \ln(\sin t) \, dt \Rightarrow I_1 = I$$

Q $\int_0^1 \ln \sin\left(\frac{\pi x}{2}\right) dx$

$\frac{\pi x}{2} = t \Rightarrow x = \frac{2}{\pi} t$

$dx = \frac{2}{\pi} dt$

$\frac{2}{\pi}\int_0^{\pi/2} \ln \sin t \, . dt = \frac{\cancel{2}}{\cancel{\pi}}\left(-\frac{\cancel{\pi}}{\cancel{2}} \ln 2\right)$

$= -\ln 2.$

Q

$\int_0^{\pi} \ln(1 - \cos x)dx$

$I = \int_0^{\pi} \ln(1-\cos x)\,dx$ —①—

King:

$I = \int_0^{\pi} \ln(1+\cos x)\,dx$ —②—

Add

$2I = \int_0^{\pi} \ln(\sin^2 x)\,dx \Rightarrow \cancel{2}I = \int_0^{\pi} \cancel{2}.\ln\sin x \, dx$

$\downarrow$ Queen

$I = 2\int_0^{\pi/2} \ln \sin x \, dx = 2\left(-\frac{\pi}{2}\ln 2\right)$

Q $\int_0^{\pi} x \ln(\sin x)\, dx$

KING & add

$$2I = \int_0^{\pi} \pi \ln(\sin x)\, dx$$

$\downarrow$ Queen.

$$\cancel{2}I = \cancel{2}\pi \int_0^{\pi/2} \ln \sin x\, dx \Rightarrow I = \pi\left(-\frac{\pi}{2}\ln 2\right)$$

Q $\int_0^{1} \frac{\sin^{-1} x}{x}\, dx$

$$\sin^{-1} x = \theta \Rightarrow x = \sin\theta.$$

$$dx = \cos\theta\, d\theta.$$

$$\int_0^{\pi/2} \frac{\theta}{\sin\theta} \cdot \cos\theta \cdot d\theta = \int_0^{\pi/2} \underset{\text{I}}{\theta} \cdot \underset{\text{II}}{\cot\theta} \cdot d\theta.$$

$$\left(\theta \cdot \ln(\sin\theta)\right)\Big|_0^{\pi/2} - \int_0^{\pi/2} \underbrace{1 \cdot (\ln \sin\theta)\, d\theta}.$$

$$0 - \lim_{\theta \to 0^+} (\theta \cdot \ln \sin\theta) - \left(-\frac{\pi}{2}\ln 2\right)$$

$$-\lim_{\theta \to 0^+} \cancelto{0}{\left(\frac{\ln \sin\theta}{(1/\theta)}\right)} + \frac{\pi}{2}\ln 2. = \frac{\pi}{2}\ln 2$$

Ans

Q $I = \int_0^{\pi} \frac{dx}{1+2\sin^2 x}$

Queen

$$2\int_0^{\pi/2} \frac{dx}{1+2\sin^2 x}$$

$$2\int_0^{\pi/2} \frac{\sec^2 x}{\underbrace{\sec^2 x}_{1+\tan^2 x} + 2\tan^2 x}\,dx$$

$$\frac{2}{3}\int_0^{\pi/2} \frac{\sec^2 x}{\tan^2 x + \left(\frac{1}{3}\right)}\,dx$$

$\tan x = t$

$$\frac{2}{3}\left(\int_0^{\infty} \frac{dt}{t^2 + \left(\frac{1}{\sqrt{3}}\right)^2}\right) = \frac{2}{3}\cdot\frac{1}{\left(1/\sqrt{3}\right)}\tan^{-1}\left(\frac{t}{1/\sqrt{3}}\right)\Bigg|_0^{\infty}$$

$$\frac{2}{\sqrt{3}}\cdot\left(\frac{\pi}{2}\right) = \frac{\pi}{\sqrt{3}}$$

Ans

Q $\int_0^{\pi} \underbrace{x}\left(\sin(\underbrace{\cos^2 x})\cos(\underbrace{\sin^2 x})\right)dx$

$\downarrow$ KING & add

$$2I = \pi\int_0^{\pi} \sin(\cos^2 x).\cos(\sin^2 x)\,dx$$

$$\cancel{2}I = \cancel{2}\pi \int_0^{\pi/2} \sin(\cos^2 x)\cos(\sin^2 x)\,dx$$

$$I = \pi \int_0^{\pi/2} \sin(\cos^2 x)\cos(\sin^2 x).\,dx \quad \text{--- 1 ---}$$

KING:

$$I = \pi \int_0^{\pi/2} \sin(\sin^2 x)\cos(\cos^2 x)\,dx \quad \text{--- 2 ---}$$

Add:

$$2I = \pi \int_0^{\pi/2} \sin(\sin^2 x + \cos^2 x)\,dx$$

$$2I = \pi \int_0^{\pi/2} (\sin 1)\,dx$$

$$I = \frac{\pi}{2}.\sin 1.\left(\frac{\pi}{2} - 0\right)$$

Q $\int_0^{2\pi} x \underbrace{\sin^4 x}\ \underbrace{\cos^6 x}\ dx$

KING & ADD

$$\cancel{2}I = \int_0^{2\pi} \cancel{2}\pi \sin^4 x \cdot \cos^6 x\, dx$$

Queen 2 times.

$$I = 4\pi \int_0^{\pi/2} \sin^4 x \cos^6 x\, dx$$

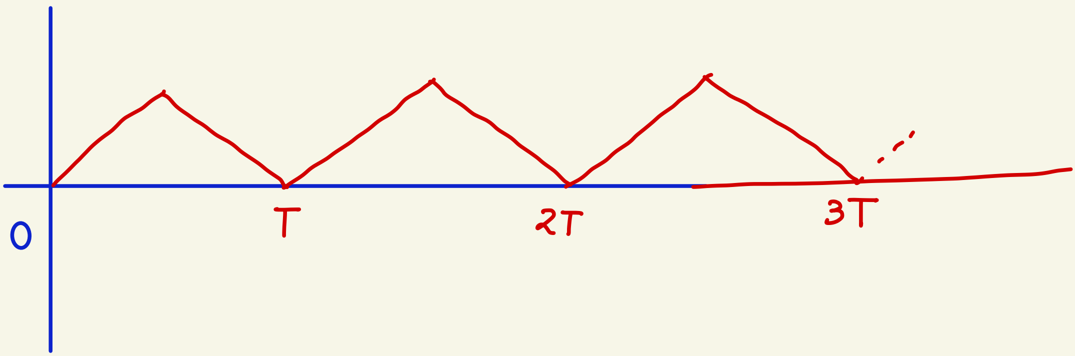
$$I = \cancel{4}\pi \left(\frac{(3\times 1)\ (\cancel{5}\times\cancel{3}\times 1)}{(\cancel{10}\times 8\times \cancel{6}\times 4\times \cancel{2})} \right) \cdot \frac{\pi}{2} = \frac{3\pi^2}{128}$$

$\cancel{2}$ 2

Ans

P – 7 Let f(x) be periodic function with period 'T' i.e. $f(T + x) = f(x)$; $T > 0$.

(i) $\int_0^{nT} f(x)dx = n\int_0^{T} f(x)\,dx$, $n \in I$ (JACK / KNAVE)



Note :- $T \to$ period, $a, b \in R$
$m, n \in I$.

① $\int_a^{T+a} f(t)dt$ will be independent of a and equal to $\int_0^{T} f(t)dt$

② $\int_a^{a+nT} f(x)dx = \int_0^{nT} f(x)dx = n\int_0^{T} f(x)\,dx$

③ $\int_{a+nT}^{b+nT} f(x)dx = \int_a^{b} f(x)dx$, where $n \in I$.

④ $\int_{mT}^{nT} f(x)dx = (n - m)\int_0^{T} f(x)dx$, where $n, m \in I$.

Q. $$\int_0^{100\pi} |\sin x|\,dx$$

$|\sin x| \longrightarrow P = \pi$

$$100\int_0^{\pi} \sin x\,dx = 100 \times 2 = 200.$$

Q. $$\int_0^{200\pi} \sqrt{1+\cos x}\,dx$$

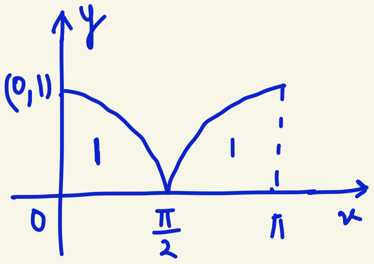
$\frac{x}{2} = t$

$dx = 2dt$

$$\int_0^{200\pi} \sqrt{2.\cos^2\frac{x}{2}}\,dx = \sqrt{2}\int_0^{200\pi} \left|\cos\frac{x}{2}\right| dx$$

$$2\sqrt{2}\int_0^{100\pi} |\cos t|.\,dt$$

$\downarrow \quad P = \pi$



$$2\sqrt{2} \times 100 \underbrace{\int_0^{\pi} |\cos t|.\,dt.}_{2}$$

$400\sqrt{2}$ Ans

Q $\int_0^{2n\pi} \left(|\sin x| - \left[\cancel{\left| \frac{\sin x}{2} \right|} \right] \right) dx$ [.] denotes greatest integer function

0

$$\underbrace{2n \int_0^{\pi} |\sin x|\, dx} = 4n.$$

Q $\int_0^{1000} e^{x-[x]} dx$; [ ] → Gif

$\{x\} \to P = 1$

$$\int_0^{1000} e^{\{x\}} dx$$

$$1000 \int_0^1 e^x dx = 1000 \cdot (e^x)\Big|_0^1$$

$$= 1000(e-1) \text{ Ans}$$

Q. $\int_0^{n\pi + v} |\cos x|\, dx$ where $\frac{\pi}{2} < v < \pi$ & $n \in N$

$$\int_0^{v} |\cos x|\,dx + \underbrace{\int_v^{v+n\pi} |\cos x|\,dx}$$

$$\underbrace{\int_0^{\pi/2} \cos x\, dx}_{} + \int_{\pi/2}^{v} (-\cos x)\, dx + \underbrace{\int_0^{n\pi} |\cos x|\, dx}$$

$$1 - (\sin x)\Big|_{\pi/2}^{v} + n(2)$$

$$2n + 1 - (\sin v - 1)$$

$$2n + 2 - \sin v$$

Ans

Q $\int_{-2.5}^{3.5} \left(2x - [2x]\right) dx$; $[\;] \to$ Gif

$\{2x\} \longrightarrow P = \left(\frac{1}{2}\right)$

$$\int_{-5\times\frac{1}{2}}^{7\times\frac{1}{2}} \{2x\}\, dx = 7-(-5) \int_0^{1/2} \{2x\}\, dx$$

$$12 \int_0^{1/2} (2x)\, dx$$

Q If $\int_0^{50\pi} \frac{x\,|\sin x|}{\{x\}+\{-x\}}\, dx = K\pi$; $K \in N$ then find K ? (Note $\{\;\} \to$ ....)

$\{x\}+\{-x\}$

Sol^n

$$I = \int_0^{50\pi} x\,|\sin x|\, dx$$

King & add

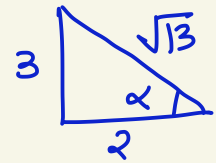
$$2I = 50\pi \int_0^{50\pi} |\sin x|\, dx$$

$$I = 25\pi\,.(50)\,(2) = 2500\pi\,.$$

Q $**$

$$\int_0^{\pi} |2\sin x + 3\cos x|\, dx$$

$$\int_0^{\pi} \sqrt{13}\ \left|\frac{2}{\sqrt{13}}\sin x + \frac{3}{\sqrt{13}}\cos x\right| dx.$$



where

$$\cos\alpha = \frac{2}{\sqrt{13}}$$

$$\sqrt{13}\int_0^{\pi} |\sin(x+\alpha)|\, dx$$

$$x+\alpha = t$$
$$dx = dt$$

$$\sqrt{13}\int_{\alpha}^{\pi+\alpha} \underbrace{|\sin t|}_{0}\, dt = \sqrt{13}\int_0^{\pi} |\sin t|\, dt$$

$$= 2\sqrt{13}.$$

Ans

Q
$$\int_0^{[x]} \frac{2^t}{2^{[t]}}\,dt, \quad x \geq 0$$

$[\ ] \rightarrow$ Gif

$$\int_0^{[x]} 2^{t-[t]}\,dt = \int_0^{[x]} 2^{\{t\}}\,dt$$

$[x] \rightarrow$ I x 1.

$\{\ \} \rightarrow P = 1$

$$[x]\int_0^1 2^{\{t\}}\,dt$$

$$[x]\int_0^1 2^t\,dt = [x]\left(\frac{2^t}{\ln 2}\right)\Bigg|_0^1$$

$$= [x]\left(\frac{2}{\ln 2} - \frac{1}{\ln 2}\right)$$

$$= \frac{[x]}{\ln 2} \quad \text{Ans}$$

DERIVATIVE OF ANTIDERIVATIVES (Newton-Leibnitz Formula) :

If h(x) & g(x) are differentiable functions of x and f is continuous function then,

$$\boxed{\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)dt = f(h(x)).h'(x) - f(g(x)).g'(x)}$$

Rem

Proof : Let $\int f(t)\,dt = F(t) + C \Rightarrow F'(t) = f(t)$

$$\text{Now } y = \int_{g(x)}^{h(x)} f(t)dt = F(h(x)) - F(g(x))$$

$$\Rightarrow \frac{dy}{dx} = F'(h(x))h'(x) - F'(g(x))g'(x) \Rightarrow \frac{dy}{dx} = f(h(x))h'(x) - f(g(x))g'(x)$$

Q Let $G(x) = \int_2^{x^2} \frac{dt}{1+\sqrt{t}}$ $(x > 0)$. Find $G'(9)$.

Sol^n diff wrt 'x' Using Leibnitz Rule

$$G'(x) = \frac{1}{1+\sqrt{x^2}} \cdot 2x - 0.$$

$$G'(x) = \frac{2x}{1+|x|}$$

$$G'(x) = \frac{2x}{1+x} \Rightarrow G'(9) = \frac{18}{10} = 1.80$$

Q. $f(x) = \int_{e^{2x}}^{e^{3x}} \frac{t}{\ell n\, t} dt \quad x > 0$. Find derivative of $f(x)$ w.r.t. $ln\, x$ when $x = ln\, 2$

$$f'(x) = \frac{e^{3x}}{\ln e^{3x}} \cdot 3e^{3x} - \frac{e^{2x}}{\ln e^{2x}} \cdot 2 \cdot e^{2x}$$

$$= \frac{\cancel{3} e^{6x}}{\cancel{3} x} - \frac{\cancel{2} e^{4x}}{\cancel{2} x} = \frac{e^{6x} - e^{4x}}{x}$$

$$g(x) = \ln x \longrightarrow g'(x) = \left(\frac{1}{x}\right)$$

$$\left.\frac{f'(x)}{g'(x)}\right|_{x=\ln 2} = \left.\left(e^{6x} - e^{4x}\right)\right|_{x=\ln 2} = 2^6 - 2^4$$

$$64 - 16$$
$$= 48.$$

Q. $x = \int_0^y \frac{dt}{\sqrt{1+4t^2}}$, If $\frac{d^2 y}{dx^2} = ky$, find k.

diff wrt 'y' :-

$$\frac{dx}{dy} = \frac{1}{\sqrt{1+4y^2}} \Rightarrow \frac{dy}{dx} = \sqrt{1+4y^2} \quad -(1)-$$

diff wrt 'x' :-

$$\frac{d^2y}{dx^2} = \frac{1}{2\sqrt{1+4y^2}} \cdot (8y)\left(\frac{dy}{dx}\right)$$

$$= 4y.$$

$$\therefore \boxed{K = 4}.$$

Q $\displaystyle \operatorname*{Limit}_{x \to 0} \frac{\int_0^{x^2} \cos t^2 \, dt}{x \sin x}$ $\left(\frac{0}{0} \text{ form}\right)$

LH Rule.

$$\lim_{x \to 0} \frac{(\cos x^4) \cdot (2x) - 0}{x \cos x + \sin x} = 1.$$

$$\overbrace{\hphantom{xxxxxxxxxxxx}}^{0}$$

* Q $\displaystyle \operatorname*{Limit}_{x \to 0} \frac{\int_0^{x} x e^{t^2} \, dt}{1 - e^{x^2}}$ $\displaystyle \lim_{x \to 0} \frac{x \int_0^x e^{t^2} \, dt}{1 - e^{x^2}}$

LH Rule

$$\lim_{x \to 0} \frac{x \left(e^{x^2}\right) + 1 \cdot \int_0^x e^{t^2} \, dt}{0 - 2x e^{x^2}}$$

$$\lim_{x\to 0} \frac{\cancel{x e^{x^2}}}{-2\cancel{x e^{x^2}}} + \lim_{x\to 0} \frac{\int_0^x e^{t^2}\,dt}{-2xe^{x^2}}$$

L'H Rule.

$$-\frac{1}{2} + \lim_{x\to 0} \frac{\cancel{e^{x^2}}}{-2\left(2x^2\cdot \cancel{e^{x^2}} + \cancel{e^{x^2}}\right)}$$

$$-\frac{1}{2} + \left(\frac{1}{-2}\right) = -1.$$

Q

$$\lim_{x\to 0} \frac{\int_0^x (t^2+e^{t^2})^{\frac{1}{1-\cos t}}\,dt}{(e^x-1)}.$$

$\left(\frac{0}{0}\text{ form}\right)$

LH Rule

$$\lim_{x\to 0} \frac{(x^2+e^{x^2})^{\frac{1}{1-\cos x}}}{\underbrace{e^x}} = e^{\lim_{x\to 0} \frac{1}{(1-\cos x)}(x^2+e^{x^2}-1)}$$

$$e^{\lim_{x\to 0} \left(\frac{x^2}{1-\cos x}\right)\left(\frac{x^2}{\underbrace{x^2}} + \frac{e^{x^2}-1}{\underbrace{x^2}}\right)}$$

$$e^{2(1+1)} = e^4.$$ Ans

Q Let f (x) is a derivable function satisfying $f(x)=\int_0^x e^t \sin(x-t)\,dt$ and $g(x)=f''(x)-f(x)$. Find the range of g (x).

Sol^n $$f(x)=\int_0^x e^t \sin(x-t)\,dt$$

KING : $$f(x)=\int_0^x e^{x-t} \sin(x-(x-t))\cdot dt$$

$$f(x)=e^x \int_0^x e^{-t} \sin t \cdot dt$$

$$e^{-x} f(x)=\int_0^x e^{-t} \sin t\, dt \xrightarrow[\text{wrt 'x'}]{\text{diff}} \cancel{e^{-x}} f'(x)-\cancel{e^{-x}} f(x)=\cancel{e^{-x}} \sin x$$

$$\left.\begin{aligned} f'(x)-f(x) &= \sin x \\ f''(x)-f'(x) &= \cos x \end{aligned}\right\} \Rightarrow \text{add} \qquad \underbrace{f''(x)-f(x)}_{g(x)}=\sin x+\cos x$$

$$R_g \in [-\sqrt{2}, \sqrt{2}]$$ Ans

Q $$\frac{d}{dx}\int_x^{x^2} \frac{dt}{(x^2+t^2)} \; ;(x>0)$$

$$\frac{d}{dx}\left(\left.\frac{1}{x}\tan^{-1}\left(\frac{t}{x}\right)\right|_x^{x^2}\right)=\frac{d}{dx}\left(\frac{1}{x}\left(\tan^{-1}\frac{x^2}{x}-\tan^{-1}\frac{x}{x}\right)\right)$$

$$\frac{d}{dx}\left(\frac{1}{x}\underbrace{\left(\tan^{-1}x-\frac{\pi}{4}\right)}\right)$$

$$\frac{1}{x}\left(\frac{1}{1+x^2}\right)+\left(\tan^{-1}x-\frac{\pi}{4}\right)\left(-\frac{1}{x^2}\right)$$ Ans

Q. If $x \in [0, \pi/2]$, $f(x) = \int_0^{\sin^2 x} \sin^{-1}\sqrt{t}\, dt + \int_0^{\cos^2 x} \cos^{-1}\sqrt{t}\, dt$

then find $f\left(\frac{5\pi}{12}\right)$?

diff wrt 'x' :-

$$f'(x) = \sin^{-1}\sqrt{\sin^2 x}\,.\,(2\sin x\cos x) + \cos^{-1}\sqrt{\cos^2 x}\,(-2\cos x\sin x)$$

$$f'(x) = \underbrace{\sin^{-1}\sin x}_{x}\,(\sin 2x) - \underbrace{\cos^{-1}\cos x}_{x}\,(\sin 2x)$$

$\boxed{f'(x) = 0} \Rightarrow f(x)$ is constant fns.

$$f\left(\frac{\pi}{4}\right) = \int_0^{1/2} \left(\sin^{-1}\sqrt{t} + \cos^{-1}\sqrt{t}\right) dt = \frac{\pi}{2}\left(\frac{1}{2} - 0\right)$$

$$f(\pi/4) = f(x) = \frac{\pi}{4}.$$

HW

Q1 If $x = \int_{1}^{t^2} z \ln z \, dz$ and $y = \int_{t^2}^{1} z^2 \ln z \, dz$ find $\frac{dy}{dx}$

Q2 If $y = \int_{0}^{z^2} \frac{dx}{1+x^3}$ find $\frac{d^2y}{dz^2}$ at $z = 1$

Q3 If $\lim_{x\to\infty} x^{-\left(\frac{3}{2}+n\right)} \int_{0}^{x} t^n \sqrt{t+1}\, dt = \frac{2}{11}$ then

find $n \in N$?

Q

Let $f(x)$ be a continuous function such that $f(x) > 0$ for all $x \geq 0$ and $(f(x))^{101} = 1 + \int_0^x f(t)\,dt$. The value of $(f(101))^{100}$ is

put $x=0$

diff wrt 'x'.

$(f(0))^{101} = 1.$

$\boxed{f(0)=1.}$

$$101 \cdot (f(x))^{100} \cdot f'(x) = 0 + f(x).$$

$$\underbrace{f(x)}_{\neq 0}\left(\underbrace{101 (f(x))^{99} f'(x) - 1}_{=0}\right) = 0.$$

$$(f(x))^{99} f'(x) = \frac{1}{101}$$

Integrate wrt 'x' :-

$$\int (f(x))^{99} f'(x)\,dx = \int \frac{dx}{101}$$

$$\boxed{\frac{(f(x))^{100}}{100} = \frac{x}{101} + c.}$$

put $x=0$

$$\boxed{\frac{1}{100} = c}$$

DETERMINATION OF FUNCTION :

E(1) Find a function f, continuous for all x (and not zero every where), such that

$$f^2(x) = \int_0^x \frac{f(t)\sin t}{2+\cos t}\,dt$$

put $x=0$

$f^2(0)=0$

$\boxed{f(0)=0}$

Sol^n diff wrt 'x' :-

$$2f(x)f'(x) = \frac{f(x)\sin x}{2+\cos x}$$

$$\underbrace{f(x)}_{\neq 0}\left(\underbrace{2f'(x) - \frac{\sin x}{2+\cos x}}\right) = 0.$$

$$f'(x) = \frac{\sin x}{2(2+\cos x)}$$

Integrate wrt 'x' :-

$2+\cos x = t$

$-\sin x\,dx = dt$

$$f(x) = \frac{1}{2}\int \frac{\sin x}{\underbrace{2+\cos x}}\,dx$$

$$f(x) = -\frac{1}{2}\ln\underbrace{|2+\cos x|} + C$$

put $x=0$

$$f(0) = 0 = -\frac{1}{2}\ln 3 + C$$

$$\therefore \boxed{C = \frac{1}{2}\ln 3}$$

Q*

If $f(x) = x + \int_0^{\pi/2} \sin(x+y).f(y)\,dy$ where x and y are independent variables, find f(x).

Sol^n

$$f(x) = x + \int_0^{\pi/2} \left(\sin x \cos y\, f(y) + \cos x \sin y\, f(y)\right) dy$$

$$f(x) = x + \sin x \underbrace{\int_0^{\pi/2} \cos y\, f(y)\, dy}_{A} + \cos x \underbrace{\int_0^{\pi/2} \sin y\, f(y)\, dy}_{B}.$$

$$\boxed{f(x) = x + A \sin x + B \cos x} \quad \text{— (1) —}$$

$$f(y) = y + A \sin y + B \cos y.$$

$$* \quad A = \int_0^{\pi/2} \cos y\left(y + A\sin y + B\cos y\right) dy$$

$$* \quad B = \int_0^{\pi/2} \sin y\left(y + A\sin y + B\cos y\right) dy.$$

get A & B and put in (1) to get f(x)

Q Let 'g' be continuous function on $\mathbb{R}$ and satisfies

$$g(x) + 2\int_0^{\pi/2} \sin x \cos t \, g(t)\, dt = \sin x$$

then find the value of $g'(\pi/3)$?

Sol^n

$$g(x) + 2\sin x \underbrace{\int_0^{\pi/2} \cos t \, g(t)\, dt}_{A.} = \sin x$$

$$g(x) + 2A\sin x = \sin x$$

$$\boxed{g(x) = \sin x(1-2A)} \quad \text{— (1) —}$$

$$g'(x) = \cos x\,(1-2A)$$

$$g'(\pi/3) = \left(\frac{1-2A}{2}\right)$$

$$A = \int_0^{\pi/2} \cos t . \sin t\,(1-2A)\, dt$$

$$A = (1-2A)\left.\frac{(\sin t)^2}{2}\right|_0^{\pi/2}$$

$$2A = (1-2A)(1-0) \Rightarrow \boxed{A = \frac{1}{4}}$$

Reduction Formula :-

Q If $u_n = \int_0^{\pi/2} x(\sin x)^n \, dx$, $n > 0$, then prove that $u_n = \frac{n-1}{n} u_{n-2} + \frac{1}{n^2}$

Sol^n

$$u_n = \int_0^{\pi/2} x \cdot \underbrace{(\sin x)}_{\text{II}} (\sin x)^{n-1} \cdot dx$$

$$u_n = \left(\underbrace{x \cdot (\sin x)^{n-1}}_{0} \cdot (-\cos x)\right)\Big|_0^{\pi/2} - \int_0^{\pi/2} \left((\sin x)^{n-1} + x \cdot (n-1)(\sin x)^{n-2} \cdot \cos x\right)(-\cos x)\, dx$$

$$u_n = \int_0^{\pi/2} \left(\underbrace{(\sin x)^{n-1} \cos x}_{0} + (n-1)\, x \cdot (\sin x)^{n-2} \cdot \cos^2 x\right) dx$$

$$u_n = \frac{(\sin x)^n}{n}\Big|_0^{\pi/2} + (n-1) \int_0^{\pi/2} x \cdot (\sin x)^{n-2} \cdot (1 - \sin^2 x)\, dx$$

$$u_n = \frac{1}{n} + (n-1)\left(u_{n-2} - u_n\right)$$

$$u_n(\cancel{1} + n - \cancel{1}) = \frac{1}{n} + (n-1) u_{n-2}$$

$$u_n = \frac{1}{n^2} + \left(\frac{n-1}{n}\right) u_{n-2}.$$

Q Let $I_n = \int_0^1 (1-x^4)^n dx$, then prove that $\frac{I_n}{I_{n-1}} = \frac{4n}{4n+1}$

Sol^n

$$I_n = \int_0^1 \underbrace{(1-x^4)^n}_{I} \cdot \underbrace{1}_{II} \, dx$$

$$I_n = \left(\cancel{(1-x^4)^n \cdot x}\right)\Big|_0^1 - \int_0^1 n(1-x^4)^{n-1}(-4x^3) \cdot x \cdot dx.$$

$$I_n = 4n \int_0^1 (1-x^4)^{n-1} x^4 \, dx$$

$$I_n = -4n \int_0^1 (1-x^4)^{n-1} (1-x^4-1) \, dx$$

$$I_n = -4n \left(I_n - I_{n-1} \right)$$

$$(4n+1) I_n = 4n I_{n-1}$$

$$\boxed{\frac{I_n}{I_{n-1}} = \frac{4n}{4n+1}}$$

$$\lim_{n \to \infty} \frac{I_n}{I_{n-1}} = ?$$

Rem

$$I_n = \int \frac{\sin nx}{\sin x}\,dx \; ; \; n \in N$$

Rem

$$\boxed{\int_0^{\pi} \frac{\sin nx}{\sin x}\,dx \rightarrow \begin{cases} \pi \; ; \; \text{if } n \in \text{odd} \\ 0 \; ; \; \text{if } n \in \text{even.} \end{cases}}$$

$$I_n - I_{n-2} = \int \frac{\sin nx - \sin(n-2)x}{\sin x}\,dx$$

$$I_n - I_{n-2} = \int \frac{2\cancel{\sin x}.\cos(n-1)x}{\cancel{\sin x}}\,dx$$

$$\boxed{I_n - I_{n-2} = 2.\frac{\sin(n-1)x}{(n-1)}}$$

$$I_n - I_{n-2} = 2\left.\frac{\sin(n-1)x}{(n-1)}\right|_0^{\pi} = 0.$$

$$I_n = I_{n-2} \Rightarrow I_{n-2} = I_{n-4}$$

$$I_n = I_{n-2} = I_{n-4} = I_{n-6} = \cdots\cdots$$

(i) If $\underline{\underline{n \in \text{odd}}}$

$$I_n = I_{n-2} = \cdots = I_1$$

$$I_n = I_1 = \int_0^{\pi} \frac{\cancel{\sin x}}{\cancel{\sin x}}\,dx = \pi.$$

(ii) If $\underline{\underline{n \in \text{even}}}$

$$I_n = I_{n-2} = \cdots = I_2$$

$$I_n = I_2 = \int_0^{\pi} \frac{\sin 2x}{\sin x}\,dx = 2\int_0^{\pi} \cos x\,dx$$

$$= 2.\,(\sin x)\Big|_0^{\pi}$$

$$= 0.$$

** $\displaystyle\int_0^{\pi/2} \frac{\sin nx}{\sin x}\,dx = \frac{\pi}{2}$ if $\underline{\underline{n \in \text{odd}}}$

$\displaystyle\int_0^{\pi} \frac{\sin 20x}{\sin x}\,dx = 0$; $\displaystyle\int_0^{\pi/2} \frac{\sin 91x}{\sin x}\,dx = \frac{\pi}{2}$; $\displaystyle\int_0^{\pi} \frac{\sin 15x}{\sin x}\,dx = \pi$

SUM OF SERIES USING DEFINITE INTEGRATION :

Using definite integration as limit of sum $\lim_{n\to\infty} h[f(a)+f(a+h)+f(a+2h)+\ldots..+f\left(a+\overline{n-1}h\right)] = \int_a^b f(x)dx$

or $\quad \lim_{h\to\infty} h\sum_{r=0}^{n-1} f(a+rh) = \int_a^b f(x)dx$, where $b-a=nh$

If $a=0$ & $b=1$ then, $\lim_{n\to\infty} h\sum_{r=0}^{n-1} f(rh) = \int_0^1 f(x)dx$; where $nh = 1$

Hence we have $\int_0^1 f(x)dx = \lim_{n\to\infty}\frac{1}{n}\sum_{r=0}^{n-1} f\left(\frac{r}{n}\right)$

Steps to express the infinite series as definite integral :

Step I : Express the given series in the form $\sum\frac{1}{n}f\left(\frac{r}{n}\right)$

Step II : Then the limit is its sum when $n\to\infty$, i.e. $\lim_{n\to\infty}\frac{1}{n}f\left(\frac{r}{n}\right)$

Step III : Replace $\frac{r}{n}$ by x and $\frac{1}{n}$ by dx and $\lim_{n\to\infty}\sum$ by the sign of $\int$

Step IV : The lower and the upper limit of integration are the limiting values of $\frac{r}{n}$ for the first and the last term of r respectively.

Q $\lim_{n\to\infty}\frac{\sqrt{1}+\sqrt{2}+\sqrt{3}+\ldots\ldots.+\sqrt{n}}{n\sqrt{n}}$

$$\lim_{n\to\infty}\sum_{K=1}^{n}\frac{1\sqrt{K}}{n\sqrt{n}} = \int_0^1\sqrt{x}\,dx$$

$$= \left.\frac{x^{3/2}}{(3/2)}\right|_0^1 = \frac{2}{3}$$

Q $\lim_{n\to\infty}\left[\frac{n+1}{n^2+1^2}+\frac{n+2}{n^2+2^2}+\frac{n+3}{n^2+3^2}+\ldots\ldots+\frac{3}{5n}\right]$

$$\frac{3n}{5n^2} \qquad \frac{n+(2n)}{n^2+(2n)^2}$$

$$\lim_{n\to\infty}\sum_{r=1}^{2n}\left(\frac{n+r}{n^2+r^2}\right) = \lim_{n\to\infty}\sum_{r=1}^{2n}\frac{\cancel{n}}{n^{\cancel{2}}}\left(\frac{1+r/n}{1+r^2/n^2}\right)$$

$$\int_0^2\left(\frac{1+x}{1+x^2}\right)dx$$

$$= \tan^{-1}2+\frac{1}{2}\ln 5$$ Ans

Q $\lim_{n\to\infty} \frac{[(n+1)(n+2)\ldots\ldots\ldots(n+n)]^{1/n}}{n}$

$$l = \lim_{n\to\infty} \left(\left(\frac{n+1}{n}\right) \left(\frac{n+2}{n}\right) \left(\frac{n+3}{n}\right) \cdots \cdot \left(\frac{n+n}{n}\right) \right)^{\frac{1}{n}}$$

$$\ln l = \lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} \ln\left(1+\frac{r}{n}\right)$$

$$\underbrace{\ln l} = \int_0^1 \ln(1+x)\,dx \Rightarrow \quad \ln l = \ln\left(\frac{4}{e}\right)$$

$$\therefore\ l = \left(\frac{4}{e}\right) \text{ Ans}$$

Q $\operatorname*{Limit}_{n\to\infty}\left[\left(1+\frac{1}{n^2}\right)^{\frac{2}{n^2}}\cdot\left(1+\frac{2^2}{n^2}\right)^{\frac{4}{n^2}}\cdot\left(1+\frac{3^2}{n^2}\right)^{\frac{6}{n^2}}\cdots\cdots\underbrace{\left(2\right)^{\frac{2}{n}}}_{\left(1+\frac{n^2}{n^2}\right)^{\frac{2n}{n^2}}}\right]$

l

$$\ln l = \lim_{n\to\infty}\sum_{r=1}^{n}\frac{2r}{n^2}\ln\left(1+\frac{r^2}{n^2}\right)$$

$$= \lim_{n\to\infty}\sum_{r=1}^{n}\frac{1}{n}\,2\left(\frac{r}{n}\right)\ln\left(1+\left(\frac{r}{n}\right)^2\right)$$

$$\ln l = \int_0^1 2x\ln(1+x^2)\,dx \qquad 1+x^2=t$$

$$\ln l = \ln\left(\frac{4}{e}\right) \Rightarrow l = \frac{4}{e}$$ Ans

HW

Q

$$\lim_{n\to\infty} \left({}^{2n}C_n\right)^{1/n}.$$

Q HW

$$\lim_{n\to\infty}\left(\tan^{-1}\frac{1}{n}\right)\left(\sum_{k=1}^{n}\frac{1}{1+\tan(k/n)}\right)$$ has the value equal to

Q HW

$$\lim_{n \to \infty} \sum_{r=1}^{n} \left(\frac{2n-1}{n^2+r^2} \right)$$

Q

Let $S_n = \sum_{k=1}^{n} \frac{n}{n^2+kn+k^2}$ and $T_n = \sum_{k=0}^{n-1} \frac{n}{n^2+kn+k^2}$, for n = 1, 2, 3, ... Then,

(A) $S_n < \frac{\pi}{3\sqrt{3}}$ (B) $S_n > \frac{\pi}{3\sqrt{3}}$ (C) $T_n < \frac{\pi}{3\sqrt{3}}$ (D) $T_n > \frac{\pi}{3\sqrt{3}}$ [JEE 2008, 4]

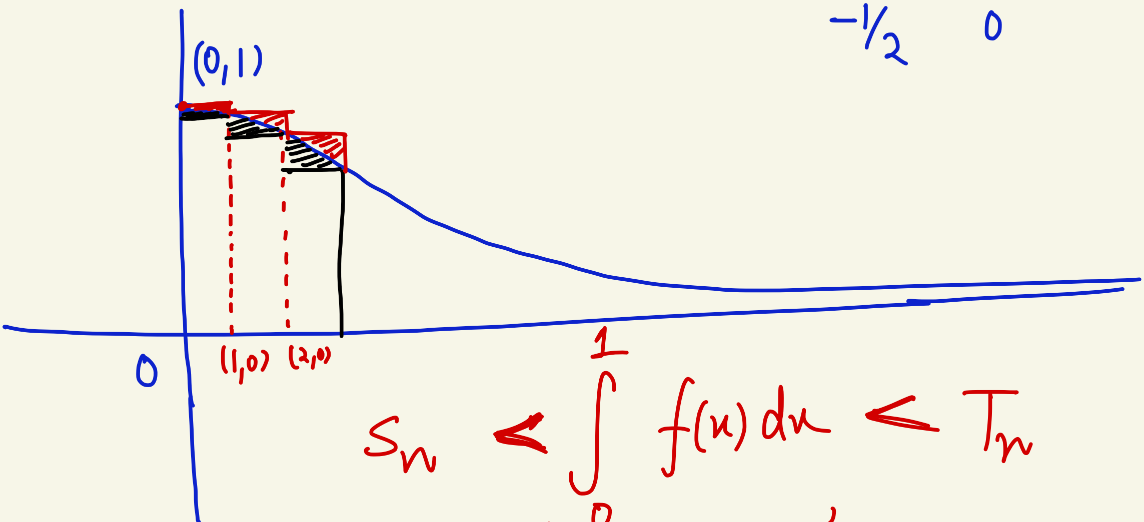
Sol^n

$$S_n = \sum_{k=1}^{n} \frac{\cancel{n}}{n^{\cancel{2}}\left(1+\frac{k}{n}+\frac{k^2}{n^2}\right)}$$

$$\boxed{f(x) = \frac{1}{1+x+x^2}} \rightarrow f'(x) = \frac{-1}{(1+x+x^2)^2}\cdot(1+2x)$$

$+$ $-$

$-1/2$ 0



$$S_n < \int_0^1 f(x)\,dx < T_n$$

Q

Find the sum of the series

$$\underbrace{\frac{x^2}{1.2}-\frac{x^3}{2.3}+\frac{x^4}{3.4}-\ldots\ldots+(-1)^{n+1}\frac{x^{n+1}}{n(n+1)}+\ldots\ldots\ldots}\quad ; \quad |x|<1$$

$x=0 \Rightarrow \boxed{S=0}$

Sol^n

$$S = \frac{x^2}{1.2} - \frac{x^3}{2.3} + \frac{x^4}{3.4} - \cdots \qquad |x|<1$$

$$\frac{dS}{dx} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$

$$\frac{dS}{dx} = \ln(1+x) \Rightarrow \int dS = \int \ln(1+x)\,dx$$

$$S = (1+x)\ln(1+x) - (1+x) + C$$

put $x=0$

$$0 = 1.\cancel{\ln 1}^{\,0} - 1 + C$$

$$\therefore \boxed{C=1}$$

EVALUATING INTEGRALS DEPENDENT ON A PARAMETER (NOT IN IIT JEE SYLLABUS)

Differentiate I w.r.t. the parameter within the sign of integrals taking variable of the integrand as constant. Now evaluate the integral so obtained as usual as a function of the parameter and then integrate the result to get I. Constant of integration to be computed by giving some arbitrary values to the parameter so that I can be evaluated.

$$I = \int_0^{\pi/2} \sin(x+c)\,dx \quad ; \text{ c is parameter.}$$

$c = 0$
$I = 1$

$$\frac{dI}{dc} = \int_0^{\pi/2} \cos(x+c)\,dx$$

$$\frac{dI}{dc} = \sin(x+c)\Big|_0^{\pi/2} = \sin\left(\frac{\pi}{2}+c\right) - \sin(0+c)$$

$$\boxed{\frac{dI}{dc} = \cos c - \sin c} \Rightarrow \int dI = \int(\cos c - \sin c)\,dc$$

$$I = (\sin c + \cos c) + \lambda$$

Const of int.

put $c = 0$
$I = 1$

$$1 = 0 + 1 + \lambda$$

$$\boxed{\lambda = 0}$$

Q. Evaluate $\int_0^{\infty} \frac{\tan^{-1} ax - \tan^{-1} x}{x} dx$ where a is a parameter. ; $(a>0)$

$a=1$, $I=0$

$$I = \int_0^{\infty} \frac{\tan^{-1} ax - \tan^{-1} x}{x} dx$$

$$\frac{dI}{da} = \int_0^{\infty} \frac{1}{\cancel{x}}\left(\frac{1}{1+a^2x^2}\right)\cdot \cancel{x} \cdot dx$$

$$= \frac{1}{a^2} \int_0^{\infty} \frac{dx}{x^2+\left(\frac{1}{a^2}\right)} = \frac{1}{a^2}(a)\tan^{-1}\left(\frac{x}{1/a}\right)\Big|_0^{\infty}$$

$$\frac{dI}{da} = \frac{\cancel{a}}{a^{\cancel{2}}}\left(\tan^{-1}(ax)\Big|_0^{\infty}\right) = \frac{1}{a}\left(\frac{\pi}{2} - 0\right)$$

$$\int dI = \frac{\pi}{2} \cdot \int \frac{da}{a}$$

$$\boxed{I = \frac{\pi}{2} \ln a + C}$$

$a=1$
$I=0$

$$0 = \frac{\pi}{2} \ln 1 + C \Rightarrow \boxed{C=0}$$

Estimation of D.I & general inequalities in Integration

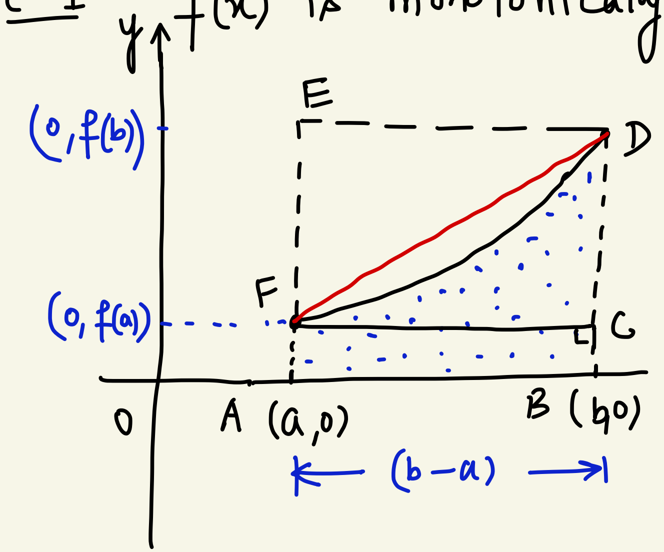
(i) If $f(x) \le g(x) \le h(x)$ in $[a, b]$, then $\int_a^b f(x)dx \le \int_a^b g(x)dx \le \int_a^b h(x)dx$

(ii) $\left|\int_a^b f(x)dx\right| \le \int_a^b |f(x)|dx$ equality holds when f(x) is entirely of the same sign on [a,b].

(iii) If m & M are respectively the least and greatest value of $f(x)$ in $[a, b]$, then

$$m(b-a) \le \int_a^b f(x)dx \le M(b-a)$$

C-I $f(x)$ is monotonically increasing fun. in (a,b)

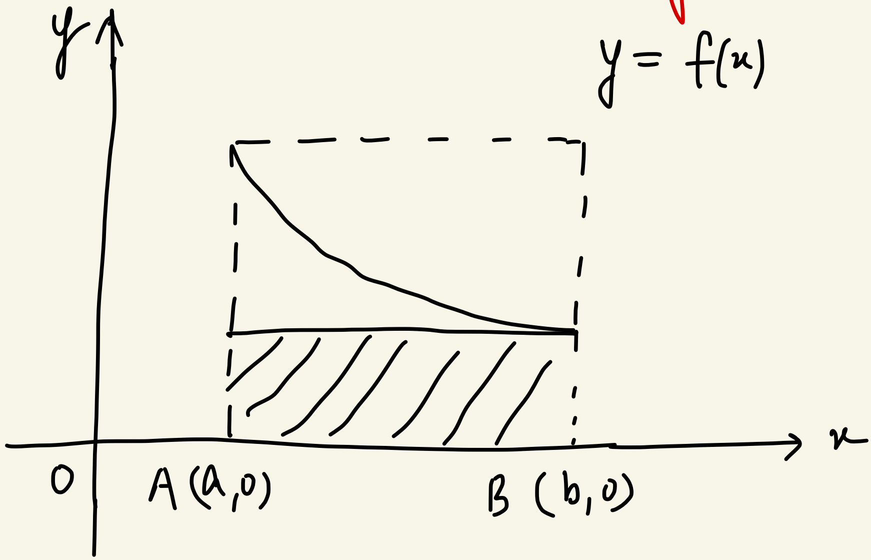


$y = f(x)$

$$(b-a)f(a) < \int_a^b f(x)dx < (b-a)f(b)$$

$$\int_a^b f(x)dx < \text{Ar of Trap } ABDF.$$

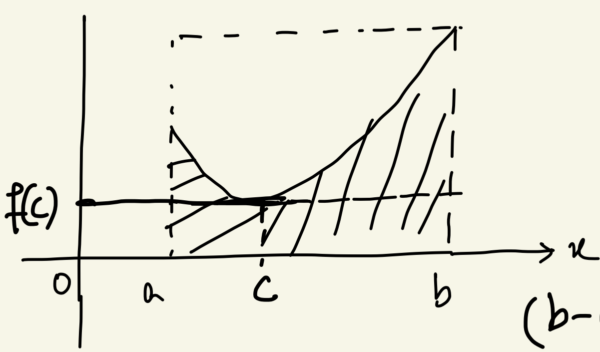
C–II $f(x)$ is monotonically dec in (a,b)



$$(b-a)f(b) < \int_a^b f(x)\,dx < (b-a)f(a)$$

*

C–III $f(x)$ is non-monotonic in (a,b).



y = f(x)

$$(b-a)f(c) < \int_a^b f(x)\,dx < (b-a)f(b)$$

$\boxed{f'(x)=0} \Rightarrow x=c$

HW

Q $\lim_{n\to\infty} \left({}^{2n}C_n\right)^{1/n}$.

$$l = \lim_{n\to\infty}\left(\frac{(2n)!}{n!\,n!}\right)^{\frac{1}{n}}$$

$$l = \lim_{n\to\infty}\left(\frac{\cancel{n!}\,(n+1)(n+2)(n+3)\ldots(n+n)}{\cancel{n!}\;n!}\right)^{\frac{1}{n}}$$

$$\ln l = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\ln\left(1+\frac{n}{r}\right)$$

$$\ln l = \int_0^1 \ln\left(1+\frac{1}{x}\right)dx$$

$$\ln l = \ln 4 \Rightarrow \boxed{l = 4} \text{ Ans}$$

Q HW

$\lim_{n\to\infty}\left(\tan^{-1}\frac{1}{n}\right)\left(\sum_{k=1}^{n}\frac{1}{1+\tan(k/n)}\right)$ has the value equal to

Sol^n

$$\lim_{n\to\infty}\underbrace{\left(\frac{\tan^{-1}\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)}\right)}_{1}\cdot\underbrace{\frac{1}{n}\sum_{K=1}^{n}\left(\frac{1}{1+\tan\frac{K}{n}}\right)}_{\int_0^1 \frac{1}{1+\tan x}dx}$$

$$\int_0^1 \frac{1}{1+\tan x}dx = \int_0^1 \frac{\cos x}{\sin x+\cos x}dx$$

$$= \frac{1+\ln(\sin 1+\cos 1)}{2} \text{ Ans}$$

Q^HW $\lim_{n\to\infty} \sum_{r=1}^{n} \left(\frac{2n-1}{n^2+r^2}\right)$

$$\lim_{n\to\infty} \sum_{r=1}^{n} \underbrace{\left(\frac{2n-1}{n}\right)}_{\downarrow 2} \left(\frac{1}{n}\right)\left(\frac{1}{1+\frac{r^2}{n^2}}\right)$$

$$2\int_0^1 \frac{1}{1+x^2}\,dx = 2\tan^{-1}x\Big|_0^1$$

$$= 2\left(\frac{\pi}{4}\right) = \frac{\pi}{2}\text{. Ans}$$

HW

Q1 If $x = \int_1^{t^2} z\ln z\,dz$ and $y = \int_{t^2}^{1} z^2\ln z\,dz$ find $\frac{dy}{dx}$ Ans → $-t^2$

Q2 If $y = \int_0^{z^2} \frac{dx}{1+x^3}$ find $\frac{d^2y}{dz^2}$ at $z=1$ Ans → -2

Q3 If $\lim_{x\to\infty} x^{-\left(\frac{3}{2}+n\right)} \int_0^x t^n\sqrt{t+1}\,dt = \frac{2}{11}$ then

find $n \in N$?

Ans → $n = 4$

Q Prove that :-

(1) (a) $\dfrac{\pi}{128} < \displaystyle\int_{\pi/4}^{\pi/2} (\sin x)^{10}\, dx < \dfrac{\pi}{4}$, (b) $1 < \displaystyle\int_{0}^{\pi/2} \dfrac{\sin x}{x}\, dx < \dfrac{\pi}{2}$; HW (2) $\dfrac{e-1}{3} < \displaystyle\int_{1}^{e} \dfrac{dx}{2+\ln x} < \dfrac{e-1}{2}$

(a) $f(x) = (\sin x)^{10}$ is increasing in $\underbrace{\left(\frac{\pi}{4}, \frac{\pi}{2}\right)}$

$$\left(\frac{\pi}{2} - \frac{\pi}{4}\right) f(x)_{min} < I < \left(\frac{\pi}{2} - \frac{\pi}{4}\right)\left(f(x)\right)_{max}$$

$$\frac{\pi}{4} \cdot f(\pi/4) < \int_{\pi/4}^{\pi/2} (\sin x)^{10}\, dx < \frac{\pi}{4} \cdot f(\pi/2)$$

$$\frac{\pi}{4} \cdot \left(\frac{1}{\sqrt{2}}\right)^{10} < I < \frac{\pi}{4} \underbrace{\left(\sin\frac{\pi}{2}\right)^{10}}$$

(b) $\boxed{f(x) = \dfrac{\sin x}{x}} \longrightarrow f'(x) = \dfrac{x\cos x - \sin x}{x^2}$

$$f'(x) = \underbrace{\left(\frac{\cos x}{x^2}\right)}_{>0}\left(\underbrace{x - \tan x}_{<0}\right)$$

−ve

0 π/2

$\therefore f(x)$ is decreasing in $[0, \pi/2)$

$$\left(\frac{\pi}{2}-0\right) f(x)_{min} < I < \left(\frac{\pi}{2}-0\right)\left(f(x)\right)_{max}$$

$$\frac{\pi}{2} f\left(\pi/2\right) < I < \frac{\pi}{2} \lim_{x\to 0^+}\left(\frac{\sin x}{x}\right)$$

$$\frac{\pi}{\cancel{2}} \cdot \frac{1}{\left(\pi/\cancel{2}\right)} < I < \frac{\pi}{2}(1)$$

$$\boxed{1 < I < \frac{\pi}{2}}$$

Q Prove that $\frac{1}{2} < \int_0^1 \frac{dx}{\sqrt{4-x^2+x^5}} < \frac{\pi}{6}$

sol^n

$x \in (0,1)$ $\quad$ $x^2 > x^5$

$$\sqrt{4-x^2} < \sqrt{4-x^2+x^5}$$

$$\frac{1}{\sqrt{4-x^2}} > \frac{1}{\sqrt{4-x^2+x^5}}$$

$$\underbrace{\int_0^1 \frac{1}{\sqrt{4-x^2+x^5}}\,dx}_{I} < \int_0^1 \frac{1}{\sqrt{4-x^2}}\,dx$$

$$I < \sin^{-1}\left(\frac{x}{2}\right)\Big|_0^1 \Rightarrow \boxed{I < \frac{\pi}{6}}$$

$$\sqrt{4-(\underbrace{x^2-x^5}_{>0})} < \sqrt{4}$$

$$\int_0^1 \frac{dx}{\sqrt{4}} < \int_0^1 \frac{1}{\sqrt{4-x^2+x^5}}\,dx$$

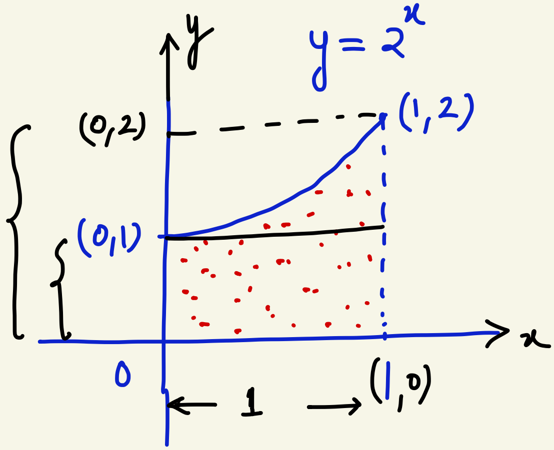
$$\frac{1}{2} < I$$

Q

Prove that $\left[\int_0^1 2^{x^2}\,dx\right] = 1$, where [.] GIF

$$1\times 1 < \int_0^1 2^{x^2}\,dx < 1\times 2$$

$$1 < I < 2$$



Q Which is greater ?

$$I_1 = \int_0^1 e^{-x}\,dx \quad \text{OR} \quad I_2 = \int_0^1 e^{-x^2}\,dx$$

$$x \in (0,1)$$

$$x > x^2$$

$$-x < -x^2$$

$$e^{-x} < e^{-x^2}$$

$$\int_0^1 e^{-x}\,dx < \int_0^1 e^{-x^2}\,dx$$

$$\boxed{I_1 < I_2}$$

Q If $f'(x) > 2 \ \forall x \in R$, $f(0) = 1$ then prove that

$$\int_0^1 f(x)\,dx > 2.$$

Sol^n

$$f'(x) > 2 \Rightarrow \int_0^x f'(x)\,dx > \int_0^x 2\,dx$$

$$f(x)\Big|_0^x > 2x$$

$$f(x) - f(0) > 2x$$

$$\boxed{f(x) > 2x+1}$$

$$\int_0^1 f(x)\,dx > \int_0^1 (2x+1)\,dx$$

Q $\displaystyle\int_1^e \frac{\sin x}{x^2}\,dx < 1.$

(i) $$\int_1^e \frac{\sin x}{x^2}\,dx < \int_1^e \frac{x}{x^2}\,dx \Rightarrow I < \int_1^e \frac{1}{x}\,dx$$

$$I < \ln(x)\Big|_1^e$$

$$\boxed{I < 1}$$

(ii) $$\frac{\sin x}{x^2} < \frac{1}{x^2}$$

$$\int_1^e \frac{\sin x}{x^2}\,dx < \int_1^e \frac{1}{x^2}\,dx \Rightarrow I < \left(\frac{-1}{x}\right)\Big|_1^e$$

Q Given that f satisfies $|f(u)-f(v)| \le |u-v|$ for u and v in [a, b] then prove that

$$\left|\int_a^b f(x)\,dx-(b-a)f(a)\right| \le \frac{(b-a)^2}{2}$$

$**$

Sol^n

$$\left|\int_a^b f(x)\,dx - f(a)\int_a^b dx\right|$$

$$\left|\int_a^b (f(x)-f(a))\,dx\right| \le \int_a^b |f(x)-f(a)|\,dx$$

$$\le \int_a^b |x-a|\,dx$$

$$\int_a^b |x-a|\,dx = \int_a^b (x-a)\,dx = \left.\frac{(x-a)^2}{2}\right|_a^b$$

$$= \frac{(b-a)^2}{2}.$$

Q

Let a & b be two distinct positive real numbers, then the value of $\int_a^b \frac{f\left(\frac{x}{a}\right)-f\left(\frac{b}{x}\right)}{x}dx$ (where $f(x)$ is a continuous function)

(A) $\frac{1}{a}+\frac{1}{b}$ (B) ab (C) $2\left(\frac{1}{a}-b\right)$ (D) 0

Sol^n

$$I = \int_a^b \frac{f\left(\frac{x}{a}\right)-f\left(\frac{b}{x}\right)}{x}dx \quad -(i)-$$

$$x = \frac{ab}{t} \Rightarrow dx = -\frac{ab}{t^2}dt$$

$$I = \int_b^a \frac{f\left(\frac{b}{t}\right)-f\left(\frac{t}{a}\right)}{\left(\frac{\cancel{ab}}{\cancel{t}}\right)}\cdot\left(-\frac{\cancel{ab}}{t^{\cancel{2}}}\right).dt$$

$$I = -\int_b^a \frac{f(b/t)-f(t/a)}{t}dt$$

$$I = \int_b^a \frac{f(t/a)-f(b/t)}{t}dt$$

$$\boxed{I = -I} \Rightarrow \boxed{I = 0}$$

Q $\displaystyle\int_{\pi/2}^{3\pi/2} \frac{(x+1)\cos x}{1+\sin^2 x}\,dx$

$$I = \frac{1}{2}\left(\int_{\pi/2}^{3\pi/2}\left(\frac{\overbrace{(x+1)}^{e}\cos x}{1+\sin^2 x} + \frac{\overbrace{(2\pi-x+1)}^{e}\cos x}{1+\sin^2 x}\right)dx\right.$$

$$= \frac{1}{2}\,(2\pi+2)\int_{\pi/2}^{3\pi/2}\frac{\cos x}{1+\sin^2 x}\,dx$$

$$= (\pi+1)\left(-\frac{\pi}{2}\right)$$ Ans

Q $\displaystyle\int_{-10}^{10} \frac{dx}{\sqrt{\{x\}\{-x\}}}$; $\{\ \} \rightarrow$ fractional part function.

$x \neq I$

$\{x\} + \{-x\} = 1$; $x \notin I$

$$\int_{-10}^{10} \frac{dx}{\sqrt{\{x\}(1-\{x\})}}$$

10×1

-10×1

$\boxed{P = 1}$

$$20 \int_0^1 \frac{dx}{\sqrt{x(1-x)}} = 20\pi.$$

Ans

Q $\displaystyle\int_1^3 \frac{dx}{(e^{2-x}+1)(x^2-4x+8)}$

$$I = \int_1^3 \frac{dx}{(e^{2-x}+1)\left(x(x-4)+8\right)}$$

$$I = \frac{1}{2}\left(\int_1^3 \frac{dx}{(e^{2-x}+1)\left(x(x-4)+8\right)} + \int_1^3 \frac{dx}{e^{2-(4-x)}\cdot\left((4-x)(4-x-4)+8\right)}\right)$$

$$I = \frac{1}{2}\int_1^3 \frac{dx}{x^2-4x+8} = \frac{1}{2}\int_1^3 \frac{dx}{(x-2)^2+2^2}$$

$$\frac{1}{2}\left(\frac{1}{2}\cdot \tan^{-1}\left(\frac{x-2}{2}\right)\Big|_1^3\right)$$

$$\frac{1}{4}\left(\tan^{-1}\frac{1}{2} - \tan^{-1}\left(-\frac{1}{2}\right)\right)$$

$$\frac{2}{4}\cdot\tan^{-1}\left(\frac{1}{2}\right) = \frac{1}{2}\cot^{-1}(2)$$

Ans

Q If $f(x+2) = f(x-1) \ \forall x$ and $\int_0^3 f(x)\,dx = 2$

then $\int_0^6 f(4x)\,dx = ?$

Sol^n $x-1=t$ $f(t+3) = f(t)$

$\downarrow \boxed{P=3}$

$\int_0^6 f(4x)\,dx \Rightarrow$ $4x = z \Rightarrow dx = \frac{1}{4}dz$.

$\frac{1}{4}\int_0^{24} f(z).\,dz = \frac{1}{\cancel{4}} \times \cancel{8}^{2} \underbrace{\int_0^3 f(z)\,dz}$ $= 2 \times 2$

$= 4$ Ans

Q $\dfrac{d}{dx}\displaystyle\int_0^{2x}(t-x)\cos t\,dt$

Sol^n

$$\frac{d}{dx}\left(\underbrace{\int_0^{2x} t\cos t\,dt}\; - \underbrace{x\int_0^{2x}\cos t\,dt}\right)$$

$$2(2x\cos 2x) - \left(x(\cos 2x)2 + 1.\underbrace{\int_0^{2x}\cos t.dt}\right)$$

$$2x\cos 2x - (\sin t)\Big|_0^{2x}$$

$$2x\cos 2x - \sin 2x$$

Ans

Q $\lim_{x\to 0^+} x \int_0^{x^2} \frac{\sin 2\sqrt{t}}{\sin x - \tan x}\, dt$

$$\lim_{x\to 0^+} \frac{\int_0^{x^2} \sin 2\sqrt{t}\, dt}{\underbrace{\left(\frac{\sin x - \tan x}{x}\right)}_{0}}$$

LH Rule.

$$\lim_{x\to 0^+} \frac{\int_0^{x^2} \sin 2\sqrt{t}\, dt}{\underbrace{\left(\frac{\tan x}{x}\right)}_{(1)}\underbrace{\left(\frac{\cos x - 1}{x^2}\right)}_{(-1/2)} \cdot x^2}$$

$$\lim_{x\to 0^+} \frac{\sin 2\sqrt{x^2} \cdot \cancel{2x}}{\left(-\frac{1}{2}\right) \cdot \cancel{(2x)}}$$

0.

Q If $\int_0^{x^2} \frac{dt}{1+t^3} + \int_y^{x} \frac{dt}{1+t^2} = \frac{\pi}{4} \quad \forall x, y$

then $\left.\frac{dy}{dx}\right|_{x=0} = ?$

put $x=0$.

$$\cancel{\int_0^0 (\;)\,dt} + \int_y^0 \frac{dt}{1+t^2} = \frac{\pi}{4}$$

$\boxed{y=-1}$

Sol^n diff wrt 'x' :-

$$\left(\frac{1}{1+x^6}\cdot 2x - 0\right) + \left(\frac{1}{1+x^2}\cdot 1 - \frac{1}{1+y^2}\cdot\frac{dy}{dx}\right) = 0.$$

put $x=0$ & $y=-1$.

$$0 + 1 - \frac{1}{2}\cdot\left(\frac{dy}{dx}\right) = 0.$$

$$\boxed{\frac{dy}{dx} = 2}$$ Ans

Q $\displaystyle\int_0^\infty \frac{\ln x}{\underbrace{x^2}+2x+\underbrace{4}}\,dx$

$x = 2t \Rightarrow dx = 2dt$

$$2\int_0^\infty \frac{\ln(2t)}{4t^2+4t+4}\cdot dt = \frac{\cancel{2}}{\cancel{4}}\int_0^\infty\left(\underbrace{\frac{\ln 2}{t^2+t+1}}+\cancel{\underbrace{\frac{\ln t}{t^2+t+1}}}\right)dt$$

$\cancel{\frac{\ln t}{t^2+t+1}} \to 0.$

$$= \frac{1}{2}\ln 2\int_0^\infty \frac{dt}{t^2+t+1}$$

$$= \frac{\pi\ln 2}{3\sqrt{3}}$$ Ans

Q $$\int_0^{\infty} \frac{\ln\left(x+\frac{1}{x}\right)}{1+x^2}\, dx$$

$x = \tan\theta \Rightarrow dx = \sec^2\theta\, d\theta$

$$\int_0^{\pi/2} \frac{\ln(\tan\theta + \cot\theta)}{\cancel{\sec^2\theta}} \cdot \cancel{\sec^2\theta} \cdot d\theta$$

$$\int_0^{\pi/2} \ln\left(\frac{2}{2\sin\theta\cos\theta}\right) d\theta = \underbrace{\int_0^{\pi/2} \ln 2\, d\theta} - \underbrace{\int_0^{\pi/2} \ln(\sin 2\theta)\, d\theta}$$

$$\ln 2 \cdot \left(\frac{\pi}{2}\right) - \left(-\frac{\pi}{2}\ln 2\right)$$

$\pi \ln 2$ Ans

Q Let 'f' be an even function such that $\underbrace{\int_0^\infty f(x)\,dx = \frac{\pi}{2}}$ then find the value of $\int_0^\infty f\left(x-\frac{1}{x}\right)dx$?

Sol^n

$$I = \int_0^\infty f\left(x - \underbrace{\frac{1}{x}}\right) dx \quad -(1)- \qquad x = \frac{1}{t} \Rightarrow dx = -\frac{1}{t^2}dt.$$

$$I = \int_\infty^0 f\left(\frac{1}{t} - t\right)\cdot\left(-\frac{1}{t^2}\right)dt$$

$$I = \int_0^\infty f\left(\underbrace{\frac{1}{t} - t}\right)\frac{1}{t^2}\,dt = \int_0^\infty f\left(t - \frac{1}{t}\right)\frac{1}{t^2}dt \quad -(2)-$$

Add (1) & (2)

$$2I = \int_0^\infty f\left(x - \frac{1}{x}\right)\left(1 + \frac{1}{x^2}\right)dx$$

$$\boxed{x - \frac{1}{x} = z} \Rightarrow \left(1 + \frac{1}{x^2}\right)dx = dz$$

$$2I = \int_{-\infty}^\infty f(z).dz \Rightarrow \cancel{2}I = \cancel{2}\int_0^\infty f(z)dz$$

$$I = \pi/2$$

Q $\lim_{n\to\infty} \dfrac{(1^2+2^2+3^2+\cdots+n^2)(1^3+2^3+\cdots+n^3)}{(1^6+2^6+\cdots+n^6)}$

$$\lim_{n\to\infty} \frac{\frac{1}{n}\left(\sum_{r=1}^{n}\frac{r^2}{n^2}\right)\frac{1}{n}\left(\sum_{r=1}^{n}\frac{r^3}{n^3}\right)}{\frac{1}{n}\left(\sum_{r=1}^{n}\frac{r^6}{n^6}\right)} = \frac{\left(\int_0^1 x^2\,dx\right)\left(\int_0^1 x^3\,dx\right)}{\left(\int_0^1 x^6\,dx\right)}$$

$$= \frac{7}{12}.$$ Ans

Q $\lim_{n\to\infty}\left((1+n)^{\frac{1}{n}}\left(1+\frac{n}{2}\right)^{\frac{2}{n}}\left(1+\frac{n}{3}\right)^{\frac{3}{n}}\cdots\cdots 2\right)^{\frac{1}{n}}$

Q

$$\int_0^1 \left((e-1)\sqrt{\ln(1+ex-x)} + e^{x^2} \right) dx$$

Q If $f(x) = x^3 + 3x^2 + 4x - 8$ then find the value of $\int_{-1}^{2} f(x)\,dx + \int_{0}^{20} f^{-1}(x)\,dx$