Q HW If $\hat{i}\times\left((\vec{a}-\hat{j})\times\hat{i}\right)+\hat{j}\times\left((\vec{a}-\hat{k})\times\hat{j}\right)$
$+\hat{k}\times\left((\vec{a}-\hat{i})\times\hat{k}\right)=0$ and $\boxed{\vec{a}=x\hat{i}+y\hat{j}+z\hat{k}}$

then value of $8(x^3-xy+zx)=\,?$

$x=y=z=\frac{1}{2}$

Sol^n

$$(\hat{i}\cdot\hat{i})(\vec{a}-\hat{j})-(\hat{i}\cdot(\vec{a}-\hat{j}))\hat{i}$$
$$(\hat{j}\cdot\hat{j})(\vec{a}-\hat{k})-(\hat{j}\cdot(\vec{a}-\hat{k}))\hat{j}$$
$$(\hat{k}\cdot\hat{k})(\vec{a}-\hat{i})-(\hat{k}\cdot(\vec{a}-\hat{i}))\hat{k}$$

$$3\vec{a}-(\hat{i}+\hat{j}+\hat{k})-(\underbrace{x\hat{i}+y\hat{j}+z\hat{k}}_{\vec{a}})=0\Rightarrow 2\vec{a}=\hat{i}+\hat{j}+\hat{k}$$

$$\therefore\ \boxed{\vec{a}=\frac{\hat{i}+\hat{j}+\hat{k}}{2}}$$

Scalar Product of Four Vector

$$(\vec{a}\times\vec{b}).(\vec{c}\times\vec{d}) = (\vec{a}.\vec{c})(\vec{b}.\vec{d})-(\vec{a}.\vec{d})(\vec{b}.\vec{c}) = \begin{vmatrix} \vec{a}.\vec{c} & \vec{a}.\vec{d} \\ \vec{b}.\vec{c} & \vec{b}.\vec{d} \end{vmatrix}$$

Proof:

$$\underbrace{(\vec{a}\times\vec{b})}_{\vec{u}}.(\vec{c}\times\vec{d}) = \vec{u}.(\vec{c}\times\vec{d}) \quad \text{where } \vec{u}=\vec{a}\times\vec{b}$$

$$\Downarrow$$

$$(\vec{u}\times\vec{c}).\vec{d}$$

$$\left(\underbrace{(\vec{a}\times\vec{b})\times\vec{c}}\right).\vec{d} = \left(\underbrace{(\vec{a}.\vec{c})}\vec{b}-(\vec{b}.\vec{c})\vec{a}\right).\vec{d}$$

$$(\vec{a}.\vec{c})(\vec{b}.\vec{d})-(\vec{b}.\vec{c})(\vec{a}.\vec{d}) \quad \text{(H.P.)}$$

Q If $\vec{a} = \hat{i} - 2\hat{j} + 3\hat{k}$; $\vec{b} = \hat{i} + \hat{j} - 4\hat{k}$; $\vec{c} = 4\hat{i} - 3\hat{j} + 6\hat{k}$; $\vec{d} = 3\hat{i} - 6\hat{j} - 5\hat{k}$
then :

$\vec{a}\cdot\vec{d} = 3 + 12 - 15$
$= 0.$

① $\{(\vec{a} \times \vec{b}) \times (\vec{a} \times \vec{c})\}.\vec{d} = ?$

② $\vec{d}.[\vec{a} \times \{\vec{b} \times (\vec{c} \times \vec{d})\}] = ?$ Ans→ -1190

HW

$\vec{u} = \vec{a} \times \vec{b}$

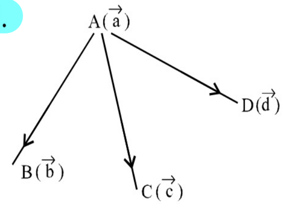
① $((\underbrace{\vec{a} \times \vec{b}}_{\vec{u}}) \times (\vec{a} \times \vec{c})) \cdot \vec{d}$

$(\vec{u} \times (\vec{a} \times \vec{c})) \cdot \vec{d} = ((\vec{u}\cdot\vec{c})\,\vec{a} - \cancel{(\vec{u}\cdot\vec{a})}^{\,0}\,\vec{c}) \cdot \vec{d}$

$(\vec{u}\cdot\vec{c})\,(\vec{a}\cdot\vec{d}) = [\vec{a}\ \vec{b}\ \vec{c}]\,\underbrace{(\vec{a}\cdot\vec{d})}_{0} = 0$ Ans

Condition for coplanarity of four points

4 points with pv's $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ are coplanar iff $\exists$ scalars x, y, z and t not all simultaneously zero and satisfying $x\vec{a} + y\vec{b} + z\vec{c} + t\vec{d} = \vec{0}$ where $x + y + z + t = 0$.



Case I : Let the four points A, B, C, D are in the same plane

$\Rightarrow$ the vectors $\vec{b} - \vec{a}, \vec{c} - \vec{a}$ and $\vec{d} - \vec{a}$ are in the same plane.

hence $\vec{d} - \vec{a} = l(\vec{b} - \vec{a}) + m(\vec{c} - \vec{a})$

or $\underbrace{(l + m - 1)}_{x}\vec{a} - \underbrace{l}_{y}\vec{b} - \underbrace{m}_{z}\vec{c} + \underbrace{1}_{t}\vec{d} = 0 \Rightarrow x\vec{a} + y\vec{b} + z\vec{c} + t\vec{d} = 0$ where, $x + y + z + t = 0$ and x, y, z, t not all simultaneous zero.

Case II : Let $x\vec{a} + y\vec{b} + z\vec{c} + t\vec{d} = 0$ where $x + y + z + t = 0$ and not all simultaneously zero

Let $t \neq 0$ $\quad (-y - z - t)\vec{a} + y\vec{b} + z\vec{c} + t\vec{d} = 0$ $\quad$ [putting $x = -y - z - t$]

$(\vec{d} - \vec{a})t + y(\vec{b} - \vec{a}) + z(\vec{c} - a) = 0$

$\Rightarrow$ $\vec{d} - \vec{a}, \vec{b} - \vec{a}$ and $\vec{c} - a$ are coplanar $\Rightarrow$ points A, B, C, D are coplanar

* Vector Product of Four Vector

(1) $\vec V = (\vec a\times\vec b)\times(\vec c\times\vec d)$

$\vec u\times(\vec c\times\vec d) \quad = [\vec a\,\vec b\,\vec d]\vec c-[\vec a\,\vec b\,\vec c]\vec d \qquad ....(1) \qquad$ (where $\vec u = \vec a\times\vec b$)

again $\vec V = (\vec a\times\vec b)\times\underbrace{(\vec c\times\vec d)}_{\vec v} = (\vec a.\vec v)\vec b-(\vec b.\vec v)\vec a = [\vec a\,\vec c\,\vec d]\vec b-[\vec b\,\vec c\,\vec d]\vec a \qquad ....(2)$

from (1) and (2) $\qquad [\vec a\,\vec b\,\vec d]\vec c-[\vec a\,\vec b\,\vec c]\vec d = [\vec a\,\vec c\,\vec d]\vec b-[\vec b\,\vec c\,\vec d]\vec a \qquad ....(3)$

Note that $(\vec a\times\vec b)\times(\vec c\times\vec d) = 0 \Rightarrow$ plane containing the values $\vec a$ & $\vec b$ and $\vec c$ & $\vec d$ are parallel.

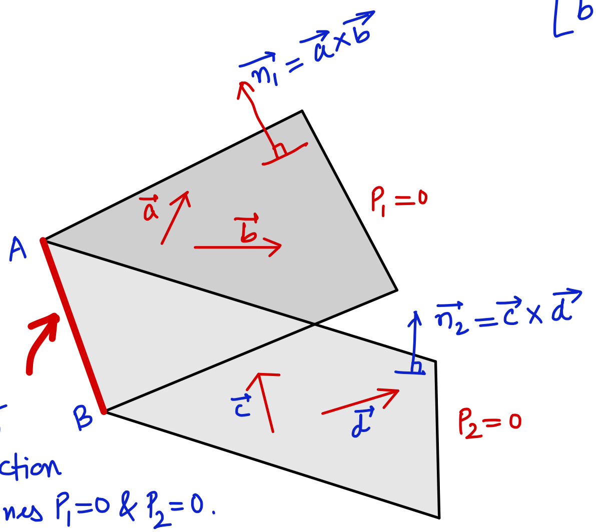
|||ly $\quad (\vec a\times\vec b)\cdot(\vec c\times\vec d) = 0 \Rightarrow$ the two planes are perpendicular.

(i) equation (3) is suggestive that if $\vec a,\vec b,\vec c,\vec d$ are four vectors no 3 three of them are coplanar then each one of them can be expressed as a linear combination of other.

(ii) If $\vec a,\vec b,\vec c,\vec d$ are p.v.'s of four points then these four points are in the same plane if

$[\vec a\,\vec b\,\vec d]-[\vec a\,\vec b\,\vec c]=[\vec a\,\vec c\,\vec d]-[\vec b\,\vec c\,\vec d]$

$$\overbrace{[\vec b\,\vec c\,\vec d]}^{x}\vec a + \overbrace{[\vec a\,\vec d\,\vec c]}^{y}\vec b + \underbrace{[\vec a\,\vec b\,\vec d]}_{z}\vec c + \overbrace{[\vec a\,\vec c\,\vec b]}^{t}\vec d = \vec 0$$



Line of intersection of 2 planes $P_1=0$ & $P_2=0$.

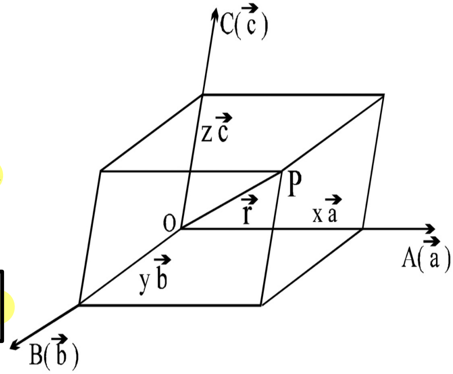
$\therefore$ direction vector of line AB $= \vec n_1\times\vec n_2$

Linear combination/linear dependence and independence (Base vectors)

Linear combination

(1) Theorem in plane (already done)

(2) Theorem in space : If $\vec{a},\vec{b},\vec{c}$ are 3 non zero non coplanar vectors then any vector $\vec{r}$ can be expressed as a linear combination $\boxed{\vec{r}=x\vec{a}+y\vec{b}+z\vec{c}}$



$\vec{b}\times\vec{c} = x\vec{a}+y\vec{b}+z\vec{c}$ —①—

① dot with $\vec{a}\times\vec{b}$ $\qquad (\vec{a}\times\vec{b})\cdot(\vec{b}\times\vec{c}) = 0+0+z[\vec{c}\,\vec{a}\,\vec{b}]$

① " " $\vec{b}\times\vec{c}$

① Express the non coplanar vectors $\vec{a},\vec{b},\vec{c}$ in terms of $\vec{b}\times\vec{c},\ \vec{c}\times\vec{a},\ \vec{a}\times\vec{b}$.

② Express $\vec{b}\times\vec{c},\ \vec{c}\times\vec{a},\ \vec{a}\times\vec{b}$ in terms of 3 non coplanar vectors $\vec{a},\vec{b},\vec{c}$.

① $\quad \vec{a} = x(\vec{b}\times\vec{c}) + y(\vec{c}\times\vec{a}) + z(\vec{a}\times\vec{b})$ —①—

① dot with $\vec{a}$:-

$$\vec{a}\cdot\vec{a} = x[\vec{a}\,\vec{b}\,\vec{c}] + 0 + 0 \Rightarrow x = \frac{a^2}{[\vec{a}\,\vec{b}\,\vec{c}]}$$

① dot with $\vec{b}$:-

$$\vec{a}\cdot\vec{b} = 0 + y[\vec{c}\,\vec{a}\,\vec{b}] + 0 \Rightarrow y = \frac{\vec{a}\cdot\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}$$

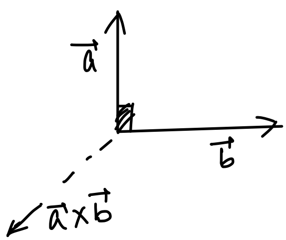
① dot with $\vec{c}$:-

$$\vec{a}\cdot\vec{c} = 0+0+z[\vec{a}\,\vec{b}\,\vec{c}] \Rightarrow z = \frac{\vec{a}\cdot\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}$$

Q* Given the vector $\vec{a}$ and $\vec{b}$ orthogonal to each other find the vector $\vec{V}$ in terms of $\vec{a}$ and $\vec{b}$ satisfying $\vec{V}.\vec{a} = 0$; $\vec{V}.\vec{b} = 1$ and $[\vec{V}\ \vec{a}\ \vec{b}] = 1$ $\Rightarrow$ $\vec{V}, \vec{a}, \vec{b}$ non-coplanar vectors.

Sol^n $\quad \vec{a}\cdot\vec{b} = 0$ —(1)— $\quad \vec{V}\cdot\vec{a} = 0$; $\vec{V}\cdot\vec{b} = 1$; $[\vec{V}\ \vec{a}\ \vec{b}] = 1$

$$\therefore \quad \vec{V} = x\vec{a} + y\vec{b} + z(\vec{a}\times\vec{b}) \quad \text{—(1)—}$$



(1) dot with $\vec{a}$:-

$$\underbrace{\vec{V}\cdot\vec{a}}_{0} = x a^2 + y\,\underbrace{\vec{a}\cdot\vec{b}}_{0} + 0 \Rightarrow \boxed{x = 0}$$

(1) dot with $\vec{b}$:-

$$\underbrace{\vec{V}\cdot\vec{b}}_{1} = y b^2 + z(0) \Rightarrow y = \frac{1}{b^2}$$

(1) dot with $\vec{a}\times\vec{b}$:-

$$z = \frac{1}{|\vec{a}\times\vec{b}|^2} \Leftarrow \underbrace{[\vec{V}\ \vec{a}\ \vec{b}]}_{1} = 0 + 0 + z\,|\vec{a}\times\vec{b}|^2$$

Real definition of linearly independence

If $\vec{V}_1, \vec{V}_2, \ldots\ldots\ldots \vec{V}_n$ are vectors and $\lambda_1, \lambda_2, \ldots\ldots\ldots \lambda_n$ are scalar and if the linear combination $\lambda_1\vec{V}_1 + \lambda_2\vec{V}_2 + \ldots\ldots + \lambda_n\vec{V}_n = 0$, necessarily implies $\lambda_1 = \lambda_2 = \ldots\ldots\lambda_n = 0$, we say that $\vec{V}_1, \vec{V}_2, \ldots\ldots\ldots \vec{V}_n$ are said to constitutes a linearly independent set of vectors.

Note:

(1) 2 non zero , non collinear vectors are linearly independent.

(2) Three non zero , non coplanar vectors are linearly independent i.e. $[\vec{a}\,\vec{b}\,\vec{c}] \neq 0$.

(3) Four or more vectors in 3D space are always linearly dependent.

Reciprocal system of vectors

(a) If $\vec{a}, \vec{b}, \vec{c}$ and $\vec{a}', \vec{b}', \vec{c}'$ are 2 sets of non coplanar vectors such that $\vec{a}.\vec{a}' = \vec{b}.\vec{b}' = \vec{c}.\vec{c}' = 1$, then $\vec{a}, \vec{b}, \vec{c}$ and $\vec{a}', \vec{b}', \vec{c}'$ are said to be constitute a reciprocal system of vectors.

(b) Reciprocal system of vectors exists only in case of dot product.

(c) It is possible to define $\vec{a}', \vec{b}', \vec{c}'$ in terms of $\vec{a}, \vec{b}, \vec{c}$ as.

Rem

$$\boxed{\vec{a}' = \frac{\vec{b}\times\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]} \;;\; \vec{b}' = \frac{\vec{c}\times\vec{a}}{[\vec{a}\,\vec{b}\,\vec{c}]} \;;\; \vec{c}' = \frac{\vec{a}\times\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}} \;;$$

where $([\vec{a}\,\vec{b}\,\vec{c}] \neq 0)$

$\vec{a}'.\vec{b} = \vec{a}'.\vec{c} = 0$
$\vec{b}'.\vec{a} = \vec{b}'.\vec{c} = 0$
$\vec{c}'.\vec{a} = \vec{c}'.\vec{b} = 0.$

Q Find the set of vector reciprocal to $\underbrace{2\hat{i}+3\hat{j}-\hat{k}}_{\vec{a}}$, $\underbrace{\hat{i}-\hat{j}-2\hat{k}}_{\vec{b}}$ and $\underbrace{-\hat{i}+2\hat{j}+2\hat{k}}_{\vec{c}}$

Do yourself

Ans: $\frac{1}{3}(2\hat{i}+\hat{k})$; $-\frac{1}{3}(8\hat{i}-3\hat{j}+7\hat{k})$; $\frac{1}{3}(-7\hat{i}+3\hat{j}-5\hat{k})$

Note: (i) $\vec{a}\times\vec{a}'+\vec{b}\times\vec{b}'+\vec{c}\times\vec{c}'=0$ i. e. $\dfrac{\vec{a}\times(\vec{b}\times\vec{c})+\vec{b}\times(\vec{c}\times\vec{a})+\vec{c}\times(\vec{a}\times\vec{b})}{[\vec{a}\,\vec{b}\,\vec{c}]} = 0.$

(ii) $(\vec{a}+\vec{b}+\vec{c}).(\vec{a}'+\vec{b}'+\vec{c}')=3$ (as $\vec{a}\cdot\vec{b}'=\vec{a}\cdot\vec{c}'=0$ etc)

(iii) If $[\vec{a}\,\vec{b}\,\vec{c}]=V$ then $[\vec{a}'\,\vec{b}'\,\vec{c}']=\dfrac{1}{V}$ $\Rightarrow$ $[\vec{a}\,\vec{b}\,\vec{c}]\,[\vec{a}'\,\vec{b}'\,\vec{c}']=1$

(iv) $\vec{a}'\times\vec{b}'+\vec{b}'\times\vec{c}'+\vec{c}'\times\vec{a}' = \dfrac{\vec{a}+\vec{b}+\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}$, $[\vec{a}\,\vec{b}\,\vec{c}]\neq 0$

(iii)
$$[\vec{a}'\ \vec{b}'\ \vec{c}'] = \left[\underbrace{\frac{\vec{b}\times\vec{c}}{[\vec{a}\,\vec{b}\,\vec{c}]}}_{R_1}\quad \underbrace{\frac{\vec{c}\times\vec{a}}{[\vec{a}'\,\vec{b}'\,\vec{c}']}}_{R_2}\quad \underbrace{\frac{\vec{a}\times\vec{b}}{[\vec{a}\,\vec{b}\,\vec{c}]}}_{R_3}\right]$$

$$= \frac{1}{[\vec{a}\,\vec{b}\,\vec{c}]^3}\left[\vec{b}\times\vec{c}\quad \vec{c}\times\vec{a}\quad \vec{a}\times\vec{b}\right]$$

$$= \frac{[\vec{a}\,\vec{b}\,\vec{c}]^2}{[\vec{a}\,\vec{b}\,\vec{c}]^3} = \frac{1}{[\vec{a}\,\vec{b}\,\vec{c}]} = \frac{1}{V}.$$

Isolating an known vectors / Vector Equations :-
Satisfying a given relationship with some known vectors:

There is no general method for solving such equations, however dot or cross with known or unknown vectors or dot with $\vec{a}\times\vec{b}$, generally isolates the unknown vector. Use of linear combination also proves to be advantageous.

(1) Solve for $\vec{x}$

$\vec{x}.\vec{a} = c$ ....(1) where c is a non-zero scalar ; $\vec{a}\,\&\,\vec{b}$ are non-zero vectors.

and $\vec{a}\times\vec{x} = \vec{b}$ ....(2)

Sol^n

$$\boxed{\vec{x}.\vec{a} = c}$$

$$\vec{a}\times\vec{x} = \vec{b}.$$

$\downarrow$ cross with $\vec{a}$

$$\vec{a}\times(\vec{a}\times\vec{x}) = \vec{a}\times\vec{b}$$

$$(\vec{a}.\vec{x})\vec{a} - a^2\vec{x} = \vec{a}\times\vec{b}$$

$$c\vec{a} - a^2\vec{x} = \vec{a}\times\vec{b} \Rightarrow \vec{x} = \frac{c\vec{a} - \vec{a}\times\vec{b}}{a^2}$$

Ans

(2) Find the unknown vector $\vec{R}$ satisfying $K\vec{R} + \vec{A}\times\vec{R} = \vec{B}$; $K \neq 0$

(3) Solve the following simultaneous equations for $\vec{x}$ & $\vec{y}$

$\vec{x}+\vec{y} = \vec{a}$ ....(1) ; $\vec{x}\times\vec{y} = \vec{b}$ ....(2) ; $\vec{x}.\vec{a} = 1$ ....(3)

(2) $K\vec{R} + \vec{A}\times\vec{R} = \vec{B}$; $K \neq 0$. — (1) —

$\vec{A}\times\vec{R} = \vec{B} - K\vec{R}$

Cross with $\vec{A}$:-

$$K(\vec{A}\times\vec{R}) + \vec{A}\times(\vec{A}\times\vec{R}) = \vec{A}\times\vec{B}$$

$$K(\underbrace{\vec{A}\times\vec{R}}) + (\vec{A}\cdot\vec{R})\vec{A} - (A^2)\vec{R} = \vec{A}\times\vec{B}$$

dot with $\vec{A}$:-

$$K\,\vec{A}\cdot\vec{R} + 0 = \vec{A}\cdot\vec{B} \Rightarrow \vec{A}\cdot\vec{R} = \frac{\vec{A}\cdot\vec{B}}{K}$$

(3) $\vec{x}+\vec{y} = \vec{a}$ — (1) — ; $\vec{x}\times\vec{y} = \vec{b}$ — (2) —

$\underbrace{\vec{x}\cdot\vec{a}} = 1$ — (3) —

(1) dot with $\vec{a}$

$$\vec{x}\cdot\vec{a} + \vec{a}\cdot\vec{y} = a^2$$

$$\boxed{\vec{a}\cdot\vec{y} = a^2 - 1} \text{ — (4) —}$$

(2) Cross with $\vec{a}$

$$\vec{a}\times(\vec{x}\times\vec{y}) = \vec{a}\times\vec{b}$$

$$\underbrace{(\vec{a}\cdot\vec{y})}\,\vec{x} - \underbrace{(\vec{a}\cdot\vec{x})}\,\vec{y} = \vec{a}\times\vec{b}$$

$$(a^2-1)\,\vec{x} - \vec{y} = \vec{a}\times\vec{b} \quad \text{—(5)—}$$

$$\vec{x} + \vec{y} = \vec{a} \quad \text{—(1)—}$$

add to get $\vec{x}$.

HW

0-2 Q8 to 16

S-1 Q9 to 12