01/04/22

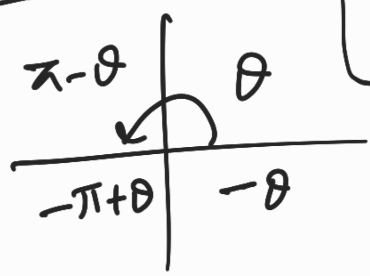
Q① $z = \dfrac{(1+i)(1+2i)(1+3i)}{(1-i)(2-i)(3-i)}$ find $|z|$ & $amp(z)$

$$|z| = \frac{|(1+i)|\,|1+2i|\,|1+3i|}{|(1-i)|\,|2-i|\,|3-i|} = \frac{\sqrt{2}\sqrt{5}\sqrt{10}}{\sqrt{2}\sqrt{5}\sqrt{10}} = 1$$

$$amp(z) = amp\left[\frac{\qquad}{\qquad}\right]$$

$$= amp(1+i) + amp(1+2i) + amp(1+3i) - amp(1-i) - amp(2-i) - amp(3-i) + 2k\pi$$

$k \in I$



$\theta = \tan^{-1}\left|\frac{y}{x}\right|$

$$amp(1+i) = \pi/4$$
$$amp(1+2i) = \tan^{-1}(2)$$
$$amp(1+3i) = \tan^{-1}(3)$$
$$amp(1-i) = -\pi/4$$
$$amp(2-i) = -\tan^{-1}\left(\tfrac{1}{2}\right)$$
$$amp(3-i) = -\tan^{-1}\left(\tfrac{1}{3}\right)$$

$$\boxed{-\pi < amp\, z \le \pi}$$

$$amp(z) = \frac{\pi}{4} + \tan^{-1}2 + \tan^{-1}3 + \frac{\pi}{4} + \tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{3}\right) + 2k\pi$$

$$= \frac{\pi}{2} + \frac{\pi}{2} + \frac{\pi}{2} + 2k\pi$$

$$= \frac{3\pi}{2} + 2k\pi$$

$k = -1$ $\quad amp(z) = \frac{3\pi}{2} - 2\pi = -\frac{\pi}{2}$

<u>Ex:-</u> find $amp(z)$, $|z|$

$$z = \left[\frac{(8+i)(7+i)}{(5-i)}\right]^6$$

$$|z| = \left(\frac{|8+i|\,|7+i|}{|5-i|}\right)^6$$

$$|z| = \left(\sqrt{65}\sqrt{50}/\sqrt{26}\right)^6$$

$$|z| = \left(\frac{\sqrt{5}\,\cancel{\sqrt{13}}\;5\cancel{\sqrt{2}}}{\cancel{\sqrt{2}}\,\cancel{\sqrt{13}}}\right)^6 = 5^9$$

$$amp(z) = amp\left(\frac{(8+i)(7+i)}{(5-i)}\right)^6$$

$$= 6\left[amp(8+i) + amp(7+i) - amp(5-i)\right] + 2k\pi$$

$$= 6\left[\tan^{-1}\left(\frac{1}{8}\right) + \tan^{-1}\left(\frac{1}{7}\right) + \tan^{-1}\left(\frac{1}{5}\right)\right] + 2k\pi$$

$$= \underbrace{6\tan^{-1}\left(\frac{1}{2}\right)}_{(-\pi,\,\pi]} + 2k\pi \quad (k=0)$$

$$amp(z) = 6\tan^{-1}\left(\frac{1}{2}\right)$$

$$\boxed{\begin{gathered}\frac{1}{2} < \frac{1}{\sqrt{3}} \\ \tan^{-1}\left(\frac{1}{2}\right) < \frac{\pi}{6} \\ 6\tan^{-1}\left(\frac{1}{2}\right) < \pi\end{gathered}}$$

Ex:- If $|z_1| = |z_2| = |z_3| = \cdots |z_n| = 1$

then prove that

$$|z_1 + z_2 + z_3 + \cdots + z_n| = \left|\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} + \cdots + \frac{1}{z_n}\right|$$

Sol^n

$$|z_k| = 1 \quad \forall\ k = 1, 2, 3, \ldots n$$

$$|z_k|^2 = 1 \Rightarrow z_k \bar{z}_k = 1$$

$$\boxed{z_k = \frac{1}{\bar{z}_k}}$$

$$z_1 = \frac{1}{\bar{z}_1}; \quad z_2 = \frac{1}{\bar{z}_2}; \quad \cdots \quad z_n = \frac{1}{\bar{z}_n}$$

$$|z_1 + z_2 + \cdots + z_n| = \left|\frac{1}{\bar{z}_1} + \frac{1}{\bar{z}_2} + \frac{1}{\bar{z}_3} + \cdots + \frac{1}{\bar{z}_n}\right|$$

$$= \left|\overline{\left(\frac{1}{z_1}\right)} + \overline{\left(\frac{1}{z_2}\right)} + \cdots + \overline{\left(\frac{1}{z_n}\right)}\right|$$

$$= \left|\overline{\frac{1}{z_1} + \frac{1}{z_2} + \frac{1}{z_3} + \cdots + \frac{1}{z_n}}\right|$$

$$\boxed{|\bar{z}| = |z|}$$

$$= \left|\frac{1}{z_1} + \frac{1}{z_2} + \cdots + \frac{1}{z_n}\right|$$

Ex:- If z_1, z_2, z_3 are distinct C.N.

Such that $\dfrac{3}{|z_1|} = \dfrac{4}{|z_2|} = \dfrac{5}{|z_3|}$ and

$$\left|\frac{9}{z_1} + \frac{16}{z_2} + \frac{25}{z_3}\right| = K \left|\frac{z_1 + z_2 + z_3}{z_1^2}\right| \text{ then}$$

find K

Soln

$$\frac{3}{|z_1|} = \frac{4}{|z_2|} = \frac{5}{|z_3|}$$

Squaring

$$\frac{9}{|z_1|^2} = \frac{16}{|z_2|^2} = \frac{25}{|z_3|^2} = \lambda$$

$$\frac{9}{z_1\bar{z}_1} = \frac{16}{z_2\bar{z}_2} = \frac{25}{z_3\bar{z}_3} = \lambda \text{ (Let)}$$

$\frac{9}{z_1} = \lambda\bar{z}_1$	$\frac{16}{z_2} = \lambda\bar{z}_2$	$\frac{25}{z_3} = \lambda\bar{z}_3$

Also

$$\left|\frac{9}{z_1} + \frac{16}{z_2} + \frac{25}{z_3}\right| = k\left|\frac{z_1+z_2+z_3}{z_1^2}\right|$$

$$\left|\lambda\bar{z}_1 + \lambda\bar{z}_2 + \lambda\bar{z}_3\right| = k\frac{|z_1+z_2+z_3|}{|z_1|^2}$$

$$\lambda\left|\bar{z}_1+\bar{z}_2+\bar{z}_3\right| = \frac{k}{z_1\bar{z}_1}|z_1+z_2+z_3|$$

$$\lambda z_1\bar{z}_1\cancel{\left|\overline{z_1+z_2+z_3}\right|} = k\cancel{|z_1+z_2+z_3|}$$

$\because \frac{9}{z_1} = \lambda\bar{z}_1 \Rightarrow$ $k=9$ Ans

Ex:- If $\left(\frac{z_1+z_2}{1+z_1z_2}\right)$ is purely real

then prove that $|z_1| = |z_2| = 1$ $(z_1z_2 \neq -1)$

$$\boxed{z = \bar{z}}$$
$\Downarrow$
z is purely real

Sol^n $\left(\frac{z_1+z_2}{1+z_1z_2}\right) = \overline{\left(\frac{z_1+z_2}{1+z_1z_2}\right)}$

$$(z_1+z_2)(1+\bar{z}_1\bar{z}_2) = (1+z_1z_2)(\bar{z}_1+\bar{z}_2)$$

$$= z_1 + \underline{z_1\bar{z}_1}\,\bar{z}_2 + z_2 + \bar{z}_1\underline{z_2\bar{z}_2} = \bar{z}_1 + \bar{z}_2 + \underline{z_1\bar{z}_1}z_2 + \underline{z_2\bar{z}_2}z_1$$

$$= z_1 + |z_1|^2\bar{z}_2 + z_2 + \bar{z}_1|z_2|^2$$
$$-\bar{z}_1 - \bar{z}_2 - |z_1|^2 z_2 - |z_2|^2 z_1 = 0$$

$$= (z_1-\bar{z}_1) + (z_2-\bar{z}_2) + |z_1|^2(\bar{z}_2 - z_2) + |z_2|^2(\bar{z}_1 - z_1) = 0$$

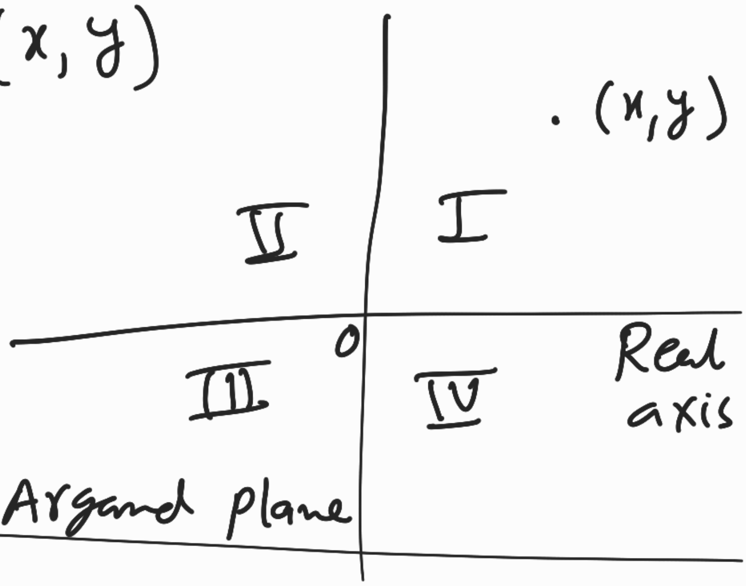
$$\Rightarrow \underbrace{(z_1-\bar{z}_1)(1-|z_2|^2)}_{\Downarrow} + \underbrace{(z_2-\bar{z}_2)(1-|z_1|^2)}_{\Downarrow} = 0$$

$$|z_1| = 1 = |z_2|$$

Representation of C.N.

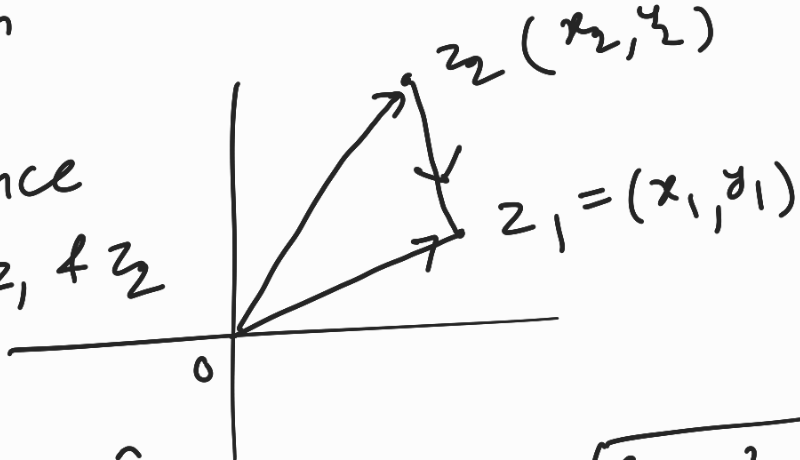
① $z = (x+iy) = (x, y)$

Cartisian form



② <u>vector form</u>

$|z_1 - z_2| =$ distance betⁿ z_1 & z_2



$z_1 = x_1 + iy_1 = x_1\hat{i} + y_1\hat{j}$

$z_2 = x_2 + iy_2 = x_2\hat{i} + y_2\hat{j}$

$\Big\}\; |z_1 - z_2| = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$

* ③ polar form / Trigonomentric form

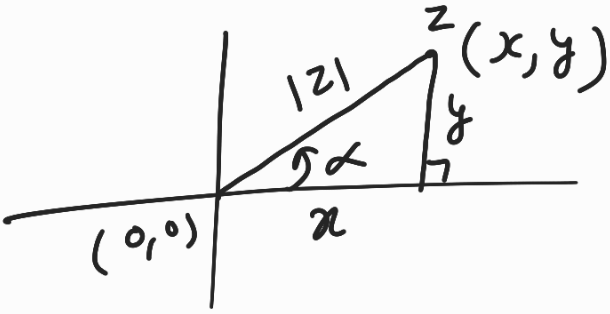
$z = |z|(\cos\alpha + i\sin\alpha)$

$\alpha = amp(z)$

Let $z = x + iy$

$$|z| \cos\alpha = x$$

$$|z| \sin\alpha = y$$



$$z = x + iy = |z| \cos\alpha + i\,|z| \sin\alpha$$

$$= |z| (\cos\alpha + i \sin\alpha)$$

(4) Eular form (Exponential form)

$$z = |z|\, e^{i\alpha} \qquad \alpha = \operatorname{amp}(z)$$

$$\boxed{e^{i\alpha} = \cos\alpha + i \sin\alpha} = \operatorname{cis}(\alpha)$$

$$e^{x} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \text{----}$$

$$e^{ix} = 1 + ix - \frac{x^2}{2!} - \frac{ix^3}{3!} + \frac{x^4}{4!} \text{----}$$

$$\underbrace{\left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} \cdots\right)}_{\cos x} + i \underbrace{\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots\right)}_{\sin x}$$

$$e^{ix} = \cos x + i \sin x$$

Ex:- write polar form

$z_1 = -1 - i\sqrt{3}$

$|z_1| = \sqrt{1+3} = 2$

$z_1 \equiv (-1, -\sqrt{3})$ 3rd

$\theta = \tan^{-1}\left|\frac{-\sqrt{3}}{-1}\right| = \frac{\pi}{3}$

$\alpha = -\pi + \theta = -\frac{2\pi}{3}$

$z_1 = |z_1| \, e^{i\alpha}$

$= 2 \, e^{-i\,2\pi/3}$

$= 2\left[\cos\left(\frac{-2\pi}{3}\right) + i \sin\left(\frac{-2\pi}{3}\right)\right]$

$= 2\left[\cos\left(\frac{2\pi}{3}\right) - i \sin\left(\frac{2\pi}{3}\right)\right]$

Ans.

$z_2 = -4\sqrt{3} + 4i$

$|z_2| = \sqrt{48+16} = 8$

$z_2 = (-4\sqrt{3}, 4)$ 2nd

$\theta = \tan^{-1}\left|\frac{4}{-4\sqrt{3}}\right| = \pi/6$

$\alpha = \pi - \pi/6 = 5\pi/6$

$z_2 = 8\left[\cos\left(\frac{5\pi}{6}\right) + i\sin\left(\frac{5\pi}{6}\right)\right]$

$z_2 = 8\, e^{i\frac{5\pi}{6}}$

Ex:- If z and w are two non zero C.N. such that $|z| = |w|$ and $amp(z) + amp(w) = \pi$ then P.T. $z = -\bar{w}$

$\underline{\text{Sol}^n}$ Let $z = |z| e^{i\alpha_1}$; $w = |w| e^{i\alpha_2}$

$$\Downarrow$$

$$\overline{w} = |w| e^{-i\alpha_2}$$

$$\boxed{|z| = |w| \,,\, \alpha_1 + \alpha_2 = \pi}$$

$$z = |w| e^{i(\pi - \alpha_2)}$$

$$= |w| e^{i\pi} e^{-i\alpha_2}$$

$$\Downarrow$$

$$= |w| \underbrace{(\cos\pi + i\sin\pi)}_{\Downarrow} e^{-i\alpha_2}$$

$$z = |w| (-1) e^{-i\alpha_2} = -\overline{w}$$

$\underline{\text{Ex:-}}$ ① $z = 1 + \cos\left(\frac{6\pi}{5}\right) + i\sin\left(\frac{6\pi}{5}\right)$

find $|z|$, $amp(z)$, Polar form of z

② $z = 6(\cos 310^\circ - i\sin 310^\circ)$

$\underline{\text{Sol}^n}$ ① $z = \left(1 + \cos\frac{6\pi}{5}, \sin\frac{6\pi}{5}\right)$

$$|z| = \sqrt{2 + 2\cos\frac{6\pi}{5}}$$

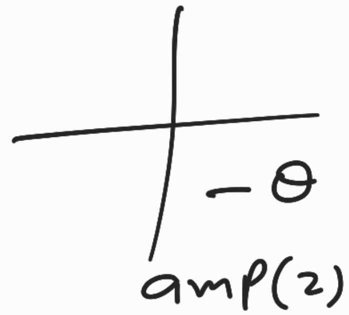
$$= \sqrt{2 \cdot 2\cos^2 3\pi/5} = 2\left|\cos\frac{3\pi}{5}\right|$$

$\Uparrow$ 2nd ques.

$$= -2\cos\left(\frac{3\pi}{5}\right) = 2\cos\left(\pi - \frac{3\pi}{5}\right)$$

$$\boxed{|z| = 2\cos\left(2\pi/5\right)}$$

$$z\left(\underbrace{1+\cos\frac{6\pi}{5}}_{+ve},\ \underbrace{\sin\frac{6\pi}{5}}_{-ve}\right) = 4^{th} \text{ quadrant}$$



$$\tan\theta = \left|\frac{\sin 6\pi/5}{1+\cos 6\pi/5}\right|$$

$$\tan\theta = \left|\frac{2\sin 3\pi/5 \cos 3\pi/5}{2\cos^2 3\pi/5}\right|$$

$$\tan\theta = \left|\tan 3\pi/5\right| = -\tan\left(\frac{3\pi}{5}\right)$$

$$\tan\theta = \tan\left(\pi - 3\pi/5\right) \Rightarrow \theta = 2\pi/5$$

$$\boxed{amp(z) = -2\pi/5}$$

$$z = 2\cos\left(2\pi/5\right) e^{-i\frac{2\pi}{5}}$$

Ans

(2) $z = 6\cos 310^\circ - i\,6\sin 310^\circ$

$z = (\underbrace{6\cos 310^\circ}_{+ve}, \underbrace{-6\sin 310^\circ}_{+ve}) = \text{I}^{st}$ quadr.

$|z| = 6$

$$\tan\theta = \frac{-\cancel{6}\sin 310^\circ}{\cancel{6}\cos 310^\circ}$$

$$\tan\theta = -\tan 310^\circ$$

$$= -\tan(360^\circ - 50^\circ)$$

$$= \tan 50^\circ \Rightarrow \theta = 50^\circ$$

$$\text{amp}(z) = \theta = \frac{5\pi}{18}$$

$$\boxed{z = 6e^{i\,5\pi/18}}$$

<u>Exr</u> $z = e^{e^{i\theta}}$ $|z|$, $\text{amp}(z)$??

$$z = e^{\cos\theta + i\sin\theta}$$

$$z = e^{\cos\theta}\, e^{i\sin\theta}$$

$$z = e^{\cos\theta}\left(\cos(\sin\theta) + i\sin(\sin\theta)\right)$$

$|z| = e^{\cos\theta}$ $\quad\quad$ $\text{amp}(z) = \sin\theta$

Some More properties on modulus

① $|z_1+z_2|^2 = |z_1|^2+|z_2|^2 + 2|z_1||z_2|\cos(\theta_1-\theta_2)$

where $\theta_1 = \text{amp}(z_1)$

$\theta_2 = \text{amp}(z_2)$

② $|z_1-z_2|^2 = |z_1|^2+|z_2|^2 - 2|z_1||z_2|\cos(\theta_1-\theta_2)$

③ If $|z_1+z_2| = |z_1|+|z_2|$

then $\text{amp}(z_1) = \text{amp}(z_2)$ and

z_1, z_2 lie on same of origin

$\Rightarrow |z_1+z_2|^2 = (|z_1|+|z_2|)^2$

$$|z_1|^2+|z_2|^2+2|z_1||z_2|\cos(\theta_1-\theta_2)$$

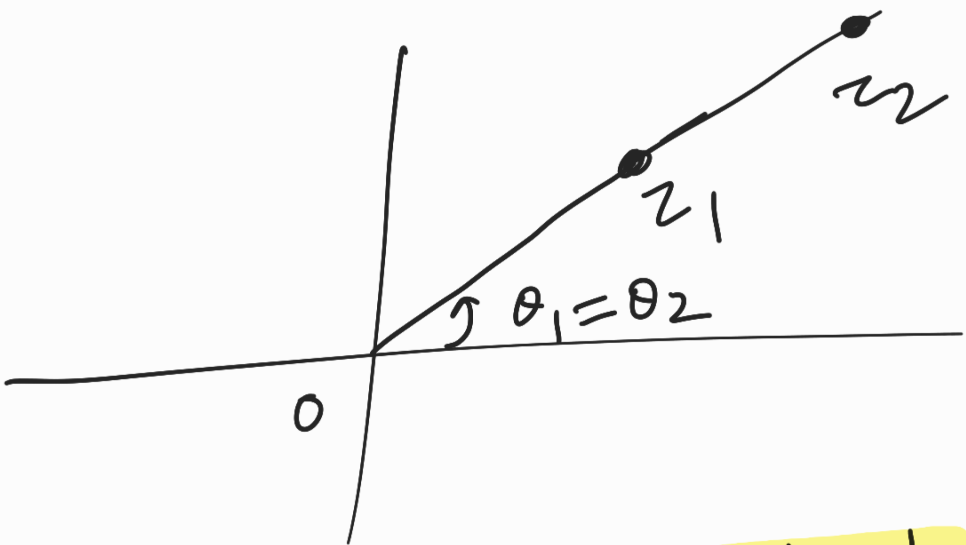
$$=|z_1|^2+|z_2|^2+2|z_1||z_2|$$

$$\Rightarrow \cos(\theta_1-\theta_2)=1$$

$$\theta_1-\theta_2=0$$

$$\theta_1=\theta_2$$

$$amp(z_1)=amp(z_2)$$



(4) If $|z_1-z_2|=|z_1|+|z_2|$ then

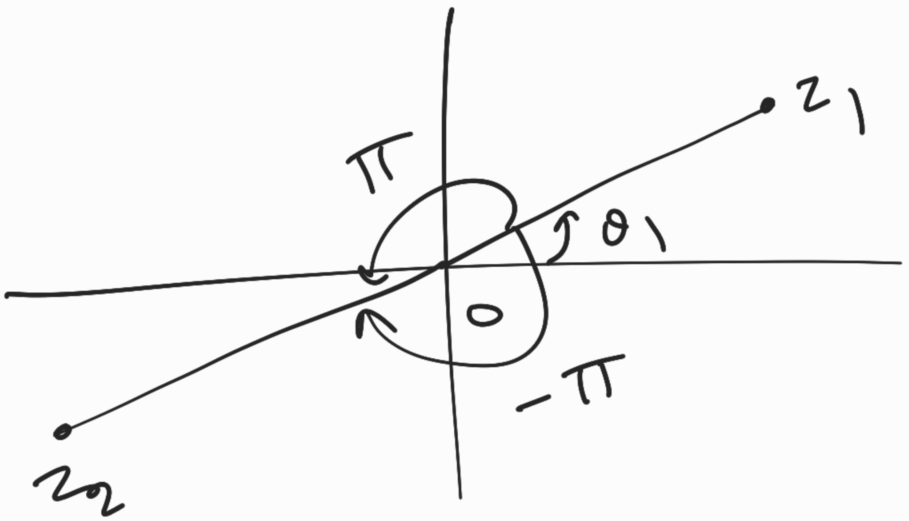
$|amp(z_1)-amp(z_2)|=\pi$ and

z_1 & z_2 lie on opposite of origin

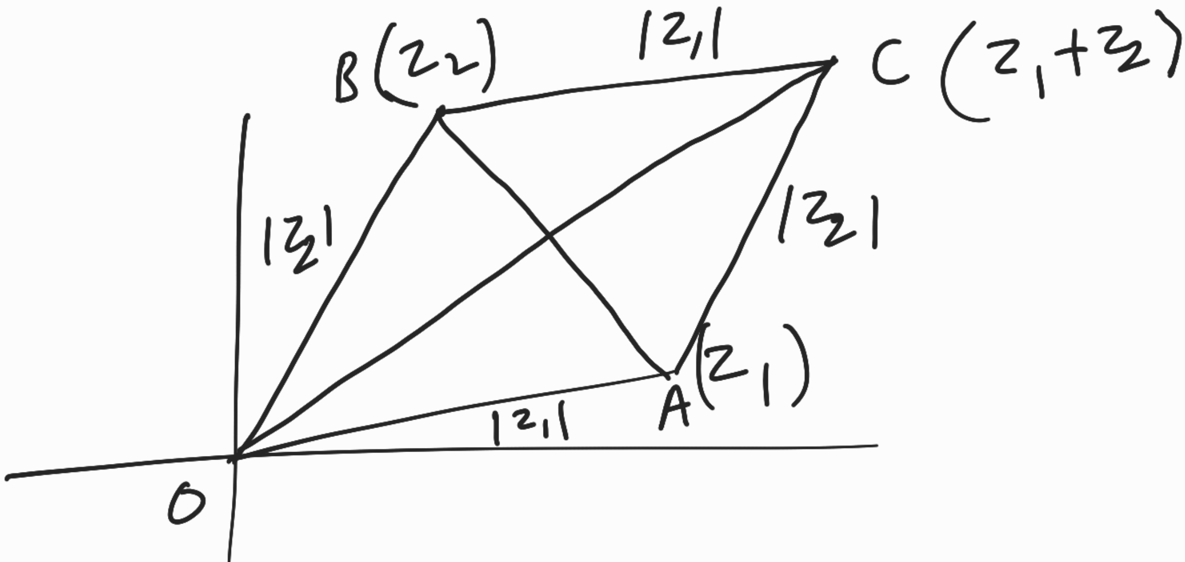
$$\Rightarrow |z_1 - z_2|^2 = (|z_1| + |z_2|)^2$$

$$\cos(\theta_1 - \theta_2) = -1$$

$$|\theta_1 - \theta_2| = \pi$$



(5) $$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\left[|z_1|^2 + |z_2|^2\right]$$



OACB = ||gram

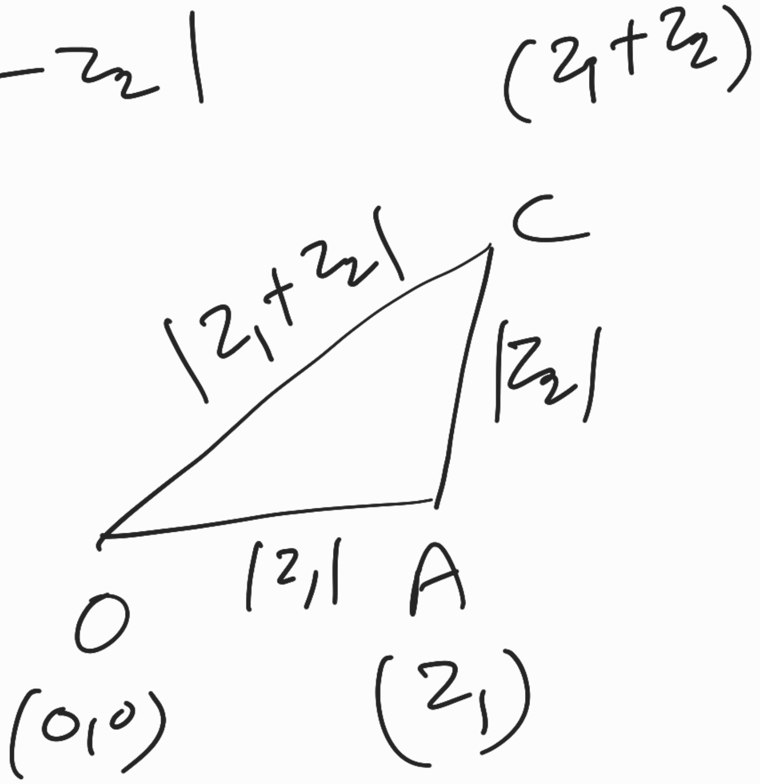
$OA = BC = |z_1|$

$OB = AC = |z_2|$

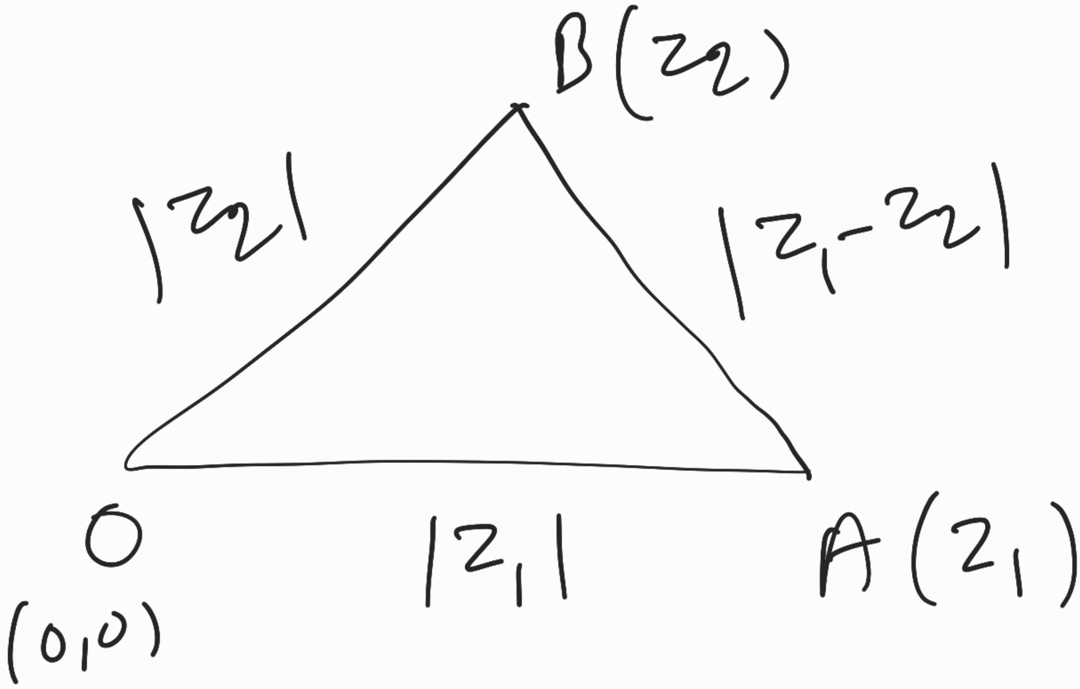
$OC = |z_1 + z_2|$

$AB = |z_1 - z_2|$

⑥ Triangular inequality



$$\big||z_1| - |z_2|\big| \le |z_1 + z_2| \le |z_1| + |z_2|$$



$$\big||z_1|-|z_2|\big| \leq |z_1-z_2| \leq |z_1|+|z_2|$$