Determinant

$$D = \begin{vmatrix} 1 & 4 \\ 0 & 8 \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \cdot a_{22} - a_{21} a_{12} = 8.$$

$a_{ij} \rightarrow$ element of i^{th} row & j^{th} column.

$$D = \begin{array}{c|ccc|} & C_1 & C_2 & C_3 \\ R_1 \rightarrow & a_{11} & a_{12} & a_{13} \\ R_2 \rightarrow & a_{21} & a_{22} & a_{23} \\ R_3 \rightarrow & a_{31} & a_{32} & a_{33} \end{array}_{3 \times 3}$$

No. of Rows = No. of Columns in determinant

$\boxed{\text{order} = 3}$

No of elements in det of order-2 $= 4$.

No. of elements in det of order-3 $= 9$.

No. of rows × No of column.

How to expand a det of order-3

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

3 Rows & 3 Column

It can expanded in 6 ways

$\longrightarrow R_1$, $\longrightarrow R_2$, $\longrightarrow R_3$ OR $\downarrow C_1$, $\downarrow C_2$, $\downarrow C_3$

Expanding by R_1:

$$D = a_{11}\underbrace{\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}}_{M_{11}} - a_{12}\underbrace{\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}}_{M_{12}} + a_{13}\underbrace{\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}}_{M_{13}}$$

$$D = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13} = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

Expanding by C_2:

$$D = -a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{22}\begin{vmatrix} a_{11} & a_{13} \\ a_{31} & a_{33} \end{vmatrix} - a_{32}\begin{vmatrix} a_{11} & a_{13} \\ a_{21} & a_{23} \end{vmatrix}$$

$$D = -a_{12}M_{12} + a_{22}M_{22} - a_{32}M_{32}$$

$$D = a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32}$$

$$a_{11}C_{12} + a_{21}C_{22} + a_{31}C_{32} = 0.$$

COFACTOR AND MINORS OF AN ELEMENT :

Minors : Minors of an element is defined as the minor determinant obtained by deleting a particular row & column in which that element lies. e.g. in the determinant

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \text{ minor of } a_{12} \text{ denotes as } M_{12} = \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} \text{ and so on.}$$

Cofactor : It has no separate identity and is related to the minors as $\boxed{C_{ij} = (-1)^{i+j} M_{ij}}$, where 'i' denotes the row and 'j' denotes the column.

Hence the value of a determinant of order three in terms of 'Minor' & 'Cofactor' can be written as :

$$D = a_{11}M_{11} - a_{12}M_{12} + a_{13}M_{13} \quad \text{or}$$
$$= a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13}$$

Note that a determinant of order 3 will have 9 minors each minor will be a determinant of order 2 and a determinant of order 4 will have 16 minors each minor will be determinant of order 3.

E(1) Find the minor and cofactors of the elements a_{23} in the determinant $\begin{vmatrix} -1 & -2 & 3 \\ -4 & -5 & -6 \\ -7 & 8 & 9 \end{vmatrix}$

$$M_{23} = \begin{vmatrix} -1 & -2 \\ -7 & 8 \end{vmatrix} = -8 - (14) = -22.$$

$$C_{23} = (-1)^{2+3} M_{23} = 22.$$

Note :

(i) **Sum of product of elements of any row (column) with their corresponding cofactors is equal to the value of DETERMINANT.**

i.e. $D = a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} = \sum_{j=1}^{3} a_{ij}C_{ij}$ etc. for i = 1, 2, 3 for $D = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$

(ii) **Sum of product of elements of any row (column) with cofactors of corresponding elements of any other row (column) is ZERO.**

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$= a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2)$$

$$D = (a_1b_2c_3 + a_3b_1c_2 + a_2b_3c_1) - (a_3b_2c_1 + a_2b_1c_3 + a_1b_3c_2)$$

Q

If $\Delta(x) = \begin{vmatrix} 0 & 2x-2 & 2x+8 \\ x-1 & 4 & x^2+7 \\ 0 & 0 & x+4 \end{vmatrix}$ and $f(x) = \sum_{j=1}^{3}\sum_{i=1}^{3} a_{ij}c_{ij}$, where a_{ij} is the element of i^{th} and j^{th} column in $\Delta(x)$ and c_{ij} is the cofactor a_{ij} $\forall$ i and j, then find the greatest value of f(x), where $x \in [-3, 18]$.

Sol^n

$$f(x) = \sum_{j=1}^{3}\sum_{i=1}^{3} a_{ij}c_{ij}$$

$$= \sum_{j=1}^{3} \underbrace{\left(a_{1j}c_{1j} + a_{2j}c_{2j} + a_{3j}c_{3j}\right)}_{\Delta(x)}$$

$$f(x) = \sum_{j=1}^{3} \Delta(x) = 3\Delta(x)$$

$$= 3\left(-(x-1)(2x-2)(x+4)\right)$$

$$f(x) = -6(x-1)^2(x+4)$$

Greatest value in $[-3,18]$ is 0 at $\underline{\underline{x=1}}$

SARRUS METHOD :-

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \Rightarrow (a_1 b_2 c_3 + b_1 c_2 a_3 + a_2 b_3 c_1) - (a_3 b_2 c_1 + a_2 b_1 c_3 + b_3 c_2 a_1)$$

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

$$\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} = 3abc - (a^3 + b^3 + c^3)$$

Note

Let $D = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$ and $D' = \begin{vmatrix} A_1 & A_2 & A_3 \\ B_1 & B_2 & B_3 \\ C_1 & C_2 & C_3 \end{vmatrix}$ where $A_1, A_2, A_3 \ldots\ldots, C_1, C_2, C_3$ are cofactors of $a_1, a_2, a_3, \ldots\ldots c_1, c_2, c_3$ respectively then prove that $\begin{vmatrix} B_2 & B_3 \\ C_2 & C_3 \end{vmatrix} = a_1 D$.

$$\begin{vmatrix} B_2 & B_3 \\ C_2 & C_3 \end{vmatrix} = B_2 C_3 - B_3 C_2$$

$$= B_2 (a_1 b_2 - b_1 a_2) + B_3 (a_1 b_3 - b_1 a_3)$$

$$= a_1 (b_2 B_2 + b_3 B_3 + b_1 B_1) - (B_2 b_1 a_2 + B_3 b_1 a_3 - a_1 b_1 B_1)$$

$$= a_1 D - b_1 (\underbrace{a_2 B_2 + a_3 B_3 + a_1 B_1}_{0})$$

$$= a_1 D.$$

Q If $a^2+b^2+c^2+ab+bc+ca \le 0 \ \forall a,b,c \in \mathbb{R}$

then value of $\begin{vmatrix} (a+b+2)^2 & a^2+b^2 & 1 \\ 1 & (b+c+2)^2 & b^2+c^2 \\ c^2+a^2 & 1 & (c+a+2)^2 \end{vmatrix}$

Sol^n

$$2a^2+2b^2+2c^2+2ab+2bc+2ca \le 0$$

$$(a+b)^2+(b+c)^2+(c+a)^2 \le 0.$$

$$\Rightarrow \left.\begin{aligned} a+b &= 0 \\ b+c &= 0 \\ c+a &= 0. \end{aligned}\right\} \text{ add } \quad a+b+c=0$$

$$\left.\begin{aligned} c &= 0 \\ a &= 0 \\ b &= 0 \end{aligned}\right\}$$

$$\begin{vmatrix} 4 & 0 & 1 \\ 1 & 4 & 0 \\ 0 & 1 & 4 \end{vmatrix} = 65$$

Ans

Q If $\begin{vmatrix} k+3 & 1 & -2 \\ 3 & -2 & 1 \\ -k & -3 & 3 \end{vmatrix} = 0$, find K.

$$-6(K+\cancel{3}) + (-K) + \cancel{18} - \left(-4K + (-3)(K+3) + 9\right) = 0.$$

$$-7K + 4K + 3(K+3) - 9 = 0.$$

$$0 = 0 \Rightarrow \text{Identity.}$$

$$\therefore K \in \mathbb{R}$$

PROPERTIES OF DETERMINANTS :

P-1 : The value of a determinant remains unaltered, if the rows & columns are interchanged.

e.g. if $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = D' = D^T$ Transpose of D.

D & D' are transpose of each other.

P-2 : If any two rows (or columns) of a determinant be interchanged, the value of determinant is changed in sign only.

e.g. Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ & $D' = \begin{vmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$ Then $D' = -D$.

P-3 : If a determinant has any two rows (or columns) identical, then its value is zero.

e.g. Let $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{vmatrix}$ then it can be verified that $D = 0$.

P-4 : If all the elements of any row (or column) be multiplied by the same number, then the determinant is multiplied by that number.

e.g. If $D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$ and $D' = \begin{vmatrix} Ka_1 & Kb_1 & Kc_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$. Then $D' = KD$

Example :

E(1) If $\begin{vmatrix} x^n & x^{n+2} & x^{2n} \\ 1 & x^p & p \\ x^{n+5} & x^{p+6} & x^{2n+5} \end{vmatrix} = 0$, then p is given by -

(A) x^n (B) $n + 1$ (C) both A and B (D) none of these

$p = x^n$

$$\begin{vmatrix} x^n & x^{n+2} & x^{2n} \\ 1 & x^{x^n} & x^n \\ x^{n+5} & x^{x^n+6} & x^{2n+5} \end{vmatrix} = x^{n+5} \begin{vmatrix} x^n & x^{n+2} & x^n \\ 1 & x^{x^n} & 1 \\ x^n & x^{x^n+1} & x^n \end{vmatrix} = 0$$

x^5 x^n

identical

Let $\boxed{p = n+1}$

$$x^5 \Leftarrow \begin{vmatrix} x^n & x^{n+2} & x^{2n} \\ 1 & x^{n+1} & n+1 \\ x^{n+5} & x^{n+7} & x^{2n+5} \end{vmatrix}$$

$$\downarrow$$

$$x^{n+1}$$

$$x^5 \cdot x^{n+1} \begin{vmatrix} x^n & x & x^{2n} \\ 1 & 1 & n+1 \\ x^n & x & x^{2n} \end{vmatrix} = 0.$$

identical