Allen Career Institute
Kota

Sheet Sol- monotonicity of function

EXERCISE 01

1. If the function $f(x) = 2x^2 - kx + 5$ is strictly increasing in $[1, 2]$, then complete set of values of k is

(A) $(-\infty, 4)$ (B) $(4, \infty)$ (C) $(-\infty, 8]$ (D) $(8, \infty)$

Solⁿ $f'(x) = 4x - K$

$f''(x) = 4 \Rightarrow f''(x) > 0 \;\forall\; x \in [1, 2]$

$\Rightarrow f'(x)$ is increasing in $[1, 2]$

and $f'(1)$ is its least value in $[1, 2]$

$\therefore f'(1) \geq 0$

$\Rightarrow 4 - K \geq 0 \Rightarrow K \leq 4$

Most appropriate answer: (A)

2. The function x^x strictly decreases on the interval-

(A) $(0, e)$ (B) $(0, 1)$ (C) $\left(0, \frac{1}{e}\right)$ (D) None of these

2. $f(x) = x^x \Rightarrow f'(x) = x^x(1 + \ln x)$, $x > 0$ (domain)

$f'(x)$ $\quad \underset{0}{|} \;-\; \underset{1/e}{|} \;+\quad \Rightarrow f(x) \downarrow$ in $x \in (0, 1/e)$

3\. Function $f(x) = x^2(x-2)^2$ is-

(A) increasing in $[0, 1]; [2, \infty)$

(B) decreasing in $[0, 1]; [2, \infty]$

3\. $f(x) = x^2(x-2)^2$

$\Rightarrow f'(x) = 2x(x-2)^2 + x^2 \cdot 2(x-2)$

$\Rightarrow f'(x) = 2x(x-2)(x-2+x)$

$\Rightarrow f'(x) = 4x(x-2)(x-1)$

$\Rightarrow f(x) \uparrow : [0,1]; [2,\infty)$

$-\ \ +\ \ -\ \ +$

$0\ \ 1\ \ 2$

4. If $f(x) = x^3 - 10x^2 + 200x - 10$, then f(x) is-

(A) strictly decreasing in $(-\infty, 10]$ and strictly increasing in $[10, \infty)$

(B) strictly increasing in $(-\infty, 10]$ and strictly decreasing in $[10, \infty)$

(C) strictly increasing for every value of x

(D) strictly decreasing for every value of x

4. $f(x) = x^3 - 10x^2 + 200x - 10$

$\Rightarrow f'(x) = 3x^2 - 20x + 200.$

$a > 0,\ D < 0 \Rightarrow f'(x) > 0\ \forall x \in R$

$\Rightarrow f(x)$ is strictly increasing $\forall x$.

5. Which one of the following statements does not hold good for the function $f(x) = \cos^{-1}(2x^2 - 1)$?

(A) f is not differentiable at $x = 0$

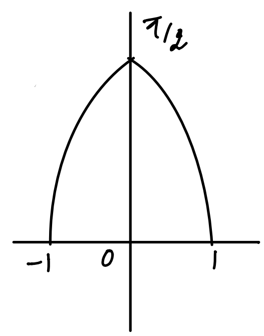
(B) f is monotonic

(C) f is even

(D) f has an extremum

Sol

$$\cos^{-1}(2x^2-1) = \begin{cases} 2\cos^{-1}x & x \in [0,1] \\ 2\pi - 2\cos^{-1}x & x \in [-1,0] \end{cases}$$



ND at $x=0$, non monotonic

even function

extremum at $x=0$

6\. Complete set of values of K in order that $f(x) = \sin x - \cos x - Kx + b$ decreases for all real values is given by-

(A) $K < 1$ (B) $K \geq 1$ (C) $K \geq \sqrt{2}$ (D) $K < \sqrt{2}$

6\. $f(x) = \sin x - \cos x - Kx + b$. $\because f(x)$ {decreases}

$f'(x) = \cos x + \sin x - K \leq 0$

$\Rightarrow \underbrace{\cos x + \sin x}_{\text{Max} = \sqrt{2}} \leq K. \Rightarrow K \geq \sqrt{2}$.

7. When $0 \le x \le 1$, $f(x) = |x| + |x - 1|$ is-

(A) strictly increasing (B) strictly decreasing (C) constant (D) None of these

7. $f(x) = |x| + |x-1|$, $x \in [0, 1]$.

$\Rightarrow f(x) = x + 1 - x \Rightarrow f(x) = 1$ (Constant)

8. Let $f(x)$ and $g(x)$ be two continuous functions defined from $R \to R$, such that $f(x_1) > f(x_2)$ and $g(x_1) < g(x_2)$, $\forall\ x_1 > x_2$, then solution set of $f\left(g(\alpha^2 - 2\alpha)\right) > f\left(g(3\alpha - 4)\right)$ is

(A) R (B) ϕ (C) (1, 4) (D) $R - [1, 4]$

8. $f(x_1) > f(x_2) \Rightarrow x_1 > x_2$; $g(x_1) < g(x_2) \Rightarrow x_1 > x_2$.

$\therefore f(g(\alpha^2 - 2\alpha)) > f(g(3\alpha - 4)) \Rightarrow g(\alpha^2 - 2\alpha) > g(3\alpha - 4)$

$\Rightarrow \alpha^2 - 2\alpha < 3\alpha - 4 \Rightarrow \alpha^2 - 5\alpha + 4 < 0 \Rightarrow (\alpha - 1)(\alpha - 4) < 0$

$\Rightarrow \alpha \in (1, 4)$ Ans.

9. If $2a + 3b + 6c = 0$, then at least one root of the equation $ax^2 + bx + c = 0$ lies in the interval-

(A) $(0, 1)$ (B) $(1, 2)$ (C) $(2, 3)$ (D) none

Sol^n consider $f(x) = \frac{ax^3}{3} + \frac{bx^2}{2} + cx$

continuous and derivable function

$f(0) = 0$; $f(1) = \frac{a}{3} + \frac{b}{2} + c$

$= \frac{2a + 3b + 6c}{6} = 0$

$\therefore$ for atleast one $x \in (0,1)$, $f'(x) = 0$

$\Rightarrow$ $ax^2 + bx + c = 0$

10. A value of c for which the conclusion of Mean values theorem holds for the function $f(x) = \log_e x$ on the interval [1, 3] is-

(A) $2\log_3 e$ (B) $\frac{1}{2}\log_e 3$ (C) $\log_3 e$ (D) $\log_e 3$

10. $f'(c) = \frac{f(3) - f(1)}{3-1} \Rightarrow \left.\frac{1}{x}\right|_{x=c} = \frac{\ln 3 - \ln 1}{3-1}$.

$\Rightarrow \frac{1}{c} = \frac{\log_e 3}{2} \Rightarrow c = \frac{2}{\log_e 3} \Rightarrow c = 2\log_3 e$.

11. The value of c in Lagrange's theorem for the function $f(x) = \begin{cases} x\cos\left(\frac{1}{x}\right), & x \neq 0 \\ 0, & x = 0 \end{cases}$ in the interval $[-1, 1]$ is-

(A) 0

(B) $\frac{1}{2}$

(C) $-\frac{1}{2}$

(D) Non existent in the interval

11. for LMVT to be applicable, $f(x)$ must be continuous in $[-1,1]$ and derivable in $(-1,1)$

CHECK : at $x = 0$: $\lim_{x\to 0} f(x) = \lim_{x\to 0} x\cos(1/x) = 0.$

$\therefore$ $f(x)$ is cont. at $x=0$, and in $[-1,1]$.

Also, at $x=0$:

$$f'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}.$$

$$\Rightarrow f'(0) = \lim_{h\to 0} \frac{\cancel{h}\cos(1/h) - 0}{\cancel{h}.} = \lim_{h\to 0} \cos(1/h) = \text{DNE}$$

$\therefore$ $f(x)$ is ND at $x=0$ in $(-1, 1)$

$\therefore$ LMVT is not applicable.

12. If the function $f(x) = x^3 - 6x^2 + ax + b$ defined on [1, 3], satisfies the rolle's theorem for $c = \frac{2\sqrt{3}+1}{\sqrt{3}}$, then-

(A) $a = 11, b = 6$ (B) $a = -11, b = 6$ (C) $a = 11, b \in R$ (D) None of these

12. $f(x)$ is cubic polynomial $\Rightarrow$ Cont & Derivable

$f(1) = f(3) \Rightarrow 1 - 6 + a + b = 27 - 54 + 3a + b$

$\Rightarrow a = 11.$

Also, $f'(c) = 0 \Rightarrow 3x^2 - 12x + a \Big|_{x = \frac{2\sqrt{3}+1}{\sqrt{3}}} = 0.$

13. The function $f : [a, \infty) \to R$ where R denotes the range corresponding to the given domain, with rule $f(x) = 2x^3 - 3x^2 + 6$, will have an inverse provided

(A) $a \geq 1$ (B) $a \geq 0$ (C) $a \leq 0$ (D) $a \leq 1$

13. function will be invertible if it is strictly increasing or strictly decreasing in its domain

$f : [a, \infty) \to R$. (R is corresponding range)

$f(x) = 2x^3 - 3x^2 + 6$

$\Rightarrow f'(x) = 6x^2 - 6x \Rightarrow f'(x) = 6x(x-1)$

+ − +

0 1

$\therefore$ $f(x)$ can be invertible if $x \in (-\infty, 0]$ or $x \in [0, 1]$ or $x \in [1, \infty)$

$\because x \in [a, \infty)$ is given $\Rightarrow$ $a \geq 1$. Ans.

(For any value of $a \geq 1$, the function will be increasing in $[a, \infty) \Rightarrow f(x)$ is invertible)

14. The function $f(x)=\tan^{-1}\left(\dfrac{1-x^2}{1+x^2}\right)$ is -

(A) strictly increasing in its domain

(B) strictly decreasing in its domain

(C) strictly decreasing in $(-\infty, 0)$ and strictly increasing in $(0, \infty)$

(D) strictly increasing in $(-\infty, 0)$ and strictly decreasing in $(0, \infty)$

14. $f(x) = \tan^{-1}\left(\dfrac{1-x^2}{1+x^2}\right) \Rightarrow f(x) = \tan^{-1}\left(\dfrac{1-x^2}{1+1\cdot x^2}\right)$

$\Rightarrow f(x) = \tan^{-1}1 - \tan^{-1}x^2 \Rightarrow f(x) = \pi/4 - \tan^{-1}x^2$

$\Rightarrow f'(x) = 0 - \dfrac{2x}{1+x^4}$.

$+ \quad | \quad - \quad f'(x)$
$\quad\quad 0$

$\Rightarrow f(x)$: st. $\uparrow$ in $(-\infty, 0)$; st. $\downarrow$ in $(0, \infty)$.

15. If the function $f(x) = 2x^2 + 3x + 5$ satisfies LMVT at $x = 2$ on the closed interval $[1, a]$, then the value of 'a' is equal to

(A) 3 (B) 4 (C) 6 (D) 1

15. $f(x) = 2x^2 + 3x + 5$.

$f'(2) = \dfrac{f(a) - f(1)}{a-1} \Rightarrow 4x+3\big|_{x=2} = \dfrac{f(a) - 10}{a-1}$

$\Rightarrow 11(a-1) = f(a) - 10.$

$\Rightarrow 2a^2 + 3a + 5 - 10 = 11a - 11$

$\Rightarrow \cancel{2a^2+3a} \quad 2a^2 - 8a + 6 = 0$

$\Rightarrow a^2 - 4a + 3 = 0 \Rightarrow \underset{\text{reject}}{a = 1}; \; a = 3$ Ans.

EXERCISE 02.

1. The equation sin x + x cos x = 0 has at least one root in

(A) $\left(-\frac{\pi}{2}, 0\right)$ (B) $(0, \pi)$ (C) $\left(\pi, \frac{3\pi}{2}\right)$ (D) $\left(0, \frac{\pi}{2}\right)$

1. B

Sol. **Ans. (B)**

Let $f(x) = \sin x + x \cos x$

consider $g(x) = \int_0^x (\sin t + t \cos t)\, dt = t \sin t\big]_0^x = x \sin x$

$g(x) = x \sin x$ which is differentiable

now $g(0) = 0$ and $g(\pi) = 0$, using Rolles Theorem

hence $\exists$ atleast one $x \in (0, \pi)$ where $g'(x) = 0$

i.e. $x \cos x + \sin x = 0$ for atleast one $x \in (0, \pi)$

2. If $f : R \to R^+$ be a differentiable function and $g(x) = e^{2x}(2f(x) - 3(f(x))^2 + 2(f(x))^3)\ \forall\ x \in R$, then which of the following is/are always correct -

(A) g(x) is strictly increasing wherever $f(x)$ is strictly increasing

(B) g(x) is strictly increasing wherever $f(x)$ is strictly decreasing

(C) g(x) is strictly decreasing wherever $f(x)$ is strictly decreasing

(D) g(x) is strictly decreasing wherever $f(x)$ is strictly increasing

Ans. (A)

Sol^n

$$g'(x) = \underbrace{2e^{2x}}_{+ve} f(x)\underbrace{(2(f(x))^2 - 3f(x) + 2)}_{\substack{a>0 \\ D<0 \\ \therefore \text{ Always} \\ +ve}} + e^{2x} f'(x)\underbrace{(6(f(x))^2 - 6f(x) + 2)}_{\substack{a>0 \\ D<0 \\ \therefore \text{ Always} \\ +ve}}$$

If $f'(x)$ is +ve then $g'(x)$ is +ve

$\therefore$ $g(x)$ is strictly increasing

$\therefore$ (A) is correct.

3. Let a, b, c, d are non-zero real numbers such that $6a + 4b + 3c + 3d = 0$, then the equation $ax^3 + bx^2 + cx + d = 0$ has

(A) atleast one root in [–2, 0]

(B) atleast one root in [0, 2]

(C) atleast two roots in [–2, 2]

(D) no root in [–2, 2]

3. B

Sol^n

Consider $h(x) = \frac{ax^4}{4} + \frac{bx^3}{3} + \frac{cx^2}{2} + dx$

$h(x)$ is continuous and derivable

$h(0) = 0$

$h(2) = \frac{a \cdot 16}{4} + \frac{b \cdot 8}{3} + \frac{c \cdot 4}{2} + d \cdot 2$

$= \frac{2}{3}(6a + 4b + 2c + 2c)$

$\Rightarrow h(2) = 0$

$\therefore$ Atleast one root of $h(x) = 0$ in $[0, 2]$

Q4 Let $F(x) = \int_{\sin x}^{\cos x} e^{(1+\arcsin t)^2} dt$ on $\left[0, \frac{\pi}{2}\right]$, then

(A) $F''(c) = 0$ for all $c \in \left(0, \frac{\pi}{2}\right)$

(B) $F''(c) = 0$ for some $c \in \left(0, \frac{\pi}{2}\right)$

(C) $F'(c) \neq 0$ for all $c \in \left(0, \frac{\pi}{2}\right)$

(D) $F(c) = 0$ for some $c \in \left(0, \frac{\pi}{2}\right)$

Solⁿ. $F'(x) = e^{(1+\sin^{-1}(\cos x))^2}(-\sin x) - e^{(1+\sin^{-1}(\sin x))^2}\cos x$

$= -\sin x\, e^{(1+(\pi/2-x))^2} - \cos x\, e^{(1+x)^2}$

$\Rightarrow F'(c) < 0 \quad \forall\, x \in [0, \pi/2]$.

$\therefore F'(c) \neq 0$ for all $c \in (0, \pi/2)$.

Also. $F(\pi/4) = \int_{1/\sqrt{2}}^{1/\sqrt{2}} e^{(1+\sin^{-1}(t))^2} dt = 0$.

$F'(0) = -e$

$F'(\pi/2) = -e$.

$\Rightarrow F''(c) = 0$ for some $c \in (0, \pi/2)$.

5. For the equation $\frac{e^{-x}}{x+1} = a$; which of the following statement(s) is/are correct ?

(A) If $a \in (0,\infty)$, then equation has 2 real and distinct roots

(B) If $a \in (-\infty,-e^2)$, then equation has 2 real & distinct roots.

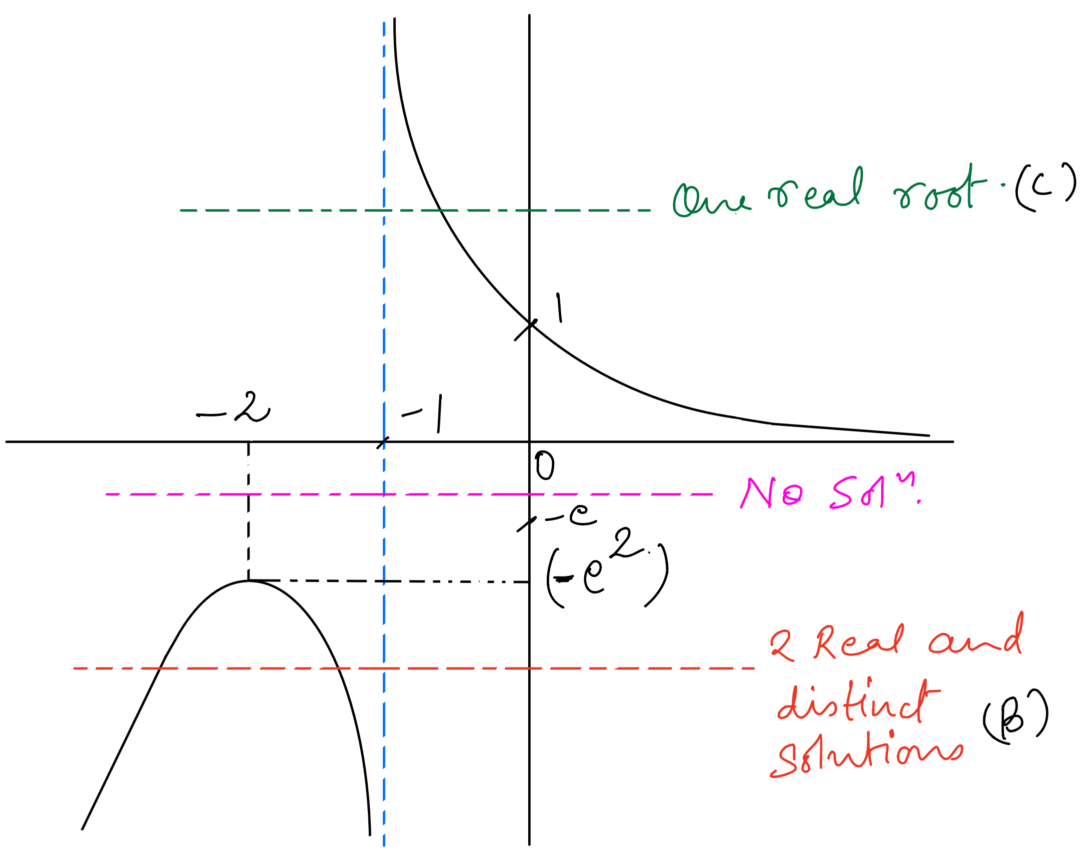
(C) If $a \in (0,\infty)$, then equation has 1 real root

(D) If $a \in (-e,0)$, then equation has no real root.

Solⁿ

$$y = \frac{e^{-x}}{x+1} = f(x)$$

We plot the graph of $f(x)$



6. Let $f:\left[0,\frac{\pi}{2}\right]\to[0,1]$ be a differentiable function such that $f(0)=0$, $f\left(\frac{\pi}{2}\right)=1$, then

(A) $f'(\alpha)=\sqrt{1-f^2(\alpha)}$ for all $\alpha\in\left(0,\frac{\pi}{2}\right)$.

(B) $f'(\alpha)=\frac{2}{\pi}$ for all $\alpha\in\left(0,\frac{\pi}{2}\right)$.

(C) $f(\alpha)f'(\alpha)=\frac{1}{\pi}$ for atleast one $\alpha\in\left(0,\frac{\pi}{2}\right)$

(D) $f'(\alpha)=\frac{8\alpha}{\pi^2}$ for atleast one $\alpha\in\left(0,\frac{\pi}{2}\right)$.

Solⁿ. (A). Not always true as $f(x)=\frac{2x}{\pi}$ also satisfies the condition in the question but not the option

B). $f(x)=\sin x$ satisfies the condition in the question but $f'(\alpha)\neq\frac{2}{\pi}$. $\forall\,\alpha\in\left(0,\frac{\pi}{2}\right)$

c). Let $g(x)=f^2(x)-\frac{2x}{\pi}$.

$\because$ $g(0)=0$ and $g(1)=0$.

Also. $g(x)$ is continous and differentiable in $[0,\pi/2]$

$\therefore$ using Rolle's theorem we can say that

$g'(\alpha)=0$ for some $\alpha\in(0,\pi/2)$

or $f'(\alpha)f(\alpha)=\frac{1}{\pi}$ for some $\alpha\in(0,\pi/2)$.

D). Let $g(x)=f(x)-\frac{4x^2}{\pi^2}$.

$g(0)=g(1)=0$.

using Rolle's theorem

$g'(\alpha)=f'(\alpha)-\frac{8\alpha}{\pi^2}=0$ for some $\alpha\in(0,\pi/2)$.

EXERCISE (S-1)

1. Find the intervals of monotonicity for the following functions & represent your solution set on the number line.

(a) $f(x) = 2.\ e^{x^2-4x}$ (b) $f(x) = e^x/x$ (c) $f(x) = x^2 e^{-x}$ (d) $f(x) = 2x^2 - \ln|x|$

Also plot the graphs in each case & state their range.

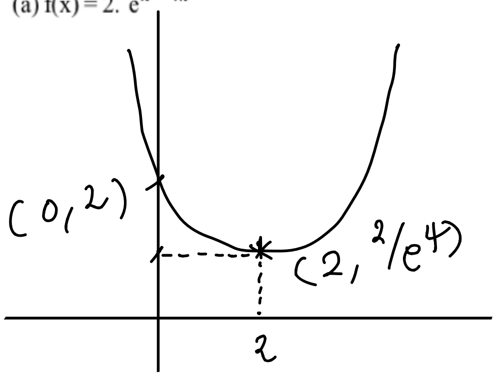
(a). $f(x) = 2e^{x^2-4x} \Rightarrow f'(x) = 2e^{x^2-4x}(2x-4)$

$\Rightarrow f'(x) = 4e^{x^2-4x}(x-2)$

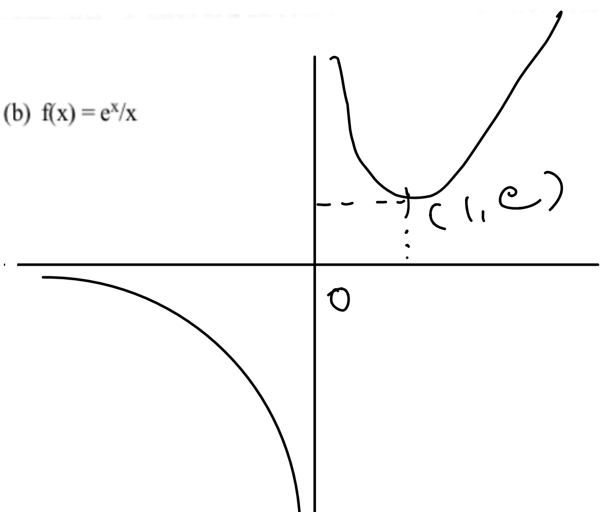
$f'(x) \geq 0 \Rightarrow x \in [2,\infty)$ for st. $\uparrow$.

$f'(x) \leq 0 \Rightarrow x \in (-\infty, 2]$ for st. $\downarrow$

(a) $f(x) = 2.\ e^{x^2-4x}$



(b) $f(x) = e^x/x$



(b) $f(x) = \dfrac{e^x}{x} \Rightarrow f'(x) = \dfrac{xe^x - e^x.1}{x^2}$

$\Rightarrow f'(x) = \dfrac{e^x(x-1)}{x^2}$ Domain: $x \neq 0$.

$-\ -\ +$
0 1

st. $\uparrow$: $x \in [1, \infty)$

st. $\downarrow$: $x \in (-\infty, 0)\ ;\ (0, 1]$

2. Let $f(x) = 1 - x - x^3$. Find all real values of x satisfying the inequality, $1 - f(x) - f^3(x) > f(1 - 5x)$

2. $f(x) = 1 - x - x^3 \Rightarrow f(x) = -1 - 3x^2 < 0 \ \forall\, x \in R$

$\therefore$ $f(x)$ is st. $\downarrow$

Now $\quad 1 - f(x) - f^3(x) > f(1-5x)$

$\Rightarrow \quad f(f(x)) > f(1-5x)$

$\Rightarrow \quad f(x) < 1 - 5x \quad (\because f(x) \text{ is } \downarrow)$

$\Rightarrow \quad \cancel{1} - x - x^3 < \cancel{1} - 5x \Rightarrow x^3 - 4x > 0$

$\Rightarrow \quad x(x-2)(x+2) > 0$

$- \quad + \quad - \quad +$
$-2 \quad 0 \quad 2$

$\therefore x \in (-2, 0) \cup (2, \infty)$

3. Find the intervals of monotonocity of the functions in $[0, 2\pi]$

(a) $f(x) = \sin x - \cos x$ in $x \in [0, 2\pi]$ (b) $g(x) = 2\sin x + \cos 2x$ in $(0 \le x \le 2\pi)$.

(c) $f(x) = \dfrac{4\sin x - 2x - x\cos x}{2 + \cos x}$

3. (a) $f(x) = \sin x - \cos x \Rightarrow f'(x) = \cos x + \sin x$.

$\Rightarrow f'(x) = \sqrt{2}\sin(x + \pi/4)$.

$+ \quad - \quad +$

$0 \quad 3\pi/4 \quad 7\pi/4 \quad 2\pi$

St. ↑ : $x \in \left[0, \frac{3\pi}{4}\right]; \left[\frac{7\pi}{4}, 2\pi\right]$

St. ↓ : $x \in \left[\frac{3\pi}{4}, \frac{7\pi}{4}\right]$.

(b) $g(x) = 2\sin x + \cos 2x \Rightarrow g'(x) = 2\cos x - 2\sin 2x$

$\Rightarrow g'(x) = 2\cos x - 4\cos x \sin x$

$\Rightarrow g'(x) = 2\cos x(1 - 2\sin x)$

$+ \quad - \quad + \quad - \quad +$

$0 \quad \pi/6 \quad \pi/2 \quad \frac{5\pi}{6} \quad \frac{3\pi}{2} \quad 2\pi$

St. ↑ : $x \in [0, \pi/6]; [\pi/2, 5\pi/6]; [3\pi/2, 2\pi]$.

St. ↓ : $x \in [\pi/6, \pi/2]; [5\pi/6, 3\pi/2]$.

(c). $f(x) = \dfrac{4\sin x - 2x - x\cos x}{2 + \cos x}$ $x \in [0, 2\pi]$

$\Rightarrow f'(x) = \dfrac{\cos x(4 - \cos x)}{(2 + \cos x)^2}$

$(2 + \cos x)^2$; $4 - \cos x$ are always positive

$\therefore f'(x)$ $+ \quad - \quad +$

$0 \quad \pi/2 \quad 3\pi/2 \quad 2\pi$

$\therefore$ St. ↑ : $x \in [0, \pi/2]; [3\pi/2, 2\pi]$

St ↓ : $x \in [\pi/2, 3\pi/2]$

Sol^n. $f'(x) = 3x^2-2x+1 > 0 \quad \forall\ x \in R. \quad [\because D<0]$

$$\therefore \quad g(x) = \begin{cases} x^3-x^2+x+1 & 0 \le x \le 1 \\ 3-x & 1 < x \le 2. \end{cases}$$

$$g'(x) = \begin{cases} 3x^2-2x+1 & 0 \le x \le 1 \\ -1 & 1 < x \le 2. \end{cases}$$

$g(1^-) = g(1) = 2.$

$g(1^+) = 2.$

$\Rightarrow g(x)$ is continuous at $x=1$.

$g'(1^-) = 2, \quad g'(1^+) = -1.$

$\therefore g(x)$ is not differentiable at $x=1$.

5. Find the greatest & the least values of the following functions in the given interval if they exist.

(a) $f(x) = \sin^{-1}\dfrac{x}{\sqrt{x^2+1}} - \ln x$ in $\left[\dfrac{1}{\sqrt{3}}, \sqrt{3}\right]$

(b) $f(x) = 12x^{4/3} - 6x^{1/3}$, $x \in [-1, 1]$

(c) $y = x^5 - 5x^4 + 5x^3 + 1$ in $[-1, 2]$

5. (a) $f(x) = \sin^{-1}\dfrac{x}{\sqrt{x^2+1}} - \ln x$, $x > 0$.

$\Rightarrow f'(x) = \dfrac{1}{\sqrt{1 - \left(\dfrac{x^2}{x^2+1}\right)}} \cdot \left(\dfrac{\sqrt{x^2+1} - \dfrac{x \cdot \cancel{2}x}{\cancel{2}\sqrt{x^2+1}}}{x^2+1}\right) - \dfrac{1}{x}$

$\Rightarrow f'(x) = (\cancel{\sqrt{x^2+1}}) \cdot \dfrac{1}{\cancel{\sqrt{x^2+1}}} \cdot \dfrac{1}{x^2+1} - \dfrac{1}{x}$

$\Rightarrow f'(x) = \dfrac{1}{1+x^2} - \dfrac{1}{x} = \dfrac{x-1-x^2}{x(1+x^2)}$

$\Rightarrow f'(x) = -\dfrac{(x^2-x+1)}{x(x^2+1)}$

for $x > 0$: $f'(x) < 0 \Rightarrow f(x)$ is st. $\downarrow$

$\therefore f(x)_{max}$ at $x = 1/\sqrt{3} \Rightarrow f(1/\sqrt{3}) = \pi/6 + \frac{1}{2}\ln 3$

$f(x)_{min}$ at $x = \sqrt{3} \Rightarrow f(\sqrt{3}) = \pi/6 - 1/2 \ln 3$

(b) $f(x) = 12x^{4/3} - 6x^{1/3}$, $x \in [-1, 1]$

$\Rightarrow f'(x) = 16x^{1/3} - 2x^{-2/3}$

$f'(x) = 0 \Rightarrow x = 1/8$.

$\therefore f(1) = 6$; $f(-1) = 18$.

$f(1/8) = -9/4$.

$\therefore m = -9/4$; $M = 18$.

(c) $f(x) = x^5 - 5x^4 + 5x^3 + 1$, $x \in [-1, 2]$

$f'(x) = 5x^4 - 20x^3 + 15x^2 = 5x^2(x^2 - 4x + 3)$

$= 5x^2(x-1)(x-3)$

$f'(x) = 0 \Rightarrow x = 0;\ x = 1;\ x = 3$

$\therefore f(-1) = -10$; $f(0) = 1$; $f(1) = 2$

$f(2) = -7 \Rightarrow m = -10;\ M = 2$.

6. If $f(x) = 2e^x - ae^{-x} + (2a+1)x - 3$ increases for every $x \in R$ then find the range of values of 'a'.

6. $f(x) = 2e^x - ae^{-x} + (2a+1)x - 3$.

$f'(x) = 2e^x + ae^{-x} + (2a+1) > 0 \ \forall x \in R$

$= \dfrac{2(e^x)^2 + (2a+1)e^x + a}{e^x} > 0$

$\Rightarrow 2(e^x)^2 + (2a+1)e^x + a > 0 \ \forall x \in R$

$\Rightarrow 2(e^x)^2 + 2a \cdot e^x + e^x + a > 0$

$\Rightarrow 2e^x(e^x + a) + 1(e^x + a) > 0$

$\Rightarrow \underbrace{(2e^x + 1)}_{+ve}(e^x + a) > 0$

$\Rightarrow e^x + a > 0$

$\Rightarrow a > -e^x \Rightarrow a \geq 0.$

7. Find the set of values of x for which the inequality $ln\,(1+x) > x/(1+x)$ is valid.

7. $\ln(1+x) > \dfrac{x}{1+x}$, $x > -1$ (domain)

$\Rightarrow \ln(1+x) - \dfrac{x}{1+x} > 0.$

Let $h(x) = \ln(1+x) - \dfrac{x}{1+x}$

$\Rightarrow h'(x) = \dfrac{1}{1+x} - \left[\dfrac{1+x-x}{(1+x)^2}\right] = \dfrac{1+x-1}{(1+x)^2}$

$\Rightarrow h'(x) = \dfrac{x}{(1+x)^2}$

$h'(x)$ $\quad -1 \;\; - \;\; 0 \;\; +$

$\qquad\quad\;\; \downarrow \quad\;\; \uparrow$

$\therefore h(x)$ has a least value at $x=0$

$h(0) = 0 \Rightarrow h(x) \geq h(0) \;\forall\, x \in D$

$\Rightarrow \ln(1+x) - \dfrac{x}{1+x} \geq 0. \;\forall\, x \in D$

Equality to zero exists at $x=0$

$\therefore \ln(1+x) > \dfrac{x}{1+x} \;\forall\, x \in D - \{0\}$

$\Rightarrow \ln(1+x) > \dfrac{x}{1+x} \;\forall\, x \in (-1,0) \cup (0,\infty)$

8. (a) Let f, g be differentiable on R and suppose that f(0) = g(0) and f '(x)≤g'(x) for all x≥ 0. Show that f(x) ≤ g(x) for all x ≥ 0.

(b) Show that exactly two real values of x satisfy the equation x^2 = xsinx + cosx.

(c) Prove that inequality $e^x > (1 + x)$ for all $x \in R_0$ and use it to determine which of the two numbers e^π and π^e is greater.

8 (a). TPT $f(x) \leq g(x) \Rightarrow f(x) - g(x) \leq 0$.

$\Rightarrow$ Let $h(x) = f(x) - g(x)$.

$\Rightarrow h'(x) = f'(x) - g'(x)$.

Given: $f'(x) \leq g'(x) \Rightarrow f'(x) - g'(x) \leq 0$.

$\therefore h'(x) = f'(x) - g'(x) \leq 0$.

$\Rightarrow h(x) \downarrow \Rightarrow h(x) \leq h(0)$ (starting pt. of domain)

$\Rightarrow f(x) - g(x) \leq f(0) - g(0)$

$\Rightarrow f(x) - g(x) \leq 0$ (given $f(0) = g(0)$)

8(b). $x^2 = x\sin x + \cos x$.

$\Rightarrow x^2 - x\sin x - \cos x = 0$

$\Rightarrow$ Let $h(x) = x^2 - x\sin x - \cos x$

$\Rightarrow h'(x) = 2x - x\cos x - \sin x + \sin x$.

$h'(x) = x(2 - \cos x)$

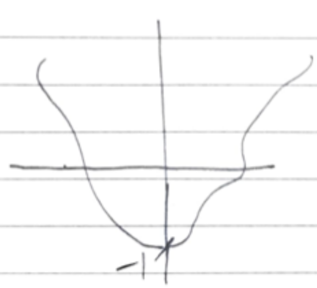
$h'(x)$: $-$, $+$; $\downarrow$ 0 $\uparrow$.

$h(0) = -1$.

Also $h(x)$ is a continuous function and.

$\lim_{x \to \infty} h(x) \to \infty$; $\lim_{x \to -\infty} h(x) \to \infty$

$\therefore$ We can estimate a graph of $h(x)$ as:



Decreasing for $x < 0$

Increasing for $x > 0$.

-1 at $x = 0$.

As $x \to \infty$, $h(x) \to \infty$ and

As $x \to -\infty$, $h(x) \to \infty$

$\therefore$ Exactly 2 distinct roots.

8.

(c) $e^x > 1 + x \quad \forall\, x \in \mathbb{R}_0$ $\left\{ \text{Notice that at } x = 0,\ LHS = RHS \right\}$.

$h(x) = e^x - x - 1 > 0 \Rightarrow h'(x) = e^x - 1$.

$h'(x)$ $\quad - \quad \underset{0}{|} \quad +$

$\downarrow \quad 0 \quad \uparrow \qquad \therefore\ h(x)_{min}$ at $x = 0$

$\therefore\ h(x) > h(0)$ {No equality $\because$ $h(0)$ is not in dom.)

$\Rightarrow e^x - x - 1 > 0 \Rightarrow e^x > x + 1$

Now for larger of e^π or π^e: Put $x = \pi/e - 1$ in above

$\Rightarrow e^{\frac{\pi}{e} - 1} > \frac{\pi}{e} - 1 + 1 \Rightarrow e^{\pi/e - 1} > \pi/e \Rightarrow e^{\pi/e} > \pi$

$\Rightarrow e^\pi > \pi^e$ UP.

9. Let f (x) = $4x^3 - 3x^2 - 2x + 1$, use Rolle's theorem to prove that there exist c, $0< c <1$ such that $f(c) = 0$.

Solⁿ

Let $g(x) = x^4 - x^3 - x^2 + x$

continuous and derivable fⁿ.

$g(0) = 0$; $g(1) = 0$

$\therefore$ by RT, atleast once in $x \in (0,1)$,

$g'(x) = 0$

$\Rightarrow \quad 4x^3 - 3x^2 - 2x + 1 = 0$ in $(0,1)$

10. Assume that f is continuous on [a, b], $a > 0$ and differentiable on an open interval (a, b). Show that if $\frac{f(a)}{a} = \frac{f(b)}{b}$, then there exist $x_0 \in (a, b)$ such that $x_0 f'(x_0) = f(x_0)$.

10. Given : $\frac{f(a)}{a} = \frac{f(b)}{b}$ $[a, b]$, $a > 0$

TPT : $\exists\, x_0 \in (a, b)$ such that $x_0 f'(x_0) = f(x_0)$

From above information,

Let $h(x) = \frac{f(x)}{x}$

$h(x)$ is cont. in $[a, b]$ & derivable in (a, b)

and $h(a) = h(b)$

$\Rightarrow \exists$ atleast one $x_0 \in (a, b)$ for which $h'(x) = 0$

$\Rightarrow h'(x) = \left.\frac{x \cdot f'(x) - f(x) \cdot 1}{x^2}\right|_{x = x_0} = 0$

$\Rightarrow x_0 \cdot f'(x_0) - f(x_0) = 0.$ H.P.

11. $f(x)$ and $g(x)$ are differentiable functions for $0 \le x \le 2$ such that $f(0) = 5$, $g(0) = 0$, $f(2) = 8$, $g(2) = 1$. Show that there exists a number c satisfying $0 < c < 2$ and $f'(c) = 3g'(c)$.

11. Let $h(x) = f(x) - 3g(x)$

$h(x)$ is cont. and derivable in $[0, 2]$

$h(0) = f(0) - 3g(0) = 5 - 3(0) = 5$

$h(2) = f(2) - 3g(2) = 8 - 3(1) = 5$

$\because h(0) = h(2) \Rightarrow \exists$ atleast one $c \in (0, 2)$ for which $h'(x) = 0$

$\Rightarrow f'(x) - 3g'(x)\big|_{x=c} = 0$

$\Rightarrow f'(c) - 3g'(c) = 0$.

12. For what value of a, m and b does the function $f(x) = \begin{cases} 3 & x = 0 \\ -x^2 + 3x + a & 0 < x < 1 \\ mx + b & 1 \le x \le 2 \end{cases}$

satisfy the hypothesis of the mean value theorem for the interval [0, 2].

12. For mean value theorem in $[0,2]$, $f(x)$ must be continuous in $[0,2]$ and derivable in $(0,2)$

$\therefore$ Cont. at $x=0$: $f(0^+) = f(0) =$

$\Rightarrow 0+0+a = 3 \Rightarrow \boxed{a=3}$.

Cont. at $x=1$: $f(1^+) = f(1^-) = f(1^-)$

$\Rightarrow m(1) + b = -(1)^2 + 3(1) + a$.

$\Rightarrow m+b = a+2 \Rightarrow m+b = 5 \longrightarrow (1)$

Derivable at $x=1$: $f'_-(1) = f'_+(1)$

$\Rightarrow -2x+3 \big|_{x=1} = m \Rightarrow \boxed{m=1}$.

$\therefore$ by (1) : $\boxed{b=4}$

EXERCISE S2.

1. Let $f(x)$ be a strictly increasing function defined on $(0, \infty)$. If $f(2a^2 + a + 1) > f(3a^2 - 4a + 1)$. Find the range of a.

1. $(0, 1/3) \cup (1, 5)$

Sol^n $\because$ $f(x)$ is st. increasing in $(0, \infty)$

$\therefore$ $f(x_1) < f(x_2) \Rightarrow 0 < x_1 < x_2$

$\therefore$ $f(3a^2 - 4a + 1) < f(2a^2 + a + 1)$

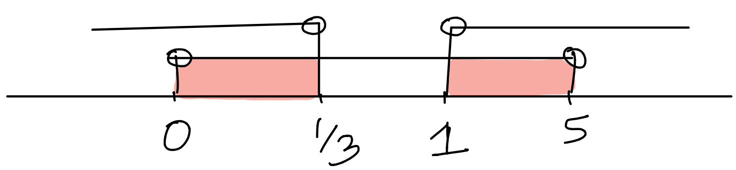
$\Rightarrow$ $0 < 3a^2 - 4a + 1 < 2a^2 + a + 1$

$3a^2 - 4a + 1 > 0$

$(3a-1)(a-1) > 0$

$a^2 - 5a < 0$

$\Rightarrow a(a-5) < 0$



$\therefore$ $a \in (0, 1/3) \cup (1, 5)$

EXERCISE (S-2)

2\. Find the values of 'a' for which the function $f(x) = \sin x - a\sin 2x - \frac{1}{3}\sin 3x + 2ax$ strictly increases throughout the number line.

2\. $[1, \infty)$

Sol^n $f(x) = \sin x - a\sin 2x - \frac{1}{3}\sin 3x + 2ax$

For st. increasing, $f'(x) \geq 0$

$\Rightarrow f'(x) = \cos x - 2a\cos 2x - \cos 3x + 2a \geq 0$

$\Rightarrow \cos x - \cos 3x - 2a\cos 2x + 2a \geq 0$

$\Rightarrow \cos x - (4\cos^3 x - 3\cos x) + 2a(1 - \cos 2x) \geq 0$

$\Rightarrow 4\cos x - 4\cos^3 x + 2a(2\sin^2 x) \geq 0$

$\Rightarrow 4\cos x(1 - \cos^2 x) + 4a\sin^2 x \geq 0$

$\Rightarrow 4\cos x\sin^2 x + 4a\sin^2 x \geq 0$

$\Rightarrow 4\sin^2 x(\cos x + a) \geq 0$

$\Rightarrow \cos x + a \geq 0 \ \forall x \in R$

$\Rightarrow a \geq 1$

3. Find the minimum value of the function $f(x) = x^{3/2} + x^{-3/2} - 4\left(x + \frac{1}{x}\right)$ for all permissible real x.

3. -10

Sol. $f(x) = x^{\frac{3}{2}} + x^{\frac{-3}{2}} - 4\left(x + \frac{1}{x}\right)$

$$= x\sqrt{x} + \frac{1}{x\sqrt{x}} - 4\left(x + \frac{1}{x}\right)$$

$\boxed{\text{Put } t = \sqrt{x}}$

$$f(t) = t^2 . t + \frac{1}{t^2 . t} - 4\left(t^2 + \frac{1}{t^2}\right)$$

$$= t^3 + \frac{1}{t^3} - 4\left(t^2 + \frac{1}{t^2}\right)$$

$$f(t) = \left(t + \frac{1}{t}\right)^3 - 3.t.\frac{1}{t}\left(t + \frac{1}{t}\right) - 4\left[\left(t + \frac{1}{t}\right)^2 - 2.t.\frac{1}{t}\right]$$

$$= \left(t + \frac{1}{t}\right)^3 - 3\left(t + \frac{1}{t}\right) - 4\left[\left(t + \frac{1}{t}\right)^2 - 2\right]$$

$$= \left(t + \frac{1}{t}\right)^3 - 3\left(t + \frac{1}{t}\right) - 4\left(t + \frac{1}{t}\right)^2 + 8$$

$$= \left(t + \frac{1}{t}\right)^3 - 4\left(t + \frac{1}{t}\right)^2 - 3\left(t + \frac{1}{t}\right) + 8$$

$$\boxed{\begin{array}{l} \text{put } t + \frac{1}{t} = a \\ \because t > 0 \Rightarrow a > 2 \end{array}}$$

$f(a) = a^3 - 4a^2 - 3a + 8$

$f'(a) = 3a^2 - 8a - 3$

$= 3a^2 - 9a + a - 3$

$= 3a(a - 3) + (a - 3)$

$= (3a + 1)(a - 3)$

$-$ $+$

$-1/3$ 2 3

Hence f(a) has minimum value at $a = 3$

$\therefore$ $f(a)_{min} = f(3)$

$= 3^3 - 4.3^2 - 3.3 + 8$

$= 27 - 36 - 9 + 8$

$= 35 - 45$

$= -10$ **Ans.**

Sol^n.

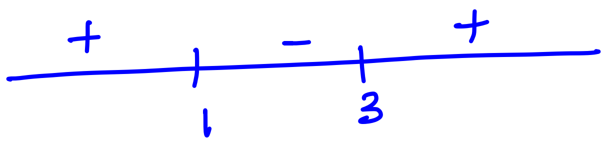
$$F'(x) = \frac{1}{x^2} \ln\left(\frac{x^2-1}{32}\right) \cdot 2x - \frac{1}{x} \ln\left(\frac{x-1}{32}\right)$$

$$= \frac{1}{x}\left[2\ln\left(\frac{x^2-1}{32}\right) - \ln\left(\frac{x-1}{32}\right)\right] = 0$$

$$\Rightarrow \quad \left(\frac{x^2-1}{32}\right)^2 = \left(\frac{x-1}{32}\right).$$

$$\Rightarrow \quad \frac{(x-1)}{32}\left[\frac{(x-1)(x+1)^2}{32} - 1\right] = 0.$$

$\Rightarrow$.



$\therefore$ $F(x)$ is decreasing in $(1, 3]$ and strictly increasing in $[3, \infty)$

5. Prove that, $x^2 - 1 > 2x\, ln\, x > 4(x - 1) - 2\, ln\, x$ for $x > 1$.

Sol^n

$$x^2 - 1 > 2x \ln x > 4(x-1) - 2\ln x$$

Part 1

Part 2

Part 1:

$$x^2 - 1 > 2x\ln x \Rightarrow x^2 - 1 - 2x\ln x > 0$$

Let $P(x) = x^2 - 1 - 2x\ln x$

$P'(x) = 2x - 2\left(x \cdot 1/x + \ln x\right)$

$P'(x) = 2(\underbrace{x - 1 - \ln x}_{Q(x)}) \longrightarrow (1)$

Let $Q(x) = x - 1 - \ln x$

$Q'(x) = 1 - 1/x = \dfrac{x-1}{x} \Big\} \longrightarrow +ve$

$\therefore Q(x)$ is increasing

$\therefore Q(x) > Q(1) \Rightarrow Q(x) > 0$

$\therefore P'(x) > 0 \Rightarrow P(x)$ is increasing (by (1))

$\Rightarrow P(x) > P(1) \Rightarrow \boxed{x^2 - x - 2x\ln x > 0}$

6. Let $a > 0$ and f be continuous in $[-a, a]$. Suppose that $f'(x)$ exists and $f'(x) \le 1$ for all $x \in (-a, a)$. If $f(a) = a$ and $f(-a) = -a$, show that $f(0) = 0$.

Sol^n LMVT on $f(x)$ in $[-a, 0]$:

$$\frac{f(0)-f(-a)}{0-(-a)} = f'(c_1) \le 1$$

$$\Rightarrow \frac{f(0)-(-a)}{a} \le 1 \Rightarrow f(0)+a \le a$$

$$\Rightarrow \boxed{f(0) \le 0} \longrightarrow (1)$$

LMVT on $f(x)$ in $[a, 0]$:

$$\frac{f(a)-f(0)}{a-0} = f'(c_2) \le 1$$

$$\Rightarrow \frac{a-f(0)}{a} \le 1 \Rightarrow a-f(0) \le a$$

$$\Rightarrow \boxed{f(0) \ge 0} \longrightarrow (11)$$

by (1) and (11):

$\therefore f(0) = 0$

Hence Proved

EXERCISE (JM)

1. A function is matched below against an interval where it is supposed to be increasing. which of the following pairs is incorrectly matched ? **[AIEEE-2005]**

interval	function
(1) $(-\infty, \infty)$	$x^3 - 3x^2 + 3x + 3$
(2) $[2, \infty)$	$2x^3 - 3x^2 - 12x + 6$
(3) $\left(-\infty, \frac{1}{3}\right]$	$3x^2 - 2x + 1$
(4) $(-\infty, -4)$	$x^3 + 6x^2 + 6$

① (1) $f(x) = x^3 - 3x^2 + 3x + 3$

$\Rightarrow f'(x) = 3x^2 - 6x + 3 = 3(x-1)^2 \geq 0 \ \forall x \in R$

(2) $f(x) = 2x^3 - 3x^2 - 12x + 6$

$\Rightarrow f'(x) = 6x^2 - 6x - 12 = 6(x^2 - x - 2)$

$\Rightarrow f'(x) = 6(x-2)(x+1)$ $+$, $-$, $+$ at -1, 2

$\therefore f(x) \uparrow$ in $[-\infty, -1]$; $[2, \infty)$

(3) $f(x) = 3x^2 - 2x + 1 \Rightarrow f'(x) = 6x - 2$

$f'(x)$: $-$, $+$ at $1/3$ $\therefore f(x) \uparrow$ in $[1/3, \infty)$

(4)

$f(x) = x^3 + 6x^2 + 6 \Rightarrow f'(x) = 3x^2 + 12x.$

$\Rightarrow f'(x) = 3x(x+4)$ $+$, $-$, $+$ at -4, 0

$\therefore f(x) \uparrow$ in $(-\infty, -4]$; $[0, \infty)$

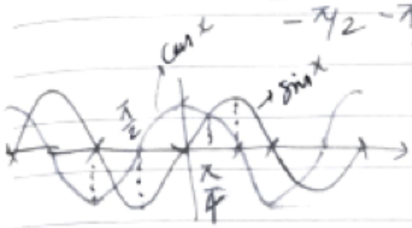
2. The function $f(x) = \tan^{-1}(\sin x + \cos x)$ is an increasing function in- **[AIEEE-2007]**

(1) $(\pi/4, \pi/2)$ (2) $(-\pi/2, \pi/4)$ (3) $(0, \pi/2)$ (4) $(-\pi/2, \pi/2)$

DATE

(2) $f(x) = \tan^{-1}(\sin x + \cos x) \Rightarrow f'(x) = \dfrac{\cos x - \sin x}{1+(\sin x + \cos x)^2}$

$-\pi/2$		$-\pi/4$		0		$\pi/4$		$\pi/2$
	+		+		+		−	



$\therefore f(x) \uparrow$ in $(-\pi/2, \pi/4)$.

3. **Statement 1 :** The function $x^2(e^x + e^{-x})$ is increasing for all $x > 0$. **[JEE-MAIN Online 2013]**

Statement 2 : The functions x^2e^x and x^2e^{-x} are increasing for all $x > 0$ and the sum of two increasing functions in any interval (a, b) is an increasing function in (a, b).

(1) Statement 1 is false; Statement 2 is true.

(2) Statement 1 is true; Statement 2 is false.

(3) Statement 1 is true; Statement 2 is true; Statement 2 is a correct explanation for Statement 1.

(4) Statement 1 is true ; Statement 2 is true; Statement 2 is not a correct explanation for Statement 1.

3) for $x > 0$: statement 2:

$h(x) = x^2 e^x \Rightarrow h'(x) = 2x e^x + x^2 e^x = e^x(x^2+2x)$

$\Rightarrow h'(x) = e^x . x(x+2) \longrightarrow$ +ve for $x > 0$.

$\Rightarrow h(x)$ is $\uparrow$.

$g(x) = x^2 e^{-x} \Rightarrow g'(x) = 2x e^{-x} - x^2 e^{-x}$

$\Rightarrow g'(x) = -x e^{-x}(x-2)$ — $-$ 0 $+$ 2 $-$

$\Rightarrow g(x)$ is non monotonic

$\therefore$ Statement 2 is false $\Rightarrow$ Ans: (2)

Note : We can verify Statement 1 is correct.

(4) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1.

4. If f and g are differentiable functions in [0, 1] satisfying $f(0) = 2 = g(1)$, $g(0) = 0$ and $f(1) = 6$, then for some $c \in]0, 1[$: **[JEE-MAIN 2014]**

(1) $2f'(c) = g'(c)$ (2) $2f'(c) = 3g'(c)$

(3) $f'(c) = g'(c)$ (4) $f'(c) = 2g'(c)$

Sol^n

Consider $h(x) = f(x) - 2g(x)$

$h(0) = 2$; $h(1) = 0$

$\therefore$ by RT, atleast once $h'(x) = 0$

$\therefore$ (4) is true

Similarly we can check other options

5. Let $f(x)=\frac{x}{\sqrt{a^2+x^2}}-\frac{d-x}{\sqrt{b^2+(d-x)^2}}$, $x\in R$, where a, b and d are non-zero real constants. Then :- **[JEE-MAIN 2019]**

(1) f is a decreasing function of x

(2) f is neither increasing nor decreasing function of x

(3) f' is not a continuous function of x

(4) f is an increasing function of x

Soln. $f'(x) = \dfrac{\sqrt{a^2+x^2} - \dfrac{x^2}{\sqrt{a^2+x^2}}}{(a^2+x^2)} - \left(\dfrac{-\sqrt{b^2+(d-x)^2} + \dfrac{(d-x)^2}{\sqrt{b^2+(d-x)^2}}}{\sqrt{b^2+(d-x)^2}^{\,2}}\right)$

$$= \frac{a^2}{(a^2+x^2)^{3/2}} + \frac{b^2}{(b^2+(d-x)^2)^{3/2}} > 0 \quad \forall\ x \in R.$$

$\therefore$ f(x) is increasing function $\forall\ x \in R$.

6. If the function f given by $f(x) = x^3 - 3(a-2)x^2 + 3ax + 7$, for some $a \in R$ is increasing in (0, 1] and decreasing in [1, 5), then a root of the equation, $\frac{f(x)-14}{(x-1)^2} = 0 (x \neq 1)$ is : **[JEE-MAIN 2019]**

(1) 6 (2) 5 (3) 7 (4) –7

6. $\frac{f(x)-14}{(x-1)^2} = 0 \Rightarrow f(x) = 14$

Also, $f(x) = x^3 - 3(a-2)x^2 + 3ax + 7$, $a \in R$

$f(x)$ is $\uparrow$ in $(0, 1]$ and $\downarrow$ in $[1, 5)$

$\Rightarrow f'(x) = 3(x^2 - 2(a-2)x + a)$

$\Rightarrow f'(x) \geq 0$ in $(0, 1]$ and $f'(x) \leq 0$ in $[1, 5)$

$\Rightarrow f'(x) = 0$ at $x = 1$:

$\Rightarrow x^2 - 2(a-2)x + a = 0$ at $x = 1$

$\Rightarrow 1 - 2(a-2) + a = 0 \Rightarrow 1 - 2a + 4 + a = 0$

$\Rightarrow$ $\Rightarrow a = 5$.

$\therefore f(x) = x^3 - 3 \times 3x^2 + 15x + 7$

$f(x) = x^3 - 9x^2 + 15x + 7$

$f(x) = 14 \Rightarrow x^3 - 9x^2 + 15x + 7 = 14$

$\Rightarrow x^3 - 9x^2 + 15x - 7 = 0$

$\Rightarrow (x-1)^2(x-7) = 0 \Rightarrow x = 1$ or 7

Ex. JA.

1. For the function $f(x) = x\cos\frac{1}{x},\ x \geq 1,$

(A) for at least one x in the interval $[1, \infty)$, $f(x+2) - f(x) < 2$

(B) $\lim_{x\to\infty} f'(x) = 1$

(C) for all x in the interval $[1, \infty)$, $f(x+2) - f(x) > 2$

(D) $f'(x)$ is strictly decreasing in the interval $[1, \infty)$ **[JEE 2009, 4M]**

Solⁿ.

$f(x) = x\cos\frac{1}{x}$

$f'(x) = \cos\frac{1}{x} + \frac{1}{x}\sin\frac{1}{x}.$

$\lim_{x\to\infty} f'(x) = \lim_{x\to\infty}\left(\cos\frac{1}{x} + \frac{1}{x}\sin\frac{1}{x}\right) = 1.$

$f''(x) = -\frac{\cos(1/x)}{x^3} < 0 \quad \forall\, x \in [1, \infty).$

$\therefore$ $f'(x)$ is decreasing $\forall\, x \in [1, \infty)$

using LMVT.

$\frac{f(x+2) - f(x)}{x+2-x} = f'(x) = \cos\frac{1}{x} + \frac{1}{x}\sin\frac{1}{x}$

$\because$ $f'(x)$ is decreasing $\Rightarrow f'(x) > \lim_{x\to\infty} f'(x).$

$\Rightarrow f(x+2) - f(x) > 2.$

2. Let f be a real-valued function defined on the interval $(0, \infty)$ by $f(x) = \ell nx + \int_0^x \sqrt{1+\sin t}\,dt$. Then which of the following statement(s) is/(are) true? **[JEE 10, 3M]**

(A) $f''(x)$ exists for all $x \in (0, \infty)$

(B) $f'(x)$ exists for all $x \in (0, \infty)$ and f' is continuous on $(0, \infty)$, but not differentiable on $(0, \infty)$

(C) there exists $\alpha > 1$ such that $|f'(x)| < |f(x)|$ for all $x \in (\alpha, \infty)$

(D) there exists $\beta > 0$ such that $|f(x)|+|f'(x)| \le \beta$ for all $x \in (0, \infty)$

Soln. $f'(x) = \frac{1}{x} + \sqrt{1+\sin x} = \frac{1}{x} + \left|\cos\frac{x}{2} + \sin\frac{x}{2}\right|$

$= \frac{1}{x} + \sqrt{2}\left|\sin(x/2 + \pi/4)\right|$.

$f''(x)$ does not exist for all x as $f'(x)$ is not differentiable.

Also $f'(x)$ is continous for all $x \in (0,\infty)$.

$\because$ $\ln x > \frac{1}{x}$ for some $x = \alpha$ $\forall$ $\alpha > 1$

& $\sqrt{1+\sin x} < \int_0^x \sqrt{1+\sin t}\,dt$ for some $x = \alpha$ $\forall \alpha > 1$

$\therefore$ $\frac{1}{x} + \sqrt{1+\sin x} < \ln x + \int_0^x \sqrt{1+\sin t}\,dt$

$\Rightarrow$ $|f'(x)| < |f(x)|$ for some α $\forall$ $\alpha > 1$.

3. Let $f : (0,1) \to R$ be defined by $f(x) = \frac{b-x}{1-bx}$, where b is a constant such that $0 < b < 1$. Then

(A) f is not invertible on (0,1)

(B) $f \neq f^{-1}$ on (0,1) and $f'(b) = \frac{1}{f'(0)}$

(C) $f = f^{-1}$ on (0,1) and $f'(b) = \frac{1}{f'(0)}$

(D) f^{-1} is differentiable on (0,1) **[JEE 2011, 4M]**

3. $x \in (0,1)$; $b \in (0,1)$; $f(x) = \frac{b-x}{1-bx}$

$\Rightarrow f'(x) = \frac{(1-bx)\cdot(-1) - (b-x)(-b)}{(1-bx)^2} = \frac{-1+bx+b^2-bx}{(1-bx)^2}$

$\Rightarrow f'(x) = \frac{b^2-1}{(1-bx)^2} \longrightarrow -ve$ $\because b \in (0,1)$. $\Rightarrow f(x)$ is $\downarrow$.

$\therefore$ Range of f: $(f(1), f(0)) \Rightarrow$ Range: $(-1, b) \neq$ Codomain

$\Rightarrow$ Non Invertible

4. The number of distinct real roots of $x^4 - 4x^3 + 12x^2 + x - 1 = 0$ is **[JEE 2011, 4M]**

4. Consider $f(x) = x^4 - 4x^3 + 12x^2 + x - 1$

$$f'(x) = 4x^3 - 12x^2 + 24x + 1$$

$$f''(x) = 12x^2 - 24x + 24$$
$$= 12\underbrace{(x^2 - 2x + 2)}_{+ve\ \forall x \in R}$$

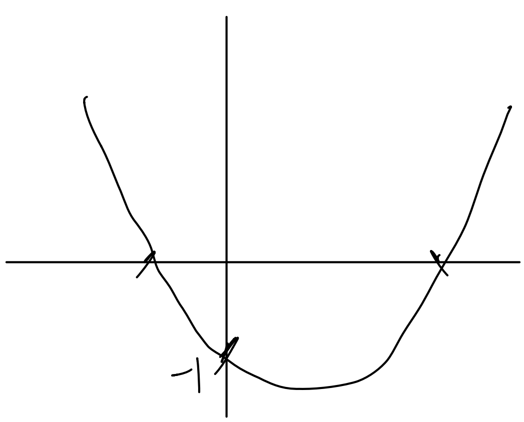
$\therefore$ $f(x)$ is open upwards $\forall x \in R$

$f(0) = -1$

Also $\lim_{x \to \infty} f(x) \to \infty$

$\lim_{x \to -\infty} f(x) \to \infty$

$\therefore$ Ex. 2 real roots



(Approx. graph)

Paragraph for Question 5 and 6

Let $f(x) = (1-x)^2\sin^2 x + x^2$ for all $x \in \mathbb{R}$, and let $g(x) = \int_1^x \left(\frac{2(t-1)}{t+1} - \ell n t\right) f(t)dt$ for all $x \in (1,\infty)$.

5. Consider the statements :

P : There exists some $x \in \mathbb{R}$ such that $f(x) + 2x = 2(1 + x^2)$

Q : There exists some $x \in \mathbb{R}$ such that $2f(x) + 1 = 2x(1 + x)$

Then **[JEE 2012, 3M, –1M]**

(A) both **P** and **Q** are true (B) **P** is true and **Q** is false

(C) **P** is false and **Q** is true (D) both **P** and **Q** are false

6. Which of the following is true ? **[JEE 2012, 3M, –1M]**

(A) g is increasing on $(1,\infty)$

(B) g is decreasing on $(1,\infty)$

(C) g is increasing on (1,2) and decreasing on $(2,\infty)$

(D) g is decreasing on (1,2) and increasing on $(2,\infty)$

P:

$(1-x)^2\sin^2 x + x^2 + 2x = 2 + 2x^2.$

$\Rightarrow \sin^2 x(1-x)^2 = x^2 - 2x + 2 = (1-x)^2 + 1$

$\Rightarrow \sin^2 x = \frac{(1-x)^2+1}{(1-x)^2} > 1 \quad \therefore$ No solution

$\Rightarrow$ P is false.

Q:

$2(1-x)^2\sin^2 x = 2x^2 + 2x - 1$

$\Rightarrow \sin^2 x = \frac{2x^2+2x-1}{2(1-x)^2}$

$0 \leq \frac{2x^2+2x-1}{2(1-x)^2} \leq 1$

Clearly we will get some values of x (eg $x = -2$)

$\therefore$ Q is true.

$\underline{\text{Sol}^n.}$ $$g'(x) = \left(\frac{2(x-1)}{x+1} - \ln x\right) f(x).$$

$$h(x) = \frac{2(x-1)}{x+1} - \ln x$$

$$= 2 - \frac{4}{x+1} - \ln x.$$

$$h'(x) = 2 + \frac{4}{(x+1)^2} - \frac{1}{x}.$$

$$= -\frac{(x-1)^2}{x(x+1)^2} < 0.$$

$\Rightarrow$ $h(x)$ is $\downarrow$.

$\Rightarrow$ $h(x) < h(1)$ $\forall\ x > 1$

$\Rightarrow$ $h(x) < 0$.

Also $f(x) > 0$ $\forall\ x > 1$

$\therefore$ $g'(x) < 0$ $\forall\ x > 1$

$\Rightarrow$ $g(x)$ is decreasing $\forall\ x > 1$

7. If $f(x)=\int_0^x e^{t^2}(t-2)(t-3)\,dt$ for all $x \in (0,\infty)$, then - **[JEE 2012, 4M]**

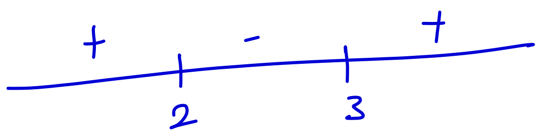
(A) f has a local maximum at $x = 2$

(B) f is decreasing on $(2,3)$

(C) there exists some $c \in (0,\infty)$ such that $f''(c) = 0$

(D) f has a local minimum at $x = 3$

<u>Solⁿ.</u> $f'(x) = e^{x^2}(x-2)(x-3)$.



$\therefore$ $f(x)$ has local max. at $x = 2$.

$f(x)$ is $\downarrow$ on $(2,3)$.

$f(x)$ has local minima at $x = 3$.

$f'(2) = f'(3) = 0$.

$\therefore$ $f''(c) = 0$ some $c \in (2,3)$.

8. The number of points in $(-\infty, \infty)$, for which $x^2 - x\sin x - \cos x = 0$, is **[JEE 2013, 2M]**

(A) 6 (B) 4 (C) 2 (D) 0

$x^2 - x\sin x - \cos x = 0$

$\Rightarrow$ Let $h(x) = x^2 - x\sin x - \cos x$

$\Rightarrow h'(x) = 2x - x\cos x - \sin x + \sin x.$

$h'(x) = x(2 - \cos x)$

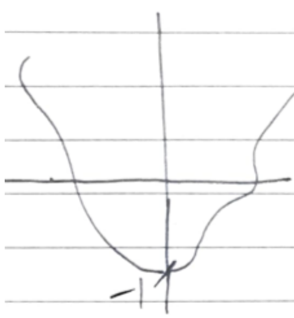
$h'(x)$: $-$, $+$; $\downarrow$ 0 $\uparrow$

$h(0) = -1.$

Also $h(x)$ is a continuous function and

$\lim_{x \to \infty} h(x) \to \infty$; $\lim_{x \to \infty} h(x) \to \infty$

$\therefore$ We can estimate a graph of $h(x)$ as:



Decreasing for $x < 0$

Increasing for $x > 0$.

-1 at $x = 0$.

As $x \to \infty$, $h(x) \to \infty$ and

As $x \to -\infty$, $h(x) \to \infty$

$\therefore$ Exactly 2 distinct roots.

8. Let $f(x) = x \sin \pi x$, $x > 0$. Then for all natural numbers n, $f'(x)$ vanishes at - **[JEE 2013, 4M, –1M]**

(A) a unique point in the interval $\left(n, n+\frac{1}{2}\right)$ (B) a unique point in the interval $\left(n+\frac{1}{2}, n+1\right)$

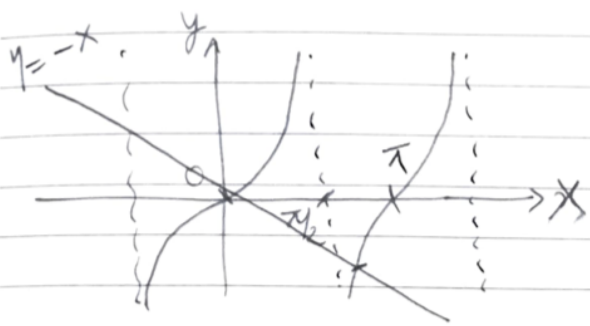
(C) a unique point in the interval $(n, n + 1)$ (D) two points in the interval $(n, n + 1)$

9. $f(x) = x \sin \pi x, \quad x > 0$

$\Rightarrow f'(x) = \pi x \cos \pi x + \sin \pi x = 0$

$\Rightarrow \tan \pi x = -\pi x.$

Let $\pi x = X \Rightarrow \tan X = -X.$



$\therefore$ Unique soln. in :

(i) $0 < X < \pi$

$\Rightarrow n\pi < X < n\pi + \pi$

$(\because \tan x$ has Period $\pi)$

$\Rightarrow n\pi < \pi x < (n+1)\pi$

$\Rightarrow n < x < n+1$ Ans(C)

Also unique soln. in $\pi/2 < X < \pi$

$\Rightarrow n\pi + \pi/2 < X < n\pi + \pi.$

$\Rightarrow \pi(n+1/2) < \pi x < \pi(n+1)$

$\Rightarrow n + 1/2 < x < n+1$ Ans (B)

10. Let $a \in \mathbb{R}$ and let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x^5 - 5x + a$. Then **[JEE(Advanced)-2014, 3]**

(A) $f(x)$ has three real roots if $a > 4$

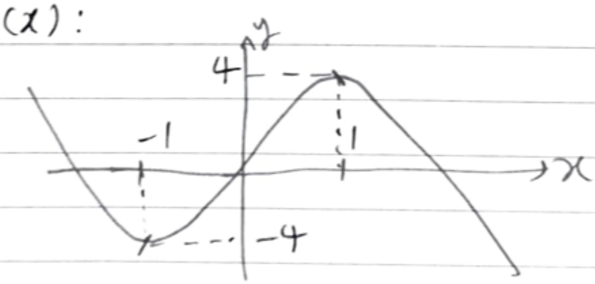
(B) $f(x)$ has only one real roots if $a > 4$

(C) $f(x)$ has three real roots if $a < -4$

(D) $f(x)$ has three real roots if $-4 < a < 4$

10. Consider $a = 5x - x^5 = g(x)$.

Graph of $g(x)$:



For only one root: $a > 4$ or $a < -4$.

For exactly three roots: $a \in (-4, 4)$

11. The total number of distinct $x \in [0, 1]$ for which $\int_0^x \frac{t^2}{1+t^4}\,dt = 2x - 1$ is

[JEE(Advanced)-2016, 3(0)]

Solⁿ.

$$g(x) = \left(\int_0^x \frac{t^2}{1+t^4}\,dt\right) - 2x + 1$$

$$g'(x) = \frac{x^2}{1+x^4} - 2 = \frac{1}{\left(x^2+1/x^2\right)} - 2 < 0.$$

$\therefore$ $g(x)$ is decreasing.

$$g(0) = 1.$$

$$g(1) = \int_0^1 \frac{dt}{\left(t^2+1/t^2\right)} - 1$$

$$\because \quad \frac{1}{\left(t^2+1/t^2\right)} < \frac{1}{2}. \quad \forall \quad t > 0.$$

$$\Rightarrow \quad \int_0^1 \frac{1}{\left(t^2+1/t^2\right)}\,dt < 1/2$$

$\Rightarrow$ $g(1)$ is $-ve$.

$\Rightarrow$ $g(x)$ will have only one root.

12. Let $f:\mathbb{R}\to(0,1)$ be a continuous function. Then, which of the following function(s) has(have) the value zero at some point in the interval (0, 1)? **[JEE(Advanced)-2017]**

(A) $e^x - \int_0^x f(t)\sin t\,dt$

(B) $x^9 - f(x)$

(C) $f(x) + \int_0^{\frac{\pi}{2}} f(t)\sin t\,dt$

(D) $x - \int_0^{\frac{\pi}{2}-x} f(t)\cos t\,dt$

Soln.

A) $g(x) = e^x - \int_0^x f(t)\sin t\,dt$

$g'(x) = e^x - f(x)\sin x > 0 \Rightarrow g(x)$ is $\uparrow$

$g(0) = 1.$

$\therefore\ g(x) > 1.$

B). $g(x) = x^9 - f(x).$

$g(0)\cdot g(1) = (-f(0))(1-f(1)) < 0.$

$\Rightarrow$ one root of $g(x)$ in $(0,1)$

C) $g(x) = f(x) + \int_0^{\pi/2} f(t)\sin(t)\,dt > 0$

D) $g(x) = x - \int_0^{\pi/2 - x} f(t)\cos t\,dt$

$g(0) = -\int_0^{\pi/2} f(t)\cos t\,dt < 0.$

$g(1) = 1 - \int_0^{\pi/2-1} f(t)\cos t\,dt > 0.$

$\Rightarrow$ $g(0)\cdot g(1) < 0 \Rightarrow$ one root of $g(x)$ in $(0,1)$

13. If $f : \mathbb{R} \to \mathbb{R}$ is a twice differentiable function such that $f''(x) > 0$ for all $x \in \mathbb{R}$, and $f\left(\frac{1}{2}\right) = \frac{1}{2}$, $f(1) = 1$ then **[JEE(Advanced)-2017]**

(A) $0 < f'(1) \le \frac{1}{2}$ (B) $f'(1) \le 0$ (C) $f'(1) > 1$ (D) $\frac{1}{2} < f'(1) \le 1$

13. $f(1/2) = 1/2$; $f(1) = 1$, $f''(x) > 0$.

Let $h(x) = f(x) - x$.

$h(1/2) = h(1) = 0. \Rightarrow$ by RT : $f'(\alpha) = 1$

for some $\alpha \in (1/2, 1)$

Also, $\because f''(x) > 0 \Rightarrow f'(x)$ is $\uparrow$.

$\Rightarrow f'(1) > f'(\alpha)$ $(\because \alpha \in (1/2, 1))$

$\Rightarrow f'(1) > 1$. Ans. (C)