Allen Career Institute Kota

Indefinite Integration

Sheet Solutions

(For Leader)

Do Yourself

Do yourself -1 :

(i) Evaluate : $\int \frac{x^2}{9+16x^6}dx$

Solution: Let $4x^3 = t$

$\Rightarrow 12x^2 dx = dt$

$$I = \int \frac{dt}{12(9+t^2)}$$

$$= \frac{1}{12} \times \frac{1}{3}\tan^{-1}\left(\frac{t}{3}\right) + C$$

$$= \boxed{\frac{1}{36}\tan^{-1}\left(\frac{4x^3}{3}\right) + C}$$

OR

Use $4x^3 = 3\tan\theta$

$\Rightarrow 12x^2 dx = 3\cdot\sec^2\theta . d\theta$

$$\therefore I = \frac{1}{12}\int \frac{3\sec^2\theta . d\theta}{9+9\tan^2\theta}$$

$$= \frac{1}{36}\int d\theta$$

$$= \frac{\theta}{36} + C$$

$$= \frac{1}{36}\tan^{-1}\left(\frac{4x^3}{3}\right) + C$$

(ii) Evaluate : $\int \cos^3 x\, dx$

Solution:

M-I

$$I = \int \cos^3 x\, dx = \int \cos^2 x \cdot \cos x\, dx$$

$$= \int (1-\sin^2 x)\cdot \cos x\, dx$$

Let $\sin x = t$

$\cos x\, dx = dt$

$$\therefore I = \int (1-t^2)\, dt$$

$$= t - \frac{t^3}{3} + C$$

$$= \boxed{\sin x - \frac{\sin^3 x}{3} + C}$$

M-II

$$I = \int \cos^3 x\, dx$$

$$\Rightarrow I = \frac{1}{4}\int 4\cos^3 x\, dx$$

$$= \frac{1}{4}\int (\cos 3x + 3\cos x)\, dx$$

$$\boxed{= \frac{1}{4}\left[\frac{\sin 3x}{3} + 3\sin x\right] + C}$$

$[\cos 3A = 4\cos^3 A - 3\cos A]$

Do yourself -2 :

(i) Evaluate : $\int \sqrt{\frac{x-3}{2-x}} dx$

Solution:

Domain: $2 \le x \le 3$

Substitution : $x = 2\cos^2\theta + 3\sin^2\theta$

$\Rightarrow dx = (-2\sin 2\theta + 3(\sin 2\theta))\, d\theta$

$\Rightarrow dx = \sin 2\theta \, d\theta = 2\sin\theta\cos\theta\, d\theta$

Substitution can be

$x = 2 + \sin^2\theta$

$\Rightarrow \theta = \sin^{-1}\sqrt{x-2}$

or $x = 3 - \cos^2\theta$

So, $I = \int \sqrt{\frac{x-3}{2-x}}\, dx$

$$\Rightarrow I = \int \sqrt{\frac{(2\cos^2\theta + 3\sin^2\theta - 3)}{(2 - (2\cos^2\theta + 3\sin^2\theta)}} \cdot 2\sin\theta\cos\theta\, d\theta$$

$$= \int \sqrt{\frac{-\cos^2\theta}{-\sin^2\theta}} \cdot 2\sin\theta\cos\theta\, d\theta$$

$$= \int 2\cos^2\theta\, d\theta = \int (1 + \cos 2\theta)\, d\theta$$

$$= \theta + \frac{\sin 2\theta}{2} + c = \theta + \sin\theta \cdot \cos\theta + C$$

$$= \sin^{-1}\sqrt{x-2} + \sqrt{x-2}\sqrt{3-x} + C$$

$$= \boxed{\sin^{-1}\sqrt{x-2} + \sqrt{(x-2)(3-x)} + C}$$

$$= \cos^{-1}\sqrt{3-x} + \sqrt{(x-2)(3-x)} + C$$

$$= \frac{\pi}{2} - \sin^{-1}\sqrt{3-x} + \sqrt{(x-2)(3-x)} + C$$

$$\boxed{= \sqrt{(x-2)(3-x)} - \sin^{-1}\sqrt{3-x} + C_1}$$

(ii) Evaluate : $\int \frac{dx}{x\sqrt{x^2+4}}$

Solution:

$$I = \int \frac{dx}{x\sqrt{x^2+4}}$$

Think : $\int \frac{1}{\sqrt{a^2+x^2}}\,dx$

Let $x = a\tan\theta$
$dx = a\sec^2\theta\, d\theta$

$$= \int \frac{a\sec^2\theta\, d\theta}{\sqrt{a^2+a^2\tan^2\theta}} = \int \sec\theta\, d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C$$

$$= \ln\left|\sqrt{1+\tan^2\theta} + \tan\theta\right| + C$$

$$= \ln\left|\sqrt{1+\frac{x^2}{a^2}} + \frac{x}{a}\right| + C$$

$$= \ln\left|\frac{\sqrt{a^2+x^2}+x}{a}\right| + C = \ln\left|x+\sqrt{a^2+x^2}\right| - \ln a + C$$

Remember : $\int \frac{1}{\sqrt{a^2+x^2}}\,dx = \ln\left|x+\sqrt{a^2+x^2}\right| + C$

M-I

$$I = \int \frac{dx}{x\cdot\sqrt{x^2}\sqrt{1+\frac{4}{x^2}}} = \int \frac{dx}{x^2\sqrt{1+\left(\frac{2}{x}\right)^2}} = -\int \frac{dt}{2\sqrt{1+t^2}}$$

Let $\frac{2}{x} = t \Rightarrow -\frac{2}{x^2}\,dx = dt$

$$= -\frac{1}{2}\ln\left|t+\sqrt{1+t^2}\right| + C$$

$$= \frac{-1}{2} \ln\left|\frac{2}{x} + \sqrt{1+\frac{4}{x^2}}\right| + c$$

$$= \frac{1}{2} \ln\left(\left|\frac{2+\sqrt{x^2+4}}{x}\right|\right)^{-1} + c$$

$$= \frac{1}{2} \ln\left|\frac{x}{2+\sqrt{x^2+4}}\right| + c$$

$$\boxed{= \frac{1}{2} \ln\left|\frac{\sqrt{x^2+4}-2}{x}\right| + C}$$

$\boxed{\text{M-II}}$

$$I = \int \frac{dx}{x\sqrt{x^2+4}}$$

Let $x = 2\tan\theta \Rightarrow dx = 2\sec^2\theta \, d\theta$

$$I = \int \frac{2\sec^2\theta \, d\theta}{2\tan\theta \cdot \sqrt{4\tan^2\theta+4}} = \int \frac{1}{2} \frac{\sec\theta}{\tan\theta} \, d\theta$$

$$= \frac{1}{2}\int \operatorname{cosec}\theta \, d\theta = \frac{1}{2} \ln\left|\operatorname{cosec}\theta - \cot\theta\right| + C$$

$$\boxed{= \frac{1}{2} \ln\left|\sqrt{1+\cot^2\theta} - \cot\theta\right| + c}$$

Do yourself -3 :

(i) Evaluate : $\int xe^x dx$

Solution: Integrating by Parts

$$\int x e^x dx = x\int e^x dx - \int\left(\frac{d}{dx}(x)\int e^x dx\right)dx$$

$$= xe^x - \int 1 \cdot e^x dx$$

$$= \boxed{xe^x - e^x + C}$$

(ii) Evaluate : $\int x^3 \sin(x^2)dx$

Solution: Let $x^2 = t$; $2x\,dx = dt$

$$I = \int x^2 \cdot \sin(x^2) \cdot x\,dx = \int t \cdot \sin t \,\frac{dt}{2}$$

$$= \frac{1}{2}\int t \cdot \sin t\, dt$$

Integrating by Parts, $I = \frac{1}{2}\left(t\int \sin t\,dt - \int\left(\frac{d}{dt}(t) \cdot \int \sin t\,dt\right)dt\right)$

$$= \frac{1}{2}\left(t(-\cos t) - \int 1 \cdot (-\cos t)\,dt\right)$$

$$= \frac{1}{2}\left(-t\cos t + \int \cos t\,dt\right)$$

$$= \frac{1}{2}\left(-t\cos t + \sin t\right) + C$$

$$\boxed{= \frac{1}{2}\left[-x^2\cos x^2 + \sin x^2\right] + C}$$

Do yourself - 4

(i) Evaluate : $\int e^x\left(\tan^{-1}x+\frac{1}{1+x^2}\right)dx$

Solution:

$$\boxed{\text{Remember: } \int e^x\left(f(x)+f'(x)\right)dx = e^x . f(x)+C}$$

Here, $f(x)=\tan^{-1}x$ & $f'(x)=\frac{1}{1+x^2}$

$$\therefore \boxed{I = e^x . \tan^{-1}x + C}$$

(ii) Evaluate : $\int xe^{x^2}\left(\sin x^2+\cos x^2\right)dx$

Solution: Let $x^2=t \Rightarrow x\,dx=\frac{dt}{2}$

$$\therefore I = \frac{1}{2}\int e^t(\sin t+\cos t)\,dt$$

Here, 'f' is '$\sin t$'

So, $I = \frac{1}{2}e^t . \sin t + C$

$$\boxed{= \frac{1}{2}e^{x^2} . \sin x^2 + C}$$

Do yourself -5 :

(i) Evaluate : $\int\left(\tan(e^x)+xe^x\sec^2(e^x)\right)dx$

Solution:

$$\boxed{\text{Remember: } \int\left(f(x)+x\,f'(x)\right)dx = x\,f(x)+C}$$

Here $f(x) = \tan(e^x)$

$f'(x) = \sec^2(e^x)\cdot e^x$

$$\therefore \boxed{I = x(\tan e^x) + C}$$

(ii) Evaluate : $\int(\ell nx+1)dx$

Solution:

Many times for $\int f(\ln x)\,dx$

Substitution $\ln x = t \Rightarrow x = e^t$ works

So, let $\boxed{\ln x = t} \Rightarrow \frac{1}{x}dx = dt \Rightarrow dx = x\,dt$

or $x = e^t$ $\Rightarrow \boxed{dx = e^t dt}$

$$\therefore I = \int (t+1)e^t dt = e^t\cdot t + C$$

$$\Rightarrow \boxed{I = x\cdot\ln x + C}$$

Do yourself -6 :

(i) Evaluate : $\int \frac{\sin^2 x}{\cos^4 x} dx$

Solution:

$$I = \int \frac{\sin^2 x}{\cos^2 x} \cdot \frac{1}{\cos^2 x} dx$$

$$= \int \tan^2 x \cdot \sec^2 x \, dx$$

Let $\tan x = t$

$$I = \int t^2 \, dt = \frac{t^3}{3} + C = \boxed{\frac{\tan^3 x}{3} + C}$$

(ii) Evaluate : $\int \frac{\sqrt{\sin x}\, dx}{\cos^{5/2} x}$

Solution: $\int \sin^{1/2} x \cdot \cos^{-5/2} x \, dx$

Observe : $\frac{1}{2} + \left(-\frac{5}{2}\right) = -2$ Mutiply & divide by $\sec^2 x$

$$I = \int \frac{\sin^{1/2} x}{\cos^{5/2} x} \cdot \frac{\sec^2 x}{\sec^2 x} dx = \int \frac{\sin^{1/2} x \cdot \sec^2 x \, dx}{\cos^{1/2} x}$$

$$= \int \sqrt{\tan x} \cdot \sec^2 x \, dx = \boxed{\frac{2}{3}(\tan x)^{3/2} + C}$$

Let $\tan x = t$; $\sec^2 x \, dx = dt$

(iii) Evaluate : $\int \sin^2 x \cos^5 x \, dx$

Solution: $I = \int \sin^2 x . \cos^5 x \, dx$

Power on $(\cos x)$ is odd, so,

$$I = \int \sin^2 x . \cos^4 x . \cos x \, dx$$

$$= \int \sin^2 x . (1 - \sin^2 x)^2 . \cos x \, dx$$

Let $\sin x = t$

$$\therefore \quad I = \int t^2 (1 - t^2)^2 \, dt = \int (t - t^3)^2 \, dt$$

$$= \int (t^2 - 2t^4 + t^6) \, dt$$

$$= \frac{t^3}{3} - \frac{2t^5}{5} + \frac{t^7}{7} + C$$

$$= \boxed{\frac{\sin^3 x}{3} - \frac{2\sin^5 x}{5} + \frac{\sin^7 x}{7} + C}$$

Do yourself -7 :

(i) Evaluate : $\int \frac{dx}{1+4\sin^2 x}$

Solution:

Denominator is homogeneous expression in $\sin x$ & $\cos x$ of two degree.
(Think $1 = \sin^2 x + \cos^2 x$)

(General Aproach) Multiply and Divide by $\sec^2 x$.

$$I = \int \frac{\sec^2 x}{(1+4\sin^2 x)\sec^2 x}\, dx$$

$$= \int \frac{\sec^2 x}{\sec^2 x + 4\tan^2 x}\, dx$$

$$= \int \frac{\sec^2 x}{1 + 5\tan^2 x}\, dx$$

Let $\sqrt{5}\tan x = t$

$$I = \int \frac{dt}{\sqrt{5}(1+t^2)} = \frac{1}{\sqrt{5}}\tan^{-1}(t) + C$$

$$= \boxed{\frac{1}{\sqrt{5}}\tan^{-1}(\sqrt{5}\tan x) + C}$$

(ii) Evaluate : $\int \frac{dx}{3\sin^2 x + \sin x \cos x + 1}$

Solution:

Denominator is homogeneous expression in $\sin x$ & $\cos x$ of two degree.
(Think $1 = \sin^2 x + \cos^2 x$)

(General Aproach) Multiply and Divide by $\sec^2 x$.

$$I = \int \frac{\sec^2 x \cdot dx}{(3\sin^2 x + \sin x \cdot \cos x + 1)\sec^2 x} = \int \frac{\sec^2 x \cdot dx}{3\tan^2 x + \tan x + \sec^2 x}$$

$$\Rightarrow I = \int \frac{\sec^2 x\, dx}{3\tan^2 x + \tan x + 1 + \tan^2 x} = \int \frac{\sec^2 x\, dx}{4\tan^2 x + \tan x + 1}$$

$$\Rightarrow I = \int \frac{\sec^2 x \quad dx}{(2\tan x)^2 + 2 \cdot 2\tan x \times \frac{1}{4} + \left(\frac{1}{4}\right)^2 + 1 - \left(\frac{1}{4}\right)^2} = \int \frac{\sec^2 x \cdot dx}{\left(2\tan x + \frac{1}{4}\right)^2 + \frac{15}{16}}$$

Let $2\tan x + \frac{1}{4} = t \Rightarrow \sec^2 x\, dx = \frac{dt}{2}$

$$\Rightarrow I = \int \frac{dt}{2\left(t^2 + \frac{15}{16}\right)} = \frac{1}{2} \times \frac{1}{\left(\frac{\sqrt{15}}{4}\right)} \tan^{-1}\left(\frac{t}{\sqrt{15}/4}\right) + C$$

$$\therefore \boxed{I = \frac{2}{\sqrt{15}} \tan^{-1}\left(\frac{8\tan x + 1}{\sqrt{15}}\right) + C}$$

Do yourself-8 :

(i) Evaluate : $\int \frac{dx}{3+\sin x}$

Solution:

$$I = \int \frac{dx}{3 + \sin x} = \int \frac{dx}{3 + \frac{2\tan\frac{x}{2}}{1+\tan^2\frac{x}{2}}}$$

$$= \int \frac{\left(1+\tan^2\frac{x}{2}\right)dx}{3 + 3\tan^2\frac{x}{2} + 2\tan\frac{x}{2}} = \int \frac{\sec^2\frac{x}{2}\,dx}{3\tan^2\frac{x}{2} + 2\tan\frac{x}{2} + 3}$$

Let $\tan\frac{x}{2} = t \Rightarrow \frac{1}{2}\sec^2\frac{x}{2}\,.dx = dt \Rightarrow \sec^2\frac{x}{2}\,dx = 2dt$

$$I = \int \frac{2dt}{3t^2 + 2t + 3} = \int \frac{2.dt}{3\left(t^2 + 2.t.\frac{1}{3} + \frac{1}{9} + 1 - \frac{1}{9}\right)}$$

$$= \int \frac{2\,dt}{3\left(\left(t+\frac{1}{3}\right)^2 + \frac{8}{9}\right)}$$

$$= \frac{2}{3} \times \frac{1}{2\sqrt{2}/3} \cdot \tan^{-1}\left(\frac{t+\frac{1}{3}}{2\sqrt{2}/3}\right) + C$$

$$= \boxed{\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{3\tan\frac{x}{2}+1}{2\sqrt{2}}\right) + C}$$

(ii) Evaluate : $\int \frac{dx}{1+4\sin x+3\cos x}$

Solution:

$$I = \int \frac{dx}{1 + \frac{4\cdot 2\tan(x/2)}{1+\tan^2(x/2)} + 3\left(\frac{1-\tan^2\left(\frac{x}{2}\right)}{1+\tan^2\left(\frac{x}{2}\right)}\right)}$$

$$= \int \frac{\sec^2\left(\frac{x}{2}\right) dx}{1+\tan^2\left(\frac{x}{2}\right) + 8\tan\frac{x}{2} + 3\left(1-\tan^2\left(\frac{x}{2}\right)\right)}$$

Let $\tan\frac{x}{2} = t$; $\frac{1}{2}\sec^2\frac{x}{2}\,dx = dt$

$$\therefore I = \int \frac{2\,dt}{1+t^2+8t+3-3t^2}$$

$$= \int \frac{2\,dt}{-2t^2+8t+4} = \int \frac{-dt}{t^2-4t-2}$$

$$= \int \frac{-dt}{(t-2)^2-6} = \frac{-1}{2\cdot\sqrt{6}} \ln\left|\frac{(t-2)-\sqrt{6}}{(t-2)+\sqrt{6}}\right| + C$$

$$\Rightarrow I = \frac{1}{2\sqrt{6}} \ln\left|\frac{t-2+\sqrt{6}}{t-2-\sqrt{6}}\right| + C$$

$$= \boxed{\frac{1}{2\sqrt{6}} \ln\left|\frac{\tan(x/2)-2+\sqrt{6}}{\tan(x/2)-2-\sqrt{6}}\right| + C}$$

Do yourself -9 :

(i) Evaluate : $\int \frac{\sin x}{\sin x + \cos x} dx$

Solution:

$$I = \int \frac{\sin x}{\sin x + \cos x} dx = \frac{1}{2}\int \frac{2\sin x}{\sin x + \cos x} dx$$

$$= \frac{1}{2}\int \frac{\sin x + \sin x}{\sin x + \cos x} dx$$

$$= \frac{1}{2}\int \frac{(\sin x + \cos x) + (\sin x - \cos x)}{(\sin x + \cos x)} dx$$

$$= \frac{1}{2}\int \left(\frac{\sin x + \cos x}{\sin x + \cos x} + \frac{\sin x - \cos x}{\sin x + \cos x}\right) dx$$

$$= \frac{1}{2}\left[\int 1\, dx + \int \frac{\sin x - \cos x}{\sin x + \cos x} dx\right]$$

Let $\sin x + \cos x = t \Rightarrow (\cos x - \sin x)\, dx = dt$

$$\therefore I = \frac{1}{2}\left[x + \int \frac{-dt}{t}\right]$$

$$= \frac{1}{2}\left[x - \ln|t|\right] + C = \boxed{\frac{1}{2}\left[x - \ln|\sin x + \cos x|\right] + C}$$

(ii) Evaluate : $\int \frac{3\sin x + 2\cos x}{3\cos x + 2\sin x}\,dx$

Solution:

$$I = \int \frac{3\sin x + 2\cos x}{3\cos x + 2\sin x}\,dx$$

Let, $3\sin x + 2\cos x = A(3\cos x + 2\sin x) + B\left(\frac{d}{dx}(3\cos x + 2\sin x)\right)$

$\Rightarrow 3\sin x + 2\cos x = A(3\cos x + 2\sin x) + B(-3\sin x + 2\cos x)$

$\Rightarrow \boxed{3 = 2A - 3B}$ and $\boxed{2 = 3A + 2B}$

Solving with get $A = \frac{12}{13}$ and $B = -\frac{5}{13}$

$$\therefore I = \int \frac{A f(x) + B f'(x)}{f(x)}\,dx$$

$$= A\int dx + B\int \frac{f'(x)}{f(x)}\,dx$$

$$= Ax + B\ln|f(x)| + C$$

$$= \boxed{\frac{12}{13}x - \frac{5}{13}\ln|3\cos x + 2\sin x| + C}$$

Do yourself - 10 :

(i) Evaluate : $\int \frac{3x+2}{(x+1)(x+3)} dx$

Solution: Say, $\frac{3x+2}{(x+1)(x+3)} = \frac{A}{x+1} + \frac{B}{x+3}$

$$\Rightarrow 3x+2 = A(x+3) + B(x+1)$$

Put $x = -3$; $-9+2 = A(0) + B(-2)$

$$\Rightarrow B = \frac{7}{2}$$

and Put $x = -1$; $-3+2 = A(2) + B(0)$

$$\Rightarrow A = -\frac{1}{2}$$

$$\therefore I = \int \left(\frac{A}{x+1} + \frac{B}{x+3} \right) dx$$

$$= A \ln|x+1| + B \ln|x+3| + C$$

$$= -\frac{1}{2} \ln|x+1| + \frac{7}{2} \ln|x+3| + C$$

(ii) Evaluate : $\int \frac{x^2-1}{(x+1)(x+2)^2}dx$

Solution: Say, $\frac{x^2-1}{(x+1)(x+2)^2} = \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}$

$\Rightarrow \quad x^2-1 = A(x+2)^2 + B(x+1)(x+2) + C(x+1)$

Put $x=-2 \Rightarrow 4-1 = A(0) + B(0) + C(-2+1)$

$\Rightarrow \boxed{C=-3}$

Put $x=-1 \Rightarrow 1-1 = A(-1+2)^2 + B(0) + C(0)$

$\Rightarrow \boxed{A=0}$

Put $x=0 \Rightarrow 0-1 = A(2)^2 + B(1)(2) + C(1)$

$\Rightarrow -1 = 4A + 2B + C$

$\Rightarrow -1 = 0 + 2B - 3$

$\Rightarrow \boxed{B=1}$

$\therefore I = \int\left(\frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{(x+2)^2}\right)dx = \int\left(\frac{1}{x+2} - \frac{3}{(x+2)^2}\right)dx$

$= \boxed{\ln|x+2| + \frac{3}{(x+2)} + C}$

Do yourself -11 :

(i) Evaluate : $\int \frac{dx}{x^2+x+1}$

Solution: $$I = \int \frac{dx}{\left(x+\frac{1}{2}\right)^2 + \frac{3}{4}} = \frac{1}{\sqrt{3}/2} \tan^{-1}\left(\frac{x+\frac{1}{2}}{\sqrt{3}/2}\right) + C$$

$$= \boxed{\frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{2x+1}{\sqrt{3}}\right) + C}$$

(ii) Evaluate : $\int \frac{5x+4}{\sqrt{x^2+4x+1}}\,dx$

Solution: Let $5x+4 = A\frac{d}{dx}(x^2+4x+1) + B \Rightarrow 5x+4 = A(2x+4) + B$

So, $5 = 2A$ and $4 = 4A + B \Rightarrow A = \frac{5}{2}$ and $B = -6$

$$I = \int \frac{A(2x+4) + B}{\sqrt{x^2+4x+1}}\,dx = A\int \frac{dt}{\sqrt{t}} + B\int \frac{dx}{\sqrt{(x+2)^2 - 3}}$$

$$\Rightarrow I = A\,\frac{t^{1/2}}{1/2} + B \ln\left|x+2+\sqrt{(x+2)^2-3}\right| + C$$

$$I = \boxed{5\sqrt{x^2+4x+1} - 6\ln\left|x+2+\sqrt{x^2+4x+1}\right| + C}$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left|x+\sqrt{x^2-a^2}\right| + C$$

Let $x = a\sec\theta$

$$I = \int \frac{a\sec\theta\tan\theta\,d\theta}{a\tan\theta} = \int \sec\theta\,d\theta$$

$$= \ln|\sec\theta + \tan\theta| + C$$

Do yourself -12 :

(i) Evaluate : $\int \frac{x^2+1}{x^4-x^2+1}dx$

Solution: $I = \int \frac{x^2+1}{x^4-x^2+1}dx$

$\Rightarrow I = \int \frac{1+\frac{1}{x^2}}{x^2-1+\frac{1}{x^2}}dx = \int \frac{dt}{t^2+1}$

Let $x-\frac{1}{x}=t \Rightarrow \left(1+\frac{1}{x^2}\right)dx = dt$ and $x^2+\frac{1}{x^2}=t^2+2$

$= \tan^{-1}(t)+C = \tan^{-1}\left(x-\frac{1}{x}\right)+C$

(ii) Evaluate : $\int \frac{1}{1+x^4}dx$

Solution: $I = \int \frac{1}{1+x^4}dx = \frac{1}{2}\int \frac{1+x^2+1-x^2}{1+x^4}dx$

$\Rightarrow I = \frac{1}{2}\left(\int \frac{1+x^2}{1+x^4}dx + \int \frac{1-x^2}{1+x^4}dx\right)$

$\Rightarrow I = \frac{1}{2}(I_1 + I_2)$

Now, $I_1 = \int \frac{1+x^2}{1+x^4}\,dx = \int \frac{\frac{1}{x^2}+1}{\frac{1}{x^2}+x^2}\,dx$

Let $x - \frac{1}{x} = t \Rightarrow \left(1+\frac{1}{x^2}\right)dx = dt$; $\left(x-\frac{1}{x}\right)^2 = t^2 \Rightarrow x^2 + \frac{1}{x^2} = t^2+2$

$$I_1 = \int \frac{dt}{t^2+2} = \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t}{\sqrt{2}}\right) + C_1$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\left(x-\frac{1}{x}\right)}{\sqrt{2}}\right) + C_1$$

$$= \boxed{\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{x^2-1}{\sqrt{2}x}\right) + C_1}$$

and $I_2 = \int \frac{1-x^2}{1+x^4}\,dx = \int \frac{\frac{1}{x^2}-1}{\frac{1}{x^2}+x^2}\,dx$

Let $x + \frac{1}{x} = \alpha \Rightarrow \left(1-\frac{1}{x^2}\right)dx = d\alpha$; $x^2 + \frac{1}{x^2} = \alpha^2 - 2$

$$I_2 = \int \frac{-d\alpha}{\alpha^2-2} = -\frac{1}{2\sqrt{2}}\ln\left|\frac{\alpha-\sqrt{2}}{\alpha+\sqrt{2}}\right| + C_2$$

$$= -\frac{1}{2\sqrt{2}}\ln\left|\frac{x+\frac{1}{x}-\sqrt{2}}{x+\frac{1}{x}+\sqrt{2}}\right| + C_2 = \frac{-1}{2\sqrt{2}}\ln\left|\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right| + C_2$$

$$\therefore I = \frac{1}{2}(I_1+I_2) = \boxed{\frac{1}{2\sqrt{2}}\left(\tan^{-1}\left(\frac{x^2-1}{\sqrt{2}x}\right) - \frac{1}{2}\ln\left|\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right|\right) + C}$$

Do yourself -13 :

(i) Evaluate : $\int \frac{x}{(x-3)\sqrt{x+1}}dx$

$\int \frac{dx}{(ax+b)\sqrt{px+q}}$ & $\int \frac{dx}{(ax^2+bx+c)\sqrt{px+q}}$; put $px+q=t^2$

$\int \frac{dx}{(ax+b)\sqrt{px^2+qx+r}}$, put $ax+b=\frac{1}{t}$; $\int \frac{dx}{(ax^2+b)\sqrt{px^2+q}}$, put $x=\frac{1}{t}$

Solution: Let $x+1=t^2$; $dx=2t\,dt$

$$\therefore I = \int \frac{(t^2-1)\,2t\,dt}{(t^2-1-3)\sqrt{t^2}}$$

$$= \int \frac{2(t^2-1)\,dt}{(t^2-4)} = 2\int \frac{(t^2-4+3)}{t^2-4}dt$$

$$= 2\int \left(1+\frac{3}{t^2-4}\right)dt$$

$$= 2\left(t+\frac{3}{2\cdot 2}\ln\left|\frac{t-2}{t+2}\right|\right)+C$$

$$= \boxed{2\sqrt{x+1}+\frac{3}{2}\ln\left|\frac{\sqrt{x+1}-2}{\sqrt{x+1}+2}\right|+C}$$

(ii) Evaluate : $\int \frac{dx}{x^2\sqrt{1+x^2}}$

<u>Solution:</u> Let $x = \frac{1}{t}$; $dx = \frac{-1}{t^2}dt$

$$I = \int \frac{\frac{-1}{t^2}dt}{\frac{1}{t^2}\sqrt{1+\frac{1}{t^2}}} = \int \frac{-t \cdot dt}{\sqrt{t^2+1}}$$

Let $t^2+1 = u^2 \Rightarrow t\,dt = u\,du$

$$\therefore I = \int \frac{-u\,du}{u} = -u + C$$

$$= -\sqrt{t^2+1} + C = -\sqrt{\frac{1}{x^2}+1} + C$$

$$\boxed{I = -\sqrt{\frac{1+x^2}{x^2}} + C}$$

Do yourself -14 :

(i) Evaluate : $\int \frac{dx}{x(x^2+1)}$

Solution: $I = \int \frac{dx}{x^3\left(1+\frac{1}{x^2}\right)}$ (Take x^2 common)

let $1+\frac{1}{x^2} = t \Rightarrow \frac{-2}{x^3}dx = dt$

$$I = \int \frac{dt}{-2\cdot t} = -\frac{1}{2}\ln|t| + C$$

$$= \boxed{\frac{-1}{2}\ln\left|1+\frac{1}{x^2}\right| + C}$$

(ii) Evaluate : $\int \frac{dx}{x^2(x^3+1)^{2/3}}$

Solution: Take x^3 Common

$$I = \int \frac{dx}{x^2\left(x^3\left(1+\frac{1}{x^3}\right)\right)^{2/3}} = \int \frac{dx}{x^4\left(1+\frac{1}{x^3}\right)^{2/3}} = \int \frac{t^2\,dt}{(t^3)^{2/3}}$$

Let $1+\frac{1}{x^3} = t^3 \Rightarrow \frac{-3}{x^4}dx = 3t^2\,dt \Rightarrow \frac{dx}{x^4} = -t^2\,dt$

$$I = -\int dt = -t + C = \boxed{-\left(1+\frac{1}{x^3}\right)^{1/3} + C}$$

(iii) Evaluate : $\int \frac{dx}{x^3(x^3+1)^{1/3}}$

Solution: Take x^3 common,

$$I = \int \frac{dx}{x^3\left(x^3\left(1+\frac{1}{x^3}\right)\right)^{1/3}} = \int \frac{dx}{x^3 \cdot (x^3)^{1/3}\left(1+\frac{1}{x^3}\right)^{1/3}}$$

$$= \int \frac{dx}{x^4\left(1+\frac{1}{x^3}\right)^{1/3}}$$

Let $1+\frac{1}{x^3} = t^3 \Rightarrow \frac{-3}{x^4}dx = 3t^2 dt \Rightarrow -\frac{dx}{x^4} = t^2 dt$

$$\therefore I = \int \frac{-t^2 dt}{(t^3)^{1/3}} = -\int t\,dt = -\frac{t^2}{2} + C$$

$$= -\frac{1}{2}\left(\left(1+\frac{1}{x^3}\right)^{1/3}\right)^2 + C$$

$$= \boxed{-\frac{1}{2}\left(1+\frac{1}{x^3}\right)^{2/3} + C}$$

EXERCISE (O-1)

1. $\int \frac{1-x^7}{x(1+x^7)} dx$ equals -

(A) $\ln x + \frac{2}{7} \ln(1+x^7) + c$

(B) $\ln x - \frac{2}{7} \ln(1-x^7) + c$

(C) $\ln x - \frac{2}{7} \ln(1+x^7) + c$

(D) $\ln x + \frac{2}{7} \ln(1-x^7) + c$

Sol:- Let $I = \int \frac{1-x^7}{1+x^7} \cdot \frac{dx}{x}$

$$\left\{ \begin{aligned} &\text{Put } x^7 = t \\ &\Rightarrow 7 \cdot x^6 \cdot dx = dt \\ &\Rightarrow 7x^7 \cdot dx = x \cdot dt \\ &\Rightarrow 7t\, dx = x \cdot dt \\ &\Rightarrow \frac{dx}{x} = \frac{dt}{7t} \end{aligned} \right\}$$

$$\Rightarrow I = \int \frac{1-t}{1+t} \cdot \frac{dt}{7t}$$

$$\Rightarrow I = \frac{1}{7} \int \frac{1-t}{t(1+t)} \cdot dt$$

$$\Rightarrow I = \frac{1}{7} \int \frac{(1+t) - 2t}{t(1+t)} \cdot dt$$

$$\Rightarrow I = \frac{1}{7} \int \frac{1}{t} - \frac{2}{1+t} \cdot dt$$

$$\Rightarrow I = \frac{1}{7} \left[\ln(t) - 2\ln(1+t) \right] + C$$

$$\Rightarrow I = \frac{1}{7} \ln(x^7) - \frac{2}{7} \ln(1+x^7) + C$$

$$\Rightarrow I = \boxed{\ln(x) - \frac{2}{7} \ln(1+x^7) + C}$$

Ans.

2 Primitive of $\frac{3x^4-1}{(x^4+x+1)^2}$ w.r.t. x is -

(A) $\frac{x}{x^4+x+1}+c$ (B) $-\frac{x}{x^4+x+1}+c$ (C) $\frac{x+1}{x^4+x+1}+c$ (D) $-\frac{x+1}{x^4+x+1}+c$

Sol:-

$$\because \int \frac{3x^4-1}{(x^4+x+1)^2} \cdot dx$$

$$= \int \frac{3x^4-1}{x^2\left(x^3+1+\frac{1}{x}\right)^2} \cdot dx$$

$$= \int \frac{3x^2-x^{-2}}{(x^3+1+x^{-1})^2} \cdot dx$$

$$\left\{ \begin{array}{l} \text{Put } x^3+1+x^{-1}=t \\ \Rightarrow (3x^2-x^{-2}).dx = dt \end{array} \right\}$$

$$= \int \frac{1}{t^2} \cdot dt$$

$$= -\frac{1}{t} + C$$

$$= \frac{-1}{x^3+1+x^{-1}} + C$$

$$= \boxed{\frac{-x}{x^4+x+1} + C}$$

Ans.

Integral of $\sqrt{1+2\cot x(\cot x+\operatorname{cosec}x)}$ w.r.t. x is

3 (A) $2\ln\cos\frac{x}{2}+c$ (B) $2\ln\sin\frac{x}{2}+c$

(C) $\frac{1}{2}\ln\cos\frac{x}{2}+c$ (D) $\ln\sin x-\ln(\operatorname{cosec}x-\cot x)+c$

Sol:- $\because \int\sqrt{1+2\cot^2(x)+2\cot x.\operatorname{cosec}x}\,.dx$

$$=\int\sqrt{1+\cot^2 x+\cot^2 x+2\cot x.\operatorname{cosec}x}\,.dx$$

$$=\int\sqrt{\operatorname{cosec}^2 x+\cot^2 x+2\cot x.\operatorname{cosec}x}\,.dx$$

$$=\int\sqrt{(\operatorname{cosec}x+\cot x)^2}\,.dx$$

$$=\int \operatorname{cosec}x+\cot x=\int\frac{1+\cos x}{\sin x}\,.dx$$

$$=\int\frac{2\cos^2(x/2)}{2\sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)}\,.dx$$

$$=\int\cot\left(\frac{x}{2}\right).dx$$

$$=\boxed{2\ln\left|\sin\left(\frac{x}{2}\right)\right|+C}$$

Ans.

4 $\int x \cdot \dfrac{\ln\left(x+\sqrt{1+x^2}\right)}{\sqrt{1+x^2}}dx$ equals -

(A) $\sqrt{1+x^2}\ \ln\left(x+\sqrt{1+x^2}\right) - x + c$

(B) $\dfrac{x}{2} \cdot \ln^2\left(x+\sqrt{1+x^2}\right) - \dfrac{x}{\sqrt{1+x^2}} + c$

(C) $\dfrac{x}{2} \cdot \ln^2\left(x+\sqrt{1+x^2}\right) + \dfrac{x}{\sqrt{1+x^2}} + c$

(D) $\sqrt{1+x^2}\ \ln\left(x+\sqrt{1+x^2}\right) + x + c$

Sol:- let $I = \displaystyle\int \frac{x \cdot \ln\left[\sqrt{1+x^2}+x\right]}{\sqrt{1+x^2}} \cdot dx$

Put $\ln\left[\sqrt{1+x^2}+x\right] = t$

$\Rightarrow \sqrt{1+x^2}+x = e^t$

$\Rightarrow 1+x^2 = (e^t - x)^2$

$\Rightarrow 1 = e^{2t} - 2xe^t$

$\Rightarrow x = \dfrac{e^t - e^{-t}}{2}$

$\Rightarrow \boxed{dx = \dfrac{e^t + e^{-t}}{2} \cdot dt}$

Now,

$\because \sqrt{1+x^2}+x = e^t$ —— 1

& $\sqrt{1+x^2}-x = e^{-t}$ —— 2

$\therefore$ 1 + 2 $\Rightarrow \boxed{\sqrt{1+x^2} = \dfrac{e^t + e^{-t}}{2}}$

1 − 2 $\Rightarrow \boxed{x = \dfrac{e^t - e^{-t}}{2}}$

$$\therefore\ I = \int \frac{\left(\frac{e^t - e^{-t}}{2}\right) \cdot t \cdot \left(\frac{e^t + e^{-t}}{2}\right) \cdot dt}{\left(\frac{e^t + e^{-t}}{2}\right)}$$

④ Continues..

$$\Rightarrow I = \int t \cdot \left(\frac{e^{t} - e^{-t}}{2}\right) \cdot dt$$

I   II   use by part

$$\Rightarrow I = t \cdot \left(\frac{e^{t} + e^{-t}}{2}\right) - \int 1 \cdot \left(\frac{e^{t} + e^{-t}}{2}\right) \cdot dt$$

$$\Rightarrow I = t\left(\frac{e^{t} + e^{-t}}{2}\right) - \left(\frac{e^{t} - e^{-t}}{2}\right) + C$$

$$\Rightarrow I = t\sqrt{1+x^2} - x + C$$

$$\Rightarrow \boxed{I = \sqrt{1+x^2} \cdot \ln\left[\sqrt{1+x^2} + x\right] - x + C}$$

Ans.

5. A function y = f (x) satisfies $f''(x) = -\frac{1}{x^2} - \pi^2 \sin(\pi x)$; $f'(2) = \pi + \frac{1}{2}$ and $f(1) = 0$. The value of $f\left(\frac{1}{2}\right)$ is

(A) $\ln 2$ (B) 1 (C) $\frac{\pi}{2} - \ln 2$ (D) $1 - \ln 2$

Sol:- $\because f''(x) = -\frac{1}{x^2} - \pi^2 \sin(\pi x)$

$$\Rightarrow f'(x) = \int -\frac{1}{x^2} - \pi^2 \sin(\pi x) \, . \, dx$$

$$\Rightarrow f'(x) = \frac{1}{x} + \pi^2 . \frac{\cos(\pi x)}{\pi} + C$$

$$\left\{ \begin{array}{l} \because f'(2) = \pi + \frac{1}{2} \\ \frac{1}{2} + \pi \cos(2\pi) + C = \pi + \frac{1}{2} \\ \Rightarrow C = 0 \end{array} \right\}$$

$$\Rightarrow f'(x) = \frac{1}{x} + \pi \cos(\pi x)$$

$$\Rightarrow f(x) = \int \frac{1}{x} + \pi \cos(\pi x) \, . \, dx$$

$$\Rightarrow f(x) = \ln(x) + \frac{\pi \sin(\pi x)}{\pi} + C_1$$

$$\left\{ \begin{array}{l} \because f(1) = 0 \\ \Rightarrow \sin \pi + C_1 = 0 \\ \Rightarrow C_1 = 0 \end{array} \right\}$$

$$\Rightarrow f(x) = \ln(x) + \sin(\pi x)$$

$$\therefore f\left(\frac{1}{2}\right) = \ln\left(\frac{1}{2}\right) + \sin\left(\frac{\pi}{2}\right) = \boxed{1 - \ln(2)}$$

Ans.

6

Consider $f(x)=\frac{x^2}{1+x^3}$; $g(t)=\int f(t)dt$. If $g(1)=0$ then $g(x)$ equals-

(A) $\frac{1}{3}\ell n(1+x^3)$ (B) $\frac{1}{3}\ell n\left(\frac{1+x^3}{2}\right)$ (C) $\frac{1}{2}\ell n\left(\frac{1+x^3}{3}\right)$ (D) $\frac{1}{3}\ell n\left(\frac{1+x^3}{3}\right)$

Sol:-

$$f(x)=\frac{x^2}{1+x^3} \; ; \; g(1)=0 \text{ given}$$

$$\because g(t)=\int f(t)\,.dt$$

$$'' = \int \frac{t^2}{1+t^3}\,.dt$$

$$\Rightarrow g(t)=\int \frac{1}{u}\cdot\frac{du}{3}$$

$$\Rightarrow '' = \frac{1}{3}\ln(u)$$

$$\Rightarrow g(t)=\frac{1}{3}\ln(1+t^3)+c$$

$$\left\{\begin{array}{l}\because g(1)=0 \\ \Rightarrow c=-\frac{1}{3}\ln(2)\end{array}\right\}$$

$$\Rightarrow g(x)=\frac{1}{3}\ln(1+x^3)-\frac{1}{3}\ln(2)$$

$$\Rightarrow \boxed{g(x)=\frac{1}{3}\ln\left(\frac{1+x^3}{2}\right)}$$

Ans.

⑦ $\int \frac{e^{\sqrt{x}}}{\sqrt{x}}\left(x+\sqrt{x}\right)dx$

(A) $2e^{\sqrt{x}}\left[x-\sqrt{x}+1\right]+C$

(B) $e^{\sqrt{x}}\left[x-2\sqrt{x}+1\right]$

(C) $e^{\sqrt{x}}\left(x+\sqrt{x}\right)+C$

(D) $e^{\sqrt{x}}\left(x+\sqrt{x}+1\right)+C$

Sol:- $\because \int \frac{e^{\sqrt{x}}}{\sqrt{x}}(x+\sqrt{x}).dx$

$\left\{ \begin{array}{l} \text{Put } x = t^2 \\ \Rightarrow dx = 2t.dt \end{array} \right\}$

$$= \int \frac{e^t}{t}(t^2+t).2t\,dt$$

$$= \int e^t(2t^2+2t).dt = \int e^t\left[(2t^2+4t)-2t\right].dt$$

$$= \int e^t(\underbrace{2t^2}_{f(x)}+\underbrace{4t}_{f'(x)})dt - 2\int \underset{\mathrm{I}}{t}.\underset{\mathrm{II}}{e^t}.dt$$

$$= e^t.2t^2 - 2\left[t.e^t - \int 1.e^t.dt\right]$$

$$= 2.e^t.t^2 - 2\left[e^t.t - e^t\right] + C$$

$$= 2e^t.t^2 - 2e^t.t + 2e^t + C$$

$$= e^t\left[2t^2 - 2t + 2\right] + C$$

$$= \boxed{2e^{\sqrt{x}}\left[x-\sqrt{x}+1\right]+C}$$

Ans.

⑧ $\int \frac{dx}{\sqrt[3]{x^{5/2}(x+1)^{7/2}}}$

(A) $-\left(\frac{x+1}{x}\right)^{1/6}+C$ (B) $6\left(\frac{x+1}{x}\right)^{-1/6}+C$ (C) $\left(\frac{x}{x+1}\right)^{5/6}+C$ (D) $-\left(\frac{x}{x+1}\right)^{5/6}+C$

Sol:- $\because \int \frac{dx}{\sqrt[3]{x^{5/2}\cdot(x+1)^{7/2}}} = \int \frac{dx}{\sqrt[3]{x^{5/2}\cdot x^{7/2}\cdot\left(1+\frac{1}{x}\right)^{7/2}}}$

$$= \int \frac{x^{-2}}{\left(1+\frac{1}{x}\right)^{7/6}}\cdot dx$$

$\left\{ \begin{array}{l} \text{Put } 1+\frac{1}{x}=t \\ \Rightarrow x^{-2}\cdot dx = dt \end{array} \right\}$

$$= \int \frac{dt}{t^{7/6}}$$

$$= 6\cdot\frac{1}{t^{1/6}} + C$$

$$= \boxed{6\left(\frac{x+1}{x}\right)^{-\frac{1}{6}} + C}$$

Ans.

9. $\int e^x\left(\frac{x^2-3}{(x-1)^2}\right)dx$ is equal to -

(A) $e^x\frac{(x+3)}{(x-1)}+C$ (B) $e^x\left(\frac{x-3}{x-1}\right)+C$ (C) $e^x\left(\frac{x+1}{x-1}\right)+C$ (D) $e^x\left(\frac{1}{x-1}\right)^2+C$

(where C is constant of integration)

Sol:-

$$\because \int e^x\left(\frac{x^2-3}{(x-1)^2}\right)\cdot dx$$

$$= \int e^x\left[\frac{x^2-1}{(x-1)^2}-\frac{2}{(x-1)^2}\right]\cdot dx$$

$$= \int e^x\left[\frac{x+1}{x-1}+\frac{-2}{(x-1)^2}\right]\cdot dx$$

$$\left\{\because \int e^x[f(x)+f'(x)]\,dx = e^x\cdot f(x)+C\right\}$$

$$= \boxed{e^x\cdot\left(\frac{x+1}{x-1}\right)+C}$$

Ans.

10 $\int \frac{x^3}{(2x^2+1)^3}dx$ is equal to-

(A) $\frac{1}{4}\left(2+\frac{1}{x^2}\right)^{-2}+C$

(B) $-\frac{1}{4}\left(2+\frac{1}{x^2}\right)^{-2}+C$

(C) $\frac{1}{2}\left(2+\frac{1}{x^2}\right)^{-2}+C$

(D) $\frac{1}{4}\left(2+\frac{1}{x^2}\right)^{2}+C$

(where 'C' is integration constant)

Sol:- $\because \int \frac{x^3}{(2x^2+1)^3} \cdot dx$

$$= \int \frac{x^3}{x^6(2+x^{-2})^3} \cdot dx$$

$$= \int \frac{x^{-3} \cdot dx}{(2+x^{-2})^3}$$

$$\left\{ \begin{array}{l} \text{Put } 2+x^{-2} = t \\ \Rightarrow x^{-3} \cdot dx = \frac{dt}{-2} \end{array} \right\}$$

$$= \frac{-1}{2}\int \frac{dt}{t^3}$$

$$= -\frac{1}{2}\left(\frac{-1}{2} \cdot \frac{1}{t^2}\right) + C$$

$$= \boxed{\frac{1}{4}\left(2+\frac{1}{x^2}\right)^{-2} + C}$$

Ans.

EXERCISE (O-2)

1. $\int \left(\sin(101x)\cdot\sin^{99}x\right)dx$ equals

(A) $\dfrac{\sin(100x)(\sin x)^{100}}{100}+C$ (B) $\dfrac{\cos(100x)(\sin x)^{100}}{100}+C$

(C) $\dfrac{\cos(100x)(\cos x)^{100}}{100}+C$ (D) $\dfrac{\sin(100x)(\sin x)^{101}}{101}+C$

$$\sin(100x+x)\sin^{99}x$$

$$= \sin(100x)\cdot\cos x\cdot\sin^{99}x + \cos(100x)\cdot(\sin x)^{100}$$

$$I = \int \underbrace{\sin(100x)}_{I}\cdot\underbrace{\cos x\cdot\sin^{99}x}_{II}\,dx + \int \cos(100x)\cdot(\sin x)^{100}\,dx$$

Use I.B.P. on first integration

$$I = \frac{(\sin x)^{100}}{100}\cdot\sin(100x) - \cancel{\int(\sin x)^{100}\cos(100x)\,dx} + \cancel{\int\cos(100x)(\sin x)^{100}\,dx}$$

$$= \frac{\sin(100x)(\sin x)^{100}}{100}+C$$

(A)

2

The integral $\int \sqrt{\cot x}\, e^{\sqrt{\sin x}} \sqrt{\cos x}\, dx$ equals

(A) $\dfrac{\sqrt{\tan x}\, e^{\sqrt{\sin x}}}{\sqrt{\cos x}} + C$ (B) $2e^{\sqrt{\sin x}} + C$ (C) $-\dfrac{1}{2}e^{\sqrt{\sin x}} + C$ (D) $\dfrac{\sqrt{\cot x}\, e^{\sqrt{\sin x}}}{2\sqrt{\cos x}} + C$

$$I = 2\int \frac{e^{\sqrt{\sin x}} \cos x}{2\sqrt{\sin x}}\, dx$$

put $\sqrt{\sin x} = t$

$$\Rightarrow \frac{1}{2\sqrt{\sin x}} \times \cos x\, dx = dt$$

$$I = 2\int e^t\, dt = 2e^{\sqrt{\sin x}} + c$$

B

Multiple Correct :

3 Which one of the following is **FALSE** ?

(A) $x.\int \frac{dx}{x} = x\ln|x| + C$

(B) $x.\int \frac{dx}{x} = x\ln|x| + Cx$

(C) $\frac{1}{\cos x}.\int \cos x \; dx = \tan x + C$

(D) $\frac{1}{\cos x}.\int \cos x \; dx = x + C$

$$x \cdot \int \frac{dx}{x} = x \cdot \ln|x| + cx$$

$$\frac{1}{\cos x} \int \cos x \, dx$$

$$= \frac{1}{\cos x}\left[(\sin x) + c\right]$$

Ans: A, C, D

4 Let $f(x) = \sin^3 x + \sin^3\left(x + \frac{2\pi}{3}\right) + \sin^3\left(x + \frac{4\pi}{3}\right)$ then the primitive of $f(x)$ w.r.t. x is

(A) $-\frac{3\sin 3x}{4} + C$ (B) $\frac{1}{2}\cos^2\left(\frac{3x}{2}\right) + C$ (C) $\frac{\sin 3x}{4} + C$ (D) $\frac{\cos 3x}{4} + C$

where C is an arbitrary constant.

$$\sin 3\theta = 3\sin\theta - 4\sin^3\theta$$

$$\Rightarrow \sin^3\theta = \frac{3\sin\theta - \sin 3\theta}{4}$$

Also

$$\sin x + \sin\left(x + \frac{2\pi}{3}\right) + \sin\left(x + \frac{4\pi}{3}\right)$$

$$= \sin x + 2\sin(x+\pi)\cdot\cos\frac{\pi}{3} = 0$$

Now

$$f(x) = \frac{3\sin x + \sin\left(x + \frac{2\pi}{3}\right) + \sin\left(x + \frac{4\pi}{3}\right)}{4}$$

$$- \frac{1}{4}\left[\sin 3x + \sin(3x + 2\pi) + \sin(3x + 4\pi)\right]$$

$$= 0 - \frac{3}{4}\sin 3x$$

$$\int f(x)\,dx = \frac{3}{4 \times 3}\cos 3x = \frac{\cos 3x}{4} + C$$

Ans. (B), (D)

5 Suppose $J = \int \frac{\sin^2 x + \sin x}{1 + \sin x + \cos x} dx$ and $K = \int \frac{\cos^2 x + \cos x}{1 + \sin x + \cos x} dx$. If C is an arbitrary constant of integration then which of the following is/are correct?

(A) $J = \frac{1}{2}(x - \sin x + \cos x) + C$

(B) $J = K - (\sin x + \cos x) + C$

(C) $J = x - K + C$

(D) $K = \frac{1}{2}(x - \sin x + \cos x) + C$

$$J + K = \int \frac{\sin^2 x + \sin x + \cos^2 x + \cos x}{1 + \sin x + \cos x} dx$$

$$= \int dx = x + C_1$$

$$\Rightarrow J + K = x + C_1 \quad --- \quad (1)$$

$$K - J = \int \frac{\cos 2x + \cos x - \sin x}{1 + \sin x + \cos x} dx$$

$$= \int \frac{(\cos x - \sin x)\cancel{(\cos x + \sin x + 1)}}{\cancel{(1 + \cos x + \sin x)}} dx$$

$$= \int (\cos x - \sin x) dx = \sin x + \cos x + C_2$$

$$\Rightarrow K - J = \sin x + \cos x + C_2 \quad --- \quad (2)$$

Ans. B C

6. $\int \frac{\cot^{-1}(e^x)}{e^x} dx$ is equal to -

(A) $\frac{1}{2}\ell n\left(e^{2x}+1\right)-\frac{\cot^{-1}(e^x)}{e^x}+x+c$

(B) $\frac{1}{2}\ell n\left(e^{2x}+1\right)+\frac{\cot^{-1}(e^x)}{e^x}+x+c$

(C) $\frac{1}{2}\ell n\left(e^{2x}+1\right)-\frac{\cot^{-1}(e^x)}{e^x}-x+c$

(D) $\frac{1}{2}\ell n\left(e^{2x}+1\right)+\frac{\cot^{-1}(e^x)}{e^x}-x+c$

$$I = \int \tan^{-1}(e^{-x})\, e^{-x}\, dx$$

Put $e^{-x} = t \Rightarrow e^{-x} dx = -dt$

$$I = -\int \tan^{-1}(t)\, dt$$

$$= -t \cdot \tan^{-1}(t) + \frac{1}{2}\int \frac{2t}{1+t^2}\, dt$$

$$= -t \cdot \tan^{-1}(t) + \frac{1}{2}\ln|1+t^2| + C$$

$$= -\frac{\cot^{-1}(e^x)}{e^x} + \frac{1}{2}\ln|1+e^{2x}| - x + C$$

C

7 $\int \sec^2\theta(\sec\theta+\tan\theta)^2 d\theta$

(A) $\frac{(\sec\theta+\tan\theta)}{2}[2+\tan\theta(\sec\theta+\tan\theta)]+C$ (B) $\frac{(\sec\theta+\tan\theta)}{3}[2+4\tan\theta(\sec\theta+\tan\theta)]+C$

(C) $\frac{(\sec\theta+\tan\theta)}{3}[2+\tan\theta(\sec\theta+\tan\theta)]+C$ (D) $\frac{3(\sec\theta+\tan\theta)}{2}[2+\tan\theta(\sec\theta+\tan\theta)]+C$

use I.B.P.

$$I = \int \underbrace{(\sec\theta+\tan\theta)^2}_{I} \cdot \underbrace{\sec^2\theta}_{II}\, d\theta$$

$$= \tan\theta(\sec\theta+\tan\theta)^2 - \int \tan\theta \cdot 2(\sec\theta+\tan\theta) \times (\sec\theta\cdot\tan\theta+\sec^2\theta)\, d\theta$$

$$= \tan\theta\cdot(\sec\theta+\tan\theta)^2 - 2\int(\sec\theta+\tan\theta)^2 \sec\theta\cdot\tan\theta\, d\theta$$

$$\Rightarrow I = \tan\theta(\sec\theta+\tan\theta)^2 - 2I_1 \quad \cdots (1)$$

where $I_1 = \int \underbrace{(\sec\theta+\tan\theta)^2}_{I} \underbrace{\sec\theta\cdot\tan\theta}_{II}\, d\theta$

use I.B.P. to evaluate I_1

7 Continues...

$$\Rightarrow I_1 = \sec\theta \cdot (\sec\theta + \tan\theta)^2 - \int \sec\theta \cdot 2(\sec\theta + \tan\theta) \times (\sec\theta \cdot \tan\theta + \sec^2\theta)\, d\theta$$

$$= \sec\theta(\sec\theta + \tan\theta)^2 - 2\int (\sec\theta + \tan\theta)^2 \sec^2\theta \cdot d\theta$$

$$= \sec\theta(\sec\theta + \tan\theta)^2 - 2I \quad \text{---} (2)$$

From (1) & (2)

$$I = \tan\theta(\tan\theta + \sec\theta)^2 - 2(\sec\theta + \tan\theta)^2 \sec\theta + 4I$$

$$\Rightarrow 3I = 2\sec\theta(\sec\theta + \tan\theta)^2 - \tan\theta(\tan\theta + \sec\theta)^2$$

$$= (\sec\theta + \tan\theta)\left[2\sec^2\theta + 2\sec\theta \cdot \tan\theta - \tan^2\theta - \sec\theta \cdot \tan\theta\right]$$

$$= (\sec\theta + \tan\theta)(2 + \tan^2\theta + \sec\theta \cdot \tan\theta)$$

$$\Rightarrow I = \frac{\sec\theta + \tan\theta}{3}\left[2 + \tan\theta(\tan\theta + \sec\theta)\right] + C$$

Ans. (c)

8. If $f'\left(x^2\right)=\frac{\ell nx}{x^2}$ and $f\left(1\right)=-\frac{1}{4}$, then -

(A) $f(e)=0$ (B) $f'\left(e\right)=\frac{1}{2e}$ (C) $f''(e)=f(e)$ (D) $f''(e)=f'(e)$

Solution:

$$f'(x^2) = \frac{\ln x}{x^2} = \frac{1}{2}\frac{\ln x^2}{x^2}$$

$$\Rightarrow f'(t) = \frac{1}{2}\frac{\ln t}{t}$$

$$\Rightarrow f(x) = \int \frac{1}{2}\frac{\ln x}{x}\,dx$$

$$= \frac{(\ln x)^2}{4} + c$$

at $x=1$; $f(1) = -\frac{1}{4} \Rightarrow c = -\frac{1}{4}$

$$\therefore f(x) = \frac{(\ln x)^2 - 1}{4}$$

So, $f(e) = 0$ and $f'(e) = \frac{1}{2e}$

and $f''(x) = \frac{1}{2}\,\frac{t\cdot\frac{1}{t} - \ln t\cdot 1}{t^2} = \frac{1-\ln t}{t^2}$

$\Rightarrow f''(e) = 0$ $\boxed{A,\ B,\ C}$

9. $I=\int \frac{2x-1-x^2}{(1+x^2)^2}dx$ is equal to

(A) $\alpha-\frac{1}{1+x^2}-\tan^{-1}x$

(B) $\cot^{-1}x-\frac{1}{1+x^2}+\beta$

(C) $\frac{x^2}{1+x^2}-\tan^{-1}x+\gamma$

(D) $\frac{2x^2+1}{1+x^2}-\tan^{-1}x+\delta$

(where $\alpha, \beta, \gamma, \delta$ are arbitrary constants)

Solution:

$$I = \int \frac{2x-1-x^2}{(1+x^2)^2}dx = \int\left(\frac{(2x)}{(1+x^2)^2}-\frac{(1+x^2)}{(1+x^2)^2}\right)dx$$

$$= \int \frac{2x}{(1+x^2)^2}dx - \int \frac{1}{(1+x^2)}dx$$

$$= \frac{-1}{1+x^2} - \tan^{-1}x + \alpha = 1-\frac{1}{1+x^2}-\tan^{-1}x+\alpha-1 = \frac{x^2}{1+x^2}-\tan^{-1}x+\gamma$$

$$= \cot^{-1}x - \frac{1}{1+x^2} + \beta$$

$$= 2-\frac{1}{1+x^2} - \tan^{-1}x + \alpha - 2$$

$$= \frac{2x^2+1}{1+x^2} - \tan^{-1}x + \delta$$

$$\boxed{A, B, C, D}$$

10. $\int \frac{x+1}{2x^{3/2}}dx$ is equal to-

(A) $x^{\frac{1}{2}} - x^{-\frac{1}{2}} + C$

(B) $\frac{x^{\frac{3}{2}} - x^{\frac{1}{2}}}{x} + C$

(C) $\frac{x^{\frac{3}{2}} + \sqrt{x}\left(\sqrt{x}-1\right)}{x} + C$

(D) $\frac{x^2-1}{x^{\frac{3}{2}} + x^{\frac{1}{2}}} + C$

(where C is constant of integration)

Solution:

$$\int \frac{x+1}{2x^{3/2}}dx = \frac{1}{2}\int\left(\frac{1}{x^{1/2}} + \frac{1}{x^{3/2}}\right)dx$$

$$= \frac{1}{2}\left(\frac{x^{-1/2+1}}{-1/2+1} + \frac{x^{-3/2+1}}{-3/2+1}\right) + C$$

$$= x^{1/2} - x^{-1/2} + C \quad \text{A}$$

$$= \sqrt{x} - \frac{1}{\sqrt{x}} + C = \frac{x-1}{\sqrt{x}} + C$$

$$= \underbrace{\frac{x^{3/2} - x^{1/2}}{x}}_{\boxed{B}} + C = \left(\frac{x^{3/2} - \sqrt{x} + 1}{x}\right) + C - 1 \quad \text{C}$$

$$= \frac{x-1}{\sqrt{x}} + C = \frac{(x-1)(x+1)}{\sqrt{x}(x+1)} + C \quad \text{D}$$

$\boxed{A, B, C, D}$

EXERCISE (S-1)

① $\int \frac{(\sqrt{x}+1)(x^2-\sqrt{x})}{x\sqrt{x}+x+\sqrt{x}}\,dx$

Solⁿ $\int \frac{(\sqrt{x}+1)\cancel{\sqrt{x}}(x^{3/2}-1)}{\cancel{\sqrt{x}}(x+1+\sqrt{x})}\,dx$

$= \int \frac{(\sqrt{x}+1)(\sqrt{x}-1)(x+1+\sqrt{x})}{(x+1+\sqrt{x})}\,dx$

$= \int (\sqrt{x}+1)(\sqrt{x}-1)\,dx = \int (x-1)\,dx$

$= \frac{x^2}{2} - x + C$ Ans.

(2)

A function g defined for all positive real numbers, satisfies $g'(x^2) = x^3$ for all $x > 0$ and $g(1) = 1$. Compute $g(4)$.

Sol^n

$$g'(x^2) = x^3$$

$$\Rightarrow 2x\, g'(x^2)\, dx = 2x^4\, dx$$

$$\Rightarrow \int 2x\, g'(x^2)\, dx = \int 2x^4\, dx$$

$$\Rightarrow g(x^2) = \frac{2}{5}x^5 + C$$

Now $x = 1$

$$g(1) = \frac{2}{5} + C \Rightarrow 1 = \frac{2}{5} + C$$

$$\Rightarrow C = 3/5$$

$$\Rightarrow g(x^2) = \frac{2}{5}x^5 + 3/5$$

put $x = 2$

$$g(4) = \frac{64}{5} + \frac{3}{5} = \frac{67}{5} \text{ Ans}$$

③ $\int\left[\sin\alpha\sin(x-\alpha)+\sin^2\left(\frac{x}{2}-\alpha\right)\right]dx$

Solⁿ $\frac{1}{2}\int\left\{2\sin\alpha\sin(x-\alpha)+2\sin^2\left(\frac{x}{2}-\alpha\right)\right\}dx$

$=\frac{1}{2}\int\left\{\cancel{\cos(2\alpha-x)}-\cos(x)+1-\cancel{\cos(x-2\alpha)}\right\}dx$

$=\frac{1}{2}\int(1-\cos x)\,dx$

$=\frac{1}{2}(x-\sin x)+C$

④ $\int\left[\left(\frac{x}{e}\right)^x+\left(\frac{e}{x}\right)^x\right]\ln x\,dx$

Sol^n $\int\left(1+\frac{1}{\left(\frac{x}{e}\right)^{2x}}\right)\cdot\left(\frac{x}{e}\right)^x \ln x\,dx$

put $\left(\frac{x}{e}\right)^x=t \Rightarrow x(\ln x-1)=\ln t$

on differentiati

$\left((\ln x-1)(1)+(x)\left(\frac{1}{x}\right)\right)dx=\frac{1}{t}dt$

$\Rightarrow \left(\frac{x}{e}\right)^x \ln x\,dx=dt$

$I=\int\left(1+\frac{1}{t^2}\right)dt=t-\frac{1}{t}+C$

$=\left(\frac{x}{e}\right)^x-\left(\frac{e}{x}\right)^x+C$ Ans.

⑤ $$\int \frac{(\sqrt{x}+1)dx}{\sqrt{x}\left(\sqrt[3]{x}+1\right)}$$

Sol^n put $x = t^6 \Rightarrow dx = 6t^5 dt$

$$I = \int \frac{(t^3+1)}{t^3(t^2+1)} \cdot 6t^5 dt$$

$$I = 6\int \frac{(t^3+1)t^2 dt}{(1+t^2)} = 6\int \frac{t^5}{1+t^2}dt + 6\int \frac{t^2}{1+t^2}dt$$

$$= 3\int \frac{2t \cdot t^4}{1+t^2}dt + 6\int \frac{t^2+1-1}{1+t^2}dt$$

$$= 3\int \underbrace{\frac{2t \cdot (t^2)^2}{1+t^2}dt} + 6\int dt - 6\int \frac{dt}{1+t^2}$$

put $1+t^2 = u$

$\Rightarrow 2t\,dt = du$

$$\int \frac{(u-1)^2}{u}du \Rightarrow \int \frac{u^2+1-2u}{u}du$$

$$= \frac{u^2}{2} - 2u + \ln u$$

$$I = \frac{3}{2}(1+t^2)^2 - 6(1+t^2) + 3\ln(1+t^2) + 6t - 6\tan^{-1}t + C$$

$$= \frac{3}{2}\left\{1+t^4+2t^2-4-4t^2+4t\right\} + 3\ln(1+t^2) - 6\tan^{-1}t + C$$

$$= 6\left\{\frac{t^4}{4} - \frac{t^2}{2} + t\right\} + 3\ln(1+t^2) - 6\tan^{-1}t + C$$

⑥ $\int \sin^{-1}\sqrt{\frac{x}{a+x}}\,dx$

sol^n put $x = a\tan^2\theta$

$\Rightarrow dx = 2a\tan\theta\sec^2\theta\,d\theta$

$$I = \int \sin^{-1}\sqrt{\frac{a\tan^2\theta}{a+a\tan^2\theta}}\,(2a\tan\theta\sec^2\theta)\,d\theta$$

$$I = \int \sin^{-1}(\sin\theta)\,(2a\tan\theta\sec^2\theta)\,d\theta$$

$$I = 2a\int \underbrace{\theta}_{I}\cdot(\underbrace{\tan\theta\sec^2\theta}_{II})\,d\theta$$

$$I = 2a\left\{\theta\cdot\frac{\tan^2\theta}{2} - \int\frac{\tan^2\theta}{2}\,d\theta\right\}$$

$$I = a\left\{\theta\tan^2\theta - (\tan\theta - \theta)\right\} + C$$

$$I = a\left\{\theta(1+\tan^2\theta) - \tan\theta\right\} + C$$

$$I = a\left\{\left(1+\frac{x}{a}\right)\tan^{-1}\sqrt{\frac{x}{a}} - \sqrt{\frac{x}{a}}\right\} + C$$

$$I = (x+a)\tan^{-1}\sqrt{\frac{x}{a}} - \sqrt{ax} + C$$

Ans:

⑦ $\int \frac{x \ell n x}{(x^2-1)^{3/2}} . dx$

Solⁿ $I = \int \underbrace{\frac{x}{(x^2-1)^{3/2}}}_{II} \cdot \underbrace{\ln x}_{I} \, dx$

$= (\ln x)\left(\frac{-1}{\sqrt{x^2-1}}\right) - \int \left(\frac{1}{x}\right)\left(\frac{-1}{\sqrt{x^2-1}}\right) dx$

$= \frac{-\ln x}{\sqrt{x^2-1}} + \sec^{-1} x + C$

Ans

$\int \frac{x}{(x^2-1)^{3/2}} dx = \frac{1}{2} \int \frac{2x \, dx}{(x^2-1)^{3/2}}$

put $x^2 - 1 = t \Rightarrow 2x dx = dt$

$I = \frac{1}{2} \int \frac{dt}{t^{3/2}} = \frac{-1}{\sqrt{t}} = \frac{-1}{\sqrt{x^2-1}}$

8. $\int\left[\frac{\sqrt{x^2+1}\,\left[\ell n(x^2+1)-2\ell n\,x\right]}{x^4}\right]dx$

Sol^n $\int \frac{\sqrt{x^2+1}}{x^4}\ln\left(\frac{x^2+1}{x^2}\right)dx$

$$= \int \frac{1}{x^3}\sqrt{1+\frac{1}{x^2}}\ln\left(1+\frac{1}{x^2}\right)dx$$

put $1+\frac{1}{x^2} = t^2 \Rightarrow \frac{-2}{x^3}dx = 2t\,dt$

$$\Rightarrow \frac{dx}{x^3} = -t\,dt$$

$$I = \int t\ln(t^2)(-t)\,dt$$

$$I = -2\int \underbrace{t^2}_{II}\,\underbrace{\ln t}_{I}\,dt \quad \text{(By parts)}$$

$$= -2\left\{(\ln t)\left(\frac{t^3}{3}\right) - \int \frac{t^3}{3}\cdot\frac{1}{t}\,dt\right\}$$

$$= -2\left\{\frac{t^3\ln t}{3} - \frac{t^3}{9}\right\} + C$$

$$= \frac{2}{3}\left\{\frac{t^3}{3} - t^3\ln t\right\} + C$$

P.T.O.

8....

$$= \frac{2}{3} t^3 \left\{ \frac{1}{3} - \ln t \right\} + C$$

$$= \frac{2}{3} \left(1 + \frac{1}{x^2}\right)^{3/2} \left\{ \frac{1}{3} - \ln\left(1 + \frac{1}{x^2}\right)^{\frac{1}{2}} \right\} + C$$

$$= \frac{2}{3} \frac{(1+x^2)\sqrt{1+x^2}}{x^3} \left\{ \frac{1}{3} - \frac{1}{2} \ln\left(1 + \frac{1}{x^2}\right) \right\} + C$$

$$= \frac{1}{9} \frac{(1+x^2)(\sqrt{1+x^2})}{x^3} \left(2 - 3\ln\left(1 + \frac{1}{x^2}\right)\right) + C$$

Ans.

9. $\int \frac{3x^2+1}{(x^2-1)^3}dx$

Sol^n

$(x+1)^3 - (x-1)^3 = 6x^2+2 = 2(3x^2+1)$

So $I = \frac{1}{2}\int \frac{(x+1)^3-(x-1)^3}{(x^2-1)^3}dx = \frac{1}{2}\int \frac{(x+1)^3-(x-1)^3}{(x+1)^3(x-1)^3}dx$

$= \frac{1}{2}\int \frac{dx}{(x-1)^3} - \frac{1}{2}\int \frac{dx}{(x+1)^3}$

$= \frac{-1}{4.(x-1)^2} + \frac{1}{4.(x+1)^2} + C$

Ans.

(10) $\int \frac{(ax^2-b)\ dx}{x\sqrt{c^2x^2-(ax^2+b)^2}}$

Sol^n

$$\int \frac{(ax^2-b)\,dx}{x^2\sqrt{c^2-\left(ax+\frac{b}{x}\right)^2}}$$

$$= \int \frac{\left(a-b/x^2\right)dx}{\sqrt{c^2-\left(ax+\frac{b}{x}\right)^2}}$$

put $ax+\frac{b}{x}=t \Rightarrow \left(a-\frac{b}{x^2}\right)dx = dt$

$$I = \int \frac{dt}{\sqrt{c^2-t^2}} = \sin^{-1}\left(\frac{t}{c}\right)+K$$

$$= \sin^{-1}\left(\frac{ax+\frac{b}{x}}{c}\right)+K$$

$$= \sin^{-1}\left(\frac{ax^2+b}{cx}\right)+K$$

Ans.

11 $\int \frac{e^{\cos x}(x\sin^3 x+\cos x)}{\sin^2 x}dx$

Sol^n $\int \underbrace{x}_{I}(\underbrace{e^{\cos x}\sin x}_{II})dx + \int \underbrace{e^{\cos x}}_{I}\cdot(\underbrace{\cot x \operatorname{cosec} x}_{II})dx$ (By parts)

$= (x)(-e^{\cos x}) - \cancel{\int (1)(-e^{\cos x})}dx + e^{\cos x}(-\operatorname{cosec} x)$

$- \int \cancel{(-\sin x\, e^{\cos x})}(-\operatorname{cosec} x)dx$

$= -e^{\cos x}(x + \operatorname{cosec} x) + C$

Ans.

(12) $$\int \frac{5x^4+4x^5}{(x^5+x+1)^2}dx$$

Soln

$$I=\int \frac{5x^4+4x^5}{(x^5+x+1)^2}dx = \int \frac{5x^4+4x^5}{x^{10}\left(1+\frac{1}{x^5}+\frac{1}{x^4}\right)^2}dx$$

$$I=\int \frac{\left(\frac{5}{x^6}+\frac{4}{x^5}\right)dx}{\left(\frac{1}{x^5}+\frac{1}{x^4}+1\right)^2}$$

put $\frac{1}{x^5}+\frac{1}{x^4}+1=t \Rightarrow \left(\frac{5}{x^6}+\frac{4}{x^5}\right)dx=-dt$

$$I=\int \frac{-dt}{t^2}=\frac{1}{t}+C=\frac{1}{\left(\frac{1}{x^5}+\frac{1}{x^4}+1\right)}+C$$

$$=\frac{x^5}{x^5+x+1}+C \quad \text{Ans.}$$

or

$$=\frac{x^5}{x^5+x+1}-1+\underbrace{1+C}\to C'$$

$$=\frac{x^5-x^5-x-1}{x^5+x+1}+C'$$

$$=-\left(\frac{x+1}{x^5+x+1}\right)+C' \quad \text{Ans.}$$

$$\int (\sin x)^{-11/3} (\cos x)^{-1/3}\, dx$$

13

Solⁿ $I = \int \dfrac{(\sin x)^{-11/3}}{(\cos x)^{-11/3}} \times (\cos x)^{-\frac{11}{3}} \cdot (\cos x)^{-1/3}\, dx$

$$I = \int \frac{1}{(\tan x)^{11/3}} \sec^4 x\, dx = \int \frac{1+\tan^2 x}{(\tan x)^{11/3}} \cdot \sec^2 x\, dx$$

put $\tan x = t \Rightarrow \sec^2 x\, dx = dt$

$$I = \int \frac{1+t^2}{t^{11/3}} dt = \int \left(t^{-11/3} + t^{-5/3}\right) dt$$

$$= \frac{t^{-8/3}}{-8/3} + \frac{t^{-2/3}}{-2/3} + C$$

$$= -\frac{3}{8} \cdot \frac{1}{(\tan x)^{8/3}} - \frac{3}{2} \cdot \frac{1}{(\tan x)^{2/3}} + C$$

$$= -\frac{3}{8} \left\{ \frac{1+4\tan^2 x}{(\tan x)^{8/3}} \right\} + C$$

Ans.

14 $\int \frac{dx}{\sin x + \sec x}$

Sol^n)

$$I = \int \frac{dx}{\sin x + \sec x} = \int \frac{\cos x \, dx}{1 + \sin x \cos x}$$

$$= \int \frac{2\cos x}{2 + 2\sin x \cos x} dx$$

$$= \int \frac{(\cos x + \sin x) + (\cos x - \sin x)}{2 + 2\sin x \cos x} dx$$

$$I = \int \frac{\cos x + \sin x}{3 - (\sin x - \cos x)^2} dx + \int \frac{\cos x - \sin x}{1 + (\sin x + \cos x)^2} dx$$

$I = I_1 + I_2$ (Say)

put $u = \sin x - \cos x$

$\Rightarrow du = (\cos x + \sin x)$

$$I_1 = \int \frac{du}{3 - u^2}$$

$$I_1 = \frac{1}{2\sqrt{3}} \ln\left(\frac{\sqrt{3} + u}{\sqrt{3} - u}\right)$$

put $v = \sin x + \cos x$

$dv = (\cos x - \sin x) dx$

$$I_2 = \int \frac{dv}{1 + v^2}$$

$$I_2 = \tan^{-1} v$$

$$\text{So} \quad I = \frac{1}{2\sqrt{3}} \ln\left(\frac{\sqrt{3} + \sin x - \cos x}{\sqrt{3} - \sin x + \cos x}\right) + \tan^{-1}(\sin x + \cos x) + C$$ Ans

15

$$\int \frac{4x^5 - 7x^4 + 8x^3 - 2x^2 + 4x - 7}{x^2(x^2+1)^2}\,dx$$

Sol^n

$$= \int \frac{4x(x^2+1)^2 - 7(x^2+1)^2 + 12x^2}{x^2(x^2+1)^2}\,dx$$

$$= \int \frac{4}{x}\,dx - \int \frac{7}{x^2}\,dx + \int \frac{12}{(x^2+1)^2}\,dx$$

$$I = 4\ln x + \frac{7}{x} + 12\int \frac{dx}{(x^2+1)^2} \quad \text{——} \quad (1)$$

Let $I_1 = \int \frac{dx}{(x^2+1)^2}$ put $x = \tan\theta$

$$I_1 = \int \frac{\sec^2\theta\, d\theta}{\sec^4\theta} = \int \cos^2\theta\, d\theta = \int \frac{1+\cos 2\theta}{2}\,d\theta$$

$$= \frac{1}{2}\left(\theta + \frac{\sin 2\theta}{2}\right) = \frac{1}{2}\tan^{-1}x + \frac{1}{4}\left\{\frac{2x}{1+x^2}\right\}$$

$$= \frac{1}{2}\tan^{-1}x + \frac{x}{2(1+x^2)}$$

So, $I = 4\ln x + \frac{7}{x} + 6\tan^{-1}x + \frac{6x}{1+x^2} + C$

Ans.

16

If the value $\int \frac{1-(\cot x)^{2008}}{\tan x+(\cot x)^{2009}}dx = \frac{1}{k} \ln|\sin^k x+\cos^k x|+C$, then find k.

Soln

$$I = \int \frac{(\sin x)^{2008} - (\cos x)^{2008}}{(\sin x)^{2010} + (\cos x)^{2010}} \times (\cos x)(\sin x)\, dx$$

$$I = \int \frac{\cos x(\sin x)^{2009} - \sin x(\cos x)^{2009}}{(\sin x)^{2010} + (\cos x)^{2010}}\, dx$$

Put $(\sin x)^{2010} + (\cos x)^{2010} = t$

$\Rightarrow \left(\cos x(\sin x)^{2009} - \sin x(\cos x)^{2009}\right)dx = dt$

$$I = \int \frac{dt}{t} = \ln|t| + C$$

$$= \ln|\sin^{2010} x + \cos^{2010} x| + C$$

Ans.

(17) $\int \frac{dx}{(x-\alpha)\sqrt{(x-\alpha)(x-\beta)}}$

Sol^n $\int \frac{dx}{(x-\alpha)^2 \sqrt{\frac{x-\beta}{x-\alpha}}}$

put $\frac{x-\beta}{x-\alpha} = t^2 \Rightarrow \frac{(x-\alpha)-(x-\beta)}{(x-\alpha)^2} dx = 2t\,dt$

$\Rightarrow \frac{dx}{(x-\alpha)^2} = \frac{2}{(\beta-\alpha)} t\,dt$

$I = \frac{2}{(\beta-\alpha)} \int \frac{t\,dt}{t} = \frac{2}{(\beta-\alpha)} \int dt$

$= \frac{2}{(\beta-\alpha)} t + C = \frac{2}{(\beta-\alpha)} \sqrt{\frac{x-\beta}{x-\alpha}} + C$

Ans.

EXERCISE (S-2)

EXERCISE (S-2)

1. $\int \frac{\tan(\ln x)\tan\left(\ln\frac{x}{2}\right)\tan(\ln 2)}{x}dx$

Solution

$$\int \frac{\tan(\ln x)\tan(\ln x - \ln 2)\tan(\ln 2)}{x}dx$$

Let $\ln x = t \Rightarrow \frac{1}{x}dx = dt$

$$\int \tan t \cdot \tan(t - \ln 2)\cdot \tan(\ln 2)\, dt$$

$\star$ $$\tan(t - \ln 2) = \frac{\tan t - \tan(\ln 2)}{1 + \tan t \tan(\ln 2)}$$

$$\Rightarrow \tan t \cdot \tan(t - \ln 2)\cdot \tan(\ln 2) = \tan t - \tan(\ln 2) - \tan(t - \ln 2)$$

$$\int \{\tan t - \tan(\ln 2) - \tan(t - \ln 2)\}\, dt$$

$$\ln \sec t - t\tan(\ln 2) - \ln(\sec(t - \ln 2)) + K$$

$$\ln\left(\frac{\sec(\ln x)}{\sec(\ln \frac{x}{2})}\right) - \frac{\tan(\ln 2)}{x} + K$$

Ans

2

$$\int \frac{e^x(2-x^2)}{(1-x)\sqrt{1-x^2}}dx$$

Solution

$$\int e^x \left(\frac{1+1-x^2}{(1-x)\sqrt{1-x^2}}\right) dx$$

$$\int e^x\left(\frac{1}{(1-x)\sqrt{1-x^2}} + \sqrt{\frac{1+x}{1-x}}\right) dx$$

$$\int e^x (f'(x) + f(x))\, dx = e^x f(x) + C$$

$$f(x) = \sqrt{\frac{1+x}{1-x}} \Rightarrow f'(x) = \frac{1}{\sqrt{1+x}\,(1-x)^{3/2}}$$

$$\Rightarrow e^x\sqrt{\frac{1+x}{1-x}} + C$$

Ans

③ $$\int \frac{dx}{\left(x+\sqrt{x(1+x)}\right)^2}$$

Solution:-
$$\int \frac{dx}{\left(x+x\sqrt{\frac{1+x}{x}}\right)^2}$$

$$\Rightarrow \int \frac{dx}{x^2\left(1+\sqrt{1+\frac{1}{x}}\right)^2} \quad ; \quad 1+\frac{1}{x}=t^2$$

$$-\frac{dx}{x^2}=2t\,dt$$

$$-\int \frac{2t\,dt}{(1+t)^2} = -2\int\left(\frac{1}{1+t}-\frac{1}{(1+t)^2}\right)dt$$

$$\Rightarrow -2\ln|1+t| - \frac{2}{1+t} + C$$

$$-2\ln\left|1+\sqrt{1+\frac{1}{x}}\right| - \frac{2}{1+\sqrt{1+\frac{1}{x}}} + K$$

4 $\displaystyle\int \frac{dx}{\sqrt{\sin^3 x \sin(x+\alpha)}}$

Solution:

$$\int \frac{dx}{\sqrt{\sin^3 x(\sin x \cos\alpha + \cos x \sin\alpha)}}$$

$$\Rightarrow \int \frac{\operatorname{cosec}^2 x \, dx}{\sqrt{\cos\alpha + \cot x \sin\alpha}}$$

$$\cos\alpha + \cot x \sin\alpha = t^2$$

$$-\operatorname{cosec}^2 x \cdot \sin\alpha \cdot dx = 2t\,dt$$

$$\Rightarrow -\int \frac{2t\,dt}{\sin\alpha \cdot t} \Rightarrow -2\operatorname{cosec}\alpha \cdot t + C$$

$$C - 2\operatorname{cosec}\alpha \cdot \sqrt{\frac{\sin(x+\alpha)}{\sin x}}$$

5 $\displaystyle\int \frac{(1+x^2)dx}{1-2x^2\cos\alpha+x^4}\ \alpha\in(0,\pi)$

Solution

divide by x^2

$$\int \frac{\left(1+\frac{1}{x^2}\right)dx}{\frac{1}{x^2}-2\cos\alpha+x^2}\ ; \quad x-\frac{1}{x}=t$$

$$\left(1+\frac{1}{x^2}\right)dx=dt$$

$$\int \frac{dt}{t^2+2-2\cos\alpha}=\int \frac{dt}{t^2+4\sin^2\frac{\alpha}{2}}$$

$$=\frac{1}{2\sin\frac{\alpha}{2}}\int \frac{dt}{\left(\frac{t}{2\sin\frac{\alpha}{2}}\right)^2+1}$$

$$\Rightarrow \frac{1}{2\sin\frac{\alpha}{2}}\cdot\tan^{-1}\frac{t}{2\sin\frac{\alpha}{2}}$$

$$\frac{1}{2}\operatorname{cosec}\frac{\alpha}{2}\tan^{-1}\left(\frac{1}{2}\cdot\operatorname{cosec}\frac{\alpha}{2}\cdot\left(x-\frac{1}{x}\right)\right)+K$$

6. Let f(x) is a quadratic function such that f(0) = 1 and $\int \frac{f(x)dx}{x^2(x+1)^3}$ is a rational function, find the value of f '(0)

Solution:- Let $f(x) = ax^2 + bx + c$

To find: $f'(0) = b$

$f(0) = 1 \Rightarrow c = 1$

$\int \frac{ax^2 + bx + 1}{x^2(x+1)^3} dx$ is a rational fun.

$$= \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x+1} + \frac{D}{(x+1)^2} + \frac{E}{(x+1)^3}$$

becomes log fun $\Rightarrow A = C = 0$ (one of the possibility)

$\Rightarrow ax^2 + bx + 1 = B(x+1)^3 + Dx^2(x+1) + Ex^2$

$x = 0 \Rightarrow \boxed{B = 1}$

Compare Coeff. of $x \Rightarrow$

$b = 3B \Rightarrow \boxed{b = 3}$

$f'(0) = 3$

(7) $\int \frac{\cos x - \sin x}{7 - 9\sin 2x}\,dx$

Solution:- Let $\sin x + \cos x = t$

$\Rightarrow (\cos x - \sin x)\,dx = dt$

$\Rightarrow 1 + \sin 2x = t^2$

$$\int \frac{dt}{7 - 9(t^2 - 1)} = \int \frac{dt}{16 - 9t^2} = \frac{1}{9}\int \frac{dt}{\left(\frac{4}{3}\right)^2 - t^2}$$

$$\frac{1}{9} \cdot \frac{1}{2 \cdot \frac{4}{3}} \ln \left| \frac{\frac{4}{3} + t}{\frac{4}{3} - t} \right| + K$$

$$\frac{1}{24} \ln \left| \frac{4 + 3(\sin x + \cos x)}{4 - 3(\sin x + \cos x)} \right| + K$$

8) $\int \cos 2\theta . \ell n \frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta} d\theta$

Solution

II I

integration by Parts

$$\ln \frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta} \cdot \frac{\sin 2\theta}{2} - \int \frac{c-s}{c+s} \frac{((c-s)(-s+c)-(c+s)(-s-c))}{(\cos\theta-\sin\theta)^2} \cdot \frac{\sin 2\theta}{2} d\theta$$

$$\frac{1}{2}\sin 2\theta \cdot \ln\left|\frac{c+s}{c-s}\right| - \int \frac{(c-s)^2+(c+s)^2}{c^2-s^2} \cdot \frac{\sin 2\theta}{2} d\theta$$

$$- \int \tan 2\theta \, d\theta$$

$$\frac{1}{2}\sin 2\theta \ln\left|\frac{\cos\theta+\sin\theta}{\cos\theta-\sin\theta}\right| - \frac{1}{2}\ln|\sec 2\theta| + C$$

EXERCISE (JM)

EXERCISE (JM)

1. If the integral $\int \frac{5\tan x}{\tan x-2}dx = x + a\ln|\sin x - 2\cos x| + k$ then a is equal to : **[AIEEE-2012]**

(1) 2 (2) –1 (3) –2 (4) 1

Sol:-

$$\because \int \frac{5\tan(x)}{\tan(x)-2}\cdot dx$$

$$= \int \frac{5\sin(x)}{\sin(x)-2\cos(x)}\cdot dx$$

$$= \int \frac{(\sin(x)-2\cos(x))+2(\cos(x)+2\sin(x))}{\sin(x)-2\cos x}dx$$

$$= \int 1\cdot dx + 2\int \frac{\cos(x)+2\sin(x)}{\sin(x)-2\cos(x)}\cdot dx$$

$$= x + 2\ln|\sin(x)-2\cos(x)| + C$$

Ans.

2. If $\int f(x)dx = \Psi(x)$, then $\int x^5 f(x^3)dx$ is equal to : **[JEE-MAIN-2013]**

(1) $\frac{1}{3}\left[x^3\Psi(x^3) - \int x^2\Psi(x^3)dx\right] + C$ (2) $\frac{1}{3}x^3\Psi(x^3) - 3\int x^3\Psi(x^3)dx + C$

(3) $\frac{1}{3}x^3\Psi(x^3) - \int x^2\Psi(x^3)dx + C$ (4) $\frac{1}{3}\left[x^3\Psi(x^3) - \int x^3\Psi(x^3)dx\right] + C$

Sol:- $\int x^5 . f(x^3) . dx$

$= \int x^3 . f(x^3) . x^2 . dx$ $\left\{ \begin{array}{l} \text{put } x^3 = t \\ 3x^2 . dx = dt \end{array} \right\}$

$= \frac{1}{3}\int \underset{\text{I}}{t} . \underset{\text{II}}{f(t)} . dt$ use by part

$= \frac{1}{3}\left[t \int f(t).dt - \int \left\{ 1 . \int f(t) . dt \right\} \right]$

$\because$ given $\int f(t) = \psi(x)$

$= \frac{1}{3} . t\,\psi(t) - \frac{1}{3}\int \psi(t) . dt$

$= \frac{1}{3} x^3 \psi(x^3) - \frac{1}{3}\int \psi(x^3) . 3x^2 . dx$

$= \boxed{\frac{1}{3} x^3 \psi(x^3) - \int x^2 . \psi(x^3) . dx + C}$

Ans.

3. The integral $\int\left(1+x-\frac{1}{x}\right)e^{x+\frac{1}{x}}dx$ is equal to : **[JEE-MAIN-2014]**

(1) $(x-1)e^{x+\frac{1}{x}}+c$ (2) $xe^{x+\frac{1}{x}}+c$ (3) $(x+1)e^{x+\frac{1}{x}}+c$ (4) $-xe^{x+\frac{1}{x}}+c$

Sol:- $\int\left(1+x-\frac{1}{x}\right)e^{x+\frac{1}{x}}\cdot dx$

$$= \int\left[e^{x+\frac{1}{x}} + e^{x+\frac{1}{x}}\cdot\left(x-\frac{1}{x}\right)\right]dx$$

$$= \int\left[\underbrace{e^{x+\frac{1}{x}}}_{f(x)} + x\cdot\underbrace{e^{x+\frac{1}{x}}\left(1-\frac{1}{x^2}\right)}_{f'(x)}\right]$$

$$= x\cdot f(x) + c$$

$$= x\cdot e^{x+\frac{1}{x}} + c \;\; \underline{\text{Ans}}.$$

4. The integral $\int \frac{dx}{x^2(x^4+1)^{\frac{3}{4}}}$ equals : **[JEE-MAIN-2015]**

(1) $-(x^4+1)^{\frac{1}{4}}+c$ (2) $-\left(\frac{x^4+1}{x^4}\right)^{\frac{1}{4}}+c$ (3) $\left(\frac{x^4+1}{x^4}\right)^{\frac{1}{4}}+c$ (4) $(x^4+1)^{\frac{1}{4}}+c$

Sol:- $\int \frac{dx}{x^2(x^4+1)^{3/4}}$

$$= \int \frac{dx}{x^5(1+x^{-4})^{3/4}}$$

$$\left\{ \begin{array}{l} \text{Put } 1+x^{-4} = t \\ -4 \cdot \frac{1}{x^5} \cdot dx = dt \end{array} \right\}$$

$$= -\frac{1}{4}\int \frac{dt}{t^{3/4}}$$

$$= -\frac{1}{4} 4 \cdot t^{1/4} + C$$

$$= -\left(\frac{x^4+1}{x^4}\right)^{1/4} + C$$

Ans.

5. The integral $\int \frac{2x^{12}+5x^{9}}{(x^{5}+x^{3}+1)^{3}}dx$ is equal to :- **[JEE-MAIN 2016]**

(1) $\frac{-x^{10}}{2(x^{5}+x^{3}+1)^{2}}+C$ (2) $\frac{-x^{5}}{(x^{5}+x^{3}+1)^{2}}+C$ (3) $\frac{x^{10}}{2(x^{5}+x^{3}+1)^{2}}+C$ (4) $\frac{x^{5}}{2(x^{5}+x^{3}+1)^{2}}+C$

where C is an arbitrary constant.

Sol:-

$$\int \frac{2x^{12}+5x^{9}}{(x^{5}+x^{3}+1)^{3}}\,dx$$

$$= \int \frac{2x^{12}+5x^{9}}{x^{15}\left(1+\frac{1}{x^{2}}+\frac{1}{x^{5}}\right)^{3}}\cdot dx$$

$$= \int \frac{\frac{2}{x^{3}}+\frac{5}{x^{6}}}{\left(1+\frac{1}{x^{2}}+\frac{1}{x^{5}}\right)^{3}}\cdot dx$$

$$\left\{\begin{array}{l}\text{Put } 1+x^{-2}+x^{-5}=t \\ \Rightarrow (-2\cdot x^{-3}-5x^{-6})\,dx=dt \\ \Rightarrow \left(\frac{2}{x^{3}}+\frac{5}{x^{6}}\right)dx=-dt\end{array}\right\}$$

$$= -\int \frac{1}{t^{3}}\cdot dt$$

$$= -\frac{t^{-3+1}}{-3+1}+c$$

$$= \frac{1}{2}\cdot\frac{1}{t^{2}}+c$$

$$= \frac{1}{2}\,\frac{x^{10}}{(x^{5}+x^{3}+1)^{2}}+C \;\; \underline{1}.$$

6. Let $I_n = \int \tan^n x\,dx, (n > 1)$. $I_4 + I_6 = a\tan^5 x + bx^5 + C$, where C is a constant of integration, then the ordered pair (a, b) is equal to :- **[JEE-MAIN 2017]**

(1) $\left(-\frac{1}{5}, 0\right)$ (2) $\left(-\frac{1}{5}, 1\right)$ (3) $\left(\frac{1}{5}, 0\right)$ (4) $\left(\frac{1}{5}, -1\right)$

Sol:-

$\because I_n = \int \tan^n(x).dx$

$\therefore I_4 + I_6 = \int \left(\tan^4(x) + \tan^6(x)\right) dx$

$= \int \tan^4(x).(1+\tan^2 x)\,dx$

$= \int \tan^4(x).\sec^2(x).dx$

$\left\{ \begin{array}{l} \underline{\underline{\text{Put}}}\ \tan(x) = t \\ \sec^2(x).dx = dt \end{array} \right\}$

$= \int t^4.dt$

$= \frac{1}{5} t^5 + C$

$= \frac{1}{5}\tan^5(x) + C$

Through Comparision

$a = \frac{1}{5}$ & $b = 0$ $\underline{1}$

7. The integral $\int \frac{\sin^2 x \cos^2 x}{(\sin^5 x + \cos^3 x \sin^2 x + \sin^3 x \cos^2 x + \cos^5 x)^2} dx$ is equal to **[JEE-MAIN 2018]**

(1) $\frac{-1}{3(1+\tan^3 x)} + C$ (2) $\frac{1}{1+\cot^3 x} + C$ (3) $\frac{-1}{1+\cot^3 x} + C$ (4) $\frac{1}{3(1+\tan^3 x)} + C$

(where C is a constant of integration)

Sol. Let $I = \int \frac{\sin^2 x \cos^2 x \qquad dx}{[(\sin^2 x + \cos^2 x)(\sin^3 x + \cos^3 x)]^2}$

$$= \int \frac{\sin^2 x \cos^2 x}{(\sin^3 x + \cos^3 x)^2} dx$$

$\div$ by $\cos^6(x)$ in N^r & D^r

$$= \int \frac{\tan^2 x \cdot \sec^2 x}{(1 + \tan^3 x)^2} dx$$

put $(1 + \tan^3 x) = t$

$3 \tan^2 x \sec^2 x \, dx = dt$

$$I = \frac{1}{3} \int \frac{dt}{t^2} = -\frac{1}{3t} + C$$

Hence, $I = \frac{-1}{3(1+\tan^3 x)} + C$

8. For $x^2 \neq n\pi + 1$, $n \in N$ (the set of natural numbers), the integral $\int x\sqrt{\frac{2\sin(x^2-1)-\sin 2(x^2-1)}{2\sin(x^2-1)+\sin 2(x^2-1)}}dx$ is equal to :
(where c is a constant of integration)

(1) $\log_e\left|\sec\left(\frac{x^2-1}{2}\right)\right|+c$

(2) $\log_e\left|\frac{1}{2}\sec^2\left(x^2-1\right)\right|+c$

(3) $\frac{1}{2}\log_e\left|\sec^2\left(\frac{x^2-1}{2}\right)\right|+c$

(4) $\frac{1}{2}\log_e\left|\sec(x^2-1)\right|+c$

[JEE-MAIN 2019]

Sol. Put $(x^2 - 1) = t$

$\Rightarrow 2xdx = dt$

$$\therefore \quad I=\frac{1}{2}\int\sqrt{\frac{1-\cos t}{1+\cos t}}\,dt$$

$$=\frac{1}{2}\int \tan\left(\frac{t}{2}\right)dt$$

$$=\ln\left|\sec\frac{t}{2}\right|+c$$

$$I=\ln\left|\sec\left(\frac{x^2-1}{2}\right)\right|+c$$

9. If $f(x)=\int \frac{5x^8+7x^6}{\left(x^2+1+2x^7\right)^2}dx,\left(x\geq 0\right)$ and $f(0) = 0$, then the value of $f(1)$ is : **[JEE (Main) 2019]**

(1) $-\frac{1}{2}$ (2) $\frac{1}{2}$ (3) $-\frac{1}{4}$ (4) $\frac{1}{4}$

Solution:

$$f(x) = \int \frac{5x^8+7x^6}{(x^2+1+2x^7)^2}dx$$

$$= \int \frac{5x^8+7x^6}{x^{14}(x^{-5}+x^{-7}+2)^2}dx$$

$$= \int \frac{5x^{-6}+7x^{-8}}{(x^{-5}+x^{-7}+2)^2}dx$$

Let $x^{-5}+x^{-7}+2 = t$

$\Rightarrow (-5x^{-6}-7x^{-8})dx = dt$

$$\therefore f(x) = \int \frac{-dt}{t^2} = \frac{1}{t}+c$$

$$\Rightarrow f(x) = \frac{1}{(x^{-5}+x^{-7}+2)}+c = \frac{x^7}{x^2+2x^7+1}+c$$

$f(0)=0 \Rightarrow c=0$

$$\therefore f(1) = \frac{1}{1+2+1} = \boxed{\frac{1}{4}} \longrightarrow \text{(4) Option}$$

10 If $\int \frac{\sqrt{1-x^2}}{x^4}dx = A(x)\left(\sqrt{1-x^2}\right)^m + C$, for a suitable chosen integer m and a function A(x), where C is a constant of integration then $(A(x))^m$ equals : **[JEE-MAIN 2019]**

(1) $\frac{-1}{3x^3}$ (2) $\frac{-1}{27x^9}$ (3) $\frac{1}{9x^4}$ (4) $\frac{1}{27x^6}$

Sol. $\int \frac{\sqrt{1-x^2}}{x^4}dx = A(x)\left(\sqrt{1-x^2}\right)^m + C$

$$\int \frac{|x|\sqrt{\frac{1}{x^2}-1}}{x^4}dx,$$

Put $\frac{1}{x^2}-1 = t \Rightarrow \frac{dt}{dx} = \frac{-2}{x^3}$

Case-1 $x \geq 0$

$$-\frac{1}{2}\int \sqrt{t}\ dt \Rightarrow -\frac{t^{3/2}}{3}+C$$

$$\Rightarrow -\frac{1}{3}\left(\frac{1}{x^2}-1\right)^{3/2}$$

$$\Rightarrow \frac{\left(\sqrt{1-x^2}\right)^3}{-3x^2}+C$$

$A(x) = -\frac{1}{3x^3}$ and $m = 3$

$$(A(x))^m = \left(-\frac{1}{3x^3}\right)^3 = -\frac{1}{27x^9}$$

Case-II $x \leq 0$

We get $\frac{\left(\sqrt{1-x^2}\right)^3}{-3x^3}+C$

$A(x) = \frac{1}{-3x^3}$, $m = 3$

$$(A(x))^m = \frac{-1}{27x^9}$$

11. The integral $\int \frac{3x^{13}+2x^{11}}{\left(2x^4+3x^2+1\right)^4}dx$ is equal to : (where C is a constant of integration) **[JEE (Main) 2019]**

(1) $\frac{x^4}{\left(2x^4+3x^2+1\right)^3}+C$ (2) $\frac{x^{12}}{6\left(2x^4+3x^2+1\right)^3}+C$

(3) $\frac{x^4}{6\left(2x^4+3x^2+1\right)^3}+C$ (4) $\frac{x^{12}}{\left(2x^4+3x^2+1\right)^3}+C$

Solution:

$$I = \int \frac{3x^{13}+2x^{11}}{(2x^4+3x^2+1)^4}\,dx$$

$$= \int \frac{3x^{13}+2x^{11}}{x^{16}(2+3x^{-2}+x^{-4})^4}\,dx$$

$$I = \int \frac{3x^{-3}+2x^{-5}}{(2+3x^{-2}+x^{-4})^4}\,dx$$

Let $2+3x^{-2}+x^{-4} = t$

$\Rightarrow (-6x^{-3}-4x^{-5})\,dx = dt$

$$\therefore I = \int \frac{-dt}{2t^4} = \frac{1}{2}\cdot\frac{t^{-3}}{3} + C$$

$$= \frac{1}{6}\frac{1}{t^3} + C = \frac{1}{6(2+3x^{-2}+x^{-4})^3} + C$$

$$= \boxed{\frac{x^{12}}{6(2x^4+3x^2+1)^3} + C} \rightarrow \text{② Option}$$

12. $\int \frac{\sin\frac{5x}{2}}{\sin\frac{x}{2}}dx$ is equal to : (where c is a constant of integration) **[JEE (Main) 2019]**

(1) $2x + \sin x + 2\sin 2x + c$

(2) $x + 2\sin x + 2\sin 2x + c$

(3) $x + 2\sin x + \sin 2x + c$

(4) $2x + \sin x + \sin 2x + c$

Solution:

$$\int \frac{\sin\frac{5x}{2}}{\sin\frac{x}{2}}dx = \int \frac{2\sin\frac{5x}{2}\cos\frac{x}{2}}{2\sin\frac{x}{2}\cos\frac{x}{2}}dx$$

$$= \int \frac{\sin\left(\frac{5x}{2}+\frac{x}{2}\right) + \sin\left(\frac{5x}{2}-\frac{x}{2}\right)}{\sin x}dx$$

$$= \int \frac{\sin 3x + \sin 2x}{\sin x}dx$$

$$= \int \frac{3\sin x - 4\sin^3 x + 2\sin x\cos x}{\sin x}dx$$

$$= \int (3 - 4\sin^2 x + 2\cos x)\,dx$$

$$= \int (3 - 2(1-\cos 2x) + 2\cos x)\,dx$$

$$= \int (1 + 2\cos 2x + 2\cos x)\,dx$$

$$= \boxed{x + \sin 2x + 2\sin x + C} \longrightarrow \text{(3) option}$$

13. The integral $\int \sec^{2/3} x \operatorname{cosec}^{4/3} x \, dx$ is equal to (Hence C is a constant of integration) **[JEE (Main) 2019]**

(1) $3\tan^{-1/3}x + C$ (2) $-\frac{3}{4}\tan^{-4/3}x + C$ (3) $-3\cot^{-1/3}x + C$ (4) $-3\tan^{-1/3}x + C$

Solution:

$$I = \int \sec^{2/3} x \cdot \operatorname{Cosec}^{4/3} x \, dx \qquad \left(\frac{4}{3} + \frac{2}{3} = 2\right)$$

$$= \int \frac{\sec^2 x \cdot}{\sin^{4/3} x \cdot \cos^{2/3} x} \frac{dx}{\sec^2 x}$$

$$= \int \frac{\sec^2 x}{\tan^{4/3} x} dx$$

$$= \frac{\tan^{-4/3+1} x}{-\frac{4}{3}+1} + C$$

$$= \boxed{-3\tan^{-1/3} x + C} \longrightarrow \text{(4) option}$$

14. Let $\alpha \in (0, \pi/2)$ be fixed. If the integral $\int \frac{\tan x + \tan \alpha}{\tan x - \tan \alpha} dx = A(x) \cos 2\alpha + B(x) \sin 2\alpha + C$, where C is a constant of integration, then the functions A(x) and B(x) are respectively : **[JEE (Main) 2019]**

(1) $x - \alpha$ and $\log_e|\cos(x - \alpha)|$

(2) $x + \alpha$ and $\log_e|\sin(x - \alpha)|$

(3) $x - \alpha$ and $\log_e|\sin(x - \alpha)|$

(4) $x + \alpha$ and $\log_e|\sin(x + \alpha)|$

Solution:

$$\int \frac{\tan x + \tan\alpha}{\tan x - \tan\alpha} dx = \int \frac{\frac{\sin x}{\cos x} + \frac{\sin\alpha}{\cos\alpha}}{\frac{\sin x}{\cos x} - \frac{\sin\alpha}{\cos\alpha}} dx$$

$$= \int \frac{\sin(x+\alpha)}{\sin(x-\alpha)} dx$$

$$= \int \frac{\sin(x-\alpha+2\alpha)}{\sin(x-\alpha)} dx = \int \frac{[\sin(x-\alpha)\cos 2\alpha + \cos(x-\alpha)\sin 2\alpha]}{\sin(x-\alpha)} dx$$

$$= \int [\cos 2\alpha + \cot(x-\alpha)\sin 2\alpha] dx$$

$$= x\cos 2\alpha + (\sin 2\alpha)\log_e|\sin(x-\alpha)| + C$$

$$\boxed{\text{Answer: (3) or (2)}}$$

EXERCISE (JA)

EXERCISE (JA)

1. $\int \frac{x^2-1}{x^3\sqrt{2x^4-2x^2+1}}dx$ is equal to - **[JEE 2006, (3M, –1M)]**

(A) $\frac{\sqrt{2x^4-2x^2+1}}{x^2}+c$ (B) $\frac{\sqrt{2x^4-2x^2+1}}{x^3}+c$

(C) $\frac{\sqrt{2x^4-2x^2+1}}{x}+c$ (D) $\frac{\sqrt{2x^4-2x^2+1}}{2x^2}+c$

1) $\int \frac{x^2-1}{x^3\sqrt{2x^4-2x^2+1}} \cdot dx$

$= \int \frac{x^2-1}{x^5\sqrt{2-\frac{2}{x^2}+\frac{1}{x^4}}} dx$

$= \int \frac{\left(\frac{1}{x^3}-\frac{1}{x^5}\right)dx}{\sqrt{2-\frac{2}{x^2}+\frac{1}{x^4}}}$

$$\left\{\begin{array}{l} \text{Put } 2-\frac{2}{x^2}+\frac{1}{x^4}=t^2 \\ \Rightarrow 4\left(\frac{1}{x^3}-\frac{1}{x^5}\right)dx = 2t \cdot dt \end{array}\right\}$$

$= \frac{1}{2}\int \frac{t}{\sqrt{t^2}} \cdot dt$

$= \frac{t}{2} + C$

$= \frac{\sqrt{2x^4-2x^2+1}}{2x^2} + C$

Ans.

2. Let $f(x) = \dfrac{x}{(1+x^n)^{1/n}}$ for $n \ge 2$ and $g(x) = \underbrace{(f \circ f \circ \ldots \circ f)}_{\text{f occurs n times}}(x)$. Then $\int x^{n-2} g(x)dx$ equals.

(A) $\dfrac{1}{n(n-1)}(1+nx^n)^{1-\frac{1}{n}} + K$

(B) $\dfrac{1}{n-1}(1+nx^n)^{1-\frac{1}{n}} + K$ **[JEE 2007, 3]**

(C) $\dfrac{1}{n(n+1)}(1+nx^n)^{1+\frac{1}{n}} + K$

(D) $\dfrac{1}{n+1}(1+nx^n)^{1+\frac{1}{n}} + K$

Sol:) $f(x) = \dfrac{x}{(1+x^n)^{1/n}}$

$f(f(x)) = \dfrac{f(x)}{[1+(f(x))^n]^{1/n}}$ Put $\{f(x)\}$

$\boxed{f(f(x)) = \dfrac{x}{(1+2.x^n)^{1/n}}}$

again $f(f(f(x))) = \dfrac{f(x)}{[1+2.(f(x))^n]^{1/n}}$

$= \dfrac{x}{(1+3x^n)^{1/n}}$

$\vdots$

$(fofof\cdots of)(x) = \dfrac{x}{(1+n.x^n)^{1/n}} = g(x)$

$\therefore \int x^{n-2} \cdot g(x)$

$= \int x^{n-2} \cdot \dfrac{x}{(1+nx^n)^{1/n}} \cdot dx$

$= \int \dfrac{(x)^{n-1}}{(1+n.x^n)^{1/n}} \cdot dx$

$\left\{ \begin{array}{l} \text{Put } 1+n.x^n = t \\ n^2.(x)^{n-1} \cdot dx = dt \end{array} \right\}$

$= \dfrac{1}{n^2} \int (t)^{-1/n} \cdot dt$

$= \dfrac{1}{n^2} \cdot \dfrac{(t)^{1-1/n}}{1-\frac{1}{n}} + C$

$= \boxed{\dfrac{1}{n(n-1)} \cdot (1+nx^n)^{1-\frac{1}{n}} + C}$

Ans.

3. Let F(x) be an indefinite integral of $\sin^2 x$. **[JEE 2007, 3]**

Statement-1 : The function F(x) satisfies $F(x + \pi) = F(x)$ for all real x.

because

Statement-2 : $\sin^2(x + \pi) = \sin^2 x$ for all real x.

(A) Statement-1 is True, Statement-2 is True ; Statement-2 is a correct explanation for Statement-1.

(B) Statement-1 is True, Statement-2 is True; Statement-2 is NOT a correct explanation for Statement-1.

(C) Statement-1 is True, Statement-2 is False.

(D) Statement-1 is False, Statement-2 is True.

Sol:- $F(x) = \int \sin^2(x) \cdot dx$

$\Rightarrow F(x) = \frac{1}{4}\left[2x - \sin 2x\right] + C$

So, $F(x+\pi) \neq F(x)$

$\therefore$ Statement 1 is FALSE

But Statement 2 is TRUE, because period of $\sin^2(x)$ is π. Ans.

4. Let $I = \int \frac{e^x}{e^{4x}+e^{2x}+1}dx$, $J = \int \frac{e^{-x}}{e^{-4x}+e^{-2x}+1}dx$. Then, for an arbitrary constant c, the value of J – I equals **[JEE 2008, 3 (–1)]**

(A) $\frac{1}{2}\ell n\left(\frac{e^{4x}-e^{2x}+1}{e^{4x}+e^{2x}+1}\right)+c$ (B) $\frac{1}{2}\ell n\left(\frac{e^{2x}+e^{x}+1}{e^{2x}-e^{x}+1}\right)+c$

(C) $\frac{1}{2}\ell n\left(\frac{e^{2x}-e^{x}+1}{e^{2x}+e^{x}+1}\right)+c$ (D) $\frac{1}{2}\ell n\left(\frac{e^{4x}+e^{2x}+1}{e^{4x}-e^{2x}+1}\right)+c$

4.) $J = \int \frac{e^{-x}}{e^{-4x}+e^{-2x}+1} \cdot dx$

$= \int \frac{e^{3x}}{1+e^{2x}+e^{4x}} dx$

and

$I = \int \frac{e^{x}}{e^{4x}+e^{2x}+1} \cdot dx$

$\therefore J - I = \int \frac{e^{x}(e^{2x}-1)}{e^{4x}+e^{2x}+1} dx$

$\left\{ \text{Put } e^{x} = t \quad e^{x}\cdot dx = dt \right\}$

$= \int \frac{t^2-1}{t^4+t^2+1} \cdot dt$

$= \int \frac{1-\frac{1}{t^2}}{t^2+\frac{1}{t^2}+1} \cdot dt$

$= \int \frac{\left(1-\frac{1}{t^2}\right)\cdot dt}{\left(t+\frac{1}{t}\right)^2-1}$ $\left\{ \text{Put } t+\frac{1}{t} = z \Rightarrow \left(1-\frac{1}{t^2}\right)dt = dz \right\}$

$= \int \frac{dz}{z^2-1}$

$= \frac{1}{2\times1} \ln\left(\frac{z-1}{z+1}\right) + c$

$= \frac{1}{2} \ln\left(\frac{t^2-t+1}{t^2+t+1}\right) + c$

$= \boxed{\frac{1}{2} \ln\left(\frac{e^{2x}-e^{x}+1}{e^{2x}+e^{x}+1}\right) + c}$ Ans.

5. The integral $\int \frac{\sec^2 x}{(\sec x + \tan x)^{9/2}} dx$ equals (for some arbitrary constant K) **[JEE 2012, 3M, –1M]**

(A) $-\frac{1}{(\sec x + \tan x)^{11/2}}\left\{\frac{1}{11} - \frac{1}{7}(\sec x + \tan x)^2\right\} + K$

(B) $\frac{1}{(\sec x + \tan x)^{11/2}}\left\{\frac{1}{11} - \frac{1}{7}(\sec x + \tan x)^2\right\} + K$

(C) $-\frac{1}{(\sec x + \tan x)^{11/2}}\left\{\frac{1}{11} + \frac{1}{7}(\sec x + \tan x)^2\right\} + K$

(D) $\frac{1}{(\sec x + \tan x)^{11/2}}\left\{\frac{1}{11} + \frac{1}{7}(\sec x + \tan x)^2\right\} + K$

Sol:- let $I = \int \frac{\sec(x) \cdot \sec(x) \cdot dx}{(\sec x + \tan x)^{9/2}}$

Put $\sec x + \tan x = t$

$\Rightarrow (\sec(x)\tan(x) + \sec^2(x)) \cdot dx = dt$

$\Rightarrow \sec(x)[\sec(x) + \tan(x)] \cdot dx = dt$

$\Rightarrow \sec(x) \cdot t \cdot dx = dt$

$\Rightarrow \boxed{\sec(x) \cdot dx = \frac{1}{t} \cdot dt}$

& $\because \sec x + \tan x = t$

$\Rightarrow \sec x - \tan x = \frac{1}{t}$

add $2\sec(x) = t + \frac{1}{t}$

$\Rightarrow \boxed{\sec(x) = \frac{1}{2}\left(t + \frac{1}{t}\right)}$

$$\therefore \quad I = \frac{1}{2}\int \frac{\left(t+\frac{1}{t}\right)\cdot\frac{dt}{t}}{(t)^{9/2}}$$

$$= \frac{1}{2}\left[\int t^{-9/2}\cdot dt + \int t^{-13/2}\cdot dt\right]$$

$$= \left[\frac{t^{-7/2}}{-7} + \frac{t^{-11/2}}{-11}\right] + C$$

$$= -t^{-11/2}\left[\frac{t^2}{7} + \frac{1}{11}\right] + C$$

$$= \frac{-1}{(\sec(x)+\tan(x))^{11/2}}\left[\frac{1}{11} + \frac{(\sec x+\tan x)^2}{7}\right] + C$$

Ans.