Q Let $\vec{a} = \hat{i} + 4\hat{j} + 2\hat{k}$; $\vec{b} = 3\hat{i} - 2\hat{j} + 7\hat{k}$ and $\vec{c} = 2\hat{i} - \hat{j} + 4\hat{k}$. Find the vector $\vec{d}$ which is perpendicular to both $\vec{a}$ and $\vec{b}$ and satisfy $\vec{c} \cdot \vec{d} = 15$.

Sol^n

$$\vec{d} = \lambda(\vec{a} \times \vec{b}) \quad \text{—①—}$$

$$\vec{c} \cdot \vec{d} = 15$$

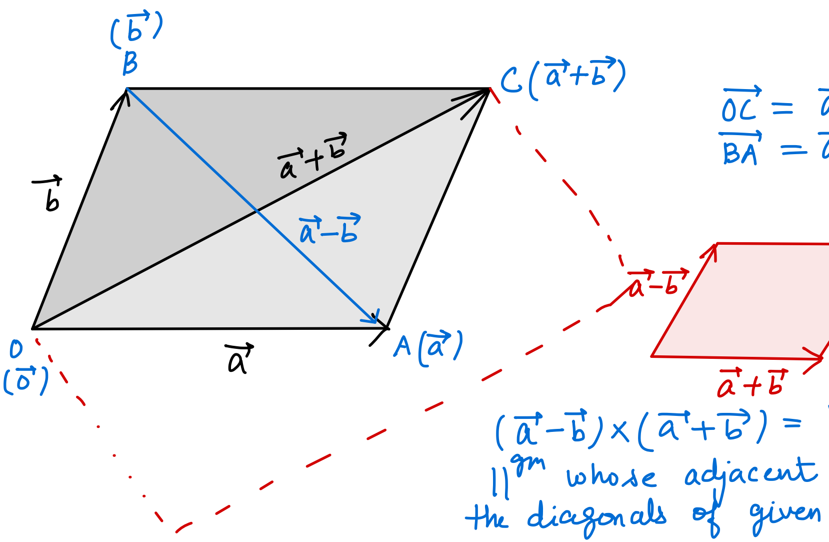
Ans: $\frac{5}{3}(32\hat{i} - \hat{j} - 14\hat{k})$

Q Prove the identity $(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) = 2(\vec{a} \times \vec{b})$ and give its geometrical interpretation.

Sol^n

LHS: $(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b}) = \vec{a} \times \vec{a} + \vec{a} \times \vec{b} - \vec{b} \times \vec{a} - \vec{b} \times \vec{b}$

$= 0 + \vec{a} \times \vec{b} + \vec{a} \times \vec{b} = 2(\vec{a} \times \vec{b})$



$\vec{OC} = \vec{a} + \vec{b}$
$\vec{BA} = \vec{a} - \vec{b}$

$(\vec{a} - \vec{b}) \times (\vec{a} + \vec{b})$ = Vector area of $\|^{gm}$ whose adjacent sides are the diagonals of given $\|$ OACB.

Q Let $\vec{OA}=\vec{a}$, $\vec{OB}=10\vec{a}+2\vec{b}$ and $\vec{OC}=\vec{b}$ where O, A & C are non–collinear points. Let 'p' denote the area of the quadrilateral OABC, and let 'q' denote the area of the parallelogram with OA and OC as adjacent sides. If p = kq. Find k.

Sol^n

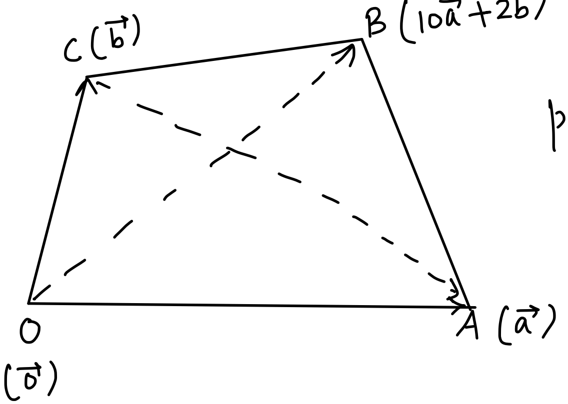
Area of quad OABC $= p = \frac{1}{2}\left|\vec{OB}\times\vec{AC}\right|$

$= \frac{1}{2}\left|(10\vec{a}+2\vec{b})\times(\vec{b}-\vec{a})\right|$

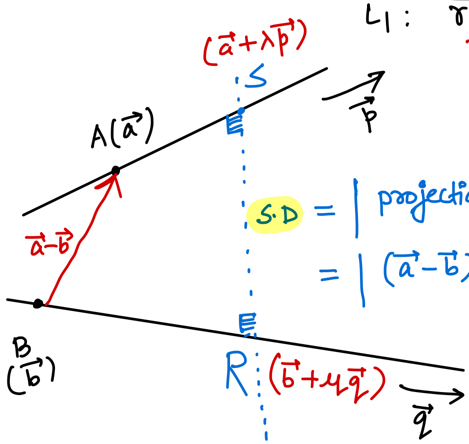
$p = \frac{1}{2}\left|12(\vec{a}\times\vec{b})\right| = 6\left|\vec{a}\times\vec{b}\right|$

$q = \left|\vec{a}\times\vec{b}\right|$

$p = Kq \Rightarrow \boxed{K=6}$



$L_1 : \vec{r} = \vec{a}+\lambda\vec{p}$



S.D $= \left|\text{projection } \vec{BA} \text{ on } \vec{n}\right|$

$= \left|(\vec{a}-\vec{b})\cdot\hat{n}\right| = \dfrac{\left|(\vec{a}-\vec{b})\cdot(\vec{p}\times\vec{q})\right|}{\left|\vec{p}\times\vec{q}\right|}$

$L_2 : \vec{r} = \vec{b}+\mu\vec{q}$

L = 0

dir^n vector : $\vec{p}\times\vec{q} = \vec{n}$

M-2

$\vec{RS} = \longleftrightarrow$

$\left.\begin{array}{l}\vec{RS}\cdot\vec{p}=0\\ \vec{RS}\cdot\vec{q}=0\end{array}\right\} \rightarrow \begin{array}{l}\lambda\\ \mu\end{array}$

SHORTEST DISTANCE BETWEEN 2 SKEW LINES :

Note that

(i) 2 lines in a plane if not || must intersect and 2 lines in a plane if not intersecting must be parallel. Conversely 2 intersecting or parallel lines must be coplanar.

(ii) In space, however we come across situation when two lines neither intersect nor ||. Two such lines in space are known as **skew lines** or **non coplanar lines**. S.D. between two such skew lines is the segment intercepted between the two lines which is perpendicular to both lines.

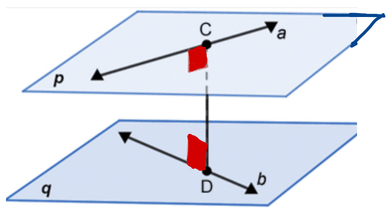
Method 1: Two ways to determine the S.D.

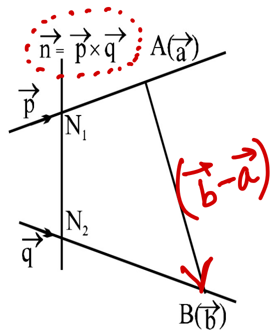
$L_1 : \vec{r} = \vec{a} + \lambda\vec{p}$

$L_2 : \vec{r} = \vec{b} + \mu\vec{q}$

$\vec{n} = \vec{p} \times \vec{q}$

$\overrightarrow{AB} = (\vec{b} - \vec{a})$





$$\text{S.D.} = \left|\text{Projection of } \overrightarrow{AB} \text{ on } \vec{n}\right| = \left|\frac{\overrightarrow{AB} \cdot \vec{n}}{|\vec{n}|}\right| = \left|\frac{(\vec{b} - \vec{a}) \cdot (\vec{p} \times \vec{q})}{|\vec{p} \times \vec{q}|}\right|$$

If S.D. = 0 $\Rightarrow$ lines are intersecting and hence coplanar.

Method 2 : p.v. of $N_1 = \vec{a} + \lambda\vec{p}$; p.v. of $N_2 = \vec{b} + \mu\vec{q}$

$\overrightarrow{N_1N_2} = (\vec{b} - \vec{a}) + (\mu\vec{q} - \lambda\vec{p})$

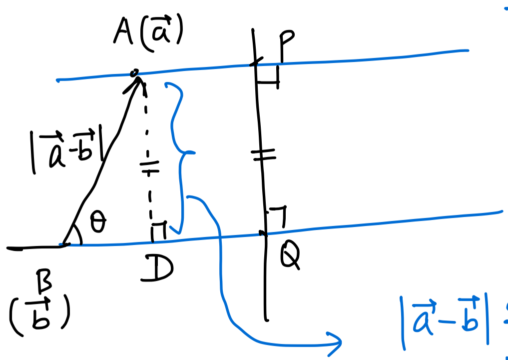
now $\overrightarrow{N_1N_2} \cdot \vec{p} = 0$ and $\overrightarrow{N_1N_2} \cdot \vec{q} = 0$ (two linear equations to get the unique values of λ and μ.)

One p.v's of N_1 and N_2 are known we can also determine the equation to the line of shortest distance and the S.D.

eg $\vec{r}=\hat{i}+2\hat{j}+\hat{k}+\lambda(\hat{i}-\hat{j}+\hat{k})$; $\vec{r}=2\hat{i}-\hat{j}-\hat{k}+\mu(2\hat{i}+\hat{j}+2\hat{k})$. Find SD.

Sol^n

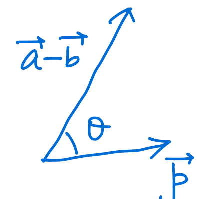
$$S.D=\left|(\hat{i}-3\hat{j}-2\hat{k})\cdot\frac{(\hat{i}-\hat{j}+\hat{k})\times(2\hat{i}+\hat{j}+2\hat{k})}{\left|(\hat{i}-\hat{j}+\hat{k})\times(2\hat{i}+\hat{j}+2\hat{k})\right|}\right|=\frac{3}{\sqrt{2}}$$



$\vec{r}=\vec{a}+\lambda\vec{p}$

$\vec{r}=\vec{b}+\mu\vec{p}$

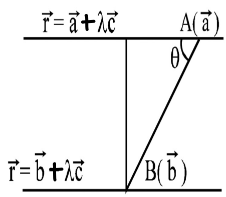
$$|\vec{a}-\vec{b}|\sin\theta=\,?=\frac{|(\vec{a}-\vec{b})\times\vec{p}|}{|\vec{p}|}$$



$$|(\vec{a}-\vec{b})\times\vec{p}|=|\vec{a}-\vec{b}|\,|\vec{p}|\sin\theta$$

Shortest Distance between two parallel lines

$$d=|\vec{a}-\vec{b}|\sin\theta\Rightarrow\left|\frac{(\vec{a}-\vec{b})\times\vec{c}}{|\vec{c}|}\right|$$



HW

Q Find the distance between the lines L_1 and L_2 given by

$$\vec{r}=(\hat{i}+2\hat{j}-4\hat{k})+\lambda(2\hat{i}+3\hat{j}+6\hat{k})$$

and $$\vec{r}=(3\hat{i}+3\hat{j}-5\hat{k})+\mu(2\hat{i}+3\hat{j}+6\hat{k})$$

Ans : $\dfrac{\sqrt{293}}{7}$

PRODUCT OF 3 OR MORE VECTORS :

When 3 vectors are involved with a dot or a cross between them, then 6 different symbols are

(1) $(\vec{a}.\vec{b})\vec{c}$ ~~(2)~~ $(\vec{a}.\vec{b}).\vec{c}$ ~~(3)~~ $(\vec{a}.\vec{b})\times\vec{c}$ ~~(4)~~ $(\vec{a}\times\vec{b})\vec{c}$ (5) $(\vec{a}\times\vec{b}).\vec{c}$ (6) $(\vec{a}\times\vec{b})\times\vec{c}$

(5) → STP (6) → VTP.

Scalar Triple product (or Box product) :

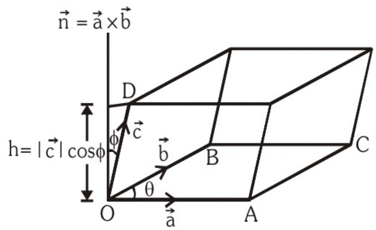
(a) **Definition :** $(\vec{a}\times\vec{b}).\vec{c} = |\vec{a}||\vec{b}|\sin\theta\hat{n}.\vec{c}$

$= |\vec{a}||\vec{b}||\vec{c}|\sin\theta\cos\phi = [\vec{a}\ \vec{b}\ \vec{c}]$

where $\theta = \vec{a} \wedge \vec{b}$; $\phi = \vec{n} \wedge \vec{c}$

but $|\vec{a}||\vec{b}|\sin\theta$ = area of $||^{gm}$ OACB and $|\vec{c}|\cos\phi = h$.

Hence $(\vec{a}\times\vec{b}).\vec{c}$ geometrically describes the volume of the parallelopiped whose three coterminous edges are the vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$.



(b) $(\vec{a}\times\vec{b}).\vec{c}$ is also known as box product and is written $[\vec{a}\,\vec{b}\,\vec{c}]$

(c) If the vector $\vec{c}$ also line in the plane of $\vec{a},\vec{b}$ then ϕ (angle between $\hat{n}$ and $\vec{c}$) is 90° and $[\vec{a}\,\vec{b}\,\vec{c}] = 0$.

Hence for three vectors, if $[\vec{a}\,\vec{b}\,\vec{c}] = 0 \Rightarrow \vec{a},\vec{b},\vec{c}$ are coplaner or linearly dependent & conversely.

(d) If $\hat{a},\hat{b},\hat{c}$ are unit vectors such that their box product is unity i.e. $[\hat{a}\ \hat{b}\ \hat{c}] = 1$

$\Rightarrow$ $\sin\theta\cos\phi = 1$. This is possible only if $\theta = 90°$ and $\phi = 0°$ i.e. $\hat{a}\ \hat{b}\ \hat{c}$ are mutually perpendicular to each other and conversely true e.g. $[\hat{i}\,\hat{j}\,\hat{k}] = 1$

General expression for $[\vec{a}\ \vec{b}\ \vec{c}]$: When $\vec{a},\vec{b},\vec{c}$ are expressed in terms of $\hat{i},\hat{j},\hat{k}$

$$[\vec{a}\ \vec{b}\ \vec{c}] = (\vec{a}\times\vec{b}).\vec{c} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} . (c_1\hat{i}+c_2\hat{j}+c_3\hat{k}) = \begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Q If $\vec{u}=2\hat{i}-\hat{j}+\hat{k}$; $\vec{v}=\hat{i}+\hat{j}+\hat{k}$ and $\vec{w}$ is a unit vector then the maximum value of $[\vec{u}\ \vec{v}\ \vec{w}]$ is

Sol^n

$$[\vec{u}\ \vec{v}\ \vec{w}] = (\vec{u}\times\vec{v}).\vec{w}$$

$$= |\vec{u}\times\vec{v}|\ \underbrace{|\vec{w}|}_{1}\ \underbrace{\cos\theta}_{max\,=\,1}$$

$$[\vec{u}\ \vec{v}\ \vec{w}]_{max} = \sqrt{6.3-(2-\cancel{1}+\cancel{1})^2}$$

$$= \sqrt{14}$$

Properties of STP :

(i) Scalar triple product of three vectors when two of them are collinear / linearly dependent or equal is zero. (two rows identical $\Rightarrow$ determinant is zero).

(ii) If the cyclic order of vector retains then the value of the STP does not change

i.e. $[\vec{a}\,\vec{b}\,\vec{c}] = [\vec{b}\,\vec{c}\,\vec{a}] = [\vec{c}\,\vec{a}\,\vec{b}]$ and if the cyclic order is changed, then the value of STP changes in sign (Prove considering properties of determinants i.e. $[\vec{a}\,\vec{c}\,\vec{b}] = -[\vec{a}\,\vec{b}\,\vec{c}]$

(iii) The position of dot and cross can be interchanged provided the cyclic order of the vectors $\vec{a},\vec{b},\vec{c}$ remains undisturbed.

we have $(\vec{a}\times\vec{b}).\vec{c} = (\vec{b}\times\vec{c}).\vec{a} = (\vec{c}\times\vec{a}).\vec{b}$

$$\vec{a}.(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b}).\vec{c}$$

Also
$$\left.\begin{aligned}(\vec{a}\times\vec{b}).\vec{c} &= \vec{c}.(\vec{a}\times\vec{b})\\ (\vec{b}\times\vec{c}).\vec{a} &= \vec{a}.(\vec{b}\times\vec{c})\\ (\vec{c}\times\vec{a}).\vec{b} &= \vec{b}.(\vec{c}\times\vec{a})\end{aligned}\right]$$
As dot is commutative

(iv) $[\vec{a}+\vec{b}\ \vec{c}\ \vec{d}] = [\vec{a}\ \vec{c}\ \vec{d}] + [\vec{b}\ \vec{c}\ \vec{d}]$

$$[\vec{a}+\vec{b}\quad \vec{c}\quad \vec{d}] = [\vec{a}\ \vec{c}\ \vec{d}] + [\vec{b}\ \vec{c}\ \vec{d}]$$

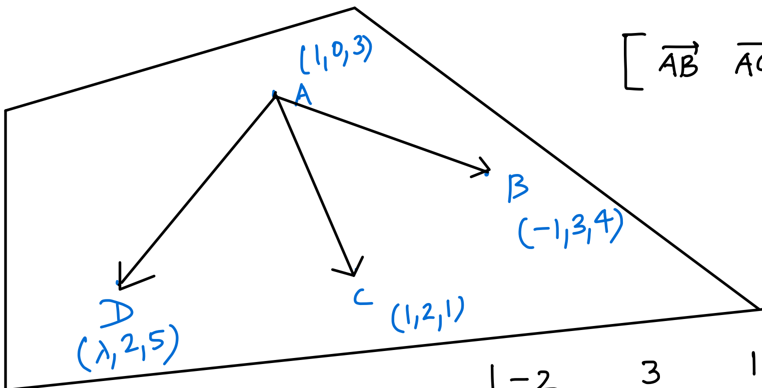
Proof : Let $\vec{a} = a_1\hat{i}+a_2\hat{j}+a_3\hat{k}$ and so on

$$\text{Now L.H.S.}=\begin{vmatrix} a_1+b_1 & a_2+b_2 & a_3+b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = \begin{vmatrix} a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix} + \begin{vmatrix} b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \\ d_1 & d_2 & d_3 \end{vmatrix} = [\vec{a}\,\vec{c}\,\vec{d}]+[\vec{b}\,\vec{c}\,\vec{d}]$$

(v) For right handed system $[\vec{a}\,\vec{b}\,\vec{c}] > 0$ and for left handed system $[\vec{a}\,\vec{b}\,\vec{c}] < 0$ $\Bigg]$ where $\vec{a}, \vec{b}, \vec{c}$ are non-coplanar

Q Find the value of λ for which points with p.v.'s A(1,0,3) ; B(–1,3,4) ; C(1,2,1) and D $(\lambda, 2,5)$ are in the same plane.

Solⁿ



$$[\overrightarrow{AB}\ \ \overrightarrow{AC}\ \ \overrightarrow{AD}] = 0.$$

$$\begin{vmatrix} -2 & 3 & 1 \\ 0 & 2 & -2 \\ \lambda-1 & 2 & 2 \end{vmatrix} = 0$$

Q Find the value of p for which the vectors $(p+1)\hat{i} - 3\hat{j} + p\hat{k}$; $p\hat{i} + (p+1)\hat{j} - 3\hat{k}$ and $-3\hat{i} + p\hat{j} + (p+1)\hat{k}$ are linearly dependent/coplanar .

$$\begin{vmatrix} p+1 & -3 & p \\ p & p+1 & -3 \\ -3 & p & p+1 \end{vmatrix} = 0.$$

Note :-

$[\vec{a}+\vec{b}, \vec{b}+\vec{c}, \vec{c}+\vec{a}] = 2[\vec{a}\,\vec{b}\,\vec{c}]$. This identity can be geometrically interpreted as: (Volume of a ||piped whose three coterminous edges are the face diagonals of the ||piped is twice the volume of the ||piped, whose three coterminous edges are the vectors $\vec{a}, \vec{b}, \vec{c}$). This is also conclusive that if $\vec{a}, \vec{b}, \vec{c}$ are coplanar then $\vec{a}+\vec{b}$, $\vec{b}+\vec{c}$, $\vec{c}+\vec{a}$ are also coplanar.

Rem

$$\boxed{\left[\vec{a}+\vec{b} \quad \vec{b}+\vec{c} \quad \vec{c}+\vec{a}\right] = 2\left[\vec{a}\ \vec{b}\ \vec{c}\right]}$$

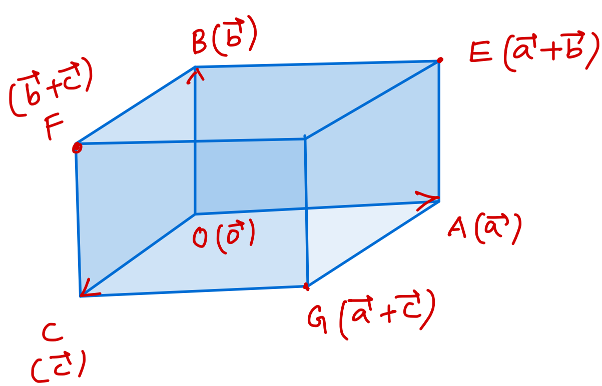
Proof :

LHS : $\left((\vec{a}+\vec{b}) \times (\vec{b}+\vec{c})\right) \cdot (\vec{c}+\vec{a})$

$$(\vec{a}\times\vec{b} + \vec{a}\times\vec{c} + \vec{b}\times\vec{c}) \cdot (\vec{c}+\vec{a})$$

$$[\vec{a}\ \vec{b}\ \vec{c}] + \cancelto{0}{[\vec{a}\ \vec{b}\ \vec{a}]} + \cancelto{0}{[\vec{a}\ \vec{c}\ \vec{c}]} + \cancelto{0}{[\vec{a}\ \vec{c}\ \vec{a}]}$$

$$+ \cancelto{0}{[\vec{b}\ \vec{c}\ \vec{c}]} + [\vec{b}\ \vec{c}\ \vec{a}] = 2[\vec{a}\,\vec{b}\,\vec{c}]$$



Rem

* $[\vec{a}-\vec{b}\ \ \vec{b}-\vec{c}\ \ \vec{c}-\vec{a}]$ is always zero.

$\Rightarrow$ $\vec{a}-\vec{b}$, $\vec{b}-\vec{c}$, $\vec{c}-\vec{a}$ are always Coplanar whether $\vec{a}, \vec{b}, \vec{c}$ are Coplanar or <u>not</u>.

Rem

$$\boxed{\vec{a}\times\vec{b} = [\hat{i}\ \vec{a}\ \vec{b}]\hat{i} + [\hat{j}\ \vec{a}\ \vec{b}]\hat{j} + [\hat{k}\ \vec{a}\ \vec{b}]\hat{k}}$$

We know,

$$\vec{r} = (\vec{r}\cdot\hat{i})\hat{i} + (\vec{r}\cdot\hat{j})\hat{j} + (\vec{r}\cdot\hat{k})\hat{k}$$

put $\vec{r} = \vec{a}\times\vec{b}$

<u>HW</u> :

O–1 Q21 to 32. & 36 to 39.

S–1 Q1 to 8.