5. If P (1, 2), Q (4, 6), R (5, 7) & S (a, b) are the vertices of a parallelogram PQRS, then : **[JEE 98, 2M]**

(A) a = 2, b = 4 (B) a = 3, b = 4 (C) a = 2, b = 3 (D) a = 3, b = 5

6. The diagonals of a parallelogram PQRS are along the lines $x + 3y = 4$ and $6x - 2y = 7$. Then PQRS must be a : **[JEE 98, 2M]**

(A) rectangle (B) square (C) cyclic quadrilateral (D) rhombus

8. The incentre of the triangle with vertices $\left(1, \sqrt{3}\right)$, (0, 0) and (2, 0) is - **[JEE 2000 (Scr.) 1 M]**

(A) $\left(1, \dfrac{\sqrt{3}}{2}\right)$ (B) $\left(\dfrac{2}{3}, \dfrac{1}{\sqrt{3}}\right)$ (C) $\left(\dfrac{2}{3}, \dfrac{\sqrt{3}}{2}\right)$ (D) $\left(1, \dfrac{1}{\sqrt{3}}\right)$

9. Let PS be the median of the triangle with vertices, P (2, 2), Q (6, – 1) and R (7, 3). The equation of the line passing through (1, – 1) and parallel to PS is : **[JEE 2000 (Scr.) 1 M]**

(A) $2x - 9y - 7 = 0$ (B) $2x - 9y - 11 = 0$ (C) $2x + 9y - 11 = 0$ (D) $2x + 9y + 7 = 0$

11. Let P = (– 1, 0), Q = (0, 0) and R = $(3, 3\sqrt{3})$ be three points. Then the equation of the bisector of the angle PQR is **[JEE 2002 (Scr.) 3M]**

(A) $\dfrac{\sqrt{3}}{2}x + y = 0$ (B) $x + \sqrt{3}y = 0$ (C) $\sqrt{3}x + y = 0$ (D) $x + \dfrac{\sqrt{3}}{2}y = 0$

15 Let O(0, 0), P (3, 4), Q(6, 0) be the vertices of the triangle OPQ. The point R inside the triangle OPQ is such that the triangles OPR, PQR, OQR are of equal area. The coordinates of R are

[JEE 2007, 3M]

(A) $(4/3, 3)$ (B) $(3, 2/3)$ (C) $(3, 4/3)$ (D) $(4/3, 2/3)$

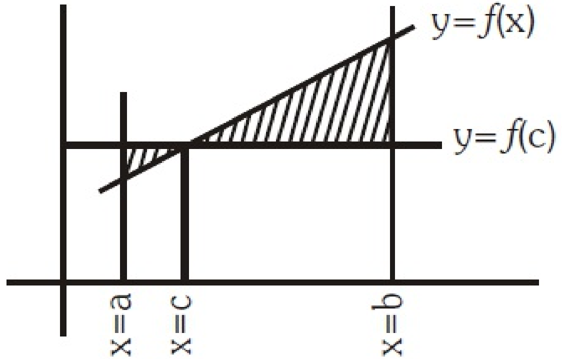
D

CONCEPT OF VARIABLE AREA (greatest and least value) :

If $y = f(x)$ is a monotonic function in (a, b) then the area bounded by the ordinates at $x = a$, $x = b$, $y = f(x)$ and $y = f(c)$, [where $c \in (a, b)$] is minimum when $c = \dfrac{a+b}{2}$.

$$A = \int_a^c (f(c) - f(x))dx + \int_c^b (f(x) - f(c))dx$$

differentiate w.r.to c



$$= f(c)(c-a) - f(c)(b-c) - \int_a^c f(x)dx - \int_b^c f(x)dx$$

Let $\quad g(c) = f(c)(2c - a - b) - \int_a^c f(x)dx - \int_b^c f(x)dx$

$\quad g'(c) = f'(c)(2c - a - b)$

area is min. when $c = \dfrac{a+b}{2}$ & max at $c = a$ or $c = b$

Q. $O(0,0)$, $A(2,0)$, $B(1, \frac{1}{\sqrt{3}})$. $\triangle OAB$

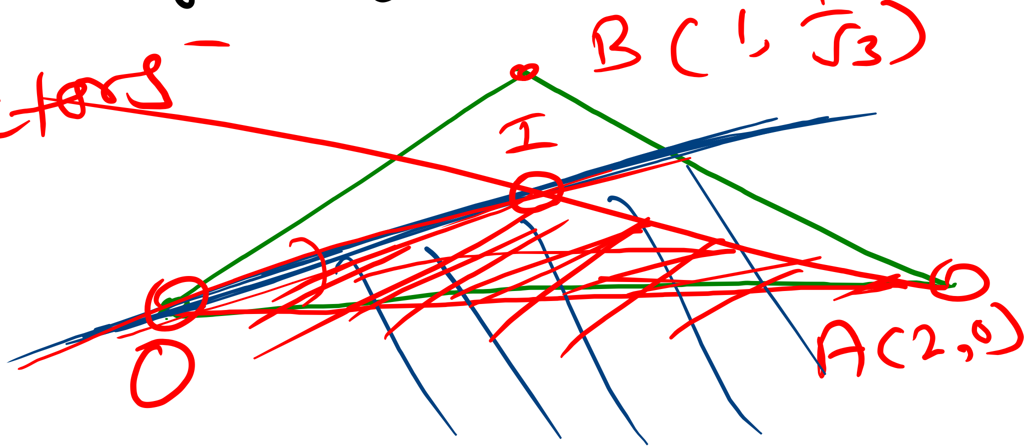
Let R be the region consisting of all those points P inside $\triangle OAB$ that satisfy

$$d(P, OA) \le \min\{d(P, OB), d(P, AB)\}$$

where d denotes the distance of P from the corresponding line

P.O.I of angle bisectors

$\Rightarrow$ I



Q. For a point P in the plane, let $d_1(P)$ and $d_2(P)$ be the distance of the point P from the lines $x-y=0$ and $x+y=0$ respectively. The area of the region R consisting of all points P lying in the first quadrant of the plane and satisfying $\boxed{2 \le d_1(P)+d_2(P) \le 4}$, is *(JEE Adv. 2014)*

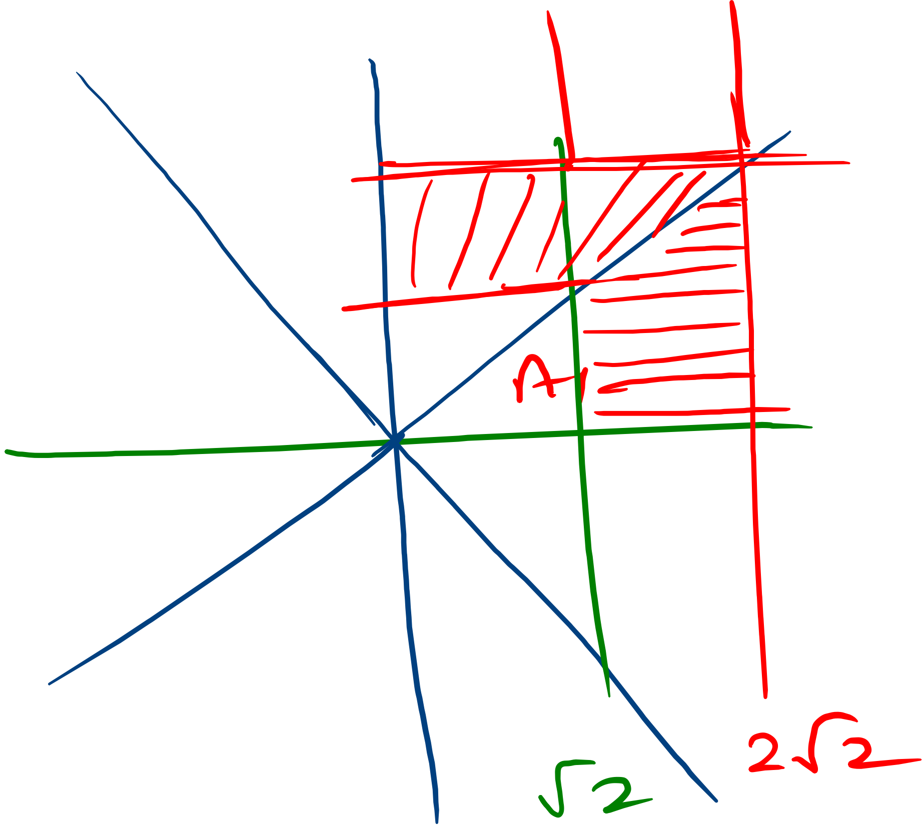
$$2 \le \frac{|x-y|}{\sqrt{2}} + \frac{|x+y|}{\sqrt{2}} \le 4$$

$$\Rightarrow 2\sqrt{2} \le |x-y| + |x+y| \le 4\sqrt{2}$$

For A_1

$$2\sqrt{2} \le x - \cancel{y} + x + \cancel{y} \le 4\sqrt{2}$$

$$\Rightarrow \boxed{\sqrt{2} \le x \le 2\sqrt{2}}$$



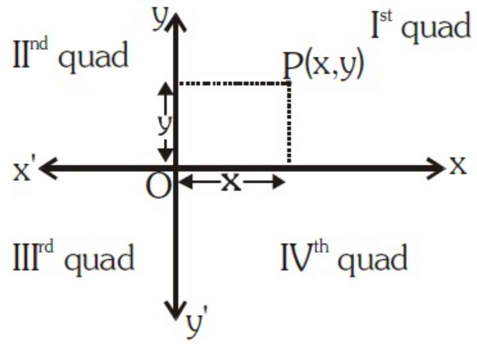
Required $A = (2\sqrt{2})^2 - (\sqrt{2})^2$

$= 8 - 2 = 6$

Basic

CO-ORDINATE SYSTEM :

(a) Rectangular Cartesian co-ordinates system :



x-coordinate $\rightarrow$ abscissa

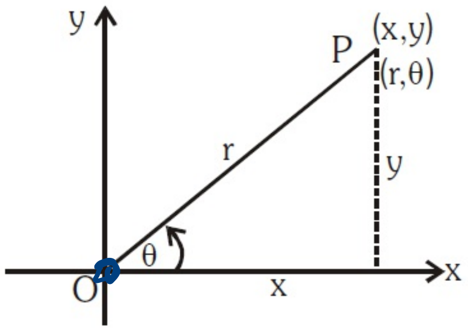
y-coordinate $\rightarrow$ ordinate

(b) Polar system :

If (x,y) are cartesian co-ordinates of a point P, then : $x = r\cos\theta$, $y = r\sin\theta$

and $\quad r = \sqrt{x^2 + y^2}$, $\qquad \theta = \tan^{-1}\left(\frac{y}{x}\right)$

where r & θ are both variable. 'r' is called radius vector of point P



(r, θ)

$\left(-5, \frac{\pi}{4}\right)$

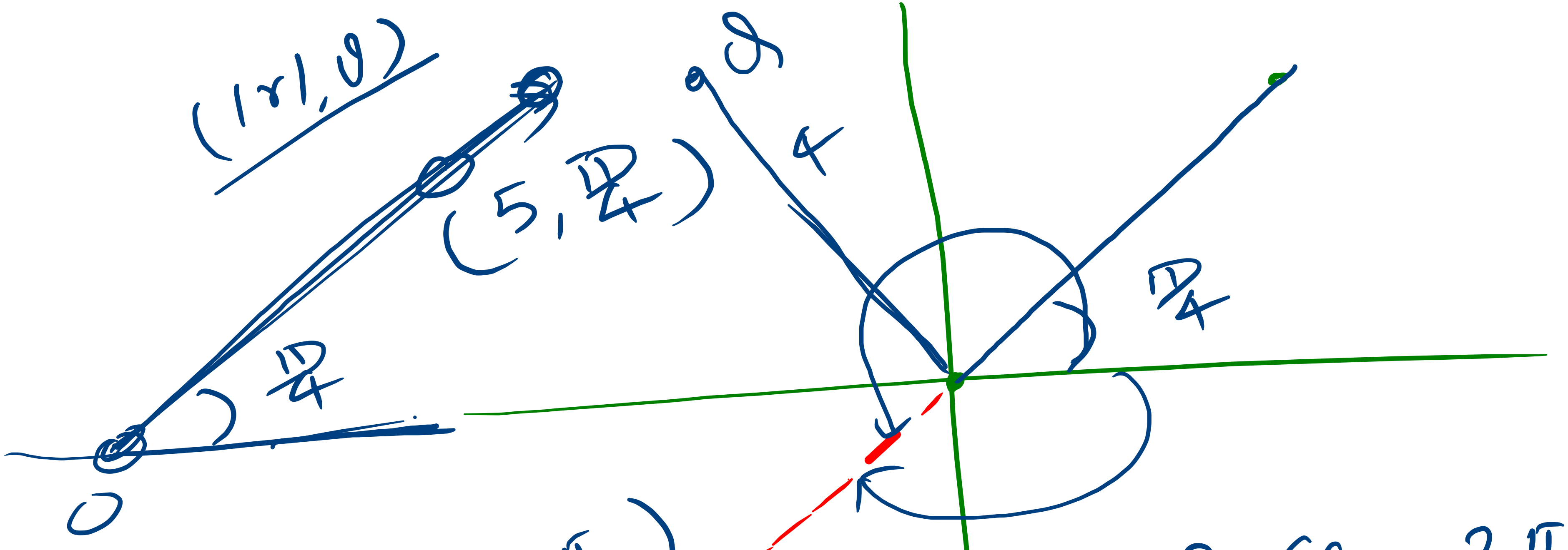
Note :

1. Radius vector is positive if measured from O along line bounding θ and negative if measured in opposite direction.

2. Polar coordinates of a point are not unique e.g. $\left(-3, \frac{\pi}{6}\right) \equiv \left(3, \frac{7\pi}{6}\right) \equiv \left(3, \frac{-5\pi}{6}\right)$

2D

$(|r|, \theta)$



$(5, \frac{5\pi}{4}) = (-5, \frac{\pi}{4}) = (5, \frac{-3\pi}{4})$

$Q(4, \frac{2\pi}{3})$

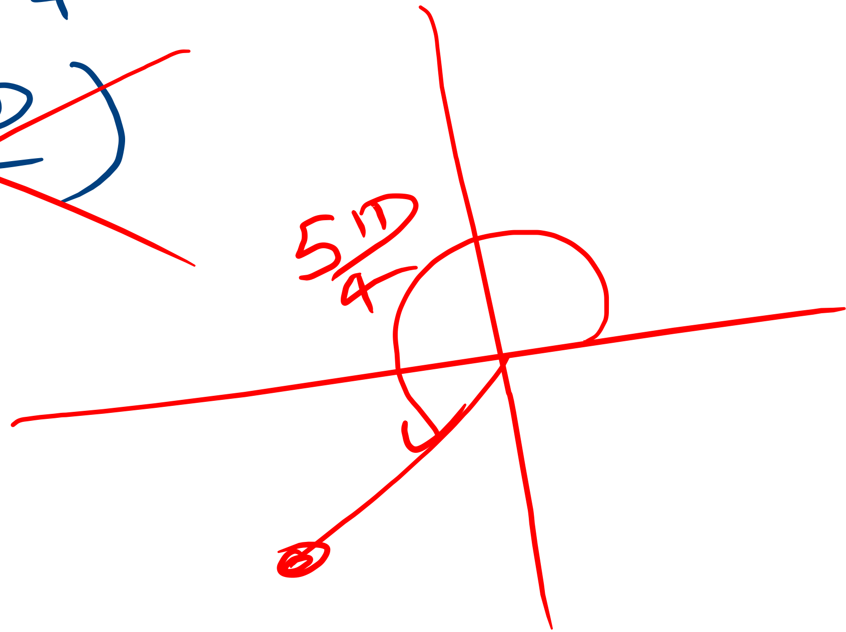
$(-1, -1)$ convert into polar system

$r = \sqrt{1+1} = \sqrt{2}$,

$\theta = \frac{5\pi}{4}$

$\left(\sqrt{2}, \frac{5\pi}{4}\right)$ ✓

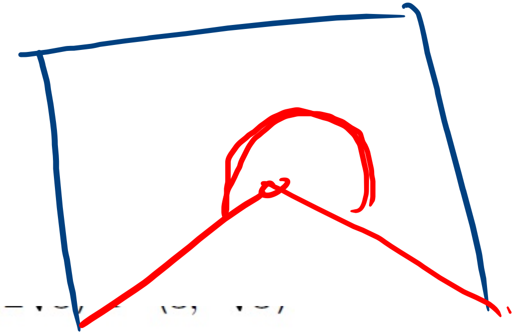
$\cancel{\left(\sqrt{2}, \frac{\pi}{4}\right)}$



DISTANCE FORMULA :

If $A(x_1,y_1)$ and $B(x_2,y_2)$ are two points, then

$$AB = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$



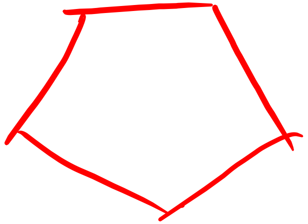
Concave

(c) Nature of quadrilaterals :

Convex polygon : A polygon that has all interior angle less than 180°

Concave polygon : A polygon that has atleast one interior angle greater than 180°.

Brief description of euclidian figures - convex polygon :



Convex

Family of Quadrilateral
(Convex Figures)

- Parallelogram
  - Rectangle
  - Rhombus
    - Square (from Rectangle and Rhombus)
- Trapezium
  ↓
  Isosceles/Equilateral Trapezium (legs are equal)
- Kite
  (diagonals perp. but only one diagonal divides the figure into two congruent triangles)
- Cyclic Quad
  opposite angles are supplementary

Parallelogram - Diagonals bisect each other and area = p_1p_2 cosecθ, where p_1 & p_2 are distances between two pairs of opposite sides & θ is the angle between any two adjacent sides .

Rectangle - Diagonals are equal & bisect each other.

Rhombus - Diagonals perpendicular bisectors $\left(\text{Area} = \frac{d_1d_2}{2}\right)$, where d_1 and d_2 are diagonals.

Square - Diagonals are equal & perpendicular bisectors and area = $d^2/2$ (where d is the diagonal)

Trapezium - Exactly one pair of parallel sides.

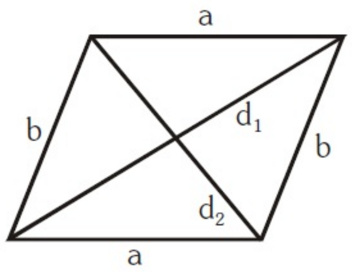
Note :

(i) If the distance between the pair of opposite sides of a parallelogram is equal $\Rightarrow$ it is a rhombus.

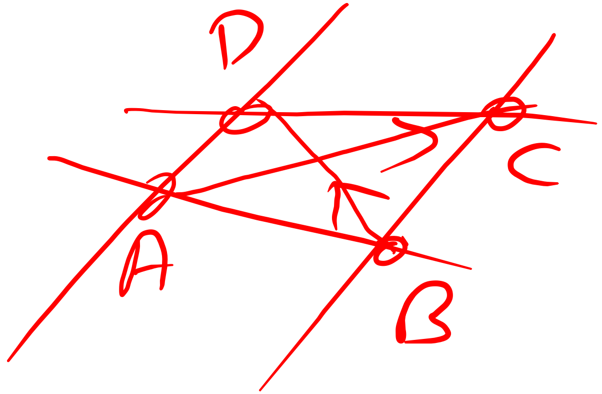
(ii) Diagonals of an isosceles trapezium are equal.

(iii) In a parallelogram $2(a^2 + b^2) = d_1^2 + d_2^2$

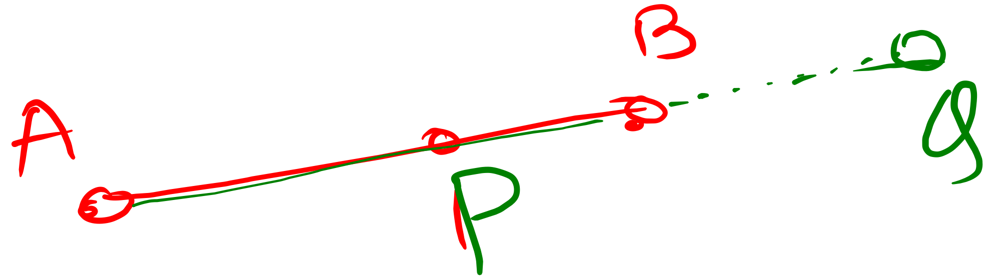
(sum of the squares of the sides of parallelogram is equal to the sum of the squares of its diagonal.)



Area = $\frac{1}{2}|\bar{d}_1 \times \bar{d}_2|$



SECTION FORMULA :



(a) Internal division : $C \equiv \left(\dfrac{mx_2 + nx_1}{m+n}, \dfrac{my_2 + ny_1}{m+n} \right)$

$$\text{Int-}\quad \frac{AP}{PB} = \frac{m}{n}$$

(b) External division : $C \equiv \left(\dfrac{mx_2 - nx_1}{m-n}, \dfrac{my_2 - ny_1}{m-n} \right)$

$AC : BC = m : n$

If $\dfrac{AC}{BC} < 1 \Rightarrow B$ lies between A and C.

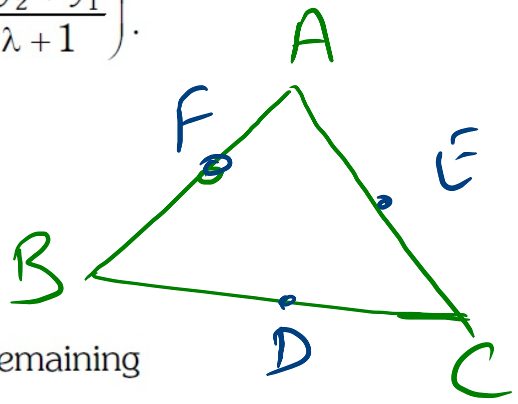
$$\text{Ex}\quad \frac{AQ}{QB} = \frac{m}{n}$$

If $\dfrac{AC}{BC} > 1 \Rightarrow A$ lies between B and C.

(c) **Harmonic conjugate :** If P divides AB internally in the ratio m : n & Q divides AB externally in the ratio m : n then P & Q are said to be harmonic conjugate of each other w.r.t. AB.

Mathematically ; $\dfrac{\mathbf{2}}{\mathbf{AB}} = \dfrac{\mathbf{1}}{\mathbf{AP}} + \dfrac{\mathbf{1}}{\mathbf{AQ}}$ i.e. AP, AB & AQ are in **H.P.**

(i) Coordinates of general point dividing AB in the ratio $\lambda : 1$ (internally or externally) are $\left(\frac{\lambda x_2 + x_1}{\lambda + 1}, \frac{\lambda y_2 + y_1}{\lambda + 1}\right)$.



(ii) The segment AB divided by line $ax + by + c = 0$ in ratio $\frac{\lambda}{1} = -\frac{(ax_1 + by_1 + c)}{ax_2 + by_2 + c}$

If λ comes out (+)ve $\Rightarrow$ internal division

If λ comes out (–)ve $\Rightarrow$ external division

(iii) **Mid point theorem :** If a line joining mid point of any two sides of triangle, then it is parallel to remaining side and half of it.

(iv) Figure formed by joining the middle points of a quadrilateral is a parallelogram.

(v) If diagonals of a quadrilateral bisect at right angle then it is a rhombus.

Q: Co-ordinates of three points A, B and C are (4, 1); (5 – 2) and (3, 7). Find the possible co-ordinates of 'D' so that figure formed is a parallelogram. **[Ans.** (2, 10); (4, 4); (6, –8)] How-

$9-3,\ 1-2-7$

$(6, -8)$

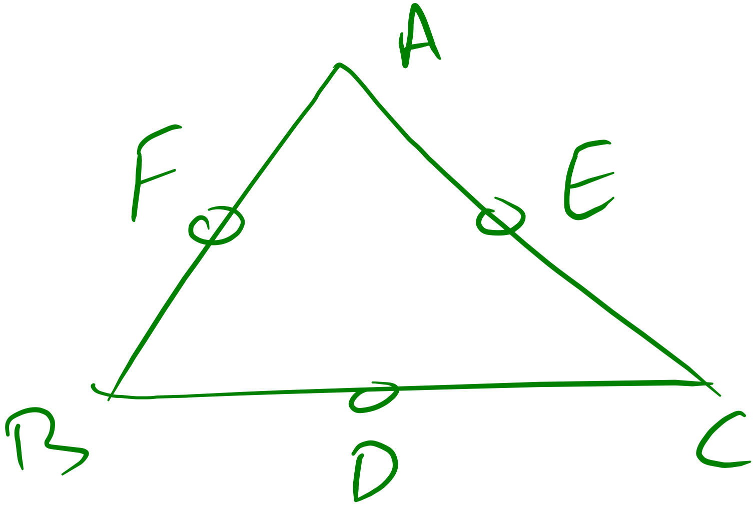
$7-5,\ 8+2$

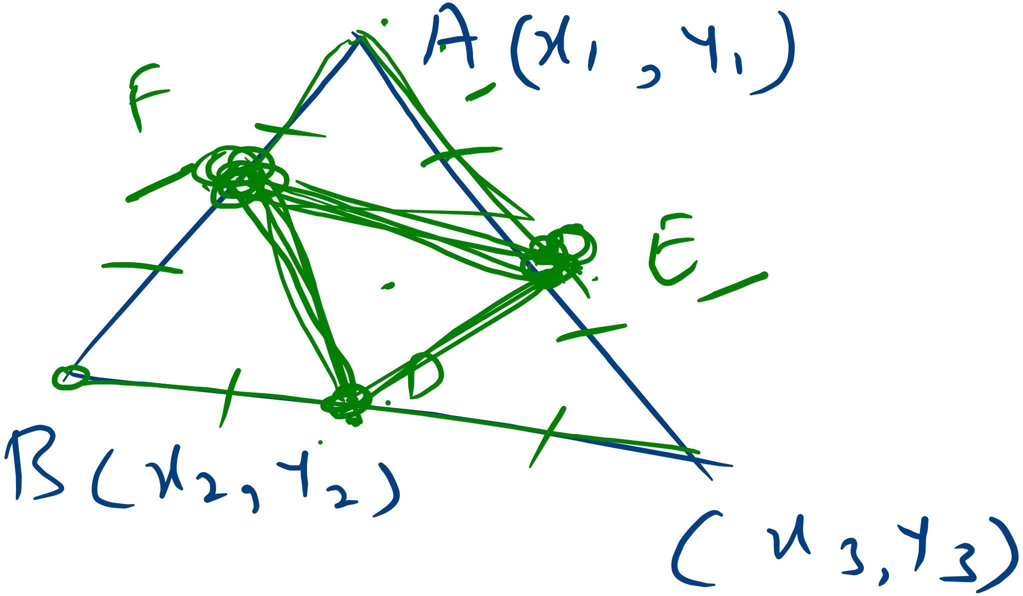
$(2, 10)$

$D(x_4, y_4)$

$E(x_5, y_5)$

$F(x_6, y_6)$





$$x_6 + x_5 - x_1 = x_4$$

$$\left.\begin{aligned} x_F + x_E - x_D &= x_A \\ x_F + x_E - x_A &= x_D \end{aligned}\right\}$$

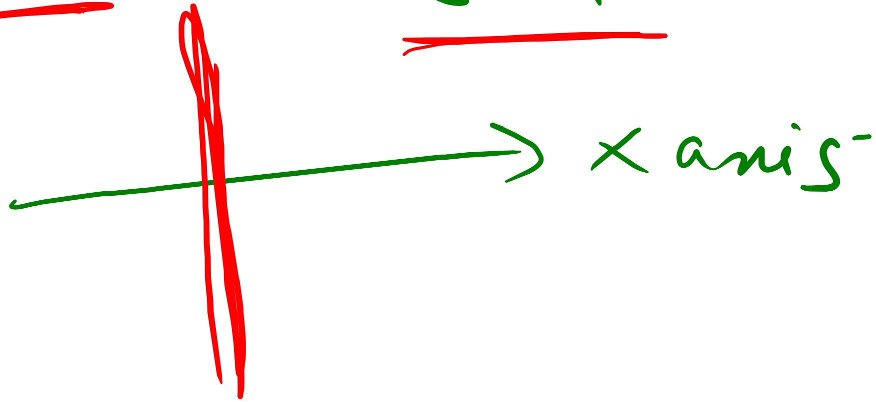
Q. Four points (x_1,y_1), (x_2,y_2), (x_3,y_3) & (x_4,y_4) are such that $\sum_{i=1}^{4}(x_i^2+y_i^2)\le 2(x_1x_3+x_2x_4+y_1y_2+y_3y_4)$, then prove that points are vertices of a rectangle.

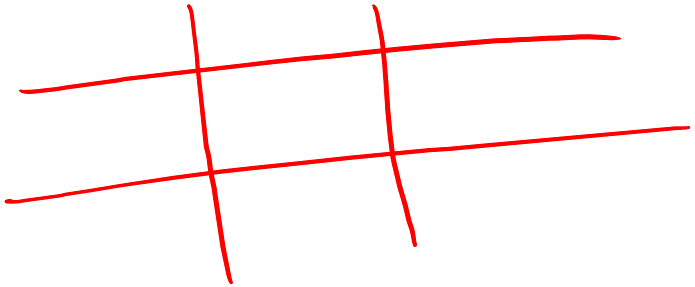
$$x_1^2+y_1^2+x_2^2+y_2^2+x_3^2+y_3^2+x_4^2+y_4^2 \le 2x_1x_3+2x_2x_4+2y_1y_2+2y_3y_4$$

$$(x_1-x_3)^2+(x_2-x_4)^2+(y_1-y_2)^2+(y_3-y_4)^2\le 0$$

$x_1-x_3=0$, $x_2-x_4=0$, $y_1-y_2=0$, $y_3-y_4=0$

x co-ordinate same $\Rightarrow$

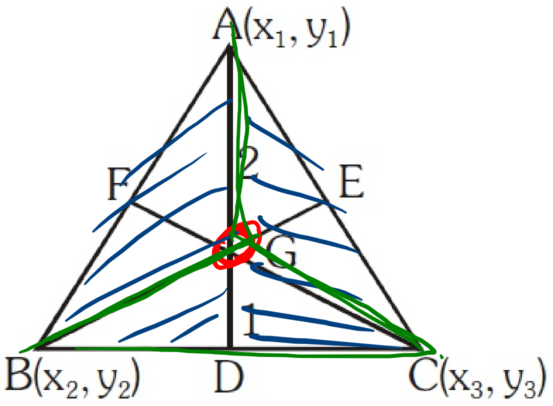




CO-ORDINATES OF SOME PARTICULAR POINTS :

(a) Centroid :

Point of intersection of the medians (line joining the mid point of sides and opposite vertices) is the centroid. Centroid divides the median in the ratio of 2:1.



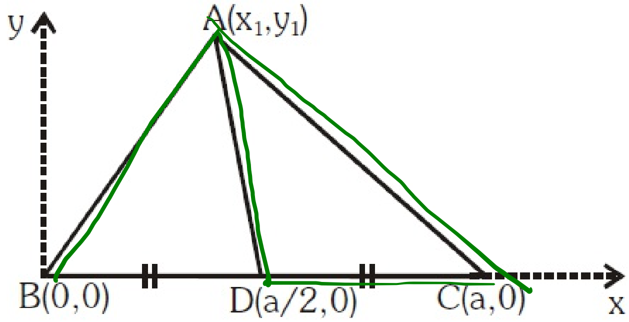
Co-ordinates of centroid $G\left(\dfrac{x_1 + x_2 + x_3}{3}, \dfrac{y_1 + y_2 + y_3}{3}\right)$

i.e. AM of x & y co-ordinates of the vertices

Note :

(i) **Apollonius theorem :**

$$\boxed{AB^2 + AC^2 = 2AD^2 + 2DC^2}$$

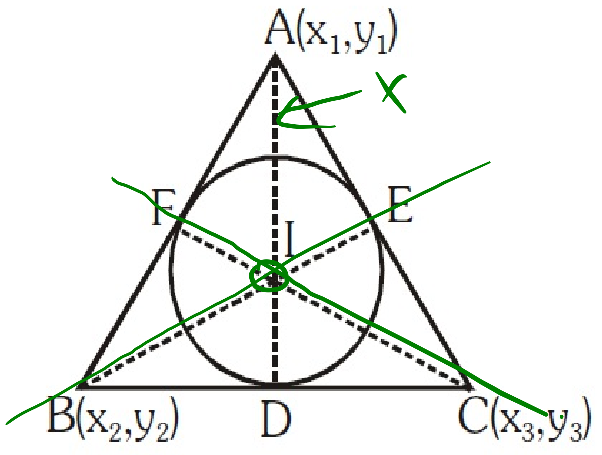


(ii) If ℓ_1, ℓ_2, ℓ_3 are length of medians, then $\ell_1^2 + \ell_2^2 + \ell_3^2 = \dfrac{3}{4}(a^2 + b^2 + c^2)$

(iii) Centroid of ΔABC is centroid of triangle formed by joining the points dividing the three sides in same ratio.

(b) **Incentre :**

The incentre is the point of intersection of internal bisectors of the angles of a triangle. Also it is a centre of a circle touching all the sides of a triangle.



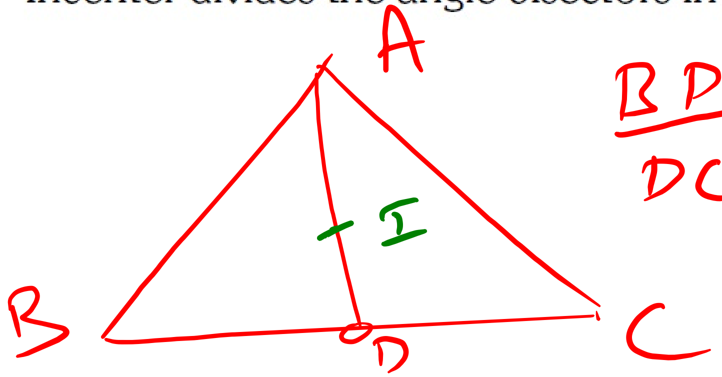
Co-ordinates of incentre $I\left(\dfrac{ax_1 + bx_2 + cx_3}{a+b+c}, \dfrac{ay_1 + by_2 + cy_3}{a+b+c}\right)$

where a, b, c are the sides of triangle ABC.

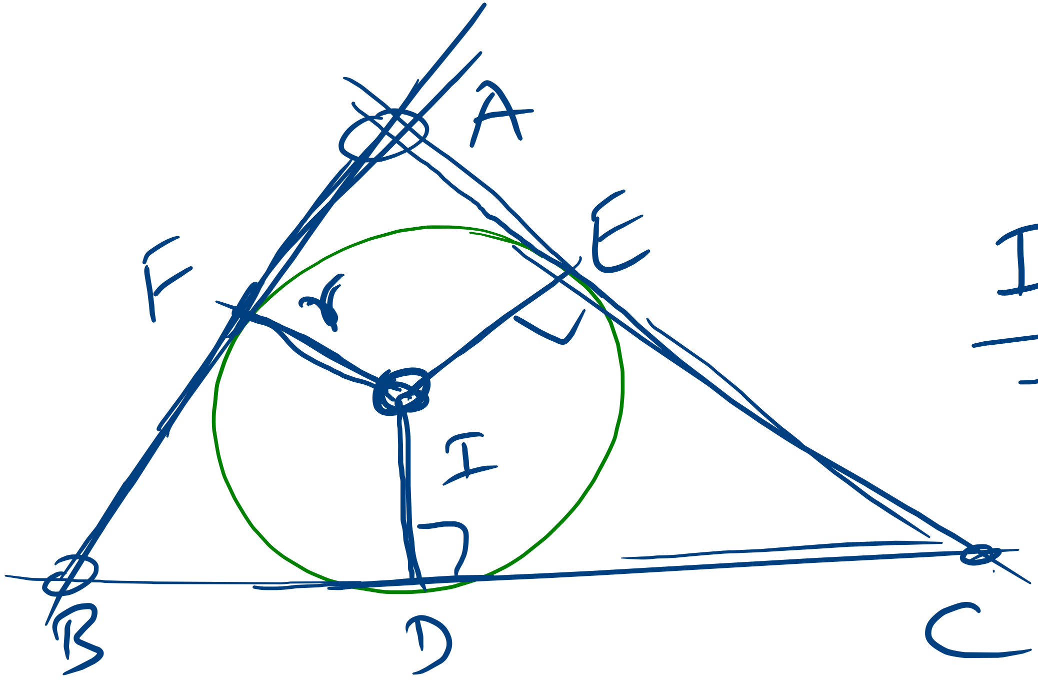
Note :

(i) Angle bisector divides the opposite sides in the ratio of remaining sides. e.g. $\dfrac{BD}{DC} = \dfrac{AB}{AC} = \dfrac{c}{b}$

(ii) Incenter divides the angle bisectors in the ratio $(b+c) : a$, $(c+a) : b$, $(a+b) : c$



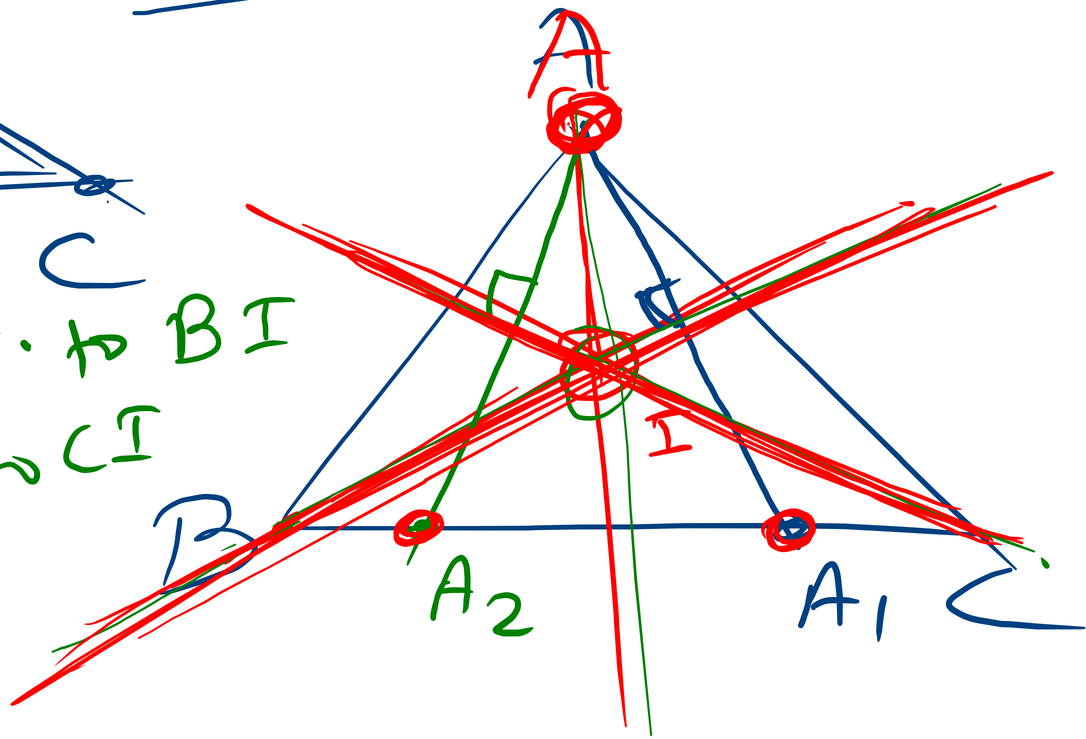
$\dfrac{BD}{DC} = \dfrac{AB}{AC}$, $\dfrac{AI}{ID} = \dfrac{b+c}{a}$

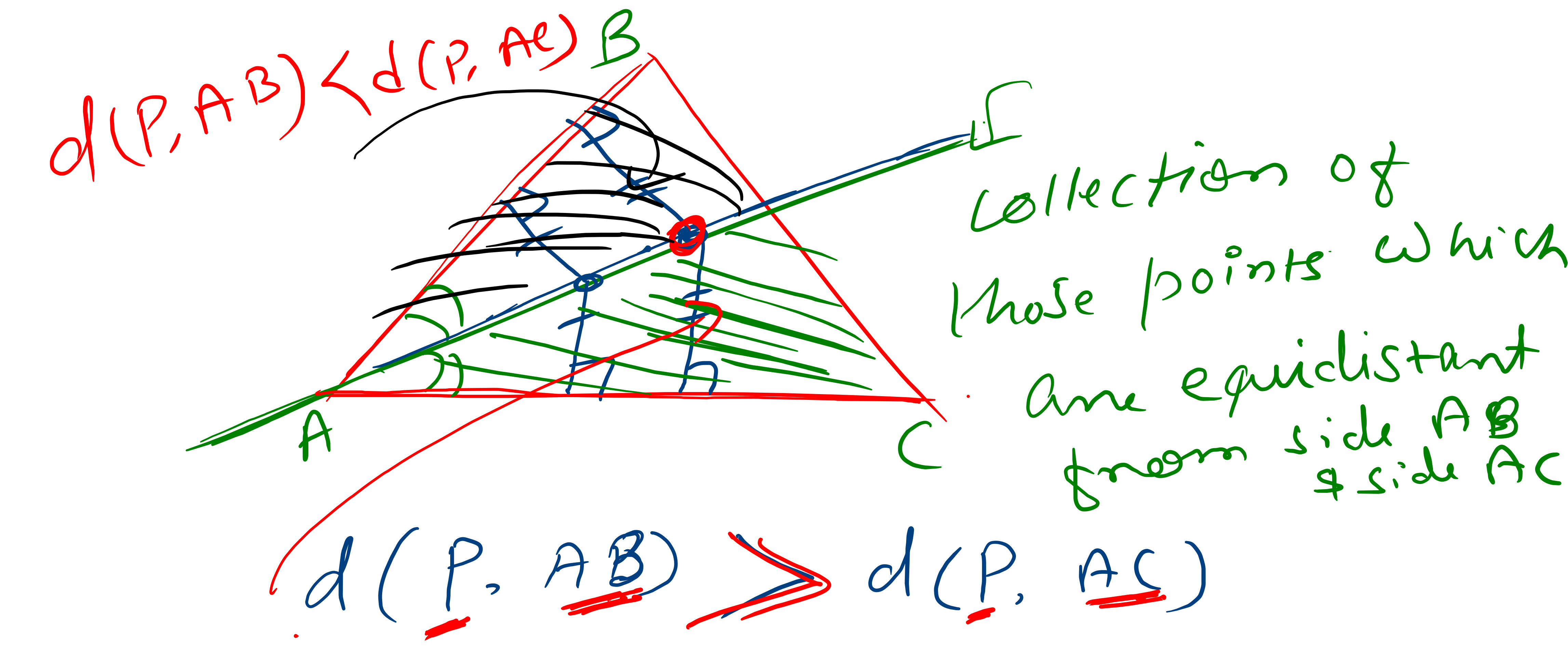


$$ID = IE = IF = r$$

$A_1 \equiv$ Image of A w.r.t BI

$A_2 \equiv$ Image of A w.r.t CI

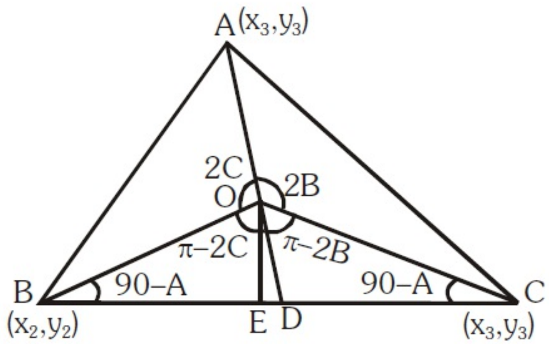



d(P, AB) < d(P, AC)
B
A
C
Collection of those points which are equidistant from side AB & side AC
d(P, AB) > d(P, AC)

Ans. $(2-\sqrt{2}, 2-\sqrt{2})$

(c) Circumcentre :

It is the point of intersection of perpendicular bisectors of the sides of a triangle. If O is the circumcentre of any triangle ABC, then $OA^2 = OB^2 = OC^2$. Also it is a centre of a circle passing through the vertices of a triangle.



Coordinates of circumcentre is

$$\left(\frac{x_1 \sin 2A + x_2 \sin 2B + x_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}, \frac{y_1 \sin 2A + y_2 \sin 2B + y_3 \sin 2C}{\sin 2A + \sin 2B + \sin 2C}\right)$$

Proof : (Desirable) $\angle BAE = \frac{2A}{2} = A$

In ΔBOD, $\frac{\sin 2C}{BD} = \frac{\cos A}{OD}$ & in ΔCOD, $\frac{\sin 2B}{CD} = \frac{\cos A}{OD}$

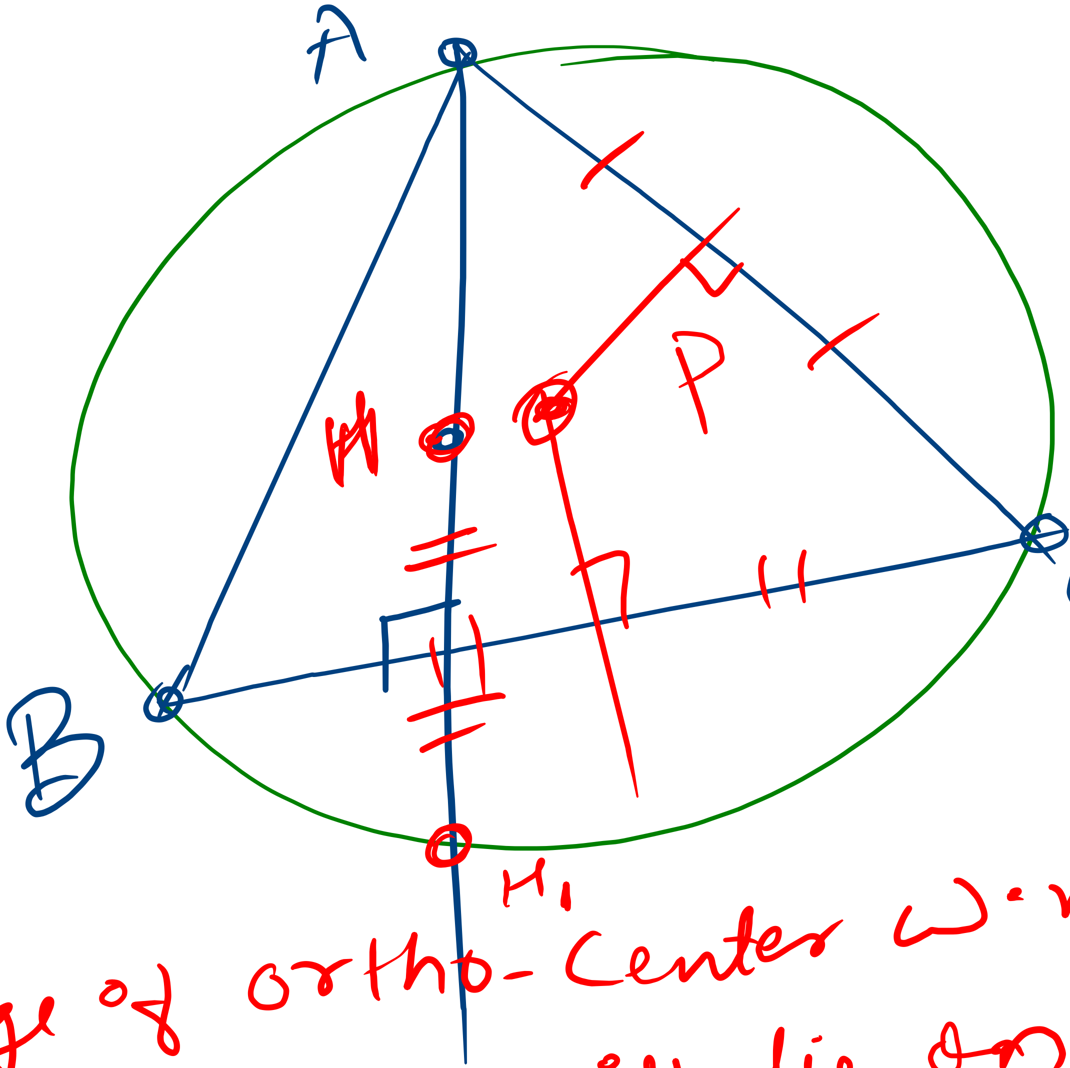
$$\therefore \quad \frac{\sin 2C}{\sin 2B} = \frac{BD}{CD}$$

$$D \equiv \left(\frac{x_2 \sin 2B + x_3 \sin 2C}{\sin 2B + \sin 2C}, \frac{y_2 \sin 2B + y_3 \sin 2C}{\sin 2B + \sin 2C}\right)$$

$P(\alpha, \beta)$

$(PA)^2 = (PB)^2$

$(PB)^2 = (PC)^2$



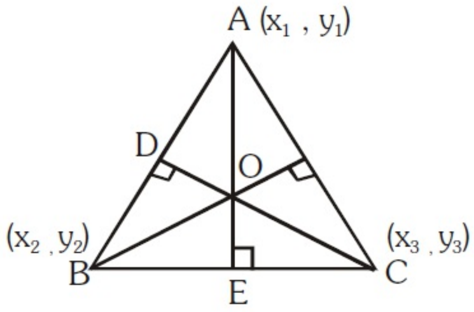
$\perp^r$ bisector of any two sides

$\Rightarrow$ P.O.I

Circum-Center

Image of ortho-Center w.r.to any side will lie on **Circum-Circle**.

(d) Orthocentre :



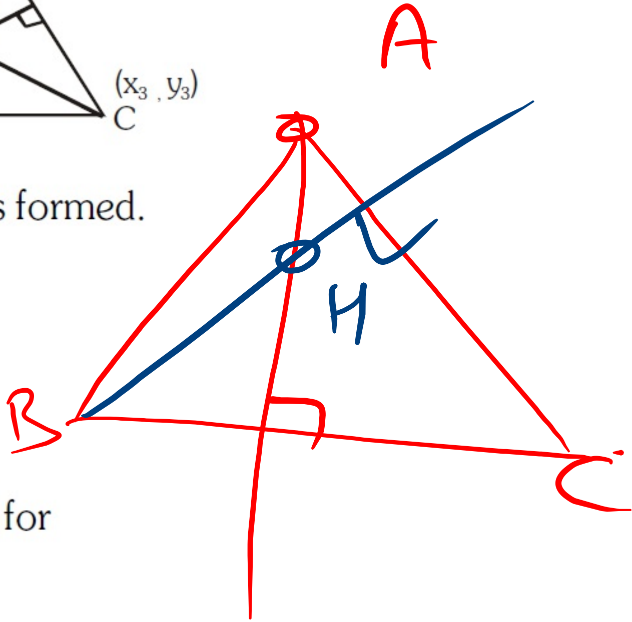
It is the point of intersection of perpendicular drawn from vertices on opposite sides (altitudes) of a triangle and can be obtained by solving the equation of any two altitudes.

Note :

(i) If a triangle is right angled triangle, then orthocenter is the point where right angle is formed.

(ii) Orthocentre is $\left(\dfrac{x_1 \tan A + x_2 \tan B + x_3 \tan C}{\tan A + \tan B + \tan C}, \dfrac{y_1 \tan A + y_2 \tan B + y_3 \tan C}{\tan A + \tan B + \tan C}\right)$

(iii) Triangle formed by joining feet of altitudes is the pedal triangle.

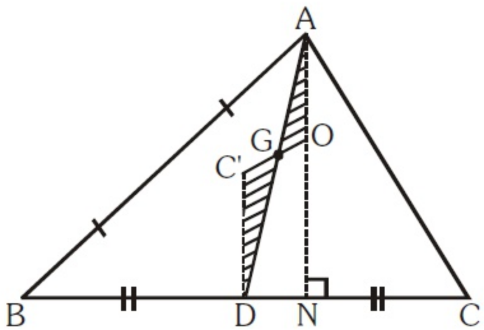


Remarks :

(i) Incentre and centroid always lie inside the triangle. Circumcentre & orthocentre lie inside for acute angle triangle and outside for obtuse angle triangle.

(ii) If the triangle is equilateral, then centroid, incentre, orthocentre, circumcentre, coincides.

(iii) Orthocentre, centroid and circumcentre are always collinear and centroid divides the line joining. orthocentre and circumcentre in the ratio 2 : 1 i.e. $\dfrac{C'G}{GO} = \dfrac{2}{1}$

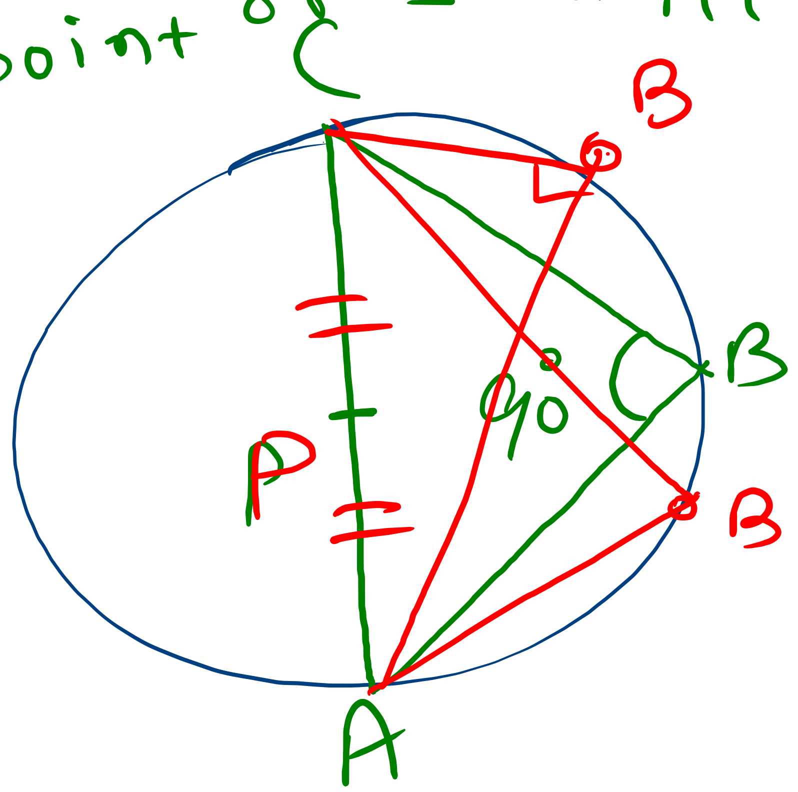


(iv) In an isosceles triangle centroid, orthocentre, incentre, circumcentre lies on the same line.

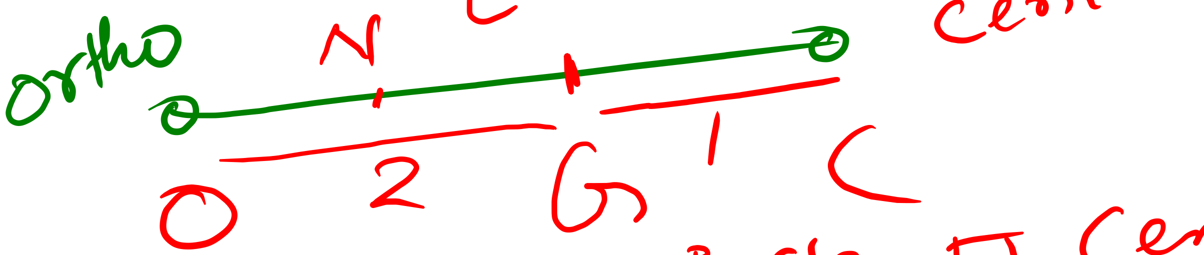
Right angled Δ

Circum-centre $\equiv$ mid point of side AC

Ortho-Center $\equiv$ Vertex B



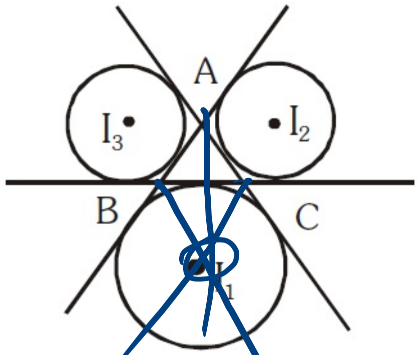
In General



N $\equiv$ Nine point circle का Center

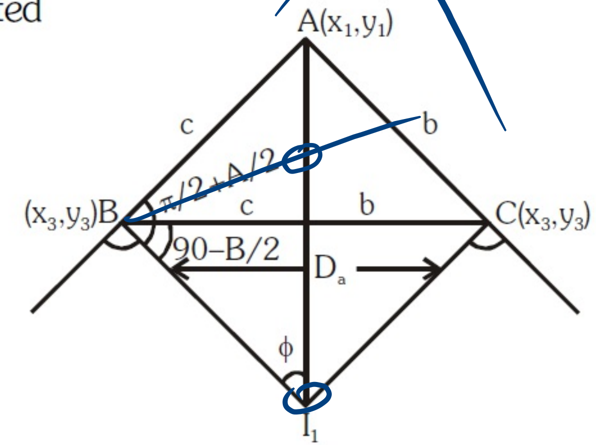
(e) Ex-centers :

The centre of the circle which touches side BC and the extended portions of sides AB and AC is called the ex-centre of ΔABC with respect to the vertex A. It is denoted by I_1 and its coordinates are



$$I_1 \equiv \left(\frac{-ax_1 + bx_2 + cx_3}{-a+b+c}, \frac{-ay_1 + by_2 + cy_3}{-a+b+c} \right)$$

Similarly ex-centers of ΔABC with respect to vertices B and C are denoted by I_2 and I_3 respectively , and



$$I_2 \equiv \left(\frac{ax_1 - bx_2 + cx_3}{a-b+c}, \frac{ay_1 - by_2 + cy_3}{a-b+c} \right),$$

$$I_3 \equiv \left(\frac{ax_1 + bx_2 - cx_3}{a+b-c}, \frac{ay_1 + by_2 - cy_3}{a+b-c} \right)$$

Note :

(i) For I_1 replace a by $-a$; for I_2 replace b by $-b$ & for I_3 replace c by $-$ c.

(ii) Incentre and excentres are harmonic conjugates of each other

Explanation :

I divides AD internally in the ratio (b + c) : a

& I_1 divides AD externally in the ratio (b + c) : a.

Hence, I and I_1 are harmonic conjugates of each other w.r.t. segment AB

AREA OF A TRIANGLE AND CONDITION FOR COLLINEARITY :

$$\Delta = \left|\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}\right| = \frac{1}{2}\left|\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix} + \begin{vmatrix} x_2 & y_2 \\ x_3 & y_3 \end{vmatrix} + \begin{vmatrix} x_3 & y_3 \\ x_1 & y_1 \end{vmatrix}\right| ; = \left|\frac{1}{2}\begin{vmatrix} x_1 - x_3 & x_2 - x_3 \\ y_1 - y_3 & y_2 - y_3 \end{vmatrix}\right|$$

Note :

Examples :

E(1) Prove that all coordinates of the vertices of an equilateral Δ can't be integer.

E(2) Find the values of K for which the points (K + 1, 2 – K) ; (1 – K, –K) and (2 + K, 3 – K) are collinear.

[**Ans.** K = 1]

E(3) Show that (b, c + a), (c, a + b) and (a, b + c) are collinear.

Let H be the orthocenter of an acute angled $\triangle ABC$

orthocenter of $\triangle ABH$ will be C

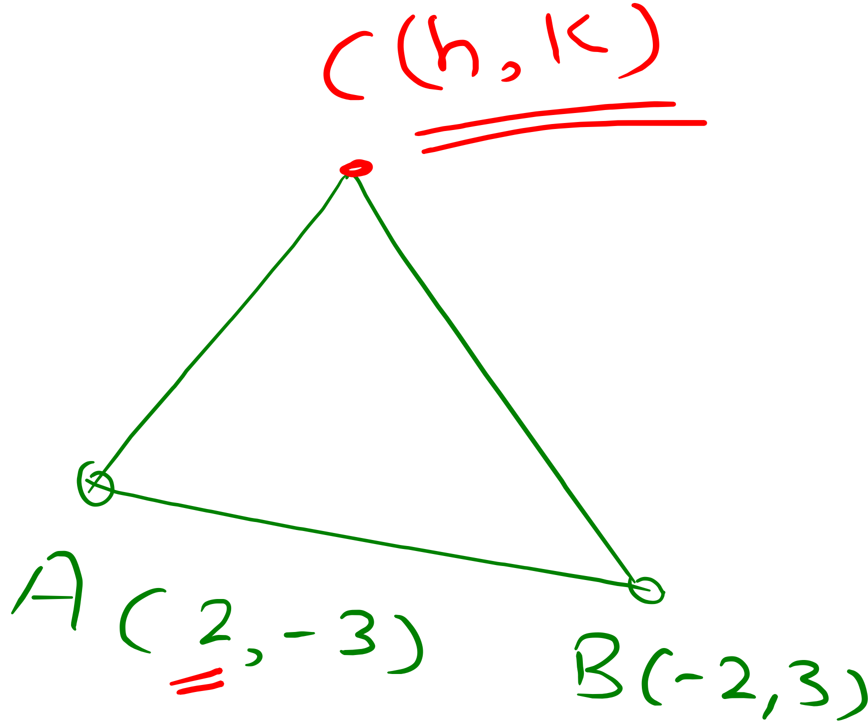
orthocenter of $\triangle ACH$ will be B

BRIEF DESCRIPTION OF ELEMENTARY LOCUS

Q: Let $A(2,-3)$ and $B(-2, 3)$ be vertices of a triangle ABC. If the centroid of this triangle moves on the line $2x+3y=1$, then the locus of the vertex C is the line

(a) $3x-2y=3$ (b) $2x-3y=7$ **[2004]**

(c) $3x+2y=5$ (d) $2x+3y=\cancel{9}\ 3$

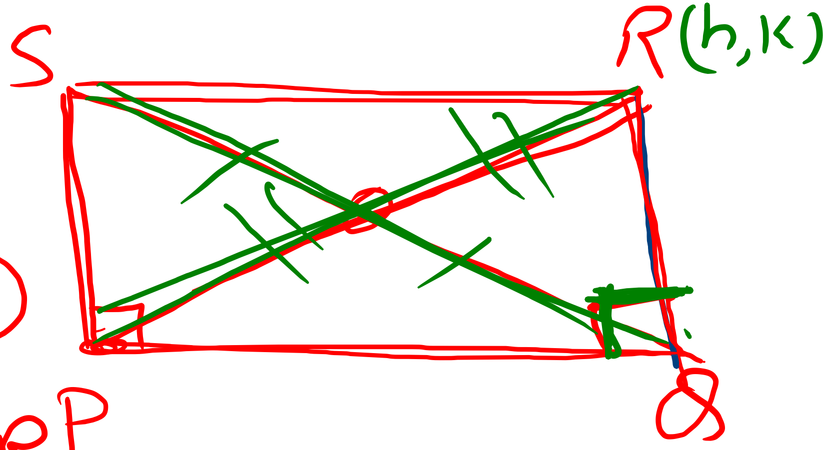


$$G\left(\frac{h+2-2}{3}, \frac{k-3+3}{3}\right)$$

lies on $2x+3y=1$

$$\Rightarrow \frac{2h}{3} + 3\times\frac{k}{3} = 1 \Rightarrow \boxed{2x+3y=3}$$

Q. A rectangle $PQRS$ has its side PQ parallel to the line $y = mx$ and vertices P, Q and S on the lines $y = a$, $x = b$ and $x = -b$, respectively. Find the locus of the vertex R. *(1996 - 2 Marks)*



$P(-b, a)$ $\quad Q(b, \beta)$, $S(-b, \ )$

mid point of QS ≡ mid point of PR

$R(h, k)$

$m_{PQ} = m$

$m_{PQ} = \dfrac{\beta - a}{b + b} = m$

$\Rightarrow \boxed{\beta = m(b+b) + a}$

$\boxed{\beta = (b-h)\left(-\frac{1}{m}\right) + k}$

Eliminate β

$m_{RQ} = \dfrac{\beta - k}{b - h} = -\dfrac{1}{m}$

$\boxed{a + m(b+b) = \dfrac{(h-b)}{m} + k}$

H.W.

The end A, B of a straight line segment of constant length c slide upon the fixed rectangular axes OX, OY respectively. If the rectangle $OAPB$ be completed, then show that the locus of the foot of the perpendicular drawn from P to AB is

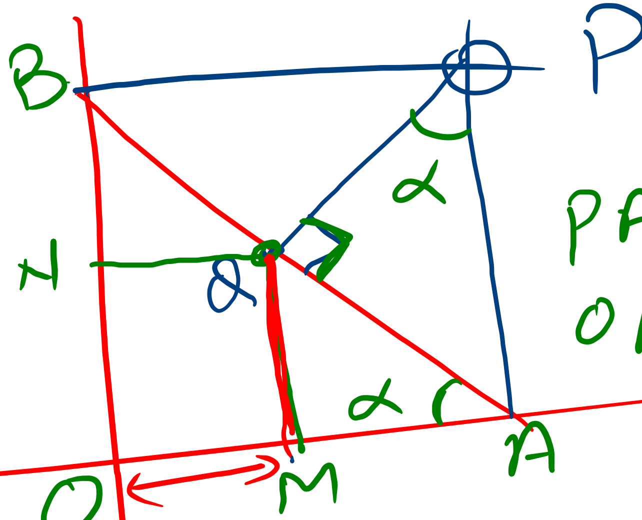
$$x^{\frac{2}{3}} + y^{\frac{2}{3}} = c^{\frac{2}{3}}$$

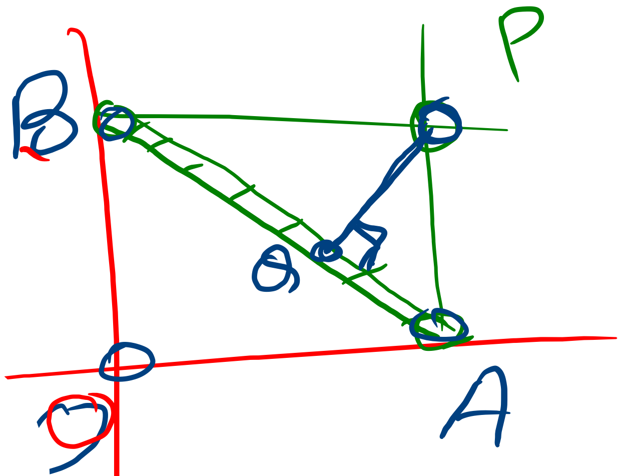
(1983 - 2 Marks)

$$\boxed{AB = c}$$

$(AB)^2 = c^2$

Locus of Q is required $\boxed{Q(h, k)}$





$\Rightarrow \boxed{h = c\cos^3\alpha}$

$\boxed{k = c\sin^3\alpha}$

$PA = c\sin\alpha$

$OA = c\cos\alpha$

$MA = (QA)\cos\alpha = (PA)\sin\alpha \cdot \cos\alpha$

$= c\sin^2\alpha \cdot \cos\alpha$

$QM = QA\sin\alpha$

$= PA\sin^2\alpha$

$= c\sin^3\alpha$

$h = OA - MA = c\cos\alpha - c\sin^2\alpha\cos\alpha =$

To eliminate α

$$h = c\cos^3\alpha \Rightarrow \quad \cos\alpha = \left(\frac{h}{c}\right)^{1/3}$$

$$k = c\sin^3\alpha \Rightarrow \quad \sin\alpha = \left(\frac{k}{c}\right)^{1/3}$$

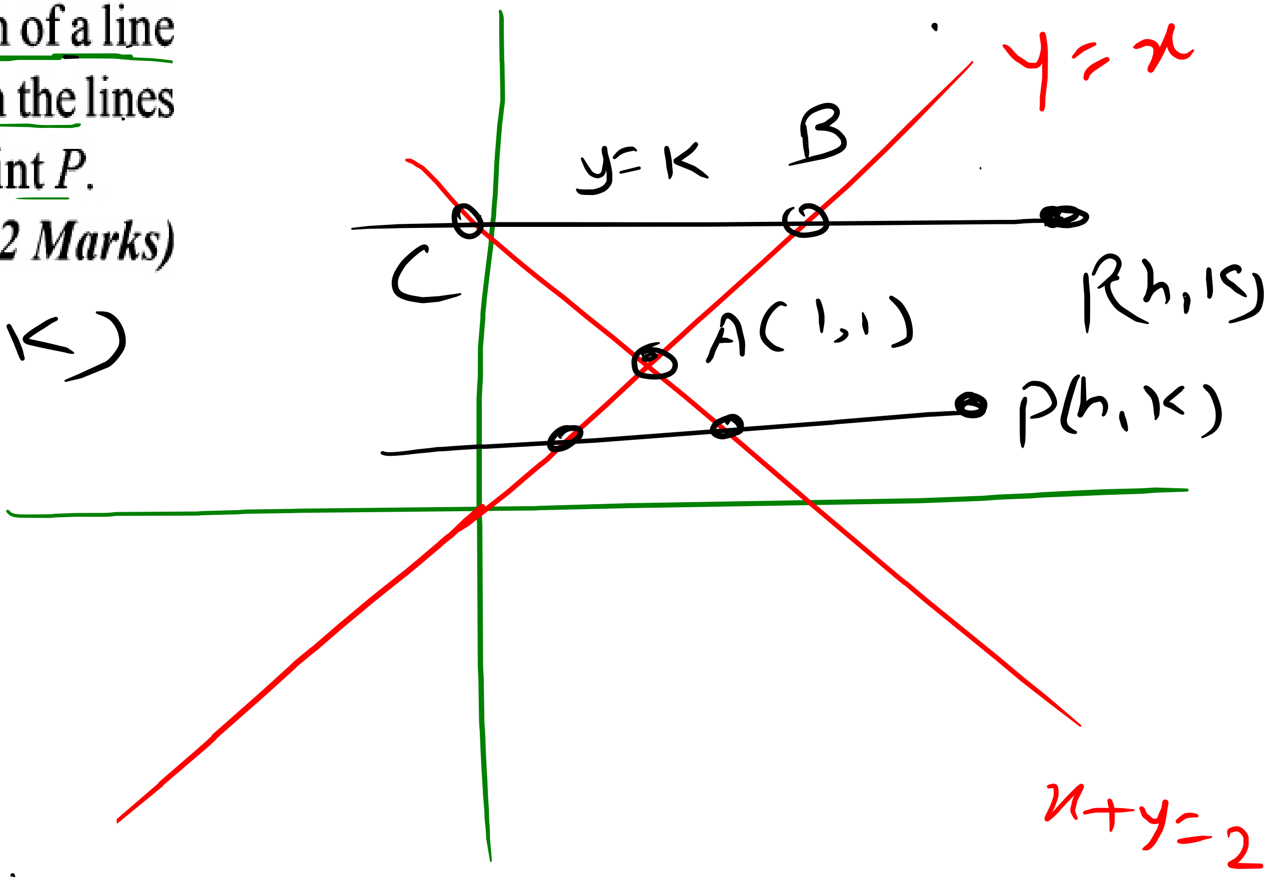
$$\sin^2\alpha + \cos^2\alpha = 1$$

$$\left(\frac{k}{c}\right)^{2/3} + \left(\frac{h}{c}\right)^{2/3} = 1$$

$$\Rightarrow \boxed{x^{2/3} + y^{2/3} = c^{2/3}}$$

The area of the triangle formed by the intersection of a line parallel to x-axis and passing through $P(h, k)$ with the lines $y = x$ and $x + y = 2$ is $4h^2$. Find the locus of the point P.

(2005 - 2 Marks)



$B(k, k)$ $\quad$ $C(2-k, k)$

$\triangle ABC$

$$4h^2 = \frac{1}{2}\left|\begin{matrix} 1 & 1 & 1 \\ k & k & 1 \\ 2-k & k & 1 \end{matrix}\right|$$

Q. A straight line L through the origin meets the lines $x + y = 1$ and $x + y = 3$ at P and Q respectively. Through P and Q two straight lines L_1 and L_2 are drawn, parallel to $2x - y = 5$ and $3x + y = 5$ respectively. Lines L_1 and L_2 intersect at R. Show that the locus of R, as L varies, is a straight line.

(2002 - 5 Marks)

$mx + x = 1$

$x = \left(\frac{1}{m+1}\right)$

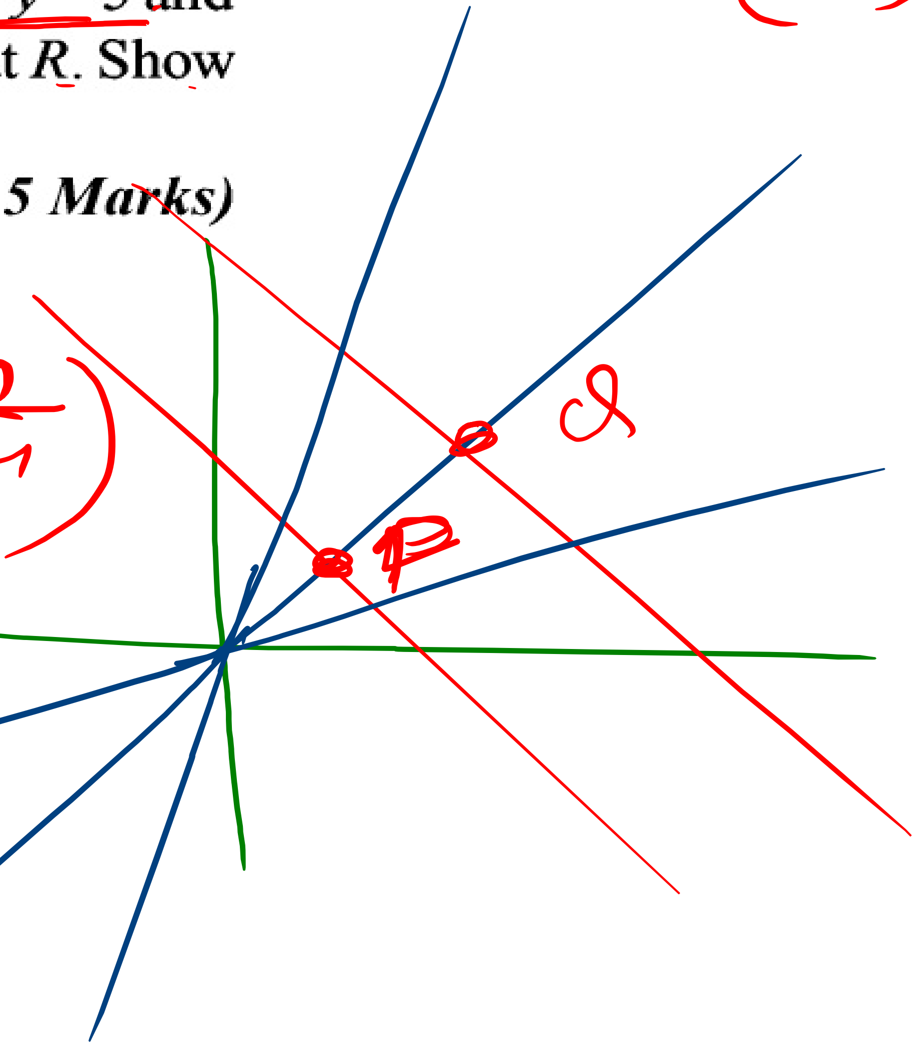
$L: \; y = mx$

$P\left(\frac{1}{m+1}, \frac{m}{m+1}\right), \; Q\left(\frac{3}{m+1}, \frac{3m}{m+1}\right)$

$L_1 \equiv 2x - y = \frac{2}{m+1} - \frac{m}{m+1}$

$L_2 \equiv 3x + y = \frac{9}{m+1} + \frac{3m}{m+1}$

Locus $\Rightarrow$ p.o.I. of L_1 & L_2



BRIEF DESCRIPTION OF ELEMENTARY LOCUS

ते FOCUS

Locus of a point is the path traced by moving point under restricted conditions.

Equation of locus : A unique relation between x & y coordiantes of any point on the curve which must besatisfied by every point on the curve and by no other point.

Four basic steps :

P(h, K)

1. Assume the coordinates of the point, say (h,k) whose locus is to be found.

2. Write the given condition in mathematical form involving h,k.

condition(s)

3. Eliminate the variables/parametres if any.

unwanted unknowns

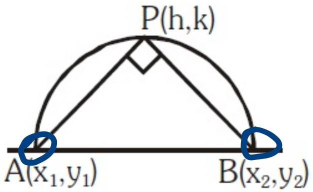
4. Replace h by x and k by y to obtain equation of locus.

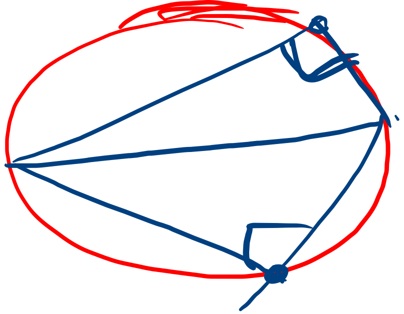
Eliminate unknowns → Eqn in h & K

Standard Locii :

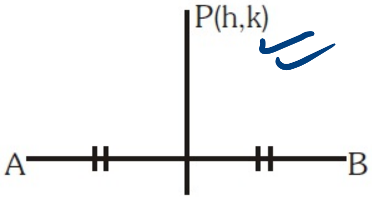
(a) Locus of P if $\angle APB = 90^\circ$

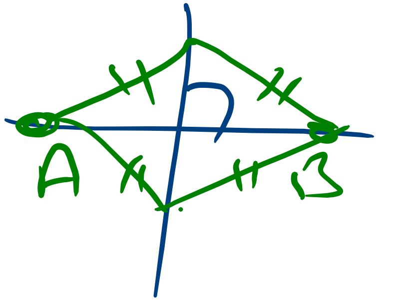
(A circle with diameter AB, excluding points A & B)





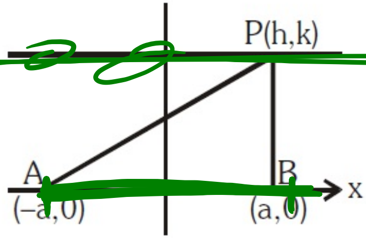
(b) Locus of P if P is equidistant from A and B. (Perpendicular bisector of AB)





(c) Point A moves so that Ar. $\Delta PAB = ab = \text{const.}$

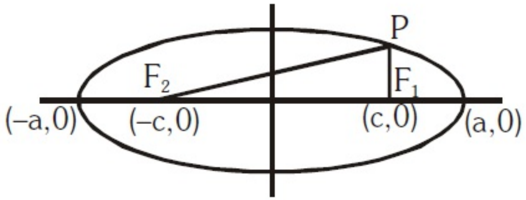
(Locus of P is a line | | to x-axis/base of the Δ)



(d) Point P moves such that it is equidistant from two given lines (Angle bisector)

(e) Parametric equation of locus :

$$|PF_1| + |PF_2| = 2a \Rightarrow \frac{x^2}{a^2} + \frac{y^2}{a^2 - c^2} = 1 \quad (a > c)$$



A & B are 2 given points.
P is a moving point s.t.

$PA + PB = \text{constant} = d$

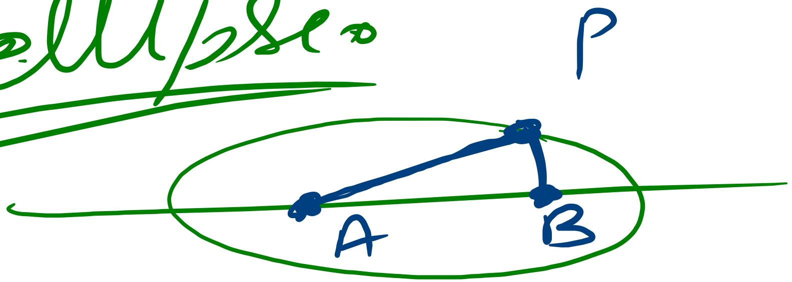
locus of P is.....

Case I: $d = AB \Rightarrow$ locus of P is Line-Segment

Case II: $d < AB \Rightarrow$ locus of P is Empty set.

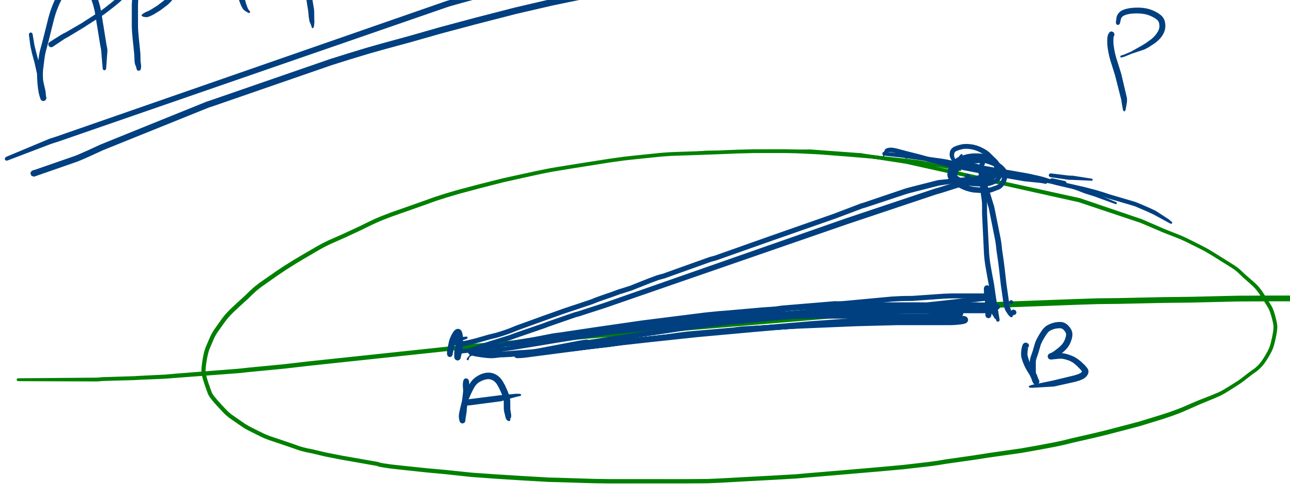
Case III: $d > AB$

$\Rightarrow$ locus of P is an ellipse.



A & B are 2 foci of the ellipse $\boxed{AB = 2ae}$

$AP + PB > AB$



$AP + PB = 2a$

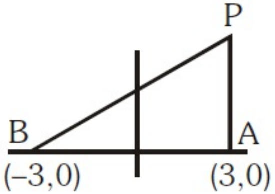
$AB = 2ae$

$AP + PB < AB$

collinear $\Longleftarrow$ $AP + PB = AB$

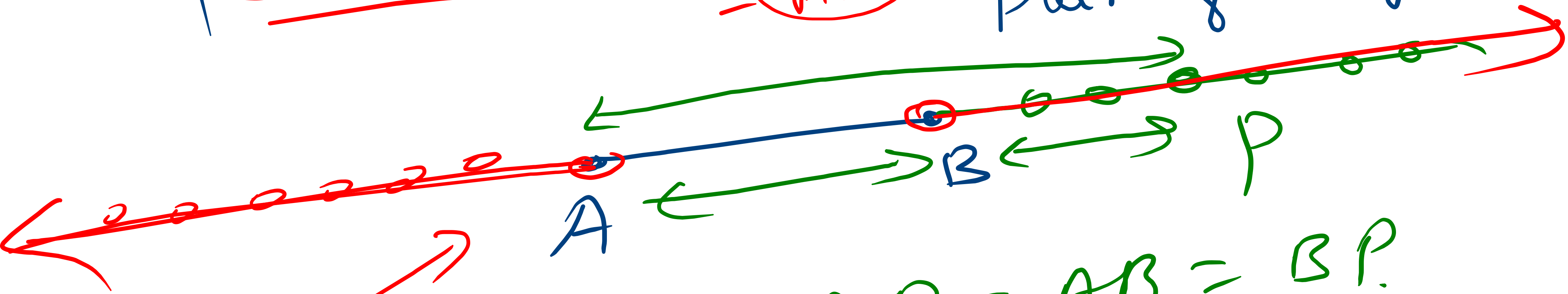
Q.

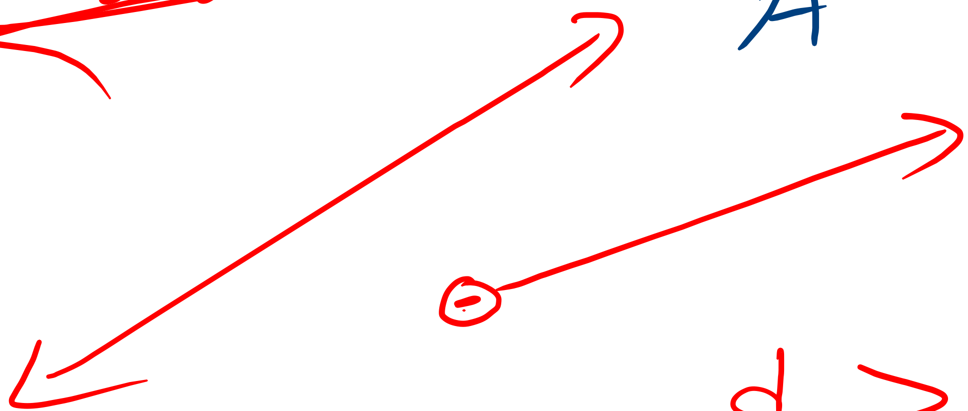
$||PA| - |PB|| = 4$



$$\frac{x^2}{4} - \frac{y^2}{5} = 1$$

$|PA - PB| = d = AB \Rightarrow$ locus of P is Pair of rays.





$AP - AB = BP$

$AP - BP = AB$

$d > AB$ No locus, $d < AB$ Hyperbola

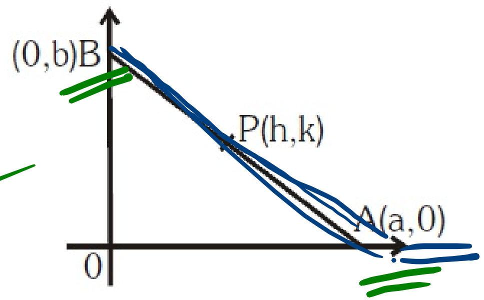
E(1) If A and B are variable points on x and y-axis respectively such that AB = ℓ, then find :

(i) Locus of middle point of AB

or locus of the circumcenter of the ΔAOB [**Ans.** $(x^2 + y^2 = \ell^2)$]

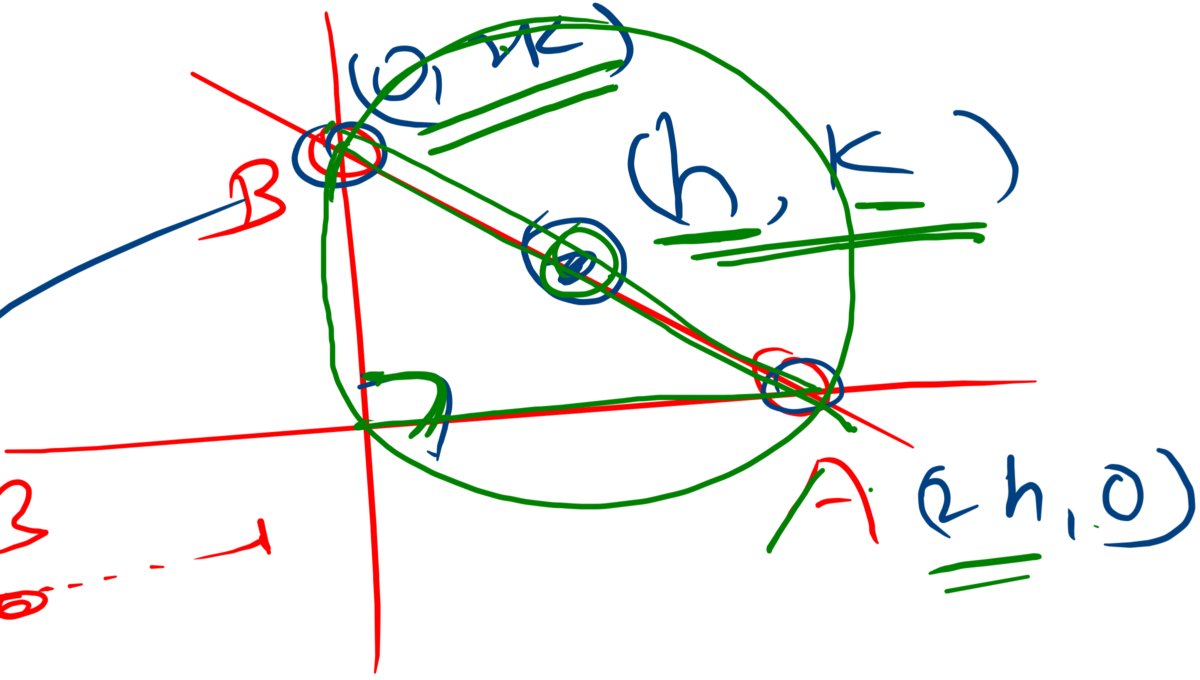
(ii) locus of the centroid of the triangle AOB [**Ans.** $9(x^2 + y^2) = 4\ell^2$]

(iii) If $\ell = 4$, then the locus of the point which divides segment AB in ratio 1 : 2. [**Ans.** $9x^2 + 36y^2 = 64$]



$$AB^2 = \ell^2$$

$$\boxed{4h^2 + 4k^2 = \ell^2}$$

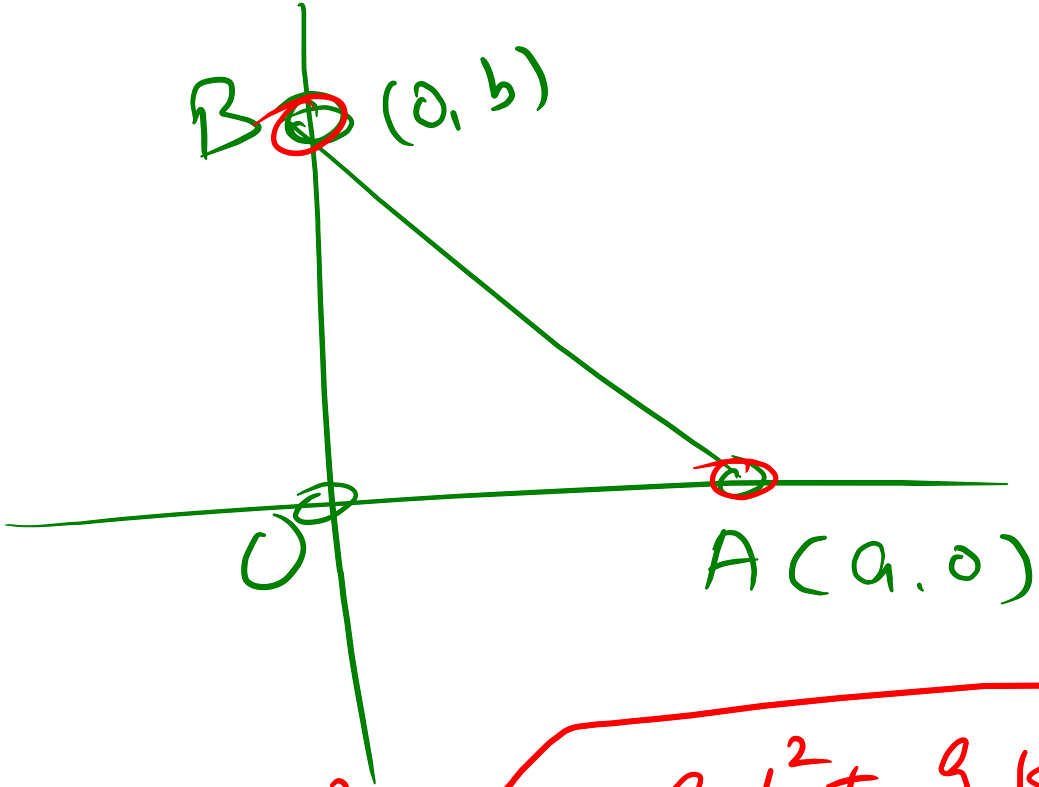


$G(h, k)$

$3h = 0 + a + 0$

$a = 3h$

$b = 3k$



$$a^2 + b^2 = l^2 \Rightarrow \boxed{9h^2 + 9k^2 = l^2}$$

$$\frac{2a+0}{3} = h$$

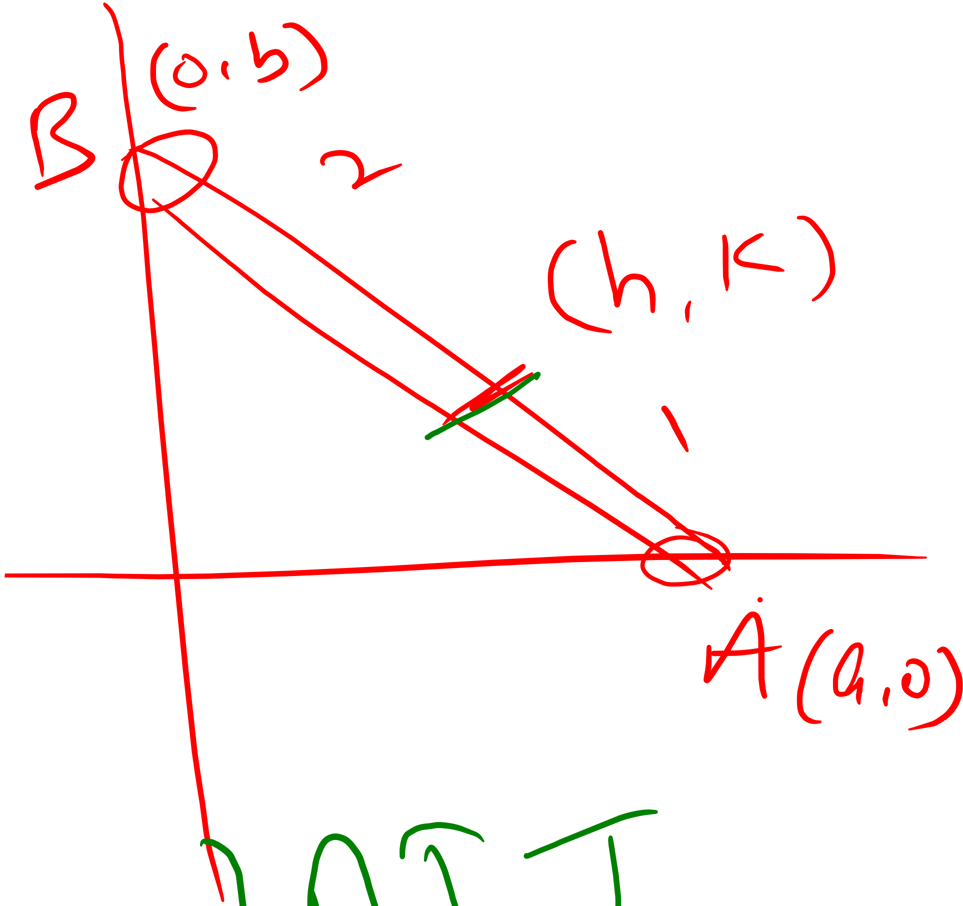
$$\Rightarrow a = \frac{3h}{2}$$

$$\frac{0+b}{3} = K$$

$$\Rightarrow b = 3K$$

$$a^2 + b^2 = l^2$$

$$\boxed{\frac{9h^2}{4} + 9K^2 = l^2}$$



WAIT

E(2) Find the equation of the locus of a point whose distance from the x-axis exceeds its distance from y-axis by 4. **[Ans.** $|y| - |x| = 4$]

E(3) A(1, 2) is a fixed point. A variable point B lies on a locus whose equation is $x^2 + y^2 = 4$. Find the locus of the mid point of AB. **[Ans.** $4x^2 + 4y^2 - 4x - 8y + 1 = 0$]

$B(2\cos\theta, 2\sin\theta)$

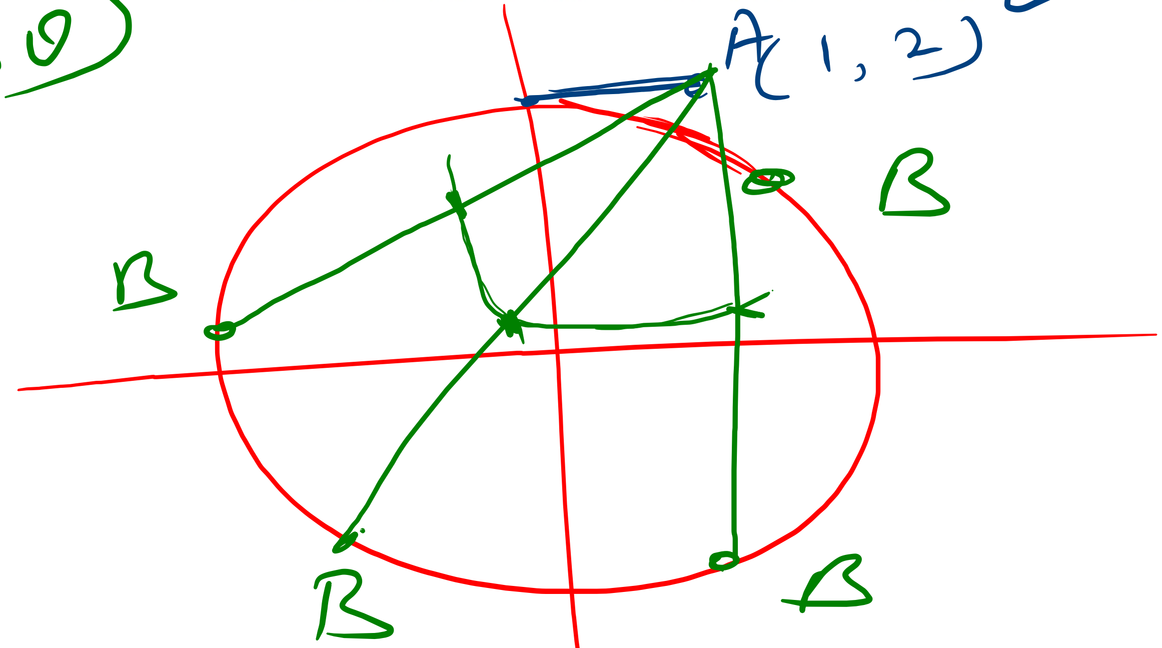
$P(h, k)$

$2\cos\theta + 1 = 2h$

$2\cos\theta = (2h - 1)$

$2\sin\theta + 2 = 2k$

$2\sin\theta = 2(k - 1)$



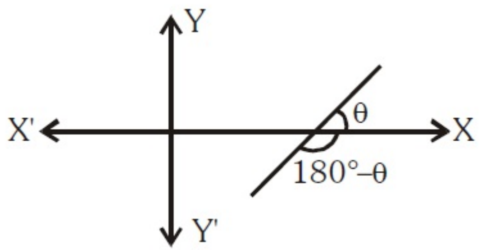
$$\boxed{4 = (2h-1)^2 + 4(k-1)^2}$$

STRAIGHT LINE :

Definition : It is defined as the locus of a point such that if any two points of this locus are joined, they define a unique direction. To understand the concept of direction, we should know inclination of a line and slope of a line.

(i) **Inclination of a line :** If a straight line intersects the x-axis, the inclination of the line is defined as the measure of the smallest non-negative angle which the line makes with the positive direction of the x-axis.

Slope (or Gradient) of a line :



If the inclination of a line (i.e. non vertical line) is θ and $\theta \neq \frac{\pi}{2}$,

then the slope of a line is defined to be tanθ.

If θ is the angle between at which a straight line is inclined to the positive direction of x-axis and $0° \leq \theta < 180°$, $\theta \neq 90°$, $(0 \leq \theta < \pi,\ \theta \neq \frac{\pi}{2})$, then the slope of the line, denotes by m, is defined by $m = \tan\theta$.

If $\theta = 90°$, m does not exist, but the line is parallel to the y-axis.

If $\theta = 0$, then $m = 0$ and the line is parallel to the x-axis (i.e. line has zero slope).

Angle between lines :

(a) If $A(x_1, y_1)$ and $B(x_2, y_2)$, $x_1 \neq x_2$, are points on a straight line, then the slope m of the line is given by $m = \frac{y_1 - y_2}{x_1 - x_2}$.

(Since slope of a non-vertical line is the slope of any segment contained in the line)

(b) If angle between line L_1 & L_2 is θ, then

$$\tan\theta = \left|\frac{m_1 - m_2}{1+m_1 m_2}\right| \quad \text{hence} \quad m_1 = m_2 \Rightarrow L_1 \parallel L_2$$

$$m_1 m_2 = -1 \Rightarrow L_1 \perp L_2$$

$m_1 + m_2 = 0 \Rightarrow L_1$ & L_2 are equally inclined to co-ordinate axes.

and $m_1 m_2 = 1 \Rightarrow$ lines make complementary angles with the x-axis.

Intercept of lines : In a cartesian co-ordinate system, it is the distance from the origin to the point at which a line cuts a given axis. It may be positive or negative

(a) If line has equal intercepts, then $m = -1$.

If line is equally inclined with coordinate axes./Line cuts off equal nonzero distances from origin, then $m = \pm 1$.

EQUATION OF STRAIGHT LINE :

A relation between x and y which is satisfied by co-ordinates of every point lying on a line is called equation of the straight line. Here remember that every one degree equation in variable x and y always represents a straight line i.e. $ax + by + c = 0$; a & $b \neq 0$ simultaneously.

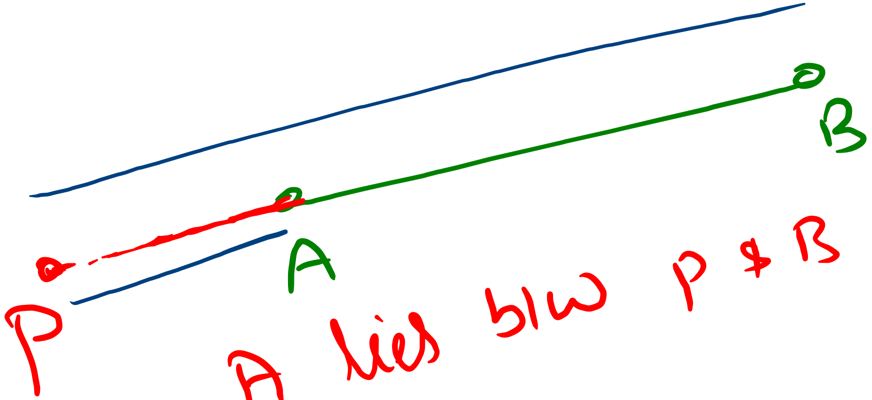
(a) Equation of a line parallel to x-axis at a distance a is $y = a$ or $y = -a$

(b) Equation of x-axis is $y = 0$

(c) Equation of line parallel to y-axis at a distance b is $x = b$ or $x = -b$

(d) Equation of y-axis is $x = 0$

(e) Any line parallel to $ax + by + c = 0$ is $ax + by = \lambda$ & perpendicular to $ax + by + c = 0$ is $bx - ay = \lambda$



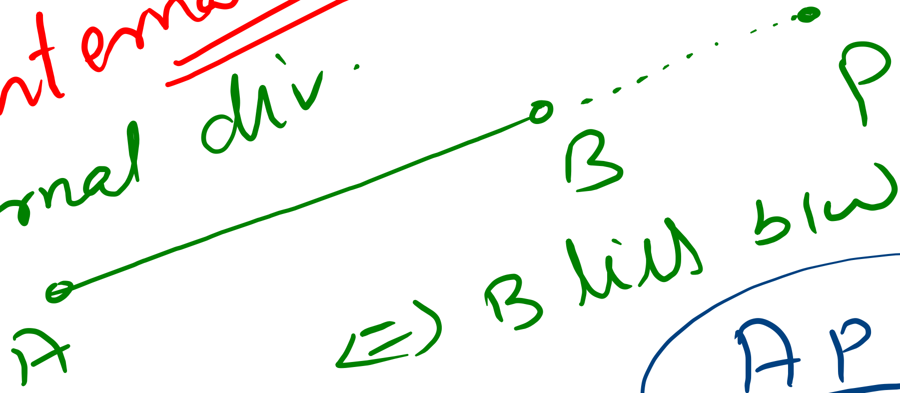
A lies b/w P & B

$$\frac{AP}{BP} < 1$$

Internal div.

P lies b/w A & B

External div.



$\Leftarrow$ B lies b/w A & P

$$\frac{AP}{BP} > 1$$

STRAIGHT LINE :

Definition : It is defined as the locus of a point such that if any two points of this locus are joined, they define a unique direction. To understand the concept of direction, we should know inclination of a line and slope of a line.

Inclination of a line

Slope (or Gradient) of a line :

Intercept of lines

Angle between lines :

If line has equal intercepts, then $m = -1$.

If line is equally inclined with coordinate axes./Line cuts off equal nonzero distances from origin, then $m = \pm 1$.

$m_1 + m_2 = 0 \Rightarrow L_1$ & L_2 are equally inclined to co-ordinate axes.

$m_1 m_2 = 1 \Rightarrow$ lines make complementary angles with the x-axis.

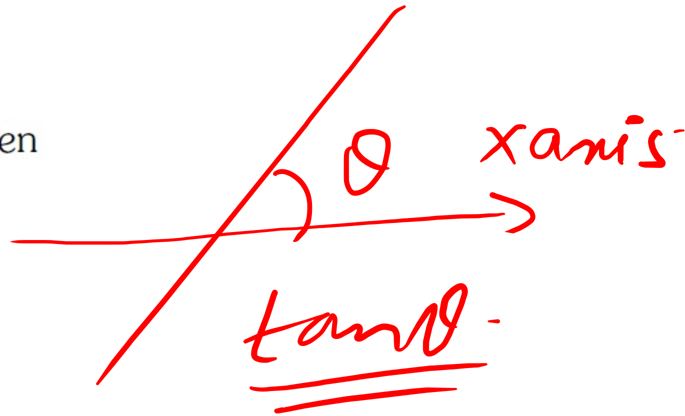
If angle between line L_1 & L_2 is θ, then

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| \quad \text{hence} \quad m_1 = m_2 \Rightarrow L_1 \parallel L_2$$

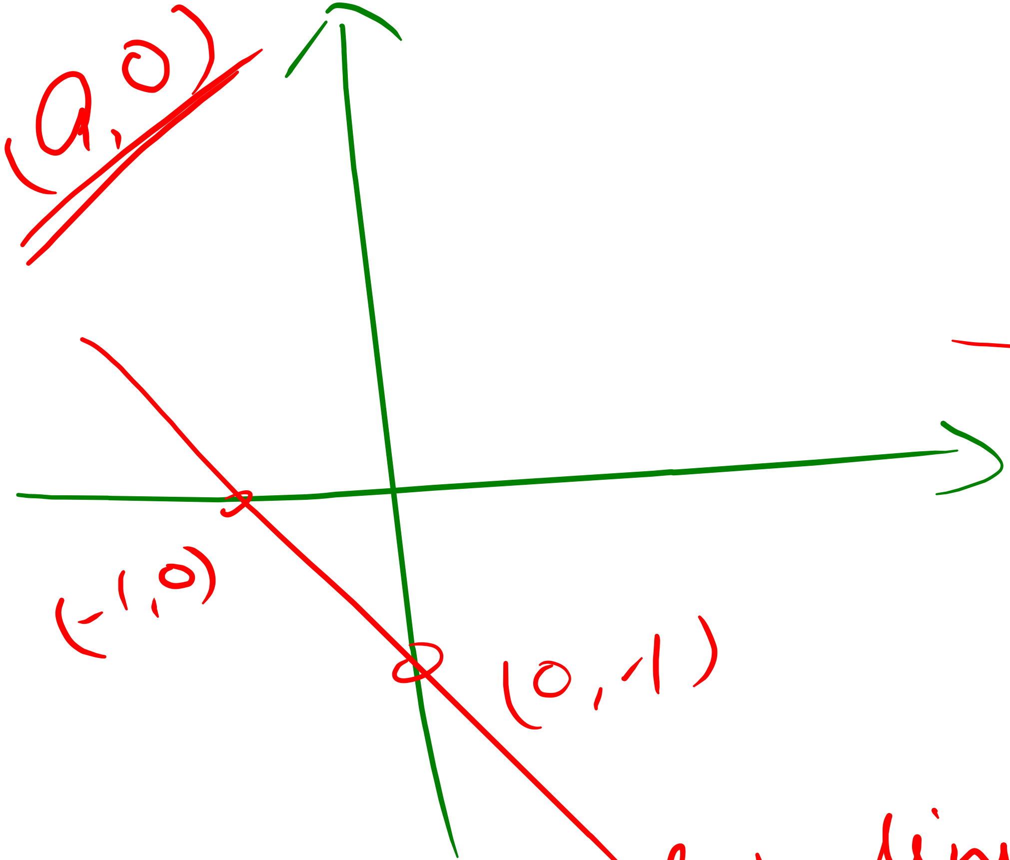
$$m_1 m_2 = -1 \Rightarrow L_1 \perp L_2$$

If $A(x_1, y_1)$ and $B(x_2, y_2)$, $x_1 \neq x_2$, are points on a straight line, then the slope m of the line is given by $m = \dfrac{y_1 - y_2}{x_1 - x_2}$.

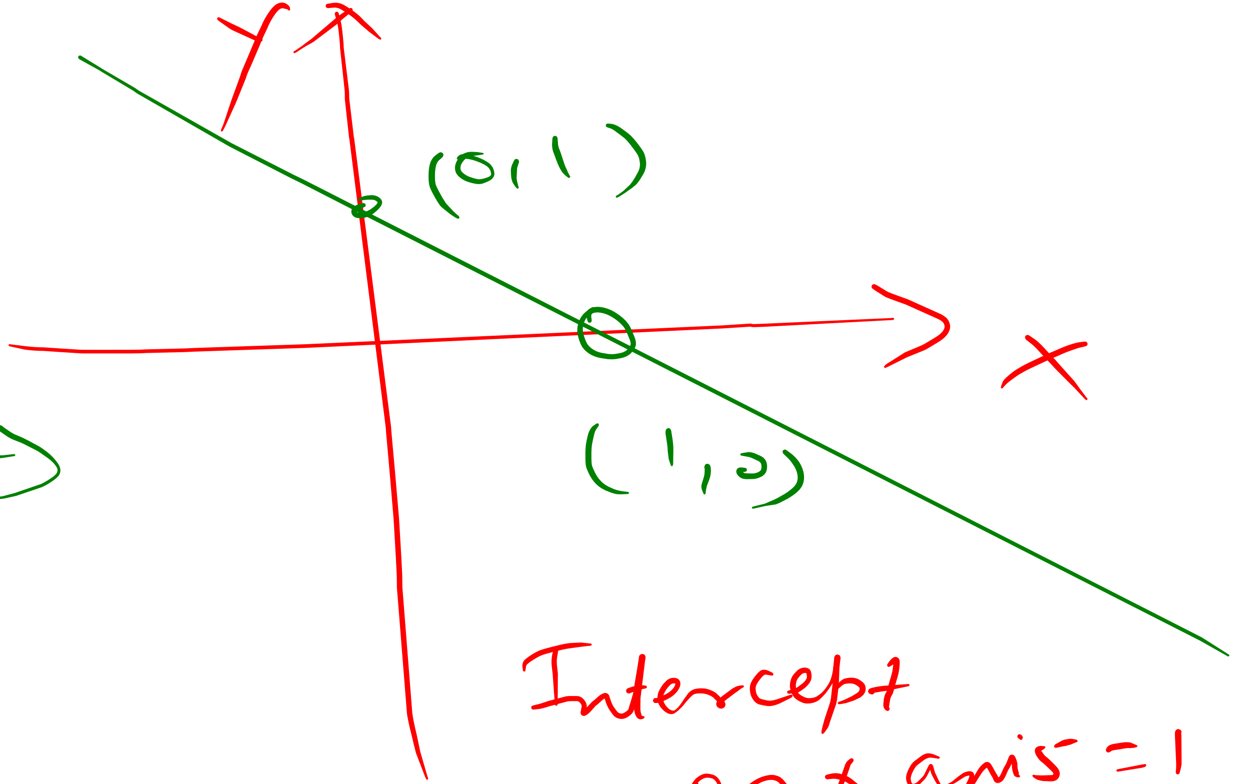
(Since slope of a non-vertical line is the slope of any segment contained in the line)



~~(a, 0)~~



Intercept made by line on
x axis = -1
y axis = -1

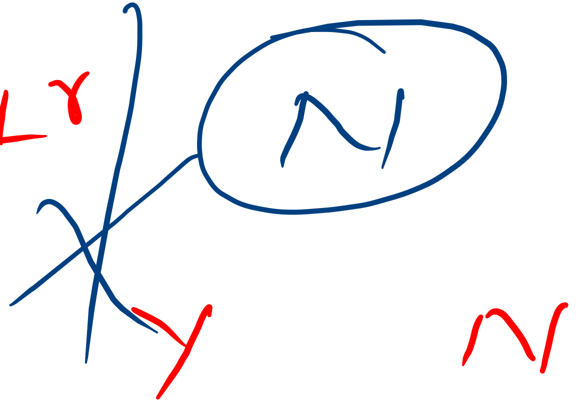


Intercept
on x axis = 1
y axis = 1

Angle b/w lines :

Lines are $\perp^r$

$\Rightarrow \quad m_1 m_2 = -1$

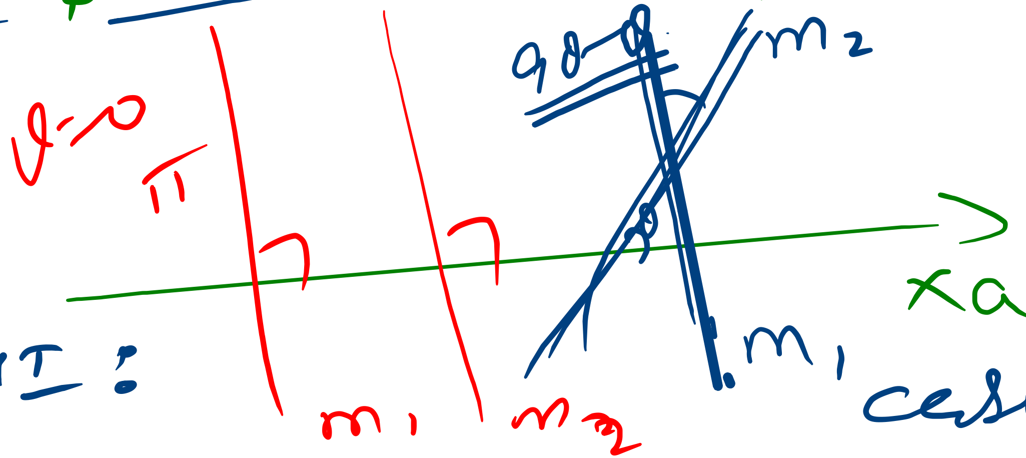


$$\boxed{\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|}$$

is used only when both the slopes m_1 & m_2 are defined.

otherwise we sketch.

Case I :



Case II :

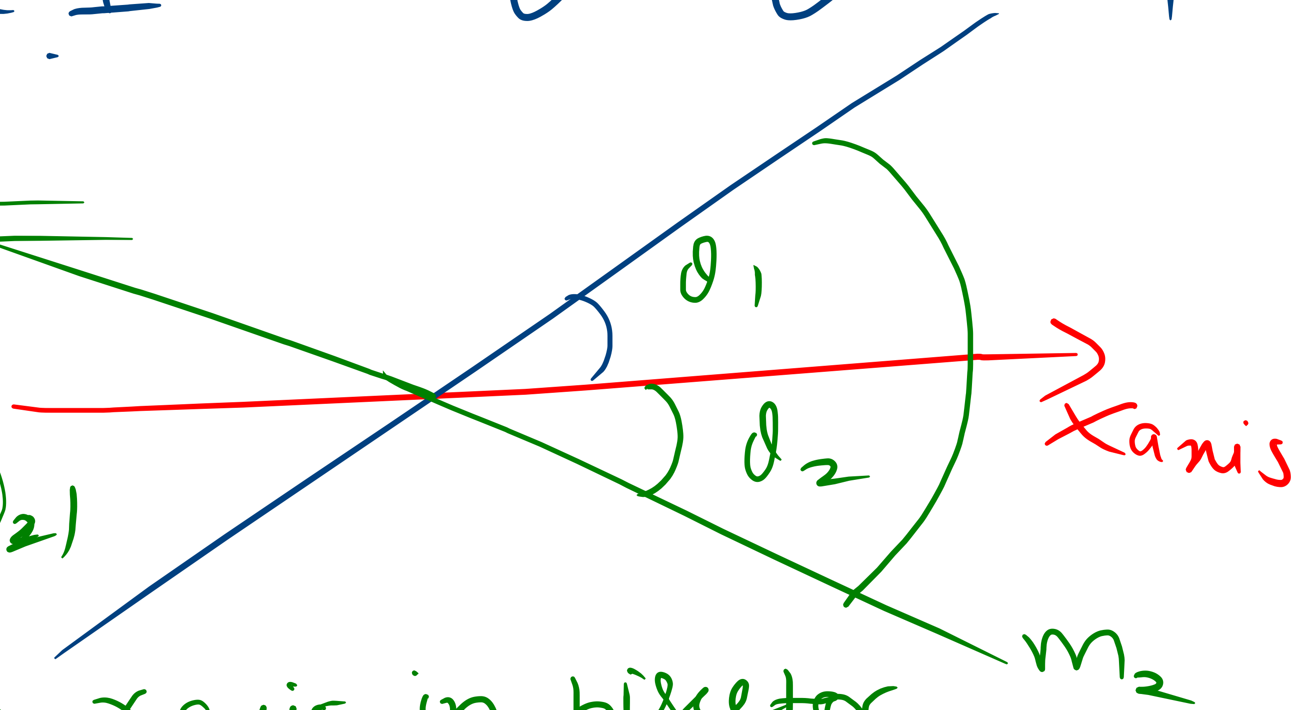
$m_1 m_2 = -1$

$\Rightarrow$ Lines are $\perp^r$

When $m_1 + m_2 = 0$

$m_1 + m_2 = 0$

$\Rightarrow |\theta_1| = |\theta_2|$



$\Rightarrow$ x axis is bisector

EQUATION OF STRAIGHT LINE :

A relation between x and y which is satisfied by co-ordinates of every point lying on a line is called equation of the straight line. Here remember that every one degree equation in variable x and y always represents a straight line i.e. $ax + by + c = 0$; $a \& b \neq 0$ simultaneously.

$0 \cdot x + 0 \cdot y + 0 = 0$

(a) Equation of a line parallel to x-axis at a distance a is $y = a$ or $y = -a$

(b) Equation of x-axis is $y = 0$

(c) Equation of line parallel to y-axis at a distance b is $x = b$ or $x = -b$

(d) Equation of y-axis is $x = 0$

(e) Any line parallel to $ax + by + c = 0$ is $ax + by = \lambda$ & perpendicular to $ax + by + c = 0$ is $bx - ay = \lambda$

Note : Every first degree equation in x, y represents a straight line and converse is also true.

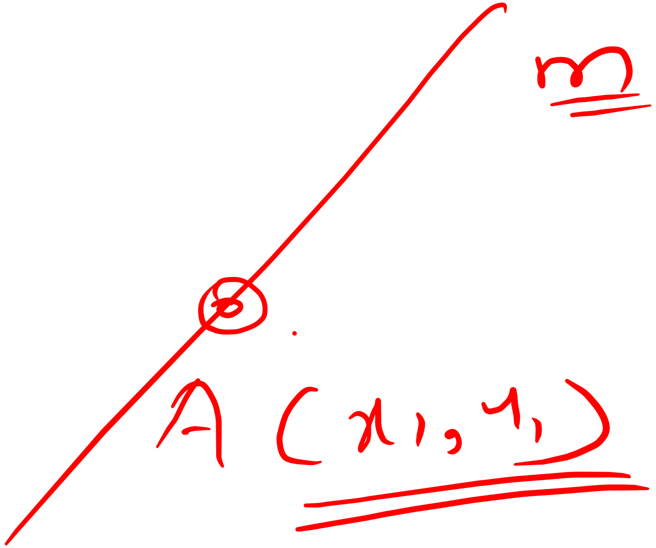
DIFFERENT FORMS OF STRAIGHT LINES :

(a) Point slope : $y - y_1 = m(x - x_1)$

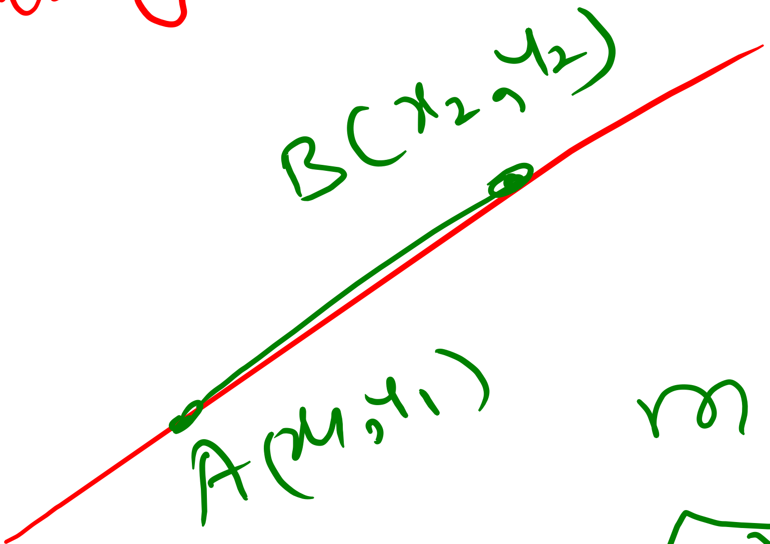
Double intercept form : $\frac{x}{a} + \frac{y}{b} = 1$

(b) Two point form : $y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1)$ or $\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0$

$$\boxed{y - y_1 = m(x - x_1)}$$



Passing through 2 points



$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\boxed{y - y_1 = m(x - x_1)}$$

Slope intercept form : y = mx + c (c is y-intercept); y = mx (line passes through origin)

$$\frac{x}{a} + \frac{y}{b} = 1$$

$$a = 2b$$

$$\frac{x}{2b} + \frac{y}{b} = 1$$

For line ax + by + c = 0, slope $= -\frac{a}{b}$ & y intercept $= -\frac{c}{b}$

(1) A line passing through the point (1, 0) and (2, 1) is rotated anticlockwise about the point (1, 0) by an angle of 15°. Find the equation is its new position. [**Ans.** $y = \sqrt{3}(x-1)$]

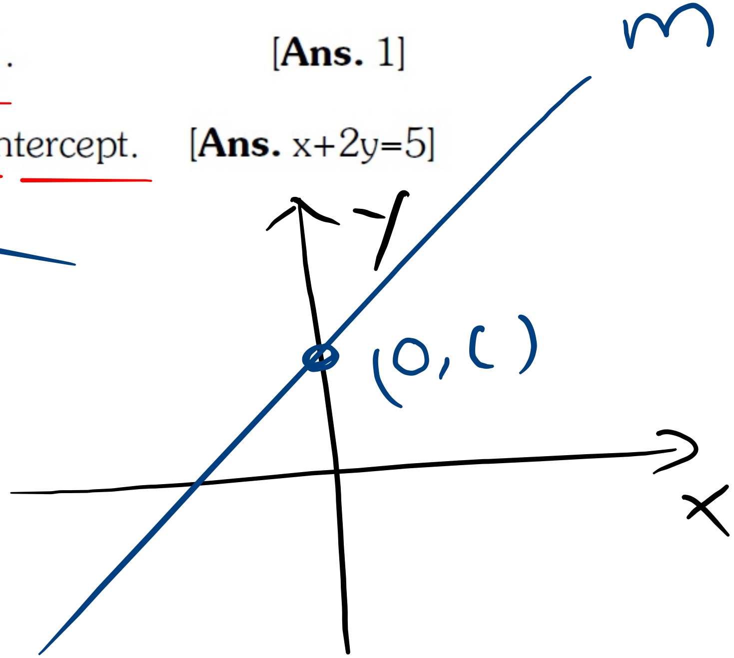
(2) If mid points of sides of triangle are (2,1), (–5,7) & (–5,–5), then find equation of sides of triangle. [**Ans.** x = 2, 7y = 6x + 79, 7y + 6x + 65 = 0]

(3) If the points (a, 0), (0, b) and (1, 1) are collinear then find the value of $\frac{1}{a} + \frac{1}{b}$. [**Ans.** 1]

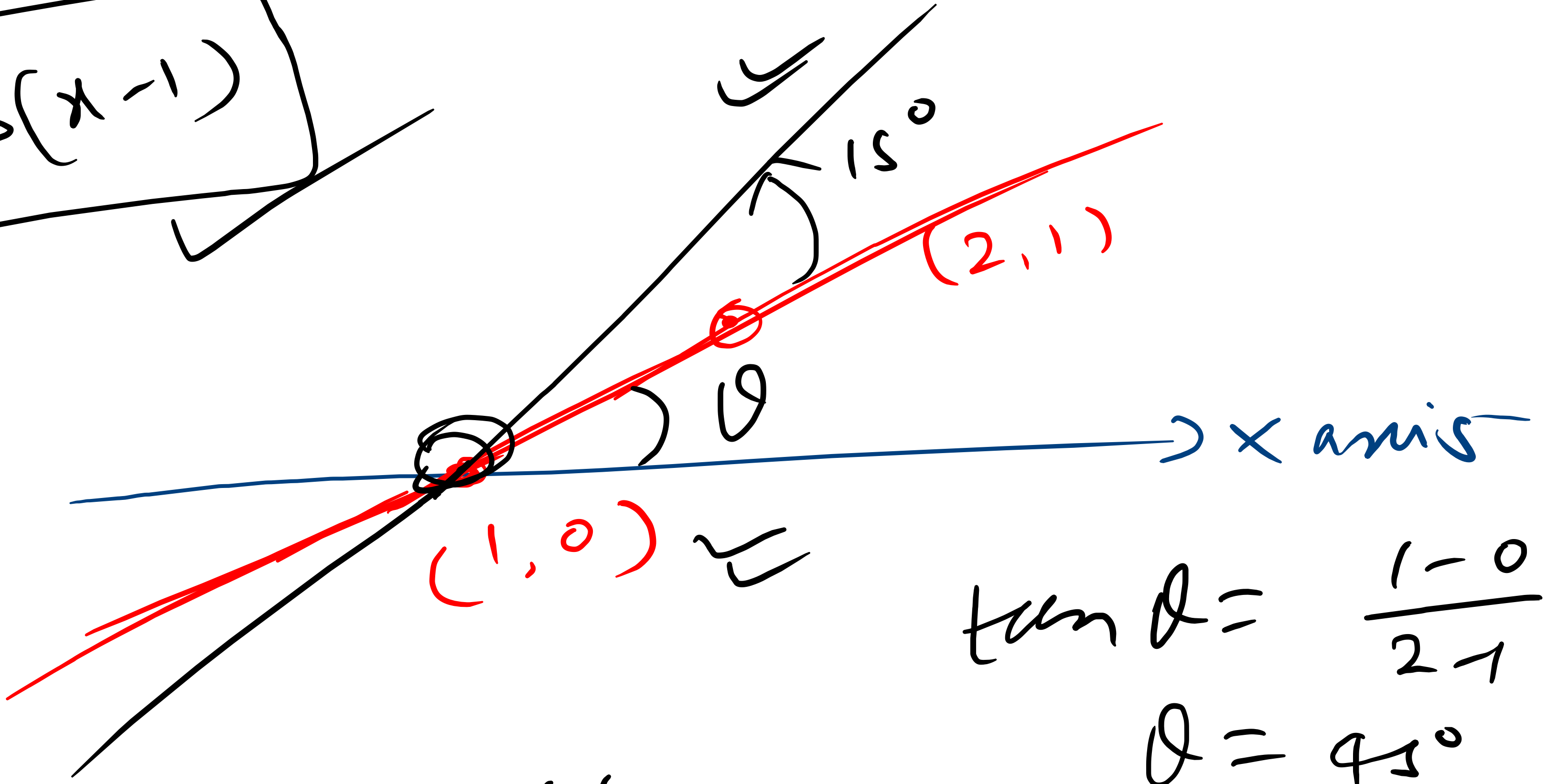
(4) Equation of straight line passing through (1, 2) if x intercepts is twice the y-intercept. [**Ans.** x+2y=5]

Passing through (0, 0) $\Rightarrow$ $\boxed{y = mx}$

$$\frac{x}{a} + \frac{y}{b} = 1 \Rightarrow \frac{1}{a} + \frac{1}{b} = 1$$



$$\boxed{y - 0 = \sqrt{3}(x-1)}$$



$$\tan\theta = \frac{1-0}{2-1} = 1$$

$$\theta = 45^\circ$$

slope of the required line

$$\tan 60^\circ \quad = (45^\circ + 15^\circ) = 60^\circ$$

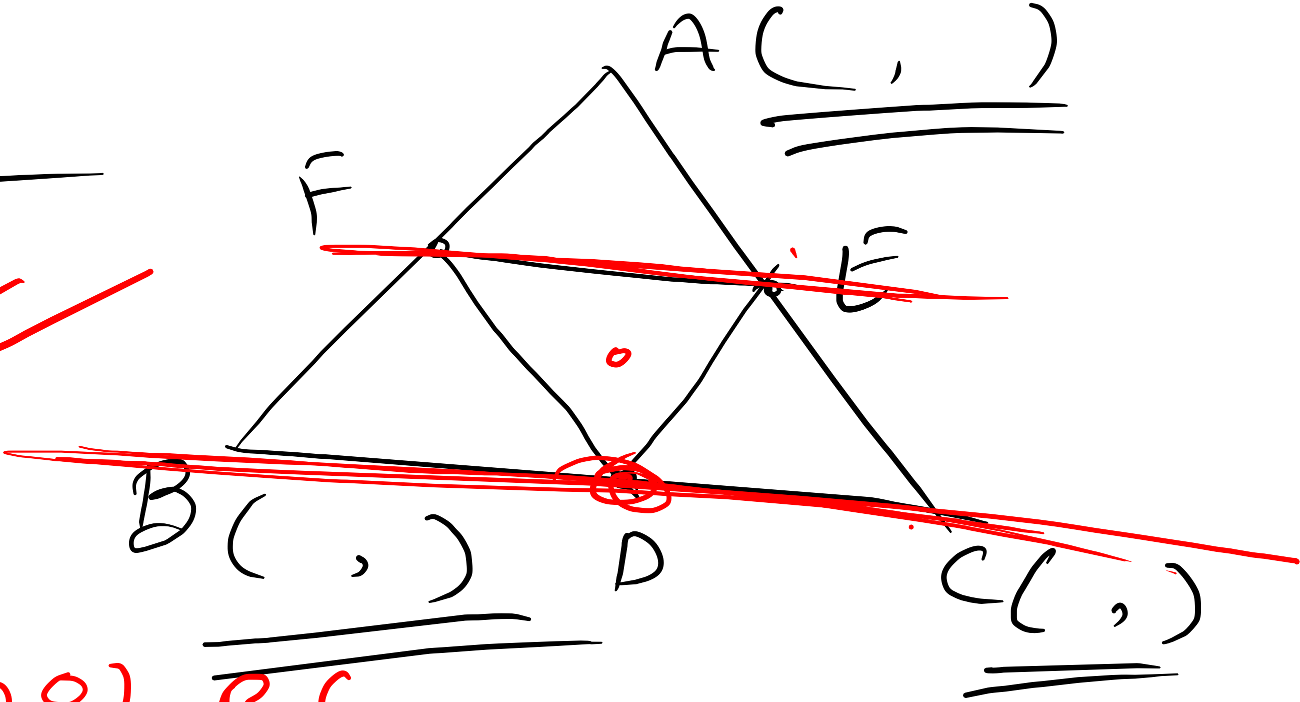
$$= \sqrt{3}$$

$$x_F + x_E - x_D = x_A$$

$$m_{BC} = m_{FE} = \checkmark$$

Point D

$\Rightarrow$ equation of BC



(5) Line through (4, –3) if the sum of the intercept is equal to 5. [**Ans.** $\frac{x}{10} - \frac{y}{5} = 1, \frac{x}{2} + \frac{y}{3} = 1$]

(6) Portion of a straight line intercepted between the co-ordinate axes is bisected at the point (x_1, y_1). Find the equation of line. [**Ans.** $\frac{x}{2x_1} + \frac{y}{2y_1} = 1$]

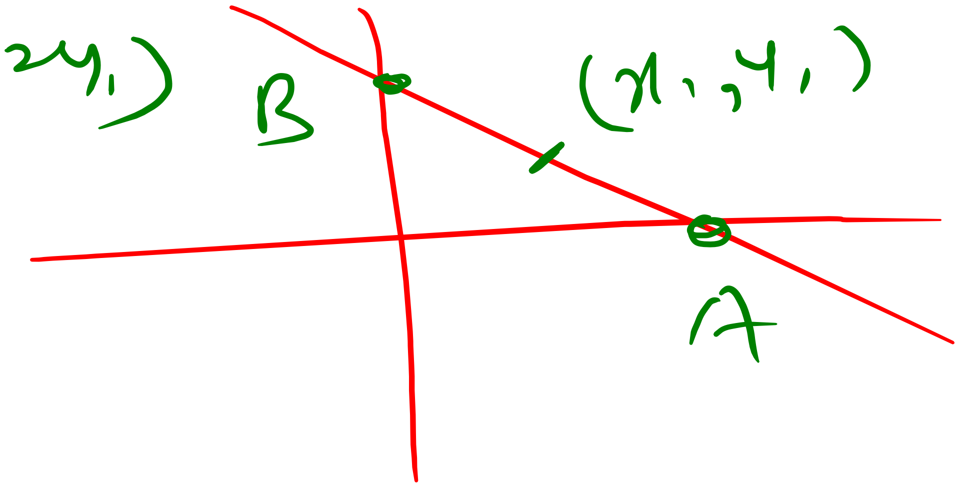
(7) Locus of the middle point of AB/circumcentre of ΔAOB, if the line passes through (a, b). (where A & B are points on co-ordinate axes) [**Ans.** $\frac{a}{2x} + \frac{b}{2y} = 1$]

E(8) Number of straight lines passing through (2, 4) & forming a triangle of area 16 square units with the co-ordinate axes. [**Ans.** 3]

$\frac{x}{a} + \frac{y}{b} = 1$, $a + b = 5$ $\boxed{\frac{4}{a} - \frac{3}{b} = 1}$ $\Rightarrow$ ✓

$A(2x_1, 0)$, $B(0, 2y_1)$

$\frac{x}{2x_1} + \frac{y}{2y_1} = 1$



E(9) Consider two points A(1,2) & B(3,4). Find point C on x-axis such that AC + BC is minimum.

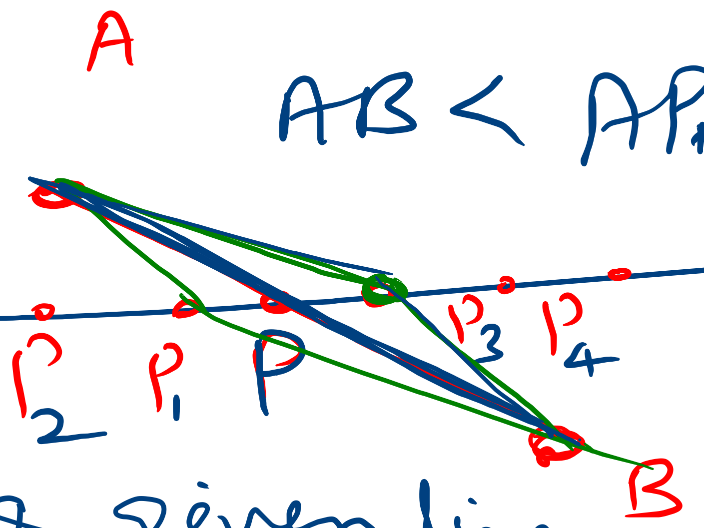
[**Ans.** (5/3, 0)]

Find P for which

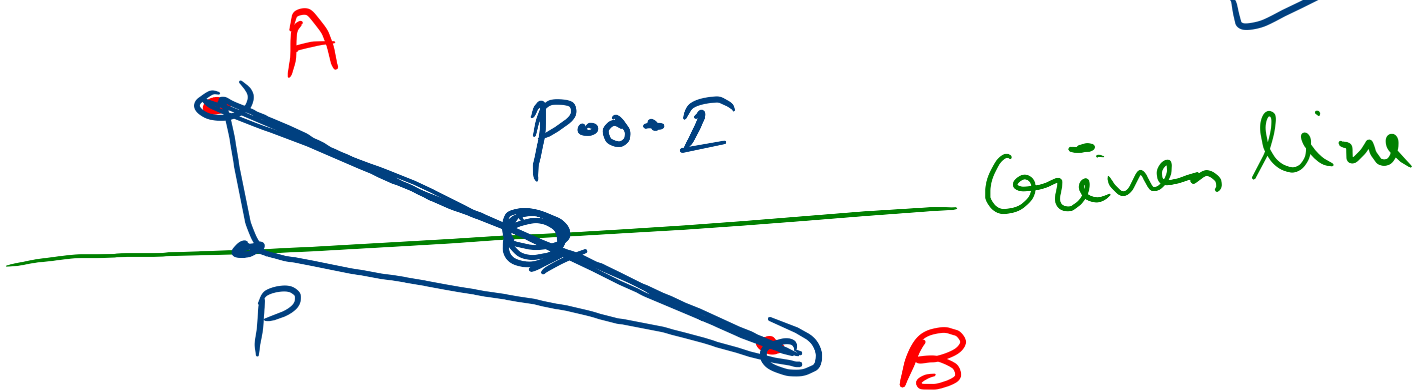
$AP + PB$ is minm

$AB < AP_1 + P_1B$

$ax+by+c=0$



P.O.I. of AB & given line $\Rightarrow$ P



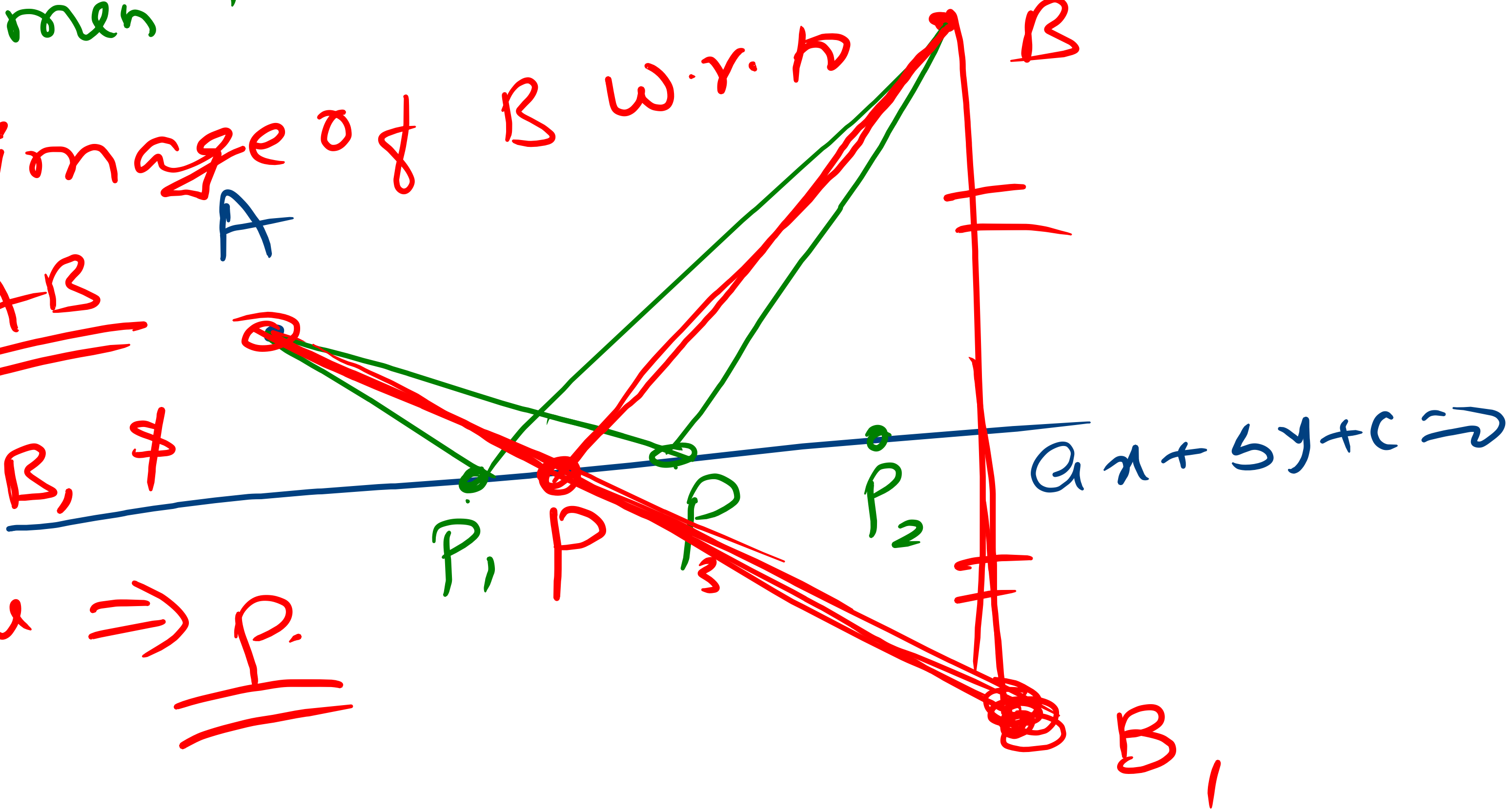
Case II :

Both A & B are lying same - side w.r.t $ax+by+c=0$

$(AP+PB)$ is min^m

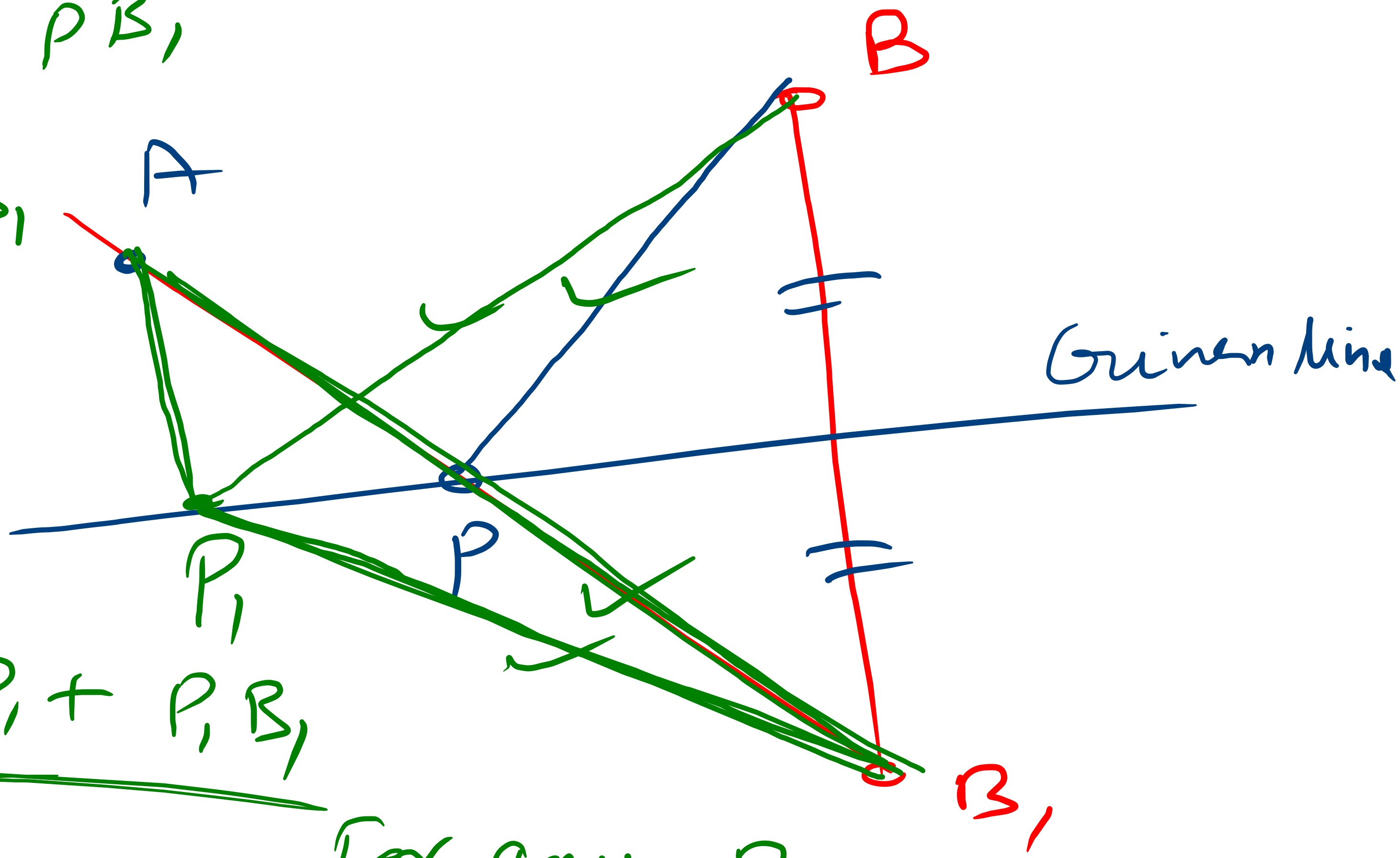
$B_1 \equiv$ mirror image of B w.r.t Line $\underline{AB}$

P.O.I of AB_1 & given line $\Rightarrow \underline{P}$



$AP + PB = AP + PB_1$

$AP_1 + P_1B = AP_1 + P_1B_1$



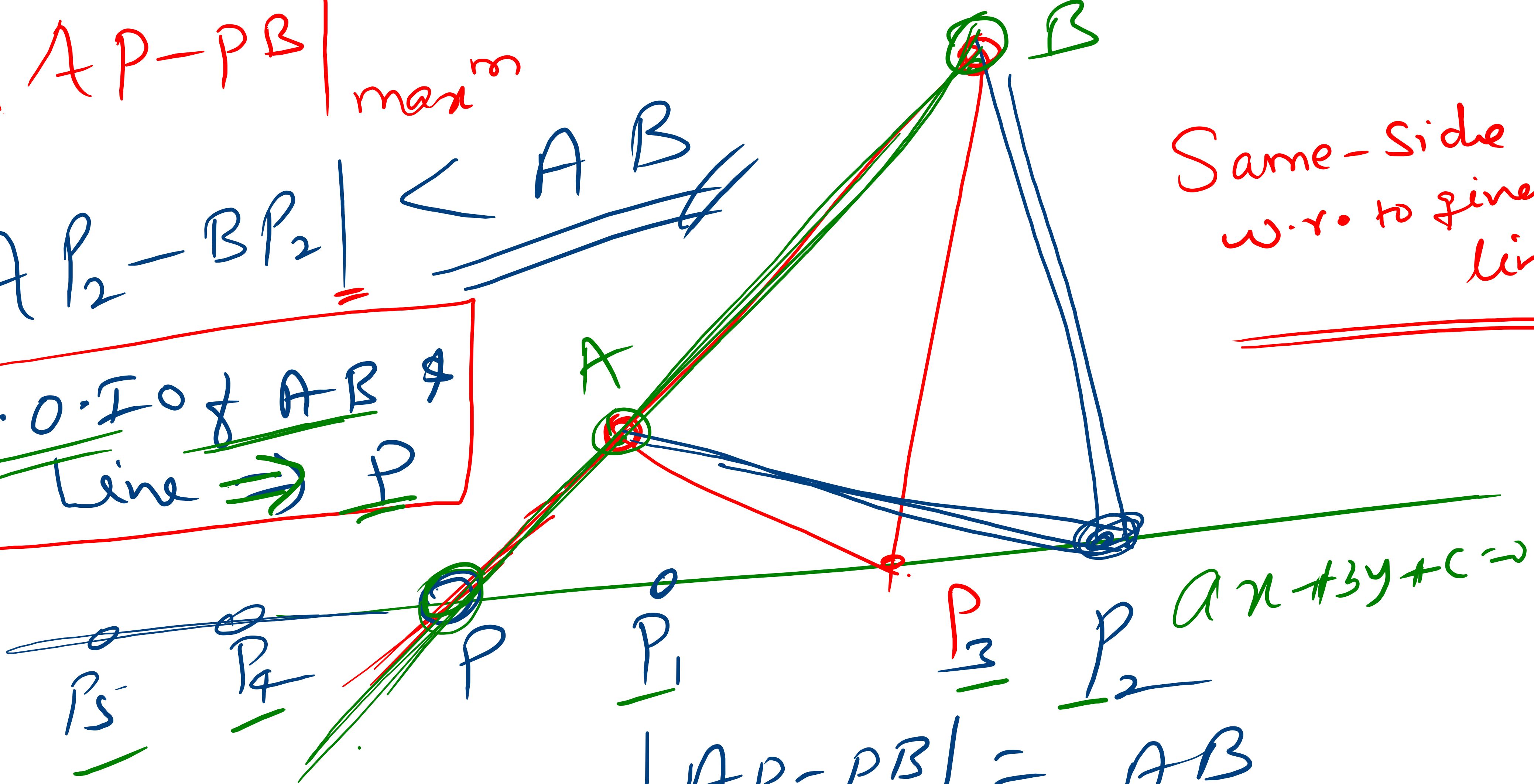
$\underline{AP + PB_1} < \underline{AP_1 + P_1B_1}$

For any P_1

$|AP - PB|_{max^m}$

$|AP_2 - BP_2| < AB$

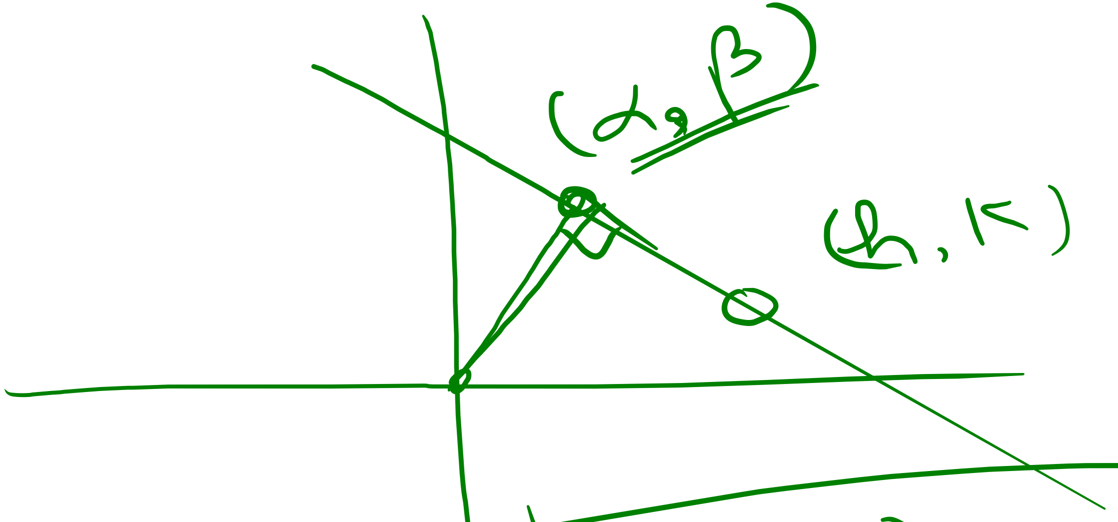
$\boxed{\text{P.O.I of } AB \text{ \& Line} \Rightarrow P}$

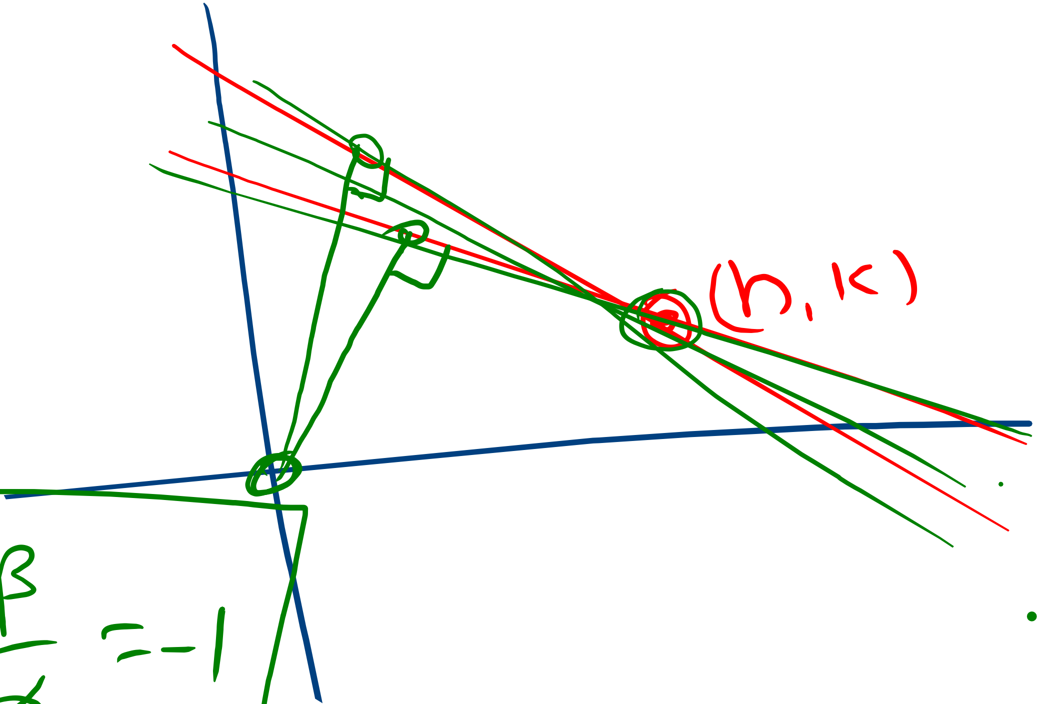


Same-Side w.r.to given line.

$|AP - PB| = AB$

E(10) Locus of foot of perpendicular drawn from origin on a variable line through point (h,k).





$$\boxed{\left(\frac{\beta - k}{\alpha - h}\right) \times \frac{\beta}{\alpha} = -1}$$

$$y^2 - ky + x^2 - hx = 0$$

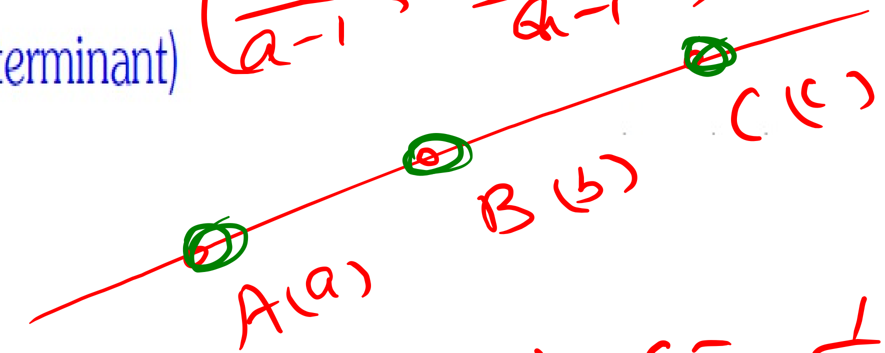
$$\boxed{x^2 + y^2 - hx - ky = 0}$$

E(11) If a, b, c are all different and the points $\left(\dfrac{r^3}{r-1}, \dfrac{r^2-3}{r-1}\right)$ where r = a, b, c are collinear, then prove that

$3(a+b+c) = ab + bc + ca - abc.$

(Use theory of equations instead of conventional determinant)

$\left(\dfrac{a^3}{a-1}, \dfrac{a^2-3}{a-1}\right)$



$y = mx + \lambda$

$\dfrac{r^2-3}{r-1} = \dfrac{m r^3}{r-1} + \lambda$

$\Rightarrow r^2 - 3 = mr^3 + \lambda r - \lambda$

$\Rightarrow mr^3 - r^2 + \lambda r + 3 - \lambda = 0$ — roots a, b, c

$a+b+c = \dfrac{1}{m}$

$abc = \dfrac{\lambda - 3}{m}$

$ab+bc+ca = \dfrac{\lambda}{m}$

$abc = (ab+bc+ca) - \dfrac{3}{m}$

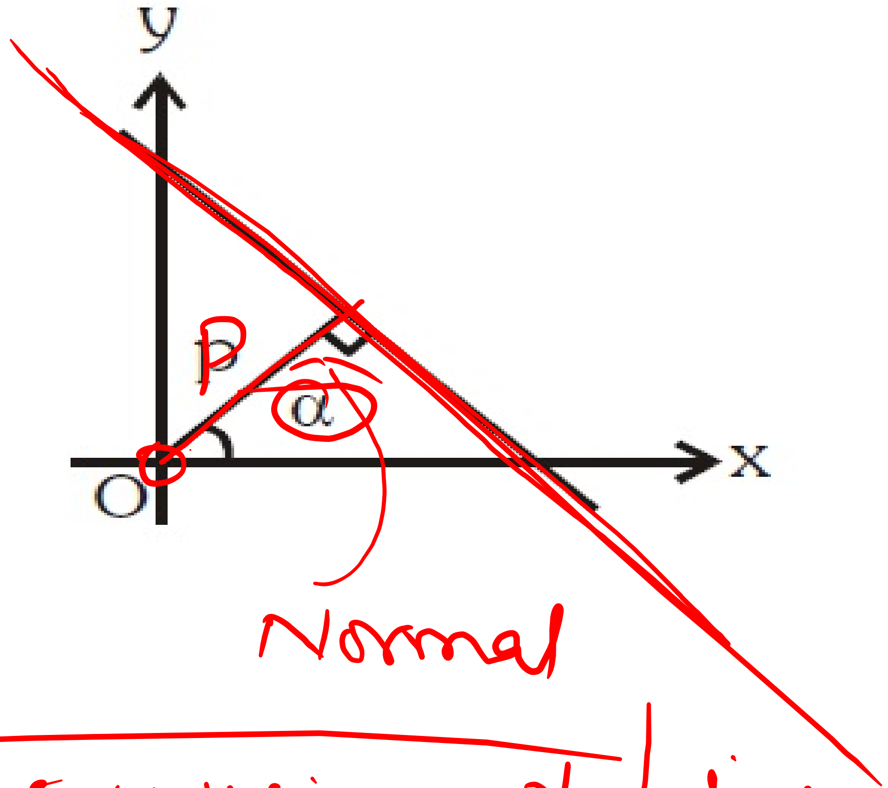
$\Rightarrow (ab+bc+ca) - abc = 3 \times \left(\dfrac{1}{m}\right)$

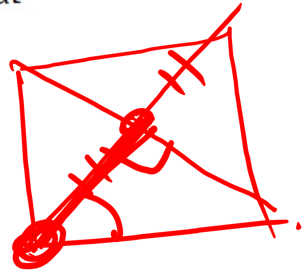
Normal form : $x\cos\alpha + y\sin\alpha = p;\ p > 0$

$0 \le \alpha < 2\pi$

The normal form can be identified by checking that

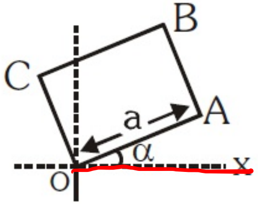
$(\text{coeff. of } x)^2 + (\text{coeff. of } y)^2 = 1\ \ \&\ p > 0$





Q. Change line $5x - 12y + 39 = 0$ in normal form.

Q.



Find equation of diagonal AC of square OABC.

$5x - 12y = -39$

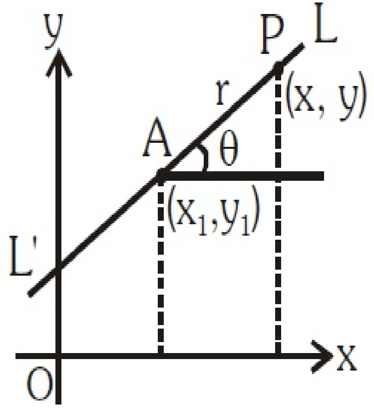
$\frac{5}{13}x - \frac{12}{13}y = \frac{-39}{13}$

$\left(-\frac{5}{13}\right)x + \left(\frac{12}{13}\right)y = 3$

$\boxed{x\cos\alpha + y\sin\alpha = p}$

$x\cos(\alpha + 45^\circ) + y\sin(\alpha + 45^\circ) = \left(\frac{1}{2}\sqrt{2}a\right)$

Parametric form : $\left[\left(\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r\right);\ 0\le\theta<\pi\right]$



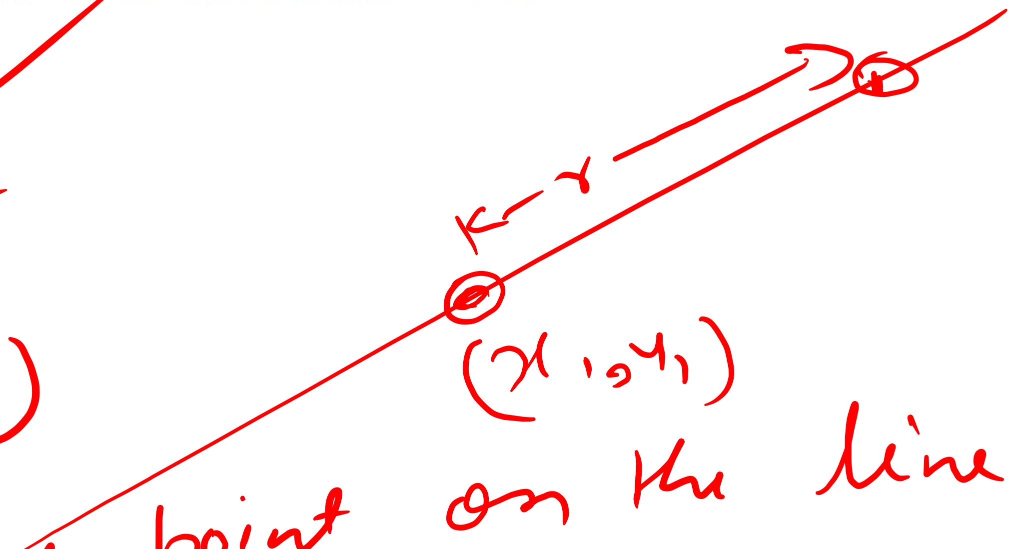
$x-x_1=r\cos\theta,\ \ y-y_1=r\sin\theta$

(i) 'r' may be positive or negative.

(ii) Whenever a question involves distances, parametric form may be used.

$y-y_1=(x-x_1)\tan\theta$

$\frac{x-x_1}{\cos\theta}=\frac{y-y_1}{\sin\theta}=r$



$P(x_1+r\cos\theta,\ y_1+r\sin\theta)$

can be taken at any point on the line at a distance $|r|$ from (x_1,y_1)

Q. A line through $A(-5, -4)$ meets the line $x + 3y + 2 = 0$, $2x + y + 4 = 0$ and $x - y - 5 = 0$ at the points B, C and D respectively. If $(15/AB)^2 + (10/AC)^2 = (6/AD)^2$, find the equation of the line. *(1993 - 5 Marks)*

$\underline{\underline{Q}}$

$\tan\theta = \underline{\underline{-2/3}}$

$B(-5 + r_1\cos\theta, -4 + r_1\sin\theta)$, $C(-5 + r_2\cos\theta, -4 + r_2\sin\theta)$

$D(-5 + r_3\cos\theta, -4 + r_3\sin\theta)$ $AB = |r_1|$, $AC = |r_2|$

$AD = |r_3|$

B lies on $x + 3y + 2 = 0 \Rightarrow -5 + r_1\cos\theta - 12 + 3r_1\sin\theta + 2 = 0$

$\Rightarrow r_1(\cos\theta + 3\sin\theta) = 15 \Rightarrow \left(\frac{15}{r_1}\right)^2 = (\cos\theta + 3\sin\theta)^2$

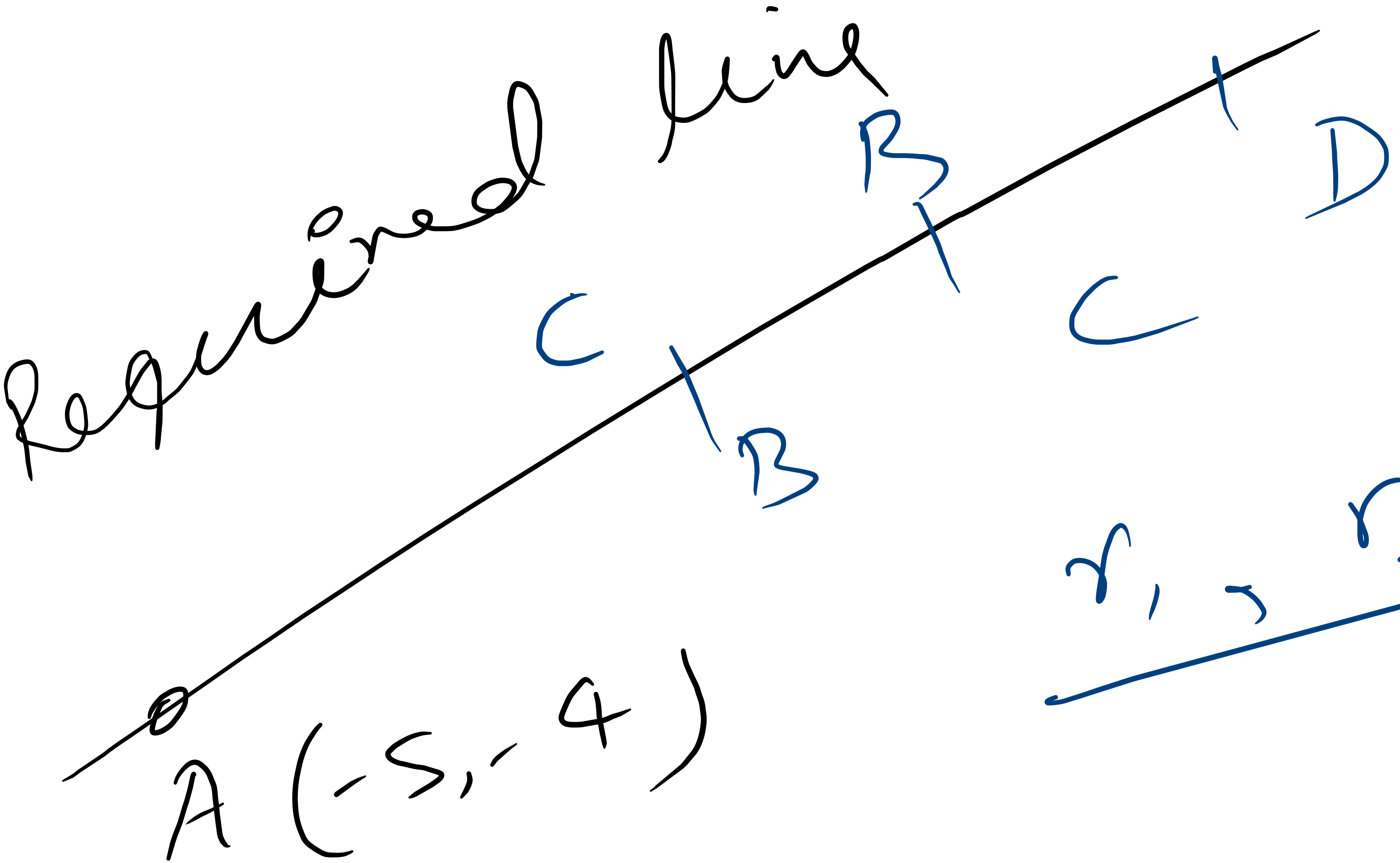
C lies on $2x + y + 4 = 0$

$\Rightarrow -10 + 2r_2\cos\theta - 4 + r_2\sin\theta + 4 = 0$

$r_2(2\cos\theta + \sin\theta) = 10 \Rightarrow \left(\frac{10}{r_2}\right)^2 = (2\cos\theta + \sin\theta)^2$

D lies on $x - y - 5 = 0 \Rightarrow -5 + r_3\cos\theta + 4 - r_3\sin\theta - 5 = 0$

$\Rightarrow \left(\frac{6}{r_3}\right)^2 = (\cos\theta - \sin\theta)^2$



r_1, r_2, r_3

For points $P = (x_1, y_1)$ and $Q = (x_2, y_2)$ of the co-ordinate plane, a new distance $d(P, Q)$ is defined by $d(P, Q) = |x_1 - x_2| + |y_1 - y_2|$. Let $O = (0, 0)$ and $A = (3, 2)$. Prove that the set of points in the first quadrant which are equidistant (with respect to the new distance) from O and A consists of the union of a line segment of finite length and an infinite ray. Sketch this set in a labelled diagram.

Q7. The area of the region $\{(x, y) : xy \le 8,\ 1 \le y \le x^2\}$ is **[JEE(Advanced)-2019, 3(–1)]**

(1) $8\log_e 2 - \frac{14}{3}$ (2) $16\log_e 2 - \frac{14}{3}$ (3) $16\log_e 2 - 6$ (4) $8\log_e 2 - \frac{7}{3}$

$\boxed{xy = 8}$

$xy - 8 = 0$

$0 - 8 < 0$

$xy - 8 \le 0$

$A_1 = \int_1^2 (x^2 - 1)\,dx = \frac{8-1}{3} - 1 = \frac{4}{3}$

$A_2 = \int_2^8 \left(\frac{8}{x} - 1\right) dx$

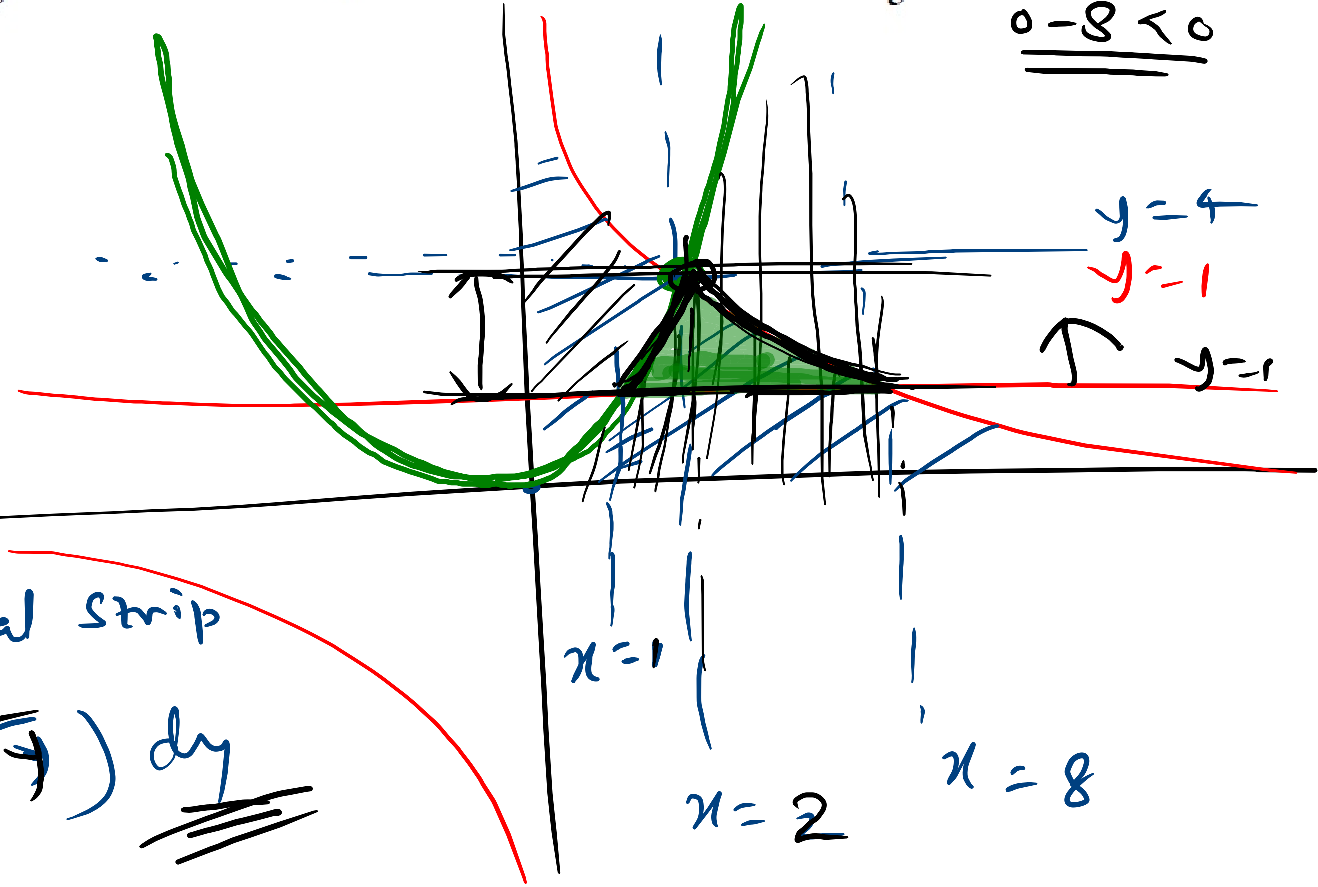
$= 8\ln 4 - 6$

$= 16\ln 2 - 6$

$A_1 + A_2 = 16\ln 2 - \frac{14}{3}$

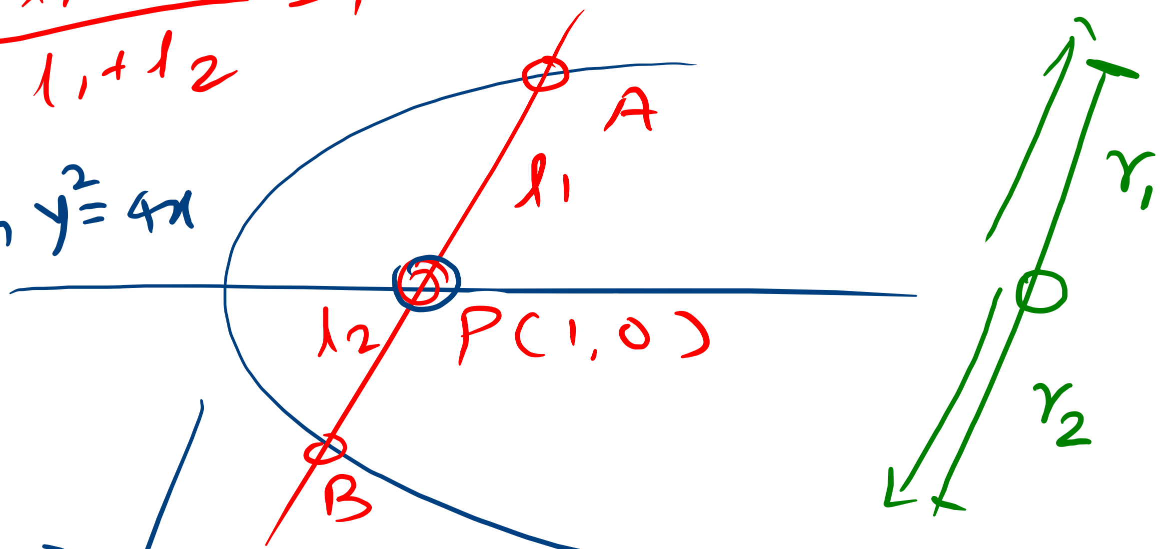
Horizontal strip

$\int_1^4 \left(\frac{8}{y} - \sqrt{y}\right) dy$



Q: A line passing through P(1,0) intersects the curve $y^2 = 4x$ at A & B. Find the value of $\frac{1}{PA}+\frac{1}{PB}$. [**Ans.** 1]

$$\frac{2l_1 l_2}{l_1+l_2} = 2 \Rightarrow \frac{l_1 l_2}{l_1+l_2} = 1$$



$A(1+r\cos\theta,\ r\sin\theta)$ lies on $y^2=4x$

$$\Rightarrow r^2\sin^2\theta = 4(1+r\cos\theta)$$
$$= 4+4r\cos\theta$$
$$\Rightarrow (\sin^2\theta)r^2 - (4\cos\theta)r - 4 = 0$$

$$r_1 r_2 = \frac{-4}{\sin^2\theta}$$
$$r_1+r_2 = \frac{4\cos\theta}{\sin^2\theta}$$

$$\frac{1}{PA}+\frac{1}{PB} = \frac{1}{r_1}+\frac{1}{r_2}$$

$$= \frac{1}{|r_1|}+\frac{1}{|r_2|} = \frac{|r_1|+|r_2|}{|r_1 r_2|} = 1$$

$$(|r_1|+|r_2|)^2 = r_1^2+r_2^2+2|r_1 r_2|$$

Q: In what direction a line through the point A(1, 2) must be drawn so that its intersection point P with line $x + y = 4$ may be at a distance of $\frac{\sqrt{6}}{3}$ from A. ($\theta = 15°$ or $75°$ finding θ)

$P(1 + r\cos\theta,\ 2 + r\sin\theta)$ lies on $x + y = 4$

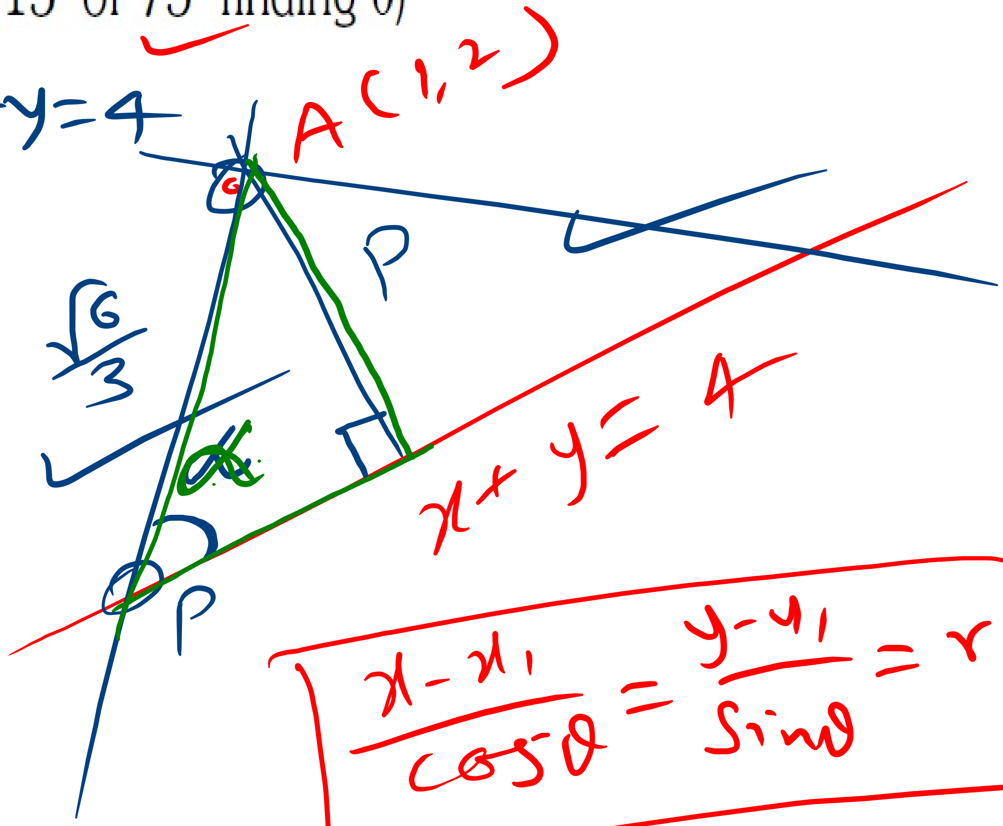
$\Rightarrow 1 + r\cos\theta + 2 + r\sin\theta = 4$

$r(\cos\theta + \sin\theta) = 1$

$\frac{1}{\sqrt{2}}\cos\theta + \frac{\sin\theta}{\sqrt{2}} = \frac{3}{\sqrt{6}} \times \frac{1}{\sqrt{2}}$

$\sin(\theta + \frac{\pi}{4}) = \frac{\sqrt{3}}{2}$

$\theta + \frac{\pi}{4} = 60° \Rightarrow \theta = 15°$



$$\boxed{\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r}$$

θ is angled made by the required line with the x axis

Q. A variable line through origin meets the line $a_1x + b_1y + c_1 = 0$ and $a_2x + b_2y + c_2 = 0$ at P and Q. On it is taken a point R. If $\frac{2}{OR} = \frac{1}{OP} + \frac{1}{OQ}$, then prove that locus of R is also a straight line.

there 2 lines are given

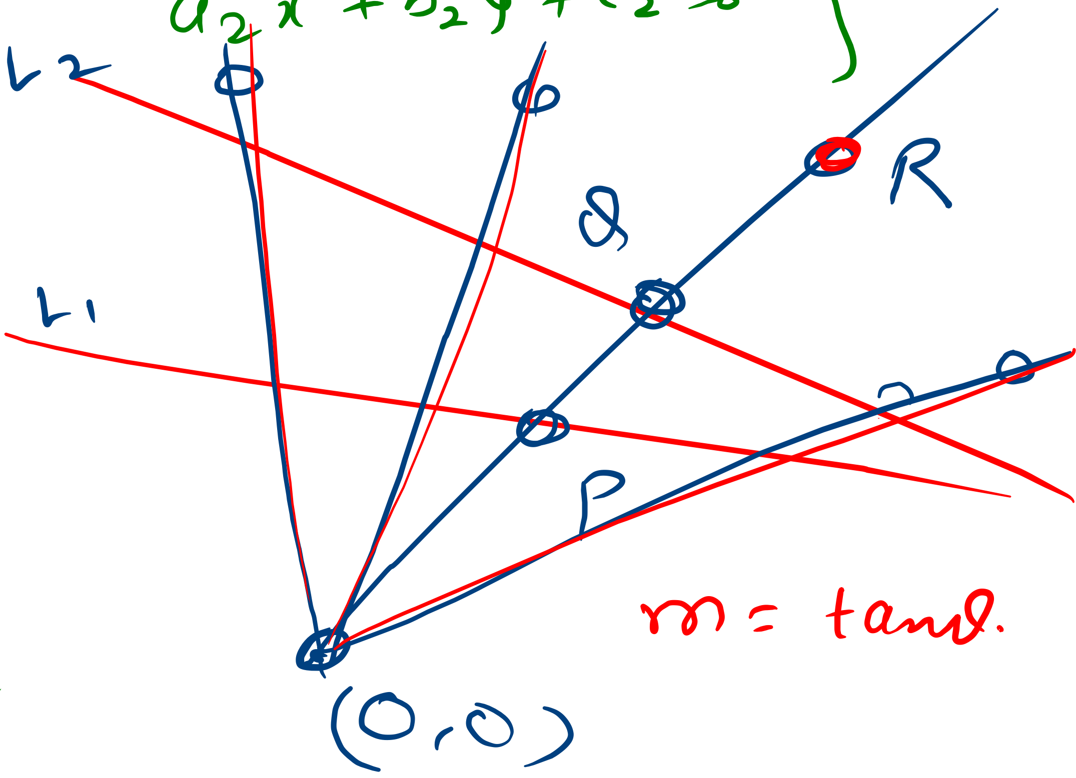
$a_1x + b_1y + c_1 = 0$
$a_2x + b_2y + c_2 = 0$

Locus of R $\equiv (h, k)$

$\frac{x-0}{\cos\theta} = \frac{y-0}{\sin\theta} = r$

$P(r_1\cos\theta,\ r_1\sin\theta)$
$Q(r_2\cos\theta,\ r_2\sin\theta)$
$R(r\cos\theta,\ r\sin\theta)$

$R \equiv (h, k)$



$P(r_1\cos\theta, r_1\sin\theta)$ Lies on $a_1x+b_1y+c_1=0$

$$\Rightarrow \quad a_1r_1\cos\theta + b_1r_1\sin\theta + c_1 = 0$$

$$(a_1\cos\theta + b_1\sin\theta)r_1 = -c_1 \Rightarrow \frac{1}{r_1} = \frac{a_1\cos\theta + b_1\sin\theta}{-c_1}$$

$Q(r_2\cos\theta, r_2\sin\theta)$ lies on $a_2x+b_2y+c_2=0$

$$\Rightarrow \quad \frac{1}{r_2} = \frac{a_2\cos\theta + b_2\sin\theta}{-c_2}$$

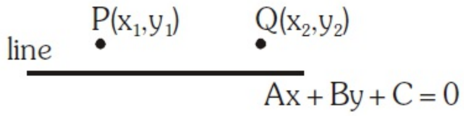
$$\frac{2}{r} = \frac{1}{r_1} + \frac{1}{r_2} = \left(\frac{a_1\cos\theta}{-c_1} + \frac{a_2\cos\theta}{-c_2}\right) + \frac{b_1\sin\theta}{-c_1} + \frac{b_2\sin\theta}{-c_2}$$

$$\Rightarrow \quad 2 = \overbrace{r\cos\theta}^{h}\left(-\frac{a_1}{c_1} - \frac{a_2}{c_2}\right) + \underbrace{r\sin\theta}_{K}\left(-\frac{b_1}{c_1} - \frac{b_2}{c_2}\right)$$

$$\Rightarrow \quad \boxed{2 = xA + yB}$$

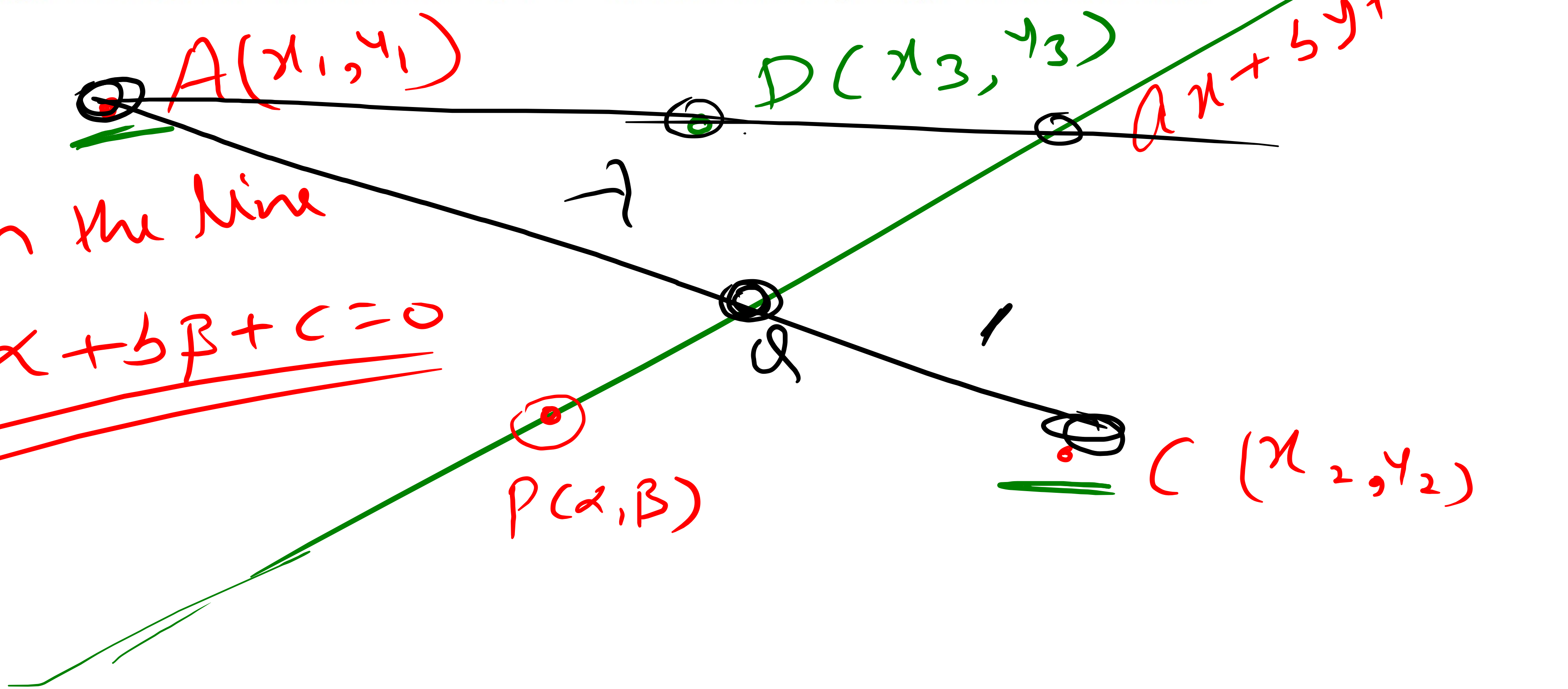
POSITION OF A POINT W.R.T. A LINE :

If the point $P(x_1, y_1)$ and $Q(x_2, y_2)$ lies on the same side of the line then the expression $Ax_1 + By_1 + C$ and $Ax_2 + By_2 + C$ will be of the same sign and if



(x_1, y_1) and (x_2, y_2) are on the opposite side of the line then the expressions $Ax_1 + By_1 + C$ and $Ax_2 + By_2 + C$ will be of opposite side.

Note : If point (x_1,y_1) is on origin side of the line $Ax + By + C = 0$, then C and $Ax_1 + By_1 + C$ are of same sign & vice versa.



P lies on the line

$\Leftrightarrow a\alpha+b\beta+c=0$

$$(ax_1+by_1+c)(ax_2+by_2+c) > 0$$

Same nature

$\Longleftrightarrow$ Same-side

$$(ax_1+by_1+c)(ax_2+by_2+c) < 0$$

opp. nature

$\Longleftrightarrow$ opp. to each other

Q. If the point (a, a^2) lies between the lines $x + y - 2 = 0$ and $4x + 4y - 3 = 0$, then find the range of value of a.

[**Ans.** $(-2, -3/2) \cup (1/2, 1)$]

$(a+a^2-2)(4a+4a^2-3) < 0$ $\Rightarrow$ $(\ )(\ ) > 0$

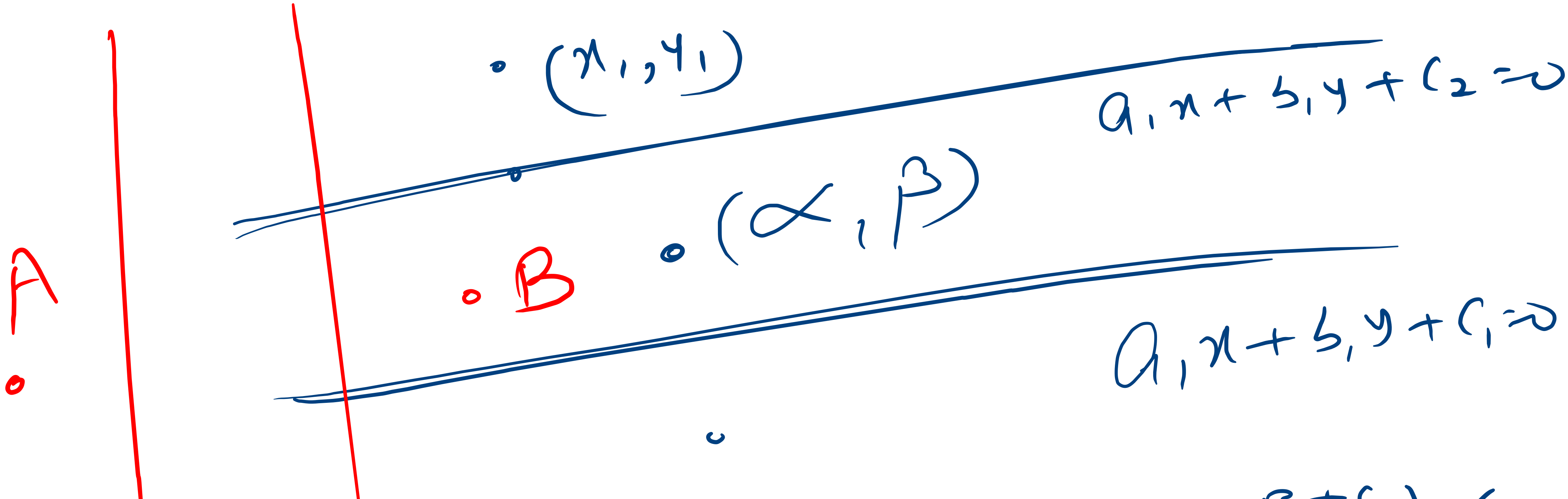
$(\ )(\ ) < 0$ $\quad (a, a^2)$

$x+y-2=0$

$(\ )(\ ) > 0$

$(\ )(\ ) < 0$

$\boxed{4(2<0}$



$$(a_1\alpha+b_1\beta+c_2)(a_1\alpha+b_1\beta+c_1)<0$$

$$(a_1x_1+b_1y_1+c_1)(a_1x_1+b_1y_1+c_2)>0$$

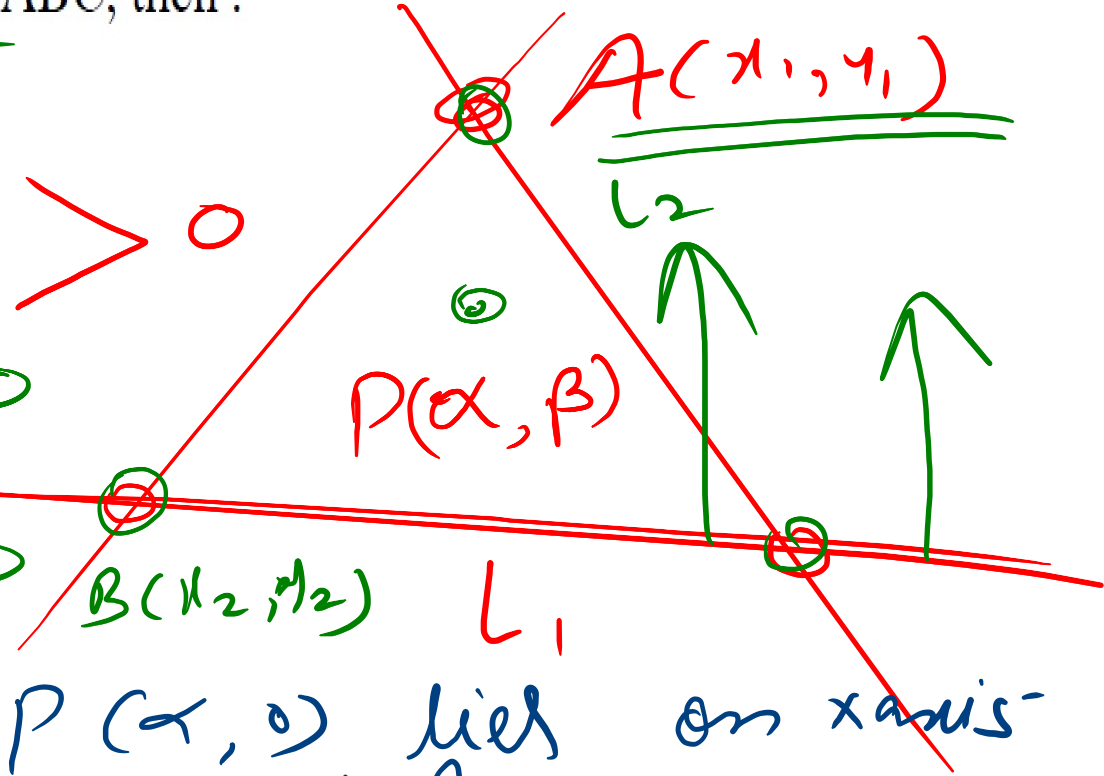
Q: A triangle ABC is formed by the lines $2x - 3y - 6 = 0$; $3x - y + 3 = 0$ and $3x + 4y - 12 = 0$. If the points $P(\alpha,0)$ and $Q(0,\beta)$ always lie on or inside the ΔABC, then :

Basic method:

$$(L_1(\alpha,\beta))(L_1(x_1,y_1)) > 0$$

$$(L_2(\alpha,\beta))(L_2(x_2,y_2)) > 0$$

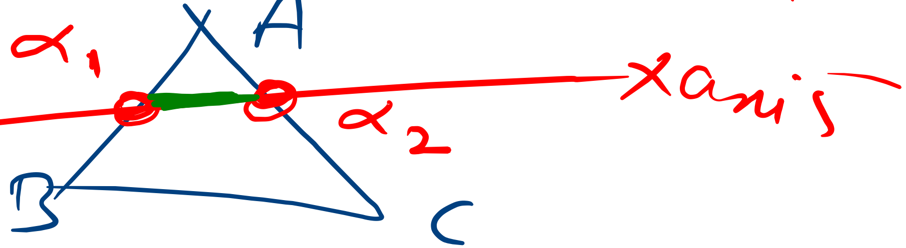
$$(L_3(\alpha,\beta))(L_3(x_3,y_3)) > 0$$



Smart approach:

$$\boxed{\alpha_1 < \alpha < \alpha_2}$$

Point $P(\alpha, 0)$ lies on x axis

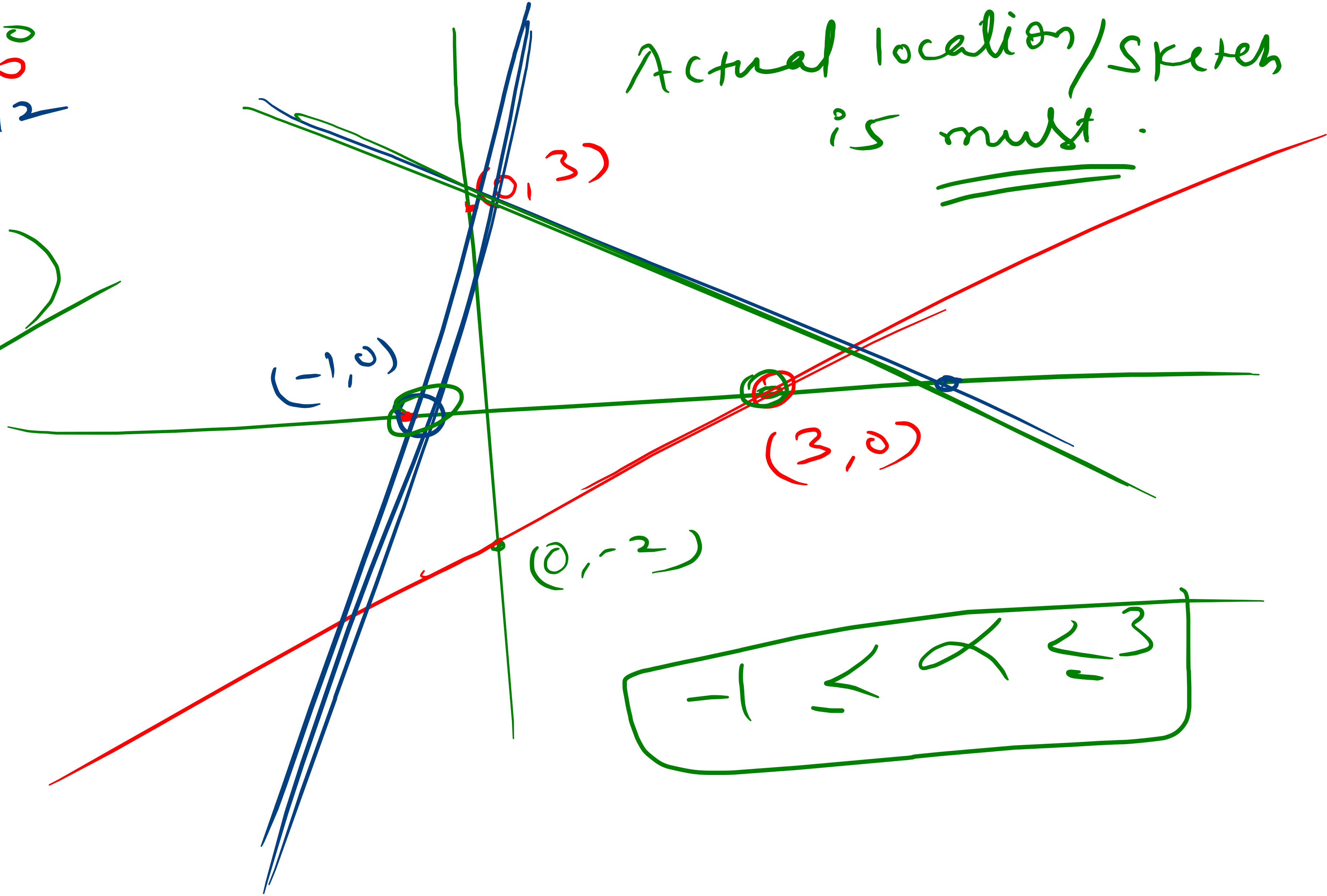


$2x - 3y - 6 = 0$

$3x - y + 3 = 0$

$3x + 4y = 12$

(α, α^2)



Actual location/sketch is must.

$$\boxed{-1 \le \alpha \le 3}$$

STRAIGHT LINE MAKING A GIVEN ANGLE WITH A LINE :

Equation of line passing through a point (x_1, y_1) and making an angle α, with the line $y = mx + c$ is written as :

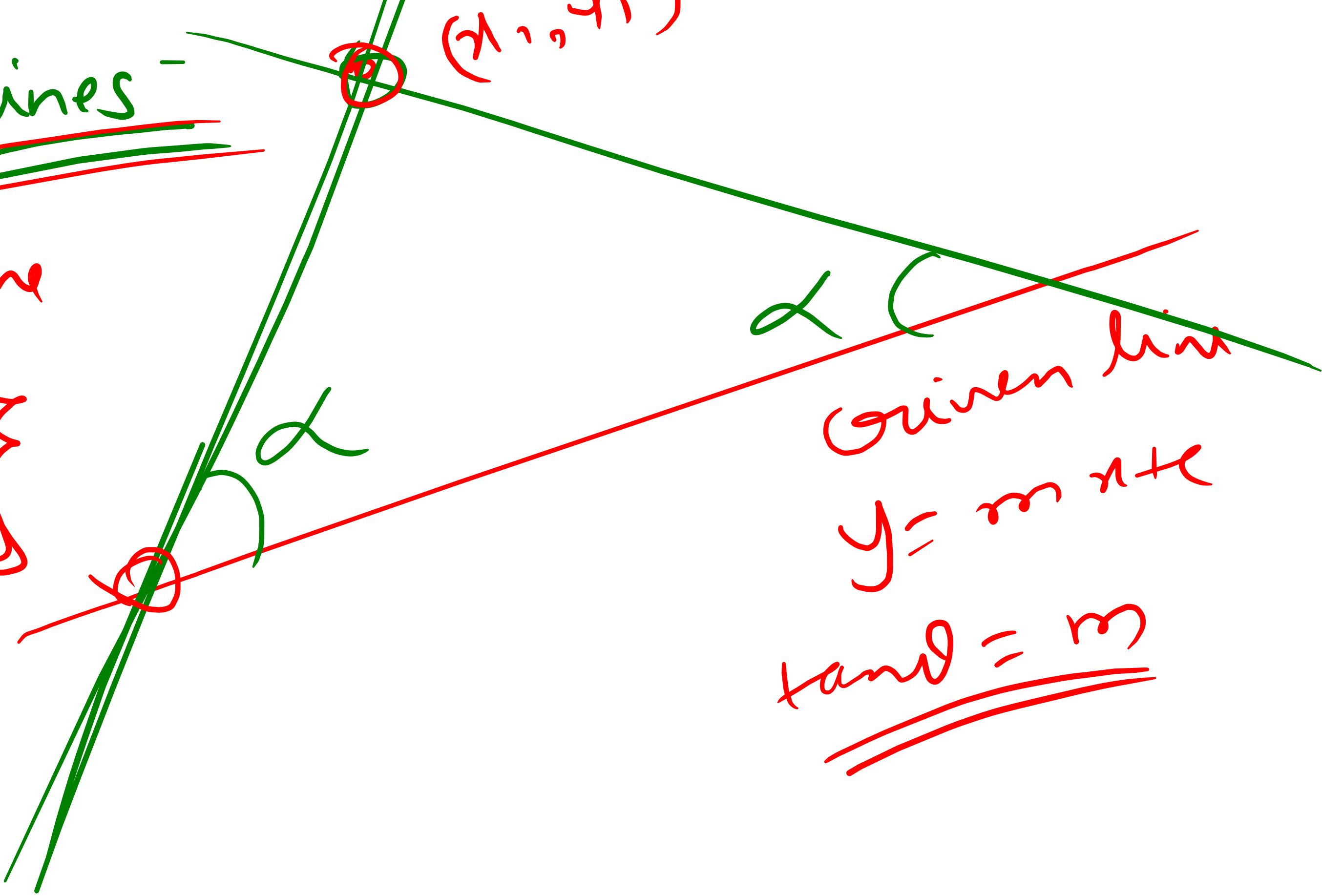
$$y - y_1 = \frac{m \pm \tan\alpha}{1 \mp m\tan\alpha}(x - x_1)$$

Q. Find the equation to the sides of an isosceles right-angled triangle, the equation of who hypotenuse is $3x + 4y = 4$ and the opposite vertex is the point $(2, 2)$.

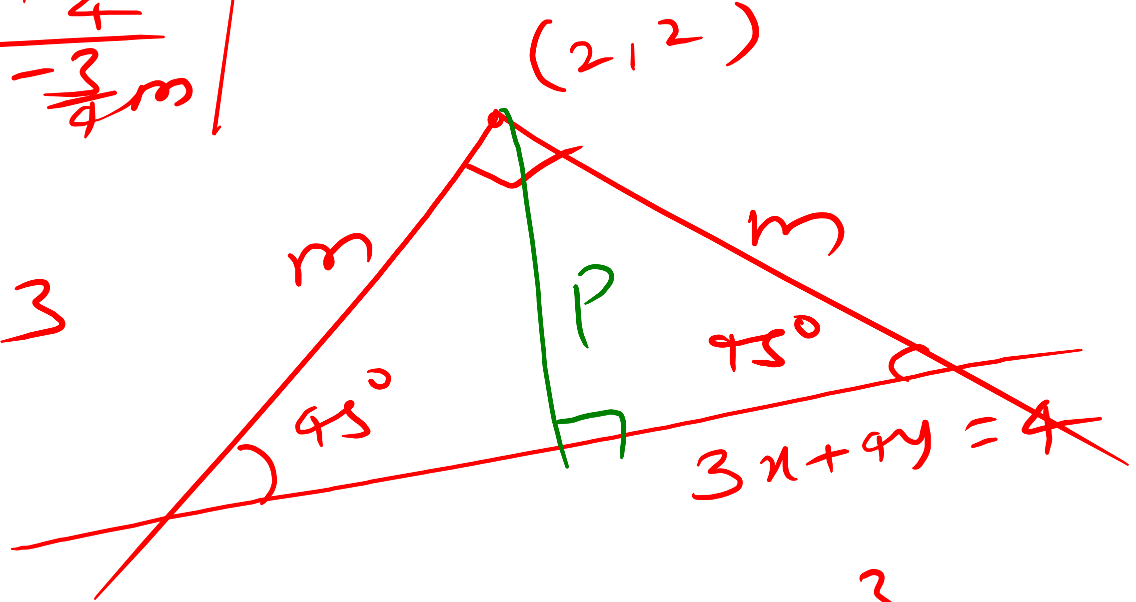
$\alpha \neq 90^\circ \Rightarrow$ 2 such lines

Slope of the required line $= \tan(\theta + \alpha)$, $\tan(\theta - \alpha)$

$$\boxed{y - y_1 = \text{slope}\,(x - x_1)}$$



$$3x+4y=4, \quad (2,2)$$



Method 1: $\tan 45^\circ = \left|\dfrac{m+\frac{3}{4}}{1-\frac{3}{4}m}\right|$

$$\Rightarrow |4-3m| = |4m+3|$$

$$4-3m = 4m+3, \quad -4m-3$$

$$\Rightarrow 7m = 1, \quad m=-7$$

$$\Rightarrow \boxed{m=\frac{1}{7}} \qquad \boxed{m=-7}$$

Method 2: $m = \tan(\theta+45^\circ) = \dfrac{\tan\theta+1}{1-\tan\theta} = \dfrac{1-\frac{3}{4}}{1+\frac{3}{4}} = \dfrac{1}{7}$

$$\tan(\theta-45^\circ) = \frac{\tan\theta-1}{1+\tan\theta} = \frac{-\frac{3}{4}-1}{1-\frac{3}{4}} = -7$$

Lines $\parallel$ & $\perp$ to $ax+by+c=0$

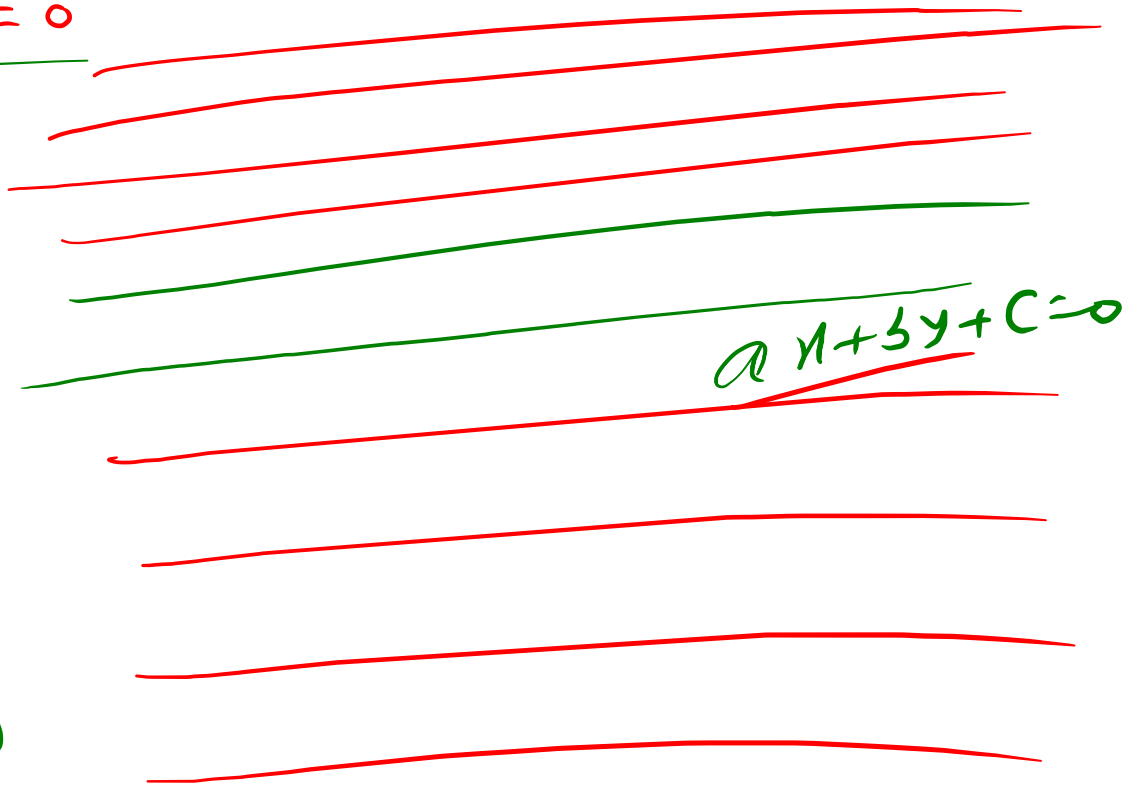
$$\boxed{ax+by=\lambda}$$

Can be taken as any line $\parallel$ to $ax+by+c=0$

$$3x+4y=5$$

Parallel & passing through $(0,0)$

$$\boxed{3x+4y=3\times 0+4\times 0}$$



Family of lines $\parallel$ to a given line

$\perp^r$ to a given line $ax+by+c=0$

$$\boxed{bx-ay=\lambda}$$

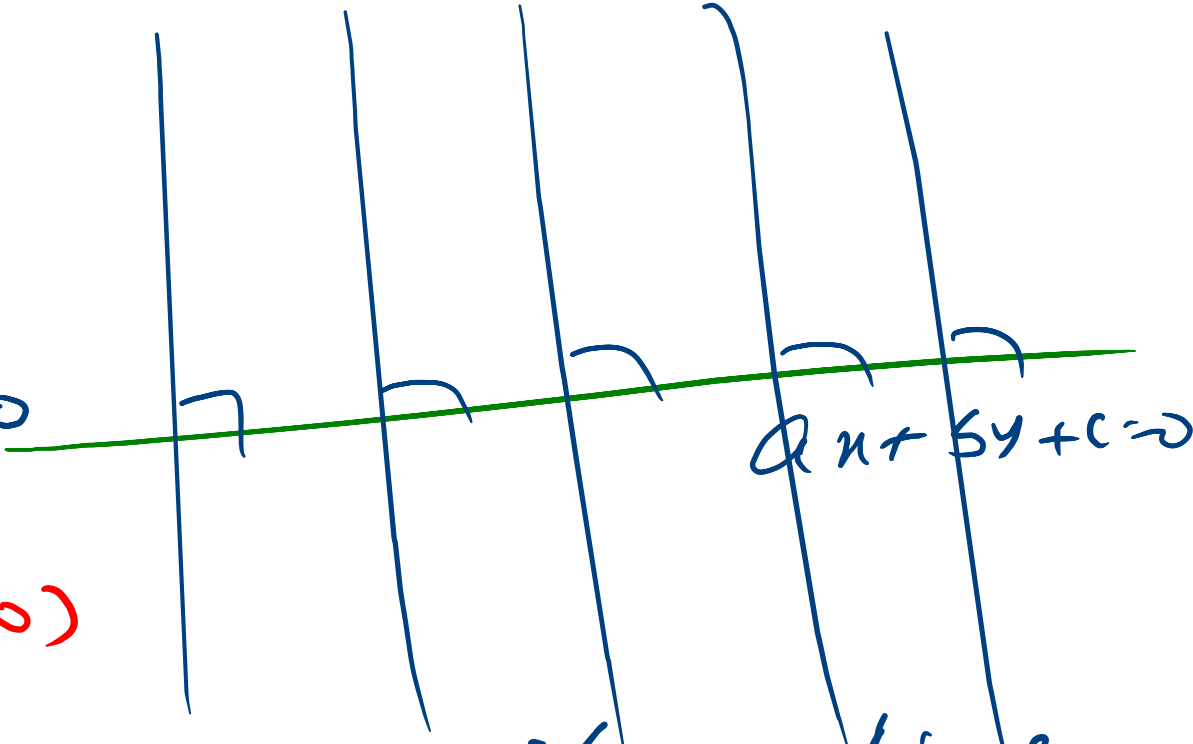
Can be taken as any line $\perp^r$ to $ax+by+c=0$

$3x+4y=5$

$\perp^r$ & passing through $(0,0)$

$4x-3y = 4\times 0 - 3\times 0$

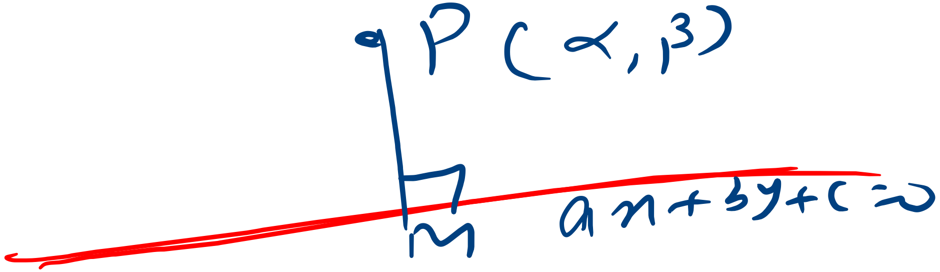
$$\boxed{4x-3y=0}$$



Family of lines

LENGTH OF PERPENDICULAR :

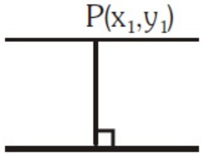
(a) **The length of perpendicular** from (x_1, y_1) on $ax + by + c = 0$;



Proof : analytical approach; vector approach $= \left|\dfrac{ax_1 + by_1 + c}{\sqrt{a^2+b^2}}\right|$

(b) Distance between the || lines $ax + by + c_1 = 0$ and

$ax + by + c_2 = 0$ is $\left|\dfrac{c_1 - c_2}{\sqrt{a^2+b^2}}\right|$



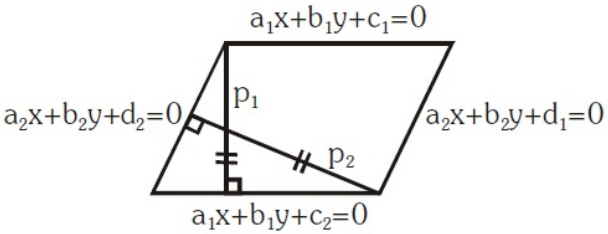
$$PM = \frac{|a\alpha + b\beta + c|}{\sqrt{a^2+b^2}}$$

[Take (x_1, y_1) on any one line and draw a perpendicular on the other line]

Note :

Parallelogram becomes a rhombus

if $\left|\dfrac{c_1 - c_2}{\sqrt{a_1^2+b_1^2}}\right| = \left|\dfrac{d_1 - d_2}{\sqrt{a_2^2+b_2^2}}\right|$



Examples :

Find equation of line parallel to $ax + by + c_1 = 0$ & $ax + by + c_2 = 0$ & dividing distance between them in ratio m : n.

$2x + 2y = 6$
$x + y = 10$

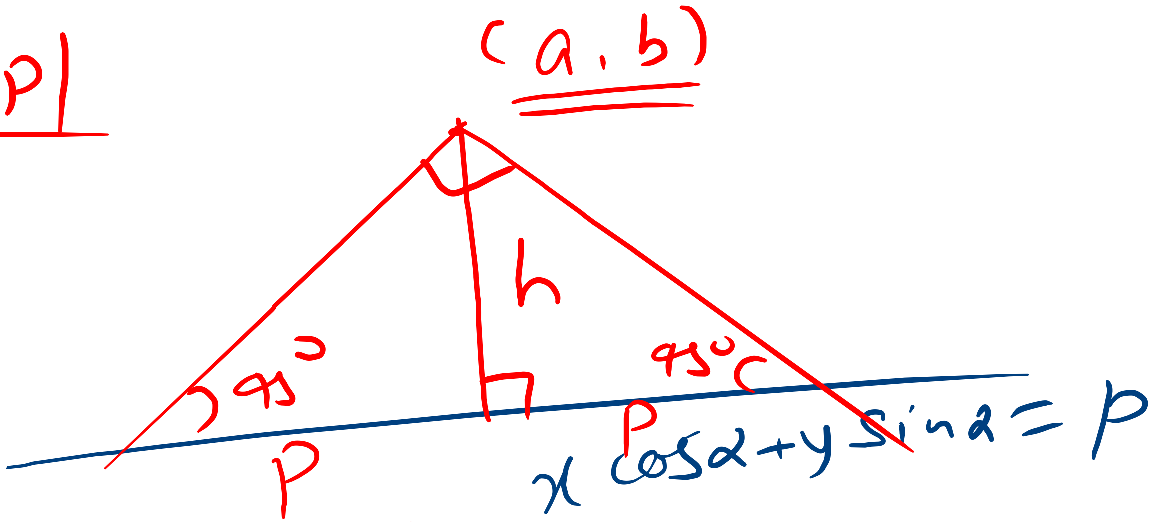
$\Rightarrow$ $x + y = 3$
$x + y = 10$

$$\frac{|10-3|}{\sqrt{2}} = \frac{7}{\sqrt{2}}$$

Two mutually ⊥ lines are drawn through the point (a, b) and enclose an isosceles Δ together with the line $x\cos\alpha + y\sin\alpha = p$. Find the area of Δ. [**Ans.** $(a\cos\alpha + b\sin\alpha + p)^2$]

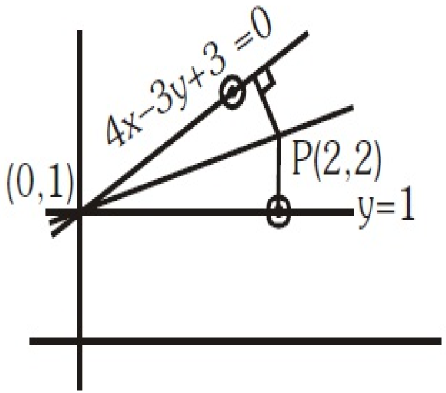
$$h = \frac{|a\cos\alpha + b\sin\alpha - p|}{\sqrt{\cos^2\alpha + \sin^2\alpha}}$$

$$= |a\cos\alpha + b\sin\alpha - p|$$



$$\Delta = h^2 = (a\cos\alpha + b\sin\alpha - p)^2$$

Equation of two straight lines passing through the point (0, 1) such that the perpendicular distance from (2, 2) on it is equal to unity. Find also the equation of the line through the feet of these perpendiculars.



$y = mx + 1$

$mx - y + 1 = 0$

$\dfrac{|2m - 2 + 1|}{\sqrt{m^2+1}} = 1$

$\Rightarrow (2m-1)^2 = (m^2+1)$

$\Rightarrow 4m^2 - 4m + \cancel{1} = m^2 + \cancel{1}$

$\Rightarrow 3m^2 - 4m = 0$

$\Rightarrow$ $m = 0$

$m = \frac{4}{3}$

$m = 4/3$

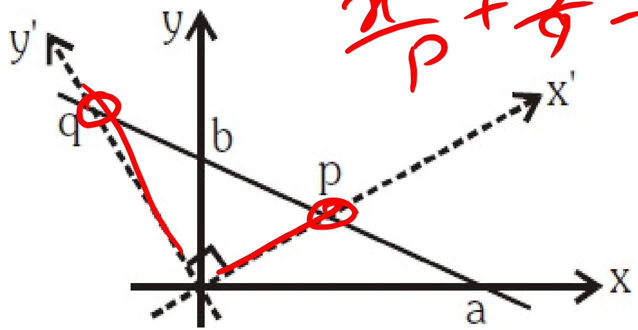
(2,2)

(0,1)

$m = 0$

$\boxed{y = 1}$

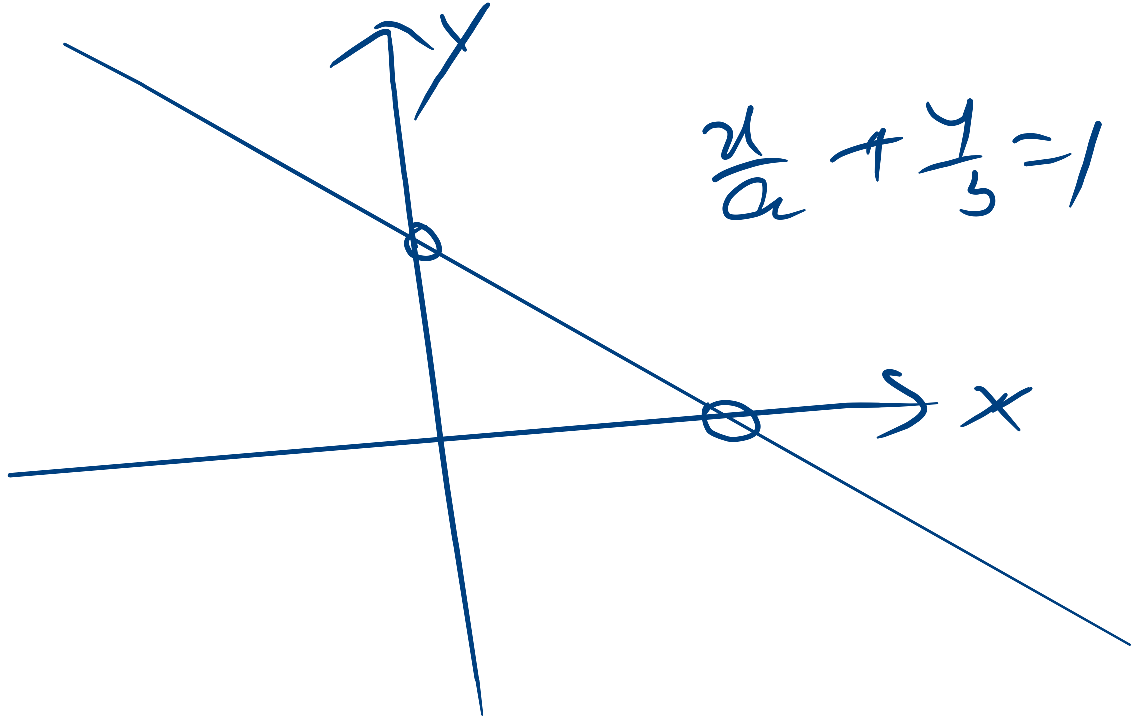
Q. Intercept made by a line L on the co-ordinate axis is a and b; if the axes are rotated about the origin, new intercepts are p, q then prove that $\frac{1}{p^2}+\frac{1}{q^2}=\frac{1}{a^2}+\frac{1}{b^2}$



$\frac{x}{p}+\frac{y}{q}=1$

$$\frac{1}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}}}=\frac{1}{\sqrt{\frac{1}{p^2}+\frac{1}{q^2}}}$$

$$\Rightarrow \boxed{\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{p^2}+\frac{1}{q^2}}$$



$\frac{x}{a}+\frac{y}{b}=1$

REFLECTION OF A POINT :

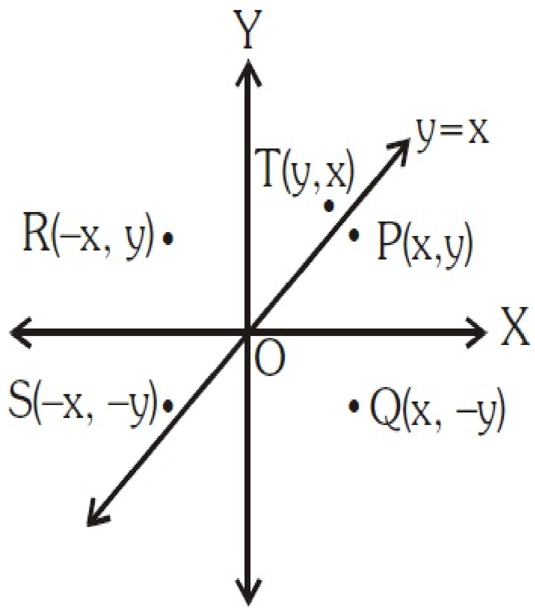
Let P(x, y) be any point, then its image with respect to

(a) x-axis is **Q(x, –y)**

(b) y-axis is **R(–x, y)**

(c) origin is **S(–x,–y)**

(d) line y = x is **T(y, x)**

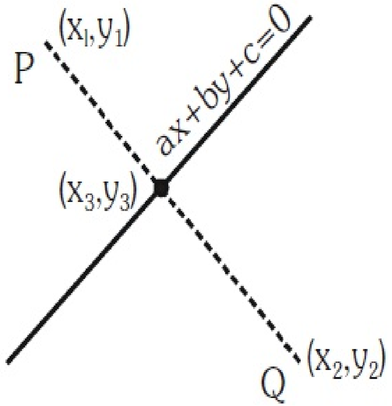


(e) Reflection of a point about any arbitrary line : The image of a point $P(x_1, y_1)$ about the line $ax + by + c = 0$ is

$$\frac{x_2 - x_1}{a} = \frac{y_2 - y_1}{b} = -2\frac{(ax_1 + by_1 + c)}{a^2 + b^2}$$

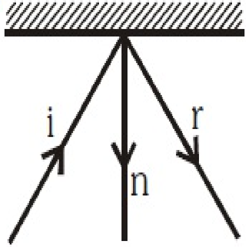
and the foot of perpendicular from a point (x_1, y_1) on the line

$ax + by + c = 0$ is $$\frac{x_3 - x_1}{a} = \frac{y_3 - y_1}{b} = -\frac{ax_1 + by_1 + c}{a^2 + b^2}$$



E(1) Find image of $(4,-13)$ w.r.t. $5x + y + 6 = 0$. [**Ans.** $(-1,-14)$]

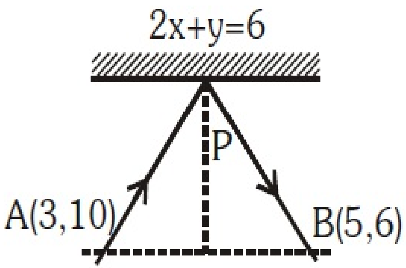
E(2) A ray of light incident along the line $x - 2y - 3 = 0$ and strikes a line mirror. If the equation of normal on the line mirror at the point of incidence is $2x + 3y + 1 = 0$, then find the equation of the reflected ray



[**Ans.** $29x - 2y - 31 = 0$]

E(3) Find the equation of incident ray AP and reflected ray PB. (Check if slope of AB is equal to the slope of the line mirror.)

[**Ans.** $3y = 4x + 18$, $y = 6$]



D(4) Equation of the perpendicular bisector of the sides AB and AC of a triangle ABC are $x - y + 5$ and $x + 2y = 0$, if the vertex is $A(1, -2)$. Find the equation of BC. [**Ans.** $14x + 23y = 40$]

Find a point P on $3x + 2y + 10 = 0$ such that $|PA - PB|$ is maximum, where A is $(4,2)$ & B is $(2,4)$. [**Ans.** $(-22,28)$]

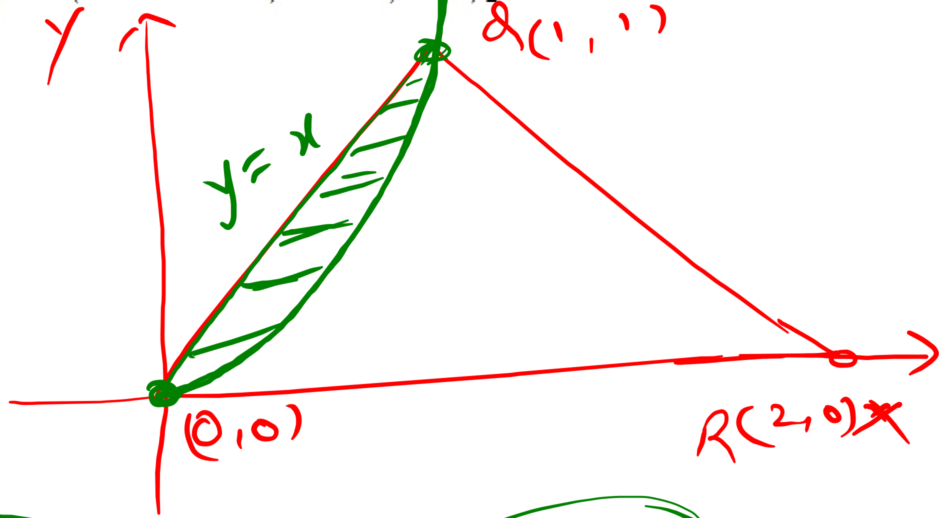
Q. A farmer F_1 has a land in the shape of a triangle with vertices at P(0, 0), Q(1, 1) and R(2, 0). From this land, a neighbouring farmer F_2 takes away the region which lies between the side PQ and a curve of the form $y = x^n$ $(n > 1)$. If the area of the region taken away by the farmer F_2 is exactly 30% of the area of ΔPQR, then the value of n is ____. [JEE(Advanced)-2018, 3(0)]

$\Delta A = \frac{30}{100} \Delta PQR$

$\Rightarrow \frac{30}{100} \times \frac{1}{2} \times 2 \times 1 = \int_0^1 (x - x^n)\, dx$

$\frac{3}{10} = \frac{1}{2} - \frac{1}{n+1}$

$\Rightarrow \frac{1}{n+1} = \frac{5}{10} - \frac{3}{10} = \frac{1}{5} \Rightarrow n = 4$



$y = x^n$

passes through (0,0) & (1,1)

REFLECTION OF A POINT :

Let P(x, y) be any point, then its image with respect to

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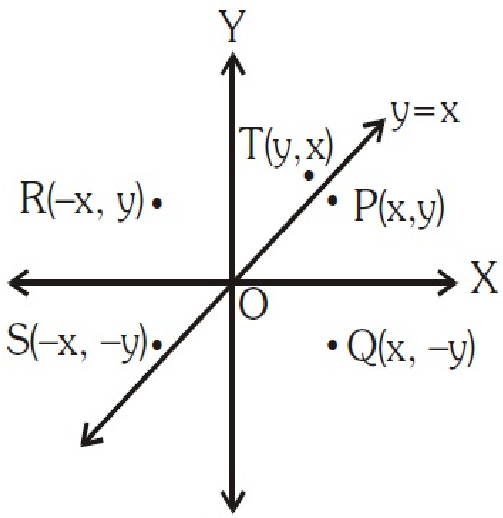
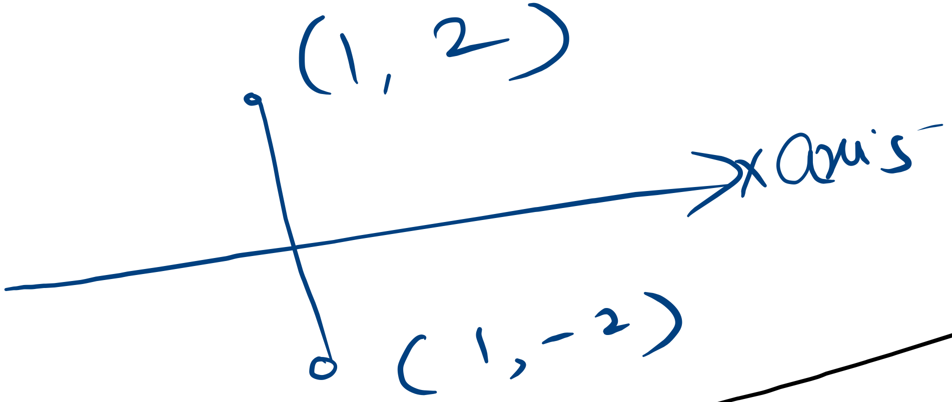
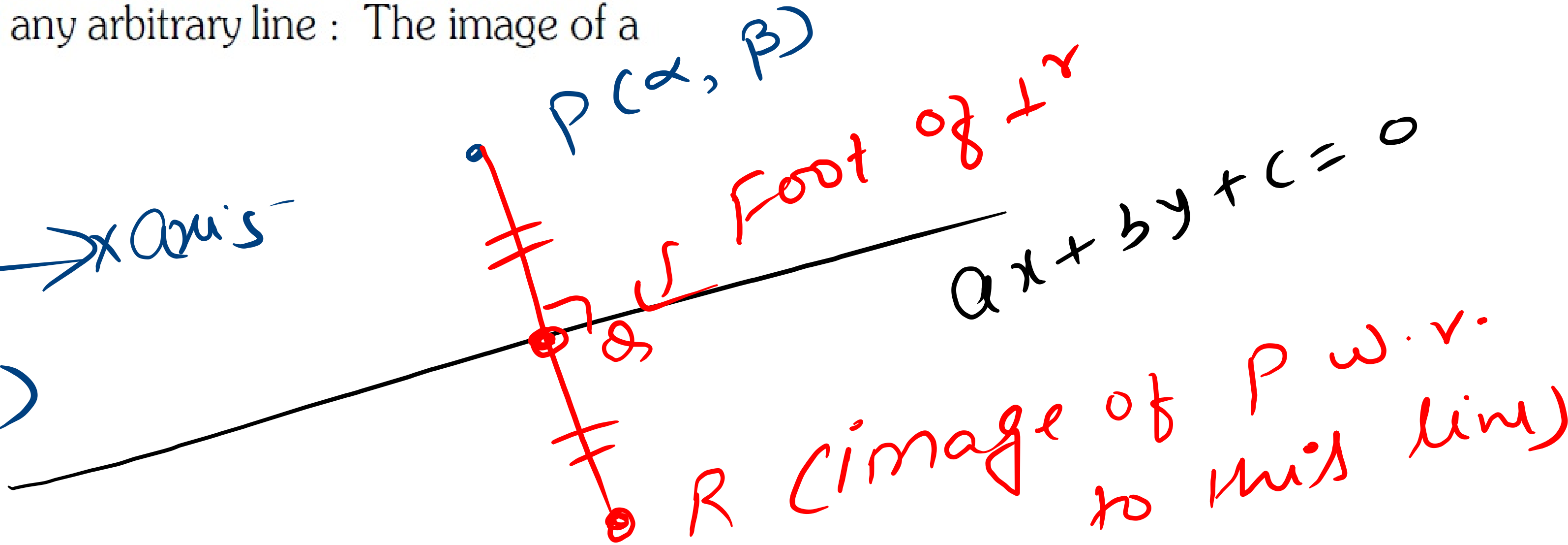


(e) Reflection of a point about any arbitrary line : The image of a





point $P(x_1, y_1)$ about the line $ax + by + c = 0$ is

$$\frac{x_2 - x_1}{a} = \frac{y_2 - y_1}{b} = -2\frac{(ax_1 + by_1 + c)}{a^2 + b^2}$$

and the foot of perpendicular from a point (x_1, y_1) on the line

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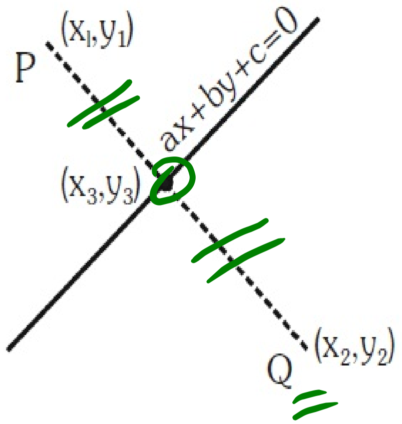

$ax + by + c = 0$

Foot of $\perp^r$ from (α, β)

$$\frac{x-\alpha}{a} = \frac{y-\beta}{b} = -\frac{(a\alpha+b\beta+c)}{a^2+b^2}$$

Image of (α, β)

$$\frac{x-\alpha}{a} = \frac{y-\beta}{b} = -2\frac{(a\alpha+b\beta+c)}{(a^2+b^2)}$$

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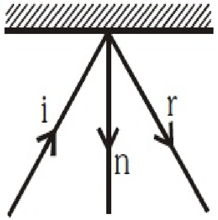
[Ans. $29x - 2y - 31 = 0$]



$$\frac{x-4}{5} = \frac{y+13}{1} = -\frac{2(5\times4-13+6)}{25+1} = -1$$

$x = -1$, $y+13 = -1 \Rightarrow y = -14$

(2)

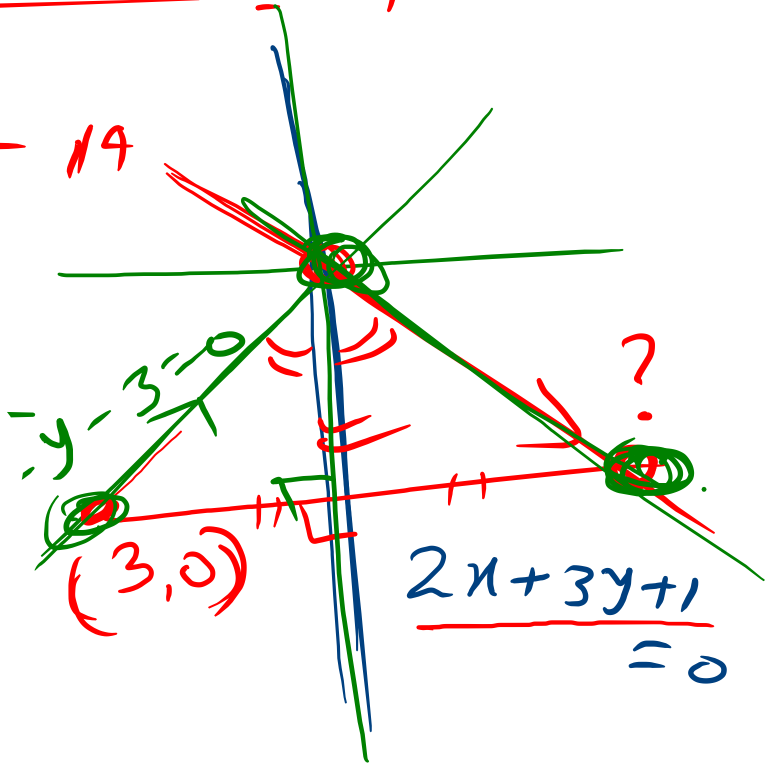
P.O.I.

$x - 2y - 3 = 0$

$2x + 3y + 1 = 0$

$(1,-1)$

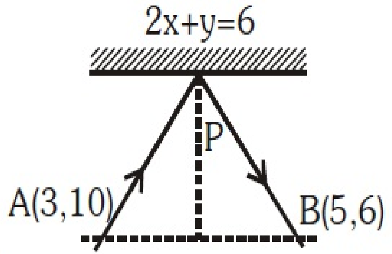
$$\frac{x-3}{2} = \frac{y-0}{3} = -\frac{2(6+0+1)}{13}$$



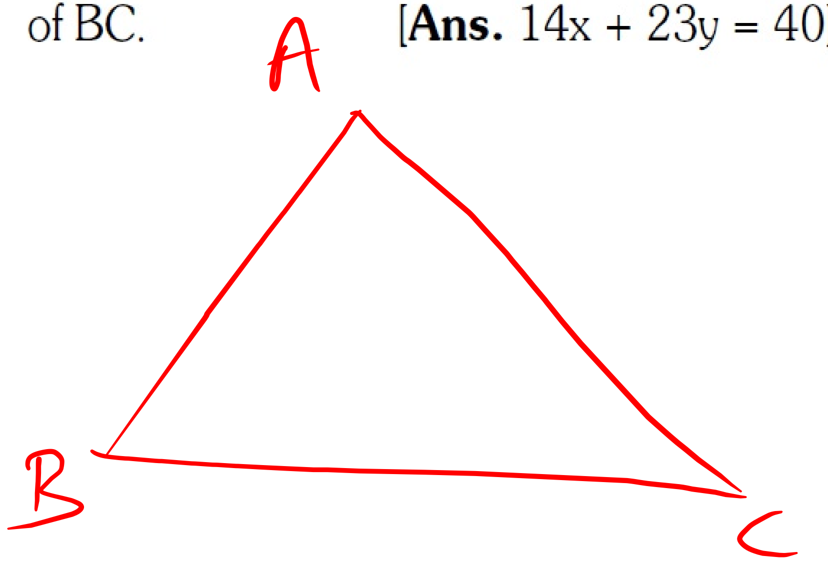
[Ans. $29x - 2y - 31 = 0$]

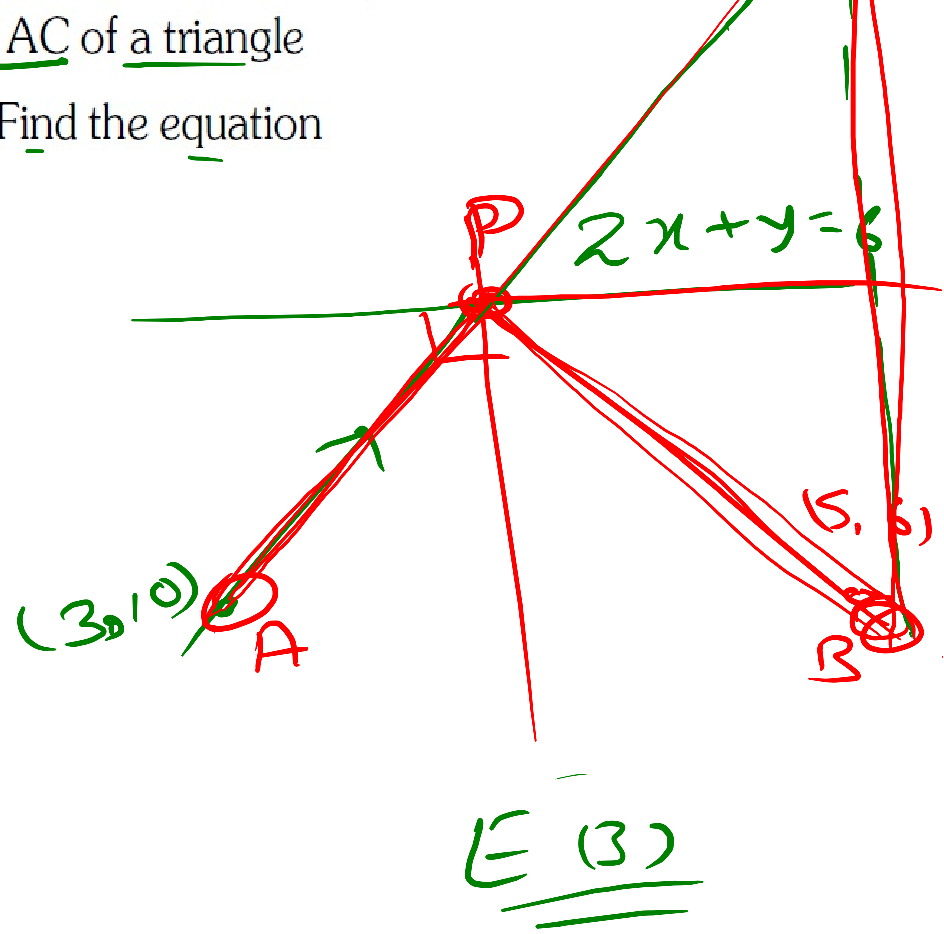
E(3) Find the equation of incident ray AP and reflected ray PB. (Check if slope of AB is equal to the slope of the line mirror.)

[**Ans.** $3y = 4x + 18$, $y = 6$]

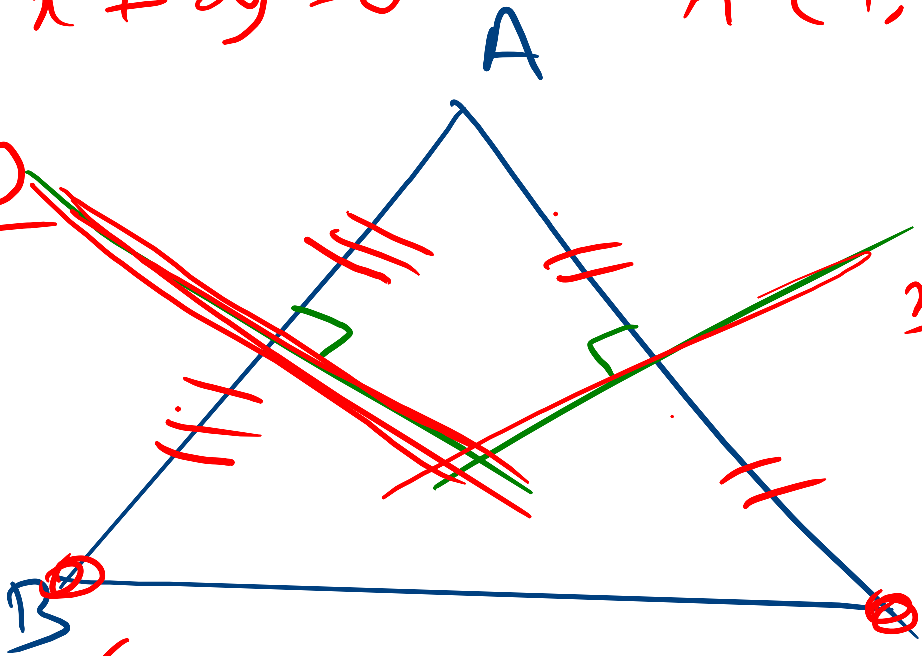


D(4) Equation of the perpendicular bisector of the sides AB and AC of a triangle ABC are $x - y + 5$ and $x + 2y = 0$, if the vertex is $A(1, -2)$. Find the equation of BC. [**Ans.** $14x + 23y = 40$]





$x - y + 5 = 0$, $x + 2y = 0$ $A(1, -2)$



$B \equiv$

$$\frac{x-1}{1} = \frac{y+2}{-1} = -2\frac{(1+2+5)}{1+1}$$

$$= -8$$

$$x = -8+1 = -7$$

$$y = -2+8 = 6$$

$B(-7, 6)$

$$\frac{x-1}{1} = \frac{y+2}{2} = -2\frac{(1-4)}{5} = \frac{6}{5}$$

$C\left(\frac{11}{5}, \frac{2}{5}\right)$

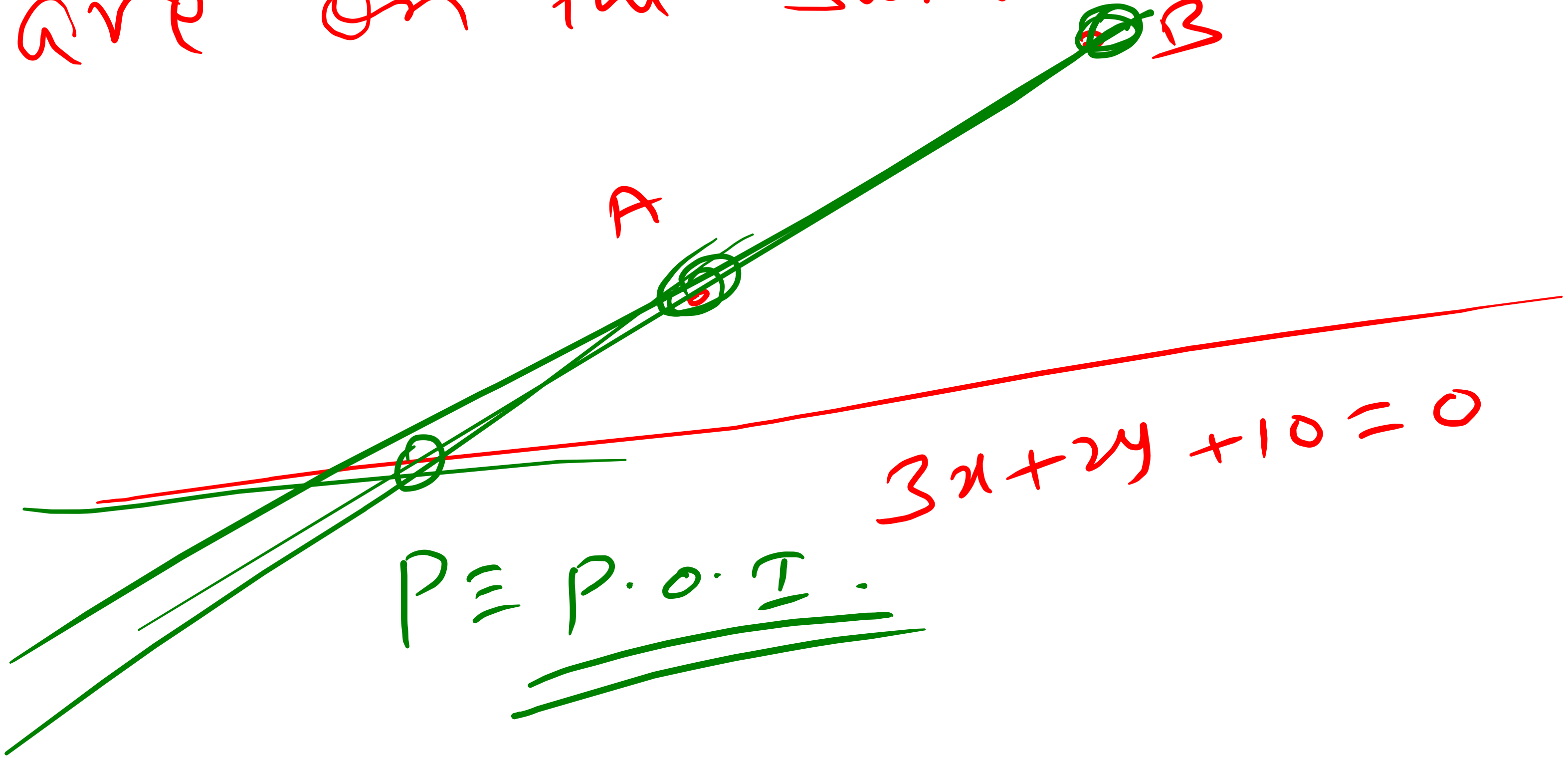
$$x = 1 + \frac{6}{5} = \frac{11}{5}$$

$$y = -2 + \frac{12}{5} = \frac{2}{5}$$

$$\frac{x-\alpha}{a} = \frac{y-\beta}{b} = -2\frac{(a\alpha+b\beta+c)}{a^2+b^2}$$

Q. Find a point P on $3x + 2y + 10 = 0$ such that $|PA - PB|$ is maximum, where A is (4,2) & B is (2,4). [**Ans.** (–22,28)]

A & B are on the same side

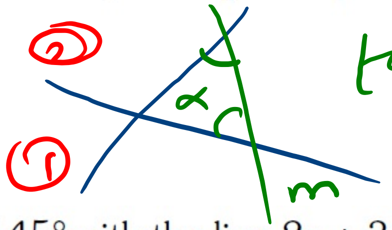


LINE INCLINED AT AN ANGLE TO OTHER LINE(S) :

(a) Slope of line equally inclined (Let θ) with line $y = mx + c$ is $\dfrac{m+\tan\theta}{1-m\tan\theta}, \dfrac{m-\tan\theta}{1+m\tan\theta}$

(b) Slope of a line (Let m) making same angle with two lines $y = m_1x + c_1$ & $y = m_2x + c_2$ is obtained by

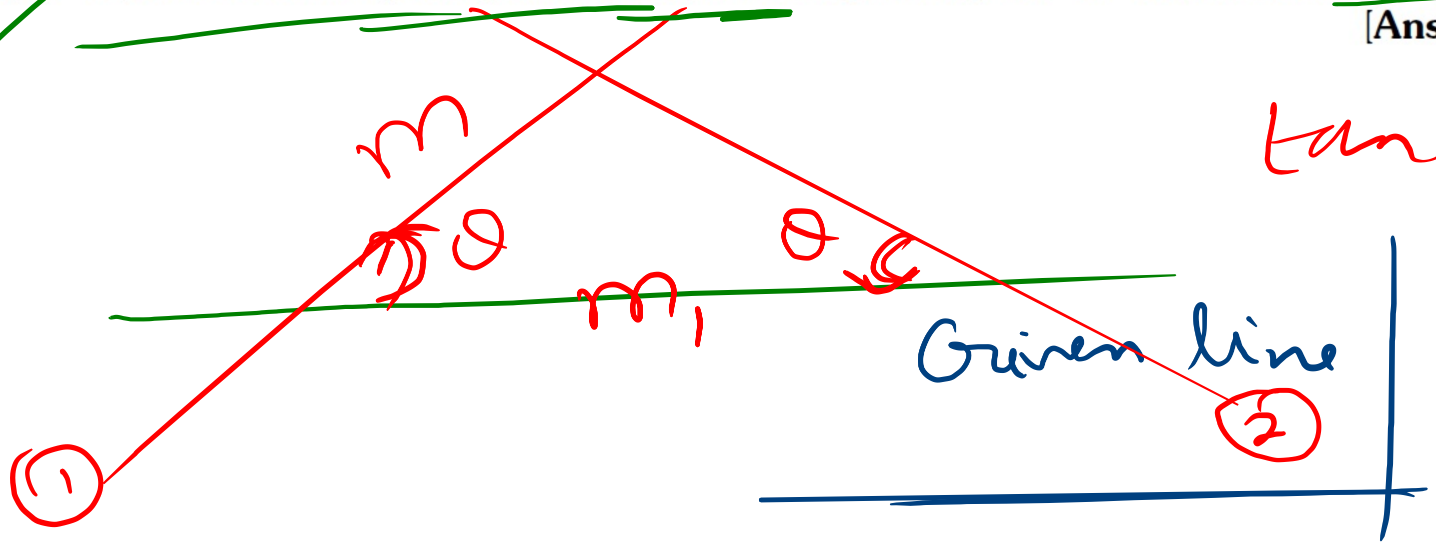
solving $\dfrac{m-m_1}{1+mm_1} = \dfrac{m_2-m}{1+mm_2}$



$$\tan\alpha = \left|\frac{m-m_1}{1+mm_1}\right| = \left|\frac{m-m_2}{1+mm_2}\right| \Rightarrow m$$

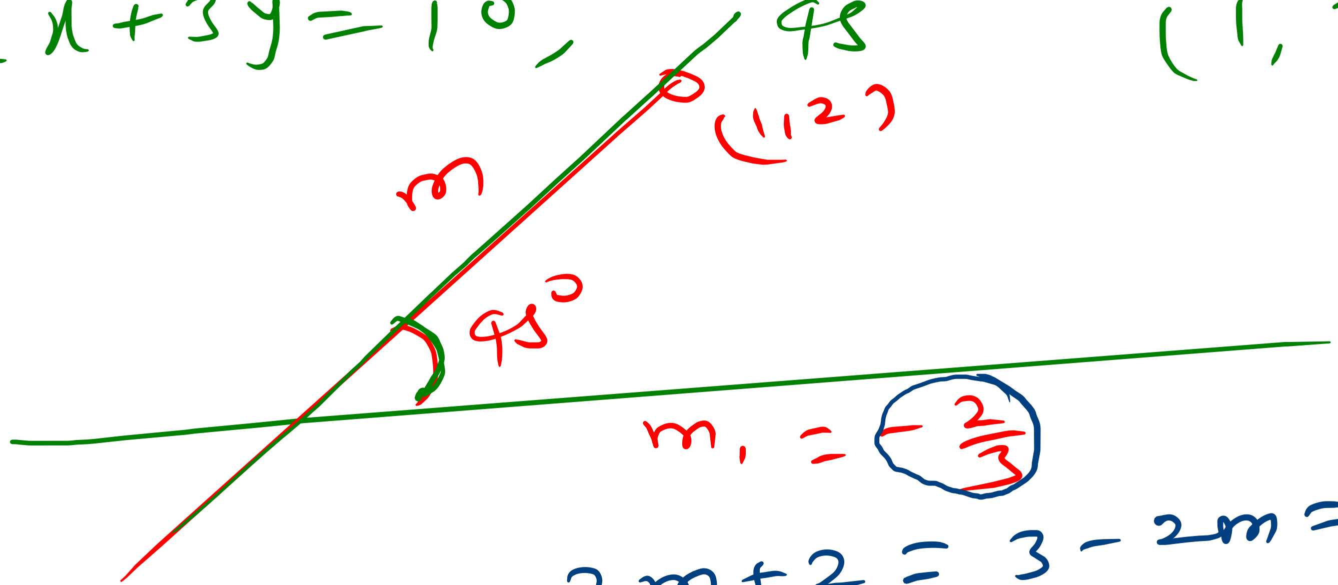
E(1) Equation of a line passing through (1, 2) making an angle of 45° with the line $2x + 3y = 10$.

[**Ans.** $m = -5,\ 1/5$]



$$\tan\theta = \left|\frac{m-m_1}{1+mm_1}\right|$$

$2x + 3y = 10, \quad 45^\circ \qquad (1, 2)$



$m_1 = -\frac{2}{3}$

$$\tan 45^\circ = \left| \frac{m + \frac{2}{3}}{1 - \frac{2}{3}m} \right| \Rightarrow 3m + 2 = 3 - 2m \Rightarrow m = \frac{1}{5}$$

OR $\quad 3m + 2 = -3 + 2m \Rightarrow m = -5$

$$\Rightarrow 1 = \left| \frac{3m + 2}{3 - 2m} \right|$$

equation:

$y - 2 = \frac{1}{5}(x - 1)$

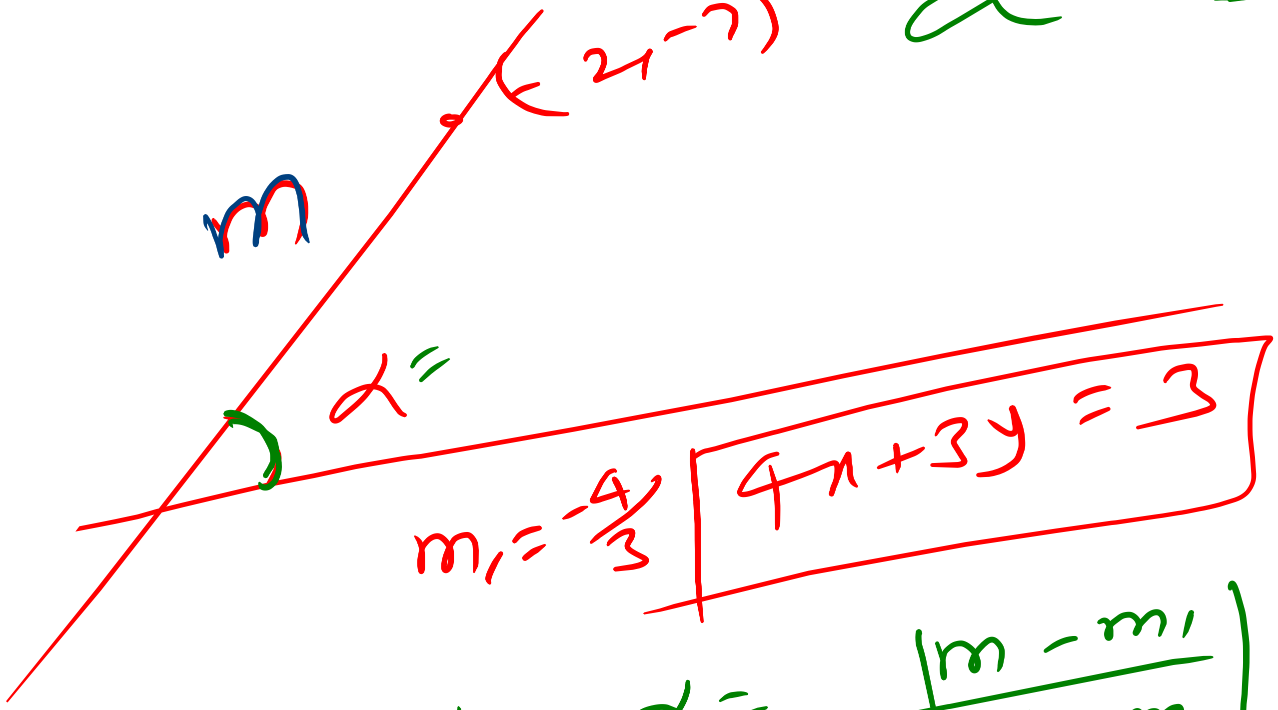
$y - 2 = -5(x - 1)$

Q: Equation of a line passing through (–2, –7) making an angle of $\tan^{-1}\frac{3}{4}$ with 4x + 3y = 3.

α

$$\boxed{m_1 = -\frac{4}{3}}$$

$$\tan\alpha = \frac{3}{4}$$



$$\tan\alpha = \left|\frac{m-m_1}{1+mm_1}\right|$$

$$\Rightarrow \frac{3}{4} = \left|\frac{m+\frac{4}{3}}{1-\frac{4}{3}m}\right|$$

$$\Rightarrow \frac{3}{4} = \left|\frac{3m+4}{3-4m}\right|$$

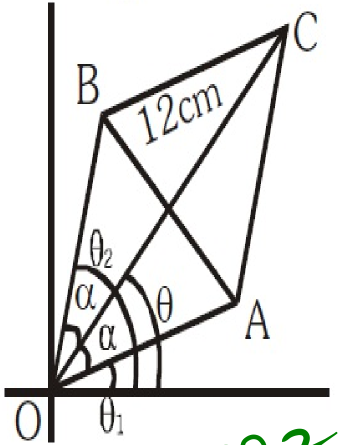
$$-\frac{3}{4} = \frac{3m+4}{3-4m}$$

$$\Rightarrow -9 + \cancel{12m} = \cancel{12m} + 16$$

$$\cancel{-9 = 16} \Rightarrow m \text{ is not defined}$$

$$\boxed{x = -2}$$

Q: Two sides of a rhombus lying in the first quadrant are given by $y = (3/4)x$ and $y = (4/3)x$. If the length of the longer diagonal OC = 12, find the equation of the other two sides.



$\theta_1 + \theta_2 = 90^\circ$

[**Ans.** $4y - 3x - 6\sqrt{2} = 0$, $3y - 4x + 6\sqrt{2} = 0$]

[$\theta - \theta_1 = \alpha = \theta_2 - \theta \Rightarrow 2\theta = \theta_1 + \theta_2 \Rightarrow \theta = 45°$]

Equation of BC, CA ?

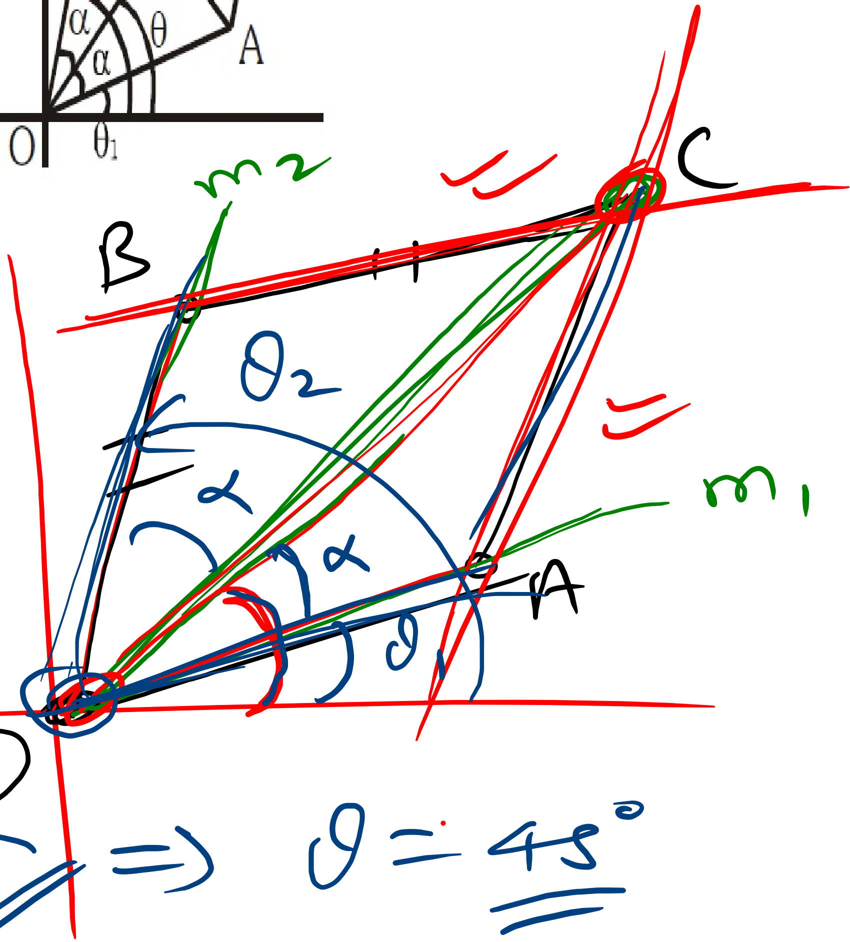
$y = \left(\frac{3}{4}\right)x, \quad y = \left(\frac{4}{3}\right)x$

$m_1 = \tan\theta_1 = \frac{3}{4}$ | $m_2 = \tan\theta_2 = \frac{4}{3}$

$C(12\cos\theta, 12\sin\theta)$

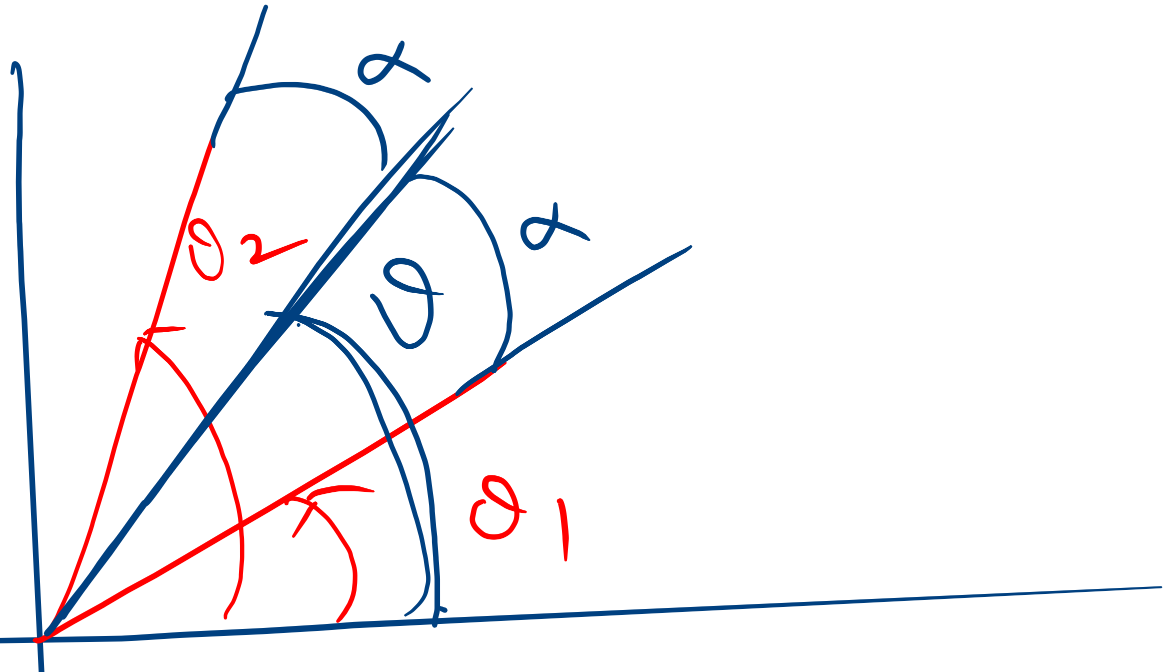
$\theta - \theta_1 = \alpha = \theta_2 - \theta$

$\Rightarrow \boxed{\theta_1 + \theta_2 = 2\theta} \Rightarrow \theta = 45^\circ$



$\tan\theta_1 \cdot \tan\theta_2 = 1$

$\theta_1 + \theta_2 = 90^\circ$



$\Rightarrow \boxed{\theta = 45^\circ}$

$\alpha = \theta_2 - \theta$

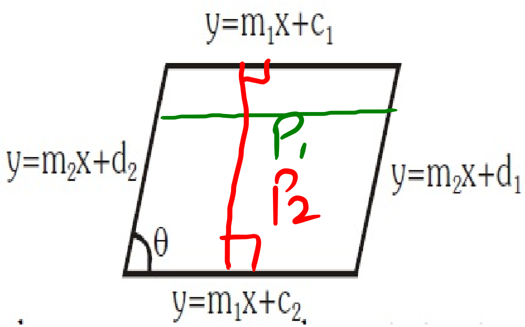
$\alpha = \theta - \theta_1$

$\theta_2 - \theta = \theta - \theta_1$

$\theta_1 + \theta_2 = 2\theta$

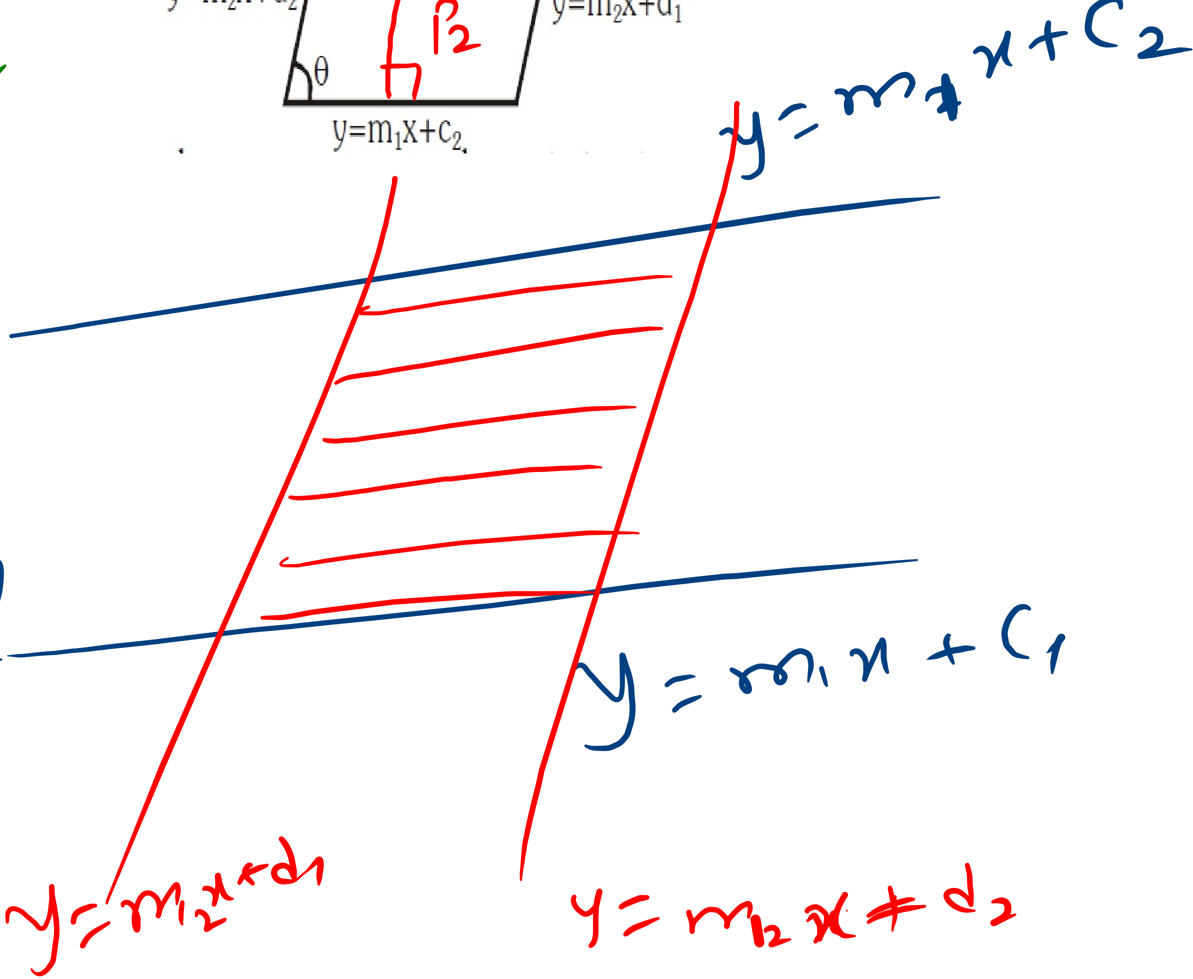
Area of the || gm whose 4 sides are as shown in the figure using

$$A = p_1p_2\operatorname{cosec}\theta \text{ is given by } \left|\frac{(c_1-c_2)(d_1-d_2)}{m_1-m_2}\right| = \left(\frac{p_1p_2}{\sin\theta}\right)$$



CDMA

$$\left|\frac{(c_1-c_2)(d_1-d_2)}{(m_1-m_2)}\right| = A$$



CONDITION FOR CONCURRENCY :

Method I : Lines L_1, L_2 & L_3 meet in a point if point of intersection of any two lines lies on the third line

Method II : 3 lines $a_r x + b_r y + c_r = 0$ $(r = 1, 2, 3)$ are concurrent then $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$

Note : If $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$, then lines may be parallel, coincident or concurrent.

$$\left.\begin{aligned} a_1 x + b_1 y + c_1 = 0 \\ a_2 x + b_2 y + c_2 = 0 \end{aligned}\right\}$$

$\boxed{\frac{a_1}{a_2} = \frac{b_1}{b_2}} \Rightarrow$ parallel

$\boxed{\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2}} \Rightarrow$ parallel & distinct

$\frac{a_1}{a_2} = \frac{b_1}{b_2} = \frac{c_1}{c_2} \Rightarrow$ Same line

$\boxed{\frac{a_1}{a_2} \neq \frac{b_1}{b_2}} \Rightarrow$ Intersecting

L_1 L_2

L_1 L_2

3 lines:

Method 1: P.O.I of 2 lines

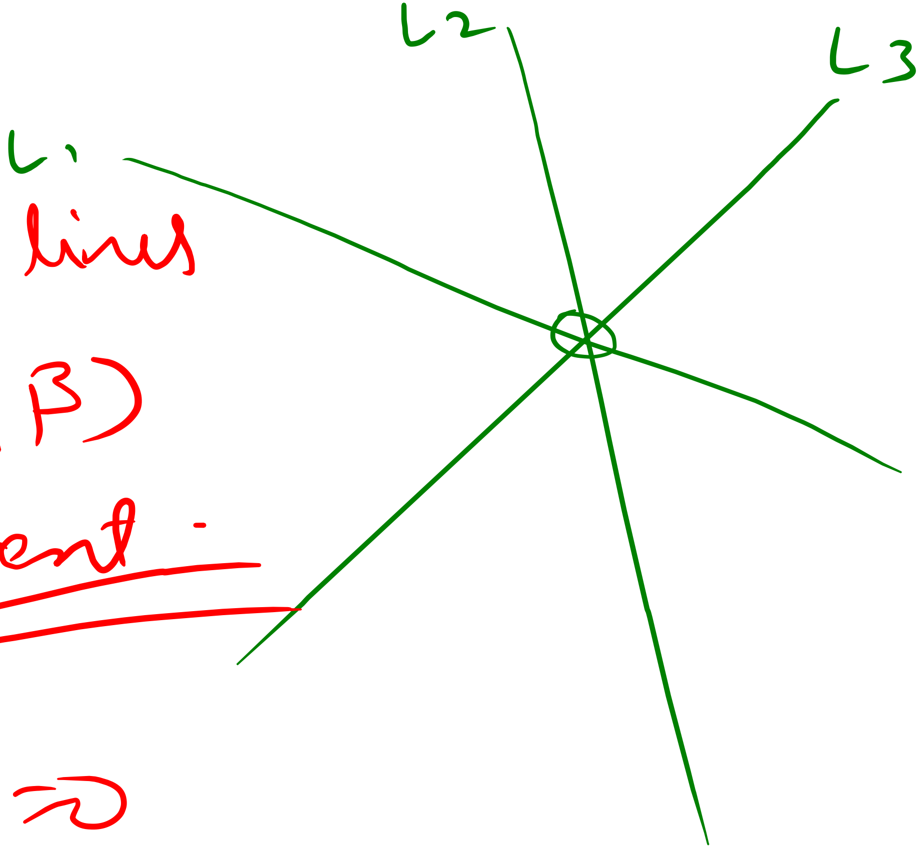
$$\left.\begin{matrix} L_1 = 0 \\ L_2 = 0 \end{matrix}\right] \Rightarrow \text{P.O.I } (\alpha, \beta)$$

$$L_3(\alpha, \beta) = 0 \Rightarrow \text{concurrent.}$$

Method 2:

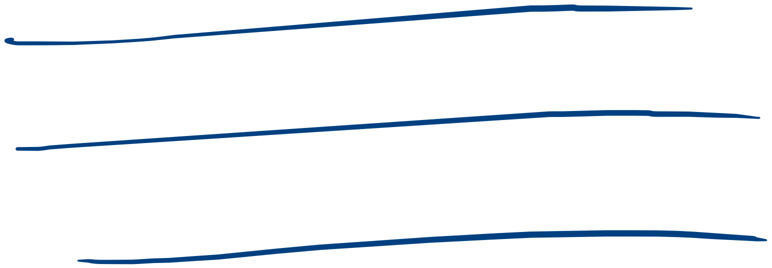
$$\text{concurrent} \Rightarrow \left| \quad \right| = 0$$

$$\left| \quad \right| = 0 \Rightarrow \text{concurrent lines.}$$



Concurrent lines: passing through a common point

$$\left.\begin{aligned} x+y&=1 \\ x+y&=2 \\ x+y&=3 \end{aligned}\right\}$$



$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 1 & 2 \\ 1 & 1 & 3 \end{vmatrix} = 0 \Rightarrow$$ Lines are ~~concurrent~~

may or may not be concurrent

$$\left.\begin{aligned} a_1x+b_1y+c_1&=0 \\ a_2x+b_2y+c_2&=0 \\ a_3x+b_3y+c_3&=0 \end{aligned}\right\}$$ concurrent

$$\Rightarrow \left|\quad\quad\right| = 0$$

E(1) Find the value of 'm' for the lines $2x - y - 1 = 0$, $x + my - 2 = 0$ & $mx + y + 1 = 0$ such that

(i) no two lines are parallel. [**Ans.** $m \neq -2, -1, -\frac{1}{2}, 1$]

(ii) the lines are concurrent. [**Ans.** no such m is possible]

$m_1 = 2$, $m_2 = -\frac{1}{m}$, $m_3 = -m$

L_1 & L_2 are ∥ $\Rightarrow 2 = -\frac{1}{m} \Rightarrow \boxed{m \neq -\frac{1}{2}}$

L_2 & L_3 are ∥
$\Rightarrow -\frac{1}{m} = -m \Rightarrow \boxed{m \neq \pm 1}$

L_3 & L_1 are ∥
$\Rightarrow -m = 2 \Rightarrow \boxed{m \neq -2}$

(ii) $| \; | = 0 \Rightarrow m = -2$

When $m = -2$
L_3 & L_1 are ∥ & distinct
No possible value

$\frac{a_1}{a_2} = \frac{b_1}{b_2} \neq \frac{c_1}{c_2} \Rightarrow$ ∥ & distinct

FAMILY OF STRAIGHT LINE :

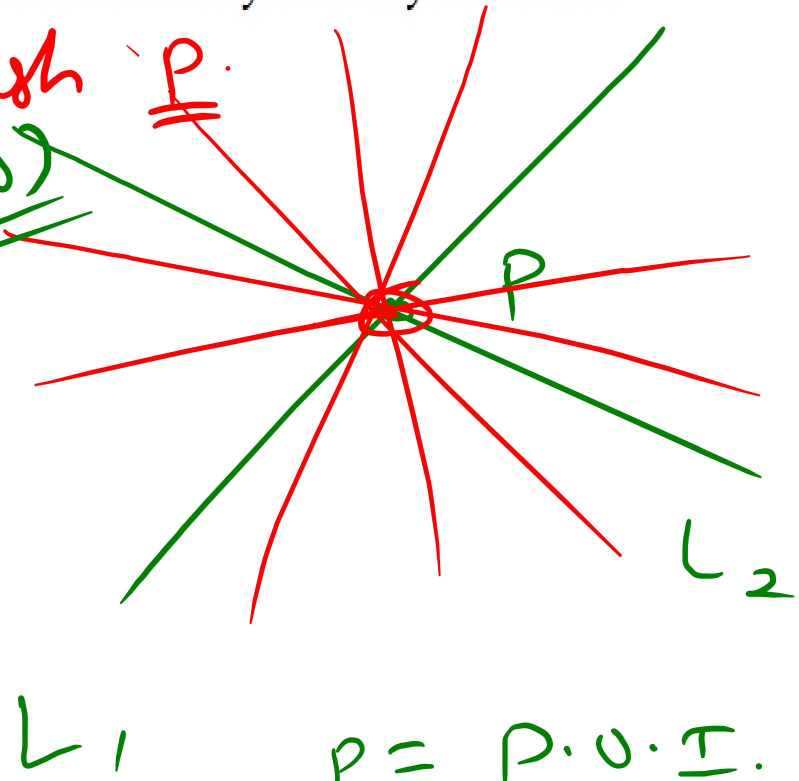
If A be the common point of two straight lines $a_1x + b_1y + c_1 = 0$ and $a_2x+b_2y+c_2=0$, then equation $a_1x+b_1y+c_1+\lambda(a_2x+b_2y+c_2)=0$ is satisfied by coordinates of common point A, where λ is any arbitrary constant.

Family of lines passing through P.

$\boxed{\mu L_1 + \lambda L_2 = 0}$ $(\mu, \lambda) \neq (0,0)$

$\boxed{L_1 + K L_2 = 0}$ can be taken as a line passing through P

In this case, L_2 is not considered.



$P \equiv$ P.O.I.

a, b, c in A.P $\quad (a, b, c) \neq (0,0,0)$

$$\left.\begin{aligned} ax + by + c &= 0 \\ 2020x + 2021y + 2022 &= 0 \end{aligned}\right\}$$

Find P.O.I $\quad (1, -2)$

$2b = a + c$

$$ax + \frac{(a+c)}{2}y + c = 0 \Rightarrow a(2x+y) + c(y+2) = 0$$

Family of lines passing through P.O.I. of

$$\left.\begin{aligned} 2x + y &= 0 \\ y + 2 &= 0 \end{aligned}\right\} (+1, -2)$$

Q: $L_1 : 3x + 4y + 6 = 0$ & $L_2 : x + y + 2 = 0$

equation of line through the point of intersection A of $L_1 = 0$ & $L_2 = 0$ and

E(1) passes through (2, –3). **[Ans.** $\lambda = 0$]

E(2) parallel to a line $x + 2y + 3 = 0$. **[Ans.** $\lambda = -2$]

E(3) perpendicular to the line $2x - 3y + 1 = 0$. **[Ans.** $\lambda = -6$]

$3x + 4y + 6 + \lambda(x + y + 2) = 0$

passes through $(2, -3)$

$\Rightarrow 6 - 12 + 6 + \lambda(2 - 3 + 2) = 0$

$\Rightarrow \lambda = 0$

$\boxed{3x + 4y + 6 = 0}$

$(x + y + 2) + K(3x + 4y + 6) = 0$

passes through $(2, -3)$

$\Rightarrow (2 - 3 + 2) + K \times 0 = 0$

$\Rightarrow \boxed{1 = 0}$

$(3 + \lambda)x + (4 + \lambda)y + 6 + 2\lambda = 0$

$\parallel^{l}$ to $x + 2y + 3 = 0$

$-\frac{3 + \lambda}{4 + \lambda} = -\frac{1}{2}$

$6 + 2\lambda = 4 + \lambda$

$\boxed{\lambda = -2}$

$L_1 : 3x + 4y + 6 = 0$ & $L_2 : x + y + 2 = 0$ situated at a maximum distance from the point P(2, 3).

Number of values of K for which the line $Kx + 5y = 9$ will not belong to the family [**Ans.** $K \neq 7$]

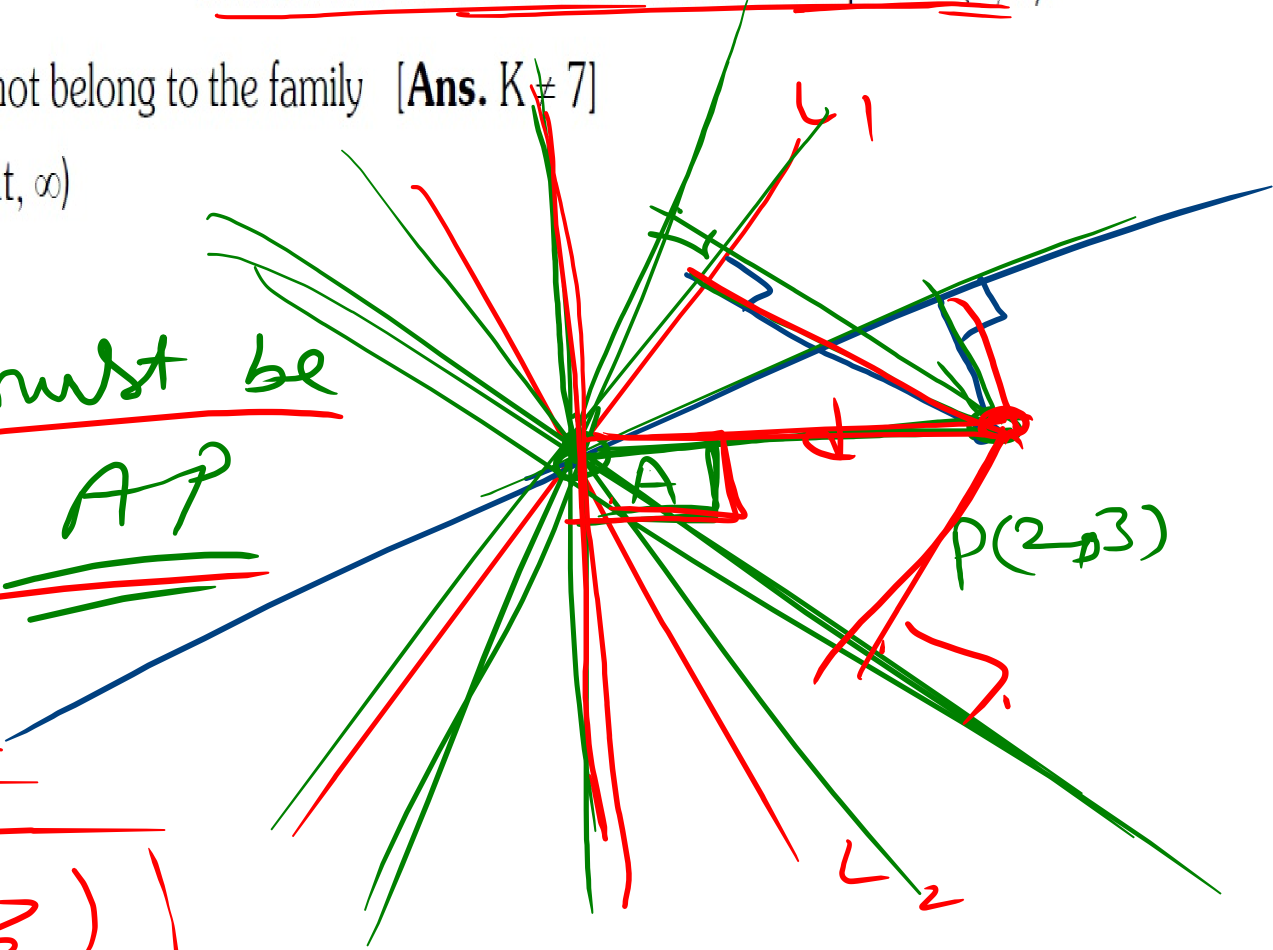
$5x + 3y - 7 + \lambda(3x + 10y + 4)$ (For K = 7 concurrent, ∞)

A (−2, 0)

Required line must be $\perp^r$ to segment AP

$m_{AP} = \frac{3-0}{2+2} = \frac{3}{4}$

$\boxed{y - 0 = (x+2)\left(-\frac{4}{3}\right)}$



Next

Type 2 : Converse of type-I i.e. $L_1 + \lambda L_2 = 0$ is a line which passes through a fixed point.

E(1) The family of lines $x(a + 2b) + y(a + 3b) = a + b$ passes through a fixed point for all values of a and b. Find the fixed point. **[Ans.** (2,–1)]

E(2) If a, b, c are A.P. $\Rightarrow$ variable line $ax + by + c = 0$ passes through a fixed point. [**Ans.** (1,–2)]

E(3) If a, b, c are in H.P. $\Rightarrow$ $bcx + cay + ab = 0$ passes through a fixed point. [**Ans.** (1,–2)]

E(4) If $a^2 + 9b^2 = 6ab + 4c^2$ $\Rightarrow$ $ax + by + c = 0$ passes through one or the other of the two fixed point.

EQUATION OF BISECTORS OF ANGLES BETWEEN TWO LINES :

If equation of two intersecting lines are $a_1x+b_1y+c_1=0$ and $a_2x + b_2y+c_2=0$, then equation of bisectors of the angles between these lines are written as :

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = \pm\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}} \quad ....(1)$$

Note : In case we have to find angle bisector containing the point (α, β), then use +ve sign in equation (1) if both $a_1\alpha + b_1\beta + c_1$ and $a_2\alpha + b_2\beta + c_2$ are of same sign and vice versa.

Equation of bisector of acute/obtuse angles :

To find the equation of the bisector of the acute or obtuse angle

If sign of $a_1a_2 + b_1b_2$	For obtuse angle bisector	For acute angle bisector
+	use + sign in eq. (1)	use − sign in eq. (1)
−	use − sign in eq. (1)	use + sign in eq. (1)

Find the bisectors between the lines.

$4x + 3y - 7 = 0$ and $24x + 7y - 31 = 0$

Identify Acute/ obtuse or origin containing/not containing.

[**Ans.** $x - 2y + 1 = 0$ & $2x + y = 3$]

The vertices of a ΔABC are

$A(-1, 11)$; $B(-9, -8)$ and $C(15, -2)$.

Find the equation of the bisector of the angle at A.

Bisectors between the lines $x+\sqrt{3}y=6+2\sqrt{3}$ and $x-\sqrt{3}y=6-2\sqrt{3}$

10. If $f : \mathbb{R} \to \mathbb{R}$ is a differentiable function such that $f'(x) > 2f(x)$ for all $x \in \mathbb{R}$, and $f(0) = 1$, then **[JEE(Advanced)-2017, 3(–2)]**

(A) $f(x) > e^{2x}$ in $(0,\infty)$
(B) $f(x)$ is decreasing in $(0,\infty)$
(C) $f(x)$ is increasing in $(0,\infty)$
(D) $f'(x) < e^{2x}$ in $(0,\infty)$

$H(x) = e^{-2x} f(x)$, $\uparrow$

$0 < x$

$H(0) < H(x)$

$1 < e^{-2x} f(x)$

$\Rightarrow e^{2x} < f(x)$, $\forall x \in (0,\infty)$

$1 < f(x)$

$f'(x) - 2f(x) > 0$ — 1

$e^{-2x} f'(x) + f(x) \times e^{-2x} > 0$

$\frac{d}{dx}\left(e^{-2x} f(x)\right) > 0$

$\boxed{f'(x) > 2f(x)}$

$f'(x) > 0$

6. For what values of a $\in [0, 1]$ does the area of the figure bounded by the graph of the function $y = f(x)$ and the straight lines $x = 0$, $x = 1$ & $y = f(a)$ is at a minimum & for what values it is at a maximum if $f(x) = \sqrt{1-x^2}$. Find also the maximum & the minimum areas.

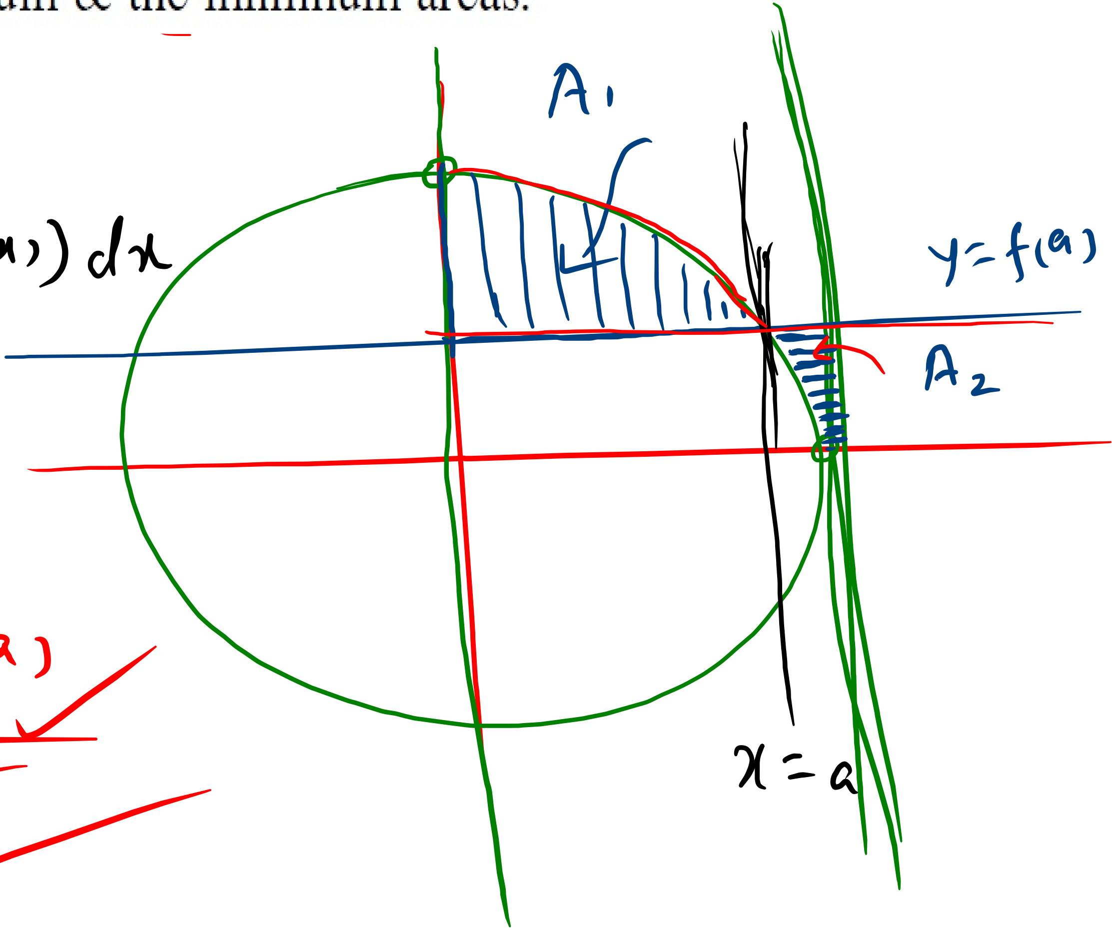
$A = A_1 + A_2$

$= \int_0^a (f(x) - f(a))\,dx + \int_a^1 (f(a) - f(x))\,dx$

$A = \int_0^a f(x)\,dx - \int_a^1 f(x)\,dx$

$+ (-a) f(a) + (1-a) f(a)$

$\frac{dA}{da} = 0 \Rightarrow$



Type 1 :

$L_1 : 3x + 4y + 6 = 0$ & $L_2 : x + y + 2 = 0$

equation of line through the point of intersection A of $L_1 = 0$ & $L_2 = 0$ and

E(1) passes through $(2, -3)$. **[Ans.** $\lambda = 0$]

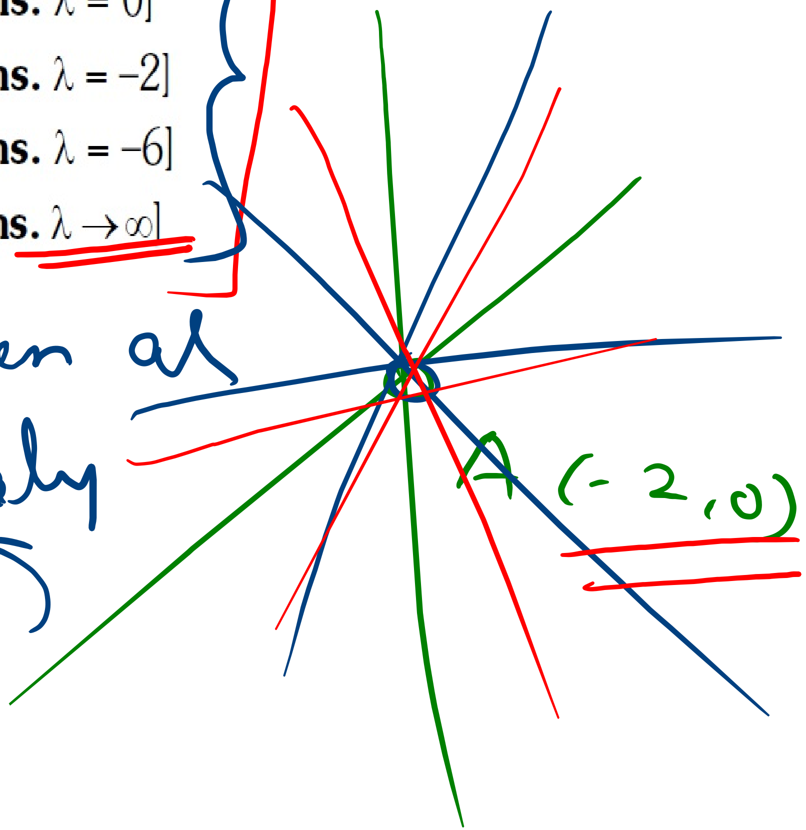
E(2) parallel to a line $x + 2y + 3 = 0$. **[Ans.** $\lambda = -2$]

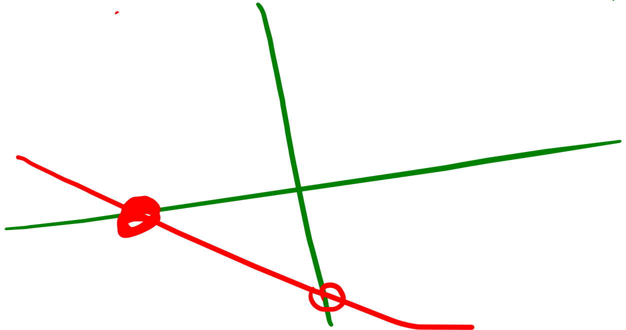
E(3) perpendicular to the line $2x - 3y + 1 = 0$. **[Ans.** $\lambda = -6$]

E(4) has equal intercepts on the coordinate axes $(m = -1)$. **[Ans.** $\lambda \to \infty$]

$L_1 + kL_2 = 0$ can be taken as any member of this family

For $k \in R$, L_2 is not included





$L_1 : 3x + 4y + 6 = 0$ & $L_2 : x + y + 2 = 0$

equation of line through the point of intersection A of $L_1 = 0$ & $L_2 = 0$ and

E(5) cutting of nonzero intercept equal in magnitude from the coordinate axes. ($m = \pm 1$).

± 1

E(6) at a distance of 2 units from the origin. (or dividing the circumference of ⊙ in the ratio of 1 : 2).

[**Ans.** ~~$\lambda = -4$~~]

E(7) situated at a maximum distance from the point P(2, 3). [**Ans.** ~~$\lambda = -7$~~]

(Write the line through A and ⊥ to AP)

E(8) is parallel to coordinate axis. [**Ans.** ~~$\lambda = -3, -4$~~]

$x + y + 2 = 0$

$m = -1$

Number of values of K for which the line $Kx + 5y = 9$ will not belong to the family $\boxed{5x + 3y - 7 + \lambda(3x + 10y + 4) = 0}$

$K(3x + 4y + 6) + x + y + 2 = 0$

$\Rightarrow \boxed{(3K+1)x + (4K+1)y + 6K + 2 = 0}$

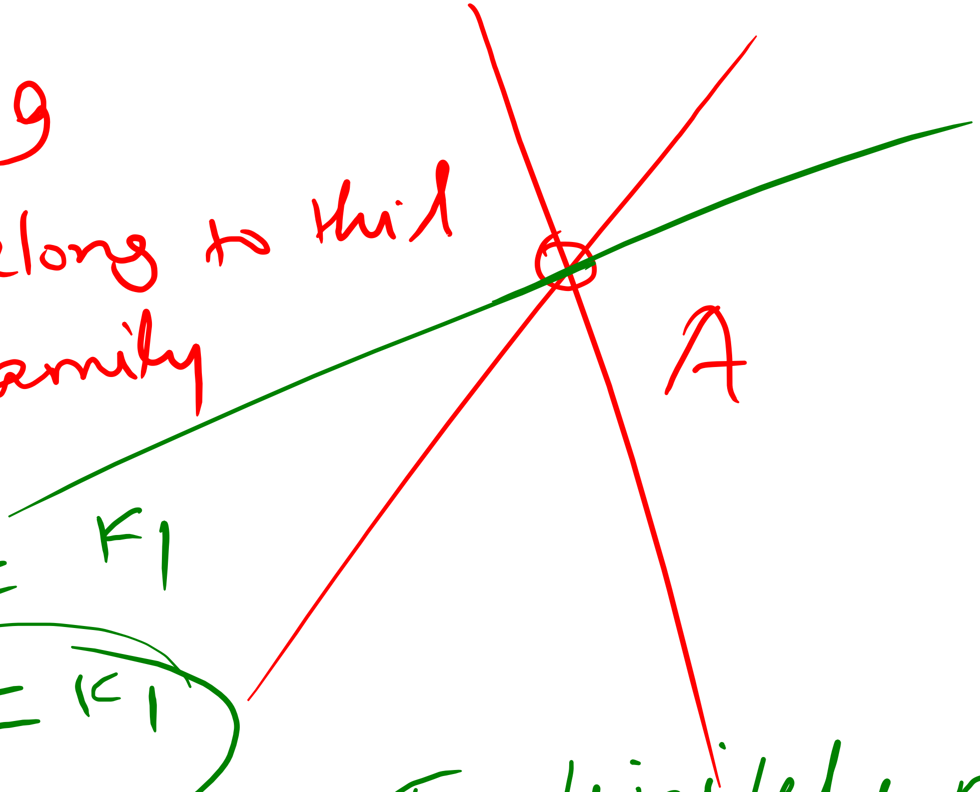
$K = -\frac{2}{7}$

$-\frac{3K+1}{4K+1} = -1 \Rightarrow 3K+1 = 4K+1 \Rightarrow \boxed{K = 0}$

$-\frac{(3K+1)}{(4K+1)} = 1 \Rightarrow 4K+1 + 3K+1 = 0 \Rightarrow K = -\frac{2}{7}$

$Kx + 5y = 9$

Should not belong to this family



$\Leftarrow$

$K \neq K_1$

Infinitely many values of K.

Type 2 : Converse of type-I i.e. $L_1 + \lambda L_2 = 0$ is a line which passes through a fixed point.

E(1) The family of lines $x(a + 2b) + y(a + 3b) = a + b$ passes through a fixed point for all values of a and b. Find the fixed point. [**Ans.** (2,–1)]

E(2) If a, b, c are A.P. $\Rightarrow$ variable line $ax + by + c = 0$ passes through a fixed point. [**Ans.** (1,–2)]

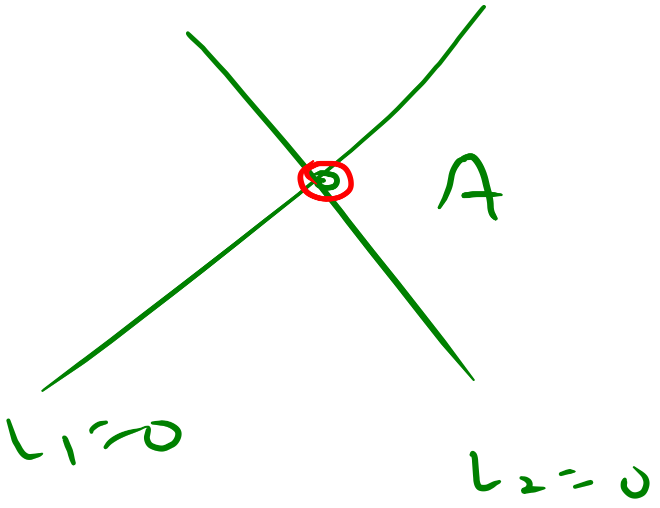
$L_1 + \lambda L_2 = 0 \Rightarrow$ This family of lines will pass through point A

(1)

$a(x+y-1) + b(2x+3y-1) = 0$

$\Rightarrow$ will pass through A

$$\left.\begin{aligned} x+y-1 &= 0 \\ 2x+3y-1 &= 0 \end{aligned}\right\} \boxed{x = 2,\ y = -1}$$



E(3) If a, b, c are in H.P. $\Rightarrow$ bcx + cay + ab = 0 passes through a fixed point. [**Ans.** (1,–2)]

E(4) If $a^2 + 9b^2 = 6ab + 4c^2$ $\Rightarrow$ ax + by + c = 0 passes through one or the other of the two fixed point.

$bcx + cay + ab = 0$ $\qquad$ $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$ in A.P.

$\Rightarrow \frac{1}{a}x + \frac{1}{b}y + \frac{1}{c} = 0 \Rightarrow (1, -2)$

$ax + by + c = 0$ $\qquad$ a, b, c are in A.P.

$a + c = 2b \Rightarrow a + (-2)b + c = 0$

$P(1, -2)$

E(4) $a^2 + 9b^2 - 6ab - 4c^2 = 0 \Rightarrow (a - 3b)^2 - (2c)^2 = 0$

$\Rightarrow a - 3b + 2c = 0$ $\qquad$ $a - 3b - 2c = 0$

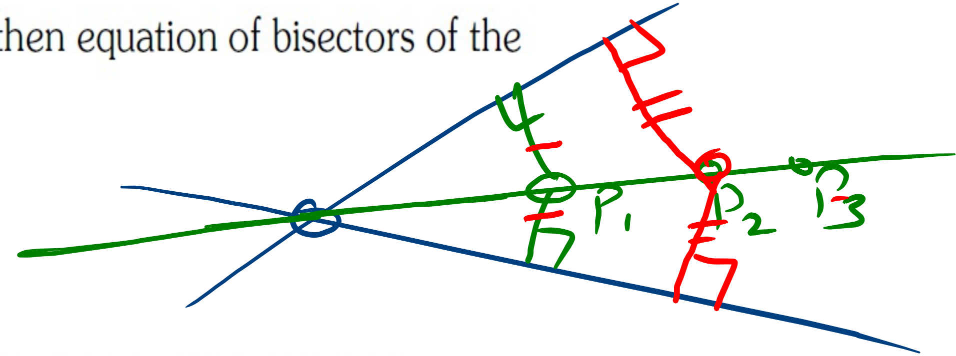
$a \cdot \frac{1}{2} + b\left(-\frac{3}{2}\right) + c = 0$ $\qquad$ $a\left(-\frac{1}{2}\right) + 3\left(\frac{3}{2}\right) + c = 0$

$\left(\frac{1}{2}, -\frac{3}{2}\right)$ $\qquad$ $\left(-\frac{1}{2}, \frac{3}{2}\right)$

EQUATION OF BISECTORS OF ANGLES BETWEEN TWO LINES :

If equation of two intersecting lines are $a_1x+b_1y+c_1=0$ and $a_2x + b_2y+c_2=0$, then equation of bisectors of the angles between these lines are written as :

$$\frac{a_1x+b_1y+c_1}{\sqrt{a_1^2+b_1^2}} = \pm\frac{a_2x+b_2y+c_2}{\sqrt{a_2^2+b_2^2}} \quad \ldots\ldots\ldots\ldots\ldots(1)$$



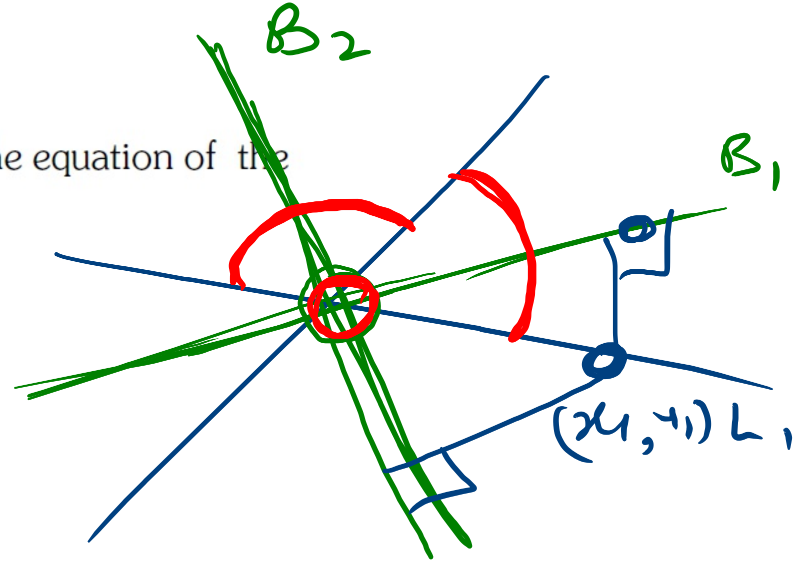
Note : In case we have to find angle bisector containing the point (α, β), then use +ve sign in equation (1) if both $a_1\alpha + b_1\beta + c_1$ and $a_2\alpha + b_2\beta + c_2$ are of same sign and vice versa.

Equation of bisector of angle containing origin :

If the equation of the lines are written with constant terms c_1 and c_2 positive, then the equation of the bisectors of the angle containing the origin is obtained by taking positive sign in (1)

Equation of bisector of acute/obtuse angles :

To find the equation of the bisector of the acute or obtuse angle



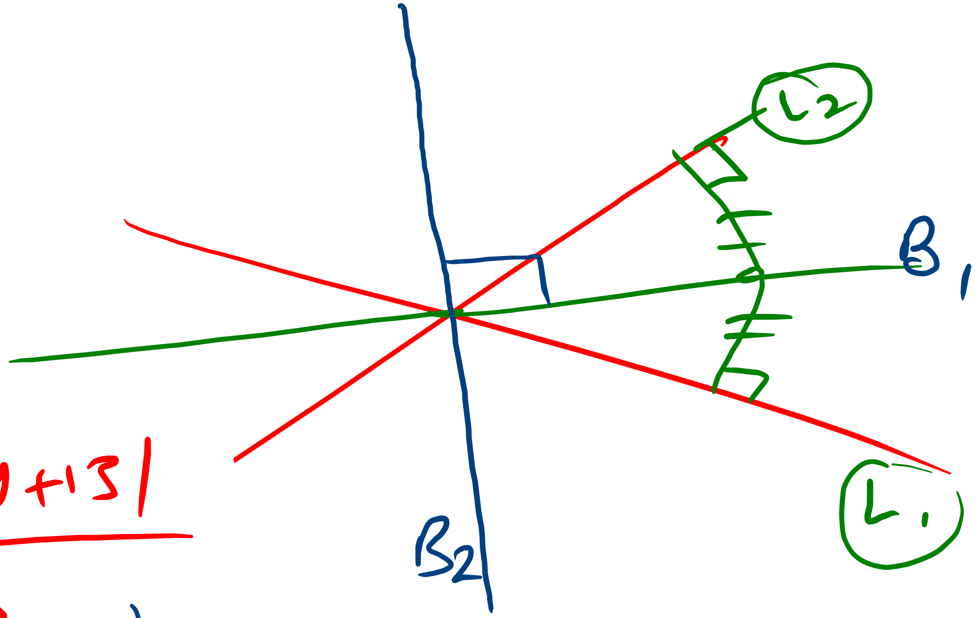
$$\frac{a_1}{b_1} \times \frac{a_2}{b_2} = -1 \Rightarrow a_1a_2 + b_1b_2 = 0$$

$$\Rightarrow \perp^r$$

$$3x+4y+5=0$$
$$12x-5y+13$$

Angle bisector

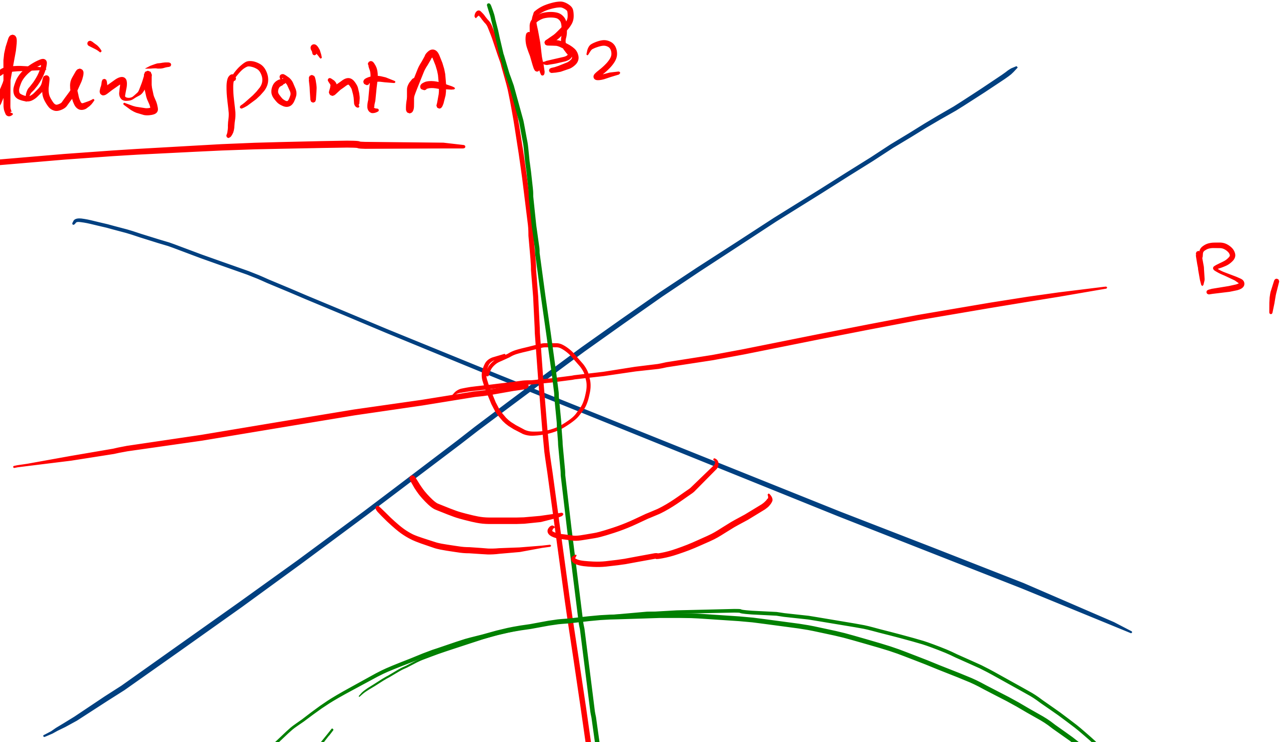
$$\frac{|3x+4y+5|}{5}=\frac{|12x-5y+13|}{13}$$



B_1: $\frac{3x+4y+5}{5}=+\frac{12x-5y+13}{13}$

B_2: $\frac{3x+4y+5}{5}=-\frac{12x-5y+13}{13}$

Region that contains point A



A(α, β)

$\left(L_1(\alpha,\beta)\right)\left(L_2(\alpha,\beta)\right) > 0$ For '+'

< 0 For '−'

Make the constant terms c_1 and c_2 positive in the given equations.

Determine the sign of $a_1a_2 + b_1b_2$.

If sign of $a_1a_2 + b_1b_2$	For obtuse angle bisector	For acute angle bisector
+	use + sign in eq. (1)	use − sign in eq. (1)
−	use − sign in eq. (1)	use + sign in eq. (1)

$B_1 \equiv (0,0)$ containing bisector

$(-2, 5)$

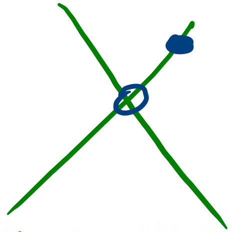
$\Rightarrow$

$\frac{|-2-10+11|}{\sqrt{5}}, \quad \frac{|-4+5-3|}{\sqrt{5}}$

$\frac{11}{\sqrt{5}}, \quad \frac{2}{\sqrt{5}}$

Q. Find the bisectors between the lines.

$4x + 3y - 7 = 0$ and $24x + 7y - 31 = 0$



Identify Acute/ obtuse or origin containing/not containing.

$$\frac{4x+3y-7}{5} = \pm \frac{24x+7y-31}{25}$$

$$100x + 75y - 175 = \pm(120x + 35y - 155)$$

$$20x - 40y + 20 = 0$$

B_1 $\boxed{x - 2y + 1 = 0}$ obtuse

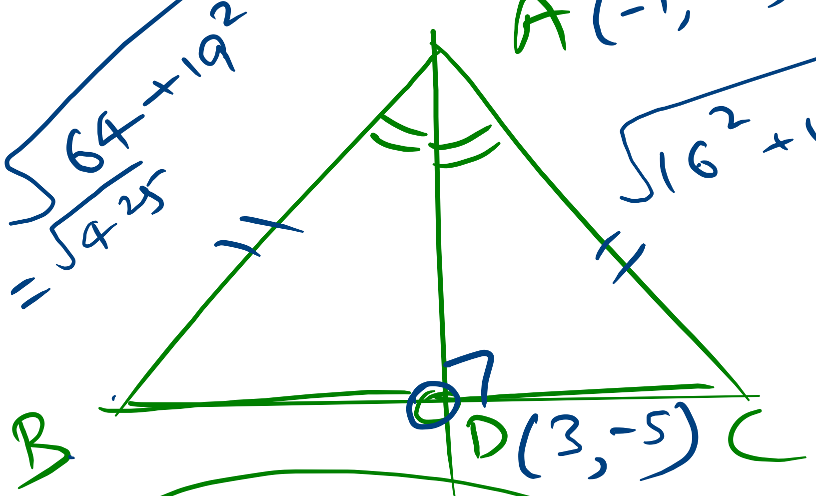
$$220x + 110y - 330 = 0$$

B_2 $\boxed{2x + y - 3 = 0}$ Acute

Q: The vertices of a ΔABC are

$A(-1, 11)$; $B(-9, -8)$ and $C(15, -2)$.

Find the equation of the bisector of the angle at A.



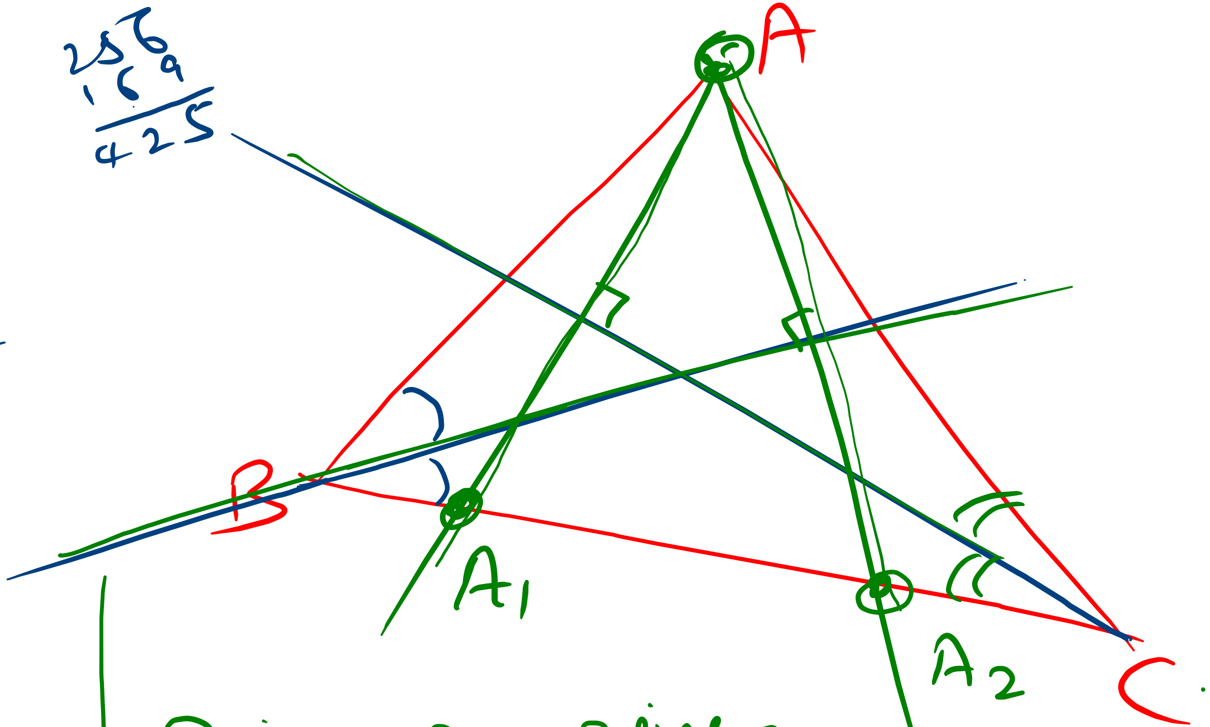
$\sqrt{64+19^2} = \sqrt{425}$

$\sqrt{16^2+13^2}$

$$\frac{BD}{DC} = \frac{AB}{AC}$$

$$m_{AD} = \frac{16}{-4} = -4$$

$$\boxed{y+5 = -4(x-3)}$$



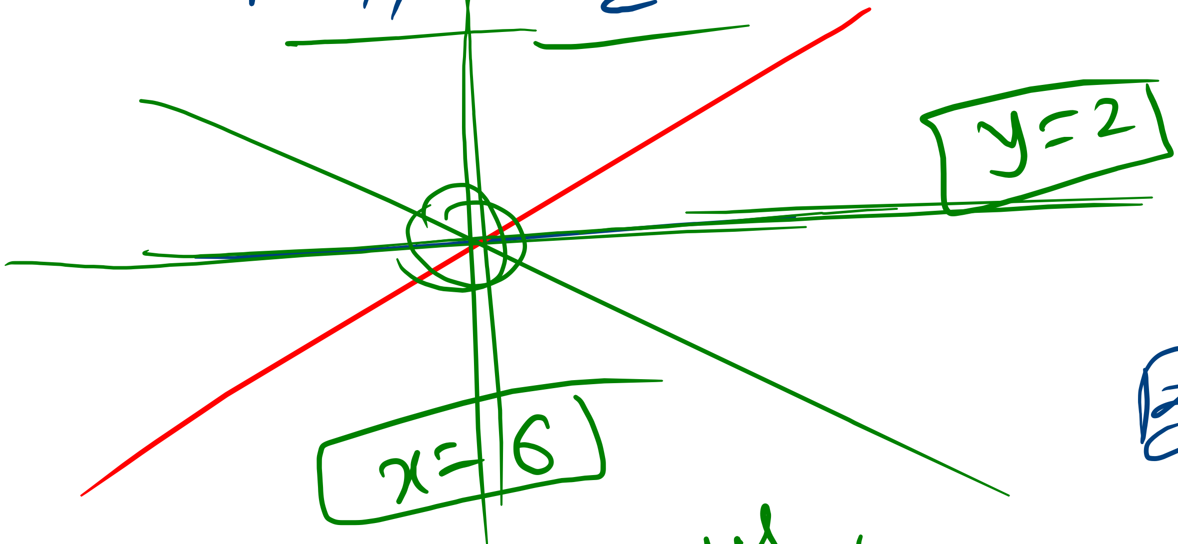
Point A given

Bisectors ∠B & ∠C are given

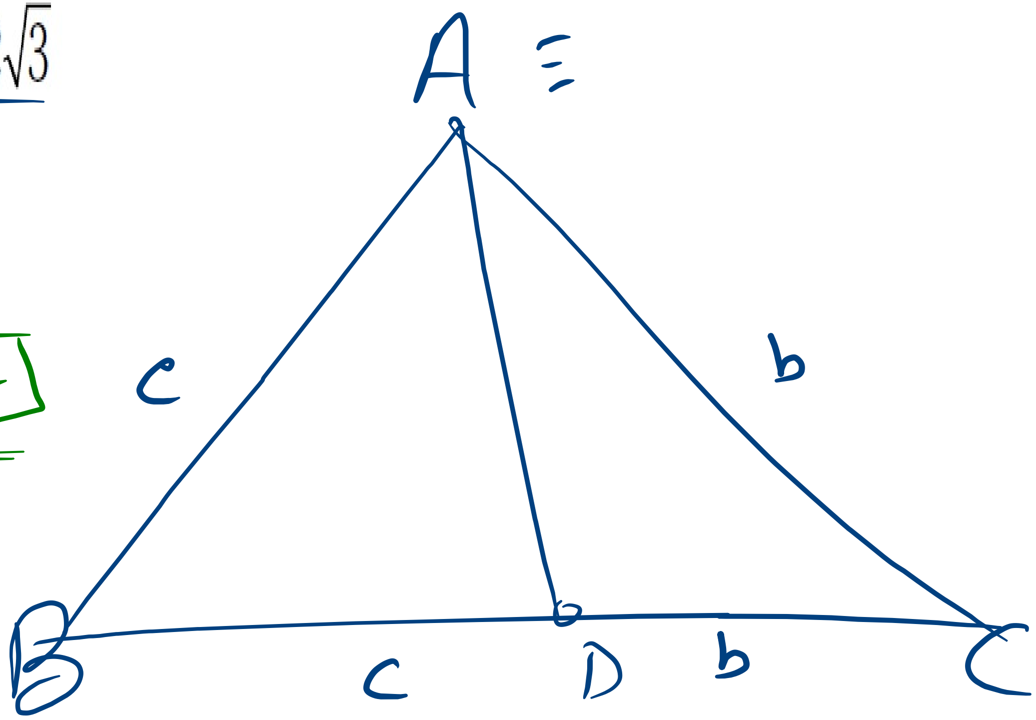
Equation of BC =

Q: Bisectors between the lines $x+\sqrt{3}y=6+2\sqrt{3}$ and $x-\sqrt{3}y=6-2\sqrt{3}$

$m_1 + m_2 = 0$



Bisectors are $\|^{l}$ to co-ordinate axes.



Section formula

$$D \equiv \left(\frac{c x_3 + b x_2}{b+c}, \frac{c y_3 + b y_2}{b+c}\right)$$

Q: Find the bisector of the angle between the lines $3x - 4y + 2 = 0$ and $5x + 12y - 8 = 0$ containing the point $(-1,2)$.

$$B_1 : \frac{3x - 4y + 2}{5} = \frac{5x + 12y - 8}{13}$$

$$B_2 : \frac{3x - 4y + 2}{5} = -\frac{5x + 12y - 8}{13}$$

$$\left(3 \times (-1) - 4 \times (2) + 2\right)\left(5 \times (-1) + 12 \times (2) - 8\right)$$

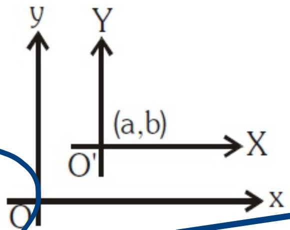
$$= (-ve)(+ve) = -ve.$$

$\Rightarrow B_2$ is the required bisector.

$(0,0)$ containing $(0 - 0 + 2)(0 + 0 - 8) < 0 \Rightarrow$ -ve वाला

TRANSFORMATION OF AXES :

(a) Shifting of origin without rotation of axes :



If the origin O(0, 0) is shifted to O'(a, b), then we have for the point O'.

x = a and X = 0; y = b and Y = 0

Hence x – a = X and y – b = Y

$X = x - \alpha = 3 - 1 = 2$

$Y = y - \beta = -4 - 2 = -6$

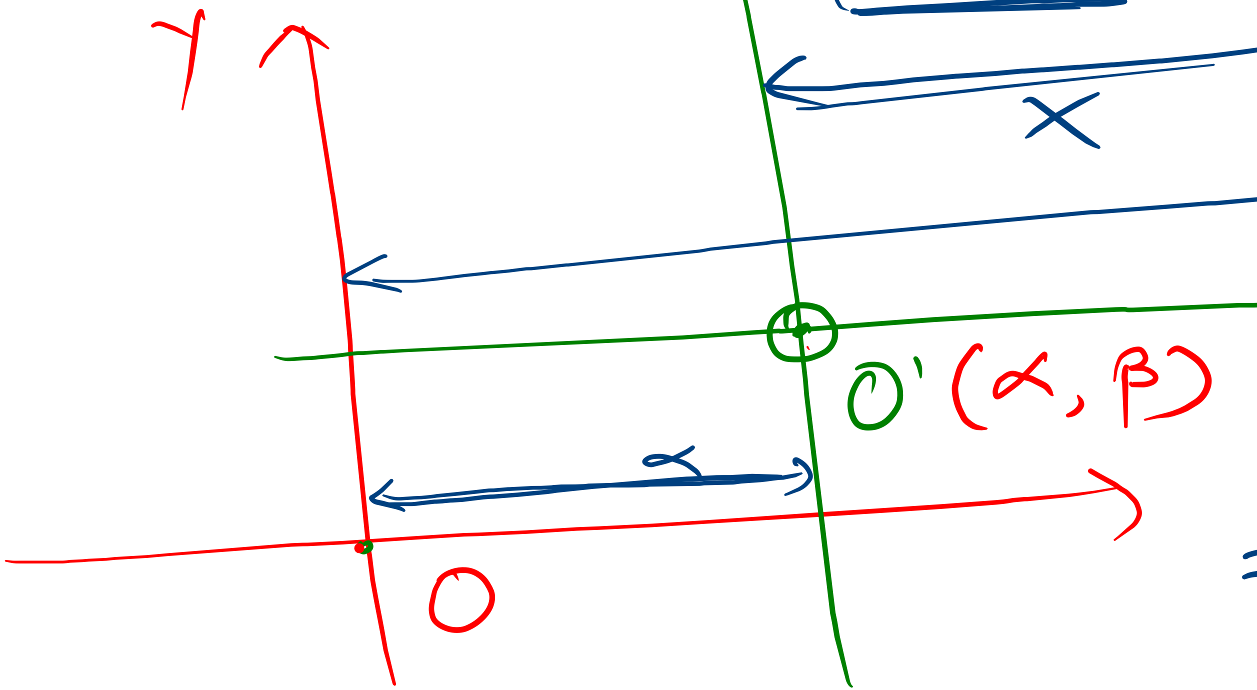
$$\boxed{\begin{aligned} x &= X + \alpha \\ y &= Y + \beta \end{aligned}}$$

E(1) Find the new co-ordinates of the point (3, –4) if the origin is shifted to (1, 2).

(Here x = 1 and y = 2 ⇒ x – 1 = X and y – 2 = Y

New co-ordinates are (2, –6))

E(2) Find the transformed equation of the straight line 2x – 3y + 5 = 0 if origin is shifted to (3, –1).



$x = X + 3,\ y = Y - 1$

$2x - 3y + 5 = 0$

$\Rightarrow 2X + 6 - 3Y + 3 + 5 = 0$

$\boxed{2X - 3Y + 14 = 0}$

When the origin is shifted to (1, 1), then what do the following equations become :

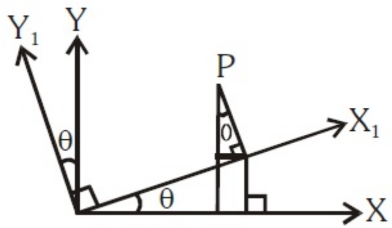
E(a) $xy - x - y + 1 = 0 \rightarrow XY = 0$

D(b) $x^2 - y^2 - 2x + 2y = 0 \rightarrow X^2 - Y^2 = 0$

D(c) $x^2 + xy - 3y^2 - y + 2 = 0 \rightarrow X^2 - 3Y^2 + XY + 3X - 6Y + 1 = 0$

Rotation of axes without shifting the origin :

To change the direction of axes of coordinates without changing the origin, both systems of coordinates being rectangular. Let OX, OY be the original system of axes and OX_1, OY_1 be the new system of axes. Let the angle XOX_1, through which the axes are turned be θ. Let P(x, y) w.r.t. a frame OX, OY also P(x', y') w.r.t. a frame OX_1, OY_1, then we have relation between as : $x = x'\cos\theta - y'\sin\theta$; $y = x'\sin\theta + y'\cos\theta$

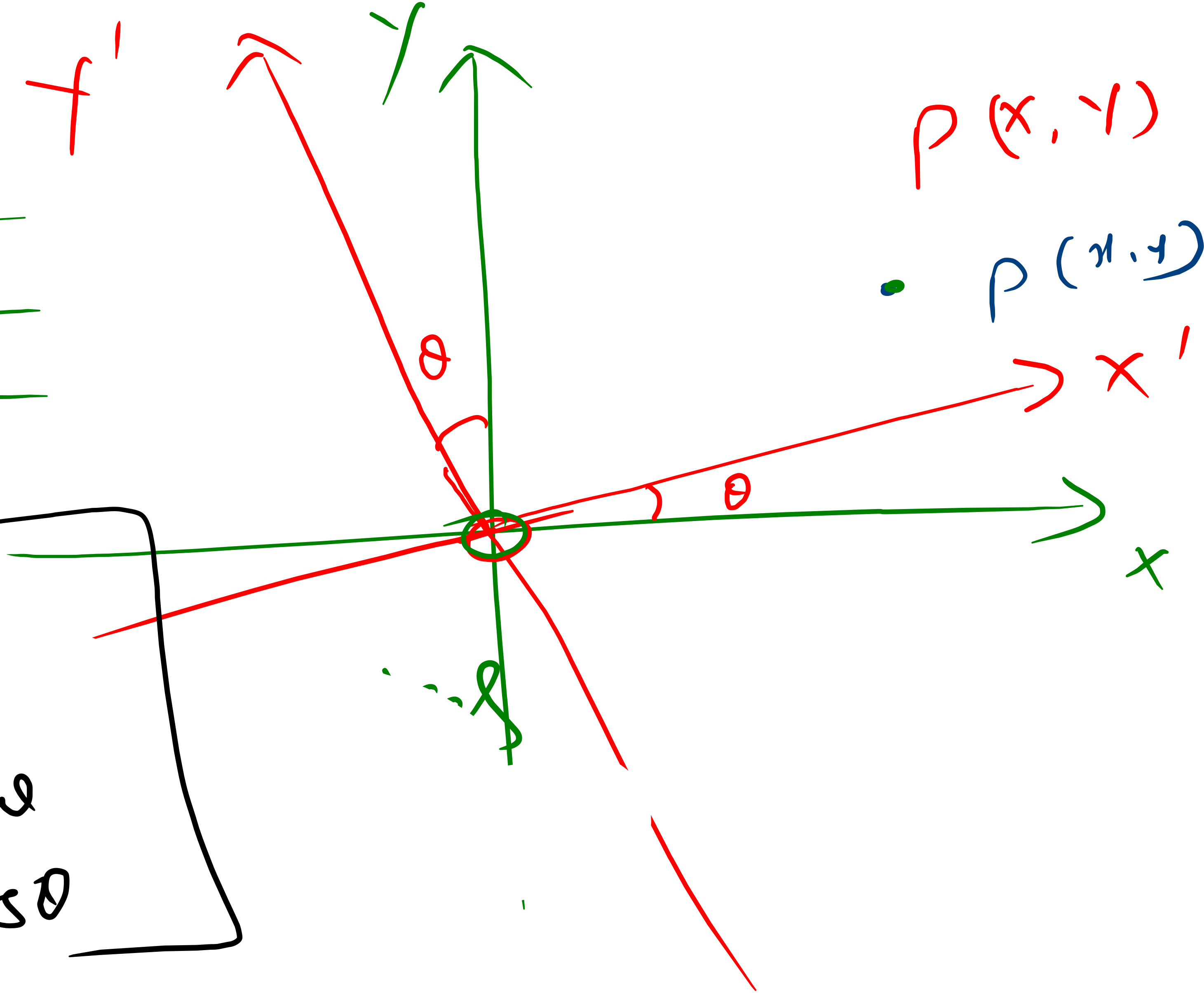


New \ Old	x ↓	y ↓
x' →	$\cos\theta$	$\sin\theta$
y' →	$-\sin\theta$	$\cos\theta$

If the axes are rotated through an angle of 30° in the anticlockwise direction about the origin, find the co-ordinates of a point $(4, -2\sqrt{3})$ in the new system w.r.t the old system. [**Ans.** $(\sqrt{3}, -5)$]

	x	y
X	$\cos\theta$	$\sin\theta$
Y	$-\sin\theta$	$\cos\theta$

$$X = x\cos\theta + y\sin\theta$$
$$Y = -x\sin\theta + y\cos\theta$$
$$x = X\cos\theta - Y\sin\theta$$
$$y = X\sin\theta + Y\cos\theta$$



If the axes are rotated through an angle of 45° in the clockwise direction about the origin, find the new equation of $x^2 - y^2 = a^2$. [**Ans.** $xy = \cancel{2a^2}$]

$\theta = -45^\circ$

$x = X\cos\theta - Y\sin\theta = \frac{X}{\sqrt{2}} + \frac{Y}{\sqrt{2}}$

$y = X\sin\theta + Y\cos\theta = \frac{-X+Y}{\sqrt{2}}$

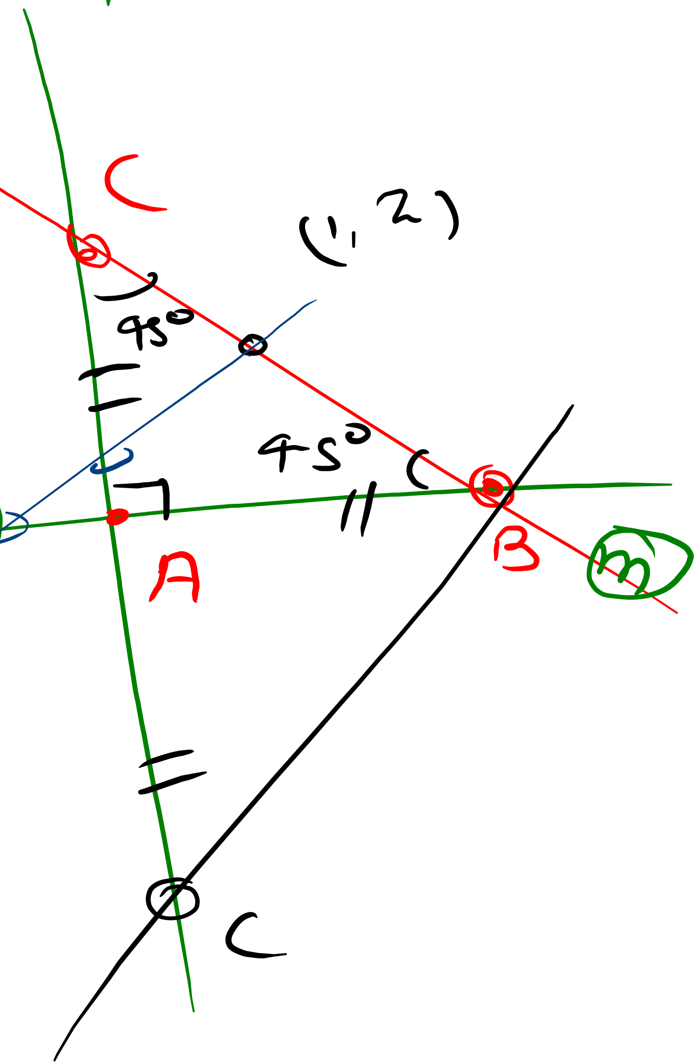
$\frac{(X+Y)^2}{2} - \frac{(-X+Y)^2}{2} = a^2$

$4XY = 2a^2$

$\boxed{XY = \frac{a^2}{2}.}$

Q. Straight lines $3x + 4y = 5$ and $4x - 3y = 15$ intersect at the point A. Points B and C are chosen on these two lines such that $AB = AC$. Determine the possible equations of the line BC passing through the point $(1, 2)$. *(1990 - 4 Marks)*

$m_1 m_2 = -\frac{3}{4} \times \frac{4}{3} = -1$



$\tan 45^\circ = \left|\frac{m - m_2}{1 + m m_2}\right| = \left|\frac{m - \frac{4}{3}}{1 + \frac{4m}{3}}\right|$

$m_2 = 4/3$

$1 = \left|\frac{3m - 4}{3 + 4m}\right| \Rightarrow \frac{3m-4}{3+4m} = 1, -1$

$\Rightarrow 3 + 4m = 3m - 4$

$\Rightarrow \boxed{m = -7}$

$3 + 4m = -3m + 4$

$\Rightarrow 7m = 1$

$\boxed{m = \frac{1}{7}}$

GENERAL EQUATION OF SECOND DEGREE REPRESENTING A PAIR OF STRAIGHT LINES :

(i) $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ represents a pair of straight lines if :

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0, \quad \text{i.e. if} \quad \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0.$$

(ii) The angle θ between the two lines representing by a general equation is the same as that between the two lines represented by its homogeneous part only .

Example :

E(1) Find the value of p & q such that $x^2 + y^2 + 2pxy + 2qx + 6y + 8 = 0$ represents two parallel straight lines.

[**Ans.** p=1, q=3; p=–1, q=–3]

$a=1,\ b=1,\ h=p\quad g=q,\ f=3,\ c=8$

$abc+2fgh-af^2-bg^2-ch^2=0$

$\Rightarrow \boxed{8+2\times3\times q\times p-9-1\times q^2-8p^2=0}$

$h^2=ab$

$p^2=1$

$\Rightarrow \boxed{p=1,-1}$

$q=$

$$\underbrace{ax^2+2hxy+by^2+2gx+2fy+c}_{\text{2nd degree general equation}}=0$$

(*) Pair of St. lines – $\Rightarrow \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} = 0$

$\Rightarrow abc+2fgh-af^2-bg^2-ch^2=0$

Circle

Parabola

Ellipse

Hyperbola

Conic-section

Central conic.

Q: Find the point of intersection of the lines $y^2 - xy - 6x^2 + 13x - y - 6 = 0$. Also find the angle between them. [**Ans.** (1,1); 45°]

To find P.O.I. of the lines represented by $f(x, y) = 0$, where $f(x, y) = ax^2 + 2hxy + by^2 + 2gx + 2fy + c$

Method I: use partial derivative method

$\frac{\partial f}{\partial x} = 0 \quad \ldots \text{(1)}, \quad \frac{\partial f}{\partial y} = 0 \quad \ldots \text{(2)}$

Solve (1) & (2) $\Rightarrow$ P.O.I.

$-y - 12x + 13 = 0 \quad \ldots \text{(1)}$

$2y - x - 1 = 0 \quad \ldots \text{(2)}$

$\boxed{x = 1, \; y = 1}$

P.O.I. (1, 1)

Caution: (1) & (2) are not the separate lines.

P.O.I of $y^2 - xy - 6x^2 + 13x - y - 6 = 0$

method 2:

$$(y+2x-3)(y-3x+2) = 0$$

$$y^2 - xy - 6x^2$$
$$= y^2 - 3xy + 2xy - 6x^2$$
$$= y(y-3x) + 2x(y-3x)$$
$$= (y+2x)(y-3x)$$

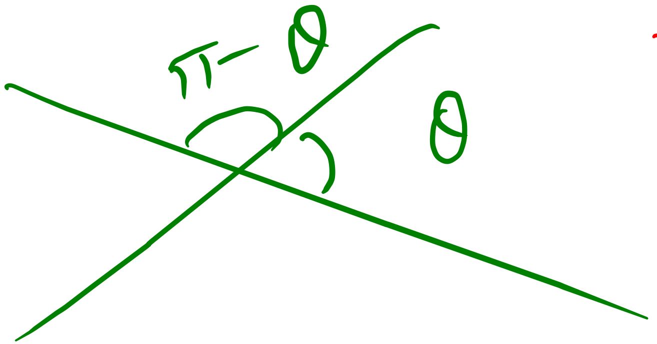
$$y+2x-3=0$$
$$y-3x+2=0$$

$m_1 = -2$

$m_2 = 3$

P.O.I. $(1,1)$

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$



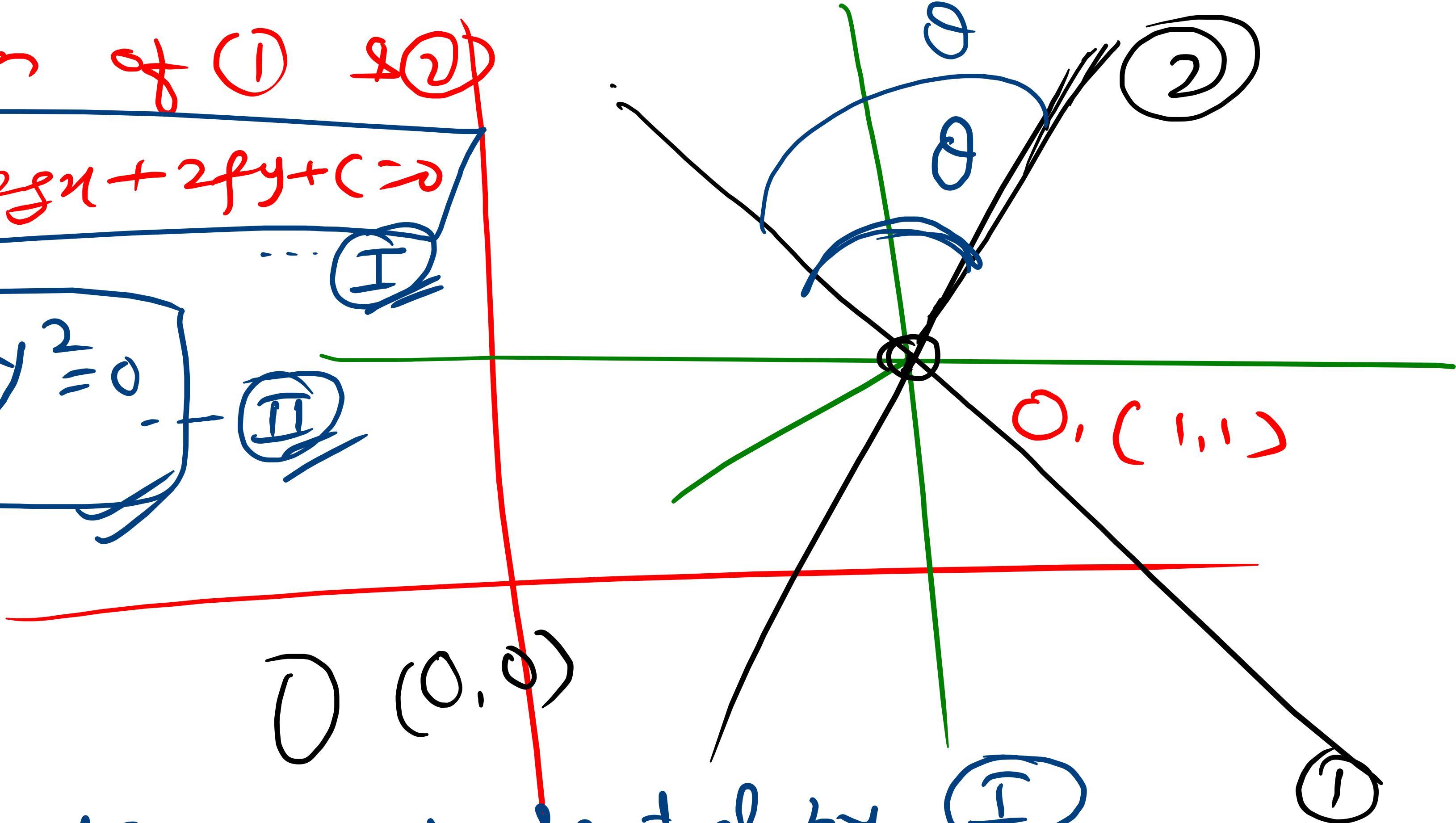
Shift $(0,0)$ at $(1,1)$, write down the above joint equation in new system: $x = (X+1)$, $y = (Y+1)$

$$\boxed{Y^2 - XY - 6X^2 = 0}$$

Joint equation of (1) & (2)

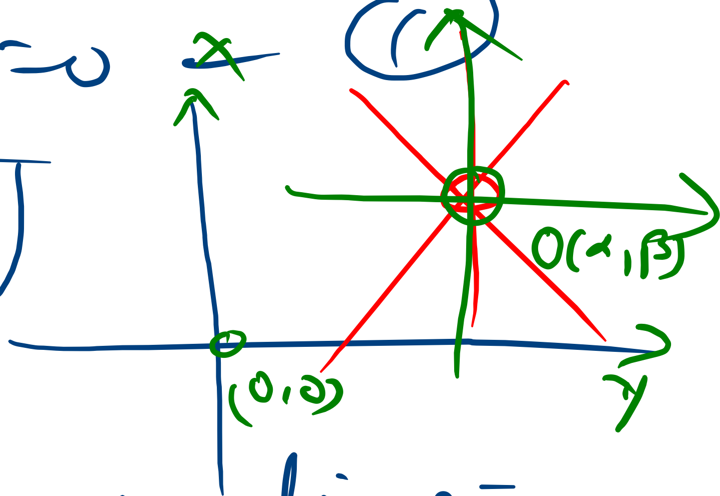
$$ax^2+2hxy+by^2+2gx+2fy+c=0 \quad \text{--- (I)}$$

$$ax^2+2hxy+by^2=0 \quad \text{--- (II)}$$



Angle b/w the lines represented by (I)
= Angle b/w the lines represented by (II)

$$\boxed{ax^2+2hxy+by^2}+2gx+2fy+c=0 \quad (1)$$



Lines are $\perp^r$ $\Longrightarrow$ $\boxed{a+b=0}$

Lines are $\parallel^l$ $\Longrightarrow$ $h^2=ab$

Let θ be the acute angle b/w the lines represented by (1)

then $\tan\theta = \left|\dfrac{2\sqrt{h^2-ab}}{a+b}\right|$

(*) $abc+2fgh-af^2-bg^2-ch^2=0$

Homogenisation : (V. V. I.)

The joint equation of a pair of straight lines joining origin to the points of intersection of the line given by

$\ell x + my + n = 0$ .... (i)

& the 2nd degree curve : $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ .... (ii)

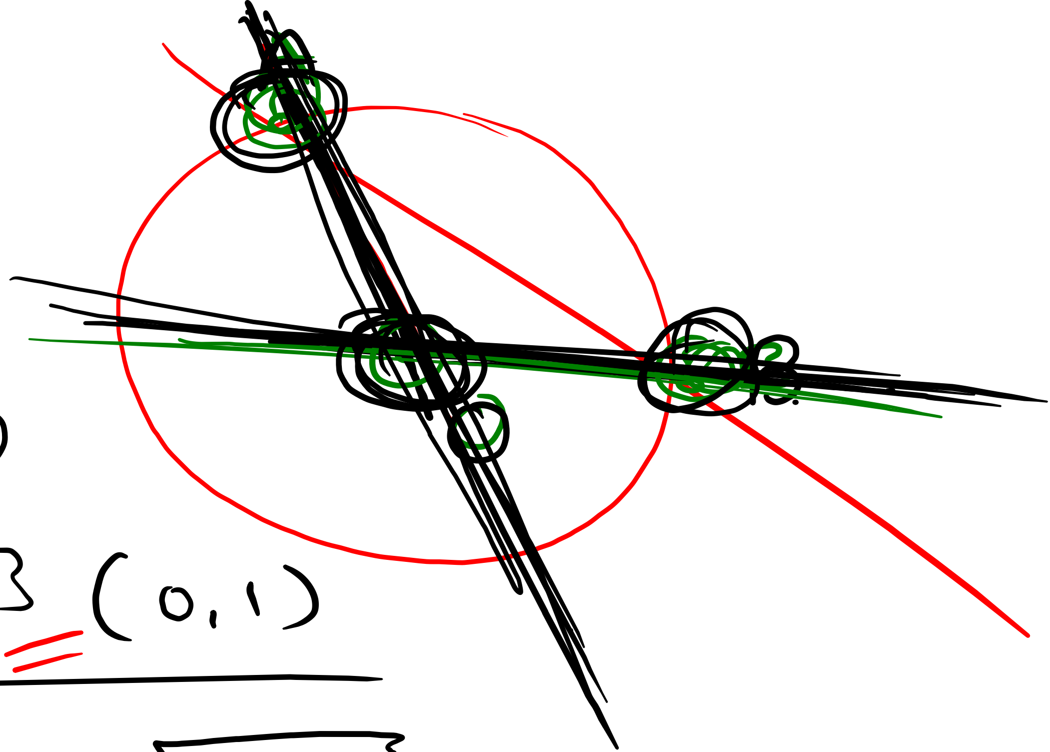
is $ax^2 + 2hxy + by^2 + 2gx\left(\frac{\ell x + my}{-n}\right) + 2fy\left(\frac{\ell x + my}{-n}\right) + c\left(\frac{\ell x + my}{-n}\right)^2 = 0$ .... (iii)

(iii) is obtained by homogenizing (ii) with the help of (i), by writing (i) in the form : $\left(\frac{\ell x + my}{-n}\right) = 1.$

Note: Any second degree curve through the four point of intersection of $f(x,y) = 0$ & $xy = 0$ is given by $f(x\,y) + \lambda\, xy = 0$ where $f(x,y) = 0$ is also a second degree curve.

Points O & A Line ①

Points O & B Line ②



Joint equation of Line ①

Line ② A (1, 0), B (0, 1)

$x = 0$ ... ①

$y = 0$ ... ②

Joint $\Rightarrow \boxed{xy = 0}$

$x^2 + y^2 = 1$

$x + y = 1$

Using Homogenization

$\boxed{x^2 + y^2 - 1 = 0}$

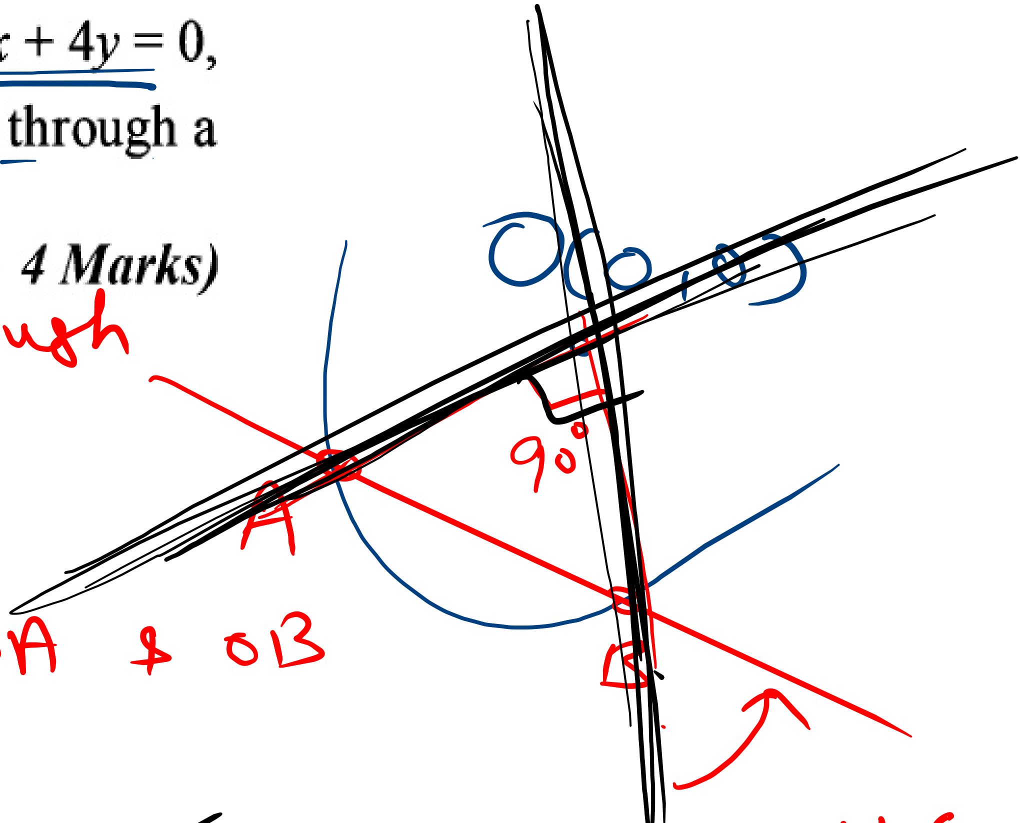
$x^2 + y^2 - 1^2 = 0$

$\boxed{x^2 + y^2 - (x+y)^2 = 0} \Rightarrow xy = 0$

Q. Show that all chords of the curve $3x^2 - y^2 - 2x + 4y = 0$, which subtend a right angle at the origin, pass through a fixed point. Find the coordinates of the point.

(1991 - 4 Marks)

Chord AB will pass through a fixed point.



Joint equation of OA & OB

$3x^2 - y^2 + (4y - 2x)\cdot 1 = 0$

$3x^2 - y^2 + (4y - 2x)\left(\frac{y - mx}{c}\right) = 0$

$y = mx + c$

$\frac{y - mx}{c} = 1$

$\boxed{A + B = 0} \Rightarrow 3 + \frac{2m}{c} + (-1) + \frac{4}{c} = 0$

$\Rightarrow 2c + 2m + 4 = 0 \Rightarrow \boxed{-2 = m + c}$ $(1, -2)$

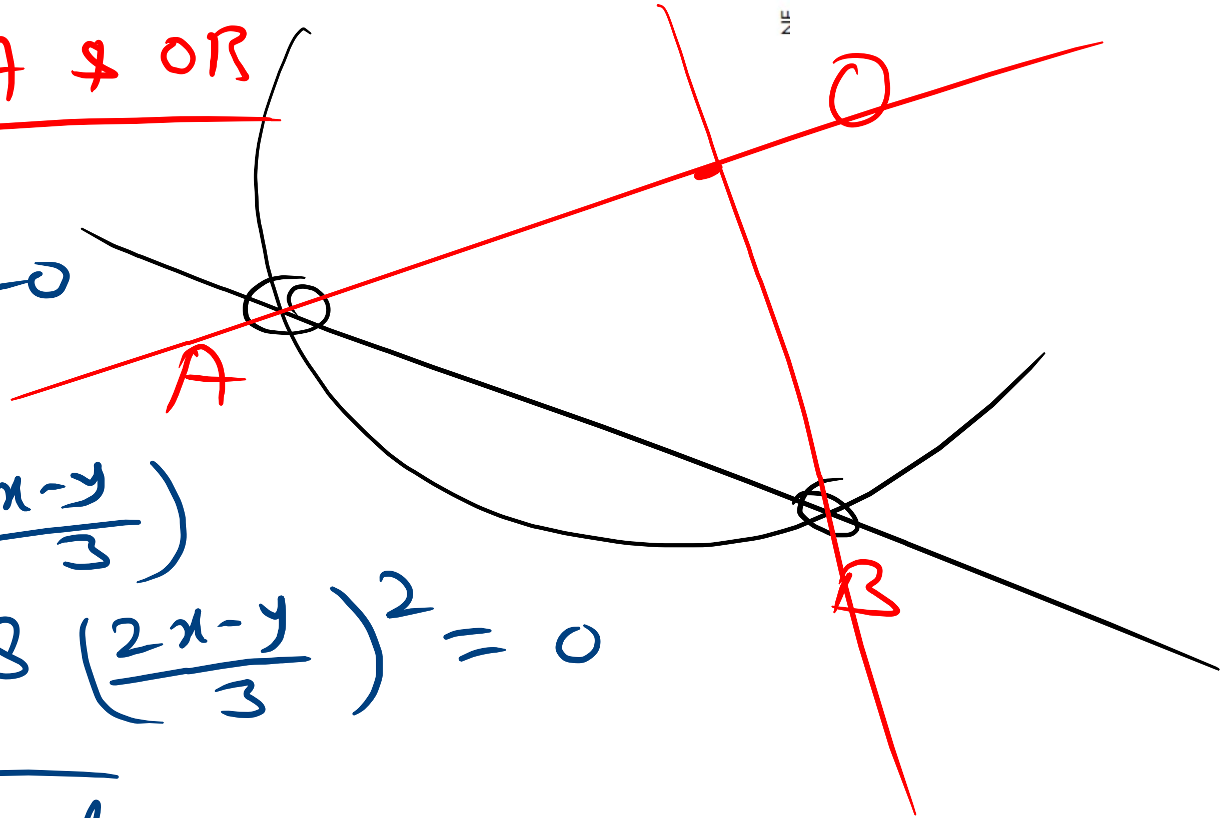
$y = mx + c, \; -2 = m + c$

E(a) Find the equation of the line pair joining origin and the point of intersection of the line $2x - y = 3$ and the curve $x^2 - y^2 - xy + 3x - 6y + 18 = 0$. Also find the angle between these two lines.

[**Ans.** $11x^2 - 14xy + 3y^2 = 0$; $\tan^{-1}4/7$]

$$\frac{2x-y}{3} = 1$$

Joint equation of OA & OB



$$x^2 - y^2 - xy + 3(x-2y)\cdot 1 + 18\cdot 1^2 = 0$$

$$\Rightarrow x^2 - y^2 - xy + 3(x-2y)\left(\frac{2x-y}{3}\right) + 18\left(\frac{2x-y}{3}\right)^2 = 0$$

$$\tan\theta = \frac{2\sqrt{h^2 - ab}}{|a+b|}$$

Q. Find the value of m if the lines joining the origin to the points common to $x^2 + y^2 + x - 2y - m = 0$ & $x + y = 1$ are at right angles. [m = 1/2]

$$x^2 + y^2 + (x - 2y)\cdot 1 - m \cdot 1^2$$

$$\Rightarrow \quad x^2 + y^2 + (x - 2y)(x+y) - m(x+y)^2 = 0$$

$$A + B = 0 \quad \Rightarrow \quad m = \frac{1}{2}$$

Q: If angle between the lines joining origin and the point of intersection of the line $x - y = 2$ and the curve $x^2 - 4xy + 2y^2 - 2x + y + k = 0$ is $45°$. Find k. [**Ans.** $k = -4, -10$]

$$\frac{x-y}{2} = 1$$

$$x^2 - 4xy + 2y^2 + (y - 2x)\cdot 1 + k \cdot 1^2 = 0$$

$$\Rightarrow x^2 - 4xy + 2y^2 + (y - 2x)\left(\frac{x-y}{2}\right) + k\left(\frac{x-y}{2}\right)^2 = 0$$

$$|a+b| = 2\sqrt{h^2 - ab}$$

Q. A line L passing through the point (2, 1) intersects the curve $4x^2 + y^2 - x + 4y - 2 = 0$ at the points A, B. If the lines joining origin and the points A, B are such that the coordinate axis are the bisectors between them then find the equation of line L. [**Ans.** $4y - x = 2$, $2y + x = 4$]

$$y - 1 = m(x-2) = mx - 2m$$

$$y = mx - 2m + 1 \Rightarrow \left(\frac{y + 2m - mx}{1}\right) = 1$$

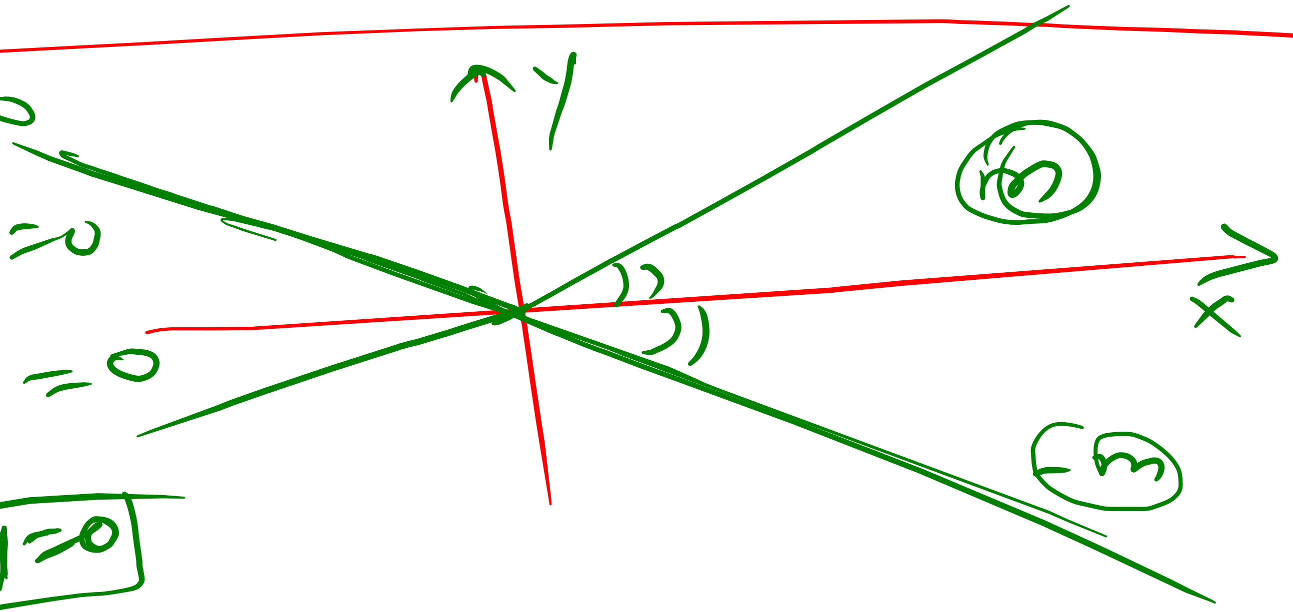
$$4x^2 + y^2 + (4y - x)(y + 2m - mx) - 2\cdot(y + 2m - mx)^2 = 0$$

$$Ax^2 + 2Hxy + By^2 = 0$$

$$\Rightarrow Bm^2 + 2Hm + A = 0$$

$$m_1 + m_2 = -\frac{2H}{B} = 0$$

$$\Rightarrow \boxed{H = 0}$$



If $a_r x + b_r y + c_r = 0$; $r = 1, 2, 3$ denotes the equations to the sides of the triangle.

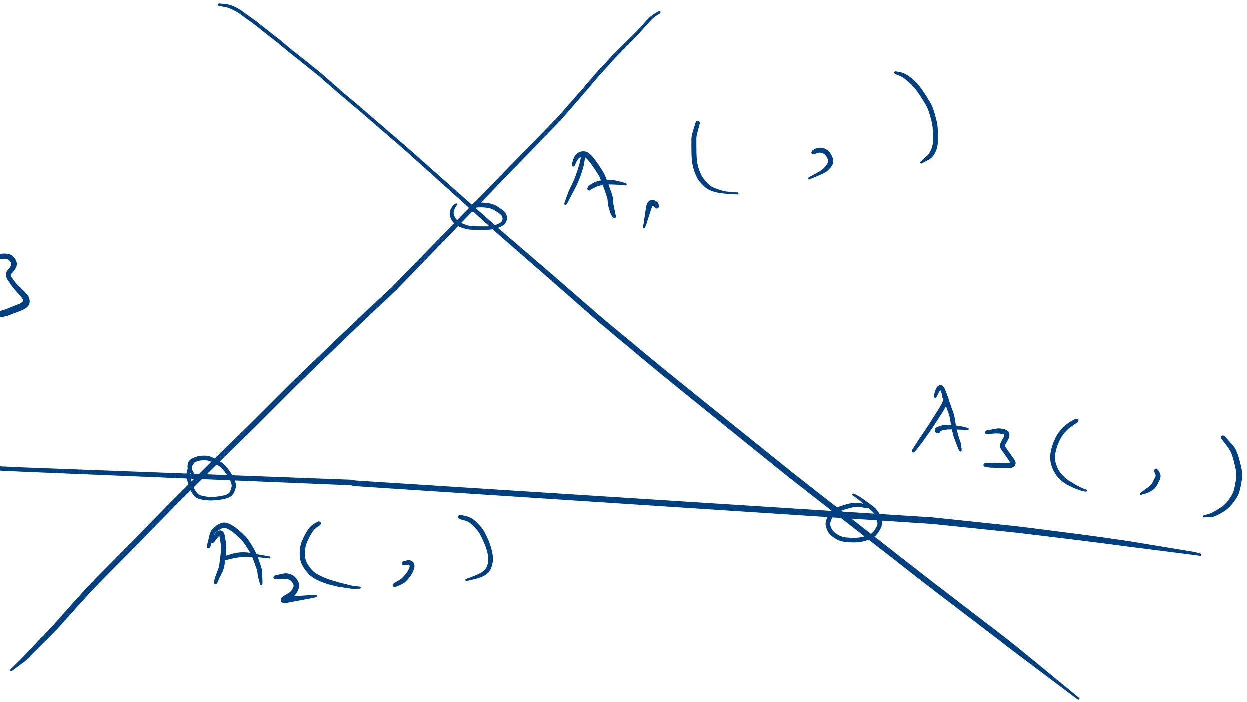
$$A = \frac{1}{2C_1C_2C_3}\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}^2$$, where C_1, C_2, C_3 are cofactors of c_1, c_2, c_3 i.e. $C_1 = a_2b_3 - b_2a_3$ etc.

or $$A = \frac{1}{2\prod(a_1b_2 - a_2b_1)}\begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}$$, where A_i, B_i, C_i $(i = 1, 2, 3)$ are co-factors.

$$D = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

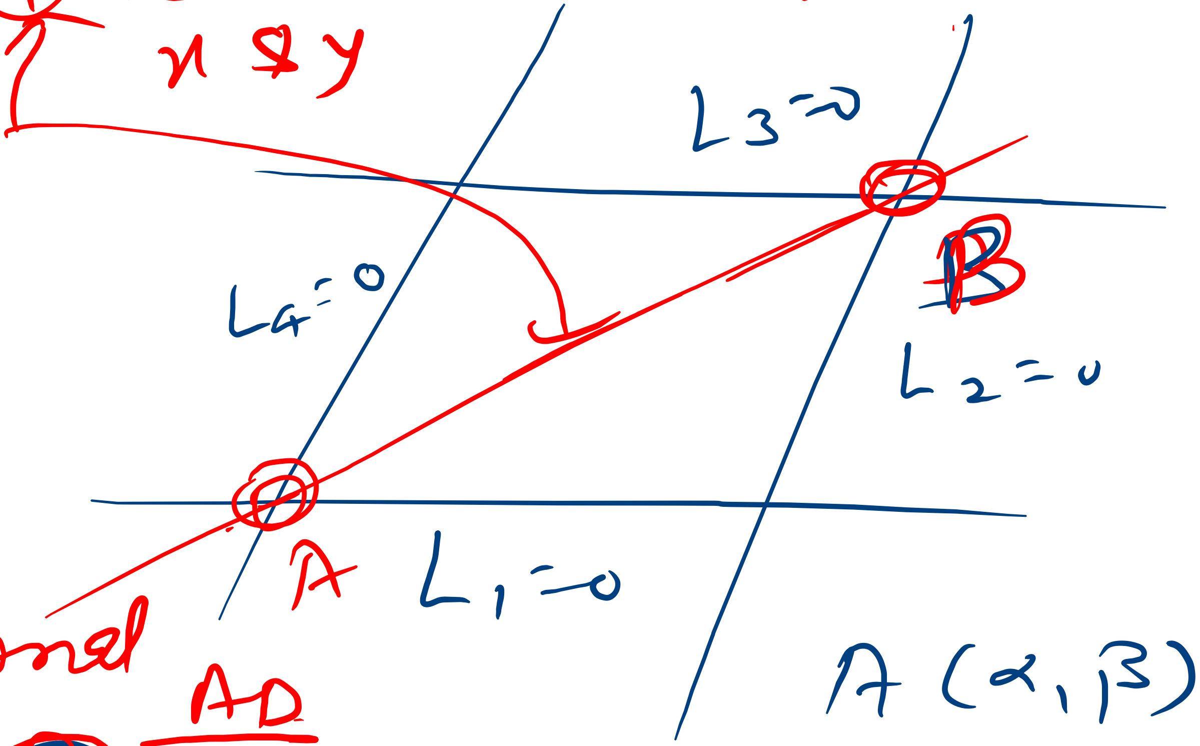
C_1, C_2, C_3

C_1 C_2 C_3



$$\left.\begin{aligned} L_1 &\equiv a_1x+b_1y+c_1 \\ L_3 &\equiv a_1x+b_1y+c_2 \\ L_2 &\equiv a_2x+b_2y+d_1 \\ L_4 &\equiv a_2x+b_2y+d_2 \end{aligned}\right\}$$

① is a linear equation in x & y



Equation of diagonal AD

$$\boxed{L_1L_2 - L_3L_4 = 0} \quad ①$$

$L_1(A)=0$, $L_4(A)=0$

$L_2(B)=0$, $L_3(B)=0$

point A lies on ①

point B lies on ①

$A(\alpha,\beta)$

$B(x_1,y_1)$

$$(x+y+1)(x+y-1)=0$$

$$\Rightarrow (x+y)^2+x+y-(x+y)-1=0$$

$$(x+y)^2-1=0$$

$$\underbrace{ax^2+2hxy+by^2}+2gx+2fy+c=0$$

perfect square

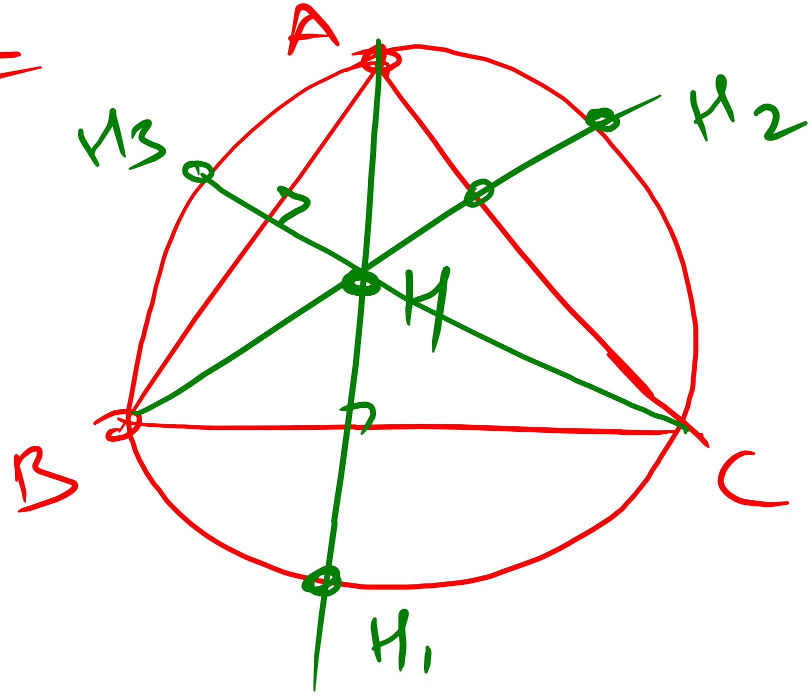
parallel lines -

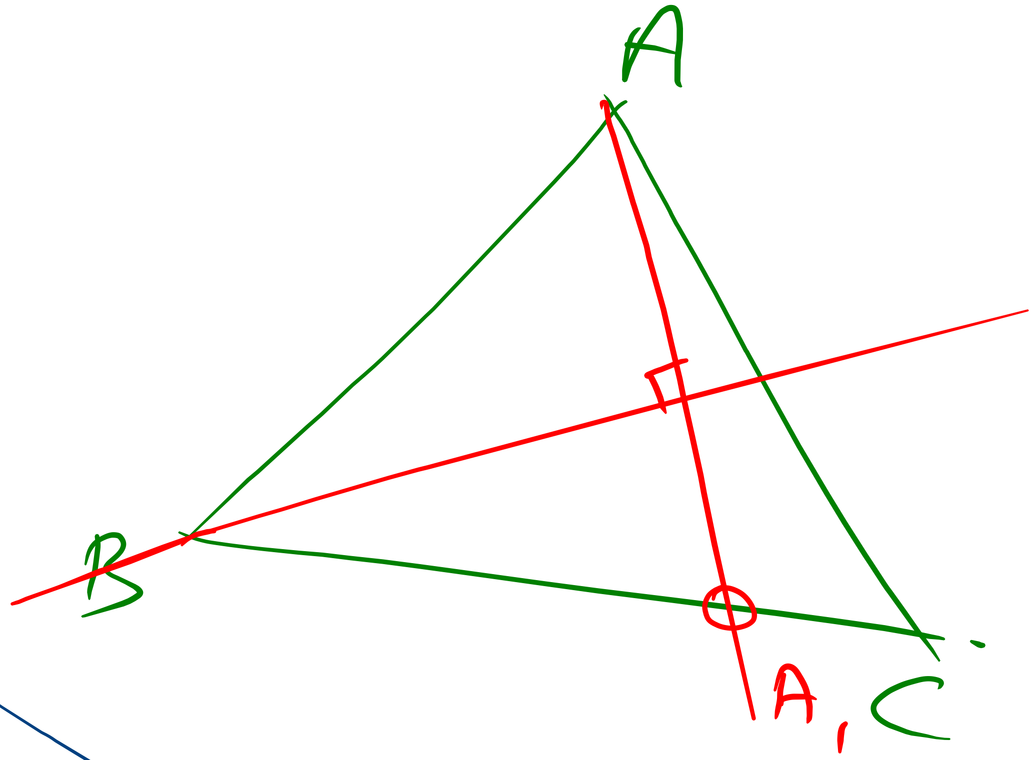
$$h^2=ab$$

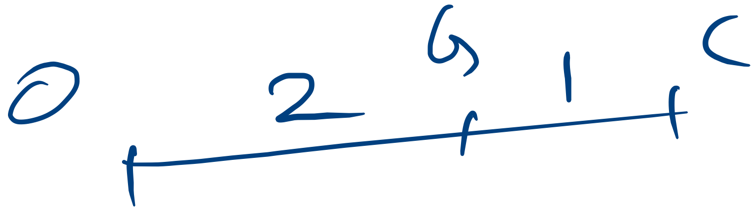
distance b/w ||' lines

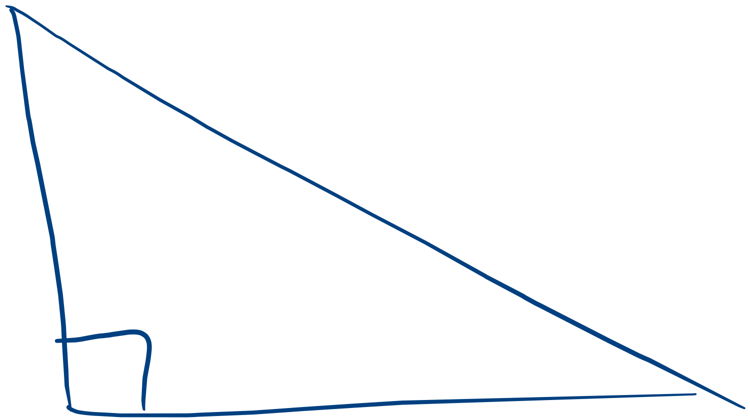
$$(\sqrt{a}x+\sqrt{b}y)^2+2gx+2fy+c=0$$

Point:







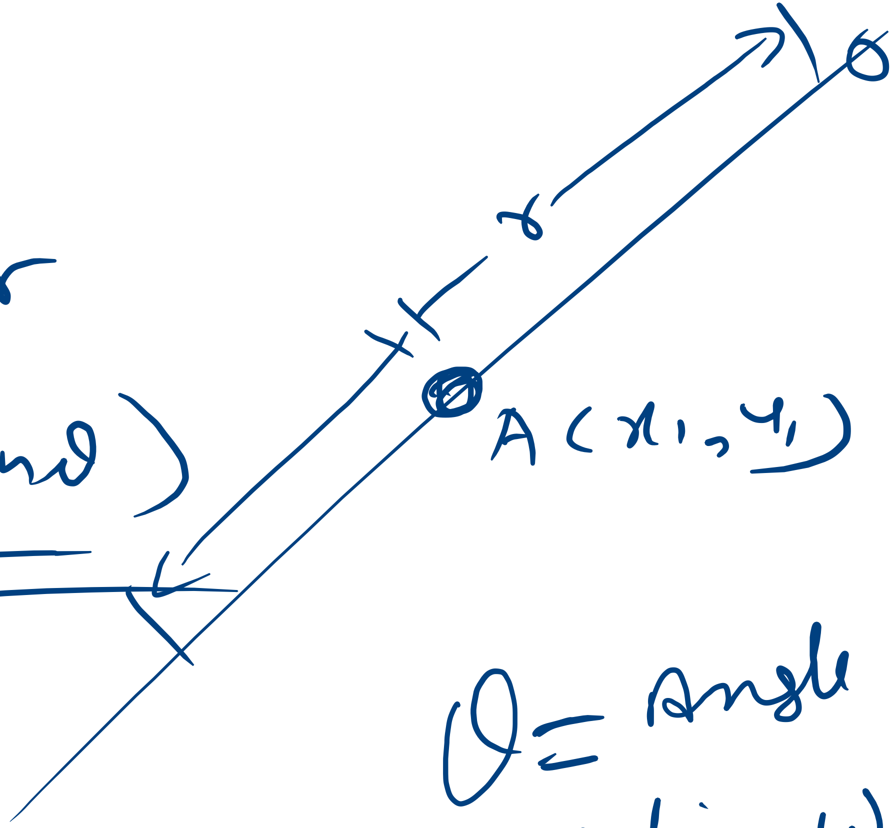


$$\boxed{y - y_1 = m(x - x_1)}$$

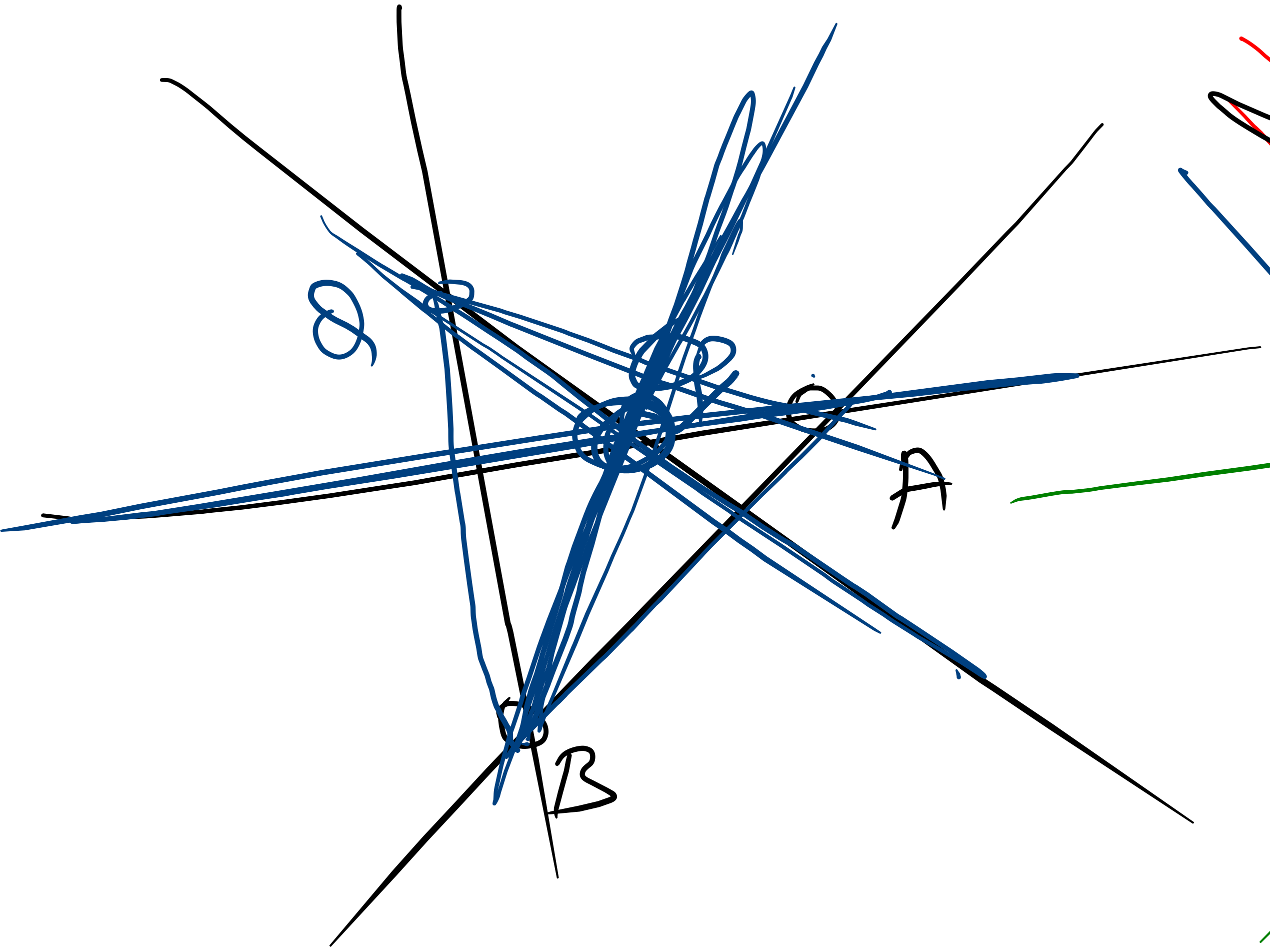
$$\frac{x - x_1}{\cos\theta} = \frac{y - y_1}{\sin\theta} = r$$

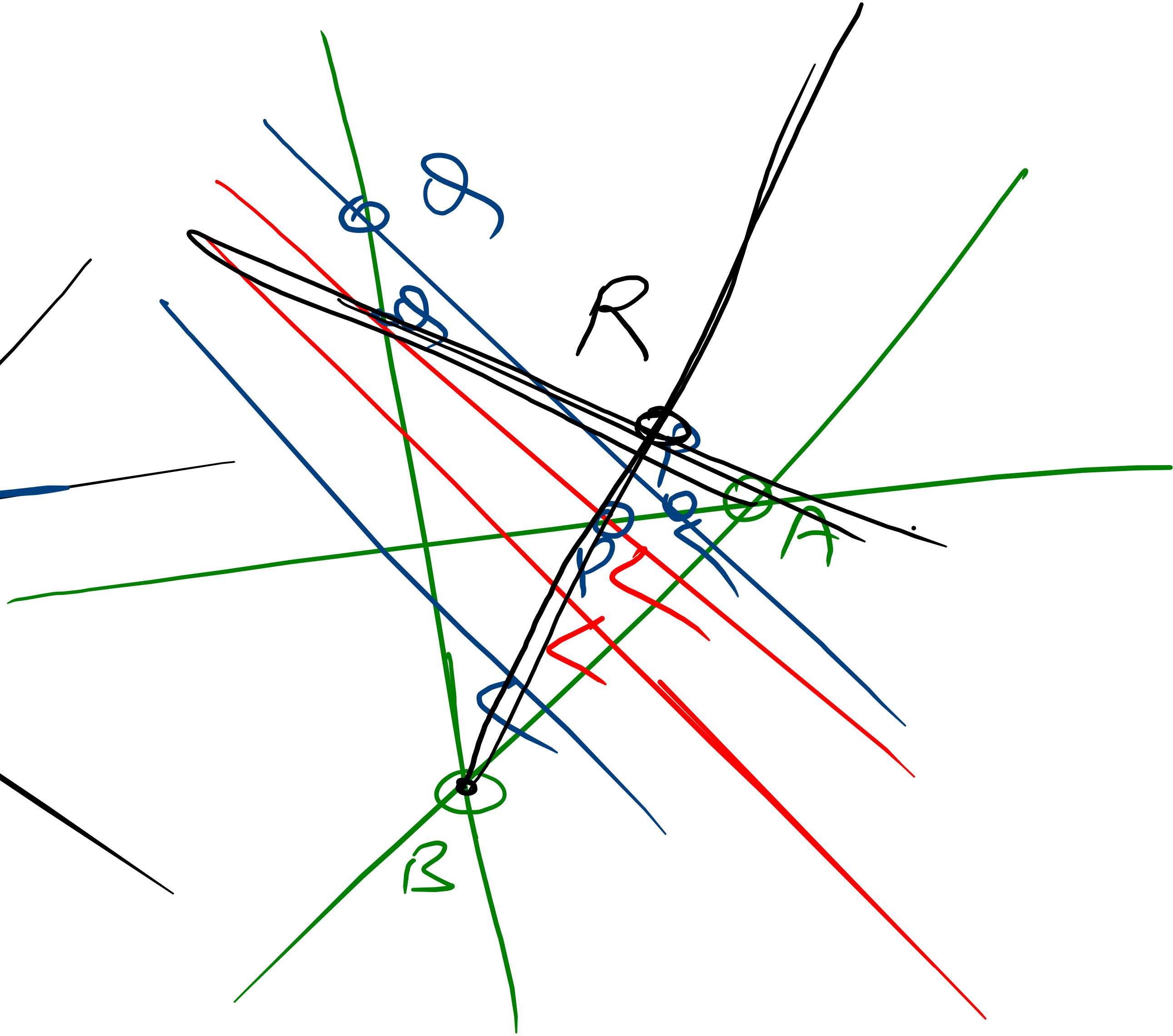
$$(x_1 \pm r\cos\theta,\ y_1 \pm r\sin\theta)$$

$$\boxed{\frac{x}{a} + \frac{y}{b} = 1}$$



θ = Angle made by this line with the x axis


A
B


R
A
B

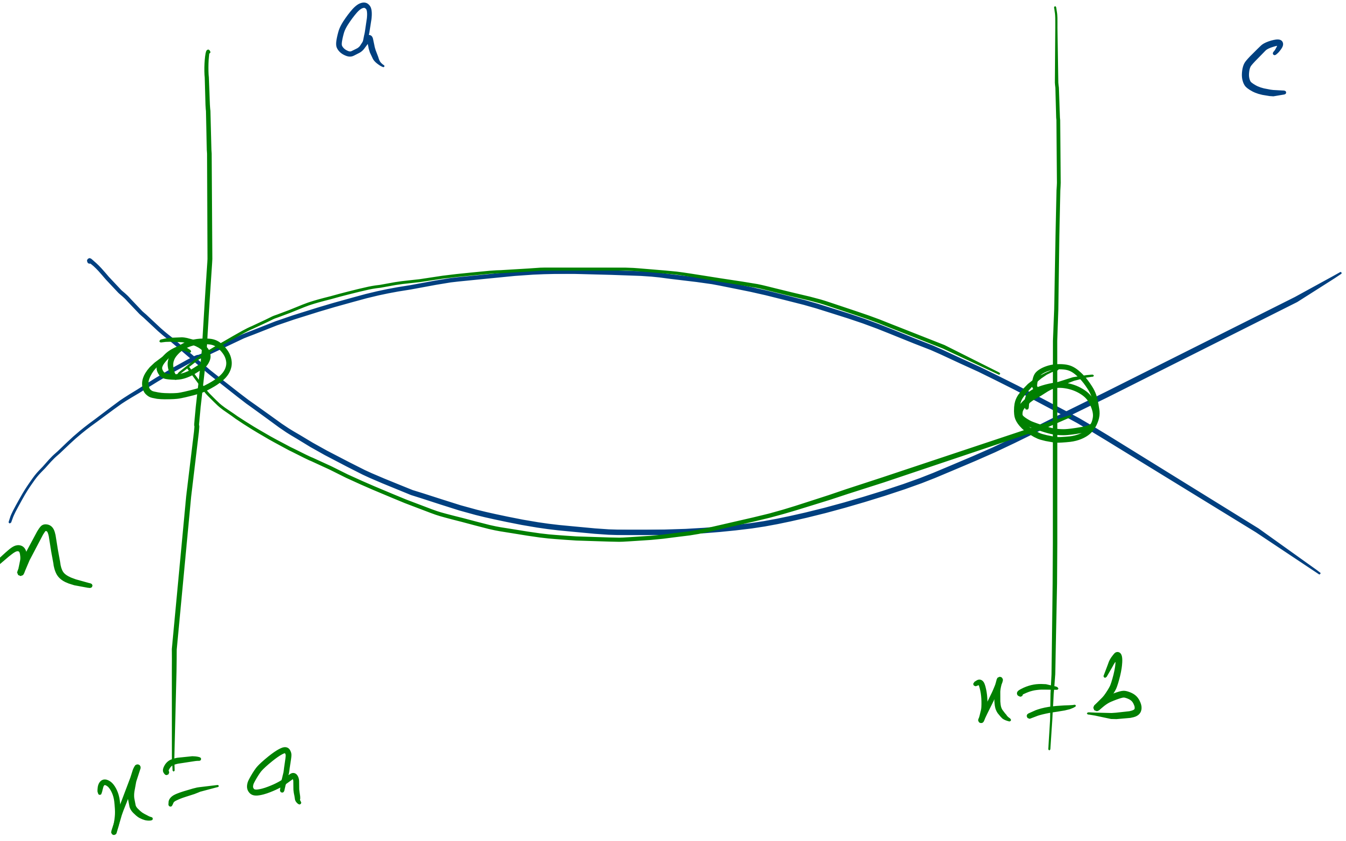
AUC.

$$\int_a^b \bigcirc \, dx$$

$$\int_c^d \boxed{(x)} \, dy$$

$$\int_a^b |f(x)| \, dx$$

$$\int_a^b |f(x) - g(x)| \, dx$$



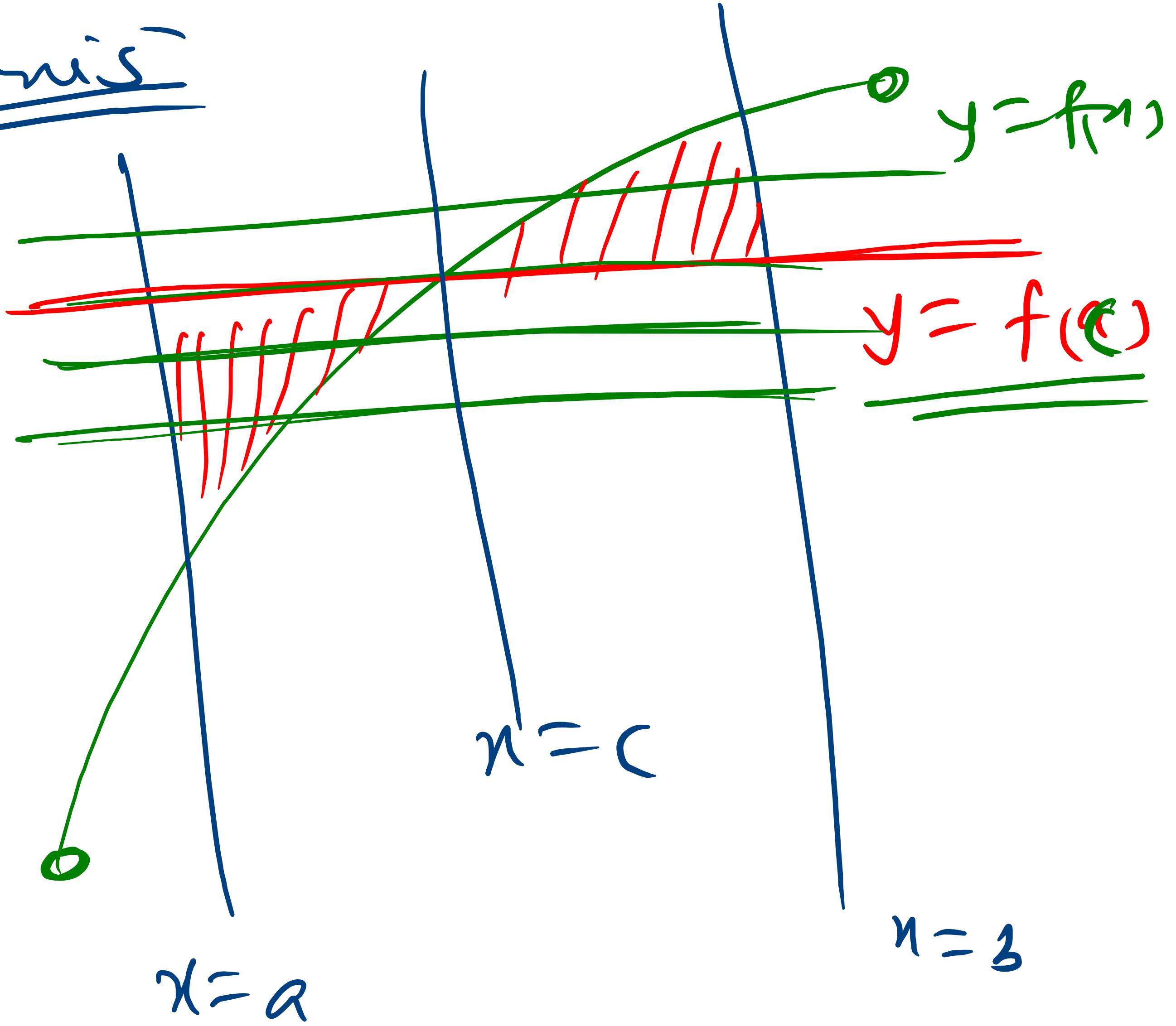
x axis

$y = f(c)$

$\boxed{c = \frac{a+b}{2}}$

$f(c) = 0$

$f\left(\frac{a+b}{2}\right) = 0$



$$\int_a^b |f(x)|\,dx$$

$$x = g(t), \quad y = h(t)$$

$$dx = g'(t)\,dt$$

$$= \int_a^b |y|\,dx$$

$$= \int_{t_1}^{t_2} |h(t)|\,g'(t)\,dt$$

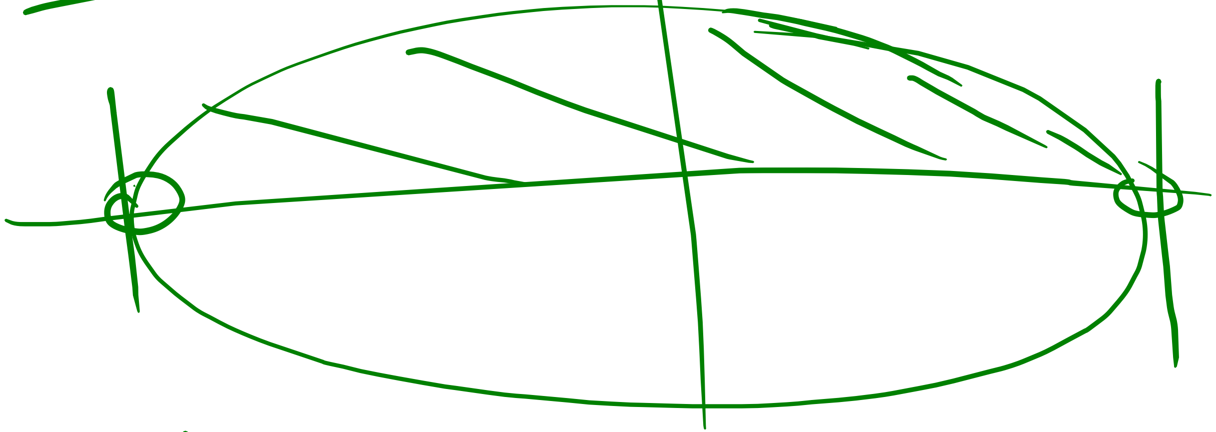
when $x = a$, $a = g(t) \Rightarrow t = t_1$

$x = b$, $b = g(t) \Rightarrow t = t_2$

$x = a$ $\quad$ $x = b$

$x = a\cos\theta$

$y = b\sin\theta \implies dx = -a\sin\theta \, d\theta$



$$\int_{-a}^{a} (y)\, dx$$

$$= \int_{\pi}^{0} |b\sin\theta| (-a\sin\theta)\, d\theta$$

$$= 2ab \int_{0}^{\pi/2} \sin^2\theta \, d\theta$$

$$= 2ab \times \frac{\pi}{4}$$

$$= \frac{\pi ab}{2}$$

$x = -a$

$\implies a\cos\theta = -a$

$\cos\theta = -1$

$x=0$
$x=\frac{\pi}{4}$

$$\sqrt{\frac{1+\sin x}{\cos x}} \qquad \sqrt{\frac{1-\sin x}{\cos x}}$$

$$\int_0^{\pi/4}\left(\sqrt{\frac{1+\sin x}{\cos x}}-\sqrt{\frac{1-\sin x}{\cos x}}\right)dx$$

$$=$$

$$\sin x = \frac{2\tan x/2}{1+\tan^2 x/2}$$

$$1+\sin x = \frac{(1+\tan x/2)^2}{(\sec^2 x/2)}$$