AOD

- Tangent & Normal
  - Rate Measure
  - Approximation
- Monotonicity of function
- Maxima & Minima.

Tangent & Normal

TANGENT & NORMAL : (Define) ; $\tan\phi = \left.\frac{dy}{dx}\right|_P$

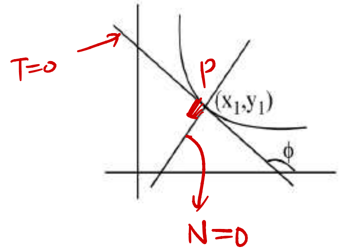
(1) Equation of a tangent at P (x_1, y_1)

$$y - y_1 = \left.\frac{dy}{dx}\right|_{x_1,y_1}(x - x_1)$$

(2) Equation of normal at (x_1, y_1)

$$y - y_1 = -\frac{1}{\left(\frac{dy}{dx}\right)_{x_1,y_1}}(x - x_1)$$

NOTE : *



* If $\left.\frac{dy}{dx}\right]_{x_1,y_1}$ exists. However in some cases $\frac{dy}{dx}$ fails to exist but still a tangent can be drawn *e.g.* case of vertical tangent. Also (x_1, y_1) must ~~le~~ lie on the tangent, normal line as well as on the curve.

Q A line is drawn touching the curve $y - \frac{2}{3-x} = 0$. Find the line if its slope/gradient is 2.

Sol^n

$m = 2$

$$\boxed{y = \frac{2}{3-x}} \rightarrow P(x_1, y_1)$$

$$y_1 = \frac{2}{3-x_1}$$

$$\left.\frac{dy}{dx}\right|_P = \left.\frac{-2}{(3-x)^2} \times (-1)\right|_{(x_1,y_1)} = \frac{2}{(3-x_1)^2}$$

$$\frac{2}{(3-x_1)^2} = 2$$

$$(3-x_1)^2 = 1 \Rightarrow 3 - x_1 = \pm 1$$

$(+)$: $x_1 = 2 \longrightarrow y_1 = 2$

$(-)$: $x_1 = 4 \longrightarrow y_1 = -2.$

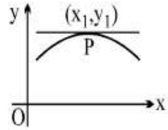
T_1: $y - 2 = 2(x-2)$

T_2: $y + 2 = 2(x-4)$ } Ans

IMPORTANT NOTES TO REMEMBER:

(1) If $\left.\frac{dy}{dx}\right|_{x_1,y_1} = 0 \Rightarrow$ tangent is parallel to x-axis and converse.

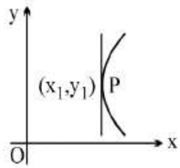
If tangent is parallel to $ax + by + c = 0 \Rightarrow \frac{dy}{dx} = -\frac{a}{b}$



(2) If $\left.\frac{dy}{dx}\right|_{x_1,y_1} \to \infty$ or $\left.\frac{dx}{dy}\right|_{x_1,y_1} = 0 \Rightarrow$ tangent is perpendicular to x-axis. / parallel to y-axis

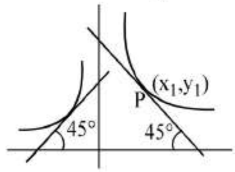
If tangent with a finite slope is perpendicular to $ax + by + c = 0$

$$\Rightarrow \left.\frac{dy}{dx}\right|_{x_1,y_1} \cdot \left(-\frac{a}{b}\right) = -1.$$

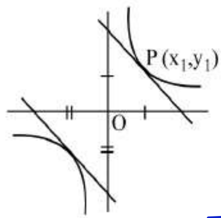


(3) If the tangent at $P(x_1, y_1)$ on the curve is equally inclined to the coordinate axes

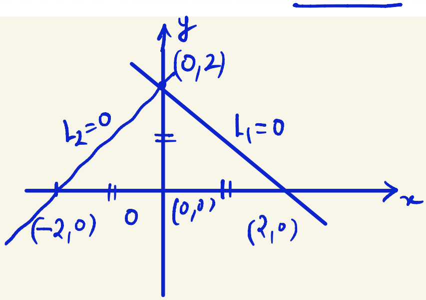
$$\Rightarrow \boxed{\left.\frac{dy}{dx}\right|_{x_1,y_1} = \pm 1.}$$



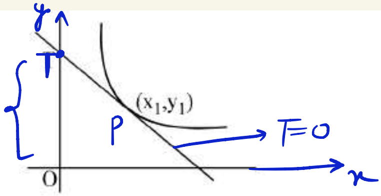
(4) If the tangent makes equal non zero intercept on the coordinate axes then $\boxed{\left.\frac{dy}{dx}\right|_{x_1,y_1} = -1}$ ✓



(5) If tangent cuts off from the coordinate axes equal distance from the origin $\Rightarrow \boxed{\frac{dy}{dx} = \pm 1.}$ ✓



(7) OT is called the initial ordinate of the tangent



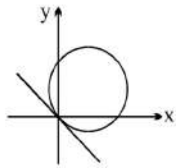
T: $Y - y = \frac{dy}{dx}(X - x)$

put $X = 0$ to get

$\therefore \quad \boxed{Y = OT = y - x\frac{dy}{dx}}$ (It is the y intercept of a tangent at P)

(8) If a curve passes through the origin, then the equation of the tangent at the origin can be directly written by equating to zero the lowest degree terms appearing in the equation of the curve.

e.g. in



(i) $x^2 + y^2 + 2gx + 2fy = 0$

equation of tangent is $gx + fy = 0$

(ii) $x^3 + y^3 - 3x^2y + 3xy^2 + x^2 - y^2 = 0$

equation of tangents at origin are $x^2 - y^2 = 0$

(iii) Equation of tangents to $x^3 + y^3 - 3xy = 0$ are $xy = 0$

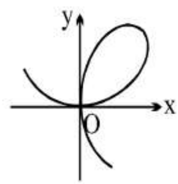
$x = 0 \Rightarrow$ y-axis

$y = 0 \Rightarrow$ x-axis.

(9) Same line could be the tangent as well as normal to a given curve at a given point.

e.g. in, $x^3 + y^3 - 3xy = 0$ (Folium of descartes)

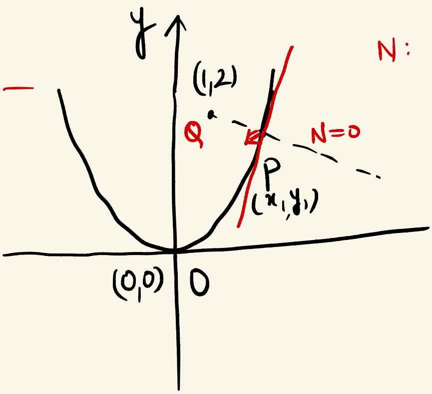
the line pair $xy = 0$ is both the tangent as well as normal at $x = 0$.



Q Equation of the normal to the curve $x^2 = 4y$ which passes through (1, 2).

Sol^n (1,2) does not lie on $x^2 = 4y$. $\xrightarrow[\text{wrt 'x'}]{\text{diff}}$ $2x = 4\frac{dy}{dx}$

$\left.\frac{dy}{dx}\right|_{(x_1,y_1)} = \frac{x_1}{2}$



N: $y - 2 = m(x - 1)$

$\frac{-1}{\left.\frac{dy}{dx}\right|_{(x_1,y_1)}} = \frac{2-y_1}{1-x_1}$

$x_1^2 = 4y_1$ —(1)—

$\frac{-2}{x_1} = \frac{2-y_1}{1-x_1}$

$-2 + \cancel{2x_1} = \cancel{2x_1} - x_1y_1$

$x_1y_1 = 2$

$x_1 \cdot \frac{x_1^2}{4} = 2 \Rightarrow \boxed{x_1 = 2} \Rightarrow \boxed{y_1 = 1}$

N: $\boxed{x + y = 3}$ Ans

Q

Tangent to the curve $y = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$ at $x = \sqrt{3}$

Sol^n

for $x > 1$

$$y = \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \pi - 2\tan^{-1}x$$

when $x = \sqrt{3}$ then $y = \pi - 2\tan^{-1}\sqrt{3}$

$$= \pi - 2\cdot\left(\frac{\pi}{3}\right) = \frac{\pi}{3}$$

$$\left.\frac{dy}{dx}\right|_{\left(\sqrt{3}, \frac{\pi}{3}\right)} = 0 - 2\left.\left(\frac{1}{1+x^2}\right)\right|_{x=\sqrt{3}} = \frac{-2}{4} = \frac{-1}{2}.$$

T: $\boxed{y - \frac{\pi}{3} = \frac{-1}{2}\left(x - \sqrt{3}\right)}$ Ans

Q

Find equation of normal to the curve $y(x) = x + \int_x^0 y(t)\,dt$ at the point where it crosses x-axis.
also find $y(2)$?

i.e $y = 0$.
in given equ
then $x = 0$.
$P(0,0)$

Sol^n

Normal at $(0,0) \rightarrow$?

$$y(x) = x + \int_x^0 y(t)\,dt$$

diff wrt 'x' :-

$$y'(x) = 1 + (0 - y(x))$$

$$\frac{dy}{dx} = 1 - y \Rightarrow \int \frac{dy}{1-y} = \int dx$$

$$-\ln(1-y) = x + c$$

$$\ln(1-y) = -x + c_1 \qquad \underline{e^{c_1} = K}$$

$$1 - y = e^{-x+c_1}$$

$$\boxed{y = 1 - Ke^{-x}}$$

$(0,0)$

$$0 = 1 - K \Rightarrow \boxed{K = 1}$$

Curve : $\boxed{y = 1 - e^{-x}} \xrightarrow[\text{wrt } 'x']{\text{diff}} \left.\frac{dy}{dx}\right|_{(0,0)} = \left. e^{-x}\right|_{x=0}$

N: $\boxed{y - 0 = -1(x - 0)}$ $\quad \boxed{m_N = -1} \Leftarrow \boxed{m_T = 1}$

Q

A differentiable function $y = f(x)$ is defined by a functional rule $f(a + b) = f(a) + f(b) + 2ab - 1$ $\forall$ a,b $\in$ R, $f'(1) = 2$, then find equation of tangent to $y = f(x)$ whose slope is 2.

Sol^n

$$f'(x) = \lim_{h\to 0}\left(\frac{f(x+h)-f(x)}{h}\right) = \lim_{h\to 0}\left(\frac{\cancel{f(x)}+f(h)+2xh-1-\cancel{f(x)}}{h}\right)$$

$$f'(x) = \underbrace{\lim_{h\to 0}\left(\frac{f(h)-\overset{f(0)}{1}}{h}\right)}_{f'(0)} + 2x$$

$f(a+b) = f(a)+f(b)+2ab-1$

$a=b=0$

$\cancel{f(0)} = \cancel{f(0)} + f(0) - 1$

$$\boxed{f(0)=1}$$

$$\boxed{f'(x) = f'(0) + 2x}$$

$x=1$

$f'(1) = f'(0) + 2 \Rightarrow \boxed{f'(0) = 0}$

$$\boxed{f'(x) = 2x} \xrightarrow{\text{integrate}} \boxed{f(x) = x^2 + c}$$

$$\boxed{1 = c}$$

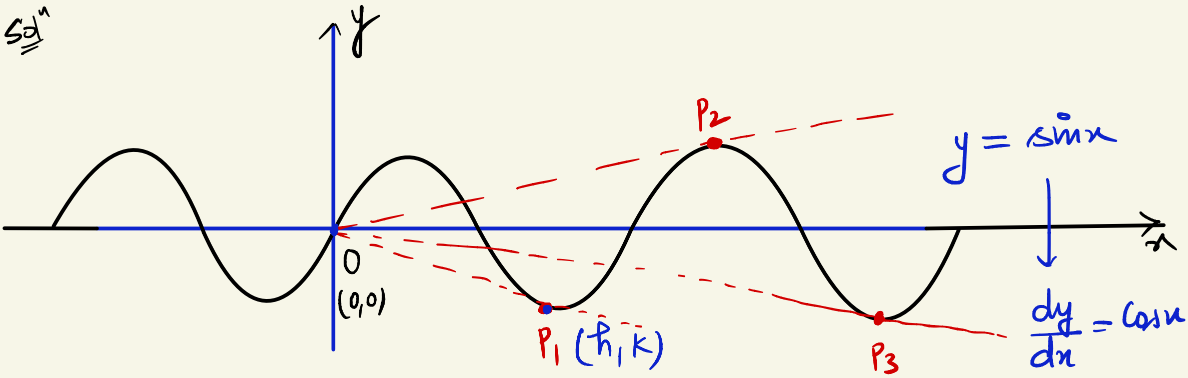
$$\therefore \boxed{f(x) = x^2+1} \longrightarrow f'(x)\Big|_{(x_1,y_1)} = 2x_1 = 2$$

P: $\boxed{x_1 = 1}$ $\boxed{y_1 = 2}$

T: $\boxed{y-2 = 2(x-1)} \Rightarrow \boxed{y = 2x}$ Ans

Q Tangent are drawn from the origin to the curve y = sin x. Prove that their point of contact lie on the curve $x^2y^2 = x^2 - y^2$.

Sol^n



$$\boxed{k = \sin h} \quad —① —$$

$$m_T = \frac{k-0}{h-0} = \left.\frac{dy}{dx}\right|_{(h,k)} \Rightarrow \boxed{\frac{k}{h} = \cos h} \quad — ② —$$

square & add

$$k^2 + \frac{k^2}{h^2} = \sin^2 h + \cos^2 h = 1$$

$$\frac{k^2h^2 + k^2}{h^2} = 1$$

$$\boxed{x^2y^2 = x^2 - y^2}$$

(H.P)

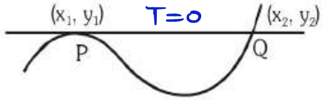
EQUATION OF TANGENT/NORMAL IN SOME SPECIAL CASES :

(a) Parametric form : If the equation of the curve is represented parametrically i.e. $x = f(t)$ and $y = g(t)$ where $\frac{dx}{dt} = f'(t)$ and $\frac{dy}{dt} = g'(t)$, $\{f'(t) \neq 0\}$ then $\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{g'(t)}{f'(t)}$

and the equation of the tangent is $y - g(t) = \frac{dy/dt}{dx/dt}\{(x - f(t))\}$.

** **

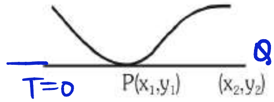
(b) If the tangent at P meeting the curve $y = f(x)$ again at Q.



$\Rightarrow \left.\frac{dy}{dx}\right|_P = \frac{y_2 - y_1}{x_2 - x_1}$ ...... (i) & $y_1 = f(x_1)$ ....(ii)

from (i) & (ii) we get equation of tangent.

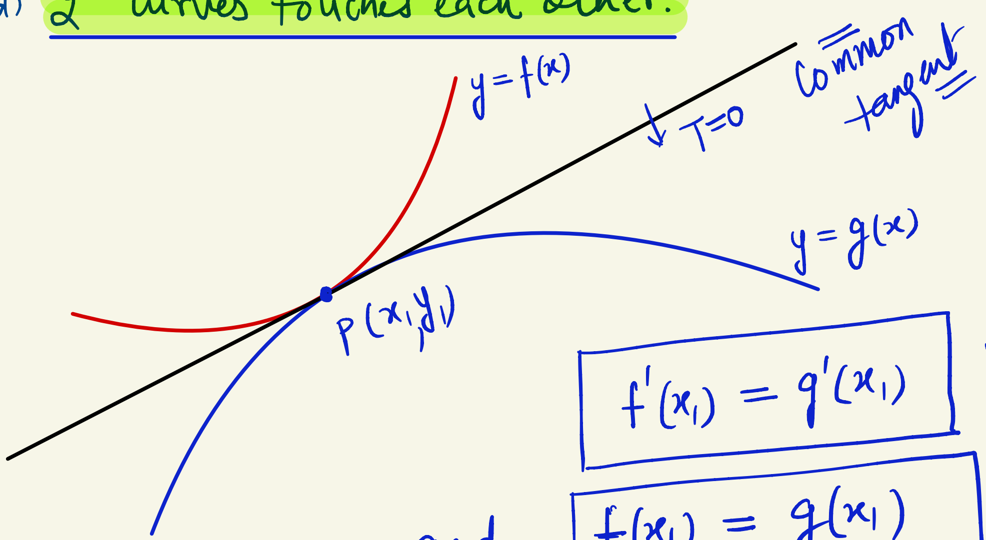
(c) If from external point tangent is drawn to the curve



$\Rightarrow \left.\frac{dy}{dx}\right|_P = \frac{y_2 - y_1}{x_2 - x_1}$ ...... (i) & $y_1 = f(x_1)$ ....(ii)

from (i) & (ii) we get equation of tangent.

(d) 2 curves touches each other.



$$\boxed{f'(x_1) = g'(x_1)}$$ ✓

and $$\boxed{f(x_1) = g(x_1)}$$ ✓

Q

Find equation of tangent to the curve $y = f(x)$ defined parametricaly $x = \dfrac{2t^2}{1+t^2}, y = \dfrac{2t^3}{1+t^2}$ at $\left(\dfrac{2}{5}, \dfrac{1}{5}\right)$.

$\downarrow x \quad \downarrow y$

Sol^n $\quad x = \dfrac{2t^2}{1+t^2} \quad ; \quad y = \dfrac{2t^3}{1+t^2}$

$$\boxed{\frac{y}{x} = t} \Rightarrow \frac{1/5}{2/5} = t \Rightarrow t = \frac{1}{2}$$

$$m_T = \left.\frac{dy}{dx}\right|_{t=\frac{1}{2}} = \left.\frac{dy/dt}{dx/dt}\right|_{t=1/2} = \frac{13}{16}.$$

$$T: \quad y - \frac{1}{5} = \frac{13}{16}\left(x - \frac{2}{5}\right)$$ Ans

Q Find equation of tangent to the curve $y = f(x)$ at $t = 0$ defined parametrically as

$$x = \begin{cases} \dfrac{e^t - 1}{t} & t \neq 0 \\ 1 & t = 0 \end{cases}, \qquad y = \begin{cases} \dfrac{1 - \cos t}{t} & t \neq 0 \\ 0 & t = 0 \end{cases}$$

Sol^n

$$\left.\frac{dy}{dx}\right|_{t=0} = \left.\frac{dy/dt}{dx/dt}\right|_{t=0}$$

** $$\frac{dx}{dt} = \lim_{t\to 0}\left(\frac{\frac{e^t-1}{t} - 1}{t}\right) = \lim_{t\to 0}\left(\frac{e^t - t - 1}{t^2}\right) = \frac{1}{2}$$

** $$\frac{dy}{dt} = \lim_{t\to 0}\left(\frac{\frac{1-\cos t}{t} - 0}{t}\right) = \lim_{t\to 0}\left(\frac{1-\cos t}{t^2}\right) = \frac{1}{2}$$

$$\boxed{\frac{dy}{dx} = 1}$$

$$T : \boxed{y - 0 = 1(x - 1)}$$ Ans

Q A curve in the plane is defined by the parametric eqautions $x = e^{2t} + 2e^{-t}$ and $y = e^{2t} + e^{t}$. An equation for the line tangent to the curve at the point $t = \ln 2$ is

HW

Q

Let the parabola $y = x^2 + ax + b$ and $y = x(c - x)$ touches each other at point (1,0), then find the a, b & c.

Sol^n

(1,0) will lie on both curves.

$0 = 1 + a + b$ —①— $\Rightarrow \boxed{a + b = -1}$

$0 = 1(c-1) \Rightarrow \boxed{c = 1}$ *

$2 + a = -1$

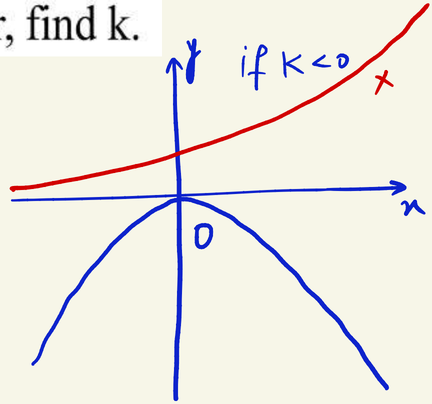
$\therefore \boxed{a = -3}$

$\boxed{b = 2}$

$$\left.\frac{dy}{dx}\right|_{(1,0)}^{C_1} = (2x + a)\Big|_{(1,0)} = 2 + a$$

$$\left.\frac{dy}{dx}\right|_{(1,0)}^{C_2} = (c - 2x)\Big|_{(1,0)} = c - 2 = -1$$

Q If $y = e^x$ and $y = kx^2$ touches each other, find k.



Sol^n $C_1 : y = e^x$; $C_2 : y = kx^2$

$\boxed{k>0}$

$k = 0$ XX

$C_2 : y = 0 \Rightarrow$ x-axis

Let $P(x_1, y_1)$ be the point where C_1 & C_2 touches each other

$$\left.\frac{dy}{dx}\right|_P^{C_1} = \left.\frac{dy}{dx}\right|_P^{C_2}$$

$$\boxed{e^{x_1} = 2kx_1} \quad -①-$$

$y_1 = e^{x_1}$

$y_1 = kx_1^2$

$$\boxed{e^{x_1} = kx_1^2} \quad -②-$$

$kx_1^2 = 2kx_1$

$$\underbrace{kx_1}_{\neq 0}(x_1 - 2) = 0 \Rightarrow \boxed{x_1 = 2} \, *$$

$$k = \frac{e^{x_1}}{x_1^2} = \frac{e^2}{4}$$

Ans

Q HW

If tangent to the curve $xy = x^2 + 1$ at (α,β) is normal to the curve $x^2 + y^2 + 2fy + c = 0$, then $|f.\alpha|$ is

Q[HW] If $y = 1 + \frac{x^2}{a^3}$ and $y = 4\sqrt{x}$ have only a common point then find 'a' ?

Some Common Parametric Representation of Curves :-

① $y^2 = 4ax \longrightarrow x = at^2;\ y = 2at.$

② $x^2 + y^2 = r^2 \longrightarrow x = r\cos\theta;\ y = r\sin\theta.$

③ $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1 \longrightarrow x = a\cos\theta;\ y = b\sin\theta$

④ $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1 \longrightarrow x = a\sec\theta;\ y = b\tan\theta.$

⑤ $y^2 = x^3 \longrightarrow x = t^2\ \&\ y = t^3.$

$y^3 = x^2 \longrightarrow y = t^2\ \&\ x = t^3.$

⑥ $xy = c^2 \longrightarrow x = ct\ \&\ y = \dfrac{c}{t}.$

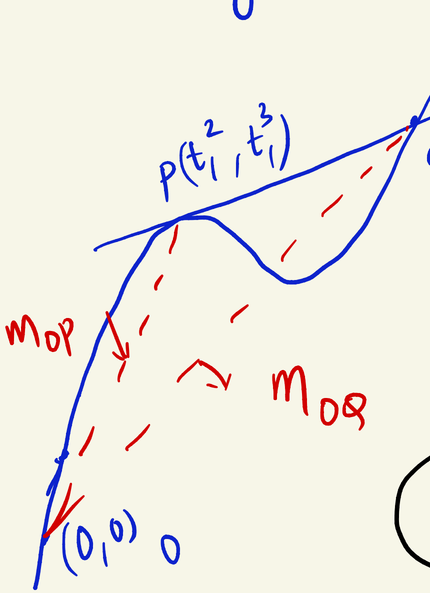
⑦ $\sqrt{x} + \sqrt{y} = \sqrt{a}\ ;\ \left\{\begin{array}{l} x = a\cos^4\theta \\ y = a\sin^4\theta. \end{array}\right\}$

Q If the tangent drawn at a point P on curve $y^2 = x^3$ meets it again at Q then find $\left(\frac{m_{OP}}{m_{OQ}}\right)$ where O is origin & m denotes slope of line

Sol^n

$$y^2 = x^3 \longrightarrow P(t_1^2, t_1^3)$$
$$Q(t_2^2, t_2^3)$$

$$** \; t_1 \neq t_2$$



$$m_{OP} = \frac{t_1^3 - 0}{t_1^2 - 0} = t_1$$

$$m_{OQ} = t_2$$

$$\frac{m_{OP}}{m_{OQ}} = \frac{t_1}{t_2} = ?$$

$$y^2 = x^3$$
$$\downarrow \text{diff}$$
$$2y \cdot \frac{dy}{dx} = 3x^2$$
$$\left.\frac{dy}{dx}\right|_P = \frac{3x_1^2}{2y_1}$$

$$\left.\frac{dy}{dx}\right|_P = \frac{3t_1^{\cancel{4}}}{2\cancel{t_1^3}} = \frac{3t_1}{2}$$

$$\left.\frac{dy}{dx}\right|_P = \frac{t_2^3 - t_1^3}{t_2^2 - t_1^2} = \frac{\cancel{(t_2 - t_1)}(t_2^2 + t_1^2 + t_1 t_2)}{\cancel{(t_2 - t_1)}(t_1 + t_2)}$$

$$\frac{3t_1}{2} = \frac{t_1^2 + t_2^2 + t_1 t_2}{(t_1 + t_2)}$$

$$3t_1(t_1+t_2) = 2t_1^2+2t_2^2+2t_1t_2$$

$$t_1^2+t_1t_2-2t_2^2=0$$

$$t_1^2+2t_1t_2-t_1t_2-2t_2^2=0$$

$$t_1(t_1+2t_2)-t_2(t_1+2t_2)=0$$

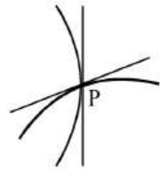
$$\underset{\neq 0}{(t_1-t_2)}(t_1+2t_2)=0$$

$$\boxed{\frac{t_1}{t_2}=-2}$$

Ans

Q HW If the tangent at any point $P(4m^2, 8m^3)$ of $x^3 - y^2 = 0$ is normal to the curve $x^3 - y^2 = 0$ then find the value of $9m^2$?

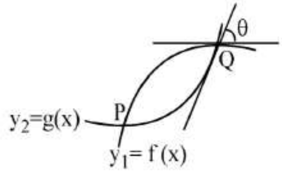
ANGLE OF INTERSECTION OF TWO CURVES :



Definition **:** The angle of intersection of two curves at a point P is defined as the angle between the two tangents to the curve at their point of intersection.

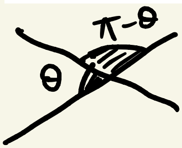
If the curves are orthogonal then

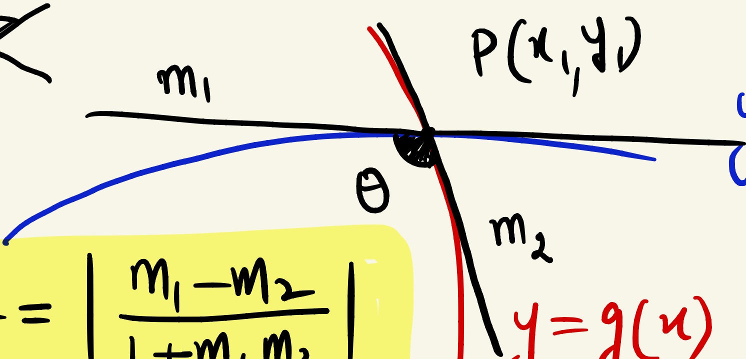
$$\left(\frac{dy_1}{dx}\right)\left(\frac{dy_2}{dx}\right) = -1 \text{ everywhere whereever they intersect.}$$



If $\left(\frac{dy_1}{dx}\right)_P\left(\frac{dy_2}{dx}\right)_P = -1$ but $\left(\frac{dy_1}{dx}\right)_Q\left(\frac{dy_2}{dx}\right)_Q \neq -1$

then the two curves are orthogonal at P but not at Q hence they are not orthogonal.





$$\begin{cases} m_1 = f'(x_1) \\ m_2 = g'(x_1) \end{cases}$$

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$\theta \to$ acute ; $\pi - \theta \to$ obtuse

Q Check whether the curves are orthogonal or not

(i) $y^2 = 4ax$ and $y = e^{-x/2a}$.

HW
(ii) $xy = c^2$ and $x^2 - y^2 = a^2$. ✓

HW
(iii) $y^2 = 4ax$ and $x^2 = 4by$. ✓

Note: $\tan\theta = \left| \dfrac{m_1 - m_2}{1 + m_1 m_2} \right|$

suppose $m_1 \to \infty$.

$$\tan\theta = \left| \frac{\cancel{m_1}\left(1 - \frac{m_2}{m_1}\right)}{\cancel{m_1}\left(\frac{1}{m_1} + m_2\right)} \right|$$

Rem

$$\boxed{\tan\theta = \left| \frac{1}{m_2} \right|}$$

$\theta \to$ acute

$\pi - \theta \to$ obtuse

① $y^2 = 4ax$ and $y = e^{-x/2a}$

Let $P(x_1, y_1)$ be the general point where 2 curves intersect.

$y_1^2 = 4ax_1$ and $y_1 = e^{-x_1/2a}$ **

$$y^2 = 4ax \xrightarrow[\text{wrt } 'x']{\text{diff}} 2y \cdot \frac{dy}{dx} = 4a$$

$$m_1 = \left.\frac{dy}{dx}\right|_P = \frac{2a}{y_1}$$

$$y = e^{-\frac{x}{2a}} \xrightarrow[\text{wrt } 'x']{\text{diff}} m_2 = \left.\frac{dy}{dx}\right|_P = e^{-\frac{x_1}{2a}} \cdot \left(\frac{-1}{2a}\right)$$

$$m_1 m_2 = \left(\frac{\cancel{2a}}{y_1}\right)\left(e^{-\frac{x_1}{2a}} \cdot \frac{-1}{\cancel{2a}}\right) = -\left(\frac{e^{-\frac{x_1}{2a}}}{y_1}\right)$$

Orthogonal curves $\Leftarrow$ $\boxed{m_1 m_2 = -1}$ *

Q

Find the angle between the curve $2y^2 = x^3$ and $y^2 = 32x$ at the point of intersection in 1^{st} quadrant ?

Sol^n

$2(32x) = x^3 \Rightarrow x(x^2-64) = 0$

$x=0$; $x=8$; $x=-8$

$y^2 = 32(8)$

$y^2 = 2^5 \cdot 2^3 = 2^8$

$y^2 = (16)^2 \Rightarrow \boxed{y=16}$

$y = -16$

$\therefore$ Point of intersection in 1^{st} Quad $(8,16)$ P

$C_1 : 2y^2 = x^3$

diff wrt 'x'

$2(2y \cdot y') = 3x^2$

$m_1 = y'\Big|_{(8,16)} = \dfrac{3x^2}{4y} = \dfrac{3 \times \cancel{64}}{\cancel{4 \times 16}} = 3$

$\boxed{m_1 = 3}$

C_2: $y^2 = 32x$

diff wrt 'x' :-

$$2y \cdot \frac{dy}{dx} = 32 \Rightarrow \left.\frac{dy}{dx}\right|_{(8,16)} = \frac{16}{16} = 1$$

$$\boxed{m_2 = 1}$$

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$$

$$\tan\theta = \left|\frac{3-1}{1+3}\right| = \frac{1}{2}$$

$\theta = \tan^{-1}\left(\frac{1}{2}\right) \longrightarrow$ acute angle

$\pi - \theta = \pi - \tan^{-1}\left(\frac{1}{2}\right) \longrightarrow$ obtuse angle

Q

Prove that the angle between the curves $y^2 = x$ and $x^3 + y^3 = 3xy$ at the point other than origin is $\tan^{-1}(16)^{1/3}$.

If the curve $y^2 = 4ax$, $a > 0$ cuts the curve $xy = b$, $b > 0$ at right angle, then find the value of $\dfrac{b^2}{a^4}$

Compute the intervals of monotonicity for the following :

(i) $f(x) = x^2(x-2)^2$

(ii) $f(x) = x + \ell n(1-4x)$

(iii) $f(x) = x^2 e^{-x}$

(iv) $f(x) = 3\cos^4 x + 10\cos^3 x + 6\cos^2 x - 3$ in $[0, \pi]$

(ii) $f(x) = x + \ln(1-4x)$

Domain: $1-4x > 0 \Rightarrow 4x-1 < 0 \Rightarrow \boxed{x < \frac{1}{4}}$ — (1) **

$$f'(x) = 1 + \frac{(-4)}{1-4x} = \frac{1-4x-4}{1-4x} = \frac{-4x-3}{1-4x}$$

$$f'(x) = \frac{4x+3}{4x-1}$$

+ $\quad -\frac{3}{4} \quad$ − $\quad \frac{1}{4} \quad$ X+XX

$\uparrow$ in $\left(-\infty, -\frac{3}{4}\right]$

$\downarrow$ in $\left[-\frac{3}{4}, \frac{1}{4}\right)$

(iii) $f(x) = x^2 e^{-x}$; $D_f \in R$

$$f'(x) = 2x e^{-x} + x^2(-e^{-x})$$

$$= e^{-x} . x(2-x)$$

$$f'(x) = -\underbrace{e^{-x}}_{+ve} \cdot x(x-2)$$

$$\begin{array}{ccccc} - & & + & & - \\ \hline & 0 & & 2 & \end{array}$$

$\uparrow$ in $[0, 2]$

$\downarrow$ in $(-\infty, 0]$; $[2, \infty)$

(iv) $f(x) = 3\cos^4 x + 10\cos^3 x + 6\cos^2 x - 3$ in $[0, \pi]$

$$f'(x) = 12\cos^3 x(-\sin x) + 30\cos^2 x(-\sin x)$$
$$+ 12\cos x(-\sin x)$$

$$f'(x) = 6\cos x \sin x\,(-2\cos^2 x - 5\cos x - 2)$$

$$= -6\underbrace{\cos x}\,\underbrace{\sin x}\,(\underbrace{\cos x + 2})(\underbrace{2\cos x + 1})$$

$$\begin{array}{ccccccc} \bullet & - & & + & & - & \bullet \\ \hline 0 & & \frac{\pi}{2} & & \frac{2\pi}{3} & & \pi \end{array}$$

$\downarrow$ in $[0, \pi/2]$; $\left[\frac{2\pi}{3}, \pi\right]$ & $\uparrow$ in $\left[\frac{\pi}{2}, \frac{2\pi}{3}\right]$

Q

Prove that $f(x) = \frac{2}{3}x^9 - x^6 + 2x^3 - 3x^2 + 6x - 1$ is always strictly increasing.

Sol^n $D_f \in R$

$$f'(x) = \frac{2}{3} \cdot 9x^8 - 6x^5 + 6x^2 - 6x + 6$$

$$= 6\left(x^8 - x^5 + x^2 - x + 1\right)$$

C–I $x < 0$

$f'(x) > 0.$ ✓

C–II $0 \le x \le 1$

$$f'(x) = 6\left(x^8 + (x^2 - x^5) + (1 - x)\right)$$

$f'(x) > 0$ ✓

C–III $x > 1$

$$f'(x) = 6\left((x^8 - x^5) + (x^2 - x) + 1\right)$$

$f'(x) > 0$ ✓

$\therefore f'(x) > 0 \ \forall x \in \mathbb{R}$

Q

If the function $f(x) = (a + 2)x^3 - 3ax^2 + 9ax - 1$ is always strictly decreasing $\forall\, x \in R$, find 'a'.

Sol^n $f'(x) = 3(a+2)x^2 - 6ax + 9a \le 0 \quad \forall x \in R$

$$\left.\begin{array}{l} a+2 < 0 \quad \text{— (1) —} \\ D \le 0 \quad \text{— (2) —} \end{array}\right\} \cap$$

Q/

*(i) $f(x) = 4 - e^x$, find x satisfying $4 - e^{f(x)} > f(e^x)$

*(ii) If f is strictly decreasing with domain $(0, \infty)$, then solve $f(x^2 - 4) > f(-3x)$

(i) $D \in R$

$f(x) = 4 - e^x \xrightarrow{\text{diff}} f'(x) = -e^x < 0 \Rightarrow f$ is $\downarrow$ fn.

$\underbrace{4 - e^{f(x)}}_{\downarrow} > f(e^x) \Rightarrow f(f(x)) > f(e^x).$

$f(f(x))$

$\Downarrow$

$f(x) < e^x. \Rightarrow 4 - e^x < e^x$

$4 < 2e^x \Rightarrow e^x > 2.$

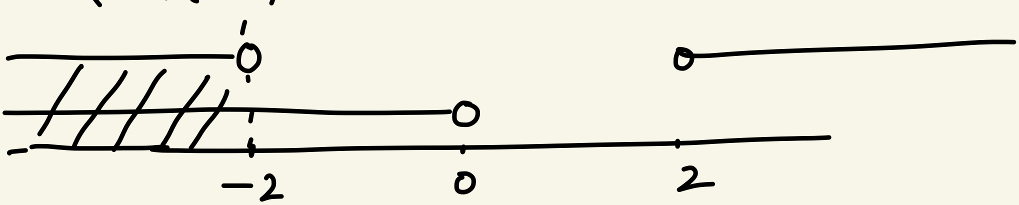
$\ln e^x > \ln 2$

$\boxed{x > \ln 2}$ Ans

(ii) $f(x^2 - 4) > f(-3x)$

$x^2 - 4 > 0$ and $-3x > 0$

$(x-2)(x+2) > 0$ and $x < 0$



$\boxed{x \in (-\infty, -2)}$ — ① —

$$f(x^2-4) > f(-3x)$$

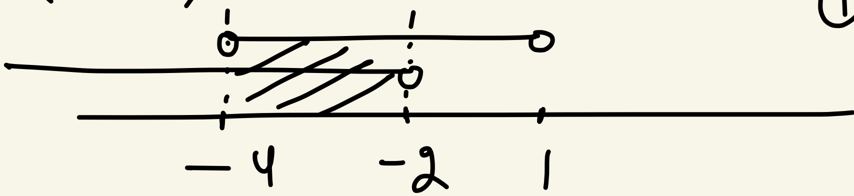
$$x^2-4 < -3x$$

$$x^2+3x-4 < 0$$

$$x^2+4x-x-4 < 0$$

$$(x+4)(x-1) < 0 \quad — ② —$$

$① \cap ②$



$$\therefore \quad x \in (-4, -2)$$ Ans

Q. If $f(x) = 2e^{x} + (2a+1)x - a e^{-x}$ is monotonically increasing $\forall x \in \mathbb{R}$ then find range of 'a' ?

Sol^n

$$f'(x) = 2e^{x} + (2a+1) + ae^{-x} \geq 0 \quad \forall x \in \mathbb{R}$$

$$f'(x) = 2(a+e^{x}) + 1 + ae^{-x}$$

$$= 2(a+e^{x}) + e^{-x}(e^{x}+a)$$

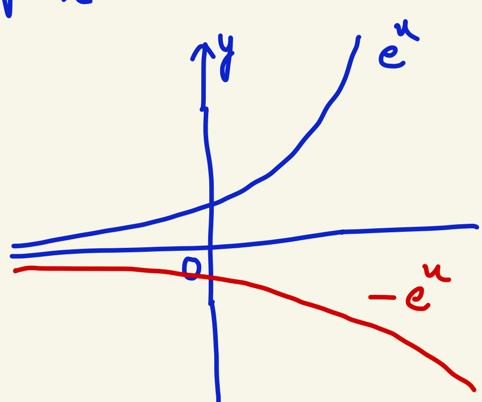
$$f'(x) = \underbrace{(2+e^{-x})}_{>0}\underbrace{(a+e^{x})}_{\geq 0} \geq 0 \quad \forall x \in \mathbb{R}$$

$$\therefore \quad a+e^{x} \geq 0 \quad \forall x \in \mathbb{R}$$

$$a \geq -e^{x} \quad \forall x \in \mathbb{R}$$

$$\therefore \boxed{a \geq 0}$$

$$-e^{x} \in (-\infty, 0)$$



Q Find number of solutions of equations

(i) $\int_0^x \sec^4 t\,dt = \frac{x+1}{3}$ in $x \in (0, 1)$

(ii) $x\sin x + \cos x = \frac{3}{2}$ in $x \in (0, \pi)$

(iii) $\frac{x^3+1}{x^2+1} = 5$ in $[0, 2]$

HW

(iv) $\frac{1}{(1+x)^3} = 3x - \sin x$

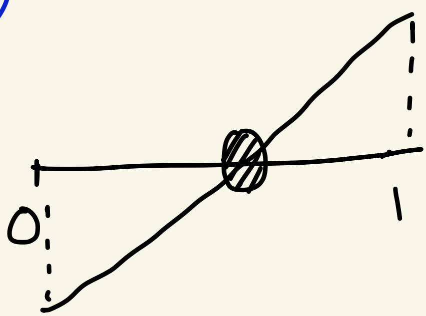
(i) $\int_0^x \sec^4 t\,dt - \left(\frac{x+1}{3}\right) = 0.$ in $(0,1)$

$$f(x) = \int_0^x \sec^4 t\,dt - \left(\frac{x+1}{3}\right)$$

$$f'(x) = \underbrace{\sec^4 x}_{>1} - \frac{1}{3} > 0 \Rightarrow f \text{ is } \uparrow \text{ in } (0,1).$$

$$f(0) = \cancel{\int_0^0 \sec^4 t\,dt}^{\,0} - \frac{1}{3} = -\text{ve}.$$

$$f(1) = \underbrace{\int_0^1 \underbrace{\sec^4 t}_{>1}\,dt}_{>1} - \left(\frac{2}{3}\right) = +\text{ve}$$



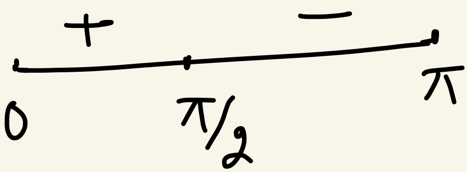
$\therefore$ 1 soln in $(0,1)$ Ans

(ii) $x\sin x + \cos x = \frac{3}{2}$ in $(0, \pi)$.

$$f(x) = x\sin x + \cos x - \frac{3}{2}.$$

$$f'(x) = x\cos x + \cancel{\sin x} - \cancel{\sin x}$$

$$f'(x) = \underbrace{x}_{>0}\,\underbrace{\cos x}_{\downarrow}$$

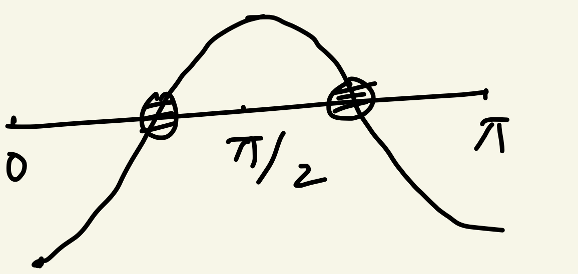


$$f(0) = 0 + 1 - \frac{3}{2} = -\frac{1}{2} < 0.$$

$$f(\pi/2) = \frac{\pi}{2}(1) + 0 - \frac{3}{2} > 0.$$

$$f(\pi) = 0 + (-1) - \frac{3}{2} < 0.$$

$\therefore$ 2 sol^n in $(0, \pi)$



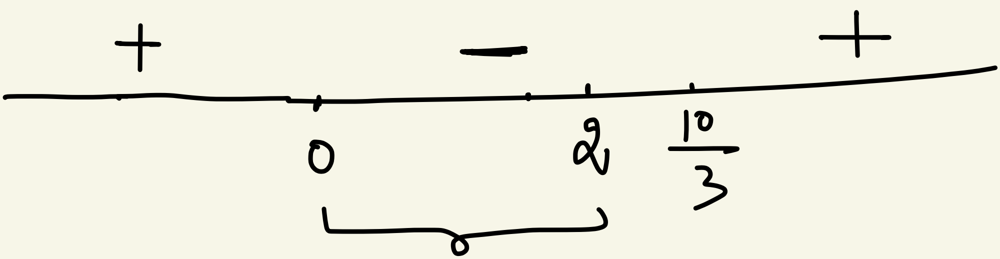
(iii) $\dfrac{x^3+1}{x^2+1} = 5$ in $[0,2]$

$$x^3+1 = 5x^2+5$$

$$f(x) = x^3-5x^2-4 \quad \text{in } [0,2].$$

$$f'(x) = 3x^2-10x$$

$$= 3x\left(x-\frac{10}{3}\right)$$



$\therefore$ $f'(x) < 0$ in $(0,2]$

f is $\downarrow$ in $[0,2]$

$f(0) = -4$

$\therefore$ No sol^n in $[0,2]$. Ans

Q

Consider the function,

$$f(x) = x^3 - 9x^2 + 15x + 6 \text{ for } 1 \le x \le 6 \text{ and } g(x) = \begin{cases} \min. f(t) & \text{for } 1 \le t \le x,\ 1 \le x \le 6 \\ x - 18 & \text{for } x > 6 \end{cases}$$

then which of the following hold(s) good?

(A) g(x) is differentiable at x = 1

(B) g(x) is discontinuous at x = 6

(C) g(x) is continuous and derivable at x = 5

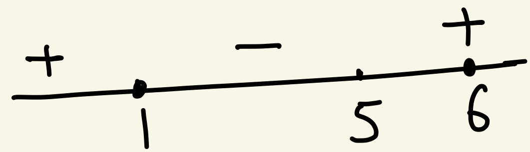
(D) g(x) is monotonic in (1, 5)

Sol^n

$\boxed{f(x) = x^3 - 9x^2 + 15x + 6}$; $1 \le x \le 6.$

$f'(x) = 3x^2 - 18x + 15$

$= 3(x^2 - 6x + 5) = 3(x-1)(x-5)$

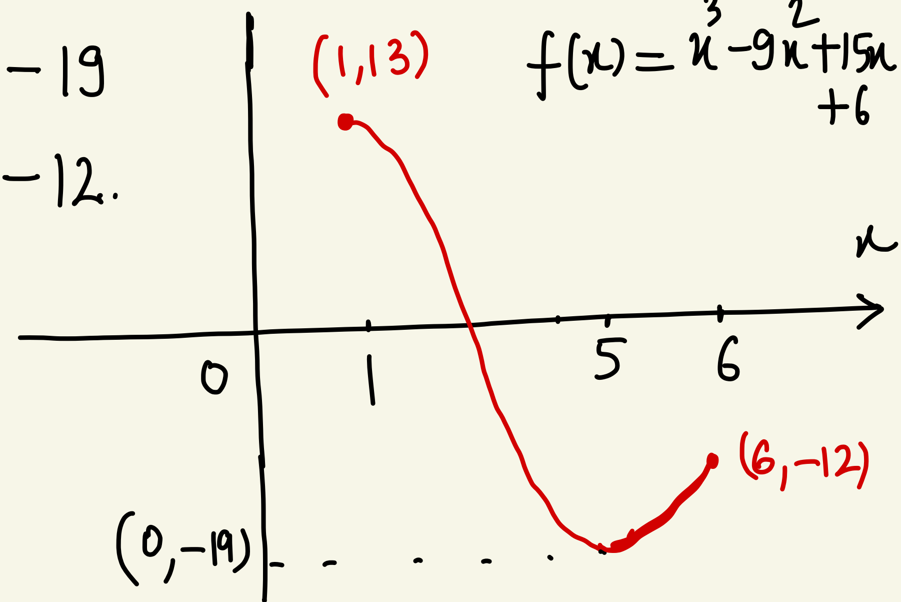


$f(1) = 1 - 9 + 15 + 6 = 13$

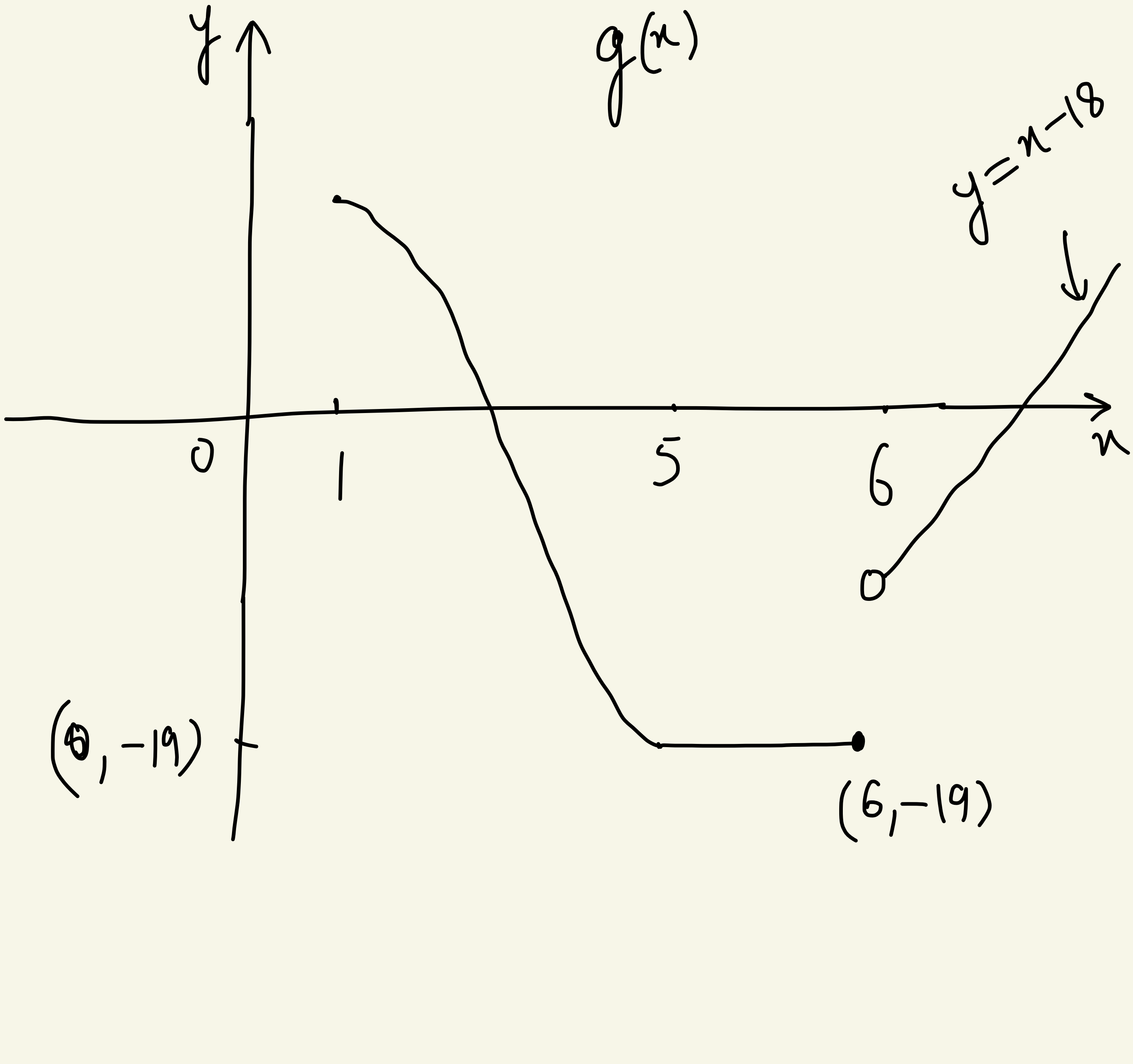
$f(5) = -19$

$f(6) = -12.$

$f(x) = x^3 - 9x^2 + 15x + 6$



$$g(x) \rightarrow \begin{cases} f(x) & ; \quad 1 \le x \le 5 \\ -19 & ; \quad 5 < x \le 6 \\ (x-18) & ; \quad x > 6 \end{cases}$$



GREATEST AND LEAST VALUE OF A FUNCTION :

(a) If a continuous function $y = f(x)$ is increasing in the closed interval [a, b] then $f(a)$ is the least value and $f(b)$ is the greatest value of $f(x)$ in [a, b] (figure-1)

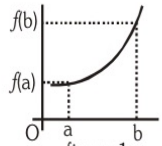


figure-1

(b) If a continuous function $y = f(x)$ is decreasing in [a, b] then $f(b)$ is the least and $f(a)$ is the greatest value of $f(x)$ in [a, b]. (figure-2)

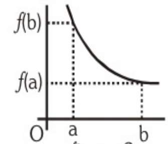


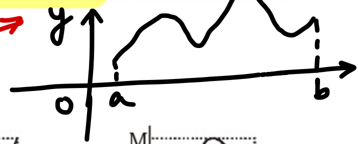
figure-2

(c) If function is continuous & non-monotonic in [a, b], then greatest & least value of function exist where

(i) $f'(x)$ is zero

(ii) $f'(x)$ does not exist

(iii) or at the end pts. of the interval



(d) If a continuous function $y = f(x)$ has least value m and greatest value M over the interval [a, b] then range is [m, M] (figure-3 & 4)

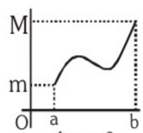


figure-3

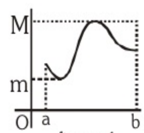


figure-4

(e) In an open interval, a continuous function need not have either a maximum or a minimum value. The function $f(x) = x$ has neither a largest nor a smallest value in (0, 1). (figure-5 & 6)

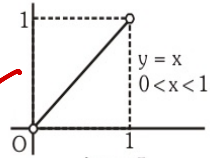


figure-5
(monotonic)

$R_f \in (0,1)$

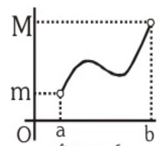
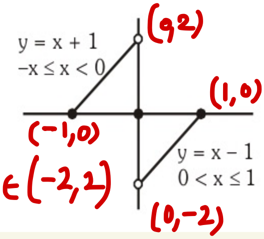


figure-6
(non-monotonic)

e.g. If $f(x) = \begin{cases} x+1, & -1 \le x < 0 \\ 0 & \text{if } x = 0 \\ x-1, & \text{if } 0 < x \le 1 \end{cases}$, then check the greatest & least value in [−1, 1]



$R_f \in (-2,2)$

Note : Even a single point of discontinuity can prevent a function from taking greatest or least value in a closed interval.

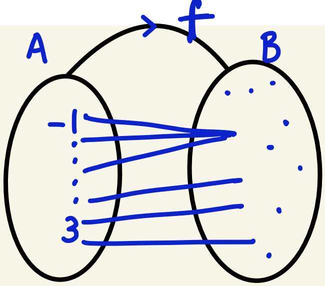
V. Imp

Note: If a function is discontinuous then we generally draw graph to determine greatest and least value.

Q

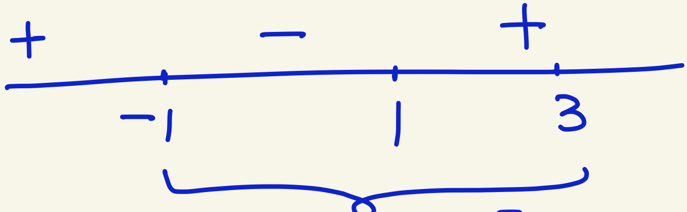
Find the image of interval [−1, 3] under the mapping specified by the function $f(x) = 4x^3 - 12x$.

Solⁿ



Range of $f(x)$ in $[-1, 3]$

$$f'(x) = 12x^2 - 12 = 12(x-1)(x+1)$$



$f(x)$ is Non-monotonic in $[-1, 3]$

$$f(1) \rightarrow \text{least} \Rightarrow f(1) = 4 - 12 = -8$$

$$f(-1) = -4 + 12 = 8$$

$$f(3) = 4(27) - 12(3) = 108 - 36 = 72$$

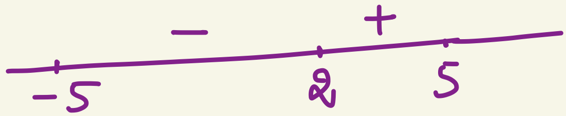
$$R_f \in [-8, 72]$$ Ans

Q. Find the greatest and least value of the following function:

① $f(x) = e^{x^2-4x+3}$ in $[-5, 5]$

$f(x) \to$ Continuous.

$$f'(x) = \underbrace{e^{x^2-4x+3}}_{>0} \cdot (2x-4)$$



$f(-5) = e^{48}$

$f(2) = e^{-1}$

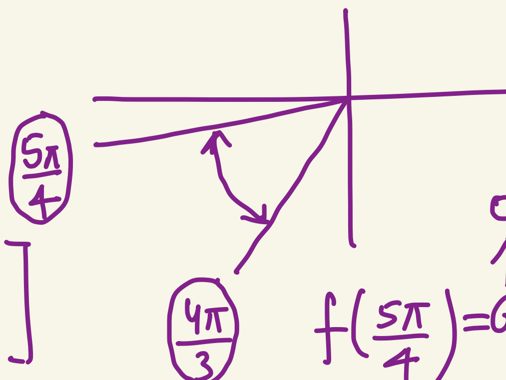
$f(5) = e^{8}$

$G = e^{48}$

$L = e^{-1}$ Ans

(2) $y = \int_{\frac{5\pi}{4}}^{x} (3\sin t + 4\cos t)\, dt$ in $\left[\frac{5\pi}{4}, \frac{4\pi}{3}\right]$

$$\frac{dy}{dx} = 3\sin x + 4\cos x$$



$\frac{dy}{dx} < 0$ in $\left[\frac{5\pi}{4}, \frac{4\pi}{3}\right]$

0 ↑

$f\left(\frac{5\pi}{4}\right) = G$

$\therefore f(x)$ is $\downarrow$ in $\left[\frac{5\pi}{4}, \frac{4\pi}{3}\right]$

$f\left(\frac{4\pi}{3}\right) = L$

$$f(4\pi/3) = \int_{5\pi/4}^{4\pi/3} (3\sin t + 4\cos t)\, dt = \frac{3}{2} + \frac{1}{\sqrt{2}} - 2\sqrt{3}$$

Ans

(iii) $f(x) = \cos 3x - 15\cos x + 8$ in $\left[\frac{\pi}{3}, \frac{3\pi}{2}\right]$

HW

Q Which of the following is (are) TRUE?

(A) $e^{\pi} > \pi^{e}$

(B) $(2020)^{2021} > (2021)^{2020}$

(C) $\left(\frac{7}{4}\right)^{5/3} > \left(\frac{5}{3}\right)^{7/4}$

(D) $(519)^{623} < (623)^{519}$

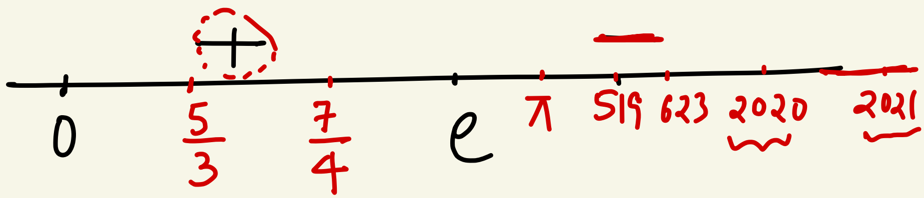
Sol^n

$$e^{\pi} > \pi^{e} \Rightarrow e^{\frac{1}{e}} > \pi^{\frac{1}{\pi}}$$

Consider $f(x) = (x)^{\frac{1}{x}}$; $(x > 0)$

$$f(x) = e^{\frac{1}{x}\ln x} = e^{\frac{\ln x}{x}}$$

$$f'(x) = (x)^{\frac{1}{x}}\left(\frac{\cancel{x}\cdot\frac{1}{\cancel{x}} - \ln x\cdot 1}{x^2}\right) = (x)^{\frac{1}{x}}\frac{(1-\ln x)}{x^2}$$



$f(e) =$ Greatest value

$f(e) > f(\pi)$

$f(7/4) > f(5/3)$

$f(2020) > f(2021)$

$f(519) > f(623)$

Q/ HW

$$f(x)=\begin{cases} x^3+9, & -3 \le x < -2 \\ |x-1|, & -2 \le x \le 2 \\ (x-3)^2, & 2 < x < 5 \end{cases}$$, find greatest and least value of $f(x)$

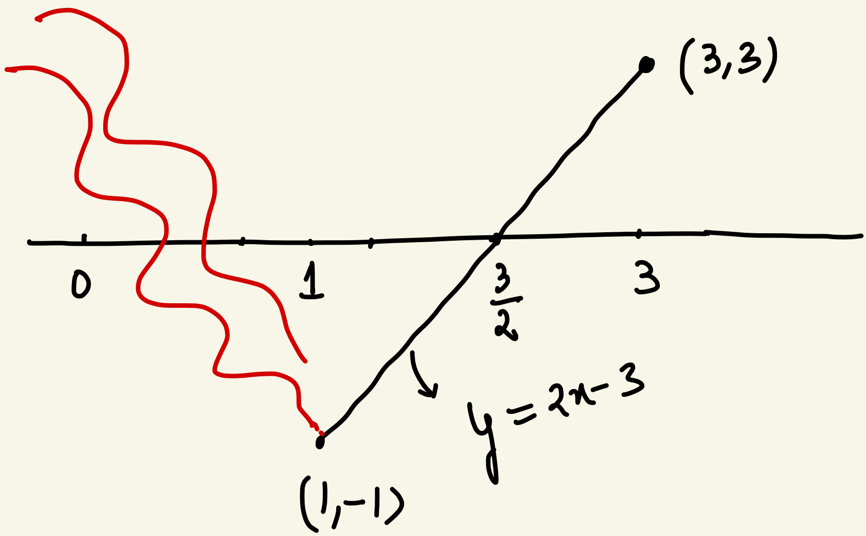
Q

Let $f(x) = \begin{cases} -x^3 + \dfrac{(b^3 - b^2 + b - 1)}{(b^2 + 3b + 2)} & ,0 \le x < 1 \\ 2x - 3 & ,1 \le x \le 3 \end{cases}$

Find all possible real values of b such that f(x) has the smallest value at x = 1.

Sol^n

$$f(x) \begin{cases} -x^3 + K \;; & 0 \le x < 1 \quad \longrightarrow f'(x) < 0 \\ 2x - 3 \;; & 1 \le x \le 3 \end{cases}$$

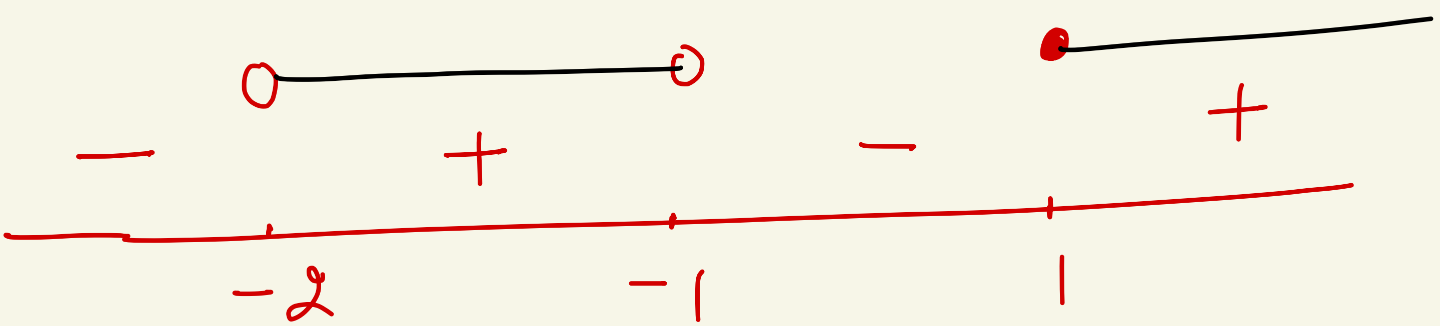


$$f(1) \le \lim_{x \to 1^-} f(x)$$

$$\cancel{-1} \le \cancel{-1} + \frac{b^3 - b^2 + b - 1}{b^2 + 3b + 2}$$

$$\frac{b^3 - b^2 + b - 1}{b^2 + 3b + 2} \geq 0$$

$$\frac{(b-1)(b^2+1)}{(b+1)(b+2)} \geq 0$$



$\therefore b \in (-2, -1) \cup [1, \infty)$

Q HW

$$f(x)=\begin{cases} a^2+a-1-|x|, & x<1 \\ \log_e x, & x\ge 1 \end{cases}$$. If $f(x)$ is increasing at $x = 1$, then find 'a'.

Q (HW) Find the least value of

$$f(x) = x^{3/2} + x^{-3/2} - 4\left(x + \frac{1}{x}\right)$$

for all permissible value of x?

OR

$$g(x) = 8^{x} + 8^{-x} - 4\left(4^{x} + 4^{-x}\right)$$

Estabilishing inequalities using monotonicity ..

Prove that

① $2\sin x + \tan x \geq 3x \quad \left(0 \leq x < \frac{\pi}{2}\right)$

TPT: $f(x) \geq 0$; $x \in [0, \pi/2)$

$f(x) = 2\sin x + \tan x - 3x$

$f'(x) = 2\cos x + \sec^2 x - 3$

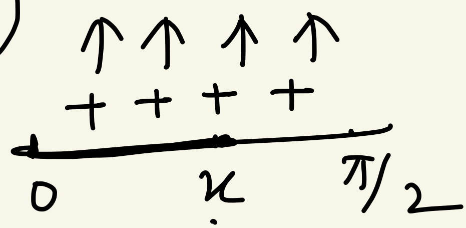
M-1 AM $\geq$ GM

$$\frac{\cos x + \cos x + \frac{1}{\cos^2 x}}{3} \geq \left(\cancel{\cos x} \cdot \cancel{\cos x} \cdot \frac{1}{\cancel{\cos^2 x}}\right)^{1/3}$$

$$2\cos x + \sec^2 x \geq 3$$

$\boxed{f'(x) \geq 0}$ for $x \in [0, \pi/2)$

f is $\uparrow$ in $[0, \pi/2)$



$f(x) \geq f(0)$

$\boxed{f(x) \geq 0}$ (H.P.)

M-2

$$f'(x) = 2\cos x + \frac{1}{\cos^2 x} - 3$$

$$= \frac{2\cos^3 x + 1 - 3\cos^2 x}{(\cos^2 x)}$$

$$= \frac{(\cos x - 1)^2 (2\cos x + 1)}{\cos^2 x}$$

$$f'(x) \geq 0 \quad \forall x \in \left[0, \pi/2\right)$$

Find number of solutions of equations

(i) $\int_0^x \sec^4 t\,dt = \frac{x+1}{3}$ in $x \in (0, 1)$

(ii) $x\sin x + \cos x = \frac{3}{2}$ in $x \in (0, \pi)$

(iii) $\frac{x^3+1}{x^2+1} = 5$ in $[0, 2]$

HW

(iv) $\frac{1}{(1+x)^3} = 3x - \sin x$

(iv)

$$f(x) = \frac{1}{(1+x)^3} - 3x + \sin x$$

$$f'(x) = \frac{-3}{(1+x)^4} - 3 + \cos x$$

$f'(x) < 0$

$\therefore f$ is $\downarrow$ fn

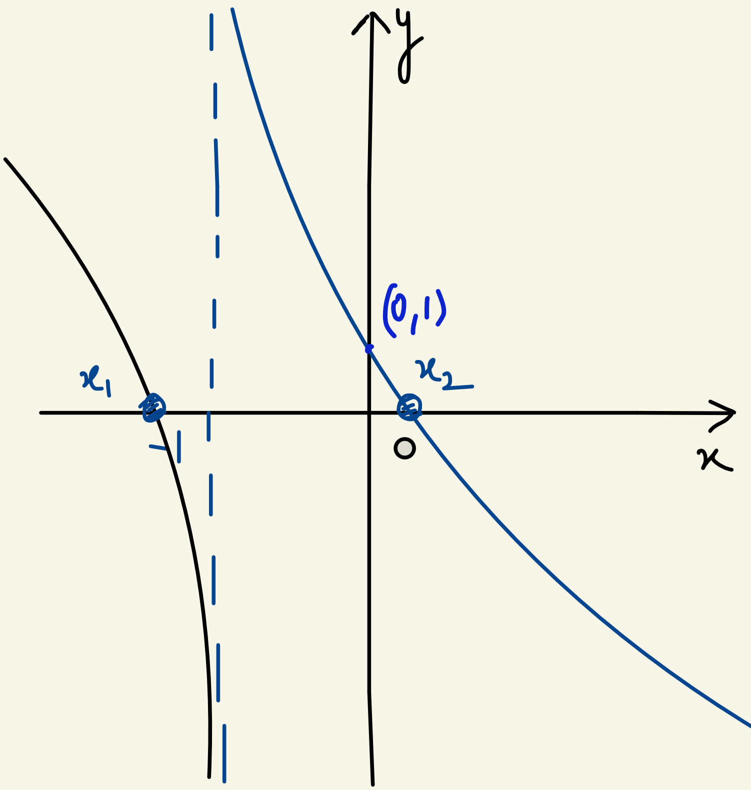
$D_f \in R - \{-1\}$

$\lim_{x \to -1^-} f(x) = -\infty$;

$\lim_{x \to -1^+} f(x) = \infty$

$\lim_{x \to (-\infty)} f(x) = \infty$

$\lim_{x \to (\infty)} f(x) = -\infty$

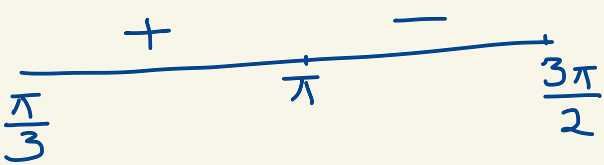


'2 solutions'

Q Find greatest & least value :

(iii) $f(x) = \cos 3x - 15\cos x + 8$ in $\left[\frac{\pi}{3}, \frac{3\pi}{2}\right]$

$$f'(x) = -3\sin 3x + 15\sin x$$
$$= -3\left(3\sin x - 4\sin^3 x\right) + 15\sin x$$
$$= 12\sin^3 x + 6\sin x$$
$$= 6\sin x\left(2\sin^2 x + 1\right)$$



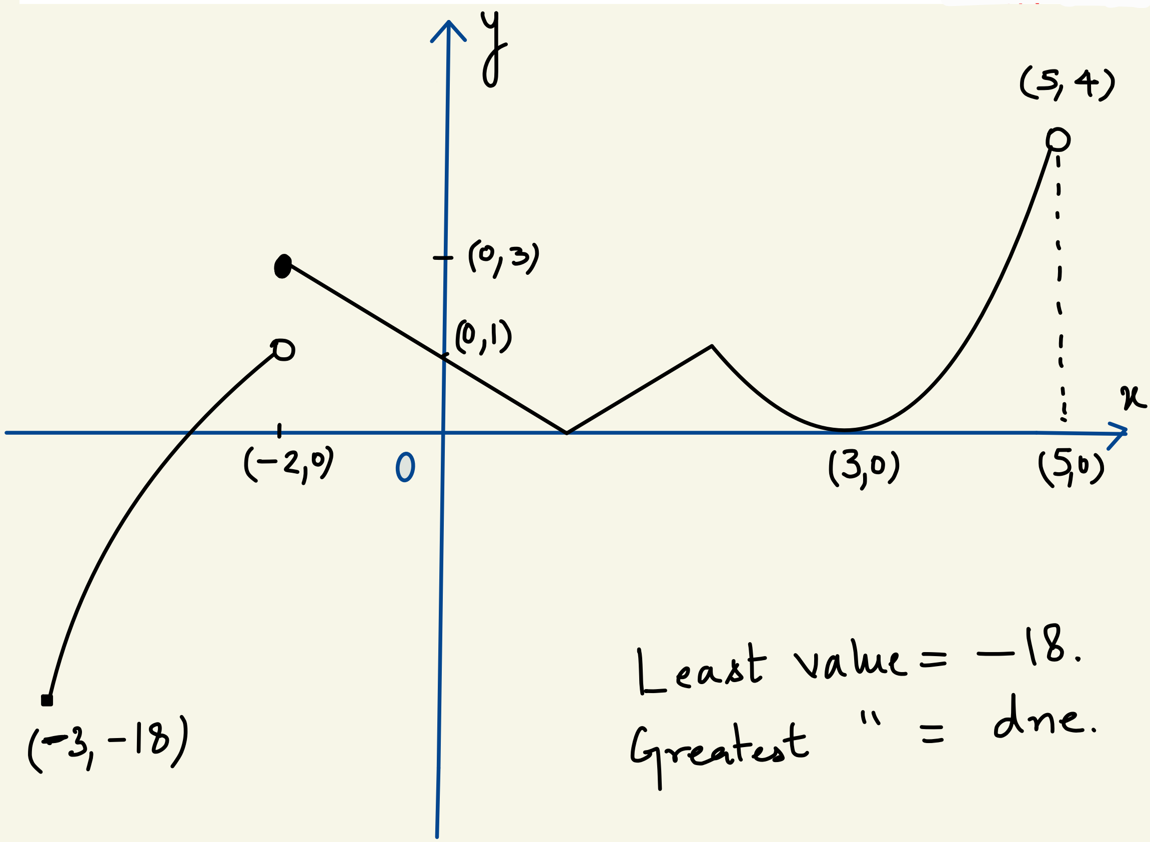
$f(\pi/3) = -1 - \frac{15}{2} + 8 = \frac{-1}{2} \rightarrow$ least.

$f(\pi) = 22 \longrightarrow$ Greatest.

$f\left(\frac{3\pi}{2}\right) = 8.$

Q

$$f(x) = \begin{cases} x^3 + 9, & -3 \le x < -2 \\ |x-1|, & -2 \le x \le 2 \\ (x-3)^2, & 2 < x < 5 \end{cases}$$, find greatest and least value of $f(x)$



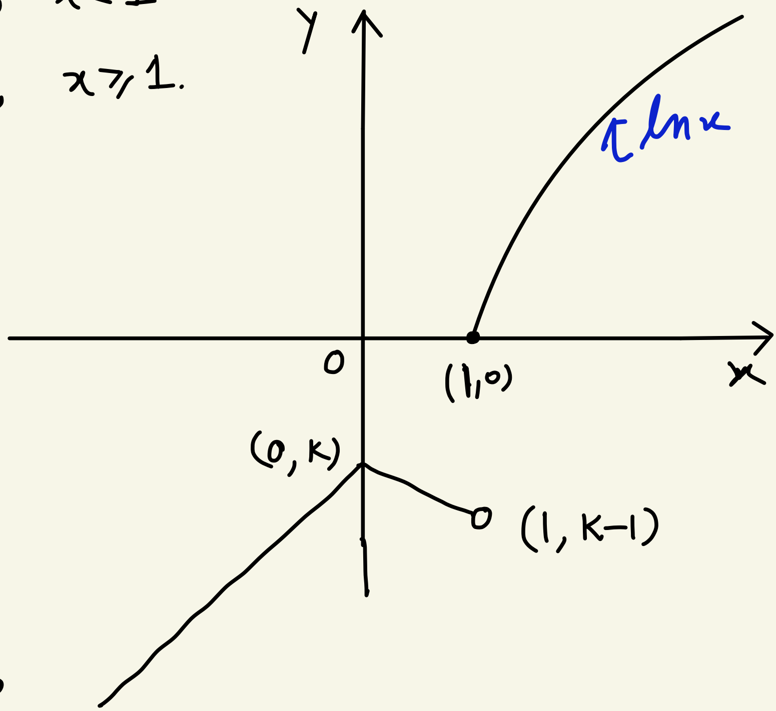
Least value = −18.

Greatest " = dne.

Q

$f(x) = \begin{cases} \underbrace{a^2 + a - 1}_{k} - |x|, & x < 1 \\ \log_e x, & x \geq 1 \end{cases}$. If $f(x)$ is increasing at x = 1, then find 'a'.

$f(x) \rightarrow \begin{cases} k - |x| \; ; & x < 1 \\ \ln x \; ; & x \geq 1. \end{cases}$



$k - 1 < 0$

$a^2 + a - 1 - 1 < 0$

$a^2 + a - 2 < 0$

$a^2 + 2a - a - 2 < 0$

$(a+2)(a-1) < 0$

$\therefore a \in (-2, 1)$ Ans

Q Find the least value of

$$f(x) = x^{3/2} + x^{-3/2} - 4\left(x + \frac{1}{x}\right)$$ for all permissible value of x ?

OR $$g(x) = 8^{x} + 8^{-x} - 4\left(4^{x} + 4^{-x}\right)$$

Sol^n $$f(x) = \left(\sqrt{x}\right)^3 + \left(\frac{1}{\sqrt{x}}\right)^3 - 4\left(\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 - 2\right)$$

$$f(x) = \left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)\left(x + \frac{1}{x} - 1\right) - 4\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 + 8.$$

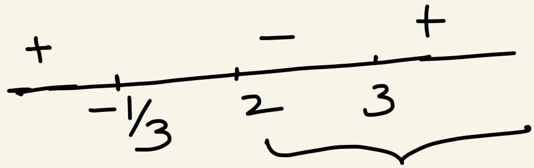
let $\sqrt{x} + \frac{1}{\sqrt{x}} = t$; $t \geq 2$

$$g(t) = t\left(t^2 - 3 - 4t^2 + 8\right) ; \quad t \geq 2$$

$$g(t) = t^3 - 4t^2 - 3t + 8.$$

$$g'(t) = 3t^2 - 8t - 3 = 3t^2 - 9t + t - 3$$

$$g'(t) = (3t+1)(t-3)$$



$\therefore$ least value at $t = 3$

Least value $= 27 - 36 - 9 + 8$

$= -10.$ Ans

(ii) PT $\dfrac{2}{2x+1} < \ell n\left(1+\dfrac{1}{x}\right) < \dfrac{1}{x}$ for $x > 0$

① $\dfrac{2}{2x+1} - \ln\left(1+\dfrac{1}{x}\right) < 0$ for $x > 0$

HW

② $\ln\left(1+\dfrac{1}{x}\right) - \dfrac{1}{x} < 0$ for $x > 0$.

Consider $f(x) = \dfrac{2}{2x+1} - \ln\left(1+\dfrac{1}{x}\right)$; $x > 0$.

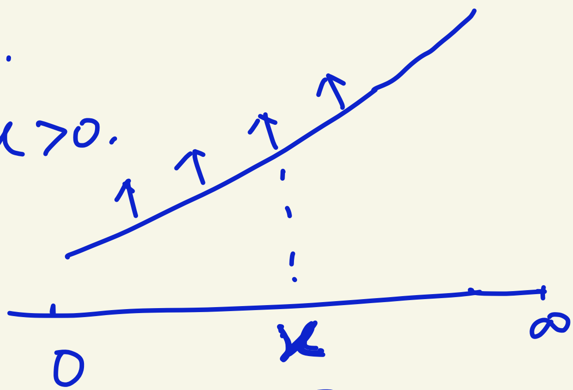
$$f'(x) = \frac{-4}{(2x+1)^2} - \frac{1}{\left(1+\frac{1}{x}\right)} \cdot \left(\frac{-1}{x^2}\right)$$

$$= \frac{-4}{(2x+1)^2} + \frac{1}{(x+1)x} = \frac{-4(x^2+x)+(2x+1)^2}{x(x+1)(2x+1)^2}$$

$$f'(x) = \frac{\cancel{-4x^2}\cancel{-4x}+\cancel{4x^2}+1+\cancel{4x}}{x(x+1)(2x+1)^2} = \frac{1}{x(x+1)(2x+1)^2}$$

$f'(x) > 0$ for $x > 0$.

$\therefore f(x)$ is $\uparrow$ for $x > 0$.



$$f(x) < \lim_{x\to\infty} f(x) \Rightarrow \boxed{f(x) < 0} \quad (\text{H.P})$$

Q

Find the set of values of x for which $\ell n(1 + x) > \frac{x}{1+x}$

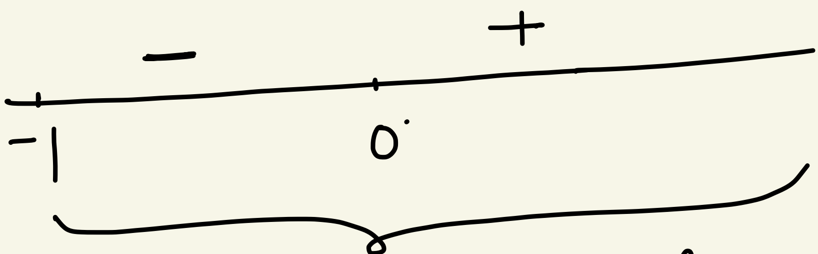
Sol^n

$$f(x) = \ln(1+x) - \frac{x}{1+x} \quad ; \quad f(x) > 0 \;??$$

$$D_f \in (-1, \infty) \qquad\qquad 1+x > 0$$

$$f'(x) = \frac{1}{1+x} - \left(\frac{(1+\cancel{x}) - \cancel{x}}{(1+x)^2}\right)$$

$$f'(x) = \frac{(1+x) - 1}{(1+x)^2} = \frac{x}{(1+x)^2}$$



$f(0) \rightarrow$ least value. $\Rightarrow f(0) = \ln 1 = 0.$

$\therefore x \in (-1, 0) \cup (0, \infty)$

Ans

Q

Find the smallest positive constant A such that $\ln x \le Ax^2$ for all $x > 0$.

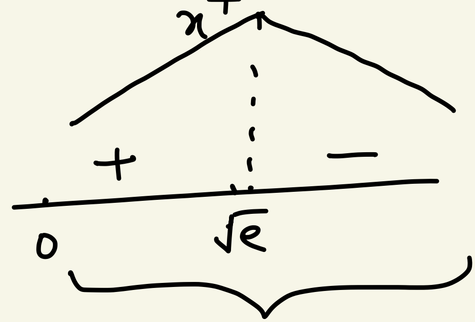
Sol^n

$$\boxed{A \geq \left(\frac{\ln x}{x^2}\right)} \; ; \text{ for } \underline{\underline{x > 0}}$$

$$f(x) = \frac{\ln x}{x^2} \; ; \quad x > 0$$

$$f'(x) = \frac{x^{\cancel{2}} \cdot \left(\frac{1}{\cancel{x}}\right) - (\ln x)(2x)}{x^4} = \frac{x(1 - 2\ln x)}{x^4}$$

$$f'(x) = \left(\frac{1 - 2\ln x}{x^3}\right)$$



$2 \ln x = 1$

$\ln x = \frac{1}{2} \Rightarrow x = \sqrt{e}$

$f(\sqrt{e}) =$ Greatest value

$$f(\sqrt{e}) = \frac{\ln \sqrt{e}}{(\sqrt{e})^2} = \frac{1}{2e}.$$

$\therefore \; A \geq \frac{1}{2e}$ $\quad \therefore A_{\text{smallest}} = \frac{1}{2e}$

Q

Let f is differentiable function $\forall$ x such that $f'(x) \geq x, \forall x \in R$. If $f(0) = 2$, then minimum value of $f(3)$

Sol^n

$$f'(x) \geqslant x \quad \forall x \in R \; ; \quad f(0) = 2$$

$$(f(3))_{min} = ?$$

$$\int_0^x f'(x)\,dx \geqslant \int_0^x x\,dx$$

$$f(x)\Big|_0^x \geqslant \frac{x^2}{2}\Big|_0^x$$

$$f(x) - f(0) \geqslant \frac{x^2}{2}$$

$$f(x) \geqslant \frac{x^2}{2} + 2$$

$$f(3) \geqslant \frac{9}{2} + 2 = 6.50$$

Ans

Q. Suppose f is a differentiable real function such that $f'(x) + f(x) \le 1$ for all real x and $f(0) = 0$. Find largest possible value of $f(1)$?

sol^n

$$f'(x) + f(x) \le 1 \qquad \forall\, x \in \mathbb{R}$$

$$\frac{dy}{dx} + y \le 1.$$

$$\boxed{\text{"LDE"} \quad IF = e^{\int dx} = e^{x}}$$

$$\underbrace{e^{x} \cdot \frac{dy}{dx} + e^{x} y}_{0} \le e^{x}$$

$$\frac{d}{dx}\left(e^{x} \cdot y\right) \le e^{x}$$

$$\int_0^1 d\left(e^{x} \cdot y\right) \le \int_0^1 e^{x}\, dx$$

$$e^{x} \cdot f(x) \Big|_0^1 \le e^{x} \Big|_0^1$$

$$e\, f(1) - 1 \cdot \cancel{f(0)}^{\,0} \le e - 1.$$

$$e\, f(1) \le e - 1 \Rightarrow \boxed{f(1) \le \frac{e-1}{e}}$$

Q If $g(x) = x - \frac{x^3}{6} + \frac{x^5}{120} - \sin x$ then show that

$g(x) > 0 \quad \forall \; x \in (0, \pi/2)$?

Sol^n

$$g(x) = x - \frac{x^3}{6} + \frac{x^5}{120} - \sin x \quad ; \quad x \in (0, \pi/2)$$

$$g'(x) = 1 - \frac{3x^2}{6} + \frac{5x^4}{120} - \cos x$$

$$g'(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \cos x$$

$$h(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} - \cos x \quad ; \quad x \in (0, \pi/2)$$

$$h'(x) = -x + \frac{4x^3}{24} + \sin x$$

$$h'(x) = -x + \frac{x^3}{6} + \sin x$$

$$w(x) = -x + \frac{x^3}{6} + \sin x \quad ; \quad x \in (0, \pi/2)$$

$$w'(x) = -1 + \frac{3x^2}{6} + \cos x$$

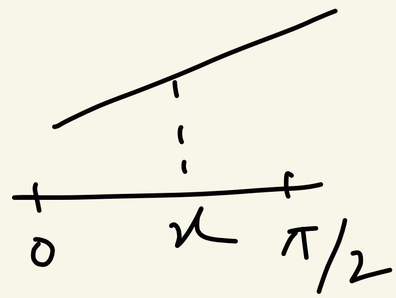
$$w'(x) = -1 + \frac{x^2}{2} + \cos x$$

$$K(x) = -1 + \frac{x^2}{2} + \cos x \quad ; \quad x \in (0, \pi/2)$$

$$K'(x) = (x - \sin x)$$

$K'(x) > 0$ for $x \in (0, \pi/2)$

$\Rightarrow K(x)$ is $\uparrow$ for $x \in (0, \pi/2)$



$$K(x) > \lim_{x \to 0} K(x)$$

$\boxed{K(x) > 0} \overset{*}{\Rightarrow} W'(x) > 0 \Rightarrow W(x)$ is $\uparrow$ fns

$$W(x) > \lim_{x \to 0} W(x) \Rightarrow W(x) > 0$$

$h'(x) > 0 \Rightarrow h(x) \uparrow$ fns

$$h(x) > \lim_{x \to 0} h(x) \Rightarrow h(x) > 0 \Rightarrow g'(x) > 0$$

$g(x) \uparrow$ fns $\Rightarrow \lim_{x \to 0} g(x) < g(x)$

$\boxed{g(x) > 0}$ H.P

Note :-

$$y = f(g(x))$$

$$\frac{dy}{dx} = \underbrace{f'(\underbrace{g(x)})}\cdot\underbrace{g'(x)}.$$

$\frac{dy}{dx} > 0$ if f' & g' are of same sign

$\frac{dy}{dx} < 0$ if f' & g' are of opp sign.

Q $f(x) = \tan^{-1}(\sin x + \cos x)^3$ is an increasing function in

(A) $(0, \pi/4)$ ✓ (B) $(0, \pi/2)$ ✗

(C) $(-\pi/4, \pi/15)$ ✓ (D) $\left(\frac{\pi}{4}, \frac{5\pi}{12}\right)$ ✗

Sol^n $f(x) = g(h(x))$

$\Rightarrow g(x) = \tan^{-1}x \longrightarrow \uparrow$ fns.

$h(x) = (\sin x + \cos x)^3$

$h'(x) = 3(\sin x + \cos x)^2 \times (\cos x - \sin x)$

$g' > 0$ & $h' > 0$

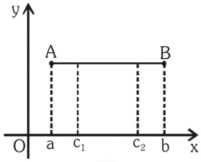
ROLLE'S THEOREM : Imp

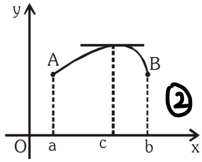
Let $f(x)$ be a function of x subject to the following conditions :

(i) $f(x)$ is a continuous function of x in [a, b]

(ii) $f(x)$ is derivable in (a, b)

(iii) $f(a) = f(b)$.

1




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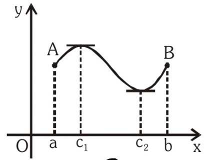


3

Then there exists at least one point $x = c$ such that $a < c < b$ where $f'(c) = 0$.

Converse of Rolle's theorem is **Not true**

Geometrical interpretation :

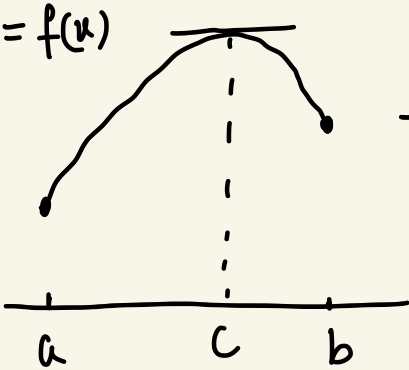
Geometrically, the Rolle's theorem says that somewhere between A and B the curve has at least one tangent parallel to x-axis.

Remarks:

1. Converse of Rolle's theorem is **Not true** i.e. $f'(x)$ may vanish at a point within (a, b) without satisfying all the three conditions of Rolle's Theorem.
2. The three conditions are sufficient but not necessary for $f'(x) = 0$ for some x in (a, b)
3. If the function $y = f(x)$ defined over [a, b] does not satisfy even one of the 3 conditions then Rolle's Theorem fails i.e. there may or may not exist point in (a, b) where $f'(x) = 0$.

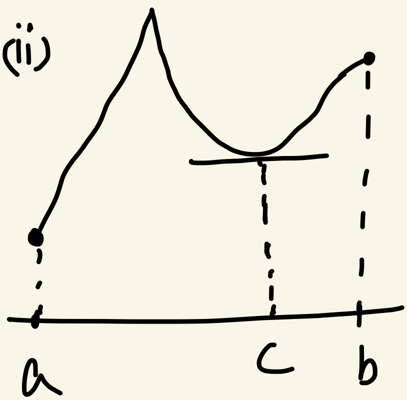
$y = f(x)$

(i)



$f'(c) = 0$

(ii)



1

Verify Rolle's Theorem for $f(x) = x(x+3)e^{-x/2}$ in $[-3, 0]$ & if valid find c.

HW

2

Verify Rolle's Theorem for $f(x) = \frac{\sin x}{e^x}$ in $[0, \pi]$ & if valid find c.

HW

3 Verify Rolle's for $f(x) = 1 - x^{2/3}$ in $[-1, 1]$?

(i) $f(x) = x(x+3) \cdot e^{-x/2}$ in $[-3, 0]$

Cont in $[-3, 0]$ $\qquad f(-3) = f(0) = 0.$

der in $(-3, 0)$

Rolle's is applicable.

$f'(c) = 0 \quad ; \quad c \in (-3, 0)$ *

$$f'(x) = (x^2+3x)\cancel{e^{-x/2}} \cdot \left(-\frac{1}{2}\right) + (2x+3)\cancel{e^{-x/2}} = 0$$

$$\Rightarrow \quad -x^2 - 3x + 4x + 6 = 0.$$

$$x^2 - x - 6 = 0.$$

$$x^2 - 3x + 2x - 6 = 0$$

$$(x-3)(x+2) = 0 \Rightarrow \boxed{x=3} \times ; \; x = -2$$

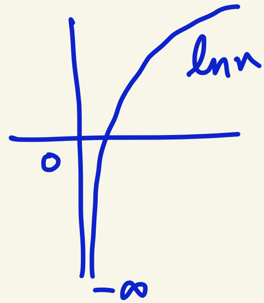
$\therefore \boxed{c = -2}$

Q

Let $f(x) = \begin{cases} x^{\alpha} \ln x, & x > 0 \\ 0, & x = 0 \end{cases}$. Rolle's theorem is applicable to f for $x \in [0, 1]$, if $\alpha =$

(A) –2 (B) –1 (C) 0 (D) 1/2 (E) 5

$$f(0) = \lim_{x \to 0^+} f(x)$$



$$0 = \lim_{x \to 0^+} \left(x^{\alpha} \ln x \right) \quad ; \boxed{\alpha > 0}$$

$$0 = \lim_{x \to 0^+} \left(\frac{\ln x}{x^{-\alpha}} \right) \quad \left(\frac{\infty}{\infty}\right)$$

LH Rule

$$0 = \lim_{x \to 0^+} \frac{\left(\frac{1}{x}\right)}{(-\alpha)\, x^{-\alpha-1}}$$

$$0 = \lim_{x \to 0^+} \left(-\frac{1}{\alpha}\right) \left(x^{\alpha} \right)$$

$$\boxed{\alpha > 0}$$

Q

If f is a differentiable function such that $f(a) + f(b) = f(b) + f(c) = f(c) + f(d) = 0$, where $a < b < c < d$, then find the minimum number of the roots of equation $f(x).f'(x) = 0$

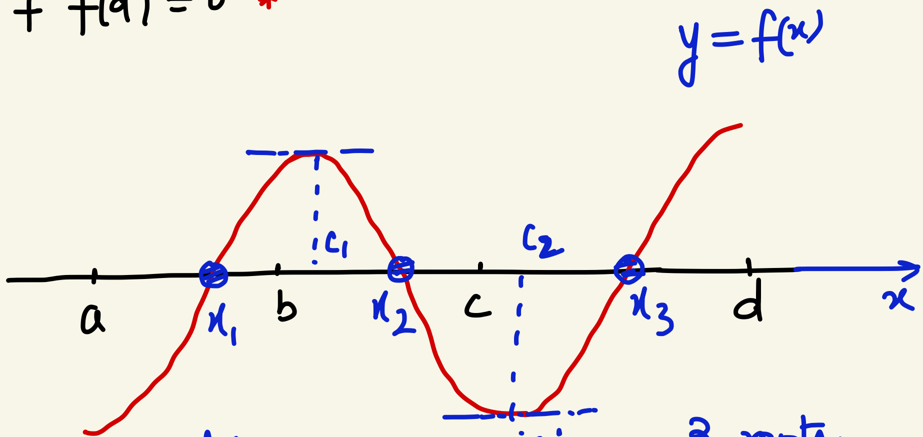
Sol^n

$f(a) + f(b) = 0$ *

$f(b) + f(c) = 0$ *

$f(c) + f(d) = 0$ *

$a < b < c < d$



$f(x) = 0 \longrightarrow$ minimum 3 roots.

$f'(x) = 0 \longrightarrow$ " 2 roots.

$f(x) \cdot f'(x) = 0 \Rightarrow$ minimum 5 roots.

Ans

Q

Let $f(x)$ be a non-constant twice differentiable function defined on $(-\infty, \infty)$ such that $f(x) = f(1-x)$ and $f'(1/4) = 0$. Then

(i) Prove that $f''(x)$ vanishes at least twice on $[0, 1]$

(ii) Find $f'(1/2)$

Sol^n

$$f(x) = f(1-x) \quad \text{and} \quad f'(1/4) = 0$$

diff

$$f'(x) = -f'(1-x) \quad \text{—} ① \text{—}$$

$x = \frac{1}{2}$ in ① $\Rightarrow$ $f'(1/2) = -f'(1/2)$

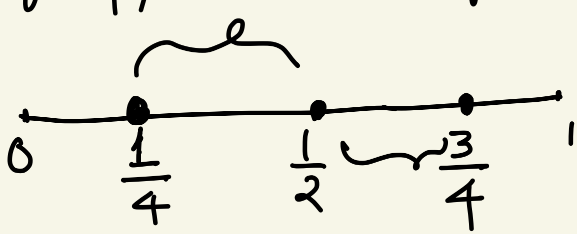
$$\boxed{f'(1/2) = 0} \; **$$

put $x = \frac{1}{4}$ in ①

$$f'\left(\frac{1}{4}\right) = -f'\left(\frac{3}{4}\right) \Rightarrow \boxed{f'(3/4) = 0}$$

Consider $g(x) = f'(x)$ in $[0,1]$

$$g\left(\frac{1}{4}\right) = 0 \; ; \; g\left(\frac{1}{2}\right) = 0 \; ; \; g\left(\frac{3}{4}\right) = 0$$



Apply Rolle's theorem on $g(x)$ in $\left[\frac{1}{4}, \frac{1}{2}\right]$

$g'(c_1) = 0$ for atleast one $c_1 \in \left(\frac{1}{4}, \frac{1}{2}\right)$

$g'(c_1) = \boxed{f''(c_1) = 0}$.

Apply Rolle's theorem on $g(x)$ in $\left[\frac{1}{2}, \frac{3}{4}\right]$

$g'(c_2) = 0$ for atleast one $c_2 \in \left(\frac{1}{2}, \frac{3}{4}\right)$

$\Downarrow$

$\boxed{f''(c_2) = 0}$

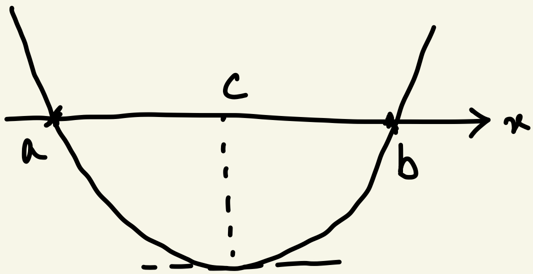
Note: Imp

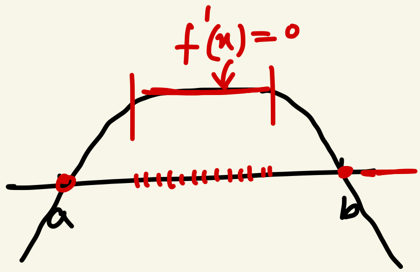
(i) If $f(x)$ is differentiable function then between any two roots of the equation $f(x) = 0$ there will be atleast one root of the equation $f'(x) = 0$.

(ii) If f is a differentiable function in (a, b) and continuous in $[a, b]$ such that it has exactly n roots in $[a, b]$, then f' will have atleast $(n - 1)$ roots $\in (a, b)$.

(iii) If $f'(x)$ has exactly m roots in (a, b), then $f(x)$ will have at most $(m + 1)$ roots in $[a, b]$

(i)





①
Verify Rolle's Theorem for $f(x) = x(x+3)e^{-x/2}$ in $[-3, 0]$ & if valid find c.

HW
②
Verify Rolle's Theorem for $f(x) = \dfrac{\sin x}{e^x}$ in $[0, \pi]$ & if valid find c. ✓ Rolle's applicable
$c = \pi/4$

HW
③ Verify Rolle's for $f(x) = 1 - x^{2/3}$ in $[-1, 1]$?

Rolle's not applicable.
$\because$ it is non-der at $x = 0$

Q

$f(x)$ and $g(x)$ are differentiable functions such that $f(1) = g(1)$, $f(2) = 4$, $g(2) = \frac{1}{2}$, then prove that $f'(x) = 3x^2 g(x) + x^3 g'(x)$ for some $x \in (1, 2)$

Solⁿ

$$\underbrace{f'(x) - (3x^2 g(x) + x^3 g'(x))}\ = 0 \quad \text{for some } x \in (1,2)$$

$$\boxed{H(x) = f(x) - x^3 g(x)} \qquad x \in [1,2]$$

$H(1) = f(1) - g(1) = 0.$

$H(2) = f(2) - 8g(2) = 4 - 8\left(\frac{1}{2}\right) = 0.$

$H(x)$ is Cont in $[1,2]$ & der in $(1,2)$

∴ Rolle's is applicable on $H(x)$

$H'(x) = 0$ for some $x \in (1,2)$

$\downarrow$

$f'(x) - (3x^2 g(x) + x^3 g'(x)) = 0$

for atleast one $x \in (1,2)$.

(H.P)

Q If $P(x) = 2013x^{2012} - 2012x^{2011} - 16x + 8$

then $\boxed{P(x) = 0}$ for $x \in \left[0, 8^{\frac{1}{2011}}\right]$ has

(A) exactly one real root
(B) No Real root
(C) atleast one & atmost 2 real roots
(D) atleast 2 real roots ✓

Sol^n

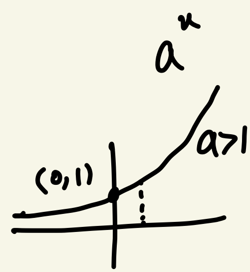
$$\int P(x)\,dx = F(x) \nearrow$$

$$F(x) = \int \left(2013x^{2012} - 2012x^{2011} - 16x + 8\right) dx$$

$$\boxed{F(x) = x^{2013} - x^{2012} - 8x^2 + 8x + c.}$$

Cont in $\left[0, 8^{\frac{1}{2011}}\right]$

& dev



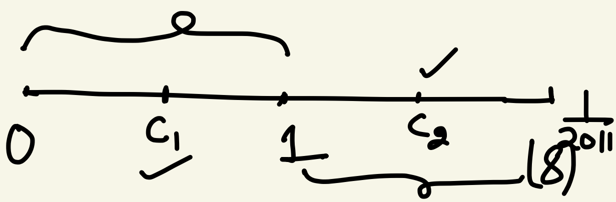
$$F(x) = x^{2012}(x-1) - 8x(x-1) + c$$

$$F(x) = x(x-1)\left(x^{2011} - 8\right) + c$$

$F(0) = c$

$F(1) = c$

$F\left(8^{\frac{1}{2011}}\right) = c$



Q

Show that between any two roots of the equation $e^x \cos x = 1$ there exists at least one root of $e^x \sin x - 1 = 0$.

Sol^n

$$e^x \cos x - 1 = 0 \Rightarrow e^x (\cos x) = 1$$

$$\cos x = e^{-x}$$

$$\cos x - e^{-x} = 0 \quad \begin{cases} \alpha \\ \beta \end{cases}$$

$$\boxed{F(x) = \cos x - e^{-x}}$$

$$F(\alpha) = F(\beta) = 0$$

From Rolle's thr exist atleast one $c \in (\alpha, \beta)$ such that $F'(c) = 0$

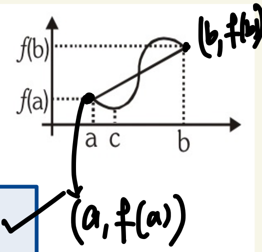
$$-\sin c + e^{-c} = 0$$

$$e^{-c} = \sin c$$

$$1 = e^c \sin c$$

$$\boxed{e^c \sin c - 1 = 0} \quad \text{(H.P.)}$$

LMVT THEOREM : (LAGRANGE'S MEAN VALUE THEOREM) :



Let $f(x)$ be a function of x subject to the following conditions :

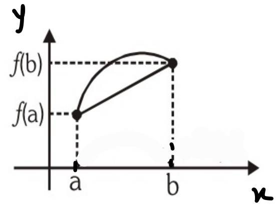
(i) $f(x)$ is a continuous function of x in [a, b]

(ii) $f(x)$ is derivable in (a, b)

Rem

Then there exists at least one point x = c such that a < c < b where $\boxed{f'(c) = \dfrac{f(b) - f(a)}{b - a}}$.

Proof :



Equation line joining end points is

$$y - f(a) = \frac{f(b) - f(a)}{b - a}(x - a)$$

$$\text{Let } G(x) = \frac{f(b) - f(a)}{b - a}(x - a) + f(a)$$

Let $H(x) = f(x) - G(x)$

H(x) satisfies Rolles condition

By Rolles on H(x) in [a, b]

$\Rightarrow \quad H'(x) = 0$ for atleast one $c \in (a, b)$

$$f'(x) - \frac{f(b) - f(a)}{b - a} = 0 \quad \Rightarrow \quad f'(x) = \frac{f(b) - f(a)}{b - a}$$

Geometrical interpretation :

Geometrically, the slope of the secant line joining the curve at x = a & x = b is equal to the slope of the tangent line drawn to the curve at x = c.

Q Find c of LMVT $f(x) = \sqrt{x-1}$ in [1, 3].

$$f'(x) = \frac{1}{2\sqrt{x-1}}$$

$$f'(c) = \frac{f(3) - f(1)}{3-1} = \frac{\sqrt{2} - 0}{2}$$

$$f'(c) = \frac{1}{\sqrt{2}} \Rightarrow \frac{1}{\sqrt{2}} = \frac{1}{2\sqrt{c-1}}$$

Q

$$f(x) = \begin{cases} \dfrac{e^x - x - a}{x}, & -1 \le x < 0 \\ \dfrac{b}{2}, & x = 0 \\ x^2 + \dfrac{x}{c}, & 0 < x \le 1 \end{cases}$$

. If LMVT is applicable for $f(x)$ in $x \in [-1, 1]$, find a, b, c.

$\boxed{c \neq 0}$

Solⁿ **Cont at $x = 0$**

$$\frac{b}{2} = f(0^+) \Rightarrow \frac{b}{2} = 0 \Rightarrow \boxed{b = 0}$$

$$f(0) = \lim_{x \to 0^-} f(x) \Rightarrow \lim_{x \to 0^-} \frac{e^x - x - a}{x} = 0$$

$$\lim_{x \to 0^-} \frac{\left(1 + \cancel{x} + \frac{x^2}{2!} + \cdots\right) - \cancel{x} - a}{x} = 0$$

$$\lim_{x \to 0^-} \frac{\overbrace{(1-a)}^{\to 0} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots}{x} = 0.$$

$\therefore \boxed{a = 1}$

Apply differentiability to get $\boxed{c = 2}$

Q Use LMVT to prove that $\tan x > x$ for $x \in \left(0, \frac{\pi}{2}\right)$

Sol^n

Consider $f(x) = \tan x$; $[0, x]$

where $x \in (0, \pi/2)$

From LMVT,

$$f'(c) = \frac{f(x) - f(0)}{x - 0}.$$ where $c \in (0, x)$

$$\underbrace{\sec^2 c}_{>1} = \frac{\tan x - 0}{x - 0}$$

$$\boxed{\frac{\tan x}{x} > 1}$$ (H.P)

Q

(1) If a < b the prove that there exist one c such that $3c^2 = a^2 + ab + b^2$, where $a < c < b$

(2) If $f(x)$ is twice differentiable function such that $f(1) = 1$, $f(2) = 2$ and $f(3) = 3$, then prove that $f''(c) = 0$ for atleast one $c \in (1, 3)$

(1) Consider $f(x) = x^3$ in $[a, b]$

From LMVT, $f'(c) = \dfrac{f(b) - f(a)}{b-a}$

for atleast one $c \in (a, b)$

$$3c^2 = \frac{b^3 - a^3}{b-a} = \frac{\cancel{(b-a)}(b^2+ab+b^2)}{\cancel{(b-a)}}$$

(2) $f(1) = 1$; $f(2) = 2$; $f(3) = 3$

Consider $[1, 2]$

Apply LMVT, $f'(c_1) = \dfrac{f(2) - f(1)}{2-1} = 1$

For atleast one $c_1 \in (1, 2)$

Consider $[2, 3]$

Apply LMVT, $f'(c_2) = \dfrac{f(3) - f(2)}{3-2} = 1$

for atleast one $c_2 \in (2, 3)$

Consider $g(x) = f'(x)$ in $[c_1, c_2]$

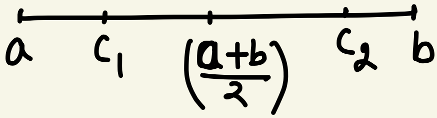
From Rolle's $g'(c) = 0$ for atleast one $c \in (c_1, c_2)$

8

If $y = f(x)$ is twice differentiable function such that $f(a) = f(b) = 0$, and $f(x) > 0 \quad \forall$ $x \in (a, b)$, then

(A) $f''(c) < 0$ for some $c \in (a, b)$ ✓
(B) $f''(c) > 0 \quad \forall\ c \in (a, b)$
(C) $f(c) = 0$ for some $c \in (a, b)$
(D) none

Sol^n



Consider $f(x)$ in $\left[a, \frac{a+b}{2}\right]$

$$f'(c_1) = \frac{f\left(\frac{a+b}{2}\right) - f(a)}{\frac{a+b}{2} - a}$$

for atleast one $c_1 \in \left(a, \frac{a+b}{2}\right)$

$$f'(c_1) = \frac{f\left(\frac{a+b}{2}\right) - 0}{\left(\frac{b-a}{2}\right)} \quad \text{— (1) —}$$

Consider $f(x)$ in $\left[\frac{a+b}{2}, b\right]$

$$f'(c_2) = \frac{\overset{0}{\cancel{f(b)}} - f\left(\frac{a+b}{2}\right)}{b - \left(\frac{a+b}{2}\right)}$$

$$f'(c_2) = \frac{-f\left(\frac{a+b}{2}\right)}{\frac{b-a}{2}} \quad \text{— (2) —}$$

Consider $\boxed{g(x) = f'(x)}$ in $[c_1, c_2]$

From LMVT

$$g'(c) = \frac{g(c_2) - g(c_1)}{c_2 - c_1}$$

$$g'(c) = f''(c) = \frac{f'(c_2) - f'(c_1)}{c_2 - c_1}$$

$$f''(c) = \frac{\dfrac{-f\left(\frac{a+b}{2}\right)}{\left(\frac{b-a}{2}\right)} - \dfrac{f\left(\frac{a+b}{2}\right)}{\left(\frac{b-a}{2}\right)}}{(c_2 - c_1)}$$

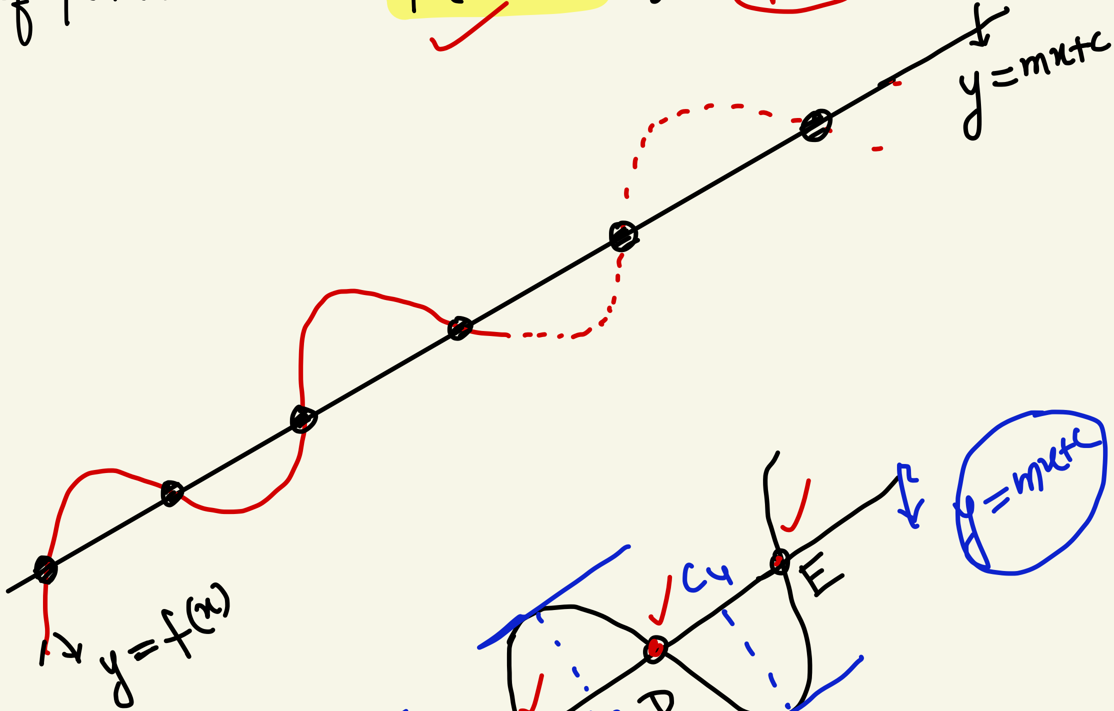
$$f''(c) = \frac{-2^2 f\left(\frac{a+b}{2}\right)}{(b-a)\,(c_2-c_1)} < 0$$

+ve

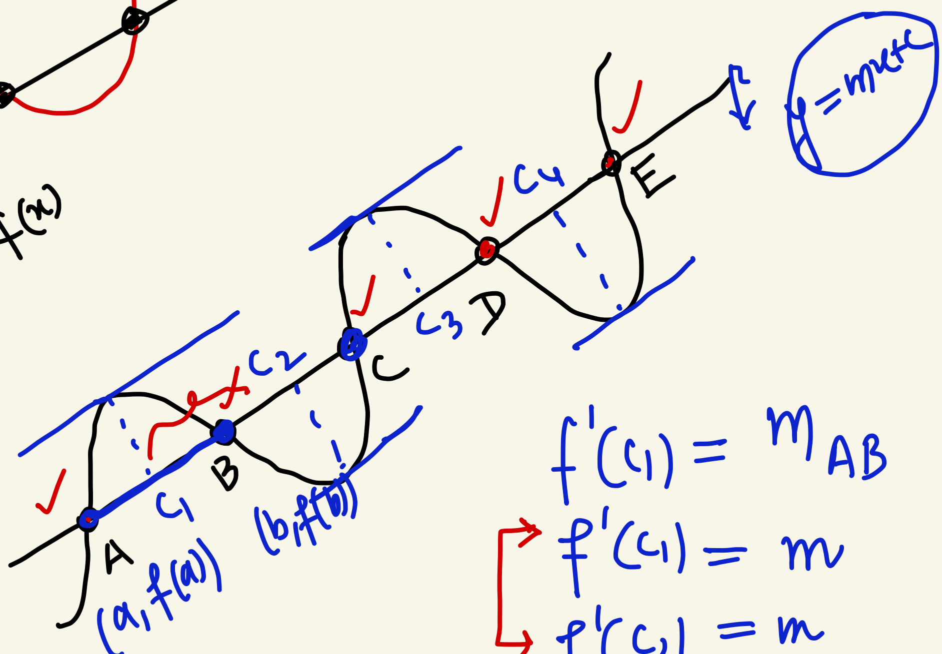
Q [HW] Let f be continuous in $[a, b]$ and derivable in (a, b). If $f(a) = a$; $f(b) = b$ then prove that thr exist distinct c_1 & c_2 in (a, b) such that $f'(c_1) + f'(c_2) = 2$.

Q A continuous & twice derivable function $y=f(x)$ is such that its graph cuts the line $y=mx+c$ at 'p' distinct points then find the minimum no. of points where $f''(x)=0$?

$\boxed{p-2}$ Ans



eg:



$f'(c_1) = m_{AB}$

$f'(c_1) = m$

$f'(c_2) = m$

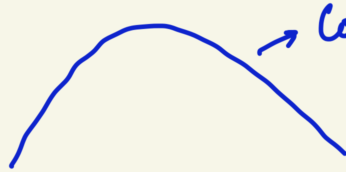
$f'(c_3) = m$

$f'(c_4) = m.$

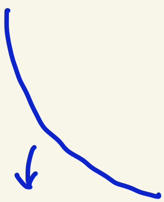
$g(x) = f'(x)$

Concavity :-

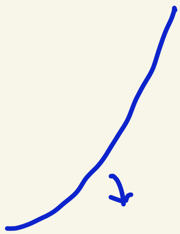
Concave up



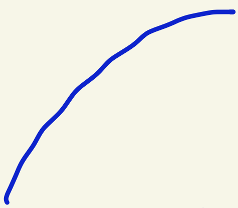
Concave down



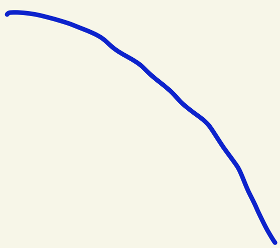
Concave up & dec.



Concave up & inc



Concave down & inc



Concave down & dec.

The points in the domain of fns where $f''(x) \geq 0$, the concavity will be upwards & where $f''(x) \leq 0$, the concavity will be downwards.

✓ $f''(x) \geq 0 \Rightarrow f'(x)$ is ↑ fns.

✓ $f''(x) \leq 0 \Rightarrow f'(x)$ is ↓ fns.

(i) straight line i.e $f(x) = mx + c$

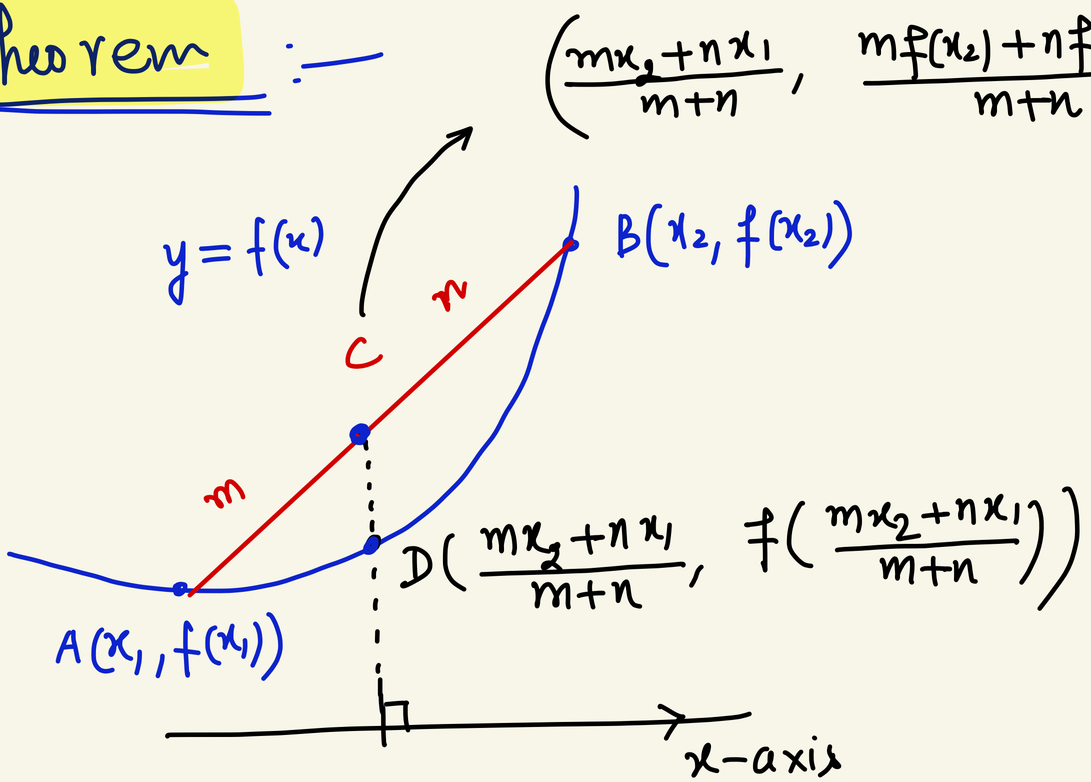
$f'(x) = m$

$f''(x) = 0.$

is both concave up as well as Concave down

Concavity theorem :—

(i) Concave up



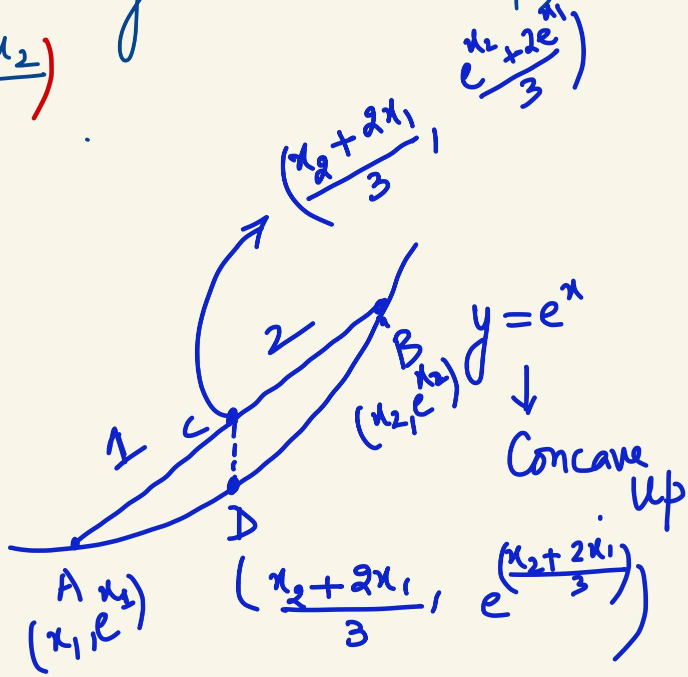
$$\boxed{\frac{mf(x_2)+nf(x_1)}{m+n} \geq f\left(\frac{mx_2+nx_1}{m+n}\right)}$$

(ii) For Concave down: $\frac{mf(x_2)+nf(x_1)}{m+n} \leq f\left(\frac{mx_2+nx_1}{m+n}\right)$

Q Prove that for any 2 numbers x_1, x_2

$$\frac{2e^{x_1}+e^{x_2}}{3} > e^{\left(\frac{2x_1+x_2}{3}\right)}.$$

Sol^n $f(x) = e^x$



(HP)

GRAPHS OF MISCELLANEOUS FUNCTIONS :

Curve-tracing

General tips for plotting the graph :

(a) Compute points where the curve crosses the x-axis and also where it cuts the y-axis by putting y = 0 and x = 0 respectively and mark points accordingly and if no real

(b) Compute $\frac{dy}{dx}$ and find the intervals where $f(x)$ is increasing or decreasing and also where it has horizontal tangent.

(c) In regions where curve is monotonic compute y if $x \to \infty$ or $x \to -\infty$ to find whether y is asymptotic or not.

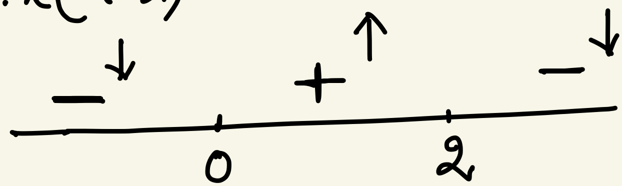
(d) If denominator vanishes say at x = a then examine $\lim_{x \to a^+} f(x)$ and $\lim_{x \to a^-} f(x)$ to find whether f approaches ∞ or $-\infty$.

① $f(x) = x^2 e^{-x}$

Domain : $x \in \mathbb{R}$

$$f'(x) = 2xe^{-x} + x^2(-e^{-x}) = e^{-x} \cdot x(2-x)$$

$$f'(x) = -(e^{-x}) \cdot x(x-2)$$



$f(0) = 0$ ✓

$f(2) = \frac{4}{e^2}$ ✓

f is ↑ in $[0, 2]$

f is ↓ in $(-\infty, 0]$; $[2, \infty)$

$\lim_{x \to \infty} \left(\frac{x^2}{e^x}\right) \to 0^+$ ✓ & $\lim_{x \to (-\infty)} \left(\frac{x^2}{e^x}\right) \to \infty$ ✓

$$f'(x) = e^{-x}(2x - x^2)$$

$$f''(x) = e^{-x}(2-2x) - e^{-x}(2x - x^2)$$

$$= e^{-x}(2 - 2x - 2x + x^2)$$

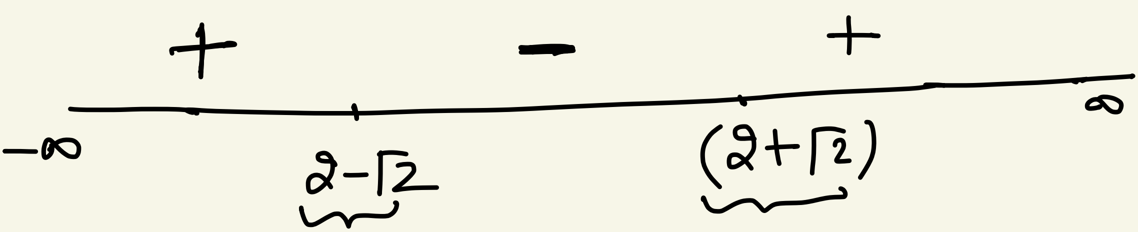
$$= e^{-x}(x^2 - 4x + 2)$$

$$= e^{-x}\left((x^2 - 4x + 4) - 2\right)$$

$$= e^{-x}\left((x-2)^2 - (\sqrt{2})^2\right)$$

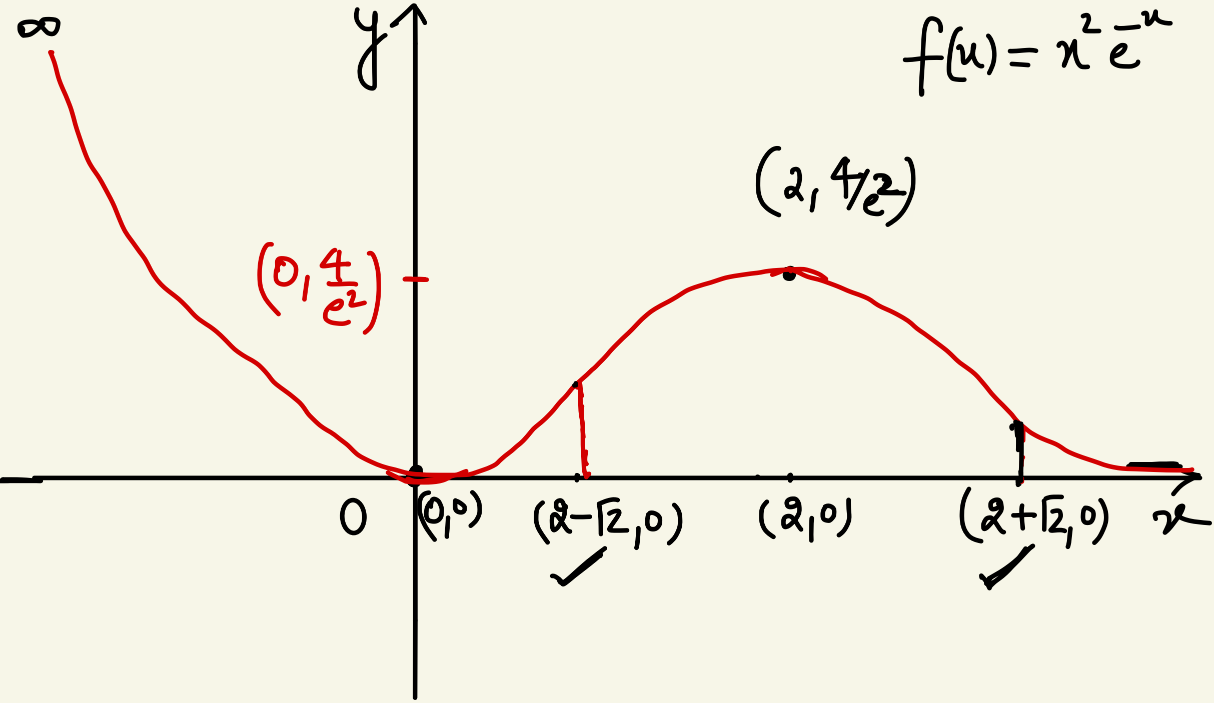
$$= e^{-x}(x - 2 - \sqrt{2})(x - 2 + \sqrt{2})$$

$$= e^{-x}\left(x - (2+\sqrt{2})\right)\left(x - (2-\sqrt{2})\right)$$



Concave up in $(-\infty, 2-\sqrt{2}]$; $[2+\sqrt{2}, \infty)$

Concave down in $[2-\sqrt{2}, 2+\sqrt{2}]$.



Q. Find K for which the equation

$Ke^x = x^2$ has

(i) No sol^n $\rightarrow K \in (-\infty, 0)$

(ii) 1 sol^n $\rightarrow K \in \left(\frac{4}{e^2}, \infty\right) \cup \{0\}$

$K \in \{4/e^2\} \leftarrow$ (iii) 2 distinct sol^n.

$K \in \left(0, \frac{4}{e^2}\right) \leftarrow$ (iv) 3 " " .

(v) 4 " " $\rightarrow K \in \phi$

Sol^n

$$\boxed{K = e^{-x} x^2}$$

$C_1 :\ y = K$

$C_2 :\ y = \underbrace{e^{-x} \cdot x^2}.$

(2) $f(x) = \dfrac{x^2 - x + 1}{x^2 + x + 1}$; $\boxed{f(0) = 1}$

$D \in \mathbb{R}$

$$f'(x) = \frac{(x^2+x+1)(2x-1) - (x^2-x+1)(2x+1)}{(x^2+x+1)^2}$$

$$f'(x) = \frac{2(x^2-1)}{(x^2+x+1)^2} = \frac{2(x-1)(x+1)}{(x^2+x+1)^2}$$

$$\begin{array}{ccccc} + & & - & & + \\ \hline & -1 & & 1 & \end{array}$$

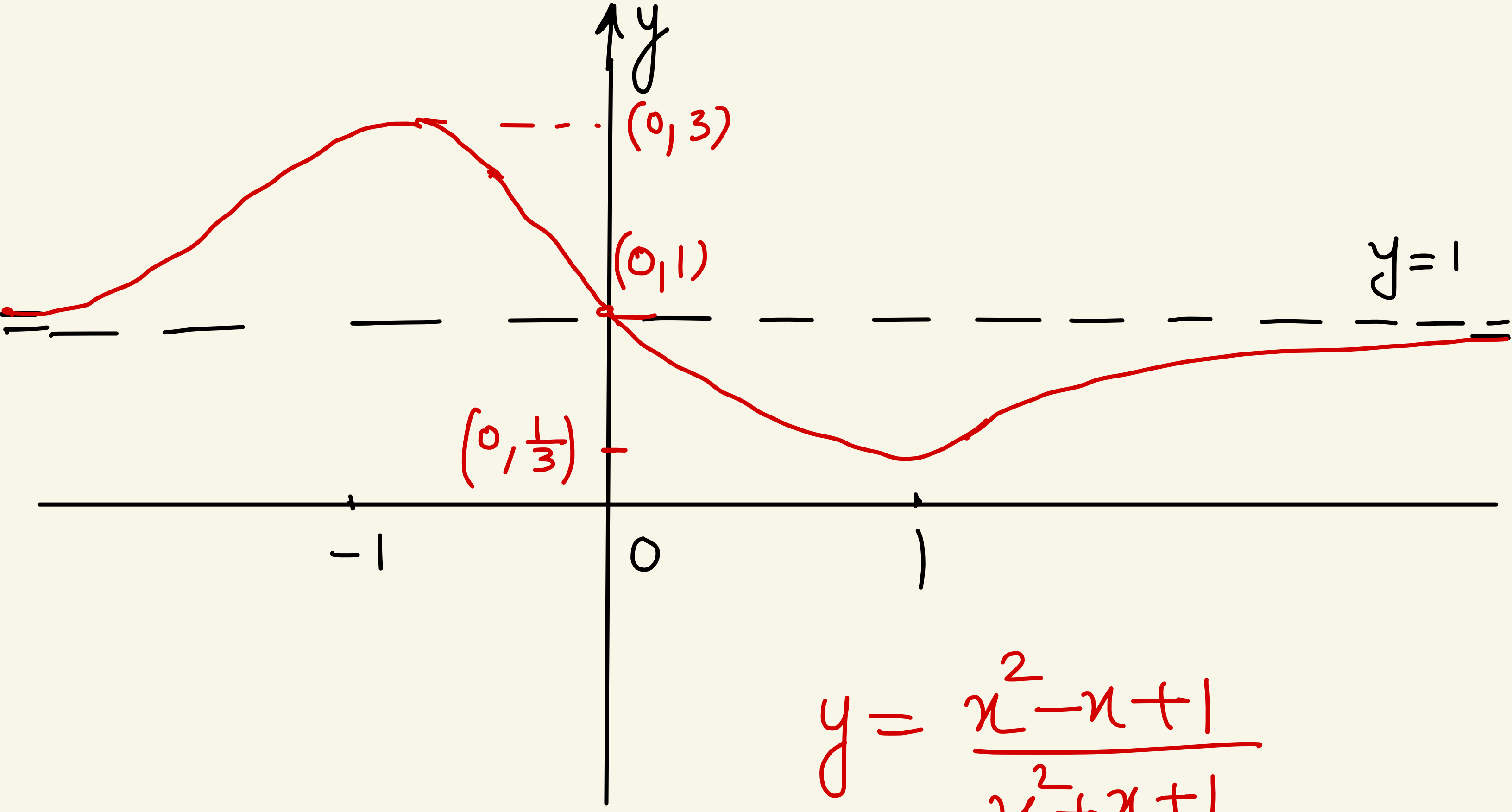
f is $\uparrow$ in $(-\infty, -1]$; $[1, \infty)$

f is $\downarrow$ in $[-1, 1]$

$f(-1) = 3$; $f(1) = \frac{1}{3}$

$$\lim_{x \to \infty} \frac{\cancel{x^2}\left(1 - \frac{1}{x} + \frac{1}{x^2}\right)}{\cancel{x^2}\left(1 + \frac{1}{x} + \frac{1}{x^2}\right)} = 1^-$$

$$\lim_{x \to (-\infty)} \frac{\cancel{x^2}\left(1 - \frac{1}{x} + \frac{1}{x^2}\right)}{\cancel{x^2}\left(1 + \frac{1}{x} + \frac{1}{x^2}\right)} = 1^+$$



$$y = \frac{x^2 - x + 1}{x^2 + x + 1}$$

Q

Sum of integers in range of $\dfrac{(4t^2+1)-[(4t^2-1)\cos 2\theta+2t\sin 2\theta]}{(4t^2+1)+[(1-4t^2)\cos 2\theta+2t\sin 2\theta]}$, $t \in R, \theta \in \left(0, \dfrac{\pi}{2}\right)$ is

(A) less than 3 (B) 3 (C) 6 (D) more than 6

Sol^n

$$\frac{4t^2(1-\cos 2\theta)-2t\sin 2\theta+(1+\cos 2\theta)}{4t^2(1-\cos 2\theta)+2t\sin 2\theta+(1+\cos 2\theta)}$$

$$\frac{4t^2(\cancel{2}\sin^2\theta)-\cancel{2}t(2\sin\theta\cos\theta)+(\cancel{2}\cos^2\theta)}{4t^2(\cancel{2}\sin^2\theta)+\cancel{2}t(2\sin\theta\cos\theta)+(\cancel{2}\cos^2\theta)}$$

$$\frac{4t^2(\tan^2\theta)-2t(\tan\theta)+1}{4t^2(\tan^2\theta)+2t(\tan\theta)+1}$$

Let $\boxed{2t\tan\theta = z}$; $t \in R$; $\theta \in (0, \pi/2)$

$$\frac{z^2-z+1}{z^2+z+1} \; ; \quad z \in R$$

$\to \left[\frac{1}{3}, 3\right]$

$\downarrow$

1, 2, 3

Sum = 6

(3) HW $f(x) = \dfrac{x^2-5x+4}{x^2+2x-3}$. ✓

(4) $f(x) = x^3-3x^2+2$. ✓

(5) $f(x) = x \quad 1x$. ✓

(6) $f(x) = \dfrac{x}{\ln x}$. ✓

Q

If $f : \mathbb{R} \to \mathbb{R}$ is a twice differentiable function such that $f''(x) > 0$ for all $x \in \mathbb{R}$, and $f\left(\frac{1}{2}\right) = \frac{1}{2}$, $f(1) = 1$, then **[JEE(Advanced)-2017]**

(A) $0 < f'(1) \leq \frac{1}{2}$

(B) $f'(1) \leq 0$

(C) $f'(1) > 1$

(D) $\frac{1}{2} < f'(1) \leq 1$

Q

For the function $f(x) = x\cos\frac{1}{x},\ x \ge 1$,

(A) for at least one x in the interval $[1, \infty)$, $f(x + 2) - f(x) < 2$

(B) $\lim_{x\to\infty} f'(x) = 1$

(C) for all x in the interval $[1, \infty)$, $f(x + 2) - f(x) > 2$

(D) $f'(x)$ is strictly decreasing in the interval $[1, \infty)$ **[JEE 2009, 4M]**

8

Let $f(x)=\lim\limits_{n\to\infty}\left(\dfrac{n^n(x+n)\left(x+\frac{n}{2}\right).....\left(x+\frac{n}{n}\right)}{n!(x^2+n^2)\left(x^2+\frac{n^2}{4}\right).....\left(x^2+\frac{n^2}{n^2}\right)}\right)^{x/n}$, for all $x>0$. Then

[JEE(Advanced)-2016, 4(–2)]

(A) $f\left(\frac{1}{2}\right)\geq f(1)$ (B) $f\left(\frac{1}{3}\right)\leq f\left(\frac{2}{3}\right)$ (C) $f'(2)\leq 0$ (D) $\dfrac{f'(3)}{f(3)}\geq\dfrac{f'(2)}{f(2)}$

Q. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be two non-constant differentiable functions. If $f'(x) = (e^{(f(x) - g(x))})g'(x)$ for all $x \in \mathbb{R}$, and $f(1) = g(2) = 1$, then which of the following statement(s) is (are) TRUE ?

[JEE-2018]

(A) $f(2) < 1 - \log_e 2$

(B) $f(2) > 1 - \log_e 2$

(C) $g(1) > 1 - \log_e 2$

(D) $g(1) < 1 - \log_e 2$

Q. Let b be a nonzero real number. Suppose $f:\mathbb{R}\to\mathbb{R}$ is a differentiable function such that $f(0)=1$. If the derivative f' of f satisfies the equation $f'(x)=\dfrac{f(x)}{b^2+x^2}$

for all $x\in\mathbb{R}$, then which of the following statements is/are TRUE?

(A) If $b>0$, then f is an increasing function

(B) If $b<0$, then f is a decreasing function

(C) $(x)\,(-x)=1$ for all $x\in\mathbb{R}$

(D) $(x)-f(-x)=0$ for all $x\in\mathbb{R}$

[JEE - 2020]

Q Let f be continuous in $[a,b]$ and derivable in (a,b). If $f(a)=a$; $f(b)=b$ then prove that thr exist distinct c_1 & c_2 in (a,b) such that $f'(c_1)+f'(c_2)=2$.

Sol^n Consider $f(x)$ on $\left[a, \frac{a+b}{2}\right]$

From LMVT,

$$f'(c_1) = \frac{f\left(\frac{a+b}{2}\right)-f(a)}{\frac{b-a}{2}} \quad \text{— (1) —}$$

Consider $f(x)$ on $\left[\frac{a+b}{2}, b\right]$

From LMVT,

$$f'(c_2) = \frac{f(b)-f\left(\frac{a+b}{2}\right)}{\frac{b-a}{2}} \quad \text{— (2) —}$$

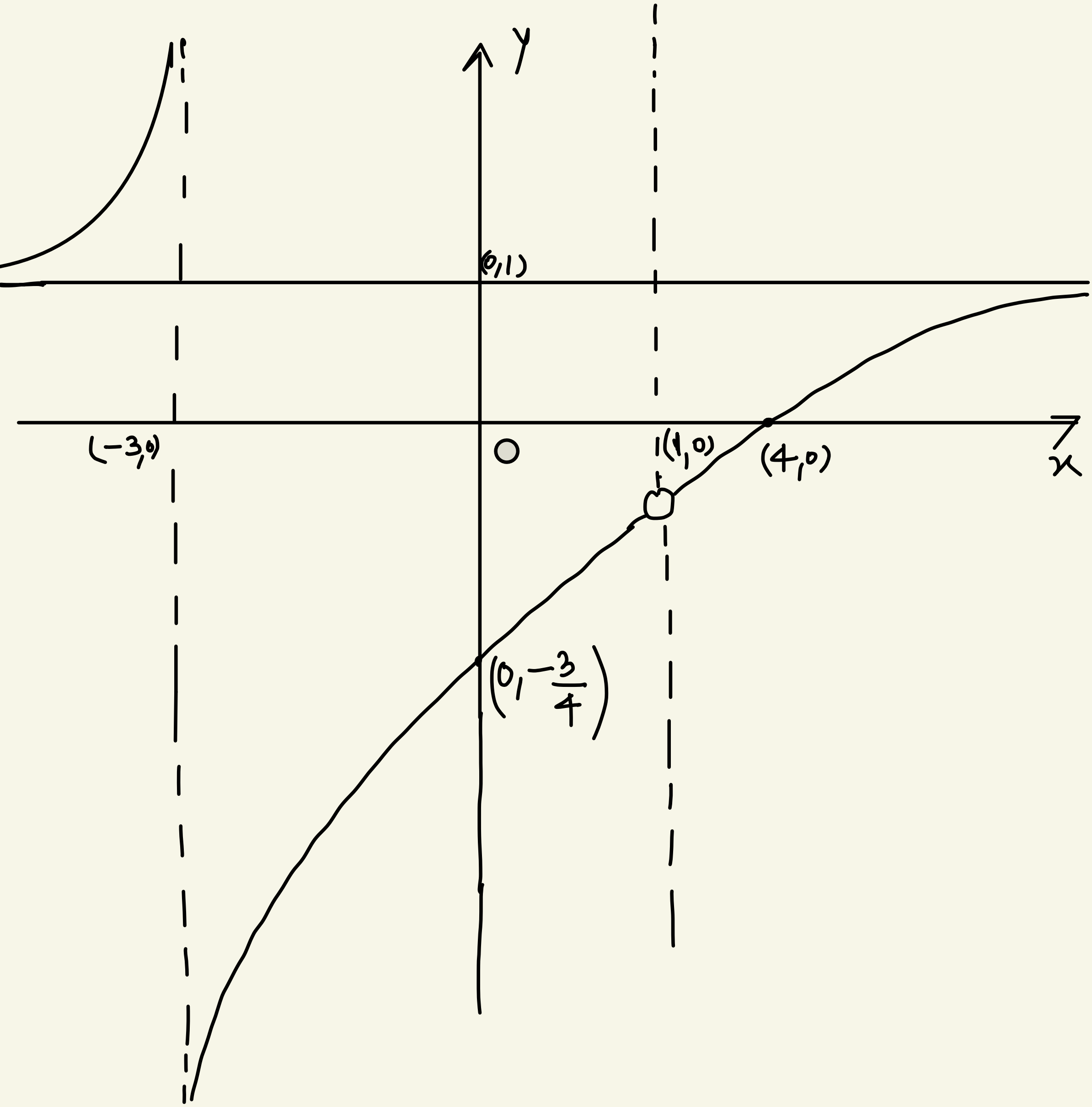
$$f'(c_1)+f'(c_2) = \frac{f(b)-f(a)}{\frac{b-a}{2}} = \frac{\cancel{(b-a)}\cdot 2}{\cancel{(b-a)}}$$

(H.P.)

(3) $f(x) = \dfrac{x^2-5x+4}{x^2+2x-3}$.

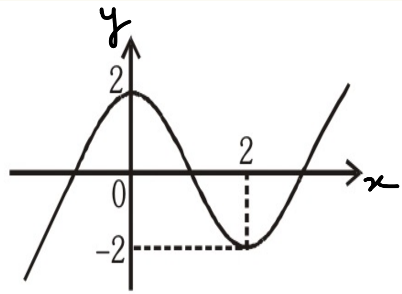
$D_f \in \mathbb{R} - \{-3, 1\}$

$f(x) = \dfrac{\text{Linear poly}_1}{\text{Linear poly}_2}$



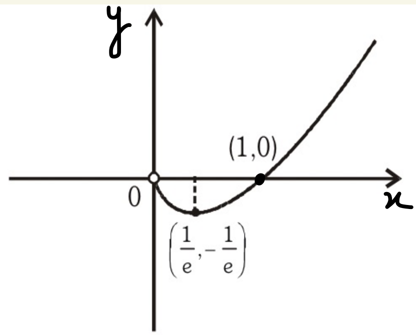
(4)

$f(x) = x^3 - 3x^2 + 2$



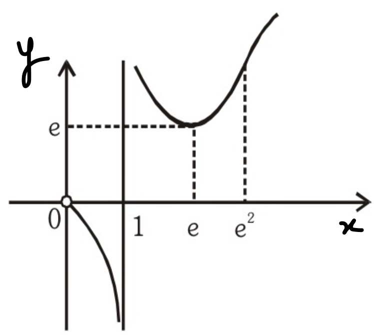
(5)

$f(x) = x\ell nx$



(6)

$f(x) = \frac{x}{\ell nx}$



Q

If $f:\mathbb{R}\rightarrow\mathbb{R}$ is a twice differentiable function such that $f''(x)>0$ for all $x\in\mathbb{R}$, and $f\left(\frac{1}{2}\right)=\frac{1}{2}$, $f(1)=1$, then **[JEE(Advanced)-2017]**

(A) $0<f'(1)\le\frac{1}{2}$ (B) $f'(1)\le 0$ (C) $f'(1)>1$ (D) $\frac{1}{2}<f'(1)\le 1$

Sol^n

$$f''(x)>0 \quad \forall\, x\in R$$

$\Rightarrow$ Concave up & $f'(x)$ is $\uparrow$ fns $\forall x\in R$

Apply LMVT on $f(x)$ on $\left[\frac{1}{2},1\right]$

$$f'(c)=\frac{f(1)-f(1/2)}{1-(1/2)} \quad \text{for atleast one } c\in\left(\frac{1}{2},1\right)$$

$$f'(c)=\frac{1-1/2}{1-1/2}=1$$

Since $f'(x)$ is a $\uparrow$ fns

hence $f'(1)>1$.

Q

For the function $f(x) = x\cos\frac{1}{x},\ x \geq 1,$

(A) for at least one x in the interval $[1, \infty)$, $f(x+2) - f(x) < 2$

✓(B) $\lim\limits_{x\to\infty} f'(x) = 1$

✓(C) for all x in the interval $[1, \infty)$, $f(x+2) - f(x) > 2$

✓(D) $f'(x)$ is strictly decreasing in the interval $[1, \infty)$ **[JEE 2009, 4M]**

Sol^n

$$f'(x) = \cancel{x}\cdot\left(\sin\frac{1}{x}\right)\left(\frac{-1}{x^{\cancel{2}}}\right) + \cos\frac{1}{x}\ ;\ x \geq 1.$$

$$f'(x) = \frac{1}{x}\sin\frac{1}{x} + \cos\frac{1}{x} > 0 \quad \forall x \geq 1$$

$x \geq 1 \Rightarrow \frac{1}{x} \in (0, 1]$

$$\lim_{x\to\infty} f'(x) = \lim_{x\to\infty} \frac{1}{x}\sin\left(\frac{1}{x}\right) + \cos\left(\frac{1}{x}\right) = 1.$$

$$f''(x) = \frac{1}{x}\cdot\left(\cos\frac{1}{x}\right)\left(-\frac{1}{x^2}\right) \cancel{- \frac{1}{x^2}\sin\frac{1}{x}} \cancel{+ \left(\sin\frac{1}{x}\right)\left(\frac{1}{x^2}\right)}$$

$$f''(x) = -\frac{1}{x^3}\cdot\cos\left(\frac{1}{x}\right) < 0 \quad \forall x \in [1, \infty).$$

$\Rightarrow f'(x)$ is strictly decreasing $\forall x \in [1, \infty)$.

$$f'(x) > \lim_{x\to\infty} f'(x) \Rightarrow \boxed{f'(x) > 1}$$

from LMVT on $f(x)$ in $[x, x+2]$

$$\underbrace{f'(c)}_{>1} = \frac{f(x+2) - f(x)}{(x+2) - x}$$

$$\Rightarrow \boxed{f(x+2) - f(x) > 2}$$

Q

Let $f(x)=\lim\limits_{n\to\infty}\left(\dfrac{n^n(x+n)\left(x+\frac{n}{2}\right)\cdots\cdots\left(x+\frac{n}{n}\right)}{n!(x^2+n^2)\left(x^2+\frac{n^2}{4}\right)\cdots\cdots\left(x^2+\frac{n^2}{n^2}\right)}\right)^{x/n}$, for all $x>0$. Then

[JEE(Advanced)-2016, 4(–2)]

(A) $f\left(\frac{1}{2}\right)\geq f(1)$ (B) $f\left(\frac{1}{3}\right)\leq f\left(\frac{2}{3}\right)$ (C) $f'(2)\leq 0$ (D) $\dfrac{f'(3)}{f(3)}\geq\dfrac{f'(2)}{f(2)}$

(B, C)

Sol. $\ell n f(x)=\lim\limits_{n\to\infty}\dfrac{x}{n}\ell n\left[\dfrac{\prod\limits_{r=1}^{n}\left(x+\frac{1}{r/n}\right)}{\prod\limits_{r=1}^{n}\left(x^2+\frac{1}{(r/n)^2}\right)}\dfrac{1}{\prod\limits_{r=1}^{n}(r/n)}\right]$

$$=x\lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n}\ell n\left(\frac{x\frac{r}{n}+1}{\left(x\frac{r}{n}\right)^2+1}\right)$$

$$=x\int_0^1\ell n\left(\frac{1+tx}{1+t^2x^2}\right)dt \qquad \text{put } tx=z$$

$$\ell n f(x)=\int_0^x\ell n\left(\frac{1+z}{1+z^2}\right)dz$$

$$\Rightarrow\quad \frac{f'(x)}{f(x)}=\ell n\left(\frac{1+x}{1+x^2}\right)$$

sign scheme of $f'(x)$ $\quad \overset{+}{\longrightarrow}\ 1\ \overset{-}{\longrightarrow}$ $\quad$ also $f'(1)=0$

$$\Rightarrow\quad f\left(\frac{1}{2}\right)<f(1),\ f\left(\frac{1}{3}\right)<f\left(\frac{2}{3}\right),\ f'(2)<0$$

Also $\dfrac{f'(3)}{f(3)}-\dfrac{f'(2)}{f(2)}=\ell n\left(\dfrac{4}{10}\right)-\ell n\left(\dfrac{3}{5}\right)$

$$=\ell n\left(\frac{4}{6}\right)<0\Rightarrow\frac{f'(3)}{f(3)}<\frac{f'(2)}{f(2)}$$

Q. Let $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ be two non-constant differentiable functions. If $f'(x) = (e^{(f(x) - g(x))})g'(x)$ for all $x \in \mathbb{R}$, and $f(1) = g(2) = 1$, then which of the following statement(s) is (are) TRUE ?

(A) $f(2) < 1 - \log_e 2$

(B) $f(2) > 1 - \log_e 2$

(C) $g(1) > 1 - \log_e 2$

(D) $g(1) < 1 - \log_e 2$

[JEE-2018]

Ans. (B,C)

Sol. $f'(x) = e^{(f(x) - g(x))}\, g'(x)\ \forall\ x \in \mathbb{R}$

$$\Rightarrow\ e^{-f(x)}.\ f'(x) - e^{-g(x)}g'(x) = 0$$

$$\Rightarrow\ \int\left(e^{-f(x)} f'(x) - e^{-g(x)}.g'(x)\right)dx = C$$

$$\Rightarrow\ -e^{-f(x)} + e^{-g(x)} = C$$

$$\Rightarrow\ -e^{-f(1)} + e^{-g(1)} = -e^{-f(2)} + e^{-g(2)}$$

$$\Rightarrow\ -\frac{1}{e} + e^{-g(1)} = -e^{-f(2)} + \frac{1}{e}$$

$$\Rightarrow\ e^{-f(2)} + e^{-g(1)} = \frac{2}{e}$$

$$\therefore\ e^{-f(2)} < \frac{2}{e} \text{ and } e^{-g(1)} < \frac{2}{e}$$

$$\Rightarrow\ -f(2) < \ln 2 - 1 \text{ and } -g(1) < \ln 2 - 1$$

$$\Rightarrow\ f(2) > 1 - \ln 2 \text{ and } g(1) > 1 - \ln 2$$

Q

Let b be a nonzero real number. Suppose $f:\mathbb{R}\to\mathbb{R}$ is a differentiable function such that $f(0)=1$. If the derivative f' of f satisfies the equation $f'(x)=\dfrac{f(x)}{b^2+x^2}$

for all $x\in\mathbb{R}$, then which of the following statements is/are TRUE?

(A) If $b>0$, then f is an increasing function ✓

(B) If $b<0$, then f is a decreasing function

(C) $f(x)f(-x)=1$ for all $x\in\mathbb{R}$ ✓

(D) $f(x)-f(-x)=0$ for all $x\in\mathbb{R}$

[JEE - 2020]

Soln

$$\int\frac{f'(x)}{f(x)}dx=\int\frac{dx}{x^2+b^2}$$

$$\Rightarrow \ell n|f(x)|=\frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)+c$$

Now $f(0)=1$

$\therefore\ c=0$

$$\therefore\ |f(x)|=e^{\frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)}$$

$$\Rightarrow f(x)=\pm e^{\frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)}$$

since $f(0)=1\ \therefore\ f(x)=e^{\frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)}$

$x\to -x$

$$f(-x)=e^{-\frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)}$$

$\therefore f(x).f(-x)=e^0=1$ (option C)

and for $b>0$

$$f(x)=e^{\frac{1}{b}\tan^{-1}\left(\frac{x}{b}\right)}$$

$\Rightarrow f(x)$ is increasing for all $x\in R$ (option A)

Maxima and Minima

MAXIMA & MINIMA :

(i) **Local Maxima/Relative maxima :** A function $f(x)$ is said to have a local maxima at x = a if

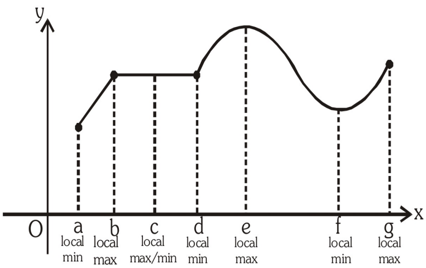
$f(a) \geq f(x)\ \forall\ x \in (a - h, a + h) \cap D_{f(x)}$

Where h is some positive real number.

(ii) **Local Minima/Relative minima :** A function $f(x)$ is said to have a local minima at x = a if

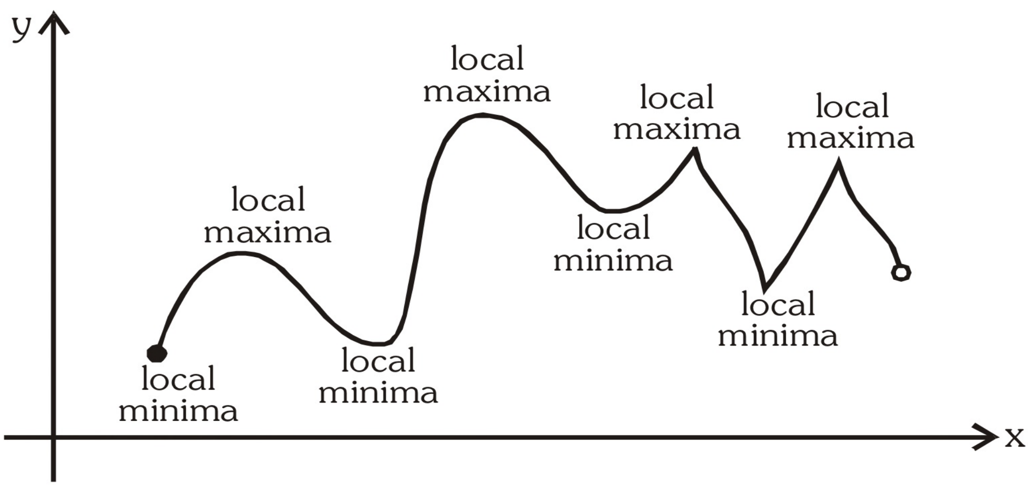
$f(a) \leq f(x)\ \forall\ x \in (a - h, a + h) \cap D_{f(x)}$

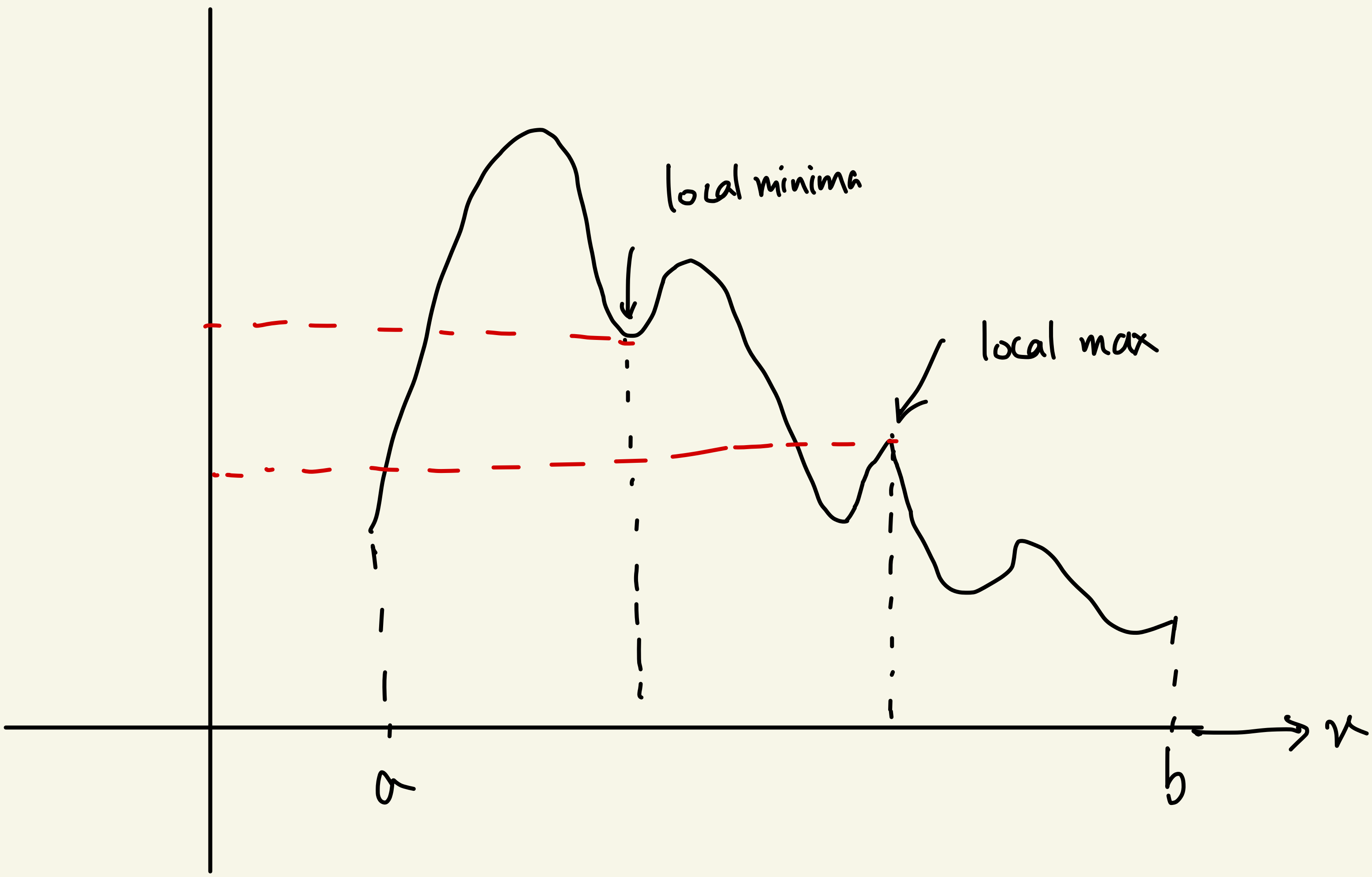
Where h is some positive real number.



(iii) **Absolute maxima (Global maxima) :**
A function f has an absolute maxima (or global maxima) at c if f(c) ≥ f(x) for all x in D, where D is the domain of f. The number f(c) is called the maximum value of f on D.

(iv) **Absolute minima (Global minima) :**
A function f has an absolute minima at c if f(c) ≤ f(x) for all x in D and the number f(c) is called the minimum value of f on D.





local minima
local max
a
b
x

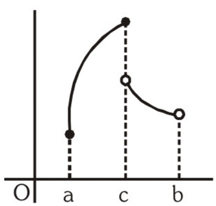
Note : or extremal or turning value or extremum

(i) The term 'extrema' is used for both maxima or minima.

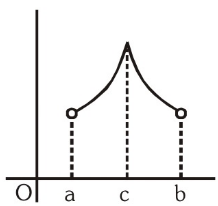
(ii) A local maximum (minimum) value of a function may not be the greatest (least) value in a finite interval.

(iii) A function can have several extreme values such that local minimum value may be greater than a local maximum value.

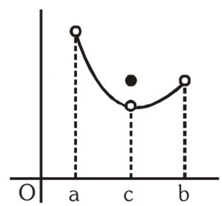
(iv) It is not necessary that $f(x)$ always has local maxima/ minima at end points of the given interval when they are included.



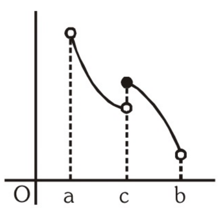
absolute maxima at x = c
absolute minima at x = a
local maxima at x = c
local minima at x = a



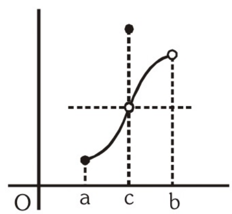
absolute maxima at x = c
no absolute minima
local maxima at x = c



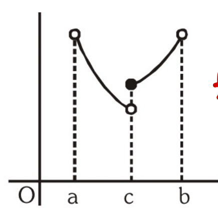
Local maxima at x = c
no absolute maxima / minima



local maxima at x = c
no absolute maxima/minima



absolute maxima at x = c
absolute minima at x = a
local maxima at x = c
local minima at x = a



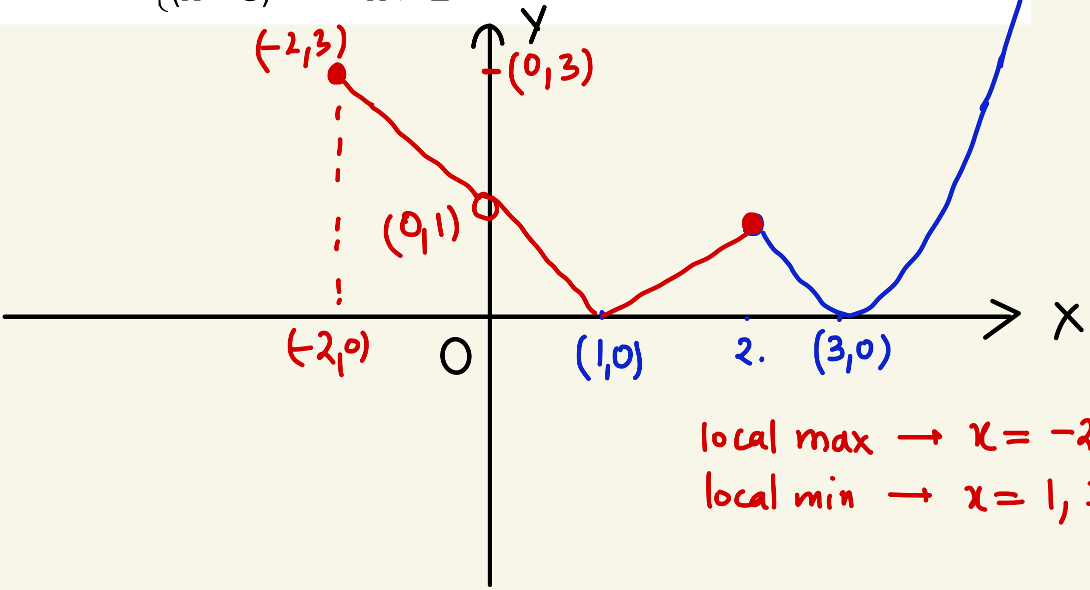
neither maxima nor minima at x = c

f(x) is ↑ at x = c

Q.

If $f(x) = \begin{cases} |x-1| & 0 < |x| \le 2 \\ (x-3)^2 & x > 2 \end{cases}$ find points of maxima/minima.

$-2 \le x \le 2 \; ; \; x \ne 0.$

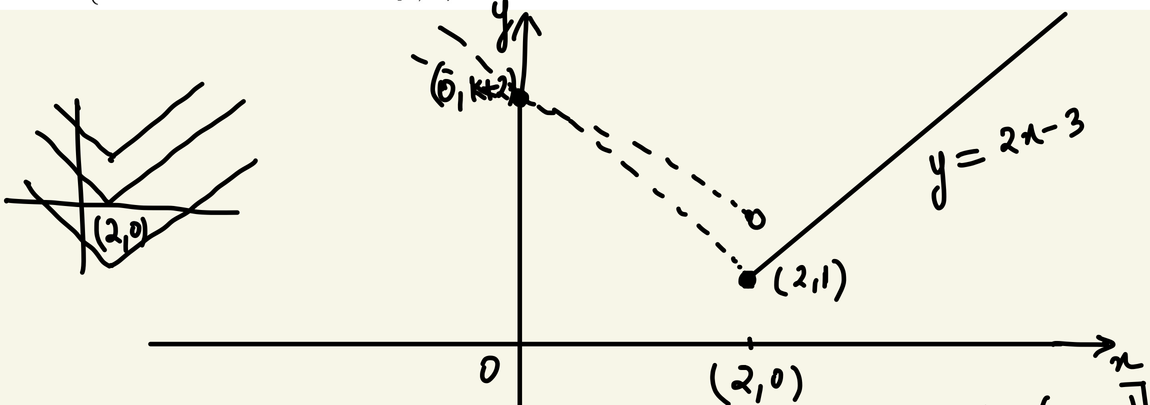


local max → $x = -2, 2$

local min → $x = 1, 3.$

Q.

If $f(x) = \begin{cases} |x-2| + \underbrace{a^2 - 9a - 9}_{k} & x \in (-\infty, 2) \\ 2x - 3 & x \in [2, \infty) \end{cases}$ find the set of values of 'a' such that x = 2 is a local minima.



$a \in (-\infty, -1] \cup [10, \infty)$ Ans

$\lim_{x \to 2^-} \left(|x-2| + k \right) = k$

$k \geq 1$

$a^2 - 9a - 9 \geq 1 \Rightarrow a^2 - 9a - 10 \geq 0$

Q Find points of extrema :

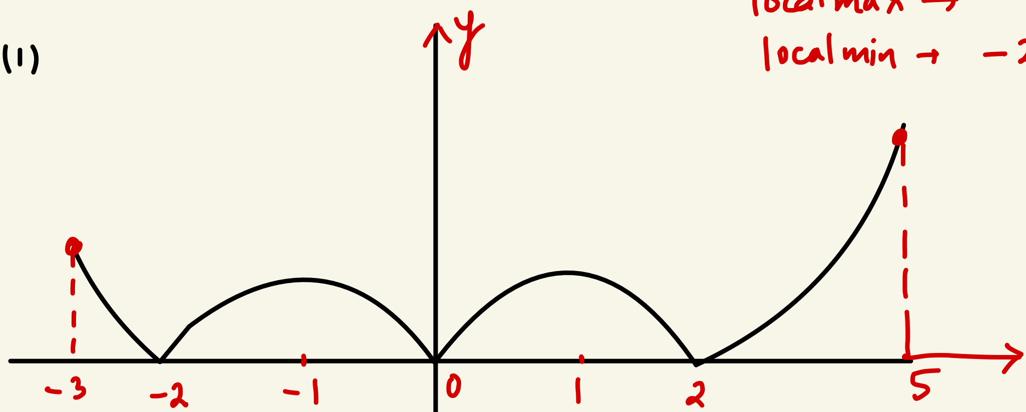
(1) $f(x) = \left| x^2 - 2|x| \right|$ in $[-3, 5]$.

(2) $f(x) \to \begin{cases} \{-x\} \; ; & x \in [-1, 0) \\ 1 - x^2 \; ; & x \in [0, 1] \\ [x] \; ; & x \in (1, 2] \end{cases}$

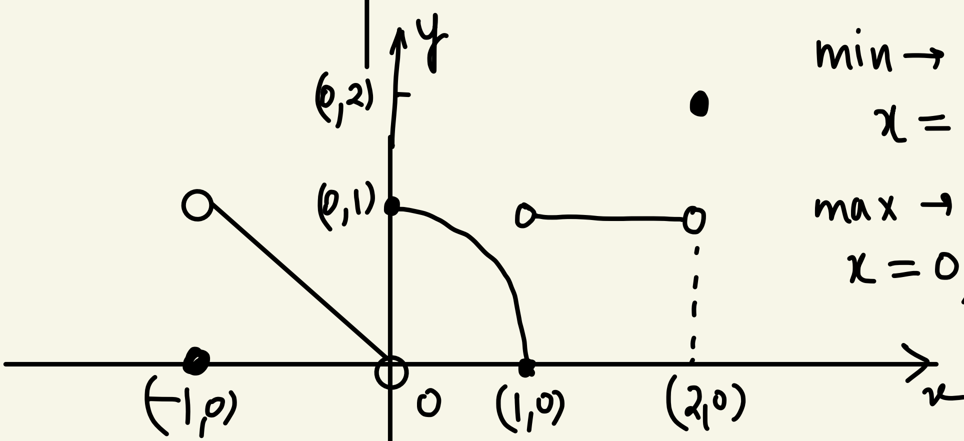
$[\ ] \to$ Gif & $\{\ \} \to$ ....——

local max $\to -3, -1, 1, 5$
local min $\to -2, 0, 2$.

(1)



(2)



min $\to$
$x = -1; 1$

max $\to$
$x = 0; 2$

Let $f(x) = \begin{cases} 3 - x & 0 \le x < 1 \\ x^2 + \ell n b & x \ge 1 \end{cases}$. Find the set of values of b such that $f(x)$ has a local minima at x = 1.

If $f(x) = |px - q| + r|x|$, $x \in (-\infty, \infty)$, p, q, r > 0, assumes its minimum value only at one point then find the relation between p, q, r.

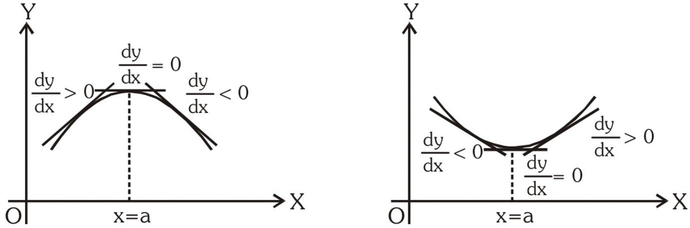
DERIVATIVE TEST FOR ASCERTAINING MAXIMA/MINIMA :

(a) First derivative test (For differentiable function) :

The point (say x = a) where f'(x) = 0 and

(i) If f'(x) changes sign from positive to negative in the neighbourhood of x = a, then x = a is said to be a point **local maxima.**

(ii) If f'(x) changes sign from negative to positive in the neighbourhood of x = a, then x = a is said to be a point **local minima.**



eg: $f(x) = \frac{x^3}{3} - x^2 - 3x + 7$?

$D_f \in \mathbb{R}$

$f'(x) = x^2 - 2x - 3$

$= (x+1)(x-3)$

+   −   +

−1   3

max   min

(b) Second derivative test :

If f(x) is continuous and differentiable at x = a where f'(a) = 0 and f''(a) also exists then for ascertaining maxima/minima at x = a, 2nd derivative test can be used -

(i) If $f''(a) > 0 \Rightarrow$ x = a is a point of local minima

(ii) If $f''(a) < 0 \Rightarrow$ x = a is a point of local maxima

(iii) If $f''(a) = 0 \Rightarrow$ second derivative test fails. To identify maxima/minima at this point either first derivative test or higher derivative test can be used.

(c) n^{th} derivative test :

Let f(x) be a function such that $f'(a) = f''(a) = f'''(a) = \dots\dots\dots = f^{n-1}(a) = 0$ & $f^n(a) \neq 0$, then

(i) If n is even &

$$\begin{cases} f^n(a) > 0 \Rightarrow \text{Minima} \\ f^n(a) < 0 \Rightarrow \text{Maxima} \end{cases}$$

(ii) If n is odd then neither maxima nor minima at x = a.

i.e. $f(x) = \frac{1}{4!}(x-5)^4$ has minima at x = 5

eg: $\boxed{f(x) = -x^4}$

$\boxed{f'(x) = -4x^3} = 0 \Rightarrow x = 0$

$f''(x) = -12x^2$

$f''(0) = 0.$

$f' > 0$ $\quad$ $f' < 0$

x = 0

max

eg: $f(x) = x^{5/3} \Rightarrow \frac{5}{3}(x^2)^{1/3}$

$f'(x) = \frac{5}{3} x^{2/3} = 0 \Rightarrow x = 0$

$f''(x) = \frac{5}{3} \cdot \frac{2}{3} x^{-1/3} \Rightarrow f''(0) = \underline{\underline{dne}}$

$f' > 0$ $\quad$ $f' > 0$

0

increasing at $\underline{\underline{x = 0}}$

Q Examine the points of local maxima/minima for

HW

(a) $f(x) = x\ell nx$ (b) $f(x) = x.e^{x-x^2}$

(a) $f(x) = x\ln x$; $D_f \in (0, \infty)$

$f'(x) = x \cdot \frac{1}{x} + \ln x \cdot 1 = (1 + \ln x) = 0 \Rightarrow x = \frac{1}{e}$

$-$ $+$

0 $\frac{1}{e}$ ∞

minima

$\therefore$ $\boxed{\text{minima at } x = \frac{1}{e}}$

Q

Find all the points of local maxima and minima and the corresponding maximum and minimum values of the function, $f(x) = -\frac{3}{4}x^4 - 8x^3 - \frac{45}{2}x^2 + 105.$

Solⁿ

$D_f \in \mathbb{R}$

$$f'(x) = -\frac{3}{\cancel{4}} \cdot \cancel{4}x^3 - 24x^2 - 45x$$

$$= -3x(x^2 + 8x + 15)$$

$$= -3x(x+3)(x+5)$$

$+$		$-$		$+$		$-$
	-5		-3		0	

max $\rightarrow$ $x = -5, 0$

min $\rightarrow$ $x = -3.$

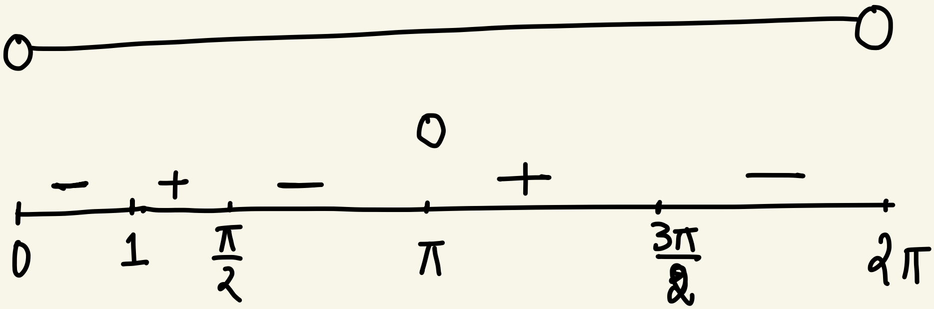
Q

$f(x)=\int_0^x (t^2-1)\cot t\,dt$, $x \in (0,2\pi)$. $f(x)$ attains local maximum value at :

(A) $x=\frac{\pi}{2}$ (B) $x = 1$ (C) $x=\frac{3\pi}{2}$ (D) None

Sol^n

$$f'(x) = \underbrace{(x^2-1)}\ \underbrace{\cot x}$$



Q HW

Find the values of a and b for which the function $y = a\ \ell nx + bx^2 + x$ has extrema at the points $x_1 = 1$ and $x_2 = 2$. Show that for the found values of a and b the given function has a minimum at the point x_1 and a maximum at x_2.

USEFUL FORMULAE OF MENSURATION TO REMEMBER :

(a) Volume of a cuboid = ℓbh.

(b) Surface area of a cuboid = $2(\ell b + bh + h\ell)$.

(c) Volume of a prism = area of the base x height.

(d) Lateral surface of a prism = perimeter of the base x height.

(e) Total surface of a prism = lateral surface + 2 area of the base

(Note that lateral surfaces of a prism are all rectangles).

(f) Volume of a pyramid = $\frac{1}{3}$ (area of the base) × (height).

(g) Curved surface of pyramid = $\frac{1}{2}$ (perimeter of the base) x slant height.

(Note that slant surfaces of a pyramid are triangles).

(h) Volume of a cone = $\frac{1}{3}\pi r^2 h$.

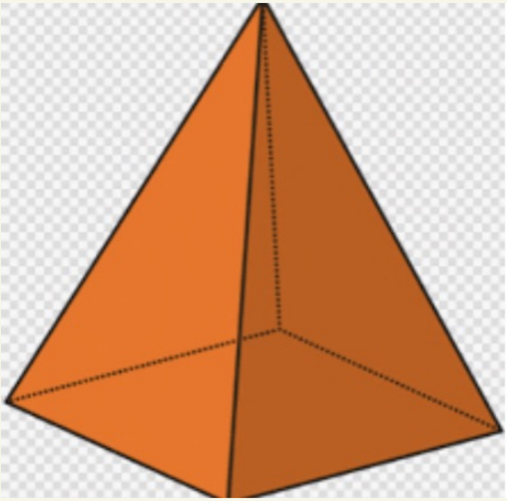
(i) Curved surface of a cylinder = $2\pi rh$.

(j) Total surface of a cylinder = $2\pi rh + 2\pi r^2$.

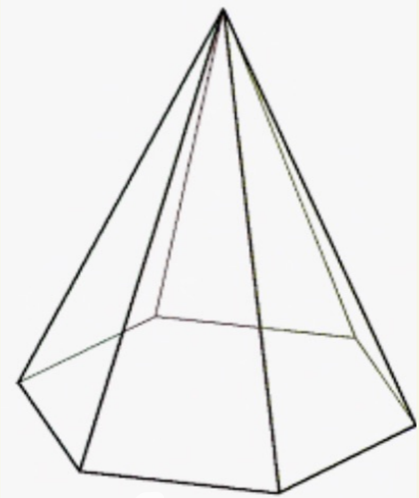
(k) Volume of a sphere = $\frac{4}{3}\pi r^3$.

(l) Surface area of a sphere = $4\pi r^2$.

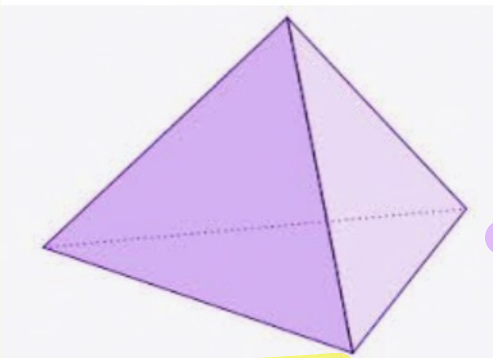
(m) Area of a circular sector = $\frac{1}{2}r^2\theta$, when θ is in radians.



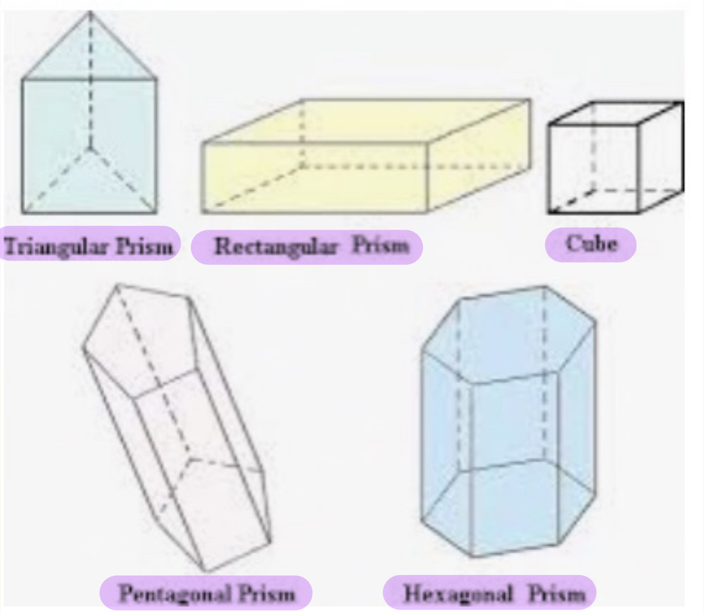
Rectangular pyramid.



Hexagonal pyramid.



Triangular pyramid



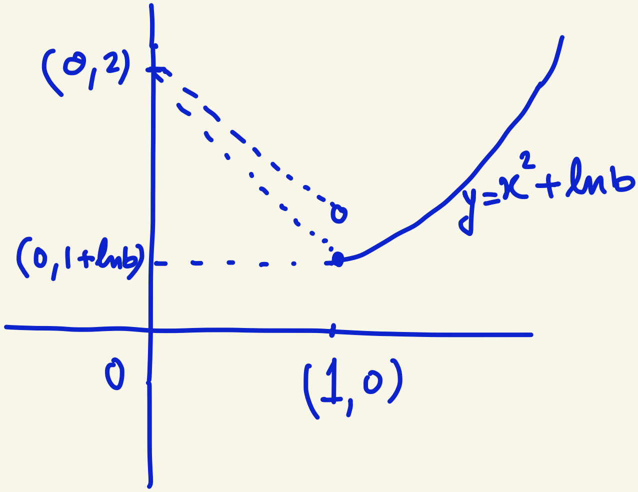
Q

Let $f(x) = \begin{cases} 3 - x & 0 \le x < 1 \\ x^2 + \ell n b & x \ge 1 \end{cases}$. Find the set of values of b such that $f(x)$ has a local minima at $x = 1$.

Q

If $f(x) = |px - q| + r|x|$, $x \in (-\infty, \infty)$, $p, q, r > 0$, assumes its minimum value only at one point then find the relation between p, q, r.

(1)



$f(1) = 1 + \ln b.$

$b > 0$ —(1)—

$1 + \ln b \le \lim_{x \to 1^-} f(x)$

$1 + \ln b \le 2$

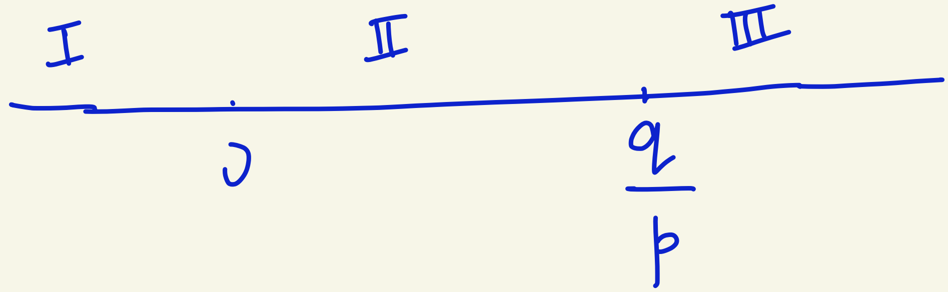
$\ln b \le 1$

$b \le e$ —(2)—

(1) ∩ (2)

$\therefore b \in (0, e]$ Ans

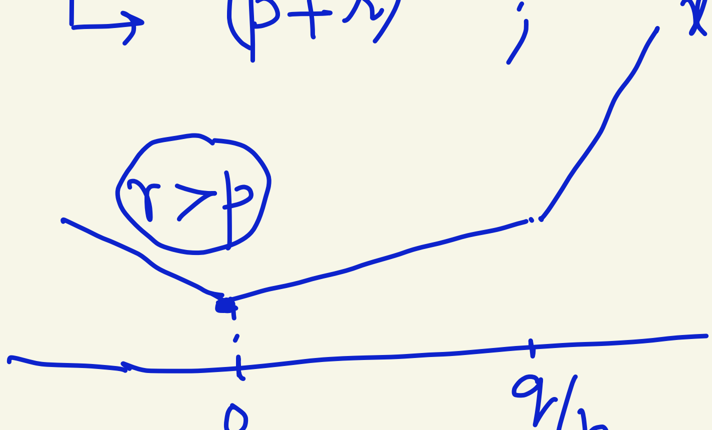
② $f(x) = |px - q| + r|x|$; $x \in \mathbb{R}$; $\underline{\underline{p, q, r > 0}}$



$$f(x) \begin{cases} q - (p+r)x & ; x < 0 \\ q - px + rx & ; 0 \le x < \frac{q}{p} \quad ✓ \\ (p+r)x - q & ; x \geq \frac{q}{p} \quad ✓ \end{cases}$$

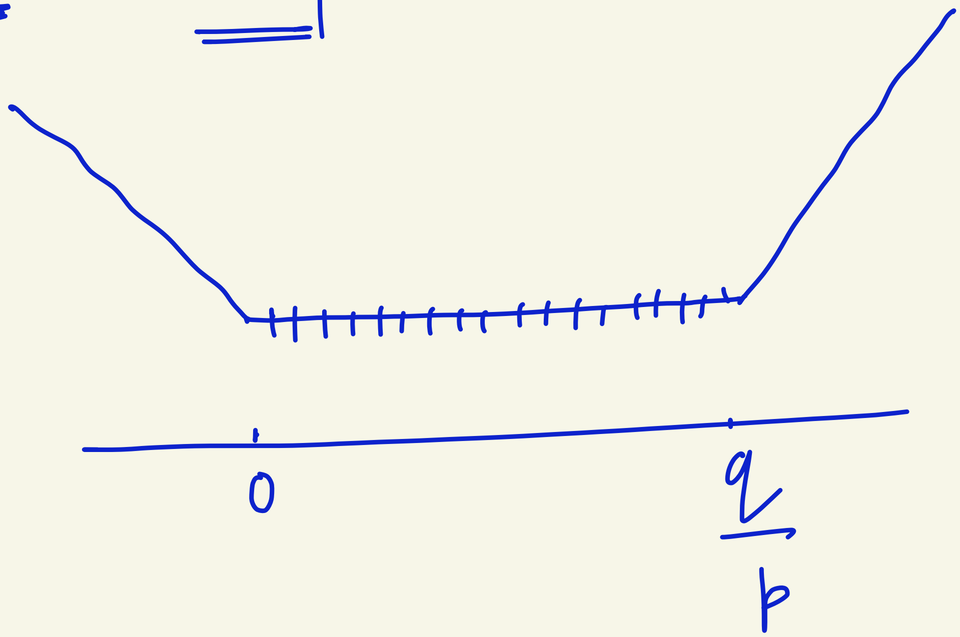
$$f'(x) \begin{cases} -(p+r) & ; x < 0 \\ * \; (r-p) & ; 0 < x < \frac{q}{p} \\ (p+r) & ; x > \frac{q}{p} \quad *. \end{cases}$$

C-I

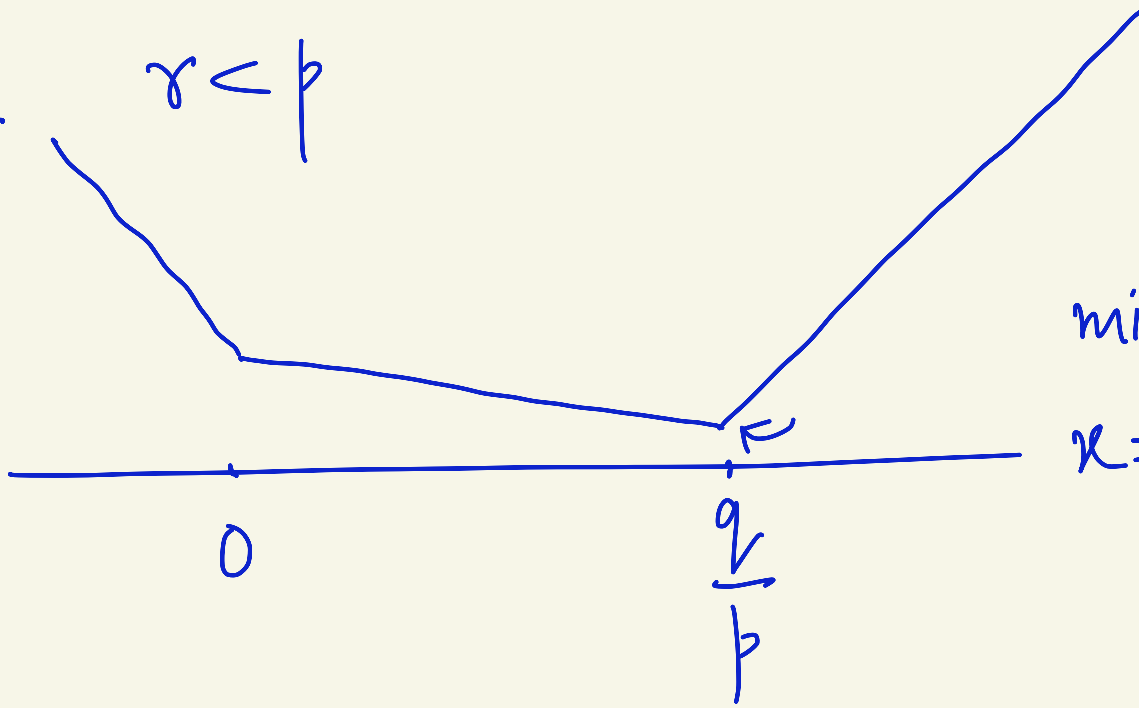


✓ minima at $x = 0$ only

C-II $r = p$ XX



C-III $r < p$ ✓



minima at $x = \frac{q}{p}$ only $\Rightarrow$

$\therefore$ $\boxed{r \neq p}$

Q Examine the points of local maxima/minima for

(a) $f(x) = x\ell nx$ HW (b) $f(x) = x.e^{x-x^2}$

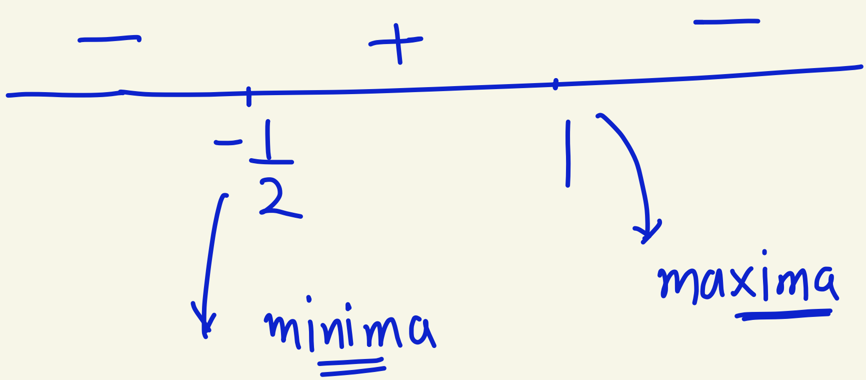
(b) $f(x) = x \cdot e^{x-x^2}$; $D_f \in \mathbb{R}$

$$f'(x) = e^{x-x^2} + x \cdot e^{x-x^2}(1-2x)$$

$$= e^{x-x^2}\left(1 + x(1-2x)\right)$$

$$= -e^{x-x^2}(2x^2 - x - 1)$$

$$= -e^{x-x^2}(2x+1)(x-1)$$



Find the values of a and b for which the function $y = a\,\ell nx + bx^2 + x$ has extrema at the points $x_1 = 1$ and $x_2 = 2$. Show that for the found values of a and b the given function has a minimum at the point x_1 and a maximum at x_2.

$a = \frac{-2}{3}$ & $b = -\frac{1}{6}$.

Q. Let $f(x)$ be a cubic polynomial which has local maximum at $x=-1$ and $f'(x)$ has a local minimum at $x=1$. If $f(-1)=10$ and $f(3)=-22$, then find the distance between its two horizontal tangents.

$\boxed{32}$ Ans

Soln

$$f''(x) = a(x-1)$$

$$f'(x) = \frac{a(x-1)^2}{2} + c.$$

$$\boxed{f'(-1)=0} \Rightarrow 0 = \frac{a}{2}(4) + c$$

$$\therefore \boxed{c = -2a}$$

$$f'(x) = \frac{a(x-1)^2}{2} - 2a.$$

$$f(x) = \frac{a(x-1)^3}{6} - 2ax + K$$

$$f(-1) = \frac{a(-8)}{6} + 2a + K = 10 \quad -①-$$

$$f(3) = \frac{a}{6}(8) - 6a + K = -22 \quad -②-$$

$x=-1 \longrightarrow$ horizontal tangent. $\Big\}$ $\left.\begin{array}{l} f'(-1)=0 \\ f(3)=0 \end{array}\right]$

$x = 3 \longrightarrow$ " "

9

Let P(x) be a polynomial of degree atmost four. It has zeros and also minima at $x_1 = -3$ and $x_2 = 5$. Given P(x) has a local maximum value 256, then find the value of definite integral

$$\int_0^1 (P(x) - P(-x))\,dx. \; = 58 \quad \text{Ans}$$

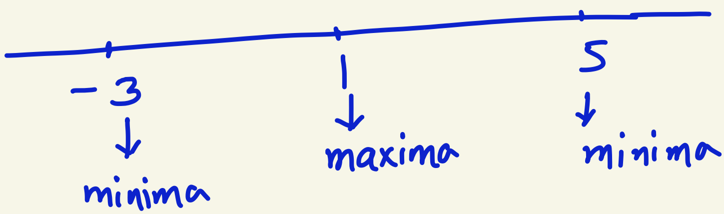
Sol^n

$$P(x) = a(x+3)^2(x-5)^2$$

$$P'(x) = a\left((x+3)^2\,2(x-5) + 2(x+3)(x-5)^2\right)$$

$$= 2a(x+3)(x-5)(x+3+x-5)$$

$$= 2a(x+3)(x-5)(2x-2)$$



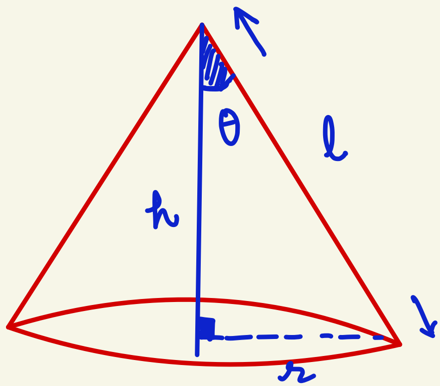
** $P(1) = 256. \Rightarrow a(16)(16) = 256$

$$\therefore \boxed{a = 1}$$

$$P(x) = (x+3)^2(x-5)^2$$

$$P(-x) = (-x+3)^2(x+5)^2.$$

Q Prove that, for a given slant height, the volume of conical tent is maximum if semi vertical angle is $\tan^{-1}\sqrt{2}$



$l \to$ given.

$r = l\sin\theta.$
$h = l\cos\theta.$

$V = \frac{1}{3}\pi r^2 h.$

$V = \frac{1}{3}\pi (l\sin\theta)^2 (l\cos\theta)$

$V = \frac{\pi l^3}{3} \underbrace{(\sin^2\theta \cdot \cos\theta)}_{f(\theta)}$

$f(\theta) = \sin^2\theta \cdot \cos\theta.$

$f'(\theta) = -\sin^3\theta + 2\sin\theta\cos^2\theta.$

$f'(\theta) = \sin\theta(2\cos^2\theta - \sin^2\theta) = 0$

xx $\sin\theta = 0$; $\tan^2\theta = 2$ **

$\boxed{\theta = n\pi}$ X

$\boxed{\tan\theta = \sqrt{2}}$ Ans

OR

$\boxed{\tan\theta = -\sqrt{2}}$ XX

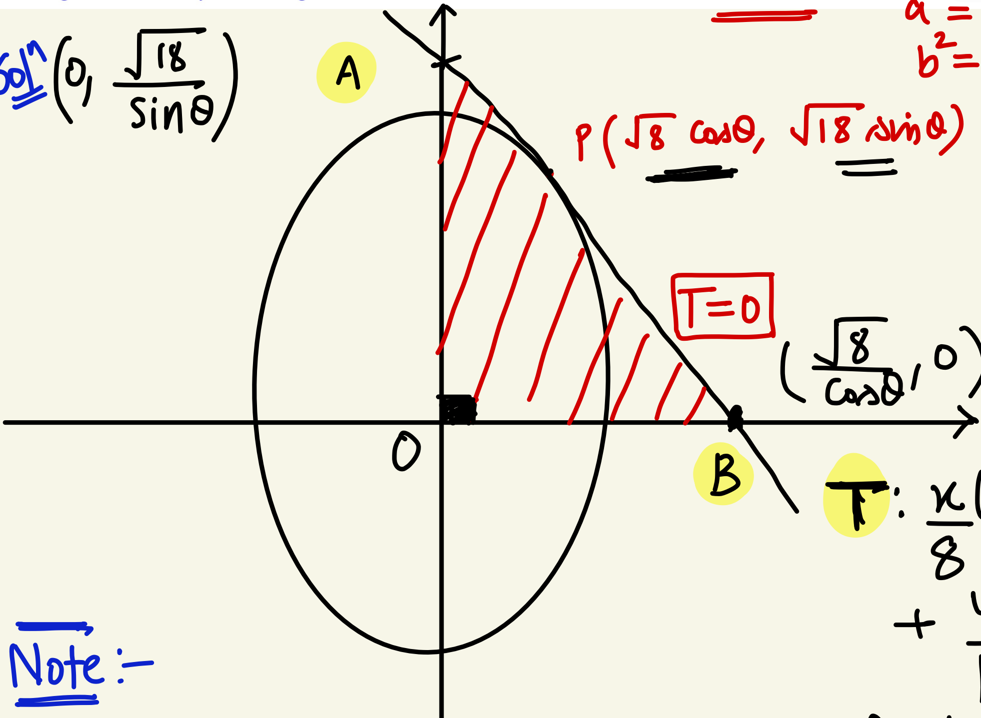
Q

Find the coordinates of the point P on the curve $\frac{x^2}{8}+\frac{y^2}{18}=1$ in the 1st quadrant so that the area of the triangle formed by the tangent at P and the coordinates axes is minimum.

$a^2 = 8$

$b^2 = 18$

Sol^n $\left(0, \frac{\sqrt{18}}{\sin\theta}\right)$



$P(\sqrt{8}\cos\theta, \sqrt{18}\sin\theta)$

$\theta \in \left(0, \frac{\pi}{2}\right)$

$\boxed{T=0}$

$\left(\frac{\sqrt{8}}{\cos\theta}, 0\right)$

$T: \frac{x(\sqrt{8}\cos\theta)}{8} + \frac{y(\sqrt{18}\sin\theta)}{18} = 1$

Note :-

For general 2^{nd} degree curve of the type

$ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$

the equ of tangent at (x_1, y_1) can be directly obtained by :

$x^2 \to xx_1$; $y^2 \to yy_1$

$2x \to x + x_1$; $2y \to y + y_1$

$2xy \to xy_1 + x_1y$; $c \to c$.

$$\Delta_{AOB} = \frac{1}{2}\left(\frac{\sqrt{8}}{\cos\theta} \cdot \frac{\sqrt{18}}{\sin\theta}\right)$$

$$\Delta_{AOB} = \frac{\sqrt{8 \times 18}}{\underbrace{\sin 2\theta}}$$

$$(\Delta_{AOB})_{min} = \sqrt{8 \times 18} \quad \text{when } \boxed{\theta = \frac{\pi}{4}}$$

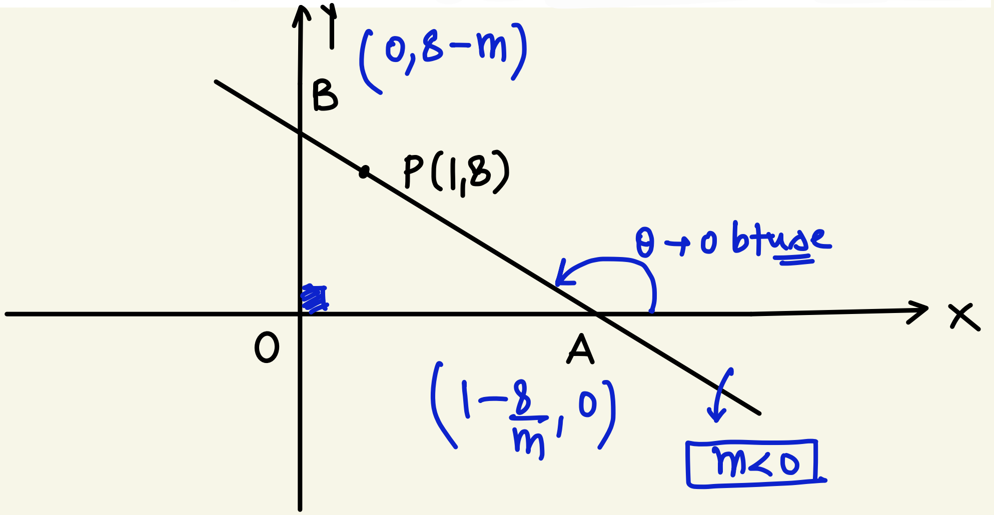
$$P\left(\frac{\sqrt{8}}{\sqrt{2}}, \frac{\sqrt{18}}{\sqrt{2}}\right) \equiv (2, 3) \text{ Ans}$$

8. Find the equation of a line through (1, 8) cutting the positive semi axes at A and B if

(i) the area of ΔOAB is minimum

(ii) its intercept between the coordinate axes is minimum.

(iii) its sum of intercepts minimum



$$L: \boxed{y-8 = m(x-1)}$$

(i) $$\Delta_{OAB} = \frac{1}{2}\left(1-\frac{8}{m}\right)(8-m).$$

$$f(m) = (\quad)(\quad)$$

(ii) $l_{AB} = \sqrt{\left(1-\frac{8}{m}\right)^2 + (8-m)^2}$.

$f(m) = \left(1-\frac{8}{m}\right)^2 + (8-m)^2 \longrightarrow$

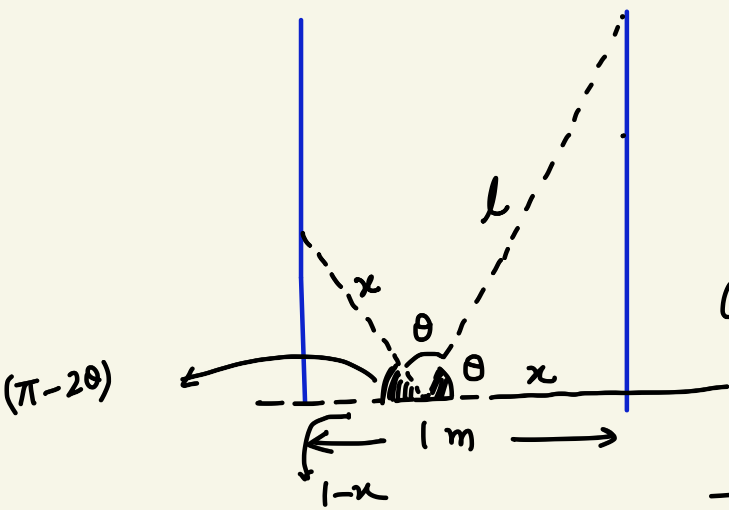
(iii) $f(m) = \left(1-\frac{8}{m}\right) + (8-m)$.

Q

One corner of a rectangular sheet of paper of width 1m, is folded over so as to reach the opposite edge of the sheet.

Find the minimum length of crease. $\frac{3\sqrt{3}}{4}$ Ans

$l \rightarrow$ minimize



$\cos\theta = \frac{x}{l}$

$\cos(\pi-2\theta) = \frac{1-x}{x}$

$\downarrow$

$-\cos 2\theta = \frac{1}{x} - 1$

$1 - \cos 2\theta = \frac{1}{x}$

$x = \frac{1}{1-\cos 2\theta}$

$x = \frac{1}{2\sin^2\theta}$

$l = \frac{x}{\cos\theta}$

$$\boxed{l = \frac{1}{\underbrace{2\sin^2\theta \cdot \cos\theta}}}$$

$f(\theta) = 2\sin^2\theta\cos\theta$

$f(\theta)_{max}$ then l_{min}

$f'(\theta) = -2\sin^3\theta + 4\sin\theta\cos^2\theta$

$\boxed{\tan\theta = \sqrt{2}}$ $\rightarrow \sin\theta$, $\cos\theta$

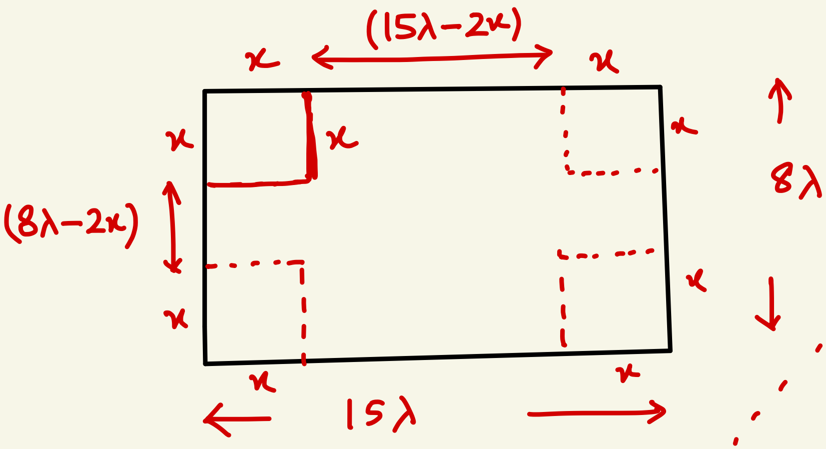
Q

A rectangular sheet of fixed perimeter with sides having their lengths in the ratio of 8 : 15 is converted into an open rectangular box by folding after removing squares of equal area from all four corners. If the total area of removed squares is 100, the resulting box has maximum volume. Then the lengths of the sides of the rectangular sheet are

(A) 24 (B) 32 (C) 45 (C) 60

$4x^2 = 100$

$\boxed{x = 5}$



$$\boxed{V = \underbrace{(15\lambda - 2x)}\,\underbrace{(8\lambda - 2x)}\cdot \underbrace{x}}$$

$\lambda \in$ Constant

$\frac{dV}{dx} = 0$ when $x = 5$ ***

$\lambda = 3$ OR $\lambda = \frac{5}{6}$.

maxima

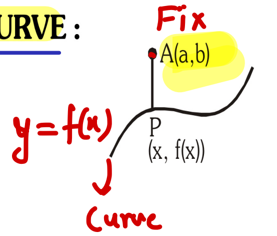
minima

Q
HW
Find the altitude of the right circular cylinder of maximum volume that can be inscribed in a given right circular cone of height 'h'.

HW

Q The lateral edge of a regular rectangular pyramid is 'a' cm long. The lateral edge makes an angle α with the plane of the base. The value of α for which the volume of the pyramid is greatest, is

MINIMUM AND MAXIMUM DISTANCE OF FIXED POINT FROM A GIVEN CURVE :



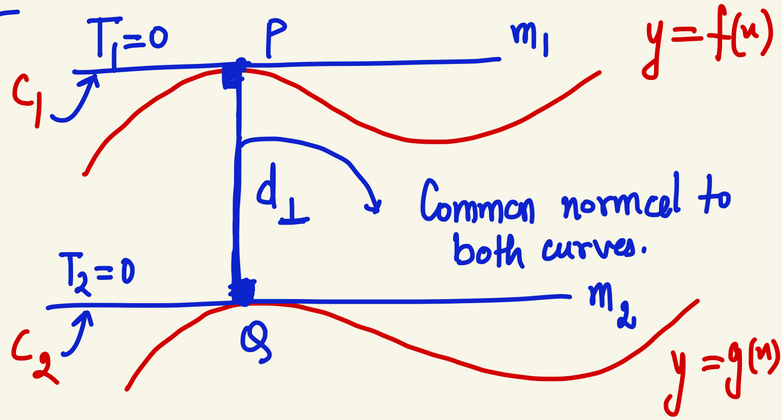
Given a fixed point A(a, b) and a moving point P(x, f(x)) on the curve y = f(x). Then AP will be maximum or minimum if it is normal to the curve at P.

Proof : $F(x) = (x-a)^2 + (f(x)-b)^2 \Rightarrow F'(x) = 2(x-a) + 2(f(x)-b) \cdot f'(x)$

$$\therefore \quad f'(x) = -\left(\frac{x-a}{f(x)-b}\right). \text{ Also } m_{AP} = \frac{f(x)-b}{x-a}. \text{ Hence } f'(x) \cdot m_{AP} = -1.$$

Note : Greatest/least distance between two curves usually occur along common normal.

Between 2 Curves:-



$m_1 \cdot m_{PQ} = -1$

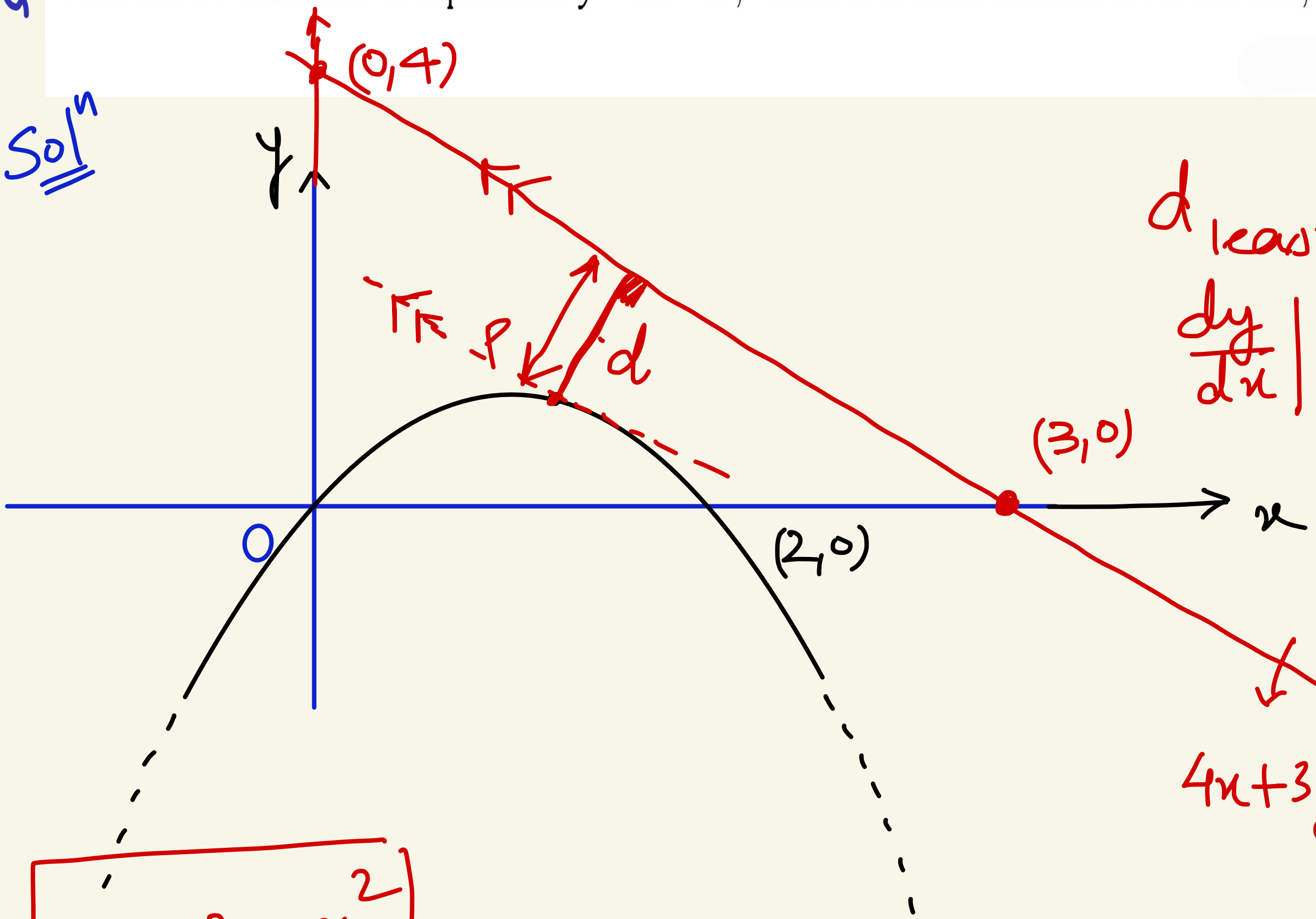
$m_2 \cdot m_{PQ} = -1$

$\Rightarrow \quad m_1 = m_2.$

$$\left.\frac{dy}{dx}\right|_{P}^{C_1} = \left.\frac{dy}{dx}\right|_{Q}^{C_2}$$

Q Find the co-ordinates of a point on $y = 2x - x^2$, which is at least distance from the line, $4x + 3y = 12$.

Sol^n



d_{least} when $\left.\frac{dy}{dx}\right|_P = -\frac{4}{3}$.

$$\boxed{y = 2x - x^2}$$

$$\left.\frac{dy}{dx}\right|_{P(x_1,y_1)} = \left.(2-2x)\right|_{(x_1,y_1)} = 2 - 2x_1$$

$$2 - 2x_1 = -\frac{4}{3}$$

$$6 - 6x_1 = -4$$

$$10 = 6x_1 \Rightarrow \boxed{x_1 = \frac{5}{3}}$$

$$y_1 = 2x_1 - x_1^2$$

$$= 2\cdot\left(\frac{5}{3}\right) - \frac{25}{9}$$

$$= \frac{10}{3} - \frac{25}{9}$$

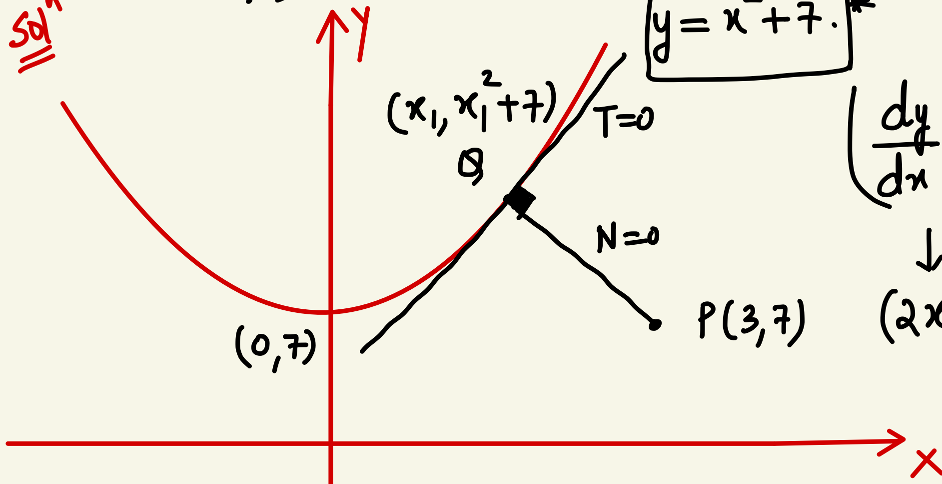
$$= +\frac{5}{9}$$

Q

A helicopter of enemy is flying along the curve given by $y = x^2 + 7$. A soldier placed at (3, 7) wants to shoot down the helicopter when it is nearest to him. Find the nearest distance. $\sqrt{5}$ Ans

$\boxed{y = x^2 + 7.}$ *

Sol^n



$$\left(\frac{dy}{dx}\Big|_Q\right) \cdot m_{PQ} = -1$$

$$\downarrow$$

$$(2x_1)\left(\frac{x_1^2+7-7}{x_1-3}\right) = -1$$

$$\downarrow$$

$$\frac{dy}{dx} = 2x$$

Q

Find the minimum value of $(x_1 - x_2)^2 + \left(\sqrt{1-x_1^2} - \frac{4}{x_2}\right)^2$, where $x_1 \in (0,1)$ and $x_2 \in R - \{0\}$.

$(PQ_{min})^2 = (2\sqrt{2}-1)^2$

$8+1-4\sqrt{2}$

$9-4\sqrt{2}$

Ans

Sol^n

PQ = distance formula = $\sqrt{(x_1-x_2)^2 + (\underbrace{\sqrt{1-x_1^2}}_{y_1} - \underbrace{\tfrac{4}{x_2}}_{y_2})^2}$

$A(x_1, \underbrace{\sqrt{1-x_1^2}}_{y_1})$; $B(x_2, \underbrace{\tfrac{4}{x_2}}_{y_2})$

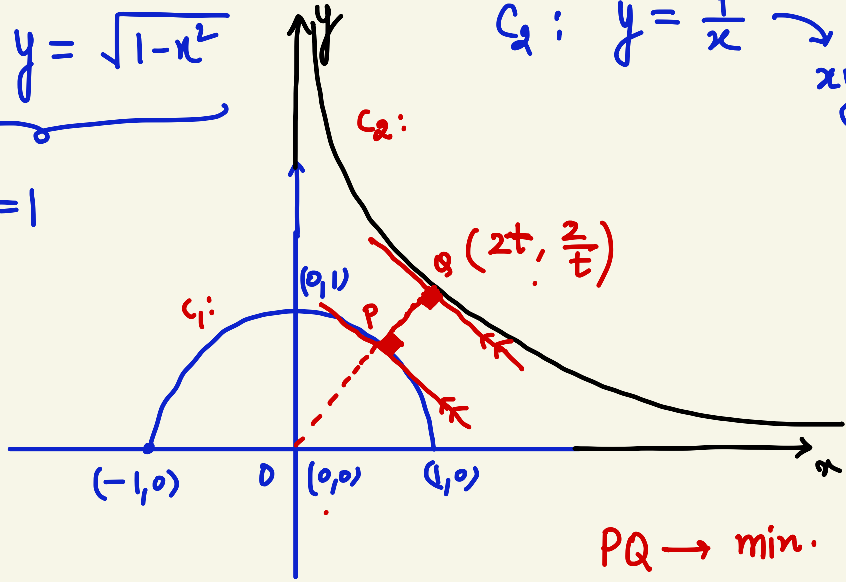
$C_1 : \underbrace{y = \sqrt{1-x^2}}_{x^2+y^2=1}$

$C_2 : y = \frac{4}{x} \rightarrow xy = 4$

$xy = c^2$

$x = ct$

$y = \frac{c}{t}$



$PQ \rightarrow$ min.

$OQ - OP = PQ$

$\sqrt{4t^2 + \frac{4}{t^2}} - 1 = PQ$

$2\sqrt{t^2 + \frac{1}{t^2}} - 1 = PQ$

$\boxed{PQ_{min} = 2\sqrt{2} - 1}$

$\frac{4}{x} = \sqrt{1-x^2}$

$16 = x^2 - x^4 \Rightarrow x^4 - x^2 + 16 = 0$

No Real Root.

SOME SPECIAL POINTS ON A GIVEN CURVE :

(a) **Stationary points:** The stationary points are the points of domain where $f'(x) = 0$.

(b) **Critical points :** There are three kinds of critical points as follows :

(i) The point at which $f'(x) = 0$

(ii) The point at which $f'(x)$ does not exists

(iii) The end points of interval (if included)

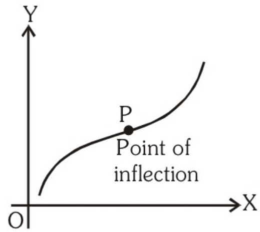
These points belongs to domain of the function.

Note : Local maxima and local minima occurs at critical points only but not all critical points will correspond to local maxima/local minima.

(c) **Point of inflection :** A point where the graph of a function has a tangent line and where the concavity changes is called a point of inflection. For finding point of inflection of any function, compute the solutions of $\frac{d^2y}{dx^2} = 0$ or does not exist. Let the solution is x = a,

if $\frac{d^2y}{dx^2} = 0$ at x = a and sign of $\frac{d^2y}{dx^2}$ changes about this point then it is called point of inflection.

if $\frac{d^2y}{dx^2}$ does not exist at x = a and sign of $\frac{d^2y}{dx^2}$ changes about this point and tangent exist at this point then it is called point of inflection.



eg: $y = x^2 \ln x$ Find point of inflection

$D_f \in (0,\infty)$

$$\frac{dy}{dx} = x^{\cancel{2}} \cdot \frac{1}{\cancel{x}} + 2x \cdot \ln x$$

$$\frac{dy}{dx} = x + 2x\ln x$$

$$\frac{d^2y}{dx^2} = 1 + 2\left(\cancel{x} \cdot \frac{1}{\cancel{x}} + \ln x\right)$$

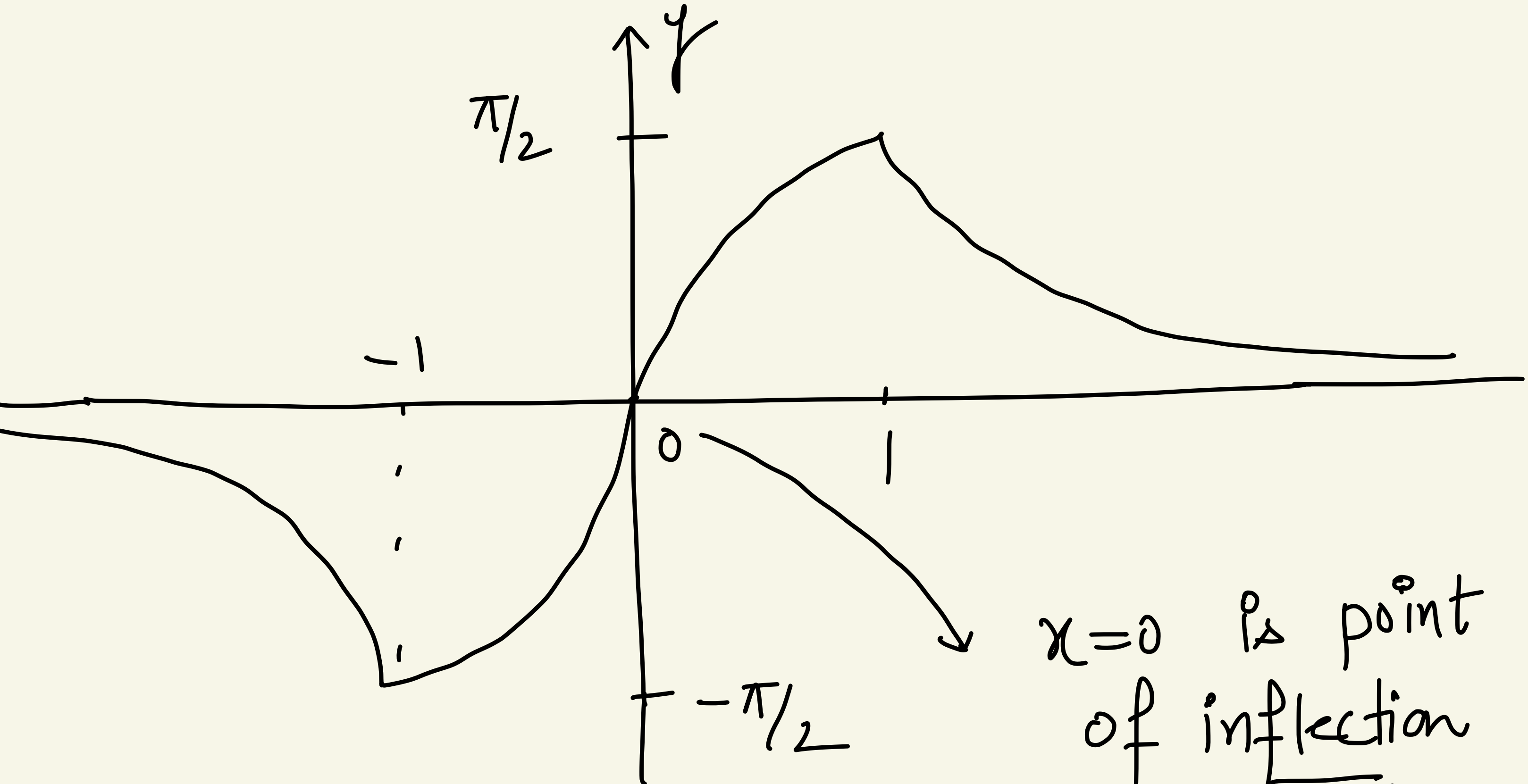
$$\frac{d^2y}{dx^2} = 3 + 2\ln x = 0 \Rightarrow \ln x = -\frac{3}{2}$$

$$x = e^{-3/2}$$

f''_{-} $\qquad$ f''_{+}

$e^{-3/2}$ inflection point

eg:
$$f(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right)$$



$x=0$ is point of inflection

eg: $f(x) = x^4 - 4x^3 + 10$??

point of inflection = ? $x=0$ and $x=2$.

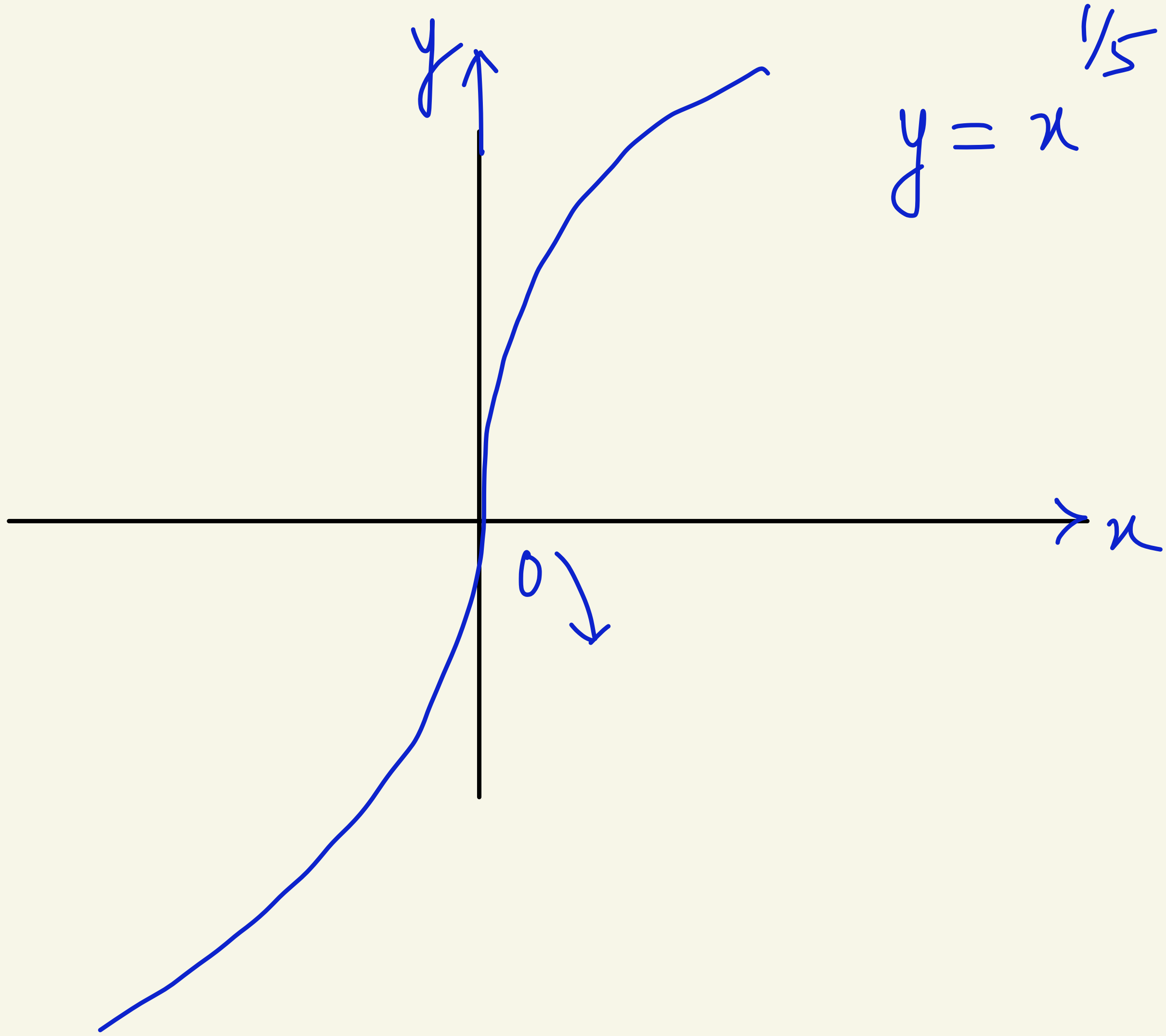
eg: $f(x) = x^{1/5}$; $D_f \in \mathbb{R}$ → $x=0$ point of inflection

$f'(x) = \frac{1}{5}x^{-4/5}$ dne at $x=0$

$f''(x) = \frac{1}{5}\left(-\frac{4}{5}\right)(x)^{-9/5}$

$f''>0$ $f''<0$

0


y
x
0
$y = x^{1/5}$

Note:- Every cubic polynomial has one point of inflection.

$$f(x) = ax^3 + bx^2 + cx + d$$

$$f'(x) = 3ax^2 + 2bx + c$$

$$f''(x) = 6ax + 2b = 0$$

$$x = -b/3a$$

f'' $\qquad$ f''

$-b/3a$

eg: $$f(x) = 2x^3 - 6x^2 + 7x + 11$$

$$f'(x) = 6x^2 - 12x + 7$$

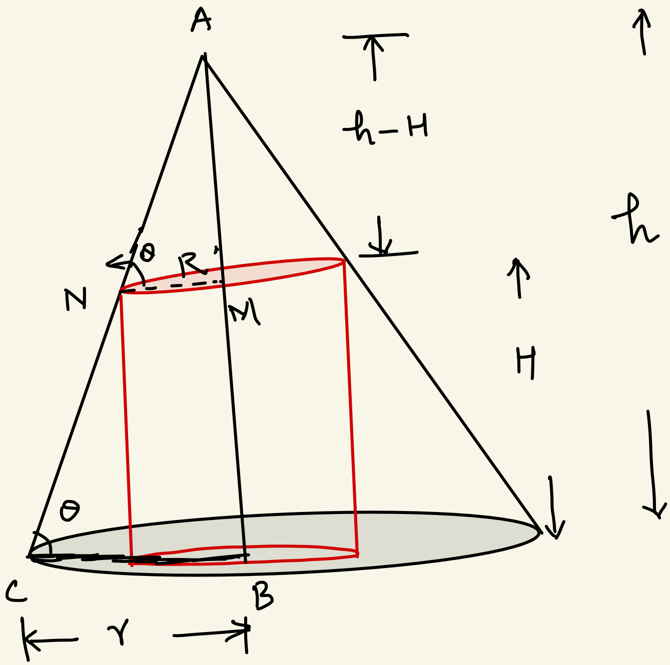
$$f''(x) = 12x - 12 = 0$$

$f'' < 0$ $\qquad$ $f'' > 0$

$1 \rightarrow$ point of inflection

Q

Find the altitude of the right circular cylinder of maximum volume that can be inscribed in a given right circular cone of height 'h'.



$$\tan\theta = \frac{h-H}{R} = \frac{h}{r}$$

$$\therefore H = h\left(\frac{r-R}{r}\right) \quad —(1)—$$

$$V = \pi R^2 H$$

$$V = \pi R^2 \cdot h\left(\frac{r-R}{r}\right)$$

$$\frac{dV}{dR} = \pi h(2R) - \frac{\pi h(3R^2)}{r} = 0$$

$$\therefore \boxed{2r = 3R}$$

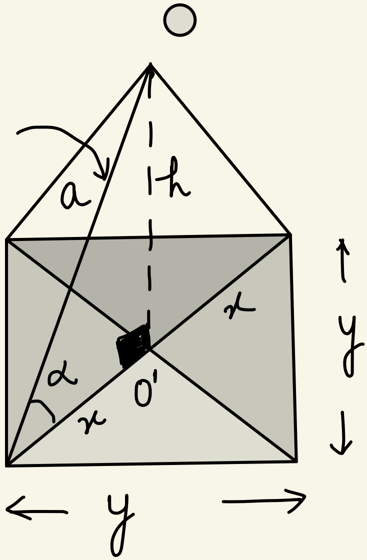
put in (1) to get H

$$H = h\left(1 - \frac{R}{r}\right)$$

$$H = h\left(1 - \frac{2}{3}\right) = \frac{h}{3}$$

$$\therefore \text{Height of cylinder} = \frac{h}{3}$$

Q The lateral edge of a regular rectangular pyramid is 'a' cm long. The lateral edge makes an angle α with the plane of the base. The value of α for which the volume of the pyramid is greatest, is



$$\frac{h}{a} = \sin\alpha \Rightarrow h = a\sin\alpha.$$

$$\frac{x}{a} = \cos\alpha \Rightarrow x = a\cos\alpha.$$

$$x^2 + h^2 = a^2.$$

$$2y^2 = 4x^2$$
$$y^2 = 2x^2$$

$$V = \frac{1}{3}.y^2.h$$

$$V = \frac{2}{3}a^2\cos^2\alpha.\,a\sin\alpha$$

$$V = \frac{2}{3}a^3\left(\cos^2\alpha\sin\alpha\right)$$

$V'(\alpha) = 0 \Rightarrow \tan\alpha = \frac{1}{\sqrt{2}}$ for maxima.

$$\therefore \boxed{V_{max} = \frac{4\sqrt{3}}{27}a^3}$$ Ans

Q Find critical points and stationary points of $f(x) = (x-2)^{2/3}(2x+1)$?

Sol^n $D_f \in \mathbb{R}$

$$f'(x) = (x-2)^{2/3}(2) + \left(\frac{2}{3}\right)(x-2)^{-1/3}(2x+1).$$

$$= 2(x-2)^{2/3} + \frac{2}{3}\frac{(2x+1)}{(x-2)^{1/3}}$$

$$= \frac{6(x-2)+2(2x+1)}{3(x-2)^{1/3}} = \frac{10x-10}{3(x-2)^{1/3}}$$

$$f'(x) = \frac{10}{3}\frac{(x-1)}{(x-2)^{1/3}}$$

$f'(x) = 0 \Rightarrow x = 1$

$\therefore x = 1$ is only stationary point

$f'(x) = \text{dne} \Rightarrow x = 2.$

$x = 1$ & $x = 2$ are Critical points.

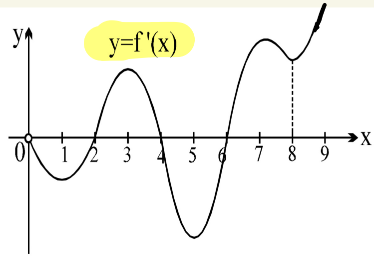
Q The graph of the first derivative f' of a function f is shown.

(a) On what intervals is f increasing? Explain.

(b) At what values of x does f have a local maximum or minimum?

(c) On what intervals is f concave upwards or concave downwards?

(d) What are the x-coordinates of the inflection points of f?



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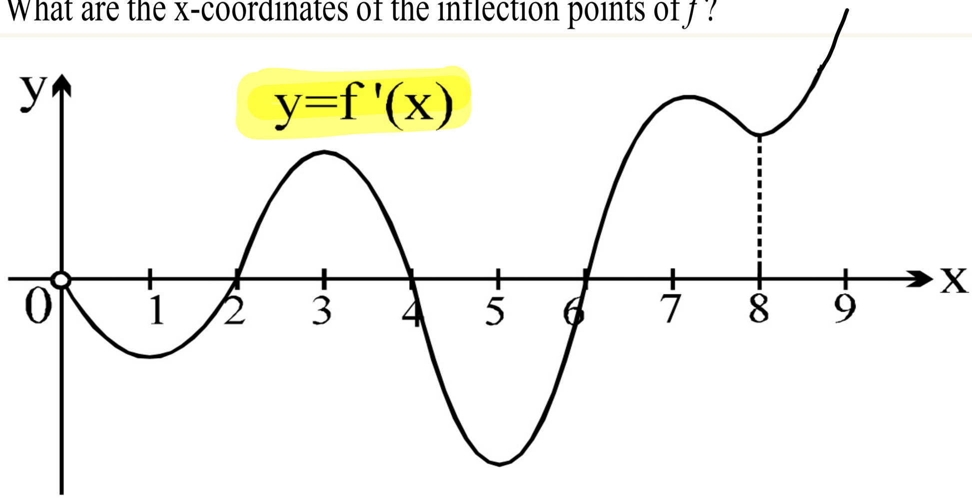
2 4 6



(a) $[2,4]$; $[6,\infty)$

(b) local min at $x=2$ & $x=6$.
local max at $x=4$.

(c) Concavity :
Concave up : $[1,3] \cup [5,7] \cup [8,\infty)$
Concave down : $(0,1] \cup [3,5] \cup [7,8]$.

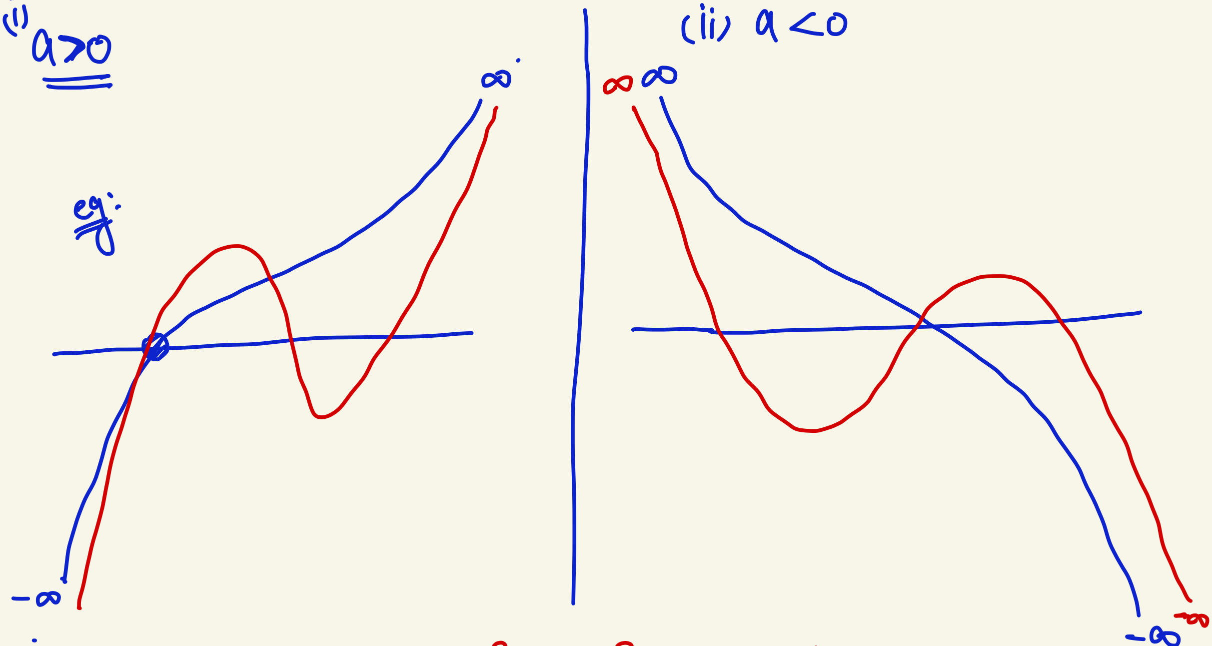
(d) inflection at $x=1,3,5,7,8$.

Nature of roots of cubic equation:

$f(x) = ax^3 + bx^2 + cx + d$; $a \neq 0$. $a, b, c, d \in \mathbb{R}$

(i) $a > 0$ (ii) $a < 0$

eg:



Note: $f(x) = ax^3 + bx^2 + cx + d$; $a, b, c, d \in \mathbb{R}$

$a > 0$ ✓

$$\boxed{f'(x) = 3ax^2 + 2bx + c.}$$

$D = 4b^2 - 12ac$

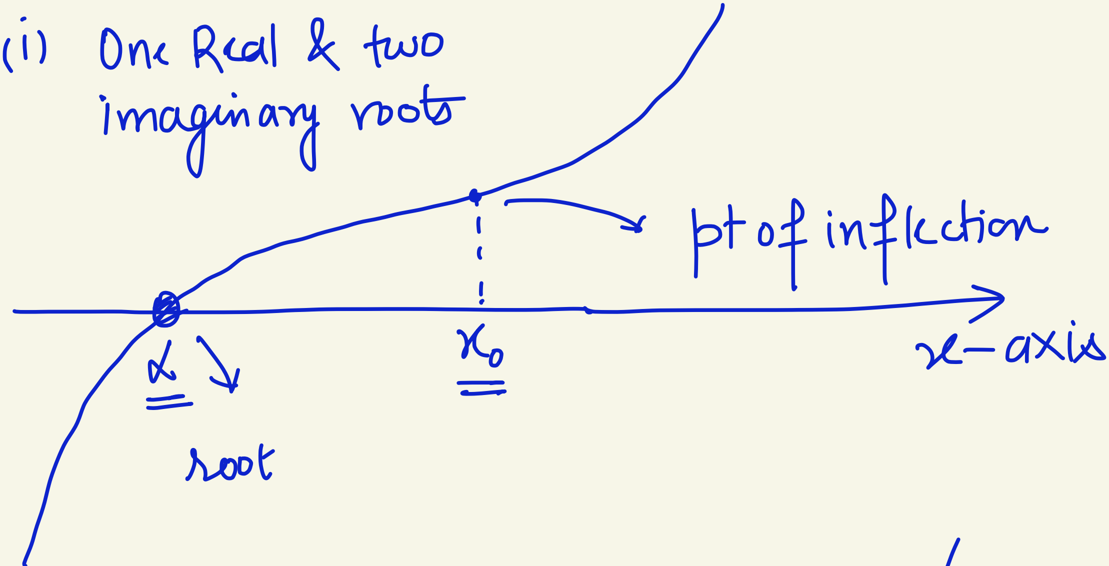
$D < 0$ $D \not< 0$

$f'(x) > 0 \ \forall x \in \mathbb{R}$ Non-monotonic

Monotonically ↑

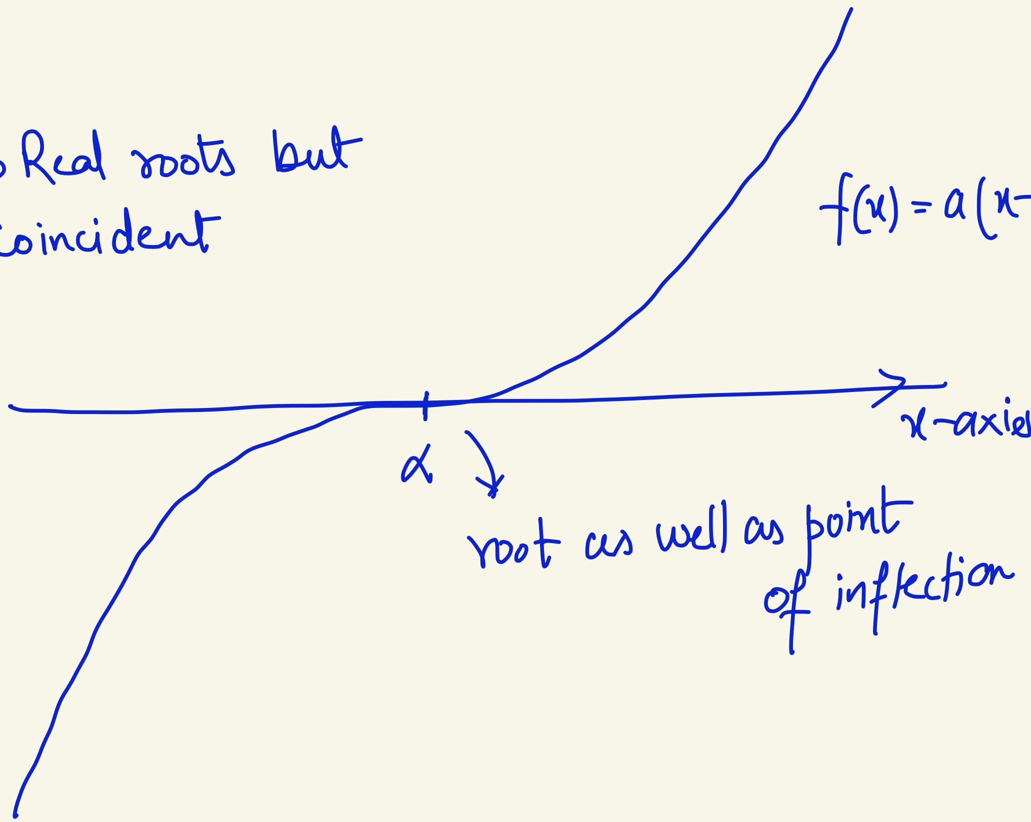
If $f(x)$ is monotonically increasing.

(i) One Real & two imaginary roots



(ii) 3 Real roots but Coincident

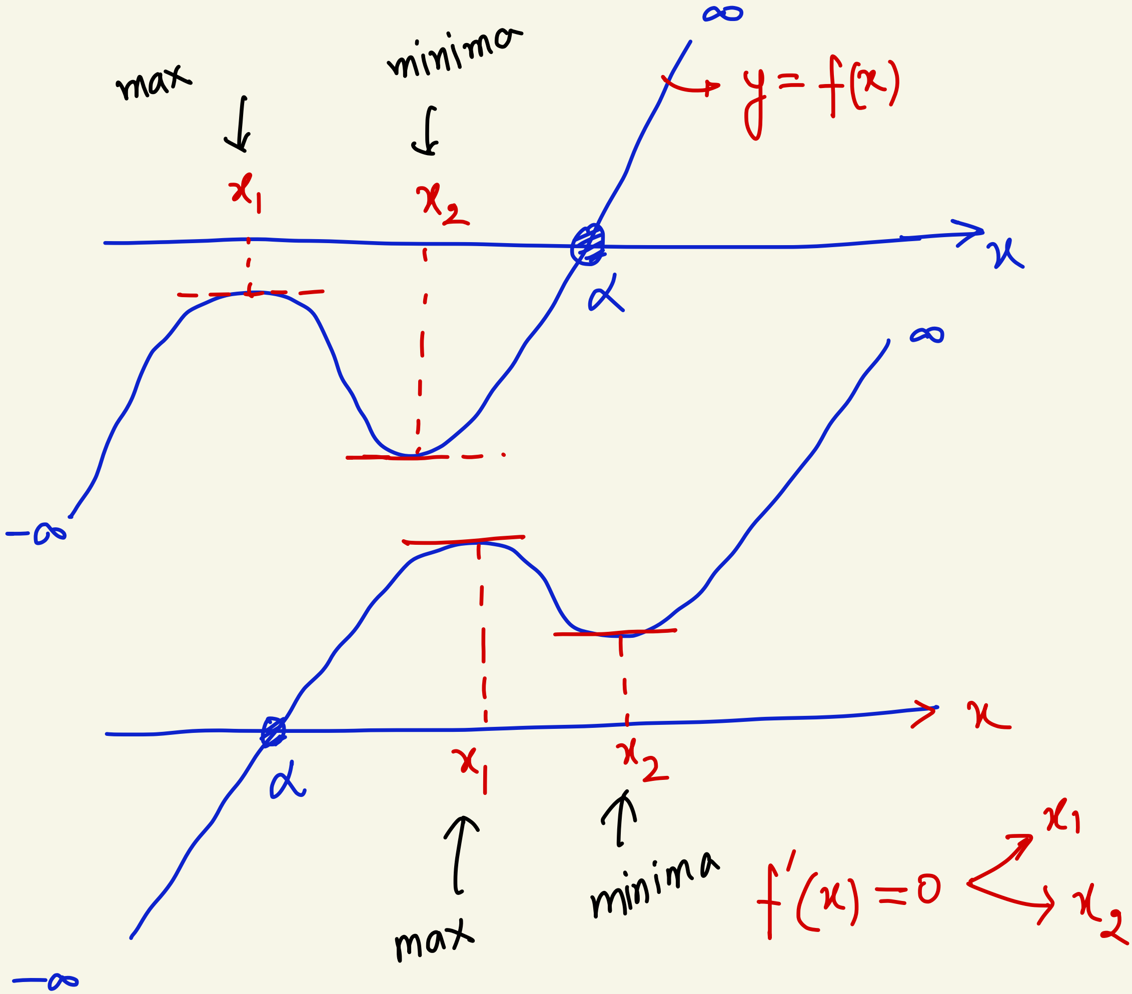
$$f(x) = a(x-\alpha)^3$$



$\underline{a>0}$

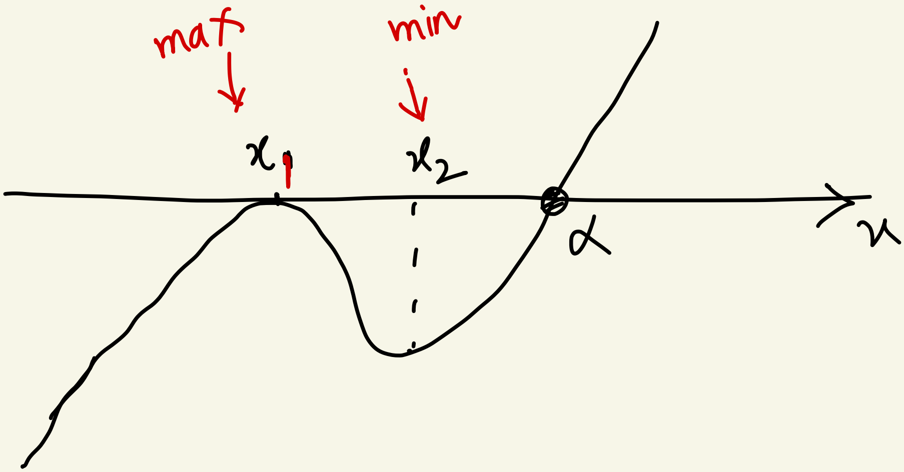
If $f(x)$ is Non-monotonic in Nature

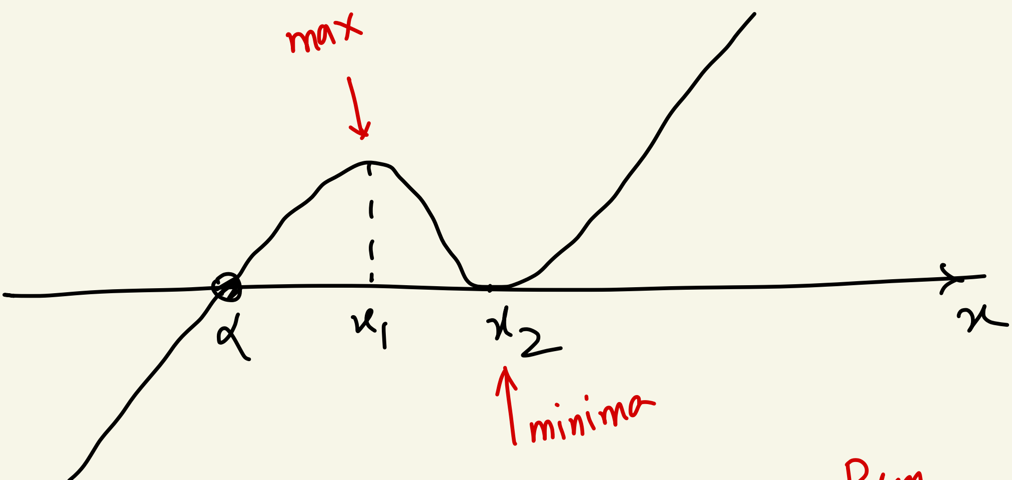
(i) One Real & 2 imaginary roots



$\therefore \boxed{f(x_1)\cdot f(x_2) > 0}$ Rem

(ii) All 3 real roots but not distinct :-

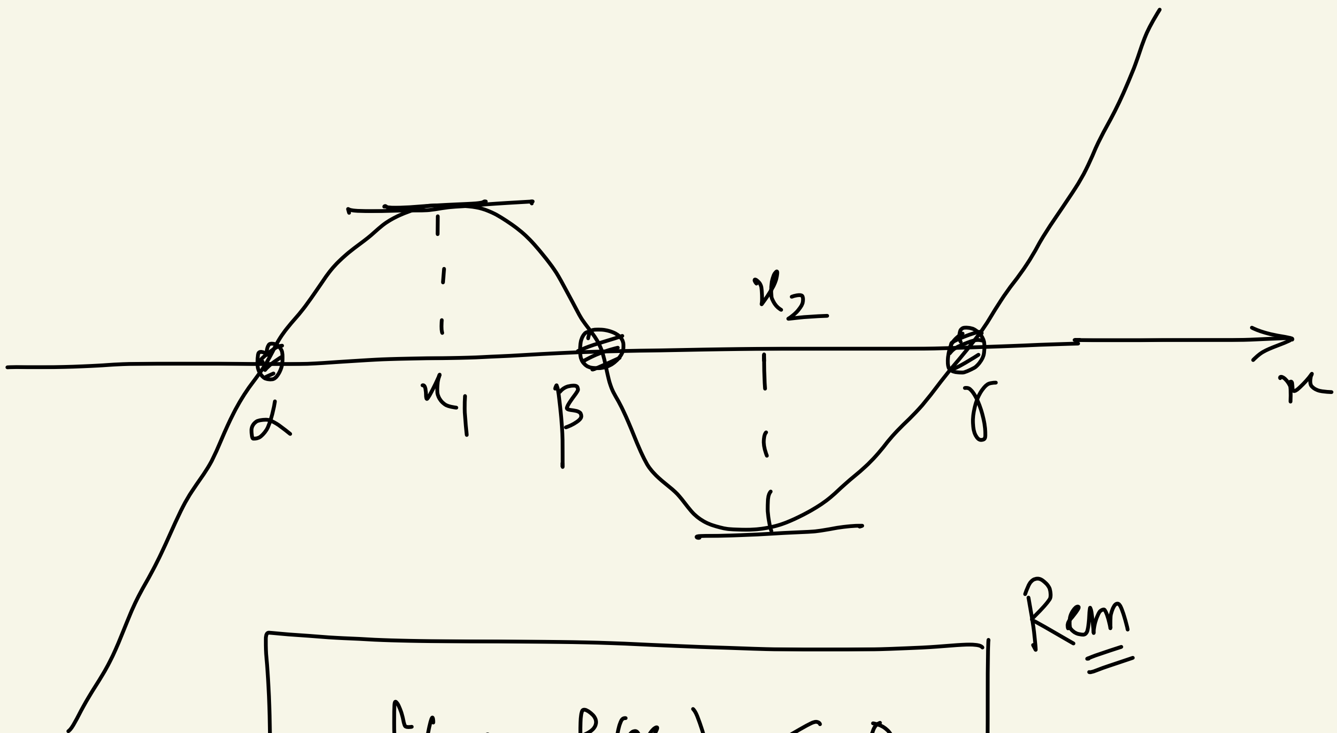




Rem

$$\boxed{f(x_1)\cdot f(x_2) = 0}$$

(iii) <u>all 3 real & distinct roots</u> :-



Rem

$$\boxed{f(x_1)\cdot f(x_2) < 0}$$

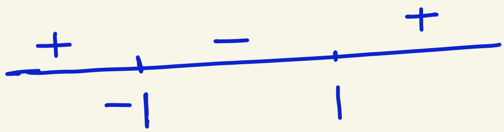
Q Find the value of 'a' if $x^3 - 3x + a = 0$ has :

(1) 3 distinct real roots.
(2) one real root exactly.
(3) all 3 real roots but not distinct.

$$\boxed{f(x) = x^3 - 3x + a}$$

Sol^n

$$f'(x) = 3x^2 - 3 = 3(x-1)(x+1).$$



$f'(x) = 0 \Rightarrow x = -1, \; x = 1$

$f(-1) = a+2$

$f(1) = a-2$

(i) $f(-1) \cdot f(1) < 0 \Rightarrow$

$(a+2)(a-2) < 0 \Rightarrow a \in (-2, 2)$

(2) $f(-1)\, f(1) > 0$

$a \in (-\infty, -2) \cup (2, \infty)$

(3) $f(-1) \cdot f(1) = 0 \Rightarrow a = \pm 2.$

Q If $f(x) = x^3 - 3x + a$ where $a \in (0,2)$ has 3 distinct real roots x_1, x_2, x_3 find $\{x_1\} + \{x_2\} + \{x_3\}$?

(<u>Note</u>:- $\{\ \}$ denotes fractional part function)

Sol^n $f(x) = x^3 - 3x + a$; $f(-2) = -8+6+a$
$= a-2 < 0$

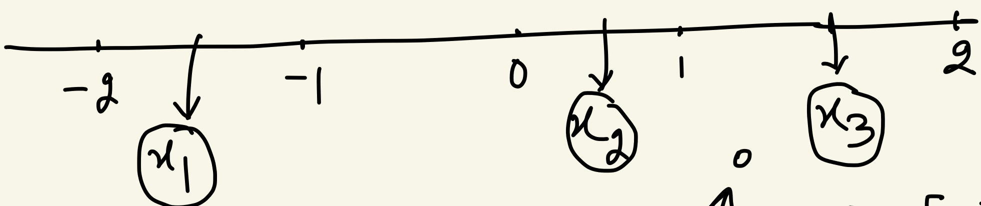
$f(-1) = -1+3+a = a+2 > 0$

$$\boxed{\begin{array}{l} x^3-3x+a=0 \begin{cases} x_1 \\ x_2 \\ x_3 \end{cases} \\ \Sigma x_1 = 0 \end{array}}$$

$$\left\{\begin{array}{l} f(0) = a > 0. \\ f(1) = 1-3+a = a-2 < 0 \end{array}\right.$$

$f(2) = 8-6+a = a+2 > 0.$

$$\boxed{\{x_1\} = x_1 - [x_1]}$$



* $x_1 \in (-2,-1)$
* $x_2 \in (0,1)$
* $x_3 \in (1,2)$

$\Sigma \{x_1\} = \Sigma x_1 - \Sigma [x_1]$
$= 0 - (-2+0+1)$
$= 1$ Ans

Q. For cubic $f(x) = x^3 + 3(a-7)x^2 + 3(a^2-9)x - 1$

Find 'a' for which $f(x)$ has :

① positive point of inflection

② positive point of maxima.

③ positive point of minima.

Sol^n

$$g(x) \leftarrow f'(x) = 3x^2 + 6(a-7)x + 3(a^2-9) = 0 \quad \begin{cases} x_1 \to \text{max} \\ x_2 \to \text{min} \end{cases}$$

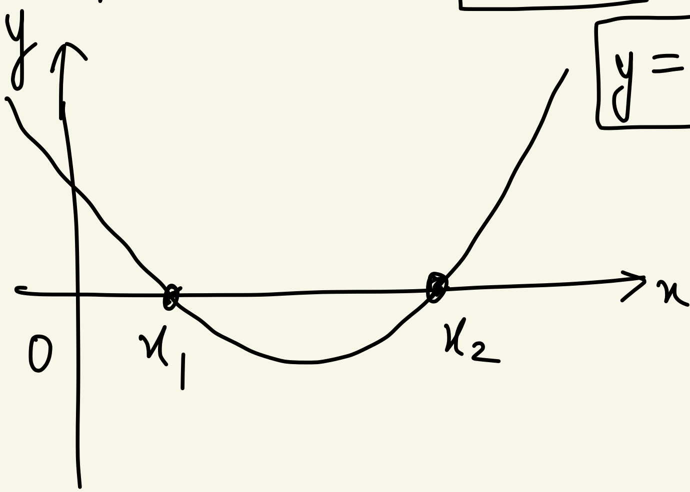
$x_2 > x_1$

$$f''(x) = \cancel{6}x + \cancel{6}(a-7) = 0.$$

$$x = (7-a)$$

① $7 - a > 0 \Rightarrow \boxed{a < 7}$

②



$\boxed{y = g(x)}$

① $D > 0$

② $g(0) > 0.$

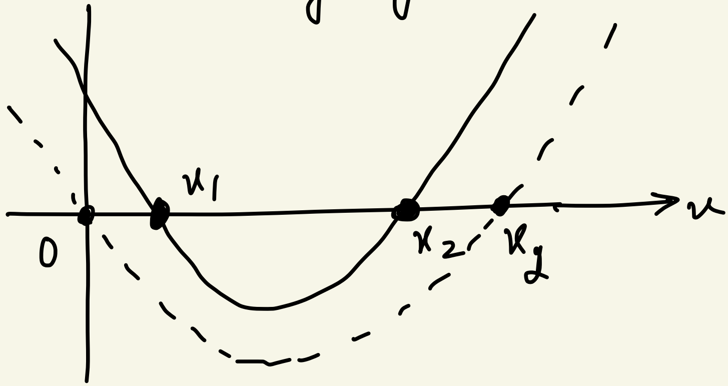
③ $-\dfrac{b}{2a} > 0.$

$\bigcap$

③

$y = g(x)$

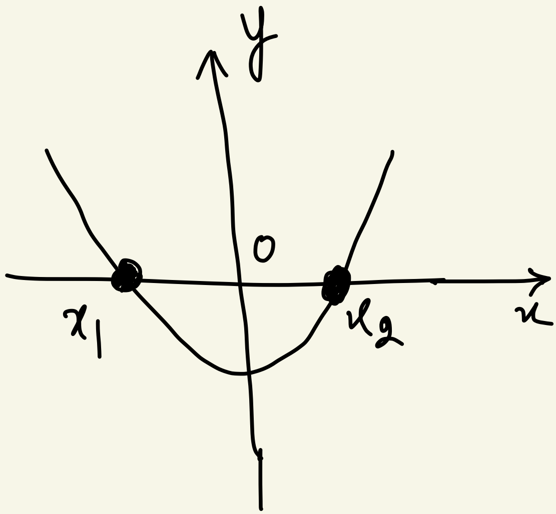
I



$$\left.\begin{array}{ll} ① & D > 0 \\ ② & g(0) \geq 0 \\ ③ & -\dfrac{b}{2a} > 0 \end{array}\right\} \cap$$

Ⓐ

II



$\boxed{g(0) < 0}$

Ⓑ

Ⓐ Union Ⓑ .

Q

Let $f(x) = 2x^3 - 6x + 1$. Then number of real roots of equation $f(f(x)) = 0$ are

Sol^n $f'(x) = 6x^2 - 6 = 6(x^2-1)$

$= 6(x-1)(x+1)$

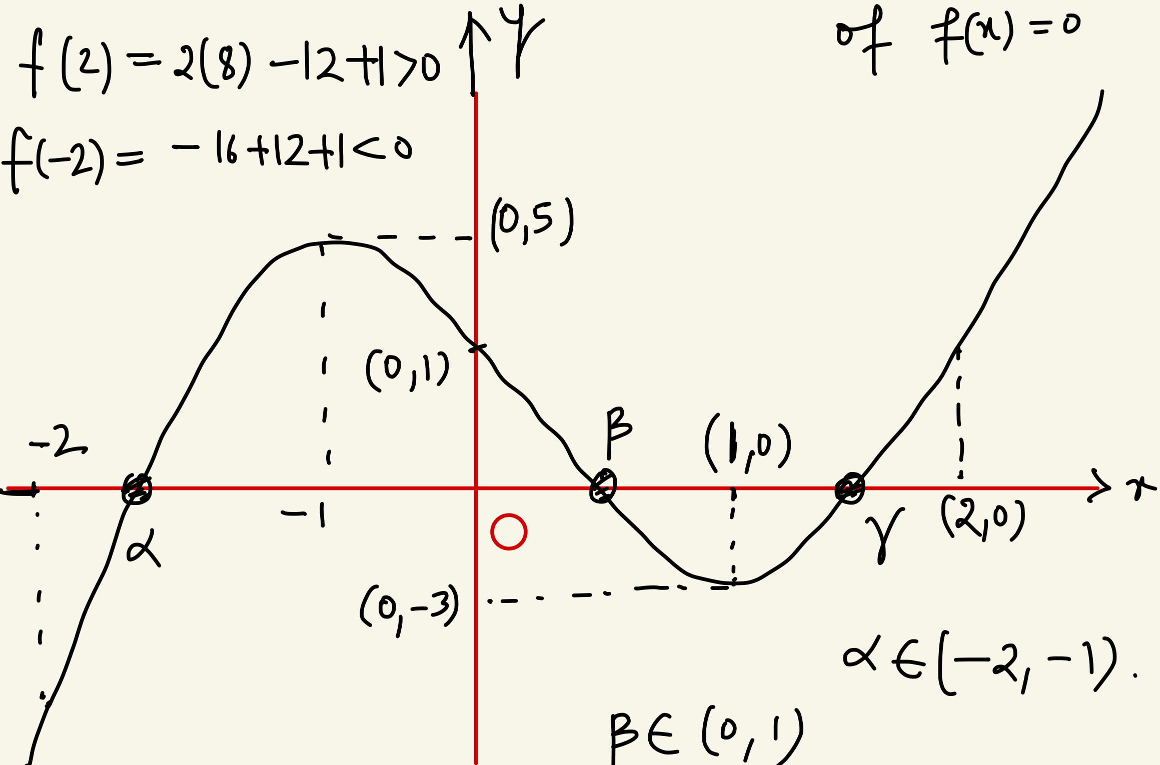
$f'(x) = 0 \Rightarrow x = -1; 1$

$\left.\begin{aligned} f(-1) &= -2+6+1 = 5 \\ f(1) &= -3 \end{aligned}\right\} \Rightarrow$ 3 distinct real roots of $f(x)=0$

$f(-1)f(1) < 0.$

$f(2) = 2(8) - 12 + 1 > 0$

$f(-2) = -16 + 12 + 1 < 0$



$\alpha \in (-2, -1).$

$\beta \in (0, 1)$

$\gamma \in (1, 2)$

$$f(f(x)) = 0$$

$$f(x) = t \begin{cases} \alpha \\ \beta \\ \gamma \end{cases}$$

$$f(t) = 0$$

$$f(\alpha) = 0 \longrightarrow 3 \text{ sol}^n$$

$$f(\beta) = 0 \longrightarrow 3 \text{ sol}^n$$

$$f(\gamma) = 0 \longrightarrow 3 \text{ sol}^n$$

$$\therefore \quad f(f(x)) = 0 \longrightarrow 9 \text{ solutions.}$$

Q

Let $f(x)=|x|+|x+1|+||x|-|x+1||$, then-

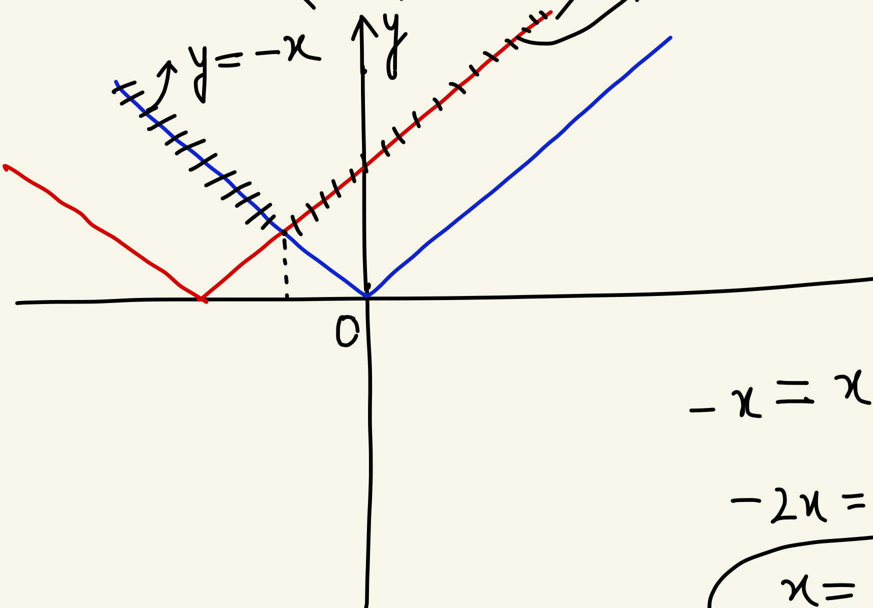
(1) $f(x)$ has local minima at $x=0$

(2) $f(x)$ has local minima at $x=-\frac{1}{2}$

(3) $f(x)$ has local maxima at $x=-\frac{1}{2}$

(4) $f(x)$ has local minima at $x=-1$

$$f(x) = 2\max\left(|x|, |x+1|\right)$$



$$-x = x+1$$

$$-2x = 1$$

$$x = -\frac{1}{2}$$

minima.

Q

If $f(1) = e^2$ and $\frac{d}{dx}(f(x)) > 3x^2 f(x) \forall x \geq 1$ then f(x) cannot take values

~~(A) e~~ ✓ ~~(B) $\frac{9}{2}$~~ ✓ (C) 8 (D) 10

Sol^n

$$f'(x) - 3x^2 f(x) > 0.$$

$$IF = e^{\int -3x^2 dx}$$

$$e^{-x^3}$$

$$\frac{dy}{dx} - 3x^2 y > 0$$

$$\frac{d}{dx}\left(\underbrace{e^{-x^3} f(x)}_{g(x)}\right) > 0$$

$$\int_1^x \frac{d}{dx}\left(e^{-x^3} f(x)\right) dx > \int_1^x 0\, dx$$

$$e^{-x^3} f(x) \Big|_1^x > 0 \quad \text{for } x \geq 1$$

$$e^{-x^3} f(x) - e^{-1} \cdot f(1) > 0 \Rightarrow \boxed{f(x) > e^{1+x^3}}$$

$$\boxed{f(x) \geq e^2} \quad \forall\, x \geq 1$$

Q

If f (x) is non-increasing function such that $f(x)+f''(x) \geq 0 \ \forall x \in R \ \& f(0)=f'(0)=0$ then f (x) is not strictly decreasing in

(A) $[-4,-2]$ ✓(B) $[0,2]$ ✓(C) $[3,4]$ ✓(D) $[2,4]$

Solⁿ

$$f'(x) \leq 0$$

$$f(x) + f''(x) \geq 0 \quad \forall x \in \mathbb{R}$$

$$f'(x)f(x) + f'(x)f''(x) \leq 0 \qquad (\because f'(x) \leq 0)$$

$$\frac{d}{dx}\Big(\underbrace{f^2(x) + (f'(x))^2}_{g(x)}\Big) \leq 0.$$

$g(x) = f^2(x) + (f'(x))^2$ is a decreasing fnc.

$$g(0) = 0 + 0 = 0$$

$$\therefore \quad f^2(x) + (f'(x))^2 \leq 0 \quad \forall x \geq 0$$

$$\Rightarrow \quad f(x) = 0 \ \& \ f'(x) = 0 \text{ for } \underline{x \geq 0}$$

Q If $f(x) = \int_{\sin x}^{\cos x} \frac{dt}{e^t \sqrt{1-t^2}}$, $x \in [0, \pi/2]$

(A) $f(x)$ is decreasing in given interval.

(B) $f(x)$ is increasing " " " .

(C) Number of local extremum for $f(x)$ in given interval is 1.

(D) If g is inverse of f then $g''(0)$ is 0.

Q

A cylindrical container is to be made from certain solid material with the following constraints. It has a fixed inner volume of V mm^3, has a 2 mm thick solid wall and is open at the top. The bottom of the container is a solid circular disc of thickness 2 mm and is of radius equal to the outer radius of the container.

If the volume of the material used to make the container is minimum when the inner radius of the container is 10mm, then the value of $\frac{V}{250\pi}$ is **[JEE 2015, 4M, 0M]**

Q Let $f: [0, 8] \rightarrow R$ be a differentiable function such that $f(0) = 0$; $f(4) = 1$; $f(8) = 1$

then : (A) there exist some $c_1 \in (0, 8)$ where $f'(c_1) = \frac{1}{4}$.

(B) there exist some $c \in (0, 8)$ where $f'(c) = \frac{1}{12}$.

(C) there exist $c_1, c_2 \in [0, 8]$ where $8 f'(c_1) f(c_2) = 1$.

(D) there exist some $\alpha, \beta \in (0, 2)$ such that

$$\int_0^8 f(t)\,dt = 3\left(\alpha^2 f(\alpha^3) + \beta^2 f(\beta^3)\right).$$

Q.

If two distinct tangents can be drawn from the point $(2h - 7, 125 - 24h)$ $\forall$ $h \in R - \{5\}$ to the curve $f(x) = x^3 - 9x^2 + \alpha x + \beta$, then

(A) $\alpha + \beta = 29$

(B) $(3, 5)$ is its point of local minima

(C) $(3, 5)$ is its point of inflection

(D) $\alpha^2 - \beta^2 = 29$

Q For every twice differentiable function $f : \mathbb{R} \to [-2, 2]$ with $(f(0))^2 + (f'(0))^2 = 85$, which of the following statement(s) is (are) TRUE ? **[JEE(Advanced)-2018]**

(A) There exist r, s $\in \mathbb{R}$, where r < s, such that f is one-one on the open interval (r, s).

(B) There exists $x_0 \in (-4, 0)$ such that $|f'(x_0)| \leq 1$

(C) $\lim_{x\to\infty} f(x) = 1$

(D) There exists $\alpha \in (-4, 4)$ such that $f(\alpha) + f''(\alpha) = 0$ and $f'(\alpha) \neq 0$