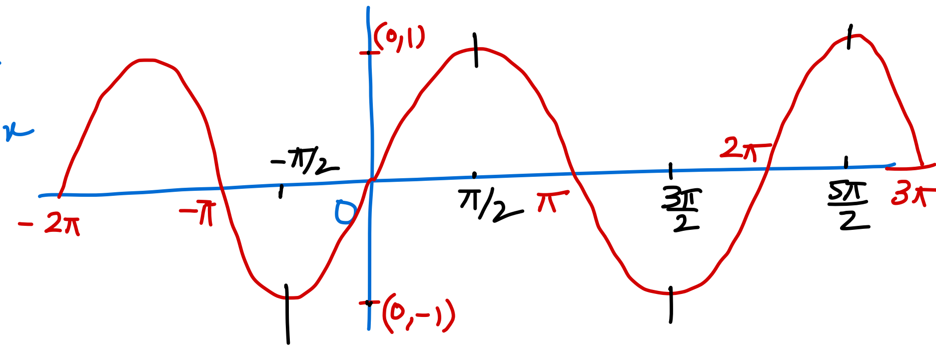
Inverse Trigonometric function

(ITF). (ϕ – 4)

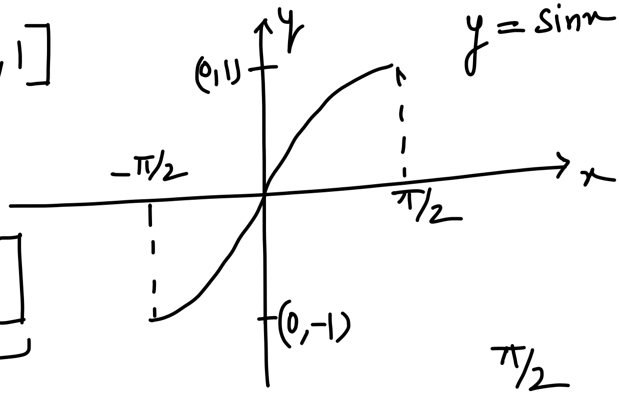
$f: R \to R$

$f(x) = \sin x$



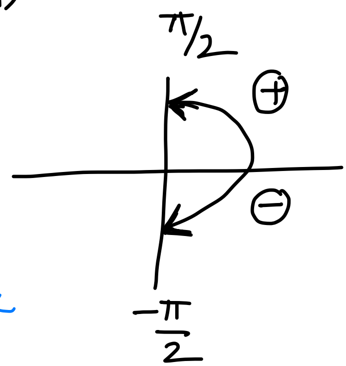
$f: \underbrace{\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]} \to [-1, 1]$

$f(x) = \sin x$



$f^{-1}: [-1, 1] \to \underbrace{\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]}$

$f^{-1}(x) = \sin^{-1} x$ OR arc sin x



* $\sin^{-1} x = \text{arc} \sin x$

↳ denotes angle whose sine is x

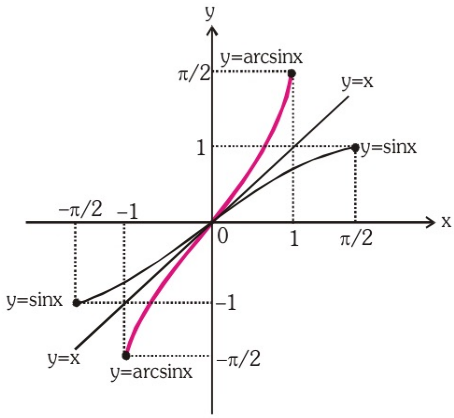
$\sin^{-1}\underbrace{\left(\frac{1}{2}\right)} = \frac{\pi}{6}$.

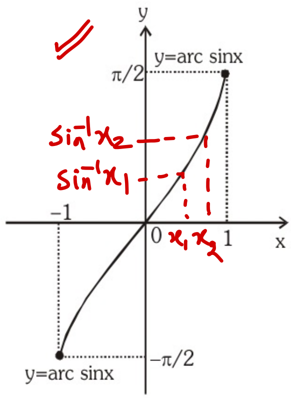
$\sin^{-1}\left(\frac{\sqrt{5}-1}{4}\right) = \frac{\pi}{10}$.

DOMAIN, RANGE & GRAPH OF INVERSE TRIGONOMETRIC FUNCTIONS :

(a) $f : \left[\frac{-\pi}{2}, \frac{\pi}{2}\right] \to [-1, 1]$ $\qquad$ $f^{-1} : [-1, 1] \to [-\pi/2, \pi/2]$

$f(x) = \sin x$ $\qquad$ $f^{-1}(x) = \sin^{-1}(x)$





$\sin^{-1}(1) = \frac{\pi}{2}$

$\sin^{-1}(0) = 0$

$\sin^{-1}(-1) = -\frac{\pi}{2}$

$\sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi}{4}$

$\sin^{-1}\left(\frac{-1}{\sqrt{2}}\right) = -\frac{\pi}{4}$

(Taking image of sin x about y = x to get $\sin^{-1}x$)

Note : $\sin^{-1}x$ is θ , then θ is the numerically smallest angle, either negative or positive whose sine is equal to x.

Comments :

(1) $\sin^{-1}x$ is monotonical increasing in its domain

If $x_2 > x_1 \Leftrightarrow \sin^{-1}x_2 > \sin^{-1}x_1$

(2) It is a bounded function

(3) It is an odd function i.e $f(x) = -f(-x)$

$\sin^{-1}(-x) = -\sin^{-1}(x)$

(4) It is aperiodic

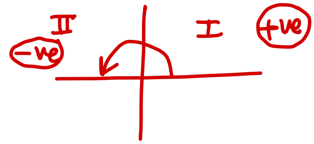
(5) It is continuous

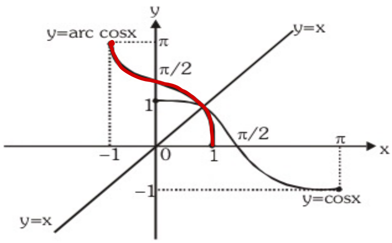
(b) $f : [0, \pi] \to [-1, 1]$

$f(x) = \cos x$

$f^{-1} : [-1, 1] \to [0, \pi]$

$f^{-1}(x) = \cos^{-1} x$





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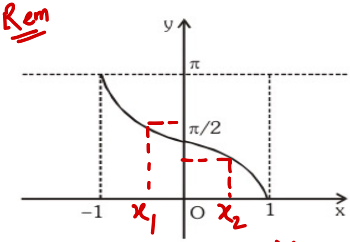


$\cos^{-1}(-1) = \pi$

$\cos^{-1}(0) = \dfrac{\pi}{2}$

$\cos^{-1}(1) = 0$

$\cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}.$

(Taking image of cos x in y = x)

Comments :

(1) $\cos^{-1}x$ is monotonic decreasing in its domain $\Rightarrow$ $x_2 > x_1 \Leftrightarrow \cos x_2 < \cos x_1$

(2) It is a bounded function

(3) It is aperiodic

*(4) It is neither even nor odd

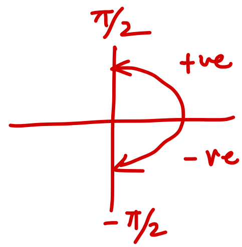
(5) It is continuous

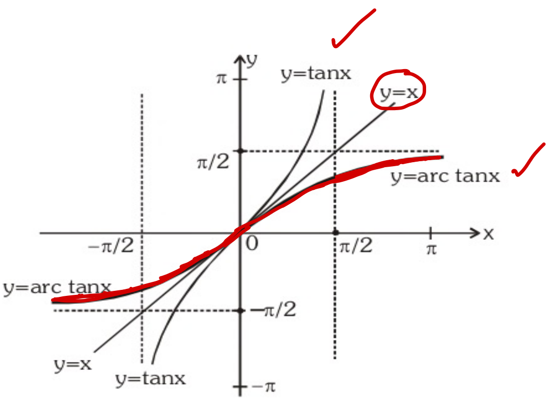
(c) $f : \left(\dfrac{-\pi}{2}, \dfrac{\pi}{2}\right) \to R$

$f(x) = \tan x$

$f^{-1} : R \to (-\pi/2, \pi/2)$

$f^{-1}(x) = \tan^{-1} x$





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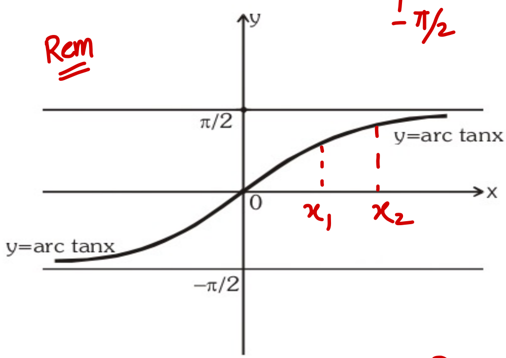


$\tan^{-1}(1) = \frac{\pi}{4}$

$\tan^{-1}(-\sqrt{3}) = -\pi/3.$

(Taking image of tan x in y = x)

Comment :

(1) $\tan^{-1}x$ is monotonic increasing

(2) It is an odd function i.e $\tan^{-1}(-x) = -\tan^{-1}(x)$

(3) It is a continuous function

(4) It is aperiodic

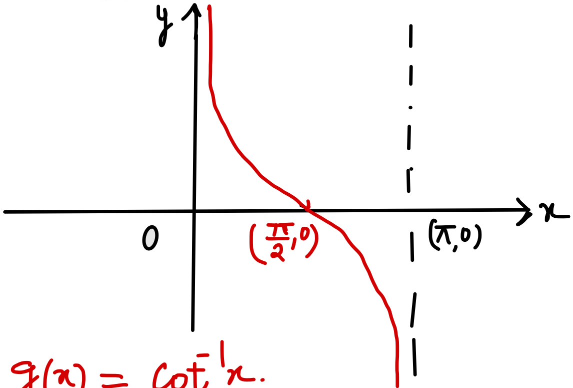
(5) It is a bounded function

(d) $f : (0, \pi) \rightarrow R$

$f(x) = \cot x$

$f^{-1} : R \rightarrow (0, \pi)$

$f^{-1}(x) = \cot^{-1} x$



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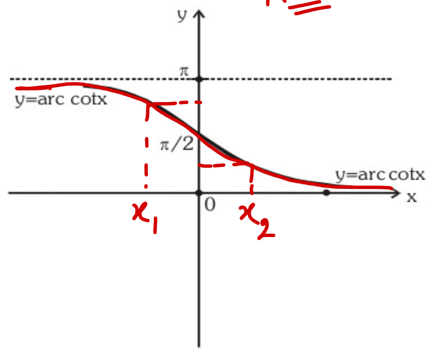


$g(x) = \cot^{-1}x.$

Comments :

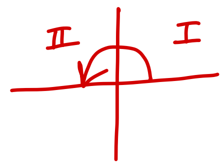
(1) It is monotonic decreasing function

(2) It is bounded function

(3) It is aperiodic

(4) It is continuous everywhere

* (5) It is neither even nor odd

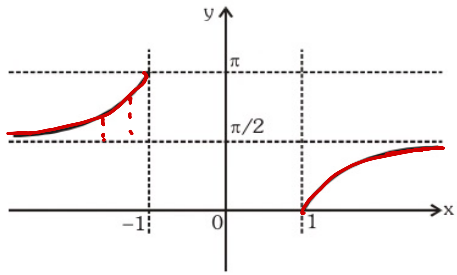


(e) $f : [0, \pi/2) \cup (\pi/2, \pi] \rightarrow (-\infty, -1] \cup [1, \infty)$

$f(x) = \sec x$

$f^{-1} : (-\infty, -1] \cup [1, \infty) \rightarrow [0, \pi/2) \cup (\pi/2, \pi]$

$f^{-1}(x) = \sec^{-1} x$



Comments

(1) It is increasing in $(-\infty, -1]$ and then again it is increasing $[1, \infty)$.

(2) It is aperiodic

(3) It is neither even nor odd

(4) It is continuous, wherever it is defined for $|x| \geq 1$

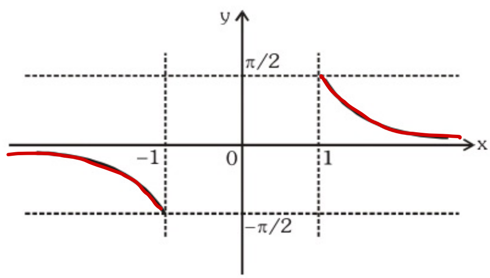
(5) It is bounded function

(f) $f : [-\pi/2, 0) \cup (0, \pi/2] \rightarrow (-\infty, -1] \cup [1, \infty)$

$f(x) = \text{cosec } x$

$f^{-1} : (-\infty, -1] \cup [1, \infty) \rightarrow [-\pi/2, 0) \cup (0, \pi/2]$

$f^{-1}(x) = \text{cosec}^{-1} x$



Note:

(1) It is decreasing in $(-\infty, -1]$ and then again it is decreasing $[1, \infty)$.

(2) It is aperiodic function

(3) It is a bounded function

(4) It is an odd function

(5) It is continuous, wherever it is defined

S.No	$f(x)$	Domain	Principal Value Range
(1)	$\sin^{-1}x$	$\lvert x \rvert \le 1$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$
(2)	$\cos^{-1}x$	$\lvert x \rvert \le 1$	$[0, \pi]$
(3)	$\tan^{-1}x$	$x \in R$	$\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$
(4)	$\sec^{-1}x$	$\lvert x \rvert \ge 1$	$[0, \pi] - \left\{\frac{\pi}{2}\right\}$ or $\left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$
(5)	$\text{cosec}^{-1}x$	$\lvert x \rvert \ge 1$	$\left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$
(6)	$\cot^{-1}x$	$x \in R$	$(0, \pi)$

Note : From the above discussions following IMPORTANT points can be concluded.

(i) All the inverse trigonometric functions represent an angle.

(ii) If $x > 0$, then all six inverse trigonometric functions viz $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$, $\sec^{-1}x$, $\text{cosec}^{-1}x$, $\cot^{-1}x$ represent an acute angle, i.e. all six have their range in I^{st} quadrant.

(iii) If $x < 0$, then $\sin^{-1}x$, $\tan^{-1}x$ & $\text{cosec}^{-1}x$ represent an angle from $-\pi/2$ to 0 (IV^{th} quadrant)

(iv) If $x < 0$, then $\cos^{-1}x$, $\cot^{-1}x$ & $\sec^{-1}x$ represent an obtuse angle. (II^{nd} quadrant)

(v) III^{rd} quadrant is never used in the range of any inverse trigonometric function.

Q1 Simplify :-

$$\sin^{-1}\left(\frac{1-\sqrt{3}}{2\sqrt{2}}\right) + \underbrace{\cos^{-1}(1)} + \cot^{-1}(\sqrt{2}+1) + \tan^{-1}(2+\sqrt{3})$$

$$-\sin^{-1}\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right) + 0 + \frac{\pi}{8} + \frac{5\pi}{12}.$$

$$-\frac{\pi}{12} + \frac{\pi}{8} + \frac{5\pi}{12} =$$

Evaluate :-

① $\sin\left(2\sin^{-1}\frac{3}{5}\right) =$

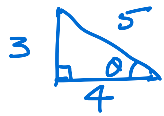
② $\sin\left(\underbrace{\arcsin\frac{3}{5}}_{\alpha} - \underbrace{\arccos\frac{3}{5}}_{\beta}\right) =$

③ $\sin\left(\tan^{-1}\cos\cot^{-1}\left(-\frac{1}{\sqrt{3}}\right)\right) =$

④ $\tan\left(\frac{1}{2}\cos^{-1}\frac{\sqrt{5}}{3}\right)$

①

$$\sin\left(\underbrace{2\sin^{-1}\frac{3}{5}}_{\theta}\right) = \sin 2\theta.$$



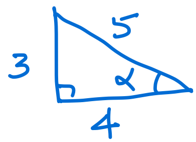
where $\underbrace{\theta = \sin^{-1}\left(\frac{3}{5}\right)}$ $\in$ 1st Qua

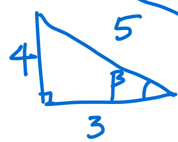
$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$= 2\left(\frac{3}{5}\right)\left(\frac{4}{5}\right) = \frac{24}{25}.$$

②

$\sin(\alpha-\beta)$; $\alpha \in$ 1st Q, $\beta \in$ 1st Q





$$\sin\alpha\cos\beta - \cos\alpha\sin\beta$$

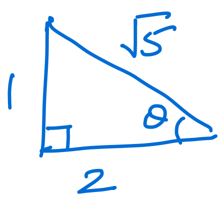
$$\left(\frac{3}{5}\right)\left(\frac{3}{5}\right) - \left(\frac{4}{5}\right)\left(\frac{4}{5}\right) = \frac{9-16}{25}$$

③

$$\sin\left(\tan^{-1}\cos\underbrace{\cot^{-1}\left(-\frac{1}{\sqrt{3}}\right)}_{\left(\frac{2\pi}{3}\right)}\right)$$

$$\sin\left(\tan^{-1}\cos\left(\frac{2\pi}{3}\right)\right)$$

$$\sin\left(\tan^{-1}\left(-\frac{1}{2}\right)\right)$$



$$\sin\left(-\tan^{-1}\left(\frac{1}{2}\right)\right)$$

$$-\sin\left(\underbrace{\tan^{-1}\left(\frac{1}{2}\right)}_{\theta}\right)$$

$$-\sin\theta. = -\frac{1}{\sqrt{5}}.$$

④

$$\tan\left(\frac{1}{2}\underbrace{\cos^{-1}\frac{\sqrt{5}}{3}}_{\theta}\right)$$

$\tan\frac{\theta}{2}$; $\cos\theta = \frac{\sqrt{5}}{3}$

$\boxed{\theta \in 1^{st} Q}$

$$\tan^2\frac{\theta}{2} = \frac{1-\cos\theta}{1+\cos\theta}$$

$$\tan^2\frac{\theta}{2} = \frac{1-\frac{\sqrt{5}}{3}}{1+\frac{\sqrt{5}}{3}}$$

$\tan\frac{\theta}{2} = \sqrt{\frac{3-\sqrt{5}}{3+\sqrt{5}}}$; $\frac{\theta}{2} \in$ 1st Q

Q2 [ ] → Gif and { } → fractional part fns.

(a) Domain & range of $\cos^{-1}[x]$

(b) Domain & range of $y = \sin^{-1}(e^x)$

(c) Domain & range of $\cos^{-1}\{x\}$

(d) Domain & range of $\tan^{-1}\sqrt{\dfrac{\ln^2 x - \ln x + 1}{\ln^2 x + \ln x + 1}}$?

(a) $f(x) = \cos^{-1}[x]$

Domain: $-1 \le [x] \le 1 \Rightarrow x \in [-1, 2)$

$[x] = -1; 0; 1$

Range: $\{0, \frac{\pi}{2}, \pi\}$

(b) $f(x) = \sin^{-1}(e^x)$

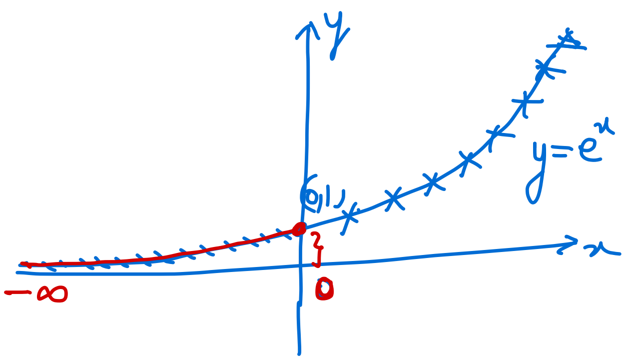
Domain:

$-1 \le e^x \le 1$

$x \in (-\infty, 0]$.

$e^x \in (0, 1]$

$R_f \in (0, \frac{\pi}{2}]$



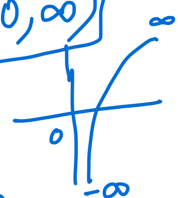
③ $f(x) = \cos^{-1}\{x\}$ $\longrightarrow$ Domain is $\underline{\underline{R}}$.

$\{x\} \in [0, 1)$

Range : $\left(0, \frac{\pi}{2}\right]$ Ans

④ $f(x) = \tan^{-1}\sqrt{\dfrac{\ln^2 x - \ln x + 1}{\ln^2 x + \ln x + 1}}$

$\boxed{D_f \in (0, \infty)}$



$$\frac{\ln^2 x - \ln x + 1}{\ln^2 x + \ln x + 1} \geq 0.$$

$\ln x = t \in (-\infty, \infty)$

$$\left(\frac{t^2 - t + 1}{t^2 + t + 1}\right) \in \left[\frac{1}{3}, 3\right]$$

$R_f \in \left[\frac{\pi}{6}, \frac{\pi}{3}\right]$ Ans

Q If $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = 3\pi$ then find the value of $x^{2020} + y^{2021} + z^{2022}$?

Sol^n $(\cos^{-1}x)_{max} = \pi$

$\cos^{-1}x = \cos^{-1}y = \cos^{-1}z = \pi$.

$\boxed{x = y = z = -1}$

$\cancel{(-1)}^{2020} + \cancel{(-1)}^{2021} + (-1)^{2022}$

$= 1$.

Q If $(\sin^{-1}x + \sin^{-1}y)(\sin^{-1}w + \sin^{-1}z) = \pi^2$ then compute the value of $(x + y)(w + z)$.

$(\sin^{-1}x)_{max} = \frac{\pi}{2}$ $\qquad$ $(\sin^{-1}x)_{min} = \frac{-\pi}{2}$

$x = y = w = z = 1$

OR

$x = y = w = z = -1$.

$(x+y)(w+z) = 4$

Ans

Q If $\operatorname{sgn}(\underbrace{\cot^{-1}x}_{+ve}) = x^2 - 3x + 3$ then find x

Sol^n $\cot^{-1}x \in (0, \pi)$; $x \in R$.

$$1 = x^2 - 3x + 3$$

$$x^2 - 3x + 2 = 0 \Rightarrow x = 1 \text{ or } 2.$$

Q Solve for x :-

① $\sin^{-1}(x-1) > \sin^{-1}(2x-3)$.

② $\cos^{-1}(x^2+1) < \cos^{-1}(-2x)$

① $-1 \le x-1 \le 1$ and $-1 \le 2x-3 \le 1$

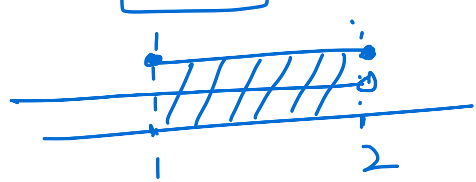
$0 \le x \le 2$ (and) $1 \le x \le 2$

Finally, $x \in [1, 2]$. —①—

$\sin^{-1}(x-1) > \sin^{-1}(2x-3)$

$x - 1 > 2x - 3. \Rightarrow x - 2 < 0.$

$\boxed{x < 2}$ —②—



① ∩ ②

$x \in [1, 2)$

Ans

② $\cos^{-1}(x^2+1) < \cos^{-1}(-2x)$.

$-1 \le x^2+1 \le 1$ $\qquad\qquad$ $-1 \le -2x \le 1$

$x^2 \le 0$ $\qquad\qquad$ $-\frac{1}{2} \le x \le \frac{1}{2}$

$x \in \{0\}$ $\quad \cap \quad$

$\underline{\underline{x \in \{0\}}}$ ✓ $\longrightarrow$ LHS < RHS

LHS: $\cos^{-1}(1) = 0$.

RHS: $\cos^{-1}(0) = \frac{\pi}{2}$

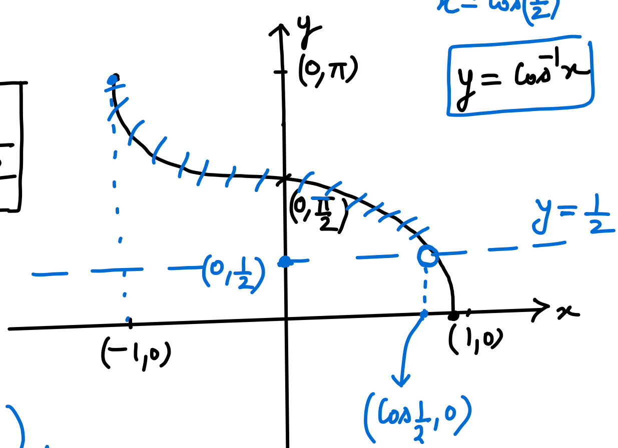
$\therefore\ x \in \{0\}$ Ans

③ $\log_{\frac{1}{2}}(\cos^{-1}x) < 1$.

$$\boxed{\cos^{-1}x > \frac{1}{2}}$$

$\cos^{-1}x = \frac{1}{2}$

$x = \cos\left(\frac{1}{2}\right)$

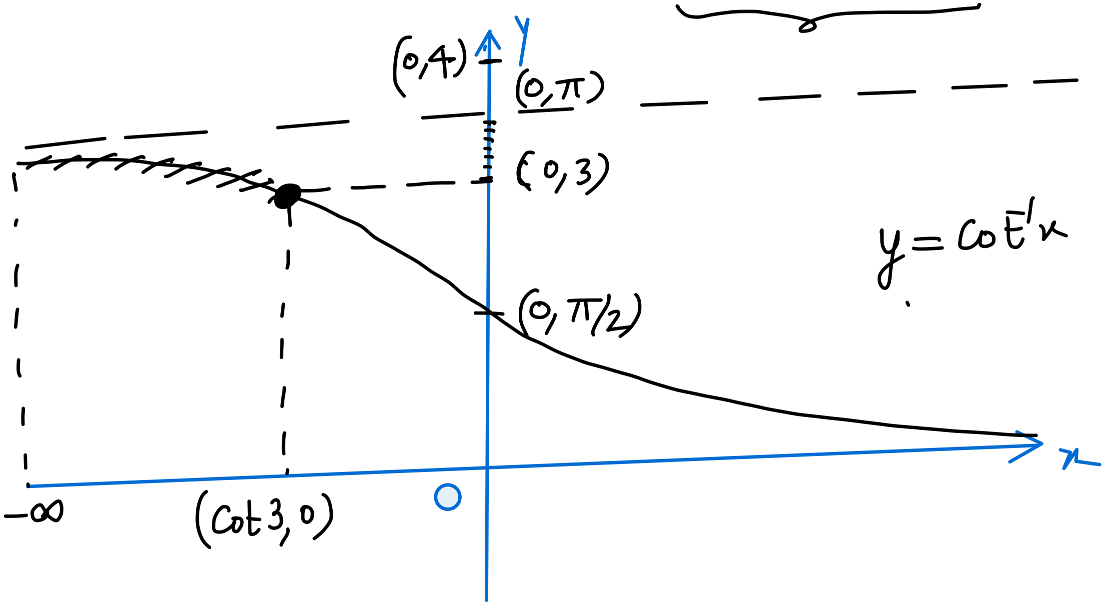


$x \in \left[-1, \cos\frac{1}{2}\right)$ Ans

④ $[\cot^{-1}x]^2 - 6[\cot^{-1}x] + 9 \le 0$; $[\ ] \to$ Gif

$$([\cot^{-1}x] - 3)^2 \le 0. \Rightarrow [\cot^{-1}x] = 3$$

$$\underbrace{3 \le \cot^{-1}x < \pi.}$$



$x \in (-\infty, \cot 3]$ Ans

PROPERTIES OF INVERSE TRIGONOMETRIC FUNCTION :

P-1 (a)

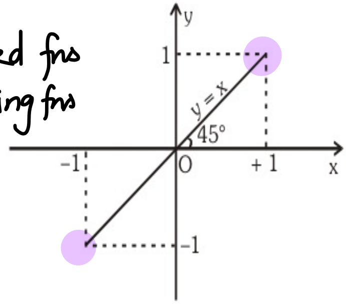
(i) $y = \sin(\underbrace{\sin^{-1}x}) = x$

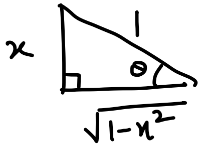
$x \in [-1,1]$, $y \in [-1,1]$, y is aperiodic

$D_f \in [-1, 1]$

$y = \sin(\underbrace{\sin^{-1}x}_{\theta}) = \sin\theta = x$.

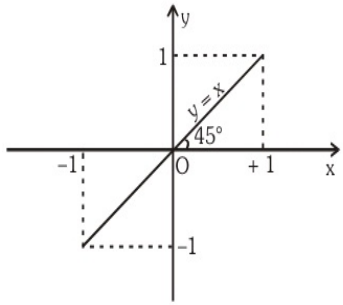
① Bounded fns
② increasing fns
③ Aperiodic





(ii) $y = \cos(\underbrace{\cos^{-1}x}) = x$

$x \in [-1,1]$, $y \in [-1,1]$, y is aperiodic



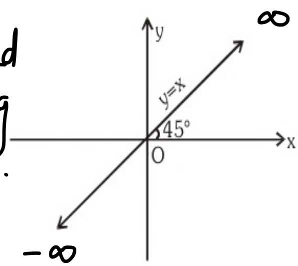
(i) & (ii) are identical fns

(iii) $y = \tan(\underbrace{\tan^{-1}x}) = x$

$x \in R$, $y \in R$, y is aperiodic

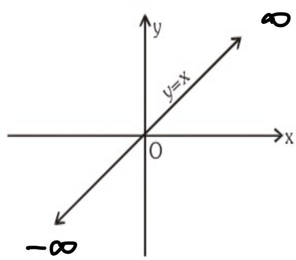
$D_f \in R$.

① Unbounded
② Increasing
③ Aperiodic.



(iv) $y = \cot(\cot^{-1}x) = x$,

$x \in R$; $y \in R$, y is aperiodic



(iii) & (iv) are identical fns.

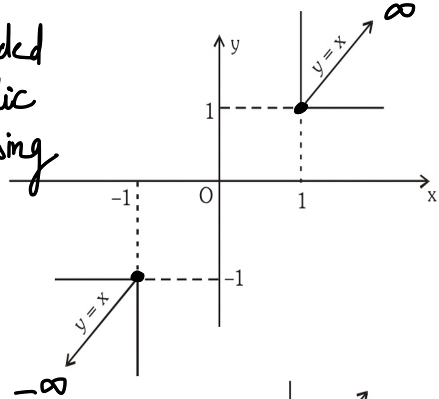
(v) $y = \operatorname{cosec}(\underbrace{\operatorname{cosec}^{-1} x}) = x,$

$|x| \geq 1, \ |y| \geq 1$, y is aperiodic

$D_f: \ |x| \geqslant 1.$

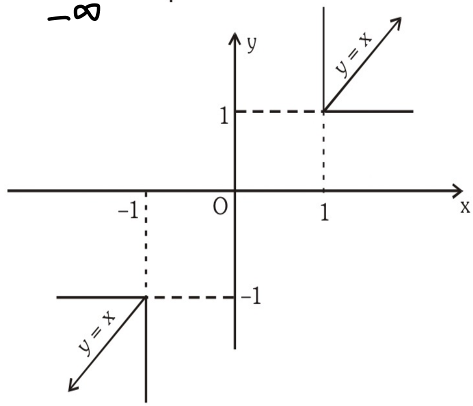
$x \in (-\infty, -1] \cup [1, \infty)$

① Unbounded
② Aperiodic
③ Increasing



(vi) $y = \sec(\sec^{-1} x) = x$

$|x| \geq 1 \ ; \ |y| \geq 1$, y is aperiodic



(v) & (vi) are identical fns.

Note: $\sin x = \theta$

if $\downarrow$ $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$x = \sin^{-1}(\theta)$.

(Not always)

$$\boxed{\sin^{-1}\theta = x}$$

$\Downarrow$ always

$$\boxed{\theta = \sin x}$$

P-1 (b)

(1) $f(x) = \sin^{-1}(\underbrace{\sin x})$ $\qquad D_f \in R$

let $\underbrace{\sin x = t}$; $t \in [-1, 1]$. $\qquad \sin x \in [-1, 1]$

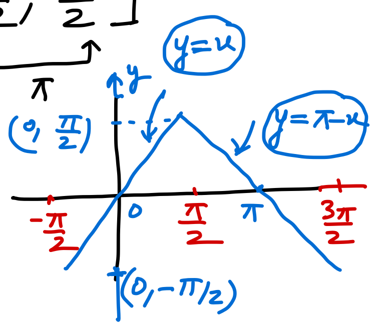
$x = \sin^{-1}(t)$; $^{*}x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ $\qquad R_f \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$y = f(x) = \sin^{-1}(\underbrace{\sin x}) = x$; $x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

π

$\sin x = t$; $t \in [-1, 1]$

$\sin(\underbrace{\pi - x}) = t \Rightarrow (\pi - x) = \sin^{-1}(t)$



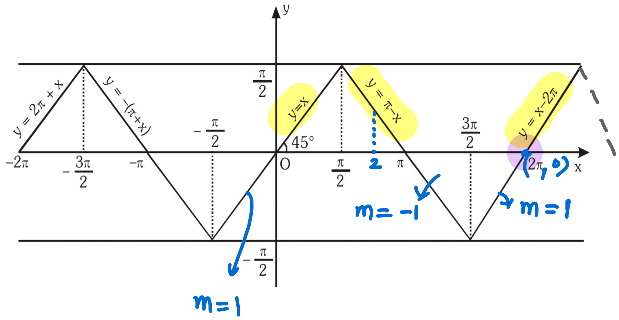
$$-\frac{\pi}{2} \leq \pi - x \leq \frac{\pi}{2}.$$

$$-\frac{3\pi}{2} \leq -x \leq -\frac{\pi}{2} \Rightarrow \frac{\pi}{2} \leq x \leq \frac{3\pi}{2}$$

(i) $y=\sin^{-1}(\sin x),\ x\in R,\ y\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ periodic with period 2π and it is an odd function.

Note : To draw the graph, plot only between $x\in[0,\pi]$ and draw rest of the graph using periodicity and odd function property.

$$\sin^{-1}(\sin x)=\begin{cases} -\pi-x, & -\frac{3\pi}{2}\le x\le -\frac{\pi}{2}\\ x, & -\frac{\pi}{2}\le x\le \frac{\pi}{2}\\ \pi-x, & \frac{\pi}{2}\le x\le \frac{3\pi}{2}\\ x-2\pi, & \frac{3\pi}{2}\le x\le \frac{5\pi}{2}\\ \vdots & \end{cases}$$



$\sin(\theta-2\pi)=\sin\theta$

$\sin^{-1}\sin(\underbrace{2}) = (\pi-2)$

$\sin^{-1}\underbrace{\sin(\underbrace{30})} = \sin^{-1}\sin(\underbrace{30-10\pi})$

$\underbrace{10\pi = 31.4}$

$= (30-10\pi)$

31.4
30.0
1.4

$\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

$5\pi = 5\times 3.14$

$\sin^{-1}\sin(\underbrace{17}) = \sin^{-1}\sin(\underbrace{5\pi-17})$

$=(5\pi-17).$

$\underbrace{15.70}$

$17-15.7 = 1.3$

$\sin^{-1}(\sin 21) = (7\pi-21)$

3.14
7
$\underbrace{21.98}$

$\sin^{-1}(\sin 61) = (19\pi-61)$

$\sin^{-1}\sin 62 = (62-20\pi).$

3.14×20

31.4×2

1.8 $= 62.8-61$

① $\sin^{-1}\sin\left(\underbrace{\frac{5\pi}{3}}\right) = \sin^{-1}\sin\left(\frac{6\pi-\pi}{3}\right)$

$= \sin^{-1}\sin\left(2\pi - \frac{\pi}{3}\right)$

$= -\sin^{-1}\sin\left(\frac{\pi}{3}\right) = -\frac{\pi}{3}.$

② $\sin^{-1}\sin\left(\underbrace{\frac{7\pi}{2}}\right) = \sin^{-1}\sin\left(\frac{6\pi}{2}+\frac{\pi}{2}\right)$

$= \sin^{-1}\sin\left(3\pi + \pi/2\right)$

$= -\sin^{-1}\sin\left(\frac{\pi}{2}\right) = -\frac{\pi}{2}.$

③ $\sin^{-1}\sin\left(\frac{48\pi}{7}\right) = \sin^{-1}\sin\left(\frac{49\pi-\pi}{7}\right)$

$= \sin^{-1}\sin\left(7\pi - \frac{\pi}{7}\right)$

$= \sin^{-1}\sin\left(\frac{\pi}{7}\right) = \left(\frac{\pi}{7}\right)$

④ $\sin^{-1}\cos\left(\underbrace{\frac{7\pi}{3}}\right) = \sin^{-1}\sin\left(\frac{\pi}{2} - \frac{7\pi}{3}\right)$

$= \sin^{-1}\sin\left(\frac{3\pi-14\pi}{6}\right)$

$= \sin^{-1}\sin\left(\frac{-11\pi}{6}\right)$

$- \sin^{-1}\sin\left(2\pi - \frac{\pi}{6}\right) = \sin^{-1}\sin\frac{\pi}{6}$

$\pi/6$

Q Find range of :

(1) $f(x) = |\sin^{-1}x| - \sec^{-1}|x|.$

$D_f \in \{-1, 1\}$

$R_f \in \left\{\frac{\pi}{2}\right\}$

(2) $f(x) = \cos^{-1}x - \underbrace{\tan^{-1}x}$

$\tan^{-1}x \downarrow$ fns

$\cos^{-1}x \downarrow$ fns

$\therefore f(x) \downarrow$ fns.

$D_f \in [-1, 1]$

$R_f \in \left[-\frac{\pi}{4}, \frac{5\pi}{4}\right]$

(3) $f(x) = \dfrac{\pi}{\underbrace{2\cot^{-1}x + \pi}}$.

$D_f \in R$

$\cot^{-1}x \in (0, \pi)$

$2\cot^{-1}x \in (0, 2\pi)$

$\pi + 2\cot^{-1}x \in (\pi, 3\pi)$

$\dfrac{1}{\pi + 2\cot^{-1}x} \in \left(\frac{1}{3\pi}, \frac{1}{\pi}\right)$

$R_f \in \left(\frac{1}{3}, 1\right)$ Ans

(2) $f(x) = \cos^{-1} \underbrace{\cos x}.$

$D_f \in R$

$R_f \in [0, \pi]$ *

Let $\cos x = t$; $t \in [-1, 1]$

$f(x) = \cos^{-1}(t)$

$\boxed{f(x) = x}$ ✓

$x = \cos^{-1}(t)$

when $x \in [0, \pi]$

✗ $\cos(\pi - \theta) = -\cos\theta$

✗ $\cos(\theta - \pi) = -\cos\theta$

✓ $\cos(2\pi - \theta) = \cos\theta$

✓ $\cos(\theta - 2\pi) = \cos\theta$

$\cos x = t$

$\cos(2\pi - x) = t$;

$2\pi - x = \cos^{-1} t$

$\boxed{2\pi - x = f(x)}$ ✓

$2\pi - x \in [0, \pi]$

$-x \in [-2\pi, -\pi]$

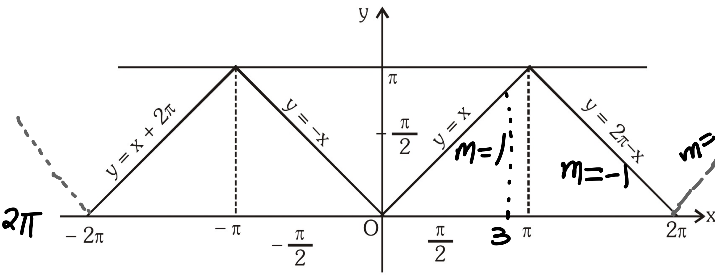
$x \in [\pi, 2\pi]$

(ii)

$y = \cos^{-1}(\cos x)$, $x \in R$, $y \in [0,\pi]$, periodic with period 2π and it is an even function.

Note : To draw the graph, plot only between $x \in [0, \pi]$ and draw rest of the graph using periodicity and even function property.

$$\cos^{-1}(\cos x) = \begin{cases} \vdots \\ -x \ , & -\pi \le x < 0 \\ x \ , & 0 \le x < \pi \\ 2\pi - x \ ; & \pi \le x < 2\pi \\ x - 2\pi \ ; & 2\pi \le x < 3\pi \\ 4\pi - x \ ; & 3\pi \le x < 4\pi \\ x - 4\pi \ ; & 4\pi \le x < 5\pi . \\ \vdots \end{cases}$$



$\cos^{-1}\cos 3 = 3$

$\cos^{-1}\cos 5 = \cos^{-1}\cos(\underbrace{2\pi - 5}_{[0,\pi]}) = 2\pi - 5.$

$\cos^{-1}\cos 7 = \cos^{-1}\cos(7 - 2\pi) = 7 - 2\pi.$

$\cos^{-1}\cos 25 = (8\pi - 25).$

$\cos^{-1}\cos 34 = (34 - 10\pi).$

$\cos^{-1}\cos\left(-\frac{2\pi}{3}\right) = \cos^{-1}\cos\left(\frac{2\pi}{3}\right) = \frac{2\pi}{3}.$

$$\cos^{-1}\cos\left(\frac{17\pi}{3}\right) = \cos^{-1}\cos\left(\frac{18\pi-\pi}{3}\right)$$
$$= \cos^{-1}\cos\left(6\pi - \frac{\pi}{3}\right)$$
$$= \cos^{-1}\cos\left(\frac{\pi}{3}\right) = \frac{\pi}{3}.$$

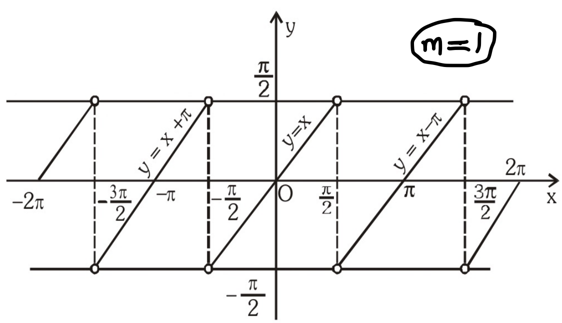
$$\cos^{-1}\sin\left(\frac{9\pi}{4}\right) = \cos^{-1}\cos\left(\frac{\pi}{2} - \frac{9\pi}{4}\right)$$
$$= \cos^{-1}\cos\left(\frac{-7\pi}{4}\right)$$
$$= \cos^{-1}\cos\left(\frac{7\pi}{4}\right) = \cos^{-1}\cos\left(\frac{8\pi-\pi}{4}\right)$$
$$= \cos^{-1}\cos\left(2\pi - \frac{\pi}{4}\right)$$
$$= \cos^{-1}\cos\frac{\pi}{4} = \frac{\pi}{4}.$$

(iii) $y = \tan^{-1}(\tan x)$

$x \in R - \left\{(2n-1)\frac{\pi}{2}, n \in I\right\}$; $y \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, periodic with period π and it is an odd function.

Note : To draw the graph, plot only between $x \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and draw rest of the graph using periodicity.

$$\tan^{-1}(\tan x) = \begin{cases} x+\pi, & -\frac{3\pi}{2} < x < -\frac{\pi}{2} \\ x, & -\frac{\pi}{2} < x < \frac{\pi}{2} \\ x-\pi, & \frac{\pi}{2} < x < \frac{3\pi}{2} \end{cases}$$



$\tan^{-1}\tan 3 = (3-\pi)$.

$\tan^{-1}\tan 5 = (5-2\pi)$.

$\tan^{-1}\tan 11 = (11-4\pi)$.

$\tan^{-1}\tan 19 = (19-6\pi)$.

×× $\tan(\pi-\theta) = -\tan\theta$

✓ $\tan(\theta-\pi) = \tan\theta$

×× $\tan(2\pi-\theta) = -\tan\theta$

✓ $\tan(\theta-2\pi) = \tan\theta$

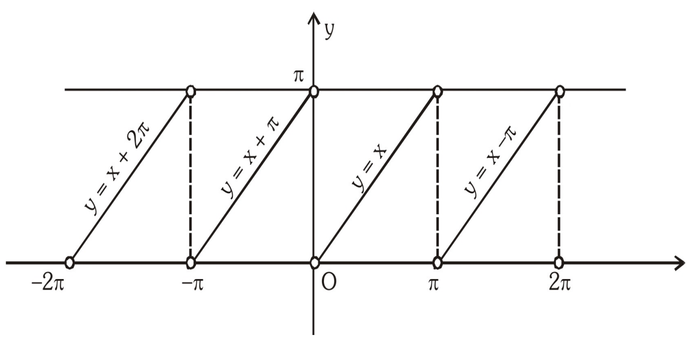
$$\begin{array}{r} 3.14 \\ 4 \\ \hline 12.56 - 11 \\ \underline{\underline{1.56}} \end{array}$$

(iv)

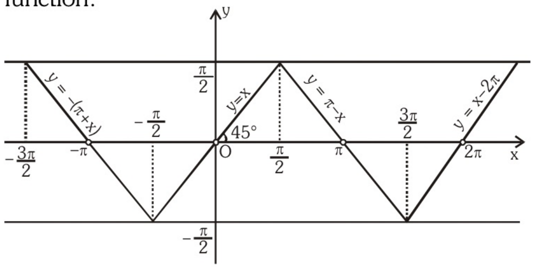
$y = \cot^{-1}(\cot x)$, $x \in R - \{n\pi\}$, $y \in (0, \pi)$, periodic with period π and neither even nor odd function.

Note : To draw the graph, plot only between $x \in (0, \pi)$ and draw rest of the graph using periodicity.

$$\cot^{-1}(\cot x) = \begin{cases} x+\pi, & -\pi < x < 0 \\ x, & 0 < x < \pi \\ x-\pi, & \pi < x < 2\pi \end{cases}$$

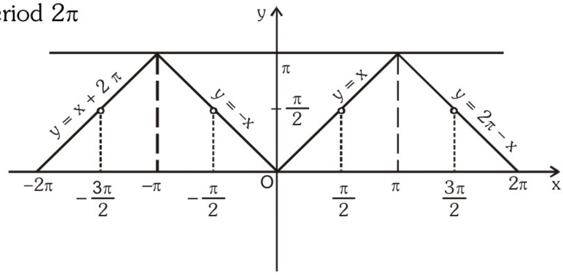


(v) $y = \text{cosec}^{-1}(\text{cosec } x)$, $x \in R - \{n\pi, n \in I\}$ $y \in \left[-\frac{\pi}{2}, 0\right) \cup \left(0, \frac{\pi}{2}\right]$, is periodic with period 2π and it is an odd function.



(vi) $y=\sec^{-1}(\sec x)$, y is periodic with period 2π

$x \in R - \left\{(2n-1)\frac{\pi}{2} n \in I\right\}$, $y \in \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right]$



① $\sin^{-1}\left(-\cos\frac{17\pi}{8}\right) = -\frac{3\pi}{8}$.

② $\sin^{-1}\left(-\sin\frac{101\pi}{7}\right) = -\frac{3\pi}{7}$.

③ $\cos^{-1}\cos\left(\frac{10\pi}{7}\right) = \frac{4\pi}{7}$

④ $\cot^{-1}\cot\left(\frac{26\pi}{5}\right) = \cot^{-1}\cot\left(5\pi + \frac{\pi}{5}\right)$

$= \cot^{-1}\cot\left(\frac{\pi}{5}\right) = \frac{\pi}{5}$.

Q If $0 < \cos^{-1} x < 1$ and $1 + \sin(\cos^{-1} x) + \sin^2(\cos^{-1} x) + \sin^3(\cos^{-1} x) + \ldots\ldots \infty = 2$, then x equals

(A) $\frac{1}{2}$ (B) $\frac{1}{\sqrt{2}}$ (C) $\frac{\sqrt{3}}{2}$ ✓ (D) 0

Sol^n

$$\frac{1}{1 - \sin(\cos^{-1} x)} = 2$$

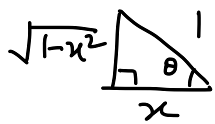
$$\frac{1}{2} = 1 - \sin(\cos^{-1} x)$$

$$\sin(\cos^{-1} x) = \frac{1}{2}$$

$\cos^{-1} x = \theta$

$x = \cos\theta$

$$\sin\theta = \frac{1}{2}$$



$$\sqrt{1-x^2} = \frac{1}{2}$$

$$x^2 = \frac{3}{4} \Rightarrow x = \frac{\sqrt{3}}{2} \checkmark \text{ or } -\frac{\sqrt{3}}{2} \times\times$$

$0 < \cos^{-1} x < 1.$

$\boxed{\cos 1 < x < 1} \Rightarrow$

P-2 **(i)** $\operatorname{cosec}^{-1} x = \sin^{-1}\frac{1}{x}$; $|x| \geq 1$ and $\sin^{-1} x = \operatorname{cosec}^{-1}\frac{1}{x}$, $|x| \leq 1$ $x \neq 0$

(ii) $\sec^{-1} x = \cos^{-1}\frac{1}{x}$; $|x| \geq 1$ and $\cos^{-1}x = \sec^{-1}\frac{1}{x}$, $|x| \leq 1, x \neq 0$

** **(iii)**
$$\cot^{-1} x = \begin{cases} \tan^{-1}\frac{1}{x} & ; \quad x > 0 \\ \pi + \tan^{-1}\frac{1}{x} & ; \quad x < 0 \end{cases}$$

Proof: (i) $\operatorname{cosec}^{-1} x = \sin^{-1}\left(\frac{1}{x}\right)$; $|x| \geq 1$

LHS: $\underbrace{\operatorname{cosec}^{-1} x} = \theta \Rightarrow x = \operatorname{cosec}\theta.$

$x = \frac{1}{\sin\theta}$

** $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] - \{0\}$

$\sin\theta = \frac{1}{x}$

$$\boxed{\theta = \sin^{-1}\left(\frac{1}{x}\right)}$$

(iii)

$\cot^{-1} x$ — $\rightarrow \tan^{-1}\left(\frac{1}{x}\right)$; $x > 0$

$\rightarrow \underbrace{\pi + \tan^{-1}\left(\frac{1}{x}\right)}$; $x < 0.$

Proof: $y = \cot^{-1} x = \theta$; $\theta \in (0, \pi).$

$\boxed{x = \cot\theta} = \frac{1}{\tan\theta}$ $\quad \left(0, \frac{\pi}{2}\right)$ $\quad \left(\frac{\pi}{2}, \pi\right)$

$$\boxed{\tan\theta = \frac{1}{x}.}$$

C–I $\quad \theta \in (0, \pi/2)$; $x = \cot\theta > 0$

$$\tan\theta = \frac{1}{x} \Rightarrow \theta = \tan^{-1}\left(\frac{1}{x}\right) ; \quad x > 0 .$$

C–II $\quad \theta \in \left(\frac{\pi}{2}, \pi\right)$ ✓ $\qquad \theta - \pi \in \left(-\frac{\pi}{2}, 0\right)$

$$\tan\theta = \frac{1}{x}$$

$$\tan(\theta - \pi) = \frac{1}{x} \qquad \cot\theta = x$$

$$\theta - \pi = \tan^{-1}\left(\frac{1}{x}\right) \qquad x < 0$$

$$\theta = \pi + \tan^{-1}\left(\frac{1}{x}\right)$$

$$\cot^{-1}x = \pi + \tan^{-1}\left(\frac{1}{x}\right) ; \quad x < 0 .$$

P-3

(i) $\sin^{-1}(-x) = -\sin^{-1}x$, $|x| \le 1$

(ii) $\tan^{-1}(-x) = -\tan^{-1}x$, $x \in R$

(iii) $\cos^{-1}(-x) = \pi - \cos^{-1}x$, $|x| \le 1$

(iv) $\cot^{-1}(-x) = \pi - \cot^{-1}x$, $x \in R$

(v) $\sec^{-1}(-x) = \pi - \sec^{-1}x$, $|x| \ge 1$

(vi) $\text{cosec}^{-1}(-x) = -\text{cosec}^{-1}x$, $|x| \ge 1$

Proof: ① $\boxed{\sin^{-1}(-x) = \theta}$; $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$-\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$$-x = \sin\theta.$$

$$x = -\sin\theta = \sin(-\theta)$$

$$-\theta = \sin^{-1}x$$

$$\theta = -\sin^{-1}(x)$$

② $\cos^{-1}(-x) = \pi - \cos^{-1}(x)$; $x \in [-1, 1]$.

$\boxed{\cos^{-1}(-x) = \theta}$; $\theta \in [0, \pi]$

$-\theta \in [-\pi, 0]$

$\pi - \theta \in [0, \pi]$

$$-x = \cos\theta.$$

$$x = -\cos\theta = \cos(\pi - \theta)$$

$$\cos^{-1}x = \pi - \theta.$$

$\boxed{\theta = \pi - \cos^{-1}x}$ (H.P)

P-4

(i) $\sin^{-1} x + \cos^{-1} x = \dfrac{\pi}{2}$, $|x| \le 1$

(ii) $\tan^{-1} x + \cot^{-1} x = \dfrac{\pi}{2}$, $x \in R$

(iii) $\operatorname{cosec}^{-1} x + \sec^{-1} x = \dfrac{\pi}{2}$, $|x| \ge 1$

Note : $\tan^{-1}x + \tan^{-1}\dfrac{1}{x} = \begin{cases} \dfrac{\pi}{2} , & x > 0 \\ -\dfrac{\pi}{2} , & x < 0 \end{cases}$

Proof: $\sin^{-1}x + \cos^{-1}x = \dfrac{\pi}{2}$

let $\sin^{-1}x = \theta$; $\theta \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

$-\theta \in \left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

$x = \sin\theta$

$x = \cos\left(\underbrace{\dfrac{\pi}{2} - \theta}\right)$ $\qquad \dfrac{\pi}{2} - \theta \in [0, \pi]$

$\cos^{-1}x = \dfrac{\pi}{2} - \theta.$

$\boxed{\cos^{-1}x + \theta = \dfrac{\pi}{2}}$ (H.P)

Q Find x if

$$5\tan^{-1}x + 3\cot^{-1}x = \frac{7\pi}{4}.$$

Sol^n

$$2\tan^{-1}x + 3\frac{\pi}{2} = \frac{7\pi}{4} \quad ; \quad \underline{\underline{x \in R}}$$

$$2\tan^{-1}x = \frac{\pi}{4}$$

$$\tan^{-1}x = \frac{\pi}{8} \Rightarrow x = \tan\frac{\pi}{8} = \sqrt{2}-1$$

Ans

Q Find range:

① $f(x) = \sin^{-1}x + \tan^{-1}x + \cos^{-1}x.$

$D_f \in [-1, 1]$

$$\boxed{f(x) = \frac{\pi}{2} + \underline{\underline{\tan^{-1}x}}}$$ *

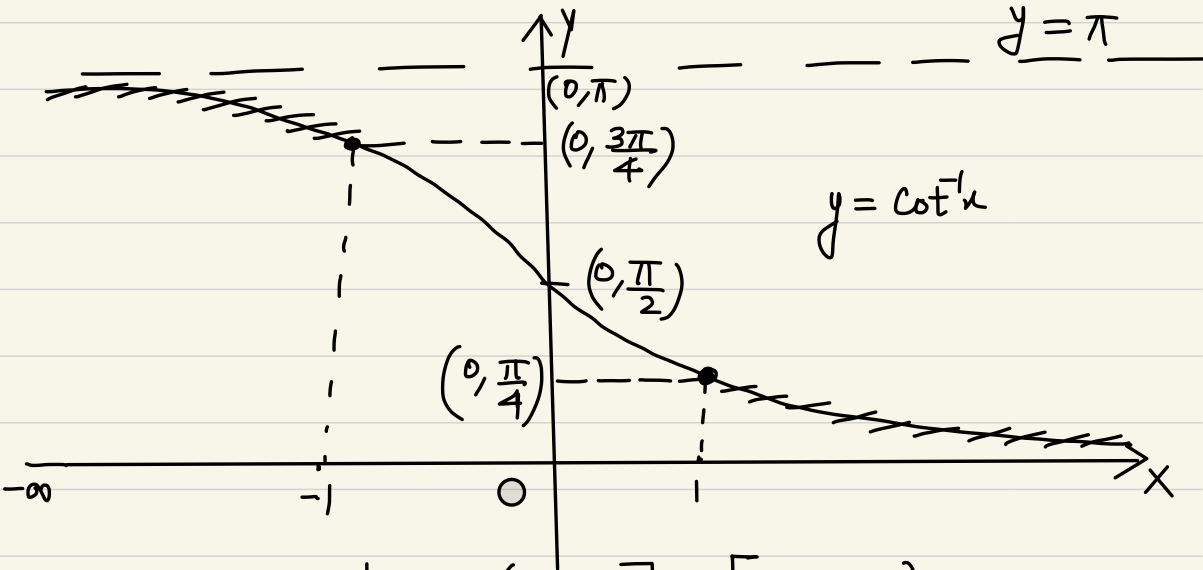
$\tan^{-1}x \uparrow$ fun

$$R_f \in \left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$$

(2) $f(x) = \sec^{-1}x + \operatorname{cosec}^{-1}x + \cot^{-1}x$

Domain : $x \in (-\infty, -1] \cup [1, \infty)$

$$\boxed{f(x) = \frac{\pi}{2} + \cot^{-1}x}$$



$\cot^{-1}x \in \left(0, \frac{\pi}{4}\right] \cup \left[\frac{3\pi}{4}, \pi\right)$

$R_f \in \left(\frac{\pi}{2}, \frac{3\pi}{4}\right] \cup \left[\frac{5\pi}{4}, \frac{3\pi}{2}\right)$ Ans

③ $f(x) = \sin^{-1}x + \underbrace{\sec^{-1}x + \cos^{-1}x}$.

$D_f \in \{-1, 1\}$

$f(x) = \frac{\pi}{2} + \sec^{-1}x$

$R_f \in \left\{\frac{\pi}{2}, \frac{3\pi}{2}\right\}$

Q If n be the number of distinct solutions of the equation

$\cos^{-1}|x| + \cos^{-1}|2x| = \pi$, then find '$n$'?

Sol^n C–I $x \geq 0$

$\cos^{-1}x + \cos^{-1}2x = \pi$.

$\cos^{-1}2x = \pi - \cos^{-1}x$

$\cos^{-1}2x = \cos^{-1}(-x)$

$2x = -x \Rightarrow \boxed{x = 0}$ ✓

C–II $x < 0$. *

$$\cos^{-1}(-x) + \cos^{-1}(-2x) = \pi.$$

$$\pi - \cos^{-1}x + \cancel{\pi} - \cos^{-1}(2x) = \cancel{\pi}$$

$$\boxed{\pi = \underbrace{\cos^{-1}x}_{} + \underbrace{\cos^{-1}(2x)}_{}}$$

Not possible

$\therefore$ Only 1 sol^n.

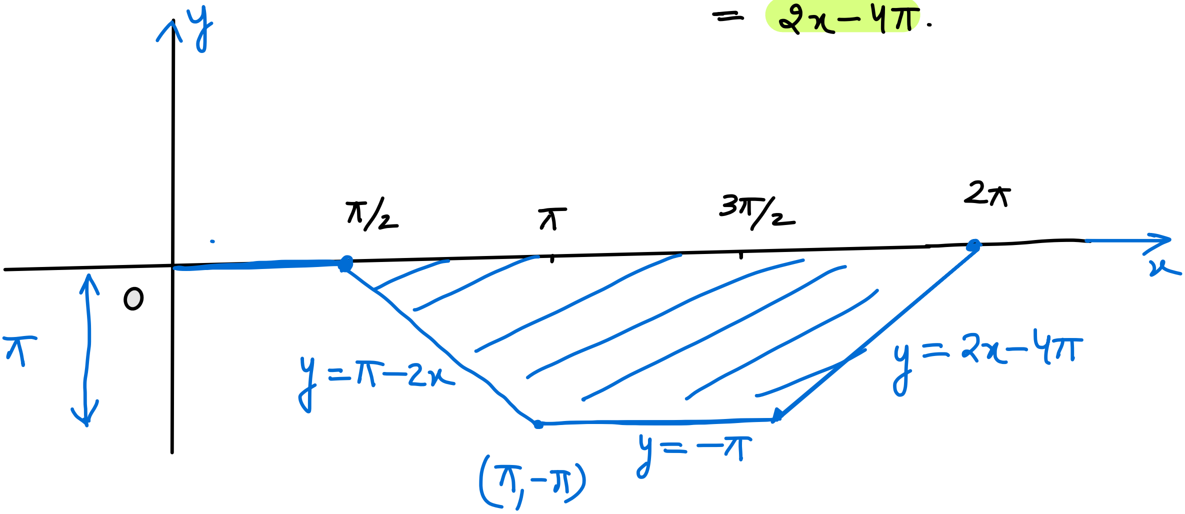
Q Find the area bounded by

$f(x) = \sin^{-1}\sin x - \cos^{-1}\cos x$ for

$x \in [0, 2\pi]$ with x–axis ?

$$f(x) = \sin^{-1}\sin x - \cos^{-1}\cos x \to \begin{cases} x - x = 0 & ;\ x \in [0, \frac{\pi}{2}] \\ (\pi - x) - x = \pi - 2x & ;\ x \in [\frac{\pi}{2}, \pi] \\ (\pi - \cancel{x}) - (2\pi - \cancel{x}) = -\pi & ;\ x \in [\pi, \frac{3\pi}{2}] \\ (x - 2\pi) - (2\pi - x) = 2x - 4\pi. & ;\ x \in [\frac{3\pi}{2}, 2\pi] \end{cases}$$

$[0, 2\pi]$



$$A = \frac{1}{2}\left(\frac{\pi}{2} + \frac{3\pi}{2}\right) \times \pi = \frac{1}{2}(2\pi) \cdot \pi = \pi^2.$$

Ans

Q Find the range of values of 'a' for which the equation $(\sin^{-1}x)^3 + (\cos^{-1}x)^3 = a\pi^3$ posses real sol?

Sol^n

$$f(x) = (\sin^{-1}x)^3 + (\cos^{-1}x)^3 \quad ; \quad \boxed{a^3+b^3=(a+b)(a^2+b^2-ab)}$$

$$= \left(\frac{\pi}{2}\right)\left(\underbrace{(\sin^{-1}x+\cos^{-1}x)^2}_{} - 3\sin^{-1}x\cos^{-1}x\right).$$

Let $\sin^{-1}x = t$

$t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ ★★

$$= \frac{\pi}{2}\left(\frac{\pi^2}{4} - 3t\left(\frac{\pi}{2}-t\right)\right)$$

$$= \frac{\pi^3}{8} - 3t\cdot\frac{\pi^2}{4} + \frac{3\pi}{2}t^2.$$

$$= \frac{\pi^3}{8} + \frac{3\pi}{2}\left(t^2 - \frac{\pi}{2}t\right) = \frac{\pi^3}{8} + \frac{3\pi}{2}\left(\left(t-\frac{\pi}{4}\right)^2 - \frac{\pi^2}{16}\right)$$

$$= \frac{\pi^3}{8} - \frac{3\pi^3}{32} + \frac{3\pi}{2}\left(t-\frac{\pi}{4}\right)^2$$

$$= \frac{\pi^3}{32} + \frac{3\pi}{2}\left(t-\frac{\pi}{4}\right)^2 \qquad \left(t-\frac{\pi}{4}\right)^2 \in \left[0, \frac{9\pi^2}{16}\right]$$

$$f(x)_{min} = \frac{\pi^3}{32} \quad ; \quad f(x)_{max} = \frac{\pi^3}{32} + \frac{3\pi}{2}\left(\frac{9\pi^2}{16}\right)$$

$$= \frac{28\pi^3}{32} = \frac{7\pi^3}{8}$$

$$a\pi^3 \in \left[\frac{\pi^3}{32}, \frac{7\pi^3}{8}\right]$$

$$a \in \left[\frac{1}{32}, \frac{7}{8}\right]$$ Ans

P-5

v.v. imp

(i) $\tan^{-1}x + \tan^{-1}y = \begin{cases} \tan^{-1}\dfrac{x+y}{1-xy}, & x>0, y>0 \text{ and } xy<1 \\ \pi + \tan^{-1}\dfrac{x+y}{1-xy}, & x>0, y>0 \text{ and } xy>1 \\ \dfrac{\pi}{2}, & x>0, y>0 \text{ and } xy=1 \end{cases}$ **(remember)**

(ii) $\tan^{-1}x - \tan^{-1}y = \tan^{-1}\dfrac{x-y}{1+xy} \quad x>0,\ y>0$

Note :

(i) $\tan^{-1}1 + \tan^{-1}2 + \tan^{-1}3 = \pi$ **(remember)** $= \cot^{-1}1 + \cot^{-1}\frac{1}{2} + \cot^{-1}\frac{1}{3}$.

(ii) $\cot^{-1}1 + \cot^{-1}2 + \cot^{-1}3 = \dfrac{\pi}{2}$ **(remember)** $= \tan^{-1}1 + \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{3}$.

Proof:

$$\tan^{-1}1 + \underbrace{\tan^{-1}\overset{x}{2} + \tan^{-1}\overset{y}{3}}_{\checkmark} = \tan^{-1}1 + \pi + \tan^{-1}\left(\frac{2+3}{1-6}\right)$$
$$= \cancel{\tan^{-1}1} + \pi + \tan^{-1}\cancel{(-1)}$$
$$= \pi.$$

(iii)

$$\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \begin{cases} \tan^{-1}\left[\dfrac{x+y+z-xyz}{1-(xy+yz+zx)}\right], & \text{where } x>0, y>0, z>0 \text{ and } xy+yz+zx<1 \\ \pi + \tan^{-1}\left[\dfrac{x+y+z-xyz}{1-(xy+yz+zx)}\right], & \text{where } x>0, y>0, z>0 \text{ and } xy+yz+zx>1 \end{cases}$$

Proof: $\tan^{-1}x + \tan^{-1}y$; $x, y > 0$.

Let $\tan^{-1}x = \alpha$ and $\tan^{-1}y = \beta$; $\alpha \in (0, \pi/2)$

$\beta \in (0, \pi/2)$.

$\alpha + \beta \in (0, \pi)$.

$(0, \frac{\pi}{2})$ $(\frac{\pi}{2}, \pi)$

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = \left(\frac{x+y}{1-xy}\right)$$

C–I $\alpha + \beta \in (0, \frac{\pi}{2})$

$\alpha + \beta < \frac{\pi}{2}$. $\Rightarrow$ $\alpha < \frac{\pi}{2} - \beta$

$\tan\alpha < \tan\left(\frac{\pi}{2} - \beta\right)$

$\tan\alpha < \cot\beta$.

$x < \frac{1}{y} \Rightarrow \boxed{xy < 1.}$

$\tan(\alpha+\beta) = \frac{x+y}{1-xy} \Rightarrow \alpha + \beta = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

$\tan^{-1}x + \tan^{-1}y = \tan^{-1}\left(\frac{x+y}{1-xy}\right)$

$x, y > 0$ and $xy < 1$.

C–II $\frac{\pi}{2} < \alpha + \beta < \pi$

$\alpha + \beta > \frac{\pi}{2} \Rightarrow \alpha > \frac{\pi}{2} - \beta$.

$\tan\alpha > \cot\beta \Rightarrow x > \frac{1}{y} \Rightarrow xy > 1$

$\tan(\alpha+\beta) = \tan(\alpha + \beta - \pi) = \frac{x+y}{1-xy}$

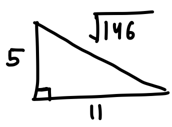
$\alpha + \beta = \pi + \tan^{-1}\left(\frac{x+y}{1-xy}\right)$; $x, y > 0$; $xy > 1$

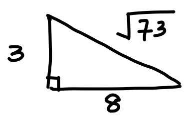
Q

If $\alpha = \tan^{-1}5 - \tan^{-1}3 + \tan^{-1}\frac{7}{9}$ and $\beta = \tan^{-1}\frac{2}{11} + \cot^{-1}\frac{24}{7} + \tan^{-1}\frac{1}{3}$, then

✓(A) $\alpha = \beta$ (B) $\alpha > \beta$ (C) $\alpha < \beta$ ✓(D) $\alpha + \beta = \pi/2$

$\alpha = \frac{\pi}{4}$; $\beta = \frac{\pi}{4}$.





Q $\sin^{-1}\left(\frac{3}{\sqrt{73}}\right) + \cos^{-1}\left(\frac{11}{\sqrt{146}}\right) + \cot^{-1}(\sqrt{3})$.

$\underbrace{\tan^{-1}\left(\frac{3}{8}\right) + \tan^{-1}\left(\frac{5}{11}\right)} + \left(\frac{\pi}{6}\right) = \frac{5\pi}{12}$

Ans

Q If $\tan^{-1}4 + \tan^{-1}5 = \cot^{-1}(\lambda)$, then find λ

Sol^n

$$\pi + \tan^{-1}\left(\frac{4+5}{1-20}\right) = \cot^{-1}(\lambda)$$

$$\pi - \tan^{-1}\left(\frac{9}{19}\right) = \cot^{-1}(\lambda).$$

$$** \quad \underbrace{\pi - \cot^{-1}\left(\frac{19}{9}\right)} = \cot^{-1}(\lambda).$$

$$\cot^{-1}\left(-\frac{19}{9}\right) = \cot^{-1}(\lambda) \Rightarrow \lambda = -\frac{19}{9} \text{ Ans}$$

Q $$\underbrace{2\cos^{-1}\frac{3}{\sqrt{13}}}_{\theta} + \underbrace{\cot^{-1}\frac{16}{63}}_{\alpha} + \frac{1}{2}\underbrace{\cos^{-1}\frac{7}{25}}_{\beta} = \tan^{-1}\left(\frac{12}{5}\right) + \tan^{-1}\left(\frac{63}{16}\right) + \tan^{-1}\left(\frac{3}{4}\right) = \pi \text{ Ans}$$

$$2\theta + \alpha + \frac{\beta}{2} = ?$$

where $\cos\theta = \frac{3}{\sqrt{13}}$

$\cot\alpha = \frac{16}{63} \Rightarrow \alpha = \tan^{-1}\frac{63}{16}.$

$$\cos 2\theta = 2\cos^2\theta - 1$$
$$= 2\cdot\frac{9}{13} - 1 = \frac{5}{13}$$
$$2\theta = \cos^{-1}\left(\frac{5}{13}\right) = \tan^{-1}\left(\frac{12}{5}\right).$$

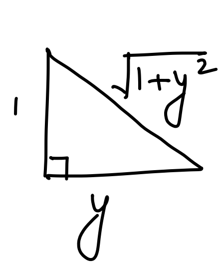
$\cos\beta = \frac{7}{25}$; $\beta \in$ 1st Quad

$$2\cos^2\frac{\beta}{2} - 1 = \frac{7}{25}$$
$$\cancel{2}\cos^2\frac{\beta}{2} = \frac{\cancel{32}^{16}}{25} \Rightarrow \cos\frac{\beta}{2} = \frac{4}{5}$$

Q Find the positive integral value(s) of x and y which satisfy the equation $\tan^{-1}x + \cos^{-1}\frac{y}{\sqrt{1+y^2}} = \sin^{-1}\left(\frac{3}{\sqrt{10}}\right)$

Sol^n

$$\tan^{-1}x + \tan^{-1}\left(\frac{1}{y}\right) = \tan^{-1}\left(\frac{3}{1}\right)$$



$$\tan^{-1}\left(\frac{x+\frac{1}{y}}{1-x\left(\frac{1}{y}\right)}\right) = \tan^{-1}(3).$$

$$\frac{xy+1}{y-x} = 3. \Rightarrow xy+1 = 3y-3x.$$

$$1+3x = y(3-x)$$

$$y = \frac{1-3(3-x)+9}{3-x} \Leftarrow y = \frac{3x+1}{3-x}.$$

$$y = \left(\frac{10}{3-x}\right) - 3.$$

$x, y \in I^+$

$x=1;\ y=2.$
$x=2;\ y=7$ Ans

Q Show that the roots $r, s,$ and t of the cubic $x(x-2)(3x-7)=2$, are real and positive. Also compute the value of $\tan^{-1}(r)+\tan^{-1}(s)+\tan^{-1}(t)$.

Sol^n

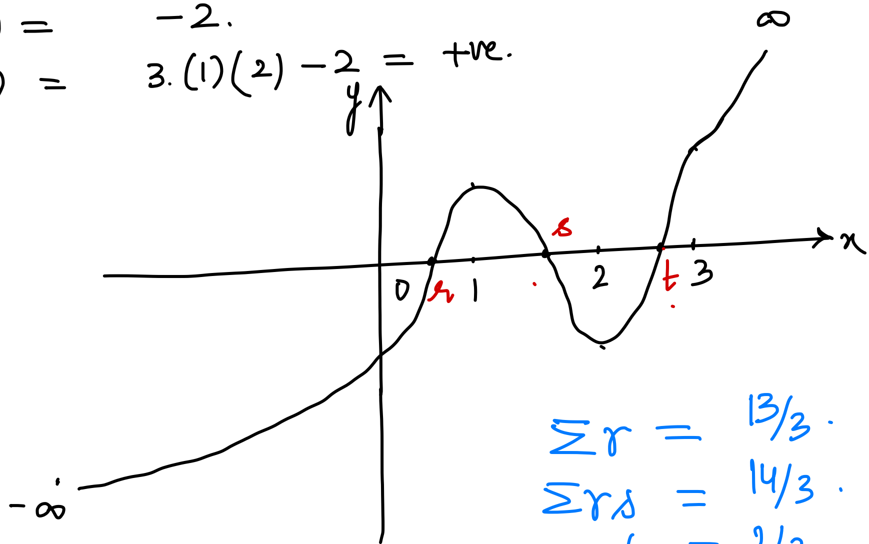
$$f(x) = x(x-2)(3x-7) - 2.$$

$$f(0) = -ve.$$

$$f(1) = 1(-1)(-4) - 2 = +ve.$$

$$f(2) = -2.$$

$$f(3) = 3.(1)(2) - 2 = +ve.$$



$$\Sigma r = 13/3.$$

$$\Sigma rs = 14/3.$$

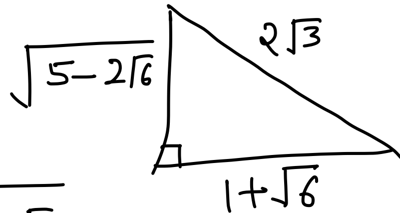
$$rst = 2/3.$$

$$\left(\tan^{-1} r + \tan^{-1} s + \tan^{-1} t\right) = \pi + \tan^{-1}\left(\frac{\Sigma r - rst}{1 - \Sigma rs}\right)$$

$$= \frac{3\pi}{4} \quad \text{Ans}$$

Q Simplify :- $\cos^{-1}\sqrt{\frac{2}{3}} - \cos^{-1}\left(\frac{\sqrt{6}+1}{2\sqrt{3}}\right)$?

Sol^n $\tan^{-1}\frac{1}{\sqrt{2}} - \tan^{-1}\left(\frac{\sqrt{5-2\sqrt{6}}}{1+\sqrt{6}}\right)$



$$\sqrt{5-2\sqrt{6}} = \sqrt{(\sqrt{3})^2+(\sqrt{2})^2-2\cdot\sqrt{3}\cdot\sqrt{2}}$$

$$= \sqrt{(\sqrt{3}-\sqrt{2})^2} = |\sqrt{3}-\sqrt{2}|$$

$$= \sqrt{3}-\sqrt{2}.$$

$$\tan^{-1}\frac{1}{\sqrt{2}} - \tan^{-1}\left(\frac{\sqrt{3}-\sqrt{2}}{1+\sqrt{3}\cdot\sqrt{2}}\right) = \cot^{-1}(\sqrt{2}) - \underbrace{(\tan^{-1}\sqrt{3}-\tan^{-1}\sqrt{2})}$$

$$\frac{\pi}{2}-\frac{\pi}{3}$$

$$\frac{\pi}{6} \text{ Ans}$$

** $\tan^{-1}\left(\frac{x-y}{1+xy}\right) = \tan^{-1}(x) - \tan^{-1}(y)$
**

Summation of series

$$\tan^{-1}\left(\frac{x-y}{1+xy}\right) = \tan^{-1}x - \tan^{-1}y$$

1. $\tan^{-1}\dfrac{2}{2+1^2+1^4} + \tan^{-1}\dfrac{4}{2+2^2+2^4} + \tan^{-1}\dfrac{6}{2+3^2+3^4} + \ldots\ldots\ \infty$ terms.

Sol^n

$$T_n = \tan^{-1}\left(\frac{2n}{2+n^2+n^4}\right) \quad ; \quad n = 1, 2, 3, \ldots\ldots$$

$$= \tan^{-1}\left(\frac{2n}{1+(n^4+n^2+1)}\right)$$

$$= \tan^{-1}\left(\frac{2n}{1+\underbrace{(n^2+n+1)}_{x}\underbrace{(n^2-n+1)}_{y}}\right)$$

$$T_n = \tan^{-1}(n^2+n+1) - \tan^{-1}(n^2-n+1)$$

$$T_1 = \cancel{\tan^{-1}(3)} - \tan^{-1}(1) \quad ✓$$

$$T_2 = \cancel{\tan^{-1}(7)} - \cancel{\tan^{-1}(3)}$$

$$\vdots$$

$$T_n = \tan^{-1}(n^2+n+1) - \cancel{\tan^{-1}(n^2-n+1)}$$

$$S_n = \tan^{-1}(\underbrace{n^2+n+1}) - \tan^{-1}(1)$$

as $n \to \infty$

$$S_\infty = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}.$$

2) $S = \lim_{n \to \infty} \sum_{n=1}^{n} \tan^{-1}\left(\frac{4n}{n^4 - 2n^2 + 2}\right)$

$$T_n = \tan^{-1}\left(\frac{4n}{1 + (n^4 - 2n^2 + 1)}\right) = \tan^{-1}\left(\frac{4n}{1 + (n^2-1)^2}\right)$$

$$= \tan^{-1}\left(\frac{(n+1)^2 - (n-1)^2}{1 + (n+1)^2 (n-1)^2}\right)$$

$$T_n = \tan^{-1}(n+1)^2 - \tan^{-1}(n-1)^2.$$

$$T_1 = \cancel{\tan^{-1}(4)} - \tan^{-1} 0$$

$$T_2 = \tan^{-1}(9) - \tan^{-1} 1.$$

$$T_3 = \tan^{-1}(16) - \cancel{\tan^{-1}(4)}.$$

$$\vdots \qquad \vdots \qquad \vdots$$

$$T_{n-1} = \tan^{-1} n^2 - \tan^{-1}(n-2)^2$$

$$T_n = \tan^{-1}(n+1)^2 - \tan^{-1}(n-1)^2$$

$$\lim_{n \to \infty} S_n = \left(\frac{\pi}{2} + \frac{\pi}{2}\right) - \left(0 + \frac{\pi}{4}\right) = \pi - \frac{\pi}{4} = \frac{3\pi}{4}.$$

3) $\operatorname{cosec}^{-1}\sqrt{5} + \operatorname{cosec}^{-1}\sqrt{65} + \operatorname{cosec}^{-1}\sqrt{325} + \operatorname{cosec}^{-1}\sqrt{1025} + \operatorname{cosec}^{-1}\sqrt{2501} + \dots \infty.$

$$\tan^{-1}\left(\frac{1}{2}\right) + \tan^{-1}\left(\frac{1}{8}\right) + \tan^{-1}\left(\frac{1}{18}\right) + \dots\dots \infty.$$

$$T_n = \tan^{-1}\left(\frac{1}{2n^2}\right) = \cancel{\tan^{-1}\left(\frac{1}{1+\underbrace{(2n^2-1)}_{(\sqrt{2}n-1)(\sqrt{2}n+1)}}\right)}$$

$$= \tan^{-1}\left(\frac{2}{1+4n^2-1}\right)$$

$$= \tan^{-1}\left(\frac{2}{1+(2n+1)(2n-1)}\right)$$

$$T_n = \tan^{-1}(2n+1) - \tan^{-1}(2n-1)$$

4) $$\sum_{K=1}^{n} \tan^{-1}\left(\frac{2^{K}}{2+2^{2K}}\right)$$

$$\sum_{K=1}^{n} \tan^{-1}\left(\frac{2^{K-1}}{1+2^{2K-1}}\right) = \sum_{K=1}^{n} \tan^{-1}\left(\frac{2^{K-1}}{1+2^{K}\cdot 2^{K-1}}\right)$$

$$\sum_{K=1}^{n} \tan^{-1}\left(\frac{2^{K}-2^{K-1}}{1+2^{K}\,2^{K-1}}\right) = \sum_{K=1}^{n}\left(\tan^{-1}2^{K}-\tan^{-1}2^{K-1}\right)$$

5) $$\lim_{n\to\infty} \sum_{r=1}^{n} \tan^{-1}\left(\frac{4}{r^2+3}\right)$$

$$\lim_{n\to\infty} \sum_{r=1}^{n} \tan^{-1}\left(\frac{1}{1+\frac{3}{4}+\frac{r^2}{4}-1}\right)$$

$$\lim_{n\to\infty} \sum_{r=1}^{n} \tan^{-1}\left(\frac{1}{1+\frac{r^2}{4}-\frac{1}{4}}\right)$$

$$\lim_{n\to\infty} \sum_{r=1}^{n} \left(\tan^{-1}\left(\frac{r+1}{2}\right) - \tan^{-1}\left(\frac{r-1}{2}\right)\right)$$

$$S_\infty = \pi - \tan^{-1}\left(\frac{1}{2}\right)$$ Ans

P-6 (a) $\underbrace{\sin^{-1}x}_{\alpha} + \underbrace{\sin^{-1}y}_{\beta} = \begin{cases} \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right) \text{ if } x \ge 0; y \ge 0 \text{ and } x^2 + y^2 \le 1 \\ \pi - \sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right) \text{ if } x \ge 0; y \ge 0 \text{ and } x^2 + y^2 > 1 \end{cases}$ ✓ ✓

(b) $\sin^{-1}x - \sin^{-1}y = \sin^{-1}\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right)$, $x > 0$; $y > 0$ ✓

(c) $\cos^{-1}x + \cos^{-1}y = \cos^{-1}[xy - \sqrt{1-x^2}\sqrt{1-y^2}]$ where $x > 0, y > 0$ ✓

(d) $\underset{\alpha}{\cos^{-1}x} - \underset{\beta}{\cos^{-1}y} = \begin{cases} \cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right) & ; \quad x < y, \quad x, y > 0 \\ -\cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right) & ; \quad x > y, \quad x, y > 0 \end{cases}$ ✓ ✓

Q If $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z = \pi$ then find the value of $x^2 + y^2 + z^2 + 2xyz$?

Sol^n $\cos^{-1}x + \cos^{-1}y = \pi - \cos^{-1}z = \cos^{-1}(-z)$

$\cos^{-1}\left(xy - \sqrt{1-x^2}\sqrt{1-y^2}\right) = \cos^{-1}(-z)$.

$xy - \sqrt{1-x^2}\sqrt{1-y^2} = -z$

$(xy+z)^2 = (1-x^2)(1-y^2)$

$x^2 + y^2 + z^2 + 2xyz = 1$ Ans

Equations involving ITF :-

1) $\cos^{-1}x - \sin^{-1}x = \cos^{-1}x\sqrt{3}$

$$\cancel{\frac{\pi}{2}} - \sin^{-1}x - \sin^{-1}x = \cancel{\frac{\pi}{2}} - \sin^{-1}(x\sqrt{3})$$

$$2\underbrace{\sin^{-1}x}_{\alpha} = \underbrace{\sin^{-1}(\sqrt{3}x)}_{\beta} \Rightarrow \boxed{2\alpha = \beta}$$

where $\sin\alpha = x$
$\sin\beta = \sqrt{3}x$.

$$\sin 2\alpha = \sin\beta.$$

$$2\sin\alpha\cos\alpha = \sin\beta \Rightarrow 2x\sqrt{1-x^2} = \sqrt{3}x$$

$$x\left(\underbrace{2\sqrt{1-x^2} - \sqrt{3}}\right) = 0 \Rightarrow \boxed{x = 0}$$ check !!!

$$4 - 4x^2 = 3$$

$x = \frac{1}{2}$ or $x = -\frac{1}{2}$

Check !!!

$$x \in \left\{-\frac{1}{2}, 0, \frac{1}{2}\right\}$$

Ans

2) $\underbrace{\tan^{-1}\frac{x+1}{x-1}+\tan^{-1}\frac{x-1}{x}}=\underbrace{\tan^{-1}(-7)}$

$$\tan^{-1}\left(\frac{\frac{x+1}{x-1}+\frac{x-1}{x}}{1-\frac{(x+1)(x-1)}{(x-1)(x)}}\right)=\tan^{-1}(-7).$$

$$\tan^{-1}\left(\frac{x^2+x+x^2-2x+1}{(x^2-x)-(x^2-1)}\right)=\tan^{-1}(-7)$$

$$\tan^{-1}\left(\frac{2x^2-x+1}{1-x}\right)=\tan^{-1}(-7)$$

$$2x^2-x+1=7x-7.$$

$$2x^2-8x+8=0.$$

$$x^2-4x+4=0 \Rightarrow x=2$$

Check

Rejected

$\therefore x\in\phi$ Ans

3) $\sin[2\cos^{-1}\{\cot(2\tan^{-1}x)\}] = 0$

$\underbrace{2\cos^{-1}\{\cot(2\tan^{-1}x)\}}_{\theta}$

$\sin\theta = 0 \Rightarrow \theta = n\pi \; ; \; n \in I$

$2\cos^{-1}(\cot(2\tan^{-1}x)) = n\pi$

$\underbrace{\cos^{-1}(\cot(2\tan^{-1}x))}_{[0,\pi]} = \frac{n\pi}{2} \; ; \; n \in I$ $\quad \boxed{n = 0; 1; 2}$

$0, \frac{\pi}{2}, \pi$

$\cos^{-1}\cot(2\tan^{-1}x) = 0 \; ; \; \frac{\pi}{2} \; ; \; \pi.$

$\cot(\underbrace{2\tan^{-1}x}_{\alpha}) = 1; \; 0; \; -1.$

$\cot\alpha \rightarrow -1, \; 0, \; 1 \Rightarrow \alpha \rightarrow n\pi + \frac{3\pi}{4} \; ; \; n \in I, \quad n\pi + \frac{\pi}{2} \; ; \; n \in I, \quad n\pi + \frac{\pi}{4} \; ; \; n \in I.$

$\underbrace{\tan^{-1}x}_{\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)} \rightarrow$

$\frac{n\pi}{2} + \frac{3\pi}{8} \; ; \; n = -1, 0.$ ✓

$\frac{n\pi}{2} + \frac{\pi}{4} \; ; \; n = -1; 0.$ ✓

$\frac{n\pi}{2} + \frac{\pi}{8} \; ; \; n = -1; 0$ ✓

$x = -(\sqrt{2}-1);\ \sqrt{2}+1;\ -1;\ 1;\ \sqrt{2}-1;$

$-(\sqrt{2}+1)$

Ans

Q

Solve the equation:

$$\cos^{-1}\left(\sqrt{6}x\right)+\cos^{-1}\left(3\sqrt{3}x^2\right)=\frac{\pi}{2}$$

Sol^n

$$\cos^{-1}\left(\underset{\substack{\downarrow\\ \sqrt{2}\cdot\sqrt{3}}}{\sqrt{6}x}\cdot 3\sqrt{3}x^2-\sqrt{1-6x^2}\sqrt{1-27x^4}\right)=\frac{\pi}{2}.$$

$$9\sqrt{2}x^3-\sqrt{1-6x^2}\sqrt{1-27x^4}=0.$$

$$\left(9\sqrt{2}x^3\right)^2=(1-6x^2)(1-27x^4)$$

$$\cancel{162x^6}=1-27x^4-6x^2+\cancel{162x^6}$$

$$27x^4+6x^2-1=0.$$

$$27x^4+9x^2-3x^2-1=0$$

$$(9x^2-1)(3x^2+1)=0.$$

$$x^2=\frac{1}{9}\Rightarrow x=\frac{1}{3} \checkmark \text{ or } x=-\frac{1}{3} \text{ XX}$$

Check !!!

Q The product of all real values of x satisfying the equation

$$\sin^{-1}\cos\left(\frac{2x^2+10|x|+4}{x^2+5|x|+3}\right)=\cot\left(\underbrace{\cot^{-1}\left(\frac{2-18|x|}{9|x|}\right)}\right)+\frac{\pi}{2} \text{ is}$$

$$\cancel{\frac{\pi}{2}} - \cos^{-1}\cos\left(\frac{2x^2+10|x|+4}{\underbrace{x^2+5|x|+3}}\right) = \frac{2-18|x|}{9|x|} + \cancel{\frac{\pi}{2}}.$$

$$-\cos^{-1}\cos\left(\frac{2(x^2+5|x|+3)}{x^2+5|x|+3} - \frac{2}{x^2+5|x|+3}\right) = \frac{2-18|x|}{9|x|}$$

$$-\cos^{-1}\cos\left(2 - \frac{2}{x^2+5|x|+3}\right) = \frac{2-18|x|}{9|x|}$$

$$-\cancel{2} + \frac{2}{x^2+5|x|+3} = \frac{2}{9|x|} - \cancel{2}.$$

$$9|x| = x^2+5|x|+3$$

$$x^2-4|x|+3=0$$

$$|x|^2-4|x|+3=0$$

$$(|x|-1)(|x|-3)=0.$$

$$\therefore \boxed{x = \pm 1; \quad x = \pm 3} \text{ Ans}$$

Q Complete solution set of equation $[\cos^{-1}x] + 2[\sin^{-1}x] = 0$ is (where [.] is greatest integer function)

(A) $(\cos 1, \sin 1)$ (B) $[-\sin 1, 0) \cup (\cos 1, \sin 1)$

(C) $[0,1]$ ✓(D) $[-\sin 1, \cos 2] \cup (\cos 1, \sin 1)$

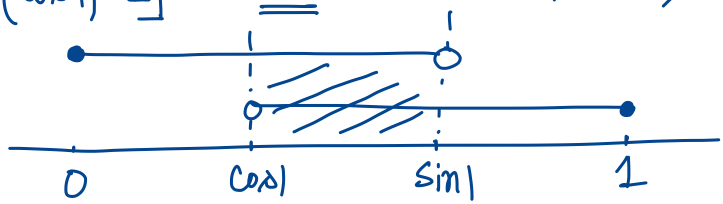
Sol^n $\cos^{-1}x \in [0, \pi]$ $\sin^{-1}x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$[\cos^{-1}x] = 0, 1, 2, 3.$ $[\sin^{-1}x] = -1, 0, 1.$

C–I $[\cos^{-1}x] = 0$ and $[\sin^{-1}x] = 0.$

$0 \le \cos^{-1}x < 1$ and $0 \le \sin^{-1}x < 1.$

$x \in (\cos 1, 1]$ and $x \in [0, \sin 1)$



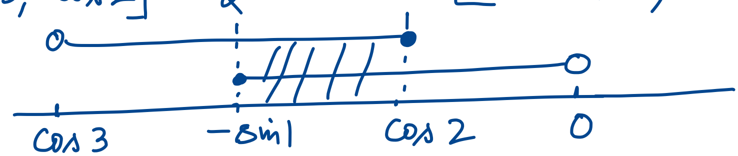
$x \in (\cos 1, \sin 1)$ — ① — ✓

C–II $[\cos^{-1}x] = 2$ & $[\sin^{-1}x] = -1.$

$\downarrow$

$2 \le \cos^{-1}x < 3$ & $-1 \le \sin^{-1}x < 0$

$x \in (\cos 3, \cos 2]$ & $x \in [-\sin 1, 0)$



$x \in [-\sin 1, \cos 2]$ — ②

Finally ① ∪ ②.

Q Solve inequality :- ① $[\sin^{-1}x] > [\cos^{-1}x]$

$[\ ] \to$ Gif

② $(\sec^{-1}x)^2 - 6\sec^{-1}x + 8 > 0.$

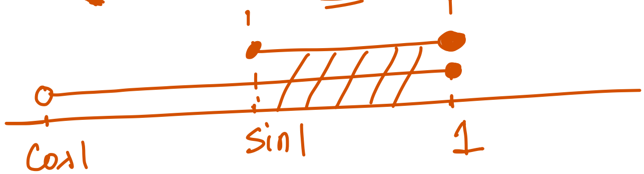
① $[\sin^{-1}x] = -1, 0, 1.$

$[\cos^{-1}x] = 0, 1, 2, 3.$

$[\sin^{-1}x] = 1$ and $[\cos^{-1}x] = 0.$

$1 \le \sin^{-1}x \le \frac{\pi}{2}$ and $\underbrace{0 \le \cos^{-1}x < 1.}$

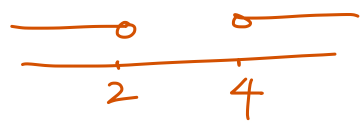
$\sin 1 \le x \le 1$ and $\cos 1 < x \le 1.$



$x \in [\sin 1, 1]$ Ans

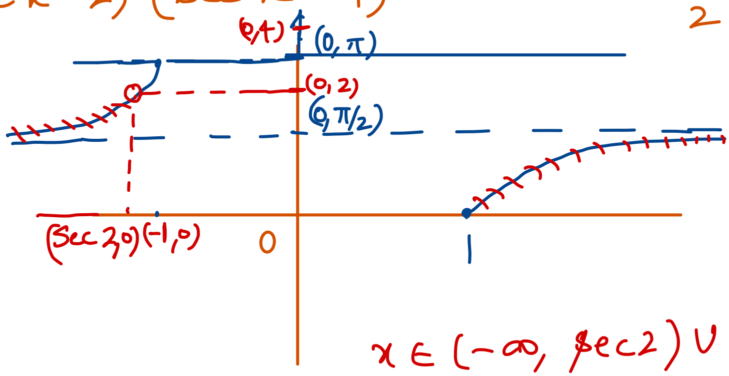
② $(\sec^{-1}x)^2 - 6\sec^{-1}x + 8 > 0$

$(\sec^{-1}x - 2)(\sec^{-1}x - 4) > 0.$



$\sec^{-1}x = 2$

$x = \sec 2$



$y = \sec^{-1}x$

$x \in (-\infty, \sec 2) \cup [1, \infty)$

Q If $\sin^2x + \sin^2y < 1$ for all $x, y \in R$ then prove that $\sin^{-1}(\underbrace{\tan x \,.\, \tan y}_{\in})\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$.

$(-1, 1)$

$$2\sin^2 x + 2\sin^2 y < 2$$

$$\cancel{1} - \cos 2x + \cancel{1} - \cos 2y < \cancel{2}$$

$$-\left(\frac{1-\tan^2 x}{\underbrace{1+\tan^2 x}}\right) - \left(\frac{1-\tan^2 y}{\underbrace{1+\tan^2 y}}\right) < 0$$

$$\tan^2 x \tan^2 y - 1 < 0.$$

$$\tan x \tan y \in (-1, 1)$$

(H.P)

Q Let $f : R \to R$ defined by $f(x) = \cos^{-1}(-\{-x\})$, where $\{x\}$ is fractional part function, then which of the following is/are true ?

(A) $f(x)$ is many one function

(B) period of $f(x)$ is 1

(C) graph of $f(x)$ lies above x-axis

(D) $f(x)$ is neither even nor odd function

ABCD

Q Let $h(x)=\tan\left(\frac{\cos^{-1}\left(\sin\left(\cos^{-1}x\right)\right)+\sin^{-1}\left(\cos\left(\sin^{-1}x\right)\right)}{2}\right)$, then choose correct option(s)

•(A) $\tan^{-1}\left(h\left(\frac{x}{2}\right)\right)=\frac{\pi}{4}\ \forall\ x\in[-2,2]$

•(B) $\sum_{x=1}^{4}h\left(\frac{x}{4}\right)=4$

•(C) $\sin^{-1}\left(h(2x)\right)=\frac{\pi}{2}\ \forall\ x\in\left[-\frac{1}{2},\frac{1}{2}\right]$

(D) $\sin^{-1}\left(h\left(\frac{x}{4}\right)\right)+\cos^{-1}\left(h\left(\frac{x}{3}\right)\right)=\frac{\pi}{4}\ \forall\ x\in[-3,3]$

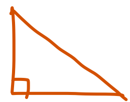
Sol^n

$$M=\frac{\cos^{-1}\sin(\underbrace{\cos^{-1}x}_{\theta})+\sin^{-1}\cos(\underbrace{\sin^{-1}x}_{\phi})}{2}$$

$\cos\theta=x$

$\sin\phi=x$

$$M=\frac{\cos^{-1}\sqrt{1-x^2}+\sin^{-1}\sqrt{1-x^2}}{2}=\frac{\pi/2}{2}$$



$$M=\pi/4$$

$$h(x)=\tan\frac{\pi}{4}=1.$$

Q Find the number of solution of the equation $\tan^{-1}x^3 + \cot^{-1}(e^x) = \frac{\pi}{2}$.

Sol^n $x^3 = e^x$ $\qquad f(x) = x^3 e^{-x}$.

$x^3 e^{-x} = 1.$

Q Point P(x, y) satisfying the equation $\sin^{-1}x + \cos^{-1}y + \cos^{-1}(2xy) = \frac{\pi}{2}$ lies on

(A) the bisector of the first and third quadrant $\longrightarrow$ $y = x$

(B) bisector of the second and fourth quadrant. $\longrightarrow$ $y = -x.$

(C) the rectangle formed by the lines $x = \pm 1$ and $y = \pm 1$.

✓(D) a unit circle with centre at the origin.

Sol^n

$$\cos^{-1}y + \cos^{-1}(2xy) = \cos^{-1}x$$

$$\cos^{-1}\left(y(2xy) - \sqrt{1-y^2}\sqrt{1-4x^2y^2}\right) = \cos^{-1}x$$

$$2xy^2 - \sqrt{}\sqrt{} = x.$$

$$(2xy^2 - x)^2 = 1 - 4x^2y^2 - y^2 + 4x^2y^4.$$

$$\cancel{4x^2y^4} + x^2 - \cancel{4x^2y^2} = 1 - \cancel{4x^2y^2} - y^2 + \cancel{4x^2y^4}$$

$$\boxed{x^2 + y^2 = 1}$$

Q Consider $f(x) = \tan^{-1}\left(\frac{(\sqrt{12}-2)x^2}{x^4+2x^2+3}\right)$ and m and M are respectively minimum and maximum values of f(x) and x = a (a > 0) is the point in the domain of f(x) where f(x) attains its maximum value.

1. If $\sin^{-1}2\sqrt{x} = 3\tan^{-1}(\tan(m+M))$ then 8x equals 1
2. The value of $\tan\left(\sec^{-1}\left(\frac{2}{a^2}\right)+M\right)$ equals 1

① $\sin^{-1}2\sqrt{x} = \cancel{3}\cdot\frac{\pi}{\cancel{12}\,4}$

$2\sqrt{x} = \frac{1}{\sqrt{2}} \Rightarrow 2\sqrt{2} = \frac{1}{\sqrt{x}}$

$\boxed{8x = 1}$ Ans

Sol^n $E = (\sqrt{12}-2)\left(\frac{x^2}{x^4+2x^2+3}\right) = \frac{(\sqrt{12}-2)}{\left(x^2+2+\frac{3}{x^2}\right)}$

$$\boxed{E_{min} = 0 = m}$$

$$E_{max} = \frac{2\sqrt{3}-2}{2\sqrt{3}+2} = \frac{\sqrt{3}-1}{\sqrt{3}+1}$$

$$\frac{x^2+\frac{3}{x^2}}{2} \geq \sqrt{3}.$$

$$x^2+\frac{3}{x^2} \geq 2\sqrt{3}$$

$$x^2 = \frac{3}{x^2} \Rightarrow x^4 = 3$$

$$M = \tan^{-1}(2-\sqrt{3})$$

$$\boxed{M = \frac{\pi}{12}}$$

$$a^4 = 3.$$

$$a^2 = \sqrt{3}$$

Q If the equation $\sin^{-1}\sqrt{x}+\cos^{-1}\sqrt{x^2-1}+\tan^{-1}(\tan y)=a$ has at least one solution then the possible integral value which 'a' can take is

(A) 1 ✓(B) 2 ✓(C) 3 ✓(D) 4

Solⁿ $0 \le x \le 1$ — ① —

$0 \le x^2-1 \le 1$ — ② — $\Big\} \cap$ $x=1$ only

$$\sin^{-1}1+\cos^{-1}0+\tan^{-1}\tan y = a.$$

$$\pi+\underbrace{\tan^{-1}\tan y}_{\left(-\frac{\pi}{2},\frac{\pi}{2}\right)} = a.$$

LHS: $\left(\frac{\pi}{2},\frac{3\pi}{2}\right)$

Q Determine the integral value of K for which the system $(\text{arc } \tan x)^2 + (\text{arc } \cos y)^2 = \pi^2 K$ and $\tan^{-1}x + \cos^{-1}y = \dfrac{\pi}{2}$

Sol^n

Let $\tan^{-1}x = a$ & $\cos^{-1}y = b$.

$a \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ $\qquad b \in [0, \pi]$

$$\underbrace{a^2 + b^2}_{} = \pi^2 K \quad \text{—}①\text{—} \quad \begin{cases} a^2 \in \left[0, \frac{\pi^2}{4}\right) \\ b^2 \in [0, \pi^2] \end{cases}$$

$$0 \le \pi^2 K < \frac{5\pi^2}{4} \qquad a^2 + b^2 \in \left[0, \frac{5\pi^2}{4}\right)$$

$$* \quad \boxed{0 \le K < \frac{5}{4}} \quad \text{—}②\text{—}$$

$$a + b = \frac{\pi}{2} \Rightarrow a^2 + b^2 + 2ab = \frac{\pi^2}{4}$$

$$2ab = \frac{\pi^2}{4} - \pi^2 K.$$

$$\boxed{ab = \frac{\pi^2}{8} - \frac{\pi^2 K}{2}} \quad \text{—}③\text{—}$$

$$x^2 - \frac{\pi}{2}x + \frac{\pi^2}{8} - \frac{\pi^2 K}{2} = 0. \quad \begin{matrix} a \\ b \end{matrix}$$

$$a, b = \frac{\frac{\pi}{2} \pm \sqrt{\frac{\pi^2}{4} - 4\left(\frac{\pi^2}{8} - \frac{\pi^2 K}{2}\right)}}{2}$$

$$a, b = \frac{\frac{\pi}{2} \pm \sqrt{2\pi^2 K - \frac{\pi^2}{4}}}{2}$$

$$2\cancel{\pi^2}K - \frac{\cancel{\pi^2}}{4} \geq 0 \Rightarrow \boxed{K \geq \frac{1}{8}} \quad (4)$$

$$(2) \cap (4) \quad \boxed{K_{int} = 1}$$

$$a, b = \frac{\frac{\pi}{2} \pm \sqrt{\frac{7\pi^2}{4}}}{2} \qquad \cos^{-1} y = \frac{\frac{\pi}{2} + \frac{\sqrt{7}\pi}{2}}{2}$$

Basic Substitutions :-

(1) $\sqrt{a^2 - x^2}$; $x = a\cos\theta / a\sin\theta$

(2) $\sqrt{a^2 + x^2}$; $x = a\tan\theta / a\cot\theta$

(3) $\sqrt{x^2 - a^2}$; $x = a\sec\theta / a\,\mathrm{cosec}\,\theta$

(4) $\sqrt{\frac{a+x}{a-x}}$; $x = a\cos 2\theta$

$\sqrt{\frac{a-x}{a+x}}$

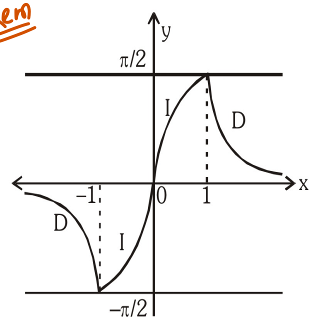
SIMPLIFICATION & TRANSFORMATION OF INVERSE FUNCTIONS BY ELEMENTARY SUBSTITUTION AND THEIR GRAPHS :

(1) $y = f(x) = \sin^{-1}\left(\frac{2x}{1+x^2}\right) = \begin{cases} -(\pi + 2\tan^{-1}x) & \text{if } x < -1 \\ 2\tan^{-1}x & \text{if } |x| \le 1 \\ \pi - 2\tan^{-1}x & \text{if } x > 1 \end{cases}$

(where, I = increasing and D = Decreasing)

Domain : R, **Range :** $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

Rem



Proof: $x = \tan\theta$; $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$; $2\theta \in (-\pi, \pi)$

$\theta = \tan^{-1}x$

$f(x) = \sin^{-1}\left(\frac{2\tan\theta}{1+\tan^2\theta}\right) = \sin^{-1}\sin(2\theta)$

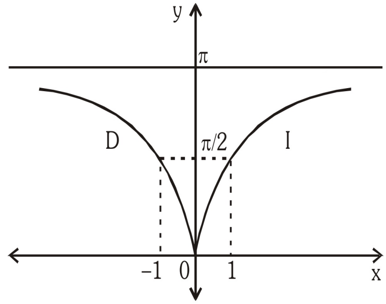
$\left(-\pi, -\frac{\pi}{2}\right)$ $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ $\left(\frac{\pi}{2}, \pi\right)$

$\sin^{-1}\sin(2\theta) \begin{cases} -\pi - 2\theta & ; 2\theta \in \left(-\pi, -\frac{\pi}{2}\right) \Rightarrow \theta \in \left(-\frac{\pi}{2}, -\frac{\pi}{4}\right) \\ 2\theta & ; 2\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right] \Rightarrow \theta \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right] \\ \pi - 2\theta & ; 2\theta \in \left(\frac{\pi}{2}, \pi\right) \end{cases}$

$f(x) \begin{cases} -\pi - 2\tan^{-1}x & ; -\infty < x < -1 \\ 2\tan^{-1}x & ; -1 \le x \le 1 \\ \pi - 2\tan^{-1}x & ; 1 < x < \infty. \end{cases}$

② ✓

$$y = f(x) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \begin{cases} -2\tan^{-1}x & \text{if} \quad x < 0 \\ 2\tan^{-1}x & \text{if} \quad x \geq 0 \end{cases}$$

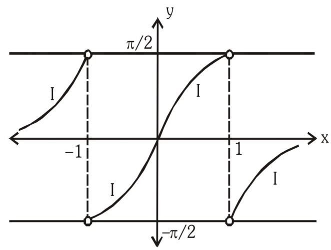


Domain : R , **Range :** $[0, \pi)$

$x = \tan\theta$; $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

3

$$y = f(x) = \tan^{-1}\frac{2x}{1-x^2} = \begin{cases} \pi + 2\tan^{-1}x & \text{if} \quad x < -1 \\ 2\tan^{-1}x & \text{if} \quad |x| < 1 \\ -(\pi - 2\tan^{-1}x) & \text{if} \quad x > 1 \end{cases}$$

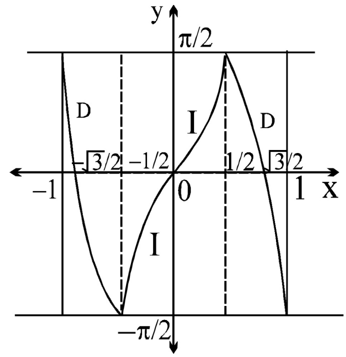


Domain : $R - \{-1, 1\}$, **Range :** $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

$x = \tan\theta$; $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$

4

$$\sin^{-1}(3x-4x^3)=\begin{cases}-(\pi+3\sin^{-1}x) & \text{if } -1\le x\le -1/2\\ 3\sin^{-1}x & \text{if } -1/2\le x\le 1/2\\ \pi-3\sin^{-1}x & \text{if } 1/2\le x\le 1\end{cases};$$



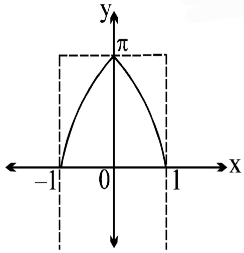
$x=\sin\theta$; $\theta\in\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$

$3\theta\in\left[-\frac{3\pi}{2},\frac{3\pi}{2}\right]$

$\sin^{-1}\sin 3\theta$

$\left[-\frac{3\pi}{2},-\frac{\pi}{2}\right)$ $\left[-\frac{\pi}{2},\frac{\pi}{2}\right]$ $\left(\frac{\pi}{2},\frac{3\pi}{2}\right]$

$$\cos^{-1}(2x^2-1)=\begin{cases} 2\cos^{-1}x & \text{if } x\in[0,1] \\ 2\pi-2\cos^{-1}x & \text{if } x\in[-1,0] \end{cases}$$



$\boxed{x=\cos\theta}$; $\theta\in[0,\pi]$

$2\theta\in[0,2\pi]$

$[0,\pi]$ $\quad$ $[\pi,2\pi]$

$\cos^{-1}\cos 2\theta$

$$\cos^{-1}\cos 2\theta \begin{cases} \to 2\theta ; & 2\theta\in[0,\pi]\Rightarrow\theta\in\left[0,\frac{\pi}{2}\right] \\ \to 2\pi-2\theta ; & 2\theta\in(\pi,2\pi]\Rightarrow\theta\in\left(\frac{\pi}{2},\pi\right] \end{cases}$$

E(1) If $f(x)=\sin^{-1}\left(\dfrac{2x}{1+x^2}\right)+2\tan^{-1}x$, then find $\dfrac{f(5)+f(10)+f(3)}{f(-7)+f(-10.5)+f(-4.5)}$ $= -1$

$-\pi-2\tan^{-1}x$ $\quad$ $x<-1$

$2\tan^{-1}x$ $\quad$ $x\in[-1,1]$

$\pi-2\tan^{-1}x$ $\quad$ $x>1$

$$\frac{\pi+\pi+\pi}{-\pi-\pi-\pi}=-1$$

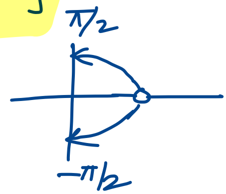
E(2) Find solution of $\tan^{-1}\dfrac{\sqrt{1+x^2}-1}{x} = \dfrac{1}{2}\tan^{-1}x$

*

Sol^n

put * $\boxed{x = \tan\theta}$; * $\theta \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) - \{0\}$

*



LHS: $\tan^{-1}\left(\dfrac{|\sec\theta| - 1}{\tan\theta}\right) = \tan^{-1}\left(\dfrac{1-\cos\theta}{\sin\theta}\right)$

$= \tan^{-1}\left(\dfrac{\cancel{2}\sin^2\theta/2}{\cancel{2}\cancel{\sin\frac{\theta}{2}}\cos\frac{\theta}{2}}\right) = \tan^{-1}\tan\left(\dfrac{\theta}{2}\right) = \dfrac{\theta}{2}$

$\dfrac{\theta}{2} \in \left(-\frac{\pi}{4}, \frac{\pi}{4}\right) - \{0\}$

$LHS = \frac{1}{2}\tan^{-1}x$

$RHS = \frac{1}{2}\tan^{-1}x.$

$\therefore \quad x \in R - \{0\}$ Ans

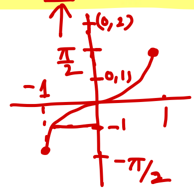
Q Complete solution set of equation $[\cos^{-1}x] + 2[\sin^{-1}x] = 0$ is (where [.] is greatest integer function)

(A) $(\cos 1, \sin 1)$ (B) $[-\sin 1, 0) \cup (\cos 1, \sin 1)$

(C) $[0,1]$ ✓(D) $[-\sin 1, \cos 2] \cup (\cos 1, \sin 1)$

Sol^n $\cos^{-1}x \in [0, \pi]$ $\sin^{-1}x \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

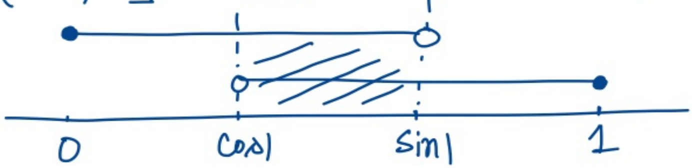
$[\cos^{-1}x] = 0, 1, 2, 3.$ $[\sin^{-1}x] = \cancel{-2}, -1, 0, 1.$



C-I $[\cos^{-1}x] = 0$ and $[\sin^{-1}x] = 0.$

$0 \le \cos^{-1}x < 1$ and $0 \le \sin^{-1}x < 1.$

$x \in (\cos 1, 1]$ and $x \in [0, \sin 1)$



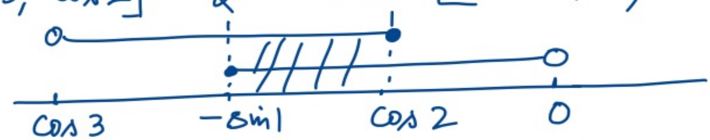
$x \in (\cos 1, \sin 1)$ — ① — ✓

C-II $[\cos^{-1}x] = 2$ & $[\sin^{-1}x] = -1.$

$\downarrow$

$2 \le \cos^{-1}x < 3$ & $-1 \le \sin^{-1}x < 0$

$x \in (\cos 3, \cos 2]$ & $x \in [-\sin 1, 0)$



$x \in [-\sin 1, \cos 2]$ — ②

Finally ① ∪ ②.

Q Solve inequality :- (1) $[\sin^{-1}x] > [\cos^{-1}x]$

$[\ ] \rightarrow$ Gif

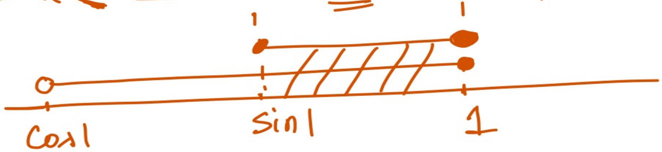
(1) $[\sin^{-1}x] = -2, -1, 0, 1.$

$[\cos^{-1}x] = 0, 1, 2, 3.$

$[\sin^{-1}x] = 1$ and $[\cos^{-1}x] = 0.$

$1 \le \sin^{-1}x \le \frac{\pi}{2}$ and $\underbrace{0 \le \cos^{-1}x < 1.}$

$\sin 1 \le x \le 1$ and $\cos 1 < x \le 1.$



$x \in [\sin 1, 1]$ Ans

Find solution of the following equation

(a) $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = 2\sin^{-1}x$

$1-x^2 \geq 0$

$x^2-1 \leq 0$

$x \in [-1, 1]$

$\boxed{x = \cos\theta}$

(b) $\cos^{-1}\left(2x^2-1\right) = 2\cos^{-1}x$

(a) put $x = \sin\theta$; $\theta \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ * $\Rightarrow$ $2\theta \in [-\pi, \pi]$

$\left[-\pi, -\frac{\pi}{2}\right)$; $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$; $\left(\frac{\pi}{2}, \pi\right]$.

LHS: $\sin^{-1}\left(2\sin\theta\,|\cos\theta|\right) = \sin^{-1}\sin 2\theta$

C-I $\quad \theta \in \left[-\frac{\pi}{2}, -\frac{\pi}{4}\right) \Rightarrow \sin\theta \in \left[-1, -\frac{1}{\sqrt{2}}\right)$

$\sin^{-1}\sin 2\theta = 2\sin^{-1}x$

$-\pi - 2\theta = 2\sin^{-1}x$

$-\pi = 4\sin^{-1}x \Rightarrow x = -\frac{1}{\sqrt{2}}$ (rejected) — (1) —

C-II $\quad$ * $\theta \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right] \Rightarrow \sin\theta \in \left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$

$\sin^{-1}\sin 2\theta = 2\sin^{-1}x$

$2\theta = 2\sin^{-1}x \Rightarrow 2\sin^{-1}x = 2\sin^{-1}x$

$x \in \left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$. — (2) —

C-III $\quad \theta \in \left(\frac{\pi}{4}, \frac{\pi}{2}\right]$

$\sin^{-1}\sin 2\theta = 2\sin^{-1}x$

$\pi - 2\theta = 2\sin^{-1}x \Rightarrow x = \frac{1}{\sqrt{2}}$ (rejected)

$\therefore$ finally $x \in \left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$ Ans

Q * $\sin^{-1}\left(\sqrt{1-x^2}\right) + \tan^{-1}\left(\frac{\sqrt{1-x^2}}{x}\right) = 0.$

$x = \cos\theta$; $\theta \in [0, \pi] - \{\pi/2\}$ **

$[0, \pi/2)$; $(\frac{\pi}{2}, \pi]$

$\sin^{-1}\sin\theta + \tan^{-1}\tan\theta = 0.$

C-I $\quad \theta \in [0, \pi/2)$

$\theta + \theta = 0 \Rightarrow \boxed{\theta = 0} \Rightarrow x = \cos\theta$

$\boxed{x = 1}$ —①—

C-II $\quad \theta \in (\frac{\pi}{2}, \pi]$ * $\longrightarrow x = \cos\theta$

$x \in [-1, 0)$

$(\pi - \theta) + (\theta - \pi) = 0$

$\boxed{0 = 0}$

$\therefore x \in [-1, 0)$ —②—

finally ; $x \in [-1, 0) \cup \{1\}$ Ans

Q. Find the number of solution of the equation $\tan^{-1}x^3 + \cot^{-1}(e^x) = \frac{\pi}{2}$.

Solⁿ

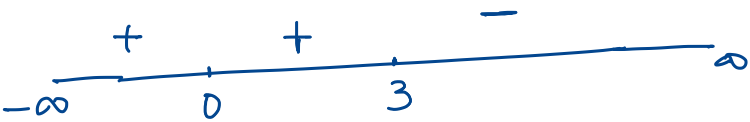
$$x^3 = e^x \Rightarrow \underbrace{x^3 e^{-x}}_{f(x)} = \underbrace{1}_{g(x)}.$$

$g(x) = 1$

$$\boxed{f(x) = x^3 e^{-x}}$$

$$f'(x) = x^3(-e^{-x}) + 3x^2 e^{-x}$$

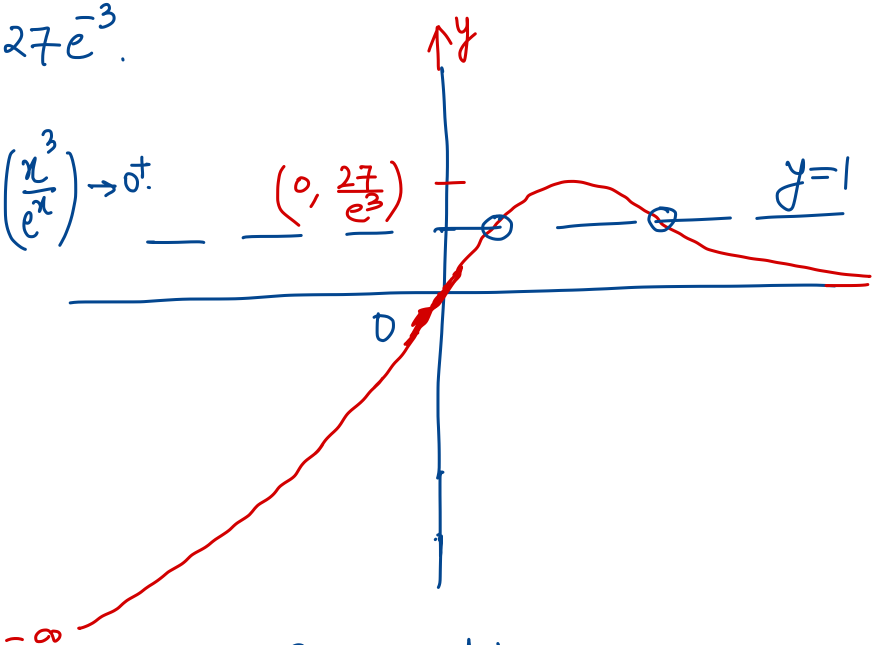
$$= e^{-x}(-x^3 + 3x^2) = -x^2 \underbrace{e^{-x}}_{>0}(x-3)$$



$$f(3) = 27e^{-3}.$$

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}\left(\frac{x^3}{e^x}\right) \to 0^+.$$

$$\lim_{x\to(-\infty)} f(x) = (-\infty)$$



$\therefore$ 2 solutions

Q A polynomial function $g(x)$ satisfies

$g(x)\cdot g\left(\frac{1}{x}\right) = g(x) + g\left(\frac{1}{x}\right)$ $\forall\ x \in R - \{0\}$ then

$g(x)$ can be

(A) $1 + x^{10}$ (B) $1 - x^{7}$

(C) 2 (D) 0.

Sol^n

$g(x) = 2 \rightarrow d = 0$

$g(x) = 0 \rightarrow$ degree dne.

$g(x) = 1 + x^n$; $n \in W$

$g(x) = 1 - x^n$; $n \in W$

Q A polynomial $P(x)$ satisfies $P(x^2) = x^2 P(x-1) + 3$ for all $x \in R$ then $P(x) = ?$

$x = 0$

$\boxed{P(0) = 3}$

$x = 1$

$P(1) = 1.(3) + 3$

$\boxed{P(1) = 6.}$

Sol^n Let $P(x)$ be poly of degree 'n'

$P(x^2) \longrightarrow$ degree $= 2n$

$P(x-1) \longrightarrow$ degree $= n$

$x^2 P(x-1) \longrightarrow$ degree $= n+2.$

$2n = n + 2. \Rightarrow \boxed{n = 2}$

Let $P(x) = ax^2 + bx + c$

$P(0) = c \Rightarrow \boxed{c = 3}$

⋮ ⋮ ⋮

Q. A polynomial $P(x)$ satisfies $P(x) - P(x-1) = 2x+1$ for all $x \in R$ and $P(0) = 1$ then $P(x) = ?$

Sol^n

$$\underbrace{P(x) - P(x-1)} = 2x+1.$$

$$P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0.$$

$$P(x-1) = a_n (x-1)^n + a_{n-1} (x-1)^{n-1} + \cdots + a_1 (x-1) + a_0.$$

$$n-1 = 1 \Rightarrow \boxed{n=2}$$

$$P(x) = ax^2 + bx + c$$

$\vdots \quad \vdots$

Domain :-

** $f(x) = (g(x))^{h(x)}$

$g(x) \to > 0$

eg: $f(x) = (x)^{\sqrt{x+1}}$

$x > 0$ & $x+1 \geq 0$.

$D_f : \boxed{x > 0}$

Limits

① Limit of what ? FUNCTION ✓

② Why Limit ?

③ How to evaluate limit ?
- → Rationalisation
- → Double - " ".
- → Factorisation
- → Binomial theorem
- ⋮

Concept of limit is the concept of immediate neighbourhood.

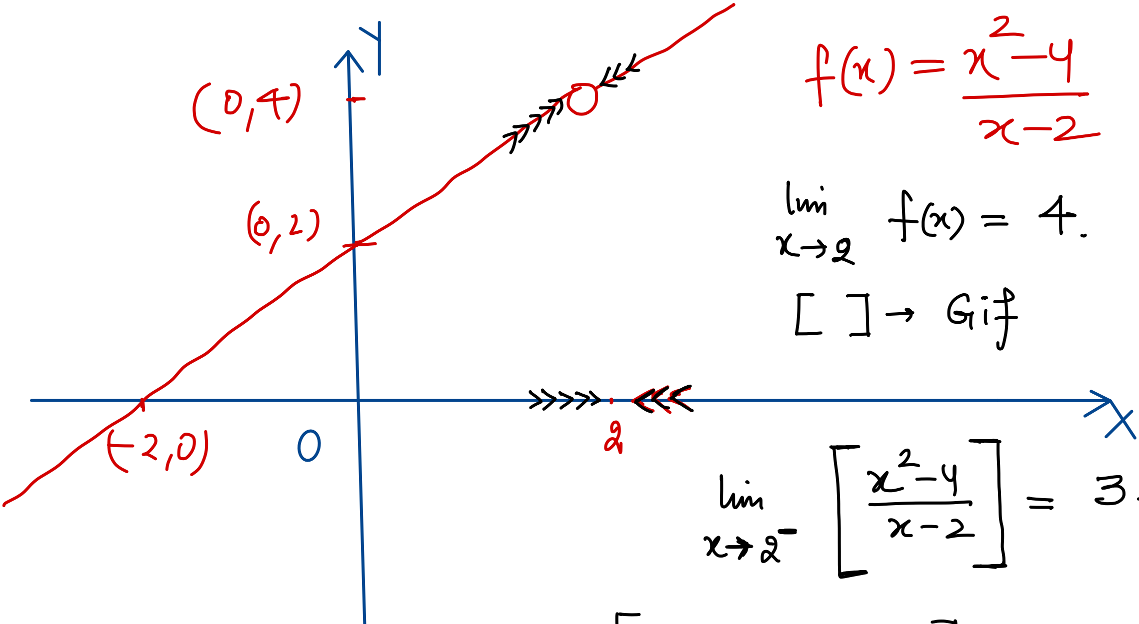
$\lim\limits_{x \to a} f(x)$ is said to exist if LHL = RHL = a finite quantity irrespective of the fact whether $f(x)$ is defined at $x = a$ or not.

eg: $f(x) = \dfrac{x^2 - 4}{x - 2}$; $D_f \in R - \{2\}$

$f(2) = $ dne

$f(x) = \dfrac{\cancel{(x-2)}(x+2)}{\cancel{(x-2)}}$.

x	$f(x)$
2.0001	4.0001
2.000001	4.000001.
2.000000...01	4.00000....01
1.9	3.9
1.999	3.999
1.999...9	3.999.....9



$$f(x) = \frac{x^2-4}{x-2}$$

$$\lim_{x\to 2} f(x) = 4.$$

[ ] → Gif

$$\lim_{x\to 2^-}\left[\frac{x^2-4}{x-2}\right] = 3.$$

$$\left[\underbrace{\lim_{x\to 2^-}\left(\frac{x^2-4}{x-2}\right)}_{o}\right] = 4.$$

LHL = Left hand limit

$$\lim_{x\to a^-} f(x) = \lim_{h\to 0} f(a-h)$$; $h \to$ is sufficiently <u>small</u> positive <u>quantity</u>

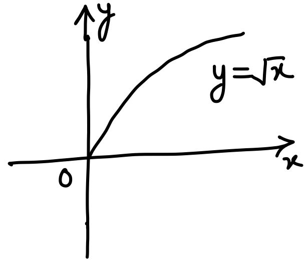
RHL = Right hand limit

$$\lim_{x\to a^+} f(x) = \lim_{h\to 0} f(a+h)$$;

* $\boxed{\lim_{x\to a} f(x) \not\Rightarrow x = a}$

eg: $\lim_{x \to 0} \sqrt{x}$; $f(x) = \sqrt{x}$

$D_f \in [0, \infty)$



$\text{RHL} = \lim_{x \to 0^+} \sqrt{x} = 0 = \underline{\underline{\text{finite}}}.$

Limit = 0.

$\underline{\underline{\text{Concept}}}$ of one-$\underline{\underline{\text{sided}}}$ limit

eg: $\lim_{x \to a} \left(\sqrt{x-a} + \sqrt{a-x}\right)$ = $x = a$ is isolated point

$f(x) = \sqrt{x-a} + \sqrt{a-x}$; $D_f \in \underline{\underline{\{a\}}}$

We should not talk about limit in this case.

eg: $\lim_{x \to 1} [x]$; $[x] \to$ Gif

LHL: put $x = 1-h$

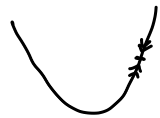
$\lim_{h \to 0} [1-h] = \underline{\underline{0}}.$

RHL: put $x = 1+h$

$\lim_{h \to 0} [1+h] = \underline{\underline{1}}$

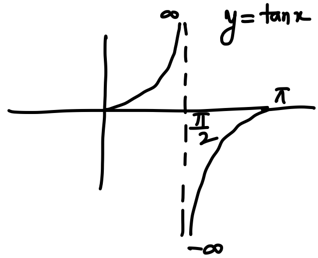
LHL $\neq$ RHL $\therefore$ Limit $\underline{\text{dne}}$

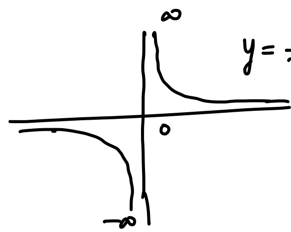
eg: $\lim_{x \to 2} (x^2+2x+5) = 4+4+5$
$= 13.$



eg: $\lim_{x \to \frac{\pi}{2}} \left(e^{\tan x}\right) = \underline{\underline{dne}}$

LHL $\to \infty$; RHL $\to 0$





eg: $\lim_{x \to 0} \tan^{-1}\left(\frac{1}{x}\right) = dne.$

LHL $\to -\pi/2$; RHL $\to \frac{\pi}{2}.$

eg: $\{x\} = x - [x]$; $\{\ \} \to$ fractional part $\underline{\underline{\text{function}}}$

$\cancel{\{\pi\} = \{3.14\}}$

$\{\pi\} = \pi - [\pi]$

$= (\pi - 3)$

There are 7 indeterminant forms :-

$$\underbrace{\frac{0}{0},\ \frac{\infty}{\infty},\ 0\times\infty,\ \infty-\infty,\ 1^{\infty}}_{\text{(tending values)}},\ 0^{0},\ \infty^{0}.$$

Five Fundamental Theorems

Before we learn how evaluate limit of a function, following 5 fundamental theorems should be remembered

If $\underset{x\to c}{\text{Limit}}\, f(x) = L$ and $\underset{x\to c}{\text{Limit}}\, g(x) = M$ where L and M are finite quantities then

(a) **Sum rule :** $\underset{x\to c}{\text{Limit}}\left(f(x) + g(x)\right) = L + M$

(b) **Difference rule :** $\underset{x\to c}{\text{Limit}}\left(f(x) - g(x)\right) = L - M$

(c) **Product rule :** $\underset{x\to c}{\text{Limit}}\left(f(x) \,.\, g(x)\right) = L \,.\, M$

* (d) **Quotient rule :** $\underset{x\to c}{\text{Limit}}\,\dfrac{f(x)}{g(x)} = \dfrac{L}{M}$, $M \neq 0$

(e) **Constant Multiple rule :** $\underset{x\to c}{\text{Limit}}\, K.\, f(x) = K\,L$

i.e. sum, difference, quotient and product rule.

where f(x) & g(x) are defined on the same neighbourhood of 'c'.

NOTE:

1. If $\underset{x\to c}{\text{Lim}}\, f(x)$ and $\underset{x\to c}{\text{Lim}}\, g(x)$ both exists then limit of $f(x) \pm g(x)$; $f(x)\cdot g(x)$ and $\dfrac{f(x)}{g(x)}$ as $x \to c$ must exists. $\left(\text{where } \underset{x\to c}{\text{Lim}}\, g(x) \neq 0\right)$,

2. If $\underset{x\to a}{\text{Lim}}\, f(x)$ exist and $\underset{x\to a}{\text{Lim}}\, g(x)$ does not exist then $\underset{x\to a}{\text{Lim}}\left(f(x) \pm g(x)\right)$ can not exist.

Proof: Let $\underset{x\to a}{\text{Lim}}\, F(x) = \underset{x\to a}{\text{Lim}}\left(f(x) + g(x)\right)$ exist where $F(x) = f(x) + g(x)$

hence $\underset{x\to a}{\text{Lim}}\left(F(x) - f(x)\right)$ must exist (by theorem)

$\Rightarrow$ $\underset{x\to a}{\text{Lim}}\, g(x)$ must exist which is a contradiction of what is given

$\Rightarrow$ $\underset{x\to a}{\text{Limit}}\,(f \pm g)$ does not exist.

* however nothing definite can be said for product or quotient

e.g. $f(x) = x$; $g(x) = [x]$ (where f exist and g does not at $x \in I$)

(i) $\underset{x\to 0}{\text{Lim}}\, x[x]$ exists and is 0 and $\underset{x\to 1}{\text{Lim}}\, x[x]$ does not exist

(ii) $\underset{x\to 0}{\text{Lim}}\,\dfrac{x}{\text{sgn } x}$ exists and is 0 ; $\underset{x\to 2}{\text{Lim}}\,\dfrac{x}{[x]}$ does not exist

3. If both $\underset{x\to a}{\text{Lim}}\, f(x)$ and $\underset{x\to a}{\text{Lim}}\, g(x)$ does not exist then nothing definite can be said about the sum and product., division, difference.

e.g. (i) $f(x) = \dfrac{1}{\sin x}$ and $g(x) = \dfrac{1}{\tan x}$ at $x = 0$

we find that $\underset{x\to 0}{\text{Lim}}\left(\dfrac{1}{\sin x} - \dfrac{1}{\tan x}\right) = \underset{x\to 0}{\text{Lim}}\left(\dfrac{1}{\sin x} - \dfrac{\cos x}{\sin x}\right)$

$$= \underset{x\to 0}{\text{Lim}}\left(\frac{1-\cos x}{\sin x}\right) = \underset{x\to 0}{\text{Lim}}\left(\frac{\sin^2 x}{\sin x(1+\cos x)}\right) = \underset{x\to 0}{\text{Lim}}\left(\frac{\sin x}{1+\cos x}\right) = 0 \quad \text{(exist)}$$

Q $\lim_{x \to \frac{\pi}{4}} \left(\frac{-1+\cot^3 x}{-2+\cot x+\cot^3 x} \right)$ $\left(\frac{0}{0} \text{ form}\right)$ as $x \to \pi/4$, $\cot x \to 1$.

$\cot x = t$

$$\lim_{t \to 1} \left(\frac{t^3-1}{t^3+t-2} \right) = \lim_{t \to 1} \frac{\cancel{(t-1)}(t^2+t+1)}{\cancel{(t-1)}(2+t+t^2)} = \frac{3}{4} \text{ Ans}$$

Q $\lim_{x \to 9} \left(\frac{3-\sqrt{x}}{4-\sqrt{2x-2}} \right)$ $\left(\frac{0}{0} \text{ form}\right)$

$$\lim_{x \to 9} \frac{(3-\sqrt{x})(3+\sqrt{x})}{(3+\sqrt{x})} \cdot \frac{4+\sqrt{2x-2}}{(4-\sqrt{2x-2})(4+\sqrt{2x-2})}$$

$$\lim_{x \to 9} \frac{(9-x)}{(3+\sqrt{x})} \frac{(4+\sqrt{2x-2})}{(\underbrace{16-2x+2}_{})} = \lim_{x \to 9} \frac{4+\sqrt{2x-2}}{2(3+\sqrt{x})}$$

$$= \frac{\cancel{8}\,4}{\cancel{2}(6)} = \frac{2}{3} \text{ Ans}$$

Q $\lim_{x \to 2} \left(\frac{2^x+2^{3-x}-6}{\sqrt{2^{-x}}-2^{1-x}} \right)$ $\left(\frac{0}{0} \text{ form}\right)$

$2^x = t$

$$\lim_{t \to 4} \frac{\left(t+\frac{8}{t}-6\right)}{\left(\frac{1}{\sqrt{t}}-\frac{2}{t}\right)} = \lim_{t \to 4} \left(\frac{t^2-6t+8}{(\sqrt{t}-2)} \right)$$

$$\lim_{t \to 4} \frac{(t-2)\cancel{(t-4)}\,(\sqrt{t}+2)}{\cancel{(\sqrt{t}-2)(\sqrt{t}+2)}}$$

$$(2)(4) = 8 \quad \text{Ans}$$

Q $\lim_{x \to 5} \left(\frac{x^2 - 9x + 20}{x - [x]} \right)$; $[\,] \to$ Gif

$$\lim_{x \to 5} \left(\frac{(x-4)(x-5)}{x - [x]} \right) \qquad \left(\frac{0}{0} \text{ form} \right)$$

LHL: $x = 5 - h$

$$\lim_{h \to 0} \left(\frac{\cancel{(5-h-4)}\,(5-h-5)}{\cancel{(5-h-4)}} \right) = 0.$$

RHL: $x = 5 + h$

$$\lim_{h \to 0} \frac{(5+h-4)\,\cancel{(5+h-5)}}{\cancel{(5+h-5)}} = 1.$$

Limit dne.