Q. A variable line has dc's ℓ, m, n and $\ell + \delta\ell$, m + δm, n + δn in two adjacent positions. If δθ be the angle between the lines in these two positions then prove that $(\delta\theta)^2 = (\delta\ell)^2 + (\delta m)^2 + (\delta n)^2$. ↳ very small.

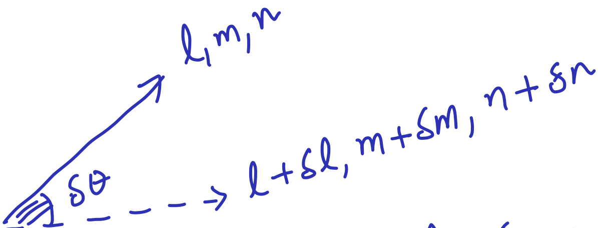
Sol^n

$$\ell^2 + m^2 + n^2 = 1 \quad \text{— ① —}$$

$$(\ell + \delta\ell)^2 + (m + \delta m)^2 + (n + \delta n)^2 = 1. \quad \text{— ② —}$$

$$\cancel{\underbrace{(\ell^2 + m^2 + n^2)}_{1}} + (\delta\ell)^2 + (\delta m)^2 + (\delta n)^2 + 2\ell(\delta\ell) + 2m(\delta m) + 2n(\delta n) = \cancel{1}.$$

$$\boxed{(\delta\ell)^2 + (\delta m)^2 + (\delta n)^2 = -2\ell(\delta\ell) - 2m(\delta m) - 2n(\delta n)} \quad \text{— ③ —}$$



$$(\ell\hat{i} + m\hat{j} + n\hat{k}) \cdot ((\ell + \delta\ell)\hat{i} + (m + \delta m)\hat{j} + (n + \delta n)\hat{k})$$
$$= \sqrt{\ell^2 + m^2 + n^2}\,\sqrt{(\ell + \delta\ell)^2 + (m + \delta m)^2 + (n + \delta n)^2}\,\cos\delta\theta$$

$$\underbrace{(\ell^2 + m^2 + n^2)}_{1} + \Sigma\,\ell(\delta\ell) = \cos\delta\theta$$

$$1 - \cos\delta\theta = -\Sigma\,\ell(\delta\ell)$$
$$1 - \left(1 - 2\sin^2\frac{\delta\theta}{2}\right) = -\Sigma\,\ell(\delta\ell)$$

Since δθ is small so $\sin\delta\theta \simeq \delta\theta$

$$2\cdot\left(\frac{\delta\theta}{2}\right)^2 = \frac{\Sigma(\delta\ell)^2}{2} \Rightarrow (\delta\theta)^2 = \Sigma(\delta\ell)^2$$

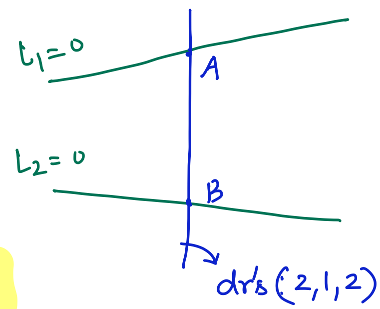
(H.P.)

Q Find the intercept made by lines

$l_1: \quad \vec{r} = \hat{i} + \hat{j} + \hat{k} + \lambda(3\hat{i} - \hat{j})$

$l_2: \quad \vec{r} = 4\hat{i} - \hat{k} + \mu(2\hat{i} + 3\hat{k})$

on a line with dr's $(2, 1, 2)$?



Sol^n General pt on l_1: $A(3\lambda+1,\ 1-\lambda,\ 1)$

" " " l_2: $B(4+2\mu,\ 0,\ 3\mu-1)$

$$\vec{AB} = (3+2\mu-3\lambda)\hat{i} - (1-\lambda)\hat{j} + (3\mu-2)\hat{k}$$

Now,

$$\frac{3+2\mu-3\lambda}{2} = \frac{\lambda-1}{1} = \frac{3\mu-2}{2}$$

$3+2\mu-3\lambda = 2\lambda-2 \Rightarrow 2\mu - 5\lambda = -5$ —①—

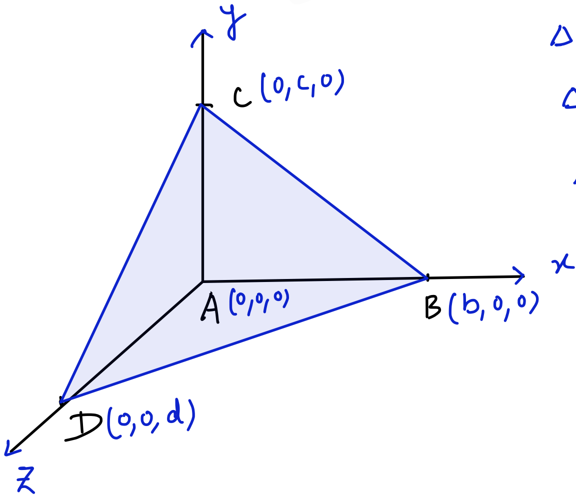
$2\lambda - \cancel{2} = 3\mu - \cancel{2} \Rightarrow 3\mu - 2\lambda = 0$ —②—

$\lambda = \frac{15}{11}$; $\mu = \frac{10}{11}$

$|\vec{AB}| = \frac{12}{11}$ Ans

Q Let ABCD be a tetrahedron such that edges AB, AC, AD are mutually $\perp$. Let the areas of $\triangle ABC$, $\triangle ACD$, $\triangle ADB$ are 3, 4, 5 sq. units respectively. then find the area of $\triangle BCD$?

Sol^n



$\triangle ABC = \frac{1}{2} bc = 3 \Rightarrow bc = 6.$ ✓

$\triangle ACD = \frac{1}{2} cd = 4 \Rightarrow cd = 8.$ ✓

$\triangle ADB = \frac{1}{2} bd = 5 \Rightarrow bd = 10.$ ✓

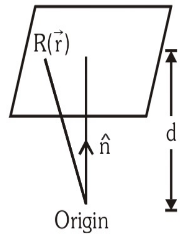
$$\triangle BCD = \frac{1}{2}\sqrt{(bc)^2 + (cd)^2 + (bd)^2}$$

Normal form of the equation of plane :

Projection of $\vec{r}$ on $\hat{n} = d$

Rem

$\boxed{\vec{r}.\hat{n} = d}$ ...... (i) $(d > 0)$



$\hat{n} \rightarrow$ unit vector pointing from origin towards plane & perpendicular to it.

$d \rightarrow$ perpendicular distance of the plane from the origin.

$d\hat{n} \rightarrow$ p.v. of foot of perpendicular from origin in plane.

i.e. from (i) $\ell x + my + nz = d$ (cartesian form)

equation (i) helps us to know the distance of the plane from the origin and also the dc's of the normal vector.

Example : Convert given equation into normal form & find D.C. & distance from origin

(1) $\vec{r}.(6\hat{i} - 3\hat{j} - 2\hat{k}) + 1 = 0$

$$\vec{r}.\left(\frac{6\hat{i} - 3\hat{j} - 2\hat{k}}{7}\right) = -\frac{1}{7}$$

$$\vec{r}.\left(\frac{-6\hat{i} + 3\hat{j} + 2\hat{k}}{7}\right) = \frac{1}{7}$$

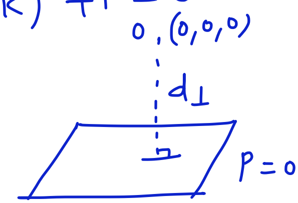
distance of $(0,0,0)$ from plane is $\frac{1}{7}$

dc's of normal : $\left(-\frac{6}{7}, \frac{3}{7}, \frac{2}{7}\right)$

alt :- $(x\hat{i} + y\hat{j} + z\hat{k}).(6\hat{i} - 3\hat{j} - 2\hat{k}) + 1 = 0$

$P: \; 6x - 3y - 2z + 1 = 0$

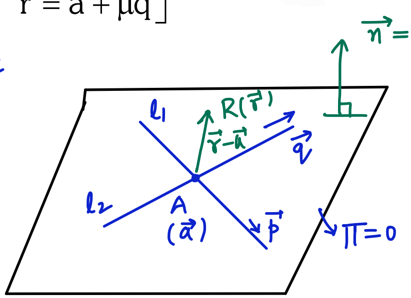
$$d_\perp = \frac{|0 - 0 - 0 + 1|}{\sqrt{6^2 + 3^2 + 2^2}} = \frac{1}{7}$$



Q. Find equation of plane containing two intersecting lines :

$$\left.\begin{array}{l}\vec{r}=\vec{a}+\lambda\vec{p}\\ \vec{r}=\vec{a}+\mu\vec{q}\end{array}\right]$$

Sol^n



$(\vec{r}-\vec{a})\cdot\vec{n}=0.$

$(\vec{r}-\vec{a})\cdot(\vec{p}\times\vec{q})=0$

$\Downarrow$

$[\vec{r}-\vec{a}\ \ \vec{p}\ \ \vec{q}]=0.$

$\Downarrow$

$[\vec{r}\ \vec{p}\ \vec{q}]=[\vec{a}\ \vec{p}\ \vec{q}]$

$\vec{r}-\vec{a}=\lambda\vec{p}+\mu\vec{q}$

$\lambda,\mu\in$ scalars.

Parametric Equation of Plane :—

$$\boxed{\vec{r}=\vec{a}+\lambda\vec{p}+\mu\vec{q}}$$

This denotes a plane containing a point $\vec{a}$ & is parallel to two collinear vectors $\vec{p}$ & $\vec{q}$.

Q Express the equation of a plane $\vec{r} = \hat{i} - 2\hat{j} + \lambda(2\hat{i} + \hat{j} + 3\hat{k}) + \mu(3\hat{i} + 4\hat{j} - \hat{k})$ in

(i) scalar triple product forms

(ii) cartesian form

(i) $\left[\vec{r} - (\hat{i} - 2\hat{j}) \quad 2\hat{i} + \hat{j} + 3\hat{k} \quad 3\hat{i} + 4\hat{j} - \hat{k}\right] = 0.$

(ii) point on plane $(1, -2, 0)$

$\vec{n} = (2\hat{i} + \hat{j} + 3\hat{k}) \times (3\hat{i} + 4\hat{j} - \hat{k}) = a\hat{i} + b\hat{j} + c\hat{k}$

$P: \quad a(x-1) + b(y+2) + c(z-0) = 0$

Q $\vec{r} = 2\hat{j} + \hat{k} + \lambda(\hat{i} - \hat{j} + \hat{k})$

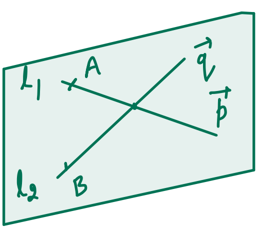
$\vec{r} = 2\hat{i} + 3\hat{j} + 6\hat{k} + \mu(2\hat{i} + \hat{j} + 5\hat{k})$

Check whether two lines are co-planar or not, if yes find the equation of plane containing them.

Sol^n $L_1 \to A(0, 2, 1)$

$L_2 \to B(2, 3, 6)$

$\vec{AB} = 2\hat{i} + \hat{j} + 5\hat{k}$



If $\left[\vec{AB} \quad \vec{p} \quad \vec{q}\right] = 0$ then the 2 lines are intersecting hence coplanar.

$\vec{p} = \hat{i} - \hat{j} + \hat{k}$.

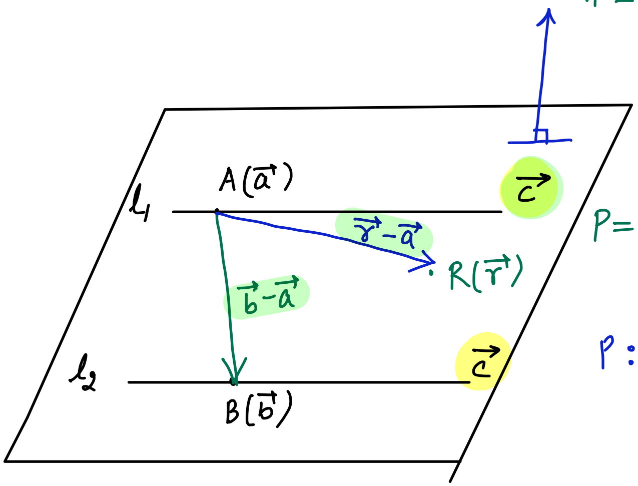
$\vec{q} = 2\hat{i} + \hat{j} + 5\hat{k}$.

Q Find equation of plane containing two parallel lines :

$\vec{r} = \vec{a} + \lambda\vec{c}$ and $\vec{r} = \vec{b} + \mu\vec{c}$

$\vec{n} = (\vec{b} - \vec{a}) \times \vec{c}$

Sol



$P = 0$

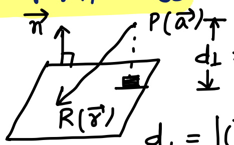
$P: \ (\vec{r} - \vec{a}) \cdot \vec{n} = 0$

$P: \ [\vec{r} - \vec{a} \quad \vec{b} - \vec{a} \quad \vec{c}] = 0$

PERPENDICULAR DISTANCE OF A POINT 'P' FROM A PLANE Ax + By + Cz + D = 0 :

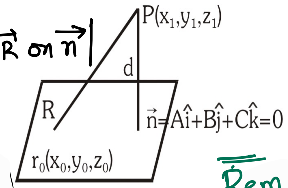
$P: \vec{r}\cdot\vec{n} = d$ —(1)—

d = | Projection of $\overrightarrow{RP}$ on $\vec{n}$ |



$d_\perp = |\text{proj of } \overrightarrow{PR} \text{ on } \vec{n}|$

$d_\perp = \dfrac{|(\vec{r}-\vec{a})\cdot\vec{n}|}{|\vec{n}|}$



$$d = \left|\frac{\overrightarrow{RP}.\vec{n}}{|\vec{n}|}\right| = \left|\frac{Ax_1+By_1+Cz_1+D}{\sqrt{A^2+B^2+C^2}}\right|$$

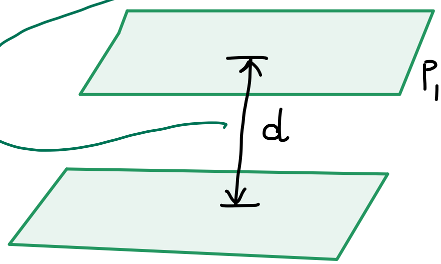
Rem

$$d_\perp = \frac{|\vec{a}\cdot\vec{n} - d|}{|\vec{n}|}$$

Note : Distance between the parallel planes $\left|\dfrac{d_1-d_2}{\sqrt{a^2+b^2+c^2}}\right| = d$

(Take any point 'P' $(x_1y_1z_1)$ on one plane and from P draw perpendicular on the other plane)

** Coeff of x, y, z must be equal in eqn of 2 planes & Constant term must be one same side of eqn of plane.



$P_1: ax+by+cz+d_1 = 0$

$P_2: ax+by+cz+d_2=0.$

ANGLE BETWEEN TWO PLANES :

(a) The angle θ, between the two planes $\vec{r}.\vec{n}_1 = q_1$ and $\vec{r}.\vec{n}_2 = q_2$, being equal to the angle between the vectors $\vec{n}_1$ and $\vec{n}_2$ which are normal to the plane, we have $\theta = \cos^{-1}\dfrac{n_1.n_2}{|\vec{n}_1||\vec{n}_2|}$.

(b) Let the plane $a_1x + b_1y + c_1z + d_1 = 0$ and $a_2x + b_2y + c_2z + d_2 = 0$ are

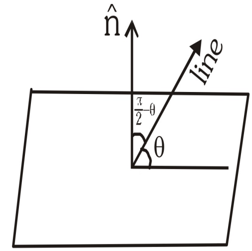
$$\cos\theta = \frac{a_1a_2+b_1b_2+c_1c_2}{\sqrt{a_1^2+b_1^2+c_1^2}\sqrt{a_2^2+b_2^2+c_2^2}}$$

(i) Parallel & distinct if $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2} \neq \dfrac{d_1}{d_2}$

(ii) Perpendicular if $a_1a_2 + b_1b_2 + c_1c_2 = 0$

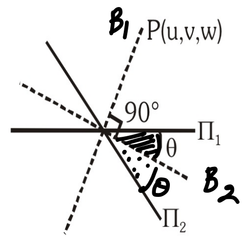
(iii) Identical if $\dfrac{a_1}{a_2} = \dfrac{b_1}{b_2} = \dfrac{c_1}{c_2} = \dfrac{d_1}{d_2}$

ANGLE BETWEEN A LINE AND A PLANE :



The angle θ, between any line $\vec{r} = \vec{a} + \vec{b}\,\vec{t}$ and any plane $\vec{r}.\vec{n} = q$, being equal to the complement of the angle between the normal vector $\vec{n}$, of the plane and the direction vectors $\vec{b}$ of the line, we have $\theta = \sin^{-1}\left(\frac{\vec{n}.\vec{b}}{|\vec{n}|\,|\vec{b}|}\right)$.

EQUATION OF THE BISECTOR PLANES BETWEEN THE PLANES :



$\Pi_1 : a_1x + b_1y + c_1y + d_1 = 0$ and .... (i)

$\Pi_1 : a_2x + b_2y + c_2y + d_2 = 0$ is .... (ii)

$$\boxed{\frac{a_1x + b_1y + c_1z + d_1}{\sqrt{a_1^2 + b_1^2 + c_1^2}} = \pm\frac{a_2x + b_2y + c_2z + d_2}{\sqrt{a_2^2 + b_2^2 + c_2^2}}}$$

Acute/Obtuse angle bisectors can be easily isolated by finding Cos θ

where θ is the angle between any one of the two given planes and any one of the two bisector planes.

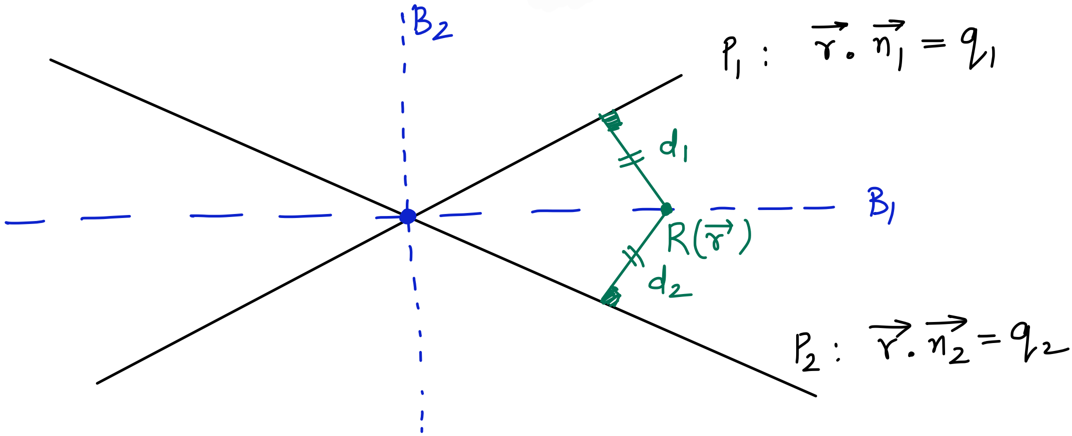
$$\boxed{\text{if } \frac{1}{\sqrt{2}} < |\cos\theta| < 1 \Rightarrow \theta \text{ is acute ; } \quad \text{if } 0 < |\cos\theta| < \frac{1}{\sqrt{2}} \Rightarrow \theta \text{ is obtuse}}$$

Vectorially : (Eqn of bisector planes)

Let $\vec{r}.\vec{n}_1 = q_1$ and $\vec{r}.\vec{n}_2 = q_2$ be the given planes. Perpendicular distance of any point $\vec{r}$ on either bisecting planes from the two given planes being equal, hence

$$\frac{|\vec{r}.\vec{n}_1 - q_1|}{|\vec{n}_1|} = \frac{|\vec{r}.\vec{n}_2 - q_2|}{|\vec{n}_2|} \Rightarrow \frac{\vec{r}.\vec{n}_1 - q_1}{|\vec{n}_1|} = \pm \frac{\vec{r}.\vec{n}_2 - q_2}{|\vec{n}_2|} \text{ or } \vec{r}.\left(\frac{\vec{n}_1}{|\vec{n}_1|} \pm \frac{\vec{n}_2}{|\vec{n}_2|}\right) = \left(\frac{q_1}{|\vec{n}_1|} \pm \frac{q_2}{|\vec{n}_2|}\right)$$

where same sign is to be taken throughout.



$$\frac{|\vec{r}.\vec{n}_1 - q_1|}{|\vec{n}_1|} = \frac{|\vec{r}.\vec{n}_2 - q_2|}{|\vec{n}_2|}$$

Q Find the equation of the plane which is parallel to the plane x + 5y – 4z + 5 = 0 and the sum of whose intercepts on the co-ordinate axes is 19 units. Also find the distance between these planes.

Sol^n

$P: \ x + 5y - 4z + \lambda = 0 \quad — (1) —$

For x_{int} put $y = z = 0 \Rightarrow x_{int} = -\lambda$

$x_{int} + y_{int} + z_{int} = 19$

||^{ly} $y_{int} = -\frac{\lambda}{5}$

$z_{int} = \frac{\lambda}{4}$

$-\lambda - \frac{\lambda}{5} + \frac{\lambda}{4} = 19$

$\lambda\left(\frac{-20 - 4 + 5}{20}\right) = 19 \Rightarrow \lambda = -20$

$$\left.\begin{aligned} P: &\ x + 5y - 4z - 20 = 0 \\ P_1: &\ x + 5y - 4z + 5 = 0 \end{aligned}\right\}$$

$$d_\perp = \frac{|25|}{\sqrt{1^2 + 5^2 + 4^2}} \quad ✓$$

Q Find the equation of the plane parallel to 2x – 6y + 3z = 0 and at a distance of 2 from the point (1, 2, –3).

Sol^n

$P: \ 2x - 6y + 3z + d = 0.$

$$d_\perp = \frac{|2(1) - 6(2) + 3(-3) + d|}{\sqrt{2^2 + 6^2 + 3^2}} = 2$$

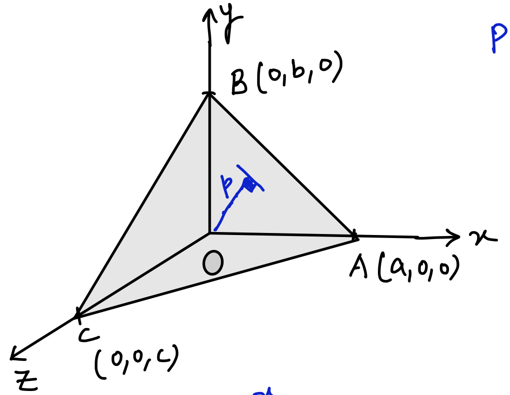
$|2 - 12 - 9 + d| = 2 \times 7$

$d - 19 = \pm 14$

$d = 33 \ ; \quad d = 5$

Q A plane which always remains at a constant distance p from the origin cuts the co-ordinate axes at A, B, C. Find the locus of

(i) Centroid of the plane face ABC (ii) Centre of the tetrahedron OABC



$P: \frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$

$$\frac{|0+0+0-1|}{\sqrt{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}}} = p$$

$$\boxed{\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2} = \frac{1}{p^2}} \quad \text{— (1) —}$$

(i) $G(\alpha, \beta, \gamma)$ ($\alpha \to x$, $\beta \to y$, $\gamma \to z$)

$\alpha = \frac{a+0+0}{3} \Rightarrow a = 3\alpha$; $\|^{ly}$ $b = 3\beta$, $c = 3\gamma$

$$\boxed{\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2} = \frac{9}{p^2}}$$

(ii) $G(\alpha, \beta, \gamma)$ ($\alpha \to x$, $\beta \to y$, $\gamma \to z$)

$\alpha = \frac{a+0+0+0}{4} \Rightarrow a = 4\alpha$; $\|^{ly}$ $b = 4\beta$; $c = 4\gamma$.

CONDITION FOR LINE TO LIE COMPLETELY IN PLANE :

Let plane $\vec{r}.\vec{n} = d$ & line is $\vec{r} = \vec{a} + \lambda\vec{b}$.

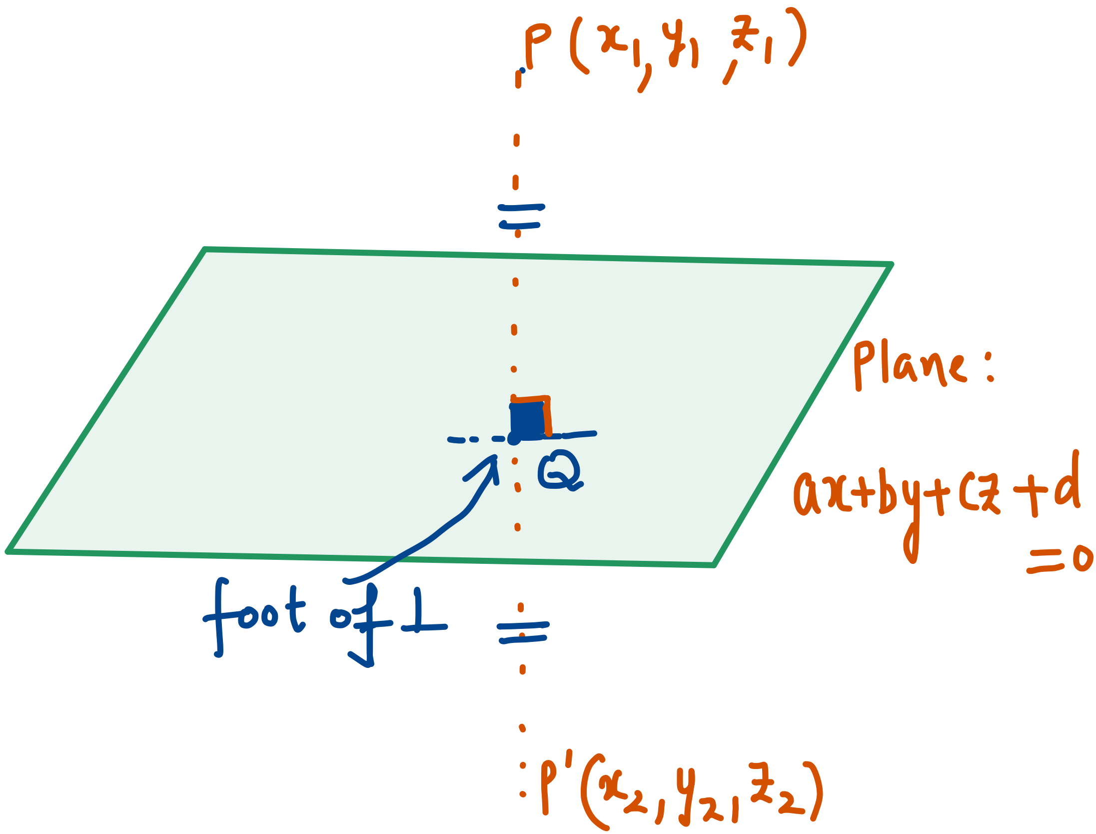
$\vec{n}.\vec{b} = 0 \quad \Rightarrow \quad$ line is parallel to plane.

Note :

(i) If a line is parallel to the plane then normal of the plane & the line are perpendicular.

(ii) If a line is perpendicular to the plane then the line & normal are parallel.

Image of a point in a Plane :-



Rem

$$\boxed{\frac{x_2-x_1}{a} = \frac{y_2-y_1}{b} = \frac{z_2-z_1}{c} = \frac{-2\,(ax_1+by_1+cz_1+d)}{a^2+b^2+c^2}}$$

HW

Vector JA (Remaining)