M_SOT_Ex-DY

Do yourself - 1 :

(i) If in a ΔABC, $\angle A = \frac{\pi}{6}$ and $b : c = 2 : \sqrt{3}$, find $\angle B$.

(ii) Show that, in any ΔABC : $a\sin(B-C) + b\sin(C-A) + c\sin(A-B) = 0$.

(iii) If in a ΔABC, $\frac{\sin A}{\sin C} = \frac{\sin(A-B)}{\sin(B-C)}$, show that a^2, b^2, c^2 are in A.P.

Solution.

(i). $\angle A = \pi/6$ $\quad \frac{b}{c} = \frac{2}{\sqrt{3}}$

Applying C&D:-

$$\frac{b+c}{b-c} = \frac{2+\sqrt{3}}{2-\sqrt{3}}$$

$$\frac{\sin B + \sin C}{\sin B - \sin C} = \frac{2+\sqrt{3}}{2-\sqrt{3}}$$

$$\tan\frac{B+C}{2}\cot\frac{B-C}{2} = \frac{2+\sqrt{3}}{2-\sqrt{3}}$$

$$\Rightarrow \frac{B-C}{2} = 15^\circ$$

Also $B + C = 120^\circ$

$$\Rightarrow \boxed{B = 75^0}$$

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(ii). $\sum a \sin(B-C)$

$\Rightarrow K \sin A \sin B-C \Rightarrow K \sin(B+C).\sin(B-C)$

$\Rightarrow K\{\sin^2 B - \sin^2 C\}$

Similarly,

$K\{\sin^2 C - \sin^2 A\}$,

and $K\{\sin^2 A - \sin^2 B\}$

$\therefore \boxed{a \sin(B-C) + b \sin(C-A) + c \sin(A-B) = 0}$

(iii). $\sin A.\sin(B-C) = \sin C.\sin(A-B)$

$\Rightarrow \sin(B+C).\sin(B-C) = \sin(A+B).\sin(A-B)$

$\Rightarrow \sin^2 B - \sin^2 C = \sin^2 A - \sin^2 B$

$\Rightarrow 2\sin^2 B = \sin^2 A + \sin^2 C$

$\Rightarrow 2b^2 = a^2 + c^2$

$\boxed{\therefore a^2, b^2, c^2 \text{ are in A.P.}}$

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Do yourself - 2 :

(i) If a : b : c = 4 : 5 : 6, then show that $\angle C = 2\angle A$.

(ii) In any ΔABC, prove that

(a) $$\frac{\cos A}{a}+\frac{\cos B}{b}+\frac{\cos C}{c}=\frac{a^2+b^2+c^2}{2abc}$$

(b) $$\frac{b^2}{a}\cos A+\frac{c^2}{b}\cos B+\frac{a^2}{c}\cos C=\frac{a^4+b^4+c^4}{2abc}$$

Solution.

(i). $a:b:c = 4:5:6$

$$\cos C = \frac{a^2+b^2-c^2}{2ab} = \frac{\cancel{k^2}[16+25-36]}{2.4.5.\cancel{k^2}}$$

$$\therefore \cos C = 1/8$$

$$\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{\cancel{k^2}[25+36-16]}{2.5.6.\cancel{k^2}}$$

$$\therefore \cos A = 3/4$$

$$\cos 2A = 2\cos^2 A - 1 = 2.\,9/16 - 1 = 1/8$$

$$= \cos C$$

$$\Rightarrow \boxed{2\angle A = \angle C}$$

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a). $$\frac{\cos A}{a} = \frac{1}{a}\left(\frac{b^2+c^2-a^2}{2bc}\right)$$

Similarly, $$\frac{\cos B}{b} = \frac{a^2+c^2-b^2}{2abc}$$

$$\frac{\cos C}{c} = \frac{a^2+b^2-c^2}{2abc}$$

$$\therefore \boxed{\sum \frac{\cos A}{a} = \frac{a^2+b^2+c^2}{2abc}}$$

b). $$\frac{b^2}{a}.\cos A = \frac{b^2}{a}.\frac{b^2+c^2-a^2}{2bc}$$

Similarly, $$\frac{c^2}{b}\cos B = \frac{c^2(a^2+c^2-b^2)}{2abc}$$

$$\frac{a^2}{c}\cos C = \frac{a^2(a^2+b^2-c^2)}{2abc}$$

$$\therefore \frac{b^2}{a}\cos A + \frac{c^2}{b}\cos B + \frac{a^2}{c}\cos C$$

$$\Rightarrow \boxed{\frac{a^4+b^4+c^4}{2abc}}$$

M_SOT_Ex-DY

Do yourself - 3 :

(i) In a ΔABC, if $\angle A = \frac{\pi}{4}$, $\angle B = \frac{5\pi}{12}$, show that $a + c\sqrt{2} = 2b$.

(ii) In a ΔABC, prove that : (a) $b(a\cos C - c\cos A) = a^2 - c^2$ (b) $2\left(b\cos^2\frac{C}{2} + c\cos^2\frac{B}{2}\right) = a + b + c$

Solution.

(i). $\angle A = \pi/4$, $\angle B = 5\pi/12$ $\angle C = \pi/3$

$$a + c\sqrt{2} = K[\sin A + \sqrt{2}\sin C]$$

$$= K\left[\frac{1}{\sqrt{2}} + \sqrt{2}.\frac{\sqrt{3}}{2}\right]$$

$$= K\left[\frac{\sqrt{3}+1}{\sqrt{2}}\right] \qquad \left\{\because \sin\frac{5\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}}\right\}$$

$$= 2\underline{K\sin B}$$

$$\therefore \boxed{a + c\sqrt{2} = 2b}$$

(ii).

a). $b(a\cos C - c\cos A) = b.k(\sin A\cos C - \sin C\cos A)$

$$\Rightarrow bk\sin(A-C) = K^2\sin B.\sin(A-C)$$

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$$\Rightarrow K^2 \sin(A+C).\sin(A-C) = K^2(\sin^2 A - \sin^2 C)$$

$$\Rightarrow a^2 - c^2$$

$$\therefore \boxed{b(a\cos C - c\cos A) = a^2 - c^2}$$

b). $$2\left(b\cos^2 \tfrac{C}{2} + c\cos^2 \tfrac{B}{2}\right) = b.2\cos^2 \tfrac{C}{2} + c.2\cos^2 \tfrac{B}{2}$$

$$\Rightarrow b(1+\cos C) + c(1+\cos B)$$

$$\Rightarrow b + c + \underline{b.\cos C + c.\cos B}$$

$$\Rightarrow b + c + a$$ {Using Projection formula}

$$\therefore \boxed{2\left(b\cos^2 \tfrac{C}{2} + c\cos^2 \tfrac{B}{2}\right) = a+b+c}$$

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Do yourself - 4 :

(i) In any ΔABC, prove that $\frac{b-c}{b+c} = \frac{\tan\left(\frac{B-C}{2}\right)}{\tan\left(\frac{B+C}{2}\right)}$

(ii) If ΔABC is right angled at C, prove that : (a) $\tan\frac{A}{2} = \sqrt{\frac{c-b}{c+b}}$ (b) $\sin(A-B) = \frac{a^2-b^2}{a^2+b^2}$

Solution.

(i).

$$\frac{b-c}{b+c} = \frac{\sin B - \sin C}{\sin B + \sin C}$$

$$= \frac{2.\cos\frac{B+C}{2}\sin\frac{B-C}{2}}{2.\sin\frac{B+C}{2}\cos\frac{B-C}{2}}$$

$$\frac{b-c}{b+c} = \cot\frac{B+C}{2} . \tan\frac{B-C}{2}$$

$$\therefore \quad \boxed{\frac{b-c}{b+c} = \frac{\tan\frac{B-C}{2}}{\tan\frac{B+C}{2}}}$$

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(ii).

a). $\tan\frac{A}{2} = \sqrt{\frac{c-b}{c+b}}$

$\angle C = \pi/2 \ , \ \angle A + \angle B = \pi/2$

R.H.S:-

$$\sqrt{\frac{c-b}{c+b}} = \sqrt{\frac{2\cos\frac{C+B}{2}\sin\frac{C-B}{2}}{2\sin\frac{C+B}{2}\cos\frac{C-B}{2}}}$$

$$= \sqrt{\cot\frac{C+B}{2}\tan\frac{C-B}{2}}$$

$$= \sqrt{\tan\frac{A}{2}\tan\left(\frac{\pi/2-(\pi/2-A)}{2}\right)}$$

$$= \sqrt{\tan^2 A/2}$$

$$\therefore \quad \boxed{\sqrt{\frac{c-b}{c+b}} = \tan A/2}$$

M_SOT_Ex-DY

b). $\sin(A-B) = \dfrac{a^2-b^2}{a^2+b^2}$

$\angle C = \pi/2, \quad \angle A + \angle B = \pi/2$

R.H.S:-

$$\frac{a^2-b^2}{a^2+b^2} = \frac{\sin^2 A - \sin^2 B}{\sin^2 A + \sin^2 B}$$

$$= \frac{\sin(A+B).\sin(A-B)}{\sin^2 A + \cos^2 A}$$

$$= \frac{1.\sin(A-B)}{1}$$

$$\therefore \boxed{\frac{a^2-b^2}{a^2+b^2} = \sin(A-B)}$$

M_SOT_Ex-DY

Do yourself - 5 :

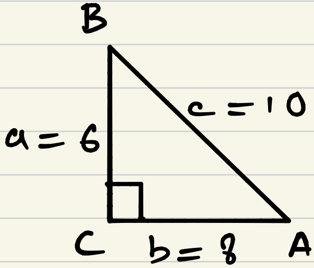
(i) Given a = 6, b = 8, c = 10. Find

(a) sinA (b) tanA (c) $\sin\frac{A}{2}$ (d) $\cos\frac{A}{2}$ (e) $\tan\frac{A}{2}$ (f) Δ

(ii) Prove that in any ΔABC, $(abcs)\ \sin\frac{A}{2}.\sin\frac{B}{2}.\sin\frac{C}{2} = \Delta^2$.

Solution.

(i). Given triangle is a right angled triangle:-



$$s = \frac{6+8+10}{2} = 12$$

a). $$\sin A = 6/10 = 3/5$$

b). $$\tan A = 6/8 = 3/4$$

c). $$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}$$

$$\therefore \sin A/2 = \sqrt{\frac{(12-8).(12-10)}{8.10}} = \frac{1}{\sqrt{10}}$$

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d). $$\cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}}$$

$$\therefore \cos A/2 = \sqrt{\frac{12.(12-6)}{8.10}} = \frac{3}{\sqrt{10}}$$

e). $$\tan A/2 = 1/3$$

f). $$\Delta = 1/2 \times AC \times BC$$

$$= 1/2 \times 8 \times 6 = 24$$

(ii). $$(abcs).\sin A/2 . \sin B/2 . \sin C/2 = \Delta^2$$

L.H.S:-

$$(abcs).\sqrt{\frac{(s-b)(s-c)}{bc}.\frac{(s-c)(s-a)}{ca}.\frac{(s-a)(s-b)}{ab}}$$

$$\Rightarrow (\cancel{abc}s).\frac{(s-a)(s-b)(s-c)}{\cancel{abc}} = \Delta^2$$

$$\boxed{(abcs).\sin A/2 . \sin B/2 . \sin C/2 = \Delta^2}$$

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Do yourself - 6 :

(i) If in ΔABC, $a = 3$, $b = 4$ and $c = 5$, find

(a) Δ (b) R (c) r

(ii) In a ΔABC, show that :

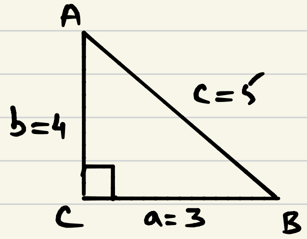
(a) $\dfrac{a^2 - b^2}{c} = 2R\sin(A - B)$ (b) $r\cos\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2} = \dfrac{\Delta}{4R}$ (c) $a + b + c = \dfrac{abc}{2Rr}$

(iii) Let Δ & Δ' denote the areas of a Δ and that of its incircle. Prove that

$\Delta : \Delta' = \left(\cot\dfrac{A}{2}.\cot\dfrac{B}{2}.\cot\dfrac{C}{2}\right) : \pi$

Solution.

(i). Given triangle is a right angled triangle:-



a). $\Delta = \frac{1}{2} BC \times AC = \frac{1}{2} \times 3 \times 4 = 6$

b). $R = \frac{c}{2} = \frac{5}{2}$

c). $r = \frac{\Delta}{s} = \frac{6}{6} = 1$ $\quad s = \frac{3+4+5}{2} = 6$

(ii).

a). $\dfrac{a^2 - b^2}{c} = 2R\sin(A - B)$

L.H.S:- $\dfrac{K^2(\sin^2 A - \sin^2 B)}{\cancel{K}\sin C} = K\left(\dfrac{\cancel{\sin(A+B)}.\sin(A-B)}{\cancel{\sin C}}\right)$

$\Rightarrow 2R\sin(A - B) \quad \{\because K = 2R\}$

$\therefore \boxed{\dfrac{a^2 - b^2}{c} = 2R\sin(A - B)}$

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b). $r \cos A/_2 \cos B/_2 \cos C/_2 = \dfrac{\Delta}{4R}$

L.H.S:-

$$r.\sqrt{\frac{s(s-a)}{bc}}.\sqrt{\frac{s(s-b)}{ac}}.\sqrt{\frac{s(s-c)}{ab}}$$

$$r.\sqrt{\frac{s^2.\ s(s-a)(s-b)(s-c)}{a^2b^2c^2}}$$

$$r.\ \frac{s\ .\ \Delta}{abc} = \frac{\Delta^2}{4\Delta R} \qquad \left\{\because r = \frac{\Delta}{s} \text{ and } \frac{abc}{4R} = \Delta\right\}$$

$$\therefore \boxed{r \cos A/_2 \cos B/_2 \cos C/_2 = \frac{\Delta}{4R}}$$

c). $a+b+c = \dfrac{abc}{2Rr}$

R.H.S:- $\qquad \left\{\because r = \dfrac{\Delta}{s} \text{ and } \dfrac{abc}{4R} = \Delta\right\}$

$$\frac{abc}{2Rr} = \frac{4\cancel{R}\Delta}{2\cancel{R}r}$$

$$= \frac{2\Delta}{r} = \frac{2\cancel{r}s}{\cancel{r}} = 2s$$

$$\therefore \boxed{a+b+c = \frac{abc}{2Rr}}$$

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(iii). $$\frac{\Delta}{\Delta'} = \frac{r.s}{\pi r^2}$$

$$= \frac{s}{r.\pi} \Rightarrow \frac{a+b+c}{2r\pi}$$

$$= \frac{2R(\sin A + \sin B + \sin C)}{2r\pi}$$

$$= \frac{R\left(4\cos A/_2 . \cos B/_2 . \cos C/_2\right)}{\pi\left(4R\sin A/_2 . \sin B/_2 . \sin C/_2\right)}$$

$$\boxed{\frac{\Delta}{\Delta'} = \frac{\cot A/_2 . \cot B/_2 . \cot C/_2}{\pi}}$$

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Do yourself - 7 :

(i) In an equilateral ΔABC, $R = 2$, find

(a) r (b) r_1 (c) a

(ii) In a ΔABC, show that

(a) $r_1r_2 + r_2r_3 + r_3r_1 = s^2$ (b) $\frac{1}{4}r^2s^2\left(\frac{1}{r}-\frac{1}{r_1}\right)\left(\frac{1}{r}-\frac{1}{r_2}\right)\left(\frac{1}{r}-\frac{1}{r_3}\right) = R$

(c) $\sqrt{rr_1r_2r_3} = \Delta$

Solution.

(i). $\triangle$ABC is an equilateral triangle

$$\therefore \frac{a}{\sin A} = 2R \Rightarrow a = 2\sqrt{3}$$

a). $$r = \frac{\Delta}{s} = \frac{\frac{\sqrt{3}}{4}a^2}{\frac{3}{2}a} = 1$$

b). $$r_1 = \frac{\Delta}{s-a} = \frac{\frac{\sqrt{3}}{4}a^2}{\frac{3}{2}a - a} = 3$$

c). $$a = 2\sqrt{3}$$

(ii).

a). $$r_1.r_2 + r_2.r_3 + r_3.r_1 = s^2$$

L.H.S:- $$r_1.r_2 = \frac{\Delta}{s-a}\cdot\frac{\Delta}{s-b} = \frac{\Delta^2}{(s-a).(s-b)}$$

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Similarly, $r_2 . r_3 = \dfrac{\Delta^2}{(s-b).(s-c)}$, $r_3 . r_1 = \dfrac{\Delta^2}{(s-c).(s-a)}$

$$\therefore r_1 . r_2 + r_2 . r_3 + r_3 . r_1 = \frac{\Delta^2 (s-a+s-b+s-c)}{(s-a)(s-b)(s-c)}$$

$$= \frac{\Delta^2 (3s-2s)}{(s-a)(s-b)(s-c)}$$

$$= \frac{\Delta^2 . s}{(s-a)(s-b)(s-c)} \times \frac{s}{s}$$

$$= \frac{\cancel{\Delta^2} . s^2}{\cancel{\Delta^2}}$$

$$\therefore \boxed{r_1 . r_2 + r_2 . r_3 + r_3 . r_1 = s^2}$$

b). $\dfrac{1}{4} r^2 . s^2 \left(\dfrac{1}{r} - \dfrac{1}{r_1}\right)\left(\dfrac{1}{r} - \dfrac{1}{r_2}\right)\left(\dfrac{1}{r} - \dfrac{1}{r_3}\right) = R$

L.H.S:-

$$\frac{1}{4} r^2 . s^2 \left(\frac{s}{\Delta} - \frac{s-a}{\Delta}\right)\left(\frac{s}{\Delta} - \frac{s-b}{\Delta}\right)\left(\frac{s}{\Delta} - \frac{s-c}{\Delta}\right)$$

$$\Rightarrow \frac{1}{4} r^2 . s^2 \left(\frac{a}{\Delta}\right)\left(\frac{b}{\Delta}\right)\left(\frac{c}{\Delta}\right)$$

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$$\Rightarrow \frac{1}{4} r^2 \cdot s^2 \quad \frac{abc}{\Delta^3} \Rightarrow \frac{1 \cancel{r^2 \cdot s^2}}{\cancel{\Delta^2}} \quad \frac{abc}{4\Delta} \Rightarrow \frac{abc}{4\Delta}$$

$$\therefore \boxed{\frac{1}{4} r^2 \cdot s^2 \left(\frac{1}{r} - \frac{1}{r_1}\right)\left(\frac{1}{r} - \frac{1}{r_2}\right)\left(\frac{1}{r} - \frac{1}{r_3}\right) = R}$$

c). $$\sqrt{r \cdot r_1 \cdot r_2 \cdot r_3} = \Delta$$

L.H.S:-

$$\sqrt{\frac{\Delta}{s} \cdot \frac{\Delta}{s-a} \cdot \frac{\Delta}{s-b} \cdot \frac{\Delta}{s-c}} = \sqrt{\frac{\Delta^4}{\Delta^2}}$$

$$\therefore \boxed{\sqrt{r \cdot r_1 \cdot r_2 \cdot r_3} = \Delta}$$

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Do yourself - 8 :

(i) If x, y, z are the distance of the vertices of ΔABC respectively from the orthocentre, then prove that $\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=\frac{abc}{xyz}$

(ii) If p_1, p_2, p_3 are respectively the perpendiculars from the vertices of a triangle to the opposite sides, prove that

(a) $p_1p_2p_3 = \frac{a^2b^2c^2}{8R^3}$ (b) $\Delta = \sqrt{\frac{1}{2}Rp_1p_2p_3}$

(iii) In a ΔABC, AD is altitude and H is the orthocentre prove that AH : DH = (tanB + tanC) : tanA

(iv) In a ΔABC, the lengths of the bisectors of the angle A, B and C are x, y, z respectively.

Show that $\frac{1}{x}\cos\frac{A}{2}+\frac{1}{y}\cos\frac{B}{2}+\frac{1}{z}\cos\frac{C}{2}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}$.

Solution.

(i). $$\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=\frac{abc}{xyz}$$

L.H.S:- $$\frac{2R\sin A}{2R\cos A}+\frac{2R\sin B}{2R\cos B}+\frac{2R\sin C}{2R\cos C}$$

$$\Rightarrow \tan A+\tan B+\tan C=\tan A.\tan B.\tan C$$

$$\Rightarrow \frac{2R\sin A}{2R\cos A}\times\frac{2R\sin B}{2R\cos B}\times\frac{2R\sin C}{2R\cos C}\Rightarrow\frac{a}{x}.\frac{b}{y}.\frac{c}{z}$$

$$\therefore \boxed{\frac{a}{x}+\frac{b}{y}+\frac{c}{z}=\frac{abc}{xyz}}$$

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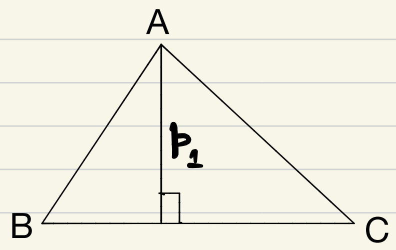
(ii).



a). $p_1 . p_2 . p_3 = \dfrac{a^2 b^2 c^2}{8R^3}$

L.H.S:-

$p_1 = c \sin B \;,\; p_2 = a \sin C \;,\; p_3 = b \sin A$

$\therefore abc . \sin A . \sin B . \sin C = abc . \dfrac{a}{2R} . \dfrac{b}{2R} . \dfrac{c}{2R}$

$$\therefore \boxed{p_1 . p_2 . p_3 = \frac{a^2 b^2 c^2}{8R^3}}$$

b). $\Delta = \sqrt{\dfrac{1}{2} \dfrac{R p_1 . p_2 . p_3}{}}$

R.H.S:- $\sqrt{\dfrac{1}{2} \dfrac{R p_1 . p_2 . p_3}{}} = \sqrt{\dfrac{1}{2} \dfrac{abc}{4\Delta} . \dfrac{a^2 . b^2 . c^2}{8R^3}}$

$$\Rightarrow \sqrt{\frac{a^3 b^3 c^3}{64 R^3 \Delta}} \Rightarrow \sqrt{\left(\frac{abc}{4R}\right)^3 . \frac{1}{\Delta}}$$

$$\therefore \boxed{\sqrt{\frac{1}{2} \frac{R p_1 . p_2 . p_3}{}} = \Delta}$$

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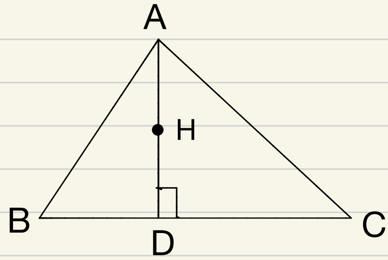

(iii). $\dfrac{AH}{DH} = \dfrac{2R\cos A}{2R\cos B.\cos C}$

$\Rightarrow \dfrac{\cos A}{\cos B.\cos C}$

Multiply and Divide by $\sin A$

$\Rightarrow \dfrac{\cos A}{\sin A} \times \dfrac{\sin A}{\cos B.\cos C} \Rightarrow \dfrac{1}{\tan A} \times \dfrac{\sin(B+C)}{\cos B.\cos C}$

$\Rightarrow \dfrac{1}{\tan A} \times \dfrac{\sin B\cos C + \cos B\sin C}{\cos B.\cos C}$

$\therefore \boxed{\dfrac{AH}{DH} = \dfrac{\tan B + \tan C}{\tan A}}$

(iv). $x = \dfrac{2bc}{b+c}\cos A/2,\ y = \dfrac{2ac}{a+c}\cos B/2$ & $z = \dfrac{2ab}{a+b}\cos C/2$

$\therefore \dfrac{1}{x}\cos A/2 + \dfrac{1}{y}\cos B/2 + \dfrac{1}{z}\cos C/2$

$= \dfrac{b+c}{2bc} + \dfrac{a+c}{2ac} + \dfrac{a+b}{2ab} \Rightarrow \dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c}$

$\therefore \boxed{\dfrac{1}{x}\cos A/2 + \dfrac{1}{y}\cos B/2 + \dfrac{1}{z}\cos C/2 = 1/a + 1/b + 1/c}$

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Do yourself - 9 :

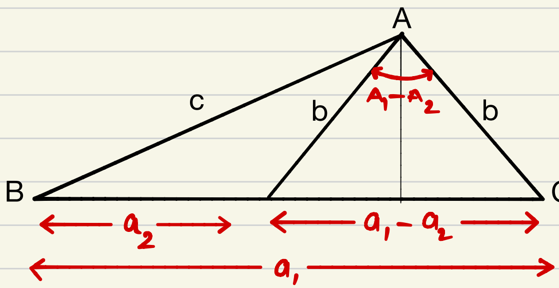
(i) If b,c,B are given and b<c, prove that $\sin\left(\frac{A_1 - A_2}{2}\right) = \frac{a_1 - a_2}{2b}$

(ii) In a ΔABC, b,c,B (c > b) are gives. If the third side has two values a_1 and a_2 such that

$a_1 = 3a_2$, show that $\sin B = \sqrt{\frac{4b^2 - c^2}{3c^2}}$.

Solution.

(i). From figure:-



Using Cosine Rule:-

$$\cos(A_1 - A_2) = \frac{2b^2 - (a_1 - a_2)^2}{2b^2}$$

As, $$\boxed{\sin\left(\frac{A_1 - A_2}{2}\right) = \sqrt{\frac{1 - \cos(A_1 - A_2)}{2}}}$$

$$\therefore \quad \boxed{\sin\left(\frac{A_1 - A_2}{2}\right) = \frac{a_1 - a_2}{2b}}$$

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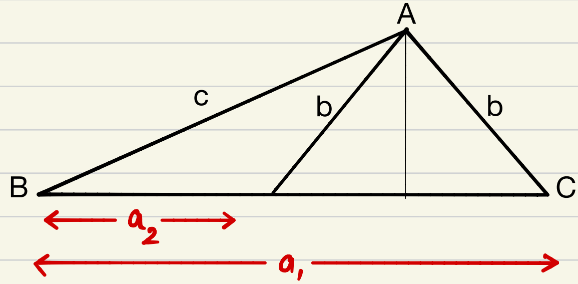
(ii). Using Cosine law:-



$$\cos B = \frac{c^2 + a^2 - b^2}{2ac}$$

$$\Rightarrow \quad a^2 - 2ac\cos B + c^2 - b^2 = 0 \quad \begin{matrix} \nearrow a_1 \\ \searrow a_2 \end{matrix}$$

$$\therefore a_1 + a_2 = 2c\cos B$$

and $\quad a_1 . a_2 = c^2 - b^2$

Also, $\boxed{a_1 = 3a_2}$

$$\Rightarrow \quad 4a_2 = 2c\cos B \text{ and } 3a_2^2 = c^2 - b^2$$

Solving, we get

$$3\left(\frac{4c^2\cos^2 B}{16}\right) = c^2 - b^2$$

$$\Rightarrow \quad \sin^2 B = \frac{4b^2 - c^2}{3c^2}$$

$$\Rightarrow \quad \boxed{\sin B = \sqrt{\frac{4b^2 - c^2}{3c^2}}}$$

$\boxed{\text{M\_SOT\_Ex-DY}}$

Do yourself - 10 :

(i) If the perimeter of a circle and a regular polygon of n sides are equal, then

prove that $\dfrac{\text{area of the circle}}{\text{area of polygon}} = \dfrac{\tan\frac{\pi}{n}}{\frac{\pi}{n}}$.

(ii) The ratio of the area of n-sided regular polygon, circumscribed about a circle, to the area of the regular polygon of equal number of sides inscribed in the circle is 4 : 3. Find the value of n.

(i). $\boxed{2\pi r = n.a}$ {where a = side length of polygon & r = radius of circle}

$\Rightarrow \boxed{a = \dfrac{2\pi r}{n}}$

Now, Area of polygon = $n \times \dfrac{1}{2} \times a \times \dfrac{a}{2\tan\pi/n} = \dfrac{na^2}{4\tan\pi/n}$ ———— (1)

Put value of a in (1) ,

$\therefore$ Area of Polygon = $\dfrac{n\,4\pi^2 r^2}{4n^2\tan\pi/n} = \dfrac{\pi^2 r^2}{n\tan\pi/n}$

$\Rightarrow \dfrac{\text{area of circle}}{\text{area of polygon}} = \dfrac{\pi r^2}{\pi^2 r^2} \times n\tan\pi/n$

$\therefore \boxed{\dfrac{\text{area of circle}}{\text{area of polygon}} = \dfrac{\tan\pi/n}{\pi/n}}$

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(ii). $\dfrac{\text{Area of regular polygon circumscribed}}{\text{Area of regular polygon inscribed}} = \dfrac{nr^2 \tan\dfrac{\pi}{n}}{\dfrac{1}{2}nr^2 \sin\dfrac{2\pi}{n}}$

$$\Rightarrow \frac{2\tan \pi/n}{\sin 2\pi/n} = \frac{4}{3}$$

$$\Rightarrow \cos^2 \pi/n = \frac{3}{4}$$

$$\Rightarrow \frac{\pi}{n} = \frac{\pi}{6}$$

$$\Rightarrow \boxed{n = 6}$$

ELEMENTARY EXERCISE

1. Angles A, B and C of a triangle ABC are in A.P. If $\frac{b}{c} = \sqrt{\frac{3}{2}}$, then $\angle A$ is equal to

(A) $\frac{\pi}{6}$ (B) $\frac{\pi}{4}$ (C) $\frac{5\pi}{12}$ (D) $\frac{\pi}{2}$

$\because$ Angles A, B, C are in A.P

$\Rightarrow \quad 2B = A + C \quad$ —(1)

$\because$ In a triangle $A + B + C = 180^\circ$ —(2)

From (1) & (2) $\Rightarrow \quad 3B = 180^\circ \Rightarrow \boxed{B = 60^\circ}$

From sine rule :-

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

$\because \quad \frac{b}{c} = \sqrt{\frac{3}{2}} \Rightarrow \frac{\sin B}{\sin C} = \sqrt{\frac{3}{2}}$

$\Rightarrow \frac{\sqrt{3}}{2\sin C} = \sqrt{\frac{3}{2}} \quad \left(\because B = 60^\circ\right)$

$\Rightarrow \sin C = \frac{1}{\sqrt{2}}$

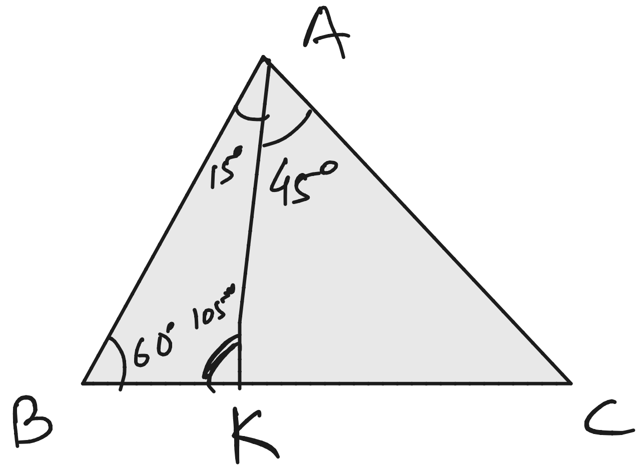
$\Rightarrow \angle C = \pi/4$

$\because \quad A + B + C = \pi$

$\Rightarrow \quad A + \frac{\pi}{3} + \frac{\pi}{4} = \pi \quad \Rightarrow \angle A = \frac{5\pi}{12}$

2. If K is a point on the side BC of an equilateral triangle ABC and if $\angle BAK = 15°$, then the ratio of lengths $\frac{AK}{AB}$ is

(A) $\frac{3\sqrt{2}(3+\sqrt{3})}{2}$ (B) $\frac{\sqrt{2}(3+\sqrt{3})}{2}$ (C) $\frac{\sqrt{2}(3-\sqrt{3})}{2}$ (D) $\frac{3\sqrt{2}(3-\sqrt{3})}{2}$



$\angle AKB = 105°$

Applying sine rule in triangle AKB:-

$$\frac{AB}{\sin 105°} = \frac{AK}{\sin 60°}$$

$$\Rightarrow \frac{AK}{AB} = \frac{\sin 60°}{\sin 105°}$$

$$\left(\because \sin 105° = \frac{\sqrt{3}+1}{2\sqrt{2}} \ \& \ \sin 60° = \frac{\sqrt{3}}{2}\right)$$

$$\Rightarrow \frac{AK}{AB} = \frac{\sqrt{3}(2\sqrt{2})}{2(\sqrt{3}+1)} = \frac{\sqrt{6}}{\sqrt{3}+1} \times \frac{\sqrt{3}-1}{\sqrt{3}-1}$$

$$\Rightarrow \frac{AK}{AB} = \frac{\sqrt{6}(\sqrt{3}-1)}{2} = \frac{\sqrt{2}(3-\sqrt{3})}{2}$$

3. 14. In a triangle ABC, $\angle A = 60°$ and $b : c = \left(\sqrt{3}+1\right) : 2$ then $(\angle B - \angle C)$ has the value equal to

(A) 15° (B) 30° (C) 22.5 ° (D) 45°

From napier's analogy :-

$$\tan\left(\frac{B-C}{2}\right) = \frac{b-c}{b+c}\cot A/2 \quad \text{— } ①$$

Given : $b : c$

$\sqrt{3}+1 : 2$

Let $b = (\sqrt{3}+1)k$ and $c = 2k$

$\angle A = 60°$

putting these values in ① $\Rightarrow$

$$\tan\left(\frac{B-C}{2}\right) = \frac{(\sqrt{3}+1)k - 2k}{(\sqrt{3}+1)k + 2k}\cot\left(\frac{60°}{2}\right)$$

$$\tan\left(\frac{B-C}{2}\right) = \frac{\sqrt{3}-1}{\sqrt{3}+3}(\sqrt{3}) = \frac{\sqrt{3}-1}{\sqrt{3}+1} \times \frac{(\sqrt{3}-1)}{(\sqrt{3}-1)}$$

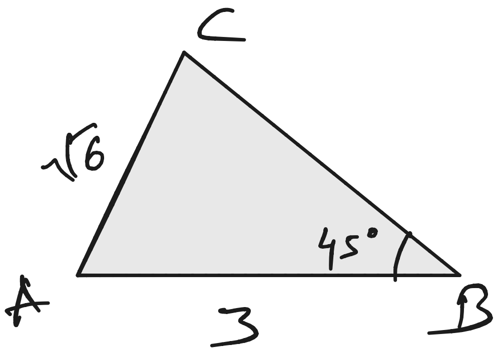
$$\Rightarrow \tan\left(\frac{B-C}{2}\right) = 2-\sqrt{3}$$

$$(\because \tan 15° = 2-\sqrt{3})$$

$$\Rightarrow \frac{B-C}{2} = 15° \Rightarrow \boxed{B - C = 30°}$$

9. In an acute triangle ABC, $\angle ABC = 45°$, AB = 3 and AC = $\sqrt{6}$. The angle $\angle BAC$, is

(A) 60° (B) 65° (C) 75° (D) 15° or 75°



From sine rule:-

$$\frac{AC}{\sin B} = \frac{AB}{\sin C} \Rightarrow \frac{\sqrt{6}}{\sin 45°} = \frac{3}{\sin C}$$

$$\Rightarrow \sqrt{12} = \frac{3}{\sin C} \Rightarrow \sin C = \frac{3}{2\sqrt{3}} = \frac{\sqrt{3}}{2}$$

$$\Rightarrow \angle C = 60°$$

$$\because A + B + C = 180°$$

$$\Rightarrow A + 45° + 60° = 180°$$

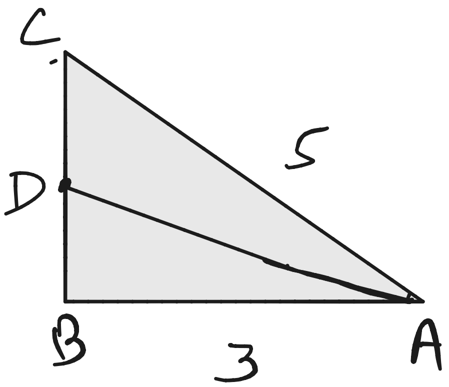
$$\Rightarrow \boxed{A = 75°}$$

3. Let ABC be a right triangle with length of side AB = 3 and hypotenuse AC = 5.

5

If D is a point on BC such that $\dfrac{BD}{DC} = \dfrac{AB}{AC}$, then AD is equal to

(A) $\dfrac{4\sqrt{3}}{3}$ ✓(B) $\dfrac{3\sqrt{5}}{2}$ (C) $\dfrac{4\sqrt{5}}{3}$ (D) $\dfrac{5\sqrt{3}}{4}$



$\because \quad AB^2 + BC^2 = AC^2$

$\Rightarrow \quad 9 + BC^2 = 25 \Rightarrow \boxed{BC = 4}$

Given that $\dfrac{BD}{DC} = \dfrac{AB}{AC} = \dfrac{3}{5}$

Let $BD = 3K$ & $DC = 5K$

$\because \quad BD + DC = BC = 4$

$\Rightarrow \quad 3K + 5K = 4 \Rightarrow K = \frac{1}{2}$

$\therefore \quad BD = 3/2$

In $\triangle ABD$:- $\quad AD^2 = AB^2 + BD^2$

$\Rightarrow \quad AD = \sqrt{9 + \frac{9}{4}} = \frac{\sqrt{45}}{2}$

Hence $AD = \dfrac{3\sqrt{5}}{2}$

6 18. In a triangle ABC, if a = 6, b = 3 and $\cos(A - B) = \frac{4}{5}$, the area of the triangle is

(A) 8 (B) 9 (C) 12 (D) $\frac{15}{2}$

Given $\cos(A-B) = 4/5$

$$\left(\because \cos\theta = \frac{1-\tan^2\theta/2}{1+\tan^2\theta/2}\right)$$

$$\Rightarrow \frac{1-\tan^2\left(\frac{A-B}{2}\right)}{1+\tan^2\left(\frac{A-B}{2}\right)} = \frac{4}{5}$$

$$\Rightarrow 5 - 5\tan^2\left(\frac{A-B}{2}\right) = 4 + 4\tan^2\left(\frac{A-B}{2}\right)$$

$$\Rightarrow \tan\left(\frac{A-B}{2}\right) = \frac{1}{3}$$

From Napier's Analogy :-

$$\tan\left(\frac{A-B}{2}\right) = \frac{a-b}{a+b}\cot C/2$$

$$\Rightarrow \frac{1}{3} = \frac{6-3}{6+3}\cot C/2$$

7. In ΔABC, if $a = 2b$ and $A = 3B$, then the value of $\frac{c}{b}$ is equal to

(A) 3 (B) $\sqrt{2}$ (C) 1 (D) $\sqrt{3}$

Given $a = 2b$ and $A = 3B$

From sine rule:- $\frac{a}{\sin A} = \frac{b}{\sin B}$

$$\Rightarrow \frac{2b}{\sin 3B} = \frac{b}{\sin B} \Rightarrow \frac{\sin 3B}{\sin B} = 2$$

$$\Rightarrow \frac{3\sin B - 4\sin^3 B}{\sin B} = 2 \quad (\because \sin 3\theta = 3\sin\theta - 4\sin^3\theta)$$

$$\Rightarrow 3 - 4\sin^2 B = 2 \Rightarrow \sin B = \frac{1}{2} \Rightarrow \angle B = 30^\circ$$

$$\therefore A = 3B = 90^\circ$$

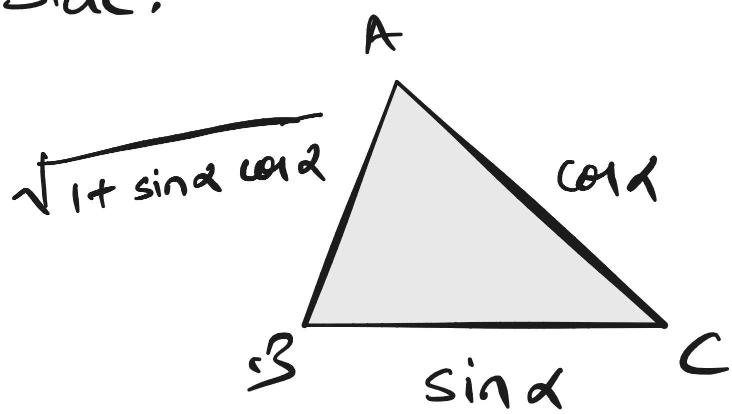
$$\because A + B + C = 180^\circ \Rightarrow \angle C = 60^\circ$$

$$\frac{c}{b} = \frac{\sin C}{\sin B} = \frac{\sin 60^\circ}{\sin 30^\circ} = \sqrt{3}$$

13. If the sides of a triangle are $\sin\alpha$, $\cos\alpha$, $\sqrt{1+\sin\alpha\cos\alpha}, 0<\alpha<\frac{\pi}{2}$, the largest angle is

(A) 60° (B) 90° (C) 120° (D) 150°

$\because$ Largest angle lies opposite to largest side.



Let $a=\sin\alpha$, $b=\cos\alpha$ & $c=\sqrt{1+\sin\alpha\cos\alpha}$

Largest side $=c \Rightarrow \angle C$ is largest angle

$$\cos C = \frac{a^2+b^2-c^2}{2ab}$$

$$\cos C = \frac{\sin^2\alpha+\cos^2\alpha-(1+\sin\alpha\cos\alpha)}{2(\sin\alpha)(\cos\alpha)}$$

$\Rightarrow \cos C = -\frac{1}{2} \Rightarrow \boxed{C=\frac{2\pi}{3}}$

5. If the angle A, B and C of a triangle are in an arithmetic progression and if a, b and c denote the lengths of the sides opposite to A, B and C respectively, then the value of expression $E = \left(\frac{a}{c}\sin 2C + \frac{c}{a}\sin 2A\right)$, is

(A) $\frac{1}{2}$ (B) $\frac{\sqrt{3}}{2}$ (C) 1 (D) $\sqrt{3}$

Given : A, B, C are in A.P. $\Rightarrow 2B = A + C$ — (1)

$\because A + B + C = 180^\circ$ — (2)

From (1) & (2) $\Rightarrow 3B = 180^\circ \Rightarrow \boxed{B = 60^\circ}$

$$E = \frac{a}{c}\sin 2C + \frac{c}{a}\sin 2A$$

$$= \frac{a}{c}\, 2\sin C\cos C + \frac{c}{a}(2\sin A\cos A)$$

From sine rule $\frac{c}{\sin C} = \frac{a}{\sin A}$

$$\Rightarrow E = 2\left(a\frac{\sin C}{c}\cos C + c\frac{\sin A}{a}\cos A\right)$$

$$= 2\left(a\frac{\sin A}{a}\cos C + c\frac{\sin C}{c}\cos A\right)$$

$$= 2(\sin A\cos C + \cos A\sin C)$$

$$E = 2\sin(A+C) = 2\sin B = 2\left(\frac{\sqrt{3}}{2}\right)$$

$$\Rightarrow \boxed{E = \sqrt{3}}$$

7. If in a triangle $\sin A : \sin C = \sin(A - B) : \sin(B - C)$, then a^2, b^2, c^2

(A) are in A.P. (B) are in G.P. (C) are in H.P. (D) none of these

Given:- $$\frac{\sin A}{\sin C} = \frac{\sin(A-B)}{\sin(B-C)}$$

$$\left(\because \sin A = \sin(\pi - (B+C)) = \sin(B+C)\right)$$

$$\Rightarrow \frac{\sin(B+C)}{\sin(A+B)} = \frac{\sin(A-B)}{\sin(B-C)}$$

$$\Rightarrow \sin(B+C) . \sin(B-C) = \sin(A-B)\sin(A+B)$$

$$\left(\because \sin(A+B)\sin(A-B) = \sin^2 A - \sin^2 B\right)$$

$$\Rightarrow \sin^2 B - \sin^2 C = \sin^2 A - \sin^2 B$$

$$\Rightarrow 2\sin^2 B = \sin^2 A + \sin^2 C$$

$$\because \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = K$$

$$\Rightarrow 2\left(\frac{b}{K}\right)^2 = \left(\frac{a}{K}\right)^2 + \left(\frac{c}{K}\right)^2$$

$$\Rightarrow 2b^2 = a^2 + c^2$$

Hence a^2, b^2, c^2 are in A.P.

16. In triangle ABC, if $\cot\frac{A}{2} = \frac{b+c}{a}$, then triangle ABC must be

[Note: All symbols used have usual meaning in ΔABC.]

(A) isosceles (B) equilateral (C) right angled (D) isoceles right angled

Given:- $\cot A/2 = \dfrac{b+c}{a}$

From sine rule:- $b = k\sin B$, $c = k\sin C$

& $a = k\sin A$

$$\cot A/2 = \frac{k(\sin B + \sin C)}{k \sin A}$$

$$\frac{\cos A/2}{\sin A/2} = \frac{2\sin\left(\frac{B+C}{2}\right) \cdot \cos\left(\frac{B-C}{2}\right)}{2\sin A/2 \cdot \cos A/2}$$

$$(\because B + C = \pi - A)$$

$$\cancel{2}\cos^2 A/2 = \cancel{2}\sin\left(\frac{\pi - A}{2}\right)\cos\left(\frac{B-C}{2}\right)$$

$$\cos^2 A/2 = \cos A/2 \cos\left(\frac{B-C}{2}\right)$$

$$\Rightarrow \cos A/2 = \cos\left(\frac{B-C}{2}\right)$$

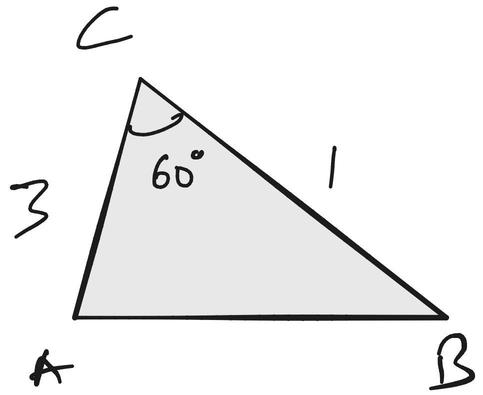
$$\Rightarrow A = B - C \Rightarrow A + C = B$$

$$\because A + B + C = 180^\circ \Rightarrow 2B = 180^\circ$$

$$\Rightarrow B = 90^\circ$$

10. Consider a triangle ABC and let a, b and c denote the lengths of the sides opposite to vertices A, B and C respectively. If a = 1, b = 3 and C = 60°, then $\sin^2 B$ is equal to

(A) $\frac{27}{28}$ (B) $\frac{3}{28}$ (C) $\frac{81}{28}$ (D) $\frac{1}{3}$



AB = c

From cosine rule :- $\cos C = \dfrac{a^2+b^2-c^2}{2ab}$

$$\Rightarrow \cos 60° = \frac{(1)^2+(3)^2-c^2}{2(1)(3)}$$

$$\Rightarrow 3 = 1+9-c^2 \Rightarrow c^2 = 7 \Rightarrow \boxed{c=\sqrt{7}}$$

From sine rule :-

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

$$\Rightarrow \frac{3}{\sin B} = \frac{\sqrt{7}}{\sin 60°} \Rightarrow \sin B = \frac{3\sqrt{3}}{2\sqrt{7}}$$

$$\Rightarrow \sin^2 B = \frac{27}{28}$$

Q.3 The ratio of the sides of a triangle ABC is $1:\sqrt{3}:2$. Then ratio of A : B : C is

(A) 3 : 5 : 2 (B) $1:\sqrt{3}:2$ (C) 3 : 2 : 1 (D) 1 : 2 : 3

Given $a : b : c$

$1 : \sqrt{3} : 2$

Let $a = K,\ b = \sqrt{3}K,\ c = 2K$

By cosine rule: $\cos A = \dfrac{b^2+c^2-a^2}{2bc} = \dfrac{3K^2+4K^2-K^2}{2(\sqrt{3}K)(2K)}$

$\Rightarrow \cos A = \dfrac{6K^2}{4\sqrt{3}K^2} = \dfrac{\sqrt{3}}{2} \Rightarrow \boxed{\angle A = 30^\circ}$

Similarly $\cos B = \dfrac{a^2+c^2-b^2}{2ac} = \dfrac{K^2+4K^2-3K^2}{2(K)(2K)}$

$\Rightarrow \cos B = \dfrac{1}{2} \Rightarrow \angle B = 60^\circ$

$\because \angle A + \angle B + \angle C = 180^\circ$

$\Rightarrow \angle C = 90^\circ$

$\Rightarrow A : B : C = 30^\circ : 60^\circ : 90^\circ$

$= 1 : 2 : 3$

11. 14 In triangle ABC, If $s = 3+\sqrt{3}+\sqrt{2}$, $3B - C = 30°$, $A + 2B = 120°$, then the length of longest side of triangle is

[**Note:** All symbols used have usual meaning in triangle ABC.]

(A) 2 (B) $2\sqrt{2}$ (C) $2(\sqrt{3}+1)$ (D) $\sqrt{3}-1$

Given:- $3B - C = 30° \longrightarrow (1)$

$A + 2B = 120°$

$\because A + B + C = 180°$

$\underbrace{A+B}_{\downarrow} + B = 120°$

$180° - C + B = 120° \Rightarrow C - B = 60° \longrightarrow (2)$

$(1) + (2) \Rightarrow 2B = 90° \Rightarrow B = 45°$

$\Rightarrow A = 30° \;\&\; C = 105°$

From sine rule:-

$$\frac{a}{\sin 30°} = \frac{b}{\sin 45°} = \frac{c}{\sin 105°} = k$$

$$\Rightarrow a = \frac{k}{2}, \quad b = \frac{k}{\sqrt{2}} \;\&\; c = \frac{k(\sqrt{3}+1)}{2\sqrt{2}}$$

Given $S = 3+\sqrt{3}+\sqrt{2} = \dfrac{a+b+c}{2}$

$$\Rightarrow a+b+c = \frac{k}{2} + \frac{k}{\sqrt{2}} + \frac{k(\sqrt{3}+1)}{2\sqrt{2}} = 6+2\sqrt{3}+2\sqrt{2}$$

$$k\sqrt{2} + 2k + k(\sqrt{3}+1) = 2\sqrt{2}(6+2\sqrt{3}+2\sqrt{2})$$

$$k\sqrt{2} + 3k + \sqrt{3}k = 12\sqrt{2} + 4\sqrt{6} + 8$$

$$k(\sqrt{2}+3+\sqrt{3}) = 4\sqrt{2}(3+\sqrt{3}+\sqrt{2})$$

$\Rightarrow K = 4\sqrt{2}$

$\because$ Largest angle is $C \Rightarrow$

Largest side $= c = \dfrac{K(\sqrt{3}+1)}{2\sqrt{2}}$

$$c = \frac{4\sqrt{2}(\sqrt{3}+1)}{2\sqrt{2}}$$

$$\Rightarrow \boxed{c = 2(\sqrt{3}+1)}$$

1.5

8. In a triangle tan A : tan B : tan C = 1 : 2 : 3, then $a^2 : b^2 : c^2$ equals

(A) 5 : 8 : 9 (B) 5 : 8 : 12 (C) 3 : 5 : 8 (D) 5 : 8 : 10

Given $\tan A : \tan B : \tan C$

$1 : 2 : 3$

Let $\tan A = K$, $\tan B = 2K$ & $\tan C = 3K$

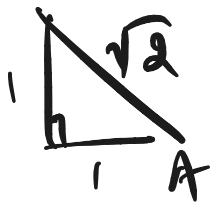
We know that in a triangle

$$\tan A + \tan B + \tan C = \tan A \cdot \tan B \cdot \tan C$$

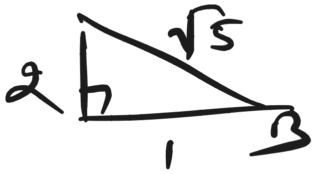
$$\Rightarrow \quad K + 2K + 3K = (K)(2K)(3K)$$

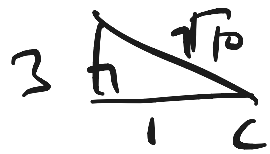
$$\Rightarrow \quad K = 1$$

$\Rightarrow \quad \tan A = 1$, $\tan B = 2$ & $\tan C = 3$



$\sin A = \dfrac{1}{\sqrt{2}}$, $\sin B = 2/\sqrt{5}$ & $\sin C = \dfrac{3}{\sqrt{10}}$





$\because \quad a^2 : b^2 : c^2 = \sin^2 A : \sin^2 B : \sin^2 C$

(From sine rule)

$$\Rightarrow \quad a^2 : b^2 : c^2 = \frac{1}{2} : \frac{4}{5} : \frac{9}{10} = 5 : 8 : 9$$

~~17.~~ 16 In ΔABC, if a,b,c (taken in that order) are in A.P. then $\cot\frac{A}{2}\ \cot\frac{C}{2} =$

[**Note:** All symbols used have usual meaning in triangle ABC.]

(A) 1 (B) 2 (C) 3 (D) 4

Given a, b, c are in A.P.

$$\Rightarrow \quad 2b = a + c$$

$$\because \cot\frac{A}{2} = \sqrt{\frac{s(s-a)}{(s-b)(s-c)}}$$

$$\& \cot\frac{C}{2} = \sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$$

$$\cot\frac{A}{2}\cdot\cot\frac{C}{2} = \sqrt{\frac{s(s-a)}{(s-b)(s-c)}}\sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$$

$$\cot\frac{A}{2}\cdot\cot\frac{C}{2} = \frac{s}{s-b} \quad \text{--- (1)}$$

$$\because 2b = a + c$$

$$\& \quad s = \frac{a+b+c}{2} = \frac{3b}{2} \quad \text{--- (2)}$$

From (1) & (2) $\Rightarrow \cot\frac{A}{2}\cdot\cot\frac{C}{2} = \frac{3b/2}{\frac{3b}{2} - b}$

$$\boxed{\cot\frac{A}{2}\cdot\cot\frac{C}{2} = 3}$$

12. In ΔABC if a = 8, b = 9, c = 10, then the value of $\frac{\tan C}{\sin B}$ is

17.

(A) $\frac{32}{9}$ (B) $\frac{24}{7}$ (C) $\frac{21}{4}$ (D) $\frac{18}{5}$

$$\frac{\tan C}{\sin B} = \frac{\sin C}{\cos C \sin B} \quad \text{--- } (1)$$

From sine rule :-
$$\frac{\sin C}{\sin B} = \frac{c}{b} = \frac{10}{9} \quad \text{--- } (2)$$

From cosine rule :-

$$\cos C = \frac{a^2+b^2-c^2}{2ab}$$

$$\cos C = \frac{64+81-100}{2(8)(9)} = \frac{145-100}{2(8)(9)}$$

$$\Rightarrow \cos C = \frac{\cancel{45}\,5}{2(8)(\cancel{9})} = \frac{5}{16} \quad \text{--- } (3)$$

Putting these values in eqn (1)

$$\frac{\sin C}{\cos C \sin B} = \frac{10 \cdot 16}{9 \cdot 5} = \frac{32}{9}$$

R rVv Vv vvvvvcvvvv r

15\. In triangle ABC, if $\Delta = a^2 - (b - c)^2$, then $\tan A =$

[**Note:** All symbols used have usual meaning in triangle ABC.]

(A) $\frac{15}{16}$ (B) $\frac{1}{2}$ (C) $\frac{8}{17}$ (D) $\frac{8}{15}$

Given $\Delta = a^2 - (b-c)^2$

$$\Rightarrow \Delta = (a-b+c)(a+b-c)$$

$$\sqrt{s(s-a)(s-b)(s-c)} = (2s-2b)(2s-2c)$$

$$(\because a-b+c = 2s-2b)$$

$$\sqrt{s(s-a)(s-b)(s-c)} = 4(s-b)(s-c)$$

$$\Rightarrow 1 = 4\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$$

$$\left(\because \tan A/2 = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}}\right)$$

$$\Rightarrow 1 = 4\tan A/2 \Rightarrow \tan A/2 = 1/4$$

$$\tan A = \frac{2\tan A/2}{1-\tan^2 A/2} = \frac{2(1/4)}{1-1/16} = \frac{8}{15}$$

~~20.~~ 19. In a triangle ABC, if the sides a, b, c are roots of $x^3 - 11x^2 + 38x - 40 = 0$. If $\sum\left(\frac{\cos A}{a}\right) = \frac{p}{q}$, then find the least value of $(p + q)$ where $p, q \in N$.

$$\frac{\cos A}{a} + \frac{\cos B}{b} + \frac{\cos C}{c}$$

$$= \frac{b^2+c^2-a^2}{(2bc)a} + \frac{a^2+c^2-b^2}{(2ac)b} + \frac{a^2+b^2-c^2}{(2ab)c}$$

(From cosine rule)

$$\Rightarrow \sum\left(\frac{\cos A}{a}\right) = \frac{a^2+b^2+c^2}{2abc} \quad \text{— } (1)$$

$$\because \quad x^3 - 11x^2 + 38x - 40 = 0 \rightarrow a, b, c$$

$a+b+c = 11$ & $ab+bc+ca = 38$

$$\Rightarrow a^2+b^2+c^2+2(ab+bc+ca) = 121$$

$$\Rightarrow a^2+b^2+c^2 = 121-2(38) = 45$$

& $abc = 40$

Putting these values in 1 $\Rightarrow$

$$\sum \frac{\cot A}{a} = \frac{45}{2(40)} = \frac{9}{16} = \frac{p}{q}$$

$$\Rightarrow \quad p+q = 25$$

20. 19. ABC is a triangle such that $\sin(2A + B) = \sin(C - A) = -\sin(B + 2C) = \frac{1}{2}$. If A, B, C are in A.P., find A, B, C.

Given A, B, C are in A.P.

$\Rightarrow \quad 2B = A + C$

$\because$ In a Δ :- $\quad A + B + C = 180^\circ$

$\Rightarrow \quad 3B = 180^\circ$

$\Rightarrow \quad \boxed{B = 60^\circ}$

$\because \quad \sin(2A + B) = \frac{1}{2} = \sin(150^\circ)$

$\Rightarrow \quad 2A + B = 150^\circ$

$\Rightarrow \quad 2A = 150^\circ - 60^\circ = 90^\circ$

$\Rightarrow \quad \boxed{A = 45^\circ}$

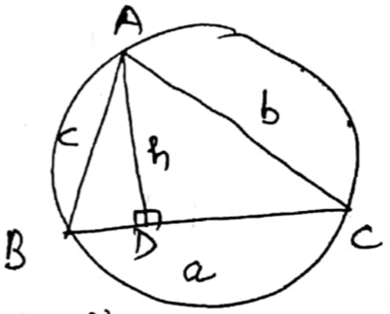
Hence $\quad C = 180^\circ - (45^\circ + 60^\circ)$

$\boxed{C = 75^\circ}$

EXERCISE (O-1)

1. A triangle has vertices A, B and C, and the respective opposite sides have lengths a, b and c. This triangle is inscribed in a circle of radius R. If b = c = 1 and the altitude from A to side BC has length $\sqrt{\frac{2}{3}}$, then R equals

(A) $\frac{1}{\sqrt{3}}$ (B) $\frac{2}{\sqrt{3}}$ (C) $\frac{\sqrt{3}}{2}$ (D) $\frac{\sqrt{3}}{2\sqrt{2}}$



Solution:

AD = altitude = h

Now using the concept of area.

$ar\,\Delta ABC = \frac{1}{2} \times \text{Base} \times h = \frac{1}{2}ah$ —— (1)

Also $ar\,\Delta ABC = \frac{1}{2}bc\sin A = \frac{1}{2}bc \cdot \frac{a}{2R}$ $\left\{\frac{a}{\sin A} = 2R\right\}$

$ar\,\Delta ABC = \frac{abc}{4R}$ —— (2)

$\frac{1}{2}ah = \frac{abc}{4R}$

$h = \frac{bc}{2R}$ or $R = \frac{bc}{2h} = \frac{1 \times 1}{2 \cdot \sqrt{\frac{2}{3}}} = \frac{\sqrt{3}}{2\sqrt{2}}$

(Hence D)

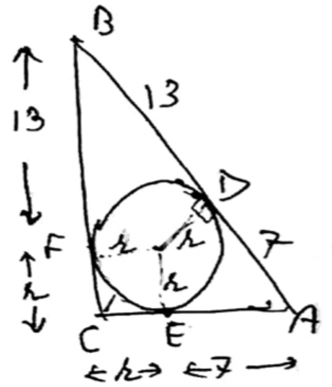
2.

A circle is inscribed in a right triangle ABC, right angled at C. The circle is tangent to the segment AB at D and length of segments AD and DB are 7 and 13 respectively. Area of triangle ABC is equal to

(A) 91 (B) 96 (C) 100 (D) 104

Solution:

V. Imp :



$BF = BD = 13$

$AE = AD = 7$

let $CE = CF = r$

Area of $\Delta ABC = \frac{1}{2} AC \times BC$

$= \frac{1}{2}(7+r)(13+r)$

ar $\Delta ABC = \frac{1}{2}[r^2 + 20r + 91]$ —— (1)

Also $(r+7)^2 + (13+r)^2 = 20^2$

$2r^2 + 40r + 218 = 400$

$2r^2 + 40r = 182$

$r^2 + 20r = 91$ —— (2)

put (2) in (1)

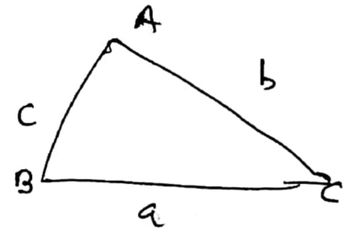
ar $\Delta ABC = \frac{1}{2}[91 + 91] = 91$ (Hence A)

3 In a triangle ABC, if a = 13, b = 14 and c = 15, then angle A is equal to

(All symbols used have their usual meaning in a triangle.)

(A) $\sin^{-1}\frac{4}{5}$ (B) $\sin^{-1}\frac{3}{5}$ (C) $\sin^{-1}\frac{3}{4}$ (D) $\sin^{-1}\frac{2}{3}$

Solution: $a = 13,\ b = 14,\ c = 15$



area of $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$

$s = \frac{13+14+15}{2} = \frac{42}{2} = 21$

area $= \sqrt{21(8)(7)(6)} = 84$

Now $\frac{1}{2} bc \sin A = 84$

$\frac{1}{2} \times 14 \times 15 \cdot \sin A = 84$

$\sin A = \frac{84}{7 \times 15} = \frac{12}{15} = \frac{4}{5}$

$A = \sin^{-1}\left(\frac{4}{5}\right)$ (Hence A)

4. In a triangle ABC, if $b = (\sqrt{3}-1)\,a$ and $\angle C = 30°$, then the value of (A – B) is equal to

(All symbols used have usual meaning in a triangle.)

(A) 30° (B) 45° (C) 60° (D) 75°

Solution: $b = (\sqrt{3}-1)\,a$ and $\angle C = 30°$

$$\frac{a}{b} = \frac{1}{\sqrt{3}-1}$$

$$\frac{a-b}{a+b} = \frac{1-(\sqrt{3}-1)}{1+\sqrt{3}-1} = \frac{2-\sqrt{3}}{\sqrt{3}}$$

Now using tangent Rule

$$\tan\left(\frac{A-B}{2}\right) = \frac{a-b}{a+b}\cot\frac{C}{2}$$

$$\tan\left(\frac{A-B}{2}\right) = \frac{2-\sqrt{3}}{\sqrt{3}}.\cot\left(\frac{30°}{2}\right)$$

$$\tan\left(\frac{A-B}{2}\right) = \frac{2-\sqrt{3}}{\sqrt{3}} \times 2+\sqrt{3} = \frac{1}{\sqrt{3}}$$

$$\frac{A-B}{2} = 30° \text{ OR } A-B = 60°$$

(Hence C)

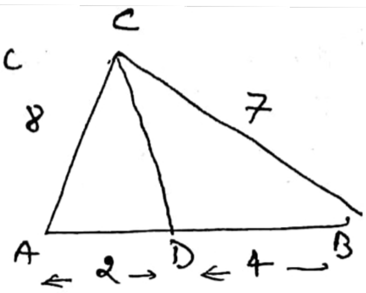
5. In triangle ABC, if AC = 8, BC = 7 and D lies between A and B such that AD = 2, BD = 4, then the length CD equals

(A) $\sqrt{46}$ (B) $\sqrt{48}$ (C) $\sqrt{51}$ (D) $\sqrt{75}$

Solution:

using cosine rule in $\triangle ABC$

$$\cos A = \frac{8^2+6^2-7^2}{2\cdot 8\cdot 6} = \frac{51}{2\cdot 8\cdot 6} \quad \text{—(1)}$$



cosine rule in $\triangle ADC$

$$\cos A = \frac{8^2+2^2-CD^2}{2\cdot 8\cdot 2} \quad \text{—(2)}$$

from (1) and (2)

$$\frac{8^2+2^2-CD^2}{2\cdot 8\cdot 2} = \frac{51}{2\cdot 8\cdot 6}$$

$$68 - CD^2 = 17$$

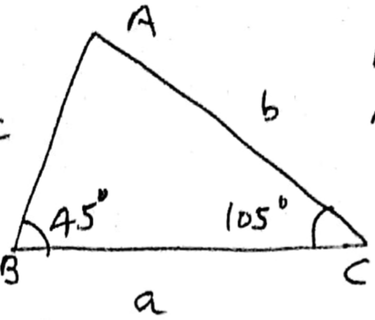
$CD^2 = 51$ or $CD = \sqrt{51}$ (hence C)

6 In a triangle ABC, if $\angle C = 105°$, $\angle B = 45°$ and length of side AC = 2 units, then the length of the side AB is equal to

(A) $\sqrt{2}$ (B) $\sqrt{3}$ (C) $\sqrt{2}+1$ (D) $\sqrt{3}+1$

solution:

using sine Rule



$AB = c$

$AC = b = 2$

$$\frac{c}{\sin 105°} = \frac{b}{\sin 45°}$$

$$c = \frac{b \cdot \sin 105°}{\sin 45°} = \frac{2 \cos 15°}{\sin 45°}$$

$$c = 2 \cdot \frac{(\sqrt{3}+1)}{2\sqrt{2}} \times \frac{\sqrt{2}}{1} = \sqrt{3}+1 \quad (D)$$

7

[redacted] In a triangle ABC, if $(a + b + c)(a + b - c)(b + c - a)(c + a - b) = \dfrac{8a^2b^2c^2}{a^2 + b^2 + c^2}$, then the triangle is

[**Note:** All symbols used have usual meaning in triangle ABC.]

(A) isosceles (B) right angled (C) equilateral (D) obtuse angled

Solution:

$a+b+c = 2s$

$a+b-c = 2s-2c$; $b+c-a = 2s-2a$; $c+a-b = 2s-2b$

Now :

$(a+b+c)(a+b-c)(b+c-a)(c+a-b) = \dfrac{8a^2b^2c^2}{a^2+b^2+c^2}$

$2s(2s-2c)(2s-2a)(2s-2b) = \dfrac{8a^2b^2c^2}{a^2+b^2+c^2}$

$16 \cdot s(s-a)(s-b)(s-c) = \dfrac{8a^2b^2c^2}{a^2+b^2+c^2}$

$16\Delta^2 = \dfrac{8a^2b^2c^2}{a^2+b^2+c^2}$ $\left\{ R = \dfrac{abc}{4\Delta} \right.$

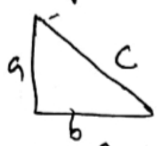
$a^2+b^2+c^2 = 8\left(\dfrac{abc}{4\Delta}\right)^2$

$a^2+b^2+c^2 = 8R^2$

and Thus triangle is Right triangle

(Hence B)

$\left\{ \right.$ If $a^2+b^2+c^2 = 8R^2$ triangle is Right angled Δ



$\dfrac{c}{\sin 90^\circ} = 2R$

$c = 2R$

$a^2+b^2 = c^2$

$a^2+b^2+c^2 = 2c^2$

$= 2(2R)^2$

$a^2+b^2+c^2 = 8R^2$

8

9. In triangle ABC, if 2b = a + c and A – C = 90°, then sin B equals
[**Note:** All symbols used have usual meaning in triangle ABC.]

(A) $\frac{\sqrt{7}}{5}$ (B) $\frac{\sqrt{5}}{8}$ (C) $\frac{\sqrt{7}}{4}$ (D) $\frac{\sqrt{5}}{3}$

sol: $2b = a + c$ and $A - C = 90°$

using sine rule $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R$

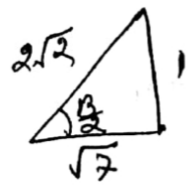
$$2(2R\sin B) = 2R\sin A + 2R\sin C$$

$$2(2R\sin B) = 2R\left[2\sin\left(\frac{A+C}{2}\right)\cos\left(\frac{A-C}{2}\right)\right]$$

$$\sin B = \sin\left(90-\frac{B}{2}\right)\cos\left(\frac{90°}{2}\right)$$

$$2\sin\frac{B}{2}\cos\frac{B}{2} = \cos\frac{B}{2}\cdot\frac{1}{\sqrt{2}} \quad \left[\cos\frac{B}{2} \neq 0\right]$$

$$\sin\frac{B}{2} = \frac{1}{2\sqrt{2}}$$



$$\sin B = 2\sin\frac{B}{2}\cos\frac{B}{2}$$

$$= 2\cdot\frac{1}{2\sqrt{2}}\times\frac{\sqrt{7}}{2\sqrt{2}} = \frac{\sqrt{7}}{4} \quad (\text{Hence C})$$

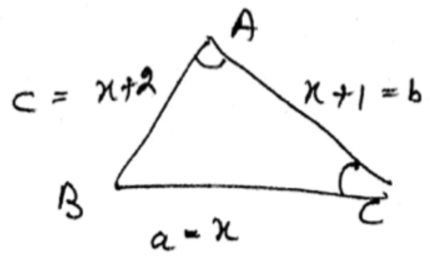
Q. The sides of a triangle are three consecutive integers. The largest angle is twice the smallest one. The area of triangle is equal to

(A) $\frac{5}{4}\sqrt{7}$ (B) $\frac{15}{2}\sqrt{7}$ (C) $\frac{15}{4}\sqrt{7}$ (D) $5\sqrt{7}$

Solution:

Let the sides $a = x$, $b = x+1$, $c = x+2$

$\angle C = 2\angle A$



using sine rule

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

$$\frac{x}{\sin A} = \frac{x+2}{\sin C}$$

OR $$\frac{\sin C}{\sin A} = \frac{x+2}{x}$$

$$\frac{\sin 2A}{\sin A} = \frac{x+2}{x}$$

$$2\cos A = \frac{x+2}{x} \quad \text{——— (1)}$$

using cosine Rule:

$$\cos A = \frac{(x+2)^2 + (x+1)^2 - x^2}{2(x+1)(x+2)}$$

$$\cos A \Rightarrow \frac{x^2+6x+5}{2(x^2+3x+2)} \quad \text{——— (2)}$$

from (1) and (2)

$$\frac{x^2+6x+5}{x^2+3x+2} = \frac{x+2}{x}$$

on solving $x^2 - 3x - 4 = 0$

$(x-4)(x+1) = 0$

$x = 4$

sides of triangle 4, 5, 6 $s = \frac{4+5+6}{2} = \frac{15}{2}$

$$\text{Area} = \sqrt{\tfrac{15}{2}\left(\tfrac{15}{2}-4\right)\left(\tfrac{15}{2}-5\right)\left(\tfrac{15}{2}-6\right)} = \frac{15\sqrt{7}}{4} \text{ (C)}$$

(Hence C)

10

~~12.~~ The sides a, b, c (taken in that order) of triangle ABC are in A.P.

If $\cos\alpha = \frac{a}{b+c}, \cos\beta = \frac{b}{c+a}, \cos\gamma = \frac{c}{a+b}$ then $\tan^2\left(\frac{\alpha}{2}\right) + \tan^2\left(\frac{\gamma}{2}\right)$ is equal to

[**Note:** All symbols used have usual meaning in triangle ABC.]

(A) 1 (B) $\frac{1}{2}$ (C) $\frac{1}{3}$ (D) $\frac{2}{3}$

Solution:

$\cos\alpha = \frac{a}{b+c}$, $\cos\beta = \frac{b}{c+a}$; $\cos\gamma = \frac{c}{a+b}$

To find : $\tan^2\frac{\alpha}{2} + \tan^2\frac{\gamma}{2}$

Also a, b c are in A.P

$\therefore 2b = a+c$

$$\tan^2\frac{\alpha}{2} + \tan^2\frac{\gamma}{2}$$

$$\Rightarrow \frac{1-\cos\alpha}{1+\cos\alpha} + \frac{1-\cos\gamma}{1+\cos\gamma}$$

$$= \frac{1-\frac{a}{b+c}}{1+\frac{a}{b+c}} + \frac{1-\frac{c}{a+b}}{1+\frac{c}{a+b}}$$

$$\Rightarrow \frac{b+c-a}{a+b+c} + \frac{a+b-c}{a+b+c}$$

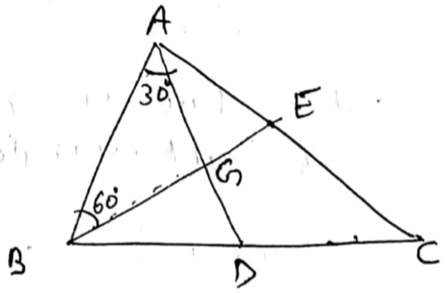
$$\Rightarrow \frac{2b}{a+b+c} = \frac{2b}{b+2b} = \frac{2}{3} \quad \{a+c=2b\}$$

Hence D

12. AD and BE are the medians of a triangle ABC. If AD = 4, $\angle DAB = \frac{\pi}{6}, \angle ABE = \frac{\pi}{3}$, then area of triangle ABC equals

(A) $\frac{8}{3}$ (B) $\frac{16}{3}$ (C) $\frac{32}{3}$ (D) $\frac{32}{9}\sqrt{3}$

Solution:



given $AD = 4$

$\angle DAB = \frac{\pi}{6} = 30°$

$\angle ABE = \frac{\pi}{3} = 60°$

Let medians intersect in G

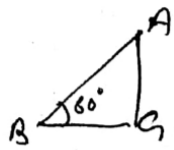
Thus : $AG = \frac{2}{3} AD = \frac{2}{3} \times 4 = \frac{8}{3}$

$AG = \frac{8}{3}$ —— (1)

Also in $\triangle AGB$ ∴ $\angle GAB + \angle ABG + \angle AGB = 180°$

$30° + 60° + \angle AGB = 180°$

$\boxed{\angle AGB = 90°}$



Thus $\triangle AGB$ = Right angled Triangle

$\frac{BG}{AG} = \cot 60°$

$BG = \frac{8}{3} \times \frac{1}{\sqrt{3}} = \frac{8}{3\sqrt{3}}$

ar $\triangle ABC = 3$ ar $\triangle AGB = 3\left[\frac{1}{2} \cdot BG \cdot AG\right] = \frac{\cancel{3}}{2} \cdot \frac{8}{3\sqrt{3}} \times \frac{8}{3} = \frac{32\sqrt{3}}{9}$

12. In triangle ABC, if $\sin^3 A + \sin^3 B + \sin^3 C = 3\sin A.\sin B.\sin C$, then triangle is

(A) obtuse angled (B) right angled (C) obtuse right angled (D) equilateral

Soln. We have

$x^3 + y^3 + z^3 - 3xyz = 0$

$\Rightarrow$ Either $x + y + z = 0$ OR $x = y = z$

$\sin A = \sin B = \sin C \Rightarrow$ (D)

$\sin A + \sin B + \sin C = 0$

$\Rightarrow \cos A/2 \cos B/2 \cos C/2 = 0$

Not possible.

13

~~15.~~ For right angled isosceles triangle, $\frac{r}{R} =$

[**Note:** All symbols used have usual meaning in triangle ABC.]

(A) $\tan\frac{\pi}{12}$ (B) $\cot\frac{\pi}{12}$ (C) $\tan\frac{\pi}{8}$ (D) $\cot\frac{\pi}{8}$

Solution:

$a+b+c = 2s$

$a+b-c = 2s-2c$; $b+c-a = 2s-2a$; $c+a-b = 2s-2b$

Now :

$(a+b+c)(a+b-c)(b+c-a)(c+a-b) = \frac{8a^2b^2c^2}{a^2+b^2+c^2}$

$2s(2s-2c)(2s-2a)(2s-2b) = \frac{8a^2b^2c^2}{a^2+b^2+c^2}$

$16 \cdot s(s-a)(s-b)(s-c) = \frac{8a^2b^2c^2}{a^2+b^2+c^2}$

$16\Delta^2 = \frac{8a^2b^2c^2}{a^2+b^2+c^2}$ $\left\{ R = \frac{abc}{4\Delta} \right.$

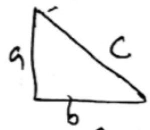
$a^2+b^2+c^2 = 8\left(\frac{abc}{4\Delta}\right)^2$

$a^2+b^2+c^2 = 8R^2$

and Thus triangle is Right triangle.

(Hence B)

$\left\{ \right.$ If $a^2+b^2+c^2 = 8R^2$ triangle is Right angled Δ



$\frac{c}{\sin 90^\circ} = 2R$

$c = 2R$

$a^2+b^2 = c^2$

$a^2+b^2+c^2 = 2c^2$

$= 2(2R)^2$

$a^2+b^2+c^2 = 8R^2$

14

~~9.~~ In triangle ABC, If $\dfrac{1}{a+c}+\dfrac{1}{b+c}=\dfrac{3}{a+b+c}$ then angle C is equal to

[**Note:** All symbols used have usual meaning in triangle ABC.]

(A) 30° (B) 45° (C) 60° (D) 90°

Sol^n:

$$\frac{1}{a+c}+\frac{1}{b+c}=\frac{3}{a+b+c}$$

$$\frac{a+b+c}{a+c}+\frac{a+b+c}{b+c}=3$$

$$1+\frac{b}{a+c}+\frac{a}{b+c}+1=3$$

$$\frac{b}{a+c}+\frac{a}{b+c}=1$$

$$b(b+c)+a(a+c)=(a+c)(b+c)$$

$$b^2+bc+a^2+ac=ab+ac+bc+c^2$$

$$b^2+a^2=ab+c^2$$

or $\dfrac{a^2+b^2-c^2}{2ab}=\dfrac{1}{2}$ (By transforming

$\cos C=\dfrac{1}{2}$ or $\angle C=60°$ (Hence C

1.

8. In a triangle ABC, let $2a^2 + 4b^2 + c^2 = 2a(2b + c)$, then which of the following holds good?
[**Note:** All symbols used have usual meaning in a triangle.]

(A) $\cos B = \frac{-7}{8}$ (B) $\sin(A - C) = 0$

(C) $\frac{r}{r_1} = \frac{1}{5}$ (D) $\sin A : \sin B : \sin C = 1 : 2 : 1$

$$2a^2 + 4b^2 + c^2 - 4ab - 2ac = 0$$

$$\Rightarrow (a - 2b)^2 + (a - c)^2 = 0 \Rightarrow a = 2b, \; a = c$$

$$\Rightarrow b = \frac{a}{2}, \; c = a \Rightarrow \angle C = \angle A$$

$$\Rightarrow A - C = 0 \Rightarrow \sin(A - C) = 0$$

$\Rightarrow$ (B) is correct

$$\text{(A)} \quad \cos B = \frac{a^2 + c^2 - b^2}{2ac} = \frac{a^2 + a^2 - \frac{a^2}{4}}{2(a).(a)} = \frac{7a^2}{8a^2} = \frac{7}{8}$$

$\Rightarrow$ (A) is not correct

$$\text{(C)} \quad \frac{r}{r_1} = \frac{(s-a)\tan\frac{A}{2}}{s\tan\frac{A}{2}} = 1 - \frac{a}{s} = 1 - \frac{a}{\frac{5a}{4}}$$

$$\left(\because s = \frac{a + \frac{a}{2} + a}{2}\right)$$

$$\Rightarrow \frac{r}{r_1} = \frac{1}{5}$$

$\Rightarrow$ (C) is correct.

(D) From sine law

$$\sin A : \sin B : \sin C = a : b : c$$
$$= a : \frac{a}{2} : a$$
$$= 1 : \frac{1}{2} : 1$$
$$= 2 : 1 : 2$$

$\Rightarrow$ (D) is not correct.

$\boxed{B, C}$

2

In a triangle ABC, if a = 4, b = 8 $\angle C = 60°$, then which of the following relations is (are) correct?
[**Note:** All symbols used have usual meaning in triangle ABC.]

(A) The area of triangle ABC is $8\sqrt{3}$

(B) The value of $\sum \sin^2 A = 2$

(C) Inradius of triangle ABC is $\frac{2\sqrt{3}}{3+\sqrt{3}}$

(D) The length of internal angle bisector of angle C is $\frac{4}{\sqrt{3}}$

Given $a = 4$, $b = 8$, $\angle C = 60°$

$$\cos C = \frac{1}{2} = \frac{16 + 64 - c^2}{2.(4).(8)} \Rightarrow c^2 = 48 \Rightarrow c = 4\sqrt{3}$$

Use Sine Rule $\frac{a}{\sin A} = \frac{c}{\sin C}$

$$\Rightarrow \frac{4}{\sin A} = \frac{4\sqrt{3}}{\sqrt{3}/2} \Rightarrow A = 30°$$

As $\angle C = 60°$, $\angle A = 30° \Rightarrow \angle B = 90°$

(A) Area of $\Delta ABC = \frac{1}{2} ac \sin B = \frac{1}{2}(4)(4\sqrt{3}) \sin 90°$

$$= 8\sqrt{3}$$

(B) $\sum \sin^2 A = \sin^2 A + \sin^2 B + \sin^2 C$

$$= \sin^2(30°) + \sin^2(90°) + \sin^2(60°)$$

$$= \frac{1}{4} + 1 + \frac{3}{4} = 2$$

(C) Now $r = \frac{\Delta}{s} = \frac{\frac{1}{2} ac \sin B}{\frac{a+b+c}{2}} = \frac{ac}{a+b+c}$

$$= \frac{4(4\sqrt{3})}{4 + 4\sqrt{3} + 8} = \frac{16\sqrt{3}}{12 + 4\sqrt{3}} = \frac{4\sqrt{3}}{3 + \sqrt{3}}$$

(D) Length of internal angle bisector of Angle C

$$= \frac{2ab \cos\left(\frac{C}{2}\right)}{a+b} = \frac{2(4)(8) \cos 30°}{4+8} = \frac{8}{\sqrt{3}}$$

$\boxed{A, B}$

3 In which of the following situations, it is possible to have a triangle ABC?
(All symbols used have usual meaning in a triangle.)

(A) $(a + c - b)(a - c + b) = 4bc$

(B) $b^2 \sin 2C + c^2 \sin 2B = ab$

(C) $a = 3, b = 5, c = 7$ and $C = \frac{2\pi}{3}$

(D) $\cos\left(\frac{A-C}{2}\right) = \cos\left(\frac{A+C}{2}\right)$

(A) $(a+c-b)(a-c+b) = 4bc$

$$(a+(c-b))(a-(c-b)) = 4bc$$

$$a^2 - (c-b)^2 = 4bc$$

$$a^2 - (c^2+b^2-2bc) = 4bc$$

$$a^2 = b^2+c^2+2bc$$

$$\Rightarrow b^2+c^2-a^2 = -2bc$$

$$\Rightarrow \frac{b^2+c^2-a^2}{2bc} = -1$$

$\Rightarrow \cos A = -1$ which is not possible.

(B) $b^2 \sin 2C + c^2 \sin 2B = ab$

$$\Rightarrow (4R^2\sin^2 B)2\sin C.\cos C + (4R^2\sin^2 C)2\sin B\cos B = ab$$

$$\Rightarrow 8R^2.\sin B.\sin C.[\sin B.\cos C + \sin C.\cos B] = (2R\sin A)(2R\sin B)$$

$$\Rightarrow 2\sin C.\ \sin\underbrace{(B+C)}_{\pi - A} = \sin A$$

$$\Rightarrow 2\sin C = 1$$

$$\Rightarrow \sin C = \frac{1}{2}$$

$\Rightarrow \angle C = \frac{\pi}{6}$ or $\frac{5\pi}{6}$ so Δ is possible.

(C) For $a=3$, $b=5$, $c=7$ and $C=\frac{2\pi}{3}$

$$\cos A = \frac{b^2+c^2-a^2}{2bc} = \frac{5^2+7^2-3^2}{2(5)(7)} = \frac{65}{70} < 1$$

Similarly $\cos B = \frac{a^2+c^2-b^2}{2ac} = \frac{3^2+7^2-5^2}{2(3)(7)} = \frac{33}{42} < 1$

So in this case triangle is possible.

(D) As, $\cos\left(\frac{A-C}{2}\right) = \cos\left(\frac{A+C}{2}\right)$

$$\Rightarrow \cos\left(\frac{A-C}{2}\right) - \cos\left(\frac{A+C}{2}\right) = 0$$

$$\Rightarrow 2\sin\frac{A}{2}\cdot\sin\frac{C}{2} = 0$$

which is not possible in ΔABC

$\boxed{B, C}$

In a triangle ABC, which of the following quantities denote the area of the triangle?

(A) $\frac{a^2-b^2}{2}\left(\frac{\sin A\sin B}{\sin(A-B)}\right)$

(B) $\frac{r_1r_2r_3}{\sqrt{\sum r_1r_2}}$

(C) $\frac{a^2+b^2+c^2}{\cot A+\cot B+\cot C}$

(D) $r^2\cot\frac{A}{2}\cdot\cot\frac{B}{2}\cot\frac{C}{2}$

(A) $\left(\frac{a^2-b^2}{2}\right)\left(\frac{\sin A.\sin B}{\sin(A-B)}\right) = \frac{4R^2(\sin^2 A-\sin^2 B)}{2}\cdot\frac{\left(\frac{a}{2R}\right)\cdot\left(\frac{b}{2R}\right)}{\sin(A-B)}$

$= \frac{\sin(A+B)\,.\,ab}{2} = \frac{1}{2}ab\sin C$

$= \Delta$

(B) $\frac{r_1r_2r_3}{\sqrt{r_1r_2+r_2r_3+r_3r_1}} = \frac{\frac{\Delta^3 s}{s(s-a)(s-b)(s-c)}}{\sqrt{\frac{\Delta^2(s(s-c)+s(s-b)+s(s-a))}{s(s-a)(s-b)(s-c)}}}$

Using $(s(s-a)(s-b)(s-c)=\Delta^2)$

$= \frac{\Delta s}{\sqrt{s(s-c)+s(s-b)+s(s-a)}}$

$= \frac{\Delta s}{\sqrt{3s^2-s(a+b+c)}} = \frac{\Delta s}{\sqrt{3s^2-2s^2}} = \Delta$

(C) $\frac{a^2+b^2+c^2}{\cot A+\cot B+\cot C} = \frac{a^2+b^2+c^2}{\frac{\cos A}{\sin A}+\frac{\cos B}{\sin B}+\frac{\cos C}{\sin C}}$

$= \frac{a^2+b^2+c^2}{\sum\left(\frac{b^2+c^2-a^2}{2bc}\right)\cdot\frac{2R}{a}}$

(Using $\sum b^2+c^2-a^2 = a^2+b^2+c^2$) $= \frac{a^2+b^2+c^2}{\frac{R}{abc}\sum b^2+c^2-a^2} = \frac{abc}{R}\cdot\frac{(a^2+b^2+c^2)}{(a^2+b^2+c^2)}$

$= 4\Delta$

(D) $r^2\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$

$= \frac{\Delta^2}{s^2}\cdot\frac{s(s-a)}{\Delta}\cdot\frac{s(s-b)}{\Delta}\cdot\frac{s(s-c)}{\Delta}$

$= \frac{s(s-a)(s-b)(s-c)}{\Delta} = \frac{\Delta^2}{\Delta} = \Delta$

$\boxed{A, B, D}$

5. In ΔABC, angle A, B and C are in the ratio 1 : 2 : 3, then which of the following is (are) correct? (All symbol used have usual meaning in a triangle.)

(A) Circumradius of ΔABC = c

(B) a : b : c = 1 : $\sqrt{3}$: 2

(C) Perimeter of ΔABC = $3+\sqrt{3}$

(D) Area of ΔABC = $\frac{\sqrt{3}}{8}c^2$

Given $A : B : C = 1 : 2 : 3$

Let $A = x$, $B = 2x$, $C = 3x$

$\Rightarrow A+B+C = \pi \Rightarrow 6x = \pi \Rightarrow x = \frac{\pi}{6}$

$\Rightarrow A = \frac{\pi}{6}$, $B = \frac{\pi}{3}$, $C = \frac{\pi}{2}$

(A) $\frac{c}{\sin C} = 2R \Rightarrow R = \frac{1}{2}\cdot\frac{c}{\sin\frac{\pi}{2}} = \frac{c}{2}$

(B) $a : b : c = \sin A : \sin B : \sin C$

$= \sin\frac{\pi}{6} : \sin\frac{\pi}{3} : \sin\frac{\pi}{2}$

$= \frac{1}{2} : \frac{\sqrt{3}}{2} : 1$

$= 1 : \sqrt{3} : 2$

(C) Let $a = K$, $b = \sqrt{3}K$, $c = 2K$

$\Rightarrow$ Perimetre $= (a+b+c) = K + \sqrt{3}K + 2K$

$= (3+\sqrt{3})K$

(D) Area of $\Delta ABC = \frac{1}{2}ab$

$= \frac{1}{2}(K).(\sqrt{3}K) = \frac{\sqrt{3}}{2}K^2$

$= \frac{\sqrt{3}}{2}\left(\frac{c}{2}\right)^2 \quad \left(\because K = \frac{c}{2}\right)$

$= \frac{\sqrt{3}}{8}c^2$

$\boxed{B, D}$

6. In triangle ABC, let $b = 10$, $c = 10\sqrt{2}$ and $R = 5\sqrt{2}$ then which of the following statement(s) is (are) correct?

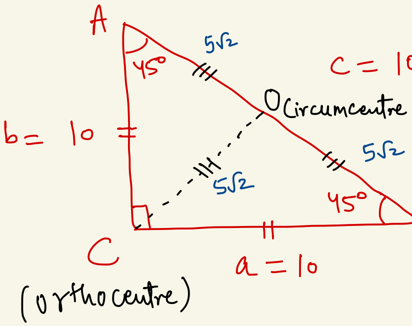
[**Note:** All symbols used have usual meaning in triangle ABC.]

(A) Area of triangle ABC is 50.

(B) Distance between orthocentre and circumcentre is $5\sqrt{2}$

(C) Sum of circumradius and inradius of triangle ABC is equal to 10

(D) Length of internal angle bisector of $\angle ACB$ of triangle ABC is $\frac{5}{2\sqrt{2}}$



$\because \frac{b}{\sin B} = 2R$

$\therefore \sin B = \frac{b}{2R} = \frac{10}{2(5\sqrt{2})}$

$\sin B = \frac{1}{\sqrt{2}}$

$\Rightarrow B = 45°$

(Here $B \neq 135°$ as $c > b \Rightarrow \angle C > \angle B$)

Also

$\frac{c}{\sin C} = 2R$

$\Rightarrow \sin C = \frac{c}{2R}$

$\sin C = \frac{10\sqrt{2}}{2(5\sqrt{2})}$

$\sin C = 1$

$\Rightarrow C = 90°$

(A) Area of triangle ABC $= \frac{1}{2} \times 10 \times 10 = 50$

(B) Distance b/w orthocentre and circumcentre $= 5\sqrt{2}$

(C) Circumradius$(R) = 5\sqrt{2}$ (Given)

Inradius $(r) = \frac{\Delta}{s} = \frac{50}{\frac{10\sqrt{2}+10+10}{2}} = \frac{100}{20+10\sqrt{2}} = \frac{10}{2+\sqrt{2}}$

$= \frac{10(2-\sqrt{2})}{2}$

$= 5(2-\sqrt{2})$

$\therefore$ Sum of $R + r = 5\sqrt{2} + 5(2-\sqrt{2})$

$= 10$

(D) Length of internal Angle Bisector of $\angle ACB$ is

$= \frac{2ab\cos\frac{C}{2}}{a+b} = \frac{2 \times 10 \times 10 \times \cos 45°}{10+10} = \frac{10}{\sqrt{2}} = 5\sqrt{2}$

$\boxed{A, B, C}$

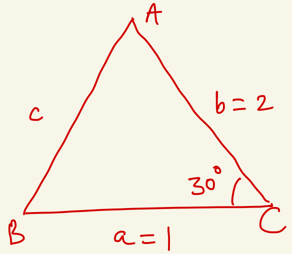
7 In a triangle ABC, let BC = 1, AC = 2 and measure of angle C is 30°. Which of the following statement(s) is (are) correct?

(A) 2 sin A = sin B

(B) Length of side AB equals $5-2\sqrt{3}$

(C) Measure of angle A is less than 30°

(D) Circumradius of triangle ABC is equal to length of side AB



(A) Use Sine law $\dfrac{a}{\sin A} = \dfrac{b}{\sin B} \Rightarrow \dfrac{1}{\sin A} = \dfrac{2}{\sin B}$

$$\Rightarrow 2\sin A = \sin B$$

(B) Use cosine law in ΔABC

$$\cos 30^\circ = \frac{1^2 + 2^2 - c^2}{2(1)(2)} \Rightarrow \frac{\sqrt{3}}{2} = \frac{5 - c^2}{4}$$

$$\Rightarrow c = \sqrt{5 - 2\sqrt{3}}$$

(C) As $c > a \Rightarrow \angle C > \angle A$

$\therefore$ Angle B must be obtuse

$\Rightarrow$ Measure of angle A is less than 30°

(D) $\Delta = \dfrac{1}{2} ab \sin C = \dfrac{1}{2}(1)(2)\sin 30^\circ = \dfrac{1}{2}$

$$\because R = \frac{abc}{4\Delta} = \frac{(1)(2)(c)}{4\left(\frac{1}{2}\right)} = c$$

$\Rightarrow$ Circumradius of ΔABC is equal of side AB.

$$\boxed{A, C, D}$$

EXERCISE 02

8 Given an acute triangle ABC such that $\sin C = \frac{4}{5}$, $\tan A = \frac{24}{7}$ and AB = 50. Then-

(A) centroid, orthocentre and incentre of ΔABC are collinear

(B) $\sin B = \frac{4}{5}$

(C) $\sin B = \frac{4}{7}$

(D) area of $\Delta ABC = 1200$

$$\sin C = \frac{4}{5} \Rightarrow \cos C = \frac{3}{5}$$

Also $\tan A = \frac{24}{7} \Rightarrow \sin A = \frac{24}{25}$ & $\cos A = \frac{7}{25}$

Using Sine Law in ΔABC

$$\frac{a}{\sin A} = \frac{c}{\sin C} \Rightarrow \frac{a}{\left(\frac{24}{25}\right)} = \frac{50}{\left(\frac{4}{5}\right)}$$

(Given $c = 50$)

$$\Rightarrow a = 60$$

Now $\sin B = \sin(\pi - (A+C)) = \sin(A+C)$

$$= \sin A . \cos C + \cos A . \sin C$$

$$= \left(\frac{24}{25}\right) . \left(\frac{3}{5}\right) + \left(\frac{7}{25}\right) . \left(\frac{4}{5}\right) = \frac{100}{125} = \frac{4}{5}$$

Now area of $\Delta ABC = \frac{1}{2} ac \sin B$

$$= \frac{1}{2} . (60) . (50) . \frac{4}{5}$$

$$= 1200$$

Using $\frac{a}{\sin A} = \frac{b}{\sin B} \Rightarrow \frac{60}{\left(\frac{24}{25}\right)} = \frac{b}{(4/5)} \Rightarrow b = 50$

So ΔABC is an isosceles.

In isosceles Δ, circumcentre, centroid, orthocentre and incentre are collinear.

$\boxed{A, B, D}$

9 In ΔABC, angle A is 120°, BC + CA = 20 and AB + BC = 21, then

(A) AB > AC (B) AB < AC

(C) ΔABC is isosceles (D) area of ΔABC = $14\sqrt{3}$

Given $\angle A = 120^\circ$ $\quad a+b=20 \longrightarrow ①$ and $a+c=21 \longrightarrow ②$

Subtract ① from ②

$$a+c-a-b=1$$
$$c-b=1$$
$$c=b+1 \Rightarrow c>b$$

$\Rightarrow AB > AC$

$\Rightarrow AB \neq AC$

$\Rightarrow \angle C \neq \angle B$

$\Rightarrow$ So it is not isosceles Δ

$$\cos A = \frac{b^2+c^2-a^2}{2bc}$$

$$\cos 120^\circ = \frac{b^2+(b+1)^2-(20-b)^2}{2(b)(b+1)}$$

$$-\frac{1}{2} = \frac{b^2+(b+1)^2-(20-b)^2}{2(b)(b+1)}$$

$$-b^2-b = 2b^2+1+2b-400-b^2+40b$$

$$2b^2+43b-399=0$$

$$b = \frac{-43 \pm \sqrt{5041}}{4} = \frac{-43\pm 71}{2}$$

$$\therefore\ b = \frac{-43+71}{4} = 7$$

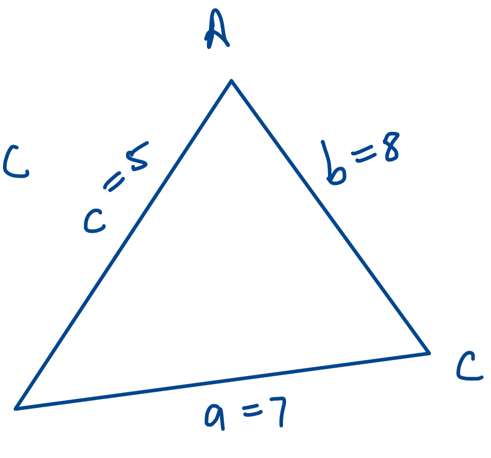
Area of $\Delta = \frac{1}{2} bc \sin A = \frac{1}{2} \times 7 \times 8 \times \sin 120^\circ = 28 \times \frac{\sqrt{3}}{2}$

$= 14\sqrt{3}$

$\boxed{A, D}$

1. Given a triangle ABC with sides a = 7, b = 8 and c = 5. If the value of the expression $\left(\sum \sin A\right)\left(\sum \cot \frac{A}{2}\right)$ can be expressed in the form $\frac{p}{q}$ where p, q ∈ N and $\frac{p}{q}$ is in its lowest form find the value of (p + q).

Solution:



$$\sum \sin A = \sin A + \sin B + \sin C$$

$$= \frac{a}{2R} + \frac{b}{2R} + \frac{c}{2R}$$

$$= \frac{a+b+c}{2R}$$

$$= \frac{s}{R} = \boxed{\frac{s \cdot 4\Delta}{abc}}$$

$$\sum \cot \frac{A}{2} = \cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2}$$

$$= \frac{s(s-a)}{\Delta} + \frac{s(s-b)}{\Delta} + \frac{s(s-c)}{\Delta}$$

$$= \frac{s}{\Delta}\left[3s - (a+b+c)\right]$$

$$= \frac{s}{\Delta}\left[3s - 2s\right] = \boxed{\frac{s^2}{\Delta}} \; ; \; \boxed{s = \frac{a+b+c}{2} = 10}$$

$$\therefore \left(\sum \sin A\right)\left(\sum \cot \frac{A}{2}\right) = \frac{4s^3}{abc} = \frac{4(10)^3}{7 \cdot 8 \cdot 5}$$

$$= \frac{4 \times 1000}{7 \times 40} = \frac{100}{7} = \frac{p}{q}$$

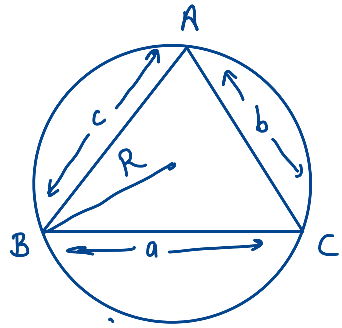
$$\therefore \boxed{p + q = 100 + 7 = 107}$$

EXERCISE (S-1)

2. If two times the square of the diameter of the circumcircle of a triangle is equal to the sum of the squares of its sides then prove that the triangle is right angled.

Solution:—

Given: $2(2R)^2 = a^2+b^2+c^2$



To Prove: ΔABC is right angled

Proof:

$$2(2R)^2 = (2R\sin A)^2 + (2R\sin B)^2 + (2R\sin C)^2$$

$$\Rightarrow 2 = \sin^2 A + \sin^2 B + \sin^2 C$$

$$\Rightarrow 4 = 2\sin^2 A + 2\sin^2 B + 2\sin^2 C$$

$$\Rightarrow 4 = (1-\cos 2A) + (1-\cos 2B) + (1-\cos 2C)$$

$$\Rightarrow \cos 2A + \cos 2B + \cos 2C + 1 = 0$$

$$\boxed{* \text{ For } \Delta ABC,\ \cos 2A + \cos 2B + \cos 2C = -1 - 4\cos A \cos B \cos C\ *}$$

$$\boxed{\Rightarrow [-1 - 4\cos A \cdot \cos B \cdot \cos C] + 1 = 0}$$

$$\Rightarrow \boxed{\cos A \cdot \cos B \cdot \cos C = 0}$$

$\cos A = 0$ or $\cos B = 0$ or $\cos C = 0$

$\Rightarrow \angle A = \frac{\pi}{2}$ or $\angle B = \frac{\pi}{2}$ or $\angle C = \frac{\pi}{2}$

Only one of them will be true

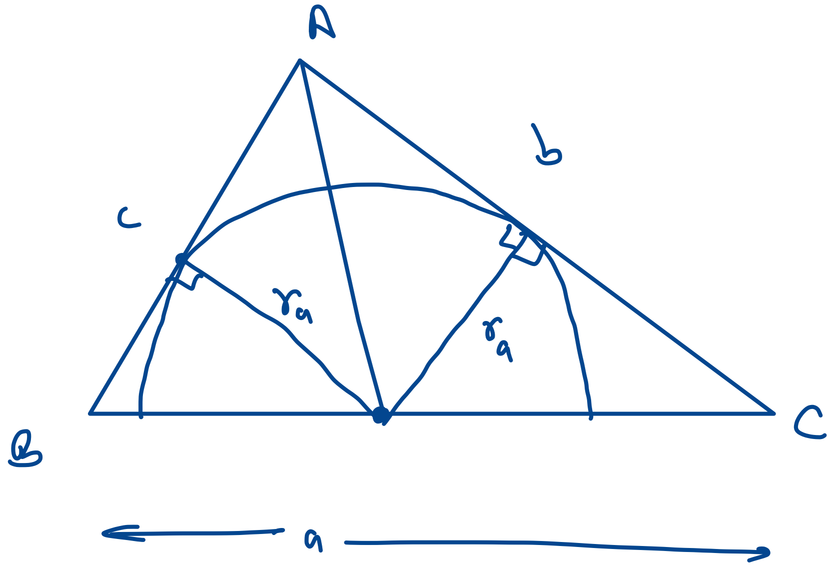
[Hence Proved]

3. In acute angled triangle ABC, a semicircle with radius r_a is constructed with its base on BC and tangent to the other two sides. r_b and r_c are defined similarly. If r is the radius of the incircle of triangle ABC then prove that, $\frac{2}{r} = \frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c}$.

Solution:

Area =

$$\Delta = \frac{1}{2} c r_a + \frac{1}{2} b r_a$$



$$\Rightarrow \boxed{r_a = \frac{2\Delta}{b+c}}$$

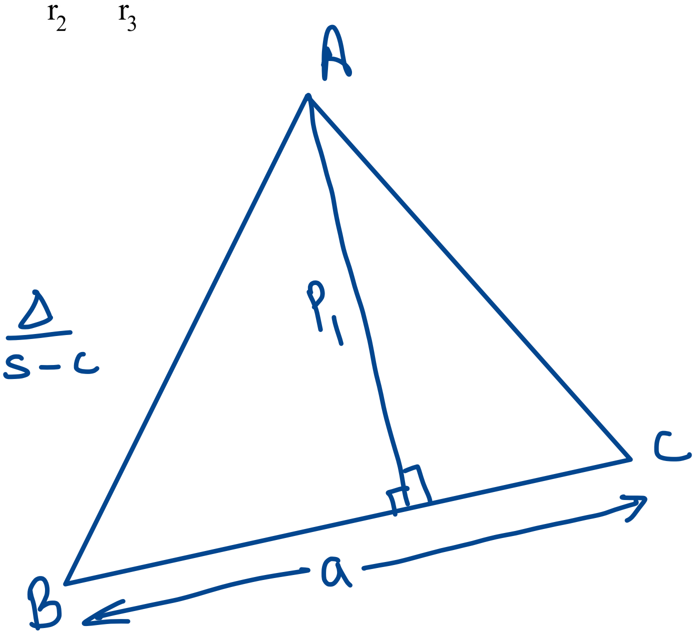
Similarly, $\boxed{r_b = \frac{2\Delta}{a+c} \quad \text{and} \quad r_c = \frac{2\Delta}{a+b}}$

Now, $$\frac{1}{r_a}+\frac{1}{r_b}+\frac{1}{r_c} = \frac{b+c}{2\Delta} + \frac{a+c}{2\Delta} + \frac{a+b}{2\Delta} = \frac{2(a+b+c)}{2\Delta}$$

$$= \frac{2s}{\Delta} = \frac{2}{r}$$

[Hence Proved]

4. If the length of the perpendiculars from the vertices of a triangle A, B, C on the opposite sides are p_1, p_2, p_3 then prove that $\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}=\frac{1}{r}=\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}$.



Solution:

$$r_1=\frac{\Delta}{s-a};\quad r_2=\frac{\Delta}{s-b};\quad r_3=\frac{\Delta}{s-c}$$

$$\therefore \boxed{\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}}$$

$$=\frac{s-a}{\Delta}+\frac{s-b}{\Delta}+\frac{s-c}{\Delta}=\frac{3s-(a+b+c)}{\Delta}$$

$$=\frac{3s-2s}{\Delta}=\frac{s}{\Delta}=\boxed{\frac{1}{r}}$$

Now, Area of $\Delta ABC=\Delta=\frac{1}{2}aP_1$

$$\Rightarrow \frac{1}{P_1}=\frac{a}{2\Delta}$$

Similarly, $\frac{1}{P_2}=\frac{b}{2\Delta}$ and $\frac{1}{P_3}=\frac{c}{2\Delta}$

$$\therefore \boxed{\frac{1}{P_1}+\frac{1}{P_2}+\frac{1}{P_3}}=\frac{a}{2\Delta}+\frac{b}{2\Delta}+\frac{c}{2\Delta}=\frac{a+b+c}{2\Delta}=\frac{2s}{2\Delta}$$

$$=\boxed{\frac{1}{r}}$$ [Hence Proved]

5. With usual notations, prove that in a triangle ABC

Rr (sin A + sin B + sin C) = Δ

Solution: LHS = $Rr(\sin A + \sin B + \sin C)$

$$= R \cdot \frac{\Delta}{s} \cdot \left(\frac{a}{2R} + \frac{b}{2R} + \frac{c}{2R}\right)$$

$$= R \cdot \frac{\Delta}{s} \cdot \frac{(a+b+c)}{2R}$$

$$= R \cdot \frac{\Delta}{s} \cdot \frac{2s}{2R} = \Delta = \text{RHS}$$ [Hence Proved]

6 With usual notations, prove that in a triangle ABC

$$\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\frac{s^2}{\Delta}$$

Solution:

$$LHS = \cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}$$

$$= \frac{s(s-a)}{\Delta}+\frac{s(s-b)}{\Delta}+\frac{s(s-c)}{\Delta}$$

$$= \frac{s}{\Delta}\left[(s-a)+(s-b)+(s-c)\right]$$

$$= \frac{s}{\Delta}\left[3s-(a+b+c)\right]$$

$$= \frac{s}{\Delta}\left[3s-2s\right] = \frac{s^2}{\Delta} = RHS$$

[Hence Proved]

7. With usual notations, prove that in a triangle ABC

$$\cot A + \cot B + \cot C = \frac{a^2 + b^2 + c^2}{4\Delta}$$

Solution:

$$\cot A = \frac{\cos A}{\sin A} = \frac{b^2+c^2-a^2}{2bc\sin A} = \frac{b^2+c^2-a^2}{2\times 2\Delta}$$

$$= \frac{b^2+c^2-a^2}{4\Delta}$$

$$\left[\frac{1}{2}bc\sin A = \Delta \Rightarrow bc\sin A = 2\Delta\right]$$

Similarly, $\cot B = \dfrac{a^2+c^2-b^2}{4\Delta}$,

$$\cot C = \frac{a^2+b^2-c^2}{4\Delta}$$

$\therefore$ Adding all,

$\cot A + \cot B + \cot C$

$$= \frac{(b^2+c^2-a^2) + (a^2+c^2-b^2) + (a^2+b^2-c^2)}{4\Delta}$$

$$= \frac{a^2+b^2+c^2}{4\Delta} = \text{RHS} \quad [\text{Hence Proved}]$$

EXERCISE (S-1)

8 1. If a,b,c are the sides of triangle ABC satisfying $\log\left(1+\frac{c}{a}\right)+\log a-\log b=\log 2$.

Also $a(1-x^2)+2bx+c(1+x^2)=0$ has two equal roots. Find the value of sinA + sinB + sinC.

Solution:

Given: $\log\left(1+\frac{c}{a}\right)+\log a-\log b=\log 2$

$$\Rightarrow \log\left(\frac{\left[\left(1+\frac{c}{a}\right)\times a\right]}{b}\right)=\log 2$$

$$\Rightarrow \frac{a+c}{b}=2 \Rightarrow \boxed{a+c=2b} \quad \text{---} (1)$$

$a(1-x^2)+2bx+c(1+x^2)=0$ has equal roots

$\Rightarrow (c-a)x^2+2bx+(c+a)=0$ has equal roots

$$D=0 \Rightarrow (2b)^2-4(c-a)(c+a)=0$$
$$\Rightarrow b^2-(c^2-a^2)=0$$
$$\Rightarrow \boxed{b^2+a^2=c^2} \quad \text{---} (2)$$

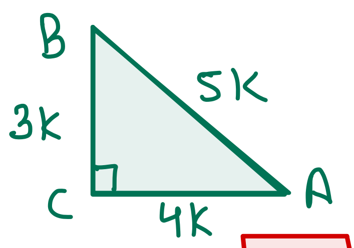
(Triangle is right angled at C)

from (1), $b=\frac{a+c}{2}$

$\therefore$ In (2) $b^2=(c-a)(c+a) \Rightarrow \frac{(a+c)^2}{4}=(c-a)(c+a)$

$$\Rightarrow a+c=4c-4a \Rightarrow \boxed{5a=3c}$$

$$\Rightarrow \boxed{\frac{a}{3}=\frac{c}{5}=K} \text{ (say)} \Rightarrow a=3K,\ c=5K$$



$$\therefore \boxed{b=\frac{a+c}{2}=4K.}$$

$$\therefore \sin A+\sin B+\sin C=\frac{3}{5}+\frac{4}{5}+1=\boxed{\frac{12}{5}}$$

9. With usual notations, prove that in a triangle ABC

$$\frac{b-c}{r_1}+\frac{c-a}{r_2}+\frac{a-b}{r_3}=0$$

Solution:

$$LHS = \frac{b-c}{r_1} + \frac{c-a}{r_2} + \frac{a-b}{r_3}$$

$$= b\left(\frac{1}{r_1}-\frac{1}{r_3}\right) + c\left(\frac{1}{r_2}-\frac{1}{r_1}\right) + a\left(\frac{1}{r_3}-\frac{1}{r_2}\right)$$

Now, $\frac{1}{r_2}-\frac{1}{r_1} = \frac{s-b}{\Delta} - \frac{s-a}{\Delta} = \frac{a-b}{\Delta}$

Similarly, $\frac{1}{r_3}-\frac{1}{r_2} = \frac{b-c}{\Delta}$ and $\frac{1}{r_1}-\frac{1}{r_3} = \frac{c-a}{\Delta}$

$$\therefore LHS = \frac{b(c-a)+c(a-b)+a(b-c)}{\Delta}$$

$$= \frac{bc-ba+ca-cb+ab-ac}{\Delta}$$

$$= 0 = RHS$$

[Hence Proved]

10 With usual notations, prove that in a triangle ABC

$$\frac{r_1}{(s-b)(s-c)}+\frac{r_2}{(s-c)(s-a)}+\frac{r_3}{(s-a)(s-b)}=\frac{3}{r}$$

<u>Solution:</u>

$$\frac{r_1}{(s-b)(s-c)} = \frac{\Delta}{(s-a)(s-b)(s-c)}$$

Similarly,

$$\frac{r_2}{(s-c)(s-a)} = \frac{\Delta}{(s-a)(s-b)(s-c)}$$

$$\frac{r_3}{(s-a)(s-b)} = \frac{\Delta}{(s-a)(s-b)(s-c)}$$

$$\therefore \text{LHS} = \frac{3\Delta}{(s-a)(s-b)(s-c)} = \frac{3s\Delta}{s(s-a)(s-b)(s-c)}$$

$$= \frac{3s\Delta}{\Delta^2} = \frac{3s}{\Delta} = \boxed{\frac{3}{r}} = \text{RHS}$$

17. With usual notations, prove that in a triangle ABC

$$\frac{abc}{s}\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}=\Delta$$

Solution: $LHS = \frac{abc}{s}\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

$$= \frac{abc}{s}\sqrt{\frac{s(s-a)}{bc}}\sqrt{\frac{s(s-b)}{ac}}\sqrt{\frac{s(s-c)}{ab}}$$

$$= \frac{abc}{s}\frac{\sqrt{s^3(s-a)(s-b)(s-c)}}{\sqrt{a^2b^2c^2}}$$

$$= \frac{abc}{s}\cdot\frac{\sqrt{s^2}\sqrt{s(s-a)(s-b)(s-c)}}{abc}$$

$$= \Delta = RHS \quad \text{[Hence Proved]}$$

Aliter:

$$\frac{abc}{s}\cos\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\cos\frac{C}{2}$$

$$= \frac{2abc}{(a+b+c)}\cos\frac{A}{2}\cdot\cos\frac{B}{2}\cdot\cos\frac{C}{2}$$

$$= \frac{2abc\cdot 4\cos A/2\cdot\cos B/2\cdot\cos C/2}{2R(\sin A+\sin B+\sin C)\,4}$$

$$= \frac{abc}{4R} = \Delta$$

12 ~~13~~. With usual notations, prove that in a triangle ABC

$$\frac{1}{r^2}+\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}=\frac{a^2+b^2+c^2}{\Delta^2}$$

Solution: $\frac{1}{r^2}+\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}$

$$=\frac{s^2}{\Delta^2}+\frac{(s-a)^2}{\Delta^2}+\frac{(s-b)^2}{\Delta^2}+\frac{(s-c)^2}{\Delta^2}$$

$$=\frac{s^2+(s-a)^2+(s-b)^2+(s-c)^2}{\Delta^2}$$

$$=\frac{s^2+(s^2-2as+a^2)+(s^2-2bs+b^2)+(s^2-2cs+c^2)}{\Delta^2}$$

$$=\frac{4s^2-2s(a+b+c)+a^2+b^2+c^2}{\Delta^2}$$

$$=\frac{4s^2-2s(2s)+a^2+b^2+c^2}{\Delta^2}$$

$$=\frac{a^2+b^2+c^2}{\Delta^2}=RHS$$ [Hence Proved]

13 ~~14~~. If $r_1 = r + r_2 + r_3$ then prove that the triangle is a right angled triangle.

Solution: Given that: $r_1 = r + r_2 + r_3$

$$\Rightarrow \frac{\Delta}{s-a} = \frac{\Delta}{s} + \frac{\Delta}{s-b} + \frac{\Delta}{s-c} \Rightarrow \frac{1}{s-a} = \frac{1}{s} + \frac{1}{s-b} + \frac{1}{s-c}$$

$$\Rightarrow \frac{1}{s-a} - \frac{1}{s} = \frac{1}{s-b} + \frac{1}{s-c} \Rightarrow \frac{s-(s-a)}{s(s-a)} = \frac{(s-c)+(s-b)}{(s-b)(s-c)}$$

$$\Rightarrow \frac{a}{s(s-a)} = \frac{2s-(b+c)}{(s-b)(s-c)} \Rightarrow \frac{a}{s(s-a)} = \frac{a}{(s-b)(s-c)}$$

$$\Rightarrow \frac{(s-b)(s-c)}{s(s-a)} = 1 \Rightarrow \tan^2\frac{A}{2} = 1 \Rightarrow \frac{A}{2} = 45^\circ \Rightarrow A = 90^\circ$$

OR

$$(s-b)(s-c) = s(s-a) \Rightarrow s^2-(b+c)s+bc = s^2-as$$

$$\Rightarrow bc = s(b+c-a) \Rightarrow bc = s(2s-a-a)$$

$$\Rightarrow \frac{s(s-a)}{bc} = \frac{1}{2} \Rightarrow \cos^2\frac{A}{2} = \frac{1}{2} \Rightarrow \frac{A}{2} = 45^\circ \Rightarrow A = 90^\circ$$

OR

$$4R\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = 4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} + 4R\cos\frac{A}{2}\sin\frac{B}{2}\cos\frac{C}{2} + 4R\cos\frac{A}{2}\cos\frac{B}{2}\sin\frac{C}{2}$$

$$\Rightarrow \sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} = \sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} + \cos\frac{A}{2}\left[\sin\frac{B}{2}\cos\frac{C}{2} + \cos\frac{B}{2}\sin\frac{C}{2}\right]$$

$$\Rightarrow \sin\frac{A}{2}\left[\cos\frac{B}{2}.\cos\frac{C}{2} - \sin\frac{B}{2}\sin\frac{C}{2}\right] = \cos\frac{A}{2}\sin\left(\frac{B+C}{2}\right)$$

$$\Rightarrow \sin\frac{A}{2}\cos\left(\frac{B+C}{2}\right) = \cos\frac{A}{2}.\cos\frac{A}{2}$$

$$\Rightarrow \tan^2\frac{A}{2} = 1 \Rightarrow \frac{A}{2} = 45^\circ \Rightarrow A = 90^\circ$$

9. With usual notation, if in a Δ ABC, $\frac{b+c}{11}=\frac{c+a}{12}=\frac{a+b}{13}$; then prove that, $\frac{\cos A}{7}=\frac{\cos B}{19}=\frac{\cos C}{25}$.

Given $\Rightarrow \frac{b+c}{11} = \frac{c+a}{12} = \frac{a+b}{13}$

$\Rightarrow \frac{b+c}{11} = \frac{c+a}{12} = \frac{a+b}{13} = \frac{a+b+c}{18}$

$\Rightarrow a = 7\lambda,\ b = 6\lambda,\ c = 5\lambda$

Using cosine formula

$\cos A = \frac{36+25-49}{60} = \frac{1}{5} = \frac{7}{35}$

$\cos B = \frac{49+25-36}{70} = \frac{19}{35} = \frac{19}{35}$

$\cos C = \frac{49+36-25}{84} = \frac{5}{7} = \frac{25}{35}$

$\Rightarrow \frac{\cos A}{7} = \frac{\cos B}{19} = \frac{\cos C}{25}$

EXERCISE (S-2)