The efficiency costs of strategy-proofness and ex post stability in school choice

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School choice

- School choice deals with matching students with schools
- Centralized mechanisms are more efficient than decentralized mechanisms
- Old mechanisms had strategic and stability concerns
- There is a mechanism that solves these concerns (Abdulkadiroglu and Sönmez, AER 2003)
  - based on Gale and Shapley (AMM 1962)'s Deferred Acceptance (DA) algorithm
- The solution has an efficiency cost (Erdil and Ergin, AER 2008)
- This paper: Measures the size of the costs

School choice

- Students are strategic agents
- School seats are goods to be consumed
- Priorities at schools are exogenous and there are ties
  - e.g. Boston has only 2 priority tiers for 11,000 students
- To resolve ties, a random tie-breaking is used
- The current mechanism:
  - *Step 1:* Randomly break ties to construct a strict priority order
  - *Step 2:* Run DA
- The random tie-breaking causes a loss of efficiency

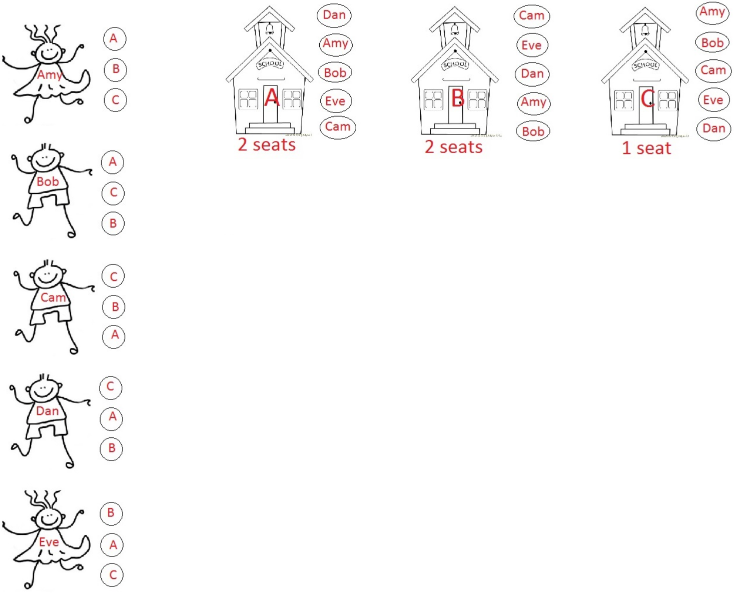
School choice

DA algorithm:

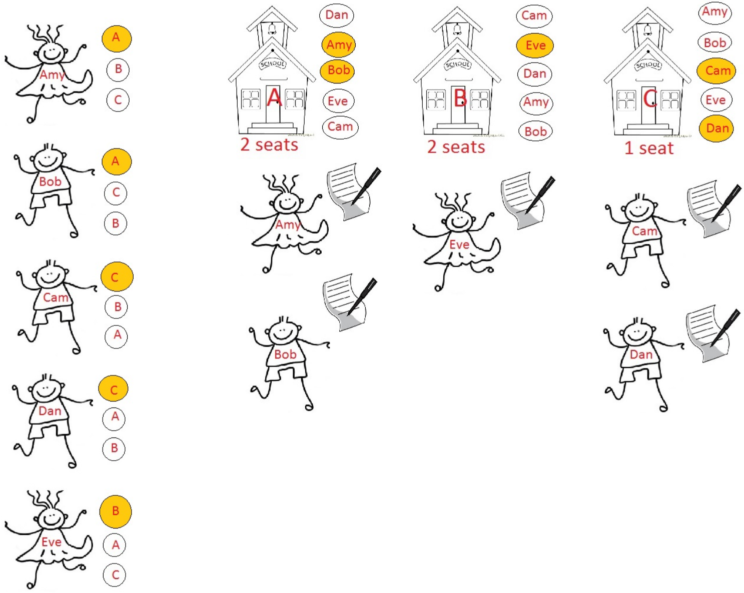
- *Step 1:* Each student applies to her top choice. Each school tentatively assigns its seats to its appliers one at a time following its priority order. Once a school reaches its quota, it rejects any remaining applicants.
- *Step k:* Any student who was rejected in the previous step applies to her next best choice. Each school considers the students tentatively assigned to it from the previous step with the new applicants, and tentatively assigns its seats to these students one at a time following its priority order. Once a school reaches its quota, it rejects any remaining applicants.

The algorithm terminates when no student is rejected.

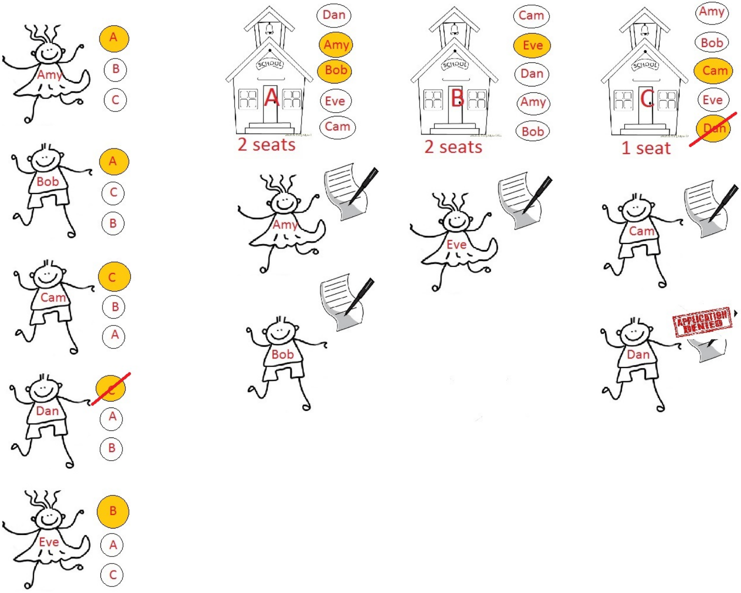
DA algorithm



DA algorithm



DA algorithm



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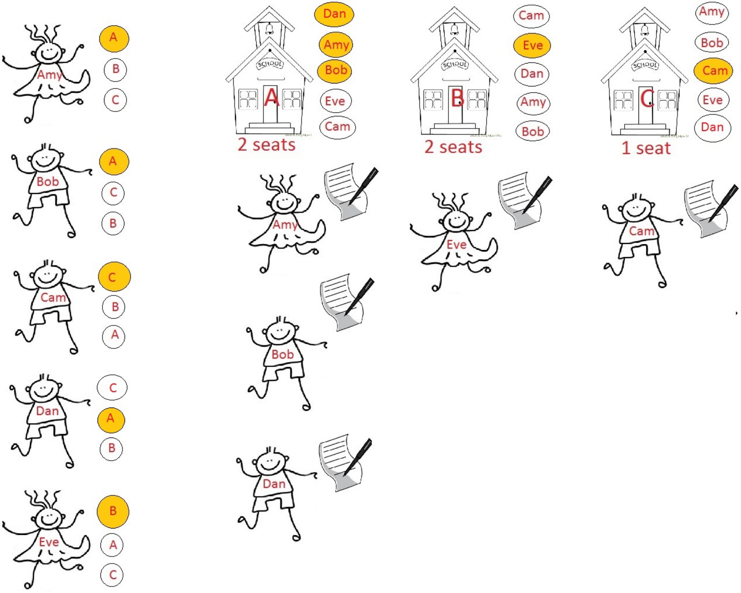

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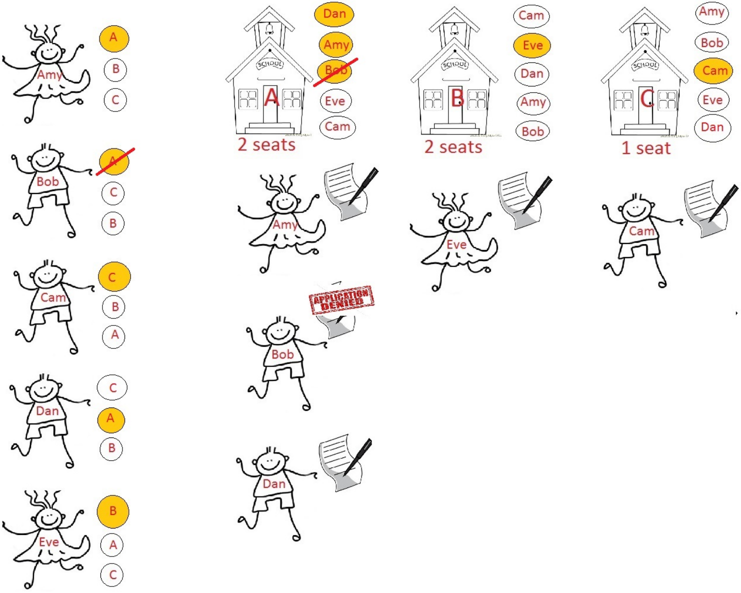

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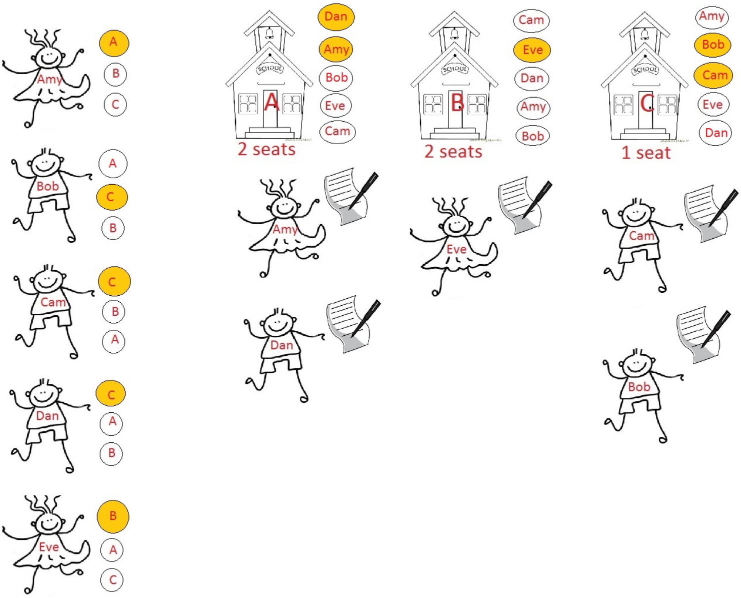

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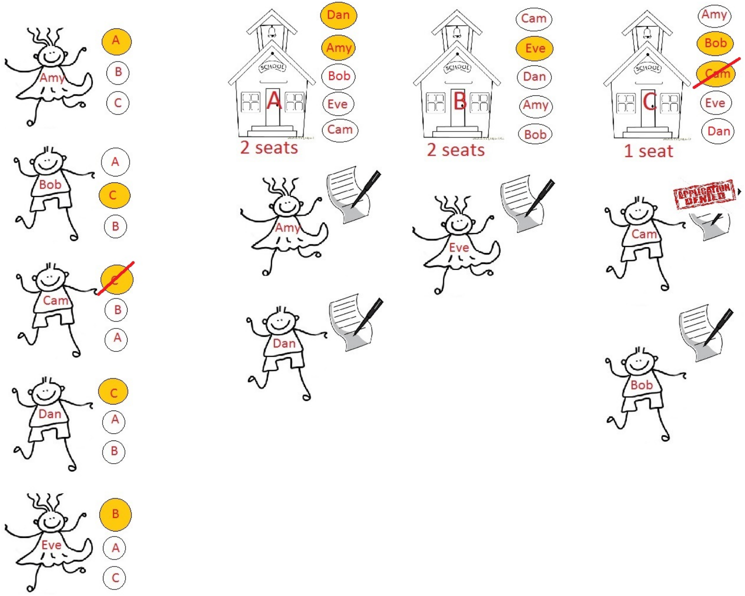
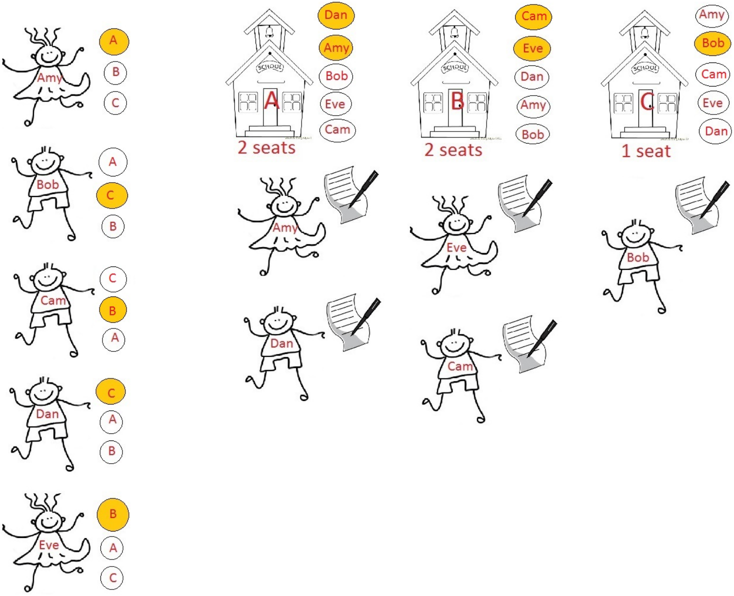
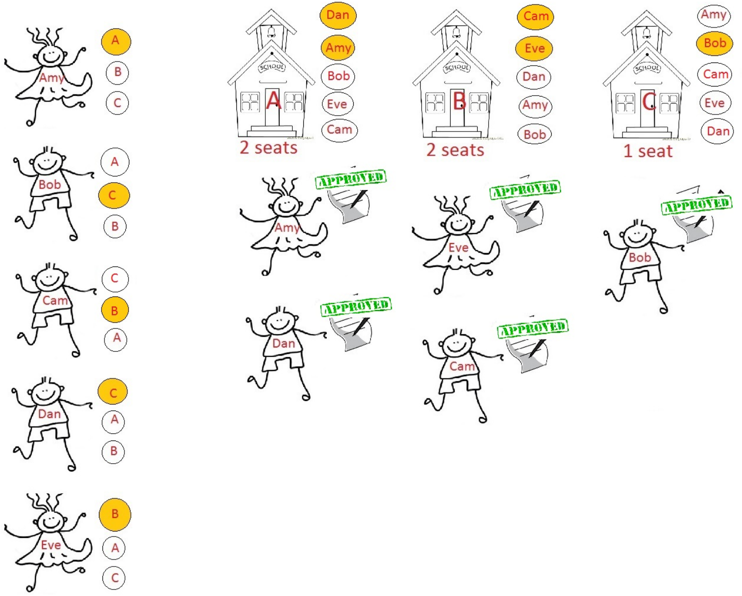


DA algorithm



DA algorithm



DA algorithm

- A matching is stable if the following does not happen:

  > Amy is matched to school B but prefers school A more. However, Bob is matched to A but has lower priority than Amy at A.

  Amy is not going to A because of Bob, who has lower priority than her at $A \Rightarrow$ not stable.
- The DA outcome is the most efficient matching that stability would allow when priorities are strict

Our contribution

- Characterize stability with weak priorities
- Characterize constrained (ordinal, rank) efficient ex post stable random matchings
- Introduce an integer programming problem that finds the most efficient matching that stability would allow
- Quantify the efficiency costs of strategy-proofness and ex post stability in Boston

Today's talk

- Introduce the problem
- How to measure?
- The setup: The model
- The constraint set: Ex post stability
- The objective: Constrained ordinal efficiency
- Simulation results

Example 1

- $I = \{1, 2, 3, 4\}$
- $C = \{a, b, c, d\}$, $q_a = q_b = q_c = q_d = 1$
- The student preferences and priority ordering are:

P_1	P_2	P_3	P_4
a	a	b	b
b	b	a	a
c	c	d	d
d	d	c	c

$\succeq_a$	$\succeq_b$	$\succeq_c$	$\succeq_d$
$1, 3$	$1, 2, 3, 4$	$1, 2$	$3, 4$
$2, 4$		$3, 4$	$1, 2$

Example 1

- One way to break it gives the following subproblem:

P_1	P_2	P_3	P_4
a	a	b	b
b	b	a	a
c	c	d	d
d	d	c	c

$\succeq_a$	$\succeq_b$	$\succeq_c$	$\succeq_d$
1	1	1	3
3	2	2	4
2	3	3	1
4	4	4	2

- The DA outcome of this subproblem is:

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ a & b & d & c \end{pmatrix}$$

- **Theorem** (Gale and Shapley) The DA outcome is the student optimal stable outcome for the subproblem as specified above

Example 1

$\succeq_a$	$\succeq_b$	$\succeq_c$	$\succeq_d$
1	1	1	3
3	2	2	4
2	3	3	1
4	4	4	2

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ a & b & d & c \end{pmatrix}$$

$\succeq_a$	$\succeq_b$	$\succeq_c$	$\succeq_d$
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$\succeq_a$	$\succeq_b$	$\succeq_c$	$\succeq_d$
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3	1	1	4
2	3	3	2
4	4	4	1

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ a & b & c & d \end{pmatrix}$$

$\succeq_a$	$\succeq_b$	$\succeq_c$	$\succeq_d$
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Example 1

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$\succeq_a$	$\succeq_b$	$\succeq_c$	$\succeq_d$
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$\succeq_a$	$\succeq_b$	$\succeq_c$	$\succeq_d$
3	3	2	3
1	2	1	4
2	1	3	2
4	4	4	1

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$\succeq_a$	$\succeq_b$	$\succeq_c$	$\succeq_d$
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$\succeq_a$	$\succeq_b$	$\succeq_c$	$\succeq_d$
1	4	2	4
3	2	1	3
4	1	4	2
2	3	3	1

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ a & c & d & b \end{pmatrix}$$

$\succeq_a$	$\succeq_b$	$\succeq_c$	$\succeq_d$
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2	1	3	1

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ d & c & a & b \end{pmatrix}$$

$\succeq_a$	$\succeq_b$	$\succeq_c$	$\succeq_d$
3	4	1	4
1	3	2	3
4	1	4	1
2	2	3	2

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ c & d & a & b \end{pmatrix}$$

$\succeq_a$	$\succeq_b$	$\succeq_c$	$\succeq_d$
3	4	2	4
1	3	1	3
4	2	4	2
2	1	3	1

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ d & c & a & b \end{pmatrix}$$

Example 1

$$\rho^{DA} = \begin{array}{c|c|c|c|c} & a & b & c & d \\ \hline 1 & \frac{3}{4} & 0 & \frac{1}{6} & \frac{1}{12} \\ 2 & 0 & \frac{1}{3} & \frac{5}{8} & \frac{1}{24} \\ 3 & \frac{1}{4} & \frac{1}{3} & \frac{1}{6} & \frac{1}{4} \\ 4 & 0 & \frac{1}{3} & \frac{1}{24} & \frac{5}{8} \end{array}$$

Example 1

$\succeq_a$	$\succeq_b$	$\succeq_c$	$\succeq_d$
1	1	1	3
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1	1	1	3
3	3	2	4
2	2	3	1
4	4	4	2

$$\boxed{\begin{pmatrix} 1 & 2 & 3 & 4 \\ a & c & b & d \end{pmatrix}} \times \frac{1}{3}$$

$\succeq_a$	$\succeq_b$	$\succeq_c$	$\succeq_d$
1	1	1	3
3	3	2	4
4	4	3	1
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2	1	3	2
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$$\boxed{\begin{pmatrix} 1 & 2 & 3 & 4 \\ c & b & a & d \end{pmatrix}} \times \frac{1}{4}$$

$\succeq_a$	$\succeq_b$	$\succeq_c$	$\succeq_d$
3	2	2	3
1	3	1	4
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$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ c & b & a & d \end{pmatrix}$$

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3	4	1	3
2	1	4	2
4	3	3	1

$$\boxed{\begin{pmatrix} 1 & 2 & 3 & 4 \\ a & b & c & d \end{pmatrix}} \times \frac{1}{12}$$

$\succeq_a$	$\succeq_b$	$\succeq_c$	$\succeq_d$
3	2	2	4
1	4	1	3
2	3	4	2
4	1	3	1

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ c & b & a & d \end{pmatrix}$$

Example 1

$$\pi = \begin{array}{c|c|c|c|c} & a & b & c & d \\ \hline 1 & \frac{3}{4} & 0 & \frac{1}{4} & 0 \\ 2 & 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ 3 & \frac{1}{4} & \frac{1}{3} & \frac{1}{12} & \frac{1}{3} \\ 4 & 0 & \frac{1}{3} & 0 & \frac{2}{3} \end{array}$$

Example 1

P_1	P_2	P_3	P_4
a	a	b	b
b	b	a	a
c	c	d	d
d	d	c	c

$$\rho^{DA} = \begin{array}{c|c|c|c|c c} & a & b & c & d & \sum \\ \hline 1 & \frac{3}{4} & 0 & \frac{1}{6} & \frac{1}{12} & \frac{3}{4} \\ 2 & 0 & \frac{1}{3} & \frac{5}{8} & \frac{1}{24} & 0 \\ 3 & \frac{1}{4} & \frac{1}{3} & \frac{1}{6} & \frac{1}{4} & \frac{1}{3} \\ 4 & 0 & \frac{1}{3} & \frac{1}{24} & \frac{5}{8} & \frac{1}{3} \end{array}$$

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Example 1

P_1	P_2	P_3	P_4
a	a	b	b
b	b	a	a
c	c	d	d
d	d	c	c

$\rho^{DA} =$

	a	b	c	d	$\sum$
1	$\frac{3}{4}$	0	$\frac{1}{6}$	$\frac{1}{12}$	$\frac{3}{4}$
2	0	$\frac{1}{3}$	$\frac{5}{8}$	$\frac{1}{24}$	$\frac{1}{3}$
3	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{6}$	$\frac{1}{4}$	$\frac{7}{12}$
4	0	$\frac{1}{3}$	$\frac{1}{24}$	$\frac{5}{8}$	$\frac{1}{3}$

$\pi =$

	a	b	c	d	$\sum$
1	$\frac{3}{4}$	0	$\frac{1}{4}$	0	$\frac{3}{4}$
2	0	$\frac{1}{3}$	$\frac{2}{3}$	0	$\frac{1}{3}$
3	$\frac{1}{4}$	$\frac{1}{3}$	$\frac{1}{12}$	$\frac{1}{3}$	$\frac{7}{12}$
4	0	$\frac{1}{3}$	0	$\frac{2}{3}$	$\frac{1}{3}$

Example 1

P_1	P_2	P_3	P_4
a	a	b	b
b	b	a	a
c	c	d	d
d	d	c	c

$$\rho^{DA} = \begin{array}{c|c|c|c|c c} & a & b & c & d & \sum \\ \hline 1 & \frac{3}{4} & 0 & \frac{1}{6} & \frac{1}{12} & \frac{11}{12} \\ 2 & 0 & \frac{1}{3} & \frac{5}{8} & \frac{1}{24} & \frac{23}{24} \\ 3 & \frac{1}{4} & \frac{1}{3} & \frac{1}{6} & \frac{1}{4} & \frac{5}{6} \\ 4 & 0 & \frac{1}{3} & \frac{1}{24} & \frac{5}{8} & \frac{23}{24} \end{array}$$

$$\pi = \begin{array}{c|c|c|c|c c} & a & b & c & d & \sum \\ \hline 1 & \frac{3}{4} & 0 & \frac{1}{4} & 0 & 1 \\ 2 & 0 & \frac{1}{3} & \frac{2}{3} & 0 & 1 \\ 3 & \frac{1}{4} & \frac{1}{3} & \frac{1}{12} & \frac{1}{3} & \frac{11}{12} \\ 4 & 0 & \frac{1}{3} & 0 & \frac{2}{3} & 1 \end{array}$$

Example 1

- π first-order stochastically dominates ρ^{DA}
- π is ex post stable.
- No stable outcome of a tie-breaker allows for a stable improvement cycle

How to measure the costs?

- Find the stable probability distribution where no student is worse off than any stable distribution: Call it π
- Measure the efficiency difference between π and DA
- Measure the efficiency difference between π and the ordinally efficient outcome

A school choice problem

- A finite set of students I
- A finite set of schools C
- A quota vector for schools $q = (q_c)_{c \in C}$
- A strict preference vector for students $P = (P_i)_{i \in I}$
- A weak priority order for schools $\succeq = (\succeq_c)_{c \in C}$

Outcome

- A random matching is a matrix $\rho \in R^{|I| \times |C|}$ where $\rho_{i,c}$ is the probability i is matched with c such that
  - $\forall i, c$, $0 \leq \rho_{i,c} \leq 1$
  - $\forall i$, $\sum_c \rho_{i,c} = 1$
  - $\forall c$, $\sum_i \rho_{i,c} = q_c$
- A random matching is a deterministic matching if $\forall i, c$, $\rho_{i,c} \in \{0, 1\}$
- A lottery is a probability distribution over deterministic matchings
- A school choice mechanism φ selects a random matching for a given school choice problem $(P, \succeq)$.

Example 1 (continued)

P_1	P_2	P_3
b	c	b
c	b	a
a	a	c

$\succeq_a$	$\succeq_b$	$\succeq_c$
3	1, 2, 3	3
1, 2		1, 2

$$\rho^{DA} = \begin{array}{c|c|c|c} & a & b & c \\ \hline 1 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 2 & \frac{1}{6} & \frac{1}{12} & \frac{3}{4} \\ 3 & \frac{7}{12} & \frac{5}{12} & 0 \end{array}$$

Lottery λ^ρ induces random matching ρ^{DA}:

$$\lambda^\rho = \frac{1}{12}\begin{pmatrix} 1 & 2 & 3 \\ c & b & a \end{pmatrix} + \frac{1}{6}\begin{pmatrix} 1 & 2 & 3 \\ c & a & b \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 1 & 2 & 3 \\ a & c & b \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 1 & 2 & 3 \\ b & c & a \end{pmatrix}$$

Desirable properties: Equal treatment of equals

- Equal students: Preferences and priorities are the same
- A random matching ρ treats equals equally if equal students receive the same distribution

In Example 1, since no pair of students has the same preferences, both ρ^{DA} and π treat equals equally

Desirable properties: Stability

- A deterministic matching μ is stable if there are no $i, j \in I$ such that
$$\mu(j) P_i \mu(i) \text{ and } i \succ_{\mu(j)} j$$
- A random matching is ex post stable if there exists a lottery inducing it that assigns positive weight to only stable matchings

Example 1 (continued)

P_1	P_2	P_3
b	c	b
c	b	a
a	a	c

$$\mu_1 = \begin{pmatrix} 1 & 2 & 3 \\ a & b & c \end{pmatrix}$$

$\succeq_a$	$\succeq_b$	$\succeq_c$
3	$1, 2, 3$	3
$1, 2$		$1, 2$

$aP_3c, \;\; 3 \succ_a 1 \Rightarrow \mu_1$ is not stable.

Example 1 (continued)

P_1	P_2	P_3
b	c	b
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$$\mu_1 = \begin{pmatrix} 1 & 2 & 3 \\ a & b & c \end{pmatrix}$$

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3	$1, 2, 3$	3
$1, 2$		$1, 2$

$aP_3c, \;\; 3 \succ_a 1 \Rightarrow \mu_1$ is not stable.

$\eta =$

	a	b	c
1	$\frac{1}{2}$	0	$\frac{1}{2}$
2	$\frac{1}{2}$	$\frac{1}{2}$	0
3	0	$\frac{1}{2}$	$\frac{1}{2}$

Example 1 (continued)

P_1	P_2	P_3
b	c	b
c	b	a
a	a	c

$$\mu_1 = \begin{pmatrix} 1 & 2 & 3 \\ a & b & c \end{pmatrix}$$

$\succeq_a$	$\succeq_b$	$\succeq_c$
3	$1, 2, 3$	3
$1, 2$		$1, 2$

$aP_3c, \;\; 3 \succ_a 1 \Rightarrow \mu_1$ is not stable.

$$\eta = \begin{array}{c|c|c|c} & a & b & c \\ \hline 1 & \frac{1}{2} & 0 & \frac{1}{2} \\ 2 & \frac{1}{2} & \frac{1}{2} & 0 \\ 3 & 0 & \frac{1}{2} & \frac{1}{2} \end{array}$$

$\Rightarrow \eta$ is ex post unstable

Example 1 (continued)

P_1	P_2	P_3
b	c	b
c	b	a
a	a	c

$\succeq_a$	$\succeq_b$	$\succeq_c$
3	$1, 2, 3$	3
$1, 2$		$1, 2$

$$\pi = \begin{array}{c|c|c|c} & a & b & c \\ \hline 1 & \frac{1}{4} & \frac{7}{12} & \frac{1}{6} \\ 2 & \frac{1}{6} & 0 & \frac{5}{6} \\ 3 & \frac{7}{12} & \frac{5}{12} & 0 \end{array}$$

Example 1 (continued)

P_1	P_2	P_3
b	c	b
c	b	a
a	a	c

$\succeq_a$	$\succeq_b$	$\succeq_c$
3	$1,2,3$	3
$1,2$		$1,2$

$$\pi = \begin{array}{c|c|c|c} & a & b & c \\ \hline 1 & \frac{1}{4} & \frac{7}{12} & \frac{1}{6} \\ 2 & \frac{1}{6} & 0 & \frac{5}{6} \\ 3 & \frac{7}{12} & \frac{5}{12} & 0 \end{array}$$

$$\lambda^\pi = \tfrac{1}{6}\begin{pmatrix} 1 & 2 & 3 \\ c & a & b \end{pmatrix} + \tfrac{1}{4}\begin{pmatrix} 1 & 2 & 3 \\ a & c & b \end{pmatrix} + \tfrac{7}{12}\begin{pmatrix} 1 & 2 & 3 \\ b & c & a \end{pmatrix}$$

Example 1 (continued)

P_1	P_2	P_3
b	c	b
c	b	a
a	a	c

$\succeq_a$	$\succeq_b$	$\succeq_c$
3	1, 2, 3	3
1, 2		1, 2

$\pi =$

	a	b	c
1	$\frac{1}{4}$	$\frac{7}{12}$	$\frac{1}{6}$
2	$\frac{1}{6}$	0	$\frac{5}{6}$
3	$\frac{7}{12}$	$\frac{5}{12}$	0

$$\lambda^{\pi} = \frac{1}{6}\begin{pmatrix} 1 & 2 & 3 \\ c & a & b \end{pmatrix} + \frac{1}{4}\begin{pmatrix} 1 & 2 & 3 \\ a & c & b \end{pmatrix} + \frac{7}{12}\begin{pmatrix} 1 & 2 & 3 \\ b & c & a \end{pmatrix}$$

i.e. π is ex post stable.

Desirable properties: Ordinal efficiency

- A random matching π ordinally dominates random matching ρ if
$$\forall i \in I, c \in C, \sum_{aR_ic} \pi_{i,a} \geq \sum_{aR_ic} \rho_{i,a}$$
$$\exists j \in I, b \in C, \sum_{aR_jb} \pi_{j,a} > \sum_{aR_jb} \rho_{j,a}$$
- A random matching ρ is ordinally efficient if it is not ordinally dominated by any other random matching

Example 1 (continued)

P_1	P_2	P_3
b	c	b
c	b	a
a	a	c

$$\rho^{DA} = \begin{array}{c|c|c|c} & a & b & c \\ \hline 1 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 2 & \frac{1}{6} & \frac{1}{12} & \frac{3}{4} \\ 3 & \frac{7}{12} & \frac{5}{12} & 0 \end{array} \qquad \begin{array}{c} \sum \\ \hline \frac{1}{2} \\ \frac{3}{4} \\ \frac{5}{12} \end{array}$$

$$\pi = \begin{array}{c|c|c|c} & a & b & c \\ \hline 1 & \frac{1}{4} & \frac{7}{12} & \frac{1}{6} \\ 2 & \frac{1}{6} & 0 & \frac{5}{6} \\ 3 & \frac{7}{12} & \frac{5}{12} & 0 \end{array} \qquad \begin{array}{c} \sum \\ \hline \frac{7}{12} \\ \frac{5}{6} \\ \frac{5}{12} \end{array}$$

Example 1 (continued)

P_1	P_2	P_3
b	c	b
c	b	a
a	a	c

$$\rho^{DA} = \begin{array}{c|c|c|c} & a & b & c \\ \hline 1 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 2 & \frac{1}{6} & \frac{1}{12} & \frac{3}{4} \\ 3 & \frac{7}{12} & \frac{5}{12} & 0 \end{array} \qquad \begin{array}{c} \sum \\ \frac{3}{4} \\ \frac{5}{6} \\ 1 \end{array}$$

$$\pi = \begin{array}{c|c|c|c} & a & b & c \\ \hline 1 & \frac{1}{4} & \frac{7}{12} & \frac{1}{6} \\ 2 & \frac{1}{6} & 0 & \frac{5}{6} \\ 3 & \frac{7}{12} & \frac{5}{12} & 0 \end{array} \qquad \begin{array}{c} \sum \\ \frac{3}{4} \\ \frac{5}{6} \\ 1 \end{array}$$

Example 1 (continued)

P_1	P_2	P_3
b	c	b
c	b	a
a	a	c

$$\rho^{DA} = \begin{array}{c|c|c|c} & a & b & c \\ \hline 1 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 2 & \frac{1}{6} & \frac{1}{12} & \frac{3}{4} \\ 3 & \frac{7}{12} & \frac{5}{12} & 0 \end{array} \qquad \begin{array}{c} \sum \\ 1 \\ 1 \\ 1 \end{array}$$

$$\pi = \begin{array}{c|c|c|c} & a & b & c \\ \hline 1 & \frac{1}{4} & \frac{7}{12} & \frac{1}{6} \\ 2 & \frac{1}{6} & 0 & \frac{5}{6} \\ 3 & \frac{7}{12} & \frac{5}{12} & 0 \end{array} \qquad \begin{array}{c} \sum \\ 1 \\ 1 \\ 1 \end{array}$$

Example 1 (continued)

- π ordinally dominates ρ^{DA}
- A quick check will show that it is, in fact, ordinally efficient

Strategy-proofness

A mechanism φ is strategy-proof if for all $i \in I$, $(P, \succeq)$ and P'_i,

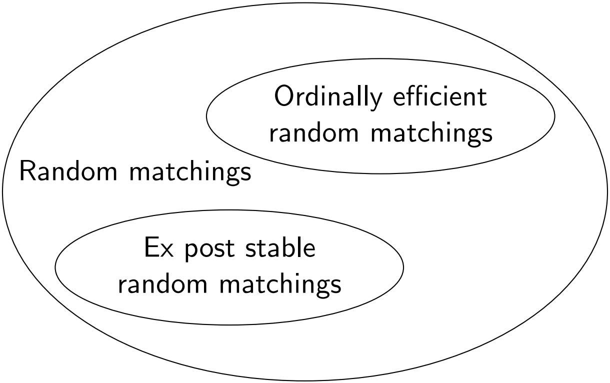
$$\sum_{aR_ic} \varphi_{i,a}(P, \succeq) \geq \sum_{aR_ic} \varphi_{i,a}(P'_i, P_{-i}, \succeq), \quad \text{for all } c \in C$$

Theorem

Let $\sum_{c \in C} q_c = |I|$. Then the outcome of the random DA is constrained ordinally efficient in the strategy-proof class.

The constraint set: Ex post stability

- Ex post stability and ordinal efficiency are incompatible (Roth, MOR 1982, Bogomolnaia and Moulin, JET 2001)
- We look for an ex post stable random matching that is not ordinally dominated by any other ex post stable random matching. $\Rightarrow$ constrained ordinal efficiency



The constraint set: Ex post stability

- Stability: For a given student-school pair, either the student is matched to a weakly better school or the school is full with weakly higher priority students
- For $i \in I$ and $c \in C$, let $B(i,c) = \{j \in I | j \succeq_c i\}$
- For each $i \in I$ and $c \in C$, let

$$\hat{q}_{i,c} = \begin{cases} q_c, & \text{if } |B(i,c)| \geq q_c \\ |B(i,c)|, & \text{otherwise} \end{cases}$$

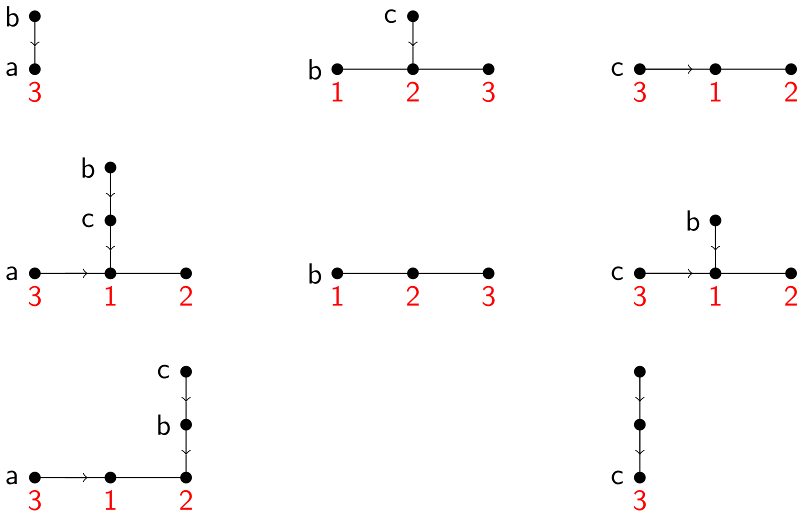
- For $i \in I$ and $c \in C$, denote the set of $\hat{q}_{i,c}$-combinations of $B(i,c)$ as $Com(i,c)$.

Comb constraint: For all $c \in C$, $i \in I$ and $T \in Com(i,c)$

$$\sum_{j \in B(i,c)} \mu_{j,c} + \sum_{j \in T} \sum_{a P_j c} \mu_{j,a} \geq \hat{q}_{i,c} \tag{1}$$

Example 1 (continued)

The comb constraint illustrated for Example 1



The constraint set: Ex post stability

Theorem

The following statements are equivalent:
(i) A deterministic matching μ is stable.
(ii) μ is an integer solution to the comb constraint.
(iii)

$$\sum_{j \in B(i,c)} \mu_{j,c} + q_c \sum_{aP_ic} \mu_{i,a} + (q_c - 1)\mu_{i,c} \geq q_c, \text{ for all } i \in I \text{ and } c \in C$$

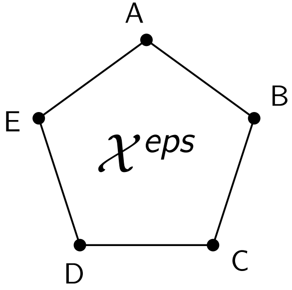
- $\mathcal{M}^s$: the set of stable matchings

The constraint set: Ex post stability

Observation

Let $\mathcal{X}^{eps}$ be set of ex post stable random matchings. Then the extreme points of $\mathcal{X}^{eps}$ are precisely equal to $\mathcal{M}^s$.

$$\mathcal{X}^{eps} = \left\{ \rho \in R^{|I| \times |C|} \mid \rho = \sum_{t=1}^{n} \lambda_t \mu^t,\ \sum_{t=1}^{n} \lambda_t = 1,\ \lambda_t \geq 0,\ \mu^t \in \mathcal{M}^s \right\}$$



A, B, C, D, E are the integer solutions to the comb constraint.

The objective: Constrained ordinal efficiency

- Stochastic improvement cycle: A cycle of trades where every student in the cycle benefits
- Ex post improvement cycle: A stochastic improvement cycle in $\mathcal{X}^{eps}$

Proposition

An ex post stable random matching ρ is constrained ordinally efficient if and only if there is no ex post stable improvement cycle at ρ.

Example 1 (continued)

P_1	P_2	P_3
b	c	b
c	b	a
a	a	c

$$\rho^{DA} = \begin{array}{c|ccc} & a & b & c \\ \hline 1 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 2 & \frac{1}{6} & \frac{1}{12} & \frac{3}{4} \\ 3 & \frac{7}{12} & \frac{5}{12} & 0 \end{array}$$

$$\pi = \begin{array}{c|ccc} & a & b & c \\ \hline 1 & \frac{1}{4} & \frac{7}{12} & \frac{1}{6} \\ 2 & \frac{1}{6} & 0 & \frac{5}{6} \\ 3 & \frac{7}{12} & \frac{5}{12} & 0 \end{array}$$

Ex post stable improvement cycle: $(1, c) \to (2, b) \to (1, c)$

The objective: Constrained ordinal efficiency

- $c(i,k)$: student i's kth most preferred school
- A weight structure w is descending if for all $i \in I$, $w_{i,c(i,k)} > w_{i,c(i,k+1)}$ for all k

$$\max_{\rho \in \mathcal{X}^{eps}} \sum_{i \in I} \sum_{c \in C} w_{i,c} \rho_{i,c} \tag{2}$$

Theorem

$\rho^{(0)}$ is constrained ordinally efficient if and only if it is a maximizer to (2) for some descending weight structure.

Example 1 (continued)

$$\pi = \begin{array}{c|c|c|c} & a & b & c \\ \hline 1 & \frac{1}{4} & \frac{7}{12} & \frac{1}{6} \\ 2 & \frac{1}{6} & 0 & \frac{5}{6} \\ 3 & \frac{7}{12} & \frac{5}{12} & 0 \end{array}$$

π is the solution of (2) for the following weight structure:

$$w = \begin{array}{c|c|c|c} & a & b & c \\ \hline 1 & 0 & 2 & 1 \\ 2 & 0 & 0.5 & 1 \\ 3 & 1 & 3 & 0 \end{array}$$

Rank efficiency

- Rank efficiency: As many students as possible maximize their chances of entering their kth (or better) preferred school, for every k.
- Rank efficient weight structure:
  (i) $w_{i,c(i,k)} > w_{i,c(i,k+1)}$ for each $i \in I$ and k
  (ii) $w_k = w_{i,c(i,k)} = w_{j,c(j,k)}$ for each $i, j \in I$ and k
- Rank efficiency is stronger than ordinal efficiency
- **Theorem** (Featherstone) A random matching $\rho^* \in \mathcal{X}$ is rank efficient if and only if it is a maximizer to

$$\max_{\rho \in \mathcal{X}} \sum_{i \in I} \sum_{c \in C} w_{i,c} \rho_{i,c} \tag{3}$$

  for some rank efficient weight structure.

Theorem

A random matching ρ is constrained rank efficient if and only if it is a maximizer to (2) for some rank efficient weight structure.

Example 1 (continued)

P_1	P_2	P_3
b	c	b
c	b	a
a	a	c

$\succeq_a$	$\succeq_b$	$\succeq_c$
3	$1,2,3$	3
$1,2$		$1,2$

There are 4 stable matchings:

	1	2	3
(i)	a	c	b
(ii)	b	c	a
(iii)	c	a	b
(iv)	c	b	a

Among any random matching constructed from $(i) - (iv)$, $\rho^{(0)}$ maximizes (2) for a rank efficient weight structure.

$$\rho^{(0)} = \begin{array}{c|c|c|c} & a & b & c \\ \hline 1 & 0 & 1 & 0 \\ 2 & 0 & 0 & 1 \\ 3 & 1 & 0 & 0 \end{array}$$

The objective: Equal treatment of equals

- *Step 1:* Find a set of equal students, $I_1^e \subseteq I$ if there is any. Using $\rho^{(0)}$, give everyone the same probability distribution. Call the new random matching $\rho^{(1)}$.
  In general,
- *Step k:* Find a set of equal students, $I_k^e \subseteq I \backslash \{I_1^e, ..., I_{k-1}^e\}$, if any. Using $\rho^{(k-1)}$, give everyone the same probability distribution. Call the new random matching $\rho^{(k)}$.

The procedure ends when no equal students are left.
Let K be the step at which the procedure ends. Then the final random matching is $\rho^{(K)}$.

The solution

Theorem

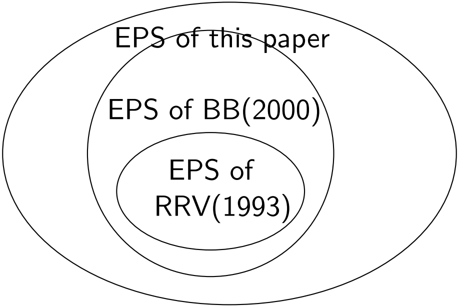
$\rho^{(K)}$ is a constrained ordinally efficient random matching that treats equals equally.

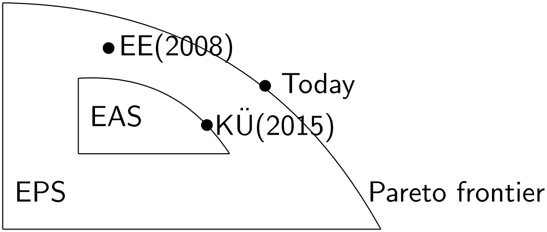
In Example 1, $\rho^{(K)} = \rho^{(0)}$.

Related literature

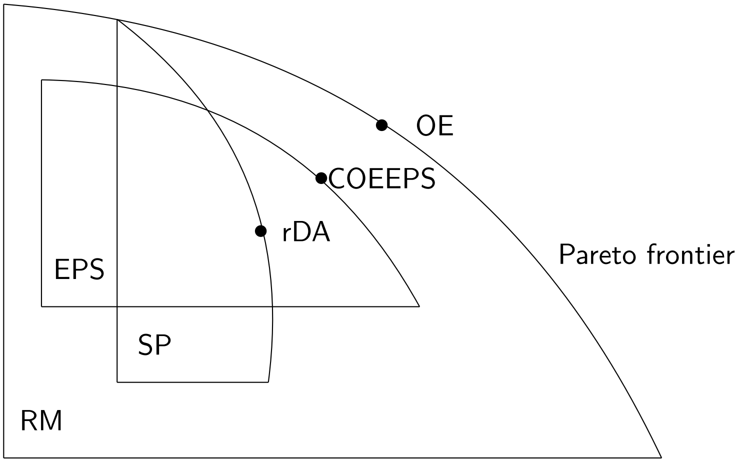
- School choice: Gale and Shapley (1962 AMM), Abdulkadiroglu and Sönmez (AER 2003), ...
- Efficiency notions: Bogomolnaia and Moulin (JET 2001), Abdulkadiroglu and Sönmez (JET 2003), Featherstone (2014)
- Ex post stability: Roth, Rothblum and Vande Vate (MOR 1993), Baïou and Balinski (MP 2000)
- Constrained efficiency: Erdil and Ergin (AER 2008), Kesten and Ünver (TE 2015)

Related literature





Efficiency costs



Boston high school data

- Aggregate data from 2008-2011
- On average, 26 schools, 2705 students per year
- On average, each school had neighborhood priority for 208 students and 2 schools per neighborhood
- 60% of the students have siblings
- Students ranked
  - a sibling's neighborhood school 3% of the time,
  - a sibling's nonneighborhood school 3% of the time,
  - a neighborhood school without sibling priority 30% of the time
  - a nonneighborhood school without sibling priority 64% of the time

Simulation assumptions: Priorities

- 20 schools in 10 neighborhoods and 400 students
- Each neighborhood has 2 schools and 40 students on average
- 30% of the students have sibling priority,
    - 1/3 have sibling priority at a neighborhood school,
    - 2/3 have sibling priority at a nonneighborhood school
- A student has sibling priority at only one school

Simulation assumptions: Preferences

- Each student with sibling priority at a neighborhood school
  - first-ranks the neighborhood school that her sibling is attending with probability 0.3,
  - first-ranks the other neighborhood school with probability 0.15,
  - first-ranks the remaining nonneighborhood schools with equal probability
- Each student with sibling priority at a nonneighborhood school
  - first-ranks the nonneighborhood school that her sibling is attending with probability 0.15,
  - first-ranks each neighborhood schools with probability 0.15,
  - first-ranks the remaining nonneighborhood schools with equal probability
- Each student without sibling priority
  - first-ranks each neighborhood schools with probability 0.15,
  - first-ranks the remaining nonneighborhood schools with equal probability

Simulation assumptions: Preferences

- Once the first choices were randomly determined, the conditional probabilities for each student's remaining schools were updated and then the second choices were determined using the same process
- This process was iterated until every student ranked every school in the market
- No school is less preferred than being unmatched for any student

Simulation assumptions: Optimization

$$\max \sum_{i \in I} \sum_{c \in C} w_{i,c} \rho_{i,c} \tag{4}$$

such that

$$\rho_{i,c} \in \{0,1\}, \quad \forall i \in I, c \in C,$$

$$\sum_{a \in C} \rho_{i,a} \leq 1, \quad \forall i \in I,$$

$$\sum_{j \in I} \rho_{j,c} \leq q_c, \ \forall c \in C,$$

$$\sum_{j \in B(i,c)} \rho_{j,c} + q_c \sum_{a P_i c} \rho_{i,a} + (q_c - 1)\rho_{i,c} \geq q_c, \ \forall i \in I, c \in C$$

Simulation assumptions: Optimization

- For each $i \in I$ and if school $c \in C$ is her kth ranked school, then its weight $w_{i,c}$ is

$$w_{i,c} = 21 - k$$

- Randomly generate 100 markets, and computed the corresponding outcomes of the random DA mechanism, programs (4) and (3)
- 200 random single tie-breaking priority lists were generated for the random DA mechanism

Simulation results: Efficiency costs

Choices	random DA outcome	constrained rank efficient outcome	rank efficient outcome
1st	77.78 (2.84)	90.44 (1.74)	91.20 (1.58)
2nd	12.93 (1.70)	9.25 (1.64)	8.75 (1.56)
3rd	4.77 (0.79)	0.31 (0.31)	0.05 (0.14)
4th	2.20 (0.44)		
5th	1.12 (0.30)		
6th	0.58 (0.21)		
7th	0.30 (0.12)		
8th	0.16 (0.07)		
9th	0.09 (0.06)		
10th	0.04 (0.03)		
11th	0.02 (0.02)		

Table: Comparison of random DA outcomes with constrained rank efficient outcomes and rank efficient outcomes in the simulations (the simulation standard errors are given in parentheses)

Simulation results: Large markets

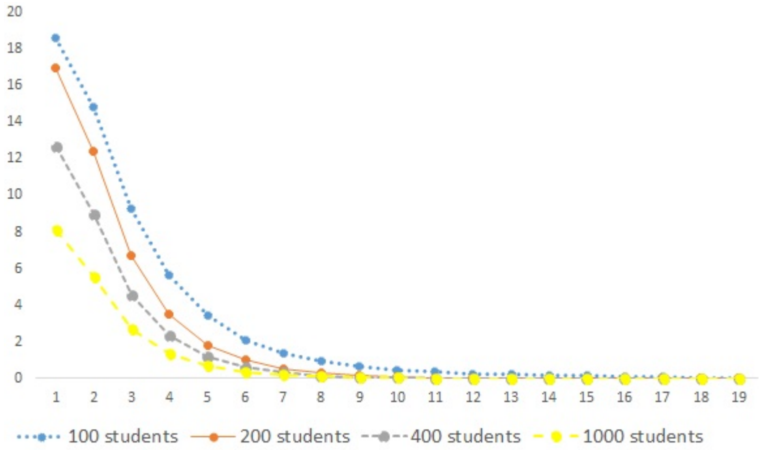


Figure: The efficiency differences between random DA outcomes and constrained rank efficient ex post stable outcomes

Simulation results: Large markets

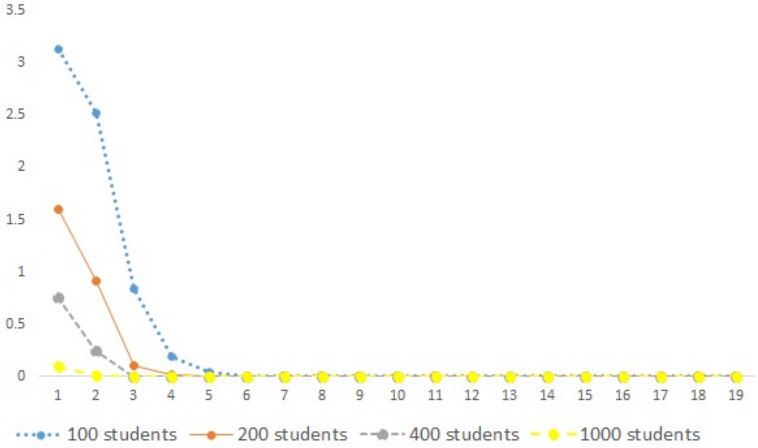


Figure: The efficiency differences between constrained rank efficient ex post stable outcomes and rank efficient outcomes

Concluding remarks

- Quantified the relative costs of strategy-proofness and ex post stability in Boston using integer programming
  - Characterization of stability as a solution of a system of inequalities
  - Characterization of constrained efficiency as a solution to an integer programming problem
- Large market implications (future work)
  - With Boston priorities, it seems random DA does very well. How fast are the convergences?
  - What priority structure allows convergence?