$\mathcal{D}$ is the set of all assignments and $\mathcal{M}$ is the set of all matchings.

A lottery $\lambda = \sum_{s\in S} \lambda_s \mu_s$ is a probability distribution over matchings such that

(i) Index set $S$ is nonempty and finite,

(ii) $\sum_{s\in S} \lambda_s = 1$,

(iii) $\forall s \in S$, $\lambda_s \in (0,1] \cap \mathbb{Q}$, and

(iv) $\forall s \in S$, $\mu_s \in \mathcal{D}$

where $\mu_S = (\mu_s)_{s\in S}$ is the support of lottery $\lambda$. Moreover, it has equal weights if for each $s \in S$, $w_s = 1/|S|$; it is feasible if instead of (iv), we have (iv'): $\forall s \in S$, $\mu_s \in \mathcal{M}$.

Let $(P, \succeq)$ be a problem and let $S$ be an index set. We rename $I$ as the set of types of students and $C$ as the set of school types. In the $|S|-$fold replica problem, for each type $i \in I$, $c \in C$, there are $|S|$ agents; for each school type $c \in C$, the quota is $q_c|S|$; for each student type $i \in I$, all $|S|$ agents of that type share the common preferences $P_i$ on $C$. For each $s \in S$, let $i^s$ denote the agent of student type $i \in I$ indexed by $s \in S$ and let $I^s$ denote the set of all agents of students types $I$ indexed by $s \in S$. Moreover, let $c^s$ denote the agent of school type $c \in C$ indexed by $s \in S$ and let $C^s$ denote the set of agents of school types $C$ indexed by $s \in S$. Let $I^S = \cup_{s\in S} I^s$ and $C^S = \cup_{s\in S} C^s$. For each $s \in S$, the student priorities $(\succeq_{c^s})_{c^s\in C^s}$ satisfy the following:

1. For each $s \in S$, if $i \succeq_c j$, then $i^s \succeq_{c^s} j^s$ for every $i, j \in I$ and $c \in C$,

2. for each $s, s' \in S$, $i^s \succeq_{c^s} l^{s'}$ for every $i, l \in I$ and $c \in C$, and

3. for each $s \in S$ and $s', s'' \in S\backslash\{s\}$, $s' \neq s''$, $i^{s'} \sim_{c^s} i^{s''}$ for every $i \in I$

Let $P_{I^s} \equiv (P_{i^s})_{i^s\in I^s}$ and $\succeq_{C^s} \equiv (\succeq_{c^s})_{c^s\in C^s}$. Then, $(P_{I^s}, \succeq_{C^s})$ denotes the $s-$replica problem and $(P_S, \succeq_S) \equiv (P_{I^s}, \succeq_{C^s})_{s\in S}$ denotes the $|S|-$fold replica problem.

An $|S|-$fold replica assignment is a function $\mu_S : I_S \to C_S$ such that for each $s \in S$, $c^s \in C^s$ $i^s \in I^s$, $|\mu_S^{-1}(c^s)| \leq q_c|S|$ and $|\mu_S(i^s)| \leq |S|$.

Given $\mu_S \in \mathcal{D}$ and $s \in S$, an $s-$replica assignment is a function $\mu_s : I^s \to C^s$ such that for each $i^s \in I^s$, $\mu_s(i^s) = \mu_S(i^s)$. Denote $\mu_S = (\mu_s)_{s\in S}$.

For every problem $(P, \succeq)$ and lottery $\lambda(P, \succeq)$, there is a corresponding $|S|-$fold replica problem and a matching that consists of the matchings in the support of $\lambda(P, \succeq)$. Let $\mu_S^\lambda = (\mu_s^\lambda)_{s\in S}$ denote the support of lottery $\lambda$ with index set $S$.

**Lemma 1.** *Let $(P, \succeq)$ be a problem and let $\lambda$ be a lottery with an index set $S$. Then $\lambda$ is ex post stable if and only $\mu_S^\lambda$ is stable at $(P_S, \succeq_S)$.*

*Proof.* "Only if." Suppose $\mu_S^\lambda$ is unstable at $(P_S, \succeq_S)$. Then there are two cases to consider:

- There exist an index $s \in S$ and a pair of student types $i, j \in I$ such that $\nu_s(j^s) P_{i^s} \nu_s(i^s)$ and $i^s \succ_{\nu_s(j^s)} j^s$. Since $\nu_s$ is in the support of $\lambda$, it is ex post unstable.

- There exist indexes $s, s' \in S$ and a pair of student types $i, j \in I$ such that $\nu_s(j^s)P_{i^{s'}}\nu_{s'}(i^{s'})$ and $i^{s'} \succ_{\nu_s(j^s)} j^s$. However, it contradicts the construction of $\succeq_S$.

"If." Suppose $\lambda$ is ex post unstable, i.e. there is a matching $\nu_s$ with index $s \in S$ in its support that is unstable. Then there exists a pair of students $i, j \in I$ such that $\nu_s(j)P_i\nu_s(i)$ and $i \succ_{\nu_s(j)} j$. Consider $(P_S, \succeq_S)$. Then by the construction of $\succeq_S$, $i \succ_{\nu_s(j)} j$ implies $i^s \succ_{\nu_s(j^s)} j^s$. Moreover, by the construction of $P_S$, $\nu_s(j)P_i\nu_s(i)$ implies $\nu_s(j^s)P_{i^s}\nu_s(i^s)$. Therefore, the support of $\lambda$ is unstable. $\square$

**Theorem 1.** *Let $(P, \succeq)$ be a problem and let $\lambda$ be a lottery with an index set $S$. Then $\lambda$ is constrained ordinally efficient ex post stable if and only the support of $\lambda$ does not allow any stable improvement cycles at $(P_S, \succeq_S)$.*

*Proof.* Let lottery $\lambda$ be with support $\mu_S^\lambda$. "Only if." Suppose

"If." Suppose that $\lambda$ is wasteful. ...

Suppose $\lambda$ is a non-wasteful and constrained ordinally inefficient ex post stable lottery. Let $\rho^\lambda$ be the random matching induced by lottery $\lambda$. Then there is an ex post stable random matching $\pi$ that ordinally dominates $\rho^\lambda$ at $(P, \succeq)$. Let $I' = \{i \in I | \rho_i^\lambda \neq \pi_i\}$. Clearly, $I' \neq \varnothing$. Note that for all $i \in I'$, $\pi_i$ ordinally dominates $\rho_i^\lambda$. Moreover, since $\lambda$ is non-wasteful, if $\pi_{i,a} > \rho_{i,a}^\lambda$ for some $a \in C$, then there exists $j \in I'$ such that $\pi_{j,a} < \rho_{j,a}^\lambda$; as $\pi_j$ ordinally dominates $\rho_j^\lambda$, there exists $b \in C$ such that $bP_ja$ and $\pi_{j,b} > \rho_{j,b}^\lambda$. Let $C' = \{c \in C | \exists i \in I', \pi_{i,c} > \rho_{i,c}^\lambda\}$. Clearly, $C' \neq \varnothing$.

Since lottery $\lambda$ with index set $S$ induces $\rho^\lambda$, there exists a subset of indexes $S' \in S$ such that for every student-school type pair $(i, c) \in I' \times C'$, there exists $s \in S'$ such that $\mu_s(i^s) = c^s$. Let $\tilde{I}^{S'}$ denote the set of agents with student types $I'$ and let $\tilde{C}^{S'}$ denote the set of agents with school types $C'$.

Consider the following directed graph where each student-school agents pair $(i^s, c^s) \in \tilde{I}^{S'} \times \tilde{C}^{S'}$ is represented by a node. For each $s \in S'$ and a school agent $c^s \in \tilde{C}^{S'}$, let each student-school agents pair $(i^s, c^s) \in \tilde{I}^{S'} \times \tilde{C}^{S'}$ in this graph containing school agent $c^s$ be pointed to by every student-school agents pair $(j^{s'}, d^{s'}) \in \tilde{I}^{S'} \times \tilde{C}^{S'}$ with $c^s P_{j^{s'}} d^{s'}$ and has the highest $\succeq_{c^s}$ $-$priority among such student agents in $\tilde{I}^{S'}$. We repeat this for every $\tilde{C}^{S'}$.

Note that no node in this graph points to itself and each node is pointed to by at least one other node. Moreover, each node can only be pointed to by a node that contains a different school type. Then there is at least one cycle of student-school agents pairs. By construction, each student agent in this graph prefers the school agent she is pointing to more than the school she is trading away; and each student agent not in this graph has her match unaltered. Denote this cycle Cyc. Then Cyc is an improvement cycle. Denote the resulting assignment in $(P_S, \succeq_S)$, $\nu_S$.

It remains to show that Cyc is a stable improvement cycle. Suppose to the contrary, for some school agent $c^s \in \tilde{C}^{S'}$, the student agent who traded for $c^s$ is not the highest $\succeq_{c^s}$ $-$priority agent in $I^S$ who pointed to $c^s$. Then there exists a student-school type pair $(j, d)$ with $\rho^{j,d} > 0$ such that $cP_jd$. By the construction of Cyc, $j \notin I'$; hence no agent of $j$ is in the graph constructed above. Note that by the construction of $(P^S, \succeq_S)$, $(j, d)$ must be indexed $s$, as otherwise it would violate the stability of $\mu_S$; i.e. $mu_s(j^s) = d^s$. Then we have $j^s \succ_{c^s} k$ for $\nu_s(k) = c^s$ which implies that $\nu_S$ is unstable. By Lemma 1, the lottery corresponding to $\nu_S$ is ex post unstable, which contradicts the ex post stability of $\pi$.

$\square$