The efficiency losses of random tie-breaking and ex post stability in school choice

Manlai Amarsaikhan

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Abstract

Deferred Acceptance with random tie-breaking is a very attractive solution concept in school choice. However, it is well known in the literature that it has an efficiency cost. We design an integer program that solves for constrained ordinal efficiency in the ex post stable class to measure the magnitude of the cost of random tie-breaking and ex post stability. In the process, we characterize stability, constrained ordinal efficiency and constrained rank efficiency in the ex post stable class as solutions to integer programming problems. Our simulation results show that random tie-breaking has a significantly larger efficiency cost than ex post stability. In fact, as the market grows larger, the cost of ex post stability shrinks faster than that of random tie-breaking.

1 Introduction

School choice has grown to be the method of choice to allocate students to public schools. In many cities around the world, a centralized mechanism is used to allocate school seats to students. Beginning with Abdulkadiroglu and Sönmez (2003b), the literature on the mechanism design aspects of school choice has grown rapidly.

Studies have shown that a centralized procedure is more efficient and generally "better" at allocation than decentralized procedures (Abdulkadiroglu et al., 2016). In a centralized procedure, students are assigned priorities at each school by the school district administration and all the students have to do is to rank the schools and submit their ranked preferences over the schools. The school district matches students with schools based on their preferences and their priorities. The main advantage of centralized mechanisms is that the mechanism designer can implement a mechanism that satisfies a certain number of desirable properties.

One of the most important properties in school choice is stability. A matching is stable if the following does not occur:

> Amy is matched to school B but prefers school A more. However, Ben is matched to A but has a lower priority than Amy at A.

In other words, Amy is not going to A because of Ben, who has a lower priority than her at A. Therefore, in the context of school choice, stability is used as a fairness notion (Balinski and

Sönmez, 1999). The main problem of school choice has been to match the students with their most preferred schools without violating stability.

When the priorities are strict, there is a solution to the problem: the student proposing deferred acceptance mechanism (DA, henceforth). When priorities are strict, the DA mechanism yields an optimal stable allocation for the students. Moreover, the DA outcome is strategy-proof. However, in practice, priorities are never strict and there are many ties, in many cases, hundreds of students have the same priority. The DA algorithm works only when priorities are strict, so in practice, an explicit tie-breaking lottery is used. Many school districts, including Boston and New York City in the US, implemented this mechanism (Abdulkadiroglu et al., 2005a, 2005b). We call it DA with random tie-breaking mechanism (random DA, henceforth).

A mechanism is ex post stable if it induces a lottery over stable matchings. As the DA outcome is stable and strategy-proof, it follows that the outcome of random DA is ex post stable and strategy-proof. We show that when there is sufficient number of school seats for students, the outcome of the random DA mechanism is ordinally Pareto undominated in the strategy-proof class (Theorem 1).

It is well documented in the literature that the outcome of random DA can lead to significant efficiency losses. There are two main sources of this efficiency loss. The first source of the efficiency loss is the incompatibility between ex post stability and ex post efficiency (Roth, 1982). The second source comes from random tie-breaking (Erdil and Ergin, 2008; Abdulkadiroglu et al., 2009). Another way to interpret this source of efficiency loss is as follows. When the priorities are strict, the DA outcome is Pareto dominant within the set of stable matchings (Gale and Shapley, 1962) and strategy-proofness comes for free. However, there is a cost in ensuring strategy-proofness with random tie-breaking. In this paper, we measure these efficiency losses.

The approach we take to measure them is as follows. We find a constrained ordinally efficient random matching within the set of ex post stable random matchings and compare it with the random DA outcome and an ordinally efficient random matching. Example 1 elaborates on how we do this. The challenge of this approach is to characterize ex post stability. We overcome this challenge by first characterizing stability.

We characterize stability in the domain where priorities allow ties. We show that it is described by the integer solutions to a linear programming problem (Theorem 2). Using the characterization of stability in this way, it is straightforward that ex post stability is a polytope spanned by the set of stable matchings.

Consider any ex post stable random matching. On it, the assignment probability of a student to a school can be traded for an equal amount of probability at a better school for the student as long as the trade does not violate ex post stability. Such trading opportunities are called ex post stable improvement cycles. We show that a random matching is constrained ordinally efficient within the set of ex post stable random matchings if and only if there is no ex post stable improvement cycle (Proposition 1). Then, we show that after choosing appropriate weights, the solutions to a linear programming problem characterizes the set of ordinally Pareto undominated random matchings within the set of ex post stable random matchings (Theorem 3). Since constrained ordinal efficiency is characterized via linear programming, the outcome may be a matching rather than a random matching. To ensure equal treatment of equals, i.e. equal students receive equal distributions, we present a procedure. Essentially, we identify equal students and equalize their outcomes. We show that after choosing the weights carefully, a solution to the linear programming problem treats equals equally as well as constrained ordinal efficiency, once it runs through this procedure (Proposition 4).

A mechanism is rank efficient if it matches as many students as possible to their best choice among the remaining choices. Rank efficiency implies ordinal efficiency. Boston mechanism is rank efficient for the stated preferences. We also show that for certain weights, the solutions to the linear programming problem characterize constrained rank efficiency within the set of ex post stable random matchings (Proposition 3).

By recognizing that a matching is stable if and only if it is an integer solution to a linear programming problem, we provide a more computationally tractable way to find specific ordinally (rank) efficient random matchings. The procedure is essentially an integer programming problem where the constraint set is significantly easier to define than that used to characterize constrained ordinal (rank) efficiency (Theorem 2). This theorem is vital for the purposes of our simulation, and more importantly, it is computationally feasible in practical applications.

Since the outcome of random DA is on the ordinal Pareto frontier of the set of strategy-proof random matchings, the only constraint causing the efficiency loss in the ex post stable class is its strategy-proofness. Therefore, the efficiency difference between the outcome of random DA and a constrained rank efficient random matching is the efficiency loss of strategy-proofness. Moreover, if we further relax ex post stability, for appropriate weights, a solution to the linear programming problem is rank efficient (Featherstone, 2014). The efficiency difference between a constrained rank efficient random matching and a rank efficient random matching is the efficiency loss of ex post stability.

Using the aggregate statistics of the Boston school choice data from Abdulkadiroglu et al. (2006) via Kesten and Ünver (2015), we compare the overall efficiencies of a random DA outcome, a constrained rank efficient outcome and a rank efficient outcome. Our simulation results show that the efficiency cost of strategy-proofness is very large compared to that of ex post stability. In fact, the cost of ex post stability shrinks very fast as the size of the schools increase. The efficiency cost of strategy-proofness also decrease as the size of the schools increase but not as fast as that of ex post stability. Moreover, neighborhood priority creates a lot more efficiency cost as opposed to sibling priority.

Since the mechanism adopted in Boston and New York City was chosen due to its stability rather than its efficiency,[1] we believe that stability plays a key role for a school choice mechanism in practice. Consequently, this paper focuses on constrained ordinal efficiency within the ex post stable class. Moreover, our Corollary 2 provides a practically useful way to find constrained ordinally efficient random matchings within the ex post stable class.

[1] Abdulkadiroglu and Sönmez (2003b) propose top trading cycles mechanism as a possible school choice mechanism. It is strategy-proof and Pareto efficient but it is not ex post stable.

The rest of the paper is organized as follows. The next subsection touches on the related literature. Section 2 introduces the model, and Section 3 sets up the problem and illustrates what is coming in the paper. Section 4 characterizes stability and constrained ordinal efficiency. Section 5 shows simulation results, and Section 6 concludes. All proofs are collected in the Appendix.

1.1 Related Literature

Our paper is closely related to a number of strands of the matching literature. Specifically, it is related to a host of papers that study the set of ex post stable random matchings. It is also related to papers on ordinal efficiency and constrained efficiency in various contexts. In the context of school choice, it is related to more recent papers that propose various forms of constrained efficiency in various classes.

Vande Vate (1989) first showed that the matchings in the core of the marriage model can be described by a linear programming problem whose extreme points are precisely the stable matchings of the marriage problem. It is further elaborated in Rothblum (1992) and Roth, Rothblum and Vande Vate (1993). They characterize the set of ex post stable random matchings in a marriage market framework. Their models are a special case of our model where each school has one seat and no two students have the same priority at any school.

Baïou and Balinski (2000) consider ex post stable random matchings in the context of college admissions. They characterize the set of ex post stable random matchings of the college admissions problem by showing that it is described by a linear programming problem whose extreme points are the stable matchings of the college admissions problem. In particular, they define a system of inequalities that they call the comb constraint to characterize stability. We generalize their comb constraint to the case with weak priorities. Sethuraman, Teo and Qian (2006) further explores the set of ex post stable random matchings and generalize the lattice structure of stable matchings to ex post stable random matchings. Their models are a special case of our model where no two students have the same priority at any school.

Bogomolnaia and Moulin (2001) first introduced ordinal efficiency in the context of resource allocation. In their model, each student has the same priority at each school. They characterized ordinal efficiency and proposed a new mechanism, probabilistic serial mechanism that finds the ordinally efficient outcome. Since every feasible matching is stable in their context, they also find the "constrained" ordinally efficient outcome in their domain. Ordinal efficiency is further characterized in Abdulkadiroglu and Sönmez (2003a). They show that the reason that an ex post efficient random matching is not ordinally efficient is due to the fact that an ex post efficient random matchings may be induced by set-dominated assignments. Their models are a special case of our model where every school has one seat and every pair of students has the same priority at every school.

Erdil and Ergin (2008) and Abdulkadiroglu et al. (2009) showed that when ties are broken randomly, there may be efficiency losses in the DA mechanism. Erdil and Ergin (2008) showed that random tie-breaking before applying deferred acceptance algorithm could result in serious efficiency losses. They introduce a procedure to characterize ex post efficiency in the ex post stable class. They prove that a matching is constrained ex post efficient in the ex post stable class if and only if it does not admit a stable improvement cycle. As ex post efficiency implies ordinal efficiency, and we are dealing with random matchings, the outcome of their procedure is not on the ordinal Pareto frontier of the ex post stable random matchings.

In dealing with weak priorities, previous research on school choice has decidedly relied on the deterministic approach to break the ties randomly. A notable exception is Kesten and Ünver (2015). They study the inefficiency in school choice from a different angle by introducing a new notion, ex ante stability and a new algorithm, fractional deferred acceptance and trading algorithm to find a constrained ordinally efficient random matching in the ex ante stable class. Their approach, much like ours, does not rely on deterministic outcomes based on random tie-breaking but rather on random matchings, which allows them to superior levels of efficiency. However, as ex ante stability implies ex post stability, they consider a smaller class than ours.

Using simulations, Abdulkadiroglu et al. (2009) measured the efficiency costs of strategy-proofness and stability in a similar setting as ours. However, the efficiency notion they use is ex post efficiency, which is weaker than ours. They found that strategy-proofness has a significant efficiency cost. However, we found that when comparing to an allocation on the ordinal Pareto frontier of ex post stability, the efficiency costs of strategy-proofness is even larger than what they have found.

2 Model

A school choice problem is a five-tuple $(I, C, q, P, \succeq)$, where I is a finite set of students, C is a finite set of schools, $q = (q_c)_{c \in C}$ is a quota vector for schools where $q_c > 0$ is the maximum quota of school c, $P = (P_i)_{i \in I}$ is a strict preference vector for students where P_i is the strict preference relation of student i over schools, and $\succeq = (\succeq_c)_{c \in C}$ is a weak priority order for schools where $\succeq_c$ is the weak priority order of school c over students.

We assume that there is enough space for all students, i.e. $\sum_{c \in C} q_c = |I|$. Moreover, let R_i be the weak preference relation associated with P_i. R_i is a complete, transitive, and antisymmetric binary relation, while $\succeq_c$ is a reflexive, complete, and transitive binary relation. Let $\succ_c$ be the acyclic portion and $\sim_c$ the cyclic portion of $\succeq_c$, i.e. $i \succeq_c j$ means that student i has at least as high priority as student j at school c, $i \succ_c j$ means student i has a strictly higher priority than student j at school c, and $i \sim_c j$ means that students i and j have equal priority at school c. I, C and q are fixed throughout the paper, so a list of preferences and priorities defines a school choice problem $(P, \succeq)$.

A random matching $\rho = (\rho_{i,c})_{i \in I, c \in C}$ is a matrix where $\rho_{i,c}$ is the probability i is matched with c such that (i) for all $i \in I$ and $c \in C$, $\rho_{i,c} \in [0,1]$, (ii) for all $i \in I$, $\sum_{c \in C} \rho_{i,c} = 1$, (iii) for all $c \in C$, $\sum_{i \in I} \rho_{i,c} = q_c$. Moreover, the stochastic row vector $\rho_i = (\rho_{i,c})_{c \in C}$ denotes the random matching vector of student i at ρ and the stochastic column vector $\rho_c = (\rho_{i,c})_{i \in I}$ denotes the random matching vector of school c at ρ. A random matching is a deterministic matching if

$\forall i \in I, c \in C$, $\rho_{i,c} \in \{0,1\}$. Let $\mathcal{X}$ be the set of random matchings and let $\mathcal{M} \subseteq \mathcal{X}$ be the set of deterministic matchings. We will also represent a deterministic matching $\mu \in \mathcal{M}$ as a list $\mu = \left(\begin{array}{cccc} i_1 & i_2 & \dots & i_{|I|} \\ c_1 & c_2 & \dots & c_{|I|} \end{array} \right)$ where for each $l \in [1, |I|]$, $\mu_{i_l, c_l} = 1$. We interpret each student i_l as matched with school c_l in this list and by slightly abusing notation, we denote it as $\mu_{i_l,c_l} = 1$ and $\mu(i_l) = c_l$.

A lottery λ is a probability distribution over matchings, i.e. $\lambda = (\lambda_\mu)_{\mu \in \mathcal{M}}$ such that (i) for all $\mu \in \mathcal{M}$, $\lambda_\mu \in [0,1]$, (ii) $\sum_{\mu \in \mathcal{M}} \lambda_\mu = 1$. Let $\Delta\mathcal{M}$ denote the set of lotteries. For any $\lambda \in \Delta\mathcal{M}$, we say that a random matching $\rho^\lambda \in \mathcal{X}$ is induced by lottery λ if $\rho^\lambda_{i,c}$ is the probability that student i will be matched to school c under lottery λ, i.e. for all $i \in I$ and $c \in C$, $\rho^\lambda_{i,c} = \sum_{\mu \in \mathcal{M}: \mu_{i,c}=1} \lambda_\mu$.

It turns out that the converse holds as well. For every random matching, there exists a lottery that induces it (Birkhoff, 1946, and von Neumann, 1953). This result lets us focus on random matchings rather than lotteries safe in the knowledge that any random matching has a corresponding lottery, thus can be implemented as a deterministic outcome.

A school choice mechanism selects a random matching for a given school choice problem. For problem $(P, \succeq)$, we denote the random matching of a mechanism φ by $\varphi(P, \succeq)$ and denote the random matching vector of a student i by $\varphi_i(P, \succeq)$.

A random matching $\rho \in \mathcal{X}$ treats equals equally if for any pair of students $i, j \in I$, if $P_i = P_j$ and $i \sim_c j$ for all $c \in C$ implies $\rho_i = \rho_j$, that is, if a set of students have the same preferences and same priorities at all schools, i.e. equals, then they should have the same enrollment probability at every school at a random matching that treats equals equally. Equal treatment of equals is a very weak notion of fairness that is satisfied by virtually every school choice mechanism in practice. If it is not guaranteed, then it calls into question the validity of using lotteries.

A deterministic matching μ is stable if there is no student pair $i, j \in I$ such that $\mu(j) P_i \mu(i)$ and $i \succ_{\mu(j)} j$. Let $\mathcal{M}^s \subseteq \mathcal{M}$ denote the set of stable matchings. $\mathcal{M}^s$ is nonempty (Gale and Shapley, 1962). A random matching $\rho \in \mathcal{X}$ is ex post stable if there exists a lottery that induces ρ that assigns positive weight to only stable matchings. Let $\mathcal{X}^{eps} \subseteq \mathcal{X}$ be the set of ex post stable random matchings. A random matching $\rho \in \mathcal{X}$ is ex ante stable if for all $i, j \in I$, $a, c \in C$, $i \succ_c j$, cP_ia, $\rho_{i,a} > 0$ implies $\rho_{j,c} = 0$. Ex ante stability implies ex post stability, while the converse is not true (Kesten and Ünver, 2015).

A random matching $\pi \in \mathcal{X}$ ordinally dominates random matching $\rho \in \mathcal{X}$ if π first-order stochastically dominates ρ, i.e.

$$\sum_{aR_ic} \pi_{i,a} \geq \sum_{aR_ic} \rho_{i,a}, \quad \text{for all } i \in I \text{ and } c \in C, \text{ and}$$
$$\sum_{aR_jb} \pi_{j,a} > \sum_{aR_jb} \rho_{j,a}, \quad \text{for some } j \in I \text{ and } b \in C$$

A random matching is ordinally efficient if it is not ordinally dominated by any other random matching. A random matching is ex post efficient if there exists a lottery that induces it and has only Pareto efficient matchings in its support.

Ordinal efficiency implies ex post efficiency, while the converse is not true (Bogomolnaia and Moulin, 2001). Moreover, there does not exist any ex post efficient and ex post stable mechanism (Roth, 1982). Therefore, there does not exist any ordinally efficient and ex post stable mechanism. Since we take stability as given, we will focus on constrained ordinal efficiency. Specifically, we will focus on constrained ordinal efficiency in the ex post stable class.

Finally, a mechanism φ is strategy-proof if for each student, his random matching vector obtained through φ via truth-telling either ordinally dominates or is equal to any vector obtained through φ via manipulation, i.e. for all $i \in I$, $(P, \succeq)$ and P'_i,

$$\sum_{aR_ic} \varphi_{i,a}(P, \succeq) \geq \sum_{aR_ic} \varphi_{i,a}(P'_i, P_{-i}, \succeq), \quad \text{for all } c \in C$$

3 A motivating example

We will start this section by explaining the school choice mechanism currently prevalent in many cities in the US. The following algorithm based on Gale and Shapley (1962)'s Deferred Acceptance (DA) algorithm was first introduced in Abdulkadiroglu and Sönmez (2003):

- Randomly break the ties to induce strict priorities.

- *Step 1:* Each student applies to her top choice. Each school tentatively assigns its seats to its appliers one at a time following their priorities. Once a school reaches its quota, it rejects any remaining appliers.

  In general,

- *Step k:* Any student who was rejected in the previous step applies to her next best choice. Each school considers the students tentatively assigned to it from the previous step with the new appliers, and tentatively assigns its seats to these students one at a time following their priorities. Once a school reaches its quota, it rejects any remaining appliers.

The algorithm terminates when no student is rejected. The DA outcome is stable (Gale and Shapley, 1962).

If we break the ties in every possible way and run the DA algorithm, we obtain a set of stable matchings. We call a mechanism that constructs a random matching from a uniform distribution over the tie-breaking rules Deferred Acceptance with random tie-breaking (random DA, henceforth) and denote it as ψ. Abdulkadiroglu et al. (2009) showed that for any tie-breaking rule τ, there is no mechanism that is strategy-proof and dominates the DA outcome at τ. Their result can be generalized to a random environment as well.

Theorem 1. *ψ is constrained ordinally efficient in the strategy-proof class.*

Proof. See Appendix. □

As $\psi(P, \succeq)$ is ex post stable for each $(P, \succeq)$, the following corollary follows immediately from Theorem 1.

Corollary 1. *ψ is constrained ordinally efficient in the ex post stable and strategy-proof class.*

3.1 A motivating example

In this section, we provide a motivating example to illustrate the contents of this paper.

Example 1. Consider the following problem with three students $\{1, 2, 3, 4, 5\}$ and four schools $\{a, b, c, d\}$ where a, b and d have one seat and c has two seats. The priority lists and student preferences are:

$\succeq_a$	$\succeq_b$	$\succeq_c$	$\succeq_d$
$1, 2, 3$	4	5	3
$4, 5$	$1, 2, 3$	$1, 2, 3$	$2, 4$
	5	4	$1, 5$

P_1	P_2	P_3	P_4	P_5
c	d	c	d	b
a	c	d	c	c
b	b	a	b	a
d	a	b	a	d

There are 120 distinct single tie-breaking rules and 1728 distinct multiple tie-breaking rules. However, in the current example, under both tie-breaking rules, the outcome of the random DA mechanism is the following random matching:

$$\psi(P, \succeq) = \begin{array}{c|c|c|c|c} & a & b & c & d \\ \hline 1 & \frac{2}{3} & 0 & \frac{1}{3} & 0 \\ 2 & \frac{1}{3} & \frac{1}{12} & \frac{1}{3} & \frac{1}{4} \\ 3 & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 4 & 0 & \frac{3}{4} & 0 & \frac{1}{4} \\ 5 & 0 & \frac{1}{6} & \frac{5}{6} & 0 \end{array}$$

By Corollary 1, $\psi(P, \succeq)$ is constrained ordinally efficient in the ex post stable and strategy-proof class. However, if we relax strategy-proofness, $\psi(P, \succeq)$ may not be constrained ordinally efficient in the ex post stable class. Indeed, $\psi(P, \succeq)$ is ordinally dominated by $\rho \in \mathcal{X}^{eps}$.

$$\rho = \begin{array}{c|c|c|c|c} & a & b & c & d \\ \hline 1 & \frac{2}{3} & 0 & \frac{1}{3} & 0 \\ 2 & \frac{1}{3} & 0 & \frac{5}{12} & \frac{1}{4} \\ 3 & 0 & 0 & \frac{2}{3} & \frac{1}{3} \\ 4 & 0 & \frac{7}{12} & 0 & \frac{5}{12} \\ 5 & 0 & \frac{5}{12} & \frac{7}{12} & 0 \end{array}$$

It turns out that ρ is a constrained ordinally efficient random matching in the ex post stable class. In Section 4, we formally develop a method to find constrained ordinally efficient random matchings in the ex post stable class. The difference in ordinal efficiency between $\psi(P, \succeq)$ and ρ

can be interpreted as the efficiency loss of strategy-proofness as both random matchings are on the ordinal Pareto frontier of their respective classes.

Moreover, if we relax ex post stability, ρ may be ordinally dominated by another random matching. Indeed, ρ is ordinally dominated by random matching $\pi \in \mathcal{X}$.

$$\pi = \begin{array}{c|c|c|c|c} & a & b & c & d \\ \hline 1 & \frac{2}{3} & 0 & \frac{1}{3} & 0 \\ 2 & \frac{1}{3} & 0 & \frac{1}{12} & \frac{7}{12} \\ 3 & 0 & 0 & 1 & 0 \\ 4 & 0 & 0 & \frac{7}{12} & \frac{5}{12} \\ 5 & 0 & 1 & 0 & 0 \end{array}$$

It is easy to check that π is an ordinally efficient random matching. Since ρ and π are both on the ordinal Pareto frontier of their respective classes, the difference in ordinal efficiency can be interpreted as the efficiency loss of ex post stability. In Section 5, we show simulation results and provide analysis on the efficiency losses of strategy-proofness and ex post stability. $\triangle$

4 Constrained ordinal efficiency

4.1 Ex post stability

In this section, we will characterize stability in our domain. The definition of stability may be interpreted as follows: For a matching to be stable, for an arbitrary school c and student i, if i is not matched to c then c is either filled with students who have a weakly higher priority than i at c, or if it is not, then i must be matched to a school she prefers more than c; not only i but also the students with weakly higher priority than i at c who are not matched to c should be matched to a school they prefer more than c, i.e. for each school c and each student i, the students with weakly higher priority than i (including i) at c are either matched to c or to a school they prefer more than c. To introduce the condition formally, we need to define a few notations first.

For student $i \in I$ and school $c \in C$, let $B(i,c)$ be the set of students with a weakly higher priority than i at c, i.e. $B(i,c) = \{j \in I | j \succeq_c i\}$. We will define an auxiliary quota vector for each student-school pair. Let auxiliary quota $\hat{q}_{i,c}$ for each $i \in I$ and $c \in C$ be defined as

$$\hat{q}_{i,c} = \begin{cases} q_c, & \text{if } |B(i,c)| \geq q_c \\ |B(i,c)|, & \text{otherwise} \end{cases}$$

For each $i \in I$ and $c \in C$, let $Com(i,c)$ denote the set of $\hat{q}_{i,c}$-combinations of $B(i,c)$, i.e. $Com(i,c) = \binom{B(i,c)}{\hat{q}_{i,c}}$.

We are now ready to formally characterize stability.

Theorem 2. *The following statements are equivalent:*

(i) A deterministic matching $\mu \in \mathcal{M}$ is stable.

$$(ii) \quad \sum_{j \in B(i,c)} \mu_{j,c} + \sum_{j \in T} \sum_{a P_j c} \mu_{j,a} \geq \hat{q}_{i,c}, \qquad \text{for all } i \in I,\ c \in C \text{ and } T \in Com(i,c) \tag{1}$$

$$(iii) \quad \sum_{j \in B(i,c)} \mu_{j,c} + q_c \sum_{a P_i c} \mu_{i,a} + (q_c - 1)\mu_{i,c} \geq q_c, \text{ for all } i \in I \text{ and } c \in C \tag{2}$$

Proof. See Appendix. □

We call inequalities (1) the comb constraint, and (2) the no blocking condition. By its definition, the set of ex post stable random matchings is a convex hull of stable deterministic matchings, i.e.

$$\mathcal{X}^{eps} = conv\{\mathcal{M}^s\}$$

Therefore, Theorem 2 implies that the extreme points of $\mathcal{X}^{eps}$ are the integer solutions to either the comb constraint or the no blocking condition.

4.2 Stochastic improvement cycles

A stochastic improvement cycle Cyc$= (i_1, c_1, ..., i_n, c_n)$ at ρ with $(i_{n+1}, c_{n+1}) \equiv (i_1, c_1)$ is a list of distinct student-school pairs (i_s, c_s) such that $c_{s+1} P_{i_s} c_s$ and $\rho_{i_s, c_s} > 0$, for all $s \in \{1, ..., n\}$. Cycle Cyc is satisfied with fraction $\varepsilon > 0$ at ρ if for all $s \in \{1, ..., n\}$, a fraction ε of school c_{s+1} is assigned to student i_s additionally and a fraction ε of school c_s is removed from her random matching, while no other matching probability is changed. Formally, we obtain a new random matching ρ' such that for all $i \in I$ and $c \in C$,

$$\rho'_{i,c} = \begin{cases} \rho_{i,c} + \varepsilon, & \text{if } i = i_s \text{ and } c = c_{s+1} \text{ for some } s \in \{1, ..., n\} \\ \rho_{i,c} - \varepsilon, & \text{if } i = i_s \text{ and } c = c_s \text{ for some } s \in \{1, ..., n\} \\ \rho_{i,c}, & \text{otherwise} \end{cases}$$

Let ex post stable improvement cycle Cyc from ρ to ρ' be a set of stochastic improvement cycles such that $\rho' \in \mathcal{X}^{eps}$.

Proposition 1. *An ex post stable random matching ρ is constrained ordinally efficient if and only if there is no ex post stable improvement cycle at ρ.*

Proof. See Appendix. □

4.3 Constrained optimization

We can treat the problem of finding constrained ordinally efficient random matchings as a constrained optimization problem. If we take the students' probability distribution over the schools as the objective function, we can optimize it over the set of ex post stable random matchings to find a constrained optimal distribution.

Let $c(i,k) \in C$ denote the kth most preferred school of student $i \in I$, i.e. $|\{a \in C | aR_i c(i,k)\}| = k$. Then, let $w_{i,c(i,k)}$ be the weight of the kth most preferred school for student i who lists $K(i)$ schools. Then a weight matrix $w = (w_{i,c(i,k)})_{k\in\{1,\dots,|C|\}}$ is descending if for all $i \in I$, $w_{i,c(i,k)} > w_{i,c(i,k+1)}$, respectively) for all $1 \leq k \leq K(i)$ and $w_{i,c(i,k)} = 0$ for all $k > K(i)$. With a slight abuse of notation, we denote $w_{i,c}$ as the weight of school c at student i's weight vector w.

Consider the following constrained optimization problem:

$$\max_{\rho\in\mathcal{X}^{eps}} \sum_{i\in I}\sum_{c\in C} w_{i,c}\rho_{i,c} \tag{3}$$

Theorem 3. *An ex post stable random matching is constrained ordinally efficient if and only if there exists a descending weight matrix such that it is a maximizer to (3).*

Proof. See Appendix. □

The "if" direction of the proof shows that if there is an ex post stable improvement cycle from ρ to ρ', then ρ cannot be a maximizer to (3). The "only if" direction follows a similar argument to the proof of the second efficiency theorem of economics. Since $\mathcal{X}^{eps}$ is convex, the set of random matchings that ordinally dominate it is also convex, we can separate them with a hyperplane and using the axes of the unit vector of this hyperplane, we can identify the weight matrix in such a way that for each student i, $w_{i,c(i,k)} \geq w_{i,c(i,k+1)}$. Moreover, the fact that $\mathcal{X}^{eps}$ is a polytope allows us to use the polyhedral separating hyperplane theorem of McLennan (2002), via which we show that weight matrix is indeed descending.

As (3) is a linear programming problem, there exists an extreme point solution, i.e. a matching. By Theorem 2, a stable matching is an integer solution to the comb constraint. Therefore, if we optimize over the set of stable matchings, we can find a constrained ordinally efficient random matching. Using these observations, we can state the following corollary that follows immediately from Theorem 3. It significantly reduces the computational load in our simulation and more importantly, provides a practical way to find constrained ordinally efficient random matchings.

Corollary 2. *For a given descending weight matrix w, a maximizer to the following integer* *programming problem is constrained ordinally efficient:*

$$
\begin{aligned}
\max \sum_{i \in I} \sum_{c \in C} w_{i,c} \rho_{i,c} \quad (4) \\
\text{such that} \\
\rho_{i,c} \in \{0,1\}, \;\; \text{for all } i \in I \text{ and } c \in C, \\
\sum_{a \in C} \rho_{i,a} \leq 1, \;\; \text{for all } i \in I, \\
\sum_{j \in I} \rho_{j,c} \leq q_c, \;\; \text{for all } c \in C, \\
\sum_{j \in B(i,c)} \rho_{j,c} + q_c \sum_{a P_i c} \rho_{i,a} + (q_c - 1)\rho_{i,c} \geq q_c, \;\; \text{for all } i \in I \text{ and } c \in C
\end{aligned}
$$

4.4 Constrained rank efficiency

The characterization of constrained ordinal efficiency implies that it is possible for the mechanism designer to give a priority to certain students such that these students maximize their chances at their top choice schools before other students get a chance at those schools.

Consider a refinement of ordinal efficiency as follows. Suppose we allocate as many students as possible to their top choice schools without violating ex post stability. Once every student's top choice school is full, we allocate as many students as possible to their second best choice schools without violating ex post stability. In general, we allocate as many students as possible to their top choice among the schools with an open slot without violating ex post stability. Is it possible to assign the weights in (3) such that we obtain an ex post stable random matching that matches students in a way described as above? The answer is affirmative.

First, we define a new efficiency notion. A random matching $\rho \in \mathcal{X}$ is rank-dominated by another random matching $\pi \in \mathcal{X}$ if

$$
\begin{aligned}
\sum_{j \in I} \sum_{a R_j c(j,k)} \pi_{j,a} \geq \sum_{j \in I} \sum_{a R_j c(j,k)} \rho_{j,a}, \quad \text{for all } k \\
\sum_{j \in I} \sum_{a R_j c(j,k)} \pi_{j,a} > \sum_{j \in I} \sum_{a R_j c(j,k)} \rho_{j,a}, \quad \text{for some } k
\end{aligned}
$$

A random matching $\rho \in \mathcal{X}$ is rank efficient if it is not rank-dominated by any other random matching. The following Proposition is due to Featherstone (2014).

Proposition 2. *Rank efficiency implies ordinal efficiency, while the converse is not true.*

Due to the inherent incompatibility between stability and efficiency, the best we can hope for is constrained rank efficiency if we also wish to maintain ex post stability. To state the characterization of constrained rank efficiency, we need to define a weight matrix that is crucial for the characterization. A descending weight matrix w is RE if for each pair of students $i, j \in I$ with $K(i) \leq K(j)$, $w_{i,c(i,k)} = w_{j,c(j,k)} = w_k$ for all $1 \leq k \leq K(i)$. Then, we can state the following proposition.

Proposition 3. *An ex post stable random matching is constrained rank efficient if and only if it is a maximizer to (3) for some RE weight matrix.*

Proof. See Appendix. □

The "if" direction of the proposition follows immediately from the definition of constrained rank efficiency. The "only if" direction is very similar to that in the proof of Theorem 3. The only difference is that there is an added constraint that every rank should have the same weight for every student, which alters the space. In Theorem 3, since the weights can be different across students, the proof proceeds in $\mathbb{R}^{|I||C|}$ space, whereas in proposition 3, the proof proceeds in $\mathbb{R}^{|C|}$ space.

The following corollary follows immediately from Proposition 3.

Corollary 3. *Given a RE weight matrix, a maximizer to (4) is constrained rank efficient.*

4.5 Equal treatment of equals

Since (3) is a linear programming problem, one solution is an extreme point. The solution to (4) is, by design, an integer solution. Therefore, equal treatment of equals may not be satisfied. The ensure equal treatment of equals, we sum over the equal students' distribution at the solution and then divide the summation equally among them.

Assume $\rho^{(0)}$ is an optimal solution to (3) (or (4)). Then we construct a constrained ordinally efficient random matching ρ that treats equals equally using the following procedure (ETE procedure).

- *Step 1:* Find a set of equal students, $I_1^e \subseteq I$. If I_1^e is a singleton, then end the procedure. The final outcome is $\rho^{(0)}$. If $|I_1^e| \geq 2$, then for each student $i \in I$,

$$\rho_i^{(1)} = \begin{cases} \frac{1}{|I_1^e|} \sum_{j \in I_1^e} \rho_j^{(0)}, & i \in I_1^e \\ \rho_i^{(0)}, & i \in I \backslash I_1^e \end{cases}$$

  In general,

- *Step k:* Find a set of equal students, $I_k^e \subseteq I \backslash \{I_1^e, ..., I_{k-1}^e\}$. If I_k^e is a singleton, then end the procedure. The final outcome is $\rho^{(k-1)}$. If $|I_k^e| \geq 2$, then for each student $i \in I$,

$$\rho_i^{(k)} = \begin{cases} \frac{1}{|I_k^e|} \sum_{j \in I_k^e} \rho_j^{(k-1)}, & i \in I_k^e \\ \rho_i^{(k-1)}, & i \in I \backslash I_k^e \end{cases}$$

The procedure ends when no equal students are left.

ETE procedure works only for solutions to specific weight matrices. Specifically, equal students should have equal weight vectors. More formally, ETE weight matrix w is defined as follows: for each $i, j \in I$, if $P_i = P_j$ and $i \sim_c j$ for all $c \in C$, then $w_{i,a} = w_{j,a}$ for all $a \in C$.

Proposition 4. *Let $\rho^{(0)}$ be a maximizer to (3) for an ETE weight matrix. Then, the above procedure yields a constrained ordinally efficient random matching that treats equals equally.*

Proof. See Appendix. □

Since equal students have equal priorities at every school, it will not violate the constraint set. Moreover, if equal students have equal weight matrices, then since equal students rank the schools in the same order, it will not alter the optimality.

5 Simulations

We ran simulations to measure the efficiency loss of the NYC/Boston deferred acceptance algorithm with a single tie-breaking lottery (DA henceforth) and the efficiency loss of the constrained efficient random matching using simulated data that approximately match the main characteristics of the Boston data from 2008 to 2011 (Abdulkadiroglu et al., 2006). Note that if the constraint set of the linear program (3) is the set of random matchings, $\mathcal{X}$, it finds an ordinally efficient outcome, i.e. the following linear program finds an ordinally efficient outcome

$$\max_{\rho \in \mathcal{X}} \sum_{i \in I} \sum_{c \in C} w_{i,c} \rho_{i,c} \tag{5}$$

By Corollary 1, the DA outcome is constrained ordinally efficient in the strategy-proof and ex post stable class, the efficiency loss of a DA outcome come from two sources: its strategy-proofness and its ex post stability. Integer programming problem (4) finds a constrained ordinally efficient outcome in the ex post stable class; thus the difference in efficiency between its outcome and the DA outcome may be interpreted as the efficiency loss of strategy-proofness. However, the outcome of (4) is still constrained by ex post stability. As linear program (5) finds an ordinally efficient outcome, the difference in efficiency between its outcome and the outcome of linear programming problem (4) may be interpreted as the efficiency loss of ex post stability.

In our simulations, we randomly generate 100 markets, each with $|C|$ schools and $|I|$ students, and computed the corresponding outcomes of linear programs (4), (5) and the DA mechanism, where 200 random single tie-breaking priority lists were generated for the latter.

Using the aggregate statistics, we simulated markets that satisfied the main characteristics of the Boston school choice market. The data showed that there were, on average, 26 schools and 2705 students per year applying for high school. On average, each school had neighborhood priority for 208 students and there were, on average, 2 schools per neighborhood. Moreover, the data suggested that 60% of the students have siblings in the system. However, we do not have data on the number of older siblings who generate the sibling priority for the younger siblings.

The aggregate statistics also showed that students ranked a sibling's neighborhood school 3% of the time, a sibling's nonneighborhood school 3% of the time, a neighborhood school without sibling priority 30% of the time and a nonneighborhood school without sibling priority 64% of the time.

For the purposes of our simulation, we made the following assumptions. The students were zoned in n neighborhoods each with $|C|/n$ schools and $|I|/n$ students on average. Moreover, 30% of the students have sibling priority, among which 1/3 have sibling priority at a neighborhood school and 2/3 have sibling priority at a nonneighborhood school. Moreover, we assumed that a student has sibling priority at only one school. This rules out cases where a student may have multiple older siblings attending multiple schools. As in Boston, the priorities at each school were generated such that neighborhood students with sibling priority were prioritized first, nonneighborhood students with sibling priority second, neighborhood students without sibling priority third, and nonneighborhood students without sibling priority last.

We generated the student preferences using the following randomization process: Each student with sibling priority at a neighborhood school had probability p_{sn} to first-rank the neighborhood school that her sibling is attending, probability p_n to first-rank the other neighborhood school, and the remaining probability was divided equally for each nonneighborhood school to determine its probability to be first-ranked. Each student with sibling priority at a nonneighborhood school had probability p_s to first-rank the nonneighborhood school that her sibling is attending, probability p_n to first-rank each neighborhood schools, and the remaining probability was divided equally for each nonneighborhood school to determine its probability to be first-ranked. Each student without sibling priority had probability p_n to first-rank each neighborhood schools, and the remaining probability was divided equally for each nonneighborhood school to determine its probability to be first-ranked. Once the first choices were randomly determined, the conditional probabilities for each student's remaining schools were updated and then the second choices were determined using the same process. This process was iterated until every student ranked every school in the market. This assumes that no school is less preferred than being unmatched for any student.

In our simulations, we had $|C| = 20$ schools and $n = 10$ neighborhoods, so that there were 2 schools per neighborhood. To have a manageable simulation we generated markets where there were $|I| = 400$ students, so that there were, on average, 40 students per neighborhood instead of 208.

For linear programs (3) and (5), the RE weights were assigned as follows: Given the students' preferences, we assigned weights to each assignment probability such that the higher ranked a school is for a student, the higher the weight is. Moreover, for each pair of students, their kth ranked schools have the same weight. Specifically, for each $i \in I$ and if school $c \in C$ is her kth ranked school, then its weight $w_{i,c}$ is

$$w_{i,c} = |C| - k + 1$$

We report the results of the simulations below.

Choices	random DA outcome	constrained rank efficient outcome	rank efficient outcome
1st	77.78 (2.84)	90.44 (1.74)	91.20 (1.58)
2nd	12.93 (1.70)	9.25 (1.64)	8.75 (1.56)
3rd	4.77 (0.79)	0.31 (0.31)	0.05 (0.14)
4th	2.20 (0.44)		
5th	1.12 (0.30)		
6th	0.58 (0.21)		
7th	0.30 (0.12)		
8th	0.16 (0.07)		
9th	0.09 (0.06)		
10th	0.04 (0.03)		
11th	0.02 (0.02)		

Table 1: Comparison of random DA outcomes with constrained rank efficient outcomes and rank efficient outcomes in the simulations (the simulation standard errors of the fraction of students are given in parentheses after the mean)

Table 1 show the average allocation of students in the simulations. Under the random DA mechanism, the fraction of students who receive their first choice is 77.78%, while it is 90.44% for the constrained rank efficient outcome and 91.20% for the rank efficient outcome. It is striking that ex post stability is not much of a constraint compared to strategy-proofness, at least when looking at the ordinal preferences. In fact, the distributions of constrained rank efficient random matching is statistically not different from that of rank efficient random matching (one sided, $p = 0.5$).

Our results support that of Bodoh-Creed (2016). His simulations also showed that strategy-proofness is quite restrictive and costly compared to stability. However, his notion of stability is ex ante stability, therefore it does not quite grasp the ordinal Pareto frontier of stable matchings.

The reason for the discrepancy between random DA outcome and a constrained rank efficient outcome is that to preserve strategy-proofness, under random DA mechanism, ties are broken randomly, which results in certain students being prioritized more than others which potentially hurts the overall efficiency of the students.

Figure 1 plots the distributions with only neighborhood priority and Figure 2 plots the distributions with only sibling priority. The preference ranks are on the horizontal axis, and percentages of student fractions are on the vertical axis. In the simulations to generate these distributions in Figure 1, we assumed that the students with only neighborhood priority choose their preferences in the same way as the main case, while those with siblings choose exactly like those without sibling priority in the main case. In the simulations for Figure 2, we assumed that the students with siblings choose their preferences in the same way as in the main case, while for the students without siblings, the preferences were generated as follows. A student without a sibling randomly chooses one school as her first choice, and then randomly chooses one school

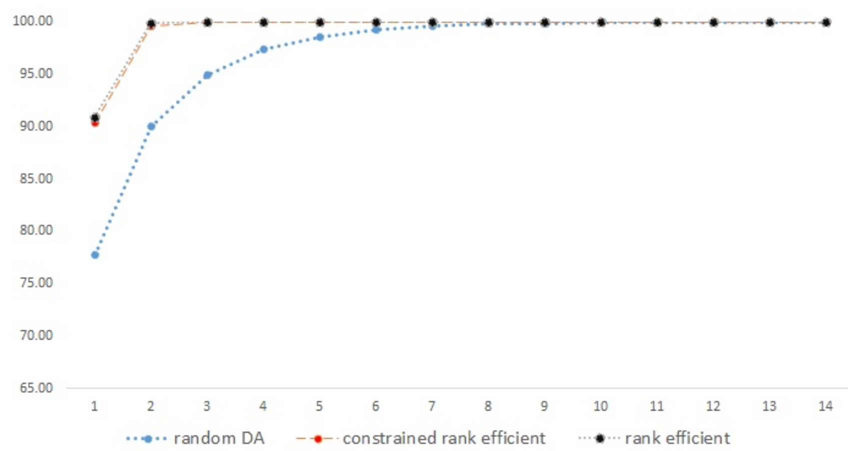

Figure 1: The distributions with only neighborhood priority

among the remaining schools as her second choice, and so on until she ranks all $|C|$ schools.

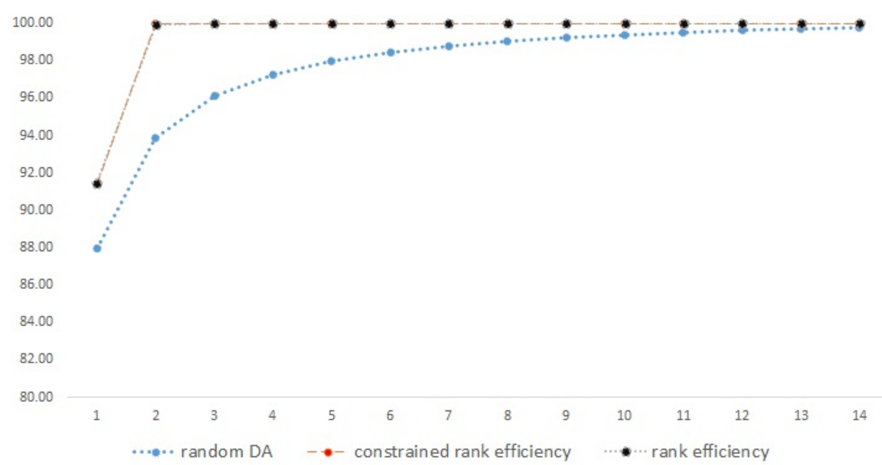

Figure 2: The distributions with only sibling priority

An interesting observation is that the distributions with neighborhood priority are essentially the same (one sided, $p = 0.5$) as the distributions of the main case. Therefore, there is still significant cost to strategy-proofness while essentially no cost to ex post stability. However, in the case with only sibling priority, while the cost of ex post stability is even smaller, that of strategy-proofness significantly shrinks compared to the other two cases.

5.1 Large market implications

There is a growing literature on large markets where difficulties associated with impossibility results shrink to make positive results possible (Kojima and Pathak, 2009; Azevedo and Budish, 2013). Simulation results show that as the size of schools increase, efficiency costs of strategy-proofness and ex post stability fall. Moreover, simulation results suggest that the cost of ex post stability shrinks faster than that of strategy-proofness.

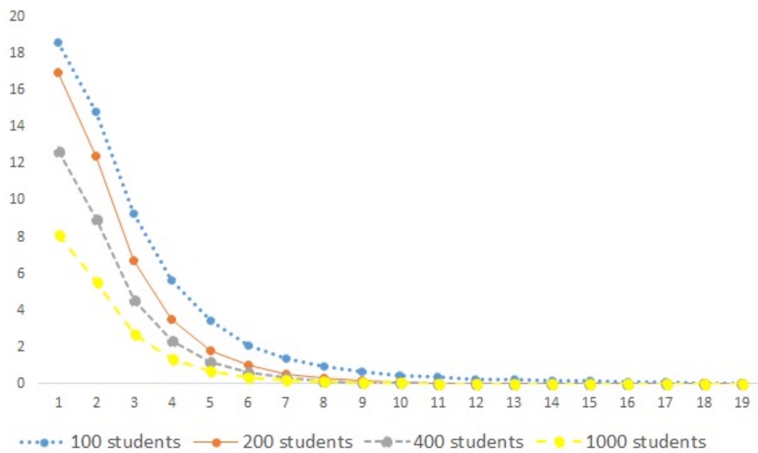

Figure 3: The efficiency differences between random DA outcomes and constrained rank efficient ex post stable outcomes

Figure 3 plots the difference between the cumulative distributions of random DA outcomes and constrained rank efficient ex post stable outcomes and Figure 4 plots the difference between the cumulative distributions of constrained rank efficient ex post stable outcome and rank efficient outcomes for $|I| = 100$, $|I| = 200$, $|I| = 400$ and $|I| = 1000$. For $|I| = 100$, 18.59% (4.60) more student fractions were matched to their first choice school at a constrained rank efficient ex post stable outcome than at a random DA outcome (with simulation standard deviation in parentheses). This difference dropped to 16.97% (3.80), 12.66% (2.05), and 8.12% (1.61) for $|I| = 200$, $|I| = 400$ and $|I| = 1000$ respectively. Moreover, for $|I| = 100$, 14.83% (3.54) more student fractions were matched to their either first or second choice schools at a constrained rank efficient ex post stable outcome than at a random DA outcome. This difference dropped to 12.40% (2.80), 8.98% (1.43) and 5.57% (1.15), for $|I| = 200$, $|I| = 400$ and $|I| = 1000$, respectively. The implication from Figure 3 is that as the size of the schools increase, the cost of strategy-proofness decreases.

For $|I| = 100$, 3.13% (2.37) more student fractions were matched to their first choice school at a rank efficient outcome than at a constrained rank efficient ex post stable outcome (with simulation standard deviation in parentheses). This difference dropped to 1.61% (1.11), 0.75% (0.54) and 0.09% (0.17), for $|I| = 200$, $|I| = 400$ and $|I| = 1000$, respectively. Moreover, for $|I| = 100$, 2.52% (2.38) more student fractions were matched to their either first or second choice schools at a rank efficient outcome than at a constrained rank efficient ex post stable outcome. This difference dropped to 0.92% (0.80), 0.25% (0.30) and 0.01% (0.04), for $|I| = 200$, $|I| = 400$ and $|I| = 1000$, respectively. The implication from Figure 4 is that as the size of the schools increase, the cost of ex post stability decreases.

As the student population becomes significantly larger than the number of schools, ex post

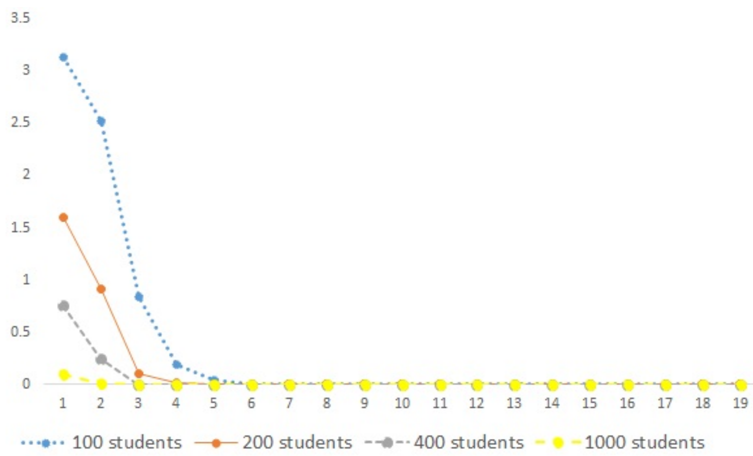

Figure 4: The efficiency differences between constrained rank efficient ex post stable outcomes and rank efficient outcomes

stability is virtually costless and while strategy-proofness still has a significant efficiency cost.

6 Conclusion

This paper takes an integer programming approach to measure the efficiency costs of strategy-proofness and ex post stability in the context of school choice. To do so, we characterized stability, and by explicitly using it as a constraint in an integer programming problem, we were able to measure the efficiency costs of strategy-proofness and ex post stability. In the simulations, we found that strategy-proofness has significant costs compared to ex post stability. However, these costs shrink as schools become larger. Therefore, our simulation results argues both against and for random DA mechanism. Even though random DA mechanism has significant efficiency costs due to its strategy-proofness, the cost is very small in real-life large school settings.

Our constrained optimization approach adds to a relatively recent strand of matching literature (Featherstone, 2014; Bodoh-Creed, 2016). Most of the literature on school choice, and matching market design in general, defines clear desiderata first and then constructs mechanisms that meet these desiderata. This is in effect, the same as optimizing an objective over constraints. However, in many of these studies, the line between constraints and objective is not clear. We however, explicitly define the constraints and the objective of our problem and use the machinery available from integer programming. As it is with any approach, this approach has both its merits and drawbacks. Since each constraint has its shadow price, this approach provides a convenient way to analyze the effects of relaxing or strengthening a constraint on the objective. This also implies that provided that certain sufficiency conditions are met, it may be convenient for conducting comparative statics. However, one drawback of the approach is that to see whether a solution satisfies a certain property or not often requires explicit characterization of that property, which in some cases is non-trivial.

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Appendix A Proofs of Sections 3

We first prove a lemma useful in some of the theorems proved here.

Lemma 1. *If $\pi \in \mathcal{X}$ ordinally dominates $\rho \in \mathcal{X}$, then there exists a stochastic improvement cycle from ρ to π.*

Proof. Suppose π ordinally dominates ρ. Since π dominates ρ, there exists some student $i_1 \in I$ and school $c_2 \in C$ such that $\pi_{i_1,c_2} > \rho_{i_1,c_2}$, while $\rho_{i_1,c_1} < \pi_{i_1,c_1}$ with $c_2 P_{i_1} c_1$. Since $\pi_{i_1,c_2} > \rho_{i_1,c_2}$, there exists a student $i_2 \in I$ such that $\pi_{i_2,c_2} < \rho_{i_2,c_2}$. Now, since π ordinally dominates ρ, there exists $c_3 \in C$ with $c_3 P_{i_2} c_2$ such that $\pi_{i_2,c_3} > \rho_{i_2,c_3}$. Notice that both i_1 and i_2 become strictly better off under π. Thus as we continue iteratively, we obtain a sequence $c_1, i_1, ..., c_n, i_n, c_{n+1}, i_{n+1}$ of schools and students where each pair (c_s, i_s) appears only once. Since C and I are finite, this sequence is finite, i.e. $(c_{n+1}, i_{n+1}) \equiv (c_1, i_1)$. The relationship between the students and the schools in the sequence is as follows:

$$\pi_{i_s,c_{s+1}} = \rho_{i_s,c_{s+1}} + \varepsilon,\ \pi_{i_s,c_s} = \rho_{i_s,c_s} - \varepsilon \text{ and } c_{s+1} P_{i_s} c_s, \quad s = 1, 2, ..., n, \quad \text{for some } \varepsilon > 0 \tag{6}$$

We call sequence (6) Cyc and we say that i_s points to c_{s+1} in Cyc. Cyc shows that there is a stochastic improvement cycle from ρ to π, so that π ordinally dominates ρ. □

Proof of Theorem 1.

Let φ be a strategy-proof mechanism. Fix $\succeq$. Suppose to the contrary, φ ordinally dominates

ψ. That is, there exists a preference profile P such that

$$\sum_{aR_ic} \varphi_{i,a}(P, \succeq) \geq \sum_{aR_ic} \psi_{i,a}(P, \succeq), \quad \text{for all } i \in I \text{ and } c \in C, \text{ and}$$

$$\sum_{aR_jb} \varphi_{j,a}(P, \succeq) > \sum_{aR_jb} \psi_{j,a}(P, \succeq), \quad \text{for some } j \in I \text{ and } b \in C \tag{7}$$

By Lemma 1, there exists a stochastic improvement cycle from $\psi(P, \succeq)$ to $\varphi(P, \succeq)$. Note that there could be many such cycles. Let Cyc be the set of cycles and let $I' \subseteq I$ be the set of students who are in a cycle. Consider student $j \in I'$. Let b be the highest ranked school in P_j such that inequality (7) holds, i.e.

$$\varphi_{j,a}(P, \succeq) = \psi_{j,a}(P, \succeq), \quad \text{for all } a \in C \text{ with } aP_jb \tag{8}$$

Consider the profile $P' = (P'_j, P_{-j})$ where P'_j ranks every school less preferred than b as unacceptable, and for all $a, c \in C$ with $\{a, c\}R_jb$, $aR'_jc \Leftrightarrow aR_jc$. Since the tie-breaking rules remain the same and ψ is induced by a lottery over DA outcomes,

$$\psi_{j,a}(P', \succeq) = \psi_{j,a}(P, \succeq), \qquad \text{for all } a \in C \text{ with } aR'_jb \tag{9}$$

$$\sum_{aR'_ic} \psi_{i,a}(P', \succeq) \geq \sum_{aR'_ic} \psi_{i,a}(P, \succeq), \quad \text{for all } c \in C \text{ and } i \in I \backslash \{j\} \tag{10}$$

Note that at P', since b is the least preferred school of j, she cannot be in any stochastic improvement cycle to improve her probability of entering b; and moreover, b is full at both $\psi(P, \succeq)$ and $\psi(P', \succeq)$, hence

$$\sum_{aR'_jb} \varphi_{j,a}(P', \succeq) = \sum_{aR'_jb} \psi_{j,a}(P', \succeq) \tag{11}$$

Consider an economy where P' is the true preference. Then since φ is strategy-proof,

$$\sum_{aR'_jc} \varphi_{j,a}(P', \succeq) \geq \sum_{aR'_jc} \varphi_{j,a}(P, \succeq), \quad \text{for all } c \in C \text{ with } cR'_jb \tag{12}$$

Combining (7), (8), (9) and (12), we obtain the following:

$$\sum_{aR'_jb} \varphi_{j,a}(P', \succeq) > \sum_{aR'_jb} \psi_{j,a}(P', \succeq) \tag{13}$$

However, this contradicts (11). Therefore φ is not strategy-proof. □

Appendix B Proofs of Section 4

To prove Theorem 2, we need to define the term, blocking pair. In a deterministic matching, a student-school pair (i,c) forms a blocking pair if $\sum_{j\in B(i,c)} \rho_{j,c} < q_c$ and $\sum_{aR_ic} \rho_{i,a} = 0$. A deterministic matching is stable if and only if there is no blocking pair.

Proof of Theorem 2.

$(ii) \Rightarrow (i)$. Suppose μ is not stable. Let (i,c) be a blocking pair, i.e. $\sum_{j\in B(i,c)} \mu_{j,c} < q_c$ and $\sum_{aR_ic} \mu_{i,a} = 0$. Let $J \subseteq B(i,c)$ be the set of students who are matched to c at μ. As μ is integer, we have $|J \cup \{i\}| \le \min\{|B(i,c)|, q_c\}$; thus $|J| < \hat{q}_{i,c}$. Then construct T as follows. Select any $(\hat{q}_{i,c} - |J \cup \{i\}|)$ students from $B(i,c)\backslash(J \cup \{i\})$ and construct T by combining these students with those in $J \cup \{i\}$. Then we have

$$\sum_{j\in B(i,c)} \mu_{j,c} + \sum_{j\in T}\sum_{aP_ic} \mu_{j,a} = \sum_{j\in T\backslash\{i\}}\sum_{aR_jc} \mu_{j,a} + \sum_{j\in B(i,c)\backslash T} \mu_{j,c} + \sum_{aR_ic} \mu_{i,a}$$

The first term on RHS is at most equal to $(\hat{q}_{i,c} - 1)$ since $|T\backslash\{i\}| = \hat{q}_{i,c} - 1$, and the second and third terms are equal to zero. Therefore, the comb constraint fails for i, c and T constructed as above.

$(i) \Rightarrow (iii)$. Suppose the no-blocking pair condition fails for some i and c, i.e.

$$\sum_{j\in B(i,c)} \mu_{j,c} + q_c \sum_{aP_ic} \mu_{i,a} + (q_c - 1)\mu_{i,c} < q_c$$

Then as μ is integer, we have

$$\sum_{j\in B(i,c)} \mu_{j,c} \le q_c - 1 \text{ and } \sum_{aR_ic} \mu_{i,a} = 0$$

i.e. (i,c) blocks μ.

$(iii) \Rightarrow (ii)$. Suppose comb constraint fails for some i, c and T, i.e.

$$\sum_{j\in T}\sum_{aR_jc} \mu_{j,a} + \sum_{j\in B(i,c)\backslash T} \mu_{j,c} < \hat{q}_{i,c}$$

As $|T| = \hat{q}_{i,c}$ and μ is integer, we have $\sum_{aR_kc} \mu_{k,a} = 0$ for some $k \in T$. Therefore,

$$q_c \sum_{aP_kc} \mu_{k,a} = 0 \tag{14}$$

$$(q_c - 1)\mu_{k,c} = 0 \tag{15}$$

As $k \in B(i,c)$, we have $B(k,c) \subseteq B(i,c)$; thus

$$\sum_{j \in B(k,c)} \mu_{j,c} \leq \sum_{j \in B(i,c)} \mu_{j,c} < \hat{q}_{i,c} \leq q_c \tag{16}$$

Inequalities (14)-(16) imply that the no blocking condition fails for k and c. $\square$

Proof of Proposition 1.

"Only if." Let ρ be an ex post stable random matching with an ex post stable improvement cycle Cyc= (Cyc^1,...,Cyc^k) where Cyc^t=$(i_1, c_1, ..., i_n, c_n)$ for $t \leq k$. Let $\pi \in \mathcal{X}^{eps}$ be the random matching obtained by satisfying Cyc. Then π ordinally dominates ρ.

"If." Suppose $\pi \in \mathcal{X}^{eps}$ ordinally dominates $\rho \in \mathcal{X}^{eps}$. Then by Lemma 1, there exists a stochastic improvement cycle. Observe that there could be many such cycles. Let Cyc= (Cyc^1,...,Cyc^k) be a set of stochastic improvement cycles that collectively construct π from ρ. Then since $\pi \in \mathcal{X}^{eps}$, Cyc is an ex post stable improvement cycle. $\square$

First, we need to define a few terms. A polyhedron $P \subset \mathbb{R}^d$ is a set that can be represented as the intersection of finitely many closed half-spaces. A proper face of a polyhedron P is a set $P \cap H$ where H is a hyperplane such that P is contained in one of its half-spaces. A face of P is a set that is either P itself or a proper face of P. A polyhedral cone $C \subseteq \mathbb{R}^d$ of a set S is a positive hull of S. The positive hull of S is the set of all positive combinations of the elements of S. A positive combination of a set of elements $\{x_1, ..., x_k\}$ of S is $\sum_l \lambda_l x_l$ where $\lambda_l \in \mathbb{R}_+$ for all l. The relative interior of a convex set S is its interior in the relative topology of its affine hull. The affine hull of S is the set of all affine combinations of the elements of S. An affine combination of a set of elements $\{x_1, ..., x_k\}$ of S is $\sum_l \lambda_l x_l$ where $\lambda_l \in \mathbb{R}$ for all l and $\sum_{l=1}^k \lambda_l = 1$.

Theorem 4. *(Motzkin, 1983). P is a polyhedron if and only if*

$$P = X + C$$

where X is a polytope and C is a polyhedral cone.

We next prove a lemma that will be useful in the proof of Theorem 3 and Proposition 3.

Lemma 2. *Let $P \subset \mathbb{R}^d$ be a polyhedron. If a point $p \in P$ is not in the relative interior of P, then there exists a hyperplane $H = \{x \in \mathbb{R}^d | v \cdot x = c\}$ such that $p \in H$, $P \subset H^-$ and $\forall l$, $v_l > 0$.*

Proof. By Theorem 2 of McLennan (2002), p is contained in the proper face of P. Let F be the smallest face of P containing p and let H be a hyperplane such that $P \subset H^-$ and $P \cap H = F$. Let $v \neq 0$ and $c \in \mathbb{R}$ be such that $H = \{x \in \mathbb{R} | v \cdot x = c\}$ and $p \in H$. Let e_l denote the unit vector on an arbitrary lth axis of $\mathbb{R}^d$. For any $\delta > 0$, we have $p - \delta e_l \in P$; thus $v \cdot (p - \delta e_l) = c - \delta v_l \leq c$, which implies that $v_l \geq 0$.

What remains to show is $\forall l$, $v_l > 0$. Suppose to the contrary, $v_l = 0$. Then $p - \delta e_l \in H$, which implies that $p - \delta e_l \in F$. Moreover, by Lemma 2 of McLennan (2002), p is in the relative interior of F; thus $p+\delta e_l \in F$ for a small enough $\delta > 0$. However, since $F \subset P$, this contradicts the assumption that p is not in the relative interior of P. $\square$

Proof of Theorem 3.

"If." By definition, $\mathcal{X}^{eps}$ is a compact set. Therefore, there exists an optimal solution, x^*. Suppose to the contrary, x^* is not constrained ordinally efficient. Then there exists a random matching $y \in \mathcal{X}^{eps}$ that ordinally dominates x^*. By Proposition 1, there exists an ex post stable improvement cycle from x^* to y, i.e. there is a list of students and schools $(i_1, c_1, ..., i_n, c_n)$ such that $c_{k+1} P_k c_k$ and $x^*_{i_k,c_k} > 0$, $k \in \{1, ..., n\}$. Since the weight matrix is descending, $w_{i_k,c_{k+1}} > w_{i_k,c_k}$ for each i_k. Then

$$\sum_{i\in\{i_1,...i_n\}} \sum_{c\in C} w_{i,c} y_{i,c} > \sum_{i\in\{i_1,...i_n\}} \sum_{c\in C} w_{i,c} x^*_{i,c}$$
$$\sum_{i\in I\setminus\{i_1,...i_n\}} \sum_{c\in C} w_{i,c} y_{i,c} = \sum_{i\in I\setminus\{i_1,...i_n\}} \sum_{c\in C} w_{i,c} x^*_{i,c}$$

The first inequality holds because the weight matrix is descending and everyone in the cycle improved their outcome. The second equality holds because the outcome of the students outside the cycle has not changed. Therefore, we obtain

$$\sum_{i\in I} \sum_{c\in C} w_{i,c} y_{i,c} > \sum_{i\in I} \sum_{c\in C} w_{i,c} x^*_{i,c}$$

which contradicts that x^* is an optimal solution.

"Only if." Assume ρ is a constrained ordinally efficient ex post stable random matching. Label the students in any random order as $\{1, ..., |I|\}$. Using the definition of ordinal dominance, define $V : \mathcal{X} \to \mathbb{R}^{|I||C|}$ as

$$V(\pi) = \left(\pi_{1,c(1,1)}, ..., \sum_{k=1}^{|C|} \pi_{1,c(1,k)}, ..., \sum_{k=1}^{K} \pi_{i,c(i,k)}, ..., \pi_{|I|,c(|I|,1)}, ..., \sum_{k=1}^{|C|} \pi_{|I|,c(|I|,k)}\right)$$

It is straightforward to see that for any pair of random matchings ρ', ρ'', ρ' ordinally dominates ρ'' if and only if $V(\rho') \geq V(\rho'')$ and $V(\rho') \neq V(\rho'')$. Let $X \equiv V(\mathcal{X}^{eps})$ and $p \equiv V(\rho)$. Then (3) can be reformulated as follows:

$$p = \arg\max_{q\in X} W \cdot q \tag{17}$$

By definition, $\mathcal{X}^{eps}$ is a polytope, and since V is linear, X is also a polytope. By Motzkin's theorem (Motzkin, 1983), $P = X - \mathbb{R}^{|I||C|}$ is a polyhedron. As ρ is constrained ordinally efficient, p is not in the relative interior of P, so by Lemma 2, there exists a vector $v \in \mathbb{R}^{|I||C|}$ such that $\forall l$, $v_l > 0$.

Consider the following weight vector W for p: for each $l \in \{1, ..., |I||C|\}$, $W_l = \sum_{j=1}^{l} v_j$. Then given W, vector p is a maximizer to (17), which implies that ρ is a maximizer to (3) for the following weight vector: for each $i \in \{1, ..., |I|\}$ and $k \in \{1, ..., |C|\}$,

$$w_{i,c(i,k)} = \sum_{j=(i-1)|C|+k}^{|I||C|} W_j$$

where $c(i,k)$ denotes the kth favorite school of student i. As $\forall l$, $v_l > 0$, $W_j < W_{j+1}$; thus $w_{i,c(i,k)} > w_{i,c(i,k+1)}$ for every $k \in \{1, ..., |C|-1\}$. □

Proof of Proposition 3.

"If." Assume ρ is a maximizer to (3) for some RE weight matrix w. Suppose to the contrary, ρ is not constrained rank efficient. Let $x \in \mathcal{X}^{eps}$ be a random matching that rank-dominates ρ. Rank-dominance and the RE weight matrix w imply the following:

$$\sum_{j \in I} \sum_{a R_j c(j,k)} w_{j,a} x_{j,a} > \sum_{j \in I} \sum_{a R_j c(j,k)} w_{j,a} \rho_{j,a}, \quad \text{for some } k$$
$$\sum_{j \in I} \sum_{a R_j c(j,k)} w_{j,a} x_{j,a} \geq \sum_{j \in I} \sum_{a R_j c(j,k)} w_{j,a} \rho_{j,a}, \quad \text{for all } k$$

Therefore, ρ is not a maximizer.

"Only if." Suppose ρ is a constrained rank efficient ex post stable random matching. First, using the definition of rank efficiency, define $R : \mathcal{X} \to \mathbb{R}^{|C|}$ as

$$R(\pi) = \Big(\sum_{i \in I} \pi_{i,c(i,1)}, ..., \sum_{l=1}^{k} \sum_{i \in I} \pi_{i,c(i,l)}, ..., \sum_{l=1}^{|C|} \sum_{i \in I} \pi_{i,c(i,l)} \Big)$$

It is obvious that for any pair of random matchings ρ', ρ'', ρ' rank-dominates ρ'' if and only if $R(\rho') \geq R(\rho'')$ and $R(\rho') \neq R(\rho'')$. Let $X^r \equiv R(\mathcal{X}^{eps})$ and $p \equiv R(\rho)$. Then (3) can be reformulated as follows:

$$p = \arg\max_{q \in X} W \cdot q \tag{18}$$

By definition, $\mathcal{X}^{eps}$ is a polytope, and since R is linear, X^r is also a polytope. By Motzkin's theorem, $P \equiv X^r - \mathbb{R}^{|C|}$ is a polyhedron. As ρ is constrained ordinally efficient, p is not in the relative interior of P, so by Lemma 2, there exists a vector $v \in \mathbb{R}^{|C|}$ such that $\forall l$, $v_l > 0$.

Consider the following weight vector W for p: for each $l \in \{1, ..., |C|\}$, $W_l = \sum_{j=1}^{l} v_j$. Then given W, vector p is a maximizer to (18), which implies that ρ is a maximizer to (3) for the following weight vector: for each $k \in \{1, ..., |C|\}$,

$$w_k = \sum_{j=k}^{|C|} W_j$$

As $\forall l$, $v_l > 0$, $W_j < W_{j+1}$; thus $w_k > w_{k+1}$ for every $k \in \{1, ..., |C|-1\}$. □

Proof of Proposition 4.

Let ρ^* be the outcome of the procedure. At a step k of the procedure, one set I_k^e, if any, of equal students are found among the remaining students and each student in I_k^e obtains an equal distribution over schools. In the other steps, these students' distributions do not change, thus ρ^* treats equals equally.

Since the solution to (3) is ex post stable, there exists a convex combination of stable matchings that induce it. Let $\{\mu^1, ..., \mu^n\} \subseteq \mathcal{M}^s$ be a set of stable matchings that induce the solution. Then at μ^t, $t \in \{1, ..., n\}$, since $i \sim_{\mu^t(j)} j$ and $i \sim_{\mu^t(i)} j$, swapping their allocations is still stable. At a step k, each stable matching in the support of $\rho^{(k-1)}$ is increased by $(|I_k^e|! - 1)$ stable matchings. Therefore, ρ^* is ex post stable.

Since equal students rank the schools in the same order, for each pair of equal students, $i, j \in I$, $w_{i,c} = w_{j,c}$ for each $c \in C$. At step 1, the following equalities hold.

$$\begin{aligned}
\sum_{i \in I_1^e} \sum_{c \in C} w_{i,c} \rho_{i,c}^{(1)} &= \sum_{i \in I_1^e} \sum_{c \in C} w_{i,c} \Big(\frac{1}{|I_1^e|} \sum_{j \in I_1^e} \rho_{j,c}^{(0)} \Big) \\
&= \frac{1}{|I_1^e|} \sum_{i \in I_1^e} \sum_{c \in C} w_{i,c} \Big(\sum_{j \in I_1^e} \rho_{j,c}^{(0)} \Big) \\
&= \sum_{c \in C} w_{i,c} \Big(\sum_{j \in I_1^e} \rho_{j,c}^{(0)} \Big) \\
&= \sum_{c \in C} \sum_{i \in I_1^e} w_{i,c} \rho_{i,c}^{(0)} \\
\sum_{i \in I \setminus I_1^e} \sum_{c \in C} w_{i,c} \rho_{i,c}^{(1)} &= \sum_{i \in I \setminus I_1^e} \sum_{c \in C} w_{i,c} \rho_{i,c}^{(0)}
\end{aligned}$$

The fourth equation on the RHS holds because $i, j \in I_1^e$, $w_{i,c} = w_{j,c}$ for each $c \in C$. In each step, the above argument holds. Therefore ρ^* is an optimal solution. □