12　三角関数の導関数, 積の微分法, 商の微分法―解答例

演習 12.1 [解答例] (1)

$$
\begin{aligned}
(\sin x)' &= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} \quad (A=x+h,\ B=-x \text{ とおいて演習 5.2 (4) を適用})\\
&= \lim_{h\to 0}\frac{2\sin\frac{h}{2}\cos\left(x+\frac{h}{2}\right)}{h} \quad \left(\widetilde{h}=\frac{h}{2} \text{ とおく. } h\to 0 \text{ のとき } \widetilde{h}\to 0\right)\\
&= \lim_{\widetilde{h}\to 0}\frac{\sin\widetilde{h}}{\widetilde{h}}\cos(x+\widetilde{h}) \quad (\text{ここで基本事項にあった極限の公式を使う})\\
&= \cos x.
\end{aligned}
$$

(2)

$$
\begin{aligned}
(\cos x)' &= \lim_{h\to 0}\frac{\cos(x+h)-\cos x}{h} \quad (A=x+h,\ B=x \text{ とおいて演習 5.2 (6) を適用})\\
&= \lim_{h\to 0}\frac{-2\sin\left(x+\frac{h}{2}\right)\sin\frac{h}{2}}{h} \quad \left(\widetilde{h}=\frac{h}{2} \text{ とおく. } h\to 0 \text{ のとき } \widetilde{h}\to 0\right)\\
&= \lim_{\widetilde{h}\to 0}(-\sin(x+\widetilde{h}))\frac{\sin\widetilde{h}}{\widetilde{h}} \quad (\text{ここで基本事項にあった極限の公式を使う})\\
&= -\sin x.
\end{aligned}
$$

演習 12.2 [解答例] (1)

$$
\begin{aligned}
\{f(x)g(x)\}' &= \lim_{x_1\to x}\frac{f(x_1)g(x_1)-f(x)g(x)}{x_1-x}\\
&= \lim_{x_1\to x}\frac{f(x_1)g(x_1)-f(x)g(x_1)+f(x)g(x_1)-f(x)g(x)}{x_1-x}\\
&= \lim_{x_1\to x}\frac{f(x_1)-f(x)}{x_1-x}g(x_1)+\lim_{x_1\to x}f(x)\frac{g(x_1)-g(x)}{x_1-x}\\
&= f'(x)g(x)+f(x)g'(x).
\end{aligned}
$$

(2) まず,

$$
\begin{aligned}
\left\{\frac{1}{g(x)}\right\}' &= \lim_{x_1\to x}\frac{\frac{1}{g(x_1)}-\frac{1}{g(x)}}{x_1-x} = \lim_{x_1\to x}\frac{\frac{g(x)-g(x_1)}{g(x_1)g(x)}}{x_1-x}\\
&= \lim_{x_1\to x}\left(-\frac{1}{g(x_1)g(x)}\cdot\frac{g(x_1)-g(x)}{x_1-x}\right)\\
&= -\frac{g'(x)}{\{g(x)\}^2}.
\end{aligned}
$$

次に, 上記と積の微分法を用いれば,

$$\begin{aligned}\left\{\frac{f(x)}{g(x)}\right\}' &= f'(x)\frac{1}{g(x)} + f(x)\left\{\frac{1}{g(x)}\right\}' \\ &= \frac{f'(x)}{g(x)} - \frac{f(x)g'(x)}{\{g(x)\}^2} \\ &= \frac{f'(x)g(x) - f(x)g'(x)}{\{g(x)\}^2}.\end{aligned}$$

演習 12.3 [解答例] 演習 12.2 (2) (商の微分法) より,

$$(\tan x)' = \left(\frac{\sin x}{\cos x}\right)' = \frac{(\sin x)'\cos x - \sin x(\cos x)'}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}.$$

演習 12.4 [解答例] (1) $y' = \dfrac{(2x-3)'(x^2+1) - (2x-3)(x^2+1)'}{(x^2+1)^2} = \dfrac{-2x^2+6x+2}{(x^2+1)^2}$.

(2) $y' = -\dfrac{(x^2+x+1)'}{(x^2+x+1)^2} = -\dfrac{2x+1}{(x^2+x+1)^2}$.

(3) $y' = \dfrac{(1-x^2)'(1+x^2) - (1-x^2)(1+x^2)'}{(1+x^2)^2} = \dfrac{-4x}{(1+x^2)^2}$.

(4) $y' = \dfrac{(x^3-3x^2+x)'x^2 - (x^3-3x^2+x)(x^2)'}{x^4} = \dfrac{x^2-1}{x^2} = 1 - \dfrac{1}{x^2}$.

(5) $y' = \left(\dfrac{\cos x}{\sin x}\right)' = \dfrac{(\cos x)'\sin x - \cos x(\sin x)'}{\sin^2 x} = \dfrac{-\sin^2 x - \cos^2 x}{\sin^2 x} = -\dfrac{1}{\sin^2 x}$.

(6) $y' = \dfrac{(\cos x)'(3+\sin x) - \cos x(3+\sin x)'}{(3+\sin x)^2} = \dfrac{-3\sin x - \sin^2 x - \cos^2 x}{(3+\sin x)^2} = \dfrac{-3\sin x - 1}{(3+\sin x)^2}$.

(7) $y' = (\sin x)'\cos x + \sin x(\cos x)' = \cos^2 x - \sin^2 x = \cos 2x$.

(8) $y' = (x^2)'\tan x + x^2(\tan x)' = 2x\tan x + \dfrac{x^2}{\cos^2 x}$.