13 対数関数・指数関数の導関数, 合成関数の微分法, 逆関数の微分法—解答例

演習 13.1 [解答例] (1)

$$\frac{dy}{dx} = \frac{1}{\frac{dx}{dy}} = \frac{1}{ny^{n-1}} = \frac{1}{n(x^{\frac{1}{n}})^{n-1}} = \frac{1}{n}x^{-\frac{n-1}{n}} = \frac{1}{n}x^{\frac{1}{n}-1}.$$

(2) ($r \geqq 0$ の場合) $r = \dfrac{m}{n}$ (m は非負整数, n は自然数) とおく. $u = x^{\frac{1}{n}}$, および $y = x^r = u^m$ とすると, 合成関数の微分法と (1) により,

$$(x^r)' = \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = (mu^{m-1})(\frac{1}{n}x^{\frac{1}{n}-1}) = \frac{m}{n}x^{\frac{m-1}{n}} \cdot x^{\frac{1}{n}-1} = rx^{r-1}.$$

($r < 0$ の場合) $r = -\dfrac{m}{n}$ (n, m は自然数) とおく. 商の微分法と上記により,

$$(x^r)' = \left(\frac{1}{x^{\frac{m}{n}}}\right)' = -\frac{(x^{\frac{m}{n}})'}{(x^{\frac{m}{n}})^2} = -\frac{\frac{m}{n}x^{\frac{m}{n}-1}}{x^{\frac{2m}{n}}} = -\frac{m}{n}x^{\frac{m}{n}-1-\frac{2m}{n}} = rx^{r-1}.$$

□

演習 13.2 [解答例] (1) $u = 3x$ とおくと, $y = \log u$ で,

$$y' = \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{u} \cdot 3 = \frac{1}{x}.$$

または, 対数法則を使って,

$$y' = (\log 3x)' = (\log 3 + \log x)' = \frac{1}{x}.$$

(2) $u = -4x$ とおくと, $y = \log_{10} u$ で,

$$y' = \frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} = \frac{1}{u \log 10} \cdot (-4) = \frac{1}{x \log 10}.$$

または, 対数法則を使って,

$$y' = (\log_{10}(-4x))' = (\log_{10}(-4) + \log_{10} x)' = \frac{1}{x \log 10}.$$

(3) $u=x^2+1$ とおくと, $y=\log u$ で,

$$y' = \frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = \frac{1}{u}\cdot(2x) = \frac{2x}{x^2+1}.$$

(4) $u=6x$ とおくと, $y=e^u$ で,

$$y' = \frac{dy}{dx} = \frac{dy}{du}\cdot\frac{du}{dx} = e^u\cdot 6 = 6e^{6x}.$$

(5) $y' = (e^x)'\cos x + e^x(\cos x)' = e^x\cos x - e^x\sin x = e^x(\cos x - \sin x)$.

(6) $y' = (x^{\frac{1}{2}})' = \dfrac{1}{2}x^{\frac{1}{2}-1} = \dfrac{1}{2\sqrt{x}}$.

(7) $y' = \dfrac{(\cos x)'\sqrt{x} - \cos x(\sqrt{x})'}{(\sqrt{x})^2} = \dfrac{-\sin x\sqrt{x} - \frac{\cos x}{2\sqrt{x}}}{x} = \dfrac{-2x\sin x - \cos x}{2x\sqrt{x}}$.

(8) まず $(\sin^2 x)'$ を求める. $u=\sin x$, および $v=\sin^2 x = u^2$ とすると,

$$(\sin^2 x)' = \frac{dv}{dx} = \frac{dv}{du}\cdot\frac{du}{dx} = 2u\cdot\cos x = 2\sin x\cos x.$$

よって,

$$\begin{aligned} y' &= (\cos x)'\sin^2 x + \cos x(\sin^2 x)' = -\sin^3 x + 2\sin x\cos^2 x = -\sin x(\sin^2 x - 2\cos^2 x) \\ &= -\sin x(3\sin^2 x - 2). \end{aligned}$$