4820 HW 3

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1 Exercise 3

1.
$[x]_B = \{z : xBz\}$ and $[y]_B = \{a : yBa\}$
Proof by contradiction - $[x]_B$ and $[y]_B$ share some elements but are not equal
so, some element $q \in [x]_B \cap [y]_B$
since $q \in [x]_B$, then xBq and $q \in [y]_B$, then yBq
by symmetry $- > qbx, qby$
by transitivity $xBq, qBy \implies xBy$
a. $[x]_B \subseteq [y]_B$
let $w \in [x]_B$, then xBw. By transitivity, $yBx, xBw \implies yBw$. So, $w \in [y]_B$
b. $[y]_B \subseteq [x]_B$
Same proof as part a, but y and x switch
So, since $[x]_B \subseteq [y]_B \wedge [y]_B \subseteq [x]_B \implies [x]_B = [y]_B$ (contradiction)

2.
a.
Since C is a preorder, it is reflexive and transitive. Need to also show is symmetric to show B is an equivalence relation.
Reflexive: $x \in S, xCx$ already exists. So, $xCx \wedge xCx$ is true.
Symmetric: Assume xBy. By definition of B, then $xCy \wedge yCx$. yBx would also hold if $yCx \wedge xCy$ existed. Well, with the existence of xBy, yBx also exists.
Transitive: Assume xBy and yBz exist. $xBy \implies xCy \wedge yCx$. $yBz \implies yCz \wedge zCy$. We know C is transitive by the definition of a preorder, so $xCy \wedge yCz \implies xCz$ and $zCy \wedge yCx \implies zCx$, showing B is also transitive.
b.
Need to prove $\preccurlyeq$ is a partial order. So, when $[x]_B$ and $[y]_B$ are related by $\preccurlyeq$, then xCy.
$\preccurlyeq$ is reflexive:
$[x]_B \in S/B$. By def of $\preccurlyeq$, $[x]_B \preccurlyeq [x]_B \equiv xCx$. So, C is reflexive, therefore $[x]_B \preccurlyeq [x]_B$ holds.
$\preccurlyeq$ is transitive:
$[x]_B, [y]_B, [z]_B \in S/B$. Assume $[x]_B \preccurlyeq [y]_B$ and $[y]_B \preccurlyeq [z]_B$. By definition of $\preccurlyeq$, $[x]_B \preccurlyeq [y]_B \implies xCy$ and $[y]_B \preccurlyeq [z]_B \implies yCz$. Since C is transitive,

$xCy, yCz \implies xCz$. So, $xCz \equiv [x]_B \preccurlyeq [z]_B$.

$\preccurlyeq$ is antisymmetric:

Assume $[x]_B \preccurlyeq [y]_B$ and $[y]_B \preccurlyeq [x]_B$. $[x]_B \preccurlyeq [y]_B \implies xCy$ and $[y]_B \preccurlyeq [x]_B \implies yCx$. Therefore, $xCy \wedge yCx \implies xBy$ by the def of B. Since xBy, $y \in [x]_B$. Since B is an equivalence relation and $y \in [x]_B$ and $y \in [y]_B$, they must be identical $\implies [x]_B = [y]_B$

2 Exercise 4

1.

($\implies$) B is total

Then, for some $x, y \in S, xBy \vee yBx$

xBy, then $\langle x, y\rangle \in B$, so $\langle x, y\rangle \in B \cup B^{-1}$

yBx, then $\langle x, y\rangle \in B^{-1}$, so $\langle x, y\rangle \in B \cup B^{-1}$

Thus, $S \times S \subseteq B \cup B^{-1}$. Also we know, $B \cup B^{-1} \subseteq S \times S$. So, $S \times S = B \cup B^{-1}$.

($\impliedby$) Assume $S \times S = B \cup B^{-1}$

Let $x, y \in S$, then $\langle x, y\rangle \in S \times S$, so $\langle x, y\rangle \in B \cup B^{-1}$. If $\langle x, y\rangle \in B$, then xBy. If $\langle x, y\rangle \in B^{-1}$, then yBx. Both ways, $xBy \vee yBx$, so B is total.

2.

($\implies$) Assume B is strict partial order

Irreflexive: $\forall x$, cannot have xBx because that would mean $x \neq x$ (by definition of the strict part), so $\neg(xBx)$

Transitive: Assume xBy and yBz. We know B is transitive (by definition of partial order, but $x \neq y, y \neq z$. We need $x \neq z$. If $x = z$, then xBy and zBy and yBx. This would mean by antisymmetry, $xBy \wedge yBx \implies y = x$ which contradicts irreflexivity. So, it is transitive.

($\impliedby$) Assume B is irreflexive and transitive

To be a strict partial order, B must be:

Transitive: Already is through our assumption

Strict part of B: This is exactly what it means to be irreflexive, which already is by assumption

Antisymmetric: $\langle \forall x, y \in S :: xBy \wedge yBx \implies x = y\rangle$. $\forall x, y \in S$ if $xBy \wedge yBx$ (by transitivity) xBx

3.

Reflexive: $\forall x \in S$, $x \preccurlyeq x \equiv x \prec x \vee x = x$. Since $x = x$, this is true.

Transitive: $\forall x, y, z \in S$, $x \preccurlyeq y$ and $y \preccurlyeq z$. If $x = y$, we can replace y with x $- > x \preccurlyeq z$. If $y = z$, replace y with z $- > x \preccurlyeq z$. If $x \prec y$ and $y \prec z$, then using transitivity of $\prec, x \preccurlyeq z$

Antisymmetric: Assume $x \preccurlyeq y$ and $y \preccurlyeq x$. If $x \neq y$, then by $\preccurlyeq$ definition, $x \prec y$ and $y \prec x$. Using transitivity, then $x \prec x$, but $\prec$ is irreflexive, so there is a contradiction. Therefore, $x = y$, so $\preccurlyeq$ is antisymmetric.

4.

Irreflexive: Assume $x \prec x$, therefore $x \prec x \equiv x \preccurlyeq x \wedge \neg(x \preccurlyeq x)$. Since we know $\preccurlyeq$ is reflexive, this simplifies to true $\wedge$ $\neg$(true) = false. Therefore, $\neg(x \preccurlyeq x)$, so $\prec$ is irreflexive.

Transitive: Assume $x \prec y$ and $y \prec z$. Therefore, $x \preccurlyeq y \wedge \neg(y \preccurlyeq x)$ and $y \preccurlyeq z \wedge \neg(z \preccurlyeq y)$. From transitivity of $\preccurlyeq$, we get $x \preccurlyeq y, y \preccurlyeq z \implies x \preccurlyeq z$. Finally, need to show we also get $\neg(z \preccurlyeq x)$. Suppose by contradiction that $z \preccurlyeq x$. Then, we have $z \preccurlyeq x$ and $x \preccurlyeq y$, which by transitivity yields $z \preccurlyeq y$. But, we know $\neg(z \preccurlyeq y)$ from $y \prec z$, which is a contradiction. Therefore, $\neg(z \preccurlyeq x)$, so overall $\prec$ is transitive.

5.

($\implies$) Assume B is a strict total order

(a) B is irreflexive: Shown in part 2

(b) B is transitive: Shown in part 2

(c): Let $x, y \in S$. Let's say B is the strict part of total order $\preccurlyeq$, which therefore $x \preccurlyeq y \wedge y \preccurlyeq x$. If $x \preccurlyeq y, x \neq y$, then xBy. If $y \preccurlyeq x, y \neq x$, then yBx. If $x \preccurlyeq y \wedge y \preccurlyeq x$, then $x = y$ by antisymmetry.

($\impliedby$) Show B is a strict total order

$x \preccurlyeq y \equiv xBy \vee yBx \vee x = y$

B is a strict partial order: Yes, by part 2.

$\preccurlyeq$ is total: Yes (c).

B is the strict part of $\preccurlyeq$: $\{\langle x, y\rangle : x \preccurlyeq y \wedge x \neq y\} = \{\langle x, y\rangle : (xBy \vee x = y) \wedge x \neq y\} = \{\langle x, y\rangle : xBy\} = B$

6.

($\implies$) B is a well order

Since it's given that B is a well order, by definition of well orders, B is also well-founded. Additionally, since well orders are strict total orders, we can use Part 5 which states then in (c) that $xBy \vee yBx \vee x = y$.

($\impliedby$) B is well-founded and $\langle \forall x, y \in S :: xBy \vee yBx \vee x = y\rangle$

1. Strict total order

(a) Irreflexive:

Suppose by contradiction that $\exists x \in S, xBx$. We can then construct a sequence such that $\langle x, x, x, ...\rangle$, which is ω-decreasing, contradicting well-foundedness. Therefore, B is irreflexive.

(b) Transitive:

Assume xBy, yBz, zBx. By our initial assumption from the problem statements, $xBz \vee zBx \vee x = z$. If zBx, then we have $x- > y- > z- > x- > ...$, contradicting well-foundedness. If $x = z$, then xBy and $yBz = yBx$, so we have $xBy \wedge yBx$, leading to $x- > y- > x- > y...$, which again contradicts well-foundedness. Therefore, we must have xBz, proving transitivity.

2. Well foundedness

Given by the intial assumption from the problem statement

3 Exercise 5

1. $\langle S, < \rangle$ is well-founded structure iff all non-empty subsets of S have a minimal element under $<$

- well-founded = terminating
- if B is well-founded, $\langle S, B \rangle$ is a well-founded structure

($\Rightarrow$) $\langle S, < \rangle$ is well-founded structure
- S = set
- $<$ = terminating binary relation
- needs minimal element bc non-terminating if can't find one

Explanation: by the def of a well-founded structure, $<$ needs to terminate. All empty subsets terminate. Non-empty ones, though, to prove termination cannot be infinite, but need to be decreasing & reach some base stopping point. Therefore, w/o a minimal element, the subset wouldnt be well-founded

($\Leftarrow$) all non-empty subsets of S have a minimal element under $<$
- show $<$ is well-founded

Assume $<$ isnt well-founded, then there is some decreasing ω-sequence $\langle a_0, a_1, a_2, \dots \rangle$ where $a_{i+1} < a_i \ \forall i$
$T = \{ a_i : i \in \omega \}$, T is non-empty.
By assumption, T has a minimal element m, but T isnt finite, it continues w/o a notable ending since always smtg less.
Contradiction!

2. Prove that $\langle S, < \rangle$ is a woset iff all non-empty subsets of S have a least element

- woset = strict total order & well-founded
  - irreflexive, transitive, $\langle \forall x, y \in S :: xBy \vee yBx \vee x = y \rangle$

($\Rightarrow$) Assume $\langle S, < \rangle$ is a woset
by part 1, since woset means well-founded, $\langle S, < \rangle$ must terminate (have a least element)

($\Leftarrow$) all non-empty subsets of S have a least element
1. well-founded $\Rightarrow$ proven by part 1 already
2. strict total order
   a. irreflexive =
      by contradiction, ~~for some~~ $x < x$ for some x
      but for $\{x\}$, the least element is x. If $x < x$, then x would be less than itself, not possible

   b. transitivity
      suppose $x < y$ and $y < x$. $\{x, y, z\}$
      its least element m satisfies $m < x$, $m < y$, $m < z$
      if $m = y$, $y < x$ or $y = x$, contradicting $x < y$
      if $m = z$, $z < y$ or $z = y$, contradicting $y < z$
      so, $m = x$, and if x is least element, $x \neq z$, so $x < z$

   c. $\langle \forall x, y \in S :: xBy \vee yBx \vee x = y \rangle$
      so, for $\{x, y\}$, has least element m
      - if $m = x$, then xBy or $x = y$
      - if $m = y$, then yBx or $y = x$

      so, all holds

4 Exercise 6

Exercise 6. — $U = S \times S$

1. The reflexive relations form a closure system

~~closure system =~~

- reflexive relations = in set, need $\langle x, x\rangle\ \forall x \in S$
  but can have more
- property P picks some subsets of U (p-sets)
- closure system
  - U itself is a p-set
  - any intersection of P-sets is also a P-set

~~need $\langle x, x\rangle\ \forall x \in S$, and can have more~~

1. U itself is a p-set
   true ✓ since U is $S \times S$, must include $\forall x$, $\langle x, x\rangle \in S \times S$
2. Any intersection of P-sets is also a P-set
   for any $x \in S$, and lets P-sets (reflexive relations) of U be P. Since each set P_i in P is reflexive, they must all include $\forall x \in S$, $\langle x, x\rangle$. Therefore, at a minimum, taking $\cap$ of all P-sets will result in $\{\}$ or $\{x \in S ; \langle x, x\rangle\}$, which are both reflexive
   or are $\{\}$

2. The irreflexive relations do not form a closure set
   $\langle \forall x \in S :: \neg(xBx)\rangle$
   1. U itself is a p-set.
      Since $U = S \times S$, this includes every possible pair of numbers.
      So $\{\forall x, y \in S :: \langle x, y\rangle\}$, but if $x = y$, we get $\langle x, x\rangle$, which contradicts the irreflexive def.

3. The symmetric relations form a closure set
   $\langle \forall x, y \in S :: xBy \Rightarrow yBx\rangle$
   1. U itself is a p-set
      true. Since $U = S \times S$, $\forall x, y \in S$: $xUy \wedge yUx$
   2. Any intersection of P-sets is a P-set
      Let's say for P-set P, $\langle x, y\rangle \in P$, then by symmetry $\langle y, x\rangle \in P$, so $\forall x, y$ if $\langle x, y\rangle$ is in the intersection, $\langle y, x\rangle$ must be as well

4. The asymmetric do not form a closure set
   $\langle \forall x, y \in S :: xBy \rightarrow (\neg yBx)\rangle$
   1. U itself a p-set
      $U = S \times S$, so $\forall x, y$ U includes $\langle x, y\rangle$ and $\langle y, x\rangle$ which goes against asymmetry

5. The antisymmetric relations dont form a closure system
   $\langle \forall x, y \in S :: xBy \wedge yBx \Rightarrow x = y\rangle$
   1. U is a p-set
      $U = S \times S$. $\forall x, y$, $x \neq y$ $\langle x, y\rangle \wedge \langle y, x\rangle \in S$
      $|S| \geq 2$

6. The transitive relations form a closure system
   1. U is a p-set
      Since U includes all possible combos, everything connection is covered for $\forall x, y, z$ ~~xBy~~ $\langle x, y\rangle$, $\langle y, z\rangle$, $\langle x, z\rangle$
   2. Any intersection of p-sets is a p-set
      Assume $\langle x, y\rangle$ and $\langle y, z\rangle \in P$ (intersected p-set). Since each p-set is transitive, if every one has $\langle x, y\rangle \wedge \langle y, z\rangle$, must have $\langle x, z\rangle$ or else they themselves wouldnt be transitive

5 Exercise 9

Exercise 9

($\Rightarrow$) WFI holds $\Rightarrow$ $<$ is terminating

Prove contrapositive $\rightarrow$ $<$ isnt terminating $\Rightarrow$ WFI doesnt hold

If $<$ doesnt term., there's some decreasing ω-sequence $\sigma = \langle a_0, a_1, a_2, \ldots \rangle$ s.t. $a_{i+1} < a_i$ $\forall i \in \omega$. Let $S = \{a_0, a_1, \ldots\}$ be range of σ

Need some property P where LHS $\neq$ RHS, showing WFI fails

$P.w \equiv (w \notin S)$

1. LHS

   Since $a_0 \in S$, $\neg P.a_0$. So, not case that $P.w$ holds $\forall w \in W$,
   So $\langle \forall w \in W :: P.w \rangle$ is false

2. RHS

   Want to show $\forall w \in W$, $\langle \forall v : v < w : P.v \rangle \Rightarrow P.w$ holds

   - Case 1: $w \notin S$
     Then $P.w$ is true. Since $(\ldots \Rightarrow P.w\ (\text{true}))$, it must be true
   - Case 2: $w \in S$
     Then $P.w$ is false ($\neg P.w$). Therefore, $w = a_i$ for some $i \in \omega$.
     Know $a_{i+1} < a_i$, and $a_{i+1} \in S$, so $\neg P.a_{i+1}$. Let's say $w = a_i$, then $a_{i+1} < w$ and $\neg P.a_{i+1}$, so $\langle \forall v : v < w : P.v \rangle$ is false. $(\text{false} \Rightarrow \ldots)$ is true

   In both cases true

LHS (false) $\neq$ RHS (true), proving contrapositive

($\Leftarrow$) $<$ is terminating $\Rightarrow$ WFI holds

Must show $\forall$ properties P, LHS $\equiv$ RHS of WFI, so show LHS $\Rightarrow$ RHS + RHS $\Rightarrow$ LHS

1. LHS $\Rightarrow$ RHS

   Assume $\langle \forall w \in W :: P.w \rangle$. Since P holds $\forall w \in W$, $P.w$ is true.
   So, RHS reduced to $(\ldots \Rightarrow \text{true})$ which is true

2. RHS $\Rightarrow$ LHS

   Prove $\neg$LHS $\Rightarrow$ $\neg$RHS
   Assume $\neg$LHS, $\neg\langle \forall w \in W :: P.w \rangle$, so $\exists$ some el. of W where $P.w$ fails, lets say this is w_0
   For any w, if $\neg P.w$, then $\neg\langle \forall v : v < w : P.v \rangle$, meaning some $v < w$ w/ $\neg P.v$
   Can build dec. sequence:
   - w_0 w/ $\neg P.w_0$, so $\exists\ w_1 < w_0$ w/ $\neg P.w_1$
     , ...
     gets decreasing ω-sequence, contradicting $<$ is terminating

So $\neg$LHS $\Rightarrow$ $\neg$RHS. ✓

6 Exercise 10

Exercise 10 – A relation is well-foundeded iff its transitive closure is well-founded

Relation $= \prec$

Trans. Closure $= \prec^T$

($\Rightarrow$) $\prec$ is well-founded $\Rightarrow$ $\prec^T$ is well-founded

Prove contrapositive: assume $\prec^T$ not well-founded, so $\exists$ decreasing ω-seq. $\langle a_0, a_1, a_2 \ldots \rangle$ w/ $a_{i+1} \prec^T a_i$ $\forall i \in \omega$.

$\forall i \in \omega$, since $a_{i+1} \prec^T a_i$, by def. of trans. closure, finite # of $\prec$-steps connecting a_{i+1} & a_i:

$$a_{i+1} = c_0^i \prec c_1^i \prec \ldots \prec c_k^i = a_i$$

$$a_i \succ c_{k-1}^i \succ \ldots \succ c_2^i \succ c_1^i \succ a_{i+1}$$

if we concat these finite strings:

a_0 to a_1 – $c_{01}^0 \prec \ldots \prec c_1^0$

a_1 to a_2 – $c_{11}^0 \prec \ldots \prec c_1^1$

a_2 to a_3 – $c_{21}^0 \prec \ldots \prec c_1^2$

$\vdots$

if we string these together we get

$$a_0 \prec c_{01}^0 \prec \ldots \prec c_1^0 \prec a_1 \prec c_{11}^0 \prec \ldots \prec c_1^1 \prec a_2 \prec \ldots$$

we concat infinitely many of these, resulting in an infinite sequences in $\prec$, meaning $\prec$ not well-founded

($\Leftarrow$) $\prec^T$ is well-founded $\Rightarrow$ $\prec$ is well-founded

Contrapositive: $\prec$ not w-f $\Rightarrow$ $\prec^T$ not w-f

$\exists$ a decreasing ω-sequence $\langle a_0, a_1, \ldots \rangle$ w/ $a_{i+1} \prec a_i$ $\forall i \in \omega$

$\prec \subseteq \prec^T$, so for every instance of $a_{i+1} \prec a_i \Rightarrow a_{i+1} \prec^T a_i$

so, the same sequence $\langle a_0, a_1, \ldots \rangle$ is infinitely $\downarrow$ in $\prec^T$

7 Exercise 11

Exercise 11 If relation $<$ on S is well-founded, then so is $<_n$ on n-tuples of elements from S, where n is positive natural number $\therefore$ $<_n$ (the lexicographic ver. of $<$) is: $<_1 = <$; for $n>1$, $\langle x_n, x_{n-1} \cdots x_1 \rangle <_n \langle y_n, y_{n-1}, \ldots y_1 \rangle$ iff $x_n < y_n$

base case: $n=1$
$<_1 = <$ by def $\therefore$ $<$ is well-founded ~~by ass~~ given by problem statement

induction: ~~assume P(n-1), do P(n)~~
$<_{n-1}$ is well-founded on $(n-1)$ tuples in S^{n-1}, show $<_n$ is wf on S^n

Assume $<_n$ not well-founded. Then $\exists$

$a^0 >_n a^1 >_n \ldots$

Write each tuple as: $a^k = \langle a_n^k, a_{n-1}^k, \ldots, a_1^k \rangle$

To follow lexicographic order:

1. $a_n^{k+1} < a_n^k$

or

2. $a_n^{k+1} = a_n^k$ $\therefore$ tail $\downarrow$ under $<_{n-1}$

Case 1: inf. many $\downarrow$ in 1st coor.
Then, $a_n^0 > a_n^1 > \ldots$ (an inf $\downarrow$ on $<$)
but $<$ is wf, so contradiction

Case 2: only finitely many $\downarrow$ in 1st coor.
Then, after some index i

$a_n^k = c \quad \forall k \geq i$

So, get $\langle a_{n-1}^k, \ldots, a_1^k \rangle$ (inf $\downarrow$ on $<_{n-1}$)
but $<_{n-1}$ is wf, so contradiction.

So, $<_n$ is wf

8 Exercise 12

Exercise 12

no, the dictionary is not well-founded. a word is defined to be any sequence of letters $\{a, b, \ldots z\}$. we compare words $L \to R$, and we can build an inf $\downarrow$ sequence like below:

$a^n b \quad \forall\ n \geq 0$

$b > ab > aab > aaab > \ldots$

9 Lemma 5

Lemma 5 — ord.a iff a is transitive and well-ordered by $\in$

($\Rightarrow$) Assume ord.a

- a is an ordinal if $\langle a, \prec \rangle$ is well-ordered for some $\prec$ & $\forall \beta \in a$, $\beta = s.\beta$

1. a is transitive

from Definition 3 $\Rightarrow$ set x is trans. iff $\langle \forall y, z :: z \in y \land y \in x \Rightarrow z \in x \rangle$
$\equiv \langle \forall y \in x :: y \subset x \rangle$

so, show $\forall y \in a$: $y \subset a$
Assume $\forall y \in a$. By Lemma 4, ord.$a \land y \in a \Rightarrow y \subset a$
Therefore, a is transitive

2. a is well-ordered by $\in$

from Corollary 1: ord.$a \equiv \langle \forall \beta, \gamma \in a :: \beta = \gamma \lor \beta \in \gamma \lor \gamma \in \beta \rangle \land$
$\langle \forall \beta \in a :: s.\beta = \beta \rangle$
by Lemma 2, corollary 1 includes this in the front ½ and statement, showing ord.a is well-ordered by $\in$

($\Leftarrow$) a is transitive and well-ordered by $\in$
by definition 2, a is an ordinal if $\langle a, \prec \rangle$ is well-ordered for some $\prec$ and $\forall \beta \in a$, $\beta = s.\beta$

Let's make $\prec = \in$
- we already know a is well-ordered by $\in$ ✓

Show $\forall \beta \in a$, $\beta = s.\beta$
↳ initial segment of β under $\in$
by Definition 1 $\rightarrow \{y : y B x\} = \{y : y \in \beta\}$
$\{y : y \in \beta\} \equiv \beta$, so $\beta = s.\beta$ ✓

10 Lemma 8

Lemma 8: If A is a set of ordinals, then $\bigcup A$ is an ordinal
↳ union all ordinals in $A \rightarrow$ ordinal

Using Lemma 5, must be transitive & well-ordered under $\in$

1. Transitive

Say $x \in \bigcup A$, then $x \in a$ for some $a \in A$. Since a is an ordinal, it's transitive (L5)
If $y \in x$, then $y \in a$ because a is trans. and $x \in a$. So $y \in \bigcup A$
$y \in x \in \bigcup A \rightarrow y \in \bigcup A$ (trans.) ✓

2. well-ordered under $\in$

Using Lemma 2 $\rightarrow \langle \forall x, y \in \bigcup A :: x = y \lor x \in y \lor y \in x \}$
Assume $\forall x, y \in \bigcup A$, therefore there's some $a, \beta \in \bigcup A$ where $x \in a$ and $y \in \beta$, and Ord.a & Ord.β

Using Lemma 6:
Ord.$a \Rightarrow \langle \forall \beta : \beta \in a$: Ord.$\beta \rangle$, so Ord.x (since $x \in a$)
Ord.$\beta \Rightarrow$ " ", so Ord.y (since $\beta \in y$)

Using Lemma 7:
Ord.$x \land$ Ord.$y \Rightarrow (x \in y) \lor (y \in x) \lor (x = y)$
which is exactly what we needed prove from L2 ✓

11 Lemma 15

Lemma 15: If $\langle S, A \rangle$ is a woset, then there is a unique ordinal α s.t. $\langle S, A \rangle \cong \alpha$

If α & β are ordinals, then $\langle S, A \rangle \cong \alpha$ & $\langle S, A \rangle \cong \beta$. By lemma 12, $\alpha \cong \beta$. By lemma 14, $\alpha = \beta$

Therefore, there is a unique ordinal α

12 Exercise 14

Exercise 14

$$(\omega^{(\omega+1)^2}) \cdot (\omega^{(\omega+\omega\cdot 12)} \cdot (\omega^{\omega+1} + \omega^2 \cdot 9))$$

$\alpha^{\beta+\gamma} = \alpha^{\beta}\alpha^{\gamma}$

$\alpha^{\beta\gamma} = (\alpha^{\beta})^{\gamma}$

$$= \omega^{(\omega+1)^2 + ((\omega+\omega\cdot 12)} \cdot (\omega^{\omega+1} + \omega^2 \cdot 9))$$

$(\omega+1)^2 = (\omega+1) \cdot (\omega+1)$

$(\omega+\omega\cdot 12) = \omega \cdot 13$

$$= \omega^{(\omega+1)^2 + (\omega\cdot 13)} \cdot (\omega^{\omega+1} + \omega^2 \cdot 9)$$