

**BCA [5TH SEM]**  
**COURSE CODE : BCA-504**

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# **NUMERICAL METHODS**

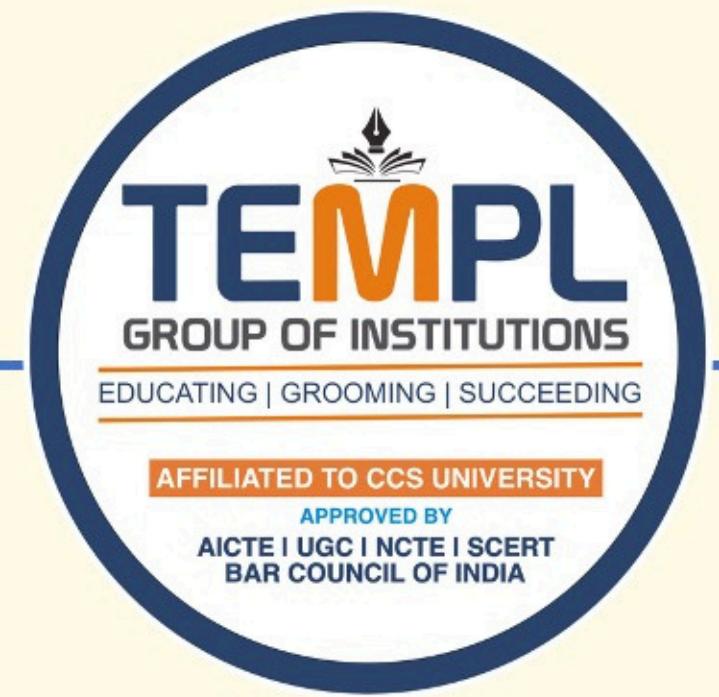
**VAISHALI**  
**[Faculty of Mathematics]**

# UNIT-5



## SOLUTION OF DIFFERENTIAL EQUATION

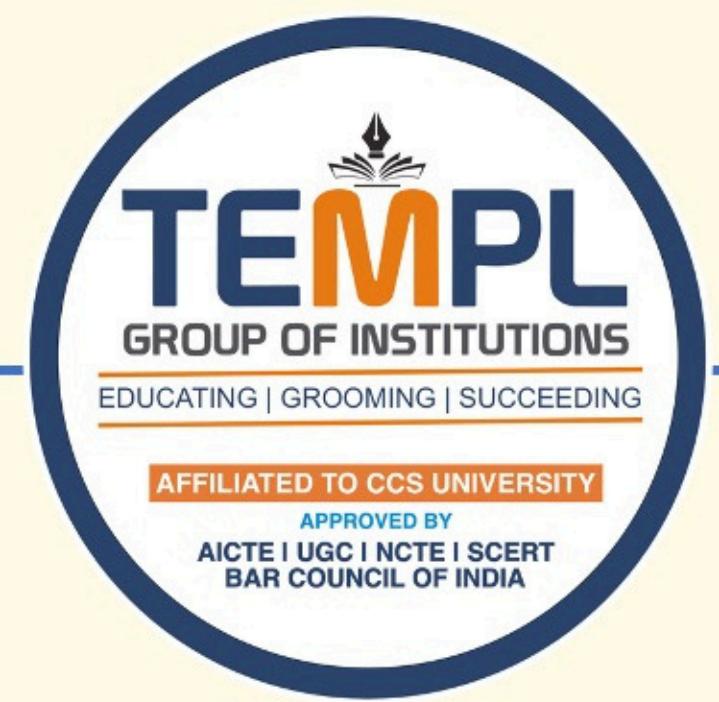
# DIFFERENTIAL EQUATION



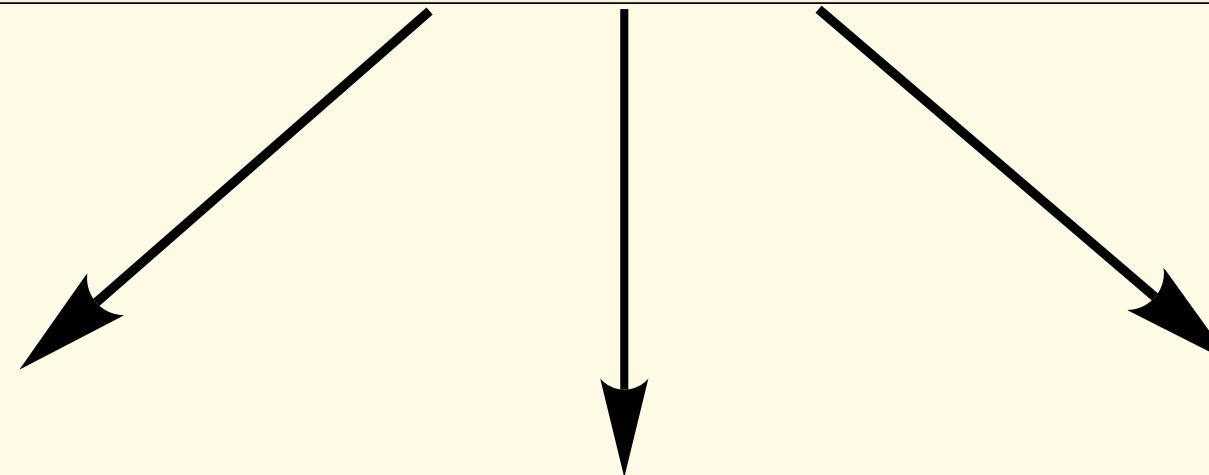
**A differential equation is an equation that involves a function and its derivative(s).**

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

# SOLUTION OF DIFFERENTIAL EQUATION



## SOLUTION OF DIFFERENTIAL EQUATION

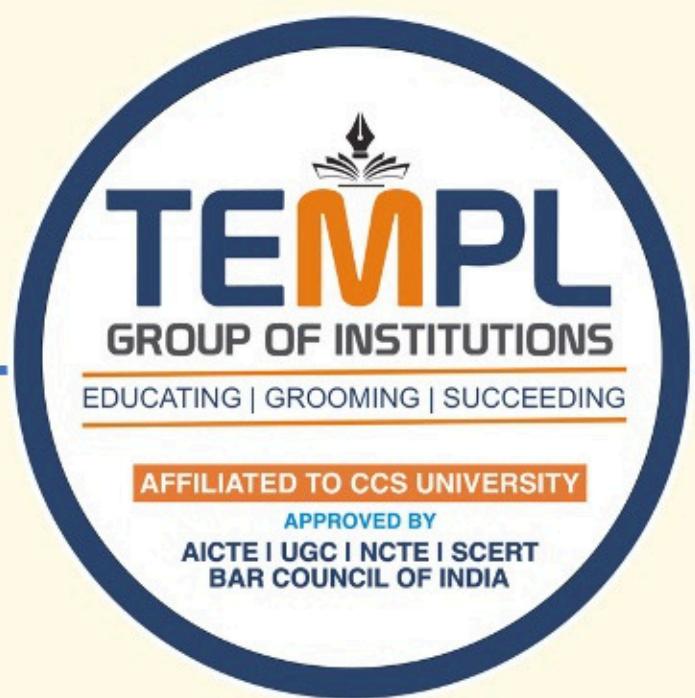


Euler's Method

Picard's Method

Runge Kutta Method

# EULER'S METHOD

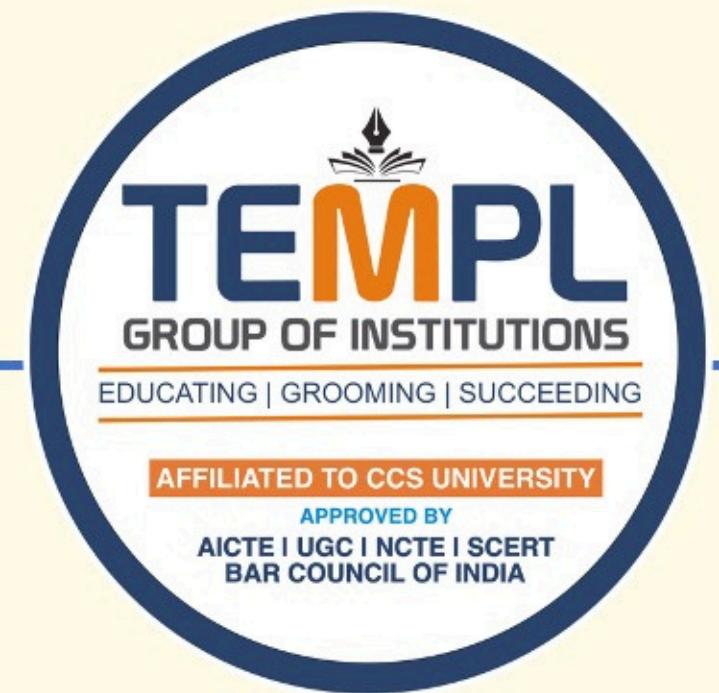


**Euler's method is the simplest numerical method to solve a first order differential equation.**

**It uses the concept of slope to predict the next value step by step.**

**The Euler Method was founded by the Swiss mathematician Leonhard Euler around the 18<sup>th</sup> century.**

# FORMULA OF EULER'S METHOD



**For the differential equation**

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

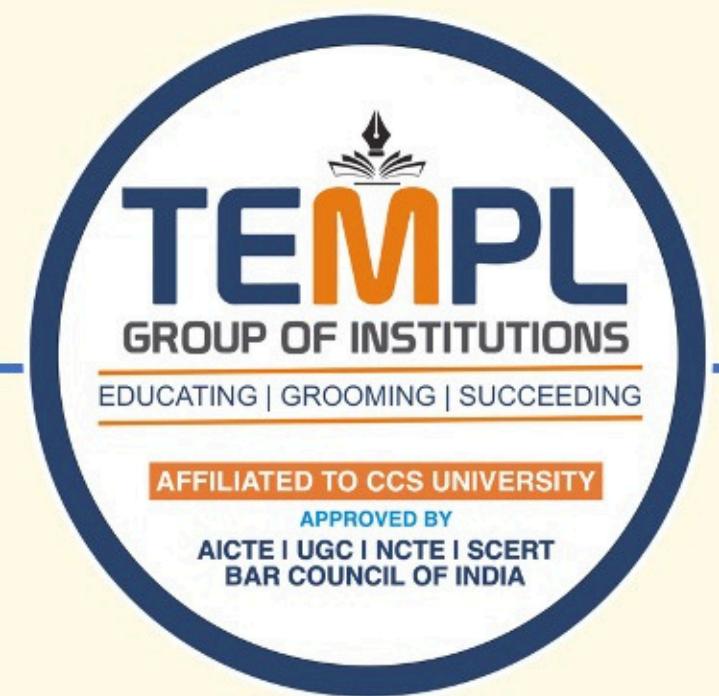
**Euler's formula is**

$$y_{n+1} = y_n + h \cdot f(x_n, y_n)$$

**where,**

- $x_0, y_0$  = initial values
- $h$  = step size
- $f(x, y)$  = given function (slope)

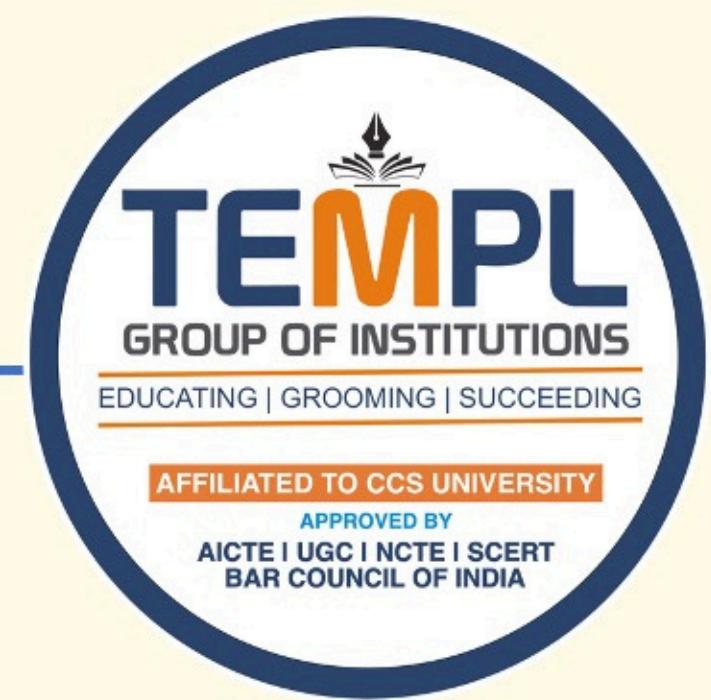
# STEPS OF EULER'S METHOD



- » Start with initial condition  $(x_0, y_0)$
- » Choose a step size  $h$ .
- » Compute slope,  $f(x_n, y_n)$ .
- » Use formula,
- » Repeat for required steps.

$$y_{n+1} = y_n + h \cdot f(x_n, y_n)$$

# EXAMPLE OF EULER'S METHOD



## The initial value problem

$$\frac{dy}{dx} = x - y, \quad y(0) = 1$$

with step size  $h = 0.1$ .

Find approximations up to  $x = 0.3$ .

Then

$$y_{n+1} = y_n + h f(x_n, y_n)$$

where  $f(x, y) = x - y$ .

$$x_0 = 0.0, \quad y_0 = 1.000.$$

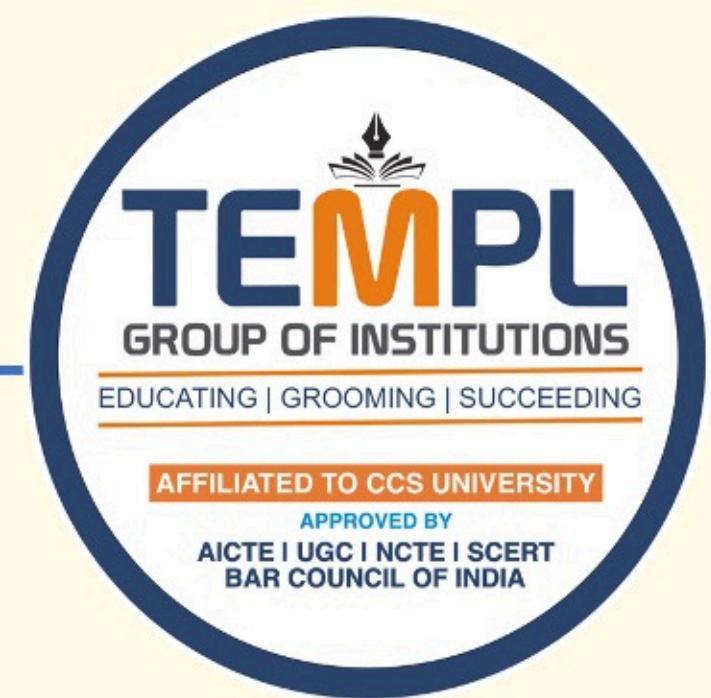
## Compute step by step

$n$	$x_n$	$y_n$	$f(x_n, y_n) = x_n - y_n$	$y_{n+1} = y_n + h f$
0	0.0	1.000	$0.0 - 1.000 = -1.000$	$1.000 + 0.1(-1.000) = 0.900$
1	0.1	0.900	$0.1 - 0.900 = -0.800$	$0.900 + 0.1(-0.800) = 0.820$
2	0.2	0.820	$0.2 - 0.820 = -0.620$	$0.820 + 0.1(-0.620) = 0.758$

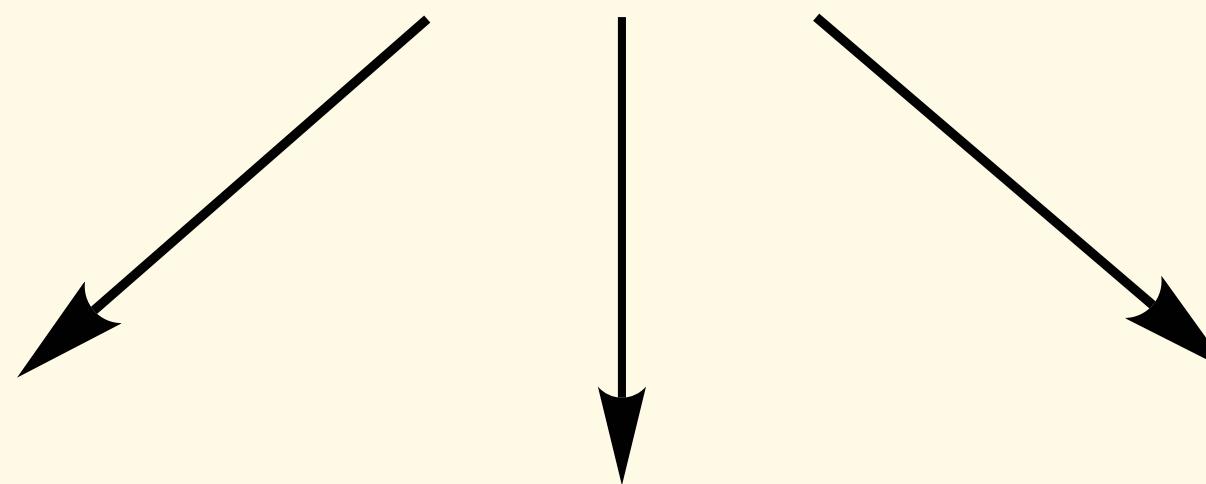
So the Euler approximations are

$$y(0.1) \approx 0.900, \quad y(0.2) \approx 0.820, \quad y(0.3) \approx 0.758$$

# **TYPES OF EULER'S METHOD**



## **TYPES OF EULER'S METHOD**



**Simple Euler's Method  
(Basic Method)**

**Improved Euler's Method  
(Heun's Method)**

**Modified Euler's Method  
(Midpoint Method)**

# SIMPLE (OR BASIC) EULER'S METHOD



Euler's method is the simplest numerical method to solve a first order differential equation of the form

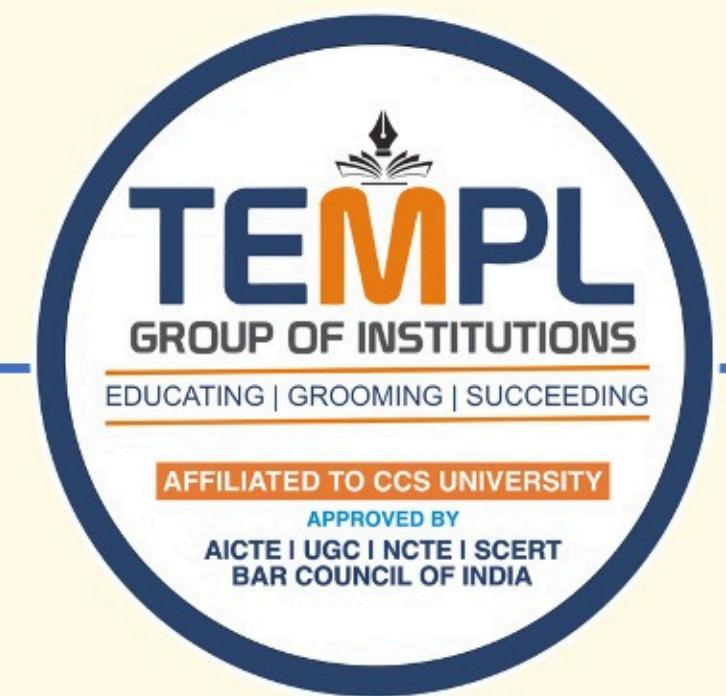
$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

It finds the approximate value of y at successive points.

$$y_{n+1} = y_n + h f(x_n, y_n)$$

Easy but less accurate (error is proportional to step size h).

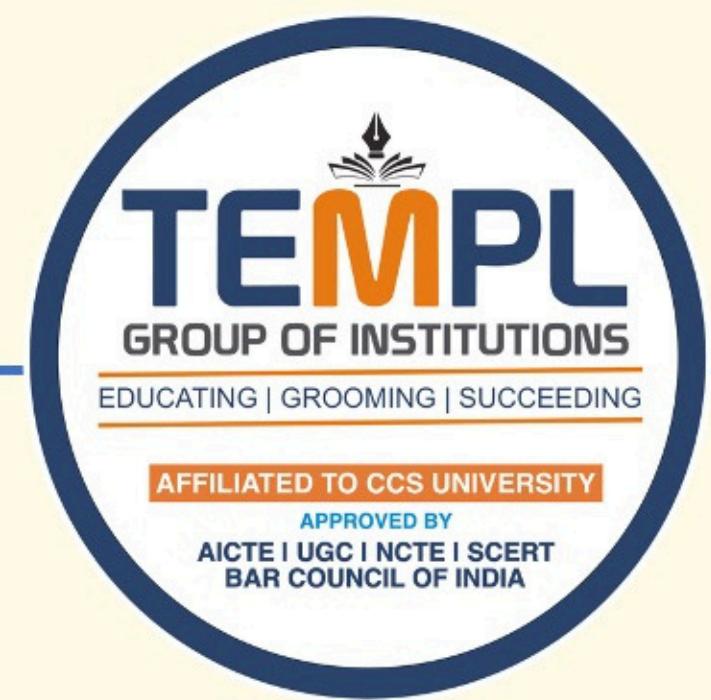
# STEPS OF EULER'S METHOD



- » Start with initial condition  $(x_0, y_0)$
- » Choose a step size  $h$ .
- » Compute slope,  $f(x_n, y_n)$ .
- » Use formula,
- » Repeat for required steps.

$$y_{n+1} = y_n + h \cdot f(x_n, y_n)$$

# EXAMPLE OF EULER'S METHOD



## The initial value problem

$$\frac{dy}{dx} = x - y, \quad y(0) = 1$$

with step size  $h = 0.1$ .

Find approximations up to  $x = 0.3$ .

Then

$$y_{n+1} = y_n + h f(x_n, y_n)$$

where  $f(x, y) = x - y$ .

$$x_0 = 0.0, \quad y_0 = 1.000.$$

## Compute step by step

$n$	$x_n$	$y_n$	$f(x_n, y_n) = x_n - y_n$	$y_{n+1} = y_n + h f$
0	0.0	1.000	$0.0 - 1.000 = -1.000$	$1.000 + 0.1(-1.000) = 0.900$
1	0.1	0.900	$0.1 - 0.900 = -0.800$	$0.900 + 0.1(-0.800) = 0.820$
2	0.2	0.820	$0.2 - 0.820 = -0.620$	$0.820 + 0.1(-0.620) = 0.758$

So the Euler approximations are

$$y(0.1) \approx 0.900, \quad y(0.2) \approx 0.820, \quad y(0.3) \approx 0.758$$

# IMPROVED EULER'S METHOD (HEUN'S METHOD)



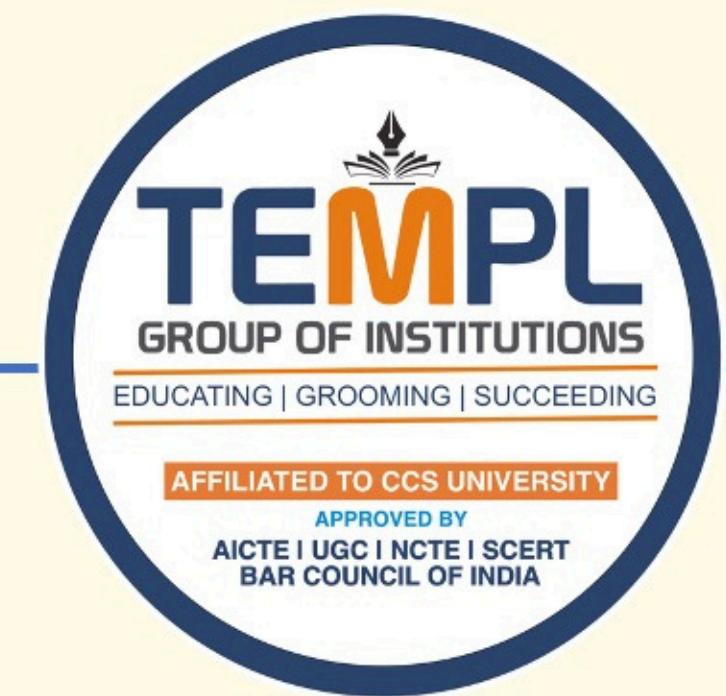
**Improved Euler's Method is a correction of the simple Euler's method.**

**It improves accuracy by taking the average slope at the beginning and end of the interval.**

$$y_{n+1} = y_n + \frac{h}{2} [f(x_n, y_n) + f(x_{n+1}, y_n + hf(x_n, y_n))]$$

**More accurate than Euler because it uses two slope values.**

# STEPS OF IMPROVED EULER'S METHOD

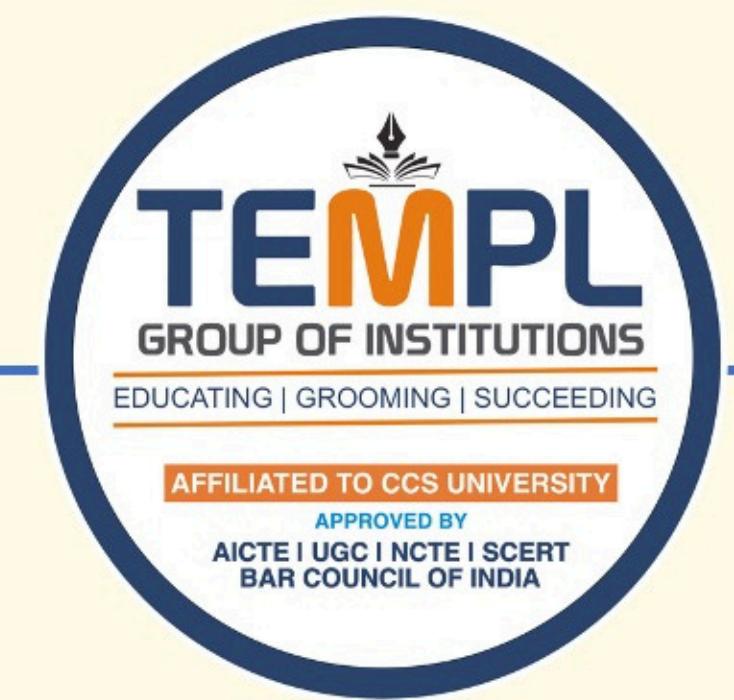


- » Start with initial condition  $(x_0, y_0)$ .
- » Choose a step size  $h$ .
- » Predictor Step (Euler's formula)
- » Corrector Step (Average slope)
- » Repeat for required steps.

$$y^* = y_n + h f(x_n, y_n)$$

$$y_{n+1} = y_n + \frac{h}{2} [f(x_n, y_n) + f(x_{n+1}, y^*)]$$

# EXAMPLE OF IMPROVED EULER'S METHOD



## The initial value problem

$$\frac{dy}{dx} = x - y, \quad y(0) = 1$$

with step size  $h = 0.1$ .

Find approximations up to  $x = 0.3$ .

Then

Predictor

$$y^* = y_n + h f(x_n, y_n)$$

Corrector

$$y_{n+1} = y_n + \frac{h}{2} [f(x_n, y_n) + f(x_{n+1}, y^*)]$$

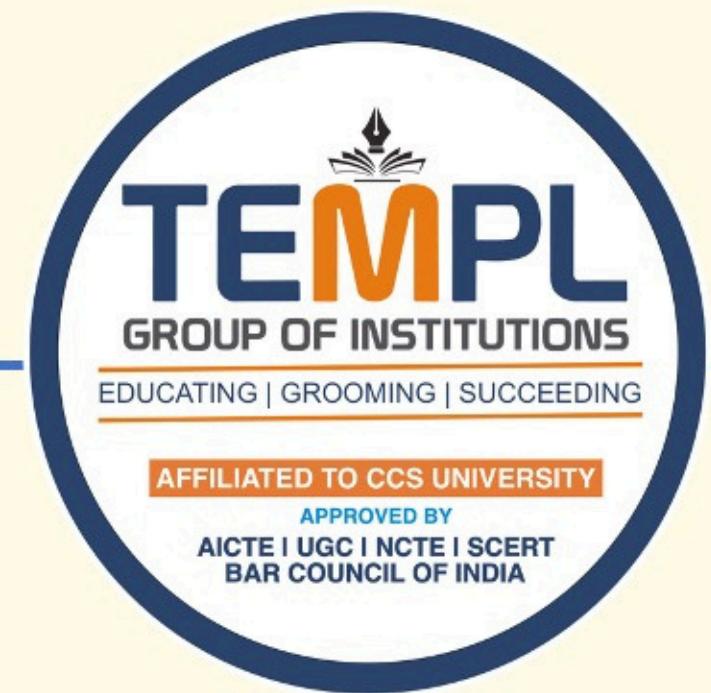
## Compute step by step

n	$x_n$	$y_n$	$f(x_n, y_n) = x_n - y_n$	Predictor $y^*$ (calc)	$f(x_{n+1}, y^*)$	Corrector $y_{n+1}$ (calc)
0	0.0	1.00000	-1.00000	$0.900000 (1 + 0.1 \cdot (-1)) = 0.900000$	-0.800000	$0.910000 (1 + \frac{0.1}{2}(-1 + -0.8)) = 0.910000$
1	0.1	0.91000	-0.81000	$0.829000 (0.91 + 0.1 \cdot (-0.81)) = 0.829000$	-0.629000	$0.838050 (0.91 + \frac{0.1}{2}(-0.81 + -0.629)) = 0.838050$
2	0.2	0.838050	-0.638050	$0.774245 (0.83805 + 0.1 \cdot (-0.63805)) = 0.774245$	-0.474245	$0.782435 (0.83805 + \frac{0.1}{2}(-0.63805 + -0.474245)) = 0.78243525 \rightarrow 0.782435$

So the Euler approximations are

$$y(0.1) \approx 0.91000, \quad y(0.2) \approx 0.83805, \quad y(0.3) \approx 0.78243$$

# MODIFIED EULER'S METHOD (MIDPOINT METHOD)



**Modified Euler's Method is another correction form of Euler's method.**

**It calculates the slope at the midpoint of the interval for better accuracy.**

$$y_{n+1} = y_n + h f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n)\right)$$

**Accuracy better than simple Euler but slightly different from Heun's method.**

# STEPS OF MODIFIED EULER'S METHOD



- » Start with initial condition  $(x_0, y_0)$ .
- » Choose a step size  $h$ .
- » Compute slope at the beginning (Predictor). 
$$k_1 = f(x_n, y_n)$$
- » Compute slope at midpoint (Corrector). 
$$k_2 = f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2}k_1\right)$$
- » Use formula to update. 
$$y_{n+1} = y_n + h \cdot k_2$$
- » Repeat for required steps.

# EXAMPLE OF MODIFIED EULER'S METHOD



## The initial value problem

$$\frac{dy}{dx} = x - y, \quad y(0) = 1$$

with step size  $h = 0.1$ .

Find approximations up to  $x = 0.3$ .

Then

Predictor

$$k_1 = f(x_n, y_n)$$

Corrector

$$y^* = y_n + \frac{h}{2} k_1$$

$$k_2 = f\left(x_n + \frac{h}{2}, y^*\right)$$

So

$$y_{n+1} = y_n + h k_2$$

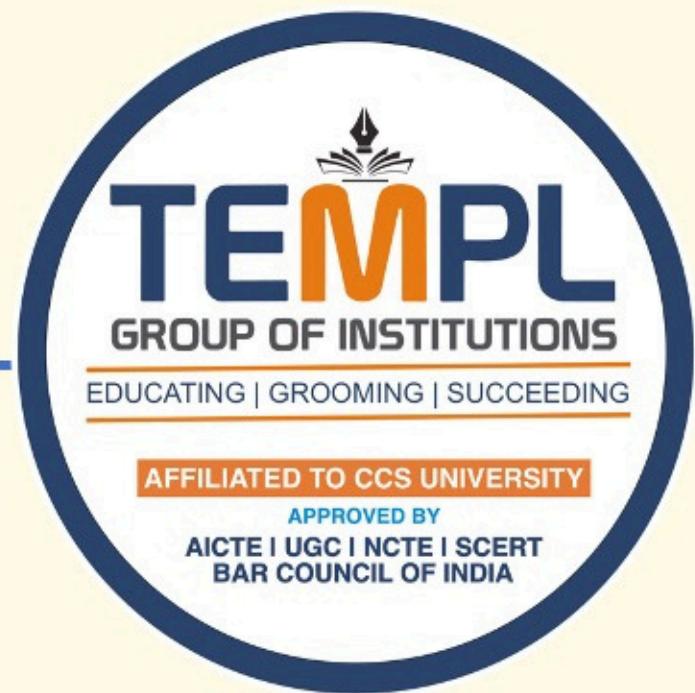
## Compute step by step

$n$	$x_n$	$y_n$	$k_1 = f(x_n, y_n) = x_n - y_n$	$y^* = y_n + \frac{h}{2} k_1$	$k_2 = f(x_n + \frac{h}{2}, y^*)$	$y_{n+1} = y_n + h k_2$
0	0.0	1.000000	-1.000000	0.950000	-0.900000	0.910000
1	0.1	0.910000	-0.810000	0.869500	-0.719500	0.838050
2	0.2	0.838050	-0.638050	0.8061475	-0.5561475	0.78243525

So the Euler approximations are

$$y(0.1) \approx 0.9100, \quad y(0.2) \approx 0.8380, \quad y(0.3) \approx 0.7824$$

# COMPARISON OF EULER, IMPROVED EULER & MODIFIED EULER METHODS



## COMPARISON OF EULER, IMPROVED EULER & MODIFIED EULER METHODS

### EULER'S METHOD

$$y_{n+1} = y_n + h f(x_n, y_n)$$

Initial slope only

Accuracy Higher

### IMPROVED EULER

$$y_{n+1} = y_n + \frac{h}{2} [f(x_n, y_n) + f(x_{n+1}, y_n + h f(x_n, y_n))]$$

Start & End slope avg.

Accuracy Higher

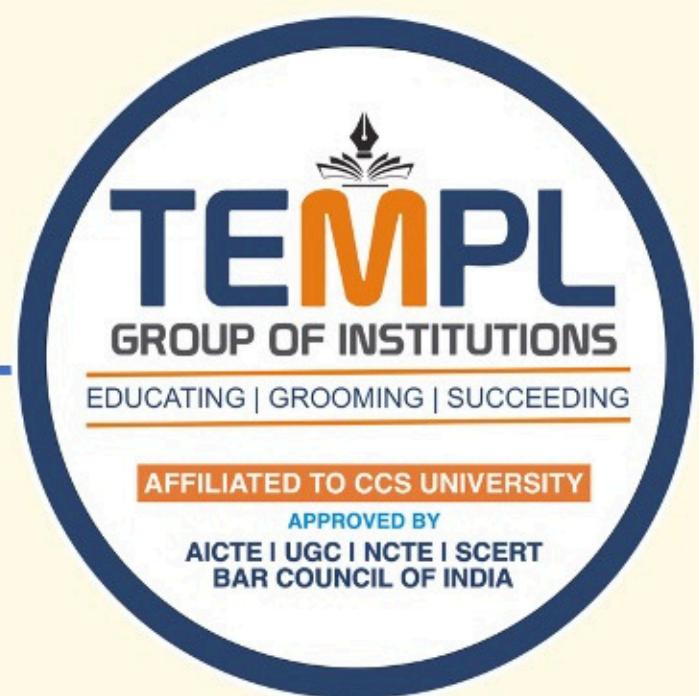
### MODIFIED EULER

$$y_{n+1} = y_n + h f\left(x_n + \frac{h}{2}, y_n + \frac{h}{2} f(x_n, y_n)\right)$$

Midpoint slope

Accuracy Low

# PICARD'S METHOD



**Picard's Method is an iterative method used to obtain an approximate solution of a first order ordinary differential equation.**

**The method is based on successive approximations.**

**The Picard's Method was founded by the French mathematician Charles Émile Picard around the 19<sup>th</sup> century.**

# FORMULA OF PICARD'S METHOD



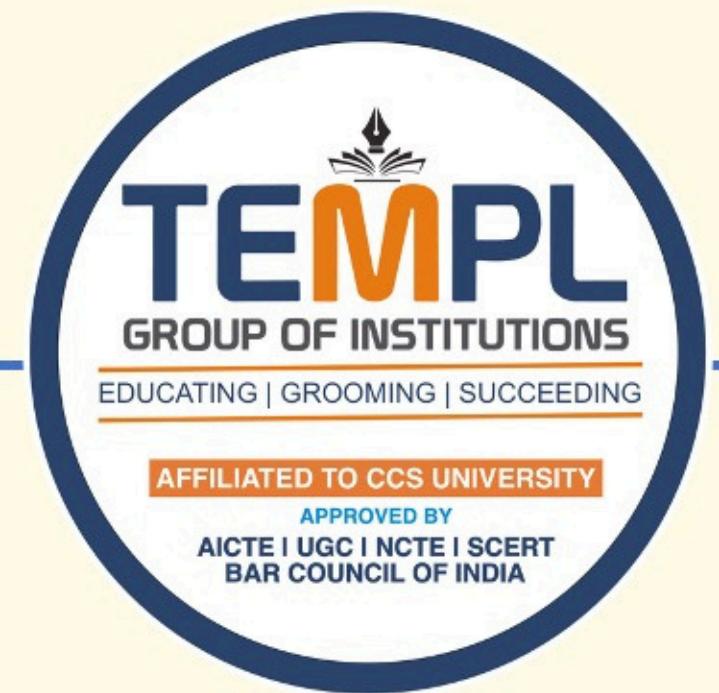
For the differential equation

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

Picard's formula is

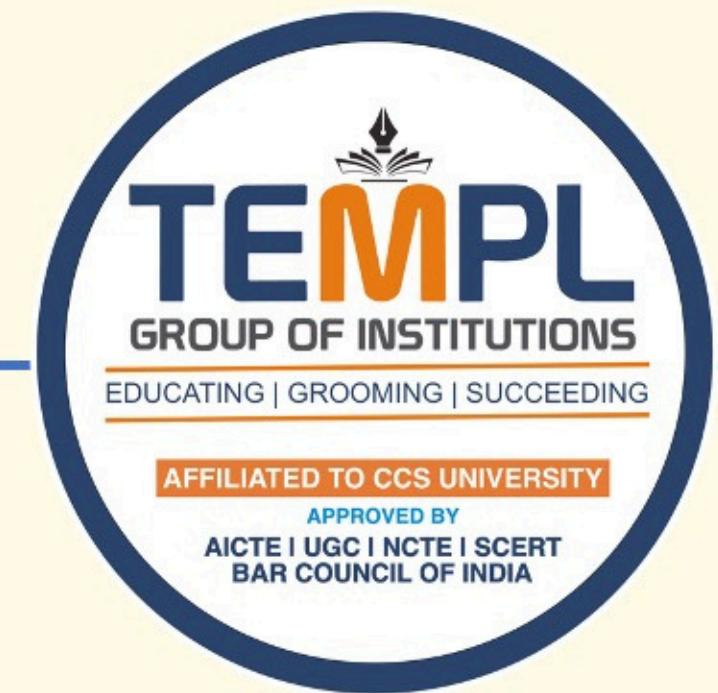
$$y_{n+1}(x) = y_0 + \int_{x_0}^x f(t, y_n(t)) dt$$

# STEPS OF PICARD'S METHOD



- » Write the differential equation in integral form.
- » Choose an initial approximation  $y_0(x)$ .
- » Substitute  $y_0(x)$  in the integral to get  $y_1(x)$ .
- » Repeat the process to obtain  $y_2(x)$ ,  $y_3(x)$ , ...
- » Continue until the required accuracy is achieved.

# RUNGE-KUTTA (RK) METHOD

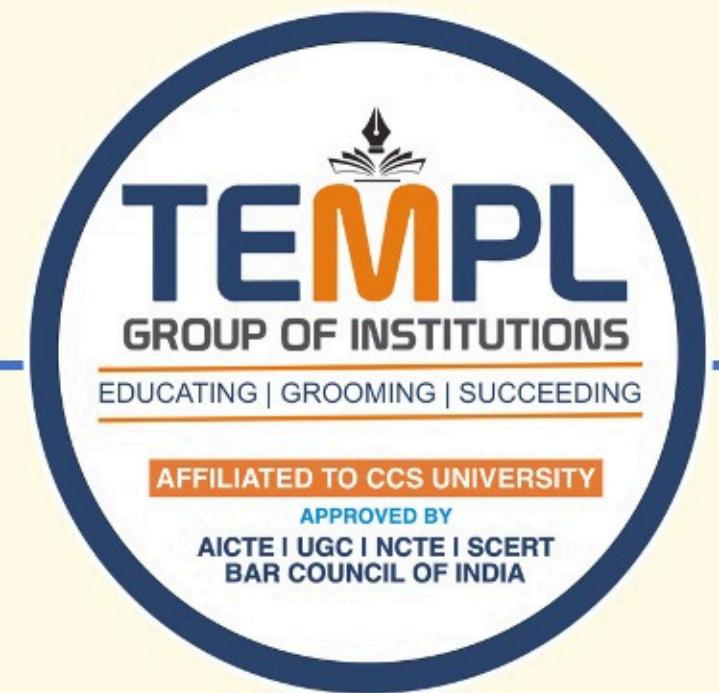


**The Runge-Kutta (RK) method is a numerical technique used to obtain an approximate solution of first order ordinary differential equations.**

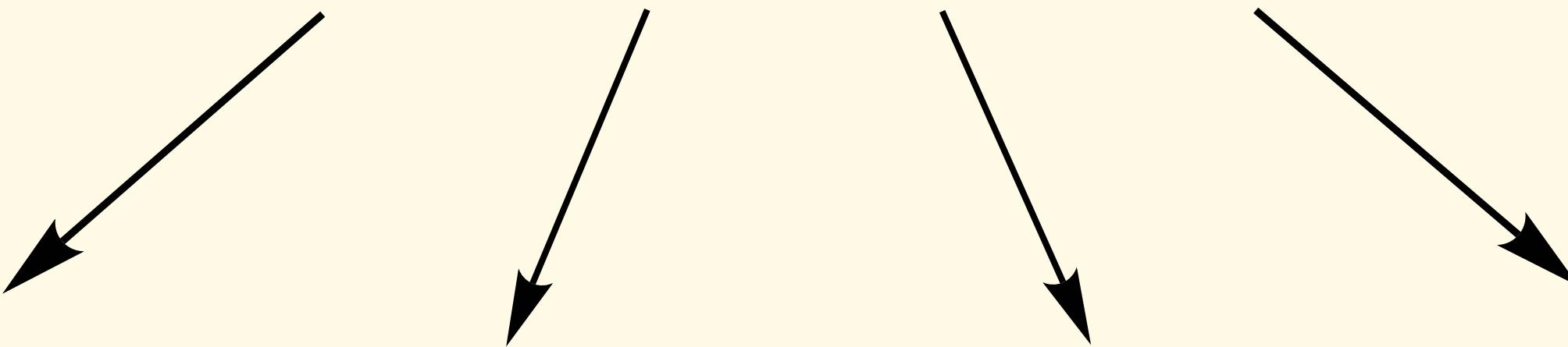
**The method improves accuracy by taking a weighted average of slopes at different points within each step.**

**The Runge-Kutta (RK) method was founded by the German mathematicians Carl Runge and Martin Kutta around the 20<sup>th</sup> century .**

# **TYPES OF RUNGE-KUTTA (RK) METHOD**



## **TYPES OF RUNGE-KUTTA (RK) METHOD**



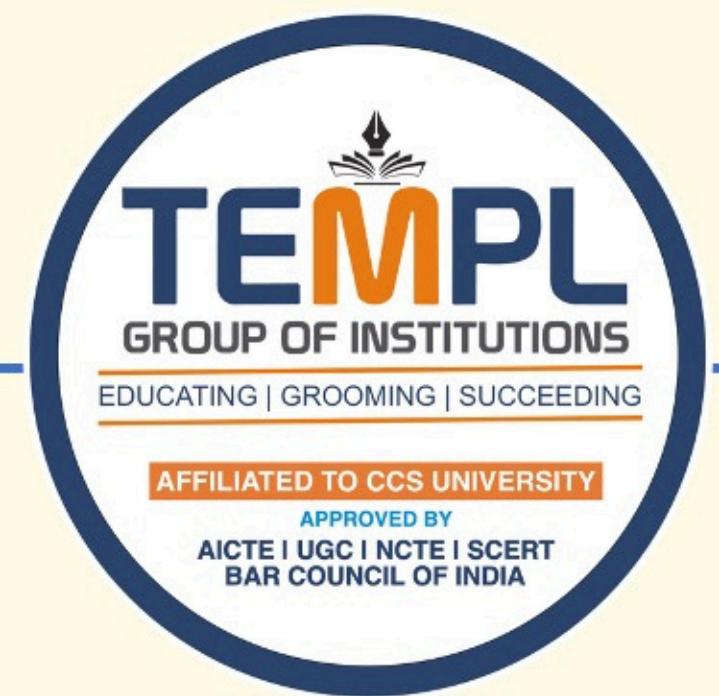
**First order  
R-K method**

**Second order  
R-K method**

**Third order  
R-K method**

**Fourth order  
R-K method**

# FORMULA OF FIRST ORDER R-K METHOD



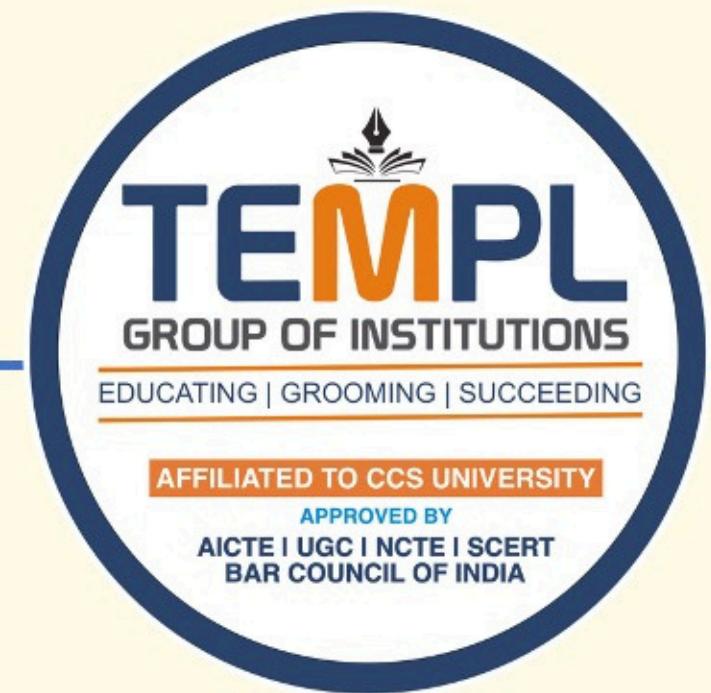
For the differential equation

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

First Order Runge-Kutta's formula is

$$y_{n+1} = y_n + h \cdot f(x_n, y_n)$$

# FORMULA OF SECOND ORDER R-K METHOD



For the differential equation

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

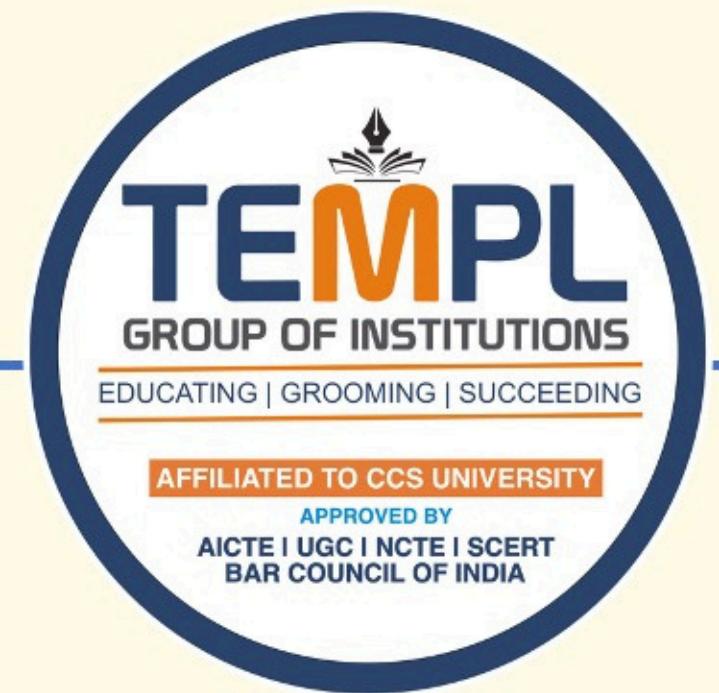
Second Order Runge-Kutta's formula is

$$k_1 = h f(x_n, y_n)$$

$$k_2 = h f\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$y_{n+1} = y_n + k_2$$

# FORMULA OF THIRD ORDER R-K METHOD



For the differential equation

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

Third Order Runge-Kutta's formula is

$$k_1 = h f(x_n, y_n)$$

$$k_2 = h f\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_3 = h f(x_n + h, y_n - k_1 + 2k_2)$$

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 4k_2 + k_3)$$

# FORMULA OF FOURTH ORDER R-K METHOD



For the differential equation

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$$

Fourth Order Runge-Kutta's formula is

$$k_1 = h f(x_n, y_n)$$

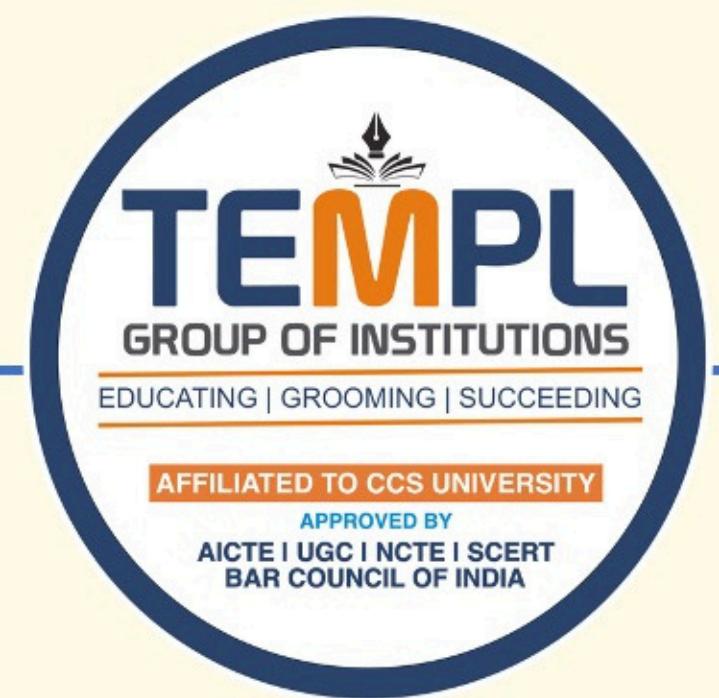
$$k_2 = h f\left(x_n + \frac{h}{2}, y_n + \frac{k_1}{2}\right)$$

$$k_3 = h f\left(x_n + \frac{h}{2}, y_n + \frac{k_2}{2}\right)$$

$$k_4 = h f(x_n + h, y_n + k_3)$$

$$y_{n+1} = y_n + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

# RUNGE-KUTTA (RK) METHOD



## QUICK EXAM ORIENTED SUMMARY TABLE

RK Method	Order	Accuracy	Syllabus Importance
RK-1	1st	Low	Basic
RK-2	2nd	Moderate	Important
RK-3	3rd	Good	Sometimes asked
RK-4	4th	Very High	Most Important