1a $A = \begin{bmatrix} -2 & -1 \\ -1 & 0 \\ 1 & 0 \\ 2 & 1 \end{bmatrix}$ $V = a'a/n$ where $a = A - (1 \cdot A) \cdot \frac{1}{n}$ $n =$ # points

$$a = \begin{bmatrix} -2 & -1 \\ -1 & 0 \\ 1 & 0 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} -2 & -1 \\ -1 & 0 \\ 1 & 0 \\ 2 & 1 \end{bmatrix} \cdot (1/4)$$

$$a = \begin{bmatrix} -2 & -1 \\ -1 & 0 \\ 1 & 0 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -2 & -1 \\ -1 & 0 \\ 1 & 0 \\ 2 & 1 \end{bmatrix}$$

$$a'a = \begin{bmatrix} -2 & -1 & 1 & 2 \\ -1 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -2 & -1 \\ -1 & 0 \\ 1 & 0 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 4 \\ 4 & 2 \end{bmatrix} \cdot 1/4 = \boxed{\begin{bmatrix} 2.5 & 1 \\ 1 & 0.5 \end{bmatrix}}$$

1b
$$\left| \begin{bmatrix} 2.5 & 1 \\ 1 & 0.5 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right| = (2.5 - \lambda)(0.5 - \lambda) - 1 = 0$$

$$\lambda^2 - 3\lambda + \frac{1}{4} = 0$$

$$\lambda = \frac{3 \pm \sqrt{9-1}}{2} = \boxed{\frac{3}{2} \pm \sqrt{2}}$$

$$\begin{bmatrix} 2.5 & 1 \\ 1 & 0.5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \lambda \begin{bmatrix} x \\ y \end{bmatrix}$$

$2.5x + y = (\frac{3}{2} + \sqrt{2})x$ $\qquad$ $2.5x + y = (\frac{3}{2} - \sqrt{2})x$

$x + 0.5y = (\frac{3}{2} + \sqrt{2})y$ $\qquad$ $x + 0.5y = (\frac{3}{2} - \sqrt{2})y$

$$\left.\begin{aligned} (1-\sqrt{2})x + y = 0 \\ x + (-1-\sqrt{2})y = 0 \end{aligned}\right\} \to \begin{bmatrix} 1-\sqrt{2} \\ 1 \end{bmatrix} \qquad \left.\begin{aligned} (1+\sqrt{2})x + y = 0 \\ x - (-1+\sqrt{2})y = 0 \end{aligned}\right\} \to \begin{bmatrix} 1+\sqrt{2} \\ 1 \end{bmatrix}$$

1c Normalize eigenvectors:

$$\frac{1}{\sqrt{4-2\sqrt{2}}} \begin{bmatrix} 1-\sqrt{2} \\ 1 \end{bmatrix}, \quad \frac{1}{\sqrt{4+2\sqrt{2}}} \begin{bmatrix} 1+\sqrt{2} \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{\sqrt{4-2\sqrt{2}}} - \frac{1}{\sqrt{2-\sqrt{2}}} \\ \frac{1}{\sqrt{4-2\sqrt{2}}} \end{bmatrix}, \quad \begin{bmatrix} \frac{1}{\sqrt{4+2\sqrt{2}}} + \frac{1}{\sqrt{2+\sqrt{2}}} \\ \frac{1}{\sqrt{4+2\sqrt{2}}} \end{bmatrix} \qquad P = \begin{bmatrix} \frac{1}{\sqrt{4-2\sqrt{2}}} - \frac{1}{\sqrt{2-\sqrt{2}}} & \frac{1}{\sqrt{4+2\sqrt{2}}} + \frac{1}{\sqrt{2+\sqrt{2}}} \\ \frac{1}{\sqrt{4-2\sqrt{2}}} & \frac{1}{\sqrt{4+2\sqrt{2}}} \end{bmatrix}$$

$$P = \begin{bmatrix} 0.9239 - 1.3066 & 0.3827 + 0.5412 \\ 0.9239 & 0.3827 \end{bmatrix} = \begin{bmatrix} -0.3827 & 0.9239 \\ 0.9239 & 0.3827 \end{bmatrix}$$

$$PP' = \begin{bmatrix} -0.3827 & 0.9239 \\ 0.9239 & 0.3827 \end{bmatrix} \begin{bmatrix} -0.3827 & 0.9239 \\ 0.9239 & 0.3827 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$P' = P^{-1}$ ✓

Principal components:

$$P_1 = \frac{1}{\sqrt{4-2\sqrt{2}}}(1-\sqrt{2})\,x + \frac{1}{\sqrt{4+2\sqrt{2}}}(1+\sqrt{2})\,y$$

$$P_2 = \frac{1}{\sqrt{4-2\sqrt{2}}}\,x + \frac{1}{\sqrt{4+2\sqrt{2}}}\,y$$

1d Class 1: $\{(-2,-1), (-1,0), (1,0)\}$

Class 2: $\{(2,1)\}$

$$\mu_1 = \frac{1}{3}\left[\begin{pmatrix} -2 \\ -1 \end{pmatrix} + \begin{pmatrix} -1 \\ 0 \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}\right] = \frac{1}{3}\begin{pmatrix} -2 \\ -1 \end{pmatrix} = \begin{pmatrix} -\frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}$$

$$\mu_2 = \frac{1}{1}\left[\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right] = 1 \cdot \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$S_1 = \frac{1}{3}\left[\left[\begin{pmatrix} -2 \\ -1 \end{pmatrix} - \begin{pmatrix} -\frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}\right]^2 + \left[\begin{pmatrix} -1 \\ 0 \end{pmatrix} - \begin{pmatrix} -\frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}\right]^2 + \left[\begin{pmatrix} 1 \\ 0 \end{pmatrix} - \begin{pmatrix} -\frac{2}{3} \\ -\frac{1}{3} \end{pmatrix}\right]^2\right]$$

$$= \frac{1}{3}\left[\left[\begin{pmatrix} -\frac{4}{3} \\ -\frac{2}{3} \end{pmatrix}\right]^2 + \left[\begin{pmatrix} -\frac{1}{3} \\ \frac{1}{3} \end{pmatrix}\right]^2 + \left[\begin{pmatrix} \frac{5}{3} \\ \frac{1}{3} \end{pmatrix}\right]^2\right]$$

$$= \frac{1}{3}\left[\begin{pmatrix} \frac{16}{9} & \frac{8}{9} \\ \frac{8}{9} & \frac{4}{9} \end{pmatrix} + \begin{pmatrix} \frac{1}{9} & -\frac{1}{9} \\ -\frac{1}{9} & \frac{1}{9} \end{pmatrix} + \begin{pmatrix} \frac{25}{9} & \frac{5}{9} \\ \frac{5}{9} & \frac{1}{9} \end{pmatrix}\right]$$

$$= \frac{1}{3}\begin{pmatrix} \frac{42}{9} & \frac{12}{9} \\ \frac{12}{9} & \frac{6}{9} \end{pmatrix} = \begin{pmatrix} \frac{14}{9} & \frac{4}{9} \\ \frac{4}{9} & \frac{2}{9} \end{pmatrix}$$

$$S_2 = 1\left[\begin{pmatrix} 2 \\ 1 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \end{pmatrix}\right]^2 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$S_w = S_1 + S_2 = \begin{pmatrix} \frac{14}{9} & \frac{4}{9} \\ \frac{4}{9} & \frac{2}{9} \end{pmatrix}$$

$$S_B = (\mu_1 - \mu_2)(\mu_1 - \mu_2)^T = \begin{pmatrix} -\frac{8}{3} \\ -\frac{4}{3} \end{pmatrix}\begin{pmatrix} -\frac{8}{3} & -\frac{4}{3} \end{pmatrix} = \begin{pmatrix} \frac{64}{9} & \frac{32}{9} \\ \frac{32}{9} & \frac{16}{9} \end{pmatrix}$$

$$S_w^{-1} S_B w = \lambda w$$

$$\left|\begin{pmatrix} \frac{14}{9} & \frac{4}{9} \\ \frac{4}{9} & \frac{2}{9} \end{pmatrix}^{-1}\begin{pmatrix} \frac{64}{9} & \frac{32}{9} \\ \frac{32}{9} & \frac{16}{9} \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right| = 0$$

$$\left|\begin{pmatrix} \frac{3}{2} & -3 \\ -3 & \frac{21}{2} \end{pmatrix}\begin{pmatrix} \frac{64}{9} & \frac{32}{9} \\ \frac{32}{9} & \frac{16}{9} \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right| = 0$$

$$\left|\begin{pmatrix} 0 & 0 \\ 16 & 8 \end{pmatrix} - \lambda\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right| = 0$$

$$\left|\begin{pmatrix} 0-\lambda & 0 \\ 16 & 8-\lambda \end{pmatrix}\right| = 0$$

$$-\lambda(8-\lambda) - 0 = 0$$

$$-\lambda(8-\lambda) = 0$$

$$\lambda = 0, 8$$

$$\lambda = 0 \Rightarrow \begin{pmatrix} 0 & 0 \\ 16 & 8 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$16x + 8y = 0$$

$$y = -2x \Rightarrow \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$$\lambda = 8 \Rightarrow \begin{pmatrix} -8 & 0 \\ 16 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$-8x = 0$$

$$16x = 0 \Rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$\boxed{\text{LDA Projection: } \begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$