Number Theory Lecture 4

Tuesday, 9 July 2024 8:20 PM

Solve $ax + by = c$

$$\left.\begin{aligned} 0 \le\ & x \le 15 \\ -300 \le\ & y \le 150 \end{aligned}\right\}$$ How many such solutions exists?

Ext. Euclidian $\to$ (x_1, y_1)

$a = 15, \quad b = 10, \quad c = 20$

$15x + 10y = 20$

$x = 4, \quad y = -4, \quad \gcd = 5$

$$5(3x+2y)=20$$

$$\Rightarrow 3x+2y=4 \checkmark$$

$$(x_1, y_1) \equiv (4, -4)$$

$$3x_1+2y_1=4 \checkmark$$

$$x, x_1, y, y_1 \in \mathbb{Z}$$

$$\Rightarrow x-x_1,\ y-y_1 \in \mathbb{Z}$$

$$\Rightarrow u, v \in \mathbb{Z}$$

$$3(\underbrace{x-x_1}_{u})+2(\underbrace{y-y_1}_{v})=0$$

$$\Rightarrow \boxed{3u+2v=0}$$

$$u=2k,\quad v=-3k,\quad k \in \mathbb{Z}$$

$k=0 \Rightarrow u=0,\ v=0$

$k=1 \Rightarrow u=2,\ v=-3$

$k=-1 \Rightarrow u=-2,\ v=+3$

$x - x_1 = 2k, \quad y - y_1 = -3k$

$\Rightarrow x = \underline{\underline{x_1}} + 2k, \quad y = \underline{\underline{y_1}} - 3k$

$\Rightarrow (x, y) \equiv (\underline{\underline{x_1 + 2k}}, \underline{\underline{y_1 - 3k}}), \quad k \in \mathbb{Z}$

$15x + 10y = 20$

$\Rightarrow \underline{\underline{3x + 2y}} = 4 \qquad x_1 = \underline{\underline{4}}, \quad y_1 = -\underline{\underline{4}}$

$(x_1 + 2k, y_1 - 3k)$

$\equiv (\underline{4 + 2k, -4 - 3k}), \quad k \in \mathbb{Z}$ ✓

$k = 0 \qquad (4, -4)$ ✓

$k = 1 \qquad (\underline{6, -7})$ ✓

$k = -1 \qquad (2, -1)$ ✓

$k = -2 \qquad (0, 2)$ ✓

$L \le x \le R$

$L \le \underline{\underline{4 + 2k}} \le R$

$4 + 2k \ge L$

$k \ge \left(\frac{L-4}{2}\right)$

$\Rightarrow k \ge \text{ceil}\left(\frac{L-4}{2}\right)$

$Ax + By = C \quad , \gcd(A, B) = g$

$g\left(\frac{A}{g}x + \frac{B}{g}y\right) = C$

$$\boxed{(x, y) \equiv \left(x_1 + \frac{B}{g}k, \quad y_1 - \frac{A}{g}k\right)}$$

Github Repo: https://github.com/jay99bansal/DSA_course

https://codeforces.com/problemset/problem/530/C

$Ax + By = C$

ex-gcd $\rightarrow$ x_1, y_1, g

$\gcd(-15, -20) = 4$

$(x, y) \equiv \left(x_1 + \frac{B}{g}k, \; y_1 - \frac{A}{g}k\right)$

$A, B, C > 0 \quad , \quad x, y > 0$

$$\boxed{x_1 + \frac{B}{g}k} > 0$$

$$\Rightarrow \frac{B}{g}k > -x_1$$

$$\Rightarrow k > \frac{-x_1}{(B/g)}$$

lower limit.

$k > \frac{12}{3} \rightarrow k > 4 \Rightarrow k \geq 5$

$k > \frac{13}{3} \Rightarrow k > 4.33 \Rightarrow k \geq 5$

$\frac{14}{3} \rightarrow 5$

$\frac{15}{3} \rightarrow 6$

$\frac{-x_1}{B/g}$

$$y_1 - \frac{A}{g}k > 0$$

$$\Rightarrow \frac{A}{g}k < y_1$$

$$\Rightarrow k < \frac{y_1}{(A/g)} \quad \left(\because A, B > 0,\ g > 0\right)$$

upper limit

$k < \frac{12}{3} \Rightarrow k < 4 \Rightarrow k \leq 3$

$k < \frac{13}{3} \Rightarrow k < 4.33 \Rightarrow k \leq 4$

$\left\{\frac{14}{3} \rightarrow 4\right.$

$\left\{\frac{15}{3} \rightarrow 4\right.$

ceil(—) − 1

```
llmit = -x1 // (B//g) + 1

k >= llmit
```

```
ulimit = y1 // (A//g)
if (y1 % (A//g) == 0)
    ulmit -= 1

k <= ulimit
```

solutions = ulimt - llimit + 1

for k in range (llimit, ulimt +1):
    print (x_1 + B//g * k , y_1 - A//g * k)

```python
def f_gcd(a, b):
    # Returns x, y, g
    if b==0:
        return 1, 0, a
    x0, y0, g = f_gcd(b, a%b)
    return y0, x0-(a//b)*y0, g

def solve(i):
    a,b,c = map(int, input().split())
    if a>b:
        x,y,g = f_gcd(a,b)
    else:
        y,x,g = f_gcd(b,a)
    if c%g == 0:
        # print(f'Case {i}: x = {(c//g)*x}, y = {(c//g)*y}')
        x1 = (c//g)*x
        y1 = (c//g)*y
        llmit = -x1//(b//g)+1
        ulimit = y1//(a//g)
        if y1%(a//g) == 0:
            ulimit -= 1
        print(ulimit-llmit+1)
        for k in range(llmit, ulimit+1):
            print(f'{x1+b//g*k} {y1-a//g*k}')
    else:
        print(0)

t = int(input())
for i in range(1, t+1):
    solve(i)
```

https://leetcode.com/problems/greatest-common-divisor-of-strings/

ABABAB | AB

ABCD ∤ AB ?

ABA ∤ AB

gcd (a, b)

|

gcd (b, a % b)

⋮

ABAB AB % ABAB

= AB

a % b < b

AAAAA % $\boxed{AA}$ = A

ABCDE % AB

= $\boxed{CDE}$

gcd(ABABAB, ABAB)

= gcd(ABAB, AB)

= gcd (AB, "")

= $\boxed{AB}$

```python
def mods(a, b):
    while len(a) >= len(b) and a[:len(b)] == b:
        a = a[len(b):]
    return a

def f_gcd(a, b):
    if b == '':
        return a
    md = mods(a, b)
    if len(md) >= len(b):
        return ''
    return f_gcd(b, md)

class Solution:
    def gcdOfStrings(self, str1: str, str2: str) -> str:
        if(len(str1) > len(str2)):
            return f_gcd(str1, str2)
        return f_gcd(str2, str1)
```

— x —

Least Common Multiple (LCM)

$a = 12, \quad b = 18$

$a = 12 = 2^2 \times 3^1 = 2 \times 2 \times 3$

$b = 18 = 2 \times 3^2 = 2 \times 3 \times 3$

$a = 2 \times 2 \times 3$ → lcm, gcd, lcm

$b = 2 \times 3 \times 3$ → lcm

gcd one time

lcm one time

$gcd = 2 \times 3 = 6$

$lcm = 2 \times 3 \times 2 \times 3 = 36$

$a * b =$

$|a * b| = gcd * lcm$

$\rightarrow O(\log b)$ — smaller number

—x—

GCD and LCM for an array of integers?

* $\gcd(a, b, c) = \gcd(\gcd(a, b), c)$ ✓
* $\mathrm{lcm}(a, b, c) = \mathrm{lcm}(\mathrm{lcm}(a, b), c)$ ✓

array of size n $\quad$ $k = \underline{\min(A)}$

↳ $\underline{\gcd} \equiv O(n \log k)$

$\underline{\mathrm{lcm}} \equiv O(n \log k)$

=x=