Dynamic Programming Lecture 5

Friday, 30 August 2024 6:09 AM

Coin Change Problem

↳ Given array of coins of different denomination and a total amount, the task is to find the number of ways to make up the amount using the available coins. Each coin can be used unlimited number of times.

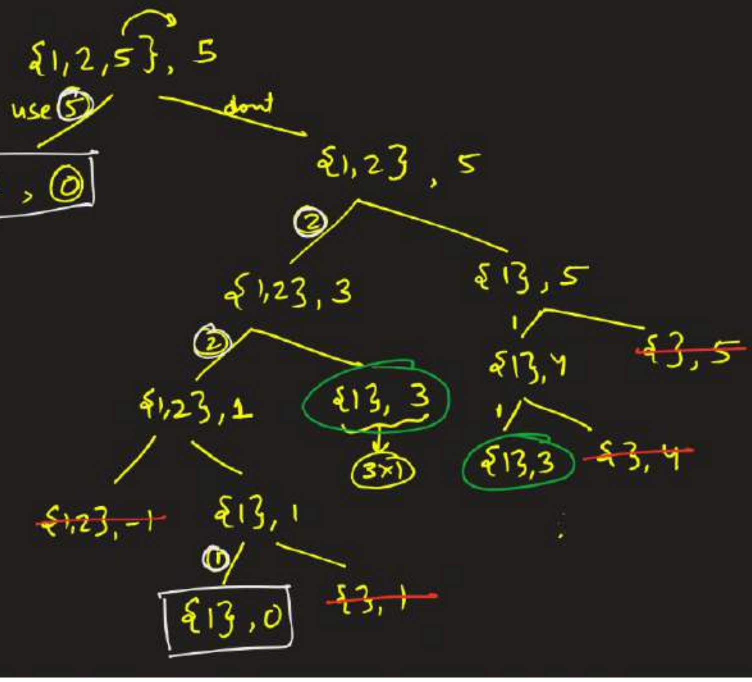
eg. Coins = {1, 2, 5}
Amount = 5

\# ways = {1,1,1,1,1}, {1,1,1,2}, {1,2,2}, {5}

count (coins, n, sum)

= count (coins, n, sum - coins[n-1])
+
count (coins, n-1, sum)

$T = O(n \cdot sum)$



=x=

Matrix Chain Multiplication

↳ Given matrices $A_1, A_2, \dots, A_n$ with dimensions $P_0 \times P_1, P_1 \times P_2 \dots P_{n-1} \times P_n$

find the minimum number of scalar multiplications needed to calculate the matrix product $A_1 A_2 \dots A_n$

$A_1 \times A_2 \times A_3 \times A_4 \dots = \underset{2\times 6}{A}$

$2 \times 3 \quad 3 \times 4 \quad 4 \times 1 \quad 1 \times 6$

$A_1 \times A_2 \neq A_2 \times A_1$

$(A_1 A_2) A_3 = A_1 (A_2 A_3)$

$A_1 \; A_2 \; A_3$

$2\times 1 \quad 1\times 3 \quad 3 \times 2$

$\begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 1 \end{bmatrix}$

$[1 \; 0 \; 2]$

$A_1 \times A_2$

$$\underset{2\times1}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} \underset{1\times3}{[1 \; 0 \; 2]} = \underset{2\times3}{\overset{A_{12}}{\begin{bmatrix} 1 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}}} \underset{3\times2}{\overset{A_3}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 1 \end{bmatrix}}} = \underset{2\times2}{\begin{bmatrix} 4 & 3 \\ 0 & 0 \end{bmatrix}}$$

$2 \times 3 \times 2$

$(A_1 A_2) A_3 : 2\times1\times3 + 2\times3\times2 = 6 + 12 = 18$

$\Rightarrow A_1 (A_2 A_3) : 1\times3\times2 + 2\times1\times2 = 6 + 4 = 10$

$$\underset{2\times1}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} \left(\underset{1\times3}{[1 \; 0 \; 2]} \underset{3\times2}{\begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 2 & 1 \end{bmatrix}} \right) = \underset{2\times1}{\begin{bmatrix} 1 \\ 0 \end{bmatrix}} \underset{1\times2}{[4 \; 3]} = \underset{2\times2}{\begin{bmatrix} 4 & 3 \\ 0 & 0 \end{bmatrix}}$$

$A_1 \times A_2$

$P_0 \times P_1 \quad P_1 \times P_2$

scalar multiplication $= P_0 \times P_1 \times P_2$

Brute :- To check all possible ways to multiply the matrices

↳ n^{th} catalan number $= \dfrac{\binom{2n}{n}}{n+1} \sim O(2^n)$

$$^{2n}C_n = \frac{(2n)!}{n!\,n!\,(n+1)} = \frac{(2n)(2n-1)\cdots(2n-(2n-1))}{n!\,(n+1)} \sim O(n^n)$$

$$\begin{array}{cccccc} A_1 & \cdot & \left[\begin{array}{ccc} A_2 \cdot & A_3 & \cdots \\ P_1 \times P_2 & P_2 \times P_3 & \end{array}\right] & A_n & = & A \\ P_0 \times P_1 & & & P_{n-1} \times P_n & & P_0 \times P_n \end{array}$$

i $\quad$ j

To multiply matrices A_i to A_j, the min. # scalar multiplication:

$$[A_i \quad A_{i+1} \quad A_{i+2}] \,|\, [\cdots \quad A_j]$$

$$\underline{m(A_i \to A_k)} \quad k \quad m(A_k \to A_j) + P_{i-1} \cdot P_k \cdot P_j$$

$$\begin{pmatrix} A_i & \cdots & A_k \\ P_{i-1} \times P_i & & P_{k-1} \times P_k \end{pmatrix} \begin{pmatrix} A_{k+1} & \cdots & A_j \\ P_k \times P_{k+1} & & A_{j-1} \times P_j \end{pmatrix}$$

$$P_k \times P_j$$

$$P_{i-1} \times P_k$$

$$P_{i-1} \times P_k \times P_j$$

$$m[i,j] = \begin{cases} \min\limits_{i \le k < j} \; m[i,k] + m[k+1,j] + P_{i-1} \times P_k \times P_j, & i \ne j \\ 0, & i = j \end{cases}$$

$A_1 \quad A_2 \quad A_n$

$P_0, P_1, \cdots P_n$

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function matrixChainMultiplication(p, n):
    dp = matrix of size (n+1)x(n+1)
         initialized to ∞

    for (i: 0 → n)
        dp[i][i] = 0                      } → size 1

    for (L: 2 → n)                        } → size 2 → n
        for (i: 1 → n-L+1):
            j = i+L-1
            for (k: i → j-1):
                mul = dp[i][k] + dp[k+1][j] + p[i-1] * p[k] * p[j]
                dp[i][j] = min(dp[i][j], mul)

    return dp[1][n]
```

→ $m[i][j]$: min scalar multiplication $A_i \cdots A_j$

$i = i \quad j = i+L-1$

$A_1\, A_2\, A_3 \cdots A_n$

$A_1 \cdots\cdots A_n$

$dp[n][n] = 0$

$i: 1 \to 4-3+1$

$1 \to 2$

$j = 3 \quad 4$

$1\times2\times3 + 1\times3\times4 + 1\times4\times3$

return $dp[1][n]$

$1\times2\times3 + 1\times3\times4 + 1\times4\times3$
$= 6 + 12 + 12$
$= 30$

$(A_1\, A_2)$

eg: $((A_1 \quad A_2) \quad A_3) \quad A_4$

$\underset{P_0\ P_1}{1\times2} \quad \underset{P_1\ P_2}{2\times3} \quad \underset{P_2\ P_3}{3\times4} \quad \underset{P_3\ P_4}{4\times3}$

dp	0	1	2	3	4
0	-	-	-	-	-
1	-	0	6	18	30
2	-	-	0	24	48
3	-	-	-	0	36
4	-	-	-	-	0

$L=2$

$dp[1][2] = 0 + 0 + 1\times2\times3 = 6$

$dp[2][3] = 0 + 0 + 2\times3\times4 = 24$

$dp[3][4] = 0 + 0 + 3\times4\times3 = 36$

$L=3$

$dp[1][3]$

$dp[2][4] = \begin{cases} 0 + 36 + 2\times3\times3 = 54 \\ 24 + 0 + 2\times4\times3 = 48 \end{cases}$

$L=4$

$dp[1][4] = \begin{cases} 0 + 48 + 1\times2\times3 = 54 \\ 6 + 36 + 1\times3\times3 = 51 \\ 18 + 0 + 1\times4\times3 = 30 \end{cases}$

$((A_1\, A_2)\, A_3)\, A_4$

$\overset{i}{A_1} \quad A_2 \quad \overset{j}{A_3}$

$\underline{P_0 P_1} \quad \underline{P_1 P_2} \quad \underline{P_2 P_3}$

$(A_1)(A_2\, A_3) \leftarrow k=1$

$(A_1\, A_2)(A_3) \leftarrow k=2$

$A_1 \cdots A_1 \quad A_2 \cdots A_3$

$A_1\, A_2 \quad A_3 \cdots A_3$

$dp[1][1] + dp[2][3] + P_0 \cdot P_1 \cdot P_3$

$0 + 24 + 1\times2\times4$

$= 32$

$dp[1][2] + dp[3][3] + P_0 \cdot P_2 \cdot P_3$

$= 6 + 0 + 1\times3\times4$

$= 18$

$\boxed{T = O(n^3), \quad S = O(n^2)}$

[GATE CS 2016 Set 2]

Let A_1, A_2, A_3 and A_4 be four matrices of dimensions 10×5, 5×20, 20×10 and 10×5, respectively. The minimum number of scalar multiplications required to find the product $A_1A_2A_3A_4$ using the basic matrix multiplication method is 1500.

$dp[1][1] = 0,\quad dp[2][2] = 0,\quad dp[3][3] = 0,\quad dp[4][4] = 0$

$A_1((A_2 A_3) A_4)$

$dp[1][2] = 10\times5\times20 = 1000$

$dp[2][3] = 5\times20\times10 = 1000$

$dp[3][4] = 20\times10\times5 = 1000$

$$(A_1)((A_2\ A_3)\ (A_4))$$

$$\underset{10\times5}{A_1}\quad \underset{5\times20}{A_2}\quad \underset{20\times10}{A_3}\quad \underset{10\times5}{A_4}$$

$$dp[1][3] = \min\begin{cases} 0 + 1000 + 10\times5\times10 = 1000 + 500 = 1500 \\ 1000 + 0 + 10\times20\times10 = 1000 + 2000 = 3000 \end{cases} = 1500$$

$$dp[2][4] = \min\begin{cases} 0 + 1000 + 5\times20\times5 = 1000 + 500 = 1500 \\ 1000 + 0 + 5\times10\times5 = 1000 + 250 = 1250 \end{cases} = 1250$$

$$dp[1][4] = \min\begin{cases} 0 + 1250 + 10\times5\times5 = 1250 + 250 = 1500 \\ 1000 + 1000 + 10\times20\times5 = 2000 + 1000 = 3000 \\ 1500 + 0 + 10\times10\times5 = 1500 + 500 = 2000 \end{cases} = \boxed{1500}$$

[GATE CS 2018]

Assume that multiplying a matrix G_1 of dimension $p \times q$ with another matrix G_2 of dimension $q \times r$ requires pqr scalar multiplications. Computing the product of n matrices $G_1G_2G_3 \ldots G_n$ can be done by parenthesizing in different ways. Define G_iG_{i+1} as an **explicitly computed pair** for a given paranthesization if they are directly multiplied. For example, in the matrix multiplication chain $G_1G_2G_3G_4G_5G_6$ using parenthesization $(G_1(G_2G_3))(G_4(G_5G_6))$, G_2G_3 and G_5G_6 are only explicitly computed pairs.

Consider a matrix multiplication chain $F_1F_2F_3F_4F_5$, where matrices F_1, F_2, F_3, F_4 and F_5 are of dimensions $2 \times 25, 25 \times 3, 3 \times 16, 16 \times 1$ and 1×1000, respectively. In the parenthesization of $F_1F_2F_3F_4F_5$ that minimizes the total number of scalar multiplications, the explicitly computed pairs is/are

A. F_1F_2 and F_3F_4 only
B. F_2F_3 only
C. F_3F_4 only
D. F_1F_2 and F_4F_5 only

$(F_1(F_2(F_3\ F_4)))\ F_5$

$2\times25 \quad 25\times3 \quad 3\times16 \quad 16\times1 \quad 1\times1000$

$(1,2) = 2\times25\times3 = 150$

$(2,3) = 25\times3\times16 = 1200$

$(3,4) = 3\times16\times1 = 48$

$(4,5) = 16\times1\times1000 = 16000$

$$(1,3) = \min\begin{cases} 0+1200+2\times25\times16 = 2000 \\ 150+0+2\times3\times16 = 246 \end{cases}$$

$$(1,4) = \min\begin{cases} 0+123+2\times25\times1 = 173 \\ 150+48+\ldots \\ 246+\ldots \end{cases}$$

$$(2,4) = \min\begin{cases} 0+48+25\times3\times1 = 123 \\ 1200+\ldots \end{cases}$$

$$(2,5) = \min\begin{cases} 0+3048+25\times3\times1000\ldots \\ 1200+16000+25\times16\times100\ldots \\ 123+0+25\times1\times1000 = 25123 \end{cases}$$

$$(3,5) = \min\begin{cases} 0+16000+\ldots \\ 48+0+3\times1\times1000 = 3048 \end{cases}$$

$$(1,5) = \min\begin{cases} 0+25123\ldots \\ \ldots \\ \ldots \\ \ldots \\ 173+0+2\times1\times1000 = \boxed{2173} \end{cases}$$