## Number Theory Lecture 6

Friday, 12 July 2024

8:14 PM

$$\phi(n) = n - \sum_{P|n} (\stackrel{>}{P}) = o(\sqrt{n})$$

$$\phi(n) = n \prod_{P|n} (\stackrel{>}{I} - \stackrel{>}{I})$$

H/w:- https://www.spoj.com/problems/ETF/

Find Euler's totient function of 
$$n$$
 $1 \le n \le 10^6$ 
 $0(T.5n)$ 
 $1 \le T \le 2 \times 10^4$ 
 $2 \times 10^7 \times 10^3 \sim 2 \times 10^7$ 

Nog log  $n$ 
 $1 \le n \le n$ 
 $2 \times 10^9 + 2 \times 10^7$ 

```
N = 1000000
phi = [i for i in range(N+1)]
for i in range(2, N+1):
    if phi[i] == i:
        for j in range(i, N+1, i):
        phi[j] -= phi[j]//i
```

```
Phi= [0,1,7,7,4,8,9,1,4]
```

```
N = 1000000
phi = [i for i in range(N+1)]
for i in range(2, N+1):
    if phi[i] == i:
        for j in range(i, N+1, i):
        phi[j] -= phi[j]//i
t = int(input())
for _ in range(t):
    n = int(input())
    print(phi[n])
```

Barel:

For each integer 
$$n \ge 1$$
,  $\sum_{d \mid n} \phi(d) = n$ 

$$\phi(1) + \beta(2) + \beta(5) + \beta(10) = 10$$

$$1 + 1 + 4 + 4 = 10$$

$$F(n) = \sum_{d|n} \phi(d)$$

$$F(P_i^{k_i}) = \sum_{d \mid P_i^{k_i}} \phi(d)$$

$$= \phi(1) + \phi(R) + \phi(R^{2}) + \phi(P^{3}) + \dots + \phi(P^{k})$$

 $\phi(a\cdot b) = \phi(a) \cdot \phi(b)$ 

if gcd (a, b)=1

$$= 1 + (p(-y)) + (p(-y))$$

```
F(\rho_{i}^{k_{1}}) = \rho_{i}^{k_{1}}
F(\rho_{i}^{k_{1}}) = \rho_{i}^{k_{1}}
F(n) = \rho_{i}^{k_{1}} \cdot \rho_{2}^{k_{2}} \cdot \cdot \cdot \rho_{r}^{k_{2}} = n

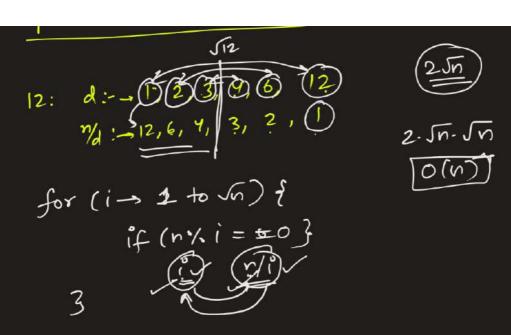
#include<br/>
F(n) = \rho_{i}^{k_{1}} \cdot \rho_{2}^{k_{2}} \cdot \cdot \cdot \rho_{r}^{k_{2}} = n

#include<br/>
```

{1,5,7,113

2 
$$gcd(k,12) = 2 \Rightarrow gcd(\frac{k}{2},6) = 1$$
 $\frac{k}{2} \in U(6) \quad g(6)$ 
 $\frac{k}{2} \in \{1,5\}$ 
 $k \in \{2,10\}$ 

3  $gcd(k,12) = 3 \Rightarrow gcd(\frac{k}{3}, 4) = 1$ 
 $\frac{k}{3} \in U_4 = \{1,3\}$ 
 $\Rightarrow k \in \{3,9\}$ 
2  $gcd(k,12) = d \Rightarrow gcd(\frac{k}{3}, \frac{n}{3}) = 1$ 
 $\sum_{k=1}^{\infty} gcd(n,k) = \sum_{k=1}^{\infty} d\cdot\phi(\frac{n}{3})$ 
 $\sum_{k=1}^{\infty} gcd(n,k) = \sum_{k=1}^{\infty} d\cdot\phi(\frac{n}{3})$ 



Proof:- Hints: (1) 
$$lcm(a,b) = \frac{a \times b}{gcd(a,b)}$$
  
(2)  $gcd(a,n) = gcd(n-a,n) = Encledian$   
(3)  $lcm(a,n) + lcm(n-a,n) = \frac{n^2}{gcd(a,n)}$ 

**Lemma 1:**  $lcm(a, n) + lcm(n - a, n) = \frac{an}{\gcd(a, n)} + \frac{(n - a)n}{\gcd(n - a, n)} = \frac{n \times n}{\gcd(a, n)}$ .

**Lemma 2:**  $\sum \frac{n}{\gcd(a,n)} = \sum_{f \mid n} \frac{n}{f} \times \phi(\frac{n}{f}) = \sum_{d \mid n} d\phi(d)$ ,

Proof: consider what happens if gcd(a, n) = f + n. It appears  $\phi(\frac{n}{f})$  times on the LHS, and each time it has value of  $\frac{n}{f}$ . Now substitute  $d = \frac{n}{f}$ , which is also a divisor of n.

Now, to your problem, pull out lcm(n, n) = n.

We have  $2\sum_{a=1}^{n-1} \operatorname{lcm}(a,n) = \sum_{a=1}^{n-1} \operatorname{lcm}(a,n) + \operatorname{lcm}(n-a,n) = n\sum_{a=1}^{n} \operatorname{lcm}(a,n) = n \times \sum_{a \mid n} d\phi(a)$ .

Add back lcm(n, n) = n, and you get the formula in OEIS.