

Euler's Totient Function

It is also known as phi function, $\phi(n)$. It counts the number of integers between 1 and n (inclusive) which are relatively prime to n .

7, 3 are relatively prime

4, 9 are coprime

a, b are coprime
iff- $\gcd(a, b) = 1$

• 1 is coprime with every positive integer.

• $\phi(1) = 1$ $\{1\}$

• $\phi(2) = 1$ $\{1\}$

• $\phi(3) = 2$ $\{1, 2\}$

• $\gcd(a, a) = 1$ then $a = 1$
($\because \gcd(a, a) = a$)

✓ prime 2, 3, 5, 7, 11, ...

✓ Coprime a, b are coprime iff their $\gcd = 1$

$$\gcd(a, b) = 1$$

eg: $\gcd(4, 9) = 1$

$\swarrow \quad \searrow$
 $2^2 \quad 3^2$

$$\gcd(4, 6) = 2$$

$$\gcd(2, 6) = 2$$

$\checkmark \quad \times \quad \times \quad \times \quad \checkmark \quad \times$
 $\{\underline{1}, 2, 3, 4, \underline{5}, 6\}$

$$U_6 = \{1, 5\}$$

$$\phi(6) = \boxed{2}$$

✓ phi $\phi(n) = \{1, 2, 3, \dots, n\}$

$$\phi(p) = p-1 \quad \{\checkmark \checkmark \checkmark \quad \dots \quad \checkmark \quad \times\}$$

$\{1, 2, 3, \dots, p-1, p\}$

- If p is prime, $\phi(p) = p-1$
eg. $\phi(7) = 6$, $\{1, 2, 3, 4, 5, 6\}$
- $\phi(4) = 2$ $\{1, 3\}$
- $\phi(5) = 4$, $\phi(6) = 2$ $\{1, 5\}$
- $\phi(p^k) = p^k - p^{k-1} = p^{k-1}(p-1) = p^k(1 - 1/p)$, p is prime

• $\phi(n) < n$, if $n > 1$

Proof: $\phi(3^7) = 3^7 - \underline{\underline{3^6}}$

Not relatively prime with 3^7

$\{ \overset{\checkmark}{1}, \overset{\checkmark}{2}, \overset{\checkmark}{3}, \dots, \overset{\times}{p}, \overset{\checkmark}{p+1}, \dots, \overset{\checkmark}{2p}, \overset{\checkmark}{2p+1}, \dots, \dots, \overset{\times}{p^k} \}$

eg. $\phi(3^3)$

$\{ \overset{\checkmark}{1}, \overset{\checkmark}{2}, \overset{\times}{3}, \overset{\checkmark}{4}, \overset{\checkmark}{5}, \overset{\times}{6}, \overset{\checkmark}{7}, \overset{\checkmark}{8}, \overset{\times}{9}, \overset{\checkmark}{10}, \overset{\checkmark}{11}, \overset{\times}{12}, \overset{\checkmark}{13}, \overset{\checkmark}{14}, \overset{\times}{15}, \overset{\checkmark}{16}, \overset{\checkmark}{17}, \overset{\times}{18}, \overset{\checkmark}{19}, \overset{\checkmark}{20}, \overset{\times}{21}, \overset{\checkmark}{22}, \overset{\checkmark}{23}, \overset{\times}{24}, \overset{\checkmark}{25}, \overset{\checkmark}{26}, \overset{\times}{27} \}$

- All the multiples of p are NOT coprime with p^k

$$\begin{aligned}
 & p^k - \text{Multiples of } p \\
 & \quad |p, 2p, 3p, \dots, p^k| \\
 & \quad = \frac{p^k}{p} = p^{k-1}
 \end{aligned}$$

$$\begin{aligned}
 \therefore \phi(p^k) &= p^k - p^{k-1} \\
 &= p^{k-1}(p-1) \\
 &= p^k(1 - 1/p) \quad \square
 \end{aligned}$$

- $\phi(a \cdot b) = \phi(a) \cdot \phi(b)$ if $\gcd(a, b) = 1$ ✓
 > ϕ is multiplicative for coprime numbers.

$$\begin{aligned}
 \text{eg. } \phi(36) &= \phi(4 \cdot 9) = \phi(4) \cdot \phi(9) = \phi(2^2) \cdot \phi(3^2) \\
 &= 2 \times 6 = \underline{\underline{12}}
 \end{aligned}$$

- The prime factorization of n is $p_1^{k_1} p_2^{k_2} \dots p_r^{k_r}$

then $\phi(n) = \phi(p_1^{k_1}) \cdot \phi(p_2^{k_2}) \dots \phi(p_r^{k_r})$

$$= \underline{p_1^{k_1}} \left(1 - \frac{1}{p_1}\right) \cdot \underline{p_2^{k_2}} \left(1 - \frac{1}{p_2}\right) \dots \underline{p_r^{k_r}} \left(1 - \frac{1}{p_r}\right)$$

$$= \boxed{n \cdot \left(1 - \frac{1}{p_1}\right) \left(1 - \frac{1}{p_2}\right) \dots \left(1 - \frac{1}{p_r}\right)}$$

$$\therefore \phi(n) = n \cdot \prod_{\substack{p = \text{prime} \\ \text{factors} \\ \text{of } n}} \left(1 - \frac{1}{p}\right)$$

- $\phi(a \cdot b) = \phi(a) \cdot \phi(b) \cdot \frac{g}{\phi(g)}$, $g = \gcd(a, b)$

↑ For any 2 integers a, b .

Sp case :- if $\gcd(a, b) = 1$. $g = 1$, $\phi(g) = 1$

$$\therefore \phi(a \cdot b) = \phi(a) \cdot \phi(b)$$

Proof :- $\phi(n) = n \cdot \prod_{p|n} \left(1 - \frac{1}{p}\right)$

$$\phi(a \cdot b) = a \cdot b \prod_{p|a \cdot b} \left(1 - \frac{1}{p}\right)$$

$$= \textcircled{a} \textcircled{b} \frac{\prod_{p|a} \left(1 - \frac{1}{p}\right) \prod_{p|b} \left(1 - \frac{1}{p}\right)}{\prod_{p|\gcd(a,b)} \left(1 - \frac{1}{p}\right)} \cdot \frac{\gcd(a,b)}{\gcd(a,b)}$$

$$= \frac{\phi(a) \cdot \phi(b) \cdot \gcd(a,b)}{\phi(\gcd(a,b))}$$

$$= \frac{\phi(a) \cdot \phi(b) \cdot g}{\phi(g)}$$

==x==

eg: $\phi(10 \times 15) = \phi(150)$

$$\phi(150) = 150 \cdot \frac{\prod_{p|10} \left(1 - \frac{1}{p}\right) \prod_{p|15} \left(1 - \frac{1}{p}\right)}{\prod_{p|5} \left(1 - \frac{1}{p}\right)}$$

$\frac{2 \cdot 5}{2 \cdot 5} \checkmark$ $\frac{3 \cdot 5}{3 \cdot 5} \checkmark$

Q: Given an integer n , find $\phi(n)$

```
int phi(n) {  
    ans ← 0  
    for (i: 1 → n) {  
        if gcd(i, n) == 1  
            ans++  
    }  
    return ans  
}
```

$$\phi(n) = n \cdot \prod_{p|n} \left(1 - \frac{1}{p}\right)$$

}

Brute force

$$T = O(\underline{n \log n})$$


```

float ans = n;
for (int i = 2; i * i <= n; i++) {
    if (n % i == 0) {
        while (n % i == 0) {
            n = n / i;
        }
        ans = ans * (1 - 1.0 / i);
    }
}
if (n > 1) {
    ans = ans * (1 - 1.0 / n);
}

```

for (int i = 2; i * i <= n; i++) { $\leftarrow O(\sqrt{n})$

if (n % i == 0) {
 while (n % i == 0) {
 n = n / i;
 }
}

$O(\sqrt{n})$

$n = 2^3 \times 3^2 \times 5^2$
 $\rightarrow 2^2 \times 3^2 \times 5^2$
 $\rightarrow 2 \times 3^2 \times 5^2$
 $\rightarrow 3^2 \times 5^2$
 $\rightarrow 3 \times 5^2$
 $\rightarrow 5^2$
 $\rightarrow 5$
 $\rightarrow 1$

$n = 2^3 \times 3^2 \times 5^2$
 $\rightarrow 3^2 \times 5^2$
 $\rightarrow 5$
 $\rightarrow 5^2$

ans = ans * (1 - 1.0 / i); \rightarrow ans = ans - ans / i;

}

if (n > 1) ans = ans * (1 - 1.0 / n); \rightarrow ans = ans - ans / n;

}

}

$$ans = n \times (1 - \frac{1}{2}) (1 - \frac{1}{3}) (1 - \frac{1}{5})$$

$$ans = n \cdot \prod_{p|n} (1 - \frac{1}{p})$$

```
int phi(int n) {
    int ans = n;
    for(int i=2; i*i<=n; i++) {
        if(n%i == 0) {
            while(n%i == 0)
                n /= i;
            ans = ans - ans/i;
        }
    }
    if(n>1) ans = ans - ans/n;
    return ans;
}
```

$$2 \times 3^2$$

$$\{1, \dots, 36\} \quad ans = 36$$

$$\{1, 3, 5, \dots, 35\} \quad ans = 18$$

$$- \{3, 9, 15, 21, 27, 33\}$$

$$\{1, 5, \dots, 35\} \quad ans = 12$$

$$n = 2^3 \times 5 \times 7^2 \times 11^2$$

$$\begin{aligned} & \rightarrow 5 \times 7^2 \times 11^2 \\ & \rightarrow 7^2 \times 11^2 \\ & \rightarrow 11^2 \\ & \rightarrow 1 \end{aligned}$$

$$i = 2$$

$$\begin{aligned} & 3 \\ & 4 \\ & 5 \\ & 6 \end{aligned}$$

$$\begin{aligned} & 7 \\ & 8 \\ & 9 \\ & 10 \\ & 11 \end{aligned}$$

$$2 \times 5 = 10$$

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$$

$$11, 12, 13, 14, 15, 16, 17, 18, 19, 20\}$$

$$\{1, 3, 7, 9, 11, 13, 17, 19\}$$

$$ans = 10$$

$$\{1, 3, 7, 9, 11, 13, 17, 19\}$$

$$1 \leq B \leq A$$

$\{1, 5, \dots, 35\}$ ans = 12

<https://www.codechef.com/problems/COZIE>

$$\boxed{1 \leq B \leq A}$$

$$\gcd(A, B) * \boxed{\text{lcm}(A, B) = A * B}$$

$$\Rightarrow \underline{\underline{\gcd(A, B) = 1}}$$

$N = 8$

$\{30, 20, \boxed{37}, \boxed{20}, \boxed{19}, \underline{2}, \underline{4}, \underline{10}\}$

Sweetness factor = $\phi(A)$

$\phi(30) \quad \phi(20) \quad \phi(37)$

prime

Q Queries

$\{30, 20, 37, 20, 19, 2, 4, 10\} \leftarrow N$

$N \times \sqrt{A}$

$\{ \phi(30), \phi(20), \phi(37), \phi(20), \phi(19), \phi(2), \phi(4), \phi(10) \}$

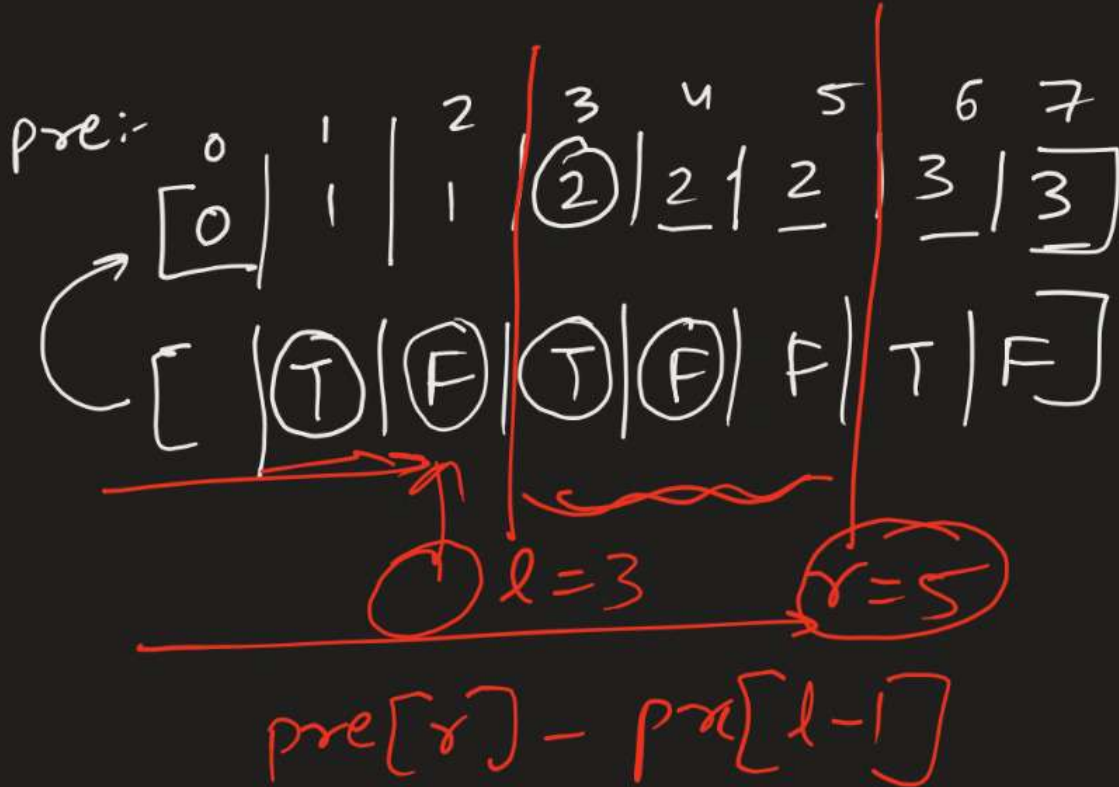
$\{ 8, 8, 36, 8, 18, 1, 2, 4 \} \leftarrow O(N \times \sqrt{A})$

$\{ F, F, F, F, F, F, T, F \} \leftarrow O(N \times \sqrt{A})$

$\{ 0, 0, 0, 0, 0, 0, 1, 1 \}$

Prefix Sum

ϕ Prime or not



```
#include<bits/stdc++.h>
using namespace std;
#define endl '\n'
#define FOR(i,a,b) for(int i=(a); i<(b); i++)
#define FORk(i,a,b,k) for(int i=(a); i<(b); i+=k)
#define RFOR(i,a,b) for(int i=(a); i>=(b); i--)
#define RFORk(i,a,b,k) for(int i=(a); i>=(b); i-=k)
#define pb push_back
typedef vector<int> vi;
typedef vector<string> vs;
typedef long long int ll;
typedef unsigned long long int ull;
typedef vector<ll> vll;
typedef vector<ull> vull;
int phi(int n) {
    int ans = n;
    for(int i=2; i*i<=n; i++) {
        if(n%i == 0) {
            while(n%i == 0)
                n /= i;
            ans = ans - ans/i;
        }
    }
    if(n>1) ans = ans - ans/n;
    return ans;
}
```

```

bool is_prime(int n) {
    if(n==1) return false;
    if(n==2) return true;
    for(int i=2; i*i<=n; i++)
        if(n%i == 0)
            return false;
    return true;
}

void solve() {
    int n, q;
    cin >> n >> q;
    vi a(n+1);
    FOR(i,1,n+1) cin >> a[i];
    vi pre(n+1, 0);
    FOR(i,1,n+1) {
        pre[i] = pre[i-1];
        if(is_prime(phi(a[i])))
            pre[i]++;
    }
    int l,r;
    FOR(i,0,q) {
        cin >> l >> r;
        cout << pre[r] - pre[l-1] << endl;
    }
}

int main() {
    ios_base::sync_with_stdio(false);
    cin.tie(NULL);
    cout.tie(NULL);
    int t = 1;
    // cin >> t;
    while(t--) {
        solve();
    }
    return 0;
}

```

H/w:- Find $\phi(n)$ for all $1 \leq n \leq N$