

Polyhedra

Lecturer - Andrei Jorza

There are a few properties of polyhedra that are worth knowing, and among these the Euler formula is by far the most versatile.

Common notations are f for the number of faces, e for the number of edges and v for the number of vertices. If the polyhedron is a convex one (convexity is not necessary but we will only use Euler's formula for convex polyhedra) then we have

$$f + v = e + 2$$

Take a stereographic projection of the polyhedron with respect to a point right above the center of a certain face \mathcal{F} . The figure obtained is a planar graph. The face \mathcal{F} becomes the outside of the planar graph, so in the case of planar graphs Euler's formula becomes

$$f + v = e + 1$$

Common notations include dv, d^+v, d^-v that mean the same things as for graphs, i.e. the number of edges meeting at a vertex, the number of edges going into a vertex and the number of edges going out of a vertex respectively.

Problems

1. Is there a convex polyhedron with a triangular section so that any vertex has at least 5 edges going into it? (Galperin, Tournament of Towns, 1989)
 2. Prove that for any convex polyhedron there is a vertex v so that $dv \leq 5$.
 3. Prove that for any convex polyhedron there is either a vertex in which enter exactly 3 edges or a triangular face.
 4. In a convex polyhedron with m triangular faces (and possibly some other types of faces too) exactly four edges meet at each vertex. Find the smallest possible value of m . (Belarus 2000)
 5. Find all convex polyhedra so that there are no three faces with the same number of edges. (Andrei Jorza, AMM 2001)
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1. Prove that for any convex polyhedron the perimeter is more than 3 times the diameter. The diameter of the polyhedron is the greatest distance between any two vertices. (Pimsner, Popa)
 2. Prove that for any convex polyhedron there are three edges that are the sides of a triangle. (Romania, 1999)

3. Let $SA_1A_2 \dots A_n$ be a pyramid ($n \geq 5$). Consider points $P_i \in (SA_i)$ for $i = 1, 2, \dots, n$. Let $T_{i,j} = A_iP_j \cap A_jP_i$. Prove that if at least $\binom{n-1}{2} + 1$ of the points $T_{i,j}$ are in the same plane then all of them are. (Iani Mirciov, Romania 1999)

1. What is the smallest number of vertices of a dodecahedron that need to be colored so that any face has at least one vertex colored black? How about for the icosahedron? (Galperin, Tournament of Towns, 1990)

2. Consider a polyhedron with exactly three edges going into any vertex. Prove that one can assign complex numbers to the vertices (one for each) so that the product of the numbers assigned to the vertices of each face is 1. (China, 1991)

3. Let P be a polyhedron. On each edge of P we put an arrow in either direction, so that for each vertex v we have $d^+v, d^-v \geq 1$. Prove that there exists a face of P so that one may trace its entire perimeter in the direction of the arrows.

4. A convex polyhedron has 100 edges. What is the maximum number of edges that can be intersected by a plane? If the polyhedron is considered non-convex, prove that this number can be as high as 96 but cannot be 100. (Andjans, Tournament of Towns, 1992)