$X + Y$ International Mathematical Olympiad

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Solutions

1. *A $2n \times 2n$ board is divided into $4n^2$ small squares in the manner of a chessboard. Each small square is painted with one of four colours so that every 2×2 block of four small squares involves all four colours. Prove that the four corner squares of the board are painted with different colours.*

   Note that the punchline could instead be either "How many such configurations are possible?"" or "Show that there are $3 \times 2^{4n-1}$ such paintings of the square". The arguments are similar to the one below.

   Suppose that there is a 3×1 horizontal (without loss of generality, vertical is similar) run of three different colours. The painting of that row determines the paining of the rest of the square, since the colours of squares adjacent to the central square (of the three) are determined, and then this information propagates to the rest of the board. The manner of the propagation ensures that horizontal rows alternate between two patterns, so in the vertical direction each column will consist of alternating colours. Also, considering a corner four (say the top left corner coloured A, and the other three squares coloured B, C, D in anticlockwise order. The third column must begin AB or BA, the fourth column must begin CD or DC etc. So the final column must begin CD or DC. The first and final columns must alternate between A and B and between C and D respectively, so we are done in this case.

   Finally suppose that there is no consecutive run of three colours (horizontal or vertical) anywhere on the board. Therefore the painting consists n^2 copies of the top left 2×2 square tiling the board. In this case too, we are done.

2. *Which positive integers n have the property that $\{1, 2, \ldots, n\}$ can be partitioned into two subsets A and B so that the sum of the squares of the elements of A is the sum of the squares of the elements of B?*

   Notice that $m^2 - (m-1)^2 = (m-2)^2 - (m-3)^2 - 5$ for each integer m, so

   $$m^2-(m-1)^2-(m-2)^2+(m-3)^2-(m-4)^2+(m-5)^2+(m-6)^2-(m-7)^2 = 0.$$

   Therefore if the result holds for N, then it holds for $N+8$.

   The sum of the first n squares is even if, and only if, n is 0 or 3 modulo 4. This is a necessary condition for the condition to hold. By inspection, the result does not hold when $n = 3, 4$. It holds when $n = 7$ by putting $m = 7$ in the displayed formula above, and discarding the irrelevant 0^2 term. It holds when $n = 8$ by putting $m = 8$ in the displayed formula above.

   When $n = 12$, the result holds because half the sum of the first 12 squares is

   $$325 = 12^2 + 10^2 + 8^2 + 4^2 + 1^2.$$

   The result holds when $n = 11$ because half the sum of the first 11 squares is

   $$253 = 11^2 + 9^2 + 5^2 + 4^2 + 3^2 + 1^2.$$

   Therefore the result holds when n is at least 8 and is divisible by 4, and when n is at least 7 and leaves remainder 3 on division by 4, and does not hold for any other positive integer n.

3. *This problem concerns polynomials in X with real coefficients. Let $f(X) = 2013X + 1$. Suppose that $g(X)$ and $h(X)$ are polynomials such that $f(g(X)) = g(f(X))$ and $f(h(X)) = h(f(X))$. Prove that $g(h(X)) = h(g(X))$.*

   Let $k = 2013$, and $c = -1/2012$ be the unique fixed point of f (i.e. $f(c) = c$).

   First we study g. Now $g(c) = g(f(c)) = f(g(c))$ so $g(c) = c$. Therefore $g = (X-c)q+c$ where q (or $q(X)$) is a real polynomial. Now $f(g(X)) = g(f(X))$ and we can write f as $k(X-c)+c$. Therefore

   $$k(X-c)q + c = k(X-c)q(f(X)) + c$$

and so $q(X) = q(f(X))$. Now $1 < f(1) = 2014$, and inductively $f^i(1) < f^{i+1}(1)$ for every positive integer i. It follows that $q(X) - q(1)$ has infinitely many roots and so is the zero polynomial. Thus $q(X)$ is u, a constant polynomial. Therefore $g = u(X - c) + c$.

Conversely if $g = u(X - c) + c$ for some constant u, then $f(g(X))$ is $uk(X - c) + c$ and this is also $g(f(X))$. The polynomials g which composition-commute with f are precisely the polynomials of degree at most 1 such that $g(c) = c$.

Now suppose that g is $u(X - c) + c$ and $h = v(X - c) + c$ are any two such polynomials, then $g(h(X)) = uv(X - c) + c = h(g(X))$ as required.