Diophantine Equations

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1 Useful Facts

- Everything from Handout 1.
- Sums of squares.
- Sandwiching: e.g. if you want to prove that some expression X cannot be a perfect kth power, show that $n^k < X < n^{k+1}$ for some n. This method generalizes.
- Pythagorean triples.
- Pell's equation/recurrences/infinite descent
- If you're looking to construct a solution, try clever algebraic substitutions.
- Don't be afraid to use the quadratic formula!
- Quadratic Reciprocity
- Look beyond $\mathbb{Z}$: factorizations in $\mathbb{Z}[i]$ and $\mathbb{Z}[\omega]$.

1 (IMO 1982)**.** Prove that if n is a positive integer such that the equation

$$x^3 - 3xy^2 + y^3 = n$$

has a solution in integers x, y, then it has at least three such solutions. Show that the equation has no solutions in integers for $n = 2891$.

2 (Crux)**.** Prove that the product of five consecutive integers is never a perfect square.

3 (IMO 1996)**.** The positive integers a and b are such that the numbers $15a + 16b$ and $16a - 15b$ are both squares of positive integers. What is the least possible value that can be taken on by the smaller of these two squares?

4 (IMO Shortlist 2002)**.** Let P be a cubic polynomial given by $P(x) = ax^3 + bx^2 + cx + d$, where a, b, c, d are integers and $a \neq 0$. Suppose that $xP(x) = yP(y)$ for infinitely many pairs x, y of integers with $x \neq y$. Prove that the equation $P(x) = 0$ has an integer root.

5 (IMO Shortlist 2001)**.** Consider the system

$$x + y = z + u, \quad 2xy = zu.$$

Find the greatest value of the real constant m such that $m \leq x/y$ for any positive integer solution (x, y, z, u) of the system, with $x \geq y$.

6 (IMO Shortlist 2002)**.** Is there an integer n such that the equation $\frac{1}{a}+\frac{1}{b}+\frac{1}{c} = \frac{m}{a+b+c}$ has infinitely many solutions in positive integers a, b, c?

7. Prove that there exists an integer $m \geq 2002$ and m distinct positive integers $a_1, a_2, \ldots, a_m$ such that

$$\prod_{i=1}^{m} a_i^2 - 4\sum_{i=1}^{m} a_i^2$$

is a perfect square.

8 (TST 2001)**.** Find all pairs of non-negative integers m, n such that $(m + n - 5)^2 = 9mn$.

9 (TST 2002)**.** Find in explicit form all ordered pairs of positive integers m, n such that $mn - 1$ divides $m^2 + n^2$.

10. Suppose that x, y are positive integers such that both $x(y + 1)$, $y(x + 1)$ are perfect squares. Show that exactly one of x, y is a perfect square.

11 (IMO Shortlist 2000)**.** Show that for infinitely many n, there exists a triangle with integer sidelengths such that semiperimeter is n times its inradius.

12 (Bulgaria '01)**.** Let p be a prime number congruent to 3 modulo 4, and consider the equation

$$(p + 2)x^2 - (p + 1)y^2 + px + (p + 2)y = 1.$$

Prove that this equation has infinitely many solutions in positive integers, and show that if $(x, y) = (x_0, y_0)$ is a solution of the equation in positive integers, then $p \mid x_0$.

13. Suppose that x, y are positive integers such that both $x(y + 1)$, $y(x + 1)$ are perfect squares. Show that exactly one of x, y is a perfect square.