Contents

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1. Problem 1.2.2 (USAMO 1994/3)

Let $\theta = \angle ACB$, $\alpha = \angle BDC$, $\beta = \angle DFE$, $\gamma = \angle FBA$. Then $\angle EPA = \angle EDB = \angle CPD = 2\theta + \gamma$ and $\angle PAE = \angle DBE = \angle DCP = \beta$, so $\triangle EPA \sim \triangle EDB \sim \triangle DPC$. Therefore

$$\frac{CP/CD}{PE/AE} = \frac{AP/AE}{PE/AE} = \frac{AP}{PE} = \frac{BD}{DE}.$$

Also $\angle ECA = \angle DOC = \angle EDO = \theta + \gamma$ and $\angle AEC = \angle CDO = \angle OED = \theta + \alpha$, so $\triangle ACE \sim \triangle COD \sim \triangle ODE$. (In fact, all six triangles given by O and two adjacent vertices of hexagon $ABCDEF$ are similar to ACE, by analogous angle-chasing.) Finally, $\triangle ACE \cong \triangle BDF$ as $ABCD$, $CDEF$, $EFAB$ are all isosceles trapezoids. Therefore

$$\frac{CP}{PE} = \frac{CD}{AE}\frac{BD}{DE} = \frac{OD}{CE}\frac{AC}{DE} = \frac{AC}{CE}\frac{OD}{DE} = \left(\frac{AC}{CE}\right)^2.$$

2. Problem 1.2.3 (IMO 1990/1)

Let N be the second intersection of the circle through A, B, C, D with the circle through D, E, M. Note $\angle NEG = \angle NDE = \angle NDC = \angle NBC = 180 - \angle NAC = \angle NAG$; therefore N, G, A, and E are concyclic, so $\angle NGE = \angle NAE = \angle NAM$. We also have $\angle NMA = \angle NME = \angle NEG$, so $\triangle NAM \sim \triangle NGE$; therefore

$$\frac{EG}{AM} = \frac{NG}{NA}.$$

As $\angle NBF = \angle NBC = \angle NEG = \pi - \angle NEF$, N, B, E, F are concyclic, so $\angle NFG = \angle NFE = \angle NBE = \angle NBA$; as $\angle NGF = \angle NGE = \angle NAE = \angle NAB$, $\triangle NGF \sim \triangle NAB$, so

$$\frac{GF}{AB} = \frac{NG}{NA}.$$

These two equations give us $EG/GF = AM/AB = 1/t$; simple algebra gives $EG/EF = t/(1-t)$.

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3. Problem 1.3.2

Note that $\angle DMC = \angle MDC + \angle DCM = \angle MDB + \angle BCM = \angle DAB + \angle BAC = \angle DAC$, so points A, C, D, and M are concyclic. Let $P = AM \cap CD$; then $\angle KAB = \angle CAB = \angle MCB = \angle MCP = \angle KPC = \angle KPB$, so points A, K, B, P are concyclic. Now

$$\angle KBD = \angle KBP = \angle KAP = \angle CAM = \angle CDM = \angle BDM = \angle BAD;$$

therefore BK is tangent to the second circle.

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4. Problem 1.3.3

Suppose P_1, P_2, P_3, P_4 lie on a line or circle; then $\angle P_4P_1P_2 = \angle P_4P_3P_2$, so $\angle P_4P_1P_2 + \angle P_2P_3P_4 = 0$. We have

$$\begin{aligned}\angle Q_1Q_2Q_3 &= \angle Q_1Q_2P_2 + \angle P_2Q_2Q_3 = \angle Q_1P_1P_2 + \angle P_2P_3Q_3\\ \angle Q_3Q_4Q_1 &= \angle P_4Q_4Q_1 + \angle Q_3Q_4P_4 = \angle P_4P_1Q_4 + \angle Q_3P_3P_4\end{aligned}$$

so $\angle Q_1Q_2Q_3 + \angle Q_3Q_4Q_1 = \angle P_4P_1P_2 + \angle P_2P_3P_4 = 0$. Therefore Q_1, Q_2, Q_3, Q_4 lie on a line or circle.

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5. Problem 1.4.1 (IMO 1994/2)

First, suppose $OQ \perp EF$. Then $\angle EBO = \angle EQO = \angle FQO = \angle FCO = \pi/2$, so quadrilaterals $BQOE$ and $FQOC$ are cyclic. Therefore $\angle FEO = \angle QEO = \angle QBO = \angle CBO = \angle BCO = \angle QCO = \angle QFO = \angle EFO$, so $OE = OF$; since $OQ \perp EF$, $QE = QF$.

Now suppose $QE = QF$, but OQ is not perpendicular to EF. Construct $E'F'$ through Q perpendicular to OQ with E' on the ray AB and F' on the ray AC; then by the first part $QE' = QF'$. Since $QE = QF$ and $\angle EQE' = \angle FQF'$, $\triangle QEE' \cong \triangle QFF'$. But then $\angle EE'F' = \angle EE'Q = \angle FF'Q = \angle FF'E'$, so $EE' \parallel FF'$, impossible as then $AB \parallel AC$. So $OQ \perp EF$.

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6. Problem 2.1.1

Let $K = [ABC]$. Then $TP/AP = [TBC]/K$, $TQ/BQ = [TCA]/K$, $TR/CR = [TAB]/K$, so

$$\frac{TP}{AP} + \frac{TQ}{BQ} + \frac{TR}{CR} = \frac{[TBC] + [TCA] + [TAB]}{K} = \frac{[ABC]}{K} = 1.$$

7. Problem 2.1.3 (Hungary-Israel, 1997)

Let A_1 be the foot of the perpendicular from A'' to AB, and C_1 the foot of the perpendicular from A'' to BC; then

$$\frac{\sin\angle ABA''}{\sin\angle A''BC} = \frac{A''A_1/BA''}{A''C_1/BA''} = \frac{A''A_1}{A''C_1} = \frac{b\cos A}{b\sqrt{2}\cos(C+\pi/4)} = \frac{\cos A}{\cos C - \sin C}.$$

(We take $A''A_1 > 0$ when A'' and C are on the same side of A_1, otherwise $A''A_1 < 0$; similarly for $A''C_1$.) Similarly

$$\frac{\sin\angle BCA'}{\sin\angle A'CA} = \frac{c\sqrt{2}\cos(B+45)}{c\cos A} = \frac{\cos B - \sin B}{\cos A}.$$

Finally, let C_2 be the foot of the perpendicular from P to AC and B_2 the foot of the perpendicular from P to AB; then

$$\frac{\sin\angle CAP}{\sin\angle PAB} = \frac{PC_2/AP}{PB_2/AP} = \frac{PC_2}{PB_2} = \frac{(a/\sqrt{2})\cos(C+45)}{(a/\sqrt{2})\cos(B+45)} = \frac{\cos C - \sin C}{\cos B - \sin B}.$$

Therefore

$$\frac{\sin\angle ABA''}{\sin\angle A''BC}\frac{\sin\angle BCA'}{\sin\angle A'CA}\frac{\sin\angle CAP}{\sin\angle PAB} = \frac{\cos A(\cos B - \sin B)(\cos C - \sin C)}{(\cos C - \sin C)\cos A(\cos B - \sin B)} = 1,$$

so AP, BA'', CA' concur by Trig Ceva.

8. Problem 2.1.4 (Răzvan Gelca)

From M drop perpendiculars MR, MQ to AB, AC respectively. Then $\triangle FRM \sim \triangle EQM$, as $\angle RFM = \angle AFE = \angle FDE = \angle FEA = \angle MEQ$; therefore

$$\frac{\sin\angle BAM}{\sin\angle MAC} = \frac{RM/MA}{QM/MA} = \frac{RM}{QM} = \frac{FM}{EM}.$$

Therefore

$$\frac{\sin\angle BAM}{\sin\angle MAC}\frac{\sin\angle ACP}{\sin\angle PCB}\frac{\sin\angle CBN}{\sin\angle NBA} = \frac{FM}{ME}\frac{EP}{PD}\frac{DN}{NF},$$

so DM, EN, FP concur if and only if AM, BN, CP do.

9. Problem 2.1.5 (USAMO 1995/3)

Let G be the centroid and H the orthocenter of $\triangle ABC$. Then $\angle OAA_2 = \angle OA_1A = \angle A_1AH$, and $\angle BAO = \pi/2 - C = \angle HAC$, so $\angle BAA_2 = \angle A_1AC$. Similarly $\angle AA_2C = \angle BAA_2$, etc., so

$$\frac{\sin\angle BAA_2}{\sin\angle A_2AC}\frac{\sin\angle ACC_2}{\sin\angle C_2CB}\frac{\sin\angle CBB_2}{\sin\angle B_2BA} = \frac{\sin\angle A_1AC}{\sin\angle BAA_1}\frac{\sin\angle B_1BA}{\sin\angle CBB_1}\frac{\sin\angle C_1CB}{\sin\angle ACC_1} = 1$$

by Trig Ceva, since AA_1, BB_1, CC_1 concur at G. Therefore AA_2, BB_2, CC_2 concur as well. (Their point of concurrence is called the *isogonal conjugate* of G; see section 5.5.)

10. Problem 2.1.6

Let $\alpha = \angle ABZ = \angle XBC$, $\beta = \angle BCX = \angle YCA$, $\gamma = \angle CAY = \angle ZAB$. Drop perpendiculars XP, XQ from X to AB, AC respectively. Then

$$\frac{\sin\angle BAX}{\sin\angle XAC} = \frac{PX/XA}{QX/XA} = \frac{PX}{QX} = \frac{BX\sin(B-\beta)}{CX\sin(C-\gamma)} = \frac{\sin\gamma\sin(B-\beta)}{\sin\beta\sin(C-\gamma)}$$

by the Law of Sines. So

$$\frac{\sin\angle BAX}{\sin\angle XAC}\frac{\sin\angle ACZ}{\sin\angle ZCB}\frac{\sin\angle CBY}{\sin\angle YBA} = \frac{\sin\gamma\sin(B-\beta)}{\sin\beta\sin(C-\gamma)}\frac{\sin\beta\sin(A-\alpha)}{\sin\alpha\sin(B-\beta)}\frac{\sin\alpha\sin(C-\gamma)}{\sin\gamma\sin(A-\alpha)} = 1,$$

and AX, BY, CZ concur by Trig Ceva.

11. Problem 2.2.2

Let $F = CE \cap AD$, $G = AE \cap CD$. Then AG, DB, CF concur (at E), so by Ceva's Theorem

$$\frac{AB}{BC}\frac{CG}{GD}\frac{DF}{FA} = 1.$$

Applying Menelaos to the points P, G, F on the sides of triangle ACD gives

$$\frac{AP}{PC}\frac{CG}{GD}\frac{DF}{FA} = -1.$$

Therefore $AB/BC = -AP/PC$, so $AC/PC = 1 + AP/PC = 1 - AB/BC$, and $PC = AC/(1 - AB/BC)$; therefore P depends only on A, B, and C.

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12. Problem 2.2.3

Apply Menelaos to the triples (A, B, C) and (D, E, F) on the sides of triangle GHI, giving

$$\frac{HA}{AI}\frac{IB}{BG}\frac{GC}{CH} = -1, \qquad \frac{HD}{DI}\frac{IE}{EG}\frac{GF}{FH} = -1.$$

Now $AI = HD$ and $CH = GF$, so $DI = AI - AD = HD - AD = HA$ and similarly $FH = GC$; therefore

$$1 = \left(\frac{HA}{AI}\frac{IB}{BG}\frac{GC}{CH}\right)\left(\frac{HD}{DI}\frac{IE}{EG}\frac{GF}{FH}\right) = \frac{IB}{BG}\frac{IE}{EG}.$$

So $BG \cdot GE = BI \cdot IE$, or $BG(BE - BG) = BI(BE - BI)$. Since $I \neq G$, we must have $BE - BG = BI$, or $BI = GE$.

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13. Problem 2.3.3

The perpendiculars to MN, NL, LM through A, B, C are the lines AL, BM, CN, which are parallel and therefore "concur". Therefore by the observation at the end of this section, the lines through BC, CA, AB perpendicular to L, M, N concur.

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14. Problem 2.4.1 (USAMO 1997/2)

Solution 1: By the observation at the end of this section it suffices to show that the lines through D, E, F perpendicular to BC, CA, AB are concurrent. But these lines are exactly the perpendicular bisectors of BC, CA, AB, which concur at the circumcenter of triangle ABC.

Solution 2: Let P be the intersection of the line through A perpendicular to EF and the line through B perpendicular to FD. Then $PE^2 - PF^2 = AE^2 - AF^2$ and $PF^2 - PD^2 = BF^2 - BD^2$, so $PE^2 - PD^2 = AE^2 - AF^2 + BF^2 - BD^2 = CE^2 - CD^2$ and PC is perpendicular to DE.

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15. Problem 2.4.2 (MOP 1997)

We want to show

$$\frac{\sin\angle TFC}{\sin\angle CFD}\frac{\sin\angle FDE}{\sin\angle EDT}\frac{\sin\angle DTQ}{\sin\angle QTF} = 1.$$

Drop perpendiculars CX, CY from C to FE, FD respectively. Then

$$\frac{\sin\angle TFC}{\sin\angle CFD} = \frac{CX/CF}{CY/CF} = \frac{CX}{CY} = \frac{CE\sin\angle XEC}{CD\sin\angle CDY} = \frac{\sin\angle AEF}{\sin\angle FDB}.$$

Since $EQ \parallel DT$, by the Law of Sines,

$$\frac{\sin\angle FDE}{\sin\angle EDT} = \frac{\sin\angle QDE}{\sin\angle QED} = \frac{QE}{QD} \quad \text{and} \quad \frac{\sin\angle DTQ}{\sin\angle QTF} = \frac{\sin\angle TQE}{\sin\angle QTE} = \frac{TE}{QE}.$$

Now $TE/QD = TF/FD = \sin\angle TDF/\sin\angle DTF = \sin\angle DFB/\sin\angle EFA$, so

$$\frac{\sin\angle TFC}{\sin\angle CFD}\frac{\sin\angle FDE}{\sin\angle EDT}\frac{\sin\angle DTQ}{\sin\angle QTF} = \frac{\sin\angle AEF}{\sin\angle FDB}\frac{QE}{QD}\frac{TE}{QE} = \frac{\sin\angle AEF}{\sin\angle FDB}\frac{\sin\angle DFB}{\sin\angle EFA} = 1$$

and DE, QT, CF concur.

16. Problem 2.4.3 (Stanley Rabinowitz)

We have

$$\frac{\sin \angle FEY}{\sin \angle YED} = \frac{FY}{YD} = \frac{YM}{YN} = \frac{\sin \angle MBY}{\sin \angle YBN} = \frac{\sin \angle ABP}{\sin \angle PBC},$$

so

$$\frac{\sin \angle FEY}{\sin \angle YED}\frac{\sin \angle EDX}{\sin \angle XDY}\frac{\sin \angle DFZ}{\sin \angle ZFE} = \frac{\sin \angle ABP}{\sin \angle PBC}\frac{\sin \angle CAP}{\sin \angle PAB}\frac{\sin \angle BCP}{\sin \angle PCA} = 1$$

and DX, EY, FZ concur.

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17. Problem 3.1.2 (MOP 1997)

First, suppose $MN \parallel BC$. Let ℓ be the bisector of angle BAC. Then as ABC and AMN are isosceles triangles, reflection in ℓ interchanges B and C, M and N. So $P = BN \cap CM$ maps to $CM \cap BN$, which is P again; therefore P must lie on ℓ and $\angle APM = \angle APN$. Conversely, suppose $\angle APM = \angle APN$. Let M' be the reflection of M in ℓ. Then the reflection of C in ℓ is $C' = AM' \cap CM$. But $AB' = AB = AC$, so we must have $B' = C$ and $M' = N$; therefore $AM = AN$ and MN is parallel to BC.

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18. Problem 3.1.4 (MOP 1996)

Let s be the common side length of all the triangles. Let ω_i be the circumcircle of $AB_{i+1}C_{i-1}$, let O_i be the center of ω_i, and let D_i be the second intersection of ω_{i-1} and ω_{i+1}. Let $\alpha = \angle B_2AC_3$, $\beta = \angle B_3AC_1$, $\gamma = \angle B_1AC_2$. Note $\angle AD_3B_3 = \pi - \angle AC_1B_3 = \pi - \angle AB_3C_1 = \angle AD_1C_1 = \angle AD_1B_3 + \angle B_3D_1C_1 = \pi - \angle AD_3B_3 + \angle C_1AB_3 = \pi + \beta - \angle AD_3B_3$, so $\angle AD_3B_3 = (\pi+\beta)/2$. Similarly $\angle AD_1C_1 = (\pi+\beta)/2$, $\angle AD_3C_3 = \angle AD_2B_2 = (\pi+\alpha)/2$, $\angle AD_2B_2 = \angle AD_1C_1 = (\pi+\gamma)/2$. Therefore $\angle B_2D_2C_2 = 2\pi - \angle B_2D_2A - \angle C_2D_2A = 2\pi - (\pi+\alpha)/2 - (\pi+\beta)/2 = (\pi+\gamma)/2$ as $\alpha+\beta+\gamma = \pi$. Consider a rotation around O_1 through $\angle AO_1B_2$. This clearly maps A to B_2, C_3 to A, and ω_1 to itself. Since distances are preserved, B_3 maps to C_2. Let ω be the circumcircle of $B_2D_2C_2$, and let P be the image of D_3. Then P lies on ω_1 as D_3 does, and P lies on ω since $\angle B_2PC_2 = \angle AD_3B_3 = (\pi+\beta)/2 = \angle B_2D_2C_2$. Since $D_3 \neq A$, $P \neq B_2$, so we must have $D_3 = D_2$. Therefore $\angle D_3O_1D_2 = \angle AO_1B_2$, so $D_2D_3 = B_2A = s$. Similarly, $D_1D_2 = D_3D_1 = s$, so triangle $D_1D_2D_3$ is congruent to the original three triangles.

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19. Problem 3.2.2 (USAMO 1992/4)

Let S be the sphere through A, B, C, and P, S' the sphere through A', B', C', and P, and O and O' the centers and r and r' the radii of S and S' respectively. Since S and S' are tangent and intersect at P, they are tangent at P, so O, O', and P are collinear with $O'P/OP = -r'/r$. Consider a homothety around P with ratio $-r'/r$. Then if X' is the image of X, $|O'X'| = |OX|r'/r$, so X lies on S if and only if X' lies on S'; therefore this homothety sends S to S'. So the image of A, which is collinear with A and P, must also lie on S', and must be A'. Similarly B' is the image of B, so $AP/PA' = BP/PB'$. Now A, B, A', B', and P are coplanar, and A, B, A', B' lie on a sphere; therefore $ABA'B'$ is a cyclic quadrilateral. So by the power-of-a-point theorem, $AP \cdot PA' = BP \cdot PB'$. Multiplying this by the equation above gives $AP = BP$, so $AA' = BB'$. Similarly $BB' = CC'$, so $AA' = BB' = CC'$.

Alternatively, we could begin by taking the cross-section through the plane containing A, B, A', B', and P. Then A, B, A', B' are concyclic, and the circle ω through A, B, and P is tangent to the circle ω' through A', B', and P, so if ℓ is their line of tangency, $\angle ABP = \angle(AP,\ell) = \angle(A'P,\ell) = \angle PB'A' = \angle BB'A' = \angle BAA' = \angle BAP$ and $AP = BP$. Similarly $A'P = B'P$, so $AA' = BB' = CC'$.

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20. Problem 3.2.4

Lemma: Suppose we have two noncongruent circles C_1 and C_2 whose external tangents intersect at P. Then there is a unique homothety with positive ratio sending C_1 to C_2, and its center is at P.

Proof. Any homothety with positive ratio sending C_1 to C_2 maps each of the external tangents to itself, so it maps P to itself, that is, the center must be P. Then the ratio is uniquely determined by the ratio of the radii of the two circles.

Now let C_1, C_2, C_3 be our three circles, P_i the intersection of the external tangents of C_i and C_{i+1}, and H_i the homothety with positive ratio mapping C_i to C_{i+1}. Let ℓ be the line through P_1 and P_2. Since H_i is centered at P_i by the Lemma, ℓ is fixed setwise by H_1 and H_2. Note that H_2H_1 is a homothety with positive ratio mapping C_1 to C_3; therefore it coincides with H_3^{-1}. But H_2H_1 leaves ℓ fixed, so H_3 must as well; therefore the center of H_3, P_3, must lie on ℓ. So P_1, P_2, and P_3 are collinear.

21. Problem 4.1.1

As in the proof of Theorem 4.1, triangles EAD and ECB are similar, as are triangles EAC and EDB; so $AD/BC = AE/CE$, $AC/BD = CE/BE$, and $(AC/BC)(AD/BD) = AE/BE$.

22. Problem 4.1.2 (Mathematics Magazine, Dec. 1992)

If P lies on AH, then quadrilaterals $DPHB$ and $EPHC$ are cyclic because of the right angles at D, E, and H, so $AB\cdot AD = AP\cdot AH = AC\cdot AE$, and $|AB\cdot AD-AC\cdot AE| = 0 = BC\cdot PQ$. If not, let $R = PD\cap AH$, $S = PE\cap AH$; then $DRHB$ and $ESHC$ are cyclic, so $|AB\cdot AD-AC\cdot AE| = |AR\cdot AH-AS\cdot AH| = RS\cdot AH$; since $\angle PRS = \angle DRA = \angle ABH = \angle ABC$, triangles ABC and PRS are similar, so $PQ/AH = RS/BC$ and $RS \cdot AH = BC \cdot PQ$.

23. Problem 4.1.3

Let M be the intersection of AE with OB. Then $\angle EOM = \angle COB = \angle OCA = \angle ECA = \angle OAE = \angle OAM$, so MO is tangent to the circle through O, E, and A; therefore $MO^2 = ME \cdot MA = MB^2$ and M is the midpoint of OB.

24. Problem 4.1.4 (MOP 1995)

We will use directed distances. Let O be the center of the given circle, r its radius, and H and J the feet of the perpendiculars to BC from A and O respectively. Then by power-of-a-point, $BP \cdot BA = BO^2 - r^2$, so $AP\cdot AB = AB^2 - PB\cdot AB = AB^2 - BO^2 + r^2$. Similarly $AR\cdot AD = AD^2 - DO^2 + r^2$, so $AP\cdot AB - AR\cdot AD = (AB^2 - BO^2 + r^2) - (AD^2 - DO^2 + r^2) = AH^2 + BH^2 - BJ^2 - OJ^2 - AH^2 - DH^2 + DJ^2 + OJ^2 = (BH - BJ)(BH + BJ) - (DH - DJ)(DH + DJ) = HJ \cdot (BH + BJ - DH - DJ) = 2HJ \cdot BD$. By a similar calculation $AQ \cdot AC - AS \cdot AE = 2HJ \cdot CE$, so

$$\frac{AP \cdot AB - AR \cdot AD}{AS \cdot AE - AQ \cdot AC} = \frac{2HJ \cdot BD}{2HJ \cdot EC} = \frac{BD}{EC}.$$

25. Problem 4.1.5 (IMO 1995/1)

The result is trivial if P coincides with X or Y, so suppose not. By power-of-a-point, $PB \cdot PN = PX \cdot PY = PC \cdot PM$, so quadrilateral $BCMN$ is cyclic. Then (using directed angles) $\angle MAD = \angle MAC = \pi/2 + \angle MCA = \pi/2 + \angle MCB = \pi/2 + \angle MNB = \angle MND$, so quadrilateral $ADMN$ is cyclic as well. Let $Q = AM \cap ND$, and let Y_1 and Y_2 be the intersections of QX with the circles on AC and BD respectively. Then $QX \cdot QY_1 = QA \cdot QM = QN \cdot QD = QX \cdot QY_2$, so $Y_1 = Y_2 = Y$ and Q lies on the line XY.

Alternatively, one could begin by letting $Q = AM \cap XY$. Then $QX \cdot QY = QA \cdot QM = QP \cdot QZ$ since triangles QMP and QZA are similar. This implies that Q lies on the radical axis of the circle on BD and the circumcircle of $PZDN$, namely the line ND. So AM, XY, DN concur at Q.

26. Problem 4.2.2 (MOP 1995)

Let AA' be the altitude from A, let N be the midpoint of AM, let ω_1 be the circle through B, C, B', and C', and let ω_2 be the circle through A, A', and M. Then A, B, A', B' are concyclic, so $HA\cdot HA' = HB\cdot HB'$; therefore H lies on the radical axis of ω_1 and ω_2. Also A', B', C', and M lie on the nine-point circle of triangle ABC, so $DB \cdot DC = DB' \cdot DC' = DA' \cdot DM$; therefore D also lies on the radical axis of ω_1 and ω_2. So DH is perpendicular to line NM, which is the same as line AM.

27. Problem 4.2.3 (IMO 1994 proposal)

Let X and Y be the points where circle ω is tangent to lines ℓ_1 and ℓ_2 respectively. It is easy to check that A, C, and Y are collinear, and similarly B, D, X and A, E, B are collinear. Now $\angle CYB = \angle AYB = \angle XAY = \angle XAC = \angle AEC$, so $BECY$ is cyclic. Therefore $AC \cdot AY = AE \cdot AB$, so A lies on the radical axis of ω and ω_2. In particular, since D is their point of tangency, AD is tangent to ω and ω_2. Similarly, BC is the radical axis of ω and ω_1 and is therefore tangent to these two circles. Therefore $Q = AD \cap BC$ is the radical center of ω, ω_1, and ω_2, so QC, QD, QE are tangents and $QC = QD = QE$.

28. Problem 4.2.4 (India, 1996)

Let M be the second intersection of the circumcircle of PDG with AB and N the second intersection of the circumcircle of PFE with AC. Then $\angle MBC = \angle MDG = \angle MPG = \angle MPC$, so M, P, B, C are concyclic. Similarly, N, P, B, C are concyclic, so all of these points lie on one circle; in particular $\angle MDE = \angle MBC = \angle MNC = \angle MNE$, so quadrilateral $MNDE$ is cyclic. Since $A = AB \cap AC = MD \cap NE$, A is the radical center of $MNDE$, $MPDG$, and $NPFE$, so A lies on the radical axis of PDG and PFE.

29. Problem 4.2.5 (IMO 1985/5)

By the radical axis theorem, AC, KN, and MB concur, at D, say. Then $\angle DMK = \angle BMK = \angle BNK = \angle CNK = \angle CAK = \angle DAK$, so D, M, A, K are concyclic. Next, let E be the second intersection of the line AM with the circle centered at O; then $\angle MEN = \angle AEN = \angle AKN = \angle AKD = \angle AMD = \angle AME$, so lines MD and EN are parallel; it therefore suffices to show $OM \perp EN$. But we also have $\angle MNE = \angle BMN = \angle BKN = \angle AKN = \angle AEN = \angle MEN$; therefore $ME = MN$, and $OE = ON$, so OM and EN are perpendicular.

30. Problem 4.3.1

The statement is: Let ACE be a triangle, and B, D, F the points where its inscribed circle touches sides AC, CE, EA, respectively. Then lines AD, BE, CF are concurrent.

31. Problem 4.3.2

Let $X = AC \cap BD$. Applying Brianchon's theorem to the degenerate hexagon $AMBCPD$, we see that lines AC, BD and MP concur, so line MP passes through point X. Similarly, applying Brianchon's theorem to $ABNCDQ$, lines AC, BD and NQ concur, so line NQ also passes through X. Hence lines AC, BD, MP, NQ concur at X.

32. Problem 4.3.3

Let $X = AC \cap BD$ as in the previous solution and let $Y = ME \cap NF$. By Pascal's theorem applied to hexagon $MEQNFP$, points $ME \cap NF = Y$, $EQ \cap FP = B$, $QN \cap PM = X$ are collinear; since X lies on BD, so does Y.

33. Problem 4.3.4

Let $P = AE \cap BC$; then $CDEP$ is cyclic as $\angle PED = \pi/2 = \angle PCD$. Let γ be the circumcircle of $CDEP$, and let Q and R be the second intersections of DA and DB, respectively, with γ. Let $G = CQ \cap ER$; then A, G, and B are collinear by Pascal's theorem applied to hexagon $PCQDRE$. By the Law of Sines,

$$\frac{AG}{BG} = \frac{QG}{RG}\frac{\sin \angle DQC}{\sin ERD}\frac{\sin \angle RBG}{\sin \angle GAQ} = \frac{\sin \angle QRG}{\sin \angle GQR}\frac{CD}{DE}\frac{\sin \angle DBA}{\sin \angle BAD} = \frac{\sin \angle ADE}{\sin \angle CDB}\frac{AD}{BD} = \frac{AE}{BC} = \frac{AF}{BF},$$

so in fact $G = F$. Thus $\angle FCE = \angle QCE = \angle ADE$ and $\angle FEC = \angle REC = \angle BDC$.

Alternatively, define P, γ, and Q as before, and let $G = AB \cap CH$. Then $\angle AHG = \angle DHC = \angle EHD = \angle EHA$ and $\angle BCG = \angle PCH = \angle PEH = \angle AEH$ so by the Law of Sines

$$\frac{AG}{BG} = \frac{AG \sin \angle AGH}{BG \sin \angle BGC} = \frac{AH \sin \angle AHG}{BC \sin \angle BCG} = \frac{AH \sin \angle EHA}{BC \sin \angle AEH} = \frac{AE}{BC} = \frac{AF}{BF}.$$

Hence $G = F$, so $\angle FCE = \angle GCE = \angle HCE = \angle HDE = \angle ADE$. Similarly, $\angle FEC = \angle BDC$.