Counting in two ways
(Blue)

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1. Numbers $1, 2, \ldots, 2007$ are written (not necessarily in this order) along a circle. We consider all the groups of three neighboring numbers. There are 600 such groups consisting of 3 odd numbers, 500 such groups consisting of 2 odd numbers. How many such groups consisting of all even numbers?

2. A 15×15 square is tiled with unit squares. Each vertex is colored either red or blue. There are 133 red points. Two of those red points are corners of the original square, and another 32 red points are on the sides. The sides of the unit squares are colored according to the following rule: If both endpoints are red, then it is colored red; if the points are both blue, then it is colored blue; if one point is red and the other is blue, then it is colored yellow. Suppose that there are 196 yellow sides. How many blue segments are there?

3. [AHSME 1989] Suppose that 7 boys and 13 girls line up in a row. Let S be the number of places in the row where a boy and a girl are standing next to each other. For example, for the row GBBGGGBGBGGGBGBGGBGG we have $S = 12$. Find the average value of S (if all possible orders of these 20 people are considered).

4. Each square of a 1998×2002 chess board contains either 0 or 1 such that the total number of squares containing 1 is odd in each row and each column. Prove that the number of white unit squares containing 1 is even.

5. [AHSME 1999] Let $x_1, x_2, \ldots, x_n$ be a sequence of integers in $\{-1, 0, 1, 2\}$ such that $\sum_{i=1}^n x_i = 19$ and $\sum_{i=1}^n x_i^2 = 99$. Find the smallest and largest possible values of $\sum_{i=1}^n x_i^3$.

6. Compute the sum of the greatest odd divisor of each of the numbers $2006, 2007, \ldots, 4012$.

7. [AMC12B'04] Given that 2^{2004} is a 604-digit number with leading digit 1, determine the number of elements in the set $\{2^0, 2^1, 2^2, \ldots, 2^{2003}\}$ that have leading digit 4.

8. [IMC 2002] Two hundred students participated in a mathematical contest. They had six problems to solve. It is known that each problem was correctly solved by at least 120 participants. Prove that there must be two participants such that each problem was solved by at least one of the two students.

9. [From a paper of mine] Prove for nonnegative integers i, j that $\sum_{k,m,n\geq 0} \binom{m}{i}\binom{m}{k}\binom{n}{k}\binom{n}{j} 2^{-2(m+n)} = \binom{i+j}{i} 2^{1-(i+j)}$.

10. [From a paper of mine] For nonnegative integers i, j find $\sum_{k,\ell,m,n,p\geq 0} \binom{m}{i}\binom{m}{k}\binom{n}{k}\binom{n}{\ell}\binom{p}{\ell}\binom{p}{j} 3^{-2(m+n+p)}$.

11. [Classic CS] A *vertex cover* of a graph is a set of vertices such that every edge in the graph contains at least one of the chosen vertices. An optimal vertex cover is one which contains the fewest vertices, though this is believed to be very hard to find. How can you efficiently find a *2-approximation* to the smallest vertex cover? That is, show how to find a vertex cover whose size is at most twice the size of the smallest vertex cover.

12. [Statistics] The *Poisson* distribution is a probability distribution over the non-negative integers, indexed by a parameter $\lambda \geq 0$ such that $\Pr[Poi(\lambda) = i] = \frac{e^{-\lambda}\lambda^i}{i!}$. Suppose we draw a non-negative integer i from $Poi(100)$ and then flip i fair coins. What is the distribution of the number of heads?

13. If the edges of a complete graph K_6 (six vertices and 15 edges connecting each pair of vertices) are colored in 2 colors, the graph contains two monochromatic triangles.

14. If the edges of a complete graph K_{12} (12 vertices and 66 edges connecting each pair of vertices) are colored in 2 colors, the graph contains a set A of 5 vertices and a disjoint set B of 2 vertices such that for each vertex of A, both edges to B have the same color.

15. [Iran'96] Sets $A_1, \ldots, A_{35}$ are given with the property that $|A_i| = 27$ for all i, and such that the intersection of any three of them has exactly one element. Show that there is an element that belongs to all the sets.

16. [IMO'98] In a competition, there are a contestants and b judges, where $b \geq 3$ is an odd integer. Each judge rates each contestant as either pass or fail. Suppose k is a number such that, for any two judges, their ratings coincide for at most k contestants. Prove that

$$\frac{k}{a} \geq \frac{b-1}{2b}.$$

17. [IMO'01] Twenty-one girls and twenty-one boys took part in a mathematical competition. It turned out that

    (a) each contestant solved at most six problems, and

    (b) for each pair of a girl and a boy, there was at least one problem that was solved by both the girl and the boy.

    Prove that there is a problem that was solved by at least three girls and at least three boys.

18. [IMO'05] In a mathematical competition 6 problems were posed to the contestants. Each pair of problems was solved by more than 2/5 of the contestants. Nobody solved all 6 problems. Show that there are at least 2 contestants who each solved exactly 5 problems each.