2008 BLUE MOP, POLYNOMIALS-II

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The following problems are collected from Dusan Djukic's article named Polynomial Equations.

(1) Determine the polynomials P for which $16P(x^2) = P(2x)^2$.

(2) If $P(x)^2$ is a polynomial in x^2, then show that so is either $P(x)$ or $P(x)/x$.

(3) Find all polynomials P such that

$$P(x)^2 + P\left(\frac{1}{x}\right)^2 = P(x^2)P\left(\frac{1}{x^2}\right).$$

(4) Are there non-linear polynomials P and Q such that

$$P(Q(x)) = (x-1)(x-2)...(x-15)?$$

(5) Determine all polynomials P for which

$$P(x)^2 - 2 = 2P(2x^2 - 1).$$

(6) Find all polynomials satisfying

$$P(x)^2 - 1 = 4P(x^2 - 4x + 1).$$

(7) Find all polynomials P satisfying

$$P(x^2 + 1) = P(x)^2 + 1.$$

(8) (IMO-04) Find all polynomials $P(x)$ with real coefficients that satisfy

$$P(a-b) + P(b-c) + P(c-a) = 2P(a+b+c)$$

for all triples a, b, c of real numbers such that $ab + bc + ca = 0$.

Problem 1, Solution by Brian Hamrick: We claim that P is a monomial. If not, then let's write $P(x) = c_a x^a + c_b x^b + ...$ where the two written terms are the ones with smallest powers. Now, let's look at both sides of $16P(x^2) = P(2x)^2$. In the expansion of $16P(x^2)$, the two lowest degree terms will have x^{2a} and x^{2b}, while in the expansion of $P(2x)^2$, the two lowest degree terms will have x^{2a} and x^{a+b}, a contradiction since $a \neq b$. We conclude that P has to be a monomial, as claimed. Then it is easy to see that $P(x)$ is of the form $16\left(\frac{x}{4}\right)^n$ for some non-negative integer n $\square$

Problem 2, Solution by John Berman: Suppose $P(x)^2$ is a polynomial in x^2 and not all the powers of x in $P(x)$ have the same parity. Let a and b be the smallest odd and even powers respectively appearing in $P(x)$ with non-zero coefficients c_a and c_b. Then $a+b$ is odd and the coefficient of x^{a+b} in $P(x)^2$ is $2c_a c_b \neq 0$, contradiction the fact that $P(x)^2$ is a polynomial in x^2. We conclude that all the powers of x in $P(x)$ have the same parity $\square$

Problem 3, Solution by Justin Brereton: Let $P(x) = a_n x^n + ... + a_1 x + a_0$, where $a_n \neq 0$. Comparing the coefficient os x^{2n} on both sides of $P(x)^2 + P\left(\frac{1}{x}\right)^2 = P(x^2)P\left(\frac{1}{x^2}\right)$, we see that $a_n^2 = a_0 a_n$, hence $a_0 = a_n$ is also non-zero. Now, we will prove by induction, on k, that 2^k divides the degree of each term of $P(x)$ for all non-negative integers k. Base case, $k = 0$ is clearly true. Assume the result for k and for $k+1$ case, suppose on the contrary that there exists a term with degree an odd number times 2^k. Let $a_j x^j$ be such a term with smallest power. Then the expansion of the LHS of the polynomial equation contains $2a_j a_0 x^j$ term but the expansion of the RHS contains only terms with powers that are multiples of 2^{k+1}, contradiction. It follows that P has to be a constant polynomial. Then $P(x) = c$ implies $c^2 + c^2 = c^2$ so $P = 0$ is the only solution $\square$

Problem 4, Solution by Nicholas Triantatillou: If the answer is yes, then $\{degP, degQ\} = \{3, 5\}$. Suppose $degP = 5$ and $degQ = 3$. If $a_1, a_2, ..., a_5$ are the roots of $P(x)$, then we have $(Q(x)-a_1)...(Q(x)-a_5) = (x-1)(x-2)...(x-15)$. So $Q(x)$ can be written as $(x-b_1)(x-b_2)(x-b_3)+d$ in 5 different ways where the 5 triples (b_1, b_2, b_3) cover the integers from 1 up to 15. We note that in each of the 5 ways, the $b_1 + b_2 + b_3$ values are the same, as well as the $b_1 b_2 + b_2 b_3 + b_3 b_1$ values. So the $b_1^2 + b_2^2 + b_3^2$ values are the same as well. But these values modulo 4 show the number of odd terms, hence the number of odd b_i's are same in all the 5 triples, which is a contradiction since we have 8 odd numbers from 1 up to 15, which is not a multiple of 5. In the other case, $degP = 3$ and $degQ = 5$, $Q(x)$ can be written as $(x-b_1)(x-b_2)...(x-b_5)+d$ in 3 different ways and as above we find that the $b_1^2 + ... + b_5^2$ values in each of these 3 ways are the same. Then this value has to be one third of $1^2 + 2^2 + ... + 15^2 = 1240$, which is not divisible by 3. From the contradictions in both cases, conclude that the answer is no $\square$

Problem 5, Solution by Joshua Pfeffer: Let $P(1) = l$. Then $l^2 - 2l - 2 = 0$. Suppose that P is not the constant polynomial. Then, by definition of l we can write $P(x) - l = (x-1)^n R(x)$ where $n \in \mathbb{Z}^+$ and R is a polynomial such that $R(1) \neq 0$. Then the given equation in terms of R becomes:

$$(x-1)^n R(x)^2 + 2lR(x) = 2^{n+1}(x+1)^n R(2x^2-1).$$

We deduce that $R(1) = 0$, a contradiction. Hence we conclude that the only polynomial solutions are the constant solutions $P(x) = l$, where l is a root of the quadratic equation $l^2 - 2l - 2 = 0$ $\square$

Problem 6, Solution by Taylor: Note that the constant polynomials $P = 2 \pm \sqrt{5}$ work. Now assume that $n = degP > 0$. Comparing first coefficients of x^{2n} terms and then x^{n+k} terms on both sides of the polynomial equation, we deduce that the coefficients of P are all rational numbers. However, we find that $P(\alpha) = 2 \pm \sqrt{5}$ where $\alpha = \frac{5+\sqrt{21}}{2}$ is a root of the equation $\alpha = \alpha^2 - 4\alpha + 1$. This is not possible because all the powers of α are rational linear combinations of 1 and $\sqrt{21}$, so is $P(\alpha)$ since the coefficients of P are rational but $2 \pm \sqrt{5}$ is not in this form. We conclude that the two constant polynomials we had earlier are the only solutions $\square$

Problem 7, Solution by Minseon Shin: Observe that $P(x)^2$ is a polynomial of x^2. So in the expansion of $P(x)^2$, the odd powers of x disappear. We find that all powers of x must have the same parity, hence $P(x)$ is a polynomial of x^2, or x times a polynomial of x^2. In the latter case, we can write $P(x) = xQ(x^2+1) \;\Rightarrow\; yQ(y^2+1) = (y-1)Q(y)^2 + 1$. Letting $y = 1$ gives, $Q(2) = 1$. Then $y = 2$ gives $Q(5) = 1$. In general, $Q(k) = 1$ implies that $Q(k^2+1) = 1$. So the polynomial $Q(x) - 1$ has infinitely many zeros, hence it is the zero polynomial and $Q(x) \equiv 1$ which gives $P(x) \equiv x$. In the former case, $P(x) = Q(x^2+1)$ and letting $y = x^2+1$ we get $Q(y^2+1) = Q(y)^2 + 1$ which is the same polynomial equation that P satisfies with a polynomial with smaller degree now since $deg(Q) = \frac{deg(P)}{2}$. For P with $deg(P) > 0$, this process will then end when $Q(x) = x$ at some point. Letting $P_1(x) = x$ and $P_{n+1}(x) = P_n(x^2+1)$, we see that P has to be one of the polynomials P_n or one of the two constant polynomials $P_0 = \frac{1\pm\sqrt{5}}{2}$ $\square$

Problem 8, Solution by Gye Hyun Baek: Let $P(x) = r_n x^n + ... + r_1 x + r_0$, where $n = degP$. Also let (a, b, c) be such that $ab + bc + ca = 0$. Then note that (ta, tb, tc) has the same property for all t so $Q(t) = \sum_{i=0}^{n} r_i((a-b)^i + (b-c)^i + (c-a)^i - 2(a+b+c)^i)t^i \equiv 0$. Hence, whenever $r_i \neq 0$, we must have $(a-b)^i + (b-c)^i + (c-a)^i - 2(a+b+c)^i = 0$. Using the triple $(a, b, c) = (2, 2, -1)$, we find that i is even. Considering the triple $(6, 3, -2)$ we get $3^i + 5^i + 8^i = 2 \times 7^i$. For $i \geq 6$ this won't be true because $8^i > 2 \times 7^i$ for $i \geq 6$. $i = 0$ doesn't satisfy the equation either. On the other hand, check that $i = 2$ and $i = 4$ satisfies the relation $(a-b)^i + (b-c)^i + (c-a)^i = 2(a+b+c)^i$ whenever $ab + bc + ca = 0$ so all the solutions are of the form $P(x) = r_4 x^4 + r_2 x^2$ $\square$