Complex Numbers in Geometry

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All geometry is algebra.

-Gabriel Dospinescu

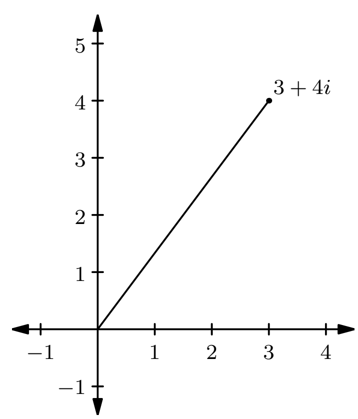
Although geometry problems are solved most elegantly using synthetic or projective means, sometimes, usually in desperation, analytic means can be consulted for a nonelegant, brute-forced solution. However, this does not mean that an analytic geometry solution must be purely analytic; synthetic and projective techniques can be incorporated in combination with algebra for a nicer finish.

1 Algebraic Properties of Complex Numbers

Definition 1.1. Define a *complex number* as a number that can be expressed in the form $a+bi$, where a and b are real numbers and $i = \sqrt{-1}$. a and b are called the *real part* and the *imaginary part* of z, respectively, and are denoted by $\Re\{z\}$ and $\Im\{z\}$. The set of complex numbers is denoted by $\mathbb{C}$.

Definition 1.2. The *complex conjugate* of a complex number $z = a + bi$ is denoted by $\overline{z}$, and is defined as $\overline{z} = a - bi$.

Definition 1.3. The *absolute value* (or *modulus*) of a complex number $z = a + bi$ is $|z| = \sqrt{a^2 + b^2}$.



Notice that complex numbers and Cartesian coordinates are similar in that the real and imaginary parts of a complex number can be thought of its "coordinates" in the two-dimensional Cartesian plane. Thus, we use the *complex plane* as the geometric interpretation of complex numbers, using the real and imaginary axes as the counterparts of the x- and y-axes of the Cartesian plane.

Note that complex conjugation is analogous to reflection over the real axis, and the modulus is the distance between a point and the origin.

It turns out that complex conjugation plays a huge role in the use of complex numbers in geometry, and as such a very important fact is required:

Lemma 1.1. *Let z and w be complex numbers. Then $\overline{z+w} = \overline{z} + \overline{w}$ and $\overline{zw} = \overline{z}\overline{w}$.*

This is very simple to prove by expanding z and w. In higher mathematics, this implies that the function mapping a complex number to its complex conjugate is a field automorphism, as it is bijective and takes identities to themselves.

From the definition of complex conjugation, we deduce that $z\overline{z} = |z|^2$. It also turns out that dealing with the unit circle is also very helpful in analytic geometry. Hence, the following lemma is often useful, which follows from the above observation:

Lemma 1.2. *Let z be a complex number. Then $\overline{z} = \frac{1}{z}$ iff $|z| = 1$.*

2 Geometric Properties of Complex Numbers

As with most brute-force algebraic techniques in solving geometry problems, the use of complex numbers relies on various lemmas. The lemmas will not be proven in this lecture, but it is encouraged that the reader prove these lemmas themselves.

2.1 Triangle Centers

If $\triangle ABC$ lies on the unit circle:

- If H is the orthocenter of $\triangle ABC$, then $h = a + b + c$.
- If O is the circumcenter of $\triangle ABC$, then $o = 0$.
- If I is the incenter of $\triangle ABC$, then $i = \pm\sqrt{ab} \pm \sqrt{bc} \pm \sqrt{ca}$.
- If N is the nine-point center of $\triangle ABC$, then $n = \frac{a+b+c}{2}$.

2.2 Point Configurations

- $|AB| = |a - b|$
- $AB \parallel CD \iff \frac{a-b}{c-d} \in \mathbb{R} \iff \frac{a-b}{\overline{a}-\overline{b}} = \frac{c-d}{\overline{c}-\overline{d}}$.
- A, B, C are collinear $\iff \frac{a-b}{b-c} \in \mathbb{R} \iff \frac{a-b}{\overline{a}-\overline{b}} = \frac{b-c}{\overline{b}-\overline{c}}$.
- $AB \perp CD \iff \frac{a-b}{c-d} \in i\mathbb{R} \iff \frac{a-b}{\overline{a}-\overline{b}} = -\frac{c-d}{\overline{c}-\overline{d}}$.
- A, B, C, D are concyclic $\iff \frac{(a-b)(c-d)}{(a-d)(b-c)} \in \mathbb{R} \iff \sum_{cyc} a\overline{a}b\overline{c} = 0$.
- X is on the angle bisector of $\angle BAC \iff \frac{(x-a)^2}{(b-a)(c-a)} \in \mathbb{R}$.
- If G is the centroid of $\triangle ABC$, then $g = \frac{a+b+c}{3}$.
- The projection P of an arbitrary point C onto chord AB of the unit circle is $p = \frac{1}{2}(a + b + c - ab\overline{c})$.
- The tangents of the unit circle at A and B intersect at $\frac{2ab}{a+b}$.
- $\triangle ABC \sim \triangle PQR \iff \frac{a-b}{a-c} = \frac{p-q}{p-r}$.
- If b is obtained by a rotation of point a centered at c of angle θ, then $b - c = e^{i\theta}(a - c)$.

2.3 Concyclicities

If $ABCD$ lies on the unit circle:

- If X is the intersection of chords AB and CD, then $x = \frac{ab(c+d)-cd(a+b)}{ab-cd}$.
- If $AD \parallel BC$, and Y is the intersection of chords AB and CD, then $y = \frac{ac-bd}{a+c-b-d}$.

2.4 Miscellaneous

If $\triangle XYZ$ has a vertex Z at the origin,

- $h = \frac{(\overline{x}y+x\overline{y})(x-y)}{x\overline{y}-\overline{x}y}$.
- $o = \frac{xy(\overline{x}-\overline{y})}{\overline{x}y-x\overline{y}}$.

Suppose that the incircle of $\triangle ABC$ is the unit circle, and it touches sides BC, CA, and AB at P, Q, and R, respectively. Then

- $a = \frac{2qr}{q+r}$, $b = \frac{2rp}{r+p}$, $c = \frac{2pq}{p+q}$.
- $h = \frac{2(p^2q^2+q^2r^2+r^2p^2+pqr(p+q+r))}{(p+q)(q+r)(r+p)}$.
- $o = \frac{2pqr(p+q+r)}{(p+q)(q+r)(r+p)}$.

If $\triangle ABC$ is inscribed in the unit circle,

- there exist $u, v, w \in \mathbb{C}$ such that $a = u^2$, $b = v^2$, and $c = w^2$, and $-uv$, $-vw$, and $-wu$ are the midpoints of minor arcs $\overset{\frown}{AB}$, $\overset{\frown}{BC}$, and $\overset{\frown}{CA}$.
- $i = -uv - vw - wu$.

The area of $\triangle ABC$ is

$$A = \frac{i}{4} \begin{vmatrix} a & \overline{a} & 1 \\ b & \overline{b} & 1 \\ c & \overline{c} & 1 \end{vmatrix}.$$

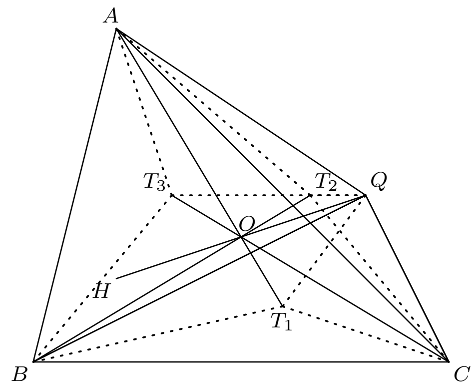
3 Examples

Notice that many of these lemmas require one or more points to be on the unit circle. Thus, it is important while solving a geometry problem to choose the orientation of the axes and the location of the unit circle on the complex plane. This motivates us to choose the unit circle so that the complex conjugate of a number will be much easier to evaluate.

Example 3.1 (Yugoslavia 1990)**.** Let O be the circumcenter and H the orthocenter of $\triangle ABC$. Let Q be the point such that O bisects HQ and denote by T_1, T_2, and T_3, respectively, the centroids of $\triangle BCQ$, $\triangle CAQ$, and $\triangle ABQ$. Prove that

$$AT_1 = BT_2 = CT_3 = \frac{4}{3}R,$$

where R denotes the circumradius of $\triangle ABC$.



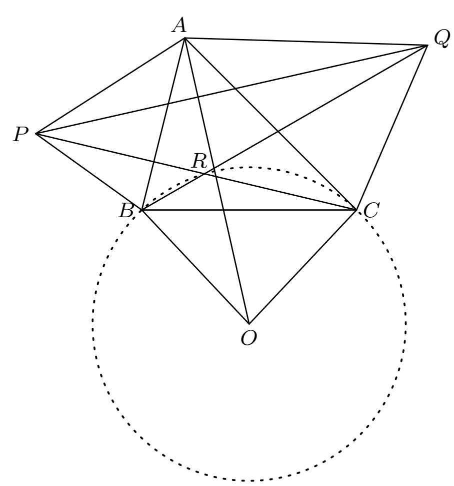
Solution. If we want to find h, it is necessary for $\triangle ABC$ to lie on the unit circle centered at the origin. Therefore, we let the circumcircle of $\triangle ABC$ be the unit circle. Then $h = a + b + c$ and the fact that O bisects HQ gives us $\frac{h+q}{2} = o = 0$, so $q = -a - b - c$. Since we have a, b, c, and q, we can easily find $t_1 = \frac{b+c+q}{3} = -\frac{a}{3}$, $t_2 = -\frac{b}{3}$, and $t_3 = -\frac{c}{3}$. Thus,

$$AT_1 = |a - t_1| = \left|a + \frac{a}{3}\right| = \frac{4}{3}|a| = \frac{4}{3}.$$

The lengths of BT_2 and CT_3 follow similarly, and now a suitable homothety with ratio R brings us to our original diagram. $\square$

Rotations are very handily dealt with using complex numbers, although most problems require a little ingenuity:

Example 3.2 (TST 2006)**.** Let ABC be a triangle. Triangles PAB and QAC are constructed outside of triangle ABC such that $AP = AB$ and $AQ = AC$ and $\angle BAP = \angle CAQ$. Segments BQ and CP meet at R. Let O be the circumcenter of triangle BCR. Prove that $AO \perp PQ$.



Solution. Although at first glance, it may seem useful to set the circumcircle of $\triangle BCR$ to be the unit circle (in order to find o), this is not necessary. We first find p and q. Note that $AB \to AP$ is a rotation centered at A with angle $\angle BAP = \theta$. Thus, $p = a + e^{-i\theta}(b - a)$ and similarly, $q = a + e^{i\theta}(c - a)$.

We can now uniquely determine O by using the fact that $\angle BOC = 2\theta$ and $BO = CO$. Thus $BO \to CO$ is a rotation centered at O with angle 2θ, so $b = o + e^{2i\theta}(c - o)$. Thus $o = \frac{b - ce^{2i\theta}}{1 - e^{2i\theta}}$.

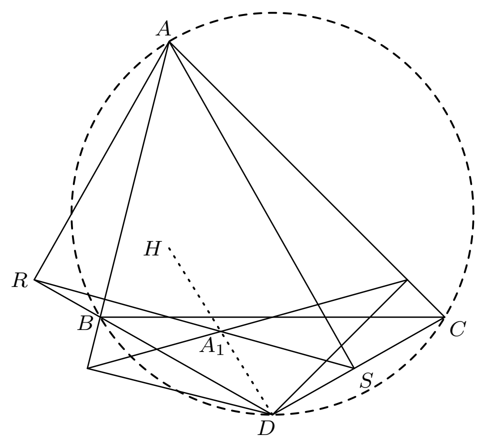
It remains to verify that $AO \perp PQ$, or that $\frac{a-o}{\overline{a}-\overline{o}} = \frac{p-q}{\overline{p-q}}$. We can start brute-forcing from the values we have obtained, or we can boil down the algebra by setting A to be the origin. In addition, we can let $z = e^{i\theta}$. A straightforward calculation shows that

$$\begin{aligned} \frac{o}{\overline{o}} &= \frac{p-q}{\overline{p-q}} \\ \iff \frac{\frac{b-cz^2}{1-z^2}}{\frac{\frac{1}{b}-\frac{1}{cz^2}}{1-\frac{1}{z^2}}} &= \frac{\frac{b}{z}-cz}{\frac{z}{b}-\frac{1}{cz}} \\ \iff \frac{bcz^2}{z^2} &= \frac{bcz}{z} \end{aligned}$$

which is true. $\square$

However, the most powerful technique of using complex numbers is to mix it up with synthetic or projective geometry.

Example 3.3 (TSTST 2013)**.** Let ABC be a triangle and D, E, F be the midpoints of arcs BC, CA, AB on the circumcircle. Line ℓ_a passes through the feet of the perpendiculars from A to DB and DC. Line m_a passes through the feet of the perpendiculars from D to AB and AC. Let A_1 denote the intersection of lines ℓ_a and m_a. Define points B_1 and C_1 similarly. Prove that triangles DEF and $A_1B_1C_1$ are similar to each other.



Solution. We prove a more general result: let D, E, and F be arbitrary points on arcs $\overset{\frown}{BC}$, $\overset{\frown}{CA}$, and $\overset{\frown}{AB}$. Notice that ℓ_a is the Simson line of A with respect to $\triangle BCD$ and m_a is that of D with respect to $\triangle ABC$.

Let H be the orthocenter of $\triangle ABC$. It is well-known that the midpoint of DH (which we will denote by P) lies on m_a. We claim that the point of intersection of ℓ_a and m_a is P. Because it is easy to find h when $\triangle ABC$ lies on the unit circle, we let the circumcircle of $\triangle ABC$ be the unit circle. Thus $p = \frac{d+h}{2} = \frac{a+b+c+d}{2}$.

We can check that this point lies on ℓ_a by confirming that R, P, and S are collinear, where R and S are the projections of A onto BD and CD, respectively. Note that $r = \frac{1}{2}(a + b + d - bd\overline{a}) = \frac{1}{2}\left(a + b + d - \frac{bd}{a}\right)$ and

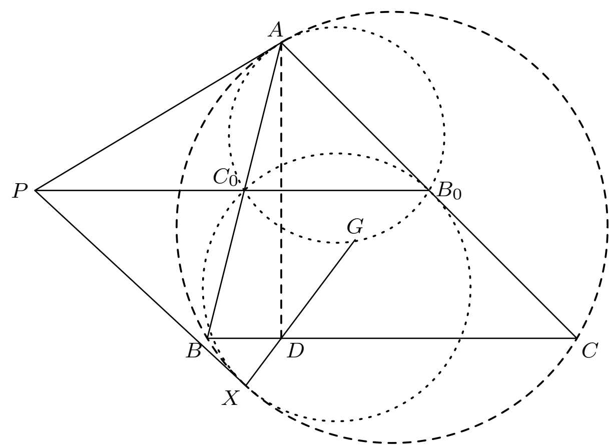
$s = \frac{1}{2}\left(a + c + d - \frac{cd}{a}\right)$. The condition that R, P, and S are collinear translates to

$$\begin{aligned} \frac{r-s}{\overline{r-s}} &= \frac{p-s}{\overline{p-s}} \\ \iff \frac{b - c - \frac{bd}{a} + \frac{cd}{a}}{\frac{1}{b} - \frac{1}{c} - \frac{a}{bd} + \frac{a}{cd}} &= \frac{b + \frac{cd}{a}}{\frac{1}{b} + \frac{a}{cd}} \\ \iff \frac{\left(1 - \frac{d}{a}\right)(b-c)}{\left(1 - \frac{a}{d}\right)\left(\frac{1}{b} - \frac{1}{c}\right)} &= \frac{bcd}{a} \\ \iff -\frac{d}{a}(-bc) &= \frac{bcd}{a}, \end{aligned}$$

which is true. Thus there exists a homothety centered at H with factor $\frac{1}{2}$ that sends $\triangle DEF$ to $\triangle A_1B_1C_1$, so $\triangle DEF \sim \triangle A_1B_1C_1$, as desired. $\square$

Remark. The last step could also have been completed using complex numbers. $\triangle DEF \sim \triangle A_1B_1C_1$ iff $\frac{d-e}{d-f} = \frac{a_1-b_1}{a_1-c_1}$, which is easy to check.

Example 3.4 (ISL 2011/G4)**.** Let ABC be an acute triangle with circumcircle Ω. Let B_0 be the midpoint of AC and let C_0 be the midpoint of AB. Let D be the foot of the altitude from A and let G be the centroid of the triangle ABC. Let ω be a circle through B_0 and C_0 that is tangent to the circle Ω at a point $X \neq A$. Prove that the points D, G and X are collinear.



Solution. Let Γ be the circle passing through A, B_0, and C_0. This circle is tangent to Ω at A. Let P be the intersection of the tangents to Ω at A and X. By the Radical Axis Theorem, the pairwise radical axes of Ω, ω, and Γ are concurrent. Note that the radical axis of Ω and ω is the tangent line to Ω at X; the radical axis of Ω and Γ is the tangent line to Ω at A; the radical axis of ω and Γ is line B_0C_0. Hence P must lie on the radical axis of ω and Γ, so P, C_0, and B_0 are collinear.

Let the circumcircle of $\triangle ABC$ be the unit circle. Then we have $p = \frac{2ax}{a+x}$, $c_0 = \frac{a+b}{2}$, $b_0 = \frac{a+c}{2}$, $g = \frac{a+b+c}{3}$, and $d = \frac{1}{2}\left(a + b + c - \frac{bc}{a}\right)$. By the collinearity of P, C_0, and B_0, we have

$$\frac{b_0 - c_0}{\overline{b_0} - \overline{c_0}} = \frac{b_0 - p}{\overline{b_0} - \overline{p}} \iff -bc = \frac{a + b - 2p}{\frac{1}{a} + \frac{1}{b} - 2\overline{p}} \iff p + bc\overline{p} = \frac{(a+b)(a+c)}{2a}.$$

$PA \perp AO$, where O is the circumcenter of Ω, so we have

$$\frac{p-a}{\overline{p}-\overline{a}} = \frac{a-o}{\overline{a}-\overline{o}} \iff p + a^2\overline{p} = 2a.$$

Solving for p in the previous equation, we obtain

$$p = \frac{a^3 + a^2b + a^2c - 3abc}{2a^2 - 2bc}.$$

Now using the fact that $p = \frac{2ax}{a+x}$, we can solve for x:

$$p = \frac{2ax}{a+x} \iff x = \frac{ap}{2a-p} = \frac{a^3 + a^2b + a^2c - 3abc}{3a^2 - bc - ab - ac}.$$

It suffices to prove that $\frac{g-d}{\overline{g}-\overline{d}} = \frac{g-x}{\overline{g}-\overline{x}}$, which is a straightforward computation. $\square$

4 Hilbert's Nullstellensatz

It is a natural question to ask whether given any geometry problem we can find an analytic solution. It turns out that Hilbert's Nullstellensatz guarantees a way to express every geometric condition into algebraic equations, although manipulating them may not always be elegant.

Theorem 4.1 (Hilbert's Nullstellensatz). *Let $P_1, P_2, \cdots, P_k$, $Q \in \mathbb{C}[x_1, \cdots, x_n]$ be polynomials such that if $(x_1, x_2, \cdots, x_n) \in \mathbb{C}^n$ is a point where $P_1, P_2, \cdots, P_k$ all vanish, then Q also vanishes at that point. Then there exists a positive integer r and polynomials $C_1, C_2, \cdots, C_k \in \mathbb{C}[x_1, x_2, \cdots, x_n]$ such that*

$$C_1P_1 + C_2P_2 + \cdots + C_kP_k = Q^r.$$

5 Problems

I'm not a basher!

-Victoria Xia, after successfully using complex numbers on TSTST 2013

1. (Varignon's Theorem) Prove that the midpoints of a quadrilateral form a parallelogram.
2. (BMO 1984) Let $ABCD$ be a cyclic quadrilateral and let H_A, H_B, H_C, H_D be the orthocenters of $\triangle BCD, \triangle CDA, \triangle DAB$, and $\triangle ABC$ respectively. Prove that the quadrilaterals $ABCD$ and $H_AH_BH_CH_D$ are congruent.
3. (Romania 1984) Let z_1, z_2, z_3 be complex numbers such that $|z_1| = |z_2| = |z_3| = R$ and $z_2 \neq z_3$. Prove that
$$\min_{a\in\mathbb{R}} |az_2 + (1-a)z_3 - z_1| = \frac{1}{2R}|z_1 - z_2| \cdot |z_2 - z_3|.$$
4. (MOP 2006) Point H is the orthocenter of triangle ABC. Points D, E, and F lie on the circumcircle of triangle ABC such that $AD \parallel BE \parallel CF$. Points S, T, and U are the respective reflections of D, E, and F across the lines BC, CA, and AB. Prove that S, T, U and H are concyclic.

5. (WOP 2004) Convex quadrilateral $ABCD$ is inscribed in circle ω. Let M and N be the midpoints of diagonals AC and BD, respectively. Lines AB and CD meet at E, and lines AD and BC meet at F. Prove that
$$\frac{2MN}{EF} = \left|\frac{AC}{BD} - \frac{BD}{AC}\right|.$$

6. (USAMO 2012) Let P be a point in the plane of $\triangle ABC$, and γ a line passing through P. Let A', B', and C' be the points where the reflections of PA, PB, and PC with respect to γ intersect lines BC, AB, and AC, respectively. Prove that A', B', and C' are collinear.

7. (Iran 2005) Let ABC be an isosceles triangle such that $AB = AC$. Let P be on the extension of side BC and X and Y on AB and AC, respectively, such that $PX \parallel AC$ and $PY \parallel AB$. Let T be the midpoint of arc BC. Prove that $PT \perp XY$.

8. (USAMO 2004) A circle ω is inscribed in quadrilateral $ABCD$. Let I be the center of ω. Suppose that
$$(AI + DI)^2 + (BI + CI)^2 = (AB + CD)^2.$$
Prove that $ABCD$ is an isosceles trapezoid.

9. (TST 2000) Let $ABCD$ be a cyclic quadrilateral and let E and F be the feet of perpendiculars from the intersection of diagonals AC and BD to AB and CD, respectively. Prove that EF is perpendicular to the line through the midpoints of AD and BC.

10. (ISL 2012/G2) Let $ABCD$ be a cyclic quadrilateral whose diagonals AC and BD meet at E. The extensions of sides AD and BC beyond A and B meet at F. Let G be the point such that $ECGD$ is a parallelogram, and let H be the image of E under the reflection in AD. Prove that D, H, F, and G are concyclic.

11. (MOP 2006) Convex quadrilateral $ABCD$ is inscribed in ω centered at O. Point O does not lie on the sides of $ABCD$. Let O_1, O_2, O_3, and O_4 denote the circumcenters of triangles OAB, OBC, OCD, and ODA, respectively. Diagonals AC and BD meet at P. Prove that O_1O_3, O_2O_4, and OP are concurrent.

12. (USAMO 2006) Let $ABCD$ be a quadrilateral and let E and F be points on sides AD and BC, respectively, such that $\frac{AE}{ED} = \frac{BF}{FC}$. Ray FE meets rays BA and CD at S and T, respectively. Prove that the circumcircles of triangles SAE, SBF, TCF, and TDE pass through a common point.

13. (China 1996) Let H be the orthocenter of $\triangle ABC$. The tangents from A to the circle with diameter BC intersect the circle at points P and Q. Prove that points P, Q, and H are collinear.

14. (ISL 1998/G8) Let ABC be a triangle such that $\angle A = 90^\circ$ and $\angle B < \angle C$. The tangent at A to the circumcircle ω of triangle ABC meets the line BC at D. Let E be the reflection of A in the line BC, let X be the foot of the perpendicular from A to BE, and let Y be the midpoint of the segment AX. Let the line BY intersect the circle ω again at Z. Prove that the line BD is tangent to the circumcircle of triangle ADZ.

15. (IMO 2000) Let AH_1, BH_2, and CH_3 be the altitudes of an acute triangle ABC. The incircle ω of triangle ABC touches sides BC, CA, and AB at T_1, T_2, and T_3, respectively. Consider the symmetric images of lines H_1H_2, H_2H_3, and H_3H_1 with respect to lines T_1T_2, T_2T_3, and T_3T_1. Prove that these images form a triangle whose vertices lie on circle ω.