Combinatorial Number Theory

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Combinatorial number theory refers to combinatorics flavored with the rich juicy arithmetical structure of the integers. At the elementary level, like many other areas of combinatorics, combinatorial number theory doesn't require a lot of deep theorems; instead it's a big hodgepodge of ideas and tricks.

A few notational conventions are useful, in particular in stating additive problems. If A and B are sets of integers, we often write $A+B$ for the set $\{a+b \mid a \in A, b \in B\}$. For c a constant, we often write $A+c$ for $\{a+c \mid a \in A\}$ and $cA = \{ca \mid a \in A\}$. Also, if we are interested in sums or products of generic sets of integers, the sum of the empty set is generally taken to be 0, and the product of the empty set is 1.

1 Problem-solving techniques

For the most part, the ideas that are useful in solving combinatorial number theory problems are the same ones that are useful in other areas of combinatorics.

- Use the pigeonhole principle (or probabilistic methods)
- Use induction
- Use greedy algorithms
- Look at prime factorizations and the divisibility lattice
- Look at largest or smallest elements
- Think about orders of magnitude
- Count things in two ways
- Use relative primality
- Look at things mod n, for conveniently chosen n
- Transform things to make them convenient to work with

- Don't be afraid of case analysis and brute force
- Use generating functions or similar algebraic techniques
- Translate the problem into graph theory
- Use actual number theory

2 Problems

1. Determine whether or not there exists an increasing sequence $a_1, a_2, \ldots$ of positive integers with the following property: for any integer k, only finitely many of the numbers $a_1 + k, a_2 + k, \ldots$ are prime.

2. [BMC, 1999] The set of positive integers is partitioned into finitely many subsets. Show that some subset S has the following property: for every positive integer n, S contains infinitely many multiples of n.

3. Given is a list of n positive integers whose sum is less than $2n$. Prove that, for any positive integer m not exceeding the sum of these integers, one can choose a sublist of the integers whose sum is m.

4. Let S be an infinite set of integers, such that every finite subset of S has a common divisor greater than 1. Show that all the elements of S have a common divisor greater than 1.

5. Show that any positive integer can be expressed as a sum of terms of the form $2^a 3^b$, where a, b are nonnegative integers, and no term is divisible by any other.

6. [Canada, 2000] Given are 2000 integers, each one having absolute value at most 1000, and such that their sum equals 1. Prove that we can choose some of the integers so that their sum equals 0.

7. [Paul Erdős] Show that if $n+1$ numbers are chosen from the set $\{1, 2, \ldots, 2n\}$, then one of these numbers divides another.

8. [BAMO, 2009] A set S of positive integers is *magic* if for any two distinct members $i, j \in S$, $(i+j)/\gcd(i,j)$ is also in S. Find all finite magic sets.

9. [IMO, 1991] Let $n > 6$ be an integer with the following property: all the integers in $\{1, 2, \ldots, n-1\}$ that are relatively prime to n form an arithmetic progression. Prove that n is either prime or a power of 2.

10. [USAMO, 1998] Prove that, for each integer $n \geq 2$, there is a set S of n integers such that ab is divisible by $(a-b)^2$ for all distinct $a, b \in S$.

11. [Reid Barton] Let $a_1 < a_2 < \cdots$ be an increasing sequence of positive integers, such that $a_{n+1} - a_n < 1000000$ for all n. Prove that there exist indices $i < j$ such that a_j is divisible by a_i.

12. [China, 2009] Find all pairs of distinct nonzero integers (a, b) such that there exists a set S of integers with the following property: for any integer n, exactly one of $n, n + a, n + b$ is in S.

13. [APMC, 1990] Let $a_1, \ldots, a_r$ be integers such that $\sum_{i \in I} a_i \neq 0$ for every nonempty set $I \subseteq \{1, \ldots, r\}$. Prove that the positive integers can be partitioned into a finite number of classes so that, whenever $n_1, \ldots, n_r$ are integers from the same class, $a_1 n_1 + \cdots + a_r n_r \neq 0$.

14. [IMO, 2003] Let $S = \{1, 2, \ldots, 10^6\}$. Prove that for any $A \subseteq S$ with 101 elements, we can find $B \subseteq S$ with 100 elements such that the sums $a+b$, for $a \in A$ and $b \in B$, are all different.

15. [Russia, 1998] A sequence $a_1, a_2, \ldots$ of positive integers contains each positive integer exactly once. Moreover, for every pair of distinct positive integers m and n,

$$\frac{1}{1998} < \frac{|a_n - a_m|}{|n - m|} < 1998.$$

Show that $|a_n - n| < 2000000$ for all n.

16. [Schur's Theorem] For any positive integer k, there exists an N with the following property: if the integers $1, 2, \ldots, N$ are colored in k colors, then there exist some three integers a, b, c of the same color such that $a + b = c$.

17. [China, 2009] Let a, b, m, n be positive integers with $a \leq m < n < b$. Prove that there exists a nonempty subset S of $\{ab, ab + 1, ab + 2, \ldots, ab + a + b\}$ such that $(\prod_{x \in S} x)/mn$ is the square of a rational number.

18. [IMO Shortlist, 1990] The set of positive integers is partitioned into finitely many subsets. Prove that there exists some subset, say A_i, and some integer m with the following property: for any k, there exist numbers $a_1 < a_2 < \cdots < a_k$ in A_i, with $a_{j+1} - a_j \leq m$ for each j.

19. [St. Petersburg, 1996] The numbers $1, 2, \ldots, 2n$ are divided into 2 sets of n numbers. For each set, we consider all n^2 possible sums $a + b$, where a, b are in that set (and may be equal). Each sum is reduced mod $2n$. Show that the n^2 remainders from one set are equal, in some order, to the n^2 remainders from the other set.

20. [IMO Shortlist, 1999] Let A be a set of N residues mod N^2. Prove that there exists a set B of N residues mod N^2 such that the set $A + B$ contains at least half of all residues mod N^2.

21. [IMO shortlist, 1999] Let x and y be odd integers with $|x| \neq |y|$. Suppose that the positive integers have been colored in four different colors. Show that there exist two different numbers of the same color whose difference is equal to x, y, $x+y$, or $x-y$.

22. [Erdős-Selfridge] For any set A of positive integers, let $\sigma_A(n)$ be the number of ways of writing n as a sum of two distinct members of A. If two different sets A and B have the property that $\sigma_A(n) = \sigma_B(n)$ for all positive integers n, prove that the number of elements in each set is a power of 2.

23. [Bulgaria, 2000] Let $p \geq 3$ be a prime number, and $a_1, \ldots, a_{p-2}$ a sequence of integers such that, for each i, neither a_i nor $a_i^i - 1$ is a multiple of p. Prove that there exists some collection of distinct terms whose product is congruent to 2 mod p.

24. [IMO, 2009] Let $a_1, a_2, \ldots, a_n$ be distinct positive integers and let M be a set of $n-1$ positive integers not containing $s = a_1 + a_2 + \cdots + a_n$. A grasshopper is to jump along the real axis, starting at the point 0 and making n jumps to the right with lengths $a_1, a_2, \ldots, a_n$ in some order. Prove that the order can be chosen in such a way that the grasshopper never lands on any point in M.

25. [Van der Waerden's Theorem] For any positive integers k and m, there exists N with the following property: if the integers $1, 2, \ldots, N$ are colored in k colors, there exists an arithmetic progression of length m, all of whose members are the same color.