New Zealand
Maths Olympiad Committee
Camp 2009
Preparation Problems

1. The diagonals AC and BD of a convex quadrilateral $ABCD$ are perpendicular. Through the midpoints of AB and AD the lines perpendicular to the corresponding opposite sides CD and BC are drawn. Prove that these lines intersect at a point on AC.

2. Prove that for any positive integer n, there exists an integer k which can be expressed as a sum of two squares and such that $n \leq k \leq n + 3\sqrt[4]{n}$.

3. Let $x_1, x_2, \ldots, x_n$ be a decreasing sequence of positive real numbers. Prove that:
$$\left(\sum_{k=1}^{n} x_k^2\right)^{\frac{1}{2}} \leq \sum_{k=1}^{n} \frac{x_k}{\sqrt{k}}.$$

4. A finite set of squares, all the same size, are drawn in the plane so that each square has a side which is parallel to a given line. Among any 2009 of the squares, there are at least two that intersect. Prove that the set of squares can be divided into no more than 4015 subsets so that all the squares in each subset have a point in common.

5. Let $ABCD$ be a quadrilateral inscribed in a semicircle with diameter AB. The lines AC and BD intersect at E and AD and BC at F. The line EF intersects the semicircle at G, and AB at H. Prove that E is the midpoint of GH if and only if G is the midpoint of FH.

6. Is it possible for the product of five consecutive positive integers to be a perfect square?

7. A collection of points in the plane are connected by arcs (possibly more than one arc connects any two given points, and there may well be points which are not connected together by an arc). There are 2008×2009 arcs. It is possible to label the arcs with the numbers from 1 to 2008 (repetitions allowed) in such a way that all the labels of arcs ending at any given point are different from one another. Show that it is also possible to do this in such a way that each label occurs exactly 2009 times.

8. Suppose that a, b, and c are positive real numbers satisfying
$$a^2 + b^2 + c^2 = 1.$$
Prove that:
$$\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \geq 3 + \frac{2(a^3 + b^3 + c^3)}{abc}.$$