Simple Synthetic Geometry

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Aqua Lecture, June 17, 2011

There were two geometry problems at the IMO every year from 1998 through 2010, and in each of those years except 2002, one day began with a geometry problem.

For all its creativity, synthetic geometry depends on only a small number of axioms. But first, a word on the diagrams. A diagram organizes axioms. One should accompany every solution, though it should not be essential to a solution except as a reminder of the conditions. However, *a good diagram helps one solve problems by revealing logical consequences.*

1 Angles, Angles, Angles!

Angle chasing is the most common and most important strategy in synthetic geometry, appearing in virutally all problems. Fortunately, amounting to a sequence of axiomatic equalities, it is probably also the easiest to master. All of my angles are directed and modulo π, and yours should be too. That means that $\angle ABC = -\angle CBA$ and directed angles differing by multiples of π are considered identical. With this convention, many basic geometric properties are equivalent to simple equations *independent of configuration.* Consider four distinct points A, B, C, and D in the plane:

- **Cyclic quadrilaterals.** Quadrilateral $ABCD$ is cyclic if and only if
$$\angle ABC = \angle ADC.$$
- **Collinearity.** Points A, B, and C are collinear if and only if
$$\angle DAB = \angle DAC.$$
- **Parallel lines.** Lines AB and CD are parallel if and only if
$$\angle ABC = \angle DCB.$$
- **Isosceles triangles.** Triangle ABC is isosceles with $AB = AC$ if and only if $\angle ABC = \angle BCA$.
- **Similar triangles.** Triangle $A'B'C'$ is *directly similar* to triangle ABC if and only if $\angle ABC = \angle A'B'C'$ and $\angle BCA = \angle B'C'A'$. If both equalities are reversed, the triangles are *inversely similar* instead.

2 Ratios

Computation involving ratios is a second essential strategy in synthetic geometry. These quantities are nearly as easy to work with as angles and, importantly, they can provide complementary information. In particular, the most important theorem about concurrence is formulated in terms of ratios. Again one has the concept of direction, as in $AB = -BA$, of distances along the same line, which helps address configuration issues.

- **Power of a point.** Let lines AB and CD intersect at P. Quadrilateral $ABCD$ is cyclic if and only if

  $$PA \cdot PB = PC \cdot PD.$$

  If ω is the circumcircle of $ABCD$, the signed quantity $PA \cdot PB$ is the power of P with respect to circle ω. Note that power of a point is weaker than the statements about similar triangles from which it derives. Namely, if $ABCD$ is cyclic, then triangles PAD and PAC are similar and oppositely oriented to triangles PCB and PDB respectively.

- **Angle bisector theorem.** Let ABC be a triangle and let point D lie on line BC. Then D lies on internal bisector of angle $\angle BAC$ if and only if

  $$\frac{BD}{DC} = \frac{AB}{AC}.$$

  Moreover, D lies on the external angle bisector if and only if

  $$-\frac{BD}{DC} = \frac{AB}{AC}.$$

- **Ceva's theorem.** Let points D, E, and F lie on lines BC, CA, and AB, respectively. Then lines AD, BE, and CF concur if and only if

  $$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1.$$

- **Menelaus' theorem.** Let points D, E, and F lie on lines BC, CA, and AB. Then points D, E, and F are collinear if and only if

  $$\frac{BD}{DC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = -1.$$

The duality between lines and points in Ceva's theorem and Menelaus' theorem is characteristic of projective geometry.

3 Lemmas

A third component required in some problems is simply recalling a suitable fact. There are too many lemmas in geometry to give a comprehensive listing; one accumulates knowledge of them over time. The centroid G, circumcenter O, incenter I, and orthocenter H of a triangle and their properties should be familiar. Fortunately, though, and by design, few obscure theorems are required for Olympiad geometry problems. We give a few, in my opinion the most important, here.

- **Radical axis theorem.** The locus of points having the same power with respect to a given pair of circles is a line, called the *radical axis* of the circles. The three radical axes determined by three circles concur at a point called their *radical center* or power center.

- **Nine point circle.** Let ABC be a triangle and H its orthocenter. Denote midpoints of segments AH, BH, CH by A_1, B_1, C_1, denote the midpoints of BC, CA, AB by A_2, B_2, C_2, and denote the feet of the altitudes from A, B, C to BC, CA, AB by A_3, B_3, C_3. Then the nine points A_i, B_i, C_i for $i = 1, 2, 3$ lie on a common circle, called the nine point circle of triangle ABC.

- **Pascal's theorem.** Suppose points A, B, C, D, E, and F lie on a conic section (notably, this can be a circle or a pair of lines.) Denote by P, Q, and R the intersections of pairs of lines AB and DE, BC and EF, and FD and CA, respectively. Then P, Q, and R are collinear.

- **Brianchon's theorem.** Let points $A_1, A_2, \ldots, A_6$ lie on a conic section (notably, a circle.) For $i = 1, \ldots, 6$, let the tangents to the conic section at A_i and A_{i+1} intersect at P_i, where $A_7 = A_1$. Then lines P_iP_{i+3} for $i = 1, 2, 3$ concur.

The duality between pole and polars in Pascal's theorem and Brianchon's theorem is also characteristic of projective geometry.

4 Transformations and Constructions

The last ingredient is creativity. Showing that two points, defined in different ways, are identical is a powerful way to demonstrate concurrence. One sometimes needs to move information around by introducing more points, lines, or circles. Sometimes transformations are appropriate. Reflections can convert angle bisection into collinearity. Spiral similarity is a general ratio-preserving transformation with an interesting center. Dilations produce many parallel lines. Certain patterns suggest inversion. A busy point, lying on many circles, lines, or participating in strange angle conditions, is a candidate for inversion; strange equalities involving angle addition can also suggest inversion. Perhaps the most difficult to master, constructing a point with the right properties can be essential. Of particular importance are methods for constructing the centers or fixed points of the transformations.

- **Dilation/Homothety.** Two dilations take a pair of points A, B to a pair of points C, D where $AB \parallel CD$. Their centers are the intersections of lines AC and AD with lines BD and BC, respectively. The dilations differ in orientation.

- **Reflection.** The perpendicular bisector of a segment AB defines the unique reflection exchanging the points A and B.

- **Spiral similarity.** A combination of rotation and dilation, there are two spiral similarities taking the pair of points A, B to the points C, D. Their centers are described in a critical exercise given below. Again, the transformations differ in orientation.

- **Inversion.** An inversion is determined by the circle it fixes. Two inversions take the set of vertices of a cyclic quadrilateral to itself nontrivially, centered at the intersections of the extensions of opposite pairs of sides. A third inversion, centered at the intersection of the diagonals, combined with rotation by π about the same point, exchanges diagonally opposed pairs of vertices. More significant than any of these is the earlier remark about busy points.

5 Problems

1. (Incenter lemma) Let I be the incenter of triangle ABC, and let ray AI intersect the circumcircle of triangle ABC again at D. Show that $DI = DB = DC$.

2. (Fermat point) Let ABC be a triangle. Equilateral triangles $A'BC$, $AB'C$, and ABC' are erected externally (not overlapping with triangle ABC.) Show that lines AA', BB', and CC' concur. Let P be the point of concurrence. What is $m\angle APB$?

3. (Simson line) Point P lies on the circumcircle of triangle ABC. Let points D, E, and F be the feet of the altitudes from P to lines BC, CA, and AB, respectively. Show that points D, E, and F are collinear.

4. (Spiral similarity lemma) Let lines AB and $A'B'$ intersect at P, and let the circumcircles of triangles PAA' and PBB' intersect again at Q. Show that Q is the center of a spiral similarity sending A and B to A' and B' respectively.

5. (Orthic triangle) Let triangle ABC be acute, and let D, E, F be the feet of the altitudes from A, B, C to lines BC, CA, AB, respectively. Show that the orthocenter of triangle ABC is the incenter of triangle DEF. What happens if triangle ABC is obtuse?

6. (Euler line) Let G, H, and O be the centroid, orthocenter, and circumcenter of triangle ABC. Show that G lies on segment OH and $GH = 2OG$.

7. (Altitude lemma) Let ABC be an acute triangle. Denote by D the foot of the altitude from A to BC, and let point P lie on line AD. Lines BP and CP intersect lines AC and AB at points E and F respectively. Show that DP bisects $\angle EDF$.

8. (MOP 2003) Acute triangle ABC is such that $AB \neq AC$. Point D is the foot of the altitude from A to BC, and point P lies on segment AD. Lines BP and CP intersect lines AC and AB at E and F respectively. Show that if quadrilateral $BFEC$ is cyclic, then P is the orthocenter of triangle ABC.

9. (Symmedian lemma) Let ABC be a triangle, let M be the midpoint of side BC, and let the tangents to the circumcircle of triangle ABC at B and C intersect at D. Show that $\angle CAD = \angle MAB$ in the sense of directed angles modulo π.

10. (Median lemma) Let the incircle of triangle ABC touch sides BC, CA, AB at D, E, F, respectively, and denote the incenter of triangle ABC by I. Lines DI and EF intersect at P, and line AP intersects segment BC at M. Show that M is the midpoint of segment BC.

11. (Ptolemy's theorem) Let $ABCD$ be a convex quadrilateral. Show that
$$AB \cdot CD + BC \cdot AD \geq AC \cdot BD,$$
with equality if and only if quadrilateral $ABCD$ is cyclic.

12. (Nine point circle) Prove the nine point circle theorem. Show further that a triangle's nine point circle bisects every segment from that triangle's orthocenter to a point on its circumcircle.

13. (Archimedes midpoint theorem) Let $AB = AC$ and suppose that D lies on minor arc AB of the circumcircle of triangle ABC. Let E be the foot of the perpendicular from A to segment CD. Show that $BD + DE = CE$.

14. (IMO 2000/1) Two circles Γ_1 and Γ_2 intersect at M and N. Let AB be the line tangent to these circles at A and B, respectively, so that M lies closer to AB than N. Let CD be the line parallel to AB and passing through M, with C on circle Γ_1 and D on Γ_2. Lines AC and BD meet at E; lines AN and CD meet at P; lines BN and CD meet at Q. Show that $EP = EQ$.

15. (IMO 2001/1) In acute triangle ABC with circumcenter O and altitude AP, $\angle C \geq \angle B + 30°$. Prove that $\angle A + \angle COP < 90°$.

16. (IMO 2003/4) Let $ABCD$ be a cyclic quadrilateral. Let P, Q, R be the feet of the perpendiculars from D to the lines BC, CA, AB, respectively. Show that $PQ = QR$ if and only if the bisectors of angles $\angle ABC$ and $\angle ADC$ are concurrent with AC.

17. (IMO 2004/1) Let ABC be an acute-angled triangle with $AB \neq AC$. The circle with diameter BC intersects the sides AB and AC at M and N, respectively. Denote by O the midpoint of BC. The bisectors of the angles BAC and MON intersect at R. Prove that the circumcircles of the triangles BMR and CNR have a common point lying on the line segment BC.

18. (IMO 2006/1) Let ABC be a triangle with incenter I. A point P in the interior of the triangle satisfies $\angle PBA + \angle PCA = \angle PBC + \angle PCB$. Show that $AP \geq AI$, and that equality holds if and only if $P = I$.

19. (IMO 2007/4) In triangle ABC the bisector of angle BCA intersects the circumcircle again at R, the perpendicular bisector of BC at P, and the perpendicular bisector of AC at Q. The midpoint of BC is K and the midpoint of AC is L. Prove that the triangles RPK and RQL have the same area.

20. (IMO 2008/1) Let H be the orthocenter of an acute-angled triangle ABC. The circle Γ_A centered at the midpoint of BC and passing through H intersects the segment BC at points A_1 and A_2. Similarly, define the points B_1, B_2, C_1, and C_2. Prove that A_1, A_2, B_1, B_2, C_1, and C_2 are concyclic.

21. (IMO 2009/4) Let ABC be a triangle with $AB = AC$. The angle bisectors of $\angle CAB$ and $\angle ABC$ meet the sides BC and CA at D and E, respectively. Let K be the incenter of triangle ADC. Suppose that $\angle BEK = 45°$. Find all possible values of $\angle CAB$.

22. (IMO 2010/4) Let P be a point interior to triangle ABC (with $CA \neq CB$). The lines AP, BP, and CP meet its circumcircle Γ again at K, L, and M, respectively. The line tangent to Γ at C meets the line AB at S. Show that if $SC = SP$, then $MK = ML$.

23. (IMO 2005/1) Six points are chosen on the sides of an equilateral triangle ABC : A_1, A_2 on BC, B_1, B_2 on CA, and C_1, C_2 on AB, such that they are the vertices of a convex hexagon $A_1A_2B_1B_2C_1C_2$ with equal side lengths. Prove that the lines A_1B_2, B_1C_2, and C_1A_2 are concurrent.

24. (IMO 2007/2) Consider five points A, B, C, D, and E such that $ABCD$ is a parallelogram and $BCED$ is a cyclic quadrilateral. Let ℓ be a line passing through A. Suppose that ℓ intersects the interior of the segment DC at F and intersects the line BC at G. Suppose also that $EF = EG = EC$. Prove that ℓ is the bisector of angle DAB.

25. (IMO 2002/2) The circle S has center O, and BC is a diameter of S. Let A be a point of S such that $\angle AOB < 120°$. Let D be the midpoint of the arc AB which does not contain C. The line through O parallel to DA meets the line AC at I. The perpendicular bisector of OA meets S at E and at F. Prove that I is the incenter of triangle CEF.

26. (IMO 2005/5) Let $ABCD$ be a fixed convex quadrilateral with $BC = DA$ and BC not parallel to DA. Let two variable points E and F lie on the sides BC and DA, respectively, and satisfy $BE = DF$. The lines AC and BD meet at P, the lines BD and EF meet at Q, the lines EF and AC meet at R. Prove that the circumcircles of the triangles PQR, as E and F vary, have a common point other than P.

27. (IMO 2009/2) Let ABC be a triangle with circumcenter O. The points P and Q are interior points of the sides CA and AB, respectively. Let K, L, and M be the midpoints of the segments BP, CQ, and PQ, respectively, and let Γ be the circle passing through K, L, and M. Suppose that the line PQ is tangent to the circle Γ. Prove that $OP = OQ$.

28. (IMO 2004/5) In a convex quadrilateral $ABCD$ the diagonal BD does not bisect the angles ABC and CDA. The point P lies inside $ABCD$ and satisfies

$$\angle PBC = \angle DBA \quad \text{and} \quad \angle PDC = \angle BDA.$$

Prove that $ABCD$ is a cyclic quadrilateral if and only if $AP = CP$.

29. (IMO 2010/2) Given a triangle ABC, with I as its incenter, Γ as its circumcircle, AI intersects Γ again at D. Let E be a point on the arc BDC, and F a point on the segment BC, such that $\angle BAF = \angle CAE < \frac{1}{2}\angle BAC$. If G is the midpoint of IF, prove that the intersection of lines EI and DG lies on Γ.

30. (IMO 2008/6) Let $ABCD$ be a convex quadrilateral with BA different from BC. Denote the incircles of triangles ABC and ADC by k_1 and k_2 respectively. Suppose that there exists a circle k tangent to ray BA beyond A and to the ray BC beyond C, which is also tangent to lines AD and CD. Prove that the common external tangents to k_1 and k_2 intersect on k.