Graph Theory 1

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Based on Paul Seymour's notes and work

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1 Minors

Definition 1.1. If $e \in E(G)$, then G/e ("G contract e") is the graph formed by deleting e and identifying its endpoints.

Definition 1.2. A graph H is a minor of a graph G iff it's obtainable from a subgraph of G by contracting edges.

That is, to get a minor, one first deletes vertices and edges, then contracts edges. Note that contraction and deletion commute, so one can do so any order.

Theorem 1.3 (Wagner's Theorem)**.** *A graph G is planar unlesss it has a K_5 or $K_{3,3}$ minor.*

Theorem 1.4 (Kuratowski's Theorem)**.** *A graph G is planar unlesss it has a subdivision of K_5 or $K_{3,3}$ (that is, with edges turned into paths) as a subgraph.*

Theorem 1.5 (Seymour)**.** *If G is an infinite set of graphs, then one is a minor of another.*

1. Prove that every 3-connected graph has a K_4 minor.

2. Prove that a graph G can be drawn in the plane with all vertices in the same region unlesss G has a K_4 or $K_{3,2}$ minor.

3. Prove that if a graph G has no K_5 minor, then it's 4-colorable. (You may assume the Four-color Theorem.)

4. (D. W. Hall) Find all simple 3-connected graphs with no $K_{3,3}$ minor.

Tutte Find all simple 3-connected graphs G such that no edge of G can be deleted or contracted to give another simple 3-connected graph.

5. Prove that every simple graph with average degree at least 2^p has a K_{p+2} minor.

6. Prove that if G is nonnull and loopless and $|E(G)| \geq 2|V(G)| - 1$, then G has a graph with three parallel edges as a minor.

7. Find all 2-connected graphs with no C_5 minor.
8. Find all 2-connected graphs with no $K_4 \setminus e$ minor.
9. (Kotzig's Theorem, also 2009 MOP K6.2/B6.4) Let G be a connected graph that has a perfect matching. Prove that if for any edge e of the perfect matching, $G \setminus e$ is connected, then G has another perfect matching.
10. Prove that if you partition the edges of K_∞ into finitely many sets, then one of them has its own K_∞.