

Advanced Lecture: Complex Numbers

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*Le plus court chemin entre deux vérités dans le domaine réel passe par le domaine complexe.*¹

– Jacques Hadamard

Arising by necessity from the solution of polynomial equations, the field \mathbb{C} of complex numbers is fascinating and incredibly useful. In applications and in pure math, the complex numbers are everywhere. Their versatility arises from the multitude of possible interpretations of their meaning. They may be considered ordered pairs of real numbers with an unusual multiplication rule, points in the Argand plane, two-dimensional vectors, spiral similarities, or even matrices. Oftentimes, as Hadamard said, the shortest path to the solution of a problem can be found by expressing it in the language of the complex numbers, even if at first glance the problem may seem to be very far removed.

1 Conjugates

For a complex number $z = a + bi$, the number $\bar{z} = a - bi$ is called the *complex conjugate* of z . You will have seen and worked with these before, but it is useful to see how some of the important facts are derived. These are:

1. $z = \bar{z}$ iff $z \in \mathbb{R}$; for all z , we have $z = \overline{\bar{z}}$; and the modulus of z is $\sqrt{z\bar{z}}$.
2. The conjugate of a sum is the sum of the conjugates: $\overline{z_1 + z_2} = \bar{z}_1 + \bar{z}_2$.
3. The conjugate of a product is the product of the conjugates: $\overline{z_1 \cdot z_2} = \bar{z}_1 \cdot \bar{z}_2$
4. If $z = re^{i\theta}$, then $\bar{z} = re^{-i\theta}$.
5. The formulas

$$\Re(z) = \frac{z + \bar{z}}{2}, \text{ and } \Im(z) = \frac{z - \bar{z}}{2i}$$

are valid for all $z \in \mathbb{C}$. Note that a special case of this is the exponential form for sin and cos.

Using these important conjugate facts, we will solve two problems.

1. Prove the identity

$$|z_1|^2 + |z_2|^2 = \frac{|z_1 + z_2|^2 + |z_1 - z_2|^2}{2}$$

for all complex numbers $z_1, z_2 \in \mathbb{C}$.

2. Prove that if $|z_1| = |z_2| = 1$ and $z_1 z_2 \neq -1$, then $\frac{z_1 + z_2}{1 + z_1 z_2}$ is a real number.

¹The shortest path between two truths in the real domain passes through the complex domain.

2 Matrices

A complex number is a 2×2 matrix of the form

$$\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$$

where a, b are real numbers. Addition of complex numbers is simply matrix addition; multiplication is just matrix multiplication. Letting 1 and i denote the complex numbers

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

respectively, every complex number z can be written just one way as $z = a \cdot 1 + b \cdot i$. Note that the modulus of a complex number is simply the square root of its determinant, as

$$|a + bi| = \sqrt{\det \begin{pmatrix} a & -b \\ b & a \end{pmatrix}} = \sqrt{a^2 + b^2}.$$

All complex numbers (their matrices) excluding 0 are therefore invertible, and we can define z/w via the map $\varphi(z/w) = zw^{-1}$. While this is a particular matrix representation of the complex numbers, it is not the only one. See if you can find another by looking at matrices J such that $J^2 = -I$.

3 Geometric Properties

When you have to resort to analytic geometry, complex numbers can sometimes greatly simplify the situation. This is a consequence of the fact that we can consider a complex number to be a vector or a geometric transformation. Recall that we add two complex numbers via vector addition ($z = a + bi, w = c + di$ gives $z + w = (a + c) + (b + d)i$). Multiplication then acts as a spiral similarity, because if we have $z = re^{i\theta}, w = \rho e^{i\phi}$, then $zw = r\rho e^{i(\theta+\phi)}$.

If we use complex numbers as our coordinates, we get a very nice formula for rotation by θ about an arbitrary point a :

$$z' = a + e^{i\theta}(z - a)$$

To see why this is true, translate a to the origin, which makes $z \rightarrow z - a$. Multiplying by $e^{i\theta}$ rotates everything by θ , and then we translate the point back by adding a . Another very useful formula is an analog of a vector version: if A, B, C are complex numbers representing the vertices of a triangle, then the centroid of the triangle is at

$$G = \frac{A + B + C}{3}$$

Even some more useful things to remember are facts about particular n -th roots of unity. Say we want to rotate an object in a geometric diagram by 60° . Multiplication by the sixth root of unity $\varepsilon = e^{i\pi/3}$ will do that, but we'll need a way of simplifying expressions involving powers of ε . That's also pretty simple, though – just factor $\varepsilon^6 - 1 = 0$ and remove the extraneous factors. For example, $\varepsilon^6 - 1 = (\varepsilon^2 - 1)(\varepsilon^4 + \varepsilon^2 + 1) = 0$, and $\varepsilon^2 - 1 \neq 0$, so $\varepsilon^4 + \varepsilon^2 + 1 = (\varepsilon^2 - \varepsilon + 1)(\varepsilon^2 + \varepsilon + 1) = 0$. Finally, $\varepsilon^2 + \varepsilon + 1 \neq 0$, so the one relation left is that $\varepsilon^2 = -\varepsilon - 1$. This is simple enough but has extremely important applications in geometry. Take any triangle and construct equilaterals on its sides. How would you prove that the centroids of these three equilateral triangles form another equilateral triangle?

4 Problems

1. Find the value of $\left(\frac{1 + i \tan \frac{\pi}{8044}}{1 - i \tan \frac{\pi}{8044}}\right)^{2011}$.

2. If x, y, z are complex numbers of modulus 1 such that

$$\begin{aligned}x + y + z &= 1 \\xyz &= 1\end{aligned}$$

then find $|(2+x)(2+y)(2+z)|$.

3. Let a, b be two distinct points in the complex plane. For all z on the line determined by a

and b , (answer **true** or **false**) we have $\begin{pmatrix} z & \bar{z} & 1 \\ a & \bar{a} & 1 \\ b & \bar{b} & 1 \end{pmatrix} = 0$.

4. Andre was walking to TJ one day, but he was given very strange directions. He was told to walk one mile to the northwest, then half a mile to the northeast, a fourth of a mile southeast, and so on. Find the distance of TJ from where Andre started walking.

5. A function f is defined on the complex numbers by $f(z) = (a+bi)z$, where a and b are positive numbers. The function has the property that the image of each point in the complex plane is equidistant from that point and the origin. Given that $|a+bi| = 8$ and that $b^2 = m/n$, where m and n are relatively prime positive integers. Find $m+n$.

6. Set $a_k = \tan\left(\sqrt{2} + \frac{k\pi}{2011}\right)$ for $k = 1, 2, 3, \dots, 2011$. Find the value of

$$\frac{a_1 + a_2 + \dots + a_{2011}}{a_1 a_2 \dots a_{2011}}$$