Sequences and Series Haitao Mao and Jacob Steinhardt

1 Basic Techniques

Finding Limits

Here's a nice warm-up problem: Find, in terms of x,

$$\sqrt{x\sqrt{x\sqrt{x\sqrt{x\sqrt{x\ldots}}}}}$$

Pretty tricky, huh? Well, first, since they are asking us to find this value, we can assume that the given value converges, so let S be what we are trying to find. Then

$$S^2 = xS$$

or $S = x$. Another way to do it is to notice that the value is

$$x^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots} = x^1 = x$$

Here's another problem: Find

$$\sqrt{x+\sqrt{x+\sqrt{x+\ldots}}}$$

Again, let S be what we are trying to find, then $S^2 = x + S$, or $S = \frac{1\pm\sqrt{1+4x}}{2}$, and since $S > 0$, we have $S = \frac{1+\sqrt{1+4x}}{2}$. Okay, now find

$$0.5^{0.5^{0.5^{\cdots}}}$$

Finding Sums

Usually the idea is to add parts of the series to itself to make it telescope, or decompose it into other, known series, or break it down into a simpler series, etc. Here is a pretty well-known example: Find

$$\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\ldots+\frac{1}{n(n-1)}$$

Telescope it please! (Hint: TPIAARL TFIROANC ODSEICTOIMOPN).

Okay, now here's a pretty tricky problem: Find

$$1^m + 2^m + 3^m + 4^m + 5^m + \ldots + n^m$$

where m is the smallest integer such that you do not already know the answer to this.

Another cool thing is to add imaginary numbers to the numerator of a series to make stuff factor or telescope, but that is advancedish so worry about getting down the basics first.

2 Advanced Techniques

Solving Recursions

All linear recursions are exponential. Except those that are not. Here is how you solve a recursion: If it is of the form

$$\sum_{i=1}^{n} a_{n+i} b_i = 0$$

for all n, then assume it is exponential and factor the polynomial

$$\sum_{i=1}^{n} b_i x^i = 0$$

then your recursion is of the form

$$a_n = \sum_i c_i r_i^n$$

where the c_i are determined by any n known terms of the sequence. If you have a double root, or triple root, or something, instead of c_i you have $(c_i + nc_{i+1})r_i^n$ and $(c_i + nc_{i+1} + n^2 c_{i+2})r_i^n$, respectively, and this generalizes (here $r_i = r_{i+1}$ and $r_i = r_{i+1} = r_{i+2}$). If you have some silly linear term at the end, like

$$a_{n+1} = 3a_n - 1$$

then you can simply write

$$a_{n+1} = 3a_n - 1$$
$$a_{n+2} = 3a_{n+1} - 1$$

and subtracting the two gives

$$a_{n+2} - a_{n+1} = 3a_{n+1} - 3a_n$$
$$a_{n+2} = 4a_{n+1} - 3a_n$$

and we now have a linear recursion that we can solve. The same method can be applied if we have any polynomial at the end.

A Little Bit of That Stuff That We Don't Know

By that, I mean Calculus, since apparently Calculus is not expected to be known by anyone in high school. However, this is blatantly FALSE as even if most problems have a non-calculus solution, some things are much easier with calculus. Example one:

$$\sum_{i=0}^{\infty} \frac{1}{i!}$$

This is just $e^1 = e$. Example two:

$$\sum_{i=0}^{\infty} \frac{(-1)^i}{i!}$$

This is $e^{-1} = \frac{1}{e}$. Example three:

$$\sum_{i=0}^{\infty} \frac{i}{5^i}$$

We can just let $f(x) = \frac{1}{1-x} = \sum_{i=0}^{\infty} x^i$. Then take the derivative, and set x to 5, and cool stuff should happen.

3 Problems

If you are at the beginning lecture, start on problem x. If you are at the advanced lecture, start on problem y, where $y > x$.

1. (Traditional) Compute

$$\sum_{k=2}^{\infty} \frac{1}{n^2 - 1}$$

2. (Traditional) Compute

$$1 + 9 + 25 + 49 + \ldots + 729$$

3. (Traditional) Simplify

$$1 \cdot 3 \cdot 5 \cdot 7 \cdot \ldots \cdot (2n + 1)$$

4. (HMMT 2006) Compute

$$\sum_{k=1}^{\infty} \frac{k^4}{k!}$$

5. (HMMT 2006) Let $a_1 = a_2 = 1$, and let $a_{n+2} = a_{n+1} + a_n$. Find

$$\sum_{n=1}^{\infty} \frac{a_n}{4^{n+1}}$$

6. (Tricky) Find the limit as m goes to infinity of

$$\sum_{i=0}^{m} \frac{i^n}{m^n}$$

7. (HMMT 2006) Compute the value of

$$\sum_{n=2}^{\infty} \frac{n^4 + 3n^2 + 10n + 10}{2^n \cdot (n_4^4)}$$

8. (HMMT 2006) Compute the value of

$$\sum_{n=0}^{\infty} \frac{n}{n^4 + n^2 + 1}$$

9. (HMMT 2001) Evaluate $\displaystyle\sum_{n=0}^{\infty} \cot^{-} 1(n^2 + n + 1)$.

10. (HMMT 2002) Determine the value of the sum

$$\frac{3}{1^2 \cdot 2^2} + \frac{5}{2^2 \cdot 3^2} + \frac{7}{3^2 \cdot 4^2} + \ldots + \frac{29}{14^2 \cdot 15^2}$$

11. (HMMT 2005) Compute

$$\sum_{k=0}^{\infty} \frac{4}{(4k)!}$$

12. (HMMT 2004) Find the positive constant c_0 such that the series

$$\sum_{n=0}^{\infty} \frac{n!}{(cn)^n}$$

converges for $c > c_0$ and diverges for $0 < c < c_0$.