Generating Functions

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> A generating function is a clothesline on which we hang a sequence of numbers up for display.
>
> –Herbert Wilf, *Generatingfunctionology*

Generating function basics

A (one-variable) *generating function* for a sequence $a_0, a_1, a_2, \ldots$ is the *formal power series* $a_0 + a_1 x + a_2 x^2 + \cdots$. Generating functions are useful for solving recurrences, counting certain combinatorial objects, and even just finding a nice formula for the generating function itself.

The classic example of a generating function identity is

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots .$$

Ordinarily, we think of this as the geometric series formula, which converges for $|x| < 1$, but in the world of generating functions we are more concerned with the coefficients of the series than with the values of x.

Definition. A **formal power series** over the variable x is simply an expression of the form $c_0 + c_1 x + c_2 x^2 + \cdots = \sum_{i=0}^{\infty} c_i x^i$ where each c_i is a complex number.[1]

We define addition and multiplication of formal power series as follows:

$$\sum_{i=0}^{\infty} a_i x^i + \sum_{i=0}^{\infty} b_i x^i = \sum_{i=0}^{\infty} (a_i + b_i) x^i$$

$$\left(\sum_{i=0}^{\infty} a_i x^i\right) \cdot \left(\sum_{i=0}^{\infty} a_i x^i\right) = \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} a_i b_{n-i}\right) x^n$$

Exercise. Use the definition of formal power series multiplication to prove that the identity

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

holds when thought of as formal power series over x.

We also define the *derivative* of a formal power series by

$$\frac{d}{dx}\left(\sum_{n=0}^{\infty} a_n x^n\right) = \sum_{i=1}^{\infty} n a_n x^{n-1}.$$

The derivative behaves exactly like ordinary derivatives from calculus.

[1] We will usually, however, encounter the case where each c_i is an integer.

Exercise. Let $F(x)$ and $G(x)$ be generating functions (formal power series over x). Use the rules for formal manipulation of power series to prove that the derivative satisfies:

- $\frac{d}{dx}(F(x)+G(x)) = \frac{d}{dx}F(x) + \frac{d}{dx}G(x)$
- $\frac{d}{dx}(F(x)G(x)) = G(x) \cdot \frac{d}{dx}F(x) + F(x) \cdot \frac{d}{dx}G(x)$
- $\frac{d}{dx}(F(x)/G(x)) = \left(G(x)\frac{d}{dx}F(x) - F(x)\frac{d}{dx}G(x)\right)/G(x)^2$.

Taking derivatives enables us to find new generating functions from old ones. For instance, taking the derivative of both sides of the geometric series formula yields

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots .$$

This, in fact, holds for $|x| < 1$, which is somewhat harder to show.

Using generating functions to solve recurrences

Suppose we wish to find an explicit formula for the nth Fibonnacci number F_n, where $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$. Consider the generating function

$$G(x) = \sum_{n=0}^{\infty} F_n x^n.$$

We can manipulate this to take advantage of the recursion: we have

$$G(x) - xG(x) - x^2G(x) = F_0 + F_1x - F_0x + \sum_{n=2}^{\infty}(F_n - F_{n-1} - F_{n-2})x^n = x.$$

Thus $G(x) = x/(1-x-x^2)$. Using partial fractions and expanding each term as a geometric series, we find that

$$G(x) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{5}} \left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n \right) x^n,$$

and so $F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$.

In general, the generating function for any linear recurrence of the form

$$A_n = c_1A_{n-1} + c_2A_{n-2} + \cdots + c_kA_{n-k}$$

can be written as a rational function of x, obtained by multiplying it by the characteristic polynomial

$$1 - c_1x - c_2x^2 - \cdots - c_kx^k$$

and using the initial conditions to solve for the generating function. We can then use partial fraction decomposition and the geometric series formula to find an explicit formula for the nth coefficient.

Exponential generating functions

The *exponential generating function* for the sequence $\{a_i\}_{i=0}^{\infty}$ is the series $\sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$. Their product behaves somewhat differently from that of ordinary generating functions:

$$\left(\sum_{n=0}^{\infty} \frac{a_n}{n!} x^n\right) \cdot \left(\sum_{n=0}^{\infty} \frac{b_n}{n!} x^n\right) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}\right) x^n$$

We define e^x, $\sin(x)$, and $\cos(x)$ to be the exponential generating functions shown below. Interpreting these as formal power series, they satisfy all the trigonometric identities one would expect.

- $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$
- $\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} x^{2n+1}$
- $\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} x^{2n}$

Exercise. Using the definition of e^x as a formal power series, show that $e^x e^y = e^{x+y}$.

Problems

1. (From Andy Niedermaier's 2009 MOP handout.) Find the generating function for each of the following sequences, and use it to find an explicit formula for a_n:
   - $a_0 = 1$, $a_1 = 5$, $a_{n+2} = 4a_{n+1} - 3a_n$
   - $a_0 = 1$, $a_1 = 6$, $a_{n+2} = 4a_{n+1} - 4a_n$
   - $a_0 = 0$, $a_1 = 5$, $a_2 = 47$, $a_{n+3} = 31a_{n+1} + 30a_n$
   - $a_0 = 0$, $a_1 = 1$, $a_{n+2} = a_{n+1} + 2a_n + 1$
   - $a_0 = a_1 = 1$, $a_{n+2} = a_{n+1} + 6a_n + n$

2. Let D_n be the number of *derangements* of n, that is, the number of permutations ϕ of $\{1, 2, \ldots, n\}$ such that $\phi(i) \neq i$ for any $1 \leq i \leq n$. Find a closed form expression for the exponential generating function of D_n, and use it to find a formula for D_n (the formula may include a finite sum.)

3. (David Savitt.) Let P_n be the number of ways a $2 \times 2 \times n$ pillar can be built out of $2 \times 1 \times 1$ bricks. Find a closed form expression for the generating function $\sum_{n=0}^{\infty} P_n x^n$.

4. Let C_n denote the nth *Catalan number*, the number of ways of parenthesizing the addition of n ones. Find a closed form expression for the generating function $C(x) = \sum_{n=0}^{\infty} C_n x^n$, and use it to show that $C_n = \frac{1}{n+1}\binom{2n}{n}$.

5. Prove that
$$\sum_{\substack{i+j=n \\ i,j \geq 0}} \binom{2i}{i}\binom{2j}{j} = 4^n.$$

6. (High school mathematics 1994/1, Qihong Xie.) Find the number of subsets of $\{1, 2, \ldots, 2000\}$, the sum of whose elements is divisible by 5.

7. (China 1996.) Let n be a positive integer. ☺ Find the number of polynomials $P(x)$ with coefficients in $\{0, 1, 2, 3\}$ such that $P(2) = n$.

8. Suppose that a finite number of arithmetic sequences $a_1 + b_1 n$, $a_2 + b_2 n$, …, $a_k + b_k n$ partition the positive integers into disjoint subsets. That is, the sequences pairwise disjoint and every positive integer is in one of the k sequences. If $b_1 \geq b_2 \geq \cdots \geq b_k$, show that $b_1 = b_2$.

9. Suppose that a finite number of arithmetic sequences $a_1 + b_1 n$, $a_2 + b_2 n$, …, $a_k + b_k n$ partition the positive integers into disjoint subsets. Show that $\sum_{i=1}^{k} \frac{a_i}{b_i} = \frac{k+1}{2}$.

10. (Richard Stanley.) Compute

$$\sum_{a_1+a_2+\cdots+a_k=n, k\geq 1} a_1 a_2 \cdots a_k.$$

11. (IMO 1995.) Let p be an odd prime. Find the number of subsets A of $\{1, 2, \ldots, 2p\}$ such that

 - A has exactly p elements, and
 - the sum of all the elements of A is divisible by p.

Some problems on partitions

1. Prove that the number of partitions of a positive integer n into distinct parts is equal to the number of partitions of n into odd parts.

2. Prove that the number of partitions of an integer n into distinct odd parts has the same parity as the total number of partitions of n.

3. Let $p(n)$ be the number of partitions of n, that is, the number of sequences $(\lambda_1, \ldots, \lambda_k)$ with $\lambda_1 \geq \cdots \geq \lambda_k$ whose sum is n. Prove that

$$\sum_{n=0}^{\infty} p(n) x^n = \left(\frac{1}{1-x}\right)\left(\frac{1}{1-x^2}\right)\left(\frac{1}{1-x^3}\right)\cdots.$$

4. Let $p(n, r)$ denote the number of partitions $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ of n (written in nonincreasing order) such that $\lambda_1 - k = r$. Let $R(z, q) = \sum_{n,r} p(n, r) z^r q^n$ be its *two-variable generating function.* Prove that

$$R(z, q) = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{\prod_{k=1}^{n}(1 - zq^k)(1 - z^{-1}q^k)}.$$

5. Prove *Euler's Pentagonal Number Theorem*, that

$$(1-x)(1-x^2)(1-x^3)\cdots = \sum_{n=-\infty}^{\infty} (-1)^n x^{n(3n-1)/2}$$

6. Prove that $p(5n + 4) \equiv 0 \pmod 5$. You may find the following identity useful:

$$\left((1-x)(1-x^2)(1-x^3)\cdots\right)^3 = \sum_{n=0}^{\infty} (-1)^n (2n+1) x^{n(n+1)/2}.$$

7. Let $Q(n)$ be the number of partitions of n into distinct parts, that is, the number of sequences $(\lambda_1, \dots, \lambda_k)$ with $\lambda_1 > \cdots > \lambda_k$ whose sum is n. Prove that

$$\sum_{n=0}^{\infty} Q(n)x^n = (1+x)(1+x^2)(1+x^3)\cdots.$$

8. Let $Q(n,r)$ be the number of partitions of n into distinct (decreasing) parts $(\lambda_1, \dots, \lambda_k)$ such that $\lambda_1 - k = r$. Let $G(z,q) = \sum_{n,r} Q(n,r)z^r q^n$ be its two-variable generating function. Prove that

$$G(z,q) = 1 + \sum_{s=1}^{\infty} \frac{q^{s(s+1)/2}}{\prod_{k=1}^{s}(1-zq^k)}.$$