Auckland Mathematical Olympiad 2009

Division 1

Questions

1. Johnny has saved

$$403^5 - 402^2 \cdot (403^3 + 2 \cdot 403^2 + 3 \cdot 403 + 4)$$

dollars to buy a new playstation whose cost is 2009.99 dollars. Has Johnny saved enough money?

2. Is it possible to write the number $1^2 + 2^2 + 3^2 + \cdots + 12^2$ as a sum of 11 distinct squares?

3. Two quadratic polynomials

$$f_1(x) = x^2 + a_1 x + b_1, \qquad f_2(x) = x^2 + a_2 x + b_2$$

have roots x_0, x_1 and x_0, x_2, respectively, with $x_1 \neq x_2$. Find the roots of the quadratic

$$f(x) = x^2 + \frac{a_1 + a_2}{2} x + \frac{b_1 + b_2}{2}.$$

4. Two players alternate making moves in a game on a 1×100 strip consisting of 100 unit squares. In one move the first player colours any four consecutive squares. The second player's move is to colour any three consecutive squares. It is not permitted to colour the same square twice. The player who cannot make his next move loses the game. Who has a winning strategy?

5. In a convex quadrilateral each vertex is connected by two line segments with the midpoints of the two opposite sides. In total eight line segments are drawn. Suppose that seven of them have the same length a. Prove that the length of the remaining segment is also a.

More maths olympiad problems and articles can be found on the New Zealand Maths Olympiad's website http://www.mathsolympiad.org.nz.

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Division 2

Questions

6. Find all triples of non-zero distinct integers a, b, c such that they can be written in an arithmetic progression and their reciprocals $1/a$, $1/b$, $1/c$ can be written in an arithmetic progression too.

7. Find all functions f for which

$$f(x-y) = f(x) + f(y) - 2xy$$

for all real numbers x and y.

8. What is the smallest positive integer n, such that there exist positive integers a and b, with b obtained from a by a rearrangement of its digits, so that

$$a - b = \underbrace{11\dots1}_{n}?$$

9. Through the incentre I of triangle ABC a straight line is drawn intersecting AB and BC at points M and N, respectively, in such a way that the triangle BMN is acute-angled. On the side AC the points K and L are chosen such that $\angle ILA = \angle IMB$ and $\angle IKC = \angle INB$. Prove that $AC = AM + KL + CN$.

10. Two players alternate making moves in a game on a 3×3 square table. In one move a player writes a number in a square of this table. The available numbers are $1, 2, 3, 4, 5, 6, 7, 8, 9$ and the same number cannot be written in the table twice so at the end of the game all of them will be written in the table. The first player wins if at the end of the game the sum of numbers in one of the rows or in one of the columns is equal to 14 and loses otherwise. Does the first player have a winning strategy?

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