TJUSAMO 2011 – Introduction to Graph Theory

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1 Terminology

Definition. *A **graph** $G = (V, E)$ comprises a set V of **vertices** and a set $E \subset V \times V$ of **edges**[1].*

Intuitively, we think of vertices as a set of points and edges linking the pairs of vertices they correspond to. If there is at most one edge between any two vertices, then we say that a graph is *simple*.

Important Graph-Theoretic Terminology

- Two vertices are *adjacent* if there is an edge that has them as endpoints.
- We say that an edge e is incident to a vertex v if v is an endpoint of e.
- A graph is *bipartite* if the vertex set can be partitioned into two sets $V_1 \cup V_2$ such that edges only run between V_1 and V_2.
- The *chromatic number* of a graph is the minimum number of colors needed to color the vertices without giving the same color to any two adjacent vertices.
- A *clique* or *complete graph* on n vertices, denoted K_n, is the simple n-vertex graph with all possible edges.
- A graph is *connected* if there is a path between every pair of distinct vertices.
- A *path* is a sequence of distinct, pairwise-adjacent vertices.
- A *walk* is a sequence of not-necessarily-distinct, pairwise-adjacent vertices.
- A *cycle* is a path for which the first and last vertices are actually adjacent.
- A *tree* is a connected graph with no cycles.
- A *forest* is collection of trees (thus, a not-necessarily-connected graph with no cycles).
- The degree $d(v)$ of a vertex v is the number of edges that are incident to v.
- An *Eulerian circuit* is a walk that traverses every edge exactly once, and returns to its starting point.
- A *Hamiltonian path* is a path that includes every vertex. A *Hamiltonian cycle* is a cycle that includes every vertex.
- A graph is *planar* if it is possible to draw it in the plane without any crossing edges.

[1] Thanks to Po-Shen Loh, who wrote an excellent MOP lecture on this topic.

2 The Basics

1. What is the maximum number of edges in a graph with n vertices? (alternately: what is the number of edges in K_n?)
2. Prove that tree contains a vertex of degree exactly 1 called a *leaf.*
3. Prove that a graph with V vertices and at least V edges contains a cycle. (alternately: prove that a graph is a tree if and only if it contains $V - 1$ vertices)
4. A *spanning tree* of a graph is a subgraph which contains the same vertices as the graph and is also a tree. Prove that every finite connected graph has a spanning tree.
5. Prove that a graph is bipartite if and only if it has no odd cycles.
6. Prove that every connected graph with all degrees even has an Eulerian circuit.
7. Let G be a tree, and let Δ be its maximum degree. Show that G has at least Δ leaves.
8. Let $d \geq 2$ be the maximum degree of a graph G. Prove that the vertices of G can be colored in $d + 1$ colors such that no two adjacent vertices have the same color.
9. Let $d \geq 2$ be the minimum degree of G. Prove that G contains a cycle of length greater than or equal to $d + 1$.

3 Problems

10. Let n be a positive integer. n people take part in a certain party. For any pair of the participants, either the two are acquainted with each other or they are not. What is the maximum possible number of the pairs for which the two are not acquainted but have a common acquaintance among the participants?
11. Let n be a positive integer. Prove that a graph with $2n$ vertices and $n^2 + 1$ edges contains a triangle.
12. A connected graph with n vertices has the property that it is connected after any edge is removed from it. Find the minimum number of edges in the graph.
13. In a community of more than six people, each member exchanges letters with precisely three other members of the community. Prove that the community can be divided into two nonempty groups so that each member exchanges letters with at least two members of the group he belongs to.
14. There are 1000 cities in the country of Euleria, and some pairs of cities are linked by dirt roads. It is possible to get from any city to any other city by traveling along these roads. Prove that the government of Euleria may pave some of the roads so that every city will have an odd number of paved roads leading out of it.
15. The following operation is allowed on a finite graph: Choose an arbitrary cycle of length 4 (if there is any), choose an arbitrary edge in that cycle, and delete it from the graph. For a fixed integer $n \geq 4$, find the least number of edges of a graph that can be obtained by repeated applications of this operation from the complete graph on n vertices (where each pair of vertices is joined by an edge).