

Polynomials

Alison Miller

June 21, 2011

Things to Know:

(This list is taken from handouts of Melanie Wood, as are some of the problems.)

- Factorization/Coefficients/Viete's identities: Let $p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_0$ be a polynomial. Then it can be written as $a_n(x-r_1)(x-r_2)\cdots(x-r_n)$ uniquely up to reordering the factors, where the r_i are the roots of p . Comparing coefficients, the k th symmetric sum of the roots is equal to $(-1)^k a_{n-k}/a_n$.
- Intermediate Value Theorem (polynomials are continuous functions).
- Division algorithm: Let $a(x)$ and $b(x)$ be polynomials. There exist unique polynomials $q(x)$ and $r(x)$ with $\deg(r) < \deg(b)$ and such that $a(x) = b(x)q(x) + r(x)$.
- Rational root theorem: Let $p(x) = a_n x^n + \cdots + a_0$ be a polynomial with integer coefficients. If $p(x)$ has a rational root $\frac{r}{s}$ with $r, s \in \mathbb{Z}$ and $(r, s) = 1$ then $r \mid a_0$ and $s \mid a_n$.
- A polynomial with of degree n has at most n roots.
- Gauss's Lemma: If $p(x) \in \mathbb{Z}[x]$ factors in $\mathbb{Q}[x]$, then it factors in $\mathbb{Z}[x]$.
- Eisenstein's Criterion: Let $p(x) = a_n x^n + \cdots + a_0$ be a polynomial with real coefficients, and let q be a prime that divides a_0, a_1, \dots, a_{n-1} but not a_n . If $q^2 \nmid a_0$ then $p(x)$ is irreducible over the integers (and thus also over the rationals, by Gauss's lemma).
- Descartes' Rule of Signs: Let $p(x) = a_n x^n + \cdots + a_0$ be a polynomial with real coefficients, and let V be the number of sign changes in the coefficient list a_n, \dots, a_0 where we ignore zero coefficients. If $p(x)$ has N positive roots (counted by multiplicity) then $N = V - 2k$ for some non-negative integer k . (Use this for $p(-x)$ to get the statement for negative roots of $p(x)$).
- Rolle's Theorem: Let $p(x)$ be a polynomial with real coefficients. Between any two zeroes of $p(x)$ lies a zero of $p'(x)$.
- Lagrange Interpolation: A polynomial of degree at most n with $p(a_k) = b_k$, $0 \leq k \leq n$ is given by

$$p(x) = \sum_{k=0}^n \frac{b_k(x-a_0)(x-a_1)\cdots(x-a_{k-1})(x-a_{k+1})\cdots(x-a_n)}{(a-a_0)(a-a_1)\cdots(a-a_{k-1})(a-a_{k+1})\cdots(a-a_n)}.$$

- Newtonian interpolation (finite differences): A polynomial p of degree at most n with $p(k) = b_k$, $0 \leq k \leq n$, is given by

$$p(x) = \sum_{k=0}^n \Delta^k p(0) \binom{x}{k},$$

where $\Delta p(x) = p(x+1) - p(x)$ and $\Delta^k p(x) = \Delta(\Delta^{k-1} p(x))$.

- Chebyshev Polynomials: The n th Chebyshev polynomial is defined so that $T_n(\cos \theta) = \cos(n\theta)$. They satisfy the recurrence

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \quad n \geq 1$$

with $T_0(x) = 1$ and $T_1(x) = x$. (There are also Chebyshev polynomials of the second kind which satisfy $U_n(\cos \theta) = \frac{\sin(n+1)\theta}{\sin \theta}$ and have a recurrence of the same form. These are much less common, though.)

Examples

1. Let A_1, A_2, \dots, A_n be points in the plane, and \overline{BC} a segment of length 2. Prove that there exists a point M on \overline{BC} such that $MA_1 \cdot MA_2 \cdots MA_n \geq \frac{1}{2^{n-1}}$.
2. Show that $x^{p-1} + x^{p-2} + \cdots + x + 1$ is irreducible.

General Problems

- 3 (St. Petersburg). Find all polynomials $P(x, y)$ in two variables such that for any x and y , $P(x+y, y-x) = P(x, y)$.
- 4 (Putnam). Let f be a polynomial with positive integer coefficients. Prove that if n is a positive integer, then $f(n)$ divides $f(f(n) + 1)$ if and only if $n = 1$.
- 5 (IMO 1993). Let $n > 1$ be an integer, and let $f(x) = x^n + 5x^{n-1} + 3$. Prove that there do not exist polynomials $g(x), h(x)$ each having integer coefficients such that $f(x) = g(x)h(x)$.
6. A point in the Cartesian coordinate plane is called a *mixed point* if one of its coordinates is rational and the other is irrational. Find all polynomials with real coefficients whose graphs do not contain any mixed point.
7. Determine all polynomials $P(x)$ with real coefficients such that $P(x)^2 + P(1/x)^2 = P(x)^2 P(1/x)^2$ for all $x \neq 0$.
- 8 (MOP '08). Let P_n be a polynomial of degree n with real coefficients, and let t be a real number with $t \geq 3$. For integers $n \geq 0$, show that

$$\max_{0 \leq k \leq n+1} |t^n - P_n(k)| \geq 1$$

- 9 (Putnam). Let $P(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_0$ be a polynomial with integer coefficients. Suppose that r is a rational number such that $P(r) = 0$. Show that the n numbers

$$\begin{aligned} &c_n r, c_n r^2 + c_{n-1} r, c_n r^3 + c_{n-1} r^2 + c_{n-2} r, \\ &\dots, c_n r^n + c_{n-1} r^{n-1} + \cdots + c_1 r \end{aligned}$$

are integers.

10. Let p, q, r be polynomials with real coefficients, such that at least one of the polynomials has degree 2 and at least one of the polynomials has degree 3. Assume that $p^2 + q^2 = r^2$: Show that at least one of the polynomials both has degree 3 and has 3 (not necessarily distinct) real roots.

11 (USAMO 2002). Prove that any monic polynomial (a polynomial with leading coefficient 1) of degree n with real coefficients is the average of two monic polynomials of degree n with n real roots.

12 (MOP 01). Let $P(x)$ be a real-valued polynomial with $P(n) = P(0)$. Show that there exist at least n distinct (unordered) pairs of real numbers $\{x, y\}$ such that $x - y$ is a real number and $P(x) = P(y)$.

What do these problems have in common?

13 (Vietnam). Find all polynomials $P(x)$ with integer coefficients such that the polynomial $Q(x) = (x^2 + 6x + 10) \cdot P^2(x) - 1$ is the square of a polynomial with integer coefficients.

14. Let $\{P_n(x)\}_{n=1}^\infty$ be a sequence of polynomials such that $P_1(x) = x^2 - 1$, $P_2(x) = 2x^2(x^2 - 1)$, and

$$P_{n-1}(x)P_{n+1}(x) = (P_n(x))^2 - (x^2 - 1)^2.$$

Let S_n denote the sum of the absolute values of the coefficients of $P_n(x)$. For each positive integer n find the largest non-negative integer k_n such that 2^k divides S_n .

15 (MOP '03). Let k be a positive integer. Prove that $\sqrt{k+1} - \sqrt{k}$ is not the real part of a complex number z such that $z^n = 1$ for some positive integer n .

16 (IMO Shortlist 2003). The sequence a_0, a_1, a_2, \dots is defined as follows: $a_0 = 2$, $a_{k+1} = 2a_k^2 - 1$ for $k \geq 0$. Prove that if an odd prime p divides a_n , then 2^{n+3} divides $p^2 - 1$.

17 (IMO Shortlist 2003). Let \mathbb{R}^+ be the set of all positive real numbers. Find all functions $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ that satisfy the following conditions:

- $f(xyz) + f(x) + f(y) + f(z) = f(\sqrt{xy})f(\sqrt{yz})f(\sqrt{zx})$ for all $x, y, z \in \mathbb{R}^+$;
- $f(x) < f(y)$ for all $1 \leq x < y$.