Demystifying Barycentric Coordinates

Cody Johnson

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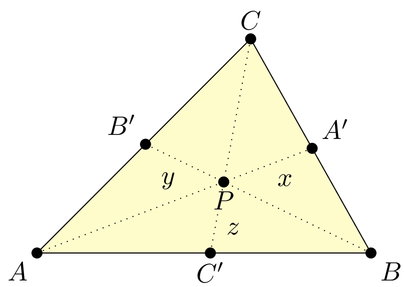
1 Introduction

Consider any triangle $\triangle ABC$ and a point P not necessarily within $\triangle ABC$. Let A', B', and C' be the intersections of lines AP, BP, and CP with BC, AC, and AB, respectively, as shown in the figure below. Now let $x = PA'$, $y = PB'$, and $z = PC'$, as also shown in the figure.



We write the *barycentric coordinate* of P as $P = (x : y : z)$, but since barycentric coordinates are homogeneous, we usually use *normalized barycentric coordinates*. Normalized barycentric coordinates are in the form of (a, b, c) such that $a + b + c = 1$. To convert from general barycentric coordinates to normalized barycentric coordinates, we use $(a, b, c) = \left(\frac{x}{x+y+z}, \frac{y}{x+y+z}, \frac{z}{x+y+z}\right)$. Note that if any one of a, b, c is negative, then the point is not within the triangle.

Exercise 1. Find the normalized barycentric coordinates of A, B, and C.

Solution 1. Clearly, if point P in the diagram above is point A, then $y = z = 0$, so the ratio is $(x : 0 : 0)$. Normalizing, we get $A = (1, 0, 0)$. Similarly, we get $B = (0, 1, 0)$ and $C = (0, 0, 1)$.

Theorem 1. *Given points $P = (a_1, b_1, c_1)$, $Q = (a_2, b_2, c_2)$, and $R = (a_3, b_3, c_3)$, the signed area*

$$[PQR] = [ABC] \cdot \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$$

Corollary 1. *Three points $P = (a_1, b_1, c_1)$, $Q = (a_2, b_2, c_2)$, and $R = (a_3, b_3, c_3)$ are collinear if and only if*

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

Exercise 2. Find the equations of lines AB, AC, and BC.

Solution 2. If a point $P(a, b, c)$ is on AB, then A, B, and P are collinear. Hence,

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ a & b & c \end{vmatrix} = c = 0$$

Thus, the equation of line AB is $c = 0$. Similarly, the equations of lines AC and BC are $b = 0$ and $a = 0$, respectively.

Exercise 3. Find the coordinates of the midpoints $M_{\overline{AB}}$, $M_{\overline{AC}}$, and $M_{\overline{BC}}$ of segments $\overline{AB}$, $\overline{AC}$, and $\overline{BC}$.

Solution 3. Since the midpoint of segment $\overline{AB}$ is on line AB, it follows that $c = 0$. Also, the distances $x = y$, so the general barycentric coordinate is $M_{\overline{AB}} = (x : x : 0)$. Normalizing, we get $M_{\overline{AB}} = \left(\frac{1}{2}, \frac{1}{2}, 0\right)$. Similarly, we get $M_{\overline{AC}} = \left(\frac{1}{2}, 0\frac{1}{2}\right)$ and $M_{\overline{BC}} = \left(0, \frac{1}{2}, \frac{1}{2}\right)$.

Exercise 4. Find the coordinates of the trisectors T_1 and T_2 of segment $\overline{AB}$, with $AT_1 < AT_2$.

Solution 4. Since the trisector of segment $\overline{AB}$ is on line AB, it follows that $c = 0$. Also, the distances $2y = x$, so the general barycentric coordinate is $T_1 = (2y : y : 0)$. Normalizing, we get $T_1 = \left(\frac{2}{3}, \frac{1}{3}, 0\right)$. Similarly, we get $T_2 = (\frac{1}{3}, \frac{2}{3}, 0)$.

Exercise 5 (Ceva's Theorem). Prove that cevians $\overline{AA'}$, $\overline{BB'}$, and $\overline{CC'}$ are concurrent if and only if

$$\frac{\overline{BA'}}{\overline{A'C}} \frac{\overline{CB'}}{\overline{B'A}} \frac{\overline{AC'}}{\overline{C'B}} = 1$$

Proof. Let the intersection of the cevians be $P = (a, b, c)$ and $A' = (0, d, 1-d)$, $B' = (1-e, 0, e)$, and $C' = (f, 1-f, 0)$. We have from collinearity

$$\begin{vmatrix} 1 & 0 & 0 \\ 0 & d & 1-d \\ a & b & c \end{vmatrix} = dc - b(1-d) = 0 \to \frac{c}{b} = \frac{1-d}{d},$$

$$\begin{vmatrix} 0 & 1 & 0 \\ 1-e & 0 & e \\ a & b & c \end{vmatrix} = ea - (1-e)c = 0 \to \frac{a}{c} = \frac{1-e}{e}, \text{ and}$$

$$\begin{vmatrix} 0 & 0 & 1 \\ f & 1-f & 0 \\ a & b & c \end{vmatrix} = fb - (1-f)a = 0 \to \frac{b}{a} = \frac{1-f}{f}$$

Multiplying these three together yields

$$\frac{1-d}{d} \frac{1-e}{e} \frac{1-f}{f} = 1$$

which is equivalent to Ceva's Theorem from homogeneity. □