Graph Theory (Teacher's Edition)

Gabriel D. Carroll

MOP 2010

Most of what I know about graph theory I learned from Kiran Kedlaya's classes at MOP. I've also made use of Bollobás, *Modern Graph Theory*, in drafting this handout. Most of the problems not credited to contests are from that book, though a couple are my own. (Bollobás also has a more introductory text.)

I haven't tried to go through graph theory in systematic detail, because (a) it's huge and (b) you probably know a lot of what I have to say already. Instead, I'll present four lists that you can use for reference: a list of common techniques for Olympiad problem-solving; a list of graph-theoretic concepts to be comfortable with; a list of good results to know; and a list of problems to practice on. Of course, feel free to add to the lists.

1 Problem-solving techniques

- Use induction
- Use the Handshake Lemma or other parity arguments
- Show that there's a cycle
- Count things cleverly (or stupidly) and pigeonhole
- Assume the graph is a tree (a general technique for proving properties that are stable under adding an extra edge)
- Look at the complement, or (for planar graphs) the dual
- Don't be afraid of case analysis
- Look at extremes (e.g. smallest-degree vertex)
- Notice when a problem that doesn't look like graph theory actually is graph theory

2 Concepts

- Subgraphs; induced subgraphs
- Degree; regular graphs
- Trees; forests
- Cycles
- Spanning trees
- Bipartite (and k-partite) graphs
- Vertex-colorings and edge-colorings
- Rooted trees; parents, children, leaves
- Paths; walks; trails

  Trail: all edges distinct; path: all edges distinct and all vertices distinct; sometimes "circuit" used for a closed trail (distinct from a cycle)
- Connectedness and components
- Complete graphs and complete k-partite graphs
- Distance between two vertices
- Eulerian paths and cycles; Hamiltonian paths and cycles
- Matchings
- Directed graphs; orientations of graphs; outdegree and indegree; tournaments
- Planar graphs; planar duals
- Minors and subdivisions
- Multigraphs; weighted graphs; hypergraphs

  Minor: graph obtained by repeatedly contracting two vertices together and then deleting redundant edges; subdivision: graph obtained by subdividing edges
- Automorphisms

3 Theorems (and other facts)

- Bipartite graphs: the vertices of a graph can be colored in two colors so that adjacent vertices always have different colors iff there are no cycles of odd length.

- Components, cycles and trees: a connected graph on n vertices has at least $n-1$ edges, with equality iff it is a tree. If a directed graph has at least one edge out of every vertex, or at least one edge into every vertex, then it has a directed cycle.

- Dirac's Theorem: A graph with n vertices, where each vertex has degree $\geq n/2$, has a Hamiltonian cycle.

  Proof: suppose not. The maximal path length is longer than the maximal cycle length (since we can take a cycle and then add one more vertex off the cycle). Now consider a maximal path. By the above, the first and last vertices aren't adjacent. Also they can't be adjacent to successive vertices along the path (else we get a cycle), and they can't both be adjacent to some common vertex off the path (else a cycle). Pigeonhole.

- Euler Characteristic: in a planar graph with F faces, E edges, and V vertices, the relation $F-E+V=2$ holds.

- Eulerian path: a finite connected graph has a trail that passes along every edge exactly once iff there are at most two vertices of odd degree. It has a cycle passing along every edge once iff there are no vertices of odd degree.

- Four-Color Theorem: a planar graph can be vertex-colored in four colors so that any two adjacent vertices have different colors.

- Hall's Marriage Lemma: Consider a bipartite graph with parts V_1 and V_2. Suppose that for every $S \subseteq V_1$, there are at least $|S|$ vertices in V_2 each adjacent to some vertex in S. Then there exists a one-one function $f : V_1 \to V_2$ such that v is adjacent to $f(v)$ for all v.

  Proof: maxflow-mincut with one source, capacity 1 to each vertex of V_1, unlimited capacities from V_1 to V_2, and capacity 1 from each vertex of V_2 to one sink.

  Alternative proof: induction. Say we've matched up a bunch of elements of V_1 and want to match up one more, v. Consider a digraphwith edges from V_1 to V_2 according to the original graph, plus reverse edges corresponding to the matching formed so far. Let V_1' be the set of vertices in V_1 reachable from v in this graph. Then at least $|V_1'|$ vertices in V_2 are reachable from v. So some vertex not already matched is reachable. By alternating edges along the relevant path from v, we get the induction step.

- Handshake Lemma: in any finite graph, the number of vertices of odd degree is even.

- Kuratowski's Theorem: a graph is planar iff it has no subgraph isomorphic to a subdivision of K_5 or $K_{3,3}$.

- Maxcut-Minflow Theorem (Ford-Fulkerson Theorem): Consider a directed graph where each edge e has a nonnegative "capacity" c_e. A *flow* from vertex v to vertex w is an assignment of numbers x_e to each edge c_e, with $0 \leq x_e \leq c_e$, such that the quantity

  $$\Delta_u = \sum_{u=tail(e)} x_e - \sum_{u=head(e)} x_e$$

  is zero for all $u \neq v, w$. The *value* of the flow is $\Delta(v)$. A *cut* is a set S of vertices containing v but not w, and the *value* of the cut is the sum of the capacities of all edges from S to its complement.

  Then, the maximum value over all flows equals the minimum value over all cuts. (Thus, the maximum flow value has the property that there's a cut that "proves" its maximality.)

  Proof: A maximal flow exists since it's the solution to a linear programming problem. Now given this flow, recursively define S as follows: if $x \in S$, and some edge (x, y) has more capacity than its flow, or there is any net flow along (y, x), then include y in S. Check this gives a cut with value equal to the flow's value.

- Minimal spanning trees: given a finite connected graph on to which every edge has been assigned a "cost," we can construct a spanning tree of lowest total cost using the greedy algorithm. That is: first choose the cheapest edge; then, given a bunch of edges, consider all the remaining edges that can be added without forming a cycle, and add the cheapest one. Keep going until no more edges can be added.

- Ramsey's Theorem (finite version): For any numbers $n_1, \ldots, n_r$, there exists N such that, whenever a complete graph on at least N vertices has its edges colored in r colors, there is some i such that there is a complete subgraph of order n_i, all colored in color i. (This extends to hypergraphs.)

- Ramsey's Theorem (infinite version): Whenever a complete graph on infinitely many vertices has its edges colored in finitely many colors, there is an infinite complete subgraph that has all its edges of the same color. (This extends to hypergraphs.)

- Turán's Theorem: for given $n \geq k$, the maximum number of edges that an n-vertex graph can have without containing a complete k-graph is achieved by the Turán graph, which is the complete $(k-1)$-partite graph whose parts' sizes are all $\lfloor n/(k-1) \rfloor$ or $\lceil n/(k-1) \rceil$. This graph is the only one that achieves the maximum.

  Proof: various ways, e.g. suppose a graph has this number of edges but no K_k; we'll show it has to be the Turan graph (which will prove the assertion). Remove a vertex of minimal degree, which is $\leq$ the minimal degree of the Turan graph. Then by induction on n, the remaining graph has to be an $(n-1, k)$ Turan graph. The removed vertex must be connected to vertices in $k-2$ distinct parts (if it's connected to all $k-1$ parts then we get a K_k), and this unqiuely determines the graph.

- Tutte's Lemma (Unisex Marriage Lemma): A graph G has a perfect matching, i.e. a set of edges such that every vertex is adjacent to exatly one edge, if and only if, for every set of vertices S, the graph $G - S$ has no more than $|S|$ components of odd order.

Some of these theorems are easy to prove. Some are harder. But almost all of them are accessible at the Olympiad level, so if there are any you don't know, try to prove them for practice. The only really hard ones are the four-color theorem (but it's not hard with 4 replaced by 5) and the planar graph theorem (but the "only if" direction is easy).

4 Problems

1. Show that every graph with average degree d contains a subgraph in which every vertex has degree at least $d/2$.

   when a vertex has degree less than $d/2$, remove it, which doesn't decrease the average degree; iterate this

2. If every face of a convex polyhedron is centrally symmetric, prove that at least six of the faces are parallelograms.

3. [FETK] G is a graph on n vertices such that, among any 4 vertices, some three are pairwise adjacent. What's the minimum number of edges of G?

   Solution: $\binom{n-1}{2}$, by ignoring one vertex and making a complete graph on the others. Otherwise, look at the complement. We just want to show every graph with $\geq n$ edges contains either a triangle or two edges with no common vertex. Just look at a cycle.

4. [BMC, 2006] There are 1000 managers in a boring corporate meeting. Each manager has exactly one boss, who may or may not be among the other managers present at the meeting. Each manager earns a strictly lower salary than his boss. A manager is *powerful* if he is the boss of at least four other managers at the meeting. What is the maximum possible number of powerful managers?

   Solution: 249 — construct a rooted tree; each powerful manager uses up 4 edges

5. Prove that in any n-tournament, it is possible to order the vertices $v_1, \ldots, v_n$ so that there is an edge from v_i to v_{i+1} for each i, $1 \leq i < n$. (That is, there's a directed Hamiltonian path.)

6. Let k and p be positive integers, with $p > 2^{k-1}$, p prime, and p congruent to -1 modulo 4. Prove that there exist integers $a_1, \ldots, a_k$, pairwise incongruent modulo p, such that $a_j - a_i$ is congruent to a square modulo p, for all $i < j$.

7. [HMMT, 2003] a people want to share b apples so that they all get equal quantities of apple. Unluckily, $a > b$. Luckily, they have a knife. Prove that at least $a - gcd(a, b)$ cuts are required.

   Solution: make a bipartite graph connecting people to apples they get pieces of; there are at most $gcd(a, b)$ components, so at least $a + b - gcd(a, b)$ edges (pieces of apple).

8. A complete graph on $6n$ vertices has its edges colored red and blue. Prove that we can find n triangles, all of whose vertices are distinct, and with all $3n$ of their edges colored in the same color.

   Solution: Get one triangle by Ramsey. Remove these vertices and induct. We eventually get $2n - 1$ vertex-disjoint triangles, and some n are the same color.

9. [BAMO, 2005] We are given a connected graph on 1000 vertices. Prove that there exists a subgraph in which every vertex has odd degree.

   Solution: symmetric differences of 500 paths, with path i connecting vertices $2i - 1$ and $2i$

10. In a government hierarchy, certain bureaucrats report to certain other bureaucrats. If A reports to B and B reports to C, then C reports to A. Also, no bureaucrat reports to himself. Prove that the bureaucrats may be divided into three disjoint sets X, Y, Z, so that the following condition holds: whenever a bureaucrat A reports to a bureaucrat B, either $A \in X$ and $B \in Y$, or $A \in Y$ and $B \in Z$, or $A \in Z$ and $B \in X$.

11. Prove that every finite graph with an even number of edges has an orientation in which every vertex has even outdegree.

    monovariant — take a random orientation and fix it

12. Given is a spanning tree of a graph G. We are allowed to remove an edge and insert another edge of G so that a new spanning tree is created. Prove that every spanning tree can be reached by a succession of such operations.

    Solution: define the distance between two spanning trees to be the number of edges in one not in the other; use cycles to show that we can always take a distance-reducing step

13. Some pairs of the 100 towns in a country are connected by two-way flights. It is given that one can reach any town from any other by a sequence of flights. Prove that one can fly around the country so as to visit every town, with a total of at most 196 flights.

    Solution: assume a tree; start at the lower-left leaf and travel up and down the tree.

14. Another country contains 2010 cities. Some pairs of cities are linked by roads. Show that the country can be divided into two states S and T so that each state contains 1005 cities, and at least half the roads connect a city in S with a city in T.

    average over all possible divisions into two states of 1004 cities; each edge crosses state boundaries more than half the time

15. Prove that one can write 2^n numbers around a circle, each equal to 0 or 1, so that any string of n 0's and 1's can be obtained by starting somewhere on the circle and reading the next n digits in clockwise order.

    Solution: digraph on the $(n-1)$-words with edges given by successibility; just use an Eulerian tour

16. For every positive integer n, prove that there exists a finite graph with exactly n automorphisms.

17. [Russia, 1997] We start with an $m \times n$ grid, where m and n are odd, and remove one corner square. The rest of the grid is arbitrarily covered with dominoes. Now we are allowed to move the dominoes by successively sliding a domino into the empty square. Prove that by a succession of such moves, we can get any corner square to become empty.

    Solution: a graph whose vertices are odd-coordinate points; edges correspond to dominoes covering these vertices. Want to show the given corner is in the same component as another corner. If not, consider the "boundary" of the component — it stretches from edge to edge and covers an odd number of squares. That can't happen if it's made up of dominoes.

18. [MOP, 2001] Let G be a connected graph on n vertices. You are playing a game against the devil. Each of you colors the vertices of G in black and white, without seeing the other's coloring. Afterwards, you compare colorings. You score a point for each vertex that is the same color in the two colorings. You score an additional point for each pair of adjacent vertices that are the same color (as each other) in the devil's coloring. Prove that you can color the graph so as to be certain of receiving at least $\lfloor n/2 \rfloor$ points.

    can assume a tree; induct on n by taking the lowest leaf, and either it's on a branch of length 1 in which case it has a sibling and we color these two in different colors (and remove them, then apply induction hypothesis); or it's on a branch of length at least 2, in which case we can color the last two nodes in the same color (and remove them, then apply induction hypothesis)

19. [USAMO, 1995] Given is an n-vertex graph having q edges and containing no triangles. Prove that some vertex has the property that, among the vertices not adjacent to it, there are at most $q(1 - 4q/n^2)$ edges.

    Solution: summing $deg(v) + deg(w)$ over adjacent pairs vw gives $\sum deg(v)^2$. For each vw the number of vertices adjacent to neither is $n - deg(v) - deg(w)$ (since no triangles). Summing over all edges gives $qn - \sum deg(v)^2$ sets of three vertices with exactly one edge among them. This is $\leq qn - 4q^2/n$. Now pigeonhole.

20. [Birkhoff-von Neumann theorem] An $n \times n$ matrix of nonnegative numbers has the property that every row and column sums to 1. Prove that the matrix can be written as a weighted average of permutation matrices. (A permutation matrix is one where every entry is 0 or 1, with one 1 in each row and each column.)

21. [Putnam, 2007] Fix a positive integer n. Prove that there is an integer M_n with the following property: if an n-sided polygon is triangulated (using vertices of the original polygon and vertices in its interior), so that each edge of the polygon is an edge of exactly one triangle, and every vertex in the interior of the polygon belongs to at least 6 triangles, then the total number of triangles is at most M_n.

    Solution: Let a_i be the number of edges of the triangulation at vertex i. Euler's formula and some manipulation gives $\sum a_i \leq 4n - 6$. Now set $M_3 = 1$ and $M_n = M_{n-1} + 2n - 3$; we'll show this works by induction. If some $a_i = 2$ then remove that vertex and get a triangulation of an $(n-1)$-gon. Otherwise, since the average a_i is < 4, there must be some sequence of consecutive vertices with $3, 4, 4, \ldots, 4, 3$ values; these correspond to a "strip" of triangles. Remove the strip and get an $(n-1)$-gon tiling, and use induction.

22. [TST, 2009] Let $N > M > 1$ be fixed integers. N people play a chess tournament; each pair plays once, with no draws. It turns out that for each sequence of $M + 1$ distinct players $P_0, P_1, \ldots, P_M$ such that P_{i-1} beat P_i for each $i = 1, \ldots, M$, player P_0 also beat P_M. Prove that the players can be numbered $1, 2, \ldots, N$ in such a way that, whenever $a \geq b + M - 1$, player a beat player b.

    Ricky's solution: ignore the condition $N > M$ (the case $N \leq M$ is easy). Proof by induction on M, then on N for M fixed. $M = 2$ is easy. Otherwise, can assume there's some cycle of M players (otherwise just apply the induction hypothesis for $M - 1$). Then show that everyone either is in the cycle, beat the whole cycle, or was beaten by the whole cycle; now use the induction hypothesis on N to number each piece.

23. [Shapley-Scarf housing markets] There are n people in a city, each owning a different house. They are considering trading houses. Each person has a ranking of the n houses, with no ties: he chooses a favorite house, a second favorite, and so on. Any allocation X of the houses (one to each person) is *blocked* by a nonempty subset S of people if it is possible for the members of S to exchange their houses among themselves such that each member of S gets a house at least as good as he would get from X, and at least one of them gets a strictly better house than from X. Prove that there is exactly one allocation of houses that is not blocked by any set.

24. In an infinite graph, a *one-way infinite Eulerian trail* is defined the way you would expect. Let G be a connected infinite graph with countably many edges and with

just one vertex of odd degree. (So the degrees of the other vertices may be even and finite, or they may be infinite.) Show that G has a one-way infinite Eulerian trail if and only if, for every finite set E of edges, $G - E$ has only one infinite component.

Given an edge starting from the odd vertex, we can use it in a trail starting from the odd vertex: if $G - \{e\}$ has one component, just start with that edge; otherwise, take an Eulerian cycle of the finite component, followed by e. We can keep going in this manner. But how do make sure every edge gets used? Enumerate the edges in an infinite "target" sequence, and at each step, choose the next edge so as to get closer to the lowest-numbered of the edges not yet used.

25. [Thomason's Theorem] Consider a graph in which every vertex has odd degree. Prove that for any given edge, the number of Hamiltonian cycles containing that edge is even.

    Solution: Let xy be the edge. Consider the graph on (ordered) Hamiltonian paths starting at x, where two cycles are "adjacent" if one is obtained from the other by reversing a final segment. The number of neighbors of any cycle equals the degree of the final vertex (in G) minus one. So every vertex has even degree, except the ones corresponding to paths ending in y. Handshake.

26. [Sperner's Lemma] The vertices of an n-dimensional simplex are assigned $n + 1$ different colors. The simplex is triangulated (using points anywhere on the boundary or in the interior of the simplex). The vertices of the triangulation are colored, subject to the constraint that a point on any face of the original simplex must be assigned the same color as one of the vertices of that face. Points on the interior may have any color. Prove that there exists a simplex of the triangulation, all of whose vertices are different colors.

    Proof: By induction on dimension, we show that there are an odd number of such simplices. Draw a graph whose vertices are each small simplex, plus the outside world. Connect two vertices if they share a face whose labels are $1, \ldots, n$. By induction, the outside world gets odd degree; by handshake, there are an odd number of such simplices inside.

27. Given $2^{2010} + 1$ points in the plane, prove that some three of them determine an angle of at least $2009\pi/2010$.

    Proof: split the pairs into 2010 classes, according to the orientation of the line between them. Too many vertices for the complete graph to be the union of 2010 bipartite graphs, so there's an odd cycle within one class, and this gives the angle we want.

28. Prove the following strengthening of Turán's theorem (due to Erdős): given any graph G containing no K_k, there exists a $(k - 1)$-partite graph H on the same vertex set, such that no vertex has lower degree in H than in G.

Solution: consider the vertex of maximal degree. Let W be its set of neighbors. By induction, construct a $(k-2)$-partite graph on W. Now connect everything not in W to everything in W.

29. [Russia, 1998] Given a connected graph on 1998 vertices such that each vertex has degree 3, prove that it is possible to choose 200 vertices, no two adjacent, so that when these 200 vertices are deleted (along with their adjoining edges), the graph remains connected.

    Solution: Delete vertices one by one; at each step we want to show we can delete a vertex that's still of degree 3 and not lose connectedness. Proof: first we'll show that if we can't do this, the graph is planar and can be drawn so that every vertex is on the "outside." If it's a tree it's obvious. Otherwise consider a minimal cycle. Each of the things branching off it must be separate — otherwise (ie if there are two intersecting cycles) then we can delete one of the vertices where they intersect and still be connected. The lemma follows by induction on num of vertices. Now we want to show that if we've removed k vertices from the 3-regular graph so that we can't remove any more degree-3 vertices, then $k > 200$. By lemma, what's left now is planar. Euler characteristic gives $F \geq 1000 - 2k$. The bounded faces are vertex-disjoint, so $3F \leq 1998 - k$. Therefore $3000 - 6k \leq 1998 - k$ giving $k > 200$.

30. [IMO, 2007] Given a graph in which the size of the largest clique (complete subgraph) is even, show that the set of vertices can be partitioned into two disjoint subsets whose largest cliques are of equal size.

    Solution: First put the largest clique in the first set and everything else in the second. Gradually move vertices to the second set until the first set's clique number is one less than the second (if we get them to be equal, we're done). The number of vertices we've moved (L) must be less than the current maximal clique size of the second set (by the evenness hypothesis and the choice of initial partition). If any maximal clique of the second set doesn't contain all of L, we can move one vertex back and be done. Now let the maximal cliques of the second set be $L \cup M_1, \ldots, L \cup M_k$. Choose any vertex from M_1 and move it back to the first set. If it's in $\cap M_i$ then it can't have formed a new clique in the first set (because the initial clique was maximal), so we're done. Otherwise, assume M_2 was left intact. Move a vertex from M_2 not adjacent to v_1 into the first set. Keep going. At some point we've destroyed all the cliques in the second set. If this final move gets the first set back to a clique size one bigger than the second set, move the penultimate vertex back to the second set again and check that this finishes the job.