Discrete Geometry

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Homographic transformations are functions $f : \mathbb{C} \longrightarrow \mathbb{C}$ so that there are constants $a, b, c, d \in \mathbb{C}$ and

$$f(z) = \frac{az + b}{cz + d}$$

In particular all linear transformations are homographic. The set of homographic transformations is closed under compositions of functions.

In the following exercises function f is homographic, the proof being left to the reader:

1. f is either a translation, rotation or homothety.
2. a, b lines and P a point. $f : a \ni X \longmapsto PX \cap b$.
3. a, b, c lines and for $X \in a$ let c_X be the line through X parallel to c. $f : a \ni X \longmapsto c_X \cap b$.
4. a, b two lines tangent to a circle $\mathcal{C}$. Let c_X be a line through X tangent to $\mathcal{C}$. Then $f : a \ni X \longmapsto c_X \cap b$.

Problems

1. Let ABC be a triangle. Outside the triangle take points P, Q, R so that $\angle RAB = \angle CAQ = 30^\circ, \angle RBA = \angle QCA = 45^\circ, \angle PBC = \angle PCB = 15^\circ$. Prove that PQ and PR are perpendicular. Also prove that $PQ = PR$. (IMO 1975)
2. Let $ABCD$ be a rhombus and $\mathcal{C}$ the incircle. Let M, N, P, Q points on AB, BC, CD, DA so that MN and PQ are tangent to $\mathcal{C}$. Prove that $NP||MQ$. (MOSP 2001)
3. Let $ABCD$ be a regular tetrahedron and M, N on BC, CD. Prove that $\angle MAN \leq 60^\circ$.
4. Let ABC be an acute angled triangle and M a point inside it. Prove that $\min(MA, MB, MC) + MA + MB + MC < AB + BC + CA$. (Shortlist 1999)
5. Let $ABCD$ be a quadrilateral with $AB \cap CD = E, AD \cap BC = F, AC \cap BD = O, FO \cap AB = M, FO \cap CD = N$. Let U, V be the intersections of the diagonals of the quadrilaterals $ADNM, MBCN$respectively. Let $X \in AD, Y \in BC$. Prove that XU, YV, FO are concurrent if and only if $E \in XY$. (Andrei Jorza, 1998)
6. Let $A_1A_2A_3A_4$ be a unit square. Consider circles $\mathcal{C}_i(A_i, r_i)$ for all i. Find the smallest real number a so that for any radii $r_1+r_2+r_3+r_4 = a$ we can find an equilateral triangle XYZ with X, Y, Z belonging to $\mathcal{C}_i, \mathcal{C}_j, \mathcal{C}_k$ respectively, for distinct i, j, k. (Romania 2002)

7. Let $\mathcal{C}$ be a circle and a, b two lines that are tangent to $\mathcal{C}$ at antipodal points E, E'. Let $G \in a, H \in b$. The tangents from G, H to $\mathcal{C}$ meet at A. If M is the projection of A to EE' then $\angle GMA = \angle HMA$.

8. Let ABC be a triangle with incenter I. Let T be the tangency point between the incircle and BC. Let M be the projection of I on AT. Prove that $\angle BMT = \angle CMT$.

9. Consider quadrilateral $ABCD$ circumsribed to a circle of center O. Let M be the projection of the point O on AC. Prove that $\angle AMB = \angle AMD$. (Summer Balkan Math Program, 1996)