Combinatorial Geometry — Kiran Kedlaya, for MOSP 1999

As was the case with combinatorial number theory, the adjective "combinatorial" suggests that we are studying structured sets of geometric objects, rather than the structure of a fixed collection of objects (as in ordinary synthetic geometry).

Some standard topics:

- Convex sets and their properties
- Dissections of regions. Related question: if n points lie in a region, how long is the shortest segment between two of them?
- Distances between points (rational, irrational, integral)
- Geometry of lattice points
- Collections of vectors

1. (Helly's theorem) A finite set of bounded, convex subsets of the plane has the property that every three of the subsets have nonempty intersection. Prove that the intersection of all of the subsets is nonempty. (Can you generalize to higher dimensions?) Also show that the conditions "finite", "bounded" and "convex" are all necessary. (If the sets are *closed*, you can drop the finiteness condition.)

2. (Russia, 1998) In the plane are given several squares with parallel sides, such that among any n of them, there exist four having a common point. Prove that the squares can be divided ~~into at~~ most $n-3$ groups, such that all of the squares in a group have a common point.

3. (Proposed for 1998 USAMO) The region between two parallel lines in the plane is divided into 10 stripes of equal width, alternately colored white and black. A convex region of the plane contains at least one point on each line. Prove that at least 45% of the area of the region must be black.

4. Given a convex region X in the plane, show that there exists a point P in X such that for every line through P, P lies in the middle third of the intersection of X with the line.

5. (Putnam, 1990) Let S be a nonempty closed bounded convex set in the plane. Let K be a line and t a positive number. Let L_1 and L_2 be support lines for S parallel to K_1, and let $\overline{L}$ be the line parallel to K and midway between L_1 and L_2. Let $B_S(K,t)$ be the band of points whose distance from $\overline{L}$ is at most $(t/2)w$, where w is the distance between L_1 and L_2. What is the smallest t such that

$$S \cap \bigcap_K B_S(K,t) \neq \emptyset$$

for all S? (K runs over all lines in the plane.)

6. (Erdős) An infinite set of points in the plane has the property that the distance between any two of the points is an integer. Prove that all of the points are collinear.

7. (IMO1975/5) Prove that there there exist 1975 points on a unit circle such that the distance between any two is rational.

8. (IMO 1987/5) Let n be an integer greater than or equal to 3. Prove that there is a set of n points in the plane such that the distance between any two points is irrational and each set of three points determines a non-degenerate triangle with rational area.

9. (USAMO 1998/6) Let $n \geq 5$ be an integer. Find the largest integer k (as a function of n) such that there exists a convex n-gon $A_1A_2 \ldots A_n$ for which exactly k of the quadrilaterals $A_iA_{i+1}A_{i+2}A_{i+3}$ have an inscribed circle. (Here $A_{n+j} = A_j$.)

10. (Dual version of the previous problem) Let $n \geq 5$ be an integer. Find the largest integer k (as a function of n) such that there exists a convex n-gon $A_1A_2 \ldots A_n$ for which exactly k of the quadrilaterals formed by the lines A_iA_{i+1}, $A_{i+1}A_{i+2}$, $A_{i+2}A_{i+3}$, $A_{i+3}A_{i+4}$ are cyclic. (Here $A_{n+j} = A_j$.)

11. Prove that the points of the plane can be colored in 7 colors such that no two points at distance 1 are the same color. Also show that this cannot be done with only 3 colors. The exact number needed is unknown.

12. A graph is called a *unit distance graph* if it can be drawn in the plane so that each edge is a segment of unit length. (The edges are allowed to intersect each other.) Show that the hypercube Q_n is a unit distance graph. (The hypercube has vertices the n-digit binary strings, with two adjacent if they differ in only 1 position.) Can you find some other unit distance graphs?

13. (Pick's theorem) The area of a non-self-intersection (but not necessarily convex) lattice polygon equals $i + b/2 - 1$, where i is the number of interior lattice points and b is the number of boundary lattice points.

14. Prove that a convex lattice pentagon has area at least $5/2$.

15. (Russia, 1998) Prove that a convex lattice $2n$-gon has area at least $n^3/100$. (The constant 100 is not optimal; can you improve it?)

16. Let X be a lattice polytope in $\mathbb{R}^k$. For $n \in \mathbb{N}$ let nX denote the poly~~hedron~~ tope obtained from X by a homothety of ratio n about some fixed lattice point. Prove that there exists a polynomial P such that $P(n)$ equals the number of lattice points contained in nX (including its boundary). This polynomial is called the *Ehrhart polynomial* of X, and its existence is a higher-dimensional analogue of Pick's theorem.

17. With notation as in the previous problem, show that there exists a polynomial Q such that $Q(n)$ equals the number of lattice points contained in nX, but not including its boundary. Moreover, show that $Q(n) = (-1)^k P(-n)$ (Ehrhart reciprocity law).