Tilings

Lecturer - Andrei Jorza

MOP 2002, Blue/Black Groups

There are very many problems with tilings and very few organized methods of approaching them. A good categorization of the types of solutions is the following: solutions that use colorings, graphs, cuts and miscellaneous solutions.

The first part of the handout contains problems to be discussed in class, grouped into four categories, according to the types of solutions. The second part is also grouped in the same four categories and contains problems that Reid said you might have already seen, but which are a good review. If you need solutions to them, come to me.

Colorings

1. Can a 5×7 rectangle be covered by several layers of corners so that each unit square be covered by the same number of layers? (Russia 1996)

   Proof: In each of the unit squares whose both coordinates are odd put a -2 and in each of the other unit squares put a 1. Then any corner can cover at most one unit square with a -2 in it so the total sum in each corner is nonnegative.

   Assume there are k layers. Then the total sum of the number in all corners (counting multiplicities) has to be nonnegative (since the sum in each corner is nonnegative), but in the same time is k times the total sum in the table. This sum is -1 and $-k$ is not nonnegative so the tiling is not possible. ■

2. On a 10×10 square we are trying to place one 1×4, two 1×3, three 1×2 and four 1×1 rectangles. Prove that if we randomly place the rectangles on the board but in the order mentioned above then we can always fit them so that no two overlap or touch. (Tournament of Towns 1993)

   Proof: To "touch" means to have points in common. On lines 1,4,7,10 color the $1 \times \{1, 2, 3, 4\}$ and $1 \times \{7, 8, 9, 10\}$ rectangles. Then any of the rectangles that we have can "touch" at most 2 of these rectangles. Therefore not all 8 can be "touched" unless we have one 1×4 and at least 2 1×3 rectangles. Therefore we can put the first three rectangles without any problems, because we can always choose one rectangle among those colored and this will not have any points in common with the other rectangles.

   Now on lines 1,4,7,10 color the following rectangles $1 \times \{1, 2\}, 1 \times \{5, 6\}, 1 \times \{9, 10\}$. Then any rectangle among the first three can "touch" at most two of the colored ones and any domino can "touch" at most one domino. Since there are 12 dominos colored we can place the first three rectangles and at least 3 other dominos so that no two have points in common because we can always choose one of the colored rectangles that does not touch any of the rectangles placed until now. Therefore we can put the first 6 rectangles as we wish.

   In order to put the last 4 unit squares we do a reasoning similar to the one above but for the following coloring: on lines 1,4,7,10 color the unit squares $1, 4, 7, 10$. ■

3. A convex n-gon may be partitioned into $n-2$ triangles by $n-3$ of its diagonals so that any two such diagonals intersect only in vertices of the polygon. Prove that one may trace all the sides and diagonals of one such partition as a closed polygonal line if and only if $3|n$. (China 1990)

   Proof: By Euler's theorem such a closed path can be traced if and only if the degree of each vertex is even. Color one of the triangles that has two sides among the original edges of the polygon in black (its existence is a standard result in graph theory - use the Euler formula). Then choose one of the vertices of the triangle that is not a vertex of the polygon. Then alternatively color all the triangles that have this vertex into black and white so that no two neighboring (with the obvious definition of neighbor) triangle have a same color. Do this procedure for all vertices in the polygon. Since each vertex has an even degree it means that the number of triangle going out of each vertex is odd so the procedure mentioned above yields a coloring where indeed any two neighboring triangles have different colors. Clearly in such a coloring all triangles that have an edge among the edges of the original polygon has to be colored black (straightforward induction on the vertices using the fact that in each vertex we have an odd number of triangles).

   Now color the exterior infinite face into white. Since any edge is part of two faces of different colors it means that the number of edges of all the white faces is equal to the number of edges of black faces. Since there are only triangle among the black and among the white there are only triangles and one infinite face with n edges it means that n is equal to a sum of 3s minus a sum a 3s. Therefore $n \equiv 0 (\text{mod } 3)$.

   If $n \equiv 0 (\text{mod } 3)$ then we do induction. For $n = 3$ it is trivial. Now if we have the polygon $A_1 \ldots A_n$ then triangulate $A_1 A_5 A_6 \ldots A_n$ with the inductive hypothesis and then go $A_1A_3, A_3A_5, A_5A_4, A_4A_3, A_3A_2, A_2A_1$ and the induction is over. ■

Graphs

1. On the lattice plane consider the rectangle $(0,0), (m,0), (m,n), (0,n)$ with m, n both odd. The rectangle is partitioned into triangles so that:
   a. each triangle in the partition has a side on the line $x = j$ or $y = k$ ($j, k \in \mathbb{N}$) called a good side and the altitude on this side has length 1.
   b. each bad side (a side that is not good) is a common side of two triangles in the partition.
   Prove that there are two triangles in the partition that have two good sides each. (Shortlist 1990)

   Proof: Consider all center of lattice squares. Then put an edge between to such centers if they are midpoints of bad sides in a triangle in the triangulation. Note that the fact that a median line joining the midpoint of two bad sides has to be at distance $\frac{1}{2}$ from a good side which means that the two midpoints are among the centers of lattice squares.

   Consider the graph of all these points and edges. If there is a vertex of degree 0 then we are done because in this point is midpoint of some edge that is part of two triangles in the triangulation and these triangles have to have two good sides because otherwise the midpoint would be joined with some other point.

If there is a point of degree 1 then I use the fact that the sum of all degrees is even (standard result in graph theory, the sum is twice the total number of edges) and we get that there must be another point with odd degree and this degree can only be 1 (any side is part in at most two triangles). Then the triangles in which these two points lie have to have two good sides each (for the same reason as in the case of the point of degree 0). Therefore we are done.

Assume that each vertex has degree 2. Then we can partition the graph into cycles. Since we have a triangulation all the centers of lattice squares have to appear in the graph. Also any edge in the graph has to be parallel to a good side so it has to be parallel to one of the two axes. Therefore as we go along a cycle we can alternatively color all vertices visited in black and white and since we are dealing with a cycle we must have a coloring where no two adjacent vertices have a same color (this happens because we have only horizontal and vertical edges in the graph which means that the number of horizontal edges going into one direction has to be equal to the number of horizontal edges going in the other direction; similarly for the vertical ones and this means that the total number of edges and so the total number of vertices has to be even). Therefore on each cycle we have an even number of vertices and since all the vertices are parts of some cycles we get that the total number of vertices is even. But we have mn vertices and this number is odd. ■

2. What is the smallest number of unit segments that can be erased from a 2000×3000 board so that no rectangles (besides the big one) appear? (St. Petersburg 2001)

   Proof: Look at the centers of unit squares as before. But an edge between any two centers of neighboring squares if the edge between has been removed. Now any connected component in the graph with at most two vertices would either represent a unit square or a domino and these cannot exist, which means that any connected component in the graph has to have at least three vertices. Therefore the total number of connected component has to be at most 2 million since the number of vertices is 6 million.

   For any connected component the number of edges if at least the number of vertices minus 2. So the total number of edges in all connected components (and this has to be the total number of removed edges, by definition) is at least the total number of vertices (6 million) minus the total numbed of connected components (this contributes at least -2 million to the sum) and this number is at least 4 million.

   Now this can be achieved by a tiling of the rectangle with corner "swastikas" which means that we have corners (unit squares mentioned for each) $(1,2)-(1,3)-(2,3), (2,2)-(2,1)-(3,1), (3,2)-(4,2)-(4,3), (3,3)-(3,4)-(2,4)$. ■

3. Prove that one cannot tile a rectangle with pointed corners. A pointed corner is the union of the unit square $[0,1] \times [0,1]$ with the right isosceles triangles $(0,1)(1,1)(0,2)$ and $(1,0)(2,1)(1,1)$. (Belarus 1999)

Proof: Consider all the center of unit squares in the lattice plane. For each pointed corner draw a line from the center of the unit square in the pointed square to the centers of the squares from which we cut out the right isosceles triangles that are part of the pointed corner. For each pointed corner we get a small corner made of two unit segments.

Assume that we have a tiling of a rectangle with pointed corners. This means that we have a graph of associated corners to the pointed corners. Since a vertex of degree 1 can only appear when we have a right isosceles triangle that has no other right isosceles triangle that is adjacent to it it means that no vertex of degree 1 can appear since an isolated right isosceles triangle cannot appear in the tiling of a rectangle.

So the graph associated to the tiling has all vertices of degree 2 which means that it can be partitioned into disjoint cycles (clearly any vertex can have degree at most 2). Consider a smallest cycle, i.e. a cycle with no other cycle in its interior (an easy applications of the extremal principle). Consider this cycle L and let S be its area.

Now we have a cycle made of unit segments that are parallel to the axes (clearly) so as at the first problem from graphs the number of horizontal segments has to be even. Let $2k$ be this number. Now any horizontal segment is part of a pointed corner so it has a unique vertical unit segment associated to it. This means that the number of vertical segments is $2k$ as well. So the total number of points in the cycle is $4k$ since the total number of pointed corners is $2k$. Note that any center of a square inside the cycle has to be part of another cycle (we have a tiling which means that everything is covered) but this cannot be because such a point has to be part of a cycle inside L which we assumed false. By Pick's theorem $S = \frac{4k}{2} - 1 = 2k - 1$.

Each pointed corner is separated by the corner associated to it into two parts. One (the one that has the point $(0,0)$ in my description of the pointed corner) has area $\frac{5}{4}$ and the other has area $\frac{3}{4}$. Assume there are x pointed corners that have the first part inside L and y pointed corners that have the second part inside L. Then we have $x + y = 2k$ and $S = 2k - 1 = \frac{5x}{4} + \frac{3y}{4}$. Subtract the two and we get $y - x = 4$.

Now in each corner associated to a pointed corner join the two extremal vertices. Then L becomes another cycle E that is made of segments of length $\sqrt{2}$. Note that if we color the plane in checkerboard style so that one of the endpoints of a corner is on black then all the endpoints of corners have to be on black because the distance between a black and a white unit square can never be $\sqrt{2}$. Therefore the vertices of E are centers of black unit squares and these form a lattice where the unit is $\sqrt{2}$.

Now each of the x pointed corners with the big part inside L, when we replace a corner with the segment joining its endpoints we actually add a right isosceles triangle the the part of the pointed corner that is inside L. Therefore any of the x pointed corners contributes $\frac{1}{2}$ area to L when it is transformed into E. Similarly any of the y pointed corners whose smaller part is inside L contributes $-\frac{1}{2}$ to the area of L because when we join the endpoints we actually cut off a right isosceles triangle. Therefore the area of E will be $area(E) = area(L) + \frac{x}{2} - \frac{y}{2} = 2k - 1 - \frac{4}{2} = 2k - 3$.

Also, from the $4k$ vertices of L we got rid of $2k$ vertices that were vertices of the corners in L so E has $2k$ vertices. When we transform L into E the only centers of unit squares that we add are possible some that are vertices in the corners associated to some of the y pointed corners. But these lie on white squares so they do not count in the lattice on which the vertices of E lie. Therefore we may apply Pick's theorem again (but now the integer has to be multiplied with the area of the new "unit" square and that is $(\sqrt{2})^2 = 2$) so $areaE = (\sqrt{2})^2(\frac{2k}{2} - 1) = 2k - 2$.

Therefore $area(E) = 2k - 2 = 2k - 3$ and this is a contradiction. The proof actually shows that one cannot tile any union of rectangles with pointed corners. ■

Cuts

1. (Cuts) A regular hexagon is partitioned into parallelograms of equal areas. Prove that the number of parallelograms is divisible by 3. (Tournament of Towns 1989)

   Proof (not detailed)Clearly any parallelogram in the tiling has all edges parallel to pairs of edges of the regular hexagon (not too hard to show, since angles have to add up to $120°$). There are three type of edges in the regular hexagon and we call them type 1,2,3 according to whether their direction is given by the angle $0°, 120°, 240°$ respectively. Any parallelogram in the tiling if of type i if its edges are parallel to the sides of the hexagon other than i.

   Consider a line parallel to an edge of the hexagon of type i. Then consider all the parallelograms in the tiling that are not of type i. All the parallelograms that this line intersects have to have a side parallel to sides of type i in the hexagon so the line creates new parallelograms among those that are not of type i. Consider the total length of the part of the line that is inside parallelograms not of type i. This length is also the sum of all the upper (if the sides of type i are placed horizontally) sides in these parallelograms and we may go upwards with this procedure and we get that the total length has to be equal to the total length of all the upper sides of the parallelograms that neighbor a side of type i so this has to be 1 (the edge-length of the hexagon).

   If we take this line from the top side to the bottom side then we keep on covering segments of length 1 so by Cavalieri's principle the area covered is $1 \cdot \sqrt{3}$ ($\sqrt{3}$ is the height of the hexagon) and this is $\frac{2}{3}$ of the total area of the hexagon. Also the line goes through all the parallelograms that are not of type i and covers them entirely. Therefore the total area covered will be the constant area times the total number of parallelograms that are not of type i. Since the total number of parallelograms is fixed this implies that the constant area times the total number of parallelograms of type i is constant and so the total number of parallelograms is 3 times the number of parallelograms of either type, so a number divisible by 3. ■

2. A square is divided into rectangles. A chain is a subset $\mathcal{K}$ of the set of these rectangles so that there is a side of the square which is covered by the projections of the rectangles of $\mathcal{K}$ and such that no points of this side is a projection of two inner points of 2 different

rectangles of $\mathcal{K}$. Prove that any two rectangles in the division are members of a chain. (Tournament of Towns 1985)

Proof (not detailed)Consider the rectangle a, b. If we have a chain from a to b then consider the edge on which the chain projects and then draw a line through a and b parallel to this edge, take all the rectangles that meet the line and then we have a chain in the whole rectangle.

We induct on the total number of rectangles in the tiling. If there is a horizontal or vertical line intersecting both a and b then take all the rectangles that intersect this line and then we have a chain. Otherwise there is a vertex A of a that is closest to b (clearly).

It is easy to see that at least on of the rectangles (let it be c) in the tiling has to have A as a vertex. Let this rectangle be to the right of a. Then the bottom edges of a and c are on the same line. Draw the line (vertical) that separates a and c and in the big rectangle formed on the right consider all the rectangles in the tiling that are parts of rectangles from the original tiling . Then the number of rectangles in the tiling of the new rectangle is less than the original number of rectangles so we may apply an inductive hypothesis. So there is a chain from c to b.

If this chain projects to the vertical side then replace c by a and we get a new chain in the original rectangle (after replacing all severed rectangles with the original ones from tiling) so we are done. If the chain projects to the horizontal line then add a and then we have a chain in the original rectangle. Note that if in this chain we have a rectangle that was cut by the vertical line that separated a and c then at least one point of c projected on the horizontal to the same point as some point of the severed rectangles. Since we have a chain none of the rectangle in the chain came from an original rectangle by cutting so this new chain is a valid one.

Since the base case is trivial we proved the problem by induction. ■

3. (Cuts) Find the greatest number of $1 \times 10\sqrt{2}$ rectangles that can be fit into a 50×90 rectangle using cuts parallel to the sides of the rectangle. (BMO 2000)

   Proof (not detailed)It is easy to tile (greedy) this rectangle into 315 small rectangles.

   Now, put the big rectangle on the plane $(0, 0), (90, 0), (90, 50), (0, 50)$. Consider the lines
$$x + y = 10k\sqrt{2}$$
   for $k = 1, 2, \ldots$.

   It is easy to see that any rectangle $1 \times 10\sqrt{2}$ whose sides are parallel to the axes intersect these lines in one or two segments, but in either case the total length of the intersection is $\sqrt{2}$ for obvious reasons.

   One can manually check that the total length of the intersection of the big rectangle with these lines is $570\sqrt{2} - 360$ so the biggest number of $1 \times 10\sqrt{2}$ rectangles that fit into the big rectangle is at most

$$\lfloor\frac{570\sqrt{2}-360}{\sqrt{2}}\rfloor = 315$$

So 315 is the best possible number. ■

Miscellaneous

1. Consider an $m \times n$ board. We place dominoes on the board so that in the end no domino can be moved from its place by sliding into a free space on the board. Prove that the number of unit squares not covered by dominoes is less than $\frac{mn}{5}$. (Tournament of Towns, 1989)

   Proof (not detailed)It is easy to note that on each edge there can be no noncovered squares because then we may slide dominoes into them. Also note that no two noncovered squares can lie at distance of less than 3 apart.

   Look into a 2×5 rectangle. The only possible way in which there can be 3 noncovered squares in this rectangle is that the squares be (WLOG) $1 \times 1, 2 \times 3, 1 \times 5$). But then it is easy to show that one can slide a domino in a free space.

   Now divide the $m \times n$ table in strips of width 2. Since the edges can have no noncovered squares it means that we can ignore the case m odd and not affect the final inequality. Look at each strip $2 \times n$ and bend it and join its edges so as to get a cylinder of height 2. On this cylinder look at all rectangles 2×5. Each of the rectangles that is not cut by the edge line (which comes from the joint of the two edges) has at most 2 noncovered squares in it. Each of the rectangles that has a point in common with the edge line can still have at most 2 noncovered squares in it because no noncovered squares appear on the edges.

   This means the in each strip we have at most $\frac{2n}{5}$ noncovered squares so overall we have at most $\frac{mn}{5}$. ■

2. A finite set of line segments, of total length 18, belongs to a unit square. The line segments are parallel to the sides of the square and may intersect. Prove that among the regions determined by the line segments there is one of area at least 0.01. (Tournament of Towns 1980)

   Proof (not detailed)Number the regions A_i, let them have area A_i and perimeter p_i. Then $\sum p_i \leq 2 \cdot 18 + 2 = 40$ because all the internal edges may count at most twice in the total perimeter.

   Take a region A_i and border it with the smallest rectangle that contains it. Then $p_i \geq$ the perimeter of the rectangle ≥ 4 times the radical of the area of the rectangle. We also know that the area of the rectangle is at least A_i since the rectangle contains A_i. So we have $p_i \geq 4\sqrt{A_i}$.

   Now assume that $A_i < 0.01 \Longrightarrow \sqrt{A_i} < 0.1$. So we have $1 = \sum A_i = \sum(\sqrt{A_i})^2 < 0.1 \sum \sqrt{A_i} \leq 0.1 \sum \frac{p_i}{4} \leq 0.1 \frac{40}{4} = 1$ and this is a contradiction. ■

3. A regular $4n$-gon is partitioned into parallelograms. Prove that among these there are at least n rectangles. (Tournament of Towns 1983)

   Proof (not detailed)Consider a pair of opposite edges in the regular $4n$-gon. Then there is a strip of parallelograms from one to the other and for each parallelogram one pair of edges is parallel to these two edges. Consider the pair of opposite edges that is perpendicular to the first pair (can do that because we are dealing with a $4n$-gon). Then the two strips of parallelograms have to intersect and there we have a rectangle.

   Since there are n quadruples of such edge we get at least n rectangles. ■

Review Problems

1. (Graphs) Tile the plane with regular hexagons. An ant is going from point A to point B on the shortest path along the edges of the hexagons (A, B are vertices of the hexagons). Prove that the ant went along one direction at least half the distance it walked from A to B. If the distance is d then half means $\lfloor \frac{n}{2} \rfloor$.

   Hint: In each hexagon draw a big equilateral triangle and this way we get a tiling of the plane with equilateral triangles. Prove that the image of the path of the ant on the new tiling can go through at most 2 directions and then note that any two directions in the new tiling share a same direction in the old tiling.

2. (Graphs) Unit cubes are made into beads by drilling a hole along a diagonal. The beads are made into a string so as to move freely as long as two neighboring cubes have at least one common vertex. Find $p, q, r \in \mathbb{N}^*$ so that is we are given pqr unit cubes that we can make the string into a $p \times q \times r$ block. Same question if we require that the beginning and the end of the string be the same in the final block. (Shortlist 1990)

   Hint: Use Euler's theorem which says that a closed path going through all the edges once exists if and only if the degree of each vertex is even.

3. (Miscellaneous) A lattice (having vertices in lattice points) rectangle is partitioned into lattice triangles of area $1/2$. Prove that there are more right triangles among these than twice the length of the shorter side of the rectangle. (Hungary 1995)

4. (Miscellaneous) On the plane we have n rectangles with parallel sides. The sides of different rectangles lie on different lines. The boundaries of the rectangles cut the plane into connected regions. A region is called nice if its boundary has a vertex of an original rectangle. Prove that the sum of the numbers of vertices of all nice regions is less than $40n$. (Shortlist 2000).