1 (IMO 1988/6) Let a and b be positive integers such that $ab+1$ divides a^2+b^2. Show that
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$$\frac{a^2+b^2}{ab+1}$$
is the square of an integer.

Solution. Suppose that a, b are positive integers so that $ab+1$ divides a^2+b^2 and let
$$k := \frac{a^2+b^2}{ab+1}. \tag{1}$$
We have to prove that k is a perfect square.

The very fundamental idea of this and similar problems is to give up the idea of proving properties of a and b directly. Instead, we are going to prove the desired property of k (i.e. that k is a perfect square) by fixing k and considering all positive integers a, b which satisfy (1), that is, we consider
$$S(k) := \left\{ (a,b) \in \mathbb{Z}^+ \times \mathbb{Z}^+ : \frac{a^2+b^2}{ab+1} = k \right\}.$$
By defining this set, we leave the concrete values of a, b and instead take the whole "environment" of k into consideration.

The next step is to assume the required statement to be wrong (for the sake of contradiction), that is, we suppose that k is not a perfect square. The rest of the problem goes by the method of *Infinite Descent*: We take any pair $(a,b) \in S(k)$ and show the existence of another pair $(a_1, b_1) \in S(k)$ which is smaller than (a,b) where (a_1, b_1) is said to be smaller than (a,b) if $a_1 + b_1 < a + b$. This however is a contradiction because $S(k) \subset \mathbb{Z}^+ \times \mathbb{Z}^+$ implies that there exists a lower bound for $a+b$ which is also achieved by at least one pair $(a,b) \in S(k)$.

Suppose that $(a,b) \in S(k)$ is any pair which satisfies (1). Wlog assume that $a \geq b$.

Consider the equation
$$\frac{x^2+b^2}{xb+1} = k \tag{2}$$
as a quadratic equation in x. This equation is equivalent to
$$x^2 - kxb + b^2 - k = 0. \tag{3}$$
We know that $x = a$ is a root of (3) since $x = a$ solves (2). Let a_1 be the other solution of (3).

Notice that by using the fact that this quadratic equation has another solution, we have found another pair (a_1, b) that solves (1). The first step is to show that $(a_1, b) \in S(k)$.

Lemma 1. *a_1 is a positive integer.*

Proof. We know from Vieta's theorem that $a_1 = kb - a$. Thus, a_1 is an integer. We still have to prove that a_1 is positive.

First, assume that $a_1 = 0$. But this implies that $k = b^2$ since we know that $x = a_1$ solves the equation (2), a contradiction to our assumption that k is not a perfect square.

Now, assume that $a_1 < 0$. Then from (3) we infer that
$$k = a_1^2 - ka_1b + b^2 \geq a_1^2 + kb + b^2 > k,$$

clearly impossible. Notice that the last step follows from $b > 0$.

We therefore know that $a_1 > 0$ and thus $a_1 \in \mathbb{Z}^+$. □

Corollary 1. $(a_1, b) \in S(k)$.

We hence have constructed another pair in $S(k)$ from any given pair (a, b). If we are able to prove that this new pair is smaller than the old one, we can use the argument of infinite descent to reach our contradiction and we are done. The next step is to prove that the new pair is indeed smaller than (a, b).

Lemma 2. $a_1 < a$.

Proof. We know that $x = a$ and $x = a_1$ are the roots of (3). It therefore follows from Vieta's theorem that

$$a_1 = \frac{b^2 - k}{a}.$$

However, since we assumed that $a \geq b$, we infer that

$$\frac{b^2 - k}{a} < a$$

from which $a_1 < a$ follows. □

We thus have proved the existence of a pair $(a_1, b_1) \in S(k)$ that is smaller than (a, b), i.e. that $a_1 + b_1 < a + b$. Iterating this procedure for (a_1, b_1), we can construct another pair $(a_2, b_2) \in S(k)$ that is smaller than (a_1, b_1) and another pair $(a_3, b_3) \in S(k)$ that is smaller than (a_2, b_2) and so on. In other words, we can construct pairs (a_j, b_j) for $j = 1, 2, \ldots$ so that

$$a + b > a_1 + b_1 > a_2 + b_2 > a_3 + b_3 > \ldots$$

which is impossible since all $(a_j, b_j) \in S(k) \subset \mathbb{Z}^+ \times \mathbb{Z}^+$. □

Revising this method, we first assumed the existence of a pair (a, b) that does not satisfy the statement we want to prove. We then went away from this concrete pair (a, b) and instead considered pairs with the same property as (a, b). The next step is to define a "size" of a pair (a, b) which in our case was simply $a + b$. It is trivial that this size has a lower bound. Using the theorem of Vieta, we constructed another pair (a_1, b_1) from any given pair (a, b) and we proved that the new pair is smaller than the old one. This method is called *Vieta-Jumping* or *Root Flipping*. Applying the method of infinite descent, we obtain our desired contradiction.

With the same ideas, we can also prove a generalisation of the problem:

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(CRUX, Problem 1420, Shailesh Shirali) If a, b, c are positive integers such that

$$0 < a^2 + b^2 - abc \leq c,$$

show that $a^2 + b^2 - abc$ is a perfect square.

Indeed, the first problem is a special case of this one since

$$0 < a^2 + b^2 - abc = c$$

implies that

$$\frac{a^2+b^2}{ab+1} = c$$

which must be a perfect square.

Solution. Again, as in the first problem, we assume that there exist positive integers a, b, c so that

$$k := a^2 + b^2 - abc \tag{4}$$

is not a perfect square. We know that $k > 0$ and $k \le c$.

We now fix k and c and consider all pairs (a, b) of positive integers which satisfy the equation (4), that is, we consider

$$S(c,k) = \left\{(a,b) \in \mathbb{Z}^+ \times \mathbb{Z}^+ : a^2 + b^2 - abc = k\right\}.$$

Suppose that (a, b) is any pair in $S(c, k)$. Wlog assume that $a \ge b$. Consider the equation

$$x^2 - xbc + b^2 - k = 0 \tag{5}$$

as a quadratic equation in x. We know that $x = a$ is a root of this equation. Let a_1 be the other root of this equation.

Lemma 3. *a_1 is a positive integer.*

Proof. Since a_1 and a are the roots of (5), we know from the theorem of Vieta that

$$a_1 = bc - a.$$

It therefore follows that a_1 is an integer.

If $a_1 = 0$ then (5) implies that $b^2 = k$ is a perfect square, a contradiction.

If $a_1 < 0$ then (5) implies that

$$k = a_1^2 + b^2 - a_1bc \ge a_1^2 + b^2 + bc > c,$$

a contradiction to $k \le c$.

Thus, a_1 is a positive integer. □

Corollary 2. $(a_1, b) \in S(c, k)$.

Again, it remains to be proven that the new pair (a_1, b) is smaller than (a, b).

Lemma 4. $a_1 < a$.

Proof. We know that a_1 and a are the roots of (5), so by Vieta's theorem,

$$a_1 = \frac{b^2-k}{a}.$$

Since we assumed that $a \ge b$, it follows that

$$\frac{b^2-k}{a} < a$$

which implies $a_1 < a$. □

We have therefore constructed another pair (a_1, b_1) in $S(c,k)$ with $a_1 + b_1 < a + b$. However, $S(c,k) \subset \mathbb{Z}^+ \times \mathbb{Z}^+$, so using the argument of infinite descent, we obtain our desired contradiction. $\square$

Remark: There exists a bunch of problems which can be solved with these ideas. Here are some of them:

1. (IMO 2007/5) Let a, b be positive integers so that $4ab - 1$ divides $(4a^2 - 1)^2$. Show that $a = b$.

   Hint: First prove that if $4ab - 1 | (4a^2 - 1)^2$, then $4ab - 1 | (a - b)^2$.

2. (A5) Let x and y be positive integers such that xy divides $x^2 + y^2 + 1$. Show that

$$\frac{x^2 + y^2 + 1}{xy} = 3.$$

3. Let a, b be positive integers with $ab \neq 1$. Suppose that $ab - 1$ divides $a^2 + b^2$. Show that

$$\frac{a^2 + b^2}{ab - 1} = 5.$$