Symmedians

Robin Park

January 13, 2014

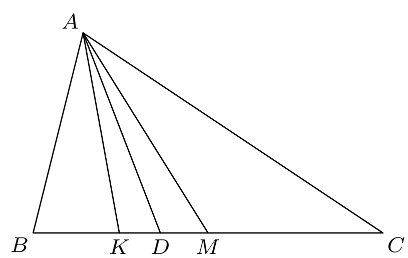
This solution is unnecessarily synthetic.

-Victor Wang, MOP 2013

Symmedians are three lines uniquely determined by a triangle. It has various properties that assist in solving Olympiad geometry problems.

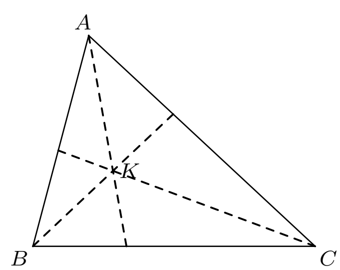
1 Properties

Definition. The A-*symmedian* of $\triangle ABC$ is defined as the reflection of the median from A over the angle bisector from A.



Like the medians, angle bisectors, altitudes, and perpendicular bisectors, the symmedians of a triangle concur at a point:

Theorem 1. *The A-, B-, and C-symmedians of $\triangle ABC$ concur.*

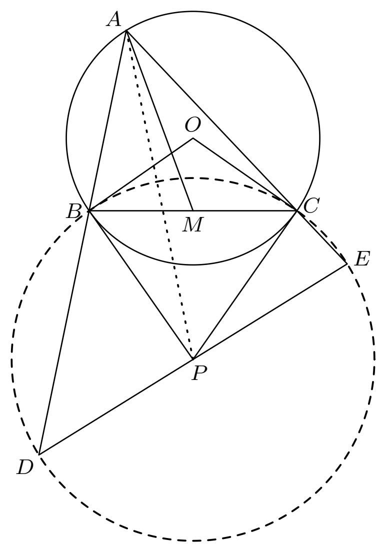


Proof. Trig Ceva implies that the concurrency of the symmedians is equivalent to the concurrency of the medians, which is true. $\square$

This point of concurrency is called the *symmedian point* (sometimes the *Lemoine point*) K. In fact, the existence of the symmedian point is a special case of the existence of the *isogonal conjugate* of a point; in a triangle, its symmedian point is the isogonal conjugate of its centroid.

The following property of the symmedian is widely known, and is usually called the "symmedian lemma."

Lemma 2. *Let ABC be a triangle, and let P be the intersection of the tangents to the circumcircle of $\triangle ABC$ at B and C. Then AP is the A-symmedian of $\triangle ABC$.*



Proof. Let ω be the circle centered at P with radius PB. This circle passes through C because $PB = PC$. Now let D and E be the intersections of ω with AB and AC, respectively. Finally, let M be the midpoint of segment BC and O the circumcenter of $\triangle ABC$.

Note that

$$\begin{aligned} \angle DBE &= \angle BAE + \angle AEB \\ &= \angle BAC + \angle CEB \\ &= \frac{1}{2}(\angle BOC + \angle CPB) = 90^\circ. \end{aligned}$$

Hence DE is a diameter of ω, and so P is the midpoint of segment DE. Observe that $\triangle ABC \sim \triangle AED$, and so $\triangle AMC \sim \triangle APD$. Thus $\angle CAM = \angle DAP = \angle BAP$, implying that AP is the A-symmedian. $\square$

The next lemma provides a nice ratio relationship between the distances from X to B and C:

Lemma 3. *Let X be a point on BC such that AX is the A-symmedian of $\triangle ABC$. Then*

$$\frac{BX}{CX} = \frac{AB^2}{AC^2}.$$

Proof. Note that

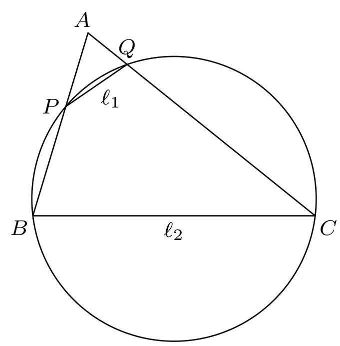
$$\frac{\sin\angle BAX}{BX} = \frac{\sin\angle AXB}{AB} \text{ and } \frac{\sin\angle CAX}{CX} = \frac{\sin\angle AXC}{AC}.$$

Dividing these two gives us

$$\frac{BX}{CX} = \frac{AB}{AC}\frac{\sin\angle BAX \sin\angle AXC}{\sin\angle AXB \sin\angle CAX} = \frac{AB}{AC}\frac{\sin\angle BAX}{\sin\angle CAX} = \frac{AB^2}{AC^2}$$

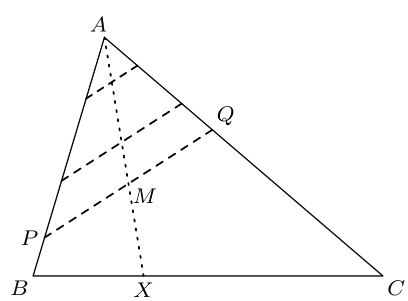
as desired. □

Another interesting property of the symmedian is that it is the locus of the midpoints of *antiparallels*. We say that two lines/segments ℓ_1 and ℓ_2 are antiparallel with respect to an angle if the angle formed by ℓ_1 with one side of the angle is equal to the angle formed by ℓ_2 with the other side.



In the diagram above, $\angle AQP = \angle ABC$ and $\angle APQ = \angle ACB$. Notice that this immediately implies that $BCQP$ is cyclic, since $\angle ABC + \angle PQC = \angle AQP + \angle PQC = 180°$.

Lemma 4. *The A-symmedian of $\triangle ABC$ is the locus of the midpoints of the antiparallels to BC with respect to $\angle BAC$.*



Proof. Let P and Q be points on AB and AC such that PQ is antiparallel to BC, and let M be the midpoint of segment PQ. Let X be the intersection of AM and BC. By the Generalized Angle Bisector Theorem,

$$1 = \frac{MP}{MQ} = \frac{AP}{AQ}\frac{\sin\angle MAP}{\sin\angle MAQ}.$$

Hence

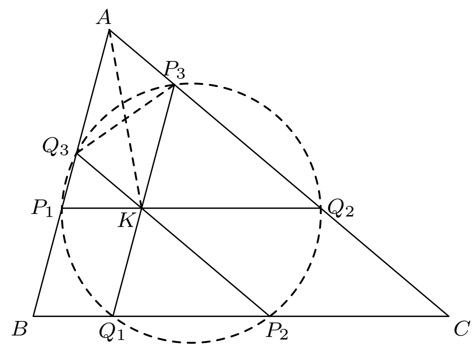
$$\frac{BX}{CX} = \frac{AB}{AC}\frac{\sin\angle XAB}{\sin\angle XAC} = \frac{AB}{AC}\frac{\sin\angle MAP}{\sin\angle MAQ} = \frac{AB}{AC}\frac{AQ}{AP} = \frac{AB^2}{AC^2}$$

and so by Lemma 3, AX is the A-symmedian. □

2 The Lemoine Circles

There are two circles that correspond to the symmedian point of a triangle:

Theorem 5 (First Lemoine Circle)**.** *Let K be the symmedian point of triangle ABC. Prove that the six intersections formed by the three parallels with respect to the sides of $\triangle ABC$ passing through K and the sides themselves lie on a circle.*

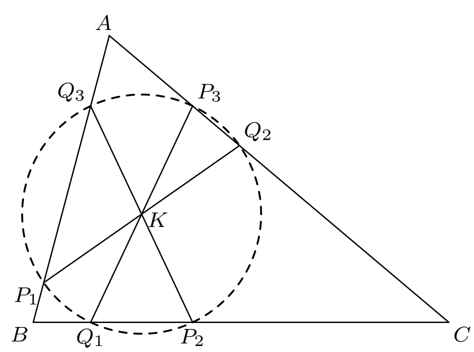


Proof. Let P_1, Q_3; P_2, Q_1; and P_3, Q_2 be points on AB, BC, and CA, respectively, such that $P_1Q_2 \parallel BC$, $P_2Q_3 \parallel CA$, and $P_3Q_1 \parallel AB$.

Notice that AP_3KQ_3 is a parallelogram, so the midpoint of P_3Q_3 lies on AK. However, AK is the A-symmedian of $\triangle ABC$, implying that P_3Q_3 is antiparallel to BC. Therefore, $\angle AP_3Q_3 = \angle ABC = \angle Q_3P_1Q_2$ and so $P_1Q_2P_3Q_3$ is cyclic. Similarly, $P_1Q_1P_2Q_3$ and $Q_1P_2Q_2P_3$ are cyclic.

Assume that the three circumcircles are distinct. Then by the Radical Axis Theorem, their pairwise radical axes concur. However, their radical axes are AB, BC, and CA, which do not concur. Hence the circumcircles are not distinct and so they coincide. □

Theorem 6 (Second Lemoine Circle)**.** *Let K be the symmedian point of triangle ABC. Prove that the six intersections formed by the three antiparallels with respect to the sides of $\triangle ABC$ passing through K and the sides themselves lie on a circle.*



Proof. Define points as in Theorem 5, except with antiparallels. By Lemma 4, K is the midpoint of P_1Q_2, P_2Q_3, and P_3Q_1. Now note that $\angle KQ_1P_2 = \angle BAC = \angle Q_3P_2Q_1$ because AC and AB are antiparallel to Q_3P_2 and Q_1P_3, respectively. Thus $KQ_1 = KP_2$ and so by symmetry the circle centered at K with radius KP_1 passes through all six points P_1, P_2, P_3, Q_1, Q_2, and Q_3. □

The First Lemoine Circle is actually a special case of the more general Tucker Circle[1].

3 The Brocard Circle

A very nice result connects various triangle centers in a way that is quite unexpected and rather amazing in its simplicity. Let us define two more triangle centers:

Definition. The *first Brocard point* Ω of a triangle ABC whose vertices are labeled in counterclockwise order is the unique point inside the triangle such that $\angle \Omega AB = \angle \Omega BC = \angle \Omega CA = \omega$. The *second Brocard point* Ω' is the unique point such that $\angle \Omega BA = \angle \Omega CB = \angle \Omega AC = \omega$. The angle ω is called the *Brocard angle.*

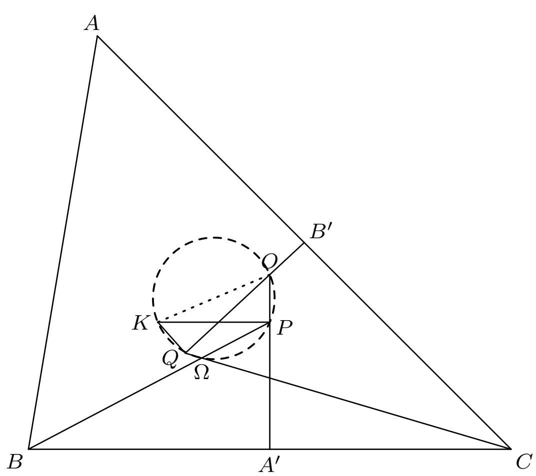
Lemma 7. *If ω is the Brocard angle of $\triangle ABC$ with area S,*

$$\cot \omega = \frac{AB^2 + BC^2 + CA^2}{4S}.$$

Lemma 8. *In $\triangle ABC$, the distance between K and a side of the triangle is $\frac{2S}{AB^2+BC^2+CA^2}$ times the length of the side.*

Theorem 9 (Brocard Circle)**.** *The points O, K, Ω, and Ω' are concyclic. Furthermore, OK is a diameter of this circumcircle.*

The following remarkable result states that four of our triangle centers are concyclic.



Proof. Let A' be the midpoint of BC, let B' be the midpoint of AC, and let $P = B\Omega \cap A'O$ and $Q = C\Omega \cap B'O$. Since $\angle PBA' = \omega$, $PA' = \frac{1}{2}BC \tan \omega$. By Lemma 7 and Lemma 8, the distances of P and K to BC are equal; that is, $KP \parallel BC$. Therefore, $\angle KPO = 90°$. Similarly, $KQ \parallel CA$ and so $\angle KQO = 90°$. It follows that K, P, O, and Q are concyclic.

Now notice that $\angle KQ\Omega = \angle AC\Omega = \angle CB\Omega = \angle \Omega PK$, so K, P, Q, Ω are concyclic. Similarly, K, P, Q, Ω' are also concyclic, implying that $KP\Omega'OQ\Omega$ is cyclic with diameter OK. □

[1] http://mathworld.wolfram.com/TuckerCircles.html

4 Other Symmedian Facts

Fact 1. The symmedian point of right triangle is the midpoint of the altitude to the hypotenuse.

Proof. Suppose that $\angle A = 90°$. Let B' be the midpoint of AC, and let M be the midpoint of the altitude AD. Since $\triangle ABC \sim \triangle DBA$, the median BB' of $\triangle ABC$ corresponds the median BM of $\triangle DBA$. Therefore, $\angle CBB' = \angle ABM$, implying that BM is indeed the B-symmedian of $\triangle ABC$. □

Fact 2. The symmedians of a triangle bisect the sides of its orthic triangle.

Fact 3. The line from the midpoint of a side of a triangle to the midpoint of the altitude to that side goes through the symmedian point.

Fact 4. The symmedian point of a triangle is the centroid of its pedal triangle.

Fact 5. The point inside a triangle which minimizes the sum of the squares of the distances to the sides is the symmedian point.

Fact 6. The symmedian from one vertex of a triangle, the median from another, and the appropriate Brocard ray from the third vertex are concurrent.

Fact 7. The symmedian point has barycentric coordinates $K = (a^2 : b^2 : c^2)$ and trilinear coordinates $K = (a : b : c)$.

Fact 8. The Gergonne point of a triangle is the symmedian point of the intouch triangle.

Fact 9. Let D be the intersection of AK with the circumcircle of ABC. Then quadrilateral $ABDC$ is harmonic.

The last and most elegant fact dealing with symmedians is very short but innately complex:

Definition. The *Brocard midpoint* Ω_m is the midpoint of $\Omega\Omega'$, or the midpoint of the two Brocard points.

Fact 10. The Brocard midpoint of the anticomplementary triangle is the isotomic conjugate of the symmedian point.

5 Problems

1. Let ABC be a triangle, and let ℓ be the A-median. Prove that the inverse of ℓ with respect to A is the A-symmedian of $\triangle AB'C'$, where B' and C' are the inverses of B and C, respectively.

2. Let PQ be a diameter of circle ω. Let A and B be points on ω on the same arc $\widehat{PQ}$, and let C be a point such that CA and CB are tangent to ω. Let ℓ be a line tangent to ω at Q. If $A' = PA \cap \ell$, $B' = PB \cap \ell$ and $C' = PC \cap \ell$, prove that C' is the midpoint of segment $A'B'$.

3. (PAMO 2013) Let $ABCD$ be a convex quadrilateral with AB parallel to CD. Let P and Q be the midpoints of AC and BD, respectively. Prove that if $\angle ABP = \angle CBD$, then $\angle BCQ = \angle ACD$.

4. (Iran 2013) Let P be a point outside of circle C. Let PA and PB be the tangents to the circle drawn from C. Choose a point K on AB . Suppose that the circumcircle of triangle PBK intersects C again at T. Let P' be the reflection of P with respect to A. Prove that $\angle PBT = \angle P'KA$.

5. (Poland 2000) Let ABC be a triangle with $AC = BC$, and a point P inside the triangle such that $\angle PAB = \angle PBC$. If M is the midpoint of AB, then show that $\angle APM + \angle BPC = 180°$.

6. (Russia 2010) Let O be the circumcenter of the acute non-isosceles triangle ABC. Let P and Q be points on the altitude AD such that OP and OQ are perpendicular to AB and AC respectively. Let M be the midpoint of BC and S be the circumcenter of triangle OPQ. Prove that $\angle BAS = \angle CAM$.

7. (Vietnam 2001) In the plane let two circles be given which intersect at two points A and B. Let PT be one of the two common tangent lines of these circles. Tangents at P and T to the circumcircle of triangle APT intersect at S. Let H be the reflection of B over PT. Show that A, S, and H are collinear.

8. (USAMO 2008, Modified) Let ABC be an acute, scalene triangle, and let M, N, and P be the midpoints of BC, CA, and AB, respectively. Let the perpendicular bisectors of AB and AC intersect ray AM in points D and E respectively, and let lines BD and CE intersect in point F, inside of triangle ABC. Prove that points A, N, F, and P all lie on one circle. Prove that AF is the A-symmedian of $\triangle ABC$.

9. Let ABC be a triangle, M the midpoint of segment BC and X the midpoint of the A-altitude. Prove that the symmedian point of $\triangle ABC$ lies on MX.

10. (TST 2007) Triangle ABC is inscribed in circle ω. The tangent lines to ω at B and C meet at T. Point S lies on ray BC such that $AS \perp AT$. Points B_1 and C_1 lie on ray ST (with C_1 in between B_1 and S) such that $B_1T = BT = C_1T$. Prove that triangles ABC and AB_1C_1 are similar to each other.

11. (ISL 2003/G2) Given three fixed pairwisely distinct points A, B, C lying on one straight line in this order. Let G be a circle passing through A and C whose center does not lie on the line AC. The tangents to G at A and C intersect each other at a point P. The segment PB meets the circle G at Q. Show that the point of intersection of the angle bisector of the angle AQC with the line AC does not depend on the choice of the circle G

12. (China TST 2010) Given acute triangle ABC with $AB > AC$, let M be the midpoint of BC. P is a point in triangle AMC such that $\angle MAB = \angle PAC$. Let O, O_1, O_2 be the circumcenters of $\triangle ABC, \triangle ABP, \triangle ACP$ respectively. Prove that line AO passes through the midpoint of O_1O_2.