New Zealand Mathematical Olympiad Committee

March Solutions

1. *Recall, that for a positive integer c, c! denotes the product of the positive integers from 1 to c inclusive (so, for example, $4! = 1 \times 2 \times 3 \times 4$.) Determine all pairs of positive integers (m, n) where $m < n$ and $n!/m!$ is a power of 2.*

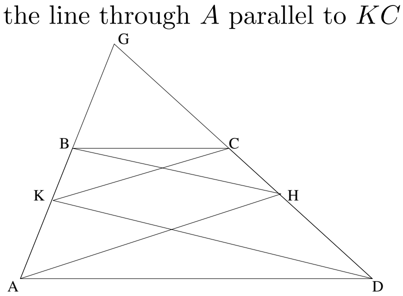
   Solution: One obvious infinite family of solutions occur when n is a power of 2, and $m = n - 1$, i.e. the pairs $(2^k - 1, 2^k)$. In fact, these are the only solutions. If $n > m + 1$, then $n!/m! = (m + 1)(m + 2)\cdots n$. One of the two numbers $m+1$ and $m+2$ is odd and greater than 1, but the only odd factor of any power of 2 is 1. So, we cannot have $n > m + 1$. Thus, the only possible solutions have $n = m + 1$, and then $n!/m! = n$, and so n must be a power of 2.

2. *Five cars leave Christchurch headed south on State Highway 1 to Dunedin, separated by various intervals, and traveling at various speeds. Before any of them reach Dunedin, five other cars leave Dunedin headed north on State Highway 1 to Christchurch. Whenever any two of these ten cars, traveling in opposite directions meet, they both stop and turn around. Whenever any of the cars reaches either Christchurch or Dunedin (regardless of where they started), they remain there. Assuming that only two cars ever meet simultaneously, what are the minimum and maximum possible numbers of meetings of cars before they all stop?*

   Solution: The minimum and maximum are both the same: 25. To see this, imagining that each car is carrying a letter, and that when cars meet, they exchange letters before turning around. Then the five letters that started in Christchurch go directly to Dunedin, crossing paths with the five letters that started in Dunedin which go directly to Christchurch. As each of the five southbound letters meets each of the five northbound letters, there must be 25 meetings.

3. *The quadrilateral $ABCD$ is a trapezoid, with $AD \parallel BC$ (and $AC \nparallel BD$.) The point K is on AB. Prove that the line through A parallel to KC and the line through B parallel to KD intersect at a point on CD.*

   Solution: Let H be the point where the line through A parallel to KC meets AD.



   The problem is equivalent to showing that BH is parallel to KD. Let G be the point where AB (extended) intersects CD (likewise.) As $\triangle GBC$ and $\triangle GAD$ are similar, $GB/GA = GC/GD$. As $\triangle GKC$ and $\triangle GAH$ are similar, $GK/GA = GC/GH$. Dividing the first equation by the second, $GB/GK = GH/GD$. Hence $\triangle GBH$ and $\triangle GKD$ are similar, and $BH \parallel KD$ as required.

4. *There is a well known construction that produces a sequence of 1000 integers, none of which is prime:*

   $$1001! + 2, 1001! + 3, \cdots 1001! + 1001$$

   as the first has 2 as a factor, the second 3, the third 4, ..., and the last 1001. Is there a sequence of 1000 integers exactly seven of which are prime?

   Solution: Starting from the given sequence, remove the largest element, and add a new smallest element (so that we always have a sequence of 1000 consecutive integers). Each time we do this, the number of primes in the sequence changes by at most 1 (we can gain or lose a prime, but not more than one). By the time we reach the sequence '1, 2, 3, ..., 1000 (which will take a while!) we certainly have a sequence containing more than seven primes. Therefore, at some intermediate point (but we don't know when!) there must have been exactly seven primes.

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