CMT 2011-2012: Introduction (Group 2)

Hello and welcome to Collaborative Math Training[1] 2011-2012. We will focus on developing mathematical creativity and problem-solving skills in the realms of algebra, geometry, combinatorics, and number theory, in contrast to the rote applications found in most American high school math classrooms. Thus you should expect to really be challenged: the hardest problems on math competitions often require lots of experimentation with various approaches, and there wouldn't be a point in doing them if your first try always worked, anyway.

On the other hand, with enough dedication and hard work, you will be able to improve significantly, learn some interesting ideas, and hopefully have a lot of fun too along the way. After all, pure math is more or less the epitome of intellectual curiosity—it is not restrained the slightest by the ugliness of reality, and so every new beautiful or surprising result is an untainted jewel of logic and even a work of art in itself! We hope that this elegance so prevalent in math will motivate you to continue seeking its endless riches.

1 Discussion Problems

1. Find a closed form expression for $1 + 2 + \cdots + n$.
2. Evaluate the sum $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{2010 \cdot 2011}$.
3. If $2^a = (1+2^1)(1+2^2)(1+2^4)(1+2^8)(1+2^{16})(1+2^{32})(1+2^{64})$, find the integer closest to a.
4. Everybody has at most 1000000 hairs on their heads. Show that there exist at least 6000 people with the same number of hairs on their heads.
5. Max is at $(0, 2)$, and his house is at $(6, 13)$. There is a river on the x-axis. If he wishes to fetch a bucket of water from the river and return to his house, find the shortest distance Max must travel.
6. A monk walks up a hill at 6:00 am one morning and reaches the top at 6:00 pm. The next day, he walks down at 6:00 am and reaches the bottom at 6:00 pm. Show that is some time between during which the monk was at the same position on both days.
7. Find the sum of the coefficients of the polynomial $(x^2 - 3x + 1)^{2011}$.
8. How many zeroes are there at the end of 2011!?
9. Compute $\frac{1}{\sqrt{1}+\sqrt{2}} + \frac{1}{\sqrt{2}+\sqrt{3}} + \cdots + \frac{1}{\sqrt{99}+\sqrt{100}}$.
10. In equilangular octagon $ABCDEFGH$, $AB^2 = 36$, $BC^2 = 50$, $CD^2 = 81$, $DE^2 = 98$, $EF^2 = 25$, and $GH^2 = 4$. Find HA^2.
11. In the context of this problem, a square is a 1×1 block, a domino is a 1×2 block, and a triomino is a 1×3 block. If N is the number of ways George can place one square, two identical dominoes, and three identical trominoes on a 1×20 chessboard such that no two overlap, find the remainder when N is divided by 1000.

[1] This is a collaborative effort among many schools nationwide.

12. Find the last three digits of the number of 7-tuples of positive integers $(a_1, a_2, a_3, a_4, a_5, a_6, a_7)$ such that a_i divides a_{i+1} for $1 \leq i \leq 6$ and $a_7 | 6468$.

13. The numbers 1 through 2011 are written on a board. At any time, we may take two numbers on the board and replace them by the absolute value of their difference. We repeat this procedure until we remain with one number on the board. Show that this last number is even.

2 Practice Problems

These problems are roughly ordered in difficulty. Some of the problems near the end are quite difficult. Feel free to work with others.

1. Find the remainder when $9 \times 99 \times 999 \times \cdots \times \underbrace{99 \cdots 9}_{999 \text{ 9's}}$ is divided by 1000.

2. Maya lists all the positive divisors of 2010^2. She then randomly selects two distinct divisors from this list. Let p be the probability that exactly one of the selected divisors is a perfect square. The probability p can be expressed in the form $\frac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

3. Given regular pentagon $ABCDE$, a circle can be drawn that is tangent to $\overline{DC}$ at D and to $\overline{AB}$ at A. Find the number of degrees in minor arc AD.

4. Given that $x^2 + y^2 = 14x + 6y + 6$, what is the largest possible value that $3x + 4y$ can have?

5. $P(n)$ is a polynomial of degree 2011 such that $P(n) = \frac{1}{n}$ for $n = 0, 1, \ldots, 2011$. Find $P(2012)$.

6. Jackie and Phil have two fair coins and a third coin that comes up heads with probability $\frac{4}{7}$. Jackie flips the three coins, and then Phil flips the three coins. Let $\frac{m}{n}$ be the probability that Jackie gets the same number of heads as Phil, where m and n are relatively prime positive integers. Find $m + n$.

7. Positive integers a, b, c, and d satisfy $a > b > c > d$, $a+b+c+d = 2010$, and $a^2-b^2+c^2-d^2 = 2010$. Find the number of possible values of a.

8. Let $P(x)$ be a quadratic polynomial with real coefficients satisfying $x^2 - 2x + 2 \leq P(x) \leq 2x^2 - 4x + 3$ for all real numbers x, and suppose $P(11) = 181$. Find $P(16)$.

9. Define an ordered triple (A, B, C) of sets to be minimally intersecting if $|A \cap B| = |B \cap C| = |C \cap A| = 1$ and $A \cap B \cap C = \emptyset$. For example, $(\{1, 2\}, \{2, 3\}, \{1, 3, 4\})$ is a minimally intersecting triple. Let N be the number of minimally intersecting ordered triples of sets for which each set is a subset of $\{1, 2, 3, 4, 5, 6, 7\}$. Find the remainder when N is divided by 1000.

10. For a real number a, let $\lfloor a \rfloor$ denote the greatest integer less than or equal to a. Let $\mathcal{R}$ denote the region in the coordinate plane consisting of points (x, y) such that $\lfloor x \rfloor^2 + \lfloor y \rfloor^2 = 25$. The region $\mathcal{R}$ is completely contained in a disk of radius r (a disk is the union of a circle and its interior). The minimum value of r can be written as $\frac{\sqrt{m}}{n}$, where m and n are integers and m is not divisible by the square of any prime. Find $m + n$.

11. Let (a, b, c) be the real solution of the system of equations $x^3 - xyz = 2$, $y^3 - xyz = 6$, $z^3 - xyz = 20$. The greatest possible value of $a^3 + b^3 + c^3$ can be written in the form $\dfrac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

12. Find the smallest positive integer n with the property that the polynomial $x^4 - nx + 63$ can be written as a product of two nonconstant polynomials with integer coefficients.

13. Positive numbers x, y, and z satisfy $xyz = 10^{81}$ and $(\log_{10} x)(\log_{10} yz) + (\log_{10} y)(\log_{10} z) = 468$. Find $\sqrt{(\log_{10} x)^2 + (\log_{10} y)^2 + (\log_{10} z)^2}$.

14. Dave arrives at an airport which has twelve gates arranged in a straight line with exactly 100 feet between adjacent gates. His departure gate is assigned at random. After waiting at that gate, Dave is told the departure gate has been changed to a different gate, again at random. Let the probability that Dave walks 400 feet or less to the new gate be a fraction $\dfrac{m}{n}$, where m and n are relatively prime positive integers. Find $m + n$.

15. Let $P(z) = z^3 + az^2 + bz + c$, where a, b, and c are real. There exists a complex number w such that the three roots of $P(z)$ are $w + 3i$, $w + 9i$, and $2w - 4$, where $i^2 = -1$. Find $|a + b + c|$.

16. Let N be the number of ordered pairs of nonempty sets $\mathcal{A}$ and $\mathcal{B}$ that have the following properties:

    - $\mathcal{A} \cup \mathcal{B} = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$,
    - $\mathcal{A} \cap \mathcal{B} = \emptyset$,
    - The number of elements of $\mathcal{A}$ is not an element of $\mathcal{A}$,
    - The number of elements of $\mathcal{B}$ is not an element of $\mathcal{B}$.

    Find N.

17. Let $ABCDEF$ be a regular hexagon. Let G, H, I, J, K, and L be the midpoints of sides AB, BC, CD, DE, EF, and AF, respectively. The segments AH, BI, CJ, DK, EL, and FG bound a smaller regular hexagon. Let the ratio of the area of the smaller hexagon to the area of $ABCDEF$ be expressed as a fraction $\dfrac{m}{n}$ where m and n are relatively prime positive integers. Find $m + n$.

18. Let m be a positive integer. Show that $1000^m - 1$ does not divide $1994^m - 1$.

19. Prove that for every positive integer n there exists an n-digit number divisible by 5^n all of whose digits are odd.

20. Find the smallest positive integer n such that if n squares of a 1000×1000 chessboard are colored, then there will exist three colored squares whose centers form a right triangle with sides parallel to the edges of the board.

21. We have $n \geq 2$ lamps $L_1, \ldots, L_n$ in a row, each of them being either on or off. Every second we simultaneously modify the state of each lamp as follows: if the lamp L_i and its neighbours (only one neighbour for $i = 1$ or $i = n$, two neighbours for other i) are in the same state, then L_i is switched off; otherwise, L_i is switched on. Initially all the lamps are off except the leftmost one which is on. Prove that (a) there are infinitely many integers n for which all the lamps will eventually be off, and (b) there are infinitely many integers n for which all the lamps will never be all off.

22. Let $a > b > 0$ be integers. Show that there exists a positive integer n such that $a^n - b^n$ is not of the form x^y for some positive integers $x, y \geq 2$.