LOGIC

One of the most important criteria in studying mathematics is having a logical mind. In the following texts, some fundamental ideas will be presented here.

1. Statement

A **statement**, or proposition, is a sentence that is readable, and is *either* true or false, but not both. For example, both “Today is Sunday” and “1+1=2” are statements. However, the sentences “Excuse me” and “What is a statement?” are not statements since we can’t say the sentences themselves are true or false.

If a statement is true, we say its **truth value** is T or 1. If it’s false, its truth value is F or 0.

2. Composition of Statements

Very often, especially in mathematics, we need to combine two or more statements into one. This is called a compound statement. Basically, there are four types of fundamental composition. They are negation, disjunction, conjunction and conditional.

The **negation** of a statement p is a statement which has truth value T when p is false, and has truth value F when p is true. It is denoted by $\sim p$, read as not p.

Illustration. Let p = “Today is Sunday”, then $\sim p$ = “Today is not Sunday”.

In showing relations between statements, truth table is often used.

p	$\sim p$
T	F
F	T

Table 1: Truth table of ~p

In table 1, we first list out all the possible truth values of p in the first column, then write down the corresponding truth values of $\sim p$ in the second column.

The **disjunction** of two statements *p* and *q* is the statement which is true when at least one of *p* and *q* is true, and is false when both *p* and *q* are false. It is denoted by $p \vee q$, read as *p* or *q*.

Illustration. Let *x* be an integer. The disjunction of "*x* is odd" and "*x* is even" is always true.

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

Table 2: Truth table of $p \vee q$

The truth table of $p \vee q$ is shown in table 2. Since there are two possible truth values for each of *p* and *q*, there are totally $2^2 = 4$ possible combinations. As a result, there are 4 rows for truth values of *p* and *q*, and the corresponding truth values of $p \vee q$ are listed in the third column.

The **conjunction** of two statements *p* and *q* is the statement which is true when both *p* and *q* is true, and is false when either *p* or *q* is false. It is denoted by $p \wedge q$, read as *p* and *q*. The truth table of $p \wedge q$ is shown in table 3.

p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

Table 3: Truth table of $p \wedge q$

The conditional of two statements p and q is the statement which is false when *p* is true and *q* is false, and is true otherwise. It is denoted by $p \rightarrow q$, read as if *p*, then *q*. The truth table of $p \rightarrow q$ is shown in table 4.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

Table 4: Truth table of $p \to q$

- "$\sim$", "$\vee$", "$\wedge$" and "$\to$" are called **logical operators**. It is worth noting that "$\vee$" and "$\wedge$" are **commutative**, while "$\to$" is not. That means $p \vee q$ and $q \vee p$ have the same truth values, so do $p \wedge q$ and $q \wedge p$, but $p \to q$ may have different truth value with $q \to p$.

- When the statement "$p \to q$" is always true, we will use the symbol **implication** "$\Rightarrow$" instead, and read as "*p* implies *q*".

Illustration. Let x is a real number, p = "x is a positive integer", q = "$x > -1$". Then $p \to q$ is always true, but $q \to p$ may not be true. We may write $p \Rightarrow q$.

Another common composition of statement is **biconditional**. The biconditional of two statements p and q is the conjunction of the statements "If p, then q" and "If q, then p". It is denoted by $p \leftrightarrow q$, read as p if and only if q. The notation "p iff q" is also used. Symbolically, it is the statement $(p \to q) \wedge (q \to p)$. The truth table of $p \leftrightarrow q$ is shown in table 5.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

Table 5: Truth table of $p \leftrightarrow q$

Readers may wonder why table 5 is the truth table of $p \leftrightarrow q$. We will illustrate it in example 2.1.

Example 2.1.

Construct the truth table of $p \leftrightarrow q$.

Solution.

Since $p \leftrightarrow q$ is the statement $(p \to q) \wedge (q \to p)$, we add the columns $p \to q$ and $q \to p$ and fill up them first. Once it is done, we can fill in the column $(p \to q) \wedge (q \to p)$ by the rules in table 4. In this way, we make the table.

p	q	$p \to q$	$q \to p$	$(p \to q) \wedge (q \to p)$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

- When the logical operators "$\sim$", "$\vee$", "$\wedge$" are used together, we must be careful about the order of operations. The precedence level follows in the order "$\sim$", "$\vee$", "$\wedge$", i.e. $\sim p \vee q$ means (not p) or q instead of not (p or q). If we want to change the statement to the second one, we can add a bracket to change it to $\sim (p \vee q)$.

- Similar to conditional, we write "$p \Leftrightarrow q$" if the statement "$p \leftrightarrow q$" is always true.

- From the table, we see that $(p \to q) \wedge (q \to p)$ is true when p and q have the same truth value, and is false otherwise.

Example 2.2.

Construct the truth table of $(\sim p \vee \sim q) \to r$.

Solution.

p	q	r	$\sim p$	$\sim q$	$\sim p \vee \sim q$	$(\sim p \vee \sim q) \to r$
T	T	T	F	F	F	T
T	T	F	F	F	F	T
T	F	T	F	T	T	T
T	F	F	F	T	T	F
F	T	T	T	F	T	T
F	T	F	T	F	T	F
F	F	T	T	T	T	T
F	F	F	T	T	T	F

Two statements, p and q are said to be **equivalent** if they share the same truth value. It is denoted by $p \equiv q$.

Example 2.3.

Prove that $p \to q \equiv \sim p \vee q$.

Solution.

In dealing with this kind of questions, one of the ways is constructing a truth table.

p	q	$\sim p$	$p \to q$	$\sim p \vee q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

It can be seen that the last two columns have the same truth values, hence they are equivalent.

Theorem 2.1.

Let p, q, r be statements, then

1. $p \vee p \equiv p$;
2. $p \wedge p \equiv p$;
3. $(p \vee q) \vee r \equiv p \vee (q \vee r)$;
4. $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$;
5. $p \vee q \equiv q \vee p$;
6. $p \wedge q \equiv q \wedge p$;
7. $(p \vee q) \wedge r \equiv (p \wedge r) \vee (q \wedge r)$;
8. $(p \wedge q) \vee r \equiv (p \vee r) \wedge (q \vee r)$;
9. $\sim (\sim p) \equiv p$.

Proof. All of these can be easily proved by constructing truth tables, and is left as an exercise to the readers.

- In theorem 2.1, (1), (2) are known as idempotent laws, (3), (4) are called associative laws, (5), (6) are commutative laws, and (7), (8) are called distributive laws.

Theorem 2.2. (De Morgan's Laws)

Let p, q be statements, then

1. $\sim(p \vee q) \equiv \sim p \wedge \sim q$;
2. $\sim(p \wedge q) \equiv \sim p \vee \sim q$.

Proof. Exercise.

Example 2.4.

Prove that $p \to q \equiv \sim q \to \sim p$.

Solution.

It can be seen from the truth table below.

p	q	$\sim q$	$\sim p$	$p \to q$	$\sim q \to \sim p$
T	T	F	F	T	T
T	F	T	F	F	F
F	T	F	T	T	T
F	F	T	T	T	T

The statement $\sim q \to \sim p$ is called the **contrapositive** statement of $p \to q$. Instead of proving a statement of the form $p \Rightarrow q$ directly, example 2.4 suggests that we may do it in the "backward" direction, i.e. its contrapositive statement is true. We will illustrate it in example 2.5.

Example 2.5.

Let $\{a_1, a_2, ..., a_{10}\}$ be a strictly increasing sequence of positive integers not exceeding 45. Prove that there exist two distinct pairs such that the differences of the two numbers in each pair are the same.

Solution.

Let p be the statement "$a_1, a_2, ..., a_{10}$ be a strictly increasing sequence of positive integers not exceeding 45" and q be "there exist two distinct pairs such that the differences of the two numbers in each pair are the same". Suppose $\sim q$ is true, i.e. the difference of the two numbers in distinct pairs are different. Then $a_{10} = a_1 + \sum_{i=1}^{9}(a_{i+1} - a_i) \geq 1 + \sum_{i=1}^{9} i = 46$, which implies $\sim p$. Hence we have shown the contrapositive statement $p \Rightarrow q$, which is equivalent to $\sim q \Rightarrow \sim p$.

Example 2.6.

Show that $\sqrt{2}$ is irrational, i.e. there doesn't exist integers *p*, *q* such that $\sqrt{2} = p/q$.

Solution.

Suppose there exists integers *p*, *q* such that $\sqrt{2} = p/q$. We may further assume that *p*, *q* are coprime, i.e. their greatest common divisor is 1. Squaring the equation we get $2q^2 = p^2$, implying that *p* is even and so we let $p = 2m$ for some integer *m*. However, if we substitute it into $2q^2 = p^2$, it would yield $q^2 = 2m^2$, meaning *q* is even again. It contradicts to our assumption p, q are coprime, so our first assumption, i.e. $\sqrt{2}$ is rational, is false.

- The method used in example 2.5 and 2.6 is called "**proof by contradiction**".

- The logic behind the proof in example 2.5 and 2.6 is a little bit different. In example 2.5, we show that the negation of the "consequence" leads to negation of the given conditions, while in example 2.6 we have shown that the assumption contradicts to itself.

Exercise

1. Finish the proof of the remaining parts of theorem 2.1.

2. Prove theorem 2.2