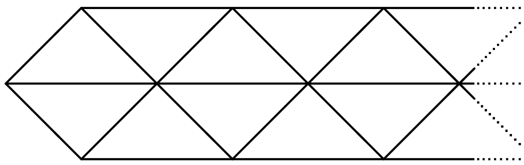
New Zealand Mathematical Olympiad Committee

2010 Squad Assignment Three

Combinatorics

Due: Thursday 18th March 2010

1. *In the road network shown below, the vertices in the middle horizontal line are labeled* $1, 4, 7, \ldots$*, the vertices in the upper row are labelled* $2, 5, 8, \ldots$*, and the vertices in the bottom row are labelled* $3, 6, 9 \ldots$.



How many paths are there from the vertex labelled 1 *to the vertex labelled* $3n+1$ *such that vertices are visited only in increasing order?*

Solution: We observe that we can only move from one vertex to another if we are moving in the rightward direction. Let a_n be the number of ways of reaching vertex $3n+1$ from vertex 1 and b_n be the number of ways of reaching vertex $3n+2$ or $3n+3$ from vertex 1 (clearly these two are the same from symmetry). Then by considering our previous possible position when we move to any vertex we can construct the following recursive formulas $a_{n+1} = a_n + 2b_n$ and $b_{n+1} = a_{n+1} + b_n$ with starting conditions $a_0 = 1$ and $b_0 = 1$. These can be solved for equations in terms of the a_i alone: we have $b_n = \frac{a_{n+1}+a_n}{2}$, so

$$b_{n+1} = \frac{a_{n+2} + a_{n+1}}{2} = a_{n+1} + \frac{a_n + a_{n-1}}{2}$$

$$a_{n+2} = 4a_{n+1} - a_n.$$

This recursion has characteristic equation $x^2 = 4x - 1$ with roots $x_{1,2} = 2 \pm \sqrt{3}$. Hence we solve the equation $a_n = A(2+\sqrt{3})^n + B(2-\sqrt{3})^n$ with $a_0 = 1$ and $a_1 = 3$ to obtain $A = \frac{3+\sqrt{3}}{6}$ and $B = \frac{-3+\sqrt{3}}{6}$. Therefore

$$a_n = \frac{3+\sqrt{3}}{6}(2+\sqrt{3})^n + \frac{-3+\sqrt{3}}{6}(2-\sqrt{3})^n.$$

2. *An odd integer is written in each cell of a* 2009×2009 *table. For* $1 \leq i \leq 2009$ *let* R_i *be the sum of the numbers in the ith row, and for* $1 \leq j \leq 2009$ *let* C_j *be the sum of the*

numbers in the jth column. Finally, let A be the product of the R_i, and B the product of the C_j.

Prove that $A + B$ is different from zero.

Solution: We consider all the integers in the cells mod 4, so each entry is ± 1. We will show that $A \equiv B \pmod 4$ and since $A \equiv \pm 1 \pmod 4$ we will have that $A + B \equiv 2 \pmod 4$ and so cannot be equal to 0. Consider R to be sum of the R_i and C to be the sum of the C_i. Then clearly both R and C are the sums of all the entries in the cells, hence $R = C$, and in particular $R \equiv C \pmod 4$.

Because R_i and C_i are a sum of 2009 odd integers, they are both odd for all i. Let there be r values of i for which $R_i \equiv -1 \pmod 4$ and c values of i for which $C_i \equiv -1 \pmod 4$. Then $R \equiv (2009 - r) - r \equiv 2009 - 2r \equiv C \equiv (2009 - c) - c \equiv 2009 - 2c \pmod 4$. Hence $2r \equiv 2c \pmod 4$, or $r \equiv c \pmod 2$.

Now we observe that $A \equiv 1^{2009-r}(-1)^r \equiv (-1)^r \pmod 4$ and $B \equiv 1^{2009-c}(-1)^c \equiv (-1)^c \pmod 4$. Hence the values of A and B mod 4 are determined by the parities of r and c, but $r \equiv c \pmod 2$, so $A \equiv B \pmod 4$ as we claimed.

3. *A number of coins have been placed at each vertex of the regular n-gon $A_1 A_2 \ldots A_n$. These coins may be re-arranged using the following move: two coins may be chosen, and each moved to an adjacent vertex, subject to the requirement that one must be moved clockwise and the other anti-clockwise. (Thus, for example, you may move a coin from each of A_1 and A_5 to vertices A_2 and A_4 respectively: each coin ends up on a vertex adjacent to the one it started on, and they move in opposite directions.)*

   Suppose that there are initially k coins at vertex A_k for each k, $1 \le k \le n$. For which n is it possible to re-arrange the coins using finitely many such moves so that there are exactly $n + 1 - k$ coins at vertex A_k, $1 \le k \le n$?

   Solution: To all coins placed on vertex A_i we assign value i. We then let S be the sum of the values of all the coins and after every move we update the value of S. We observe that after a move where a coin is not moved from A_1 to A_n or vice versa, the sum of S stays constant since one coin increases in value by 1, and another decreases by 1. If a coin is moved from A_1 to A_n, then the change in S will be $(n-1)+1 = n$, and if a coin is moved from A_n to A_1 then the change in S will be $-(n-1)-1 = -n$. Hence we see that after every move, the remainder of S upon division by n will stay constant. The initial value of S is

$$1 \cdot 1 + 2 \cdot 2 + \cdots + n \cdot n = \sum_{k=1}^{n} k^2 = \frac{1}{6} n(n+1)(2n+1)$$

   and the final desired value of S is

$$\sum_{k=1}^{n} k(n+1-k) = (n+1)\sum_{k=1}^{n} k - \sum_{k=1}^{n} k^2 = \frac{1}{6} n(n+1)(n+2).$$

   Since the remainder of these two values must be equal upon division by n, their difference is divisible by n. Hence $\frac{1}{6}n(n+1)(2n+1) - \frac{1}{6}n(n+1)(n+2) = \frac{1}{6}n(n^2-1)$ must be divisible by n, i.e. $\frac{1}{6}(n^2-1)$ must be an integer. The only values of n for which this holds are $n \equiv \pm 1 \pmod 6$.

Now we can show that any desired arrangement can be reached provided that the initial and final values of S give the same remainder when divided by n. To see this, choose a coin from each of the initial and final positions, and distinguish it from the others by painting it red. Now move every coin other than the red coin to its desired position, by moving them all in the anti-clockwise direction only, and moving the red coin clockwise at every move. We claim that once this is done the red coin must be in its desired position too. Indeed, if the sum of the coins excluding the red one is T, then the red coin must be at the unique vertex A_k such that $k \equiv S_{\text{initial}} - T \bmod n$; but since $S_{\text{initial}} - T \equiv S_{\text{final}} - T \bmod n$, this must be the position of the red coin in the desired final arrangement also. Hence any arrangement can be reached from the initial position provided that the difference of the initial and final values of S is divisible by n. Therefore a sufficient and necessary condition for n is $n \equiv \pm 1 \pmod 6$.

4. *A convex 2010-gon is partitioned into triangles using non-intersecting diagonals. One of these diagonals is painted green. The triangulation may be modified using the following move: if ABC and BCD are triangles of the partition having BC as a common side, then the diagonal BC may be replaced by the diagonal AD. Moreover, if BC is green, then it loses its colour and AD becomes green instead. Prove that an arbitrarily chosen diagonal of the polygon can be coloured green using finitely many such operations.*

   Solution: Note that for any convex n-gon all the diagonals from a given vertex determine a triangulation (i.e. all the diagonals in this triangulation are attached to the one vertex). Call such a triangulation *basic* with respect to this vertex. We first show that starting from an arbitrary triangulation of an n-gon we can reach any basic triangulation.

   For a vertex A moving anti-clockwise denote by $B_1, B_2, \ldots, B_k$ the consecutive vertices of the n-gon such that AB_i is either a side or a diagonal in the given triangulation. If the segment B_iB_{i+1} is not a side then it is a diagonal in the triangulation. By carrying out the allowed operation we obtain a new triangulation with one more diagonal from vertex A between AB_i and AB_{i+1}. Continuing this until A has the maximum number of diagonals completes the basic triangulation.

   We prove the assertion for any n-gon by induction on n. For $n = 4$ and 5 the verification is straightforward. Assume the assertion is true for all n-gons with $n \leq k$ where $k \geq 5$ and consider an arbitrary triangulation of a $(k+1)$-gon with one green diagonal. Assume that A_1A_i is the diagonal we wish to colour green. We then turn the triangulation into a basic triangulation with respect to A_1. If after doing so A_1A_i is green then we are done. So assume without loss of generality that A_1A_j is green where $j < i$ (if $j > i$ then we can simply reverse the order of the vertex labels).

   Consider the polygon $A_1A_2 \ldots A_i$. The inductive hypothesis implies that we can obtain a triangulation of this polygon with green diagonal A_1A_{i-1}. Since $k \geq 5$ and any triangulation of a $(k+1)$-gon has $k-2$ diagonals we have that there exists a diagonal different from A_1A_{i-1} and A_1A_i. This diagonal divides the $(k+1)$-gon into two convex polygons each of them having less than $k+1$ vertices. Moreover the diagonals A_1A_{i-1} and A_1A_iboth lie in the same polygon and after applying the inductive hypothesis to this polygon we see that A_1A_i can be coloured green as desired.

5. *Let $n \geq 1$ be an integer. In town X there are n girls and n boys, and each girl knows each boy. In town Y there are n girls, g_1, g_2, $\ldots$, g_n, and $2n-1$ boys, b_1, b_2, $\ldots$, b_{2n-1}. For $i = 1, 2, \ldots, n$, girl g_i knows boys b_1, b_2, $\ldots$, b_{2i-1} and no other boys.*

   Let r be an integer with $1 \leq r \leq n$. In each of the towns a party will be held, where r girls from that town are to dance with r boys from the same town in r pairs of dancers. However, each girl will only dance with a boy that she knows. Let $X(r)$ be the number of ways we can choose r pairs of dancers from town X, and let $Y(r)$ be the number of ways that we can choose r pairs of dancers from town Y.

   Show that $X(r) = Y(r)$ for $r = 1, 2, \ldots, n$.

   Solution: For a given n we will denote $X(r)$ and $Y(r)$ by $X_n(r)$ and $Y_n(r)$. We see that $X_n(r)$ is the product of the number of ways of picking r girls, r boys and then pairing them. Hence $X_n(r) = \binom{n}{r}^2 r!$.

   We first check the cases when $r = 1$ and $r = n$. For town Y, g_i can go with a total of $2i - 1$ boys. So summing over all girls $Y_n(1) = \sum_{i=1}^{n}(2i-1) = n^2 = X_n(1)$. For $r = n$ every girl must be picked for a pair. We see g_1 can go with 1 boy, then g_2 can go with 2 boys since 1 of her 3 available boys was taken by g_1, then g_3 can go with 3 boys since 2 of her 5 available boys were taken by g_1, g_2, etc. ..., g_n can go with n boys. Hence $Y_n(n) = 1 \cdot 2 \cdot 3 \cdot \cdots \cdot n = n! = X_n(n)$.

   Consider now $2 \leq r \leq n-1$ and choosing r pairs from town Y. We can either choose g_n or not choose her. If we choose her then we have $r-1$ pairs to choose from the remainder of the girls who do not know b_{2n-2}, b_{2n-1}. There are $Y_{n-1}(r-1)$ ways of choosing the remainder of the girls, and g_n is then left with $(2n-1)-(r-1) = 2n-r$ available boys. Hence the total number of ways of including g_n is $(2n-r)Y_{n-1}(r-1)$ and note that this is well defined since $2 \leq r$. If we do not include g_n then since none of the girls except her know b_{2n-2}, b_{2n-1}, we are in the position of picking r pairs from the same town but with $n-1$ girls. Hence the number of ways of not including g_n is $Y_{n-1}(r)$, and note that this is well defined since $r \leq n-1$. Now $Y_n(r)$ is the sum of these, so $Y_n(r) = (2n-r)Y_{n-1}(r-1) + Y_{n-1}(r)$. It is easy to check that $X_n(r)$ satisfies the same recursive formula. Since $X_n(r)$ and $Y_n(r)$ agree on their boundary conditions and satisfy the same recursive formula which constantly looks at smaller values of n, $X_n(r) = Y_n(r)$ for all n and $r \leq n$.

   Alternate solution. We may also give a bijective solution, as follows. We will create n towns X_1, X_2, $\ldots$, X_n, each with n girls g_1, g_2, $\ldots$, g_n, and $2n-1$ boys, b_1, b_2, $\ldots$, b_{2n-1}. In town X_k girl g_i knows only boys b_1 to b_k if $i \leq k$, and only boys b_1 to b_{2i-1} if $i > k$. Clearly, town X_1 is the same as town Y; and in town X_n every girl knows boys $b_1, \ldots, b_n$, and no girl knows boys $b_{n+1}, \ldots, b_{2n-1}$, so we may consider X_n to be the same as X. Our goal is to show that there is a bijection between acceptable choices of r dancers in towns X_k and X_{k+1} for each k, $1 \leq k \leq n-1$.

   Suppose then that we have an acceptable arrangement of r pairs of dancers from town X_k. This will also be an acceptable arrangement in town X_{k+1}, unless girl g_{k+1} dances with boy b_{2k+2-i} for some $i \leq k$, as these two do not know each other in town X_{k+1}. In this case we ask girl g_i to dance with boy b_{k+1} in their stead. If girl g_i is already partnered with some boy b_j, $j \leq k$, then we have girl g_{k+1} dance with b_j, otherwise g_{k+1} does not dance; and if boy b_{k+1} is already partnered with some girl g_l then we have boy b_{2k+2-i} dance with girl g_l, otherwise boy b_{2k+2-i} does not dance.

We claim that the result of these rearrangements will be acceptable in town X_{k+1}. Every girl in town X_{k+1} knows boy b_{k+1}, so the first pairing is certainly acceptable; and the second is too, because $i \leq k$ forces $j \leq k$, so girl g_{k+1} knows boy b_j. For the last, we must have $l \geq k+1$ in order for girl g_l to know boy b_{k+1} in town X_k, which gives $l \geq k+2$, since $l \neq k+1$. Then $2k+2-i \leq 2(k+1)-1 < 2l-1$, so girl g_l knows boy b_{2k+2-i}.

Conversely, suppose that we have an acceptable arrangement of r pairs of dancers in town X_{k+1}. This will also be an acceptable arrangement of dancers in town X_k, unless girl g_i dances with boy b_{k+1} for some $i \leq k$. In this case we simply reverse the rearrangements made above. We ask girl g_{k+1} to dance with boy b_{2k+2-i}, with girl g_i taking g_{k+1}'s partner b_j if she has one (otherwise g_i does not dance), and b_{k+1} taking boy b_{2k+2-i}'s partner g_l if he has one (otherwise boy b_{k+1} does not dance).

The result will acceptable in town X_k. We must have $j \leq k+1$, in order for girl g_{k+1} to know boy b_j, and in addition $j \neq k+1$, since b_{k+1} was already dancing with girl g_i. So girl g_i knows boy b_j in town X_k, and moreover, girl g_l knows boy b_{k+1}, because $k+1 \leq 2k+2-i$. As we have simply reversed the rearrangements made two paragraphs above, this establishes the desired bijection.

Remark: This problem can be phrased quite naturally in terms of rook polynomials. Given a "chessboard" B, the co-efficient of x^r in the rook polynomial of B counts the number of ways to place r rooks on B so that no rook attacks any other. The result of the problem then says that the "chessboards" corresponding to towns X and Y have the same rook polynomial.

6. *Let G be a finite connected graph, whose edges are labelled $1, 2, \ldots, e$ in some order. Starting from an arbitrary vertex, repeat the following process:*

   (a) Choose the edge incident to the current vertex with the largest label.

   (b) Move along the chosen edge to the adjacent vertex, relabelling the edge 1, and adding 1 to the labels of all the other edges.

   Prove that eventually each edge is traversed.

   Solution: Call a vertex *recurrent* if it is visited infinitely many times, and similarly, call an edge recurrent if it is traversed infinitely many times. Clearly, G must have at least one recurrent vertex, since it has only finitely many vertices. We will solve the problem by proving the following lemma:

   Lemma. *Any edge adjacent to a recurrent vertex must itself be recurrent.*

   Before proving the lemma we will introduce some notation and make an observation. Write $\ell_n(a)$ for the label of edge a at step n. Condition (b) implies that

$$\ell_a(n+1) - \ell_b(n+1) = \begin{cases} \ell_a(n) & \text{if } b \text{ is traversed at step } n+1; \\ -\ell_b(n) & \text{if } a \text{ is traversed at step } n+1; \\ \ell_a(n) - \ell_b(n) & \text{otherwise.} \end{cases}$$

In particular, if $\ell_a(n) - \ell_b(n)$ is positive at $n = k$, it will remain positive for $n \geq k$ unless a is traversed. We now prove the lemma.

Proof. Let v be recurrent, and let f be an edge adjacent to v that is traversed infinitely often in the direction *away* from v. Such an edge must exist, because there are only finitely many edges adjacent to v, and v is visited infinitely often.

We claim that whenever f is traversed, every other edge adjacent to v must be traversed before f is next traversed in the direction away from v. Let g be an arbitrary edge adjacent to v, and suppose that f has just been traversed (in either direction) at step n. Then $\ell_g(n) - \ell_f(n) = \ell_g(n) - 1 > 0$.

Immediately before f is next traversed in the direction away from v we must have $\ell_f > \ell_h$ for all edges h adjacent to v, by condition (a). In particular, $\ell_f > \ell_g$, so $\ell_g - \ell_f$ is now negative. By our observation above, g must have been traversed.

Since f is traversed away from v infinitely often, and every other edge adjacent to v must be traversed between any two such occurrences, every edge adjacent to v is recurrent. □

The desired result now follows from the lemma. The lemma implies that every vertex adjacent to a recurrent vertex is recurrent; since the graph is connected, and has at least one recurrent vertex, every vertex is recurrent. But then every edge is adjacent to a recurrent vertex, so by the lemma again every edge is recurrent, and so is traversed at least once.

7. *Determine the largest positive integer n for which there exist pairwise different sets S_1, S_2, …, S_n with the following properties:*

   (a) $|S_i \cup S_j| \leq 2006$ *for any two indices* $1 \leq i, j \leq n$, *and*

   (b) $S_i \cup S_j \cup S_k = \{1, 2, \ldots, 2010\}$ *for any* $1 \leq i < j < k \leq n$.

   Solution: Consider $G_{\{i,j\}} = \{1, 2, \ldots, 2010\} \setminus (S_i \cup S_j)$ for all $1 \leq i, j \leq n$. Then condition (a) implies $|G_{\{i,j\}}| \geq 4$ and condition (b) implies $G_{\{i,j\}} \cap G_{\{k,l\}} = \emptyset$ for all i, j, k, l with $i \neq j$, $k \neq l$ and $\{i, j\} \neq \{k, l\}$. It follows that

$$2010 \geq \left| \bigcup_{1 \leq i < j \leq n} G_{\{i,j\}} \right| = \sum_{1 \leq i < j \leq n} |G_{\{i,j\}}| \geq 4\binom{n}{2},$$

   hence $\binom{n}{2} \leq 502$ so $n \leq 32$.

   Now we show that $n = 32$ is possible. Partition the set $\{1, 2, \ldots, 1984\}$ arbitrarily into 494 (disjoint) 4-element subsets, denote these subsets $G_{\{i,j\}}$, $1 \leq i < j \leq n$ in an arbitrary manner. Then define $S_i = \{1, 2, \ldots, 2010\} \setminus \bigcup_{j \neq i} G_{\{i,j\}}$ for each $i = 1, 2, \ldots, 32$.

   Then $S_i \cup S_j = \{1, 2, \ldots, 2010\} \setminus G_{\{i,j\}}$ and so condition (a) is satisfied and since $S_i \cup S_j \cup S_k = (S_i \cup S_j) \cup (S_j \cup S_k) = \{1, 2, \ldots, 2010\} \setminus \left(G_{\{i,j\}} \cap G_{\{j,k\}}\right) = \{1, 2, \ldots, 2010\}$, condition (b) is also satisfied. Hence the largest n satisfying the given conditions is 32.

8. *Consider a graph with n vertices, and let k, $1 \le k \le n$, be a positive integer. It is known that among any k vertices there exists a vertex which is connected to the remaining $k-1$ vertices. Find all values of n and k for which there must always exist a vertex of degree $n-1$.*

   Solution: Consider the complement of the given graph, that is, the graph with the same vertices, and an edge between vertices u and v precisely when the original graph does not. The condition in the problem then states that among any k vertices, there is one that is not connected to any of the remaining $k-1$; more succinctly, the subgraph induced by any k vertices contains an isolated vertex. The problem is then to determine for which n and k the graph as a whole must have an isolated vertex. We will assume that the graph as a whole does not have an isolated vertex, and determine for which n and k we are able to satisfy the condition. The solution will then be those n and k for which we are never able to do this.

   The assumption that the graph does not have an isolated vertex implies that every connected component has at least two vertices. Suppose first that every component has exactly two vertices (and so that n is necessarily even). If k is odd then any choice of k vertices will intersect at least one connected component in just one vertex, and the condition will be satisfied; but if $k = 2m$ is even then we may choose k vertices that violate the condition by choosing m of the connected components. So the graph need not have an isolated vertex if n is even and k is odd.

   Suppose now that there is at least one connected component C with at least 3 vertices. We claim that in this case it is possible to choose k vertices that violate the condition if $k \ge 2$. The vertices may be chosen as follows:

   (a) Choose two adjacent vertices from C, and then two adjacent vertices from each of the other components in turn, stopping if at any point either k or $k-1$ vertices have been chosen.

   (b) If fewer than k vertices were chosen at the previous step, then either $k-1$ vertices have been chosen, or two adjacent vertices have been chosen from each component. In the former case we obtain k vertices violating the condition by choosing a third vertex from C adjacent to one of those already chosen. Otherwise, we simply continue choosing vertices adjacent to those already chosen one by one until we have a total of k vertices. This is certainly possible, since there are $n \ge k$ vertices, and each vertex belongs to a component containing a vertex already chosen.

   Since any graph satisfies the condition with $k = 1$, the above shows that the graph as a whole necessarily has an isolated vertex unless $k = 1$, or n is even and k is odd. □

April 12, 2010

www.mathsolympiad.org.nz