New Zealand Mathematical Olympiad Committee

Transformations in Geometry

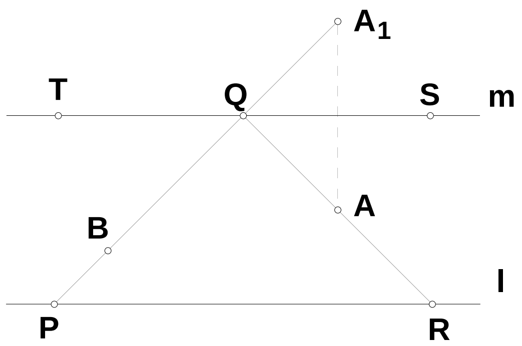
Arkadii Slinko

1 Reflections

We start with the following simple problem:

Example 1. Given two parallel lines ℓ, m and two points A, B between them, construct an isosceles triangle PQR with $PQ = QR$ such that P and R are on ℓ, Q is on m, and the two lateral sides QR and QP pass through A and B, respectively.

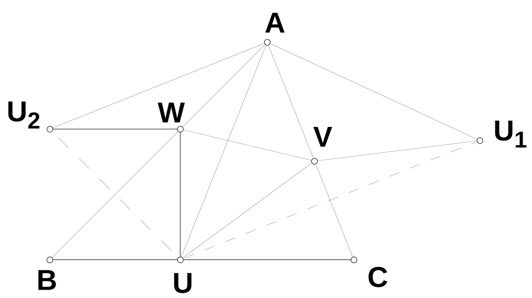
Solution. Suppose that PQR is the triangle sought. Consider the reflection of the plane in the line m. Let A_1 be the image of A.



Let T and S be two points of m situated as shown. Then $\angle A_1QS = \angle SQA = \angle QRP = \angle QPR = \angle PQT$ and PQA_1 is a straight line. We can draw this line since we know two distinct points of it, namely B and A_1. This gives the method of constructing PQ and the triangle PQR as well. ☐

Example 2 (Classical Problem of Fagnano)**.** Given an acute-angled triangle ABC, inscribe in it a triangle UVW of least possible perimeter.

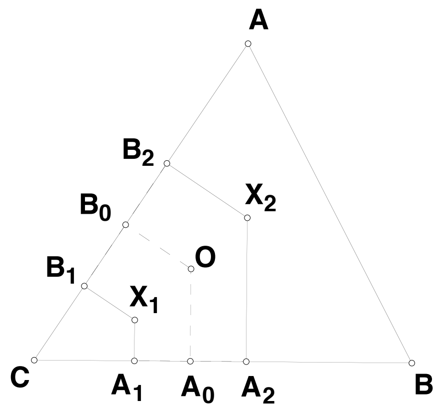
Solution. Let UVW be an arbitrary inscribed triangle of ABC (see diagram below) and let points U_1, U_2 be the reflections of U through the lines AB and AC.



The perimeter p of UVW is equal to $p = UV + VW + WU = U_1V + VW + WU_2$. For a fixed U, this sum is minimal in the case when the broken line U_1VWU_2 is a straight segment. When that happens, the value of p will be $p = 2AU \sin \angle A$ since $AU_1{=}AU_2 = AU$ and $\angle U_1AU_2 = 2\angle A$. Hence p will be minimal when AU is minimal and U_1VWU_2 is a straight line. AU is minimal, when it is the altitude of ABC. Similar arguments work for BV and CW. Hence the perimeter of $\triangle UVW$ is minimal when U, V, W are the feet of the altitudes ofABC. □

Example 3. A circle intersects the sides AB, BC and CA of a triangle ABC at points C_1, C_2, A_1, A_2, B_1, B_2 respectively. Suppose that perpendiculars drawn to the sides AB, BC, CA through the points C_1, A_1, B_1 are concurrent. Prove that perpendiculars to these sides drawn through the points C_2, A_2 and B_2 are also concurrent.

Solution. Let C_0, A_0 and B_0 be the midpoints of AB, BC and CA respectively.

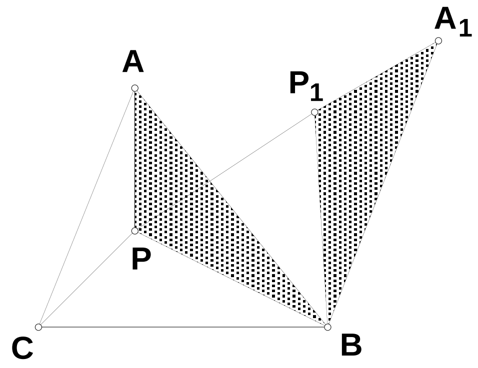


Let X_1 be the point of intersection of the perpendiculars drawn through A_1 and B_1, and let X_2 be the point of intersection of the perpendiculars drawn through A_2 and B_2. Since the line A_1X_1 is symmetric to the line A_2X_2 about OA_0 and similarly B_1X_1 is symmetric to B_2X_2 about OB_0, the point X_2 is centrally symmetric to X_1 about the point O. It is clear now that if the first set of perpendiculars intersect at X_1, then the second set of perpendiculars intersect at X_2. □

2 Rotations

Example 4 (Fermat's problem)**.** In a given acute-angled triangle ABC find a point P for which the sum $AP + BP + CP$ of segment lengths is minimal.

Solution. Let P be an arbitrary point inside ABC. Rotate the triangle ABP clockwise about B by 60°.

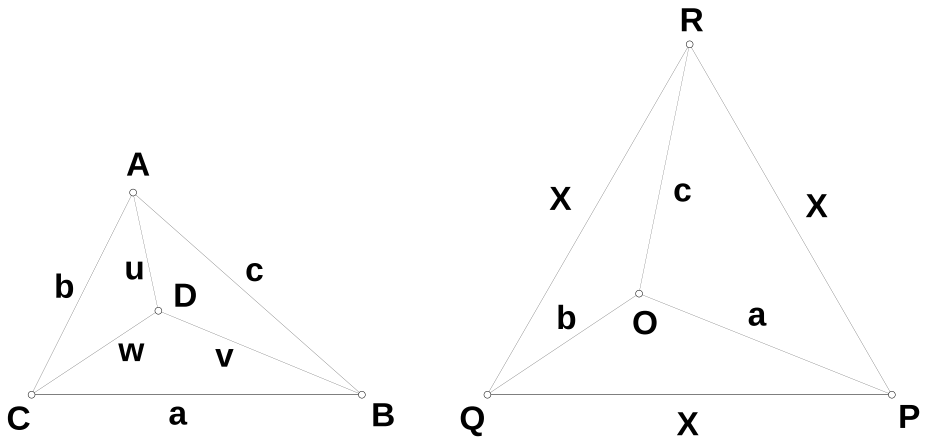


Then we obtain $\triangle A_1BP_1$. Since $AP = A_1P_1$ and $BP = BP_1 = PP_1$ it is clear that

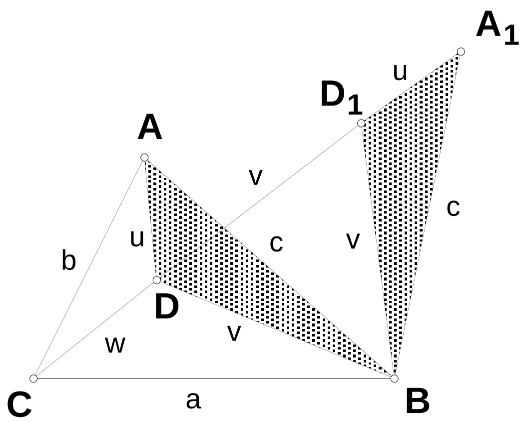
$$AP + BP + CP = A_1P_1 + P_1P + PC.$$

Thus we see that the sum $AP+BP+CP$ is minimal when $A_1P_1+P_1P+PC$ is a straight segment. This happens when $\angle BPC = 180^\circ - \angle P_1PB = 180^\circ - 60^\circ = 120^\circ$. Similar arguments show that $\angle APB = \angle CPA = 120^\circ$. The point P with these properties is known as the *Fermat point* of ABC. □

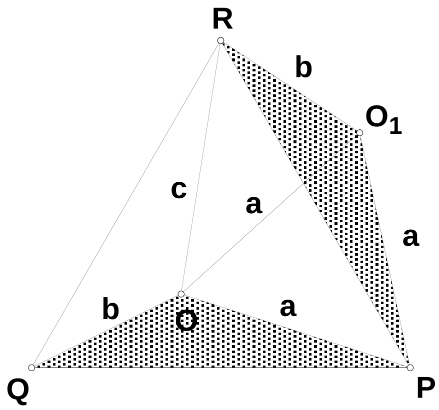
Example 5 (USAMO 1973)**.** Two triangles ABC and PQR, with associated points and lengths as shown on the figure, satisfy $\angle ADB = \angle BDC = \angle CDA = 120^\circ$. Prove that $X = u + v + w$.



Solution. Let us rotate the triangle ABD clockwise about the vertex B through the angle of 60° as shown. It will be mapped onto a triangle ABD_1:



The triangle BDD_1 thus obtained is equilateral and the length of DD_1 is v. So CDD_1A_1 is a straight segment of length $u + v + w$. It follows that a triangle A_1BC has sides a, c, $u + v + w$ and that $\angle A_1BC = \angle B + 60^\circ$. Let us turn to another triangle PQR and rotate the triangle OPQ clockwise about P through an angle of 60°. We shall obtain a triangle RO_1P:

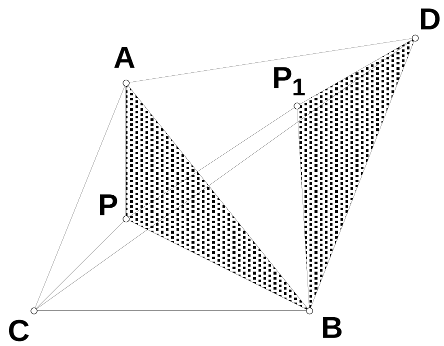


The triangle $OO'P$ is equilateral. Thus the length of OO_1 is a and $\angle O_1OP = 60°$. This means that $\triangle RO_1O = \triangle ABC$ and $\angle POR = \angle ROO_1 + \angle O_1OP = \angle B + 60°$. Now $\triangle A_1BC$ and $\triangle ROP$ are equal, whence $X = u + v + w$. □

The following variation on Fermat's problem is also quite interesting.

Example 6 (Fermat Point Inequality)**.** Let P be a point inside a triangle ABC and let ABD be an equilateral triangle erected externally on side AB. Then $PA + PB + PC \geq CD$.

Solution. Let us draw the already familiar picture obtained by rotating the triangle APB through $60°$ about vertex B. After this rotation the image of A will coincide with D.



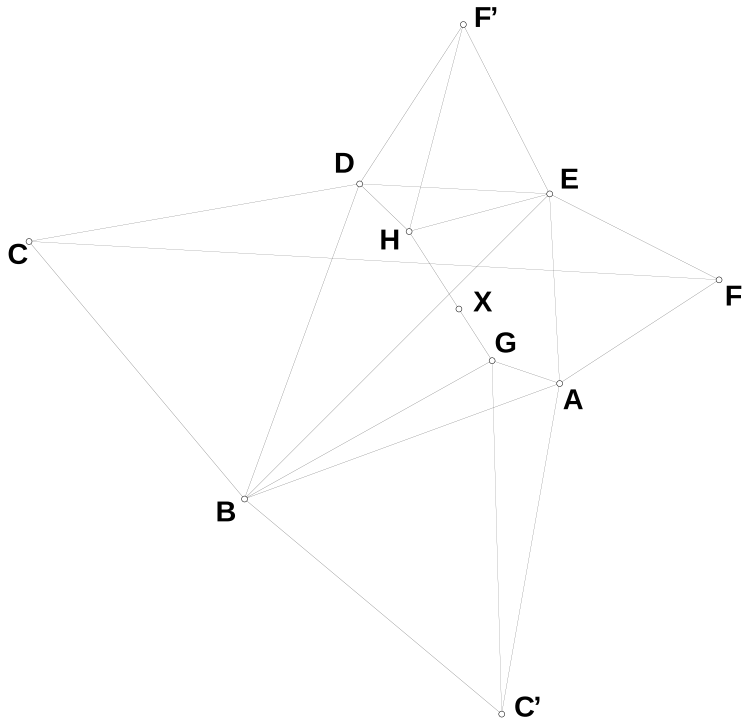
Then $PB = PP'$ and $PA = P'D$, hence $PA + PB + PC = CP + PP' + P'D \geq CD$. □

The following beautiful problem was submitted to the IMO by New Zealand; its author is Alastair McNaughton. It can also be solved using Ptolemy's inequality.

Example 7 (IMO, 1995)**.** Let $ABCDEF$ be a hexagon with $AB = BC = CD$ and $DE = EF = FA$, and $\angle BCD = \angle EFA = 60°$. Also let G and H be two arbitrary points. Then

$$AG + BG + GH + DH + EH \geq CF.$$

Solution. As BCD and AEF are equilateral triangles, $ABDE$ is a kite.

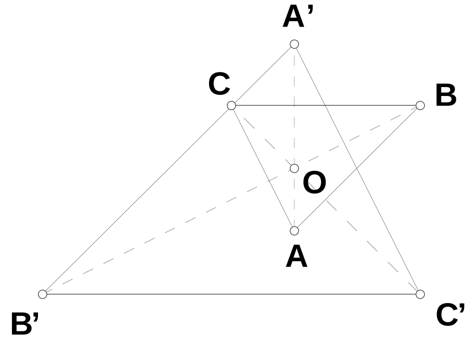


Reflect C and F about BE to get points C' and F'. Then $CF = C'F'$ and we obtain equilateral triangles DEF' and ABC'. Let X be any point on the segment GH. Applying the Fermat Point inequality to triangles XDE and XAB, we have $HX + HD + HE \geq XF'$ and $GX + GA + GB \geq XC'$, respectively. Adding these, we obtain $HG + HE + HD + GA + GB \geq XF' + XC' \geq C'F' = CF$. $\square$

3 Homotheties

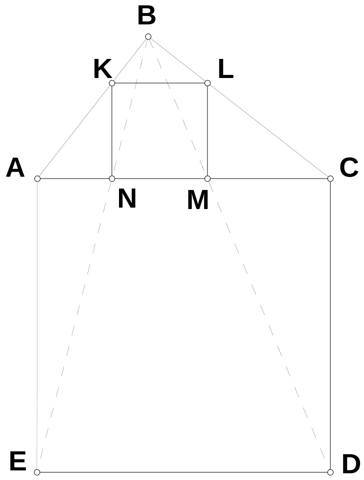
A very important family of transformations is the *homotheties* (also known as *dilations*). A homothety is defined by a fixed point O, called the *centre of homothety*, and a nonzero real number $k \neq 0$, called the *ratio of homothety*. The transformation maps a point A of the plane onto the point A_1 satisfying $\overrightarrow{OA_1} = k\overrightarrow{OA}$.

Example 8. The triangle ABC below is homothetic to triangle $A'B'C'$, since the homothety with centre O and coefficient $k = -2$ maps ABC onto $A'B'C'$.



Example 9. A square $KLMN$ is inscribed into a right-angled triangle ABC so that its side MN of it is on the hypotenuse AC, and its two other vertices K, L are on the two legs AB and BC. Given the segment AC and the vertex M, construct the rest of the triangle.

Solution. Let a square $KLMN$ be inscribed in a right-angled triangle ABC as shown on the diagram.



Consider the homothety with center B and ratio $k = AB/KB = CB/LB$. The square $KLMN$ will be mapped by this homothety onto the square $ACDE$, which is easy to construct. Then, knowing the point M and the point D, we can find B since it is the intersection point of MD and the semicircle with AC as its diameter. $\square$

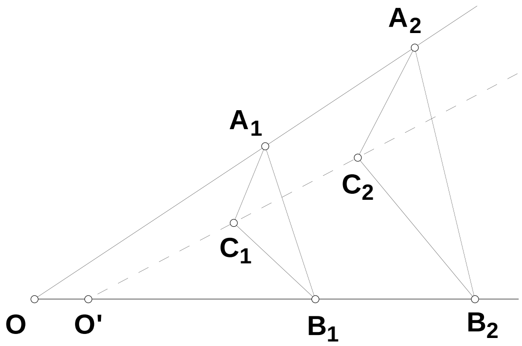
It is clear that any two homothetic triangles are similar, and moreover that their corresponding sides are parallel. The converse is also true, which gives us a very powerful tool.

Theorem 1. *If two similar triangles $A_1B_1C_1$ and $A_2B_2C_2$ have their corresponding sides parallel, then they are either homothetic or related by a translation. In particular, the lines A_1A_2, B_1B_2 and C_1C_2 are either concurrent, with intersection at the centre of homothety, or are all parallel.*

Proof. Since the triangles are similar, there is some number k such that

$$\frac{A_1B_1}{A_2B_2} = \frac{B_1C_1}{B_2C_2} = \frac{C_1A_1}{C_2A_2} = k.$$

Suppose that A_1A_2 and B_1B_2 intersect at O, while C_1C_2 and B_1B_2 intersect at $O' \neq O$.



We will show nevertheless that $OB_1 = O'B_1$, and therefore obtain a contradiction. Indeed, the similarity of triangles OA_1B_1 and OA_2B_2 implies that

$$OB_1 = \frac{A_1B_1}{A_2B_2}OB_2 = kOB_2.$$

Similarly, the similarity of triangles $O'C_1B_1$ and $O'C_2B_2$ implies that

$$O'B_1 = \frac{C_1B_1}{C_2B_2} OB_2 = kO'B_2.$$

We claim this cannot happen unless $O = O'$. Indeed, if we introduce a coordinate system on the line B_1B_2, then B_1 and B_2 will have coordinates b_1 and b_2. Let x be any other point on B_1B_2. The function $f(x) = \frac{x-b_1}{x-b_2} = 1 - \frac{b_1-b_2}{x-b_2}$ is one-to-one; that is, the equation $f(x) = k$ cannot have two solutions. Therefore

$$\frac{O'B_1}{O'B_2} = \frac{OB_1}{OB_2} = k$$

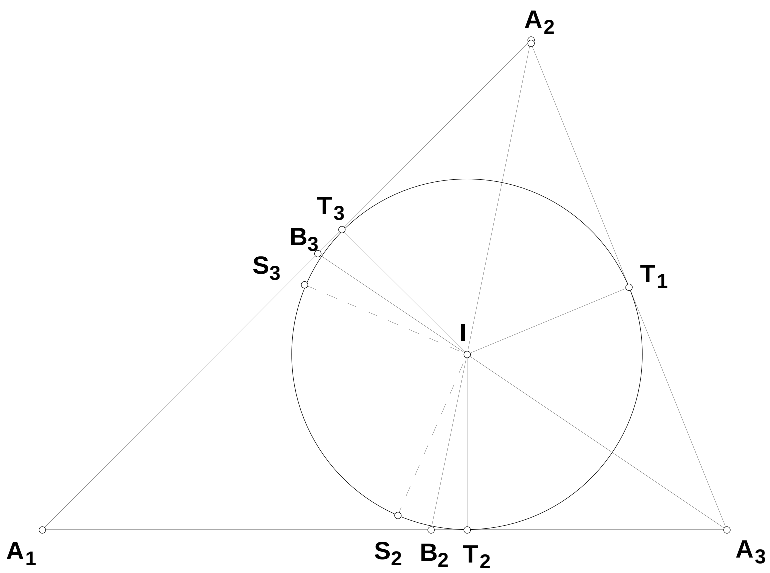
is impossible unless $O = O'$. □

Corollary 2. *Being homothetic is a transitive property; that is, if $A_1B_1C_1$ is homothetic to $A_2B_2C_2$ and $A_2B_2C_2$ is homothetic to $A_3B_3C_3$ then $A_1B_1C_1$ is homothetic to $A_3B_3C_3$.*

Proof. The triangles $A_1B_1C_1$ and $A_3B_3C_3$ are similar and have their respective sides parallel, hence they are homothetic by Theorem 1. □

Example 10 (IMO 1982)**.** A scalene triangle $A_1A_2A_3$ is given with sides a_1, a_2, a_3, where a_i is the side which is opposite to A_i. For all $i = 1, 2, 3$, let M_i be the midpoint of a_i, and T_i be the point, where the incircle touches a_i. Denote by S_i the reflection of T_i in the interior bisector of angle A. Prove that the lines M_1S_1, M_2S_2 and M_3S_3 are concurrent.

Solution. We have to prove that $S_1S_2S_3$ is homothetic to $M_1M_2M_3$. By Corollary 2, it will suffice to show that $S_1S_2S_3$ is homothetic to $A_1A_2A_3$. So by Theorem 1 it is sufficient to prove that $S_3S_1 \parallel A_3A_1$, $S_1S_2 \parallel A_1A_2$, and $S_2S_3 \parallel A_2A_3$. We will prove the last of these; the other two are similar.



Suppose that the bisectors of the angles A_1, A_2, A_3 intersect the opposite sides at B_1, B_2, B_3. Let us calculate the angle $\angle S_3IT_1$. Since $\angle T_3IA_2 = \angle T_1IA_2$, we get $\angle S_3IT_1 = 2\angle B_3IA_2 = \angle A_2 + \angle A_3$. Similarly, $\angle S_2IT_1 = \angle A_2 + \angle A_3$. Since $\angle S_3IT_1 = \angle S_2IT_1$, we have proved that $S_2S_3 \parallel A_2A_3$, as required. □

Next we will study compositions of two homotheties. To do this it is most convenient to use complex numbers. Choose a coordinate system. Then every point A obtains two coordinates, namely an x-coordinate a and a y-coordinate b. So to every point A there corresponds a coordinate pair (a, b). We can go further and assume that a complex number $a + bi$ corresponds to this pair and hence to A. Further we will identify points with complex numbers. The following two statements are easy to prove and we leave them to the reader.

Proposition 3. *Let z_1 and z_2 be two distinct complex numbers. Then the equation of the line passing through z_1 and z_2 will be*

$$z = tz_1 + (1-t)z_2,$$

where t is an arbitrary real number.

Proposition 4. *Let z_1 be a complex number and k be a nonzero real number. Then the homothety h with the centre z_0 and coefficient k will be given by*

$$h(z) = z_0 + k(z - z_0),$$

and the parallel displacement d by a vector z_0 (that takes 0 to z_0) is given by

$$d(z) = z + z_0.$$

Theorem 5. *Let h_1 be the homothety with centre z_1 and ratio k_1, and h_2 the homothety with centre z_2 and coefficient k_2. Suppose $z_1 \neq z_2$. Then, if $k_1k_2 \neq 1$, the composition $h_1 \circ h_2$ of these homotheties, defined as $(h_1 \circ h_2)(z) = h_1(h_2(z))$, is the third homothety with the centre on the line through z_1 and z_2. If $k_1k_2 = 1$, the composition $h_1 \circ h_2$ is a parallel displacement by a multiple of $z_1 - z_2$.*

Proof. The following direct calculation shows:

$$(h_1 \circ h_2)(z) = h_1(h_2(z)) = z_1 + k_1\left(z_2 + k_2(z - z_2) - z_1\right)$$

$$= z_1 + k_1(z_2 - z_1) + k_1k_2(z - z_2).$$

If $k_1k_2 = 1$, this expression becomes $(h_1 \circ h_2)(z) = z + (1-k_1)(z_2 - z_1)$ so this is a parallel displacement. If $k_1k_2 \neq 1$, this becomes

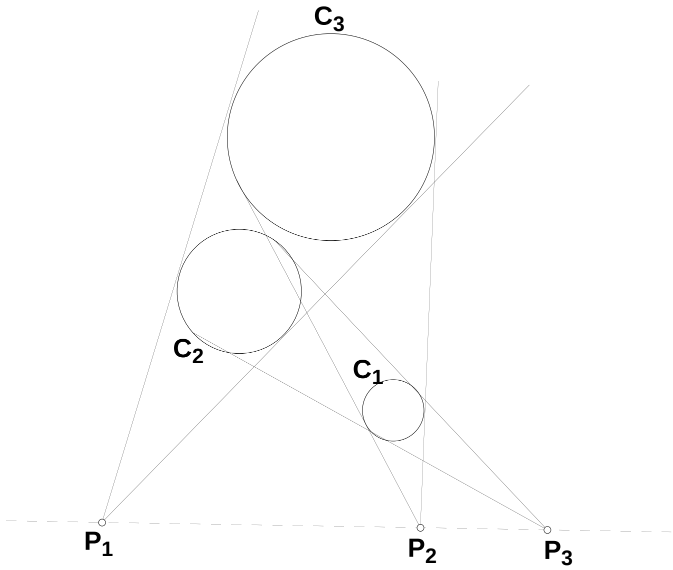
$$h_1 \circ h_2(z) = z_0 + k(z - z_0),$$

where $k = k_1k_2$ and

$$z_0 = \frac{1-k_1}{1-k_1k_2}z_1 + \frac{k_1(1-k_2)}{1-k_1k_2}z_2.$$

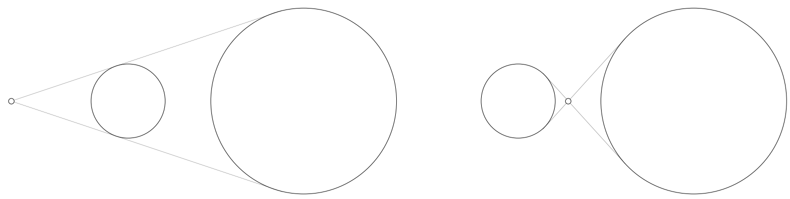
Since $\frac{1-k_1}{1-k_1k_2} + \frac{k_1(1-k_2)}{1-k_1k_2} = 1$, the point z_0 is on the line through z_1 and z_2. $\square$

Example 11. [1] Let C_1, C_2 and C_3 be three circles situated so that no one of them is inside the other and such that no two radii are equal. Let P_1 be the point of intersection of two external tangent lines drawn to the circles C_2 and C_3. Points P_2 and P_3 are defined similarly. Prove that the points P_1, P_2 and P_3 are collinear.



[1] I am grateful to Vladimir Oleinik for this problem.

Proof. First, we note that, given two nonintersecting circles C and C', there exist two homotheties which map C onto C'. In the first case the centre is the point of intersection of the two external tangents and in the second case the centre is the point of intersection of the two internal tangents.



In the first case the coefficient of homothety is positive and in the second it is negative.

Now we can start the proof. Let us consider two homotheties: one, h_1, with centre P_1, which maps C_2 onto C_3, and the second, h_2, with centre P_2, which maps C_3 onto C_1. Since, in both cases, the centres were points of intersection of external tangents, both coefficients k_1 and k_2 will be positive.

The composition $h_3 = h_2 \circ h_1$ of these two homotheties, also a homothety, will map C_2 onto C_1. Since its coefficient $k_3 = k_1 k_2 > 0$ is positive h_3, it will have centre P_3. So by Theorem 2, P_3 lies on the line P_1P_2. □

4 Problems

1. Inscribe a square $KLMN$ in a given triangle ABC so that one side of the square KL lies on one side of the triangle and the other two vertices M, N of the square lie on the other two sides of the triangle.

2. Two equal circles S_1 and S_2 are internally tangent to a circle S at points A_1 and A_2 respectively. An arbitrary point C of the circle S is connected by straight segments with the points A_1 and A_2. These segments intersect S_1 and S_2 at points B_1 and B_2. Prove that A_1A_2 is parallel to B_1B_2.

3. Choose a point O inside an equilateral triangle PQR, such that $\angle ROP = 110°$, $\angle POQ = 120°$, $\angle QOR = 130°$. Prove that there exists a triangle whose sidelengths are the lengths of OP, OQ and OR, and find the angles of this triangle.

4. Two points A and B lie inside an acute angle. Construct an isosceles triangle PQR whose base PR lies on one side of the angle, whose third vertex Q lies on the other side of the angle, and whose two lateral sides PQ and PR pass through A and B, respectively.

5. (IMO 1981) Three congruent circles have a common point O and lie inside a given triangle. Each circle touches a pair of sides of the triangle. Prove that the triangle's incentre and circumcentre are collinear with the point O.

6. (IMO 1983) Let A be one of the two distinct points of intersection of two unequal coplanar circles C_1 and C_2 with centres O_1 and O_2, respectively. One of the common tangents to the circles touches C_1 at P_1 and C_2 at P_2, while the other touches C_1 at Q_1 and C_2 at Q_2. Let M_1 be the midpoint of P_1Q_1 and M_2 be the midpoint of P_2Q_2. Prove that $\angle O_1AO_2 = \angle M_1AM_2$.

7. Let C_1, C_2, C_3 be three circles situated so that no one of them is inside the other and such that no two radii are equal. Let P_1 be the point of intersection of two external tangent lines drawn to the circles C_2 and C_3. Let points P_2 and P_3 be the point of intersection of internal tangent lines drawn to the circles C_1 and C_3 and C_1 and C_2, respectively. Prove that the points P_1, P_2 and P_3 are collinear.

April 21, 2009
http://www.mathsolympiad.org.nz