Graph Theory

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1 Definitions

- A *graph* is a pair $G = (V, E)$ of a set of *vertices* V and a set of *edges* E. An edge is an unordered pair of elements of V. Sometimes we allow a graph to have loops or multiple edges between vertices. Usually, it is assumed that the vertex and edge sets are finite.

- A *directed graph* is a pair $G = (V, E)$ of a set of vertices V and a set of edges E, where we now take the elements of E to be ordered pairs of elements of V. Loops (v, v) are generally allowed, and multiple edges between the same ordered pair of vertices may be allowed if specified. Most of the definitions below are for undirected graphs, but can be carried over to the directed case, though there may not be universal agreement on how to do so.

- The vertices belonging to an edge are called *endpoints* of the edge. A vertex v is said to be *incident* on an edge e (or vice versa) if v is an endpoint of e. Two vertices v and w in a graph G are *adjacent* if there is an edge incident on both. Two edges are adjacent if they share an endpoint.

- The *complement* of a graph G is the graph $\overline{G}$ with the same vertex set as G but containing all edges *not* present in G. The complement of the graph with n vertices and no edges is the *complete graph* on n vertices, and is denoted K_n.

- A *subgraph* of a graph $G = (V, E)$ is a graph $G' = (V', E')$ with V' contained in V and E' contained in E. In other words, G' is obtained from G by removing some number of vertices and edges. If G' has the same set of vertices as G, we call G' a *spanning subgraph* of G. If every pair of vertices of G' which are adjacent in G are also adjacent in G', then G' is an *induced subgraph* of G; this means that G' is obtained from G by removing some vertices and their incident edges.

- A *walk* in a graph is an alternating sequence of vertices and edges, beginning and ending with vertices, such that for any edge in the sequence, the vertices preceding and following it are its two endpoints. (In the directed case, we require that the edges have the "correct" orientation.) The *length* of a walk is the number of edges used. A walk is *closed* if its first and last vertices are the same, otherwise *open*.

- A *path* is an open walk which is *simple*; no vertex appears more than once. A *cycle* is a closed walk which is simple, meaning that no vertex appears more than once when we consider the common first and last vertex of the walk as one appearance.

- A *Hamiltonian* path or cycle in a graph G is one which uses every vertex of G. An *Eulerian* path or cycle is a walk which uses every edge of G exactly once (so in general it is not a path or cycle in the sense of the previous definition).

- A graph G is called *connected* if its vertex set cannot be partitioned into two non-empty subsets V_1 and V_2 such that G has no edge between a vertex in V_1 and a vertex in V_2. (For technical reasons we should also assume that the vertex set of G is nonempty.)

- A *tree* is a connected graph containing no cycles. A *forest* is any graph with no cycles.

- A *clique* in a graph is a subset of the vertices any two of which are adjacent, that is, an (induced) subgraph isomorphic to K_n for some n.

- The *degree* of a vertex is the number of edges incident on it. A graph is *regular* if all of its vertices have the same degree.

- A *coloring* of a graph G with a set of colors S is an assignment of a color to each vertex of G so that no two adjacent vertices are assigned the same color. A graph is *k-colorable* if it can be colored with a set of k colors. A 2-colorable graph is also called *bipartite*.
- The *complete bipartite* graph K_{n_1,n_2} is a graph with vertices partitioned into parts of size n_1 and n_2, and edges joining any two vertices not in the same part. The complete k-partite graph $K_{n_1,n_2,\ldots,n_k}$ is defined analogously.
- The *distance* $d(v, w)$ from a vertex v to a vertex w is the length of the shortest path from v to w, or ∞ if no such path exists. The *diameter* of a graph G is the largest distance between any two vertices of G.
- A *matching* of a graph G is a collection of edges such that any vertex is incident to at most one edge in the collection. A *perfect matching* is a matching that uses every vertex.
- A graph is *planar* if its vertices can be assigned locations in the plane and its edges curves joining the locations assigned to their endpoints such that these curves intersect only at endpoints. A *plane* graph is a graph together with a particular choice of such an embedding in the plane. The edges of such an embedding divide the plane into regions called the *faces* of G. (Warning: sometimes the unbounded region is not considered a face.) The *dual* of a plane graph G is the graph G^* with a vertex f^* for each face f of G and an edge e^* corresponding to each edge e of G; the endpoints of e^* are the faces of G separated by e.

2 Handy Facts (mostly easy)

1. A (non-empty) graph G is connected if and only if for any two vertices v and w of G, there exists a path from v to w in G.
2. Any graph can be decomposed uniquely into a disjoint union of connected graphs. These graphs are called the *connected components* of G.
3. ("Handshake Lemma") The sum of the degrees of the vertices of the graph is twice the number of edges; in particular, it is even.
4. Any two of the following conditions imply the third:
   (a) G contains no cycles.
   (b) G is connected.
   (c) $|E(G)| = |V(G)| - 1$.
5. Distances in graphs satisfy the triangle inequality: $d(u, v) + d(v, w) \geq d(u, w)$.
6. Every connected graph has a spanning tree.
7. A graph has an Eulerian cycle if and only if it is connected and every vertex has even degree. A graph has an Eulerian path if and only if it is connected and exactly two vertices have odd degree.
8. A graph is bipartite if and only if it contains no odd cycle.
9. ("Marriage Lemma" or Hall's Matching Theorem) Let G be a bipartite graph with parts A and B. Suppose that B has at least as many vertices as A. Then G has a matching using all the vertices of A if and only if for any subset S of k vertices of A, there are at least k vertices in B adjacent to some vertex of A.

   In particular, if every vertex of A has the same degree a and every vertex of B has the same degree b, then G has a matching using every vertex of A.

10. If G is a planar graph, we can draw G in the plane so that the curves joining adjacent vertices are all straight lines.

11. The dual G^* of a plane graph is again a plane graph, and $G^{**} = G$.

12. (Euler's Formula) If G is a plane graph with V vertices, E edges and F faces, then $V - E + F = 2$.

13. The average degree of a vertex of a planar graph is less than six.

14. (Turán's Theorem) If G is a graph on n vertices containing no K_3, then G has at most $\left\lfloor \frac{n^2}{4} \right\rfloor$ edges. More generally, if G is a graph on n vertices containing no K_k, the maximum number of edges of G is attained by a complete $(k-1)$-partite graph in which the part sizes are as equal as possible.

3 Problems

1. (St. Petersburg 1997) In a group of several people, some are acquainted with each other and some are not. Every evening, one person invites all of his acquaintances to a party and introduces them to each other. Suppose that after each person has arranged at least one party, some two people are still unacquainted. Prove that they will not be introduced at the next party.

2. (IMO 1983 shortlist) The edges of K_{1983} are divided into 10 sets. Prove that at least one of the sets contains an odd cycle.

3. (BAMO 2004) Find the smallest value of N with the following property: any graph with 2004 vertices and N edges is connected.

4. (BAMO 2005) Let G be a connected graph with 1000 vertices. Prove that some of the edges can be selected so that each vertex is incident on an odd number of selected edges.

5. (Iran 1998) Suppose an $n \times n$ table is filled with the numbers 0, 1, -1 in such a way that each row and column contains exactly one 1 and one -1. Prove that the rows and columns can be reordered so that in the resulting table each number has been replaced by its negative.

6. (IMO 1991) Let G be a connected graph with n edges. Prove that the edges of G can be labeled 1, 2, $\ldots$, n in such a way that at each vertex with degree two or more, the greatest common divisor of the labels of the edges incident on that vertex is 1.

7. (Russia 1998) In an $m \times n$ rectangular grid, where m and n are odd integers, 1×2 dominos are initially placed so as to exactly cover all but one of the 1×1 squares at one corner of the grid. It is permitted to slide a domino towards the empty square, thus exposing another square. Show that by a sequence of such moves, we can move the empty square to any corner of the rectangle.

8. (Russia 1998) Let G be a 3-regular connected graph with 1998 vertices. Prove that one can find a subset S of 200 of the vertices of G, no two adjacent, such that removing the vertices S from G leaves a connected graph.

9. (St. Petersburg 1998) The sides of a convex polyhedron are all triangles. At least 5 edges meet at each vertex, and no two vertices of degree 5 are connected by an edge. Prove that this polyhedron has a side whose vertices have degrees 5, 6, 6, respectively.

10. (Sperner's Theorem) Let S be a collection of subsets of $\{1, 2, \ldots, n\}$ such that no set in S contains another set in S. Show that $|S| \leq \binom{n}{\lfloor n/2 \rfloor}$.