Games

1. A house has an even number of lamps distributed among its rooms in such a way that there are at least three lamps in every room. Each lamp shares a switch with exactly one other lamp, not necessarily from the same room. Each change in the switch shared by two lamps changes their states simultaneously. Prove that for every initial state of the lamsp there exists a sequence of changes in some of the switches at the end of which each room contains lamps which are on as well as lamps which are off.

2. Numbers $1, 2, \ldots, n^2$ are written lexicographically in an $n \times n$ array, so that the entry in the i^{th} row and j^{th} column $(1 \leq i, j \leq n)$ is equal to $(i-1)n + j$. At each step, one is allowed to choose two neighboring (horizontally or vertically) numbers, add 1 to one chosen number and add 2 to the other. Determine all positive integers n such that after a finite number of steps, all the entries in the array can be made equal to n^2.

3. Mr. Fat and Ms. Taf play a game. Mr. Fat chooses a sequence of positive integers $k_1, k_2, \ldots, k_n$. Ms. Taf must guess this sequence of integers. She is allowed to give Mr. Fat a red card and a blue card, each with an integer written on it. Mr. Fat replaces the number on the red card with k_1 times the number on the red card plus the number on the blue card, and replaces the number on the blue card with the number originally on the red card. He repeats this process with number k_2. (That is, he replaces the number on the red card with k_2 times the number now on hte red card plus the number now on the blue card, and replaces the number on the blue card with the number that was just replaced on the red card.) He then repeats the process with each of the numbers $k_3, \ldots, k_n$, in this order. After he has gone through the sequence of integers, Mr. Fat then gives the cards back to Ms. Taf. How many times must Ms. Taf submit the red and blue cards in order to be able to determine the sequence of integers $k_1, k_2, \ldots, k_n$?

4. A solitaire game is played on an $m \times n$ rectangle board, using mn markers which are white on one side and black on the other. Initially, each square of the board contains a marker with its white side up, except for one corner, which contains a marker with its black side up. In each move, one may take away one marker with its black side up, but must then turn over all markers which are in squares having an edge in common with the square of the removed marker. Determine all pairs (m, n) of positive integers such that all markers can be removed from the board.

5. Let m, n be odd positive integers. 1×2 dominoes are placed onto the squares of an $m \times n$ chessboard such that all squares other than the top left square are covered exactly once. Mr. Fat picks a square on the chessboard and tries to move the dominoes so that the square is empty. He is allowed only to slide dominoes horizontally or vertically into empty squares. Prove that Mr. Fat can do this for all choices of squares.

6. There is a 7×7 square board divided into 49 unit cells, and tiles of three types: 3×1 rectangles, 3-unit-square corners, and unit squares. Jerry has infinitely many rectangles and one corner, while Tom has only one square.

   (a) Prove that Tom can put his square somewhere on the board (covering exactly one unit cell) in such a way that Jerry cannot tile the rest of the board with his tiles.

   (b) Now Jerry is given another corner. Prove that no matter where Tom puts his square (covering exactly one unit cell), Jerry can tile the rest of the board with his tiles.

7. A magician wants to determine the area of a hidden convex 2008-gon $a_1 a_2 \cdots a_{2008}$. In each step, he chooses two points on hte perimeter, where the points can be chosen to be vertices or points dividing selected sides in selected ratios. Then his helper divides the polygon into two parts by the line through these two points and announces the area of the smaller of the two parts. Show that the magician can find the area of the polygon in 2006 steps.

8. A calculator is broken so that the only keys that still work are the sin, cos, tan, $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$ buttons. The display initially shows 0. Given any positive rational number q, show that pressing some finite sequence of buttons will yield q. Assume that the calculator does real number calculations with infinite precision. All functions are in terms of radians.

9. Consider the 7×7 array $a_{ij} = (i^2 + j)(i + j^2)$, $1 \leq i, j \leq 7$. We are allowed to perform the following operation: choose an arbitrary integer (positive or negative) and seven elements in the array, one from each row and each column, then add the chosen integer to each chosen element in the array. Determine if it is possible that after applying this operation finitely many times we can transform the given array into an array all of whose rows are arithmetic progressions.

10. Let T be the set of ordered triples (x, y, z), where x, y, z are integers with $0 \leq x, y, z \leq 9$. Players A and B play the following guessing game. Player A chooses a triple (x, y, z) in T, and Player B has to discover A's triple in as few moves as possible. A *move* consists of the following: B gives A a triple (a, b, c) in T, and A replies by giving B the number

$$|x + y - a - b| + |y + z - b - c| + |z + x - c - a|.$$

Find the minimum number of moves that B needs to be sure of determining A's triple.

11. Let N be a positive integer. Two players A and B, taking turns, write numbers from the set $\{1, \ldots, N\}$ on a blackboard. A begins the game by writing 1 on his first move. Then, if a player has written n on a certain move, his adversary is allowed to write $n + 1$ or $2n$ (provided the number he writes does not exceed N). The player who writes N wins. We say that N is of type A or of type B according as A or B has a winning strategy.

   (a) Determine whether $N = 2004$ is of type A or of type B.

   (b) Find the least $N > 2004$ whose type is different from the one of 2004.

12. Magician Arutyun and his assistant Amayak perform the following trick. A circle has been drawn on the board. While the magician is away, an onlooker marks 2007 points on the circle and the assistant erases one of them. Then the magician enters the room, looks at teh picture and determines a semicircle on which the erased point was lying. How can the magician make a deal with the assistant so that the trick is always successful?

13. There are n markers, each with one side white and the other side black, aligned in a row so that their white sides are up. In each step, if possible, we choose a marker with the white side up (but not one of the outermost markers), remove it and reverse the closest marker to the left and the closest marker to the right of it. Prove that one can achieve the state with only two markers remaining if and only if $n - 1$ is not divisible by 3.

14. Numbers $1, 2, \ldots, n^2$ are written in an $n \times n$ array such that the entry in the i^{th} row and j^{th} column $(1 \leq i, j \leq n)$ is equal to $(i-1)n+j$. At each step, one is allowed to choose two neighboring (horizontally or vertically) numbers, add 1 to one of the chosen numbers and 2 to the other. Determine all positive integers n such that, after a finite number of steps, all the entries in the array can be equal to n^2.

15. A cake has the form of an $n \times n$ square composed of n^2 unit squares. Strawberries lie on some of the unit squares so that each row or column contains exactly one strawberry; call this arrangement $\mathcal{A}$.

   Let $\mathcal{B}$ be another such arrangement. Suppose that every grid rectangle with one vertex at the top left corner of the cake contains no fewer strawberries of arrangement $\mathcal{B}$ than arrangement $\mathcal{A}$. Prove that arrangement $\mathcal{B}$ can be obtained from $\mathcal{A}$ by performing a number of *switches*, defined as follows:

   > A *switch* consists in selecting a grid rectangle with only two strawberries, situated at its top right corner and bottom left corner, and moving these two strawberries to the other two corners of that rectangle

16. Let n be a natural number. A pawn is placed in a cell of an $n \times n$ table. In one move, the pawn can jump from any cell of the k^{th} column ($1 \leq k \leq n$) to any cell of the k^{th} row. Prove that it is possible in a sequence of n^2 moves for the pawn to visit all cells of the table, ending in the original cell.

17. Let $n \geq 2$ be a positive integer. Initially, there are n fleas on a horizontal line, not all at eth same point. For a positive real number λ, define a *move* as follows:

    > Choose any two fleas, at points A and B, with A to the left of B; let the flea at A jump to the point C on the line to the right of B with $BC/AB = \lambda$.

    Determine all values of λ such that, for any point M on the line and any initial positions of the n fleas, there is a finite sequence of moves that will take all the fleas to positions to the right of M.

18. There are 2010 segments $A_1B_1, A_2B_2, \ldots, A_{2010}B_{2010}$ of unit length lying in the plan. Ms. Aft and Mr. Fat play the following game with these segments: They take turns, with Mr. Fat playing first, to assign one unassigned segment a direction, changing this segment into a vector, namely to change segment A_iB_i to either $\mathbf{u}_i = \overrightarrow{A_iB_i}$ or $\mathbf{u}_i = \overrightarrow{B_iA_i}$. Ms. Aft wins the game if the vector $\mathbf{u} = \mathbf{u}_1 + \mathbf{u}_2 + \cdots + \mathbf{u}_{2010}$ lies within the square $|x| \leq \sqrt{2}$, $|y| \leq \sqrt{2}$. Otherwise, Mr. Fat wins. Who has the winning strategy?