1. Let $ABC$ be a triangle. Show that if $\sum \sqrt{a+h_b} = \sum \sqrt{a+h_c}$ (cyclic sums), then $ABC$ is isosceles.

   **Solution:** If $x = \sin A$, etc., suppose $x > y > z > 0$ for contradiction (we can't have two angles sum to $\pi$, so sin is effectively "injective" here); we get $\sum \sqrt{x+xy} = \sum \sqrt{x+xz}$, so

   $$0 = \sum \sqrt{x+y}(\sqrt{x}-\sqrt{y}) = (\sqrt{x}-\sqrt{y})(\sqrt{x+y}-\sqrt{z+x}) + (\sqrt{y}-\sqrt{z})(\sqrt{y+z}-\sqrt{z+x}),$$

   whence $(\sqrt{x}+\sqrt{y})(\sqrt{x+y}+\sqrt{z+x}) = (\sqrt{y}+\sqrt{z})(\sqrt{y+z}+\sqrt{z+x})$. But using $x > y > z$, this is clearly impossible.

2. Prove that if $t$ is a positive integer, then there exists a positive integer $n > 1$ such that $\gcd(n,t) = 1$ and none of the numbers $n^k + t$ $(k \geq 1)$ are perfect powers.

   **Solution:** Suppose $p \mid t+1$ for some prime $p$ and let $\ell = v_p(t+1)$. Now take $n = (1+mp^{\ell+1}t)^{\ell}$ for large $m$, noting that $n^k + t$ must be a $d$th power for some $d \mid \ell$ in order to be a perfect power at all.

3. Suppose $ABCD$ is a parallelogram. Consider circles $w_1, w_2$ with $w_1$ tangent to segments $AB, AD$ and $w_2$ tangent to segments $BC, CD$. Suppose circle $w_3$ is tangent to lines $AD, DC$ and externally tangent to $w_1, w_2$; prove there exists a circle $w_4$ tangent o lines $AB, BC$ and externally tangent to $w_1, w_2$.

   **Solution:** Let $x = AB = CD$, $y = AD = BC$, $\alpha = A/2 = C/2$, and $\beta = B/2 = D/2$ so that $\alpha + \beta = 90^\circ$. WLOG assume $w_3$ is "inside" the parallelogram (other is analogous). It's easy to see $\sqrt{y} = \sqrt{r_A \cot\alpha} + \sqrt{r_D \tan\alpha}$ and $\sqrt{x} = \sqrt{r_C \cot\alpha} + \sqrt{r_D \tan\alpha}$ and that $r_B$ exists iff $\sqrt{x} = -\sqrt{r_a \cot\alpha} + \sqrt{r_B \tan\alpha}$ and $\sqrt{y} = -\sqrt{r_C \cot\alpha} + \sqrt{r_B \tan\alpha}$ are compatible, which is clear.

4. Points $A, B$ are on a circle $\omega$ with center $O$. Let $C$ be the circumcenter of $\triangle AOB$, and $\ell$ be a line passing through $C$ such that the angle between $\ell$ and $OC$ is $\pi/3$. Let $\ell$ intersect $AA, BB$ at $M, N$, respectively. Suppose $(CAM) \cap (CBN) = P$, $(CAM) \cap \omega = Q$, and $(CBN) \cap \omega = R$. Prove that $OP \perp QR$.

   **Solution:** Angle chase to get that $\measuredangle BOQ = \measuredangle AOR = 60^\circ$ under some orientation (basically just find $\angle AQC$ and then $\angle BQC$). It's easy to angle chase that $P$ lies on $(OAB)$, so phantom $P'$ such that $\angle QOP' = \angle P'OR$ and trivially get isosceles trapezoids, which must be cyclic.

5. Find all integer numbers $x, y$ such that $(y^3 + xy - 1)(x^2 + x - y) = (x^3 - xy + 1)(y^2 + x - y)$.

   **Solution:** When expressions like $x^2 + x - y$ have small absolute value, we can do finite case check. Otherwise, divide both sides by $x^2 + x - y$ and $y^2 + x - y$ and do some trivial inequalities (triangle inequality helps). Also exists harder problem on AoPS thread.

6. Suppose $p$ is an odd prime. We call the polynomial $f(x) = \sum_{j=0}^{n} a_j x^j$ with integer coefficients $i$-remainder if

   $$\sum_{p-1|j, j>0} a_j \equiv i \pmod{p}.$$

   Prove that $\{f(0), \ldots, f(p-1)\}$ is a complete residue system modulo $p$ if and only if polynomials $f(x)^k$ for $1 \leq k \leq p-2$ are 0-remainder and $f(x)^{p-1}$ is 1-remainder.

   **Solution:** Rewrite the $i$-remainder condition in terms of $f(0) + \cdots + f(p-1) \pmod{p}$, and use Newton sums.

7. Let $O$ be the circumcenter of the acute triangle $ABC$. Suppose points $A', B', C'$ are on sides $BC, CA, AB$ such that $(AB'C') \cap (BC'A') \cap (CA'B') = O$. Let $\ell_a$ be the radical axis of the circle centered at $B'$ with radius $B'C$ and the circle centered at $C'$ with radius $C'B$. (Define $\ell_b, \ell_c$ analogously.) Prove that $\ell_a, \ell_b, \ell_c$ form a triangle with orthocenter $H$.

   **Solution:** This is killed by spiral similarity about $O$ and the observation that some point $A_1 \in BC$ lies on the radical axis $\ell_a$ (it turns out this is isogonal conjugates-ish, i.e. $(A'B'C')$ meets $BC$ at $A_1$). Noting that $O$ is the orthocenter of $A'B'C'$ helps to complex bash. Basically, if $a'/m = 2r$, etc. (some complex $r$), then $r + \overline{r} = 1$, and we compute $x = \frac{a(1-r)+r(b+c)}{1-r}$ where $XYZ$ is the triangle formed by the $\ell_a$.

8. A piece begins at the origin of the coordinate plane. Two players, $A$ and $B$, play the following game. In $A$'s turn, $A$ marks a lattice point in the first quadrant that the piece is not on. In $B$'s turn, $B$ moves the piece up to $k$ times, where a move is defined as moving the piece from $(x, y)$ to $(x+1, y)$ or $(x, y+1)$, as long as the lattice point that the piece moves onto is not marked. They then proceed to alternate turns, with $A$ playing first. If $B$ cannot move the piece, $A$ wins. For what values of $k$ can $A$ win?

   **Solution:** This is a special case of Conway's Angel problem for which $A$ (the devil) can always win.

   The idea is to reduce solution sets, where ultimately we'll need $B$ to be at $(x, y)$ while $(x+1, y)$ and $(x, y+1)$ are both unavailable. The "most obvious" way to guarantee this, i.e. to make sure that when $B$ is at some $(x, y)$, all of the $(x+i, y+j)$ are unavailable for $i, j \geq 0$ and $i + j = r$ (some $r \geq 1$), does not directly work if we try to get consecutive things on some fixed line $x + y = T$.

   The next dumbest thing, however, does work: we want to gradually narrow down the choices on this fixed line $x + y = T$ (call this $L$) depending on the remaining possible locations of $B$ (note that if $B$ is at $(x, y)$, then it can only go to $(x', y')$ later with $x' \geq x$ and $y' \geq y$). Hence we'll want to do this iteratively, checking at certain points where $B$ is and letting $A$ mark the remaining possible points between consecutive checks. The most natural way to do this is uniformly; for example, if we have $m$ moves before the first check, then $B$ can move up to the line $x + y = mk$ but no further, and we'll let $A$ mark things at $(x, y) \in L$ with $x \equiv 0 \pmod{\lceil T/m \rceil}$. The idea is that we'll want to do $\lceil T/m \rceil$ checks to make sure $B$ cannot get past $L$. Of course, if $m$ is too small, our checking gets little done, and if $m$ is too large, $B$ will pass $L$ before we even finish checking once (for instance, we need $m \leq T/k$ for the first time).

   Let $M = \lceil T/m \rceil$ and suppose that $B$ is at $x_i + y_i = r_i$ at the $i^{th}$ checkpoint ($i \geq 0$) and $A$ responds by making $m_i \leq 2 + (T - r_i)/M$ marks at the positions $(x, y) \in L$ with $x \equiv i \pmod{M}$ and $x \geq x_i$, $y \geq y_i$. Now suppose for contradiction that $r_n \geq T$ for $n$ minimal; since we must not have finished marking all residues $\pmod{M}$ in order for $B$ to have passed $L$ between the $(n-1)^{th}$ and $n^{th}$ checkpoints, we have $1 \leq n \leq M$ (otherwise, everything would've been marked between the $0^{th}$ and $M^{th}$ checkpoints before $B$ passed $L$). Since $r_i - r_{i-1} \leq km_{i-1} \leq 2k + k(T - r_{i-1})/M$ for $i \geq 1$ (note that $r_0 = 0$), this means that $r_{n-1} + 2k + k(T - r_{n-1})/M \geq r_n \geq T$. Setting $c = k/M$ (let's choose $M$ and $T$ large enough so that $0 < c < 1$), we get $r_{n-1} \geq T - \frac{2k}{1-c}$, and by a simple induction with geometric series we get $0 = r_0 \geq T - 2k((1-c)^{-n} + \cdots + (1-c)^{-1})$, so fixing some $0 < c < 1$ and taking $T$ sufficiently large, we get a contradiction.

9. For any given positive integer $n$, show that there are infinitely many triples of pairwise coprime positive integers $(x, y, z)$ such that $nx^2 + y^3 = z^4$.

   **Solution:** Use the standard norm trick. If $y^3 = (z^2 - x\sqrt{n})(z^2 + x\sqrt{n})$, then it's natural to set $y = u^2 - nv^2$ to get $z^2 = u(u^2 + 3nv^2)$ and $x = 3u^2v + nv^3$. It's then convenient to set $u = w^2$ so we want $w^4 + 3nv^2 = t^2$. Using norms again, if $w = \alpha^2 - \beta^2(3n)$, we want $u = (\alpha^2 - (3n)\beta^2)^2$ and $v = 4\alpha\beta(\alpha^2 + (3n)\beta^2)$ (we have $t = \alpha^4 + 6\alpha^2\beta^2(3n) + \beta^4(3n)^2$). We make $\alpha$ odd, $\beta$ even, and $\alpha \perp 3n\beta$ to get the coprimality stuff. If we fix $\beta$ and take $\alpha$ large then we'll get $u^2 > nv^2$ as $u$ is quartic in $\alpha$ and $v$ is only cubic.

10. Let $A, B$ be finite sets of real numbers and $x \in A + B$ be an element. Show that $|A \cap (x - B)| \leq |A - B|^2 / |A + B|$.

    **Solution:** It suffices to show that $|A + B|^2 \geq |X| \cdot |A - B|$, where $X = A \cap B$. But $|A + B|^2 \geq |A + X| \cdot |B + X|$, and every difference $d \in A - B$ appears as $(a + x) - (b + x)$ at least $|X|$ times.

11. Let $ABCD$ be a convex tangential quadrilateral such that $\angle ABC + \angle ADC < 180^\circ$ and $\angle ABD + \angle ACB = \angle ACD + \angle ADB$. Prove that one of the diagonals of quadrilateral $ABCD$ bisects the other.

    **Solution:** Invert about $A$ to get that $AB \cdot CD = AD \cdot BC$, and use the tangential condition to get symmetrical stuff.

12. Find the maximum possible number of kings on a $12 \times 12$ chess table so that each king attacks exactly one of the other kings (a king attacks only the squares that have a common point with the square he sits on).

**Solution:** Pair up the kings, and note that each pair borders at least 6 "grid" points, while no two pairs can share any grid points. This gets a bound of $13^2/6$, so there are at most 28 pairs. The construction is not hard using this intuition.