2011 BLUE MOP, CYCLIC QUADRILATERALS
ALİ GÜREL

(1) (Russia-1996) Points E and F are given on the side BC of a conxex quadrilateral $ABCD$ (with E closer than F to B). It is known that $\angle BAE = \angle CDF$ and $\angle EAF = \angle FDE$. Prove that $\angle CAF = \angle EDB$.

(2) (Simpson Line) The projections of a point onto the sides (or extensions) of a triangle are on a line if and only if the point is on the circumcircle of the triangle.

(3) (Prasolov, PPSG, p107) Points A, B and C lie on one line, point P lies outside this line. Prove that the centers of the circumscribed circles of triangles ABP, BCP, ACP and point P lie on one circle.

(4) Distinct points A and B are on a semicircle with diameter MN and center C. Point P lies on segment CN and $\angle CAP = \angle CBP = \alpha$ and $\angle ACM = \beta$. Express $\angle BPN$ in terms of α and β.

(5) (Ptolemy's Theorem) Let $ABCD$ be a convex quadrilateral. Prove that

$AB \cdot CD + AD \cdot BC = AC \cdot BD$ if and only if $ABCD$ is cyclic.

(6) Given a regular nonagon $COMPUTERS$, show that

$$TE + ES = SP.$$

(7) (USAMO-1993) Let $ABCD$ be a convex quadrilateral such that the diagonals AC and BD are perpendicular, and let P be their intersection. Prove that the reflections of P with respect to AB, BC, CD, and DA are concyclic.

(8) (Andreescu & Gelca, MOC, p.9) Let B and C be the endpoints and A the midpoint of a semicircle. Let M be a point on the line segment AC, and P, Q the feet of the perpendiculars from A and C to the line BM, respectively. Prove that $BP = PQ + QC$.

(9) (Hong Kong-1999) Let $PQRS$ be a cyclic quadrilateral with $\angle PSR = 90°$, and let H and K be the respective feet of perpendiculars from Q to lines PR and PS. Prove that line HK bisects $\overline{QS}$.

(10) (9-Point Circle) In a triangle ABC, let H_A, H_B, H_C be the feet of altitudes, M_A, M_B, M_C be the midpoints and K_A, K_B, K_C be the midpoints of AH, BH, CH where H is the orthocenter. Then the nine points: $H_A, H_B, H_C, M_A, M_B, M_C, K_A, K_B, K_C$ are all cyclic.

(11) (Andreescu & Enescu, MOT, p.46) In the triangle ABC, the altitude, angle bisector and median from C divide the angle $\angle C$ into four equal angles. Find the angles of the triangle.

(12) (Posamentier & Salkind, CPinG, p.35) A line drawn from vertex A of equilateral $\triangle ABC$, meets BC at D and the circumcircle at P. Prove that

$$\frac{1}{PD} = \frac{1}{PB} + \frac{1}{PC}$$

(13) (Posamentier & Salkind, CPinG, p.34) Express in terms of the sides of a cyclic quadrilateral the ratio of the diagonals.

(14) (Romania-1992) Let ABC be an acute triangle, and let T be a point in its interior such that $\angle ATB = \angle BTC = \angle CTA$. Let M, N, and P be the projections of T onto BC, CA, and AB, respectively. The circumcircle of the triangle MNP intersect the lines BC, CA, and AB for the second time at M', N', and P', respectively. Prove that the triangle $M'N'P'$ is equilateral.

(15) (Russia-1999) In triangle ABC, points D and E are chosen on side CA such that $AB = AD$ and $BE = EC$ (E lying between A and D). Let F be the midpoint of the arc BC of the circumcircle of ABC. Show that B, E, D, F lie on a circle.