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We can now recover the three classical inequalities cited in section 2 from the results of section 3 by taking the limit as $\alpha \to -1^+$. In fact, it is clear that the Fejér-Riesz inequality (Theorem 1) follows from Theorem 4, Proposition 8, and Fatou's lemma; Hardy's inequality (Theorem 2) is a consequence of Theorem 5, Proposition 8, and Fatou's lemma; and the Hardy-Littlewood inequality (Theorem 3) can be deduced from Theorem 6, Proposition 8, and Fatou's lemma.

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A Simple Proof of Descartes's Rule of Signs

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"Descartes' Rule of Signs is a staple of high school algebra, but a proof is seldom seen, even at the college level" [**2**]. A proof of the theorem is usually several pages long [**2**]. In this note we give a simple proof.

Theorem (Descartes's Rule of Signs). *Let $p(x) = a_0x^{b_0} + a_1x^{b_1} + \cdots + a_nx^{b_n}$ denote a polynomial with nonzero real coefficients a_i, where the b_i are integers satisfying $0 \le b_0 < b_1 < b_2 < \cdots < b_n$. Then the number of positive real zeros of $p(x)$ (counted with multiplicities[1]) is either equal to the number of variations in sign in the sequence $a_0, \dots, a_n$ of the coefficients or less than that by an even whole number. The number of negative zeros of $p(x)$ (counted with multiplicities) is either equal to the number of variations in sign in the sequence of the coefficients of $p(-x)$ or less than that by an even whole number.*

In the following we denote the number of variations in the signs of the sequence of the coefficients of p by $v(p)$ and the number of positive zeros of p counting multiplicities by $z(p)$. We need the following simple lemma.

Lemma. *Let $p(x) = a_0x^{b_0} + a_1x^{b_1} + \cdots + a_nx^{b_n}$ be a polynomial as in the theorem. If $a_0a_n > 0$, then $z(p)$ is even; if $a_0a_n < 0$, then $z(p)$ is odd.*

Proof. We consider only the case when $a_0 > 0$ and $a_n > 0$. The other cases can be handled similarly. Because $p(0) \ge 0$ and $p(x) \to \infty$ as $x \to \infty$, it is clear that the graph of p crosses the positive x-axis an even number of times, where we count crossings without regard to multiplicity. If $x = a$ is a point at which the graph of p touches but does not cross the positive x-axis, then the multiplicity of a is even. If the graph

[1]The theorem would be false if the positive real roots were counted without multiplicity, as the example $x^2 - 2x + 1$ shows.

of p crosses the positive x-axis at a multiple zero $x = a$, then $x = a$ is a zero of odd multiplicity. Thus the multiple zeros contribute an additional even number to $z(p)$, whence $z(p)$ is even. ■

Proof of the theorem. We need prove only the first part of the theorem. Without loss of generality we may assume that $b_0 = 0$. We argue by induction on n. It is obvious that for $n = 1$, $v(p) = z(p)$. Assume that for $n \leq k - 1$, $v(p) \geq z(p)$ and $z(p) \equiv v(p) (\bmod 2)$, and consider the situation for $n = k$. We need to treat the following cases.

Case 1: $a_0 a_1 > 0$. In this instance $v(p) = v(p')$. By the lemma, $z(p) \equiv z(p')$ $(\bmod 2)$. By the induction hypothesis, $z(p') \equiv v(p') (\bmod 2)$ and $z(p') \leq v(p')$. Thus $z(p) \equiv v(p) (\bmod 2)$. An appeal to Rolle's theorem gives $z(p') \geq z(p) - 1$. It follows that

$$v(p) = v(p') \geq z(p') \geq z(p) - 1 > z(p) - 2,$$

so $z(p) \leq v(p)$.

Case 2: $a_0 a_1 < 0$. Here $v(p') + 1 = v(p)$ and $z(p) - z(p') \equiv 1 (\bmod 2)$ in view of the lemma. By the induction hypothesis, $z(p') \equiv v(p') (\bmod 2)$ and $z(p') \leq v(p')$. Again we infer that $z(p) \equiv v(p) (\bmod 2)$. By Rolle's theorem, $z(p') \geq z(p) - 1$. Thus $v(p) = v(p') + 1 \geq z(p') + 1 \geq z(p)$. ■

Remark. All steps of the foregoing proof remain valid if we drop the requirement that the b_i be nonnegative integers and instead allow them to be arbitrary real numbers (possibly negative) satisfying $b_0 < b_1 < \cdots < b_n$. As a result, the first part of the theorem can be generalized to polynomials with real exponents.

Example. Let $p(t) = a_0 + a_1 e^{b_1 t} + a_2 e^{b_2 t}$. Letting $x = e^{b_1 t}$ yields $p(t) = a_0 + a_1 x + a_2 x^{b_2/b_1} = Q(x)$. Since Q has at most two positive zeros, p has at most two real zeros.

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