116 Problems in Algebra

Problems' Proposer: Mohammad Jafari *

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116 Problems in Algebra is a nice work of Mohammad Jafari. Tese problems have been published in a book, but it is in Persian (Farsi). Problems are very nice, so I [1] decided to collect a set of solutions for them. I should thank **pco**, **socrates**, **applepi2000**, **Potla**, **goldeneagle**, and **professordad** who solved the problems and posted the solutions [2].

Remark. A very few number of problems remained unsolved on AoPS. I will add the solutions of those problems as soon as I find the book written by Mr. Jafari. As this happened, problems' numbers are not the same as the file posted on AoPS by the author [3]: they are consecutive.

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[1] Amir Hossein Parvardi

[2] Topic in AoPS: `http://www.artofproblemsolving.com/Forum/viewtopic.php?t=444651`

[3] Topic in AoPS: `http://www.artofproblemsolving.com/Forum/viewtopic.php?t=406530`

1 Functional Equations

1. Find all functions $f(x)$ from $\mathbb{R} \to \mathbb{R}$ that satisfy:

$$f(x+y) = f(x)f(y) + xy$$

Solution. [First Solution] *[by pco [4]]* Let $P(x, y)$ be the assertion $f(x+y) = f(x)f(y) + xy$. Let $f(1) = u$. Tthe function $f(x) = 0\ \forall x$ is not a Solution. Let then a such that $f(a) \neq 0 : P(a, 0) \implies f(a)(f(0) - 1) = 0$ and so $f(0) = 1$.

- $P(-1, 1) \implies f(1)f(-1) = 2 \implies f(-1) = \frac{2}{u}$
- $P(1, 1) \implies f(2) = u^2 + 1$
- $P(-1, 2) \implies f(1) = f(-1)f(2) - 2 \implies u = \frac{2}{u}(u^2 + 1) - 2 \implies u^2 - 2u + 2 = 0$, impossible

And so no solution.

Solution. [Second Solution] *[by applepi2000]* Let $P(x, y)$ be the above assertion. Then:

$$P(0, y) \implies f(y) = f(0)f(y).$$

So, either $f(y)$ is always 0, which isn't a solution (test $x = y = 1$), or $f(0) = 1$. Thus $f(0) = 1$ and:

$$P(x, -x) \implies f(x)f(-x) - x^2 = 1.$$

Let $x = 2a$. Then:

$$1 + 4a^2 = f(-2a)f(2a) = (f(a)^2 + a^2)(f(-a)^2 + a^2) \geq a^4.$$

However, taking large a, such as $a = 42$, this is false. So, no such function exists.

[4] Here: `http://www.artofproblemsolving.com/Forum/viewtopic.php?t=378997`

2. Find all functions $f(x)$ from $\mathbb{R}\setminus\{1\}\to\mathbb{R}$ such that: $f(xy)=f(x)f(y)+xy$ $\forall x,y\in\mathbb{R}\setminus\{1\}$.

Solution. *[by pco]* According to me, domain of functional equation must also contain $xy\neq 1$.

Let then $g(x)=\frac{f(x)}{x}$ defined from $(1,+\infty)\to\mathbb{R}$.

The equation becomes $P(x,y):\ g(xy)=g(x)g(y)+1$.

Let $x>1:\ P(\sqrt{x},\sqrt{x}) \implies g(x)=g(\sqrt{x})^2+1\geq 1$.

Let then $m=\inf_{x>1}g(x)$ and a_n a sequence of reals in $(1,+\infty)$ such that $\lim_{n\to+\infty}g(a_n)=m$.

- $P(\sqrt{a_n},\sqrt{a_n}) \implies g(a_n)=g(\sqrt{a}_n)^2+1\geq m^2+1$.

Setting $n\to+\infty$ in this inequality, we get $m\geq m^2+1$, impossible. So no solution.

3. Find all functions $f:\mathbb{Z}\to\mathbb{Z}$ such that:

$$f(x)=2f(f(x))\quad \forall x\in\mathbb{Z}.$$

Solution. *[by Rust and mavropnevma [5]]* From $f(x)=2f(f(x))$ we get $2\mid f(x)$. But then $2\mid f(f(x))$, so $4\mid f(x)$. But then $4\mid f(f(x))$, so $8\mid f(x)$. Repeating this, we have $2^n\mid f(x)$ for all $n\in\mathbb{N}$, so $f(x)\equiv 0$.

4. Find all functions $f:\mathbb{Z}\to\mathbb{Z}$ and $g:\mathbb{Z}\to\mathbb{Z}$ such that:

$$f(x)=3f(g(x))\quad \forall x\in\mathbb{Z}.$$

Solution. *[by tkrass [6]]* $f(g(x))\in\mathbb{Z}$, so $3|f(x)$ for all $x\in\mathbb{Z}$. Hence $3|f(g(x))$ and $9|f(x)$ for all x. By easy induction we get that $f(x)$ is divisible by any power of 3, hence $f(x)\equiv 0$. And $g(x)$ may be any function taking integer values.

[5] Here: `http://www.artofproblemsolving.com/Forum/viewtopic.php?t=378363`
[6] Here: `http://www.artofproblemsolving.com/Forum/viewtopic.php?t=378998`

5. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that:

$$7f(x) = 3f(f(x)) + 2x \quad \forall x \in \mathbb{Z}.$$

Solution. *[by Sansa [7]]* Let $f(x) = g(x) + 2x$, substituting in the main equation we get:

$$7(g(x) + 2x) = 3f(g(x) + 2x) + 2x \implies$$

$$7g(x) + 12x = 3(g(g(x) + 2x) + 2g(x) + 4x) \implies$$

$$g(x) = 3g(g(x) + 2x)$$

Now let $t(x) = g(x) + 2x$, so we have: $g(x) = 3g(t(x))$ At first we found out that $3|g(x) \implies 3|g(t(x)) \implies 9|g(x) \cdots$. Thus $3^n \mid g(x) \quad \forall n \in \mathbb{N}$, so $g(x) \equiv 0 \implies f(x) = 2x \ \ \forall x \in \mathbb{R}$.

6. Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ that for all $x, y \in \mathbb{Q}$ satisfy:

$$f(x + y + f(x + y)) = 2f(x) + 2f(y).$$

Solution. *[by socrates]* Put $y := 0$ to get

$$f(x + f(x)) = 2f(x) + 2f(0).$$

Now, put $x := x + y$ into the last equality to get

$$f(x + y + f(x + y)) = 2f(x + y) + 2f(0).$$

This, together with the initial, gives

$$f(x + y) - f(0) = f(x) - f(0) + f(y) - f(0),$$

so $f(x) = ax + b$. Substituting into the original equation, we find the solutions:

[7] Here: `http://www.artofproblemsolving.com/Forum/viewtopic.php?t=378999`

- $f(x) = x, \ \forall x$
- $f(x) = -2x, \ \forall x.$

7. Let $f : \mathbb{R} \to \mathbb{R}$ be a function that for all $x, y \in \mathbb{R}$ satisfies:

$$f(x + f(x) + y) = x + f(x) + 2f(y).$$

Prove that f is a bijective function.

Solution. *[by socrates]* Let $P(x, y) \ : \ f(x + f(x) + y) = x + f(x) + 2f(y)$. Consider a, b such that $f(a) = f(b)$. Then $P(a, b), \ P(b, a)$ give $a = b$. So f is injective.

Putting $y := -f(x)$ we get

$$f(-f(x)) = -\frac{x}{2},$$

so f is clearly surjective.

Actually, no such function exists: Put $y := 0, x := -f(0)$ so, since $f(-f(0)) = 0, \ f(0) = 0$. So $x := 0$ gives $f \equiv 0$ which is not a solution .

8. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all real x, y we have:

$$f(x + f(x) + 2y) = x + f(f(x)) + 2f(y).$$

Solution. *[by applepi2000]* Let $P(x, y)$ be the above assertion, let a be an arbitrary real number. Then:

$$P(-2a, a) \implies f(a) = a.$$

We're done!

9. Given a function $f : \mathbb{R} \to \mathbb{R}$ such that:

$$f(x + f(x) + 2y) = x + f(x) + 2f(y).$$

Prove that f is bijective and that $f(0) = 0$.

Solution. *[by applepi2000]* Let $P(x, y)$ be the above assertion.

$$P(x, -\frac{f(x)}{2}) \implies -\frac{x}{2} = f(-\frac{f(x)}{2}).$$

So f is surjective. Now assume $f(a) = f(b)$, then from above $-\frac{a}{2} = -\frac{b}{2}$, so $a = b$ and thus f is injective. Finally, let $f(0) = n$. Then from the above, $f(-\frac{n}{2}) = 0 \implies f(0) = \frac{n}{4}$. So $n = 0$ and $f(0) = 0$ as desired.

10. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $x > y > 0$ we have:

$$f(x - y) = f(x) - f(x)f(\frac{1}{x})y.$$

Solution. *[by Dijkschneier [8]]*

- $P(x, y) : f(x - y) = f(x) - f(x)f(\frac{1}{x})y$
- $f(x - y) > 0 \implies f(x)(1 - yf(\frac{1}{x})) > 0$
- $\implies \frac{1}{y} > f(\frac{1}{x})$
- $y \to x \implies \frac{1}{x} \geq f(\frac{1}{x})$
- $\implies x \geq f(x) \forall x > 0$
- $1 > y > 0, P(1, y) \implies f(1 - y) = f(1) - f(1)^2 y$

Now take in particular $1 > y > \frac{1}{(1+f(1))}$ and so:

- $\implies 1 - y \geq f(1) - f(1)^2 y$
- $\implies 1 - f(1) \geq y(1 - f(1))(1 + f(1))$
- $\implies (1 - f(1))(1 - y(1 + f(1))) \geq 0$
- $\implies f(1) \geq 1$
- $\implies f(1) = 1$

So $f(1 - y) = 1 - y$, and hence $f(y') = y' \forall 1 > y' > 0$.

- $x - y \geq f(x - y) = f(x) - f(x)f(\frac{1}{x})y \implies x - f(x) \geq y(1 - f(x)f(1/x))$

For $0 < x < 1$, we have $x = f(x)$ and so from the inequality $\frac{1}{x} = \frac{1}{f(x)} \leq f(\frac{1}{x}) \leq \frac{1}{x}$, that is, $f(\frac{1}{x}) = \frac{1}{x}$.

[8] Here: `http://www.artofproblemsolving.com/Forum/viewtopic.php?t=424694`

Hence $f(x) = x \forall x > 1$ and so in conclusion: $f(x) = x \forall x > 0$, which conversely is a solution.

11. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(f(x + f(y))) = x + f(y) + f(x + y),$$

for all $x, y \in \mathbb{R}$.

Solution. *[by pco]* Let $P(x, y)$ be the assertion $f(f(x + f(y))) = x + f(y) + f(x + y)$

If $f(a) = f(b)$ for some a, b, then, comparing $P(x - b, a)$ and $P(x - b, b)$, we get $f(x) = f(x + a - b)\ \forall x$

But then, comparing $P(x, y)$ and $P(x + a - b, y)$, we get $x = x + a - b$ and so $a = b$ and $f(x)$ is injective.

$P(-f(x), x) \implies f(f(0))) = f(x - f(x))$ and so, since injective : $f(0) = x - f(x)$ and $f(x) = x + a$, which is never a solution.

So no solution.

12. Find all functions $f : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ such that

$$f(f(x + f(y))) = 2x + f(x + y),$$

for all $x, y \in \mathbb{R}^+ \cup \{0\}$.

Solution. *[by pco]* Let $P(x, y)$ be the assertion $f(f(x + f(y))) = 2x + f(x + y)$

If $f(a) = f(b)$ for some a, b, then, comparing $P(x, a)$ and $P(x, b)$, we get $f(x + a) = f(x + b)\ \forall x$

But then, comparing $P(x + a, y)$ and $P(x + b, y)$, we get $a = b$ and so $f(x)$ is injective.

$P(0, x) \implies f(f(f(x))) = f(x)$ and, since injective, $f(f(x)) = x$

So $P(x, 0)$ becomes $x + f(0) = 2x + f(x)$ and so $f(x) = f(0) - x$ which is never a Solution 1. (since < 0 for x great enough)

So no solution.

13. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$:

$$f(x + f(x) + 2f(y)) = x + f(x) + y + f(y).$$

Solution. *[by socrates]* Putting $x := 0$ we see that f is injective. Put $x := y,\ y := x$ to get

$$f(x + f(x) + 2f(y)) = x + f(x) + y + f(y) = f(y + f(y) + 2f(x)),$$

so $x+f(x)+2f(y) = y+f(y)+2f(x) \implies f(x)-x = f(y)-y$, that is $f(x)-x$ is constant: $f(x) = x + c$.

Substituing, we find $f(x) = x,\ \forall x$.

14. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$:

$$f(2x + 2f(y)) = x + f(x) + y + f(y).$$

Solution. *[by socrates]* As in the previous problem, we get $f(2x + 2f(y)) = f(2y + 2f(x))$ so $f(x) = x + c$. Substituing, we find $f(x) = x,\ \forall x$.

15. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$ satisfy:

$$f(f(x) + 2f(y)) = f(x) + y + f(y).$$

Solution. *[by socrates]* Putting $x := 0$ we see that f is injective. Put $y := -f(x)$ to find $f(-f(x)) = -f(x)$. So, $x := -f(y)$ gives $f(f(y)) = y$ and so f is surjective. Finally, put $y := 0$ to get $f(f(x) + 2f(0)) = f(x) + f(0)$. Since f is surjective we get $f(x) = x + c$. Substituing, we find $f(x) = x,\ \forall x$.

16. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that • $f(x^2 + f(y)) = f^2(x) + f(y)$ for all $x, y \in \mathbb{R}$ • $f(x) + f(-x) = 0$ for all $x \in \mathbb{R}^+$ • The number of the elements of the set $\{x \in \mathbb{R} | f(x) = 0\}$ is finite.

Solution. *[by socrates]* Assume $f(x) > x$ for each $x \neq 0$. Then $0 = f(x) + f(-x) > x - x = 0$ contradiction. So $f(a) \leq a$ for some $a \neq 0$.

Put $x := \sqrt{a - f(a)},\ y := a$ into the first condtion to get $f(a) = f^2(\sqrt{a - f(a)}) + f(a) \implies f(\sqrt{a - f(a)}) = 0$, so $f(c) = 0$ for some c.

Now put $x, y := c$ and we get $f(c^2) = 0$ and by induction $f(c^{2^n}) = 0$. If $c \neq 0, 1, -1$ then third condition is false. So $c = 0, 1, -1$.

Observe that if $f(k) > k$ ($f(k) < k$) then $f(-k) < -k$ ($f(-k) > -k$) so for each $x \neq 0$ we have $f(x) = x,\ x + 1,\ x - 1$.

If $f(l) = l - 1$ for some l (the case $f(l) = l + 1$ reduses to $f(-l) = -l - 1$) then $f(1) = 0$. So $f(1 + f(y)) = f(y)$.

- If $f(2) = 2$ then $f(3) = 2$ and so $f(9) = f^2(3) = 4$, impossible.
- If $f(2) = 1$ then $f(4) = f^2(2) = 1$, impossible.
- If $f(2) = 3$ then $f(4) = 9$, impossible.

Thus, $f(x) = x, \forall x \neq 0$. Finally, $x := 0$ gives $f(0) = 0$ so the only solution is $f(x) = x$.

17. Let $f : \mathbb{R} \to \mathbb{R}$ be an injective function that for all $x, y \in \mathbb{R}$ satisfies:

$$f(x + f(x)) = 2x.$$

Prove that $f(x) + x$ is a bijective function.

Solution. *[by socrates]* Let $x + f(x) = g(x)$. So $f(g(x)) = 2x$. g is clearly injective. Take $y \in \mathbb{R}$, arbitrary and let $f(y) = 2z = f(g(z))$ so by injectivity of f we get $g(z) = y$. We're done.

18. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that :

$$f(x + f(x) + 2f(y)) = 2x + y + f(y) \quad \forall x, y \in \mathbb{R}.$$

Solution. *[by mousavi [9]]* It is obvious that f is injective.

[9] Here: `http://www.artofproblemsolving.com/Forum/viewtopic.php?t=378130`

- $x = y = 0 \implies f(3f(0)) = f(0) \implies f(0) = 0$
- $y = 0 \implies f(x + f(x)) = 2x$
- $x = 0 \implies f(2f(y)) = y + f(y) \implies ff(2f(y)) = 2y$

Put $f(2f(y))$ instead of $y \implies f(x + f(x) + 4y) = 2x + f(2f(y)) + 2y$

- $x = 0 \implies f(4y) = 2y + f(2f(y))$
- $f(f(4y) - 2y) = 2y + f(y + f(y))$
- $f(4y) - f(y) = 3y$
- $y = -2x \implies f(x + f(x) + 2f(-2x)) = f(-2x)$
- $3x + f(x) + 2f(-2x) = 0$
- $\implies -6x + f(-2x) + 2f(4x) = 0$
- $2f(x) = -f(-2x)$
- $x = -2x, y = x \implies f(-2x + f(-2x) + 2f(x)) = -4x + x + f(x)$
- $f(-2x) = 3x + f(x)$

$-2f(x) = -3x + f(x)$

- $\implies f(x) = x.$

19. Let $f, g, h : \mathbb{R} \to \mathbb{R}$ be functions such that f is injective and h is bijective, satisfying $f(g(x)) = h(x)$ for all $x \in \mathbb{R}$. Prove that g is a bijective function.

Solution. *[by socrates]* Injectivity is obvious. Now take $y \in \mathbb{R}$, arbitrary. There exists $z : h(z) = f(y) = f(g(z))$ so by injectivity of f we get $g(z) = y$. We are done.

20. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$ satisfy:

$$f(2x + 2f(y)) = x + f(x) + 2y.$$

Solution. *[by socrates]* Putting $x := 0$ we see that f is bijective. We have $f(2f(0)) = f(0)$ so $f(0) = 0$. Put $y := 0$ to get $f(2x) = x + f(x)$. So the initial equation becomes $f(2x + 2f(y)) = f(2x) + f(2f(y))$ or $f(x + y) = f(x) + f(y)$ from the surjectivity of f.

So, $x + f(x) = f(2x) = 2f(x) \implies f(x) = x$ which is indeed a Solution 1.

21. Find all functions $f : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+$ such that for all $x, y \in \mathbb{R}^+ \cup \{0\}$:

$$f(\frac{x + f(x)}{2} + y) = f(x) + y.$$

Solution. *[by socrates]* Put

$$y := \frac{y + f(y)}{2}$$

to get

$$f(\frac{x + f(x)}{2} + \frac{y + f(y)}{2}) = f(x) + \frac{y + f(y)}{2}.$$

So

$$f(y) + \frac{x + f(x)}{2} = f(\frac{x + f(x)}{2} + \frac{y + f(y)}{2}) = f(x) + \frac{y + f(y)}{2}$$

so $f(x) - x = f(y) - y$, that is $f(x) - x$ is constant: $f(x) = x + c$. Substituing, we find $f(x) = x, \ \forall x$.

22. Find all functions $f : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ such that for all $x, y \in \mathbb{R}^+ \cup \{0\}$:

$$f(\frac{x + f(x)}{2} + f(y)) = f(x) + y.$$

Solution. *[by socrates]* Put $x = y := 0 \implies f(\frac{3f(0)}{2}) = f(0)$. Put $x := 0, \ y := \frac{3f(0)}{2}$: $f(\frac{3f(0)}{2}) = \frac{5f(0)}{2}$ so $f(0) = 0$.

Now, $x := 0$ and $y := 0$ give $f(f(y)) = y$ and $f(\frac{x+f(x)}{2}) = f(x)$, respectively. The former implies that f is injective and the latter $f(x) = x$, which is indeed a Solution 1.

23. Let $f : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$ be a function such that • $f(x + y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}^+ \cup \{0\}$ • The number of the elements of the set $\{x | f(x) = 0, \ x \in \mathbb{R}^+ \cup \{0\}\}$ is finite. Prove that f is injective function.

Solution. *[by socrates]* We easily find $f(0) = 0$. Suppose there exist $a, b \geq 0$ such that $a \neq b$ and $f(a) = f(b)$. Wlog assume $a > b$. Then $f(a) = f(a-b+b) = f(a-b) + f(b)$ so $f(c) = 0$ for some $c \neq 0$. So, by induction, $f(2^n c) = 0$ for each $n \in \mathbb{N}$, contradicting the second condtion. So f is injective.

24. Find all functions $f : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}$ such that **i)** $f(x + f(x) + 2y) = f(2x) + 2f(y)$, for all $x, y \in \mathbb{R}^+ \cup \{0\}$ **ii)** The number of the elements of the set $\{x | f(x) = 0,\ x \in \mathbb{R}^+ \cup \{0\}\}$ is finite.

Solution. *[by goldeneagle* [10] Let $P(a, b)$ be the assertion. Let A be the set which mentioned in the second condition. Define $\frac{k}{2} = Max(A)$.

If $x > f(x)$ then $P(x, \frac{x-f(x)}{2}) \Rightarrow f(\frac{x-f(x)}{2}) = 0 \Rightarrow f(x) \geq x - k$ so $\forall x \in \mathbb{R}^+ \cup \{0\}$: $f(x) \geq x - k$ (I).

I want to prove that f is injective. If not, then $\exists a < b : f(a) = f(b)$. Define $t = b - a$.

- $P(\frac{a}{2}, \frac{x}{2}), P(\frac{b}{2}, \frac{x}{2})$, (I) $\Rightarrow \frac{a}{2} + f(\frac{a}{2}) = \frac{b}{2} + f(\frac{b}{2})$ (II)
- $P(a, \frac{b}{2}), P(b, \frac{a}{2})$, (II) $\Rightarrow f(2a) - f(2b) = t$ (III)

$P(a, x), P(b, x)$, (III) $\Rightarrow \forall x \geq a + f(a)$: $f(x) - f(x+t) = t$. But $f(x) \geq x - k$, so this is contradiction! $(t > 0)$

Now I want to prove that $f(0) = 0$. Define $c = f(0)$. If $c < 0$, then $P(0, -\frac{c}{2}) \Rightarrow f(-\frac{c}{2}) = 0$ and since f is injective we should have $c = 0$, contradiction!

So $c \geq 0$. We have

- $P(0, 0) \Rightarrow f(c) = 3c$
- $P(0, x) \Rightarrow f(2x + c) = 2f(x) + c$ (*)
- $P(c, x) \Rightarrow f(2x + 4c) = 2f(x) + f(2c)$ (**)
- (*), (**) $\Rightarrow f(2x+4c) - f(2x+c) = f(2c) - c \Rightarrow (x = \frac{c}{2}) f(5c) = 2f(2c) - c$

Put $x = 2c$ in (*): $f(5c) = 2f(2c) + c$, so $c = 0$ and then $P(x, 0) \Rightarrow f(x) = x$.

[10] http://www.artofproblemsolving.com/Forum/viewtopic.php?t=442600

25. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that: • $f(f(x) + y) = x + f(y)$ for all $x, y \in \mathbb{R}$ • for each $x \in \mathbb{R}^+$ there exists some $y \in \mathbb{R}^+$ such that $f(y) = x$.

Solution. *[by socrates]* Put $y := 0$ to get $f(f(x)) = x + f(0)$ that is f is bijective. So, since $f(f(0)) = f(0)$ we have $f(0) = 0$. So $f(f(x)) = x$ and $f(x + y) = f(x) + f(y)$. Since f bijective, for each $x > 0$ there exists unique $y = f(x)$ such that $f(y) = x$, so second condition means $f(x) > 0$ for each $x > 0$.

It is well known that since f is Cauchy function, it is increasing so $f(x) = cx$ and substituing $f(x) = x$.

26. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that • $f(f(x) + y) = x + f(y)$ for all $x, y \in \mathbb{R}$ • The set $\{x \in \mathbb{R} | f(x) = -x\}$ has a finite number of elements.

Solution. *[by socrates]* Put $y := 0$ to get $f(f(x)) = x + f(0)$ so f is bijective. Hence, $f(0) = 0,\ f(f(x)) = x$ and $f(x + y) = f(x) + f(y)$.

Let $g(x) = f(x) + x$. Then $g(x + y) = g(x) + g(y)$. If $g(a) = 0$ for some $a \neq 0$ then $g(2^n a) = 0,\ \forall n \in \mathbb{N}$, contradiction. So $g(x) = 0 \iff x = 0.(^*)$

Now, take a, b such that $f(a) = f(b)$. Then $f(a) = f(a - b + b) = f(a - b) + f(b) \implies f(a - b) = 0 \overset{(^*)}{\implies} a = b$. Thus, g is injective.

We have $g(f(x)) = f(f(x)) + f(x) = x + f(x) = g(x) \implies f(x) = x$.

27. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y, z \in \mathbb{R}$:

$$f(f(f(x)) + f(y) + z) = x + f(y) + f(f(z)).$$

Solution. *[by socrates]* Putting $y, z := 0$ we see that f is bijective. So $f(c) = 0$ for some $c \in \mathbb{R}$. Putting $x := 0, y := c$ we get $f(f(z)) = f(z + f(f(0)))$ so, by injectivity, $f(x) = x + c$ and finally $f(x) = x$.

28. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(\frac{x + f(x)}{2} + y + f(2z)) = 2x - f(x) + f(y) + 2f(z),$$

for all $x, y, z \in \mathbb{R}$.

Solution. *[by goldeneagle]* Let $P(a,b,c)$ be the assertion. Now if $f(a)=f(b)$, then $P(0,0,\frac{a}{2}), P(0,0,\frac{b}{2}) \Rightarrow f(\frac{a}{2})=f(\frac{b}{2})$ and then $P(a,\frac{b}{2},0), P(b,\frac{a}{2},0) \Rightarrow a=b$. So f is injective. Also $P(x,x,0)$ gives us that f is surjective. Define $f(0)=t$ and $f(a)=0$.

- $P(0,0,a) \Rightarrow a=\frac{t}{2}+f(2a)$,
- $P(0,a,a) \Rightarrow f(\frac{t}{2}+a+f(2a))=-t \Rightarrow f(2a)=-t$, so $t=-2a, f(2a)=2a$,
- $P(2a,a,a) \Rightarrow f(5a)=2a \Rightarrow 5a=2a \Rightarrow a=0$,
- $P(0,0,x) \Rightarrow f(f(2x))=2f(x)$,
- $P(0,x,y): f(x+f(2y))=f(x)+2f(y)=f(x)+f(f(2y)) \Rightarrow f(x+y)=f(x)+f(y)$ ($f$is surjective!).

Now $f(f(2x))=2f(x)=f(2x) \Rightarrow f(x)=x \quad \forall x \in \mathbb{R}$.

29. Find all functions $f:[0,+\infty) \to [0,+\infty)$ such that

$$f(\frac{x+f(x)}{2}+y+f(2z))=2x-f(x)+f(y)+2f(z),$$

for all $x, y, z \geq 0$.

Solution. *[by pco]* Let $P(x,y,z)$ be the assertion $f(\frac{x+f(x)}{2}+y+f(2z)) = 2x-f(x)+f(y)+2f(z)$ Let $f(0)=2a$

- $P(0,f(2x),0) \implies f(3a+f(2x)))=f(2x)+2a$
- $P(0,f(2y),x) \implies f(3a+f(2x))=2f(x)$

And so $f(2x)=2f(x)-2a \ \forall x$

$P(\frac{x}{2},\frac{x}{2},0) \implies f(\text{something})=x+4a$ and any real $\geq 4a$ is in the image of $f(x)$ So the quantity $\frac{x+f(x)}{2}+f(2z)$ may take any value $\geq 5a$ (choosing $x=0$ and appropriate z)

So we got $f(u+y)=g(u)+f(y) \ \forall y \geq 0$ and $\forall u \geq 5a$ and for some function $g(x)$ Setting there $y=0$, we get $g(u)=f(u)-2a$ and so :

$f(x+y)=f(x)+f(y)-2a \ \forall x \geq 0$ and $\forall y \geq 5a$

Writing $h(x) = f(x) - 2a$, the two properties are : $h(2x) = 2h(x)$ $\forall x$ $h(x + y) = h(x) + h(y)$ $\forall x \geq 0$ and $\forall y \geq 5a$

But, $\forall y > 0$, $\exists n \in \mathbb{N}$ such that $2^n y > 5a$ So $h(2^n x + 2^n y) = h(2^n x) + h(2^n y)$ So $h(x + y) = h(x) + h(y)$ $\forall x, y \geq 0$

And since $h(x)$ is lower bounded, we get $h(x) = cx$ and so $f(x) = cx + 2a$

Plugging this back in original equation, we find the unique solution $f(x) = x$ $\forall x$.

30. Find all non decreasing functions $f : [0, +\infty) \to [0, +\infty)$ such that $f(\frac{x+f(x)}{2} + y) = 2x - f(x) + f(f(y))$ $\forall x, y \geq 0$.

Solution. *[by pco]* Let $P(x, y)$ be the assertion $f(\frac{x+f(x)}{2} + y) = 2x - f(x) + f(f(y))$

If $f(x) < x$ for some x, then $P(x, \frac{x-f(x)}{2}) \implies f(x) - x = \frac{1}{2} f(f(\frac{x-f(x)}{2})) \geq 0$ and so contradiction. So $f(x) \geq x$ $\forall x$

If $f(a) = f(b)$ for some $b > a$ then $f(x) = f(a)$ $\forall x \in [a, b]$ since $f(x)$ is non decreasing and then : $\forall x \in [a, b]$: $P(x, 0) \implies f(\frac{x+f(a)}{2}) = 2x - f(a) + f(f(0))$ So $\forall x \in [\frac{a+f(a)}{2}, \frac{b+f(a)}{2}]$: $f(x) = 4x - 3f(a) + f(f(0)) = 4x + c$

So $\forall x \in [\frac{a+f(a)}{2}, \frac{b+f(a)}{2}]$, $P(x, 0) \implies f(\frac{5x+c}{2}) = -2x - c + f(f(0))$ which is impossible since $f(x)$ is non decreasing So $f(x)$ is injective.

$f(x) \geq x \implies \frac{x+f(x)}{2} + y \geq x + y \implies f(\frac{x+f(x)}{2} + y) \geq f(x + y) \geq f(x)$ So $2x - f(x) + f(f(y)) \geq f(x)$ and $f(f(y)) \geq 2(f(x) - x)$

Setting $x = 0$ and $y = 0$ in this inequality, we get $f(f(0)) \geq 2f(0)$ Setting $x = f(0)$ and $y = 0$ in the same inequality, we get $2f(0) \geq f(f(0))$ And so $f(f(0)) = 2f(0)$

Then $P(f(0), x) \implies f(\frac{3f(0)}{2} + x) = f(f(x))$ and so, since injective : $f(x) = x + \frac{3f(0)}{2}$

So $f(0) = 0$ and $f(x) = x$ $\forall x$, which indeed is a solution.

31. Find all functions $f : [0,+\infty) \to [0,+\infty)$ such that $f(x+f(x)+2y) = 2x + f(2f(y))$ $\forall x, y \geq 0$.

Solution. *[by pco]* Let $P(x,y)$ be the assertion $f(x+f(x)+2y) = 2x+f(2f(y))$.

$P(0,y) \implies f(2y+f(0)) = f(2f(y))$ and $P(x,y)$ becomes: New assertion $Q(x,y)$: $f(x+f(x)+y) = 2x+f(y+f(0))$.

Let then $f(a) = f(b)$ with $a, b \geq f(0)$:

- $Q(a, b-f(0)) \implies f(a+f(a)+b-f(0)) = 2a+f(a)$
- $Q(b, a-f(0)) \implies f(b+f(b)+a-f(0)) = 2b+f(b)$

And so $a = b$ and we have a kind of "pseudo injectivity" (with limitation $a, b \geq f(0)$)

- $P(0,0) \implies f(f(0)) = f(2f(0))$

So, since both $f(0), 2f(0) \geq f(0)$ applying previous "pseudo injectivity", we get $f(0) = 2f(0)$ and so $f(0) = 0$

And pseudo injectivity above becomes injectivity : $f(x)$ is injective.

$P(0,x) \implies f(2x) = f(2f(x))$ and so, since injective : $f(x) = x$ $\forall x$ which indeed is a solution.

32. Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that $f(x+f(x)+2y) = 2x+2f(f(y))$ $\forall x, y \in \mathbb{Q}$.

Solution. *[by pco]* Let $P(x,y)$ be the assertion $f(x+f(x)+2y) = 2x+2f(f(y))$

- $P(0,0) \implies f(f(0)) = 0$
- $P(f(0),0) \implies f(0) = 0$
- $P(0,y) \implies f(2y) = 2f(f(y))$ and so $P(x,y)$ may be rewritten as : $f(x+f(x)+y) = f(y)+2x$ which implies $f(y+n(x+f(x))) = f(y)+2nx$ and so (setting $y=0$): new assertion $Q(x,n)$: $f(n(x+f(x))) = 2nx$

Let then $x+f(x) = \frac{p}{q}$ and $1+f(1) = \frac{r}{s}$ with $q, s \neq 0$

- $Q(x, rq) \implies f(pr) = 2rqx$
- $Q(1, ps) \implies f(pr) = 2ps$

And so $rqx = ps \iff \frac{r}{s}x = \frac{p}{q} = x + f(x) \implies f(x) = (\frac{r}{s} - 1)x$

Plugging then back in original equation $f(x) = ax$, we get $a = 1$ and so the unique solution $f(x) = x\ \forall x$.

33. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x - f(y)) = f(y)^2 - 2xf(y) + f(x)$ $\forall x, y \in \mathbb{R}$.

Solution. *[by pco]* Let $P(x, y)$ be the assertion $f(x - f(y)) = f(y)^2 - 2xf(y) + f(x)$.

$f(x) = 0\ \forall x$ is a solution.

So let us from now look for non allzero solutions. And so let u such that $f(u) \neq 0$.

- $P(\frac{f(u)^2 - x}{2f(u)}, u) \implies f(a) = x + f(b)$ and so $x = f(a) - f(b)$ for some a, b depending on x.
- $P(f(a), a) \implies f(f(a)) = f(a)^2 + f(0)$.
- $P(f(a), b) \implies f(f(a) - f(b)) = f(b)^2 - 2f(a)f(b) + f(f(a)) = f(b)^2 - 2f(a)f(b) + f(a)^2 + f(0) = (f(a) - f(b))^2 + f(0)$ And since $x = f(a) - f(b)$, this becomes $f(x) = x^2 + f(0)$ which indeed is a solution, whatever is $f(0)$.

Hence the second solution $f(x) = x^2 + a\ \forall x$ and for any a.

34. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all real x, y:

$$(x + y)(f(x) - f(y)) = (x - y)(f(x) + f(y)).$$

Solution. *[by applepi2000]* Simplifying, we get $\frac{f(x)}{x} = \frac{f(y)}{y} \forall x, y \neq 0$, so $f(n) = kn$, which is indeed always a Solution 1. It remains to be shown that $f(0) = 0$, which is true by substituting $x = 0, y = n$ into the above. This gives $f(0) = -f(0)$ so indeed $f(0) = 0$.

35. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x-y)(x+y) = (x-y)(f(x)+f(y))$ $\forall x, y \in \mathbb{R}$.

Solution. *[by pco]* Let $P(x,y)$ be the assertion $f(x-y)(x+y)=(x-y)(f(x)+f(y))$.

• $P(x+1,1) \implies f(x)(x+2)=xf(x+1)+xf(1)$.

• $P(x+1,x) \implies f(1)(1+2x)=f(x+1)+f(x) \implies f(1)(x+2x^2)=xf(x+1)+xf(x)$. Subtracting, we get $f(x)(x+1)=f(1)x(x+1)$.

And so $f(x)=xf(1)\ \forall x\neq -1$.

• $P(0,1) \implies f(-1)=-f(1)$

And so $f(x)=xf(1)\ \forall x$, which indeed is a solution.

Hence the answer: $f(x)=ax\ \forall x$ and for any real a

36. Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that for all real x,y:

$$f(x+y)(x-y)=f(x-y)(x+y).$$

Solution. *[by applepi2000]* Let a,b be arbitrary positive reals, and let $P(x,y)$ be the above assertion. Then:

$$P\left(\frac{a+b}{2},\frac{a-b}{2}\right) \implies \frac{f(a)}{a}=\frac{f(b)}{b}.$$

and so $f(x)=kx$ for a real constant k, and indeed this always works. (Also $f(0)=0$ from plugging in (x,x) into the above with $x\neq 0$).

37. Find all non decreasing functions $f,g:[0,+\infty)\to[0,+\infty)$ such that $g(x)=2x-f(x)\ \forall x,y\geq 0$.

Prove that f and g are continuous functions.

Solution. *[by pco]* $f(0)+g(0)=0$ and so $f(0)=g(0)=0$.

• $x\geq y \implies f(x)\geq f(y)$ and $g(x)\geq g(y)$ and so $2x-f(x)\geq 2y-f(y)$

So $x\geq y \implies 0\leq f(x)-f(y)\leq 2(x-y)$ and obviously $0\leq g(x)-g(y)\leq 2(x-y)$. This prove continuity of the two functions.

And the properties $f(0)=0$ and $0\leq f(x)-f(y)\leq 2(x-y)\ \forall x\geq y$ and $g(x)=2x-f(x)$ are sufficient to build a solution:

• $f(x)$ is non decreasing and $\geq f(0) = 0$.

• $x \geq y \implies f(x) - f(y) \leq 2(x - y) \implies 2x - f(x) \geq 2y - f(y)$ and so $g(x) \geq g(y)$ and so $g(x)$ is non decreasing and $\geq g(0) = 0$

And since there are infinitely many such f, we have infinitely many solutions and I dont think that we can find a more precise form.

If we limit our choice to differentiable, we can choose for example any function whose derivative is in $[0, 2]$. For example : $f(x) = x + \sin x$ and $g(x) = x - \sin x$ But a lot of non differentiable solutions exist too.

38. Find all bijective functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x + f(x) + 2f(y)) = f(2x) + f(2y)$.

Solution. *[by pco]* Let $P(x, y)$ be the assertion $f(x + f(x) + 2f(y)) = f(2x) + f(2y)$.

• Let a such that $f(a) = 0$.

• Let $x \in \mathbb{R}$ and y such that $f(y) = \frac{x-f(x)}{2}$.

• $P(x, y) \implies f(2x) = f(2x) + f(2y)$ and so $y = \frac{a}{2}$ and $f(\frac{a}{2}) = \frac{x-f(x)}{2}$ and so $f(x) = x - 2f(\frac{a}{2})$.

Plugging $f(x) = x + c$ in original equation, we get $c = 0$.

And so the solution $f(x) = x$.

39. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$f(x + f(x) + y) = f(2x) + f(y)$$

for all $x, y \in \mathbb{R}^+$.

Solution. *[by goldeneagle]* $P(a, b)$ means put $x = a, y = b$. If $f(a) = f(b)(a > b)$ then define $t = a - b$.

• $P(a, b), p(b, a) \Rightarrow f(2a) = f(2b)$ and then $P(a, x), P(b, x) \Rightarrow f(x) = f(x + t) \forall x > b + f(b)$ so we can find $r \in \mathbb{R}^+$ that $f(r) < r$. Now $P(r, r - f(r)) \Rightarrow f(r - f(r)) = 0$, contradiction!

• So $f(a) = f(b) \Rightarrow a = b$ now $P(x, 2y), P(y, 2x) \Rightarrow f(x) - x = c$ so $f(x) = x + c$.

40. Find all functions $f : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ such that

$$f(x + f(x) + 2f(y)) = 2f(x) + y + f(y),$$

for all $x, y \in \mathbb{R}^+ \cup \{0\}$.

Solution. *[by Babak [11]]* This is actually pretty simple;

• Let A be the set of all numbers z such that $z = x + f(x) + 2f(y)$ for some x, y non-negative.

• Now note that if z belongs to A then for some x, y we have $z = x + f(x) + 2f(y)$, so $f(f(z)) = z$. Let t be any non-negative number;

• Now $f(z + f(z) + 2f(t)) = 2f(z) + t + f(t)$ and also $f(f(z) + f(f(z)) + 2f(t)) = 2z + t + f(t)$. But the LHS are the same and so this implies that for all z in A we have that $f(z) = z$.

• Now let $z = x + f(x) + 2f(y)$. We know that $f(z) = y + f(y) + 2f(x)$ and also that $f(z) = z = x + f(x) + 2f(y)$. So $f(x) + y = f(y) + x$ for all non negative x and y. Let $y = 0$, hence $f(x) = f(0) + x$. One can easily see that $f(0) = 0$ and so $f(x) = x$.

41. Find all functions $f : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ such that $f(0) = 0$ and

$$f(x + f(x) + f(2y)) = 2f(x) + y + f(y),$$

for all $x, y \in \mathbb{R}^+ \cup \{0\}$.

Solution. *[by goldeneagle]* $P(a, b)$ means put $x = a, y = b$.

• $P(0, x) \Rightarrow f(f(2x)) = x + f(x)$ (*)

• $P(x, 0) \Rightarrow f(x + f(x)) = 2f(x)$ (**)

[11]Here: `http://www.artofproblemsolving.com/Forum/viewtopic.php?t=444477`

Now consider $f(2a) = f(2b)$ by (*) $a+f(a) = b+f(b)$ and by (**) $f(a) = f(b)$ so $a = b \Rightarrow f$ is injective.

- $P(x, x) \Rightarrow f(x + f(x) + f(2x)) = x + 3f(x)$, and $P(0, x + f(x)) \Rightarrow f(f(2x + 2f(x))) = x + 3f(x)$ so $f(2x + 2f(x)) = x + f(x) + f(2x)$.

Since $x + f(x) = f(f(2x))$, so we have $\forall a \in \mathbb{R}^+ \cup 0 : f(2f(f(a))) = f(a) + f(f(a))$. Now put $x = f(f(a))$ in (*) : $f(f(a)+f(f(a))) = f(f(a))+f(f(f(a)))$ and by (**) replace $f(f(a)+f(f(a)))$ with $2f(f(a))$ and then $f(f(a)) = f(f(f(a))) \Rightarrow a = f(a)$.

42. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that for all $x, y \in \mathbb{R}^+$:

$$f(x + y^n + f(y)) = f(x),$$

where $n \in \mathbb{N}_{n\geq 2}$.

Solution. *[by bappa1971]* Let $\exists w, z$ such that, $\frac{f(w)+w^n}{f(z)+z^n} \notin \mathbb{Q}$.

Denote $s = f(w) + w^n, t = f(z) + z^n$.

Then, $\forall \epsilon > 0$, $\exists a, b \in \mathbb{N}$ such that, $0 < |as - bt| < \epsilon$. [12]

So, $f(x) = f(x + bt - bt) = f(x + bt + (as - bt)) = f(x + bt + \in) = f(x + \in)$.

Which implies, $\lim_{\in \longrightarrow 0} f(x + \in) = f(x)$, so f in continious.

Now, if $f(x)+x^n = c$ for some constant c then, for large x, $f(x) = c-x^n < 0$.

So, take, $u = \liminf_{x\longrightarrow\infty}(f(x) + x^n)$ and $v = \limsup_{x\longrightarrow\infty}(f(x) + x^n)$, we have $v = \infty$.

So, continiuty of f implies $\forall x \geq u$, $\exists j$ such that, $x = f(j) + j^n$.

Hence $f(x) = f(x + k)$ for all $k > u$.

Now, take arbitary x, y and then take z such that $z > \max(x, y) + u$.

Then, $f(x) = f(z) = f(y)$.

So, f is constant.

Now let, $\frac{f(b)+b^n}{f(a)+a^n} \in \mathbb{Q}$ for all a, b.

[12] See here: `http://www.artofproblemsolving.com/Forum/viewtopic.php?t=432389`

Take $r = f(1) + 1 > 1$ and $g : \mathbb{Q} \longrightarrow \mathbb{Q}$, $g(x) = \frac{f(x)+x^n}{r}$.

Then we have $f(x) = rg(x) - x^n$.

So, $rg(x + rg(y)) - (x + rg(y))^n = rg(x) - x^n$.

So

$$g(x + rg(y)) - g(x) = \frac{(x + rg(y))^n - x^n}{r} = \sum_{i=1}^{n} c_i r^{i-1} g(y)^i x^{n-i} \in \mathbb{Q},$$

for all $x \in \mathbb{R}^+$.

• $x = g(y) \Longrightarrow \sum_{i=1}^{n} c_i r^i = \frac{(r+1)^n - 1}{r} \in \mathbb{Q}$ (1).

$x = r \Longrightarrow r^{n-1} \in \mathbb{Q}$ (2).

$y = 1, x = r^2 \Longrightarrow r^{n-1}((r + 1)^n - r^n) \in \mathbb{Q} \Longrightarrow (r + 1)^n - r^n = u \in \mathbb{Q}$ (3).

• (1)-(2) and (3) $\Longrightarrow \frac{u-1}{r} \in \mathbb{Q} \Longrightarrow r \in \mathbb{Q}$

Now, $y = 1$, $x = \pi \Longrightarrow \frac{(r+\pi)^n - \pi^n}{r} = v \in \mathbb{Q} \Longrightarrow (r + \pi)^n - \pi^n - rv = 0$.

The polynomial $h(x) = (x + r)^n - x^n - rv$ has π as a root as well as has all rational co-efficients. An impossibility!

So, $f(x) = c$ is the only solution.

43. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that $f(n-1) + f(n+1) < 2f(n)$ $\forall n \geq 2$.

Solution. *[by pco]* So $f(n+1)-f(n) < f(n)-f(n-1)$ and so $f(n+1)-f(n) < 0$ $\forall n$ great enough. So $f(n+1) < f(n)$ $\forall n$ great enough and so $f(n) < 0$ $\forall n$ great enough

And so no solution.

44. Find all functions $f : \{A \text{ such that } A \in \mathbb{Q}, A \geq 1\} \to \mathbb{Q}$ such that $f(xy^2) = f(4x)f(y) + \frac{f(8x)}{f(2y)}$.

Solution. *[by pco]* Let $P(x, y)$ be the assertion $f(xy^2) = f(4x)f(y) + \frac{f(8x)}{f(2y)}$

Notice that $f(x) \neq 0$ $\forall x \geq 2$.

• $P(x,2) \implies f(8x) = f(4x)(1-f(2))f(4)$ and so $f(2x) = af(x)\ \forall x \geq 4$ and some $a \neq 0$.

Let $x, y \geq 4$ and the equation becomes then $f(xy^2) = a^2 f(x)f(y) + a^2\frac{f(x)}{f(y)}$.

Setting $y \to 2y$ in this equation, we get $f(xy^2) = af(x)f(y) + \frac{f(x)}{af(y)}$.

And so (subtracting) : $a^2(a-1)f(y)^2 = (1-a^3)$ and so $a = 1$ (the case $|f(y)| =$ constant is easy to cancel).

So $f(xy^2) = f(x)f(y) + \frac{f(x)}{f(y)}\ \forall x, y \geq 4$.

Setting $y = 4$ in the above equality, we get $f(x)(f(4) + \frac{1}{f(4)} - 1) = 0\ \forall x \geq 4$ and so $f(x) = 0$, impossible.

And so no solution.