A Geometry Problem Set

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All answers are integers from 000 to 999 inclusive.

1 Docile

1. Right triangle ABC with right angle C has sidelengths $AC = 3$ and $BC = 4$. Altitude CD is constructed, with D on the hypotenuse of ABC. The length of CD can be expressed as $\frac{p}{q}$, where p and q are relatively prime positive integers. Compute $p + q$.

   Answer: **017**. Observe that $m\angle ACD = 90^\circ - m\angle CAD = m\angle ABC$. It follows that triangles ACD and ABC are similar. Now since $AB = 5$ and $\frac{CD}{CA} = \frac{BC}{BA}$, we have $CD = \frac{3\cdot 4}{5} = \frac{12}{5}$.

2. In scalene triangle ABC, D is the midpoint of BC, E is the midpoint of AC, and F is the midpoint of AB. The area of triangle DEF is 6. Compute the area of triangle ABC.

   Answer: **024**. Because D and E are the midpoints of $\overline{BC}$ and $\overline{AC}$, $\overline{DE}$ is parallel to $\overline{AB}$ and is half as long. It follows that each side of DEF is $1/2$ as long as the corresponding side of ABC. Hence, the area of DEF is $(1/2)^2 = 1/4$ of the area of ABC.

3. A, B, and C are points on a line in that order, with $BC = 6$ and $AB = 4$. Points E and F are chosen on the same side of this line such that $EC = 7, AE = 6, BF = 10,$ and $CF = 7$. Let the intersection of BF and CE be D. The value of the expression $\frac{[ABDE]}{[CDF]}$ can be expressed as $\frac{p}{q}$, where p and q are relatively prime positive integers. Compute $p + q$.

   Answer: **002**. Note that $AC = AB + BC = 10$, so that triangle AEC is congruent to triangle BCF. Now $[ABDE] = [AEC] - [BDC] = [BCF] - [BDC] = [CDF]$, so the desired ratio is $\frac{1}{1}$.

4. Three circles are mutually externally tangent. Two of the circles have radii 3 and 7. If the area of the triangle formed by connecting their centers is 84, then the area of the third circle is $k\pi$ for some integer k. Determine k.

Answer: **196**. Let r denote the radius of the third circle. Then the sides of the triangle are $10, 3+r$, and $7+r$. Using Heron's formula and equating this with the given area, we have $84 = \sqrt{(10+r)(r)(7)(3)}$ from which $r = 14, -24$. Since r is positive, it follows that the area of the third circle is 196π.

5. $ABCDEFG$ is a regular heptagon, and P is a point in its interior such that ABP is equilateral. p and q are relatively prime positive integers such that $m\angle CPE = \left(\frac{p}{q}\right)^\circ$. Compute the value of $p+q$.

   Answer: **667**. Since ABP is equilateral, $BP = BA = BC$, hence $\angle BCP \cong \angle CPB$. Let α denote the degree measure of each of the angles of $ABCDEFG$. Then $m\angle PCB = \alpha - 60^\circ$ from which $m\angle CPB = m\angle BCP = 120^\circ - \frac{\alpha}{2}$ and $m\angle PCD = \frac{3\alpha}{2} - 120^\circ$. By symmetry, P lies on the angle bisector of $\angle DEF$, thus $m\angle DEP = \frac{\alpha}{2}$. Finally, as $m\angle CDE = \alpha$, we have $m\angle EPC = 360^\circ - \frac{\alpha}{2} - \alpha - \left(\frac{3\alpha}{2} - 120^\circ\right) = 480^\circ - 3\alpha$. Computing $\alpha = 180^\circ - \frac{360^\circ}{7} = \frac{900^\circ}{7}$, we find that $m\angle EPC = \frac{660^\circ}{7}$.

6. The length of a diagonal connecting opposite vertices of a rectangular prism is 47. Determine its volume, given that one of its dimensions is 2 and that the other two dimensions differ by $\sqrt{2005}$.

   Answer: **200**. Let the other dimensions be x and $x + \sqrt{2005}$. Then by the distance formula, $47^2 = 2^2 + x^2 + (x+\sqrt{2005})^2 = 2009 + 2x(x+\sqrt{2005})$. It follows that the desired volume is $47^2 - 2009$.

7. $ABCDEF$ is a regular hexagon of area 100. $GHIJKL$ is the hexagon formed by connecting adjacent midpoints of the sides of $ABCDEF$. Compute the area of $GHIJKL$.

   Answer: **075**. Suppose G and H are the midpoints of $\overline{AB}$ and $\overline{BC}$ respectively, and let O be the center of $ABCDEF$. Note that $GHIJKL$ is also regular and shares center O. Observe that GOA is a 30-60-90 right triangle. Since any two regular hexagons are similar, we have $\frac{[ABCDEF]}{[GHIJKL]} = \frac{OA^2}{OG^2} = \frac{4}{3}$. The answer follows.

8. ABC is an equilateral triangle and D is a point on minor arc AB of the circumcircle of ABC such that $BD = 2005$ and $CD = 2006$. Compute AD.

   Answer: **001**. Since $AB = BC = CA$, Ptolemy's theorem

$$AD \cdot BC + AC \cdot BD = AB \cdot CD$$

   reduces to $AD + BD = CD$, which immediately gives AD.

2 Demanding

1. $ABCD$, a rectangle with $AB = 12$ and $BC = 16$, is the base of pyramid $\mathcal{P}$, which has a height of 24. A plane parallel to $ABCD$ is passed through $\mathcal{P}$, dividing $\mathcal{P}$ into a frustum $\mathcal{F}$ and a smaller pyramid $\mathcal{P}'$. Let X denote the center of the circumsphere of

$\mathcal{F}$, and let T denote the apex of $\mathcal{P}$. If the volume of $\mathcal{P}$ is eight times that of $\mathcal{P}'$, then the value of XT can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute the value of $m+n$.

Answer: **177**. $\mathcal{P}$ and $\mathcal{P}'$ are similar; since the volume of the former is 8 times that of the latter, if follows that the plane passes through $\mathcal{P}$ halfway up the pyramid $\mathcal{P}$. Let Z be the apex of $\mathcal{P}$, and A', B', C', and D' the midpoints of $A'Z, B'Z, C'Z$, and $D'Z$ respectively. $A'B'C'D'$, the rectangular intersection of the plane and $\mathcal{P}$, has $A'B' = C'D' = 6$ and $B'C' = D'A' = 8$. Let O and O' denote the centers of $ABCD$ and $A'B'C'D'$ respectively. Since the height of $\mathcal{P}$ is 24, $OO' = 12$. By symmetry, the circumsphere of the frustum $\mathcal{F}$ is centered on OO'. Since for any point X on OO', we have $AX = BX = CX = DX$ and $A'X = B'X = C'X = D'X$, we need only find the point X such that $AX = A'X$. Suppose that $OX = x$ and $XO' = 12 - x$. By the Pythagorean theorem in 3-space, we have

$$\begin{aligned} AX = A'X \iff 6^2 + 8^2 + x^2 &= 3^2 + 4^2 + (12-x)^2 \\ 100 + x^2 &= 25 + 144 - 24x + x^2 \\ x = \frac{69}{24} \end{aligned}$$

Then $XT = 24 - \frac{69}{24} = \frac{507}{24} = \frac{169}{8}$, so the answer is $169 + 8 = 177$.

2. ABC is a scalene triangle. Points D, E, and F are selected on sides BC, CA, and AB respectively. The cevians AD, BE, and CF concur at point P. If $[AFP] = 126, [FBP] = 63$, and $[CEP] = 24$, determine the area of triangle ABC.

Answer: **351**. Since triangles AFP and FBP share an altitude from P, we have $\frac{BF}{FA} = \frac{[FBP]}{[AFP]} = \frac{1}{2}$. Let $[EAP] = k$. By similar reasoning, $\frac{AE}{EC} = \frac{k}{24}$. By Ceva's theorem, $\frac{CD}{DB}\frac{BF}{FA}\frac{AE}{EC} = 1 \Longrightarrow \frac{CD}{DB} = \frac{48}{k}$. Now we note that $\frac{[ADC]}{[ABD]} = \frac{[PDC]}{[PBD]} = \frac{CD}{DB} = \frac{48}{k}$. Hence, $\frac{[ADC]-[PDC]}{[ABD]-[PBD]} = \frac{[APC]}{[APB]} = \frac{48}{k}$. We use the fact that $[APC] = [APE] + [EPC] = k + 24$ and $[ABP] = [AFP] + [FBP] = 126 + 63 = 189$. We have

$$\begin{aligned} \frac{24+k}{189} &= \frac{48}{k} \\ 48 \cdot 189 &= k^2 + 24k \\ k &= \frac{-24 \pm \sqrt{24^2 + 4 \cdot 48 \cdot 189}}{2} = -12 \pm \sqrt{12^2 + 48 \cdot 189} \\ &= -12 \pm 12\sqrt{1 + 63} = -108, 84 \end{aligned}$$

We take $k = 84$ since it represents an area. Now, $\frac{AE}{EC} = \frac{7}{2}$ and $\frac{CD}{DB} = \frac{4}{7}$. By Menelaus' theorem, $\frac{BF}{FA}\frac{AP}{PD}\frac{DC}{DB} = -1$ (Ceva and Menelaus use the convention of directed distances, where $XY = -YX$.) This yields $\frac{AP}{PD} = \frac{11}{2}$ from which $\frac{[ABPC]}{[PBC]} = \frac{11}{2}$. Hence, $[ABC] = \frac{13}{11} \cdot [ABPC] = \frac{13}{11} \cdot (24 + 84 + 126 + 63) = 351$.

ALTERNATE SOLUTION

Assign the weights 1, 2, and ω to A, B, and C. It must be that $[EAP] = 24\omega$, $[DCP] = 2k$, and $[BDP] = \omega k$ for some k. But we have $\frac{2}{w} = \frac{[DCP]}{[BDP]} = \frac{[DCA]}{[BDA]} = \frac{[PCA]}{[BPA]} = \frac{24(\omega+1)}{126+3} = \frac{8(\omega+1)}{63}$. We solve this quadratic for $\omega = \frac{7}{2}, -\frac{9}{2}$, and choose the former since 24ω is an area. But the weight on D is $\omega + 2$ so that $\frac{\omega+2}{1} = \frac{AP}{PD} = \frac{[ABPC]}{[PBC]}$. Substituting, $\frac{11}{2} = \frac{24+24\cdot\frac{7}{2}+126+63}{[PBC]}$ which implies that $[PBC] = 54$. Therefore, $[ABC] = [ABPC] + [PBC] = 297 + 54 = 351$.

3. $ABCD$ is a cyclic quadrilateral that has an inscribed circle. The diagonals of $ABCD$ intersect at P. If $AB = 1$, $CD = 4$, and $BP : DP = 3 : 8$, then the area of the inscribed circle of $ABCD$ can be expressed as $\frac{p\pi}{q}$, where p and q are relatively prime positive integers. Determine $p + q$.

   Answer: **049**. Because $ABCD$ has an incircle, $AD + BC = AB + CD = 5$. Suppose that $AD : BC = 1 : \gamma$. Then $3 : 8 = BP : DP = (AB \cdot BC) : (CD \cdot DA) = \gamma : 4$. We obtain $\gamma = \frac{3}{2}$, which substituted into $AD + BC = 5$ gives $AD = 2$, $BC = 3$. Now, the area of $ABCD$ can be obtained via Brahmagupta's formula: $s = \frac{1+2+3+4}{2} = 5$, $K = \sqrt{(s-a)(s-b)(s-c)(s-d)} = \sqrt{24}$ and $K = rs = 5r$, where r is the inradius of $ABCD$. Thus, $r = \frac{\sqrt{24}}{5}$ from which its area $\frac{24\pi}{25}$ yields the answer $24 + 25 = 49$.

4. ABC is an isosceles triangle with base $\overline{AB}$. D is a point on $\overline{AC}$ and E is the point on the extension of $\overline{BD}$ past D such that $\angle BAE$ is right. If $BD = 15, DE = 2$, and $BC = 16$, then CD can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Determine $m + n$.

   Answer: **225**. Draw in altitude $\overline{CF}$ and denote its intersection with $\overline{BD}$ by P. Since ABC is isosceles, $AF = FB$. Now, since BAE and BFP are similar with a scale factor of 2, we have $BP = \frac{1}{2}BE = \frac{17}{2}$, which also yields $PD = BD - BP = 15 - \frac{17}{2} = \frac{13}{2}$. Now, applying Menelaus to triangle ADB and collinear points C, P, and F, we obtain

$$\begin{aligned} \frac{AC}{CD}\frac{DP}{PB}\frac{BF}{FA} &= \frac{AC}{CD}\frac{DP}{PB} = -1 \\ \Longrightarrow |CD| = AC \cdot \frac{DP}{PB} &= 16 \cdot \frac{\left(\frac{13}{2}\right)}{\left(\frac{17}{2}\right)} = \frac{208}{17} \end{aligned}$$

   where the minus sign was a consequence of directed distances.[1] The answer is therefore $208 + 17 = 225$.

5. $\mathcal{P}$ is a pyramid consisting of a square base and four slanted triangular faces such that all of its edges are equal in length. $\mathcal{C}$ is a cube of edge length 6. Six pyramids similar to $\mathcal{P}$ are constructed by taking points P_i (all outside of $\mathcal{C}$) where $i = 1, 2, \ldots, 6$ and adjoining each to the vertices of the nearest face of $\mathcal{C}$, using each face of $\mathcal{C}$ once. The volume of the octahedron formed by the P_i (taking the convex hull) can be expressed

[1] A system of linear measure in which for any points A and B, $AB = -BA$.

as $m + n\sqrt{p}$ for some positive integers m, n, and p, where p is not divisible by the square of any prime. Determine the value of $m + n + p$.

Answer: **434**. By a Pythagoras argument, the length of the altitude of each pyramid from P_i to the nearest face of $\mathcal{C}$ is $3\sqrt{2}$. Therefore, the distance from opposite vertices of the octohedron is $6 + 2 \cdot 3\sqrt{2} = 6 \cdot \left(1 + \sqrt{2}\right)$. Let P_s and P_t be a pair of opposite vertices. The square formed by the other four vertices of the octahedron has area $\frac{1}{2} \cdot \left(6 \cdot \left(1 + \sqrt{2}\right)\right)^2 = 18\left(3 + 2\sqrt{2}\right)$. Finally, the volume is given by $\frac{1}{3}\left(6(1 + \sqrt{2})\right)\left(18(3 + 2\sqrt{2})\right) = 252 + 180\sqrt{2}$.

6. Line segments $\overline{AB}$ and $\overline{CD}$ intersect at P such $AP = 8, BP = 24, CP = 11$, and $DP = 13$. Line segments $\overline{DA}$ and $\overline{BC}$ are extended past A and C respectively until they intersect at Q. If $\overline{PQ}$ bisects $\angle BQD$, then $\frac{AD}{BC}$ can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Determine $m + n$.

   Answer: **046**. Let E and F be the projections of P onto AD and BC respectively. Note that the angle bisector condition is equivalent to $PE = PF$. It follows that $\frac{AD}{BC} = \frac{[APD]}{[BPC]} = \frac{AP \cdot DP}{BP \cdot CP} = \frac{13}{33}$.

7. Three spheres $\mathcal{S}_1$, $\mathcal{S}_2$, and $\mathcal{S}_3$ are mutually externally tangent and have radii 2004, 3507, and 4676 respectively. Plane $\mathcal{P}$ is tangent $\mathcal{S}_1$, $\mathcal{S}_2$, and $\mathcal{S}_3$ at A, B, and C respectively. The area of triangle ABC can be expressed as $m\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. Determine the remainder obtained when $m + n$ is divided by 1000.

   Answer: **707**. The radii are obviously deliberately chosen, but the reason is not yet apparent. Write r_i for the radius of S_i, and let O_i be the center of S_i. Let be P the projection of O_1 onto $\overline{O_2B}$. Then O_1PO_2 is a right triangle with $O_1O_2 = r_1 + r_2$ and $PO_2 = r_2 - r_2$. It follows that $O_1P = 2\sqrt{r_1r_2} = AB$. Similarly, $BC = 2\sqrt{r_2r_3}$ and $CA = 2\sqrt{r_3r_1}$. Now we check $2004 \cdot 3507 + 2004 \cdot 4676 = 3507 \cdot 4606$, so ABC is a right triangle. It follows that its area is $2 \cdot 2004\sqrt{3507 \cdot 4676} = 9370704\sqrt{3}$, which gives an answer of 707.

8. ABC is a triangle in which $BC = 3, CA = 4, AB = 5$. Two congruent circles ω_1 and ω_2 are mutually externally tangent such that ω_1 is also tangent to $\overline{BC}$ and $\overline{AB}$ while ω_2 is also tangent to $\overline{AC}$ and $\overline{AB}$. The radius of ω_1 can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.

   Answer: **012**. Let O_1 and O_2 be the centers of ω_1 and ω_2, and let the circles be tangent to $\overline{AB}$ at T_1 and T_2 respectively. Write r for the desired radius. $T_1T_2O_2O_1$ is a rectangle, so $T_1T_2 = 2r$. Now observe that $\overline{BO_1}$ bisects $\angle ABC$. Since $\cos(\angle ABC) = 3/5$, it follows that $BT_1/T_1O = 2$ so that $BT_1 = 2r$. Similarly, $AT_2 = 3r$. Therefore, $AB = BT_1 + T_1T_2 + T_2A = 7r$ from which $r = \frac{5}{7}$.

3 Difficult

1. Triangle ABC has sidelengths $AB = 13, BC = 14, CA = 15$. Triangle $A_1B_1C_1$ lies outside triangle ABC and has sides parallel to ABC with a distance of 2 between corresponding sides. Compute the area of $A_1B_1C_1$.

   Answer: **189**. Suppose that A_1, B_1, C_1 are near A, B, C respectively. Let P and Q be the feet of the perpendiculars from A to $\overline{A_1B}$ and $\overline{A_1C}$. Since $AP = AQ = 2$, right triangles APA_1 and AQA_1 are congruent. Therefore, AA_1 is the angle bisector of $\angle B_1A_1C_1$. Since the sides of $A_1B_1C_1$ are parallel to the sides of ABC, we have that AA_1, BB_1, CC_1 concur at the shared incenter I of both triangles.

   Let ω and ω' denote the incircles of ABC and $A_1B_1C_1$ respectively. By Heron's formula, $[ABC] = 84$, hence, $84 = rs = 21r$ so that $r = 4$. It follows that the inradius of $A_1B_1C_1$ is 6. Since the homothety centered at I that sends ω to ω' sends triangle ABC to triangle $A_1B_1C_1$, we have that $\frac{[A_1B_1C_1]}{[ABC]} = \left(\frac{6}{4}\right)^2 = \frac{9}{4}$, from which $[A_1B_1C_1] = 189$.

2. $ABCDEFG$ is a regular heptagon inscribed in a unit circle centered at O. l is the line tangent to the circumcircle of $ABCDEFG$ at A, and P is a point on l such that $\triangle AOP$ is isosceles. Let p denote value of $AP \cdot BP \cdot CP \cdot DP \cdot EP \cdot FP \cdot GP$. Determine the value of p^2.

   Answer: **113**. Overlay the complex number system with $O = 0 + 0i, A = 1 + 0i$, and $P = 1 + i$. The solutions to the equation $z^7 = 1$ are precisely the seven vertices of the heptagon. Letting a, b, c, d, e, f, and g denote the complex numbers for A, B, C, D, E, F, and G respectively, this equation rewrites as $(z-a)(z-b)(z-c)(z-d)(z-e)(z-f)(z-g) = z^7 - 1 = 0$. The magnitude of the factored product represents the product of the distances from the arbitrary point represented by z. Thus, plugging in $1 + i$, we have $AP \cdot BP \cdot CP \cdot DP \cdot EP \cdot FP \cdot GP = |(1+i)^7 - 1| = |8 - 8i - 1| = |7 - 8i| = \sqrt{7^2 + 8^2} = \sqrt{113}$. It follows that the answer is 113.

3. $ABCD$ is a cyclic quadrilateral with $AB = 8, BC = 4, CD = 1$, and $DA = 7$. Let O and P denote the circumcenter and intersection of AC and BD respectively. The value of OP^2 can be expressed as $\frac{m}{n}$, where m and n are relatively prime, positive integers. Determine the remainder obtained when $m + n$ is divided by 1000.

   Answer: **589**. Consider D' on the circumcircle of $ABCD$ such that $CD' = 7$ and $D'A = 1$. Let $m\angle D'AB = \alpha$ and $m\angle BCD' = \pi - \alpha$. Then by the Law of Cosines,

$$\begin{aligned} 1^2 + 8^2 - 2 \cdot 1 \cdot 8 \cos(\alpha) &= BD'^2 = 4^2 + 7^2 - 2 \cdot 4 \cdot 7 \cos(\pi - \alpha) \\ \implies \cos(\alpha) &= 0 \end{aligned}$$

   Hence $D'AB$ is a right triangle and the circumradius of $ABCD$ is $\frac{\sqrt{65}}{2}$. Now, by similar triangles, we have $AP : BP : CP : DP = 56 : 32 : 4 : 7$. Let $AP = 56x$ so that $AC = 60x$ and $BD = 39x$. Ptolemy's theorem applied to $ABCD$ yields $60x \cdot 39x = 1 \cdot 8 + 4 \cdot 7 = 36$ from which $x^2 = \frac{1}{65}$.

Now we apply Stewart's theorem to triangle BOD and cevian OP, obtaining

$$\begin{aligned}
OB^2 \cdot PD + OD^2 \cdot BP &= OP^2 \cdot BD + BP \cdot BD \cdot PD \\
\frac{65}{4}(32x + 7x) &= 39x \cdot OP^2 + 32x \cdot 39x \cdot 7x \\
\frac{65}{4} - 7 \cdot 32\frac{1}{65} &= OP^2 \\
OP^2 &= \frac{3329}{260}
\end{aligned}$$

It follows that the answer is $329 + 260 = 589$.

4. ABC is an acute triangle with perimeter 60. D is a point on $\overline{BC}$. The circumcircles of triangles ABD and ADC intersect $\overline{AC}$ and $\overline{AB}$ at E and F respectively such that $DE = 8$ and $DF = 7$. If $\angle EBC \cong \angle BCF$, then the value of $\frac{AE}{AF}$ can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.

   Answer: **035**. Since $BDEA$ is cyclic, $\angle EBD \cong \angle EAD$. Similarly, $\angle DCF \cong \angle DAF$. Since we are given $\angle BCF \cong \angle EBC$, we have $\angle DAB \cong \angle CAD$. Because $\overline{CD}$ and $\overline{DF}$ are intercepted by congruent angles in the same circle, $DF = CD = 7$. Similarly, $DB = 8$. Now, by the angle bisector theorem, $AC = 7x$ and $AB = 8x$. Since the perimeter of ABC is 60, $15+15x = 60$ and $x = 3$, so that $AC = 21$ and $AB = 24$. Now, by Power of a Point from B, $BF = \frac{BD \cdot BC}{BA} = \frac{8 \cdot 15}{24} = 5$ and $CE = \frac{CD \cdot CB}{CA} = \frac{7 \cdot 15}{21} = 5$. Subtracting these lengths from AB and AC respectively, we find that $AF = 19$ and $AE = 16$. It follows that the answer is $16 + 19 = 35$.

5. ABC is a scalene triangle. The circle with diameter $\overline{AB}$ intersects $\overline{BC}$ at D, and E is the foot of the altitude from C. P is the intersection of $\overline{AD}$ and $\overline{CE}$. Given that $AP = 136$, $BP = 80$, and $CP = 26$, determine the circumradius of ABC.

   Answer: **085**. It is easily seen that $P = H$, the orthocenter of ABC. Recall that $AH = 2R\cos(A)$. Thus, $\cos(A) = \frac{68}{R}, \cos(B) = \frac{40}{R}, \cos(C) = \frac{13}{R}$. Now we have

$$\begin{aligned}
\cos^2(A) + \cos^2(B) + \cos^2(C) + 2\cos(A)\cos(B)\cos(C) &= 1 \\
\iff \left(\frac{68}{R}\right)^2 + \left(\frac{40}{R}\right)^2 + \left(\frac{13}{R}\right)^2 + 2\left(\frac{68}{R}\right)\left(\frac{40}{R}\right)\left(\frac{13}{R}\right) &= 1 \\
\iff R^3 - (68^2 + 40^2 + 13^2)R - 2 \cdot 68 \cdot 40 \cdot 13 &= 0
\end{aligned}$$

   Thus, any integer solution R will need to be a factor of $2 \cdot 68 \cdot 40 \cdot 13 = 2^6 5^1 13^1 17^1$. Now $2R(\cos(A) + \cos(B) + \cos(C)) = AH + BH + CH = 242$. Since $1 < \cos(A) + \cos(B) + \cos(C) \le \frac{3}{2}$, we establish the bounds $\frac{242}{3} \le R < 121$. It is then a simple matter to compute $R = 85$.

6. A is the center of circle ω_1 and B is the center of circle ω_2 such that A lies on ω_2 and B lies on ω_1. Let C be a point of intersection of ω_1 and ω_2. Γ is the circle that is

internally tangent to ω_1, ω_2, and also tangent to $\overline{AB}$. If the length of minor arc BC is 12, then compute the circumference of Γ.

Answer: **27**. Let Γ be tangent to ω_1 and $\overline{AB}$ at T_1 and T_2 respectively, denote its center by O, and write R and r for the radii of ω_1 and Γ respectively. Due to the tangency, A, O, and T_1 are collinear. Thus, $AO = AT_1 - T_1O = R - r$. By symmetry, $AT_2 = R/2$. Now, since AT_2O is a right triangle, $(\frac{R}{2})^2 + r^2 = (R-r)^2 = R^2 - 2Rr + r^2$, from which $r = \frac{3}{8}R$. The circumference of Γ is therefore $3/8$ of the circumference of ω_1, which is 72 since ABC is equilateral, so the answer is 27.

7. A, B, and C are points on circle O such that B is the midpoint of major arc AC. D lies on minor arc AB such that $BD = 65$, and E is the foot of the perpendicular from B to $\overline{CD}$. Given that $BC = 200$ and $BE = 56$, compute AD.

   Answer: **159**. Reflect A over $\overline{BD}$ to A'. Then $m\angle BDA' = m\angle ADB = \pi - m\angle BCA = \pi - m\angle BDC$. Therefore, A' lies on line CD. It follows that $CE = EA' = ED + DA' = ED + DA$. Thus, $AD = CE - ED$. Now Pythagoras gives $AD = \sqrt{200^2 - 56^2} - \sqrt{65^2 - 56^2} = 192 - 33 = 159$.

8. $ABCDE$ is a cyclic pentagon with $AB = BC = CD = 2$ and $DE = EA = 3$. The length of AD can be expressed as $\frac{p}{q}$, where p and q are relatively prime positive integers. Compute $p + q$.

   Answer: **041**. Construct A' on minor arc AE such that $A'E = 2$ and $A'B = 3$. Now $BE = EC = CA' = x$ because each intercepts the same pair of arcs. Ptolemy on $BCEA'$ gives $x = 4$. Now Ptolemy on $BCEA$ gives $AC = 7/2$. And, since $AC = BD$, a third Ptolemy on $ABCD$ gives $AD = \frac{33}{8}$.

4 Draconian

1. $ABCD$ is a rectangular sheet of paper. E and F are points on $\overline{AB}$ and $\overline{CD}$ respectively such that $BE < CF$. If $BCFE$ is folded over $\overline{EF}$, C maps to point C' on $\overline{AD}$ and B maps to B' such that $\angle AB'C' \cong \angle B'EA$. If $AB' = 5$ and $BE = 23$, then the area of $ABCD$ can be expressed as $a + b\sqrt{c}$ square units, where a, b, and c are integers and c is not divisible by the square of any prime. Compute $a + b + c$.

   Answer: **338**. By the reflection, we have $B'E = BE = 23$. Because $ABCD$ is a rectangle, we have $m\angle C'AE = m\angle C'B'E = \frac{\pi}{2} \implies C'AB'E$ is cyclic with diameter $C'E \implies \angle B'C'A \cong \angle B'EA \cong \angle AB'C' \implies \triangle AB'C'$ is isosceles with $AB' = AC' = 5$. It would suffice to determine $C'E$ as this would eventually yield both sides of $ABCD$.

   Let ω denote the circumcircle of $AB'EC'$. Consider the point P on the minor arc $B'E$ of ω such that $AP = 23$ and $PE = 5$. $APEC'$ is an isosceles trapezoid with $m\angle C'AE = m\angle C'PE = \frac{\pi}{2}$. Let $C'E = x$. Then by Pythagoras, $C'B' = AE = \sqrt{x^2 - 25}$, but by Ptolemy's Theorem applied to this trapezoid,

$$23x + 25 = x^2 - 25$$

from which we find $x = 25$ or -2. Taking $C'E = x = 25$, we obtain $AE = \sqrt{625 - 25} = 10\sqrt{6}$ and $C'B' = \sqrt{25^2 - 23^2} = 4\sqrt{6}$.

Now we have $AB = AE + EB = 10\sqrt{6} + 23$ and $C'B' = BC = 4\sqrt{6}$ so that the area of $ABCD$ is $240 + 92\sqrt{6}$, which yields an answer of $240 + 92 + 6 = 338$.

2. Triangle ABC has an inradius of 5 and a circumradius of 16. If $2\cos B = \cos A + \cos C$, then the area of triangle ABC can be expressed as $\frac{a\sqrt{b}}{c}$, where a, b, and c are positive integers such that a and c are relatively prime and b is not divisible by the square of any prime. Compute $a + b + c$.

   Answer: **141**. It follows from $2\cos B = \cos A + \cos C$ that $\cos A, \cos B, \cos C$ is an arithmetic progression. It also follows that

$$3\cos B = \cos A + \cos B + \cos C = 1 + \frac{r}{R} = \frac{21}{16}$$

   so we may set $\cos A = \frac{7}{16} + k, \cos B = \frac{7}{16}, \cos C = \frac{7}{16} - k$. We substitute these into another famous trig identity,

$$\begin{aligned}
\cos^2 A + \cos^2 B + \cos^2 C + 2\cos A \cos B \cos C &= 1 \\
3 \cdot \left(\frac{7}{16}\right)^2 + 2k^2 + 2 \cdot \frac{7}{16}\left(\left(\frac{7}{16}\right)^2 - k^2\right) &= 1 \\
\frac{18}{16} \cdot 16^3 \cdot k^2 + 7^2(48 + 14) &= 16^3 \\
2 \cdot 3^2 \cdot 16^2 \cdot k^2 = 1058 &= 2 \cdot 23^2 \\
k &= \pm\frac{23}{48}
\end{aligned}$$

   So we have $\cos A = \frac{11}{12}, \cos B = \frac{7}{16}$, and $\cos C = \frac{-1}{24}$, which imply $\sin A = \frac{\sqrt{23}}{12}, \sin B = \frac{3}{16}\sqrt{23}$, and $\sin C = \frac{5}{24}\sqrt{23}$ respectively. Finally,

$$[ABC] = 2R^2 \sin A \sin B \sin C = 2 \cdot 16^2 \cdot \left(\frac{\sqrt{23}}{12}\right)\left(\frac{3}{16}\sqrt{23}\right)\left(\frac{5}{24}\sqrt{23}\right) = \frac{115\sqrt{23}}{3}$$

   which gives an answer of $115 + 23 + 3 = 141$.

3. $ABCDE$ is a cyclic pentagon with $BC = CD = DE$. The diagonals AC and BE intersect at M. N is the foot of the altitude from M to AB. We have $MA = 25, MD = 113$, and $MN = 15$. The area of triangle ABE can be expressed as $\frac{m}{n}$ where m and n are relatively prime positive integers. Determine the remainder obtained when $m + n$ is divided by 1000.

   Answer: **727**. Pythagoras gives $AN = 20$. We draw BD and AD, and construct the altitute MP to AD, with P on AD, and altitude MM' to AE, with M' on AE. Because $BC = CD = DE$, angles BAC, CAD, and DAE are congruent. Because P is on AD, triangles MNA and MPA are congruent by AAS, so $MP = 15$ and $PA = 20$, from which Pythagoras gives $PD = 112$, implying $AD = 132$.

Let $\alpha = m\angle BAC$, so $m\angle MAE = 2\alpha$, and $m\angle NAE = 3\alpha$. Because we have $\sin\alpha = \frac{3}{5}$ and $\cos\alpha = \frac{4}{5}$, we compute $\sin(2\alpha) = \frac{24}{25}$, and $\sin(3\alpha) = \frac{117}{125}$. We find that $MM' = 24$ using $\sin(2\alpha) = \frac{24}{25}$. By a simple Law of Sines argument $DE : EB : BD = 25 : 39 : 40$.

Let $[ABE] =$ the area of ABE. We have $[ABE] = 1/2(15 \cdot AB + 24 \cdot AE)$.

Ptolemy on $ABDE$ yields $AB \cdot DE + AE \cdot BD = AD \cdot BE$. Using the abundance of facts that we have ascertained previously, this gives:

$$\begin{aligned}
AB \cdot 25x + AE \cdot 40x &= 132 \cdot 39x \\
25AB + 40AE &= 39 \cdot 132 \\
15AB + 24AE &= \frac{39 \cdot 132 \cdot 3}{5}
\end{aligned}$$

Finally, $[ABE] = \frac{1}{2} \cdot (15AB + 24AE) = \frac{1}{2}\frac{39 \cdot 132 \cdot 3}{5} = \frac{7722}{5}$. Therefore, the answer is $722 + 5 = 727$.

4. In triangle ABC, we have $BC = 13, CA = 37$, and $AB = 40$. Points D, E, and F are selected on BC, CA, and AB respectively such that AD, BE, and CF concur at the circumcenter of ABC. The value of

$$\frac{1}{AD} + \frac{1}{BE} + \frac{1}{CF}$$

can be expressed as $\frac{m}{n}$ where m and n are relatively prime positive integers. Determine $m + n$.

Answer: **529**. Drop altitude AA'. We have $m\angle AA'B = \frac{\pi}{2} - B$, but AOB is an isosceles triangle with $m\angle AOB = 2C \iff m\angle BAO = \frac{\pi}{2} - C$. Therefore, $\cos DAA' = \cos(C - B)$. Therefore we have $AD\cos(C-B) = AA' = AC\sin(C) = 2R\sin(B)\sin(C)$ so that $\frac{2R}{AD} = \frac{\cos(C-B)}{\sin(B)\sin(C)}$. Now,

$\frac{2R}{AD} + \frac{2R}{BE} + \frac{2R}{CF} = \frac{\cos(C-B)}{\sin(B)\sin(C)} + \frac{\cos(A-C)}{\sin(C)\sin(A)} + \frac{\cos(B-A)}{\sin(A)\sin(B)}$
$\iff 2R\sin(A)\sin(B)\sin(C)\left(\frac{1}{AD} + \frac{1}{BE} + \frac{1}{CF}\right)$
$= \sin(A)\cos(B - C) + \sin(B)\cos(C - A) + \sin(C)\cos(A - B)$
$= 3\sin(A)\sin(B)\sin(C) + \sin(A)\cos(B)\cos(C) + \sin(B)\cos(A)\cos(C) +$
$\sin(C)\cos(A)\cos(B)$
$= 3\sin(A)\sin(B)\sin(C) + \sin(A + B)\cos(C) + \sin(C)\cos(A)\cos(B)$
$= 3\sin(A)\sin(B)\sin(C) + \sin(C)\left(\cos(C) + \cos(A)\cos(B)\right)$
$= 3\sin(A)\sin(B)\sin(C) + \sin(C)\left(-\cos(A + B) + \cos(A)\cos(B)\right)$
$= 4\sin(A)\sin(B)\sin(C) \implies \frac{1}{AD} + \frac{1}{BE} + \frac{1}{CF} = \frac{2}{R}$

Heron's formula yields $[ABC] = \sqrt{45 \cdot 5 \cdot 8 \cdot 32} = 240$. We substitute this into $[ABC] = \frac{abc}{4R} \iff R = \frac{abc}{4 \cdot [ABC]} = \frac{13 \cdot 37 \cdot 40}{4 \cdot 240} = \frac{13 \cdot 37}{24}$. From this we find that

$$\frac{1}{AD} + \frac{1}{BE} + \frac{1}{CF} = \frac{2}{R} = \frac{48}{481}$$

It follows that the answer is $48 + 481 = 529$.

5. Circles ω_1 and ω_2 are centered on opposite sides of line l, and are both tangent to l at P. ω_3 passes through P, intersecting l again at Q. Let A and B be the intersections of ω_1 and ω_3, and ω_2 and ω_3 respectively. AP and BP are extended past P and intersect ω_2 and ω_1 at C and D respectively. If $AD = 3, AP = 6, DP = 4$, and $PQ = 32$, then the area of triangle PBC can be expressed as $\frac{p\sqrt{q}}{r}$, where p, q, and r are positive integers such that p and r are coprime and q is not divisible by the square of any prime. Determine $p + q + r$.

   Answer: **468**. We invert about P with radius 1, mapping the circles ω_1 and ω_2 to lines ω_1' and ω_2', each parallel to l, and ω_3 to a line ω_3' that intersects ω_1' and ω_2' at A' and B' respectively. Q' is the intersection of l and ω_3', and C' and D' are the intersections of the extensions of $A'P$ and $B'P$ past P to ω_2' and ω_1' respectively.

   We have $PQ' = \frac{1}{32}, PA' = \frac{1}{6}$, and $PD' = \frac{1}{4}$. The inversive distance formula gives $A'D' = \frac{R^2 \cdot AD}{AP \cdot DP} = \frac{1}{8}$. The crossed ladders theorem asserts

$$\frac{1}{A'D'} + \frac{1}{B'C'} = \frac{1}{PQ'}$$

   from which $B'C' = \frac{1}{24}$. However, it is clear in the inverted figure that triangles $C'B'P$ and $A'D'P$ are similar. Therefore, $PC' = \frac{1}{18}$ and $PB' = \frac{1}{12}$.

   But inversion is its own inverse transformation. Hence, $PC = 18$ and $PB = 12$. The inversive distance formula gives $BC = \frac{R^2 \cdot B'C'}{PB' \cdot PC'} = \frac{18 \cdot 12}{24} = 9$. Finally, the area of PBC may be found via Heron's formula: $K = \sqrt{\frac{39}{2}\frac{21}{2}\frac{15}{2}\frac{3}{2}} = \frac{9\sqrt{455}}{4}$. The answer is therefore $455 + 9 + 4 = 468$.

6. In acute triangle ABC, $BC = 10$, $CA = 12$, and $AB = 14$. ω_1, ω_2, and ω_3 are circles with diameters $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$ respectively. Let $\mathcal{B}$ denote the boundary of the region interior to the three ω_i. Ω is the circle internally tangent to the three arcs of $\mathcal{B}$. The radius of Ω can be expressed as $\frac{m - p\sqrt{q}}{n}$, where m, p, and n are positive integers with no common prime divisor and q is a positive integer not divisible by the square of any prime. Compute $m + n + p + q$.

   Answer: **392**. Let M_A, M_B, and M_C denote the midpoints of the sides opposite A, B, and C respectively, and write P for the center of Ω. Finally, let the tangents of ω_1, ω_2, and ω_3 with Ω be denoted by T_1, T_2, and T_3.

   Note that $M_AM_B = 7$, $M_BM_C = 5$, and $M_CM_A = 6$. Take points U_1, U_2, and U_3 on the extensions of T_1M_A, T_2M_B, and T_3M_C past M_A, M_B, and M_C respectively such that $T_iU_i = 9$ for $i = 1, 2, 3$. Because T_1 lies on ω_1, $T_1M_A = 5$ so that $M_AU_1 = 4$. Analogously, $M_BU_2 = 3$ and $M_CU_3 = 2$.

Now consider circles ω_1', ω_2' and ω_3' centered at M_A, M_B, and M_C and of radii 4, 3, and 2 respectively. ω_1', ω_2', and ω_3' are mutually externally tangent and contain points U_1, U_2, and U_3 respectively. But the circumcenter of $U_1U_2U_3$ is P; ergo, the circumcircle Ω' of triangle $U_1U_2U_3$ is tangent to ω_1', ω_2', and ω_3'. Moreover, $R_\Omega' = 9 - R_\Omega$. Applying the explicit form of the Descartes Circle Theorem for the outer circle, we find

$$\begin{aligned} R_\Omega' &= \left| \frac{2 \cdot 3 \cdot 4}{2 \cdot 3 + 3 \cdot 4 + 4 \cdot 2 - 2\sqrt{2 \cdot 3 \cdot 4 \cdot (2+3+4)}} \right| \\ &= \frac{12 \cdot (13 + 6\sqrt{6})}{216 - 169} = \frac{156 + 72\sqrt{6}}{47} \end{aligned}$$

from which $R_\Omega = 9 - R_\Omega' = \frac{267 - 72\sqrt{6}}{47}$. The answer is therefore $267 + 72 + 6 + 47 = 392$.

7. Let $O = (0, 0)$ and $A = (14, 0)$ denote the origin and a point on the positive x-axis respectively. $B = (x, y)$ is a point not on the line $y = 0$. These three points determine lines l_1, l_2, and l_3. Let $P_1, \ldots, P_n$ denote all of the points that are equidistant from l_i for $i = 1, 2, 3$. Let Q_j denote the distance from P_j to the l_i for $j = 1, \ldots, n$. If

$$\begin{aligned} Q_1 + \cdots + Q_n &= \frac{103}{3} \\ \frac{1}{Q_1} + \cdots + \frac{1}{Q_n} &= \frac{2}{3} \end{aligned}$$

then the maximum possible value of $x + y$ can be expressed as $\frac{u}{v}$, where u and v are relatively prime positive integers. Determine $u + v$.

Answer: **037**. A point that is equidistant from two lines lies on one of the two lines that bisect angles formed at the intersection of the two lines. Considering this fact, it is clear that $n = 4$, where P_1 is the incenter of ABO (which we will denote I) and P_2, P_3, and P_4 are the three excenters. Let r denote the inradius of ABO and r_1, r_2, and r_3 the three exradii. We have

$$\begin{aligned} r + r_1 + r_2 + r_3 &= 4R + 2r = \frac{103}{3} \\ \frac{1}{r} + \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} &= \frac{2}{r} = \frac{2}{3} \end{aligned}$$

This pair of equations is easily solved for $r = 3$ and $R = \frac{85}{12}$, where R is the circumradius of ABO. By the Extended Law of Sines, $\sin\angle B = \frac{AO}{2R} = \frac{14}{2 \cdot \frac{85}{12}} = \frac{84}{85}$. Pythagoras yields $\cos\angle B = \pm\frac{13}{85}$. Then $\cot .5\angle B = \frac{6}{7}$ or $\frac{7}{6}$.

Let the incircle of ABO be tangent to $\overline{BO}$, $\overline{OA}$, and $\overline{AB}$ at P, Q, and R respectively. Let $AQ = X$ and $QO = 14 - X$. Since $r = IQ = 3$, we have $\cot .5\angle A = \frac{X}{3}$ and

$\cot .5\angle O = \frac{14-X}{3}$. Since $\cot .5\angle A \cot .5\angle B \cot .5\angle O = \cot .5\angle A + \cot .5\angle B + \cot .5\angle O$, we have

$$X \cdot (14 - X) = \frac{42 + 9\cot .5\angle B}{\cot .5\angle B} \leq 49$$

by substitution and AM-GM. $\cot .5\angle B = \frac{6}{7}$ leads to $58 \leq 49$, impossible, so $\cot .5\angle B = \frac{7}{6}$, which leads to $X = 5$ or 9. Thus, there are four possible B, each obtained by reflecting B over $\overline{AO}$ and the perpendicular bisector of $\overline{AO}$. Since we are maximizing the sum $x + y$, we choose $X = 5$ and assume $y > 0$. Now, $BR = \frac{7}{2}$ and $RA = 5$. Since $\cos\angle A = 2\cos .5\angle A^2 - 1 = \frac{25}{17} - 1 = \frac{8}{17}$, it must be that $x = 14 - \frac{17}{2} \cdot \frac{8}{17} = 10$ and $y = \frac{17}{2} \cdot \frac{15}{17} = \frac{15}{2}$, which gives $x + y = \frac{35}{2}$ and an answer of $35 + 2 = 37$.

8. Let G, H, and I denote the centroid, orthocenter, and incenter of triangle ABC. If $HI = 2, IG = 3$, and $GH = 4$, then the value of

$$\cos(A) + \cos(B) + \cos(C)$$

can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.

Answer: **197**. Examining the Euler line, O lies on line HG such that $GO = 2$. Now Stewart's theorem on G, H, I, O yields $4OI^2 + 8 = 54 + 48$ from which $OI^2 = \frac{47}{2}$. Another famous result of Euler is that $OI^2 = R(R-2r)$. Finally, a corollary of Feuerbach's theorem gives $-\frac{2}{3}r(R - 2r) = \vec{IG} \cdot \vec{IH} = IG \cdot IH \cos(HIG) = \frac{HG^2 - IG^2 - IH^2}{-2} = -\frac{3}{2}$. It follows that $\cos(A) + \cos(B) + \cos(C) = 1 + \frac{r}{R} = 1 + \frac{9}{94} = \frac{103}{94}$.