Bijections

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April 7, 2010

1 Bijections

A bijection involves demonstrating a one-to-one correspondence between two sets, in turn showing that the cardinalities of the two sets are the same. Since this is the advanced lecture, I'll assume you know how to count basic stuff and therefore jump right into the problems. Combinatorics, at least at the high school level, has very few "core theorems" and thus requires a higher degree of cleverness than most other areas. This means that the best way to get better is to do problems, and a lot of them.
Warm-Up: How many integer solutions exist to the system $a_1 + a_2 + a_3 + \cdots + a_k = n$ if $a_i > m$ for all i?

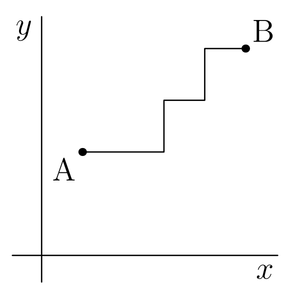
1.1 Grid-Walking

Given p and q coprime positive integers, determine the value of

$$\left\lfloor \frac{p}{q} \right\rfloor + \left\lfloor \frac{2p}{q} \right\rfloor + \left\lfloor \frac{3p}{q} \right\rfloor + \cdots + \left\lfloor \frac{(q-1)p}{q} \right\rfloor$$

Work on this problem in groups for five to ten minutes; I'll invite people who've solved it to present.

In this section we'll take a geometric outlook on some combinatorial identities you may already know (if not in the geometric form). Let's assume that in the plane with Cartesian coordinates we are given straight lines with the equations $x = r, y = s$, where $r, s \in \mathbb{Z}$. We'll now consider "walks" along the given lines in the given rectangular grid; specifically, shortest paths between A and B. This means that walks we consider will consist of a finite number of steps "to the right" and "up," like below.



How many ways are there to get from A to B if we take a shortest path?

1.2 Words

A frequently useful method for finding bijections is to encode the sequence into a "word" of some sort. For example, in the problem above, suppose we must take m steps up and n steps to the right to get from A to B. Now let's encode each walk with m letters U and n letters R. Each shortest walk corresponds to a unique such word, so obviously we have the number of words to be the number of ways of choosing n items from $n+m$ total: $\binom{m+n}{m} = \binom{m+n}{n}$ ways.

2 Problems

1. Show that $\displaystyle\binom{n+1}{k+1} = \frac{n+1}{k+1}\binom{n}{k}$ a) using a bijection, b) some other way.

2. Show that

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

   a) using a bijection, b) some other way.

3. A line of $2n$ people is waiting outside a movie theater box office. Half of these people have a five-dollar bill with them and the others each have a ten-dollar bill. A ticket costs five dollars and at the beginning of the ticket sales there is no money in the cash register. In how many ways can the moviegoers line up at the box office such that they can proceed without delays; that is, nobody has to wait until the cashier has enough change?

4. Suppose that there are m people with a five-dollar bill and n people with a ten-dollar bill. Now in how many ways can the moviegoers line up?

5. Now suppose that the cashier came prepared with q five-dollar bills for change. Now in how many ways can the moviegoers line up?

6. You have 6 square envelopes of different sizes. You may place smaller envelopes inside larger envelopes, and you may place an unlimited number of envelopes inside any one envelope (as long as the envelopes are smaller than their container). How many ways can you arrange your 6 envelopes so that exactly 3 envelopes do NOT contain any other envelopes? (AKessler)

7. Show via a bijection that

$$\binom{m+n}{m} = \sum_{i=0}^{n} \binom{k+i}{k}\binom{m+n-k-i-1}{m-k-1}$$

8. Prove that the number of paths of length n that can be made using only left or right moves of length 1, starting on the left side of a line segment of length n, is $\displaystyle\binom{n}{\lfloor \frac{n}{2} \rfloor}$.