Bijective Proofs

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Bijections

Many combinatorial problems can be solved by finding the right bijection. A map between two sets A and B is a *bijection* if it is both *injective* and *surjective*, that is, it is one-to-one and maps onto all of B. Equivalently, a bijection is a map that has a well-defined inverse.

Most bijective (a.k.a. *counting in two ways*) proofs use the following principle:

> *If there is a bijection between finite sets A and B, then A and B have the same number of elements.*

So, if we wish to find $|A|$ (the number of elements in A) and there is a bijection from A to a set B whose elements are easy to count, then we know how to count the elements of A. We can also prove two integers are equal by showing they both enumerate the same set, or that they enumerate bijective sets.

Example. Prove that

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

To prove this, we can easily use a straightforward induction, but we can we can also find a bijection between sets that each side counts. Consider a class with $2n$ students, n of whom are boys and n of whom are girls.

There is a natural bijection between

- pairs (G, B) of subsets G of the girls and B of the boys with $|G| + |B| = n$, and
- subsets of the set of all students of size n,

defined by $(G, B) \mapsto G \cup B$. The number of possible pairs (G, B) is

$$\sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} = \sum_{k=0}^{n} \binom{n}{k}^2,$$

and the number of subsets of the class of size n is $\binom{2n}{n}$. So, these quantities must be equal.

Problems

1. *Find the bijection!* For each of the following pairs, give an explicit bijection that maps one set of objects to the other.

   (a) Tilings of a $2 \times n$ grid with dominoes $\leftrightarrow$ Sequences of $n-1$ white or black dots such that no two black dots are adjacent

   (b) Increasing binary trees with nodes labeled $1, 2, \ldots, n \leftrightarrow$ Permutations of $1, 2, \ldots, n$.

   (c) Partitions[1] of n into distinct parts $\leftrightarrow$ Partitions of n into odd parts

   (d) Partitions of n into distinct odd parts $\leftrightarrow$ Partitions of n whose Young Diagram[2] is symmetric about the diagonal

2. Give a bijective proof of each of the following identities. All unspecified variables are assumed to be positive integers.

   - $\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$
   - $\sum_{k=0}^{r} \binom{n}{k}\binom{m}{r-k} = \binom{m+n}{r}$
   - $\sum_{k=0}^{n} \binom{n}{k} s^k t^{n-k} = (s+t)^n$

3. **Hall's Marriage Lemma**. Suppose there is a finite set of girls G and a finite set of boys B. Each girl likes a finite number of boys enough to marry them, and any boy would be happy to marry any girl that likes him. Suppose further that for any proper subset A of the girls, the total number of boys liked by the girls in A is at least as large as $|A|$. Prove that there is a *matching* for the girls - an injective map $G \to B$ such that each girl is matched with a boy that she likes.

4. **Sperner's Lemma**. Determine the maximum number of subsets of $\{1, 2, \ldots, n\}$ one can choose such that no subset is contained in any other.

5. How many $m \times n$ matrices of 0's, 1's, and 2's have the property that the sum of the entries of every row and column is congruent to $1 \pmod 3$?

6. Prove Fermat's Little Theorem using a combinatorial argument as follows. We wish to show that if p is prime and a is a positive integer, then $a^p - a$ is divisible by p. To do so, it suffices to find a set S with $a^p - a$ elements and sort the elements of S into disjoint subsets having p elements each.

7. (Putnam 2002.) A nonempty subset $S \subseteq \{1, 2, \ldots, n\}$ is *decent* if the average of its elements is an integer. Prove that the number of decent subsets has the same parity as n.

8. (Richard Stanley.) Using an injection, prove that for $0 \leq k < \lfloor n/2 \rfloor$, we have

$$\binom{n}{k} \leq \binom{n}{k+1}.$$

[1] A *partition* of a positive integer is a way of writing it as a sum of other integers, called the *parts* of the partition, where we list the parts in nonincreasing order.

[2] The Young Diagram of a partition is a partial grid of squares, aligned at the left, where each row has a number of squares corresponding to the size of the parts in nonincreasing order.

9. (USAMO 1996.) An n-term sequence in which every term is either 0 or 1 is called a "binary sequence" of length n. Let a_n be the number of binary sequences of length n containing no three consecutive terms equal to 0, 1, 0 in that order. Let b_n be the number of binary sequences of length n containing no four consecutive terms equal to 0, 0, 1, 1 or 1, 1, 0, 0 in that order. Prove that $b_{n+1} = 2a_n$ for all positive integers n.

10. (China 1996.) Let n be a positive integer. Find the number of polynomials $P(x)$ with coefficients in $\{0, 1, 2, 3\}$ such that $P(2) = n$.

11. Find the number of strings of n letters, each equal to A, B, or C, such that the same letter never occurs three times in succession.

12. (Richard Stanley.) Show that the number of incongruent triangles with integer sides and perimeter n is equal to the number of partitions of $n - 3$ into parts equal to 2, 3, or 4. For example, there are three such triangles with perimeter 9, the side lengths being $(3, 3, 3)$, $(2, 3, 4)$, $(1, 4, 4)$. The partitions of 6 using only the summands 2, 3, and 4 are $2 + 2 + 2 = 3 + 3 = 4 + 2$.

13. (China 1994.) Let n be a positive integer. Prove that

$$\sum_{k=0}^{n} 2^k \binom{n}{k} \binom{n-k}{\lfloor (n-k)/2 \rfloor} = \binom{2n+1}{n}.$$

14. (Hard.) Prove *Cayley's Formula*, that the number of trees (connected graphs having no cycles) whose vertices are labeled $1, 2, \ldots, n$ is n^{n-2}.

15. **The Catalan numbers:** The Catalan numbers $C_0, C_1, C_2, \ldots$ can be defined by the recurrence relation

$$C_{n+1} = C_n C_0 + C_{n-1} C_1 + C_{n-2} C_2 + \cdots + C_0 C_n$$

along with the initial value $C_0 = 1$. The nth Catalan number C_n can also be defined as the number of lattice paths from $(0, 0)$ to (n, n), formed by moving one unit right or one unit up at each step, that lie below or on the diagonal $x = y$.

   (a) Show that the lattice paths described above give a valid combinatorial interpretation[3] for C_n.

   (b) (Hard.) Find a bijective proof that the nth Catalan number C_n is equal to

$$\frac{1}{n+1}\binom{2n}{n}.$$

16. **Bonus: Create your own combinatorial identity!** Think of an interesting set and count it in two ways to come up with an equality like the ones in Problem 2.

If you come up with an identity and wish to be considered in the Create Your Own Combinatorial Identity Contest, submit it to Maria Monks by this Wednesday, June 22 at 11:00PM. The winner's identity will be announced by the end of the week!

[3]See http://math.mit.edu/~rstan/ec/ for a list of 193 different combinatorial interpretations of the Catalan numbers!