NUMBER SYSTEMS

1. The Real Numbers

Readers are assumed to be familiar with the **real numbers**: 2, 47, –3, $\frac{1}{3}$, $\sqrt{2}$, $\sqrt[3]{5}$, π, e, … Just like there are many ways to classify people (according to gender, age, personality, nationality, etc.), there are many ways to classify real numbers. In this set of notes we will study various subsets of the real numbers $\mathbb{R}$.

One of the most important subsets of $\mathbb{R}$ is the set of **natural numbers** {1, 2, 3, …}, which is usually denoted by $\mathbb{N}$. The study of the natural numbers, or more generally the **integers** $\mathbb{Z}$, forms a whole branch of mathematics known as **number theory**.

In studying subsets of $\mathbb{R}$ we are often interested in 'how large' the set is. Intuitively we feel that $\mathbb{N}$ is 'small' relative to $\mathbb{R}$. Although both sets are *infinite*, we see that there are 'many more' real numbers than natural numbers. We shall use the 'size' of $\mathbb{N}$ as a standard to compare the relative sizes of infinite subsets of $\mathbb{R}$. In the light of this, we make the following definition.

Definition 1.1. (Countable Set)

An infinite set S is said to be **countable** if there exists a bijective function $f : S \to \mathbb{N}$. Otherwise it is said to be **uncountable**.

In the light of the above definition, a countable set has the 'same size' as $\mathbb{N}$. (Some people prefer to define also all finite sets to be countable, and call infinite countable sets **countably infinite**.)

Example 1.1.

Is $\mathbb{Z}$ countable?

Solution.

The answer is yes. Indeed, We can define a bijective mapping from $\mathbb{Z}$ to $\mathbb{N}$ as follows:

$$0 \quad 1 \quad -1 \quad 2 \quad -2 \quad 3 \quad -3 \quad \cdots$$
$$\downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \downarrow \quad \cdots$$
$$1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad \cdots$$

The term 'countable' is so called because if a set S is countable, then there is a bijective mapping from S to $\mathbb{N}$. Consequently, just like we can 'enumerate' the elements of $\mathbb{N}$, we can enumerate the elements of S. This was illustrated in the previous example.

Although there are 'more' integers than natural numbers (in the sense that all natural numbers are integers but not all integers are natural numbers), they are still of the 'same size' (in the sense that both $\mathbb{N}$ and $\mathbb{Z}$ are countable). However, it can be shown $\mathbb{R}$ is uncountable. Furthermore, it can be shown that every infinite set has a countable subset. Hence uncountable sets are 'larger' than countable sets. In studying subsets of $\mathbb{R}$, we are often interested to know whether they are countable.

The set of real numbers is more than just a set. We have the familiar **addition** and **multiplication** on $\mathbb{R}$. These two operations are **closed**. That is, whenever m and n are real numbers, $m+n$ and $m \cdot n$ are also real numbers. This is also one of the properties we are interested in when studying subsets of $\mathbb{R}$.

2. The Rational Numbers

We said that $\mathbb{R}$ is closed under addition and multiplication. Moreover, every element $x \in \mathbb{R}$ has an **additive inverse** $-x$, and every *non-zero* real number has a **multiplicative inverse** x^{-1}. Thus $\mathbb{R}$ is also closed under subtraction and division (division by zero is, by convention, not allowed). These, together with many nice properties (existence of additive identity 0 and multiplicative identity 1, commutativity and associativity of addition and multiplication, the distributive laws of multiplication over addition, etc), constitute an algebraic structure which we shall call a **field**.

$\mathbb{N}$ does not have an additive identity. More precisely, there does not exist an element '0' in $\mathbb{N}$ for which $n+0=0+n=n$ for all $n \in \mathbb{N}$. Even if we include the element 0 and consider the set $\mathbb{N} \cup \{0\}$, elements in the set have no additive inverse. So, although the set is closed under addition, it is not closed under subtraction. So we may instead consider the set $\mathbb{Z}$ of integers. Then $\mathbb{Z}$ is closed under addition, subtraction and multiplication. Yet multiplicative inverses do not exist (for

instance you cannot find an integer k for which $2k = 1$, so the element '2' has no multiplicative inverse in $\mathbb{Z}$). These together with the many nice properties mentioned before make $\mathbb{Z}$ into an algebraic structure which we shall call a **ring**.

To make $\mathbb{Z}$ into a field, we need to include the multiplicative inverses of its elements. In order that division be closed, we need to include all quotients of integers. This constitute the **rational numbers**, and the set of all rational numbers is usually denoted by $\mathbb{Q}$.

Definition 2.1. (Rational Number)

A real number x is said to be **rational** if it can be written in the form

$$x = \frac{m}{n}$$

where m, n are integers with $n \neq 0$. Otherwise it is said to be **irrational**.

In the light of the above definition, it is easy to see that $\mathbb{Q}$ is closed under addition, subtraction, multiplication and division. The details are left as an exercise.

By definitions, all 'fractions' are rational numbers, and all integers are rational numbers. But if a number is not written in such form, how can we determine whether it is rational?

Clearly, all **terminating decimals** are rational, for we can write a terminating decimal as a fraction with denominator being a power of 10. For instance,

$$1.234 = \frac{1234}{1000}.$$

How about non-terminating decimals? Some non-terminating decimals are **recurring**, where the digits repeat at certain periods. For instance,

$$0.3333\ldots = 0.\dot{3}.$$

We are familiar with this number and we know that it is equal to $\frac{1}{3}$. Indeed, when a 'fraction' is written as a decimal, it is either terminating or recurring. We leave the details as an exercise.

Conversely, is every recurring decimal a rational number? The answer is affirmative. Indeed, we have the following theorem which helps us easily convert a recurring decimal into a fraction.

Theorem 2.1.

Every recurring decimal is rational. Indeed, if

$$x = \overline{0.a_1 a_2 \cdots a_k \dot{b_1} b_2 \cdots \dot{b_h}}$$

(i.e. $\overline{a_1 a_2 \cdots a_k}$ is the *non-recurring part* and $\overline{b_1 b_2 \cdots b_h}$ is the *recurring part*), then we have

$$x = \frac{\overline{a_1 a_2 \cdots a_k b_1 b_2 \cdots b_h} - \overline{a_1 a_2 \cdots a_k}}{\underbrace{99 \cdots 99}_{h \text{ digits}} \underbrace{00 \cdots 00}_{k \text{ digits}}}$$

Proof. Let $y = \overline{0.\dot{b_1} b_2 \cdots \dot{b_h}} = \overline{0.b_1 b_2 \cdots b_h b_1 b_2 \cdots b_h b_1 b_2 \cdots b_h \cdots}$. Then

$$10^h y = \overline{b_1 b_2 \cdots b_h . b_1 b_2 \cdots b_h b_1 b_2 \cdots b_h b_1 b_2 \cdots b_h \cdots} .$$

Upon subtraction, we have

$$(10^h - 1) y = \overline{b_1 b_2 \cdots b_h}$$

so that

$$y = \frac{\overline{b_1 b_2 \cdots b_h}}{10^h - 1}$$

Consequently,

$$\begin{aligned}
x &= \overline{0.a_1 a_2 \cdots a_k \dot{b_1} b_2 \cdots \dot{b_h}} \\
&= \overline{0.a_1 a_2 \cdots a_k} + \overline{0.\underbrace{00 \cdots 00}_{k \text{ digits}} \dot{b_1} b_2 \cdots \dot{b_h}} \\
&= \frac{\overline{a_1 a_2 \cdots a_k}}{10^k} + \frac{y}{10^k} \\
&= \frac{(10^h - 1) \times \overline{a_1 a_2 \cdots a_k} + \overline{b_1 b_2 \cdots b_h}}{10^k (10^h - 1)} \\
&= \frac{\overline{a_1 a_2 \cdots a_k b_1 b_2 \cdots b_h} - \overline{a_1 a_2 \cdots a_k}}{10^k (10^h - 1)} \\
&= \frac{\overline{a_1 a_2 \cdots a_k b_1 b_2 \cdots b_h} - \overline{a_1 a_2 \cdots a_k}}{\underbrace{99 \cdots 99}_{h \text{ digits}} \underbrace{00 \cdots 00}_{k \text{ digits}}}
\end{aligned}$$

Q.E.D.

Example 2.1.

Convert the following recurring decimals into fractions.

(a) $0.\dot{1}22\dot{1}$

(b) $0.12\dot{3}45\dot{6}$

Solution.

(a) $0.\dot{1}22\dot{1} = \dfrac{1221}{9999} = \dfrac{37}{303}$

(b) $0.12\dot{3}45\dot{6} = \dfrac{123456 - 12}{999900} = \dfrac{3429}{27775}$

It can be shown that all decimals which are neither terminating nor recurring (e.g. 0.12345678910111213…) are irrational. Thus we have established a simple way to classify all decimals as either rational or irrational. But there are numbers which are neither fractions nor decimals. But we may still be able to determine whether they are rational, as illustrated by the examples below.

Example 2.2

Determine whether the following numbers are rational.

(a) $\sqrt{2}$

(b) $\log_3 5$

Solution.

(a) Suppose $\sqrt{2}$ is rational and write

$$\sqrt{2} = \frac{m}{n}$$

where m, n are positive integers with $n \neq 0$. Cancelling common factors if necessary, we may assume that m and n are relatively prime. Squaring both sides and transposing terms, we get

$$2n^2 = m^2.$$

Hence m^2 is even, so m is even. Consequently m^2 is divisible by 4. This forces n^2 to be even, and hence n must also be even. This contradicts the assumption that m and n are relatively prime. It follows that $\sqrt{2}$ must be irrational.

(b) Suppose $\log_3 5$ is rational and write

$$\log_3 5 = \frac{m}{n}$$

where m, n are positive integers with $n \neq 0$. This implies

$$3^{\frac{m}{n}} = 5.$$

Raising both sides to the power n, we get

$$3^m = 5^n$$

which is clearly impossible as the left hand side is divisible by 3 but the right hand side is not. So $\log_3 5$ must be irrational.

We leave it as an exercise to show that when n and k are positive integers, $\sqrt[k]{n}$ is either an integer or irrational.

It is known that the numbers π and e are both irrational. We include in the exercises proofs of the fact that π and e are irrational.

As mentioned before, when studying subsets of $\mathbb{R}$ we are interested in knowing its 'relative size'. Indeed, although $\mathbb{Q}$ is an 'extension' of $\mathbb{Z}$, they have the 'same size'. This is given in the theorem below.

Theorem 2.2.

$\mathbb{Q}$ is countable.

Proof. To prove the theorem we need only find a bijective mapping from $\mathbb{Q}$ to $\mathbb{N}$. Equivalently, we need to think about how to 'arrange the elements of $\mathbb{Q}$ in a row'. Note also that every rational number of the form $\frac{m}{n}$ can be identified with the point (m, n) on the plane, so we are essentially trying to enumerate the lattice points on the plane. The idea is based on the diagram below.
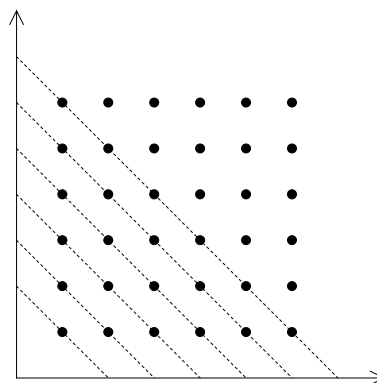
We leave it to the reader to complete the details of the proof.

<div align="right">Q.E.D.</div>

Despite the fact that $\mathbb{Q}$ is countable, hence 'much smaller' than $\mathbb{R}$, it is **dense** in $\mathbb{R}$. This is given more precisely below.

Theorem 2.3.

$\mathbb{Q}$ is dense in $\mathbb{R}$. That is, between any two real numbers a and b, with $a < b$, there is a rational number p such that $a < p < b$.

Proof. Set $d = b - a$ and choose a natural number N, large enough so that $\frac{1}{N} < d$. Then

$$Nb - Na = N(b-a) = Nd > 1,$$

so there is an integer M between Na and Nb, i.e. $Na < M < Nb$. Thus

$$a < \frac{M}{N} < b$$

and hence $\frac{M}{N}$ is the desired rational number.

Q.E.D.

3.   The Irrational Numbers

We have seen some examples of irrational numbers: $\sqrt{2}$, $\log_3 5$, π, e, etc. In this section, we will study some properties of irrational numbers.

Since $\mathbb{Q}$ is countable but $\mathbb{R}$ is not, the set of irrational numbers, being $\mathbb{R} \setminus \mathbb{Q}$, must be uncountable (why?). Hence the set of irrational numbers is 'larger' than the set of rational numbers.

Like $\mathbb{Q}$, the set of irrational numbers is dense in $\mathbb{R}$. More precisely, if a and b are two real numbers, $a < b$, there is an irrational number q such that $a < q < b$. The proof is similar to that of Theorem 2.3, and we leave it to the reader.

While $\mathbb{Q}$ is closed under addition, subtraction, multiplication and division, the set of irrational numbers is not closed under any of these operations. For instance, $\sqrt{2}$ is irrational, but $\sqrt{2} - \sqrt{2} = 0$ is rational. So the set of irrational numbers is not closed under subtraction. The reader should provide a counterexample for the other operations.

How about taking powers? If p and q are irrational, is it true that p^q must also be irrational? The answer is no, and there is a nice way of showing this. Consider the number $\sqrt{2}^{\sqrt{2}}$. Is it rational? Well, if it is, then we are done because we have provided an example where p and q are irrational but p^q is rational. If it is not, then we consider the number $\left(\sqrt{2}^{\sqrt{2}} \right)^{\sqrt{2}}$. Now both $\sqrt{2}^{\sqrt{2}}$ and $\sqrt{2}$ are irrational, yet $\left(\sqrt{2}^{\sqrt{2}} \right)^{\sqrt{2}} = \left(\sqrt{2} \right)^2 = 2$ is rational! Therefore, no matter whether $\sqrt{2}^{\sqrt{2}}$ is rational or not, we can provide a counterexample to show that p^q may not be irrational even if both p and q are irrational.

4. The Algebraic Numbers

We have seen numerous examples of irrational numbers: $\sqrt{2}$, $\sqrt[3]{5}$, $\sqrt[5]{7+\sqrt{2}}$ e, π, etc. Of these, you may feel that $\sqrt{2}$, $\sqrt[3]{5}$ and $\sqrt[5]{7+\sqrt{2}}$ are 'nicer'. At least we are able to express them in terms of surds (or 'radicals'). Because of this, we know that by 'certain operations' we can make them into a rational number. For instance, if we write $x = \sqrt[5]{7+\sqrt{2}}$, then we have

$$x = \sqrt[5]{7+\sqrt{2}}$$
$$x^5 = 7+\sqrt{2}$$
$$x^5 - 7 = \sqrt{2}$$
$$(x^5 - 7)^2 = 2$$
$$x^{10} - 14x^5 + 47 = 0$$

In other words, x is a zero of the polynomial $x^{10} - 14x^5 + 47$. This leads us to the following definition.

Definition 4.1. (Algebraic number)

A real number is said to be an **algebraic number** if it is a zero of a polynomial with rational coefficients.

Remark. The concept of algebraic numbers can be defined not only for the real numbers but also the complex numbers. For instance, the complex number i is algebraic since it is a zero of the polynomial $x^2 + 1$.

The set of all algebraic numbers is often denoted by $\mathbb{A}$. While we use the concept of 'radicals' to motivate the definition of an algebraic number, it should be noted that not all algebraic numbers can be expressed by a finite sequence of addition, subtraction, multiplication, division and taking radicals starting from rational numbers. (Incidentally, real numbers which can be such expressed are said to be **arithmetic numbers** and the set of all arithmetic numbers is often denoted by $\mathbb{B}$.) This result can be explained by theories in abstract algebra, and this in some sense explains why there can be no general formula for equations of degree 5 or higher.

It can be shown that the set of algebraic numbers is closed under addition, subtraction, multiplication and division. In other words, $\mathbb{A}$ forms a field under the usual addition and multiplication.

It can be shown that $\mathbb{A}$ is countable. Since $\mathbb{R}$ is uncountable, this implies that there are real numbers which are not algebraic, and these are known as **transcendental numbers**, which are our focus in the next section.

5. The Transcendental Numbers

In the previous section we remarked that irrational numbers like $\sqrt{2}$, $\sqrt[3]{5}$, $\sqrt[5]{7+\sqrt{2}}$ are 'nicer' than irrational numbers like e and π, and we characterized certain numbers to be algebraic. We also remarked that the countability of $\mathbb{A}$ implies the existence of numbers which are not algebraic, and which we call transcendental.

In fact, the numbers e and π were shown to be transcendental. The earliest proofs were due to Hermite in 1873 and Lindemann in 1882 respectively. Before that, Liouville established in 1851 established conditions for a number to be algebraic, with which he proved that the number

$$10^{-1!}+10^{-2!}+10^{-3!}+\cdots$$

is transcendental.

In general, it is not easy to determine whether a given number is transcendental. Despite that we know e and π are transcendental, whether $e+\pi$ is transcendental still remains an open problem.

6. Exercise

1. Prove that $\mathbb{Q}$ is closed under addition, subtraction, multiplication and division.

2. (a) Prove that $\sqrt[3]{5}$ is irrational.

   (b) Let n and k be positive integers. Prove that if $\sqrt[k]{n}$ is not an integer, then it is irrational.

3. Is $\log_{12} 72$ rational?

4. If p and q are rational numbers, must p^q be rational?

5. Show that the set of irrational numbers is not closed under addition, subtraction, multiplication and division.

6. Complete the proof of Theorem 2.2.

7. Imitating the proof of Theorem 2.3, or otherwise, show that the set of irrational numbers is dense in $\mathbb{R}$.

8. Show that in Definition 4.1, the word 'rational' can be changed to 'integral' without altering the meaning of an algebraic number.

9. Is $\mathbb{A}$ dense in $\mathbb{R}$?

10. Knowing that $\mathbb{A}$ is countable, what can you say about the countability of

    (a) $\mathbb{B}$, the set of all arithmetic numbers;

    (b) the set of all transcendental numbers?

11. Is π^2 transcendental? How about $\sqrt{\pi}$?

12. (China Hong Kong Mathematical Olympiad 1999) Determine all positive rational numbers $r \neq 1$ such that $r^{\frac{1}{r-1}}$ is rational.

13. In this exercise we will show that the number π is irrational. Assuming the contrary, we have

$$\pi = \frac{a}{b}$$

where a and b are positive integers. Define the polynomials

$$f(x) = \frac{x^n (a - bx)^n}{n!}$$
$$F(x) = f(x) - f^{(2)}(x) + f^{(4)}(x) - \ldots + (-1)^n f^{(2n)}(x)$$

where n is a certain positive integer.

(a) Show that $F(0)$ and $F(\pi)$ are integers.

(b) Considering $F'(x) \sin x - F(x) \cos x$, or otherwise, show that $\int_0^\pi f(x) \sin x \, dx$ is an integer.

(c) Establish the inequality $0 < f(x) \sin x < \dfrac{\pi^n a^n}{n!}$ for $0 < x < \pi$. Hence obtain a contradiction and conclude that π is irrational.

14. In this exercise we show that the number

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$$

is irrational. We write

$$s_n = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}$$

and so $e = \lim\limits_{n \to \infty} s_n$. Suppose e is rational, $e = \dfrac{p}{q}$, where p and q are relatively prime integers.

(a) Show that $q!(e - s_q)$ is a positive integer.

(b) Obtain a contradiction by showing $q!(e - s_q) < \dfrac{1}{q} < 1$. Hence conclude that e is irrational.

15. In this exercise we will see another proof of the fact that e is irrational. For non-negative integer n, define

$$I_n = \int_1^e \frac{(\ln x)^n}{x^2}\,dx.$$

(a) Find I_0 and establish the recurrence relation

$$I_n = nI_{n-1} - \frac{1}{e}.$$

(b) Obtain the following closed form for I_n:

$$I_n = \frac{1}{e}\left[n!e - \sum_{k=0}^n \frac{n!}{(n-k)!}\right].$$

(c) Establish the inequality

$$\frac{1}{n+1} < eI_n < \frac{e}{n+1}$$

and conclude that e cannot be rational.