New Zealand Mathematical Olympiad Committee

2011 Squad Assignment Four

Algebra

Due: Monday 28th March 2011

1. Find all polynomials $p(x)$ with real coefficients such that

$$p(a+b-2c)+p(b+c-2a)+p(c+a-2b)=3p(a-b)+3p(b-c)+3p(c-a)$$

for all $a,b,c\in\mathbb{R}$.

2. Determine if there exist non-zero real numbers $a_1,a_2,\ldots,a_{10}$ such that

$$\left(a_1+\frac{1}{a_1}\right)\cdot\ldots\cdot\left(a_{10}+\frac{1}{a_{10}}\right)=\left(a_1-\frac{1}{a_1}\right)\cdot\ldots\cdot\left(a_{10}-\frac{1}{a_{10}}\right).$$

3. Find all functions $f:\mathbb{R}\to\mathbb{R}$ which satisfy

$$f(x)=\max_{y\in\mathbb{R}}(2xy-f(y))$$

for all $x\in\mathbb{R}$.

Note: In general the expression $a=\max_{a\in S} g(s)$ means that $a\geq g(s)$ for all $s\in S$ and furthermore there exists $s\in S$ such that $a=g(s)$.

4. Given positive real numbers x,y,z, which satisfy $x^2+y^2+z^2+2xyz=1$, show that

$$2(x+y+z)\leq 3.$$

5. Let $f:\mathbb{N}\to\mathbb{N}$ be a strictly increasing function such that $f(f(n))=3n$, for all $n\in\mathbb{N}$. Find f(2011).

6. Find all monic polynomials, P, with real coefficients satisfying the following conditions:

   (a) $P(x+P(x))=x^2+P(P(x))$ for all real x and

   (b) $|P(0)|>1$.

7. Determine all finite sets A of non-negative real numbers, containing at least four distinct elements, and such that for all distinct $a,b,c,d\in A$, $ab+cd\in A$.

8. Let $a_1,a_2,\ldots,a_n$ be real numbers with $n\geq 3$, satisfying $a_1+a_2+\ldots+a_n=0$, and

$$2a_k\leq a_{k-1}+a_{k+1}\;\text{ for } k=2,3,\ldots,n-1.$$

Determine the smallest possible $\lambda(n)$ such that for any such sequence and $k\in\{1,2,\ldots,n\}$, it holds that

$$|a_k|\leq\lambda(n)\cdot\max\{|a_1|,|a_n|\}.$$

February 12, 2011

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