New Zealand Mathematical Olympiad Committee

2010 March Problems — Solutions

1. *Two players, A and B, are playing the following game. They take turns writing down the digits of a six-digit number from left to right; A writes down the first digit, which must be nonzero, and repetition of digits is not permitted. Player A wins the game if the resulting six-digit number is divisible by 2, 3 or 5, and B wins otherwise.*

   Prove that A has a winning strategy.

   Solution: Let a_1, a_2, a_3 be the digits chosen by player A, and let b_1, b_2, b_3 be the digits chosen by player B. Then the resulting six-digit number is $n = \overline{a_1 b_1 a_2 b_2 a_3 b_3}$, where $a_1 \neq 0$ and the digits are all different.

   Let $M = \{0, 2, 4, 5, 6, 8\}$ and $N = \{1, 3, 7, 9\}$. If B is to win she must choose b_3 from N, otherwise n is divisible by 2 or 5. A's goal then is to leave at most 1 and 7 from N available at the end of the game, and to choose a_3 so that $a_1 + b_1 + a_2 + b_2 + a_3 \equiv 2 \bmod 3$. If she does this then any choice of b_3 from the remaining digits in N makes the sum $a_1 + b_1 + a_2 + b_2 + a_3 + b_3$ congruent to 0 mod 3, and A will win because n will be divisible by 3.

   To this end A chooses $a_1 = 3$. This forces B to choose b_1 and b_2 from M (otherwise A may exhaust N on her next two choices), freeing A to choose $a_2 = 9$. There are now three cases, depending on B's choice of b_1 and b_2.

   - *Case 1:* $b_1 + b_2 \equiv 0 \bmod 3$. In this case $a_1 + b_1 + a_2 + b_2 \equiv 0 \bmod 3$, so A chooses a_3 from $\{2, 5, 8\}$. This is always possible, because at least one of these must still be unchosen.
   - *Case 2:* $b_1 + b_2 \equiv 1 \bmod 3$. In this case $a_1 + b_1 + a_2 + b_2 \equiv 1 \bmod 3$, and A chooses $a_3 = 1$.
   - *Case 3:* $b_1 + b_2 \equiv 2 \bmod 3$. In this last case $a_1 + b_1 + a_2 + b_2 \equiv 2 \bmod 3$, and A chooses a_3 from $\{0, 6\}$. This is always possible, because if B has chosen both then $b_1 + b_2 \equiv 0 \bmod 3$, putting us in Case 1 above.

   In all three cases A succeeds in forcing $a_1 + b_1 + a_2 + b_2 + a_3$ to be congruent to 2 mod 3, with only 1 and 7 left from N, and therefore wins the game. $\square$

2. *Prove that $n^n - n$ is divisible by 24 for all odd positive integers n.*

   Solution: Since $24 = 3 \times 8$ it's enough to show that $n^n - n$ is divisible by both 8 and 3. Since n is odd we may write $n = 2k + 1$, so $n^n - n = n(n^{n-1} - 1) = n(n^{2k} - 1)$.

   To prove divisibility by 8 we will use the fact that $m^2 - 1$ is divisible by 8 whenever m is odd, i.e., $m^2 \equiv 1 \bmod 8$ whenever $m \equiv 1 \bmod 2$. To prove this write $m = 2\ell + 1$. Then $m^2 - 1 = 4\ell^2 + 4\ell = 4\ell(\ell + 1)$, which is obviously divisible by 4; and since either ℓ or $\ell + 1$

must be even, we get a third factor of 2. Applying to this to our present problem, if n is odd then n^k is too, so $(n^k)^2 - 1$ is divisible by 8.

To prove divisibility by 3 we will use the fact that $m^2 - 1$ is divisible by 3 whenever m itself is not divisible by 3. This follows from Fermat's Little Theorem, but it can also be proved directly using the factorisation $m^2 - 1 = (m-1)(m+1)$. If m is not divisible by 3 then either $m-1$ or $m+1$ must be divisible by 3 (just consider the remainder when m is divided by 3), so $m^2 - 1$ will be divisible by 3. Applying this to our present problem, either n or $(n^k)^2 - 1$ will be divisible by 3, and in either case the product $n(n^{2k} - 1)$ has a factor of 3. $\square$

3. *Let a and b be real numbers. Prove that the inequality*

$$\frac{(a+b)^3}{a^2 b} \geq \frac{27}{4} \tag{1}$$

holds.

When does equality hold?

Solution: Since a and b are positive, the inequality is equivalent to

$$\left(\frac{a+b}{3}\right)^3 \geq \frac{a^2 b}{4}.$$

To prove this apply the arithmetic mean-geometric mean inequality to $a/2$, $a/2$, b. This gives

$$\frac{\frac{a}{2} + \frac{a}{2} + b}{3} \geq \sqrt[3]{\frac{a}{2}\frac{a}{2}b} = \sqrt[3]{\frac{a^2 b}{4}},$$

and cubing gives the desired result.

Equality holds in the AM-GM inequality when the averaged quantities are all equal, so equality holds in (1) when $b = a/2$. $\square$

4. *Let $ABCD$ be a quadrilateral. The circumcircle of the triangle ABC intersects the sides CD and DA in the points P and Q respectively, while the circumcircle of CDA intersects the sides AB and BC in the points R and S. The straight lines BP and BQ intersect the straight line RS in the points M and N respectively. Prove that the points M, N, P and Q lie on the same circle.*

Solution: By equality of angles subtended on the same chord, $\angle BAC = \angle BQC$ and $\angle CQP = \angle CBP$ (see Figure 1). In addition, quadrilateral $ACSR$ is cyclic, so $\angle RSC + \angle RAC = 180°$, and

$$\begin{aligned} \angle BSR &= 180° - \angle RSC && \text{(angles on a straight line)} \\ &= \angle RAC \\ &= \angle BAC \\ &= \angle BQC. \end{aligned}$$

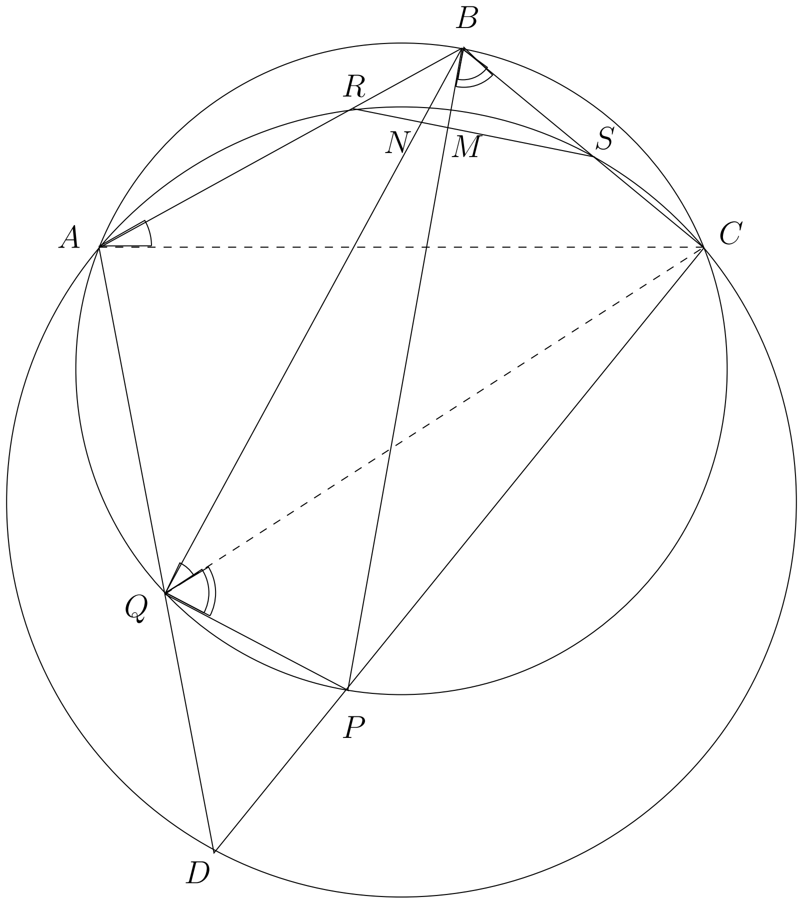

Figure 1: Diagram for Problem 4.

Using these relations we obtain

$$
\begin{aligned}
180^\circ - \angle PMN &= 180^\circ - \angle BMS && \text{(opposite angles)}\\
&= \angle SBM + \angle BSM && \text{(angles in triangle)}\\
&= \angle CBP + \angle BSR\\
&= \angle CQP + \angle BQC\\
&= \angle BQP\\
&= \angle NQP,
\end{aligned}
$$

so $\angle PMN + \angle NQP = 180^\circ$. This shows that $MNPQ$ is cyclic. $\square$

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