# CHAPTER 10

# PHYSICAL TREATMENTS OF SOME MATHEMATICAL PROBLEMS

Mathematics is the study of the relationships between numbers, quantities and space, whereas physics is a science that deals with laws of nature. For a long time these two disciplines have been closely related like twins, as evidenced by the fact that they exert strong influence on each other through their course of progress. On one hand, mathematics provides powerful tools for fuelling the advancement of physics. On the other hand, some physical laws offer fresh perspectives for solving mathematical problems in ingenious ways.

In this note we would like to focus on how physics can be applied to solve challenging mathematical problems neatly. Hopefully the reader can appreciate the elegance of the physical treatments and how well mathematics is consistent with the laws of nature.

It should be noted, however, that most of the physical arguments presented in this note are intuitive in nature and must not be deemed the correct way of problem solving. Rather they had better be seen as illuminating ways of predicting answers or seeking the right lines of attack. The reader is therefore advised to refer to the rigorous mathematical solutions of each example in this note.

## 1. Center of Mass

Center of mass is one of the important notions in statics. It is also a useful tool in tackling geometric problems.

Center of mass is, by definition, the position of the object concerned through which its weight acts, no matter how the object is positioned. In this section, we are more interested in center of mass of discrete particles (i.e. point masses, which have mass but no size).

For two particles of masses $m_1$ and $m_2$ placed at $A$ and $B$ respectively, their center of mass lies on $AB$ and divides $AB$ in the ratio $m_2 : m_1$. In general, the center of mass of particles of masses $m_1$, $m_2$, …, $m_k$ at $(x_1, y_1), (x_2, y_2), \ldots, (x_k, y_k)$ is $(x_M, y_M)$ where

$$x_M = \frac{\sum_{i=1}^{k} m_i x_i}{\sum_{i=1}^{k} m_i} \quad \text{and} \quad y_M = \frac{\sum_{i=1}^{k} m_i y_i}{\sum_{i=1}^{k} m_i}.$$

- We use $A(a)$ to denote the particle of mass $a$ at $A$.

**Example 10-1.1**

In $\Delta ABC$ (Figure 1), $D$ and $E$ are points on $BC$ and $AC$ respectively such that $BD$:$DC$ = 1:1 and $AE$:$EC$ = 2:1. $AD$ cuts $BE$ at $F$. Find $AF$:$FD$.

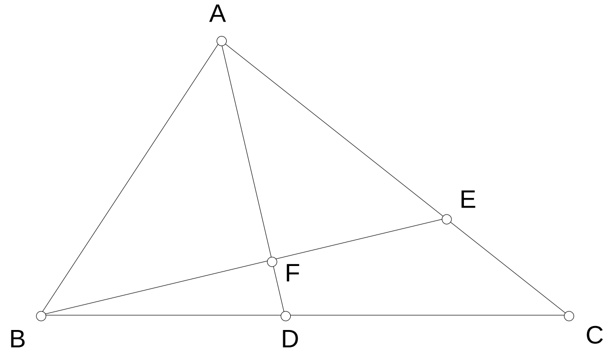


Figure 1

**Solution.**

Put three particles of masses 1, 2 and 2 at $A$, $B$ and $C$ respectively so that $D$ is the center of mass of $\{B(2), C(2)\}$ and $E$ is the center of mass of $\{A(1), C(2)\}$. Then the whole system $\{A(1), B(2), C(2)\}$ can be reduced to $\{A(1), D(4)\}$. It follows that its center of mass lies on $AD$. Similarly we have the center of mass on $BE$. So, $F$ is the center of mass of the whole system. Consider {A(1), D(4)}. We conclude that $AF$:$FD$ = 4:1

**Example 10-1.2 (Ceva's Theorem)**

Prove that in $\Delta ABC$ (Figure 2), if $D$, $E$ and $F$ are on $BC$, $AC$ and $AB$ respectively such that $AD$, $BE$ and $CF$ are concurrent, then $\frac{AF}{FB}\frac{BD}{DC}\frac{CE}{EA}=1$.

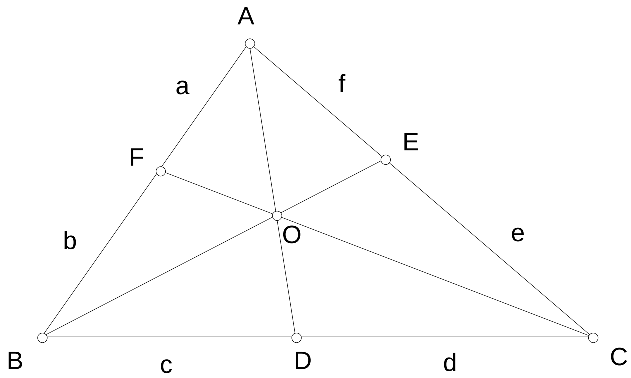


Figure 2

**Solution.**

Put three particles of masses *be*, *ea* and *bf* at *A*, *B* and *C* respectively. Then *F* and *E* are the centers of mass of {*A*(*be*), *B*(*ea*)} and {*A*(*be*), *C*(*bf*)} respectively. Also, the intersection of *CF* and *BE* is in fact the center of mass of {*A*(*be*), *B*(*ea*), *C*(*bf*)}. Denote this point by *O*. Therefore, it can be seen that the center of mass of {*B*(*ea*), *C*(*bf*)} is the intersection of *AO* and *BC*, i.e. *D*.

$$\therefore \frac{ea}{bf} = \frac{d}{c} \Rightarrow \frac{ace}{bdf} = 1,$$

or in other words

$$\frac{AF}{FB}\frac{BD}{DC}\frac{CE}{EA} = 1.$$

Center of mass can be used to solve algebraic problems, as illustrated in the following examples

**Example 10-1.3**

Evaluate $\sum_{k=1}^{n} k(a+k-1)$.

**Solution.**

Dissect the equilateral triangle *ABC* into smaller congruent equilateral triangles as shown in Figure 3.

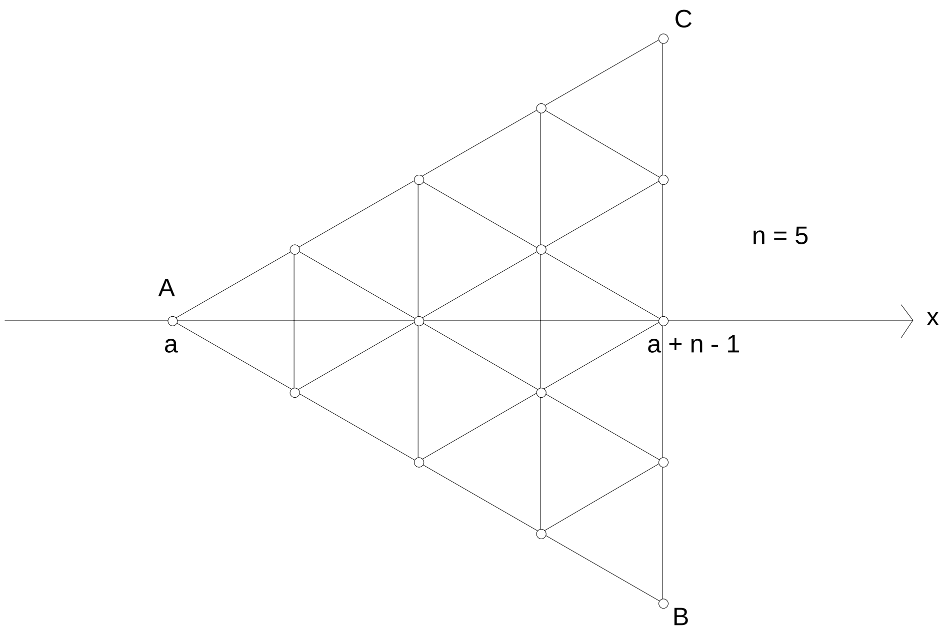


Figure 3

Put a particle of unit mass at each vertex of the smaller equilateral triangle. Let the *x*-coordinate of the particle at *A* be *a* and those of the particles on *BC* be $a + n - 1$. *Intuitively*, we assert that the center of mass of the configuration coincides with the centroid of the triangle *ABC*. Considering the *x*-coordinate of the center of mass of the whole system, we have

$$\frac{\sum_{k=1}^{n} k(a+k-1)}{\sum_{k=1}^{n} k} = a + \frac{2}{3}(n-1)$$

$$\Rightarrow \sum_{k=1}^{n} k(a+k-1) = \frac{n(n+1)}{2}[a + \frac{2}{3}(n-1)].$$

Putting $a = 1$, we obtain the well-known formula of sum of squares of the first *n* positive integers:

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

**Example 10-1.4 (Chebyshev's Inequality)**

If $0 < a_1 < a_2 < \dots < a_n$, $0 < b_1 < b_2 < \dots < b_n$, then

$$n\sum_{k=1}^{n} a_k b_k > \left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} b_k\right).$$

**Solution.**

Let $A_1, A_2, \dots, A_n$ be collinear and their coordinates on the line be $a_1, a_2, \dots, a_n$ respectively. Put at $A_1, A_2, \dots, A_n$ identical particles of mass $\dfrac{M}{n}$, where $M = \sum_{k=1}^{n} b_k$. The center of mass of the system is

$$\frac{1}{n}\sum_{k=1}^{n} a_k .$$

Consider another system obtained by putting particle of mass $b_k$ at $A_k$. Then its center of mass is

$$\frac{1}{M}\sum_{k=1}^{n} a_k b_k .$$

Now that $b_1 < \frac{M}{n} < b_n$, there exists *i* such that $b_1 < \frac{M}{n} < b_i$ but $b_{i+1} < \frac{M}{n}$. The first *i* particles of the first system are heavier than those of the second system. At the same time the total masses of the two systems are the same. Thus the center of mass of the first system should be 'biased' to the left.

$$\therefore \frac{1}{n}\sum_{k=1}^{n} a_k < \frac{1}{M}\sum_{k=1}^{n} a_k b_k ,$$

i.e.

$$n\sum_{k=1}^{n} a_k b_k > \left(\sum_{k=1}^{n} a_k\right)\left(\sum_{k=1}^{n} b_k\right).$$

## 2. Systems of Forces

**Theorem 10-2.1 (Lami's Theorem)**
If three coplanar forces $F_1$, $F_2$ and $F_3$ are in equilibrium, their lines of action are concurrent and

$$\frac{F_1}{\sin\alpha} = \frac{F_2}{\sin\beta} = \frac{F_3}{\sin\gamma}$$

where $\alpha$, $\beta$ and $\gamma$ are the angles between the lines of action of $F_2$ and $F_3$, $F_1$ and $F_3$, and, $F_2$ and $F_1$ respectively.

**Theorem 10-2.2**
If the resultant of a system of forces passes through *A*, *B*, *C*, …, then *A*, *B*, *C*, … are collinear.

The two theorems above can be used to solve geometric problems involving concurrence and collinearity.

**Example 10-2.1 (Ceva's Theorem)**
In $\Delta ABC$ (Figure 4), $\frac{\sin\alpha}{\sin\alpha'}\frac{\sin\beta}{\sin\beta'}\frac{\sin\gamma}{\sin\gamma'} = 1$ if and only if *AD*, *BE* and *CF* are concurrent.

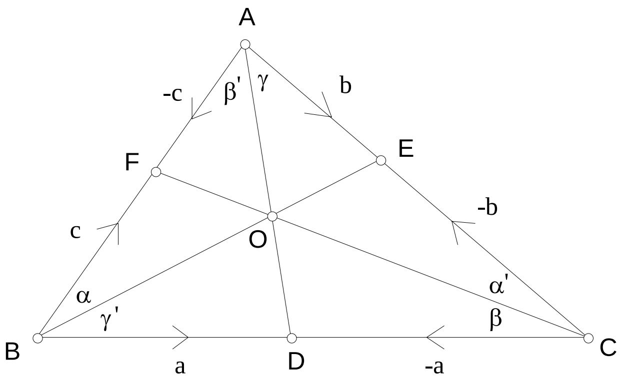


Figure 4

**Solution.**
'If' part: let two forces $\vec{a}$ and $\vec{c}$ act through *BC* and *BA* respectively such that their resultant acts through *BO*. Then

$$|\vec{c}|\sin\alpha = |\vec{a}|\sin\gamma',$$

i.e.

$$\frac{\sin\alpha}{\sin\gamma'}=\frac{|\vec{a}|}{|\vec{c}|}.$$

Similarly, let $-\vec{c}$ and $\vec{b}$ act through *AB* and *AC* respectively such that their resultant acts through *AO*. Then

$$\frac{\sin\gamma}{\sin\beta'}=\frac{|-\vec{c}|}{|\vec{b}|}=\frac{|\vec{c}|}{|\vec{b}|}.$$

Let $-\vec{a}$ and $-\vec{b}$ act through *CB* and *CA* respectively. Now, since the system of forces is in equilibrium, the resultant $-\vec{a}+(-\vec{b})$ must act through *O*. Then

$$\frac{\sin\beta}{\sin\alpha'}=\frac{|-\vec{b}|}{|-\vec{a}|}=\frac{|\vec{b}|}{|\vec{a}|}.$$

$$\frac{\sin\alpha}{\sin\gamma'}\frac{\sin\beta}{\sin\alpha'}\frac{\sin\gamma}{\sin\beta'}=\frac{|\vec{a}|}{|\vec{c}|}\frac{|\vec{b}|}{|\vec{a}|}\frac{|\vec{c}|}{|\vec{b}|}=1.$$

'Only if' part: We can always find the system of forces mentioned above such that

$$\frac{\sin\alpha}{\sin\gamma'}=\frac{|\vec{a}|}{|\vec{c}|},\frac{\sin\beta}{\sin\alpha'}=\frac{|\vec{b}|}{|\vec{a}|},\frac{\sin\gamma}{\sin\beta'}=\frac{|\vec{c}|}{|\vec{b}|}.$$

Since the system is in equilibrium, the lines of action of the three resultants are concurrent.

**Example 10-2.2**

In $\Delta ABC$ (Figure 5), the exterior angle bisector of *A* and the interior angle bisectors of *B* and *C* cut *BC*, *AC* and *AB* at *D*, *E* and *F* respectively. Prove that *D*, *E* and *F* are collinear.

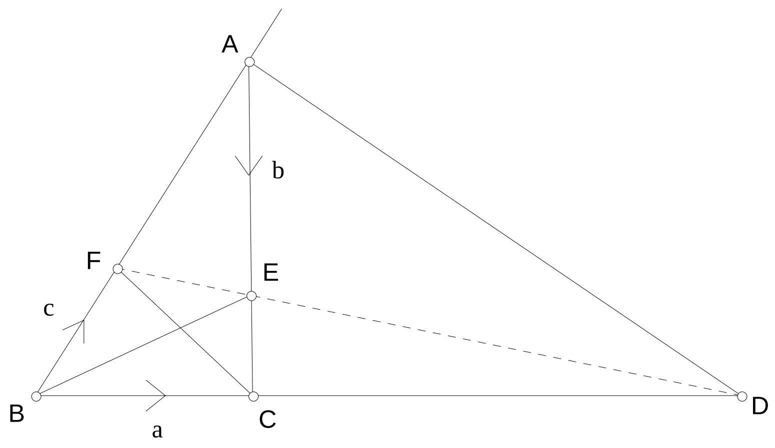


Figure 5

**Solution.**

Let three forces $\vec{a}$, $\vec{b}$ and $\vec{c}$ of equal magnitude act through *BC*, *AC* and *BA* respectively. We can see that the resultant $\vec{a}+\vec{c}$ acts through *BE*. Moreover, $\vec{b}$ acts through *AE*, so $\vec{a}+\vec{b}+\vec{c}$ acts through *E*. Similarly, we can prove that $\vec{a}+\vec{b}+\vec{c}$ also acts through *D* and *F*. Thus *D*, *E* and *F* are collinear.

**Example 10-2.3**

Four directed lines, no three of which are concurrent, intersect each other at six points called vertices. Two vertices are said to be opposite if they are not on the same line (so there are altogether three pairs of opposite vertices). For each pair of opposite vertices, a pair of angle bisectors is drawn as shown in Figure 6 and their intersection point is found. Prove that these three intersection points are collinear.

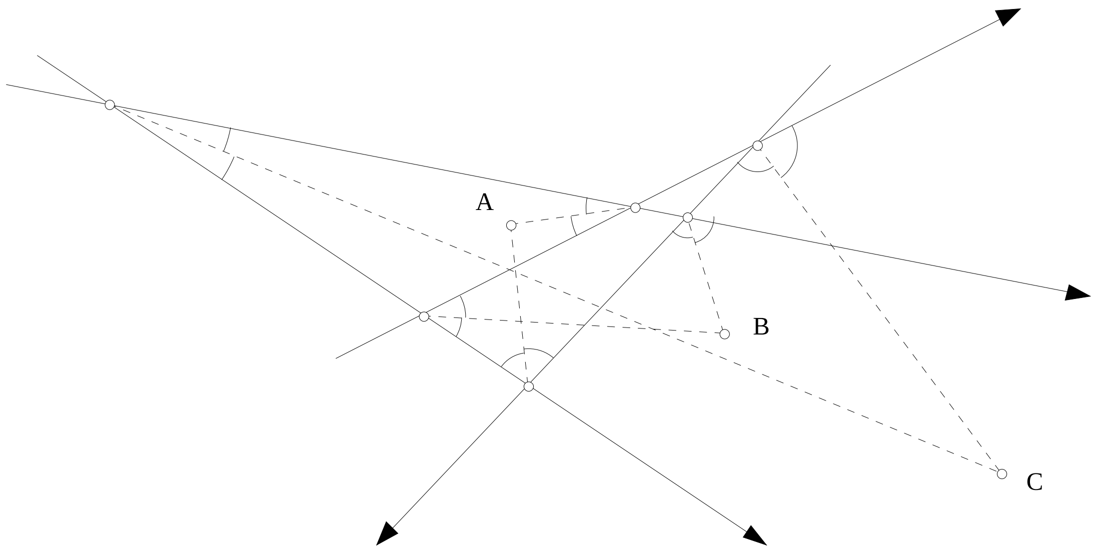


Figure 6

The four directed lines are represented by the four thin lines with arrows, while the six vertices are the unlabelled points. The dashed lines are the angle bisectors, while *A*, *B* and *C* are the three (collinear) intersection points.

**Solution.**

Four forces $\vec{a}_1, \vec{a}_2, \vec{a}_3$ and $\vec{a}_4$ of equal magnitude are arranged along the directions of the four lines. Considering their resultant and using similar arguments to those in the previous proof (more precisely, the resultant of these four forces must pass through all of *A*, *B* and *C*), we get the desired conclusion.

## 3. Stable Equilibrium and Potential Energy

A system is said to be in stable equilibrium if, after the system is subject to slight disturbance, equilibrium can be restored. As shown Figure 7, particle *A* is in stable equilibrium while particle *B* is in unstable equilibrium.

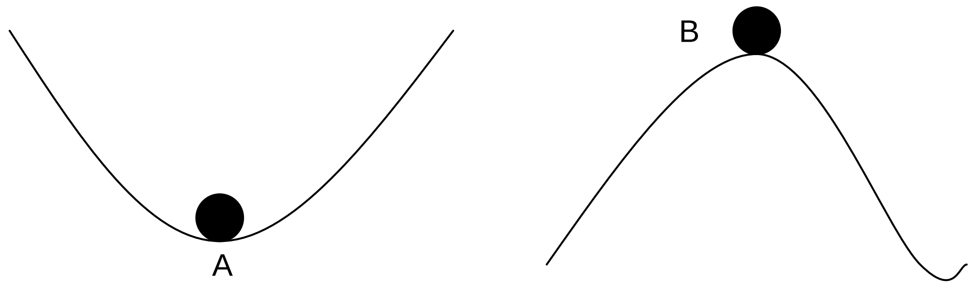


Figure 7

There is an important physical law about stable equilibrium, which is stated below.

**Theorem 10-3.1 (Dirichlet's Principle)**
If a system is in stable equilibrium, its potential energy is minimal.

**Example 10-3.1**

*A* and *B* are on two sides of a line *l*. Find a point *X* on *l* such that $\frac{AX}{p}+\frac{XB}{q}$ is minimal.

**Solution.**

Arrange the two smooth, light fixed pulleys *A* and *B* and a light smooth horizontal wire *l* on the same vertical plane (Figure 8). A small smooth ring *X* which can freely slide on *l*, is connected by two light inextensible strings, which pass through *A* and *B* and hang two particles *A*' and *B*' of masses $\frac{1}{p}$ and $\frac{1}{q}$ respectively. It can be shown experimentally that the equilibrium of the system is a stable one. When the stable equilibrium is reached, the potential energy of $A'$ and $B'$ is minimal. Then $\frac{AA'}{p}+\frac{BB'}{q}$ is maximal and $\frac{AX}{p}+\frac{XB}{q}$ is minimal. In other words, $\frac{AX}{p}+\frac{XB}{q}$ attains its minimum when *X* is at the equilibrium position. In equilibrium, the horizontal components of tensions in the two strings balance each other.

$$\therefore \frac{\sin\alpha}{p}=\frac{\sin\beta}{q}.$$

We can use the above equality to locate this equilibrium position.

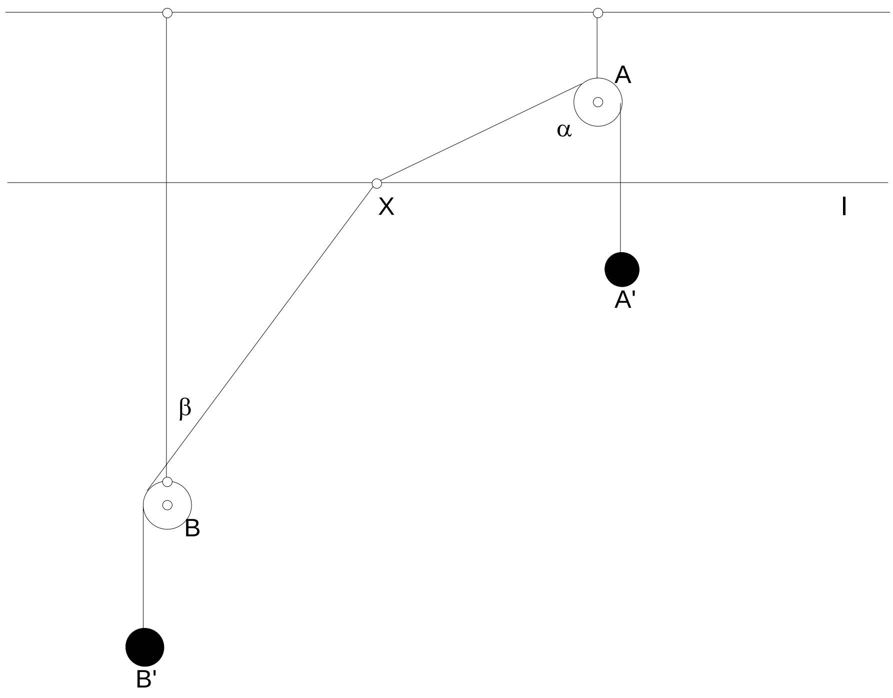


Figure 8

**Example 10-3.2**

There are respectively 40, 50 and 60 children in village *A*, *B* and *C*. The villagers propose that a school be built for these children. Where should this school be located so that the total distance traveled by the children from their own villages to the school is minimal?

**Solution.**

On a wooden board drill three holes representing the locations of the three villages *A*, *B* and *C* (Figure 9). Three identical light inextensible strings pass through *A*, *B* and *C* and hang three particles of masses 4, 5 and 6 at one end respectively, with the other ends being joined together on the board. Denote this meeting point of the three strings by *X*. Release the system from rest and it will reach a stable equilibrium. Since the potential energy of the three particles is minimal when in equilibrium, it can be shown (why?) that $4XA + 5XB + 6XC$ is also minimal when *X* is at its equilibrium position, which is therefore the required location of the school. By Lami's Theorem, *X* should be a point such that

$$\frac{4}{\sin\angle BXC}=\frac{5}{\sin\angle AXC}=\frac{6}{\sin\angle AXB}.$$

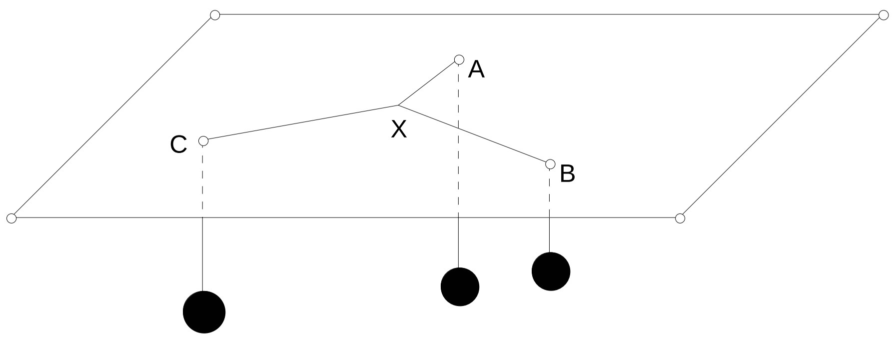


Figure 9

- The Fermat Point $O$ of $\Delta ABC$ (i.e. the point that the sum of its distance from the three vertices is minimal) can be found by using three identical particles. By Lami's Theorem, $O$ should be such that $\angle BOC = \angle AOC = \angle AOB = 120^0$.

**Example 10-3.3**

Prove that the tangent of an ellipse at $A$ makes equal angle with $F_1A$ and $F_2A$, where $F_1$ and $F_2$ are the two foci of the ellipse.

**Solution.**

Fix the two ends of a light inextensible string at $F_1$ and $F_2$ on a vertical wall. A light smooth ring $A$ can freely slide through the string. The locus of the ring is an ellipse if it moves in such a way that the string is held taut. (See the left of Figure 10) If the system is released from rest, it reaches a stable equilibrium, when the potential energy of the ring is minimal. Thus the equilibrium position of the ring is the lowest position of the ellipse. It follows that the tangent of the ellipse at $A$ is a horizontal line. On the other hand, the equilibrium of the ring requires the horizontal component of the tension of the string to vanish. This can only be achieved when $F_1A$ and $F_2A$ make equal angle with the horizontal (note that the tensions in $F_1A$ and $F_2A$ are equal in magnitude).

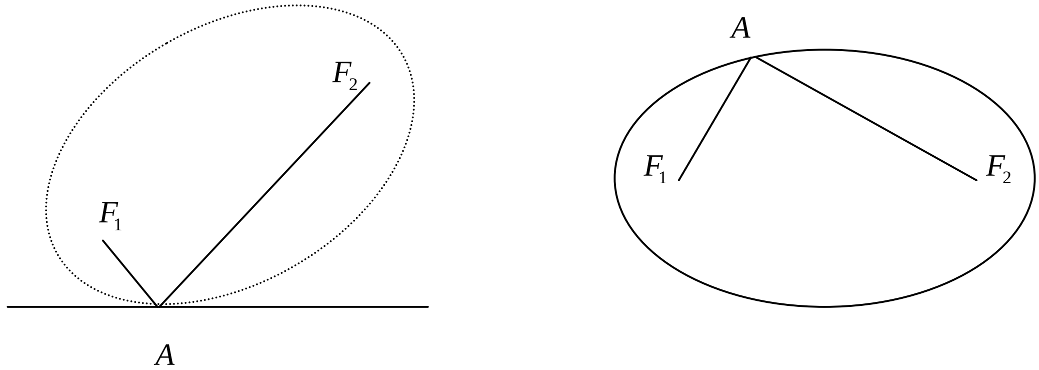


Figure 10

**Alternative Solution.**

Let a small smooth ring *A* freely slide through a wire in the shape of an ellipse. A light elastic string passes through *A*, with its two ends fixed at the two foci of the ellipse $F_1$ and $F_2$. (See the right of figure 10) When the ring moves along the wire, the string does no work to the ring as the length of the string remains unchanged.

Consider the tensions in $F_1A$ and $F_2A$. Since they are equal in magnitude, their resultant bisects $\angle F_1AF_2$. Note that this resultant does no work to the string. Thus its line of action is always perpendicular to the direction of movement of the ring, which is also the tangent of the ellipse at that point.