Combinatorics

Jacob Steinhardt and Daniel Schafer

1. Prove that one can arrange a set of n items in $n!$ ways.

   You can place the first element in any of n places. You can place the second element in any of $n-1$ places. You can place the kth element in any of $n-k+1$ places. Thus there are a total of $n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot 3 \cdot 2 \cdot 1 = n!$ possibilities.

2. Prove that one can choose k out of n items (order does matter) in $\dfrac{n!}{(n-k)!}$ ways.

   Use the same argument as before, except stop just before the $k+1$st element. Then you get $n \cdot (n-1) \cdot (n-2) \cdot \ldots \cdot (n-k+1) = \frac{n!}{(n-k)!}$ possibilities.

3. Prove that one can choose k out of n items (order doesn't matter) in $\dfrac{n!}{k!(n-k)!}$ ways.

   This is the definition of combination, so it is clearly $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. This can be easily derived; there are n choices for the first item, $(n-1)$ for the second, until eventually there are $(n-(k-1))$ for the kth. However, there are $k!$ orderings of these k items, so there must be $\frac{n(n-1)\cdots(n-k+1)}{k!} = \frac{n!}{k!(n-k)!}$ total combinations.

4. Prove that $\binom{n}{k}$ sequences $a_1, a_2, \ldots, a_k$ of positive integers exists such that $a_i < a_{i+1} \leq n$.

   Clearly, each of the a_i is distinct. Also, given any k distinct numbers, there will be exactly 1 sequence of a_i with those k numbers in it that satisfies the requirement. This creates a bijection between sequences $a_1, \ldots, a_k$ and sets of k numbers chosen from the numbers 1,2,$\ldots$,n. The number of sets for the latter is clearly $\binom{n}{k}$, so there are $\binom{n}{k}$ such sequences.

5. Prove that n distinct items can be partitioned into k groups in n^k ways.

   Consider each item individually. For each item, there are k options of what group to put it in. There are n such choices, each of which is independent, so there are n^k ways to perform the partition.

6. Prove that one can travel from $(0,0)$ to (x,y) traveling only to the right and up and only between adjacent lattice points in $\binom{x+y}{x}$ ways.

   Consider representing each path with a list of what direction was travelled, either right (R) or up (U). Clearly, any path can be represented by a distinct arrangement of the sequence $RR\cdots RUU\cdots U$, where there are x Rs and y Us. This creates a bijection, so the number of paths is the same as the number of arrangements, which is $\binom{x+y}{x}$.

7. Prove that n distinct items can be partitioned into k groups of sizes $s_1, s_2, \ldots, s_k$, where $s_1 + \ldots + s_k = n$, in $\dfrac{n!}{s_1!s_2!\cdots s_k!}$ ways.

   Consider selecting each group seperately. For the first group, there are $\binom{n}{s_1}$ options, for the second, $\binom{n-s_1}{s_2}$, etc. Multiplying these out, there is a total of $\binom{n}{s_1}\binom{n-s_1}{s_2}\binom{n-s_1-s_2}{s_3}\cdots = \frac{n!}{s_1!s_2!\cdots s_k!}$. This value is often expressed as $\binom{n}{s_1,s_2,\ldots,s_k} = \frac{n!}{s_1!s_2!\cdots s_k!}$, and is refered to as a *multinomial* coefficient. Note that these coefficients have a relationship with the term $a_1^{s_1}\cdots a_k^{s_k}$ in the expansion of $(a_1 + \ldots + a_k)^n$ similar to that of binomial coefficients in the expansion of $(x+y)^n$. Also note that if $s_1 + \ldots + s_k < n$, one can add a pseudo-set s_{k+1} which contains all of the items not in any of the first k sets (so its size will be $n - (s_1 + \ldots + s_k)$), and calculate the coefficient from there. Note that this value is also the number of ways to arrange n objects in k indistinct groups of sizes $s_1, s_2, \ldots, s_k$, where the sizes again sum to n.

8. Prove that $\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$.

   Method one. Writing out the factorial notation and simplifying quickly yields the desired reuslt. $\frac{n!}{k!(n-k)!} + \frac{n!}{(k+1)!(n-k-1)!} = \frac{n!(k+1)}{(k+1)!(n-k)!} + \frac{n!(n-k)}{(k+1)!(n-k)!} = \frac{n!(n+1)}{(k+1)!(n-k)!} = \binom{n+1}{k+1}$. This is the basis of Pascal's Triangle.

   Method two. Consider selecting $k+1$ items from $n+1$. Either the first item is chosen, in which case there are $\binom{n}{k}$ ways to select the remaining k, or the first item is not chosen, in which case there are $\binom{n}{k+1}$ ways to choose the remaining $k+1$. Thus, $\binom{n+1}{k+1} = \binom{n}{k} + \binom{n}{k+1}$.

9. Prove that $2^n = \binom{n}{0} + \binom{n}{1} + \ldots + \binom{n}{n-1} + \binom{n}{n}$.

   This can be derived easily by counting in two ways. This is the total number of subsets of a set of size n; for each element of the original set, there are two options: in the set, or not in the set. Thus, there are 2^n subsets, and the expression above simplifies to 2^n.

10. Prove that $\binom{n-1}{k-1}$ positive integer solutions exist to the system $a_1 + a_2 + \ldots + a_k = n$.

    Draw n circles in a row, and consider drawing $k-1$ lines in the $n-1$ gaps between the circles. Let a_1 be the number of circles before the first line, a_2 be the number of circles between the first and second lines, etc. Clearly, there will be k total a_i (if there are $k-1$ lines), all a_i will be at least 1 (since there is at least 1 circle between each gap) and the sum of the a_i will be n (since each circle is counted exactly once). Thus, there is a bijection between the sequence $a_1, \ldots, a_k$ and the drawing; as the drawing was generated by choosing $k-1$ lines from $n-1$ gaps, there are $\binom{n-1}{k-1}$ drawings, and thus $\binom{n-1}{k-1}$ sequences. This algorithm is sometimes refered to as "Stars and Bars."

11. Prove that $\binom{n+k-1}{k-1}$ non-negative integer solutions exist to the system $a_1 + a_2 + \ldots + a_k = n$.

    Draw $n+k-1$ circles in a row, and cross out $k-1$ of them. Let a_1 be the number of circles before the first crossed-out one, a_2 be the number of circles between the first crossed-out one and the second crossed-out one, etc. Clearly, there will be k total a_i (if there are $k-1$ crossed out circles), all a_i will be at least 0 (since there may be two consecutive crossed-out circles) and the sum of the a_i will be n (since each remaining circle is counted exactly once, and there are $(n+k-1)-(k-1) = n$ remaining circles). Thus, there is a bijection between the sequence $a_1, \ldots, a_k$ and the drawing; as the drawing was generated by choosing $k-1$ circles from $n+k-1$ choices, there are $\binom{n+k-1}{k-1}$ drawings, and thus $\binom{n+k-1}{k-1}$ sequences. Note that this algorithm can be used to determine the number of ways to partition n indistinct objects into k sets.

12. Prove that $\binom{n-km-1}{k-1}$ integer solutions exist to the system $a_1 + a_2 + \ldots + a_k = n$ if $a_i > m$.

    Method one. Use algebraic manipulation. Subtract km from both sides, so that now we have $b_1 + b_2 + \ldots + b_k = n - km$ with $b_i > 0$ (we subtracted m from each of the a_i's). Now apply the previous argument and arrive at $\binom{n-km-1}{k-1}$.

    Method two. Use a bijection. This is the same as dividing up a set of size n into k groups, all of size greater than m. But this means we can remove m items from each group and still have a positive number of items in each group, yielding a scenario in which we divide up the set of size $n - km$ into k non-empty groups. Also, for every scenario with this smaller set we can add m items to each group to get back to what we had in the larger set. This means there is a one-to-one correspondence between the number of possibilities in both cases. We count the easier quantity (the set of size $n - km$). To ensure that all sets are non-empty, we simply place $k - 1$ dividing lines between elements of the set to divide the set into k subsets. There are $\binom{n-km-1}{k-1}$ ways to do this, as there are $n - km - 1$ spaces between successive elements of the set.

13. Prove that one can arrange n of one object and k of another in $\binom{n+1}{k}$ ways if none of the k objects may be next to each other.

    First, lay out the first objects in a row of length n. There are now $n + 1$ gaps to place the k other objects in (there are $n-1$ gaps between the layed-out objects and 1 on both ends of the row). Thus, there are $\binom{n+1}{k}$ arrangements of these objects to avoid having any of the k objects touch each other.

14. Prove that one can place k indistinct items between n other indistinct items in $\binom{n+k-2}{k}$ ways. (Any number of the first type can go between the sucessive items of the second type).

    Draw $n + k$ circles in a row, and mark k of them as objects of the second type. Clearly, one cannot mark either the first or the last, as the problem specifies that the second objects go between objects of the first type. Thus, there are $n+k-2$ markable objects, and k marks, so there are $\binom{n+k-2}{k}$ arrangements.

15. Prove that $\binom{n+k+1}{n+1} = \binom{n}{n} + \binom{n+1}{n} + \ldots + \binom{n+k}{n}$.

    Note that if we have a set of size $n + k + 1$ and we wish to choose $n + 1$ elements, the first element we choose from the set can be either the first, second, third, etc., all the way up to the $k + 1$st element. Each of these corresponds to one of the elements in the sum (in general, if the first element we choose is the ith element in the set, then there are $n+k+1-i$ elements in the set left to choose from, and there are n elements left to choose, yielding $\binom{n+k+1-i}{n}$ choices). Thus, the sum is equal to $\binom{n+k+1}{n+1}$, as desired (this is known as the *Hockeystick Identity*).

16. Prove that:

$$1 = \binom{n}{0}$$

$$\sum_{i=1}^{n} 1 = \binom{n}{1}$$

$$\sum_{i=1}^{n-1} \sum_{j=i+1}^{n} 1 = \binom{n}{2}$$

$$\sum_{i=1}^{n-2} \sum_{j=i+1}^{n-1} \sum_{k=j+1}^{n} 1 = \binom{n}{3}$$

and so on.

This is a direct result of repeatedly applying the Hockeystick Identity.

17. Prove that:

$$\sum_{a+b=k} \binom{n}{a}\binom{m}{b} = \binom{m+n}{k}$$

The right-hand side is clearly the number of ways to pick k elements from an $m+n$ element set. Now consider splitting the set into two sets, one of size n and the other of size m, and picking a total of k elements between the two sets. This is of course equivalent to picking k elements from the original set, so it is equal to the right-hand side, but it is also the left-hand side of the equation, thus the above identity. Alternately, note that $(x+1)^m(x+1)^n = (x+1)^{m+n}$, thus the coefficient of x^k is equal in both, yielding the above identity.

18. Prove that the nth Catalan number $C_n = \frac{\binom{2n}{n}}{n+1}$ satisfies the recursion $C_n = \sum\limits_{a+b=n-1} C_a C_b$.

We conjure up the following problem, which we will then show obeys both the Catalan formula and its recursion:

In an n x n grid, how many paths between adjacent lattice points, going only to the right and up, start at the bottom-left and end at the top-right, and never cross the main diagonal?

Let $p(n)$ denote the number of such paths. The main diagonal can first be *touched* after $2, 4, \ldots, 2i, \ldots, 2n$ moves, splitting the problem up into identical subproblems of size i and $n-i-1$, so that the desired recursion is satisfied.

Now consider the paths that do cross the diagonal. We draw a bijection between these paths and paths in an $(n-1)$ x $(n+1)$ grid going only to the right and up, and starting in the bottom-left and going to the top right, as follows: immediately after a path crosses the diagonal, reverse the direction of all moves after that in the path. For the reverse mapping, draw a diagonal in the $(n-1)$ x $(n+1)$ grid from the point $(0,0)$ to $(n-1, n-1)$. All paths ending in the top-right must cross this diagonal at least once. Take the first such time that this happens, and reverse the direction of all moves after that move in the path. We now have the desired bijection.

The number of paths in the $(n-1)$ x $(n+1)$ grid is $\binom{2n}{n-1}$, which is also the number of paths that *do* cross the diagonal in our original grid. There are $\binom{2n}{n}$ total paths, regardless of whether they cross the diagonal or not, so subtracting out, we get $\binom{2n}{n} - \binom{2n}{n-1} = \frac{\binom{2n}{n}}{n+1}$, as desired.

19. Prove that the number of paths of length n that can be made using only left or right moves of length 1, starting on the left side of a line segment of length n, is $\binom{n}{\lfloor \frac{n}{2} \rfloor}$.

Consider the number of ways one can return to the origin with a path of length $2n$ (note that there is clearly no way to return to the origin with a path of length $2n+1$); let this value be $g(n)$. Clearly, the first move and the last move are preordained; the first move is invariably a move to the right and the last move a move to the left. Thus, if the path never returns to the origin until the last move, there are $g(n-1)$ options. If the path returns on the second move, then there are $g(n-2)$ options; on the fourth move, $g(1)g(n-3)$, etc. Thus, there are $g(n) = g(n-1)+g(n-2)+g(1)g(n-3)+g(2)g(n-4)+\ldots+g(n-4)g(2)+g(n-2)g(1)+g(n-2)+g(n-1)$ options. This is the recurrence for the Catalan numbers as shown above, so $g(n)$ is the nth Catalan number, which is $\frac{(2n)!}{(n)!(n+1)!}$.

Now, consider $f(n)$ to be the number of paths possible after n moves (without the return to the origin restriction placed on g.) A recurrance relation can be easily found; any path of length $x-1$ that does not end at the origin has two child paths of length x, and any path of length $x-1$ that does end at the origin has one child path of length x. Thus, for even numbers $(2n+2)$, the recurrance is $f(2n+2) = 2f(2n+1)$, as no odd path can end at the origin. For odd numbers $(2n+1)$, the recurrance is $f(2n+1) = 2f(2n) - g(n)$.

The proof will be completed through induction. Assume the solution is $f(n) = \binom{n}{\lfloor \frac{n}{2} \rfloor}$. The base cases are $n = 1 \rightarrow f(n) = 1$ and $n = 2 \rightarrow f(n) = 2$, so the base case is complete. For any odd number $2n+1$, $f(2n+1) = 2f(2n) - g(n)$. Testing this, $2f(2n) - g(n) = 2\binom{2n}{\lfloor \frac{2n}{2} \rfloor} - \frac{1}{n+1}\binom{2n}{n} = 2\binom{2n}{n} - \frac{1}{n+1}\binom{2n}{n} = \binom{2n}{n}(2 - \frac{1}{n+1}) = \binom{2n}{n}(\frac{2n+1}{n+1}) = \binom{2n+1}{n+1} = \binom{2n+1}{n} = f(2n+1)$, so the induction works for all $2n+1$. For any even number $2n+2$, $f(2n+2) = 2f(2n+1)$. Testing this in the induction, $2f(2n+1) = 2\binom{2n+1}{\lfloor \frac{2n+1}{2} \rfloor} = 2\binom{2n+1}{n} = \binom{2n+1}{n} + \binom{2n+1}{n+1} = \binom{2n+2}{n+1} = f(2n+2)$, so the induction works for all $2n+2$. Thus, the induction is complete, and $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ is the number of paths possible.

20. Prove that one can place in order a total of n elements of two different types in $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ ways if one can at no point have placed down more elements of the first type than of the second type.

    If one considers placing items of the second type moves to the right and items of the first type moves to the left, there is a clear bijection to the scenario of the previous problem.