35th United States of America Mathematical Olympiad

Day I   12:30 PM $-$ 5 PM EDT

April 18, 2006

1. Let p be a prime number and let s be an integer with $0 < s < p$. Prove that there exist integers m and n with $0 < m < n < p$ and

$$\left\{ \frac{sm}{p} \right\} < \left\{ \frac{sn}{p} \right\} < \frac{s}{p}$$

   if and only if s is not a divisor of $p - 1$.

   (For x a real number, let $\lfloor x \rfloor$ denote the greatest integer less than or equal to x, and let $\{x\} = x - \lfloor x \rfloor$ denote the fractional part of x.)

2. For a given positive integer k find, in terms of k, the minimum value of N for which there is a set of $2k + 1$ distinct positive integers that has sum greater than N but every subset of size k has sum at most $N/2$.

3. For integral m, let $p(m)$ be the greatest prime divisor of m. By convention, we set $p(\pm 1) = 1$ and $p(0) = \infty$. Find all polynomials f with integer coefficients such that the sequence $\{p(f(n^2)) - 2n\}_{n \geq 0}$ is bounded above. (In particular, this requires $f(n^2) \neq 0$ for $n \geq 0$.)

4. Find all positive integers n such that there are $k \geq 2$ positive rational numbers $a_1, a_2, \ldots, a_k$ satisfying $a_1 + a_2 + \ldots + a_k = a_1 \cdot a_2 \cdots \cdot a_k = n$.

5. A mathematical frog jumps along the number line. The frog starts at 1, and jumps according to the following rule: if the frog is at integer n, then it can jump either to $n+1$ or to $n + 2^{m_n+1}$ where 2^{m_n} is the largest power of 2 that is a factor of n. Show that if $k \geq 2$ is a positive integer and i is a nonnegative integer, then the minimum number of jumps needed to reach $2^i k$ is greater than the minimum number of jumps needed to reach 2^i.

6. Let $ABCD$ be a quadrilateral, and let E and F be points on sides AD and BC, respectively, such that $AE/ED = BF/FC$. Ray FE meets rays BA and CD at S and T, respectively. Prove that the circumcircles of triangles SAE, SBF, TCF, and TDE pass through a common point.