Polynomials

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Introduction

This brief set of notes contains some basic ideas and the most well-known theorems about polynomials. I have not gone into deep details on the concepts that are too easy in practice (eg: long division, synthetic division). I want you to be able to understand these ideas yourself. If you have any problems on anything in this set of notes, let me know (or pester your teacher!)

Definition 1 *A* polynomial of degree n *is a function* $f(x)$ *of the form*

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_2x^2 + a_1x + a_0,$$

where $a_n, a_{n-1}, \ldots, a_1, a_0$ *are constants and* $a_n \neq 0$.

For each i, $0 \leq i \leq n$, a_i is the *coefficient of* x^i, and a_n is the *leading coefficient of* $f(x)$. The a_is may be in $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$. We denote the set of all polynomials in the variable x whose coefficients are in $\mathbb{Z}$ by $\mathbb{Z}[x]$. If $f(x) \in \mathbb{Z}[x]$, then we say that $f(x)$ is a polynomial over $\mathbb{Z}$. Similar definitions hold when $\mathbb{Z}$ is replaced by $\mathbb{Q}$, $\mathbb{R}$ or $\mathbb{C}$. Note that $\mathbb{Z}[x] \subset \mathbb{Q}[x] \subset \mathbb{R}[x] \subset \mathbb{C}[x]$. The variable x is usually taken in $\mathbb{R}$ or $\mathbb{C}$ (if x is taken to be in $\mathbb{C}$, then z may be used instead of x).

$f(x)$ is called *monic* if $a_n = 1$.

We write $\deg(f(x))$ for the degree of $f(x)$.

$f(x)$ is a *constant polynomial* if $n = 0$, a *linear polynomial* if $n = 1$, a *quadratic* if $n = 2$, a *cubic* if $n = 3$, a *quartic* if $n = 4$, and a *quintic* if $n = 5$.

Polynomial Division

If $f(x)$ and $g(x)$ are polynomials and $\deg(g(x)) \leq \deg(f(x))$, then we may divide $g(x)$ into $f(x)$ and get a remainder polynomial $r(x)$ with $\deg(r(x)) < \deg(g(x))$. This works quite like dividing an integer into another integer to get a remainder. More precisely, if $\deg(g(x)) \leq \deg(f(x))$ and $g(x) \neq 0$, then we can find polynomials $q(x)$ and $r(x)$ such that

$$f(x) = q(x)g(x) + r(x), \quad \text{where } \deg(r(x)) < \deg(g(x)).$$

$g(x)$ is the *divisor*, $q(x)$ is the *quotient* and $r(x)$ is the *remainder*.

If $r(x) \equiv 0$, then $f(x) = q(x)g(x)$, and we say that $g(x)$ *divides* $f(x)$, or that $f(x)$ *is* *divisible by* $g(x)$.

We can usually find $q(x)$ and $r(x)$ by *long division*.

Example. Divide $g(x) = x^2 - 1$ into $f(x) = x^5 + 3x^4 + 2x^3 + x^2 + x - 3$.

Using long division,

$$
\begin{array}{rrrrrrr}
 & & & x^3 & +3x^2 & +3x & +4 \\ \cline{2-7}
x^2-1 & \multicolumn{1}{|r}{x^5} & +3x^4 & +2x^3 & +\ x^2 & +\ x & -3 \\
 & x^5 & & -\ x^3 & & & \\ \cline{2-4}
 & & 3x^4 & +3x^3 & +\ x^2 & & \\
 & & 3x^4 & & -3x^2 & & \\ \cline{3-5}
 & & & 3x^3 & +4x^2 & +\ x & \\
 & & & 3x^3 & & -3x & \\ \cline{4-6}
 & & & & 4x^2 & +4x & -3 \\
 & & & & 4x^2 & & -4 \\ \cline{5-7}
 & & & & & 4x & +1
\end{array}
$$

The quotient is $q(x) = x^3 + 3x^2 + 3x + 4$ and the remainder is $r(x) = 4x + 1$. So we have

$$f(x) = (x^3 + 3x^2 + 3x + 4)(x^2 - 1) + (4x + 1).$$

The algorithm goes as follows. We find whatever we need to multiply x^2 (the leading term of $g(x)$) by to get x^5 (the leading term of $f(x)$). That is x^3. We get $x^3(x^2 - 1) = x^5 - x^3$. Put this below $f(x)$, aligning the same powers. Put x^3 on the top, aligning it with the $-x^3$ term of $x^5 - x^3$. Then, subtract $x^5 - x^3$ from $x^5 + 3x^4 + 2x^3$ (the part of $f(x)$ that sits above $x^5 - x^3$) to get $3x^4 + 3x^3$. Then, bring down $+x^2$ from $f(x)$. Repeat the whole procedure until we get a remainder term ($4x + 1$ in this example) which has degree less than $g(x)$ ($= 2$ in this example).

Note that if $f(x)$ misses some powers, then we should put a coefficient of 0 for the missing powers before carrying out the long division. For example, we should write $3x^4 - 2x + 3$ as $3x^4 + 0x^3 + 0x^2 - 2x + 3$. This helps us to preserve the alignment of the terms with the same powers. Also, sometimes we may have to bring down more than one term of $f(x)$ in one step.

Now try exercises 1 and 2.

The Remainder Theorem

Now, when we want to divide $g(x) = x - a$ into $f(x)$, then the remainder must have degree at most 1, ie: it is a constant. We have the following very useful theorem.

Theorem 1 (The Remainder Theorem) *Let* $f(x) \in \mathbb{C}[x]$ *and* $\deg(f(x)) = n$. *Then we have* $f(x) = q(x)(x-a) + r$, *where* $\deg(q(x)) = n-1$ *and* $r = f(a)$.

Definition 2 *If* $f(x)$ *is a function (not necessarily a polynomial in this definition) and* $f(a) = 0$, *then we say that* a *is a* zero, *or a* root *of* $f(x)$.

Corollary 2 *Let* $f(x) \in \mathbb{C}[x]$. *Then* a *is a zero of* $f(x)$ *if and only if* $x-a$ *divides* $f(x)$.

Proof. If a is a zero of $f(x)$, then by the Remainder Theorem, $f(x) = q(x)(x-a) + r = q(x)(x-a)$, since $r = f(a) = 0$. So $x-a$ divides $f(x)$.

Conversely, If $x-a$ divides $f(x)$, then $f(x) = q(x)(x-a)$ for some polynomial $q(x)$. But then $f(a) = 0$. $\square$

Corollary 2 itself may also be called the Remainder Theorem. It is a very useful and simple result that can be extremely useful in many Olympiad problems. This is because if we know that a is a zero of a polynomial $f(x)$, then we may immediately conclude that $f(x) = q(x)(x-a)$ for some polynomial $q(x)$.

The Fundamental Theorem of Algebra and Synthetic Division

What do we know about the zeros of a polynomial $f(x)$? The following theorem is the most important result.

Theorem 3 (The Fundamental Theorem of Algebra) *Every polynomial* $f(x) \in \mathbb{C}[x]$ *has a zero in* $\mathbb{C}$.

Theorem 3 is difficult to prove. There is more than one way to prove Theorem 3, but every known proof uses advanced techniques which we will not go into here.

By an easy induction argument on the degree of $f(x)$, and using the Remainder Theorem, we have:

Corollary 4 *If* $f(x) \in \mathbb{C}[x]$ *and* $\deg(f(x)) = n$, *then* $f(x)$ *may be factorized into the form*

$$f(x) = a(x-\alpha_1)(x-\alpha_2)\cdots(x-\alpha_n),$$

where a *is the leading coefficient of* $f(x)$, *and* $\alpha_1, \ldots, \alpha_n \in \mathbb{C}$ *are the zeros of* $f(x)$, *not necessarily distinct.*

So, Corollary 4 tells us the following.

(a) If $\deg(f(x)) = n$, then $f(x)$ can have at most n distinct zeros in $\mathbb{C}$.

(b) If $f(x)$ is a polynomial and has infinitely many zeros, then $f(x) \equiv 0$.

Corollary 4 also tells us that any polynomial can be factorized into a product of linear polynomials in $\mathbb{C}[x]$. But if $f(x) \in \mathbb{R}[x]$, and we want to only factorize $f(x)$ into a product of polynomials, each in $\mathbb{R}[x]$, then, we may have factors which are not linear. For example, we know that, if $f(x)$ is a quadratic over $\mathbb{R}$, then we may not even be able to factorize $f(x)$ into two linear factors in $\mathbb{R}[x]$ if the discriminant of $f(x)$ is less than 0.

In particular, factorizing a polynomial $f(x) \in \mathbb{Z}[x]$ into a product of polynomials, each over $\mathbb{Z}, \mathbb{R}$ or $\mathbb{C}$, as far as possible (ie: the factors may not be factorized further; these factors are called *irreducible*), is something very standard, and very important to know.

A very useful method in doing so is *synthetic division*.

Example. Factorize $f(x) = x^3 - 5x^2 - 2x + 24$ into a product of polynomials over $\mathbb{Z}$.

We do the following. If $x - a$ is a factor of $f(x)$, then, by considering the constant terms, we see that a divides 24. We make a guess that $a = 3$ is a zero, and do the following.

$$\begin{array}{r|rrrr} 3 & 1 & -5 & -2 & 24 \\ & & 3 & -6 & -24 \\ \hline & 1 & -2 & -8 & 0 \end{array}$$

The algorithm goes as follows. Place the 3 and the coefficients of $f(x)$ along the top row as shown. Then, bring the 1 down to the third row. Then, multiply the 3 by the 1 in the bottom to get 3, and place this 3 under the -5. Then, add the -5 and 3 to get -2 and place this under the 3. Then multiply -2 by 3 to get -6, and repeat the previous procedure until we get to the bottom right.

If we used a in place of 3, then the number in the bottom right is actually $f(a)$. We see that when we use 3, we get $f(3) = 0$. So, this means that 3 is a zero of $f(x)$, and $x - 3$ is a factor of $f(x)$.

Furthermore, the numbers 1, -2, -8 that we have along the bottom row are the coefficients of the quotient polynomial. So, we have

$$f(x) = (x - 3)(x^2 - 2x - 8).$$

We can factorize the quotient easily. The full factorization is $f(x) = (x-3)(x-4)(x+2)$.

Note that using 4 or -2 instead of 3 at the start would work as well, but for example, if we used 1, then it would not have worked, since we would have $f(1) \neq 0$. It is a trial and error procedure.

Example. Factorize $f(x) = x^3 - 2x^2 + 5x + 26$ into a product of polynomials (a) over $\mathbb{Z}$, and (b) over $\mathbb{C}$.

We try synthetic division several times, using factors of 26 as possible zeros each time, until we find one. We have

$$\begin{array}{r|rrrr} 1 & 1 & -2 & 5 & 26 \\ & & 1 & -1 & 4 \\ \hline & 1 & -1 & 4 & 30 \end{array} \qquad \begin{array}{r|rrrr} -1 & 1 & -2 & 5 & 26 \\ & & -1 & 3 & -8 \\ \hline & 1 & -3 & 8 & 18 \end{array}$$

$$\begin{array}{r|rrrr} 2 & 1 & -2 & 5 & 26 \\ & & 2 & 0 & 10 \\ \hline & 1 & 0 & 5 & 36 \end{array} \qquad \begin{array}{r|rrrr} -2 & 1 & -2 & 5 & 26 \\ & & -2 & 8 & -26 \\ \hline & 1 & -4 & 13 & 0 \end{array}$$

We see that 1, -1 and 2 are not zeros of $f(x)$, but -2 is. We have

$$f(x) = (x+2)(x^2 - 4x + 13).$$

Now, we want to try and factorize $x^2 - 4x + 13$. By the quadratic formula, the solutions of $x^2 - 4x + 13 = 0$ are

$$x = \frac{4 \pm \sqrt{16 - 4 \cdot 13}}{2} = 2 \pm 3i.$$

So, $x^2 - 4x + 13 = (x - (2+3i))(x - (2-3i))$, and $x^2 - 4x + 13$ does not factorize over $\mathbb{Z}$. So the answers are:

(a) $f(x) = (x+2)(x^2 - 4x + 13)$, and

(b) $f(x) = (x+2)(x - (2+3i))(x - (2-3i))$.

Now, try exercises 3 and 4.

Relations between the zeros of a polynomial

If α_1, α_2 are the solutions of $x^2 + a_1 x + a_0 = 0$, then

$$x^2 + a_1 x + a_0 = (x - \alpha_1)(x - \alpha_2) = x^2 - (\alpha_1 + \alpha_2)x + \alpha_1\alpha_2,$$

so $a_1 = -(\alpha_1 + \alpha_2)$ and $a_0 = \alpha_1\alpha_2$.

Similarly, if $\alpha_1, \alpha_2, \alpha_3$ are the solutions of $x^3 + a_2 x^2 + a_1 x + a_0 = 0$, then

$$\begin{aligned} x^3 + a_2 x^2 + a_1 x + a_0 &= (x - \alpha_1)(x - \alpha_2)(x - \alpha_3) \\ &= x^3 - (\alpha_1 + \alpha_2 + \alpha_3)x^2 + (\alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1)x - \alpha_1\alpha_2\alpha_3, \end{aligned}$$

so $a_2 = -(\alpha_1 + \alpha_2 + \alpha_3)$, $a_1 = \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1$, and $a_0 = -\alpha_1\alpha_2\alpha_3$.

In general, if the solutions of $x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 = 0$ are $\alpha_1, \ldots, \alpha_n$, then:

$$\begin{aligned}
a_{n-1} &= -\sum_i \alpha_i, \\
a_{n-2} &= \sum_{i<j} \alpha_i\alpha_j, \\
a_{n-3} &= -\sum_{i<j<k} \alpha_i\alpha_j\alpha_k, \\
&\vdots \\
a_0 &= (-1)^n \alpha_1 \cdots \alpha_n.
\end{aligned}$$

To get a_{n-i}, we sum the products of the i-tuples of the α_js, and multiply by $(-1)^i$.

Example. If α, β, γ are the zeros of $f(x) = 2x^3 + x^2 - 5x - 3$, find a polynomial whose zeros are $\alpha^2, \beta^2, \gamma^2$.

If $f(x) = 0$, then $x^3 + \frac{1}{2}x^2 - \frac{5}{2}x - \frac{3}{2} = 0$. So we have $-(\alpha+\beta+\gamma) = \frac{1}{2}$, $\alpha\beta + \beta\gamma + \gamma\alpha = -\frac{5}{2}$ and $-\alpha\beta\gamma = -\frac{3}{2}$. So

$$\begin{aligned}
\alpha^2 + \beta^2 + \gamma^2 &= (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\beta\gamma+\gamma\alpha) = \Big(-\frac{1}{2}\Big)^2 - 2\Big(-\frac{5}{2}\Big) = \frac{21}{4}, \\
\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 &= (\alpha\beta+\beta\gamma+\gamma\alpha)^2 - 2\alpha\beta\gamma(\alpha+\beta+\gamma) \\
&= \Big(-\frac{5}{2}\Big)^2 - 2\cdot\frac{3}{2}\Big(-\frac{1}{2}\Big) = \frac{31}{4}, \\
\alpha^2\beta^2\gamma^2 &= \Big(\frac{3}{2}\Big)^2 = \frac{9}{4}.
\end{aligned}$$

So a possible polynomial is $g(x) = 4x^3 - 21x^2 + 31x - 9$.

Now try exercise 5.

Some algebraic identities

The following identities are useful when we deal with polynomials, and also in other areas as well.

(a) $x^n - y^n = (x-y)(x^{n-1} + x^{n-2}y + \cdots + xy^{n-2} + y^{n-1})$,

(b) $x^n + y^n = (x+y)(x^{n-1} - x^{n-2}y + \cdots - xy^{n-2} + y^{n-1})$ if n is odd,

(c) $x^3 + y^3 + z^3 - 3xyz = (x+y+z)(x^2+y^2+z^2-yz-zx-xy)$
$\qquad = \frac{1}{2}(x+y+z)((y-z)^2 + (z-x)^2 + (x-y)^2)$.

So, for example, (a) gives

$$x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + \cdots + x + 1).$$

Roots of Unity

The complex number $Re^{i\theta}$ has modulus R and argument θ. It has real part $R\cos\theta$ and imaginary part $R\sin\theta$. So, we have $Re^{i\theta} = R\cos\theta + iR\sin\theta$.

If $\omega = e^{2\pi i/n}$, then the polynomial $f(x) = x^n - 1$ has (complex) zeros $1, \omega, \omega^2, \ldots, \omega^{n-1}$. These are the n^{th} *roots of unity.* So over $\mathbb{C}$, $f(x)$ factorizes into

$$f(x) = (x-1)(x-\omega)(x-\omega^2)\cdots(x-\omega^{n-1}).$$

Since $x^n - 1 = (x-1)(x^{n-1} + x^{n-2} + \cdots + x + 1)$, it follows that

$$x^{n-1} + x^{n-2} + \cdots + x + 1 = (x-\omega)(x-\omega^2)\cdots(x-\omega^{n-1}),$$

that is, the zeros of $g(x) = x^{n-1} + x^{n-2} + \cdots + x + 1$ are $\omega, \omega^2, \ldots, \omega^{n-1}$.

On the Argand diagram, the complex numbers 1, ω, $\omega^2, \ldots, \omega^{n-1}$ are the vertices of a regular n-gon. We have the relation $1 + \omega + \omega^2 + \cdots + \omega^{n-1} = 0$. Also, note that $\omega^n = 1$, and hence $\omega^j = \omega^k \iff j - k \equiv 0(\bmod\ n)$.

Exercises and Problems

Standard exercises.

1. Use long division to divide $f(x) = x^5 + 2x^3 - 3x^2 - 5$ by $g(x) = x^3 + 3x$. Hence write $f(x)$ in the form $q(x)g(x) + r(x)$, where $\deg(r(x)) < \deg(g(x))$.

2. Repeat exercise 1 with $f(x) = x^4 - 3x^3 + 2$ and $g(x) = x^2 + 3x - 6$. What can you say?

3. Factorize $f(x) = x^4 + 6x^3 - 12x^2 - 88x - 96$ into a product of polynomials over $\mathbb{Z}$.

4. Factorize $f(x) = x^4 - 12x^3 + x^2 + 18x - 360$ into a product of polynomials (a) over $\mathbb{Z}$ and (b) over $\mathbb{C}$.

5. Suppose that $f(x) = x^3 + ax^2 + cx + d$ has zeros α, β and γ.

   (a) Find a polynomial whose zeros are α^2, β^2 and γ^2.

   (b) Find a polynomial whose zeros are α^3, β^3 and γ^3.

Problems.

Hint. Look through the set of notes carefully. At least one of the ideas discussed will be relevant for each problem.

There are some easy problems here and some of medium difficulty.

6. Find all polynomials $f(x)$ such that $f(1) = 1$ and $f(x^2 + 2004) = (f(x))^2 + 2004$ for all $x \in \mathbb{R}$.

7. Let α, β, γ and δ be the zeros of $f(x) = x^4 + ax^3 + bx^2 + cx + d$. Prove that if $\alpha\beta = \gamma\delta$, then $a^2 d = c^2$.

8. Suppose that $f(x) = x^2 + ax + 1$ has zeros α, β, and $g(x) = x^2 + bx + 1$ has zeros γ, δ. Prove that $(\alpha - \gamma)(\beta - \gamma)(\alpha + \delta)(\beta + \delta) = b^2 - a^2$.

9. Let $f(x) \in \mathbb{Z}[x]$. Suppose that $a, b, c, d \in \mathbb{Z}$ are distinct and we have $f(a) = f(b) = f(c) = f(d) = -4$. If $r \in \mathbb{Z}$ is a zero of $f(x)$, prove that $4r = a + b + c + d$.

10. Suppose that $f(x) \in \mathbb{Z}[x]$ satisfies $f(21) = 17$, $f(32) = -247$ and $f(37) = 33$. Prove that if $f(n) = n + 51$ for some $n \in \mathbb{Z}$, then $n = 26$.

11. Suppose that $f(x) \in \mathbb{Z}[x]$, and $f(2)$ is divisible by 5, and $f(5)$ is divisible by 2. Prove that $f(7)$ is divisible by 10.

12. Suppose that $f(x)$ is a polynomial of degree n such that $f(k) = \dfrac{1}{k}$ for $k = 1, 2, 4, 8, \ldots, 2^n$. Find the value of $f(0)$.

13. Suppose that $f(x) \in \mathbb{C}[x]$ and $\deg(f(x)) < k$. Let $\omega = e^{2\pi i/k}$. Prove that
$$\frac{1}{k}\sum_{i=0}^{k-1} f(\omega^i) = f(0).$$

14. Find all polynomials $f : \mathbb{R} \to \mathbb{R}$ such that $2(1 + f(x)) = f(x-1) + f(x+1)$ for all $x \in \mathbb{R}$.

15. Find all polynomials $f : \mathbb{R} \to \mathbb{R}$ such that $(x - 16)f(2x) = 16(x-1)f(x)$ for all $x \in \mathbb{R}$.

16. Find all polynomials $f : \mathbb{C} \to \mathbb{C}$ such that $f(x)f(x+1) = f(x^2)$ for all $x \in \mathbb{C}$.

17. Let $f(x) = x^{k-1} + x^{k-2} + \cdots + x + 1$. Find the remainder when $f(x^k)$ is divided by $f(x)$.

18. Let $f(x) = x^4 - 3x^3 + 5x^2 - 9x$. Find all pairs of integers (a, b), where $a < b$, such that $f(a) = f(b)$.

19. Find all pairs (n, k), where $n \in \mathbb{Z}$, $n > 0$ and $k \in \mathbb{R}$, such that $f(x) = (x+1)^n - k$ is divisible by $g(x) = 2x^2 + 2x + 1$.