Graph Theory 1

Adam Hesterberg
Based on Paul Seymour's notes and work

Black MOP 2011, June 16th

1 Definitions

1.1 Definitions you should already know

Graph (= multigraph), simple graph, vertex, edge, adjacent, loop, parallel edge, clique/complete graph, degree, subgraph, connected, component (= connected component), cut-vertex/-edge/-set, tree, forest, spanning tree, path, cycle, Eulerian cycle, Hamiltonian cycle, bipartite graph, stable set (= independent set), matching, planar graph, k-regular graph, digraph

1.2 Definitions you might not know

Definition 1.1. A graph G is k-connected iff $|V(G)| \geq k+1$ and for every $X \subset V(G)$ with $|X| < k$, $G \setminus X$ is connected.

Definition 1.2. The line graph $L(G)$ of a graph G is the graph with $V(L(G)) = E(G)$, with an edge for for every pair of incidences of two edges of G on the same vertex of G.

Definition 1.3. A graph G is k-edge-connected iff its line graph is k-connected. Alternately, G is k-edge-connected iff for every $X \subset E(G)$ with $|X| < k$, $G \setminus X$ is connected.

Definition 1.4. A *separation* of G is a pair (A, B) of subsets of $V(G)$ with $A \cup B = V(G)$, such that there is no edge between $A \setminus B$ and $B \setminus A$. Its *order* is $|A \cap B|$.

2 Matchings

1. (K onig's Theorem) Let G be bipartite, and $k \geq 0$ an integer. Then G has a matching of size at least k unlesss there exists $X \subset V(G)$ with $|X| < k$ such that X meets every edge of G.

2. Let G be a loopless graph in which every vertex has positive degree. Let X be the largest matching in G, and let Y be the smallest set of edges of G whose union contains $V(G)$. Show that $|X| + |Y| = |V(G)$.

3. (Tutte's Theorem) Let $odd(X)$ be the number of components of X with an odd number of vertices. Then G has a perfect matching unlesss there exists $X \subset V(G)$ with $odd(G \setminus X) > |X|$.

4. Show that every 2-edge-connected cubic graph has a perfect matching.

5. Let G be a d-regular bipartite graph. Show that $E(G)$ can be partitioned into perfect matchings.

3 Other Problems

Definition 3.1. A *circulation* on a digraph is an assignment of real weights to its edges such that the difference between every vertex's inedges' and outedges' weights is 0. An *s-t flow* is similar, but vertices s and t are allowed to have that difference nonzero; that difference is called the flow's *total value*.

1. Let s and t be vertices of a digraph G, and let φ be a real-valued s-t flow. Show that there's an integer-valued s-t flow with total value at least that of φ such that $\forall e \in E(G)$, $|\psi(e) - \varphi(e)| < 1$.

2. (Menger's Theorem) Let $Q, R \subset V(G)$, and let $k \geq 0$. Then there are k pairwise vertex-disjoint paths from Q to R unless there's a separation (A, B) of G of order $< K$ with $Q \subset A$ and $R \subset B$.

3. (Dirac/Posa) If G is simple, $|V(G)| \geq 3$, and for all $\{u, v\} \subset V(G)$, either u is adjacent to v or $\deg(u) + \deg(v) \geq n$, then G has a Hamiltonian cycle.

4. (Erdős) If G is a graph with no stable set of size t, then there's a graph H with $V(G) = V(H)$ and at most $t - 1$ components, each of which is a complete graph, such that $\forall v, \deg_H(v) \leq \deg_G(v)$.

5. (Bipartite Ramsey) Let $s, t \geq 1$ and let G be a simple graph with a bipartition (A, B) such that $|A| = |B| \geq \binom{s+t}{s} - 1$. Show that G has a $K_{s,s}$ or $\overline{K_{t+t}}$ subgraph.