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Combinatorics

1. An n -string is a string of digits formed by writing the numbers $1, 2, \dots, n$ in some order (in base 10). For example, one possible 10-string is

35728910461.

What is the smallest n greater than 1 such that there exists a palindromic n -string?

2. We have a polyhedron such that an ant can walk from one vertex to another, traveling only along edges, and traveling every edge exactly once. What is the smallest possible total number of vertices, edges, and faces of this polyhedron?
3. The member of a distinguished committee were choosing a president, and each member gave one vote to one of the 27 candidate. For each candidate, the exact percentage of votes the candidate got was smaller by at least 1 than the number of votes for that candidate. What is the smallest possible number of member of the committee?
4. A class room of a 5×5 array of desks, to be filled by anywhere from 0 to 25 students, inclusive. No student will sit at a desk unless either all other desks in its row or all others in its column are filled (or both). Considering only the set of desks that are occupied (and which student sits at each desk), how many possible arrangements are there?
5. There are 51 senators in a senate. The senate needs to be divided into n committees so that each senator is on one committee. Each senator hates exactly three other senators. (If senator A hates senator B, then senator B does *not* necessarily hate senator A.) Find the smallest n such that it is always possible to arrange the committees so that no senator hates another senator on his or her committee.

Solution: Assume that there are 7 senators A_1, \dots, A_7 such that each A_i hates A_{i+1} , A_{i+2} , and A_{i+3} (where indices are taken modulo 7). In this situation, for any different A_i, A_j , either A_i hates A_j or vice versa. The senators A_1, \dots, A_7 must be placed on a different committee. Thus, $n \geq 7$.

In order to show that $n \leq 7$, we will prove the following stronger statement by induction on $k \geq 1$: for k senators, each of whom hates *at most* 3 others, it is possible to arrange the senators into 7 committees so that no senator hates another senator on his or her committee.

For the base case $k = 1$, note that we can have 7 committees, 6 of which are empty and 1 of which contains the sole senator.

Now assume the claim is true for all $k \leq m - 1$. Suppose we are given m senators, each of whom hates at most 3 others. If each of those m senators is hated by more than 3 others, then the total number of acts of hating must be greater than $3m$, but this is not possible since each senator hates at most 3 others. Therefore, there must be at least one senator A who is hated by at most 3 others. By the induction hypothesis, we can split the $m - 1$ other senators into 7 committees satisfying the property that no senator hates another senator on the same committee. By the **Pigeonhole Principle**, one of those committees contains neither a person whom A hates nor a person who hates A . We can therefore place A in that committee. The induction is complete.

6. Given that 2^{2004} is a 604-digit number with leading digit 1. Determine the number of elements in the set $\{2^0, 2^1, 2^2, \dots, 2^{2003}\}$ with leading digit 4.

7. Let S be a set with 2002 elements, and let N be an integer with $0 \leq N \leq 2^{2002}$. Prove that it is possible to color every subset of S either blue or red so that the following conditions hold:
- (a) the union of any two red subsets is red;
 - (b) the union of any two blue subsets is blue;
 - (c) there are exactly N red subsets.

First Solution: We prove that this can be done for any n -element set, where n is a positive integer, $S_n = \{1, 2, \dots, n\}$ and for any positive integer N with $0 \leq N \leq 2^n$.

We induct on n . The base case $n = 1$ is trivial. Assume that we can color the subsets of $S_n = \{1, 2, \dots, n\}$ in the desired manner, for any integer N_n with $0 \leq N_n \leq 2^n$. We show that there is a desired coloring for $S_{n+1} = \{1, 2, \dots, n, n+1\}$ and any integer N_{n+1} with $0 \leq N_{n+1} \leq 2^{n+1}$. We consider the following cases.

- (i) $0 \leq N_{n+1} \leq 2^n$. Applying the induction hypothesis to S_n and $N_n = N_{n+1}$, we get a coloring of all subsets of S_n satisfying conditions (a), (b), (c). All uncolored subsets of S_{n+1} contain the element $n+1$; we color all of them blue. It is not hard to see that this coloring of all the subsets of S_{n+1} satisfies conditions (a), (b), (c).
- (ii) $2^n + 1 \leq N_{n+1} \leq 2^{n+1}$. By case (i), we know that there exists a coloring of the subsets of S_{n+1} satisfying (a) and (b) and having $2^{n+1} - N_{n+1}$ red subsets. Then, we switch the color of each subset: if it is blue now, we recolor it red; if it is red now, we recolor it blue. It is not hard to see that this coloring of all the subsets of S_{n+1} satisfies conditions (a), (b), (c).

Thus our induction is complete.

Second Solution: If $N = 0$, we color every subset blue; if $N = 2^{2002}$, we color every subset red. Now suppose neither of these holds. We may assume that $S = \{0, 1, 2, \dots, 2001\}$. Write N in binary representation:

$$N = 2^{a_1} + 2^{a_2} + \dots + 2^{a_k},$$

where the a_i are all distinct; then each a_i is an element of S . Color each a_i red, and color all the other elements of S blue. Now declare each nonempty subset of S to be the color of its largest element, and color the empty subset blue. If T, U are any two nonempty subsets of S , then the largest element of $T \cup U$ equals the largest element of T or the largest element of U , and if T is empty, then $T \cup U = U$. It readily follows that (a) and (b) are satisfied. To verify (c), notice that, for each i , there are 2^{a_i} subsets of S whose largest element is a_i (obtained by taking a_i in combination with any of the elements $0, 1, \dots, a_i - 1$). If we sum over all i , each red subset is counted exactly once, and we get $2^{a_1} + 2^{a_2} + \dots + 2^{a_k} = N$ red subsets.

8. An n -term sequence (x_1, x_2, \dots, x_n) in which each term is either 0 or 1 is called a *binary sequence of length n* . Let a_n be the number of binary sequences of length n containing no three consecutive terms equal to 0, 1, 0 in that order. Let b_n be the number of binary sequences of length n that contain no four consecutive terms equal to 0, 0, 1, 1 or 1, 1, 0, 0 in that order. Prove that $b_{n+1} = 2a_n$ for all positive integers n .

Solution: We refer to the binary sequences counted by (a_n) and (b_n) as “type A” and “type B”, respectively. For each binary sequence (x_1, x_2, \dots, x_n) there is a corresponding binary sequence

(y_0, y_1, \dots, y_n) obtained by setting $y_0 = 0$ and

$$y_i = x_1 + x_2 + \dots + x_i \pmod{2}, \quad i = 1, 2, \dots, n. \quad (*)$$

(Addition mod 2 is defined as follows: $0 + 0 = 1 + 1 = 0$ and $0 + 1 = 1 + 0 = 1$.) Then

$$x_i = y_i + y_{i-1} \pmod{2}, \quad i = 1, 2, \dots, n,$$

and it is easily seen that $(*)$ provides a one-to-one correspondence between the set of all binary sequences of length n and the set of binary sequences of length $n + 1$ in which the first term is 0. Moreover, the binary sequence (x_1, x_2, \dots, x_n) has three consecutive terms equal to 0, 1, 0 in that order if and only if the corresponding sequence (y_0, y_1, \dots, y_n) has four consecutive terms equal to 0, 0, 1, 1 or 1, 1, 0, 0 in that order, so the first is of type A if and only if the second is of type B. The set of type B sequences of length $n + 1$ in which the first term is 0 is exactly half the total number of such sequences, as can be seen by means of the mapping in which 0's and 1's are interchanged.

9. Some checkers placed on an $n \times n$ checkerboard satisfy the following conditions:

- (a) every square that does not contain a checker shares a side with one that does;
- (b) given any pair of squares that contain checkers, there is a sequence of squares containing checkers, starting and ending with the given squares, such that every two consecutive squares of the sequence share a side.

Prove that at least $(n^2 - 2)/3$ checkers have been placed on the board.

Solution: It suffices to show that if m checkers are placed so as to satisfy condition (b), then the number of squares they either cover or are adjacent to is at most $3m + 2$. But this is easily seen by induction: it is obvious for $m = 1$, and if m checkers are so placed, some checker can be removed so that the remaining checkers still satisfy (b); they cover at most $3m - 1$ squares, and the new checker allows us to count at most 3 new squares (since the square it occupies was already counted, and one of its neighbors is occupied).

Note. The exact number of checkers required is known for $m \times n$ checkerboards with m small, but only partial results are known in the general case. Contact the authors for more information.

10. Find the smallest positive integer n such that if n unit squares of a 1000×1000 unit-square board are colored, then there will exist three colored unit squares whose centers form a right triangle with legs parallel to the edges of the board.

First Solution: We show that $n = 1999$. Indeed, $n \geq 1999$ because we can color 1998 squares without producing a right triangle: color every square in the first row and the first column, except for the one square at their intersection.

Now assume that some squares have been colored so that no desired right triangle is formed. Call a row or column *heavy* if it contains more than one colored square, and *light* otherwise. Our assumption then states that no heavy row and heavy column intersect in a colored square.

If there are no heavy rows, then each row contains at most one colored square, so there are at most 1000 colored squares. We reach the same conclusion, if there are no heavy columns. If there is a heavy row and a heavy column, then by the initial observation, each colored square in the heavy row

or column must lie in a light column or row, and no two can lie in the same light column or row. Thus the number of colored squares is at most the number of light rows and columns, which is at most $2 \times (1000 - 1) = 1998$.

We conclude that in fact 1999 colored squares is the minimum needed to force the existence of a right triangle of the type described.

Second Solution: Assume that 1999 squares are colored and the required right triangle does not exist. By the Pigeonhole Principle, there is a row with $a_1 \geq 2$ colored squares. Interchange rows to make this the first row. Interchange columns so that the first a_1 squares in the first row are all colored. Then the first a_1 columns have no colored squares other than the ones in the first row, for otherwise we would have a right triangle.

Observe that a_1 cannot equal 1000, for then we would have no place for the remaining 999 colored squares. Also, a_1 cannot equal 999, for then the remaining 1000 colored squares must all be in the last column and we would have a right triangle, a contradiction. Hence $1000 - a_1 \geq 2$.

Throw away for now the first a_1 columns and the first row and consider the remaining $(1000 - a_1) \times 999$ rectangular grid G_2 . It contains $1999 - a_1 \geq 999 + 2 = 1001$ colored squares. Therefore, there is a row in G_2 with at least $a_2 \geq 2$ colored squares. Interchange rows and then columns so that the first a_2 squares of the first row are colored. Then the first a_2 columns have no colored squares other than the ones in the first row.

Observe that $a_1 + a_2$ cannot equal to 1000, for then we would have no place to put the remaining 999 colored squares. Also, $a_1 + a_2$ cannot equal 999, for then the remaining 1000 colored squares must all be in the last column and we would have a right triangle, a contradiction. Hence $1000 - (a_1 + a_2) \geq 2$.

The above process can be continued, but $1000 - (a_1 + a_2 + \cdots) \geq 2$ contradicts the fact that $a_1, a_2, \cdots \geq 2$. Thus, with 1999 colored squares there must be a right triangle. As in the first solution, we can find a way to arrange 1998 colored squares without obtaining a right triangle of the type described.

Third Solution: We prove a more general statement:

Lemma Let $n_{k,\ell}$ be the smallest positive integer such that if $n_{k,\ell}$ squares of a $k \times \ell$ ($k, \ell \geq 2$) board are colored, then there necessarily exist a right triangle of the type described. Define $t = t_{k,\ell} = k + \ell$ to be the total dimension of the board. Then $n_{k,\ell} = t - 1$.

Proof: As in the first solution, we can color $t - 2$ squares without producing a right triangle: fill every square in the first row and the first column, except for the one square at their intersection. Hence $n_{k,\ell} \geq t - 1$.

Now we prove by induction on t that $n_{k,\ell} = t - 1$.

For the base case $t = 4$, we have $k = \ell = 2$ and it is easy to see that $n_{2,2} = 3$.

Assume that the claim is true for $t = m$, $m \geq 4$. For $t = m + 1$, we claim that $n_{k,\ell} = m$ when $k + \ell = m + 1$, $k, \ell \geq 2$. For the sake of contradiction, suppose that there is a $k \times \ell$ board with m colored squares and no right triangles. Without loss of generality, suppose that $k \geq \ell$. Then $k > 3$. There is a row with at most 1 colored square because otherwise we will have at least $2k \geq t > m$ colored squares. Cross out that row to obtain a $(k - 1) \times \ell$ board with $t = m$, and $k - 1, \ell \geq 2$, and at least $\geq m - 1$ colored squares. By the induction hypothesis, there is right triangle, contradicting our assumption. Therefore our assumption is wrong and we conclude that $n_{k,\ell} = m = t - 1$. Our induction is complete, and this finishes our proof. ■

11. Every unit square of a 2004×2004 grid is to be filled with one of the letters A, B, C, D , so that every 2×2 subsquare contains exactly one of each letter. In how many ways can this be done?
12. Let $n \neq 0$. For every sequence of integers

$$a = a_0, a_1, a_2, \dots, a_n$$

satisfying $0 \leq a_i \leq i$, for $i = 0, \dots, n$, define another sequence

$$t(a) = t(a)_0, t(a)_1, t(a)_2, \dots, t(a)_n$$

by setting $t(a)_i$ to be the number of terms in the sequence a that precede the term a_i and are different from a_i . Show that, starting from any sequence a as above, fewer than n applications of the transformation t lead to a sequence b such that $t(b) = b$.

First Solution: Note first that the transformed sequence $t(a)$ also satisfies the inequalities $0 \leq t(a)_i \leq i$, for $i = 0, \dots, n$. Call any integer sequence that satisfies these inequalities an *index bounded sequence*.

We prove now that that $a_i \leq t(a)_i$, for $i = 0, \dots, n$. Indeed, this is clear if $a_i = 0$. Otherwise, let $x = a_i > 0$ and $y = t(a)_i$. None of the first x consecutive terms a_0, a_1, \dots, a_{x-1} is greater than $x - 1$, so they are all different from x and precede x (see the diagram below). Thus $y \geq x$, that is, $a_i \leq t(a)_i$, for $i = 0, \dots, n$.

	0	1	...	$x - 1$...	i
a	a_0	a_1	...	a_{x-1}	...	x
$t(a)$	$t(a)_0$	$t(a)_1$...	$t(a)_{x-1}$...	y

This already shows that the sequences stabilize after finitely many applications of the transformation t , because the value of the index i term in index bounded sequences cannot exceed i . Next we prove that if $a_i = t(a)_i$, for some $i = 0, \dots, n$, then no further applications of t will ever change the index i term. We consider two cases.

- In this case, we assume that $a_i = t(a)_i = 0$. This means that no term on the left of a_i is different from 0, that is, they are all 0. Therefore the first i terms in $t(a)$ will also be 0 and this repeats (see the diagram below).

	0	1	...	i
a	0	0	...	0
$t(a)$	0	0	...	0

- In this case, we assume that $a_i = t(a)_i = x > 0$. The first x terms are all different from x . Because $t(a)_i = x$, the terms $a_x, a_{x+1}, \dots, a_{i-1}$ must then all be equal to x . Consequently, $t(a)_j = x$ for $j = x, \dots, i - 1$ and further applications of t cannot change the index i term (see the diagram below).

	0	1	...	$x - 1$	x	$x + 1$...	i
a	a_0	a_1	...	a_{x-1}	x	x	...	x
$t(a)$	$t(a)_0$	$t(a)_1$...	$t(a)_{x-1}$	x	x	...	x

For $0 \leq i \leq n$, the index i entry satisfies the following properties: (i) it takes integer values; (ii) it is bounded above by i ; (iii) its value does not decrease under transformation t ; and (iv) once it stabilizes under transformation t , it never changes again. This shows that no more than n applications of t lead to a sequence that is stable under the transformation t .

Finally, we need to show that no more than $n-1$ applications of t is needed to obtain a fixed sequence from an initial $n+1$ -term index bounded sequence $a = (a_0, a_1, \dots, a_n)$. We induct on n .

For $n = 1$, the two possible index bounded sequences $(a_0, a_1) = (0, 0)$ and $(a_0, a_1) = (0, 1)$ are already fixed by t so we need zero applications of t .

Assume that any index bounded sequence (a_0, a_1, \dots, a_n) reach a fixed sequence after no more than $n-1$ applications of t . Consider an index bounded sequence $a = (a_0, a_1, \dots, a_{n+1})$. It suffices to show that a will be stabilized in no more than n applications of t . We approach indirectly by assuming on the contrary that $n+1$ applications of transformations are needed. This can happen only if $a_{n+1} = 0$ and each application of t increased the index $n+1$ term by exactly 1. Under transformation t , the resulting value of index i term will not be effected by index j term for $i < j$. Hence by the induction hypothesis, the subsequence $a' = (a_0, a_1, \dots, a_n)$ will be stabilized in no more than $n-1$ applications of t . Because index n term is stabilized at value $x \leq n$ after no more than $\min\{x, n-1\}$ applications of t and index $n+1$ term obtains value x after exactly x applications of t under our current assumptions. We conclude that the index $n+1$ term would become equal to the index n term after no more than $n-1$ applications of t . However, once two consecutive terms in a sequence are equal they stay equal and stabilize together. Because the index n term needs no more than $n-1$ transformations to be stabilized, a can be stabilized in no more than $n-1$ applications of t , which contradicts our assumption of $n+1$ applications needed. Thus our assumption was wrong and we need at most n applications of transformation t to stabilize an $(n+1)$ -term index bounded sequence. This completes our inductive proof.

Note: There are two notable variations proving the last step.

- *First variation* The key case to rule out is $t^i(a)_n = i$ for $i = 0, \dots, n$. If $a_n = 0$ and $t(a)_n = 1$, then a has only one nonzero term. If it is a_1 , then $t(a) = 0, 1, 1, \dots, 1$ and $t(t(a)) = t(a)$, so $t(t(a))_n \neq 2$; if it is a_i for $i > 1$, then $t(a) = 0, \dots, 0, i, 1, \dots, 1$ and $t(t(a)) = 0, \dots, 0, i, i+1, \dots, i+1$ and $t(t(a))_n \neq 2$. That's a contradiction either way. (Actually we didn't need to check the first case separately except for $n = 2$; if $a_n = a_{n-1} = 0$, they stay together and so get fixed at the same step.)
- *Second variation* Let b_{n-1} be the terminal value of a_{n-1} . Then a_{n-1} gets there at least as soon as a_n does (since a_n only rises one each time, whereas a_{n-1} rises by at least one until reaching b_{n-1} and then stops, and furthermore $a_{n-1} \geq 0 = a_n$ to begin with), and when a_n does reach that point, it is equal to a_{n-1} . (Kiran Kedlaya, one of the graders of this problem, likes to call this a "tortoise and hare" argument—the hare a_{n-1} gets a head start but gets lazy and stops, so the tortoise a_n will catch him eventually.)

Second Solution: We prove that for $n \geq 2$, the claim holds without the initial condition $0 \leq a_i \leq i$. (Of course this does not prove anything stronger, but it's convenient.) We do this by induction on n , the case $n = 2$ being easy to check by hand as in the first solution.

Note that if $c = (c_0, \dots, c_n)$ is a sequence in the image of t , and d is the sequence (c_1, \dots, c_n) , then the following two statements are true:

- (a) If e is the sequence obtained from d by subtracting 1 from each nonzero term, then $t(d) = t(e)$. (If there are no zero terms in d , then subtracting 1 clearly has no effect. If there is a zero term in d , it must occur at the beginning, and then every nonzero term is at least 2.)
- (b) One can compute $t(c)$ by applying t to the sequence c_1, \dots, c_n , adding 1 to each nonzero term, and putting a zero in front.

The recipe of (b) works for computing $t^i(c)$ for any i , by (a) and induction on i .

We now apply the induction hypothesis to $t(a)_1, \dots, t(a)_n$ to see that it stabilizes after $n - 2$ more applications of t ; by the recipe above, that means a stabilizes after $n - 1$ applications of t .

Note: A variation of the above approach is the following. Instead of pulling off one zero, pull off *all* initial zeroes of a_0, \dots, a_n . (Or rather, pull off all terms equal to the initial term, whatever it is.) Say there are $k + 1$ of them (clearly $k \leq n$); after $\min\{k, 2\}$ applications of t , there will be $k + 1$ initial zeroes and all remaining terms are at least k . So now $\max\{1, n - k - 2\}$ applications of t will straighten out the end, for a total of $\min\{k, 2\} + \max\{1, n - k - 2\}$. A little case analysis shows that this is good enough: if $k + 1 \leq n - 1$, then this sum is at most $n - 1$ except maybe if $3 > n - 1$, i.e., $n \leq 3$, which can be checked by hand. If $k + 1 > n - 1$ and we assume $n \geq 4$, then $k \geq n - 1 \geq 3$, so the sum is $2 + \max\{1, n - k - 2\} \leq \max\{3, n - k\} \leq n - 1$.

13. For any nonempty set S of real numbers, let $\sigma(S)$ denote the sum of the elements of S . Given a set A of n positive integers, consider the collection of all distinct sums $\sigma(S)$ as S ranges over the nonempty subsets of A . Prove that this collection of sums can be partitioned into n classes so that in each class, the ratio of the largest sum to the smallest sum does not exceed 2.

Solution: Let $A = \{a_1, a_2, \dots, a_n\}$ where $a_1 < a_2 < \dots < a_n$. For $i = 1, 2, \dots, n$ let $s_i = a_1 + a_2 + \dots + a_i$ and take $s_0 = 0$. All the sums in question are less than or equal to s_n , and if σ is one of them, we have

$$s_{i-1} < \sigma \leq s_i \quad (*)$$

for an appropriate i . Divide the sums into n classes by letting C_i denote the class of sums satisfying (*). We claim that these classes have the desired property. To establish this, it suffices to show that (*) implies

$$\frac{1}{2} s_i < \sigma \leq s_i. \quad (**)$$

Suppose (*) holds. The inequality $a_1 + a_2 + \dots + a_{i-1} = s_{i-1} < \sigma$ shows that the sum σ contains at least one addend a_k with $k \geq i$. Then since $a_k \geq a_i$, we have

$$s_i - \sigma < s_i - s_{i-1} = a_i \leq a_k \leq \sigma,$$

which together with $\sigma \leq s_i$ implies (**).

Note: The result does not hold if 2 is replaced by any smaller constant c . To see this, choose n such that $c < 2 - 2^{-(n-1)}$ and consider the set $\{1, \dots, 2^{n-1}\}$. If this set is divided into n subsets, two of $1, \dots, 2^{n-1}, 1 + \dots + 2^{n-1}$ must lie in the same subset, and their ratio is at least $(1 + \dots + 2^{n-1}) / (2^{n-1}) = 2 - 2^{-(n-1)} > c$.

14. Let a set of 2004 points in the plane be given, no three of which are collinear. Let \mathcal{S} denote the set of all lines determined by pairs of points from the set. Show that it is possible to color the points of

S with at most two colors, such that any points P and Q in the set, the number of lines in \mathcal{S} which separates P and Q is odd if and only if P and Q have the same color.

A line ℓ separates two points P and Q if P and Q lie on opposite sides of ℓ with neither point on ℓ .

15. I have an $n \times n$ sheet of stamps, from which I've been asked to tear out blocks of three adjacent stamps in a single row or column. (I can only tear along the perforations separating adjacent stamps, and each block must come out of a sheet in one piece.) Let $b(n)$ be the smallest number of blocks I can tear out and make it impossible to tear out any more blocks. Prove that there are constants c and d such that

$$\frac{1}{7}n^2 - cn \leq b(n) \leq \frac{1}{5}n^2 + dn$$

for all $n > 0$.

Solution: The upper bound requires an example of a set of $\frac{1}{5}n^2 + dn$ blocks whose removal makes it impossible to remove any further blocks. We note first that we can tile the plane, using tiles that contain one block for every five stamps, so that no more blocks can be chosen. Two such tilings are shown below with one tile outlined in heavy lines. Assume that there are x unit squares in each tile. Then there are $\frac{1}{5}x$ blocks in each tile. Choose a constant m such that the basic tile fits inside an $(m+1) \times (m+1)$ square. Given an $n \times n$ section of the tiling, take all tiles lying entirely within that section and add as many additional tiles, which lie partially in and partially out of that section, as possible. Let k denote the total number of chosen tiles. Hence there are $\frac{1}{5}kx$ blocks contained in the k chosen tiles. The $n \times n$ section is covered by all the chosen tiles, and these are all contained in a concentric $(n+2m) \times (n+2m)$ square. Then $kx \leq (n+2m)^2$, and so there are at most

$$\frac{1}{5}kx \leq \frac{1}{5}(n+2m)^2 \leq \frac{1}{5}n^2 + \frac{4m^2 + 4m}{5}n$$

blocks total. We can classify all the above blocks into three categories (i) blocks lying completely in the $n \times n$ section; (ii) blocks lying partially in the section; (iii) blocks lying completely outside of the section. Suppose there are x_1, x_2, x_3 blocks in categories (i), (ii), (iii), respectively. We do not have to worry about blocks in category (iii), and we take all the blocks in category (i). We need to deal with blocks in category (ii) with more care. By the conditions of the problem, we can not take out those blocks from the $n \times n$ section. All the blocks in category (ii) are on the border of the section. Hence there are at most $4n$ blocks in category (ii), and so these blocks contain at most $8n$ stamps in the $n \times n$ square. We might need additional blocks to deal with these stamps. Each of the additional blocks must contain one of these stamps. Thus there are at most $8n$ additional blocks. Thus there are at most

$$x_1 + 8n \leq x_1 + x_2 + x_3 + 8n \leq \frac{1}{5}n^2 + \frac{4m^2 + 4m + 40}{5}n$$

blocks needed.

In fact, we'll prove the lower bound

$$b(n) \geq \frac{1}{6}(n^2 - 2n).$$

Each block can be classified as "horizontal" or "vertical" in the obvious fashion. Given an arrangement of blocks, let H and V be the numbers of horizontal and vertical blocks, respectively. Without loss of generality, we may assume $V \leq H$.

We associate each unused stamp which is not in one of the two leftmost columns to the first block one encounters proceeding leftward from the stamp. Note that one never has to proceed leftward more than two stamps; otherwise, there would be another block to remove.

h-block	s	s
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Each block is associated to at most two stamps in each row that it occupies. In particular, each horizontal block is associated to at most two stamps. Moreover, a vertical block cannot have an unused stamp on its immediate right in each of the three rows it covers; otherwise, those three stamps would form a block. Thus a vertical block is associated to at most four stamps.

v-		
b		
l	s	s
o		
c	s	s
k		

v-	s	s
b		
l		
o		
c	s	s
k		

Thus, if we count stamps block by block (plus the extra stamps in the two leftmost columns), the total number is

$$\begin{aligned} n^2 &\leq 2n + 3H + 3V + 2H + 4V = 2n + 5H + 7V \\ &\leq 2n + 6H + 6V, \end{aligned}$$

giving the desired bound.

Note: This problem was inspired by a paper of Manjul Bhargava (*Mistilings of the plane with rectangles*, to appear), in which the improved lower bound

$$b(n) \geq \frac{4}{21}n^2 - cn$$

is obtained by a rather complicated argument. It is believed that in fact $b(n) \geq \frac{1}{5}n^2 - cn$, but the fact that there are essentially two different equality cases makes this extremely difficult to prove. The aforementioned paper also treats rectangles of other sizes for which there is only one optimal arrangement; in those cases one can achieve upper and lower bounds with the same quadratic constant.

16. A computer screen shows a 98×98 chessboard, colored in the usual way. One can select with a mouse any rectangle with sides on the lines of the chessboard and click the mouse button: as a result, the colors in the selected rectangle switch (black becomes white, white becomes black). Find, with proof, the minimum number of mouse clicks needed to make the chessboard all one color.

Solution: More generally, we show that the minimum number of selections required for an $n \times n$ chessboard is $n - 1$ if n is odd, and n if n is even. Consider the $4(n - 1)$ squares along the perimeter of the chessboard, and at each step, let us count the number of pairs of adjacent perimeter squares which differ in color. This total begins at $4(n - 1)$, ends up at 0, and can decrease by no more than 4 each turn (If the rectangle touches two adjacent edges of the board, then only two pairs can be affected. Otherwise, the rectangle either touches no edges, one edge, or two opposite edges, in which case 0, 2 or 4 pairs change, respectively). Hence at least $n - 1$ selections are always necessary.

If n is odd, then indeed $n - 1$ selections suffice, by choosing every second, fourth, sixth, etc. row and column. However, if n is even, then $n - 1$ selections cannot suffice: at some point a corner square

must be included in a rectangle (since the corners do not all begin having the same color), and such a rectangle can only decrease the above count by 2. Hence n selections are needed, and again by selecting every other row and column, we see that n selections also suffice.

17. The Y2K Game is played on a 1×2000 grid as follows. Two players in turn write either an S or an O in an empty square. The first player who produces three consecutive boxes that spell SOS wins. If all boxes are filled without producing SOS then the game is a draw. Prove that the second player has a winning strategy.

Solution: Call a partially filled board *stable* if there is no SOS and no single move can produce an SOS; otherwise call it *unstable*. For a stable board call an empty square *bad* if either an S or an O played in that square produces an unstable board. Thus a player will lose if the only empty squares available to him are bad, but otherwise he can at least be guaranteed another turn with a correct play.

Claim: A square is bad if and only if it is in a block of 4 consecutive squares of the form S – – S.

Proof: If a square is bad, then an O played there must give an unstable board. Thus the bad square must have an S on one side and an empty square on the other side. An S played there must also give an unstable board, so there must be another S on the other side of the empty square. \square

From the claim it follows that there are always an even number of bad squares. Thus the second player has the following winning strategy:

- (a) If the board is unstable at any time, play the winning move, otherwise continue as below.
- (b) On the first move, play an S at least four squares away from either end and from the first player's first move. (The board is long enough that this is possible.)
- (c) On the second move, play an S three squares away from the second player's first move, so that the squares in between are empty. (Regardless of the first player's second move, this must be possible on at least one side.) This produces two bad squares; whoever plays in one of them first will lose. Thus the game will not be a draw.
- (d) On any subsequent move, play in a square which is not bad. Such a square will always exist because if the board is stable, there will be an odd number of empty squares and an even number of bad squares.

Since there exist bad squares after the second player's second move, the game cannot end in a draw, and since the second player can always leave the board stable, the first player cannot win. Therefore eventually the second player will win.

18. A game of solitaire is played with R red cards, W white cards, and B blue cards. A player plays all the cards one at a time. With each play he accumulates a penalty. If he plays a blue card, then he is charged a penalty which is the number of white cards still in his hand. If he plays a white card, then he is charged a penalty which is twice the number of red cards still in his hand. If he plays a red card, then he is charged a penalty which is three times the number of blue cards still in his hand. Find, as a function of R, W , and B , the minimal total penalty a player can amass and all the ways in which this minimum can be achieved.

Solution: Let the integers at any time be a_1, a_2, \dots, a_n , and let ℓ be the index of the integer chosen as large in the previous step. Define the score of the position to be $S = \sum_{i \neq \ell} a_i$. On any step we will choose a new large integer a_ℓ , (which currently contributes to S but will not after the

move,) and we will replace a_ℓ (which currently does not contribute to S) by something smaller than a_ℓ , (which will contribute to new S). Thus S is decreased by at least one on every move. Since S starts with a finite values and $S \geq 0$, play must stop in a finite amount of moves.

19. A game of solitaire is played with R red cards, W white cards, and B blue cards. A player plays all the cards one at a time. With each play he accumulates a penalty. If he plays a blue card, then he is charged a penalty which is the number of white cards still in his hand. If he plays a white card, then he is charged a penalty which is twice the number of red cards still in his hand. If he plays a red card, then he is charged a penalty which is three times the number of blue cards still in his hand. Find, as a function of R, W , and B , the minimal total penalty a player can amass and all the ways in which this minimum can be achieved.

Solution: The minimum achievable penalty is

$$\min\{BW, 2WR, 3RB\}.$$

The three penalties $BW, 2WR$, and $3RB$ can clearly be obtained by playing cards in one of the three orders

- **bb**...**brr**...**rww**...**w**,
- **rr**...**rww**...**wbb**...**b**,
- **ww**...**wbb**...**brr**...**r**.

Given an order of play, let a “run” of some color denote a set of cards of that color played consecutively in a row. Then the optimality of one of the three above orders follows immediately from the following lemma, along with the analogous observations for blue and white cards.

Lemma 1 For any given order of play, we may combine any two runs of red cards without increasing the penalty.

Proof: Suppose that there are w white cards and b blue cards between the two red runs. Moving a red card from the first run to the second costs us $2w$ because we now have one more red card after the w white cards. However, we gain $3b$ because this red card is now after the b blues. If the net gain $3b - 2w$ is non-negative, then we can move all the red cards in the first run to the second run without increasing the penalty. If the net gain $3b - 2w$ is negative, then we can move all the red cards in the second run to the first run without increasing the penalty, as desired. ■

Thus there must be an optimal game where cards are played in one of the three given orders. To determine whether there are other optimal orders, first observe that **wr** can never appear during an optimal game; otherwise, playing these two cards in the order **rw** instead decreases the penalty. Similarly, **bw** and **rb** can never appear. Now we prove the following lemma.

Lemma 2 Any optimal order of play must have less than 5 runs.

Proof: Suppose that some optimal order of play had at least five runs. Assume the first card played is red; the proof is similar in the other cases. Say we first play r_1, w_1, b_1, r_2, w_2 cards of each color, where each r_i, w_i, b_i is positive and where we cycle through red, white, and blue runs. From the proof of our first lemma we must have both $3b_1 - 2w_1 = 0$ and $b_1 - 2r_2 = 0$. Hence the game starting with playing $r_1, w_1 + w_2, b_1, r_2, 0$ cards is optimal as well, so we must also have $3b_1 - 2(w_1 + w_2) = 0$, a contradiction. ■

Thus, any optimal game has at most 4 runs. Now from lemma 1 and our initial observations, any order of play of the form

$$\mathbf{rr} \cdots \mathbf{rww} \cdots \mathbf{wbb} \cdots \mathbf{brr} \cdots \mathbf{r},$$

is optimal if and only if $2W = 3B$ and $2WR = 3RB \leq WB$; and similar conditions hold for 4-run games that start with \mathbf{w} or \mathbf{b} .

20. Each of eight boxes contains six balls. Each ball has been colored with one of n colors, such that no two balls in the same box are the same color, and no two colors occur together in more than one box. Determine, with justification, the smallest integer n for which this is possible.

First Solution: The smallest such n is 23.

We first show that $n = 22$ cannot be achieved.

Assume that some color, say red, occurs four times. Then the first box containing red contains 6 colors, the second contains red and 5 colors not mentioned so far, and likewise for the third and fourth boxes. A fifth box can contain at most one color used in each of these four, so must contain 2 colors not mentioned so far, and a sixth box must contain 1 color not mentioned so far, for a total of $6+5+5+5+2+1=24$, a contradiction.

Next, assume that no color occurs four times; this forces at least four colors to occur three times. In particular, there are two colors that occur at least three times and which both occur in a single box, say red and blue. Now the box containing red and blue contains 6 colors, the other boxes containing red each contain 5 colors not mentioned so far, and the other boxes containing blue each contain 3 colors not mentioned so far (each may contain one color used in each of the boxes containing red but not blue). A sixth box must contain one color not mentioned so far, for a total of $6+5+5+3+3+1=23$, again a contradiction.

We now give a construction for $n = 23$. We still cannot have a color occur four times, so at least two colors must occur three times. Call these red and green. Put one red in each of three boxes, and fill these with 15 other colors. Put one green in each of three boxes, and fill each of these boxes with one color from each of the three boxes containing red and two new colors. We now have used $1 + 15 + 1 + 6 = 23$ colors, and each box contains two colors that have only been used once so far. Split those colors between the last two boxes. The resulting arrangement is:

1	3	4	5	6	7
1	8	9	10	11	12
1	13	14	15	16	17
2	3	8	13	18	19
2	4	9	14	20	21
2	5	10	15	22	23
6	11	16	18	20	22
7	12	17	19	21	23

Note that the last 23 can be replaced by a 22.

Now we present a few more methods of proving $n \geq 23$.

Second Solution: As in the first solution, if $n = 22$ is possible, it must be possible with no color appearing four or more times. By the Inclusion-Exclusion Principle, the number of colors (call

it C) equals the number of balls (48), minus the number of pairs of balls of the same color (call it P), plus the number of triples of balls of the same color (call it T); that is,

$$C = 48 - P + T.$$

For every pair of boxes, at most one color occurs in both boxes, so $P \leq \binom{8}{2} = 28$. Also, if $n \leq 22$, there must be at least $48 - 2(22) = 4$ colors that occur three times. Then $C \geq 48 - 28 + 4 = 24$, a contradiction.

Third Solution: Assume $n = 22$ is possible. By the **Pigeonhole Principle**, some color occurs three times; call it color 1. Then there are three boxes containing 1 and fifteen other colors, say colors 2 through 16. The other five boxes each contain at most three colors in common with the first three boxes, so they contain at least three colors from 17 through 22.

Since $5 \times 3 > 2 \times 6$, one color from 17 to 22 occurs at least three times in the last five boxes; say it's color 17. Then two balls in each of those three boxes have colors among those labeled 18 through 22. But then one of these colors must appear together with 17, a contradiction.

Fourth Solution: Label the colors $1, 2, \dots, n$, and let a_1, a_2, \dots, a_n be the number of balls of color $1, 2, \dots, n$, respectively. Then

$$\sum_{i=1}^n a_i = 48.$$

Since $\binom{a_i}{2}$ is the number of boxes sharing color i and there are $\binom{8}{2} = 28$ pairs of boxes, each of which can only share at most one color,

$$\begin{aligned} 28 &= \binom{8}{2} \geq \sum_{i=1}^n \binom{a_i}{2} = \sum_{i=1}^n \frac{a_i(a_i - 1)}{2} \\ &= \frac{1}{2} \sum_{i=1}^n a_i^2 - \frac{1}{2} \sum_{i=1}^n a_i = \frac{1}{2} \sum_{i=1}^n a_i^2 - 24, \end{aligned}$$

or $\sum_{i=1}^n a_i^2 \leq 104$. By the **RMS-AM Inequality**,

$$\left(\frac{1}{n} \sum_{i=1}^n a_i^2 \right)^{\frac{1}{2}} \geq \frac{1}{n} \sum_{i=1}^n a_i.$$

It follows that

$$104n \geq 48^2 \quad \text{or} \quad n \geq \frac{288}{13} > 22.$$

Note: For the general case of $m + 2$ boxes each containing m balls, this method leads to the lower bound of

$$n \geq \frac{m^2(m+2)}{2m+1}.$$

But for $m = 8$, this lower bound of $n \geq \frac{640}{17} = 37.647\dots$ is not good enough. The actual minimum value of n is 39, as proved in the sixth solution.

Fifth Solution: Let $m_{i,j}$ be the number of balls which are the same color as the j^{th} ball in box i (including that ball). For a fixed box i , $1 \leq i \leq 8$, consider the sums

$$S_i = \sum_{j=1}^6 m_{i,j} \quad \text{and} \quad s_i = \sum_{j=1}^6 \frac{1}{m_{i,j}}.$$

For each fixed i , since no pair of colors is repeated, each of the remaining seven boxes can contribute at most one ball to S_i . Thus $S_i \leq 13$. It follows by the **convexity** of $f(x) = 1/x$ (and consequently, by the **Jensen's Inequality**) that s_i is minimized when one of the $m_{i,j}$ is equal to 3 and the other five equal 2. Hence $s_i \geq 17/6$. Note that

$$n = \sum_{i=1}^8 \sum_{j=1}^6 \frac{1}{m_{i,j}} \geq 8 \cdot \frac{17}{6} = \frac{68}{3} = 22\frac{2}{3}.$$

Hence there must be at least 23 colors.

Sixth Solution: Let x_i be the number of colors that occur exactly i times. Then

$$x_1 + x_2 + x_3 + \cdots + x_i + \cdots = n \tag{1}$$

and

$$x_1 + 2x_2 + 3x_3 + \cdots + ix_i + \cdots = 48. \tag{2}$$

Now we count the number of pairs of like-colored balls. Each pair of boxes can contain at most one pair of like-colored balls. Hence

$$x_2 + \binom{3}{2}x_3 + \cdots + \binom{i}{2}x_i + \cdots \leq 28. \tag{3}$$

Taking $(1) - \frac{2}{3} \times (2) + \frac{1}{3} \times (3)$ gives

$$\frac{1}{3}x_1 + \frac{1}{3}x_4 + \cdots + \frac{(i-2)(i-3)}{6}x_i + \cdots \leq n - \frac{68}{3}.$$

The coefficients on the left-hand side are all nonnegative. Hence n is at least 23.

The following construction shows that 23 is indeed enough. Each line represents a box and each intersection \circ represents a color.

More generally, we have the following Lemmas.

Lemma 1. *There are b boxes with an average of a balls each box. Each ball has been colored with one of n colors, such that no two balls in the same box are the same color, and no two colors occur together in more than one box. Then*

$$n \geq \frac{2}{r+1}ab - \frac{2}{r(r+1)}\binom{b}{2},$$

where $r = \left\lceil \frac{b-1}{a} \right\rceil$.

Proof: Let x_i be the number of colors that occur exactly i times. Then we have the following linear system:

$$\begin{aligned} (4) : \quad & \sum_{i \geq 1} x_i = x_1 + x_2 + \cdots + x_i + \cdots = n, \\ (5) : \quad & \sum_{i \geq 1} ix_i = x_1 + 2x_2 + \cdots + ix_i + \cdots = ab, \\ (6) : \quad & \sum_{i \geq 1} \binom{i}{2} x_i + y = x_2 + \cdots + \binom{i}{2} x_i + \cdots + y = \binom{b}{2}, \end{aligned}$$

where y is a slack variable, denoting the number of pairs of boxes with no like-colored balls. To solve this linear system in real numbers x_i , let $r = \lceil \frac{b-1}{a} \rceil$; then combine $(4) - \frac{2}{r+1} \times (5) + \frac{2}{r(r+1)} \times (6)$. We obtain

$$\begin{aligned} & \sum_{i \geq 1} \frac{(r-i)(r-i+1)}{r(r+1)} x_i + \frac{2}{r(r+1)} y \\ &= n - \frac{2}{r+1} ab + \frac{2}{r(r+1)} \binom{b}{2}. \end{aligned}$$

Since all of the coefficients on the left-hand-side of the last equation are nonnegative, we have

$$n \geq \frac{2}{r+1} ab - \frac{2}{r(r+1)} \binom{b}{2},$$

as desired.

Since the coefficients of x_r and x_{r+1} in the last equation are 0, and that it is possible to realize this bound in reals by setting $y = x_i = 0$, where $i \neq r$ and $i \neq r+1$. Then (5) and (6) become

$$\begin{aligned} rx_r + (r+1)x_{r+1} &= ab, \\ \binom{r}{2} x_r + \binom{r+1}{2} x_{r+1} &= \binom{b}{2}. \end{aligned}$$

Solving the last system of equations in two variables leads to

$$x_r = ab - \frac{2}{r} \binom{b}{2} \quad \text{and} \quad x_{r+1} = \frac{1}{r+1} \left(2 \binom{b}{2} - ab \right).$$

Hence the lower bound is indeed obtainable. ■

Lemma 2. Suppose that each of $m+2$ boxes contains m balls. Let n_m denote the smallest integer n for which it is possible to color each ball with one of n colors, such that no two balls in the same box are the same color, and no two colors occur together in more than one box. Then $n_0 = 0, n_1 = 1$, and

$$n_m = \begin{cases} \frac{9k^2 - k}{2} & m = 3k - 1, \\ \frac{9k^2 + 5k}{2} & m = 3k, \\ \frac{9k^2 + 11k + 2}{2} & m = 3k + 1, \end{cases}$$

for positive integers k .

Proof: When $m = 0$ and $m = 1$, the results are trivial. We assume that $m \geq 3$. By Lemma 1, we have $r = 2$ and

$$n \geq \frac{2}{3}a(a+2) - \frac{1}{3}\binom{a+2}{2},$$

or

$$n \geq \frac{1}{6}(3a-1)(a+2).$$

This gives

m	2	3	4	5	6	7	8	9	10
lower bound	$\frac{10}{3}$	$\frac{20}{3}$	11	$\frac{49}{3}$	$\frac{68}{3}$	30	$\frac{115}{3}$	$\frac{141}{3}$	58
n_m	4	7	11	17	23	30	39	48	58

The values of n can indeed be achieved. We can construct our examples inductively. Again, let lines represent boxes and let their intersections \circ represent different colors.

For $m = 2$, we have

For $m = 3$, we have

For $m = 4$, we have

If G_k is the construction for $m = k$, we add

to G_k to obtain G_{k+3} the construction for $m = k+3$. (One can compare the construction for G_3 and G_6 , which appeared earlier. One can also see the construction for G_5 and G_8 .)

Therefore $n_{m+3} = n_m + 3(m+2) + 1$, for $m \geq 2$. For $m = 3k$, we have

$$\begin{aligned} n_{3(k+1)} &= n_{3k} + 9k + 7 = n_3 + 9 \cdot \frac{(k+1)k}{2} + 7k \\ &= \frac{9k^2 + 23k + 14}{2}, \end{aligned}$$

or

$$n_{3k} = \frac{9k^2 + 5k}{2}.$$

Similarly, we have

$$n_{3k+1} = \frac{9k^2 + 11k + 2}{2} \quad \text{and} \quad n_{3k-1} = \frac{9k^2 - k}{2},$$

as desired. ■

21. Each point in the plane is assigned a real number such that, for any triangle, the number at the center of its inscribed circle is equal to the arithmetic mean of the three numbers at its vertices. Prove that all points in the plane are assigned the same number.

Solution: Let A, B be arbitrary distinct points, and consider a regular hexagon $ABCDEF$ in the plane. Let lines CD and FE intersect at G . Let ℓ be the line through G perpendicular to line ED . Then A, F, E and B, C, D are symmetric to each other, respectively, with respect to line ℓ . Hence triangles CEG and DFG share the same incenter, i.e., $c + e = d + f$; triangles ACE and BDF share the same incenter, i.e., $a + c + e = b + d + f$. Therefore, $a = b$, and we are done.

22. Let n be a positive integer and let S be a set of $2^n + 1$ elements. Let f be a function from the set of two-element subsets of S to $\{0, \dots, 2^{n-1} - 1\}$. Assume that for any elements x, y, z of S , one of $f(\{x, y\})$, $f(\{y, z\})$, $f(\{z, x\})$ is equal to the sum of the other two. Show that there exist a, b, c in S such that $f(\{a, b\})$, $f(\{b, c\})$, $f(\{c, a\})$ are all equal to 0.

Solution: We prove the result by induction on n , the case $n = 1$ being obvious. Pick $w \in S$ and set $f(w, w) = 0$. We partition S into subsets U and V by putting $x \in S$ into U if $f(\{w, x\})$ is even and into V if $f(\{w, x\})$ is odd (hence w is in U). By the **Pigeonhole Principle**, at least one of U or V , say U , contains a subset of $\lceil \frac{2^n+1}{2} \rceil = 2^{n-1} + 1$ elements.

Note that the given condition implies that the sum $f(\{x, y\}) + f(\{y, z\}) + f(\{z, x\})$ is even for any $x, y, z \in S$. In particular, if x and y are both in U or both in V , then $f(\{w, x\}) + f(\{w, y\})$ is even, so $f(\{x, y\})$ is even. Hence f maps the two-element subsets of U into $\{0, \dots, 2^{n-1} - 2\}$. Of course the condition on f is preserved by dividing all values by 2; then the induction hypothesis applies to show that some $x, y, z \in U$ satisfy $f(\{x, y\}) = f(\{y, z\}) = f(\{z, x\}) = 0$, as desired.

23. For a pair of integers a and b , with $0 < a < b < 1000$, the set $S \subseteq \{1, 2, \dots, 2003\}$ is called a *skipping set* for (a, b) if for any pair of elements $s_1, s_2 \in S$, $|s_1 - s_2| \notin \{a, b\}$. Let $f(a, b)$ be the maximum size of a skipping set for (a, b) . Determine the maximum and minimum values of f .

Note: This problem caused unexpected difficulties for students. It requires two ideas: applying the greedy algorithm to obtain the minimum and applying the Pigeonhole Principle on congruence classes to obtain the maximum. Most students were successful in getting one of the two ideas and obtaining one of the extremal values quickly, but then many of them failed to switch to the other idea. In turn, their solutions for the second extremal value were very lengthy and sometimes unsuccessful.

Solution: The maximum and minimum values of f are 1334 and 338, respectively.

- (a) First, we will show that the maximum value of f is 1334. The set $S = \{1, 2, \dots, 667\} \cup \{1336, 1337, \dots, 2002\}$ is a skipping set for $(a, b) = (667, 668)$, so $f(667, 668) \geq 1334$.

Now we prove that for any $0 < a < b < 1000$, $f(a, b) \leq 1334$. Because $a \neq b$, we can choose $d \in \{a, b\}$ such that $d \neq 668$. We assume first that $d \geq 669$. Then consider the $2003 - d \leq 1334$ sets $\{1, d + 1\}, \{2, d + 2\}, \dots, \{2003 - d, 2003\}$. Each can contain at most one element of S , so $|S| \leq 1334$.

We assume second that $d \leq 667$ and that $\lceil \frac{2003}{a} \rceil$ is even, that is, $\lceil \frac{2003}{a} \rceil = 2k$ for some positive integer k . Then each of the congruence classes of $1, 2, \dots, 2003$ modulo a contains at most $2k$ elements. Therefore at most k members of each of these congruence classes can belong to S . Consequently,

$$|S| \leq ka < \frac{1}{2} \left(\frac{2003}{a} + 1 \right) a = \frac{2003 + a}{2} \leq 1335,$$

implying that $|S| \leq 1334$.

Finally, we assume that $d \leq 667$ and that $\lceil \frac{2003}{a} \rceil$ is odd, that is, $\lceil \frac{2003}{a} \rceil = 2k + 1$ for some positive integer k . Then, as before, S can contain at most k elements from each congruence

class of $\{1, 2, \dots, 2ka\}$ modulo a . Then

$$\begin{aligned}
 |S| &\leq ka + (2003 - 2ka) = 2003 - ka \\
 &= 2003 - \left(\frac{\left\lceil \frac{2003}{a} \right\rceil - 1}{2} \right) a \\
 &\leq 2003 - \left(\frac{\frac{2003}{a} - 1}{2} \right) a \\
 &= \frac{2003 + a}{2} \leq 1335.
 \end{aligned}$$

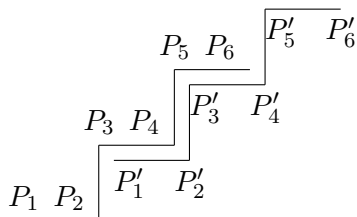
The last inequality holds if and only if $a = 667$. But if $a = 667$, then $\frac{2003}{a}$ is not an integer, and so the second inequality is strict. Thus, $|S| \leq 1334$. Therefore the maximum value of f is 1334.

- (b) We will now show that the minimum value of f is 668. First, we will show that $f(a, b) \geq 668$ by constructing a skipping set S for any (a, b) with $|S| \geq 668$. Note that if we add x to S , then we are not allowed to add x , $x + a$, or $x + b$ to S at any later time. Then at each step, let us add to S the smallest element of $\{1, 2, \dots, 2003\}$ that is not already in S and that has not already been disallowed from being in S . Then since adding this element prevents at most three elements from being added at any future time, we can always perform this step $\left\lceil \frac{2003}{3} \right\rceil = 668$ times. Thus, $|S| \geq 668$, so $f(a, b) \geq 668$. Now notice that if we let $a = 1, b = 2$, then at most one element from each of the 668 sets $\{1, 2, 3\}, \{4, 5, 6\}, \dots, \{1999, 2000, 2001\}, \{2002, 2003\}$ can belong to S . This implies that $f(1, 2) = 668$, so indeed the minimum value of f is 668.

24. Let n be an integer greater than 2, and P_1, P_2, \dots, P_n distinct points in the plane. Let \mathcal{S} denote the union of the segments $P_1P_2, P_2P_3, \dots, P_{n-1}P_n$. Determine whether it is always possible to find points A and B in \mathcal{S} such that $P_1P_n \parallel AB$ (segment AB can lie on line P_1P_n) and $P_1P_n = kAB$, where (1) $k = 2.5$; (2) $k = 3$.

Solution: The answer is negative for $k = 2.5$ and positive for $k = 3$.

- (1) Let $n = 6$, $P_1 = (0, 0)$, $P_2 = (5, 0)$, $P_3 = (5, 5)$, $P_4 = (10, 5)$, $P_5 = (10, 10)$, $P_6 = (15, 10)$.



Then for $1 \leq i \leq 6$, let $P'_i = P_i + \frac{2}{5} \overrightarrow{P_1P_6}$. Because \mathcal{S} and its image under this transformation do not intersect, there are no two points A, B in \mathcal{S} such that $\overrightarrow{P_1P_6} = \frac{5}{2} \overrightarrow{AB}$, so this is a counterexample.

- (2) The answer is yes; in fact, it is yes for all positive integers k . We approach indirectly. The statement is obviously true for $k = 1$, because we can have $A = P_1$ and $B = P_n$. Assume to the contrary that the claim is false for some positive integer $k \geq 2$; let P_1, \dots, P_n be a counterexample. Introduce a coordinate system in which $P_1 = (0, 0)$ and $P_n = (k, 0)$. Choose indices $T, B \in \{1, \dots, n\}$ so that the y -coordinate of P_T is maximal, the y -coordinate of P_B is minimal, and $|T - B|$ is as small as possible. Assume without loss of generality that

$T < B$ (because we can relabel the points backward otherwise). Let M be the region in the plane consisting of points whose y -coordinates lie between those of P_B and P_T , inclusive; then P_{T+1}, \dots, P_{B-1} lie in the interior of this region. Let \mathcal{D} be the union of the closed segments $P_T P_{T+1}, P_{T+1} P_{T_2}, \dots, P_{B-1} P_B$.

Let X be a point on \mathcal{D} with its x -coordinate no smaller than that of any point on \mathcal{D} . (On polygonal path \mathcal{D} , X is indeed one of points P_T, P_{T+1}, \dots, P_B .) For $P \in M$ not lying on \mathcal{D} , we say P is *right of* \mathcal{D} if there is a continuous curve connecting P and X not intersecting \mathcal{D} besides at X . Otherwise, we say P is to the *left of* \mathcal{D} . (That is, we split M into left and right sides using \mathcal{D} as the border line.) For any figure F , let $F^{(x)}$ denote the image of F under translation by the vector $[x, 0]$, and put $F' = F^{(1)}$. We define *right of* (or *left of*) $\mathcal{D}^{(x)}$ analogously.

Note that any polygonal path in M not intersecting \mathcal{D} consists entirely either of points right of \mathcal{D} or of points left of \mathcal{D} . (If P is a point on the path and P is right of \mathcal{D} , then there is a continuous curve connecting P and X not intersecting \mathcal{D} besides at X . If Q is an other point on the path, then there is a continuous path from Q to P to X not intersecting \mathcal{D} besides at X , and so Q is right of \mathcal{D} .) By hypothesis, \mathcal{S} and \mathcal{S}' are disjoint, so \mathcal{D} and \mathcal{S}' are disjoint. Because X' is on \mathcal{S}' and is clearly right of \mathcal{D} , so is all of \mathcal{S}' ; in particular, \mathcal{D}' is right of \mathcal{D} . By translation, $\mathcal{D}^{(2)}$ is right of \mathcal{D}' and hence of \mathcal{D} as well; by induction, $\mathcal{D}^{(i)}$ is right of \mathcal{D} for all $i = 1, 2, \dots$.

Because $k \geq 2$, $\mathcal{D}^{(k-1)}$ is right of \mathcal{D} and so $\mathcal{D}^{(k)}$ is right of \mathcal{D}' . In particular, $P_n = P_1^{(k)}$ is right of \mathcal{D}' . But that means that all of \mathcal{S} is right of \mathcal{D}' and hence of \mathcal{D} , which is impossible because $\mathcal{D} \subseteq \mathcal{S}$. Thus our assumption was wrong and the statement in the problem is true for all positive integers k .

25. At the vertices of a regular hexagon are written six nonnegative integers whose sum is 2003. Bert is allowed to make moves of the following form: he may pick a vertex and replace the number written there by the absolute value of the difference between the numbers written at the two neighboring vertices. Prove that Bert can make a sequence of moves, after which the number 0 appears at all six vertices.

Note: Let

$$\begin{array}{ccccc} & B & C & & \\ A & & & D & \\ & F & E & & \end{array}$$

denote a position, where A, B, C, D, E, F denote the numbers written on the vertices of the hexagon. We write

$$\begin{array}{ccccc} & B & C & & \\ A & & & D & \\ & F & E & & \end{array} \pmod{2}$$

if we consider the numbers written modulo 2.

This is the hardest problem on the test. Many students thought they had considerable progress. Indeed, there were only a handful of contestants who were able to find some algorithm without major flaws. Richard Stong, one of the graders of this problem, wrote the following summary.

There is an obvious approach one can take to reducing this problem, namely the greedy algorithm: reducing the largest value. As is often the case, this approach is fundamentally flawed. If the initial values are

$$\begin{array}{ccccc} & 3 & 2 & & \\ 1 & & & 5 & \\ & n & 7 & & \end{array}$$

where n is an integer greater than 7, then the first move following the greedy algorithm gives

$$\begin{array}{ccccc} & 3 & 2 & & \\ 1 & & & 5 & \\ & 6 & 7 & & \end{array}$$

No set of moves can lead from these values to the all zeroes by a parity argument. This example also shows that there is no sequence of moves which always reduces the sum of the six entries and leads to the all zeroes. A correct solution to the problem requires first choosing some parity constraint to avoid the

$$\begin{array}{ccc} & 1 & 0 \\ 1 & & 1 \\ & 0 & 1 \end{array} \pmod{2}$$

situation, which is invariant under the operation. Secondly one needs to find some moves that preserve the chosen constraint and reduce the six values.

Solution: Define the *sum* and *maximum* of a position to be the sum and maximum of the six numbers at the vertices. We will show that from any position in which the sum is odd, it is possible to reach the all-zero position.

Our strategy alternates between two steps:

- (a) from a position with odd sum, move to a position with exactly one odd number;
- (b) from a position with exactly one odd number, move to a position with odd sum and strictly smaller maximum, or to the all-zero position.

Note that no move will ever increase the maximum, so this strategy is guaranteed to terminate, because each step of type (b) decreases the maximum by at least one, and it can only terminate at the all-zero position. It suffices to show how each step can be carried out.

First, consider a position

$$\begin{array}{ccc} & B & C \\ A & & D \\ & F & E \end{array}$$

with odd sum. Then either $A + C + E$ or $B + D + F$ is odd; assume without loss of generality that $A + C + E$ is odd. If exactly one of A , C and E is odd, say A is odd, we can make the sequence of moves

$$\begin{array}{ccc} & B & 0 \\ 1 & & D \\ & F & 0 \end{array} \rightarrow \begin{array}{ccc} & 1 & 0 \\ & 1 & 0 \\ & 0 & 0 \end{array} \rightarrow \begin{array}{ccc} & 1 & 0 \\ 0 & & 0 \\ & 1 & 0 \end{array} \rightarrow \begin{array}{ccc} & 1 & 0 \\ 0 & & 0 \\ & 0 & 0 \end{array} \pmod{2},$$

where a letter or number in boldface represents a move at that vertex, and moves that do not affect each other have been written as a single move for brevity. Hence we can reach a position with exactly one odd number. Similarly, if A , C , E are all odd, then the sequence of moves

$$\begin{array}{ccc} & B & 1 \\ 1 & & D \\ & F & 1 \end{array} \rightarrow \begin{array}{ccc} & 0 & 1 \\ & 0 & 1 \\ & 0 & 1 \end{array} \rightarrow \begin{array}{ccc} & 0 & 0 \\ 1 & & 0 \\ & 0 & 0 \end{array} \pmod{2},$$

brings us to a position with exactly one odd number. Thus we have shown how to carry out step (a).

Now assume that we have a position

$$\begin{array}{ccc} & B & C \\ A & & D \\ & F & E \end{array}$$

with A odd and all other numbers even. We want to reach a position with smaller maximum. Let M be the maximum. There are two cases, depending on the parity of M .

- In this case, M is even, so one of B , C , D , E , F is the maximum. In particular, $A < M$.

We claim after making moves at B , C , D , E , and F in that order, the sum is odd and the maximum is less than M . Indeed, the following sequence

$$\begin{array}{ccc} & 0 & 0 \\ 1 & & 0 \\ & 0 & 0 \end{array} \rightarrow \begin{array}{ccc} & 1 & 0 \\ & 0 & 0 \\ & 0 & 0 \end{array} \rightarrow \begin{array}{ccc} & 1 & 1 \\ & 0 & 0 \\ & 0 & 0 \end{array} \rightarrow \begin{array}{ccc} & 1 & 1 \\ 1 & & 1 \\ & 0 & 0 \end{array} \rightarrow \begin{array}{ccc} & 1 & 1 \\ & 1 & 1 \\ & 0 & 1 \end{array} \pmod{2}.$$

shows how the numbers change in parity with each move. Call this new position $\begin{smallmatrix} A' & B' & C' \\ & F' & E' \end{smallmatrix} D'$. The sum is odd, since there are five odd numbers. The numbers A', B', C', D', E' are all less than M , since they are odd and M is even, and the maximum can never increase. Also, $F' = |A' - E'| \leq \max\{A', E'\} < M$. So the maximum has been decreased.

- In this case, M is odd, so $M = A$ and the other numbers are all less than M .

If $C > 0$, then we make moves at B, F, A , and F , in that order. The sequence of positions is

$$\begin{array}{c} 1 \quad 0 \quad 0 \\ 0 \quad 0 \quad 0 \end{array} \rightarrow \begin{array}{c} 1 \quad 0 \\ 0 \quad 0 \end{array} \rightarrow \begin{array}{c} 1 \quad 0 \\ 1 \quad 0 \end{array} \rightarrow \begin{array}{c} 1 \quad 0 \\ 0 \quad 0 \end{array} \rightarrow \begin{array}{c} 1 \quad 0 \\ 0 \quad 0 \end{array} \pmod{2}.$$

Call this new position $\begin{smallmatrix} A' & B' & C' \\ & F' & E' \end{smallmatrix} D'$. The sum is odd, since there is exactly one odd number. As before, the only way the maximum could not decrease is if $B' = A$; but this is impossible, since $B' = |A - C| < A$ because $0 < C < M = A$. Hence we have reached a position with odd sum and lower maximum.

If $E > 0$, then we apply a similar argument, interchanging B with F and C with E .

If $C = E = 0$, then we can reach the all-zero position by the following sequence of moves:

$$\begin{array}{c} A \quad B \quad 0 \\ F \quad 0 \quad D \end{array} \rightarrow \begin{array}{c} A \quad 0 \\ A \quad 0 \end{array} \rightarrow \begin{array}{c} A \quad 0 \\ A \quad 0 \end{array} \rightarrow \begin{array}{c} 0 \quad 0 \\ 0 \quad 0 \end{array}.$$

(Here 0 represents zero, not any even number.)

Hence we have shown how to carry out a step of type (b), proving the desired result. The problem statement follows since 2003 is odd.

Note: Observe that from positions of the form

$$\begin{array}{c} 1 \quad 1 \\ 0 \quad 1 \end{array} \pmod{2} \quad \text{or rotations}$$

it is impossible to reach the all-zero position, because a move at any vertex leaves the same value modulo 2. Dividing out the greatest common divisor of the six original numbers does not affect whether we can reach the all-zero position, so we may assume that the numbers in the original position are not all even. Then by a more complete analysis in step (a), one can show from any position not of the above form, it is possible to reach a position with exactly one odd number, and thus the all-zero position. This gives a complete characterization of positions from which it is possible to reach the all-zero position.

There are many ways to carry out the case analysis in this problem; the one used here is fairly economical. The important idea is the formulation of a strategy that decreases the maximum value while avoiding the “bad” positions described above.

Second Solution: (By Richard Stong) We will show that if there is a pair of opposite vertices with odd sum (which of course is true if the sum of all the vertices is odd), then we can reduce to a position of all zeros.

Focus on such a pair $\{a, d\}$ with smallest possible $\max\{a, d\}$. We will show we can always reduce this smallest maximum of a pair of opposite vertices with odd sum or reduce to the all-zero position.

Because the smallest maximum takes nonnegative integer values, we must be able to achieve the all-zero position.

To see this assume without loss of generality that $a \geq d$ and consider an arc (a, x, y, d) of the position

$$\begin{array}{ccc} & x & y \\ a & & d \\ & * & * \end{array}$$

Consider updating x and y alternately, starting with x . If $\max\{x, y\} > a$, then in at most two updates we reduce $\max\{x, y\}$. Thus, we can repeat this *alternate updating* process and we must eventually reach a point when $\max\{x, y\} \leq a$, and hence this will be true from then on.

Under this alternate updating process, the arc of the hexagon will eventually enter a unique cycle of length four modulo 2 in at most one update. Indeed, we have

$$\begin{array}{ccc} 1 & 0 & 0 \\ * & * & 0 \end{array} \rightarrow \begin{array}{ccc} 1 & 0 & 0 \\ * & * & 0 \end{array} \rightarrow \begin{array}{ccc} 1 & 1 & 0 \\ * & * & 0 \end{array} \rightarrow \begin{array}{ccc} 0 & 1 & 0 \\ * & * & 0 \end{array} \rightarrow \begin{array}{ccc} 0 & 0 & 0 \\ * & * & 0 \end{array} \pmod{2}$$

and

$$\begin{array}{ccc} 1 & 0 & 0 \\ * & * & 0 \end{array} \rightarrow \begin{array}{ccc} 0 & 0 & 0 \\ * & * & 0 \end{array} \pmod{2}; \quad \begin{array}{ccc} 1 & 0 & 0 \\ * & * & 0 \end{array} \rightarrow \begin{array}{ccc} 1 & 0 & 0 \\ * & * & 0 \end{array} \pmod{2}$$

$$\begin{array}{ccc} 1 & 1 & 0 \\ * & * & 0 \end{array} \rightarrow \begin{array}{ccc} 1 & 1 & 0 \\ * & * & 0 \end{array} \pmod{2}; \quad \begin{array}{ccc} 0 & 1 & 0 \\ * & * & 0 \end{array} \rightarrow \begin{array}{ccc} 0 & 1 & 0 \\ * & * & 0 \end{array} \pmod{2},$$

or

$$\begin{array}{ccc} 0 & 0 & 1 \\ * & * & 1 \end{array} \rightarrow \begin{array}{ccc} 1 & 1 & 1 \\ * & * & 1 \end{array} \rightarrow \begin{array}{ccc} 1 & 0 & 1 \\ * & * & 1 \end{array} \rightarrow \begin{array}{ccc} 0 & 0 & 1 \\ * & * & 1 \end{array} \rightarrow \begin{array}{ccc} 0 & 1 & 1 \\ * & * & 1 \end{array} \pmod{2}$$

and

$$\begin{array}{ccc} 0 & 0 & 0 \\ * & * & 1 \end{array} \rightarrow \begin{array}{ccc} 0 & 0 & 1 \\ * & * & 1 \end{array} \pmod{2}; \quad \begin{array}{ccc} 0 & 0 & 1 \\ * & * & 1 \end{array} \rightarrow \begin{array}{ccc} 0 & 1 & 1 \\ * & * & 1 \end{array} \pmod{2}$$

$$\begin{array}{ccc} 0 & 1 & 1 \\ * & * & 1 \end{array} \rightarrow \begin{array}{ccc} 1 & 0 & 1 \\ * & * & 1 \end{array} \pmod{2}; \quad \begin{array}{ccc} 0 & 1 & 0 \\ * & * & 1 \end{array} \rightarrow \begin{array}{ccc} 1 & 0 & 1 \\ * & * & 1 \end{array} \pmod{2}.$$

Further note that each possible parity for x and y will occur equally often.

Applying this alternate updating process to both arcs (a, b, c, d) and (a, e, f, d) of

$$\begin{array}{ccc} & b & c \\ a & & d \\ & f & e \end{array},$$

we can make the other four entries be at most a and control their parity. Thus we can create a position

$$\begin{array}{ccc} & x_1 & x_2 \\ a & & d \\ & x_5 & x_4 \end{array}$$

with $x_i + x_{i+3}$ ($i = 1, 2$) odd and $M_i = \max\{x_i, x_{i+3}\} \leq a$. In fact, we can have $m = \min\{M_1, M_2\} < a$, as claimed, unless both arcs enter a cycle modulo 2 where the values congruent to a modulo 2 are always exactly a . More precisely, because the sum of x_i and x_{i+3} is odd, one of them is not congruent to a and so has its value strictly less than a . Thus both arcs must pass through the state (a, a, a, d) (modulo 2, this is either $(0, 0, 0, 1)$ or $(1, 1, 1, 0)$) in a cycle of length four. It is easy to check that for this to happen, $d = 0$. Therefore, we can achieve the position

$$\begin{array}{ccc} & a & a \\ a & & 0 \\ & a & a \end{array}.$$

From this position, the sequence of moves

$$\begin{array}{ccc} & a & a \\ a & & 0 \end{array} \rightarrow \begin{array}{ccc} 0 & a & 0 \\ 0 & & a \end{array} \rightarrow \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \end{array}$$

completes the task.

Third Solution: (By Tiankai Liu) In the beginning, because $A + B + C + D + E + F$ is odd, either $A + C + E$ or $B + D + F$ is odd; assume without loss of generality it is the former. Perform the following steps repeatedly.

- a. In this case we assume that A, C, E are all nonzero. Suppose without loss of generality that $A \geq C \geq E$. Perform the sequence of moves

$$\begin{aligned} \begin{matrix} A & B & C & D \\ & F & E & \end{matrix} &\rightarrow \begin{matrix} A & (A-C) & C \\ & (A-E) & E \end{matrix} \begin{matrix} (C-E) \\ (C-E) \end{matrix} \\ &\rightarrow \begin{matrix} (C-E) & (A-C) & C \\ & (A-E) & (A-C) \end{matrix} \begin{matrix} (C-E) \\ (C-E) \end{matrix}, \end{aligned}$$

which decreases the sum of the numbers in positions A, C, E while keeping that sum odd.

- b. In this case we assume that exactly one among A, C, E is zero. Assume without loss of generality that $A \geq C > E = 0$. Then, because $A + C + E$ is odd, A must be strictly greater than C . Therefore, $-A < A - 2C < A$, and the sequence of moves

$$\begin{aligned} \begin{matrix} A & B & C & D \\ & F & 0 & \end{matrix} &\rightarrow \begin{matrix} A & (A-C) & C \\ & A & 0 \end{matrix} \begin{matrix} C \\ C \end{matrix} \\ &\rightarrow \begin{matrix} C & (A-C) & |A-2C| \\ & A & 0 \end{matrix} \begin{matrix} C \\ C \end{matrix}, \end{aligned}$$

decreases the sum of the numbers in positions A, C, E while keeping that sum odd.

- c. In this case we assume that exactly two among A, C, E are zero. Assume without loss of generality that $A > C = E = 0$. Then perform the sequence of moves

$$\begin{matrix} A & B & 0 \\ & F & 0 \end{matrix} D \rightarrow \begin{matrix} A & 0 \\ A & 0 \end{matrix} \begin{matrix} 0 \\ 0 \end{matrix} \rightarrow \begin{matrix} A & 0 \\ A & 0 \end{matrix} \begin{matrix} 0 \\ 0 \end{matrix} \rightarrow \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}.$$

By repeatedly applying step (a) as long as it applies, then doing the same for step (b) if necessary, and finally applying step (c) if necessary, $\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}$ can eventually be achieved.

26. Let n be a positive integer. A *corner* is a finite set C of ordered n -tuples of positive integers such that if $a_1, a_2, \dots, a_n, b_1, b_2, \dots, b_n$ are positive integers with $a_k \geq b_k$ for $k = 1, 2, \dots, n$ and $(a_1, a_2, \dots, a_n) \in C$, then $(b_1, b_2, \dots, b_n) \in C$. Prove that among any infinite collection S of corners, there exist two corners, one of which is a subset of the other one.

Solution: (by Reid Barton) If $a = (a_1, \dots, a_n)$ and $b = (b_1, \dots, b_n)$, we write $a \leq b$ if $a_i \leq b_i$ for $i = 1, \dots, n$. We first note that every sequence of n -tuples of positive integers contains a subsequence which is nondecreasing with this definition. For $n = 1$, we may simply pick the smallest term, then the smallest term that comes later in the sequence, and so on. For general n , first pick a subsequence which is nondecreasing in its first coordinate, then pick a subsequence of that which is also nondecreasing in its second coordinate, and so on.

For the sake of contradiction, suppose that there are no corners $A, B \in S$ with $A \subset B$. Let C_1 be a corner in S ; then each other corner in S fails to contain one of the n -tuples in C_1 . Since S is infinite and C_1 is finite, there exists an n -tuple a^1 in C_1 such that the set S_1 of corners not containing a^1 is infinite.

Now let C_2 be a corner in S_1 ; again, we may find an n -tuple a^2 in C_2 such that the set S_2 of corners not containing a^2 is infinite. Analogously, we recursively construct sequences C_k of corners, S_k of infinite sets of corners and a^k of n -tuples such that C_k is a corner in S_{k-1} , a^k is an n -tuple in C_k and S_k is the set of corners in S_{k-1} not containing a^k .

To conclude, simply notice that $a^i \not\leq a^j$ for $i \leq j$, since a^j is an element of a corner which does not contain a^i . This contradicts the result of the first paragraph.

Note: The assertion of the problem still holds if corners are not required to be finite; this statement is a recent theorem of Diane Maclagan, which turns out to yield a nontrivial result in algebraic geometry.

27. Suppose that r_1, \dots, r_n are real numbers. Prove that there exists $S \subseteq \{1, 2, \dots, n\}$ such that

$$1 \leq |S \cap \{i, i+1, i+2\}| \leq 2,$$

for $1 \leq i \leq n-2$, and

$$\left| \sum_{i \in S} r_i \right| \geq \frac{1}{6} \sum_{i=1}^n |r_i|.$$

Solution: Let $s = \sum_{i=1}^n |r_i|$ and for $i = 0, 1, 2$, define

$$s_i = \sum_{r_j \geq 0, j \equiv i \pmod{3}} r_j \quad \text{and} \quad t_i = \sum_{r_j < 0, j \equiv i \pmod{3}} r_j.$$

Then we have $s = s_1 + s_2 + s_3 - t_1 - t_2 - t_3$, or

$$\begin{aligned} 2s &= (s_1 + s_2) + (s_2 + s_3) + (s_3 + s_1) \\ &\quad - (t_1 + t_2) - (t_2 + t_3) - (t_3 + t_1). \end{aligned}$$

Therefore there are $i_1 \neq i_2$ such that either $s_{i_1} + s_{i_2} \geq \frac{s}{3}$ or $t_{i_1} + t_{i_2} \leq -\frac{s}{3}$ or both. Without loss of generality, we assume that $s_{i_1} + s_{i_2} \geq \frac{s}{3}$ and $|s_{i_1} + s_{i_2}| \geq |t_{i_1} + t_{i_2}|$. Thus $s_{i_1} + s_{i_2} + t_{i_1} + t_{i_2} \geq 0$. We have

$$(s_{i_1} + s_{i_2} + t_{i_1}) + (s_{i_1} + s_{i_2} + t_{i_2}) \geq s_{i_1} + s_{i_2} \geq \frac{s}{3}.$$

Therefore at least one of $s_{i_1} + s_{i_2} + t_{i_1}$ and $s_{i_1} + s_{i_2} + t_{i_2}$ is greater than or equal to $\frac{s}{6}$ and we are done.

Note: By setting $r_1 = r_2 = r_3 = 1$ and $r_4 = r_5 = r_6 = -1$, it is easy to prove that $\frac{1}{6}$ is the best value for the bound.

28. An $m \times n$ array is filled with the numbers $\{1, 2, \dots, n\}$ each being used exactly m times. Show that one can always permute the numbers within columns to arrange that each row contains every number $\{1, 2, \dots, n\}$ exactly once.

Solution: It suffices to show that we can permute the numbers within columns to arrange that top row contains every number $1, 2, \dots, n$ exactly one. The result then follows by induction on the number of rows. For this we apply the Marriage theorem. For the *boys* take the columns, for the *girls* take the numbers $\{1, 2, \dots, n\}$ and say a boy (column) likes a girl (number) if that number occurs in

the column. For each set of k columns, there are a total of km numbers in those columns. Therefore there must be at least k different distinct numbers among them. Therefore there is a marriage of the columns and the numbers they contain. Permuting these numbers to the top of their respective columns makes the first row have all n numbers.

29. Suppose that C_1, C_2, \dots, C_n are circles of radius 1 in the plane such that no two of them are tangent and the subset of the plane formed by the union of these circles is connected (i.e., for any partition of $\{1, 2, \dots, n\}$ into nonempty subsets A and B , $\bigcup_{a \in A} C_a$ and $\bigcup_{b \in B} C_b$ are not disjoint). Prove that $|S| \geq n$, where

$$S = \bigcup_{1 \leq i < j \leq n} C_i \cap C_j,$$

the set of intersection points of the circles. (Each circle is viewed as the set of points on its circumference, not including its interior.)

Solution: Let $T = \{C_1, C_2, \dots, C_n\}$. For every $s \in S$ and $C \in T$ define

$$f(s, C) = \begin{cases} 0, & \text{if } s \notin C, \\ \frac{1}{k}, & \text{if } s \in C, \end{cases}$$

where k is the number of circles passing through s (including C). Thus

$$\sum_{C \in T} f(s, C) = 1$$

for every $s \in S$.

On the other hand, for a fixed circle $C \in T$, let $s_0 \in S \cap C$ be a point such that

$$f(s_0, C) = \min\{f(s, C) \mid s \in S \cap C\}.$$

Suppose that C, C_2, \dots, C_k are the circles which pass through s_0 . Then C meets C_2, \dots, C_k again in distinct points s_2, \dots, s_k . Therefore

$$\sum_{s \in C} f(s, C) \geq \frac{1}{k} + \frac{k-1}{k} = 1.$$

We have

$$|S| = \sum_{s \in S} \sum_{C \in T} f(s, C) = \sum_{C \in T} \sum_{s \in S} f(s, C) \geq n,$$

as desired.

30. Let $S = \{x_0, x_1, \dots, x_n\} \subset [0, 1]$ be a finite set of real numbers with $x_0 = 0$, $x_1 = 1$, such that every distance between pairs of elements occurs at least twice, except for the distance 1. Prove that all of the x_i are rational.

Solution: The set S spans some finite-dimensional vector space over \mathbb{Q} ; let β_1, \dots, β_m be a basis of this vector space with $\beta_m = 1$. For each i , we write

$$x_i = q_{i1}\beta_1 + \dots + q_{im}\beta_m$$

and define the vector $v_i = (q_{i1}, \dots, q_{im})$.

Let us compare the v_i in lexicographic order: $v_i > v_j$ if in the first position where they differ, v_i has the larger component. Let v_s and v_t be the largest and smallest of the v_i , respectively, and suppose a and b are such that $x_s - x_t = x_a - x_b$; since the β_k form a basis, this also implies $v_s - v_t = v_a - v_b$. By our choice of v_s and v_t , this can only happen if $a = s$ and $b = t$. Thus for $0 < i < n$,

$$v_s = (0, 0, \dots, 1) > v_i > v_t = (0, 0, \dots, 0).$$

This is only possible if the v_i only have nonzero entries in the last component, which is to say the x_i are all rational.

31. Suppose that $S = \{1, 2, \dots, n\}$ and that A_1, A_2, \dots, A_k are subsets of S such that for every $1 \leq i_1, i_2, i_3, i_4 \leq k$, we have

$$|A_{i_1} \cup A_{i_2} \cup A_{i_3} \cup A_{i_4}| \leq n - 2.$$

Prove that $k \leq 2^{n-2}$.

Solution: For a set T , let $|T|$ denote the numbers of elements in T . We call a set $T \subseteq S$ *2-coverable* if $T \subseteq A_i \cup A_j$ for some i and j (not necessarily distinct). By the given condition, for any subset T of S at least one of the sets T and $S - T$ is not 2-coverable. Among the subsets of S that are not 2-coverable, let A be a subset with minimum $|A|$.

Consider the family of sets $S_1 = \{A \cap A_1, A \cap A_2, \dots, A \cap A_k\}$. ($A \cap A_i$ might equal $A \cap A_j$ for some distinct i and j , but we ignore any duplicate sets.) Because A is not 2-coverable, if $X \in S_1$, then $A - X \notin S_1$. Thus, at most half the subsets of $|A|$ are in S_1 , and $|S_1| \leq 2^{|A|-1}$.

On the other hand, let $B = S - A$ and consider the family of sets $S_2 = \{B \cap A_1, B \cap A_2, \dots, B \cap A_k\}$. We claim that if $X \in S_2$, then $B - X \notin S_2$. Suppose on the contrary that both $X, B - X \in S_2$ for some $X = B \cap A_\ell$ and $B - X = B \cap A_{\ell'}$. By the minimal definition of A , there are A_i and A_j such that $A_i \cup A_j = A \setminus \{m\}$ for some i, j , and m . Then

$$|A_\ell \cup A_{\ell'} \cup A_i \cup A_j| = n - 1,$$

a contradiction. Thus our assumption is false and $|S_2| \leq 2^{|B|-1} = 2^{n-|A|-1}$.

Because every set A_i is uniquely determined by its intersection with sets A and $B = S - A$, it follows that $k \leq |S_1| \cdot |S_2| \leq 2^{n-2}$.

32. Numbers $1, 2, \dots, 2004$ are arranged in a row such that each number is either greater than all of the numbers to its left or less than all of the numbers to its right. How many such arrangements are there?