Integer Polynomials

1. A polynomial $P(x)$ has integer coefficients. Prove that if the polynomials $P(x)$ and $P(P(P(x)))$ have a common zero, then they also have a common integer zero.

2. Let $f, g \in \mathbb{Z}[x]$ be relatively prime over $\mathbb{Q}$ and define the sequence
$$a_n = (f(n), g(n)).$$
Prove that $(a_n)_{-\infty}^{\infty}$ is periodic.

3. Find all polynomials $f(x) \in \mathbb{Z}[x]$ such that there exists a natural number N, such that for any prime $p > N$, $|f(p)|$ is also prime.

4. Determine all monic polynomials $p(x)$ with integer coefficients of degree two for which there exists a polynomial $q(x)$ with integer coefficients such that $p(x)q(x)$ is a polynomial having all coefficients ± 1.

5. Find all monic polynomials $f(x) \in \mathbb{Z}[x]$ such that $\{f(a) | a \in \mathbb{Z}\}$ is closed under multiplication.

6. Does there exist a sequence of natural numbers $a_0, a_1, a_2, \ldots$ such that for each $i \neq j$, $(a_i, a_j) = 1$ and for each n, the polynomial $\sum_{i=0}^{n} a_i x^i$ is irreducible in $\mathbb{Z}[x]$?

7. Let
$$f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$
be an integer polynomial of degree $n \geq 3$ such that $a_k + a_{n-k}$ is even for all $k = 1, 2, \ldots, n-1$ and a_0 is even. Suppose that $f = gh$, where $g, h \in \mathbb{Z}[x]$, $\deg g \leq \deg h$ and all the coefficients of h are odd. Prove that f has an integer root.

8. Find all triples of integers (x, y, z) such that
$$x^2 + y^2 + z^2 + 2xyz = 1$$

9. Let $p(x)$ be a polynomial with integer coefficients. Determine if there always exists a positive integer k such that $p(x) - k$ is irreducible.

10. Suppose that $f, g \in \mathbb{Z}[x]$ are monic nonconstant irreducible polynomials such that for all sufficiently large n, $f(n)$ and $g(n)$ have the same set of prime divisors. Prove that $f = g$.

11. Is there a polynomial f with integer coefficients that has no rational zeros, but nonetheless has a zero modulo any positive integer?

12. Find all polynomials p with integer coefficients such that, for integers a, b with $a + b$ a perfect square, $p(a) + p(b)$ is a perfect square.

13. Let a, b, c, d, m be positive integers such that $(m, c) = 1$. Prove that there exists a polynomial f with rational coefficients and of degree at most d such that

$$f(n) \equiv c^{an+b} \pmod{m}$$

for all positive integers n if and only if m devides $(c^a - 1)^{d+1}$.

14. Suppose that a polynomial f with integer coefficients has no double zeros. Let $r \geq 1$ be an integer. Prove that there exists an integer n such that the prime factorization of $f(n)$ contains at least r primes occurring with multiplicity 1.

15. Prove that, for every integer $n \geq 2$, there is a polynomial $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ with integer coefficients satisfying the following conditions:

    (a) the product $a_0 a_1 \cdots a_{n-1}$ is nonzero;

    (b) the polynomial $f(x)$ is irreducible in $\mathbb{Z}[x]$;

    (c) for every integer x, $|f(x)|$ is not prime.

16. Let $f \in \mathbb{Z}[x]$ be a nonconstant polynomial, and let $k \geq 2$ be an integer such that $f(n)$ is a perfect k^{th} power for each positive integer n. Prove that there exists a polynomial $g \in \mathbb{Z}[x]$ with $f = g^k$.

17. Prove that for any given positive integer n, there exists a unique polynomial $f(x)$ of degree n with integer coefficients such that $f(0) = 1$ and $(x+1)(f(x))^2 - 1$ is odd.

18. Let m and n be odd numbers with $n > m > 1$. Prove that $f(x) = x^n + x^m + x + 1$ is irreducible in $\mathbb{Z}[x]$.

19. Find all polynomials f with integer coefficients such that $f(n) | n^{n-1} - 1$ for all sufficiently large n.

20. For a non-zero integer n, let $d(n)$ denote the number of positive integer divisors of $|n|$.

    (a) For any polynomial $f(x)$ having integer coefficients and no integer roots, prove that the largest primes divisor of $d(f(n))$ is unbounded as n varies over the integers.

    (b) For polynomials $f(x)$ and $g(x)$ with integer coefficients and no integer roots, prove that if $d(f(n)) = d(g(n))$ for all integers n, then there is some constant c such that $f(x) = cg(x)$.

21. Find all integers n such that $1 + x + x^2 + \cdots + x^{100}$ can be represented as $P^2 + nQ^2$ where P and Q are polynomials with rational coefficients.