New Zealand Mathematical Olympiad Committee

2011 May Problems

1. *Let x_1 and x_2 be the distinct roots of the equation $2x^2 - 3x + 4 = 0$. Compute the value of*
$$\frac{1}{x_1^3} + \frac{1}{x_2^3}.$$

Solution: If α and β are the roots of the quadratic $x^2 - ax + b = 0$ then
$$x^2 - ax + b = (x-\alpha)(x-\beta) = x^2 - (\alpha+\beta)x + \alpha\beta,$$
so $\alpha+\beta = a$ and $\alpha\beta = b$. In our case this gives $x_1 + x_2 = 3/2$ and $x_1x_2 = 2$.
Now also $x^2 = \frac{3}{2}x - 2$, so $x^3 = \frac{3}{2}x^2 - 2x = \frac{3}{2}(\frac{3}{2}x - 2) - 2x = \frac{1}{4}x - 3$. We then get
$$\frac{1}{x_1^3} + \frac{1}{x_2^3} = \frac{x_1^3 + x_2^3}{x_1^3x_2^3} = \frac{\frac{1}{4}(x_1+x_2) - 6}{(x_1x_2)^3} = \frac{\frac{3}{8} - 6}{8} = -\frac{45}{64}.$$
$\square$

2. *Six points are given inside a square of side-length 10, such that the distance between any two of them is an integer. Prove that at least two of these distances are the same.*

Solution: The maximum possible distance between two points in the square is $10\sqrt{2}$, when the points are placed at diagonally opposite vertices. Since $10\sqrt{2} \approx 14.14$, the maximum possible integer distance between two points in the square is 14. So there are fourteen possible integer distances between two points in the square, namely $1, 2, \ldots, 14$. However, six points gives a total of $\binom{6}{2} = \frac{6\cdot 5}{2} = 15$ pairs of points, so there are a total of fifteen distances. By the pigeonhole principle two of them must be the same. $\square$

3. *Find all possible ways of expressing 2010 as a sum of (one or more) consecutive positive integers.*

Solution: Suppose that
$$2010 = \sum_{j=k+1}^{n} j.$$
Then
$$\begin{aligned} 2010 &= \sum_{j=1}^{n} j - \sum_{j=1}^{k} j \\ &= \frac{n(n+1)}{2} - \frac{k(k+1)}{2} \\ &= \frac{n^2 - k^2 + n - k}{2} \\ &= \frac{(n-k)(n+k+1)}{2}. \end{aligned}$$

Hence $(n-k)(n+k+1) = 4020$, so we are interested in factorisations of $4020 = 2^2 \cdot 3 \cdot 5 \cdot 67$. Now $n-k$ and $n+k+1$ differ by $2k+1$, and so have opposite parities; thus, one is divisible by 4, and of course $n-k$ must be the smaller of the two factors. This gives the following possibilities:

$n-k$	1	3	4	5	12	15	20	60
$n+k+1$	4020	2010	1005	804	335	268	201	67
n	2010	671	504	404	173	141	110	63
k	2009	668	500	399	161	126	90	3

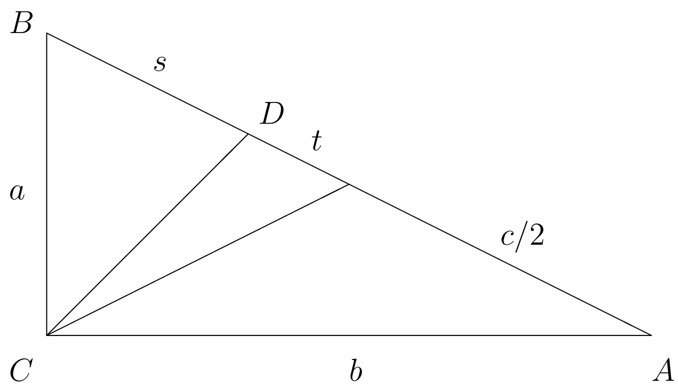
(here we have used $n = ((n-k)+(n+k+1)-1)/2$, $k = ((n+k+1)-(n-k)-1)/2$). Hence

$$\begin{aligned}
2010 &= 2010 \\
&= 669 + 670 + 671 \\
&= 501 + 502 + 503 + 504 \\
&= 400 + 401 + 402 + 403 + 404 \\
&= 162 + 163 + 164 + \cdots + 172 + 173 && \text{(twelve terms)} \\
&= 127 + 128 + 129 + \cdots + 141 && \text{(fifteen terms)} \\
&= 91 + 92 + \cdots + 110 && \text{(twenty terms)} \\
&= 4 + 5 + 6 + 7 + \cdots + 63 && \text{(sixty terms)}
\end{aligned}$$

as a sum of consecutive positive integers. $\square$

4. *In a right triangle the median and bisector of the right angle divide the hypotenuse in three parts. The lengths of these parts, in a certain order, from an arithmetic sequence. Find all possible ratios of the lengths of the legs of the triangle (i.e., of the sides adjacent to the right angle).*

Solution: Let the legs of the triangle have lengths a and b, with $a \leq b$, and let the hypotenuse have length c. Then the median divides the hypotenuse into two segments of length $c/2$, and the bisector then divides one of these into two segments of lengths s and t (see diagram). So the longest of the three segments has length $c/2$, and either $s, t, c/2$ or $t, s, c/2$ is an arithmetic progression, depending on which of s and t is the larger.



Suppose for the moment that $t < s$, as in the diagram. Then $s + t = c/2$, and also $c/2 = t + 2(s - t)$; solving, we find that $s = 2t$. So t, s and $c/2$ are in the ratio $1 : 2 : 3$ or $2 : 1 : 3$.

We now apply the angle bisector theorem, to obtain $a/b = BD/AD$. If $t < s$ then $a/b = 2/4 = 1/2$; and if $s < t$ then $a/b = 1/5$. So the possible ratios of the smaller leg over the longer are $1/2$ and $1/5$.

Note that the problem can be completed without the angle bisector theorem, essentially by proving it in this special case. Use the sine rule to obtain $a/\sin(BDC) = BD/\sin(45°)$ and $b/\sin(CDA) = DA/\sin(45°)$, and then note that $\sin(CDA) = \sin(180° - CDB) = \sin(CDB)$. $\square$

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