Logical Deductions and Applications to Combinatorics

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Logic has many mathematical implications, especially when applied to combinatorics. This is because math is logic (universal axioms) applied to a specific set of non-universal axioms. There are several "classic" logic problem types. We will discuss those most related to the rest of mathematics today.

1 Scale Problems

Scale problems are the most simplistic and, occasionally, the hardest logic problems you will encounter, as they require no special knowledge to solve. They use the concept of a two-pan balance, which gives you an inequality relating the contents on either side of the balance. Note that this can be interpreted mathematically as an algebraic inequality! Thus, if you encounter a particularly ugly looking inequality, you may want to consider this approach.

Here is a nice easy example problem to start with:

A man has n coins, where $n = 3^x$ and x is an integer greater than 0, one of which is slightly heavier than the others. Find, with proof, the minimum number of weighings on a two-pan balance scale to assure the man that he will find the heavy coin.

In this problem, we are faced with one very simple question. How many coins do we put on each side? Obviously, they must be the same amount, or else we would not get any information out of the results of the scale. This means that on either side of the scale, we can place $m \leq \frac{n}{2}$ coins. Now we have to figure out how many to place on each side.

Starting at $m = \frac{n}{2}$, we know that the scale will tip to one side, eliminating $\frac{n}{2}$ coins. Going to $\frac{n}{3}$ it will either

1. tip, eliminating $\frac{2n}{3}$ coins or
2. not tip, again eliminating $\frac{2n}{3}$ coins

In general, if we place m coins on each side, it will either

1. tip, eliminating all but m coins or
2. not tip, eliminating $n - 2m$ coins

Since we are assuming the worst at each step, we can conclude that it is impossible to guarantee the elimination of more than $\frac{2n}{3}$ coins. Thus, to eliminate the most coins, we should weigh $\frac{n}{3}$ coins on each side of the scale.

Now we use our heads for a second and realize that to get the actual least number of weighings, we use our good friend, the log base 3. (We decided to use $\frac{n}{3}$ remember.) So the answer is x weighings.

This is a fairly straightforward problem, but it demonstrates the point that with a little inspection, one can reduce seemingly nasty combinatorics problems to nice easy logic problems!

Of course, there's more than one thing you can do with a scale, but those will be practice problems later on.

2 Hat Problems

Another thing logicians love is hats. Please don't ask me why, maybe the first logician was poor and only sold hats to monkeys. However, they lead to some interesting results. Fist off, there are two kinds of hat problems, the kind where everyone knows everything but can't communicate, and the kind where only I know everything and must resort to some elegant code because there are constraints on my vocabulary. We will call these type 1 and 2, respectively.

Type 1:

In type one problems, there are generally two ways to solve them - By simple case analysis, or by recursion. There are many more type one problems than type two problems, as the type 1 problems are considerably easier. Again, rather than explaining things, it's easier to look at 2 examples.

Example 1: There are some gnomes who are cheats. They find flaws in simple games of chance that allow them to always win. This was their most famous con.

$2k$, where k is an integer greater than or equal to 5, of these mean, cheating gnomes go to the queen of GnomeLand and make a business proposal. They proposed that the queen could place hats on their heads. These hats may be either black or white. The hats are randomly distributed and the number of each hat color will vary from round to round. The gnomes may only see the colors of the other gnome's hats - that is, they cannot know the color of their own hat, only the colors of their peers' hats. After that, they are sent to different rooms and asked for the color of their hat. If they get the color of the hat right, they get a point. If they guess incorrectly, they lose a point. After the queen has asked each gnome for its hat color, the points are tallied. If the gnomes have at least one point, they get ten gold coins. If they have less than that, they must pay 100 gold coins. If all the gnomes guess incorrectly, no one pays because that means the gnomes were having bad luck, but if even one guesses their hat color correctly, then they have to pay. Prove that the gnomes can devise a strategy by which they never lose.

This problem may look difficult or impossible at first, but instead of sitting there and being intimidated, you should try to solve it! First, let us look at what we know (this is generally a good way to do all problems you ever face in life, so lets start by writing what we know down):

1. The gnome knows what the other hats are
2. There are 2 colors of hats
3. There are an odd number of hats in front of the gnomes
4. Only half the gnomes need get the answer correct OR all of them can answer incorrectly (this turns out to be the key observation)

Now we consider what we can derive from this. If this problem is possible, then the gnomes must be able to get half of the answers correct. So they could either say, 1 predetermined hat color, or a function of the hat colors in front of them. Immediately we see that saying the same hat color is wrong because I could have 2k-1 black hats and 1 white hat, forcing them to lose.

So now, what can we derive from the hats before us (yes we are all now gnomes)? Well, hopefully the obvious choice is we can say the color of the majority of the hats, or the color of the minority of the hats. Obviously we can rule out the minority, with 2 white hats and 2k-2 black hats. Thus we are left with the majority. Let us examine this case. If there are a majority of black hats in the room, then everyone will say black, and the net tally will be greater than 1, the desired result. However, if there are an equal number of black and white hats, everyone will say the wrong color. Thus, we fall into the last case of the problem, that if no one gets the answer right, it is still a winning case. A minority of black hats yields a majority of white hats, so the logic holds. Therefore, we have proved this problem by construction (offering with proof a scheme that works).

Here, we see the power of case analysis. As it turns out, there is a slightly harder problem similar to this, which requires more cases, but in general, this, or one other line of logic will hold for all problems of this type.

Example 2: The old gnome professor at the logic conservatory for the gnomes has recently had budget cuts! He originally was able to have 7 assistants and now must make due with one. He decides that he wants to keep only the most logical one, and therefore decided to set up a test. He would give each one of the gnomes a hat, either red or blue. The gnomes can see the color of all the other hats in the room, just not their own. The professor, being as clever as he was, set up the room in such a way that all the participants had an equal chance at winning the position. After placing hats on everyone's head, he provided the following rules:

1. The instant you know what your hat color is, you must say it.
2. If you see a red hat, clap until one of you has gotten your hat color.
3. Guessing one's hat color is forbidden. Should you guess, you will be banished from GnomeLand. Thus I will require your reasoning.

When the professor told the gnomes to start clapping, they all clapped. And clapped. And clapped. After about 7 minutes, one of the gnomes stood up and said his hat color. The professor, skeptical of his ability to actually get the answer asked him for the reason why. What was the color of his hat, his reason, and the total number of blue and red hats in the room? (Give proof for your answer)

Solution: First of all, we want to know how many hats are in the room. We begin by looking at a smaller sample, of say 3 (we want to keep it small, if there is only one person, there is too little information, if there are 2 people, there is too much). Here we see that for everyone to clap, there must be at least 2 red hats (this is true if there are n people as well). However, if that were the case, the people wearing blue hats would be at a disadvantage, because the people with red hats would know their hat color. Thus, to make everything fair, there must be 3 red hats. Intuitively, this should still be true if we move on to the case of 4, then 5, then 6, then 7. We can confirm this by examining the gnome's reasoning.

Here we must realize that this is some generic gnome who knew, not a specific gnome, so it doesn't really matter who (as we are guaranteed that everyone has an equal chance). So again, we visit the nice easy case of 3 people. We list all the possibilities. There are 2 * 2 * 2 or 8 total possibilities. We throw out a few because everyone is clapping. We are left with RRB (we being blue) and RRR. Here we realize that if it were RRB someone else would get the answer immediately. Because no one has, our hat must be red. So now, we look at the case of 4. If it is RRRB (the only case we need to care about) they will simply ignore us and proceed to solve the RRR case. But, as we have given them a bit of time, and they still don't know, a minute later we stand up and give the exact same reason. We apply the same logic for 5, 6, and then 7! And thus we have demonstrated that we must have a red hat. And from the point of view of someone who doesn't know that there are only red hats, we have given proof. From here we can also see that, indeed, the only fair arrangement of hats is RRRRRRR (as our deductions relied on the fact that other people could not make deductions). We can easily generalize this result with induction.

Type 2:

I have only ever come across 2 problems in this category, which are related and are both really good. Here's the easy one (because these problems are from a compilation of gnome problems I wrote, there are remnants of the story line. They are not important and can be ignored):

Problem 1: By now the queen has become an evil witch. She has decided that because all the gnomes who swindled her ran away, she will simply kill ALL THE GNOMES! Of course, the gnomes won't let that happen. They know that the code of GnomeLand says that they must have a chance to defend themselves in all cases.

The witch has decided that because she lost at a game of guessing hat colors she will kill them in the same way. She has them all form a single file line and places a hat on each gnome's head. She then starts at the back of the line and asks the gnome the color of his hat. If he gets it right, he is allowed to live, if he gets it wrong, he dies. He is only allowed to say the colors black or white, which as it turns out are the colors of the hats.

The gnomes, as always, were prepared. They had devised a strategy by which the person at the end's answer would notify the rest of the gnomes of the color of their hats. It was a code of sorts. There was no set ratio of black to white hats, and all the gnomes knew for certain what their hat colors were based entirely on the first person's response. (Clarification: the guy at the back of the line, the one asked first can see everyone else's hat color. The guy at the front of the line, who heard everyone else's response, is asked last. Everyone else has a combination of the two.) Given that there are X gnomes, where $X > 1$ is an integer, determine the greatest number of gnomes who can be saved.

That's right; you have to work this one out on your own. (Hint: the answer is more than $\frac{X}{2}$, but less than X because we can assume the first guy to speak dies, as he has no information.)

3 General Logic Problem

It is important to be able to be able to voice your intuition when writing proofs. Here is a famous problem, which forces you to be able to accurately write down what your intuition tells you and why your intuition is correct.

Given a set of non-collinear points on the xy-plane, prove that one can create a polygon in which every point lies on the edge (particularly, unless there are 3 or more collinear points in the set, each point must be at a vertex of the polygon)

Of course, there are several solutions to this problem. However, for the solution to be valid, one must prove that their solution will hold, that is, prove that it will generate a polygon for any set of points.

4 Practice Problems

These problems are substantially harder than the previous problems, but some patience, application of previous principles and a bit of cleverness will get you to the answer.

1. Scale problem (Hard): Given 12 coins, one of which is defective (heavy or light, you don't know which), find the least number of weighings needed to find the defective coin and determine if it is light or heavy. (You will want to know the number before attempting to prove it.) (This is an interesting problem if you justify your steps, we will discuss it before the session is over.)

2. Scale Problem (Really, Really, Really, Crazy, Hard): Given a scale which is accurate to ± 1 Kg (that is 1 Kg = 2 Kg = 3 Kg but 1 Kg < 3 Kg) determine, with proof, the least number of weighings needed to sort a set of weights weighing 1,2,3,4, and 5 Kgs.

3. Type 1 logic problem: The Dark Lord from The West has captured a bunch of gnomes and put them on an island. He then placed a colored hat on each gnome's head. He told the gnomes that the color of their hat could be either black or white, and there would be at least one of each. The Dark Lord

then said that all gnomes with white hats must commit suicide before the end of the week. He also caused the gnomes to be mute, so they could not communicate. If they attempted to communicate with each other nonverbally, they all would die. Prove that the gnomes could perform this seemingly impossible feat, or prove that they could not. (PS they can have a prearranged plan before they get on the island, such as town meetings every hour. Assume that the number of gnomes is less than 30 and greater than 5. Note that the bounds actually have no bearing on the problem.)

4. This is the next of the great problems of GnomeLand. There is a hallway full of switches. The first person to run down the hallways flips the first switch, the second switch, the third switch, etc. The second person to go down the hall flips the second switch, the fourth switch, the sixth switch, etc. The third person to go down the hall flips the third switch, the sixth switch, the ninth switch, etc. There are as many people as there are switches. When everyone has finished going down the hallway, what is the position of every switch? (ok, so this one is just easy, there is a very specific kind of number associated with each switch in the on or off positions. Prove that they must be that kind of number.)

5. Same as the example Type 2 problem, only now, there are 5 different colored hats. Determine, with proof, the most number of gnomes that can be saved. (Your answer to the first problem will give you a hint, you probably just need to generalize it more.)

6. Type 1, Example 1, except now there are $2k + 1$ gnomes. Prove the same things.

7. There are 7 hats: 4 Red, 3 Blue. Seven gnomes lined up (much like the line in Example 1, Type 2) and were each given a hat. Starting at the end of the line (the one who can see everything), the gnomes were asked for the color of their hat. The gnomes asked 2nd and 4th did not know the color of their hat. The rest of the people did. Determine, with proof, the hat colors of the last 3 gnomes to be asked.

8. A woman walked up to a house one day and rang the doorbell. When an elderly man answered the door, the woman asked "what are the ages of your three daughters?" The man replied that the product of their ages was 72. To this the woman said she could not find the answer. Then the man added that the sum of their ages was the house number. Again the woman could not get the answer. Finally the man said that his oldest daughter liked chocolate milk. The woman then knew the ages of the three daughters. What were their ages? (Case analysis is very helpful here)

9. Here's an easy table logic problem, which most of you would have associated with logic before this lecture.

   (a) Alan's house number and Calvin's house number are the roots of the equation y = $2x^3$ - $7x^2$ + 2x + 3.
   (b) The person in house number 1 owns the elephant.
   (c) Unlike Alan and Calvin, the ant and the cat are friends, so they are neighbors, while Alan and Calvin are not.
   (d) The ant lives in the house whose number is the units digit of the sum of the house numbers plus nine.
   (e) Dora owns the dog and Bobbie owns the bat, but no one else own animals who start with the same letter as their name.
   (f) Bobbie has two neighbors.

   (Hint: Making a table helps)

10. Solve the first example problem where n is simply an integer greater than or equal to 10 (you will need to add slightly to your answer, as one cannot have fractional weighings).

11. Construct a statement that will yield outcome a if a true statement will yield outcome b and a false statement will yield an outcome where nothing happens.