TJUSAMO 2011 - Olympiad Geometry, Part 1

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1 Introduction to Olympiad Geometry

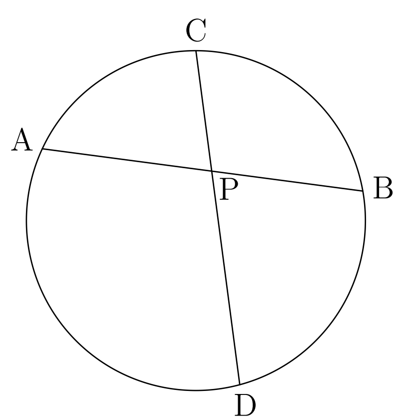
We now turn from the more computationally intensive, numerically-focused area of number theory to the highly visual, concrete, non-computationally intensive field of olympiad geometry. There are not many high-tech theorems one needs to know for the USAMO, although that of course helps. Olympiad geometry tends to involve simple concepts applied in inventive ways. As with all other areas, getting good at geometry problems requires practice. However, this is especially true for geometry, as it is a subject which most people are uncomfortable with, and a subject where intuition is very important. This is the first of at least three geometry lectures we'll have at TJUSAMO, so this one starts with the basics you should be familiar with – circles, triangles, similarity. Once we have the basics out of the way, though, we'll move to deeper waters for the later lectures, where we'll focus transformations and weird things you can do the plane.

2 Circles

Circles are simple to define, but in many ways are the essence of Euclidean geometry. In two dimensions, a circle is the set of points equidistant from the center of the circle. An open disk is the interior of the circle; a closed disk is the union of the circle and its interior.

2.1 Power of a Point

The *power* of a point is a positive real number that reflects the relative distance of a given point from a given circle. The exact definition is that the power $h = s^2 - r^2$, where s is the distance from the point P to the center of circle and r is the radius. Alternatively, given a circle ω and a point P, draw a line through P intersecting ω at A and B. Then the power of a point theorem states that the product $PA \cdot PB$ depends only on P and ω, not on the line. To prove this, draw another line through P meeting ω at C and D as below, and look for similar triangles.



3 Triangles

Three non-collinear points determine a triangle, a circle, and a plane. There is a specific set of terms associated with triangles that you need to be familiar with. Hopefully you've seen all of the following words before, but they're easy to confuse sometimes so we've included a refresher.

- Cevian - any line segment in a triangle with one endpoint on a vertex of the triangle and the other endpoint on the opposite side.
- Median - a line segment joining a vertex to the midpoint of the opposing side.
- Centroid - the intersection point of the three medians. Denoted by G.
- Incenter - the center of the circle inscribed within a triangle, also the point of intersection of the three angle bisectors.
- Circumcenter - the center of the circle circumscribed about the triangle, also the point of intersection of the perpendicular bisectors of the sides. Denoted by O.
- Orthocenter - the point at which the three altitudes intersect. Denoted by H.
- Excenter - the center of a circle escribed about a triangle (tangent to one side and the extensions of the two other sides).
- Euler Line - the line connecting the circumcenter O, centroid G, and orthocenter H.

4 Similar Triangles

$\triangle ABC$ and $\triangle DEF$ are called *similar* iff they have proportional sides and equal angles. These two conditions, however, are equivalent. The basic idea is this: when you see parallel lines, look for similar triangles and use them to prove things about ratios. Then, use those ratios to find more similar triangles.

5 Cyclic Quadrilaterals

A polygon is *cyclic* if and only if there is a circle which passes each of its vertices. A set of points is called *concyclic* if the points are vertices of a cyclic polygon. Cyclic quadrilaterals pop up all the time in Olympiad geometry problems, and they become much easier to solve once cyclic quadrilaterals can be recognized immediately. Take note of the following statements in particular:

Cyclic criteria. *If $ABCD$ is a convex quadrilateral, then the following are equivalent:*

- *$ABCD$ is cyclic.*
- $\angle ABC + \angle CDA = \pi$.
- $\angle ACB = \angle ADB$.
- *The perpendicular bisectors of AB, BC, CD are concurrent.*
- *If AC intersects BD at E, then $AE \cdot CE = BE \cdot DE$. (This is one of the many special cases of the Power of a Point theorem.)*

6 Geometry Problem Solving Tips

1. Draw a diagram. I repeat: draw a BIG, clean, somewhat-to-scale diagram. Use a full clean sheet of paper for a diagram (or more!). If there are multiple cases, draw multiple diagrams. More paper used is better than more clutter. If your diagram gets messy, draw a new one. Try to keep the mess down by using a compass/ruler (note: "ruler," not "straightedge") and avoiding needless constructions.

2. If problems contain generic triangles, try to make them as scalene as possible. Haitao recommends a $45^\circ - 60^\circ - 75^\circ$ triangle.

3. Don't just stare at the page. It's difficult to solve problems without writing things down. Write down all the givens in the problem in the simplest form you can find. You will almost always need every single given in the problem, so be aware of ones you haven't used yet.

4. Prove the converse of what you are trying to prove: take the result you want to prove and try to work back to your givens. Many times, you will find that all your steps are directly reversible, or they can be reversed by an application of "phantom points."

5. If a problem asks about a point satisfying a certain condition, find a way to construct it.

6. If a problem asks about a line, it is often helpful to think of that line as a locus.

7. If two lines "look parallel" in your diagram, draw more diagrams to confirm that these lines might always be parallel, and then try to prove it. (This technique is not limited to parallel lines: you might apply it to segments or angles which "look equal.")

8. Don't forget about trig, complex numbers, and coordinates. That said, try to avoid them unless absolutely necessary.

9. Try multiple ideas; don't just get stuck with one idea for too long. If it's clear you aren't making progress with a construction, try something else.

7 Problems

1. Let $\triangle ABC$ be a triangle with orthocenter H and circumcenter O. Prove that $\angle HAO = |\angle B - \angle C|$.

2. Let a, b, c, d satisfy the property that $2a, 2b, 2c, 2d$ are all less than $a + b + c + d$. Prove that there is a unique (up to congruence) cyclic quadrilateral $ABCD$ with $AB = a, BC = b, CD = c, DA = d$.

3. Let ABC be a triangle, and let D be on BC such that AD bisects $\angle BAC$. Prove that $\dfrac{AB}{BD} = \dfrac{AC}{CD}$.

4. Let $ABCD$ be a quadrilateral with an inscribed circle. Prove that $AB + CD = AD + BC$. Is the converse true?

5. Triangle ABC has orthocenter H and altitudes AD, BE, CF. Prove that $HE \cdot AC + HF \cdot AB = AH \cdot BC$.

8 More Problems

6. Let $ABCD$ be a quadrilateral and let O be a point in the plane of the quadrilateral. $H_{AB}, H_{BC}, H_{CD}, H_{DA}$ are the projections of O onto lines AB, BC, CD, DA respectively. H_A, H_B, H_C, H_D are the projections of O onto $H_{DA}H_{AB}, H_{AB}H_{BC}, H_{BC}H_{CD}, H_{CD}H_{DA}$ respectively. Suppose that H_A, H_B, H_C, H_D are concyclic. Prove that $ABCD$ is cyclic.

7. Let $AXYZB$ be a convex pentagon inscribed in a semicircle of diameter AB. Denote by P, Q, R, S the feet of the perpendiculars from Y onto lines AX, BX, AZ, BZ, respectively. Prove that the acute angle formed by lines PQ and RS is half the size of $\angle XOZ$, where O is the midpoint of segment AB.

8. Let A, B, C, D be points occurring in that order on circle ω and let P be the point of intersection of AC and BD. Let EF be a chord of ω passing through P, Q be the point of intersection of BC and EF, and R be the point of intersection of DA and EF. Prove that $PQ = PR$. This is known as the *Butterfly Theorem*.

9. Let the incenter of triangle $\triangle ABC$ touch side BC at D, and let DT be a diameter of the circle. If line AT meets BC at X, prove that $BD = CX$.

10. Let $ABCD$ be a cyclic quadrilateral. Let P, Q, R be the feet of the perpendiculars from D to the lines BC, CA, AB, respectively. Show that $PQ = QR$ if and only if the bisectors of $\angle ABC$ and $\angle ADC$ are concurrent with AC.

11. Let P be a point interior to triangle ABC (with $CA \neq CB$). The lines AP, BP and CP meet again its circumcircle Γ at K, L, respectively M. The tangent line at C to Γ meets the line AB at S. Show that from $SC = SP$ follows $MK = ML$.