TJUSAMO 2011 - Olympiad Geometry, Part 2

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1 Problems from Last Week

1. Let $ABCD$ be a quadrilateral and let O be a point in the plane of the quadrilateral. $H_{AB}, H_{BC}, H_{CD}, H_{DA}$ are the projections of O onto lines AB, BC, CD, DA respectively. H_A, H_B, H_C, H_D are the projections of O onto $H_{DA}H_{AB}, H_{AB}H_{BC}, H_{BC}H_{CD}, H_{CD}H_{DA}$ respectively. Suppose that H_A, H_B, H_C, H_D are concyclic. Prove that $ABCD$ is cyclic.

2. Let A, B, C, D be points occurring in that order on circle ω and let P be the point of intersection of AC and BD. Let EF be a chord of ω passing through P with P its midpoint, Q be the point of intersection of BC and EF, and R be the point of intersection of DA and EF. Prove that $PQ = PR$. This is known as the *Butterfly Theorem*.

3. Let $AXYZB$ be a convex pentagon inscribed in a semicircle of diameter AB. Denote by P, Q, R, S the feet of the perpendiculars from Y onto lines AX, BX, AZ, BZ, respectively. Prove that the acute angle formed by lines PQ and RS is half the size of $\angle XOZ$, where O is the midpoint of segment AB.

2 Concurrency and Collinearity

Often, a problem will ask you to prove concurrency or collinearity of points. Here are some basic ways to prove concurrency of lines AX, BY, CZ.

1. Turn it into a collinearity problem: C, Z, and $AX \cap BY$ (note: $\cap$ denotes intersection)

2. Describe the lines as loci. (Use of the Radical Center Theorem is a special case of this technique.)

3. Suppose that $AX \cap CZ = P$ and $BY \cap CZ = Q$, then prove that $P = Q$. (This can be proven in several ways; for example, $CP = CQ$.)

4. Use Ceva's Theorem.

Here are some basic ways to prove collinearity of A, B, C:

1. Find some point P with $\angle PAB = \angle PAC$.

2. Find some point P with $\angle PAB + \angle CAP = \pi$.

3. Use Menelaus's Theorem.

3 Ceva's Theorem

Let AD, BE, and CF be cevians of triangle $\triangle ABC$. These cevians concur if and only if

$$\frac{AE}{EC} \cdot \frac{CD}{DB} \cdot \frac{BF}{FA} = 1.$$

This is proven using the phantom point combined with area ratios. Ceva's theorem also has a very useful trigonometric form:

$$\frac{\sin \angle ABE}{\sin \angle EBC} \cdot \frac{\sin \angle CAD}{\sin \angle DAB} \cdot \frac{\sin \angle BCF}{\sin \angle FCA} = 1.$$

This is derived from the standard form of Ceva's theorem using the law of sines, and its use will often be accompanied by several uses of the law of sines.

4 Menelaus's Theorem

If and only if points X, Y, Z on the extensions of sides BC, CA, AB of triangle $\triangle ABC$ are collinear, then

$$\frac{BX}{CX} \cdot \frac{CY}{AY} \cdot \frac{AZ}{BZ} = -1.$$

The appearance of -1 on the RHS is a consequence of the use of directed lengths. This theorem occasionally appears on Olympiad geometry problems. To avoid its use, it is possible to replace collinearity problems with concurrency problems.

5 Potpourri

- Let A, B, C, D be points in the plane. Assume $A \neq B$ and $C \neq D$. Then lines AB and CD and perpendicular if and only if $AC^2 + BD^2 = AD^2 + BC^2$.

6 Radical Axis

Given two circles in the plane, their radical axis is the locus of points of equal power to the two circles. As an exercise, try to prove that this is always a line, as it so happens to be the case. If the two circles intersect, then note that the radical axis must pass through the two intersection points of the circle, so it is concurrent with their common chord.

7 Symmedian

Let ABC be a triangle and Γ be its circumcircle. Suppose the tangents to Γ at B and C meet at D. Then AD is the *symmedian*, or reflection of the median across the angle bisector of $\angle A$. Try to prove this in three ways. For the first proof, let the reflection of AD across the angle bisector of $\angle A$ meet BC at M'. Then consider $\frac{BM'}{M'C}$ and use the Law of Sines. For the second proof, draw the circle centered at D with radius DB, and look for similar triangles. For the third proof, let the tangent of Γ at A meet line BC at E. Think about a projection.

8 Isogonal Conjugates

The symmedian point is defined as the point the centroid is mapped to when it is reflected over the angle bisector. This is one example of an isogonal conjugate. In general, the isogonal conjugate $P*$ of a point P is the point constructed by reflecting the lines PA, PB, and PC about the angle bisectors of A, B, C. The three reflected lines concur at the isogonal conjugate. This is a transformation of the points inside the triangle, as can be seen in Figure 1.

9 Pascal's Theorem

If the three pairs of opposite sides in a cyclic hexagon all intersect, then the three points of intersection are collinear.

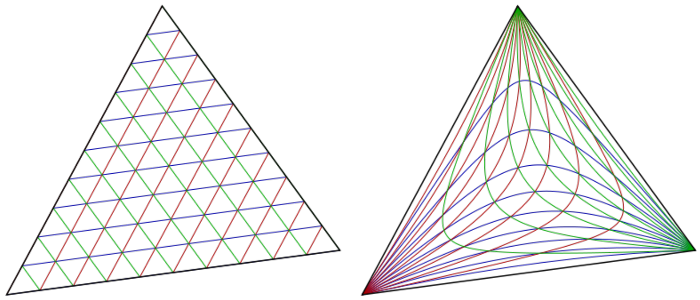

Figure 1: Isogonal conjugate transformation over points in the triangle.

10 Problems

These problems are extremely important, as much of the theoretical substance of this lecture is hidden among the problems here. This means that it is especially important for you to work on these outside of TJUSAMO time and verify your solutions with either of us.

1. Let $\triangle ABC$ be a triangle. Points X, Y, and Z lie on sides BC, CA, and AB, respectively. Prove that the circumcircles of triangles $\triangle AYZ, \triangle BXZ, \triangle CXY$ meet at a common point. This is *Miquel's Theorem*.

2. Let C_1, C_2, C_3 be circles with distinct radii. Prove that the the centers of the external homotheties taking $C_1 \mapsto C_2$, $C_2 \mapsto C_3$, and $C_3 \mapsto C_1$ are collinear. This is *Monge's Theorem*.

3. Prove that if equilateral triangles are erected externally on the sides of any triangle, their centers form an equilateral triangle. This is *Napoleon's Theorem*.

4. Consider a convex pentagon $ABCDE$ such that
$$\angle BAC = \angle CAD = \angle DAE, \qquad \angle ABC = \angle ACD = \angle ADE$$
Let P be the point of intersection of the lines BD and CE. Prove that the line AP passes through the midpoint of the side CD.

5. Let $\triangle ABC$ be a triangle, and let P be another point on its circumcircle. Let X, Y, Z be the feet of perpendiculars from P to lines BC, CA, AB respectively. Prove that X, Y, Z are collinear. The line going through X, Y, Z is known as the *Simson line*.

6. Let ABC be a triangle whose intouch triangle (that is, the triangle whose vertices are the points of tangency of the incircle of $\triangle ABC$ with the sides of $\triangle ABC$) is $\triangle A_1B_1C_1$.
   (a) Prove that AA_1, BB_1, CC_1 are concurrent. The point of concurrency is known as the *Gergonne point*.
   (b) Prove that if A_2, B_2, C_2 are points on the incircle of ABC such that AA_2, BB_2, CC_2 are concurrent, then A_1A_2, B_1B_2, C_1C_2 are also concurrent.

7. Let $\triangle Q_1Q_2Q_3$ be the cevian triangle of point P with respect to $\triangle P_1P_2P_3$. Let $\triangle R_1R_2R_3$ be the cevian triangle [1] of point Q with respect to $\triangle Q_1Q_2Q_3$. Prove that the lines P_1R_1, P_2R_2, P_3R_3 are concurrent. This is the *Cevian Nest Theorem*.

[1] The cevian triangle of a point P with respect to triangle ABC is the triangle with vertices $AP \cap BC$, $BP \cap AC$, and $CP \cap AB$, where $\cap$ denotes intersection.

8. Let $C, C_1, C_2, C_3, C_4, C_5, C_6$ be circles such that C is tangent to C_i at P_i for all i, and C_i is externally tangent to C_{i+1} for all i, with C_6 externally tangent to C_1. Prove that P_1P_4, P_2P_5, P_3P_6 are concurrent. This is the *Seven Circles Theorem*.

9. Points A_1, B_1, and C_1 are chosen on sides BC, CA, and AB of a triangle ABC, respectively. The circumcircles of triangles AB_1C_1 , BC_1A_1 , and CA_1B_1 intersect the circumcircle of triangle $\triangle ABC$ again at points A_2, B_2, and C_2 , respectively ($A2 = A, B2 = B$, and $C2 = C$). Points A_3 , B_3 , and C_3 are symmetric to A_1, B_1 , C_1 with respect to the midpoints of sides BC, CA, and AB, respectively. Prove that triangles $A_2B_2C_2$ and $A_3B_3C_3$ are similar.

10. Let $\triangle ABC$ be an acute, scalene triangle and let M, N, and P be the midpoints of BC, CA, and AB respectively. Let the perpendicular bisectors of AB and AC intersect ray AM in points D and E respectively, and let lines BD and CE intersect in point F, inside of triangle $\triangle ABC$. Prove that points A, N, F, and P all lie on one circle.

11. Circles w_1 and w_2 with centres O_1 and O_2 are externally tangent at point D and internally tangent to a circle w at points E and F respectively. Line t is the common tangent of w_1 and w_2 at D. Let AB be the diameter of w perpendicular to t, so that A, E, O_1 are on the same side of t. Prove that lines AO_1, BO_2, EF and t are concurrent.