MOSP Black Group
June 28, 2004

Test 6

1. [3pts] Let a, b and c be positive real numbers. Prove that
$$\frac{a^2+2bc}{(a+2b)^2+(a+2c)^2}+\frac{b^2+2ca}{(b+2c)^2+(b+2a)^2}+\frac{c^2+2ab}{(c+2a)^2+(c+2b)^2}\le\frac{1}{2}.$$

2. [5pts] Three distinct points A, B, and C are fixed on a line in this order. Let ω be a circle passing through A and C whose center does not lie on line AC. Denote by P the intersection of the tangents to ω at A and C. Suppose that ω meets segments PB at Q. Prove that the intersection of the bisector of $\angle AQC$ and line AC does not depend on the choice of ω.

3. In the coordinate plane, color the lattice points which have both coordinates even black and all other lattice points white. Let P be a polygon with black points as vertices. Prove that any white point on or inside P lies halfway between two black points, both of which lie on or inside P.

4. [7pts] Let $\mathbb{R}$ denote the set of real numbers. Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that
$$f(x)f(yf(x)-1)=x^2f(y)-f(x)$$
for all real numbers x and y.

b1. In the coordinate plane, color the lattice points which have both coordinates even black and all other lattice points white. Let P be a polygon with black points as vertices. Prove that any white point on or inside P lies halfway between two black points, both of which lie on or inside P.

b2. Let A_1 and B_1 be two points on the base AB of an isosceles triangle ABC ($\angle C > 60^\circ$) such that $\angle A_1CB_1 = \angle ABC$. A circle externally tangent to the circumcircle of triangle A_1B_1C is tangent also to the rays CA and CB at points A_2 and B_2, respectively. Prove that $A_2B_2 = 2AB$.

b3. Let $\mathbb{R}$ denote the set of real numbers. Find all functions $f:\mathbb{R}\to\mathbb{R}$ such that
$$f(x)f(yf(x)-1)=x^2f(y)-f(x)$$
for all real numbers x and y.

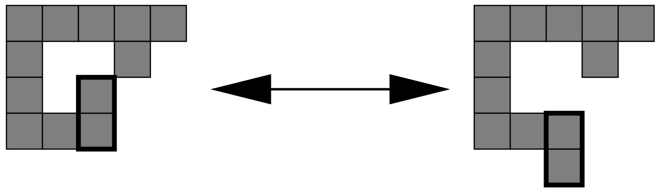
k1. A robot is placed on an infinite square grid; it is composed of a (finite) connected block of units occupying one square each. A *valid subdivision* of the robot is a partition of its units into two connected pieces which meet along a single unbroken line segment. The robot moves as follows: it may divide into a valid subdivision, then one piece may slide one square sideways so that the result is again a valid subdivision, at which point the pieces rejoin. (See diagram for examples.)

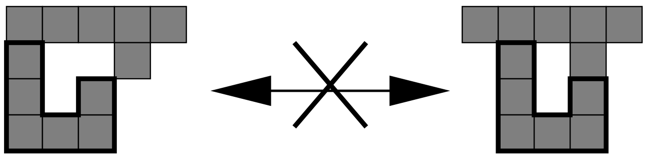
We say a position of the robot (i.e., a connected block of squares in the plane) is *row-convex* if

(a) the robot does not occupy only a single row or only a single column, and

(b) no row meets the robot in two or more separate connected blocks.

Prove that from any row-convex position in the plane, the robot can move to any other row-convex position in the plane.





k2. Given integer a with $a > 1$, an integer m is *good* if $m = 200a^k + 4$ for some integer k. Prove that, for any integer n, there is a degree n polynomial with integer coefficients such that $p(0), p(1), \ldots, p(n)$ are distinct good integers.

k3. Let ABC be a triangle with ω and I with incircle and incenter, respectively. Circle ω touches the sides AB, BC, and CA at points C_1, A_1, and B_1, respectively. Segments AA_1 and BB_1 meet at point G. Circle ω_A is centered at A with radius AB_1. Circles ω_B and ω_C are defined analogously. Circles ω_A, ω_B, and ω_C are externally tangent to circle ω_1. Circles ω_A, ω_B, and ω_C are internally tangent to circle ω_2. Let O_1 and O_2 be the centers of ω_1 and $omega_2$, respectively. Lines A_1B_1 and AB meet at C_2, and lines A_1C_1 and AC meet at B_2. Prove that points I, G, O_1, and O_2 lie on a line ℓ that is perpendicular to line B_2C_2.