Games

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Game problems

Many problems that appear on contests are games that involve one or more player and ask for a winning strategy, which is often in the form of an algorithm or process that can be applied at each step. While game problems vary widely in topic and require a lot of creativity, there are some general strategies to keep in mind when solving game problems:

- Find an invariant or monovariant. Often if some quantity can be kept invariant or always made to decrease, it will help us find a winning strategy.
- Assign numbers to the objects being played with (squares on a checkerboard, for instance) in order to create the right invariant with those numbers.
- Look at the problem from various points of view. If there are two players, pretend that you are each for a while to see which has a winning strategy.
- Get your hands dirty. Play the game. It's not wasting time; it's helping you build intuition.
- Try some small cases. If the game is played on a 100×100 board, first play it on a 2×2 and see what happens.
- Look for parity arguments or other useful congruences.
- Consider the penultimate step, and work backwards. What is the last step that must be taken before winning the game? What steps can lead to that?
- Try contradiction. Assume that one of the players has a winning strategy, and prove that the other player can actually use this to win.
- Induct. Sometimes we are playing a game with a certain number of stones or a certain size checkerboard, and it can't hurt to try induction. Sometimes a winning strategy for a size-n game can be constructed from a winning strategy for a size-$(n-1)$ game.
- Remember that you don't always have to *find* an explicit winning strategy. Sometimes you only have to prove that there *is* (or is not) one, by using induction, contradiction, invariants, etc.

Problems

1. (HMMT 2009, Tedrick Leung.) Stan has a stack of 100 blocks and starts with a score of 0, and plays a game in which he iterates the following two-step procedure:

   (a) Stan picks a stack of blocks and splits it into 2 smaller stacks each with a positive number of blocks, say a and b. (The order in which the new piles are placed does not matter.)

   (b) Stan adds the product of the two piles' sizes, ab, to his score.

   The game ends when there are only 1-block stacks left. What is the expected value of Stan's score at the end of the game?

2. (HMMT 2010, Winston Luo.) Consider the following two-player game. Player 1 starts with a number, N. He then subtracts a proper divisor of N from N and gives the result to player 2. Player 2 does the same thing with the number she gets from player 1, and gives the result back to player 1. The two players continue until a player is given a prime number or 1, at which point that player loses. For which values of N does player 1 have a winning strategy?

3. (USAMO 1999.) The Y2K Game is played on a 1×2000 grid as follows. Two players in turn write either an S or an O in an empty square. The first player who produces three consecutive boxes that spell SOS wins. If all boxes are filled without producing SOS then the game is a draw. Prove that the second player has a winning strategy.

4. (Berkeley Math Circle.) There are x red chameleons, y blue chameleons, and z yellow chameleons playing a game. At each step, two chameleons of different colors simultaneously change to the third color. Their goal is to all become the same color. For which triples (x, y, z) of nonnegative integers can the chameleons win the game?

5. (USAMO 2005.) Alice and Bob play a game on a 6 by 6 grid. On his or her turn, a player chooses a rational number not yet appearing on the grid and writes it in an empty square of the grid. Alice goes first and then the players alternate. When all squares have numbers written in them, in each row, the square with the greatest number is colored black. Alice wins if she can then draw a line from the top of the grid to the bottom of the grid that stays in black squares, and Bob wins if she can't. (If two squares share a vertex, Alice can draw a line from on to the other that stays in those two squares.) Find, with proof, a winning strategy for one of the players.

6. (Melanie Wood's 2005 MOP handout.) The numbers 1 through n are written on a blackboard. Zuming and Ian take turns erasing one of the numbers m along with any divisors of m that are still on the board. The player who erases the last number wins the game. If Zuming goes first, which of Zuming or Ian has a winning strategy?

7. (USAMO 1994.) The sides of a 99-gon are initially colored so that consecutive sides are red, blue, red, blue, $\ldots$, red, blue, yellow. We make a sequence of modifications in the coloring, changing the color of one side at a time to one of the three given colors (red, blue, yellow), under the constraint that no two adjacent sides may be the same color. By making a sequence of such modifications, is it possible to arrive at the coloring in which consecutive sides are red, blue, red, blue, red, blue, $\ldots$, red, yellow, blue?

8. (Colombia 1997.) We play the following game with an equilateral triangle of $n(n+1)/2$ pennies (with n pennies on a side). Initially, all the pennies are turned heads up. On each turn, we may turn over three pennies which are mutually adjacent; the goal is to make all the pennies tails up. For which values of n can this be achieved?

9. (USAMO 2002.) A *nice* sequence is a positive integer sequence $a_1, \ldots, a_n$ such that $a_i + a_{i+1}$ divides $a_i a_{i+1}$ for $i = 1, 2, \ldots, n-1$. Prove that for any two integers a and b both greater than 2, there exists a nice sequence connecting them.

10. (Putnam 2006.) Alice and Bob play a game in which they take turns removing stones from a heap that initially has n stones. The number of stones removed at each turn must be one less than a prime number. The winner is the player who takes the last stone. Alice plays first. Prove that there are infinitely many n such that Bob has a winning strategy. (For example, if $n = 17$, then Alice might take 6 leaving 11; then Bob might take 1 leaving 10; then Alice can take the remaining stones to win.)

11. (USAMO 1998.) A computer screen shows a 98×98 chessboard, colored in the usual way. One can select with a mouse any rectangle with sides on the lines of the chessboard and click the mouse button: as a result, the colors in the selected rectangle switch (black becomes white, white becomes black.) Find, with proof, the minimum number of mouse clicks needed to make the chessboard all one color.

12. (Alex Saltman's 2006 MOP handout.) Starting at $(1, 1)$, a stone is moved in the coordinate plane according to the following rules:

    (i) From any point (a, b), the stone can be moved to $(2a, b)$ or $(a, 2b)$.

    (ii) From any point (a, b), the stone can be moved to $(a-b, b)$ if $a > b$ or to $(a, b-a)$ if $a < b$.

    For which positive integers x, y can the stone be moved to (x, y)?

13. (BAMO 2006.) We have k switches arranged in a row, and each switch points up, down, left, or right. Whenever three successive switches all point in different directions, all three may be simultaneously turned so as to point in the fourth direction. Prove that this operation cannot be repeated infinitely many times.

14. (USAMO 1995.) A calculator is broken so that the only keys that still work are the sin, cos, tan, $\sin^{-1}$, $\cos^{-1}$, and $\tan^{-1}$ buttons. The display initially shows 0. Given any positive rational number q, show that pressing some finite sequence of buttons will yield q. Assume that the calculator does real number calculations with infinite precision. All functions are in terms of radians.

15. (Maksim Kontsevich.) There are three clones at points $(0, 0)$, $(1, 0)$, and $(0, 1)$ on an infinite coordinate plane. If a clone is standing at point (a, b) and points $(a, b+1)$ and $(a+1, b)$ both do not contain clones, then the clone can make two copies of itself and move to $(a, b+1)$ and $(a+1, b)$ (the point (a, b) then becomes vacant.) Show that no sequence of such moves can result in a configuration in which the points $(0, 0)$, $(1, 0)$, and $(0, 1)$ are all vacant.

16. (Euler Math Olympiad 2010.) You are given 100 coins, four of which are fake. The fake coins are lighter than the real coins, and they are all the same weight as each other. Prove that, using just two weighings on a balancing scale, there is a coin that you can point to and say with certainty, "this coin is real!"

17. (USAMO 2003.) At the vertices of a regular hexagon are written six nonnegative integers whose sum is 2003. Bert is allowed to make moves of the following form: he may pick a vertex and replace the number written there by the absolute value of the difference between the numbers written at the two neighboring vertices. Prove that Bert can make a sequence of moves, after which the number 0 appears at all six vertices.

18. (Israel 1995.) Tarzan and Jane play a game on an infinite board that consists of 1×1 squares. Tarzan chooses a square and marks it with an O. Then Jane chooses another square and marks it with an X. They play until one of the players marks a whole row or column of 5 consecutive squares, and this plyaer wins the game. Show that Jane can prevent Tarzan from winning.