New Zealand Mathematical Olympiad Committee

February Solutions

1. *Find the smallest three digit number N such that all the digits of $3 \times N$ are even.*

   Solution: Recall that a number is a multiple of 3 if and only if the sum of its digits is a multiple of 3. Since $3N \geq 300$, and all its digits are even, $3N \geq 400$. The smallest multiple of 3 greater than 400 is 402 and all of its digits are even, so the value of N is 134.

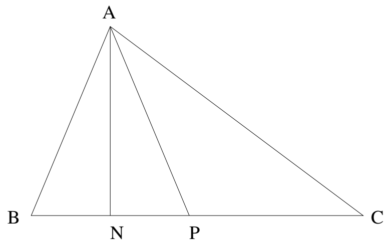
2. *Find all integer solutions to the equation $x^3 - y^3 = 91$.*

   Solution: Since $x^3 - y^3$ is to be positive, in any solution, $x > y$. Also, $x^3 - y^3 = (x-y)(x^2 + xy + y^2)$. Since the prime factorization of 91 is 7×13, there are only a few possible values for the two factors:

   - $x - y = 1$, $x^2 + xy + y^2 = 91$. Subsituting $x = y + 1$ in the second equation gives $3y^2 + 3y + 1 = 91$, or $y^2 + y - 30 = 0$, or $(y-5)(y+6) = 0$. This gives a pair of solutions for (x, y), namely $(6, 5)$ and $(-5, -6)$.
   - $x - y = 7$, $x^2 + xy + y^2 = 91$. Proceeding similarly yields two more solutions $(4, -3)$ and $(3, -4)$.
   - The remaining possibilities, $x - y = 13$ or $x - y = 91$ do not yield real solutions.

3. *In $\triangle ABC$, the median and altitude at A divide the angle at A into three equal parts. What are the angles of $\triangle ABC$?*

   Solution: Consider the following diagram, where N is the foot of the altitude from A, and P the midpoint of BC.



   We are given that $\angle BAN = \angle NAP = \angle PAC = \alpha$. Since AP bisects angle CAN, $AN/AC = NP/PC$. Also, since AN bisects angle PAB, ANP and ANB are congruent and in particular $NP/PC = NP/PB = 1/2$. Therefore $AN = (1/2)AC$. But ANC is a right angled triangle, hence (since the hypotenuse is now twice the length of one leg), $\angle ACB = 30°$, and $\angle CAN = 60°$. So, the original triangle has a right angle at A and $30°$ and $60°$ degree angles at the other two vertices.

4. *A sequence of numbers is to be written down using, at each step, one of the two rules: if the number is a perfect square, you may next write down its square root; or, in any case, you may write down the result of tripling it and adding 1. If the first number is 33, can we reach 31? If the first number is 31, can we reach 33?*

   Solution: Reaching 31 from 33 is easy: $100 = 3 \times 33 + 1$, $10 = \sqrt{100}$, $31 = 3 \times 10 + 1$. We cannot go in the opposite direction. The reason is that neither of the two rules can take a number which is not a multiple of three, and produce one which is (obviously tripling and adding one never leaves a multiple of 3, and if a number is not a multiple of 3 then neither is its square root.) So, no number which is a multiple of 3 can be produced from 31, and in particular, 33 cannot be produced.

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