The Probabilistic Method

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1 Warm-up

1. (Russia 1996/4.) In the Duma there are 1600 delegates, who have formed 16000 committees of 80 persons each. Prove that one can find two committees having at least four common members.

2. (Iran Team Selection Test 2008/6.) Suppose 799 teams participate in a tournament in which every pair of teams plays against each other exactly once. Prove that there exist two disjoint groups A and B of 7 teams each such that every team from A defeated every team from B.

3. Let G be a graph in which all vertices have nonzero degree. Prove that its vertices can be partitioned into two sets $V_1 \cup V_2$ such that the number of edges going between the V_i is at least $\frac{n}{2} + \frac{m}{6}$. Is this tight?

4. **760.** В лагерь приехало несколько пионеров, каждый из них имеет от 50 до 100 знакомых среди остальных. Докажите, что пионерам можно выдать пилотки, покрашенные в 1331 цвет так, чтобы у знакомых каждого пионера были пилотки хотя бы 20 различных цветов. (*Д.Карпов*)

2 Olympiad problems

1. (MOP 2007/7/1.) In a 100×100 array, each of the numbers $1, 2, \ldots, 100$ appears exactly 100 times. Show that there is a row or a column in the array with at least 10 distinct numbers.

2. (Russia, 1999.) In a class, each boy is friends with at least one girl. Show that there exists a group of at least half of the students, such that each boy in the group is friends with an odd number of the girls in the group.

3. (IMO Shortlist 1999/C4.) Let A be any set of n residues mod n^2. Show that there is a set B of n residues mod n^2 such that at least half of the residues mod n^2 can be written as $a + b$ with $a \in A$ and $b \in B$.

4. (MOP 2010, harder variation.) Let G be a graph with average degree d. Prove that for every $k \leq d$, there is a K_{k+1}-free induced subgraph on at least $\frac{kn}{d+1}$ vertices.

5. (Russia 2006, final problem.) A group of pioneers has arrived to summer camp. Each pioneer has at least 50 and at most 100 friends among the others. Prove that one can distribute field caps of 1331 colors among the pioneers so that the friends of each pioneer have caps of at least 20 colors.