Graph Theory: Definitions and Results Kiran Kedlaya, for MOSP 1999

- A *graph* is a pair (V, E), where V is a finite set and E is a set of unordered pairs of elements of V. The elements of V and E are called *vertices* and *edges*, respectively. Note that this phrasing forbids edges from a vertex to itself. It also forbids multiple edges between the same pair of vertices; if those are allowed, we call the result a *multigraph.*

- A *directed graph (digraph)* is the same as a graph, except that the edges are now ordered pairs of distinct vertices. (An edge is said to "come out of" the first vertex in the pair and "go into" the second vertex.) A *multidigraph* is the same with repeated edges allowed. When we say "graph", we mean an undirected graph unless otherwise specified.

- A *subgraph* of a graph (V, E) is a subset V' of V together with a subset E' of E consisting only of edges whose endpoints lie in V'. An *induced subgraph* is an edge in which *all* edges with endpoints in V' are included.

- An *isomorphism* of two (directed or undirected) graphs is a bijection of their vertices such that two vertices of one graph form an edge if and only if their images form an edge.

- Two vertices are *adjacent* if they are the endpoints of an edge. An edge is *incident* to a vertex if it has that vertex as an endpoint.

- The *degree* of a vertex is the number of edges it is an endpoint of. The sum of all degrees in a graph is even (handshake lemma). In a directed graph, the *in-degree* and *out-degree* of a vertex are the number of edges coming in and going out of, respectively, that vertex. A graph is *k-regular* if every vertex has degree k.

- A graph is *connected* if one cannot separate the vertices into two groups, such that every edge connects two vertices in the same group. Otherwise it is *disconnected.* A digraph is *strongly connected* if there is a path from any vertex to any other vertex. The *connected components* of a graph are the maximal connected induced subgraphs; likewise for the *strongly connected components* of a digraph. A graph is *k-connected* (resp. k-edge-connected) if removing any $k-1$ vertices (resp. edges) yields a connected graph.

- A *k-coloring* of a graph is a partition of the vertex set into k subsets, such that no two vertices in the same subset are adjacent. A *k-edge-coloring* is a partition of the edge set into k subsets, such that no two edges in the same subset have a common endpoint. A 2-colorable graph is also said to be *bipartite.* A graph is bipartite if and only if it has no cycles of odd length. (Beware that sometimes graphs are "colored" according to different sets of rules, as specified in the problem statements.)

- The *complete* graph K_n is the graph on n vertices in which every pair of vertices is an edge. The *complete bipartite* graph $K_{m,n}$ is the graph on $m+n$ vertices in which every pair of vertices, one from the first m and one from the other n, is an edge. Similarly, the *complete multipartite* graph $K_{m_1,\dots,m_k}$ is the graph on $m_1+\cdots+m_k$ vertices, containing every pair of vertices in distinct partitions as an edge.

- A *planar* graph is one that can be drawn in the plane, with points representing the vertices, and (polygonal) curves representing the edges, so that no two edges meet except at a common endpoint. The regions into which the edges divide the plane are called *faces*. If a connected planar graph has v vertices, e edges and f faces, then $v-e+f=2$ (Euler's formula). From this it follows that a connected planar graph on v vertices has at most $3v-6$ edges, and so in particular has a vertex of degree at most 5.

- A *walk* is a sequence of vertices such that any two consecutive vertices are adjacent. (In a digraph, there must be an edge from each vertex to the next.) A *tour* is a walk whose first and last vertices are the same. A *path/cycle* is a walk/tour with no repeated vertices; the subgraph consisting of the vertices on the path/cycle and the edges between consecutive vertices is also called a path/cycle.

- A *tree* is a connected graph with no cycles. A graph on n vertices is a tree if and only if it is connected and has $n-1$ edges. A *forest* is a general graph with no cycles, i.e. a graph whose connected components are trees.

- An *Eulerian walk/tour* is a walk/tour (a path/cycle with repeated vertices allowed) containing each edge exactly once. An Eulerian tour on a connected graph exists if and only if every vertex has even degree; an Eulerian walk exists if and only if at most two vertices have odd degree. On a strongly connected digraph, an Eulerian tour exists if and only if every vertex has equal in-degree and out-degree.

- A *Hamiltonian path/cycle* is a path/cycle containing each vertex exactly once. It is a very hard (NP-complete) problem to determine if an arbitrary graph admits a Hamiltonian path/cycle.

- The *adjacency matrix* of a graph on the vertices $\{1,\dots,n\}$ is the matrix A with $A_{ij}=1$ if vertices i and j are adjacent, and 0 otherwise. (In particular, $A_{ii}=0$.)

- (Marriage theorem) Let A, B be the color sets in a 2-coloring of a bipartite graph. We wish to choose a set of edges such that every vertex of A lies on exactly one of the edges, but no vertex of B lies no more than one. This is possible if and only if, for all $S \subseteq A$, the set of vertices adjacent to at least one vertex of S contains at least as many elements as S. (Can be deduced from MAXFLOW-MINCUT.)

- (Tutte's "partnership theorem") Given a graph G, we wish to choose a set of edges such that each vertex of G is incident to exactly one of the edges. This is possible if and only if, for each set S of vertices, the number of connected components of the induced subgraph of $V(G) - S$ which have an *odd* number of vertices is no greater than S.

- (MAXFLOW-MINCUT theorem) In a directed graph, one vertex has been designated the *source* and one the *sink*, and each edge has been assigned a nonnegative real number (its *capacity*). A *flow* is an assignment of a nonnegative real number to each edge, not exceeding its capacity, such that the inflow and outflow at every vertex other than the source and sink is equal. Given any partition of the vertices that separates the source from the sink, the *capacity* of the partition is the sum of the capacities of the edges from the source partition to the sink partition. Then for any partition of minimum capacity, there exists a flow in which all edges from the source partition to the sink partition are assigned their full capacity. (Algorithmic proof: construct "augmenting paths".)

- (Four-color theorem) Every planar graph is 4-colorable. (Proved by Appel and Haken, and again by Robertson, Seymour et al., using computers.)

- (Cayley's formula) The number of trees on n labeled vertices is n^{n-2}. This admits at least two combinatorial proofs (as well as one using generating functions). The *Prüfer encoding* maps trees to $(n-2)$-tuples of labels as follows: remove the lowest-numbered leaf, record the label of its former neighbor, and repeat. Knuth exhibited a bijection between rooted trees on $n+1$ vertices and "parking functions" (functions from $\{1, \ldots, n\}$ to itself such that the preimage of $\{1, \ldots, k\}$ contains at least k elements for $k = 1, \ldots, n$). Given a parking function, let 0 be the root of the tree, and draw an edge from n to m if person n wanted the spot behind where m ended up (the first spot is behind person 0).

- (Ramsey's theorem) Given positive integers k, m, there exists a positive integer R such that whenever the edges of a K_R are partitioned into k subsets $E_1, \ldots, E_k$, one of the E_i contains the edges of a complete subgraph on m vertices. For $k = 2, m = 3$ the smallest such R is 6; for $k = m = 3$, the smallest is 17 (IMO 1964). Note that constructing examples to show that R has to be large ordinarily requires probabilistic methods.

- (Turán's theorem) A graph on n vertices containing no K_{k+1} contains at most

$$\left\lfloor \left(\frac{n}{k}\right)^2 \binom{k}{2} \right\rfloor$$

edges, with equality only for the complete k-partite graph $K_{m_1,\ldots,m_k}$, where $m_1 + \cdots + m_k = n$ and no two of the m_i differ by more than 1. For example, every graph on n vertices with more than $n^2/4$ edges contains a 3-cycle. (Proof by induction on k: pick out an induced subgraph containing no K_n with as many vertices as possible, etc.)

- (Kuratowski's theorem) A graph is planar if and only if it does not contain a subgraph obtained from K_5 or $K_{3,3}$ by replacing each edge with a path.

- (Stable marriage theorem) Each of n men and n women ranks all n people of the opposite sex (with no ties). It is then possible to pair each man with a woman so that no man and woman not married to each other rank each other higher than they rank their respective mates. (Algorithmic proof: each man proposes to his first choice, each man not highest among the suitors of the woman he proposed to proposes to his second choice, etc.)

- (Cayley's spanning tree formula) Let A be the adjacency matrix of a graph G, let D be the diagonal matrix such that $-d_{ii}$ is the degree of vertex i, and let B be obtained from A by removing the first row and column. Then the number of spanning trees of a graph (subgraphs which are trees and which contain all of the vertices) equals the determinant of B.