New Zealand Mathematical Olympiad Committee

2010 June Problems — Solutions

These problems are intended to help students prepare for the 2010 camp selection problems, used to choose students to attend our week-long residential training camp in Christchurch in January.

In recent years the camp selection problems have been known as the “September Problems”, as they were made available in September. This year we’re going to trial moving the selection problems earlier in the year, releasing them in July and moving the due date to August. This will allow more time for pre-camp training, building up to Round One of the British Mathematical Olympiad in December.

The solutions will be posted in about two month’s time, but can be obtained before then by email if you write to me with evidence that you’ve tried the problems seriously.

Good luck!

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1. *A circle O_1 of radius 2 and a circle O_2 of radius 4 are externally tangent at the point P. Points A and B, both distinct from P, are chosen on O_1 and O_2 respectively so that A, P and B are collinear. Determine the length of the line segment PB if the length of AB is 4.*

   Solution: Refer to Figure 1. Let C_1 be the centre of O_1, and C_2 the centre of O_2. Then $\angle C_1PA$ and $\angle C_2PB$ are vertically opposite, hence equal, and in addition $\angle C_1PA = \angle C_1AP$, $\angle C_2PB = \angle C_2BP$, because triangles C_1AP and C_2BP are isosceles. It follows that the triangles C_1AP and C_2BP are similar. Hence

   $$\frac{PA}{C_1P} = \frac{PB}{C_2P},$$

   so $PA = C_1P \cdot PB/C_2P = PB/2$. Since we are given that $AB = 4 = AP + PB$ we get $3PB/2 = 4$, so $PB = 8/3$. □

2. *Prove that the number $9^n + 8^n + 7^n + 6^n - 4^n - 3^n - 2^n - 1^n$ is divisible by 10 for all non-negative integers n.*

   Solution: It’s enough to show that the number is both even and divisible by five. To do this, write it in the form

   $$(9^n - 4^n) + (8^n - 3^n) + (7^n - 2^n) + (6^n - 1^n).$$

   Each bracketed expression has the form $(k+5)^n - k^5$, and is divisible by five, because $k + 5 \equiv k \bmod 5$, so

   $$(k+5)^n - k^5 \equiv k^5 - k^5 \equiv 0 \bmod 5.$$

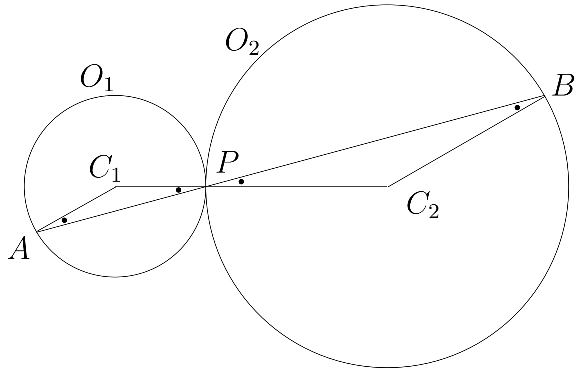

Figure 1: Figure for Problem 1.

Therefore the sum as a whole is divisible by five. In addition, each bracketed expression is odd, since each is the difference between an odd and an even number. We therefore have a sum of four odd numbers, which is even. □

3. *Suppose that each point in the plane is coloured either black or white. Show that there is a set of three points of the same colour which form the vertices of an equilateral triangle.*

Solution: Suppose that there is no such triangle, and consider a regular hexagon $ABCDEF$, with centre O (see Figure 2).

Without loss of generality we may assume that O is coloured black. At least one of the vertices of the triangle BDF must also be black, and we may assume that this is B; the triangles OBA and OBC then force A and C to be white, which in turn forces E to be black. Finally, OEF forces F to be white.

Letting G be the intersection of AB and EF we see that there is no way to colour G without creating a monochromatic equilateral triangle: if G is black then BEG has all black vertices, and if G is white then AFG has all white vertices. So there must exist an equilateral triangle whose vertices are all the same colour. □

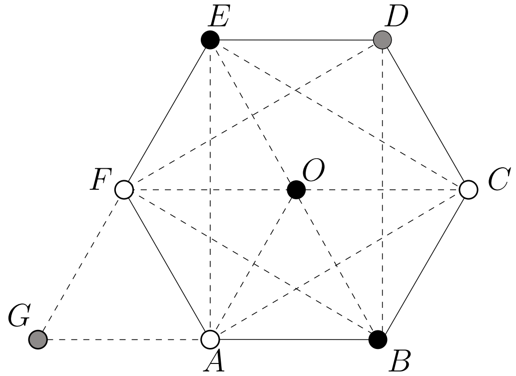

Figure 2: Figure for Problem 3.

4. *If* $1 = d_1 < d_2 < \cdots < d_k = n$ *are all the positive divisors of a positive integer* $n > 1$*, prove that*

$$d_1 + d_2 + \cdots + d_k > k\sqrt{n}.$$

Solution: We have $d_i d_{k+1-i} = n$, so by the arithmetic mean-geometric mean inequality

$$\frac{d_i + d_{k+1-i}}{2} \geq \sqrt{d_i d_{k+1-i}} = \sqrt{n}.$$

Hence $d_i + d_{k+1-i} \geq 2\sqrt{n}$. If $k = 2m$ is even then

$$\begin{aligned} d_1 + d_2 + \cdots + d_k &= (d_1 + d_{2m}) + (d_2 + d_{2m-1}) + \cdots + (d_m + d_{m+1}) \\ &\geq m \cdot 2\sqrt{n} = k\sqrt{n}. \end{aligned}$$

Otherwise, if $k = 2m + 1$ is odd, then n must be square and d_{m+1} must be the square root of n. In this case we get

$$\begin{aligned} d_1 + d_2 + \cdots + d_k &= (d_1 + d_{2m+1}) + (d_2 + d_{2m}) + \cdots + (d_m + d_{m+2}) + d_{m+1} \\ &\geq m \cdot 2\sqrt{n} + \sqrt{n} \\ &= (2m + 1)\sqrt{n} = k\sqrt{n}. \end{aligned}$$

So in either case $d_1 + d_2 + \cdots + d_k \geq k\sqrt{n}$. Moreover, the inequality is strict: since $n > 1$ we have $1 = d_1 \neq d_k = n$, so equality cannot hold in their AM-GM inequality. □

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