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A RESULT ABOUT THE PRIMES DIVIDING FIBONACCI NUMBERS

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(Submitted March 1998-Final Revision February 2001)

1. INTRODUCTION

The following theorem arose from my correspondence with Dr. Peter Neumann of Queen's College, Oxford, concerning the number of ways of writing an integer of the form $F_{n_1}F_{n_2}\dots F_{n_r}$ as a sum of two squares.

Theorem 1.1: If $m \geq 3$, then with the exception of $m = 6$ and $m = 12$, F_m is divisible by some prime p which does not divide any F_k, $k < m$.

Theorem 1.1 is similar to a theorem proved by K. Zsigmondy in 1892 (see [4]), which states that, for any natural number a and any m, there is a prime that divides $a^m - 1$ but does not divide $a^k - 1$ for $k < m$ with a small number of explicitly stated exceptions. A summary of Zsigmondy's article can be found in [2, Vol. 1, p. 195]. Since the arithmetic behavior of the sequence of Fibonacci numbers F_n is very similar to that of the sequences $a^n - b^n$ (for fixed a and b), Theorem 1.1 can be regarded as an analog of Zsigmondy's theorem for the Fibonacci sequence.

2. PRELIMINARY LEMMAS

This section includes a few lemmas that are required for the proof of Theorem 1.1.

Lemma 2.1: Let m, n be positive integers and let (a, b) denote the highest common factor of a and b. Then

$$\left(\frac{F_{mn}}{F_n}, F_n\right) \Big| \, m.$$

Proof: First, we prove by induction on m that

$$\frac{F_{mn}}{F_n} \equiv m(F_{n-1})^{m-1} \pmod{F_n}.$$

The result holds for $m = 1$. Suppose the result holds for $m = k$. Then

$$\frac{F_{kn}}{F_n} \equiv k(F_{n-1})^{k-1} \pmod{F_n}.$$

Now

$$F_{m+n+1} = F_m F_n + F_{m+1}F_{n+1} \quad \text{(see [1] or [3])}, \tag{1}$$

so $F_{(k+1)n} = F_{kn+(n-1)+1} = F_{kn}F_{n-1} + F_{kn+1}F_n$. Therefore,

$$\frac{F_{(k+1)n}}{F_n} = \frac{F_{kn}}{F_n}F_{n-1} + F_{kn+1} \equiv k(F_{n-1})^{k-1}F_{n-1} + F_{kn+1} \pmod{F_n}$$
$$\equiv k(F_{n-1})^k + F_{kn+1} \pmod{F_n}.$$

Using (1) again,

$$F_{kn+1} = F_{(k-1)n}F_n + F_{(k-1)n+1}F_{n+1} \equiv F_{(k-1)n+1}F_{n+1} \pmod{F_n}$$
$$\equiv F_{(k-1)n+1}F_{n-1} \pmod{F_n}.$$

Similarly, $F_{(k-1)n+1} \equiv F_{(k-2)n+1}F_{n-1} \pmod{F_n}$ giving us

$$F_{kn+1} \equiv F_{(k-1)n+1}F_{n-1} \equiv F_{(k-2)n+1}(F_{n-1})^2 \equiv \cdots \equiv (F_{n-1})^k \pmod{F_n}.$$

Therefore,

$$\frac{F_{(k+1)n}}{F_n} \equiv k(F_{n-1})^k + (F_{n-1})^k \equiv (k+1)(F_{n-1})^k \pmod{F_n}.$$

This completes the inductive step.

Let us define

$$d = \left(\frac{F_{mn}}{F_n}, F_n\right) = (m(F_{n-1})^{m-1} + tF_n, F_n),$$

where t is some integer. Then we have $d \mid F_n$ and $d \mid m(F_{n-1})^{m-1}$. However, $(F_n, F_{n-1}) = 1$, so d divides m and the lemma is proved. □

Lemma 2.2:

$$p_1^{\alpha_1}p_2^{\alpha_2}\dots p_n^{\alpha_n} = \frac{\displaystyle\prod_{k\text{ odd}} \frac{p_1^{\alpha_1}p_2^{\alpha_2}\dots p_n^{\alpha_n}}{p_{i_1}p_{i_2}\cdots p_{i_k}}}{\displaystyle\prod_{k\text{ even}} \frac{p_1^{\alpha_1}p_2^{\alpha_2}\dots p_n^{\alpha_n}}{p_{i_1}p_{i_2}\cdots p_{i_k}}},$$

where the numerator is the product of all numbers of the form $p_1^{\alpha_1}p_2^{\alpha_2}\dots p_n^{\alpha_n}$ divided by an odd number of distinct primes and the denominator is the product of all numbers of the form $p_1^{\alpha_1}p_2^{\alpha_2}\dots$ $p_n^{\alpha_n}$ divided by an even nonzero number of distinct primes.

Proof: The exponent of p_r on the left-hand side is α_r. The exponent of p_r in the numerator of the right-hand side is

$$\sum_{k\text{ odd}}\left(\alpha_r\binom{n}{k} - \binom{n-1}{k-1}\right),$$

as there are $\binom{n}{k}$ ways of choosing $i_1, \dots, i_k$ and, if $i_s = r$ for some s, there are $\binom{n-1}{k-1}$ ways of choosing the other i_j. Similarly, the exponent of p_r in the denominator of the right-hand side is

$$\sum_{k\text{ even}}\left(\alpha_r\binom{n}{k} - \binom{n-1}{k-1}\right),$$

so the exponent of p_r on the right-hand side is

$$\alpha_r\left(\binom{n}{1} - \binom{n}{2} + \binom{n}{3} - \cdots + (-1)^n\binom{n}{n}\right) - \left(1 - \binom{n-1}{1} + \binom{n-1}{2} - \cdots + (-1)^{n-1}\binom{n-1}{n-1}\right)$$
$$= \alpha_r(1-(1-1)^n) - (1-1)^{n-1} = \alpha_r$$

as required. □

Lemma 2.3: If $0 < a < 1$, then $\prod_{n=1}^{\infty}(1-a^n) > (1-a)^{\frac{1}{1-a}}$.

Proof: Equivalently, we must prove that

$$\sum_{n=1}^{\infty} \ln(1-a^n) > \frac{\ln(1-a)}{1-a}.$$

If $|x|<1$, then the Taylor series expansion for $\ln x$ about $x=1$ is $\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots$. Thus,

$$\ln(1-a^n) = -\left(a^n + \frac{a^{2n}}{2} + \frac{a^{3n}}{3} + \cdots\right).$$

Therefore,

$$\sum_{n=1}^{\infty} \ln(1-a^n) = -\sum_{k=1}^{\infty} \frac{1}{k}(a^k + a^{2k} + a^{3k} + \cdots)$$
$$= -\sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{a^k}{1-a^k}\right) > -\sum_{k=1}^{\infty} \frac{1}{k}\left(\frac{a^k}{1-a}\right) = \frac{\ln(1-a)}{1-a}. \quad \square$$

Lemma 2.4: If $a=(\sqrt{5}-1)/(\sqrt{5}+1)$, then

$$\prod_{\substack{n \text{ odd} \\ n\geq 1}} (1-a^n) \Big/ \prod_{\substack{n \text{ even} \\ n\geq 2}} (1-a^n) < 2.$$

Proof: Note that $1-x^2<1$ and so, for $x<1$, we have $1+x<(1-x)^{-1}$. Thus,

$$\prod_{\substack{n \text{ odd} \\ n\geq 1}} (1-a^n) \Big/ \prod_{\substack{n \text{ even} \\ n\geq 2}} (1-a^n) < (1+a)\Big/\prod_{n=2}^{\infty}(1-a^n)$$
$$= (1-a^2)\Big/\prod_{n=1}^{\infty}(1-a^n) < (1-a^2)(1-a)^{\frac{-1}{1-a}} < 2,$$

where the penultimate inequality follows from Lemma 2.3, and the final inequality holds for the value of a given. $\square$

Lemma 2.5: If $m=p_1^{\alpha_1}p_2^{\alpha_2}\dots p_n^{\alpha_n}$, then the only solutions m, $m\geq 3$, to the inequality

$$f(m) = \left(\frac{1+\sqrt{5}}{2}\right)^{(p_1^{\alpha_1}-p_1^{\alpha_1-1})\dots(p_n^{\alpha_n}-p_n^{\alpha_n-1})} \leq 2p_1\dots p_n = g(m) \tag{2}$$

are $m=3,4,5,6,10,12,14$, and 30.

We first prove the following three easy facts:

(i) If $f(m)>Cg(m)$, $C>1$, and m' is formed from m by replacing p_i in the prime factorization of m by q_i, where $q_i>p_i$ and $q_i\neq p_k$ for any k, then $f(m')>Cg(m')$.

(ii) If $f(m)>g(m)$ and p is an odd prime, then $f(pm)>g(pm)$.

(iii) If $f(m)>g(m)$ and m is even, then $f(2m)>g(2m)$. If $f(m)>2g(m)$ and m is odd, then $f(2m)>g(2m)$.

Proof of (i): $f(m)>Cg(m)\geq 4C$ so, in particular, $f(m)>\exp(1)$. Now

$$q_i > p_i \Rightarrow q_ip_i - p_i > q_ip_i - q_i \Rightarrow \frac{q_i-1}{p_i-1} > \frac{q_i}{p_i},$$

so

$$f(m') \geq f(m)^{\frac{q_i-1}{p_i-1}} > f(m)^{\frac{q_i}{p_i}} = f(m)(f(m))^{\frac{q_i}{p_i}-1} > f(m)\exp\left(\frac{q_i}{p_i}-1\right).$$

Since $\exp(x-1) > x$ for $x > 1$, we have

$$f(m') > \left(\frac{q_i}{p_i}\right) f(m) > C\left(\frac{q_i}{p_i}\right) g(m) = Cg(m').$$

Proof of (ii): Note that $p > 2$ and $g(m) \geq 4$ so

$$f(pm) \geq f(m)^{p-1} > g(m)^{p-1} \geq 4^{p-2} g(m) > pg(m) \geq g(pm).$$

Proof of (iii): If m is even and $f(m) > g(m)$, then $f(2m) > f(m) > g(m) = g(2m)$. If m is odd and $f(m) > 2g(m)$, then $f(2m) = f(m) > 2g(m) = g(2m)$.

Proof of Lemma 2.5: We call m "good" if $f(m) > 2g(m)$ or if m is even and $f(m) > g(m)$. Note that, by (ii) and (iii), if m is good, then no multiple of m may satisfy inequality (2).

Standard calculations show that $m = 11$ is good. It then follows from (i) that every prime greater than 11 is good, so any solution m of (2) must only have 2, 3, 5, and 7 as prime divisors.

It is easy to show that $m = 3^2$ and $m = (3)(7)$ are good. So, by (i), except for $m = (3)(5)$, $m = p_i^2$ and $m = p_i p_j$ are good for odd primes p_i, p_j. Hence, the only odd numbers whose multiples may satisfy inequality (2) are 3, 5, 7, and 15.

Now $m = 2^3$ is good, as is $m = 2^2(5)$. Thus, $m = 2^2(p_i)$ is good for odd primes p_i, $p_i \geq 5$. Therefore, the only possible solutions to inequality (2) are 2, 3, 5, 7, (3)(5), (2)(3), (2)(5), (2)(7), (2)(3)(5), 2^2, and $2^2(3)$. Of these, 7 and (3)(5) are not solutions and $2 < 3$, so we obtain the list as stated in the lemma. □

3. PROOF OF THE MAIN THEOREM

Suppose we choose a Fibonacci number F_m, with $m \geq 3$ and $m = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_n^{\alpha_n}$, such that all prime factors of F_m divide some previous Fibonacci number.

Then every prime dividing F_m must divide one of $F_{m[1]}, F_{m[2]}, \ldots, F_{m[n]}$, where $m[i] = m / p_i$, making use of the well-known fact that $(F_m, F_n) = F_{(m,n)}$. Now $F_m \leq p_1 p_2 \cdots p_n F_{m[1]} F_{m[2]} \cdots F_{m[n]}$, for the left-hand side divides the right-hand side, using Lemma 2.1. However, some of the factors of F_m are being double counted, such as $F_{p_1^{\alpha_1-1} p_2^{\alpha_2-1} \ldots p_n^{\alpha_n}}$, which divides both $F_{m[1]}$ and $F_{m[2]}$.

To remove repeats, the same Inclusion-Exclusion Principle idea of Lemma 2.2 can be used. This gives

$$F_m \leq p_1 p_1 \cdots p_n \frac{\prod\limits_{k \text{ odd}} F_{m[i_1, i_2, \ldots, i_k]}}{\prod\limits_{k \text{ even}} F_{m[i_1, i_2, \ldots, i_k]}}, \tag{3}$$

where $m[i_1, i_2, \ldots, i_k] = m / p_{i_1} p_{i_2} \cdots p_{i_k}$ and the i_j are all distinct. In fact, the left-hand side divides the right-hand side, but the inequality is sufficient for our purposes.

It is now necessary to simplify (3) to obtain a weaker inequality that is easier to handle.

Multiplying by the denominator in (3),

$$\prod_{k \text{ even}} F_{m[i_1, i_2, \ldots, i_k]} \leq p_1 p_2 \cdots p_n \prod_{k \text{ odd}} F_{m[i_1, i_2, \ldots, i_k]}, \tag{4}$$

where we have absorbed F_m into the product on the left-hand side.

Let us define F_n' to equal

$$\frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n.$$

By Binet's formula,

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right) \quad \text{and} \quad -1 < \frac{1-\sqrt{5}}{2} < 0,$$

so, as $n \to \infty$, $F_n \to F_n'$. Furthermore, $F_n > F_n'$ for n odd and $F_n < F_n'$ for n even.

All the Fibonacci numbers on the left-hand side of (4) are of the form $F_{m/k}$, k a product of an even number of distinct primes, and they are all distinct since, if $F_{m/k} = F_{m/k'}$, then $k = k'$ or m/k and m/k' are 1 and 2 in some order, contradicting the fact that k and k' are both products of an even number of distinct primes. Let us define γ_1 to equal

$$\prod_{n \text{ even}} \left(\frac{F_n}{F_n'}\right),$$

where the product is taken over all even integers n. The left-hand side of (4) would therefore be made even smaller if all the F_n in it were replaced by F_n' and the result were multiplied by γ_1. Similarly, the right-hand side of (4) would be made even larger if all the F_n in it were replaced by F_n' and the result were multiplied by γ_2, where γ_2 is equal to

$$\prod_{n \text{ odd}} \left(\frac{F_n}{F_n'}\right).$$

Thus, if we define $\varepsilon = \gamma_2/\gamma_1$, we obtain from (4) the weaker inequality,

$$\prod_{k \text{ even}, \geq 0} F'_{m[i_1, i_2, \ldots, i_k]} \leq \varepsilon p_1 p_2 \ldots p_n \prod_{k \text{ odd}} F'_{m[i_1, i_2, \ldots, i_k]}. \tag{5}$$

The number of terms in the product on the left-hand side of (5) is $1 + \binom{n}{2} + \binom{n}{4} + \cdots$ and on the right-hand side is $\binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots$, and these numbers are equal as their difference is $(1-1)^n = 0$. Therefore, the $1/\sqrt{5}$ factors of F_n' will cancel on both sides, leaving

$$\left[\left(\frac{1+\sqrt{5}}{2}\right)^m\right]^{(1-\frac{1}{p_1})(1-\frac{1}{p_2})\cdots(1-\frac{1}{p_n})} \leq \varepsilon p_1 p_2 \cdots p_n,$$

on rearranging. Since $m = p_1^{\alpha_1} p_2^{\alpha_2} \ldots p_n^{\alpha_n}$, this simplifies to give

$$\left(\frac{1+\sqrt{5}}{2}\right)^{(p_1^{\alpha_1} - p_1^{\alpha_1 - 1})\cdots(p_n^{\alpha_n} - p_n^{\alpha_n - 1})} \leq \varepsilon p_1 p_2 \cdots p_n. \tag{6}$$

Now, setting $a = (\sqrt{5}-1)/(\sqrt{5}+1)$,

$$\gamma_1 = \prod_{n \text{ even}} \left(\frac{F_n}{F_n'}\right) = \prod_{n \text{ even}} \left(\frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{(1+\sqrt{5})^n}\right) = \prod_{n \text{ even}} (1-a^n).$$

Similarly,

$$\gamma_2 = \prod_{n \text{ odd}} \left(\frac{F_n}{F_n'}\right) = \prod_{n \text{ odd}} \left(\frac{(1+\sqrt{5})^n - (1-\sqrt{5})^n}{(1+\sqrt{5})^n}\right) = \prod_{n \text{ odd}} (1-a^n).$$

Therefore, by Lemma 2.4,

$$\varepsilon = \gamma_2 / \gamma_1 < 2.$$

Now Lemma 2.5 gives us a list of possible m which may satisfy inequality (6). Thus, it only remains for us to check which of these m give rise to F_m, all of whose prime factors divide some previous Fibonacci number. The possible solutions, m, to (6), with $m \geq 3$, are 3, 4, 5, 6, 10, 12, 14, and 30.

Note that $2|F_3$, $3|F_4$, $5|F_5$, $11|F_{10}$, $29|F_{14}$, and $31|F_{30}$ and the respective primes do not divide any previous Fibonacci numbers. Thus, the only exceptions to the result are $F_6 = 8$ and $F_{12} = 144$. Therefore, Theorem 1.1 is proved. □

A similar result can also be proved for the Lucas numbers.

Corollary 3.1: If $m \geq 2$, then, with the exception of $m = 3$ and $m = 6$, L_m is divisible by some prime p that does not divide any L_k, $0 \leq k < m$.

Proof: Suppose $m \geq 2$ and m does not equal 3 or 6. Then, since $2m \geq 3$ and $2m$ does not equal 6 or 12, Theorem 1.1 implies the existence of a prime p such that p divides F_{2m}, but does not divide any smaller Fibonacci number. Now $F_{2m} = F_m L_m$ (see [3]), so p must divide L_m. We claim that p does not divide any L_k for $k < m$, for $p | L_k$ would imply $p | F_{2k}$, and since $2k < 2m$, this contradicts our choice of p. Hence, the corollary. □

We end with the following conjecture for the general Fibonacci-type sequence.

Conjecture 3.2: Suppose that K_1 and K_2 are positive integers and that K_n is defined recursively for $n \geq 3$ by $K_n = K_{n-1} + K_{n-2}$. Then, for all sufficiently large m, there exists a prime p that divides K_m but does not divide any K_r, $r < m$.

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SUMMATION OF CERTAIN RECIPROCAL SERIES RELATED TO THE GENERALIZED FIBONACCI AND LUCAS NUMBERS

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1. INTRODUCTION

We are interested in the generalized Fibonacci and Lucas numbers defined by

$$U_n(p,q)=\frac{\alpha^n-\beta^n}{\alpha-\beta},\quad V_n(p,q)=\alpha^n+\beta^n,$$

where

$$\alpha=\frac{p+\sqrt{\Delta}}{2},\quad \beta=\frac{p-\sqrt{\Delta}}{2},\quad \Delta=p^2-4q,\ p>0,\text{ and } q<0.$$

It is well known that $\{U_n(1,-1)\}$ and $\{V_n(1,-1)\}$ are the classical Fibonacci sequence $\{F_n\}$ and the Lucas sequence $\{L_n\}$. There are many publications dealing with summation of reciprocal series related to the classical Fibonacci and Lucas numbers (see, e.g., [2]-[5]). Backstrom [3] obtained

$$\sum_{n=0}^{\infty}\frac{1}{F_{2n+1}+F_s}=\frac{\sqrt{5}S}{2L_s}\quad (s\text{ odd}) \tag{1}$$

and André-Jeannin [2] proved that

$$\sum_{n=0}^{\infty}\frac{1}{F_{2n+1}+L_s/\sqrt{5}}=\frac{s}{2F_s}\quad (s\text{ even, } s\neq 0). \tag{2}$$

Are there results similar to (1) or (2) for the generalized Fibonacci and Lucas numbers? In this paper we will discuss the summation of reciprocal series related to the generalized Fibonacci and Lucas numbers. We will establish a series of identities involving the generalized Fibonacci and Lucas numbers and some identities of [2] and [3] will emerge as special cases of our results. In the final section, following the method introduced by Almkvist, we express four reciprocal series related to the generalized Fibonacci and Lucas numbers in terms of the theta functions and give their estimates. Some of the estimates obtained generalize the results of [1] and [2], respectively.

2. MAIN RESULTS

The following lemmas will be used later on.

Lemma 1: Let t be a real number with $|t|>1$, s and a be positive integers, and b be a nonnegative integer. Then one has that

$$\sum_{n=0}^{\infty}\frac{1}{t^{2an+b}+t^{-2an-b}+t^{as}+t^{-as}}=\frac{1}{t^{as}-t^{-as}}\sum_{n=0}^{s-1}\frac{1}{1+t^{2an+b-as}} \tag{3}$$

and

$$\sum_{n=0}^{\infty}\frac{1}{t^{2an+b}+t^{-2an-b}-(t^{as}+t^{-as})}=\frac{1}{t^{as}-t^{-as}}\sum_{n=0}^{s-1}\frac{1}{1-t^{2an+b-as}}. \tag{4}$$

Proof: Because the proof of (4) is similar to that of (3), we only give the proof of (3). One can readily verify that

$$\frac{1}{t^{2an+b}+t^{-2an-b}+t^{as}+t^{-as}}=\frac{1}{t^{-as}-t^{as}}\left(\frac{1}{1+t^{2an+b+as}}-\frac{1}{1+t^{2an+b-as}}\right)$$

holds for $n>s$. Hence, by the telescoping effect, one has that

$$\sum_{n=0}^{N}\frac{1}{t^{2an+b}+t^{-2an-b}+t^{as}+t^{-as}}=\frac{1}{t^{-as}-t^{as}}\left(\sum_{n=N-s+1}^{N}\frac{1}{1+t^{2an+b+as}}-\sum_{n=0}^{s-1}\frac{1}{1+t^{2an+b-as}}\right)$$

for all $N>s$. Letting $N\to+\infty$, we obtain equality (3) (since $|t|>1$). □

Lemma 2: Let t be a real number with $|t|>1$ and s be a positive integer. Then

$$\sum_{n=0}^{s-1}\frac{1}{1+t^{2n-s}}=\frac{s-1}{2}+\frac{1}{1+t^{-s}}, \tag{5}$$

$$\sum_{n=0}^{s-1}\frac{1}{1+t^{2n-s+1}}=\frac{s}{2}, \tag{6}$$

$$\sum_{n=0}^{2s}\frac{1}{1-t^{2n-2s-1}}=s+\frac{1}{1-t^{-2s-1}}, \tag{7}$$

and

$$\sum_{n=0}^{2s-1}\frac{1}{1-t^{2n-2s+1}}=s. \tag{8}$$

Proof: We only show that equality (5) is valid. The proofs of (6)-(8) follow the same pattern and therefore are omitted here. First,

$$\sum_{n=0}^{2m-1}\frac{1}{1+t^{2n-2m}}=\frac{1}{1+t^{-2m}}+\frac{1}{2}+\sum_{n=1}^{m-1}\left(\frac{1}{1+t^{2n}}+\frac{1}{1+t^{-2n}}\right)=m-\frac{1}{2}+\frac{1}{1+t^{-2m}}.$$

On the other hand,

$$\sum_{n=0}^{2m}\frac{1}{1+t^{2n-2m-1}}=\frac{1}{1+t^{-2m-1}}+\sum_{n=1}^{m}\left(\frac{1}{1+t^{2n-1}}+\frac{1}{1+t^{-2n+1}}\right)=m+\frac{1}{1+t^{-2m-1}}.$$

Therefore, equality (5) holds. □

The above lemmas are used to find some equalities involving the generalized Fibonacci and Lucas numbers. Using the lemmas, we calculate some reciprocal series related to $\{U_n(p,q)\}$ and $\{V_n(p,q)\}$.

Theorem 1: Assume that a and b are integers with $a\geq 1$ and $b\geq 0$. Then

$$\sum_{n=0}^{\infty}\frac{(-q)^{an+b/2}}{V_{2an+b}(p,q)+(-q)^{an+(b-a)/2}V_a(p,q)}=\frac{(-q)^{a/2}}{\sqrt{\Delta}U_a(p,q)(1+(-\alpha/\beta)^{(b-a)/2})}, \tag{9}$$

$$\sum_{n=0}^{\infty}\frac{(-q)^{an+b/2}}{U_{2an+b}(p,q)+(-q)^{an+(b-a)/2}U_a(p,q)}=\frac{(-q)^{a/2}\sqrt{\Delta}}{V_a(p,q)(1+(-\alpha/\beta)^{(b-a)/2})}, \tag{10}$$

$$\sum_{n=0}^{\infty}\frac{(-q)^{an+b/2}}{\sqrt{\Delta}U_{2an+b}(p,q)+(-q)^{an+(b-a)/2}V_a(p,q)}=\frac{(-q)^{a/2}}{\sqrt{\Delta}U_a(p,q)(1+(-\alpha/\beta)^{(b-a)/2})}, \tag{11}$$

and

$$\sum_{n=0}^{\infty}\frac{(-q)^{an+b/2}}{V_{2an+b}(p,q)+\sqrt{\Delta}(-q)^{an+(b-a)/2}U_a(p,q)}=\frac{(-q)^{a/2}}{V_a(p,q)(1+(-\alpha/\beta)^{(b-a)/2})},\tag{12}$$

where a is even in (9), (11) and odd in (10), (12), and b is even in (9), (12) and odd in (10), (11), respectively.

Proof: Putting $t=\sqrt{-\alpha/\beta}$ in (3) and noticing that $\alpha\beta=q$, we have

$$\begin{aligned}\sum_{n=0}^{\infty}&\frac{(-q)^{an+b/2}}{\alpha^{2an+b}+(-1)^b\beta^{2an+b}+(-q)^{an+(b-as)/2}(\alpha^{as}+(-1)^{as}\beta^{as})}\\&=\frac{(-q)^{as/2}}{\alpha^{as}-(-1)^{as}\beta^{as}}\sum_{n=0}^{s-1}\frac{1}{1+(-\alpha/\beta)^{an+(b-as)/2}}.\end{aligned}\tag{13}$$

Let us examine different cases according to the values of a, b, and s. With $s=1$ in (13), then (9) holds if both a and b are even and (10) holds if both a and b are odd. On the other hand, if a is even, b is odd, and $s=1$, then we have (11) from (13). If a is odd, b is even, and $s=1$, then we have (12) from (13). □

Theorem 2: Suppose that a and b are integers with $a\geq 1$, $b\geq 0$, and $a\neq b$. Then

$$\sum_{n=0}^{\infty}\frac{(-q)^{an+b/2}}{V_{2an+b}(p,q)-(-q)^{an+(b-a)/2}V_a(p,q)}=\frac{-(-q)^{a/2}}{\sqrt{\Delta}U_a(p,q)(1-(-\alpha/\beta)^{(b-a)/2})},\tag{14}$$

$$\sum_{n=0}^{\infty}\frac{(-q)^{an+b/2}}{U_{2an+b}(p,q)-(-q)^{an+(b-a)/2}U_a(p,q)}=\frac{-\sqrt{\Delta}(-q)^{a/2}}{V_a(p,q)(1-(-\alpha/\beta)^{(b-a)/2})},\tag{15}$$

$$\sum_{n=0}^{\infty}\frac{(-q)^{an+b/2}}{\sqrt{\Delta}\,U_{2an+b}(p,q)-(-q)^{an+(b-a)/2}V_a(p,q)}=\frac{-(-q)^{a/2}}{\sqrt{\Delta}\,U_a(p,q)(1-(-\alpha/\beta)^{(b-a)/2})},\tag{16}$$

and

$$\sum_{n=0}^{\infty}\frac{(-q)^{an+b/2}}{V_{2an+b}(p,q)-(-q)^{an+(b-a)/2}\sqrt{\Delta}U_a(p,q)}=\frac{-(-q)^{a/2}}{V_a(p,q)(1-(-\alpha/\beta)^{(b-a)/2})},\tag{17}$$

where a is even in (14), (16) and odd in (15), (17) and b is even in (14), (17) and odd in (15), (16), respectively.

The proof is similar to that of Theorem 1 except that (13) is replaced by

$$\begin{aligned}\sum_{n=0}^{\infty}&\frac{(-q)^{an+b/2}}{\alpha^{2an+b}+(-1)^b\beta^{2an+b}-(-q)^{an+(b-as)/2}(\alpha^{as}+(-1)^{as}\beta^{as})}\\&=\frac{-(-q)^{as/2}}{\alpha^{as}-(-1)^{as}\beta^{as}}\sum_{n=0}^{s-1}\frac{1}{1-(-\alpha/\beta)^{an+(b-as)/2}}.\end{aligned}\tag{18}$$

Equality (18) is valid by putting $t=\sqrt{-\alpha/\beta}$ in (4).

Theorem 3: Suppose that s is a positive integer. Then

$$\sum_{n=0}^{\infty}\frac{(-q)^{n}}{V_{2n}(p,q)+(-q)^{n-s/2}V_s(p,q)}=\frac{(-q)^{s/2}}{\sqrt{\Delta}U_s(p,q)}\left(\frac{s-1}{2}+\frac{1}{1+(-\beta/\alpha)^{s/2}}\right)\ (s\text{ even}),\tag{19}$$

$$\sum_{n=0}^{\infty}\frac{(-q)^{n+1/2}}{U_{2n+1}(p,q)+(-q)^{n+(1-s)/2}V_s(p,q)/\sqrt{\Delta}}=\frac{(-q)^{s/2}s}{2U_s(p,q)}\quad (s\text{ even, } s\neq 0), \tag{20}$$

$$\sum_{n=0}^{\infty}\frac{(-q)^{n}}{V_{2n}(p,q)+\sqrt{\Delta}(-q)^{n-s/2}U_s(p,q)}=\frac{(-q)^{s/2}}{V_s(p,q)}\left(\frac{s-1}{2}+\frac{1}{1+(-\beta/\alpha)^{s/2}}\right)\quad (s\text{ odd}), \tag{21}$$

$$\sum_{n=0}^{\infty}\frac{(-q)^{n+1/2}}{U_{2n+1}(p,q)+(-q)^{n+(1-s)/2}U_s(p,q)}=\frac{(-q)^{s/2}\sqrt{\Delta}s}{2V_s(p,q)}\quad (s\text{ odd}), \tag{22}$$

$$\sum_{n=0}^{\infty}\frac{(-q)^{n}}{V_{2n}(p,q)-(-q)^{n-s/2}\sqrt{\Delta}U_s(p,q)}=\frac{(-q)^{s/2}}{V_s(p,q)}\left(\frac{1-s}{2}+\frac{1}{(-\beta/\alpha)^{s/2}-1}\right)\quad (s\text{ odd}), \tag{23}$$

and

$$\sum_{n=0}^{\infty}\frac{(-q)^{n+1/2}}{U_{2n+1}(p,q)-(-q)^{n+(1-s)/2}V_s(p,q)/\sqrt{\Delta}}=\frac{-(-q)^{s/2}s}{2U_s(p,q)}\quad (s\text{ even, } s\neq 0). \tag{24}$$

Proof: Letting $a=1$ and $b=0$ in (13), we have

$$\sum_{n=0}^{\infty}\frac{(-q)^{n}}{V_{2n}(p,q)+(-q)^{n-s/2}V_s(p,q)}=\frac{(-q)^{s/2}}{\sqrt{\Delta}U_s(p,q)}\sum_{n=0}^{s-1}\frac{1}{1+(-\alpha/\beta)^{n-s/2}}\quad (s\text{ even}).$$

Due to (5), we obtain equality (19). On the other hand, if $a=b=1$ in (13), then

$$\sum_{n=0}^{\infty}\frac{(-q)^{n+1/2}}{U_{2n+1}(p,q)+(-q)^{n+(1-s)/2}V_s(p,q)/\sqrt{\Delta}}=\frac{(-q)^{s/2}}{U_s(p,q)}\sum_{n=0}^{s-1}\frac{1}{1+(-\alpha/\beta)^{n+(1-s)/2}}\quad (s\text{ even}).$$

Noticing that (6), we have equality (20).

Similarly, equalities (21) and (22) follow from (13), (5), and (6). Equalities (23) and (24) can be obtained from (18), (7), and (8). □

From the above theorems, we can obtain some results of [2] and [3] according to the values of p and q. For instance, if $p=-q=1$ in (9), we obtain Theorem V of [3]. If $p=-q=1$ in (22), we obtain equality (1). If $p=-q=1$ in (20), we have (2).

3. THE ESTIMATES OF FOUR SERIES

In this section, the summation $\sum_n$ is over all integers n. Using the method introduced by Almkvist [1], we give the estimates of four series related to the generalized Fibonacci and Lucas numbers. Putting $s=0$ in the left-hand side of (20), we have

$$\sum_{n=0}^{\infty}\frac{(-q)^{n+1/2}}{U_{2n+1}(p,q)+2(-q)^{n+1/2}/\sqrt{\Delta}}=\sqrt{\Delta}\sum_{n=0}^{\infty}\frac{t^{2n+1}}{(t^{2n+1}+1)^2},$$

where $t=(-\beta/\alpha)^{1/2}$. By a classical formula (see [1] or [6]), we know that

$$\sum_{n=0}^{\infty}\frac{t^{2n+1}}{(t^{2n+1}+1)^2}=-\frac{\vartheta_3''}{8\pi^2\vartheta_3},$$

where

$$\vartheta_3=\sqrt{-\frac{\pi}{\log t}}\sum_n e^{\pi^2n^2/\log t}$$

and

$$\vartheta_3'' = \frac{2\pi^2}{\log t}\sqrt{-\frac{\pi}{\log t}}\sum_n \left(1+\frac{2\pi^2 n^2}{\log t}\right)e^{\pi^2 n^2/\log t}.$$

By simple computation, we can obtain

$$\sum_{n=0}^{\infty}\frac{(-q)^{n+1/2}}{U_{2n+1}(p,q)+2(-q)^{n+1/2}/\sqrt{\Delta}} = \frac{\sqrt{\Delta}}{2\log(-\alpha/\beta)} - \frac{4\pi^2\sqrt{\Delta}\sum_{n=1}^{\infty} n^2 e^{-2\pi^2 n^2/\log(-\alpha/\beta)}}{(\log(-\alpha/\beta))^2\left(1+2\sum_{n=1}^{\infty} e^{-2\pi^2 n^2/\log(-\alpha/\beta)}\right)}.$$

Hence,

$$\sum_{n=0}^{\infty}\frac{(-q)^{n+1/2}}{U_{2n+1}(p,q)+2(-q)^{n+1/2}/\sqrt{\Delta}} \approx \frac{\sqrt{\Delta}}{2\log(-\alpha/\beta)} - \frac{4\pi^2\sqrt{\Delta}}{(\log(-\alpha/\beta))^2(2+e^{2\pi^2/\log(-\alpha/\beta)})}.$$

Using a similar method, we can obtain the estimates of some other series. We have that

$$\sum_{n=0}^{\infty}\frac{(-q)^n}{V_{2n}(p,q)+2(-q)^n} = \frac{1}{4}+\sum_{n=1}^{\infty}\frac{t^{2n}}{(t^{2n}+1)^2},$$

$$\sum_{n=0}^{\infty}\frac{(-q)^{n+1/2}}{U_{2n+1}(p,q)-2(-q)^{n+1/2}/\sqrt{\Delta}} = \sqrt{\Delta}\sum_{n=0}^{\infty}\frac{t^{2n+1}}{(t^{2n+1}-1)^2},$$

and

$$\sum_{n=1}^{\infty}\frac{(-q)^n}{V_{2n}(p,q)-2(-q)^n} = \sum_{n=1}^{\infty}\frac{t^{2n}}{(t^{2n}-1)^2},$$

where $t=(-\beta/\alpha)^{1/2}$. From the following facts (see [1] or [6]), i.e.,

$$\frac{\vartheta_2''}{\vartheta_2} = -\pi^2\left(1+8\sum_{n=1}^{\infty}\frac{t^{2n}}{(t^{2n}+1)^2}\right),$$

$$\frac{\vartheta_4''}{\vartheta_4} = 8\pi^2\sum_{n=0}^{\infty}\frac{t^{2n+1}}{(t^{2n+1}-1)^2},$$

and

$$\frac{1}{24\pi^2}\left(\frac{\vartheta_2''}{\vartheta_2}+\frac{\vartheta_3''}{\vartheta_3}+\frac{\vartheta_4''}{\vartheta_4}\right)+\frac{1}{24} = \sum_{n=1}^{\infty}\frac{t^{2n}}{(t^{2n}-1)^2},$$

where

$$\vartheta_2 = \sqrt{\frac{-\pi}{\log t}}\sum_n (-1)^n e^{\pi^2 n^2/\log t}$$

and

$$\vartheta_4 = \sqrt{\frac{-\pi}{\log t}}\sum_n e^{\pi^2 (n-1/2)^2/\log t},$$

we have that

$$\sum_{n=0}^{\infty}\frac{(-q)^n}{V_{2n}(p,q)+2(-q)^n} \approx \frac{1}{8}+\frac{1}{2\log(-\alpha/\beta)}+\frac{4\pi^2}{(\log(-\alpha/\beta))^2}\cdot\frac{1}{e^{2\pi^2/\log(-\alpha/\beta)}-2},$$

$$\sum_{n=0}^{\infty}\frac{(-q)^{n+1/2}}{U_{2n+1}(p,q)-2(-q)^{n+1/2}/\sqrt{\Delta}} \approx \frac{\pi^2\sqrt{\Delta}}{2(\log(-\alpha/\beta))^2} - \frac{\sqrt{\Delta}}{2\log(-\alpha/\beta)},$$

and

$$\sum_{n=1}^{\infty}\frac{(-q)^n}{V_{2n}(p,q)-2(-q)^n}\approx\frac{1}{24}-\frac{1}{2\log(-\alpha/\beta)}$$
$$+\frac{4\pi^2}{3(\log(-\alpha/\beta))^2}\left(\frac{1}{e^{2\pi^2/\log(-\alpha/\beta)}+2}-\frac{1}{e^{2\pi^2/\log(-\alpha/\beta)}-2}+\frac{1}{8}\right).$$

Clearly, some of the estimates obtained in this section are the generalizations of [1] and [2], respectively.

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Author and Title Index

The TITLE, AUTHOR, ELEMENTARY PROBLEMS, ADVANCED PROBLEMS, and KEY-WORD indices for Volumes 1-38.3 (1963-July 2000) of *The Fibonacci Quarterly* have been completed by Dr. Charles K. Cook. It is planned that the indices will be available on The Fibonacci Web Page. Anyone wanting their own disc copy should send two 1.44 MB discs and a self-addressed stamped envelope with enough postage for two discs. PLEASE INDICATE WORDPERFECT 6.1 OR MS WORD 97.

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A DYNAMICAL PROPERTY UNIQUE TO THE LUCAS SEQUENCE

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1. INTRODUCTION

A *dynamical system* is taken here to mean a homeomorphism

$$f : X \to X$$

of a compact metric space X (though the observations here apply equally well to any bijection on a set). The number of points with period n under f is

$$\mathrm{Per}_n(f) = \#\{x \in X \mid f^n x = x\},$$

and the number of points with least period n under f is

$$\mathrm{LPer}_n(f) = \#\{x \in X \mid \#\{f^k x\}_{k \in \mathbb{Z}} = n\}.$$

There are two basic properties that the resulting sequences $(\mathrm{Per}_n(f))$ and $(\mathrm{LPer}_n(f))$ must satisfy if they are finite. First, the set of points with period n is the disjoint union of the sets of points with least period d for each divisor d of n, so

$$\mathrm{Per}_n(f) = \sum_{d|n} \mathrm{LPer}_d(f). \tag{1}$$

Second, if x is a point with least period d, then the d distinct points $x, f(x), f^2(x), \ldots, f^{d-1}(x)$ are all points with least period d, so

$$0 \le \mathrm{LPer}_d(f) \equiv 0 \mod d. \tag{2}$$

Equation (1) may be inverted via the Möbius inversion formula to give

$$\mathrm{LPer}_n(f) = \sum_{d|n} \mu(n/d)\mathrm{Per}_d(f),$$

where $\mu(\cdot)$ is the Möbius function defined by

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1, \\ 0 & \text{if } n \text{ has a squared factor, and} \\ (-1)^r & \text{if } n \text{ is a product of } r \text{ distinct primes.} \end{cases}$$

A short proof of the inversion formula may be found in Section 2.6 of [6].

Equation (2) therefore implies that

$$0 \le \sum_{d|n} \mu(n/d)\mathrm{Per}_d(f) \equiv 0 \mod n. \tag{3}$$

Indeed, (3) is the only condition on periodic points in dynamical systems: define a given sequence of nonnegative integers (U_n) to be *exactly realizable* if there is a dynamical system $f : X \to X$ with $U_n = \mathrm{Per}_n(f)$ for all $n \ge 1$. Then (U_n) is exactly realizable if and only if

$$0 \le \sum_{d|n} \mu(n/d)U_d \equiv 0 \mod n \text{ for all } n \ge 1,$$

since the realizing map may be constructed as an infinite permutation using the quantities

$$\frac{1}{n}\sum_{d|n}\mu(n/d)U_d$$

to determine the number of cycles of length n.

Our purpose here is to study sequences of the form

$$U_{n+2}=U_{n+1}+U_n,\ n\geq 1,\ \ U_1=a,\ U_2=b,\ \ a,b\geq 0 \tag{4}$$

with the distinguished Fibonacci sequence denoted (F_n), so

$$U_n=aF_{n-2}+bF_{n-1}\ \text{ for } n\geq 3. \tag{5}$$

Theorem 1: The sequence (U_n) defined by (4) is exactly realizable if and only if $b=3a$.

This result has two parts: the *existence* of the realizing dynamical system is described first, which gives many modular corollaries concerning the Fibonacci numbers. One of these is used later on in the *obstruction* part of the result. The realizing system is (essentially) a very familiar and well-known system, the *golden-mean shift*.

The fact that (up to scalar multiples) the Lucas sequence (L_n) is the only exactly realizable sequence satisfying the Fibonacci recurrence relation to some extent explains the familiar observation that (L_n) satisfies a great array of congruences.

Throughout, n will denote a positive integer and p, q distinct prime numbers.

2. EXISTENCE

An excellent introduction to the family of dynamical systems from which the example comes is the recent book by Lind and Marcus [4]. Let

$$X=\{\mathbf{x}=(x_k)\in\{0,1\}^{\mathbb{Z}}\mid x_k=1\Rightarrow x_{k+1}=0 \text{ for all } k\in\mathbb{Z}\}.$$

The set X is a compact metric space in a natural metric (see [4], Ch. 6, for the details). The set X may also be thought of as the set of all (infinitely long in both past and future) itineraries of a journey involving two locations (0 and 1), obeying the rule that from 1 you must travel to 0, and from 0 you must travel to either 0 or 1. Define the homeomorphism $f:X\to X$ to be the *left shift*,

$$(f(\mathbf{x}))_k=x_{k+1}\ \text{ for all } k\in\mathbb{Z}.$$

The dynamical system $f:X\to X$ is a simple example of a *subshift of finite type*. It is easy to check that the number of points of period n under this map is given by

$$\mathrm{Per}_n(f)=\ \mathrm{trace}(A^n), \tag{6}$$

where $A=\left[\begin{smallmatrix}1&1\\1&0\end{smallmatrix}\right]$ (see [4], Prop. 2.2.12; the 0–1 entries in the matrix A correspond to the allowed transitions $0\to 0$ or 1; $1\to 0$ in the elements of X thought of as infinitely long journeys in a graph with vertices 0 and 1).

Lemma 2: If $b=3a$ in (4), then the corresponding sequence is exactly realizable.

Proof: A simple induction argument shows that (6) reduces to $\mathrm{Per}_n(f)=L_n$ for $n\geq 1$, so the case $a=1$ is realized using the golden mean shift itself. For the general case, let $\overline{X}=X\times B$, where B is a set with a elements, and define $\bar{f}: \bar{X} \to \bar{X}$ by $\bar{f}(\mathbf{x}, y) = (f(\mathbf{x}), y)$. Then $\mathrm{Per}_n(\bar{f}) = a \times \mathrm{Per}_n(f)$, so we are done. □

The relation (3) must as a result hold for (L_n).

Corollary 3: $\sum_{d|n} \mu(n/d) L_d \equiv 0 \bmod n$ for all $n \geq 1$.

This has many consequences, a sample of which we list here. Many of these are, of course, well known (see [5], §2.IV) or follow easily from well-known congruences.

(a) Taking $n = p$ gives

$$L_p = F_{p-2} + 3F_{p-1} \equiv 1 \bmod p. \tag{7}$$

(b) It follows from (a) that

$$F_{p-1} \equiv 1 \bmod p \Leftrightarrow F_{p-2} \equiv -2 \bmod p, \tag{8}$$

which will be used below.

(c) Taking $n = p^k$ gives

$$L_{p^k} \equiv L_{p^{k-1}} \bmod p^k \tag{9}$$

for all primes p and $k \geq 1$.

(d) Taking $n = pq$ (a product of distinct primes) gives

$$L_{pq} + 1 \equiv L_p + L_q \bmod pq.$$

3. OBSTRUCTION

The negative part of Theorem 1 is proved as follows. Using some simple modular results on the Fibonacci numbers, we show that, if the sequence (U_n) defined by (4) is exactly realizable, then the property (3) forces the congruence $b \equiv 3a \bmod p$ to hold for infinitely many primes p, so (U_n) is a multiple of (L_n).

Lemma 4: For any prime p, $F_{p-1} \equiv 1 \bmod p$ if $p = 5m \pm 2$.

Proof: From Hardy and Wright (see [2], Theorem 180), we have that $F_{p+1} \equiv 0 \bmod p$ if $p = 5m \pm 2$. The identities $F_{p+1} = 2F_{p-1} + F_{p-2} \equiv 0 \bmod p$ and (7) imply that $F_{p-1} \equiv 1 \bmod p$. □

Assume now that the sequence (U_n) defined by (4) is exactly realizable. Applying (3) for n a prime p shows that $U_p - U_1 \equiv 0 \bmod p$, so by (5), $aF_{p-2} + bF_{p-1} \equiv a \bmod p$. If p is 2 or 3 mod 5, Lemma 4 implies that

$$(F_{p-2} - 1)a + b \equiv 0 \bmod p. \tag{10}$$

On the other hand, for such p, (8) implies that $F_{p-2} \equiv -2 \bmod p$, so (10) gives $b = 3a \bmod p$. By Dirichlet's theorem (or simpler arguments), there are infinitely many primes p with p equal to 2 or 3 mod 5, so $b = 3a \bmod p$ for arbitrarily large values of p. We deduce that $b = 3a$, as required.

4. REMARKS

(a) Notice that the example of the golden mean shift plays a vital role here. If it were not to hand, exhibiting a dynamical system with the required properties would require *proving* Corollary

3, and *a priori* we have no way of guessing or proving this congruence without using the dynamical system.

(b) The congruence (7) gives a different proof that $F_{p-1} \equiv 0$ or $1 \bmod p$ for $p \neq 2, 5$. If $F_{p-1} \equiv \alpha \bmod p$, then (7) shows that $F_{p-2} \equiv 1-3\alpha \bmod p$, so $F_p \equiv 1-2\alpha$. On the other hand, the recurrence relation gives the well-known equality

$$F_{p-2}F_p = F_{p-1}^2 + 1$$

(since p is odd), so $1-5\alpha+6\alpha^2 \equiv \alpha^2+1$, hence $5(\alpha^2-\alpha) \equiv 0 \bmod p$. Since $p \neq 5$, this requires that $\alpha^2 \equiv \alpha \bmod p$, so $\alpha \equiv 0$ or 1.

(c) The general picture of conditions on linear recurrence sequences that allow exact realization is not clear, but a simple first step in the Fibonacci spirit is the following question: For each $k \geq 1$, define a recurrence sequence $(U_n^{(k)})$ by

$$U_{n+k}^{(k)} = U_{n+k-1}^{(k)} + U_{n+k-2}^{(k)} + \cdots + U_n^{(k)}$$

with specified initial conditions $U_j^{(k)} = a_j$ for $1 \leq j \leq k$. The subshift of finite type associated to the 0–1 $k \times k$ matrix

$$A^{(k)} = \begin{bmatrix} 1 & 1 & 1 & \dots & 1 & 1 \\ 1 & 0 & 0 & \dots & 0 & 0 \\ 0 & 1 & 0 & \dots & 0 & 0 \\ & & \ddots & & & \\ 0 & 0 & \dots & 1 & 0 & 0 \\ 0 & 0 & \dots & 0 & 1 & 0 \end{bmatrix}$$

shows that the sequence $(U_n^{(k)})$ is exactly realizable if $a_j = 2^j - 1$ for $1 \leq j \leq k$. If the sequence is exactly realizable, does it follow that $a_j = C(2^j - 1)$ for $1 \leq j \leq k$ and some constant C? The special case $k = 1$ is trivial, and $k = 2$ is the argument above. Just as in Corollary 3, an infinite family of congruences follows for each of these multiple Fibonacci sequences from the existence of the exact realization.

(d) We are grateful to an anonymous referee for suggesting the following questions. Given a dynamical system $f: X \to X$ for which the quantities $\mathrm{Per}_n(f)$ are all finite, it is conventional to define the *dynamical zeta function*

$$\zeta_f(z) = \exp\left(\sum_{n=1}^{\infty} \frac{z^n}{n} \mathrm{Per}_n(f)\right),$$

which defines a complex function on the disc of radius

$$1 / \limsup_{n\to\infty} \mathrm{Per}_n(f)^{1/n}.$$

It is a remarkable fact that for many dynamical systems—indeed, all "hyperbolic" ones—the zeta function is a rational function. For example, the golden mean subshift of finite type used above has zeta function $\frac{1}{1-z-z^2}$. There are also sharp results that determine exactly what rational functions can arise as zeta functions of irreducible subshifts of finite type or of *finitely presented* systems—these are expansive quotients of subshifts of finite type. A simple application of Theorem 6.1 in [1], which describes the possible shape of zeta functions for finitely presented systems shows that the sequence a, $3a$, $4a$, $7a$, ... can be exactly realized by an irreducible subshift of finite type if and only if $a = 1$.

It is possible that the recent deep results of Kim, Ormes, and Roush [3] may eventually provide a complete description of linear recurrence sequences that are exactly realized by subshifts of finite type.

ACKNOWLEDGMENT

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ON THE GENERALIZED LAGUERRE POLYNOMIALS

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1. INTRODUCTION

In this note we shall study two classes of polynomials $\{g_{n,m}^a(x)\}_{n\in N}$ and $\{h_{n,m}^a(x)\}_{n\in N}$. These polynomials are generalizations of Panda's polynomials (see [2], [3]). Also, these polynomials are special cases of the polynomials which were considered in [4] and [5]. For $m=1$, the polynomials $\{g_{n,m}^a(x)\}$ are the well-known Laguerre polynomials $L_n^\alpha(x)$ (see [6]), i.e.,

$$g_{n,1}^a(x) \equiv L_n^{a-1}(x). \tag{1.0}$$

In this paper the polynomials $\{g_{n,m}^a(x)\}$ and $\{h_{n,m}^a(x)\}$ are given by

$$F(x,t) = (1-t^m)^{-a} e^{-\frac{xt}{1-t^m}} = \sum_{n=0}^{\infty} g_{n,m}^a(x) t^n \tag{1.1}$$

and

$$G(x,t) = (1+t^m)^{-a} e^{-\frac{xt}{1+t^m}} = \sum_{n=0}^{\infty} h_{n,m}^a(x) t^n. \tag{1.2}$$

Using (1.1) and (1.2), we shall prove a great number of interesting relations for $\{g_{n,m}^a(x)\}$ and $\{h_{n,m}^a(x)\}$, as well as some mixed relations.

2. RECURRENCE RELATIONS AND EXPLICIT REPRESENTATIONS

First we find two recurrence relations of the polynomials $\{g_{n,m}^a(x)\}$.
Differentiating (1.1) with respect to t, we get

$$\begin{aligned}\frac{\partial F(x,t)}{\partial t} &= (1-t^m)^{-a-1} e^{-\frac{xt}{1-t^m}} (amt^{m-1} - amt^{2m-1} - x - x(m-1)t^m) \\ &= (1-t^m)\sum_{n=1}^{\infty} n g_{n,m}^a(x) t^{n-1}.\end{aligned} \tag{2.1}$$

By (2.1) and from (1.1), we obtain the following recurrence relation:

$$\begin{aligned}&n g_{n,m}^a(x) - (n-m) g_{n-m,m}^a(x) \\ &= am(g_{n-m,m}^{a+1}(x) - g_{n-2m,m}^{a+1}(x)) - x(g_{n-1,m}^{a+1}(x) + (m-1) g_{n-1-m,m}^{a+1}(x)).\end{aligned} \tag{2.2}$$

Again, from (1.1) and (2.1), we get

$$\begin{aligned}n g_{n,m}^a(x) = &-x(g_{n-1,m}^a(x) + (m-1) g_{n-1-m,m}^a(x)) \\ &+ (m(a-2)+2n) g_{n-m,m}^a(x) - (m(a-2)+n) g_{n-2m,m}^a(x), \ n \ge 2m.\end{aligned} \tag{2.3}$$

Corollary 2.1: If $m=1$, then (2.2) and (2.3) yield the corresponding relations for Laguerre polynomials:

$$nL_n^{a-1}(x) - (n-1)L_{n-1}^{a-1}(x) = (a-x)L_{n-1}^a(x) - aL_{n-2}^a(x)$$

and

$$nL_n^a(x) = (2n+a-2-x)L_{n-1}^a(x) - (n+a-2)L_{n-2}^a(x), \ n \ge 2.$$

In a similar way, from (1.2), we get the following relations:

$$nh_{n,m}^a(x) = (m-1)xh_{n-1-m,m}^{a+2}(x) - amh_{n-m,m}^{a+1}(x) - xh_{n-1,m}^{a+2}(x), \ n \ge m,$$

and

$$\begin{aligned} nh_{n,m}^a(x) &= x(m-1)h_{n-1-m,m}^a(x) - xh_{n-1,m}^a(x) \\ &\quad - (2n+am-2m)h_{n-m,m}^a(x) - (n+am-2m)h_{n-2m,m}^a(x), \ n \ge m. \end{aligned}$$

Starting from (1.1) and (1.2), we get the following explicit representations of the polynomials $\{g_{n,m}^a(x)\}$ and $\{h_{n,m}^a(x)\}$, respectively:

$$g_{n,m}^a(x) = \sum_{i=0}^{[n/m]} \frac{(-1)^{n-mi}(a+n-mi)_i}{i!(n-mi)!} x^{n-mi} \tag{2.4}$$

and

$$h_{n,m}^a(x) = \sum_{i=0}^{[n/m]} \frac{(-1)^{n-(m-1)i}(a+n-mi)_i}{i!(n-mi)!} x^{n-mi}. \tag{2.5}$$

Corollary 2.2: If $m=1$, then (2.6) is the explicit representation of the Laguerre polynomials:

$$L_n^{a-1}(x) = \sum_{i=0}^{n} \frac{(-1)^{n-i}(a+n-i)_i}{i!(n-i)!} x^{n-i}.$$

Now, differentiating (1.1) with respect to x, we get

$$Dg_{n,m}^a(x) = -g_{n-1,m}^{a+1}(x), \ n \ge 1. \tag{2.6}$$

If we differentiate (2.6), with respect to x, k times, we obtain

$$D^k g_{n,m}^a(x) = (-1)^k g_{n-k,m}^{a+k}(x), \ n \ge k. \tag{2.7}$$

Corollary 2.3: Using the idea in [1], from (2.2) and (2.6), we get

$$(n-xD)g_{n,m}^a(x) = (n-m+x(m-1)D)g_{n-m,m}^a(x) + amD(g_{n+1-2m,m}^a(x) - g_{n+1-m,m}^a(x)).$$

For $m=1$ in the last equality and from (1.0), we get

$$(n+(a-x)D)L_n^{a-1}(x) = (n-1+aD)L_{n-1}^{a-1}(x).$$

In a similar way, from (1.2), we have

$$Dh_{n,m}^a(x) = -h_{n-1,m}^{a+1}(x)$$

and

$$D^s h_{n,m}^a(x) = (-1)^s h_{n-s,m}^{a+s}(x), \ n \ge s.$$

3. SOME IDENTITIES OF THE CONVOLUTION TYPE

In this section we shall prove some interesting identities related to $\{g_{n,m}^a(x)\}$ and $\{h_{n,m}^a(x)\}$. First, from (1.1), we find

$$F(x,t)\cdot F(y,t)=(1-t^m)^{-2a}e^{-\frac{(x+y)t}{1-t^m}}=\sum_{n=0}^{\infty}g_{n,m}^{2a}(x+y)t^n, \tag{3.1}$$

whence we get

$$\sum_{i=0}^{n}g_{n-i,m}^{a}(x)g_{i,m}^{a}(y)=g_{n,m}^{2a}(x+y).$$

Theorem 3.1: The following identities hold:

$$g_{n,m}^{2a}(x)=\sum_{j=0}^{[n/m]}\sum_{i=0}^{n-mj}\frac{y^{n-i-mj}(n-i-mj)_j}{j!(n-i-mj)!}g_{i,m}^{2a}(x+y); \tag{3.2}$$

$$\sum_{i=0}^{n}D^s g_{n-i,m}^{a}(x)D^s g_{i,m}^{a}(y)=g_{n-2s,m}^{2a+2s}(x+y),\ n\geq 2s; \tag{3.3}$$

$$\sum_{i=0}^{n}D^k g_{n-i,m}^{a}(x)D^k h_{i,m}^{a}(x)=g_{n-2k,2m}^{a+k}(2x),\ n\geq 2k; \tag{3.4}$$

$$\sum_{i=0}^{[(n-k)/m]}\frac{(k)_i}{i!}g_{n-k-mi,2m}^{a}(2x)=(-1)^k\sum_{i=0}^{n}g_{n-i-k,m}^{a+k}(x)h_{i,m}^{a}(x); \tag{3.5}$$

$$\sum_{i=0}^{[(n-k)/m]}(-1)^i\frac{(k)_i}{i!}g_{n-k-mi,2m}^{a}(2x)=(-1)^k\sum_{i=0}^{n}h_{n-i-k,m}^{a+k}(x)g_{i,m}^{a}(x); \tag{3.6}$$

$$\sum_{i=0}^{n}g_{n-i,m}^{a}(x)g_{i,m}^{b}(x)=g_{n,m}^{a+b}(2x). \tag{3.7}$$

Proof: From (3.1), we have

$$(1-t^m)^{-2a}e^{-\frac{xt}{1-t^m}}=e^{\frac{yt}{1-t^m}}\sum_{n=0}^{\infty}g_{n,m}^{2a}(x+y)t^n,$$

whence

$$\sum_{n=0}^{\infty}g_{n,m}^{2a}(x)t^n=\left(\sum_{n=0}^{\infty}\frac{y^nt^n}{n!}\right)\left(\sum_{k=0}^{\infty}\binom{-n}{k}(-t^m)^k\right)\left(\sum_{n=0}^{\infty}g_{n,m}^{2a}(x+y)t^n\right).$$

Multiplying the series on the right side, then comparing the coefficients to t^n, by the last equality we get (3.2).

If we differentiate (1.1) s times, with respect to x, we find

$$\frac{\partial^s F(x,t)}{\partial x^s}=(-1)^s t^s(1-t^m)^{-a-s}e^{-\frac{xt}{1-t^m}}. \tag{a}$$

From (a), we get

$$\frac{\partial^s F(x,t)}{\partial x^s}\cdot\frac{\partial^s F(y,t)}{\partial y^s}=\sum_{n=0}^{\infty}g_{n,m}^{2a+2s}(x+y)t^{n+2s}. \tag{i}$$

Since

$$\frac{\partial^s F(x,t)}{\partial x^s}\cdot\frac{\partial^s F(y,t)}{\partial y^s}=\sum_{n=0}^{\infty}\sum_{i=0}^{n}D^s g_{n-i,m}^{a}(x)D^s g_{i,m}^{a}(y)t^n,$$

and, from (i), it follows that

$$\sum_{i=0}^{n} D^s g_{n-i,m}^a(x) D^s g_{i,m}^a(y) = g_{n-2s,m}^{2a+as}(x+y),\ n \geq 2s.$$

The last identity is the desired identity (3.3).

Differentiating (1.2) k times, with respect to x, we get

$$\frac{\partial^k G(x,t)}{\partial x^k} = (-1)^k t^k (1+t^m)^{-a-k} e^{-\frac{xt}{1+t^m}}. \tag{b}$$

Then, from (a) and (b), we find

$$\frac{\partial^k F(x,t)}{\partial x^k} \cdot \frac{\partial^k G(x,t)}{\partial x^k} = \sum_{n=0}^{\infty} g_{n,2m}^{a+k}(2x) t^{n+2k}. \tag{ii}$$

The left side of (ii) yields

$$\frac{\partial^k F(x,t)}{\partial x^k} \cdot \frac{\partial^k G(x,t)}{\partial x^k} = \sum_{n=0}^{\infty} \sum_{i=0}^{n} D^k g_{n-i,m}^a(x) D^k h_{i,m}^a(x) t^n. \tag{iii}$$

So, from (ii) and (iii), we get (3.4).

In a similar way, starting from (1.1) and (1.2), we can prove identity (3.5). From (1.1) and (b), we can prove identity (3.6).

In the proof identity (3.7), we start from

$$F^a(x,t) = (1-t^m)^{-a} e^{-\frac{xt}{1-t^m}}, \quad \text{by (1.1)},$$

and

$$F^b(x,t) = (1-t^m)^{-b} e^{-\frac{xt}{1-t^m}}, \quad \text{by (1.1)}.$$

So, we obtain

$$F^a(x,t) \cdot F^b(x,t) = \sum_{n=0}^{\infty} g_{n,m}^{a+b}(2x) t^n.$$

On the other side, we have

$$\left(\sum_{n=0}^{\infty} g_{n,m}^a(x) t^n\right)\left(\sum_{n=0}^{\infty} g_{n,m}^b(x) t^n\right) = \sum_{n=0}^{\infty} g_{n,m}^{a+b}(2x) t^n.$$

Identity (3.7) follows by the last equality and the proof of Theorem 3.1 is completed.

Corollary 3.1: If $m=1$ in (3.2), (3.3), and (3.7), then we get

$$L_n^{2a-1}(x) = \sum_{j=0}^{n} \sum_{i=0}^{n-j} \frac{y^{n-i-j}(n-i-j)_i}{j!(n-i-j)!} L_i^{2a-1}(x+y),$$

$$\sum_{i=0}^{n} D^s L_{n-i}^{a-1}(x) D^s L_i^{a-1}(y) = L_{n-2s}^{2a+2s-1}(x+y),$$

and

$$\sum_{i=0}^{n} L_{n-i}^{a-1}(x) L_i^{b-1}(x) = L_n^{a+b-1}(2x),$$

respectively.

Furthermore, we shall prove some more general results.

Theorem 3.2:

$$\sum_{i_1+\cdots+i_k=n} g_{i_1,m}^{a_1}(x_1)\cdots g_{i_k,m}^{a_k}(x_k) = g_{n,m}^{a_1+\cdots+a_k}(x_1+\cdots+x_k); \tag{3.8}$$

$$\sum_{i_1+\cdots+i_k=n} h_{i_1,m}^{a_1}(x_1)\cdots h_{i_k,m}^{a_k}(x_k) = h_{n,m}^{a_1+\cdots+a_k}(x_1+\cdots+x_k); \tag{3.9}$$

$$\begin{aligned}&\sum_{s=0}^{n}\sum_{i_1+\cdots+i_k=n-s} g_{i_1,m}^{a}(x_1)\cdots g_{i_k,m}^{a}(x_k)\cdot\sum_{j_1+\cdots+j_k=s} h_{j_1,m}^{a}(x_1)\cdots h_{i_k,m}^{a}(x_k)\\ &=\sum_{i_1+\cdots+i_k=n} g_{i_1,2m}^{a}(2x_1)\cdots g_{i_k,2m}^{a}(2x_k).\end{aligned} \tag{3.10}$$

Proof: From (1.1), we get

$$F^{a_1}(x_1,t)\cdots F^{a_k}(x_k,t)=\sum_{n=0}^{\infty} g_{n,m}^{a_1+\cdots+a_k}(x_1+\cdots+x_k)t^n.$$

Further, we have the following identity:

$$\sum_{n=0}^{\infty}\sum_{i_1+\cdots+i_k=n} g_{i_1,m}^{a_1}(x_1)\cdots g_{i_k,m}^{a_k}(x_k)t^n=\sum_{n=0}^{\infty} g_{n,m}^{a_1+\cdots+a_k}(x_1+\cdots+x_k)t^n.$$

Identity (3.8) follows immediately from the last equality. In a similar way, from (1.2), we can prove (3.9).

Now we shall prove (3.10). From (1.1) and (1.2), we have

$$F(x_1,t)\cdots F(x_k,t)\cdot G(x_1,t)\cdots G(x_k,t)=(1-t^{2m})^{-ka}e^{-\frac{2(x_1+\cdots+x_k)t}{1-t^{2m}}}.$$

So we get

$$\begin{aligned}&\left(\sum_{n=0}^{\infty}\sum_{i_1+\cdots+i_k=n} g_{i_1,m}^{a}(x_1)\cdots g_{i_k,m}^{a}(x_k)t^n\right)\cdot\left(\sum_{n=0}^{\infty}\sum_{j_1+\cdots+j_k=n} h_{j_1,m}^{a}(x_1)\cdots h_{j_k,m}^{a}(x_k)t^n\right)\\ &=\sum_{n=0}^{\infty} g_{n,2m}^{ka}(2x_1+\cdots+2x_k)t^n.\end{aligned}$$

Comparing the coefficients to t^n in the last equality, we get (3.10) and the proof of Theorem 3.2 is completed.

Corollary 3.2: If $m=1$, using (1.0), then (3.8) becomes

$$\sum_{i_1+\cdots+i_k=n} L_{i_1}^{a_1-1}(x)\cdots L_{i_k}^{a_k-1}(x)=L_n^{a_1+\cdots+a_k-1}(x_1+\cdots+x_k).$$

Corollary 3.3: If $x_1=x_2=\cdots=x_k=x$ and $a_1=a_2=\cdots=a_k=a$, then (3.8) becomes

$$\sum_{i_1+\cdots+i_k=n} g_{i_1,m}^{a}(x)\cdots g_{i_k,m}^{a}(x)=g_{n,m}^{ka}(kx). \tag{3.11}$$

Corollary 3.4: If $m=1$, then (3.11) yields

$$\sum_{i_1+\cdots+i_k=n} L_{i_1}^{a-1}(x)\cdots L_{i_k}^{a-1}(x)=L_n^{ka-1}(kx).$$

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Errata for "Generalizations of Some Identities Involving the Fibonacci Numbers"
by Fengzhen Zhao & Tianming Wang

The Fibonacci Quarterly **39.2** (2001):165-167

On page 166, (10) should be

$$\sum_{a+b+c=n} U_{ak}U_{bk}U_{ck} = \frac{U_k^2}{2(V_k^2-4q^k)^2}((n-1)(n-2)V_k^2U_{nk} - q^kV_k(4n^2-6n-4)U_{(n-1)k} + q^{2k}(4n^2-4)U_{(n-2)k}), \; n \geq 2.$$

Hence, on page 167, (13) should be

$$\sum_{a+b+c=n} F_{ak}F_{bk}F_{ck} = \frac{F_k^2}{2(L_k^2-4(-1)^k)^2}((n-1)(n-2)L_k^2F_{nk} - (-1)^kL_k(4n^2-6n-4)F_{(n-1)k} + (4n^2-4)F_{(n-2)k}), \; n \geq 2.$$

In the meantime, line 14 and line 16 of page should be, respectively,

$$\sum_{a+b+c=n} F_{2a}F_{2b}F_{2c} = \frac{1}{50}(9(n-1)(n-2)F_{2n} - 3(4n^2-6n-4)F_{2n-2} + (4n^2-4)F_{2n-4}).$$

$$\sum_{a+b+c=n} F_{2a}F_{2b}F_{2c} = \frac{1}{50}((15n^2-63n+66)F_{2n-3} + (10n^2-36n+44)F_{2n-4}).$$

Line 19 of page 167 should be: $+(4n^2-4)P_{(n-2)k}), \; n \geq 2.$

ANALYTIC CONTINUATION OF THE FIBONACCI DIRICHLET SERIES

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1. INTRODUCTION

Functions defined by Dirichlet series $\sum_{n=1}^{\infty} a_n n^{-s}$ are interesting because they often code and link properties of an algebraic nature in analytic terms. This is most often the case when the coefficients a_n are multiplicative arithmetic functions, such as the number or sum of the divisors of n, or group characters. Such series were the first to be studied, and are fundamental in many aspects of number theory. The most famous example of these is undoubtedly $\zeta(s)=\sum_{n=1}^{\infty} n^{-s}$ $(\mathrm{Re}(s)>1)$, the Riemann zeta function. Initially studied by Euler, who wanted to know the values at the positive integers, it achieved prominence with Riemann, who clarified its intimate connection with the distribution of primes, and gave it lasting notoriety with his hypothesis about the location of its zeros.

Another class of Dirichlet series arises in problems of Diophantine approximation, taking a_n to be the fractional part of $n\theta$, where θ is an irrational number. Their properties depend on how well one can approximate θ by rational numbers, and how these fractional parts are distributed modulo 1. The latter is also a dynamical question about the iterative behavior of the rotation by angle θ of the unit circle. Such functions were defined and studied by Hardy and Littlewood in [3], and also by Hecke [5], Ostrowski and others.

A Dirichlet series typically converges in a half-plane $\mathrm{Re}(s)>\sigma_0$. The first step in retrieving the information contained in it is to study its possible analytic continuation. Even its existence is not usually something that can be deduced immediately from the form of the coefficients, however simple their algebraic or analytic nature may be. For instance, as is well known, $\zeta(s)$ extends meromorphically to the whole complex plane, with only a simple pole at $s=1$. In addition, it has an important symmetry around $\mathrm{Re}(s)=1/2$, in the form of a functional equation, a hallmark of many arithmetical Dirichlet series. It has "trivial" zeros at $-2,-4,-6,\ldots$, and its values at the negative odd integers are rational, essentially given by the Bernoulli numbers.

The Diophantine series described above also extend to meromorphic functions on $\mathbb{C}$, but there is no reason to expect a symmetric functional equation. Indeed their poles form the half of a lattice in the left half-plane. Other series, more fancifully defined, are likely not to extend at all. For instance, it is known that $\sum p^{-s}$, where p runs over the primes, cannot extend beyond any point on the imaginary axis, even though it is formed from terms of $\sum_{n=1}^{\infty} n^{-s}$ (Chandrasekharan's book [1] is a nice introduction to these arithmetical connections, whereas Hardy and Riesz's book [4] is a good source for the more analytical aspects of the general theory of Dirichlet series).

The function $\varphi(s)$ we study in this paper, defined by the Dirichlet series $\sum F_n^{-s}$, where F_n is the n^{th} Fibonacci number, shares properties with both types mentioned above. We will show that it extends to a meromorphic function on all of $\mathbb{C}$ and that it has, like the Riemann zeta function, "trivial" zeros at $-2,-6,-10,\ldots$. However, it has trivial simple poles at $0,-4,-8,\ldots$. Again like $\zeta(s)$, we show that at the odd negative integers its values are rational numbers, in this case naturally expressible by Fibonacci and Lucas numbers. In addition, we derive arithmetical expressions for the values of $\varphi(s)$ at positive integers.

On the other hand, we also show that $\varphi(s)$ is analytically similar to the Diophantine series with the golden ratio as the irrational number θ. Indeed $\varphi(s)$ has the same "half-lattice" of poles. More recently, Grabner and Prodinger [2] describe a "Fibonacci" stochastic process in which there arise analytic continuations sharing yet again this kind of pole structure, thus adding a third interesting context in which similar analytic behavior arises. This can be explained by a formal similarity in the calculations in each case, but it would be interesting to study further if there are deeper connections between them.

2. FIRST STEPS

The Fibonacci numbers grow exponentially, and, in general, if $\alpha > 1$ and v_n are integers with $v_n \geq \alpha^n$, then for $\sigma = \mathrm{Re}\, s > 0$ we have the estimate

$$\sum_{n=1}^{\infty} |v_n^{-s}| \leq \sum_{n=1}^{\infty} v_n^{-\sigma} \leq \sum_{n=1}^{\infty} \alpha^{-\sigma n} = (\alpha^{\sigma} - 1)^{-1}.$$

Hence, the Dirichlet series $\sum_{n=1}^{\infty} v_n^{-s}$ defines an analytic function $f(s)$ for $\sigma > 0$, and furthermore,

$$|sf(s)| \leq |s|(\alpha^{\sigma} - 1)^{-1} = (\log \alpha)^{-1} \frac{|s|}{\sigma} + O(\sigma)$$

as $\sigma \to 0^+$, so that $sf(s)$ is bounded in every angular sector with vertex at 0 opening into the half-plane $\mathrm{Re}\, s > 0$.

Applying this to the Fibonacci numbers F_n, we get an analytic function $\varphi(s)$ defined for $\sigma = \mathrm{Re}\, s > 0$ by the Dirichlet series $\sum_{n=1}^{\infty} F_n^{-s}$. It is interesting to express this as a Mellin transform in the classical manner (see Ch. 4 of [4], for example). This is accomplished by the counting function $\Phi(x) = \#\{n \geq 1 : F_n \leq x\}$, which counts the number of Fibonacci numbers less than or equal to x, where we start with F_1 and count $F_1 = F_2 = 1$ twice. Equivalently, $\Phi(x) = \max\{n \geq 0 : F_n \leq x\}$ (but this is not the same as starting from $F_0 = 0$ and counting distinct F_n). Then

$$\varphi(s) = s\int_0^{\infty} \Phi(x) x^{-s-1} dx.$$

Note that $\Phi(x) = 0$ for $0 \leq x < 1$, so the integral actually starts at $x = 1$.

Let $N(x) = [\log_{\varphi} x\sqrt{5}]$, where $\log_{\varphi}$ means the logarithm in base φ and $[x]$ is the integer part of x. Then it is not hard to see that $\Phi(x) = N(x) + \delta(x)$, where $\delta(x) = 0, 1, -1$ and, in fact, $\delta(x) = 1$ if and only if x is in an interval of the form $[F_{2n}, \varphi^{2n}/\sqrt{5})$, $n \geq 1$, and $\delta(x) = -1$ if and only if x is in an interval of the form $[\varphi^{2n+1}/\sqrt{5}, F_{2n+1})$. Let $E \subseteq [1, \infty)$ be the union of these intervals. Then $m(E) < \infty$, where m is Lebesgue measure, and thus we have

$$\varphi(s) = s\int_1^{\infty} \left[\frac{\log(x\sqrt{5})}{\log \varphi}\right] x^{-s-1} dx + s\int_E \delta(x) x^{-s-1} dx. \tag{1}$$

The first integral may be calculated explicitly, and defines a meromorphic function on the whole complex plane. The second integral converges at least for $\sigma > -1$, since for such σ,

$$\int_E |x^{-s-1}| dx = \int_E x^{-\sigma-1} dx < m(E) < \infty.$$

In fact, we do not need to calculate the first integral once we realize that approximating Φ by N is equivalent to approximating F_n by $\varphi^n/\sqrt{5}$ in the Dirichlet series. Indeed,

$$\begin{aligned}\Delta(s) &= \sum_{n=1}^{\infty}\int_{F_{2n}}^{\varphi^{2n}/\sqrt{5}} -dx^{-s} + \int_{\varphi^{2n+1}/\sqrt{5}}^{F_{2n+1}} dx^{-s} \\ &= \sum_{n=1}^{\infty} F_{2n}^{-s} + F_{2n+1}^{-s} - \left(\frac{\varphi^{2n}}{\sqrt{5}}\right)^{-s} - \left(\frac{\varphi^{2n+1}}{\sqrt{5}}\right)^{-s} \\ &= \varphi(s) - 1 - 5^{-s/2}\frac{\varphi^{-2s}}{1-\varphi^{-s}}\end{aligned}$$

for $\sigma > 0$. Thus,

$$\varphi(s) = \Delta(s) + 1 + \frac{\varphi^{cs}}{\varphi^s - 1},$$

where $c = \log_\varphi(\sqrt{5}) - 1$. This is an analytic continuation of $\varphi(s)$ to $\sigma > -1$, and we see that φ has a simple pole at $s = 0$ with residue $1/\log\varphi$. In fact, the series expression

$$\Delta(s) = \sum_{n=2}^{\infty} F_n^{-s} - (\varphi^n/\sqrt{5})^{-s}$$

converges for $\sigma > -2$, since by the mean value theorem,

$$\begin{aligned}|F_n^{-s} - (\varphi^n/\sqrt{5})^{-s}| &\le |F_n - (\varphi^n/\sqrt{5})| \cdot \sup_{|F_n,\, \varphi^n/\sqrt{5}|} |sx^{-s-1}| \\ &= \frac{|s|}{\varphi^n\sqrt{5}} O(\varphi^{-n(\sigma+1)}) = |s| O(\varphi^{-n(\sigma+2)}).\end{aligned}$$

Note also that $\Delta(s) \to 1$ as $|s| \to \infty$ and s lies in an angular sector at 0 opening onto $\mathrm{Re}\, s > 0$. This is consistent with $\varphi(s) \to 2$ as $|s| \to \infty$ in this manner.

Now we proceed to determine the analytic continuation of $\varphi(s)$ to a meromorphic function on $\mathbb{C}$, and determine its poles. From this we will see the reason for this first "jump" from $\sigma = 0$ to $\sigma = -2$.

3. ANALYTIC CONTINUATION

Proposition 1: The Dirichlet series $\sum_{n=1}^{\infty} F_n^{-s}$ can be continued analytically to a meromorphic function $\varphi(s)$ on $\mathbb{C}$ whose singularities are simple poles at $s = -2k + \frac{\pi i(2n+k)}{\log\varphi}$, $k \ge 0$, $n \in \mathbb{Z}$, with residue $(-1)^k 5^{s/2}\binom{-s}{k}/\log(\varphi)$.

Proof: We obtain the full analytic continuation of $\varphi(s)$ by refining the approximation to F_n to a full binomial series

$$\begin{aligned}F_n^p &= \left(\frac{\varphi^n - \varphi^{*n}}{\sqrt{5}}\right)^p = 5^{-p/2}\varphi^{np}\left(1 - \left(\frac{\varphi^*}{\varphi}\right)^n\right)^p \\ &= 5^{-p/2}\varphi^{np}\left(1 - (-1)^{n+1}\frac{1}{\varphi^{2n}}\right)^p = 5^{-p/2}\sum_{k=0}^{\infty}\binom{p}{k}(-1)^{(n+1)k}\varphi^{n(p-2k)}. \qquad (2)\end{aligned}$$

This expansion is valid for any $p \in \mathbb{C}$ and principal powers since then $(xy)^p = x^p y^p$ for $x, y > 0$, and the binomial series converges since $\varphi > 1$. Substituting this into the Dirichlet series for $\varphi(s)$, we get

$$\varphi(s) = \sum_{n=1}^{\infty} F_n^{-s} = 5^{s/2} \sum_{n=1}^{\infty} \sum_{k=0}^{\infty} \binom{-s}{k} (-1)^{k(n+1)} \varphi^{-n(2+1k)}. \tag{3}$$

The double series (3) is absolutely convergent for $\sigma > 0$, for we have the estimate

$$\begin{aligned} \left| \binom{-s}{k} \right| &= \left| \frac{(-s)(-s-1)\cdots(-s-k+1)}{k!} \right| \\ &\le \left| \frac{|s|(|s|+1)\cdots(|s|+k-1)}{k!} \right| = \binom{|s|+k-1}{k} = (-1)^k \binom{-|s|}{k}, \end{aligned} \tag{4}$$

and then

$$\begin{aligned} \sum_{n\ge1,\, k\ge0} \left| \binom{-s}{k} (-1)^{k(n+1)} \varphi^{-n(s+2k)} \right| &\le \sum_{n\ge1,\, k\ge0} (-1)^k \binom{-|s|}{k} \varphi^{-n(\sigma+2k)} \\ &= \sum_{n=1}^{\infty} \varphi^{n\sigma} (1-\varphi^{-2n})^{-|s|} \le (1-\varphi^{-2})^{-|s|} \sum_{n=1}^{\infty} \varphi^{n\sigma} < \infty. \end{aligned}$$

Interchanging the order of summation, we get, since $|\varphi^{-(s+2k)}| = \varphi^{-(\sigma+2k)} < 1$ for $\sigma > 0, k \ge 0$,

$$\begin{aligned} \varphi(s) &= 5^{s/2} \sum_{k=0}^{\infty} \binom{-s}{k} (-1)^k \sum_{n=1}^{\infty} ((-1)^k \varphi^{-(s+2k)})^n \\ &= 5^{s/2} \sum_{k=0}^{\infty} \binom{-s}{k} (-1)^k \frac{(-1)^k \varphi^{-(s+2k)}}{1-(-1)^k \varphi^{-(s+2k)}} \\ &= 5^{s/2} \sum_{k=0}^{\infty} \binom{-s}{k} \frac{1}{\varphi^{s+2k} + (-1)^{k+1}}. \end{aligned} \tag{5}$$

This series converges uniformly and absolutely on compact subsets of $\mathbb{C}$ not containing any of the poles of the functions

$$f_k(s) = \binom{-s}{k} \frac{1}{\varphi^{s+2k} + (-1)^{k+1}},$$

which are at the points $s = -2k + \frac{\pi i(2n+k)}{\log \varphi}$ for $k \ge 0$ and $n \in \mathbb{Z}$. Thus, they lie on the lines $\sigma = -2k$ spaced at intervals of $\frac{2\pi i}{\log \varphi}$; $s = -2k$ is a pole when k is even, and $s = -2k + \frac{\pi i}{\log \varphi}$ is a pole when k is odd. Here we see the reason for our initial jump from $\sigma = 0$ to $\sigma = -2$. For any $s \in \mathbb{C}$, we have $|\varphi^{s+2k} + (-1)^{k+1}| \ge \varphi^{\sigma+2k} - 1 > \varphi^{\sigma+k}$ for $k \gg 0$; hence,

$$\sum_{k > k_0} |f_k(s)| \le \varphi^{-\sigma} \sum_{k=0}^{\infty} (-1)^k \binom{-|s|}{k} \varphi^{-k} = \varphi^{-\sigma}(1-\varphi^{-1})^{-|s|} < \infty$$

for $k_0 \gg 0$, and this bound is uniform when s varies in a compact set. Hence, (5) defines the analytic continuation of $\varphi(s)$ to a meromorphic function on $\mathbb{C}$ with simple poles at $s_{kn} = -2k + \frac{\pi i(2n+k)}{\log \varphi}$, $k \ge 0$, $n \in \mathbb{Z}$. The residue at s_{kn} is easily seen to be

$$\frac{5^{s_{kn}/2}\binom{-s_{kn}}{k}}{\frac{d}{ds}(\varphi^{s+2k}+(-1)^{k+1})_{s=s_{kn}}}=\frac{(-1)^k 5^{s_{kn}/2}\binom{-s_{kn}}{k}}{\log(\varphi)}. \ \square \tag{6}$$

4. VALUES AT NEGATIVE INTEGERS

Next, we discuss the values of $\varphi(s)$ at the negative integers. We already know that 0, -4, $-8, \ldots$ are simple poles.

Proposition 2: $\varphi(s)$ has "trivial" zeros at $-m$, where $m \geq 0$, $m \equiv 2 \bmod 4$, and the values at other negative integers are rational numbers, which can be expressed in terms of Fibonacci and Lucas numbers.

Proof: Let $m \geq 0$ be an integer not a multiple of 4. By (5),

$$\varphi(-m)=5^{-m/2}\sum_{k=0}^{\infty}\binom{m}{k}\frac{1}{\varphi^{-m+2k}+(-1)^{k+1}},$$

and since $m \in \mathbb{Z}^+$, all terms with $k > m$ are 0, so that this is really a finite sum belonging to $\mathbb{Q}(\sqrt{5})$. Let $\sigma_k = \binom{m}{k}(\varphi^{-m+2k}+(-1)^{k+1})^{-1}$ and $\alpha_k = \sigma_k + \sigma_{m-k}$, so that $\alpha_k = \alpha_{m-k}$ and

$$\varphi(-1)=\frac{1}{2}5^{-m/2}\sum_{k=0}^{m}\alpha_k,$$

with

$$\varphi(-m)=5^{-m/2}\sum_{k=0}^{\frac{m-1}{2}}\alpha_k$$

if m is odd. We compute

$$\begin{aligned}\alpha_k &= \binom{m}{k}\frac{1}{\varphi^{2k-m}+(-1)^{k+1}}+\binom{m}{m-k}\frac{1}{\varphi^{m-2k}+(-1)^{m-k+1}}\\ &=\binom{m}{k}\left(\frac{1}{\varphi^{2k-m}+(-1)^{k+1}}+\frac{1}{(-1)^m\varphi^{*(2k-m)}+(-1)^{m+k+1}}\right)\\ &=\binom{m}{k}\left(\frac{1}{\varphi^{2k-m}+(-1)^{k+1}}+\frac{(-1)^m}{\varphi^{*(2k-m)}+(-1)^{k+1}}\right),\end{aligned}$$

so that $\alpha_k^* = \alpha_k$ if m is even, and $\alpha_k^* = -\alpha_k$ if m is odd, where α^* denotes the Galois conjugate in $\mathbb{Q}(\sqrt{5})$. Thus, if m is even, we have $\alpha_k \in \mathbb{Q}$ for all k, and since also $5^{-m/2} \in \mathbb{Q}$, we see that $\varphi(-m) \in \mathbb{Q}$ in this case. If m is odd, then α_k if of the form $a_k\sqrt{5}$, where $a_k \in \mathbb{Q}$, as is also $5^{-m/2}$, so that again $\varphi(-m) \in \mathbb{Q}$. We get further information by carrying through the computation of α_k:

$$\begin{aligned}\alpha_k &= \binom{m}{k}\frac{(-1)^m\varphi^{2k-m}+(-1)^{m+k+1}+\varphi^{*(2k-m)}+(-1)^{k+1}}{(\varphi\varphi^*)^{2k-m}+(-1)^{k+1}(\varphi^{2k-m}+\varphi^*(2k-m)+1)}\\ &=\binom{m}{k}(-1)^{k+1}\frac{(-\varphi)^{2k-m}+\varphi^{*(2k-m)}+(-1)^{k+1}(1+(-1)^m)}{\varphi^{2k-m}+\varphi^{*(2k-m)}+(-1)^{k+1}(1+(-1)^m)}.\end{aligned} \tag{7}$$

If $m \equiv 2 \bmod 4$, then this simplifies to $\alpha_k = \binom{m}{k}(-1)^{k+1}$, and then

$$\varphi(-m) = \frac{1}{2}5^{-m/2}\sum_{k=0}^{m}\binom{m}{k}(-1)^{k+1} = 0 \quad (m \geq 0 \text{ even}). \tag{8}$$

These may be considered the "trivial" zeros of $\varphi(s)$.

If m is odd, then

$$\begin{aligned}\alpha_k &= \binom{m}{k}(-1)^{k+1}\frac{-\varphi^{2k-m}+\varphi^{*(2k-m)}}{\varphi^{2k-m}+\varphi^{*(2k-m)}}\\ &= \sqrt{5}\binom{m}{k}(-1)^k\frac{F_{2k-m}}{L_{2k-m}} = \sqrt{5}\binom{m}{k}(-1)^{k+1}\frac{F_{m-2k}}{L_{m-2k}},\end{aligned}$$

where $L_n = \varphi^n + \varphi^{*n}$ is the Lucas sequence $2, 1, 3, 4, 7, \ldots$, and both F_n and L_n are extended to all $n \in \mathbb{Z}$, so that $F_{-n} = (-1)^{n+1}F_n$ and $L_{-n} = (-1)^n L_n$; hence, $F_{-n}/L_{-n} = -F_n/L_n$ for all $n \neq 0$. Then

$$\varphi(-m) = \frac{1}{5^{(m-1)/2}}\sum_{k=0}^{\frac{m-1}{2}}\binom{m}{k}(-1)^{k+1}\frac{F_{m-2k}}{L_{m-2k}} \quad (m \geq 1 \text{ odd}). \ \square \tag{9}$$

All that has been done for the Dirichlet series $\sum_{n=1}^{\infty}F_n^{-s}$ may be carried out in an entirely analogous manner for $\sum_{n=1}^{\infty}(-1)^n F_n^{-s}$. Carrying out the corresponding calculations, which amounts to chasing sign changes in the previous ones, yields the following result.

Theorem 1: The Dirichlet series $\sum_{n=1}^{\infty}(-1)^n F_n^{-s}$ can be analytically continued to a meromorphic function $\psi(s)$ on $\mathbb{C}$ by the series

$$5^{s/2}\sum_{k=0}^{\infty}\binom{-s}{k}\frac{1}{(-1)^{k+1}-\varphi^{s+2k}}. \tag{10}$$

The singularities of $\psi(s)$ are simple poles at

$$s = -2k + \frac{\pi i(2n+k+1)}{\log\varphi}, \ k \geq 0, \ n \in \mathbb{Z},$$

with residue

$$(-1)^k 5^{s/2}\binom{-s}{k}/\log(\varphi).$$

These are "complementary" to the poles of $\varphi(s)$. Thus, $-m$ is a simple pole for integers $m \geq 0$, $m \equiv 2 \bmod 4$. Similarly, $\psi(s)$ has trivial zeros at $-m$, where $m > 0$, $m \equiv 0 \bmod 4$ (note that $\psi(0) = -1/2$). Finally,

$$\psi(-m) = \varphi(-m) = 5^{-(m-1)/2}\sum_{k=0}^{\frac{m-1}{2}}\binom{m}{k}(-1)^{k+1}\frac{F_{m-2k}}{L_{m-2k}}$$

for $m > 0$, $m \equiv 1 \bmod 2$.

In particular, $\frac{1}{2}(\varphi(s) - \varphi(s))$ analytically continues the series $\sum_{n=0}^{\infty}F_{2n+1}^{-s}$ to a function with simple poles at

$$s = -2k + \frac{\pi i n}{\log\varphi}, \ k \geq 0, \ n \in \mathbb{Z},$$

hence, at all even negative integers. The odd negative integers are trivial zeros of this function. Similarly, $\frac{1}{2}(\varphi(s)-\varphi(s))$ analytically continues the series $\sum_{n=1}^{\infty} F_{2n}^{-s}$ to a function with the same singularities, and rational values at the odd negative integers.

5. VALUES AT POSITIVE INTEGERS

Theorem 2: For $m \in \mathbb{N}$, $\varphi(m) = 5^{m/2}\sum_{l=1}^{\infty} c_l \varphi^{-l}$, where the coefficients c_l are combinations of sums of powers of divisors of l. In particular, we have the formulas

$$\begin{aligned}
\varphi(1) &= \sqrt{5}\sum_{l=1}^{\infty}(d_1(l)+(-1)^l d_3(l))\,\varphi^{-l},\\
\varphi(2) &= 5\sum_{l\equiv 0 \bmod 2}(-1)^{\frac{l}{2}+1}\sigma_1([l]_2')\,\varphi^{-l},\\
\varphi(3) &= \frac{5\sqrt{5}}{8}\sum_{l=1}^{\infty}(d_3^2(l)-d_3(l)+(-1)^l(d_1^2(l)-d_1(l)))\,\varphi^{-l}\\
\varphi(4) &= \frac{25}{6}\sum_{l\equiv 2 \bmod 4}(\sigma_1([l]_2')-\sigma_3([l]_2'))\,\varphi^{-l}\\
&\quad +25\sum_{l\equiv 0 \bmod 4}\left(\frac{1-[l]_2}{6}\sigma_1([l]_2')+\frac{([l]_2)^3-1}{42}\sigma_3([l]_2'\right)\varphi^{-l},
\end{aligned}\tag{11}$$

where $d_i^k(n) = \sum_{d|n,\, d\equiv i(4)} d^k$, $d_i = d_i^1$, $\sigma_k(n) = \sum_{d|n} d^k$, $[l]_2 = 2^{\mathrm{ord}_2(l)}$ is the 2-part of l, and $[l]_2'$ is the part of l prime to 2.

Proof: Starting from (2) and

$$(-1)^k\binom{-s}{k} = \binom{s+k-1}{k}$$

we have, for $m \in \mathbb{N}$,

$$F_n^{-m} = 5^{m/2}\sum_{k=0}^{\infty}\binom{m+k-1}{m-1}(-1)^{kn}\varphi^{-n(m+2k)}.$$

Let $d = m+2k$, which ranges over $S_m = \{d \ge m : d \equiv m \bmod 2\}$, so

$$F_n^{-m} = 5^{m/2}\sum_{d\in S_m}\binom{\frac{d+m-2}{2}}{m-1}(-1)^{n\frac{d-m}{2}}\varphi^{-nd}.$$

Let $S_m^+ = \{d \ge m : s \equiv m \bmod 4\}$ and $S_m^- = \{d \ge m : s \equiv m+2 \bmod 4\}$. Then

$$F_n^{-m} = 5^{m/2}\left(\sum_{d\in S_m^+}\binom{\frac{d+m-2}{2}}{m-1}\varphi^{-nd} + (-1)^n\sum_{d\in S_m^-}\binom{\frac{d+m-2}{2}}{m-1}\varphi^{-nd}\right).$$

To sum over n, we will collect like powers $l = nd$, so that l runs over all natural numbers and we restrict to $d \mid l$, obtaining

$$\sum_{n=1}^{\infty}F_n^{-m} = 5^{m/2}\sum_{l=1}^{\infty}\left(\sum_{d|l,\, d\in S_m^+}\binom{\frac{d+m-2}{2}}{m-1} + \sum_{d|l,\, d\in S_m^-}(-1)^{l/d}\binom{\frac{d+m-2}{2}}{m-1}\right)\varphi^{-l}. \tag{12}$$

Similarly, we may sum over subsets S of the natural numbers, letting l run over multiples of S_m by $n \in S$. For example, if m is odd, then the divisors $d \in S_m$ are odd, so if we wish to sum over odd n, then we let l run over odd numbers. If we wish to sum over even n, then l runs over even numbers. If m is even, then the divisors $d \in S_m$ are even, so l runs over even numbers. In the case $m = 1$, we have $S_1^+ = \{d \geq 1, d \equiv 1 \bmod 4\}$ and $S_1^- = \{d \geq 1, d \equiv 3 \bmod 4\}$. The binomial coefficients reduce to 1. Noting that $\sum_{d|l, d\equiv i\,(4)} (-1)^{l/d} = (-1)^l d_i(l)$, this gives us the formulas:

$$\begin{aligned}
\sum_{n=1}^{\infty} F_n^{-1} &= \sqrt{5}\sum_{l=1}^{\infty}(d_1(l)+(-1)^l d_3(l))\,\varphi^{-l};\\
\sum_{n=1}^{\infty} (-1)^n F_n^{-1} &= \sqrt{5}\sum_{l=1}^{\infty}((-1)^l d_1(l)+d_3(l))\,\varphi^{-l};\\
\sum_{n=0}^{\infty} F_{2n+1}^{-1} &= \sqrt{5}\sum_{l\equiv 1 \bmod 2}(d_1(l)-d_3(l))\,\varphi^{-l};\\
\sum_{n=1}^{\infty} F_{2n}^{-1} &= \sqrt{5}\sum_{l\equiv 0 \bmod 2}(d_1(l)+d_3(l))\,\varphi^{-l}.
\end{aligned} \tag{13}$$

Horadam [6] treats other approaches to these and other sums of reciprocals ($s = 1$) of quadratic recurrence sequences involving elliptic functions (see Proposition 3 below).

In general,

$$P_m(x) = \binom{\frac{x+m-2}{2}}{m-1}$$

is a polynomial in x of degree $m-1$, divisible by x if m is even. Write

$$P_m(x) = \sum_{k=0}^{m-1} a_{km} x^m,$$

where $a_{km} \in \mathbb{Q}$. Then $\varphi(m) = 5^{m/2}\sum_{l=1}^{\infty} c_l \varphi^{-l}$, where

$$c_l = \sum_{k=0}^{m-1} a_{km}\left(\sum_{d|l,\, d\in S_m^+} d^k + \sum_{k|l,\, d\in S_m^-} (-1)^{l/d} d^k\right). \tag{14}$$

This observation proves the theorem. □

To get the specific formulas for fixed l, m, let s_{klm} denote the expression in parentheses. Note that, for odd m, we have sums over divisor classes $d \equiv 1$, 3 mod 4, so the signs do not bother us:

$$c_l = \sum_{k=0}^{m-1} a_{km}\left(\sum_{d|l,\, d\in S_m^+} d^k + (-1)^l \sum_{k|l,\, d\in S_m^-} d^k\right)$$

and the greater difficulty is the size restriction on divisors, $d \geq m$. For even m, the signs are more of a nuisance. The classes S_m^+, S_m^- are of divisors $d \equiv 0$, 2 mod 4. We are summing over even l, and we write $l = 2^r \lambda$ with $r \geq 1$ and λ odd. Then the divisors $d|l$ with $d \equiv 2$ mod 4 are of the form $d = 2\delta$ with $\delta|\lambda$. Thus, forgetting for the moment about the restrictions on the size of d, we note that $\sum_{d|l, d\equiv 2 \bmod 4} d^k = 2^k \sigma_k(\lambda)$ and $\sum_{d|l, d\equiv 2 \bmod 4} (-1)^{l/d} d^k = (-1)^{l/2} 2^k \sigma_k(\lambda)$, since $l/d = \lambda/\delta$ is odd or even according to whether $\lambda = l/2$ is odd or even.

The divisors $d|l$ with $d \equiv 0 \bmod 4$ are nonexistent if $r = 1$; otherwise, they are of the form $d = 2^\rho \delta$, with $2 \le \rho \le r$ and $\delta | \lambda$. Thus,

$$\sum_{d|l,\, d\equiv 2 \bmod 4} d^k = \sum_{\delta|\lambda} \sum_{\rho=2}^{r} (2^\rho \delta)^k = \frac{2^{k(r+1)} - 2^{2k}}{2^k - 1} \sigma_k(\lambda)$$

and

$$\sum_{d|l,\, d\equiv 2 \bmod 4} (-1)^{l/d} d^k = \sum_{\delta|\lambda} \sum_{\rho=2}^{r} (2^\rho \delta)^k (-1)^{2^{r-\rho}\frac{\lambda}{\delta}}$$

$$= -\sum_{\delta|\lambda} 2^{rk}\delta^k + \sum_{\delta|\lambda} \sum_{\rho=2}^{r-1} (2^\rho \delta)^k = -2^{2k}\sigma_k(\lambda).$$

Putting it all together, we obtain the formulas

$$s_{klm} = \begin{cases} 2^k \sigma_k(\lambda) - \displaystyle\sum_{\substack{d|l \\ d\equiv 2 \bmod 4 \\ d<m}} d^k, & l \equiv 2 \ (4), \\ (2^k - 2^{2k})\sigma_k(\lambda) - \displaystyle\sum_{\substack{d|l \\ d\equiv 2 \bmod 4 \\ d<m}} d^k - \displaystyle\sum_{\substack{d|l \\ d\equiv 2 \bmod 4 \\ d<m}} (-1)^{l/d} d^k, & l \equiv 0 \ (4), \end{cases}$$

if $m \equiv 2 \bmod 4$, and

$$s_{klm} = \begin{cases} -2^k \sigma_k(\lambda) + \displaystyle\sum_{\substack{d|l \\ d\equiv 2 \bmod 4 \\ d<m}} d^k, & l \equiv 2 \ (4), \\ 2^k \sigma_k(\lambda) + \dfrac{2^{k(r+1)} - 2^{2k}}{2^k - 1}\sigma_k(\lambda) - \displaystyle\sum_{\substack{d|l \\ d\equiv 2 \bmod 4 \\ d<m}} d^k - \displaystyle\sum_{\substack{d|l \\ d\equiv 2 \bmod 4 \\ d<m}} d^k, & l \equiv 0 \ (4), \end{cases}$$

if $m \equiv 0 \bmod 4$, from which we get the formulas in the theorem.

A curious result may be derived from these formulas in the case $m = 1$, which is probably the subject of Landau's centenary paper [7], to which, unfortunately, the author did not have access. Let $\Theta(z) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2 z}$ denote Jacobi's theta function. We write also $\Theta(q) = \sum_{n\in\mathbb{Z}} q^{n^2}$, for $q = e^{\pi i z}$ with $\operatorname{Im} z > 0$. Then,

Proposition 3: The following formula holds for $\delta = \frac{i\pi}{\log\varphi}$:

$$\sum_{n\equiv 1 \bmod 2} F_n^{-1} = \frac{\sqrt{5}}{4}\left(\Theta\left(-\frac{1}{\delta}\right)^2 - \Theta\left(-\frac{2}{\delta}\right)^2\right). \tag{15}$$

Proof: Note that $q = \varphi^{-1}$ for $z = -1/\delta$ (δ is the minimum difference of the poles of $\varphi(s)$ along the vertical lines $\sigma = -2k$). We have $\Theta(q)^2 = \sum_{i=0}^{\infty} r_2(l) q^l$, where $r_2(l)$ is the number of representations of l as a sum of two integer squares. Since $r_2(l) = 4(d_1(l) - d_3(l))$, we have shown

$$\sum_{n\equiv 1 \bmod 2} F_n^{-1} = \frac{\sqrt{5}}{4} \sum_{l\equiv 1 \bmod 2} r_2(l)\varphi^{-l},$$

and the formula follows from noting that $r_2(2l) = r_2(l)$ since $d_1(2l) = d_1(l)$, hence

$$\Theta(q^2)^2 = \sum_{l=0}^{\infty} r_2(l) q^{2l} = \sum_{l=0}^{\infty} r_2(2l) q^{2l}$$

and so

$$\sum_{l \equiv 1 \bmod 2} r_2(l) q^l = \Theta(q)^2 - \Theta(q^2)^2. \ \square$$

6. DIOPHANTINE APPROXIMATION

The "Fibonacci zeta function" $\varphi(s)$ has much in common with the meromorphic function obtained by analytic continuation of the Dirichlet series $f(s) = \sum_{n=1}^{\infty} \frac{\{n\varphi\}}{n^s}$ (where $\{x\}$ is the fractional part of x) studied by (among others) Hecke in [5] and Hardy and Littlewood in [3] and related papers. Indeed, they show that this function has the same singularities as $\varphi(s)$, namely, simple poles at $-2k + \frac{(2n+k)\pi i}{\log \varphi}$. They work with any reduced quadratic irrational α, and it is easily seen that we have analogous results in that case also. In particular, $f(s)$ and $\varphi(s)$ differ by an entire function. The function $\varphi(s)$ is not in those papers, which have in mind the study of the distribution of the fractional parts $\{n\alpha\}$ (see also Lang [8]). Hecke mentions that $\sum_{n \in S} n^{-s}$ also has an analytic continuation when S is the set of positive integers satisfying $\{n\alpha\} < \varepsilon$ for a given $\varepsilon > 0$. Note that $F_{2n+1} \in S$ except for finitely many n, but by Weyl's equidistribution theorem there are infinitely more numbers in S, making these continued functions have an additional pole at $s = 1$. Comparing (5) with formulas in [5] and [3], we find similar summands multiplied by zeta-like functions. It would be interesting to obtain more qualitative information. Further questions about $\varphi(s)$ might involve finding nontrivial zeros and studying their distribution, and more properties of the values $\varphi(m)$ at integers m.

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SOME GENERAL FORMULAS ASSOCIATED WITH THE SECOND-ORDER HOMOGENEOUS POLYNOMIAL LINE-SEQUENCES

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In our previous report [5], we developed some methods in the study of the line-sequential properties of the polynomial sequences treated in Shannon and Horadam [9]. In this report, we work out the properties for the general case and apply them to the Pell polynomial line-sequence as an example. Some known results are included for completeness, but only new aspects will be presented in some detail.

1. THE BASIC FORMULAS

Recall that the linear homogeneous second-order recurrence relation is given by

$$cu_n + bu_{n+1} = u_{n+2},\ c, b \neq 0,\ n \in \mathbf{Z}; \tag{1.0a}$$

and a general line-sequence is expressed as

$$\bigcup_{u_0, u_1}(c, b)\colon \ldots, u_{-3}, u_{-2}, u_{-1}, [u_0, u_1], u_2, u_3, u_4, \ldots, u_n \in \mathbf{R}, \tag{1.0b}$$

where $[u_0, u_1]$ denotes the generating pair of the line-sequence.

1. Basis Pair. The basis pair for the general case, that is, without specifying the parametric coefficients b and c, is given by (4.2) and (4.3) in [8]:

$$G_{1,0}(c, b)\colon \ldots, (c+b^2)/c^2, -b/c, [1, 0], c, cb, c(c+b^2), \ldots, \tag{1.1a}$$

$$G_{0,1}(c, b)\colon \ldots, (c+b^2)/c^3, -b/c^2, 1/c, [0, 1], b, (c+b^2), \ldots. \tag{1.1b}$$

Definition 1: Two line-sequences are said to be *complementary* if they are orthogonal.

Obviously, the pair (1.1a) and (1.1b) are orthogonal and form a set of basis. When $c = b = 1$, they reduce to the complementary Fibonacci and the Fibonacci line-sequences, $F_{1,0}$ and $F_{0,1}$, respectively. It is clear that all the line-sequential properties of either a number line-sequence or a polynomial line-sequence given by the recurrence relation (1.0a) originate from the properties of this pair. Following are some of the main properties.

2. Translation. The translational relation between the basic pair is given by:

$$TG_{1,0} = cG_{0,1}, \tag{1.2a}$$

where T denotes the translation operation, see (3.1) in [8]. Let $g_n[i, j]$ denote the n^{th} term in the line-sequence $G_{i,j}$, then, in terms of the elements, (1.2a) becomes

$$g_{n+1}[1, 0] = cg_n[0, 1]. \tag{1.2b}$$

3. Parity. The parity relation of the elements in $G_{1,0}$ is found to be

$$g_{-n}[1, 0] = (-1)^n c^{-(n+1)} g_{n+2}[1, 0]. \tag{1.3a}$$

From (4.9) in [8], the parity relation of the elements in $G_{0,1}$ is given by

$$g_{-n}[0,1] = (-1)^{n+1}c^{-n}g_n[0,1]. \tag{1.3b}$$

Applying translational relation (1.2b) to (1.3b), we get (1.3a). In the nomenclature of Shannon and Horadam [9], parity relation (1.3b) reduces to (1.7) in [1] for $c=-1$ in the case of Morgan-Voyce even Fibonacci polynomials.

4. Cross Relations. Combining the translational and parity relations, we obtain the following set of *cross* relations among the elements of the two basis polynomial line-sequences:

$$g_{-n}[1,0] = (-1)^n c^{-n} g_{n+1}[0,1], \tag{1.4a}$$

$$g_{-n}[1,0] = c g_{-(n+1)}[0,1]; \tag{1.4b}$$

or

$$g_{-n}[0,1] = (-1)^{n+1} c^{-(n+1)} g_{n+1}[1,0], \tag{1.4c}$$

$$g_{-n}[0,1] = c^{-1} g_{-(n-1)}[1,0]. \tag{1.4d}$$

5. Geometrical Line-Sequences. The pair of geometrical line-sequences relating to $G_{1,0}$ is given by:

$$G_{1,\alpha}(c,b): \ldots, \alpha^{-2}, \alpha^{-1}, [1,\alpha], \alpha^2, \alpha^3, \ldots, \tag{1.5a}$$

$$G_{1,\beta}(c,b): \ldots, \beta^{-2}, \beta^{-1}, [1,\beta], \beta^2, \beta^3, \ldots, \tag{1.5b}$$

where α and β are the roots of the generating equation

$$q^2 - bq - c = 0 \tag{1.5c}$$

(ref. (1.7) in [4], with $g=0$).

6. Binet's Formula. Binet's formula for the $G_{1,0}$ basis is given by

$$G_{1,0} = (-\beta G_{1,\alpha} + \alpha G_{1,\beta}) / (\alpha - \beta), \tag{1.6a}$$

and for the $G_{0,1}$ basis is given by

$$G_{0,1} = (G_{1,\alpha} - G_{1,\beta}) / (\alpha - \beta) \tag{1.6b}$$

(ref. (4.9) in [7]).

7. (General) Lucas Pair. Recall that the line-sequence "conjugate" to $G_{0,1}$ is the "general" Lucas line-sequence generated by $[2,b]$, see (4.12) in [8]:

$$G_{2,b}(c,b): \ldots, (2c+b^2)/c^2, -b/c, [2,b], 2c+b^2, b(3c+b^2), \ldots, \tag{1.7a}$$

which reduces to the Lucas line-sequence $L_{2,1}$ if $c=b=1$. Its complement is

$$G_{b,-2}(c,b): \ldots, b(c+2+b^2)/c^2, -(2+b^2)/c, [b,-2], (c-2)b, -2c+(c-2)b^2, \ldots. \tag{1.7b}$$

For $c=b=1$, it reduces to the complementary Lucas line-sequence,

$$L_{1,-2}(1,1): \ldots, -7, 4, -3, [1,-2], -1, -3, -4, -7, \ldots. \tag{1.7c}$$

The orthogonal pair (1.7a) and (1.7b) then form a "(general) Lucas basis" spanning the same $2D$ space as does the basis pair $G_{1,0}$ and $G_{0,1}$, but with a normalization factor $(b^2+4)^{-1/2}$.

8. Decomposition Schemes. Thus, we have the following different ways of decomposing a given line-sequence, $G_{i,j}(c,b)$: The first is the "basic decomposition" resulting in the basic component expression, see (2.9) in [8]:

$$G_{i,j}(c,b) = iG_{1,0}(c,b) + jG_{0,1}(c,b). \tag{1.8a}$$

The second is the "Binet decomposition" the general formula of which is

$$G_{i,j}(c,b) = [-(\beta i - j)G_{1,\alpha} + (\alpha i - j)G_{1,\beta}]/(\alpha - \beta). \tag{1.8b}$$

Note that the Binet pair $G_{1,\alpha}$ and $G_{1,\beta}$ does not form an orthogonal pair unless $c=1$, see (2.8) in [8].

The third is the "Lucas decomposition," which produces the Lucas component expression, the formula of which is given by

$$G_{i,j}(c,b) = [(2i+bj)G_{2,b} + (bi-2j)G_{b,-2}]/(b^2+4), \tag{1.8c}$$

where the denominator accounts for the normalization factor.

Since line-sequences $G_{x,y}$ and $G_{y,-x}$ are complementary, by repeated application of the vector addition and the scalar multiplication rules, see [8], we obtain the general orthogonal decomposition formula:

$$G_{i,j}(c,b) = [(xi+yj)G_{x,y} + (yi-xj)G_{y,-x}]/(x^2+y^2). \tag{1.8d}$$

Putting $x=1$ and $y=-1$, and applying the rule for scalar multiplication by -1, we obtain (1.8a); if we put $x=2$ and $y=b$, we obtain (1.8c).

Similarly, for an arbitrary pair of line-sequences $G_{x,y}$ and $G_{z,w}$, we find

$$G_{i,j}(c,b) = [-(wi-zj)G_{x,y} + (yi-xj)G_{z,w}]/(yz-wx). \tag{1.8e}$$

This is the general decomposition formula. For convenience, we call $G_{x,y}$ and $G_{z,w}$ the pair of coordinate line-sequences, and their coefficients the respective components. Putting $z=y$ and $w=-x$, we get (1.8d); if we put $x=z=1$, $y=\alpha$, and $w=\beta$, we get (1.8b). Wang and Zhang [11] adopted a very special pair of coordinates based on their conjugation property: $G_{0,1}$ and $G_{2,b}$. Putting $x=0$, $y=1$, $z=2$, and $w=b$, we obtain

$$G_{i,j}(c,b) = [-(bi-2j)G_{0,1} + iG_{2,b}]/2, \tag{1.8f}$$

which is equivalent to equation (2) in [11]. This decomposition scheme is particularly convenient to use in treating products of terms because of the conjugation property.

9. Translational Representation. By applying the translational relation (1.2a) to (1.8a), we obtain the translational representation of a general line-sequence in terms of the first basis,

$$G_{i,j}(c,b) = (iI + jc^{-1}T)G_{1,0}(c,b), \tag{1.9a}$$

where I denotes the identity translation; or, in terms of the elements, in the second basis:

$$g_n[i,j] = cig_{n-1}[0,1] + jg_n[0,1]. \tag{1.9b}$$

10. Binet's Product. Consistent with the multiplication of the corresponding terms in two line-sequences to obtain their product, we present the following definition.

Definition 2: The product of two line-sequences is defined as the product of the two respective Binet formulas, and shall be referred to as "Binet's Product." Also, for convenience, exponentiation notation is adopted when it applies.

Note that, except for some special cases, Binet's product does not, in general, constitute a line-sequence governed by (1.0a). This question will be discussed in a later paper.

In the line-sequential format, we then have

$$G_{1,0}G_{0,1} = (-(\beta G_{1,\alpha}^2 + \alpha G_{1,\beta}^2) + b(-c)^n) / (b^2 + 4c) \tag{1.10a}$$

or, in terms of the elements,

$$g_n[1, 0]g_n[0, 1] = (cg_{2n-1}[2, b] + b(-c)^n) / (b^2 + 4c). \tag{1.10c}$$

The conjugation of $G_{0,1}$ and $G_{2,b}$ (ref. (4.7) and (4.8) in [8]) is then given by

$$G_{0,1}G_{2,b} = (G_{1,\alpha}^2 - G_{1,\beta}^2) / (\alpha - \beta) \tag{1.10c}$$

or, in terms of the elements,

$$g_n[0, 1]g_n[2, b] = g_{2n}[0, 1]. \tag{1.10d}$$

This is the general conjugation formula relating the Fibonacci and the Lucas elements. For $c = b = 1$, it reduces to the basic conjugation relation, $f_n l_n = f_{2n}$.

The Binet product of $G_{1,0}$ and $G_{b,-2}$ is somewhat more complex, that is,

$$G_{1,0}G_{b,-2} = \{[\beta(\beta b + 2)G_{1,\alpha}^2 + \alpha(\alpha b + 2)G_{1,\beta}^2] + 2b(c-1)G_{1,\alpha}G_{1,\beta}\} / (\alpha - \beta)^2 \tag{1.10e}$$

or, in terms of the elements,

$$g_n[1, 0]g_n[b, -2] = c\{bcg_{2n-2}[2, b] - 2g_{2n-1}[2, b] + 2b(-c)^{n-1}(c-1)] / (b^2 + 4c). \tag{1.10f}$$

When $c = b = 1$, it reduces to the more easily recognizable relation,

$$f_n[1, 0]l_n[1, -2] = (l_{2n-2}[2, 1] - 2l_{2n-1}[2, 1]) / 5, \tag{1.10g}$$

which is a set of even terms in the *negative* Fibonacci line-sequence,

$$F_{0,-1}: \ldots, \mathbf{8}, -5, \mathbf{3}, -2, \mathbf{1}, -1, [\mathbf{0}, -1], -\mathbf{1}, -2, -\mathbf{3}, -5, -\mathbf{8}, \ldots. \tag{1.10h}$$

11. Summation. From the recurrence relation, it is easy to show that the general consecutive terms summation formula is given by

$$(b + c - 1)\sum_{i=k}^{k+n} u_i = cu_{k+n} + u_{k+n+1} + (b-1)u_k - u_{k+1}, \tag{1.11a}$$

where $i \geq k$, $n \geq 0$; $i, k, n \in \mathbf{Z}$. We stress that this formula, like (4.3u) in [6], is translationally covariant. In the harmonic case, $b = c = 1$, it reduces to the latter. In the case of Jacobsthal numbers, it reduces to (2.7) and (2.8) in [3], respectively; in the case of Jacobsthal polynomials, it reduces to (3.7) and (3.8) in [2], respectively, and so forth.

A word about the convention. In (4.3u) in [6], the translational degree of freedom is implicit in that the zero[th] element u_0 may be assigned to any element in the line-sequence. In the current case, however, we want to assign the zero[th] element in the formula to the zero[th] element of the line-sequence in question; for example, in the Pell line-sequence, we want to assign $u_0 = p_0$, so the translational degree of freedom lies explicitly in the value of the parameter k chosen.

It is easy to show that the following two equations hold:

$$(c-1)\sum_{i=k}^{k+n} u_{2i} + b\sum_{i=k}^{k+n} u_{2i-1} = c(u_{2(k+n)} - u_{2(k-1)}), \tag{1.11b}$$

$$(c-1)\sum_{i=k}^{k+n} u_{2i+1} + b\sum_{i=k}^{k+n} u_{2i} = c(u_{2(k+n)+1} - u_{2k-1}). \tag{1.11c}$$

In the harmonic case, (1.11b) reduces to the *odd* terms summation formula (4.4u), and (1.11c) reduces to the *even* terms summation formula (4.5u) in [6], respectively.

Combining (1.11b) and (1.11c), we obtain the general even terms summation formula,

$$\sum_{i=k}^{k+n} u_{2i} = [(c-1)(u_{2(k+n)+2} - u_{2k}) - bc(u_{2(k+n)+1} - u_{2k-1})]/[(c-1)^2 - b^2], \tag{1.11d}$$

and the general odd terms summation formula,

$$\sum_{i=k}^{k+n} u_{2i+1} = [c^2(u_{2(k+n)+1} - u_{2k-1}) - (u_{2(k+n)+3} - u_{2k+1})]/[(c-1)^2 - b^2]. \tag{1.11e}$$

12. Translational Operators. By the dual relation of Section 4 in [6], corresponding to formulas (1.11a) through (1.11e), we have the following set of covariant equations of the translational operators:

$$(b+c-1)\sum_{i=k}^{k+n} T_i = cT_{k+n} + T_{k+n+1} + (b-1)T_k - T_{k+1}, \tag{1.12a}$$

$$(c-1)\sum_{i=k}^{k+n} T_{2i} + b\sum_{i=k}^{k+n} T_{2i-1} = c(T_{2(k+n)} - T_{2(k-1)}), \tag{1.12b}$$

$$(c-1)\sum_{i=k}^{k+n} T_{2i+1} + b\sum_{i=k}^{k+n} T_{2i} = c(T_{2(k+n)+1} - T_{2k-1}), \tag{1.12c}$$

$$\sum_{i=k}^{k+n} T_{2i} = [(c-1)(T_{2(k+n)+2} - T_{2k}) - bc(T_{2(k+n)+1} - T_{2k-1})]/[(c-1)^2 - b^2], \tag{1.12d}$$

$$\sum_{i=k}^{k+n} T_{2i+1} = [c^2(T_{2(k+n)+1} - T_{2k-1}) - (T_{2(k+n)+3} - T_{2k+1})]/[(c-1)^2 - b^2]. \tag{1.12e}$$

13. Simson's Formula. The general Simson formula is found to be

$$g_{n+1}[i,j]g_{n-1}[i,j] - (g_n[i,j])^2 = (-c)^{n-1}(bij + ci^2 - j^2). \tag{1.13}$$

In particular, for the general Fibonacci and the general Lucas pairs,

$$g_{n+1}[1,0]g_{n-1}[1,0] - (g_n[1,0])^2 = -(-c)^n, \tag{1.13a}$$

$$g_{n+1}[0,1]g_{n-1}[0,1] - (g_n[0,1])^2 = -(-c)^{n-1}, \tag{1.13b}$$

$$g_{n+1}[2,b]g_{n-1}[2,b]-(g_n[2,b])^2=(-c)^{n-1}(b^2+4c), \tag{1.13c}$$

$$g_{n+1}[b,-2]g_{n-1}[b,-2]-(g_n[b,-2])^2=-(-c)^{n-1}(2b^2-b^2c+4). \tag{1.13d}$$

In the case of Jacobsthal numbers, (1.13b) and (1.13c) above reduce to (2.5) and (2.6) in [3], respectively. In the case of Jacobsthal polynomials, they reduce to (3.5) and (3.6) in [2], respectively, and so forth. From (1.13), it is clear that the significance of Simson's formula lies in its independence of the index n, apart from a sign correction.

2. THE GENERAL LUCAS PAIR

The general Lucas line-sequence $G_{2,b}$ is particularly interesting, mainly owing to its being conjugate to the second basis line-sequence $G_{0,1}$. In addition to the aforementioned properties, a few more basic properties are given below.

The basis component expression of $G_{2,b}$, according to (1.8a), is given by

$$G_{2,b}=2G_{1,0}+bG_{0,1} \tag{2.1a}$$

or, in terms of the elements,

$$g_n[2,b]=2g_n[1,0]+bg_n[0,1]. \tag{2.1b}$$

Substitution of the translational relation (1.2a) into (2.1a) produces the translational representation of $G_{2,b}$ in terms of the first basis,

$$G_{2,b}=(2I+bc^{-1}T)G_{1,0}, \tag{2.1c}$$

which can also be obtained from (1.9a) by putting $i=2$ and $j=b$.

The basis component expression of $G_{b,-2}$ is given by

$$G_{b,-2}=bG_{1,0}-2G_{0,1} \tag{2.2a}$$

or, in terms of the elements,

$$g_n[b,-2]=bg_n[1,0]-2g_n[0,1]. \tag{2.2b}$$

The translational representation in terms of the first basis is then given by

$$G_{b,-2}=(bI-2c^{-1}T)G_{1,0}, \tag{2.2c}$$

which can again be obtained from (1.9a) by putting $i=b$ and $j=-2$.

Binet's formula for $G_{2,b}$, according to (1.8b), is

$$G_{2,b}(c,b)=G_{1,\alpha}+G_{1,\beta}, \tag{2.3a}$$

and Binet's formula for its complement is

$$G_{b,-2}(c,b)=[-(\beta b+2)G_{1,\alpha}+(\alpha b+2)G_{1,\beta}]/(\alpha-\beta). \tag{2.3b}$$

Substituting the geometrical line-sequences (1.5a) and (1.5b) into (2.3a) and noting that $\alpha\beta=-c$, we obtain the parity relation of the elements in $G_{2,b}$, that is,

$$g_{-n}[2,b]=(-c)^{-n}g_n[2,b].$$

Again in the nomenclature of Shannon and Horadam [9], the parity relation (2.4) reduces to (1.9) in [1] for $c=-1$ in the case of Morgan-Voyce even Lucas polynomials.

Applying the cross relation (1.4a) and the parity relation (1.3b) to the component expression (1.13b), using the parity relation (2.4), we obtain

$$g_n[2, b] = 2g_{n+1}[0, 1] - bg_n[0, 1], \tag{2.5}$$

which is the general version of the basis representation of the Lucas elements (for $c = b = 1$): $l_n = 2f_{n+1} - f_n$.

Similarly, from the component expression (2.2b), we obtain

$$g_n[b, -2] = bcg_{n-1}[0, 1] - 2g_n[0, 1], \tag{2.6a}$$

which is the basis representation of the complementary Lucas elements in terms of the second basis. Note that if we choose to express the elements in terms of the first basis, using the translational relation (1.2b), we would obtain

$$g_n[b, -2] = -2c^{-1}g_{n+1}[1, 0] + bg_n[1, 0], \tag{2.6b}$$

which is more symmetrical with (2.5).

3. THE PELL POLYNOMIAL LINE-SEQUENCES

We now apply the formulas obtained in the previous sections to the Pell polynomials and, for the sake of checking, we also calculate the results independently, that is, without using those formulas. The results are found to agree in each and every case. The order of development follows largely that of the previous sections with some minor variations.

The Pell polynomials line-sequence is defined by $b = 2x$, $c = 1$. The basic pair is given by

$$P_{1,0}(1, 2x): \ldots, -4x(1+2x^2), (1+4x^2), -2x, [1, 0], 1, 2x, (1+4x^2), \ldots, \tag{3.1a}$$

$$P_{0,1}(1, 2x): \ldots, -4x(1+2x^2), 1+4x^2, -2x, 1, [0, 1], 2x, (1+4x^2), \ldots, \tag{3.1b}$$

where the first one is referred to as the complementary P-Fibonacci line-sequence, or the $P_{1,0}$ line-sequence for short; the second is referred to as the P-Fibonacci line-sequence, or the $P_{0,1}$ line-sequence for short.

Obviously, they are translationally related, in agreement with (1.2a), that is,

$$TP_{1,0} = P_{0,1}. \tag{3.2a}$$

In terms of the elements, this becomes

$$p_{n+1}[1, 0] = p_n[0, 1]. \tag{3.2b}$$

The parity relation of the elements in $P_{1,0}$ is given by (1.3a),

$$p_{-n}[1, 0] = (-1)^n p_{n+2}[1, 0], \tag{3.3a}$$

and the parity relation of the elements in $P_{0,1}$ is given by (1.3b),

$$p_{-n}[0, 1] = (-1)^{n+1} p_n[0, 1]. \tag{3.3b}$$

Or, by applying (3.2b) to (3.3b), we also obtain (3.3a).

Combining the translational relations with the parity ones, we obtain the following set of cross relations among the elements of the two basis polynomial line-sequences, in agreement with relations (1.4a) through (1.4d):

$$p_{-n}[1, 0] = (-1)^n p_{n+1}[0, 1] \tag{3.4a}$$

$$p_{-n}[1, 0] = p_{-(n+1)}[0, 1]; \tag{3.4b}$$

or

$$p_{-n}[0, 1] = (-1)^{n+1} p_{n+1}[1, 0], \tag{3.4c}$$

$$p_{-n}[0, 1] = p_{-(n-1)}[1, 0]. \tag{3.4d}$$

From (1.5a) and (1.5b), the pair of geometrical line-sequences relating to $P_{1,0}$ is given by

$$P_{1,\alpha}(1, 2x): \ldots, \alpha^{-2}, \alpha^{-1}, [1, \alpha], \alpha^2, \alpha^3, \ldots, \tag{3.5a}$$

$$P_{1,\beta}(1, 2x): \ldots, \beta^{-2}, \beta^{-1}, [1, \beta], \beta^2, \beta^3, \ldots, \tag{3.5b}$$

respectively, where α and β are the roots of the generating equation

$$q^2 - 2xq - 1 = 0. \tag{3.5c}$$

By formulas (1.6a) and (1.6b), Binet's formula for $P_{1,0}$ is

$$P_{1,0} = (-\beta P_1, \alpha + \alpha P_1, \beta) / (\alpha - \beta), \tag{3.6a}$$

and for the $P_{0,1}$ is

$$P_{0,1} = (P_{1,\alpha} - P_{1,\beta}) / (\alpha - \beta). \tag{3.6b}$$

From (1.7a) and (1.7b), the P-Lucas line-sequence is given by

$$P_{2,2x}(1, 2x): -2x(3+4x^2), 2(1+2x^2), -2x, [2, 2x], 2(1+2x^2), 2x(3+4x^2), \ldots. \tag{3.7a}$$

Its complement is then

$$P_{2x,-2}(1, 2x): \ldots, 2x(3+4x^2), -2(1+2x^2), [2x, -2], -2x, -2(1+2x^2), \ldots. \tag{3.7b}$$

These two line-sequences are clearly orthogonal, with a normalization factor $[2(1+x^2)^{1/2}]^{-1}$.

The basis component expression for an arbitrary Pell polynomial line-sequence, according to (1.8a), is given by

$$P_{i,j}(1, 2x) = iP_{1,0}(1, 2x) + jP_{0,1}(1, 2x), \tag{3.8}$$

so we have, for the P-Lucas pair:

$$P_{2,2x}(1, 2x) = 2P_{1,0}(1, 2x) + 2xP_{0,1}(1, 2x), \tag{3.8a}$$

$$P_{2x,-2}(1, 2x) = 2xP_{1,0}(1, 2x) - 2P_{0,1}(1, 2x). \tag{3.8b}$$

It can be easily shown that the general formula of Binet decomposition, in the simpler applicable form for the Pell polynomial line-sequences, is given by

$$P_{i,j}(1, 2x) = [-i(\beta P_{1,\alpha} - \alpha P_{1,\beta}) + j(P_{1,\alpha} - P_{1,\beta})] / (\alpha - \beta). \tag{3.9}$$

Thus, we have

$$P_{2,2x}(1, 2x) = 2[-(\beta P_{1,\alpha} - \alpha P_{1,\beta}) + x(P_{1,\alpha} - P_{1,\beta})] / (\alpha - \beta). \tag{3.9a}$$

$$P_{2x,-2}(1, 2x) = 2[-x(\beta P_{1,\alpha} - \alpha P_{1,\beta}) + (P_{1,\alpha} - P_{1,\beta})] / (\alpha - \beta). \tag{3.9b}$$

The formula for the Lucas decomposition of an arbitrary Pell polynomial line-sequence, according to (1.8c), is given by

$$P_{i,j}(1, 2x) = [(i+xj)P_{2,2x} + (xi-j)P_{2x,-2}]/2(1+x^2), \tag{3.10}$$

so the component formulas of the basis pair with respect to the Lucas bases are

$$P_{1,0}(1, 2x) = [P_{2,2x} + xP_{2x,-2}]/2(1+x^2), \tag{3.10a}$$

$$P_{0,1}(1, 2x) = [xP_{2,2x} - P_{2x,-2}]/2(1+x^2). \tag{3.10b}$$

The conjugation of $P_{0,1}$ and $P_{2,b}$, by (1.10c), produces

$$P_{0,1}P_{2,2x} = (P_{1,\alpha}^2 - P_{1,\beta}^2)/(\alpha-\beta). \tag{3.11a}$$

In terms of the elements, this becomes

$$p_n[0, 1]p_n[2, 2x] = p_{2n}[0, 1], \tag{3.11b}$$

which is the P-version of the conjugation relation $f_n l_n = f_{2n}$.

The Binet product of $P_{1,0}$ and $P_{2x,-2}$, by (1.10e), is found to be

$$P_{1,0}P_{2x,-2} = 2[\beta(\beta x+1)P_{1,\alpha}^2 + \alpha(\alpha x+1)P_{1,\beta}^2]/(\alpha-\beta)^2. \tag{3.12a}$$

In terms of the elements, with $\alpha\beta = -1$, this becomes

$$p_n[1, 0]p_n[2x, -2] = (xp_{2n-2}[2, 2x] - p_{2n-1}[2, 2x])/2(1+x^2). \tag{3.12b}$$

From Binet's formula (3.9a), we obtain the following parity relation between the elements of the P-Lucas line-sequence (3.7a),

$$p_{-n}[2, 2x] = (-1)^n p_n[2, 2x], \tag{3.13}$$

which apparently holds true.

The component expression of the P-Lucas line-sequence is given by

$$P_{2,2x} = 2P_{1,0} + 2xP_{0,1}. \tag{3.14a}$$

In terms of the elements, this becomes

$$p_n[2, 2x] = 2p_n[1, 0] + 2xp_n[0, 1]. \tag{3.14b}$$

Applying (3.3b) and (3.4a) and using the parity relation (3.13), we obtain

$$p_n[2, 2x] = 2p_{n+1}[0, 1] - 2xp_n[0, 1], \tag{3.14c}$$

which is the P-version of the relation $l_n = 2f_{n+1} - f_n$.

Substituting the translation relation (3.2a) to the component expression (3.14a), we obtain the translational representation of the P-Lucas line-sequence,

$$P_{2,2x} = 2(I + xT)P_{1,0}. \tag{3.15}$$

The component expression of the complementary P-Lucas line-sequence is

$$P_{2x,-2} = 2xP_{1,0} - 2P_{0,1}. \tag{3.16a}$$

In terms of the elements, this becomes

$$p_n[2x, -2] = 2xp_n[1, 0] - 2p_n[0, 1]. \tag{3.16b}$$

Applying parity relation (3.3b) and cross relation (3.4a), we obtain

$$p_n[2x,-2] = -2p_n[0,1] + 2xp_{n-1}[0,1], \tag{3.16c}$$

which is the complement of the relation (3.14c) in terms of the second basis. Its equivalence in terms of the first basis is obtained by applying the translational relation (3.2b),

$$p_n[2x,-2] = -2p_{n+1}[1,0] + 2xp_n[1,0], \tag{3.16d}$$

which is more symmetrical with (3.14c).

Substituting the translation relation (3.2a) into the component expression (3.16a), we obtain the translational representation of the complementary P-Lucas line-sequence,

$$P_{2x,-2} = 2(xI - T)P_{1,0}. \tag{3.17}$$

The following summation formulas can be verified easily:

$$\sum_{i=k}^{k+n} p_i = [p_{k+n} + p_{k+n+1} + (2x-1)p_k - p_{k+1}]/2x, \tag{3.18a}$$

$$\sum_{i=k}^{k+n} p_{2i-1} = (p_{2(k+n)} - p_{2(k-1)})/2x, \tag{3.18b}$$

$$\sum_{i=k}^{k+n} p_{2i} = (p_{2(k+n)+1} - p_{2k-1})/2x. \tag{3.18c}$$

Formulas (1.11b) and (1.11c) in the general case reduce to (3.18b) and (3.18c), respectively.

The dual relation then gives the corresponding operators equations of translation:

$$\sum_{i=k}^{k+n} T_i = [T_{k+n} + T_{k+n+1} + (2x-1)T_k - T_{k+1}]/2x, \tag{3.19a}$$

$$\sum_{i=k}^{k+n} T_{2i-1} = (T_{2(k+n)} - T_{2(k-1)})/2x, \tag{3.19b}$$

$$\sum_{i=k}^{k+n} T_{2i} = (T_{2(k+n)+1} - T_{2k-1})/2x. \tag{3.19c}$$

For example, let $k=-3$ and $n=5$, then the left-hand side (l.h.s.) of (3.19a) gives

$$\left(\sum_{i=k}^{k+n} T_i\right)p_1[0,1] = 3 + 4x^2,$$

and its right-hand side (r.h.s.) gives $[T_2 + T_3 + (2x-1)T_{-3} - T_{-2}]p_1[0,1]/2x = 3 + 4x^2$; hence, l.h.s. = r.h.s.

Simson's formulas for the Pell polynomial line-sequence are found to be:

$$p_{n+1}[1,0]p_{n-1}[1,0] - (p_n[1,0])^2 = (-1)^{n+1}, \tag{3.20a}$$

$$p_{n+1}[0,1]p_{n-1}[0,1] - (p_n[0,1])^2 = (-1)^n, \tag{3.20b}$$

$$p_{n+1}[2,2x]p_{n-1}[2,2x] - (p_n[2,2x])^2 = (-1)^{n-1}(4)(1+x^2), \tag{3.20c}$$

$$p_{n+1}[2x,-2]p_{n-1}[2x,-2] - (p_n[2x,-2])^2 = (-1)^n(4)(1+x^2). \tag{3.20d}$$

For example, let $n=-1$ in (3.20d), then l.h.s. = r.h.s. = $-4(1+x^2)$.

Remark: A number of specific problems in this work need to be addressed. For example, as of this writing, we have not yet found the parity relation of the elements in $G_{b,-2}$, as compared to those in $G_{2,b}$, see (2.4). Also, as far as this author is aware of, the relation (3.12b) does not seem to relate to any known line-sequential relation, in contradistinguishing to relation (3.11b), which relates to the well-known conjugation relation $f_n l_n = f_{2n}$. It is also interesting to see, as is pointed out by the referee, that viewing from the bigger picture, so to say, how this piece of 2D work relates to the work in the 3D case, as, for example, in the context of [10].

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ENUMERATION OF PATHS, COMPOSITIONS OF INTEGERS, AND FIBONACCI NUMBERS

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1. INTRODUCTION

We study certain paths in the first quadrant from $(0, 0)$ to (i, j). These paths consist of segments from $(x_0, y_0) = (0, 0)$ to (x_1, y_1) to ... to $(x_k, y_k) = (i, j)$. If we write

$$p_h = x_h - x_{h-1},\ q_h = y_h - y_{h-1},\ h = 1, 2, \dots, k,$$

then $p_h \geq 0$, $q_h \geq 0$, and

$$p_1 + p_2 + \cdots + p_k = i,\ q_1 + q_2 + \cdots + q_k = j.$$

Thus, the p_h form a composition of i, the q_h a composition of j.

The paths we shall study will have some restrictions on the p_h and q_h. For example, in Section 3, we shall enumerate paths for which

$$a_x \leq p_h \leq b_x,\ a_y \leq q_h \leq b_y,\ h = 1, 2, \dots, k,\ k \geq 1,$$

where $a_x \geq 1$, $a_y \geq 1$.

2. COMPOSITIONS

A *composition* of a nonnegative integer n is a vector $(p_1, \dots, p_k)$ for which

$$p_1 + p_2 + \cdots + p_k = n,\ p_h \geq 0.$$

Note that the order in which the p_h are listed matters. Each p_h is called a *part*, and k, the *number of parts*. Let $c(n, k, a, b)$ be the number of compositions of n into k parts p_h with $a \leq p_h \leq b$. It is well known that

$$c(n, k, 0, \infty) = \binom{n+k-1}{k-1}.$$

On subtracting a from each part, it is easy to see that

$$c(n, k, a, \infty) = c(n - ka, k, 0, \infty) = \binom{n - ka + k - 1}{k-1}. \tag{1}$$

With $a = 1$, this gives the number of compositions of n into k positive parts,

$$c(n, k, 1, \infty) = \binom{n-1}{k-1},$$

and the number of compositions of n into positive parts,

$$\sum_{k=1}^{n} c(n, k, 1, \infty) = \sum_{k=1}^{n} \binom{n-1}{k-1} = 2^{n-1}.$$

3. SOME RESULTS ON PATHS

Theorem 1: Suppose a_x, b_x, a_y, b_y are integers satisfying $1 \le a_x \le b_x$ and $1 \le a_y \le b_y$. Let $T(0,0)=1$, and for $i \ge 0$ and $j \ge 0$ but not $i=j=0$, let $T(i,j)$ be the number of paths satisfying

$$a_x \le p_h \le b_x \quad \text{and} \quad a_y \le q_h \le b_y \tag{2}$$

for $h = 1, 2, \ldots, k$ and $k = 1, 2, \ldots, \min(i,j)$. Then

$$T(i,j) = \sum_{k=1}^{\min(i,j)} c(i,k,a_x,b_x)c(j,k,a_y,b_y). \tag{3}$$

Proof: Suppose $1 \le k \le \min(i,j)$. Corresponding to each path of the sort described in the introduction satisfying (2) is a composition of i into k parts and a composition of j into k parts satisfying (2), and conversely.

There are $c(i,k,a_x,b_x)$ such compositions of i and $c(j,k,a_y,b_y)$ such compositions of j, hence $c(i,k,a_x,b_x)c(j,k,a_y,b_y)$ such paths consisting of k segments. Summing over all possible numbers of segments yields (3). □

Corollary 1.1: The number of paths $T(i,j)$ with $p_h \ge a_x$, $q_h \ge a_y$ (where $a_x \ge 1$, $a_y \ge 1$) is

$$T(i,j) = \sum_{k=1}^{\min(i,j)} \binom{i-ka_x+k-1}{k-1}\binom{j-ka_y+k-1}{k-1}.$$

Proof: Let $b_x = \infty$ and $b_y = \infty$ in Theorem 1. □

Lemma 1.1: If I and J are nonnegative integers, then

$$\sum_{k=0}^{I}\binom{I}{k}\binom{J}{k} = \binom{I+J}{I} \quad \text{and} \quad \sum_{k=1}^{I}\binom{I}{k}\binom{J}{k-1} = \binom{I+J}{I-1}.$$

Proof: The assertions clearly hold for $I = J = 0$. Suppose that $I + J \ge 1$ and that both identities hold for all I' and J' satisfying $I' + J' < I + J$. Then

$$\sum_{k=0}^{I}\binom{I}{k}\binom{J}{k} = \sum_{k=0}^{I}\binom{I}{k}\binom{J-1}{k-1} + \sum_{k=0}^{I}\binom{I}{k}\binom{J-1}{k}$$
$$= \binom{I+J-1}{I-1} + \binom{I+J-1}{I} = \binom{I+J}{I}$$

and

$$\sum_{k=1}^{I}\binom{I}{k}\binom{J}{k-1} = \sum_{k=1}^{I}\binom{I-1}{k-1}\binom{J}{k-1} + \sum_{k=1}^{I}\binom{I-1}{k}\binom{J}{k-1}$$
$$= \binom{I+J-1}{I-1} + \binom{I+J-1}{I-2} = \binom{I+J}{I-1}.$$

Although both of the identities in Lemma 1.1 are needed inductively in the foregoing proof, only the first identity will be used below.

Corollary 1.2: The number $T(i,j)$ of paths satisfying $p_h \ge 1$, $q_h \ge 1$ is given by

$$T(i,j)=\binom{i+j-2}{i-1}.$$

Proof: Put $a_h=1$, $a_y=1$ in Corollary 1.1, and apply Lemma 1.1. □

Theorem 2: The number $T(i,j)$ of paths satisfying $p_h\ge 1$, $q_h\ge 0$ is given by

$$T(i,j)=\sum_{k=1}^{i}c(i,k,1,\infty)c(j,k,0,\infty)=\sum_{k=1}^{i}\binom{i-1}{k-1}\binom{j+k-1}{k-1}.$$

Proof: The method of proof is essentially the same as for Theorem 1. Here, however, the greatest k for which there is a path for which all $p_h\ge 1$ is i, rather than $\min(i,j)$. □

The array of Theorem 2 is of particular interest; for example:

(A) $T(i,0)=2^{i-1}$ for $i\ge 1$;

(B) $T(i,1)=(i+1)2^{i-2}$ for $i\ge 1$;

(C) $\sum_{\substack{i+j=n\\ i\ge 0,\, j\ge 0}}T(i,j)=F_{2n}$ for $n\ge 1$, i.e., antidiagonal sums are Fibonacci numbers;

(D) $\sum_{\substack{i+j=n\\ i\ge 0,\, j\ge 0}}(-1)^jT(i,j)=F_n$ for $n\ge 1$, i.e., alternating antidiagonal sums are Fibonacci numbers;

(E) the diagonal $T(n,n-1)=(1,3,13,63,321,\dots)$ is the Delannoy sequence, A001850 in [1];

(F) the diagonal $T(n,n)=(1,1,4,19,96,501,\dots)$ is the sequence A047781 in [1].

We leave proofs of (A)-(F) to the reader, along with the determination of the position and magnitude of the maximum number M_n in the n^{th} antidiagonal of T. The first fourteen values of M_n are 1, 2, 4, 8, 20, 48, 112, 272, 688, 1696, 4096, 10496, 26624, 66304. One wonders what can be said about $\lim_{n\to\infty}M_n/F_{2n}$. Initial terms of the sequences in (A)-(F) appear in Figure 1.

The $T(i,j)$ given in Theorem 2 are determined recursively by $T(0,0)=1$, $T(0,j)=0$ for $j\ge 1$, $T(i,0)=2^{i-1}$ for $i\ge 1$, and

$$T(i,j)=\sum_{k=0}^{i-1}\sum_{l=0}^{j}T(k,l). \tag{4}$$

To verify (4), note that each path with final segment terminating on (i,j) has penultimate segment terminating on a lattice point (h,s) in the rectangle

$$R_{i,j}:=\{0,1,\dots,i-1\}\times\{0,1,\dots,j\}.$$

Therefore, the number of relevant paths from $(0,0)$ to (i,j) is the sum of the numbers of such paths from $(0,0)$ to a point in $R_{i,j}$.

More generally, all arrays as in Theorems 1 and 2 are determined recursively by

$$T(i,j)=\sum_{(h,s)\in R_{i,j}}T(h,s)\text{ for }i\ge 2\text{ and }j\ge 1,$$

where initial values and lattice point sets $R_{i,j}$ are determined by (2) or other conditions.

0	1	10	64	328	1462	5908	22180
0	1	9	53	253	1059	4043	14407
0	1	8	43	190	743	2668	8989
0	1	7	34	138	501	1683	5336
0	1	6	26	96	321	1002	2972
0	1	5	19	63	192	552	1520
0	1	4	13	38	104	272	688
0	1	3	8	20	48	112	256
1	1	2	4	8	16	32	64

FIGURE 1. $p_h \geq 1$, $q_h \geq 0$ **(see Theorem 2)**

4. RESTRICTED HORIZONTAL COMPONENTS

Suppose $m \geq 2$. In this section, we determine $T(i, j)$ when $1 \leq p_h \leq m$ and $q_h \geq 0$. By the argument of Theorem 2,

$$T(i, j) = \sum_{k=1}^{i} c(i, k, 1, m) c(j, k, 0, \infty) = \sum_{k=1}^{i} c(i, k, 1, m) \binom{j+k-1}{k-1},$$

where $c(i, k, 1, m)$ is the number of compositions of i into k positive parts, all $\leq m$. By the remarks at the end of Section 3, the numbers $T(i, j)$ are determined recursively by $T(0, 0) = 1$, $T(0, j) = 0$ for $j \geq 1$, $T(i, 0) = 2^{i-1}$ for $i = 1, 2, \ldots, m$, and

$$T(i, j) = \sum_{k=i-m}^{i-1} \sum_{l=0}^{j} T(k, l). \tag{5}$$

Values of $T(i, j)$ for $m = 2$ are shown in Figure 2.

In Figure 2, the antidiagonal sums $(1, 1, 3, 7, 17, 41, 99, \ldots)$ comprise a sequence that appears in many guises, such as the numerators of the continued-fraction convergents to $\sqrt{2}$, (See the sequence A001333 in [1].)

Also in Figure 2, the numbers in the bottom row, $T(i, 0)$, are the Fibonacci numbers. Since the other rows are easily obtained via (5) from these, it is natural to inquire about the bottom row when $m \geq 3$; we shall see, as a corollary to Theorem 3, that the m-Fibonacci numbers then occupy the bottom row.

0	1	10	63	309	1290	4797	16335
0	1	9	52	236	918	3198	10248
0	1	8	42	175	630	2044	6132
0	1	7	33	125	413	1239	3458
0	1	6	25	85	255	701	1806
0	1	5	18	54	145	361	850
0	1	4	12	31	73	162	344
0	1	3	7	15	30	58	109
1	1	2	3	5	8	13	21

FIGURE 2. Enumeration of Paths Consisting of Segments with Horizontal Components of Lengths 1 or 2

Lemma 3.1: Suppose $m \geq 1$. The number $F(m, n) = \sum_{k=1}^{n} c(n, k, 1, m)$ of compositions of n into positive parts $\leq m$ is given by:

$$F(1, n) = 1 \text{ for } n \geq 1;$$

$$F(m, n) = 2^{n-1} \text{ for } 1 \leq n \leq m,\ m \geq 2;$$

$$F(m, n) = F(m, n-1) + F(m, n-2) + \cdots + F(m, n-m) \text{ for } n \geq m+1 \geq 3.$$

Proof: For row 1 of the array F, there is only one composition of n into positive parts all ≤ 1, namely, the n-dimensional vector $(1, 1, \ldots, 1)$, so that $F(1, n) = 1$ for $n \geq 1$. Now suppose that the row number m is ≥ 2 and $1 \leq n \leq m$. Then every composition of n has all parts $\leq m$, and $F(m, n) = 2^{n-1}$.

Finally, suppose $n \geq m+1 \geq 3$. Each composition $(p_1, p_2, \ldots, p_k)$ of n into parts all $\leq m$ has a final part p_k that will serve our purposes. For $u = 1, 2, \ldots, m$, the set

$$S_u = \left\{ (p_1, p_2, \ldots, p_k) : 1 \leq p_i \leq m \text{ for } i = 1, 2, \ldots, k;\ \sum_{i=1}^{k} p_i = n;\ p_k = u \right\}$$

is an obvious one-to-one correspondence with the set of compositions $(p_1, p_2, \ldots, p_{k-1})$ for which $1 \leq p_i \leq m$ for $i = 1, 2, \ldots, k-1$ and $\sum_{i=1}^{k-1} p_i = n - u$, of which, by the induction hypothesis, there are $F(m, n-u)$. The sets $S_1, S_2, \ldots, S_m$ partition the set of compositions to be enumerated, so that the total count is $F(m, n-1) + F(m, n-2) + \cdots + F(m, n-m)$. □

Theorem 3: Suppose $m \geq 1$. Then the bottom row of array T is given by $T(i, 0) = F(m, i+1)$ for $i \geq 0$.

Proof: We have $T(0, 0) = F(m, 1) = 1$. Suppose now that $i \geq 1$. The sum in (5) and initial values given with (5) yield

$$T(i, 0) = \begin{cases} 2^{i-1} & \text{if } 1 \leq i \leq m, \\ T(i-m, 0) + T(i-m+1, 0) + \cdots + T(i-1, 0) & \text{if } i \geq m+1, \end{cases}$$

and by Lemma 3.1,

$$F(m, i+1) = \begin{cases} 2^{i} & \text{if } 0 \leq i \leq m-1, \\ F(m, i) + F(m, i-1) + \cdots + F(m, i+1-m) & \text{if } i \geq m. \end{cases}$$

Thus, the initial values and recurrences of the sequences $\{T(i, 0)\}$ and $\{F(m, i+1)\}$ are identical, so that the sequences are equal.

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SOME IDENTITIES FOR THE GENERALIZED FIBONACCI AND LUCAS FUNCTIONS

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1. INTRODUCTION

In this paper, we consider the generalized Fibonacci and Lucas functions, which may be defined by

$$U(x)=\frac{\alpha^x-e^{i\pi x}\beta^x}{\alpha-\beta} \tag{1}$$

and

$$V(x)=\alpha^x+e^{i\pi x}\beta^x, \tag{2}$$

where $\alpha=(p+\sqrt{\Delta})/2$, $\beta=(p-\sqrt{\Delta})/2>0$, $\Delta=p^2-4q$, p and q are integers with $q>0$, and x is an arbitrary real number. It is clear that $U(x)=F(2x)$ and $V(x)=L(2x)$ when $p=3$ and $q=1$, where $F(x)$ and $L(x)$ are the Fibonacci and Lucas functions, respectively (see [2]).

In [2], R. André-Jeannin proved that well-known identities for Fibonacci and Lucas numbers are again true for $F(x)$ and $L(x)$. Basic results regarding these topics can be found in [1]. Some special cases of the functions $U(x)$ and $V(x)$ are treated in [4] and referred to in the Remark in [8]. The aim of this paper is to establish some identities for $U(x)$ and $V(x)$. We are interested in calculating the summation of reciprocals of products of $U(x)$ and $V(x)$.

2. MAIN RESULTS

From the definitions of the generalized Fibonacci and Lucas functions, we can obtain the main results of this paper.

Theorem: Assume that n, r, and s are positive integers, and x is an arbitrary real number. Then

$$\sum_{k=1}^{n}\frac{e^{i\pi(k-1)rx}q^{(k-1)rx}}{U(krx-rx+sx)U(krx+sx)}=\frac{U(nrx)}{U(rx)U(sx)U(sx+nrx)} \tag{3}$$

and

$$\sum_{k=1}^{n}\frac{e^{i\pi(k-1)rx}q^{(k-1)rx}}{V(sx+krx-rx)V(sx+krx)}=\frac{U(nrx)}{U(rx)V(sx)V(sx+nrx)}. \tag{4}$$

Proof: From (1) and (2), it is easy to verify that $U(x)$ and $V(x)$ satisfy

$$\frac{V(sx)}{U(sx)}-\frac{V(sx+rx)}{U(sx+rx)}=\frac{2e^{i\pi sx}q^{sx}U(rx)}{U(sx)U(sx+rx)} \tag{5}$$

and

$$\frac{U(sx)}{V(sx)}-\frac{U(sx+rx)}{V(sx+rx)}=\frac{-2e^{i\pi sx}q^{sx}U(rx)}{V(sx)V(sx+rx)}. \tag{6}$$

In (5) and (6), replacing s by s, $s+r$, $s+2r$, ..., $s+(n-1)r$, and adding the results we can obtain

$$\sum_{k=1}^{n}\frac{e^{i\pi(k-1)rx}q^{(k-1)rx}}{U(sx+krx-rx)U(sx+krx)}=\frac{e^{-i\pi sx}q^{-sx}}{2U(rx)}\left(\frac{V(sx)}{U(sx)}-\frac{V(sx+nrx)}{U(sx+nrx)}\right)$$

and

$$\sum_{k=1}^{n}\frac{e^{i\pi(k-1)rx}q^{(k-1)rx}}{V(sx+krx-rx)V(sx+krx)}=\frac{e^{-i\pi sx}q^{-sx}}{2U(rx)}\left(\frac{U(sx)}{V(sx)}-\frac{U(sx+nrx)}{V(sx+nrx)}\right).$$

From (5) and (6), we can prove that the equalities (3) and (4) hold. □

Remark: From (1) and (2), we can show that the following relations are valid:

$$V(2rx)-e^{i\pi(r-s)x}q^{(r-s)x}V(2sx)=\Delta U(rx-sx)U(rx+sx); \tag{7}$$

$$V(2rx)+e^{i\pi(r-s)x}q^{(r-s)x}V(2sx)=V(rx-sx)V(rx+sx); \tag{8}$$

$$U^2(rx)-e^{i\pi(r-s)x}q^{(r-s)x}U^2(sx)=U(rx-sx)U(rx+sx); \tag{9}$$

$$V^2(rx)-e^{i\pi(r-s)x}q^{(r-s)x}V^2(sx)=\Delta U(rx-sx)U(rx+sx). \tag{10}$$

From (7) and (8), we have

$$\sum_{k=0}^{n}\frac{e^{i\pi krx}q^{krx}}{V(2krx+rx+2sx)-e^{i\pi(sx+krx)}q^{sx+krx}V(rx)}=\sum_{k=0}^{n}\frac{e^{i\pi krx}q^{krx}}{\Delta U(krx+sx)U(krx+rx+sx)}$$

and

$$\sum_{k=0}^{n}\frac{e^{i\pi krx}q^{krx}}{V(2krx+rx+2sx)+e^{i\pi(sx+krx)}q^{sx+krx}V(rx)}=\sum_{k=0}^{n}\frac{e^{i\pi krx}q^{krx}}{V(krx+sx)V(krx+rx+sx)}.$$

By the method used to obtain (3) and (4), we obtain the equalities

$$\sum_{k=0}^{n}\frac{e^{i\pi krx}q^{krx}}{V(2krx+rx+2sx)-e^{i\pi(sx+krx)}q^{sx+krx}V(rx)}=\frac{U(nrx+rx)}{\Delta U(rx)U(sx)U(sx+nrx+rx)}$$

and

$$\sum_{k=0}^{n}\frac{e^{i\pi krx}q^{krx}}{V(2krx+rx+2sx)+e^{i\pi(sx+krx)}q^{sx+krx}V(rx)}=\frac{U(nrx+rx)}{U(rx)V(sx)V(sx+nrx+rx)}.$$

Using (9) and (10) and applying the method used to obtain (3), we obtain the equalities

$$\sum_{k=0}^{n-1}\frac{e^{2i\pi krx}q^{2krx}}{U^2(2krx+rx+sx)-e^{i\pi(sx+2krx)}q^{sx+2krx}U^2(rx)}=\frac{U(2nrx)}{U(2rx)U(sx)U(sx+2nrx)}$$

and

$$\sum_{k=0}^{n}\frac{e^{2i\pi krx}q^{2krx}}{V^2(2krx+rx+sx)-e^{i\pi(sx+2krx)}q^{sx+2krx}V^2(rx)}=\frac{U(2nrx)}{\Delta U(2rx)U(sx)U(sx+2nrx)}.$$

Letting x be a positive real number and $\left|\frac{\beta}{\alpha}\right|<1$, due to

$$\lim_{n\to+\infty}\frac{U(nx)}{U(nx+rx)}=\frac{1}{\alpha^{rx}} \quad\text{and}\quad \lim_{n\to+\infty}\frac{U(nx)}{V(nx+rx)}=\frac{1}{\sqrt{\Delta}\alpha^{rx}},$$

we immediately have the following corollary.

Corollary: Suppose that r and s are positive integers, and x is a positive real number. If $\left|\frac{\beta}{\alpha}\right|<1$, then we have

$$\sum_{k=1}^{\infty} \frac{e^{i\pi(k-1)rx} q^{(k-1)rx}}{U(sx+krx-rx)U(sx+krx)} = \frac{1}{\alpha^{sx}U(rx)U(sx)}$$

and

$$\sum_{k=1}^{\infty} \frac{e^{i\pi(k-1)rx} q^{(k-1)rx}}{V(sx+krx-rx)V(sx+krx)} = -\frac{1}{\sqrt{\Delta}\,\alpha^{sx}U(rx)V(sx)}.$$

We note that formulas (3.3), (3.4), (3.5), (3.6), (3.9), and (3.10) in [6] are special cases of the Theorem and the Corollary.

Valuable references connected with the main results of this paper are [3], [5], and [7].

Finally, we give some special cases of the Theorem and the Corollary. If $p=3$ and $q=1$ in (3) and (4), we obtain

$$\sum_{k=1}^{n} \frac{e^{i\pi(k-1)rx}}{F(2krx-2rx+2sx)F(2krx+2sx)} = \frac{F(2nrx)}{F(2rx)F(2sx+2nrx)}$$

and

$$\sum_{k=1}^{n} \frac{e^{i\pi(k-1)rx}}{L(2krx-2rx+2sx)L(2krx+2sx)} = \frac{F(2nrx)}{F(2rx)L(2sx)L(2sx+2nrx)}.$$

If $p=3$, $q=1$, and $x=1$ in the Corollary, we have

$$\sum_{k=1}^{\infty} \frac{(-1)^{(k-1)r}}{F(2s+2kr-2r)F(2s+2kr)} = \frac{2^s}{(3+\sqrt{5})^s F(2r)F(2s)}$$

and

$$\sum_{k=1}^{\infty} \frac{(-1)^{(k-1)r}}{L(2s+2kr-2r)L(2s+2kr)} = \frac{2^s}{\sqrt{5}(3+\sqrt{5})^s F(2r)L(2s)}.$$

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A SIMPLE PROOF OF CARMICHAEL'S THEOREM ON PRIMITIVE DIVISORS

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1. INTRODUCTION

For arbitrary positive integer n, numbers of the form $D_n = (\alpha^n - \beta^n)/(\alpha - \beta)$ are called the *Lucas numbers*, where α and β are distinct roots of the polynomial $f(z) = z^2 - Lz + M$, and L and M are integers that are nonzero. The Lucas sequence (D): $D_1, D_2, D_3, \ldots$ is called *real* when α and β are real. Throughout this paper, we assume that L and M are coprime. Each D_n is an integer. A prime p is called a *primitive divisor* of D_n if p divides D_n but does not divide D_m for $0 < m < n$. Carmichael [2] calls it a *characteristic factor* and Ward [9] an *intrinsic divisor*. As Durst [4] observed, in the study of primitive divisors, it suffices to take $L > 0$. Therefore, we assume $L > 0$ in this paper.

In 1913, Carmichael [2] established the following.

Theorem 1 (Carmichael): If α and β are real and $n \neq 1, 2, 6$, then D_n contains at least one primitive divisor except when $n = 12$, $L = 1$, $M = -1$,

In 1974, Schinzel [6] proved that if the roots of f are complex and their quotient is not a root of unity and if n is sufficiently large then the n^{th} term in the associated Lucas sequence has a primitive divisor. In 1976, Stewart [7] proved that if $n = 5$ or $n > 6$ there are only finitely many Lucas sequences that do not have a primitive divisor, and they may be determined. In 1995, Voutier [8] determined all the exceptional Lucas sequences with n at most 30. Finally, Bilu, Hanrot, and Voutier [1] have recently shown that there are no other exceptional sequences that do not have a primitive divisor for the n^{th} term with n larger than 30.

The aim of this paper is to give an elementary and simple proof of Theorem 1. To prove that Theorem 1 is true for all real Lucas sequences, it is sufficient to discuss the two special sequences, namely, the Fibonacci sequence and the so-called Fermat sequence.

2. A SUFFICIENT CONDITION THAT D_n HAS A PRIMITIVE DIVISOR

Let $n > 1$ be an integer. Following Ward [9], we call the numbers

$$Q_1 = 1,\ Q_n = Q_n(\alpha, \beta) = \prod_{\substack{1 \le r \le n \\ (r,n)=1}} (\alpha - e^{2\pi i r/n}\beta) \ \text{ for } n \geq 2$$

the cyclotomic numbers associated with the Lucas sequence, where α, β are the roots of the polynomial $f(z) = z^2 - Lz + M$ and the product is extended over all positive integers less than n and prime to n. Each Q_n is an integer, and $D_n = \prod_{d|n} Q_n$, where the product is extended over all divisors d of n. Hence, p is a primitive divisor of D_n if and only if p is a primitive divisor of Q_n.

Lemma 1 below was shown by several authors (Carmichael, Durst, Ward, and others).

Lemma 1: Let p be prime and let k be the least positive value of the index i such that p divides D_i. If $n \neq 1, 2, 6$ and if p divides Q_n and some Q_m with $0 < m < n$, then p^2 does not divide Q_n and $n = p^r k$ with $r \geq 1$.

Now suppose that n has a prime-power factorization $n = p_1^{e_1} p_2^{e_2} \ldots p_l^{e_l}$, where $p_1, p_2, \ldots, p_l$ are distinct primes and $e_1, e_2, \ldots, e_l$ are positive integers. Lemma 1 leads us to the following lemma (cf. Halton [5], Ward [9]).

Lemma 2: Let $n \neq 1, 2, 6$. A sufficient condition that D_n contains at least one primitive divisor is that $|Q_n| > p_1 p_2 \cdots p_l$.

Proof: We prove the contraposition. Suppose that D_n has no primitive divisors. If p is an arbitrary prime factor of Q_n, then p divides some Q_m with $0 < m < n$. Therefore, p divides n and p^2 does not divide Q_n. Hence, Q_n divides $p_1 p_2 \cdots p_l$, so $|Q_n| \leq p_1 p_2 \cdots p_l$. □

Our proof of Carmichael's theorem is based on the following.

Theorem 2: If $n \neq 1, 2, 6$ and if both the n^{th} cyclotomic number associated with $z^2 - z - 1$ and that associated with $z^2 - 3z + 2$ are greater than the product of all prime factors of n, then, for every real Lucas sequence, D_n contains at least one primitive divisor.

Now assume that n is an integer greater than 2 and that α and β are real, that is, $L^2 - 4M$ is positive. As Ward observed,

$$Q_n(\alpha, \beta) = \prod(\alpha - \zeta^r \beta)(\alpha - \zeta^{-r} \beta) \tag{1}$$

$$= \prod((\alpha + \beta)^2 - \alpha\beta(2 + \zeta^r + \zeta^{-r})), \tag{2}$$

where $\zeta = e^{2\pi i/n}$ and the products are extended over all positive integers less than $n/2$ and prime to n. Since $\alpha + \beta = L$ and $\alpha\beta = M$, by putting $\theta_r = 2 + \zeta^r + \zeta^{-r}$, we have

$$Q_n = Q_n(\alpha, \beta) = \prod(L^2 - M\theta_r). \tag{3}$$

Fix an arbitrary $n > 2$. Then Q_n can be considered as the function of variables L and M. We shall discuss for what values of L and M the n^{th} cyclotomic number Q_n has its least value.

Lemma 3: Let $n > 2$ be an arbitrary fixed integer. If α and β are real, then Q_n has its least value either when $L = 1$ and $M = -1$ or when $L = 3$ and $M = 2$.

Proof: Take an arbitrary θ_r and fix it. Since $n > 2$, we have $0 < \theta_r < 4$. Thus, if $M < 0$, we have $L^2 - M\theta_r \geq 1 + \theta_r$, with equality holding only in the case $L = 1$, $M = -1$. When $M > 0$, consider the cases $M = 1$, $M > 1$. In the first case we have $L \geq 3$, so that

$$L^2 - M\theta_r \geq 9 - \theta_r > 9 - 2\theta_r.$$

Now assume $M > 1$. Then, since $L^2 \geq 4M + 1$, we have

$$L^2 - M\theta_r \geq 4M + 1 - M\theta_r = 9 - 2\theta_r + (M - 2)(4 - \theta_r) \geq 9 - 2\theta_r$$

with equality holding only in the case $M = 2$, $L = 3$. Hence, by formula (3), we have completed the proof. □

Combining Lemma 2 with Lemma 3, we complete the proof of Theorem 2.

3. CARMICHAEL'S THEOREM

We call the Lucas sequence generated by $z^2 - z - 1$ the *Fibonacci sequence* and that generated by $z^2 - 3z + 2$ the *Fermat sequence*. Theorem 2 implies that to prove Carmichael's theorem it is sufficient to discuss the Fibonacci sequence and the Fermat sequence.

Now we suppose that n has a prime-power factorization $n = p_1^{e_1} p_2^{e_2} \cdots p_l^{e_l}$, and let $\Phi_n(x)$ denote the n^{th} cyclotomic polynomial.

Lemma 4: If $n > 2$ and if a is real with $|a| < 1/2$, then $\Phi_n(a) \geq 1 - |a| - |a|^2$.

Proof: We have

$$\Phi_n(a) = \prod_{d|n} (1 - a^{n/d})^{\mu(d)},$$

where μ denotes the Möbius function and the product is extended over all divisors d of n. Since $|a| < 1/2$ and $(1 - a^{n/d})^{\mu(d)} \geq 1 - |a|^{n/d}$,

$$\begin{aligned}\Phi_n(a) &\geq \prod_{i=1}^{\infty} (1 - |a|^i) \geq (1 - |a|)(1 - |a|^2 - |a|^3 - |a|^4 - \cdots) \\ &= (1 - |a|)\left(1 - \frac{|a|^2}{1 - |a|}\right) = 1 - |a| - |a|^2.\end{aligned}$$

Here we have used the fact that if $0 \leq x \leq 1$ and $0 \leq y \leq 1$ then $(1-x)(1-y) \geq 1 - x - y$. We have thus proved the lemma. □

Theorem 3: If $n \neq 1, 2, 6, 12$, then the n^{th} term of the Fibonacci sequence contains at least one primitive divisor.

Proof: Assume $n > 2$. We shall determine for what n the inequality $|Q_n| > p_1 p_2 \cdots p_l$ is satisfied, where Q_n is the n^{th} cyclotomic number associated with the Fibonacci sequence. The roots of the polynomial $z^2 - z - 1$ are $\alpha = (1 + \sqrt{5})/2$ and $\beta = (1 - \sqrt{5})/2$. Since $|\beta/\alpha| = (3 - \sqrt{5})/2 < 1/2$, Lemma 4 gives

$$\Phi_n(\beta/\alpha) \geq 1 - |\beta/\alpha| - |\beta/\alpha|^2 = 2\sqrt{5} - 4 > 2/5.$$

In addition, since $\alpha > 3/2$, we have

$$Q_n(\alpha, \beta) = \alpha^{\phi(n)} \Phi_n(\beta/\alpha) > (2/5)(3/2)^{\phi(n)},$$

where $\phi(n)$ denotes the Euler function: $\phi(n) = \prod_{i=1}^{l} p_i^{e_i - 1}(p_i - 1)$. Thus, $|Q_n| > p_1 p_2 \cdots p_l$ is true for n satisfying

$$(2/5)(3/2)^{\phi(n)} > p_1 p_2 \cdots p_l. \tag{4}$$

We first suppose $p_1 > 7$ without loss of generality. Then $(2/5)(3/2)^{\phi(p_1)} > 2p_1$ is true, and consequently $(2/5)(3/2)^{\phi(n)} > p_1 p_2 \cdots p_l$. Here we have used the fact that if x, y are real with $x > y > 3$ and if m is integral with $m > 2$ then $x^{m-1} > my$. We next suppose $p_1^{e_1} = 2^4, 3^3, 5^2$, or 7^2 without loss of generality. Therefore, $(2/5)(3/2)^{\phi(p_1^{e_1})} > 2p_1$ is true, and consequently $(2/5)(3/2)^{\phi(n)} > p_1 p_2 \cdots p_l$. Hence, inequality (4) is true unless n is of the form

$$n = 2^a 3^b 5^c 7^d, \tag{5}$$

where $0 \le a \le 3$, $0 \le b \le 2$, $0 \le c \le 1$, and $0 \le d \le 1$. By substituting (5) into (4), we verify that inequality (4) is true for $n \ne 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 18, 30$. However, by direct computation, we have

$$\begin{array}{lllll} Q_2 = 1, & Q_3 = 2, & Q_4 = 3, & Q_5 = 5, & Q_6 = 4, \\ Q_7 = 13, & Q_8 = 7, & Q_9 = 17, & Q_{10} = 11, & Q_{12} = 6, \\ Q_{14} = 29, & Q_{15} = 61, & Q_{18} = 19, & Q_{30} = 31. & \end{array}$$

Hence, $|Q_n| > p_1 p_2 \dots p_l$ holds for $n \ne 1, 2, 3, 5, 6, 12$. It follows from Lemma 2 that if $n \ne 1, 2, 3, 5, 6, 12$ then the n^{th} Fibonacci number F_n contains at least one primitive divisor. In addition, since $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$, $F_6 = 2^3$, $F_{12} = 2^4 \cdot 3^2$, the numbers F_3 and F_5 have a primitive divisor, and F_1, F_2, F_6, and F_{12} do not. □

Theorem 4: If $n \ne 1, 2, 6$, then the n^{th} term of the Fermat sequence contains at least one primitive divisor.

Proof: The roots of the polynomial $z^2 - 3z + 2$ are $\alpha = 2$ and $\beta = 1$. By Lemma 4,

$$\Phi_n(\beta / \alpha) \ge 1 - |\beta / \alpha| - |\beta / \alpha|^2 = 1/4.$$

Therefore,

$$Q_n(\alpha, \beta) = \alpha^{\phi(n)} \Phi_n(\beta / \alpha) > (1/4) \cdot 2^{\phi(n)}.$$

Now the inequality $(1/4) \cdot 2^{\phi(n)} > (2/5)(3/2)^{\phi(n)}$ is true for all $n > 2$. As shown in the proof of Theorem 3, the inequality $(2/5)(3/2)^{\phi(n)} > p_1 p_2 \dots p_l$ is true for $n \ne 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 12, 14, 15, 18, 30$. Moreover, by direct computation, we observe that $(1/4) \cdot 2^{\phi(n)} > p_1 p_2 \dots p_l$ is true for $n = 7, 8, 9, 14, 15, 18, 30$, and furthermore, we have

$$Q_3 = 7,\ Q_4 = 5,\ Q_5 = 31,\ Q_6 = 3,\ Q_{10} = 11,\ Q_{12} = 13.$$

Hence, $|Q_n| > p_1 p_2 \dots p_l$ holds for $n \ne 1, 2, 6$. It follows from Lemma 2 that if $n \ne 1, 2, 6$ then the n^{th} term of the Fermat sequence contains at least one primitive divisor. □

Now we are ready to prove Carmichael's theorem.

Proof of Carmichael's Theorem: As observed previously, for $n \ne 1, 2, 3, 5, 6, 12$, both the n^{th} cyclotomic number associated with the Fibonacci sequence and that associated with the Fermat sequence are greater than $p_1 p_2 \dots p_l$. It follows from Theorem 2 that if $n \ne 1, 2, 3, 5, 6, 12$ then D_n contains at least one primitive divisor. In addition, $Q_3 = L - M > 3$ except when $L = 1$, $M = -1$. Moreover, since $Q_5 = 5$ and $Q_{12} = 6$ when $L = 1$, $M = -1$, and $Q_5 = 31$ and $Q_{12} = 13$ when $L = 3$, $M = 2$, Lemma 3 gives $Q_5 > 5$ and $Q_{12} > 6$ except for the Fibonacci sequence.

Therefore, by Lemma 2, if $n \ne 1, 2, 6$ then D_n contains at least one primitive divisor except when $L = 1$, $M = -1$. Combining with Theorem 3, we complete the proof. □

4. APPENDIX

In 1955, Ward [9] proved the theorem below for the Lehmer numbers defined by

$$P_n = \begin{cases} (\alpha^n - \beta^n) / (\alpha - \beta), & n \text{ odd}, \\ (\alpha^n - \beta^n) / (\alpha^2 - \beta^2), & n \text{ even}, \end{cases}$$

where α and β are distinct roots of the polynomial $z^2 - \sqrt{L}z + M$, and L and M are coprime integers with L positive and M nonzero. Here a sufficient condition $n \neq 6$ was pointed out by Durst [3].

Theorem 5 (Ward): If α and β are real and $n \neq 1, 2, 6$, then P_n contains at least one primitive divisor except when $n = 12$, $L = 1$, $M = -1$ and when $n = 12$, $L = 5$, $M = 1$.

We can also give an elementary proof of this theorem. It parallels the proof of Carmichael's theorem. The essential observation is that if $n \neq 1, 2, 6$ and if both the n^{th} cyclotomic number associated with $z^2 - z - 1$ and that associated with $z^2 - \sqrt{5}z + 1$ are greater than the product of all prime factors of n then, for all real Lehmer sequences, P_n contains at least one primitive divisor.

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ON THE ORDER OF STIRLING NUMBERS AND ALTERNATING BINOMIAL COEFFICIENT SUMS

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1. INTRODUCTION

We prove that the order of divisibility by prime p of $k!S(a(p-1)p^q, k)$ does not depend on a and q is sufficiently large and k/p is not an odd integer. Here $S(n, k)$ denotes the Stirling number of the second kind; i.e., the number of partitions of a set of n objects into k nonempty subsets. The proof is based on divisibility results for p-sected alternating binomial coefficient sums. A fairly general criterion is also given to obtain divisibility properties of recurrent sequences when the coefficients follow some divisibility patterns.

The motivation of the paper is to generalize the identity [8]

$$\nu_2(k!S(n,k)) = k-1, \quad 1 \le k \le n, \tag{1}$$

where $S(n, k)$ denotes the Stirling number of the second kind, and $n = a2^q$, a is odd, and q is sufficiently large (for example, $q \ge k-2$ suffices). Here $\nu_p(m)$ denotes the order of divisibility by prime p of m, i.e., the greatest integer e such that p^e divides m. It is worth noting the remarkable fact that the order of divisibility by 2 does not depend on a and q if q is sufficiently large. We will clarify later what value is large enough.

Our objective in this paper is to analyze $\nu_p(k!S(n,k))$ for an arbitrary prime p. It turns out that identity (1) can be generalized to calculate the exact value of $\nu_p(k!S(n,k))$ if $n = a(p-1)p^q$ and k is divisible by $p-1$. The main result of this paper is

Theorem 1: If $n = a(p-1)p^q$, $1 \le k \le n$, a and q are positive integers such that $(a, p) = 1$, q is sufficiently large, and k/p is not an odd integer, then

$$\nu_p(k!S(n,k)) = \left\lfloor \frac{k-1}{p-1} \right\rfloor + \tau_p(k),$$

where $\tau_p(k)$ is a nonnegative integer. Moreover, if k is a multiple of $p-1$, then $\tau_p(k) = 0$.

Here $\lfloor x \rfloor$ denotes the greatest integer function. Note that the order of divisibility by p of $k!S(a(p-1)p^q, k)$ does not depend on a and q if q is sufficiently large. For instance, we may choose q such that $q > \frac{k}{p-1} - 2$ in this case. Numerical evidence suggests that the condition on the magnitude of q may be relaxed and it appears that $n \ge k$ suffices in many cases (cf. [8]).

The case excluded by Theorem 1, in which k/p is an odd integer, behaves somewhat differently.

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Theorem 2: For any odd prime p, if k/p is an odd integer, then $\nu_p(k!S(a(p-1)p^q, k)) > q$.

In Section 2 we prove a fundamental lemma: If $n = a(p-1)p^q$, then

$$(-1)^{k+1}k!S(n,k) \equiv G_p(k) \pmod{p^{q+1}}, \tag{2}$$

where

$$G_p(k) = \sum_{p|i}\binom{k}{i}(-1)^i.$$

All of our divisibility results for the Stirling numbers are consequences of divisibility results for the alternating binomial coefficient sums $G_p(k)$, which are of independent interest. Theorem 2 is an immediate consequence of (2) since, if k/p is an odd integer, the corresponding binomial coefficient sum is 0.

To prove Theorem 1, we prove the analogous divisibility result for $G_p(k)$. The proof is presented in Section 2, and it combines number-theoretical, combinatorial, and analytical arguments. By an application of Euler's theorem, we prove (2). We then apply p-section of the binomial expansion of $(1-x)^k$ to express $G_p(k)$ as a sum of $p-1$ terms involving roots of unity. We take a closer look at this sum from different perspectives in Sections 3 and 4 and give a comprehensive study of the special cases $p=3$ and 5. We choose two different approaches in these sections: we illustrate the use of roots of unity in the case in which $p=3$, and for $p=5$ we use known results relating $G_5(k)$ to Fibonacci and Lucas numbers.

We outline a generating function based method to analyze the sum in terms of a recurrent sequence in Section 2. A fairly general lemma (Lemma 7) is also given in order to provide the framework for proving divisibility properties. The reader may find it a helpful tool in obtaining divisibility properties of recurrent sequences when the coefficients follow some divisibility patterns (e.g., [1]). The lemma complements previous results that can be found, for example, in [11] and [13]. Theorem 1 follows by an application of Lemma 7. A similar approach yields

Theorem 3: For any odd prime p and any integer t,

$$\sum_{i \equiv t \pmod p}\binom{k}{i}(-1)^i \equiv \begin{cases} (-1)^{\frac{k}{p-1}-1}p^{\frac{k}{p-1}-1} \left(\bmod p^{\frac{k}{p-1}}\right), & \text{if } k \text{ is divisible by } p-1, \\ 0 \quad \left(\bmod p^{\lfloor \frac{k}{p-1} \rfloor}\right), & \text{otherwise.} \end{cases}$$

Fleck [4] and Kapferer [7] proved the second part of Theorem 3, and Lundell [10] obtained the first part (Theorem 1.1(ii)). Lundell has only indicated that the proof is based on a tedious induction on $\lfloor \frac{k}{p-1} \rfloor$. The case $t=0$, $k=p(p-1)$ of Theorem 3 was proposed as an *American Mathematical Monthly* problem by Evans [3].

The proofs of Lemma 7 and Theorem 3 are given in Section 5 in which an application of Theorem 3 is also presented to prove its generalization.

Theorem 4: Let p be an odd prime and let m be an integer with $0 \le m \le \min(k,p)$ such that $r = \frac{k-m}{p-1}$ is an integer. We set $r \equiv r' \pmod p$ with $1 \le r' \le p$. If $r' \ge m$, then for any integer t,

$$\sum_{i \equiv t \pmod p}\binom{k}{i}(-1)^i i^m \equiv (-1)^{m+\frac{k-m}{p-1}-1}\binom{k}{m}m!\,p^{\frac{k-m}{p-1}-1} \left(\bmod p^{\frac{k-m}{p-1}+\nu_p\left(\binom{k}{m}m!\right)}\right).$$

For example, it follows that

$$\sum_{i \equiv t \pmod{17}} \binom{135}{i}(-1)^i i^7 \equiv \binom{135}{7} 7! 17^7 \pmod{17^8},$$

independently of t. Here we have $m = 7$ and $r' = r = 8$.

Theorem 4 is a generalization of Theorem 1.7 of [10]. Note that the conditions of Theorem 4 are always satisfied for $m = 0$ and 1 provided $p-1 | k-m$. The theorem can be generalized to the case in which $p = 2$ and $m = 0$ or 1, also (see [8]), e.g.,

$$\sum_{2|i} \binom{k}{i} = \sum_{2 \nmid i} \binom{k}{i} = 2^{k-1}.$$

Some conjectures on $\tau_p(k)$ are discussed at the end of the paper.

2. TOOLS AND THE GENERAL CASE

Lemma 5: If $n = a(p-1)p^q$, then

$$(-1)^{k+1} k! S(n, k) \equiv \sum_{p|i} \binom{k}{i}(-1)^i \pmod{p^{q+1}}. \tag{3}$$

Proof: By a well-known identity for the Stirling numbers (see [2], p. 204), we have

$$k! S(n, k) = \sum_{i=0}^{k} \binom{k}{i} i^n (-1)^{k-i} \equiv \sum_{p \nmid i} \binom{k}{i} i^n (-1)^{k-i} \pmod{p^n}.$$

For $n = a(p-1)p^q$ and $(i, p) = 1$, we have

$$i^n \equiv 1 \pmod{p^{q+1}}$$

by Euler's theorem. Notice that $n \geq q+1$. By the binomial theorem, we obtain

$$(1-1)^k = \sum_{p|i} \binom{k}{i}(-1)^i + \sum_{p \nmid i} \binom{k}{i}(-1)^i;$$

therefore, we have

$$\begin{aligned} k! S(n, k) \equiv \sum_{p \nmid i} \binom{k}{i}(-1)^{k-i} &= (-1)^k \sum_{p \nmid i} \binom{k}{i}(-1)^i \\ &= (-1)^{k+1} \sum_{p|i} \binom{k}{i}(-1)^i \pmod{p^{q+1}}. \quad \square \end{aligned}$$

Lemma 6: For any odd prime p, if k is an odd multiple of p, then

$$\sum_{p|i} \binom{k}{i}(-1)^i = 0.$$

Proof: The terms $\binom{k}{i}(-1)^i$ and $\binom{k}{k-i}(-1)^{k-i}$ cancel in (3). $\square$

Theorem 2 is an immediate consequence of Lemmas 5 and 6. We note that by multisection identities (see [12], p. 131, or [2], p. 84),

$$\sum_{m|i} \binom{k}{i}(-1)^i = \frac{1}{m}\sum_{t=1}^{m-1}(1-\omega^t)^k. \tag{4}$$

where $\omega = \exp(2\pi i/m)$ is a primitive m^{th} *root of unity*. To illustrate the use of this identity, we note that identity (1) follows immediately if we set $m = p = 2$; identities (3) and (4) with $\omega = -1$, imply that

$$k!S(n,k) \equiv (-1)^{k+1}2^{k-1} \pmod{2^{q+1}}$$

if $q > k-2$. The ways of improving this lower bound on q have been discussed in [8].

In the general case, we set

$$G_m(k) = \sum_{m|i}\binom{k}{i}(-1)^i. \tag{5}$$

For example, for any prime p, we have $G_p(k) = 1$ for $0 \le k < p$, and $G_p(p) = 0$. By identity (3), we get that

$$k!S(a(p-1)p^q, k) \equiv (-1)^{k+1}G_p(k) \pmod{p^{q+1}} \tag{6}$$

holds for all $q > 0$.

Now we are going to determine the generating function of $G_p(k)$ in identity (8) and deduce recurrence (9). An application of Lemma 7 to this recurrence will imply the required divisibility properties. For any odd m, we obtain

$$\begin{aligned}\sum_{k=0}^{\infty}\left[\sum_{m|i}(-1)^i\binom{k}{i}\right]x^k &= \sum_{m|i}(-1)^i\frac{x^i}{(1-x)^{i+1}} \\ &= \sum_{j=0}^{\infty}(-1)^{mj}\frac{x^{mj}}{(1-x)^{mj+1}} = \frac{1}{1-x}\left(1-\frac{(-x)^m}{(1-x)^m}\right)^{-1} \\ &= \frac{(1-x)^{m-1}}{(1-x)^m + x^m} = 1 + x\frac{(1-x)^{m-1}-x^{m-1}}{(1-x)^m+x^m}.\end{aligned} \tag{7}$$

We note that an alternative derivation of identity (7) follows by binomial inversion [5].

From now on p denotes an odd prime. In some cases, the discussion can be extended to $p = 2$, as will be pointed out.

We set $m = p$ and subtract 1 from both sides of (7), to yield

$$\sum_{k=1}^{\infty} G_p(k)x^k = x\frac{(1-x)^{p-1}-x^{p-1}}{(1-x)^p+x^p}. \tag{8}$$

We adopt the usual notation $[x^k]f(x)$ to denote the coefficient of x^k in the formal power series $f(x)$. If we multiply both sides of (8) by the denominator of the right-hand side and equate coefficients, we get a useful recurrence that helps us in deriving divisibility properties. Note that the right side is a polynomial of degree $p-1$. For $k \ge p$, we obtain that the coefficient of x^k is zero; i.e.,

$$[x^k]((1-x)^p + x^p)\sum_{i=1}^{\infty} G_p(i)x^i = 0.$$

It follows that

$$\sum_{i=0}^{p-1}(-1)^i\binom{p}{i}G_p(k-i)=0,$$

i.e.,

$$G_p(k)=-\sum_{i=1}^{p-1}(-1)^i\binom{p}{i}G_p(k-i). \tag{9}$$

Remark: Note that, for $p=2$, identity (8) has a slightly different form as it becomes

$$\sum_{k=1}^{\infty}G_2(k)x^k=\frac{x}{1-2x},$$

and we can easily deduce that $G_2(k)=2^{k-1}$, which agrees with $\nu_2(k!S(n,k))=k-1$.

The calculation of $G_p(k)$ is more complicated for $p>2$. However, we can find a lower bound on $\nu_p(G_p(k))$ and effectively compute $G_p(k) \pmod{p^{\nu_p(G_p(k))+1}}$ if $p-1\mid k$ by making some observations about identity (9). We shall need the following general result.

Lemma 7: Let p be an arbitrary prime. Assume that the integral sequence a_k satisfies the recurrence

$$a_k=\sum_{i=1}^{d}b_ia_{k-i},\quad k\geq d+1,$$

and that, for some nonnegative m, $\nu_p(a_d)=m\geq 0$ and the initial values a_i, $i=1,2,\dots,d-1$, are all divisible by p^m. Let $\nu_p(b_d)=r\geq 1$ and suppose that the coefficients b_i, $i=1,2,\dots,d-1$, are all divisible by p^r. We write $a_d=\alpha p^m$ and $b_d=\beta p^r$, and set

$$f(k)=f_p(k,m,r)=m+\left\lfloor\frac{k-1}{d}\right\rfloor r.$$

Then $\nu_p(a_k)\geq f(k)$, and equality holds if $d\mid k$. Moreover, for any integer $t\geq 1$, we have

$$a_{td}\equiv\alpha\beta^{t-1}p^{m+(t-1)r} \pmod{p^{m+tr}}.$$

According to the lemma, there is a transparent relation between the lower bound $f(k)$ on $\nu_p(a_k)$ and the parameters $\nu_p(a_d)$, $\nu_p(b_d)$, and d provided $\nu_p(a_i)\geq m$ and $\nu_p(b_i)\geq r$ for $i=1,2,\dots,d-1$.

We prove Lemma 7 in Section 5. With its help, we can now prove Theorem 1.

Proof of Theorem 1: By identity (5), we have $a_i=G_p(i)=1$ for $i=1,2,\dots,p-1$, and by identity (9), $b_i=(-1)^{i+1}\binom{p}{i}$ for $i=1,2,\dots,p-1$; therefore, $\nu_p(a_i)=0$ and $\nu_p(b_i)=1$. We apply Lemma 7 with $d=p-1$, $m=0$, $r=1$, $\alpha=1$, $\beta=-1$, and $s=2$, and get

$$G_p(k)=a_k\equiv\begin{cases}(-1)^{\frac{k}{p-1}-1}p^{\frac{k}{p-1}-1}\left(\operatorname{mod} p^{\frac{k}{p-1}}\right), & \text{if } k \text{ is divisible by } p-1,\\[2ex] 0 \qquad\qquad\qquad\quad \left(\operatorname{mod} p^{\left\lfloor\frac{k}{p-1}\right\rfloor}\right), & \text{otherwise.}\end{cases} \tag{10}$$

It follows that $\nu_p(a_k)\geq\frac{k}{p-1}-1$, and equality holds if and only if $p-1\mid k$. We define $\tau_p(k)$ by

$$\tau_p(k)=\nu_p(a_k)-\left\lfloor\frac{k-1}{p-1}\right\rfloor.$$

By identities (6) and (10), it follows that, for all $q > \nu_p(a_k)-1$, $\nu_p(k!S(n,k)) = \nu_p(a_k)$, which concludes the proof of Theorem 1. □

In the next two sections, we study the cases $p=3$ and $p=5$ in detail.

3. AN APPLICATION, $p = 3$

We set $m = p = 3$ and $\omega^3 = 1$. By identities (3) and (4), we have

$$\sum_{3|i}\binom{k}{i}(-1)^i = \frac{1}{3}((1-\omega)^k + (1-\omega^2)^k) = \frac{1}{3}(1-\omega)^k(1+(1+\omega)^k). \tag{11}$$

Note that $1+\omega = -\omega^2$, and $(1-\omega)^2 = 1-2\omega+\omega^2 = -3\omega$. Therefore, identity (11) implies

$$\sum_{3|i}\binom{k}{i}(-1)^i = \frac{1}{3}(1-\omega)^k(1+(-\omega^2)^k) = \frac{1}{3}(-3\omega)^{k/2}(1+(-\omega^2)^k). \tag{12}$$

For $6|k$, we get $\frac{1}{3}(-3\omega)^{k/2}2 = (-1)^{k/2}2\cdot 3^{k/2-1}$, yielding $\nu_3(k!S(n,k)) = k/2-1$ for $q > k/2-2$.

For k even and $3\nmid k$, by identity (12) we have

$$\begin{aligned}(-1)^{k/2}3^{k/2-1}\omega^{k/2}(1+\omega^{2k}) &= (-1)^{k/2}3^{k/2-1}(\omega^{k/2}+\omega^{-k/2})\\ &= (-1)^{k/2+1}3^{k/2-1},\end{aligned}$$

since $\omega^{k/2}+\omega^{-k/2} = \omega+\omega^{-1} = -1$ in this case.

We are left with cases in which k is odd. For k odd and $3\nmid k$, we have two cases. If $k \equiv 1 \pmod 6$, say $k = 6l+1$ for some integer $l \ge 0$, then by identity (11) we obtain

$$\begin{aligned}\frac{1}{3}(1-\omega)^{6l}(1-\omega)(1+(-\omega^2)^{6l+1}) &= \frac{1}{3}(-3\omega)^{3l}(1-\omega)(1-\omega^2)\\ &= (-3)^{3l} = (-3)^{\frac{k-1}{2}}.\end{aligned}$$

If $k \equiv 5 \pmod 6$, say $k = 6l+5$ for some integer $l \ge 0$, then by identity (11) we obtain

$$\begin{aligned}\frac{1}{3}(1-\omega)^{6l}(1-\omega)^5(1+(-\omega^2)^{6l+5}) &= \frac{1}{3}(-3\omega)^{3l}(1-\omega)^5(1-\omega)\\ &= \frac{1}{3}(-3)^{3l}(-3\omega)^3 = (-1)^{\frac{k+1}{2}}3^{\frac{k-1}{2}}.\end{aligned}$$

In summary, we get

Theorem 8: For $q > \lfloor\frac{k-1}{2}\rfloor - 1$, $k > 0$, and $k \not\equiv 3 \pmod 6$, we have

$$\nu_3(k!S(2a3^q,k)) = \left\lfloor\frac{k-1}{2}\right\rfloor.$$

Recall that, if $k/3$ is an odd integer, then $\nu_3(k!S(n,k)) > q$ by Theorem 2.

4. AN APPLICATION, $p = 5$

For $p=5$, we can use the fact that $G_5(k)$ can be expressed explicitly in terms of Fibonacci or Lucas numbers, with the formula depending on k modulo 20, as shown by Howard and Witt [6].

(The Fibonacci numbers F_n are given by $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. The Lucas numbers L_n satisfy the same recurrence, but with the initial conditions $L_0 = 2$ and $L_1 = 1$.)

The power of 5 dividing a Fibonacci or Lucas number is determined by the following result [9].

Lemma 9: For all $n \geq 0$, we have $\nu_5(F_n) = \nu_5(n)$. On the other hand, L_n is not divisible by 5 for any n.

Theorem 10: If $k \equiv 5 \pmod{10}$, then $G_5(k) = 0$. If $k \not\equiv 5 \pmod{10}$, then

$$\nu_5(G_5(k)) = \left\lfloor \frac{k-1}{4} \right\rfloor + \tau_5(k),$$

where

$$\tau_5(k) = \begin{cases} \nu_5(k+1), & \text{if } k \equiv 9 \pmod{20}, \\ \nu_5(k), & \text{if } k \equiv 10 \pmod{20}, \\ \nu_5(k+2), & \text{if } k \equiv 18 \pmod{20}, \\ 0, & \text{otherwise.} \end{cases} \tag{13}$$

Proof: From a result of Howard and Witt (see [6], Theorem 3.2), we find that the value of $5^{-\lfloor (k-1)/4 \rfloor} G_5(k)$ is given by the following table:

k mod 20	0	1	2	3	4
$5^{-\lfloor (k-1)/4 \rfloor} G_5(k)$	$2L_{k/2}$	$F_{(k+1)/2}$	$F_{k/2+1}$	$L_{(k-1)/2}$	$L_{k/2-1}$

k mod 20	5	6	7	8	9
$5^{-\lfloor (k-1)/4 \rfloor} G_5(k)$	0	$-F_{k/2-1}$	$-L_{(k-1)/2}$	$-L_{k/2+1}$	$-F_{(k+1)/2}$

k mod 20	10	11	12	13	14
$5^{-\lfloor (k-1)/4 \rfloor} G_5(k)$	$-2F_{k/2}$	$-L_{(k+1)/2}$	$-L_{k/2+1}$	$-F_{(k-1)/2}$	$-F_{k/2-1}$

k mod 20	15	16	17	18	19
$5^{-\lfloor (k-1)/4 \rfloor} G_5(k)$	0	$L_{k/2-1}$	$F_{(k-1)/2}$	$F_{k/2+1}$	$L_{(k+1)/2}$

The result then follows easily from Lemma 9. □

We can now derive our main result on the divisibility of Stirling numbers by powers of 5.

Theorem 11: If n is divisible by $4 \cdot 5^q$, where q is sufficiently large, and $k \not\equiv 5 \pmod{10}$, then

$$\nu_5(k!S(n,k)) = \nu_5(G_5(k)) = \left\lfloor \frac{k-1}{4} \right\rfloor + \tau_5(k),$$

where $\tau_5(k)$ is given by (13).

Proof: Apply Lemma 5 to Theorem 10. □

Note that in Theorem 11 any q exceeding $\nu_5(k!S(n,k)) - 1$ will suffice; for instance, we can select the lower bound $\lfloor \frac{k-1}{4} \rfloor + \tau_5(k) - 1$.

Our proof of Theorem 10 does not generalize to other primes, so we mention another approach that in principle does generalize, though it is not easy to apply it to primes larger than 5.

The basic idea is to p-sect the generating functions $\sum_k G_p(k)x^k$. If $A(x) = N(x)/D(x)$, where $N(x)$ and $D(x)$ are polynomials, then we can determine all of the p-sections of $A(x)$ by multiplying the numerator and denominator of $A(x)$ by $D(\omega x)D(\omega^2 x)\cdots D(\omega^{p-1}x)$, where ω is a primitive p^{th} root of unity; since $D(x)D(\omega x)\cdots D(\omega^{p-1}x)$ is invariant under substituting ωx for x, it must be a polynomial in x^p. For example, we find in this way that

$$\sum_{k=1}^{\infty} G_3(k)x^k = x\frac{(1-x)^2-x^2}{(1-x)^3+x^3} = \frac{x+x^2-3x^4-9x^5-18x^6}{1+27x^6} \tag{14}$$

and

$$\sum_{k=1}^{\infty} G_5(k)x^k = x\frac{(1-x)^4-x^4}{(1-x)^5+x^5} = \frac{N_5(x)}{1+5^4x^{10}+5^5x^{20}}, \tag{15}$$

where

$$\begin{aligned} N_5(x) = {} & x+x^2+x^3+x^4-5x^6-20x^7-55x^8-125x^9-250x^{10}+175x^{11} \\ & -100x^{12}-375x^{13}-375x^{14}+500x^{16}+625x^{17}-1250x^{19}-2500x^{20}. \end{aligned} \tag{16}$$

Of course, once we have found these formulas, by whatever method, they may be immediately verified.

We note that in both of these generating functions the denominator is actually a polynomial in x^{2p} rather than just in x^p, and it is not difficult to show that this is true in general.

From (14), we can immediately derive a formula for $G_3(k)$. With somewhat more difficulty, one can use (15) and (16) to determine the exact power of 5 dividing $G_5(k)$ and thereby give a different proof of Theorem 10.

5. THE PROOFS OF LEMMA 7, THEOREM 3, AND THEOREM 4

We present the proofs of a lemma and two theorems that were stated in Section 2.

Proof of Lemma 7: We prove that the following two assertions hold, by induction on k:

(i) $\nu_p(a_k) \ge f(k)$.

(ii) If $k = td$, where t is a positive integer, then $a_k \equiv \alpha\beta^{t-1}p^{m+(t-1)r} \pmod{p^{m+tr}}$.

Note that (ii) implies that $\nu_p(a_k) = f(k)$.

If $1 \le k \le d$, then these assertions are consequences of the initial conditions. Now suppose that (i) and (ii) hold for $a_{k-d}, \ldots, a_{k-1}$. Then the induction hypothesis implies that $b_i a_{k-i}$ is divisible by $p^{r+f(k-i)}$ for $i = 1, 2, \ldots, d$. We have $r + f(k-i) \ge r + f(k-d) = f(k)$, so $b_i a_{k-i}$ is divisible by $p^{f(k)}$, and thus so is $a_k = b_1 a_{k-1} + \cdots + b_d a_{k-d}$. This proves (i).

For (ii), suppose that $k = td$. By the induction hypothesis, we have

$$a_{(t-1)d} \equiv \alpha\beta^{t-2}p^{m+(t-2)r} \pmod{p^{m+(t-1)r}}$$

and

$$\nu_p(a_{td-i}) \ge m + \left\lfloor \frac{td-i-1}{d} \right\rfloor r = m + (t-1)r \quad \text{for } 1 \le i < d.$$

Thus,

$$b_d a_{(t-1)d} \equiv \beta p^r \cdot \alpha\beta^{t-2}p^{m+(t-2)r} = \alpha\beta^{t-1}p^{m+(t-1)r} \pmod{p^{m+tr}}$$

and

$$\nu_p(b_i a_{td-i}) \geq m + tr \quad \text{for } 1 \leq i < d.$$

Then (ii) follows from the recurrence for a_k. □

We note that the lemma extends the study of situations discussed in [11] and [13] by relaxing the condition that the coefficient b_d be relatively prime to the modulus.

We can further generalize identity (10) and obtain the

Proof of Theorem 3: Analogously to the definition of (5), we set

$$G_m(k,t) = \sum_{i \equiv t \ (\mathrm{mod}\ m)} \binom{k}{i}(-1)^i$$

for $0 \leq t \leq m-1$. In a manner similar to the derivation of identity (7), for every odd m, we obtain that

$$\sum_{k=0}^{\infty}\left[\sum_{i \equiv t \ (\mathrm{mod}\ m)} (-1)^i \binom{k}{i}\right] x^k = \frac{(-x)^t (1-x)^{m-t-1}}{(1-x)^m + x^m}.$$

Note that the degree of the numerator is $m-1$. It is fairly easy to modify identities (8) and (9) for $G_m(k,t)$. An application of Lemma 7 to $G_p(k,t)$ yields Theorem 3. We note that here

$$\alpha = \binom{p-1}{t}(-1)^t;$$

hence, $\alpha \equiv 1 \ (\mathrm{mod}\ p)$. The congruence follows from the two identities

$$\binom{p}{t} \equiv 0 \ (\mathrm{mod}\ p),\ 1 \leq t \leq p-1, \quad \text{and} \quad \binom{p}{t} = \binom{p-1}{t-1} + \binom{p-1}{t}.$$

(We note that, for every prime p and positive integer n,

$$\binom{p^n - 1}{t} \equiv (-1)^t \ (\mathrm{mod}\ p)$$

also holds.) □

The interested reader may try another application of Lemma 7 to prove the following identity (cf. [1]):

$$\nu_2\left(\sum_{k=0}^{n-1}\binom{2n-1}{2k}3^k\right) = n-1,\ n = 1, 2, \ldots.$$

Finally, we note that it would be interesting to find an upper bound on $\nu_p(a_k) - f(k)$ as a function of k. The case $p = 5$ and $k \equiv 9$, 10, or 18 (mod 20) shows that the difference can be as big as C log k with some positive constant C.

We conclude this section with the

Proof of Theorem 4: Theorem 3 deals with the case in which $m = 0$, thus we may assume that $m \geq 1$. Using the identities

$$i^m = \sum_{l=0}^{m} S(m,l)\binom{i}{l} l! \quad \text{and} \quad \binom{k}{i}\binom{i}{l} = \binom{k}{l}\binom{k-l}{i-l}$$

for $l \leq i \leq k$, we have

$$\begin{aligned}\sum_{i\equiv t \pmod p}\binom{k}{i}(-1)^i i^m &= \sum_{i\equiv t \pmod p}\binom{k}{i}(-1)^i \sum_{l=0}^{m} S(m,l)\binom{i}{l}l! \\ &= \sum_{l=0}^{m} S(m,l)\binom{k}{l}l! \sum_{i\equiv t \pmod p}\binom{k-l}{i-l}(-1)^i \\ &= \sum_{l=0}^{m}(-1)^l S(m,l)\binom{k}{l}l! \sum_{i\equiv t-l \pmod p}\binom{k-l}{i}(-1)^i.\end{aligned} \tag{17}$$

Observe that Theorem 3 applies to the last sum.

We shall show that under the conditions of Theorem 4, the term with $l=m$ has the smallest exponent of p on the right side of (17). If $l=0$, then $S(m,l)=0$ in identity (17), so we need only consider the terms in which $l\geq 1$. Let $\chi_y(x)=1$ if and only if $y\mid x$. We shall show that

$$\nu_p\left(\binom{k}{l}l!\,p^{\lfloor \frac{k-l}{p-1}\rfloor-\chi_{p-1}(k-l)}\right)=\nu_p(k!)-\nu_p((k-l)!)+\left\lfloor\frac{k-l}{p-1}\right\rfloor-\chi_{p-1}(k-l),\ 1\leq l\leq m,$$

assumes its unique minimum at $l=m$; this fact, together with Theorem 3, implies Theorem 4.

By a well-known formula, we have

$$\nu_p((k-l)!)=\left\lfloor\frac{k-l}{p}\right\rfloor+\left\lfloor\frac{k-l}{p^2}\right\rfloor+\cdots.$$

The hypotheses of Theorem 4 imply that $k-m=r(p-1)\equiv -r' \pmod p$, where $1\leq r'\leq p$ and $m\leq r'$. It follows that $\lfloor\frac{k-m+i}{p}\rfloor$ is constant for $i=0,1,\ldots,r'-1$; i.e., $\lfloor\frac{k-l}{p}\rfloor$ is constant for $l=m$, $m-1,\ldots,m-r'+1$. Since $r'\geq m$, this implies that $\lfloor\frac{k-l}{p}\rfloor$ is constant for $1\leq l\leq m$. Similarly, $\lfloor\frac{k-l}{p^i}\rfloor$ is constant for $1\leq l\leq m$. Therefore, $\nu_p((k-l)!)$ is constant for $1\leq l\leq m$.

Next, we show that

$$\left\lfloor\frac{k-l}{p-1}\right\rfloor-\chi_{p-1}(k-l)>\left\lfloor\frac{k-m}{p-1}\right\rfloor-\chi_{p-1}(k-m),\ 1\leq l<m. \tag{18}$$

Since $p-1$ divides $k-m$, $k-l$ is not divisible by $p-1$ for $l=m-1,\ m-2,\ \ldots,\ m-p+2$, and since $m\leq p$, this implies all cases of (18) except $m=p$, $l=1$. In this case, we have

$$\left\lfloor\frac{k-1}{p-1}\right\rfloor-\chi_{p-1}(k-1)=1+\left\lfloor\frac{k-p}{p-1}\right\rfloor-\chi_{p-1}(k-p),$$

and thus (18) holds in this case also. The proof is now complete. □

We note that the generating function of the sum on the left-hand side of (17) can be derived by binomial inversion [5] in terms of Eulerian polynomials.

6. CONJECTURES

Empirical evidence suggests that formulas for $\tau_p(k)$ exist based on the residue of k modulo $p(p-1)$. The following conjectures have been proved only in the cases $p=3$ and $p=5$.

Conjecture 1:

(a) If k is divisible by $2p$ but not by $p(p-1)$, then $\tau_p(k)=\nu_p(k)$.

(b) If $k+1$ is divisible by $2p$ but not by $p(p-1)$, then $\tau_p(k)=\nu_p(k+1)$.

Conjecture 2: For each odd prime p, there is a set $A_p \subseteq \{1, 2, \ldots, p(p-1)-1\}$ such that, if $k \not\equiv 0$ or $-1 \pmod{2p}$ and k is not an odd multiple of p, then $\tau_p(k) > 0$ if and only if k is congruent modulo $p(p-1)$ to an element of A_p.

It usually seems to be true under the conditions of Conjecture 2 that, for each $i \in A_p$, there exists some integer $u_{p,i}$ such that, if $k \equiv i \pmod{p(p-1)}$, then $\tau_p(k) \equiv \nu_p(k + u_{p,i})$.

For example, Theorem 7 asserts that the conjectures hold for $p = 3$, with $A_3 = \emptyset$, and Theorem 8 asserts that the conjectures hold for $p = 5$ with $A_5 = \{18\}$ and $u_{5,18} = 2$. Empirical evidence suggests that $A_7 = \{16\}$, with $u_{7,16} = 75$. Here are the empirical values of A_p for primes p from 11 to 23.

$$A_{11} = \{14, 18, 73, 81, 93\},$$
$$A_{13} = \{82, 126, 148\},$$
$$A_{17} = \{37, 39, 62, 121, 179, 230, 234\},$$
$$A_{19} = \{85, 117, 119, 156, 196, 201, 203, 244, 279, 295, 299, 316, 320, 337\},$$
$$A_{23} = \{72, 128, 130, 145, 148, 170, 171, 188, 201, 210, 211, 232, 233, 234, 317, 325, 378, 466\}.$$

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THE LEAST NUMBER HAVING 331 REPRESENTATIONS AS A SUM OF DISTINCT FIBONACCI NUMBERS

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1. INTRODUCTION

Let A_n be the least number having exactly n representations as a sum of distinct Fibonacci numbers. Let $R(N)$ denote the number of representations of N as sums of distinct Fibonacci numbers, and let Zeck N denote the Zeckendorf representation of N, which is the unique representation of N as a sum of distinct, nonconsecutive Fibonacci Numbers. The sequence $\{A_n\}$ is sequence A013583 studied earlier [7], [9], where we list the first 330 terms; here, we extend our computer results by pencil and logic to calculate A_{331} and other "missing values." We list some pertinent background information.

Theorem 1: The least integer having F_k representations is $(F_k)^2-1$, and F_k is the largest value for $R(N)$ for N in the interval $F_{2k-2} \le N < F_{2k-1}$.

Theorem 2: Let N be an integer written in Zeckendorf form; if $N = F_{n+k}+K$, $F_n \le K < F_{n+1}$, we can write $R(N)$ by using the appropriate formula:

$$R(N) = R(F_{n+2p}+K) = pR(K)+R(F_{n+1}-k-2),\ k=2p; \tag{1.1}$$

$$R(N) = R(F_{n+2p+1}+K) = (p+1)R(K),\ k=2p+1; \tag{1.2}$$

$$R(N) = R(N-F_{2w})+R(F_{2w+1}-N-2),\ F_{2w} \le N < F_{2w+1}. \tag{1.3}$$

Theorem 3: Zeck A_n ends in $\ldots+F_{2c}$, $c \ge 2$. If Zeck N ends in $\ldots+F_{2c+2k+1}+F_{2c}$, $c \ge 2$, then

$$R(N) = R(N-1)R(F_{2c}) = cR(N-1).$$

If Zeck $A_n = F_m + K$, then $F_m < A_n < F_m + F_{m-2}$.

Lemma 1: If $\{b_n\}$ is a sequence of natural numbers such that $b_{n+2} = b_{n+1}+b_n$, then

$$R(b_n-1) = R(b_{n+1}-1) = k$$

for all sufficiently large n (see [8]).

Lemma 1 and Theorem 3 are useful in calculating A_n when n is composite. Theorems 2 and 3 are proved in [2] and [3], while Theorem 1 is the main result of [1].

Theorem 4: If $n = R(A_n)$ is a prime, then Zeck A_n ;is the sum of even-subscripted Fibonacci numbers only. If Zeck A_n begins with F_{2k+1}, then $n = R(A_n)$ cannot be prime.

Proof: Theorem 3 and (1.2) show that a change in parity in subscripts indicates that at least one pair of factors exists for $R(N)$.

Lemma 2: For integers N such that $F_{2k} \le A_n \le N < F_{2k+1}$ and $R(N) = n$,

$$F_{k-1} \le R(A_n - F_{2k}) \le R(F_{2k+1} - N - 2) \le R(F_{2k+1} - A_n - 2) \le F_k. \tag{1.4}$$

Proof: The pair of exterior endpoints are a consequence of Theorem 1. The pair of interior endpoints reflect symmetry about the center of the interval, since $R(N)$ is a palindromic sequence within each such interval $F_{2k} \le N < F_{2k+1} - 2$.

If Zeck $n = F_k + K$, $0 < K \le F_{k-2}$, then $A_n > F_{2k-1}$, where we note that we are relating the Zeckendorf representations of A_n and of $R(A_n)$. In our extensive tables, Zeck A_n begins with F_{2k-1}, F_{2k}, F_{2k+1}, or F_{2k+2}, while all values for n, $1 \le n = R(N) \le F_k$, appear for $N < F_{2k+1}$, but this has not been proved. The first 330 values for A_n are listed in [7], too long a table to repeat here. Our computer results conclude with $A_{466} = 229971$; there are 69 "missing values" for n between 330 and 466. We also have complete tables for $R(N)$ for all $N < F_{22}$, not included here, which shorten the work but are not essential to follow the logic in solving for A_n given n.

2. THE CALCULATION OF A_{331}

Since A_n is known for all $n \le 330$ and, for all n such that $A_n < F_{28}$, and since 331 is prime, we can find A_{331} by listing successive addends for Zeck A_n, and choosing the smallest possibility at each step. Let $N = F_{28} + K^*$, for $F_{28-2q} \le K < F_{29-2q}$. Then

$$R(N) = qR(K) + R(F_{29-2q} - K - 2) \tag{2.1}$$

by (1.1), and the maximum possible value for $R(N)$ is

$$\max R(N) = qF_{15-q} + F_{14-q}$$

by Theorem 1. Since $F_{2k} \le A_n < F_{2k} + F_{2k-2}$, $2 \le q$. We summarize in Table 1.

TABLE 1

$N = F_{28} + K,\ F_{28-2q} \le K < F_{29-2q}$

$\max R(N)$: $qF_{15-q} + F_{14-q}$

$q = 2$:	$2F_{13} + F_{12} = 466 + 144 = 610$
$q = 3$:	$3F_{12} + F_{11} = 432 + 89 = 521$
$q = 4$:	$4F_{11} + F_{10} = 356 + 55 = 411$
$q = 5$:	$5F_{10} + F_9 = 275 + 34 = 309$

Notice that maximum values for $R(N)$ for $q \ge 5$ are smaller than 331. For our purposes, the smallest possibility is $q = 4$, or $N = F_{28} + F_{20} + K$. We write Table 2 to determine the third possible even subscript in Zeck N when $q = 4$.

Start with $w = 3$ in Table 2, the smallest possibility, with $F_{14} \le K < F_{15}$. Solve the Diophantine equation $14A + 5B = 331$, $13 < A \le 21$, which has $14(19) + 5(13) = 331$. By Lemma 2, since $A_{19} = F_{14} + A_7$, $7 \le B = R(F_{15} - K - 2) \le 19 - 7 = 12$. Thus, $B \ne 13$ and $w \ne 3$.

TABLE 2

$$N = F_{28} + F_{20} + K, \; F_{20-2w} \le K < F_{21-2w}$$
$$R(N) = (5w-1)\,R(K) + 5R(F_{21-2w} - K - 2)$$
$$\max R(N):\; (5w-1)F_{11-w} + 5F_{10-w}$$

$w=1$:	$4F_{10} + 5F_9 = 220 + 170 = 390$
$w=2$:	$9F_9 + 5F_8 = 306 + 105 = 411$
$w=3$:	$14F_8 + 5F_7 = 294 + 65 = 359$
$w=4$:	$19F_7 + 5F_6 = 247 + 40 = 287$

Next take $w=2$ in Table 2, with $F_{16} \le K < F_{17}$, and solve $9A + 5B = 331$, $21 < A \le 34$, which has $9(34) + 5(5) = 331$, but $A_{34} = F_{16} + A_{13}$, so that we must have $13 \le B \le 21$; $B = 5$ is too small. We also find $9(29) + 5(14) = 331$, which is plausible since $A_{29} = 1050 = F_{16} + A_8$, and $8 \le B = 14 \le 21$. However, this combination of values does not appear in the computer printouts; only $N = 1050$, 1152, 1189 have $R(N) = 29$ for $N < F_{16} + F_{14}$, so $B = 8$, 21, 18, 11, 17, or 12, but not 14. However, we can verify that $B \ne 14$ either by assuming that the next term is F_{14} and calculating one more step, or by noting that we are solving $A = R(K) = 29$ for some K which also has $R(F_{17} - K - 2) = 14$ and $R(K - F_{16}) = 29 - 14 = 15$. We must have $K - F_{16} \ge A_{15} = F_{13} + F_8 + F_4$ or $K = F_{16} + F_{14} + K'$. Then, because $F_{17} - K - 2 = F_{13} - K' - 2 < A_{14} = F_{13} + 16$, we cannot have $R(F_{17} - K - 2) = 14 = B$, a contradiction. The last viable solution $9(24) + 5(23) = 331$ has B too large. Thus, $w \ne 2$.

Finally, take $w = 1$, with $F_{18} \le K < F_{19}$. Solve $4A + 5B = 331$ for $34 < A \le 55$, obtaining $4(49) + 5(27) = 331$ and $4(44) + 5(31) = 331$, where $4(39) + 5(35) = 331$ has B too large. From the computer printout, $A_{44} = F_{18} + A_{12} = 2744$, but $R(F_{19} - 2744 - 2) = 32$, not 31. The next occurrence of $R(K) = 44$ in our computer table is for $K = 2791$ for which $31 = R(F_{19} - 2791 - 2)$; and since 2791 is the smallest integer that satisfies all of the parameters, we have a solution. Without such a table, one could assume that F_{18} is the next term, and compute the term following F_{18}. We now have

$$A_{331} = F_{28} + F_{20} + 2791 = 327367.$$

Let us make use of our work thus far. In Table 2, $w = 3$ has $14F_8 + 5F_7 = 359$, one of the "missing values." Since we cannot write a smaller solution,

$$A_{359} = F_{28} + F_{20} + A_{21} = 317811 + 6765 + 440 = 325016.$$

Also, Table 1, $q = 4$, $N = F_{28} + (F_{20} + K)$ has $R(N) = 359$ for $4(76) + 55 = 359$, or for $N = F_{28} + A_{76} = 317811 + 7205$, which gives the same result.

3. THE CALCULATION OF A_{339}

The second missing value on our list is 339. We can find A_{339} with very little effort, although $339 = 3 \cdot 113$ is not a prime. Since $A_{113} = F_{24} + K$, $N = F_{28} + F_{23} + \cdots$ has $R(N) = 3R(F_{23} + \cdots)$, and A_{113} is too large to appear as the second factor. Now, taking $q = 4$, for $N = F_{28} + (F_{20} + K)$, $4(74) + 43 = 339$, and $A_{74} = 8187$ while $R(F_{21} - 8187 - 2) = R(2757) = 43$; in fact, $2757 = A_{43}$. Then

$$N = A_{339} = F_{28} + A_{74} = 317811 + 8187 = 325998.$$

We have also generated

$$N = F_{28} + A_{89} = 317811 + 7920 = 325731$$

which has $R(N) = 411$ from Table 1, $q = 4$, while Table 2, $w = 2$, gives $A_{411} = F_{28} + F_{20} + A_{34}$ which is the same result. Just as for 339, while we can factor $411 = 3 \cdot 137$, A_{137} is too large. Again from Table 2, $w = 2$, changing A and B slightly, we find $9F_9 + 5F_7 = 371$, also on our list. If we take $K = 1427 = F_{16} + F_{14} + F_{10} + F_6$, then $R(K) = 34$, $R(F_{21} - K - 2) = 13$; the only other value for K in this interval such that $R(K) = 34$ is A_{34} but $R(F_{21} - A_{34} - 2) = 21$, so 1427 is the smallest we can take for K. Thus, we write

$$A_{371} = F_{28} + F_{20} + 1427 = 326003.$$

We next illustrate how to use factoring to find A_n when n is composite, using Lemma 1 and Theorem 3. Let $n = 410 = 41 \cdot 10$:

$$\begin{aligned}
&A_{10} = 105 = F_{11} + F_7 + F_4,\\
&A_{41} = 2736 = F_{18} + F_{12} + F_6,\\
&41 = R(F_{18} + F_{12} + F_6 + F_1 - 1) = R(F_{28} + F_{22} + F_{16} + F_{11} - 1),\\
&41 \cdot 10 = R(F_{28} + F_{22} + F_{16} + A_{10}).
\end{aligned}$$

$N = F_{28} + F_{22} + F_{16} + F_{11} + F_7 + F_4 = 336614$ has $R(N) = 410$. Writing $R(N)$ as $205 \cdot 2$ gives the same solution, while $82 \cdot 5$ gives a slightly larger solution. $N = A_{410}$ if there is no smaller solution using the even subscript formula. We can easily see that $N \neq F_{28} + F_{20} + K$ from our earlier work, so we test out $N = F_{28} + F_{22} + K$ in Table 3.

TABLE 3

$$N = F_{28} + F_{22} + K, \; F_{22-2p} \le K < F_{23-2p}$$

$$R(N) = (4p-1)R(K) + 4R(F_{23-2p} - K - 2)$$

$$\max R(N)\text{: } (4p-1)F_{12-p} + 4F_{11-p}$$

$p = 1$:	$3F_{11} + 4F_{10} = 267 + 220 = 487$
$p = 2$:	$7F_{10} + 4F_9 = 385 + 136 = 521$
$p = 3$:	$11F_9 + 4F_8 = 374 + 84 = 458$
$p = 4$:	$15F_8 + 4F_7 = 315 + 52 = 367$
$p = 5$:	$19F_7 + 4F_6 = 247 + 24 = 271$

The smallest choice to generate 410 is $p = 3$ for $F_{16} \le K < F_{17}$ which requires that we solve $11A + 4B = 410$ for $A \le 34$ which, in turn, gives us $11(30) + 4(20) = 410$; $A_{30} = 1092 = F_{16} + 105$ and $R(F_{17} - A_{30} - 2) = 20$, so that

$$A_{410} = F_{28} + F_{22} + F_{16} + 105,$$

the same result as by factoring.

Note that Table 3 provides more "missing values" on our list. Here, $p = 4$ gives $R(N) = 367$ for $N = F_{28} + F_{22} + A_{21}$, which easily demonstrates that $N \neq F_{28} + F_{20} + K$, so $A_{367} = 335962$, the same result as $A_{367} = F_{28} + A_{97}$ by working with Table 1, $q = 3$. Furthermore,

$$A_{458} = F_{28} + A_{123} = 317811 + 18866 = 336677$$

comes from $q = 3$ of Table 1, and $A_{458} = F_{28} + F_{22} + A_{34}$ comes from $p = 3$ above.

We expect to see all "missing values" $n < F_{14} = 377$ appearing for $N = F_{28} + K$ based on our previous experience, but we have been unable to prove that all $n = R(N)$, $1 \le n \le F_k$, will appear for $N = F_{2k} + K$. Generating some of them will take patience, especially for a value such as $n = 421$ which has no solution for $A_n = F_{28} + K$. One can generate more tables such as Table 4 similarly to Tables 1 through 3, or one can list possible successive subscripts for Zeck A_n and evaluate each case.

Some results, verifiable in other ways, can be read from the tables. From Table 4. below, we have

$$A_{610} = F_{28} + F_{24} + A_{89} \quad \text{and} \quad A_{542} = F_{28} + F_{24} + A_{55}.$$

However, $A_{555} = A_{610} + 5 < F_{28} + F_{24} + A_{144}$. Table 1 gives

$$A_{610} = F_{28} + A_{233} \text{ (the same result)} \quad \text{and} \quad A_{521} = F_{28} + A_{144}.$$

Table 3 gives

$$A_{487} = F_{28} + F_{22} + A_{89} \quad \text{and} \quad A_{521} = F_{28} + F_{22} + A_{55} \text{ (the same result)}.$$

Table 4 gives $R(N) = 333$ for $N = F_{28} + F_{24} + A_{21}$, where Zeck N uses only even-subscripted Fibonacci numbers, but $A_{333} = 209668 < N$. One must verify that N is the smallest possible, especially if $R(N)$ is composite.

TABLE 4

$$N = F_{28} + F_{24} + K,\ F_{24-2p} \le K < F_{25-2p}$$

$$R(N) = (3p-1)R(K) + 3R(F_{25-2p} - K - 2)$$

$$\max R(N):\ (3p-1)F_{13-p} + 3F_{12-p}$$

$p = 1$:	$2F_{12} + 3F_{11} = 288 + 267 = 555$
$p = 2$:	$5F_{11} + 3F_{10} = 445 + 165 = 610$
$p = 3$:	$8F_{10} + 3F_9 = 440 + 102 = 542$
$p = 4$:	$11F_9 + 43F_8 = 374 + 63 = 437$
$p = 5$:	$14F_8 + 3F_7 = 294 + 39 = 333$

By constructing N taking one even-subscripted Fibonacci number at a time, one can find A_n for n prime, $n < 466$; some solutions are very short, while others take patience. Prime values for n in Table 5 can be found for $N = F_{28} + K$ except for $n = 421$, 439, and 461, which need $N = F_{30} + K$. The composites n for which $A_n > F_{28} + K$, found by considering factors of n, need $N = F_{29} + K$. Note that only the subscripts in Zeck A_n are listed in Table 5.

The calculations of A_n for n prime and of A_n, where Zeck A_n has even subscripts only agree with D. Englund [4], [5], and with computations using "Microsoft Excel" by M. Johnson. Of the composites $n = R(A_n)$, where A_n contains an odd-subscripted term, there are very many cases to consider and thus checking is more difficult. Each composite n starred in the table can be computed from its factors and has $A_n < N$, where $R(N) = n$ and Zeck N contains even-subscripted Fibonacci numbers only.

TABLE 5. "Missing Values" for n, $331 \leq n = R(N) \leq 465$

n prime

n	A_n	Zeck A_n
331	327367	28,20,18,12,10,6
347	336067	28,22,14,12,8,4
349	339528	28,22,18,16,14,10,4
353	338185	28,22,18,10,8,4
359	325016	28,20,14,10,6
367	335962	28,22,14,10,6
373	336588	28,22,16,10,8,4
379	338690	28,22,18,14,12,10,6
383	338638	28,22,18,14,12,6,4
389	336944	28,22,16,14,10,4
397	342688	28,22,20,14,8,4
401	338648	28,22,18,14,12,8
409	343476	28,22,20,16,12,10,4
419	338656	28,22,18,14,12,8,6
421	839994	30,20,16,12,10,4
431	343714	28,22,20,16,14,10,8
433	343426	28,22,20,16,12,6
439	841557	30,20,18,12,8,4
443	343447	28,22,20,16,12,8,6
449	367292	28,24,18,14,12,6
457	367923	28,24,18,16,12,8,6
461	851181	30,22,16,14,10,6,4
463	338562	28,22,18,14,10,8,4

n composite

n	A_n	Zeck A_n
339	325998	28,20,16,14,10,4
371	326003	28,20,16,14,10,6
381	339533	28,22,18,16,14,10,6
391	336674	28,22,16,12,8
394	343709	28,22,20,16,14,10,4
396*	337224	28,22,17,11,7,4
402*	336690	28,22,16,12,9,4
404*	343722	28,22,20,16,14,10,7,4
406	336661	28,22,16,12,6
407	338258	28,22,18,12,6
410*	336614	28,22,16,11,7,4
411	325731	28,20,16,12,8,4
412*	365326	28,24,16,12,7,4
413	336716	28,22,16,12,10,6
415	339300	28,22,18,16,12,10,6
417	336682	28,22,16,12,8,6
422	371960	28,24,20,16,8,6
423*	338580	28,22,18,14,11,6
425	338279	28,22,18,12,8,6
426	336949	28,22,16,14,10,6
427	372015	28,24,20,16,10,8,6
428*	372468	28,24,20,16,14,12,7,4
429*	337287	28,22,17,12,8,4
430	338635	28,22,18,14,12,6
434	339156	28,22,18,16,10,6
435*	338363	28,22,18,13,8,4
436*	338266	28,22,18,12,7,4
437	343337	28,22,20,16,10,6
438	338512	28,22,18,14,8,6
444*	339253	28,22,18,16,12,7,4
446	367957	28,24,18,16,12,10,6
447*	530063	29,21,19,15,11,6
448*	338643	28,22,18,14,12,7,4
450*	338829	28,22,18,15,11,8,4
451*	544635	29,23,17,12,6
452*	527110	29,21,17,13,11,7,4
453	371350	28,24,20,14,8,6
454*	526877	29,21,17,11,7,4
455*	340426	28,22,19,15,11,8,4
456*	338520	28,22,18,14,9,4
458	336677	28,22,16,12,8,4
459*	544580	29,23,17,11,6
460*	343434	28,22,20,16,12,7,4
462*	337389	28,22,17,13,9,4
464*	338376	28,22,18,13,9,4
465	338274	28,22,18,12,8,4

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AMS Classification Numbers: 11B39, 11B37, 11Y55

GENERALIZED HAPPY NUMBERS

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1. HAPPY NUMBERS

Let $S_2 : \mathbf{Z}^+ \to \mathbf{Z}^+$ denote the function that takes a positive integer to the sum of the squares of its digits. More generally, for $e \geq 2$ and $0 \leq a_i \leq 9$, define S_e by

$$S_e\left(\sum_{i=0}^{n} a_i 10^i\right) = \sum_{i=0}^{n} a_i^e .$$

A positive integer a is a *happy number* if, when S_2 is applied to a iteratively, the resulting sequence of integers (which we will call the S_2*-sequence* of a) eventually reaches 1. Thus a is a happy number if and only if there exists some $m \geq 0$ such that $S_2^m(a) = 1$. For example, 13 is a happy number since $S_2^2(13) = 1$.

Notice that 4 is not a happy number. Its S_2-sequence is periodic with $S_2^8(4) = 4$. It is simple to verify that every positive integer less than 100 either is a happy number or has an S_2-sequence that enters the cyclic S_2-sequence of 4. It can further be shown that, for each positive integer $a \geq 100$, $S_2(a) < a$. This leads to the following well-known theorem. (See [2] for a complete proof.)

Theorem 1: Given $a \in \mathbf{Z}^+$, there exists $n \geq 0$ such that $S_2^n(a) = 1$ or 4.

Generalizing the concept of a happy number, we say that a positive integer a is a *cubic happy number* if its S_3-sequence eventually reaches 1. We note that a positive integer can be a cubic happy number only if it is congruent to 1 modulo 3. This follows immediately from the following lemma.

Lemma 2: Given $a \in \mathbf{Z}^+$, for all m, $S_3^m(a) \equiv a \pmod 3$.

Proof: Let $a = \sum_{i=0}^{n} a_i 10^i$, $0 \leq a_i \leq 9$. Using the fact that, for each i, $a_i^3 \equiv a_i \pmod 3$ and $10^i \equiv 1 \pmod 3$, we get

$$S_3(a) = S_3\left(\sum_{i=0}^{n} a_i 10^i\right) \stackrel{\text{def}}{=} \sum_{i=0}^{n} a_i^3 \equiv \sum_{i=0}^{n} a_i \equiv \sum_{i=0}^{n} a_i 10^i = a \pmod 3.$$

Thus, by a simple induction argument, we get that, for all $m \in \mathbf{Z}^+$, $S_3^m(a) \equiv a \pmod 3$. □

The fixed points and cycles of S_3 are characterized in Theorem 3, which can be found without proof in [1].

Theorem 3: The fixed points of S_3 are 1, 153, 370, 371, and 407; the cycles are $136 \to 244 \to 136$, $919 \to 1459 \to 919$, $55 \to 250 \to 133 \to 55$, and $160 \to 217 \to 352 \to 160$. Further, for any positive integer a:

- If $a \equiv 0 \pmod 3$, then there exists an m such that $S_3^m(a) = 153$.
- If $a \equiv 1 \pmod 3$, then there exists an m such that $S_3^m(a) = 1, 55, 136, 160, 370$, or 919.
- If $a \equiv 2 \pmod 3$, then there exists an m such that $S_3^m(a) = 371$ or 407.

Note that the second part of the theorem follows from the first half and Lemma 2. Rather than prove the first part here, we state and prove a generalization of Theorems 1 and 3 in the following section.

2. VARIATIONS OF BASE

By expressing numbers in different bases, we can generalize happy numbers even further.

Fix $b \geq 2$. Let $a = \sum_{i=0}^{n} a_i b^i$ with $0 \leq a_i \leq b-1$. Let $e \geq 2$. We then define the function $S_{e,b} : \mathbf{Z}^+ \to \mathbf{Z}^+$ by

$$S_{e,b}(a) = S_{e,b}\left(\sum_{i=0}^{n} a_i b^i\right) = \sum_{i=0}^{n} a_i^e.$$

If an $S_{e,b}$ sequence reaches 1, we call a an e-power b-happy number.

Theorem 4: For all $e \geq 2$, every positive integer is an e-power 2-happy number.

Proof: Fix e. Let $a = \sum_{i=0}^{n} a_i 2^i$, $0 \leq a_i \leq 1$, $a_n > 0$. Then

$$a - S_{e,2}(a) = \sum_{i=0}^{n} a_i 2^i - \sum_{i=0}^{n} a_i^e = \sum_{i=0}^{n} a_i 2^i - \sum_{i=0}^{n} a_i = \sum_{i=0}^{n} a_i (2^i - 1).$$

Note that none of the terms can be negative. Thus, if $n \geq 1$, $a - S_{e,2}(a) > 0$. So, for $a \neq 1$, $S_{e,2}(a) < a$. With this fact, it is easy to prove by induction that every positive integer is an e-power 2-happy number. □

Again, we ask: What are the fixed points and cycles generated when these functions are iterated? We give the answers for $S_{2,b}$, $2 \leq b \leq 10$, in Table 1 and for $S_{3,b}$, $2 \leq b \leq 10$, in Table 2.

TABLE 1. Fixed points and cycles of $S_{2,b}$, $2 \leq b \leq 10$

Base	Fixed Points and Cycles
2	1
3	1, 12, 22 2 → 11 → 2
4	1
5	1, 23, 33 4 → 31 → 20 → 4
6	1 32 → 21 → 5 → 41 → 25 → 45 → 105 → 42→ 32
7	1, 13, 34, 44, 63 2 → 4 → 22 → 11 → 2 16 → 52 → 41 → 23 → 16
8	1, 24, 64 4 → 20 → 4 5 → 31 → 12 → 5 15 → 32 → 15
9	1, 45, 55 58 → 108 → 72→ 58 82 → 75 → 82
10	1 4 → 16 → 37 → 58 → 89 → 145 → 42 → 20 → 4

TABLE 2. Fixed points and cycles of $S_{3,b}$, $2 \le b \le 10$

Base	Fixed Points and Cycles
2	1
3	1, 122 2 → 22 → 121 → 101 → 2
4	1, 20, 21, 203, 313, 130, 131, 223, 332
5	1, 103, 433 14 → 230 → 120 → 14
6	1, 243, 514, 1055 13 → 44 → 332 → 142 → 201 → 13
7	1, 12, 22, 250, 251, 305, 505 2 → 11 → 2 13 → 40 → 121 → 13 23 → 50 → 236 → 506 → 665 → 1424 → 254 → 401 → 122 → 23 51 → 240 → 132 → 51 160 → 430 → 160 161 → 431 → 161 466 → 1306 → 466 516 → 666 → 1614 → 552 → 516
8	1, 134, 205, 463, 660, 661 662 → 670 → 1057 → 725 → 734 → 662
9	1, 30, 31, 150, 151, 570, 571, 1388 38 → 658 → 1147 → 504 → 230 → 38 152 → 158 → 778 → 1571 → 572 → 578 → 1308 → 660 → 530 → 178 → 1151 → 152 638 → 1028 → 638 818 → 1358 → 818
10	1, 153, 371, 407, 370 55 → 250 → 133 → 55 136 → 244 → 136 160 → 217 → 352 → 160 919 → 1459 → 919

It is easy to verify that each entry in the tables above is, indeed, a fixed point or cycle. Theorem 5 asserts that the tables are, in fact, complete.

Theorem 5: Tables 1 and 2 give all of the fixed points and cycles of $S_{2,b}$ and $S_{3,b}$, respectively, for $2 \le b \le 10$.

The proof of Theorem 5 uses the same techniques as the proof of Theorem 1 given in [2]. First, we find a value N for which $S_{e,b}(a) < a$ for all $a \ge N$. This implies that, for each $a \in \mathbf{Z}^+$, there exists some $m \in \mathbf{Z}^+$ such that $S_{e,b}^m(a) < N$. Then a direct calculation for each $a < N$ completes the process and Theorem 5 is proven. Lemma 6 provides an N for $e = 2$ and all bases $b \ge 2$ while Lemma 8 does the same for $e = 3$.

Lemma 6: If $b \ge 2$ and $a \ge b^2$, then $S_{2,b}(a) < a$.

Proof: Let $a = \sum_{i=0}^n a_i b^i$. We have

$$a - S_{2,b}(a) = \sum_{i=0}^n a_i b^i - \sum_{i=0}^n a_i^2 = \sum_{i=0}^n a_i(b^i - a_i).$$

Every term in the final sum is positive with the possible exception of the $i = 0$ term which is at least $(b-1)(1-(b-1))$. It is not difficult to show that the $i = n$ term is minimal if $a_n = 1$. From

$a \geq b^2 = 100_{(b)}$, it follows that $n \geq 2$. So the $i = n$ term is at least $1(b^2 - 1)$. Thus, $a - S_{2,b}(a) > b^2 - 1 + (b-1)(1-(b-1)) = 3b - 3 > 0$, since $b \geq 2$. Hence, for all $a \geq b^2$, $S_{2,b}(a) < a$. □

Using induction, Corollary 7 is immediate.

Corollary 7: For each $a \in \mathbf{Z}^+$, there is an $m \in \mathbf{Z}^+$ such that $S_{2,b}^m(a) < b^2$.

This completes the argument for $e = 2$. Now we consider $e = 3$.

Lemma 8: If $b \geq 2$ and $a \geq 2b^3$, then $S_{3,b}(a) < a$.

Proof: The proof of Theorem 4 gives an even stronger result for $b = 2$, so we will assume $b > 2$. Using the notation from above, we have

$$a - S_{3,b}(a) = \sum_{i=0}^{n} a_i b^i - \sum_{i=0}^{n} a_i^3 = \sum_{i=0}^{n} a_i(b^i - a_i^2).$$

The $i = 0$ term is at least $(b-1)(1-(b-1)^2)$ and the $i = 1$ term is at least $(b-1)(b-(b-1)^2)$. The remaining terms are all nonnegative. Since $a \geq 2b^3 = 2000_{(b)}$, $n \geq 3$ and if $n = 3$, then $a_3 \geq 2$. So, if $n = 3$, the a_n term is at least $2(b^3 - 4)$. If $n > 3$, then the a_n term is at least $b^4 - 1 > 2(b^3 - 4)$. Thus,

$$\begin{aligned} a - S_{3,b}(a) &\geq a_n(b^3 - a_n^2) + a_1(b - a_1^2) + a_0(1 - a_0^2) \\ &\geq 2(b^3 - 4) + (b-1)(b-(b-1)^2) + (b-1)(1-(b-1)^2) \\ &= 7b^2 - 6b - 7 > 0 \end{aligned}$$

since $b > 2$. Hence, for all $a \geq 2b^3$, $S_{3,b}(a) < a$. □

Corollary 9: For each $a \in \mathbf{Z}^+$, there is an $m \in \mathbf{Z}^+$ such that $S_{3,b}^m(a) < 2b^3$.

Theorem 5 now follows from a direct calculation of the $S_{2,b}$-sequences for all $a < b^2$ and the $S_{3,b}$-sequences for all $a < 2b^3$. These calculations are easily completed with a computer.

We conclude with two general theorems concerning congruences. If, for given e, b, and d, $S_{e,b}^m(a) \equiv a \pmod{d}$ for all a and m, then, as in Lemma 2, all e-power b-happy numbers must be congruent to 1 modulo d. Thus, the following theorems yield a great deal of information concerning generalized happy numbers. In particular, bounds on the densities of the numbers are immediate.

Theorem 10: Let p be prime and let $b \equiv 1 \pmod{p}$. Then, for any $a \in \mathbf{Z}^+$ and $m \in \mathbf{Z}^+$, $S_{p,b}^m(a) \equiv a \pmod{p}$.

Proof: Let $a = \sum_{i=0}^{n} a_i b^i$. By Fermat, $a_i^p \equiv a_i \pmod{p}$ for all i. Thus,

$$S_{p,b}(a) = S_{p,b}\left(\sum_{i=0}^{n} a_i b^i\right) = \sum_{i=0}^{n} a_i^p \equiv \sum_{i=0}^{n} a_i \equiv \sum_{i=0}^{n} a_i b^i = a \pmod{p}.$$

Using induction, we see that, for all $m \in \mathbf{Z}^+$, $S_{p,b}^m(a) \equiv a \pmod{p}$. □

Corollary 11: If a is a (2-power) b-happy number with b odd, then a must be odd. In general, if a is a p-power b-happy number with $b \equiv 1 \pmod{p}$ for some prime p, then $a \equiv 1 \pmod{p}$.

Theorem 12: Let $b \equiv 1 \pmod{\gcd(6, b-1)}$. Then, for any $a \in \mathbf{Z}^+$ and $m \in \mathbf{Z}^+$, $S_{3,b}^m(a) \equiv a \pmod{\gcd(6, b-1)}$.

Proof: Let $a = \sum_{i=0}^n a_i b^i$ and $d = \gcd(6, b-1)$. If $d = 1$, then the theorem is vacuous. For $d = 2$, note that $a^3 \equiv a \pmod 2$. Since $b \equiv 1 \pmod 2$, we have

$$S_{3,b}(a) = S_{3,b}\left(\sum_{i=0}^n a_i b^i\right) = \sum_{i=0}^n a_i^3 \equiv \sum_{i=0}^n a_i \equiv \sum_{i=0}^n a_i b^i = a \pmod 2,$$

and induction completes the argument. The case $d = 3$ is immediate from Theorem 10. Finally, $d = 6$ follows from the cases $d = 2$ and $d = 3$. □

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ELEMENTARY PROBLEMS AND SOLUTIONS

Edited by
Russ Euler and Jawad Sadek

Please submit all new problem proposals and corresponding solutions to the Problems Editor, DR. RUSS EULER, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468. All solutions to others' proposals must be submitted to the Solutions Editor, DR. JAWAD SADEK, Department of Mathematics and Statistics, Northwest Missouri State University, 800 University Drive, Maryville, MO 64468.

If you wish to have receipt of your submission acknowledged, please include a self-addressed, stamped envelope.

Each problem and solution should be typed on separate sheets. Solutions to problems in this issue must be received by May 15, 2002. If a problem is not original, the proposer should inform the Problem Editor of the history of the problem. A problem should not be submitted elsewhere while it is under consideration for publication in this Journal. Solvers are asked to include references rather than quoting "well-known results".

BASIC FORMULAS

The Fibonacci numbers F_n and the Lucas numbers L_n satisfy

$$F_{n+2} = F_{n+1} + F_n, \ F_0 = 0, \ F_1 = 1;$$
$$L_{n+2} = L_{n+1} + L_n, \ L_0 = 2, \ L_1 = 1.$$

Also, $\alpha = (1+\sqrt{5})/2$, $\beta = (1-\sqrt{5})/2$, $F_n = (\alpha^n - \beta^n)/\sqrt{5}$, and $L_n = \alpha^n + \beta^n$.

PROBLEMS PROPOSED IN THIS ISSUE

B-925 ***Proposed by José Luis Díaz & Juan J. Egozcue, Universitat Politècnica de Catalunya, Terrassa, Spain***

Prove that $\sum_{k=0}^{n} F_{k+1}^2$ divides

$$\sum_{k=0}^{n} F_{k+1}^2 [F_{k+2} + (-1)^k F_k] \ \text{ for } n \geq 0.$$

B-926 ***Proposed by Ovidiu Furdui, Western Michigan University, Kalamazoo, Michigan***

If $1 < a < \alpha$, evaluate

$$\lim_{n\to\infty} \left(a^{\frac{1}{F_1}+\frac{1}{F_2}+\cdots+\frac{1}{F_n}} - a^{\frac{1}{F_1}+\frac{1}{F_2}+\cdots\frac{1}{F_{n-1}}} \right).$$

B-927 ***Proposed by R. S. Melham, University of Technology, Sydney, Australia***

G. Candido ["A Relationship between the Fourth Powers of the Terms of the Fibonacci Series," *Scripta Mathematica* **17.3-4** (1951):230] gave the following fourth-power relation:

$$2(F_n^4 + F_{n+1}^4 + F_{n+2}^4) = (F_n^2 + F_{n+1}^2 + F_{n+2}^2)^2.$$

Generalize this relation to the sequence defined for all integers n by

$$W_n = pW_{n-1} - qW_{n-2}, \ W_0 = a, W_1 = b,$$

B-928 ***Proposed by H.-J. Seiffert, Berlin, Germany***

The Fibonacci polynomials are defined by $F_0(x)=0$, $F_1(x)=1$, $F_{n+2}(x)=xF_{n+1}(x)+F_n(x)$ for $n\geq 0$. Show that, for all complex numbers x and all nonnegative integers n,

$$F_{2n+1}(x)=\sum_{k=0}^{n}(-1)^{\lceil k/2\rceil}\binom{n-\lceil k/2\rceil}{\lfloor k/2\rfloor}(x^2+2)^{n-k},$$

where $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ denote the floor- and ceiling-function, respectively.

B-929 ***Proposed by Harvey J. Hindin, Huntington Station, NY***

Prove that:

A) $F_{2N}=(1/5^{1/2})\sum_{K=0}^{2N-1}P_K(5^{1/2}/2)P_{2N-1-K}(5^{1/2}/2)$ for $N\geq 1$

and

B) $L_{2N+1}=\sum_{K=0}^{2N}P_K(5^{1/2}/2)P_{2N-K}(5^{1/2}/2)$ for $N\geq 0$,

where $P_K(x)$ is the Legendre polynomial given by $P_0(x)=1$, $P_1(x)=x$, and the recurrence relation $(K+1)P_{K+1}(x)=(2K+1)xP_K(x)-KP_{K-1}(x)$.

SOLUTIONS

Divisible or Not Divisible; That Is, by 5

B-911 ***Proposed by M. N. Deshpande, Institute of Science, Nagpur, India***
(Vol. 39, no. 1, February 2001)

Determine whether $L_n+2(-1)^mL_{n-2m-1}$ is divisible by 5 for all positive integers m and n.

Solution by Pantelimon Stănică, Montgomery, AL

We prove that the expression is divisible by 5 for all positive integers m, n. Formula (17b) on page 177 in S. Vajda'a *Fibonacci & Lucas Numbers, and the Golden Section* (Ellis Horwood) states: $L_{p+k}-(-1)^kL_{p-k}=5F_pF_k$. Taking $p=n-m$, $k=m+1$, and $p=n-m-1$, $k=m$, we get $L_{n+1}-(-1)^{m+1}L_{n-2m-1}=5F_{n-m}F_{m+1}$ and $L_{n-1}-(-1)^mL_{n-2m-1}=5F_{n-m-1}F_m$. Subtracting the second formula from the first, and using the definition of L_n, we obtain

$$L_n+2(-1)^mL_{n-2m-1}=5(F_{n-m}F_{m+1}-F_{n-m-1}F_m),$$

which implies the claim.

Also solved by Brian D. Beasley, Paul Bruckman, L. A. G. Dresel, Ovidiu Furdui, Russell Hendel, Walther Janous, Harris Kwong, Carl Libis, H.-J. Seiffert, James Sellers, and the proposer.

From a Product to a Sum

B-912 ***Proposed by the editor***
(Vol. 39, no. 1, February 2001)

Express $F_{n+(n \bmod 2)}\cdot L_{n+1-(n \bmod 2)}$ as a sum of Fibonacci numbers.

Solution by Charles K. Cook, University of South Carolina at Sumter, Sumter, SC

The following formulas from [1] will be used:

$$(\mathrm{I}_6)\ \sum_{j=1}^{n} F_{2j} = F_{2n+1} - 1 \quad \text{and} \quad (\mathrm{I}_{23})\ F_{n+p} - F_{n-p} = F_n L_p \ \text{ if } p \text{ is odd.}$$

Case 1: n even $\Rightarrow n \bmod 2 = 0$. So

$$F_n \cdot L_{n+1} = F_{n+n+1} - F_{n-n-1} = F_{2n+1} - F_{-1} = F_{2n+1} - 1 = \sum_{j=1}^{n} F_{2j}.$$

Case 2: n odd $\Rightarrow n \bmod 2 = 1$. So

$$F_{n+1} \cdot L_n = F_{2n+1} - F_1 = F_{2n+1} - 1 = \sum_{j=1}^{n} F_{2j}.$$

Thus,

$$F_{n+(n \bmod 2)} \cdot L_{n+1-(n \bmod 2)} = \sum_{j=1}^{n} F_{2j}.$$

Reference

1. V. E. Hoggatt, Jr. *Fibonacci and Lucas Numbers.* Santa Clara, CA: The Fibonacci Association, 1979.

Also solved by Brian D. Beasley (3 solutions), Paul Bruckman, L. A. G. Dresel, Ovidiu Furdui, Pentti Haukkanen, Russell Hendel, Steve Hennagin, Walther Janous, Harris Kwong, Carl Libis, H.-J.-Seiffert, James Sellers, Pantelimon Stănică, and the proposer.

A "Constant" Search

B-913 ***Proposed by Herbert S. Wilf, University of Pennsylvania, Philadelphia, PA (Vol. 39, no. 1, February 2001)***

Fix an integer $k \geq 1$. The Fibonacci numbers satisfy an "accelerated" recurrence of the form $F_{n2^k} = \alpha_k F_{(n-1)2^k} - F_{(n-2)2^k}$ $(n = 2, 3, \ldots)$ with $F_0 = 0$ and F_{2^k} to start the recurrence. For example, when $k = 1$, we have $F_{2n} = 3F_{2(n-1)} - F_{2(n-2)}$ $(n = 2, 3, \ldots; F_0 = 0; F_2 = 1)$.

a. Find the constant α_k by identifying it as a certain member of a sequence that is known by readers of these pages.

b. Generalize this result by similarly identifying the constant β_m for which the accelerated recurrence $F_{mn+h} = \beta_m F_{m(n-1)+h} + (-1)^{m+1} F_{m(n-2)+h}$, with appropriate initial conditions, holds.

Solution by N. Gauthier, Kingston, ON

Case a is deduced from Case b by setting $h = 0$ and $m = 2^k$ for k a positive integer, so we solve Case b only. The sought answer is $\beta_m = L_m$ for values of n such that $m(n-1)+h \neq 0$; for $m(n-1)+h = 0$, β_m can be arbitrary but finite, since $F_0 = 0$. The former is of interest and we have, from the definition, that

$$\beta_m = \frac{[\alpha^{mn+h} + (-1)^m \alpha^{m(n-2)+h}] - [\beta^{mn+h} + (-1)^m \beta^{m(n-2)+h}]}{[\alpha^{m(n-1)+h} - \beta^{m(n-1)+h}]}$$

$$= \frac{[\alpha^{m(n-1)+h}(\alpha^m + (-1)^m \alpha^{-m}) - \beta^{m(n-1)+h}(\beta^m + (-1)^m \beta^{-m})]}{[\alpha^{m(n-1)+h} - \beta^{m(n-1)+h}]}$$

$$= [\alpha^m + \beta^m] = L_m$$

since $\alpha^{-m} = (-1)^m \beta^m$ and vice versa. This completes the proof.

Also solved by Brian D. Beasley, Paul Bruckman, L. A. G. Dresel, Ovidiu Furdui, Walther Janous, Harris Kwong, H.-J. Seiffert, Pantelimon Stănică, and the proposer.

A "Product and a Sum" Inequality

B-914 ***Proposed by José Luis Díaz, Universitat Politècnica de Catalunya, Terrassa, Spain (Vol. 39, no. 1, February 2001)***

Let $n \geq 2$ be an integer. Prove that

$$\prod_{k=2}^{n} \left\{ \sum_{j=1}^{k} \frac{1}{(F_{k+2} - F_j - 1)^2} \right\} \geq \frac{1}{F_2 F_{n+1}} \left(\frac{n}{F_3 F_4 \dots F_n} \right)^2.$$

Solution by H.-J. Seiffert, Berlin, Germany

We first prove that

$$kF_k F_{k+1} \geq (F_{k+2} - 1)^2 \text{ for } k \geq 1. \tag{1}$$

For $k = 1, 2, 3, 4, 5, 6,$ and 7, this can be verified directly. If $k \geq 8$, then $F_{k+2} < 2F_{k+1} < 4F_k$, and therefore $kF_k F_{k+1} > (k/8)F_{k+2}^2 \geq F_{k+2}^2 > (F_{k+2} - 1)^2$.

Let $k \geq 2$. The function

$$f(x) = \frac{1}{(F_{k+2} - x - 1)^2}, \quad 0 \leq x \leq F_k,$$

is convex, as is seen from its second derivative. Applying Jensen's Inequality gives

$$\sum_{j=1}^{k} f(F_j) \geq kf\left(\frac{1}{k} \sum_{j=1}^{k} F_j \right).$$

From (I_1) of [1], we know that $\sum_{j=1}^{k} F_j = F_{k+2} - 1$. Hence,

$$\sum_{j=1}^{k} \frac{1}{(F_{k+2} - F_j - 1)^2} \geq \left(\frac{k}{k-1} \right)^2 \frac{k}{(F_{k+2} - 1)^2},$$

which, by (1), may be weakened to

$$\sum_{j=1}^{k} \frac{1}{(F_{k+2} - F_j - 1)^2} \geq \left(\frac{k}{k-1} \right)^2 \frac{1}{F_k F_{k+1}}.$$

Taking the product over $k = 2, 3, \dots, n$, $n \geq 2$, we obtain the desired inequality.

Reference

1. V. E. Hoggatt, Jr. *Fibonacci and Lucas Numbers*. Santa Clara, CA: The Fibonacci Association, 1979.

Also solved by Paul Bruckman, Walther Janous, and the proposer.

Editor's Comment: *Walther Janous actually improved the inequality by elementary means and showed the right-hand side to be greater than* $n!/2^{2(n-1)}\prod_{k=2}^{n}(F_{k+1}-1)^2$.

A "Double Sum" Inequality

B-915 ***Proposed by Mohammad K. Azarian, University of Evansville, Evansville, IN (Vol. 39, no. 1, February 2001)***

If $|x|\le 1$, prove that

$$\left|\sum_{i=1}^{n}\sum_{j=1}^{i} i2^{-j-1}F_j x^{i-1}\right| < n^3.$$

Solution by Paul Bruckman, Sacramento, CA

A stronger result is actually true, namely:

$$\left|\sum_{i=1}^{n}\sum_{j=1}^{i} i2^{-j-1}F_j x^{i-1}\right| \le n(n+1)/2, \text{ whenever } |x|\le 1.$$

Let $G(x,n)$ denote the quantity within the absolute value bars. Then

$$|G(x,n)|\le|G(1,n)|=\sum_{i=1}^{n}\sum_{j=1}^{i} i2^{-j-1}F_j \le \sum_{i=1}^{n} i\sum_{j=1}^{\infty}2^{-j-1}F_j.$$

Now $\sum_{j=1}^{\infty}u^{j-1}F_j=(1-u-u^2)^{-1}$, provided $|u|\le\alpha^{-1}$. Setting $u=1/2$, we obtain $\sum_{j=1}^{\infty}2^{-j-1}F_j=1$. Thus,

$$|G(x,n)|\le\sum_{i=1}^{n} i = n(n+1)/2.$$

Note that $n(n+1)/2\le n^3$, with equality iff $n=1$. Since $G(x,1)=1/4$, we see that the result indicated in the statement of the problem is certainly true.

Also solved by Ovidiu Furdui, H.-J. Seiffert, and the proposer.

A Response to Gauthier's Comments on the Bruckman Conjecture

A Comment by Paul Bruckman

First, I would like to make a slight correction. Although I appreciate being referred to as "Professor Bruckman," I must regretfully inform the world that I am no longer teaching math, having returned to consulting work for a private firm.

Secondly, I am sincerely flattered to have my name associated with a certain set of polynomials (the $P_r(n)$ of Dr. Gauthier's note). Before accepting this honor, however, I would like to be sure that these polynomials are indeed new in the literature; I would be loath to usurp someone else's rightful niche in mathematical history.

I am grateful to Dr. Gauthier for pointing out the corrected version of my conjecture. I might have discovered this for myself, had I taken the time and effort to develop the correct polynomial expressions, as Dr. Gauthier has obviously done.

It should be mentioned that there is an advanced problem proposed by Dr. Gauthier (H-568) in the last issue [*The Fibonacci Quarterly* **38.5** (2000):473; corrected **39.1** (2001):91-92] which is highly interesting and bears some superficial resemblance to my problem B-871 [**37.1** (1999): 85]. However, unlike the polynomials $P_r(n)$, the "Gauthier functions" $f_m(n)$ are rational functions. I have submitted my solution for Problem H-568 to the Advanced Problems Editor.

The remainder of this letter is devoted to indicating some of my subsequent research in response to Dr. Gauthier's comments.

Following Gauthier's notation, we may define the functions $P_r(n)$ as follows:

$$P_r(n) \equiv \binom{2n}{n}^{-1} \sum_{k=0}^{2n} \binom{2n}{k} |n-k|^{2r-1}. \tag{1}$$

As Dr. Gauthier correctly pointed out, what I *should* have originally conjectures was:

$$P_r(n) \text{ is a polynomial in } n \text{ of degree } r \tag{2}$$

(that is, leaving out the erroneous modifier "monic").

Actually, we can prove a somewhat stronger result than (2), namely: $P_r(n)$ is a polynomial in n of degree r, with its first two leading terms given by

$$P_r(n) = (r-1)!n^r - (r-2)!\binom{r}{3}n^{r-1} + \cdots. \tag{3}$$

Towards this end, we first demonstrate that the $P_r(n)$'s satisfy the recurrence relation:

$$P_{r+1}(n) = n^2(P_r(n) - P_r(n-1)), \ r = 1, 2, \ldots. \tag{4}$$

By means of (4), with $P_1(n) = n$, we readily obtain the expressions for $P_r(n)$ indicated by Gauthier in his note, for $r = 1, 2, 3, 4, 5$. The proof of (4) is straightforward, following from the definition of the $P_r(n)$. In turn, (4) implies (3), as can be demonstrated by induction.

What appears to be a more difficult problem is to obtain a general expansion (for all the terms) of $P_r(n)$. Once obtained, this might reveal other properties of the $P_r(n)$, and may possibly demonstrate that they are special cases of well-studied polynomials with known properties.

If it should turn out that these are indeed new polynomials, they may be expected to yield additional research and should generate further interest in their properties. It already seems interesting enough that the special form given in the definitions of the polynomials $P_r(n)$ and $f_m(n)$ (as given in Dr. Gauthier's note and in H-568, respectively) should yield polynomial functions and rational functions, respectively.

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ADVANCED PROBLEMS AND SOLUTIONS

Edited by
Raymond E. Whitney

Please send all communications concerning ADVANCED PROBLEMS AND SOLUTIONS to RAYMOND E. WHITNEY, MATHEMATICS DEPARTMENT, LOCK HAVEN UNIVERSITY, LOCK HAVEN, PA 17745. This department especially welcomes problems believed to be new or extending old results. Proposers should submit solutions or other information that will assist the editor. To facilitate their consideration, all solutions should be submitted on separate signed sheets within two months after publication of the problems.

PROBLEMS PROPOSED IN THIS ISSUE

H-577 ***Proposed by Paul S. Bruckman, Sacramento, CA***

Define the following constant: $C \equiv \prod_p \{1-1/p(p-1)\}$ as an infinite product over all primes p.

(A) Show that

$$\sum_{n=1}^{\infty} \mu(n)/\{n\phi(n)\},$$

where $\mu(n)$ and $\phi(n)$ are the Möbius and Euler functions, respectively.

(B) Let

$$P(n) = \sum_{d|n} \mu(n/d) L_d.$$

It was shown in the solution to H-517 (see Vol. 35, no. 4 (1997), pp. 381-82) that $n|P(n)$.

Show that

$$C = \prod_{n=2}^{\infty} \{\zeta(n)\}^{-P(n)/n},$$

where ζ is the Riemann zeta function.

Note: C is the conjectured density of primes p such that $Z(p) = p-(5/p)$; see P. G. Anderson & P. S. Bruckman, "On the a-Densities of the Fibonacci Sequence," *NNTDM* **6.1** (2000):1-13. Approximately, $C = 0.37395581$.

H-578 ***Proposed by N. Gauthier & J. R. Gosselin, Royal Military College of Canada***

In Problem B-863, S. Rabinowitz gave a set of four 2×2 matrices which are particular solutions of the matrix equation

$$X^2 = X + I, \tag{1}$$

where I is the unit matrix [*The Fibonacci Quarterly* **36.5** (1998); solved by H. Kappus, **37.3** (1999)]. The matrices presented by Rabinowitz are not diagonal (i.e., they are nontrivial), have determinant -1 and trace $+1$.

a. Find the complete set $\{X\}$ of the nontrivial solutions of (1) and establish whether the properties $\det(X) = -1$ and $\text{tr}(X) = +1$ hold generally.

b. Determine the complete set $\{X\}$ of the nontrivial solutions of the generalized characteristic equation

$$X^2 = xX + yI, \tag{2}$$

for the 2×2 Fibonacci matrix sequence $X^{n+2} = xX^{n+1} + yX^n$, $n = 0, 1, 2, \ldots$, where x and y are arbitrary parameters such that $x^2/4 + y \neq 0$; obtain expressions for the determinant and for the trace.

H-579 ***Proposed by Paul S. Bruckman, Sacramento, CA***

Prove or disprove that, for all odd primes p,

$$\sum_{n=1}^{1/2(p-1)} \binom{2n}{n}(-1)^n/n \equiv 0 \pmod{p}.$$

Each quantity $1/n$ is to be interpreted as $n^{-1} \pmod{p}$.

H-580 ***Proposed by José Díaz-Barrero, Politechnic University of Cataluya, Barcelona, Spain***

Let

$$A(z) = \sum_{k=0}^{n} a_k z^k$$

be a monic complex polynomial. Show that all its zeros lie in the disk $C = \{z \in \mathbb{C} : |z| < r\}$, where

$$r = \max_{1\le k\le n}\left\{\sqrt[k]{\frac{L_{j+3n}}{C(n,k)2^k L_{j+k}}|a_{n-k}|}\right\},\ j = 0, 1, 2, \ldots.$$

SOLUTIONS

Sum Problem

H-566 ***Proposed by N. Gauthier, Royal Military College of Canada (Vol. 38, no. 4, August 2000)***

Let $\phi_n := \pi/2n$, where n is a positive integer, and set $L_n = a^n + b^n$, $F_n = (a^n - b^n)/(a-b)$, where $a = \frac{1}{2}(u + \sqrt{u^2-4})$, $b = \frac{1}{2}(u - \sqrt{u^2-4})$, $u \neq \pm 2$, and show that, for $n \geq 2$,

$$S_n(u) := \sum_{k=1}^{n-1} \frac{1}{1 + \left(\frac{u+2}{u-2}\right)tg^2(k\phi_n)} = -\frac{1}{2} + \frac{n}{2(u+2)^2 F_n}[L_{n+1} + 3L_n + 3L_{n-1} + L_{n-2}].$$

Solution by Paul S. Bruckman, Berkeley, CA

The notation "tg^2" in the statement of the problem evidently means "$\tan^2$". We make use of a general identity which was derived in this solver's solution of Problem H-559, Part (a) (Feb. 2000) by the proposer of this problem. This identity is the following, valid for all integers $n \geq 1$, complex x and y with $x^2 \neq y^2$:

$$\frac{n(x^n + y^n)}{(x^2 - y^2)(x^n - y^n)} = \sum_{k=1}^{n} (x^2 - 2xy\cos 2k\pi/n + y^2)^{-1}. \tag{1}$$

If we set $x = a$ and $y = b$ in (1), we obtain (using the proposer's notation):

$$nL_n/\{u(u^2-4)F_n\} = \sum_{k=1}^{n} (u^2 - 2 - 2\cos 2k\pi/n)^{-1}. \tag{2}$$

We may transform the sum in the right member of (2) as follows:

$$u^2-2-2\cos 2k\pi/n=u^2-4\cos^2 k\pi/n,$$

so the sum becomes:

$$1/2u\sum_{k=1}^{n}\{(u-2\cos k\pi/n)^{-1}+(u+2\cos k\pi/n)^{-1}\}.$$

Therefore,

$$\begin{aligned}2nL_n/F_n&=(u^2-4)\sum_{k=1}^{n}\{(u-2\cos k\pi/n)^{-1}+(u+2\cos k\pi/n)^{-1}\}\\&=(u^2-4)\sum_{k=1}^{n}\{(u+2-4\cos^2 k\phi_n)^{-1}+(u-2+4\cos^2 k\phi_n)^{-1}\}.\end{aligned}$$

Let D_k denote $1+\{(u+2)/(u-2)\}\tan^2 k\phi_n$. We note that the transformation $k\to n-k$ transforms $\tan k\phi_n$ to $1/\tan k\phi_n$. Then using standard trigonometric manipulations, we obtain after some effort:

$$\begin{aligned}2nL_n/F_n&=\sum_{k=1}^{n}\{u-2+4/D_k\}+\sum_{k=0}^{n-1}\{u+2-4(u+2)/(u-2)\tan^2 k\phi_n/D_k\}\\&=(u-2)n+\sum_{k=1}^{n-1}4/D_k+(u+2)n-4\sum_{k=1}^{n-1}(D_k-1)/D_k,\end{aligned}$$

since the terms for $k=0$ and $k=n$ involving D_k vanish. Then $nL_n/F_n=un-2(n-1)+4S_n(u)$. Hence, $4S_n(u)=nL_n/F_n-(u-2)n-2$, or

$$S_n(u)=-1/2+n\{L_n-(u-2)F_n\}/4F_n. \tag{3}$$

We note that $a^2=au-1$, $b^2=bu-1$, which implies $(a+1)^2=a(u+2)$, $(b+1)^2=b(u+2)$, so $(a+1)^3=a(a+1)(u+2)$, $(b+1)^3=b(b+1)(u+2)$. The sum $L_{n+1}+3L_n+3L_{n-1}+L_{n-2}$ reduces to:

$$\begin{aligned}a^{n-2}(a^3+3a^2+3a+1)+b^{n-2}(b^3+3b^2+3b+1)&=a^{n-2}(a+1)^3+b^{n-2}(b+1)^3\\&=a^{n-1}(a+1)(u+2)+b^{n-1}(b+1)(u+2)=(u+2)(L_n+L_{n-1}).\end{aligned}$$

Therefore, letting $R_n(u)$ denote the expression in the right member of the statement of the problem, we have

$$R_n(u)=n(L_n+L_{n-1})/\{2(u+2)F_n\}-1/2. \tag{4}$$

We next prove the following identity:

$$uL_n=2L_{n-1}+(u^2-4)F_n. \tag{5}$$

Proof of (5): Let $\theta=(u^2-4)^{1/2}$. Note that $a-b=\theta$, $ab=1$. Hence,

$$\begin{aligned}2L_{n-1}+(u^2-4)F_n&=a^{n-1}(2+\theta a)+b^{n-1}(2-\theta b)\\&=a^{n-1}(1+a^2)+b^{n-1}(1+b^2)=L_{n+1}+L_{n-1}=uL_n,\end{aligned}$$

since the characteristic equation of the L_n's (and of the F_n's) is $U_{n+1}=uU_n-U_{n-1}$. □

Comparing the results of (3) and (4), we see that it suffices to prove the following:

$$(L_n+L_{n-1})/(u+2)=(L_n-(u-2)F_n\}/2,$$

which (after simplification) reduces to (5). Thus, $S_n(u) = R_n(u)$, which completes the proof.

Incidentally, although not required, we could also express the result as follows:

$$S_n(u) = a_n - a_1, \text{ where } a_n = n(F_n - F_{n-1}) / 2F_n. \tag{6}$$

Also solved by H.-J. Seiffert and the proposer.

An UnEqual Problem

H-567 ***Proposed by Ernst Herrmann, Siegburg, Germany***
(Vol. 38, no. 5, November 2000)

Let F_n denote the n^{th} Fibonacci number. For any natural number $n \geq 3$, the four inequalities

$$\begin{aligned}\frac{1}{F_n} + \frac{1}{F_{n+a_1}} &< \frac{1}{F_{n-1}} \\ &\leq \frac{1}{F_n} + \frac{1}{F_{n+a_1-1}},\end{aligned} \tag{1}$$

$$\begin{aligned}\frac{1}{F_n} + \frac{1}{F_{n+a_1}} + \frac{1}{F_{n+a_1+a_2}} &< \frac{1}{F_{n-1}} \\ &\leq \frac{1}{F_n} + \frac{1}{F_{n+a_1}} + \frac{1}{F_{n+a_1+a_2-1}},\end{aligned} \tag{2}$$

determine uniquely two natural numbers a_1 and a_2. Find the numbers a_1 and a_2 dependent on n.

Solution by H.-J. Seiffert, Berlin, Germany

It is known [see A. F. Horadam & Bro. J. M. Mahon, "Pell and Pell-Lucas Polynomials," *The Fibonacci Quarterly* **23.1** (1985):7-20, Identity (3.32)] that

$$F_{m+h}F_{m+k} - F_mF_{m+h+k} = (-1)^m F_hF_k, \; m, h, k \in \mathbf{Z}.$$

With $(m, h, k) = (n, -1, 1)$, $(n-1, 1, 2)$, $(n-1, -1, 4)$, $(n-1, -1, 5)$, $(n-1, -1, 2)$, $(n, -2, 2)$, we obtain, respectively:

$$F_{n-1}F_{n+1} - F_n^2 = (-1)^n, \tag{3}$$

$$F_nF_{n+1} - F_{n-1}F_{n+2} = (-1)^{n-1}, \tag{4}$$

$$F_{n-2}F_{n+3} - F_{n-1}F_{n+2} = 3(-1)^{n-1}, \tag{5}$$

$$F_{n-2}F_{n+4} - F_{n-1}F_{n+3} = 5(-1)^{n-1}, \tag{6}$$

$$F_{n-2}F_{n+1} - F_{n-1}F_n = (-1)^{n-1}, \tag{7}$$

$$F_{n-2}F_{n+2} - F_n^2 = (-1)^{n-1}, \tag{8}$$

valid for all integers n. Also, the easily verifiable identity

$$F_{3n} = 5F_n^3 + 3(-1)^n F_n, \; n \in \mathbf{Z}, \tag{9}$$

and the following inequalities are needed below.

Lemma 1: For all integers $n \geq 3$, it holds that

$$F_{3n-4} < F_{n-1}F_nF_{n+1} < F_{3n-3}.$$

Proof: From (3) and (9), we have

$$F_{n-1}F_nF_{n+1} = F_n^3 + (-1)^n F_n = \frac{1}{5}(F_{3n} + 2(-1)^n F_n),$$

showing that the left side is equivalent to $5F_{3n-4} < F_{3n} + 2(-1)^n F_n$. But this inequality follows from $F_{3n} = 5F_{3n-4} + 3F_{3n-5}$ and, by $n \ge 3$, $F_{3n-5} > F_n$. Using (9) with n replaced by $n-1$, we see that the right side is equivalent to $F_nF_{n+1} < 5F_{n-1}^2 - 3(-1)^n$, so that, in view of (4), we must show that $F_{n-1}F_{n+2} < 5F_{n-1}^2 - 2(-1)^n$. Since $F_{n+2} = 3F_{n-1} + 2F_{n-2}$, this inequality becomes $F_{n-2}F_{n-1} < F_{n-1}^2 - (-1)^n$. Clearly, this holds for $n = 3$. For $n \ge 4$, the latter inequality follows from $1 \le F_{n-2} \le F_{n-1} - 1$. Q.E.D.

Lemma 2: For all integers $n \ge 5$, it holds that

$$F_{n+3} < \frac{F_{n-1}F_nF_{n+2}}{F_{n-2}F_n - 1} < F_{n+4}.$$

Proof: After multiplying the left side by $F_{n-2}F_n - 1 > 0$, we see that we must prove the inequality $F_n(F_{n-2}F_{n+3} - F_{n-1}F_{n+2}) < F_{n+3}$ which, by (5), reduces to $3(-1)^n F_n < F_{n+3}$. However, this follows from $F_{n+3} = 2F_{n+1} + F_n > 3F_n$. Similarly, the right side is equivalent to

$$F_{n+4} < F_n(F_{n-2}F_{n+4} - F_{n-1}F_{n+2}).$$

Using $F_{n+2} = F_{n+3} - F_{n+1}$ and (6), we see that we must prove that $F_{n+4} < 5(-1)^{n-1}F_n + F_{n-1}F_nF_{n+1}$, which follows from $F_{n+4} = 5F_n + 3F_{n-1} < 8F_n$ and $13 < F_{n-1}F_{n+1}$; note that $n \ge 5$. Q.E.D.

Let $n \ge 3$ be an integer. Since $F_n - F_{n-1} = F_{n-2}$, it is easily verified that (1) is equivalent to

$$F_{n-2}F_{n+a_1-1} \le F_{n-1}F_n < F_{n-2}F_{n+a_1}.$$

We claim that

$$a_1 = \begin{cases} 1 & \text{if } n \text{ is odd,} \\ 2 & \text{if } n \text{ is even.} \end{cases}$$

For odd n, we must show that $F_{n-2}F_n \le F_{n-1}F_n < F_{n-2}F_{n+1}$. The left side is obviously true, while the right side follows from (7). If n is even, we must show that $F_{n-2}F_{n+1} \le F_{n-1}F_n < F_{n-2}F_{n+2}$, whose left side again follows from (7). To prove the right side, observe that $1 \le F_{n-1} \le F_n - 1$, so that $F_{n-2}F_{n+2} - F_{n-1}F_n \ge F_{n-2}F_{n+2} - F_n^2 + F_n = F_n - 1 > 0$, where we have used (8). This proves the above claim. Next, we shall prove that

$$a_2 = \begin{cases} 2n-4 & \text{if n is odd,} \\ 3 & \text{if } n = 4, \\ 2 & \text{if } n > 4 \text{ is even.} \end{cases}$$

If $n \ge 3$ is odd, then $a_1 = 1$, and based on $F_n + F_{n+1} = F_{n+2}$ and (4), it is easily seen that (2) becomes $F_{n+a_2} \le F_{n-1}F_nF_{n+1} < F_{n+a_2+1}$. Applying Lemma 1, it follows that $a_2 = 2n-4$. If $n \ge 4$ is even, then $a_1 = 2$ and, since $F_n - F_{n-1} = F_{n-2}$, (2) is equivalent to

$$F_{n+a_2+1} \le \frac{F_{n-1}F_nF_{n+2}}{F_{n-2}F_{n+2} - F_{n-1}F_n} < F_{n+a_2+2}.$$

Since, by (8), $F_{n-2}F_{n+2} = F_n^2 - 1$, and since $F_n - F_{n-1} = F_{n-2}$, these inequalities are equivalent to

$$F_{n+a_2+1} \le \frac{F_{n-1}F_nF_{n+2}}{F_{n-2}F_n - 1} < F_{n+a_2+2}.$$

Direct computation shows that $a_2 = 3$ if $n = 4$. For even $n \ge 6$, from Lemma 2, we find $a_2 = 2$. This completes the solution.

Also solved by P. Bruckman, L. A. G. Dresel, R. Martin, and the proposer.

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Announcement

TENTH INTERNATIONAL CONFERENCE ON FIBONACCI NUMBERS AND THEIR APPLICATIONS

June 24–June 28, 2002

Northern Arizona University, Flagstaff, Arizona

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CALL FOR PAPERS

The purpose of the conference is to bring together people from all branches of mathematics and science who are interested in Fibonacci numbers, their applications and generalizations, and other special number sequences. For the conference *Proceedings*, manuscripts that include new, unpublished results (or new proofs of known theorems) will be considered. A manuscript should contain an abstract on a separate page. For papers not intended for the *Proceedings*, authors may submit just an abstract, describing new work, published work or work in progress. Papers and abstracts, which should be submitted in duplicate to F. T. Howard at the address below, are due no later than May 1, 2002. Authors of accepted submissions will be allotted twenty minutes on the conference program. Questions about the conference may be directed to:

Professor F. T. Howard
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BOOKS AVAILABLE
THROUGH THE FIBONACCI ASSOCIATION

Introduction to Fibonacci Discovery by Brother Alfred Brousseau, Fibonacci Association (FA), 1965. $18.00

Fibonacci and Lucas Numbers by Verner E. Hoggatt, Jr. FA, 1972. $23.00

A Primer for the Fibonacci Numbers. Edited by Marjorie Bicknell and Verner E. Hoggatt, Jr. FA, 1972. $32.00

Fibonacci's Problem Book, Edited by Marjorie Bicknell and Verner E. Hoggatt, Jr. FA, 1974. $19.00

The Theory of Simply Periodic Numerical Functions by Edouard Lucas. Translated from the French by Sidney Kravitz. Edited by Douglas Lind. FA, 1969. $6.00

Linear Recursion and Fibonacci Sequences by Brother Alfred Brousseau. FA, 1971. $6.00

Fibonacci and Related Number Theoretic Tables. Edited by Brother Alfred Brousseau. FA, 1972. $30.00

Number Theory Tables. Edited by Brother Alfred Brousseau. FA, 1973. $39.00

Tables of Fibonacci Entry Points, Part One. Edited and annotated by Brother Alfred Brousseau. FA, 1965. $14.00

Tables of Fibonacci Entry Points, Part Two. Edited and annotated by Brother Alfred Brousseau. FA, 1965. $14.00

A Collection of Manuscripts Related to the Fibonacci Sequence—18th Anniversary Volume. Edited by Verner E. Hoggatt, Jr. and Marjorie Bicknell-Johnson. FA, 1980. $38.00

Applications of Fibonacci Numbers, Volumes 1-7. Edited by G.E. Bergum, A.F. Horadam and A.N. Philippou. Contact Kluwer Academic Publishers for price.

Applications of Fibonacci Numbers, Volume 8. Edited by F.T. Howard. Contact Kluwer Academic Publishers for price.

Generalized Pascal Triangles and Pyramids Their Fractals, Graphs and Applications by Boris A. Bondarenko. Translated from the Russian and edited by Richard C. Bollinger. FA, 1993. $37.00

Fibonacci Entry Points and Periods for Primes 100,003 through 415,993 by Daniel C. Fielder and Paul S. Bruckman. $20.00

Handling charges will be $4.00 for each book in the United States and Canada. For Foreign orders, the handling charge will be $9.00 for each book.

Please write to the Fibonacci Association, P.O. Box 320, Aurora, S.D. 57002-0320, U.S.A., for more information.