A FEW OLD PROBLEMS

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Remark. This is a list of Math problems (in no particular order) that I have found in the past. They have mainly historical interest for me. Not all of them are hard, but some may be challenging, or require some ingenious idea for their solution, or have a surprising or unexpected solution. There are also a few open questions among them.—Miguel A. Lerma

1. **The Great Theorem of Humpty-Dumpty**.[1] Assume that in the equation of Fermat's Last Theorem we replace addition with numeral concatenation (in base 10), here represented '$\oplus$':

$$x^n \oplus y^n = z^n$$

where x, y and z must be positive integers. For instance $(2, 3, 7)$ would be a solution for $n = 2$, because $2^2 \oplus 3^2 = 4 \oplus 9 = 49 = 7^2$. Other solutions are $4^2 \oplus 3^2 = 169 = 13^2$, $4^2 \oplus 3^2 = 1681 = 41^2$, $6^2 \oplus 1^2 = 361 = 19^2$, $6^2 \oplus 10^2 = 36100 = 190^2$, etc. A solution will be called *primitive* if y and z are not multiples of 10 (e.g., all solutions shown above except the last one are primitive). Prove that for $n = 2$ the above equation has infinitely many primitive solutions. (Insights into the problem for $n > 2$ and in bases different from 10 are also encouraged.)

Generalization of case $n = 2$: Prove that for every positive integer x there are infinitely many positive integers y, z not multiple of 10 such that

$$x \oplus y^2 = z^2\,.$$

2. **Rational distances on the unit circle**. Prove that there are infinitely many points on the unit circle $x^2 + y^2 = 1$ such that the distance between any two of them is a rational number.

3. **Half of a ball**. (This problem is inspired by the one about a goat tied with a rope to the border of a circular grass field so that it can eat exactly half of the grass.) Let B and B' two n-dimensional balls such that the radius of B is 1, the center of B' is in the boundary of B and the (n-dimensional) volume of the intersection of B and B' is half the volume of B. Let r_n be the radius of B'. Find the limit of r_n as $n \to \infty$.

4. **Slicing an n-ball**. What is the maximum number of parts in which an n-dimensional ball can be divided by k hyperplanes? How many of them are at the boundary of the n-ball? What is the maximum number of parts in which an n-sphere (the boundary of an $n+1$-dimensional ball) can be divided by k hyperplanes?

[1]Proposed in *Carrollia* magazine No. 19 (Dec. 1988) by Josep María Albaigès; a solution (by me) appeared in *Carrollia* magazine No. 20 (Mar. 1989)—also provided solutions to the problem for cases $n = 3$ and $n = 4$.

5. Eventually constant modulo m. Prove that for any two positive integers a and m, the following sequence is eventually constant modulo m: $a, a^a, a^{a^a}, a^{a^{a^a}}, \dots$.

6. Coloring infinite maps. In 1976 Appel and Haken proved that any (finite) planar map can be colored with 4 colors so that no two regions that share a boundary have the same color. Prove that the same is true for any planar map with infinitely many regions.

7. Ackerman's function. Ackerman's function $A : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is defined for every $m, n \geq 0$ by the following double recurrence:
(a) $A(0, n) = n + 1$ for every $n \geq 0$.
(b) $A(m, 0) = A(m - 1, 1)$ if $m \geq 1$.
(c) $A(m, n) = A(m - 1, A(m, n - 1))$ if $m, n \geq 1$.
Find close-form formulas (involving any arithmetic operations, powers and iterated powers) for $A(1, n)$, $A(2, n)$, $A(3, n)$ and $A(4, n)$. Find all other values of $A(m, n)$ ($m \geq 5$) whose decimal representation can be explicitly written in our universe (assume that our universe has 10^{80} atoms.)

8. Exceptional primes. If m is an integer greater than 1 with some prime factor different from 2 and 5 then $1/m$ has a non-terminating periodic decimal representation. Let $l_{10}(m)$ be the length of the period in the decimal representation of $1/m$; e.g., $1/7 = 0.\overline{142857} \Rightarrow l_{10}(7) = 6$, $1/11 = 0.\overline{09} \Rightarrow l_{10}(11) = 2$, etc.

1) Prove that if p is a prime number different from 2 and 5, then either $l_{10}(p^2) = p\, l_{10}(p)$ or $l_{10}(p^2) = l_{10}(p)$.

2) Since available evidence seems to show that most prime numbers different from 2 and 5 verify $l_{10}(p^2) = p\, l_{10}(p)$, we will call "exceptional" any such prime if it verifies the alternative relation $l_{10}(p^2) = l_{10}(p)$. For instance 3 is exceptional because $1/3 = 0.\overline{3}$ and $1/9 = 0.\overline{1}$, hence $l_{10}(3^2) = l_{10}(3) = 1$. Find the next exceptional prime.

3) (May require using a computer) Find the third smallest exceptional prime.[2]

4) (Open) Find the fourth smallest exceptional prime.

5) According to the Prime Number Theorem the total number of primes in $[2, N]$ is asymptotically $N/\ln N$. Give an educated guess on the number of exceptional primes that we can expect to find in that interval.

9. Bags of candies. In a box we have $N+1$ bags with N candies each ($N \geq 2$). The first bag contains N orange flavored candies, the second one contains $N-1$ orange flavored candies and 1 lemon flavored one, the third one contains $N - 2$ orange flavored and 2 lemon flavored, and so on. We take one bag at random and try one of the candies,

[2] Solved in *Carrollia* No. 30

which turns out to be lemon flavored. Then we take a second candy from the same bag. What is the probability that the second candy is also lemon flavored?

10. Consulting books in a library. In a library that contains N books every day we choose one of the books at random for consultation, and return it to the shelf. What is the expected number of days it will take us to consult all the books in the library at least once?

11. N roulettes. We have N roulettes, all with the same probability p of stopping at zero. We spin them simultaneously. If any of them stops at zero we keep spinning only that ones that did not stop at zero. We keep doing the same until all of them have stopped at zero. What is the expected number of times that we must spin the roulettes until all of them stop at zero?

12. The round random table. In the border of a perfectly circular piece of wood we choose N points at random to place legs and make a table. What is the probability that the table will stand without falling?

13. Throwing pies. A group of initially N people play the following game. Each one picks another person at random as a target, and at the voice of "now!" they throw their pies at their selected targets with perfect aim. Each player hit by a pie must abandon the game; the ones not hit by a pie are called "survivors". They keep playing until all of them have been hit or only one survivor remains.

(a) If at a given stage of the game there are n survivors, what is the expected number of survivors at the next stage?

(b) If at a given stage of the game there are n survivors and $0 \leq k \leq n$, what is the probability of having exactly k survivors at the next stage?

(c) (Probably open) Study the asymptotic behavior as $N \to \infty$ of the probability of ending up with one survivor.

Generalize the problem assuming that the players' aim is not perfect. Assume that the probability p of hitting the selected target is constant and the same for everybody.

14. John's PIN. John's PIN for his teller machine consists of a 6-digit number '`abcdef`'. He has bad memory, but does not want to write it down just in case someone finds it. So he breaks the number in two 3-digit numbers '`abc`' and '`def`', and with his pocket calculator finds the quotient `abc/def`, which he writes down. A few days later he needs the PIN, and as expected he cannot remember it, but he remembers that he wrote down the quotient between the two halves of the number in a piece of paper: '`abc/def`$= 0.195323246$'. So he takes his pocket calculator (with only basic arithmetic operations and the inversion '$1/x$' key), punches the keys for a few seconds, scratches some numbers on a piece of paper for a few seconds more and gets the original 6-digit number. What is that number and how did he find it so quickly?

15. Game of 15. The game of 15 consists of a 4 by 4 square grid with 15 pieces labeled with the numbers 1 to 15 covering all of the squares of the grid except one (the 'hole'). Any piece right above, right below, right to the left or right to the right of the hole can be moved to it, covering the hole and uncovering the square previously occupied by the piece. The pieces are placed initially in increasing numeric order covering all the spaces starting at the top row from left to right and from top to bottom, with the hole at the right bottom corner. Is it possible to move the pieces so that number 1 and 2 are swapped and the rest end up in their initial positions?

16. Irreducible random fraction. What is the probability that a fraction a/b whose numerator and denominator are chosen at random among all positive integers turns out to be irreducible? (Assume that a and b are chosen with uniform probability in the interval $[1, N]$ and then let $N \to \infty$.)

17. Swapping registers.[3] I have a (rather primitive) pocket calculator with the following operation keys: $\boxed{+}$, $\boxed{-}$, $\boxed{*}$, $\boxed{/}$, $\boxed{=}$, $\boxed{M+}$ (add to memory), $\boxed{M+}$ (subtract from memory), $\boxed{MR}$ (read memory). In order to avoid rounding errors in what follows I will ignore the multiplication and division keys, so I will work with $\boxed{+}$, $\boxed{-}$, $\boxed{=}$, $\boxed{M+}$, $\boxed{M+}$ and $\boxed{MR}$ only. Assume that at a given time I have a number 'x' in the display and another number 'y' in memory, and I want to swap them, i.e., using the keys $\boxed{+}$, $\boxed{-}$, $\boxed{=}$, $\boxed{M+}$, $\boxed{M+}$ and $\boxed{MR}$ I want to end up with 'y' in the display and 'x' in memory. Is there any way to accomplish this?

18. The honest forecaster.[4] A forecaster makes predictions on possible outcomes O_1, O_2,...,O_n of a given event (such as tomorrows weather) by assigning probabilities $p_1, p_2, \ldots, p_n$ to them. Find a cheat-proof way to evaluate the accuracy of his/her predictions. (Note that some evaluation schemes are easy to defeat, for instance comparing the frequency of past occurrences of each O_k to the probability p_k assigned to it can be defeated by adjusting the predicted probabilities to the frequency of past occurrences—so the forecaster could pretend to be doing his/her job without actually forecasting anything.)

19. Series involving e. Find the sum of the following series:

$$\sum_{n=1}^{\infty} \left\{ e - \left(1 + \frac{1}{n}\right)^n \right\}$$

20. The pages of a book. The pages of a book with two parts are numbered $1, 2, 3, \ldots, N$. The sum of the page numbers of the first part equals the sum of the page numbers of the second part. How many pages does the book have?

[3] *Carrollia* No. 13, Jun 1987.
[4] Carollia No. 23.

21. Round of prisoners (a.k.a. Josephus problem).[5] The director of a prison decides to free one of the prisoners by the following method. First he arranges all the prisoners in a circle, then he starts removing every other prisoner from the circle until only one remains. The last prisoner in the circle is set free. For instance assume that there are five prisoners numbered $1, 2, 3, 4, 5$; then the director starts removing in this order prisoners 2, 4, 1, 5, so 3 is the last one and is set free. If there are initially N prisoners, which one will be the lucky one?

22. Almost perfect odd numbers.[6] A positive integer is called *perfect* if the sum of its positive proper divisors (different from n) equals n. For instance 6 is perfect because $1 + 2 + 3 = 6$. Equivalently n is perfect iff $\sigma(n)/n = 2$, where $\sigma(n)$ = sum of all positive divisors of n (including n itself). Although the structure of even perfect numbers is well known, up to this date nobody has ever found an odd perfect number, nor anybody knows whether an odd perfect number exists. Prove however that there are odd numbers as close to be perfect as we wish, in the following sense:

(a) For every $\varepsilon > 0$ there is some positive odd number n such that

$$\left| \frac{\sigma(n)}{n} - 2 \right| < \varepsilon \, .$$

Technically 2 is called an *accumulation point* of the set

$$S = \{\sigma(n)/n \mid n = 1, 3, 5, \dots \} \, .$$

(b) Find all accumulation points of S.

23. Modified Nim. This is a modified version of the game of Nim (in the following we assume that there is an unlimited supply of tokens.) Two players set several piles of tokens in a row. By turns each of them takes one token from one of the piles and adds at will as many tokens as he/she wishes to piles placed to the left of the pile from which the token was taken. Assuming that the game ever finishes, the player that takes the last token wins.

(a) Prove that, no matter how they play, the game will eventually end after finitely many steps.

(b) Find a winning strategy.

Do the same for another variant of the game in which the piles are arranged in several rows (not necessarily with the same number of piles each), and players are also allowed to add any number of piles with any number of tokens each to rows placed above the one containing the pile from which the token was taken. Generalize the problem to n-dimensional arrangements (and solve it).

24. Increasing and decreasing subsequences. Prove that every sequence $a_1, a_2, \dots, a_{n^2+1}$ of $n^2 + 1$ different numbers contains either an increasing subsequence of length $n + 1$ or a decreasing subsequence of length $n + 1$.

[5] *Carrollia* No. 29.
[6] *Carrollia* No. 30.

25. Three-colored plane. Prove that if we color all the points of the plane using three colors, there are two points with the same color placed at a mutual distance 1.

26. Fermat's Last Theorem with real exponents. Show that the set of real values of α such that the equation

$$x^\alpha + y^\alpha = z^\alpha$$

has positive integer solutions is dense in $\mathbb{R}$, i.e., every non-empty open interval of real numbers contains values of α for which the equation shown above has solutions with $x, y, z \in \mathbb{Z}^+$.

27. Sums and products of tangents squared. Find the values of the following expressions ($n \geq 3$ odd):

$$S(n) = \sum_{k=0}^{\frac{n-3}{2}} \tan^2 \left\{ \tfrac{(2k+1)\pi}{2n} \right\} ,$$

$$P(n) = \prod_{k=0}^{\frac{n-3}{2}} \tan^2 \left\{ \tfrac{(2k+1)\pi}{2n} \right\} ,$$

$$\frac{S(n)}{P(n)} = \sum_{k=0}^{\frac{n-3}{2}} \prod_{\substack{l=0 \\ l \neq k}}^{\frac{n-3}{2}} \cot^2 \left\{ \tfrac{(2l+1)\pi}{2n} \right\} .$$

Generalize the result to certain sums of products of tangents or cotangents squared.

28. A binary operation on rationals. Let $\circ$ be a binary operation defined on rational numbers with the following properties:
(a) Commutative: $a \circ b = b \circ a$.
(b) Associative: $a \circ (b \circ c) = (a \circ b) \circ c$.
(c) Idempotency of zero: $0 \circ 0 = 0$
(d) Distributivity of '+' respect to '$\circ$': $(a \circ b) + c = (a + c) \circ (b + c)$.
Prove that the operation is either $a \circ b = \max(a, b)$ or $a \circ b = \min(a, b)$.

29. Sorted Random Numbers. Pick n random numbers $x_1, x_2, ..., x_n$ in the interval $[0, 1]$ with uniform probability. Let $y_1 \leq y_2 \leq \cdots \leq y_n$ be those same numbers sorted in non-decreasing order. For each $k = 1, \dots, n$, what is the expected value of y_k?

30. Ternary Addition. We define a binary operation '$*$' on the real numbers so that for every a, b, c, $(a * b) * c = a + b + c$. Prove that $*$ is $+$.

31. Candy Jars. In a professional office there are two jars initially filled with 100 candies each. Every day ten clients come to the office an each of them takes a candy from one of the jars chosen at random (or from the only jar with candies if one of them is empty). If one of the jars gets empty, at the end of the day it is refilled with 100 candies again. What is the expected number of days it will take for both jars to get empty on the same day?