New Zealand Mathematical Olympiad Committee

Camp Selection Problems 2010

Due: 27th August 2010

Junior division

J1. We number both the rows and the columns of an 8×8 chessboard with the numbers 1 to 8. Some grains of rice are placed on each square, in such a way that the number of grains on each square is equal to the product of the row and column numbers of the square. How many grains of rice are there on the entire chessboard?

J2. AB is a chord of length 6 in a circle of radius 5 and centre O. A square is inscribed in the sector OAB with two vertices on the circumference and two sides parallel to AB. Find the area of the square.

J3. Find all positive integers n such that $n^5 + n + 1$ is prime.

J4. Find all positive integer solutions (a, b) to the equation

$$\frac{1}{a} + \frac{1}{b} + \frac{n}{\operatorname{lcm}(a,b)} = \frac{1}{\gcd(a,b)}$$

for

(i) $n = 2007$;

(ii) $n = 2010$.

Note: "$\operatorname{lcm}(a, b)$" means the least common multiple of a and b, and "$\gcd(a, b)$" means the greatest common divisor of a and b. For example, $\operatorname{lcm}(18, 30) = 90$, and $\gcd(18, 30) = 6$.

J5. The diagonals of quadrilateral $ABCD$ intersect in point E. Given that $|AB| = |CE|$, $|BE| = |AD|$, and $\angle AED = \angle BAD$, determine the ratio $|BC|/|AD|$.

J6. At a strange party, each person knew exactly 22 others.

For any pair of people X and Y who knew one another, there was no other person at the party that they both knew.

For any pair of people X and Y who did not know each other, there were exactly six other people that they both knew.

How many people were at the party?

Senior division

S1. For any two positive real numbers $x_0 > 0$, $x_1 > 0$, a sequence of real numbers is defined recursively by

$$x_{n+1} = \frac{4\max\{x_n, 4\}}{x_{n-1}} \qquad \text{for } n \geq 1.$$

Find x_{2010}.

Note: "$\max\{x, y\}$" means the maximum of x and y — that is, whichever of the two numbers x and y is the larger. For example, $\max\{2, 3\} = 3$.

S2. In a convex pentagon $ABCDE$ the areas of the triangles ABC, ABD, ACD and ADE are all equal to the same value x. What is the area of the triangle BCE?

S3. Let p be a prime number. Find all pairs (x, y) of positive integers such that

$$x^3 + y^3 - 3xy = p - 1.$$

S4. A line drawn from the vertex A of an equilateral triangle ABC meets the side BC at D and the circumcircle at P. Show that

$$\frac{1}{|PD|} = \frac{1}{|PB|} + \frac{1}{|PC|}.$$

S5. Determine the values of the positive integer n for which

$$A = \sqrt{\frac{9n-1}{n+7}}$$

is rational.

S6. Suppose a_1, a_2, ..., a_8 are eight distinct integers from $\{1, 2, \ldots, 16, 17\}$. Show that there is an integer $k > 0$ such that there are at least three different (not necessarily disjoint) pairs (i, j) such that $a_i - a_j = k$.

Also find a set of seven distinct integers from $\{1, 2, \ldots, 16, 17\}$ such that there is no integer $k > 0$ with that property.

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