Polynomial Interpolation

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## Warm-ups

1. Explain how to evaluate $\sum_{n=0}^{\infty} \frac{P(n)}{2^n}$ for any polynomial $P(n)$.

2. Let $a, b$ be roots of $x^4 + x^3 - 1 = 0$. Show that $ab$ is a root of $x^6 + x^4 + x^3 - x^2 - 1$.

3. Let $f$ be a polynomial. Show that there are only finitely many numbers $a$ for which $f(x) = a$ has a multiple root.

4. Let $\omega$ be a primitive $n$-th root of unity. Find
$$\sum_{k=1}^{n-1} \frac{1}{1-\omega^k}.$$

5. Let $f$ by a polynomial of degree less than $n-2$. Let $p = (t-a_1)(t-a_2)\cdots(t-a_n)$. Show that $\sum_{k=1}^{n} \frac{f(a_k)}{p'(a_k)} = 0$.

## Problems

6. Given a quadratic polynomial $f(x) = x^2+ax+b$, suppose that $f(f(x))$ has four distinct real roots, two of which sum to $-1$. Show that $b \leq -\frac{1}{4}$

7. Show that if the integers are partitioned into finitely many arithmetic sequences, two of these sequences have the same common difference.

8. A point in the plane with a cartesian coordinate system is called a mixed point if one of its coordinates is rational and the other one is irrational. Find all polynomials with real coefficients such that their graphs do not contain any mixed points.

9. For a polynomial $P(x)$ with integer coefficients, $r(2i-1)$ (for $i = 1, 2, 3, \ldots, 512$) is the remainder obtained when $P(2i-1)$ is divided by 1024. The sequence $(r(1), r(3), \ldots, r(1023))$ is called the remainder sequence of $P(x)$. A remainder sequence is called complete if it is a permutation of (1, 3, 5, …, 1023). Prove that there are no more than $2^{35}$ different complete remainder sequences.

10. Let $m$ and $n$ be positive integers. The polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ has integer coefficients with $\gcd(a_0, a_1, \ldots, a_n, m) = 1$ such that $m$ divides $f(j)$ for all integers $j$. Prove that $m$ divides $n!$.

11. Show that for $k \geq 4$, if $F(x)$ is a polynomial with integer coefficients and $0 \leq F(c) \leq k$ for $c = 0, 1, \ldots, k+1$, then $F(0) = F(1) = \cdots + F(k+1)$.

12. Let n be a positive integer. Consider $S = \{(x, y, z) : x, y, z \in \{0, 1, ..., n\}, x+y+z > 0\}$ as a set of $(n+1)^3-1$ points in three-dimensional space. Determine the smallest possible number of planes, the union of which contains S but does not include $(0, 0, 0)$.

13. Let $M$ be the maximum value that a polynomial $f$ attains on the interval $[0, 1]$. If $f$ is monic, what is the minimal possible value of $M$?

14. (Iran 2010) Let $p(x)$ be a quadratic polynomial such that $|p(x)| \leq 1$ for $x = -1, 0, 1$. Show that $|p(x)| \leq \frac{5}{4}$ for $-1 \leq x \leq 1$.

15. (HMMT 2010)

    Let $x, y, z, w$ be real numbers such that

$$w + x + y + z = 5, \quad (1)$$
$$2w + 4x + 8y + 16z = 7, \quad (2)$$
$$3w + 9x + 27y + 81z = 11, \quad (3)$$
$$4w + 16x + 64y + 256z = 1. \quad (4)$$

    What is the value of $5w + 25x + 125y + 625z$?

16. Let $a, b, c,$ and $d$ be distinct real numbers such that $a+b+c+d = 9$ and $a^2+b^2+c^2+d^2 = 10$. Evaluate

$$\frac{a^5}{(a-b)(a-c)(a-d)}+\frac{b^5}{(b-a)(b-c)(b-d)}+\frac{c^5}{(c-a)(c-b)(c-d)}+\frac{d^5}{(d-a)(d-b)(d-c)}.$$

17. (IMO 1996 Shortlist) Let $P(x)$ be the real polynomial function, $P(x) = ax^3 + bx^2 + cx + d$. Prove that if $|P(x)| \leq 1$ for all $x$ such that $|x| < 1$, then $|a| + |b| + |c| + |d| \leq 7$.

18. Let $p$ be prime, and let $f(x)$ be a polynomial of degree $d$ with integer coefficients such that $f(0) = 0$, $f(1) = 1$ and for every integer $n$, $f(n)$ is either 0 or 1 mod $p$. Show that $d \geq p - 1$.

## More for Fun

19. Prove Newton's identities: if $s_i = x_1^i+\cdots+x_n^i$, then $c_0s_k+c_1s_{k-1}+\cdots+c_{k-1}s_1+kc_k = 0$.

20. Let $P$ be a cubic polynomial with integer coefficients. Suppose that $xP(x) = yP(y)$ for infinitely many pairs of distinct integers $x$ and $y$. Show that $P$ has an integer root.

21. Suppose that $x^{2n} + ax^{2n-1} + ax^{2n-2} + \cdots + ax^2 + ax + 1$ factors into $n$ quadratic polynomials. Show that these quadratic polynomials all have constant coefficient 1.

22. Let $P(x)$ and $Q(x)$ be polynomials with the with the same roots (and possibly different multiplicities) such that $P(x) + 1$ and $Q(x) + 1$ have the same roots (with possibly different multiplicities). Show that $P(x) = Q(x)$.