Combinatorics 2 - More on Enumerative Combinatorics

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This set of notes acts as a sort of addendum to the previous set titled 'Combinatorics'. In the previous notes, many techniques on counting are covered. However, one method that is absent is the idea of *recurrence relations*. The idea is simple. If we have some parameter n, and a_n is the number of certain objects in relation to n, then we may be able to derive a recurrence relation for a_n. This will enable us to calculate a_n for small values of n.

Let us look at an example.

Example. *How many anagrams of the word 'orange' are there such that no letter is in its original position?*

Solution. This problem can be interpreted as follows. We would like to find the number of permutations of $\{1, 2, 3, 4, 5, 6\}$ with no *fixed points*. Recall that if S is a set and $f : S \to S$ is a permutation of S, then $i \in S$ is a *fixed point* of f if $f(i) = i$.

We consider the more general problem with n in place of 6. Let a_n be the number of permutations of $\{1, 2, \ldots, n\}$ with no fixed points. We shall derive a recurrence relation for a_n. The number of permutations fixing at least one element is $n! - a_n$. Also, for $1 \le k \le n-1$, the number of permutations fixing exactly k elements is $\binom{n}{k}a_{n-k}$, since there are $\binom{n}{k}$ choices for the positions of the k fixed elements, and there are a_{n-k} ways to position the other $n-k$ elements so that there is no fixed point among them. There is also one permutation fixing every element (the identity permutation). It follows that $n! - a_n = 1 + \sum_{k=1}^{n-1}\binom{n}{k}a_{n-k}$, and hence the required recurrence relation is

$$a_n = n! - 1 - \sum_{k=1}^{n-1}\binom{n}{k}a_{n-k}.$$

In particular, we have

$$\begin{aligned}
a_1 &= 0,\\
a_2 &= 2! - 1 - \binom{2}{1}a_1 = 2 - 1 - 0 = 1,\\
a_3 &= 3! - 1 - \binom{3}{1}a_2 - \binom{3}{2}a_1 = 6 - 1 - 3\cdot 1 - 0 = 2,\\
a_4 &= 4! - 1 - \binom{4}{1}a_3 - \binom{4}{2}a_2 - \binom{4}{3}a_1 = 24 - 1 - 4\cdot 2 - 6\cdot 1 - 0 = 9,\\
a_5 &= 5! - 1 - \binom{5}{1}a_4 - \binom{5}{2}a_3 - \binom{5}{3}a_2 - \binom{5}{4}a_1\\
&= 120 - 1 - 5\cdot 9 - 10\cdot 2 - 10\cdot 1 - 0 = 44,
\end{aligned}$$

$$\begin{aligned} a_6 &= 6! - 1 - \binom{6}{1}a_5 - \binom{6}{2}a_4 - \binom{6}{3}a_3 - \binom{6}{4}a_2 - \binom{6}{5}a_1 \\ &= 720 - 1 - 6\cdot 44 - 15\cdot 9 - 20\cdot 2 - 15\cdot 1 - 0 = 265. \end{aligned}$$

Problems

In this set of problems, some can be tackled by using recurrence relations, while others can be tackled by the methods that were discussed in 'Combinatorics'.

1. How many anagrams of the word 'mathematics' are there where no two vowels are adjacent?

2. How many integers in the set $\{1, 2, 3, \ldots, 1000000\}$ are neither perfect squares, nor perfect cubes, nor perfect fourth powers?

3. Find the number of permutations $\{a_1, a_2, a_3, a_4, a_5, a_6\}$ of $\{1, 2, 3, 4, 5, 6\}$ such that, for no integer k with $1 \le k \le 5$, we have $\{a_1, a_2, \ldots, a_k\}$ forming a permutation of $\{1, 2, \ldots, k\}$.

4. How many permutations of $\{1, 2, 3, \ldots, n\}$ are there such that no three consecutive terms are either increasing or decreasing, when

   (a) $n = 7$;

   (b) $n = 8$?

   For example, for $n = 7$, such a permutation can be $\{1, 6, 4, 7, 2, 5, 3\}$.

5. For a positive integer n, let $P(n)$ be the number of partitions of n. For example, $P(4) = 5$, with the partitions being $1+1+1+1$, $1+1+2$, $2+2$, $1+2$, and 4. The *diversity* of a partition is the number of distinct terms in the partition. Let $Q(n)$ be the sum of the diversities of all the partitions of n. For example, $Q(4) = 1+2+1+2+1 = 7$. Prove that for all positive integers n,

$$Q(n) = 1 + P(1) + P(2) + \cdots + P(n-1).$$

6. $A_1A_2A_3\cdots A_{20}$ is a regular 20-gon. How many non-isosceles triangles are there, whose vertices are three of the vertices of $A_1A_2A_3\cdots A_{20}$, but whose sides are not sides of $A_1A_2A_3\cdots A_{20}$?

7. How many ways can the number 1000000 be expressed as a product of three positive integers? Products which only differ in the order of the factors are not considered to be different.

8. How many pairwise non-congruent triangles are there with integer sides and perimeter 2010?

9. Show that the number of 3-element subsets $\{a, b, c\}$ of $\{1, 2, \ldots, 63\}$ with $a + b + c < 95$ is less than the number of those with $a + b + c \geq 95$.

10. The numbers 1 to 64 are written on an 8×8 chessboard so that 1 to 8 are written from left to right in the first row, 9 to 16 are written from left to right in the second row, and so on. Minus signs are then inserted in front of some of the numbers, so that there are exactly four in each row and exactly four in each column. Prove that the sum of all of the resulting numbers is 0.

11. Find the number of 4×4 arrays, whose entries are from the set $\{0, 1, 2, 3\}$, and which are such that the sum of the numbers in each of the four rows and in each of the four columns is divisible by 4.

12. A *type 1* sequence is a sequence with each term 0 or 1 which does not have $0, 1, 0$ as consecutive terms. A *type 2* sequence is a sequence with each term 0 or 1 which does not have $0, 0, 1, 1$ or $1, 1, 0, 0$ as consecutive terms. Show that there are twice as many type 2 sequences of length $n + 1$ as type 1 sequences of length n.

13. The numbers $1, 2, \ldots, n^2$ are arranged in an $n \times n$ array, so that the numbers in each row increase from left to right, and the numbers in each column increase from top to bottom. For $1 \leq i \leq n$, let a_i be the number of possible values of the entry in the ith row and ith column. Prove that
$$a_1 + a_2 + \cdots + a_n = \frac{1}{3}n(n^2 - 3n + 5).$$

14. Let p be a permutation of the set $S_n = \{1, 2, 3, \ldots, n\}$. An element $j \in S_n$ is called a *fixed point* of p if $p(j) = j$. Let f_n be the number of permutations having no fixed points, and g_n be the number with exactly one fixed point. Show that $|f_n - g_n| = 1$.

15. Given a permutation $\sigma = \{a_1, a_2, \ldots, a_n\}$ of $\{1, 2, \ldots, n\}$, an ordered pair (a_i, a_j) is called an *inversion* of σ if $1 \leq i < j \leq n$ and $a_i > a_j$. Let $m(\sigma)$ denote the number of inversions of the permutation σ. Find the average of $m(\sigma)$ as σ varies over all permutations.

16. Let $p_n(k)$ be the number of permutations of the set $\{1, \ldots, n\}$, $n \geq 1$, which have exactly k fixed points. Prove that
$$\sum_{k=0}^{n} k p_n(k) = n!$$
(Remark: For a set S and a permutation $f : S \to S$, an element i in S is called a *fixed point* of the permutation f if $f(i) = i$.)

17. A fair dice is thrown n times. Find the probability that the sum of the scores is divisible by 5.

18. Let $L = \{(x, y) : x, y \in \mathbb{Z}\}$ be the set of all lattice points in the plane. A path of length n is a chain $P_0, P_1, \ldots, P_n$ of points in L such that $P_{i-1}P_i = 1$ for $1 \le i \le n$. Let $F(n)$ be the number of distinct paths beginning at $P_0 = (0, 0)$ and ending at any point P_n on the line $y = 0$. Prove that
$$F(n) = \binom{2n}{n}.$$

19. On the circumference of a circle, $2n$ points are marked and equally spaced. How many ways are there to join these points in pairs with n chords so that no two chords intersect within the circle?

20. For a positive integer n, a *partition* of n is an increasing sequence of positive integers with sum n. For example, the partitions of 5 are: $1, 1, 1, 1, 1$; $1, 1, 1, 2$; $1, 1, 3$; $1, 4$; 5; $1, 2, 2$; and $2, 3$. If p is a partition, let $f(p)$ be the number of 1s in p, and $g(p)$ be the number of distinct integers in p. Show that $\sum f(p) = \sum g(p)$, where each sum is taken over all partitions of n. For example, when $n = 5$, we have $\sum f(p) = 5+3+2+1+0+1+0 = 12$, and $\sum g(p) = 1 + 2 + 2 + 2 + 1 + 2 + 2 = 12$.