Lecture 23- Linearly Recurrent Sequences

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1 Introduction and the general case

Often problems in Olympiad mathematics will focus on linearly recurrent sequences. These sequences are all in the form, for constant $a_1, a_{2,}..., a_k$,

$$x_n = a_1x_{n-1} + a_2x_{n-2} + ... + a_kx_{n-k}$$

We realize that a geometric sequence can also be written as a recurrence. Given that $x_0 = c$ and $x_n = rx_{n-1}$ for all n≥0, we can conclude that $x_n = cr^n$. This is because $x_n = r(x_{n-1}) = r(r(x_{n-2})) = r(r(r(x_{n-3}))) = ... = r(r(r(...(c)...)) = r^nc$. Interestingly enough, linear occurrences can generally be expressed as either a combination of geometric sequences or sequences related to geometric sequences. If we want to make a linear occurrence that satisfies $x_n = cr^n$, it would be of the form

$$x_n = cr^n = a_1cr^{n-1} + a_2cr^{n-2} + ... + a_kcr^{n-k} = a_1x_{n-1} + a_2x_{n-2} + ... + a_kx_{n-k}$$

rearranging gives us

$$cr^n - a_1cr^{n-1} - a_2cr^{n-2} - ... - a_kcr^{n-k} = 0$$

Dividing both sides by cr^{n-k} in order to simplify yields

$$r^k - a_1r^{k-1} - a_2r^{k-2} - ... - a_k = 0$$

Since the formula $x_n = cr^n$ satisfies the recurrence only if $r^k - a_1r^{k-1} - a_2r^{k-2} - ... - a_k = 0$, r must be a root of $x^k - a_1x^{k-1} - a_2x^{k-2} - ... - a_k$. This polynomial is known as the characteristic polynomial.

Given a general linear sequence, with terms $x_{0,}x_1,...x_{k-1}$defined, and with $x_n = a_1x_{n-1} + a_2x_{n-2} + ... + a_kx_{n-k}$for all $n \geq k$, we let its characteristic polynomial be $x^k - a_1x^{k-1} - a_2x^{k-2} - ... - a_k$. We let this polynomial have roots $r_{1,}r_{2,}...r_k$. The formula for this sequence depends on whether or not the roots $r_{1,}r_{2,}...r_k$ are distinct.

1.1 Case 1: The characteristic polynomial has distinct roots

Set $x_n = c_1 r_1^n + c_2 r_2^n + ... + c_k r_k^n$ for n=0,1,...,k-1. This gives us the following system of equations using which we can solve for $c_{1,} c_{2,} ... c_k$:

$$x_0 = c_1 + c_2 + ... + c_k$$

$$x_2 = c_1 r_1 + c_2 r_2 + ... + c_k r_k$$

$$...$$

$$x_{k-1} = c_1 r_1^{k-1} + c_2 r_2^{k-1} + ... + c_k r_k^{k-1}$$

After solving for $c_{1,} c_{2,} ... c_k$, we just substitute our result into $x_n = c_1 r_1^n + c_2 r_2^n + ... + c_k r_k^n$ in order to find the general formula for this sequence.

1.2 Case 2: The characteristic polynomial does not have distinct roots

Let root r occur m times with m>1. The term cr^nmust be replaced by

$$c_1 r^n + c_2 n r^n + ... + c_m n^{m-1} r^m$$

After making this replacement, you can solve using the same process shown when the characteristic polynomial has distinct roots.

2 Examples

2.1 Example 1

2.1.1 Question

One of the most well known linearly recurrent sequence is the Fibonacci sequence. It is defined by the formula $F_n = F_{n-1} + F_{n-2}$. Find its general formula.

2.1.2 Solution

Since $F_n - F_{n-1} - F_{n-2} = 0$, this sequence has a characteristic polynomial $x^2 - x - 1$. Using the quadratic formula the characteristic polynomial's roots can easily be found to be $x = \frac{1 \pm \sqrt{5}}{2}$. For the sake of simplicity, let $\alpha = \frac{1+\sqrt{5}}{2}$and let $\beta = \frac{1-\sqrt{5}}{2}$. We know that $F_n = c_1 \alpha^n + c_2 \beta^n$ for some constants $c_1$and c_2. We know that $F_0 = 0$ and $F_1 = 1$. Setting $n = 0$ and $n = 1$, we obtain the equations:

$$0 = c_1 + c_2$$

$$1 = c_1 \alpha + c_2 \beta$$

Solving this system of equations shows us that $c_1 = \frac{1}{\alpha - \beta} = \frac{1}{\sqrt{5}}$ and $c_2 = -\frac{1}{\alpha - \beta} = -\frac{1}{\sqrt{5}}$. This means that

$$F_n = \frac{1}{\sqrt{5}} \alpha^n - \frac{1}{\sqrt{5}} \beta^n = \frac{1}{\sqrt{5}} ((\frac{1+\sqrt{5}}{2})^n - (\frac{1-\sqrt{5}}{2})^n)$$

2.2 Example 2

2.2.1 Question

Given sequence $x_n = 14x_{n-1} - 80x_{n-2} + 238x_{n-3} - 387x_{n-4} - 324x_{n-5} + 108x_{n-6}$, and that $x_0 = 1$,$x_1 = 21$,$x_2 = 123$,$x_3 = 3221$,$x_4 = 11223$,$x_5 = 31442$, solve for x_n.

2.2.2 Solution

This sequence has characteristic polynomial $x^6 - 14x^5 + 80x^4 - 238x^3 + 387x^2 - 324x + 108$. Using the rational roots theorem, and a bit of trial and error, the roots of this polynomial can be found to be $(x-3)^3(x-2)^2(x-1)$. This means that its formula must be of the form $x_n = a3^n + bn3^n + cn^2 3^n + d2^n + en2^n + f$ for some constants a, b, c, d, e, f. Substituting 0, 1, 2, 3, 4, and 5 into this equation gives us

$$a + d + f = 1$$

$$3a + 3b + 3c + 2d + 2e + f = 21$$

$$9a + 18b + 36c + 4d + 8e + f = 123$$

$$27a + 81b + 243c + 8d + 24e + f = 3221$$

$$81a + 324b + 1296c + 16d + 64e + f = 11223$$

$$243a + 1215b + 6075c + 32d + 160e + f = 31442$$

Solving yields $a = \frac{974225}{8}$,$b = -\frac{72046}{3}$,$c = \frac{16475}{12}$,$d = -111184$,$e = -\frac{64425}{2}$, $f = -\frac{84745}{8}$. Therefore:

$$x_n = (\tfrac{974225}{8})3^n - (\tfrac{72046}{3})n3^n + (\tfrac{16475}{12})n^2 3^n - (111184)2^n - (\tfrac{64425}{2})n2^n - \tfrac{84745}{8}$$

2.3 Example 3

2.3.1 Question

Find the set of real numbers a_0 for which the infinite sequence (a_n) of real numbers defined by $a_{n+1} = 2^n - 3a_n$ for n=0, 1, 2, ... is strictly increasing, that is, $a_n < a_{n+1}$ for $n > 0$. (British Mathematical Olympiad, 1980).

2.3.2 Solution

First it is important to note that $a_{n+1} = 2^n - 3a_n$is not quite a linearly recurrent series. In order to fix this, we look at the formulae for $a_{n+1}$and a_n.

$$a_{n+1} = 2^n - 3a_n$$

$$a_n = 2^{n-1} - 3a_{n-1}$$

In order to eliminate any powers of two and find a true linear recurrence, we multiply both sides of $a_n = 2^{n-1} - 3a_{n-1}$ by 2 in order to obtain $2a_n = 2^n - 6a_{n-1}$. By subtracting this from $a_{n+1} = 2^n - 3a_n$, we find that $a_{n+1} - 2a_n = 2^n - 3a_n - (2^n - 6a_{n-1}) = -a_n + 6a_{n-1}$.Rearranging yields $a_{n+1} = -a_n + 6a_{n-1}$.This is a linear recurrence with characteristic polynomial $x^2 + x - 6$, which is equal to $(x-2)(x+3)$. Therefore $a_n = c_1 2^n + c_2(-3)^n$. In order to solve for $c_1$and c_2,we must find $a_0$and a_1. We leave $a_0$as a variable, and find that $a_1 = 1 - 3a_0$.This gives us the system of equations $c_1 + c_2 = a_0$and $2c_1 - 3c_2 = 1 - 3a_0$.Solving for c_1 and c_2 in terms of a_0 yields $c_1 = \frac{1}{5}$and $c_2 = a_0 - \frac{1}{5}$. This means that $a_n = (\frac{1}{5})(2^n) + (a_0 - \frac{1}{5})(-3)^n$. We notice that, unless $a_0 - \frac{1}{5} = 0$, $(-3)^n$will eventually eclipse 2^n.In other words, since $(-3)^n$ grows faster than 2^n, the terms in the sequence will begin to alter signs once n gets big enough unless $(-3)^n$ has a coefficient of 0. This only occurs when $a_0 = \frac{1}{5}$.

2.4 Example 4

2.4.1 Question

Given a sequence (x_n) defined by $x_0 = 1$ and $x_n = 2x_{n-1} + 3n$ for all n≥1, find x_n.

2.4.2 Solution

First we realize that the sequence in its current form is not a true ruccerence because of the presence of the term 2n. In order to eliminate this term, we create a second equations by substituting n-1 in for n. Before doing this, we rearrange the given equation to give us $x_n - 2x_{n-1} = 3n$.

$$x_n - 2x_{n-1} = 3n$$

$$x_{n-1} - 2x_{n-2} = 3n - 3$$

We subtract the second equation from the first and find that

$$x_n - 3x_{n-1} + 2x_{n-2} = 3$$

This equation is also not a true linear recurrence because of the presence of a constant 2. In order to eliminate it, we repeat a process similar to the one used above. We substitute n-1 into the given equation in order to find that

$$x_{n-1} - 3x_{n-2} + 2x_{n-3} = 3$$

Subtracting this from $x_n - 3x_{n-1} + 2x_{n-2} = 3$ gives us

$$x_n - 4x_{n-1} + 5x_{n-2} - 2x_{n-3} = 0$$

This process is called shifting the index n. Since we now have a true linear recurrence, we can finally find the characteristic polynomial to be $x^3 - 4x^2 + 5x - 2 = (x-1)^2(x-2)$. This tells us that $x_n = c_1 1^n + c_2 n 1^n + c_3 2^n = c_1 + c_2 n + c_3 2^n$. We also know that $x_0 = 1$, $x_1 = 2(1) + 3(1) = 5$, $x_2 = 2(5) + 3(2) = 16$. By substitution of n=0, 1, 2 into $x_n = c_1 + c_2 n + c_3 3^n$ equation, we get the system of equations

$$1 = c_1 + c_3$$

$$5 = c_1 + c_2 + 2c_3$$

$$16 = c_1 + 2c_2 + 4c_3$$

We can easily solve and find that $c_1 = -6$,$c_2 = -3$ and $c_3 = 7$. Therefore

$$x_n = (7)(2^n) - 6 - 3n$$

2.4.3 Trend

Any recurrence of the form $x_n - a_1x_{n-1} - a_2x_{n-2} - ... - a_kx_{n-k} = f(n)$ for any arbitrary function $f(n)$ is called an inhomogeneous recurrence. If $f(n)$ satisfies any recurrence, you can use the technique of shifting the index n to convert inhomogeneous recurrences into linear recurrence.

3 Problems

1. Let (x_n) be a sequence such that $x_0 = x_1 = 5$ and $x_n = \frac{x_{n+1}+x_{n-1}}{2}$ for all positive integers n. Prove that $\frac{(x_n+1)}{6}$ is a perfect square for all n, and find a formula for x_n.

2. Let a, b, and c be the roots of the equation $x^3 + x^2 + x^1 = 0$. Show that a, b, and c are distinct, and that $\frac{a^{1982}-b^{1982}}{a-b} + \frac{b^{1982}-c^{1982}}{b-c} + \frac{c^{1982}-a^{1982}}{c-a}$is an integer. (CMO, 1982)

3. A sequence (a_n) is dened by $a_0 = a_1 = 0, a_2 = 1$, and $a_{n+3} = a_{n+1} + 1998a_n$ for all n 0. Prove that $a_{2n-1} = 2a_na_{n+1} + 1998a_{n-1}^2$ for every positive integer n. (Komal)

4. Let $a_1 = a_2 = 1$ and $a_{n+1} = \frac{(a_n^2+4)}{a_{n-1}}$ for all n$\geq$ 2. Find a formula for a_n.

5. Let a be a positive integer, and let a_n be defined by $a_0 = 0$, and $a_{n+1} = (a_n + 1)a + (a+1)a_n + 2\sqrt{a(a+1)a_n(a_n+1)}$ for n$\geq$ 1. Show that for each positive integer n, a_n is a positive integer. (IMO Short List, 1983)

6. A sequence of numbers a_1, a_2, a_3,... satisfies $a_1 = \frac{1}{2}$, and $a_1 + a_2 + \ldots + a_n = n^2a_n$ for all $n \geq 1$. Determine the value of a_n. (CMO, 1975)

7. For which real numbers a does the sequence dened by the initial condition $u_0 = a$ and the recursion $u_{n+1} = 2u_n - n^2$ have $u_n > 0$ for all n$\geq$0? (Putnam, 1980)

8. An integer sequence is dened by $a_0 = 0, a_1 = 1$, and $a_n = 2a_{n-1} + a_{n-2}$ for all n$\geq$2. Prove that 2^k divides an if and only if 2^k divides n. (IMO Short List, 1988)

9. A sequence an is defined as follows, $a_0 = 1, a_{n+1} = \frac{1+4a_n+\sqrt{1+24a_n}}{16}$ for n $\geq$ 0. Find an explicit formula for a_n. (IMO short list 1981)

10. For each positive integer n, let $S_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$, $T_n = S_1 + S_2 + S_3 + \cdots + S_n$, $U_n = \frac{T_1}{2} + \frac{T_2}{3} + \frac{T_3}{4} + \cdots + \frac{T_n}{n+1}$. Find, with proof, integers $0 < a, b, c, d < 1000000$ such that $T_{1988} = aS_{1989} - b$ and $U_{1988} = cS_{1989} - d$. (USAMO 1989)

Please feel free to email me at karanpahil@gmail.com if you have any questions, corrections or comments.