Combinatorial Sums

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1 Warmup

Problem 1 (Vandermonde convolution)**.** Find a closed form expression for

$$\sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}.$$

2 Common Techniques

- **Combinatorial interpretation:** Counting in two ways, Inclusion-exclusion, Thinking probabilistically.
- **Algebraic manipulation:** Changing order of summation, Telescoping sums.
- **Miscellaneous:** Induction, Recognizing / Reversing a product expansion, Interpreting as a power series.

3 Useful Facts

Fact 1 (Pascal's Identity)**.** For $0 \le k \le n$, we have

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}.$$

Fact 2 (Hockeystick Identity)**.** For any k, we have that

$$\sum_{n=k}^{m} \binom{n}{k} = \binom{m+1}{k+1}.$$

Fact 3 (Principle of Inclusion-Exclusion)**.** Let $A_1, \ldots, A_n$ be sets. Then, we have

$$|A_1 \cup \cdots \cup A_n| = (|A_1| + \cdots + |A_n|) - (|A_1 \cup A_2| + \cdots + |A_{n-1} \cup A_n|) + \cdots + (-1)^n |A_1 \cup \cdots \cup A_n|.$$

Fact 4. We have

$$\int_0^1 x^n (1-x)^m dx = \frac{1}{(n+m+1)\binom{n+m}{n}}.$$

4 Problems

4.1 Pure summation

Problem 2. Evaluate the sum
$$\binom{n}{0}+\binom{n}{1}+\cdots+\binom{n}{n}.$$

Problem 3. Prove that
$$\binom{n}{0}+\binom{n}{2}+\cdots=\binom{n}{1}+\binom{n}{3}+\cdots.$$

Problem 4. Find a closed form expression for
$$\binom{3n}{0}+\binom{3n}{3}+\cdots+\binom{3n}{3n}.$$

Problem 5. Show that
$$\sum_{t=0}^{n-1-k}\frac{(-1)^t}{t+k+1}\binom{n-1-k}{t}=\frac{k!(n-1-k)!}{n!}.$$

Problem 6 (HMMT 2008)**.** Let $a_0,a_1,a_2,\ldots$ be a sequence satisfying $a_0=11$, $a_1=0$, and $30a_{n+2}=a_{n+1}+a_n$ for all $n\geq 0$. Evaluate the infinite sum
$$\sum_{n=0}^{\infty}\binom{2n}{n}a_n.$$

Problem 7 (TST 2001)**.** Express
$$\sum_{k=0}^{n}(-1)^k(n-k)!(n+k)!$$
in closed form.

Problem 8 (TST 2000)**.** Let n be a positive integer. Prove that
$$\binom{n}{0}^{-1}+\binom{n}{1}^{-1}+\cdots+\binom{n}{n}^{-1}=\frac{n+1}{2^{n+1}}\left(\frac{2}{1}+\frac{2^2}{2}+\cdots+\frac{2^{n+1}}{n+1}\right).$$

Problem 9 (TST 2010)**.** Let m,n be positive integers with $m\geq n$, and let S be the set of all ordered n-tuples $(a_1,a_2,\ldots,a_n)$ of positive integers such that $a_1+a_2+\cdots+a_n=m$. Show that
$$\sum_S 1^{a_1}2^{a_2}\cdot\cdots\cdot n^{a_n}=\binom{n}{n}n^m-\binom{n}{n-1}(n-1)^m+\cdots+(-1)^{n-2}\binom{n}{2}2^m+(-1)^{n-1}\binom{n}{1}.$$

Problem 10 (Putnam 1999)**.** Sum the series
$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m^2n}{3^m(n3^m+m3^n)}.$$

4.2 Some combinatorics

Problem 11 (HMRT 2010)**.** Amy starts with a piece of paper with the number 0 on it. On each of $\binom{n+m}{m}$ days, she starts at the lower left corner of a neighborhood that consists of a $n \times m$ grid of blocks separated by roads, and travels along the roads to the upper right corner, only going right or up at each intersection. When she gets there, she counts the number of blocks, b, that lie above here path, and adds the monomial q^b to the value on her paper. If Amy never took the same path more than once, show that, by the end of the $\binom{n+m}{m}$ days, the sum on her paper is equal to the q-binomial coefficient

$$\binom{\mathbf{n}+\mathbf{m}}{\mathbf{m}}_q = \frac{(\mathbf{n}+\mathbf{m})_q!}{(\mathbf{n})_q!(\mathbf{m})_q!}.$$

(Here $(\mathbf{a})_q!$ denotes the q-factorial $(1+q)(1+q+q^2)\cdot\dots\cdot(1+q+q^2+\dots+q^{a-1})$.)

Problem 12 (IMO 1981)**.** Take r such that $1 \le r \le n$, and consider all subsets of r elements of the set $\{1, 2, \dots, n\}$. Each subset has a smallest element. Let $F(n, r)$ be the arithmetic mean of these smallest elements. Prove that: $F(n, r) = \frac{n+1}{r+1}$.

Problem 13 (IMO 1987)**.** Let $p_n(k)$ be the number of permutations of the set $\{1, 2, 3, \dots, n\}$ which have exactly k fixed points. Prove that

$$\sum_{k=0}^{n} k p_n(k) = n!.$$

Problem 14 (Putnam 2006)**.** Let $S = \{1, 2, \dots, n\}$ for some integer $n > 1$. Say a permutation π of S has a local maximum at $k \in S$ if

1. $\pi(k) > \pi(k+1)$ for $k = 1$;
2. $\pi(k-1) < \pi(k)$ and $\pi(k) > \pi(k+1)$ for $1 < k < n$;
3. $\pi(k-1) < \pi(k)$ for $k = n$

What is the average number of local maxima of a permutation of S, averaging over all permutations of S?

Problem 15 (ISL 2005)**.** Let $n \ge 3$ be a fixed integer. Each side and each diagonal of a regular n-gon is labeled with a number from the set $\{1, 2, \dots, r\}$ in a way such that the following two conditions are fulfilled:

1. Each number from the set $\{1, 2, \dots, r\}$ occurs at least once as a label.
2. In each triangle formed by three vertices of the n-gon, two of the sides are labeled with the same number, and this number is greater than the label of the third side.

(a) Find the maximal r for which such a labeling is possible.
(b) For this maximal value of r, how many such labellings are there?

4.3 A mix

Problem 16 (HMMT 2009)**.** Let $s(n)$ denote the number of 1's in the binary representation of n. Compute

$$\frac{1}{255}\sum_{0\leq n<16} 2^n(-1)^{s(n)}.$$

Problem 17 (IMO 1974)**.** Prove that for any natural number n, the number

$$\sum_{k=0}^{n}\binom{2n+1}{2k+1}2^{3k}$$

cannot be divided by 5.

Problem 18 (IMO 1995)**.** Let p be an odd prime number. How many p-element subsets A of $\{1,2,\ldots,2p\}$ are there, the sum of whose elements is divisible by p?

Problem 19 (IMO 2008)**.** Let n and k be positive integers with $k\geq n$ and $k-n$ an even number. Let $2n$ lamps labeled $1,2,...,2n$ be given, each of which can be either on or off. Initially all the lamps are off. We consider sequences of steps: at each step one of the lamps is switched (from on to off or from off to on).
Let N be the number of such sequences consisting of k steps and resulting in the state where lamps 1 through n are all on, and lamps $n+1$ through $2n$ are all off.
Let M be number of such sequences consisting of k steps, resulting in the state where lamps 1 through n are all on, and lamps $n+1$ through $2n$ are all off, but where none of the lamps $n+1$ through $2n$ is ever switched on.
Determine $\frac{N}{M}$.

Problem 20 (TST 2006)**.** Let n be a given integer with n greater than 7, and let $\mathcal{P}$ be a convex polygon with n sides. Any set of $n-3$ diagonals of $\mathcal{P}$ that do not intersect in the interior of the polygon determine a triangulation of $\mathcal{P}$ into $n-2$ triangles. A triangle in the triangulation of $\mathcal{P}$ is an interior triangle if all of its sides are diagonals of $\mathcal{P}$. Express, in terms of n, the number of triangulations of $\mathcal{P}$ with exactly two interior triangles, in closed form.

Problem 21 (ISL 2006)**.** Let S be a finite set of points in the plane such that no three of them are on a line. For each convex polygon P whose vertices are in S, let $a(P)$ be the number of vertices of P, and let $b(P)$ be the number of points of S which are outside P. A line segment, a point, and the empty set are considered as convex polygons of 2, 1, and 0 vertices respectively. Prove that for every real number x:

$$\sum_{P} x^{a(P)}(1-x)^{b(P)}=1,$$

where the sum is taken over all convex polygons with vertices in S.

Problem 22 (Putnam 2005)**.** Let S_n denote the set of all permutations of the numbers $1,2,\ldots,n$. For $\pi\in S_n$, let $\sigma(\pi)=1$ if π is an even permutation and $\sigma(\pi)=-1$ if π is an odd permutation. Also, let $v(\pi)$ denote the number of fixed points of π. Show that

$$\sum_{\pi\in S_n}\frac{\sigma(\pi)}{v(\pi)+1}=(-1)^{n+1}\frac{n}{n+1}.$$

Problem 23 (Putnam 2004)**.** Show that for any positive integer n, there is an integer N such that the product $x_1x_2\cdot\dots\cdot x_n$ can be expressed identically in the form

$$x_1x_2\cdot\dots\cdot x_n = \sum_{i=1}^{N} c_i(a_{i1}x_1 + a_{i2}x_2 + \cdots + a_{in}x_n)^n$$

where the c_i are rational numbers and each a_{ij} is one of the numbers $-1, 0, 1$.

Problem 24 (Putnam 1997)**.** Let $a_{m,n}$ denote the coefficient of x^n in the expansion of $(1+x+x^2)^m$. Prove that for all integers $k \geq 0$,

$$0 \leq \sum_{i=0}^{\lfloor \frac{2k}{3} \rfloor} (-1)^i a_{k-i,i} \leq 1.$$