2007 Mathematical Olympiad Summer Program Tests

Edited by

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MOSP Black Group
Spring, 2007

Practice Test 1

1.1. In triangle ABC three distinct triangles are inscribed, similar to each other, but not necessarily similar to triangle ABC, with corresponding points on corresponding sides of triangle ABC. Prove that if two of these triangles share a vertex, than the third one does as well.

1.2. Let a, b, and c be positive real numbers. Prove that

$$\left(\frac{a}{a+2b}\right)^2+\left(\frac{b}{b+2c}\right)^2+\left(\frac{c}{c+2a}\right)^2 \geq \frac{1}{3}.$$

1.3. An *elevated Schröder path* of order $2n$ is a lattice path in the first quadrant of the coordinate plane traveling from the origin to $(2n, 0)$ using three kinds of steps: $[1, 1]$, $[2, 0]$, and $[1, -1]$. An *uprun* in an elevated Schröder path is a maximum string of consecutive steps of the form $[1, 1]$. Let $U(n, k)$ denote the number of Schröder paths of order $2n$ with exactly k upruns. Compute $U(n, 0), U(n, 1), U(n, n-1)$, and $U(n, n)$.

1.4. Each positive integer a undergoes the following procedure in order to obtain the number $d = d(a)$:

(i) move the last (rightmost) digit of a to the front (leftmost) to obtain the number b;

(ii) square b to obtain the number c;

(iii) move the first digit of c to the end to obtain the number d.

(All the numbers in the problem are considered to be represented in base 10). For example, for $a = 2003$, we get $b = 3200$, $c = 10240000$, and $d(2003) = 02400001 = 2400001$.

Find all numbers a for which $d(a) = a^2$.

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Practice Test 2

2.1. Let $\{a_n\}_{n=1}^{\infty} = \{2, 4, 8, 1, 3, 6, \dots\}$ be the infinite integer sequence such that a_n is the leftmost digit in the decimal representation of 2^n, and let $\{b_n\}_{n=1}^{\infty} = \{5, 2, 1, 6, 3, 1, \dots\}$ be the infinite integer sequence such that a_n is the leftmost digit in the decimal representation of 5^n. Prove that for any block of consecutive terms in $\{a_n\}$, there is a block of consecutive terms in $\{b_n\}$ in the reverse order.

2.2. Let d be a positive integer. Integers $t_1, t_2, \dots, t_d$ and real numbers that $a_1, a_2, \dots, a_d$ are given such that

$$a_1 t_1^j + a_2 t_2^j + \cdots + a_d t_d^j$$

is an integer for all integers j with $0 \le j < d$. Prove that

$$a_1 t_1^d + a_2 t_2^d + \cdots + a_d t_d^d$$

is also an integer.

2.3. Let n and k be integers with $0 \le k < \frac{n}{2}$. Initially, let A be the sequence of subsets of $\{1, 2, \dots, n\}$ with exactly k elements, and B the sequence of subsets of $\{1, 2, \dots, n\}$ with exactly $k+1$ elements, both arranged in lexicographic (dictionary) order. Now let S be the first element in A. If there is a T in B such that $S \subseteq T$, remove S from A and the first such T from B, and repeat this process as long as A is nonempty; otherwise, stop. Prove that this process terminates with A empty.

2.4. Let P be a point in the interior of acute triangle ABC. Set $R_a = PA$, $R_b = PB$, and $R_c = PC$. Let d_a, d_b and d_c denote the distances from P to sides BC, CA, and AB, respectively. Prove that

$$\frac{1}{3} \le \frac{R_a^2 \sin^2 \frac{A}{2} + R_b^2 \sin^2 \frac{B}{2} + R_c^2 \sin^2 \frac{C}{2}}{d_a^2 + d_b^2 + d_c^2} \le 1.$$

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Practice Test 3

3.1. Let n be a positive integer. Consider an $2 \times n$ chessboard. In some cells there are some coins. In each step we are allowed to choose a cell containing more than 2 coins, remove two of the coins in the cell, put one coin back into either the cell upper to the chosen cell or to the cell to the right of the chosen cell. Assume that there are at least 2^n coins on the chessboard. Prove that after a finite number of moves, it is possible for the upper right corner cell to contain a coin.

3.2. Let $x, y,$ and z be positive real numbers with $x + y + z = 1$. Prove that

$$\frac{xy}{\sqrt{xy+yz}} + \frac{yz}{\sqrt{yz+zx}} + \frac{zx}{\sqrt{zx+xy}} \le \frac{\sqrt{2}}{2}.$$

3.3. Let $a, b_1, b_2, \dots, b_n, c_1, c_2, \dots, c_n$ be real numbers such that

$$x^{2n} + ax^{2n-1} + ax^{2n-2} + \cdots + ax + 1 = (x^2 + b_1x + c_1)(x^2 + b_2x + c_2)\cdots(x^2 + b_nx + c_n)$$

for all real numbers x. Prove that $c_1 = c_2 = \cdots = 1$.

3.4. Let $ABCD$ be a cyclic quadrilateral. Diagonals AC and BD meet at E. Let P be point inside the quadrilateral. Let O_1, O_2, O_3 and O_4 be circumcenters of triangles ABP, BCP, CDP, and DAP, respectively. Prove that lines O_1O_3, O_2O_4, and OE are concurrent.

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Practice Test 4

4.1. In acute triangle ABC, $\angle A < 45^\circ$. Point D lies in the interior of triangle ABC such that $BD = CD$ and $\angle BDC = 4\angle A$. Point E is the reflection of C across line AB, and point F is the reflection of B across line AC. Prove that $AD \perp EF$.

4.2. Given 10^6 points in the space, show that the set of pairwise distances of given points has at least 79 elements.

4.3. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that for all real numbers x and y,

$$f(x^3 + y^3) = (x + y)(f(x)^2 - f(x)f(y) + f(y)^2).$$

Prove that for all real numbers x, $f(1996x) = 1996f(x)$.

4.4. Let c be a fixed positive integer, and let $\{a_n\}_{n=1}^{\infty}$ be a sequence of positive integers such that $a_n < a_{n+1} < a_n + c$ for every positive integer n. Let s denote the infinite string of digits obtained by writing the terms in the sequence consecutively from left to right, starting from the first term. For every positive integer k, let s_k denote the number whose decimal representation is identical to the k most left digits of s. Prove that for every positive integer m there exists a positive integer k such that s_k is divisible by m.

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Practice Test 5

5.1. Let ω be with center O. Convex quadrilateral $AEDB$ inscribed in ω with segment AB as a diameter of ω. Rays ED and AB meet at C. Let ω_1 denote the circumcircle of triangle OBD, and let segment OF be a diameter of ω_1. Ray CF meet ω_1 at G. Prove that A, O, G, and E lie on a circle.

5.2. Let k be a positive integer, and let $x_1, x_2, \ldots, x_n$ be positive real numbers. Prove that

$$\left(\sum_{i=1}^{n} \frac{1}{1+x_i}\right)\left(\sum_{i=1}^{n} x_i\right) \leq \left(\sum_{i=1}^{n} \frac{x_i^{k+1}}{1+x_i}\right)\left(\sum_{i=1}^{n} \frac{1}{x_i^k}\right).$$

5.3. Given positive integer n with $n \geq 2$, determine the minimum number of elements in set X such that for any n 2-element subsets $S_1, S_2, \ldots, S_n$ of X, there exists an n-element subset Y of X with $Y \cap S_i$ has at most one element for every integer $i = 1, 2, \ldots, n$.

5.4. Given positive integers a and c and integer b, prove that there exists a positive integer x such that $a^x + x \equiv b \pmod{c}$.

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Practice Test 6

6.1. Determine all positive integers a such that

$$S_n = \{\sqrt{a}\} + \{\sqrt{a}\}^2 + \cdots + \{\sqrt{a}\}^n$$

is rational for some positive integer n. (For real number x, $\{x\} = x - \lfloor x \rfloor$, where $\lfloor x \rfloor$ denote the the greatest integer less than or equal to x.)

6.2. A integer is called *good* if it can be written as the sum of three cubes of positive integers. Prove that for every $i = 0, 1, 2, 3$, there are infinitely many positive integers n such that there are exactly i good numbers among $n, n+2$, and $n+28$.

6.3. Let ABC be an acute triangle and let D, E, and F be the feet of the altitudes from A, B, and C to sides BC, CA, and AB respectively. Let P, Q, and R be the feet of the perpendiculars from A, B, and C to EF, FD, and DE respectively. Prove that

$$2(PQ + QR + RP) \geq DE + EF + FD.$$

6.4. Let $\mathcal{G}$ be a directed complete graph on n vertices having each of its edges colored either red or blue. Prove that there exists a vertex $v \in \mathcal{G}$ with the property that for every other vertex $u \in \mathcal{G}$, there exists a monochromatic directed path from v to u.

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Practice Test 7

7.1. In an $n \times n$ array, each of the numbers $1, 2, \ldots, n$ appear exactly n times. Show that there is a row or a column in the array with at least $\sqrt{n}$ distinct numbers.

7.2. Let ABC be a triangle. Circle ω passes through A and B and meets sides AC and BC at D and E, respectively. Let F be the midpoint of segment AD. Suppose that there is a point G on side AB such that $FG \perp AC$. Prove that $\angle EGF = \angle ABC$ if and only if $AF/FC = BG/GA$.

7.3. Let n be a positive integer which is not a power of a prime number. Prove that there exists an equiangular polygon whose side lengths are $1, 2, \ldots, n$ in some order.

7.4. Find all functions $f : \mathbb{R} to \mathbb{R}$ such that $f(1) = 1$ and

$$f\left(f(x)y + \frac{x}{y}\right) = xyf(x^2 + y^2)$$

for all real numbers x and y with $y \neq 0$.

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Practice Test 8

8.1. Let p be a prime great than 3. Prove that there exists integers $a_1, a_2, \ldots, a_n$ with

$$-\frac{p}{2} < a_1 < a_2 < \cdots < a_n < \frac{p}{2}$$

such that

$$\frac{(p-a_1)(p-a_2)\cdots(p-a_n)}{|a_1a_2\cdots a_n|}$$

is a perfect power of 3.

8.2. let a, b, and c be nonnegative real numbers with

$$\frac{1}{a^2+1} + \frac{1}{b^2+1} + \frac{1}{c^2+1} = 2.$$

Prove that

$$ab + bc + ca \le \frac{3}{2}.$$

8.3. Let M denote the midpoint of side BC in triangle ABC. Line AM intersects the incircle of ABC at points K and L. Lines parallel to BC are drawn through K and L, intersecting the incircle again at points X and Y, respectively. Lines AX and AY intersect BC at P and Q, respectively. Prove that $BP = CQ$.

8.4. Consider the integer lattice points in the plane, with one pebble placed at the origin. We play a game where at each step one pebble is removed from a lattice point and two new pebbles are placed at two neighboring (either horizontally or vertically, but not both) lattice points, provided that those points are unoccupied. There will be a pebble lies inside or on the boundary of the square $\mathcal{S}$ determined by the lines $|x + |y|| = k$. Determine the minimum value of k.

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Practice Test 9

9.1. Let a, b, x, y be positive integers such that $ax + by$ is divisible by $a^2 + b^2$. Prove that $\gcd(x^2 + y^2, a^2 + b^2) > 1$.

9.2. In triangle ABC, point L lies on side BC. Extend segment AB through B to M such that $\angle ALC = 2\angle AMC$. Extend segment AC through C to N such that $\angle ALB = 2\angle ANB$. Let O be the circumcenter of triangle AMN. Prove that $OL \perp BC$.

9.3. Find the maximum value of real number k such that

$$\frac{(b-c)^2(b+c)}{a} + \frac{(c-a)^2(c+a)}{b} + \frac{(a-b)^2(a+b)}{c} \geq k(a^2 + b^2 + c^2 - ab - bc - ca)$$

for all positive real numbers a, b, and c.

9.4. Given n collinear points, consider the distances between the points. Suppose each distance appears at most twice. Prove that there are at least $\lfloor n/2 \rfloor$ distances that appear once each.

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Practice Test 10

10.1. For positive integer n, Lucy and Windy play the n-game with numbers. Initially, Lucy goes first by writing number 1 on the board, and then the players alternate. On his turn, a player erases the number, say k, on the board and writes either the number $k+1$ or $k+2$, or $2k$ on the board. The player who first reaches a number greater than n losses. Find all n for which Lucy has a winning strategy.

10.2. In triangle ABC, point L lies on side BC. Extend segment AB through B to M such that $\angle ALC = 2\angle AMC$. Extend segment AC through C to N such that $\angle ALB = 2\angle ANB$. Let O be the circumcenter of triangle AMN. Prove that $OL \perp BC$.

10.3. Let $\mathbb{R}^*$ denote the set of nonzero real numbers. Find all functions $f : \mathbb{R}^* \to \mathbb{R}^*$ such that

$$f(x^2 + y) = (f(x))^2 + \frac{f(xy)}{f(x)}$$

for every pair of nonzero real numbers x and y with $x^2 + y \neq 0$.

10.4. Let n be a positive integer with $n \geq 2$. Fix $2n$ points in space in such a way that no four of them are in the same plane, and select any $n^2 + 1$ segments determined by the given points. Prove that these segments form at least n triangles.

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Practice Test 11

11.1. Let n be a given positive integer. Consider a set S of n points, with no 3 collinear, such that the distance between any pair of points in the set is least 1. We define the radius of the set, denoted by r_S, as the largest circumradius of the triangles with their vertices in S. Determine the minimum value of r_S.

11.2. Suppose that a sequence $a_1, a_2, a_3, \ldots$ satisfies

$$0 < a_n \leq a_{2n} + a_{2n+1} \qquad (*)$$

for all $n \geq 1$. Determine if the series $\sum_{n=1}^{\infty} a_n$ converges or not. What if $a_1, a_2, a_3, \ldots$ is a sequence of positive numbers satisfies

$$0 < a_n \leq a_{n+1} + a_{n^2} \qquad (**)$$

instead?

11.3. Let ABC be a triangle. Circle Ω passes through points B and C. Circle ω is tangent internally to Ω and also to sides AB and AC at T, P, and Q, respectively. Let M be midpoint of arc $\widehat{BC}$ (containing T) of Ω. Prove that lines PQ, BC, and MT are concurrent.

11.4. Suppose n coins have been placed in piles on the integers on the real line. (A "pile" may contain zero coins.) Let T denote the following sequence of operations.

(a) Move piles $0, 1, 2, \ldots$ to $1, 2, 3, \ldots$, respectively.

(b) Remove one coin from each nonempty pile from among piles $1, 2, 3, \ldots$, then place the removed coins in pile 0.

(c) Swap piles i and $-i$ for $i = 1, 2, 3, \ldots$.

Prove that successive applications of T from any starting position eventually lead to some sequence of positions being repeated, and describe all possible positions that can occur in such a sequence.

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Practice Test 12

12.1. Set $A_1, A_2, \ldots, A_{35}$ are given with the property that $|A_i| = 27$ for $1 \le i \le 35$, such that the intersection of any three of them has exactly one element. Show that there is a element belongs to all the given sets.

12.2. Let ABC be a triangle, and let O, R, and r denote its circumcenter, circumradius, and inradius. Set $AB = c$, $BC = a$, $CA = b$, and $s = \frac{a+b+c}{2}$. Point N lies inside the triangle such that

$$\frac{[NBC]}{s-a} = \frac{[NCA]}{s-b} = \frac{[NAB]}{s-c}.$$

Express ON by R and r.

12.3. If p is a prime number greater than 3 and $k = \lfloor 2p/3 \rfloor$, prove that the sum

$$\binom{p}{1} + \binom{p}{2} + \cdots + \binom{p}{k}$$

of binomial coefficients is divisible by p^2.

12.4. Let a, b, c, x, y, z be positive real numbers such that $ax + by + cz = xyz$. Prove that

$$x + y + z > \sqrt{a+b} + \sqrt{b+c} + \sqrt{c+a}.$$

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Practice Test 13

13.1. Let p be a prime number. Find all natural numbers n such that p divides $\varphi(n)$ and such that n divides $a^{\frac{\varphi(n)}{p}} - 1$ for all positive integers a relatively prime to n.

13.2. Let ABC be a triangle with circumcirlce ω. Point D lies on side BC such that $\angle BAD = \angle CAD$. Let I_A denote the excenter of triangle ABC opposite A, and let ω_A denote the circle with AI_A as its diameter. Cricles ω and ω_A meet at P other than A. The circumcle of triangle APD meet line BC again at Q (other than D). Prove that Q lies the excircle of triangle ABC opposite A.

13.3. Each positive integer is colored either red or blue. Prove that there exists an infinite increasing sequence of positive integers $\{k_n\}_{n=1}^{\infty}$ such that the sequence

$$2k_1, k_1 + k_2, 2k_2, k_2 + k_3, 3k_3, k_3 + k_4, 2k_4, \ldots$$

is monochromatic.

13.4. Let $a_1, a_2, \ldots, a_n$ be positive real numbers with $a_1 + a_2 + \cdots + a_n = 1$. Prove that

$$(a_1a_2 + a_2a_3 + \cdots + a_{n-1}a_n + a_na_1)\left(\frac{a_1}{a_2^2 + a_2} + \frac{a_2}{a_3^2 + a_3} + \cdots + \frac{a_{n-1}}{a_n^2 + a_n} + \frac{a_n}{a_1^2 + a_1}\right) \geq \frac{n}{n+1}.$$

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Practice Test 14

14.1. Let k be a given positive integer greater than 1. An k-digit integer $a_1 a_{k-1} \dots a_k$ is called *parity-monotonic* if for every integer i with $1 \le i \le k-1$,

$$\begin{cases} a_i > a_{i+1} & \text{if } a_i \text{ is odd,} \\ a_i < a_{i+1} & \text{if } a_i \text{ is even.} \end{cases}$$

How many k-digit parity-monotonic integers are there?

14.2. Four circles ω, ω_A, ω_B, and ω_C, with the same radius r, are drawn in the interior of triangle ABC such that ω_A is tangent to sides AB and AC, ω_B to BC and BA, ω_C to CA and CB, and ω (externally) to ω_A, ω_B, and ω_C. Find the possible values of ratio between r and the inradius of the triangles.

14.3. Let P be a polynomial with rational coefficients. Suppose that for any integer n, $P(n)$ is an integer. Prove that for any distinct integers m and n,

$$\operatorname{lcm}(1, 2, \dots, \deg(P)) \frac{P(m) - P(n)}{m - n}$$

is an integer.

14.4. Given a positive integer n, prove that there exists $\epsilon > 0$ such that for any n positive real numbers $a_1, a_2, \dots, a_n$, there exists $t > 0$ such that

$$\epsilon < \{ta_1\}, \{ta_2\}, \dots, \{ta_n\} < \frac{1}{2}.$$

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Practice Test 15

15.1. Given n points on the plane with no three collinear, a set of k of the points is called *k-polite* if they determine a convex k-gon that contains no other given point in its interior. Let c_k denote the number of k-polite subsets of the given points. Show that the series

$$\sum_{k=3}^{n}(-1)^k c_k$$

is independent of the configuration of the points and depends only on n.

15.2. Let ABC be an acute triangle. Circle ω_{BC} has segment BC as its diameter. Circle ω_A is tangent to lines AB and AC and is tangent externally to ω_{BC} at A_1. Points B_1 and C_1 are defined analogously. Prove that lines AA_1, BB_1, and CC_1 are concurrent.

15.3. Let p be a polynomial of degree $n \geq 2$ such that $|p(x)| \leq 1$ for all x in the interval $[-1, 1]$. Determine the maximum value of the leading coefficient of f.

15.4. A *k-coloring* of a graph G is a coloring of its vertices using k possible colors such that the end points of any edge have different colors. We say a graph G is *uniquely k-colorable* if one hand it has a k-coloring, on the other hand there do not exist vertices u and v such that u and v have the same color in one k-coloring and u and v have different colors in another k-coloring. Prove that if a graph G with n vertices ($n \geq 3$) is uniquely 3-colorable, then it has at least $2n - 3$ edges.

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TST 2002

1. Let ABC be a triangle. Prove that

$$\sin\frac{3A}{2}+\sin\frac{3B}{2}+\sin\frac{3C}{2}\le\cos\frac{A-B}{2}+\cos\frac{B-C}{2}+\cos\frac{C-A}{2}.$$

2. Let p be a prime number greater than 5. For any integer x, define

$$f_p(x)=\sum_{k=1}^{p-1}\frac{1}{(px+k)^2}.$$

   Prove that for all positive integers x and y, the numerator of $f_p(x)-f_p(y)$, when written in lowest terms, is divisible by p^3.

3. Let n be an integer greater than 2, and $P_1, P_2, \cdots, P_n$ distinct points in the plane. Let $\mathcal{S}$ denote the union of the segments $P_1P_2, P_2P_3, \dots, P_{n-1}P_n$. Determine whether it is always possible to find points A and B in $\mathcal{S}$ such that $P_1P_n \parallel AB$ (segment AB can lie on line P_1P_n) and $P_1P_n = kAB$, where (1) $k=2.5$; (2) $k=3$.

4. Let n be a positive integer and let S be a set of 2^n+1 elements. Let f be a function from the set of two-element subsets of S to $\{0,\dots,2^{n-1}-1\}$. Assume that for any elements x,y,z of S, one of $f(\{x,y\}), f(\{y,z\}), f(\{z,x\})$ is equal to the sum of the other two. Show that there exist a,b,c in S such that $f(\{a,b\}), f(\{b,c\}), f(\{c,a\})$ are all equal to 0.

5. Consider the family of non-isosceles triangles ABC satisfying the property $AC^2+BC^2=2AB^2$. Points M and D lie on side AB such that $AM=BM$ and $\angle ACD=\angle BCD$. Point E is in the plane such that D is the incenter of triangle CEM. Prove that exactly one of the ratios

$$\frac{CE}{EM},\quad\frac{EM}{MC},\quad\frac{MC}{CE}$$

   is constant (i.e., is the same for all triangles in the family).

6. Find in explicit form all ordered pairs of positive integers (m,n) such that $mn-1$ divides m^2+n^2.

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TST 2003

1. For a pair of integers a and b, with $0 < a < b < 1000$, the set $S \subseteq \{1, 2, \ldots, 2003\}$ is called a *skipping set* for (a, b) if for any pair of elements $s_1, s_2 \in S$, $|s_1 - s_2| \notin \{a, b\}$. Let $f(a, b)$ be the maximum size of a skipping set for (a, b). Determine the maximum and minimum values of f.

2. Let ABC be a triangle and let P be a point in its interior. Lines PA, PB, and PC intersect sides BC, CA, and AB at D, E, and F, respectively. Prove that
$$[PAF] + [PBD] + [PCE] = \frac{1}{2}[ABC]$$
if and only if P lies on at least one of the medians of triangle ABC. (Here $[XYZ]$ denotes the area of triangle XYZ.)

3. Find all ordered triples of primes (p, q, r) such that
$$p \mid q^r + 1, \quad q \mid r^p + 1, \quad r \mid p^q + 1.$$

4. Let $\mathbb{N}$ denote the set of positive integers. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that
$$f(m+n)f(m-n) = f(m^2)$$
for all $m, n \in \mathbb{N}$.

5. Let a, b, c be real numbers in the interval $(0, \frac{\pi}{2})$. Prove that
$$\frac{\sin a \sin(a-b)\sin(a-c)}{\sin(b+c)} + \frac{\sin b \sin(b-c)\sin(b-a)}{\sin(c+a)} + \frac{\sin c \sin(c-a)\sin(c-b)}{\sin(a+b)} \geq 0.$$

6. Let $\overline{AH_1}, \overline{BH_2}$, and $\overline{CH_3}$ be the altitudes of an acute scalene triangle ABC. The incircle of triangle ABC is tangent to $\overline{BC}, \overline{CA}$, and $\overline{AB}$ at T_1, T_2, and T_3, respectively. For $k = 1, 2, 3$, let P_i be the point on line H_iH_{i+1} (where $H_4 = H_1$) such that $H_iT_iP_i$ is an acute isosceles triangle with $H_iT_i = H_iP_i$. Prove that the circumcircles of triangles $T_1P_1T_2$, $T_2P_2T_3$, $T_3P_3T_1$ pass through a common point.

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TST 2004

1. Let $a_1, a_2, \ldots, a_n$, $b_1, b_2, \ldots, b_n$ be real numbers such that
$$(a_1^2 + a_2^2 + \cdots + a_n^2 - 1)(b_1^2 + b_2^2 + \cdots + b_n^2 - 1) > (a_1b_1 + a_2b_2 + \cdots + a_nb_n - 1)^2.$$
Show that $a_1^2 + a_2^2 + \cdots + a_n^2 > 1$ and $b_1^2 + b_2^2 + \cdots + b_n^2 > 1$.

2. Let n be a positive integer. Consider sequences $a_0, a_1, \ldots, a_n$ such that $a_i \in \{1, 2, \ldots, n\}$ for each i and $a_n = a_0$.
   (a) Call such a sequence *good* if for all $i = 1, 2, \ldots, n$, $a_i - a_{i-1} \not\equiv i \pmod{n}$. Suppose that n is odd. Find the number of good sequences.
   (b) Call such a sequence *great* if for all $i = 1, 2, \ldots, n$, $a_i - a_{i-1} \not\equiv i, 2i \pmod{n}$. Suppose that n is an odd prime. Find the number of great sequences.

3. A 2004×2004 array of points is drawn. Find the largest integer n such that it is possible to draw a convex n-sided polygon whose vertices lie on the points of the array.

4. Let ABC be a triangle and let D be a point in its interior. Construct a circle ω_1 passing through B and D and a circle ω_2 passing through C and D such that the point of intersection of ω_1 and ω_2 other than D lies on line AD. Denote by E and F the points where ω_1 and ω_2 intersect side BC, respectively, and by X and Y the intersections of lines DF, AB and DE, AC, respectively. Prove that $XY \parallel BC$.

5. Let $A = (0, 0, 0)$ be the origin in the three dimensional coordinate space. The *weight* of a point is the sum of the absolute values of its coordinates. A point is a *primitive lattice point* if all its coordinates are integers with their greatest common divisor equal to 1. A square $ABCD$ is called a *unbalanced primitive integer square* if it has integer side length and the points B and D are primitive lattice points with different weights.

   Show that there are infinitely many unbalanced primitive integer squares $AB_iC_iD_i$ such that the plane containing the squares are not parallel to each other.

6. Let $\mathbb{N}_0^+$ and $\mathbb{Q}$ be the set of nonnegative integers and rational numbers, respectively. Define the function $f : \mathbb{N}_0^+ \to \mathbb{Q}$ by $f(0) = 0$ and
$$f(3n + k) = -\frac{3f(n)}{2} + k, \quad \text{for } k = 0, 1, 2.$$
Prove that f is one-to-one, and determine its range.

MOSP Black Group
Spring, 2007

TST 2005

1. Let n be an integer greater than 1. For a positive integer m, let $S_m = \{1, 2, \ldots, mn\}$. Suppose that there exists a $2n$-element set T such that

   (a) each element of T is an m-element subset of S_m;

   (b) each pair of elements of T shares at most one common element; and

   (c) each element of S_m is contained in exactly two elements of T.

   Determine the maximum possible value of m in terms of n.

2. Let $A_1A_2A_3$ be an acute triangle, and let O and H be its circumcenter and orthocenter, respectively. For $1 \le i \le 3$, points P_i and Q_i lie on lines OA_i and $A_{i+1}A_{i+2}$ (where $A_{i+3} = A_i$), respectively, such that OP_iHQ_i is a parallelogram. Prove that

$$\frac{OQ_1}{OP_1} + \frac{OQ_2}{OP_2} + \frac{OQ_3}{OP_3} \ge 3.$$

3. For a positive integer n, let S denote the set of polynomials $P(x)$ of degree n with positive integer coefficients not exceeding $n!$. A polynomial $P(x)$ in set S is called *fine* if for any positive integer k, the sequence $P(1), P(2), P(3), \ldots$ contains infinitely many integers relatively prime to k. Prove that at least 71% of the polynomials in the set S are fine.

4. Consider the polynomials

$$f(x) = \sum_{k=1}^{n} a_k x^k \quad \text{and} \quad g(x) = \sum_{k=1}^{n} \frac{a_k}{2^k - 1} x^k,$$

   where $a_1, a_2, \ldots, a_n$ are real numbers and n is a positive integer. Show that if 1 and 2^{n+1} are zeros of g then f has a positive zero less than 2^n.

5. Find all finite sets S of points in the plane with the following property: for any three distinct points A, B, and C in S, there is a fourth point D in S such that A, B, C, and D are the vertices of a parallelogram (in some order).

6. Let ABC be a acute scalene triangle with O as its circumcenter. Point P lies inside triangle ABC with $\angle PAB = \angle PBC$ and $\angle PAC = \angle PCB$. Point Q lies on line BC with $QA = QP$. Prove that $\angle AQP = 2\angle OQB$.

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TST 2006

1. A communications network consisting of some terminals is called a *3-connector* if among any three terminals, some two of them can directly communicate with each other. A communications network contains a *windmill* with n blades if there exist n pairs of terminals $\{x_1, y_1\}, \ldots, \{x_n, y_n\}$ such that each x_i can directly communicate with the corresponding y_i and there is a *hub* terminal that can directly communicate with each of the $2n$ terminals $x_1, y_1, \ldots, x_n, y_n$. Determine the minimum value of $f(n)$, in terms of n, such that a 3-connector with $f(n)$ terminals always contains a windmill with n blades.

2. In acute triangle ABC, segments AD, BE, and CF are its altitudes, and H is its orthocenter. Circle ω, centered at O, passes through A and H and intersects sides AB and AC again at Q and P (other than A), respectively. The circumcircle of triangle OPQ is tangent to segment BC at R. Prove that $CR/BR = ED/FD$.

3. Find the least real number k with the following property: if the real numbers x, y, and z are not all positive, then
$$k(x^2 - x + 1)(y^2 - y + 1)(z^2 - z + 1) \geq (xyz)^2 - xyz + 1.$$

4. Let n be a positive integer. Find, with proof, the least positive integer d_n which cannot be expressed in the form
$$\sum_{i=1}^{n} (-1)^{a_i} 2^{b_i},$$
where a_i and b_i are nonnegative integers for each i.

5. Let n be a given integer with n greater than 7, and let $\mathcal{P}$ be a convex polygon with n sides. Any set of $n-3$ diagonals of $\mathcal{P}$ that do not intersect in the interior of the polygon determine a triangulation of $\mathcal{P}$ into $n-2$ triangles. A triangle in the triangulation of $\mathcal{P}$ is an interior triangle if all of its sides are diagonals of $\mathcal{P}$.

   Express, in terms of n, the number of triangulations of $\mathcal{P}$ with exactly two interior triangles, in closed form.

6. Let ABC be a triangle. Triangles PAB and QAC are constructed outside of triangle ABC such that $AP = AB$ and $AQ = AC$ and $\angle BAP = \angle CAQ$. Segments BQ and CP meet at R. Let O be the circumcenter of triangle BCR. Prove that $AO \perp PQ$.