Combinatorics of Sets

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1 Warm-Up

1. Let $\mathcal{F}$ be a collection of subsets $A_1, A_2, \ldots$ of $\{1, \ldots, n\}$, such that for each $i \neq j$, $A_i \cap A_j \neq \emptyset$. Prove that $\mathcal{F}$ has size at most 2^{n-1}.

2. Suppose that $\mathcal{F}$ above has size exactly 2^{n-1}. Must there be a common element $x \in \{1, \ldots, n\}$ which is contained by every A_i?

3. Let $\mathcal{F}$ be a family of sets, each of size exactly 3, such that:

   (a) Every pair of sets intersects in a single element.

   (b) Every pair of elements in the ground set $X = \bigcup_{L \in \mathcal{F}} S$ is contained in a unique set $L \in \mathcal{F}$.

   Suppose that $\mathcal{F}$ has more than one set. Prove that the ground set X has exactly 7 elements, and show that such a family $\mathcal{F}$ exists.

2 Designs

1. (TST 2005/1.) Let n be an integer greater than 1. For a positive integer m, let $X_m = \{1, 2, \ldots, mn\}$. Suppose that there exists a family $\mathcal{F}$ of $2n$ subsets of X_m such that:

   (a) each member of $\mathcal{F}$ is an m-element subset of X_m;

   (b) each pair of members of $\mathcal{F}$ shares at most one common element;

   (c) each element of X_m is contained in exactly 2 elements of $\mathcal{F}$.

   Determine the maximum possible value of m in terms of n.

2. (USAMO 2011/6.) Let X be a set with $|X| = 225$. Suppose further that there are eleven subsets $A_1, \ldots, A_{11}$ of X such that $|A_i| = 45$ for $1 \leq i \leq 11$ and $|A_i \cap A_j| = 9$ for $1 \leq i < j \leq 11$. Prove that $|A_1 \cup \cdots \cup A_{11}| \geq 165$, and give an example for which equality holds.

3. A collection of subsets $L_1, \ldots, L_m$ in the universe $\{1, \ldots, n\}$ is called a *projective plane* if:

   (a) Every pair of sets (called "lines") intersects in a single element.

   (b) Every pair of elements in the ground set $X = \bigcup_{L \in \mathcal{F}} S$ is contained in a unique set $L \in \mathcal{F}$.

   Actually, there are two families of degenerate planes which satisfy the two conditions above, but are not considered to be projective planes. They are:

   (a) $L_1 = \{1, \ldots, n\}$, $L_2 = \{1\}$, $L_3 = \{1\}$, $L_4 = \{1\}$, ...

   (b) $L_1 = \{2, 3, \ldots, n\}$, $L_2 = \{1, 2\}$, $L_3 = \{1, 3\}$, $L_4 = \{1, 4\}$, ..., $L_n = \{1, n\}$.

It is well-known that for every projective plane, there is an N (called the "order" of the plane) such that:

(a) Every line contains exactly $N+1$ points, and every point is on exactly $N+1$ lines.

(b) The total number of points is exactly N^2+N+1, which is the same as the total number of lines.

4. For every prime power p^n, there exists a projective plane of that order.

5. (Open.) What are the possible orders of projective planes? All known projective planes have prime power order, but it is unknown whether, for example, there is a projective plane of order 12.

3 Graphs and partitioning

1. Construct a bipartite graph in which all degrees are equal, and every pair of vertices on the same side has exactly 1 common neighbor. Show that this must achieve the maximum possible number of edges in any C_4-free bipartite graph with the same number of vertices.

2. Construct a non-bipartite graph in which all degrees are equal, and every pair of vertices has exactly 1 common neighbor. Show that this must achieve the maximum possible number of edges in any C_4-free graph with the same number of vertices.

3. Let n be odd. Partition the edge set of K_n into n matchings with $\frac{n-1}{2}$ edges each.

4. Let n be even. Partition the edge set of K_n into $n-1$ matchings with $\frac{n}{2}$ edges each.

5. Find (nontrivial) infinite families of t and n for which it is possible to partition the edges of K_n into disjoint copies of edges corresponding to K_t.

4 Extremal set theory

1. (Erdős-Ko-Rado.) Let $n \geq 2k$ be positive integers, and let $\mathcal{C}$ be a collection of pairwise-intersecting k-element subsets of $\{1, \ldots, n\}$, i.e., every $A, B \in \mathcal{C}$ has $A \cap B \neq \emptyset$. Prove that $|\mathcal{C}| \leq \binom{n-1}{k-1}$.

   Remark. This corresponds to the construction which takes all subsets that contain the element 1.

2. (Non-uniform Fisher's inequality.) Let $\mathcal{C} = \{A_1, \ldots, A_r\}$ be a collection of distinct subsets of $\{1, \ldots, n\}$ such that every pairwise intersection $A_i \cap A_j$ $(i \neq j)$ has size t, where t is some fixed integer between 1 and n inclusive. Prove that $|\mathcal{C}| \leq n$.

5 Combinatorics and geometry

1. (Happy ending problem.) Given any 5 distinct points in the plane, no 3 collinear, show that some 4 are in *convex position*, i.e., forming the vertices of a convex quadrilateral.

2. (Erdős-Szekeres.) For every integer n, there is some finite N such that the following holds. Given any N distinct points in the plane, no 3 collinear, some n are in convex position.

   Remark. It is conjectured that $N = 1 + 2^{n-2}$ suffices for all $n \geq 3$, and known that $N \geq 1 + 2^{n-2}$ is required. The best known upper bound is of order $4^n/\sqrt{n}$.

3. (Caratheodory.) A *convex combination* of points x_i is defined as a linear combination of the form $\sum_i \alpha_i x_i$, where the α_i are non-negative coefficients which sum to 1.

   Let X be a finite set of points in $\mathbb{R}^d$, and let $\text{cvx}(X)$ denote the set of points in the convex hull of X, i.e., all points expressible as convex combinations of the $x_i \in X$. Show that each point $x \in \text{cvx}(X)$ can in fact be expressed as a convex combination of only $d+1$ points of X.

4. (Radon.) Let A be a set of at least $d+2$ points in $\mathbb{R}^d$. Show that A can be split into two disjoint sets $A_1 \cup A_2$ such that $\text{cvx}(A_1)$ and $\text{cvx}(A_2)$ intersect.

5. (Helly.) Let $C_1, C_2, \ldots, C_n$ be sets of points in $\mathbb{R}^d$, with $n \geq d+1$. Suppose that every $d+1$ of the sets have a non-empty intersection. Show that all n of the sets have a non-empty intersection.

6 Bonus problems

1. (From Peter Winkler.) The 60 MOPpers were divided into 8 teams for Team Contest 1. They were then divided into 7 teams for Team Contest 2. Prove that there must be a MOPper for whom the size of her team in Contest 2 was strictly larger than the size of her team in Contest 1.

2. (MOP 2008.) Let $\mathcal{F}$ be a collection of 2^{n-1} subsets $A_1, A_2, \ldots$ of $\{1, \ldots, n\}$, such that for each $i \neq j \neq k$, $A_i \cap A_j \cap A_k \neq \emptyset$. Prove that there is a common element $x \in \{1, \ldots, n\}$ that is contained in every A_i.

3. (Sperner capacity of cyclic triangle, also Iran 2006.) Let A be a collection of vectors of length n from $\mathbb{Z}_3$ with the property that for any two distinct vectors $a, b \in A$ there is some coordinate i such that $b_i = a_i + 1$, where addition is defined modulo 3. Prove that $|A| \leq 2^n$.