34^{th} United States of America Mathematical Olympiad

Day I 12:30 PM – 5 PM EDT

April 19, 2005

1. Determine all composite positive integers n for which it is possible to arrange all divisors of n that are greater than 1 in a circle so that no two adjacent divisors are relatively prime.

2. Prove that the system

$$x^6 + x^3 + x^3y + y = 147^{157}$$

$$x^3 + x^3y + y^2 + y + z^9 = 157^{147}$$

has no solutions in integers x, y, and z.

3. Let ABC be an acute-angled triangle, and let P and Q be two points on side BC. Construct point C_1 in such a way that convex quadrilateral $APBC_1$ is cyclic, $QC_1 \parallel CA$, and C_1 and Q lie on opposite sides of line AB. Construct point B_1 in such a way that convex quadrilateral $APCB_1$ is cyclic, $QB_1 \parallel BA$, and B_1 and Q lie on opposite sides of line AC. Prove that points B_1, C_1, P, and Q lie on a circle.

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Day II 12:30 PM – 5 PM EDT

April 20, 2005

1. Legs L_1, L_2, L_3, L_4 of a square table each have length n, where n is a positive integer. For how many ordered 4-tuples (k_1, k_2, k_3, k_4) of nonnegative integers can we cut a piece of length k_i from the end of leg L_i $(i = 1, 2, 3, 4)$ and still have a stable table? (The table is *stable* if it can be placed so that all four of the leg ends touch the floor. Note that a cut leg of length 0 is permitted.)

2. Let n be an integer greater than 1. Suppose $2n$ points are given in the plane, no three of which are collinear. Suppose n of the given $2n$ points are colored blue and the other n colored red. A line in the plane is called a *balancing line* if it passes through one blue and one red point and, for each side of the line, the number of blue points on that side is equal to the number of red points on the same side. Prove that there exist at least two balancing lines.

3. For m a positive integer, let $s(m)$ be the sum of the digits of m. For $n \geq 2$, let $f(n)$ be the minimal k for which there exists a set S of n positive integers such that $s\left(\sum_{x \in X} x\right) = k$ for any nonempty subset $X \subset S$. Prove that there are constants $0 < C_1 < C_2$ with

$$C_1 \log_{10} n \leq f(n) \leq C_2 \log_{10} n.$$