COMPLEX NUMBERS

When we are solving quadratic equations with real roots, the roots of the equations exhibit three cases: two distinct real roots, a double root or no real roots. To accommodate the case of no real roots, i.e., to provide solutions to those quadratic equations, the concept of complex numbers was invented. One may expect that as complex numbers are never encountered in our daily routines, the studies of complex numbers are impractical. However, complex numbers are actually extremely useful tools in many applications like factorization and proofs of trigonometric identities. In this set of notes, we will illustrate the basic concepts and some of the applications of complex numbers with examples.

1. Basic Terms and Concepts

As noted, the interests in complex numbers originated from studies of the roots of quadratic equations. As an illustration, consider the equation $x^2+1=0$. This equation does not admit real roots because for all real number x, $x^2+1\geq 0+1=1>0$. Yet, if we set $i=\sqrt{-1}$, then $i^2+1=-1+1=0$ is a root of the equation.

Definition 1.1. (Imaginary unit)

The imaginary unit is defined by $i=\sqrt{-1}$.

Based on the imaginary unit, we develop a set of numbers called the complex numbers. The general form of a complex number z is $z = a + bi$, where a and b are real numbers.

Definition 1.2 (Real and imaginary parts of a complex number)

For a complex number $z = a + bi$, where $a,b\in\mathbb{R}$, a is called the **real part** of z and b is called the **imaginary part** of z. These are denoted as $\mathrm{Re}(z) = a$ and $\mathrm{Im}(z) = b$ respectively.

Illustration. If $z = 3 + 5i$, then the real part of z is 3 and the imaginary part is 5. We write $\mathrm{Re}(z) = 3$ and $\mathrm{Im}(z) = 5$.

Illustration. All real numbers x can be represented in the form of a complex number by setting a =

x and b = 0. Hence, $\mathbb{R} \subset \mathbb{C}$, where $\mathbb{R}$ represents the set of real numbers and $\mathbb{C}$ the set of complex numbers.

In particular, if Im(z) = 0, then z is a real number. On the contrary, if Re(z) = 0 and $\operatorname{Im}(z) \neq 0$, z is called a **purely imaginary number**.

In general, we cannot compare complex numbers with each other as in real numbers. For example, we say 6 > 4, but we cannot say which of 6i and 4i is larger or smaller. However, we can say that they are unequal. In fact, two complex numbers are said to be equal iff both their real parts and imaginary parts are equal.

2. Arithmetic Operations

Basic arithmetic operations on complex numbers include addition, subtraction, multiplication and division. For two complex numbers $z_1 = a_1 + b_1 i$ and $z_2 = a_2 + b_2 i$, we have

$$z_1 + z_2 = (a_1 + b_1 i) + (a_2 + b_2 i) = (a_1 + a_2) + (b_1 + b_2) i,$$

$$z_1 - z_2 = (a_1 + b_1 i) - (a_2 + b_2 i) = (a_1 - a_2) + (b_1 - b_2) i,$$

$$\begin{aligned} z_1 z_2 &= (a_1 + b_1 i)(a_2 + b_2 i) = a_1 a_2 + a_1 b_2 i + b_1 a_2 i + b_1 i b_2 i \\ &= a_1 a_2 + a_1 b_2 i + a_2 b_1 i + b_1 b_2 i^2 = (a_1 a_2 - b_1 b_2) + (a_1 b_2 + a_2 b_1) i, \end{aligned}$$

and if $z_2 \neq 0$,

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{a_1 + b_1 i}{a_2 + b_2 i} = \frac{(a_1 + b_1 i)(a_2 - b_2 i)}{(a_2 + b_2 i)(a_2 - b_2 i)} = \frac{(a_1 a_2 + b_1 b_2) + (a_2 b_1 - a_1 b_2) i}{a_2^{\,2} - (b_2 i)^2} \\ &= \frac{(a_1 a_2 + b_1 b_2) + (a_2 b_1 - a_1 b_2) i}{a_2^{\,2} + b_2^{\,2}} = \frac{a_1 a_2 + b_1 b_2}{a_2^{\,2} + b_2^{\,2}} + \frac{a_2 b_1 - a_1 b_2}{a_2^{\,2} + b_2^{\,2}} i. \end{aligned}$$

Example 2.1.

Suppose $z_k = 3^{-k} + 2^{-k} i$ for k = 0, 1, 2, … . Evaluate $\sum_{k=0}^{\infty} z_k$.

Solution.

Evaluating the real and imaginary parts separately and applying the formula for geometric series, we have

$$\sum_{k=0}^{\infty} z_k = \sum_{k=0}^{\infty}\left(3^{-k}+2^{-k}i\right)=\sum_{k=0}^{\infty}3^{-k}+i\sum_{k=0}^{\infty}2^{-k}=\frac{1}{1-\frac{1}{3}}+\frac{1}{1-\frac{1}{2}}i=\frac{3}{2}+2i\,.$$

Example 2.2.

Suppose $z_k = a_k + b_k i$ for k = 1, 2, where $a_k, b_k \in \mathbb{R}$. Let $z_1 = 3z_2$ and $z_1 = (1+2i)^4$. Find a_2 and b_2.

Solution.

We shall evaluate z_1 first:

$$\begin{aligned} z_1 &= (1+2i)^4 = \left[(1+2i)^2\right]^2 = \left(1+4i+4i^2\right)^2 = (1-4+4i)^2 = (-3+4i)^2 \\ &= 9-24i+16i^2 = 9-24i-16 = -7-24i \end{aligned}$$

Then, $-7-24i = z_1 = 3z_2 = 3(a_2+b_2 i) = 3a_2 + 3b_2 i$. Equating the real and imaginary parts, we have

$$a_2 = -\frac{7}{3} \text{ and } b_2 = -8\,.$$

Example 2.3.

Express the complex number $\dfrac{1+i\tan\theta}{1-i\tan\theta}$ in the general form.

Solution.

$$\begin{aligned} \frac{1+i\tan\theta}{1-i\tan\theta} &= \frac{(1+i\tan\theta)(1+i\tan\theta)}{(1-i\tan\theta)(1+i\tan\theta)} = \frac{1+2i\tan\theta+(i\tan\theta)^2}{1^2-(i\tan\theta)^2} = \frac{1-\tan^2\theta+2i\tan\theta}{1+\tan^2\theta} \\ &= \frac{1-\tan^2\theta+2i\tan\theta}{\sec^2\theta} = \frac{1-\tan^2\theta}{\sec^2\theta}+\frac{2i\tan\theta}{\sec^2\theta} = \left(\cos^2\theta-\sin^2\theta\right)+2i\sin\theta\cos\theta \\ &= \cos 2\theta + i\sin 2\theta \end{aligned}$$

Exercise

1. Express $\dfrac{1}{\cos\theta - i\sin\theta}$ in the general form.

2. Suppose $z = a + 2i$ is a root of the equation $x^2+6x+k=0$, where both a and k are real. Find a

and k.

3. The Argand Diagram, Modulus and Argument

Maybe it is too abstract to just talk about complex numbers. A better understanding of this number system can be obtained by drawing diagrams. In this section, we will discuss the Argand diagrams of complex numbers. The Argand diagrams are also of core importance in the application of complex numbers on some geometry problems.

An Argand diagram consists of two axes: a horizontal one, like the x-axis in the rectangular coordinate system, called the real axis, and a vertical one, like the y-axis, called the imaginary axis. The point (x, y) in an Argand diagram represents the complex number $x + yi$.

In figure 1, the complex numbers represented by the points A, B, C and D are $1 + i$, $2 - 3i$, $3 - 2i$ and $-3 + i$ respectively. By convention the number zero is represented by the origin O. Another way to interpret Argand diagrams is by way of vectors. For example, the complex numbers mentioned above are said to be associated with the vectors $\overrightarrow{OA}$, $\overrightarrow{OB}$, $\overrightarrow{OC}$ and $\overrightarrow{OD}$ respectively. Readers might have noticed that $1 + i + 2 - 3i = 3 - 2i$, which in the notation of vectors is represented by $\overrightarrow{OA} + \overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AC} = \overrightarrow{OC}$.

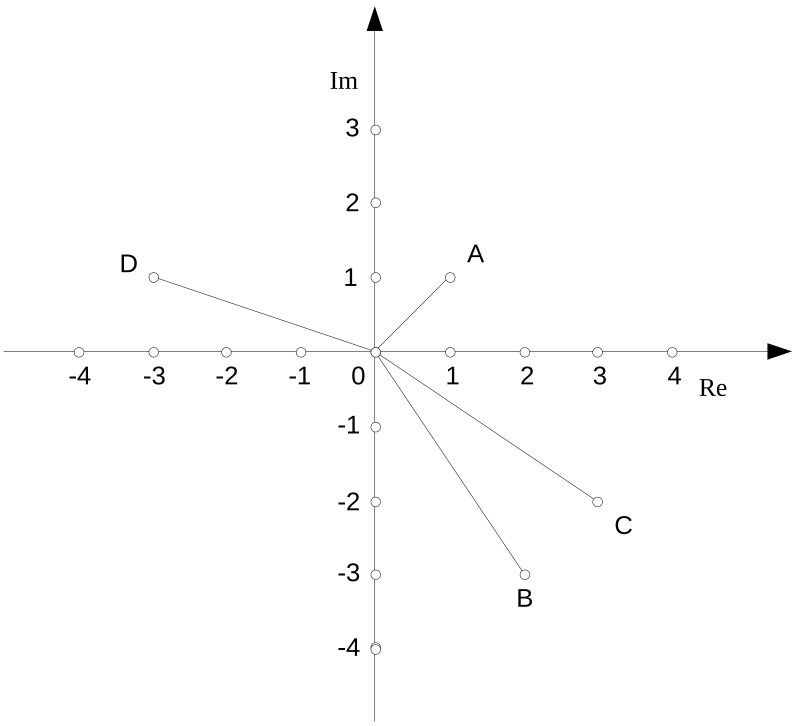

Figure 1: Argand Diagram

From the perspectives of the Argand diagram and vectors, we can define some more properties of complex numbers. The length of the vector in the Argand diagram, *r*, is called the modulus of the associated complex number, while the angle the vector made with positive real axis, θ, is said to be an argument of the complex number. Hence we have, if $z \neq 0$,

$$|x+yi| = r = \sqrt{x^2+y^2} ,$$

$$x = r\cos\theta \text{ and } y = r\sin\theta .$$

An immediate result of the above is that $\tan\theta = \dfrac{y}{x}$.

Since $\cos\theta = \cos(\theta+2k\pi)$ and $\sin\theta = \sin(\theta+2k\pi)$ for any integer *k*, there are infinitely many possible values for the argument of a complex number. And this leads to the following definition.

Definition 3.1.

The principal value of the argument of a complex number *z* = *x* + *yi* is the angle θ such that $r\cos\theta = x$, $r\sin\theta = y$ and $-\pi < \theta \leq \pi$.

- We usually denote the principal argument by Arg *z*, and the set of arguments by arg *z* = Arg *z* + $2k\pi$.

Example 3.1.

Find the modulus and the principal value of the argument of the complex number 1 + *i*.

Solution.

We now have *x* = *y* = 1. Hence, modulus $= r = \sqrt{1^2+1^2} = \sqrt{2}$.

Besides, $\tan^{-1}\left(\dfrac{y}{x}\right) = \tan^{-1}1$ has only two possible values, $\dfrac{\pi}{4}$ and $-\dfrac{3\pi}{4}$ in the interval $(-\pi,\pi]$. We have $r\cos\dfrac{\pi}{4} = 1 = x$ and $r\sin\dfrac{\pi}{4} = 1 = y$, but $r\cos\left(-\dfrac{3\pi}{4}\right) = -1 \neq x$ and $r\sin\left(-\dfrac{3\pi}{4}\right) = -1 \neq y$.

Hence, the principal value of the argument of the complex number 1 + *i* is $\dfrac{\pi}{4}$.

Exercise

1. Find the moduli and principal arguments of the following complex numbers:
   (a) $\sqrt{3}+i$
   (b) $1-\sqrt{3}i$
   (c) $-3-3i$
   (d) $\sin\theta - i\cos\theta$, where $0 \le \theta \le \frac{\pi}{2}$.

2. (a) Evaluate $\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$.

   (b) Suppose $z_1=\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}$ and $z_2=\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}$. Draw the points Z_1 and Z_2, representing z_1 and z_2 respectively, on the Argand diagram. What is the relationship between OZ_1 and OZ_2?

4. Other Forms of Representation

The concepts of modulus and argument are important in dealing with representations of complex numbers. In this section, we will introduce the polar form and the exponential form, both of which are associated with representing complex numbers in terms of their moduli and arguments.

The Polar Form

Definition 4.1. (The polar form)

Suppose r and θ are respectively the modulus and argument of the complex number z. Then the polar form of z is $r\left(\cos\theta+i\sin\theta\right)$, or abbreviated as $r\operatorname{cis}\theta$.

Illustration. The polar form of the complex number 1 + i is $\sqrt{2}\left(\cos\frac{\pi}{4}+\sin\frac{\pi}{4}\right)$, or in short, $\sqrt{2}\operatorname{cis}\frac{\pi}{4}$.

Example 4.1.

Express, in polar form, the following complex numbers:

(a) $\sqrt{3}+i$

(b) $1+\sin\theta - i\cos\theta$ for $-\frac{\pi}{2}\le\theta\le\frac{3\pi}{2}$.

Solution.

(a) Let $z=\sqrt{3}+i$. Then $r=\sqrt{\left(\sqrt{3}\right)^2+1^2}=\sqrt{3+1}=2$.

$$\tan\theta=\frac{1}{\sqrt{3}}\Rightarrow\theta=-\frac{5\pi}{6}\text{ (rejected) or }\frac{\pi}{6}$$

Hence, the polar form of z is $r\operatorname{cis}\theta=2\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right)$.

(b) Let $z=1+\sin\theta-i\cos\theta$. Then,

$$\begin{aligned}z&=1+\cos\left(\frac{\pi}{2}-\theta\right)-i\sin\left(\frac{\pi}{2}-\theta\right)=1+\left[2\cos^2\left(\frac{\pi}{4}-\frac{\theta}{2}\right)-1\right]-2i\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\\&=2\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\left[\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)-i\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\right]=2\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\left[\cos\left(\frac{\theta}{2}-\frac{\pi}{4}\right)+i\sin\left(\frac{\theta}{2}-\frac{\pi}{4}\right)\right].\end{aligned}$$

Since $-\frac{\pi}{2}\le\theta\le\frac{3\pi}{2}$, we have $-\frac{\pi}{2}\le\frac{\pi}{4}-\frac{\theta}{2}\le\frac{\pi}{2}$, i.e., $2\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\ge 0$. Therefore, the polar form of z is $2\cos\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\left[\cos\left(\frac{\theta}{2}-\frac{\pi}{4}\right)+i\sin\left(\frac{\theta}{2}-\frac{\pi}{4}\right)\right]$.

The polar form of representation is especially useful when we perform multiplication and division of complex numbers.

Theorem 4.1.

Let z_1 and z_2 be two complex numbers. Then for $\arg(z_i)\ne 0$,

1. $|z_1z_2|=|z_1||z_2|$ and $\arg(z_1z_2)=\arg(z_1)+\arg(z_2)$.
2. If $z_2\ne 0$, $\left|\frac{z_1}{z_2}\right|=\frac{|z_1|}{|z_2|}$ and $\arg\left(\frac{z_1}{z_2}\right)=\arg(z_1)-\arg(z_2)$.

Proof. Let $z_1=r_1\operatorname{cis}\theta_1$ and $z_2=r_2\operatorname{cis}\theta_2$ be the polar form representations.

For property 1, using the above notations,

$$
\begin{aligned}
z_1 z_2 &= r_1\left(\cos\theta_1 + i\sin\theta_1\right)\cdot r_2\left(\cos\theta_2 + i\sin\theta_2\right)\\
&= r_1 r_2\left(\cos\theta_1\cos\theta_2 + i\sin\theta_1\cos\theta_2 + i\cos\theta_1\sin\theta_2 + i^2\sin\theta_1\sin\theta_2\right)\\
&= r_1 r_2\left[\left(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\right) + i\left(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2\right)\right]\\
&= r_1 r_2\left[\cos\left(\theta_1+\theta_2\right) + i\sin\left(\theta_1+\theta_2\right)\right] = r_1 r_2 \operatorname{cis}\left(\theta_1+\theta_2\right).
\end{aligned}
$$

Hence, $|z_1 z_2| = r_1 r_2 = |z_1||z_2|$ and $\arg\left(z_1 z_2\right) = \theta_1 + \theta_2 = \arg\left(z_1\right) + \arg\left(z_2\right)$.

For property 2, let $z_1' = \dfrac{z_1}{z_2}$. Then using the results of the above proof, we have

$$|z_1' z_2| = |z_1'||z_2| = \left|\frac{z_1}{z_2}\right||z_2| \text{ and } \arg\left(z_1' z_2\right) = \arg\left(z_1'\right) + \arg\left(z_2\right) = \arg\left(\frac{z_1}{z_2}\right) + \arg\left(z_2\right).$$

On the other hand, $z_1' z_2 = \dfrac{z_1}{z_2}\cdot z_2 = z_1$. Hence,

$$\left|\frac{z_1}{z_2}\right| = \frac{|z_1' z_2|}{|z_2|} = \frac{|z_1|}{|z_2|} \text{ and } \arg\left(\frac{z_1}{z_2}\right) = \arg\left(z_1' z_2\right) - \arg\left(z_2\right) = \arg\left(z_1\right) - \arg\left(z_2\right).$$

Q.E.D.

Definition 4.2. (The exponential form)

Suppose r and θ are respectively the modulus and argument of the complex number z. Then the exponential form of z is $re^{i\theta}$.

Theorem 4.1. can be applied to the exponential form easily with the law of indices. For property 1, by definition 4.2, $z_1 z_2 = r_1 e^{\theta_1} \bullet r_2 e^{\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}$. Hence, $|z_1 z_2| = r_1 r_2 = |z_1||z_2|$ and $\arg\left(z_1 z_2\right) = \theta_1 + \theta_2 = \arg\left(z_1\right) + \arg\left(z_2\right)$ Similarly, for property 2, $\dfrac{z_1}{z_2} = \dfrac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \dfrac{r_1}{r_2} e^{i(\theta_1-\theta_2)}$. So, $\left|\dfrac{z_1}{z_2}\right| = \dfrac{r_1}{r_2} = \dfrac{|z_1|}{|z_2|}$ and $\arg\left(\dfrac{z_1}{z_2}\right) = \theta_1 - \theta_2 = \arg\left(z_1\right) - \arg\left(z_2\right)$.

Theorem 4.2.

1. $e^{i\theta} + e^{-i\theta} = 2\cos\theta$.
2. $e^{i\theta} - e^{-i\theta} = 2i\sin\theta$.

Proof. For property 1, $e^{i\theta}+e^{-i\theta}=\left(\cos\theta+i\sin\theta\right)+\left[\cos\left(-\theta\right)+i\sin\left(-\theta\right)\right]=2\cos\theta$.

For property 2, $e^{i\theta}-e^{-i\theta}=\left(\cos\theta+i\sin\theta\right)-\left[\cos\left(-\theta\right)+i\sin\left(-\theta\right)\right]=2i\sin\theta$.

Q.E.D.

Example 4.2.

Simplify the fraction $\dfrac{1+\sin\theta+i\cos\theta}{1-\sin\theta+i\cos\theta}$.

Solution.

Notice that

$$\sin\theta+i\cos\theta=\cos\left(\frac{\pi}{2}-\theta\right)+i\sin\left(\frac{\pi}{2}-\theta\right)=e^{i\left(\frac{\pi}{2}-\theta\right)},$$

$$\sin\theta-i\cos\theta=\cos\left(\frac{\pi}{2}-\theta\right)-i\sin\left(\frac{\pi}{2}-\theta\right)=\cos\left(\theta-\frac{\pi}{2}\right)+i\sin\left(\theta-\frac{\pi}{2}\right)=e^{i\left(\theta-\frac{\pi}{2}\right)}.$$

Hence,

$$\begin{aligned}\frac{1+\sin\theta+i\cos\theta}{1-\sin\theta+i\cos\theta}&=\frac{1+e^{i\left(\frac{\pi}{2}-\theta\right)}}{1-e^{i\left(\theta-\frac{\pi}{2}\right)}}=\frac{e^{i\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}\left[e^{i\left(\frac{\theta}{2}-\frac{\pi}{4}\right)}+e^{i\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}\right]}{e^{i\left(\frac{\theta}{2}-\frac{\pi}{4}\right)}\left[e^{i\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}-e^{i\left(\frac{\theta}{2}-\frac{\pi}{4}\right)}\right]}=e^{i\left(\frac{\pi}{2}-\theta\right)}\frac{2\cos\left(\frac{\theta}{2}-\frac{\pi}{4}\right)}{2i\sin\left(\frac{\pi}{4}-\frac{\theta}{2}\right)}\\&=e^{i\left(\frac{\pi}{2}-\theta\right)}\frac{\cos\left(\frac{\theta}{2}-\frac{\pi}{4}\right)}{-i\sin\left(\frac{\theta}{2}-\frac{\pi}{4}\right)}=e^{i\left(\frac{\pi}{2}-\theta\right)}\cdot i\cot\left(\frac{\theta}{2}-\frac{\pi}{4}\right)=e^{i\left(\frac{\pi}{2}-\theta\right)}\cdot e^{i\cdot\frac{\pi}{2}}\cot\left(\frac{\theta}{2}-\frac{\pi}{4}\right)\\&=e^{i(\pi-\theta)}\cot\left(\frac{\theta}{2}-\frac{\pi}{4}\right)=\cot\left(\frac{\theta}{2}-\frac{\pi}{4}\right)\operatorname{cis}\left(\pi-\theta\right).\end{aligned}$$

Exercise

1. Simplify the fraction $\dfrac{3+3\sqrt{3}i}{1-i}$.

2. Prove that $\left(\cos\theta - i\sin\theta\right)^3 = \cos 3\theta - i\sin 3\theta$.

5. Complex Conjugates

Definition 5.1. (Complex conjugate)

Suppose $z = a + bi$ is a complex number, where $a, b \in \mathbb{R}$. Then the complex conjugate of z is defined as $\overline{z} = a - bi$.

Thus the complex conjugate of a complex number is the mirror image of the original number across the real axis. There are some nice properties associated with complex conjugate pairs.

Theorem 5.1.

For any complex number z, we have

1. $\left|\overline{z}\right| = \left|z\right|$.
2. $\arg\left(\overline{z}\right) = -\arg\left(z\right)$.
3. $z + \overline{z} = 2\operatorname{Re}\left(z\right)$.
4. $z - \overline{z} = 2i\operatorname{Im}\left(z\right)$.
5. $z\overline{z} = \left|z\right|^2$.
6. $\overline{\overline{z}} = z$.

Proof. The theorem follows easily from the definition and its proof is left as an exercise.

Q.E.D.

An implication of this theorem is that if z is real, then $z - \overline{z} = 2i\operatorname{Im}\left(z\right) = 0$. Likewise, if z is purely imaginary, then $z + \overline{z} = 2\operatorname{Re}\left(z\right) = 0$. The converses of these are also true, with the only exception that the number zero is real but 2Re(0) = 0. These facts are often useful in proving some of the properties of some numbers in questions.

Example 5.1.

Suppose $|w|=1$ and $z=w+\dfrac{1}{w}$. Prove that *z* is a real number.

Solution.

By property 5 of theorem 5.1, we have $w\overline{w}=|w|^2=1^2=1$. Hence, $\overline{w}=\dfrac{1}{w}$ as $w\neq 0$. Therefore, $z=w+\dfrac{1}{w}=w+\overline{w}=2\operatorname{Re}(w)$ must be a real number.

Theorem 5.2.

For complex numbers z_1 and z_2, we have

1. $\overline{z_1+z_2}=\overline{z}_1+\overline{z}_2$.
2. $\overline{z_1-z_2}=\overline{z}_1-\overline{z}_2$.
3. $\overline{z_1\cdot z_2}=\overline{z}_1\cdot\overline{z}_2$.
4. $\overline{\left(\dfrac{z_1}{z_2}\right)}=\dfrac{\overline{z}_1}{\overline{z}_2}$, if $z_2\neq 0$.

Proof. The proofs of the first two properties are left as an exercise.

For property 3, we let $z_1=a_1+b_1i$ and $z_2=a_2+b_2i$. Then $\overline{z}_1=a_1-b_1i$ and $\overline{z}_2=a_2-b_2i$. Hence,

$$\begin{aligned}\overline{z}_1\cdot\overline{z}_2&=(a_1-b_1i)(a_2-b_2i)=(a_1a_2-b_1b_2)-(a_1b_2+a_2b_1)i\\&=\overline{(a_1a_2-b_1b_2)+(a_1b_2+a_2b_1)i}=\overline{(a_1+b_1i)(a_2+b_2i)}=\overline{z_1z_2}.\end{aligned}$$

For property 4, using previous results,

$$\overline{z}_2\cdot\overline{\left(\frac{z_1}{z_2}\right)}=\overline{z_2\cdot\frac{z_1}{z_2}}=\overline{z}_1.$$

And the theorem follows.

Q.E.D.

Example 5.2.

Let *z* = *x* + *yi*. If $|z-6|=5$ and $|z|=5$, find the possible values of *x* and *y*.

Solution.

From property 5 of theorem 5.1 and property 3 of theorem 5.2,

$$5^2 = |z-6|^2 = (z-6)\left(\overline{z-6}\right) = (z-6)(\overline{z}-6),$$

$$\begin{aligned} 25 &= z\overline{z} - 6z - 6\overline{z} + 36 = |z|^2 - 6(z+\overline{z}) + 36 \\ &= 5^2 - 6\cdot 2\operatorname{Re}(z) + 36. \end{aligned}$$

Hence, $12\operatorname{Re}(z) = 36 \Rightarrow x = \operatorname{Re}(z) = 3$.

On the other hand, $5 = |z| = \sqrt{x^2+y^2} = \sqrt{3^2+y^2}$. Thus, $y^2 = 5^2 - 3^2 = 4^2$. The possible values of y are therefore ± 4.

We started our discussion of complex numbers on solving algebraic equations. In fact, there is a nice relationship between the complex conjugates in solving equations.

Theorem 5.3.

If z is a zero of a real polynomial (polynomial with real coefficients), then so is $\overline{z}$.

Proof. Suppose z is a zero of the real polynomial $a_n x^n + a_{n-1}x^{n-1} + \cdots a_0$. From property 3 of theorem 5.2, $(\overline{z})^k = \overline{z}\cdot\overline{z}\cdot\cdots\cdot\overline{z} = \overline{z\cdot z\cdot\cdots\cdot z} = \overline{z^k}$ for positive integer k. Besides, since the a_k 's are real, we have $a_k = \overline{a_k}$. Then,

$$\begin{aligned} a_n(\overline{z})^n + a_{n-1}(\overline{z})^{n-1} + \cdots a_0 &= \overline{a_n}\,\overline{z^n} + \overline{a_{n-1}}\,\overline{z^{n-1}} + \cdots \overline{a_0} = \overline{a_n z^n} + \overline{a_{n-1}z^{n-1}} + \cdots \overline{a_0} \\ &= \overline{a_n z^n + a_{n-1}z^{n-1} + \cdots a_0} = \overline{0} = 0 \end{aligned}$$

Thus, $\overline{z}$ is also a zero of the polynomial.

Q.E.D.

This theorem implies that for a real polynomial, either both members of a complex conjugate pair are the zeros, or both are not. In fact, a real polynomial can only have an even number of complex zeros.

Example 5.3.

Given that 3 + i is a root of the equation $x^4 - 5x^3 + 10x^2 + ax + b = 0$, where a and b are real numbers. Find a and b.

Solution.

Since 3 + i is a root, its conjugate, 3 – i, is also a root. By factor theorem, $x^4-5x^3+10x^2+ax+b$ is divisible by both x – (3 + i) and x – (3 – i), i.e., by $(x-3-i)(x-3+i)=x^2-6x+10$.

Upon long division by $x^2-6x+10$, the remainder of $x^4-5x^3+10x^2+ax+b$ is $(a+26)x+(b-60)$. Thus, we have a + 26 = 0 and b – 60 = 0, i.e., a = –26 and b = 60.

Exercise

1. Prove theorem 5.1.

2. Prove the first two properties of theorem 5.2.

3. Let k be a real constant and z be a complex number with $|z|=1$. Prove that $|z+k|=|kz+1|$.

4. Given that 1 + 3i is a root of the equation $x^4+3x^2+ax+b=0$. Find a and b and factorize x^4+3x^2+ax+b.

6. De Moivre's Theorem

Performing multiplications and divisions on complex numbers can be very tedious and error-prone, as we have to deal with both the real and imaginary parts. If we employ the polar form and exponential form, the operations will be much simpler. Furthermore, if the operations involve rational powers of complex numbers, the operations can be eased with the De Moivre's theorem.

Theorem 6.1. (De Moivre's Theorem for Integral Index)
For any integer n, $(\cos\theta+i\sin\theta)^n=\cos n\theta+i\sin n\theta$.

Proof. We shall prove the theorem for non-negative n by induction first. For n = 0 or 1, the result is obvious.

Suppose for $n = k$, $(\text{cis}\theta)^k = \text{cis}k\theta$. Then when $n = k+1$, by theorem 4.1,

$$(\text{cis}\theta)^{k+1} = (\text{cis}\theta)^k \text{cis}\theta = \text{cis}k\theta\text{cis}\theta = \text{cis}(k+1)\theta .$$

So, by mathematical induction, the theorem is true for all non-negative integer n.

For negative n, let $n = -m$. So m is positive. Then,

$$(\text{cis}\theta)^n = (\text{cis}\theta)^{-m} = \frac{1}{(\text{cis}\theta)^m} = \frac{1}{\text{cis}m\theta} = \frac{\text{cis}0}{\text{cis}m\theta} = \text{cis}(0-m\theta) = \text{cis}(-m\theta) = \text{cis}n\theta .$$

Thus the theorem is true for all integer n.

Q.E.D.

Example 6.1.

Redo Example 2.3 using De Moivre's theorem.

Solution.

If $\cos\theta = 0$, then $\theta = k\pi + \frac{\pi}{2}$. Hence, we have $\tan\theta = \infty$, $\cos 2\theta = \cos(2k\pi + \pi) = -1$ and $\sin 2\theta = \sin(2k\pi + \pi) = 0$. So,

$$\frac{1+i\tan\theta}{1-i\tan\theta} = -1 = \cos 2\theta + i\sin 2\theta .$$

For $\cos\theta \neq 0$,

$$\begin{aligned}\frac{1+i\tan\theta}{1-i\tan\theta} &= \frac{(1+i\tan\theta)(\cos\theta)}{(1-i\tan\theta)(\cos\theta)} = \frac{\cos\theta + i\sin\theta}{\cos\theta - i\sin\theta} = \frac{\cos\theta + i\sin\theta}{\cos(-\theta) + i\sin(-\theta)} \\ &= \frac{\text{cis}\theta}{\text{cis}(-\theta)} = \text{cis}\theta\left[\text{cis}(-\theta)\right]^{-1} = \text{cis}\theta(\text{cis}\theta) = (\text{cis}\theta)^2 = \text{cis}2\theta \\ &= \cos 2\theta + i\sin 2\theta.\end{aligned}$$

The De Moivre's theorem for integral index is often used in proving trigonometric identities. In fact, many identities can be proved with the help of some direct results of the theorem: for $z = \text{cis}\theta$,

$$z^n + \frac{1}{z^n} = \text{cis}n\theta + \text{cis}(-n\theta) = 2\cos n\theta ,$$

$$z^n - \frac{1}{z^n} = \text{cis}n\theta - \text{cis}(-n\theta) = 2i\sin n\theta .$$

Example 6.2.

Show that $\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$.

Hence prove that $\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$.

Solution.

According to the De Moivre's theorem, $(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$. However, by the binomial theorem,

$$\begin{aligned}
&(\cos\theta + i\sin\theta)^5 \\
&= \cos^5\theta + 5\cos^4\theta(i\sin\theta) + 10\cos^3\theta(i\sin\theta)^2 + 10\cos^2\theta(i\sin\theta)^3 + 5\cos\theta(i\sin\theta)^4 + (i\sin\theta)^5 \\
&= \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta.
\end{aligned}$$

Equating the real parts of both equations, we get $\cos 5\theta = \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta$.

Hence, using the identity $\sin^2\theta + \cos^2\theta = 1$, we have

$$\begin{aligned}
\cos 5\theta &= \cos^5\theta - 10\cos^3\theta\sin^2\theta + 5\cos\theta\sin^4\theta \\
&= \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1-\cos^2\theta)^2 \\
&= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta.
\end{aligned}$$

Example 6.3.

Let $z = \operatorname{cis}\theta$. By using the relations $z + \frac{1}{z} = 2\cos\theta$ and $z - \frac{1}{z} = 2i\sin\theta$, deduce that

$$\sin^2\theta\cos^4\theta = -\frac{1}{32}\cos 6\theta - \frac{1}{16}\cos 4\theta + \frac{1}{32}\cos 2\theta + \frac{1}{16}.$$

Solution.

Substituting with the given relations,

$$\begin{aligned}
(2i\sin\theta)^2(2\cos\theta)^4 &= \left(z - \frac{1}{z}\right)^2\left(z + \frac{1}{z}\right)^4 \\
-64\sin^2\theta\cos^4\theta &= z^6 + 2z^4 - z^2 - 4 - z^{-2} + 2z^{-4} + z^{-6} \\
&= (z^6 + z^{-6}) + 2(z^4 + z^{-4}) - (z^2 + z^{-2}) - 4 \\
&= 2\cos 6\theta + 4\cos 4\theta - 2\cos 2\theta - 4.
\end{aligned}$$

Hence,

$$\sin^2\theta\cos^4\theta = -\frac{1}{32}\cos 6\theta - \frac{1}{16}\cos 4\theta + \frac{1}{32}\cos 2\theta + \frac{1}{16}.$$

The De Moivre's theorem can be generalized to rational index.

Theorem 6.2. (De Moivre's Theorem for Rational Index)

For any positive integer *n*,

$$(\text{cis}\,\theta)^{\frac{1}{n}} = \text{cis}\,\frac{2k\pi+\theta}{n}$$

where $k = 0, 1, 2, \dots, n-1$.

Proof. We let $r\,\text{cis}\,\alpha = (\text{cis}\,\theta)^{\frac{1}{n}}$, where r is the modulus of the complex number. Then $r^n\text{cis}\,n\alpha = (r\,\text{cis}\,\alpha)^n = \text{cis}\,\theta$. Taking the modulus on both sides, we get $r^n = 1$, i.e., $r = 1$ as r must be positive. So $\text{cis}\,\theta = \text{cis}\,n\alpha$, implying $\cos\theta = \cos n\alpha$ and $\sin\theta = \sin n\alpha$.

Therefore, $n\alpha = \theta + 2k\pi$, where $k = 0, 1, 2, \dots, n-1$. Thus, $\alpha = \frac{2k\pi+\theta}{n}$, i.e.,

$$(\text{cis}\,\theta)^{\frac{1}{n}} = r\,\text{cis}\,\alpha = \text{cis}\,\frac{2k\pi+\theta}{n}.$$

Q.E.D.

Furthermore, if *m* and *n* are relatively prime integers and *n* is positive, we have

$$(\text{cis}\,\theta)^{\frac{m}{n}} = \left[(\text{cis}\,\theta)^m\right]^{\frac{1}{n}} = (\text{cis}\,m\theta)^{\frac{1}{n}} = \text{cis}\,\frac{2k\pi+m\theta}{n}$$

where $k = 0, 1, 2, \dots, n-1$.

With the De Moivre's theorem for rational index, we can easily evaluate the n^{th} roots of complex numbers without dealing with tedious multiplications. We will illustrate this with an example.

Example 6.4.

Find the cube roots of the complex number $1+\sqrt{3}i$.

Solution.

First of all, we transform the complex number into its polar form:

$$1+\sqrt{3}i=2\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)=2\text{cis}\frac{\pi}{3}.$$

Thus,

$$\left(1+\sqrt{3}i\right)^{\frac{1}{3}}=\left(2cis\frac{\pi}{3}\right)^{\frac{1}{3}}=2^{\frac{1}{3}}\left(cis\frac{\pi}{3}\right)^{\frac{1}{3}}=2^{\frac{1}{3}}cis\frac{2k\pi+\frac{\pi}{3}}{3}$$

for k = 0, 1, 2, i.e.,

$$\left(1+\sqrt{3}i\right)^{\frac{1}{3}}=2^{\frac{1}{3}}\text{cis}\frac{\pi}{9}\text{ or }2^{\frac{1}{3}}\text{cis}\frac{7\pi}{9}\text{ or }2^{\frac{1}{3}}\text{cis}\frac{13\pi}{9}.$$

The n^{th} roots of the number 1 are given the special name the n^{th} roots of unity.

Definition 6.1. (n^{th} roots of unity)

The n distinct roots of the equation $z^n=1$ are called the n^{th} roots of unity.

Suppose w is a n^{th} root of unity. Thus $w^n=1=\text{cis}0$. Hence, $w=\text{cis}\frac{2k\pi+0}{n}=\text{cis}\frac{2k\pi}{n}$, where k = 0, 1, 2, … , n – 1. Furthermore, if $w\neq 1$,

$$1+w+w^2+\cdots w^{n-1}=\frac{w^n-1}{w-1}=\frac{0}{w-1}=0.$$

Example 6.5.

Factor the polynomial z^7-1 into a product of linear and quadratic polynomials with real coefficients.

Solution.

The roots of the equation $z^7-1=0$ are the 7^{th} roots of unity:

$$1,\ \text{cis}\frac{2\pi}{7},\ \text{cis}\frac{4\pi}{7},\ \text{cis}\frac{6\pi}{7},\ \text{cis}\frac{8\pi}{7},\ \text{cis}\frac{10\pi}{7}\text{ and }\text{cis}\frac{12\pi}{7}.$$

Then,

$$\begin{aligned}
z^7-1&=(z-1)\left(z-\operatorname{cis}\frac{2\pi}{7}\right)\left(z-\operatorname{cis}\frac{4\pi}{7}\right)\left(z-\operatorname{cis}\frac{6\pi}{7}\right)\left(z-\operatorname{cis}\frac{8\pi}{7}\right)\left(z-\operatorname{cis}\frac{10\pi}{7}\right)\left(z-\operatorname{cis}\frac{12\pi}{7}\right)\\
&=(z-1)\left(z-\operatorname{cis}\frac{2\pi}{7}\right)\left(z-\operatorname{cis}\frac{4\pi}{7}\right)\left(z-\operatorname{cis}\frac{6\pi}{7}\right)\left(z-\operatorname{cis}\frac{-6\pi}{7}\right)\left(z-\operatorname{cis}\frac{-4\pi}{7}\right)\left(z-\operatorname{cis}\frac{-2\pi}{7}\right)\\
&=(z-1)\prod_{k=1}^{3}\left(z-\operatorname{cis}\frac{2k\pi}{7}\right)\left(z-\operatorname{cis}\frac{-2k\pi}{7}\right)=(z-1)\prod_{k=1}^{3}\left(z^2-2z\cos\frac{2k\pi}{7}+1\right).
\end{aligned}$$

Exercise

1. Solve the equation $z^4+4z^2+5=0$.

2. Find the fourth roots of the number $\sqrt{3}-i$.

3. By considering the real parts of the equation $\cos 2n\theta+i\sin 2n\theta=(\operatorname{cis}\theta)^{2n}$, prove that $\cos 2n\theta=\sum_{k=0}^{n}(-1)^k C_{2k}^{2n}\cos^{2n-2k}\theta\sin^{2k}\theta$.

7. Applications

As shown in the last section, complex numbers find their use in proving trigonometric identities. Besides, it is also useful in solving geometry problems. This application arises from the geometric representation of complex numbers on the Argand diagram.

For instance, the complex number $z = x + yi$ can represent the vector $\overrightarrow{OZ}=x\vec{i}+y\vec{j}$. On the other hand, the resultant vector from the subtraction of vectors can also be represented by the subtraction of the corresponding complex numbers. The modulus of the resulting complex number is the length of the resultant vector and the argument represents the angle between the resultant vector and the *x*-axis.

In this section, we will illustrate the applications of complex numbers with a couple of examples.

Example 7.1.

If the points A_1, A_2, B_1 and B_2 are associated with the complex numbers a_1, a_2, b_1 and b_2 respectively, express the angle between the line segments A_1B_1 and A_2B_2 in terms of a_1, a_2, b_1 and b_2.

Solution.

Refering to Figure 2,

$$\begin{aligned}
&\text{angle between line segments } A_1B_1 \text{ and } A_2B_2 \\
&= \theta = \theta_1 - \theta_2 \\
&= \text{angle between } A_1B_1 \text{ and } x\text{-axis} - \text{ angle between } A_2B_2 \text{ and } y\text{-axis} \\
&= \arg(a_1 - b_1) - \arg(a_2 - b_2) \\
&= \arg\left(\frac{a_1 - b_1}{a_2 - b_2}\right)
\end{aligned}$$

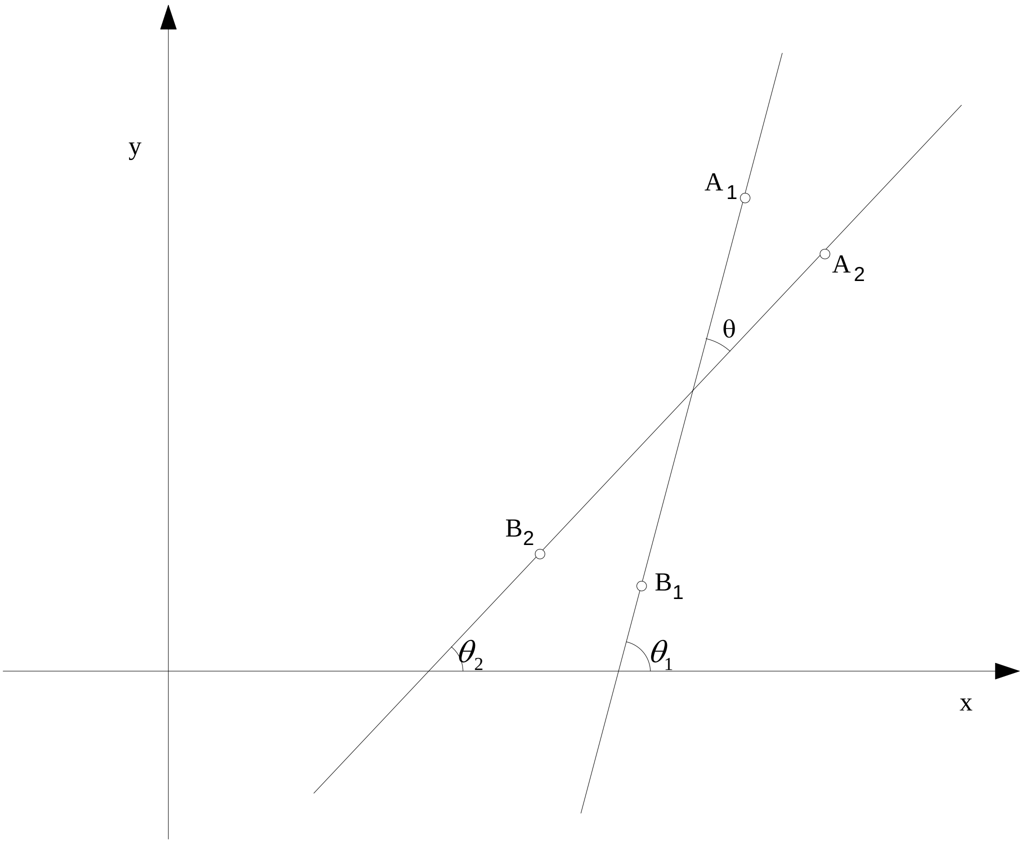

Figure 2

Example 7.2.

Let O = (0, 0) and A = (6, 8). Prove that for every point P (other than O and A) on the circle represented by $x^2 + y^2 - 6x - 8y = 0$, we have $OP \perp PA$.

Solution.

Let $p = x + yi$. Then $\overrightarrow{OP}$ is represented by p and $\overrightarrow{PA}$ is represented by $p - (6 + 8i)$. Then, following from the result of Example 7.1,

$$\begin{aligned}
& OP \perp PA \\
\Leftrightarrow\ & \text{angle between } OP \text{ and } PA \text{ is } \frac{\pi}{2} \text{ or } \frac{3\pi}{2} \\
\Leftrightarrow\ & \arg\left(\frac{p}{p-(6+8i)}\right) = \frac{\pi}{2} \text{ or } \frac{3\pi}{2} \\
\Leftrightarrow\ & \frac{p}{p-(6+8i)} \text{ is purely imaginary} \\
\Leftrightarrow\ & \frac{p}{p-(6+8i)} + \overline{\left[\frac{p}{p-(6+8i)}\right]} = 2\operatorname{Re}\left(\frac{p}{p-(6+8i)}\right) = 0.
\end{aligned}$$

Notice that

$$\begin{aligned}
\frac{p}{p-(6+8i)} + \overline{\left[\frac{p}{p-(6+8i)}\right]} &= \frac{p}{p-(6+8i)} + \frac{\overline{p}}{\overline{\left[p-(6+8i)\right]}} = \frac{x+yi}{(x-6)+(y-8)i} + \frac{x-yi}{(x-6)-(y-8)i} \\
&= \frac{2x(x-6)-2y(y-8)i^2}{(x-6)^2+(y-8)^2} = \frac{2\left(x^2+y^2-6x-8y\right)}{(x-6)^2+(y-8)^2} = 0.
\end{aligned}$$

Hence, $OP \perp PA$.

We conclude this set of notes with a discussion on the triangle inequality, a well-known theorem which can be readily applied to complex numbers as well.

Theorem 7.1 (Triangle Inequality)

For any two complex numbers z_1 and z_2, we have $\left|z_1+z_2\right| \le \left|z_1\right| + \left|z_2\right|$. Equality holds iff Z_1, Z_2 and *O* are collinear, and Z_1 and Z_2 are on the "same side" of *O*.

Proof. In a triangle Z_1Z_2O, the sum of lengths of any two sides is greater than the length of the third side, i.e., we have $Z_1O + Z_2'O \ge Z_1Z_2'$, where equality holds iff Z_1, Z_2' and *O* are collinear, and Z_1 and Z_2' are on the opposite side of *O*. In terms of complex numbers,

$$\left|z_1\right| + \left|z_2'\right| \ge \left|z_1 - z_2'\right|.$$

Taking $z_2' = -z_2$, we have $\left|z_1+z_2\right| \le \left|z_1\right| + \left|-z_2\right| = \left|z_1\right| + \left|z_2\right|$. Note that Z_2, Z_2' and *O* are always collinear and Z_2 and Z_2' are on the opposite side of *O*. Hence, equality holds iff Z_1, Z_2 and *O* are collinear and Z_1 and Z_2 are on the "same side" of *O*.

Q.E.D.

➢ In fact, for $n \geq 2$, we have $|z_1 + z_2 + \cdots + z_n| \leq |z_1| + |z_2| + \cdots + |z_n|$. This follows from induction on *n* and theorem 7.1.

Example 7.3.

Suppose $w^n z + w^{n-1} z^2 + \cdots + wz^n = 1$ for some *w* such that $|w| \leq 1$. Prove that $|z| > \frac{1}{2}$.

Solution.

Suppose $|z| \leq \frac{1}{2}$, applying the Triangle Inequality,

$$\begin{aligned} 1 = |1| = |w^n z + w^{n-1} z^2 + \cdots + wz^n| &\leq |w^n z| + |w^{n-1} z^2| + \cdots + |wz^n| \\ &= |w^n||z| + |w^{n-1}||z^2| + \cdots + |w||z^n| = |w|^n |z| + |w|^{n-1} |z|^2 + \cdots + |w||z|^n \\ &\leq 1^n \cdot \frac{1}{2} + 1^{n-1} \cdot \frac{1}{2^2} + \cdots + 1 \cdot \frac{1}{2^n} = \frac{1}{2} + \frac{1}{2^2} + \cdots + \frac{1}{2^n} \\ &= \frac{\frac{1}{2}\left(1 - \frac{1}{2^n}\right)}{1 - \frac{1}{2}} = 1 - \frac{1}{2^n} < 1. \end{aligned}$$

which is a contradiction. Therefore, we must have $|z| > \frac{1}{2}$.

Exercise

1. For each of the following conditions, find the locus of *z* (take $z = x + yi$):
   a. $|z - (2 + 3i)| = 6$.
   b. $\frac{2z}{3z+1}$ is purely imaginary, where $z \neq -\frac{1}{3}$.
   c. $|z - 1| = |z + 1| + 1$.

2. Prove that $\Delta ABC \sim \Delta XYZ$ iff $\dfrac{a-b}{c-b}=\dfrac{x-y}{z-y}$ or $\dfrac{a-b}{c-b}=\overline{\dfrac{x-y}{z-y}}$.

3. Suppose z_0, z_1, … , z_{n-1} are complex numbers such that $z_k = \operatorname{cis}\dfrac{2k\pi}{n}$ for k = 0, 1, …, n – 1. Prove that for any complex number z, $\sum_{k=0}^{n-1}|z-z_k| \ge n$.

8. Solutions to Exercise

Arithmetic Operations

1. $\cos\theta + i\sin\theta$.

2. Since a + 2i is a root of $x^2+6x+k=0$,

$$\begin{aligned}0&=(a+2i)^2+6(a+2i)+k=(a^2+4ai+4i^2)+(6a+12i)+k\\&=(a^2-4+6a+k)+(4a+12)i.\end{aligned}$$

Equating the real and imaginary parts of both sides, we have $a^2-4+6a+k=0$ and 4a + 12 = 0. Therefore, a = –3 and k = 13.

The Argand diagram, modulus and argument

1. (a) modulus = 2, principal value of argument = $\dfrac{\pi}{6}$.

(b) modulus = 2, principal value of argument = $-\dfrac{\pi}{3}$.

(c) modulus = $3\sqrt{2}$, principal value of argument = $-\dfrac{3\pi}{4}$.

(d) modulus = 1, principal value of argument = $\theta-\dfrac{\pi}{2}$.

2. (a) i

(b) OZ_2 is obtained by rotating OZ_1 by $\dfrac{\pi}{3}$ about O anticlockwise.

Other forms of representation

1. Converting to the polar form and applying theorem 4.1,

$$\frac{3+3\sqrt{3}i}{1-i}=\frac{6\left(\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)}{\sqrt{2}\left(\frac{1}{\sqrt{2}}-\frac{1}{\sqrt{2}}i\right)}=\frac{6\operatorname{cis}\frac{\pi}{3}}{\sqrt{2}\operatorname{cis}\left(-\frac{\pi}{4}\right)}=3\sqrt{2}\operatorname{cis}\left[\frac{\pi}{3}-\left(-\frac{\pi}{4}\right)\right]=3\sqrt{2}\operatorname{cis}\frac{7\pi}{12}.$$

2. Converting into the exponential form,

$$\begin{aligned}(\cos\theta-i\sin\theta)^3&=\left[\cos(-\theta)+i\sin(-\theta)\right]^3=\left[e^{i(-\theta)}\right]^3=e^{i(-3\theta)}\\&=\cos(-3\theta)+i\sin(-3\theta)=\cos 3\theta-i\sin 3\theta.\end{aligned}$$

Complex conjugates

1. Applying property 5 of theorem 5.1,

$$\begin{aligned}|z+k|^2&=(z+k)\left(\overline{z+k}\right)=(z+k)(\overline{z}+k)=z\overline{z}+k(z+\overline{z})+k^2\\&=|z|^2+k(z+\overline{z})+k^2\cdot 1=1+k(z+\overline{z})+k^2|z|^2\\&=1+k(z+\overline{z})+k^2z\overline{z}=(kz+1)(k\overline{z}+1)=(kz+1)\left(\overline{kz+1}\right)\\&=|kz+1|^2.\end{aligned}$$

2. Since 1 + 3*i* is a root, 1 – 3*i* is also a root. Then x^4+3x^2+ax+b is divisible by

$$\left[x-(1+3i)\right]\left[x-(1-3i)\right]=x^2-2x+10.$$

However, $x^4+3x^2+ax+b=(x^2-2x+10)(x^2+2x-3)+(a-26)x+(b+30)$. Hence, a = 26 and b = –30. Thus, $x^4+3x^2+ax+b=(x^2-2x+10)(x^2+2x-3)=(x-1)(x+3)(x^2-2x+10)$.

De Moivre's theorem

1. First, consider the equation as a quadratic equation in z^2, i.e., $(z^2)^2+4z^2+5=0$. Then by the quadratic formula, we get $z^2=-2+i$ or $-2-i$. So the roots of the equation are the square roots of the complex numbers – 2 + *i* and – 2 – *i*. Applying theorem 6.2 to find these square roots, we get $z=5^{\frac{1}{4}}\operatorname{cis}1.34$ or $5^{\frac{1}{4}}\operatorname{cis}1.80$ or $5^{\frac{1}{4}}\operatorname{cis}4.48$ or $5^{\frac{1}{4}}\operatorname{cis}4.94$.

2. Note $\sqrt{3}-i=2\left(\frac{\sqrt{3}}{2}-\frac{1}{2}i\right)=2\operatorname{cis}\left(\frac{-\pi}{6}\right)$. Hence,

$$\left(\sqrt{3}-i\right)^{\frac{1}{4}}=2^{\frac{1}{4}}\operatorname{cis}\left(\frac{\frac{-\pi}{6}+2k\pi}{4}\right)$$

for k = 0, 1, 2, 3. Thus,

$$\left(\sqrt{3}-i\right)^{\frac{1}{4}}=2^{\frac{1}{4}}\operatorname{cis}\left(-\frac{\pi}{24}\right)\text{ or }2^{\frac{1}{4}}\operatorname{cis}\left(\frac{11\pi}{24}\right)\text{ or }2^{\frac{1}{4}}\operatorname{cis}\left(\frac{23\pi}{24}\right)\text{ or }2^{\frac{1}{4}}\operatorname{cis}\left(\frac{35\pi}{24}\right).$$

3. $\cos 2n\theta+i\sin 2n\theta=\left(\operatorname{cis}\theta\right)^{2n}=\sum_{k=0}^{2n}C_k^{2n}\left(\cos\theta\right)^k\left(i\sin\theta\right)^{2n-k}$.

$$\begin{aligned}\cos 2n\theta+i\sin 2n\theta&=\sum_{k=0}^{n}C_{2k}^{2n}\left(\cos\theta\right)^{2k}\left(i\sin\theta\right)^{2n-2k}+\sum_{k=1}^{n}C_{2k-1}^{2n}\left(\cos\theta\right)^{2k-1}\left(i\sin\theta\right)^{2n-2k+1}\\&=\sum_{k=0}^{n}C_{2k}^{2n}\cos^{2k}\theta\sin^{2n-2k}\theta\left(-1\right)^{n-k}+i\sum_{k=1}^{n}C_{2k-1}^{2n}\cos^{2k-1}\theta\sin^{2n-2k+1}\theta\left(-1\right)^{n-k}.\end{aligned}$$

Equating the real parts on both sides,

$$\cos 2n\theta=\sum_{k=0}^{n}\left(-1\right)^{n-k}C_{2k}^{2n}\cos^{2k}\theta\sin^{2n-2k}\theta=\sum_{k=0}^{n}\left(-1\right)^{k}C_{2k}^{2n}\cos^{2n-2k}\theta\sin^{2k}\theta.$$

<u>Applications</u>

1. (a) $x^2+y^2-4x-6y-23=0$.

(b) $\frac{2z}{3z+1}$ is purely imaginary iff $z\neq-\frac{1}{3}$, $z\neq 0$, and

$$0=2\operatorname{Re}\left(\frac{2z}{3z+1}\right)=\frac{2z}{3z+1}+\overline{\left(\frac{2z}{3z+1}\right)}=\frac{2x+2yi}{\left(3x+1\right)+3yi}+\frac{2x-2yi}{\left(3x+1\right)-3yi}=\frac{4\left(3x^2+3y^2+x\right)}{\left(3x+1\right)^2+\left(3y\right)^2}$$

Thus $3x^2+3y^2+x=0$ excluding the points $\left(x,y\right)=\left(-\frac{1}{3},0\right)$ or $\left(0,0\right)$.

(c) From the definition of modulus, we have

$$|(x-1)+yi| = |(x+1)+yi|+1$$
$$\sqrt{(x-1)^2+y^2} = \sqrt{(x+1)^2+y^2}+1$$
$$(x-1)^2+y^2 = (x+1)^2+y^2+2\sqrt{(x+1)^2+y^2}+1$$
$$4x+1 = -2\sqrt{(x+1)^2+y^2}$$
$$16x^2+8x+1 = 4(x+1)^2+4y^2.$$

Thus, the locus of z is $12x^2-4y^2-3=0$.

2. Note that $\Delta ABC \sim \Delta XYZ \Leftrightarrow \dfrac{AB}{CB}=\dfrac{XY}{ZY}$ and $\angle ABC = \angle XYZ$. However,

$$\frac{AB}{CB}=\frac{XY}{ZY} \Leftrightarrow \left|\frac{a-b}{c-b}\right| = \frac{|a-b|}{|c-b|} = \frac{|x-y|}{|z-y|} = \left|\frac{x-y}{z-y}\right|,$$

$$\angle ABC = \angle XYZ \Leftrightarrow \arg\left(\frac{a-b}{c-b}\right) = \pm\arg\left(\frac{x-y}{z-y}\right).$$

These two conditions together implies $\dfrac{a-b}{c-b}=\dfrac{x-y}{z-y}$ or $\dfrac{a-b}{c-b}=\overline{\dfrac{x-y}{z-y}}$.

Hence $\Delta ABC \sim \Delta XYZ$ iff $\dfrac{a-b}{c-b}=\dfrac{x-y}{z-y}$ or $\dfrac{a-b}{c-b}=\overline{\dfrac{x-y}{z-y}}$.

3. The key to this problem is to use the formulae for geometric series to obtain

$$\sum_{k=0}^{n-1} e^{-\frac{2k\pi}{n}i} = \frac{1-\left(e^{-\frac{2\pi}{n}i}\right)^n}{1-e^{-\frac{2\pi}{n}i}} = \frac{1-e^{-2\pi i}}{1-e^{-\frac{2\pi}{n}i}} = \frac{1-1}{1-e^{-\frac{2\pi}{n}i}} = 0.$$

With this and the triangle inequality in mind, we have

$$\begin{aligned}\sum_{k=0}^{n-1}|z-z_k| &= \sum_{k=0}^{n-1}\left|z-e^{\frac{2k\pi}{n}i}\right| = \sum_{k=0}^{n-1}\left|z-e^{\frac{2k\pi}{n}i}\right|\bullet 1 = \sum_{k=0}^{n-1}\left|z-e^{\frac{2k\pi}{n}i}\right|\bullet\left|e^{-\frac{2k\pi}{n}i}\right| \\ &= \sum_{k=0}^{n-1}\left|\left(z-e^{\frac{2k\pi}{n}i}\right)\bullet e^{-\frac{2k\pi}{n}i}\right| = \sum_{k=0}^{n-1}\left|ze^{-\frac{2k\pi}{n}i}-1\right| \ge \left|\sum_{k=0}^{n-1}\left(ze^{-\frac{2k\pi}{n}i}-1\right)\right| \\ &= \left|\sum_{k=0}^{n-1} ze^{-\frac{2k\pi}{n}i} - n\right| = \left|z\sum_{k=0}^{n-1} e^{-\frac{2k\pi}{n}i} - n\right| = |0-n| = n.\end{aligned}$$