$\lfloor x \rfloor$ and $\lceil x \rceil$

1. A large rectangle in the plane is partitioned into smaller rectangles, each of which has either integer height or integer width (or both). Prove that the large rectangle also has this property.

2. Given positive integers a, b, and m with $\gcd(a, b) = 1$. Let A be a nonempty set of positive integers such that for every positive integer n either $an \in A$ or $bn \in A$. Determine the minimum number of elements the set $A \cap \{1, 2, \ldots, m\}$ can have.

3. In the Cartesian coordinate plan define the strip
$$S_n = \{(x, y) : n \leq x < n + 1\}$$
for every integer n. Assume that each strip S_n is colored either red or blue, and let a and b be two distinct positive integers. Prove that there exists a rectangle with side lengths a and b such that its vertices have the same color.

4. What is the units digit of
$$\left\lfloor \frac{10^{20000}}{10^{100} + 3} \right\rfloor?$$

5. Let a, b, c be positive real numbers such that
$$\lfloor an \rfloor + \lfloor bn \rfloor = \lfloor cn \rfloor$$
for all positive integers n. Prove that at least one of a, b is an integer.

6. Find all integers a with the property that for infinitely many positive integers n,
$$\left\lfloor \frac{2n^2}{\lfloor n\sqrt{2} \rfloor} \right\rfloor = \left\lfloor n\sqrt{2} \right\rfloor + a.$$

7. For a prime p and a positive integer n, denote by $\nu_p(n)$ the exponent of p in the prime factorization of $n!$. Given a positive integer d and a finite set $\{p_1, \ldots, p_k\}$ of primes, show that there are infinitely many positive integers n such that $d | \nu_p(n)$ for all $1 \leq i \leq k$.

8. Prove that the sequence $(\lfloor n\sqrt{2003} \rfloor)_{n \geq 1}$ contains arbitrarily long geometric progressions with arbitrarily large ratio.

9. Consider a positive integer k and a real number a such that $\log a$ is irrational. For each $n \geq 1$ let x_n be the number formed by the first k digits of $\lfloor a^n \rfloor$. Prove that the sequence $(x_n)_{n \geq 1}$ is not eventually periodic.

10. For a pair (a, b) of real numbers let $F(a, b)$ denote the sequence of general term $c_n = \lfloor an + b \rfloor$. Find all pairs (a, b) such that $F(x, y) = F(a, b)$ implies $(x, y) = (a, b)$.

11. Let α and β be positive real irrational numbers with
$$\frac{1}{\alpha} + \frac{1}{\beta} = 1.$$
Prove that the sets $\{\lfloor n\alpha \rfloor\}_{n \geq 1}$ and $\{\lfloor n\beta \rfloor\}_{n \geq 1}$ comprise a partition of the positive integers.

12. Prove that for every k one can find distinct positive integers $n_1, n_2, \ldots, n_k$ such that $\lfloor n_1\sqrt{2} \rfloor, \lfloor n_2\sqrt{2} \rfloor, \ldots, \lfloor n_k\sqrt{2} \rfloor$ and $\lfloor n_1\sqrt{3} \rfloor, \lfloor n_2\sqrt{3} \rfloor, \ldots, \lfloor n_k\sqrt{3} \rfloor$ are both geometric sequences.

13. A flea moves in the positive direction along the x-axis, starting from the origin. It can only jump over distances equal to $\sqrt{2}$ and $\sqrt{2005}$. Prove that there exists an n_0 such that the flea will be able to arrive in any interval $[n, n+1]$ for each $n \geq n_0$.

14. Let k be a positive integer. Prove that there exist polynomials $P_0(n), P_1(n), \ldots, P_{k-1}(n)$ (which may depend on k) such that for any integer n,
$$\left\lfloor \frac{n}{k} \right\rfloor^k = P_0(n) + P_1(n) \left\lfloor \frac{n}{k} \right\rfloor + \cdots + P_{k-1}(n) \left\lfloor \frac{n}{k} \right\rfloor^{k-1}.$$

15. Suppose that α and β are real numbers such that for no rationals a, b, c not all zero $a\alpha + b\beta + c = 0$. Let $I_\alpha, I_\beta \subseteq [0, 1)$ be intervals of positive length in $[0, 1)$. Prove that there is a positive integer n with $\{n\alpha\} \in I_\alpha$ and $\{n\beta\} \in I_\beta$.

16. Find a nonzero polynomial $P(x, y)$ such that $P(\lfloor a \rfloor, \lfloor 2a \rfloor) = 0$ for all real numbers a.

17. Find necessary and sufficient conditions on positive integers m and n so that
$$\sum_{i=0}^{mn-1} (-1)^{\lfloor i/m \rfloor + \lfloor i/n \rfloor} = 0.$$

18. Let a, b, c be positive real numbers. Prove that the sets
$$A = \{\lfloor na \rfloor | n \geq 1\}, \quad B = \{\lfloor nb \rfloor | n \geq 1\}, \quad C = \{\lfloor nc \rfloor | n \geq 1\}$$
cannot form a partition of the set of positive integers.