

ARML 2011 – Advanced Combinatorics: Principle of Inclusion-Exclusion

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Combinatorics is unlike algebra, calculus, and number theory in that it does not have an explicitly named “fundamental theorem.” Instead, there are probably about nine fundamental *principles*¹. Some of the principles or methods may seem almost trivial, but all have quite deep implications in multiple areas of mathematics. Strictly speaking, the principle of inclusion-exclusion is part of a more general group of *sieve methods*. The general idea of a sieve method is that we can approximate what we want to enumerate by overcounting, then subtract off an overcounted estimate of our error in counting, and repeat until we have converged to the exact answer. This is really the essence of the principle of inclusion-exclusion, although it turns out that much more sophisticated results are possible. In fact, in the most abstract sense, the statement of the principle of inclusion-exclusion is simply the computation of the inverse of a certain matrix.

While it is virtually certain that you won’t ever need to know that for a high school math competition, it illustrates an important fact: the principle of inclusion exclusion is *very* general and *widely* applicable. There are many ways you can manipulate the simple statement into other assertions that are more useful in particular situations, and it is important to know when it’s useful to do so. Thus, the purpose of this lecture is to demonstrate how to best apply the powerful tool of the principle of inclusion-exclusion and to teach you when to use it.

1 The Principle Itself

Suppose we have n finite sets $A_1, A_2, A_3, \dots, A_n$. Then

$$\begin{aligned} |A_1 \cup A_2 \cup \dots \cup A_n| &= |A_1| + |A_2| + \dots + |A_n| \\ &\quad - |A_1 \cap A_2| - |A_1 \cap A_3| - \dots - |A_{n-1} \cap A_n| \\ &\quad + |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + \dots + |A_{n-2} \cap A_{n-1} \cap A_n| \\ &\quad \dots \\ &\quad + (-1)^{n+1} |A_1 \cap A_2 \cap \dots \cap A_n| \\ &= \sum (-1)^{r+1} |A_{j_1} \cap A_{j_2} \cap \dots \cap A_{j_r}|, \end{aligned}$$

where the sum is taken over all nonempty subsets $\{j_1, j_2, \dots, j_r\}$ of the set $\{1, 2, 3, \dots, n\}$. Note that since row r of the sum contains the summands associated with the r -subsets; thus, the total number of summands on the right hand side of this equation is $2^n - 1$. For large n , the number of terms in the formula grows too quickly for it to be all that useful. However, as we shall see, it is frequently possible to simplify this to a number of summands on the order of n .

To prove this, consider an arbitrary element $a \in A_1 \cup A_2 \cup \dots \cup A_n$ and denote by $A_{k_1}, A_{k_2}, \dots, A_{k_m}$ those of the n given sets that contain the element a (clearly, $m \in \{1, 2, 3, \dots, n\}$). Which of the sets of the type $A_{j_1} \cap A_{j_2} \cap \dots \cap A_{j_m}$ contain element a ? Clearly, this only occurs for sets of j_i such that $\{j_1, j_2, \dots, j_m\} \subset \{k_1, k_2, \dots, k_m\}$ and $\{j_1, j_2, \dots, j_m\}$ is not empty. For each r in the range $1 \dots m$, the element a is counted in exactly $\binom{m}{r}$ summands, so the final contribution of a single

¹Specifically: addition principle, multiplication principle, pigeonhole principle, parity, proof by bijection, method of induction/recursion, invariants/monovariants, principle of inclusion-exclusion, and generating functions.

element a to the right-hand side of the equation is

$$\binom{m}{1} - \binom{m}{2} + \cdots + (-1)^{m+1} \binom{m}{m} = 1,$$

where the last step can be verified in any number of ways.

The principle of inclusion-exclusion is the most useful in problems where we are asked to determine the number of objects that have at least one of a given set of properties.

Example 1. How many ways can we rearrange the letters in “CATHAT” if no two of the same letters are to be next to each other?

Solution. Let n be the number we’re trying to find, and let n_0 be the total number of ways of rearranging the letters. Let A_1 be the set of all arrangements in which the two letters ‘A’ are next to each other and let A_2 be the set of all arrangements in which the two letters ‘T’ are next to each other. By the principle of inclusion-exclusion, we know that

$$n = n_0 - |A_1 \cup A_2| = n_0 - |A_1| - |A_2| + |A_1 \cap A_2|$$

We can easily obtain $n_0 = \frac{6!}{2 \cdot 2}$, $|A_1| = |A_2| = \frac{5!}{2}$, and $|A_1 \cap A_2| = 4!$. Therefore, we have

$$n = \frac{6!}{2^2} - 2 \cdot \frac{5!}{2} + 4! = 84.$$

2 Special Cases

Quite frequently, the summands for each row r in the statement of the principle of inclusion-exclusion are all the same; say, some value $A(r)$. This collapses a sum with an exponential number of summands into a sum with n summands; specifically,

$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{r=1}^n (-1)^{r+1} \binom{n}{r} A(r).$$

Alternately, we can count the complement (“all things that don’t have the properties A_1, A_2, \dots, A_n ”) to get the formula

$$A(0) - |A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{r=0}^n (-1)^r \binom{n}{r} A(r).$$

It’s important to note that the principle of inclusion-exclusion cannot *always* be reduced to this form. However, it is often the case that at least some of the summands can be simplified or ignored in this manner.

Example 2. Find the number of arrangements with repetitions of $AABBCCDD \cdots$, where there are n different pairs of two letters, such that no two letters that are the same are adjacent to each other.

Solution. Let a_n be the desired number of arrangements, and let A_i be the set of all arrangements in which the elements of the i -th type are adjacent. Then clearly

$$a_n = \frac{(2n)!}{2^n} - |A_1 \cup A_2 \cup \cdots \cup A_n|.$$

To determine the right-hand side specifically, we use the simplified form of the principle of inclusion-exclusion. Note that the set $A_{j_1} \cap A_{j_2} \cap \cdots \cap A_{j_r}$ contains the arrangements in which a fixed set of r letter pairs are adjacent to each other. Each such pair can be considered as a single object, and the rest of the objects can be arranged in any arbitrary configuration. Our choice of the r letters is clearly irrelevant, so

$$\begin{aligned} A(r) &= |A_{j_1} \cap A_{j_2} \cap \cdots \cap A_{j_r}| \\ &= \frac{(2n-r)!}{2^{n-r}}, \end{aligned}$$

as the problem is now really finding the number of arrangements with repetitions of $(n-r)$ pairs of elements of the same type and r different new objects. Using the simplified inclusion-exclusion formula, we get that

$$a_n = \frac{(2n)!}{2^n} - \sum_{r=1}^n (-1)^{r+1} \binom{n}{r} \frac{(2n-r)!}{2^{n-r}} = \sum_{r=1}^n (-1)^r \binom{n}{r} \frac{(2n-r)!}{2^{n-r}}.$$

Example 3. In how many ways can we select k elements, possibly with repetition, from a set of n different types such that every selection contains at least one element of each type?

Solution. We'll call this this number $v_0(k, n)$. Clearly, if $k < n$, then $v_0(k, n) = 0$, so we can assume that $k \geq n$ and let A_i denote the set of all of these selections of k elements with repetitions that contain NO elements of the i -th type. Without the condition that each selection has at least one element of each type, there are clearly n^k ways to make the selection. Thus,

$$v_0(k, n) = n^k - |A_1 \cup A_2 \cup \cdots \cup A_n|.$$

This situation seems familiar...convince yourself that $A(r) = (n-r)^k$. This implies that

$$\begin{aligned} v_0(k, n) &= n^k - |A_1 \cup A_2 \cup \cdots \cup A_n| \\ &= n^k - \sum_{r=1}^n (-1)^{r+1} \binom{n}{r} (n-r)^k \\ &= \sum_{r=0}^n (-1)^r \binom{n}{r} (n-r)^k. \end{aligned}$$

and we're done with the problem. However, we can consider some special cases for $v_0(k, n)$. As we noted earlier, if $k < n$, then $v_0(k, n) = 0$. Thus "for free" we get the identity

$$\sum_{r=0}^n (-1)^r \binom{n}{r} (n-r)^k = 0, \quad k < n.$$

Additionally, if $k = n$, then clearly there can be absolutely no repetitions in the selection for $v_0(k, n)$, so $v_0(n, n) = n!$ and we also obtain the curious result

$$n! = \sum_{r=0}^n (-1)^r \binom{n}{r} (n-r)^n.$$

3 Problems

Most problems involving the principle of inclusion-exclusion are not hard, if you know what you're looking for. Generally, the principle is good at dealing with problems involving conditions and restrictions such as "there must be at least..." etc. However, if you don't think of it, you might be reduced to trying to enumerate by hand - which often takes much longer - and you'll probably fail anyways if the problem is decent. You should work on these problems so you can get comfortable with the steps of a "PIE-bash"; it's a very useful and pretty widely applicable technique.

1. How many of the positive integers less than 2011 are either a multiple of 3 or 17?
2. In how many ways can we place 3 mathletes, 3 debaters, and 3 football players in a row such that no three persons doing the same activity are next to each other?
3. How many perfect powers are less than 2011?
4. Six people are in an elevator. In how many ways can they get off at four floors if at each floor at least one person gets off?
5. In how many ways can seven different cows be placed into four separate barns, if each barn must contain at least one cow?
6. A caravan consisting of 9 camels travels through the desert. In how many ways can the caravan be assembled after a rest stop such that in front of each of the last 8 camels there is a different one than before?
7. Kin Sim, Zuqing Yang, Xictoria Via, Leronica Vee, Poeseph Jark, Rilly Bieger, and Lemy Ree are riding on a carousel. They agree that for the next ride they will change their places in such a way that in front of each person there is a different one than before. In how many ways can this be done?
8. In how many ways can 8 rooks be placed on a standard 8×8 chessboard such that each free field is threatened by at least one of the rooks?
9. Find the number of all placements of n rooks on an $n \times n$ chessboard such that no two rooks threaten each other, and none is on the main diagonal.
10. A $150 \times 324 \times 375$ rectangular solid is made by gluing together $1 \times 1 \times 1$ cubes. An internal diagonal of this solid passes through the interiors of how many of the $1 \times 1 \times 1$ cubes?
11. How many six-digit integers have exactly three different digits?
12. We are given the nonnegative integers $\alpha_1, \alpha_2, \dots, \alpha_n$. Find the number b_n of all k -element combinations with repetitions from elements of n different types such that for each $i = 1, 2, \dots, n$ the elements of the i -th type are repeated no more than α_i times in the combinations.
13. King Arthur invites n pairs of mutually hostile knights for dinner at his round table. In how many ways can the royal butler place the knights around the table such that no two enemies sit next to each other?
14. Let $\{a_i\}_0^n$ and $\{b_i\}_0^n$ be two sequences of complex numbers such that

$$a_m = \sum_{k=0}^m \binom{m}{k} b_k \quad \text{for all } 0 \leq m \leq n.$$

Determine a formula for b_m in terms of a_0, a_1, \dots, a_m .