Combinatorics

Jacob Steinhardt

Combinatorics is one of the most creative areas of mathematics, essentially because there is very little structure involved. As such, since this is the advanced lecture, I'm going to assume you know some stuff and rather than telling you how to do anything, demand that you figure it out on your own, since that is the best way to learn.

1 Symmetry

The idea is that if two things are equivalent, and the same operations are applied to them, then they are still equivalent. Alternately, if the initial conditions of a situation are the same if we interchange a and b, and some operations are applied to it to get $f(a, b)$, then $f(a, b) = f(b, a)$. So, for example, if I flip n coins, what is the expected number of times I will get heads? If you ask anyone this question, they will tell you that it is $\frac{n}{2}$ without even doing any calculations. Why is this? Because nothing distinguishes heads and tails, so the expected number of heads is the expected number of tails, and they must sum to n, so the expected number of either one is $\frac{n}{2}$. We can generalize this to weighted coins with a probability p of turning up heads as follows: if p is rational, $p = \frac{a}{a+b}$, then this is equivalent to rolling a die with a red faces and b blue faces. Using the same symmetry argument as before, the expected number of times any given face comes up is $\frac{n}{a+b}$. There are a red faces, so the expected number of red faces is $\frac{na}{a+b} = pn$. Then, we claim that expected value is a continuous function with respect to probability, and thus since we have a function that is equal to pn across the rationals, it is pn across the reals. Of course, I just used a bunch of space to prove something that was already completely obvious to you, so now I'm going to use much less space to state a problem that may not be as obvious:

If A flips n coins, and B flips n coins, what is the probability that A has more heads than B?

Here's another problem:

Let $S = \{1, 2, \ldots, 2n\}$ and select a non-empty subset T of S at random. What is the probability that the smallest element of T is even? There's at least three ways to do this, see if you can find them all.

2 Doing Stuff

If you want to talk about a set under some given constraints, often it is necessary to mess with the set until it becomes nice. For lack of a better word, I call this *doing stuff*, though I'm sure somebody has come up with a better name. I don't really want to give anything away, so instead I'm just going to hammer you with a bunch of problems until you get it. I'll start you guys off with an easy problem to warm up, and then they'll get progressively harder.

3 Problems

1. (2007 AMC 12A Problem 25) Call a set of integers *spacy* if it contains no more than one out of any three consecutive integers. How many subsets of $\{1, 2, 3, \ldots, 12\}$, including the empty set, are spacy?

2. (HMMT Guts) Consider the set S of the first 8 positive integers. How many sets can be formed such that each element of the set is a 2-element subset of S and no two sets overlap by more than 1 element?

3. (AIME) Given the set S of the first 10 positive integers, how many unordered pairs exist such that each of the elements in the pair are non-empty subsets of S with no overlap?

4. Pick 3 integers at random on the interval $[0, 6]$. What is the probability that none of them are within a distance of 1 of each other?

5. Given a standard 52-card deck, you draw cards until you have picked up n ($n < 26$) black cards. You then set aside all of the cards you have drawn. What is the probability that the top card of the deck is black?

   Suppose you draw hearts until you pick of n ($n < 13$) hearts. What is the probability that your next card is a heart?

   In the first scenario, what happens if we first remove all of the diamonds? The spades?

6. (Gary Sivek 2006) $S_1 \subseteq S_2 \subseteq \ldots \subseteq S_k \subseteq \{1, 2, \ldots, n\}$. How many ordered k-tuples of integer sets exist that satisfy this condition?