New Zealand Mathematical Olympiad Committee

Symmetric Polynomials

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1 Introduction

These notes begin with two basic results about symmetric polynomials: the Fundamental Theorem of Symmetric Polynomials, which states that any symmetric polynomial can be expressed in terms of the appropriate *elementary symmetric polynomials*, and Muirhead's inequality, which gives comparisions between the sizes of symmetric polynomials in a given set of variables.

Further on, we analyse relationships between properties of the roots of a cubic and properties of its coefficients (which are symmetric polynomials of the roots). This analysis is then used to derive a number of triangle-geometric inequalities.

We restrict ourselves throughout to polynomials in three variables. The results of Sections 2-3, however, hold (with appropriately generalized statements and proofs) for symmetric polynomials in arbitrarily many variables.

2 The Fundamental Theorem

A polynomial $F(x, y, z)$ is said to be *symmetric* if

$$F(x, y, z) = F(x, z, y) = F(y, x, z) = F(y, z, x) = F(z, x, y) = F(z, y, x).$$

Example 1. $F(x, y, z) = x^3yz + xy^3z + xyz^3$.

The three polynomials

$$\begin{aligned}
\sigma_1(x, y, z) &= x + y + z, \\
\sigma_2(x, y, z) &= xy + yz + zx, \\
\sigma_3(x, y, z) &= xyz.
\end{aligned}$$

are the best-known examples. For instance, they appear in Vieta's theorem expressing the coefficients of a polynomial in terms of its roots. They are called *elementary symmetric polynomials.* Of course there are many other important symmetric polynomials, e.g.

$$\Delta(x, y, z) = (x - y)^2(y - z)^2(z - x)^2,$$

but the importance of the elementary symmetric polynomials stems from the main theorem of this section, which will assert that all symmetric polynomials can be expressed as polynomials in the elementary symmetric polynomials $\sigma_1, \sigma_2, \sigma_3$.

To start on the proof of this theorem, let us consider firstly the monomials $x^{k_1}y^{k_2}z^{k_3}$ of a fixed degree $n = k_1 + k_2 + k_3$. We consider these monomials as words in the alphabet of the three letters x, y, z. Let us write down all the monomials in the lexicographic (vocabulary) order. For instance, for $n = 3$ we shall have:

$$x^3 > x^2y > x^2z > xy^2 > xyz > xz^2 > y^3 > y^2z > yz^2 > z^3,$$

where the sign ">" means lexicographically earlier. In general, for two monomials of degree n we write

$$x^{k_1}y^{k_2}z^{k_3} > x^{m_1}y^{m_2}z^{m_3}$$

if $k_1 > m_1$ or if $k_1 = m_1$ and $k_2 > m_2$ and say that the first monomial is earlier than the second one.

Let $F(x, y, z)$ be a symmetric polynomial which monomials are all of the same degree n. Among them there is a monomial that are earlier of all other monomials of $F(x, y, z)$. We shall call it the *leading monomial* of $F(x, y, z)$.

Example 2. The leading monomial of

$$F(x, y, z) = x^3yz + xy^3z + xyz^3$$

is x^3yz.

Lemma 1. *A monomial $x^{k_1}y^{k_2}z^{k_3}$ is a leading monomial of some symmetric polynomial if and only if $k_1 \geq k_2 \geq k_3$. If $k_1 \geq k_2 \geq k_3$ the monomial $x^{k_1}y^{k_2}z^{k_3}$ is the leading monomial of $\sigma_1^{k_1-k_2}\sigma_2^{k_2-k_3}\sigma_3^{k_3}$.*

Proof. If, for instance, $k_1 < k_2$, then the monomial $x^{k_1}y^{k_2}z^{k_3}$ cannot be the leading monomial of a symmetric polynomial $F(x, y, z)$, because $F(x, y, z) = F(y, x, z)$ contains also a monomial $x^{k_2}y^{k_1}z^{k_3}$ which is earlier than $x^{k_1}y^{k_2}z^{k_3}$. The case $k_2 < k_3$ is similar.

The leading monomial of $\sigma_1^{k_1-k_2}\sigma_2^{k_2-k_3}\sigma_3^{k_3}$ is equal to the product of the leading monomials of $\sigma_1^{k_1-k_2}, \sigma_2^{k_2-k_3}$ and $\sigma_3^{k_3}$, that is $x^{k_1-k_2} \cdot (xy)^{k_2-k_3} \cdot (xyz)^{k_3} = x^{k_1}y^{k_2}z^{k_3}$. □

Now we can prove the main theorem of the section.

Theorem 2 (Fundamental Theorem)**.** *For any symmetric polynomial $F(x, y, z)$ there exists a (not necessarily symmetric) polynomial $f(u, v, w)$ such that*

$$F(x, y, z) = f(\sigma_1, \sigma_2, \sigma_3).$$

Proof. We sketch an algorithm for constructing f. Let $x^{k_1}y^{k_2}z^{k_3}$ be the leading monomial of $F(x, y, z)$. Then $k_1 \geq k_2 \geq k_3$ and the polynomial $\sigma_1^{k_1-k_2}\sigma_2^{k_2-k_3}\sigma_3^{k_3}$ has the same leading monomial. Consider

$$F_1(x, y, z) = F(x, y, z) - \sigma_1^{k_1-k_2}\sigma_2^{k_2-k_3}\sigma_3^{k_3}.$$

This polynomial is also symmetric but its leading monomial is later than that of $F(x, y, z)$. This process of reducing of leading monomials can be continued. Since we have only finite number of monomials of a fixed degree, finally we shall obtain the required presentation of $F(x, y, z)$. □

Example 3. In this example we shall not use the complete algorithm exposed in the proof of Theorem 2 since an easier way is possible. Consider

$$\delta(x, y, z) = (x + y - z)(y + z - x)(z + x - y).$$

Then

$$\delta(x, y, z) = -\sigma_1^3 + 4\sigma_1\sigma_2 - 8\sigma_3.$$

The following computation shows that this is true:

$$\begin{aligned} \delta(x, y, z) &= (\sigma_1 - 2z)(\sigma_1 - 2x)(\sigma_1 - 2y) \\ &= \sigma_1^3 - 2(x + y + z)\sigma_1^2 + 4(xy + yz + zx)\sigma_1 - 8xyz \\ &= -\sigma_1^3 + 4\sigma_1\sigma_2 - 8\sigma_3. \end{aligned}$$

Example 4. Here the complete algorithm cannot be avoided. We leave the routine computation to the reader.

$$\Delta(x, y, z) = -4\sigma_1^3\sigma_3 + \sigma_1^2\sigma_2^2 + 18\sigma_1\sigma_2\sigma_3 - 4\sigma_2^3 - 27\sigma_3^2.$$

Exercise 1. Express the following symmetric polynomials as polynomials of $\sigma_1, \sigma_2, \sigma_3$:

(a) $x^2 + y^2 + z^2$;

(b) $x^3+y^3+z^3$;

(c) $x^4+y^4+z^4$;

(d) $x^2y+y^2z+z^2x+xy^2+yz^2+zx^2$;

(e) $x^2y^2+y^2z^2+z^2x^2$.

Exercise 2. Check Example 4.

Exercise 3. Express as a function of $\sigma_1, \sigma_2, \sigma_3$:

(a) $\dfrac{1}{x^2}+\dfrac{1}{y^2}+\dfrac{1}{z^2}$;

(b) $\dfrac{1}{xy}+\dfrac{1}{yz}+\dfrac{1}{zx}$.

3 Muirhead's theorem

Let us consider the set S_n of sequences $(\alpha)=(\alpha_1,\ldots,\alpha_n)$ with the following two properties:

- $\alpha_1+\alpha_2+\ldots+\alpha_n=1$,
- $\alpha_1\geq\alpha_2\geq\ldots\geq\alpha_n\geq 0$.

For any two sequences (α) and (β) from S we say that (α) *majorises* (β) if

$$\alpha_1+\alpha_2+\ldots+\alpha_r\geq\beta_1+\beta_2+\ldots+\beta_r$$

for all $1\leq r<n$. We denote this as $(\alpha)\succeq(\beta)$. If $(\alpha)\succeq(\beta)$ and $(\alpha)\neq(\beta)$, we will write $(\alpha)\succ(\beta)$.

Example 5. $(1,0,0)\succ(\frac{1}{2},\frac{1}{2},0)\succ(\frac{1}{2},\frac{1}{3},\frac{1}{6})\succ(\frac{1}{3},\frac{1}{3},\frac{1}{3})$.

Now we introduce one more notation: for (α) from S_n we denote

$$x^{(\alpha)}=\frac{1}{n!}(x_1^{\alpha_1}x_2^{\alpha_2}\ldots x_n^{\alpha_n}+\ldots),$$

where the dots denote all $n!-1$ terms obtained by permutations of α's.

Example 6. Let $n=3$. Then

$$\begin{aligned}
x^{(1,0,0)} &= \frac{1}{6}(x_1^1x_2^0x_3^0+x_1^1x_3^0x_2^0+x_1^0x_2^1x_3^0+x_1^0x_2^1x_3^0+x_1^0x_2^0x_3^1+x_1^0x_2^0x_3^1)\\
&= \frac{1}{3}(x_1+x_2+x_3);\\
x^{(\frac{1}{2},\frac{1}{2},0)} &= \frac{1}{6}(x_1^{\frac{1}{2}}x_2^{\frac{1}{2}}x_3^0+x_1^{\frac{1}{2}}x_2^{\frac{1}{2}}x_3^0+x_1^0x_2^{\frac{1}{2}}x_3^{\frac{1}{2}}+x_1^0x_2^{\frac{1}{2}}x_3^{\frac{1}{2}}+x_1^{\frac{1}{2}}x_2^0x_3^{\frac{1}{2}}+x_1^{\frac{1}{2}}x_2^0x_3^{\frac{1}{2}})\\
&= \frac{1}{3}(\sqrt{x_1x_2}+\sqrt{x_1x_3}+\sqrt{x_2x_3});\\
x^{(\frac{1}{3},\frac{1}{3},\frac{1}{3})} &= \sqrt[3]{x_1x_2x_3}.
\end{aligned}$$

Theorem 3 (Muirhead)**.** *If $(\alpha)\succeq(\beta)$, then the inequality*

$$x^{(\alpha)}\geq x^{(\beta)}$$

holds for all non-negative $x_1,\ldots,x_n$. There is equality only when either $(\alpha)=(\beta)$ or all the x_i are equal.

Proof. We will prove this theorem for the case $n = 3$. The general case can be proved similarly. We assume that $(\alpha) \neq (\beta)$ and not all the x_i are equal. Let us consider the following three partial cases from which the general case will follow.

(a) Let $(\alpha) = (\alpha_1, \alpha_2, \alpha_3)$, and ρ be a positive real number such that $\rho < \alpha_1 - \alpha_2$ and $(\alpha') = (\alpha_1 - \rho, \alpha_2 + \rho, \alpha_3)$. Then

$$\begin{aligned} 3!\left(x^{(\alpha)} - x^{(\alpha')}\right) &= (x_1^{\alpha_1}x_2^{\alpha_2} + x_1^{\alpha_2}x_2^{\alpha_1} - x_1^{\alpha'_1}x_2^{\alpha'_2} - x_1^{\alpha'_2}x_2^{\alpha'_1})x_3^{\alpha_3} \\ &\quad +(x_1^{\alpha_1}x_3^{\alpha_2} + x_1^{\alpha_2}x_3^{\alpha_1} - x_1^{\alpha'_1}x_3^{\alpha'_2} - x_1^{\alpha'_2}x_3^{\alpha'_1})x_2^{\alpha_3} \\ &\quad +(x_2^{\alpha_1}x_3^{\alpha_2} + x_2^{\alpha_2}x_3^{\alpha_1} - x_2^{\alpha'_1}x_3^{\alpha'_2} - x_2^{\alpha'_2}x_3^{\alpha'_1})x_1^{\alpha_3}. \end{aligned}$$

We claim the right-hand side is positive. Indeed, certainly for any x_i and x_j,

$$x_i^{\alpha_1}x_j^{\alpha_2} + x_i^{\alpha_2}x_j^{\alpha_1} - x_i^{\alpha'_1}x_j^{\alpha'_2} - x_i^{\alpha'_2}x_j^{\alpha'_1} = (x_ix_j)^{\alpha_2}\left[(x_i^{\alpha_1-\alpha_2-\rho} - x_j^{\alpha_1-\alpha_2-\rho})(x_i^{\rho} - x_j^{\rho})\right] \geq 0,$$

since the two round-bracketed differences are either both non-negative or both non-positive. Moreover, since not all the x_i are equal, at least one of $x_1 \neq x_2$, $x_1 \neq x_3$, $x_2 \neq x_3$ will hold, securing strict inequality. So $x^{(\alpha)} > x^{(\alpha')}$ as required.

(b) Similarly, if $\rho < \alpha_2 - \alpha_3$ and $(\alpha'') = (\alpha_1, \alpha_2 - \rho, \alpha_3 + \rho)$, then $x^{(\alpha)} > x^{(\alpha'')}$.

(c) The same argument also shows that if $\alpha_2 = \alpha'_2$, then, for any $\rho \leq (\alpha_1 - \alpha_2)$, and $(\alpha''') = (\alpha_1 - \rho, \alpha_2, \alpha_3 + \rho)$, then $x^{(\alpha)} > x^{(\alpha''')}$.

Suppose now that $\alpha_2 < \beta_2$. Then $\alpha_2 < \beta_1$, and $\rho = \alpha_1 - \beta_1 < \alpha_1 - \alpha_2$. Then by (a), $x^{(\alpha)} > x^{(\alpha')}$, where $(\alpha') = (\alpha_1 - \rho, \alpha_2 + \rho, \alpha_3) = (\beta_1, \alpha_2 + \rho, \alpha_3)$. As $\alpha_3 < \beta_3$, $\alpha_2 + \rho > \beta_2$, hence $x^{(\alpha')} > x^{(\beta)}$ by (b). Thus $x^{(\alpha)} > x^{(\beta)}$, as required. The case $\alpha_2 > \beta_2$ is dealt with similarly using (b) and (a). If $\alpha_2 = \beta_2$, then the statement follows straight from (c). $\square$

Example 7. For all non-negative x_1, x_2, x_3

$$\frac{1}{3}(x_1 + x_2 + x_3) \geq \frac{1}{3}(\sqrt{x_1x_2} + \sqrt{x_1x_3} + \sqrt{x_2x_3}) \geq \sqrt[3]{x_1x_2x_3}$$

or, after substitution $a = \sqrt{x_1}$, $b = \sqrt{x_2}$, $c = \sqrt{x_3}$,

$$\frac{1}{3}(a^2 + b^2 + c^2) \geq \frac{1}{3}(ab + ac + bc) \geq \sqrt[3]{a^2b^2c^2}.$$

Exercise 4. Prove that for all non-negative a, b, c the following two inequalities hold:

$$\begin{aligned} a^3 + b^3 + c^3 &\geq a^2(b + c) + b^2(c + a) + c^2(a + b), \\ a^3 + b^3 + c^3 &\geq a^2b + b^2c + c^2a. \end{aligned}$$

Establish when these are equalities.

4 Cubic polynomials and their roots

Theorem 4. *Let a, b, c be the roots (real or complex) of a cubic equation*

$$x^3 - \sigma_1x^2 + \sigma_2x - \sigma_3 = 0. \tag{1}$$

Then any symmetric function of the roots a, b, c is a polynomial function of the coefficients $\sigma_1, \sigma_2, \sigma_3$.

Proof. Follows from Vieta's theorem and the Fundamental Theorem. $\square$

The value

$$\Delta(a,b,c) = (a-b)^2(b-c)^2(c-a)^2,$$

is known as the *discriminant* of the equation (1). We know it is equal to

$$\Delta(a,b,c) = -4\sigma_1^3\sigma_3 + \sigma_1^2\sigma_2^2 + 18\sigma_1\sigma_2\sigma_3 - 4\sigma_2^3 - 27\sigma_3^2.$$

Theorem 5. *Let σ_1, σ_2, σ_3 be real. Then the equation (1) has exactly three (possibly repeated) real roots if and only if*

$$-4\sigma_1^3\sigma_3 + \sigma_1^2\sigma_2^2 + 18\sigma_1\sigma_2\sigma_3 - 4\sigma_2^3 - 27\sigma_3^2 \geq 0. \tag{2}$$

If $\sigma_1 \geq 0, \sigma_2 \geq 0, \sigma_3 \geq 0$ all these roots are positive. These roots are the lengths of sides of a (nondegenerate) triangle if and only if in addition

$$-\sigma_1^3 + 4\sigma_1\sigma_2 - 8\sigma_3 > 0. \tag{3}$$

Proof. Let a, b, c be the roots of (1). If all of them are real, then obviously

$$\Delta(a,b,c) = -4\sigma_1^3\sigma_3 + \sigma_1^2\sigma_2^2 + 18\sigma_1\sigma_2\sigma_3 - 4\sigma_2^3 - 27\sigma_3^2 \geq 0.$$

Suppose now that (2) holds, i.e $\Delta(a,b,c) \geq 0$. It is known that either all roots a, b, c are real or one of them is real and the other two are complex conjugates. We will show that the later case cannot occur. Indeed, if a is real, $b = z, c = \overline{z}$, then

$$\begin{aligned}
\Delta(a,b,c) &= (a-z)^2(z-\overline{z})^2(\overline{z}-a)^2 \\
&= \left[(a-z)(a-\overline{z})\right]^2 (2\Im(z)z)^2 \cdot i^2 \\
&= (a^2 - 2\Re(z)\cdot a + |z|^2)^2 \cdot 4(\Im(z)z)^2 \cdot (-1) \\
&< 0,
\end{aligned}$$

where $z = \Re(z) + i\Im(z)$. Moreover, (3) together with Example 3 shows that

$$\delta(a,b,c) = (a+b-c)(b+c-a)(c+a-b) > 0.$$

Either all three factors are positive and we can construct a triangle or two factors are negative. The later case cannot happen since if, for instance, $a+b-c < 0$ and $b+c-a < 0$, adding these two inequalities we obtain $2b < 0$. $\square$

Corollary 6. *A cubic equation*

$$x^3 + px + q = 0, \tag{4}$$

where p and q are real, has three real roots if and only if $\Delta = -4p^3 - 27q^2 \geq 0$.

Note that any cubic with real coefficients can be reduced to a cubic of the form (4) by a simple substitution.

5 Triangle geometry

Recall that the lengths of the sides a, b, c of a triangle ABC are the roots of the cubic equation

$$x^3 - 2px^2 + (p^2 + r^2 + 4rR)x - 4prR = 0,$$

where p, r and R are the semiperimeter, inradius and circumradius respectively of ABC.

Theorem 7. *Three positive numbers p, r, R are the semiperimeter, inradius and circumradius respectively of some triangle, if and only if*

$$(p^2 - 2R^2 - 10Rr + r^2)^2 \leq 4R(R-2r)^3. \tag{5}$$

Proof. We want to show that the roots of the cubic

$$x^3 - 2px^2 + (p^2 + r^2 + 4rR)x - 4prR = 0$$

are the sides of a triangle. To do this we use Theorem 5 about the relationship between properties of a cubic's roots and properties of its coefficients.

First, we need to show that the roots of this cubic are real. Theorem 5 says this happens if and only if the cubic's coefficients satisfy condition (2). Substituting and manipulating, we find this to be equivalent to the condition (5) specified.

Next we need to show that all three roots of the cubic are positive; this is clear by Theorem 5, since $2p$, $p^2 + r^2 + 4rR$ and $4prR$ are all positive.

Finally we will check that the roots of our cubic are the sides of a triangle, using Theorem 5's further condition (3). Indeed, for every positive p, r, R we have

$$-(2p)^3 + 4(2p)(p^2 + r^2 + 4rR) - 8 \cdot 4prR = 8pr^2 > 0.$$

$\square$

Corollary 8. *For any triangle ABC,*

(a) $27r^2 \leq 16Rr - 5r^2 \leq p^2 \leq 4R^2 + 4Rr + 3r^2 \leq \frac{27}{7}R^2$.

(b) $3\sqrt{3}r \leq p \leq \frac{3\sqrt{3}}{2}R$.

These inequalities become equalities if and only if the triangle ABC is equilateral.

Proof. By Theorem 7,

$$2R^2 + 10Rr - r^2 - 2(R - 2r)\sqrt{R(R - 2r)} \leq p^2 \leq 2R^2 + 10Rr - r^2 + 2(R - 2r)\sqrt{R(R - 2r)}.$$

It follows that

$$\begin{aligned} 2R^2 + 10rR - r^2 - 2(R - 2R)\sqrt{R(R - 2r)} &= 16rR - 5r^2 + \left[(R - 2r) - \sqrt{R(R - 2r)}\right]^2 \\ &\geq 16rR - 5r^2 \end{aligned}$$

and

$$\begin{aligned} 2R^2 + 10rR - r^2 + 2(R - 2r)\sqrt{R(R - 2r)} &= 4R^2 + 4rR + 3r^2 - \left[(R - 2r) - \sqrt{R(R - 2r)}\right]^2 \\ &\leq 4R^2 + 4rR + 3r^2. \end{aligned}$$

In both cases equality occurs if $R = 2r$ which means by Euler's theorem that ABC is equilateral. Using $R \geq 2r$ we conclude further

$$\begin{aligned} &16rR - 5r^2 \geq 27r^2, \\ &4R^2 + 4rR + 3r^2 \leq \frac{27}{4}R^2. \end{aligned}$$

This proves (a). Part (b) follows directly. $\square$

Theorem 9. *If S is the area of a triangle ABC, then*

$$r \leq \frac{\sqrt[4]{3}}{3}\sqrt{S} \leq \frac{\sqrt{3}}{9}p \leq \frac{1}{2}R.$$

If equality occurs in any of these, then ABC is equilaterial.

Proof. By the second part of Theorem 8,

$$r \le \frac{\sqrt{3}}{9}p \le \frac{1}{2}R,$$

so we need only prove the inequalities involving S. Indeed, applying the second part of Theorem 8 again gives

$$S = pr \ge 3\sqrt{3}r^2,$$

that is, $r \le \frac{1}{3}\sqrt{\sqrt{3}S} = \frac{\sqrt[4]{3}}{3}\sqrt{S}$. Also by Theorem 8's first part,

$$S = pr \le \frac{3\sqrt{3}}{2}Rr = \frac{1}{3\sqrt{3}} \cdot \frac{27}{2}Rr \le \frac{1}{3\sqrt{3}}(16Rr - 5r^2) \le \frac{p^2}{3\sqrt{3}},$$

whence $\frac{1}{3}\sqrt{\sqrt{3}S} \le \frac{\sqrt{3}}{9}p$. □

Example 8 (IMO, 1962)**.** Let a, b, c be the sides of a triangle of an area S. Prove that

$$a^2 + b^2 + c^2 \ge 4\sqrt{3}S.$$

Solution. Using both parts of Theorem 8,

$$\begin{aligned}
a^2 + b^2 + c^2 &= \sigma_1^2 - 2\sigma_2 \\
&= (2p)^2 - 2(p^2 + r^2 + 4rR) \\
&= 2p^2 - 2r^2 - 8rR \\
&\ge 32rR - 12r^2 - 8rR \\
&= 24rR - 12r^2 \\
&\ge 18rR \\
&\ge 18r \cdot \frac{2}{3\sqrt{3}}p \\
&= 4\sqrt{3}S.
\end{aligned}$$

□

Many other problems can be solved by this technique.

Exercise 5. Prove that

(a) $36r^2 \le 20rR - 4r^2 \le ab + bc + ca \le 4(r + R)^2 \le 9R^2$;

(b) $24\sqrt{3}r^3 \le 12\sqrt{3}r^2R \le abc \le 6\sqrt{3}rR^2 \le 3\sqrt{3}R^3$;

(c) $\frac{\sqrt{3}}{R} \le \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \le \frac{\sqrt{3}R}{4r^2}$;

(d) $\frac{1}{R^2} \le \frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} \le \frac{1}{4r^2}$;

(e) $\frac{1}{R^2} \le \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \le \frac{1}{4r^2}$;

(f) $6 \le \frac{a}{p-a} + \frac{b}{p-b} + \frac{c}{p-c} = \frac{4R - 2r}{r}$;

(g) $9r^2 \le h_a + h_b + h_c \le \frac{9}{2}R$;

(h) $\dfrac{2}{R} \leq \dfrac{1}{h_a} + \dfrac{1}{h_6} + \dfrac{1}{h_c} = \dfrac{1}{r}$;

(i) $9r \leq \rho_a + \rho_b + \rho_c \leq \dfrac{9}{2}R$;

(j) $\dfrac{2}{R} \leq \dfrac{1}{\rho_a} + \dfrac{1}{\rho_b} + \dfrac{1}{\rho_c} = \dfrac{1}{r}$;

(k) $3\sqrt{3}\dfrac{r}{R} \leq \sin\alpha + \sin\beta + \sin\gamma \leq \dfrac{3\sqrt{3}}{2}$;

(l) $\dfrac{3r}{R} \leq \cos\alpha + \cos\beta + \cos\gamma \leq \dfrac{3}{2}$.

Exercise 6. Prove that it is possible to construct a triangle from segments of lengths

(a) $\sin\alpha$, $\sin\beta$, $\sin\gamma$;

(b) $\cos^2\frac{\alpha}{2}$, $\cos^2\frac{\beta}{2}$, $\cos^2\frac{\gamma}{2}$.

Exercise 7. Prove the following inequalities:

(a) $(2a-p)(b-c)^2 + (2b-p)(c-a)^2 + (2c-p)(a-b)^2 \geq 0$;

(b) $(a+b)(b+c)(c+a) \leq 8pR(R+2r)$

(c) $ab(a+b) + bc(b+c) + ca(c+a) \leq 8pR(R+r)$

(d) $a^3 + b^3 + c^3 \leq 8p(R^2 - r^2)$

(e) $(p-a)(p-b)(p-c) \geq (2a-p)(2b-p)(2c-p)$

(f) $20S^2 \leq (a^2+b^2)(b+c-a) + (b^2+c^2)(c+a-b) +$ $(c^2+a^2)(a+b-c) \leq 8\left[(p-a)^4 + (p-b)^4 + (p-c)^4\right] + 12S^2$;

(g) if ABC is acute-angled, then

$$2abc(abc + p(a^2+b^2+c^2) - (a^3+b^3+c^3)) \geq$$
$$5(a^2+b^2-c^2)(b^2+c^2-a^2)(c^2+a^2-b^2);$$

(h) $a^2 + b^2 + c^2 \leq \dfrac{72R^4}{9R^2 - 4r^2}$;

(i) $(1-\cos\alpha)(1-\cos\beta)(1-\cos\gamma) \geq \cos\alpha \cdot \cos\beta \cdot \cos\gamma$;

(j) $(1+\cos 2\alpha)(1+\cos 2\beta)(1+\cos 2\gamma) + \cos 2\alpha \cdot \cos 2\beta \cdot \cos 2\gamma \geq 0$.

6 Problems

1. (USA, 1977-79) Prove that

   (a) $\dfrac{1}{a} + \dfrac{1}{b} + \dfrac{1}{c} \geq \dfrac{9}{2p}$;

   (b) $a^2 + b^2 + c^2 \geq \dfrac{4p^2}{3}$;

   (c) $p^2 \geq 3\sqrt{3}S$;

   (d) $a^2 + b^2 + c^2 \geq 4\sqrt{3}S$;

(e) $a^3+b^3+c^3 \geq \frac{8}{g}p^3$;

(f) $a^3+b^3+c^3 \geq \frac{8\sqrt{3}}{3}Sp$;

(g) $a^4+b^4+c^4 \geq 16S^2$

2. (GDR, 1965) Prove that
$$\cos\alpha+\cos\beta+\cos\gamma \leq \frac{3}{2}$$
and determine when the equality happens.

3. (England, 1967) Prove that
$$\cos^2\alpha+\cos^2\beta+\cos^2\gamma \geq \frac{3}{4}$$
and determine when the equality occurs. Prove that $\cos^2\alpha+\cos^2\beta+\cos^2\gamma$ does not attain its maximal value.

4. (Vietnam, 1977) Prove that
$$\frac{ab+ac+bc}{4S} \geq \sqrt{3}.$$

5. (Yugoslavia, 1976) Prove that for angles of any nonobtuse-angled triangle
$$\sin\alpha+\sin\beta+\sin\gamma > \cos\alpha+\cos\beta+\cos\gamma.$$

6. (Bulgaria, 1968) Find relations between the lengths of sides of a triangle if
$$\frac{a\cos\alpha+b\cos\beta+c\cos\gamma}{a\sin\beta+b\sin\gamma+c\sin\alpha} = \frac{2p}{9R}$$

7. (USSR, 1989) Prove that if $a+b+c=1$, then
$$a^2+b^2+c^2+4abc < \frac{1}{2}.$$

8. (IMO 1961) Let a, b, c be the sides of a triangle of area S. Prove that
$$a^2+b^2+c^2 \geq 4\sqrt{3}\cdot S.$$

9. (IMO 1962, also known as Euler's theorem) Given an isosceles triangle ABC, R is the radius of the circumscribed circle, r is the radius of inscribed circle. Prove that the distance d between their centers is $d=\sqrt{R(R-2r)}$.

10. (IMO 1964) Let a, b, c be the sides of a triangle. Prove that
$$a^2(b+c-a)+b^2(c+a-b)+c^2(a+b-c) \leq 4abc.$$

11. (IMO 1983) Let a, b, c be the sides of a triangle. Prove that
$$a^2b(a-b)+b^2c(b-c)+c^2a(c-a) \geq 0.$$

12. (IMO) If the lengths a, b, c of the sides of a triangle satisfy
$$2(bc^2+ca^2+ab^2) = b^2c+c^2a+a^2b+3abc,$$
prove that the triangle is equilateral. Prove also that the equation can be satisfied by positive real numbers which are not lengths of sides of a triangle.

13. (IMO) Three roots of the equation

$$x^4 - px^3 + qx^2 - rx + s = 0$$

are $\tan A, \tan B, \tan C$, where A, B, C are the angles of a triangle. Determine the fourth root as a function of p, q, r, s (only).

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