TJUSAMO 2011 – Olympiad Geometry, Part 3

Mitchell Lee, Andre Kessler

1 Reflection, rotation, and translation

It's useful to know the basic properties of reflection, rotation, and translation, because they have the capacity to trivialize a good number of Olympiad geometry problems very quickly. (This is not to say that these problems are trivial.) Most of these properties are not very difficult, and are left as an exercise to the reader – see the problems section.

2 Homothety

This is one of the most useful nontrivial geometric transformations, and is extremely important in problems dealing with circles. A homothety from a point P with ratio r is simply a dilation from the point P by the factor r. The concept of a homothety with negative ratio is also defined - a homothety with ratio $-r$ from point P is the composition of a homothety with ratio r from point P and a reflection about point P. Any two triangles whose corresponding sides are parallel are homothetic - that is, there is a homothety bringing one to the other. In the case of triangles ABC and DEF, this homothety is from point $AD \cap BE$. Additionally, it is easily seen that homothety maps parallel lines to parallel lines.
This is most commonly applied to circles. Any two circles are homothetic in two ways - one homothety has positive ratio (this is called the "external homothety") and one has a negative ratio (this is called the "internal homothety"). Two tangent circles are homothetic with a center at their point of tangency. The external center of homothety of two circles external to each other is the intersection of their common external tangents, and the internal center of homothety between these two circles is the intersection of their common internal tangents.

3 Spiral similarity

A spiral similarity is the composition of a homothety from a point with a rotation about the same point. All homotheties are also spiral similarities. The important fact about spiral similarities is that any two segments AB, CD in the plane are related by two spiral similarites: an "internal" and an "external" one. This center of spiral similarity is the intersection of the circumcircles of PAC, PBD, where P is the intersection of the two segments, as can be proven by an angle chase.

4 Problems

Here are some problems on which these geometric transformations can be used.

1. Prove that the composition of two rotations (not necessarily about the same point) is a rotation. How can the center of the resulting rotation be constructed?

2. Circles ω_1 and ω_2 are internally tangent at A (with ω_2 inside ω_1), and line ℓ is tangent to ω_2 at B and intersects ω_1 at P and Q. Prove that A, B, and the midpoint of arc PQ not containing A are collinear.

3. Let $ABCD$ be a cyclic quadrilateral, with $AB \cap CD = E$ and $BC \cap AD = F$. Prove that the circumcircles of EAC, EBD, FAB, FCD are concurrent.

4. ω_1 and ω_2 are two circles such that the intersection of their external common tangents is P. Let ℓ_1 and ℓ_2 be lines through P. Let A be the intersection of ℓ_1 with ω_1 farthest from P, let B be the intersection of ℓ_2 with ω_1 farthest from P, let C be the intersection of ℓ_2 with ω_2 closest to P, and let D be the intersection of ℓ_1 with ω_2 closest to P. Prove that $ABCD$ is a cyclic quadrilateral.

5. Circle ω contains circles ω_1 and ω_2 which are interally tangent to ω, and line ℓ is tangent to ω_1 and ω_2 and intersects ω at P and Q. Prove that the midpoint of one of the arcs PQ lies on the radical axis of ω_1 and ω_2.

5 More Problems

These problems, unlike the previous ones, do not necessarily involve the geometric transformations listed above. (Don't forget about radical axes, Ceva's theorem, and Menelaus's theorem when trying the below problems.)

1. Suppose that $\omega_1, \omega_2, \omega_3$ are circles. Prove that the centers of the three external homotheties taking ω_1 to ω_2, ω_2 to ω_3, and ω_3 to ω_1 are collinear. (This is known as Monge's theorem.)

2. Let ABC be an acute-angled triangle with $AB \neq AC$. The circle with diameter BC intersects the sides AB and AC at M and N respectively. Denote by O the midpoint of the side BC. The bisectors of the angles $\angle BAC$ and $\angle MON$ intersect at R. Prove that the circumcircles of the triangles BMR and CNR have a common point lying on the side BC.

3. Two circles ω_1 and ω_2 with radii r_1 and r_2 are externally tangent at point A. Line ℓ meets ω_1 at P, Q and ω_2 at R, S, such that P, Q, R, S are on the line in that order, and $\frac{PQ}{RS} = \frac{r_1}{r_2}$. Prove that $\angle QAS$ is right.

4. Let $ABCD$ be a quadrilateral, and let E and F be points on sides AD and BC, respectively, such that $\frac{AE}{ED} = \frac{BF}{FC}$. Ray FE meets rays BA and CD at S and T, respectively. Prove that the circumcircles of triangles SAE, SBF, TCF, and TDE pass through a common point.

5. Let A, B, C, D, E, F be points such that A, B, C are collinear and D, E, F are collinear. Prove that $AE \cap BD$, $AF \cap CD$, $BF \cap CE$ are collinear. (This is known as Pappus's theorem.)