Sequences

1 General

1. Let $a_1 = 2, a_2 = 5$ and
$$a_{n+2} = (2 - n^2)a_{n+1} + (2 + n^2)a_n$$
for $n \geq 1$. Do there exist p, q, r so that $a_p a_q = a_r$? (Czech-Slovak Match 1995 [16])

2. Prove that for all natural numbers $n \geq 3$ there exist odd natural numbers x_n, y_n such that $7x_n^2 + y_n^2 = 2^n$. (Bulgaria 1996 [17])

3. The sequence (a_n) is defined by $a_1 = 1, a_{n+1} = \frac{a_n}{n} + \frac{n}{a_n}$ for $n \geq 1$. Prove that for $n \geq 4, \lfloor a_n^2 \rfloor = n$. (Bulgaria 1996 [17])

4. Let a, b be positive integers with a odd. Define the sequence u_n as follows: $u_0 = b$, and for $n \in \mathbb{N}$, $u_{n+1} = \frac{1}{2}u_n$ if u_n is even and $u_{n+1} = u_n + a$ otherwise.
i. Show that $u_n \leq a$ for some $n \in \mathbb{N}$.
ii. Show that the sequence u_n is periodic from some point onwards. (Vietnam 1996 [17])

5. The polynomials $P_n(x)$ are defined by $P_0(x) = 0, P_1(x) = x$ and $P_n(x) = xP_{n-1} + (1 - x)P_{n-2}(x)$ for $n \geq 2$. Find the roots of P_n. (Austrian-Polish Mathematical Competition 1996 [17])

6. The positive integers $x_1, x_2, \ldots, x_7$ satisfy the conditions $x_6 = 144, x_{n+3} = x_{n+2}(x_{n+1} + x_n)$ for $n = 1, 2, 3, 4$. Find x_7. (Poland 1997 [18])

7. Define a sequence by $x_0, x_1 \in \mathbb{R}$ and
$$x_{n+2} = \frac{1 + x_{n+1}}{x_n}$$
for $n \geq 0$. Find x_{1998}. (Ireland 1998 [19])

2 Inequalities for Sequences

8. Suppose that $2n$ real numbers $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ $(n \geq 3)$ satisfy the following relations:
i. $\sum a_i = \sum b_i$.
ii. $0 < a_1 = a_2$ and $a_i + a_{i+1} = a_{i+2}$ for $i = 1, 2, \ldots, n-2$.
iii. $0 < b_1 \leq b_2$ and $b_i + b_{i+1} = b_{i+2}$ for $i = 1, 2, \ldots, n-2$.
Prove that $a_{n-1} + a_n \leq b_{n-1} + b_n$ (China 1995 [16]).

9. The sequence a_n satisfies $a_{m+n} + a_{m-n} = \frac{1}{2}(a_{2m} + a_{2n})$ for all $m \geq n \geq 0$. If $a_1 = 1$ find a_{1995}. (Russia 1995 [16])

10. Let $x_1, x_2, \ldots, x_{1997}$ be real numbers satisfying the following conditions:
i. $-\frac{1}{\sqrt{3}} \leq x_n \leq \sqrt{3}$.
ii. $\sum x_i = -318\sqrt{3}$.
Find the maximum value of $\sum x_i^{12}$. (China 1997 [18])

11. Let $a_1, a_2, \ldots$ be nonnegative integers satisfying $a_{n+m} \leq a_n + a_m$ for $(m, n \in \mathbb{N})$. Prove that
$$a_n \leq m a_1 + \left(\frac{n}{m} - 1\right) a_m$$
(China 1997 [18])

12. Let $n \geq 3$ be an integer, and suppose that the sequence $a_1, a_2, \ldots, a_n$ satisfies $a_{i-1} + a_{i+1} = k_i a_i$ for positive integers k_i. Prove that $2n \leq \sum k_i \leq 3n$. (Taiwan 1997 [18])

13. Let n be a positive integer. Determine if there exist positive integers $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ so that $\sum a_i = \sum b_i$ and
$$n - 1 > \sum \frac{a_i - b_i}{a_i + b_i} > n - 1 - \frac{1}{1998}$$
(China 1998 [19])

14. Let n_i be a sequence of positive integers so that the first digits of n_i are not n_j for any i, j. Prove that $\sum \frac{1}{n_i} \leq 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{9}$. (Iran 1998 [19])

15. Define sequences $x_1, x_2, \ldots y_1, y_2, \ldots$ by $x_1 = y_1 = \sqrt{3}$ and
$$x_{n+1} = x_n + \sqrt{1 + x_n^2}, y_{n+1} = \frac{y_n}{1 + \sqrt{1 + y_n^2}}$$
. Prove that for $n \geq 2$ we have $2 < x_n y_n < 3$. (Belarus 1999 [20])

16. Consider a finite sequence $(a_n) \subset \mathbb{N}$ so that any two distinct subsequences have different sums. Prove that $\sum_{k=1}^{n} \frac{1}{a_k} < 2$. (Romania 1999 [20])

17. Prove that for an integer $n \geq 3$ there exists an arithmetic progression $a_1, a_2, \ldots, a_n$ and a geometric progression $b_1, b_2, \ldots, b_n$ so that $b_1 < a_1 < b_2 < a_2 < \ldots < b_n < a_n$. (Romania 1999 [20])

18. Let $x_1 > 0$ and $x_{n+1} \geq (n+2)x_n - \sum_{k=1}^{n-1} k x_k$ for $n \geq 2$. Prove that for any $a \in \mathbb{R}$ the sequence x_n eventually gets bigger than a. (Romania 1999 [20])

19. Define the sequence $(a_n)_{n \geq 1}$ by $a_{m+n} \leq a_m + a_n$. Prove that for any n we have
$$\sum_{k=1}^{n} \frac{a_k}{k} \geq a_n$$
(Asian Pacific Mathematical Olympiad 1999 [20])

20. Suppose that the real numbers $a_1, a_2, \ldots, a_{100}$ satisfy $a_1 \geq a_2 \geq \ldots \geq a_{100} \geq 0, a_1 + a_2 \leq 100$ and $a_3 + a_4 + \ldots + a_{100} \leq 100$. Determine the maximum possible value of $a_1^2 + \ldots + a_{100}^2$, and find all possible sequences $a_1, a_2, \ldots, a_{100}$ for which this maximum is achieved. (Canada 2000 [21])

21. Define the infinite sequence $a_1, a_2, \ldots$ recursively as follows: $a_1 = 0, a_2 = 1$ and $a_n = \frac{1}{2}na_{n-1} + \frac{1}{2}n(n-1)a_{n-2} + (-1)^n \left(1 - \frac{n}{2}\right)$ for all $n \geq 3$. Find an explicit formula for
$$f_n = \sum_{k=1}^{n} k a_{n+1-k} \binom{n}{k-1}$$

22. For any integer $a_1 > 5$ consider the sequence $a_1, a_2, \ldots$ where $a_{n+1} = a_n^2 - 5$ if a_n is odd and $a_{n+1} = \frac{a_n}{2}$ if a_n is even. Prove that this sequence is not bounded. (Russia 2000 [21])

23. Find all sequences $a_1, a_2, \ldots, a_{2000}$ of real numbers such that $\sum_{n=1}^{2000} a_n = 1999$ and for any $n \geq 1$ we have $\frac{1}{2} < a_n < 1$ and $a_{n+1} = a_n(2 - a_n)$. (Turkey 2000 [21])

24. Consider the sequence if nonnegative real numbers so that $a_k - 2a_{k+1} + a_{k+2} \geq 0$ and $\sum a_i \leq 1$ for $k \geq 1$. Prove that for any k we have $0 \leq a_k - a_{k+1} < \frac{2}{k^2}$. (Shortlist 1988 [3])

25. For every integer $n \geq 2$ determine the minimum value that the sum $a_0 + a_1 + \ldots + a_n$ can take for nonnegative numbers $a_0, \ldots, a_n$ satisfying $a_0 = 1$ and for $i = 0, 1, \ldots, n-2$ we have $a_i \leq a_{i+1} + a_{i+2}$. (Shortlist 1997 [8])

26. The positive real numbers $x_0, \ldots, x_{1995}$ satisfy $x_0 = x_{1995}$ and for $i = 1, 2, \ldots, 1995$ we have
$$x_{i-1} + \frac{2}{x_{i-1}} = 2x_i + \frac{1}{x_i}$$
(IMO 1995 [9])

27. Suppose $x_1, x_2, \ldots$ are positive real numbers so that
$$x_n^n = \sum_{i=0}^{n-1} x_n^j$$
Prove that
$$2 - \frac{1}{2^{n-1}} \leq x_n < 2 - \frac{1}{2^n}$$
(Shortlist 1995 [6])

28. The nonnegative integers $a_1, a_2, \ldots, a_{1997}$ satisfy $a_i + a_j \leq a_{i+j} \leq a_i + a_j + 1$ for $i + j \leq 1997$. Prove that there exists $x \in \mathbb{R}$ so that $a_n = \lfloor nx \rfloor$ for all n. (USAMO 1997 [12])

29. The sequence $a_0, a_1, a_2, \ldots, a_n$ of real numbers satisfies $a_0 = a_n = 0$ and for $1 \leq k \leq n-1$ we have $a_k = c + \sum_{i=k}^{n-1} a_{i-k}(a_i + a_{i+1})$. Prove that $c \leq \frac{1}{4n}$. (Shortlist 1989 [4])

30. Let $a_0 = 1994$ and $a_{n+1} = \frac{a_n^2}{a_n+1}$ for $n \geq 0$. Prove that $\lfloor a_n \rfloor = 1994 - n$ for $0 \leq n \leq 998$. (Shortlist 1994 [5])

31. Let $a > 2$. Define

$$a_0 = 1, a_1 = 1, a_{n+1} = \left(\frac{a_n^2}{a_{n-1}^2} - 2 \right) a_n$$

Show that for all k we have

$$\frac{1}{a_0} + \ldots + \frac{1}{a_k} \leq \frac{1}{2} \left(2 + a - \sqrt{a^2 - 4} \right)$$

(Shortlist 1996 [?])

32. A sequence of positive integers (a_n) contains each positive integers exactly once. If $m \neq n$ then

$$\frac{1}{1998} < \frac{|a_m - a_n|}{|m - n|} < 1998$$

Prove that $|a_n - n| < 2000000$ for all n. (Russia 1998 [19])

3 Integer Sequences

33. Prove that for any positive integer a_1 there is an increasing sequence of positive integers $a_1, a_2, \ldots$ so that for any natural number k we have $a_1 + \ldots a_k | a_1^2 + \ldots a_k^2$. (Russia 1995 [16])

34. Let p be an odd prime. The sequence $(a_n)_{n \geq 1}$ is defined as follows: $a_0 = 0, a_1 = 1, \ldots, a_{p-2} = p - 2$. For $n \geq p - 1$ a_n is the smallest integer greater than a_{n-1} that does not form an arithmetic progression of length p with any of the previously defined terms of the sequence. Prove that for all n, a_n is the number obtained by writing n in base $p - 1$ and reading the result in base $p - 2$. (USAMO 1995 [11])

35. The sequence (a_n) is defined by $a_0 = 1, a_1 = 3$ and $a_{n+2} = a_{n+1} + 9a_n$ if n is even and $a_{n+2} = 9a_{n+1} + 5a_n$ if n is odd. Show that $\sum_{k=1995}^{2000} a_k^2$ is divisible by 20. Also a_{2n+1} is not a perfect square for every $n \geq 0$. (Vietnam 1995 [16])

36. Consider the sequence of positive integers which satisfies $a_n = a_{n-1}^2 + a_{n-2}^2 + a_{n-3}^2$. Prove that is $a_k = 1997$ then $k \leq 3$. (Austria 1997 [18])

37. The sequence a_n is defined by $a_1 = 0$ and $a_n = a_{\lfloor \frac{n}{2} \rfloor} + (-1)^{\frac{n(n+1)}{2}}$ for $n > 1$. For every k find the number of n so that $2^k \leq n < 2^{k+1}$ and $a_n = 0$. (Poland 1997 [18])

38. $Let f : \mathbb{N} \longrightarrow \mathbb{Z}$ be the function defined by $f(0) = 2, f(1) = 503, f(n+2) = 503f(n+1) - 1996f(n)$. For $k \in \mathbb{N}$ take integers $s_1, s_2, \dots, s_k$ not less than k and let p_i be a prime divisor of $f(2^{s_i})$. Prove that $\sum p_i | 2^t$ if and only if $k | 2^t$. (Vietnam 1997 [18]).

39. Let m be a positive integer. Define the sequence a_n by $a_0 = 0, a_1 = m$ and $a_{n+1} = m^2 a_n - a_{n-1}$. Prove that $a \leq b$ is the solution to $\frac{a^2+b^2}{ab+1} = m^2$ if and only if $(a, b) = (a_{n-1}, a_n)$ for some n.

40. Let F_n be the Fibonacci sequence. Determine all pairs of integers (k, m) with $m > k \geq 0$ so that the sequence defined by $x_0 = \frac{F_k}{F_m}$ and

$$x_{n+1} = \frac{2x_n - 1}{1 - x_n}$$

contains the number 1. (Poland 1998 [19])

41. Prove that the sequence defined by $a_1 = 1$ and $a_n = a_{n-1} + a_{\lfloor \frac{n}{2} \rfloor}$ contains infinitely many terms divisible by 7. (Poland 1997 [18])

42. (a_n) is a sequence of integers so that

$$(n-1)a_{n+1} = (n+1)a_n - 2(n-1)$$

If $2000 | a_{1999}$ then find the smallest n so that $2000 | a_n$. (Bulgaria 1999 [20])

43. Define the sequence (a_n) by $a_0 = 0$ and $a_n = a_{n-1} + \frac{3^{r+1}-1}{2}$ if $n = 3^r(3k+1)$ and $a_n = a_{n-1} + \frac{3^{r+1}+1}{2}$ if $n = 3^r(3k+2)$ $(r, k \geq 0)$. Prove that every integer appears exactly once in the sequence. (Iran 1999 [20])

44. Show that for any positive integer n the number

$$S_n = \binom{2n+1}{0} 2^{2n} + \binom{2n+1}{2} 2^{2n-2} \cdot 3 + \ldots + \binom{2n+1}{2n} 3^n$$

is the sum of two consecutive squares. (Romania 1999 [20])

45. Find all infinite bounded sequences $a_1, a_2, \dots$ of positive integers so that

$$a_n = \frac{a_{n-1} + a_{n-2}}{\gcd(a_{n-1}a_{n-2})}$$

(Russia 1999 [20])

46. Define sequences $(x_n), (y_n)$ by $x_1 = 1, y_1 = 2$ and $x_{n+1} = 22y_n - 15x_n, y_{n+1} = 17y_n - 12x_n$. Prove that the sequences are nonzero. Prove that each sequence contains infinitely many positive and negative terms. For $n = 1999^{1945}$ determine whether x_n, y_n are divisible by 7 or not. (Vietnam 1999 [20])

47. Define $(a_n) \subset \mathbb{Z}$ by $a_{n+1} = a_n^3 + 1999$. Prove that at most one of the terms of the sequence is a perfect square. (Austrian Polish Mathematics Competition 1999 [20])

48. Define the sequence of positive integers (x_n) by $x_1 = 10^{999}+1$ and for $n \geq 2$ the number x_n is obtained from $11x_{n-1}$ by deleting the first digit. Is the sequence bounded? (St. Petersburg City Mathematical Olympiad 1999 [20]).

49. Let $a_1, a_2, \ldots$ be a sequence such that $a_1 = 43, a_2 = 142$ and $a_{n+1} = 3a_n + a_{n-1}$ for all $n \geq 2$. Prove that
i. a_n and a_{n+1} are relatively prime for all $n \geq 1$
ii. for every natural number m, there exist infinitely many natural numbers n such that $a_n - 1$ and $a_{n+1} - 1$ are both divisible by m. (Bulgaria 2000 [21])

50. Let $r(1) = 1$ and for $k > 1$ let $r(k)$ equal the product of the prime divisors of k. A sequence of natural numbers $a_1, a_2, \ldots$ with arbitrary first term a_1 is defined recursively by the relation $a_{n+1} = a_n + r(a_n)$. Show that for any positive integer m, the sequence $a_1, a_2, \ldots$ contains m consecutive terms in arithmetic progression. (Mongolia 2000 [21])

51. A sequence $p_1, p_2, \ldots$ of prime numbers satisfies the following condition: for $n \geq 3$, p_n is the greatest prime divisor of $p_{n-1}+p_{n-2}+2000$. Prove that the sequence is bounded. (Poland 2000 [21])

52. Let $a_1, a_2, \ldots$ be a sequence with $a_1 = 1$ satisfying the recursion $a_{n+1} = a_n - 2$ if $a_n - 2 \notin \{a_1, a_2, \ldots, a_n\}$ and $a_n - 2 > 0$ and $a_{n+1} = a_n + 3$ otherwise. Prove that for every positive integer k there is a positive integer n so that $a_n = k^2 = a_{n-1} + 3$. (Russia 2000 [21])

53. Consider the sequence $(a_n)_{n \leq 0}$ defined by $a_0 = a_1 = 1$ and $a_{n+1} = 14a_n - a_{n-1}, \forall n \geq 1$. Prove that for any $n \geq 0, 2a_n - 1$ is a perfect square. (Romania 2002 [15])

54. Consider the sequence $(a_n)_{n \geq 1}$ as follows so that $a_1 = 20, a_2 = 30$ and $a_{n+1} = 3a_n - a_{n-1}$ for $n \geq 2$. Find all n so that $1 + 5a_n a_{n+1}$ is a perfect square. (Balkan Mathematical Olympiad 2002 [13])

55. An integer sequence is defined by $a_0 = 0, a_1 = 1, a_{n+1} = 2a_n + a_{n-1}$ for $n \geq 1$. Prove that $2^k | a_n$ if and only if $2^k | n$. (Shortlist 1988 [3])

56. The sequence of integers (a_n) is defined by $a_1 = 2, a_2 = 7$ and

$$-\frac{1}{2} < a_{n+1} - \frac{a_n^2}{a_{n-1}} \leq \frac{1}{2}$$

Prove that a_n is odd. (Shortlist 1988 [3])

57. In a sequence of positive integers the number a_{n+1} is obtained from a_n by the following rule. If the last digit of a_n is ≤ 5 then remove it to get a_{n+1}. Otherwise $a_{n+1} = 9a_n$. Can one choose a_0 so that we never get to 0? (USSR 1991 [22])

58. The sequence of positive integers (x_n) is defined by $x_1 = 1, x_{n+1} = n + x_1^2 + \ldots + x_n^2$. Prove that there are no squares of natural numbers in this sequence except x_1. (CIS 1992 [22])

59. Let $a_0, a_1 \in \mathbb{Z}$. Define

$$a_{n+1} = \frac{a_n^2 + 1}{a_{n-1}}$$

for $n \geq 1$. Show that for any $n \geq 2$ the denominator of the (irreducible) fraction a_n has no prime factors other than those of a_0, a_1. (MOP 2001)

60. Let $a_0 = 4, a_1 = 22$ and define $a_{n+1} = 6a_n - a_{n-1}$. Prove that there exist sequences $(x_n), (y_n)$ of positive integers so that

$$a_n = \frac{y_n^2 + 7}{y_n - x_n}$$

(MOP 2001)

61. Define $a_2 = 2001$ and for $n \geq 3$ we have $a_n = a_{n-1}a_{n-2} - 1$. Prove that there are infinitely many values of a_1 so that $a_n = 2002$ for some n. (Rookie Team Contest MOP 2001)

62. For an integer $x \geq 1$ let $p(x)$ be the least prime that does not divide x and define $q(x)$ to be the product of all primes less than $p(x)$. In particular $p(1) = 2$. If $p(x) = 2$ then define $q(x) = 1$. Consider the sequence $x_1, x_2, \ldots$ defined by $x_1 = 1$ and $x_{n+1} = \frac{x_n p(x_n)}{q(x_n)}$ for $n \geq 0$. Find all n so that $x_n = 1995$. (Shortlist 1995 [6])

63. Define (a_n) by

$$\sum_{d|n} a_d = 2^n$$

Show that $n|a_n$. (Shortlist 1989 [**?**])

64. Let $c \in \mathbb{N}^*$. Define (f_n) by $f_1 = 1, f_2 = c, f_{n+1} = 2f_n - f_{n-1} + 2$ for $n \geq 2$. Show that for any k there is a positive integer r so that $f_k f_{k+1} = f_r$. (Some Shortlist)

65. For $x_0 \in \mathbb{N}^*$ define sequences $(x_n), (y_n), (z_n)$ by $y_0 = 4, z_0 = 1$. If x_n is even then $x_{n+1} = \frac{x_n}{2}, y_{n+1} = 2y_n, z_{n+1} = z_n$. Otherwise $x_{n+1} = x_n - \frac{y_n}{2} - z_n, y_{n+1} = y_n, z_{n+1} = y_n + z_n$. The integer x_0 is good if $x_n = 0$ for some $n \geq 1$. Find the number of good integers ≤ 1994. (Shortlist 1994 [5]).

66. Let x_1, x_2 be coprime integers. Define $x_{n+1} = x_n x_{n-1} + 1$ for $n \geq 2$. Prove that for any $i > 1$ there is a j so that $x_i^i | x_j^j$. Does this hold for $i = 1$? (Shortlist 1994 [5])

67. Let p, q, n be three positive integers so that $p + q < n$. Let $(x_0, x_1, x_2, \ldots, x_n)$ be an $(n+1)$-tuple of integers so that $x_0 = x_n = 0$ and for $1 \leq i \leq n$ we have $x_i - x_{i-1} \in \{p, -q\}$. Show that there is a pair $(i, j) \neq (0, n)$ so that $x_i = x_j$. (IMO 1996 [10])

68. Let $a_0, a_1, a_2, \ldots$ be an increasing sequence of nonnegative integers so that every nonnegative integer can be expressed uniquely in the form $a_i + 2a_j + 4a_k$ for i, j, k not necessarily distinct. Determine a_n. (Shortlist 1998 [?])

69. Let $a_1 = 19, a_2 = 98$. Define $a_{n+2} = (a_n + a_{n+1}) \bmod 100$. Find $a_1^2 + \ldots + a_{1998}^2 \bmod 8$. (UK 1998 [19])

70. Show that there is a unique sequence of positive integers defined by $a_1 = 1, a_2 = 2, a_4 = 12$ and $a_{n+1}a_{n-1} = a_n^2 \pm 1$ for $n \geq 2$. (UK 1998 [19])

71. Let $a, b \in \mathbb{Z}$. Define $a_0, a_1, a_2, \ldots$ by $a_0 = a, a_1 = b, a_2 = 2b - a + 2, a_{n+3} = 3a_{n+2} - 3a_{n+1} + a_n$. Find the general term of the sequence and find a, b so that a_n is a perfect square for $n \geq 1998$. (Vietnam 1998 [19])

4 Analysis and Sequences

72. Define a sequence of reals by $x_1 = 1$ and

$$x_{n+1} = x_n + \sqrt[3]{x_n}$$

for $n \geq 1$. Prove that there exist $a, b \in \mathbb{R}$ so that $\lim_{n \longrightarrow \infty} \frac{x_n}{an^b} = 1$. (Turkey 1995 [16])

73. Find the largest real number α for which there exists an infinite sequence a_n of positive integers satisfying the following properties:
i. For each $n \in \mathbb{N}$, $a_n > 1997n$.
ii. For every $n \geq 2$, a_n^α does not exceed the greatest common divisor of the set $\{a_i + a_j | i + j = n\}$. (Vietnam 1997 [18])

74. Given a real number $c > 2$, a sequence $x_1, x_2, \ldots$ of real numbers is defined recursively by $x_1 = 0$ and $x_{n+1} = \sqrt{c - \sqrt{c + x_n}}$ for all $n \geq 1$. Prove that the sequence $x_1, x_2, \ldots$ is defined for all n and has a finite limit. (Vietnam 2000 [21])

75. Let $e > 0$ and b_n a decreasing sequence in $(0, 1)$. The sequence a_n satisfies

$$e + a_{n+1} \leq a_n \left(1 + \frac{b_n}{n}\right)$$

Prove that $\liminf a_n \leq 0$. (Longlist 1984 [1])

76. Define $a_0 = 1, a_1 = \frac{64}{15}, a_2 = \frac{143}{30}$ and $a_{n+3}^2 = \frac{3}{2}a_{n+2}^2 + \frac{3}{4}a_{n+1}^2 - \frac{1}{8}a_n^2$. Find $\sum_{k=0}^{\infty} \frac{a_k}{\sqrt{5}^k}$. (Rookie Team Contest MOP 2001)

77. The sequences $a_0, a_1, a_2, \ldots$ and $b_0, b_1, b_2, \ldots$ are defined by $a_0 = \frac{\sqrt{2}}{2}, a_{n+1} = \frac{\sqrt{1 - \sqrt{1 - a_n^2}}}{\sqrt{2}}$ and $b_0 = 1, b_{n+1} = \frac{\sqrt{1 + b_n^2} - 1}{b_n}$. Prove that $2^{n+2}a_n < \pi < 2^{n+2}b_n$. (Longlist 1989 [2])

78. Define $x_0, x_1, x_2, \ldots$ by $x_0 = 1989, x_n = \frac{1989}{n}\sum_{k=0}^{n-1} x_k$. Find $\sum_{k=0} 19892^n x_n$. (Longlist 1989 [2])

79. Find all $a \in \mathbb{R}$ for which there is no infinite sequence $x_0 = a$ and $x_{n+1} = \frac{x_n+\alpha}{\beta x_n+1}$ for $n \geq 0$ and $\alpha\beta > 0$. (Longlist 1989 [2])

80. Consider the sequence $(x_n)_{n\geq 1}$ of real numbers so that $x_1 = 1, x_2 = 0, x_3 = \frac{1}{3}$ and for any $n \geq 2$ we have $(n+2)x_{n+2} + (2n+1)x_{n+1} + (n-1)x_n = 0$. Find x_n.

81. Let $a > 1$ and define $x_1, x_2, \ldots$ by $x_1 = a$ and

$$x_{n+1} = 1 + \log \frac{x_n(x_n^2+3)}{3x_n^2+1}$$

Prove that it has a limit and find it. (Vietnam 1998 [19])

82. Let $x_0 \in \mathbb{R} \setminus \mathbb{Q}$. Define the sequence (x_n) by

$$x_{n+1} \in \left\{ \frac{x_n+1}{x_n}, \frac{x_n+2}{2x_n-1} \right\}$$

Find the cases when this sequence has a limit and in that case find it. (Romania 2000 [14], [21])

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