Enumeration Techniques

Gabriel Carroll

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Often you want to find the number of objects of some type; find an upper or lower bound for this number; find its value modulo n for some n; or compare the number of objects of one type with the number of objects of another type. There are a lot of methods for doing all of these things.

I'm going to focus on methods rather than on knowing formulas, but I've attached a short list of useful formulas at the end. As an exercise, you can try to prove whichever ones you don't already know.

If you're looking for reading or reference materials, a general-purpose source for a lot of enumeration techniques is Graham, Knuth, and Patashnik's *Concrete Mathematics*. The bible of the subject (but much more advanced) is Stanley's *Enumerative Combinatorics*. Andreescu and Feng's book *A Path to Combinatorics for Undergraduates* is a more accessible, problem-solving-oriented treatment.

1 Counting techniques

A typical counting problem is as follows: you're given the definition of a quagga of order n, and told what it means for a quagga to be blue. How many blue quaggas of order n are there?

Here are some general-purpose techniques to approach such a problem:

- Write down a recurrence relation
- Count the non-blue quaggas
- Find a bijection with something you know how to count
  - If you only need a lower or upper bound, find a surjection or injection to something you know how to count
- Put all quaggas into groups of size n, such that there's one blue quagga in each group
- Count incarnations of blue quaggas, then show that each quagga has n incarnations

- To find out the number of quaggas mod n, find a way to put most of the blue quaggas into groups of size n and see how many are left over
- Use generating functions
- Attach variables to parts of quaggas, then use algebra to count quaggas

Here are a couple more specialized techniques:

- The Inclusion-Exclusion Principle: if $A_1, \ldots, A_n$ are finite subsets of some big set A, then
$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{k=1}^{n} (-1)^{k-1} \sum_{i_1 < \cdots < i_k} |A_{i_1} \cap \cdots \cap A_{i_k}|.$$
  In fact, more is true: if we consider just the first l terms of the sum, where $1 \leq l < n$, then we'll have an overestimate of $|A_1 \cup \cdots \cup A_n|$ if l is odd and an underestimate if l is even. (These latter inequalities are sometimes called Bonferroni's inequalities.)

  The Inclusion-Exclusion principle is a special case of the general Möbius inversion formula.

- Burnside's Lemma: Suppose you have a group G acting on a finite set S. In simpler language, this means that G consists of a bunch of bijections from S to itself, so that the composition of any two bijections in G is also in G. You want to count the orbits — the number of equivalence classes, where two elements of S are equivalent if you can get from one to the other by applying maps in G. For example, you may want to count the number of ways of coloring an $n \times n$ grid in k colors, where rotations and reflections are not considered distinct. (So S is the original set of all colorings, and G is the set of rotations and reflections.)

  For each $g \in G$, let n_g be the number of fixed points of g. Then, Burnside's Lemma says that the number of equivalence classes is equal to $(\sum_g n_g)/|G|$.

2 Problems

1. Given are positive integers n and m. Put $S = \{1, 2, \ldots, n\}$. How many ordered sequences are there of m subsets $T_1, \ldots, T_m$ of S, such that $T_1 \cup T_2 \cup \cdots \cup T_m = S$?
2. [Putnam, 1990] How many ordered pairs (A, B) are there, where A, B are subsets of $\{1, 2, \ldots, n\}$ such that every element of A is larger than $|B|$ and every element of B is larger than $|A|$?
3. [Putnam, 2002] A nonempty subset $S \subseteq \{1, 2, \ldots, n\}$ is *decent* if the average of its elements is an integer. Prove that the number of decent subsets has the same parity as n.

4. Let $k \leq n$ be positive integers. How many permutations of the set $\{1, 2, \ldots, n\}$ have the property that every cycle contains at least one of the numbers $1, 2, \ldots, k$?

5. [HMMT, 2002] Find the number of pairs of subsets (A, B) of $\{1, 2, \ldots, 2008\}$ with the property that exactly half the elements of A are in B.

6. [China, 2000] Let n be a positive integer, and M be the set of integer pairs (x, y) with $1 \leq x, y \leq n$. Consider functions f from M to the nonnegative integers such that

   - $\sum_{y=1}^{n} f(x, y) = n - 1$ for each x;
   - if $f(x_1, y_1)f(x_2, y_2) > 0$ then $(x_1 - x_2)(y_1 - y_2) \geq 0$.

   Find the number of functions f satisfying these conditions.

7. Prove that the number of partitions of a positive integer n into distinct parts equals the number of partitions into odd parts.

8. Let $f(a, b, c)$ be the number of ways of filling each cell of an $a \times b$ grid with a number from the set $\{1, \ldots, c\}$ so that every number is greater than or equal to the number immediately above it and the number immediately to its left. Prove that $f(a, b, c) = f(c - 1, a, b + 1)$.

9. [CGMO, 2008] On a 2010×2010 chessboard, each unit square is colored in red, blue, yellow, or green. The board is *harmonic* if each 2×2 subsquare contains each color once. How many harmonic colorings are there?

10. [Romania, 2003] Let n be a given positive integer. A permutation of the set $\{1, 2, \ldots, 2n\}$ is *odd-free* if there are no cycles of odd length. Show that the number of odd-free permutations is a square.

11. [Iran, 1999] In a deck of $n > 1$ cards, each card has some of the numbers $1, 2, \ldots, 8$ written on it. Each card contains at least one number; no number appears more than once on the same card; and no two cards have the same set of numbers. For every set containing between 1 and 7 numbers, the number of cards showing at least one of those numbers is even. Determine n, with proof.

12. [China, 2006] d and n are positive integers such that $d \mid n$. Consider the ordered n-tuples of integers $(x_1, \ldots, x_n)$ such that $0 \leq x_1 \leq \cdots \leq x_n \leq n$, and $x_1 + \cdots + x_n$ is divisible by d. Prove that exactly half of these n-tuples satisfy $x_n = n$.

13. Consider partitions of a positive integer n into (not necessarily distinct) powers of 2. Let $f(n)$ be the number of such partitions with an even number of parts, and let $g(n)$ be the number of partitions with an odd number of parts. For which values of n do we have $f(n) = g(n)$?

14. [Putnam, 2005] For positive integers m, n, let $f(m, n)$ be the number of n-tuples of integers $(x_1, \ldots, x_n)$ such that $|x_1| + \cdots + |x_n| \leq m$. Prove that $f(m, n) = f(n, m)$.

15. [St. Petersburg, 1998] 999 points are marked on a circle. We want to color each point red, yellow, or green so that on any arc between two points of the same color, the number of other points is even. How many colorings have this property?

16. [IMO Shortlist, 2008] For every positive integer n, determine the number of permutations $a_1, \ldots, a_n$ of the numbers $1, \ldots, n$, such that

$$2(a_1 + \cdots + a_k) \text{ is divisible by } k \qquad \text{for each } k = 1, \ldots, n.$$

17. [IMO, 1989] A permutation π of $\{1, 2, \ldots, 2n\}$ has property P if $|\pi(i) - \pi(i+1)| = n$ for some i. For any given $n \geq 1$, prove that there are more permutations with property P than without it.

18. [IMO, 1995] Let p be an odd prime. Find the number of subsets A of $\{1, 2, \ldots, 2p\}$ such that

    - A has exactly p elements;
    - the sum of the elements of A is divisible by p.

19. [China, 2008] Let S be a set with n elements, and let $A_1, \ldots, A_k$ be k distinct subsets of S ($k \geq 2$). Prove that the number of subsets of S that don't contain any of the A_i is greater than or equal to $2^n \prod_{i=1}^{k} (1 - 1/2^{|A_i|})$.

20. [IMO Shortlist, 2002] Let n be a positive integer. Find the number of sequences of n positive integers with the following property: for each $k \geq 2$, if k appears in the sequence then $k - 1$ appears in the sequence, and moreover the first occurrence of $k - 1$ comes before the last occurrence of k.

21. [TST, 2004] Let N be a positive integer. Consider sequences $a_0, a_1, \ldots, a_n$ with each $a_i \in \{1, 2, \ldots, n\}$ and $a_n = a_0$.

    (a) If n is odd, find the number of such sequences satisfying $a_i - a_{i-1} \not\equiv i \bmod n$ for all i.

    (b) If n is an odd prime, find the number of such sequences satisfying $a_i - a_{i-1} \not\equiv i, 2i \bmod n$ for all i.

22. You have a necklace consisting of $2n$ beads on a loop of string, and n different colors of paint. In how many ways can you paint the beads so that every color is used exactly twice? Rotations and reflections are *not* considered to be different colorings.

23. [TST, 2010] Let T be a finite set of positive integers greater than 1. A subset S of T is called *good* if, for every $t \in T$, there exists some $s \in S$ with $\gcd(s, t) > 1$. Prove that the number of good subsets of T is odd.

24. [ARML, 2004] If s is a sequence of integers, not necessarily distinct, let $S(s)$ denote the number of *distinct* subsequences that may be obtained by taking terms from s in order, including possibly the empty sequence and all of s. The terms taken to form a subsequence need not be distinct.

   (a) If s and t are sequences such that $S(s)$ and $S(t)$ are odd, prove that $S(st)$ is also odd. (st is the concatenation of s and t.)

   (b) Write s^k for the sequence obtained by concatenating s to itself k times. For any sequence s of length n, prove that at least one of the numbers

   $$S(s), S(s^2), \ldots, S(s^{n+1})$$

   is odd.

25. [IMO, 1997] For each positive integer n, let $f(n)$ denote the number of partitions of n into powers of 2. Prove that for every $n \geq 3$,

$$2^{n^2/4} \leq f(2^n) \leq 2^{n^2/2}.$$

26. There are n parking spaces in a row, initially empty. There are n drivers, numbered $1, \ldots, n$, each of whom has a favorite parking space. Different drivers may have the same favorite parking space. Drivers $1, 2, \ldots, n$ arrive at the row of parking spaces in order. Each driver first drives up to his favorite parking space. If it is empty, he parks there; if not, he continues down the row until he finds an empty space and parks there. If he gets to the end of the row without parking, he goes home and cries.

   Of the n^n possible choices of a favorite space for each driver, how many will allow everyone to park?

27. Given n vertices labeled $1, \ldots, n$, how many trees are there on these vertices?

Useful Counting Facts
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- Number of subsets of an n-element set: 2^n
- Number of permutations of n objects: $n!$
- Number of k-element subsets of an n-element set: $\binom{n}{k} = n!/k!(n-k)! \quad (0 \le k \le n)$
- Binomial coefficient identities:
  - $\binom{n}{k} = \binom{n}{n-k}$
  - $\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$
  - $\binom{n}{m}\binom{m}{k} = \binom{n}{k}\binom{n-k}{m-k}$
  - $\sum_{m=k-1}^{n} \binom{m}{k-1} = \binom{n+1}{k}$
  - $k\binom{n}{k} = n\binom{n-1}{k-1}$
  - $\sum_{i=0}^{k} \binom{n}{i}\binom{m}{k-i} = \binom{n+m}{k}$ (Vandermonde convolution)
  - $(a+b)^n = \sum_{k=0}^{n} \binom{n}{k} a^k b^{n-k}$
  - $\sum_{k=0}^{n} k\binom{n}{k} = 2^{n-1}n$
  - $\sum_{m=0}^{n} \binom{m}{j}\binom{n-m}{k} = \binom{n+1}{j+k+1}$
  - $\sum_{i=0}^{n} (-1)^i \binom{n}{i} = 0$ for $n > 0$
  - more generally $\sum_{i=0}^{n} (-1)^i \binom{n}{i} P(x+i) = 0$ if P is a polynomial of degree $< n$

  All of these, except maybe the last statement, can be checked by direct counting arguments. They can also be proven algebraically.

- Number of functions from $\{1, 2, \ldots, n\}$ to $\{1, 2, \ldots, m\}$: m^n
- Number of choices of k elements of $\{1, 2, \ldots, n\}$, without regard to ordering and with repetitions allowed: $\binom{n+k-1}{k}$
- Number of paths from $(0,0)$ to (m,n) using steps $(1,0)$ and $(0,1)$: $\binom{n+m}{m}$
- Number of ordered r-tuples of positive integers with sum n: $\binom{n-1}{r-1}$
- Number of ways of dividing $\{1, 2, \ldots, kn\}$ into k subsets of size n: $(kn)!/(n!)^k k!$
- Number of Dyck paths of length $2n$ or ways of triangulating a regular $(n+2)$-gon by diagonals (see main handout for more): $C_n = \binom{2n}{n}/(n+1)$ (nth Catalan number)

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