(Structural) Graph Theory

Adam Hesterberg
Based on Paul Seymour's notes and work

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1 Definitions

1.1 Definitions you must know

Graph (= multigraph), simple graph, vertex, edge, adjacent, loop, parallel edge, clique/complete graph, degree, subgraph, connected, component (= connected component), tree, forest, spanning tree, path, cycle, Eulerian cycle, Hamiltonian cycle, bipartite graph, stable set (= independent set), matching, perfect matching, planar graph, k-regular graph, digraph.

1.2 Other useful definitions

Definition 1.1. A *cut-vertex* of a connected graph is a vertex whose deletion disconnects the graph. A *cut-edge* or *cut-set* (of vertices or edges, usually the former) is similar.

Definition 1.2. A graph G is k-connected iff $|V(G)| \geq k+1$ and for every $X \subset V(G)$ with $|X| < k$, $G \setminus X$ is connected.

Definition 1.3. The line graph $L(G)$ of a graph G is the graph with $V(L(G)) = E(G)$, with an edge for for every pair of incidences of two edges of G on the same vertex of G.

Definition 1.4. A graph G is k-edge-connected iff its line graph is k-connected. Alternately, G is k-edge-connected iff for every $X \subset E(G)$ with $|X| < k$, $G \setminus X$ is connected.

Definition 1.5. A *separation* of G is a pair (A, B) of subsets of $V(G)$ with $A \cup B = V(G)$, such that there is no edge between $A \setminus B$ and $B \setminus A$. Its *order* is $|A \cap B|$.

2 Useful Theorems

Theorem 2.1. *(Erdős) If G is a graph with no stable set of size t, then there's a graph H with $V(G) = V(H)$ and at most $t-1$ components, each of which is a complete graph, such that $\forall v, \deg_H(v) \leq \deg_G(v)$.*

Theorem 2.2. *(Menger's Theorem) Let $Q, R \subset V(G)$, and let $k \geq 0$. Then there are k pairwise vertex-disjoint paths from Q to R unlesss there's a separation (A, B) of G of order $< K$ with $Q \subset A$ and $R \subset B$.*

Theorem 2.3. *(Tutte's Theorem) Let odd(X) be the number of components of X with an odd number of vertices. Then G has a perfect matching unlesss there exists $X \subset V(G)$ with odd$(G \setminus X) > |X|$.*

3 Matchings

1. (König's Theorem) Let G be bipartite, and $k \geq 0$ an integer. Then G has a matching of size at least k unlesss there exists $X \subset V(G)$ with $|X| < k$ such that X meets every edge of G.
2. Let G be a loopless graph in which every vertex has positive degree. Let X be the largest matching in G, and let Y be the smallest set of edges of G whose union contains $V(G)$. Show that $|X|+|Y| = |V(G)$.
3. Show that every 2-edge-connected cubic graph has a perfect matching.
4. Let G be a d-regular bipartite graph. Show that $E(G)$ can be partitioned into perfect matchings.

4 Minors

Definition 4.1. If $e \in E(G)$, then G/e ("G contract e") is the graph formed by deleting e and identifying its endpoints.

Definition 4.2. A graph H is a minor of a graph G iff it's obtainable from a subgraph of G by contracting edges.

That is, to get a minor, one first deletes vertices and edges, then contracts edges. Note that contraction and deletion commute, so one can do so any order.

Theorem 4.3 (Wagner's Theorem)**.** *A graph G is planar unlesss it has a K_5 or $K_{3,3}$ minor.*

Theorem 4.4 (Kuratowski's Theorem)**.** *A graph G is planar unlesss it has a subdivision of K_5 or $K_{3,3}$ (that is, with edges turned into paths) as a subgraph.*

Theorem 4.5 (Seymour)**.** *If G is an infinite set of graphs, then one is a minor of another.*

1. Prove that every 3-connected graph has a K_4 minor.
2. Prove that a graph G can be drawn in the plane with all vertices in the same region unlesss G has a K_4 or $K_{3,2}$ minor.
3. Prove that if a graph G has no K_5 minor, then it's 4-colorable. (You may assume the Four-color Theorem.)
4. Prove that every simple graph with average degree at least 2^p has a K_{p+2} minor.
5. Prove that if G is nonnull and loopless and $|E(G)| \geq 2|V(G)| - 1$, then G has a graph with three parallel edges as a minor.
6. Find all 2-connected graphs with no C_5 minor.
7. Find all 2-connected graphs with no $K_4 \setminus e$ minor.
8. (Kotzig's Theorem, also 2009 MOP K6.2/B6.4) Let G be a connected graph that has a perfect matching. Prove that if for any edge e of the perfect matching, $G \setminus e$ is connected, then G has another perfect matching.