TJUSAMO 2011 – Polynomials II

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1 Division

The *division algorithm* states that for any polynomials A, B in one variable where B is nonconstant, there are polynomials Q, R with $\deg R < \deg B$ and $A(x) = B(x)Q(x) + R(x)$. This is seen frequently with induction (or the equivalent "infinite descent").

2 Interpolation

The most important result here is that if two polynomials of degree at at most n agree at $n+1$ points, then they are identical. Suppose we have some pairwise distinct real numbers $x_0, x_1, \ldots, x_n$ and some arbitrary real numbers $y_0, y_1, \ldots, y_n$. Then we can find exactly one polynomial with real coefficients such that $f(x_i) = y_i$ for $0 \le i \le n$. This polynomial is given by the Lagrange interpolation formula, which says

$$f(x) = \sum_{i=0}^{n} y_i \prod_{\substack{0 \le j \le n \\ j \ne i}} \frac{x - x_j}{x_i - x_j}$$

While it might not be obvious where to come up with this at first glance, it should be clear that the polynomial takes on precisely the values advertised, and, by the first fact stated, is the *only* such polynomial.

3 Problems

1. Let $P(x)$ be a polynomial of degree n. Knowing that
$$P(k) = \frac{k}{k+1}$$
for $k = 0, 1, 2, \ldots, n$, find $P(m)$ for $m > n$. (Note: this problem or very simple variants appear at least three times per year on normal contests and seemingly no one can solve them in time. Know how to solve it).

2. A polynomial $P(x)$ of degree n satisfies $P(k) = 1/k$ for $k = 1, 2, 4, 8, \ldots, 2^n$. Find $P(0)$.

3. Let $a_1, a_2, ..., a_{100}$ and $b_1, b_2, ..., b_{100}$ be 200 distinct real numbers. Consider an 100×100 table and put the number $a_i + b_j$ in the (i, j) position. Suppose that the product of the entries in each column is 1. Prove that the product of the entries in each row is -1

4. Let α be a real number. Let Q be a polynomial with integer coefficients with minimal degree satisfying $Q(\alpha) = 0$. (This is called the minimal polynomial of α.) If P a polynomial with integer coefficients satisfying $P(\alpha) = 0$, prove that $Q|P$.

5. A polynomial p of degree n satisfies $p(k) = 2^k$ for all $0 \le k \le n$. Find its value at $n+1$.

6. Let P be a polynomial with integer coefficients. Let n, k be integers with $n > 0$, and let P^n be the n-fold application of P (that is, $\underbrace{P \circ P \circ \cdots \circ P}_{n}$). Suppose that $P^n(k) = k$. Prove that $P^2(k) = k$.

7. Let $P(x)$ be a polynomial with real coefficients. Prove that $P(x) - x | P(P(x)) - x$.

8. Let $P(x), Q(x)$ be polynomials such that the points $(P(1), Q(1)), (P(2), Q(2)), \cdots, (P(n), Q(n))$ are the vertices of a regular n-gon. Prove that at least one of P, Q has degree at least $n - 1$.

9. Let $F_1 = F_2 = 1$, $F_{n+2} = F_n + F_{n+1}$ and let f be a polynomial of degree 990 such that $f(k) = F_k$ for $k \in \{992, \dots, 1982\}$. Show that $f(1983) = F_{1983} - 1$.

10. Let f be a polynomial with integer coefficients, and let p be a prime such that $f(0) = 0$, $f(1) = 1$, and $f(k)$ is congruent to either 0 or 1 modulo p, for all positive integers k. Show that the degree of f is at least $p - 1$.

11. Prove that for any real number a we have the following identity:

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} (a-k)^n = n!$$

12. Let $a, b, c, d \in \mathbb{R}$ such that $|ax^3 + bx^2 + cx + d| \leq 1$ for all $x \in [-1, 1]$. Prove that

$$|a| + |b| + |c| + |d| \leq 7$$

13. Prove that for any real numbers $x_1, x_2 \dots, x_n \in [-1, 1]$ the following inequality is true:

$$\sum_{i=1}^{n} \frac{1}{\prod\limits_{\substack{1 \leq j \leq n \\ j \neq i}} |x_i - x_j|} \geq 2^{n-2}$$