Invariants and Monovariants

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1. Some people are in a building with several rooms. Each minute a person leaves a room and enters one with at least as many people. Show that eventually all the people are in one room.

2. Given n red points and n blue points in the plane, show that we can draw n nonintersecting line segments each of which has one red endpoint and one blue endpoint.

3. On an $n \times n$ board there are n^2 unit squares, $n-1$ of which are infected. Every second any square adjacent to at least two infected squares becomes infected itself. Show that at least one square will remain free of infection.

4. A checker is placed at every lattice point (x, y) with $y \leq 0$. A move consists of jumping one checker over an adjacent checker onto an empty lattice point, then removing the jumped checker. Prove that no checker can ever reach a point (x, y) with $y \geq 5$.

5. A set of n^2 checkers is arranged in an $n \times n$ square lattice. A move consists of jumping one checker over an adjacent checker onto an empty lattice point, then removing the jumped checker. Find all n such that a series of jumps can be made after which only one checker remains.

6. (IMO '98 SL) A solitaire game is played on an $m \times n$ rectangular board, using mn markers each of which are white on one side and black on the other. Initially, each square of the board contains a marker with its white side up, except for one corner, which contains a marker with its black side up. In each move, one may take away one marker with its black side up, but must then turn over all markers in adjacent squares. Determine all pairs (m, n) of positive integers such that all markers can be removed from the board.

7. (a) Let P be a polygon. A *flip-turn* is the following operation: take a line ℓ which passes through two non-adjacent vertices A and B of P such that the interior of P lies entirely on one side of ℓ, and reflect the section of P between A and B through the midpoint of AB, yielding a new polygon P'. Prove that any sequence of flip-turns starting with a given polygon P eventually terminates (because the polygon becomes convex).

   (b) (Erdős-Nagy Theorem) Same problem, but with *flips* instead of flip-turns; a flip is performed by instead reflecting across the line ℓ.

8. A deck contains n cards labeled 1, 2, …, n. Starting with an arbitrary ordering of the cards, repeat the following operation: if the top card is labeled k, reverse the order of the top k cards. Prove that eventually the top card will be labeled 1.

9. A deck of 50 cards contains two cards labeled n for each $n = 1, 2, \ldots, 25$. There are 25 people seated at a table, each holding two of the cards in this deck. Each minute every person passes the lower-numbered card of the two they are holding to the right. Prove that eventually someone has two cards of the same number.

10. (MOP '96) There are $n+1$ fixed positions in a row, labeled 0 to n in increasing order from right to left. Cards numbered 0 to n are shuffled and dealt, one in each position. The object of the game is to have the card i in the ith position for $0 \leq i \leq n$. If this has not been achieved, the following move is eqecuted. Determine the smallest k such that the kth position is occupied by a card $l > k$. Remove this card, slide all cards from the $(k+1)$st position to the lth position one place to the right, and place the card l in the lth position.

    (a) Prove that the game lasts at most $2^n - 1$ moves.

    (b) Prove that there exists a unique initial configuration for which the game lasts exactly $2^n - 1$ moves.

11. (IMO '00) Let $n \geq 2$ be a positive integer. Initially, there are n fleas on a horizontal line, not all at the same point. For a positive real number λ, define a *move* as follows:

   choose any two fleas, at points A and B, with A to the left of B; let the flea at A jump to the point C on the line to the right of B with $BC/AB = \lambda$.

   Determine all values of λ such that, for any point M on the line and any initial positions of the n fleas, there is a finite sequence of moves that will take all the fleas to positions to the right of M.

12. (IMO '86) To each vertex of a regular pentagon an integer is assigned, such that the sum of all five numbesr is positive. If three consecutive vertices are assigned the numbers x, y, z respectively, and $y < 0$, then the following operation is allowed: x, y, z are replaced by $x+y$, $-y$, $z+y$ respectively. Such an operation is performed repeatedly as long as at least one of the five numbers is negative. Determine whether this procedure necessarily comes to an end after a finite number of steps.

13. (IMO '01 SL) Let $a_1 = 11^{11}$, $a_2 = 12^{12}$, $a_3 = 13^{13}$, and

$$a_n = |a_{n-1} - a_{n-2}| + |a_{n-2} - a_{n-3}|, \qquad n \geq 4.$$

   Determine $a_{14^{14}}$.

14. (USAMO '03) A positive integer is written at each vertex of a hexagon. A move is to replace a number by the absolute value of the difference of the two adjacent numbers. If the original numbers sum to 2003^{2003}, prove that it is possible to make a series of moves after which all the numbers are zero.