Combinations and Permutations

In our daily life, we always count things. For example, when we are dressing, we may count the number of possible ways to choose a pair of trousers, a shirt and a jacket from the wardrobe for a proper match. Sometimes, the number we count is so huge that the counting job is not easy. However, some kind of mathematics may help us do it in a systematic way. This powerful tool is called *combinatorics*, a branch of mathematics that mainly involves counting things. In this set of notes, we shall mainly deal with combinations and permutations, which is the core of complicated counting.

1. Additive Rule and Multiplicative Rule

Here are the two most basic rules of combinatorics:

Theorem 1.1. (Additive Rule)

Suppose there are p and q ways to choose an object satisfying conditions P and Q respectively. If there is no object satisfying both P and Q, then the number of ways to choose an object satisfying P or Q is $p+q$.

Theorem 1.2. (Multiplicative Rule)

Suppose there are p and q ways to choose an object satisfying conditions P and Q respectively. The number of ways to choose 2 objects, where the first satisfies P and the second satisfies Q, is pq.

Example 1.1.

There are 18 boys and 20 girls in a class.

(a) In how many ways can we choose a representative from the class?

(b) In how many ways can we choose a monitor and a monitress from the class?

Solution.

Let P be the condition 'the student is a boy' and Q be the condition 'the student is a girl'.

(a) In this case we are choosing an object (a student) which satisfies either P or Q, but not both. Using the additive rule, we can see that there are $18+20=38$ ways to choose a representative from the class.

(b) This is simply equivalent to choosing an object (a student) satisfying P and then an object satisfying Q. Using the multiplicative rule, we see that there are $18 \times 20 = 360$ ways to do so.

Example 1.2.

How many 2-digit numbers have an odd tens digit while the unit digit is even or equal to 5?

Solution.

We can form a 2-digit integer by specifying its tens digit and unit digit. First of all, there are 5 ways to choose its tens digit (i.e. 1, 3, 5, 7 or 9). For the unit digit, we have $5+1=6$ choices by the additive rule. Finally, we see that there are $5 \times 6 = 30$ ways to choose the 2 digits by the multiplicative rule, and hence the answer is 30.

Actually, the additive rule and the multiplicative rule can be generalized to situations with more conditions as follows:

Theorem 1.3. (Generalized Additive Rule)

Suppose there are p_1, p_2, …, p_n ways to choose an object satisfying conditions P_1, P_2, …, P_n respectively. If there is no object satisfying more than one of P_1, P_2, …, P_n, then the number of ways to choose an object satisfying P_1, P_2, … or P_n is $p_1 + p_2 + \cdots + p_n$.

Theorem 1.4. (Generalized Multiplicative Rule)

Suppose there are p_1, p_2, …, p_n ways to choose an object satisfying conditions P_1, P_2, …, P_n respectively. The number of ways to choose n objects such that the i-th object satisfies P_i $(1 \le i \le n)$ is $p_1 \times p_2 \times \cdots \times p_n$.

Example 1.3.

The numbers of students in different classes of a school are shown in Figure 1 below.

Class	A	B	C	D
Boys	20	15	12	14
Girls	15	22	25	26

Figure 1

In how many ways can we choose

(a) one representative from the school?

(b) one male and one female representative from each class?

(c) one representative from each class?

Solution.

(a) By the generalized additive rule, there are $20+15+15+22+12+25+14+26=149$ ways to choose the representative.

(b) By the generalized multiplicative rule, there are

$$20\times15\times15\times22\times12\times25\times14\times26=10810800000$$

ways to choose the representatives.

(c) By the additive rule, there are $(20+15)$, $(15+22)$, $(12+25)$ and $(14+26)$ ways to choose a representative from Class A, B, C and D respectively. Using the generalized multiplicative rule, there are $(20+15)\times(15+22)\times(12+25)\times(14+26)=1916600$ ways to choose the representatives.

2. The P_r^n and C_r^n Notations

In the previous section, we have been counting the number of ways to choose only *one* object from each category. How about choosing *2 or more* objects from the same category? In this case, the situation is more complicated. We need some more considerations.

2.1 Allowing Duplicated Choices

Consider the following situation. There are 8 sons and 9 daughters in a big family. In each day of the week, one of them is responsible for cooking. In how many different ways can we assign the duties in a week?

Here we are encountering something different from the cases in Section 1. We have to choose more than one boy and/or more than one girl, and we can choose the same object (child) more than once (or even seven times). We say the children are chosen <u>with replacement</u>. Besides, we have to consider <u>whether the order of choice is important</u>.

What is meant by 'whether the order of choice is important'? When we choose some objects, if different orders of choice lead to different choices, we say that <u>the order of choice is important</u>. Otherwise, the order is not important. For example, in this question, different situations arise when the order of choosing the seven children is different (see Figure 2). Therefore, the order of choice is important here.

Day	Child on duty
Sunday	Amy
Monday	***Ben***
Tuesday	***Carl***
Wednesday	Dora
Thursday	Eric
Friday	Eric
Saturday	Eric

Day	Child on duty
Sunday	Amy
Monday	***Carl***
Tuesday	***Ben***
Wednesday	Dora
Thursday	Eric
Friday	Eric
Saturday	Eric

Figure 2: The situation is different when the order of choice is different.

How can we compute the number of ways to choose the children? Let there be p_0, p_1, …, p_6 choices for the child responsible for the duties on Sunday, Monday, …, Saturday respectively. Obviously, $p_1 = p_2 = \cdots = p_7 = 17$. In this sense, using the generalized multiplicative rule, there are 17^7 ways to choose the children on duty. This rule can be summarized as follows:

Theorem 2.1.1.

When we choose r objects from n distinct objects <u>with replacement</u> and <u>the order of choice is important</u>, there are altogether n^r ways.

Example 2.1.1.

Mr. Chan is going to visit 5 different tour spots in his 10-day journey. For each spot he will spend at most one day, and he may visit more than one spoton one day. How many possible schedules can Mr. Chan arrange?

Solution.

Mr. Chan has to choose 5 out of the 10 days (with replacement, as he may visit more than one spot on one day) and the order of choice is important (as swapping the days for two spots possibly leads to a different schedule). By Theorem 2.1.1, the answer is $10^5 = 100000$.

2.2 No Duplicated Choices

Let us return to the family in Section 2.1 again. The family finds that it is unfair to let a child work for more than 1 day while some of them need not work. Therefore, they decide that a child may work for at most one day in a week. That means if a child is chosen to be on duty on some day, he/she must not be chosen for another day. We say that the children are chosen <u>without replacement</u>. In this case, in how many ways can the family schedule the duties in a week?

We shall look at the number of choices of children for each day. On Sunday, there are, of course, 17 choices. After choosing the child to cook on Sunday, no matter which child is chosen, there will only be 16 children available on Monday as duplicated choices are not allowed. On Tuesday, only 15 choices remain as two children have already been working hard on Sunday and Monday. Similarly, there are 14, 13, 12 and 11 choices for Wednesday, Thursday, Friday and Saturday respectively. By the generalized multiplicative rule, there are altogether $17\times16\times15\times14\times13\times12\times11 = 98017920$ ways to schedule the duties.

Based on this example, we have the following theorem.

Theorem 2.2.1.

When we choose r objects from n distinct objects <u>without replacement</u> and <u>the order of choice is important</u>, the number of choices is

$$n(n-1)(n-2)\cdots(n-r+1).$$

This expression can be abbreviated as P_r^n.

- Sometimes P_r^n is also written as ${}_nP_r$, ${}^nP_r$ or $P(n,r)$.
- The letter 'P' in the notation 'P_r^n' stands for 'permutation'.
- By writing $1\times2\times3\times\cdots\times k$ as $k!$ (read as 'k factorial'), we may write

  $$P_r^n = \frac{n!}{(n-r)!}$$

  as

  $$n(n-1)\cdots(n-r+1) = \frac{n(n-1)\cdots(n-r+1)(n-r)(n-r-1)\cdots(2)(1)}{(n-r)(n-r-1)\cdots(2)(1)} = \frac{n!}{(n-r)!}.$$

  Note, however, that the original expression is easier to compute in general.

Example 2.2.1.

In the horse racing game in Hong Kong, there is a kind of betting called *Tierce*. People have to bet which 3 horses come first, second and third, including their order. Assuming there are 14 horses in a race, how many different possible bets are available?

Solution.

The problem is simply choosing 3 horses from 14 horses without replacement and the order of choice is important. Hence there are $P_3^{14} = 14 \times 13 \times 12 = 2184$ different possible bets, i.e. the answer is 2184.

Example 2.2.2.

In Example 2.1.1, if Mr. Chan decides not to visit more than one tour spot on each day, how many possible schedules can he arrange?

Solution.

In this case, Mr. Chan should choose 5 *different* days among the 10 days (it means the days are chosen without replacement), and the order of choice is important (as swapping 2 chosen days leads to 2 different schedules). Therefore, there are $P_5^{10} = 10 \times 9 \times 8 \times 7 \times 6 = 30240$ possible schedules.

Example 2.2.3.

5 children are standing in a row. How many different orders of their positions are possible?

Solution.

There are 5 different positions for the 5 children to stand. Obviously, they cannot stand in the same position (i.e. the positions are chosen without replacement) and the order is important. Therefore, the children can stand in $P_5^5 = 5 \times 4 \times 3 \times 2 \times 1 = 5! = 120$ different ways.

The result of the previous example can be summarized as a new theorem.

Theorem 2.2.2.

There are $n!$ ways to permute the orders of n objects.

2.3 What if the order of choice is unimportant?

Let us return to the family once again. The family further thinks that it is too strict to assign the days of duties for the children. Therefore, as a better method to assign the duties, they only choose 7 children for the duties every week, so that the children can arrange the duties among themselves. How many different choices can be made under this method?

It is somehow different from the case in Section 2.2. Two choices are now regarded to be the same even if the order of choice is different, as illustrated in Figure 3.

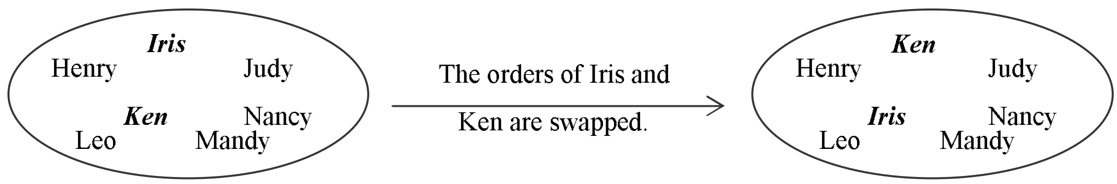

Figure 3: The choice remains the same even if the order of choice is different.

Suppose there are x different ways to choose the 7 children in the new method. We now pick up one of the possible ways, assuming Henry, Iris, Judy, Ken, Leo, Mandy and Nancy are chosen. They need to plan their days of duties among themselves. By Theorem 2.2.2, there are 7! ways to do so. Since there are x possible ways to choose the 7 children (without specifying the days of the duties, i.e. the order of choice is unimportant), there are ($x \times 7!$) ways to choose the children with the duties of each day assigned.

On the other hand, we know from Theorem 2.2.1 that the latter number should be equal to P_7^{17}. Hence we have $P_7^{17} = x \times 7!$ and so

$$x = P_7^{17} \div 7! = \frac{17\times16\times15\times14\times13\times12\times11}{7\times6\times5\times4\times3\times2\times1} = \left(\frac{17!}{10!}\right) \div 7! = \frac{17!}{7!10!}.$$

This can be generalized to the following theorem.

Theorem 2.3.1.

When we choose r objects from n distinct objects <u>without replacement</u> and <u>the order of choice is unimportant</u>, the number of choices is

$$\frac{n(n-1)(n-2)\cdots(n-r+1)}{r!} = \frac{n!}{(n-r)!r!}.$$

This expression can be abbreviated as C_r^n.

- Sometimes C_r^n is also written as ${}_nC_r$, ${}^nC_r$ $C(n,r)$ or $\begin{pmatrix} n \\ r \end{pmatrix}$.
- In mainland China, it is a common practice to write C_r^n as C_n^r. This should not cause any confusion at this stage as we always have $n \geq r$. However, it is no longer true in more advanced cases, where we may talk about C_r^n for $r > n$.
- The letter 'C' in the notation 'C_r^n' stands for 'combination'.
- In general, the expression $\dfrac{n(n-1)(n-2)\cdots(n-r+1)}{r!}$ is easier to compute than $\dfrac{n!}{(n-r)!r!}$.

Example 2.3.1.

There are 20 members in an organization. They now choose 3 people as their representatives. How many different choices are there?

Solution.

They are now choosing 3 objects among 20 objects without replacement and the order of choice is unimportant. Hence the number of different choice is

$$C_3^{20} = \frac{20 \times 19 \times 18}{3 \times 2 \times 1} = 1140.$$

Example 2.3.2.

There is another kind of betting called *Trio* in the horse racing game in Hong Kong. People have to bet which 3 horses come first, without guessing their order. Assuming there are 14 horses in a race, how many different possible bets are available?

Solution.

The problem is simply choosing 3 horses from 14 horses without replacement and the order of choice is unimportant. Hence there are $C_3^{14} = \dfrac{14 \times 13 \times 12}{3!} = 364$ different possible bets, i.e. the answer is 364. (Compare this with Example 2.2.1.)

2.4 More complicated cases

In many cases, the counting problems we encounter in daily life are not of one of the types we have mentioned, but rather of a combination of them. We will look at some further examples in this section.

Example 2.4.1.

20 different balls are to be packed into 5 different boxes so that each box contains 4 balls. Find the number of ways of packing balls.

Solution.

For convenience, we name the boxes as A, B, C, D and E. First of all, we can choose 4 balls into A. By Theorem 2.3.1, there are C_4^{20} ways to do so. Next we consider B, and there are C_4^{16} ways to choose 4 of the remaining balls to put in. Similarly, there are C_4^{12} and C_4^{8} ways for C and D. The remaining 4 balls must be for E. By the generalized multiplicative rule, the total number of combinations is $C_4^{20} \times C_4^{16} \times C_4^{12} \times C_4^{8}$.

Example 2.4.2.

Consider the family in Section 2.1 again. They now want to choose 4 boys and 3 girls to be responsible for the cooking in a week. In how many ways can the children on duty be chosen if

(a) the days of duties are not specified?

(b) the days of duties are specified?

Solution.

(a) We are simultaneously choosing 4 boys out of 8 and 3 girls out of 9 without replacement, where the order of choice is unimportant. Therefore, there are $C_4^8 \times C_3^9 = 5880$ ways to do so.

(b) We first fix 4 days which are assigned for boys, say from Sunday to Wednesday. Then 4 boys are chosen to work on these 4 days. Note that the choices are without replacement and, of course, the order of choice is important. A similar case occurs for girls. Therefore, there are $P_4^8 \times P_3^9$ ways to choose the children in this particular choice of 4 days. Next, we count the number of ways in which the 4 days for boys can be chosen. We have to choose 4 out of 7 days without replacement and the order of choice is unimportant. Hence there are C_4^7 ways to do so. By the generalized multiplicative rule, there are altogether $P_4^8 \times P_3^9 \times C_4^7 = 26935200$ different ways to choose the children.

Remark. The answer 26935200 in (b) turns out to be 7! times the answer in (a). In the light of this, can you work out an alternative solution to (b)?

Example 2.4.3.

'Mark Six' is a lottery in Hong Kong. A person has to choose 6 distinct integers from 1 to 49 to be put on a ticket. The lottery committee then draws 6 winning numbers and another 'extra number' randomly from 1 to 49. Here is the table on how the tickets win:

A ticket contains		Prize
Winning numbers	Extra number	
6	0	First Prize
5	1	Second Prize
5	0	Third Prize
4	1	Fourth Prize
4	0	Fifth Prize
3	1	Sixth Prize
3	0	Seventh Prize

(a) How many different tickets can win the third prize?

(b) How many different tickets can win the sixth prize?

Solution.

(a) To win the third prize, there must be 5 winning numbers in the ticket. Therefore, there are $C_5^6 = 6$ such combinations of winning numbers. For the last number, it must not be a winning number nor the extra number (for otherwise the ticket would win a first or second prize). Therefore, there are $49 - 7 = 42$ choices. Hence the answer is $6 \times 42 = 252$.

(b) To win the sixth prize, the extra number must be included in the ticket. Of the remaining 5 numbers on the ticket, there must be exactly 3 winning numbers and exactly 2 non-winning numbers. As there are C_3^6 ways to choose 3 winning numbers and C_2^{42} ways to choose 2 non-winning numbers, the answer is $C_3^6 \times C_2^{42} = 17220$.

3. The H_r^n Notation

Note that in Sections 2.1 to 2.3, we have been considering the following counting problems:

Section	Counting problems considered
2.1	Allowing duplicated choices, order of choice important
2.2	No duplicated choices, order of choice important
2.3	No duplicated choices, order of choice unimportant

Naturally, we ask what happens when the order of choice is unimportant and duplicated choices are allowed.

Consider the family again. Suppose the father has got 5 cash coupons at a bookstore, which he plans to give to some of the 8 boys as gift. The father decides that a boy may get more than one coupon (i.e. duplicated choices of the boys are allowed). Certainly, the order by which the boys are chosen is unimportant, as the 5 coupons are identical. Now, deciding how to distribute the 5 coupons is equivalent to deciding how many coupons each boy will get, in a way such that the number of coupons to be given to each of the 8 boys must be a non-negative integer, and that the sum of all such integers must be 5. This leads us to the following definition.

Definition 3.1.
Consider an equation $a_1 + a_2 + a_3 + \cdots + a_n = r$ where n and r are positive integers. The number of non-negative integer solutions to the equation is denoted by H_r^n.

- In P_r^n and C_r^n, $n \ge r$ is needed. However, it is not required in H_r^n.

Therefore, the answer to the above problem is denoted by H_5^8. Dividing the coupons among the 8 boys is equivalent to solving the equation $a_1 + a_2 + \cdots + a_8 = 5$ in non-negative integers.

How to compute H_5^8? Well, suppose 5 balls (representing the 5 coupons) are lined up in a straight line. To distribute the 5 balls among the 8 sons, we insert 7 vertical slits. The following is one example:

O O | | | O | O | O | |

In the above example, the 7 vertical slits separate the space into 8 portions, representing the 8 sons, and the number of balls in each represents the number of coupons the corresponding son gets. Hence, in this example, the first son gets 2 coupons, while each of the fourth, fifth and sixth sons gets 1 coupon. This also corresponds to the non-negative integer solution (2, 0, 0, 1, 1, 1, 0, 0) to the equation $a_1 + a_2 + \cdots + a_8 = 5$.

Some more examples of the correspondence between placing the vertical slits and the corresponding solution to the equation $a_1 + a_2 + \cdots + a_8 = 5$ are shown below:

Way of placing slits	Corresponding solution
| | | | O O O | | O O |	(0, 0, 0, 0, 3, 0, 2, 0)
| | | | | | | O O O O O	(0, 0, 0, 0, 0, 0, 0, 5)
| O | O | O | O | O | |	(0, 1, 1, 1, 1, 1, 0, 0)

In this way we see that there is a one-to-one correspondence between ways of placing 7 slits on 5 balls and non-negative integer solutions to the equation $a_1 + a_2 + \cdots + a_8 = 5$. The former can be considered as ways of lining up 12 objects (7 identical slits and 5 identical balls). Each way can be specified by 5 positions out of 12 for the balls, where duplication of positions is not allowed and the order in which the positions are chosen are unimportant. Clearly, there are $C_5^{12} = 792$ ways to do so. It follows that the value of H_5^8 is also equal to 792, i.e. there are 792 non-negative integer solutions to the equation $a_1 + a_2 + \cdots + a_8 = 5$, which also means that there are 792 ways for the father to distribute the 5 coupons among his 8 sons.

The above example thus shows that $H_5^8 = C_5^{12}$. Certainly, this can be generalized. Clearly, we have equality of the lower indices (5 in this example) in general. For the upper index, the number 12 in the above example is the sum of 5 (number of balls) and 7 (number of slits), which in turn is the sum of 5 (number of coupons) and 8 (number of sons) less 1 (as 7 slits can already separate the space into 8 portions). Hence the result of this example can be generalized to the following theorem.

Theorem 3.1.

When we choose *r* objects from *n* distinct objects <u>with replacement</u> and <u>the order of choice is unimportant</u>, the number of choices is

$$H_r^n = C_r^{r+n-1}.$$

Example 3.1.

How many non-negative integer solutions are there to the equation $a_1 + a_2 + a_3 = 20$?

Solution.

By Theorem 3.1 and the definition of H_r^n, the answer is $H_{20}^3 = C_{20}^{3+20-1} = 231$.

Example 3.2.

How many positive integer solutions are there to the equation $a_1 + a_2 + a_3 + a_4 = 15$?

Solution.

This example is a variation of Example 3.1. If we let $b_i = a_i - 1$ for $1 \le i \le 4$, then each b_i is a non-negative integer (as each a_i is a positive integer), so we are back to what we already know. After this substitution and some simplification, the original equation becomes

$$b_1 + b_2 + b_3 + b_4 = 11.$$

Obviously, each non-negative integer solution to this new equation corresponds to a positive integer solution to the original equation. For example, $(b_1, b_2, b_3, b_4) = (2,3,6,0)$ corresponds to $(a_1, a_2, a_3, a_4) = (2,3,7,1)$. In view of Theorem 3.1, the number of non-negative integer solution to this new equation, which is also the number of positive integer solutions to the original equation $a_1 + a_2 + a_3 + a_4 = 15$, is equal to $H_{11}^4 = C_{11}^{14} = 364$.

Example 3.3.

If the following constraints are given: $a_1 \geq 3$, $a_2 \geq 1$, $a_3 \geq -2$ and $a_4 \geq 5$, how many integer solutions are there to the equation $a_1 + a_2 + a_3 + a_4 = 10$?

Solution.

Once again, we try to bring the problem back to what we already know. If we let $b_1 = a_1 - 3$, $b_2 = a_2 - 1$, $b_3 = a_3 + 2$ and $b_4 = a_4 - 5$, then we get $b_1 + b_2 + b_3 + b_4 = 3$, where each b_i is a non-negative integer. Hence the answer is $H_3^4 = C_3^6 = 20$.

Example 3.4.

Suppose there are 10 red marbles, 10 yellow marbles, 10 green marbles and 10 blue marbles. If 5 marbles are chosen,

(a) how many different colour combinations are possible?

(b) how many colour combinations with more red marbles than yellow marbles are there?

Solution.

(a) Note that the number 10 is not important as long as there are 'enough' marbles of each colour. Hence, as far as colour combination is concerned, we may assume that there is only one marble of each colour, and we are to choose 5 marbles where duplication is allowed and the order is unimportant (as different orders of choice do not lead to different colour combinations). By Theorem 3.1, the answer is $H_5^4 = C_5^{4+5-1} = 56$.

(b) We first count the number of colour combinations in which there is an equal number (say x) of red and yellow marbles. Clearly, there are 2 such combinations if $x = 2$ (corresponding to adding 1 more green marbles or adding 1 more blue marble), 4 such combinations if $x = 1$ (in this case the numbers of green and blue marbles may be (0, 3), (1, 2), (2, 1) or (3, 0)) and similarly 6 such combinations if $x = 0$. It follows that there are $2 + 4 + 6 = 12$ combinations in which there is an equal number of red and yellow marbles, and hence $56 - 12 = 44$ combinations in which the numbers of red and yellow marbles are not the same.

Of these 44 colour combinations, there should be as many 'more red than yellow' cases as 'more yellow than red' cases, owing to the symmetry of the roles of different colours. It follows that the answer is $44 \div 2 = 22$.

4. A Summary

The results of Section 2 and Section 3 can be summarized by the following table, which shows the number of ways of choosing r objects from n distinct objects under different situations:

	Order of choice important	Order of choice unimportant
Duplications allowed (Choosing with replacement)	n^r	$H_r^n = C_r^{n+r-1}$
Duplications not allowed (Choosing without replacement)	$P_r^n = \dfrac{n!}{(n-r)!}$	$C_r^n = \dfrac{n!}{r!(n-r)!}$

5. Exercise

1. A dinner set consists of a soup, a main dish and a drink. If 6 soups, 12 main dishes and 5 drinks are available, how many different choices of the dinner set are there?

2. This is a pool called 'Six Up' in the horse racing game in Hong Kong, in which one bets on a horse in each of the 6 given races. The biggest prize 'Six Win Bonus' in this pool can be won if someone guesses the winning horses in all 6 races correctly. Suppose there are 12, 13, 14, 13, 14, 14 horses in the races respectively. How many different bets should be placed so that one can be sure to win the 'Six Win Bonus'?

3. Prior to the introduction of 'Mark Six', an old lottery was played in a similar fashion. There were 14 numbers, and 6 numbers were drawn each time. One has to bet on the 6 numbers drawn, including their order. How many different possible bets were there?

4. In the current Mark Six lottery in Hong Kong,
   (a) how many different bets of 6 numbers are available?
   (b) how many different tickets can win the fourth prize?
   (c) how many different tickets can win the fifth prize?
   (d) what is the probability that a ticket wins the seventh prize?

5. (a) How many non-negative integer solutions are there to the equation $a_1 + a_2 + a_3 = 16$?

   (b) How many positive integer solutions are there to the equation $a_1 + a_2 + a_3 = 16$?

6. 30 pencils are to be distributed to 5 students. If each student must get at least 2 pencils, how many different ways are there to distribute the pencils?

7. Follow the steps below to find the number of non-negative integer solutions to the inequality

$$a_1 + a_2 + a_3 \le 20 .$$

   Step 1: What properties does the value $20-(a_1 + a_2 + a_3)$ have?

   Step 2: Let $a_4 = 20-(a_1 + a_2 + a_3)$. A new equation $a_1 + a_2 + a_3 + a_4 = 20$ is created. What constraints does this equation have?

   Step 3: How many non-negative integer solutions are there to the equation in Step 2? How is it related to the solutions to the inequality?

   Step 4: Find the answer using Theorem 3.1.

8. Generalize the method in Question 7 to the inequality $a_1 + a_2 + \cdots + a_n \le r$.