ARML 2011 – Advanced Algebra

2/3/11

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This lecture will attempt to expose you to some of the more advanced areas of algebra that may appear on math contests such as ARML or the AMC. First, we will treat symmetric polynomials and the fundamental Newton Identities. Then we will take a seemingly abrupt change of course by discussing the Cauchy-Schwarz Inequality. Understand that this lecture is driven primarily by the problems at the end, so you should be sure to solve them all on your own time. Each problem has appeared at least once – and usually multiple times – in math competitions, so you would do well to solve the problems and understand the algebraic techniques involved in their solution.

1 Symmetric Polynomials

Symmetric polynomials are polynomials such that $f(x, y) = f(y, x)$ for all x, y. The building blocks of the symmetric polynomials are the elementary symmetric polynomials

$$\begin{aligned} \sigma_1 &= x + y \\ \sigma_2 &= xy \end{aligned}$$

Likewise, for three variables, the elementary symmetric polynomials are

$$\begin{aligned} \sigma_1 &= x + y + z \\ \sigma_2 &= xy + yz + zx \\ \sigma_3 &= xyz \end{aligned}$$

All power sums $x_1^k + x_2^k + \cdots + x_n^k$ can be expressed in terms of the elementary symmetric polynomials[1]. We can do this in the two-variable case by constructing a recurrence. Letting $T_n(x, y) = x^n + y^n$, we find that

$$\begin{aligned} T_n &= x^n + y^n \\ &= (x + y)(x^{n-1} + y^{n-1}) - xy(x^{n-2} + y^{n-2}) \\ &= \sigma_1 T_{n-1} - \sigma_2 T_{n-2} \end{aligned}$$

Via a nearly identical argument, we find the recurrence for $T_n(x, y, z)$ to be

$$T_n = \sigma_1 T_{n-1} - \sigma_2 T_{n-2} + \sigma_3 T_{n-3}.$$

This is suggestive of a general rule – even more suggestive when we notice that this works for the one-variable case $T_n = \sigma_1 T_{n-1}$. Indeed, the "general rule" is one of the celebrated Newton Identities and finds applications in combinatorics, Galois theory, and even general relativity.

2 Cauchy-Schwarz

Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$ be sequences of real numbers. Then the Cauchy-Schwarz inequality states that

$$(a_1 b_1 + \cdots + a_n b_n)^2 \leq \left(a_1^2 + \cdots + a_n^2\right)\left(b_1^2 + \cdots + b_n^2\right)$$

[1] In fact, we can do even better and claim the *Fundamental Theorem of Symmetric Polynomials:* every symmetric polynomial can be represented as a polynomial in the elementary symmetric polynomials. However, this is unnecessary for the purposes of this lecture.

If we write the sequences as n-dimensional vectors $\mathbf{a} = \langle a_1, \ldots, a_n \rangle$ and $\mathbf{b} = \langle b_1, \ldots, b_n \rangle$, we get

$$(\mathbf{a} \cdot \mathbf{b})^2 \leq |\mathbf{a}|^2 \, |\mathbf{b}|^2$$

Equality holds if and only if $a_1/b_1 = a_2/b_2 = \cdots = a_n/b_n$, which is the same thing as saying that $\mathbf{a}$ and $\mathbf{b}$ are linearly dependent. For a quick proof, consider the discriminant of the quadratic polynomial

$$\begin{aligned} P(x) &= (a_1 x + b_1)^2 + (a_2 x + b_2)^2 + \cdots + (a_n x + b_n)^2 \\ &= \left(\sum_{k=1}^{n} a_k^2 \right) x^2 + \left(2 \sum_{k=1}^{n} a_k b_k \right) x + \left(\sum_{k=1}^{n} b_k^2 \right) \end{aligned}$$

Observe that $P(x)$ is always nonnegative, so it has as most one real root. Thus, its discriminant is non-positive and the Cauchy-Schwarz inequality follows.

$$\begin{aligned} B^2 - 4AC &\leq 0 \\ \left(2 \sum_{k=1}^{n} a_k b_k \right)^2 - 4 \left(\sum_{k=1}^{n} a_k^2 \right) \left(\sum_{k=1}^{n} b_k^2 \right) &\leq 0 \\ \Leftrightarrow \left(\sum_{k=1}^{n} a_k b_k \right)^2 &\leq \left(\sum_{k=1}^{n} a_k^2 \right) \left(\sum_{k=1}^{n} b_k^2 \right) \end{aligned}$$

3 Identities

This section is included for reference. Note that when a Vieta bash exists, there is often a more elegant and less error-prone solutions using some basic calculus techniques.

The Canonical Identity

$$\begin{aligned} x^3 + y^3 + z^3 - 3xyz &= (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx) \\ &= \frac{1}{2} (x + y + z) \left((x - y)^2 + (y - z)^2 + (z - x)^2 \right) \geq 0 \end{aligned}$$

Newton's Identities

$$T_m(x_1, x_2, \ldots, x_n) = \sum_{k=1}^{n} (-1)^{k+1} \sigma_k T_{m-k}$$

Formal Derivative of a Polynomial: Let $P(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$. Then the formal derivative of the polynomial $P'(x)$ is defined to be

$$P'(x) = n a_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + \cdots + a_1$$

The Log Trick: Let $x_1, x_2, \ldots, x_n$ be the zeros of the polynomial $P(x)$. Then

$$\frac{d}{dx} \left(\log P(x) \right) = \frac{P'(x)}{P(x)} = \frac{1}{x - x_1} + \frac{1}{x - x_2} + \frac{1}{x - x_3} + \cdots + \frac{1}{x - x_n}$$

Sophie Germain Identity

$$a^4 + 4b^4 = \left(a^2 - 2ab + 2b^2 \right) \left(a^2 + 2ab + 2b^2 \right)$$

Brahmagupta's Identity

$$\begin{aligned}\left(a^2+b^2\right)\left(c^2+d^2\right) &= (ac-bd)^2+(ad+bc)^2 \\ &= (ac+bd)^2+(ad-bc)^2\end{aligned}$$

Lagrange Identity

$$\sum_{k=1}^{n} a_k^2 \sum_{k=1}^{n} b_k^2 - \left(\sum_{k=1}^{n} a_k b_k\right)^2 = \sum_{i<k} (a_i b_k - a_k b_i)^2$$

Crazy Complex Arctan: Let $x_1, x_2, \ldots, x_n$ be the zeros of the polynomial $P(x)$. Then

$$\arctan x_1 + \arctan x_2 + \cdots + \arctan x_n = \operatorname{Im}\left[\ln\left(iP(i)\right)\right]$$

as long as the sum on the left hand side of the equation is no larger than 2π.

Hermite Identity

$$\lfloor x \rfloor + \left\lfloor x + \frac{1}{n} \right\rfloor + \left\lfloor x + \frac{2}{n} \right\rfloor + \cdots + \left\lfloor x + \frac{n-1}{n} \right\rfloor = \lfloor nx \rfloor$$

4 Problems

1. Solve the system

$$\begin{aligned} x^5 + y^5 &= 33 \\ x + y &= 3 \end{aligned}$$

2. (a) Find the maximum value of the function $5\sin x + 7\cos x$ over the interval $\left(0, \frac{\pi}{2}\right)$.

   (b) What about $a\sin x + b\cos x$ for any $a, b > 0$?

3. Find the maximum possible value of $(x+y+z)^2$ if $x^2+y^2+z^2 = 2011$.

4. If $x, y, z > 0$, find the minimum possible value of $(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$.

5. Find all solutions to the equation

$$\sqrt[4]{97-x} + \sqrt[4]{x} = 5$$

6. Find the sum of all real numbers x such that

$$\sqrt[3]{x-1} + \sqrt[3]{x} + \sqrt[3]{x+1} = 0$$

7. Compute the sum $\displaystyle\sum_{k=1}^{n} \frac{k^2 - 1/2}{k^4 + 1/4}$.

8. Find a polynomial with integer coefficients that has the zero $\sqrt{2} + \sqrt[3]{3}$.

9. There exists a polynomial $P(x)$ such that

$$(x+1)P(x) = (x-10)P(x+1)$$

   Find the sum of the roots of $P(x)$.

10. Given that the polynomial

$$p(x) = x^5 + yx^3 + \frac{1}{25}$$

has a double root, determine the value of y.

11. (Classic) Let $P(x)$ be a polynomial of degree n. Knowing that

$$P(k) = \frac{k}{k+1}$$

for $k = 0, 1, 2, \ldots, n$, find $P(m)$ for $m > n$.

12. Let r be a positive real number such that

$$\sqrt[6]{r} + \frac{1}{\sqrt[6]{r}} = 6$$

Find the maximum possible value of $\sqrt[4]{r} - \frac{1}{\sqrt[4]{r}}$.

13. Find the maximum value of the function $f(x, y, z) = 5x - 6y + 7z$ on the ellipsoid $2x^2 + 3y^2 + 4z^2 = 1$ without resorting to calculus.

14. Give a method for constructing a segment of length $\sqrt[4]{a^4 + b^4}$ with a straightedge.

15. (ARML '87) If $x, y, z > 0$ and $x + y + z = 6$, determine the minimum possible value of

$$\left(x + \frac{1}{y}\right)^2 + \left(y + \frac{1}{z}\right)^2 + \left(z + \frac{1}{x}\right)^2.$$

16. Solve for real numbers x, y in the system of equations

$$(3x + y)(x + 3y)\sqrt{xy} = 14$$
$$(x + y)(x^2 + 14xy + y^2) = 36$$

17. Prove all unproven assertions in this lecture.