Bijective Proofs

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Bijections!

Many combinatorial problems can be solved by finding the right bijection. A map between two sets A and B is a *bijection* if it is both *injective* and *surjective*, that is, it is one-to-one and maps onto all of B. Equivalently, a bijection is a map that has a well-defined inverse.

Most bijective (a.k.a. *counting in two ways*) proofs use the following principle:

> *If there is a bijection between finite sets A and B, then A and B have the same number of elements.*

So, if we wish to find $|A|$ (the number of elements in A) and there is a bijection from A to a set B whose elements are easy to count, then we know how to count the elements of A. We can also prove two integers are equal by showing they count sets that have a bijection between them.

Example. Prove that

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

To prove this, we can easily use a straightforward induction argument, but it is more exciting to find a bijection between sets that each side counts. Consider a class with $2n$ students, n of whom are boys and n of whom are girls.

There is a natural bijection between

- pairs (G, B) of subsets G of the girls and B of the boys with $|G| + |B| = n$, and
- subsets of the set of all students of size n,

defined by $(G, B) \mapsto G \cup B$. A simple counting argument shows that the number of possible pairs (G, B) is $\sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} = \sum_{k=0}^{n} \binom{n}{k}^2$, and the number of subsets of the class of size n is $\binom{2n}{n}$. So, these quantities must be equal!

Problems

1. *Find the bijection!* For each of the following pairs of mathematical objects, give a description of a bijection that maps one set of objects to the other.

   (a) Binary sequences of length n $\leftrightarrow$ Subsets of $\{1, 2, \ldots, n\}$

(b) Lattice paths from $(0,0)$ to (m,n) that only travel right or up at each step $\leftrightarrow$ Choices of n blocks from a pile of m blue and n red blocks

(c) Tilings of a $2 \times n$ grid with dominoes $\leftrightarrow$ Sequences of $n-1$ white or black dots such that no two black dots are adjacent

(d) Partitions[1] of n into distinct parts $\leftrightarrow$ Partitions of n into odd parts

(e) Partitions of n into distinct odd parts $\leftrightarrow$ Partitions of n whose Young Diagram[2] is symmetric about the diagonal

(f) Increasing binary trees with nodes labeled $1, 2, \ldots, n \leftrightarrow$ Permutations of $1, 2, \ldots, n$.

2. Give a bijective proof of each of the following identities. All unspecified variables are assumed to be positive integers.

- $\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$
- $\sum_{k=0}^{r} \binom{n}{k}\binom{m}{r-k} = \binom{m+n}{r}$
- $\sum_{i=0}^{n} \binom{x+i}{i} = \binom{x+n+1}{n}$
- $\sum_{k=0}^{n} \binom{n}{k} s^k t^{n-k} = (s+t)^n$

3. Let $w = a_1 a_2 \cdots a_n$ be a permutation of $1, 2, \ldots, n$. We say that i is a *fixed point* of w if $a_i = i$. Show that the total number of fixed points of all possible permutations w is $n!$.

4. Show that, for positive integers $n > 2$, the number of integers $x \in \{0, 1, \ldots, n-1\}$ for which $x^2 \equiv 1 \pmod{n}$ is even.

5. How many $m \times n$ matrices of 0's and 1's have the property that every row and column contains an odd number of 1's?

6. (AIME 1983.) For $\{1, 2, \ldots, n\}$ and each of its nonempty subsets a unique *alternating sum* is defined as follows: Arrange the numbers in the subset in decreasing order and then, beginning with the largest, alternately add and subtract successive numbers. (For example, the alternating sum for $\{1, 2, 4, 6, 9\}$ is $9 - 6 + 4 - 2 + 1 = 6$.) Find the sum of all alternating sums of the nonempty subsets of $\{1, 2, \ldots, n\}$.

7. Prove Fermat's Little Theorem using a combinatorial argument as follows. We wish to show that if p is prime and a is a positive integer, then $a^p - a$ is divisible by p. To do so, it suffices to find a set S with $a^p - a$ elements and sort the elements of S into disjoint subsets having p elements each.

8. (Putnam 2002.) A nonempty subset $S \subseteq \{1, 2, \ldots, n\}$ is *decent* if the average of its elements is an integer. Prove that the number of decent subsets has the same parity as n.

9. (AIME 1998.) Find the number of ordered quadruples (x_1, x_2, x_3, x_4) of positive odd integers that satisfy $x_1 + x_2 + x_3 + x_4 = 98$.

[1] A *partition* of a positive integer is a way of writing it as a sum of other integers, called the *parts* of the partition, where we list the parts in nonincreasing order.

[2] The Young Diagram of a partition is a partial grid of squares, aligned at the left, where each row has a number of squares corresponding to the size of the parts in nonincreasing order.

10. (USAMO 1996.) An n-term sequence in which every term is either 0 or 1 is called a "binary sequence" of length n. Let a_n be the number of binary sequences of length n containing no three consecutive terms equal to 0, 1, 0 in that order. Let b_n be the number of binary sequences of length n containing no four consecutive terms equal to 0, 0, 1, 1 or 1, 1, 0, 0 in that order. Prove that $b_n + 1 = 2a_n$ for all positive integers n.

11. (China 1996.) Let n be a positive integer. Find the number of polynomials $P(x)$ with coefficients in $\{0, 1, 2, 3\}$ such that $P(2) = n$.

12. Find the number of strings of n letters, each equal to A, B, or C, such that the same letter never occurs three times in succession.

13. (Putnam 1996.) Given a finite string S of symbols X and O, we write $\Delta(S)$ for the number of X's in S minus the number of O's. For example, $\Delta(XOOXOOX) = -1$. We call a string S *balanced* if every substring T of (consecutive symbols of) S has $-2 \leq \Delta(T) \leq 2$. Thus, $XOOXOOX$ is not balanced, since it contains the substring $OOXOO$. Find, with proof, the number of balanced strings of length n.

14. **The Catalan numbers:** The Catalan numbers[3] $C_0, C_1, C_2, \ldots$ can be defined by the recurrence relation

$$C_{n+1} = C_nC_0 + C_{n-1}C_1 + C_{n-2}C_2 + \cdots + C_0C_n$$

along with the initial value $C_0 = 1$. The nth Catalan number C_n can also be defined as:

- The number of lattice paths from $(0, 0)$ to (n, n), formed by moving one unit right or one unit up at each step, that lie below or on the diagonal $x = y$
- The number of ways to fully parenthesize the addition $1 + 1 + \cdots + 1$ of $n + 1$ ones. For instance, $1 + 1 + 1 + 1$ can be parenthesized in five ways:

$$\begin{array}{l} ((1+1)+1)+1 \\ (1+(1+1))+1 \\ 1+((1+1)+1) \\ 1+(1+(1+1)) \\ (1+1)+(1+1) \end{array}$$

- The number of rooted binary trees having n leaves labeled $1, 2, \ldots, n$
- The number of ways of triangulating a regular $n + 2$-gon by drawing $n - 1$ diagonals (different triangulations that are congruent are considered distinct.)
- The number of ways of connecting $2n$ points on a circle with n nonintersecting chords

Show that each of these sets satisfies the Catalan recurrence. Can you find bijections between each of these pairs of sets?

15. (Hard.) Find a bijective proof that the nth Catalan number C_n is equal to $\frac{1}{n+1}\binom{2n}{n}$.

[3]See http://math.mit.edu/ rstan/ec/ for a list of 188 different combinatorial interpretations of the Catalan numbers.