Combinatorial Geometry

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What is combinatorial geometry? Basically any problem involving collections of geometric objects, whether they be lattice points, lines, vectors, or convex sets, can be described as combinatorial geometry. This is sort of a broad definition, so let's jump into a few examples.

The problems below are roughly organized according to subtopic.

1. (Pick) The area of a lattice polygon is $i + b/2 - 1$, where i is the number of interior lattice points and b is the number of boundary points. (Note: an analogous formula does not exist in higher dimensions. For instance, exhibit a family of lattice tetrahedra which contain no lattice points other than their vertices but whose volume is unbounded.)

2. In the plane, let those lattice points with both coordinates even be colored black, and let all other lattice points be colored white. Let P be a lattice polygon with black vertices. Show that any white point on or inside P is the midpoint of two black points, both of which lie on or inside P. What about the analogous question in higher dimensions?

3. (Minkowski) A convex region in the plane contains exactly one lattice point and is symmetric about the origin. Prove that its area is at most 4.

4. (TST 2004) Draw a 2004×2004 array of points. What is the largest integer n for which it is possible to draw a convex n-gon whose vertices are chosen from points of the array?

5. (Russia 1998) Prove that a convex lattice $2n$-gon has area at least $n^3/100$. (Note: this bound is not strict).

6. Show that by dissection and translation of the pieces, any polygon can be transformed into any other polygon of the same area.

7. Show that any triangle can be dissected into n isosceles triangles for any $n \geq 4$.

8. Find all triangles that can be dissected into five smaller triangles, each similar to the original.

9. (USAMO 2004) For what real values of k is it possible to dissect a $1 \times k$ rectangle into two similar, but noncongruent, polygons?

10. (Putnam 1997) Let the *diameter* of a region be the least upper bound of the distances between pairs of points in the region, and let the diameter of a dissection be the maximum diameter of any piece. Find the minimum possible diameter of a dissection of a 3–4–5 triangle into four pieces.

11. (Putnam 1994) Prove that the points of a right isosceles triangle with side length 1 cannot be colored in four colors such that any two points of the same color lie less than a distance of $2 - \sqrt{2}$ apart.

12. Prove that the points of the plane can be colored in 7 colors such that any two points at distance 1 are different colors. Also show that this cannot be done with 3 colors.

13. Let R consist of a regular hexagon of side length 1 and its interior. Find the minimum value of d such that it is possible to color the points of R in three colors such that any two points of the same color lie less than a distance of d apart.

14. Let a sphere be covered by finitely many open hemispheres. Show that there exist four of these hemispheres that cover the sphere.

15. (Poland 1997) Given $n \geq 2$ points on a unit circle, show that at most $n^2/3$ of the segments with endpoints among the given points have length greater than $\sqrt{2}$.

16. (IMO 1973) Let $P_1, \ldots, P_{2n+1}$ be points on a semicircle centered at O. Prove that the sum of the vectors $OP_1, \ldots, OP_{2n+1}$ has length at least 1.

17. Given 111 unit vectors in the plane whose sum is 0, prove that there exist 55 of them whose sum has length at most 1.

18. (Sylvester) Given a finite set of points in the plane, not all on a line, show that there is a line containing exactly two of the points.
19. Given a set of $n \geq 3$ points in the plane, not all on a line, show that there are at least n lines containing at least two of the points.
20. Given a finite set of points in the plane, not all on a line, with each point colored either black or white, show that there exists a line containing at least two points of one color and none of the other color.
21. (IMO 1969) Given n points in the plane, no three collinear, prove that the number of convex quadrilaterals with vertices among the n points is at least $\binom{n-2}{2}$. (In fact, show that it is at least $\frac{1}{n-4}\binom{n}{5}$).
22. Show that the boundary of the intersection of $n \geq 2$ discs consists of at most $2n-2$ arcs.
23. (IMO 1999) Find all finite sets S of at least three points in the plane such that for all distinct points $A, B \in S$, the perpendicular bisector of AB is an axis of symmetry for S. (What about in three dimensions?)
24. (Erdős-Szekeres) Prove that for any n there exists N such that among any N points in the plane, no three collinear, n of them form the vertices of a convex n-gon.
25. In the plane lie n rays such that the endpoint of any ray does not lie on any other ray. Suppose that the rays bound a finite region and that each endpoint lies inside this region. Find the minimum possible value of n.
26. Given $n \geq 3$ points in the plane, not all on a line, show that the lines joining any two of them determine at least $n-1$ different slopes. For which n can equality occur?
27. A collection of $n \geq 2$ lines, no two parallel and no three concurrent, divides the plane into several regions. Prove that there are at least $n-2$ triangular regions.
28. (St. Petersburg 1997) Given $2n+1$ lines in the plane, prove that there are at most $n(n+1)(2n+1)/6$ acute triangles with sides on the lines.
29. (IMO Shortlist 2005) Let M be a convex n-gon, with $n \geq 4$. Some $n-3$ of its diagonals are colored green, and some other $n-3$ diagonals are colored red, so that no two diagonals of the same degree meet inside M. Find the maximum possible number of intersection points between red and green diagonals inside M.
30. (IMO Shortlist 2000) Ten gangsters are standing in a field such that the distances between any two are distinct. When the clock strikes noon, each gangster shoots their nearest neighbor dead. What is the fewest number of gangsters that can be shot?
31. (IMO 1975) Prove that there exist 1975 points on a unit circle such that the distance between any two is rational.
32. (IMO 1987) Let $n \geq 3$ be an integer. Prove that there is a set of n points in the plane such that the distance between any two is irrational, but any three of them determine a non-degenerate triangle with rational area.
33. (Erdős) An infinite set of points in the plane has the property that the distance between any two of them is an integer. Prove that all the points are collinear.
34. The region between two parallel lines in the plane is divided into ten stripes of equal width alternately colored white and black. A convex region of the plane contained in the strip contains at least one point from each line. Prove that at least 45% of the region must be black.
35. (Helly) A finite set of bounded, convex subsets of the plane has the property that every three have nonempty intersection. Prove that the intersection of all of the subsets is nonempty.
36. Given a convex region X in the plane, show that there exists a point P in X such that for every line through P, P lies in the middle third of the intersection of X with the line.
37. (Russia 1998) In the plane are given several squares with parallel sides such that among any n of them, there exist four having a common point. Prove that the squares can be divided into at most $n-3$ groups such that all of the squares in a group have a common point.