Polynomials

Alison Miller

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Things to Know:

(This list is taken from handouts of Melanie Wood, as are some of the problems.)

- Coefficients: Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ be a polynomial. Then the kth symmetric sum of the roots is equal to $(-1)^k a_{n-k}/a_n$.

- Division algorithm: Let $a(x)$ and $b(x)$ be polynomials. There exist unique polynomials $q(x)$ and $r(x)$ with $\deg(r) < \deg(d)$ and such that $a(x) = b(x)q(x) + r(x)$.

- Rational root theorem: Let $p(x) = a_n x^n + \cdots + a_0$ be a polynomial with integer coefficients. If $p(x)$ has a rational root $\frac{r}{s}$ with $r, s \in \mathbb{Z}$ and $(r, s) = 1$ then $r \mid a_0$ and $s \mid a_n$.

- Gauss's Lemma: If $p(x) \in \mathbb{Z}[x]$ factors in $\mathbb{Q}[x]$, then it factors in $\mathbb{Z}[x]$.

- Eisenstein's Criterion: Let $p(x) = a_n x^n + \cdots + a_0$ be a polynomial with real coefficients, and let q be a prime that divides $a_0, a_1, \cdots, a_{n-1}$ but not a_n. If $q^2 \nmid a_0$ then $p(x)$ is irreducible over the integers (and thus also over the rationals, by Gauss's lemma).

- Descartes' Rule of Signs: Let $p(x) = a_n x^n + \cdots + a_0$ be a polynomial with real coefficients, and let V be the number of sign changes in the coefficient list $a_n, \ldots, a_0$ where we ignore zero coefficients. If $p(x)$ has N positive roots (counted by multiplicity) then $N = V - 2k$ for some non-negative integer k. (Use this for $p(-x)$ to get the statement for negative roots of $p(x)$.

- Rolle's Theorem: Let $p(x)$ be a polynomial with real coefficients. Between any two zeroes of $p(x)$ lies a zero of $p'(x)$.

- Lagrange Interpolation: A polynomial of degree at most n with $p(a_k) = b_k$, $0 \le k \le n$ is given by
$$p(x) = \sum_{k=0}^{n} \frac{b_k(x-a_0)(x-a_1)\cdots(x-a_{k-1})(x-a_{k+1})\cdots(x-a_n)}{(a-a_0)(a-a_1)\cdots(a-a_{k-1})(a-a_{k+1})\cdots(a-a_n)}.$$

- Newtonian interpolation (finite differences): A polynomial p of degree at most n with $p(k) = b_k$, $0 \le k \le n$, is given by
$$p(x) = \sum_{k=0}^{n} \Delta^k p(0)\binom{x}{k},$$
where $\Delta p(x) = p(x+1) - p(x)$ and $\Delta^k p(x) = \Delta(\Delta^{k-1} p(x))$.

- Chebyshev Polynomials: The nth Chebyshev polynomial is defined so that $T_n(\cos\theta) = \cos(n\theta)$. They satisfy the recurrence

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \quad n \geq 1$$

  with $T_0(x) = 1$ and $T_1(x) = x$. (There are also Chebyshev polynomials of the second kind which satisfy $U_n(\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}$ and have a recurrence of the same form. These are much less common, though.)

- Maximum Modulus Theorem: Let $p(x)$ be a polynomial with complex coeffcients. Let D be an open region in the complex plane, and let ∂D be the boundary of D. Then $\max_{z\in D\cap\partial D} |p(z)| = \max_{z\in\partial D} |p(z)|$.

What do these problems have in common?

1. Let $A_1, A_2, \ldots, A_n$ be points in the plane, and $\overline{BC}$ a segment of length 2. Prove that there exists a point M on $\overline{BC}$ such that $MA_1 \cdot MA_2 \cdots MA_n \geq \frac{1}{2^{n-1}}$.

2 (Vietnam)**.** Find all polynomials $P(x)$ with integer coefficients such that the polynomial $Q(x) = (x^2 + 6x + 10) \cdot P^2(x) - 1$ is the square of a polynomial with integer coefficients.

3. Let $\{P_n(x)\}_{n=1}^{\infty}$ be a sequence of polynomials such that $P_1(x) = x^2 - 1$, $P_2(x) = 2x^2(x^2 - 1)$, and

$$P_{n-1}(x)P_{n+1}(x) = (P_n(x))^2 - (x^2 - 1)^2.$$

Let S_n denote the sum of the absolute values of the coefficients of $P_n(x)$. For each positive integer n find the largest non-negative integer k_n such that 2^k divides S_n.

4 (MOP '03)**.** Let k be a positive integer. Prove that $\sqrt{k+1} - \sqrt{k}$ is not the real part of a complex number z such that $z^n = 1$ for some positive integer n.

5 (IMO Shortlist 2003)**.** The sequence $a_0, a_1, a_2, \ldots$ is defined as follows: $a_0 = 2$, $a_{k+1} = 2a_k^2 - 1$ for $k \geq 0$. Prove that if an odd prime p divides a_n, then 2^{n+3} divides $p^2 - 1$.

6 (IMO Shortlist 2003)**.** Let $\mathbb{R}^+$ be the set of all positive real numbers. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ that satisfy the following conditions:

- $f(xyz) + f(x) + f(y) + f(z) = f(\sqrt{xy})f(\sqrt{yz})f(\sqrt{zx})$ for all $x, y, z \in \mathbb{R}^+$;
- $f(x) < f(y)$ for all $1 \leq x < y$.

General Problems

7. Determine all polynomials $P(x)$ with real coefficients such that $P(x)^2 + P(1/x)^2 = P(x)^2 P(1/x)^2$ for all $x \neq 0$.

8 (MOP '08)**.** Let P_n be a polynomial of degree n with real coecients, and let t be a real number with $t \geq 3$. For integers $n \geq 0$, show that

$$\max_{0\leq k\leq n+1} |t^n - P_n(k)| \geq 1$$

9. Let p, q, r be polynomials with real coefficients, such that at least one of the polynomials has degree 2 and at least one of the polynomials has degree 3. Assume that $p^2 + q^2 = r^2$: Show that at least one of the polynomials both has degree 3 and has 3 (not necessarily distinct) real roots.

10 (USAMO 2002)**.** Prove that any monic polynomial (a polynomial with leading coefficient 1) of degree n with real coefficients is the average of two monic polynomials of degree n with n real roots.

11 (MOP 01)**.** Let $P(x)$ be a real-valued polynomial with $P(n) = P(0)$. Show that there exist at least n distinct (unordered) pairs of real numbers $\{x, y\}$ such that $x - y$ is a real number and $P(x) = P(y)$.

12 (USAMO 88)**.** A certain polynomial product of the form

$$(1 - z^{b_1})(1 - z^{b_2}) \cdots (1 - z^{32})^{b_{32}}$$

where the b_k are positive integers, has the surprising property that if we multiply it out and discard all terms involving z to a power larger than 32, we are left with $1 - 2z$. Find b_{32}.