VECTOR

- as an essential tool for 3-dimensional coordinate geometry -

1. Introduction

We begin with some basic definitions about vectors.

1. Scalar:
   A scalar is a quantity which is specified by magnitude only. (e.g. mass, length, volume…) We can describe it by a real number.
   In this text, the term 'scalar' can be viewed as the same as 'real number'.

2. Vector:
   A vector is a quantity which possesses both magnitude and direction. (e.g. force, velocity, acceleration…) We can describe it by stating a real number for its magnitude and its direction with respect to some reference.
   In this text, vectors can be viewed as 'arrows', while the magnitude of a vector refers to its 'length', and the direction of it refers to the direction of the arrow.

You may have noticed the difference in essence between scalars and vectors is that vectors possess directions while scalars do not. Two vectors are defined to be *equal* if they have the same magnitude and direction. Note also that there is no specific order among vectors, i.e., we cannot tell which vector is larger than the other. This point will be made clearer in later discussions.

2. Basic Notations and Operations

Representation of Vectors

In the *n*-dimensional space $\mathbb{R}^n$, there are *n* basic vectors, namely $\hat{e}_1$, $\hat{e}_2$, …, $\hat{e}_n$, so that every vector $\vec{v}$ in $\mathbb{R}^n$ can be written as $\vec{v} = k_1\hat{e}_1 + k_2\hat{e}_2 + \cdots + k_n\hat{e}_n$ and the choice of each $k_1, k_2, \ldots, k_n$ is unique (such a collection of vectors is called a *basis* for the space). The sign "^" denotes a vector with unit (1 unit) length. The directions of the vectors correspond to those of the *n* axes. These *n* vectors are said to be mutually *perpendicular*.

Generally, a vector in $\mathbb{R}^n$ can be represented in the following manner.

Vectors can be viewed as ‘arrows’. An arrow has a ‘head’ and a ‘tail’, by which a vector is specified. The “tail” is called the *initial point*, while the “head” is called the *terminal point* of the vector. Suppose the initial and the terminal point of a vector are A and B respectively, then the vector is denoted by $\overrightarrow{AB}$.

Note that there are two main differences between vectors and lines in a space. First, a line is fixed in the space while a vector is movable. Second, a line has infinite length while a vector has a finite fixed length.

As vectors are movable, one can fix the initial point *A* to be the origin *O* of the coordinate system. Let, correspondingly, the terminal point *B* is moved to *P* by maintaining the direction and length of $\overrightarrow{AB}$, then we have $\overrightarrow{OP} = \overrightarrow{AB}$. Let the coordinates of *P* be $(p_1, p_2, \ldots, p_n)$, then $\overrightarrow{OP} = p_1\vec{e}_1 + p_2\vec{e}_2 + p_3\vec{e}_3 + \ldots + p_n\vec{e}_n$ is called the *position vector* of *P*. In such a manner, with every vector $\vec{v}$ in $\mathbb{R}^n$ associates uniquely coordinates $(p_1, p_2, \ldots, p_n)$, and then we write:

$$\overrightarrow{AB} = (p_1, p_2, \ldots, p_n)$$

One can show that if $A = (a_1, a_2, \ldots, a_n)$ and $B = (b_1, b_2, \ldots, b_n)$, then

$$\overrightarrow{AB} = (b_1 - a_1, b_2 - a_2, \ldots, b_n - a_n).$$

Its magnitude (or length) is defined as $\overrightarrow{AB} = \sqrt{p_1{}^2 + p_2{}^2 + \cdots + p_n{}^2}$.

Note that the notation $(p_1, p_2, p_3, ..., p_n)$ may be used to represent the point *P* or the vectors $\overrightarrow{AB}$ and $\overrightarrow{OP}$. Readers have to take care of its meaning in different occasions.

Operations of Vectors

Basically, there are four basic operations between vectors. Consider the vectors $\vec{a} = (a_1, a_2, a_3, ..., a_n)$, $\vec{b} = (b_1, b_2, b_3, ..., b_n)$.

Operations:

1. Vector addition:

$$\vec{a}+\vec{b} = (a_1, a_2, a_3, \ldots, a_n) + (b_1, b_2, b_3, \ldots, b_n) = (a_1+b_1, a_2+b_2, a_3+b_3, \ldots, a_n+b_n)$$

(Figure 1)

2. Scalar multiplication:

$c\vec{a} = (c)\vec{a} = (ca_1, ca_2, ca_3, \ldots, ca_n)$ where c is a scalar.

3. Dot product:

$\vec{a}\cdot\vec{b} = a_1b_1 + a_2b_2 + \cdots + a_nb_n = |\vec{a}||\vec{b}|\cos\theta$ where θ is the inclined angle between $\vec{a}$ and $\vec{b}$ with $0 \le \theta \le \pi$ (Figure 2).

4. Cross product (For vectors in the 3-dimensional space $\mathbb{R}^3$ only):

$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{e}_1 & \vec{e}_2 & \vec{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = |\vec{a}||\vec{b}|\sin\theta\hat{e}$ where $\hat{e}$ is the unit vector with direction perpendicular to both $\vec{a}$ and $\vec{b}$, and obeys the right-hand rule (Figure 3).

Note that a dot product is a scalar while the cross product of two vectors is still a vector which is perpendicular to them.

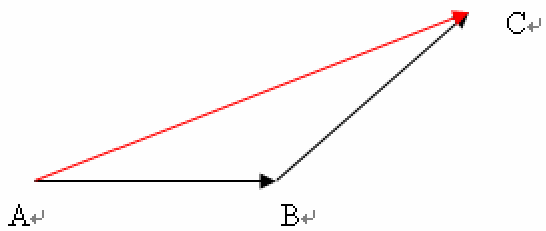

Figure 1: Addition of vectors: $\overrightarrow{AB}+\overrightarrow{BC}=\overrightarrow{AC}$

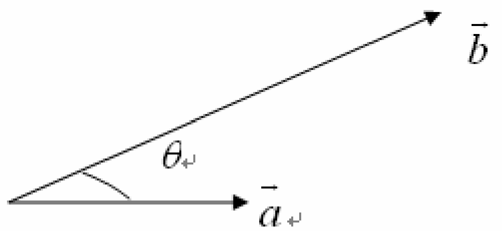

Figure 2: The inclined angle between $\vec{a}$ and $\vec{b}$

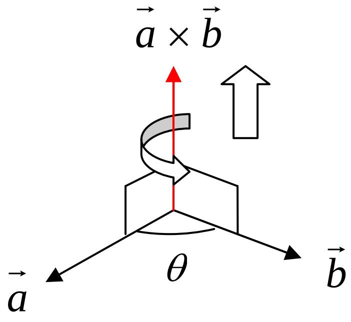

Figure 3: Cross product: $\vec{a} \times \vec{b} = |\vec{a}||\vec{b}| \sin \theta \hat{e}$

Direction Vector and Position Vector

From the geometric point of view, vectors can be categorized into two types: *direction vector* and *position vector*. As its name suggests, a direction vector represents a certain direction in space, whereas a position vector represents a specific position in space. Both vectors have n components in an n-dimensional space.

The difference lies in the interpretation of the vectors. If a vector is used to represent a certain direction in the space only, its magnitude does not matter at all. In other words, a positive constant k can always be multiplied to this vector by the scalar multiplication without changing the direction it represents. If the constant k is negative, the direction will change to the opposite. The same vector can be regarded as a position vector. But in this case, each component of this vector cannot be altered; otherwise, the position represented will be changed.

The above difference can be clarified by the following example. Let $A = (1, 1)$, $B = (2, 2)$ and $C = (3, 3)$. To represent the point A in the space by a vector, $\overrightarrow{OA}$ is used and it is viewed as a position vector. By direct computation, it can be seen that $\overrightarrow{BC} = \overrightarrow{OC} - \overrightarrow{OB} = \overrightarrow{OA}$. Hence, $\overrightarrow{BC}$ is viewed as a position vector which represents the position of A, and it does not represent either the position of B or C. Note that $\overrightarrow{OA}$ can also be viewed as a direction vector with the same direction as $\overrightarrow{BC}$. Indeed, for $k \neq 0$, $(k, k) = k(1, 1) = k\overrightarrow{OA}$ is always parallel to $\overrightarrow{OA}$ (the direction will be opposite if $k<0$). If $k\overrightarrow{OA}$ is viewed as a position vector, then $k\overrightarrow{OA}$ and $\overrightarrow{OA}$ represent the same position only if $k = 1$. That is, $k\overrightarrow{OA}$ represents different positions from $\overrightarrow{OA}$ for all $k \neq 1$ although they are parallel (Figure 4).

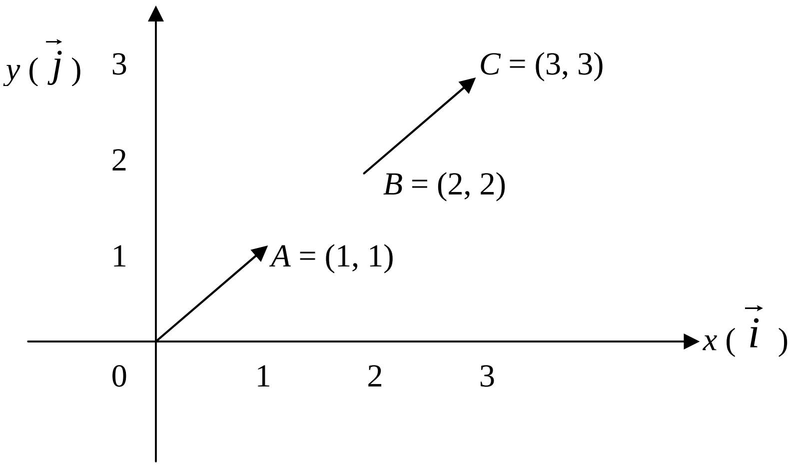

Figure 4: The position that $\overrightarrow{OA}$ represents is (1, 1)

while the direction that $\overrightarrow{OA}$ represents is $\vec{i} + \vec{j}$

The following sections will focus on the operations between vectors in the <u>3-dimensional space</u> P^3.

3. Geometry in 3-dimensional space

In the 3-dimensional space P^3, the most elementary quantity is point. Any two points determine a line. Any two intersecting lines determine a plane. We are interested in the relations between points, lines and planes. These relations mainly include distance, angle and intersection. Points, lines and planes can be represented well by means of vectors. In addition, the relations between them can be easily evaluated through the notion of vectors easily.

Geometric quantities

There are 13 main relations between points, lines, and planes; and they are categorized as follow:

i) <u>Distance:</u>
1. Distance between two points.
2. Distance between a point and a line.
3. Distance between a point and a plane.

4. Distance between two parallel lines.
5. Distance between a pair of skew lines.
6. Distance between two planes.
7. Distance between a line and a plane.

ii) <u>Angle:</u>

8. Angle between two lines.
9. Angle between two planes.
10. Angle between a line and a plane.

iii) <u>Intersection:</u>

11. Intersection point of two lines.
12. Intersection point of a line and a plane.
13. Intersection line of two planes.

Before we begin discussing the above quantities, let us familiarize ourselves with the representation of points, lines and planes by means of vectors.

Representation

1. <u>Point</u>

For a given point $P(x_0, y_0, z_0)$ in P^3, its representation is $\overrightarrow{OP} = x_0\vec{i} + y_0\vec{j} + z_0\vec{k}$ or simply $P = (x_0, \; y_0, \; z_0)$, where O is the origin. (Figure 5)

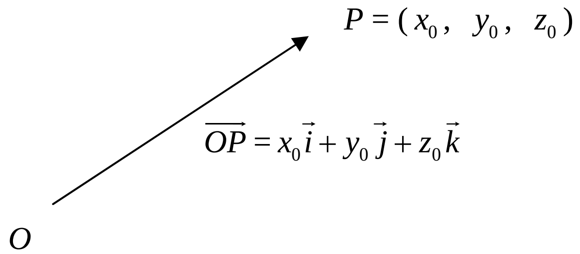

Figure 5: A position vector representing the point P

2. <u>Line</u>

A line is completely determined by its direction and any point which it contains.

Suppose point $P(p_1, p_2, p_3)$ is on the line L and a direction vector $\vec{v} = (v_1, v_2, v_3)$ is parallel to L. Then, any point R with position vector $\overrightarrow{OR}$ on the line can be represented by $\overrightarrow{OR} = \overrightarrow{OP} + k\vec{v}$ where k is any real number (Figure 6). Therefore, every point R on the line L can be represented by the formula $\overrightarrow{OR} = \overrightarrow{OP} + k\vec{v}$, where k specifies the position of the point along the line.

Let $R = (r_1, r_2, r_3)$. After writing them explicitly, we have $(r_1, r_2, r_3) = (p_1, p_2, p_3) + k(v_1, v_2, v_3)$. By equating the corresponding components, we have three equations:

$$\begin{cases} r_1 = p_1 + kv_1 \\ r_2 = p_2 + kv_2 \\ r_3 = p_3 + kv_3 \end{cases}$$

The equation of the line expressed in this way is called the *parametric form*. After some simple manipulations, we get the following equations:

$$\begin{cases} \dfrac{r_1 - p_1}{v_1} = k \\ \dfrac{r_2 - p_2}{v_2} = k \\ \dfrac{r_3 - p_3}{v_3} = k \end{cases} \quad \text{or} \quad \frac{r_1 - p_1}{v_1} = \frac{r_2 - p_2}{v_2} = \frac{r_3 - p_3}{v_3} = k\,.$$

This is called the *symmetric form*.

Note that r_i are variables while p_i and v_i are the given constants for i = 1, 2, 3. Sometimes, the equation of a line can be represented by the intersection line of two planes. We shall return to this point after introducing of equations of planes.

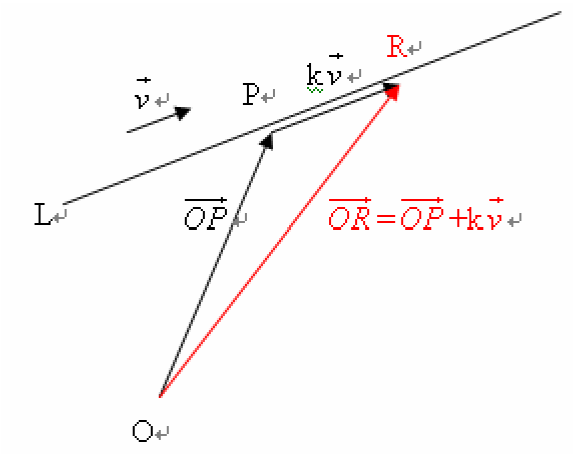

Figure 6: An arbitrary point R on line L

3. Plane

We first introduce a new term – *normal vector*. It is a direction vector perpendicular to a given plane. To represent a plane, we use the property that every vector lying in a plane is perpendicular to any normal vector of the plane. Hence, to determine the equation of a plane, we have to know a point on the plane and a normal vector to the plane. Let $P\,(p_1, p_2, p_3)$ be a given point on the plane and $\vec{n}$ = (n_1, n_2, n_3) the normal vector of the plane. For any point $R = (r_1, r_2, r_3)$ (Figure 7) with position vector $\overrightarrow{OR}$, the vector $\overrightarrow{PR}$ lies in the plane, and thus is perpendicular to $\vec{n}$. Therefore, $\overrightarrow{PR} \cdot \vec{n} = 0$ (or $(\overrightarrow{OR} - \overrightarrow{OP}) \cdot \vec{n} = 0$, or $r_1n_1 + r_2n_2 + r_3n_3$ $= p_1n_1 + p_2n_2 + p_3n_3$).

Note that r_i are variables while p_i and n_i are the given constants for i = 1, 2, 3. In general, $ax + by + cz = \rho$ is the equation of a plane with normal vector = (a, b, c), where (x, y, z) is any point on the plane and ρ is a constant.

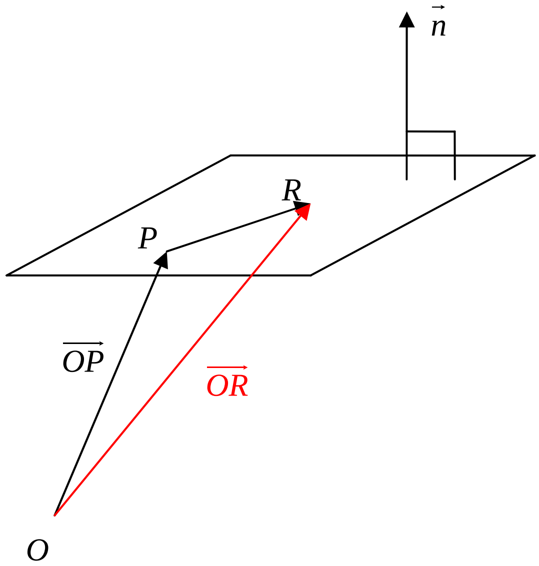

Figure 7: An arbitrary point R on the plane

<u>Summary:</u>

i) <u>Different forms of representations:</u>

1. <u>Point:</u>
   $P = (p_1, p_2, p_3)$.

2. Line:

   i) $\overrightarrow{OR} = \overrightarrow{OP} + k\vec{v}$ (Vector form)

   ii) $\begin{cases} r_1 = p_1 + kv_1 \\ r_2 = p_2 + kv_2 \\ r_3 = p_3 + kv_3 \end{cases}$ (Parametric form)

   iii) $\dfrac{r_1 - p_1}{v_1} = \dfrac{r_2 - p_2}{v_2} = \dfrac{r_3 - p_3}{v_3} = k$ (Symmetric form)

3. Plane:

   i) $\overrightarrow{PR} \cdot \vec{n} = 0$ (Vector form)

   ii) $(\overrightarrow{OR} - \overrightarrow{OP}) \cdot \vec{n} = 0$

   iii) $r_1n_1 + r_2n_2 + r_3n_3 = p_1n_1 + p_2n_2 + p_3n_3$

   iv) $ax + by + cz = \rho$

ii) Conditions for determining lines and planes:

Any of the followings is a necessary and sufficient condition for finding the equation of a line:

1. Given two distinct points (two-point form):

   Let the points be P and Q. Then the vector $\overrightarrow{PQ}$ is a direction vector parallel to the line. We can choose either P or Q as a point on the line. Then, the equation can be found by vector form.

2. Given a point and a direction vector parallel to the line (point-vector form): Use vector form directly.

Any of the followings is a necessary and sufficient condition for finding the equation of a plane:

1. Given three distinct points:
   Let the three points be P, Q, and R. The normal vector to the plane can be found by the cross product $\overrightarrow{PQ} \times \overrightarrow{PR}$. We can choose either one of P, Q or R

as the given point. Use vector form.

2. Given two intersecting lines:

   Let $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$ be the two direction vectors. Then a normal of the plane is $\overrightarrow{v_1} \times \overrightarrow{v_2}$. Choose the intersection point of the lines as a point on the plane, and then use vector form.

3. Given one line and one point not lying on the given line:
   Let P be the given point, Q be a point on the line, $\vec{v}$ be the direction vector of the line. Then a normal vector to the plane is $\overrightarrow{PQ} \times \vec{v}$. Finally, use vector form.

In general, if two lines do not intersect, there does not exist a plane containing both lines. As we can use two distinct points to represent each line, there are a total of four distinct points in order to represent two lines. As there is always a unique plane containing any 3 points, there is an additional point for which it may not lie on the plane.

Relations between points, lines and planes

With the above knowledge, we can now explore the relationships between points, lines and planes.

Distance

1. Distance between two points

In $\mathbb{R}^2$, the distance, denoted by *d*, between two points (x_1, y_1) and (x_2, y_2) can be found by Pythagoras' Theorem: $d = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2}$. Similarly, given two points $A\ (x_1, y_1, z_1)$ and $B\ (x_2, y_2, z_2)$ in $\mathbb{R}^3$, the distance *AB* is given by

$\sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2 + (z_1 - z_2)^2}$.

2. <u>Distance between a point and a line</u>

Given a point $P = (p_1, p_2, p_3)$ and a line L: $(r_1, r_2, r_3) = (q_1, q_2, q_3) + k\,(v_1, v_2, v_3)$. where $Q\,(q_1, q_2, q_3)$ is a point on the line L and $\vec{v} = (v_1, v_2, v_3)$ is a direction vector parallel to L. The perpendicular distance between P and L is $\left|\overrightarrow{PQ}\right|\sin\theta$, where θ is the inclined angle between $\overrightarrow{PQ}$ and $\vec{v}$ (Figure 8). By cross product, we have

$\left|\overrightarrow{PQ}\times\vec{v}\right| = \left|\overrightarrow{PQ}\right|\left|\vec{v}\right|\sin\theta$. Hence, $d = \left|\overrightarrow{PQ}\right|\sin\theta = \dfrac{\left|\overrightarrow{PQ}\times\vec{v}\right|}{\left|\vec{v}\right|}$.

Another approach is to find the normal projection R from P to L, and then determine PR. Since R is on L, $R = (r_1, r_2, r_3) = (q_1, q_2, q_3) + k\,(v_1, v_2, v_3) = (q_1 + kv_1, q_2 + kv_2, q_3 + kv_3)$ for some constant k. To find the coordinates of R, we need to determine the parameter k. Since PR is perpendicular to L, $\overrightarrow{PR}\cdot\vec{v} = 0$. That is,

$$(q_1 + kv_1 - p_1, q_2 + kv_2 - p_2, q_3 + kv_3 - p_3)\cdot(v_1, v_2, v_3) = 0$$

$$(q_1 + kv_1 - p_1)v_1 + (q_2 + kv_2 - p_2)v_2 + (q_3 + kv_3 - p_3)v_3 = 0$$

Hence,

$$k = \frac{(p_1 - q_1)v_1 + (p_2 - q_2)v_2 + (p_3 - q_3)v_3}{v_1^2 + v_2^2 + v_3^2}.$$

A third approach is to express PR explicitly with a parameter k, and then determine the smallest value by differentiation. This is left as an exercise to the readers.

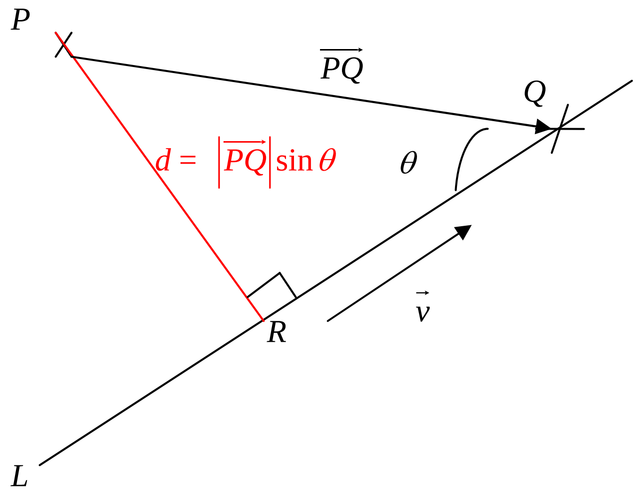

Figure 8: The shortest distance d from P to L

3. Distance between a point and a plane

Given a point $P = (p_1, p_2, p_3)$ and a plane π: $r_1n_1 + r_2n_2 + r_3n_3 = q_1n_1 + q_2n_2 + q_3n_3$ where $\vec{n} = (n_1, n_2, n_3)$ is a normal vector and $Q\,(q_1, q_2, q_3)$ is on π. The shortest distance between P and π is $|PQ \cos\theta|$ (Figure 9). By dot product, $\left|\overrightarrow{PQ}\cdot\vec{n}\right| = \left|\overrightarrow{PQ}\right|\left|\vec{n}\right|\left|\cos\theta\right|$. Hence, $d = |PQ\cos\theta| = \dfrac{\left|\overrightarrow{PQ}\cdot\vec{n}\right|}{\left|\vec{n}\right|}$.

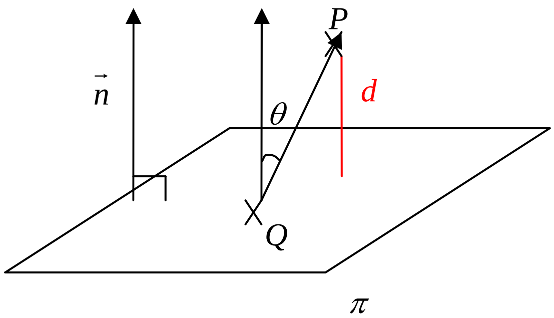

Figure 9: The shortest distance d from P to π

4. Distance between two parallel lines

Let $L_1 : \overrightarrow{OR} = \overrightarrow{OP} + k_1\vec{v}$ and $L_2 : \overrightarrow{OR} = \overrightarrow{OQ} + k_2\vec{v}$ be a pair of parallel lines. Take the point P on L_1 as a given point. Then the problem is then reduced to finding the shortest distance d, between a point P and a line L_2. Therefore, $d = \dfrac{\left|\overrightarrow{PQ}\times\vec{v}\right|}{\left|\vec{v}\right|}$.

5. Distance between a pair of skew lines

A pair of skew lines is a pair of lines which do not intersect. Let the pair of skew lines be $L_1 : \overrightarrow{OR} = \overrightarrow{OP} + k_1\,\overrightarrow{v_1}$ and $L_2 : \overrightarrow{OR} = \overrightarrow{OQ} + k_2\,\overrightarrow{v_2}$. The cross product $\overrightarrow{v_1}\times\overrightarrow{v_2}$ is perpendicular to $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$. Together with the point P, we can form a plane π with a normal vector $\vec{n} = \overrightarrow{v_1}\times\overrightarrow{v_2}$ containing the line L_1 (Figure 10). The problem is reduced to finding the distance between a point Q and the plane π. Hence,

$$d = \frac{\left|\overrightarrow{PQ}\cdot\overrightarrow{v_1}\times\overrightarrow{v_2}\right|}{\left|\overrightarrow{v_1}\times\overrightarrow{v_2}\right|}.$$

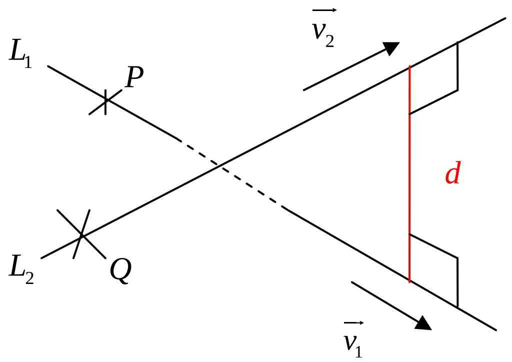

Figure 10: The shortest distance between a pair of skew lines

6. Distance between two planes

If the normals of the two planes are not parallel, the planes intersect and their shortest distance is 0. Suppose the two planes are parallel. Their normals will then be parallel. Let the equations of the two planes be π_1: $r_1n_1 + r_2n_2 + r_3n_3 = p_1n_1 + p_2n_2 + p_3n_3$ and π_2: $r_1n_1 + r_2n_2 + r_3n_3 = q_1n_1 + q_2n_2 + q_3n_3$. Let $P = (p_1, p_2, p_3)$, $Q = (q_1, q_2, q_3)$ and $\vec{n} = (n_1, n_2, n_3)$. The problem is reduced to finding the shortest distance d, between the point P and the plane π_2. Therefore, $d = \dfrac{\left|\overrightarrow{PQ} \cdot \vec{n}\right|}{\left|\vec{n}\right|}$.

7. Distance between a line and a plane

If the line intersects the plane, the shortest distance between the line and the plane is 0. Suppose a line L is parallel to a plane π. Let L be $\overrightarrow{OR} = \overrightarrow{OQ} + k\vec{v}$ and π be $\overrightarrow{OR} \cdot \vec{n} = \overrightarrow{OP} \cdot \vec{n}$. The problem is reduced to finding the shortest distance d, between a point Q and the plane π. Therefore, $d = \dfrac{\left|\overrightarrow{PQ} \cdot \vec{n}\right|}{\left|\vec{n}\right|}$.

Angle

8. Angle between two lines

The angle between two lines is same as the acute angle between their direction vectors. Let the direction vectors of the two lines be $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$. By dot product, $\overrightarrow{v_1} \cdot \overrightarrow{v_2} = \left|\overrightarrow{v_1}\right| \left|\overrightarrow{v_2}\right| \cos\theta$ where θ is the inclined angle between $\overrightarrow{v_1}$ and $\overrightarrow{v_2}$. Hence, the acute angle between the direction vectors = $\cos^{-1} \dfrac{\left|\overrightarrow{v_1} \cdot \overrightarrow{v_2}\right|}{\left|\overrightarrow{v_1}\right|\left|\overrightarrow{v_2}\right|}$.

9. Angle between two planes

Let $\overrightarrow{n_1}$ and $\overrightarrow{n_2}$ be the normal vectors of the two planes. Then, the inclined angle between $\overrightarrow{n_1}$ and $\overrightarrow{n_2}$ is $\theta = \cos^{-1} \dfrac{\overrightarrow{n_1} \cdot \overrightarrow{n_2}}{\left|\overrightarrow{n_1}\right|\left|\overrightarrow{n_2}\right|}$. Hence, the inclined angle between the two planes is $\pi - \cos^{-1} \dfrac{\overrightarrow{n_1} \cdot \overrightarrow{n_2}}{\left|\overrightarrow{n_1}\right|\left|\overrightarrow{n_2}\right|}$ (Figure 11).

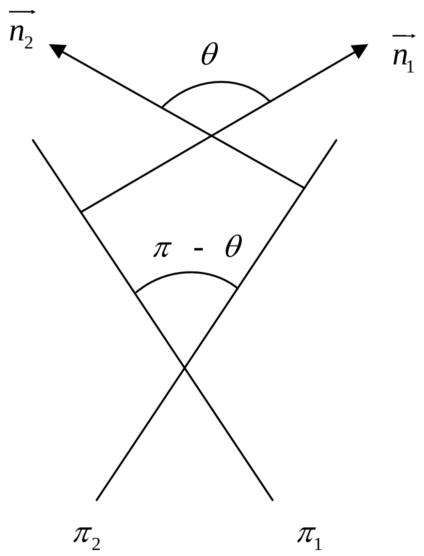

Figure 11: The angle between two planes

10. <u>Angle between a line and a plane.</u>

Let L: $\overrightarrow{OR} = \overrightarrow{OP} + k\vec{v}$ be a line and π_1: $\overrightarrow{OR} \cdot \vec{n} = \overrightarrow{OQ} \cdot \vec{n}$ be a plane. The inclined angle between $\vec{v}$ and $\vec{n}$ is $\theta = \cos^{-1}\dfrac{|\vec{v}\cdot\vec{n}|}{|\vec{v}||\vec{n}|}$. Hence, the angle between L and π_1 is $\dfrac{\pi}{2} - \cos^{-1}\dfrac{|\vec{v}\cdot\vec{n}|}{|\vec{v}||\vec{n}|}$ (Figure 12).

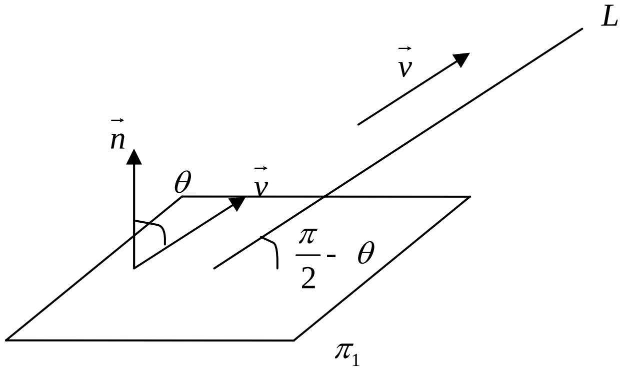

Figure 12: The angle between a line and a plane

Intersection

11. Intersection point of two lines

Let the two lines be $L_1: \begin{cases} r_1 = p_1 + kv_1 \\ r_2 = p_2 + kv_2 \\ r_3 = p_3 + kv_3 \end{cases}$ and $L_2: \begin{cases} r_1 = q_1 + mw_1 \\ r_2 = q_2 + mw_2 \\ r_3 = q_3 + mw_3 \end{cases}$. Let R = (r_1, r_2, r_3) be the intersection point between L_1 and L_2. By equating the corresponding three equations in L_1 and L_2, we have three equations:

$$\begin{cases} q_1 + mw_1 = p_1 + kv_1 \\ q_2 + mw_2 = p_2 + kv_2 \\ q_3 + mw_3 = p_3 + kv_3 \end{cases} \text{ or } \begin{cases} mw_1 - kv_1 = p_1 - q_1 \\ mw_2 - kv_2 = p_2 - q_2 \\ mw_3 - kv_3 = p_3 - q_3 \end{cases}.$$

Now, there are three equations and two unknowns (k and m). The system may have no solution, which implies the two lines have no intersection. If the system has infinitely many solutions, then L_1 and L_2 represent the same line. The two lines have an intersection point if and only if there is a unique solution for k and m. Once k (or m) is found, by substituting it into $L_1: \begin{cases} r_1 = p_1 + kv_1 \\ r_2 = p_2 + kv_2 \\ r_3 = p_3 + kv_3 \end{cases}$ (or $L_2: \begin{cases} r_1 = q_1 + mw_1 \\ r_2 = q_2 + mw_2 \\ r_3 = q_3 + mw_3 \end{cases}$), R can be found.

12. Intersection point of a line and a plane

Let L: $\begin{cases} x = p_1 + kv_1 \\ y = p_2 + kv_2 \\ z = p_3 + kv_3 \end{cases}$ be a line and $\pi: ax + by + cz = \rho$ be a plane. Let X (x, y, z) be the intersection point of L and π. By substituting the three equations of L into π, we have $a(p_1 + kv_1) + b(p_2 + kv_2) + c(p_3 + kv_3) = \rho$ or $(av_1 + bv_2 + cv_3)k = \rho - (ap_1 + bp_2 + cp_3)$. Clearly, $k = \dfrac{\rho - (ap_1 + bp_2 + cp_3)}{av_1 + bv_2 + cv_3}$ if $av_1 + bv_2 + cv_3 \neq 0$. In this case, by substituting $k = \dfrac{\rho - (ap_1 + bp_2 + cp_3)}{av_1 + bv_2 + cv_3}$ into L:

$$\begin{cases} x = p_1 + kv_1 \\ y = p_2 + kv_2 \\ z = p_3 + kv_3 \end{cases}$$, there is a unique solution. That means there is one and only one intersection point X between L and π.

Note that $av_1 + bv_2 + cv_3 = 0 \Leftrightarrow (a, b, c) \sqcup (v_1, v_2, v_3) = 0 \Leftrightarrow (a, b, c)$ and (v_1, v_2, v_3) are perpendicular to each other $\Leftrightarrow$ The direction vector of L is perpendicular to the normal vector of π $\Leftrightarrow$ L is parallel to π.

If $av_1 + bv_2 + cv_3 = 0$, there are two possible cases:

Case (i): $ap_1 + bp_2 + cp_3 = \rho$.

Since $\rho = aq_1 + bq_2 + cq_3$ for some point Q (q_1, q_2, q_3) on the plane, $ap_1 + bp_2 + cp_3 = \rho \Leftrightarrow ap_1 + bp_2 + cp_3 = aq_1 + bq_2 + cq_3 \Leftrightarrow a(p_1 - q_1) + b(p_2 - q_2) + c(p_3 - q_3) = 0 \Leftrightarrow (a, b, c) \sqcup (p_1 - q_1,\ p_2 - q_2,\ p_3 - q_3) = 0 \Leftrightarrow (a, b, c)$ is perpendicular to $(p_1 - q_1,\ p_2 - q_2,\ p_3 - q_3)$. Since Q lies on π and PQ is perpendicular to (a, b, c), P lies on π as well. Since L is parallel to π, L lies on π. That is, every point on L lies on π.

Case (ii): $ap_1 + bp_2 + cp_3 \neq \rho$

In this case, there is no solution for k. That implies there is no intersection point of L and π. From the geometric meaning, $ap_1 + bp_2 + cp_3 \neq \rho$ implies PQ is not perpendicular to (a, b, c). Hence, P does not lie on L. Since L is parallel to π, all the points on L do not lie on π. In other words, there is no intersection point of L and π.

13. Intersection line of two planes

Let π_1: $r_1n_1 + r_2n_2 + r_3n_3 = p_1n_1 + p_2n_2 + p_3n_3$ and π_2: $r_1m_1 + r_2m_2 + r_3m_3 = q_1m_1 + q_2m_2 + q_3m_3$ be two distinct planes. Let $\vec{n} = (n_1, n_2, n_3)$ and $\vec{m} = (m_1, m_2, m_3)$. If $\vec{n} = k\vec{m}$ for some non-zero real number k, the normal vectors of the planes are parallel. Therefore the planes are parallel and they have no intersection.

Suppose $\vec{n}$ and $\vec{m}$ are not parallel. Since the two equations of π_1 and π_2

have three variables, we can easily find an intersection point, say $H\ (h_1, h_2, h_3)$, of the planes (e.g. Put $r_1 = 0$ and solve for r_2 and r_3). By cross product, $\vec{n} \times \vec{m}$ will be a vector parallel to the intersection line of the planes. Hence, using point-vector form with the point H, the intersection line is $\overrightarrow{OR} = \overrightarrow{OH} + k\vec{n} \times \vec{m}$.

Another approach is to find one more intersection point, say J, of π_1 and π_2 (e.g. Put $r_2 = 0$ and solve for r_1 and r_3) and then use two-point form. The intersection line is $\overrightarrow{OR} = \overrightarrow{OH} + k\overrightarrow{HJ}$.