2008 BLUE MOP, POLYNOMIALS-III

ALİ GÜREL

(1) Prove that for any polynomial $P(x)$ with degree n, we have the relation
$$P(x+n+1) = \sum_{j=0}^{n} (-1)^{n-j} \binom{n+1}{j} P(x+j).$$

(2) Polynomial P of degree n satisfies $P(j) = \binom{n+1}{j}^{-1}$ for $j = 0, 1, ..., n$. Evaluate $P(n+1)$.

(3) If P is a polynomial of an even degree n with $P(0) = 1$ and $P(j) = 2^{j-1}$ for $j = 1, ..., n$, prove that $P(n+2) = 2P(n+1) - 1$.

(4) Let $a_1, a_2, ..., a_n$ be non-negative real numbers and assume that the polynomial
$$P(x) = x^n + a_{n-1}x^{n-1} + ... + a_1 x + 1$$
has n real roots. Prove that $P(2) \geq 3^n$.

(5) (BMO-89) If $a_n...a_1a_0$ is the decimal representation of a prime number and $a_n > 1$, prove that the polynomial $P(x) = a_n x^n + ... + a_1 x + a_0$ is irreducible in $\mathbb{Z}[x]$.

(6) Let $p > 2$ be a prime number and $P(x) = x^p - x + p$. Prove that the polynomial $P(x)$ is irreducible in $\mathbb{Z}[x]$.

(7) For polynomials $P(x)$ and $Q(x)$ and an arbitrary $k \in \mathbb{C}$, denote
$$P_k = \{z \in \mathbb{C} \,|\, P(z) = k\}, \; Q_k = \{z \in \mathbb{C} \,|\, Q(z) = k\}$$
Prove that $P_0 = Q_0$ and $P_1 = Q_1$ imply that $P \equiv Q$.

(8) Prove that the polynomial $P(x) = x^n + 4$ is irreducible over $\mathbb{Z}[x]$ if and only if n is not a multiple of 4.

Problem 1, Solution by Joshua Pfeffer: Let $P_0(x) = P(x)$ and $P_k(x) = P_{k-1}(x+1) - P_{k-1}(x)$ for $k \in \mathbb{N}$.

Lemma. For all integers $k \geq 0$: $P_k(x) = \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}P(x+j)$.
The proof of the lemma is done by induction on k and using the identity $\binom{k}{j-1} + \binom{k}{j} = \binom{k+1}{j}$.

Now since $degP_k = degP - k$ for $k \leq degP$, we have $degP_n = 0$. Hence, $P_{n+1} \equiv 0$ and $\sum_{j=0}^{n+1}(-1)^{n+1-j}\binom{n+1}{j}P(x+j) = 0$. Thus we conclude that $P(x+n+1) = \sum_{j=0}^{n}(-1)^{n-j}\binom{n+1}{j}P(x+j)$ $\square$

Problem 2, Solution by Matthew Superdock: Using the result from Problem 1,

$$P(n+1) = \sum_{j=0}^{n}(-1)^{n-j}\binom{n+1}{j}P(j) = \sum_{j=0}^{n}(-1)^{n-j}\binom{n+1}{j}\binom{n+1}{j}^{-1}.$$

Hence, we deduce that $P(n+1)$ is 1 if n is even and 0 if n is odd $\square$

Problem 3, Solution by Taylor Han: Let the given polynomial with degree n be P_n. We will induct on the even integer n. When $n = 2$ we show the result by finding that $P_2(x) = \frac{x^2-x+2}{2}$. Assume the result for up to n. Let $Q(x) = P_{n+2}(x+1) - P_{n+2}(x)$ and let $R(x) = Q(x+1) - Q(x)$. Note that R has the same values with P_n at $j = 0, 1, ..., n$ and $degR \leq n = degP_n$. We deduce that $R \equiv P_n$. So $R(n+1) = 2^n$ and $R(n+2) = 2^{n+1} - 1$ by the induction hypothesis. Now we go back and using the values of R prove that $P_{n+2}(n+3) = 2^{n+2}$ and $P_{n+2}(n+4) = 2^{n+3} - 1$, which completes the induction $\square$

Problem 4, Solution by Justin Brereton: Note that all the roots are negative numbers. So $P(x) = (x+r_1)...(x+r_n)$ where r_j are positive and their product is 1. By AM-GM,

$$P(2) = (2+r_1)...(2+r_n) \geq 3^n \ \square$$

Problem 5, Solution by David B. Rush: On the contrary, suppose that $P(x) = Q(x)R(x)$ with $0 < degQ < degP$. Since $P(10)$ is prime, w.l.o.g. let $Q(0) = \pm 1$. Let $\alpha_1, ..., \alpha_k$ be the roots of Q. We will proceed by using the following Theorem:

Theorem. Let $P(x) = a_n x^n + ... + a_0$ with $degP = n$. Also let $M = max_{0 \leq j \leq n}\left|\frac{a_j}{a_n}\right|$ and k denote the number of zero coefficients following a_n. Then any root of P has norm at most $1 + \sqrt[k+1]{M}$.

Note. For a proof of the Theorem above, write down $P(\alpha)$ as a sum and estimate the norm using the triangle inequality.

It follows from the Theorem above that

$$P(\alpha) = 0 \Rightarrow |\alpha| \leq 1 + \left(\frac{9}{2}\right)^{\frac{1}{k+1}} < 9,$$

where k is the number of zero coefficients of P following a_n. Then

$$|Q(10)| = |(10-\alpha_1)...(10-\alpha_k)| \geq (10-|\alpha_1|)...(10-|\alpha_k|) > 1$$

a contradiction. □

Problem 6, Solution by David B. Rush: Assume for the sake of contradiction that $P(x) = Q(x)R(x)$, where $0 < degQ < p$. Then w.l.o.g $Q(0) = \pm p$. Let the roots of Q be $\alpha_1, ..., \alpha_k$. So $\alpha_1...\alpha_k = \pm p$. One of the roots, call it α, will have norm at least the geometric average of the product which is $p^{\frac{1}{k}} \geq p^{\frac{1}{p-1}}$. However, $\alpha^m - \alpha = -p$ implies that $p \geq |\alpha|^p - |\alpha| = |\alpha|\left(|\alpha|^{p-1} - 1\right)$. Then $p \geq p^{\frac{1}{p-1}}(p-1)$ which implies that $3 > \left(1 + \frac{1}{p-1}\right)^{p-1} \geq p$. So only possibility left is that $p = 2$ in which case irreducibility is easily checked □

Problem 7, Solution by Gye Hyun Baek: Let $P_0 = Q_0 = \{z_1, ..., z_n\}$ and $P_1 = Q_1 = \{w_1, ..., w_m\}$. Then, we can write

$$\begin{aligned} P(x) &= \alpha \prod_{j=1}^{n} (x - z_j)^{e_j} = \alpha \prod_{j=1}^{m} (x - w_j)^{h_j} + 1, \\ Q(x) &= \beta \prod_{j=1}^{n} (x - z_j)^{g_j} = \beta \prod_{j=1}^{m} (x - w_j)^{h_j} + 1, \end{aligned}$$

where $e_j, f_j, g_j, h_j \in \mathbb{N}$, and $\sum e_j = \sum f_j = degP$, $\sum g_j = \sum h_j = degQ$. W.L.O.G. let $degP \geq degQ$. Notice that $(x - z_j)^{e_j - 1}$ and $(x - w_j)^{f_j - 1}$ divide $P'(x)$ so $degP' \geq \sum(e_j - 1) + \sum(f_j - 1) = 2degP - n - m$ which implies $n + m \geq degP + 1$. Let $R = P - Q$. Then $degR \leq degP$. However, z_j and w_j are roots of R so R has at least $n + m \geq degP$ roots. Hence, we conclude that $R \equiv 0$ and $P \equiv Q$ □

Problem 8, Solution by Brian Hamrick: First note that if $4|n$, then $P(x) = x^{4k} + 4 = (x^{2k} + 2x^k + 2)(x^{2k} - 2x^k + 2)$. Now, suppose that n is not a multiple of 4 and $P(x) = Q(x)R(x)$ with $0 < degQ < n$. All the roots of P have norm $\sqrt[n]{4}$. Since the product of the roots of Q, $4^{\frac{degQ}{n}}$, is an integer, $n|2degQ$ but $2degQ < 2n$, hence $n = 2degQ$ is even. However, n is not a multiple of 4, so $degQ$ is odd which means that Q has a real root which is a root of P as well but P doesn't have any real roots. We conclude that P is irreducible when n is not a multiple of 4 □