Polynomials

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- Roots of polynomials; Viète's formulas

  A polynomial of degree n has at most n roots, so if two polynomials $f(x)$, $g(x)$ of degree less than n agree at n values of x, then $f = g$ as polynomials. As a result, we have the Lagrange interpolation formula: given n distinct points $x_1, \ldots, x_n$ and corresponding values $y_1, \ldots, y_n$, the unique polynomial $f(x)$ of degree less than n such that $f(x_i) = y_i$ for each i is given by

$$f(x) = \sum_{i=1}^{n} y_i \frac{(x - x_1) \cdots (x - x_{i-1})(x - x_{i+1}) \cdots (x - x_n)}{(x_i - x_1) \cdots (x_i - x_{i-1})(x_i - x_{i+1}) \cdots (x_i - x_n)}.$$

1. Prove this.

2. (USAMO 1975) Suppose $P(x)$ is a polynomial of degree $n \geq 1$ such that $P(k) = k/(k+1)$ for $k = 0$, $1, \ldots, n$. Find $P(n+1)$. (You can make up many more problems of this form.)

The Fundamental Theorem of Algebra says that every polynomial can be written as a product of linear factors over the complex numbers. In particular, a polynomial of degree n has n complex roots, counting multiplicity. Viète's formulas give relations between the roots and the coefficients of a polynomial f: they are obtained by simply multiplying out the equation

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = a_n (x - r_1)(x - r_2) \cdots (x - r_n).$$

When f is *monic*, that is, $a_n = 1$, Viète's formulas state that a_{n-k} is $(-1)^k$ times the *kth elementary symmetric polynomial* s_k, the sum of all products of the r_i taken k at a time.

To analyze the *real* roots of a polynomial, the following results may be useful:

- the Intermediate Value Theorem: If a polynomial (or other continuous) function has opposite signs at the endpoints of an interval, then it has at least one root in the interval.
- Rolle's Theorem: If f is a polynomial, then between any two roots of f there is a root of f'.
- Descartes's Rule of Signs: If there are m sign changes in the series of coefficients of f, then the number of positive real roots (counting multiplicity) of f has the form $m - 2k$ for some nonnegative integer k. (To count negative real roots, apply this rule to $f(-x)$.)

And never forget about roots of unity: $x^n - 1 = (x-1)(x-\zeta)(x-\zeta^2) \cdots (x - \zeta^{n-1})$ where $\zeta = e^{2\pi i/n}$.

3. Let $P_1 P_2 \cdots P_n$ be a regular n-gon inscribed in the unit circle. Find the product of the lengths of all sides and diagonals of the polygon.

4. If $x < y < z$ are real numbers such that $x + y + z = 5$, $x^2 + y^2 + z^2 = 11$, and $x^3 + y^3 + z^3 = 26$, find y.

5. ("Kiran's root trick") Given a set of n numbers $x_1, \ldots, x_n$, define the *kth symmetric average* by $d_k = s_k / \binom{n}{k}$, where s_k is the kth elementary symmetric function of the x_i. Prove that if $x_1, \ldots, x_n$ are n (positive) reals, and $m < n$, then there exist m (positive) reals $x'_1, \ldots, x'_m$ such that the first m symmetric averages are equal for the x_i and the x'_i.

6. Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be $2n$ distinct complex numbers, where n is even. Form the $n \times n$ matrix whose (i,j)-entry is $a_i + b_j$. If the product of the elements in each row is 1, prove that the product of the elements in each column is -1.

7. (Bulgaria?) Find all polynomials f such that $f(x^2) = f(x) f(x-1)$.

8. Let p be a prime. Suppose that $P(x)$ is a nonconstant polynomial such that for each $0 \leq i \leq p-1$, $P(i)$ equals 0 or 1. Prove that the degree of P is at least $p-1$.

- Polynomials as generating functions

9. Prove the Vandermonde convolution formula: for any integer n and real numbers x and y,
$$\sum_{i=0}^{n} \binom{x}{i}\binom{y}{n-i} = \binom{x+y}{n}.$$

10. Let n be a positive integer. Evaluate $\binom{n}{0}^2 - \binom{n}{1}^2 + \binom{n}{2}^2 - \cdots + (-1)^n \binom{n}{n}^2$.

11. Is it possible to label a pair of dice in a nonstandard way so that for each integer n, the probability of obtaining a sum of n is the same as for a pair of standard dice? (The two dice must be fair in that each side is equally likely to come up, but the sides of the dice may be labeled with any positive integers, and the dice need not be labeled in the same way.)

12. (MOP 1999) Let n and $a_1, a_2, \ldots, a_m$ be positive integers. Define the function f as follows: for each integer k let $f(k)$ be the number of ordered m-tuples $(c_1, \ldots, c_m)$ of integers such that $c_1 + \cdots + c_m \equiv k \pmod{n}$ and $1 \leq c_i \leq a_i$ for each i. Show that f is constant if and only if n divides at least one of $a_1, \ldots, a_n$.

13. (IMO 1995: the "Nikolai Nikolov trick") Let p be an odd prime number. How many p-element subsets of $\{1, 2, \ldots, 2p\}$ are there, the sum of whose elements is divisible by p?

- Integer polynomials; divisibility and irreducibility

  Even though divisibility and irreducibility are defined in terms of integer polynomials, don't be afraid to use more general tools; in particular, studying the real or complex roots of the polynomials in question can often be useful.

  Useful fact: if f is a polynomial with integer coefficients, then for distinct integers a and b, $a-b$ divides $f(a) - f(b)$.

14. (IMO 1993) Let $n > 1$ be an integer. Prove that the polynomial $x^n + 5x^{n-1} + 3$ is irreducible.

15. (MOP 1999) Find all positive integers n such that the polynomial $x^n + 64$ is reducible.

16. Let f be a polynomial with integer coefficients, and starting with an integer a_1, define $a_2 = f(a_1)$, $a_3 = f(a_2)$, ... Suppose that $a_n = a_1$ for some integer $n > 1$. Show that $a_3 = a_1$.

17. Let $x_1, \ldots, x_n$ be distinct integers. Find all polynomials of the form $P(x) = (x - x_1) \cdots (x - x_n) \pm 1$ that are reducible.

- Assorted hard problems

18. (MOP 1998) Let $a_1, \ldots, a_n$ and $b_1, \ldots, b_n$ be two sequences of distinct numbers such that $a_i + b_j \neq 0$ for all i, j. Suppose c_{jk} are n^2 numbers such that, for fixed i and k, we have the relation
$$\sum_{j=1}^{n} \frac{c_{jk}}{a_i + b_j} = \begin{cases} 1, & \text{if } i = k \\ 0, & \text{otherwise.} \end{cases}$$
Show that the sum of all the c_{jk} is $a_1 + \cdots + a_n + b_1 + \cdots + b_n$.

19. (MOP 1999) Several points are given on a unit circle so that the product of the distances from any point on the circle to the given points does not exceed 2. Prove that the given points are the vertices of a regular polygon.

20. (Romania 1998) Let n be a positive integer. Prove that the polynomial $(x^2 + x)^{2^n} + 1$ is irreducible.

21. (IMO 2003) Find all pairs of integers $m > 2$, $n > 2$ such that there are infinitely many positive integers a for which $a^n + a^2 - 1$ divides $a^m + a - 1$.