New Zealand Mathematical Olympiad Committee

April Solutions

1. *Call a number* mystical *if it is possible to begin with the sum of its digits, and then by a sequence of operations where we replace $n = a + b$, by $m = a \times b$ (where a and b are positive integers) eventually obtain the number again. For example, 35 is mystical because, starting from $8 = 3 + 5$ we can write $8 = 6 + 2$, getting $12 = 6 \times 2$, and then $12 = 7 + 5$ getting $35 = 7 \times 5$. Is 2009 mystical?*

   Solution: Yes it is. Working backwards, we see that $2009 = 49 \times 41$. Then, $49 + 41 = 90 = 15 \times 6$. Then, $15 + 6 = 21 = 3 \times 7$. Finally, $3 + 7 = 10 = 10 \times 1$ and $10 + 1 = 11$ which is the sum of the digits of 2009. In proper forward order:

$$\begin{aligned} 11 &= 10 + 1 \\ 10 \times 1 &= 3 + 7 \\ 3 \times 7 &= 15 + 6 \\ 15 \times 6 &= 49 + 41 \\ 49 \times 41 &= 2009. \end{aligned}$$

2. *There are five paper triangles on a table – each is equilateral of side length 10 cm. Show that, no matter how they are oriented, it is possible to cover any one of them up using the other four, simply by sliding them (i.e. without any rotation). Does this remain true if the triangles are all congruent, but not necessarily equilateral?*

   Solution: Observe that when an equilteral triangle is divided into four smaller equilateral triangles by connecting the midpoints of its sides, that the incircle of the large triangle is equal to the circumcircle of the small triangle in the centre. In particular, the large triangle covers that circumcircle. Hence, by translation only the large triangle can cover any equilateral triangle of side length 5 cm. Given one of the triangles, divide it into four in the manner just described, and then use the other four large triangles to cover each of these.

   This result does not remain true for arbitrary triangles – just think of an isosceles triangle with a very large obtuse angle (i.e. a triangle which is almost a line segment.) Take one of these in horizontal orientation, and four others in vertical orientation – each of them can cover only a small fraction of the horizontal one, so they cannot cover the whole thing.

3. *A 20 kg block of cheese was on display at the local A and P show. Towards the end of the day, they started selling off bits of it. After each of the first ten customers bought a chunk the exhibitor announced "if everyone buys an amount equal to the average amount sold to each customer so far, there's just enough left for 10 more pieces". Could that announcement be correct (in all ten cases)? If so, how much cheese was left after the first 10 purchases?*

   Solution: As a working assumption, suppose that it could be correct. After k people have bought cheese then, the average amount sold to each customer so far would have to be $1/(10 + k)$ of the total, and $10/(10 + k)$ would have to be left over. Then, what proportion did customer k buy? The amount purchased before his/her purchase was $(k - 1)/(10 + (k - 1)) = (k - 1)/(9 + k)$, and as noted $k/(10 + k)$ had been purchased after. So he/she would have to buy:

$$\frac{k}{10 + k} - \frac{k - 1}{9 + k} = \frac{10}{(10 + k)(9 + k)}.$$

   Since this is positive for all k, the announcements are indeed consistent. After 10 purchases, $10/(10+10) = 1/2$ of the cheese, i.e. 10 kg remains.

4. *Can the product of two consecutive positive integers equal the product of two consecutive even positive integers?*

Solution: Suppose that this were possible. Then there would be positive integers n and k such that

$$n(n+1) = 2k(2k+2)$$

Hence,

$$4n^2 + 4n = 16k^2 + 16k$$

and thus

$$(2n+1)^2 - 1 = (4k+2)^2 - 4$$

or

$$(2n+1)^2 + 3 = (4k+2)^2$$

But, the only positive squares that differ by 3 are 1 and 4, and these do not give a positive solution (they would lead to $n = k = 0$.)

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