PROBABILITY

In everyday life, we would frequently encounter questions like "What is the chance that our team will win in today's football match?" or "Is it likely to rain in the afternoon?" In answering these questions, we need to determine the chance that an event occurs. In Mathematical language, we use the concept of probability to answer such questions.

1. What is Probability?

Consider a simple "tossing coin experiment". Given a fair coin, with head and tail on the two different sides, we repeat flipping it for many times. If we record the number of times that the head appears after n tosses, we would get results similar to that in table 1:

Number of tosses, n	4	10	20	30	50	100
Number of head appears, H	1	3	8	13	24	51
Ratio of H to n, $\frac{H}{n}$	$\frac{1}{4}=0.25$	$\frac{3}{10}=0.3$	$\frac{8}{20}=0.4$	$\frac{14}{30}=0.43$	$\frac{24}{50}=0.48$	$\frac{51}{100}=0.51$

Table 1: Result of a tossing coin experiment

From the table, it can be seen that the ratio becomes closer to 0.5 when the number of tosses, n, is large. There should be no surprise that the ratio is about 0.5, since the chance of getting a "head" and a "tail" should be the same. Moreover, as the number of tosses increases, the ratio is getting closer to 0.5.

Probability is about measuring the *relative frequency* that an **event** occurs. One way to think of it is to consider it as the ratio:

$$P(E)=\frac{\text{number of time of the event } E \text{ occurs}}{\text{number of repetitions of the experiment}}\text{, for large number of repetitions.}$$

However, this interpretation may be quite confusing. First, the result depends on experiment. Moreover, what is the meaning of "large number of repetitions"? As shown in the table 1 of the "throwing coin experiment", repeating the experiment for 100 times seems enough, yet it still cannot give our expected result 0.5. Due to randomness, the answer may vary. So, we need another definition.

Definition 1.1. (Probability)
If an event E can happen in m different ways out of totally n *equally likely* possible outcomes, then the probability of this event E is defined to be:

$$P(E)=\frac{m}{n}$$

- The readers are reminded that although we use the definition 1.1. here, many would prefer defining probability in terms of experiment, with the concept of **limit,** in advanced level.

- The ratios 0.25, 0.3, … 0.51 in table 1.1 are called **empirical probability**, since they are obtained purely from experiment.

Illustration. The probability that a number picked from {2,4,6,8} is even is 1, while the probability of throwing a head from the fair coin described in this section is 0.5.

Illustration. P(6 is obtained from throwing a fair dice) $=\frac{1}{6}$.

It should be note that in the definition 1.1, all the n possible outcomes must happen equally likely. Consider the example P(6 is obtained from throwing a fair dice), one may consider that there are totally two possible outcomes only, namely "6 is obtained" and "6 is not obtained". If the word "equally likely" is dropped from the definition, we may regard m = 1, n = 2, giving P(6 is obtained from throwing a fair dice) $=\frac{1}{2}$, which is of course not sensible.

2. Some Basic Properties of Probabilities

Theorem 2.1.

For any event E, we have the following two properties:

1. $0\le P(E)\le 1$
2. $P(\text{not } E) = 1 - P(E)$

Proof. For property 1, from the definition 1.1, it is clear that m, the number of ways of the event E can happen is smaller than n, the total number of possible outcomes. Therefore, $0\le P(E)\le 1$.

For property 2, if an event E can occur in m different ways out of a total of n equally likely possible outcomes, E cannot occur in $n - m$ different outcomes. For this reason, we have

$P(\text{not } E) = \frac{n-m}{n} = 1 - \frac{m}{n} = 1 - P(E)$.

Q.E.D.

- $P(E) = 1$ if E *always* occurs while $P(E) = 0$ if E *never* occurs.

If among two or more events, no two of them can happen at the same time, then they are said to be **mutually exclusive**.

Theorem 2.2. (Addition Principle)
If k events, E_1, E_2, …, E_k are *mutually exclusive* events, then

$$P(E_1 \text{ or } E_2 \text{ or ... or } E_k) = P(E_1) + P(E_2) + ... + P(E_k).$$

Proof. Suppose there are totally n different equally likely possible outcomes. Let m_1, m_2, …, m_k be the number of different possible outcomes that E_1, E_2, …, E_k happens respectively. Since the k events are mutually exclusive, the total number of ways that the event "E_1 or E_2 or ... or E_k" happen is $m_1 + m_2 + ... + m_k$. As a result,

$$P(E_1 \text{ or } E_2 \text{ or ... or } E_k) = \frac{m_1 + m_2 + ... + m_k}{n} = \frac{m_1}{n} + \frac{m_2}{n} + ... + \frac{m_k}{n} = P(E_1) + P(E_2) + ... + P(E_k).$$

Q.E.D.

- In terms of probability, k events, E_1, E_2, …, E_k are mutually exclusive if and only if for any $1 \le i < j \le k$, $P(E_i \text{ and } E_j) = 0$.

Illustration. Consider throwing a fair dice. By theorem 2.2, we know that P(greater than 1) = P(2) + P(3) + P(4) + P(5) + P(6) $= \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} + \frac{1}{6} = \frac{5}{6}$. Alternately, we may apply theorem 2.1 to get P(greater than 1) = 1 − P(not greater than 1) = $1 - \frac{1}{6} = \frac{5}{6}$.

Example 2.1.

When two dice are thrown, what are the probabilities of getting a sum of 2 and 4 respectively?

Solution.

We must be careful in determining the probabilities. There are totally 11 different outcomes for the sum of the numbers, which are 2, 3, …, 12. However, $P(2) = \frac{1}{36} \neq \frac{1}{11}$. It is because the 11 different outcomes are not equally likely. The correct way is to consider the 36 different outcomes, {1, 1}, {2, 1}, …{6, 1}, {2, 1}, …{6, 6}, where {m, n} represents the outcome with the first dice giving the number m and the second giving the number n. Clearly these outcomes are equally likely, so we have the probabilities $P(2) = P(1,1) = \frac{1}{36}$, $P(4) = P(1,3) + P(2,2) + P(3,1) = \frac{3}{36} = \frac{1}{12}$.

We will often encounter events which the occurrence of each would not affect that of the others. For example the events "Tomorrow is Sunday" and "It will rain in the afternoon" clearly would not affect the occurrence of each other. These events are said to be **independent events**. On the contrary, if the occurrence of one of the events would affect the outcomes of the others, we will call them **dependent events**.

For simplicity, we will denote the event "E_1 and E_2 and… and E_k" by the notation $E_1E_2...E_k$.

Theorem 2.3. (Multiplication Principle)
If k events, E_1, E_2, …, E_k are independent events, then

$$P(E_1E_2...E_k) = P(E_1)P(E_2)...P(E_k).$$

Proof. Suppose that for i = 1, 2, 3, … , k, the event E_i can occur in m_i ways out of a total of n_i equally likely possible outcomes. Take into consideration all the different cases, there are totally $n_1n_2n_3...n_k$ possible outcomes, each of them being equally likely. For the events E_1, E_2, …, E_k to occur together, there are totally $m_1m_2m_3...m_k$ different ways out of all the possible outcomes. Therefore, $P(E_1E_2...E_k) = \dfrac{m_1m_2...m_k}{n_1n_2...n_k} = \dfrac{m_1}{n_1}\cdot\dfrac{m_2}{n_2}\cdot...\cdot\dfrac{m_k}{n_k} = P(E_1)P(E_2)...P(E_k)$

Q.E.D.

Example 2.2.

Suppose that there are 3 boxes A, B, C, and each of them contains 1 black ball and 4 white balls. Now, one ball is drawn from each box randomly. What is the probability that all the balls drawn are white? How about the case that exactly two balls drawn are white?

Solution.

For each box, the probability that a white ball is drawn is $\frac{4}{5}$. Since drawing a white ball from each box are independent events, by theorem 2.3, P(all are white)=$\frac{4}{5}\cdot\frac{4}{5}\cdot\frac{4}{5} = \frac{64}{125}$.

If we require that exactly two balls drawn are white, then by Theorem 2.2, and Theorem 2.3, we have P(exactly two are white) = P(BWW) + P(WBW) + P(WWB) $= \frac{1}{5}\cdot\frac{4}{5}\cdot\frac{4}{5}+\frac{4}{5}\cdot\frac{1}{5}\cdot\frac{4}{5}+\frac{4}{5}\cdot\frac{4}{5}\cdot\frac{1}{5}=\frac{48}{125}$.

Example 2.3.

Three fair dice are thrown. Let the product of the three numbers thrown be k. What is P(k is odd)?

Solution.

If we do this problem by writing down all the possible outcomes and calculate the product in each case, it will be very lengthy. Instead, we may using the fact that k is odd if and only if all the three numbers thrown are odd. With this observation, we can easily compute that $P(k$ is odd) = P(1st number is odd)P(2nd number is odd)P(3rd number is odd) $=\frac{3}{6}\cdot\frac{3}{6}\cdot\frac{3}{6}=\frac{1}{8}$.

Exercise

1. What is the probability of getting a sum of 5 when throwing 3 fair dices?

2. When tossing a fair coin three times, what is the probability of getting 2 heads and 1 tails?

3. Conditional Probabilities

Before going to the next theorem, we first introduce a notation. If A, B are two events, then the **conditional probability**, $P(A|B)$, denotes the probability that A would occur, given that B occurs.

Illustration. Since there are totally 9 positive numbers smaller than 10, and among them, only 2, 3, 5 and 7 are prime, the probability P(n is prime | n is a positive number smaller than 10) $=\frac{4}{9}$.

Theorem 3.1.
Let E_1, E_2, …, E_k be k events, then

$$P(E_1E_2...E_k)=P(E_1)P(E_2\mid E_1)P(E_3\mid E_1E_2)...P(E_k\mid E_1E_2...E_{k-1})$$

In particular, if k = 2, and A, B are two events, then we have

$$P(AB)=P(A)P(B\mid A)$$

Proof. We will only prove the special case k = 2, since it will then be easy to induce the general result from this simple case.

Let A, B be two events, and AB can occur in m different ways out of totally n equally likely possible outcomes. Suppose there are l ways in which A, but not B occurs. Then totally A can occur in $m + l$ different ways, and $P(A) = \frac{m+l}{n}$. Now, if A is a given condition, then our consideration is restricted to $m + l$ outcomes only. Among these outcomes, m of them are with B occurs. So, $P(B \mid A) = \frac{m}{m+l}$. Thus, we have proved $P(A)P(B \mid A) = \frac{m+l}{n} \cdot \frac{m}{m+l} = \frac{m}{n} = P(AB)$.

Q.E.D.

One may notice that $P(B \mid A) = P(B)$ if A, B are independent events. Substitute this into Theorem 3.1, it would lead to the result of Theorem 2.3, the multiplication principle, that $P(AB) = P(A)P(B)$.

Example 3.1.

A fair coin is tossed twice. It is known that the head appears at least once. What is the probability that the head appear in both two tosses?

Solution.

We have to eliminate the case "TT" from all possible cases. Thus, "HT", "TH", "HH" are three equally likely, and all possible outcomes. Therefore, $P(\text{HH} \mid \text{H appear at least once}) = \frac{1}{3}$.

Example 3.2.

In a Mark Six game, 6 balls are drawn from a total of 49 balls, numbering from 1 to 49. One will get the first prize if the 6 numbers he picked are the 6 numbers drawn. How can we calculate the probability that a person can win the first prize with a single pick?

Solution.

Suppose 6 numbers have been picked. Let E_i be the event that the i-th number drawn is in the pick. Clearly, $P(E_1) = \frac{6}{49}$. Once E_1 occurs, there are 48 possible outcomes for the next number and 5 numbers left in the pick. Thus, $P(E_2 \mid E_1) = \frac{5}{48}$. By similar reasons, we have $P(E_3 \mid E_2 E_1) = \frac{4}{47}$, … , $P(E_6 \mid E_5 E_4 ... E_1) = \frac{1}{44}$. Finally, by Theorem 3.1, we get $P(E_1 E_2 ... E_6) = \frac{6}{49} \cdot \frac{5}{48} \cdot ... \cdot \frac{1}{44} = \frac{1}{13983816}$, which is the required probability.

Example 3.3.

There are 3 white balls and 1 black ball in a box. Two people play a game in the following way: Each person draws a ball from the box alternatively without replacement, until the black ball is drawn, and the one gets the black ball will be the winner. In order to maximize your winning chance, would you choose drawing the first ball?

Solution.

In answering such a question, splitting it into different cases would be helpful. Let A, B play the game, with A drawing the first ball. By observation, A can win in only two ways, either by drawing the black ball in his first or second turn. The probability that he can win in the first turn is $\frac{1}{4}$. If he wins it through another way, then both A and B must draw white balls in their first turns. The probability of this condition is $\frac{3}{4}\cdot\frac{2}{3}=\frac{1}{2}$. Once the two white balls are drawn, there are one white ball and one black ball left. Therefore, by Theorem 3.1, the probability that A can win it in his second turn is $\frac{1}{2}\cdot\frac{1}{2}=\frac{1}{4}$. By addition principle, $P(\text{A win})=\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$. Therefore, the winning probability is the same, independent of the order of the draw.

Example 3.4.

In a 40-student class, what is the probability that at least two students have the same birthday? We assume that there are 365 days in each year to simplify our questions.

Solution.

Number the 40 students by 1, 2, …,40, and for i = 2, 3, …, 40, let E_i be the event that the i-th students does not have same birthday with any one of the first (i − 1) students. It is easy to see that $P(E_2)=\frac{364}{365}$, $P(E_3\mid E_2)=\frac{363}{365}$, $P(E_4\mid E_3E_2)=\frac{362}{365}$ and so on. Hence, by Theorem 3.1, we can immediately get $P(\text{no two students have same birthday})=P(E_2E_3...E_{40})=\frac{364}{365}\cdot\frac{363}{365}\cdot...\cdot\frac{326}{365}\approx 0.109$. Therefore, the answer is 1 − 0.109 = 0.891.

To conclude this section, we will give a theorem that should be of no surprise to the readers.

Theorem 3.2. (Total Probability theorem)
Let A_1, A_2, …, A_k be k events such that for any possible outcomes, exactly one of them occurs. Then for any event B, we have

$$P(B)=P(A_1)P(B\mid A_1)+P(A_2)P(B\mid A_2)+...+P(A_k)P(B\mid A_k)$$

In particular, if k = 2, then for any events A, B,

$$P(B)=P(A)P(B\mid A)+P(\text{not }A)P(B\mid \text{not }A)$$

Proof. The condition implies that A_1, A_2, …, A_k are mutually exclusive, so are A_1B, A_2B, …, A_kB. Since B occurs if and only if the event (A_1B or A_2B or …or A_kB) occur, we have $P(B)=P(A_1B\text{ or }A_2B\text{ or ... or }A_kB)$. By Theorem 2.2, and Theorem 3.1, we conclude

$$P(B) = P(A_1B) + P(A_2B) + \ldots + P(A_kB) = P(A_1)P(B \mid A_1) + P(A_2)P(B \mid A_2) + \ldots + P(A_k)P(B \mid A_k)$$

Q.E.D.

- The condition "exactly one of them occurs" in theorem 3.2 actually means that the k events are mutually exclusive, with $P(A_1 \text{ or } A_2 \text{ or } \ldots \text{ or } A_k) = P(A_1) + P(A_2) + \ldots + P(A_k) = 1$.

Example 3.5.

Suppose there are fair coins and two types of biased coins in a bag. The numbers of them in the bag are 4, 5 and 6, and the probabilities of tossing a head using each types are 0.5, 0.3, and 0.6 respectively. If a coin is drawn randomly from the bag and tossed, what is the probability of tossing a head?

Solution.

If a coin is drawn randomly from the bag, the coin must be of one of the three types above. Hence, by theorem 3.2, the probability that a head is thrown is $\frac{4}{15}(0.5) + \frac{5}{15}(0.3) + \frac{6}{15}(0.6) = 0.473$.

Example 3.6.

Suppose two positive numbers are chosen randomly from 1 to 50. What is the probability that their difference is divisible by 3?

Solution.

In tackling this type of problems, it's useful to divide them into groups. In this case, we divide the 50 numbers into $A = \{1, 4, \ldots, 49\}$, $B = \{2, 5, \ldots, 50\}$ and $C = \{3, 6, \ldots, 48\}$. Note the difference of any two numbers chosen will be divisible by 3 if and only if they are in the same group. Consider that the two numbers are chosen one by one. The probability that the first number chosen is in group A, B, C are $\frac{17}{50}$, $\frac{17}{50}$, $\frac{16}{50}$ respectively. If the first number chosen is in group A, then the probability that the second is also from group A is $\frac{16}{49}$. The corresponding probabilities for group B and C are $\frac{16}{49}$ and $\frac{15}{49}$. Hence, by theorem 3.2, the required probability is $\frac{17}{50} \cdot \frac{16}{49} + \frac{17}{50} \cdot \frac{16}{49} + \frac{16}{50} \cdot \frac{15}{49} = \frac{8}{25}$.

Exercise

1. There are 5 pairs of identical socks in a bag. Suppose 2 socks are drawn, what is the probability that they form a good pair?

2. David and Ryan play a game with the following rules. There are 4 red balls, 1 white ball and 1 black ball in a box. They draw a ball from the box alternately, without replacement, until a white or black ball is drawn. If one draws a white ball, he will win. But if he draws a black ball, he will lose. What is the probability that David will win the game if he is the first drawer?

3. A and B play a game under the following rule: They throw a fair dice alternately, with A being the first thrower, until one can throw a "six". The one throw the "six" can win the game. What is the probability that B can win the game?

4. Consider 3 boxes. Box A contains 2 white balls and 4 red balls; Box B contains 8 white balls and 4 red balls; and Box C contains 1 white ball and 3 red balls. If 1 ball is selected from each box, and it is found that exactly 2 white balls were selected, what is the probability that the ball chosen from Box A was white?

5. This is a modification of the example 3.6. Suppose two positive numbers are chosen randomly from 1 to 50. What is the probability that their sum is divisible by 3?

4. Counting techniques

In solving probability problem, one important thing is to determine the number of possible outcomes correctly. In doing this, we must consider the following three questions.

1. Did we include all the possible outcomes?
2. Did we count any possible outcomes for more than once?
3. Did we include any unwanted outcomes in our consideration?

It is a good practice to ask ourselves these questions when we calculate the number of possible outcomes.

Here, we first introduce two terms.

Definition 4.1. (Factorial)

Let n be a non-negative integer. Then $n!$, read as n factorial, is defined by

$$n! = \begin{cases} n(n-1)(n-2)...(2)(1) & , n \geq 1 \\ 1 & , n = 0 \end{cases}$$

Factorial n! gives the number of **permutations** of n different objects. A permutation of n objects is an *ordered* combination of them.

Illustration. If we have 3 objects, A, B and C, then there are 3! = 6 permutations, namely, {A, B, C}, {A, C, B}, {B, A, C}, {B, C, A}, {C, A, B} and {C, B, A}.

In considering permutation, *order is important*. Otherwise {A, B, C} and {A, C, B} represent the same thing.

Definition 4.2. (C_r^n and P_r^n)

For any positive integer n, non-negative integer r, with $r \le n$, we define C_r^n and P_r^n by

$$C_r^n = \frac{n!}{r!(n-r)!} \qquad \text{and} \qquad P_r^n = \frac{n!}{(n-r)!}$$

The number C_r^n is equal to the number of ways of choosing r objects from n different objects, while P_r^n is equal to the number of permutations with r objects choosing from n different objects. The reader may verify them. For example, if two letters are chosen from A, B, C and D, there are totally $C_2^4 = 6$ combinations. ({A, B}, {A, C}, {A, D}, {B, C}, {B, D}, {C, D}) If order is important, that is, {A, B} and {B, A} are considered to be different, then there are $P_2^4 = 12$ different outcomes, including the six permutations listed above and those with their letters reversed.

Illustration. Consider the Mark Six game as in example 3.2 again. Among the numbers 1 to 49, we pick 6 of them to form a combination. As there are totally C_6^{49} equally likely 6-number combinations, the probability of first prize with one pick is $\frac{1}{C_6^{49}} = \frac{1}{13983816}$.

For the Theorem 4.1, a new notation |A| is used to represent the number of members of a group that satisfy the property A.

Illustration. Suppose there are 3 red balls, 5 green balls and 7 yellow balls in a box. Then these 15 balls is the “group” we consider. We can see |“green”| = 5, |“not red”| = 12, |“not red” and “not green”| = 7, and |“not red” or “not green”| = 15.

Again, for simplicity, the property “A_1 and A_2 and … and A_k” will be abbreviated as “$A_1\, A_2\ \dots\ A_k$”.

Theorem 4.1. (Inclusion-exclusion Principle)
For any group considered, let A_1, A_2, …, A_k be *k* properties. Then

$$|A_1 \text{ or } A_2 \text{ or...or } A_k| = \sum_{i=1}^{k} |A_i| - \sum_{i<j} |A_i A_j| + ... + (-1)^{k+1} |A_1 A_2 ... A_k|$$

Proof. Here we will give the proof for the case *k* = 2 only. The general case may be proved by similar fashion.

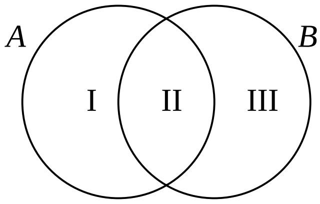

Figure 1: Venn diagram

Let we place all the members of the group with property *A*, *B* into the circle *A*, *B* respectively, with those possessing both properties placed in the common region. |*A* or *B*| = number of elements in region I, II, III = number of element in I, II + number of element in II, III – number of element in II = |*A*| + |*B*| – |*A* and *B*|.

Q.E.D.

- The diagram in figure 1 is **Venn diagram**. It is often used in mathematics to show relationships between **sets**.

In terms of probability, we get Theorem 4.2, which is the immediate result of Theorem 4.1.

Theorem 4.2. (Inclusion-exclusion Principle)
Let E_1, E_2, …, E_k be *k* events. Then

$$P(E_1 \text{ or } E_2 \text{ or...or } E_k) = \sum_{i=1}^{k} P(E_i) - \sum_{i<j} P(E_i E_j) + ... + (-1)^{k+1} P(E_1 E_2 ... E_k)$$

Example 4.1.

What is the number of positive numbers that is divisible by 3 or 5, and is smaller than 100?

Solution.

Define [*x*] to be the greatest integer not exceeding *x*. The number of positive numbers smaller than 100, which are divisible by 3 or 5, by theorem 4.1, is

|“divisible by 3”| + |“divisible by 5”| − |“divisible by 3 and 5”| $=[\frac{99}{3}]+[\frac{99}{5}]-[\frac{99}{3\times 5}]=33+19-6=46$.

Example 4.2.

This is a famous probability problem. Suppose there are *n* letters and *n* envelopes. We number them from 1 to *n* separately. If the letters are put in these envelopes randomly, what is the probability that for $1\le i\le n$, the letter numbered *i* is not in the envelope numbered *i*?

Solution.

For $1\le i\le n$, let E_i be the event that the letter numbered *i* is in the envelope number *i*. Obviously, $P(E_i)=\frac{(n-1)!}{n!}$. For any distinct *i*, *j*, since there are totally (n − 2)! possible outcomes such that E_iE_j happens, $P(E_iE_j)=\frac{(n-2)!}{n!}$. By similar reason, we have for any distinct *i*, *j*, *k*, $P(E_iE_jE_k)=\frac{(n-3)!}{n!}$ and so on. Hence, by theorem 4.2, we know that

$$P(E_1 \text{ or } E_2 \text{ or...or } E_k)=C_1^n\cdot\frac{(n-1)!}{n!}-C_2^n\cdot\frac{(n-2)!}{n!}+\ldots+(-1)^{k+1}C_n^n\cdot\frac{1}{n!}=\frac{1}{1!}-\frac{1}{2!}+\ldots+(-1)^{k+1}\cdot\frac{1}{n!}.$$

Therefore, the probability required is $1-P(E_1 \text{ or } E_2 \text{ or...or } E_k)=\frac{1}{2!}-\frac{1}{3!}\ldots+(-1)^k\cdot\frac{1}{n!}$.

Exercise

1. Find the number of positive integers from 1 to 1000 that is divisible by 3, 5 or 7.

2. Determine the number of triangles with (*x*, *y*) as vertices on *xy*-plane, where *x* and *y* are integers, with $1\le x, y\le 4$.

3. What is the number of positive factors of 64000?

5. Solutions to Selected Exercise

<u>Some Basic Properties of Probabilities</u>

1. $\frac{1}{36}$

2. $\frac{3}{8}$

Conditional Probabilities

1. $\frac{5}{9}$

2. We can solve this problem without using those conditional probability theorems stated in that section. Observe that the "chance" of drawing a white ball or a black ball is the same each time, the winning and losing probability of David should be the same. Since there is no drawing game, we conclude that the probability is $\frac{1}{2}$.

3. If B wins in the *i*-th round, then A cannot throw a six in his first *i* round, and B cannot throw a six in his first (*i*-1) round. By theorem 2.2 and then 3.1, *P*(B win) = *P*(B win in 1st round) + *P*(B win in 2nd round) + *P*(B win in 3rd round) + … = $\frac{5}{6}\cdot\frac{1}{6}+(\frac{5}{6})^3\cdot\frac{1}{6}+(\frac{5}{6})^5\cdot\frac{1}{6}+...=\frac{5}{11}$ by summation of this infinite geometric series.

4. $\frac{7}{11}$

5. $\frac{409}{1225}$

Counting techniques

1. 543

2. Three points can form the three vertices of a triangle if and only if they are not collinear. In the coordinate system we considered, if the three points are collinear, they can only be located on 10 lines, namely the four rows, four columns and the two diagonal. Hence, number of combinations of the three points $=C_3^{16}-10C_3^4=560-40=520$.

3. $64000=2^9\times 5^3$ in prime factorization. Hence, any positive factors of 64000 is of the form $2^i\times 5^j$, where $0\le i\le 9$, $0\le j\le 3$. Therefore, there are totally (9+1)(3+1) = 40 positive factors for 64000.