New Zealand Mathematical Olympiad Committee

2010 April Problems

These problems are intended to help students prepare for the 2010 camp selection problems, used to choose students to attend our week-long residential training camp in Christchurch in January.

In recent years the camp selection problems have been known as the "September Problems", as they were made available in September. This year we're going to trial moving the selection problems earlier in the year, releasing them in July and moving the due date to August. This will allow more time for pre-camp training, building up to Round One of the British Mathematical Olympiad in December.

The solutions will be posted in about two month's time, but can be obtained before then by email if you write to me with evidence that you've tried the problems seriously.

Good luck!

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1. *A coin has been placed at each vertex of a regular 2008-gon. These coins may be rearranged using the following move: two coins may be chosen, and each moved to an adjacent vertex, subject to the requirement that one must be moved clockwise and the other anti-clockwise. Decide whether, using this move, it is possible to rearrange the coins into*

   (a) 8 heaps of 251 coins each;

   (b) 251 heaps of 8 coins each.

   Solution: We show that (a) is possible, but (b) is not. To assist in doing so, we label the vertices consecutively as $v_1, v_2, \ldots, v_{2008}$.

   To achieve (a), we first of all move the coins on vertices v_1 to v_{251} into a single heap at vertex v_{251}, compensating for their movements by symmetric movements of the coins at v_{2008} to $v_{2008-250} = v_{1758}$ onto v_{1758}. This gives us the first two heaps of coins.

   We continue by moving the coins at vertices v_{252} to v_{502} into a single heap at v_{502}, again compensating for their movements with symmetric movements of the coins at v_{1757} to $v_{1757-250} = v_{1507}$ into a single heap at v_{1507}. Repeat this procedure two more times results in eight heaps of 251 coins each, located at vertices v_{251}, v_{502}, v_{753}, v_{1004}, v_{1005}, v_{1256}, v_{1507} and v_{1558}.

   To show that (b) is not possible, we assign value i to each coin currently placed at vertex v_i. We then let S be the sum of the values of all the coins, and we update the value of

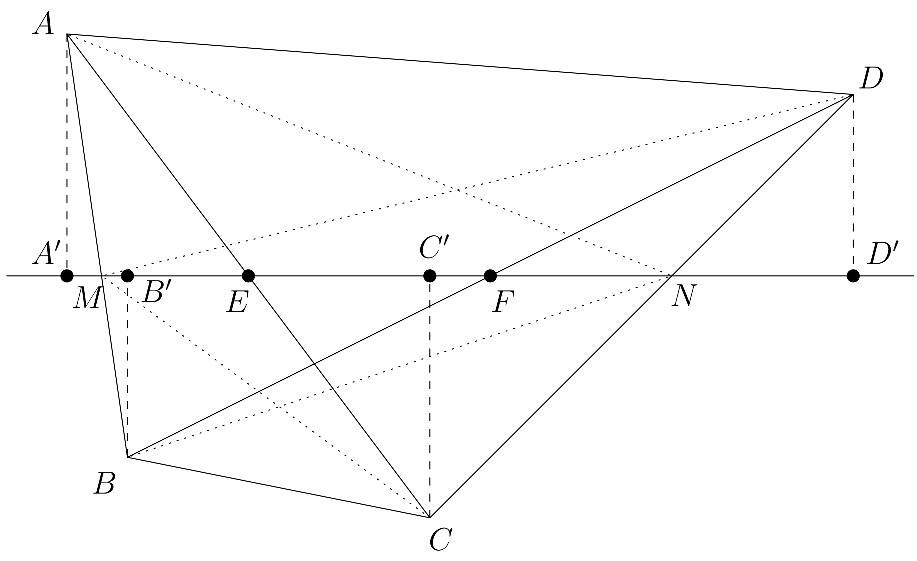

Figure 1: Diagram for Problem 2.

S after each move. We observe that after a move where a coin is not moved from v_1 to v_{2008} or vice versa, the sum of S stays constant, since one coin increases in value by 1, and another decreases by 1. If a coin is moved from v_1 to v_{2008}, then the change in S will be $(2008 - 1) + 1 = 2008$, and if a coin is moved from v_{2008} to v_1 then the change in S will be $-(2008 - 1) - 1 = -2008$. Hence we see that after every move, the remainder of S upon division by 2008 will stay constant.

The initial value of S is

$$1 + 2 + 3 + \cdots + 2007 + 2008 = \frac{2009 \cdot 2008}{2} = 1004 \cdot 2009.$$

If it were possible to arrange the coins into 251 piles of eight coins each, located at the vertices $v_{i_1}, v_{i_2}, \ldots, v_{i_{251}}$, then we would have

$$S = 8(i_1 + i_2 + \cdots + i_{251}) = 1004 \cdot 2009 + 2008k = 4(251 \cdot 2009 + 502k),$$

for some integer k. But this is impossible, because the lefthand side is divisible by 8, while the righthand side is not. □

2. *Let $ABCD$ be a convex quadrilateral that is not a parallelogram. The straight line passing through the midpoints of the diagonals of $ABCD$ intersects the sides AB and CD in the points M and N respectively. Prove that the triangles ABN and CDM have the same area.*

   Solution: Let E, F be the midpoints of the diagonals AC and BD, and let A', B', C' and D' be the feet of the perpendiculars from A, B, C, D to the straight line EF (see Figure 1). Since $|AE| = |CE|$ and $\angle AEA' = \angle CEC'$, the right triangle $AA'E$ is congruent to $CC'E$. Therefore $|AA'| = |CC'|$, and by the same reasoning, $|BB'| = |DD'|$.

   Now, triangles ANM and CMN have the same area, because they have the same base and altitude; and similarly, triangles BNM and DMN have the same area, because they have the same base and altitude too. Putting these triangles together we find that ANB and CMD have equal areas. $\square$

3. *Find all positive integers m and n such that $6^m + 2^n + 2$ is the square of an integer.*

Solution: Suppose that $6^m + 2^n + 2 = z^2$. If m and n are both at least 2 then $z^2 \equiv 2 \bmod 4$, which is impossible because squares are congruent to 0 or 1 mod 4. Hence exactly one of m and n must be one.

Suppose first that $m = 1$. Then $2^n + 8 = z^2$. For $n = 1$ and $n = 2$ we get $2^n + 8 = 10$ and $2^n + 8 = 12$ respectively, which are not squares, so we may assume $n \geq 3$. Then $z^2 = 8(2^{n-3} + 1)$, and for the right hand side to be square the bracketed must be even. This happens only for $n = 3$, and this gives the solution $6^1 + 2^3 + 2 = 16 = 4^2$.

Suppose now that $n = 1$. Then $z^2 = 6^m + 4$, and mod 7 we have $z^2 \equiv (-1)^m + 4 \equiv 3, 5$. This is impossible, because a square is congruent to 0, 1, 2 or 4 mod 7, so we get no solutions in this case.

Answer: The only solution is $m = 1$, $n = 3$. $\square$

4. *Determine all functions $f : \mathbb{N} \to \mathbb{N}$ satisfying*

$$\frac{f(x+y) + f(x)}{2x + f(y)} = \frac{2y + f(x)}{f(x+y) + f(y)},$$

for all $x, y \in \mathbb{N}$.

Solution: Letting $y = x$ we get

$$(f(2x) + f(x))^2 = (2x + f(x))^2.$$

Since $f(2x) + f(x)$ and $2x + f(x)$ are both positive we may take the square root of both sides to get

$$f(2x) + f(x) = 2x + f(x),$$

or $f(2x) = 2x$. The function f therefore satisfies $f(x) = x$ for x even.

Let x be odd, and let $y = 1$. Then $x + 1$ is even, so $f(x + 1) = x + 1$, and

$$\begin{aligned}(x + 1 + f(x))(x + 1 + f(1)) &= (2 + f(x))(2x + f(1)) \\ (x + 1)^2 + (x + 1)(f(x) + f(1)) + f(x)f(1) &= 4x + 2f(1) + 2xf(x) + f(x)f(1) \\ (x - 1)^2 + (x - 1)f(1) &= (x - 1)f(x).\end{aligned}$$

When $x > 1$ we may divide by $x - 1$ and to get

$$f(x) = x + f(1) - 1, \tag{1}$$

for x odd, $x \geq 3$.

Now let x be even, and let $y = 1$. Then $f(x) = x$, and $x + 1$ is odd and at least 3, so $f(x + 1) = x + f(1)$. Hence

$$\begin{aligned}\frac{2x + f(1)}{2x + f(1)} &= \frac{2 + x}{x + 2f(1)} \\ x + 2f(1) &= 2 + x,\end{aligned}$$

so $f(1) = 1$. Combining this with (1) and the result for x even we get $f(x) = x$ for all x. It remains to check that our solution $f(x) = x$ actually satisfies the given functional equation. But

$$\frac{f(x+y)+f(x)}{2x+f(y)} = \frac{2x+y}{2x+y} = 1 = \frac{2y+x}{2y+x} = \frac{2y+f(x)}{f(x+y)+f(y)},$$

completing the proof. $\square$

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