The Probabilistic Method

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1 Warm-up

1. (Russia 1996/4.) In the Duma there are 1600 delegates, who have formed 16000 committees of 80 persons each. Prove that one can find two committees having at least four common members.

   Solution: Simply select a random 2-set of committees. Calculate expected number of people in both committees, and use convexity of $\sum x_i^2 \geq n(\text{avg})^2$.

2. (Iran Team Selection Test 2008/6.) Suppose 799 teams participate in a tournament in which every pair of teams plays against each other exactly once. Prove that there exist two disjoint groups A and B of 7 teams each such that every team from A defeated every team from B.

   Solution: Sample A as a random 7-set. Let X be the number of guys that are totally dominated by A. Letting d_v^- denote the in-degree of v, we have $\mathbb{E}[X] = \sum_v \binom{d_v^-}{7} / \binom{799}{7}$. But $\sum_v d_v^- = \binom{799}{2}$, which means that the average in-degree is exactly 399. By convexity, $\mathbb{E}[X] \geq 799 \cdot \binom{399}{7} / \binom{799}{7} \approx 800 \cdot (1/2)^7 \approx 6.25$, which is enough since X is an integer. Pick 7 teams B from the dominated group.

3. Let G be a graph in which all vertices have nonzero degree. Prove that its vertices can be partitioned into two sets $V_1 \cup V_2$ such that the number of edges going between the V_i is at least $\frac{n}{2} + \frac{m}{6}$. Is this tight?

   Solution: Analyze the greedy partitioning algorithm, which puts each new vertex onto the side which maximizes the number of crossing edges to already-placed vertices. Observe that we gain $\frac{1}{2}$ per vertex which had odd back-degree. But if we take a random ordering of the vertices, then the expected number of vertices with odd back-degree is at least $m/3$, with sharpness on vertices of degree 2 (since their back-degree is 0, 1, or 2).

4. **760.** В лагерь приехало несколько пионеров, каждый из них имеет от 50 до 100 знакомых среди остальных. Докажите, что пионерам можно выдать пилотки, покрашенные в 1331 цвет так, чтобы у знакомых каждого пионера были пилотки хотя бы 20 различных цветов. (*Д.Карпов*)

2 Olympiad problems

1. (MOP 2007/7/1.) In a 100×100 array, each of the numbers $1, 2, \ldots, 100$ appears exactly 100 times. Show that there is a row or a column in the array with at least 10 distinct numbers.

   Solution: Let $n = 100$. Choose a random row or column ($2n$ choices). Let X be the number of distinct entries in it. Now $X = \sum I_i$, where each I_i is the indicator variable of i appearing (possibly more than once) in our random row or column. Clearly, each $\mathbb{E}[I_i] = \mathbb{P}[I_i \geq 1]$. To lower-bound this, observe that the worst-case is if all n appearances of i are in some $\sqrt{n} \times \sqrt{n}$ submatrix, which gives $\mathbb{P}[I_i \geq 1] \geq 2\sqrt{n}/(2n) = 1/\sqrt{n}$. Hence by linearity, $\mathbb{E}[X] \geq \sqrt{n}$.

2. (Russia, 1999.) In a class, each boy is friends with at least one girl. Show that there exists a group of at least half of the students, such that each boy in the group is friends with an odd number of the girls in the group.

   Solution: Choose girls independently with probability $1/2$, and then let the set of boys be all of those who have an odd number of friends in the girl group. Let X be the number of boys and girls selected, and break this into the sum of indicators. For each girl, obviously the indicator adds $1/2$ to the sum. For each boy, the probability that he joins is precisely the probability that $\text{Bin}\,[k, 1/2]$ is odd, where k was the number of girls he knew. To see that this probability is $1/2$, note that it is the parity of the sum of k independent coin flips. In particular, the final flip independently flips or retains the final parity, hence odd with probability $1/2$.

3. (IMO Shortlist 1999/C4.) Let A be any set of n residues mod n^2. Show that there is a set B of n residues mod n^2 such that at least half of the residues mod n^2 can be written as $a + b$ with $a \in A$ and $b \in B$.

   Solution: Make n independent uniformly random choices from the n^2 residues, and collect them into a set B. Note that since we use independence, this final set may have size $< n$. But if we still have $A + B$ occupying at least half of the residues, then this is okay (we could arbitrarily augment B to have the full size n).

   Let X be the number of residues achievable as $a + b$. For each potential residue i, there are exactly n ways to choose some b for which $A + b \ni i$, since $|A| = n$. Therefore, the probability that a given residue i appears in $A + B$ is precisely $1 - \left(1 - \frac{n}{n^2}\right)^n$. Then $\mathbb{E}\,[X]$ is exactly n^2 times that, because there are n^2 total residues. Hence it suffices to show that $1 - \left(1 - \frac{n}{n^2}\right)^n \geq 1/2$. But this follows from the bound $1 - \frac{1}{n} \leq e^{-1/n}$, using $e \approx 2.718$.

4. (MOP 2010, harder variation.) Let G be a graph with average degree d. Prove that for every $k \leq d$, there is a K_{k+1}-free induced subgraph on at least $\frac{kn}{d+1}$ vertices.

   Solution: Randomly permute the vertices. Use the greedy algorithm, taking each vertex if it can be added without making any K_{k+1}. Observe that we will actually take every vertex v with the property that the permutation induced on $\{v\} \cup N(v)$ has v in position $1 \ldots k$. (We might also take more.) This is because v would only have degree at most $k - 1$ back to the previously selected guys, making at most a K_k. Now the expected size of the selected set is at least

$$\sum_v \frac{k}{d_v + 1} \geq n \cdot \frac{k}{d + 1}.$$

5. (Russia 2006, final problem.) A group of pioneers has arrived to summer camp. Each pioneer has at least 50 and at most 100 friends among the others. Prove that one can distribute field caps of 1331 colors among the pioneers so that the friends of each pioneer have caps of at least 20 colors.

   Solution: We prove a much stronger bound. Let $C = 49$ be the total number of colors, and give each person an independent, uniformly random color. We will apply the Lovász Local Lemma. For each vertex, let B_v be the event that $N(v)$ receives 19 or fewer colors. We have:

$$\mathbb{P}\,[B_v] \leq \binom{C}{19} \left(\frac{19}{C}\right)^{50}.$$

   Now we build the dependency graph. Consider all vertices in $N(v)$, together with those adjacent to $N(v)$. Connect each vertex to v in the dependency graph. It is clear that all other vertices do not need to be connected to v. Hence the dependency is below 10^4. We therefore have a solution when

$$e \cdot \binom{C}{19} \left(\frac{19}{C}\right)^{50} \cdot 10^4 < 1,$$

Note in particular that $\binom{C}{19} \leq \left(\frac{eC}{19}\right)^{19}$, so it suffices to have:

$$\begin{aligned} e\left(\frac{eC}{19}^{19}\right)\left(\frac{19}{C}\right)^{50} 10^4 &< 1 \\ e^{20} 19^{31} 10^4 &< C^{31} \\ 48.75 \approx 19 e^{\frac{20}{31}} 10^{\frac{4}{31}} &< C\,. \end{aligned}$$