MATHEMATICAL OLYMPIAD SUMMER PROGRAM 1999

ROTATIONS AND VECTORS

ZVEZDELINA STANKOVA–FRENKEL
UC BERKELEY

1. (G260) A point M and a circle k are given in the plane. If $ABCD$ is an arbitrary square inscribed in k, prove that the sum $MA^4 + MB^4 + MC^4 + MD^4$ is independent of the positioning of the square. Replace now the square by a regular n-gon $A_1A_2...A_n$. Let $S_m = \sum_i MA_i^m$. For what natural m is S_m independent of the position of the n-gon (still inscribed in k)?
2. (G262) $\triangle ABC$ is rotated to $\triangle A'B'C'$ around its circumcenter O by angle α. Let A_1, B_1 and C_1 be the intersection points of lines BC and $B'C'$, CA and $C'A'$, and AB and $A'B'$, respectively. Prove that $\triangle A_1B_1C_1$ and $\triangle ABC$ are similar, and find the ratio of their sides.
3. (G263) The quadrilateral $ABCD$ is inscribed in a circle k with center O, and the quadrilateral $A'B'C'D'$ is obtained by rotating $ABCD$ around O by some angle. Let A_1, B_1, C_1, D_1 be the intersection points of the lines $A'B'$ and AB, $B'C'$ and BC, $C'D'$ and CD, and $D'A'$ and DA. Prove that $A_1B_1C_1D_1$ is a parallelogram.
4. (G264) In quadrilateral $ABCD$, the diagonals intersect in point O. Quadrilateral $A'B'C'D'$ is obtained by rotating $ABCD$ around O by some angle. Let A_1, B_1, C_1, D_1 be the intersection points of the lines $A'B'$ and AB, $B'C'$ and BC, $C'D'$ and CD, and $D'A'$ and DA. Prove that $A_1B_1C_1D_1$ is cyclic if and only $AC \perp BD$.
5. The composition of two rotations $\rho_1(O_1, \alpha_1)$ and $\rho_2(O_2, \alpha_2)$ about different centers O_1 and O_2 is:
   (a) rotation if $\alpha_1 + \alpha_2 \neq k\pi$ $(k \in \mathbb{Z})$;
   (b) translation if $\alpha_1 + \alpha_2 = 2k\pi$ $(k \in \mathbb{Z})$;
   (c) central symmetry if $\alpha_1 + \alpha_2 = (2k+1)\pi$ $(k \in \mathbb{Z})$.
6. (G265) On the sides of a convex quadrilateral draw externally squares. Prove that the quadrilateral with vertices the centers of the squares has equal perpendicular diagonals.
7. (G266) Given two equally oriented equilateral triangles AB_1C_1 and AB_2C_2 with centers O_1 and O_2, respectively, let M be the midpoint of B_1C_2. Prove that $\triangle O_1MB_2 \sim \triangle O_2MC_1$.
8. (G269) A hexagon $ABCDEF$ is inscribed in a circle of radius r so that $AB = CD = EF = r$. Let the midpoints of BC, DE, FA be L, M, N respectively. Prove that $\triangle LMN$ is equilateral.

9. (Napoleon) If three equilateral triangles ABC_1, BCA_1 and CAB_1 are constructed off the sides of $\triangle ABC$, show that the centers of these equilateral triangle form another equilateral triangle. Prove also that AA_1, BB_1 and CC_1 are concurrent and have same lengths. Can you identify the medicenter of $\triangle O_1O_2O_3$ with some distinguished point of $\triangle ABC$?

10. (G270) Points $A_1, A_2, \dots, A_n$ ($n \geq 3$) lie on a circle with center O. Drop a perpendicular through the centroid of every $n-2$ of these points towards the line determined by the remaining two points. Prove that the $\binom{n}{2}$ thus drawn lines are all concurrent.

11. (G271) Points $A_1, A_2, \dots, A_n$ ($n \geq 2$) lie on a sphere. Drop a perpendicular through the centroid of every $n-1$ of these points towards the plane, tangent to the sphere at the remaining n-th point. Prove that the n drawn lines are all concurrent.

12. (Kazanluk'95 X) Given $\triangle ABC$ with sides $AB = 22$, $BC = 19$, $CA = 13$.
    (a) If M is the medicenter of $\triangle ABC$, prove that $AM^2 + CM^2 = BM^2$.
    (b) Find the locus of points P in the plane such that $AP^2 + CP^2 = BP^2$.
    (c) Find the minimum and maximum of BP if $AP^2 + CP^2 = BP^2$.

13. (G272) Given $\triangle ABC$, find the locus of points M in the plane such that $MA^2 + MB^2 = MC^2$.

14. (G273) Given tetrahedron $ABCD$, find the locus of points M in such that $MA^2 + MB^2 + MC^2 = MD^2$. How about $MA^2 + MB^2 = MC^2 + MD^2$?

15. (a) (Leibnitz) Let M be the medicenter of $\triangle ABC$, Q be an arbtrary point in the plane. Prove that

$$QA^2 + QB^2 + QC^2 = 3QM^2 + MA^2 + MB^2 + MC^2.$$

    (b) (Stuard) Prove that if point D lies on the side BC of $\triangle ABC$, and $BC = a$, $CA = b$, $AB = c$, $BD = m$, $CD = n$, $AD = d$, then $d^2a = b^2m + c^2n - amn$.

16. (Kazanluk'97 X) Point F on the base AB of trapezoid $ABCD$ is such that $DF = CF$. Let E be the intersection point of the diagonals AC and BD, and O_1 and O_2 be the circumcenters of $\triangle ADF$ and $\triangle BCF$, respectively. Prove that the lines FE and O_1O_2 are perpendicular.

17. (UNESCO'95) Given a fixed segment AB and a constant $k > 0$, find the locus of points C in the plane such that in $\triangle ABC$ the ratio of some side to the altitude dropped to this side equals k.

18. (UNESCO'95) We are given $\triangle ABC$ in the plane. A rectangle $MNPQ$ is called *circumscribed* around $\triangle ABC$ if on each side of the reactangle there is at least one vertex of the triangle. Find the locus of all centers O of the rectangles $MNPQ$ circumscribed around $\triangle ABC$.

18. (Sylvester's theorem) A finite set of points in the plane has the property that the line through any two of them passes through a third one. Prove that all of the points are collinear. (Note: this fails in the *complex* projective plane, e.g. for the 3-torsion points of an elliptic curve.)

19. (IMO 1969/5) Given n points in the plane, no three collinear, prove that the number of convex quadrilaterals with vertices among the n points is at least $\binom{n-3}{2}$. (In fact, it is easy to prove the much better lower bound $\frac{1}{n-4}\binom{n}{5}$. Can you improve this?)

20. (Erdös-Szekeres) Prove that for any n, there exists N such that among any N points in the plane, no three collinear, there exist n which are the vertices of a convex n-gon. (There are two different reductions of this statement to Ramsey's theorem.)

21. (IMO 1973/1) Let $P_1, \ldots, P_{2n+1}$ be points on a semicircle centered at O. Prove that the sum of the vectors $OP_1, \ldots, OP_{2n+1}$ has length not less than 1.

22. Given 111 unit vectors in the plane whose sum is 0, prove that there exist 55 of them whose sum has length at most 1.

23. Given n unit vectors in the plane whose sum is 0, prove that there exists a permutation of the vectors such that the sum of the first k vectors has length at most $\cancel{2}$ $\sqrt{2}$ for $k = 1, \ldots, n$. Can you improve this constant?

24. (Austrian-Polish, 1995) Consider the cube consisting of points (x, y, z) with $|x|, |y|, |z| \leq 1$. Let $V_1, \ldots, V_{95}$ be points in the cube, and let v_i denote the vector from $(0, 0, 0)$ to V_i. Prove that there exist $s_i \in \{+1, -1\}$ such that $s_1 v_1 + \cdots + s_{95} v_{95}$ has length not greater than $\sqrt{48}$. Can you improve this constant? (In the 1995-1996 packet, I got it down to $\sqrt{12}$.)

25. Given three points chosen uniformly at random on a circle, what is the probability that the triangle they form is acute?

26. (St. Petersburg, 1997) Given $2n + 1$ lines in the plane, prove that there are at most $n(n + 1)(2n + 1)/6$ acute triangles with sides on the lines.

27. (IMO 1970/6) Given 100 points in the plane, no three collinear, prove that at most 70% of the triangles with vertices among the given points are acute. (Can you improve this?)

28. (Putnam, 1992) Four points are chosen uniformly at random on the surface of a sphere. What is the probability that the center of the sphere lies inside the tetrahedron whose vertices are at the four points?

29. (Asian Pacific, 1999) Given $2n + 1$ points in the plane, no four concyclic, prove that the number of circles through 3 of the points containing exactly $n - 1$ of the other points has the same parity as n.

30. (Poland, 1997) Given $n \geq 2$ points on a unit circle, show that at most $n^2/3$ of the segments with endpoints among the given points have length greater than $\sqrt{2}$. (This should have been included as a "You call this graph theory?!" problem on Tuesday's handout.)

31. Find the smallest real number r such that a unit square can be covered by three disks of radius r.

32. (Putnam, 1994) Prove that the points of an isosceles triangle of side length 1 cannot be colored in four colors such that no two points at distance at least $2 - \sqrt{2}$ from each other receive the same color.

33. (Putnam, 1997) Find the least diameter of a dissection of a 3–4–5 triangle into four parts. (The diameter of a dissection is the least upper bound of the distances between pairs of points belonging to the same part.)

34. Prove that a collection of squares of total area at most 1 can be fit into a square of area 4. (The optimal result in this direction is a theorem of Richard Stong. I think the minimum area of the square is 2, so as to accommodate two squares of area 1/2, but I'm not certain.)