New Zealand Mathematical Olympiad Committee

May Solutions

1. *Alice and Bob each chose a positive integer, and told it to Eve. Eve wrote the sum of these numbers on one card, the product on another, hid one card and showed the other to Alice and Bob. Alice looked at the number on the card, which was 2010, and declared that she was unable to determine Bob's number. Knowing this, Bob said he was also unable to determine Alice's number.*

   What was Bob's number? Can you determine Alice's number?

   Solution: Let a and b be Alice's and Bob's numbers respectively. Then a must divide 2010, otherwise Alice would have been able to recover Bob's number as $b = 2010 - a$. In addition, a cannot be 2010, because in this case the condition $a > 0$ implies a must be 1. So a divides 2010, and is at most 1005.

   The exact same argument applies to b, so $b \leq 1005$. However, we also know that Bob was unable to determine a, knowing that $a \leq 1005$. This tells us that $b \geq 1005$, as otherwise Bob would know that $a = 2010 - b$ is impossible. Therefore Bob's number must be 1005; we are unable to determine Alice's number, as it could still be either 2 or 1005.

2. *The increasing sequence 1, 3, 4, 9, 10, 12, 13,... consists of all those positive integers which are powers of 3 or sums of distinct powers of 3. Of the first googol (10^{100}) terms in the sequence, how many are powers of 3?*

   Solution: A number belongs to the given sequence if and only if its representation in base three uses the digits 0 and 1 only. For example, $37 = 27 + 9 + 1 = 1 \times 3^3 + 1 \times 3^2 + 0 \times 3^1 + 1 \times 3^0$ belongs to the sequence, because its representation in base three is 1101. Consequently, the nth term of the sequence is found by re-interpreting the binary expansion of n as a number in base three; thus, for example, 37 is the 13th term of the sequence, because 13 is written 1101 in base two. (Check that the first seven terms 1, 3, 4, 9, 10, 12, 13, are written as 1, 10, 11, 100, 101, 110, 111 in base three.)

   To count the number of powers of three among the first googol terms we therefore need to count the number of powers of two less than or equal to a googol. This is most easily done by taking the log to base two:

   $$\log_2 10^{100} = 100 \log_2 10 = 100 \frac{\log_{10} 10}{\log_{10} 2} = \frac{100}{\log_{10} 2} \approx 332.19.$$

   The largest power of 2 less than or equal to a googol is therefore 2^{332}, so, remembering to count $1 = 3^0$, there are 333 powers of three in the first googol terms.

3. *Quadrilateral $ABCD$ is circumscribed about a circle, which is tangent to the sides AB, BC, CD and DA at K, L, M, N respectively. Let S be the point of intersection of the line segments KM and LN, and suppose that the quadrilateral $SKBL$ is cyclic. Prove that the quadrilateral $SNDM$ must be too.*

   Solution: Refer to Figure 1. A quadrilateral is cyclic if and only if its opposite angles are supplementary (add to 180°), so we must show that $\angle MDN$ and $\angle MSN$ are supplementary. We will use the fact that the angle subtended on a circle by a chord is equal to the angle between the chord and a tangent to the circle at its endpoint. This tells us that $\angle KML = \angle KLB = \angle BKL = \theta$, and $\angle MLN = \angle MND = \angle DMN = \phi$. Now $\angle LSK$ is exterior to triangle MLS, so $\angle LSK = \angle LMS + \angle SLM = \theta + \phi$; but in addition we have $\angle LSK = 180° - \angle LBK = 2\theta$, since $SKBL$ is cyclic and LBK is isosceles. It follows that $\theta = \phi$. Therefore $\angle MDN = \angle LBK$, and $\angle MDN + \angle MSN = \angle LBK + \angle LSK = 180°$, so $SNDM$ is cyclic.

4. *Fifteen rooks are placed on a 15×15 chess board in such a way that no rook is attacking any other. Each rook then makes a single knight's move. Prove that after this has taken place there must be a pair of rooks that are attacking each other.*

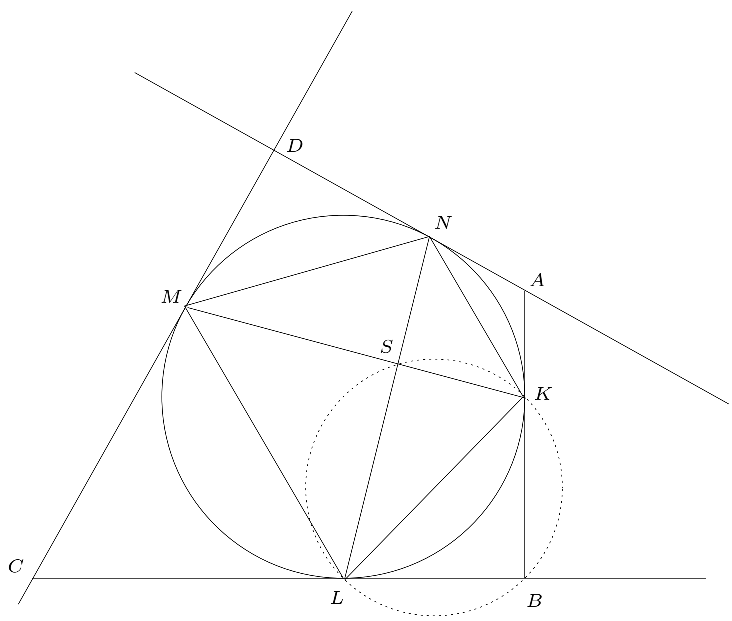

Figure 1: Diagram for Problem 3. The diagram was drawn by beginning with B, K and L, and choosing S to lie on the circumcircle of BKL. M and N were then found by extending KS and LS, and finally the remaining vertices A, C D were found by drawing the tangents at M and N. This approach gives a diagram in which $SKBL$ is cyclic, which we are unlikely to get if we start with $ABCD$ instead.

Solution: The rooks are non-attacking if there is exactly one in each row and each column of the board. Suppose that the ith rook is in column x_i, row y_i. Then each of the sums $\sum_{i=1}^{15} x_i$ and $\sum_{i=1}^{15} y_i$ must equal $\sum_{i=1}^{15} i$, so

$$\sum_{i=1}^{15}(x_i + y_i) = 2\sum_{i=1}^{15} i = 15 \times 16,$$

an even number. We will show that there must be a pair of attacking rooks after the knight's moves by showing that this sum must become odd.

A knight's move consists of two steps in one direction followed by a single step at right angles. Suppose that the ith rook's position after its move is (x'_i, y'_i). Then either x'_i differs from x_i by ± 2 and y'_i differs from y_i by ± 1, or else x'_i differs from x_i by ± 1 and y'_i differs from y_i by ± 2; in either case $x'_i + y'_i$ differs from $x_i + y_i$ by an *odd* number. Since there are 15 rooks the sum $\sum_{i=1}^{15}(x'_i + y'_i)$ differs from $\sum_{i=1}^{15}(x_i + y_i)$ by an odd number also, and in particular, is not equal to 15×16. It follows that there must either be two rooks in the same row, or two rooks in the same column, and therefore a pair of mutually attacking rooks.

Alternate solution. Here's an alternate solution that uses essentially the same idea, but perhaps doesn't hide where it came from quite so much. Suppose that the rooks are non-attacking both before and after the knight's moves. Paint the rows alternately black and white, starting with black, so that there are eight black rows and seven white rows. Since the rooks are non-attacking there must be exactly one in each row, so there are eight rooks on a black row, and seven on a white row. This must be true after the knight's moves, so there must be an even number of rooks that change colour: m that go from black to white, balanced by m that go from white to black, making $2m$ that change colour altogether. This means that there must be an even number of rooks whose knight's moves involve one step in the vertical direction and two in the horizontal, leaving an odd number that move one step horizontally and two steps vertically. However, the exact same argument applies to the columns, so there must be an even number that move one step horizontally and two steps vertically. This contradiction shows that it is impossible to arrange the knight's moves so that the rooks are non-attacking both before and after the moves.

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