Complex Numbers

Andre Kessler

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1 Definitions

This section is included mainly as a reference, since most of you will know this. There are many polynomials with real coefficients that do not have real solutions; the classic example is $x^2 + 1 = 0$. The way to deal with equations like these is to set $i = \sqrt{-1}$. The complex numbers are all numbers of the form $a + bi$, where a and b are real. Real numbers are a subset of the complex numbers because they are of the form $a + 0i$. Addition works componentwise, meaning that $(a + bi) + (c + di) = (a + c) + (b + d)i$. Multiplication works just as it normally does, but with the condition that $i^2 = -1$. For example, $(a + bi)(c + di) = (ac - bd) + (ad + bc)i$.

To visualize complex numbers, we can plot them in the Gaussian (or Argand) plane, with the x axis being the real axis and y axis being the imaginary axis. Once we've plotted them in the plane, we can identify them with different characteristics: a magnitude and a direction, just like vectors. This means that we can write any complex number $a + bi$ in the form $z = r(\cos\theta + i\sin\theta) = re^{i\theta}$, where $\theta = \arctan\frac{b}{a}$ and $r = |a + bi| = \sqrt{a^2 + b^2}$. We define the conjugate of a complex number $z = a + bi$ to be the number $\bar{z} = a - bi$. Some useful properties of conjugates are that $z\bar{z} = |z|^2$, $\bar{z}\bar{w} = \overline{zw}$, and $\bar{z} + \bar{w} = \overline{z + w}$.

A complex number has n nth roots, which can be found using DeMoivre's Theorem. This theorem states that if we have a complex number $z = re^{i\theta}$, then $z^{p/q} = r^{p/q}e^{i(p\theta/q + 2\pi k/q)}$, for $k \in \mathbb{Z}, 0 \leq k < q$.

2 Algebra

2.1 Roots of Unity

An nth root of unity is a complex number satisfying the equation $z^n - 1 = 0$. Obviously, by the fundamental theorem of algebra, there are n nth roots of unity. Additionally, since $1 = e^{2\pi i}$, we can write an nth root as $\omega = e^{2\pi i/n}$. All other nth roots are given by the integer powers of this root between 0 and $n - 1$: the roots are $1, \omega, \omega^2, \ldots, \omega^{n-1}$.
By factoring the equation $z^n - 1 = 0$, we can obtain a useful identity:

$$z^n - 1 = (z - 1)(z^{n-1} + z^{n-2} + \cdots + z + 1) = 0$$

This means that all nth roots of unity other than 1 satisfy the equation

$$1 + \omega + \omega^2 + \cdots + \omega^{n-1} = 0$$

One other nice property of roots of unity is that they are periodic, which is due to the fact that $\omega^n = 1$. If we have a power of ω that is greater than n, it is equal to ω to that power mod n. For example, if ω is a third root of unity, then $\omega^5 + \omega^4 + 1 = \omega^2 + \omega + 1 = 0$.

2.2 Trigonometry

The key to connecting trigonometry to complex numbers is Euler's formula:

$$e^{i\theta} = \cos\theta + i\sin\theta$$

With this, we can quickly find that $\cos\theta = \mathrm{Re}(e^{i\theta}) = \frac{e^{i\theta}+e^{-i\theta}}{2}$ and that $\sin\theta = \mathrm{Im}(e^{i\theta}) = \frac{e^{i\theta}-e^{-i\theta}}{2i}$

2.3 Problems

1. If $x + x^{-1} = 2\cos\theta$, find $x^n + x^{-n}$ in terms of n and θ.
2. Evaluate
$$\sum_{n=0}^{\infty} \frac{\cos n\theta}{2^n}$$
where $\cos\theta = \frac{1}{5}$.
3. Find a closed form for $1 + \cos\theta + \cos 2\theta + \cdots + \cos n\theta$.

3 Number Theory

Although it may seem strange, complex numbers can play a very useful role when dealing with integers. The complex numbers that we will be using most in relation to number theory are the nth roots of unity. The first example is a problem that appeared on a TJML some time ago:

The number $1,280,000,401$ can be written as the product of distinct primes pq, where $p < q$. Find p.

Note that the number given can be written in the form $20^7 + 20^2 + 1$. In addition, the polynomial $x^7 + x^2 + 1$ has the factor $x^2 + x + 1$, because if $\omega = e^{2i\pi/3}$ then $\omega^7 = \omega^{3\cdot 2+1} = \omega$ which implies $\omega^7 + \omega^2 + 1 = \omega^2 + \omega + 1 = 0$. Thus $p = 20^2 + 20 + 1 = 421$.

3.1 Cyclotomic Polynomials

Cyclotomic means "circle-cutting." The nth cyclotomic polynomial is defined to be $\Phi_n(x) = \prod(x - \zeta)$, where ζ ranges over all the primitive nth roots of unity. For example, the first few are:

- $\Phi_1(x) = x - 1$
- $\Phi_2(x) = x + 1$
- $\Phi_3(x) = x^2 + x + 1$
- $\Phi_4(x) = x^2 + 1$

There is a formula for the nth cyclotomic polynomial, though it's a bit unwieldy (if the Möbius function means nothing to you, you may ignore this part):

$$\Phi_n(x) = \prod_{d|n} (x^d - 1)^{\mu(n/d)}$$

If you want to try to derive this formula, start by proving that $x^n - 1 = \prod_{d|n} \Phi_d(x)$. Then, take logs, and perform a Möbius inversion.
So why are cyclotomic polynomials useful? Well, they're all irreducible, and all of their coefficients are integers (must they be ± 1?). Think about $\Phi_p(x)$ for p prime - what form must it take? Cyclotomic polynomials are a relatively easy concept to understand, but they offer deep connections with other areas of math, especially field theory. Also, their name is pretty cool.

3.2 Other Number-Theoretic Uses

What are the solutions to the polynomial $x^2 + 1 \equiv 0 \pmod{13}$? Well, you can quickly see that $x = 5, 8$ are the solutions by inspection. However, you could also have written the polynomial as $x^2 \equiv -1 \pmod{13} \Leftrightarrow x \equiv i \pmod{13}$. For all intents and purposes, you can replace 5 with i in mod 13 and all of your equations will still work - for example, $5^{-1} \equiv i^{-1} \equiv -i \equiv -5 \equiv 8 \pmod 5$. However, while this works for mod 13, it doesn't work for mod 7. In general, $x^2 + 1 \equiv 0 \pmod p$ will have two solutions in mod p iff $p = 4k + 1$.

3.3 Problems

1. If p is a prime and $p = 4k + 1$, find a formula for i in $\mathbb{Z}/p\mathbb{Z}$ in terms of p.
2. If p is a prime, find solutions to $x^2 + y^2 = p$.
3. Let n be a positive integer. Prove that if $4^n + 2^n + 1$ is prime, then n is a power of 3.

4 Combinatorics

4.1 Roots of Unity Filter

Finding the sum of the coefficients of a polynomial is quite easy - if the polynomial is $f(x)$, then the sum of the coefficients is $f(1)$. What if we only want the coefficients whose terms have even exponents? After a little more thought, it's pretty easy to see that the answer is $\frac{f(1)+f(-1)}{2}$. What if we only want the terms whose exponents are multiples of three? To do this, think about why the $\frac{f(1)+f(-1)}{2}$ trick works. It works because the two square roots of unity (± 1) have the property that

$$\frac{1^n + (-1)^n}{2} = \begin{cases} 1 & \text{if } 2 \mid n \\ 0 & \text{otherwise} \end{cases}$$

Well, this property holds for kth roots of unity as well; that is

$$\frac{1}{k} \sum_{\omega^k = 1} \omega^n = \begin{cases} 1 & \text{if } k \mid n \\ 0 & \text{otherwise} \end{cases}$$

How does this apply to combinatorics? The following problem should convince you of the usefulness of the roots of unity filter.

The value of the sum
$$\sum_{k=0}^{33} \binom{99}{3k}$$
can be expressed in the form $\frac{a^b+c}{d}$, where a, c, and d are prime. Find $a+b+c+d$.

This problem is asking you to find the sum of the coefficients of the polynomial $f(x) = (x+1)^{99}$ for all powers of x that are divisible by 3. Well, we know how to do that, using the roots of unity filter... we just need to evaluate
$$\frac{f(1) + f(\omega) + f(\omega^2)}{3}$$
where $\omega = e^{2\pi i/3}$. We simply write out the polynomial and see that we have
$$\begin{aligned} \frac{2^{99} + (1+\omega)^{99} + (1+\omega^2)^{99}}{3} &= \frac{2^{99} + (-\omega^2)^{99} + (-\omega)^{99}}{3} \\ &= \frac{2^{99} - 2}{3} \end{aligned}$$
so the solution is $2 + 99 - 2 + 3 = \boxed{102}$.

4.2 Other Applications of Roots of Unity

There are a number of different ways we can apply complex numbers to solve combinatorial problems. In tiling problems, for example, the idea is usually to place a complex number in each square of a table and then to reformulate the problem in terms of the complex numbers.

4.3 Problems

1. You have 100 presents to give out tommorrow, and you plan to give 4 presents to each child that asks you for one (until you have no more presents to give or no more children ask). In how many ways can your presents be given out?

2. Prove the identity
$$\binom{n}{0} + \binom{n}{k} + \binom{n}{2k} + \cdots = \frac{2^n}{k} \sum_{j=1}^{k} \cos^n \frac{j\pi}{k} \cos \frac{nj\pi}{k}$$

3. Compute the sum $\binom{n}{1}\cos x + \binom{n}{2}\cos 2x + \cdots + \binom{n}{n}\cos nx$.

4. Consider a rectangle that can be tiled by a finite combination of $1 \times m$ and $n \times 1$ rectanles, where m, n are positive integers. Prove that it is possible to tile this rectangle using only $1 \times m$ rectangles or only $n \times 1$ rectangles. (BMC '00)

5. We roll a regular die n times. What is the probability that the sum of the numbers shown is a multiple of 5? (IMC '99)

5 Geometry

5.1 Complex Numbers and Transformations

When you have to resort to analytic geometry, complex numbers can sometimes greatly simplify the situation. This is a consequence of the fact that we can consider a complex number to be a vector or a geometric transformation.
If we use complex numbers as our coordinates, we get a very nice formula for rotation by $\frac{\pi}{2}$ about an arbitrary point a:

$$z' = a + i(z - a)$$

To see why this is true, translate a to the origin, which makes $z \to z - a$. Multiplying by i rotates everything by $\frac{\pi}{2}$, and then we translate the point back by adding a. To see if you understand this concept, try the following problems:

An old treasure map reads as follows: *Go to the island X, start at the gallows, go to the elm tree, and count the steps. Then turn left by* $90°$*, and go the same number of steps until point* g'*. Again, go from the gallows to the fig tree, and count the steps. Then turn right by* $90°$*, and go the same number of steps to point* g''*. A treasure is buried in the midpoint of* $g'g''$*.*
A treasure-hunter went to the island and found the elm tree at point $(0, 0)$, and the fig tree at $(10, 0)$. The gallows was somewhere on the line $y = 2009$. Find the location of the treasure.

Andre was walking to ARML practice one day, but he was given very strange directions. He was told to walk one mile to the northwest, then half a mile to the northeast, a fourth of a mile southeast, and so on. Find the distance of ARML practice from where Andre started walking.

5.2 More Problems

1. A function f is defined on the complex numbers by $f(z) = (a+bi)z$, where a and b are positive numbers. The function has the property that the image of each point in the complex plane is equidistant from that point and the origin. Given that $|a + bi| = 8$ and that $b^2 = m/n$, where m and n are relatively prime positive integers. Find $m + n$. (AIME '99)

2. Prove that no equilateral triangle exists with all of its vertices at lattice points.

3. Squares are erected outwards on the sides of a convex quadrilateral $ABCD$. If $AB = 2$, $AD = 10$, and the distance between the center of the square on side A and the square on side C is 15, find the distance between the center of the square on side D and the square on side B.

4. A trapezoid $ABCD$ is inscribed in a circle of radius $BC = DA = r$ and center O. Show that the midpoints of the radii OA, OB, and the midpoint of the side CD are vertices of an equilateral triangle.