MOP experiment ("geometry" and combinatorics)

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(See Dropbox: `https://www.dropbox.com/sh/1pd5bjm3038gku2/AACyg9LWs6xFGafWYRAG4lc0a` (or if/once that link breaks, my website, `http://web.mit.edu/vywang/www/`) for the latest version. Email me at vywang (at) mit.edu for errors/comments/suggestions or to discuss anything, e.g. where to look for more of these kinds of problems or topics.)

Recommendation: For your two sessions with me, work on the topic/problems you think would help you the most.[1] Of course, if you don't feel like doing geometry that's fine; there's an algebra class tomorrow anyways.

Also, while it's important to be able to find certain ideas on your own, I encourage you to occasionally work in small groups, both in and out of class; I think you can gain a surprising amount of intuition just by talking to others. This might be easier to coordinate out of class (especially for the harder problems), but just throwing up a possibility that could work well for some of you.

1 A touch of geometry

1. (MOP 2007, for James Tao) In triangle ABC, point L lies on side BC. Extend segment AB through B to M such that $\angle ALC = 2\angle AMC$. Extend segment AC through C to N such that $\angle ALB = 2\angle ANB$. Let O be the circumcenter of triangle AMN. Prove that $OL \perp BC$.

2. (Yang Liu, David Yang, based on USAMO 2014) Construct $P_0, P_1, P_{-1}, P_2, P_{-2}, P_3, P_{-3}, P_4, P_{-4}, P_5, P_{-5}$ in order such that

   (i) P_0, P_1, P_{-1} are not collinear;

   (ii) $P_2 \in P_0P_{-1}$ only (i.e. it doesn't lie on any other line so far except the ones specified, in this case P_0P_{-1});

   (iii) $P_{-2} \in P_1P_2$ only;

   (iv) $P_3 \in P_0P_{-2}$ only;

   (v) $P_{-3} \in P_1P_3$ only;

   (vi) $P_4 \in P_0P_{-3}, P_{-1}P_{-2}$ only;

   (vii) $P_{-4} \in P_1P_4, P_2P_3$ only;

   (viii) $P_5 \in P_0P_{-4}, P_{-1}P_{-3}$ only;

   (ix) $P_{-5} \in P_1P_5, P_2P_4$ only.

   Prove that lines P_0P_{-5}, $P_{-1}P_{-4}$, $P_{-2}P_{-3}$ have nonempty intersection.

3. (Putnam 1996; also in analysis) Let $(a_1, b_1), (a_2, b_2), \ldots, (a_n, b_n)$ be the vertices of a convex polygon which contains the origin in its interior. Prove that there exist positive reals x, y such that $\sum (a_i, b_i)x^{a_i}y^{b_i} = (0, 0)$.

[1] I remember last year I felt like I wasn't getting anything out of around half of the classes. It's a pity if you're not learning at least one really new/interesting thing every day.

4. ("tan9p", Math StackExchange; also in analysis) ABC is an equilateral triangle, and $AD = BE = CF$ for some distinct points D, E, F inside, with A, D, E collinear, B, E, F collinear, and C, F, D collinear. Prove that DEF is an equilateral triangle.

5. (Motzkin-Rabin) Let $\mathcal{S}$ be a finite set of points in the plane (not all collinear), each colored red or blue. Show that there exists a monochromatic line passing through at least two points of $\mathcal{S}$.

6. (Romania TST 2004; also in analysis) Let D be a closed disk in the complex plane. Prove that for all positive integers n, and for all complex numbers $z_1, z_2, \dots, z_n \in D$ there exists a $z \in D$ such that $z^n = z_1 z_2 \cdots z_n$.

7. (China 2012; also in analysis) Find the smallest possible value of a real number c such that for any 2012-degree monic polynomial $P(x) = x^{2012} + a_{2011}x^{2011} + \cdots + a_1 x + a_0$ *with real coefficients*, we can obtain a new polynomial $Q(x)$ by multiplying some of its coefficients by -1 such that every root z of $Q(x)$ satisfies the inequality $|\Im z| \le c|\Re z|$.

8. (V. Galperin and G. Galperin, also various olympiads) Darkness has descended on the plane of Mopok. All electricity has been cut-off by the Trydan Corporation. All they have to light the plane are k lighthouses that run on cooking oil where $k \ge 1$ is a positive integer. They can be kept going indefinitely, the Mopoks love their french fries, but each lighthouse can only illuminate a sector of $360/k$ degrees. The lamp of each lighthouse can be rotated but it must be fixed before the light is turned on. Can the lights be rotated so that the whole plane is covered?

   (The lighthouses positions are given, you don't get to choose where to put them.)

   (Assuming that k is even makes the problem a bit easier.)

2 Combinatorial stuff

1. (Algebraic and additive combinatorics, vaguely)

   (a) (Boris Bukh) Suppose $A_1, \dots, A_m$ are $n \times n$ matrices with $m > n$ and $A_1 + \cdots + A_m$ invertible. Show that for some proper subset $S \subset \{1, 2, \dots, m\}$ of indices, $\sum_{i \in S} A_i$ is also invertible.

   (b) (TSTST 2011) Let n be a positive integer. Suppose we are given $2^n + 1$ distinct sets, each containing finitely many objects. Place each set into one of two categories, the red sets and the blue sets, so that there is at least one set in each category. We define the *symmetric difference* of two sets as the set of objects belonging to exactly one of the two sets. Prove that there are at least 2^n different sets which can be obtained as the symmetric difference of a red set and a blue set.

   (c) (China 2011) Let ℓ, m, n be positive integers, and $A_1, A_2, \dots, A_m, B_1, \dots, B_n$ be $m + n$ pairwise distinct subsets of the set $\{1, 2, \dots, \ell\}$. Suppose that the symmetric differences $A_i \Delta B_j := (A \cup B) \setminus (A \cap B)$ (for $1 \le i \le m, 1 \le j \le n$) cover each nonempty subset of $\{1, 2, \dots, \ell\}$ exactly once. Find all possible values of m, n, in terms of ℓ.

   (d) (TST 2014, W.) Find all functions $f : \mathbb{N} \to \mathbb{Z}$ such that $(f(m) - f(n))(m - n)$ is always a square.

   (e) (China?) Let p be a prime and $a_1, \dots, a_k$ be $k \ge 1$ distinct *nonzero* residues modulo p. Prove that there are at most $(p-1)/k$ numbers $n \in \{1, 2, \dots, p-1\}$ such that $[na_1]_p < [na_2]_p < \cdots < [na_k]_p$, where $[x]_p$ denotes the (nonnegative) remainder of x modulo p.

   (f) Let p be a prime and $b_1, \dots, b_k$ be $k \ge 1$ *nonzero* residues modulo p. Prove that there are at most $(p-1)/k$ numbers $n \in \{1, 2, \dots, p-1\}$ such that $[nb_1]_p + \cdots + [nb_k]_p = [n(b_1 + \cdots + b_k)]_p$, where $[x]_p$ denotes the (nonnegative) remainder of x modulo p.

   (g) (Putnam 2012) Let q and r be integers with $q > 0$, and let A and B be intervals on the real line. Let T be the set of all $b + mq$ where b and m are integers with b in B, and let S be the set of all integers a in A such that ra is in T. Show that if the product of the lengths of A and B is less than q, then S is the intersection of A with some arithmetic progression.

(h) (RMM 2012, Ben Elliott; also in analysis) Each positive integer is coloured red or blue. A function f from the set of positive integers to itself has the following two properties:
(a) if $x \le y$, then $f(x) \le f(y)$; and
(b) if x, y and z are (not necessarily distinct) positive integers of the same colour and $x + y = z$, then $f(x) + f(y) = f(z)$.
Prove that there exists a positive number a such that $f(x) \le ax$ for all positive integers x.

(i) (China 2012) Prove that there exists a positive real number C with the following property: for any integer $n \ge 2$ and any subset X of the set $\{1, 2, \ldots, n\}$ such that $|X| \ge 2$, there exist $x, y, z, w \in X$(not necessarily distinct) such that $0 < |xy - zw| < C\alpha^{-3}$, where $\alpha = \frac{|X|}{n}$. (The original problem asks for the weaker $C\alpha^{-4}$.)

(j) (China 2011) Let $n > 1$ be an integer, and let k be the number of distinct prime divisors of n. Prove that there exists an integer a, $1 < a < \frac{n}{k} + 1$, such that $n \mid a^2 - a$.

(k) (ELMO 2013, W. [2]) Let $m_1, m_2, \ldots, m_{2013} > 1$ be 2013 pairwise relatively prime positive integers and $A_1, A_2, \ldots, A_{2013}$ be 2013 (possibly empty) sets with $A_i \subseteq \{1, 2, \ldots, m_i - 1\}$ for $i = 1, 2, \ldots, 2013$. Prove that there is a positive integer N such that $N \le (2|A_1| + 1) \cdots (2|A_{2013}| + 1)$ and for each $i = 1, 2, \ldots, 2013$, there does *not* exist $a \in A_i$ such that m_i divides $N - a$.

2. (ELMO Shortlist 2012, Linus Hamilton) Form the infinite graph A by taking the set of primes p congruent to 1 (mod 4), and connecting p and q if they are quadratic residues modulo each other. Do the same for a graph B with the primes 1 (mod 8). Show A and B are isomorphic to each other.

3. (a) (RMM 2012, Ilya Bogdanov) Given a positive integer $n \ge 3$, colour each cell of an $n \times n$ square array with one of $\lfloor (n + 2)^2/3 \rfloor$ colours, each colour being used at least once. Prove that there is some 1×3 or 3×1 rectangular subarray whose three cells are coloured with three different colours.

   (b) Construct an example showing that $n^2/3 + O(n)$ colors are needed.

4. (Math Prize Olympiad 2010) Let S be a set of n points in the coordinate plane. Say that a pair of points is *aligned* if the two points have the same x-coordinate or y-coordinate. Prove that S can be partitioned into disjoint subsets such that (a) each of these subsets is a collinear set of points, and (b) at most $n^{3/2}$ unordered pairs of distinct points in S are aligned but not in the same subset.

5. Simple processes.

   (a) (Russia 2003) Ana and Bora start with the letters A and B, respectively. Every minute, one of them either prepends or appends to his/her own word the other person’s word (not necessarily operating one after another). Prove that Ana’s word can always be partitioned into two palindromes.

   (b) (Bulgarian solitaire) Suppose we have $N = 1 + 2 + \cdots + n$ cards total among some number of stacks. In each move, Bob takes one card from each stack and forms a new stack with them. Show that Bob eventually ends up with $1, 2, \ldots, n$ in some order.

   (c) (MOP 2005?) Suppose n coins have been placed in piles on the integers on the real line. (A pile may contain zero coins.) Let T denote the following sequence of operations.
   (a) Move piles $0, 1, 2, \ldots$ to $1, 2, 3 \ldots$, respectively.
   (b) Remove one coin from each nonempty pile from among piles $1, 2, 3, \ldots$, then place the removed coins in pile 0.
   (c) Swap piles i and $-i$ for $i = 1, 2, 3, \ldots$.
   Prove that successive applications of T from any starting position eventually lead to some sequence of positions being repeated, and describe all possible positions that can occur in such a sequence.

6. Some graph theory.

[2] For extensions/comments see the solutions file here.

(a) (TST 2014, Zoltán Füredi) Let n be an even positive integer, and let G be an n-vertex graph with exactly $\frac{n^2}{4}$ edges, where there are no loops or multiple edges (each unordered pair of distinct vertices is joined by either 0 or 1 edge). An unordered pair of distinct vertices $\{x, y\}$ is said to be *amicable* if they have a common neighbor (there is a vertex z such that xz and yz are both edges). Prove that G has at least $2\binom{n/2}{2}$ pairs of vertices which are amicable.

(b) (MOP 2010) Fix n points in space in such a way that no four of them are in the same plane, and choose any $\lfloor n^2/4 \rfloor + 1$ segments determined by the given points. Determine the least number of points that are the vertices of a triangle formed by the chosen segments.

7. Consider a directed graph G with n vertices, where 1-cycles and 2-cycles are permitted. For any set S of vertices, let $N^+(S)$ denote the out-neighborhood of S (i.e. set of successors of S), and define $(N^+)^k(S) = N^+((N^+)^{k-1}(S))$ for $k \geq 2$.

For fixed n, let $f(n)$ denote the maximum possible number of distinct sets of vertices in $\{(N^+)^k(X)\}_{k=1}^{\infty}$, where X is some subset of $V(G)$.

(a) (ELMO Shortlist 2012, Linus Hamilton[3]) Show that $f(n)$ is *sub-exponential*, in the sense that for any $c > 1$, we have $f(n) < c^n$ for all sufficiently large n (certainly depending on c).

(b) (Linus) Let $g(n)$ denote Landau's function: the largest least common multiple of any partition of n. Prove that $g(n) \leq f(n) \leq (n + c)g(n) + n^d$ for some constants $c, d > 0$ independent of n.

(c) (W.) Prove that $g(n) \leq f(n) \leq g(n) + n^d$ for some constant $d > 0$ independent of n.

[3]Linus said he thought of this problem with Mitchell Lee and David Yang by trying to optimize their answer to the “half(L)” problem on this page, which I haven't looked at but may be interesting.