CHAPTER 2

POINTS, LINES, AND TRIANGLES

In this chapter we will highlight a small core of basic results related to triangles that proves useful in a large number of problems in Mathematics Olympiads. Those results are simple, but readers should pay attention to their applications since the most important and difficult thing is how we make use of them. The examples, mainly came from Olympiads, are not easy in general and readers should not be discouraged if one cannot figure out the solution in a short while.

2.1 Four special points

To a triangle are associated literally many special points. We will study but a few of the more important ones, namely centroid, circumcenter, orthocenter and incenter. Recall that the converse of Ceva's theorem proved in chapter 1 guarantees three medians (the segment joining a vertex to the midpoint of the opposite side) of a triangle are concurrent. The point of concurrency is called the **centroid**（重心）of the triangle and it is usually denoted by the letter G.

Theorem 2.1-1

The centroid G of a triangle ABC trisects each median. More precisely, if D is the midpoint of BC then $AG : GD = 2 : 1$.

Although one can easily prove Theorem 2.1-1 by using the converse of Ceva's theorem, there are many alternative solutions that are equally elegant; one by performing an affine transformation making the triangle equilateral; another one applies Theorem 1.1-1. These are left as exercises.

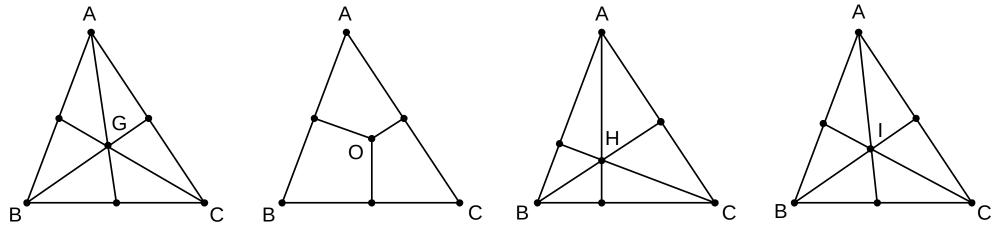


Figure 1

We all know the perpendicular bisectors of the sides of a triangle are concurrent and the point of concurrency, usually denoted by the letter *O*, is called the **circumcenter**（外心）of the triangle. Three altitudes of a triangle are also concurrent at a point called **orthocenter**（垂心）which we denote it by the letter *H*. Finally, three internal angle bisectors concurrent at the **incenter**（內心）and most of the times it is denoted by the letter *I*. Let us see some examples involving these notions.

We start with a simple example that will be used in the next section for proving the existence of Euler line.

Example 2.1-1

Let *G* be the centroid of ΔABC, *L* be a straight line. Prove that

$$GG' = \frac{AA' + BB' + CC'}{3},$$

where P' denotes the foot of perpendicular of a point *P* to the line *L*.

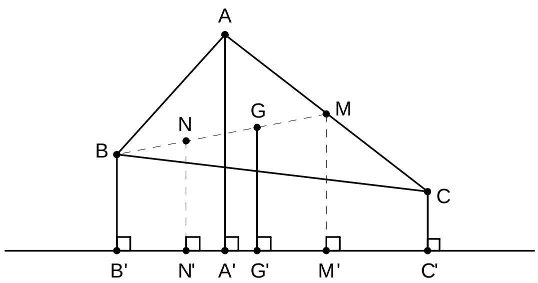


Figure 2

Solution

Let *M* be the midpoint of *CA*, *N* be the midpoint of *BG* (see Figure 2). Then

$$2GG' = MM' + NN',$$

and therefore

$$\begin{aligned} 4GG' &= 2MM' + 2NN' \\ &= (AA' + CC') + (BB' + GG') \end{aligned}$$

i.e. $3GG' = AA' + BB' + CC'$.

Q.E.D.

- In general, if *P*, *Q*, *R* are collinear points such that $PR = \lambda PQ$, then $RR' = (1-\lambda)PP' + \lambda QQ'$ (here we allows λ to be negative). This fact can be used to give a simpler solution to Example

2.1-1, without introducing the point *N*. It is left to the reader as exercise.

The importance of Example 2.1-1 is revealed by the following corollary.

Corollary

Let *G* be the centroid of ΔABC and *P* be any point. Then

(1.1) $$\overrightarrow{PG} = \frac{1}{3}(\overrightarrow{PA} + \overrightarrow{PB} + \overrightarrow{PC}) .$$

Moreover, if a rectangular coordinate system is setup on the plane such that $A = (a_1, a_2)$, $B = (b_1, b_2)$, and $C = (c_1, c_2)$, then

$$G = \left(\frac{a_1 + b_1 + c_1}{3}, \frac{a_2 + b_2 + c_2}{3} \right).$$

The existence of orthocenter (or other three special points) can be used to prove concurrent lines. Here is an example.

Example 2.1-2 (IMO 1995-1)

Let *A*, *B*, *C* and *D* be four distinct points on a line, in that order. The circles with diameters *AC* and *BD* intersect at the points *X* and *Y*. The line *XY* meets *BC* at the point *Z*. Let *P* be a point on the line *XY* different from *Z*. The line *CP* intersects the circle with diameter *AC* at the points *C* and *M*, and the line *BP* intersects the circle with diameter *BD* at the points *B* and *N*. Prove that the lines *AM*, *DN* and *XY* are concurrent.

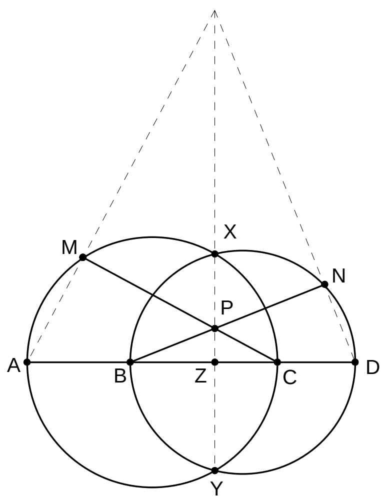


Figure 3

Sketch of Proof

Draw DE parallel to CM meets XY at E, and draw AE' parallel to BN meets XY at E' (see Figure 4). It can be proved that $E = E'$. We will prove this in chapter 3 but not here, for one reason we want to focus on the technique for proving concurrent lines and for another reason we need intersecting chords theorem discussed in chapter 3. Now, AM, DN and XY are the altitudes of triangle ADE, therefore they are concurrent.

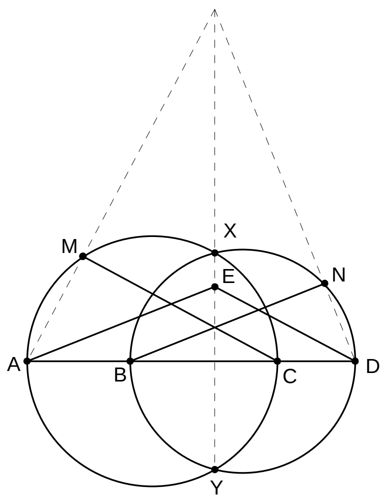


Figure 4

Example 2.1-3

Two circles, C_1 and C_2, centerd at O_1 and O_2 respectively, meet at A and B. O_1B is produced to meet C_2 at E. O_2B is produced to meet C_1 at F. A straight line is constructed through B parallel to EF cutting C_1 and C_2 at M and N. Prove that $MN = AE + AF$.

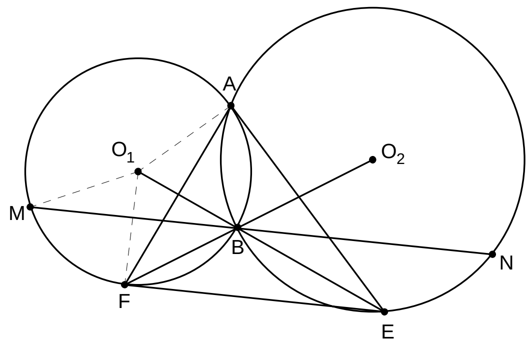


Figure 5

Analysis

Faced with such a problem, a very natural way of thought is to try to prove that $AF = MB$ (and by the same argument we can prove $AE = NB$, thereby completing the proof). To prove this conjecture, we need only prove that $\angle AO_1F = \angle MO_1B$, which is equivalent to $\angle AO_1B = \angle MO_1F$.

On the other hand, we notice that $\angle AO_1B = 2\angle AFB$ and $\angle MO_1F = 2\angle MBF = 2\angle EFB$. Consequently, the problem has now been reduced to proving that B lies on the internal bisector of $\angle AFE$. If this can be done, then by the same argument B also lies on the internal bisector of $\angle AEF$. That is, B is the incenter of ΔAEF.

The above analysis shows that a possible way to solve the problem is to prove that B is the incenter of ΔAEF. Note that this is independent of the points M and N, so we can remove them from the figure and obtain the simplified Figure 6.

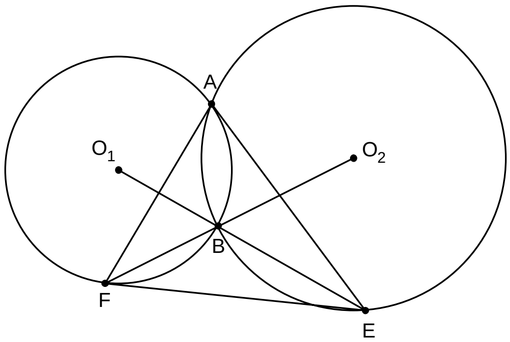


Figure 6

Example 2.1-3 is a good example illustrating a common situation in geometry: the four special points (centroid, circumcenter, orthocenter and incenter) often come up even if the original question seems irrelevant to them. They are the core part of elementary geometry and it is beneficial for us to learn and grasp them. Incidentally, it is worthwhile to note that in Example 2.1-3 we tackle the problem by first assuming the conclusion, and by some analysis concluding B is the incenter of a certain triangle. Then we realize that proving B is the incenter would be essentially the same as proving the desired conclusion. Therefore the problem can be simplified. Tracing backward is a very useful technique.

Example 2.1-4 (Continuation of Example 2.1-3)

Two circles, C_1 and C_2, centerd at O_1 and O_2 respectively, meet at A and B. O_1B is produced to meet C_2 at E. O_2B is produced to meet C_1 at F (see Figure 6). Prove that B is the incenter of ΔAEF.

Analysis

If B really is the incenter then $\angle AFE = 2\angle AFB = \angle AO_1B = \angle AO_1E$, and so the points A, O_1, F and E are concyclic (and by symmetry we know that O_2 also lies on this circle).

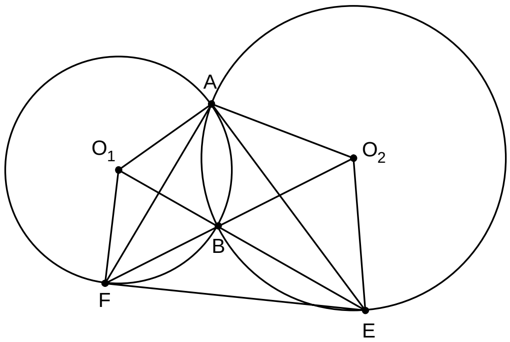


Figure 7

Conversely, proving that the points A, O_1, F and E are concyclic would be essentially the same as having proved that B is the incenter of ΔAEF.

Solution

Note that $\Delta O_1AO_2 \cong \Delta O_1BO_2$, so $\angle O_1AO_2 = \angle O_1BO_2 = 180° - \angle O_1BF = 180° - \angle O_1FO_2$ which implies A, O_1, F, O_2 are concyclic. By symmetry, E also lies on this circle and the proof is completed.

Q.E.D.

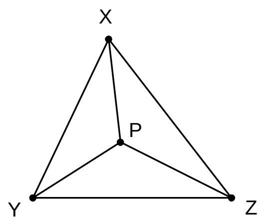
There are different ways to prove a point is the incenter of a given triangle. We would like to introduce two of these techniques. The first one makes use of the concept of locus to determine the incenter. Having mastered this technique, problems involving orthocenter can be dealt with similarly. This technique is heavily based on the following lemma.

Lemma 2.1-2

A point P inside ΔXYZ is the incenter of the triangle if and only if it satisfies:

(i) P lies on the internal angle bisector of angle $\angle ZXY$; and

(ii) $\angle YPZ = 90° + \dfrac{\angle ZXY}{2}$.



It is clear that the incenter of the triangle satisfies (i) and (ii) in Lemma 2.1-2. But why a point that satisfies (i) and (ii) must be the incenter? We can understand this using the concept of locus. The set of points satisfying (i) is a line segment inside the triangle, while the set of points satisfying (ii) is a circular arc. It is clear that these two loci have a unique intersection, so a point satisfying both (i) and (ii) must be the incenter of the triangle. Now let's return to Example 2.1-4.

Alternative solution to Example 2.1-4

Let $\alpha = \angle O_1FB = \angle O_1BF = \angle O_2BE$. Then we have $\angle BO_1F = 180° - 2\alpha$ and $\angle BAF = 90° - \alpha$. By the same argument $\angle BAE = 90° - \alpha$. Hence $\angle BAE = \angle BAF$, i.e. *B* lies on the bisector of $\angle EAF$. The condition (i) of the Lemma 2.1-2 is satisfied. On the other hand, note that $\angle EBF = 180° - \alpha$ and $\angle EAF = 180° - 2\alpha$. It is easy to see that condition (ii) is also satisfied.

Q.E.D.

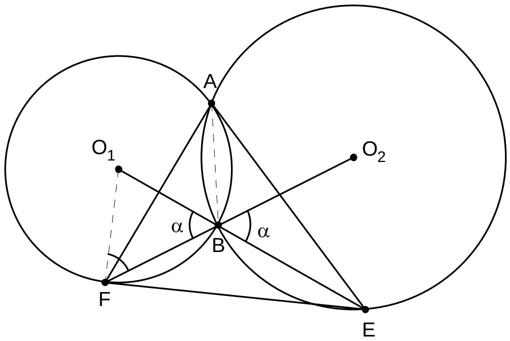
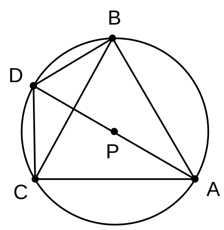


Figure 8

Another technique in proving incenter is based on Lemma 2.1-3 below. This is also standard that the reader should memorize.

Lemma 2.1-3

Let *D* be a point on the circumcircle of ΔABC such that *AD* bisects $\angle CAB$. A point *P* on the segment *AD* is the incenter of ΔABC if and only if *DB* = *DP* (or, equivalently, *DC* = *DP*). In that case, *D* is the circumcenter of ΔBPC.



Example 2.1-5 (IMO 2002)

Let *BC* be a diameter of the circle Γ with centre *O*. Let *A* be a point on Γ such that $\angle AOC > 60°$. Let *D* be the midpoint of arc *AB* not containing *C*. The line through *O* parallel to *DA* meets the line

AC at *J*. The perpendicular bisector of *OA* meets Γ at *E* and at *F*. Prove that *J* is the incentre of triangle *CEF*.

Solution

Since $EA = EO$ and $OE = OA$, ΔOAE is equilateral. Since $\angle AOC > 60^\circ$, *F* lies on the minor arc *AC*. It follows that $\angle ACF = \angle AEF = \frac{1}{2}\angle AEO = 30^\circ$ and $\angle ACE = \frac{1}{2}\angle AOE = 30^\circ = \angle ACF$, so *J* lies on the bisector of $\angle ECF$. In the view of Lemma 2.1-3, it remains to prove $AJ = AE$.

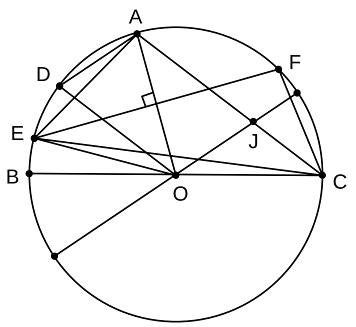


Figure 9

Note that $\angle ACB = \frac{1}{2}\angle AOB = \angle DOB$, we have $AC \parallel DO$ and therefore *ADOJ* is a parallelogram. It follows that $AJ = DO = AO = AE$. Lemma 2.1-3 implies that *J* is the incenter of ΔCEF.

Q.E.D.

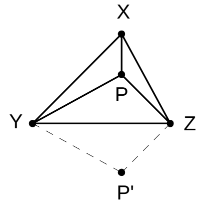
Another example applying Lemma 2.1-3 will be given in the next section (see Example 2.2-1). Let us depart from the discussion about incenter and go ahead to a lemma which is parallel to Lemma 2.1-2 and can be used to prove a point is the orthocenter of a given triangle.

Lemma 2.1-4

A point *P* inside an acute triangle *XYZ* is the orthocenter if and only if

(i) $XP \perp YZ$; and

(ii) $\angle ZXY + \angle ZPY = 180^\circ$.



Note that Lemma 2.1-4 said that if *P* is the orthocenter then the reflection of *P* about the line *YZ* lies on the circumcircle of ΔXYZ. This is true even if ΔXYZ is not assumed to be acute.

Example 2.1-7 (CMO 1999)

In acute triangle ABC, $\angle C > \angle B$. D is a point on BC such that $\angle ADB$ is obtuse. H is the orthocenter of ΔABD. Point F is in the interior of ΔABC and on the circumcircle of ΔABD. Prove that F is the orthocenter of ΔABC if and only if HD is parallel to CF and H lies on the circumcircle of ΔABC.

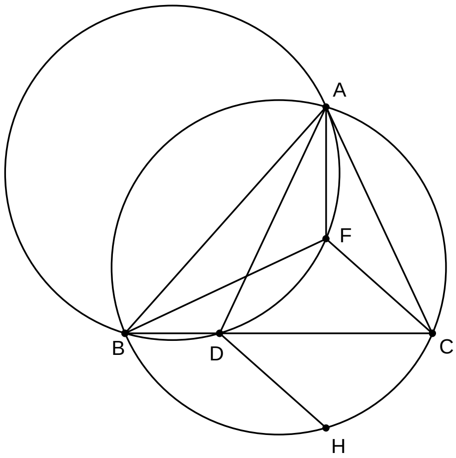


Figure 10

Solution

By Lemma 2.1-4, F is the orthocenter of ΔABC if and only if the following conditions are satisfied:

(i) $CF \perp AB$;

(ii) $\angle BCA + \angle BFA = 180°$.

Since H is the orthocenter of ΔABD, one has $HD \perp AB$ and therefore (i) is equivalent to $HD \parallel CF$. It remains to prove (ii) holds if and only if H lies on the circumcircle of ΔABC. Note that $\angle AFB = \angle ADB = 180° - \angle AHB$. So, (ii) becomes $\angle ACB = \angle AHB$ which is equivalent to saying that H lies on the circumcircle of ΔABC.

Q.E.D.

Example 2.1-8 (Three circle theorem)

Three unit circles meet at a common point P, they intersect each other at points A, B and C. Prove that P is the orthocenter of ΔABC and the circumradius of ΔABC is 1.

Solution

In Figure 11, D, E, F are points on the circles which diametrically opposite to P. Note that $\angle PAE = \angle PAF = 90°$ and $PE = PF = 2$, these imply A lies on the segment EF and PA is perpendicular bisector of EF. Similarly PB, PC are perpendicular bisectors of FD, DE respectively.

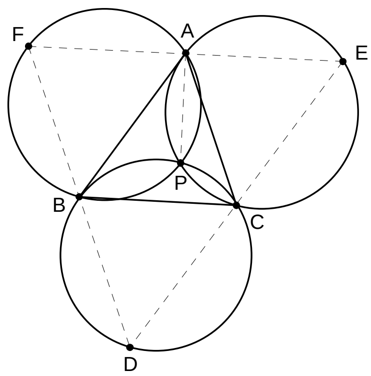


Figure 11

Furthermore, ΔABC is the medial triangle of ΔDEF, so its circumradius is half of the circumradius of ΔDEF. Since P is the circumcenter of ΔDEF, P is the orthocenter of the medial triangle ABC and the circumradius of ΔDEF is 2. Hence, the circumradius of ΔABC is 1.

Q.E.D.

Exercise

1. Prove Theorem 2.1-1.

2. Show that three altitudes of a triangle are concurrent.

3. Show that three angle bisectors of a triangle are concurrent.

4. Simplify the solution of Example 2.1-1 by using the remark following it.

5. Find an alternative solution to Example 2.1-2.

6. Redo Example 2.1-7 without using Lemma 2.1-4.

7. Given a triangle ABC, prove that there exists a unique point P on the same plane such that

$$AP^2 + BC^2 = BP^2 + CA^2 = CP^2 + AB^2 = 4R^2,$$

where R denotes the circumradius of triangle ABC.

8. (Mathematics Magazine, problem 1506) Let I and O be the incenter and circumcenter of ΔABC, respectively. Assume ΔABC is not equilateral (so $I \neq O$). Prove that

$$\angle AIO \leq 90° \quad \text{if and only if} \quad 2BC \leq AB + CA.$$

9. (IMO 1996) Let P be a point inside triangle ABC such that

$$\angle APB - \angle ACB = \angle APC - \angle ABC.$$

Let D, E be the incenters of triangles APB, APC respectively. Show that AP, BD and CE meet at a point.

10. (IMO 1988) ABC is a triangle right-angled at A, and D is the foot of the altitude from A. The straight line joining the incenters of the triangles ABD, ACD intersects the sides AB, AC at the points K, L respectively. S and T denote the areas of the triangles ABC and AKL respectively. Show that $S \geq 2T$.

11. (IMO 1981) Three congruent circles have a common point O and lie inside a given triangle. Each circle touches a pair of sides of the triangle. Prove that the incenter and the circumcenter of the triangle and the point O are collinear.

12. (Iran, 1995) Let M, N, P be the points of intersection of the incircle of ΔABC with sides BC, CA, AB respectively. Prove that the orthocenter of ΔMNP, the incenter of ΔABC, and the circumcenter of ΔABC are collinear.

2.2 Euler line and Euler theorem

The results given in this section may not be very useful in problem solving. Truly speaking, they appear in here mainly because of their beauty and history. Anyway, they are worth mentioning since later it is possibly, although not probably, we will make use of the materials in this section. Also, Euler line is an extremely beautiful object; it relates three of our four special points together in a very unexpected way!

Theorem 2.2-1 (Euler line)

In any triangle *ABC*, the circumcenter *O*, centroid *G* and orthocenter *H* are collinear. Moreover, we have $OG : GH = 1 : 2$.

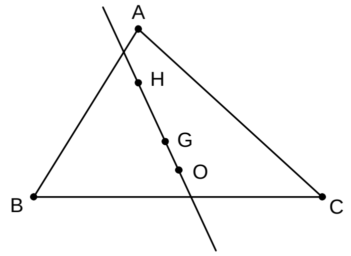


Figure 12

- The line *OGH* is called the **Euler line**（歐拉線）of triangle *ABC*.

We will give two proofs to this theorem, one uses vectors and the other by homothety（位似變換）. Since homothety is a topic in chapter 5, readers may skip the second proof at the first reading.

The first proof to Theorem 2.2-1

Extend *BO* to meet the circumcircle of ΔABC at *D*. Since both *AH* and *DC* are perpendicular to *BC*, one has *AH* // *DC*. Similarly we have *DA* // *CH*. Therefore, *AHCD* is a parallelogram.

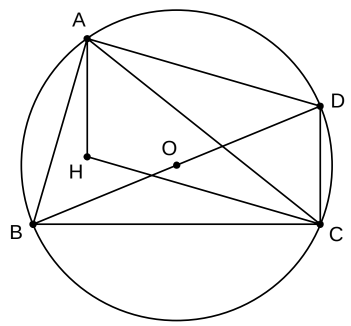


Figure 13

It follows that

$$\overrightarrow{OH} = \overrightarrow{OA} + \overrightarrow{AH} = \overrightarrow{OA} + \overrightarrow{DC} = \overrightarrow{OA} + (\overrightarrow{DO} + \overrightarrow{OC}) = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}\ .$$

Hence, by (1.1) we have $\overrightarrow{OH} = 3\overrightarrow{OG}$. It means that *O*, *G*, *H* are collinear and *OG* : *GH* = 1 : 2.

Q.E.D.

The second proof to Theorem 2.2-1

Consider the homothety *T* about *G* with ratio −1/2. *T* maps triangle *ABC* to its medial triangle *DEF* (see Figure 14). Since $DO \perp BC$ and *BC* // *FE*, we have $DO \perp EF$. Similarly, $EO \perp FD$ and so *O* is the orthocenter of the medial triangle *DEF*.

Under the action of *T*, the orthocenter of ΔABC should be mapped to the orthocenter of ΔDEF, i.e. *T* sends *H* to *O*. We have proved that *O*, *G*, *H* are collinear and *OG* : *GH* = 1 : 2.

Q.E.D.

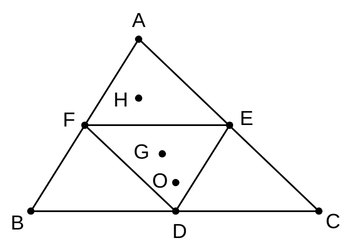


Figure 14

Theorem 2.2-2 (Euler theorem)

Let *ABC* be a triangle, *O*, *I* be respectively the circumcenter and incenter. Then

$$OI^2 = R^2 - 2Rr,$$

where *R* denotes the circumradius and *r* denotes the inradius.

- As a corollary of Theorem 2.2-2 we have $R \geq 2r$. It is easy to prove that the equality holds if and only if the triangle is equilateral.

Analysis

Extend *AI* to meet the circumcircle of ΔABC at *D*. The equality $OI^2 = R^2 - 2Rr$ holds if and only if $2Rr = AI \times ID$ because the quantity $AI \times ID$ is equal to $R^2 - OI^2$. This quantity, in the sense of absolute value, is equal to the **power** of *I* with respect to the circumcircle of ΔABC. The concept of "power" will be discussed in chapter 3. Of course, at this stage the fact $AI \times ID = R^2 - OI^2$ is by no means obvious. Readers may prove it themselves. It would not be difficult.

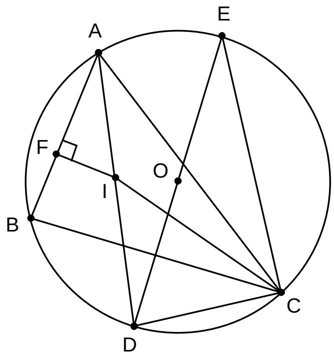


Figure 15

Proof of Theorem 2.2-2

Let E be the point on the circumcircle of ΔABC which diametrically opposite to D, let also F be the foot of perpendicular from I to AB. Since $\angle AFI = \angle ECD = 90°$ and $\angle IAF = \angle DAC = \angle DEC$, the triangles AFI and ECD are similar. Therefore,

$$\frac{FI}{AI} = \frac{CD}{ED},$$

i.e. $2Rr = AI \times CD$. We complete the proof by showing that $CD = ID$.

Recall that I is the incenter of ΔABC. So,

$$\angle DCI = \angle DCB + \angle BCI = \angle DAB + \angle ICA = \angle IAC + \angle ICA = \angle DIC$$

and hence $CD = ID$.

Q.E.D.

Example 2.2-1

Let C_1, C_2 be two circles having no intersection and C_2 lies inside C_1. For any point P on C_1, let Q, R be points on C_1 such that PQ, PR are tangent to C_2 (thus Q, R depend on P). The point P is said to be "good" if QR is also tangent to C_2. Prove that either none of the points on C_1 is good or all the points on C_1 are good.

Solution

Let O, I be respectively the centers of C_1 and C_2; let R, r be respectively the radii of C_1 and C_2. We suppose there exists a good point and prove that all points on C_1 are good. Since good points exist, Euler theorem implies $OI^2 = R^2 - 2Rr$.

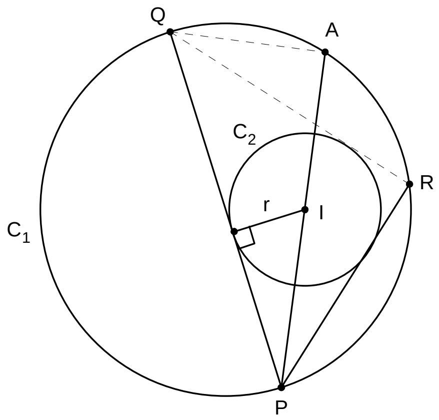


Figure 16

Fix an arbitrary point P on C_1 and let QR be the corresponding tangent (our abuse of the letter R will make no confusion). As in the analysis of Euler theorem, $AI \times PI = R^2 - OI^2 = 2Rr$. So,

$$AI = 2R \times \frac{r}{PI} = 2R \sin \angle APQ = AQ.$$

Obviously, I lies on the bisector of $\angle RPQ$. Lemma 2.1-3 implies I is the incenter of ΔPQR.

Q.E.D.

- The philosophy behind Example 2.2-1 is that everything is degree 2; if there exist three good points then every point on C_1 have to be good (coincidentally, a degree 2 polynomial having three roots must be identically zero). On the other hand, the existence of a good point guarantees that there are three. To be precise, we let D be the foot of perpendicular from I to QR. It can be proved that when P moves around the circle the locus of D is a circle (i.e. a degree 2 object!). Being a circle, it can have at most two intersections with C_2 unless it coincides with C_2. However, the existence of a good point implies that there are at least three intersections. All these constitute an alternative solution to Example 2.2-1.

Theorem 2.2-3 (Nine-point circle)

In any triangle ABC, the feet of three altitudes, the midpoints of three sides, and the midpoints of the segments from the vertices to the orthocenter, all lie on the same circle. This circle is of radius half as the circumradius of triangle ABC.

- The circle in Theorem 2.2-3 is called the **nine-point circle**（九點圓）of triangle *ABC*. Its center is therefore naturally called **nine-point center** of triangle *ABC*. We denote the nine-point center by the letter *N*. We will see later *N* lies on the Euler line and is the midpoint of *OH*.

- The triangles *ABC*, *HBC*, *AHC*, *ABH* all have the same nine-point circle, provided that the orthocenter *H* does not coincide with each of the vertices *A*, *B*, *C*.

Proof

Refer to Figure 17, we let *D*, *E*, *F* be the midpoints of *BC*, *CA*, *AB* respectively, *L*, *M*, *N* be the feet of altitudes, and *P*, *Q*, *R* be the midpoints from the orthocenter *H* to the vertices. We will prove *D*, *E*, *F*, *L*, *M*, *N*, *P*, *Q*, *R* lie on the same circle. Note that it suffices to show *L* and *P* lie on the circumcircle of triangle *DEF*. If this can be done, then *M*, *N*, *Q*, *R* also lie on that circle by symmetry. Moreover, the second part of Theorem 2.2-3 is clear because nine-point circle is the circumcircle of the medial triangle.

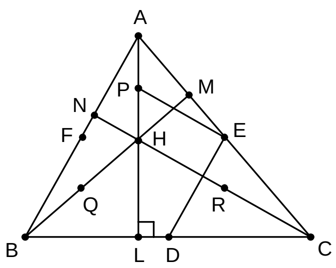


Figure 17

Consider the circle ω having *PD* as diameter. *L* lies on ω since $\angle DLP = 90°$. On the other hand, we note that $PE \parallel HC$, $DE \parallel BA$, and $HC \perp BA$, these imply $\angle DEP = 90°$ and therefore *E* (and by symmetry *F*) also lies on ω. We have proved that *D*, *L*, *E*, *P*, *F* lie on the same circle.

Q.E.D.

Theorem 2.2-4

The nine-point center *N* lies on the Euler line, and it is the midpoint between the circumcenter *O* and the orthocenter *H*.

The proof of Theorem 2.2-4 is left to the reader as exercise.

Exercise

1. Prove Theorem 2.2-4. If Theorem 2.2-1 is used in your solution, try to find an alternative proof without using it. (It is just for practice. There is no reason for which we can't use Theorem 2.2-1.)

2. Refer to Figure 17, let *X*, *Y* be the intersection points between the tangent line of the nine-point circle at *D* with *CA*, *AB* respectively. Prove that *B*, *C*, *X*, *Y* are concyclic.

3. Recall that the external bisectors of any two angles of a triangle are concurrent with the internal bisector of the third angle. The point of intersection is called an **excenter**（旁心）of the triangle. The circle centered at the excenter which touches three sides of the triangle is called an **excircle**（旁切圓）of that triangle . Note that a triangle has three excenters and excircles. Prove that an excircle of a triangle is tangent to the nine-point circle of the same triangle.

2.3 Geometric calculation

Up to now we always concentrate on proofs. Practically it is necessary to calculate the geometric quantities such as length, angle, area, circumradius, inradius, … etc. The objective of this section is to provide the fundamental tools for calculating the quantities related to triangles.

This section can also be viewed as some sort of appendix of this chapter. All the calculations are straightforward and no tricky example is presented. Throughout this section, we adopt the notations a, b, c, m_a, m_b, m_c, h_a, h_b, h_c, t_a, t_b, t_c … defined in the very beginning of chapter 1. Most of these definitions are standardized and we do not explain them anymore.

We all know that for a triangle *ABC*, if the length of its three sides are given then the shape of the triangle is fixed. In principle, any quantity associated with ΔABC can be expressed in terms of a, b, c. Our job in this section is to find out these expressions. We begin with sine law and cosine law which readers are supposed to be familiar with them.

Theorem 2.3-1 (Sine law)

In any triangle *ABC*, one has

$$\frac{a}{\sin\angle A}=\frac{b}{\sin\angle B}=\frac{c}{\sin\angle C}=2R.$$

Theorem 2.3-2 (Cosine law)

With the same notations, we have

$$\cos\angle A=\frac{b^2+c^2-a^2}{2bc}.$$

Similarly,

$$\cos\angle B=\frac{c^2+a^2-b^2}{2ca} \quad \text{and} \quad \cos\angle C=\frac{a^2+b^2-c^2}{2ab}.$$

- Cosine law gives expressions for $\angle A$, $\angle B$, $\angle C$ in terms of *a*, *b*, *c*.

Theorem 2.3-3 (Area of a triangle)

The area of triangle *ABC* is given by any one of the following expressions:

$$\frac{1}{2}ah_a \qquad \frac{1}{2}ab\sin\angle C \qquad \sqrt{p(p-a)(p-b)(p-c)}$$

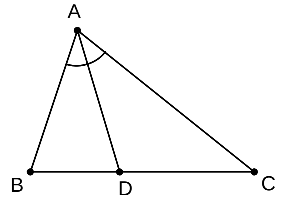
$$rp \qquad \frac{abc}{4R} \qquad 2R^2\sin\angle A\sin\angle B\sin\angle C$$

$$\frac{a^2\sin\angle B\sin\angle C}{2\sin(\angle B+\angle C)}$$

- Recall that $p = (a + b + c) / 2$ is the semi-perimeter. Some authors use the letter *s* instead.

Theorem 2.3-4 (Angle bisector theorem)

Suppose the internal angle bisector of $\angle CAB$ intersects *BC* at point *D*.
Then $\frac{BD}{CD}=\frac{AB}{AC}$.



Proof

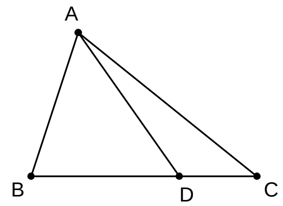
$$\frac{BD}{DC}=\frac{(ABD)}{(ADC)}=\frac{\frac{1}{2}\times AB\times AD\times\sin\angle DAB}{\frac{1}{2}\times AC\times AD\times\sin\angle CAD}=\frac{AB}{AC}$$

Q.E.D.

Theorem 2.3-5 (Stewart Theorem)

Let D be a point on the side BC of triangle ABC. Then

$$AD^2=\frac{AB^2\times CD+AC^2\times BD}{BC}-BD\times CD.$$



Proof

Note that $\angle ADB$ = 180° $-\angle ADC$, so $\cos\angle ADB = -\cos\angle ADC$. Apply cosine law to both sides, we obtain

$$\frac{AD^2+BD^2-AB^2}{2\times AD\times BD}=-\frac{AD^2+CD^2-AC^2}{2\times AD\times CD},$$

the result follows by straightforward algebra.

Q.E.D.

Example 2.3-1

The inradius r of a triangle is given by

$$r=\frac{(ABC)}{p}=\frac{\sqrt{p(p-a)(p-b)(p-c)}}{p}=\frac{1}{2}\sqrt{\frac{(a+b-c)(b+c-a)(c+a-b)}{a+b+c}}$$

and the circumradius R is given by

$$R=\frac{abc}{4(ABC)}=\frac{abc}{4\sqrt{p(p-a)(p-b)(p-c)}}=\frac{abc}{\sqrt{(a+b+c)(a+b-c)(b+c-a)(c+a-b)}}.$$

Exercise

1. Prove the following formulae:

   (a) $m_a=\frac{1}{2}\sqrt{2b^2+2c^2-a^2}$.

(b) $t_a = \dfrac{2}{b+c}\sqrt{bcp(p-a)} = \sqrt{bc\left(1-\left(\dfrac{a}{b+c}\right)^2\right)}$.

(c) $h_a = \dfrac{2}{a}\sqrt{p(p-a)(p-b)(p-c)}$.

2. Show that $\cos\dfrac{\angle A}{2} = \sqrt{\dfrac{p(p-a)}{bc}}$.

3. (a) (Leibnitz Theorem) Let G be the centroid of triangle ABC and P is an arbitrary point. Prove that

$$PA^2 + PB^2 + PC^2 = 3PG^2 + \frac{1}{3}(a^2 + b^2 + c^2).$$

(b) Hence, or otherwise, find the formula of OG in terms of a, b, c, where O is the circumcenter.