MOP 2004 Team Contest #4

June 25, 1:00 PM

Problems

1. Let n be a positive integer, and let $A = \{n, n+1, \ldots, n+17\}$. Find all values of n for which we can partition A into two subsets B and C such that the product of the elements of B equals the product of the elements of C.

2. Let $ABCD$ be a parallelogram. Let M and N be points on sides AB and BC respectively such that $AM = CN$. Lines AN and CM meet at Q. Prove that line DQ bisects angle ADC.

3. The incircle of triangle ABC is tangent to sides AB, BC, CA at P, Q, R respectively. Prove that
$$\frac{BC}{PQ} + \frac{CA}{QR} + \frac{AB}{RP} \geq 6.$$

4. Let P be the set of all prime numbers. Let M be a subset of P with at least three elements. Suppose that for any nonempty proper subset A of M, all prime divisors of the integer $\prod_{p \in A} p - 1$ are in M. Prove that $M = P$.

5. Find all ordered triples (x, y, z) of real numbers which satisfy the following system of equations:
$$\begin{aligned} xy &= z - x - y \\ yz &= x - y - z \\ zx &= y - z - x \end{aligned}$$

6. How many ways can 8 mutually non-attacking rooks be placed on a 9×9 chessboard so that all 8 rooks are on squares of the same color?

7. Let A, B, C, D be four points on a circle (occurring in clockwise order) with $AB < AD$ and $BC > CD$. Let the bisector of angle BAD meet the circle at X and the bisector of angle BCD meet the circle at Y. Consider the hexagon formed by these six points on the circle. If four of the six sides of the hexagon have equal length, prove that BD must be a diameter of the circle.

8. Let p be an odd prime. Prove that
$$\sum_{k=1}^{p-1} k^{2p-1} \equiv \frac{p(p+1)}{2} \pmod{p^2}.$$

9. Let T be the set of all positive integer divisors of 2004^{100}. What is the largest possible number of elements that a subset S of T can have if no element of S is an integer multiple of any other element of S?

10. Let I be the incenter of triangle ABC and let A_1, B_1, C_1 be arbitrary points on segments AI, BI, CI, respectively. The perpendicular bisectors of AA_1, BB_1, CC_1 intersect at A_2, B_2, C_2. Prove that the circumcenter of triangle $A_2B_2C_2$ coincides with the the circumcenter of triangle ABC if and only if I is the orthocenter of triangle $A_1B_1C_1$.

11. Find all primes $p \geq 3$ with the following property: for any prime $q < p$, the number
$$p - \left\lfloor \frac{p}{q} \right\rfloor$$
is squarefree.

12. Let G be a graph with n vertices containing no triangles. Suppose that for every partition of the vertices of G into two sets A and B, either A or B contains two adjacent vertices. Prove that there exists a vertex of G with degree at most $\frac{2}{5}n$.

13. An $n \times n$ table is filled with real numbers such that no two rows are identical. Prove that it is possible to remove a column of the table such that the resulting $n \times (n-1)$ table also has pairwise distinct rows.

14. Let a, b, c, d be positive integers such that the set $\{(x, y) \mid 0 < x, y < 1,\ ax + by \in \mathbb{Z},\ cx + dy \in \mathbb{Z}\}$ contains 2004 elements. If $\gcd(a, c) = 6$, find $\gcd(b, d)$.

15. Find all real values of α for which there exists exactly one function $f : \mathbb{R} \to \mathbb{R}$ satisfying the equation

$$f(x^2 + y + f(y)) = f(x)^2 + \alpha y$$

for all x, $y \in \mathbb{R}$.