TJUSAMO Practice #2: Introductory Geometry

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Geometry is one of the four major topics of the USAMO. There is always a geometry problem on the USAMO, and often even two problems. It is true that there are numerous methods, theorems, and special points any geometry master must know, but there are only a few ideas you need to know to start tackling geometry problems.

To figure out where to start, let us examine 10 geometry problems from the USAMOs of the third millennium:

1. (USAMO 2000 #2) Let S be the set of all triangles ABC for which

$$5\left(\frac{1}{AP}+\frac{1}{BQ}+\frac{1}{CR}\right)-\frac{3}{\min\{AP,BQ,CR\}}=\frac{6}{r},$$

   where r is the inradius and P, Q, R are the points of tangency of the incircle with sides AB, BC, CA, respectively. Prove that all triangles in S are isosceles and similar to one another.

   Official solution involves:

   - incenter
   - angle chasing
   - lots of trig

2. (USAMO 2000 #5) Let $A_1A_2A_3$ be a triangle and let ω_1 be a circle in its plane passing through A_1 and A_2. Suppose there exist circles $\omega_2, \omega_3, \ldots, \omega_7$ such that for $k = 2, 3, \ldots, 7$, ω_k is externally tangent to ω_{k-1} and passes through A_k and A_{k+1}, where $A_{n+3} = A_n$ for all $n \geq 1$. Prove that $\omega_7 = \omega_1$.

   Official solution involves:

   - angle chasing

3. (USAMO 2001 #2) Let ABC be a triangle and let ω be its incircle. Denote by D_1 and E_1 the points where ω is tangent to sides BC and AC, respectively. Denote by

D_2 and E_2 the points on sides BC and AC, respectively, such that $CD_2 = BD_1$ and $CE_2 = AE_1$, and denote by P the point of intersection of segments AD_2 and BE_2. Circle ω intersects segment AD_2 at two points, the closer of which to the vertex A is denoted by Q. Prove that $AQ = D_2P$.

Official solution involves:

- Menelaus
- vertex contraction
- length chasing

4. (USAMO 2001 #4) Let P be a point in the plane of triangle ABC such that the segments PA, PB, and PC are the sides of an obtuse triangle. Assume that in this triangle the obtuse angle opposes the side congruent to PA. Prove that $\angle BAC$ is acute.

   Official solution involves:

   - Ptomely's inequality
   - Cauchy-Schwarz
   - Law of Cosines

5. (USAMO 2001 #6) Each point in the plane is assigned a real number such that, for any triangle, the number at the center of its inscribed circle is equal to the arithmetic mean of the three numbers at its vertices. Prove that all points in the plane are assigned the same number.

   Official solution involves:

   - incircles
   - similar triangles
   - multiple clever constructions

6. (USAMO 2002 #2) Let ABC be a triangle such that

$$\left(\cot \frac{A}{2}\right)^2 + \left(2\cot \frac{B}{2}\right)^2 + \left(3\cot \frac{C}{2}\right)^2 = \left(\frac{6s}{7r}\right)^2,$$

   where s and r denote its semiperimeter and its inradius, respectively. Prove that triangle ABC is similar to a triangle T whose side lengths are all positive integers with no common divisors and determine these integers.

   Official solution involves:

   - Cauchy-Schwarz
   - algebra
   - trig

7. (USAMO 2003 #2) A convex polygon $\mathcal{P}$ in the plane is dissected into smaller convex polygons by drawing all of its diagonals. The lengths of all sides and all diagonals of the polygon $\mathcal{P}$ are rational numbers. Prove that the lengths of all sides of all polygons in the dissection are also rational numbers.

   Official solution involves:

   - coordinates
   - trig
   - algebra

8. (USAMO 2003 #4) Let ABC be a triangle. A circle passing through A and B intersects segments AC and BC at D and E, respectively. Lines AB and DE intersect at F while lines BD and CF intersect at M. Prove that $MF = MC$ if and only if $MB \cdot MD = MC^2$.

   Official solution involves:

   - cyclic quadrilaterals
   - similar triangles
   - Ceva

9. (USAMO 2004 #1) Let $ABCD$ be a quadrilateral circumscribed about a circle, whose interior and exterior angles are at least 60°. Prove that

$$\frac{1}{3}|AB^3 - AD^3| \leq |BC^3 - CD^3| \leq 3|AB^3 - AD^3|.$$

   When does equality hold?

   Official solution involves:

   - Law of Cosines
   - incircles

10. (USAMO 2004 #6) A circle ω is inscribed in a quadrilateral $ABCD$. Let I be the center of ω. Suppose that

$$(AI + DI)^2 + (BI + CI)^2 = (AB + CD)^2.$$

   Prove that $ABCD$ is an isosceles trapezoid.

   Official solution involves:

   - incircles
   - AM-GM
   - Ptolemy

As you can see, most USAMO-level geometry problems do not require advanced techniques. Familiarizing yourself with topics such as angle chasing(using similar triangles and cyclic quadrilaterals), special points(namely the incenter), coordinates, trigonometry, and common equations and inequalities will allow you make decent headway into these problems. Let's start with the most basic but fundamental topics in geometry.

1 Circles

A circle is the set of all points that are a positive distance r from a point O. r is the **radius**. O is the **center**. Most geometry problems involving circles also involve a polygon either **inscribed** inside a circle(all its vertices are on the circle), or **circumscribed** around a circle(all its sides are tangent to the circle).

1.1 Angles

Circles have $360°$, or 2π radians. The measure of an inscribed angle is half of the length of the arc it intercepts, no matter where on the circle the vertex lies. The measure of a central angle is equal to the length of its arc.

1.2 Power of a Point

To find the power of a point P relative to a circle ω, choose any point on the circle A, and draw line PA. Let B be the point of intersection of PA with ω other than A. If A is the only point of intersection, the power is PA^2. Otherwise the power is $PA * PB$. The power of a point with respect to a given circle is constant. Also, the point can be inside, outside, or on the circle. Note that a point on the circle will have a power of 0 with respect to that circle.

1.3 Cyclic Quadrilaterals

A **cyclic quadrilateral** is a polygon with four sides that can be inscribed in a circle. All cyclic quads are convex, since all their angles are less than $180°$. Opposite angles in a cyclic quad add up to $180°$. The converse is also true. If a quadrilateral's opposite angles add up to $180°$, then it is cyclic. This is the easiest way to find a cyclic quad. Cyclic quads are usually tools to ease angle chasing, but sometimes, proving a quadrilateral is cyclic will be the key step in a problem.

2 Triangles

A triangle is defined by three non-collinear points. Triangles are the simplest shapes, and also the most important ones. Over half of the geometry problems you'll see are based on a triangle(probably $\triangle ABC$). There are several key triangle terms one must recognize.

Let $\triangle ABC$ be a triangle. Its vertexes are A, B, and C. Its angles are $\angle A$, $\angle B$, and $\angle C$. Its sides are AB, BC, and AC.

- **cevian**: A line segment connecting a vertex with a point on the opposing side.

- **median**: A line segment connecting a vertex with the midpoint of the opposing side.
- **centroid**: The intersection point of the three medians.
- **incircle**: A circle inscribed within a polygon. Its center is the **incenter**.
- **excircle**: A circle escribed about a triangle. It is tangent to one side as well as the extensions of other two sides. Its center is the **excenter**.
- **circumcircle**: A circle circumscribed around a polygon. Its center is the **circumcenter**.
- **altitude**: A line segment connecting a vertex with a point on the opposite side and perpendicular to that opposite side.
- **orthocenter**: The point at which the three altitudes intersect. This is not the center of an "orthocircle".

2.1 Similar Triangles

When two triangles are **similar**, all their corresponding angles are equal and all their corresponding side lengths are in the same ratio. When two triangles are **congruent**, all their corresponding side lengths and angles are equal. There are multiple ways to prove two triangles are similar/congruent.

- **AA similarity**: two pairs of angles are equal. Clearly the last pair is also equal.
- **SAS similarity**: two pairs of side lengths have the same ratio, and the angle between the two sides is the same in both triangles. If both pairs of sides are equal, then the triangles are congruent.
- **SSS similarity**: all corresponding side lengths have the same ratio. If all corresponding side lengths are the same, then the triangles are congruent.
- **SSA similarity**: this one is a bit tricky. When two pairs of side lengths have the same ratio, and one of the angle pairs not between the two sides is the same, this does not guarentee similarity. There are usually still two possible angles for the angle in between the two sides, but if you can somehow eliminate one of these possibilities, then you can still prove similarity or congruence.

2.2 Angle Chasing

Now that we are armed with tools, we can start angle chasing. This is the most important skill a geometer must learn. Here are some tips for how one should go about angle chasing.

- Angle chasing is not making trivial observations such as: "hey, these two angles are vertical, they must be the same!" Instead, you must use simialar triangles, cyclic quads, angle bisectors, etc to discover new angle measures.

- Figure out what you want to prove first, then prove it. Don't angle chase purposelessly.
- Try splitting an angle into two different angles.
- If you know two angles add up to 180°, try making a cyclic quad out of them.
- Try constructing a parallel line to find more angles.
- Likewise, try constructing a perpendicular line or a perpendicular bisector.
- Sometimes, if an angle can't be changed in any other way, try using $\angle ABC = 180^\circ - \angle BAC - \angle ACB$.
- Identify which angles can be easily expressed in terms of $\angle A, \angle B, \angle C$.

3 Common Well-Known Theorems

What's a well-known theorem? It probably is NOT a theorem that most people know. However, you are allowed to cite any of these theorems on the USAMO or in any proof you write. Well-known theorems sometimes are named about famous people, and it's imperative that you know these names to use the theorems. Note however, that if a proof is nearly identical to one of these theorems, you should prove it yourself so that you do not trivialize a problem by using one theorem.

3.1 Law of (insert trig function)s

3.1.1 Law of Sines(Extended)

In a triangle ABC with sides a, b, c and circumradius R:

$$\frac{a}{\sin \angle A} = \frac{b}{\sin \angle B} = \frac{c}{\sin \angle C} = 2R$$

3.1.2 Law of Cosines

For a triangle ABC with sides a, b, c:

$$c^2 = a^2 + b^2 - 2ab\cos \angle C$$

If $\angle C = 90^\circ$, this reduces to the Pythagorean theorem.

3.1.3 Law of Tangents

For a triangle ABC with sides a, b, c:

$$\frac{a-b}{a+b} = \frac{\tan(\frac{\angle A - \angle B}{2})}{\tan(\frac{\angle A + \angle B}{2})}$$

This theorem is rarely used, but included here for completeness.

3.2 Area of a triangle

Let brackets denote area of a triangle ABC with sides a, b, c, inradius r, semiperimeter s, circumradius R, altitude d on BC, exradius r_a on the excircle opposite A, then:

$$[ABC] = \frac{ad}{2} = \frac{ab\sin\angle C}{2} = rs = \sqrt{s(s-a)(s-b)(s-c)} = \frac{abc}{4R} = (s-a) * r_a$$

3.3 Ceva's Theorem

Let AD, BE, CF be cevians of triangle ABC. These cevians concur iff

$$\frac{AE}{EC} * \frac{CD}{DB} * \frac{BF}{FA} = 1$$

3.3.1 Trigonometric Ceva

Let AD, BE, CF be cevians of triangle ABC. These cevians concur iff

$$\frac{\sin\angle BAD}{\sin\angle CAD} * \frac{\sin\angle CBE}{\sin\angle ABE} * \frac{\sin\angle ACF}{\sin\angle BCF} = 1$$

3.4 Angle Bisector Theorem

Let AD be an angle bisector of $\angle BAC$ such that D is on BC. Then,

$$\frac{AB}{BD} = \frac{AC}{CD}$$

3.5 Stewart's Theorem

Let D be a point on side BC of triangle ABC with sides a, b, c. Let m be BD and n be DC. Then,

$$dad + man = bmb + cnc$$

This is often pronounced "dad plus man equals bomb plus sink" as a mnemonic.

3.6 Nine point circle

For any triangle, the feet of the three altitudes, the midpoints of the line segments connecting the three vertices to the orthocenter, and the midpoints of each side lie on the same circle. The center of this circle is the nine point center.

3.7 Euler Line

For any triangle with orthocenter H, nine point center N, centroid G, and circumcenter O, the four special points are collinear in that order, and

$$2 * HN = 6 * NG = 3 * GO$$

To remember this easily, think of an isosceles right triangle.

3.8 Radical Axis

Given two circles with distinct centers X and Y, the locus of all points that have equal powers with respect to the two circles is a line perpendicular to XY. This line is called the radical axis. Given three circles with non-collinear centers, there exists one point with equal powers to all three circles called the radical center. The radical center is the intersection of the three radical axes formed by the three pairs of circles.

3.9 Triangle Inequality

A triangle has sides a, b, c. Then,

$$a + b > c, \quad b + c > a, \quad c + a > b$$

If the triangle is acute,

$$a^2 + b^2 > c^2, \quad b^2 + c^2 > a^2, \quad a^2 + c^2 > b^2$$

Else, exactly one of the last three inequalities is false.

3.10 Ptolemy's Inequality

In a quadrilateral $ABCD$,

$$AB * CD + BC * DA \geq AC * BD$$

with equality iff $ABCD$ is cyclic.

3.11 Trichotomy

For two reals a, b, exactly one of the following three conditions is true:

$$a < b, \quad a = b, \quad a > b$$

This is obvious, but can be a powerful tool. When you're trying to prove two angles or two lines are equal, assume one is greater, then find a contradiction. This is considered an "indirect" proof, but it can be much simpler than any other method.

4 The Art of Geometry Problem Solving

Geometry is a special and distinct field of mathematics, so our methods of approaching a geometry problem are also different.

- Write down all the givens of the problem in their simplest forms. These facts will most likely be used in one way or another.
- Write down what you're trying to prove, and some facts equivalent to this result.
- Try starting from your result, and work backwards. Try to get closer to your givens.
- Keep separate tabs of what you've figured out working forwards and working backwards.
- If you get stuck, look back at your givens. Chances are that you haven't used one of them.
- Draw a big, neat diagram. This cannot be stressed enough. More on this later.
- Don't start by brute forcing with coordinates or complex numbers. This might give you a headache in the long run, and it is likely that it won't work. Instead, try to identify what ideas the problem might involve.
- Don't let the problem intimidate you. When looking at a complicated problem, don't think: "ugh, I probably don't know the background necessary to see the clever/fast solution, so I'll try brute force." Once again, geoemtry problems usually don't require much complicated knowledge. Try to play around with the problem until something dawns on you. This way, if you have a chance of solving it, you don't wasting time not thinking.
- Don't stick with one idea too long. It it's clear you aren't making any progress angle chasing, try another way, such as a construction or a theorem.
- Remember that common theorems have a higher chance of being applicable to the problem.

- If a problem has multiple parts, the key step of the first part often involves using a well-known theorem or result.
- The only way you'll ever get good at geometry is by doing problems. You must have seen enough methods and results so that you are familiar with any problem you are given.

4.1 Diagrams

"Hah a diagram, I don't need a diagram!" Don't be foolish like that kid. Every problem needs several diagrams. In fact, there's almost an "art" to drawing a diagram. Get it? Anyways, here are some guidelines.

- Always draw big. Give enough space to make markings and constructions.
- Draw your diagram multiple times until you are satisfied with it.
- Make sure all the givens look somewhat to scale in your diagram.
- Draw multiple cases to familiarize yourself with the problem, such as drawing an acute triangle and an obtuse triangle.
- Don't clog up the diagram with needless lines or repetitive marks. By keeping the diagram clear, you'll keep your mind clear. Don't make constructions without motivation. If it gets messy, draw a new one.
- When you attempt to draw a polygon inscribed in a circle, draw the circle first. Compasses help for this.
- Keep your lines straight. A straightedge will help.
- When you aren't sure if two segments will intersect or if several lines should concur, etc, try to draw the diagram to scale using compass and ruler. This will give you a better idea of what you're trying to prove.
- If a problem introduces a generic triangle, try to make it very scalene. A $45°$-$60°$-$75°$ triangle usually does the trick.

5 Problems

As always, each problem will be followed by a difficulty rating between 1 and 5, inclusive. First, a key to the rating system:

1. easy - few steps
2. pre-USAMO

3. USAMO #1/#4
4. USAMO #2/#5
5. USAMO #3/#6 and beyond

Here they are:

1. {1-1.5} Show the concurrency of the medians, the altitudes, and the internal angle bisectors.
2. {1-2.5} Prove every theorem in the theorems section. This is an excellent exercise.
3. {1} Prove that the sum of the angles in a triangle is $180°$ without using any information related to a triangle or a circle.
4. {1} Let the incircle of triangle ABC touch BC at D, CA at E, and AB at F. Prove that AD, BE, and CF are concurrent.
5. {1.5} Let $\triangle ABC$ have an altitude AD and orthocenter H. Let X be the intersection of AD with the circumcircle of ABC, such that A and X lie on opposite sides of D. Prove that $HD = DX$.
6. {2} Let ABC be a triangle with orthocenter H and circumcenter O. Prove that $\angle HAO = |\angle B - \angle C|$.
7. {2} Let point P be inside triangle ABC. Let A_1, B_1, and C_1 be the foot of the perpendicular from P to BC, AC, and AB, respectively. Let A_2, B_2, and C_2 be the foot of the perpendicular from P to B_1C_1, A_1C_1, and A_1B_1, respectively. Let A_3, B_3, and C_3 be the foot of the perpendicular from P to B_2C_2, A_2C_2, and A_2B_2, respectively. Prove that $\triangle ABC$ is similar to $\triangle A_3B_3C_3$.
8. (Coxeter/Greitzer) {2} Let P be a point on the minor arc of CD in the circumcircle of square $ABCD$. Prove that

$$PA(PA + PC) = PB(PB + PD)$$

9. (Coxeter/Greitzer) {2} Cevian AQ is extended to meet the circumcircle of an equilateral triangle ABC at point P. Prove that

$$\frac{1}{PB} + \frac{1}{PC} = \frac{1}{PQ}$$

10. (Coxeter/Greitzer) {2} Let $\angle PAB = \angle PBA = 15°$. Square ABCD is drawn so that P lies within. Prove that $PC = CD$.
11. {2} A triangle has sides 12, 15, 18. Find the ratio of the largest angle to the smallest angle.

12. $\{2\}$ Let $\triangle ABC$ have excenters X, Y, Z. Prove that the circumcircle of $\triangle ABC$ is the nine point circle of $\triangle XYZ$.

13. (Coxeter/Greitzer) $\{2.5\}$ Let PT and PB be two tangents to a circle with diameter AB. Let H be the foot of the perpendicular from T to AB. Let X be the intersection of AP and TH. Prove that $HX = TX$.

14. $\{2.5\}$ Find a construction to trisect a line segment.

15. $\{2.5\}$ Let ABC be a triangle. Let D be the midpoint of side BC. Let E be a point on BC such that AE is the angle bisector of $\angle BAC$. The circumcircle of DEA intersects AB at X and AC at Y. Prove that $BX = CY$.

16. (MOP '05) $\{2.5\}$ Circles ω_1 and ω_2 intersect at P and Q. A line is tangent to ω_1 at A and ω_2 at B. A different line is tangent to ω_1 at C and ω_2 at D. A third line connects the centers of ω_1 and ω_2, and intersects AC at M and BD at N. Prove that PNQM is a rhombus.

17. (MOP '05) $\{3\}$ Let ABCD by a convex quadrilateral. If $\angle C = \angle D = 120°$, prove that
$$(AD + CD)^3 + (BC + CD)^3 \leq 2AB^3$$

18. $\{3.5\}$ Find the minimum integer n such that the following condition holds: any triangle can be dissected into k isosceles triangles, where k is an integer greater than n.

19. (MOP '05) $\{3.5\}$ A fish is trapped in a circular pool ω_1. There is a shark that can only swim on ω_1(the edge of the circle). The shark swims four times faster than the fish. The fish can swim anywhere inside ω_1. If the shark uses his best strategy to stop it, determine if the fish can ever swim to the edge of the pool ω_1 without the shark being there.

20. $\{3\text{-}5\}$ Solve the 10 USAMO problems used in the introduction section. You might need more information than provided in this lecture.