New Zealand
Maths Olympiad Committee
Camp 2009
Friday Test

Instructions

- You have four and one half hours to work on this test.
- Each question is worth seven points.
- You may ask for clarifications *in writing* only during the first half hour.

Questions

1. Let a triangle ABC be given. Let E be the midpoint of AC, F be the midpoint of CB, and G the foot of the perpendicular from C to AB (or its extension.) Prove that ABC is an isosceles triangle if and only if EFG is also isosceles.

2. Find all positive integers whose first digit is 6 and which have the property that when this digit is deleted, the number is reduced by a factor of 25. Show that no positive integer has the property that when its first digit is deleted it is reduced by a factor of 35.

3. Let $a_n = 1 + 1/n - 1/n^2 - 1/n^3$. Find the least positive integer k such that
$$a_2 a_3 \cdots a_k > 1000.$$

4. Let n be a positive integer. Prove that the number:
$$\sum_{k=n}^{2n} \frac{1}{k}$$
is not an integer.

5. How many sequences $a_1, a_2, \ldots, a_{2009}$ are there such that each of the numbers $1, 2, \ldots, 2009$ occurs in the sequence, and $i \in \{a_1, a_2, \ldots, a_i\}$ for all $i \geq 2$?

6. Let $ABCD$ be a square, and let M be the point of intersection of its diagonals. Given a point P inside BMC, let E be the intersection of line DP and CM, and F the intersection of line CP and BM. Determine the set of such points P for which EF is parallel to AD.

7. The squares of an $n \times n$ chessboard are coloured black and white in the usual way, with a black square in the upper left hand corner. A series of re-colourings are then applied according to the following rule: a 2×3 or 3×2 rectangle containing three white squares is chosen, and the white squares are coloured black. For which values of n, if any, is it possible to colour all the white squares black in this way?

8. For which positive integers n does there exist a polynomial $P(x)$ with integer coefficients, such that $P(d) = n/d$ for all positive divisors d of n?