MATHEMATICAL OLYMPIAD SUMMER PROGRAM 1999

TRIGONOMETRIC EQUALITIES, EQUATIONS AND INEQUALITIES

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Note: For $\triangle ABC$ we denote by α, β, γ, a, b, c, h_a, h_b, h_c, l_a, l_b, l_c, r, R, r_a, r_b, r_c and S its angles, sides, altitudes, angle bisectors, inradius, circumradius, exradii and area. If a problem does not refer to a triangle, then we use x, y, z, etc. to denote arbitrary angles.

1. Prove the equalities:
   (a) $\cos\frac{\pi}{19}+\cos\frac{3\pi}{19}+\cdots+\cos\frac{17\pi}{19}=\frac{1}{2}$.
   (b) $\cos\frac{2\pi}{21}+\cos\frac{4\pi}{21}+\cdots+\cos\frac{20\pi}{21}=-\frac{1}{2}$.
   (c) $\mathrm{tg}1^\circ+\mathrm{tg}5^\circ+\mathrm{tg}9^\circ+\cdots+\mathrm{tg}177^\circ=45$.
   (d) $\mathrm{tg}x+2\mathrm{tg}2x+4\mathrm{tg}4x+8\mathrm{ctg}8x=\mathrm{ctg}x$.
   (e) $4\cos x\cos y\cos z=\cos(x+y+z)+\sum\cos(-x+y+z)$.

2. If $\alpha+\beta+\gamma=\pi$, prove that
   (a) $\sin\alpha+\sin\beta+\sin\gamma=4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$.
   (b) $\sum\mathrm{tg}\frac{\alpha}{2}\mathrm{tg}\frac{\beta}{2}=1$, $\sum\mathrm{ctg}\frac{\alpha}{2}=\prod\mathrm{ctg}\frac{\alpha}{2}$.
   (c) $\sum\mathrm{ctg}\,\alpha\,\mathrm{ctg}\,\beta=1$ $\sum\mathrm{tg}\,\alpha=\prod\mathrm{tg}\,\alpha$.
   (d) (Revisited) For $x,y,z>0$:
   $$xa^2+yb^2+zc^2\geq 4S\sqrt{xy+yz+zx}.$$

3. If $0<x,y,z<\pi$ and $\mathrm{tg}\frac{x}{2}, \mathrm{tg}\frac{y}{2}, \mathrm{tg}\frac{z}{2}$ are roots of the equation $t^3+pt^2+t+q=0$, then
$$\mathrm{tg}x+\mathrm{tg}y+\mathrm{tg}z=\mathrm{tg}x\,\mathrm{tg}y\,\mathrm{tg}z.$$

4. $\cos^2x+\cos^2y+\cos^2z+2\cos x\cos y\cos z=1$ iff $x\pm y\pm z=(2k+1)\pi$ for $k\in\mathbb{Z}$.

5. Show that the given number is a root of the equation and find the other two roots.
   (a) $x^3-5x^2+6x-1=0$, $4\cos^2\left(\frac{2\pi}{7}\right)$;
   (b) $x^3-33x^2+27x-3=0$, $\mathrm{tg}^2 80^\circ$.

11. Let $k \in \mathbb{R}$ and $n \in \mathbb{Z}$.

(a) Prove that $8k \sum \sin n\alpha \leq 12k^2 + 9$.

(b) Find for which values of k (a) becomes equality. Further, show that

$$|\sin n\alpha| \leq \frac{3\sqrt{3}}{2}.$$

12. Let T be $\triangle ABC$, and let P be in the plane of T, different from the vertices of T. Prove that there exists triangle $T_0 = T_0(P)$, possibly degenerate, with sides $a \cdot PA$, $b \cdot PB$ and $c \cdot PC$. If $R_0 = R_0(p)$ is the circumradius of T_0, find the set of points P for which $PA \cdot PB \cdot PC \leq R \cdot R_0$. When is equality attained?

13. Let P be a point inside $\triangle ABC$. Lines through P, parallel to the sides of the triangle, intersect the other sides in points B_1 and B_2, A_1 and A_2, and C_1 and C_2, with B_1, $A_2 \in AB$; C_1, $B_2 \in BC$; and A_1, $C_2 \in AC$. Prove that

(a) $S_{A_1B_1C_1} \leq \frac{1}{3} S_{ABC}$.

(b) $S_{A_1C_2B_1A_2C_1B_2} \geq \frac{2}{3} S_{ABC}$.

14. For $\triangle ABC$ let $M = (R - 2r)/2r$. An inequality $P \geq Q$ for elements of $\triangle ABC$ is called *strong* (*weak*) if $P - Q \leq M$ $(P - Q \geq M)$.

(a) Prove that the inequality $\sum \sin^2 \frac{\alpha}{2} \geq \frac{3}{4}$ is strong.

(b) Prove that the inequality $\sum \cos^2 \frac{\alpha}{2} \geq \sum \sin \beta \sin \gamma$ is weak.

15. Consider the known inequalities: $\sum \operatorname{tg}^2 \frac{\alpha}{2} \geq 1$; $2 - 8 \prod \sin \frac{\alpha}{2} \geq 1$. Prove or disprove:

$$\operatorname{tg}^2 \frac{\alpha}{2} + \operatorname{tg}^2 \frac{\beta}{2} + \operatorname{tg}^2 \frac{\gamma}{2} \geq 2 - 8 \sin \frac{\alpha}{2} \sin \frac{\beta}{2} \sin \frac{\gamma}{2}.$$

16. Prove the following inequalities:

(a) $(1 - \cos\alpha)(1 - \cos\beta)(1 - \cos\gamma) \geq \cos\alpha \cos\beta \cos\gamma$;

(b) $(1 + \cos 2\alpha)(1 + \cos 2\beta)(1 + \cos 2\gamma) + \cos 2\alpha \cos 2\beta \cos 2\gamma \geq 0$.

17. Let $\triangle ABC$ be such that $\sum \operatorname{tg}^2(\frac{\alpha}{2}) = l$, for some $1 \leq l < 2$. Prove that

$$\operatorname{tg} \frac{\gamma}{2} < \frac{\cos \frac{\alpha - \beta}{2}}{\sin \frac{\alpha + \beta}{2}}.$$

6. Prove that for an arbitrary triangle:

(a) $\sin\frac{\alpha}{2} = \sqrt{\frac{a^2-(b-c)^2}{4bc}}$; $\cos\frac{\alpha}{2} = \sqrt{\frac{p(p-a)}{bc}}$.

(b) $\frac{r}{a} = \frac{\sin\frac{\beta}{2}\sin\frac{\gamma}{2}}{\cos\frac{\alpha}{2}}$; $\frac{r}{4R} = \sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}$.

(c) $bc\cos^2\frac{\alpha}{2} + ca\cos^2\frac{\beta}{2} + ab\cos^2\frac{\gamma}{2} = p^2$.

(d) $\frac{\cos\frac{\alpha}{2}}{l_a} + \frac{\cos\frac{\beta}{2}}{l_b} + \frac{\cos\frac{\gamma}{2}}{l_c} = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$.

(e) $\mathrm{tg}^2\frac{\alpha}{2} + \mathrm{tg}^2\frac{\beta}{2} + \mathrm{tg}^2\frac{\gamma}{2} = \frac{r_a^2 + r_b^2 + r_c^2}{r_a r_b + r_b r_c + r_c r_a}$.

(f) $\left(\mathrm{tg}\frac{\alpha}{2} + \mathrm{tg}\frac{\beta}{2} + \mathrm{tg}\frac{\gamma}{2}\right)^2 = \frac{(r_a + r_b + r_c)^2}{r_a r_b + r_b r_c + r_c r_a}$.

7. Solve the trigonometric equations:

(a) $\mathrm{tg}x + \mathrm{tg}2x + \mathrm{tg}3x + \mathrm{tg}4x = 0$.

(b) $d\sin x + \mathrm{tg}x + 1 = \frac{1}{\cos x}$ for a fixed $d \in \mathbb{R}$.

(c) $\frac{1}{\cos x\cos 2x} + \frac{1}{\cos 2x\cos 3x} + \cdots + \frac{1}{\cos 100x\cos 101x} = 0$.

(d) $1 + 2\sum_{k=1}^{2^n-1}\cos 2kx = 0$ for $n \in \mathbb{N}$.

(e) $\mathrm{ctg}2x + 2\sum_{k=0}^{n}\frac{1}{2^k}\mathrm{tg}\frac{x}{2^k} = 0$ for $n \in \mathbb{N}$.

8. Solve the system of trigonometric equations provided $\cos x\cos y\cos z \neq 0$:

$$\begin{aligned}
\sin x\sin y &= \sin z + 3\cos x\cos y\\
\sin y\sin z &= \sin x - 5\cos y\cos z\\
\sin z\sin x &= \sin y - 3\cos z\cos x
\end{aligned}$$

9. Eliminate x, y, z from the following system provided $\cos x\cos y\cos z \neq 0 \neq \sin x\sin y\sin z$:

$$\begin{aligned}
\sin y\sin z\sin(y+z) &= a\cos^2 y\cos^2 z\\
\sin z\sin x\sin(z+x) &= b\cos^2 z\cos^2 x\\
\sin x\sin y\sin(x+y) &= c\cos^2 x\cos^2 y\\
\sin x\sin y\sin z &= d\cos x\cos y\cos z
\end{aligned}$$

10. Prove that if for all x $\sum_{k=1}^{n} a_1\cos kx \geq -1$, then $\sum_{k=1}^{n} a_k \leq n$.

11. Prove the inequalities:

(a) $4\sin 3x + 5 \geq 4\cos 2x + 5\sin x$.

(b) $8\cos x\cos 3x \leq 5 + 5\cos 2x + 8\cos x\sin 2x$.

(c) $\cos x + n\cos nx + 2n\cos 2nx + 1 + \frac{33}{16}n \geq 0$ for all integer $n \geq 0$.

(d) $\mathrm{tg}x + \mathrm{tg}2x + 2\mathrm{tg}4x + 4\mathrm{ctg}8x \geq 1$ when $\mathrm{tg}x > 0$.

(e) $\sin\frac{1}{n-1} - 2\sin\frac{1}{n} + \sin\frac{1}{n+1} > 0$ for all natural $n \geq 2$.

(f) $\frac{\sin^{n+2}x}{\cos^n x} + \frac{\cos^{n+2}x}{\sin^n x} \geq 1$ for all $x \in (0, \pi/2)$ and all integer $n \geq 0$.

(f) $\sin x + \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x + \cdots + \frac{1}{n}\sin nx > 0$ for all $x \in (0, \pi)$ and all $n \in \mathbb{N}$.

(g) $\sqrt{\frac{\sin(x-z)\sin z}{\cos x\cos^2 z}} + \sqrt{\frac{\sin(y-z)\sin z}{\cos y\cos^2 z}} \leq \sqrt{\mathrm{tg}x\,\mathrm{tg}y}$ if $\mathrm{tg}x$ and $\mathrm{tg}y \geq \mathrm{tg}z \geq 0$.

(h) $\prod_{k=0}^{n-1}\sin\frac{(2k+1)\pi}{2n} \geq \frac{1}{\sqrt{n^n}}$ for all $n \in \mathbb{N}$.

12. For an arbitrary triangle show the inequalities:

(a) $\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2} \leq \frac{1}{8}$.

(b) $\cos\alpha + \cos\beta + \cos\gamma \leq \frac{3}{2}$.

(c) $\sin^2\frac{\alpha}{2} + \sin^2\frac{\beta}{2} + \sin^2\frac{\gamma}{2} \geq \frac{3}{4}$.

(d) $\cos^2\frac{\alpha}{2} + \cos^2\frac{\beta}{2} + \cos^2\frac{\gamma}{2} \leq \frac{9}{4}$.

(e) $\cos\frac{\alpha}{2} + \cos\frac{\beta}{2} + \cos\frac{\gamma}{2} \leq \frac{3\sqrt{3}}{2}$.

(f) $\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2} \leq \frac{3\sqrt{3}}{8}$.

(g) $\sin\alpha + \sin\beta + \sin\gamma \leq \frac{3\sqrt{3}}{2}$.

(h) $\sin^2\alpha + \sin^2\beta + \sin^2\gamma \leq \frac{9}{4}$.

(i) $\cos^2\alpha + \cos^2\beta + \cos^2\gamma \geq \frac{3}{4}$.

(j) $\frac{1}{\sin^2\alpha + \sin^2\beta} + \frac{1}{\sin^2\beta + \sin^2\gamma} + \frac{1}{\sin^2\gamma + \sin^2\alpha} \geq 2$.

(k) $\operatorname{tg}\frac{\alpha}{2} + \operatorname{tg}\frac{\beta}{2} + \operatorname{tg}\frac{\gamma}{2} \geq \sqrt{3}$.

(l) $\operatorname{tg}^2\frac{\alpha}{2} + \operatorname{tg}^2\frac{\beta}{2} + \operatorname{tg}^2\frac{\gamma}{2} \geq 1$.

(m) $\sin\alpha \sin\beta \sin\gamma \leq \frac{3\sqrt{3}}{8}$.

(n) $\sqrt[n]{\sin\alpha} + \sqrt[n]{\sin\beta} > \sqrt[n]{\sin\gamma}$.

(o) $\cos\frac{-\alpha+\beta+\gamma}{2} + \cos\frac{\alpha-\beta+\gamma}{2} + \cos\frac{\alpha+\beta-\gamma}{2} \leq \frac{3\sqrt{3}}{2}$.

(p) $\dfrac{\sin^5\alpha + \sin^5\beta + \sin^5\gamma - (\sin\alpha + \sin\beta + \sin\gamma)^5}{\sin^3\alpha + \sin^3\beta + \sin^3\gamma - (\sin\alpha + \sin\beta + \sin\gamma)^3} \leq \frac{15}{2}$.

(q) $\sqrt{5 + \operatorname{tg}\frac{\alpha}{2}\operatorname{tg}\frac{\beta}{2}} + \sqrt{5 + \operatorname{tg}\frac{\beta}{2}\operatorname{tg}\frac{\gamma}{2}} + \sqrt{5 + \operatorname{tg}\frac{\gamma}{2}\operatorname{tg}\frac{\alpha}{2}} \leq 4\sqrt{3}$.

(r) $\operatorname{tg}^2\frac{\alpha}{2} + \operatorname{tg}^2\frac{\beta}{2} + \operatorname{tg}^2\frac{\gamma}{2} - \operatorname{tg}^2\frac{\alpha}{2}\operatorname{tg}^2\frac{\beta}{2}\operatorname{tg}^2\frac{\gamma}{2} \geq \frac{26}{27}$.

(s) $\sin\alpha + \sin\beta + \sin\gamma \geq \sin 2\alpha + \sin 2\beta + \sin 2\gamma$.

13. For an arbitrary triangle show the inequalities:

(a) $\operatorname{tg}\frac{\alpha}{2} + \operatorname{tg}\frac{\beta}{2} + \operatorname{tg}\frac{\gamma}{2} \leq \frac{9R^2}{4S}$.

(b) $\dfrac{\cos\frac{\alpha}{2}}{l_a} + \dfrac{\cos\frac{\beta}{2}}{l_b} + \dfrac{\cos\frac{\gamma}{2}}{l_c} \geq \dfrac{9}{2p}$.

(c) $\dfrac{\cos^2\frac{\alpha}{2}}{a} + \dfrac{\cos^2\frac{\beta}{2}}{b} + \dfrac{\cos^2\frac{\gamma}{2}}{c} \geq \dfrac{27r}{8S}$.

(d) $\sqrt{a^2 + b^2 - h_c^2} + \sqrt{b^2 + c^2 - h_a^2} + \sqrt{c^2 + a^2 - h_b^2} \leq 6R$.

(e) $\frac{r}{R} \leq \frac{1}{2}$.

(e) $S \leq \frac{3\sqrt{3}}{4}R^2$.

(f) $\dfrac{l_a}{b+c} + \dfrac{l_b}{c+a} + \dfrac{l_c}{a+b} \leq \dfrac{3\sqrt{3}}{4}$.

(f) $\dfrac{a}{l_b + l_c} + \dfrac{b}{l_c + l_a} + \dfrac{c}{l_a + l_b} \geq \sqrt{3}$.

(g) $\dfrac{a^2}{h_b^2 + h_c^2} + \dfrac{b^2}{h_c^2 + h_a^2} + \dfrac{c^2}{h_a^2 + h_b^2} \geq 2$.

14. If $\alpha + \beta + \gamma = \pi$, prove that for all $n \in \mathbb{N}$:

$$(-1)^{n+1}\left(\sum \cos n\alpha\right) \leq \frac{3}{2} \quad \text{and} \quad -1 \leq (-1)^{n+1}\prod \cos n\alpha \leq \frac{1}{8}.$$