Lecture 25 - Mass Points

Kevin An

July 2, 2011

1 Introduction

Mass points is a technique used to calculate ratios between the lengths of segments formed by cevians and transversals of a triangle. Questions answerable by mass points can also be solved with vectors or area ratios, but it presents a much simpler and faster solution. This makes it an important tool for math contests, during which speed is a must. It became popular as a result of the many new math competitions in the 20th century.

2 How does it work?

In its most basic form, mass points is simply a formula involving levers which should be familiar to most students of physics:

$$Wd = \tau$$

in which d is equal to the distance between the object and the fulcrum, W is equal to the magnitude of the weight of the object, and τ is equal to the torque vector (what is causing the lever to move about the fulcrum). When two objects are placed at 2 opposite ends of a lever, they are balanced if the two torque forces are equal, giving the equation

$$W_1 d_1 = W_2 d_2$$

Dividing through by g (gravity), we have

$$m_1 d_1 = m_2 d_2$$

As a result, we can calculate the ratios of lengths of segments by dividing:

$$\frac{d_1}{d_2} = \frac{m_2}{m_1}$$

Also, note the total mass at the fulcrum of the lever is $m_1 + m_2$. This allows us to "chain" together fulcrums as the endpoints of levers.

3 Definitions and Properties

*Note that while the definitions and properties do not explicitly mention triangles, our aim is to be able to use mass points to solve otherwise difficult problems involving triangles.

Def 1. A **mass point** is can be represented as a conjunction mP. This corresponds to a mass of m placed on the point P.

Def 2. Two mass points $nP{=}mQ$ (coincide) iff $n = m$ and $P = Q$. This definition is mainly useful for proving that a specific mass point in the problem is actually a well known point with exploitable properties (ie. proving that a mass point is actually the incenter of a triangle).

Property 1. Suppose $A - F - B$. If the masses are distributed such that mA, mB, and $(m+n)F$, then F is the balancing point of $\overline{AB}$ and $\frac{AF}{FB} = \frac{n}{m}$. This property is often times expressed as $mA + nB = (m + n)F$. The converse of this property is also true.

Property 2. The above mass point addition obeys communtativity, associativity, and distributivity of a scalar accross mass points. Note however, that if one wishes to distribute a scalar to simplify calculations, the scalar **MUST BE DISTRIBUTED OVER ALL MASS POINTS IN THE SYSTEM**.

4 Examples

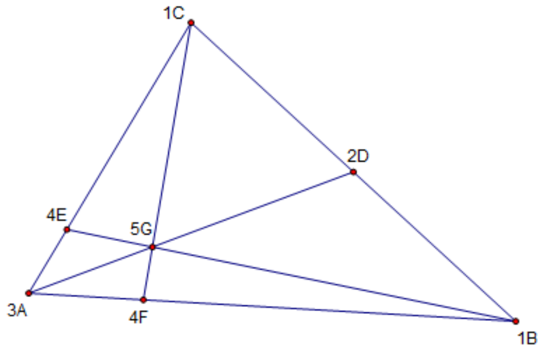
Pure Cevians

Problems involving cevians in a triangle have the simplest mass point solutions.

In $\triangle ABC$, D is on $\overline{BC}$ such that $BD = DC$ and E is on $\overline{AC}$ such that $AE = 3EC$. $\overline{AD}$ and $\overline{BE}$ intersect at a point G. If F is a point on $\overline{AB}$ such that $C - G - F$, then calculate $\frac{AF}{FB}$ and $\frac{AG}{GD}$.

Solution:

We begin by assigning a mass of 1 to vertex C. Then, A and B will automatically aquire masses of 3 and 1, respectively. Summing masses D,F, and E will have masses of 2, 4 and 4 respectively. Summing the masses again at G, we see that it has a mass of 5.



We then have:

$$\frac{AF}{FB} = \frac{1}{3}$$

since there is a mass of 3 on A and 1 on B.

Also, we have

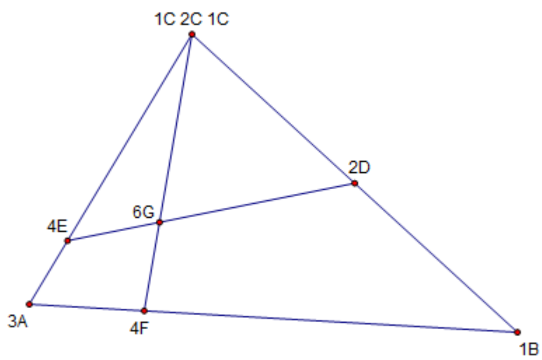
$$\frac{AG}{GD} = \frac{2}{3}$$

since there exists a mass of 3 on A and a mass of 2 on D.

Cevian and Transversal

In $\triangle ABC$, D is on $\overline{BC}$ such that $BD = DC$, E is on $\overline{AC}$ such that $AE = 3EC$, and F is on $\overline{AB}$ such that $3AF = 3FB$. $\overline{ED}$ and $\overline{B}$ intersect at a point G. Compute $\frac{EG}{GD}$ and $\frac{CG}{GF}$.

Solution:



We begin by assigning a mass of 3 to vertex A and 1 to vertex B. Then, by summing masses, F has a mass of 4. Here, the method of attack changes from the previous problem. Because we are dealing with a transversal, we must use splitting masses on vertex C. This involves calculating the mass at C from both $\overline{AC}$ and $\overline{BC}$. When dealing with these two segments, we use the corresponding mass on C. However, when dealing with $\overline{CF}$, we must sum the two individual masses at C. On $\overline{AC}$, we can assign a mass of 1 to C, which results in a mass of 4 at E. On $\overline{BC}$, we can also assign a mass of 1 to C, which results in a mass of 2 on D. We now compute the necessary ratios.

$$\frac{EG}{GD} = \frac{2}{4} = \frac{1}{2}$$

$$\frac{CG}{GF} = \frac{1+1}{4} = \frac{1}{2}$$

5 Problems

Unfortunately, with mass points widely gaining popularity in contest participants, geometry problems with pure cevians or one cevian and one transversal are rarely seen. Instead, competition make enough manipulations so that either mass points alone is not enough to solve a problem or making the use of mass points *seem* impropable. In the space below, I will include these more tricky problems. Also, instead of copying problems from AIME, I will simply post the years and question numbers. Note that these questions can be found at Art of Problem Solving (link).

Problem 1: AIME 1992 #14

In $\triangle ABC$, A', B', and C' are on $\overline{BC}$, $\overline{AC}$, and $\overline{AB}$, respectively. Given that $\overline{AA'}$, $overline BB'$, and $\overline{CC'}$ are concurrent at P, and that $\frac{AO}{OA'} + \frac{BO}{OB'} + \frac{CO}{OC'} = 92$, find $\frac{AO}{OA'} \cdot \frac{BO}{OB'} \cdot \frac{CO}{OC'}$.

Problem 2: AIME 1989 #15

P is inside $\triangle ABC$. $\overline{APD}$, $\overline{BPE}$, and $\overline{CPF}$ are drawn with D, E, and F on $\overline{BC}$, $\overline{AC}$, and $\overline{BC}$ respectively. Given that $AP = 6$, $BP = 9$, $PD = 6$,

$PE = 3$, and $CF = 20$, find the area of $\triangle ABC$.

Problem 3: AIME 2003I #15

In $\triangle ABC$, $AB = 360$, $BC = 507$, and $CA = 780$. Let M be the midpoint of $\overline{CA}$, and let D be the point on $\overline{CA}$ such that $\overline{BD}$ bisects $\angle ABC$. Let F be the point on $\overline{BC}$ such that $\overline{DF} \perp \overline{BD}$. Suppose that $\overline{DF}$ meets $\overline{BM}$ at E. The ratio $DE : EF$ can be written in the form $fracmn$, where m and n are relatively prime positive integers. Find $m + n$.

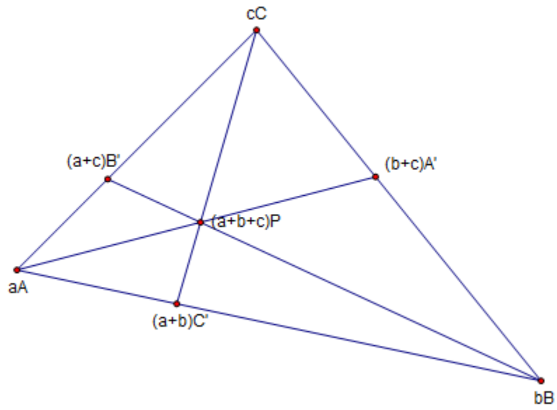
Problem 4: In $\triangle ABC$, D is on $\overline{BC}$ such that $BD = DC$, E is on $\overline{AC}$ such that $AE = 3EC$, and F and G are on$\overline{AC}$ such that $AF : FG : GB = 2 : 3 : 3$. $\overline{CF}$ and $\overline{ED}$ intersect at a point H and $\overline{CG}$ and $\overline{ED}$ intersect at a point I. Compute $EH : HI : ID$.

Problem 5: Let $\overline{AA_1}$, $\overline{BB_1}$, and $\overline{CC_1}$ be three concurrent cevians in $\triangle ABC$ and let this point of intersection be G. If $BH = 2HB_1$, prove that $BE = B_1E$, where E is the intersection point between $\overline{BB'}$ and $\overline{A'C'}$.

Problem 6: Let a point P be randomly selected in $\triangle ABC$. There exist $\overline{DE}$, $\overline{FG}$, and $\overline{HI}$ such that they are concurrent at P and parallel to $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$, respectively. Compute $\frac{DE}{AB} + \frac{FG}{BC} + \frac{HI}{CA}$.

6 Solutions

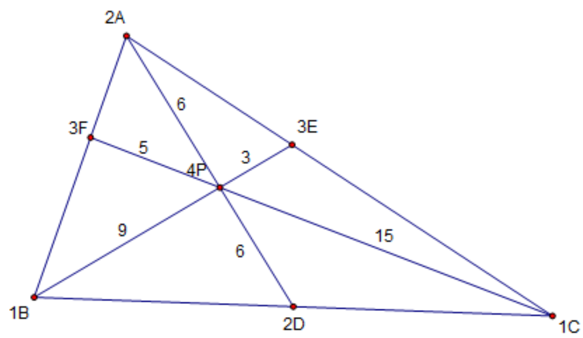
Problems 1 Solution: We can assign masses of a, b, and c to the vertices A, B, and C, respectively. By summing masses, the weights at A', B', an C' are $b+c$, $a+c$, and $a+b$, respectively. Thus, we have $\frac{AO}{OA'} = \frac{b+c}{a}$, $\frac{BO}{OB'} = \frac{a+c}{b}$, and $\frac{CO}{OC'} = \frac{a+b}{c}$.



We then have:

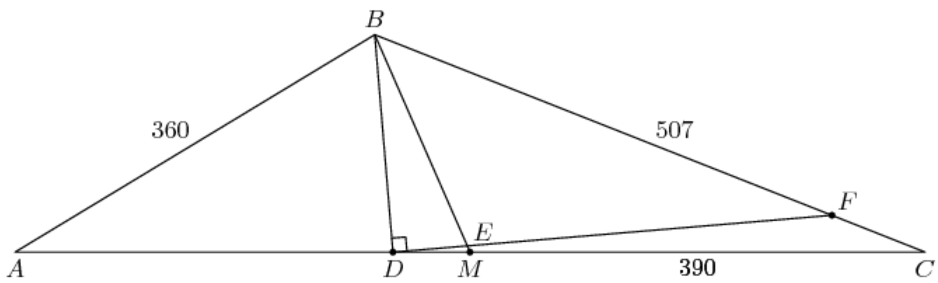
$$\frac{AO}{OA'} \cdot \frac{BO}{OB'} \cdot \frac{CO}{OC'} = \frac{b+c}{a} \cdot \frac{c+a}{b} \cdot \frac{a+b}{c}$$
$$\frac{AO}{OA'} \cdot \frac{BO}{OB'} \cdot \frac{CO}{OC'} = \frac{2abc + b^2c + bc^2 + c^2a + ca^2 + a^2b + ab^2}{abc}$$
$$\frac{AO}{OA'} \cdot \frac{BO}{OB'} \cdot \frac{CO}{OC'} = 2 + \frac{bc(b+c)}{abc} + \frac{ca(c+a)}{abc} + \frac{ab(a+b)}{abc}$$
$$\frac{AO}{OA'} \cdot \frac{BO}{OB'} \cdot \frac{CO}{OC'} = 2 + \frac{b+c}{a} + \frac{c+a}{b} + \frac{a+b}{c}$$
$$\frac{AO}{OA'} \cdot \frac{BO}{OB'} \cdot \frac{CO}{OC'} = 2 + 92 = 94$$

Problem 2 Solution: We can immediately assign a mass of 3 to E, a mass of 1 to B, and masses of 2 to both A and D. As a result, P has a mass of 4. Then, we know that there is a mass of 1 on C (note that since A and B both have masses of 1, D is the midpoint of $\overline{BC}$) and a mass of 3 on F. Thus, $\frac{CP}{PF} = 3$, so $CP = 15$ and $PF = 5$.



Now, we apply Stewart's Theorem on cevian $\overline{BC}$ of $\triangle BCP$. This gives us that $BC^2 + 12^2 = 2(15^2 + 9^2)$, or $BC = 6\sqrt{13}$. Now, notice that $2[BCP] = [ABC]$, since they share the same base in $\overline{BC}$ and $h_{\triangle ABC} = 2h_{\triangle BCP}$ $(2PD = 2AD)$. By using Heron's Formula to find $[BCP]$, which has side lengths 15, 9, and $BC = 6\sqrt{13}$. Notice here that the calculations will not be bad because there will different of squares involved. We then get $[BPC] = 54$, so $[ABC] = 2[BPC] = 2 \cdot 54 = 108$.

Problem 3 Solution:



For computation, instead consider the triangle as above except $AB = 120$, $BC = 169$, $CA = 260$. In the following, let a point represent the mass located there.

By the Angle Bisector Theorem, we can place mass points on C, D, and A of 120, 289, 169 respectively. Thus, a mass of $\frac{289}{2}$ belongs at F (seen by reflecting F across $\overline{BD}$, to an image which lies on $\overline{AB}$). Having determined $\frac{CB}{CF}$, we reassign mass points to determine $\frac{FE}{FD}$. This setup involves $\triangle CFD$ and $\overline{MEB}$. For simplicity, put masses of 240 and 289 at C and F. To find the mass we should put at D, we compute $\frac{CM}{MD}$: applying the Angle Bisector Theorem again and using the fact M is a midpoint, we find:

$$\frac{CM}{MD} = \frac{169 \cdot \frac{260}{289} - 130}{130} = \frac{49}{289}$$

$$\frac{DE}{EF} = \frac{F}{D}$$

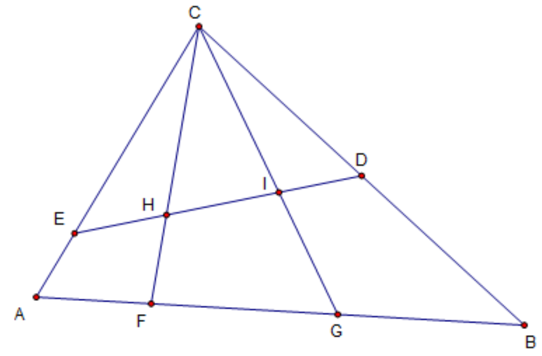
$$\frac{DE}{EF} = \frac{F}{C} \cdot \frac{C}{D}$$

$$\frac{DE}{EF} = \frac{289}{240} \cdot \frac{49}{289}$$

$$\frac{DE}{EF} = \frac{49}{240}$$

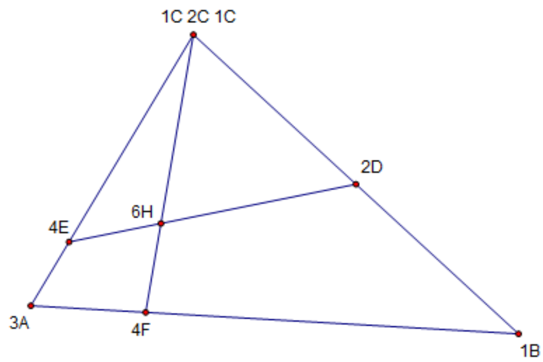
Since 49 and 240 are relatively prime, our answer will be $49 + 240 = 289$.

Problem 4 Solution:



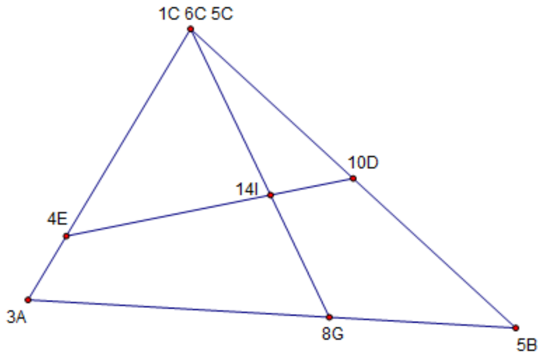
This problem is different than the others because it seems to generate two centers of mass, H and I. Instead of assigning masses right off the bat, we must realize that this is a major difference, as mass points only works when we have one center of mass, but in this case there are two fulcrum points along $\overline{DE}$. This suggests the idea of breaking up the problem into two parts, one concerning $\overline{CF}$ and the other concerning $\overline{CG}$.

Part 1: We have $\frac{AF}{FB} = \frac{2}{3+3} = \frac{1}{3}$. So, we assign a mass of 3 to vertex A and 1 to vertex B. Then, by summing masses, F has a mass of 4. On $\overline{AC}$, we can assign a mass of 1 to C, which results in a mass of 4 at E. On $\overline{BC}$, we can also assign a mass of 1 to C, which results in a mass of 2 on D. Which gives $\frac{EH}{DE} = \frac{1}{3}$.



Part 2: We have $\frac{AG}{GB} = \frac{2+3}{3} = \frac{5}{3}$. So, we assign a mass of 3 to vertex A and 5 to vertex B. Then, by summing masses, G has a mass of 8. On $\overline{AC}$, we

can assign a mass of 1 to C, which results in a mass of 4 at E. On $\overline{BC}$, we must assign a mass of 5 on C, which results in a mass of 10 on D. This gives $\frac{ID}{DE} = \frac{4}{14} = \frac{2}{7}$.

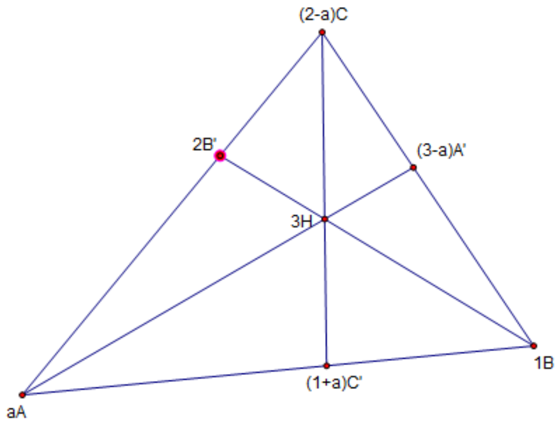


We are now ready to solve for $\frac{HI}{DE}$.

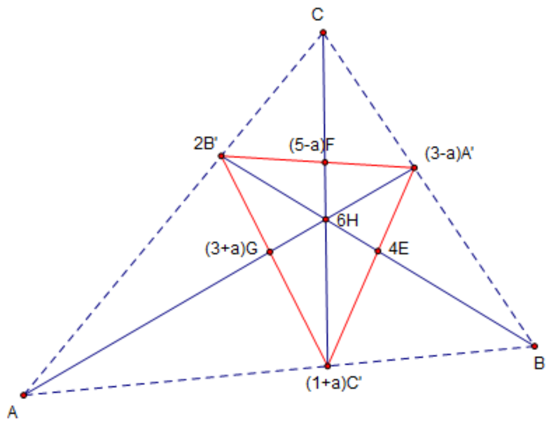
$$\frac{HI}{DE} = \frac{DE - EH - ID}{DE}$$
$$\frac{HI}{DE} = \frac{DE}{DE} - \frac{EH}{DE} - \frac{ID}{DE}$$
$$\frac{HI}{DE} = 1 - \frac{1}{3} - \frac{2}{7}$$
$$\frac{HI}{DE} = \frac{8}{21}$$

$$EH : HI : ID = \frac{EH}{DE} : \frac{HI}{DE} : \frac{ID}{DE} = \frac{1}{3} : \frac{8}{21} : \frac{2}{7} = 7 : 8 : 6$$

Problem 5 Solution: Problem 6 demonstrates the idea of maintaining the same center of mass but shifting the perspective of the triangle being used. The first step is to assign some masses to $\triangle ABC$. We know $BH = 2HB_1$, so we can place a mass of 2 on B' and a mass of 1 on B. Then, we assign A an arbitrary mass of a. As a result, C' has a mass of $1 + a$, C has a mass of $2 - a$, and A' has a mass of $3 - a$. Note that this makes H the center of mass, with a mass of 3.



Then, we draw $\overline{B'C'}$ and $\overline{A'B'}$ and label their intersections with $\overline{CC'}$ and $\overline{AA'}$ F and G, respectively. Now, we focus on $\triangle A'B'C'$ while trying to keep H the center of mass. We can do this, but the mass at H will be double of what it when focusing on $\triangle ABC$ (can you prove this in the general case?). By adding masses, E has a mass of 4. Looking at the masses on E and B', we know that $\frac{B'H}{B'E} = \frac{2}{3}$ and $\frac{EH}{B'E} = \frac{1}{3}$.



We then have:

$$BH = 2B'H$$

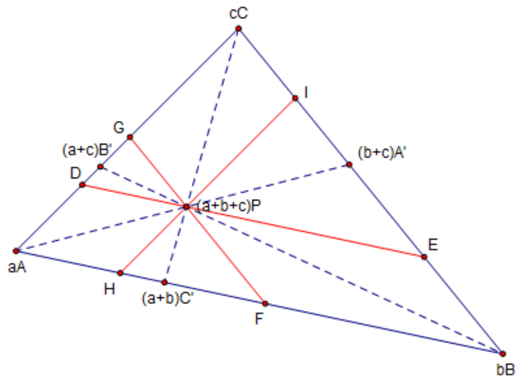
$$BE + EH = 2B'H$$

$$BE = 2B'H - EH$$

$$BE = \frac{4}{3}B'E - \frac{1}{3}B'E$$

$$BE = B'E$$

Problem 6 Solution: While using mass points might not be obvious, it is definitely the method that trivializes the problem. We can create 3 cevians by extending $\overline{AP}$, $\overline{BP}$, and $\overline{CP}$ to the 3 sides of $\triangle ABC$, thus creating $\overline{AA'}$, $\overline{BB'}$, and $\overline{CC'}$. We can then assign masses of a, b, and c to the vertices A, B, and C, respectively. By summing masses in respect to P, we have a mass of $b+c$ at A,$a+c$ at B, and $a+b$ at C, which means there is a mass of $a+b+c$ at P. Notice that $\triangle AFG$ $\triangle ABC$ since $\overline{FG} \parallel \overline{BC}$. Also, $\overline{AP}$ and $\overline{AA'}$ are corresponding segments in similar triangles, which means their ratio will be the same as any other pairs of corresponding segments of the two similar triangles. By similar reasoning, we can also set up the same relationship between $\overline{BP}$ and $\overline{BB'}$ and $\overline{CP}$ and $\overline{CC'}$.



We then have:

$$\frac{DE}{AB}+\frac{FG}{BC}+\frac{HI}{CA}=\frac{AP}{AA'}+\frac{BP}{BB'}+\frac{CP}{CC'}$$
$$\frac{DE}{AB}+\frac{FG}{BC}+\frac{HI}{CA}=\frac{b+c}{a+b+c}+\frac{a+c}{a+b+c}+\frac{a+b}{a+b+c}$$
$$\frac{DE}{AB}+\frac{FG}{BC}+\frac{HI}{CA}=\frac{2(a+b+c)}{a+b+c}=2$$