NUMBER THEORY

UNIT 5 CONTINUED FRACTIONS

1. Introduction

What is a continued fraction? Two examples are

$$\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{2}}} \text{ and } 4+\cfrac{1}{3+\cfrac{1}{2+\cfrac{1}{1}}}.$$

In this unit, we shall see how a number could be changed to a continued fraction and how this method is related to the Euclidean algorithm we learnt in Unit 1. Furthermore, we will look into some properties of continued fractions.

2. Definitions

Definition 2.1.

A **continued fraction** x is a number expressed in the form

$$x = a_0 + \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{\ddots + \cfrac{1}{a_n}}}}$$

where $a_i > 0$ for all $i > 0$ and $a_0 \geq 0$.

If the series $a_1, \ldots, a_n$ is finite, the above representation of x is called a **finite continued fraction**. If the series is not finite, it is called an **infinite continued fraction**.

We denote the continued fraction by $[a_0, a_1, \ldots, a_n]$ for a finite continued fraction and $[a_0, a_1, \ldots, a_n, \ldots]$ for an infinite continued fraction.

Illustration: $\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{3}}} = [0, 1, 2, 3]$ is a finite continued fraction.

$$1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\ddots}}}=[1,2,2,2,\ldots]$$ is an infinite continued fraction.

3. Continued Fractions and Euclidean Algorithm

Before we proceed to the relationship between Euclidean algorithm and continued fractions, let us look at the following example.

Example 3.1.

Convert $\frac{7}{11}$ and $\frac{11}{8}$ into continued fractions.

Solution.

$$\frac{7}{11}=0+\cfrac{1}{\cfrac{11}{7}}=0+\cfrac{1}{1+\cfrac{4}{7}}=0+\cfrac{1}{1+\cfrac{1}{\cfrac{7}{4}}}=0+\cfrac{1}{1+\cfrac{1}{1+\cfrac{3}{4}}}=0+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\cfrac{4}{3}}}}=0+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{3}}}}$$

Thus we can write $\frac{7}{11}=[0,1,1,1,3]$.

Also, we have $\frac{11}{8}=1+\frac{3}{8}=1+\cfrac{1}{\cfrac{8}{3}}=1+\cfrac{1}{2+\cfrac{2}{3}}=1+\cfrac{1}{2+\cfrac{1}{\cfrac{3}{2}}}=1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2}}}=[1,2,1,2]$.

As you may notice, the process of finding the continued fraction involves dividing the denominator by the numerator and after taking the quotient, the numerator becomes the denominator in the next step.

This is the idea of Euclidean algorithm, which has been discussed as Theorem 2.3 of Unit 1. It is stated below for reference.

Theorem 3.1. (Euclidean algorithm)

Let a and b be positive integers, $a > b$. Then we apply a series of divisions as follows.

$$\begin{aligned} a &= bq_0 + r_1 & 0 &< r_1 < b \\ b &= r_1 q_1 + r_2 & 0 &< r_2 < r_1 \\ r_1 &= r_2 q_2 + r_3 & 0 &< r_3 < r_2 \\ &\vdots & &\vdots \\ r_{n-2} &= r_{n-1} q_{n-1} + r_n & 0 &< r_n < r_{n-1} \\ r_{n-1} &= r_n q_n + r_{n+1} & & \end{aligned}$$

The process of division comes to an end when $r_{n+1} = 0$. The integer r_n is the G.C.D. of a and b.

The equations in the Euclidean algorithm can be rewritten as

$$\begin{aligned} a &= bq_0 + r_1 & \frac{a}{b} &= q_0 + \frac{r_1}{b} \\ b &= r_1 q_1 + r_2 & \frac{b}{r_1} &= q_1 + \frac{r_2}{r_1} \\ r_1 &= r_2 q_2 + r_3 & \frac{r_1}{r_2} &= q_2 + \frac{r_3}{r_2} \\ &\vdots & &\vdots \\ r_{n-2} &= r_{n-1} q_{n-1} + r_n & \frac{r_{n-2}}{r_{n-1}} &= q_{n-1} + \frac{r_n}{r_{n-1}} \\ r_{n-1} &= r_n q_n + r_{n+1} & \frac{r_{n-1}}{r_n} &= q_n \end{aligned}$$

With the Euclidean algorithm, we have

$$\begin{aligned} \frac{a}{b} &= q_0 + \frac{r_1}{b} = q_0 + \cfrac{1}{\cfrac{b}{r_1}} = q_0 + \cfrac{1}{q_1 + \cfrac{r_2}{r_1}} \\ &= q_0 + \cfrac{1}{q_1 + \cfrac{1}{\cfrac{r_1}{r_2}}} = q_0 + \cfrac{1}{q_1 + \cfrac{1}{q_2 + \cfrac{r_3}{r_2}}} = \cdots = q_0 + \cfrac{1}{q_1 + \cfrac{1}{q_2 + \cfrac{1}{\ddots + \cfrac{1}{q_{n-1} + \cfrac{1}{q_n}}}}}. \end{aligned}$$

Thus any rational number $x = \dfrac{a}{b}$ (a and b are relatively prime) can be written as the continued fraction $[q_0, q_1, \ldots, q_n]$.

4. Convergents of Continued Fractions

Definition 4.1.

Let $x=[q_0, q_1, \ldots, q_n]$ be the continued fraction representation of a rational number *x*.

The integers q_0, q_1, …, q_n are called the **partial quotients** of *x*. The fractions

$$\delta_0=[q_0]=q_0,\ \delta_1=[q_0,q_1]=q_0+\frac{1}{q_1},\ \delta_2=[q_0,q_1,q_2]=q_0+\cfrac{1}{q_1+\cfrac{1}{q_2}}$$

$$,\ \ldots,\ \delta_n=[q_0,q_1,\ldots,q_n]=q_0+\cfrac{1}{q_1+\cfrac{1}{q_2+\cfrac{1}{\ddots+\cfrac{1}{q_{n-1}+\cfrac{1}{q_n}}}}}$$

are called **convergents** of *x*, where δ_i is the *i*-th convergent of *x*.

Illustration: Referring to Example 3.1, we have

$$\frac{7}{11}=0+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{3}}}}$$

and from the above definitions,

$$\delta_0=0,\ \delta_1=0+\frac{1}{1}=1,\ \delta_2=0+\cfrac{1}{1+\cfrac{1}{1}}=\frac{1}{2},\ \delta_3=0+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1}}}=\frac{2}{3}$$

$$\text{and }\ \delta_4=0+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{3}}}}=\frac{7}{11}.$$

Theorem 4.1.

Let $x=[q_0, q_1, \ldots, q_n]$ be a rational number with the *i*-th convergent $\delta_i=[q_0, q_1, \ldots, q_i]$. Let $P_{-1}=1$, $P_0=q_0$, $Q_{-1}=0$, $Q_0=1$ and define $P_k=q_kP_{k-1}+P_{k-2}$ and $Q_k=q_kQ_{k-1}+Q_{k-2}$ for $k>0$. Then for $0\le i\le n$, $\frac{P_i}{Q_i}=\delta_i$.

Proof. We shall prove this theorem by mathematical induction. Let $S(i)$ be the statement '$\frac{P_i}{Q_i}=\delta_i$' for $0\le i\le n$.

For $i=0$, we have $\delta_0=[q_0]=\frac{q_0}{1}=\frac{P_0}{Q_0}$.

For $i=1$, we have $\delta_1=[q_0,q_1]=q_0+\frac{1}{q_1}=\frac{q_1q_0+1}{q_1\cdot 1+0}=\frac{q_1P_0+P_{-1}}{q_1Q_0+Q_{-1}}=\frac{P_1}{Q_1}$.

Thus $S(0)$ and $S(1)$ are both true.

Suppose $S(k)$ is true for some $0\le k<n$, i.e. $\delta_k=\frac{P_k}{Q_k}=\frac{q_kP_{k-1}+P_{k-2}}{q_kQ_{k-1}+Q_{k-2}}$. Then replacing q_k by $\left(q_k+\frac{1}{q_{k+1}}\right)$ gives δ_{k+1}. Thus we have

$$\delta_{k+1}=[q_0,q_1,\ldots,q_k,q_{k+1}]=q_0+\cfrac{1}{q_1+\cfrac{1}{\ddots+\cfrac{1}{q_k+\cfrac{1}{q_{k+1}}}}}$$

$$=\left[q_0,q_1,\ldots,\left(q_k+\frac{1}{q_{k+1}}\right)\right]=\frac{\left(q_k+\frac{1}{q_{k+1}}\right)P_{k-1}+P_{k-2}}{\left(q_k+\frac{1}{q_{k+1}}\right)Q_{k-1}+Q_{k-2}}$$

$$=\frac{(q_kq_{k+1}+1)P_{k-1}+q_{k+1}P_{k-2}}{(q_kq_{k+1}+1)Q_{k-1}+q_{k+1}Q_{k-2}}=\frac{q_{k+1}(q_kP_{k-1}+P_{k-2})+P_{k-1}}{q_{k+1}(q_kQ_{k-1}+Q_{k-2})+Q_{k-1}}$$

$$=\frac{q_{k+1}P_k+P_{k-1}}{q_{k+1}Q_k+Q_{k-1}}.$$

Thus $S(k+1)$ is also true.

This completes the induction.

Q.E.D.

With the above convergents, we can compute the continued fraction in the form of a table.

n	-1	0	1	$\ldots$	$n-1$	n
q_k		q_0	q_1	$\ldots$	q_{n-1}	q_n
P_k	1	P_0	P_1	$\ldots$	P_{n-1}	P_n
Q_k	0	1	Q_1	$\ldots$	Q_{n-1}	Q_n

Example 4.1.

Find the continued fraction and the convergents for $\frac{81}{35}$.

Solution.

By Euclidean algorithm, we have

$$\begin{aligned}81&=35\cdot 2+11\\35&=11\cdot 3+2\\11&=2\cdot 5+1\\2&=1\cdot 2\end{aligned}$$

Thus the continued fraction of $\frac{81}{35}$ is $2+\cfrac{1}{3+\cfrac{1}{5+\cfrac{1}{2}}}$.

We have the following table.

n	-1	0	1	2	3
q_k		2	3	5	2
P_k	1	2	7	37	81
Q_k	0	1	3	16	35

Thus the convergents are $\frac{2}{1}$, $\frac{7}{3}$, $\frac{37}{16}$ and $\frac{81}{35}$.

Theorem 4.2.

For $k>0$, the difference between consecutive convergents is $\delta_k-\delta_{k-1}=\frac{(-1)^{k-1}}{Q_kQ_{k-1}}$.

Proof. For $k>0$, we have $\delta_k-\delta_{k-1}=\frac{P_k}{Q_k}-\frac{P_{k-1}}{Q_{k-1}}=\frac{P_kQ_{k-1}-P_{k-1}Q_k}{Q_kQ_{k-1}}=\frac{h_k}{Q_kQ_{k-1}}$, where $h_k=P_kQ_{k-1}-P_{k-1}Q_k$.

Then

$$\begin{aligned} h_k &= P_k Q_{k-1} - P_{k-1} Q_k \\ &= (q_k P_{k-1} + P_{k-2}) Q_{k-1} - P_{k-1}(q_k Q_{k-1} + Q_{k-2}) \\ &= P_{k-2} Q_{k-1} - P_{k-1} Q_{k-2} \\ &= -h_{k-1}. \end{aligned}$$

Thus $h_k = (-1)^k h_0$ and we have $h_0 = P_0 Q_{-1} - P_{-1} Q_0 = -1$ and so $h_k = (-1)^{k-1}$.

Finally, $\delta_k - \delta_{k-1} = \dfrac{h_k}{Q_k Q_{k-1}} = \dfrac{(-1)^{k-1}}{Q_k Q_{k-1}}$.

Q.E.D.

Corollary 4.3.

Let $x = [q_0, q_1, \ldots, q_n]$. Then for $0 \le k < n$,

$$\begin{cases} \delta_k < x & \text{if } k \text{ is even} \\ \delta_k > x & \text{if } k \text{ is odd} \end{cases}.$$

Corollary 4.4.

Let $x = [q_0, q_1, \ldots, q_n]$. Then $|\delta_k - x| \le \frac{1}{Q_k Q_{k-1}}$ for $k > 0$.

Note that since $P_k Q_{k-1} - Q_k P_{k-1} = h_k = (-1)^k$, $(P_k, Q_k) = 1$ and hence the convergent $\frac{P_k}{Q_k}$ is in the lowest term.

Theorem 4.5.

Let $x = [q_0, q_1, \ldots, q_n]$. Then $|\delta_k - x| < |\delta_{k-1} - x|$ for $k > 0$.

Proof. We have

$$\begin{aligned} x &= [q_0, q_1, \ldots, q_{k-1}, q_k, \ldots, q_n] \\ &= q_0 + \cfrac{1}{q_1 + \cfrac{1}{\ddots + \cfrac{1}{q_{k-1} + \cfrac{1}{q_k + \cfrac{1}{\ddots + \cfrac{1}{q_n}}}}}} \\ &= q_0 + \cfrac{1}{q_1 + \cfrac{1}{\ddots + \cfrac{1}{q_{k-1} + \cfrac{1}{y}}}} \end{aligned}$$

where

$$y = q_k + \cfrac{1}{\ddots + \cfrac{1}{q_n}} .$$

Now with the definition of P_k and Q_k ,

$$x = q_0 + \cfrac{1}{q_1 + \cfrac{1}{\ddots + \cfrac{1}{q_{k-1} + \cfrac{1}{y}}}} = [q_0, q_1, \ldots, q_{k-1}, y] = \frac{yP_{k-1} + P_{k-2}}{yQ_{k-1} + Q_{k-2}}$$

Thus

$$\begin{aligned} x(yQ_{k-1} + Q_{k-2}) &= yP_{k-1} + P_{k-2} \\ y(xQ_{k-1} - P_{k-1}) &= P_{k-2} - xQ_{k-2} \\ yQ_{k-1}\left(x - \frac{P_{k-1}}{Q_{k-1}}\right) &= Q_{k-2}\left(\frac{P_{k-2}}{Q_{k-2}} - x\right) \end{aligned}$$

Since $y > 1$, $Q_{k-1} = q_{k-1}Q_{k-2} + Q_{k-2} > Q_{k-2} > 0$, we have $yQ_{k-1} > Q_{k-2}$ and hence $|x - \frac{P_{k-1}}{Q_{k-1}}| < |\frac{P_{k-2}}{Q_{k-2}} - x|$ and $\left(x - \frac{P_{k-1}}{Q_{k-1}}\right)$ and $\left(\frac{P_{k-2}}{Q_{k-2}} - x\right)$ have the same sign.

Q.E.D.

5. Infinite Continued Fraction

It has been shown that a rational number x can be expressed as a finite continued fraction and it is obvious that a finite continued fraction is a rational number. For irrational numbers, the idea of taking quotients can be applied and we would get an infinite continued fraction. In this section, we will see some properties of such continued fractions.

Let $x=[q_0, q_1, \ldots]$ be an infinite continued fraction. Then Theorem 4.5 also holds. As $k \to \infty$, $Q_k \to \infty$ as well and hence we have

$$\lim_{k\to\infty}\left(\frac{P_k}{Q_k}-\frac{P_{k-1}}{Q_{k-1}}\right)=0 .$$

Also, from Corollary 4.3 and 4.4, it can be shown that $\frac{P_0}{Q_0}<\frac{P_2}{Q_2}<\frac{P_4}{Q_4}<\cdots$ and $\frac{P_1}{Q_1}>\frac{P_3}{Q_3}>\frac{P_5}{Q_5}>\cdots$ are two sequences converging to *x* and hence we have $x=\lim_{k\to\infty}\frac{P_k}{Q_k}$.

Furthermore, Corollary 4.4 holds for infinite continued fraction and we can use this for writing the first few convergents of an irrational number, as well as convergents to a certain degree of accuracy. Below is an example.

Example 5.1.

Find a convergent of $\sqrt{2}$ with maximum error 0.001.

Solution.

We first find the convergents of the *approximate* value of $\sqrt{2}$ and we shall then stop when we get the required accuracy.

We shall take 9 decimal places approximation, i.e. 1.414213562.

Then we have

$$\begin{aligned}1.41423562 &= 1+0.41423562 = 1+\frac{1}{2.41413563}\\ &= 1+\frac{1}{2+0.41413563} = 1+\frac{1}{2+\frac{1}{2.41421356}}\\ &= 1+\frac{1}{2+\frac{1}{2+0.41421356}} = 1+\frac{1}{2+\frac{1}{2+\frac{1}{2.414213576}}}\\ &= 1+\frac{1}{2+\frac{1}{2+\frac{1}{2+0.414213576}}}\end{aligned}$$

We have the following table for convergents:

n	-1	0	1	2	3	4	5
q_k		1	2	2	2	2	2
P_k	1	1	3	7	17	41	99
Q_k	0	1	2	5	12	29	70

Thus the first six convergents are $\delta_0 = 1$, $\delta_1 = \frac{3}{2}$, $\delta_2 = \frac{7}{5}$, $\delta_3 = \frac{17}{12}$, $\delta_4 = \frac{41}{29}$ and $\delta_5 = \frac{99}{70}$.

Now we are going to show that indeed of the required accuracy.

Corollary 4.4 also applies to infinite continued fractions (try to prove this yourself!) and thus for $k > 0$, $|\delta_k - x| \le \frac{1}{Q_k Q_{k-1}}$.

Putting $k = 5$, we get $|\delta_5 - \sqrt{2}| \le \frac{1}{29 \cdot 70} < 0.001$.

6. Exercises

1. Convert $\frac{14}{17}$ and $\frac{28}{9}$ into continued fractions.

2. Suppose a and b are two relatively prime positive integers. Convert

$$\frac{2a^2b + a^2 + ab + 2a + 1}{2ab + a + 2}$$

into its continued fraction.

3. Prove that there exists a unique representation for any positive rational number in the form of a continued fraction.

4. Find the convergents of $\frac{16}{27}$.

5. Show that Corollary 4.4 applies to infinite continued fractions.

6. (a) Determine an integer n for which the n th convergent of $x = 0.5 + \sqrt{7}$ approximates x with a maximum error of 0.005.

   (b) For the number n found in part (a), compute the n th convergent of x.

7. In this exercise we will consider the continued fraction

$$x = 2 + \cfrac{1}{2 + \cfrac{1}{2 + \cdots}} = [2, 2, 2, ...].$$

(a) Using mathematical induction or otherwise, prove that

$$P_k = \left(1 + \frac{3\sqrt{2}}{4}\right)\left(1+\sqrt{2}\right)^k + \left(1 - \frac{3\sqrt{2}}{4}\right)\left(1-\sqrt{2}\right)^k \text{ and}$$

$$Q_k = \left(1 + \frac{3\sqrt{2}}{4}\right)\left(1+\sqrt{2}\right)^{k-1} + \left(1 - \frac{3\sqrt{2}}{4}\right)\left(1-\sqrt{2}\right)^{k-1}.$$

(b) Using $x = \lim_{k\to\infty} \frac{P_k}{Q_k}$, or otherwise, show that $x = 1+\sqrt{2}$.