Lecture 16 — Pigeon Hole Principle

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3/26/11

1 Introduction

1.1 Definitions

Pigeon Hole principle, or sometimes known as the "Box Principal" is an extremely intuitive and easy concept. The hard part is identifying when to use this problem, and we shall see later in identifying the the "pigeons" and sometimes even the hole.

For $Nk+1$ "pigeons" in N "holes", one of the holes will have at least $k+1$ "pigeons", or in other words $\left\lceil \frac{nk+1}{h} \right\rceil$ which is read as "The ceiling of $nk+1$ divided by h rounded up to the lowest integer. for example $\left\lceil \frac{1}{4} \right\rceil = 1$ and $\left\lceil \frac{\pi}{1} \right\rceil = 4$

A pigeon is defined as the quantity of objects being counted, while "holes" are the amount of groups they can be organized in. The following are obvious examples, to give you a feel before we move on to more challenging problems.

You need 13 people, in order to guarantee a common birth month amongst them. You need 367 people, to guarantee at least two people with common birthday (since there are 366 days different days, including leap year, so we need $n+1$ pigeons, to guarantee at least 2 people in one "hole".)

1.2 Problems

As mentioned earlier, the hardest part of a pigeonhole problem usually is just recognizing that it's a pigeonhole problem. Below are two "similar" problems, looking at the 2 problems below, determine which one is harder.

1 Five points are chosen in a $\sqrt{14}$ by $\sqrt{14}$ square. Show that two points are within a distance of $\sqrt{7}$ units from other.

2 Take 10 points inside the circle with diameter 5. Prove that for any these 10 points there exist two points whose distance is less than 2.

Solutions: The first one is by far the easier one. We try to maximize the distance of 4 of the points, to create a worst case scenario. So we place a point on each of the vertices of the square. Using pythagorean theorem, the diagonal has length of

$$\sqrt{14}^2 + \sqrt{14}^2 = 28 = c^2$$
$$c = 2\sqrt{7}$$

Thus, the mid-center of the square is the midpoint of the diagonal or $\frac{2\sqrt{7}}{2} = \sqrt{7}$ hence if any of the points move, they will only get closer to each other and we are done.

The second problem is trickier and was an Olympiad problem in Japan, 1997.

When we try to use 9 points on the circle, and a point in the center, the maximum radius is 2.5, oh no! Turns out it's not the end of the world, but this can happen a lot in Pigeonhole and we must figure a way to manipulate the problems to make it more flexible. So we try something different.

We recreate a concentric circle, with a radius of one. So any two points inside the circle would be within a distance, so we in essence forced a restriction on ourselves (or on the holes) which is very useful. So we divide the remaining area into 8 congruent parts with equal radius (essentially 8 slices of the 10 unit diameter circle, with a concentric circle of diameter 2). Then the distances formulas show us $\sqrt{1^2 + (5/2)^2 - (5/2)\sqrt{2}} = \sqrt{\frac{29-10\sqrt{2}}{4}} < 2$, hence the distance between any two points is ≤ 2

Prove that amongst N friends, there exists at least two people have same amount of friends.

Sol: We note that the choice of friends are 0 through $(N-1)$ friends, but that means there are N choices for N friends. In this case it's obvious, but in other problems it will be less so, common sense let's us know that if one person has zero friends, then it's impossible for another person to be friends with EVERYONE ELSE. So really there are $N-1$ holes amongst N pigeons.

1.3 Problem Set 1

1. (A. Soifer, Slobodnik) 41 Castles or Rooks are placed on a 10×10 chessboard. Prove that there must exist 5 rooks, none of which attack each other. (Note: Rooks moves vertically and horizontally on the board) This seems like an easy problem, because $\lceil \frac{41}{10} \rceil = 5$ but that is not a solution nor is it even close. We notice that through pigeonhole principle, with 41 rooks and 10 Rows, at least one row must have 5 rooks. What does this mean for the other rows? At the most the first could have had 10 rooks (completely filled up) hence we can only assert that the 2nd row will have $\lceil \frac{31}{9} \rceil = 4$ again, in the worst case the 2nd row can have at most 10, so we assert for 3rd row $\lceil \frac{21}{8} \rceil = 3$ and for 4th row, $\lceil \frac{11}{7} \rceil = 2$ and $\lceil \frac{1}{1} \rceil = 1$. Doing backward pigeonhole, looking at the 5th row with at least one rook, we compare it to the 4th row which has two rooks. At least one of the rooks in the 4th row will not attack the rook of the first row, and so forth for the 3rd row concerning the 4th, and 4th row concerning the 5th row and we are done!

   Let A be any set of exactly 20 distinct integers chosen from the arithmetic progression, $1, 4, 7, \ldots, 97, 100$ Prove there must be at least two distinct integers in A whose sum add up to 104. (Putnam 1978)

   Solution: we start by counting the amount of pairs that would add up to 104 in hope that some pigeons will be eliminated. we notice the pairs $(4, 100)$, $(7, 97)$, $(10, 94) \ldots (49, 55)$. 1 and 52 cannot be used, and even in the case where you choose 1 and 52, there are 18 numbers to choose out of possible pairs 16 pairs, hence we are done. Note: The problem could have told us to select 19 distinct integers, rather than 20. It did this, in order to not reveal it's true nature or quantity so to speak.

2. This is a much harder one, level 6 Putnam style question (was used on a practice exam, am not aware of any official use)

The Fibanocci Sequence, $1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...$ defined as $F_1 = F_2 = 1$ for $n \geq 3$

Show that there are infinitely many Fibonacci numbers ending in a zero (that there exists a F_n which ends with k zeroes.

I want you to take a step back, and appreciate this problem. In all of the other problems we did, there was a hint of Pigeonhole Principle, because they had key phrase i.e., "show their exists", while here, there is nothing that indicates pigeonhole. On the contrary, one may think the solution involves some sort of induction or generating function, especially since we are dealing with infinity here.

Solution: A term F_p ends in k zeroes, if it is divisible by 10^k or in other words $F_p \equiv 0$ (mod10^k. Thus we consider the Fibanocci Sequence modulo 10^k, and we prove that the term 0 will occur in the sequence. Take $(10^{2k} + 1)$ terms of the sequence $(F_1, F_2), (F_2, F_3)$.., however the pair (0,0) cannot occur (eliminated a hole), hence there are only $(10^{2k} - 1)$ hence one pair will repeat. So the period length is at most $(10^{2k} - 1)$ possible pairs. The first pair to repeat it (1,1). Then $F_p = 1 - 1 = 0$, this it will occur in the sequence and be last as well.

I am posting a slight variation of this problem, which will make the above much more obvious.

3. In the sequence $1, 1, 2, 3, 5, 8, 3, 1, 4, 5, ...$ which is the Fibanocci sequence but *mod* 10. Prove that the sequence is purely periodic 9hence there are finite amount of holes). What is the maximum possible length of the period?

   Sol: Any two consecutive terms of the sequence determine all succeeding terms and all preceding terms. Thus the sequence will become periodic if any pair (a, b) of successive terms repeat, and the first repeating pair will be $(1, 1)$. Consider 101 successive $1, 1, 2, 3, 5, 8,$ They form a 100 pairs. $(1, 1), (1, 2), (2, 3)$ Since the pair $(0, 0)$ cannot appear, there are only 99 distinct pairs, thus 2 pairs will repeat, and the period of the sequence is at most 99.

   Here are some fun problems to try out on your own, many of them will be ranked high, but you should expect to find them easy.

4. Two permutations $a_1, a_2, \ldots, a_{2010}$ and $b_1, b_2, \ldots, b_{2010}$ of the numbers $1, 2, \ldots, 2010$ are said to intersect if $a_k = b_k$ for some value of k in the range $1 \leq k \leq 2010$. Show that there exist 1006 permutations of the numbers $1, 2, \ldots, 2010$ such that any other such permutation is guaranteed to intersect at least one of these 1006 permutations. (USAJMO 5, 2009)

5. Prove that from a set of ten distinct two-digit numbers, it is always possible to find two disjoint subsets whose members have the same sum. (IMO 1972)

6. Find the greatest positive integer n for which there exist n nonnegative integers $x_1, x_2, \cdots, x_n$ not all zero, such that for any sequence $\epsilon_1, \epsilon_2, \epsilon_3, \cdots, \epsilon_n$ of elements of $1, 0, -1$, not all zero, n^3 does not divide $\epsilon_1 x_1 + \epsilon_2 x_2 + \cdots + \epsilon_n x_n$ (Romania, 1996)

References

[1] Engel, Arthur: *Problem-Solving Strategies.* Springer, Frankfurt De, 1998.