Combinatorial Sums

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1 Common Techniques

- Generating functions (multiplication, plugging in roots of unity)
- Combinatorial interpretation (can go in both directions: sometimes the algebra is simpler and sometimes the combinatorics is simpler)
- Counting in 2 ways (special case: interchanging order of summation)
- Induction (algebraically, this usually requires manipulating factorials; combinatorially, this is usually a matter of figuring out the right way to group things)

2 Basic Identities

- Binomial theorem:
$$\sum_{k=0}^{n} \binom{n}{k} x^k = (1+x)^n.$$
- Negative exponent version of the binomial theorem:
$$\sum_{n=k}^{\infty} \binom{n}{k} x^n = \frac{x^k}{(1-x)^{k+1}}.$$
- Hockey-stick identity:
$$\sum_{n=k}^{m} \binom{n}{k} = \binom{m+1}{k+1}.$$
- Partial sums of polynomials (use finite differences + hockey-stick identity to make this explicit): for any polynomial P of degree d, there is a polynomial Q of degree $d+1$ such that
$$\sum_{n=1}^{m} P(n) = Q(m).$$
- Finite differences:
$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} f(k) = (-1)^n (\Delta^n f)(0).$$
- Vandermonde convolution:
$$\sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k} = \binom{m+n}{r}.$$

3 Problems

1. Find a closed form expression for
$$\binom{3n}{0}+\binom{3n}{3}+\cdots+\binom{3n}{3n}.$$

2. (IMO 1981) Consider all r-element subsets of the set $\{1,2,\ldots,n\}$, where $1\le r\le n$. Prove that the arithmetic mean of the smallest elements of the subsets is $\frac{n+1}{r+1}$.

3. Let p be an odd prime and let $0\le a\le b$ be nonnegative integers. How many ap-element subsets of $\{1,2,\ldots,bp\}$ have sum divisible by p?

4. (Putnam 1999) Sum the series
$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m^2n}{3^m(n3^m+m3^n)}.$$

5. Let p be an odd prime. Consider the p^2 points (i,j) for $1\le i,j\le p$. How many subsets of these points have x-coordinates and y-coordinates both summing to multiples of p?

6. (TST 2001) Express
$$\sum_{k=0}^{n}(-1)^k(n-k)!(n+k)!$$
in closed form.

7. (ELMO 200?) A *cycle* of a permutation π of $\{1,\ldots,n\}$ is a cyclically ordered subset $(a_1a_2\cdots a_k)$ of $\{1,\ldots,n\}$ satisfying $\pi(a_i)=a_{i+1}$ for each i (indices taken mod k). Let $z(\pi)$ denote the number of cycles in a permutation π. Prove that
$$\sum_{\pi}2^{z(\pi)}=(n+1)!,$$
where the sum runs over all $n!$ permutations of $\{1,\ldots,n\}$.

8. Let n be a positive integer, let a be any nonzero real, and let b and c be any real numbers. Show that
$$(a+b)^n=a\sum_{k=0}^{n}\binom{n}{k}(a-kc)^{k-1}(b+kc)^{n-k}.$$

9. Prove that for any positive integer n,
$$\sum_{k=0}^{n}\binom{2k}{k}\binom{2n-2k}{n-k}=4^n.$$

10. (TST 2000) Let n be a positive integer. Prove that
$$\binom{n}{0}^{-1}+\binom{n}{1}^{-1}+\cdots+\binom{n}{n}^{-1}=\frac{n+1}{2^{n+1}}\left(\frac{2}{1}+\frac{2^2}{2}+\cdots+\frac{2^{n+1}}{n+1}\right).$$

11. Show that
$$\sum_{0\le k<l\le n}\frac{(-1)^k\binom{n}{k}+(-1)^l\binom{n}{l}}{(k+1)(l+1)}=0$$
for any positive integer n.

12. (Putnam 2005) For a permutation π, let $f(\pi)$ denote the number of fixed points of π. (A fixed point of π is a number i such that $\pi(i)=i$.) Also let $s(\pi)$ be equal to 1 if π is an even permutation and -1 if π is an odd permutation. Show that
$$\sum_{\pi}\frac{s(\pi)}{1+f(\pi)}=(-1)^{n+1}\frac{n}{n+1},$$
where the sum runs over all $n!$ permutations of $\{1,\ldots,n\}$.