TJUSAMO 2011 – Induction

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"This is easy to inductable." – Patrick Yang

1 Problems from last week

1. Show that every infinite sequence of reals has an infinite monotone subsequence.

2. A point D is placed on side BC of triangle ABC. Circles are inscribed in ABD and ACD. Their common exterior tangent (other than BC) meets AD at K. Prove that the length of AK is independent of D.

2 Induction

Induction is a technique which is used to prove statements about a natural number n. The general technique is to prove that the statement holds for $n = 1$ (called the *base case*), and that if the statement holds for $n = k - 1$, then it holds for $n = k$ (called the *inductive step*). [1]
More formally, if a proposition $P(n)$ satisfies the following properties:

(i) $P(1)$ is true.

(ii) $P(n-1)$ implies $P(n)$ for all $n > 1$.

then $P(n)$ is true for all natural numbers n. This is called *(weak) induction on* n.

3 An Example

Problem: Show that $1 + 2 + \ldots + n = \dfrac{n(n+1)}{2}$.

Solution: We proceed by induction on n. For the base case, if $n = 1$, then $\dfrac{1(1+1)}{2} = \dfrac{2}{2} = 1$, as desired. For the inductive step, suppose that for some integer n, $1 + 2 + \ldots + (n-1) = \dfrac{n(n-1)}{2}$. Consider the sum $1 + 2 + \ldots + (n-1) + n$. It will equal $\dfrac{n(n-1)}{2} + n = n\left(1 + \dfrac{n-1}{2}\right) = n\left(\dfrac{n+1}{2}\right) = \dfrac{n(n+1)}{2}$ and the induction is complete.

4 Variants of Induction

There are several variants of induction, most notably strong induction and double induction. Strong induction is used in cases where $P(n-1)$ does not imply $P(n)$: if $P(1)$ is true and $P(1), P(2), \cdots, P(n-1)$ together imply $P(n)$ for all n, then $P(n)$ is true for all n. Double induction is used when proving statements $P(m, n)$ about two variables m, n: if $P(1, 1)$ is true and $P(m, n)$ implies $P(m+1, n)$ and $P(m, n+1)$ for all m, n, then $P(m, n)$ is true for all m, n.

[1] The standard way of visualizing this is to consider a row of dominoes corresponding to $P(1), P(2), P(3), \cdots$. Knocking over the $P(0)$ domino will cause all of them to fall down.

5 Tips

Here are some things to keep in mind when using induction:

1. Always keep induction in mind on any problem involving a positive integer n. What this essentially means is that you should assume that $P(n-1)$ is true whenever you're proving a statement $P(n)$.

2. Sometimes, you won't want to apply induction directly to the problem statement. Maybe $P(n)$ isn't provable by induction but there is some statement $Q(n)$, provable by induction, which implies $P(n)$.

3. By the same token, sometimes a problem will ask you to prove a specific case $P(k)$ of a general statement $P(n)$. Then you might want to try to prove $P(n)$ for all n by induction. Make sure, however, to first verify that $P(n)$ is true for all n by testing small cases.

4. When writing inductive proofs, make clear which variable you're inducting on.

5. There are many, many (in fact, infinitely many) other variants of induction that could possibly be useful. For example, perhaps $P(1)$ and $P(2)$ are true and $P(n-2)$ implies $P(n)$ for all $n > 2$. You don't need to *know* all of these variants of induction beforehand to be able to use them.

6 Problems

The best way to recognize which problems can be solved using induction is to practice. As with last week, you should try these problems alone for a while and then begin to solve them in groups.

1. Let n be a positive integer. Prove that $1 + 4 + 9 + \cdots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$.

2. Let F_n be the sequence defined by $F_0 = 0, F_1 = 1$, and $F_{n+2} = F_{n+1} + F_n$ for all $n \geq 0$. Prove that $F_{m+n+1} = F_{m+1}F_{n+1} + F_mF_n$ for all nonnegative integers m, n.

3. Let a_1, a_2, a_3, a_4, a_5 be real numbers with $0 < a_1 < a_2 < a_3 < a_4 < a_5 < 1$. Prove that $a_1a_2a_3a_4a_5 \geq a_1 + a_2 + a_3 + a_4 + a_5 - 4$.

4. Prove that for all positive integers n, the numbers $1, 2, \cdots, n$ can be arranged in some order in such a way that for any a, b with $1 \leq a, b \leq n$, the number $\dfrac{a+b}{2}$ does not appear between a and b.

5. Let $a_0, a_1, a_2, \cdots$ be a sequence of real numbers such that $a_{m+n} + a_{m-n} = \dfrac{1}{2}(a_{2m} + a_{2n})$ and $a_1 = 1$. Find a_{100}.

6. Consider a chess tournament with n players. They have played some number of chess games with each other, and there is no sequence $P_1, P_2, \cdots, P_k$ of players such that P_1 beat P_2, P_2 beat P_3, $\cdots$, P_{k-1} beat P_k, and P_k beat P_1. Prove that it is possible to rank the players in such a way that each player has only beaten players of lower rank. (In other words, prove that a finite directed acyclic graph can be topologically sorted.)

7 More Problems

7. Let n be a positive integer. Prove that $\frac{1}{2} \cdot \frac{3}{4} \cdot \frac{5}{6} \cdots \frac{2n-1}{2n} \leq \frac{1}{\sqrt{3n}}$.

8. Prove that every positive integer can be expressed as $\pm 1^2 \pm 2^2 \pm 3^2 \pm \ldots \pm n^2$ for some positive integer n and some choice of signs.

9. There are n points inside a triangle. The triangle is split into smaller triangles using these points as vertices. Prove that you always end up with $2n+1$ triangles.

10. There are n students standing in a circle, one behind the other. The students have heights $h_1 < h_2 < \cdots < h_n$ (but not necessarily in that order). If a student with height h_k is standing directly behind a student with height h_{k-2} or less, the two students are permitted to switch places. Prove that it is not possible to make more than $\binom{n}{3}$ such switches before reaching a position in which no further switches are possible.

11. Let $a_1, a_2, \ldots, a_n$ be distinct positive integers and let M be a set of $n-1$ positive integers not containing $s = a_1 + a_2 + \ldots + a_n$. A grasshopper is to jump along the real axis, starting at the point 0 and making n jumps to the right with lengths $a_1, a_2, \ldots, a_n$ in some order. Prove that the order can be chosen in such a way that the grasshopper never lands on any point in M.