Cyclic Quadrilaterals

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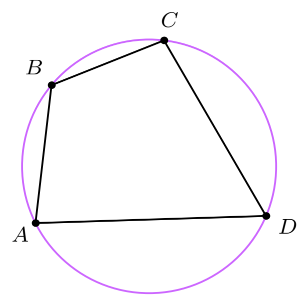
1 Review from Last Week

1. A country has 2000 cities and a complete lack of roads. Show that it is possible to join pairs of cities by (two-way) roads so that for $n = 1, ..., 1000$, there are exactly two cities where exactly n roads meet.

2. Call a finite sequence $a_1, a_2, ..., a_k$ (with $k \geq 1$) of positive integers *progressive* if $1 = a_1 < a_2 < \cdots < a_k$ and a_i divides a_{i+1} for $1 \leq i \leq k-1$. Let $S(n)$ denote the number of progressive sequences whose sum equals n. Prove that $S(n) \leq n^2$.

3. Let S be a set containing $n^2 + n - 1$ elements, for some positive integer n. Suppose that the n-element subsets of S are partitioned into two classes. Prove that there are at least n pairwise disjoint sets in the same class.

A *cyclic quadrilateral* is a quadrilateral that is inscribable in a circle. This circle is called the *circumcircle* of the cyclic quadrilateral. Its prevalence in Olympiad geometry makes it an indispensable topic. Generally, cyclic quadrilaterals are assumed to be convex, but self-intersecting cyclic quadrilaterals do exist.



2 Properties

Determining whether a quadrilateral is cyclic or not is the crux for many problems. The following lemma summarizes the angle relationships in cyclic quadrilaterals.

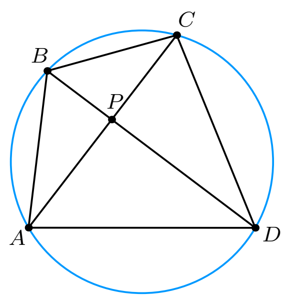
Lemma 1. *Let $ABCD$ be a convex quadrilateral. Then the following statements are equivalent:*

- *$ABCD$ is cyclic.*
- *$\angle BAD + \angle BCD = 180°$.*
- *$\angle ABD = \angle ACD$.*

Proof. If $ABCD$ is cyclic, both angles ABD and ACD intercept arc AD, so they are equal. Angles BAD and BCD intercept the minor and major arcs BD, so their sum is equal to $180°$. □

Of course, there are many more ways to prove that a quadrilateral is inscribed. The following theorem relates concyclicity to the product of the lengths of segments:

Theorem 2 (Power of a Point)**.** *Let $ABCD$ be a quadrilateral, and let $P = AC \cap BD$. Then $ABCD$ is cyclic if and only if $AP \cdot CP = BP \cdot DP$.*

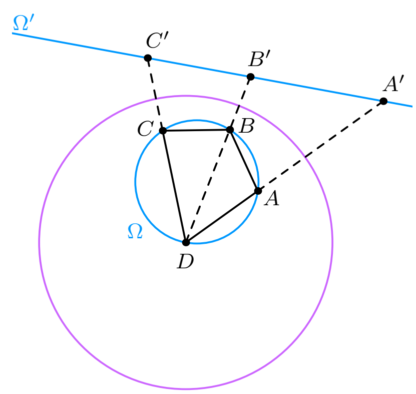


Proof. Note that if $ABCD$ is cyclic, then $\angle ABP = \angle DCP$ and $\angle BAP = \angle CDP$. Hence $\triangle ABP \sim \triangle DCP$, so $\frac{AP}{BP} = \frac{DP}{CP}$. The reverse direction follows similarly. □

Ptolemy's Theorem also provides a nice biconditional statement for which a quadrilateral is cyclic:

Theorem 3 (Ptolemy)**.** *Let $ABCD$ be a quadrilateral. Then $AB \cdot CD + BC \cdot DA = AC \cdot BD$ if and only if $ABCD$ is cyclic.*

Proof. Let Ω be the circumcircle of $ABCD$. We now invert with respect to a circle centered at D with arbitrary radius:



Note that Ω passes through the center of the circle of inversion. Hence Ω is mapped to a line not passing through D. As a result, A', B', and C' are collinear, so $A'B' + B'C' = A'C'$. Thus,

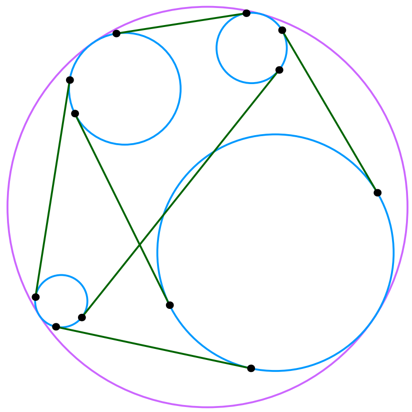
$$\begin{aligned} A'B' + B'C' = A'C' &\iff \frac{r^2}{DA \cdot DB} AB + \frac{r^2}{DB \cdot DC} BC = \frac{r^2}{DA \cdot DC} AC \\ &\iff AB \cdot CD + BC \cdot AD = AC \cdot BD. \end{aligned}$$

The last equality follows from multiplying both sides by $\frac{DA \cdot DB \cdot DC}{r^2}$. □

There is actually a generalization of Ptolemy's Theorem called Casey's Theorem:

Theorem 4 (Casey)**.** *Let Ω be a circle, and let ω_1, ω_2, ω_3, and ω_4 be non-intersecting circles that are internally tangent to Ω. If $t_{i,j}$ is the length of the exterior common tangent to circles ω_i and ω_j for $i, j \in \{1, 2, 3, 4\}$, then*

$$t_{1,2}t_{3,4} + t_{1,4}t_{2,3} = t_{1,3}t_{2,4}.$$



Usually, in solving problems, we let some of the tangent circles have radius 0 (that is, make it a point). Then the length of the exterior common tangent to the point and a circle is just the length of the tangent segment from the point. In fact, note that if we let all of the tangent circles ω_i be degenerate, we have Ptolemy's Theorem.

Theorem 5 (Bretschneider)**.** *The area of a convex quadrilateral with sidelengths a, b, c, and d and two opposite angles α and β is given by the formula*

$$A = \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd\cos^2\left(\frac{\alpha+\beta}{2}\right)}$$

where $s = \frac{a+b+c+d}{2}$.

From this we find the extension of Heron's Formula for quadrilaterals; the similarity is quite obvious.

Corollary 6 (Brahmagupta)**.** *The area of a cyclic quadrilateral with sidelengths a, b, c, and d is given by the formula*

$$A = \sqrt{(s-a)(s-b)(s-c)(s-d)}$$

where $s = \frac{a+b+c+d}{2}$.

Proof. In a cyclic quadrilateral, $\alpha + \beta = 180^\circ$, where α and β are two opposite angles. Hence by Theorem 5 we have that

$$\begin{aligned} A &= \sqrt{(s-a)(s-b)(s-c)(s-d) - abcd\cos^2 90^\circ} \\ &= \sqrt{(s-a)(s-b)(s-c)(s-d)}. \end{aligned}$$

□

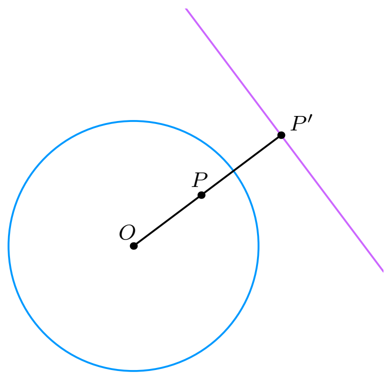
Finally, using the previous theorem, we can deduce a nice unique areal property of the cyclic quadrilateral:

Lemma 7. *Given four sidelengths of a quadrilateral, the quadrilateral that maximizes its area is cyclic.*

Proof. The minimum of $\cos^2\left(\frac{\alpha+\beta}{2}\right)$, where α and β are two opposite angles, is reached when $\cos^2\left(\frac{\alpha+\beta}{2}\right) = 0$, or $\alpha + \beta = 180°$; hence, by Theorem 5, the area of the quadrilateral is maximal when it is cyclic. $\square$

3 Poles and Polars

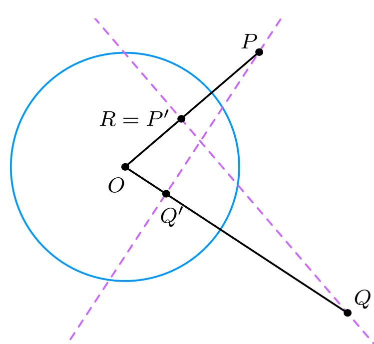
We briefly go over some lemmas in projective geometry. The *pole* of a line ℓ with respect to a circle is the inverse of the point P on ℓ that is closest to the circle. Conversely, the *polar* of a point P with respect to a circle with center O is the line passing through its inverse P' and perpendicular to line OP.



A *pole-polar transformation* is a transformation in which poles are turned into polars, and polars are turned into poles. Of course, the pole of the polar of a point is itself, and the polar of the pole of a line is itself; hence, a pole-polar transformation is a reciprocation.

The following theorem, despite sounding rather simple, lends itself to many applications in solving problems.

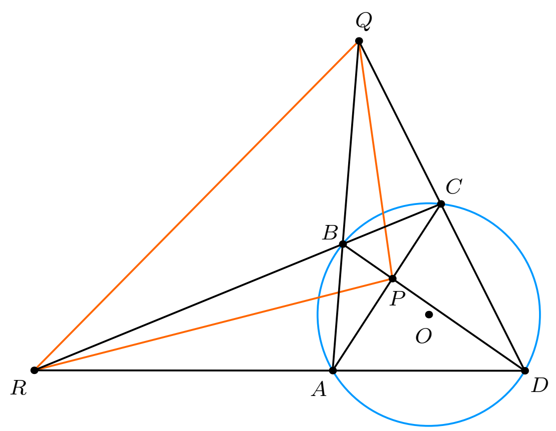
Theorem 8 (La Hire)**.** *If P lies on the polar of Q, then Q lies on the polar of P.*



Proof. Let O be the center of the circle with radius r; let P' and Q' be the inverses of P and Q. Suppose that P lies on the polar of Q. Then $PQ' \perp OQ$. Now let R be a point on OP such that $QR \perp OP$. Then $\triangle OPQ' \sim \triangle OQR$, so $\frac{OP}{OQ} = \frac{OQ'}{OR}$, so $OP \cdot OR = OQ \cdot OQ' = r^2$; this implies that $R = P'$, as desired. $\square$

A triangle ABC is said to be *self-polar* with respect to a circle if C is the polar of line AB, A is the polar of BC, and B is the polar of CA. If a self-polar triangle is found, it is often helpful to use La Hire's Theorem to find collinearities and concurrencies. The next theorem is usually very important in solving problems that begin with "Let $ABCD$ be a cyclic quadrilateral."

Theorem 9 (Brokard)**.** *Let $ABCD$ be a cyclic quadrilateral with circumcenter O, and let $P = AC \cap BD$, $Q = AB \cap CD$, and $R = AD \cap BC$. Then $\triangle PQR$ is self-polar; furthermore, O is the orthocenter of $\triangle PQR$.*

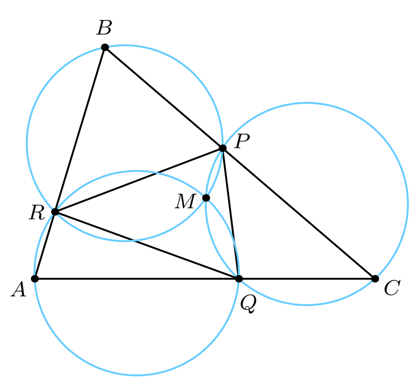


Proof. Let $PQ \cap BC = X$ and $PQ \cap AD = Y$. Since $(B; X, A; D) = -1$, we have that $XY = PQ$ is the polar of R. Similarly, PR is the polar of Q, and so by La Hire's Theorem QR must be the polar of P. In addition, $OQ \perp PR$ and $OR \perp PQ$, so O is the orthocenter of $\triangle PQR$. $\square$

4 Miquel Point

Before going into more advanced theorems regarding cyclic quadrilaterals, we first introduce the *Miquel point* of a triangle. The Miquel point in a triangle deals with a nice concurrency of circumcircles, and we can extend this notion to cyclic quadrilaterals:

Theorem 10 (Miquel)**.** *Let ABC be a triangle, and let P, Q, and R be points on segments BC, CA, and AB, respectively. Then the circumcircles of $\triangle AQR$, $\triangle BRP$, and $\triangle CPQ$ concur at the Miquel point.*



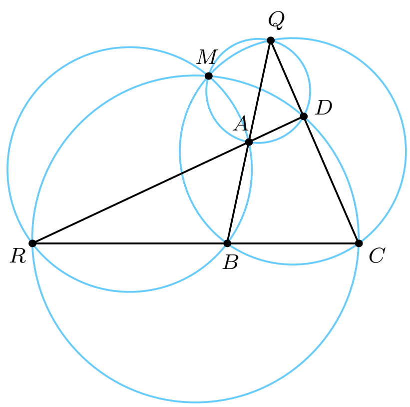
Proof. Let M be the intersection of the circumcircles of $\triangle AQR$ and $\triangle BRP$. Then

$$\angle CQM = \angle ARM = \angle BPM = 180^\circ - \angle CPM,$$

implying that $PCQM$ is cyclic. Hence the three circumcircles concur at M. $\square$

Before we discuss Miquel's Theorem for cyclic quadrilaterals, we introduce a new definition: A *complete quadrilateral* consists of four noncollinear points and the six lines connecting each pair of points. In other words, we take a convex quadrilateral and extend their sides until they meet another side. This configuration has another beautiful concurrency:

Theorem 11 (Miquel)**.** *Let $ABCD$ be a convex quadrilateral, and let $AB \cap CD = Q$ and $AD \cap BC = R$. Then the circumcircles of $\triangle ADQ$, $\triangle BCQ$, $\triangle ABR$, and $\triangle CDR$ concur.*



Proof. Let M be the intersection of the circumcircles of QAD and QBC. Then $\angle MAR = \angle MQD = \angle MBR$, so $MABR$ is cyclic. In addition, $\angle QMC = \angle QBC = \angle RMA$, so $\angle RMC = \angle RMA + \angle CMA = \angle QMC + \angle CMA = \angle QMA = \angle RDC$, so $RMDC$ is also cyclic. $\square$

If $ABCD$ is cyclic, its Miquel point has an interesting property:

Lemma 12. *Let $ABCD$ be a convex quadrilateral, and let $AB \cap CD = Q$ and $AD \cap BC = R$; let M be the Miquel point of this complete quadrilateral. Then M lies on QR if and only if $ABCD$ is cyclic.*

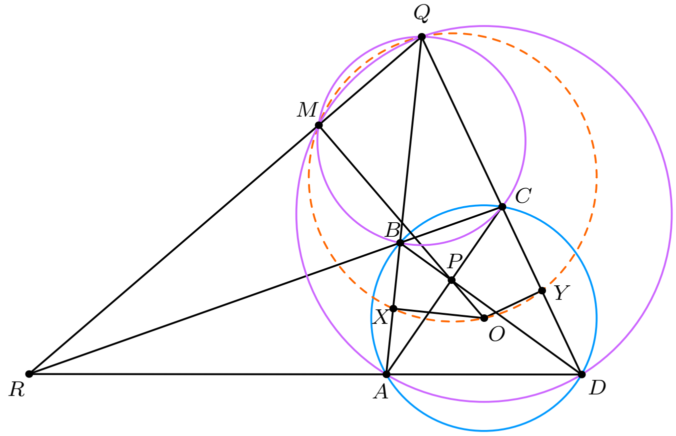
Proof. If $ABCD$ is cyclic, by Miquel's Theorem

$$\angle AMQ = \angle ADC = \angle ABR = 180^\circ - \angle AMR,$$

so M lies on QR. The reverse direction follows similarly. $\square$

The last property of the Miquel point is even nicer:

Lemma 13. *Let $ABCD$ be a cyclic quadrilateral with circumcenter O, and let $AC \cap BD = P$, $AB \cap CD = Q$, and $AD \cap BC = R$; let M be the Miquel point of this complete quadrilateral. Then M lies on QR and $OM \perp QR$; furthermore, M is the inverse of P with respect to the circumcircle.*



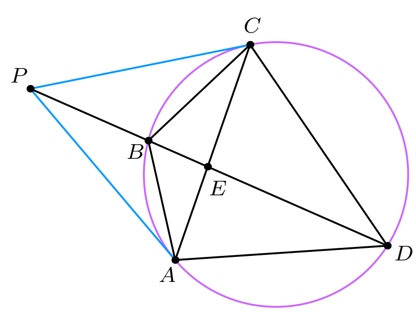
Proof. There exists a spiral similarity centered at M that takes A to D and B to C. Let X and Y be the midpoints of segments AB and CD, respectively. Then the spiral similarity takes X to Y. Hence M, Q, Y, and X are concyclic. Now OQ is the diameter of the circumcircle of $MQYX$, so it follows that $\angle OMQ = \angle OXQ = 90^\circ$.

Since $\triangle PQR$ is self-polar, the polar of P is QR. By La Hire's Theorem, P lies on the polar of M. Since $OM \perp QR$, it follows that P is the inverse of M. □

5 Harmonic Quadrilaterals

A convex cyclic quadrilateral $ABCD$ is *harmonic* if $\frac{BA}{BC} = \frac{DA}{DC}$. This quadrilateral has numerous useful projective properties that often show up in Olympiad problems. One can take a projectivity of a harmonic quadrilateral onto a line, provided that the center of projectivity lies on the circle.

Lemma 14. *Let $ABCD$ be a harmonic quadrilateral with circumcircle ω. Let P be a point outside of ω such that PA and PC are tangent to ω. Then P, B, and D are collinear. Furthermore, if $E = AC \cap BD$, then $(P, E; B, D) = -1$; that is, $\frac{BP}{BE} / \frac{DP}{DE} = -1$.*



Proof. We have $\frac{BA}{BC} = \frac{DA}{DC}$, so $\frac{BA}{DA} = \frac{BC}{DC} = \frac{\sin \angle BDC}{\sin \angle CBD} = \frac{\sin \angle PCB}{\sin \angle PBC} = \frac{PB}{PC} = \frac{PB}{PD}$. Hence $\triangle PAB \sim \triangle PDA$, implying that P, B, and D are collinear.

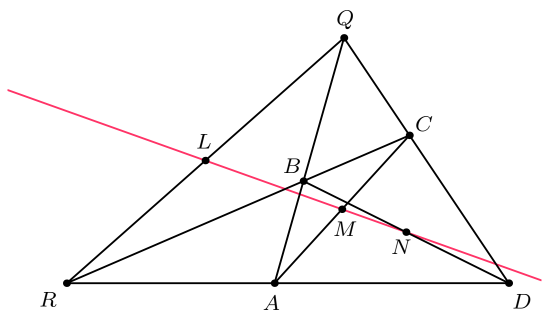
Now consider the projectivity centered at C that projects A, B, C, and D onto line PB; we have that $(C, A; B, D) = (P, E; B, D) = -1$. □

Note that because P is the intersection of the tangents to the circumcircle of ABC at A and C, by the so-called Symmedian Lemma, the previous lemma also implies that PB is the B- and D-symmedians of $\triangle ABC$ and $\triangle ADC$, respectively. Hence, if we find a symmedian of a triangle that intersects its circumcircle at a different point, we immediately know that the cyclic quadrilateral formed by the triangle and the intersection is harmonic.

6 Newton-Gauss Line

The *Newton-Gauss Line* is a line that is uniquely determined by a convex quadrilateral.

Theorem 15 (Newton-Gauss)**.** *Let $ABCD$ be a convex quadrilateral, and let $AB \cap CD = Q$ and $AD \cap BC = R$; let M, N, and L be the midpoints of segments AC, BD, and QR, respectively. Then M, N, and L are collinear on the Newton-Gauss Line.*



Proof. Let X, Y, and Z be the midpoints of AB, BR, and AR. Then

$$\frac{ZL}{LY} \cdot \frac{YN}{NX} \cdot \frac{XM}{MZ} = \frac{AQ}{QB} \cdot \frac{RD}{DA} \cdot \frac{BC}{CR} = 1$$

by Menelaus' Theorem. Hence L, M, and N are collinear. □

7 Miscellaneous Lemmas and Theorems

Theorem 16 (Pascal)**.** *Let $ABCDEF$ be a hexagon inscribed in a conic, and let $X = AB \cap DE$, $Y = BC \cap EF$, and $Z = CD \cap FA$. Then X, Y, and Z are collinear.*

Theorem 17 (Brianchon)**.** *Let $ABCDEF$ be a hexagon circumscribed about a conic. Then AD, BE, and CF concur.*

Lemma 18. *Let $ABCD$ be a quadrilateral such that $AB + CD = BC + DA$. Then $ABCD$ has an inscribed circle.*

Theorem 19 (Euler Line)**.** *Let ABC be a triangle, and let G, H, O, and N be its centroid, orthocenter, circumcenter, and nine-point center, respectively. Then all four points lie on a line called the* Euler Line*. Furthermore, $GO = \frac{1}{2}HG$ and $ON = \frac{1}{2}OH$.*

8 Problems

1. (USAJMO 2011) Points A, B, C, D, and E lie on a circle ω and point P lies outside the circle. The given points are such that

   (a) lines PB and PD are tangent to ω,

   (b) P, A, C are collinear, and

   (c) $DE \parallel AC$.

   Prove that BE bisects AC.

2. Let $ABCD$ be a cyclic quadrilateral, and let $Q = AB \cap CD$ and $R = AD \cap BC$. Prove that the angle bisectors of $\angle P$ and $\angle Q$ are perpendicular to each other.

3. (ELMO Shortlist 2013) Let $ABCD$ be a cyclic quadrilateral inscribed in circle ω whose diagonals meet at F. Lines AB and CD meet at E. Segment EF intersects ω at X. Lines BX and CD meet at M, and lines CX and AB meet at N. Prove that MN and BC concur with the tangent to ω at X.

4. Let $ABCD$ be a cyclic quadrilateral, and let $Q = AB \cap CD$. Let M and N be the midpoints of AC and BD, respectively. Find $\frac{[MNQ]}{[ABCD]}$.

5. (IMO 1985) A circle with center O passes through the vertices A and C of the triangle ABC and intersects the segments AB and BC again at distinct points K and N respectively. Let M be the point of intersection of the circumcircles of triangles ABC and KBN (apart from B). Prove that $\angle OMB = 90^\circ$.

6. Let $ABCD$ be a cyclic quadrilateral whose circumcircle is the unit circle centered at the origin; let $P = AC \cap BD$, $Q = AB \cap CD$, and $R = AD \cap BC$, and let M be the Miquel point of the complete quadrilateral. Denote the complex coordinate of a point by its lowercase letter.

   (a) Prove that $p = \frac{ac(b+d)-bd(a+c)}{ac-bd}$. Find similar expressions for r and s.

   (b) Prove that $m = \frac{ad-bc}{a-b-c+d}$.

7. (BMO 1984) Let $ABCD$ be a cyclic quadrilateral and let H_A, H_B, H_C, and H_D be the orthocenters of $\triangle BCD$, $\triangle CDA$, $\triangle DAB$, and $\triangle ABC$ respectively. Prove that the quadrilaterals $ABCD$ and $H_AH_BH_CH_D$ are congruent.

8. Let $ABCD$ be a cyclic quadrilateral with circumcircle ω, and let $Q = AB \cap CD$. Let X be a point on BC such that XA is tangent to ω, and let Y be a point on AD such that YB is tangent to ω. Prove that Q, X, and Y are collinear.

9. Let $ABCD$ be a cyclic quadrilateral and let N_A, N_B, N_C, and N_D be the nine-point centers of $\triangle BCD$, $\triangle CDA$, $\triangle DAB$, and $\triangle ABC$, respectively. Find $\frac{[N_AN_BN_CN_D]}{[ABCD]}$.

10. (PUMaC 2012) Cyclic quadrilateral $ABCD$ has side lengths $AB = 2$, $BC = 3$, $CD = 5$, and $AD = 4$. Find $\sin A \sin B \left(\cot \frac{A}{2} + \cot \frac{B}{2} + \cot \frac{C}{2} + \cot \frac{D}{2}\right)^2$.

11. (China 1992) Convex quadrilateral $ABCD$ is inscribed in circle ω with center O. Diagonals AC and BD meet at P. The circumcircles of triangles ABP and CDP meet at P and Q. Assume that points O, P, and Q are distinct. Prove that $\angle OQP = 90^\circ$.

12. (APMO 2013) Let $ABCD$ be a quadrilateral inscribed in a circle ω, and let P be a point on the extension of AC such that PB and PD are tangent to ω. The tangent at C intersects PD at Q and the line AD at R. Let E be the second point of intersection between AQ and ω. Prove that B, E, R are collinear.

13. (ISL 2012/G2) Let $ABCD$ be a cyclic quadrilateral whose diagonals AC and BD meet at E. The extensions of the sides AD and BC beyond A and B meet at F. Let G be the point such that $ECGD$ is a parallelogram, and let H be the image of E under reflection in AD. Prove that D, H, F, and G are concyclic.

14. (ISL 2004/G8) Given a cyclic quadrilateral $ABCD$, let M be the midpoint of the side CD, and let N be a point on the circumcircle of triangle ABM. Assume that the point N is different from the point M and satisfies $\frac{AN}{BN} = \frac{AM}{BM}$. Prove that the points E, F, N are collinear, where $E = AC \cap BD$ and $F = BC \cap DA$.