Polynomials Problems

Version 2 - Amir Hossein Parvardi *

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1. Find all polynomials $P(x)$ with real coefficient such that:

$$P(0) = 0, \quad \text{and} \quad \lfloor P \lfloor P(n) \rfloor \rfloor + n = 4 \lfloor P(n) \rfloor \quad \forall n \in \mathbb{N}.$$

2. Find all functions $f : \mathbb{R} \to R$ such that

$$f(x^n + 2f(y)) = (f(x))^n + y + f(y) \quad \forall x, y \in \mathbb{R}, \quad n \in \mathbb{Z}_{\geq 2}.$$

3. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$x^2 y^2 (f(x+y) - f(x) - f(y)) = 3(x+y) f(x) f(y)$$

4. Find all polynomials $P(x)$ with real coefficients such that

$$P(x)P(x+1) = P(x^2) \quad \forall x \in \mathbb{R}.$$

5. Find all polynomials $P(x)$ with real coefficient such that

$$P(x)Q(x) = P(Q(x)) \quad \forall x \in \mathbb{R}.$$

6. Find all polynomials $P(x)$ with real coefficients such that if $P(a)$ is an integer, then so is a, where a is any real number.

7. Find all the polynomials $f \in \mathbb{R}[X]$ such that

$$\sin f(x) = f(\sin x), \ (\forall) x \in \mathbb{R}.$$

8. Find all polynomial $f(x) \in \mathbb{R}[x]$ such that

$$f(x) f(2x^2) = f(2x^3 + x^2) \quad \forall x \in \mathbb{R}.$$

9. Find all real polynomials f and g, such that:

$$(x^2 + x + 1) \cdot f(x^2 - x + 1) = (x^2 - x + 1) \cdot g(x^2 + x + 1),$$

for all $x \in \mathbb{R}$.

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10. Find all polynomials $P(x)$ with integral coefficients such that $P(P'(x)) = P'(P(x))$ for all real numbers x.

11. Find all polynomials with integer coefficients f such that for all $n > 2005$ the number $f(n)$ is a divisor of $n^{n-1} - 1$.

12. Find all polynomials with complec coefficients f such that we have the equivalence: for all complex numbers z, $z \in [-1, 1]$ if and only if $f(z) \in [-1, 1]$.

13. Suppose f is a polynomial in $\mathbb{Z}[X]$ and m is integer .Consider the sequence a_i like this $a_1 = m$ and $a_{i+1} = f(a_i)$ find all polynomials f and alll integers m that for each i:

$$a_i | a_{i+1}$$

14. $P(x), Q(x) \in \mathbb{R}[x]$ and we know that for real r we have $p(r) \in \mathbb{Q}$ if and only if $Q(r) \in \mathbb{Q}$ I want some conditions between P and Q.My conjecture is that there exist ratinal a, b, c that $aP(x) + bQ(x) + c = 0$

15. Find the gcd of the polynomials $X^n + a^n$ and $X^m + a^m$, where a is a real number.

16. Find all polynomials p with real coefficients that if for a real a,$p(a)$ is integer then a is integer.

17. $\mathfrak{P}$ is a real polynomail such that if α is irrational then $\mathfrak{P}(\alpha)$ is irrational. Prove that $\deg[\mathfrak{P}] \leq 1$

18. Show that the odd number n is a prime number if and only if the polynomial $T_n(x)/x$ is irreducible over the integers.

19. P, Q, R are non-zero polynomials that for each $z \in \mathbb{C}$, $P(z)Q(\bar{z}) = R(z)$. a) If $P, Q, R \in \mathbb{R}[x]$, prove that Q is constant polynomial. b) Is the above statement correct for $P, Q, R \in \mathbb{C}[x]$?

20. Let P be a polynomial such that $P(x)$ is rational if and only if x is rational. Prove that $P(x) = ax + b$ for some rational a and b.

21. Prove that any polynomial $\in \mathbb{R}[X]$ can be written as a difference of two strictly increasing polynomials.

22. Consider the polynomial $W(x) = (x-a)^k Q(x)$, where $a \neq 0$, Q is a nonzero polynomial, and k a natural number. Prove that W has at least $k+1$ nonzero coefficients.

23. Find all polynomials $p(x) \in \mathbb{R}[x]$ such that the equation

$$f(x) = n$$

has at least one rational solution, for each positive integer n.

24. Let $f \in \mathbb{Z}[X]$ be an irreducible polynomial over the ring of integer polynomials, such that $|f(0)|$ is not a perfect square. Prove that if the leading coefficient of f is 1 (the coefficient of the term having the highest degree in f) then $f(X^2)$ is also irreducible in the ring of integer polynomials.

25. Let p be a prime number and f an integer polynomial of degree d such that $f(0) = 0, f(1) = 1$ and $f(n)$ is congruent to 0 or 1 modulo p for every integer n. Prove that $d \geq p - 1$.

26. Let $P(x) := x^n + \sum_{k=1}^{n} a_k x^{n-k}$ with $0 \leq a_n \leq a_{n-1} \leq \dots a_2 \leq a_1 \leq 1$. Suppose that there exists $r \geq 1$, $\varphi \in \mathbb{R}$ such that $P(re^{i\varphi}) = 0$. Find r.

27. Let $\mathcal{P}$ be a polynomail with rational coefficients such that

$$\mathcal{P}^{-1}(\mathbb{Q}) \subseteq \mathbb{Q}.$$

Prove that $\deg \mathcal{P} \leq 1$.

28. Let f be a polynomial with integer coefficients such that $|f(x)| < 1$ on an interval of length at least 4. Prove that $f = 0$.

29. prove that $x^n - x - 1$ is irreducible over $\mathbb{Q}$ for all $n \geq 2$.

30. Find all real polynomials $p(x)$ such that

$$p^2(x) + 2p(x)p\left(\frac{1}{x}\right) + p^2\left(\frac{1}{x}\right) = p(x^2)p\left(\frac{1}{x^2}\right)$$

For all non-zero real x.

31. Find all polynomials $P(x)$ with odd degree such that

$$P(x^2 - 2) = P^2(x) - 2.$$

32. Find all real polynomials that

$$p(x + p(x)) = p(x) + p(p(x))$$

33. Find all polynomials $P \in \mathbb{C}[X]$ such that

$$P(X^2) = P(X)^2 + 2P(X).$$

34. Find all polynomials of two variables $P(x, y)$ which satisfy

$$P(a, b)P(c, d) = P(ac + bd, ad + bc), \forall a, b, c, d \in \mathbb{R}.$$

35. Find all real polynomials $f(x)$ satisfying

$$f(x^2) = f(x)f(x-1) \forall x \in \mathbb{R}.$$

36. Find all polynomials of degree 3, such that for each $x, y \geq 0$:

$$p(x+y) \geq p(x) + p(y).$$

37. Find all polynomials $P(x) \in \mathbb{Z}[x]$ such that for any $n \in \mathbb{N}$, the equation $P(x) = 2^n$ has an integer root.

38. Let f and g be polynomials such that $f(Q) = g(Q)$ for all rationals Q. Prove that there exist reals a and b such that $f(X) = g(aX + b)$, for all real numbers X.

39. Find all positive integers $n \geq 3$ such that there exists an arithmetic progression $a_0, a_1, \dots, a_n$ such that the equation $a_n x^n + a_{n-1}x^{n-1} + \dots + a_1 x + a_0 = 0$ has n roots setting an arithmetic progression.

40. Given non-constant linear functions $p_1(x), p_2(x), \dots p_n(x)$. Prove that at least $n-2$ of polynomials $p_1 p_2 \dots p_{n-1} + p_n, p_1 p_2 \dots p_{n-2} p_n + p_{n-1}, \dots p_2 p_3 \dots p_n + p_1$ have a real root.

41. Find all positive real numbers $a_1, a_2, \dots, a_k$ such that the number $a_1^{\frac{1}{n}} + \dots + a_k^{\frac{1}{n}}$ is rational for all positive integers n, where k is a fixed positive integer.

42. Let f, g be real non-constant polynomials such that $f(\mathbb{Z}) = g(\mathbb{Z})$. Show that there exists an integer A such that $f(X) = g(A + x)$ or $f(x) = g(A - x)$.

43. Does there exist a polynomial $f \in \mathbb{Q}[x]$ with rational coefficients such that $f(1) \neq -1$, and $x^n f(x) + 1$ is a reducible polynomial for every $n \in \mathbb{N}$?

44. Suppose that f is a polynomial of exact degree p. Find a rigurous proof that $S(n)$, where $S(n) = \sum\limits_{k=0}^{n} f(k)$, is a polynomial function of (exact) degree $p + 1$ in varable n .

45. The polynomials P, Q are such that $\deg P = n$,$\deg Q = m$, have the same leading coefficient, and $P^2(x) = (x^2 - 1)Q^2(x) + 1$. Prove that $P'(x) = nQ(x)$

46. Given distinct prime numbers p and q and a natural number $n \geq 3$, find all $a \in \mathbb{Z}$ such that the polynomial $f(x) = x^n + ax^{n-1} + pq$ can be factored into 2 integral polynomials of degree at least 1.

47. Let F be the set of all polynomials Γ such that all the coefficients of $\Gamma(x)$ are integers and $\Gamma(x) = 1$ has integer roots. Given a positive intger k, find the smallest integer $m(k) > 1$ such that there exist $\Gamma \in F$ for which $\Gamma(x) = m(k)$ has exactly k distinct integer roots.

48. Find all polynomials $P(x)$ with integer coefficients such that the polynomial

$$Q(x) = (x^2 + 6x + 10) \cdot P^2(x) - 1$$

is the square of a polynomial with integer coefficients.

49. Find all polynomials p with real coefficients such that for all reals a, b, c such that $ab + bc + ca = 1$ we have the relation

$$p(a)^2 + p(b)^2 + p(c)^2 = p(a + b + c)^2.$$

50. Find all real polynomials f with $x, y \in \mathbb{R}$ such that

$$2yf(x + y) + (x - y)(f(x) + f(y)) \geq 0.$$

51. Find all polynomials such that $P(x^3 + 1) = P((x + 1)^3)$.

52. Find all polynomials $P(x) \in \mathbb{R}[x]$ such that $P(x^2 + 1) = P(x)^2 + 1$ holds for all $x \in \mathbb{R}$.

53. Problem: Find all polynomials $p(x)$ with real coefficients such that

$$(x + 1)p(x - 1) + (x - 1)p(x + 1) = 2xp(x)$$

for all real x.

54. Find all polynomials $P(x)$ that have only real roots, such that

$$P(x^2 - 1) = P(x)P(-x).$$

55. Find all polynomials $P(x) \in \mathbb{R}[x]$such that:

$$P(x^2) + x \cdot (3P(x) + P(-x)) = (P(x))^2 + 2x^2 \quad \forall x \in \mathbb{R}$$

56. Find all polynomials f, g which are both monic and have the same degree and

$$f(x)^2 - f(x^2) = g(x).$$

57. Find all polynomials $P(x)$ with real coefficients such that there exists a polynomial $Q(x)$ with real coefficients that satisfy

$$P(x^2) = Q(P(x)).$$

58. Find all polynomials $p(x, y) \in \mathbb{R}[x, y]$ such that for each $x, y \in \mathbb{R}$ we have

$$p(x + y, x - y) = 2p(x, y).$$

59. Find all couples of polynomials (P, Q) with real coefficients, such that for infinitely many $x \in \mathbb{R}$ the condition

$$\frac{P(x)}{Q(x)} - \frac{P(x+1)}{Q(x+1)} = \frac{1}{x(x+2)}$$

Holds.

60. Find all polynomials $P(x)$ with real coefficients, such that $P(P(x)) = P(x)^k$ (k is a given positive integer)

61. Find all polynomials

$$P_n(x) = n!x^n + a_{n-1}x^{n-1} + ... + a_1x + (-1)^n(n+1)n$$

with integers coefficients and with n real roots $x_1, x_2, ..., x_n$, such that $k \le x_k \le k+1$, for $k = 1, 2..., n$.

62. The function $f(n)$ satisfies $f(0) = 0$ and $f(n) = n - f(f(n-1))$, $n = 1, 2, 3\cdots$. Find all polynomials $g(x)$ with real coefficient such that

$$f(n) = [g(n)], \qquad n = 0, 1, 2\cdots$$

Where $[g(n)]$ denote the greatest integer that does not exceed $g(n)$.

63. Find all pairs of integers a, b for which there exists a polynomial $P(x) \in \mathbb{Z}[X]$ such that product $(x^2 + ax + b) \cdot P(x)$ is a polynomial of a form

$$x^n + c_{n-1}x^{n-1} + ... + c_1x + c_0$$

where each of $c_0, c_1, ..., c_{n-1}$ is equal to 1 or -1.

64. There exists a polynomial P of degree 5 with the following property: if z is a complex number such that $z^5 + 2004z = 1$, then $P(z^2) = 0$. Find all such polynomials P

65. Find all polynomials $P(x)$ with real coefficients satisfying the equation

$$(x+1)^3P(x-1) - (x-1)^3P(x+1) = 4(x^2-1)P(x)$$

for all real numbers x.

66. Find all polynomials $P(x, y)$ with real coefficients such that:

$$P(x, y) = P(x+1, y) = P(x, y+1) = P(x+1, y+1)$$

67. Find all polynomials $P(x)$ with reals coefficients such that

$$(x-8)P(2x) = 8(x-1)P(x).$$

68. Find all reals α for which there is a nonzero polynomial P with real coefficients such that

$$\frac{P(1) + P(3) + P(5) + \cdots + P(2n-1)}{n} = \alpha P(n) \quad \forall n \in \mathbb{N},$$

and find all such polynomials for $\alpha = 2$.

69. Find all polynomials $P(x) \in \mathbb{R}[X]$ satisfying

$$(P(x))^2 - (P(y))^2 = P(x+y) \cdot P(x-y), \quad \forall x, y \in \mathbb{R}.$$

70. Find all $n \in \mathbb{N}$ such that polynomial

$$P(x) = (x-1)(x-2)\cdots(x-n)$$

can be represented as $Q(R(x))$, for some polynomials $Q(x), R(x)$ with degree greater than 1.

71. Find all polynomials $P(x) \in R[x]$ such that $P(x^2 - 2x) = (P(x) - 2)^2$.

72. Find all non-constant real polynomials $f(x)$ such that for any real x the following equality holds

$$f(\sin x + \cos x) = f(\sin x) + f(\cos x).$$

73. Find all polynomials $W(x) \in \mathbb{R}[x]$ such that

$$W(x^2)W(x^3) = W(x)^5 \quad \forall x \in \mathbb{R}.$$

74. Find all the polynomials $f(x)$ with integer coefficients such that $f(p)$ is prime for every prime p.

75. Let $n \geq 2$ be a positive integer. Find all polynomials $P(x) = a_0 + a_1 x + \cdots + a_n x^n$ having exactly n roots not greater than -1 and satisfying

$$a_0^2 + a_1 a_n = a_n^2 + a_0 a_{n-1}.$$

76. Find all polynomials $P(x), Q(x)$ such that

$$P(Q(X)) = Q(P(x)) \forall x \in \mathbb{R}.$$

77. Find all integers k such that for infinitely many integers $n \geq 3$ the polynomial

$$P(x) = x^{n+1} + kx^n - 870x^2 + 1945x + 1995$$

can be reduced into two polynomials with integer coefficients.

78. Find all polynomials $P(x), Q(x), R(x)$ with real coefficients such that

$$\sqrt{P(x)} - \sqrt{Q(x)} = R(x) \quad \forall x \in \mathbb{R}.$$

79. Let $k = \sqrt[3]{3}$. Find a polynomial $p(x)$ with rational coefficients and degree as small as possible such that $p(k + k^2) = 3 + k$. Does there exist a polynomial $q(x)$ with integer coefficients such that $q(k + k^2) = 3 + k$?

80. Find all values of the positive integer m such that there exists polynomials $P(x), Q(x), R(x, y)$ with real coefficient satisfying the condition: For every real numbers a, b which satisfying $a^m - b^2 = 0$, we always have that $P(R(a, b)) = a$ and $Q(R(a, b)) = b$.

81. Find all polynomials $p(x) \in \mathbb{R}[x]$ such that $p(x^{2008} + y^{2008}) = (p(x))^{2008} + (p(y))^{2008}$, for all real numbers x, y.

82. Find all Polynomials $P(x)$ satisfying $P(x)^2 - P(x^2) = 2x^4$.

83. Find all polynomials p of one variable with integer coefficients such that if a and b are natural numbers such that $a+b$ is a perfect square, then $p(a)+p(b)$ is also a perfect square.

84. Find all polynomials $P(x) \in \mathbb{Q}[x]$ such that

$$P(x) = P\left(\frac{-x+\sqrt{3-3x^2}}{2}\right) \quad \text{for all} \quad |x| \leq 1.$$

85. Find all polynomials f with real coefficients such that for all reals a, b, c such that $ab + bc + ca = 0$ we have the following relations

$$f(a-b) + f(b-c) + f(c-a) = 2f(a+b+c).$$

86. Find All Polynomials $P(x, y)$ such that for all reals x, y we have

$$P(x^2, y^2) = P\left(\frac{(x+y)^2}{2}, \frac{(x-y)^2}{2}\right).$$

87. Let n and k be two positive integers. Determine all monic polynomials $f \in \mathbb{Z}[X]$, of degree n, having the property that $f(n)$ divides $f\left(2^k \cdot a\right)$, forall $a \in \mathbb{Z}$, with $f(a) \neq 0$.

88. Find all polynomials $P(x)$ such that

$$P(x^2 - y^2) = P(x+y)P(x-y).$$

89. Let $f(x) = x^4 - x^3 + 8ax^2 - ax + a^2$. Find all real number a such that $f(x) = 0$ has four different positive solutions.

90. Find all polynomial $P \in \mathbb{R}[x]$ such that: $P(x^2 + 2x + 1) = (P(x))^2 + 1$.

91. Let $n \geq 3$ be a natural number. Find all nonconstant polynomials with real coefficients $f_1(x), f_2(x), \ldots, f_n(x)$, for which

$$f_k(x) f_{k+1}(x) = f_{k+1}(f_{k+2}(x)), \quad 1 \leq k \leq n,$$

for every real x (with $f_{n+1}(x) \equiv f_1(x)$ and $f_{n+2}(x) \equiv f_2(x)$).

92. Find all integers n such that the polynomial $p(x) = x^5 - nx - n - 2$ can be written as product of two non-constant polynomials with integral coefficients.

93. Find all polynomials $p(x)$ that satisfy

$$(p(x))^2 - 2 = 2p(2x^2 - 1) \quad \forall x \in \mathbb{R}.$$

94. Find all polynomials $p(x)$ that satisfy

$$(p(x))^2 - 1 = 4p(x^2 - 4X + 1) \quad \forall x \in \mathbb{R}.$$

95. Determine the polynomials P of two variables so that:

a.) for any real numbers t, x, y we have $P(tx, ty) = t^n P(x, y)$ where n is a positive integer, the same for all t, x, y;

b.) for any real numbers a, b, c we have $P(a+b, c)+P(b+c, a)+P(c+a, b) = 0$;

c.) $P(1, 0) = 1$.

96. Find all polynomials $P(x)$ satisfying the equation

$$(x + 1)P(x) = (x - 2010)P(x + 1).$$

97. Find all polynomials of degree 3 such that for all non-negative reals x and y we have

$$p(x + y) \leq p(x) + p(y).$$

98. Find all polynomials $p(x)$ with real coefficients such that

$$p(a + b - 2c) + p(b + c - 2a) + p(c + a - 2b) = 3p(a - b) + 3p(b - c) + 3p(c - a)$$

for all $a, b, c \in \mathbb{R}$.

99. Find all polynomials $P(x)$ with real coefficients such that

$$P(x^2 - 2x) = (P(x - 2))^2$$

100. Find all two-variable polynomials $p(x, y)$ such that for each $a, b, c \in \mathbb{R}$:

$$p(ab, c^2 + 1) + p(bc, a^2 + 1) + p(ca, b^2 + 1) = 0.$$

Solutions

1. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=392562.

2. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=385331.

3. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=337211.

4. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=395325.

5. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=396236.

6. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=392444.

7. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=392115.

8. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=391333.

9. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=381485.

10. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=22091.

11. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=21897.

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13. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=16684.

14. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=18474.

15. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=398232.

16. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=6122.

17. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=78454.

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19. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=111404.

20. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=85409.

21. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=66713.

22. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=54236.

23. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=53450.

24. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=53271.

25. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=49788.

26. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=49530.

27. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=47243.

28. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=48110.

29. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=68010.

30. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=131296.

31. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=397716.

32. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=111400.

33. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=136814.

34. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=145370.

35. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=151076.

36. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=151408.

37. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=26076.

38. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=1890.

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42. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=16783.

43. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=28770.

44. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=35998.

45. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=37142.

46. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=37593.

47. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=38449.

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50. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=249173.

51. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=57623.

52. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=39570.

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61. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=91220.

62. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=97498.

63. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=82906.

64. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=100806.

65. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=107523.

66. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=112983.

67. http://www.artofproblemsolving.com/Forum/viewtopic.php?t=175482.

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