TJUSAMO Practice #12: Intermediate Geometry

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This lecture will cover some of the topics in geometry that are more sophisticated, without entering the realm of obscure mathematics. Within this area of geometry lies numerous interesting theorems and results, along with new methods. After you familiarize yourself with with this material, you should be able to solve most of the USAMO level geometry problems.

1 Theorems and Ideas

1.1 Menelaus's Theorem

Iff points X, Y, Z on the extensions of sides BC, CA, AB of $\triangle ABC$ are collinear, then

$$\frac{BX}{CX} * \frac{CY}{AY} * \frac{AZ}{BZ} = 1$$

1.2 Brahmagupta's Formula

For any cyclic quadrilateral with side lengths a, b, c, d, area A, and semiperimeter s:

$$A = \sqrt{(s-a)(s-b)(s-c)(s-d)}$$

Note that if d=0, then this simpifies to Heron's Formula.

1.3 The Butterfly Theorem

Let AB be a chord of circle ω with midpoint M. Let CD and EF be two other chords of ω through M. Let X and Y be the intersections of AB with CE and DF, respectively. In this construction, M is also the midpoint of XY.

1.4 Simson's Theorem

Let D, E, F be the feet of the perpendiculars from a point P to $\triangle ABC$. Iff P lies on the circumcircle of $\triangle ABC$, D, E, F are collinear and the line is known as the Simson line of P with respect to $\triangle ABC$.

1.5 Hexagons

In a hexagon $ABCDEF$, two vertices or two sides can be either adjacent, alternate, or opposite. For example, A and B are adjacent, C and F are opposite, and DE and FA are alternate, and BC and EF are opposite. A, C, E is a triad of alternating vertices, while AB, CD, EF are three alternating sides. The interior angles of a hexagon sum to 720°.

Note that in a cyclic hexagon, any three alternating angles sum up to 360°, but the converse is not always true. Also note that the following two theorems only go one way; their converses are much more complicated.

1.5.1 Pascal's Theorem

If the three pairs of opposite sides in a cyclic hexagon all intersect, then the three points of intersection are collinear.

1.5.2 Brianchon's Theorem

If a hexagon $ABCDEF$ can be circumscribed about a circle, then sides AD, BE, CF are either concurrent or parallel.

1.6 Coordinates

There are many different types of coordinates that have been discovered. Here are four of the most common systems and how they work. The first two have been around since ancient times, but the last two were both introduced in the early 19th century. Beware: it is usually unwise to start off a problem by attempting to use coordinates. They are time consuming to use and often do not do the trick.

1.6.1 Cartesian Coordinates AKA Rectangular Coordinates

You should know this system, so here are some tips:

- A triangle can be represented using three variables. Use $A = (0, 0), B = (0, a), C = (b, c)$.
- The distance between (a, b) and (c, d) is $\sqrt{(c-a)^2 + (d-b)^2}$.

- If a problem introduces a circle, start with a unit circle: a circle of radius 1 centered at the origin.

1.6.2 Polar Coordinates

The use of this system is limited to simple problems, usually ones involving primarily circles and angles. Polar coordinates are (r, θ), where r is the distance from the origin, and θ is the angle a line connecting the point to the origin makes with the positive x-axis, going counterclockwise from the positive x-axis.

We use the substitution $x = r\cos\theta,\ y = r\sin\theta$ to convert to Cartesian coordinates, and we use $r = \sqrt{x^2 + y^2},\ \theta = \tan^{-1}(\frac{y}{x})$ to convert from Cartesian coordinates.

1.6.3 Trilinear Coordinates

Trilinear Coordinates denote the ratio of the distances from a point $P = (x, y, z)$ to each of three sides of a triangle. Note that since this is a ratio, (x, y, z) and $(-5x, -5y, -5z)$ represent the same point. However, the coordinates are often normalized so that $x^2 + y^2 + z^2 = 1$.

Using trilinear coordinates, many triangle centers can be expressed with ease. For example, here is a list of common triangle center functions: (note that a, b, c are the side lengths of the triangle, while α, β, γ are the angles of the triangle)

- incenter: $(1, 1, 1)$
- circumcenter: $(\cos\alpha, \cos\beta, \cos\gamma)$
- orthocenter: $(\cos\beta\cos\gamma, \cos\alpha\cos\gamma, \cos\alpha\cos\beta)$
- nine point center: $(\cos\beta - \cos\gamma, \cos\alpha - \cos\gamma, \cos\alpha - \cos\beta)$
- centroid: $(\frac{1}{a}, \frac{1}{b}, \frac{1}{c})$

  Note that all triangle centers(not just the above) should have symmetric trilinear coordinates.

1.6.4 Barycentric Coordinates

Barycentric coordinates (x, y, z) represent ratios of masses placed at the vertices of a triangle, which determines some center of mass. When these coordinates are normalized, $x+y+z = 1$. Note that the centroid has barycentric coordinates $(1, 1, 1)$, and the incenter has coordinates (a, b, c) (where a, b, c are side lengths of the triangle). Other centers have slightly less elegant barycentric coordinates.

1.7 Vectors/Complex Numbers

These tools can be very useful for brute forcing a geometry problem. However, a lot of practice is necesary to use these effectively. This will not be covered here. Try finding a specialized book if you wish to learn about computational geometry.

2 Problems

1. {1.5-3.5} Prove all the theorems in this lecture. Remember, this is always an excellent exercise.

2. {1} Find a simplified formula for the area of a cyclic quadrilateral with side lengths a, a, b, c.

3. {1.5} Let pentagon $ABCDE$ be circumscribed about a circle ω. Let point X be on line AE and ω. Let Y be the intersection of AD and BE. Prove that C, X, Y are collinear.

4. {1.5} Let $ABCD$ be a quadrilateral with no parallel sides that is inscribed in circle ω. Let W be the intersection of AC and BD. Let X be the intersection of AB and CD. Let Y be the intersection of the lines tangent to ω at A and D. Similarly, let Z be the intersection of the lines tangent to ω at B and C. Prove that the points W, X, Y, Z lie on a line.

5. {1.5} Let P and Q be two points of the circumcircle of $\triangle ABC$ such that PQ is perpendicular to AB. Prove that the line CP is parallel to the Simson line of Q with respect to $\triangle ABC$.

6. {2} (PDiao) A circle inscribed in an ellipse is also centered at one focus of that ellipse. How many points of tangency do the two conics share?

7. {2} (Coxeter/Greitzer) In a scalene triangle ABC, let the external bisectors of A, B, C meet sides BC, AC, AB, respectively, at points D, E, F, respectively. Prove that D, E, F are collinear.

8. {2.5} Let $ABCEF$ be a convex cyclic hexagon. Prove that segments AD, BE, CF are concurrent iff
$$AB * CD * EF = BC * DE * FA$$

9. {2.5} (ZFeng) Let M and N be the midpoints of sides AD and BC of rectangle $ABCD$, respectively. Let P be a point on ray CD but not on side CD. Let point Q be the intersection of AC and PM. Prove that $\angle MNQ = \angle MNP$.

10. {3} (Coxeter/Greitzer) Let $\triangle ABC$ have incenter I. Let the incircle of $\triangle ABC$ be tangent to BC at D. Let M be the midpoint of BC. Prove that line DI bisects AD.

11. $\{3\}$ (IMO Shortlist 2003 G1) Let $ABCD$ be a cyclic quadrilateral. Let points P, Q, R be the feet of the perpendiculars from D to lines BC, CA, AB, respectively. Prove that iff the bisectors of $\angle ABC$ and $\angle ADC$ are concurrent with AC, then $PQ = QR$.

12. $\{3.5\}$ (IMO Shortlist 2003 G3) Let point P be in the interior of $\triangle ABC$. Let points D, E, F be the feet of the perpendiculars from point P to lines BC, CA, AB, respectively. Let X, Y, Z be the three excenters of $\triangle ABC$. Prove that if

$$AP^2 + PD^2 = BP^2 + PE^2 = CP^2 + PF^2$$

then P is the circumcenter of $\triangle XYZ$.

13. $\{3.5\}$ (MOP '05) Let $ABCDEF$ be a cyclic hexagon such that $AB = CD = EF$. Let P, Q, R be the intersections of AC and BD, CE and DF, and AE and BF, respectively. Prove that $\triangle BDF$ is similar to $\triangle PQR$.

14. $\{4\}$ (MOP '05) Let $\triangle ABC$ be acute and scalene with circumcenter O. Let points D, E, F be the midpoints of sides BC, CA, AB, respectively. Prove that the circumcircles of $\triangle ADO, \triangle BEO, \triangle CFO$ pass through two common points.

15. $\{5\}$ (USAMO '06 #6) Let points E, F be on sides AD and BC of quadrilateral $ABCD$, respectively, such that $AE * CF = ED * BF$. Ray FE meets rays BA and CD at S and T, respectively. Prove that the circumcircles of $\triangle SAE$, $\triangle SBF$, $\triangle TCF$, $\triangle TDE$ pass through a common point.