Geometry in a Nutshell

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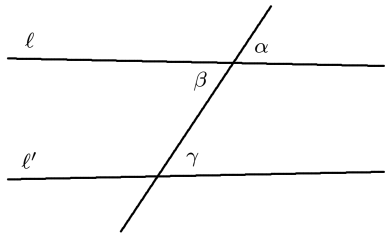
This short handout is a list of some of the very basic ideas and results in pure geometry. *Draw your own diagrams with a pencil, ruler and compass where necessary, and make sure you understand every idea and result!* Then, try the short list of problems at the end.

0. A fundamental axiom

Before we talk about objects like triangles and circles, we mention a fundamental axiom about equal angles.

Theorem 1



In the figure, $\alpha = \beta$, and they are opposite angles.

Also, if ℓ and ℓ' are parallel lines, then $\alpha = \gamma$ and they are corresponding angles*, and $\beta = \gamma$ and they are* alternate angles.

Conversely, if either $\alpha = \gamma$, or $\beta = \gamma$, then ℓ and ℓ' are parallel

1. Ideas and results about triangles

Standard Notation.

If $\triangle ABC$ is a triangle, then the measure of the interior angle at A is often denoted by A itself, and similarly for B and C. The measure of the side opposite to A is often denoted by a, and similarly for B and C, giving rise to b and c. A similar thing happens if the triangle is labelled by other letters.

Types of triangles, and relations between sides and angles.

Let $\triangle ABC$ be a triangle. We say that $\triangle ABC$ is *equilateral* if $a = b = c$, that $\triangle ABC$ is *isosceles* if two of a, b, c are equal, and that $\triangle ABC$ is *scalene* if a, b, c are distinct.

We have a result which relates these types of triangles with their interior angles.

Theorem 2 *Let $\triangle ABC$ be a triangle.*

(a) If $\triangle ABC$ is equilateral, then $A = B = C = 60°$, and conversely.

(b) If $\triangle ABC$ is isosceles, with $b = c$, then $B = C$, and conversely.

(c) If $\triangle ABC$ is scalene, then A, B, C, are distinct, and conversely.

Note that in Theorem 5 below, we will state that the sum of the interior angles of a triangle is always 180°. So in (a) above, $A = B = C$ itself forces each interior angle to be equal to 60°.

Somewhat in the direction of Theorem 2, we have the following result.

Theorem 3 *Let $\triangle ABC$ be a triangle. If $A \leq B \leq C$, then $a \leq b \leq c$, and conversely.*

In other words, Theorem 3 simply says that, in any triangle, the smallest angle must be opposite to the shortest side, the second smallest angle must be opposite to the second shortest side, and the largest angle must be opposite to the longest side.

We have one more, very well-known property.

Theorem 4 (Triangle inequalities) *Let $\triangle ABC$ be a triangle. Then, we have $b + c > a$, $c + a > b$, and $a + b > c$.*

Four well-known points and three others.

We now define four well-known points associated with a triangle $\triangle ABC$, as well as three other less well-known ones.

- A *median* of $\triangle ABC$ joins a vertex to the midpoint of the opposite side. The three medians of any triangle always concur. The point of concurrence is the *centroid*, often denoted by the letter G. The centroid lies $\frac{2}{3}$ of the way along each median, going from a vertex to the opposite side. That is, if $\triangle ABC$ has centroid G and AA', BB', CC' are the medians, then $\frac{AG}{AA'} = \frac{BG}{BB'} = \frac{CG}{CC'} = \frac{2}{3}$.

- The three perpendicular bisectors of the sides of $\triangle ABC$ are concurrent. The point of concurrence is the *circumcentre*, often denoted by the letter O. The circumcentre is equidistant from the three vertices A, B, C, and this common distance is the *circumradius* of $\triangle ABC$, often denoted by the letter R. The circle with centre O, radius R, is the *circumcircle* of $\triangle ABC$. This circle is the unique circle which passes through A, B, C.

- An *altitude* of $\triangle ABC$ is a line segment joining a vertex to the opposite side, and is perpendicular to that opposite side (produced if necessary). The three altitudes of any triangle are concurrent, and the point of concurrence is the *orthocentre*, often denoted by the letter H.

To define the other points, we must first define what a *tangent* is. If Γ is a circle, then a straight line ℓ is a *tangent* of Γ if ℓ "touches" Γ at exactly one point. Note that ℓ *does not* cut across the circumference of Γ, and that, if it is extended infinitely far on both sides, it always meets Γ at one point.

The point where a tangent meets a circle is the *point of contact* of the tangent to the circle.

- The three interior angle bisectors of $\triangle ABC$ are concurrent. The point of concurrence is the *incentre*, often denoted by the letter I. The incentre is equidistant from the three sides of $\triangle ABC$. The common distance is the *inradius*, often denoted by the lower case letter r. The circle with centre I, radius r is the *incircle*. The three sides of $\triangle ABC$ are each tangent to the incircle, and the incircle is the unique circle with this property.

The centroid, circumcentre, orthocentre and incentre are points of a triangle which one must be familiar with. We now define the *excentres* of a triangle. A triangle has three excentres. These three points generally do not appear as often in olympiad problems, but should be understood, nevertheless.

- In $\triangle ABC$, the interior angle bisector of A, and the bisectors of the exterior angles at B and C, are concurrent. The point of concurrence is the *excentre opposite* A. It may be denoted by E_A. The excentre E_A is equidistant from the side BC and the extensions of AB and AC. The common distance is the *exradius opposite* A, which can be denoted by r_A. The circle with centre E_A and radius r_A is the *excircle opposite* A. The side BC, and the extensions of AB and AC are each tangent to this excircle.
  Similarly, we can construct the other two excentres E_B and E_C, opposite B and C respectively, giving a further two excircles.

Some basic results.

Here are some basic results associated with a triangle.

Theorem 5 *Let $\triangle ABC$ be a triangle. We have*

(a) $A + B + C = 180°$.

(b) $A + B =$ *measure of the exterior angle at C. Similar statements hold for the other two pairs of interior angles.*

Theorem 6 (Pythagoras' Theorem) *Let $\triangle ABC$ be a triangle. If $C = 90°$, then $a^2 + b^2 = c^2$. Conversely, if $a^2 + b^2 = c^2$, then $C = 90°$.*

Theorem 7 (The Sine Rule) *Let $\triangle ABC$ be a triangle, with circumradius R. Then, we have*

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R.$$

Theorem 8 (The Cosine Rule) *Let $\triangle ABC$ be a triangle. Then, we have*

$$a^2 = b^2 + c^2 - 2bc\cos A.$$

Note that we get two other similar equations by cycling the above equation by $a \to b \to c \to a$ and $A \to B \to C \to A$.

If P is a figure in the plane, then we write $[P]$ for the area of P. In particular, $[\triangle ABC]$ denotes the area of a triangle $\triangle ABC$. This is a widely understood notation.

Theorem 9 (Area of a triangle) *Let $\triangle ABC$ be a triangle. Then*

(a) $[\triangle ABC] = \frac{1}{2}ah_a = \frac{1}{2}bh_b = \frac{1}{2}ch_c$, *where h_a is the length of the altitude passing through A, and h_b, h_c are defined similarly.*

(b) $[\triangle ABC] = \frac{1}{2}bc\sin A = \frac{1}{2}ca\sin B = \frac{1}{2}ab\sin C$.

(c) (Heron's formula) $[\triangle ABC] = \sqrt{s(s-a)(s-b)(s-c)}$, *where $s = \frac{1}{2}(a+b+c)$ is the* semi-perimeter *of $\triangle ABC$.*

Theorem 10 *Let $\triangle ABC$ be a triangle, let $\triangle = [\triangle ABC]$, let r be the inradius, and let $s = \frac{1}{2}(a+b+c)$, the* semi-perimeter *of $\triangle ABC$. Then, we have $\triangle = rs$.*

Theorem 11 (The angle bisector theorem) *Let $\triangle ABC$ be a triangle, and let D be the point on BC such that $\angle BAD = \angle CAD$. Then, we have $\frac{BA}{BD} = \frac{CA}{CD}$.*

Similarity and congruence of triangles.

Similarity and congruence are two very important concepts relating two triangles, $\triangle ABC$ and $\triangle XYZ$ say.

- We say that $\triangle ABC$ and $\triangle XYZ$ are *similar* if we have $\frac{a}{x} = \frac{b}{y} = \frac{c}{z} = k$, for some real number k. We write $\triangle ABC \sim \triangle XYZ$.
- We say that $\triangle ABC$ and $\triangle XYZ$ are *congruent* if we have $\frac{a}{x} = \frac{b}{y} = \frac{c}{z} = 1$. We write $\triangle ABC \cong \triangle XYZ$.

Note that when we write $\triangle ABC \sim \triangle XYZ$ or $\triangle ABC \cong \triangle XYZ$, the corresponding letters must match. That is, the correspondences $A \leftrightarrow X$, $B \leftrightarrow Y$ and $C \leftrightarrow Z$ are implied, (and indeed, as we shall see, $A = X$, $B = Y$ and $C = Z$ in both notations). For example, if we wrote $\triangle CAB \cong \triangle ZYX$, then we would be saying $\frac{c}{z} = \frac{a}{y} = \frac{b}{x} = 1$, rather than $\frac{a}{x} = \frac{b}{y} = \frac{c}{z} = 1$.

There are many conditions which can tell us if two triangles are similar or congruent.

Theorem 12 *Let $\triangle ABC$ and $\triangle XYZ$ be two triangles. Then, we have $\triangle ABC \sim \triangle XYZ$ if any of the following properties hold.*

- $\frac{a}{x} = \frac{b}{y} = \frac{c}{z} = k$, *for some real number* k.
- $A = X$, *and* $\frac{b}{y} = \frac{c}{z} = k$, *for some real number* k *(or a similar relationship obtained by cycling* $a \to b \to c \to a$, $A \to B \to C \to A$, $x \to y \to z \to x$ *and* $X \to Y \to Z \to X$, *simultaneously).*
- *Two of the equalities* $A = X$, $B = Y$, $C = Z$ *hold.*

Conversely, if $\triangle ABC \sim \triangle XYZ$, *then each of the above three properties holds.*

Theorem 13 *Let* $\triangle ABC$ *and* $\triangle XYZ$ *be two triangles. Then, we have* $\triangle ABC \cong \triangle XYZ$ *if any of the following properties hold.*

- $\frac{a}{x} = \frac{b}{y} = \frac{c}{z} = 1$. *This is known as the* SSS (side-side-side) law.
- $A = X$, *and* $\frac{b}{y} = \frac{c}{z} = 1$ *(or a similar relationship obtained by cycling* $a \to b \to c \to a$, $A \to B \to C \to A$, $x \to y \to z \to x$ *and* $X \to Y \to Z \to X$, *simultaneously). This is known as the* SAS (side-angle-side) law.
- $A = X$, $B = Y$, *and* $c = z$ *(or a similar relationship obtained by cycling* $a \to b \to c \to a$, $A \to B \to C \to A$, $x \to y \to z \to x$ *and* $X \to Y \to Z \to X$, *simultaneously). This is known as the* ASA (angle-side-angle) law.

Conversely, if $\triangle ABC \cong \triangle XYZ$, *then each of the above three properties holds.*

In many geometry problems, it is a common tactic to establish the similarity or congruence between two triangles. Doing so can enable us to deduce that certain lengths or angles are equal.

2. Circle theorems

We now consider some of the most fundamental results associated with a circle. We begin with those involving just one circle.

One circle.

Theorem 14 (Angles in the same segment theorem) *Let* Γ *be a circle, and let* A, C, D, B *be four distinct points on* Γ, *in that order. Then, we have* $\angle ACB = \angle ADB$.

Theorem 14 has a consequence, which itself has a converse.

Theorem 15 *Let* Γ *be a circle, and let* AB *and* DE *be two chords. Let* C *and* F *be points on the major arcs of* AB *and* DE *respectively. If* $AB = DE$, *then* $\angle ACB = \angle DFE$. *That is, equal chords subtend equal angles on* Γ. *A similar statement holds if 'major' is replaced by 'minor'.*

Conversely, let A, B, C *be three distinct points of* Γ, *and let* D, E, F *be another three distinct points (any other coincidences are allowed). If* $\angle ACB = \angle DFE$, *then* $AB = DE$. *That is, equal angles are subtended by equal chords in* Γ.

Theorem 16 ("Angle at centre is twice the angle on the arc" theorem) *Let Γ be a circle with centre O, and let A, B, C be three distinct points on Γ. Then, we have $2\angle ACB = \angle AOB$, where $\angle AOB$ is taken to be the non-reflex angle if C and O are on the same side of AB, the reflex angle if C and O are on opposite sides of AB, and the 180° angle outside $\triangle ACB$ if AB is a diameter.*

Theorem 16 does not have a name. But it is very well-known and widely understood. It has the following consequence.

Theorem 17 (Angle in a semi-circle is a right angle) *Let Γ be a circle, and let A, B, C be three distinct points on Γ, where AB is a diameter. Then, we have $\angle ACB = 90^\circ$.*

Theorem 18 (Intersecting chords theorem) *Let Γ be a circle, and let A, B, C, D be four distinct points on Γ. Let AB meet CD meet at X, which can be inside or outside of Γ. Then, we have $AX \cdot BX = CX \cdot DX$.*

Next, we have four well-known results, all of which involve *tangents*.

Theorem 19 *Let Γ be a circle with centre O. Let ℓ be a tangent to Γ with point of contact P. Let X be any point on ℓ, different from P. Then $\angle OPX = 90^\circ$.*

Theorem 20 *Let Γ be a circle with centre O, and let A be an exterior point. Let ℓ and ℓ' be the two tangents to Γ passing through A, with points of contact P and P' respectively. Then $\triangle APO \cong \triangle AP'O$, so that $AP = AP'$, $\angle AOP = \angle AOP'$, and $\angle OAP = \angle OAP'$.*

Theorem 21 (Alternate segment theorem) *Let Γ be a circle. Let ℓ be a tangent to Γ with point of contact P. Let X be a point on ℓ different from P. Let A and B be two distinct points on Γ, both different from P, and such that B and X are on opposite sides of AP. Then, we have $\angle XPA = \angle PBA$.*

The next theorem involves a *secant* as well. A *secant* of a circle Γ is a line which intersects Γ at exactly two points.

(Note: A secant is generally not a chord, since a chord is a line segment, but a secant can be an infinite line).

Theorem 22 (Tangent-secant theorem) *Let Γ be a circle. Let ℓ be a tangent to Γ with point of contact P. Let X be a point on ℓ different from P. Let a secant passing through X meet Γ at A and B. Then, we have $AX \cdot BX = PX^2$.*

Next, we consider some results about *cyclic quadrilaterals*. A quadrilateral is *cyclic* if there exists a circle which passes though all four of its vertices.

Also, a quadrilateral is *convex* if all four of its interior angles are less than 180°.

It is easy to see that a cyclic quadrilateral must be convex.

Theorem 23 *Let $ABCD$ be a cyclic quadrilateral. Then,*

(a) $A + C = B + D = 180°$,

(b) *A is equal in measure to the exterior angle at C, and similar statements hold for B, C and D (by cycling $A \to B \to C \to D \to A$).*

Conversely, if $ABCD$ is a quadrilateral such that either

- $A + C = 180°$, *or* $B + D = 180°$, *or*
- *$ABCD$ is convex, and A is equal in measure to the exterior angle at C, (or if a similar statement holds after some cycling of $A \to B \to C \to D \to A$).*

Then, $ABCD$ is cyclic.

The following result is a sort of converse to Theorem 14.

Theorem 24 *Let $ABCD$ be a convex quadrilateral such that $\angle ACB = \angle ADB$. Then $ABCD$ is a cyclic quadrilateral.*

Theorem 25 (Ptolemy's theorem) *Let $ABCD$ be a cyclic quadrilateral. Then*

$$AB \cdot CD + BC \cdot DA = AC \cdot BD.$$

Two circles.

We will consider the ways of how two circles can 'interact' with each other.
Let Γ_1 and Γ_2 be two circles, with centres O_1 and O_2 respectively. Observe that

- Γ_1 and Γ_2 may possibly not meet each other. In this case, either one circle lies entirely within the other, or the two circles are external to each other.
- Or, Γ_1 and Γ_2 may meet at one point. In this case, the two circles *touch* each other, or are *tangent* to each other. There are two ways that this can happen. Either one circle lies entirely inside the other (with the sole exception of the touching point); in this case, they *touch internally.* Or, the two circles are outside of each other, (again with the sole exception of the touching point); in this case, they *touch externally.*
- Or, Γ_1 and Γ_2 may meet at two points.

In any case, the line passing through O_1 and O_2 is an axis of symmetry of the figure created by Γ_1 and Γ_2.

What can we say about tangents which are common to Γ_1 and Γ_2 (that is, lines which are tangent to both Γ_1 and Γ_2)?

Theorem 26 *Let Γ_1 and Γ_2 be two circles.*

(a) If Γ_1 and Γ_2 do not meet, then they have no common tangents if one circle lies inside the other, and four common tangents if they are external to each other.

(b) If Γ_1 and Γ_2 meet at one point, then they have one common tangent if the circles touch internally, and three common tangents if the circles touch externally.

(c) If Γ_1 and Γ_2 meet at two points, then they have two common tangents.

3. Problems

Here are some standard geometry problems. The difficulty of these are approximately the same as those found in BMO1 or the Senior Mentoring Scheme.

1. Let $\triangle ABC$ be a triangle with $AC > AB$. Let D be the point on the side AC such that $AD = AB$. Given that $\angle ABC - \angle ACB = 30°$, find the measure of $\angle CBD$.

2. Let $\triangle ABC$ be a triangle with $B = C = 40°$. Let D be the point on AC such that BD bisects B. Prove that $AD + BD = BC$.

3. Let Γ_1 be a circle and let AB be a chord of Γ_1 with midpoint C. Let DE be another chord of Γ_1 which passes through C. Let Γ_2 be a semi-circle with diameter DE, and let F be a point on Γ_2 such that CF is perpendicular to DE. Prove that $AC = CF$.

4. Let $\triangle ABC$ be a triangle, where none of its interior angles is a right angle. Let H be the orthocentre of $\triangle ABC$. Prove that the circumradii of $\triangle BHC$, $\triangle CHA$ and $\triangle AHB$ are equal.

5. Let Γ be a circle with centre O, and let A be an exterior point. The tangents to Γ passing through A touch Γ at M and N. Let P be a point on the minor arc MN, and let the tangent to Γ at P meet AM and AN at B and C respectively. Let D be the circumcentre of $\triangle ABO$. Prove that the points A, B, C and D lie on a circle.

6. An equilateral triangle has the property that there exists an interior point whose distance from the vertices of the triangle are 3, 4 and 5. Find the exact area of the triangle.