Graph Theory

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Some Theorems

- **Cayley's formula**: The number of trees on n labeled vertices is n^{n-2}.
- **Turán's theorem**: The maximum number of edges in a graph on n vertices containing no complete subgraph on k vertices is attained by the complete $(k-1)$-partite graph $K_{n_1,n_2,\ldots,n_{k-1}}$ where $n_1+n_2+\cdots+n_{k-1}=n$ and no two of the n_i differ by more than 1.
- **Hall's matching theorem** or **the marriage lemma**: Let A and B be the parts of a bipartite graph. We wish to choose a set of edges such that every vertex of A lies on exactly one edge, but no vertex of B lies on more than one edge. This is possible if and only if, for all $S \subset A$, the number of vertices of B adjacent to at least vertex in S is at least the number of elements of S.
- **Euler's formula**: If a connected planar graph has V vertices, E edges, and F faces (including the unbounded one), then $V - E + F = 2$.

Problems

1. A graph with $n \geq 3$ vertices in which every vertex has degree at least $n/2$ has a Hamiltonian cycle.
2. A *tournament graph* is a directed graph in which there is exactly one edge between pair of vertices (in one direction or the other but not both). Prove that every tournament graph contains a Hamiltonian path. Which tournament graphs contain a Hamiltonian cycle?
3. Given a simple connected graph with an even number of edges, show that the edges can be directed so that each vertex has even out-degree.
4. Given a graph with $2n+1$ vertices, such that for any n vertices there exists a vertex adjacent to all of them, prove that there exists a vertex adjacent to all other vertices.
5. Let n be a positive integer. Find the largest integer N with the following property: Every graph with N vertices such that removing any vertex leaves a graph containing a complete subgraph on n vertices itself contains a complete subgraph on $n+1$ vertices.
6. A planar graph with n vertices has at most $3n-6$ edges. In particular, it must contain a vertex of degree at most 5.
7. An $n \times n$ table filled with numbers has the property that no two columns are identical. Prove that there exists a row which can be removed so that the property is maintained.
8. Cities $C_1, C_2, \ldots, C_N$ are served by airlines $A_1, A_2, \ldots, A_n$. There is direct non-stop service between any two cities (by at least one airline), and all airlines provide service in both directions. If $N \geq 2^n+1$, prove that at least one of the airlines can offer a round trip with an odd number of landings.
9. A tromino is a figure obtained from a 2×2 square by removing one of its corners. Suppose that we are given 2005 trominos in the plane such that the square removed from each tromino is covered by another tromino. Show that we can remove a tromino and preserve this property.
10. Suppose a graph with n vertices and q edges contains no 3-cycles. Prove that there is a vertex v such that there are at most $q(1-4q/n^2)$ edges neither of whose vertices is v or any vertex adjacent to v.
11. An $n \times n$ square table ($n \geq 2$) is filled with 0's and 1's so that any subset of n cells, no two of which lie in the same row or column, contains at least one 1. Prove that there exist i rows and j columns with $i + j \geq n+1$ whose intersection contains only 1's.

12. At a meeting of $12k$ people, each person exchanges greetings with exactly $3k+6$ others. For any two people, the number who exchange greetings with both is the same. How many people are at the meeting? (You must also show that your answer(s) can be realized.)

13. I have an $n \times n$ sheet of stamps, from which I am asked to tear out as many blocks as I can in the shape of a T (the figure formed from a 1×3 rectangle by adding one stamp adjacent to the middle stamp of the rectangle). I can only tear along the perforations separating adjacent stamps, and each block must come out of the sheet in one piece. However, I'm in a bad mood and decide to tear out as *few* blocks as I can while making it impossible to tear out any more blocks. Let c_n denote the smallest number of blocks I can tear out in this fashion. Prove that there exist constants a, b such that $|c_n - an^2| \leq bn$ for all n, and determine the value of a.

14. There are 51 senators in a senate. The senate needs to be divided into n committees so that each senator is on one committee. Each senator hates exactly three other senators. (If senator A hates senator B, then senator B does *not* necessarily hate senator A.) Find the smallest n such that it is always possible to arrange the committees so that no senator hates another senator on his or her committee.

15. The edges of a complete graph on 10 vertices are colored in k colors, such that for any k points, at least k different colors appear among the edges joining those points. Show that $k \geq 5$.

16. Among a group of 120 people, some pairs are friends. A *weak quartet* is a set of four people containing exactly one pair of friends. Determine the maximum possible number of weak quartets.