Functional Equations

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Functional equations are equations in which a function is specified implicitly. Functional equations are different from algebraic equations in that they generally cannot be reduced into solving an expression in terms of $f(x)$. As functional equations are mainly based on recursion, various substitutions can be used to find the solutions of a functional equation, but there are various clever manipulations involved in them.

1 Definitions

Definition 1.1. In a function $f : X \to Y$, X is called the *codomain* and Y is called the *image* of f. We also denote $Y = \text{Im}(f)$.

Definition 1.2. In a function $f : X \to Y$, the *kernel* of f denoted by $\text{Ker}(f)$ is the subset of X that satisfy $f(x) = 1$.

Definition 1.3. A function $f : X \to Y$ is said to be *surjective* if, for every element y in Y, there exists an element x in X such that $f(x) = y$.

Definition 1.4. A function f is *injective* if $f(a) = f(b)$ implies $a = b$.

Definition 1.5. A function f is *bijective* if it is both surjective and injective.

Definition 1.6. A function f is *monotonic* if it is either nonincreasing or nondecreasing. It is *monotonically increasing* if, for all x and y in the codomain of f such that $x \leq y$, then $f(x) \leq f(y)$. It is *monotonically decreasing* if $f(x) \geq f(y)$.

Definition 1.7. A function f is *continuous at c* if for any $\epsilon > 0$, there exists $\delta > 0$ such that for all x in the codomain of f with $|x - c| < \delta$, the value of f satisfies $|x - f(c)| < \epsilon$. A function is *continuous* if it is continuous everywhere.

2 Techniques

Unfortunately, functional equations can only be mastered through practice. However, there are various techniques that one can use in solving functional equations.

- Prove that f is injective by assuming that $f(a) = f(b)$ for some $a, b \in X$, and then proving that $a = b$.
- Prove that f is surjective by letting the value of a function equal a variable.
- If $f(0) = 0$, we can let c be some number such that $f(c) = 0$. Substituting c in the functional equation may prove that c is unique.
- Transform the original functional equation into a special one by considering a new function.

3 Substitutions

Most Olympiad functional equation problems will not fall due to pure substitutions. As such, cleverer manipulations must be made, often implicitly. A technique discussed above is to use the fact that there exists c such that $f(c) = 0$, and then substituting this new variable in the equation. This is exemplified in the following example.

Example 3.1 (MOP 2013)**.** Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$,

$$f(f(x+y)f(x-y)) = x^2 - yf(y).$$

Solution. Let $P(x, y)$ be the above assertion. Note that $P(0, 0)$ implies that $f(f(0)^2) = 0$, so there exists a constant c such that $f(c) = 0$. Now $P(2c, c)$ and $P(c, 0)$ give $f(0) = 4c^2$ and $f(0) = c^2$, so $c = 0$. Hence $f(0) = 0$.

Now from $P(x, x)$ we obtain $f(f(2x)f(0)) = x^2 - xf(x)$, so $xf(x) = x^2$ for all real x. It follows that $f(x) = x$ for all real x. □

Other substitutions can boil down functional equations easily.

Example 3.2. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$(x-y)f(x+y) - (x+y)f(x-y) = 4xy(x^2 - y^2)$$

for all $x, y \in \mathbb{R}$.

Solution. Let $u = x + y$ and $v = x - y$. The original equation gives us $vf(u) - uf(v) = (u+v)(u-v)uv$, so dividing through by uv yields

$$\frac{f(u)}{u} - \frac{f(v)}{v} = u^2 - v^2,$$

which rearranges to

$$\frac{f(u)}{u} - u^2 = \frac{f(v)}{v} - v^2$$

for all $u, v \in \mathbb{R}$. Hence the value $\frac{f(x)}{x} - x^2$ is constant for all $x \in \mathbb{R}$, so $\frac{f(x)}{x} - x^2 = c$ implies that $f(x) = x^3 + cx$ for some constant $c \in \mathbb{R}$. □

4 Special Functional Equations

4.1 Cauchy's Functional Equation

Cauchy's functional equation is the functional equation

$$f(x+y) = f(x) + f(y).$$

Functions that satisfy this condition are called *additive functions*. By induction, it is not difficult to prove that $f\left(\sum x\right) = \sum f(x)$ where the sum is taken over any variables. It is also clear that $f(x) = cx$ for some constant c if x is an integer. This notion can be extended to the rationals:

Lemma 4.1. *If $f : \mathbb{Q} \to \mathbb{R}$ is an additive function, then $f(x) = cx$ for some constant $c \in \mathbb{R}$.*

Unfortunately, with only the fact that f is additive, we cannot conclude that f is a linear function over the reals. In fact, there are extremely complicated additive functions that do not satisfy $f(x) = cx$. However, with additional information, we can deduce that f is linear:

Theorem 4.2. *If $f : \mathbb{R} \to \mathbb{R}$ is an additive function, and it is monotonic, continuous at a point, or bounded on some interval, then $f(x) = cx$ for some constant $c \in \mathbb{R}$.*

In Olympiad mathematics, functional equations that involve Cauchy's functional equation will likely not require the continuity case, as continuity is technically a concept taught in Calculus. Moreover, new functions can be defined in order to satisfy Cauchy's functional equation.

Example 4.1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(f(x)^2 + y) = x^2 + f(y).$$

Solution. Let $P(x, y)$ be the above assertion. Suppose that $f(a) = f(b)$ for $a, b \in \mathbb{R}$. Then $P(a, y)$ and $P(b, y)$ gives

$$a^2 + f(y) = f(f(a)^2 + y) = f(f(b)^2 + y) = b^2 + f(y),$$

and so $a^2 = b^2$. Consider the function $g : (0, \infty) \to (0, \infty)$ such that $g(x) = f(\sqrt{x})^2$. The given condition becomes $f(g(x) + y) = x + f(y)$ for $x > 0$. Now notice that

$$f(g(p) + g(q) + y) = p + f(g(q) + y) = p + q + f(y) = f(g(p + q) + y)$$

so that $g(p) + g(q) + y = \pm(g(p + q) + y)$ for all positive reals p and q and reals x and y. However, $g(p)+g(q)+y = -(g(p+q)+y)$ is impossible to hold for all reals y, so it follows that $g(p)+g(q)+y = g(p+q)+y$, or that g is additive. Since the image of g is positive, it is evident that g is bounded below, and thus $g(x) = cx$ for some positive real c. It follows that $f(x) = \pm\sqrt{c}x$.

If $f(y) = -\sqrt{c}y$ for some y, then $c(cx^2 + y)^2 = f(cx^2 + y)^2 = (x^2 - \sqrt{c}y)^2$ which is clearly not true for all x and y. Hence, $f(y) = \sqrt{c}y$ and a quick check reveals that $c = 1$. Therefore, the only solution is $f(x) = x$. □

Functions that satisfy

$$f(xy) = f(x)f(y)$$

are called *multiplicative functions*. Note that we can derive the so-called multiplicative Cauchy's functional equation by taking the logarithm of Cauchy's functional equation and setting $g(x) = \log f(x)$. When a function is both additive and multiplicative, then an interesting result occurs:

Theorem 4.3. *If f is both additive and multiplicative, then f is the identity function.*

Note that we can also derive the exponential and logarithmic Cauchy's functional equations by setting $g(x) = f(\log x)$ and $g(x) = e^{f(x)}$.

4.2 Jensen's Functional Equation

Jensen's functional equation is the functional equation

$$f\left(\frac{x + y}{2}\right) = \frac{f(x) + f(y)}{2}.$$

This functional equation is similar to Jensen's Inequality in the two-variable case. Like Cauchy's functional equation, this equation alone cannot give a solution for all reals. However, given additional information about f, we can deduce that f must be linear.

Theorem 4.4. *If $f : I \to \mathbb{R}$ satisfies $f\left(\frac{x+y}{2}\right) = \frac{f(x)+f(y)}{2}$ and it is monotonic, continuous at a point, or bounded on some interval, then $f(x) = cx + d$ for some constants $c, d \in \mathbb{R}$.*

Example 4.2. Find all functions $f : (0,\infty) \to (0,\infty)$ such that for all $x, y \in (0,\infty)$,

$$f\left(\frac{2xy}{x+y}\right) = \frac{2f(x)f(y)}{f(x)+f(y)}.$$

Solution. Consider the function $g(x) = \frac{1}{f(\frac{1}{x})}$. The given condition then becomes $g\left(\frac{x+y}{2}\right) = \frac{g(x)+g(y)}{2}$. Since f is bounded, g is also bounded, and thus by Theorem 4.4 it follows that $f(x) = cx + d$ for some constants $c, d \in (0,\infty)$. Therefore, $g(x) = \frac{1}{\frac{c}{x}+d} = \frac{x}{c+dx}$. □

4.3 Pexider's Functional Equation

Pexider's functional equation is the functional equation

$$f(x+y) = g(x) + h(y).$$

This is an obvious generalization to Cauchy's functional equation, and it is clear that Pexider's equation alone will not yield any desirable results. However, if the condition that f, g, and h are all continuous is added, then f, g and h are all linear:

Theorem 4.5. *If $f, g, h : \mathbb{R} \to \mathbb{R}$ are continuous functions such that*

$$f(x+y) = g(x) + h(y)$$

for all $x, y \in \mathbb{R}$, then $f(x) = ax + b + c$, $g(x) = ax + b$, and $h(x) = ax + c$ where $a, b, c \in \mathbb{R}$ are constants.

Example 4.3. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy

$$f(f(x+y)) = f(x) + f(y)$$

for all $x, y \in \mathbb{R}$.

Solution. Applying Pexider's functional equation to functions $f \circ f, f, f$ gives us $(f \circ f)(x) = ax + b + c$, $f(x) = ax + b$, and $f(x) = ax + c$. Hence $b = c$ and so $a = 0, 1$, implying that $f(x) = 0$ and $f(x) = x + b, b \in \mathbb{R}$. □

4.4 D'Alembert's Functional Equation

D'Alembert's functional equation is the functional equation

$$f(x+y) + f(x-y) = 2f(x)f(y).$$

The motivation behind this functional equation is the fact that *cosx* satisfies the identity

$$\cos(x+y) + \cos(x-y) = 2\cos x \cos y.$$

Note, however, that the hyperbolic cosine function $\cosh x = \frac{e^x + e^{-x}}{2}$ also satifies D'Alembert's functional equation. In fact:

Theorem 4.6. *If $f : \mathbb{R} \to \mathbb{R}$ is a continuous function that satsifies $f(x+y) + f(x-y) = 2f(x)f(y)$, then either $f(x) = 0$, $f(x) = \cos ax$, or $f(x) = \cosh ax$, where $a \in \mathbb{R}$ is a constant.*

5 Extremal Element Method

Although the extremal principle is a technique that is most effectively used in combinatorics, some functional equation problems can also be solved using the extremal principle. The most common case in which the extremal principle is used in functional equations is when the range of a function is bounded, and thus has a minimal/maximal element.

Example 5.1. Find all functions $f : \mathbb{Z} \to \mathbb{N}$ such that

$$6f(k+3) - 3f(k+2) - 2f(k+1) - f(k) = 0$$

for any $k \in \mathbb{Z}$.

Solution. The most important difference between $\mathbb{Z}$ and $\mathbb{N}$ is that $\mathbb{N}$ is bounded, and thus every subset of $\mathbb{N}$ has a minimal element. Let $a = \min\{\text{Im}(f)\}$, and $f(x) = a$. Letting $k = x - 3$ yields

$$6a = 6f(x) = 3f(x-1) + 2f(x-2) + f(x-3) \geq 3a + 2a + a = 6a$$

and so equality occurs everywhere where $f(x-1) = f(x-2) = f(x-3) = a$. A simple induction shows that all constant functions satisfy the condition. $\square$

6 Injectivity, Surjectivity, and Bijectivity

Surjectivity is generally easier to prove than injectivity. If a variable is allowed to vary freely while the other side of an equation is the image of a function, then the function is surjective. As an example, the functional equation

$$f(yf(x)) = x - y$$

is an example of such an equation because we may let x span the real numbers, and thus every real number has an inverse image; that is, f is surjective.

Injectivity is proven by assuming that $f(a) = f(b)$ for some a and b, and then proving that $a = b$. In the previous example, f is also injective because

$$a - y = f(y(f(a))) = f(y(f(b))) = b - y,$$

implying that $a = b$. Since f is both surjective and injective, f is bijective.

The fact that a function is bijective provides crucial information. If f is bijective, then an inverse of the function exists; that is, the inverse of any value is unique. In addition, it says that such an inverse always exists. In a problem, you generally cannot assume something like "assume that there exists t such that $f(t) = 2014$." However, if f has been found to be bijective, then we can make such an assumption.

Example 6.1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(f(x) + yz) = x + f(y)f(z)$$

for all $x, y, z \in \mathbb{R}$.

Suppose that $f(a) = f(b)$ for some $a, b \in \mathbb{R}$. Then

$$a + f(y)f(z) = f(f(a) + yz) = f(f(b) + yz) = b + f(y)f(z),$$

so $a = b$, implying that f is injective. f is also surjective because fixing y and z and letting x vary shows that f can attain any value. Therefore, f is bijective.

Let $P(x, y, z)$ be the above assertion. Let $q \in \mathbb{R}$ be such that $f(q) = 1$. $P(0, y, q)$ yields $f(f(0)+yq) = f(y)$, so $f(0) + yq = y$ for all $y \in \mathbb{R}$, which is impossible unless $f(0) = 0$ and $q = 1$. Hence $f(0) = 0$ and $f(1) = 1$. The assertion $P(x, 0, z)$ now yields $f(f(x)) = x$. Finally, $P(f(u), y, 1)$ yields $f(u + v) = f(u) + f(y)$, and $P(0, y, z)$ yields $f(yz) = f(y)f(z)$. Since f is both additive and multiplicative, $f(x) = x$.

7 Polynomials

In some problems, it is given that the functional equation is a polynomial. This provides a significant amount of information, such as the function cannot equal some value infinitely many times, and in the case of integer polynomials, the fact that the difference of two integers divides the difference of the values of the function evaluated at the two integers. In addition, polynomials are continuous and infinitely differentiable. Most importantly, we can compare coefficients, as polynomials are finite.

Example 7.1. Find all nonconstant polynomials $P \in \mathbb{R}[x]$ such that

$$P(x^3 + 1) = P(x + 1)^3$$

for all $x \in \mathbb{R}$.

Solution 1. Let $Q(x) = P(x + 1) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0x^0$. The given condition states that $Q(x^3) = Q(x)^3$, or

$$a_nx^{3n} + a_{n-1}x^{3n-3} + \cdots + a_0x^0 = (a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0x^0)^3.$$

Let a_m be a nonzero coefficient of Q such that $m < n$ is maximal. Comparing the coefficient of x^{2n+m} yields $0 = 3a_k^2a_m$, which is a contradiction. Thus $a_nx^{3n} = Q(x^3) = Q(x0^3 = a_n^3x^{3n}$, so $a_n = \pm 1$. Hence $P(x) = (x - 1)^k$ or $P(x) = -(x - 1)^k$. □

Solution 2. We have $P(3^{3^k}+1) = P(3^{3^{k-1}}+1)^3 = \cdots = P(4)^{3^k}$ for all $k \in \mathbb{N}$. Let $P(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_0x^0$. Since P is continuous, the limit

$$\lim_{x\to\infty} \frac{P(x)}{a_nx^n} = 1$$

is well-defined. Therefore,

$$1 = \lim_{k\to\infty} \frac{P(3^{3^k} + 1)}{a_n(3^{3^k} + 1)} = \lim_{k\to\infty} \frac{P(3^{3^k} + 1)}{a_n3^{3^k}} = \frac{1}{a_n} \lim_{k\to\infty} \left(\frac{P(4)}{3^n}\right)^{3^k}.$$

Comparing the leading coefficients of the given condition yields $a_n = \pm 1$. We thus have $P(4) = \pm 3^k$, and so $P(3^{3^k} + 1) = P(4)^{3^k} = a_n(3^{3^k})^n$. Thus $P(x) = (x - 1)^k$ or $P(x) = -(x - 1)^k$ for all $x = 3^{3^k} + 1$, $k \in \mathbb{N}$. The fact that P is a polynomial extends this to all real numbers. □

Regardless of what type of polynomial P is, the codomain of P is all complex numbers. That is, even if $P \in \mathbb{Z}[x]$, we may still substitute complex numbers in x (although this may rarely help).

Example 7.2. If P, Q, R, and S are polynomials such that

$$P(x^5) + xQ(x^5) + x^2R(x^5) = (x^4 + x^3 + x^2 + x + 1)S(x),$$

prove that $P(1) = 0$.

Solution. Let ω be a fifth root of unity not equal to 1. Substituting $x = 1, \omega, \omega^2, \omega^3, \omega^4$ in $P(x^5) + xQ(x^5) + x^2R(x^5) = (x^4 + x^3 + x^2 + x + 1)S(x)$ and adding the resulting equations gives us $5P(1) = 5S(1)$. Now substituting them in $xP(x^5) + x^2Q(x^5) + x^3R(x^5) = x(x^4 + x^3 + x^2 + x + 1)S(x)$ yields $0 = 5S(1)$, so $P(1) = 0$, as desired. $\square$

8 Miscellaneous Theorems/Lemmas

Theorem 8.1 (Brouwer). *For any continuous function $f : I \to I$, where I is a subset of $\mathbb{R}$, there exists $x \in I$ such that $f(x) = x$.*

Theorem 8.2 (Mikusiński). *Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that*

$$f(x + y) = f(x) + f(y)$$

for all $x, y \in \mathbb{R}$ with $f(x + y) \neq 0$. Then f is additive.

Theorem 8.3 (Intermediate Value Theorem). *Consider an interval $I = [a, b] \subset \mathbb{R}$ and a continuous function $f : I \to \mathbb{R}$. If $f(a) < u < f(b)$, then there exists a $c \in (a, b)$ such that $f(c) = u$.*

Theorem 8.4 (Bézout). *If P and Q are coprime polynomials, there exist polynomials R and S such that $PR + QS = 1$.*

Theorem 8.5 (Mason-Stothers). *Let a, b, and c be pairwise coprime polynomials such that $a + b + c = 0$. Then their degrees are not greater than $N(abc) = 1$, where $N(abc)$ is the number of the distinct zeros of the polynomial abc.*

Theorem 8.6 (Fermat). *The equation $f^n + g^n = h^n$, $n \geq 3$ has no solution is pairwise coprime polynomials at least one of which is not a constant.*

Theorem 8.7 (Davenport). *Let f and g be coprime polynomials of positive degree and $h = f^3 - g^2 \neq 0$. Then* $\deg f \leq 2 \deg h - 2$.

9 Problems

1. Find all polynomials P that such that

$$P(x^2 + 1) = P(x)^2 + 1.$$

2. Find all polynomials $P \in \mathbb{R}[x]$ such that

$$P(x)P(x + 1) = P(x^2).$$

3. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(f(x)) = x^2 - 2$$

for all $x \in \mathbb{R}$.

4. Find all functions $f : \mathbb{N} \to \mathbb{N}$ such that

$$f(f(n)) + f(n + 1) = n + 2$$

for all $n \in \mathbb{N}$.

5. (Vietnam 2003) Let F be the set of all functions $f : (0, \infty) \to (0, \infty)$ which satisfy the inequality $f(3x) \geq f(f(2x)) + x$ for all positive x. Find the largest positive number α such that for all functions $f \in F$, we have $f(x) \geq \alpha x$.

6. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x + y) + f(x)f(y) = f(xy) + f(x) + f(y)$$

for all $x, y \in \mathbb{R}$.

7. (USAMO 2002) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x^2 - y^2) = xf(x) - yf(y)$$

for all $x, y \in \mathbb{R}$.

8. Find all functions $f : \mathbb{Q} \to \mathbb{Q}$ such that $f(1) = 2$ and

$$f(xy) = f(x)f(y) - f(x + y) + 1.$$

9. (BMO 2007) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(xf(x) + f(y)) = y + f(x)^2$$

for all $x, y \in \mathbb{R}$.

10. (USAMO 2000) Call a function $f : \mathbb{R} \to \mathbb{R}$ *very convex* if

$$\frac{f(x) + f(y)}{2} \geq f\left(\frac{x + y}{2}\right) + |x - y|$$

holds for all real numbers x and y. Prove that no very convex function exists.

11. (IMO 2008) Find all functions $f : (0, \infty) \to (0, \infty)$ such that

$$\frac{(f(w))^2 + (f(x))^2}{f(y^2) + f(z^2)} = \frac{w^2 + x^2}{y^2 + z^2}$$

for all positive real numbers w, x, y, and z satisfying $wx = yz$.

12. Let $f : \mathbb{R} \to \mathbb{R}$ be an additive function such that

$$f\left(\frac{1}{x}\right) = \frac{f(x)}{x^2}$$

for all $x \neq 0$. Prove that $f(x) = cx$ for some constant $c \in \mathbb{R}$.

13. Find all polynomials $P \in \mathbb{R}[x]$ such that

$$P(x^2 - 2x) = P(x - 2)^2.$$

14. Find all continuous functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(xy) = xf(y) + yf(x).$$

15. (IMO 1990) Construct a function $f : \mathbb{Q}^{\geq 0} \to \mathbb{Q}^{\geq 0}$ such that

$$f(xf(y)) = \frac{f(x)}{y}$$

for all $x, y \in \mathbb{Q}^{\geq 0}$.

16. Given a positive integer n, let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function satisfying $f(0) = 0$, $f(1) = 1$, and $f^{(n)}(x) = x$ for every $x \in [0, 1]$. Prove that $f(x) = x$ for all $x \in [0, 1]$.

17. Let $f : \mathbb{R} \to \mathbb{R}$ be an additive function such that $f(1) = 1$ and

$$f(x)f\left(\frac{1}{x}\right) = 1$$

for all $x \neq 0$. Prove that $f(x) = x$.

18. (ISL 2002/A1) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(f(x) + y) = 2x + f(f(y) - x)$$

for all $x, y \in \mathbb{R}$.

19. (USAMO 2012) Find all functions $f : \mathbb{Z}^+ \to \mathbb{Z}^+$ (where $\mathbb{Z}^+$ is the set of positive integers) such that $f(n!) = f(n)!$ for all positive integers n and such that $m - n$ divides $f(m) - f(n)$ for all distinct positive integers m and n.

20. (ISL 2012/A1) Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ such that, for all integers a, b, and c that satisfy $a+b+c = 0$, the following equality holds:

$$f(a)^2 + f(b)^2 + f(c)^2 = 2f(a)f(b) + 2f(b)f(c) + 2f(c)f(a).$$

21. (MOP 2013) The function $f : \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ satisfies the following properties for all $a, b \in \mathbb{R}^{\geq 0}$:

    (a) $f(a) = 0$ if and only if $a = 0$,

    (b) $f(ab) = f(a)f(b)$, and

    (c) $f(a + b) \leq 2 \max\{f(a), f(b)\}$.

    Prove that for all $a, b \in \mathbb{R}^{\geq 0}$ we have $f(a + b) \leq f(a) + f(b)$.

22. (Bulgaria 1994) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$xf(x) - yf(y) = (x - y)f(x + y)$$

for all $x, y \in \mathbb{R}$.

23. (ISL 1999/A5) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1$$

for all $x, y \in \mathbb{R}$.

24. (China 1996) Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that for all real numbers x and y,

$$f(x^3 + y^3) = (x + y)(f(x)^2 - f(x)f(y) + f(y)^2).$$

Prove that for all real numbers x, $f(1996x) = 1996f(x)$.

25. (ISL 2005/A2) Find all functions $f : (0, \infty) \to (0, \infty)$ such that

$$f(x)f(y) = 2f(x + yf(x))$$

for all $x, y \in (0, \infty)$.

26. (Romania 2001) Find all polynomials $P \in \mathbb{R}[x]$ such that

$$P(x)P(2x^2 - 1) = P(x^2)P(2x - 1)$$

for all $x \in \mathbb{R}$.

27. (ISL 2007/A2) Consider the functions $f : \mathbb{N} \to \mathbb{N}$ which satisfy the condition

$$f(m + n) \geq f(m) + f(f(n)) - 1$$

for all $m, n \in \mathbb{N}$. Find all possible values of $f(2007)$.

28. (Romania 1986) Let $f, g : \mathbb{N} \to \mathbb{N}$ be functions such that f is surjective, g is injective and $f(n) \geq g(n)$ for all $n \in \mathbb{N}$. Prove that $f(n) = g(n)$ for all $n \in \mathbb{N}$.

29. (IMO 2009) Determine all functions f from the set of positive integers to the set of positive integers such that, for all positive integers a and b, there exists a non-degenerate triangle with sides of lengths

$$a, f(b), \text{ and } f(b + f(a) - 1).$$

30. (ELMO Shortlist 2013/A3) Find all $f : \mathbb{R} \to \mathbb{R}$ such that for all $x, y \in \mathbb{R}$, $f(x) + f(y) = f(x + y)$ and $f(x^{2013}) = f(x)^{2013}$.

31. (ISL 2009/A5) Let f be any function that maps the set of real numbers into the set of real numbers. Prove that there exist real numbers x and y such that

$$f(x - f(y)) > yf(x) + x.$$

32. (ISL 2012/A5) Find all functions $f : \mathbb{R} \to \mathbb{R}$ that satisfy the conditions

$$f(1 + xy) - f(x + y) = f(x)f(y)$$

for all $x, y \in \mathbb{R}$ and $f(-1) \neq 0$.

33. (TSTST 2013) Find all functions $f : \mathbb{N} \to \mathbb{N}$ that satisfy the equation

$$f^{abc-a}(abc) + f^{abc-b}(abc) + f^{abc-c}(abc) = a + b + c$$

for all $a, b, c \geq 2$. (Here $f^1(n) = f(n)$ and $f^k(n) = f(f^{k-1}(n))$ for every integer k greater than 1.)

34. (ISL 2009/A7) Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that

$$f(xf(x + y)) = f(yf(x)) + x^2$$

for all $x, y \in \mathbb{R}$.

35. Find all continuous functions $f, g : \mathbb{R} \to \mathbb{R}$ such that

$$f(x - y) = f(x)f(y) + g(x)g(y)$$

for all $x, y \in \mathbb{R}$.