TJUSAMO 2011 – Inequalities

Mitchell Lee, Andre Kessler

1 Trivial Inequality

The trivial inequality states that $x^2 \geq 0$ for all x. All polynomial inequalities can be reduced to this, but most of the time you don't want to try that.

2 Dumbassing

If, when faced with a complicated three-variable homogenous polynomial inequality, your first instinct is to clear all denominators and expand all expressions, then you're a dumbass. This is not taken as an insult by sophisticated dumbass-ers, who value this technique highly, as it is very often applicable to simpler inqualities. In fact, sophisticated dumbassing methods exist ("Chinese dumbass notation"). You should remember that this is an option, but not one that you should choose readily. Unless you are careful, you're likely to make a mistake somewhere and spend a long time trying to prove something incorrect (or trivial but unapplicable). Additionally, a dumbassing solution will likely make your grader angry at you and generally harsher on errors.

3 Convexity & Jensen's Inequality

Let I be the interval $[a, b]$ and I' be the interval (a, b). A function f is said to be *convex* on I if and only if $\lambda f(x) + (1-\lambda)f(y) \geq f(\lambda x + (1-\lambda)y)$ for all $x, y \in I$ and $0 \leq \lambda \leq 1$. Functions with the inequality flipped are said to be *concave* on I.

(Weighted) Jensen's inequality states that for any convex function f, reals $x_1, x_2, \cdots, x_n$, and positive reals $\omega_1, \omega_2, \cdots, \omega_n$ with $\sum_{i=1}^{n} \omega_i = 1$, we have

$$\omega_1 f(x_1) + \omega_2 f(x_2) + \cdots + \omega_n f(x_n) \geq f(\omega_1 x_1 + \omega_2 x_2 + \cdots + \omega_n x_n).$$

This is called "unweighted Jensen's" or just "Jensen's" if all the weights ω_i are $1/n$.

3.1 How do I know a function is convex?

You can immediately tell if you know calculus by taking the second derivative. Sometimes convex functions are called "concave up" and concave functions are called "concave down." It's well known that if $f''(x)$ is greater than or equal to 0 for all x in an interval I, then f is convex on I. If the inequality is reversed, the function is concave. Note also that if $-f$ is convex, then f is concave. Some functions you should immediately be able to recognize and claim as convex or concave are: constant functions (both convex and concave); powers of x so $f(x) = x^r$ are convex on the interval $0 < x < \infty$ when $r \geq 1$, concave when $0 < r \leq 1$; powers of $\frac{1}{x}$ are convex on $0 < x < \infty$ for $r > 0$; the exponential function $f(x) = e^x$ is convex everywhere; and the logarithm $f(x) = \log x$ is concave on $0 < x < \infty$.

4 Sturm's Principle aka Smoothing

Sometimes, it is most fruitful to turn to results from analysis when trying to prove an inequality. Sturm's principle states that given a function f defined on a set M and a point $x_0 \in M$, if f has a maximum on M, and if no other point x in M is a maximum of f, then x_0 is the maximum of f (replace all the instances of "maximum" there with "minimum" and it's still true). To decide if a function f has a maxima or minima, we fall to the Theorem of Weierstrass: a continuous function on a bounded and closed domain assumes its maximum and minimum. As an example, let $\alpha_1, \alpha_2, \ldots, \alpha_n$ be positive real numbers, $n \geq 2$, such that $\alpha_1 + \alpha_2 + \cdots + \alpha_n = 1$. Then we want to prove that

$$\frac{\alpha_1}{1+\alpha_2+\cdots+\alpha_n} + \frac{\alpha_2}{1+\alpha_1+\cdots+\alpha_n} + \cdots + \frac{\alpha_n}{1+\alpha_1+\cdots+\alpha_{n-1}} \geq \frac{n}{2n-1}$$

If not all of the α_is were equal, then one of them (say α_j) must be less than $\frac{1}{n}$ and one of them (say α_k) must be greater. Add a smallest number ϵ so that either $\alpha_j + \epsilon = \frac{1}{n}$ or $\alpha_k - \epsilon = \frac{1}{n}$. Then stop and choose another pair. Continue this algorithm until all numbers become $\frac{1}{n}$. At this point, the value of the expression is $\frac{1}{n}\frac{1}{2-\frac{1}{n}} \cdot n = \frac{n}{2n-1}$. Since the process kept decreasing the value of the expression, initially it must have been greater than or equal to $\frac{n}{2n-1}$, which proves the inequality.

5 The AM-GM Inequality

$$\omega_1 a_1 + \omega_2 a_2 + \cdots + \omega_n a_n \geq a_1^{\omega_1} a_2^{\omega_2} \cdots a_n^{\omega_n}$$

for any positive $x_1, x_2, \cdots, x_n$ and $\omega_1, \omega_2, \cdots, \omega_n$ with $\sum_{i=1}^{n} \omega_i = 1$. This is a direct result of weighted Jensen on the exponential function, and we have equality when all the variables are equal.
This is called the weighted AM-GM inequality. When all the ω_i are $1/n$, it becomes the unweighted AM-GM inequality. An expression of the form x^x is a big hint to use the weighted AM-GM inequality.

6 The AM-HM Inequality

$$\omega_1 a_1 + \omega_2 a_2 + \cdots + \omega_n a_n \geq \frac{1}{\frac{\omega_1}{a_1} + \frac{\omega_2}{a_2} + \cdots + \frac{\omega_n}{a_n}}$$

for any positive $x_1, x_2, \cdots, x_n$ and $\omega_1, \omega_2, \cdots, \omega_n$ with $\sum_{i=1}^{n} \omega_i = 1$. This is proven by applying weighted AM-GM twice.

7 The Cauchy-Schwarz Inequality

The following inequality has far more uses than its statement seems to allow. $\|u\|\|v\| \geq |u \cdot v|$ for every u, v. In other words,

$$\left(a_1^2 + a_2^2 + \cdots + a_n^2\right)\left(b_1^2 + b_2^2 + \cdots + b_n^2\right) \geq \left(a_1 b_1 + a_2 b_2 + \cdots + a_n b_n\right)^2.$$

This holds for *any* real numbers $a_1, a_2, \cdots, a_n, b_1, b_2, \cdots, b_n$.

8 Schur's Inequality

Let a, b, c be nonnegative reals and $r > 0$. Then

$$a^r(a-b)(a-c) + b^r(b-c)(b-a) + c^r(c-a)(c-b) \geq 0$$

with equality if and only if $a = b = c$ or some two of a, b, c are equal and the other is 0. This inequality can be made considerably sharper, but such results are nonstandard.

9 Techniques

9.1 Induction

Don't forget about induction when working with inequalities in n variables. Remember that when the problem is about $x_1, x_2, \cdots, x_n$, you don't always have to apply the inductive hypothesis to $x_1, x_2, \cdots, x_{n-1}$ - sometimes you might want to apply it to different numbers.

9.2 Equality cases

Several of the inequalities you will prove will involve $\geq$ and $\leq$ relations rather than $>$ and $<$ relations. Find when the two sides of the inequality are equal and make sure only to apply inequalities that respect this equality case.

9.3 Homogenization

Polynomial inequalities of several variables with certain constraints can be homogenized. For example, $x^2 + y^2 + z^2 \geq 3$ for $x + y + z = 3$ can be replaced with the homogeneous $x^2 + y^2 + z^2 \geq \frac{(x+y+z)^2}{3}$. Note that in the homogeneous inequality, the constraint can be dropped.

9.4 Substitution

Often, constraints can be rewritten as substitutions. For example, the constraint that $abc = 1$ can be rewritten as $a = x/y, b = y/z, c = z/x$ for some x, y, z. Also, if a, b, c are the sides of a triangle, then there are positive p, q, r with $a = p + q, b = q + r, c = r + p$.

9.5 Assuming an order

When given variables in a symmetric inequality, it is sometimes beneficial to assume (without loss of generality) that the variables are in sorted order.

10 Problems

1. Show that for real a, b, c, we have $a^2 + b^2 + c^2 \geq ab + bc + ca$.

2. Prove *Nesbitt's Inequality*, which states

$$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \geq \frac{3}{2}$$

for all positive real numbers a, b, c.

3. Let a, b, c, d be fixed positive reals, and let x, y satisfy $ax + by = 1$. What is the maximum value of $x^c y^d$?

4. Prove that for all positive real numbers a, b, c:
$$\frac{9}{a+b+c} \leq 2\left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a}\right)$$

5. Prove that for all $0 \leq a, b, c \leq 1$,
$$\frac{a}{1+b+c} + \frac{b}{1+a+c} + \frac{c}{1+a+b} + (1-a)(1-b)(1-c) \leq 1$$

6. Let $0 < p \leq a, b, c, d, e \leq q$. Show that
$$(a+b+c+d+e)\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d} + \frac{1}{e}\right) \leq 25 + 6\left(\sqrt{\frac{p}{q}} - \sqrt{\frac{q}{p}}\right)^2.$$

7. Prove *Aczel's inequality*, which states if $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ are real numbers such that
$$(a_1^2 + a_2^2 + \cdots + a_n^2 - 1)(b_1^2 + b_2^2 + \cdots + b_n^2 - 1) > (a_1 b_1 + a_2 b_2 + \cdots + a_n b_n - 1)^2,$$
then $a_1^2 + a_2^2 + \cdots + a_n^2 > 1$ and $b_1^2 + b_2^2 + \cdots + b_n^2 > 1$.

8. Let $a, b, c > 0, a + b + c = 1$. Prove that
$$0 \leq ab + bc + ca - 2abc \leq \frac{7}{27}$$

9. Let $a_1, a_2, \ldots$ be an infinite sequence of real numbers, for which there exists a real number c with $0 \leq a_i \leq c$ for all i, such that
$$|\, a_i - a_j \,| \geq \frac{1}{i+j}.$$
for all i, j with $i \neq j$. Prove that $c \geq 1$.

10. Let $x_1, x_2, ..., x_n$ be positive real numbers. Prove that
$$(1+x_1)(1+x_1+x_2)\cdots(1+x_1+x_2+\cdots+x_n) \geq \sqrt{(n+1)^{n+1}}\sqrt{x_1 x_2 ... x_n}$$

11. Let $a_1, a_2 ... a_n$ be non-negative reals, not all zero. Show that that
    (a) The polynomial $p(x) = x^n - a_1 x^{n-1} + ... - a_{n-1}x - a_n$ has precisely 1 positive real root R.
    (b) Let $A = \sum_{i=1}^{n} a_i$ and $B = \sum_{i=1}^{n} i a_i$. Show that $A^A \leq R^B$.

12. Let $x_1, x_2, \ldots, x_n$ be arbitrary real numbers. Prove the inequality
$$\frac{x_1}{1+x_1^2} + \frac{x_2}{1+x_1^2+x_2^2} + \cdots + \frac{x_n}{1+x_1^2+\cdots+x_n^2} < \sqrt{n}.$$

13. Let $x_1, x_2, x_3, \ldots, x_n$ be positive real numbers such that $x_1 x_2 x_3 \cdots x_n = 1$. Prove that
$$\sum_{i=1}^{n} \frac{1}{n-1+x_i} \leq 1.$$