The Š Point

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Abstract

This article presents basic geometries properties of the midpoint of arc also known[1] as the Š point. Significance of this point in olympiad geometry cannot be emphasized enough, as it appears at the IMO almost every other year. A set of exercises is included.

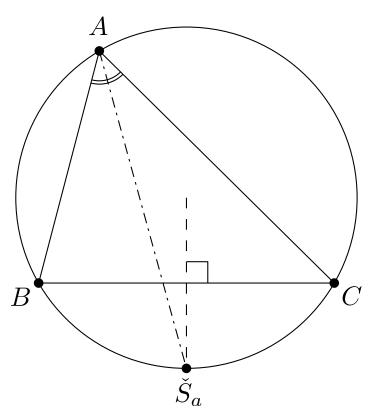
In order to introduce properties of the Š point we need to assume some knowledge. Namely, basic concyclicity criteria for quadrilaterals and basic angle-chasing. We hope the reader is familiar with these topics.

Now let's get started!

Definition. Let ABC be a triangle inscribed in a circle ω. Denote by $\check{S}_a$ the midpoint of arc BC which does not contain A. This point is called the $\check{S}$ point of $\triangle ABC$ with respect to the vertex A. Points $\check{S}_b$, $\check{S}_c$ are defined similarly.

Straight from the definition it is clear that $\check{S}_aB = \check{S}_aC$ and thus $\check{S}_a$ lies on the perpendicular bisector of BC. Also as arcs $\check{S}_aB$ and $\check{S}_aC$ are equal, the corresponding angles must also be equal. Hence $\angle BA\check{S}_a = \angle \check{S}_aAC$ and so $\check{S}_a$ lies on the angle bisector.

We have derived the first important property of the $\check{S}$ point.



Proposition 1. *In ABC the angle bisector of $\angle A$, the perpendicular bisector of BC and the circumcircle ω are concurrent. The point of concurrency is $\check{S}_a$.*

We are consistently going to use the following notation.

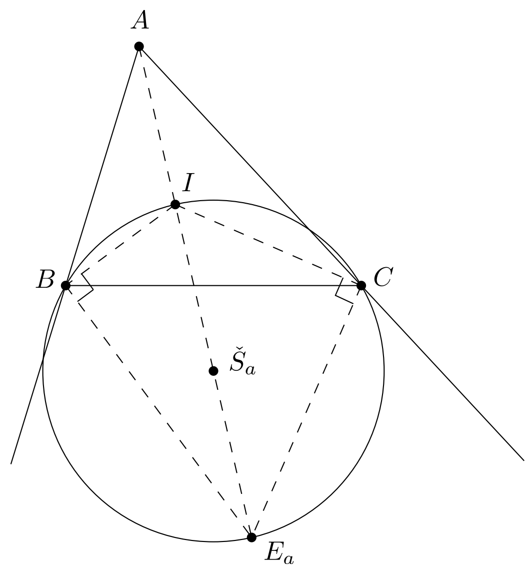
Notation. In $\triangle ABC$ let I be the incenter, $\check{S}_a$, $\check{S}_b$, $\check{S}_c$ the corresponding Š-points, E_a, E_b, E_c the corresponding excenters and let AD, BE, CF be angle bisectors in ABC, where $D \in BC$, $E \in AC$, $F \in AB$.

[1] At least in the Czech Republic. In USAMO I would be careful for a few more years.

Fundamental properties

The key to understanding the Š point is to realize that it produces many circles and many pairs of similar triangles.

Proposition 2. *In $\triangle ABC$ the points B, C, I, E_a are concyclic and $\check{S}_a$ is the center of this circle. In particular $\check{S}_a I = \check{S}_a B = \check{S}_a C = \check{S}_a E_a$.*



Proof. First we use the fact that the incenter (excenter) lie on the interior (exterior) angle bisectors of $\angle B$ and $\angle C$. We obtain

$$\angle IBE_a = \angle IBC + \angle CBE_a = \frac{\angle B}{2} + \frac{180^\circ - \angle B}{2} = 90^\circ$$

and similarly $\angle ICE_a = 90^\circ$. Hence the points B, C, I, E_a are indeed concyclic.

We already know that $\check{S}_a C = \check{S}_a B$ so it suffices to prove $\check{S}_a I = \check{S}_a B$, since then $\check{S}_a$ will be the circumcenter of $\triangle IBC$ and thus also the circumcenter of BE_aCI.

We angle-chase to obtain

$$\angle BI\check{S}_a = 180^\circ - \angle BIA = \frac{\angle A}{2} + \frac{\angle B}{2}$$

and

$$\angle \check{S}_a BI = \angle \check{S}_a BC + \angle CBI = \angle \check{S}_a AC + \frac{\angle B}{2} = \frac{\angle A}{2} + \frac{\angle B}{2}.$$

Hence the triangle $IB\check{S}_a$ is isosceles with $\check{S}_a I = \check{S}_a B$ and we may conclude the proof. $\square$

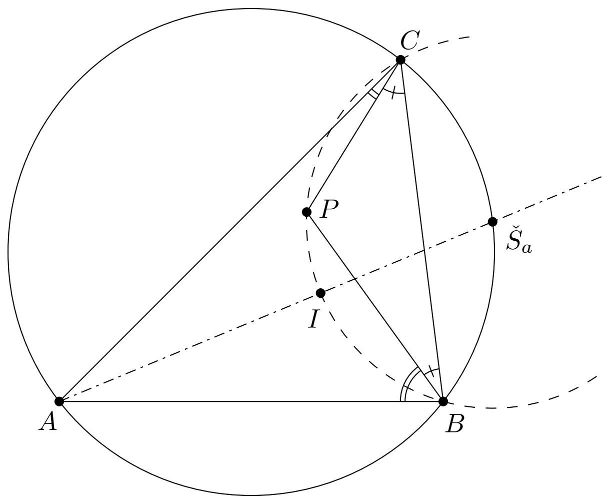
With this knowledge the following example is very easy!

Example (IMO 2006)**.** *Let ABC be a triangle with incenter I. A point P in the interior of the triangle satisfies*

$$\angle PBA + \angle PCA = \angle PBC + \angle PCB.$$

Show that $AP \geq AI$, and that equality holds if and only if $P = I$.

Proof. If $P = I$ then the proposition is true (note that point I satisfies $\angle IBA + \angle ICA = \frac{\angle B}{2} + \frac{\angle C}{2} = \angle IBC + \angle ICB$). We are to show that if $P \neq I$ then $AP > AI$. First, we get rid of the condition.



Since $(\angle PBA + \angle PCA) + (\angle PBC + \angle PCB) = \angle B + \angle C$, simple angle chase gives us that $\angle BPC = 180^\circ - (\angle PBC + \angle PCB) = 180^\circ - \frac{1}{2}(\angle B + \angle C)$. This should look familiar to us. Indeed, $\angle BIC = 180^\circ - (\frac{\angle B}{2} + \frac{\angle C}{2})$ and hence the points B, C, I, P lie on one circle.

The key is to identify the center of this circle. It has to be a circumcenter of triangle BIC which we already know to be $\check{S}_a$. Once we recall that $\check{S}_a$ lies on internal angle bisector, the conclusion should be clear. One way to express it formally is to write down triangle inequality for $A\check{S}_aP$ to obtain

$$AP + P\check{S}_a > A\check{S}_a = AI + I\check{S}_a,$$

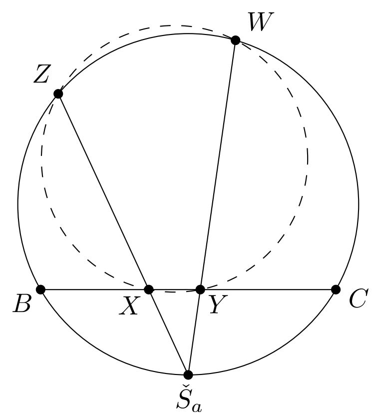
from which the result follows. $\square$

Proposition 3. *Let BC be a chord of circle ω and $\check{S}_a$ midpoint of arc BC. Let p, q be two lines passing through $\check{S}_a$. Let X,Y be their intersections with chord BC, respectively, and let Z, W be their second intersections with ω. Then X,Y,Z,W are concyclic.*

Proof. To prove that $XYWZ$ is cyclic, it suffices to show $\angle XZW + \angle XYW = 180^\circ$ or equivalently $\angle XZW = \angle CYW$.

Now using that arcs $B\check{S}_a$ and $\check{S}_aC$ are equal we obtain

$$\angle XZW = \angle \check{S}_aZC + \angle CZW = \angle \check{S}_aCB + \angle C\check{S}_aW.$$



We conclude by observing that $\angle CYW$ is an exterior angle in $\triangle \check{S}_a YC$ hence $\angle CYW = \angle \check{S}_a CB + \angle C\check{S}_a W$ and the proof is finished.

Note that the proposition remains valid for the lines p, q intersecting line BC not necessarily at segment BC. The proof is analogous to the one provided above. $\square$

Proposition 4. *In $\triangle ABC$ we have $\check{S}_a D \cdot \check{S}_a A = \check{S}_a I^2 = \check{S}_a C^2 = \check{S}_a B^2$.*

Proof. Since $\angle \check{S}_a AC = \angle BC\check{S}_a$ the triangles $A\check{S}_a C$ and $C\check{S}_a D$ are similar (AA). Thus

$$\frac{\check{S}_a A}{\check{S}_a C} = \frac{\check{S}_a C}{\check{S}_a D}$$

and then $\check{S}_a I^2 = \check{S}_a C^2 = \check{S}_a D \cdot \check{S}_a A$. $\square$

We may use these metric identities to form alternative definitions of the incenter of a triangle. These are often useful, especially in problems, where only one angle bisector is drawn.

Proposition 5. *Let X be a point on segment $A\check{S}_a$. The following statements are equivalent*

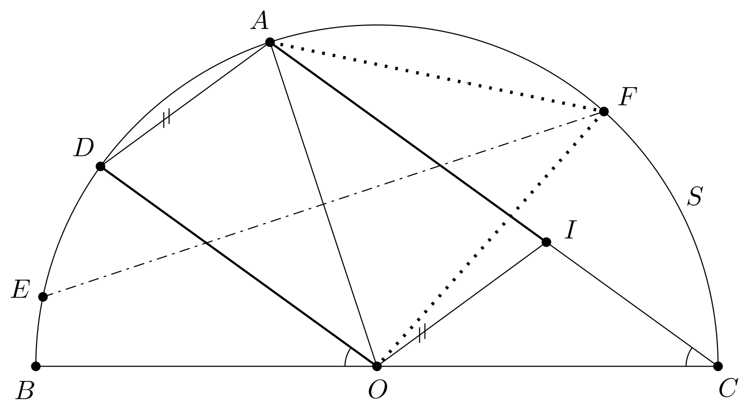
(i) $X = I$.

(ii) $\check{S}_a X = \check{S}_a I$.

(iii) $\check{S}_a D \cdot \check{S}_a A = \check{S}_a X^2$.

Proof. We only need to realize that I is the unique point on segment $A\check{S}_a$ with properties (ii) and (iii). $\square$

Example (IMO 2002)**.** *Let BC be a diameter of circle S centered at O. Let A be a point of S such that $\angle AOB < 120°$. Let D be the midpoint of the arc AB which does not contain C. The line through O parallel to DA meets the line AC at I. The perpendicular bisector of OA meets S at E and at F. Prove that I is the incenter of the triangle CEF.*



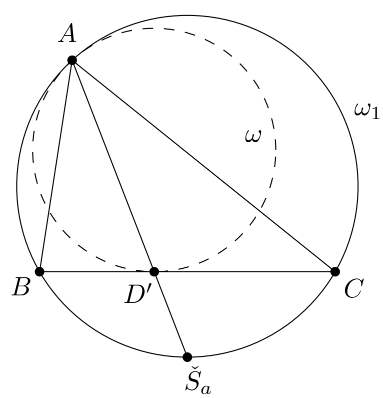
Proof. Thanks to condition $\angle AOB < 120°$, point A is a midpoint of arc EF which does not contain C. Hence line CA is an angle bisetor of $\angle ECF$. Using previous proposition it is enough to prove that $AI = AF$. We will show that both lengths are in fact equal to the radius of the circle S.

This assertion is obvious for AF because as F lies on a perpendicular bisector of segment AO, we have $AF = OF$.

Moreover, since D is midpoint of arc AB we have $\angle BOD = 2 \cdot \angle BCD = \angle BCA$, so $OD \| CA$. But this means that quadrilateral $DOIA$ is a parallelogram ($DA \| OI$ by problem statement). Thus $AI = DO$ and we are done. □

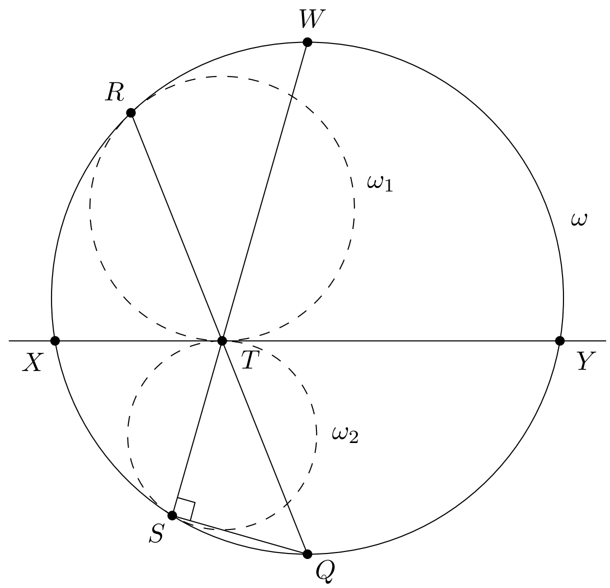
The last property covered in this article will concern tangent circles. The following proposition is integral part of a deeper concept called *homothety of circles*, which is beyond the scope of this article. Yet, the proposition has many applications by itself.

Proposition 6. *Let circle ω be internally tangent to the circumcircle ω_1 of $\triangle ABC$ at A and tangent to BC at D'. Then A,D', $\check{S}_a$ are collinear.*



Proof. Take homothety with center A which maps ω to ω_1. The line BC is mapped to a parallel line, which is tangent to ω_1. But this must be a tangent at the point $\check{S}_a$ (recall symmetry). Hence A, D' and $\check{S}_a$ are collinear. □

Example (Slovak contest). *Two circles ω_1 and ω_2 are externally tangent at T and both internally tangent to circle ω at points R and S, respectively. Let Q be the second intersection of RT and ω. Show that $\angle QST = 90°$.*



Proof. Denote by X,Y the intersections of ω and common internal tangent of ω_1 and ω_2. Further, let W be the second intersection of ST and ω. By Proposition 6 both Q and W are midpoints of the respective arcs XY. Hence they are antipodal and form a diameter. The proof follows. □

Problems

Problem 1 (Junior Balkan 2010). *Let AL and BK be angle bisectors in the non-isosceles triangle ABC (L lies on the side BC, K lies on the side AC). The perpendicular bisector of BK intersects the line AL at point M. Point N lies on the line BK such that LN is parallel to MK. Prove that $LN = NA$.*

Problem 2. *In $\triangle ABC$ prove the following metric identities*

(i) $A\check{S}_a \cdot ID = \check{S}_a I \cdot AI$.

(ii) $A\check{S}_a \cdot AD = AI \cdot AE_a = AB \cdot AC$.

(iii) $IA \cdot E_a D = E_a A \cdot ID$.

Problem 3 (IMO 2004). *Let ABC be an acute-angled triangle with $AB \neq AC$. The circle with diameter BC intersects the sides AB and AC at M and N respectively. Denote by O the midpoint of the side BC. The bisectors of the angles $\angle BAC$ and $\angle MON$ intersect* *at R. Prove that the circumcircles of the triangles BMR and CNR have a common point lying on the side BC.*

Problem 4 (IMO Shortlist 2005)**.** *Given a triangle ABC satisfying $AC + BC = 3 \cdot AB$. The incircle of triangle ABC has center I and touches the sides BC and CA at the points D and E, respectively. Let K and L be the reflections of the points D and E with respect to I. Prove that the points A, B, K, L lie on one circle.*

Problem 5 (IMO 2010)**.** *Given a triangle ABC, with I as its incenter and Γ as its circumcircle, AI intersects Γ again at D. Let E be a point on the arc BDC, and F a point on the segment BC, such that $\angle BAF = \angle CAE < \frac{1}{2}\angle BAC$. If G is the midpoint of IF, prove that the meeting point of the lines EI and DG lies on Γ.*

Problem 6. *Let K be a point on the shorter arc BC of the circumcircle of $\triangle ABC$. Two circles are tangent to the circumcircle of a triangle ABC at K and one of them is tangent to the side AB at a point M, and the other is tangent to AC at a point N. Prove that the incenter of ABC lies on the line MN.*

Problem 7. *Line ℓ intersects circle Γ at points A, B. Two externally tangent circles Γ_1, Γ_2 are inscribed in circular segment corresponding to shorter arc AB. Show that their common internal tangent passes through a fixed point if the two circles move inside the circular segment.*

Problem 8 (Lemma for Sawayama-Thebault theorem)**.** *Let ABC be a triangle inscribed in circle ω and D on side BC. Let ω_1 be a circle tangent to AD at F, to BC at E and to ω at K. Prove that the incenter I of $\triangle ABC$ lies on EF.*

Problem 9 (Asian-Pacific MO 2000)**.** *Let ABC be a triangle. Let M and N be the points in which the median and the angle bisector, respectively, at A meet the side BC. Let Q and P be the points in which the perpendicular at N to NA meets MA and BA, respectively. Finally, let O be the point in which the perpendicular at P to BA meets line AN. Prove that QO is perpendicular to BC.*

Hints

1 Use Proposition 1 to interpret M as $\check{S}$ point of smoe triangle. Angle chase to show that N is also $\check{S}$ point for some triangle.

2 Use similarites, expressing distances in terms of basic elements of $\triangle ABC$ and keep in mind Proposition 4

3 Use Proposition 1 to say that R is $\check{S}$ point of $\triangle AMN$. Angle chase!

4 Guess where the center of the circle will be! Reduce the problem into proving a metric relation (equal tangents may be useful).

5 Draw E_a to get rid of the midpoint, then use result of Problem 2 (and possibly some angle-chasing). More approaches are possible you may use Menelaus theorem, Proposition 3 and angle-chasing.

6 Make use of Proposition 6 and Pascal theorem.

7 Use Proposition 3, Proposition 6 and the existence of radical center.

8 Intersect angle-bisector of $\angle A$ with EF (draw also the $\check{S}$ point!) and use alternative definition of the incenter from Proposition 5(iii). By power of a point and Proposition 6 this reduces the problem into angle-chasing.

9 Draw B', C' such that we are in fact proving that O is $\check{S}$ point for $\triangle AB'C'$. Then proceed indirectly (be careful about your logic!), take O' as this $\check{S}$ point and show $\angle O'PA$ is right. Use angle-chasing.