Generating Functions

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> A generating function is a clothesline on which we hang a sequence of numbers up for display.
>
> –Herbert Wilf, *Generatingfunctionology*

Generating function basics

Generating functions are a useful tool for solving recurrences and counting certain combinatorial objects.

A classic example of a generating function identity is

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots .$$

The term *function* is misleading - here x is just a formal symbol, and the coefficients of the series are the important part. Let's make this all rigorous.

Definition. The *generating function* of the sequence $c_0, c_1, c_2, \ldots$ with variable x is the expression

$$c_0 + c_1 x + c_2 x^2 + \cdots .$$

We abbreviate this series as

$$\sum_{i=0}^{\infty} c_i x^i.$$

Generating functions are not functions, but they can still be added and multiplied together. They can also be differentiated!

The following are the *definitions* of addition, multiplication, and differentiation of generating functions (we're starting from the beginning here - no calculus allowed.)

- **Addition:** $\sum_{i=0}^{\infty} a_i x^i + \sum_{i=0}^{\infty} b_i x^i = \sum_{i=0}^{\infty} (a_i + b_i) x^i$
- **Multiplication:** $\left(\sum_{i=0}^{\infty} a_i x^i\right) \cdot \left(\sum_{i=0}^{\infty} a_i x^i\right) = \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n} a_i b_{n-i}\right) x^n$
- **Differentiation:** $\frac{d}{dx}\left(\sum_{n=0}^{\infty} a_n x^n\right) = \sum_{i=1}^{\infty} n a_n x^{n-1}$

Exercise. Is there a generating function that behaves like an "additive identity"? A "multiplicative identity"? Can subtraction and division of generating functions be defined? What about composition?

Exercise. Use the definitions above to prove the generating function identity

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots .$$

(Notice that $1 - x = 1 - x + 0 \cdot x^2 + 0 \cdot x^3 + \cdots$ is a generating function as well.)

Tricks for manipulating generating functions

Let $F(x) = \sum_{n=0}^{\infty} a_n x^n$ and $G(x) = \sum_{n=0}^{\infty} b_n x^n$ be generating functions over x. Try your hand at proving the following identities, using only the definitions above.

- $xF(x) = \displaystyle\sum_{n=1}^{\infty} a_{n-1}x^n$
- $\dfrac{F(x) - a_0}{x} = \displaystyle\sum_{n=0}^{\infty} a_{n+1}x^n$
- $\dfrac{d}{dx}(F(x) + G(x)) = \dfrac{d}{dx}F(x) + \dfrac{d}{dx}G(x)$
- $\dfrac{d}{dx}(F(x)G(x)) = G(x) \cdot \dfrac{d}{dx}F(x) + F(x) \cdot \dfrac{d}{dx}G(x)$
- If $b_0 \neq 0$, $G(x)$ has a multiplicative inverse:
$$G(x)^{-1} = b_0^{-1} - b_0^{-1}b_1x + (b_0^{-3}b_1^2 - b_0^{-2}b_2)x^2 + \cdots .$$
- If $b_0 \neq 0$, $\dfrac{d}{dx}\left(\dfrac{F(x)}{G(x)}\right) = \dfrac{G(x)\frac{d}{dx}F(x) - F(x)\frac{d}{dx}G(x)}{G(x)^2}$.

Example. Show that

$$\frac{1}{(1-x)^2} = 1 + 2x + 3x^2 + 4x^3 + 5x^4 + \cdots .$$

Using generating functions to solve recurrences

Suppose we wish to find an explicit formula for the nth Fibonnacci number F_n, where $F_0 = 0$, $F_1 = 1$, and $F_n = F_{n-1} + F_{n-2}$ for all $n \geq 2$. Consider the generating function

$$G(x) = \sum_{n=0}^{\infty} F_n x^n.$$

We can manipulate this to take advantage of the recursion: we have

$$G(x) - xG(x) - x^2G(x) = F_0 + F_1x - F_0x + \sum_{n=2}^{\infty}(F_n - F_{n-1} - F_{n-2})x^n = x.$$

Thus $G(x) = x/(1-x-x^2)$. Using partial fractions and expanding each term as a geometric series, we find that

$$G(x) = \sum_{n=0}^{\infty} \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)x^n,$$

and so $F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$.

In general, the generating function for any linear recurrence of the form

$$A_n = c_1A_{n-1} + c_2A_{n-2} + \cdots + c_kA_{n-k}$$

can be written as a rational function of x, obtained by multiplying it by the *characteristic polynomial*

$$1 - c_1 x - c_2 x^2 - \cdots - c_k x^k$$

and using the initial conditions to solve for the generating function. We can then use partial fraction decomposition and the geometric series formula to find an explicit formula for the nth coefficient.

Exponential generating functions

The *exponential generating function* for the sequence $\{a_i\}_{i=0}^{\infty}$ is the series $\sum_{n=0}^{\infty} \frac{a_n}{n!} x^n$. Their product behaves somewhat differently from that of ordinary generating functions:

$$\left(\sum_{n=0}^{\infty} \frac{a_n}{n!} x^n\right) \cdot \left(\sum_{n=0}^{\infty} \frac{b_n}{n!} x^n\right) = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\sum_{k=0}^{n} \binom{n}{k} a_k b_{n-k}\right) x^n$$

Exercise. What do addition and differentiation do to exponential generating functions?

We can define e^x, $\sin(x)$, and $\cos(x)$ to be the exponential generating functions shown below.

- $e^x = \sum_{n=0}^{\infty} \frac{1}{n!} x^n$
- $\sin(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n+1)!} x^{2n+1}$
- $\cos(x) = \sum_{n=0}^{\infty} (-1)^n \frac{1}{(2n)!} x^{2n}$

Exercise. Using the definition of e^x as a generating function, show that $e^x e^y = e^{x+y}$ and that $\frac{d}{dx} e^x = e^x$.

Problems!

1. (Andy Niedermaier.) Find a closed form for the generating function for each of the following sequences, and use it to find an explicit formula for a_n:
   (a) $a_0 = 1$, $a_1 = 5$, $a_{n+2} = 4a_{n+1} - 3a_n$
   (b) $a_0 = 1$, $a_1 = 6$, $a_{n+2} = 4a_{n+1} - 4a_n$
   (c) $a_0 = 0$, $a_1 = 5$, $a_2 = 47$, $a_{n+3} = 31a_{n+1} + 30a_n$
   (d) $a_0 = a_1 = 1$, $a_{n+2} = a_{n+1} + 6a_n + n$
2. Prove the following combinatorial identities using generating functions:
   (a) $\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$
   (b) $\sum_{k=0}^{n} k\binom{n}{k} = n \cdot 2^{n-1}$
   (c) $\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \binom{n}{5} + \cdots = 2^{n-1}$
3. Find an explicit formula for

$$\binom{n}{0} + \binom{n}{3} + \binom{n}{6} + \binom{n}{9} + \cdots$$

   in terms of n.

4. **Derangements:** Let D_n be the number of *derangements* of n, that is, the number of permutations ϕ of $\{1, 2, \ldots, n\}$ such that $\phi(i) \neq i$ for any $1 \leq i \leq n$. Find a closed form expression for the exponential generating function of D_n, and use it to find a formula for D_n (the formula may include a finite sum.)

5. (China 1996.) Let n be a positive integer. Find the number of polynomials $P(x)$ with coefficients in $\{0, 1, 2, 3\}$ such that $P(2) = n$.

6. (High school mathematics 1994/1, Qihong Xie.) Find the number of subsets of $\{1, 2, \ldots, 2000\}$, the sum of whose elements is divisible by 5.

7. Let C_n denote the nth *Catalan number*, the number of ways of parenthesizing the addition of n ones. Find a closed form expression for the generating function $C(x) = \sum_{n=0}^{\infty} C_n x^n$, and use it to show that $C_n = \frac{1}{n+1}\binom{2n}{n}$.

8. Prove that
$$\sum_{\substack{i+j=n\\ i,j\geq 0}} \binom{2i}{i}\binom{2j}{j} = 4^n.$$

9. **Delannoy numbers:** Let $P_{m,n}$ denote the number of paths from $(0, 0)$ to (m, n) using only the moves $(0, 1)$, $(1, 0)$, and $(1, 1)$ at each step. For example, one valid path from $(0, 0)$ to $(3, 4)$ is
$$(0, 0), (0, 1), (1, 2), (2, 2), (3, 3), (3, 4).$$
   (a) Find a closed form expression for the generating function
   $$\sum_{m,n\geq 0} P_{m,n} x^m y^n.$$
   (b) Find a closed form expression for the generating function of the "central" Delannoy numbers:
   $$\sum_{n=0}^{\infty} P_{n,n} x^n.$$

10. (Richard Stanley.) Compute
$$\sum_{a_1+a_2+\cdots+a_k=n, k\geq 1} a_1 a_2 \cdots a_k.$$

11. (102 Combinatorial Problems.) Let $A_1, A_2, \ldots, B_1, B_2, \ldots$ be sets such that $A_1 = \emptyset$, $B_1 = \{0\}$,
$$A_{n+1} = \{x + 1 \mid x \in B_n\}, B_{n+1} = A_n \cup B_n - A_n \cap B_n,$$
for all positive integers n. Determine all positive integers n such that $B_n = \{0\}$.

Some problems on partitions

1. Prove that the number of partitions of a positive integer n into distinct parts is equal to the number of partitions of n into odd parts.

2. Prove that the number of partitions of an integer n into distinct odd parts has the same parity as the total number of partitions of n.

3. Let $p(n)$ be the number of partitions of n, that is, the number of sequences $(\lambda_1, \ldots, \lambda_k)$ with $\lambda_1 \geq \cdots \geq \lambda_k$ whose sum is n. Prove that

$$\sum_{n=0}^{\infty} p(n)x^n = \left(\frac{1}{1-x}\right)\left(\frac{1}{1-x^2}\right)\left(\frac{1}{1-x^3}\right)\cdots.$$

4. Let $p(n,r)$ denote the number of partitions $(\lambda_1, \lambda_2, \ldots, \lambda_k)$ of n (written in nonincreasing order) such that $\lambda_1 - k = r$. Let $R(z,q) = \sum_{n,r} p(n,r)z^r q^n$ be its *two-variable generating function.* Prove that

$$R(z,q) = 1 + \sum_{n=1}^{\infty} \frac{q^{n^2}}{\prod_{k=1}^{n}(1-zq^k)(1-z^{-1}q^k)}.$$

5. Prove **Euler's Pentagonal Number Theorem:**

$$(1-x)(1-x^2)(1-x^3)\cdots = \sum_{n=-\infty}^{\infty} (-1)^n x^{n(3n-1)/2}.$$

6. Prove that $p(5n+4) \equiv 0 \pmod 5$. You may find the following identity useful:

$$\big((1-x)(1-x^2)(1-x^3)\cdots\big)^3 = \sum_{n=0}^{\infty}(-1)^n(2n+1)x^{n(n+1)/2}.$$

7. Let $Q(n)$ be the number of partitions of n into distinct parts, that is, the number of sequences $(\lambda_1, \ldots, \lambda_k)$ with $\lambda_1 > \cdots > \lambda_k$ whose sum is n. Prove that

$$\sum_{n=0}^{\infty} Q(n)x^n = (1+x)(1+x^2)(1+x^3)\cdots.$$

8. Let $Q(n,r)$ be the number of partitions of n into distinct (decreasing) parts $(\lambda_1, \ldots, \lambda_k)$ such that $\lambda_1 - k = r$. Let $G(z,q) = \sum_{n,r} Q(n,r)z^r q^n$ be its two-variable generating function. Prove that

$$G(z,q) = 1 + \sum_{s=1}^{\infty} \frac{q^{s(s+1)/2}}{\prod_{k=1}^{s}(1-zq^k)}.$$