New Zealand Maths Olympiad Committee
2009 Maths Gymnastics
Christchurch, Thursday 14 January

1. Simplify
$$\sin^2 70^o \times \sin^2 50^o \times \sin^2 10^o$$

2. Simplify
$$\sin \frac{3\pi}{10} - \sin \frac{\pi}{10}$$

3. Suppose a, b and c are the sides of a triangle such that
$$\frac{1}{a+b} + \frac{1}{b+c} = \frac{3}{a+b+c}$$
Find one of the angles of this triangle.

4. Prove that
$$\frac{1}{2} \times \frac{3}{4} \times \frac{5}{6} \times \cdots \times \frac{99}{100} < \frac{1}{10}$$

5. Find all pairs of real numbers (x, y) such that the following two conditions both hold.
$$\begin{aligned} x^4 + y^4 &= 17 \\ x + y &= 3 \end{aligned}$$

6. Solve the equation
$$26 \sin^2 x^2 + 12 \cos 2x + 5 \sin 2x = 13$$

7. $ABCD$ is a trapezium with $BC||AD$, $AB = 9$ and $CD = 5$. The bisector of $\angle D$ intersects the bisectors of $\angle A$ and $\angle C$ at points M and N respectively. The bisector of $\angle B$ intersects the bisectors of $\angle A$ and $\angle C$ at points L and K respectively. Suppose that K lies on AD. Let P be the intersection of lines LN and AB, and let Q be the intersection of lines KM and BC.
   - Find the ratios $AP : PB$ and $BQ : QC$.
   - If $LM : KN = 3 : 7$, find the ratio $MN : KL$.

8. Find one of the roots of the following polynomial
$$x^3 + 2\sqrt{3}x^2 + 3x + \sqrt{3} - 1 = 0$$

9. Solve the equation:
$$x^2 + \frac{25x^2}{(5+2x)^2} = \frac{74}{49}$$

10. Prove the inequality
$$\cos(x) + x \sin(x) > 1$$
where $0 < x \leq \frac{\pi}{2}$

11. Suppose α and β are two angles of a given triangle and
$$\cos \alpha + \cos \beta - \cos(\alpha + \beta) = \frac{3}{2}$$
Find the angles of the triangle.