Combinatorial Geometry

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1 Techniques

- Try to rephrase the problem in graph-theoretical terms, then use results such as Ramsey's theorem (to prove that certain configurations exist) and Turán's theorem (if you know that certain configurations cannot exist).
  - Ramsey's theorem: Let n and k be positive integers. Then there exists an integer N such that if the edges of a complete graph on N vertices are colored in k colors, there exists a complete subgraph of n vertices all of whose edges are colored in the same color.
  - Turán's theorem: Let G be a graph on n vertices which contains no complete subgraph of k vertices. Then G has at most as many edges as G_0, where G_0 is the complete $(k-1)$-partite graph whose part sizes sum to n with no two differing by more than one. In particular G has at most
  $$\frac{k-2}{2(k-1)}n^2$$
  edges.
- Arrange in order.
- Extremal arguments: look at the closest two points, or the smallest triangle, etc.

2 Coloring

1. (a) Color the plane in 3 colors. Prove that there are two points of the same color 1 unit apart. What if we color in 4 colors? 5? 6? 7?

   (b) Color the plane in 2 colors. Prove that one of these colors contains points at *every* mutual distance.

   (c) Color the plane in 2 colors. Show that given any non-equilateral triangle T, there exists a triangle congruent to T with all vertices the same color.

   (d) Show that the previous result is false for an equilateral triangle.

   (e) Color the plane in 2 colors. Prove that there will always exist an equilateral triangle with all its vertices of the same color.

   (f) Color the circumference of a circle in 2 colors. Show that there must exist three points X_1, X_2, X_3 on the circle, all of the same color, such that $X_1X_2 = X_2X_3$.

   (g) Color the plane in 2 colors. Show that there exists a rectangle, all of whose vertices are the same color.

   (h) Color the plane in k colors. Show that there exists a right triangle, all of whose vertices are the same color.

2. The last four parts of the previous problem are all consequences of the following two-dimensional version of van der Waerden's theorem:

   > Let n, k be positive integers. Then there exists an integer N such that for any coloring of the lattice points $\{\,(i,j) \mid 1 \le i,j \le N\,\}$ with k colors, there exists an $n \times n$ square grid of lattice points all of the same color. (An $n \times n$ square grid is the image of the set $\{\,(i,j) \mid 1 \le i,j \le n\,\}$ under a homothety.)

   Prove this theorem.

3. (Russia '04) Each lattice point in the plane is colored in one of three colors, with each color used at least once. Show that there exists a right triangle whose vertices all have different colors.
4. Determine all values of r for which the points inside a circle of radius r can be colored in two colors such that every pair of points at distance 1 have different colors.
5. (MOP '97) The points on the sides of a right isosceles triangle with hypotenuse $\sqrt{2}$ are colored in four colors. Prove that there are two points of the same color at distance at least $2-\sqrt{2}$.
6. Let S denote the set of points inside and on the boundary of a regular hexagon with side length 1 unit. Find the smallest value of r such that the points of S can be colored in three colors in such a way that any two points of the same color are less than r units apart.
7. (Putnam '97) The boundary and interior points of a right triangle with legs of lengths 3 and 4 are colored in four colors. Find the greatest r such that for any such coloring, there exist two points of the same color at distance at least r.

3 Convexity and Collinearity

1. (Sylvester's Theorem) Let S be a finite collection of points in the plane, not all collinear. Then there exists a line which passes through exactly two points of S.
2. (Erdős-Szekeres Theorem) Let $n \geq 3$ be an integer. Then there exists an integer N such that if S is a set of N points in the plane with no three points collinear, then S contains n points which form a convex n-gon. (Take $N = \binom{2n-4}{n-2} + 1$.)
3. (Helly's Theorem) Let F_1, F_2, $\ldots$, F_n be convex sets in the plane. If every three of them have a common point, then all n sets have a common point.
4. Given 2005 points in the plane, show that one can form 401 disjoint convex quadrilaterals whose vertices are among the given points.
5. Let A and B be two disjoint finite non-empty sets of points in the plane, with the property that every segment joining two points in the same set contains a point in the other set. Prove that all the points of $A \cup B$ lie on a single line.
6. (IMO '96 SL) Determine whether or not there exist two disjoint infinite sets A and B of points in the plane satisfying the following conditions:
   (a) No three points in $A \cup B$ are collinear, and the distance between any two points in $A \cup B$ is at least 1.
   (b) There is a point of A inside any triangle whose vertices lie in B, and there is a point of B inside any triangle whose vertices lie in A.
7. Let S be a set of points in the plane satisfying the following conditions:
   (a) there are seven points in S that form a convex heptagon; and
   (b) for any five points in S, if they form a convex pentagon, then there is a point of S that lies in the interior of the pentagon.

   Determine the minimum possible number of elements in S.
8. Let S be a set of $n \geq 5$ points in the plane, no three of which are collinear. Prove that at least $\binom{n-3}{2}$ convex quadrilaterals can be formed with vertices among the points of S.
9. (IMO '99 SL) A circle is called a *separator* for a set of five points in the plane if it passes through three of these points, it contains a fourth point inside and the fifth point is outside the circle. Let S be a set of five points such that no three points of S are collinear and no four points of S are concyclic. Prove that S has exactly four separators.

10. (APMO '99) Let S be a set of $2n+1$ points in the plane such that no three are collinear and no four concyclic. A circle will be called *good* if it has 3 points of S on its circumference, $n-1$ points of S in its interior and $n-1$ points of S in its exterior. Prove that the number of good circles has the same parity as n.

11. (IMO '00 SL) Let $n \geq 4$ be a fixed positive integer. Let $S = \{P_1, P_2, \ldots, P_n\}$ be a set of n points in the plane such that no three are collinear and no four concyclic. Let a_t, $1 \leq t \leq n$, denote the number of unordered triples $\{i, j, k\}$ such that the circumcircle of $P_iP_jP_k$ contains P_t, and let

$$m(S) = a_1 + a_2 + \cdots + a_n.$$

Prove that there exists a positive integer $f(n)$, depending only on n, such that the points of S are the vertices of a convex polygon if and only if $m(S) = f(n)$.

12. (IMO '91 SL) Let S be a set of $n \geq 3$ points in the plane with no three collinear. Show that there exists a set T of at most $2n-5$ points such that every triangle determined by three points of S contains a point in T.

13. (IMO '02) Let n be an integer with $n \geq 3$. Let Γ_1, Γ_2, …, Γ_n be unit circles in the plane, with centers O_1, O_2, …, O_n respectively. If no line meets more than two of the circles, prove that

$$\sum_{1 \leq i < j \leq n} \frac{1}{O_iO_j} \leq \frac{(n-1)\pi}{4}.$$

4 Covering

1. Let S be a set of circles in the plane with the following properties:

   (a) the interiors of any two circles of S are disjoint;

   (b) every circle of S has radius at least one;

   (c) every circle of S is tangent to six other circles of S.

   Prove that all circles of S have the same radius.

2. (a) Given n points in the plane such that any three are contained in a disc of radius 1, show that all the points are contained in a disc of radius 1.

   (b) Consider a convex polygon with the property that no two vertices are more than 1 unit apart. What is the smallest r such that any polygon with this property is contained within a disc of radius r?

3. (a) Given a finite collection of squares of total area 3, prove that they can be arranged to cover the unit square.

   (b) Given a finite collection of squares of total area $\frac{1}{2}$, prove that they can be arranged so as to fit inside a unit square (with no overlaps).

4. A finite set of points in the plane has the property that the triangle determined by any three of its points has area at most 1. Prove that there exists a triangle of area 4 which contains all of the points of this set.

5. Prove that every plane figure of diameter 1 can be covered by a regular hexagon of side length $\frac{1}{\sqrt{3}}$.

6. Let C be a circle of radius 1. Let P_1, P_2, …, P_n be points inside C, with P_1 the center of C. For each j let X_j be the smallest distance from P_j to another point P_i. Prove that $X_1^2 + X_2^2 + \cdots + X_n^2 \leq 9$.

7. (Putnam '98) Let D_1, …, D_n be a collection of open discs in the plane whose union contains a given set E. For $i = 1$, …, n, let D_i' be the disc concentric with D_i of three times the radius. Prove that one can find a subset S of $\{1, \ldots, n\}$ such that D_i and D_j are disjoint for i, $j \in S$ and $E \subset \bigcup_{i \in S} D_i'$.

8. (IMO '03 SL) Let D_1, D_2, $\ldots$, D_n be closed discs in the plane. Suppose that every point in the plane is contained in at most 2003 of these discs. Prove that there exists a disc which intersects at most 14020 other discs.

5 Constraints

1. (a) Let S be a set of $n \geq 3$ points in the plane such that the distance between any two points is at least 1. Show that there are at most $3n-6$ pairs of points at distance exactly one.

   (b) Let S be a set of n points in the plane of diameter 1 (the distance between any two points of S is at most 1). Find the maximum possible number of pairs of points of S at distance 1.

   (c) Let S be a set of n points in the plane of diameter 1. Find the maximum possible number of pairs of points at distance greater than $\frac{1}{\sqrt{2}}$.

2. Let S be a collection of points in the plane such that every three points in S are the vertices of an isosceles (or equilateral) triangle. Prove that S has at most 6 points.

3. (a) Show that there exists a set of 2005 points in the plane such that the distance between any two of these points is an integer.

   (b) Does there exist an infinite set with the same property?

4. (IMO '95) Determine all integers $n > 3$ for which there exist n points A_1, $\ldots$, A_n in the plane, no three collinear, and real numbers r_1, $\ldots$, r_k such that for $1 \leq i < j < k \leq n$, the area of triangle $A_iA_jA_k$ is $r_i + r_j + r_k$.

5. (St. Pete '98) A convex $2n$-gon has its vertices at lattice points. Prove that its area is not less than $n^3/100$.

6. (USA TST '04) A convex polygon has its vertices among the points of a 2004×2004 square grid. Find the maximum number of sides of such a polygon.

7. (IMO '00 SL) Ten students stand on a flat surface, with the distances between them all distinct. At twelve o'clock, when the church bells start chiming, each of them throws a water balloon at their closest neighbor. What is the minimum number of students who will be hit by water?