VECTORS IN PHYSICS

Vectors represent directions and magnitudes. Indeed, many concepts in Physics can be defined or represented by vectors. This set of notes will serve as an introduction of some physical concepts related to vector applications. We will not go into details of the basic properties of vectors as they will be discussed in another set of notes which is dedicated to vectors.

1. Centre of mass

A frequently encountered concept in mechanics is the centre of mass, also known as the centre of gravity. This is the point at which the resultant of all gravity forces at an object is acted on. The position of the centre of mass can be defined as a weighted sum of position vectors.

Definition 1.1. (Centre of mass)

Suppose an object consists of a finite number of mass points. Let the mass and position vector of the i^{th} mass point be m_i and $\vec{r}_i$. Then the position vector of the centre of mass of the object, $\vec{r}$, is given by

$$\vec{r} = \frac{\sum m_i \vec{r}_i}{\sum m_i}$$

Therefore, the position vector of the centre of mass is a mass-weighted average of position vectors of component mass points. As a corollary, if all mass points of an object have equal masses, the position vector of the centre of mass is the simple average of the position vectors of the mass points. On the other hand, if the object consists of finite number of mass points, the above sum will be evaluated by means of integration instead of summation.

Example 1.1.

Suppose an object consists of 3 mass points, with masses, 2, 3, and 5, at positions $4\vec{i} + \vec{j}$, $\vec{i} + 3\vec{j}$ and $-\vec{i} + 6\vec{j}$ respectively. Determine the position of the centre of mass of the object.

Solution.

The position vector of the centre of mass of the object is given by

$$\vec{r}=\frac{1}{2+3+5}\left[2\left(4\vec{i}+\vec{j}\right)+3\left(\vec{i}+3\vec{j}\right)+5\left(-\vec{i}+6\vec{j}\right)\right]=\frac{1}{10}\left(6\vec{i}+41\vec{j}\right)=0.6\vec{i}+4.1\vec{j}$$

Example 1.2.

Consider a circular plate centred at the origin with unit radius. Suppose the density at the point (x, y) is given by $m(x,y)=k|y|$, where k is a positive constant. Determine the position of the centre of mass of the ring.

Solution.

The position vector of the point (x, y) is $x\vec{i}+y\vec{j}$.

Thus, the position vector of the centre of mass is given by

$$\begin{aligned}\vec{r}&=\frac{\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}k|y|\left(x\vec{i}+y\vec{j}\right)dydx}{\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}k|y|\,dydx}=\frac{\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}k|y|\,xdydx}{\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}k|y|\,dydx}\vec{i}+\frac{\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}k|y|\,ydydx}{\int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}k|y|\,dydx}\vec{j}\\&=\frac{2k\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}}yxdydx}{2k\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}}ydydx}\vec{i}+\frac{\int_{-1}^{1}0dx}{2k\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}}ydydx}\vec{j}=\frac{\int_{-1}^{1}x\int_{0}^{\sqrt{1-x^2}}ydydx}{\int_{-1}^{1}\int_{0}^{\sqrt{1-x^2}}ydydx}\vec{i}\\&=\frac{\frac{1}{2}\int_{-1}^{1}x\left(1-x^2\right)dx}{\frac{1}{2}\int_{-1}^{1}\left(1-x^2\right)dx}\vec{i}=\frac{0}{\frac{1}{2}\int_{-1}^{1}\left(1-x^2\right)dx}\vec{i}=\vec{0}\end{aligned}$$

Note that $\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}}k|y|\,ydy=0$ because $f(y)=k|y|\,y$ is an odd function. Similarly, $\int_{-1}^{1}x\left(1-x^2\right)dx$ =0. The reader may also note that the circular plate is symmetrical, in terms of mass distribution, with respect to both x- and y- axes. Therefore, the centre of mass is intuitively the origin.

In a system of objects, the centre of mass of the system corresponds to the position vector given by a mass-weighted average of the position vectors of the centres of mass of the objects.

Definition 1.2. (Centre of mass of a system of objects)

Suppose a system of n objects. Let the mass and position vector of the centre of mass of the i^{th} object be m_i and $\vec{r}_i$. Then the position vector of the centre of mass of the system, $\vec{r}$, is given by

$$\vec{r} = \frac{\sum_{i=1}^{n} m_i \vec{r}_i}{\sum_{i=1}^{n} m_i}$$

Indeed, this definition is consistent with Definition 1.1. This can be shown as follows. Suppose the i^{th} object is consisted of k_i mass points. The k_i mass points have masses $m_{i,1}, m_{i,2}, \cdots, m_{i,k_i}$ and position vectors $\vec{r}_{i,1}, \vec{r}_{i,2}, \cdots, \vec{r}_{i,k_i}$ correspondingly. Since the i^{th} object has mass m_i and centre of mass $\vec{r}_i$ by assumption, we have $m_i = m_{i,1} + m_{i,2} + \cdots + m_{i,k_i} = \sum_{j=1}^{k_i} m_{i,j}$. And further, by Definition 1.1, we have

$$\vec{r}_i = \frac{\sum_{j=1}^{k_i} m_{i,j} \vec{r}_{i,j}}{\sum_{j=1}^{k_i} m_{i,j}} = \frac{\sum_{j=1}^{k_i} m_{i,j} \vec{r}_{i,j}}{m_i}$$

From Definition 1.1, if we look at the system of objects as a number of mass points, the position of centre of mass of the system can be calculated as a mass-weighted mean of the position vectors of the mass points:

$$\vec{r} = \frac{\sum_{i=1}^{n} \sum_{j=1}^{k_i} m_{i,j} \vec{r}_{i,j}}{\sum_{i=1}^{n} \sum_{j=1}^{k_i} m_{i,j}} = \frac{\sum_{i=1}^{n} \left(\sum_{j=1}^{k_i} m_{i,j} \cdot \frac{\sum_{j=1}^{k_i} m_{i,j} \vec{r}_{i,j}}{\sum_{j=1}^{k_i} m_{i,j}} \right)}{\sum_{i=1}^{n} m_i} = \frac{\sum_{i=1}^{n} \left(\sum_{j=1}^{k_i} m_{i,j} \cdot \vec{r}_i \right)}{\sum_{i=1}^{n} m_i} = \frac{\sum_{i=1}^{n} m_i \vec{r}_i}{\sum_{i=1}^{n} m_i}$$

which is precisely what Definition 1.2 reads.

Exercise

1. Prove that if equal masses are put at points *A*, *B* and *C*, the centre of mass of the system is the centroid of triangle *ABC*.

2. Suppose $\overrightarrow{OA} = \vec{i} + 2\vec{j}$, $\overrightarrow{OB} = 3\vec{i} + \vec{j}$. Let M_P denote the mass at point *P*. If $M_A = 2$, $M_B = 1$ and $M_C = 3$. What is the locus of *C* if centre of mass lie on the *x*-axis?

2. Resultant of Forces

To study the motion of objects, we are most concerned with the forces acting on them. Forces are vectors: they have directions as well as magnitudes. They affect the direction and speed with which objects move.

If there are no forces acting on an object, the object will maintain its motion, for example, moving at uniform speed in along one direction, or remain stationary. With a single force acting on an object, the object will accelerate along the direction of the acting force. How about two or more forces acting on an object? The resulting translational motion of the centre of mass of the object will be identical to a point mass, which has the same mass as the object under investigation, subject to a force that is equivalent to the resultant of all the forces acting on the object.

The resultant of a number of forces is determined by addition of the vectors representing the forces.

Example 2.1.

An object is being pulled by three man X, Y and Z, with forces of magnitude x, y and z respectively, where $x \geq y \geq z$. What is the minimum magnitude of the resultant force on the object and when is the minimum achieved?

Solution.

We will separate our analysis into two cases:

(1) $x \leq y + z$

(2) $x > y + z$

Case 1: $x \leq y + z$

From triangle inequality, we can form a triangle with three sides equal to x, y and z (the triangle is degenerate in the case $x = y + z$). Construct one such triangle on the ground. When X, Y and Z pull at the direction dictated by the corresponding side of triangle, the resultant of the forces, i.e., the sum of the forces, will be zero. This is the minimum magnitude of resultant.

Case 2: $x > y + z$

If Y and Z pull at the same direction, and X pull at the opposite direction, the magnitude of the resultant will equal $x - y - z$. We can prove that this is the minimum magnitude possible.

The magnitude of the resultant is:

$$\left|\vec{X}+\vec{Y}+\vec{Z}\right| \geq \left|\vec{X}\right| - \left|-\vec{Y}-\vec{Z}\right| = \left|\vec{X}\right| - \left|\vec{Y}+\vec{Z}\right| \geq \left|\vec{X}\right| - \left(\left|\vec{Y}\right| + \left|\vec{Z}\right|\right) = x - y - z$$

One useful technique in solving physical problems involving coplanar forces is resolution into components. This means to break a force vector into two perpendicular component vectors. Indeed if we represent a point (x, y) as $x\vec{i} + y\vec{j}$, we can also represent a force with two perpendicular vectors, one along the horizontal axis and the other along the vertical one. If a force has a magnitude of r and makes an angle of θ with the horizontal axis, then resolving the force into two components can give us $r\cos\theta\vec{i}$ and $r\sin\theta\vec{i}$.

Example 2.2.

A boat is sailing across a river in perpendicular direction with the parallel banks. River flows exert a force of 5 to the boat perpendicular to its course. The boat has an engine that can provide a force of 7. In what direction should the engine force be exerted to maintain the direction of course?

Solution.

Let the river bank be in the x-direction and the engine force be $x\vec{i} + y\vec{j}$. Then we have $\sqrt{x^2 + y^2} = 7$ and the resultant of the two forces is $(x+5)\vec{i} + y\vec{j}$. The y-direction component of the resultant reinforces the movement of the boat in the current course, but the x-direction component will divert the course. Therefore, for the boat to maintain its course, the x-direction component must be zero. Thus x = -5, and $y = \pm\sqrt{7^2 - 5^2} = \pm 2\sqrt{6}$, sign depending on the whether the boat's course is clockwise or anticlockwise $90°$ from the river flow.

Example 2.3.

A mass (O) of 5/g (g being gravitational accelaration) is being hung with two strings from the ends A and B of two rods. OA and OB make angles α and β with the horizontal respectively. Let $\vec{F_A}$ and $\vec{F_B}$ be the tensions in the two strings. If tanα=2 tanβ and the resultant on the mass is zero, prove that

$$F_A{}^2 = F_B{}^2 + \frac{25}{3},$$

where F_A and F_B denote the magnitudes of the forces in the two strings respectively.

Solution.

We resolve the resultant into the vertical and horizontal directions. Since the resultant on the mass is zero, the components are zero as well.

For the vertical component:

$$F_A \sin\alpha + F_B \sin\beta - 5 = 0 .$$

For the horizontal component:

$$F_A \cos\alpha - F_B \cos\beta = 0$$

Therefore,

$$\begin{aligned} F_A \sin\alpha &= F_A\left(\cos\alpha \cdot \tan\alpha\right) = \left(F_A \cos\alpha\right) \cdot \tan\alpha \\ &= \left(F_B \cos\beta\right) \cdot 2\tan\beta = 2F_B\left(\cos\beta \cdot \tan\beta\right) = 2F_B \sin\beta \end{aligned}$$

Solving this simultaneously with the first equation, we have

$$F_A \sin\alpha = \frac{10}{3} \text{ and } F_B \sin\beta = \frac{5}{3}.$$

Finally,

$$\begin{aligned} F_A{}^2 - F_B{}^2 &= \left(F_A{}^2 \cos^2\alpha + F_A{}^2 \sin^2\alpha\right) - \left(F_B{}^2 \cos^2\beta + F_B{}^2 \sin^2\beta\right) \\ &= \left(F_A{}^2 \cos^2\alpha - F_B{}^2 \cos^2\beta\right) + \left(F_A{}^2 \sin^2\alpha - F_B{}^2 \sin^2\beta\right) \\ &= \left(F_A \cos\alpha - F_B \cos\beta\right)\left(F_A \cos\alpha + F_B \cos\beta\right) + \left[\left(\frac{10}{3}\right)^2 - \left(\frac{5}{3}\right)^2\right] \\ &= 0 + \frac{75}{9} = \frac{25}{3} \end{aligned}$$

and the result follows.

Exercise.

1. (Angles measured from the positive x-axis) The force $\vec{F} = x\vec{i} + y\vec{j}$ and a force of magnitude 2 at 30° is resulting in a force at -60°. Determine the relationship between x and y.

3. Moments

Another important concept in physics related to vectors is moment. Moment can be viewed as a turning strength around a point. Intuitively, the stronger the turning force, the moment of turning will be greater. On the other hand, if the line of turning force goes along with the pivotal point, there will not be a turning effect. Therefore, the moment depends on both the direction and magnitude of the turning force.

In line with our intuition, the moment is defined as the cross product of the vectors of force and the position vector of a point on the line of force.

Definition 3.1.

The moment of a vector $\vec{F}$ about a point O is given by

$$\vec{M} = \vec{r} \times \vec{F} \text{ ,}$$

where $\vec{r}$ is the position where $\vec{F}$ is applied.

For all forces acting on an object, if the resultant of the moments of these forces about a point P is non-zero, the object will rotate around P at an increasing rate in the direction dictated by the resultant of the moment about that point. For instance, when you draw the position vector and the acting force on a piece of paper and if the moment is out of the paper, the object will rotate about P at an increasing rate in an anticlockwise direction. On the contrary, if the moment is into the paper, the rotation will increase its rate in the clockwise direction.

As the moment is the cross product of the position vector and the acting force, the moment is perpendicular to the plane containing the position vector and acting force.

Example 3.1.

A circular cylinder lies in space with only three coplanar horizontal forces acting on it, with the circular faces facing sideways. The three forces all lie on the same vertical plane. Two equal forces, $\overrightarrow{F_1}$ and $\overrightarrow{F_2}$, act through the top (T) and bottom (B) points of the cylinder, whilst the third force, $\overrightarrow{F_3}$, acts through the center (C). Prove that the cylinder has no translational acceleration iff the moment of any point on the cylinder is zero.

Solution.

First we shall prove that the cylinder has no translational acceleration if the moment of any point on the cylinder is zero. Consider the point B. Since C is the mid-point of T and B, $\overrightarrow{BT} = 2\overrightarrow{BC}$. The moment at this point is zero. Therefore, $\overrightarrow{BT} \times \overrightarrow{F_1} + \overrightarrow{BC} \times \overrightarrow{F_3} = \vec{0}$.

$$\vec{0} = 2\overrightarrow{BC} \times \overrightarrow{F_1} + \overrightarrow{BC} \times \overrightarrow{F_3} = \overrightarrow{BC} \times \left(2\overrightarrow{F_1} + \overrightarrow{F_3}\right)$$

As $\overrightarrow{BC}$ is vertical but $\left(2\overrightarrow{F_1} + \overrightarrow{F_3}\right)$ is horizontal, the only possibility that the cross product to be zero is either $\left(2\overrightarrow{F_1} + \overrightarrow{F_3}\right)$ is zero vector or $\overrightarrow{BC}$ is zero, which is not possible. Thus $\vec{0} = 2\overrightarrow{F_1} + \overrightarrow{F_3} = \overrightarrow{F_1} + \overrightarrow{F_2} + \overrightarrow{F_3}$, ie., the resultant force on the cylinder is zero, meaning there is no translational acceleration.

Suppose there is no translational acceleration, then the sum of horizontal forces is zero, i.e., $\overrightarrow{F_1} + \overrightarrow{F_2} + \overrightarrow{F_3} = \vec{0}$. Then for any point, O, on the cylinder, the moment is

$$\begin{aligned}\overrightarrow{OB} \times \overrightarrow{F_1} + \overrightarrow{OC} \times \overrightarrow{F_3} + \overrightarrow{OT} \times \overrightarrow{F_2} &= \overrightarrow{OB} \times \overrightarrow{F_1} + \left[\frac{1}{2}\left(\overrightarrow{OB} + \overrightarrow{OT}\right)\right] \times \left(-\overrightarrow{F_1} - \overrightarrow{F_2}\right) + \overrightarrow{OT} \times \overrightarrow{F_2} \\ &= \overrightarrow{OB} \times \overrightarrow{F_1} + \frac{1}{2}\left(\overrightarrow{OB} + \overrightarrow{OT}\right) \times \left(-2\overrightarrow{F_1}\right) + \overrightarrow{OT} \times \overrightarrow{F_1} = \vec{0}\end{aligned}$$

And the results follow.

Exercise.

1. A door of width 1 is hinged at one end on a fixed frame. A force of 5 is applied to the door. In what direction and location should the force be applied so that the moment on the door is maximized / minimized?

4. Static equilibrium

The motion of an object can be divided into two parts: translational and rotational. Translational motions are characterized by the displacement of the centre of mass of the object, while rotational motions are characterized by revolving around the centre of mass.

An object is in equilibrium when there is no acceleration: either it performs uniform motion or is stationary. No acceleration means that the object must not have linear acceleration nor rotate.

The above gives an intuitive idea of what equilibrium is. There is a more vigorous way of defining equilibrium. Indeed, since an object under equilibrium does not accelerate in any direction, the resultant of all forces acting on it must be zero. Besides, the object will not rotate, because otherwise the various parts of an object will be changing their direction of motion continuously. Therefore the moment about every point on the object is also zero.

Definition 4.1. (Static equilibrium)

An object is said to be in static equilibrium if the resultant of all forces acting on it is zero and the sum of moments at any points is zero.

In fact, we need not prove that all the moments of all mass points of the object to be zero in order to demonstrate equilibrium. This is illustrated in the following example.

Example 4.1.

Prove that if the resultant of all forces acting on an object, which has finitely many mass points, is zero and the moment about one of the mass points is zero, then the moment about any point on the object in also zero.

Solution.

Let the forces acting on the object be $\vec{F}_1, \vec{F}_2, \cdots, \vec{F}_n$ acting through the points represented by $\vec{r}_1, \vec{r}_2, \cdots, \vec{r}_n$. Suppose the moment on the point $\vec{r}_0$ is zero. Therefore we have

$$\sum_{i=1}^{n} \vec{F}_i = \vec{0} \text{ and } \sum_{i=1}^{n} (\vec{r}_i - \vec{r}_0) \times \vec{F}_i = \vec{0}.$$

For any other point $\vec{r}$ on the object the moment on the point is

$$\sum_{i=1}^{n} (\vec{r}_i - \vec{r}) \times \vec{F}_i = \sum_{i=1}^{n} (\vec{r}_i - \vec{r}) \times \vec{F}_i - \sum_{i=1}^{n} (\vec{r}_i - \vec{r}_0) \times \vec{F}_i = \sum_{i=1}^{n} (\vec{r}_0 - \vec{r}) \times \vec{F}_i = (\vec{r}_0 - \vec{r}) \times \sum_{i=1}^{n} \vec{F}_i = (\vec{r}_0 - \vec{r}) \times \vec{0} = \vec{0}.$$

Exercise.

1. Consider object *ABCD*, where *A* = (0,0), *B* = (1,0), *C* = (1,1) and *D* = (0,1). Three forces, $\vec{i}$, $3\vec{j}$ and $-\vec{i} - 3\vec{j}$ are acting on the object through the points (1, 0.5), (0.5, 0.7) and (0.4, 0.2)

respectively. Is the object in equilibrium?

5. Other Forces

The above sections briefly mention a few mechanical concepts that can be expressed as vectors. However, the application of vector is not limited to mechanical topics. Indeed, vectors can represent any object that consists of a direction and a magnitude, for example, electric forces and magnetic field. While these are out of scope of this set of notes, readers are encouraged to explore on their own the roles vectors can play in problems involving these concepts.

6. Solutions to Exercise

Centre of mass

1. Assume the points in *xyz*-space and apply Theorem 1.1.

2. Let $\overrightarrow{OC} = x\vec{i} + y\vec{j}$. Then the centre of mass is

$$\frac{2\left(\vec{i}+2\vec{j}\right)+1\left(3\vec{i}+\vec{j}\right)+3\left(x\vec{i}+y\vec{j}\right)}{2+1+3}=\frac{5+3x}{6}\vec{i}+\frac{5+3y}{6}\vec{j}\,.$$

This represents a point that lies on the *x*-axis. So $\frac{5+3y}{6}=0$, forcing $y=-\frac{5}{3}$. So the locus is the horizontal line $y=-\frac{5}{3}$. Checking confirms this.

Resultant of forces

1. The force of magnitude 2 is $\overrightarrow{F_1} = 2\cos 30°\vec{i} + 2\sin 30°\vec{j} = \sqrt{3}\vec{i} + \vec{j}$. Thus the resultant of the two forces is $\overrightarrow{F_R} = \vec{F} + \overrightarrow{F_1} = \left(\sqrt{3}+x\right)\vec{i} + \left(1+y\right)\vec{j}$.

For this resultant to be at $-60°$, we have $-\sqrt{3} = \tan\left(-60°\right) = \frac{1+y}{\sqrt{3}+x}$, i.e., $y+\sqrt{3}x+4=0$. Besides, we must have $1+y<0$ and $\sqrt{3}+x>0$, i.e., $y<-1$ and $x>-\sqrt{3}$.

Moments

1. Maximum: applied perpendicular to the door at the other end of the door from the hinge

   Minimum: applied in the plane of the door at any point on the door or applied on the hinge in any direction

Static equilibrium

1. Obviously the resultant of the forces is zero.

   For the moment, consider the point (0.5, 0.5). It can be easily checked that all three forces go through this point. Thus the moment at this point must be zero. According to Example 4.1, the object is in equilibrium.