New Zealand Mathematical Olympiad Committee

Camp Selection Problems 2011

*Due: 10 August 2011*

# Junior division

J1. *A three by three square is filled with positive integers. Each row contains three different integers, the sums of each row are all the same, and the products of each row are all different. What is the smallest possible value for the sum of each row?*

**Solution**: Let $n$ be the smallest possible value. It must be possible to write $n$ as a sum of three different sets of distinct positive integers (since the row products differ, the sets of elements appearing in each row must be distinct). Now, by trial and error we determine that 9 is the smallest positive integer with this property as we can use $\{1, 2, 6\}$, $\{1, 3, 5\}$, and $\{2, 3, 4\}$. Since these three sets of numbers also have different products, the answer is 9. □

J2. *Let an acute angled triangle $ABC$ be given. Prove that the circles whose diameters are $AB$ and $AC$ have a point of intersection on $BC$.*

**Solution**: Let $X$ be the foot of the altitude from $A$ to $BC$. Since $AXB$ is a right angle, $X$ lies on the circle with diameter $AB$, and similarly it lies on the circle with diameter $AC$. □

J3. *There are 16 competitors in a tournament, all of whom have different playing strengths and in any match between two players the stronger player always wins. Show that it is possible to find the strongest and second strongest players in 18 matches.*

**Solution**: The strongest player can be found in 15 matches by a simple knock out competition. The second strongest player must have been one of the four players whom the strongest player beat, and so can be found in three more matches. □

J4. *Find all pairs of positive integers $m$ and $n$ such that:*

$$(m + 1)! + (n + 1)! = m^2 n^2.$$

*(Note: $k! = 1 \times 2 \times 3 \times \cdots \times k$.)*

**Solution**: Assume that we have a solution with $m \leq n$. Then

$$(n + 1)! = (n - 1)! \times n \times (n + 1) \geq n^2 (n - 1)!$$

So $n^2(n - 1)! \leq m^2 n^2$ and hence $(n - 1)! \leq m^2 \leq n^2$. But, for $n \geq 6$, $(n - 1)! > n^2$. So $1 \leq n \leq 5$. Now by trial and error we can determine that the only possibility is $m = 3$, $n = 4$ (or by symmetry the reverse of this pair). □

J5. *Let a square $ABCD$ with sides of length 1 be given. A point $X$ on $BC$ is at distance $d$ from $C$, and a point $Y$ on $CD$ is at distance $d$ from $C$. The extensions of: $AB$ and $DX$ meet at $P$, $AD$ and $BY$ meet at $Q$, $AX$ and $DC$ meet at $R$, and $AY$ and $BC$ meet at $S$. If points $P$, $Q$, $R$ and $S$ are collinear, determine $d$.*

**Solution**: Put $ABCD$ in the plane with $A$ at $(0, 0)$, and $B$ at $(1, 0)$. Then $X = (1, 1-d)$, and $Y = (1 - d, 1)$. From similar triangles $CDX$ and $APD$ we see that $P = (1/d, 0)$. Similarly $Q = (0, 1/d)$. So, the line $PQ$ is given by $x + y = 1/d$. The line $AX$ is given by $y = (1 - d)x$, and the line $DC$ by $y = 1$. So, $R = (1/(1 - d), 1)$. If the collinearity holds then we know that $R$ lies on $PQ$ (this will suffice by symmetry), that is:

$$1 + 1/(1 - d) = 1/d$$

which gives $d = (3 - \sqrt{5})/2$. □

J6. *Find all pairs of non-negative integers $m$ and $n$ that satisfy*

$$3 \times 2^m + 1 = n^2.$$

**Solution**: Suppose that we have a solution. Since the left hand side is not a multiple of 3 it must be the case that $n = 3k + 1$, or $n = 3k + 2$. We consider the two possibilities in turn.

If $n = 3k + 1$ then:

$$3 \times 2^m + 1 = (3k + 1)^2 = 9k^2 + 6k + 1.$$

So $2^m = 3k^2 + 2k = k(3k + 2)$. So, both $k$ and $3k + 2$ must be powers of 2. Obviously $k = 1$ does not work, while $k = 2$ does. But, if $k$ is any larger power of 2 then $3k + 2$ is a multiple of 2 but not of 4 and so that doesn't work. So, we have one solution arising in this case, $m = 4$, $n = 7$.

If $n = 3k + 2$ then:

$$3 \times 2^m + 1 = (3k + 2)^2 = 9k^2 + 12k + 4.$$

So $2^m = 3k^2 + 4k + 1 = (k + 1)(3k + 1)$. So, both $k + 1$ and $3k + 1$ must be powers of 2. But $3k + 1 = 3(k + 1) - 2$ and so the same argument as above rules out $k + 1 \geq 4$. Thus, the only posibilities are $k + 1 = 1$ and $k + 1 = 2$. The first gives $m = 0$, $n = 2$, while the second gives $m = 3$, $n = 5$.

So, the possible solutions are:

$$(m, n) = (0, 2), (3, 5), (4, 7).$$

□

# Senior division

S1. *Find all pairs of positive integers $m$ and $n$ such that*

$$m! + n! = m^n.$$

**Solution**: We first claim that in any solution, $m \leq n$. For, if $m > n$ then the left hand side is

$$n!(1 + (n+1)(n+2)\cdots m)$$

and the second factor is greater than 1 and relatively prime to $m$, hence the left hand side cannot be $m^n$.

Since $m \leq n$, $m - 1$ is a factor of the left hand side (unless $m = 1$ which is plainly impossible). But again $m - 1$ is relatively prime to $m$, so it must be the case that $m - 1 = 1$, i.e. $m = 2$. Now we have:

$$2 + n! = 2^n.$$

If $n \geq 4$ then the left hand side is a multiple of 2 but not of 4, hence not a power of 2. So the only possible pairs are $(2, 2)$ and $(2, 3)$ and these both work. $\square$

S2. *In triangle $ABC$, the altitude from $B$ is tangent to the circumcircle of $ABC$. Prove that the largest angle of the triangle is between $90°$ and $135°$. If the altitudes from* both *$B$ and from $C$ are tangent to the circumcircle, then what are the angles of the triangle?*

**Solution**: Certainly the triangle must have an obtuse angle at either $A$ or at $C$ (otherwise the altitude from $B$ lies inside the triangle). Without loss of generality suppose that $\alpha = \angle CAB > 90°$, and let the altitude from $B$ meet the extension of $CA$ at $D$. By the tangent-secant theorem $\angle ABD = \angle ACB = \gamma$. Further $\alpha = \angle CAB = \angle ABD + \angle BDA = \gamma + 90°$ since it is exterior to triangle $ABD$.

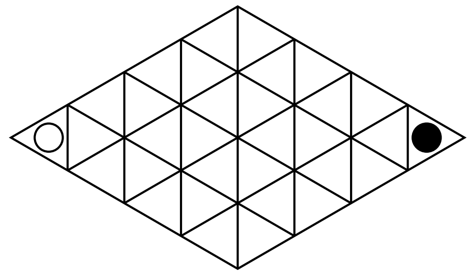
Now

$$\begin{aligned} 180° &= \alpha + \beta + \gamma \\ &= 2\alpha + \beta - 90°. \end{aligned}$$

So $2\alpha < 270°$, and $\alpha < 135°$ as required.

For the second part, we have by symmetry $\alpha = \beta + 90°$, which gives $3\alpha - 180° = 180°$ so $\alpha = 120°$ and $\beta = \gamma = 30°$. $\square$

S3. *Chris and Michael play a game on a board which is a rhombus of side length n (a positive integer) consisting of two equilateral triangles, each of which has been divided into equilateral triangles of side length 1. Each has a single token, initially on the leftmost and rightmost squares of the board, called the "home" squares (the illustration shows the case $n = 4$).*



*A move consists of moving your token to an adjacent triangle (two triangles are adjacent only if they share a side). To win the game, you must either capture your opponent's token (by moving to the triangle it occupies), or move on to your opponent's home square. Supposing that Chris moves first, which, if any, player has a winning strategy?*

**Solution**: Chris has a winning strategy. Every move moves a token from a left pointing triangle to a right pointing one. Since the two tokens initially sit on differently pointing triangles, they will do so after each of Michael's moves and so Michael can never capture Chris's token. So Chris can simply take the shortest path to Michael's home square — on the way he must either capture Michael, or reach his home square before Michael can do the same. □

S4. *Let a point $P$ inside a parallelogram $ABCD$ be given such that $\angle APB + \angle CPD = 180°$. Prove that*

$$AB \cdot AD = BP \cdot DP + AP \cdot CP$$

*(here $AB$, $AD$ etc. refer to the lengths of the corresponding segments).*

**Solution**: Place $P'$ outside the parallelogram so that triangle $BP'A$ is congruent to $CPD$. Since opposite angles of this quadrilateral are supplementary we have, by Ptolemy's theorem:

$$AB \cdot P'P = BP \cdot AP' + AP \cdot BP'.$$

But, by construction $AP' = CP$ and $BP' = DP$. Furthermore, $ADPP'$ is a parallelogram, so $P'P = AD$. Making all these substitutions yields the required equality. □

S5. *Prove that for any three distinct positive real numbers $a$, $b$ and $c$:*

$$\frac{(a^2-b^2)^3 + (b^2-c^2)^3 + (c^2-a^2)^3}{(a-b)^3 + (b-c)^3 + (c-a)^3} > 8abc.$$

**Solution**: Let $x = a - b$, $y = b - c$, so $c - a = -(x+y)$. Then the denominator on the LHS is:

$$x^3 + y^3 - (x+y)^3 = -3x^2y - 3xy^2 = -3xy(x+y) = 3(a-b)(b-c)(c-a)$$

Similarly, the numerator is

$$3(a^2 - b^2)(b^2 - c^2)(c^2 - a^2)$$

and so the LHS is equal to

$$(a + b)(b + c)(c + a)$$

Now the result follows from the AM-GM inequality $a+b > 2\sqrt{ab}$ etc. (never equal because the three values are distinct). $\square$

S6. *Consider the set $G$ of $2011^2$ points $(x, y)$ in the plane where $x$ and $y$ are both integers between 1 and 2011 inclusive. Let $A$ be any subset of $G$ containing at least $4 \times 2011 \times \sqrt{2011}$ points. Show that there are at least $2011^2$ parallelograms whose vertices lie in $A$ and all of whose diagonals meet at a single point.*

**Solution**: Let $n = 2011$, and $m = |A|$. Let $S$ be the set of all segments whose endpoints lie in $A$, so $|S| = m(m - 1)/2$. The midpoint of a segment in $S$ has both coordinates an integer multiple of $1/2$ lying between 1 and $n$. There are fewer than $4 \times n^2$ such points. So, there is a point $B$ that is the midpoint of at least $m(m - 1)/8n^2$ segments. Moreover,

$$\frac{m(m-1)}{8n^2} \geq \frac{4n\sqrt{n}(4n\sqrt{n} - 1)}{8n^2} = 2n - 1/2\sqrt{n} > 2n - 1.$$

Thus $B$ is the midpoint of at least $2n$ segments.

Now, two of these segments determine a parallelogram unless their endpoints are collinear. Divide the segments whose midpoints are $B$ up in to groups of collinear points. Since each segment shares a common midpoint, each group contains at most $n/2$ segments. Say we have $k$ groups of sizes $a_1$ through $a_k$. We know $S = \sum a_i \geq 2n$. We want to give a lower bound for $\sum_{i<j} a_i a_j$. But:

$$\begin{aligned}
\sum_{i<j} a_i a_j &= \frac{1}{2}\left(\left(\sum a_i\right)^2 - \sum a_i^2\right) \\
&\geq \frac{1}{2}\left(S^2 - \sum a_i n/2\right) \\
&= \frac{1}{2} S\left(S - \frac{n}{2}\right) \\
&\geq n(2n - n/2) = 3n^2/2 > n^2.
\end{aligned}$$

$\square$