Catalan Numbers
(Green/Blue)

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Much of this is from Richard Stanley's book *Enumerative Combinatorics*. See this for much more about Catalan numbers and many other topics.

1 Background

The nth *Catalan number* is defined as $C_n = \frac{1}{n+1}\binom{2n}{n}$.

1.1 Basics

1. Show that C_n is always an integer (without appealing to what is below?)
2. Express C_n as a linear combination of binomial coefficients, with integer coefficients.
3. Prove that C_n satisfies the recurrence relation $C_0 = 1$ and $C_{n+1} = \sum_{i=0}^{n} C_i C_{n-i}$ for $n \geq 0$.
4. What is the generating function expression for C_n? (That is, what is $\sum_{i=0}^{\infty} x^i C_i$?)

1.2 Examples

Show that each of the following are in fact the Catalan numbers. You can show these directly via counting arguments, via recurrence relations, induction, relating one sequence to another sequence that has already been shown to equal the Catalan numbers, etc. Try approaching each example several ways. These are very rich objects and there are many fruitful ways to deal with them.

1. C_n is the number of ways of writing n pairs of matched parentheses. For $n = 3$ we have

$$((())) \quad ()(()) \quad ()()() \quad (())() \quad (()()).$$

2. C_n is the number of ways of completely paranthesizing $n+1$ numbers. For $n = 3$ we have

$$((ab)c)d \quad (a(bc))d \quad (ab)(cd) \quad a((bc)d) \quad a(b(cd)).$$

3. The previous question can be seen as asking for the number of *full binary trees* with $n+1$ leaves. A full binary tree is where every vertex has either 2 children, or is a leaf. We consider "left children" to be different from "right children" when counting trees.
4. C_n is the number of binary trees on n vertices.

5. C_n is the number of trees on $n+1$ vertices, where the children of each vertex are in a fixed order.

6. C_n is the number of *monotonic* paths on an $n \times n$ grid of squares that do not cross above the diagonal. A path is monotonic if all moves are either up or right.

7. C_n is the number of monotonic paths on an $n+1 \times n+1$ grid that do not cross above the diagonal, where every (maximal) vertical segment that ends on the diagonal has odd length.

8. C_n is the number of monotonic paths on an $n+1 \times n+1$ grid that do not cross above the diagonal, and where no "left turn" occurs on the line two below the diagonal.

9. C_n is the number of (unordered) pairs of monotonic length-$n+1$ paths on the lattice that begin and end at the same point but do not intersect otherwise.

10. C_n is the number of (unordered) pairs of monotonic length-$n-1$ paths on the lattice that begin and end at the same point and where the bottom path never crosses "above" the top path.

11. C_n is the number of ways a convex $n+2$-gon can be cut into triangles vertices by non-intersecting edges.

12. C_n is the number of permutations of $\{1, \ldots, n\}$ with no length-3 increasing subsequence.

13. C_n is the number of Young Tableaus for the $2 \times n$ rectangle, that is, the number of ways of arranging the numbers $1, \ldots, 2n$ in a $2 \times n$ rectangle such that every row and column is increasing.

14. C_n is the number of sequences of n integers such that $1 \leq a_{i-1} \leq a_i \leq i$.

15. C_n is the number of sequences of $n-1$ integers satisfying $1 \leq a_{i-1} < a_i \leq 2i$.

16. C_n is the number of sequences of n integers such that $a_1 = 0$ and $0 \leq a_{i+1} \leq a_i + 1$.

17. C_n is the number of sequences of $n-1$ integers such that $a_i \leq 1$ and all partial sums are non-negative.

18. C_n is the number of sequences of n integers such that $a_i \geq -1$, all partial sums are non-negative, and $a_1 + \ldots + a_n = 0$.

19. C_n is the number of ways to "stack coins" in the plane, where the bottom row consists of n consecutive coins.