Mock AIME Series

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The following are five problem sets designed to be used for preparation for the American Invitation Math Exam. Part of my philosophy is that one should train by working problems that are more difficult than one is likely to encounter, so I have made these mock contests extremely difficult. The idea is that, once you become acclimated to them, the real AIMEs will seem easier, and you will approach them with justifiable confidence. Therefore, *do not be discouraged when working your way through this document. It is expected that you will find these problems and solutions extremely challenging.*

Although there is no fixed set of rules for practicing, you might try working each batch of problems under standard AIME conditions. Essentially, that means no calculators are allowed, the testing period is 3 consecutive hours, all answers are integers from 000 to 999 inclusive, and there are no penalties for guessing. An appoximation of the cover of the actual AIME phamphlet preceeds each problem set[1]; the cover will list the official testing parameters, including any slight changes from past years. Although in the AMC could change the rules, as they have for the AMC-10/12 contests, they have been consistent when arbitrating the AIME. That said, the rules for USAMO qualification have recently been in flux. For further information, see the AMC website http://www.unl.edu/amc/.

The five contests begin on pages 3, 17, 29, 40, and 50. Within each contest, I have provided the problems, followed by the answers without solutions, followed by complete solutions for every problem.[2] I believe this format should allow for maximum flexibility in individual practice style. I have checked most of the document fairly thoroughly, but it is possible that mistakes remain. This is a work in progress distributed for personal educational use only. In particular, any publication of all or part of this manuscript without prior contest of the author is strictly prohibited. Please send comments, suggestsions, or corrections to the author at `tmildorf@mit.edu`.

Without further ado, the math.

[1] The dates listed are when the contests were administered through the Art of Problem Solving forums. The real AIMEs will be given in March and/or April

[2] Except for contest #5, although references to where these solutions can be found are given

Mock AIME #1

3:00-6:00 PM EST
July 31, 2004

1. DO NOT OPEN THIS BOOKLET UNTIL YOU ARE TOLD TO DO SO BY YOUR PROCTOR.

2. This is a 15-questions, 3-hour examination. All answers are integers ranging from 000 to 999, inclusive. Your score will be the number of correct answers; i.e., there is neither partial credit nor a penalty for wrong answers.

3. No aids other than scratch paper, ruler, compass, and protractor are permitted. In particular, CALCULATORS ARE NOT PERMITTED.

4. A combination of the AIME and the American Mathematics Contest 10 or 12 scores are used to determine eligibility for participation in the U.S.A. Mathematical Olympiad (USAMO).

5. Record all of your answers, and certain other information, on the AIME answer form. Only the answer form will be collected from you.

1 Mock AIME 1: Problems

1. Let S denote the sum of all of the three digit positive integers with three distinct digits. Compute the remainder when S is divided by 1000.

2. If $x^2 + y^2 - 30x - 40y + 24^2 = 0$, then the largest possible value of $\frac{y}{x}$ can be written as $\frac{m}{n}$, where m and n are relatively prime, positive integers. Determine $m + n$.

3. A, B, C, D, and E are collinear in that order such that $AB = BC = 1$, $CD = 2$, and $DE = 9$. If P can be any point in space, what is the minimum possible value of $AP^2 + BP^2 + CP^2 + DP^2 + EP^2$?

4. When $1 + 7 + 7^2 + \cdots + 7^{2004}$ is divided by 1000, a remainder of N is obtained. Determine the value of N.

5. Let a and b be the two real values of x for which
$$\sqrt[3]{x} + \sqrt[3]{20 - x} = 2$$
The smaller of the two values can be expressed as $p - \sqrt{q}$, where p and q are integers. Compute $p + q$.

6. A paperboy delivers newpapers to 10 houses along Main Street. Wishing to save effort, he doesn't always deliver to every house, but to avoid being fired he never misses three consecutive houses. Compute the number of ways the paperboy could deliver papers in this manner.

7. Let N denote the number of permutations of the 15-character string AAAABBBBBCCCCCC such that

None of the first four letters is an A. (1)

None of the next five letters is a B. (2)

None of the last six letters is a C. (3)

Find the remainder when N is divided by 1000.

8. $ABCD$, a rectangle with $AB = 12$ and $BC = 16$, is the base of pyramid $\mathcal{P}$, which has a height of 24. A plane parallel to $ABCD$ is passed through $\mathcal{P}$, dividing $\mathcal{P}$ into a frustum $\mathcal{F}$ and a smaller pyramid $\mathcal{P}'$. Let X denote the center of the circumsphere of $\mathcal{F}$, and let T denote the apex of $\mathcal{P}$. If the volume of $\mathcal{P}$ is eight times that of $\mathcal{P}'$, then the value of XT can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute the value of $m + n$.

9. p, q, and r are three non-zero integers such that $p + q + r = 26$ and
$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} + \frac{360}{pqr} = 1$$
Compute pqr.

10. $ABCDEFG$ is a regular heptagon inscribed in a unit circle centered at O. l is the line tangent to the circumcircle of $ABCDEFG$ at A, and P is a point on l such that $\triangle AOP$ is isosceles. Let p denote value of $AP \cdot BP \cdot CP \cdot DP \cdot EP \cdot FP \cdot GP$. Determine the value of p^2.

11. Let S denote the value of the sum

$$\sum_{n=0}^{668}(-1)^n\binom{2004}{3n}$$

Determine the remainder obtained when S is divided by 1000.

12. $ABCD$ is a rectangular sheet of paper. E and F are points on $\overline{AB}$ and $\overline{CD}$ respectively such that $BE < CF$. If $BCFE$ is folded over $\overline{EF}$, C maps to point C' on $\overline{AD}$ and B maps to B' such that $\angle AB'C' \cong \angle B'EA$. If $AB' = 5$ and $BE = 23$, then the area of $ABCD$ can be expressed as $a + b\sqrt{c}$ square units, where a, b, and c are integers and c is not divisible by the square of any prime. Compute $a + b + c$.

13. A sequence $\{R_n\}_{n\geq 0}$ obeys the recurrence $7R_n = 64 - 2R_{n-1} + 9R_{n-2}$ for any integers $n \geq 2$. Additionally, $R_0 = 10$ and $R_1 = -2$. Let

$$S = \sum_{i=0}^{\infty}\frac{R_i}{2^i}$$

S can be expressed as $\frac{m}{n}$ for two relatively prime positive integers m and n. Determine the value of $m + n$.

14. Wally's Key Company makes and sells two types of keys. Mr. Porter buys a total of 12 keys from Wally's. Determine the number of possible arrangements of Mr. Porter's 12 new keys on his keychain (Where rotations are considered the same and any two keys of the same type are identical.)

15. Triangle ABC has an inradius of 5 and a circumradius of 16. If $2\cos B = \cos A + \cos C$, then the area of triangle ABC can be expressed as $\frac{a\sqrt{b}}{c}$, where a, b, and c are positive integers such that a and c are relatively prime and b is not divisible by the square of any prime. Compute $a + b + c$.

2 Mock AIME 1: Answers

1. 680
2. 161
3. 110
4. 801
5. 118
6. 504
7. 320
8. 177
9. 576
10. 113
11. 006
12. 338
13. 443
14. 352
15. 141

3 Mock AIME 1: Solutions

1. Let S denote the sum of all of the three digit positive integers with three distinct digits. Compute the remainder when S is divided by 1000.

   Answer: **680**. Consider independently the sums from each digit in ABC. Each digit $\{1, 2, \ldots, 9\}$ appears as A exactly $9 \cdot 8 = 72$ times since exactly 9 and 8 distinct digits are available to be B and C respectively. Each of these digits also appears as B and C $8 \cdot 8 = 64$ times since there are eight choices for A and C respectively. We may ignore the sum of the 0's, and we have $S = (1+2+\cdots+9)(100)(72)+(1+\cdots+9)(10)(64)+(1+\cdots+9)(1)(64) = 45 \cdot 7904 = 355,680$. Since we are told to divide S by 1000, the answer is 680.

2. If $x^2 + y^2 - 30x - 40y + 24^2 = 0$, then the largest possible value of $\frac{y}{x}$ can be written as $\frac{m}{n}$, where m and n are relatively prime, positive integers. Determine $m + n$.

   Answer: **161**. Notice that the equation can be rewritten at $(x-15)^2+(y-20)^2 = 15^2+20^2-24^2 = 7^2$. It is clear that the possible (x, y) lie on a circle of radius 7 centered at $(15, 20)$. Consider $\frac{y}{x} = k$. This can be rewritten as $y = kx$. Thus, finding the maximum $\frac{y}{x}$ is equivalent to finding the line of maximum slope that passes through the origin and intersects the circle. This is the tangent to the circle that is nearer to the $+y$-axis. Let O denote the origin, P the center of the circle and T the point of tangency. By Pythagoras, $OP = 25$. OT has a length of $\sqrt{25^2 - 7^2} = 24$ since it is part of right triangle OTP. Let α be the angle formed by OP and the $+x$-axis and β the angle TOP. Then $\frac{m}{n}$ is

$$\tan(\alpha + \beta) = \frac{\tan(\alpha) + \tan(\beta)}{1 - \tan(\alpha)\tan(\beta)} = \frac{\frac{4}{3} + \frac{7}{24}}{1 - \frac{4}{3}\frac{7}{24}} = \frac{117}{44}$$

3. A, B, C, D, and E are collinear in that order such that $AB = BC = 1$, $CD = 2$, and $DE = 9$. If P can be any point in space, what is the minimum possible value of $AP^2 + BP^2 + CP^2 + DP^2 + EP^2$?

   Answer: **110**. Without loss of generality, A, B, C, D, and E are points on the x-axis. We can assume further that P is in the plane, and that $A = (0, 0), B = (1, 0), C = (2, 0), D = (4, 0)$, and $E = (13, 0)$. Let $P = (x, y)$. Then

$$\begin{aligned} AP^2 + BP^2 + CP^2 + DP^2 + EP^2 &= x^2 + (x-1)^2 + (x-2)^2 + (x-4)^2 + (x-13)^2 + 5y^2 \\ &= 5x^2 - 40x + 190 + 5y^2 \\ &= 5(x-4)^2 + 110 + 5y^2 \end{aligned}$$

   since squares are non-negative, we choose $x = 4$ and $y = 0$ to give the sum its minimum of 110.

4. When $1 + 7 + 7^2 + \cdots + 7^{2004}$ is divided by 1000, a remainder of N is obtained. Determine the value of N.

Answer: **801**. Note that $S = 7^0+7^1+\cdots+7^{2004} = (7^0+7^1+7^2+7^3)(7^0+7^4+\cdots+7^{2000})+7^{2004}$. But $1+7+49+343 = 400$, so that when we divide S by 1000 we care only about $\sum_{k=0}^{500} 7^{4k}$ modulo 10 and the extra term 7^{2004}. Since the sum contains $7^{4k} = (2401)^k \equiv 1 \pmod{10}$ for 501 values of k, $(7^0+7^1+7^2+7^3)(7^0+7^4+\cdots+7^{2000}) \equiv 400 \cdot 501 \equiv 400 \pmod{1000}$. To determine $7^{2004} \pmod{1000}$, we note that $\phi(1000) = 400$ so that $7^{2004} \equiv 7^4 \equiv 401 \pmod{1000}$. Adding the two yields $S \equiv 801 \pmod{1000}$.

5. Let a and b be the two real values of x for which

$$\sqrt[3]{x} + \sqrt[3]{20-x} = 2$$

The smaller of the two values can be expressed as $p-\sqrt{q}$, where p and q are integers. Compute $p+q$.

Answer: **118**. Let $a = \sqrt[3]{x}$ and $b = \sqrt[3]{20-x}$. We have $a+b = 2$ and $a^3+b^3 = 20 = (a+b)^3 - 3(a+b)(ab) = 8 - 6ab$. We find that $ab = -2 = a(2-a)$. We solve this for $a = 1 \pm \sqrt{3} = \sqrt[3]{x}$. Cubing both sides, we have $x = 10 \pm \sqrt{108}$. Hence, the answer is $10+108 = 118$.

ALTERNATE SOLUTION

Cube the given equation, and substitute the given recursively:

$$\begin{aligned} \left(\sqrt[3]{x} + \sqrt[3]{20-x}\right)^3 &= 20 + 3\left(\sqrt[3]{x} + \sqrt[3]{20-x}\right)\sqrt[3]{x(20-x)} = 8 \\ 20 + 6\sqrt[3]{x(20-x)} &= 8 \\ x(20-x) &= -8 \end{aligned}$$

This is a quadratic and can easily be solved for $x = 10 \pm \sqrt{108}$, which gives the answer.

6. A paperboy delivers newpapers to 10 houses along Main Street. Wishing to save effort, he doesn't always deliver to every house, but to avoid being fired he never misses three consecutive houses. Compute the number of ways the paperboy could deliver papers in this manner.

Answer: **504**. Let a_n be the number of ways the paperboy could deliver papers to n houses. We want to find a_{10}. We work out the small cases $a_1 = 2$, $a_2 = 4$, and $a_3 = 7$. Now consider the case $n \geq 4$. Either the paperboy delivers to the first house, after which there are a_{n-1} possible routes, or he skips the first house. If he skips the first house he may deliver to the second house, after which there are a_{n-2} routes, or he may skip the second house. If he skips the first and second houses, he must deliver to the third house, which leaves a_{n-3} possible routes. Hence, $a_n = a_{n-1} + a_{n-2} + a_{n-3}$. Now we have

$$\begin{aligned} a_4 &= a_3 + a_2 + a_1 = 7+4+2 = 13 \\ a_5 &= a_4 + a_3 + a_2 = 13+7+4 = 24 \\ a_6 &= a_5 + a_4 + a_3 = 24+13+7 = 44 \\ a_7 &= a_6 + a_5 + a_4 = 44+24+13 = 81 \end{aligned}$$

$$
\begin{aligned}
a_8 &= a_7 + a_6 + a_5 = 81 + 44 + 24 = 149 \\
a_9 &= a_8 + a_7 + a_6 = 149 + 81 + 44 = 274 \\
a_{10} &= a_9 + a_8 + a_7 = 274 + 149 + 81 = 504
\end{aligned}
$$

7. Let N denote the number of permutations of the 15-character string AAAABBBBBCCCCCC such that

$$\text{None of the first four letters is an A.} \tag{4}$$
$$\text{None of the next five letters is a B.} \tag{5}$$
$$\text{None of the last six letters is a C.} \tag{6}$$

Find the remainder when N is divided by 1000.

Answer: **320**. Suppose that k of the C's become A's and $4-k$ of the B's become A's. Then $6-k$ of the C's become B's and $1+k$ of the B's become C's. We have $k-1$ B's and $5-k$ C's to replace the 4 A's. This can be accomplished in

$$
\begin{aligned}
\sum_{k=0}^{4} \binom{6}{k}\binom{5}{4-k}\binom{4}{k-1} &= 1\cdot 5\cdot 0 + 6\cdot 10\cdot 1 + 15\cdot 10\cdot 4 + 20\cdot 5\cdot 6 + 15\cdot 1\cdot 4 \\
&= 0 + 60 + 600 + 600 + 60 = 1320
\end{aligned}
$$

ways. Dividing through by 1000 leaves a remainder of 320.

8. $ABCD$, a rectangle with $AB = 12$ and $BC = 16$, is the base of pyramid $\mathcal{P}$, which has a height of 24. A plane parallel to $ABCD$ is passed through $\mathcal{P}$, dividing $\mathcal{P}$ into a frustum $\mathcal{F}$ and a smaller pyramid $\mathcal{P}'$. Let X denote the center of the circumsphere of $\mathcal{F}$, and let T denote the apex of $\mathcal{P}$. If the volume of $\mathcal{P}$ is eight times that of $\mathcal{P}'$, then the value of XT can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute the value of $m+n$.

Answer: **177**. $\mathcal{P}$ and $\mathcal{P}'$ are similar; since the volume of the former is 8 times that of the latter, if follows that the plane passes through $\mathcal{P}$ halfway up the pyramid $\mathcal{P}$. Let Z be the apex of $\mathcal{P}$, and A', B', C', and D' the midpoints of $A'Z, B'Z, C'Z$, and $D'Z$ respectively. $A'B'C'D'$, the rectangular intersection of the plane and $\mathcal{P}$, has $A'B' = C'D' = 6$ and $B'C' = D'A' = 8$. Let O and O' denote the centers of $ABCD$ and $A'B'C'D'$ respectively. Since the height of $\mathcal{P}$ is 24, $OO' = 12$. By symmetry, the circumsphere of the frustum $\mathcal{F}$ is centered on OO'. Since for any point X on OO', we have $AX = BX = CX = DX$ and $A'X = B'X = C'X = D'X$, we need only find the point X such that $AX = A'X$. Suppose that $OX = x$ and $XO' = 12 - x$. By the Pythagorean theorem in 3-space, we have

$$
\begin{aligned}
AX = A'X \iff 6^2 + 8^2 + x^2 &= 3^2 + 4^2 + (12-x)^2 \\
100 + x^2 &= 25 + 144 - 24x + x^2 \\
x = \frac{69}{24}
\end{aligned}
$$

Then $XT = 24 - \frac{69}{24} = \frac{507}{24} = \frac{169}{8}$, so the answer is $169 + 8 = 177$.

9. p, q, and r are three non-zero integers such that $p+q+r=26$ and

$$\frac{1}{p}+\frac{1}{q}+\frac{1}{r}+\frac{360}{pqr}=1$$

Compute pqr.

Answer: **576**. Consider the following algebra:

$$\begin{aligned}\frac{1}{p}+\frac{1}{q}+\frac{1}{r}+\frac{360}{pqr} &= 1\\ pq+qr+rp+360 &= pqr\\ 359 &= pqr-(pq+qr+rp)+((p+q+r)-26)-1\\ 385 &= (p-1)(q-1)(r-1)\end{aligned}$$

Now consider the factorization $385=5\cdot 7\cdot 11$. Each term in the product $(p-1)(q-1)(r-1)$ has to divide 385. If any of $p-1$, $q-1$, $r-1$ contains two of the prime factors, then the sum $p+q+r$ cannot be 26 unless (WLOG) $p-1=35, q-1=-11, r-1=-1$, but this is invalid since $r=0$. Hence, $p-1=\pm 5, q-1=\pm 7, r-1=\pm 11 \Longrightarrow p+q+r=3\pm 5\pm 7\pm 11=26$. By inspection, the only possibility is $p=6, q=8, r=12$, which gives $pqr=576$.

10. $ABCDEFG$ is a regular heptagon inscribed in a unit circle centered at O. l is the line tangent to the circumcircle of $ABCDEFG$ at A, and P is a point on l such that $\triangle AOP$ is isosceles. Let p denote value of $AP\cdot BP\cdot CP\cdot DP\cdot EP\cdot FP\cdot GP$. Determine the value of p^2.

Answer: **113**. Overlay the complex number system with $P=0+0i, A=1+0i$, and $O=1+i$. The solutions to the equation $z^7=i$ are seven points equally spaced around the unit circle centered at P. To translate these points to the heptagon $ABCDEFG$, we replace z with $(z-(1+i))$, obtaining $(z-(1+i))^7=z^7-\cdots+8(-1+i)=i$. The product we are interested is the magnitude of the product of the roots of this equation. Since this is a monic polynomial in z, the product of the solutions z_i is $8-7i$. Hence, we have $AP\cdot BP\cdot CP\cdot DP\cdot EP\cdot FP\cdot GP=|z_1z_2z_3z_4z_5z_6z_7|=|8-7i|=\sqrt{113}$. It follows that the answer is 113.

11. Let S denote the value of the sum

$$\sum_{n=0}^{668}(-1)^n\binom{2004}{3n}$$

Determine the remainder obtained when S is divided by 1000.

Answer: **006**. Consider the polynomial $f(x)$ defined by

$$f(x)=(x-1)^{2004}=\sum_{n=0}^{2004}\binom{2004}{n}\cdot(-1)^n x^{2004-n}$$

Let $\omega^3 = 1$ with $\omega \neq 1$. We have

$$\begin{aligned}\frac{f(1)+f(\omega)+f(\omega^2)}{3} &= \frac{(1-1)^{2004}+(\omega-1)^{2004}+(\omega^2-1)^{2004}}{3} \\ &= \frac{1}{3}\sum_{n=0}^{2004}\binom{2004}{n}\cdot(-1)^n\cdot\left(1^{2004-n}+\omega^{2004-n}+(\omega^2)^{2004-n}\right) \\ &= \sum_{n=0}^{668}(-1)^n\binom{2004}{3n}\end{aligned}$$

where the last step follows in part from the fact that the only integers n for which $1^n+\omega^n+\omega^{2n}$ is non-zero are multiples of three, where the expression is always equal to 3. WLOG, $\omega - 1 = \sqrt{3}\cdot\frac{-\sqrt{3}+i}{2}$ and $\omega^2 - 1 = \sqrt{3}\cdot\frac{-\sqrt{3}-i}{2}$. Both expressions, when raised to the 2004-th power, become 3^{1002}, as their complex factors are two of the 12-th roots of unity and $2004 = 12\cdot 167$. Hence,

$$S = \sum_{n=0}^{668}(-1)^n\binom{2004}{3n} = 2\cdot 3^{1001}$$

In finding $2\cdot 3^{1001} \pmod{1000}$, we note that $3^{\phi(500)} \equiv 3^{200} \equiv 1 \pmod{500}$ so that $3^{1001} \equiv 3 \pmod{500}$. Hence, we may write $2\cdot 3^{1001} = 2\cdot(3+500k) \equiv 6 \pmod{1000}$ for some integer k. It follows that the answer is 6.

12. $ABCD$ is a rectangular sheet of paper. E and F are points on $\overline{AB}$ and $\overline{CD}$ respectively such that $BE < CF$. If $BCFE$ is folded over $\overline{EF}$, C maps to point C' on $\overline{AD}$ and B maps to B' such that $\angle AB'C' \cong \angle B'EA$. If $AB' = 5$ and $BE = 23$, then the area of $ABCD$ can be expressed as $a + b\sqrt{c}$ square units, where a, b, and c are integers and c is not divisible by the square of any prime. Compute $a + b + c$.

Answer: **338**. By the reflection, we have $B'E = BE = 23$. Because $ABCD$ is a rectangle, we have $m\angle C'AE = m\angle C'B'E = \frac{\pi}{2} \implies C'AB'E$ is cyclic with diameter $C'E \implies \angle B'C'A \cong \angle B'EA \cong \angle AB'C' \implies \triangle AB'C'$ is isosceles with $AB' = AC' = 5$. It would suffice to determine $C'E$ as this would eventually yield both sides of $ABCD$.

Let ω denote the circumcircle of $AB'EC'$. Consider the point P on the minor arc $B'E$ of ω such that $AP = 23$ and $PE = 5$. $APEC'$ is an isosceles trapezoid with $m\angle C'AE = m\angle C'PE = \frac{\pi}{2}$. Let $C'E = x$. Then by Pythagoras, $C'B' = AE = \sqrt{x^2-25}$, but by Ptolemy's Theorem applied to this trapezoid,

$$23x + 25 = x^2 - 25$$

from which we find $x = 25$ or -2. Taking $C'E = x = 25$, we obtain $AE = \sqrt{625-25} = 10\sqrt{6}$ and $C'B' = \sqrt{25^2 - 23^2} = 4\sqrt{6}$.

Now we have $AB = AE + EB = 10\sqrt{6} + 23$ and $C'B' = BC = 4\sqrt{6}$ so that the area of $ABCD$ is $240 + 92\sqrt{6}$, which yields an answer of $240 + 92 + 6 = 338$.

13. A sequence $\{R_n\}_{n\geq 0}$ obeys the recurrence $7R_n = 64 - 2R_{n-1} + 9R_{n-2}$ for any integers $n \geq 2$. Additionally, $R_0 = 10$ and $R_1 = -2$. Let

$$S = \sum_{i=0}^{\infty} \frac{R_i}{2^i}$$

S can be expressed as $\frac{m}{n}$ for two relatively prime positive integers m and n. Determine the value of $m + n$.

Answer: **443**. We have $R_0 = 10, R_1 = -2$ and $R_2 = \frac{1}{7} \cdot (64 - 2(-2) + 9(10)) = \frac{158}{7}$. We solve for 64 in terms of the sequence, obtaining

$$64 = 7R_{n+3} + 2R_{n+2} - 9R_{n+1} = 7R_{n+2} + 2R_{n+1} - 9R_n$$

for all integers $n \geq 0$. The characteristic equation of $\{R_n\}_{n\geq 0}$ is now seen to be $7x^3+2x^2-9x = 7x^2 + 2x - 9$ or $7x^3 - 5x^2 - 11x + 9 = 0$. The rational root test can be applied to facilitate guess and check, which produces the factorization $(x-1)^2(7x+9) = 0$. This implies that we have $R_n = a \cdot n + b + c \cdot \left(\frac{-9}{7}\right)^n$. We solve for a, b, c by checking the first three terms:

$$\begin{aligned} R_0 &= b + c = 10 \\ R_1 &= a + b - \frac{9c}{7} = -2 \\ R_2 &= 2a + b + \frac{81c}{49} = \frac{158}{7} \end{aligned}$$

This can be accomplished by algebra, but we should always check for simple solutions. Intuitively, it seems that c should be a multiple of 7. Plugging in $c = 7$ makes it easy to find the unique solution $(a, b, c) = (4, 3, 7)$. Hence, we are asked to compute the sum

$$\begin{aligned} S = \sum_{i=0}^{\infty} \frac{R_i}{2^i} &= \sum_{n=0}^{\infty} \frac{4n + 3 + 7 \cdot \left(\frac{-9}{7}\right)^n}{2^n} \\ &= 4\sum_{n=0}^{\infty} \frac{n}{2^n} + 3\sum_{n=0}^{\infty} \frac{1}{2^n} + 7\sum_{n=0}^{\infty} \left(\frac{-9}{14}\right)^n \end{aligned}$$

By the formula for the sum of an infinite geometric sequence, we have $\sum_{n=0}^{\infty} \frac{1}{2^n} = 2$ and $\sum_{n=0}^{\infty} \left(\frac{-9}{14}\right)^n = \frac{14}{23}$. Let $T = \sum_{n=0}^{\infty} \frac{n}{2^n}$. We telescope T with itself, finding

$$\begin{aligned} T &= 2T - T \\ &= \left(\frac{1}{1} + \frac{2}{2} + \frac{3}{4} + \frac{4}{8} + \cdots\right) - \left(\frac{0}{1} + \frac{1}{2} + \frac{2}{3} + \frac{3}{8} + \cdots\right) \\ &= \frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots \\ &= 2 \end{aligned}$$

so that $S = 8 + 6 + \frac{98}{23} = \frac{420}{23}$, which gives an answer of $420 + 23 = 443$.

ALTERNATE SOLUTION (Due to Yoni Levy)

Divide the given by 2^n, obtaining $7\frac{R_n}{2^n} = \frac{64}{2^n} - \frac{R_{n-1}}{2^{n-1}} + \frac{9}{4}\frac{R_{n-2}}{2^{n-2}}$. Let us sum this equation from 2 to ∞. That is,

$$\begin{aligned}
\sum_{n=2}^{\infty} 7\frac{R_n}{2^n} &= \frac{64}{2^n} - \frac{R_{n-1}}{2^{n-1}} + \frac{9}{4}\frac{R_{n-2}}{2^{n-2}} \\
7\sum_{n=2}^{\infty} \frac{R_n}{2^n} &= 64\sum_{n=2}^{\infty} - \sum_{n=1}^{\infty} \frac{R_n}{2^n} + \frac{9}{4}\sum_{n=0}^{n} \frac{R_n}{2^n} \\
7(S-9) &= 32 - (S-10) + \frac{9}{4}S
\end{aligned}$$

This equation can readily be solved for $S = \frac{420}{23}$, which leads to the correct answer.

ALTERNATE SOLUTION (Due to Daniel J. Hermes / Pork_Chop8)

Note that we may write

$$\begin{aligned}
7R_{n+2} + 9R_{n+1} &= 64 + 7R_{n+1} + 9R_n \\
&= 64 + (64 + 7R_n + 9R_{n-1}) \\
&\cdots \\
&= 64 \cdot (n+1) + 7R_1 + 9R_0 = 64n + 140
\end{aligned}$$

Therefore,

$$\begin{aligned}
23S = 23\sum_{i=0}^{\infty} \frac{R_i}{2^i} &= 2 \cdot 7\sum_{i=0}^{\infty} \frac{R_i}{2^i} + 9\sum_{i=0}^{\infty} \frac{R_i}{2^i} \\
&= 2 \cdot 7\left(\left(\sum_{i=2}^{\infty} \frac{R_i}{2^i}\right) + \frac{R_1}{2^1} + \frac{R_0}{2^0}\right) + 9\left(\left(\sum_{i=1}^{\infty} \frac{R_i}{2^i}\right) + \frac{R_0}{2^0}\right) \\
&= 2 \cdot 7\sum_{i=0}^{\infty} \frac{R_{i+2}}{2^{i+2}} + 126 + 9\sum_{i=0}^{\infty} \frac{R_{i+1}}{2^{i+1}} + 90 \\
&= 7\sum_{i=0}^{\infty} \frac{R_{i+2}}{2^{i+1}} + 9\sum_{i=0}^{\infty} \frac{R_{i+1}}{2^{i+1}} + 216 = \sum_{i=0}^{\infty} \frac{7R_{i+2} + 9R_{i+1}}{2^{i+1}} + 216 \\
&= \sum_{i=0}^{\infty} \frac{64i + 140}{2^{i+1}} + 216 \\
&= 32\sum_{i=0}^{\infty} \frac{i}{2^i} + 70\sum_{i=0}^{\infty} \frac{1}{2^i} + 216 = 32 \cdot 2 + 70 \cdot 2 + 216 = 420
\end{aligned}$$

From which it follows that $S = \frac{420}{23}$ and the answer is $420 + 23 = 443$.

14. Wally's Key Company makes and sells two types of keys. Mr. Porter buys a total of 12 keys from Wally's. Determine the number of possible arrangements of Mr. Porter's 12 new keys on his keychain (Where rotations are considered the same and any two keys of the same type are identical.)

Answer: **352**. Suppose that the two types of keys are A and B. Let the 12 character string $X = X_1X_2X_3\ldots X_{12}$ represent a generic keychain. Define $R_i(X_1\ldots X_{12}) = X_{i+1}\ldots X_{12}X_1\ldots X_i$ for $i \in \mathbb{Z}_{12}$ to represent a rotation of i keys. We will argue that the answer should be given by

$$\frac{\sum_{i=0}^{11} c_i}{12}$$

where c_i is the number of strings that remain fixed under R_i. Suppose the string X repeats every n (where n is minimal) characters. Obviously $n|12$ or else the string repeats every $\gcd(n, 12) < n$ characters. We want to show that

$$\sum_{i=0}^{11} c_i$$

counts X exactly $\frac{12}{n}$ times, since there are exactly this many rotations R_i that fix X. This is the case, however, since R_{kn} fixes X for integers k, and exactly $\frac{12}{n}$ such k exist for which $kn \in \mathbb{Z}_{12}$. Based on our definition of c_i, we know that the string X is counted in each c_{kn} and only these c_i. But c_i is the number of strings that are fixed under R_i, which means that these strings must repeat every i, which implies that they *must* repeat every $\gcd(i, 12)$. Since each such string is a block of $\gcd(i, 12)$ characters copied $\frac{12}{\gcd(i,12)}$ times, there are exactly $c_i = 2^{\gcd(i,12)}$ such strings.

Thus the summation $\sum_{i=0}^{11} c_i$ counts every string 12 times and there are

$$\frac{2^{12} + 2^1 + 2^2 + 2^3 + 2^4 + 2^1 + 2^6 + 2^1 + 2^4 + 2^3 + 2^2 + 2^1}{12} = \frac{4224}{12} = 352$$

different keychains. This is the basic idea behind the Pólya-Redfield method of counting distinct, rotationally-independent strings.

ALTERNATE SOLUTION

Let the types of keys be A and B. Consider the keychain X, represented by a string of 12 A's and B's. We will count the number of distinct keychains with $0, 1, 2, \ldots, 11$, and 12 type B keys. Since there exists a bijection (A's to B's and B's to A's) between the case with n B's and $12 - n$ B's, we need only consider 0, 1, 2, 3, 4, 5, and 6 type B's.

Let S_n denote the number of 12-character strings with n B's and $12 - n$ A's. Let $A_{n,m} \subseteq S_n$ denote the subset of S_n that contains strings fixed under rotation[3] by m (not necessarily minimal) characters. We will call a rotation that leaves a string unchanged a *fixing rotation.*

[3] By "rotation" we mean taking the string $X_1X_2\ldots X_{12}$ to $X_{m+1}\ldots X_{12}X_1\ldots X_m$.

Obviously, there is one string with 0 B's, and there is one string with a single B that has 12 rotational positions. The case with 2 B's is a question of how far apart the B's are, which has 6 possibilities. The case with 5 B's has a rotation iff all of the string is all B's, which is a contradiction as there are only 5 B's. Hence, any string with 5 B's has 12 non-identical rotational positions, and it follows that there are $\frac{C(12,5)}{12} = 66$ such rotationally independent strings.

In the case with 3 B's, the possible fixing rotations are the indentity rotation and a rotation of 4 characters. There are $|S_3| - |A_{3,4}| = C(12,3) - C(4,1) = 220 - 4 = 216$ strings fixed only under the identity rotation and $|A_{3,4}| = C(4,1) = 4$ strings with 3 B's fixed only under a rotation of 4 characters. Hence, there are $\frac{216}{12} + \frac{4}{4} = 18 + 1 = 19$ rotationally independent strings with 3 B's.

In the case with 4 B's, the possible fixing rotations are the indentity rotation and rotations by 3 and 6 characters. There are $|S_4| - |A_{4,6}| = C(12,4) - C(6,2) = 495 - 15 = 480$ strings fixed only under the identity rotation, $|A_{4,6}| - |A_{4,3}| = C(6,2) - C(3,1) = 15 - 3 = 12$ strings fixed only under a rotation by 6 characters, and $|A_{4,3}| = C(3,1) = 3$ strings fixed only under rotation by 3 characters. Hence, there are $\frac{480}{12} + \frac{12}{6} + \frac{3}{3} = 40 + 2 + 1 = 43$ rotationally independent strings with 4 B's.

For the case with 6 B's, the possible fixing rotations are the identity rotation and rotations of 2, 4, and 6. A rotation of 3 characters cannot hold a string fixed since this would require that there were a multiple of 4 B's, a contradiction. The number of strings that are fixed only under the identity rotation is given by $|S_6| - |A_{6,6}| - |A_{6,4}| + |A_{6,2}| = C(12,6) - C(6,3) - C(4,2) + C(2,1) = 924 - 20 - 6 + 2 = 900$. The number of strings fixed under 6-character rotation is given by $|A_{6,6}| - |A_{6,2}| = C(6,3) - C(2,1) = 20 - 2 = 18$, and the numbers of strings fixed under rotations of 4 and 2 characters are given by $C(4,2) - C(2,1) = 6 - 2 = 4$ and $C(2,1) = 2$ respectively. Hence, $|S_6| = \frac{900}{12} + \frac{18}{6} + \frac{4}{4} + \frac{2}{2} = 75 + 3 + 1 + 1 = 80$.

Therefore, there are $2 \cdot (1 + 1 + 6 + 19 + 43 + 66) + 80 = 352$ distinct possible keychains.

15. Triangle ABC has an inradius of 5 and a circumradius of 16. If $2\cos B = \cos A + \cos C$, then the area of triangle ABC can be expressed as $\frac{a\sqrt{b}}{c}$, where a, b, and c are positive integers such that a and c are relatively prime and b is not divisible by the square of any prime. Compute $a + b + c$.

Answer: **141**. It follows from $2\cos B = \cos A + \cos C$ that $\cos A, \cos B, \cos C$ is an arithmetic progression. It also follows that

$$3\cos B = \cos A + \cos B + \cos C = 1 + \frac{r}{R} = \frac{21}{16}$$

so we may set $\cos A = \frac{7}{16} + k, \cos B = \frac{7}{16}, \cos C = \frac{7}{16} - k$. We substitute these into another famous trig identity,

$$\begin{aligned} \cos^2 A + \cos^2 B + \cos^2 C + 2\cos A \cos B \cos C &= 1 \\ 3 \cdot \left(\frac{7}{16}\right)^2 + 2k^2 + 2 \cdot \frac{7}{16}\left(\left(\frac{7}{16}\right)^2 - k^2\right) &= 1 \\ \frac{18}{16} \cdot 16^3 \cdot k^2 + 7^2(48 + 14) &= 16^3 \end{aligned}$$

$$\begin{aligned} 2 \cdot 3^2 \cdot 16^2 \cdot k^2 = 1058 &= 2 \cdot 23^2 \\ k &= \pm\frac{23}{48} \end{aligned}$$

So we have $\cos A = \frac{11}{12}, \cos B = \frac{7}{16}$, and $\cos C = \frac{-1}{24}$, which imply $\sin A = \frac{\sqrt{23}}{12}, \sin B = \frac{3}{16}\sqrt{23}$, and $\sin C = \frac{5}{24}\sqrt{23}$ respectively. Finally,

$$[ABC] = 2R^2 \sin A \sin B \sin C = 2 \cdot 16^2 \cdot \left(\frac{\sqrt{23}}{12}\right)\left(\frac{3}{16}\sqrt{23}\right)\left(\frac{5}{24}\sqrt{23}\right) = \frac{115\sqrt{23}}{3}$$

which gives an answer of $115 + 23 + 3 = 141$.

Mock AIME #2

2:00-5:00 PM EST
August 8, 2004

1. DO NOT OPEN THIS BOOKLET UNTIL YOU ARE TOLD TO DO SO BY YOUR PROCTOR.

2. This is a 15-questions, 3-hour examination. All answers are integers ranging from 000 to 999, inclusive. Your score will be the number of correct answers; i.e., there is neither partial credit nor a penalty for wrong answers.

3. No aids other than scratch paper, ruler, compass, and protractor are permitted. In particular, CALCULATORS ARE NOT PERMITTED.

4. A combination of the AIME and the American Mathematics Contest 10 or 12 scores are used to determine eligibility for participation in the U.S.A. Mathematical Olympiad (USAMO).

5. Record all of your answers, and certain other information, on the AIME answer form. Only the answer form will be collected from you.

4 Mock AIME 2: Problems

1. Compute the largest integer k such that 2004^k divides $2004!$.

2. x is a real number with the property that $x + \frac{1}{x} = 3$. Let $S_m = x^m + \frac{1}{x^m}$. Determine the value of S_7.

3. In a box, there are 4 green balls, 4 blue balls, 2 red balls, a brown ball, a white ball, and a black ball. These balls are randomly drawn out of the box one at a time (without replacement) until two of the same color have been removed. This process requires that at most 7 balls be removed. The probability that 7 balls are drawn can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.

4. Let $S := \{5^k | k \in \mathbb{Z}, 0 \leq k \leq 2004\}$. Given that $5^{2004} = 5443 \cdots 0625$ has 1401 digits, how many elements of S begin with the digit 1?

5. Let S be the set of integers $n > 1$ for which $\frac{1}{n} = 0.d_1d_2d_3d_4\ldots$, an infinite decimal that has the property that $d_i = d_{i+12}$ for all positive integers i. Given that 9901 is prime, how many positive integers are in S? (The d_i are digits.)

6. ABC is a scalene triangle. Points D, E, and F are selected on sides BC, CA, and AB respectively. The cevians AD, BE, and CF concur at point P. If $[AFP] = 126, [FBP] = 63$, and $[CEP] = 24$, determine the area of triangle ABC.

7. Anders, Po-Ru, Reid, and Aaron are playing Bridge. After one hand, they notice that all of the cards of two suits are split between Reid and Po-Ru's hands. Let N denote the number of ways 13 cards can be dealt to each player such that this is the case. Determine the remainder obtained when N is divided by 1000. (Bridge is a card game played with the standard 52-card deck.)

8. Determine the remainder obtained when the expression

$$2004^{2003^{2002^{2001}}}$$

is divided by 1000.

9. Let

$$\left(1 + x^3\right)\left(1 + 2x^{3^2}\right)\cdots\left(1 + kx^{3^k}\right)\cdots\left(1 + 1997x^{3^{1997}}\right) = 1 + a_1x^{k_1} + a_2x^{k_2} + \cdots + a_mx^{k_m}$$

where $a_i \neq 0$ and $k_1 < k_2 < \cdots < k_m$. Determine the remainder obtained when a_{1997} is divided by 1000.

10. $ABCDE$ is a cyclic pentagon with $BC = CD = DE$. The diagonals AC and BE intersect at M. N is the foot of the altitude from M to AB. We have $MA = 25, MD = 113$, and $MN = 15$. The area of triangle ABE can be expressed as $\frac{m}{n}$ where m and n are relatively prime positive integers. Determine the remainder obtained when $m + n$ is divided by 1000.

11. α, β, and γ are the roots of $x(x-200)(4x+1) = 1$. Let

$$\omega = \tan^{-1}(\alpha) + \tan^{-1}(\beta) + \tan^{-1}(\gamma)$$

The value of $\tan(\omega)$ can be written as $\frac{m}{n}$ where m and n are relatively prime positive integers. Determine the value of $m+n$.

12. $ABCD$ is a cyclic quadrilateral with $AB = 8, BC = 4, CD = 1$, and $DA = 7$. Let O and P denote the circumcenter and intersection of AC and BD respectively. The value of OP^2 can be expressed as $\frac{m}{n}$, where m and n are relatively prime, positive integers. Determine the remainder obtained when $m+n$ is divided by 1000.

13. $P(x)$ is the polynomial of minimal degree that satisfies

$$P(k) = \frac{1}{k(k+1)}$$

for $k = 1, 2, 3, \ldots, 10$. The value of $P(11)$ can be written as $-\frac{m}{n}$, where m and n are relatively prime positive integers. Determine $m+n$.

14. 3 Elm trees, 4 Dogwood trees, and 5 Oak trees are to be planted in a line in front of a library such that

i) No two Elm trees are next to each other.
ii) No Dogwood tree is adjacent to an Oak tree.
iii) All of the trees are planted.

How many ways can the trees be situated in this manner?

15. In triangle ABC, we have $BC = 13, CA = 37$, and $AB = 40$. Points D, E, and F are selected on BC, CA, and AB respectively such that AD, BE, and CF concur at the circumcenter of ABC. The value of

$$\frac{1}{AD} + \frac{1}{BE} + \frac{1}{CF}$$

can be expressed as $\frac{m}{n}$ where m and n are relatively prime positive integers. Determine $m+n$.

5 Mock AIME 2: Answers

1. 012
2. 843
3. 437
4. 604
5. 255
6. 351
7. 000[4]
8. 704
9. 280
10. 727
11. 167
12. 589
13. 071
14. 152
15. 529

[4] 400 was also accepted, due to allegedly ambiguous wording.

6 Mock AIME 2: Solutions

1. Compute the largest integer k such that 2004^k divides $2004!$.

Answer: **012**. The number of 2's in the prime factorization of 2004! is $\left\lfloor \frac{2004}{2^1} \right\rfloor + \left\lfloor \frac{2004}{2^2} \right\rfloor + \left\lfloor \frac{2004}{2^3} \right\rfloor + \cdots = 1002 + 501 + 250 + \cdots > 1000$. There are 2 2's in the prime factorization of 2004; hence $(2^2)^k | 2004!$ for all integers $k \leq 250$. Similarly, $3^k | 2004!$ for all integers $k \leq 200$, but there are only 12 167's in the prime factorization of 2004!. Hence, the answer is 12.

2. x is a real number with the property that $x + \frac{1}{x} = 3$. Let $S_m = x^m + \frac{1}{x^m}$. Determine the value of S_7.

Answer: **843**. Notice that $\left(x^m + \frac{1}{x^m}\right)\left(x + \frac{1}{x}\right) = x^{m+1} + \frac{1}{x^{m+1}} + x^{m-1} + \frac{1}{x^{m-1}}$ so that $S_m S_1 = S_{m+1} + S_{m-1}$ or, equivalently, $S_{m+1} = S_m S_1 - S_{m-1}$. Therefore,

$$\begin{aligned}
S_2 &= S_1S_1 - S_0 = 3 \cdot 3 - 2 = 7 \\
S_3 &= S_2S_1 - S_1 = 7 \cdot 3 - 3 = 18 \\
S_4 &= S_3S_1 - S_2 = 18 \cdot 3 - 7 = 47 \\
S_5 &= S_4S_1 - S_3 = 47 \cdot 3 - 18 = 123 \\
S_6 &= S_5S_1 - S_4 = 123 \cdot 3 - 47 = 322 \\
S_7 &= S_6S_1 - S_5 = 322 \cdot 3 - 123 = 843
\end{aligned}$$

ALTERNATE SOLUTION

Solve for $x = \frac{3 \pm \sqrt{5}}{2}$. These two values are reciprocals; WLOG we take $x = \frac{3+\sqrt{5}}{2}$ so that

$$\begin{aligned}
S_7 = \left(\frac{3+\sqrt{5}}{2}\right)^7 + \left(\frac{3-\sqrt{5}}{2}\right)^7 &= \frac{(3+\sqrt{5})^7 + (3-\sqrt{5})^7}{2^7} \\
&= \frac{3^7 + \binom{7}{2} \cdot 3^5 \cdot 5 + \binom{7}{4} \cdot 3^3 \cdot 5^2 + \binom{7}{6} \cdot 3^1 \cdot 5^3}{2^6} \\
&= \frac{2187 + 21 \cdot 243 \cdot 5 + 35 \cdot 27 \cdot 25 + 7 \cdot 3 \cdot 125}{64} \\
&= \frac{2187 + 25515 + 23625 + 2625}{64} \\
&= \frac{53952}{64} = 843
\end{aligned}$$

3. In a box, there are 4 green balls, 4 blue balls, 2 red balls, a brown ball, a white ball, and a black ball. These balls are randomly drawn out of the box one at a time (without replacement) until two of the same color have been removed. This process requires that at most 7 balls be removed. The probability that 7 balls are drawn can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.

Answer: **437**. Note that the probability we want is equivalent to the probability that among 6 balls drawn out simultaneously, no two have the same color. This can be accomplished only by choosing exactly one of each color, which leaves $4 \cdot 4 \cdot 2 \cdot 1 \cdot 1 \cdot 1$ possibilities out of $\binom{13}{6}$ total possibilities. Hence, the desired probability is

$$\frac{4 \cdot 4 \cdot 2 \cdot 1 \cdot 1 \cdot 1}{\binom{13}{6}} = \frac{8}{429}$$

therefore, the answer is $8 + 429 = 437$.

4. Let $S := \{5^k | k \in \mathbb{Z}, 0 \leq k \leq 2004\}$. Given that $5^{2004} = 5443 \cdots 0625$ has 1401 digits, how many elements of S begin with the digit 1?

Answer: **604**. We note that if 5^k has n digits and begins with 1, then 5^{k+1} has n digits and does not start with 1. If 5^k does not start with 1, then 5^{k+1} has $n+1$ digits. If every power of 5 starting at 5^1 started with a digit other than 1, then 5^k would have k digits. Since 5^{2004} has 1401 digits, we reason that 603 powers of 5 between 5^1 and 5^{2003} begin with 1. 5^{2004} begins with a 5, but we add in 1 for 5^0 and obtain the answer.

5. Let S be the set of integers $n > 1$ for which $\frac{1}{n} = 0.d_1 d_2 d_3 d_4 \ldots$, an infinite decimal that has the property that $d_i = d_{i+12}$ for all positive integers i. Given that 9901 is prime, how many positive integers are in S? (The d_i are digits.)

Answer: **255**. If a number has a 12-digit repeating decimal, its fractional part can be expressed as $\frac{abcdefghijkl}{999999999999}$, where each of $a - l$ is a digit 0 - 9. If this is to reduce to $\frac{1}{n}$, then $abcdefghijkl$ must divide $999999999999 = 10^{12} - 1$. Hence, the possible n are $\frac{10^{12}-1}{abcdefghijkl}$. We factor:

$$\begin{aligned} 10^{12} - 1 &= \left(10^6 + 1\right)\left(10^6 - 1\right) \\ &= \left(10^2 + 1\right)\left(10^4 - 10^2 + 1\right)\left(10^3 - 1\right)\left(10^3 + 1\right) \\ &= 101 \cdot 9901 \cdot 9 \cdot 111 \cdot 11 \cdot 91 \\ &= 3^3 \cdot 7 \cdot 11 \cdot 13 \cdot 37 \cdot 101 \cdot 9901. \end{aligned}$$

Therefore, there are $(3+1)(1+1)^6 = 256$ factors of $10^{12} - 1$, but one of these corresponds to $n = 1$, which is disallowed. Hence, the answer is 255.

6. ABC is a scalene triangle. Points D, E, and F are selected on sides BC, CA, and AB respectively. The cevians AD, BE, and CF concur at point P. If $[AFP] = 126, [FBP] = 63$, and $[CEP] = 24$, determine the area of triangle ABC.

Answer: **351**. Since triangles AFP and FBP share an altitude from P, we have $\frac{BF}{FA} = \frac{[FBP]}{[AFP]} = \frac{1}{2}$. Let $[EAP] = k$. By similar reasoning, $\frac{AE}{EC} = \frac{k}{24}$. By Ceva's theorem, $\frac{CD}{DB}\frac{BF}{FA}\frac{AE}{EC} = 1 \Longrightarrow \frac{CD}{DB} = \frac{48}{k}$. Now we note that $\frac{[ADC]}{[ABD]} = \frac{[PDC]}{[PBD]} = \frac{CD}{DB} = \frac{48}{k}$. Hence, $\frac{[ADC]-[PDC]}{[ABD]-[PBD]} = \frac{[APC]}{[APB]} = \frac{48}{k}$. We use the fact that $[APC] = [APE] + [EPC] = k + 24$ and $[ABP] =$

$[AFP]+[FBP]=126+63=189$. We have

$$\begin{aligned}\frac{24+k}{189} &= \frac{48}{k}\\ 48\cdot 189 &= k^2+24k\\ k &= \frac{-24\pm\sqrt{24^2+4\cdot 48\cdot 189}}{2} = -12\pm\sqrt{12^2+48\cdot 189}\\ &= -12\pm 12\sqrt{1+63} = -108, 84\end{aligned}$$

We take $k=84$ since it represents an area. Now, $\frac{AE}{EC}=\frac{7}{2}$ and $\frac{CD}{DB}=\frac{4}{7}$. By Menelaus' theorem, $\frac{BF}{FA}\frac{AP}{PD}\frac{DC}{DB}=-1$ (Ceva and Menelaus use the convention of directed distances, where $XY=-YX$.) This yields $\frac{AP}{PD}=\frac{11}{2}$ from which $\frac{[ABPC]}{[PBC]}=\frac{11}{2}$. Hence, $[ABC]=\frac{13}{11}\cdot[ABPC]=\frac{13}{11}\cdot(24+84+126+63)=351$.

ALTERNATE SOLUTION

Assign the weights 1, 2, and ω to A, B, and C. It must be that $[EAP]=24\omega, [DCP]=2k$, and $[BDP]=\omega k$ for some k. But we have $\frac{2}{w}=\frac{[DCP]}{[BDP]}=\frac{[DCA]}{[BDA]}=\frac{[PCA]}{[BPA]}=\frac{24(\omega+1)}{126+3}=\frac{8(\omega+1)}{63}$. We solve this quadratic for $\omega=\frac{7}{2},-\frac{9}{2}$, and choose the former since 24ω is an area. But the weight on D is $\omega+2$ so that $\frac{\omega+2}{1}=\frac{AP}{PD}=\frac{[ABPC]}{[PBC]}$. Substituting, $\frac{11}{2}=\frac{24+24\cdot\frac{7}{2}+126+63}{[PBC]}$ which implies that $[PBC]=54$. Therefore, $[ABC]=[ABPC]+[PBC]=297+54=351$.

7. Anders, Po-Ru, Reid, and Aaron are playing Bridge. After one hand, they notice that all of the cards of two suits are split between Reid and Po-Ru's hands. Let N denote the number of ways 13 cards can be dealt to each player such that this is the case. Determine the remainder obtained when N is divided by 1000. (Bridge is a card game played with the standard 52-card deck.)

   Answer: **000**. Note that if two complete suits are in the union Po-Ru and Reid's hands, then the other two complete suits are in Anders and Aaron's hands. There are $\binom{4}{2}=6$ ways that the pairs of suits can be distributed. For each pair, one player has some 13 of the 26 cards, so the number of possible deals is $\binom{4}{2}\binom{26}{13}^2=6\cdot\left(\frac{26\cdot25\cdot24\cdots15\cdot14}{13!}\right)^2$. Note that $\binom{26}{13}$ is divisible by 2 and 25, hence N is divisible by $2\cdot(2\cdot25)^2=5000$. Therefore, the last three digits of N are 000.

8. Determine the remainder obtained when the expression

$$2004^{2003^{2002^{2001}}}$$

   is divided by 1000.

   Answer: **704**. Obviously, the last three digits of 2004^k are the same as 4^k. It is also clear that $k=2003^{2002^{2001}}$ exceeds 2, so that $4^{2003^{2002^{2001}}}\equiv 0 \pmod 8$. Let us determine $4^{2003^{2002^{2001}}}$ (mod 125).

Because $\phi(125) = 100$, we have $4^{2003^{2002^{2001}}} \equiv 4^{3^{2002^{2001}}} \pmod{125}$. We are interested in $3^{2002^{2001}} \pmod{100}$.

We play the same card again, that is, $\phi(100) = 40$ so that $3^{2002^{2001}} \equiv 3^{2^{2001}} \pmod{100}$. We are also interested in $2^{2001} \pmod{40}$. Clearly, 8 divides 2^{2001} so that $2^{2001} \equiv 0 \pmod 8$. We also have $2^{2001} \equiv 2 \pmod 5$ by Fermat's little theorem. By the Chinese remainder theorem, it must be that $2^{2001} \equiv 32 \pmod{40}$. Now we need $3^{32} \pmod{100}$. This can be quickly found: $3^4 = 81 \Longrightarrow 3^8 \equiv (81)^2 \equiv 61 \pmod{100} \Longrightarrow 3^{16} \equiv (61)^2 \equiv 61^2 \equiv 21 \pmod{100} \Longrightarrow 3^{32} \equiv (21)^2 \equiv 41 \pmod{100}$.

Therefore, $4^{2003^{2002^{2001}}} \equiv 4^{41} \pmod{125}$. $2^7 = 128 \equiv 3 \pmod{125}$, hence $4^{41} \equiv 2^{82} \equiv 3^{11} \cdot 2^5 \pmod{125}$. $3^7 = 2187 \equiv 62 \pmod{125}$ so that $3^{11} \cdot 2^5 \equiv 81 \cdot 32 \cdot 62 \equiv 79 \pmod{125}$. We apply the Chinese remainder theorem again, and determine that the unique residue r such that $r \equiv 0 \pmod 8$ and $r \equiv 79 \pmod{125}$ is $r \equiv 704 \pmod{1000}$.

9. Let

$$\left(1 + x^3\right)\left(1 + 2x^{3^2}\right) \cdots \left(1 + kx^{3^k}\right) \cdots \left(1 + 1997x^{3^{1997}}\right) = 1 + a_1 x^{k_1} + a_2 x^{k_2} + \cdots + a_m x^{k_m}$$

where $a_i \neq 0$ and $k_1 < k_2 < \cdots < k_m$. Determine the remainder obtained when a_{1997} is divided by 1000.

Answer: **280**. It is known that $3^{k+1} > 3^k + 3^{k-1} + \cdots + 3 + 1$. In the expansion of the product, each k_i consists only of 1's and 0's when written in trinary since the exponent is compiled from distinct powers of three from certain binomials. Since $1997 = 11111001101_2$, we have $k_{1997} = 111110011010_3$. It follows that $a_{1997} = 1 \cdot 3 \cdot 4 \cdot 7 \cdot 8 \cdot 9 \cdot 10 \cdot 11$. This can be multiplied out (obtaining 665,280) or kept modulo 1000; either method done correctly gives a remainder of 280 under division by 1000.

10. $ABCDE$ is a cyclic pentagon with $BC = CD = DE$. The diagonals AC and BE intersect at M. N is the foot of the altitude from M to AB. We have $MA = 25, MD = 113$, and $MN = 15$. The area of triangle ABE can be expressed as $\frac{m}{n}$ where m and n are relatively prime positive integers. Determine the remainder obtained when $m + n$ is divided by 1000.

Answer: **727**. Pythagoras gives $AN = 20$. We draw BD and AD, and construct the altitute MP to AD, with P on AD, and altitude MM' to AE, with M' on AE. Because $BC = CD = DE$, angles BAC, CAD, and DAE are congruent. Because P is on AD, triangles MNA and MPA are congruent by AAS, so $MP = 15$ and $PA = 20$, from which Pythagoras gives $PD = 112$, implying $AD = 132$.

Let $\alpha = m\angle BAC$, so $m\angle MAE = 2\alpha$, and $m\angle NAE = 3\alpha$. Because we have $\sin\alpha = \frac{3}{5}$ and $\cos\alpha = \frac{4}{5}$, we compute $\sin(2\alpha) = \frac{24}{25}$, and $\sin(3\alpha) = \frac{117}{125}$. We find that $MM' = 24$ using $\sin(2\alpha) = \frac{24}{25}$. By a simple Law of Sines argument $DE : EB : BD = 25 : 39 : 40$.

Let $[ABE] =$ the area of ABE. We have $[ABE] = 1/2(15 \cdot AB + 24 \cdot AE)$.

Ptolemy on $ABDE$ yields $AB \cdot DE + AE \cdot BD = AD \cdot BE$. Using the abundance of facts that we have ascertained previously, this gives:

$$\begin{aligned}
AB \cdot 25x + AE \cdot 40x &= 132 \cdot 39x \\
25AB + 40AE &= 39 \cdot 132 \\
15AB + 24AE &= \frac{39 \cdot 132 \cdot 3}{5}
\end{aligned}$$

Finally, $[ABE] = \frac{1}{2} \cdot (15AB + 24AE) = \frac{1}{2}\frac{39 \cdot 132 \cdot 3}{5} = \frac{7722}{5}$. Therefore, the answer is $722 + 5 = 727$.

11. α, β, and γ are the roots of $x(x-200)(4x+1) = 1$. Let

$$\omega = \tan^{-1}(\alpha) + \tan^{-1}(\beta) + \tan^{-1}(\gamma)$$

The value of $\tan(\omega)$ can be written as $\frac{m}{n}$ where m and n are relatively prime positive integers. Determine the value of $m + n$.

Answer: **167**. It is known that $\tan(A+B) = \frac{\tan(A)+\tan(B)}{1-\tan(A)\tan(B)}$. If we set $A = \tan^{-1}(a)$ and $B = \tan^{-1}(b)$, then we may write $\tan^{-1}(a) + \tan^{-1}(b) = \tan^{-1}\left(\frac{a+b}{1-ab}\right)$. Using this, we may write

$$\begin{aligned}
\omega &= \alpha + \beta + \gamma \\
&= \tan^{-1}\left(\frac{\alpha+\beta}{1-\alpha\beta}\right) + \tan^{-1}(\gamma) \\
&= \tan^{-1}\left(\frac{\left(\frac{\alpha+\beta}{1-\alpha\beta}\right) + \gamma}{1 - \left(\frac{\alpha+\beta}{1-\alpha\beta}\right)\gamma}\right) \\
&= \tan^{-1}\left(\frac{\alpha+\beta+(1-\alpha\beta)\gamma}{(1-\alpha\beta) - \gamma(\alpha+\beta)}\right) \\
&= \tan^{-1}\left(\frac{\alpha+\beta+\gamma-\alpha\beta\gamma}{1-(\alpha\beta+\beta\gamma+\gamma\alpha)}\right) \\
\Longrightarrow \tan(\omega) &= \frac{\alpha+\beta+\gamma-\alpha\beta\gamma}{1-(\alpha\beta+\beta\gamma+\gamma\alpha)}
\end{aligned}$$

We expand the given equation, obtaining $4x^3 - 799x^2 - 200x - 1 = 4(x-\alpha)(x-\beta)(x-\gamma) = 0$. We have $\alpha+\beta+\gamma = \frac{799}{4}$, $\alpha\beta+\beta\gamma+\gamma\alpha = -50$, and $\alpha\beta\gamma = \frac{1}{4}$. Therefore, $\tan(\omega) = \frac{\frac{799}{4} - \frac{1}{4}}{1+50} = \frac{399}{102} = \frac{133}{34}$. It follows that the answer is $133 + 34 = 167$.

12. $ABCD$ is a cyclic quadrilateral with $AB = 8, BC = 4, CD = 1$, and $DA = 7$. Let O and P denote the circumcenter and intersection of AC and BD respectively. The value of OP^2 can be expressed as $\frac{m}{n}$, where m and n are relatively prime, positive integers. Determine the remainder obtained when $m + n$ is divided by 1000.

Answer: **589**. Consider D' on the circumcircle of $ABCD$ such that $CD' = 7$ and $D'A = 1$. Let $m\angle D'AB = \alpha$ and $m\angle BCD' = \pi - \alpha$. Then by the Law of Cosines,

$$\begin{aligned}
1^2 + 8^2 - 2 \cdot 1 \cdot 8\cos(\alpha) &= BD'^2 = 4^2 + 7^2 - 2 \cdot 4 \cdot 7\cos(\pi - \alpha) \\
\Longrightarrow \cos(\alpha) &= 0
\end{aligned}$$

Hence $D'AB$ is a right triangle and the circumradius of $ABCD$ is $\frac{\sqrt{65}}{2}$. Now, by similar triangles, we have $AP : BP : CP : DP = 56 : 32 : 4 : 7$. Let $AP = 56x$ so that $AC = 60x$ and $BD = 39x$. Ptolemy's theorem applied to $ABCD$ yields $60x \cdot 39x = 1 \cdot 8 + 4 \cdot 7 = 36$ from which $x^2 = \frac{1}{65}$.

Now we apply Stewart's theorem to triangle BOD and cevian OP, obtaining

$$\begin{aligned} OB^2 \cdot PD + OD^2 \cdot BP &= OP^2 \cdot BD + BP \cdot BD \cdot PD \\ \frac{65}{4}(32x + 7x) &= 39x \cdot OP^2 + 32x \cdot 39x \cdot 7x \\ \frac{65}{4} - 7 \cdot 32 \frac{1}{65} &= OP^2 \\ OP^2 &= \frac{3329}{260} \end{aligned}$$

It follows that the answer is $329 + 260 = 589$.

13. $P(x)$ is the polynomial of minimal degree that satisfies

$$P(k) = \frac{1}{k(k+1)}$$

for $k = 1, 2, 3, \ldots, 10$. The value of $P(11)$ can be written as $-\frac{m}{n}$, where m and n are relatively prime positive integers. Determine $m + n$.

Answer: **071**. Consider the polynomial $Q(x) = x \cdot (x+1) \cdot P(x) - 1$. The given implies that $Q(x) = 0$ for $x = 1, 2, 3, \cdots, 10$. Therefore, we may write $Q(x) = R(x)(x-1)(x-2)(x-3)\cdots(x-10)$ for some polynomial $R(x)$. But $Q(0) = Q(-1) = -1$, so that $R(x)$ is non-constant. Hence, the minimum degree $Q(x)$ that corresponds to the minimum degree $P(x)$ must be of the form $(ax+b)(x-1)(x-2)\cdots(x-10)$,. Setting $x = 0$, we find that $-1 = 10! \cdot b \iff b = \frac{-1}{10!}$. Setting $x = -1$ yields $-1 = 11! \cdot (b - a) \iff a = b + \frac{1}{11!} = \frac{-10}{11!}$. Therefore, $Q(11) = \left(11 \cdot \frac{-10}{11!} + \frac{-1}{10!}\right) \cdot 10 \cdot 9 \cdot 8 \cdots 1 = -11 = 11 \cdot 12 \cdot P(11) - 1$. We solve for $P(11) = \frac{-10}{132} = \frac{-5}{66}$ from which it follows that the answer is 71.

ALTERNATE SOLUTION

Define $\Delta^k(n) = \Delta^{k-1}(n+1) - \Delta^{k-1}(n)$ where $\Delta^0(n) = P(n)$. We argue that $\Delta^k(n) = \frac{(-1)^k(k+1)!}{n(n+1)\cdots(n+k+1)}$ for all positive integers n and k for which $n + k \leq 10$. We induct on k; obviously the base case $k = 0$ is true. If we assume this identity for row k, then

$$\begin{aligned} \Delta^{k+1}(n) &= \Delta^k(n+1) - \Delta^k(n) = \frac{(-1)^k(k+1)!}{(n+1)(n+2)\cdots(n+k+2)} - \frac{((-1)^k(k+1)!}{n(n+1)\cdots(n+k+1)} \\ &= \frac{(-1)^k(k+1)!}{(n+1)(n+2)\cdots(n+k+1)} \cdot \left(\frac{1}{n+k+2} - \frac{1}{n}\right) \\ &= \frac{(-1)^{k+1}(k+1)!}{(n+1)\cdots(n+k+1)} \left(\frac{k+2}{n(n+k+2)}\right) \\ &= \frac{(-1)^{k+1}(k+2)!}{n(n+1)\cdots(n+k+2)} \end{aligned}$$

The first k such that $\Delta^k(n)$ is constant for all integers n must be at least $k = 9$; hence P is at least 9th degree. Since P is of minimal degree, we may assert that $\Delta^9(n)$ is constant. We may now retrace our subtractions to find $\Delta^0(11)$. Specifically,

$$\begin{aligned}
\Delta^0 P(11) &= \Delta^0(10) + \Delta^1(10) = \Delta^0(10) + \big(\Delta^1(9) + \Delta^2(9)\big) = \cdots \\
&= \sum_{k=0}^{9} \Delta^k(10-k) = \sum_{k=0}^{9} \frac{(-1)^k (k+1)!}{(10-k)(11-k)\cdots(11)} \\
&= \frac{1!}{10\cdot 11} - \frac{2!}{9\cdot 10\cdot 11} + \frac{3!}{8\cdot 9\cdot 10\cdot 11} - \cdots - \frac{10!}{11!} \\
&= \frac{1!9! - 2!8! + 3!7! - \cdots - 10!0!}{11!} \\
&= \frac{5!(3024 - 672 + 252 - 144 + 120 - 144 + 252 - 672 + 3024 - 30240)}{11!} \\
&= \frac{-25200}{11\cdot 10\cdot 9\cdot 8\cdot 7\cdot 6} \\
&= \frac{-5}{66}
\end{aligned}$$

From which it follows that the answer is $5 + 66 = 71$.

14. 3 Elm trees, 4 Dogwood trees, and 5 Oak trees are to be planted in a line in front of a library such that

i) No two Elm trees are next to each other.
ii) No Dogwood tree is adjacent to an Oak tree.
iii) All of the trees are planted.

How many ways can the trees be situated in this manner?

Answer: **152**. We will tackle the analgous problem of a string of 12 characters consisting of 3 E's, 4 D's, and 5 O's such that no two E's are adjacent and no D is next to an O. Write the three E's, that is, we consider 1E2E3E4. We must separate the E's with solid blocks of D's and O's in slots 2 and 3, but slots 1 and 4 can be empty or contain a solid block. We consider three cases.

Case I - 1 and 4 are blank. Then either 2 is 4 D's and 3 is 5 O's or vice versa. There are two possible arrangements.

Case II - 1 is not blank, but 4 is blank. Since for each of these arrangements, we could swap 1 and 4, we need not consider the case 1 blank and 4 not-blank separately, and merely double the number of strings in this case. Either there are 2 blocks of D's and 1 of O's or 1 block of D's and two of O's. In the first subcase, there are 3 ways to choose slots according to type, and 3 ways to distribute the 4 D's among two non-empty slots. In the latter subcase, there are again three ways to choose slot types but there are 4 ways to distribute 5 O's into two non-empty slots. Hence there are $9 + 12 = 21$ strings in this case, but via the bijection we count this as 42.

Case III - 1 and 4 are both non-blank. If three of the four slots contain only D's, there are 4 type arrangements and 3 quantity arrangements for a total of 12 possibile strings. If exactly two of $\{1, 2, 3, 4\}$ are D's and the other two are O's, then there are 6 possible type arrangements. $x + y = n$ has $n - 1$ solutions in positive integers, hence this subcase has $6 \cdot 3 \cdot 4 = 72$ possible strings. Finally, if there is one slot filled with D's and three filled with O's, then there are 4 type arrangements. $x+y+z = n$ has $\binom{n-1}{2}$ solutions in positive integers, so this gives $4 \cdot \binom{4}{2} = 24$ strings. Adding, we have $12 + 72 + 24 = 108$ such strings.

Therefore, the answer is $2 + 42 + 108 = 152$.

15. In triangle ABC, we have $BC = 13, CA = 37$, and $AB = 40$. Points D, E, and F are selected on BC, CA, and AB respectively such that AD, BE, and CF concur at the circumcenter of ABC. The value of

$$\frac{1}{AD} + \frac{1}{BE} + \frac{1}{CF}$$

can be expressed as $\frac{m}{n}$ where m and n are relatively prime positive integers. Determine $m+n$.

Answer: **529**. Drop altitude AA'. We have $m\angle AA'B = \frac{\pi}{2} - B$, but AOB is an isosceles triangle with $m\angle AOB = 2C \iff m\angle BAO = \frac{\pi}{2} - C$. Therefore, $\cos DAA' = \cos(C - B)$. Therefore we have $AD \cos(C - B) = AA' = AC \sin(C) = 2R \sin(B) \sin(C)$ so that $\frac{2R}{AD} = \frac{\cos(C-B)}{\sin(B)\sin(C)}$. Now,

$\frac{2R}{AD} + \frac{2R}{BE} + \frac{2R}{CF} = \frac{\cos(C-B)}{\sin(B)\sin(C)} + \frac{\cos(A-C)}{\sin(C)\sin(A)} + \frac{\cos(B-A)}{\sin(A)\sin(B)}$
$\iff 2R \sin(A) \sin(B) \sin(C) \left(\frac{1}{AD} + \frac{1}{BE} + \frac{1}{CF}\right)$
$= \sin(A) \cos(B - C) + \sin(B) \cos(C - A) + \sin(C) \cos(A - B)$
$= 3 \sin(A) \sin(B) \sin(C) + \sin(A) \cos(B) \cos(C) + \sin(B) \cos(A) \cos(C) + \sin(C) \cos(A) \cos(B)$
$= 3 \sin(A) \sin(B) \sin(C) + \sin(A + B) \cos(C) + \sin(C) \cos(A) \cos(B)$
$= 3 \sin(A) \sin(B) \sin(C) + \sin(C) (\cos(C) + \cos(A) \cos(B))$
$= 3 \sin(A) \sin(B) \sin(C) + \sin(C) (-\cos(A + B) + \cos(A) \cos(B)) = 4 \sin(A) \sin(B) \sin(C)$
$\implies \frac{1}{AD} + \frac{1}{BE} + \frac{1}{CF} = \frac{2}{R}$

Heron's formula yields $[ABC] = \sqrt{45 \cdot 5 \cdot 8 \cdot 32} = 240$. We substitute this into $[ABC] = \frac{abc}{4R} \iff R = \frac{abc}{4 \cdot [ABC]} = \frac{13 \cdot 37 \cdot 40}{4 \cdot 240} = \frac{13 \cdot 37}{24}$. From this we find that

$$\frac{1}{AD} + \frac{1}{BE} + \frac{1}{CF} = \frac{2}{R} = \frac{48}{481}$$

It follows that the answer is $48 + 481 = 529$.

Mock AIME #3

3:00-6:00 PM EST
November 7, 2004

1. DO NOT OPEN THIS BOOKLET UNTIL YOU ARE TOLD TO DO SO BY YOUR PROCTOR.

2. This is a 15-questions, 3-hour examination. All answers are integers ranging from 000 to 999, inclusive. Your score will be the number of correct answers; i.e., there is neither partial credit nor a penalty for wrong answers.

3. No aids other than scratch paper, ruler, compass, and protractor are permitted. In particular, CALCULATORS ARE NOT PERMITTED.

4. A combination of the AIME and the American Mathematics Contest 10 or 12 scores are used to determine eligibility for participation in the U.S.A. Mathematical Olympiad (USAMO).

5. Record all of your answers, and certain other information, on the AIME answer form. Only the answer form will be collected from you.

7 Mock AIME 3: Problems

1. Three circles are mutually externally tangent. Two of the circles have radii 3 and 7. If the area of the triangle formed by connecting their centers is 84, then the area of the third circle is $k\pi$ for some integer k. Determine k.

2. Let N denote the number of 7 digit positive integers have the property that their digits are in increasing order. Determine the remainder obtained when N is divided by 1000. (Repeated digits are allowed.)

3. A function $f(x)$ is defined for all real numbers x. For all non-zero values x, we have

$$2f(x) + f\left(\frac{1}{x}\right) = 5x + 4$$

Let S denote the sum of all of the values of x for which $f(x) = 2004$. Compute the integer nearest to S.

4. ζ_1, ζ_2, and ζ_3 are complex numbers such that

$$\begin{aligned} \zeta_1 + \zeta_2 + \zeta_3 &= 1 \\ {\zeta_1}^2 + {\zeta_2}^2 + {\zeta_3}^2 &= 3 \\ {\zeta_1}^3 + {\zeta_2}^3 + {\zeta_3}^3 &= 7 \end{aligned}$$

Compute ${\zeta_1}^7 + {\zeta_2}^7 + {\zeta_3}^7$.

5. In Zuminglish, all words consist only of the letters M, O, and P. As in English, O is said to be a vowel and M and P are consonants. A string of M's, O's, and P's is a word in Zuminglish if and only if between any two O's there appear at least two consonants. Let N denote the number of 10-letter Zuminglish words. Determine the remainder obtained when N is divided by 1000.

6. Let S denote the value of the sum

$$\sum_{n=1}^{9800} \frac{1}{\sqrt{n + \sqrt{n^2 - 1}}}$$

S can be expressed as $p + q\sqrt{r}$, where p, q, and r are positive integers and r is not divisible by the square of any prime. Determine $p + q + r$.

7. $ABCD$ is a cyclic quadrilateral that has an inscribed circle. The diagonals of $ABCD$ intersect at P. If $AB = 1$, $CD = 4$, and $BP : DP = 3 : 8$, then the area of the inscribed circle of $ABCD$ can be expressed as $\frac{p\pi}{q}$, where p and q are relatively prime positive integers. Determine $p + q$.

8. Let N denote the number of 8-tuples $(a_1, a_2, \ldots, a_8)$ of real numbers such that $a_1 = 10$ and

$$\begin{aligned} \left|{a_1}^2 - {a_2}^2\right| &= 10 \\ \left|{a_2}^2 - {a_3}^2\right| &= 20 \\ &\cdots \\ \left|{a_7}^2 - {a_8}^2\right| &= 70 \\ \left|{a_8}^2 - {a_1}^2\right| &= 80 \end{aligned}$$

Determine the remainder obtained when N is divided by 1000.

9. ABC is an isosceles triangle with base $\overline{AB}$. D is a point on $\overline{AC}$ and E is the point on the extension of $\overline{BD}$ past D such that $\angle BAE$ is right. If $BD = 15, DE = 2$, and $BC = 16$, then CD can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Determine $m + n$.

10. $\{A_n\}_{n \geq 1}$ is a sequence of positive integers such that

$$a_n = 2a_{n-1} + n^2$$

for all integers $n > 1$. Compute the remainder obtained when a_{2004} is divided by 1000 if $a_1 = 1$.

11. ABC is an acute triangle with perimeter 60. D is a point on $\overline{BC}$. The circumcircles of triangles ABD and ADC intersect $\overline{AC}$ and $\overline{AB}$ at E and F respectively such that $DE = 8$ and $DF = 7$. If $\angle EBC \cong \angle BCF$, then the value of $\frac{AE}{AF}$ can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.

12. Determine the number of integers n such that $1 \leq n \leq 1000$ and $n^{12} - 1$ is divisible by 73.

13. Let S denote the value of the sum

$$\left(\frac{2}{3}\right)^{2005} \cdot \sum_{k=1}^{2005} \frac{k^2}{2^k} \cdot \binom{2005}{k}$$

Determine the remainder obtained when S is divided by 1000.

14. Circles ω_1 and ω_2 are centered on opposite sides of line l, and are both tangent to l at P. ω_3 passes through P, intersecting l again at Q. Let A and B be the intersections of ω_1 and ω_3, and ω_2 and ω_3 respectively. AP and BP are extended past P and intersect ω_2 and ω_1 at C and D respectively. If $AD = 3, AP = 6, DP = 4$, and $PQ = 32$, then the area of triangle PBC can be expressed as $\frac{p\sqrt{q}}{r}$, where p, q, and r are positive integers such that p and r are coprime and q is not divisible by the square of any prime. Determine $p + q + r$.

15. Let Ω denote the value of the sum

$$\sum_{k=1}^{40} \cos^{-1}\left(\frac{k^2 + k + 1}{\sqrt{k^4 + 2k^3 + 3k^2 + 2k + 2}}\right)$$

The value of $\tan(\Omega)$ can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.

8 Mock AIME 3: Answers

1. 196
2. 435
3. 601
4. 071
5. 936
6. 121
7. 049
8. 472
9. 225
10. 058
11. 035
12. 164
13. 115
14. 468
15. 041

9 Mock AIME 3: Solutions

1. Three circles are mutually externally tangent. Two of the circles have radii 3 and 7. If the area of the triangle formed by connecting their centers is 84, then the area of the third circle is $k\pi$ for some integer k. Determine k.

   Answer: **196**. Let r denote the radius of the third circle. Then the sides of the triangle are $10, 3+r$, and $7+r$. Using Heron's formula and equating this with the given area, we have $84 = \sqrt{(10+r)(r)(7)(3)}$ from which $r = 14, -24$. Since r is positive, it follows that the area of the third circle is 196π.

2. Let N denote the number of 7 digit positive integers have the property that their digits are in increasing order. Determine the remainder obtained when N is divided by 1000. (Repeated digits are allowed.)

   Answer: **435**. Consider placing 7 X's and 8 $+$'s into a 15 character string. Replace each X with 1 plus the number of $+$'s that appear to its left. It follows that the result is a 7 digit number with its digits in increasing order. This bijection establishes that the number N is $\binom{15}{7} = 1435$, and the answer follows.

3. A function $f(x)$ is defined for all real numbers x. For all non-zero values x, we have

   $$2f(x) + f\left(\frac{1}{x}\right) = 5x + 4$$

   Let S denote the sum of all of the values of x for which $f(x) = 2004$. Compute the integer nearest to S.

   Answer: **601**. Substituting $\frac{1}{x}$ produces $2f\left(\frac{1}{x}\right) + f(x) = \frac{5}{x} + 4$. Subtract this from twice the given to obtain $3f(x) = 10x + 4 - \frac{5}{x}$. The non-zero solutions to $f(x) = 2004$ are therefore solutions to $10x^2 - 6008x - 5 = 0$. It follows that their sum is 600.8. (We don't care about $f(0)$ since if $f(0) = 2004$, the value of S is unchanged.) 601 is the nearest integer.

4. ζ_1, ζ_2, and ζ_3 are complex numbers such that

   $$\begin{aligned} \zeta_1 + \zeta_2 + \zeta_3 &= 1 \\ {\zeta_1}^2 + {\zeta_2}^2 + {\zeta_3}^2 &= 3 \\ {\zeta_1}^3 + {\zeta_2}^3 + {\zeta_3}^3 &= 7 \end{aligned}$$

   Compute ${\zeta_1}^7 + {\zeta_2}^7 + {\zeta_3}^7$.

   Answer: **071**. Consider the following algebra:

   $$\begin{aligned} 1 = (\zeta_1 + \zeta_2 + \zeta_3)^2 &= \left({\zeta_1}^2 + {\zeta_2}^2 + {\zeta_3}^2\right) + 2(\zeta_1\zeta_2 + \zeta_2\zeta_3 + \zeta_3\zeta_1) \\ &= 3 + 2(\zeta_1\zeta_2 + \zeta_2\zeta_3 + \zeta_3\zeta_1) \\ \Longrightarrow \zeta_1\zeta_2 + \zeta_2\zeta_3 + \zeta_3\zeta_1 &= -1 \end{aligned}$$

$$\begin{aligned}
3 = \left({\zeta_1}^2 + {\zeta_2}^2 + {\zeta_3}^2\right)(\zeta_1 + \zeta_2 + \zeta_3) &= \left({\zeta_1}^3 + {\zeta_2}^3 + {\zeta_3}^3\right) + \sum_{Sym} {\zeta_1}^2\zeta_2 \\
&= 7 + \sum_{Sym} {\zeta_1}^2\zeta_2 \\
\Longrightarrow \sum_{Sym} {\zeta_1}^2\zeta_2 &= -4 \\
1 = (\zeta_1 + \zeta_2 + \zeta_3)^3 &= \left({\zeta_1}^3 + {\zeta_2}^3 + {\zeta_3}^3\right) + 3\sum_{Sym} {\zeta_1}^2\zeta_2 + 6\zeta_1\zeta_2\zeta_3 \\
&= 7 - 12 + 6\zeta_1\zeta_2\zeta_3 \\
\Longrightarrow \zeta_1\zeta_2\zeta_3 &= 1
\end{aligned}$$

But now, $(x - \zeta_1)(x - \zeta_2)(x - \zeta_3) = x^3 - (x^2 + x + 1)$. If we write $S_n = {\zeta_1}^n + {\zeta_2}^n + {\zeta_3}^n$, we have $S_{n+3} = S_{n+2} + S_{n+1} + S_n$. With this recursion, we find $S_4 = 7 + 3 + 1 = 11$, $S_5 = 11 + 7 + 3 = 21$, $S_6 = 21 + 11 + 7 = 39$, and $S_7 = 39 + 21 + 11 = 71$.

5. In Zuminglish, all words consist only of the letters M, O, and P. As in English, O is said to be a vowel and M and P are consonants. A string of M's, O's, and P's is a word in Zuminglish if and only if between any two O's there appear at least two consonants. Let N denote the number of 10-letter Zuminglish words. Determine the remainder obtained when N is divided by 1000.

   Answer: **936**. A ten letter word in Zuminglish contains between 0 and 4 (inclusive) O's. If there are no O's, each letter is either M or P. There are 1024 such words. If there is one O, then we have 10 places to insert O and 9 choices between M and P. There are $10 \cdot 512 = 5120$ such words. If there are $2 \leq k \leq 4$ O's, then the k O's partition the word into $k+1$ blocks (some possibly empty) of M's and P's. Since between any two O's there must be at least two letters, we have only $10 - k - 2 \cdot (k-1) = 12 - 3k$ consonants to insert by choice into $k+1$ slots. This can be accomplished in $\binom{12-2k}{k}$ ways. Since k O's corresponds to $10 - l$ choices between M and P, for $k = 2, 3, 4$ we have $\binom{8}{2} \cdot 2^8 = 7168$, $\binom{6}{3} \cdot 2^7 = 2560$, and $\binom{4}{4} \cdot 2^6 = 64$ different Zuminglish words with k O's repsectively. Adding, there are $1024 + 5120 + 7168 + 2560 + 64 = 15936$ such Zuminglish words.

6. Let S denote the value of the sum

$$\sum_{n=1}^{9800} \frac{1}{\sqrt{n + \sqrt{n^2 - 1}}}$$

   S can be expressed as $p + q\sqrt{r}$, where p, q, and r are positive integers and r is not divisible by the square of any prime. Determine $p + q + r$.

   Answer: **121**. The key lies in noticing that $\sqrt{n + \sqrt{n^2 - 1}} = \frac{1}{\sqrt{2}} \cdot \sqrt{2n + 2\sqrt{n^2 - 1}} = \frac{1}{\sqrt{2}} \cdot \left(\sqrt{n+1} + \sqrt{n-1}\right)$, since we may now write

$$\sum_{n=1}^{9800} \frac{1}{\sqrt{n + \sqrt{n^2 - 1}}} = \sqrt{2}\sum_{n=1}^{9800} \frac{1}{\sqrt{n+1} + \sqrt{n-1}}$$

$$
\begin{aligned}
&= \frac{1}{\sqrt{2}}\sum_{n=1}^{9800}\sqrt{n+1}-\sqrt{n-1} \\
&= \frac{1}{\sqrt{2}}\cdot\left(\sqrt{9801}+\sqrt{9800}-\sqrt{1}-\sqrt{0}\right) \\
&= 70+49\sqrt{2}
\end{aligned}
$$

And it follows that the answer is $70 + 49 + 2 = 121$.

7. $ABCD$ is a cyclic quadrilateral that has an inscribed circle. The diagonals of $ABCD$ intersect at P. If $AB = 1$, $CD = 4$, and $BP : DP = 3 : 8$, then the area of the inscribed circle of $ABCD$ can be expressed as $\frac{p\pi}{q}$, where p and q are relatively prime positive integers. Determine $p + q$.

   Answer: **049**. Because $ABCD$ has an incircle, $AD + BC = AB + CD = 5$. Suppose that $AD : BC = 1 : \gamma$. Then $3 : 8 = BP : DP = (AB \cdot BC) : (CD \cdot DA) = \gamma : 4$. We obtain $\gamma = \frac{3}{2}$, which substituted into $AD + BC = 5$ gives $AD = 2$, $BC = 3$. Now, the area of $ABCD$ can be obtained via Brahmagupta's formula: $s = \frac{1+2+3+4}{2} = 5$, $K = \sqrt{(s-a)(s-b)(s-c)(s-d)} = \sqrt{24}$ and $K = rs = 5r$, where r is the inradius of $ABCD$. Thus, $r = \frac{\sqrt{24}}{5}$ from which its area $\frac{24\pi}{25}$ yields the answer $24 + 25 = 49$.

8. Let N denote the number of 8-tuples $(a_1, a_2, \ldots, a_8)$ of real numbers such that $a_1 = 10$ and

$$
\begin{aligned}
\left|{a_1}^2 - {a_2}^2\right| &= 10 \\
\left|{a_2}^2 - {a_3}^2\right| &= 20 \\
&\cdots \\
\left|{a_7}^2 - {a_8}^2\right| &= 70 \\
\left|{a_8}^2 - {a_1}^2\right| &= 80
\end{aligned}
$$

   Determine the remainder obtained when N is divided by 1000.

   472. Removing the absolute value signs, we have $a_1^2 = \pm 10 + a_2^2 = \pm 10 \pm 20 + a_3^2 = \cdots = \pm 10 \pm 20 \pm 30 \pm \cdots \pm 80 + a_1^2$. We see the constraint that the signs of each $\pm$ must be chosen such that $100 \pm 10 \pm \cdots$ is non-negative at each step, since at each step it corresponds to the value of the square of some a_i, in addition to the sum $\pm 10 \pm 20 \pm \cdots \pm 80$ being 0. Dividing out a factor of 10, we are interested in the subsets of $\{1, 2, 3, \ldots, 8\}$ with a sum of 18. Each subset corresponds to a zero-sum choice of signs in $10 \pm 1 \pm \cdots \pm n$ for $n = 1, 2, \cdots, 8$. The number of 8-tuples that correspond to each of these subsets is 0 if for any n the partial sum is negative, otherwise 2^k, where k is the number of values of $1 \leq n < 8$ for which $10 \pm 1 \pm \cdots \pm n$ is positive. This follows from recalling that each partial sum is a tenth of the square of some a_i, which allows for a choice of signs in each value except in a_1.

   The subset $\{3, 7, 8\}$ corresponds to $10 - 1 - 2 + 3 - 4 - 5 - 6 + 7 + 8 = 10$, but the partial sum is negative for $n = 6$, which corresponds to $a_7^2 = -10 - 20 + 30 - 40 - 50 - 60 + a_1^2 = -\cancel{10,}$
   The subset $\{4, 6, 8\}$, however, corresponds to $10 - 1 - 2 - 3 + 4 - 5 + 6 - 7 + 8 = 10$, which

is positive for all of its partial sums. Thus, there are 128 ordered 8-tuples corresponding to this subset. Continuing in this fashion:

$$\begin{array}{cclcclccl}
\{5,6,7\} & \to & 64. & \{1,2,7,8\} & \to & 0. & \{1,3,6,8\} & \to & 128. \\
\{1,4,5,8\} & \to & 128. & \{2,3,5,8\} & \to & 128. & \{1,4,6,7\} & \to & 128. \\
\{2,3,6,7\} & \to & 128. & \{2,4,5,7\} & \to & 128. & \{3,4,5,6\} & \to & 128. \\
\{1,2,3,4,8\} & \to & 128. & \{1,2,3,5,7\} & \to & 128. & \{1,2,4,5,6\} & \to & 128.
\end{array}$$

Where the subset $\{5,6,7\}$ is the only one for which we encounter an $a_i^2 = 0$, which halves the number of corresponding 8-tuples. Adding these numbers, we find that there are 1472 such 8-tuples, hence the answer.

9. ABC is an isosceles triangle with base $\overline{AB}$. D is a point on $\overline{AC}$ and E is the point on the extension of $\overline{BD}$ past D such that $\angle BAE$ is right. If $BD = 15$, $DE = 2$, and $BC = 16$, then CD can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Determine $m+n$.

   Answer: **225**. Draw in altitude $\overline{CF}$ and denote its intersection with $\overline{BD}$ by P. Since ABC is isosceles, $AF = FB$. Now, since BAE and BFP are similar with a scale factor of 2, we have $BP = \frac{1}{2}BE = \frac{17}{2}$, which also yields $PD = BD - BP = 15 - \frac{17}{2} = \frac{13}{2}$. Now, applying Menelaus to triangle ADB and collinear points C, P, and F, we obtain

$$\begin{aligned}
\frac{AC}{CD}\frac{DP}{PB}\frac{BF}{FA} &= \frac{AC}{CD}\frac{DP}{PB} = -1 \\
\Longrightarrow |CD| = AC \cdot \frac{DP}{PB} &= 16 \cdot \frac{\left(\frac{13}{2}\right)}{\left(\frac{17}{2}\right)} = \frac{208}{17}
\end{aligned}$$

   where the minus sign was a consequence of directed distances.[5] The answer is therefore $208 + 17 = 225$.

10. $\{A_n\}_{n\geq 1}$ is a sequence of positive integers such that

$$a_n = 2a_{n-1} + n^2$$

   for all integers $n > 1$. Compute the remainder obtained when a_{2004} is divided by 1000 if $a_1 = 1$.

   Answer: **058**. $a_{2004} = 2a_{2003} + 2004^2 = 2\left(2a_{2002} + 2003^2\right) + 2004^2 = \cdots = 2^{2003} \cdot 1^2 + 2^{2002} \cdot 2^2 + 2^{2001} \cdot 3^2 + \cdots + 2^0 \cdot 2004^2$. We subtract a_{2004} from twice itself two times to telescope this sum:

$$\begin{aligned}
a_{2004} = 2a_{2004} - a_{2004} &= \left(2^{2004} \cdot 1 + 2^{2003} \cdot 4 + \cdots + 2 \cdot 2004^2\right) - \left(2^{2003} \cdot 1 + \cdots + 2^0 \cdot 2004^2\right) \\
&= 2^{2004} + 3 \cdot 2^{2003} + \cdots + 4007 \cdot 2^1 - 2004^2 \\
a_{2004} = 2a_{2004} - a_{2004} &= \left(2^{2005} + 3 \cdot 2^{2004} + \cdots + 4007 \cdot 2^2 - 2 \cdot 2004^2\right) \\
&\quad - \left(1 \cdot 2^{2004} + 3 \cdot 2^{2003} + \cdots + 4007 \cdot 2^1 - 2004^2\right) \\
&= 2^{2005} + 2 \cdot \left(2^{2004} + 2^{2003} + \cdots + 2^2\right) - 2 \cdot 4007 - 2004^2 \\
a_{2004} &\equiv 2^5 + 2\left(2^{2005} - 4\right) - 14 - 16 \pmod{1000} \\
&\equiv 32 + 64 - 8 - 30 = 58 \pmod{1000}
\end{aligned}$$

[5] A system of linear measure in which for any points A and B, $AB = -BA$.

11. ABC is an acute triangle with perimeter 60. D is a point on $\overline{BC}$. The circumcircles of triangles ABD and ADC intersect $\overline{AC}$ and $\overline{AB}$ at E and F respectively such that $DE = 8$ and $DF = 7$. If $\angle EBC \cong \angle BCF$, then the value of $\frac{AE}{AF}$ can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m + n$.

Answer: **035**. Since $BDEA$ is cyclic, $\angle EBD \cong \angle EAD$. Similarly, $\angle DCF \cong \angle DAF$. Since we are given $\angle BCF \cong \angle EBC$, we have $\angle DAB \cong \angle CAD$. Because $\overline{CD}$ and $\overline{DF}$ are intercepted by congruent angles in the same circle, $DF = CD = 7$. Similarly, $DB = 8$. Now, by the angle bisector theorem, $AC = 7x$ and $AB = 8x$. Since the perimeter of ABC is 60, $15 + 15x = 60$ and $x = 3$, so that $AC = 21$ and $AB = 24$. Now, by Power of a Point from B, $BF = \frac{BD \cdot BC}{BA} = \frac{8 \cdot 15}{24} = 5$ and $CE = \frac{CD \cdot CB}{CA} = \frac{7 \cdot 15}{21} = 5$. Subtracting these lengths from AB and AC respectively, we find that $AF = 19$ and $AE = 16$. It follows that the answer is $16 + 19 = 35$.

12. Determine the number of integers n such that $1 \le n \le 1000$ and $n^{12} - 1$ is divisible by 73.

Answer: **164**. We are interested in values of n for which $n^{12} - 1 = (n^6 + 1)(n^6 - 1) = (n^2 + 1)(n^4 - n^2 + 1)(n + 1)(n - 1)(n^4 + n^2 + 1)$ is divisible by 73. We note that 73 is prime, so that at most two distinct residues satisfy $r^2 \equiv k \pmod{73}$. First, we need to solve for the square roots of -1, since they are the zeros of $n^2 + 1 \equiv 0 \pmod{73}$. If we do not notice that $27^2 = 729 \equiv -1 \pmod{73}$, then we can work from $3^2 + 8^2 = 73$, since $8^2 \equiv -(3^2) \pmod{73} \iff \left(8 \cdot 3^{-1}\right)^2 \equiv -1 \pmod{73}$. 3^{-1} is the unique residue r such that $3r \equiv 1 \pmod{73}$. It is the only integer in the set $\{\frac{1}{3}, \frac{74}{3}, \frac{147}{3}\}$, or 49. Thus, $8 \cdot 49 \equiv 27 \pmod{73}$ is a square root of -1. The other is $73 - 27 = 46$. (We will omit writing $\pmod{73}$ understanding that all of the following algebra is in this numeric system.)

In solving $n^2 - n + 1 = n(n-1) + 1 \equiv 0$, if we do not notice that $8 \cdot 9 = 72 \equiv -1$, we can proceed by completing the square. Since $n^2 - n + 1 \equiv n^2 - 74n + 1$, $(n - 37)^2 \equiv 1368 \equiv 1295 \equiv \cdots \equiv 784$. We stop at 784 because it is the square of 28. Thus, $n - 37 \equiv \pm 28$, or $n \equiv 9, 65$. The solutions to $n^2 + n + 1 = (n+1) \cdot n + 1 \equiv 0$ are now easy to find since we are merely substituting $n + 1$ in place of n. Thus, $n^2 + n + 1 \equiv 0 \iff n \equiv 8, 64$.

To solve $n^4 - n^2 + 1 = n^2(n^2 - 1) + 1 \equiv 0$, we take the square roots of the solutions to $n(n-1) + 1 \equiv 0$, since $n^2 \equiv 9, 65$. For $n^9 \equiv 9$, we have $n \equiv 3, 70$. The solutions to $n^2 \equiv 65 \equiv -81 \equiv (27 \cdot 9)^2 \equiv 24^2$ are $n \equiv 24, 49$.

In solving $n^4 + n^2 + 1 \equiv 0$, we play the same trick once more, noting that the solutions to $n^2 \equiv 64$ are $n \equiv 8, 65$. The solutions to $n^2 \equiv 8 \equiv 81$ are $n \equiv 9, 64$.

Combining these with the trivial solutions, we have $n \equiv 1, 3, 8, 9, 24, 27, 46, 49, 64, 65, 70, 72 \pmod{73}$. Now, $1000 = 13 \cdot 73 + 51$, so we count all 12 of these solutions 13 times, and the 8 residues less than or equal to 51 one more time. This gives a total of $13 \cdot 12 + 8 = 164$ solutions.

13. Let S denote the value of the sum

$$\left(\frac{2}{3}\right)^{2005} \cdot \sum_{k=1}^{2005} \frac{k^2}{2^k} \cdot \binom{2005}{k}$$

Determine the remainder obtained when S is divided by 1000.

Answer: **115**. We note the combinatorial identity $k\binom{n}{k} = n\binom{n-1}{k-1}$ and write

$$\begin{aligned} k^2\binom{2005}{k} = 2005k\binom{2004}{k-1} &= 2005\left((k-1)\binom{2004}{k-1} + \binom{2004}{k-1}\right) \\ &= 2005\left(2004\binom{2003}{k-2} + \binom{2004}{k-1}\right) \end{aligned}$$

Employing this result,

$$\begin{aligned} S &= \left(\frac{2}{3}\right)^{2005} \cdot \sum_{k=1}^{2005} \frac{k^2}{2^k}\binom{2005}{k} \\ &= \left(\frac{2}{3}\right)^{2005} \cdot 2005 \sum_{k=1}^{2005} \frac{1}{2^k}\left(2004\binom{2003}{k-2} + \binom{2004}{k-1}\right) \\ &= \left(\frac{2}{3}\right)^{2005} \cdot 2005\left(501 \cdot \sum_{k=0}^{2003} \frac{1}{2^k}\binom{2003}{k} + \frac{1}{2}\sum_{k=0}^{2004} \frac{1}{2^k}\binom{2004}{k}\right) \\ &= 2005\left(501 \cdot \frac{4}{9} \cdot \sum_{k=0}^{2003} \frac{2^{2003-k}}{3^{2003-k}}\frac{1^k}{3^k}\binom{2003}{k} + \frac{1}{2} \cdot \frac{2}{3} \cdot \sum_{k=0}^{2004} \frac{2^{2004-k}}{3^{2004-k}}\frac{1^k}{3^k}\binom{2004}{k}\right) \\ &= 2005\left(\frac{668}{3} \cdot \left(\frac{2}{3}+\frac{1}{3}\right)^{2003} + \frac{1}{3} \cdot \left(\frac{2}{3}+\frac{1}{3}\right)^{2004}\right) = 447115. \end{aligned}$$

14. Circles ω_1 and ω_2 are centered on opposite sides of line l, and are both tangent to l at P. ω_3 passes through P, intersecting l again at Q. Let A and B be the intersections of ω_1 and ω_3, and ω_2 and ω_3 respectively. AP and BP are extended past P and intersect ω_2 and ω_1 at C and D respectively. If $AD = 3, AP = 6, DP = 4$, and $PQ = 32$, then the area of triangle PBC can be expressed as $\frac{p\sqrt{q}}{r}$, where p, q, and r are positive integers such that p and r are coprime and q is not divisible by the square of any prime. Determine $p+q+r$.

Answer: **468**. We invert about P with radius 1, mapping the circles ω_1 and ω_2 to lines ω_1' and ω_2', each parallel to l, and ω_3 to a line ω_3' that intersects ω_1' and ω_2' at A' and B' respectively. Q' is the intersection of l and ω_3', and C' and D' are the intersections of the extensions of $A'P$ and $B'P$ past P to ω_2' and ω_1' respectively.

We have $PQ' = \frac{1}{32}, PA' = \frac{1}{6}$, and $PD' = \frac{1}{4}$. The inversive distance formula gives $A'D' = \frac{R^2 \cdot AD}{AP \cdot DP} = \frac{1}{8}$. The crossed ladders theorem asserts

$$\frac{1}{A'D'} + \frac{1}{B'C'} = \frac{1}{PQ'}$$

from which $B'C' = \frac{1}{24}$. However, it is clear in the inverted figure that triangles $C'B'P$ and $A'D'P$ are similar. Therefore, $PC' = \frac{1}{18}$ and $PB' = \frac{1}{12}$.

But inversion is its own inverse transformation. Hence, $PC = 18$ and $PB = 12$. The inversive distance formula gives $BC = \frac{R^2 \cdot B'C'}{PB' \cdot PC'} = \frac{18 \cdot 12}{24} = 9$. Finally, the area of PBC may be found via Heron's formula: $K = \sqrt{\frac{39}{2}\frac{21}{2}\frac{15}{2}\frac{3}{2}} = \frac{9\sqrt{455}}{4}$. The answer is therefore $455 + 9 + 4 = 468$.

15. Let Ω denote the value of the sum

$$\sum_{k=1}^{40} \cos^{-1}\left(\frac{k^2+k+1}{\sqrt{k^4+2k^3+3k^2+2k+2}}\right)$$

The value of $\tan(\Omega)$ can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Compute $m+n$.

Answer: **041**. The cosine inverse subtraction formula,

$$\cos^{-1}(a) - \cos^{-1}(b) = \cos^{-1}\left(ab + \sqrt{1-a^2}\sqrt{1-b^2}\right)$$

will be the vehicle for telescoping this sum. It can be shown via AM-GM that $x^4 + 2x^3 + 3x^2 + 2x + 2$ has no real roots, so we inspect for imaginary solutions among Gaussian integers. Finding that $x = \pm i$ are solutions, we factor accordingly: $x^4 + 2x^3 + 3x^2 + 2x + 2 = (x^2 + 1)(x^2 + 2x + 2) = (x^2 + 1)((x+1)^2 + 1)$ and note that the other two roots are $-1 \pm i$. If this sum is going to telescope, it ought to be due to

$$\cos^{-1}\left(\frac{k^2+k+1}{\sqrt{k^4+2k^3+3k^2+2k+2}}\right) = \cos^{-1}\left(f\left((k+1)^2+1\right)\right) - \cos^{-1}\left(f(k^2+1)\right)$$

for some function f. Because there is a square root in the denominator on the left, we conjecture that $f(x) = \frac{1}{\sqrt{x}}$. Checking this reveals remarkable simplification:

$$\begin{aligned}
& \cos^{-1}\left(\frac{1}{\sqrt{(k+1)^2+1}}\right) - \cos^{-1}\left(\frac{1}{\sqrt{k^2+1}}\right) \\
= \quad & \cos^{-1}\left(\frac{1}{\sqrt{k^2+1}\sqrt{(k+1)^2+1}} + \sqrt{1-\frac{1}{k^2+1}}\sqrt{1-\frac{1}{(k+1)^2+1}}\right) \\
= \quad & \cos^{-1}\left(\frac{1}{\sqrt{k^4+2k^3+3k^2+2k+2}} + \frac{k(k+1)}{\sqrt{k^2+1}\sqrt{(k+1)^2+1}}\right) \\
= \quad & \cos^{-1}\left(\frac{k^2+k+1}{\sqrt{k^4+2k^3+3k^2+2k+2}}\right)
\end{aligned}$$

Hence, $\Omega = \cos^{-1}\left(\frac{1}{\sqrt{41^2+1}}\right) - \cos^{-1}\left(\frac{1}{\sqrt{1^2+1}}\right) = \cos^{-1}\left(\frac{1+41}{\sqrt{2\cdot(41^2+1)}}\right) = \cos^{-1}\left(\frac{42}{\sqrt{2\cdot 1682}}\right) = \cos^{-1}\left(\frac{21}{29}\right)$. Recalling that a triangle of sides 20, 21, and 29 is a right triangle, it follows that $\tan(\Omega) = \frac{20}{21}$, whence the answer $20 + 21 = 41$.

Mock AIME #4

3:00-6:00 PM EST
December 30, 2004

1. DO NOT OPEN THIS BOOKLET UNTIL YOU ARE TOLD TO DO SO BY YOUR PROCTOR.

2. This is a 15-questions, 3-hour examination. All answers are integers ranging from 000 to 999, inclusive. Your score will be the number of correct answers; i.e., there is neither partial credit nor a penalty for wrong answers.

3. No aids other than scratch paper, ruler, compass, and protractor are permitted. In particular, CALCULATORS ARE NOT PERMITTED.

4. A combination of the AIME and the American Mathematics Contest 10 or 12 scores are used to determine eligibility for participation in the U.S.A. Mathematical Olympiad (USAMO).

5. Record all of your answers, and certain other information, on the AIME answer form. Only the answer form will be collected from you.

10 Mock AIME 4: Problems

1. For how many positive integers $n > 1$ is it possible to express 2005 as the sum of n distinct positive integers?

2. $a_1, a_2, \ldots$ is a sequence of real numbers where a_n is the arithmetic mean of the previous $n-1$ terms for $n > 3$ and $a_{2004} = 7$. $b_1, b_2, \ldots$ is a sequence of real numbers in which b_n is the geometric mean of the previous $n-1$ terms for $n > 3$ and $b_{2005} = 6$. If $a_i = b_i$ for $i = 1, 2, 3$ and $a_1 = 3$, then compute the value of $a_2^2 + a_3^2$.

3. Compute the largest integer n such that $2005^{2^{100}} - 2003^{2^{100}}$ is divisible by 2^n.

4. $ABCDEFG$ is a regular heptagon, and P is a point in its interior such that ABP is equilateral. There exists a unique pair $\{m, n\}$ of relatively prime positive integers such that $m\angle CPE = \left(\frac{m}{n}\right)^\circ$. Compute the value of $m + n$.

5. Compute, to the nearest integer, the area of the region enclosed by the graph of $13x^2 - 20xy + 52y^2 - 10x + 52y = 563$.

6. Determine the remainder obtained when 1000! is divided by 2003.

7. $\mathcal{P}$ is a pyramid consisting of a square base and four slanted triangular faces such that all of its edges are equal in length. $\mathcal{C}$ is a cube of edge length 6. Six pyramids similar to $\mathcal{P}$ are constructed by taking points P_i (all outside of $\mathcal{C}$) where $i = 1, 2, \ldots, 6$ and using the nearest face of $\mathcal{C}$ as the base of each pyramid exactly once. The volume of the octahedron formed by the P_i (taking the convex hull) can be expressed as $m + n\sqrt{p}$ for some positive integers m, n, and p, where p is not divisible by the square of any prime. Determine the value of $m + n + p$.

8. A single atom of Uranium-238 rests at the origin. Each second, the particle has a 1/4 chance of moving one unit in the negative x direction and a 1/2 chance of moving in the positive x direction. If the particle reaches $(-3, 0)$, it ignites a fission that will consume the earth. If it reaches $(7, 0)$, it is harmlessly diffused. The probability that, eventually, the particle is safely contained can be expressed as $\frac{m}{n}$ for some relatively prime positive integers m and n. Determine the remainder obtained when $m + n$ is divided by 1000.

9. The value of the sum
$$\sum_{n=1}^{\infty} \frac{(7n+32) \cdot 3^n}{n \cdot (n+2) \cdot 4^n}$$
can be expressed in the form $\frac{p}{q}$, for some relatively prime positive integers p and q. Compute the value of $p + q$.

10. 100 blocks are selected from a crate containing 33 blocks of each of the following dimensions: $13 \times 17 \times 21$, $13 \times 17 \times 37$, $13 \times 21 \times 37$, and $17 \times 21 \times 37$. The chosen blocks are stacked on top of each other (one per cross section) forming a tower of height h. Compute the number of possible values of h.

11. 10 lines and 10 circles divide the plane into at most n disjoint regions. Compute n.

12. Determine the number of permutations of $1, 2, 3, 4, \ldots, 32$ such that if m divides n, the mth number divides the nth number.

13. x, y, and z are distinct non-zero integers such that $-7 \leq x, y, z \leq 7$. Compute the number of solutions (x, y, z) to the equation

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{x+y+z}$$

14. In triangle ABC, $BC = 27, CA = 32$, and $AB = 35$. P is the unique point such that the perimeters of triangles BPC, CPA, and APB are equal. The value of $AP + BP + CP$ can be expressed as $\frac{p+q\sqrt{r}}{s}$, where p, q, r, and s are positive integers such that there is no prime divisor common to p, q, and s, and r is not divisible by the square of any prime. Determine the value of $p + q + r + s$.

15. $ABCD$ is a convex quadrilateral in which $\overline{AB} \parallel \overline{CD}$. Let U denote the intersection of the extensions of $\overline{AD}$ and $\overline{BC}$. Ω_1 is the circle tangent to line segment $\overline{BC}$ which also passes through A and D, and Ω_2 is the circle tangent to $\overline{AD}$ which passes through B and C. Call the points of tangency M and S. Let O and P be the points of intersection between Ω_1 and Ω_2. Finally, $\overline{MS}$ intersects $\overline{OP}$ at V. If $AB = 2$, $BC = 2005$, $CD = 4$, and $DA = 2004$, then the value of UV^2 is some integer n. Determine the remainder obtained when n is divided by 1000.

11 Mock AIME 4: Answers

1. 061
2. 180
3. 103
4. 667
5. 075
6. 002
7. 434
8. 919
9. 035
10. 595
11. 346
12. 240
13. 504
14. 171
15. 039

12 Mock AIME 4: Solutions

1. For how many positive integers $n > 1$ is it possible to express 2005 as the sum of n distinct positive integers?

   Answer: **061**. The sum of n distinct positive integers is at least $1+2+3+\cdots+n = \frac{n(n+1)}{2}$, but because we can exchange n with $n+k$ for any integer $k \geq 0$, the sum of n distinct positive integers can be *any* integer at least $\frac{n(n+1)}{2}$. We have $1+2+\cdots+62 = \frac{62\cdot 63}{2} = 1953$ and $1+2+\cdots+63 = \frac{63\cdot 64}{2} = 2016 > 2005$. Therefore, 2005 can be expressed as the sum of $n > 1$ integers for $n = 2, 3, \ldots, 61, 62$; 61 distinct values.

2. $a_1, a_2, \ldots$ is a sequence of real numbers where a_n is the arithmetic mean of the previous $n-1$ terms for $n > 3$ and $a_{2004} = 7$. $b_1, b_2, \ldots$ is a sequence of real numbers in which b_n is the geometric mean of the previous $n-1$ terms for $n > 3$ and $b_{2005} = 6$. If $a_i = b_i$ for $i = 1, 2, 3$ and $a_1 = 3$, then compute the value of $a_2^2 + a_3^2$.

   Answer: **180**. Note that $a_{2004} = \frac{a_1+a_2+a_3+\frac{a_1+a_2+a_3}{3}+\cdots}{2004}$ is symmetric with respect to a_1, a_2, and a_3. Therefore, it is the arithmetic mean of the first three terms. This implies that $a_2 + a_3 = 18$. By similar reasoning, b_{2005} is the geometric mean of b_1, b_2, and b_3, from which $b_2 b_3 = 72$. Since $b_2 b_3 = a_2 a_3$, we have $a_2^2 + a_3^2 = (a_2+a_3)^2 - 2a_2a_3 = 18^2 - 2\cdot 72 = 180$.

3. Compute the largest integer n such that $2005^{2^{100}} - 2003^{2^{100}}$ is divisible by 2^n.

   Answer: **103**. The expression factors as

   $$\left(2005^{2^{99}} + 2003^{2^{99}}\right)\cdots\left(2005^{2^{k}} + 2003^{2^{k}}\right)\cdots\left(2005^{2^{0}} + 2003^{2^{0}}\right)\left(2005^{2^{0}} - 2003^{2^{0}}\right)$$

   Each term is even, but since all odd squares are equivalent to 1 modulo 4, the only term that contains more than one factor of 2 is $2005^{2^0} + 2003^{2^0} = 4008 = 8\cdot 501$. Thus, treating each of the 101 terms as once divisible by 2 undercounts the number of factors by 2, giving an answer of 103.

4. $ABCDEFG$ is a regular heptagon, and P is a point in its interior such that ABP is equilateral. There exists a unique pair $\{m, n\}$ of relatively prime positive integers such that $m\angle CPE = \left(\frac{m}{n}\right)^\circ$. Compute the value of $m+n$.

   Answer: **667**. Since ABP is equilateral, $BP = BA = BC$, hence $\angle BCP \cong \angle CPB$. Let α denote the degree measure of each of the angles of $ABCDEFG$. Then $m\angle PCB = \alpha - 60^\circ$ from which $m\angle CPB = m\angle BCP = 120^\circ - \frac{\alpha}{2}$ and $m\angle PCD = \frac{3\alpha}{2} - 120^\circ$. By symmetry, P lies on the angle bisector of $\angle DEF$, thus $m\angle DEP = \frac{\alpha}{2}$. Finally, as $m\angle CDE = \alpha$, we have $m\angle EPC = 360^\circ - \frac{\alpha}{2} - \alpha - \left(\frac{3\alpha}{2} - 120^\circ\right) = 480^\circ - 3\alpha$. Computing $\alpha = 180^\circ - \frac{360^\circ}{7} = \frac{900^\circ}{7}$, we find that $m\angle EPC = \frac{660^\circ}{7}$.

5. Compute, to the nearest integer, the area of the region enclosed by the graph of $13x^2 - 20xy + 52y^2 - 10x + 52y = 563$.

Answer: **075**. We apply the algebra

$$\begin{aligned} 13x^2 + \left[52\left(y+\frac{1}{2}\right)^2 - 13\right] - 20x\left(y+\frac{1}{2}\right) &= 563 \\ 13\left(\frac{x}{2}\right)^2 + 13\left(y+\frac{1}{2}\right)^2 - 10\left(\frac{x}{2}\right)\left(y+\frac{1}{2}\right) &= 144 \\ 13x_0^2 + 13y_0^2 - 10x_0y_0 &= 144 \end{aligned}$$

where $x_0 = \frac{x}{2}$ and $y_0 = y + \frac{1}{2}$. It is clear that the transformation $(x, y) \to (x_0, y_0)$ shifts (x, y) up half of a unit and then scales this image by a factor of $\frac{1}{2}$ along the x direction. Therefore, the area of the region enclosed by the original equation is twice that enclosed by the new region.

Setting $x_0 = y_0$, we find the points $\pm(3, 3)$. Setting $x_0 = -y_0$, we find the points $\pm(-2, 2)$. The new graph encloses an ellipse that has a semimajor-axis length of $3\sqrt{2}$ and semiminor-axis length of $2\sqrt{2}$. Its area is then $\left(2\sqrt{2}\right)\left(3\sqrt{2}\right)\pi = 12\pi$. Therefore, the original graph encloses an ellipse of area $24\pi = 75.398\ldots$

6. Determine the remainder obtained when 1000! is divided by 2003.

Answer: **002**. Note that 2003 is prime. Now, by Wilson's Theorem, $2002! \equiv -1 \pmod{2003}$, but $2002! \equiv (1 \cdot 2 \cdots 1001)(-1001 \cdots -1) \equiv -(1001!)^2 \equiv -1 \pmod{2003}$. Hence, $1001! \equiv 1 \pmod{2003}$ or $1001! \equiv 2002 \pmod{2003}$. Dividing by 1001, two possible answers are 2001 and 2 respectively. Because the answer must be less than 1000, it must be that the remainder is 2.

7. $\mathcal{P}$ is a pyramid consisting of a square base and four slanted triangular faces such that all of its edges are equal in length. $\mathcal{C}$ is a cube of edge length 6. Six pyramids similar to $\mathcal{P}$ are constructed by taking points P_i (all outside of $\mathcal{C}$) where $i = 1, 2, \ldots, 6$ and using the nearest face of $\mathcal{C}$ as the base of each pyramid exactly once. The volume of the octahedron formed by the P_i (taking the convex hull) can be expressed as $m + n\sqrt{p}$ for some positive integers m, n, and p, where p is not divisible by the square of any prime. Determine the value of $m+n+p$.

Answer: **434**. By a Pythagoras argument, the height of each pyramid from P_i to the nearest face of $\mathcal{C}$ is $3\sqrt{2}$. Therefore, the distance from opposite vertices of the octohedron is $6 + 2 \cdot 3\sqrt{2} = 6 \cdot \left(1 + \sqrt{2}\right)$. Let P_s and P_t be a pair of opposite vertices. The square formed by the other four vertices of the octahedron has area $\frac{1}{2} \cdot \left(6 \cdot \left(1 + \sqrt{2}\right)\right)^2 = 18\left(3 + 2\sqrt{2}\right)$. Finally, the volume is given by $\frac{1}{3}\left(6(1+\sqrt{2})\right)\left(18(3+2\sqrt{2})\right) = 252 + 180\sqrt{2}$.

8. A single atom of Uranium-238 rests at the origin. Each second, the particle has a $1/4$ chance of moving one unit in the negative x direction and a $1/2$ chance of moving in the positive x direction. If the particle reaches $(-3, 0)$, it ignites a fission that will consume the earth. If it reaches $(7, 0)$, it is harmlessly diffused. The probability that, eventually, the particle is safely contained can be expressed as $\frac{m}{n}$ for some relatively prime positive integers m and n. Determine the remainder obtained when $m + n$ is divided by 1000.

Answer: **919**. Let $p(n)$ be the probability that the atom is safely contained if released from $(n, 0)$. In this notation, $P(-3) = 0, P(7) = 1$. Now, since the particle is twice as likely to move right as it is likely to move left, $P(n) = \frac{2}{3}P(n+1) + \frac{1}{3}P(n-1)$ or equivalently $P(n+1) = \frac{3P(n)-P(n-1)}{2}$ for $-2 \leq n \leq 6$. Let $P(-2) = r$. Then $P(-1) = \frac{3r}{2}, P(0) = \frac{7r}{4}, P(1) = \frac{15r}{8}, P(2) = \frac{31r}{16}, P(3) = \frac{63r}{32}, P(4) = \frac{127r}{64}, P(5) = \frac{255r}{128}$, and $P(6) = \frac{511r}{256}$. But now $\frac{511r}{256} = \frac{2}{3} + \frac{1}{3}\frac{255r}{128}$ which gives $r = \frac{512}{1023}$ from which $\frac{m}{n} = \frac{7}{4}\frac{512}{1023} = \frac{896}{1023}$.

9. The value of the sum

$$\sum_{n=1}^{\infty} \frac{(7n+32) \cdot 3^n}{n \cdot (n+2) \cdot 4^n}$$

can be expressed in the form $\frac{p}{q}$, for some relatively prime positive integers p and q. Compute the value of $p + q$.

Answer: **035**. Note that $\frac{7n+32}{n \cdot (n+2)} = \frac{16}{n} - \frac{9}{n+2}$ so that

$$\begin{aligned}
\sum_{n=1}^{\infty} \frac{(7n+32)}{n \cdot (n+2)} \frac{3^n}{4^n} &= \sum_{n=1}^{\infty} \frac{16}{n} \frac{3^n}{4^n} - \sum_{n=1}^{\infty} \frac{9}{n+2} \frac{3^n}{4^n} \\
&= \sum_{n=1}^{\infty} \frac{16}{n} \frac{3^n}{4^n} - \sum_{n=1}^{\infty} \frac{16}{n+2} \frac{3^{(n+2)}}{4^{(n+2)}} \\
&= \sum_{n=1}^{\infty} \frac{16}{n} \frac{3^n}{4^n} - \sum_{n=3}^{\infty} \frac{16}{n} \frac{3^n}{4^n} \\
&= \frac{16}{1} \frac{3}{4} + \frac{16}{2} \frac{9}{16} = \frac{33}{2}
\end{aligned}$$

10. 100 blocks are selected from a crate containing 33 blocks of each of the following dimensions: $13 \times 17 \times 21$, $13 \times 17 \times 37$, $13 \times 21 \times 37$, and $17 \times 21 \times 37$. The chosen blocks are stacked on top of each other (one per cross section) forming a tower of height h. Compute the number of possible values of h.

Answer: **595**. Subtract 13 from each dimension and then divide everything by 4. This changes each height of h to a new height of $h' = \frac{h-1300}{4}$. This injective mapping guarantees that the number of possible h is equal to the number of possible h'. Now we are working with the sum $h' = x_1 + x_2 + \cdots + x_{100}$ where $x_i \in \{0, 1, 2, 6\}$. Obviously, h' is bounded by $0 \leq s \leq 600$. If we initially take $k \geq 97$ 6's and $97 - k$ 0's, we have three more choices. With three choices from $\{0, 1, 2\}$, we can make any number in $\{0, 1, 2, 3, 4, 5, 6\}$, so all of the sums $0 \leq h' \leq 97 \cdot 6 + 6 = 588$ are possible. For h' to exceed 588, we must take 98 6's. Two choices from $\{0, 1, 2, 6\}$, we can make $\{0, 1, 2, 3, 4, 6, 7, 8, 12\}$, which, when added to 98 6's, give the values $h' \in \{588, 589, 590, 591, 592, 594, 595, 596, 600\}$. The fact that we are actually choosing 100 blocks from the crate disqualifies towers built with 100 of the same dimension since each dimension appears on 99 blocks. Although there are many towers that give $h' = 100$ and $h' = 200$, the towers giving $h' = 0$ and $h' = 600$ are uniquely built from 100 13's and 100 37's. Thus, the possible h' are $1 \leq h' \leq 592$, $h' = 594, h' = 595$, and $h' = 596$. A total of 595 values.

11. 10 lines and 10 circles divide the plane into at most n disjoint regions. Compute n.

Answer: **346**. Any arrangement of 10 lines and 10 circles can be constructed in any order. Ten lines such that no two are parallel and no three have a common intersection divide the plane into $1 + (1 + 2 + 3 + \cdots + 10) = 56$ regions. Now, each new circle creates additional regions equal in number to the number of new points of intersection between itself and the other lines and circles (or 1 if it intersects no other objects, but this is clearly not maximal.) Thus, the assumption that the lines divide the plane into as many rregions as possible is valid. Furthermore, we know that the answer is given by $56 + V_{\max}$, where $V_{\max}$ is the number of intersections between two shapes such that at least one is a circle. Since a circle can intersect a line in at most two places, there are at most $2 \cdot 10 \cdot 10 = 200$ circle-line intersections. A pair of circles intersect at no more than two points, so there are at most $2 \cdot \binom{10}{2} = 90$ circle-circle points of intersection. Therefore, the optimal configuration yields $56+200+90 = 346$ regions.

12. Determine the number of permutations of $1, 2, 3, 4, \ldots, 32$ such that if m divides n, the mth number divides the nth number.

Answer: **240**. Let $\pi(m)$ denote the mth number of the permutation. Because there there are exactly as many instances where $\pi(i)|\pi(j)$ as there are $i|j$ for $i \neq j$, it must be that m divides n if and only if $\pi(m)$ divides $\pi(n)$. Therefore, $\pi(m)$ must divide exactly $\lfloor \frac{32}{m} \rfloor$ numbers in the set $\{1, 2, 3, \ldots, 32\}$. It follows that $\pi(m)$ must have as many factors as m. This, in turn, implies that the permutation π can only shuffle sets S with the property that for every x and y in S, the number of factors of x equals the number of factors of y and $\lfloor \frac{32}{x} \rfloor = \left\lfloor \frac{32}{y} \right\rfloor$. The only such sets containing more than one element are $\{11, 13\}$, $\{14, 15\}$, $\{17, 19, 23, 29, 31\}$, $\{18, 20, 28\}$, $\{21, 22, 26, 27\}$, and $\{24, 30\}$. All m not in these sets must have $\pi(m) = m$. With the exception of $11, 13, 22$, and 26, all of these numbers have the property that their proper divisors d must obey $\pi(d) = d$. It follows that the only permutable sets are $\{11, 13\}$, $\{22, 26\}$, and $\{17, 19, 23, 29, 31\}$. The first two are linked - either both pairs of numbers are swapped or neither is swapped; two possibilities. The primes can be arranged in any order; 120 possibilities. All of these permutations are easily seen to satisfy the constraints.

13. x, y, and z are distinct non-zero integers such that $-7 \leq x, y, z \leq 7$. Compute the number of solutions (x, y, z) to the equation

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} = \frac{1}{x+y+z}$$

Answer: **504**. Obviously, $x + y + z \neq 0$. We clear the denominators and simplify:

$$\begin{aligned} yz(x+y+z) + zx(x+y+z) + xy(x+y+z) &= xyz \\ x^2(y+z) + y^2(z+x) + z^2(x+y) + 2xyz &= 0 \\ (x+y)(y+z)(z+x) &= 0 \end{aligned}$$

It must be that $x = -y$, $y = -z$, or $z = -x$. Because x, y, and z are distinct, these three cases do not overlap. The first case has 14 possible pairs (x, y) and 12 choices of z for each of these pairs. Therefore, there are $3 \cdot 14 \cdot 12 = 504$ possible (x, y, z).

14. In triangle ABC, $BC = 27, CA = 32$, and $AB = 35$. P is the unique point such that the perimeters of triangles BPC, CPA, and APB are equal. The value of $AP + BP + CP$ can be expressed as $\frac{p+q\sqrt{r}}{s}$, where p, q, r, and s are positive integers such that there is no prime divisor common to p, q, and s, and r is not divisible by the square of any prime. Determine the value of $p + q + r + s$.

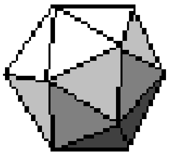
Answer: **171**. Algebra implies that $AP = k + 27$, $BP = k + 32$, and $CP = k + 35$ for some k. Let the points of tangency between the incircle of ABC and $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$ be D, E, and F respectively. Circles ω_1, ω_2, and ω_3 of radii $s - 27$, $s - 32$, and $s - 35$ ($s = 47$, the semiperimeter) and centered at A, B, and C respectively, are pairwise externally tangent at D, E, and F. Let Ω be the circle tangent to ω_1, ω_2, and ω_3, at points T_1, T_2, and T_3 respectively, such that the ω_i lie in the interior of Ω. Let ζ denote the radius of Ω. By the common tangency, lines T_1A, T_2B, and T_3C concur at the center of Ω. P is the center of Ω because if we let $k = \zeta - s$, we have $AP = \zeta - (s - 27) = k + 27$, $BP = k + 32$, and $CP = k + 35$. By the Descartes Circle Theorem,

$$\begin{aligned}
2\left(\frac{1}{12^2} + \frac{1}{15^2} + \frac{1}{20^2} + \frac{1}{\zeta^2}\right) &= \left(\frac{1}{12} + \frac{1}{15} + \frac{1}{20} - \frac{1}{\zeta}\right)^2 \\
2\left(25 + 16 + 9 + \frac{3600}{\zeta^2}\right) &= \left(5 + 4 + 3 - \frac{60}{\zeta}\right)^2 \\
100 + \frac{7200}{\zeta^2} &= 144 - \frac{1440}{\zeta} + \frac{3600}{\zeta^2} \\
44\zeta^2 - 1440\zeta - 3600 &= \\
\zeta &= \frac{180 \pm 30\sqrt{47}}{11}
\end{aligned}$$

We take the positive solution. Computing $AP + BP + CP = 3\zeta - s = \frac{23+90\sqrt{47}}{11}$, which gives an answer of $23 + 90 + 47 + 11 = 171$.

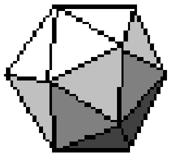
15. $ABCD$ is a convex quadrilateral in which $\overline{AB} \parallel \overline{CD}$. Let U denote the intersection of the extensions of $\overline{AD}$ and $\overline{BC}$. Ω_1 is the circle tangent to line segment $\overline{BC}$ which also passes through A and D, and Ω_2 is the circle tangent to $\overline{AD}$ which passes through B and C. Call the points of tangency M and S. Let O and P be the points of intersection between Ω_1 and Ω_2. Finally, $\overline{MS}$ intersects $\overline{OP}$ at V. If $AB = 2$, $BC = 2005$, $CD = 4$, and $DA = 2004$, then the value of UV^2 is some integer n. Determine the remainder obtained when n is divided by 1000.

Answer: **039**. WLOG $M \in \overline{BC}$. Because $\triangle UAB \sim \triangle UDC$, $UA = 2004$ and $UB = 2005$. Now, by power of a point from U, $UM^2 = 2 \cdot 2004^2$ and $US^2 = 2 \cdot 2005^2$. Hence, $\frac{UM}{US} = \frac{2004}{2005} = \frac{UA}{UB}$, implying that $\triangle UMS \sim \triangle UAB \sim \triangle UDC$. Also, $\frac{UA}{UM} = \frac{1}{\sqrt{2}} = \frac{UM}{UD}$ implying that $\triangle UAM \sim \triangle UMD$. Analogously, $\triangle UBS \sim \triangle USC$. Now $m\angle UMS = m\angle BAU = \pi - m\angle SAB$, hence, $ABMS$ is cyclic. Similarly, $SMCD$ is cyclic. Now, by the Radical Axis Theorem, $\overline{AM}, \overline{BS}$, and $\overline{OP}$ concur at T_1. Similarly, $\overline{CS}, \overline{DM}$, and $\overline{OP}$ concur at T_2. But $\angle USB \cong \angle SCU \cong \angle UDM$, so $\overline{ST_1} \parallel \overline{DM}$. Similarly, $\overline{MT_1} \parallel \overline{CS}$. T_1MT_2S is a parallelogram, hence V is the midpoint of $\overline{MS}$. Using similar triangles again, we find that $MS = 2\sqrt{2}$, from which $UV^2 = \frac{1}{4}\left(2UM^2 + 2US^2 - MS^2\right) = 2004^2 + 2005^2 - 2 \equiv 39 \pmod{1000}$.



Mock AIME #5

3:00-6:00 PM EST
March 5, 2005



1. DO NOT OPEN THIS BOOKLET UNTIL YOU ARE TOLD TO DO SO BY YOUR PROCTOR.

2. This is a 15-questions, 3-hour examination. All answers are integers ranging from 000 to 999, inclusive. Your score will be the number of correct answers; i.e., there is neither partial credit nor a penalty for wrong answers.

3. No aids other than scratch paper, ruler, compass, and protractor are permitted. In particular, CALCULATORS ARE NOT PERMITTED.

4. A combination of the AIME and the American Mathematics Contest 10 or 12 scores are used to determine eligibility for participation in the U.S.A. Mathematical Olympiad (USAMO).

5. Record all of your answers, and certain other information, on the AIME answer form. Only the answer form will be collected from you.

13 Mock AIME 5: Problems

1. The length of a diagonal connecting opposite vertices of a rectangular prism is 47. Determine its volume, given that one of its dimensions is 2 and that the other two dimensions differ by $\sqrt{2005}$.

2. Compute the sum of the prime divisors of $1^2 + 2^2 + 3^2 + \cdots + 2005^2$.

3. x, y, and z are positive integers. Let N denote the number of solutions of $2x + y + z = 2004$. Determine the remainder obtained when N is divided by 1000.

4. The walkway to the front of Euler Elementary School is a 6×15 grid. ACME Sidewalk company has been chartered to tile it with 15 non-overlapping 2×3 concrete slabs. Compute the number of tilings ACME can choose from for this task.

5. If $\frac{1}{0!10!} + \frac{1}{1!9!} + \frac{1}{2!8!} + \frac{1}{3!7!} + \frac{1}{4!6!} + \frac{1}{5!5!}$ is written as a common fraction reduced to lowest terms, the result is $\frac{m}{n}$. Compute the sum of the prime divisors of m plus the sum of the prime divisors of n.

6. Let F be a sequence of positive integers defined by $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for integers $n \geq 1$. Determine the greatest common factor of F_{1495} and F_{1989}.

7. Line segments $\overline{AB}$ and $\overline{CD}$ intersect at P such $AP = 8, BP = 24, CP = 11$, and $DP = 13$. Line segments $\overline{DA}$ and $\overline{BC}$ are extended past A and C respectively until they intersect at Q. If $\overline{PQ}$ bisects $\angle BQD$, then $\frac{AD}{BC}$ can be expressed as $\frac{m}{n}$, where m and n are relatively prime positive integers. Determine $m + n$.

8. Determine the smallest positive integer n for which
$$\frac{2\sin(16^\circ)\cos(32^\circ)}{\sin(29^\circ)} = \sqrt{2} - 2\sin(n^\circ)$$

9. Three spheres $\mathcal{S}_1$, $\mathcal{S}_2$, and $\mathcal{S}_3$ are mutually externally tangent and have radii 2004, 3507, and 4676 respectively. Plane $\mathcal{P}$ is tangent $\mathcal{S}_1$, $\mathcal{S}_2$, and $\mathcal{S}_3$ at A, B, and C respectively. The area of triangle ABC can be expressed as $m\sqrt{n}$, where m and n are positive integers and n is not divisible by the square of any prime. Determine the remainder obtained when $m + n$ is divided by 1000.

10. p, q, and r are positive real numbers such that
$$\begin{aligned} p^2 + pq + q^2 &= 211 \\ q^2 + qr + r^2 &= 259 \\ r^2 + rp + p^2 &= 307 \end{aligned}$$
Compute the value of $pq + qr + rp$.

11. x_1, x_2, and x_3 are complex numbers such that
$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ {x_1}^2 + {x_2}^2 + {x_3}^2 &= 16 \\ {x_1}^3 + {x_3}^3 + {x_3}^3 &= -24 \end{aligned}$$

Let $\gamma = \min(|x_1|, |x_2|, |x_3|)$, where $|a+bi| = \sqrt{a^2+b^2}$ and $i = \sqrt{-1}$. Determine the value of $\gamma^6 - 15\gamma^4 + \gamma^3 + 56\gamma^2$.

12. ABC is a scalene triangle. The circle with diameter $\overline{AB}$ intersects $\overline{BC}$ at D, and E is the foot of the altitude from C. P is the intersection of $\overline{AD}$ and $\overline{CE}$. Given that $AP = 136$, $BP = 80$, and $CP = 26$, determine the circumradius of ABC.

13. Let T_n be defined by

$$T_n := \sum_{i=1}^{n-1} \left\lfloor \frac{i}{n-i} \right\rfloor$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. Compute $T_{2009} - T_{2005}$.

14. In acute triangle ABC, $BC = 10$, $CA = 12$, and $AB = 14$. ω_1, ω_2, and ω_3 are circles with diameters $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$ respectively. Let $\mathcal{B}$ denote the boundary of the region interior to the three ω_i. Ω is the circle internally tangent to the three arcs of $\mathcal{B}$. The radius of Ω can be expressed as $\frac{m-p\sqrt{q}}{n}$, where m, p, and n are positive integers with no common prime divisor and q is a positive integer not divisible by the square of any prime. Compute $m+n+p+q$.

15. Let $O = (0,0)$ and $A = (14,0)$ denote the origin and a point on the positive x-axis respectively. $B = (x, y)$ is a point not on the line $y = 0$. These three points determine lines l_1, l_2, and l_3. Let $P_1, \ldots, P_n$ denote all of the points that are equidistant from l_i for $i = 1, 2, 3$. Let Q_j denote the distance from P_j to the l_i for $j = 1, \ldots, n$. If

$$\begin{aligned} Q_1 + \cdots + Q_n &= \frac{103}{3} \\ \frac{1}{Q_1} + \cdots + \frac{1}{Q_n} &= \frac{2}{3} \end{aligned}$$

then the maximum possible value of $x + y$ can be expressed as $\frac{u}{v}$, where u and v are relatively prime positive integers. Determine $u + v$.

14 Mock AIME 5: Answers

1. 200
2. 680
3. 001
4. 028
5. 057
6. 233
7. 046
8. 013
9. 707
10. 253
11. 056
12. 085
13. 024
14. 392
15. 037

15 Mock AIME 5: Solutions

I have provided full solutions for only the last two problems, which are extraordinarily difficult. See if you can solve the others![6]

1. The length of a diagonal connecting opposite vertices of a rectangular prism is 47. Determine its volume, given that one of its dimensions is 2 and that the other two dimensions differ by $\sqrt{2005}$.

2. Compute the sum of the prime divisors of $1^2 + 2^2 + 3^2 + \cdots + 2005^2$.

3. x, y, and z are positive integers. Let N denote the number of solutions of $2x + y + z = 2004$. Determine the remainder obtained when N is divided by 1000.

4. The walkway to the front of Euler Elementary School is a 6×15 grid. ACME Sidewalk company has been chartered to tile it with 15 non-overlapping 2×3 concrete slabs. Compute the number of tilings ACME can choose from for this task.

5. If $\frac{1}{0!10!} + \frac{1}{1!9!} + \frac{1}{2!8!} + \frac{1}{3!7!} + \frac{1}{4!6!} + \frac{1}{5!5!}$ is written as a common fraction reduced to lowest terms, the result is $\frac{m}{n}$. Compute the sum of the prime divisors of m plus the sum of the prime divisors of n.

6. Let F be a sequence of positive integers defined by $F_1 = F_2 = 1$ and $F_{n+2} = F_{n+1} + F_n$ for integers $n \geq 1$. Determine the greatest common factor of F_{1495} and F_{1989}.

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$$\frac{2\sin(16^\circ)\cos(32^\circ)}{\sin(29^\circ)} = \sqrt{2} - 2\sin(n^\circ)$$

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Compute the value of $pq + qr + rp$.

[6] If you really need help, look for "Mock AIME 7" on the AoPS AMC forum. Upon compilation of this document, the relevant URL was http://www.artofproblemsolving.com/Forum/topic-29263.html.

11. x_1, x_2, and x_3 are complex numbers such that

$$\begin{aligned} x_1 + x_2 + x_3 &= 0 \\ {x_1}^2 + {x_2}^2 + {x_3}^2 &= 16 \\ {x_1}^3 + {x_3}^3 + {x_3}^3 &= -24 \end{aligned}$$

Let $\gamma = \min\left(|x_1|, |x_2|, |x_3|\right)$, where $|a+bi| = \sqrt{a^2+b^2}$ and $i = \sqrt{-1}$. Determine the value of $\gamma^6 - 15\gamma^4 + \gamma^3 + 56\gamma^2$.

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$$T_n := \sum_{i=1}^{n-1} \left\lfloor \frac{i}{n-i} \right\rfloor$$

where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x. Compute $T_{2009} - T_{2005}$.

14. In acute triangle ABC, $BC = 10$, $CA = 12$, and $AB = 14$. ω_1, ω_2, and ω_3 are circles with diameters $\overline{BC}$, $\overline{CA}$, and $\overline{AB}$ respectively. Let $\mathcal{B}$ denote the boundary of the region interior to the three ω_i. Ω is the circle internally tangent to the three arcs of $\mathcal{B}$. The radius of Ω can be expressed as $\frac{m-p\sqrt{q}}{n}$, where m, p, and n are positive integers with no common prime divisor and q is a positive integer not divisible by the square of any prime. Compute $m+n+p+q$.

Answer: **392**. Let M_A, M_B, and M_C denote the midpoints of the sides opposite A, B, and C respectively, and write P for the center of Ω. Finally, let the tangents of ω_1, ω_2, and ω_3 with Ω be denoted by T_1, T_2, and T_3.

Note that $M_AM_B = 7$, $M_BM_C = 5$, and $M_CM_A = 6$. Take points U_1, U_2, and U_3 on the extensions of T_1M_A, T_2M_B, and T_3M_C past M_A, M_B, and M_C respectively such that $T_iU_i = 9$ for $i = 1, 2, 3$. Because T_1 lies on ω_1, $T_1M_A = 5$ so that $M_AU_1 = 4$. Analogously, $M_BU_2 = 3$ and $M_CU_3 = 2$.

Now consider circles ω_1', ω_2' and ω_3' centered at M_A, M_B, and M_C and of radii 4, 3, and 2 respectively. ω_1', ω_2', and ω_3' are mutually externally tangent and contain points U_1, U_2, and U_3 respectively. But the circumcenter of $U_1U_2U_3$ is P; ergo, the circumcircle Ω' of triangle $U_1U_2U_3$ is tangent to ω_1', ω_2', and ω_3'. Moreover, $R_\Omega' = 9 - R_\Omega$. Applying the explicit form of the Descartes Circle Theorem for the outer circle, we find

$$\begin{aligned} R_\Omega' &= \left| \frac{2\cdot 3\cdot 4}{2\cdot 3 + 3\cdot 4 + 4\cdot 2 - 2\sqrt{2\cdot 3\cdot 4\cdot(2+3+4)}} \right| \\ &= \frac{12\cdot(13+6\sqrt{6})}{216-169} = \frac{156+72\sqrt{6}}{47} \end{aligned}$$

from which $R_\Omega = 9 - R_\Omega' = \frac{267-72\sqrt{6}}{47}$. The answer is therefore $267 + 72 + 6 + 47 = 392$.

15. Let $O = (0,0)$ and $A = (14,0)$ denote the origin and a point on the positive x-axis respectively. $B = (x,y)$ is a point not on the line $y = 0$. These three points determine lines l_1, l_2, and l_3. Let $P_1, \ldots, P_n$ denote all of the points that are equidistant from l_i for $i = 1,2,3$. Let Q_j denote the distance from P_j to the l_i for $j = 1, \ldots, n$. If

$$\begin{aligned} Q_1 + \cdots + Q_n &= \frac{103}{3} \\ \frac{1}{Q_1} + \cdots + \frac{1}{Q_n} &= \frac{2}{3} \end{aligned}$$

then the maximum possible value of $x+y$ can be expressed as $\frac{u}{v}$, where u and v are relatively prime positive integers. Determine $u+v$.

Answer: **037**. A point that is equidistant from two lines lies on one of the two lines that bisect angles formed at the intersection of the two lines. Considering this fact, it is clear that $n = 4$, where P_1 is the incenter of ABO (which we will denote I) and P_2, P_3, and P_4 are the three excenters. Let r denote the inradius of ABO and r_1, r_2, and r_3 the three exradii. We have

$$\begin{aligned} r + r_1 + r_2 + r_3 &= 4R + 2r = \frac{103}{3} \\ \frac{1}{r} + \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} &= \frac{2}{r} = \frac{2}{3} \end{aligned}$$

This pair of equations is easily solved for $r = 3$ and $R = \frac{85}{12}$, where R is the circumradius of ABO. By the Extended Law of Sines, $\sin\angle B = \frac{AO}{2R} = \frac{14}{2 \cdot \frac{85}{12}} = \frac{84}{85}$. Pythagoras yields $\cos\angle B = \pm\frac{13}{85}$. Then $\cot .5\angle B = \frac{6}{7}$ or $\frac{7}{6}$.

Let the incircle of ABO be tangent to $\overline{BO}$, $\overline{OA}$, and $\overline{AB}$ at P, Q, and R respectively. Let $AQ = X$ and $QO = 14 - X$. Since $r = IQ = 3$, we have $\cot .5\angle A = \frac{X}{3}$ and $\cot .5\angle O = \frac{14-X}{3}$. Since $\cot .5\angle A \cot .5\angle B \cot .5\angle O = \cot .5\angle A + \cot .5\angle B + \cot .5\angle O$, we have

$$X \cdot (14 - X) = \frac{42 + 9\cot .5\angle B}{\cot .5\angle B} \leq 49$$

by substitution and AM-GM. $\cot .5\angle B = \frac{6}{7}$ leads to $58 \leq 49$, impossible, so $\cot .5\angle B = \frac{7}{6}$, which leads to $X = 5$ or 9. Thus, there are four possible B, each obtained by reflecting B over $\overline{AO}$ and the perpendicular bisector of $\overline{AO}$. Since we are maximizing the sum $x+y$, we choose $X = 5$ and assume $y > 0$. Now, $BR = \frac{7}{2}$ and $RA = 5$. Since $\cos\angle A = 2\cos .5\angle A^2 - 1 = \frac{25}{17} - 1 = \frac{8}{17}$, it must be that $x = 14 - \frac{17}{2} \cdot \frac{8}{17} = 10$ and $y = \frac{17}{2} \cdot \frac{15}{17} = \frac{15}{2}$, which gives $x+y = \frac{35}{2}$ and an answer of $35 + 2 = 37$.