Weights & Coloring

1. Let n be a given integer greater than 2. We wish to label each side and each diagonal of a regular n-gon $P_1P_2\cdots P_n$ with a positive integer less than or equal to r so that

   (a) every integer between 1 and r occurs as a label;

   (b) in each triangle $P_iP_jP_k$ two of the labels are equal and greater than the third.

   Given these conditions:

   (1) Determine the largest positive integer r for which this can be done.

   (2) For that maximum value of r, how many such labelings are there?

2. In an $m \times n$ rectangular board of mn unit squares, two squares are *adjacent* if they share a common edge. A *path* is a sequence of squares in which any two consecutive squares are adjacent. Each square of the board can be colored black or white. Let N denote the number of coloring schemes of the board such that there exists at least one black path from the left edge of the board to its right edge, and let M denote the number of coloring schemes in which there exist at least two non-intersecting black paths from the left edge to the right edge. Prove that $N^2 \geq M \cdot 2^{mn}$.

3. A k-coloring of a graph G is a coloring of its vertices using k possible colors such that the end points of any edge have different colors. We say a graph G is *uniquely k-colorable* if on the one hand it has a k-coloring, but on the other hand there do not exist vertices u and v such that u and v have the same color in one k-coloring and u and v have different colors in another k-coloring. Prove that if a graph G with n vertices ($n \geq 3$) is uniquely 3-colorable, then it has at least $2n-3$ edges.

4. Let $a_1, a_2, \ldots, a_8$ be eight distinct points on the circumference of a circle such that no three chords, each joining a pair of points, are concurrent. Every four of the eight points form a quadrilateral which is called a **quad**. If two chords, each joining a pair of the eight points, intersect, the point of intersection is called a **bullet**. Suppose some of the bullets are colored red. For each pair (i,j), with $1 \leq i < j \leq 8$, let $r(i,j)$ be the number of quads containing a_i, a_j as vertices and whose diagonals intersect at a red bullet. Determine the smallest positive integer n such that it is possible to color n of the bullets red so that $r(i,j)$ is a constant for all pairs (i,j).

5. Points $P_1, P_2, \ldots, P_n$, with no three collinear, are given in the plane. Each point is colored in eaither red or blue. A nonempty set S of the triangles with $P_1, P_2, \ldots, P_n$ as their vertices is called *good* if for every pair of segments P_iP_j and P_uP_v, the number of triangles in S containing side P_iP_j is equal to the number of triangles in S containing side P_uP_v. A triangle is called *monochromatic* if all its vertices are of the same color. Determine the minimum value of n such that there are always two monochromatic triangles in any good set.

6. There is a set of n coins with distinct integer weights $w_1, w_2, \ldots, w_n$. It is known that if any coin with weight w_k, $1 \leq k \leq n$, is removed from the set, the remaining coins can be split into two groups of the same weight. (The number of coins in the two groups can be different.) Find all n for which such a set of coins exists.

7. Can we tile a 13×13 table from which we remove the central unit square using only 1×4 or 4×1 rectangles?

8. In the Cartesian corrdinate plane define the strip

$$S_n = \{(x, y) : n \leq x \leq n + 1\}$$

for every integer n. Assume that each strip S_n is colored either red or blue, and let a and b be two distinct positive integers. Prove that there exists a rectangle with side lengths a and b such that its vertices have the same color.

9. A (simple and finite) graph is called *balanced* if it has n edges, and it is possible to assign each of the numbers $1, 2, \ldots, n$ to an edge (one number per edge) in such a way that teh sum of the numbers assigned to the edges connecting to each vertex is the same. Determine if every balanced graph is connected.

10. Find all positive integers n, for which the numbers in the set $S = \{1, 2, \ldots, n\}$ can be colored red and blue, with the following condition being satisfied: the set $S \times S \times S$ contains exactly 2007 ordered triples (x, y, z) such that (i) x, y, z are of the same color and (ii) the integer $x + y + z$ is divisible by n.

11. Every positive integer is colored in either red or blue. Prove that there exists an infinite sequence $a_1 < a_2 < \cdots$ such that the sequence

$$a_1, \frac{a_1 + a_2}{2}, a_2, \frac{a_2 + a_3}{2}, a_3, \frac{a_3 + a_4}{2}, \ldots$$

is a monochromatic integer sequence.

12. A collection of $\frac{n^2+n}{2}$ mathematicians are arranged in a series of piles. A move consists in taking one mathematician from each pile and forming a new pile from these. (A pile whose last element has been removed is no longer a pile.) Prove that after some number of moves, the piles will have sizes $1, 2, \ldots, n$.

13. Given a square board of size $n \times n$, where n is an integer greater than 1, we label some of the squares by distinct numbers from the set $\{1, 2, \ldots, n^2\}$. What is the largest number of board squares we can label in this way without creating a difference of n or higher between any two labels on neighboring squares on the board?

14. Determine the minimum positive integer n satisfying the following condition: If each of the vertices of an n-sided regular polygon is colored in one of the three colors (say, red, green, and blue), then there is a monochromatic isosceles trapezoid. (Note that a parallelogram is not a trapezoid).

15. For a given prime p, find the greatest positive integer n with the following property: the edges of the complete graph on n vertices can be colored with $p + 1$ colors so that:

    (a) at least two edges have different colors;

    (b) if A, B and C are any three vertices and the edges AB and AC are of the same color, then BC has the same color as well.

16. Every integer is marked with one of 100 colors so that there is at least one number of each color. If $[a, b]$ and $[c, d]$ are two intervals of the same length such that a and c have the same color and also b and d have the same color, then $a + x$ and $c + x$ have the same color for each $x \in [0, b - a]$. Prove that -2001 and 2001 have different colors.

17. For what values of n is it possible to color every square in an $n \times n$ grid red, blue, yellow, green, or orange, so that for all i, j, k between 1 and n with $i \neq j$ and $j \neq k$, the square in row i and column$j$ is assigned a different color from the square in row j and column k?

18. Let $a_1 \geq a_2 \geq \cdots \geq a_k$ and $b_1 \geq b_2 \geq \cdots \geq b_m$ be nonnegative integers such that

$$\min(a_1, d) + \min(a_2, d) + \cdots + \min(a_k, d) \geq b_1 + b_2 + \cdots + b_d$$

for all $d \leq m$ and

$$\min(b_1, d) + \min(b_2, d) + \cdots + \min(b_m, d) \geq a_1 + a_2 + \cdots + a_d$$

for all $d \leq k$. Prove that we can mark some of the squares of a $k \times m$ table so that the numbers of marked squares in the rows are exactly $a_1, a_2, \ldots, a_k$ and in the columns are exactly $b_1, b_2, \ldots, b_m$.