Geometric Calculations

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Some of these problems may already be familiar to you. If they are, try to find a computational answer.

1. Let AB be a diameter of a circle ω with center O. Suppose P lies on ω such that AOP is isoceles. Let ω_1 have diameter OB. Suppose circle ω_2 is tangent to ω, ω_1 and PO. Show that the center of ω_2 lies on the perpendicular bisector of AB.

2. Prove Descartes' four circle theorem.

3. For any point A inside a circle, we define a transformation f_A rom the circle to itself that takes P on the circle to the other intersection of AP with the circle. Show that the composition of f_A and f_B for two different points A, B is f_C for some C followed by a rotation.

4. Consider five points A, B, C, D, E such that $ABCD$ is a parallelogram and $BCED$ is a cyclic quadrilateral. Let l be a line passing through A. Suppose that l intersects the interior of the segment DC at F and intersects line BC at G. Suppose also that $EF = EG = EC$. Prove that l is the bisector of $\angle DAB$.

5. Let I be the incenter of triangle ABC. Let K, L, and M be the points of tangency of the incircle of ABC with AB, BC, and CA, respectively. The line t passes through B and is parallel to KL. The lines MK and ML intersect t at the points R and S. Prove that $\angle RIS$ is acute.

6. Two circles O_1 and O_2 touch internally the circle O in M and N, and the center of O_2 is on O_1. The common chord of the circles O_1 and O_2 intersects O in A and B. MA and MB intersect O_1 in C and D. Prove that O_2 is tangent to CD.

7. The point M inside the convex quadrilateral $ABCD$ is such that $MA = MC$, $\angle AMB = \angle MAD + \angle MCD$, $\angle CMD = \angle MCB + \angle MAB$. Prove that $AB \cdot CM = BC \cdot MD$ and $BM \cdot AD = MA \cdot CD$.

8. $A_1A_2A_3$ is an acute-angled triangle. The foot of the altitude from A_i is K_i, and the incircle touches the side opposite A_i at L_i. The line K_1K_2 is reflected in the line L_1L_2. Similarly, the line K_2K_3 is reflected in L_2L_3 and K_3K_1 is reflected in L_3L_1. Show that the three new lines form a triangle with vertices on the incircle.

9. The tangents at B and A to the circumcircle of an acute-angled triangle ABC meet the tangent at C at T and U respectively. AT meets BC at P, and Q is the midpoint of AP; BU meets CA at R, and S is the midpoint of BR. Prove that $\angle ABC = \angle BAS$. Determine, in terms of ratios of side lengths, the triangles for which this angle is a maximum.

10. Let ABC be a triangle with $\angle BAC = 60°$. Let AP bisect $\angle BAC$ and let BQ bisect $\angle ABC$, with P on BC and Q on AC. If $AB + BP = AQ + QB$, what are the angles of the triangle?

11. The incircle ω of the acute-angled triangle ABC is tangent to BC at K. Let AD be an altitude of triangle ABC and let M be the midpoint of AD. If N is the other common point of ω and KM, prove that ω and the circumcircle of triangle BCN are tangent at N.

12. A convex hexagon has the property that for any pair of opposite sides the distance between their midpoints is $\sqrt{3}/2$ times the sum of their lengths Show that all the hexagons angles are equal.

13. Let S_1 and S_2 be circles meeting at the points A and B. A line through A meets S_1 at C and S_2 at D. Points M, N, K lie on the line segments CD, BC, BD respectively, with MN parallel to BD and MK parallel to BC. Let E and F be points on those arcs BC of S_1 and BD of S_2 respectively that do not contain A. Given that EN is perpendicular to BC and FK is perpendicular to BD, prove that $\angle EMF = 90°$.

14. Let ABC be an acute triangle with ω, Ω, and R being its incircle, circumcircle, and circumradius, respectively. Circle ω_A is tangent internally to Ω at A and tangent externally to ω. Circle Ω_A is tangent internally to Ω at A and tangent internally to ω. Let P_A and Q_A denote the centers of ω_A and Ω_A, respectively. De?ne points P_B, Q_B, P_C , Q_C analogously. Prove that

$$8P_AQ_A \cdot P_BQ_B \cdot P_CQ_C = R^3.$$

15. Prove Feuerbach's theorem, which says that the nine-point circle is tangent to the incircle and all three excircles. The nine-point circle of a triangle passes through the midpoints of the sides and the feet of the altitudes.