Complex Numbers

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1 Problems

1. The following evaluates to $\frac{a^b+c}{d}$ find the ordered quadruple (a, b, c, d) for a, c, d prime.

$$\sum_{k=0}^{33} \binom{99}{3k}$$

2. The following evaluates to $\frac{n+1}{n} \frac{m+1}{m}$ for $x = \frac{\pi}{3}$ find (n, m).

$$\sum_{k=0}^{2006} \frac{\cos(kx)}{3^k}$$

2 Complex Numbers Basics

Complex numbers are all numbers,z, of the form $a + bi$, where a, b is real and $i = \sqrt{-1}$. $\bar{z} = a - bi$ and $\mid z \mid = z \cdot \bar{z}$. There is an interesting property of complex numbers called Euler's Formula and it says the following.

$$e^{i\theta} = \cos(\theta) + i \sin(\theta)$$

θ is called the argument of z sometimes called $\text{Arg}(z)$. There is also a consequence of this formula called Demoivre's Theorem.

$$e^{ni\theta} = (\cos(\theta) + i \sin(\theta))^n$$

3 Solution to Problems

1. This problem is disguised as a combinatorics problem, but it is really a complex numbers problem. This problem asks to find the sum of the coefficients of $(1 + x)^{99}$ for all powers of x

divisible by 3. For a polynomial, $P(x)$, the sum of the coefficients for all powers divisible by 1 is just $P(1)$ and the sum of the coefficients for all powers divisible by 2 is just $\frac{P(1)+P(-1)}{2}$, but what is it it for powers divisible by 3 or more? This brings complex numbers onto the field.

It can be shown that $\displaystyle\sum_{k=0}^{m-1} e^{2ni\pi k/m} = 0$ for $m \nmid n$. Therefore in our problem $1^n + e^{n2i\pi/3} + e^{n4i\pi/3} = 0$ for $3 \nmid n$ and $1^n + e^{n2i\pi/3} + e^{n4i\pi/3} = 3$ for $3 \mid n$. Then the sum of the coefficients for all powers divisible by 3 is just $\frac{P(1)+P(e^{2i\pi/3})+P(e^{4i\pi/3})}{3}$ and in this problem $P(x) = (1+x)^{99}$.

$$\frac{2^{99} + (1+e^{2i\pi/3})^9 9 + (1+e^{4i\pi/3})^9 9}{3} = \frac{2^{99} + (e^{i\pi/3})^{99} + (e^{-i\pi/3})^{99}}{3} =$$
$$\frac{2^{99} + 2\cos(33\pi)}{3} = \frac{2^{99} - 2}{3} \to (2, 99, -2, 3).$$

2. This problem when using complex numbers nicely evaluates to a geometric series, because $\frac{\cos(kx)}{3^k} = \Re(\frac{e^{ix}}{3})^k$. Therefore the sum is $\Re\frac{\frac{e^{2007ix}}{3^{2007}} - 1}{\frac{e^{ix}}{3} - 1} = \Re\frac{(\frac{e^{2007ix}}{3^{2007}} - 1)(\cos(x)/3 - 1 - i\sin(x)/3}{(\cos(x)/3-1)^2 + (\sin(x)/3)^2} =$ $\frac{\frac{\cos(2006x)}{3^{2008}} - \frac{\cos(x)}{3} - \frac{\cos 2007x}{3^{2007}} + 1}{(\cos(x)/3-1)^2 + (\sin(x)/3)^2} = \frac{\frac{5}{6} + \frac{5}{2\cdot 3^{2008}}}{7/9} = \frac{15}{14}\frac{3^{2007}+1}{3^{2007}} \to (14, 3^{2007})$

4 Problems

1. Evaluate the following.

$$\sum_{n=0}^{\infty} \frac{n^2+n+1}{(n^2+1)(n^2+2n+2)}$$

2. Compute the number of z, such that $0 \le \text{Arg}(z) \le 2\pi$ and $\mid z \mid= 1$, satisfying the following.

$$z^{29} - z^7 - 1 = 0$$

3. For regular heptagon $ABCDEFG$ inscribed in a unit circle compute $AB^2 + AC^2 + AD^2 + AE^2 + AF^2 + AG^2$.