Triangle Geometry

Hojoo Lee

In this short note, we give some well-known formulae in Triangle Geomtry :

1. Law of Sines

$$\frac{a}{sinA} = \frac{b}{sinB} = \frac{c}{sinC} = 2R$$

2. Law of Cosines

$$a = bcosC + ccosB$$
$$cosA = \frac{b^2 + c^2 - a^2}{2bc}$$

3. Area S, semiperimeter s, $x = s - a$, $y = s - b$, $z = s - c$

$$\begin{aligned}
S &= \sqrt{s(s-a)(s-b)(s-c)} \\
&= \sqrt{xyz(x+y+z)} \\
&= \frac{1}{4}\sqrt{2\sum_{\text{cyclic}} a^2b^2 - \sum_{\text{cyclic}} a^4} \\
&= \frac{1}{2}bcsinA = \frac{1}{2}casinB = \frac{1}{2}absinC \\
&= 2R^2 sinAsinBsinC \\
&= rs = (s-a)r_A = (s-b)r_B = (s-c)r_C \\
&= \sqrt{rr_Ar_Br_C} \\
&= \frac{abc}{4R}
\end{aligned}$$

4. $\cos\frac{A}{2}$, $\sin\frac{A}{2}$, $\tan\frac{A}{2}$

$$cos\frac{A}{2} = \sqrt{\frac{s(s-a)}{bc}},\ sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}},\ tan\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \frac{r}{s-a}$$

5. $R,\ r,\ r_A,\ r_B,\ r_C$

$$4R + r = r_A + r_B + r_C,\ \frac{1}{r} = \frac{1}{r_A} + \frac{1}{r_B} + \frac{1}{r_C}$$

$$r^2 = \frac{xyz}{x+y+z},\ R^2 = \frac{(x+y)^2(y+z)^2(z+x)^2}{16xyz(x+y+z)}$$

$$\frac{r}{R} = \frac{xyz}{2(x+y)(y+z)(z+x)},\ r = 4Rsin\frac{A}{2}sin\frac{B}{2}sin\frac{C}{2}$$

$$1 + \frac{r}{R} = cosA + cosB + cosC,\ Rr = \frac{abc}{4s} = \frac{(x+y)(y+z)(z+x)}{2(x+y+z)}$$

6. O(circumcenter), G(centroid), H(orthocenter), I(incenter)

$$Euler\ Line\ OGH\ :\ \vec{OH} = 3\vec{OG} = \vec{OA} + \vec{OB} + \vec{OC}$$

$$XG^2 = \frac{1}{3}\sum_{\text{cyclic}} XA^2 - \frac{1}{9}\sum_{\text{cyclic}} BC^2$$

$$3\left(XA^2 + XB^2 + XC^2\right) \geq \left(BC^2 + CA^2 + AB^2\right)$$

$$3(GA^2 + GB^2 + GC^2) = BC^2 + CA^2 + AB^2$$

$$HG^2 = 4R^2 - \frac{4}{9}(BC^2 + CA^2 + AB^2)$$

$$OH^2 = 9R^2 - (BC^2 + CA^2 + AB^2)$$

$$XI^2\sum_{\text{cyclic}} a + abc = \sum_{\text{cyclic}} aXA^2$$

$$aXA^2 + bXB^2 + cXC^2 \geq abc$$

$$\frac{IA^2}{bc} + \frac{IB^2}{ca} + \frac{IC^2}{ab} = 1$$

$$Euler\ :\ OI^2 = R^2 - 2rR$$

$$\vec{OI} = \frac{1}{a+b+c}\sum_{\text{cyclic}} a\vec{OA}$$

$$IG^2 = r^2 + \frac{1}{36}\left(5\sum_{\text{cyclic}} a^2 - 6\sum_{\text{cyclic}} ab\right)$$

$$AI = \frac{b+c}{a+b+c}\sqrt{bc\left(1 - \left(\frac{a}{b+c}\right)^2\right)}$$

$$AH = 2RcosA$$

$$\vec{IH}\cdot\vec{IG} = -\frac{2}{3}r(R-2r)$$

7. Trigonometric Identities

$$sinx + siny + sinz - sin(x+y+z) = 4sin\frac{x+y}{2}sin\frac{y+z}{2}sin\frac{z+x}{2}$$
$$cosx + cosy + cosz - cos(x+y+z) = 4cos\frac{x+y}{2}cos\frac{y+z}{2}cos\frac{z+x}{2}$$

$$\sum_{\text{cyclic}} sinA = 4cos\frac{A}{2}cos\frac{B}{2}cos\frac{C}{2}$$
$$\sum_{\text{cyclic}} sin2A = 4sinAsinBsinC$$
$$\sum_{\text{cyclic}} cosA = 1 + 4sin\frac{A}{2}sin\frac{B}{2}sin\frac{C}{2}$$
$$\sum_{\text{cyclic}} cos2A = -1 - 4cosAcosBcosC$$
$$\sum_{\text{cyclic}} cos^2A = 1 - 2cosAcosBcosC$$
$$\sum_{\text{cyclic}} sin^2A = 2 + 2cosAcosBcosC$$
$$\sum_{\text{cyclic}} tanA = tanAtanBtanC$$
$$\sum_{\text{cyclic}} cotAcotB = 1$$
$$\sum_{\text{cyclic}} cot\frac{A}{2} = cot\frac{A}{2}cot\frac{B}{2}cot\frac{C}{2}$$