Numbers in Tables
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Problems
Integer Numbers

1. A rectangular array of real numbers is given. In each row and each column the sum of the numbers is an integer. Prove that each noninteger number x can be replaced by $\lfloor x \rfloor$ or $\lceil x \rceil$ so that the row-sums and column-sums remain the same. (Shortlist 1998)

   Proof: In each unit square we place a + if the number was changed to the upper integer part and a - if the number was changed to a lower integer part (leave blank if the number is an integer). Start with a table filled with +'s.

   One can easily place +'s and -'s so that in each column we preserve the sum of the numbers. Let s be the sum of the absolute value of the difference between the initial row sum and the final row sum. The problem is equivalent to showing that one can place the +'s and -'s in the table so that $s = 0$.

   We start from a determined configuration and assume that in this case $s > 0$ (otherwise we are done). Then we shall decrease s by 2 at a time.

   We call row a connected to row b if there is a column c so that $a \cap c = +$ and $b \cap c = -$. We call a row overrated (underrated) if the sum of the numbers in the row is bigger (smaller) than the initial row sum.

   Now what we are trying to show is that we can place +'s and -'s in rows so that the row sums are also preserved. Note that if in a column we swap a + and a - then the column sum remains unchanged. So we need to decrease s by using such transformations.

   Consider an overrated row a. Then there is a + in it. Consider the column containing this +. Since the column sum is preserved then there is a - on this column. If the row b containing this - is underrated then swap the + with the - and then the difference between the row sum on a and the initial row sum of a decreases by 1 as well as the difference between the row sum on b and the initial row sum of b. Therefore s decreases by 2.

   This means that if a is connected to b and a is overrated and b is underrated then we can get to a configuration with a smaller s. If b is also overrated then there must be a + on this row since we already have a - on it. The we may apply the same procedure to row b. If there is an underrated row c to which b is connected then we are done. By continuing this procedure (going from one overrated row to another so that one is connected to the one immediately after it) we get to certain rows that we call accessible from a.

   Now assume this cannot always happen. Let a be a row with a sum bigger than the initial sum. Let A be the set of all rows that are accessible from a and let B be the set of all the other rows (not including a). Consider a column C.

If $A \cap C \neq +$ then the sum in $A \cap C$ has not increased. If $A \cap C = +$ then $B \cap C \neq -$ because then some row in B would be accessible from a and that is not possible. So the sum in $B \cap C$ has not decreased. But in that case it means that the sum in $A \cap C$ has not increased since the sum in C is preserved. Since C was chosen arbitrarily this means that the sum in A has not increased. Since $a \in A$ and this row is overrated it means that there is an underrated row in A. But by the previously mentioned argument this means that we can decrease s.

Apply the decreasing procedure several times and then s has to get to 0 by the nonexistence of infinite descent.

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2. The numbers from 1 to n^2 are randomly arranged in the cells of a $n \times n$ square ($n \geq 2$). For any two numbers in the same row or column we calculate the ratio of the greater one to the smaller one. We call the characteristic of the table the smallest of these $n^2(n-1)$ numbers. What is the highest possible characteristic? (Shortlist 1999)

   Proof: Look at the numbers $A = \{n^2 - n + 1, \dots, n^2\}$. If any two of these numbers are on a same row or column then the characteristic is $\leq \frac{n^2}{n^2-n+1} < \frac{n+1}{n}$.

   If each of these numbers is on different row and column then look at number $n^2 - n$. Then this is on a row and on a column where one of the numbers in A is. At least one of these two numbers is not equal to n^2. Then choose this number $a \in A$ (so $a > n^2 - n$) so the characteristic of the table is $\leq \frac{a}{n^2-n} \leq \frac{n^2-1}{n^2-n} = \frac{n+1}{n}$.

   Now consider the following table (all numbers are in base n)

$$\begin{pmatrix} \overline{(n-1)1} & 1 & \overline{11} & \overline{21} & \dots & \overline{(n-2)1} \\ \overline{(n-2)2} & \overline{(n-1)2} & 2 & \overline{31} & \dots & \overline{(n-3)2} \\ \overline{(n-3)3} & \overline{(n-2)3} & \overline{(n-1)3} & 3 & \dots & \overline{(n-4)3} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \overline{(n-1)1} & \overline{2(n-1)} & \overline{3(n-1)} & \overline{4(n-1)} & \dots & n-1 \\ \overline{10} & \overline{20} & \overline{30} & \dots & \overline{(n-1)0} & \overline{100} \end{pmatrix}$$

   Now in each row and column the difference between any two numbers is at least n so the characteristic is at least $\frac{n+1}{n}$. Since I already proved the upper bound it means that $\frac{n+1}{n}$ is the highest value of the characteristic of the table. ■

3. The 2^n rows of a $2^n \times n$ table are filled with all the different n-tuples of the numbers $+1, -1$. Then some numbers were replaced by 0-s. Prove that one can choose a nonempty set of rows so that the sum of all the numbers written in them is 0. (Tournament of Towns 1996)

   Proof: Let $a_1 = (1, 1, \dots, 1), a_2, \dots, a_{2^n-1}, a_{2^n} = (-1, -1, \dots, -1)$ be the initial rows and for a row x let $f(x)$ be the row with some 0's in it. Also for a row x with some zeros in it let $g(x)$ be the corresponding row in the initial table so that for any 1 in

x $g(x)$ has a -1 in the same position and for any 0 or -1 in x $g(x)$ has a 1 in that position.

Define $b_1 = f(a_1), b_{n+1} = b_n + f(g(b_n))$. It is easy to see that this sequence is well defined. I prove by induction that b_n consists only of 0's and 1's. The base case is obvious since b_1 is just $a_1 = (1, 1, \dots, 1)$ with some 0's in it.

Assume that b_n has only 0's and 1's. Then $f(g(b_n))$ can only have -1's where b_n has a 1. Since 1 goes to -1 through g it means that the only way to get a -1 in $f(g(b_n))$ is from a 1 in b_n so b_{n+1} will have nonnegative elements. Also, we need that no two 1's occur on the same position in the addition. But a 1 in $f(g(b_n))$ can only occur where a 0 was in b_n so by induction b_n consists only of 0's and 1's.

If we put $b_0 =^{\text{def}} 0$ then we have 2^n numbers b_i that can take 2^n values (the number of n-sequences of 0's and 1's). So either we have a $b_i = 0 \Longrightarrow b_i = b_0$ or by the pigeonhole principle we have $i \neq j$ with $b_i = b_j$. Either way we get to $i > j$ with $b_i = b_j$ which means that $f(g(b_j)) + f(g(b_{j+1})) + \dots + f(g(b_{i-1})) = b_i - b_j = 0$ and this is what we had to prove since this is a sum of rows in the transformed table. ■

Real Numbers

1. Consider two lattice figures $\mathcal{F}_1, \mathcal{F}_2$ made of unit squares in the lattice plane. These figures can be translated in any direction. Real numbers are written in each lattice unit square so that for any translation of $\mathcal{F}_1$ the sum of the numbers in it is positive. Prove that there is a translation of $\mathcal{F}_2$ so that the sum of the numbers written in it is positive. (Romania 2000)

   Proof: Let $\mathcal{F}_1 = \{a_1, \dots, a_n\}$ and $\mathcal{F}_2 = \{b_1, \dots, b_m\}$ be the unit squares (identified with the vectors pointing to the centers of the unit squares) that make up the two figures.

   Consider the following figure (overlapping unit squares count several times) $\mathcal{F} = \{a_i + b_j | i = 1, \dots, n, j = 1, \dots, m\}$. Then note that $\mathcal{F} = (\mathcal{F}_1 + b_1) \cup \dots \cup (\mathcal{F}_1 + b_m) = (\mathcal{F}_2 + a_1) \cup \dots \cup (\mathcal{F}_2 + a_n)$. The first union implies that the sum of all the numbers in $\mathcal{F}$ is positive (counting multiplicities for the numbers of course). This means that the sum of the numbers written in $(\mathcal{F}_2 + a_1) \cup \dots \cup (\mathcal{F}_2 + a_n)$ is also positive so for at least one i we have that the sum of the numbers in $\mathcal{F}_2 + a_i$ is positive and this is what we had to prove.

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2. Consider the table $(a_{i,j})_{i,j=1,2,\dots,n}$ so that $a_{i,j} = \frac{1}{i+j-1}$. Choose any n entries of the table so that no two lie on the same row or column. Prove that the sum of all these numbers is ≥ 1. (Tournament of Towns 1992)

   Proof: Let the rows be $1, 2, \dots, n$ (WLOG) and then the columns are $\sigma(1), \dots, \sigma(n)$ for a $\sigma \in S_n$. Then the sum of the numbers is $\sum \frac{1}{i+\sigma i-1}$. By Cauchy Schwarz we get that

$$\left(\sum (i + \sigma(i) - 1)\right) \left(\sum \frac{1}{i + \sigma i - 1}\right) \geq n^2$$

. Note that $\sum(i+\sigma(i)-1) = n(n+1)-n = n^2$ so the conclusion follows. ■

3. On a $m \times n$ sheet of paper is drawn a grid dividing the sheet into unit squares. The two sides of length $n$are taped together to form a cylinder. Prove that it is possible to write a real number in each square, not all zero, so that each number is the sum of the numbers in the neighboring squares, if and only if there exist integers k, l such that $n+1$ does not divide k and

$$\cos\frac{2l\pi}{m} + \cos\frac{k\pi}{n+1} = \frac{1}{2}$$

(Romania 1998)

Proof: Number the rows $1, 2, \ldots, n$ downwards and the columns $1, 2, \ldots, m$ anticlockwise. Associate the following polynomial to each row

$$P_i(X) = \sum_{k=0}^{m-1} a_{i,k+1} X^k$$

where a_{ij} is the number written on row i and column j. We consider $P_0 = P_{m+1} = 0$. The condition in the problem becomes (clearly)

$$P_i \equiv P_{i-1} + P_{i+1} + (X + X^{-1})P_i (\text{mod } X^m - 1)$$

Now $X^{-1} \equiv X^{m-1} (\text{mod } X^m - 1)$. Let $Q_i = P_i/P_1$. Then we have $Q_0 = 0, Q_1 = 1$ and

$$Q_{i+1} = (1 - X - X^{m-1})Q_i - Q_{i-1} (\text{mod } X^m - 1)$$

This means that all the Q_i are polynomials. The condition $P_{m+1} = 0$ becomes $P_1 Q_{m+1} = 0 (\text{mod } X^m - 1)$. The numbers in the table are not all zero if P_1 is not the zero polynomial of course. And the equation above has a nonzero solution if and only if Q_{m+1} and $X^m - 1$ are not coprime.

This means that there is a $\varepsilon^m = 1$ so that $Q_{m+1}(\varepsilon) = 0$. Let $x_k = Q_k(\varepsilon)$. Then $x_0 = 0, x_1 = 1$ and $x_{k+1} = a x_k - x_{k-1}$ where $a = 1 - \varepsilon - \varepsilon^{m-1}$. Let r_1, r_2 be the roots of the characteristic equation $x^2 - ax + 1 = 0$ of the linear recurrence satisfied by x_k. If $r_1 = r_2$ then $x_k = k r_1^{k-1}$ and this can never be zero because $r_1 r_2 = 1$.

So $r_1 \neq r_2$ and then $x_k = \frac{r_1^k - r_2^k}{r_1 - r_2}$. Now $x_{n+1} = 0 \Longleftrightarrow r_2 = r_1\omega$ where $\omega^{m+1} = 1$. This is equivalent to $r_1 + r_2 = r_1(1+\omega) = a$ and $r_1 r_2 = r_1^2\omega = 1$ and this is equivalent to

$$\frac{(1+\omega)^2}{\omega} = a^2 = (1 - \varepsilon - \overline{\varepsilon})^2$$

because $\varepsilon^{m-1} = \overline{\varepsilon}$.

The relation becomes $\overline{\omega}+2+\omega=(1-\varepsilon-\overline{\varepsilon})^2$. So there exist k,l integers so that $n+1$ does not divide k and $(\omega=e^{2k\pi i/(n+1)},\varepsilon=e^{2l\pi i/m})$

$$2+2\cos\frac{2k\pi}{n+1}=\left(1-2\cos\frac{2l\pi}{m}\right)^2$$

$\Longleftrightarrow$

$$4\cos^2\frac{k\pi}{n+1}=\left(1-2\cos\frac{2l\pi}{m}\right)^2$$

$\Longleftrightarrow$ (may replace k by $n+1-k$ is the signs are not good)

$$\cos\frac{k\pi}{n+1}+\cos\frac{2l\pi}{m}=\frac{1}{2}$$

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Configuration Problems

1. A square table $n\times n$ ($n\geq 2$) is filled with 0's and 1's so that any subset of n cells, no two of which lie in the same row or column, contains at least one 1. Prove that there exist i rows and j columns with $i+j\geq n+1$ whose intersection contains only 1's. (Bulgaria 1998)

   Hint: Use the Marriage theorem, a standard result in graph theory.

2. Let n and k be positive integers so that $n/2<k\leq 2n/3$. Find the least m for which it is possible to place m pawns on a square of an $n\times n$ chessboard so that no column or row contains a block of k adjacent unoccupied squares. (Shortlist 2000)

   Proof (not detailed)It is easy to see that if we put diagonal strips of pawns at distance m then we get that $4(n-k)$ pawns are enough.

   Now divide the table into nine subtables

$$\begin{pmatrix} A & B & C\\ D & E & F\\ H & I & J\end{pmatrix}$$

   with A,C,H,I of dimensions $(n-k)\times(n-k)$. Then B and I have dimensions $(n-k)\times(2k-n)$, D and F have dimensions $(2k-n)\times(n-k)$ and E has dimensions $(2k-n)\times(2k-n)$.

   Assume there are b rows in B with no pawns. There these rows extended to the whole table will have at least one pawn in A and at least one pawn in C. This means that (if there are x empty rows/columns in the subtable X) the total number of pawns in A,C,H,J is $2(b+d+i+f)$ each pawn being counted twice so in all at least $b+d+i+f$ pawns. Also in B,D,F,I there are at least $\sum(n-k-b)$. This means that the total number of pawns is at least $\sum b+\sum(n-k-b)=4(n-k)$. ■

3. On the lattice plane shade the squares $(1,1),(1,2),(2,1),(1,3),(2,2),(3,1)$. One some of these there are pieces and initially there are no pieces on nonshaded squares. For a piece, if the neighboring squares to the right and above are without pieces then we may remove that piece and put one piece in each of the two squares mentioned above (i.e. above and to the right of the square where the removed piece lied). The goal is to have all the shaded square free of any pieces. Can this be done if each shaded square has one piece? How about if there is only one piece on the square $(1,1)$? (Tournament of Towns 1980)

   Proof (not detailed)Assign number $2^{-(i+j)}$ to a square of coordinates (i,j) (starting from $(0,0)$ though). Then the sum of all the numbers in the squares where there are pieces is invariant.

   If all the shaded squares have numbers on them then this sum is bigger than the sum of all the numbers in the unshaded squares so this cannot be done.

   Look at the column $1 \times \{4,5,\ldots\}$ and row $\{4,5,\ldots\} \times 1$. Then if we start with one shaded square (0,0) then we always have at most one piece on the row and at most one token on the column. This means that these pieces contribute at most $\frac{1}{8}$ to the total sum that has to be 1. Also, outside the row, the column and the shaded region the sum of the numbers if $\frac{3}{4}$. This means that in order to get all the pieces out of the shaded region we need to fill the entire region besides the row, the column and the shaded region. But this cannot be done in a finite number of steps. So the answer is no to both questions.

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Changing Configurations

1. On an $m \times n$ board there are mn cards with one side white and the other black. At first all cards have the white side upwards except for one card in the upper left corner. In a move one can take out a card with black side up and turn all adjacent cards upside-down (adjacent means having a common edge). Find all m and n for which all cards may be removed from the board after a finite number of moves. (Shortlist 1998)

   Proof (not detailed)Consider all centers of squares and draw edges between adjacent centers. Assign -1 to all edges and to all white centers and 1 to all black centers. Consider the product of all the numbers, both vertices and edges. If we remove a black then we divide by 1. Also for any removed edge we divide the number at its end with -1, so we divide by $(-1)^2$ and this means that removing does not change the product.

   In the end the product has to be 1, and in the beginning is is $(-1)^{mn-1}(-1)^{m(n-1)+n(m-1)} = (-1)^{(m-1)(n-1)}$ because we have 1 black vertex exactly. This means that $(m-1)(n-1)$ is even so at least one of m, n is odd. It is easy to see that this is also sufficient. ■

2. A table has m rows and n columns. The following permutations if its mn elements are permitted: an arbitrary permutation leaving each element in the same row (horizontal move) and an arbitrary permutation leaving each element in the same column (vertical

move). Find the smallest number number k so that any permutation of the mn elements can be obtained after k moves. (Tournament of Towns 1992)

Hint: Prove that 2 moves are not sufficient. Then prove that 3 moves are sufficient by the following argument: Color each number is colored with the row on which it has to end up. Now prove that one can choose in each row one number so that we have distinct numbers in the end. Then permute each row so that these are in the last column and then we may permute the last column so that each element ends up in the right row. Do this in parallel (by induction) for all columns and then, we after two moves you have a configuration where each element is on the right row add a third move to create the right permutation in each row. In order to prove the fact that one can choose a number in each row so that all the numbers are distinct use an extremal argument.

3. The numbers in an $n \times n$ table may be changed by adding 1 to each number along an arbitrary closed non-selfintersecting polygonal line of unit squares whose side are parallel to the sides of the square. Originally 1-s stand on one of the diagonals and 0-s in all the other cells of the table. Can one get after a finite number of such transformations a configuration where all the numbers are equal? (Tournament of Towns 1992)

   Proof (not detailed)Color in a checkerboard pattern. Then the sum of the black squares minus the sum of the white ones is constant, since any closed path has alternating squares along it. This implies that n is odd since in the beginning this number is n. If n is odd then choose one number on the main diagonal and then partition the rest of the table into disjoint closed paths and increment by 1 all the numbers along them. Do this for all the numbers along the main diagonal. This way we increment all the numbers skipping each of the numbers on the main diagonal exactly once, which means that in the end all the numbers will be equal. ∎

Proposed Problems - no solutions

1. The numbers from 1 to 100 are arranged in a 10×10 table so that no two adjacent numbers have sum less than S. Find the smallest value of S for which this is possible. (Russia 1997)

2. A figure composed of 1×1 squares has the property that if the squares of a (fixed) $m \times n$ rectangle are filled with numbers the sum of all of which is positive, the figure can be placed on the rectangle (possibly after being rotated) so that the numbers it covers also have positive sum. (The figure may not be placed so that any of its squares fails to lie over the rectangle.) Prove that a number of such figures can be put on the $m \times n$ rectangle so that each square is covered by the same number of figures. (Russia 1998)

3. On a 5×5 board two players mark alternatively numbers 1 and 0 respectively until the square is filled. For each 3×3 square the sum of all nine numbers in the square is computed. Let A be the maximum of all these sums when the square ranges over all 3×3 subsquares of the 5×5 square. How big can the first player make A no matter what the strategy of the second layer is? (Shortlist 1994)

4. We are given a positive integer r and a board $ABCD$ of dimensions $AB = 20, BC = 12$. The rectangle is divided into a grid of 20×12 unit squares. The following moves are permitted on the board: one can move from a square to another only if the distance between the centers of the two squares is $\sqrt{r}$. The task is to find a sequence of moves leading from A to B. Show that the task cannot be done is r is divisible by 2 or 3. Prove that the task is possible when $r = 73$. Is there a solution for $r = 97$? (IMO 1996)

5. An $n \times n$ matrix with entries $\{1, 2, \ldots, 2n - 1\}$ is called a coveralls matrix if for each i the union of the i^{th} row and the i^{th} column contains $2n - 1$ distinct entries. Show that there is no coveralls matrix for $n = 1997$. Show that coveralls exist for infinitely many values of n. (IMO 1997)

6. The numbers $1, 2, \ldots, n^2$ are placed on the squares of an $n \times n$ chessboard. Prove that there exist two neighboring (with common edge) squares such that their numbers differ by at least n. (Shortlist 1988)

7. In each square of a $m \times n$ chessboard we write a nonnegative integer. One may add the same integer k to the numbers written in two squares with a common edge. Find the necessary and sufficient condition that after a finite number of steps we can make all the numbers equal to 0. (Shortlist 1989)

8. In an $n \times n$ array of numbers, all rows are different (meaning that they differ in at least one entry). Prove that one may remove a column so that the rows are still distinct. (Tournament of Towns, 1980)

9. The digits $0, 1, \dots, 9$ are written in the squares of a 10×10 square, each digit appearing 10 times. Is it possible to write them in such a way that in any row or column there be no more that 4 different digits? Prove that there must be a row or a column containing more than 3 different digits. (Tournament of Towns 1985)

10. Each square of a finite chessboard is painted either red or blue. Prove that there is a color so that a queen may tour all of them. The queen may pass through each square several times, she can stand on a square only of one color but she may jump over squares of the other color too. (Tournament of Towns 1986)

11. In the diagonal row of a right isosceles triangular table we have the numbers

$$1, \frac{1}{2}, \frac{1}{3}, \dots, \frac{1}{1993}$$

Under each pair of a row we write the absolute value of the difference between the two number. Any new row will have one less elements than the row immediately above it. For example next two rows will be

$$\frac{1}{2}, \frac{1}{6}, \dots, \frac{1}{1992 \cdot 1993}$$

$$\frac{1}{3}, \frac{1}{12}, \dots$$

We continue this procedure until we get to a row with one element. Find this element. (Tournament of Towns 1993)

12. Suppose an $n \times n$ table is filled with the numbers $0, 1, -1$ in such a way that every row and column contains exactly one 1 and one -1. Prove that the rows and columns can be reordered so that in the resulting table each number has been replaced with its negative. (Iran 1998)