

# Orthopoles and the Pappus Theorem

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**Abstract.** If the vertices of a triangle are projected onto a given line, the perpendiculars from the projections to the corresponding sidelines of the triangle intersect at one point, the orthopole of the line with respect to the triangle. We prove several theorems on orthopoles using the Pappus theorem, a fundamental result of projective geometry.

## 1. Introduction

Theorems on orthopoles are often proved with the help of coordinates or complex numbers. In this note we prove some theorems on orthopoles by using a well-known result from projective geometry, the Pappus theorem. Notably, we need not even use it in the general case. What we need is a simple affine theorem which is a special case of the Pappus theorem. We denote the intersection of two lines  $g$  and  $g'$  by  $g \cap g'$ . Here is the Pappus theorem in the general case.

**Theorem 1.** *Given two lines in a plane, let  $A, B, C$  be three points on one line and  $A', B', C'$  three points on the other line. The three points*

$$BC' \cap CB', \quad CA' \cap AC', \quad AB' \cap BA'$$

*are collinear.*

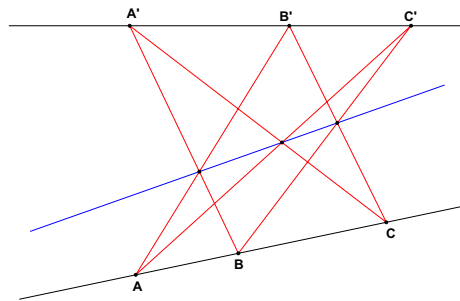


Figure 1

Theorem 1 remains valid if some of the points  $A, B, C, A', B', C'$  are projected to infinity, even if one of the two lines is the line at infinity. In this paper, the only case we need is the special case if the points  $A', B', C'$  are points at infinity. For the sake of completeness, we give a proof of the Pappus theorem for this case.

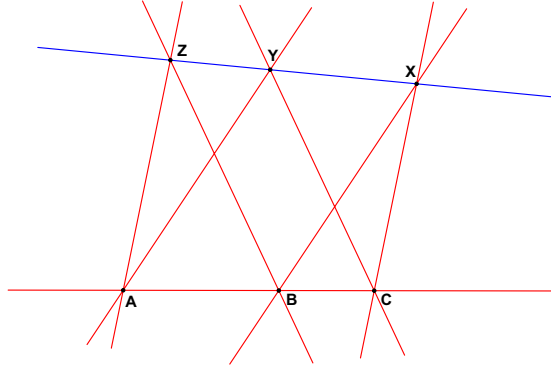


Figure 2

Let  $X = BC' \cap CB'$ ,  $Y = CA' \cap AC'$ ,  $Z = AB' \cap BA'$ . The points  $A'$ ,  $B'$ ,  $C'$  being infinite points, we have  $CY \parallel BZ$ ,  $AZ \parallel CX$ , and  $BX \parallel AY$ . See Figure 2. We assume the lines  $ZX$  and  $ABC$  intersect at a point  $P$ , and leave the easy case  $ZX \parallel ABC$  to the reader. In Figure 3, let  $Y' = ZX \cap AY$ . We show that  $Y' = Y$ . Since  $AY \parallel BX$ , we have  $\frac{PA}{PB} = \frac{PY'}{PX}$  in signed lengths. Since  $AZ \parallel CX$ , we have  $\frac{PC}{PA} = \frac{PX}{PZ}$ . From these,  $\frac{PC}{PB} = \frac{PY'}{PZ}$ , and  $CY' \parallel BZ$ . Since  $CY \parallel BZ$ , the point  $Y'$  lies on the line  $CY$ . Thus,  $Y' = Y$ , and the points  $X, Y, Z$  are collinear.

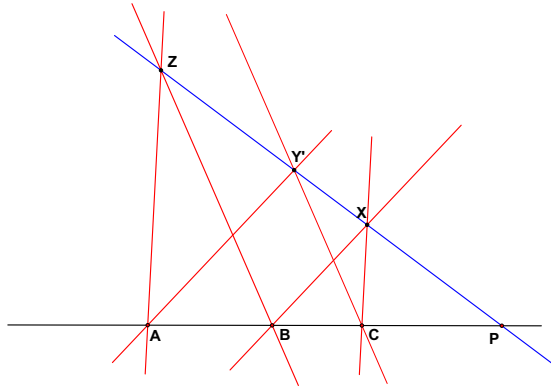


Figure 3

## 2. The orthocenters of a fourline

We denote by  $\Delta abc$  the triangle bounded by three lines  $a, b, c$ . A complete quadrilateral, or, simply, a fourline is a set of four lines in a plane. The fourline consisting of lines  $a, b, c, d$ , is denoted by  $\square abcd$ . If  $g$  is a line, then all lines perpendicular to  $g$  have an infinite point in common. This infinite point will be called  $\bar{g}$ . With this notation,  $P\bar{g}$  is the perpendicular from  $P$  to  $g$ . Now, we establish the well-known Steiner's theorem.

**Theorem 2 (Steiner).** *If  $a, b, c, d$  are any four lines, the orthocenters of  $\Delta bcd$ ,  $\Delta acd$ ,  $\Delta abd$ ,  $\Delta abc$  are collinear.*

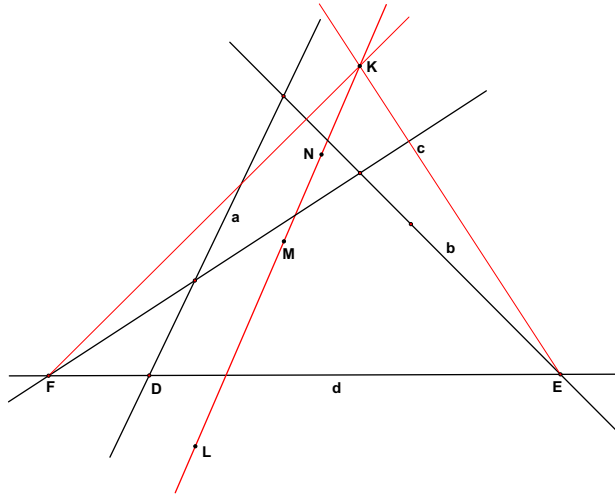


Figure 4

*Proof.* Let  $D, E, F$  be the intersections of  $d$  with  $a, b, c$ , and  $K, L, M, N$  the orthocenters of  $\Delta bcd$ ,  $\Delta acd$ ,  $\Delta abd$ , and  $\Delta abc$ . Note that  $K = E\bar{c} \cap F\bar{b}$ , being the intersection of the perpendiculars from  $E$  to  $c$  and from  $F$  to  $b$ . Similarly,  $L = F\bar{a} \cap D\bar{c}$  and  $M = D\bar{b} \cap E\bar{a}$ . The points  $D, E, F$  being collinear and the points  $\bar{a}, \bar{b}, \bar{c}$  being infinite, we conclude from the Pappus theorem that  $K, L, M$  are collinear. Similarly,  $L, M, N$  are collinear. The four orthocenters lie on the same line.  $\square$

The line  $KLMN$  is called the Steiner line of the fourline  $\square ABCD$ . Theorem 2 is usually associated with Miquel points [6, §9] and proved using radical axes. A consequence of such proofs is the fact that the Steiner line of the fourline  $\square abcd$  is the radical axis of the circles with diameters  $AD, BE, CF$ , where  $A = b \cap c$ ,  $B = c \cap a$ ,  $C = a \cap b$ ,  $D = d \cap a$ ,  $E = d \cap b$ ,  $F = d \cap c$ . Also, the Steiner line is the directrix of the parabola touching the four lines  $a, b, c, d$ . The Steiner line is also called four-orthocenter line in [6, §11] or the orthocentric line in [5], where it is studied using barycentric coordinates.

### 3. The orthopole and the fourline

We prove the theorem that gives rise to the notion of orthopole.

**Theorem 3.** *Let  $\triangle ABC$  be a triangle and  $d$  a line. If  $A', B', C'$  are the pedals of  $A, B, C$  on  $d$ , then the perpendiculars from  $A', B', C'$  to the lines  $BC, CA, AB$  intersect at one point.*

This point is the orthopole of the line  $d$  with respect to  $\triangle ABC$ .

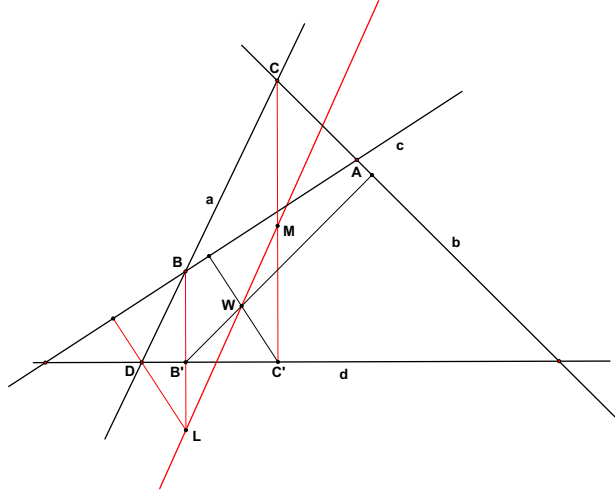


Figure 5

*Proof.* Denote by  $a, b, c$  the lines  $BC, CA, AB$ . By Theorem 2, the orthocenters  $K, L, M, N$  of triangles  $\triangle bcd, \triangle acd, \triangle abd, \triangle abc$  lie on a line. Let  $D = d \cap a$ , and  $W = B'\bar{b} \cap C'\bar{c}$ . The orthocenter  $L$  of  $\triangle acd$  is the intersection of the perpendiculars from  $D$  to  $c$  and from  $B$  to  $d$ . Since the perpendicular from  $B$  to  $d$  is also the perpendicular from  $B'$  to  $d$ ,  $L = D\bar{c} \cap B'\bar{d}$ . Analogously,  $M = D\bar{b} \cap C'\bar{d}$ . By the Pappus theorem, the points  $W, M, L$  are collinear. Hence,  $W$  lies on the line  $KLMN$ . Since  $W = B'\bar{b} \cap C'\bar{c}$ , the intersection  $W$  of the lines  $KLMN$  and  $B'\bar{b}$  lies on  $C'\bar{c}$ . Similarly, this intersection  $W$  lies on  $A'\bar{a}$ . Hence, the point  $W$  is the common point of the four lines  $A'\bar{a}, B'\bar{b}, C'\bar{c}$ , and  $KLMN$ . Since  $A'\bar{a}, B'\bar{b}, C'\bar{c}$  are the perpendiculars from  $A', B', C'$  to  $a, b, c$  respectively, the perpendiculars from  $A', B', C'$  to  $BC, CA, AB$  and the line  $KLMN$  intersect at one point. This already shows more than the statement of the theorem. In fact, we conclude that the orthopole of  $d$  with respect to triangle  $\triangle ABC$  lies on the Steiner line of the complete quadrilateral  $\square abcd$ .  $\square$

The usual proof of Theorem 3 involves similar triangles ([1], [10, Chapter 11]) and does not directly lead to the fourline. Theorem 4 originates from R. Goormaghtigh, published as a problem [7]. It was also mentioned in [5, Proposition 6], with reference to [2]. The following corollary is immediate.

**Corollary 4.** *Given a fourline  $\square abcd$ , the orthopoles of  $a, b, c, d$  with respect to  $\triangle bcd, \triangle acd, \triangle abd, \triangle abc$  lie on the Steiner line of the fourline.*

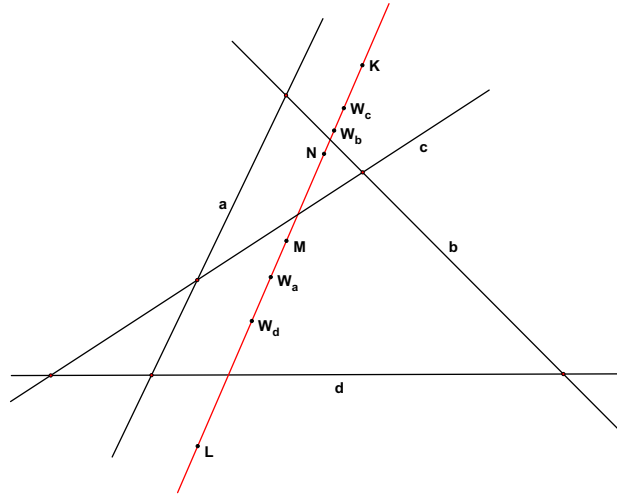


Figure 6

#### 4. Two theorems on the collinearity of quadruples of orthopoles

**Theorem 5.** *If  $A, B, C, D$  are four points and  $e$  is a line, then the orthopoles of  $e$  with respect to triangles  $\triangle BCD, \triangle CDA, \triangle DAB, \triangle ABC$  are collinear.*

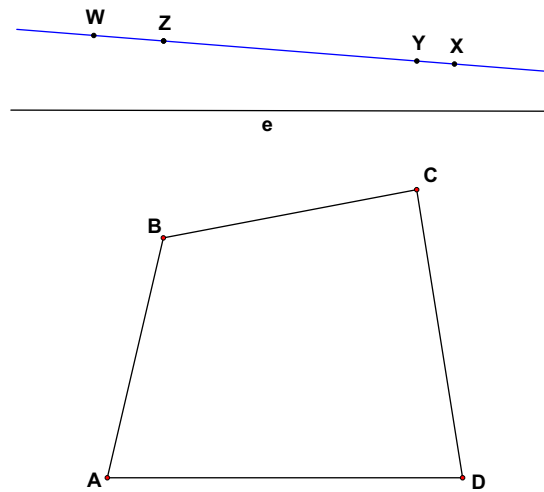


Figure 7

*Proof.* Denote these orthopoles by  $X, Y, Z, W$  respectively. If  $A', B', C', D'$  are the pedals of  $A, B, C, D$  on  $e$ , then  $X = \overline{B'CD} \cap \overline{C'DD}$ . Similarly,  $Y = \overline{C'AD} \cap \overline{A'CD}$ ,  $Z = \overline{A'BD} \cap \overline{B'AD}$ . Now,  $A', B', C'$  lie on one line, and  $\overline{AD}, \overline{BD}, \overline{CD}$  lie on the line at infinity. By Pappus' theorem, the points  $X, Y, Z$  are collinear. Likewise,  $Y, Z, W$  are collinear. We conclude that all four points  $X, Y, Z, W$  are collinear.  $\square$

Theorem 5 was also proved using coordinates by N. Dergiades in [3] and by R. Goormaghtigh in [8, p.178]. A special case of Theorem 5 was shown in [11] using the Desargues theorem.<sup>1</sup> Another theorem surprisingly similar to Theorem 5 was shown in [9] using complex numbers.

**Theorem 6.** *Given five lines  $a, b, c, d, e$ , the orthopoles of  $e$  with respect to  $\triangle bcd$ ,  $\triangle acd$ ,  $\triangle abd$ ,  $\triangle abc$  are collinear.*

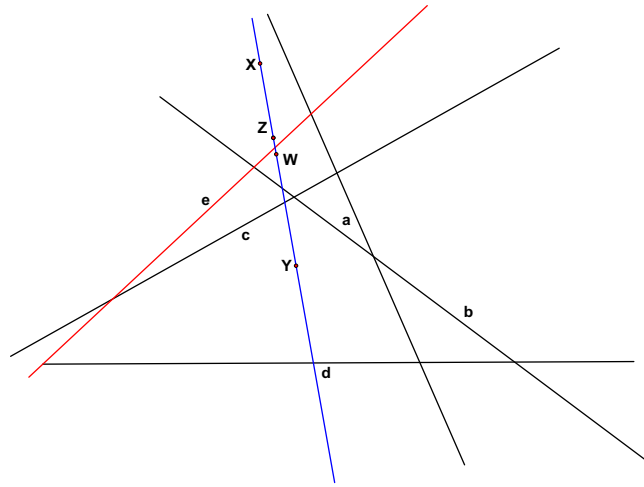


Figure 8

*Proof.* Denote these orthopoles by  $X, Y, Z, W$  respectively. Let the line  $d$  intersect  $a, b, c$  at  $D, E, F$ , and let  $D', E', F'$  be the pedals of  $D, E, F$  on  $e$ .

Since  $E = b \cap d$  and  $F = c \cap d$  are two vertices of triangle  $\triangle bcd$ , and  $E'$  and  $F'$  are the pedals of these vertices on  $e$ , the orthopole  $X = \overline{E'c} \cap \overline{F'b}$ . Similarly,  $Y = \overline{F'a} \cap \overline{D'c}$ , and  $Z = \overline{D'b} \cap \overline{E'a}$ . Since  $D', E', F'$  lie on one line, and  $\overline{a}, \overline{b}, \overline{c}$  lie on the line at infinity, the Pappus theorem yields the collinearity of the points  $X, Y, Z$ . Analogously, the points  $Y, Z, W$  are collinear. The four points  $X, Y, Z, W$  are on the same line.  $\square$

<sup>1</sup>In [11], Witczyński proves Theorem 5 for the case when  $A, B, C, D$  lie on one circle and the line  $e$  crosses this circle. Instead of orthopoles, he equivalently considers Simson lines. The Simson lines of two points on the circumcircle of a triangle intersect at the orthopole of the line joining the two points.

## References

- [1] A. Bogomolny, Orthopole, <http://cut-the-knot.com/Curriculum/Geometry/Orthopole.shtml>.
- [2] J. W. Clawson, The complete quadrilateral, *Annals of Mathematics*, Ser. 2, 20 (1919) 232–261.
- [3] N. Dergiades, Hyacinthos message 3352, August 5, 2001.
- [4] A. Dixit, Hyacinthos message 3340, August 4, 2001.
- [5] J.-P. Ehrmann, Steiner's theorems on the complete quadrilateral, *Forum Geom.*, 4 (2004) 35–52.
- [6] W. Gallatly, *The Modern Geometry of the Triangle*, 2nd ed. 1913, Francis Hodgson, London.
- [7] R. Goormaghtigh, Question 2388, *Nouvelles Annales de Mathématiques*, Série 4, 19 (1919) 39.
- [8] R. Goormaghtigh, A study of a quadrilateral inscribed in a circle, *Amer. Math. Monthly*, 49 (1942) 174–181.
- [9] C. Hsu, On a certain collinearity of orthopoles, and of isopoles, *Soochow J. Math.*, 10 (1984) 27–31.
- [10] R. Honsberger, *Episodes of 19th and 20th Century Euclidean Geometry*, Math. Assoc. America, 1995.
- [11] K. Witczyński, On collinear Griffiths points, *Journal of Geometry*, 74 (2002) 157–159.

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