Functional Equations

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1. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that $f(x + y) = f(x) + f(y)$ and $f(xy) = f(x)f(y)$ for all x, $y \in \mathbb{R}$.

2. Let $f : [0, 1] \to \mathbb{R}$ be a function such that

   (a) $f(1) = 1$,

   (b) $f(x) \geq 0$ for all $x \in [0, 1]$,

   (c) if x, y and $x + y$ all lie in $[0, 1]$, then $f(x + y) \geq f(x) + f(y)$.

   Prove that $f(x) \leq 2x$ for all $x \in [0, 1]$.

3. Let $n > 2$ be an integer and let $f : \mathbb{R}^2 \to \mathbb{R}$ be a function scuh that for any regular n-gon $A_1A_2 \cdots A_n$,
$$f(A_1) + f(A_2) + \cdots + f(A_n) = 0.$$
   Prove that f is the zero function.

4. Find all polynomials $p(x)$ such that for all x,
$$(x - 16)p(2x) = 16(x - 1)p(x).$$

5. Find all functions $f : \mathbb{R} \to [0, \infty)$ such that for all x, $y \in \mathbb{R}$,
$$f(x^2 + y^2) = f(x^2 - y^2) + f(2xy).$$

6. Find all pairs of functions f, $g : \mathbb{R} \to \mathbb{R}$ such that

   (a) if $x < y$, then $f(x) < f(y)$;

   (b) for all x, $y \in \mathbb{R}$, $f(xy) = g(y)f(x) + f(y)$.

7. For which α does there exist a nonconstant function $f : \mathbb{R} \to \mathbb{R}$ such that $f(\alpha(x + y)) = f(x) + f(y)$ for all x, $y \in \mathbb{R}$?

8. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that the equality $f(f(x) + y) = f(x^2 - y) + 4f(x)y$ holds for all pairs of real numbers x, y.

9. Let $f : \mathbb{R} \to \mathbb{R}$ be a function such that for all x, $y \in \mathbb{R}$,
$$f(x^3 + y^3) = (x + y)(f(x)^2 - f(x)f(y) + f(y)^2).$$
   Prove that for all $x \in \mathbb{R}$, $f(1996x) = 1996f(x)$.

10. Find all functions $u : \mathbb{R} \to \mathbb{R}$ for which there exists a strictly monotonic function $f : \mathbb{R} \to \mathbb{R}$ such that
$$f(x + y) = f(x)u(y) + f(y) \quad \text{for any } x,\ y \in \mathbb{R}.$$

11. Let $\mathbb{R}^+$ be the set of positive real numbers. Prove that there does not exist a function $f : \mathbb{R}^+ \to \mathbb{R}^+$ such that
$$f(x)^2 \geq f(x + y)(f(x) + y) \quad \text{for any } x,\ y \in \mathbb{R}^+.$$

12. Find all nondecreasing functions $f : \mathbb{R} \to \mathbb{R}$ such that

    (a) $f(0) = 0$ and $f(1) = 1$;

    (b) $f(a) + f(b) = f(a)f(b) + f(a + b - ab)$ for all real numbers a and b with $a < 1 < b$.

13. Let $\mathbb{R}^+$ be the set of all positive real numbers. Find all functions $f : \mathbb{R}^+ \to \mathbb{R}^+$ that satisfy the following conditions:

    (a) $f(xyz) + f(x) + f(y) + f(z) = f(\sqrt{xy})f(\sqrt{yz})f(\sqrt{zx})$ for all x, y and z in $\mathbb{R}^+$;

    (b) $f(x) < f(y)$ for all $1 \leq x < y$.

14. Find all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all real numbers x, y, z, t,

$$(f(x) + f(z))(f(y) + f(t)) = f(xy - zt) + f(xt + yz).$$

15. Find all pairs of functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ such that for all x, $y \in \mathbb{R}$,

$$f(x + g(y)) = xf(y) - yf(x) + g(x).$$

16. Determine all functions $f : \mathbb{R} \to \mathbb{R}$ such that for all x, $y \in \mathbb{R}$,

$$f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1.$$

17. Let $f : \mathbb{N} \to \mathbb{N}$ be a function satisfying

    (a) For every $n \in \mathbb{N}$, $f(n + f(n)) = f(n)$.

    (b) For some $n_0 \in \mathbb{N}$, $f(n_0) = 1$.

    Show that $f(n) = 1$ for all $n \in \mathbb{N}$.

18. Find all functions $f : \mathbb{Z} \to \mathbb{Z}$ which satisfy $f(m + f(n)) = f(m) + n$ for all m, $n \in \mathbb{Z}$.

19. Let S denote the set of nonnegative integers. Find all function $f : S \to S$ such that

$$f(m + f(n)) = f(f(m)) + f(n) \quad \text{for all } m, n \in S.$$

20. Let $\mathbb{Q}^+$ denote the set of positive rational numbers. Find all functions $f : \mathbb{Q}^+ \to \mathbb{Q}^+$ such that for all $x \in \mathbb{Q}^+$

$$f(x + 1) = f(x) + 1 \quad \text{and} \quad f(x^2) = f(x)^2.$$

21. For which integers k does there exist a function $f : \mathbb{N} \to \mathbb{Z}$ such that

    (a) $f(1995) = 1996$, and

    (b) $f(xy) = f(x) + f(y) + kf(\gcd(x, y))$ for all x, $y \in \mathbb{N}$?

22. Let S denote the set of nonnegative integers. Find a bijective function $f : S \to S$ such that for all m, $n \in S$,

$$f(3mn + m + n) = 4f(m)f(n) + f(m) + f(n).$$

23. Determine all functions $f : \mathbb{N} \to \mathbb{N}$ such that for all $n \in \mathbb{N}$,

$$f(n) + f(n + 1) = f(n + 2)f(n + 3) - 1996.$$

24. Consider all functions $f : \mathbb{N} \to \mathbb{N}$ such that $f(t^2 f(s)) = sf(t)^2$ for all s, $t \in \mathbb{N}$. Determine the least possible value of $f(1998)$.