1 Cross-Ratios

1. Definition: Let A, B, C three collinear points in this order. Then there exists exactly one other point $D \in AC$ so that $\frac{\overrightarrow{AB}}{\overrightarrow{BC}} = \frac{\overrightarrow{AD}}{\overrightarrow{CD}}$. In this case one says that B and D are harmonic conjugates with respect to A and C.

2. Definition: Let A, B, C, D be four collinear points in this order. Then one defines the cross-ratio $(A, B, C, D) = \frac{\overrightarrow{AB}}{\overrightarrow{BC}} : \frac{\overrightarrow{AD}}{\overrightarrow{CD}} = \frac{AB}{BC} : \frac{AD}{CD}$.

3. Pappus's Theorem: Let $(Ox_i$ for $i = 1, \ldots, 4$ be four rays (forming a total angle of < 180) and $A_i, B_i \in (Ox_i$ so that A_1, A_2, A_3, A_4 and B_1, B_2, B_3, B_4 are collinear respectively. Then $(A_1, A_2, A_3, A_4) = (B_1, B_2, B_3, B_4)$.

2 Polar Transformations

1. Let $\mathcal{C}$ be a circle and A a point outside it. Let l be a line containing A, $l \bigcap \mathcal{C} = \{M, N\}$, and let B be the harmonic conjugate of A with respect to M and N. Find the locus of B, and denote it by p_A. This is called the polar of A with respect to $\mathcal{C}$.

2. $M \in p_A \Longleftrightarrow MO^2 - MA^2 = 2R^2 - OA^2$, where O is the center of $\mathcal{C}$ and R is its radius.

3. Prove that $B \in p_A \Longleftrightarrow A \in p_B$.

4. Prove that one can define the polar for any point other than O.

5. Let l be a line not containing O. Then there is a point A so that $l = p_A$. This point is called the pole of l.

6. Prove that A, B, C are collinear $\Longleftrightarrow p_A, p_B, p_C$ are concurrent.

7. Dual tranformation. Let $\mathcal{C}$ be a projective configuration of points and lines (projective means that the only things that count are collinearities and concurrences). Choose a circle whose center is not among the points of $\mathcal{C}$. Then take the polar of each point and the pole of each line in $\mathcal{C}$. Then in the new configuration, all previous collinearities become concurrences and vice-versa. This is a very efficient method of simplifying problems.

Problems

1. Let $\mathcal{C}$ be a circle and P a point outside it. A line l through P intersects $\mathcal{C}$ in M and N. The tangents through M and N to $\mathcal{C}$ intersect at Q. Prove that Q describes a line.

2. Let $ABCD$ be a cyclic quadrilateral with circumcircle $\mathcal{C}$. Let $AB \bigcap CD = \{U\}, BC \bigcap AD = \{V\}, AC \bigcap BD = \{T\}$. Prove that $UT = p_V$.

3. Let ABC be an acute triangle. The interior bisectors of $\angle B$ and $\angle C$ meet the opposite sides in L, M respectively. Prove that there is a point $K \in (BC)$ such that KLM is equilateral $\iff \angle A = 60$. (Romanian Selection Test, 1999)

3 Projective Transformation

The set of all points of intersections of parallel lines is called the line at infinity of the Euclidian plane. Let $\prod$ be the projective plane, i.e. the union of the Euclidian plane with the line at infinity.

Let $l \in \prod$ and $O \in \prod \setminus l$. Consider a point V outside the projective plane. Take d a line containing O. Let a plane π through d, parallel to the plane determined by l and V. . For each point $X \in \prod$ let $Y = f(X) = VX \bigcap \pi$. Then this is called the projective transformation that sends l to infinity. The reason for this is quite obvious, i.e. for any $X \in l$ then $f(Y)$ will be on the line to infinity of π. In this case one just says that one "throws l to infinity"

This method is very efficient, because the following properties imply that any two lines intersecting on l will be transformed into two parallel lines.

1. Prove that f takes lines to lines.

2. Prove that f invaries cross-ratios.

Problems

1. Let ABC be a triangle and M, N, P be three points on sides BC, CA, AB respectively. Let D, E, F be the harmonic conjugates of M, N, P with respect to the endpoints of the sides on which they are. Prove that D, E, F are collinear $\iff AM, BN, CP$ are concurrent.

2. Pappus's Theorem (*come on, how many are there?*). Let $a \bigcap b = \{P\}$ be two intersecting lines. Let $A_i \in a, B_i \in b$ for $i = 1, 2, 3$. Prove that A_iB_i intersect $\iff (P, A_1, A_2, A_3) = (P, B_1, B_2, B_3)$.

3. Guess what? Pappus Theorem, again... Let d_1, d_2 be two lines and $A_1, A_2, A_3 \in d_1$ and $B_1, B_2, B_3 \in d_2$ distinct points. If $\{U_i\} = A_jB_k \bigcap A_kB_j$ for any permutation $\{i, j, k\} = \{1, 2, 3\}$ then U_1, U_2, U_3 are collinear.

4. Let $U, V \in BC$ in triangle ABC. A line parallel to AC intersects AB, AU, AV in P, Q, R respectively. Find $\frac{PQ}{QR}$.

5. Let ABCD be a convex quadrilateral with $O = AD \bigcap BC, T = AC \bigcap BD$. We consider points P and Q on OT. Let be U and V the intersection points of the diagonals of AQPD and BQPC respectively. Let M and N be two points on (AD) and (BC). Prove that OT,MU si NV are concurrent if and only if AB,DC and MN are concurrent.(*my problem, i.e.*)

6. (*My favorite...*) Let $VA_1 \dots A_n$ be a pyramid and $B_k \in (VA_k)$. Let $\{T_{i,j}\} = A_iB_j \bigcap A_jB_i$. Prove that if at least $1 + \binom{n-1}{2}$ of the points $T_{i,j}$ are coplanar then all points $T_{i,j}$ are coplanar. (*County olympiad, Arad 1999. Proposed by Iani Mirciov*)

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