

# TJUSAMO 2011 - Introduction to Proofs

Mitchell Lee and Andre Kessler

Hello and welcome to TJUSAMO 2011. The purpose of these meetings is to prepare you for the USA Mathematical Olympiad. If you haven't been to TJUSAMO before, you may be wondering what exactly all of this is about. The USA(J)MO is a nine-hour, six-problem math contest split over two days – so that's 4.5 hours, 3 problems per day. Solutions to the problems must be given in the form of proofs, unlike most problems you deal with in math team or math class; the answers themselves are worth almost no credit whatsoever. Scores are integers between 0 and 7, inclusive. By far the most common scores are 0, 1, 6, and 7. Partial credit is not easy to obtain. Throughout the year, we will teach you the requisite mathematical concepts and provide you will ample amounts of practice. However, we only have two hours per week in which to do this, and that's not enough. If you wish to substantially improve, you will need to put in a lot of work on your own time. Simply matching the amount of time spent at TJUSAMO with time spent at home solving problems will double your rate of improvement; spending a good 4-5 hours extra outside of practice, per week, will lead to phenomenal gains. We cannot emphasize this enough.

## 1 Tips and Meta-Techniques

Here are some general tips and strategies for solving problems. Throughout the year, we will come back to these time and time again.

1. First and foremost, practice. Although 48 hours of TJUSAMO this year may seem like a lot, it's impossible to become good at Olympiad math in 48 hours. For this reason, it is important that you practice at home. There will be more problems on our lectures than we can solve during the two-hour TJUSAMO. When you get home, allot time to sit down and work on the left-over problems. If you have questions, feel free to contact either of us via email. Additionally, the best way to practice on your own is to just find a good source of problems and solve them. We recommend getting a problem book, such as the *Art and Craft of Problem Solving* by Paul Zeitz or *Problem-Solving Strategies* by Engel. Also, the Art of Problem Solving website (<http://artofproblemsolving.com>) is a very useful resource.
2. Look at special cases of the problem. When a problem asks you to prove a statement for all positive integers, try to prove it for small positive integers first.
3. Draw pictures. This is especially important in geometry problems. It is also often useful for non-geometry problems.
4. Don't just stare at the page. It's difficult to solve problems without writing things down.
5. If you get stuck, write down your progress and work on a different problem. When you return to the problem, you might notice something that wasn't obvious before. That said, don't hop back and forth between problems for no reason.
6. Try many different techniques. Sometimes, the first thing you think of won't help to solve the problem. Sometimes, a completely silly technique will solve the problem.
7. Use all of the information given in the problem. If you get stuck, think about what information you haven't used yet.
8. If there doesn't seem to be much information in the problem, you might want to try using an indirect approach, as that will give you a new piece of information to work with.

9. When trying to prove that something satisfying certain properties exists constructively, narrow your search by proving other properties.
10. Work backwards and forwards. Find statements that would imply what you are trying to prove, and then prove those statements.
11. Ask yourself, “what makes the problem hard?” This goes with wishful thinking - try “wish away” the difficulty. Then, find ways to get around that.
12. If all else fails, don’t be afraid to use brute force. Be able to recognize which problems are “brutalizable<sup>1</sup>” and which aren’t.

## 2 Problems

Here are some problems. Work on these alone for a while, then start to solve them in groups if you want. If you can’t solve them immediately, “give up you shall not.” Remember that on the USAMO you have 4.5 hours for only three problems, and the problems we will give you at TJUSAMO will be olympiad-level. If you solve or understand the solutions to a majority of the problems we give you, you’ll notice a substantial improvement in your math skill throughout the year. Note that practicing USAMO-level problems will help you on the AMC and AIME, but the converse is not necessarily true.

1. Prove that there are infinitely many prime numbers.
2. Prove that  $\sqrt{2}$  is irrational. If you already know how to do this, find another way.
3. Prove that in a set of  $n$  elements, there exists a subset with sum divisible by  $n$ .
4. Show that every infinite sequence of real numbers has an infinite monotone subsequence.
5. Prove Pascal’s Identity, which states that  $\binom{n+1}{k+1} = \binom{n+1}{k} + \binom{n}{k}$ .
6. Prove that no arithmetic sequence contains only squares.
7. Show that for any positive integer  $n$ , there exists a nonconstant arithmetic progression of length  $n$  with no perfect squares.
8. Every person who has ever lived has, up to this moment, made a certain number of handshakes with other people. Prove that the number of people who have made an odd number of handshakes is even.
9. Show that if  $p$  and  $q$  are any two consecutive odd primes, then  $p + q$  is a product of at least three (not necessarily distinct) primes.
10. Two non-overlapping squares fit entirely within a circle of radius 1. Prove that at least one of them has a side length less than or equal to  $\frac{2}{\sqrt{5}}$ .
11. Prove that the sum of the dihedral angles of a tetrahedron is not constant.
12. Show that among any six people, there are always three who know each other or three who are complete strangers.

---

<sup>1</sup>This is a term coined by Po-Shen Loh in his 2003 lecture “Brutal Force.”

### 3 More Problems

13. Your calculator cannot perform multiplications, but it can add, subtract, or compute the reciprocal  $\frac{1}{x}$  of any number  $x$ . Show that you can multiply any two given reals using this calculator.
14. Let  $m$  and  $n$  be two positive integers. Let  $a_1, a_2, \dots, a_m$  be  $m$  different numbers from the set  $\{1, 2, \dots, n\}$  such that for any two indices  $i$  and  $j$  with  $1 \leq i \leq j \leq m$  and  $a_i + a_j \leq n$ , there exists an index  $k$  such that  $a_i + a_j = a_k$ . Show that

$$\frac{a_1 + a_2 + \dots + a_m}{m} \geq \frac{n+1}{2}.$$

15. A point  $D$  is placed on side  $BC$  of triangle  $ABC$ . Circles are inscribed in  $ABD$  and  $ACD$ . Their common exterior tangent (other than  $BC$ ) meets  $AD$  at  $K$ . Prove that the length of  $AK$  is independent of  $D$ .
16. The function  $F$  is defined on the set of nonnegative integers and takes nonnegative integer values satisfying the following conditions: for every  $n \geq 0$ ,
- (a)  $F(4n) = F(2n) + F(n)$ ,
  - (b)  $F(4n+2) = F(4n) + 1$ ,
  - (c)  $F(2n+1) = F(2n) + 1$ .

Prove that for each positive integer  $m$ , the number of integers  $n$  with  $0 \leq n < 2^m$  and  $F(4n) = F(3n)$  is  $F(2^{m+1})$ .