2. PLANAR GEOMETRY WITH COMPLEX NUMBERS

2.1 Distance and Angle

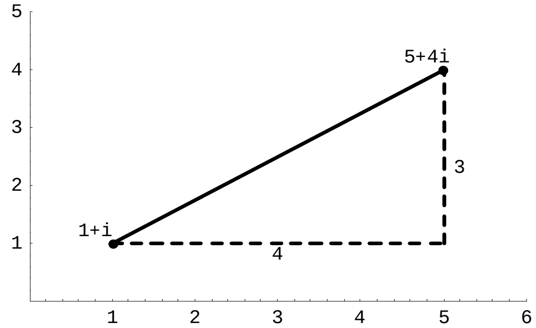
Definition 16 *Given two complex numbers $z, w \in \mathbb{C}$ then the* ***distance*** *between them, by Pythagoras' Theorem, is*

$$\sqrt{(z_1 - w_1)^2 + (z_2 - w_2)^2} = |(z_1 - w_1) + i(z_2 - w_2)| = |z - w|.$$

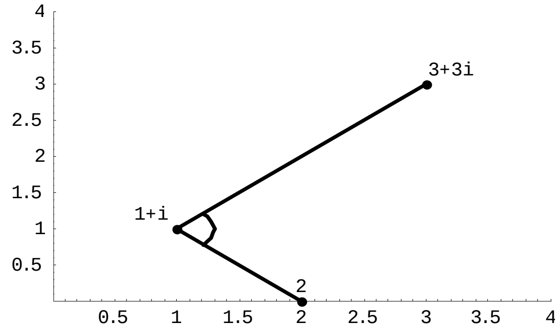
Definition 17 *Given three complex numbers a, b, c then the* ***angle*** *$\measuredangle cab$ equals*

$$\arg(c - a) - \arg(b - a) = \arg\left(\frac{c - a}{b - a}\right)$$

Notice that this is a "signed" angle in that the angle is measured in an anti-clockwise fashion and the above formula would give negative this result if the roles of b and c were swapped. Note also that the angle is defined only up to multiples of 2π.



The distance here is $\sqrt{3^2 + 4^2} = 5$



The angle here is $\arg\left(\frac{2+2i}{1-i}\right) = \arg 2i = \frac{1}{2}\pi$

Example 18 *Find the smaller angle $\measuredangle bac$ where $a = 1 + i$, $b = 3 + 2i$, and $c = 4 - 3i$.*

Solution. The angle $\measuredangle bac$ is given by

$$\arg\left(\frac{b - a}{c - a}\right) = \arg\left(\frac{2 + i}{3 - 4i}\right) = \arg\left(\frac{2 + 11i}{25}\right) = \tan^{-1}\left(\frac{11}{2}\right).$$

with the answer taken in the range $(0, \pi/2)$. ■

Proposition 19 *(**Triangle Inequality**) Given complex numbers z, w then*

$$|z + w| \leq |z| + |w|,$$

with equality if and only if z and w are real positive multiples of one another.

Proof. From a geometric point of view the triangle inequality simply states the well known fact that one side of a triangle is shorter than the sum of the other sides, with equality only when the triangle degenerates into a line.

For an algebraic proof, note that for any complex number $z + \bar{z} = 2\operatorname{Re} z$ and $\operatorname{Re} z \leq |z|$. So for $z, w \in \mathbb{C}$,

$$\frac{z\bar{w} + \bar{z}w}{2} = \operatorname{Re}(z\bar{w}) \leq |z\bar{w}| = |z|\,|\bar{w}| = |z|\,|w|\,.$$

Then

$$\begin{aligned}
|z+w|^2 &= (z+w)\,\overline{(z+w)} \\
&= (z+w)\,(\bar{z}+\bar{w}) \\
&= z\bar{z} + z\bar{w} + \bar{z}w + w\bar{w} \\
&\leq |z|^2 + 2\,|z|\,|w| + |w|^2 = (|z| + |w|)^2\,,
\end{aligned}$$

to give the required result. There is equality only if $z\bar{w} = \lambda > 0$ or equivalently $z = \mu w$ where $\mu = \lambda / |w|^2 > 0$. ■

2.2 Regions and Maps of the Argand Diagram

Given now definitions of distance and angle we can use these to describe various regions of the plane.

Example 20 *Describe the following regions of the Argand diagram*

- $|z - a| = r$, where $a \in \mathbb{C}$ and $r > 0$;

This is clearly the locus of points at distance r from a — i.e. a circle with centre a and radius r. The regions

$$|z - a| < r \quad \text{and} \quad |z - a| > r$$

are then the interior and exterior of the circle respectively.

- $|z - a| = |z - b|$, where $a, b \in \mathbb{C}$;

This is the set of points equidistant from a and b and so is the line which is the perpendicular bisector of the line segment connecting a and b. The regions

$$|z - a| < |z - b| \quad \text{and} \quad |z - a| > |z - b|$$

are then the half-planes either side of this line and which respectively contain a and b.

- $\arg(z - a) = \theta$;

This is a half-line emanating from the point a and making an angle θ with the positive real axis. Note that it doesn't include the point a itself.

- $|z-a|+|z-b|=r$ where $a,b\in\mathbb{C}$ and $r\in\mathbb{R}$;

From the Geometry I course we know that this is an ellipse with foci a and b provided $r>|a-b|$.

- $|z-a|-|z-b|=r$ where $a,b\in\mathbb{C}$ and $r\in\mathbb{R}$;

Again from the Geometry I course we know that this one branch of a hyperbola with foci a and b. If $r>0$ then this is the branch closest to b and if $r<0$ this is the branch closest to a.

Exercise 13 *On separate Argand diagrams sketch the following sets*

$$\begin{array}{llll} (i) & -\pi/4<\arg z<\pi/4; & (ii) & \operatorname{Re}(z+1)=|z-1|\,; \\ (iii) & \operatorname{Re}((1+i)\,z)=1; & (iv) & \operatorname{Im}(z^3)>0. \end{array}$$

An important set of maps of the complex plane are the isometries, i.e. the distance preserving maps. We can use our knowledge of the isometries of $\mathbb{R}^2$ from the Michaelmas term Geometry I course to help us describe the isometries of $\mathbb{C}$ in terms of the complex co-ordinate.

Proposition 21 *Let $f:\mathbb{C}\to\mathbb{C}$ be an isometry so that for any $z,w\in\mathbb{C}$ we have*

$$|f(z)-f(w)|=|z-w|\,.$$

If f is orientation preserving then f has the form $f(z)=az+b$ for some $a,b\in\mathbb{C}$ with $|a|=1$. If f reverses orientation then f has the form $f(z)=a\bar{z}+b$ for some $a,b\in\mathbb{C}$ with $|a|=1$.

Proof. From the Michaelmas Term Geometry I course we know that orientation-preserving isometries of $\mathbb{R}^2$ can be written in the form

$$\begin{pmatrix} x \\ y \end{pmatrix}\mapsto\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}+\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

for some $\theta,b_1,b_2\in\mathbb{R}$. If we write $z=x+iy$, $b=b_1+ib_2$, and $a=\cos\theta+i\sin\theta$, which has modulus 1, then the above equation reads as $z\mapsto az+b$.

Similarly orientation reversing maps are of the form

$$\begin{pmatrix} x \\ y \end{pmatrix}\mapsto\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}+\begin{pmatrix} b_1 \\ b_2 \end{pmatrix}$$

which reads as $z\mapsto a\bar{z}+b$ for the same choice of z,a,b. ∎

Example 22 *Of particular note are the following three isometries:*

$$\begin{array}{lll} z & \mapsto & e^{i\theta}z \quad \textit{which is rotation anticlockwise by } \theta \textit{ about } 0; \\ z & \mapsto & z+k \quad \textit{which is translation by } \operatorname{Re}k \textit{ to the right, and } \operatorname{Im}k \textit{ up;} \\ z & \mapsto & \bar{z} \quad \textit{which is reflection in the real axis.} \end{array}$$

Example 23 *Express in the form $f(z)=a\bar{z}+b$ reflection in the line $x+y=1$.*

Solution. Knowing from the proposition that the reflection has the form $f(z) = a\bar{z} + b$ we can find a and b by considering where two points go to. As 1 and i both lie on the line of reflection then they are both fixed. So

$$\begin{aligned} a1 + b &= a\bar{1} + b = 1, \\ -ai + b &= a\bar{i} + b = i. \end{aligned}$$

Substituting $b = 1 - a$ into the second equation we find

$$a = \frac{1-i}{1+i} = -i,$$

and $b = 1 + i$. Hence

$$f(z) = -i\bar{z} + 1 + i.$$

■

Exercise 14 *What is the centre of rotation of the map $z \mapsto az + b$ where $|a| = 1, a \neq 1$? What is the invariant line of the reflection $z \mapsto a\bar{z} + b$ where $|a| = 1$?*

Exercise 15 *Write in the form $z \mapsto az + b$ the rotation through $\pi/3$ radians anti-clockwise about the point $2 + i$.*

Exercise 16 *Let $f : \mathbb{C} \to \mathbb{C}$ be given by $f(z) = iz + 3 - i$. Find a map $g : \mathbb{C} \to \mathbb{C}$ of the form $g(z) = az + b$ where $|a| = 1$ such that*

$$g(g(z)) = f(z).$$

How many such maps g are there? Geometrically what transformations do these maps g and the map f represent?

There are other important maps of the complex plane which aren't isometries. In due course we will meet the Möbius transfomrations in some detail. Here are some examples which involve determining the images of certain regions under maps. Mainly the methods used either involve relying on the fact that the map is a bijection or use a method of parametrization.

Example 24 *Find the image of each given region $A_i \subseteq \mathbb{C}$ under the given map $f_i : A_i \to \mathbb{C}$.*

- A_1 is the line $\operatorname{Re} z = 1$ and $f_1(z) = \sin z$;

A general point on the line has the form $z = 1 + it$ and we have

$$f_1(z) = \sin(1 + it) = \sin 1 \cos it + \cos 1 \sin it = \sin 1 \cosh t + i \cos 1 \sinh t.$$

So the image is given parametrically by

$$x(t) = \sin 1 \cosh t, \qquad y(t) = \cos 1 \sinh t.$$

(Note in particular that $\sin z$ is unbounded here!) Eliminating t by means of the identity $\cosh^2 t - \sinh^2 t = 1$ we see the image is one branch of the hyperbola

$$\frac{x^2}{\sin^2 1} - \frac{y^2}{\cos^2 1} = 1.$$

- A_2 is the region $\operatorname{Im} z > \operatorname{Re} z > 0$ and $f_2(z) = z^2$;

A general point in A_2 can be written in the form $z = re^{i\theta}$ where $\pi/4 < \theta < \pi/2$ and $r > 0$. As $z^2 = r^2 e^{i2\theta}$ has double the argument of z then

$$f_2(A_2) = \{z \in \mathbb{C} : \pi/2 < \arg z < \pi\}.$$

- A_3 is the region $0 < \operatorname{Re} z < \pi$ and $f_3(z) = e^{iz}$;

A general point in A_3 can be written in the form $z = x + iy$ where $0 < x < \pi$. Then $f_3(z) = e^{-y}e^{ix}$ is a complex number with arbitrary positive modulus and argument in the range $(0, \pi)$. Hence the image $f_3(A_3)$ is the upper half plane.

- A_4 is the unit disc $|z| < 1$ and $f_4(z) = (1+z)/(1-z)$.

The map f_4 is a bijection from the set $\mathbb{C}\backslash\{1\}$ to $\mathbb{C}\backslash\{-1\}$, with $f_4^{-1}(z) = (z-1)/(z+1)$. So

$$\begin{aligned} z \in f_4(A_4) &\iff f_4^{-1}(z) \in A_4 \\ &\iff \left|\frac{z-1}{z+1}\right| < 1 \\ &\iff |z-1| < |z+1| \end{aligned}$$

which is the half-plane of points closer to 1 than -1, or equivalently the half-plane $\operatorname{Re} z > 0$.

2.3 Complex Co-ordinates

Usually the hypotheses of a geometric theorem consist of a situation something along the lines of "A triangle has vertices $A, B, C, \ldots$" or "A circle with diameter $PQ \ldots$". The actual theorem makes no mention of complex numbers or of Cartesian co-ordinates and takes place in a generally featureless *Euclidean plane*, devoid of an origin and axes. The theorems we will prove each have a certain level of generality but a proof of them would be just a valid if length were measured in feet or furlongs rather than metres and just as valid wherever in the plane we place our origin and set our perpendicular axes.

So if we are faced with proving a theorem involving a triangle ABC then we can place our origin at A so that the point is represented by the complex number 0; we can further direct AB down the positive real axis and take AB as our unit length so that B is represented by the complex number 1. But at this point we have largely used up our degrees of freedom and C will be represented by a general complex number z — if it helps we can without loss of generality consider C to be in the upper half-plane so that $\operatorname{Im} z > 0$, but z will remain otherwise arbitrary. A fourth point would similarly have to be treated generally unless further specific facts were given.

- **With the introduction of complex co-ordinates geometric proofs become algebraic identitites in a handful of complex variables.**

We now prove a selection of basic geometric facts. Here is a quick reminder of some identities which will prove useful in their proofs.

$$\operatorname{Re} z = \frac{z+\bar{z}}{2}, \qquad z\bar{z} = |z|^2, \qquad \cos\arg z = \frac{\operatorname{Re} z}{|z|}, \qquad \sin\arg z = \frac{\operatorname{Im} z}{|z|}.$$

Theorem 25 *(**Cosine Rule**). Let ABC be a triangle. Then*

$$|BC|^2 = |AB|^2 + |AC|^2 - 2|AB||AC|\cos\hat{A}. \tag{2.1}$$

Proof. We can choose our co-ordinates in the plane so that A is at the origin and B is at 1. Let C be at the point z. So in terms of our co-ordinates:

$$|AB| = 1, \qquad |BC| = z - 1, \qquad |AC| = |z|, \qquad \hat{A} = \arg z.$$

So

$$\begin{aligned}
\text{RHS of (2.1)} &= |z|^2 + 1 - 2|z|\cos\arg z \\
&= z\bar{z} + 1 - 2|z| \times \frac{\operatorname{Re} z}{|z|} \\
&= z\bar{z} + 1 - 2 \times \frac{(z+\bar{z})}{2} \\
&= z\bar{z} + 1 - z - \bar{z} \\
&= (z-1)(\bar{z}-1) \\
&= |z-1|^2 = \text{LHS of (2.1)}.
\end{aligned}$$

■

The sine rule is similarly easy.

Theorem 26 *(**Sine Rule**) Let ABC be a triangle. Then*

$$\frac{|AB|}{\sin\hat{C}} = \frac{|BC|}{\sin\hat{A}} = \frac{|CA|}{\sin\hat{B}}.$$

Proof. With A, B, C represented by complex numbers $0, 1, z$ we have

$$\frac{|CA|}{\sin\hat{B}} = \frac{|z||z-1|}{\operatorname{Im}(z-1)} = \frac{|z||z-1|}{\operatorname{Im} z} = \frac{|BC|}{\sin\hat{A}}.$$

■

Here is a further example before we describe some more general theory of lines, triangles and circles.

Example 27 *On each side of an arbitrary quadrangle, draw a square externally. Show that the two line segments joining the opposite squares' centres are perpendicular to each other and of the same length.*

Solution. Suppose that the four vertices A, B, C, D are represented by complex numbers a, b, c, d, taken anti-clockwise about the quadrangle. (There is no need to use any of our degrees of freedom as the required algebraic identities follow readily.) The square that is based on AB has vertices represented by

$$a, \quad a - i(b-a), \quad b - i(b-a), \quad b$$

(taken anti-clockwise) and so has centre

$$\frac{a + [a - i(b-a)] + [b - i(b-a)] + b}{4} = \frac{(a+b) + i(a-b)}{2}.$$

So the line segment from this centre, to the centre of the opposite square is given by

$$\alpha = \frac{(c+d-a-b) + i(c-d-a+b)}{2}$$

and the line segment between the centres of the other two squares is represented by

$$\beta = \frac{(d+a-b-c) + i(d-a-b+c)}{2}.$$

As

$$\beta = i\alpha$$

then these line segments are of the same length and perpendicular to one another. ■

2.4 Some Geometric Theory 1 — Lines

Consider the equation

$$Az + B\bar{z} = C \tag{2.2}$$

where $A, B, C \in \mathbb{C}$ and A, B are not both zero. By conjugating this we get

$$\bar{A}\,\bar{z} + \bar{B}z = \bar{C}.$$

We can uniquely solve these two equations to get

$$z = \frac{C\bar{A} - \bar{C}B}{|A|^2 - |B|^2}$$

provided that $|A| \neq |B|$. This is not that surprising as equation (2.2), though only one equation in the complex co-ordinate z represent two linear equations in $\operatorname{Re} z$ and $\operatorname{Im} z$.

On the other hand if $|A| = |B| \neq 0$ then we need to have $C\bar{A} = \bar{C}B$ to have any solutions at all. In this event, provided $C \neq 0$, then equation (2.2) is equivalent to

$$\bar{C}Az + C\bar{A}\,\bar{z} = C\bar{C}.$$

If we set $\alpha = C\bar{A}$ and $k = C\bar{C} = |C|^2$ then we have rewritten (2.2) in the form

$$\bar{\alpha}z + \alpha\bar{z} = k, \qquad (\alpha \in \mathbb{C}, k \in \mathbb{R})$$

which is the equation of a line (as this is a non-unique, non-empty set of solutions to two linear equations in the plane). But we can see this more readily if we rewrite the equation again as

$$\operatorname{Re}(\bar{\alpha}z) = \frac{k}{2}.$$

This is the image of the line $\operatorname{Re} z = k/2$ under the map $z \mapsto z/\bar{\alpha}$ which takes the line and performs an enlargement of order $1/|\alpha|$ on it and then a rotation by $\arg\alpha$ anticlockwise — both of which will preserve it as a line.

Proposition 28 *Three points z_1, z_2, z_3 are collinear if and only if*

$$\begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix} = 0.$$

Proof. The proof is trivial if any of the three points agree. So assuming the three points are distinct then

$$\begin{aligned}
& \begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 & \bar{z}_2 & 1 \\ z_3 & \bar{z}_3 & 1 \end{vmatrix} = 0 \\
\iff \quad & \begin{vmatrix} z_1 & \bar{z}_1 & 1 \\ z_2 - z_1 & \bar{z}_2 - \bar{z}_1 & 0 \\ z_3 - z_1 & \bar{z}_3 - \bar{z}_1 & 0 \end{vmatrix} = 0 \\
\iff \quad & \begin{vmatrix} z_2 - z_1 & \bar{z}_2 - \bar{z}_1 \\ z_3 - z_1 & \bar{z}_3 - \bar{z}_1 \end{vmatrix} = 0 \\
\iff \quad & \left(\frac{z_3 - z_1}{z_2 - z_1}\right) = \overline{\left(\frac{z_3 - z_1}{z_2 - z_1}\right)} \\
\iff \quad & \frac{z_3 - z_1}{z_2 - z_1} \text{ is real} \\
\iff \quad & z_1, z_2, z_3 \text{ are collinear}
\end{aligned}$$

as required. ■

Corollary 29 *The line connecting the points p and q has equation*

$$\begin{vmatrix} z & \bar{z} & 1 \\ p & \bar{p} & 1 \\ q & \bar{q} & 1 \end{vmatrix} = 0.$$

Corollary 30 *The perpendicular bisector of p and q has equation*

$$\begin{vmatrix} z & \bar{z} & 1 \\ p & \bar{q} & 1 \\ q & \bar{p} & 1 \end{vmatrix} = 0.$$

Proof. By taking independent combinations of the last two lines of the determinant we see that the above equation can be rewritten as

$$\begin{vmatrix} z & \bar{z} & 1 \\ \frac{p+q}{2} & \frac{\bar{p}+\bar{q}}{2} & 1 \\ \frac{(1+i)p+(1-i)q}{2} & \overline{\frac{(1+i)p+(1-i)q}{2}} & 1 \end{vmatrix} = 0.$$

The point $(p+q)/2$ is clearly on the perpendicular bisector being the midpoint, as is

$$\frac{(1+i)\,p+(1-i)\,q}{2} = \frac{p+q}{2} + i\left(\frac{p-q}{2}\right)$$

because it is removed from the midpoint in a perpendicular direction, namely $i\,(p-q)$. ■

2.5 Some Geometric Theory 2 — Triangles

As promised we apply the use of complex co-ordinates to the theory of triangles. One particular advantage that complex number methods have in the plane, over more general vector methods, is the ease with which rotations can be described.

Proposition 31 *The triangle abc in $\mathbb{C}$ (with the vertices taken in anticlockwise order) is equilateral if and only if*

$$a + \omega b + \omega^2 c = 0$$

where $\omega = e^{2\pi i/3}$ is a cube root of unity.

Proof. Firstly note that $1+\omega+\omega^2 = 0$, as

$$0 = \omega^3 - 1 = (\omega - 1)\left(\omega^2 + \omega + 1\right).$$

The triangle abc is equilateral if and only if $c-b$ is the side $b-a$ rotated through $2\pi/3$ anticlockwise — i.e. if and only if

$$\begin{aligned} c-b &= \omega\,(b-a) \\ &\iff \omega a + (-1-\omega)\,b + c = 0 \\ &\iff \omega a + \omega^2 b + c = 0 \\ &\iff a + \omega b + \omega^2 c = 0. \end{aligned}$$

■