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MONOVARIANTS

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Imagine a system on which you can perform various operations. You would like to analyze the behavior of the system, to determine what positions can be reached from what other positions. One of the most important mathematical tools for analyzing this is the notion of an "invariant," a property of the system (often numerical) which does not change under various operations. Invariants can be used to show that one configuration cannot be reached from another. But if you wanted to hear about invariants, you came to the wrong place, because I'm talking instead about a closely related topic: monovariants. A monovariant is a property of the system — most often a real number, especially an integer, though any sort of ordered set provides a meaningful context for monovariants — that may vary, but only in one direction. In general, this means that with each operation, the monovariant either always increases or always decreases.

That was pretty vague, so here are a few examples of monovariants in everyday life. Perhaps the simplest is your age. As time passes, you only get older (sadly). Another is the physical concept of entropy — the second law of thermodynamics states that it is a monovariant. If you're trapped in a room full of fine china and you proceed to drop it on the concrete floor, the number of pieces of china in the room will always increase; this is a monovariant. These quantities can be used to study the attainability of certain configurations, starting from other configurations. For example, because the number of pieces increases, you cannot reassemble a plate from a bunch of shards by repeatedly dropping them on the floor. However, monovariants have much more substantial mathematical uses, as we shall see. I classify these broadly into two categories...

1 What must eventually occur

One often shows that a system must eventually attain some particular configuration because a monovariant cannot go on changing forever. This typically happens with *strict* monovariants, i.e. quantities which must increase (or always decrease), never staying the same; usually we know our process terminates because the monovariant can only assume finitely many values or is otherwise limited in the extent of possible change, such as a positive integer that always decreases. But enough generality. Concrete examples are needed. So, here's the stereotypical monovariant problem:

Example 1 2000 *people are distributed among the rooms of a* 115-*room mansion. Each minute, as long as not all the people are in the same room, somebody walks from one room into a different room with at least as many people. Prove that eventually all the people will be gathered in one room.*

This result seems intuitively plausible. The heavily populated rooms will get even more popular, while the rooms with few people will come to have even fewer, so we should expect that the outcome is for all people to be in one room. But how do we make this rigorous? A numerical monovariant is the key.

Solution: For each room, consider the square of the number of people in that room. Let the sum of these squares be S. We claim that S increases with each move. To see this, suppose that a person from a room with n people walks into a room with $m \geq n$ people. Then the squares of the populations of these rooms change from n^2 and m^2 to $(n-1)^2$ and $(m+1)^2$, respectively, while all the other rooms remain passive, so the net change in S is

$$[(n-1)^2 + (m+1)^2] - [n^2 + m^2] = [n^2 + m^2 - 2n + 2m + 2] - [n^2 + m^2] = 2(m-n) + 2 \geq 2 > 0.$$

Thus S always increases.

Now, the number of people is clearly always 2000 (an invariant!). There are finitely many possible distributions of 2000 people among the various rooms, so there are finitely many possible values of S. This means that S cannot increase forever. But as long as at least two rooms have people in them, our operation can be performed and S will increase, so when the process terminates, all the people will be in the same room. ■

That wasn't so bad, was it? Many different expressions can turn out to be monovariants. In fact, sometimes using more than one monovariant helps. In this next example, we let one monovariant get lazy and not change for a while, during which a different quantity does its work. (If you like to be needlessly complicated, you can think of these two numbers as one ordered pair, which is a monovariant under lexicographical ordering.)

Example 2 *On an infinite square grid, several squares are colored black. A new square grid is produced according to the following rule: a square is black iff at least three of its four neighbors were black in the previous stage. This process is repeated. Prove that eventually there are no black squares left.*

Solution: Consider all those rows that contain some black square. Look at the "range" of these rows: the vertical distance between the top of the highest row and the bottom of the lowest. We claim that this range never increases. In fact, we can show a stronger statement: any row that is devoid of black squares will remain so. This is because, at each stage, every square in such a row has at most two black neighbors (since its left and right neighbors are white), so it will remain white at the next stage.

So the range of not-all-white rows cannot increase. But what if it becomes constant at some value greater than zero? In that case, consider the range of black squares in the topmost such row (the distance from the leftmost edge to the rightmost edge). This distance not only cannot increase; it always decreases: any square to the left of the leftmost black one, and this black square itself, will have at least two white neighbors (in the upward and leftward directions) and so becomes white. A similar argument holds on the right side. Thus the range of this row decreases until it becomes zero, at which point the row is entirely white, so that the range of not-all-white rows decreases.

This shows that, as long as there is at least one row with some black square, the range of such rows will eventually decrease. This decrease cannot happen forever, so eventually we reach a state with no rows containing black squares — i.e. the whole grid is white. ■

An even more significant use of monovariants is as follows: We may wish to show that a configuration with certain properties exists, and no moves are given. We set up moves so that the final configuration necessarily has the desired properties. Here's an example...

Example 3 *(Quantum, 1994) Given are n red points and n blue points in the plane, no three collinear. Show that we can draw n nonintersecting segments connecting the blue points to the red points.*

Solution: Consider any arbitrary pairing of the blue points with the red points. We consider the following operation: If any two such segments cross, their endpoints form a convex quadrilateral with vertices in the order R, B, B, R. Replace the two R-B diagonals (which the segments must be, since they intersect) with the two R-B sides. It follows from the triangle inequality that the sum of the lengths of these sides is less than the sum of the diagonals, so our operation decreases the total length of the n segments. This total length, then, is our monovariant. Since there are only finitely many possible pairings, iteration of the above operation must eventually lead us somewhere from which we can proceed no further, i.e. no two segments intersect, as desired. ■

It is worth noting that this use of monovariants appears in number theory, in the form of "infinite descent" arguments. For example, one often shows that an equation has no integer solutions by the following method: one constructs a function of a solution (e.g. the sum of all the variables) whose value is a positive integer; then, given a solution, one constructs a new solution for which this function has a smaller value. One then iterates this process, obtaining an infinite decreasing sequence of positive integers; this is an impossibility, which proves that the original sequence cannot exist. Several examples of results obtainable by this method are in the problems.

This selection of problems expands on many of the ideas presented here. In general, the solution to a "show we eventually get somewhere" problem is as follows: find the monovariant and show it must change under a certain operation; show that it can change only finitely many times; then prove that it can only stop changing once "somewhere" has been achieved. If the moves are not provided for you, you need to construct them appropriately. Many different monovariants can be constructed — sums, products, maxima, minima, and some really ineffably weird things can turn out to be useful monovariants. See for yourself.

1. (St. Petersburg, 1996) Several positive integers are written on a blackboard. One can erase any two distinct integers and write their greatest common divisor and least common multiple instead. Prove that eventually the numbers will stop changing.

2. (Russia, 1961, adapted) An $m \times n$ array of real numbers is given. When the sum of the numbers in any row or column is negative, we may switch the signs of all the numbers in that row or column. If this operation is iterated, prove that all of the row or column sums eventually become nonnegative.

3. (USAMO, 1997) Let $p_1, p_2, p_3, \ldots$ be the prime numbers listed in increasing order, and let x_0 be a real number between 0 and 1. For positive integer k, define

$$x_k = \begin{cases} 0 & \text{if } x_{k-1} = 0, \\ \left\{ \frac{p_k}{x_{k-1}} \right\} & \text{if } x_{k-1} \neq 0, \end{cases}$$

where $\{x\}$ denotes the fractional part of x. Find, with proof, all x_0 satisfying $0 < x_0 < 1$ for which the sequence $x_0, x_1, x_2, \ldots$ eventually becomes 0.

4. (USAMO, 1993) Let a, b be odd positive integers. Define the sequence (f_n) by putting $f_1 = a, f_2 = b$, and letting f_n for $n \geq 3$ be the greatest odd divisor of $f_{n-1} + f_{n-2}$. Show that f_n is constant for n sufficiently large and determine the eventual value as a function of a and b.

5. (MOP 1998) Find all solutions to the equation $x^2 + y^2 = 3(z^2 + w^2)$ in positive integers.

6. (via Paul Zeitz) An arbitrary finite graph is given. Prove that the vertices can be colored in black and white so that, for each vertex, at least half of its neighbors are the opposite color from the vertex itself.

7. (J. Cofman, *What to Solve?*) Place four nonnegative integers a_0, b_0, c_0, d_0 around a circle. For any two consecutive numbers take their absolute difference and place those numbers in order on a new circle: $a_1 = |a_0 - b_0|, b_1 = |b_0 - c_0|, c_1 = |c_0 - d_0|, d_1 = |d_0 - a_0|$. Iterate this process. Is it true that the process always eventually leads to a circle with four 0s? (After solving this, try to generalize your result by replacing 4 with any power of 2 greater than one. - Ed.)

8. (MOP 1998) A circle has been cut into 2000 sectors. There are 2001 frogs inside these sectors. There will always be some two frogs in the same sector; each second, two such frogs jump to the two sectors adjacent to their original sector (in opposite directions). Prove that, at some point, at least 1001 sectors will be inhabited.

9. (Classical) Prove that the equation $a^4 + b^4 = c^2$ has no solutions in nonzero integers.

10. (Moscow, 1964) King Arthur summoned $2n$ knights to his court. Each knight has at most $n-1$ enemies among the other knights present. Prove that the knights can sit at the Round Table so that no two enemies sit next to each other. (Assume that the relation of "enmity" is symmetric.)

11. (IMO, 1986) To each vertex of a regular pentagon an integer is assigned so that the sum of all five numbers is positive. If three consecutive vertices are assigned the numbers x, y, z with $y < 0$, then the following operation is allowed: the numbers x, y, z are replaced by $x + y, -y, z + y$, respectively. Such an operation is performed repeatedly as long as at least one of the five numbers is negative. Determine whether this procedure necessarily comes to an end in a finite number of steps.

2 What can never occur

The second, and perhaps more intuitively obvious, use of monovariants is to show that a system can never attain a particular configuration. The example of dropping plates, from a part of this lecture that now lies in the distant past, represents this sort of situation. In a certain sense, showing unattainability of a particular position is easier than what we did before: the monovariant need not be forced to change only finitely many times; also, the monovariant does not need to be strict — it can remain constant under some moves. (From this point of view, invariants are a special kind of monovariant.) All we need is to show that the monovariant can only change in one direction if it changes at all, and that reaching one configuration from another would require a change in the opposite direction.

Here's an example of this technique, of highly contemporary vintage:

Example 4 *(IMO, 2000, adapted) Let $n \geq 2$ be a positive integer. Initially, there are n fleas on a horizontal line. For any positive number $\lambda < 1/(n-1)$, define a move as follows: choose any two fleas, at points A and B, with A to the left of B; let the flea at A jump to the point C on the line to the right of B with $BC/AB = \lambda$. Show that there exists some initial position of the n fleas and a point M on the line so that it is not possible to get all the fleas to the right of M.*

Solution: The natural way to start is to coordinatize the line and identify the fleas with their respective coordinates. Thus a move consists of changing a flea A to $C = B + \lambda(B - A)$ for some flea $B > A$. We will show a stronger result than required: for *any* initial position, we can choose M so that *none* of the fleas can become $> M$.

Consider the following quantity: 1 times the rightmost flea, minus λ times the sum of all other fleas. We claim this quantity cannot increase. Indeed, consider a jump by the flea A over B, and pretend that A increases gradually until the jump is finished. Let the rightmost flea be D. As A moves rightward, our quantity only decreases (since the coefficient of A is $-\lambda < 0$) as long as A is to the left of D. If A passes by D, then the quantity has decreased (so far) by $\lambda(D - A)$. After the passing occurs, A is the rightmost flea, so as A increases, our expression also increases, by exactly the distance that A travels. But how far past D can A go? Certainly the way to make A jump the farthest is to take $B = D$. But in this case, the increase beyond D is simply equal to $\lambda(D - A)$. Thus, at best, our quantity decreases by $\lambda(D - A)$ and increases again by $\lambda(D - A)$; if A stops somewhere in the middle, our quantity strictly decreases. In any event, it can never increase.

On the other hand, each of the fleas other than D is $\leq D$, so our expression is $D - \lambda(n-1$ fleas farther left$)$ $\geq D - \lambda(n-1)D = \mu D$ where $\mu = 1 - (n-1)\lambda > 0$ (positivity is important). Now, if we choose M large enough so that μM exceeds the initial value of our monovariant, then the monovariant will always be less than μM. The above inequalities imply that $\mu D < \mu M \Rightarrow D < M$. So even the rightmost flea can never land to the right of M. ■

The idea of showing the unattainability of a final configuration can be equivalently thought of as stating that a fixed relation holds between the initial and final configurations. Sometimes we are given the monovariant, and we know the initial and final configurations; we need to construct moves that show that the expression monovaries. The clearest use of this technique is in proving inequalities. The following classical inequality is a key example.

Example 5 *(Rearrangement Inequality) Suppose $x_1 \geq x_2 \geq \cdots \geq x_n$ and $y_1 \geq y_2 \geq \cdots \geq y_n$ are real numbers, and π is any permutation of $\{1, 2, \ldots, n\}$. Show that*

$$x_1y_1 + x_2y_2 + \cdots + x_ny_n \geq x_1y_{\pi(1)} + x_2y_{\pi(2)} + \cdots + x_ny_{\pi(n)}.$$

Solution: We use induction on n. We need to do two base cases. The case $n = 1$ is trivial. If $n = 2$, there are two possibilities: either π is the identity, in which case the statement is again trivial, or π switches 1 and 2, in which case we have

$$(x_1y_1 + x_2y_2) - (x_1y_2 + x_2y_1) = (x_1 - x_2)(y_1 - y_2) \geq 0$$

which proves the claim.

Now consider $n > 2$; pretend for notational convenience that n is large. By the two-variable case, we have $x_1y_1 + x_2y_2 \geq x_1y_2 + x_2y_1$, so

$$x_1y_1 + x_2y_2 + x_3y_3 + \cdots + x_ny_n \geq x_1y_2 + x_2y_1 + x_3y_3 + \cdots + x_ny_n.$$

But applying the two-variable case again gives $x_1y_2 + x_3y_3 \geq x_1y_3 + x_3y_2$, so that

$$x_1y_2 + x_2y_1 + x_3y_3 + x_4y_4 + \cdots + x_ny_n \geq x_1y_3 + x_2y_1 + x_3y_2 + x_4y_4 + \cdots + x_ny_n.$$

We can apply this method again, repeatedly changing $x_1y_k + x_{k+1}y_{k+1}$ to $x_1y_{k+1} + x_{k+1}y_k$ until x_1 is paired with $y_{\pi(1)}$. (If $\pi(1) = 1$, no switches are necessary.) Combining all these inequalities gives

$$\begin{aligned} x_1y_1 + x_2y_2 + \cdots + x_ny_n \quad &\geq \quad x_1y_{\pi(1)} + x_2y_1 + x_3y_2 + \cdots + x_{\pi(1)}y_{\pi(1)-1} \\ &\qquad + x_{\pi(1)+1}y_{\pi(1)+1} + x_{\pi(1)+2}y_{\pi(1)+2} + \cdots + x_ny_n. \end{aligned}$$

Now using the induction hypothesis tells us that all of the right-hand side except the first term, i.e. $x_2y_1 + x_3y_2 + \cdots + x_ny_n$, is $\geq x_2y_{\pi(2)} + x_3y_{\pi(3)} + \cdots + x_ny_{\pi(n)}$. This completes the induction step and the proof. ■

What did we do? In effect, we had a variable permutation, ϕ, which gradually changed from the identity to π in such a way that the quantity $x_1y_{\phi(1)} + \cdots + x_ny_{\phi(n)}$ never increased. This is where the monovariant appears. This method of proving inequalities is not at all uncommon.

Most often, in using monovariance to prove inequalities, one uses a method sometimes called "smoothing": if the equality case occurs when all variables are equal, then we bring the variables gradually closer together until (after some finite number of steps) they coincide and show that the value of some appropriate function is a monovariant. A related concept is that of "unsmoothing," bringing variables farther apart in order to reach the equality case of an inequality. Some examples of inequalities which can be proven by these methods appear in the ensuing problems.

We have seen that a variety of monovariants can be used to show that a configuration can never be attained, generally by simply constructing a monovariant and observing that it really is one. Instead of being given moves, we could also be given the monovariant and have to find the moves, an approach used for proving inequalities. These problems provide some more applications of these concepts.

1. n boxes in an infinite square grid are colored black; the rest are colored white. When a square is the opposite color from 2 or more of its 4 neighbors, its color may be switched. Eventually, we get to having 2000 black boxes, no two of which border along an edge, and all other boxes white. Prove that $n \geq 2000$.

2. (*Quantum*, 1994) A 1 and nine 0's are written on the board. We may replace any two numbers each by their average. What smallest number can eventually replace the 1?

3. (Balkan Olympiad, 1998) If $n \geq 2$ is an integer and $0 < a_1 < a_2 < \cdots < a_{2n+1}$ are real numbers, prove that
$$\sqrt[n]{a_1} - \sqrt[n]{a_2} + \sqrt[n]{a_3} - \cdots - \sqrt[n]{a_{2n}} + \sqrt[n]{a_{2n+1}} < \sqrt[n]{a_1 - a_2 + a_3 - \cdots - a_{2n} + a_{2n+1}}.$$

4. (AM-GM inequality) Let $x_1, x_2, \ldots, x_n \geq 0$. Show that
$$\sqrt[n]{x_1x_2\cdots x_n} \leq \frac{x_1 + x_2 + \cdots + x_n}{n},$$
with equality iff all x_i are equal.

5. (Rearrangement inequality, "dual" version) Suppose $x_1 \geq x_2 \geq \cdots \geq x_n$ and $y_1 \geq y_2 \geq \cdots \geq y_n$ are real numbers, and π is any permutation of $\{1, 2, \ldots, n\}$. Show that
$$x_1y_n + x_2y_{n-1} + \cdots + x_ny_1 \leq x_1y_{\pi(1)} + x_2y_{\pi(2)} + \cdots + x_ny_{\pi(n)}.$$

6. (USAMO, 1999) Let $a_1, a_2, \ldots, a_n$ ($n > 3$) be real numbers such that
$$a_1 + a_2 + \cdots + a_n \geq n \quad \text{and} \quad a_1^2 + a_2^2 + \cdots + a_n^2 \geq n^2.$$
Prove that $\max(a_1, a_2, \ldots, a_n) \geq 2$.

7. (MOP 1998) On an infinite square grid, a horizontal line is chosen. Checkers may be placed below the line, at most one per square. The following move may be performed: A checker may jump over an adjacent checker (horizontally or vertically) into an adjacent empty square; the checker jumped over is then removed. Prove that no checker can ever arrive more than four squares above the chosen line.

That's it for today. Tune in next week for another exciting episode of BMC!