The Pigeonhole Principle (Teacher's Edition)

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1 Overview

The pigeonhole principle is sometimes stated as follows: If you have n pigeons and $n+1$ holes, then some pigeon has more than one hole in it.

In fact, a pigeonhole is neither a part of a pigeon nor a container for pigeons, but rather a box used for filing papers (such as the faculty mailboxes that might line the walls of your school's main office). However, that fact is boring, so you may as well ignore it like everyone else does.

The pigeonhole principle is a special case of the following more general idea: If you have some collection of numbers, and you know their average is $\geq x$, then there must be at least one of the numbers that is $\geq x$. Similarly, if you have a collection of numbers and their average is $\leq y$, then one of the numbers must be $\leq y$.

The pigeonhole principle is generally useful when you're trying to prove something exists. In particular, if you need to prove the existence of some object satisfying an inequality, that should set your pigeonhole alert on high (although it can also call for other methods, such as extremal arguments).

2 Problems

1. Given are n integers. Prove that there is some nonempty subset of them whose sum is divisible by n.

2. Let α be an arbitrary real number. Prove that for any positive integer n, there exists an integer k with $0 < |k| \leq n$ and $\lfloor k\alpha \rfloor < 1/n$.

3. (a) Seven different real numbers are chosen. Prove that there are some two of them, say a and b, such that
$$0 < \frac{a-b}{1+ab} < \frac{\sqrt{3}}{3}.$$

(b) Seven different real numbers are chosen. Prove that there are some two of them, say a and b, such that

$$0 < \frac{a-b}{3+ab} < \frac{1}{3}.$$

4. [MOP, 2004] The unit squares of a 5×41 grid are colored in red and blue. Prove that there are 3 rows and 3 columns such that the 9 squares where they intersect are all the same color.

5. [St. Petersburg, 1998] On each of 10 sheets of paper are written several powers of 2. A given number may be written multiple times on the same sheet, and may be written on more than one sheet. Show that some number appears at least 6 times among the 10 sheets.

6. Ten different 10-element subsets of $\{1, 2, \ldots, 20\}$ are chosen. Prove that some two of the subsets have at least five elements in common.

7. The Fibonacci numbers are defined by $F_1 = F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 3$. If p is a prime number, prove that some one of the first $p + 1$ Fibonacci numbers must be divisible by p.

8. [Putnam, 1980] Let X be a finite set. Suppose subsets $A_1, A_2, \ldots, A_{2010}$ of X are given, each containing more than half the elements of X. Prove that there exist ten elements $x_1, x_2, \ldots, x_{10} \in X$ such that each A_i contains at least one of $x_1, \ldots, x_{10}$.

9. [MOP, 2004] Nonnegative numbers $a_1, \ldots, a_7$ and $b_1, \ldots, b_7$ are given, with $a_i + b_i \leq 2$ for each i. Prove that there exist distinct indices i, j with $|a_i - a_j| + |b_i - b_j| \leq 1$.

10. There are 2010 cities in a country. Some pairs of cities are connected by roads. Prove that it is possible to partition the cities into two sets A and B, containing 1005 cities each, such that more than half the roads connect a city in A with a city in B.

11. [MOP, 2004] 16 numbers are chosen from the set $\{1, 2, \ldots, 100\}$. Prove that among these chosen numbers are four distinct values a, b, c, d such that $a + b = c + d$.

12. A set S of 10 positive integers is given, whose sum is less than 250. Prove that there exist two disjoint, nonempty subsets $A, B \subseteq S$, having the same size, and such that the sum of the elements of A equals the sum of the elements of B.

13. [Putnam, 2001] Let B be a set of more than $2^{n+1}/n$ distinct points in n-dimensional space with coordinates of the form $(\pm 1, \pm 1, \ldots, \pm 1)$, where $n \geq 3$. Show that there are three distinct points in B which are the vertices of an equilateral triangle.

14. [IMO, 1998] In a contest, there are m candidates and n judges, where $n \geq 3$ in an odd integer. Each candidate is evaluated by each judge as either pass or fail. It turns out that each pair of judges agrees on at most k candidates. Prove that

$$\frac{k}{m} \geq \frac{n-1}{2n}.$$

15. [Russia, 1999] In a class, each boy is friends with at least one girl. Show that there exists a group of at least half of the students, such that each boy in the group is friends with an odd number of the girls in the group.

    for each set of girls, find all the boys who are friends with an odd number of them; average over sets of girls

16. [Po-Shen's handout, 2010] An $n^2 \times n^2$ array is filled with the numbers $\{1, 2, \ldots, n^2\}$, each appearing n^2 times. Prove that some row or column contains at least n different numbers.

17. [Paul Erdős] Prove that, if $n+1$ integers are chosen from the set $\{1, 2, \ldots, 2n\}$, one of them must be divisible by another.

18. [Putnam, 1993] Let $x_1, \ldots, x_{19}$ be positive integers less than or equal to 93. Let $y_1, \ldots, y_{93}$ be positive integers less than or equal to 19. Prove that there exists a (nonempty) sum of some x_i's equal to a sum of some y_j's.

    assume wlog $\sum y_j \geq \sum x_i$. for each initial collection of x's, consider the shortest initial segment of y's that has greater sum. the differences $\sum y_j - \sum x_i$ must lie in $(0, 19]$. there are 20 of them. pigeonhole.

19. Let n be a positive odd integer, and let $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ be nonnegative numbers such that $x_1 + \cdots + x_n = y_1 + \cdots + y_n$. Prove that there exists a proper, nonempty subset of indices $J \subseteq \{1, 2, \ldots, n\}$ such that

$$\frac{n-1}{n+1} \sum_{j \in J} x_j \leq \sum_{j \in J} y_j \leq \frac{n+1}{n-1} \sum_{j \in J} x_j.$$

    there's some j such that $x_j \leq 2/(n+1)$ and $y_j \leq 2/(n+1)$ (if not, either $\sum_j x_j > 1$ or $\sum_j y_j > 1$). now just let J consist of all indices except this j.

20. [Iran, 1999] Let $r_1, r_2, \ldots, r_n$ be real numbers. Prove that there exists $S \subseteq \{1, 2, \ldots, n\}$ such that

$$1 \leq |S \cap \{i, i+1, i+2\}| \leq 2$$

    for each i, $1 \leq i \leq n-2$, and

$$\left| \sum_{i \in S} r_i \right| \geq \frac{1}{6} \sum_{i=1}^{n} |r_i|.$$

wlog sum of all numbers is positive; consider including the numbers whose indices are i mod 3 and the positive numbers whose indices are j mod 3, for each pair of distinct (i, j). by averaging check that we get at least $1/3$ the sum of all positive numbers, which in turn is at least $1/6$ the sum of all absolute values.

21. [USAMO, 1995] Among n people, any two are either friends or strangers. There are q pairs of friends, and there are no three people who are all friends with each other. Prove that some person has the following property: among all the other people who are not friends with him, there are at most $q(1 - 4q/n^2)$ pairs of friends.

22. [Erdős-Szekeres Theorem] The numbers $1, 2, 3, \ldots, mn + 1$ are arranged in some order. Prove that there exists either a subsequence of $m + 1$ terms in increasing order, or a subsequence of $n + 1$ terms in decreasing order. (Terms in a subsequence do not have to be consecutive.)