Combinatorial Geometry (Black Group)

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Combinatorial geometry problems often have deceptively simple statements. Ironically, this is precisely what makes them difficult: there are so many possible approaches. While many problems have somewhat ad hoc solutions, there are a few (very) general methods to keep in mind.

- **Extremal arguments.** A good example is *Sylvester's theorem*: Given a finite number of points in the plane, either (1) all the points are collinear; or (2) there is a line which contains exactly two of the points.
- **The convex hull.** For any finite set of points in the plane, there exists a subset forming a convex polygon that contains all the points.
- **Helly's theorem.** Suppose that $X_1, X_2, \ldots, X_n$ is a finite collection of convex subsets of $\mathbb{R}^d$, where $n > d$. If the intersection of every $d+1$ of these sets is nonempty, then the whole collection has a nonempty intersection. (This theorem is used somewhat infrequently but worth knowing.)

Puzzling Problems

1. True or false? If convex polygon A contains convex polygon B, then the perimeter of A is at least the perimeter of B.
2. Let $\mathcal{S}$ be the set of all polygonal areas in the plane. Prove that there is a function $f : \mathcal{S} \to (0,1)$ which satisfies
$$f(S_1 \cup S_2) = f(S_1) + f(S_2)$$
for any $S_1, S_2 \in \mathcal{S}$ which have common points only on their borders.
3. A unit square is dissected into $n > 1$ rectangles such that their sides are parallel to the sides of the square. Any line, parallel to a side of the square and intersecting its interior, also intersects the interior of some rectangle. Prove that in this dissection, there exists a rectangle having no point on the boundary of the square.
4. A set of 101 points no three of which are collinear are chosen inside a unit square. Prove that some three of them form a triangle with area less than $\frac{1}{180}$.
5. In the Cartesian coordinate plane define the strip
$$S_n = \{(x, y) \ : \ n \le x < n+1\}$$

for every integer n. Assume that each strip S_n is colored either red or blue, and let a and b be two distinct positive integers. Prove that there exists a rectangle with side lengths a and b such that its vertices have the same color.

6. Let k and n be integers with $0 \leq k \leq n-2$. Consider a set L of n lines in the plane such that no two of them are parallel and no three have a common point. Denote by I the set of intersection points of lines in L. Let O be a point in the plane not lying on any line of L. A point $X \in I$ is colored red if the open line segment OX intersects at most k lines in L. Prove that I contains at least
$$\frac{1}{2}(k+1)(k+2)$$
red points.

7. In the plane, we are given a set S of equilateral triangles each of which is a dilation of another. Assume that each pair of triangles from S have at least one common point; that is, for each pair of triangles, there is at least a common point lies within the boundary of each triangle. Prove that there exist three points such that for each triangle in S, at least one of the three points lies within the boundary of the triangle.

8. Let $(P_1, P_2, \ldots, P_{2n})$ be a permutation of the vertices of a regular polygon. Prove that the closed polygonal line segment $P_1P_2 \ldots P_{2n}$ contains a pair of parallel segments.

9. Given $2n+1$ segments of on a line such that each segment intersects at least n other segments, prove that one of the segments intersects all other segments.

10. The vertices of a convex polygon are colored by at least three colors such that no two consecutive vertices have the same color. Prove that one can dissect the polygon into triangles by diagonals that do not cross and whose endpoints have different colors.

11. Prove that any n points in the plane can be covered by finitely many disks with the sum of the diameters less than n and the distance between any two disks greater than 1.

12. Given a convex polygon $\mathcal{P}$, prove that there exists a homothety $\mathbf{H}$ with ratio $-\frac{1}{2}$ such that the image of $\mathcal{P}$ under $\mathbf{H}$ is inside $\mathcal{P}$.

13. A large rectangle in the plane is partitioned into smaller rectangles, each of which has either integer height or integer width (or both). Prove that the large rectangle also has this property.

14. An equilateral triangle ABC is divided into n^2 congruent equilateral triangles. What is the greatest number of vertices of small triangles that can be chosen so that no two of them lie on a line that is parallel to any of the sides of triangle ABC?

15. Let $\mathcal{P}$ be a convex polygon. A segment connecting two points on (distinct) sides of $\mathcal{P}$ is called a *chord*. Assume that if a chord bisects the area of the polygonal region, then the chord has length at most 1. Prove that the area of $\mathcal{P}$ is at most $\frac{\pi}{4}$.

16. Let n be an integer greater than 2. Prove that among any n given points in the plane, there are three of them, denoted by A, B, C, such that
$$1 \leq \frac{AB}{AC} < \frac{n+1}{n-1}.$$