Sums & Products

1. Let n be a positive integer. Evaluate
$$\sum_{k=1}^{n} \sum_{1\le i_1<\cdots<i_k\le n} \frac{2^k}{(i_1+1)(i_2+1)\cdots(i_k+1)}.$$

2. Let k be an odd positive integer, and let x be an arbitrary real number. Prove that
$$\tan(kx) = (-1)^{\frac{k-1}{2}} \prod_{j=0}^{k-1} \tan\left(x - \frac{j\pi}{k}\right)$$

3. Prove that for any positive integer n, $n!$ is a divisor of
$$\prod_{k=0}^{n-1}(2^n - 2^k).$$

4. Let $f(n)$ be a function defined on the set of positive integers such that $f(1) = 2$ and
$$f(n+1) = (f(n))^2 - f(n) + 1 \qquad \text{for every positive integer } n.$$
Prove that for $n = 2, 3, \ldots,$
$$1 - \frac{1}{2^{2^{n-1}}} < \frac{1}{f(1)} + \frac{1}{f(2)} + \cdots + \frac{1}{f(n)} < 1 - \frac{1}{2^{2^n}}.$$

5. Let n be an integer greater than 1, and let A be a set of real numbers with less than n elements. Suppose that $2^0, 2^1, 2^2, \ldots, 2^{n-1}$ can be written as the sum of distinct elements in A. Prove that A contains at least one negative number.

6. Prove that for every positive integer n,
$$\sum_{k=1}^{n} \frac{1}{k\binom{n}{k}} = \frac{1}{2^{n-1}} \sum_{\substack{k=1 \\ k \text{ odd}}}^{n} \frac{\binom{n}{k}}{k}.$$

7. For a nonnegative integer N, let $u(N)$ denote the number of ones in N when expressed in binary (so that for example $u(10) = u(1010_2) = 2$). Let $m, n \ge 0$ be integers and $k \ge mn$ a third integer. Express
$$\sum_{i=0}^{2^k-1} (-1)^{u(i)} \binom{\binom{i}{m}}{n}$$
in closed form.

8. For a finite set S of numbers, let $|S|$ denote its cardinality, $\sigma(S)$ the sum of its elements and $\pi(S)$ the product. Suppose that S is a finite set of nonnegative integers, and let $N \geq \sigma(S)$ be a nonnegative integer. Prove that

$$\sum_{T \subseteq S} (-1)^{|T|} \binom{N - \sigma(T)}{|S|} = \pi(S).$$

9. Given a positive integer n, express the sum

$$\frac{1}{\binom{2n}{1}} - \frac{1}{\binom{2n}{2}} + \frac{1}{\binom{2n}{3}} - \cdots - \frac{1}{\binom{2n}{2n-2}} + \frac{1}{\binom{2n}{2n-1}}$$

in closed form.

10. Let $a_1, a_2, \ldots, a_n$ be real numbers satisfying the system of equations

$$\begin{aligned} \frac{a_1}{2} + \frac{a_2}{3} + \cdots + \frac{a_n}{n+1} &= \frac{4}{3}, \\ \frac{a_1}{3} + \frac{a_2}{4} + \cdots + \frac{a_n}{n+2} &= \frac{4}{5} \\ &\vdots \\ \frac{a_1}{n+1} + \frac{a_2}{n+2} + \cdots + \frac{a_n}{2n} &= \frac{4}{2n+1}. \end{aligned}$$

Express $\sum_{i=1}^{n} \frac{a_i}{2i+1}$ in closed form.

11. Let $S = \{x_1, x_2, \ldots, x_{k+\ell}\}$ be a $(k+\ell)$-element set of real numbers contained in the interval $[0,1]$; k and ℓ are positive integers. A k-element subset $A \subset S$ is called *nice* if

$$\left| \frac{1}{k} \sum_{x_i \in A} x_i - \frac{1}{\ell} \sum_{x_j \in S \setminus A} x_j \right| \leq \frac{k+\ell}{2k\ell}.$$

Prove that the number of nice subsets is at least

$$\frac{2}{k+\ell} \binom{k+\ell}{k}.$$

12. Prove that for every pair m, k of natural numbers, m has a unique representation in the form

$$m = \binom{a_k}{k} + \binom{a_{k-1}}{k-1} + \cdots + \binom{a_t}{t},$$

where

$$a_k > a_{k-1} > \cdots > a_t \geq t \geq 1.$$

13. Let n be a positive integer and let $a_1 \le a_2 \le \cdots \le a_n$ and $b_1 \le b_2 \le \cdots \le b_n$ be two nondecreasing sequences of real numbers such that

$$a_1 + \cdots + a_i \le b_1 + \cdots + b_i \text{ for every } i = 1, 2, \ldots, n-1$$

and

$$a_1 + a_2 + \cdots + a_n = b_1 + b_2 + \cdots + b_n.$$

Suppose that for any real number m, the number of pairs (i, j) with $a_i - a_j = m$ equals the number of pairs (k, ℓ) with $b_k - b_\ell = m$. Prove that $a_i = b_i$ for $i = 1, 2, \ldots, n$.

14. Let A and B be two sets each with n positive real numbers, such that the sum of the elements in A equals the sum of the elements in B. Show that there exists an $n \times n$ array of nonnegative real numbers such that

    (a) The set of row sums is A;

    (b) The set of column sums is B;

    (c) There are at least $(n-1)^2 + |A \cap B|$ zeroes in the array.

15. Let n be a given integer greater than 1, and let $a_1, a_2, \ldots, a_n$ be real numbers (not all zero). Prove that the following statements are equivalent:

    (a) There exist integers $0 < x_1 < x_2 < \cdots < x_n$ such that

    $$a_1x_1 + a_2x_2 + \cdots + a_nx_n \ge 0.$$

    (b) There exists k, $1 \le k \le n$, such that $a_k + a_{k+1} + \cdots + a_n > 0$.

16. (Tricky) Let $\mathcal{P}$ be a convex polygon in the plane. A real number is assigned to each point in the plane so that the sum of the numbers assigned to the vertices of any polygon similar to $\mathcal{P}$ is equal to 0. Prove that all the assigned numbers are equal to 0.

17. Let S denote the set of rational numbers in the interval $(0, 1)$. Determine, with proof, if there exists a subset T of S such that every element of S can be uniquely written as a sum of finitely many distinct elements in T.