# Bijective Proofs

Maria Monks

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## Bijections

Many combinatorial problems can be solved by finding the right bijection. A map between two sets $A$ and $B$ is a *bijection* if it is both *injective* and *surjective*, that is, it is one-to-one and maps onto all of $B$. Equivalently, a bijection is a map that has a well-defined inverse.

Most bijective (a.k.a. *counting in two ways*) proofs use the following principle:

> *If there is a bijection between finite sets $A$ and $B$, then $A$ and $B$ have the same number of elements.*

So, if we wish to find $|A|$ (the number of elements in $A$) and there is a bijection from $A$ to a set $B$ whose elements are easy to count, then we know how to count the elements of $A$. We can also prove two integers are equal by showing they count sets that have a bijection between them.

**Example.** Prove that

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}.$$

To prove this, we can easily use a straightforward induction argument, but it is more exciting to find a bijection between sets that each side counts. Consider a class with $2n$ students, $n$ of whom are boys and $n$ of whom are girls.

There is a natural bijection between

- pairs $(G, B)$ of subsets $G$ of the girls and $B$ of the boys with $|G| + |B| = n$, and
- subsets of the set of all students of size $n$,

defined by $(G, B) \mapsto G \cup B$. The number of possible pairs $(G, B)$ is

$$\sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k} = \sum_{k=0}^{n} \binom{n}{k}^2,$$

and the number of subsets of the class of size $n$ is $\binom{2n}{n}$. So, these quantities must be equal.
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Here is another example, that brings in several bijections to solve it:

**Example.** Prove *Cayley's Formula*, that the number of trees (connected graphs having no cycles) whose vertices are labeled $1, 2, \ldots, n$ is $n^{n-2}$.

The $n^{n-2}$ in Cayley's Formula is rather strange, and $n^n$ seems much easier to work with. So, let's first consider a set that is $n^2$ times as large as the one we wish to count.

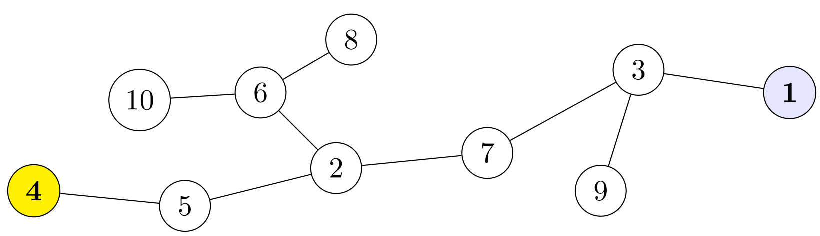
Let $T_n$ be the number of trees on vertices labeled $1, 2, \ldots, n$. Define a *vertebrate* to be a labeled tree along with one node designated as the *head* and another node designated as the *tail* (the head and the tail can be the same node). There is a unique path between the head and the tail (why?) and we call this path the *spine* of the vertebrate. The remaining edges and vertices form the *limbs*. ☺

Given a labeled tree on $n$ vertices, there are $n^2$ ways to choose the head and tail to make it a vertebrate. So, the number of vertebrates $V_n$ on $n$ vertices is equal to $n^2 T_n$, and we want to show that $V_n = n^n$.

We now construct a bijection $\phi$ between $V_n$ and the set of all maps $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$. Let $G$ be a vertebrate. To construct the map $f = \phi(G)$, first consider the sequence $s$ of nodes along the spine of $G$, from the head to the tail. Place commas after some of the elements of $s$ in such a way that a list of smaller sequences $s_1, s_2, \ldots, s_n$ is formed that satisfies the following two properties:

- The smallest element of $s_i$ is the first number in $s_i$ for $1 \le i \le n$.
- The smallest element of $s_i$ is greater than the smallest element of $s_{i+1}$ for $1 \le i \le n-1$.

It is easy to see that there is always a unique such way to place the commas. For instance, in the tree below with 4 as the head and 1 as the tail, the sequence $s$ is 452731, and the resulting sequence of sequences is $45, 273, 1$.




The map $f := \phi(G)$ then sends each element of each smaller sequence $s_i$ to the next element, or to the first element if it is the last element. In our example of the spine 452731, the map $f$ sends $4 \to 5 \to 4$, it sends $2 \to 7 \to 3 \to 2$, and it maps 1 to itself.

To define $f$ on the labels corresponding to nodes on the limbs, orient the edges of the limbs so that they point towards the spine. If $x$ is the label of a node having an edge pointing to a node labeled $y$, define $f(x) = y$. For instance, in the tree above, $f$ sends $8 \to 6 \to 2$, $10 \to 6$, and $9 \to 3$. It is easy to see that this gives a well-defined map $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$.

Finally, it suffices to show that the construction of the map $f$ is reversible. Given a map $f : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$, let $S$ be the set of all elements of $\{1, 2, \ldots, n\}$ that map to themselves after a finite number of successive applications of $f$. There is then a unique way to make $S$ the spine of a vertebrate $G'$ and then connect the limbs to the spine in such a way that $\phi(G') = f$ agree with $f$. (Can you see why this is?) This shows that $\phi$ is a bijection.

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## Problems

1. *Find the bijection!* For each of the following pairs of mathematical objects, give a description of a bijection that maps one set of objects to the other.

   (a) Binary sequences of length $n \leftrightarrow$ Subsets of $\{1, 2, \ldots, n\}$

   (b) Lattice paths from $(0, 0)$ to $(m, n)$ that only travel right or up at each step $\leftrightarrow$ Choices of $n$ blocks from a pile of $m$ blue and $n$ red blocks

   (c) Tilings of a $2 \times n$ grid with dominoes $\leftrightarrow$ Sequences of $n-1$ white or black dots such that no two black dots are adjacent

   (d) Partitions[1] of $n$ into distinct parts $\leftrightarrow$ Partitions of $n$ into odd parts

   (e) Partitions of $n$ into distinct odd parts $\leftrightarrow$ Partitions of $n$ whose Young Diagram[2] is symmetric about the diagonal

   (f) Increasing binary trees with nodes labeled $1, 2, \ldots, n \leftrightarrow$ Permutations of $1, 2, \ldots, n$.

2. Give a bijective proof of each of the following identities. All unspecified variables are assumed to be positive integers.

   - $\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$
   - $\sum_{k=0}^{r} \binom{n}{k}\binom{m}{r-k} = \binom{m+n}{r}$
   - $\sum_{i=0}^{n} \binom{x+i}{i} = \binom{x+n+1}{n}$
   - $\sum_{k=0}^{n} \binom{n}{k} s^k t^{n-k} = (s+t)^n$

3. Let $w = a_1 a_2 \cdots a_n$ be a permutation of $1, 2, \ldots, n$. We say that $i$ is a *fixed point* of $w$ if $a_i = i$. Show that the total number of fixed points of all possible permutations $w$ is $n!$.

4. How many $m \times n$ matrices of 0's and 1's have the property that every row and column contains an odd number of 1's?

5. (AIME 1983.) For $\{1, 2, \ldots, n\}$ and each of its nonempty subsets a unique *alternating sum* is defined as follows: Arrange the numbers in the subset in decreasing order and then, beginning with the largest, alternately add and subtract successive numbers. (For example, the alternating sum for $\{1, 2, 4, 6, 9\}$ is $9 - 6 + 4 - 2 + 1 = 6$.) Find the sum of all alternating sums of the nonempty subsets of $\{1, 2, \ldots, n\}$.

6. Prove Fermat's Little Theorem using a combinatorial argument as follows. We wish to show that if $p$ is prime and $a$ is a positive integer, then $a^p - a$ is divisible by $p$. To do so, it suffices to find a set $S$ with $a^p - a$ elements and sort the elements of $S$ into disjoint subsets having $p$ elements each.

7. (Putnam 2002.) A nonempty subset $S \subseteq \{1, 2, \ldots, n\}$ is *decent* if the average of its elements is an integer. Prove that the number of decent subsets has the same parity as $n$.

[1] A *partition* of a positive integer is a way of writing it as a sum of other integers, called the *parts* of the partition, where we list the parts in nonincreasing order.

[2] The Young Diagram of a partition is a partial grid of squares, aligned at the left, where each row has a number of squares corresponding to the size of the parts in nonincreasing order.

8. (AIME 1998.) Find the number of ordered quadruples $(x_1, x_2, x_3, x_4)$ of positive odd integers that satisfy $x_1 + x_2 + x_3 + x_4 = 98$.

9. (USAMO 1996.) An $n$-term sequence in which every term is either 0 or 1 is called a "binary sequence" of length $n$. Let $a_n$ be the number of binary sequences of length $n$ containing no three consecutive terms equal to 0, 1, 0 in that order. Let $b_n$ be the number of binary sequences of length $n$ containing no four consecutive terms equal to 0, 0, 1, 1 or 1, 1, 0, 0 in that order. Prove that $b_{n+1} = 2a_n$ for all positive integers $n$.

10. (China 1996.) Let $n$ be a positive integer. ☺ Find the number of polynomials $P(x)$ with coefficients in $\{0, 1, 2, 3\}$ such that $P(2) = n$.

11. Find the number of strings of $n$ letters, each equal to $A$, $B$, or $C$, such that the same letter never occurs three times in succession.

12. (Richard Stanley.) Let $F_r$ denote the $r$th Fibonnacci number. Show that

$$\sum (2^{a_1-1} - 1) \cdots (2^{a_k-1} - 1) = F_{2n-2},$$

where the sum is over all compositions $a_1 + a_2 + \cdots + a_k = n$.

13. (China 1994.) Let $n$ be a positive integer. Prove that

$$\sum_{k=0}^{n} 2^k \binom{n}{k} \binom{n-k}{\lfloor (n-k)/2 \rfloor} = \binom{2n+1}{n}.$$

14. (Putnam 1996.) Given a finite string $S$ of symbols $X$ and $O$, we write $\Delta(S)$ for the number of $X$'s in $S$ minus the number of $O$'s. For example, $\Delta(XOOXOOX) = -1$. We call a string $S$ *balanced* if every substring $T$ of (consecutive symbols of) $S$ has $-2 \le \Delta(T) \le 2$. Thus, $XOOXOOX$ is not balanced, since it contains the substring $OOXOO$. Find, with proof, the number of balanced strings of length $n$.

15. Let $E(n)$ be the number of partitions of the natural number n into an even number of parts, and let $O(n)$ be the number of partitions of $n$ into an odd number of parts. Prove that $|E(n) - O(n)|$ equals the number of partitions of $n$ into distinct odd parts.

16. **The Catalan numbers:** The Catalan numbers[3] $C_0, C_1, C_2, \ldots$ can be defined by the recurrence relation

$$C_{n+1} = C_n C_0 + C_{n-1} C_1 + C_{n-2} C_2 + \cdots + C_0 C_n$$

along with the initial value $C_0 = 1$. The $n$th Catalan number $C_n$ can also be defined as:

- The number of lattice paths from $(0, 0)$ to $(n, n)$, formed by moving one unit right or one unit up at each step, that lie below or on the diagonal $x = y$
- The number of ways to fully parenthesize the addition $1 + 1 + \cdots + 1$ of $n + 1$ ones. For instance, $1 + 1 + 1 + 1$ can be parenthesized in five ways:

[3] See http://math.mit.edu/ rstan/ec/ for a list of 188 different combinatorial interpretations of the Catalan numbers.

$$
\begin{array}{l}
((1+1)+1)+1 \\
(1+(1+1))+1 \\
1+((1+1)+1) \\
1+(1+(1+1)) \\
(1+1)+(1+1)
\end{array}
$$

- The number of rooted binary trees having $n+1$ leaves labeled $1, 2, \ldots, n+1$
- The number of ways of triangulating a regular $(n+2)$-gon by drawing $n-1$ diagonals (different triangulations that are congruent are considered distinct.)
- The number of ways of connecting $2n$ points on a circle with $n$ nonintersecting chords

Show that each of these sets satisfies the Catalan recurrence. Can you find bijections between each of these pairs of sets?

17. (Hard.) Find a bijective proof that the $n$th Catalan number $C_n$ is equal to $\frac{1}{n+1}\binom{2n}{n}$.