NT Practice

I guess the number 720 is just awesome.

1. (JSteinhardt) Let $\diamond(k)$ be the sum of k's digits in base $720+1$. How many positive integers m exist such that $\diamond(k) \equiv k \pmod{m}$ for all k?

   We will separate digits base 721 by grouping them with parentheses for clarification purposes. First we can see that any divisor of 720 works, since $721^n \equiv 1 \pmod{m}$ if $m|720$. In addition, we can see that it is necessary that m is a divisor of 720 by noting that in particular $(1)(1)_{721} \equiv 2_{721} \pmod{m}$, so $(720)_{721} \equiv 0 \pmod{m} \implies m|720$. Then it only remains to find the number of divisors of $720 = 6! = 6*5*4*3*2 = 3*2*5*2*2*3*2 = 5*3^2*2^4$, so there can be between 0 and 1 powers of 5, 0 and 2 powers of 3, and 0 and 4 powers of 2 that divide any factor of 720, leading to $2*3*5 = \boxed{30}$ factors.

2. (JSteinhardt) Find the number of divisors of the sum of the squares of the divisors of 720.

   The sum of the squares of the divisors of 720 can be found similarly to the sum of the divisors of 720. That is, since $720 = 5*3^2*2^4$ (shown above), we can use the factorization $(1^2+5^2)(1^2+3^2+9^2)(1^2+2^2+4^2+8^2+16^2)$ (do you see why this works?) and simplify this to $6*91*341 = 2*3*7*13*11*31$ and use the same logic as before to deduce that there are $\boxed{32}$ divisors to this number.

3. (JSteinhardt) Find the sum of the square divisors of 720.

   We again use a factorization, this time ignoring any factors that aren't squares. The resulting form is $(1)*(1+3^2)*(1+2^2+2^4) = 10*21 = \boxed{210}$.

Actually, 210 is pretty cool as well.

4. (Traditional) Find the right-most non-zero digit of 210! in base 210.

   First we note that $210 = 2*3*5*7$. The largest factor of 210 that divides 210! will therefore correspond to the largest factor of 7 that divides 210!, which is $\lfloor\frac{210}{7}\rfloor + \lfloor\frac{210}{49}\rfloor + \ldots = 30+4 = 34$. Then taking $\frac{210!}{210^{34}}$ mod 2, 3, 5, and 7, we obviously have that it is congruent to 0 for the first 3. This only leaves mod 7.

$$\frac{210!}{210^{34}} \equiv 6!^{30} * \frac{1*2*3*4*5*6*1*8*\ldots*26*27*4*29*30}{30^{34}}$$
$$\equiv (-1)^{30} * \frac{6!^4*1*2*(1*2*3*4)}{2^{34}} \equiv \frac{(-1)^4*48}{2^{34}}$$
$$\equiv (-1)*4^{34} \equiv (-1)*4^4 \equiv (-1)*16^2 \equiv (-1)*2^2 \equiv -4 \equiv 3 \pmod{7}$$

   By the Chinese Remainder Theorem there is a unique number in mod 210 that is 0 mod 2, 3, and 5 and 3 mod 7. Starting with 5 and 7, we get that it is 10 mod 35. Bringing in 3, we have that it is 45 mod 105. Finally, bringing in 2 gives that it is 150 mod 210, so the answer is $\boxed{150}$.

5. (Traditional) Find the length of the repeating part of $\frac{1}{17}$ in base 210.

   Call the length l. If $\frac{1}{17}*(210^n-1)$ is an integer, then $l|n$ (consider that multiplying $\frac{1}{17}$ by (210^l-1) causes the repeating part to telescope, as is the case for any multiple of l. This is equivalent to $210^n \equiv 1 \pmod{17}$, which by Fermat's theorem says that the smallest possible n such that this is true must divide 16. We can then test 1, 2, 4, and 8: $210^n \equiv 6^n \pmod{17}$, so 210^1, 210^2, 210^4, and 210^8 correspond to 6, $36 \equiv 2$, 4, and $16 equiv - 1$. It is then clear that the smallest value of n is 16, so the length of the repeating decimal is $\boxed{16}$.

But 47 is the coolest of all.

6. (Traditional) If $\sum_{i=1}^{46} \frac{1}{i} = \frac{p}{q}$, and r exists such that $47|p-qr$, then find $r - 47\lfloor \frac{r}{47} \rfloor$.

   We first note that the problem is basically asking for the value of the given sum mod 47, which happens to be prime. We then note that $\{frac11, \ldots, \frac{1}{p-1}\}$ is equivalent to $\{1, \ldots, p-1\}$ in a prime mod, because multiplicative inverses are unique in prime mods (prove it!), so mapping each number to its multiplicative inverse simply constitutes a permutation of the set. So we can actually take $\sum_{i=1}^{46} i = \frac{46 * (46+1)}{2}$ mod 47, which happens to be a factor of 47, so the answer is $\boxed{0}$.

7. (Traditional) Find the smallest positive integer n such that $1^n + 2^n + \ldots + 46^n$ is not divisible by 47.

   Let g be a generator mod 47. Then $\{g^1, \ldots, g^{46}\}$ is equivalent to the set $\{1, \ldots, 46\}$ be definition. Thus, $\sum_{i=1}^{46} i^n = \sum_{i=1}^{46} g^{in}$, which is now an easily evaluatable geometric series that is in particular congruent to 0 mod 47 for any $n < 46$. It is then clear from Fermat's theorem that it is not congruent to 0 for $n = 46$, so the answer is $\boxed{46}$.

And finally, Haitao has asked permission to give you all a problem, and I said yes.

8. Let $p > 2$ be a prime and let $P(x) = a_0 x^{p-1} + \ldots + a_{p-1}$ be a polynomial with integer coefficients. If for all integral x,y such that if p does not divide $x - y$, then p also does not divide $P(x) - P(y)$. Prove that $p|a_0$.