New Zealand Mathematical Olympiad Committee

Camp Selection Problems 2012

Due: 15 August 2012

Junior division

J1. *From a square of side length 1, four identical triangles are removed, one at each corner, leaving a regular octagon. What is the area of the octagon?*

Solution: Let x be the side length of the octagon. The removed triangles are isosceles right triangles with hypotenuse x, and legs of length $(1-x)/2$. So:

$$\begin{aligned}
2\left((1-x)/2\right)^2 &= x^2 \\
(1-x)^2 &= 2x^2 \\
0 &= x^2+2x-1 \\
0 &= (x+1)^2-2 \\
2 &= (x+1)^2 \\
\sqrt{2} &= x+1 \\
\sqrt{2}-1 &= x
\end{aligned}$$

Note that the four triangles removed fit together to form a square of side length x and so the total area removed is $x^2 = 3-2\sqrt{2}$. Therefore, the total area remaining, which is the area of the octagon is $2\sqrt{2}-2$. $\square$

J2. *Show the the sum of any three consecutive positive integers is a divisor of the sum of their cubes.*

Solution: Let the integers be $a-1$, a, and $a+1$. Their sum is $3a$ and the sum of their cubes is:

$$(a-1)^3 + a^3 + (a+1)^3 = 3a^3 + 6a = 3a(a^2 + 2a)$$

which is clearly a multiple of $3a$. $\square$

J3. *Find all triples of positive integers (x, y, z) with*

$$\frac{xy}{z} + \frac{yz}{x} + \frac{zx}{y} = 3.$$

Solution: The only possible triple is $(1, 1, 1)$. To see this assume without loss of generality that $x \leq y \leq z$. If all three are equal then $3x = 3$, and we get the given solution. Suppose that we had some other solution. Then, $z \geq x+1$ and in particular:

$$x + 1 \leq \frac{yz}{x} < 3$$

and so $x = 1$. Now we have
$$\frac{y}{z} + yz + \frac{z}{y} = 3$$
so $yz < 3$ and (remember we are assuming that not all three are equal) we must have $y = 1$, $z = 2$, but this does not give a solution.

Alternate solutions. Other solutions are possible. Multiplying through by xyz gives

$$x^2y^2 + y^2z^2 + z^2x^2 = 3xyz,$$

which we may rearrange to a quadratic in x:

$$x^2(y^2 + z^2) - (3yz)x + y^2z^2 = 0.$$

For this to have real solutions the discriminant must be positive, so we must have

$$9y^2z^2 - 4(y^2 + z^2)y^2z^2 = (9 - 4(y^2 + z^2))y^2z^2 \geq 0.$$

This implies $y^2 + z^2 \leq 9/4$, so we must have $y, z \leq 1$, and by symmetry $x \leq 1$ also.

Alternately, from $(a - b)^2 \geq 0$ we get $a^2 + b^2 \geq 2ab$. Hence

$$\begin{aligned} 6xyz &= ((xy)^2 + (zx)^2) + ((xy)^2 + (yz)^2) + ((yz)^2 + (zx)^2) \\ &\geq 2(x^2yz + xy^2z + xyz^2) \\ &= 2xyz(x + y + z). \end{aligned}$$

Dividing by $2xyz$ we're left with $3 \geq x + y + z$, implying $x = y = z = 1$ since x, y, z are positive integers.

To complete these two approaches we check that $x = y = z = 1$ does indeed satisfy the given equation. □

J4. *A pair of numbers are* twin primes *if they differ by two, and both are prime. Prove that, except for the pair* $\{3, 5\}$, *the sum of any pair of twin primes is a multiple of 12.*

Solution: Let the two primes be $n - 1$ and $n + 1$. Their sum is $2n$ so we seek to show that n must be a multiple of 6. Certainly n must be even, as both the primes are odd. Also it must be a multiple of 3 since otherwise either $n - 1$ or $n + 1$ would be a multiple of three (and we know this is not the case). So, it is a multiple of 6 and we're done. □

J5. *Let* $ABCD$ *be a quadrilateral in which every angle is smaller than* $180°$. *If the bisectors of angles* $\angle DAB$ *and* $\angle DCB$ *are parallel, prove that* $\angle ADC = \angle ABC$.

Solution: The bisector of $\angle DAB$ meets either BC or DC, in which case that of $\angle DCB$ meets AD or AB respectively. The two cases are symmetric (just switch the labels of A and C) so suppose that the bisector of $\angle DAB$ meets BC at E and that of $\angle DCB$ meets AD at F. From corresponding angles on parallel lines we get $\angle DFC = \angle DAE$, and because it is an angle bisector $\angle DAE = \angle BAE$. Similarly, $\angle BEA = \angle BCF = \angle DCF$. So the triangles DFC and BAE are similar, and thus $\angle ADC = \angle FDC = \angle ABE = \angle ABC$ as required. □

J6. *The vertices of a regular 2012-gon are labelled with the numbers 1 through 2012 in some order. Call a vertex a* peak *if its label is larger than the label of its two neighbours, and a* valley *if its label is smaller than the label of its two neighbours. Show that the total number of peaks is equal to the total number of valleys.*

Solution: Call a vertex *extreme* if it is either a peak or a valley. Note that the vertex labelled 1 must be a valley. Reading clockwise from that vertex, the next extreme vertex encountered must be a peak, and carrying on paying attention only to extreme vertices, peaks and valleys must alternate. So each valley can be paired with the next peak in clockwise order, and the number of each type must be the same.

Alternate inductive solution. We will prove by induction that the total number of peaks equals the total number of valleys for any labelling of the vertices of a regular n-gon by the numbers 1 to n. For the base case, consider $n = 3$. There is essentially only one labelling, and for this labelling there is one peak (the number 3), and one valley (the number 1). So the assertion is true for $n = 3$.

Now suppose that the assertion is true for labellings of a regular n-gon, and consider a labelling of a regular $(n + 1)$-gon. If we remove the vertex labelled $n + 1$ we may regard what is left as a labelling of an n-gon, for which the number of peaks equals the number of valleys, by our inductive hypothesis. We now consider what happens when $n + 1$ is re-inserted, necessarily becoming a peak. Suppose its neighbours are a and b, and suppose further without loss of generality that $a < b$.

Since $n + 1 > a$ also, there is no change to whether a is a peak or a valley. Consider then the number c on the other side of b. If $c < b$ then b was a peak; after $n + 1$ is inserted it is no longer a peak or a valley, but $n + 1$ is a peak in its place, so there is no net change to the number of peaks or valleys. On the other hand, if $c > b$, then b was neither a peak nor a valley, but becomes a valley when $n + 1$ is inserted, balancing the creation of the new peak at $n + 1$. So in this case the number of peaks and valleys remains equal also. The result follows by induction.

Alternate approach. Another promising approach is to start from the consecutive labelling 1, 2, . . . , 2012, which has one peak and one valley, and consider how the number of peaks and valleys changes as small changes are made to this labelling (for example, if two adjacent numbers are swapped). Generally speaking solutions that took this approach didn't consider enough cases to be able to rearrange the consecutive labelling to an arbitrary labelling, which is what is needed to solve the problem in this way. $\square$

Senior division

S1. *Find all real numbers x such that*

$$x^3 = \{(x + 1)^3\}$$

where $\{y\}$ denotes the fractional part *of y, i.e. the difference between y and the largest integer less than or equal to y.*

Solution: Since $\{t\} < 1$ for all t it must be the case that $x^3 < 1$ and hence $x < 1$. Also $x \geq 0$ since the right hand side is non-negative. Now $(x + 1)^3 = x^3 + 3x^2 + 3x + 1$, so if the fractional part of this is x^3, $3x^2 + 3x + 1$ must be an integer, and so $3x^2 + 3x$ must be an integer. The maximum integer value of $3x^2 + 3x$ for $x < 1$ is 5 so we simply solve:

$$3x^2 + 3x - k = 0$$

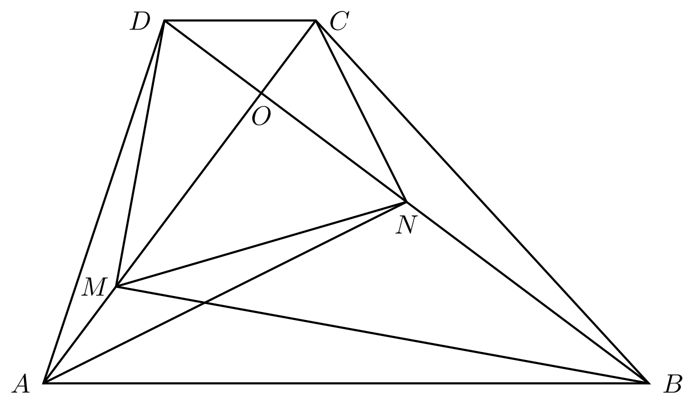
for $k = 0, 1, 2, 3, 4, 5$, giving the values:

$$x = \frac{-3 + \sqrt{9 + 12k}}{6}$$

as the solutions. $\square$

S2. *Let $ABCD$ be a trapezoid, with $AB \parallel CD$ (the vertices are listed in cyclic order). The diagonals of this trapezoid are perpendicular to one another and intersect at O. The base angles $\angle DAB$ and $\angle CBA$ are both acute. A point M on the line segment OA is such that $\angle BMD = 90°$, and a point N on the line segment OB is such that $\angle ANC = 90°$. Prove that triangles OMN and OBA are similar.*

Solution: Consider the diagram below:



In a right angled triangle, the square of the length of the altitude to the hypotenuse is equal to the product of the lengths of the two parts of the hypotenuse it forms. So:

$$ON^2 = OA \cdot OC, \quad OM^2 = OB \cdot OD.$$

Also AOB and COD are similar (because $AB \parallel CD$) and so:

$$\frac{OA}{OC} = \frac{OB}{OD} = \frac{AB}{DC}.$$

Now:

$$\frac{ON^2}{OA^2} = \frac{OA \cdot OC}{OA^2} = \frac{OC}{OA} = \frac{DC}{AB}$$

and

$$\frac{OM^2}{OB^2} = \frac{OB \cdot OD}{OB^2} = \frac{OD}{OB} = \frac{DC}{AB}$$

So $ON/OA = OM/OB$, and since the two triangles OMN and OBA also share the common angle $\angle MON = \angle BOA$, they are similar. $\square$

S3. *Two courier companies offer services in the country of Old Aland. For any two towns, at least one of the companies offers a direct link in both directions between them. Additionally, each company is willing to chain together deliveries (so if they offer a direct link from A to B, and B to C, and C to D for instance, they will deliver from A to D.) Show that at least one of the two companies must be able to deliver packages between* any *two towns in Old Aland.*

Solution: Let A be one of the towns, and suppose that the first company cannot deliver between A and some other town in Old Aland. Let $\mathcal{S}$ be the set of towns that the first company *can* reach from A, and $\mathcal{T}$ be the remaining towns. Then both $\mathcal{S}$ and $\mathcal{T}$ are non-empty and the direct link between any town in $\mathcal{S}$ and any town in $\mathcal{T}$ must only be provided by the second company. Therefore, the second company can deliver between any two towns (directly if one is in $\mathcal{S}$ and the other in $\mathcal{T}$, or at worst via a single relay through the opposite set if both belong to $\mathcal{S}$ or both belong to $\mathcal{T}$).

Note that we've actually proved a stronger result: either both companies can deliver everywhere, or one of them can deliver between any two towns with at most one intermediate stop.

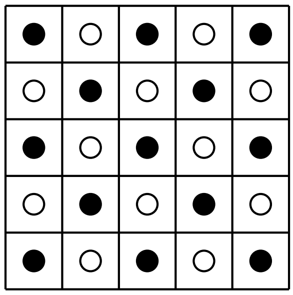
Alternate inductive solution. If there are only two towns in Old Aland, then by the conditions of the problem one of the companies must offer a direct link between these two towns, and so this company can deliver between any two towns in Old Aland.

Suppose then that the result is true for any collection of n towns, and consider a network of $n + 1$ towns. Ignoring the $(n + 1)$th town for the moment, one of the companies — the first, say — must be able to deliver between any two of the remaining n towns. If the first company also offers a direct link between any one of these towns and the $(n+1)$th, then the first company is able to deliver between any two of the towns and we are done. Otherwise, only the second company offers direct links to the $(n + 1)$th town. But then every town has a direct link to the $(n + 1)$th town by the second company, so the second company is able to deliver between any two towns in Old Aland, and we are again done. □

S4. *Let $p(x)$ be a polynomial with integer coefficients, and let a, b and c be three distinct integers. Show that it is not possible to have $p(a) = b$, $p(b) = c$, and $p(c) = a$.*

Solution: Suppose that the three equalities held. Since $x - y$ is a divisor of $x^k - y^k$ for all $k > 0$, $x - y|p(x) - p(y)$. The equalities would then imply: $b - c = q_1(a - b)$, $c - a = q_2(b - c)$ and $a - b = q_3(c - a)$ for some integers q_1, q_2 and q_3. Multiplying the three equations together we get $q_1q_2q_3 = 1$ and hence $q_1, q_2, q_3 = \pm 1$. But, no $q_i = -1$ as that would imply two of a, b and c are equal. So $a - b = b - c = c - a$, but this easily implies that all three are equal (for example add $a - b = c - a$ to $b - c = c - a$ to get $a - c = 2(c - a)$, hence $3a = 3c$, hence $a = c$, and symmetry), and so we have a contradiction. Thus it is impossible for the three equalities to hold. □

S5. *Chris and Michael play a game on a 5×5 board, initially containing some black and white counters as shown below:*



Chris begins by removing any black counter, and sliding a white counter from an adjacent square onto the empty square. From that point on, the players take turns. Michael slides a black counter onto an adjacent empty square, and Chris does the same with white counters (no more counters are removed). If a player has no legal move, then he loses.

(a) *Show that, even if Chris and Michael play cooperatively, the game will come to an end.*

(b) *Which player has a winning strategy?*

Solution: Imagine that the board is coloured, with the squares initially containing black counters coloured black, and those with white counters, white. At Chris's turn to slide a counter it is always a black square that is vacant. Likewise at Michael's turn it is always a white square. So Chris can only ever move white counters off white squares and onto black ones, while Michael can only ever move black counters off black squares onto white ones. In particular, once a counter is moved, it can never be moved again. Since there are only 12 white, and 12 black counters in play, the game must end.

Michael has a winning strategy. When Chris first chooses a counter to remove, Michael should imagine the rest of board tiled with 2×1 rectangles (or dominoes) – it's easy to see that this is always possible. Now whenever Chris makes a slide move, Michael takes the other counter from the corresponding domino and moves it into the position which has just been vacated. This guarantees that Michael will always have a move available and since we know that the game ends, it must be Chris who loses. □

S6. *Let a, b and c be positive integers such that $a^{b+c} = b^c c$. Prove that b is a divisor of c, and that c is of the form d^b for some positive integer d.*

Solution: If $a = 1$ we're done. Supposing $a > 1$, let p be a prime dividing a, in fact assume that p^α, p^β and p^γ are the largest powers of p dividing a, b and c respectively. Suppose first that $\gamma = 0$. Then from the given equation:

$$\alpha(b + c) = \beta c.$$

So

$$c(\beta - \alpha) = \alpha b$$

Now p^β is a divisor of b but coprime to c, so p^β is a divisor of $\beta - \alpha$. But $\beta - \alpha < \beta$ while $p^\beta > \beta$, so we have a contradiction. So $\gamma > 0$.

Now again from

$$\alpha(b + c) = \beta c + \gamma$$

and noting p^γ is not a divisor of γ but is of c we conclude that p^γ is not a divisor of b, that is $\beta < \gamma$. This is true for all primes, so b is a divisor of c. Moreover, in the equation above, every term except possibly γ is now divisible by b, so γ must be as well. That means $c = d^b$ for some positive integer $d \geq 2$. $\square$