All About Excircles!

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1 Introduction

The excenters and excircles of a triangle seem to have such a beautiful relationship with the triangle itself. Drawing a diagram with the excircles, one finds oneself riddled with concurrences, collinearities, perpendicularities and cyclic figures everywhere.

Not only this, but a triangle ABC and the triangle formed by the excenters, I_A, I_B, and I_C ,share several triangle centers. For example, the incenter of ABC is the orthocenter of $I_AI_BI_C$, the circumcenter of ABC is the nine point center of $I_AI_BI_C$ and so on. In the following article, we will look into these properties and many more.

2 The Basics

Before we get into any real theory, let us properly define the excircle:

Definition 1. *The A-excircle of Triangle ABC is the circle that is tangent to the side BC and to the rays AB and AC beyond B and C respectively.*

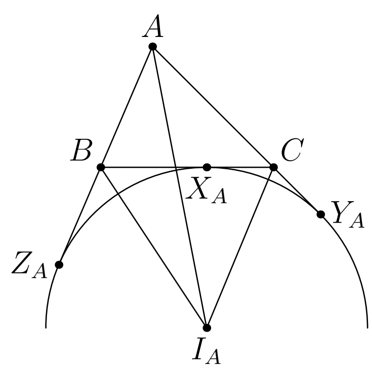

Figure 1: The Excircle

Let us quickly see why such a circle must exist. By the trigonometric form of Ceva's theorem, the internal angle bisector of A must be concurrent with the external angle bisectors of B and C. Denote this concurrency point by I_A. Let the feet of the perpendiculars from I_A to BC, AC and AB be X_A, Y_A and Z_A respectively.

As the triangles AI_AY_A and AI_AZ_A are congruent, $I_AY_A = I_AZ_A$. Similarly, by looking at triangles CI_AY_A and CI_AX_A, we find that $I_AX_A = I_AY_A = I_AZ_A$. Thus, if we take the circle centered at I_A passing trough X_A, Y_A and Z_A, we get a circle which is tangent to the line BC and to the rays AB and AC beyond B and C respectively. Thus, it is the A-excircle and I_A is the A-excenter.

2.1 One Excircle

Of course, there is a vast amount of theory that comes from the relationship of the excircles with each other, but to begin with, we should break down the diagram to just one excircle and see what we can do with that.

First and foremost, there's a little more that we can say about the points X_A, Y_A and Z_A. Notice that because of the tangents, $AY_A = AZ_A$, $BX_A = BZ_A$ and $CX_A = CY_A$. But then,

$$\begin{aligned} AY_A + AZ_A &= AB + BZ_A + AC + CY_A \\ &= AB + BX_A + AC + CX_A \\ &= AB + AC + BC \\ &= 2s \end{aligned}$$

Hence, each of AY_A, AZ_A is a semi perimeter. So we get the following:

Theorem 1. *Using the notation already described,*

$$AY_A = AZ_A = s,\ BX_A = BZ_A = s - c \text{ and } CX_A = CY_A = s - b$$

Note that if the incircle is tangent to BC at D, then D and X_A are symmetric about the midpoint of BC (Since $BD = CX_A = s - b$).

These results are vital to most excenter problems. Knowing these lengths, which repeat often, we can compute ratios and identify similar triangles (Problem 4, as an example).

We can also observe the relationship between an excenter and the incenter:

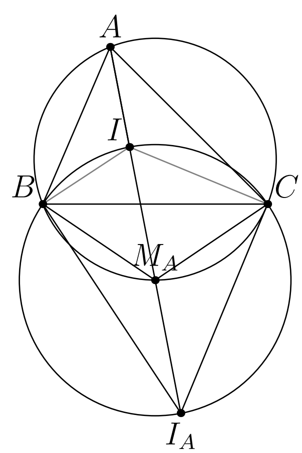

Figure 2: Theorem 2

Theorem 2. *Let M_A be the midpoint of arc BC not containing A in the circumcircle of triangle ABC. Then I, I_A, B, C all lie on a circle that is centered at M_A.*

Proof. To begin with, we know that the internal and external bisectors of an angle are perpendicular. Hence, $\angle IBI_A = \angle ICI_A = 90$. This implies that I, I_A, B, C all lie on a circle whose center is the midpoint of II_A (The internal angle bisector of A). We are required to prove that this center is M_A.

Let the center be called M'_A. We know that:

$$\angle CM'_A A = \angle CM'_A I = 2\angle CBI = \angle CBA$$

Hence M'_A lies on the circumcircle of ABC. However, II_A only intersects the circumcircle at A and M_A, so $M'_A = M_A$.

□

A notable corollary of this is that M_A is the midpoint of II_A.

This lemma turns out to be extremely useful in olympiad problems. Most of its use is as a way to prove that something is an incenter. If it can be proven that a point P lies on the internal angle bisector of A, and also that $M_A P = M_A B$, $M_A P = M_A C$ or $M_A P = M_A I_A$, then P has no choice but to be the incenter of the triangle!

The use of this lemma, however, rarely involves excircles, so we'll refrain from investigating problems that involve it in this paper. However, if the reader would like to see it being applied, the problem 2 from 2002 IMO and problem 4 from 2006 IMO respectively can be solved using this lemma.

We introduce another lemma which proves useful to many olympiad problems as well:

Theorem 3. *Let the incircle of ABC touch BC at D. if P is the Q antipode of D with respect to the incircle and Q reflection of D about the midpoint of BC, then A, P, and Q are collinear.*

Proof. This result has bearing on excircle problems because by Theorem 1, Q is the tangency point of the A-excircle with BC (we have that $BD = CQ = s - b$). Note that there is a homothety centered at A that maps the incircle to the A-excircle, and as $IP \parallel I_A Q$, this homothety must map P to Q. Hence A, P and Q are indeed collinear. □

[2] has an excellent set of problems that utilize this lemma, but we will only go through one in the problem section.

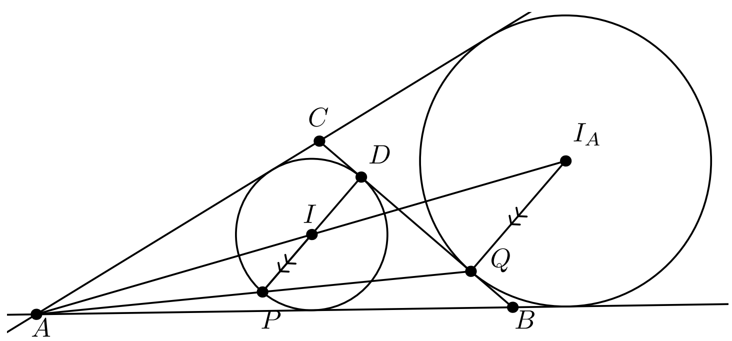

Figure 3: Theorem 3

2.2 Two Excircles

Let's quickly see at what happens when we look at just a pair of excircles through a result that is quite similar to Theorem 2:

Theorem 4. *Let N_A be the midpoint of the arc BC containing A in the circumcircle of triangle ABC. Then I_B, I_C, B, C all lie on a circle that is centered at N_A.*

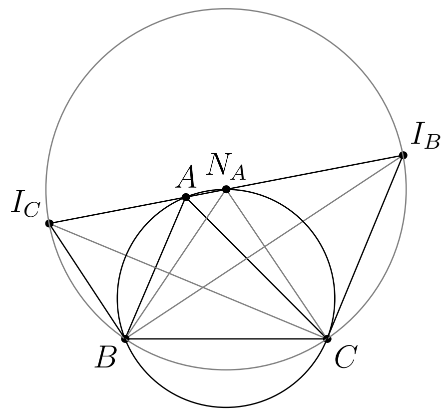

Figure 4: Theorem 4

Proof. Again, we know that the internal and external bisectors of an angle are perpendicular. Hence, $\angle I_B B I_C = \angle I_B C I_C = 90$. This implies that I_B, I_C, B, C all lie on a circle whose center is the midpoint of $I_B I_C$ (The external angle bisector of A). We are required to prove that this center is N_A.

Let the center be called N'_A. We know that:

$$\angle C N'_A I_B = 2\angle C B I_B = \beta \text{ and } \angle B N'_A I_B = 2\angle B C I_B = \gamma$$

So then $\angle B N'_A C = 180 - (\beta + \gamma) = \alpha$. Hence N'_A lies on the circumcircle of ABC. However, $I_B I_C$ only intersects the circumcircle at A and N_A, so $N'_A = N_A$.

□

Again, we find that a corollary of this theorem is that N_A is the midpoint of $I_B I_C$.

We're finally ready to look at what happens when all three circles come into play, and thus the entire triangle $I_A I_B I_C$!

2.3 Where did that nine point circle come from?!

We've mentioned several times before that the internal and external bisectors are perpendicular to each other. That is, $AI_A \perp I_B I_C$, $BI_B \perp I_A I_C$, $CI_C \perp I_A I_B$. So then AI_A, BI_B, CI_C are the altitudes of $I_A I_B I_C$ which makes ABC its orthic triangle. But let's not forget what the circumcircle of the orthic triangle is- the nine point circle!

We can use this fact to see what points related to $I_A I_B I_C$ actually lie on the circumcircle of ABC.

Of course, we've already noted the points A, B, and C, the feet of the altitudes. Because of Theorem 4, we've looked at the midpoints of the sides of $I_A I_B I_C$ (N_A, N_B and N_C), and because of Theorem 2, we've already looked at the midpoints of II_A, II_B, II_C (M_A, M_B and M_C). It seems that we've already gone through all nine points! Perhaps then, we can look at this nine point circle as a different approach for proving Theorems 2 and 4.

Nonetheless, we can still think about properties of the triangle $I_A I_B I_C$ and see how they relate to ABC. For example, where is the Euler line? We know that the orthocenter is I and the nine point center is O. It

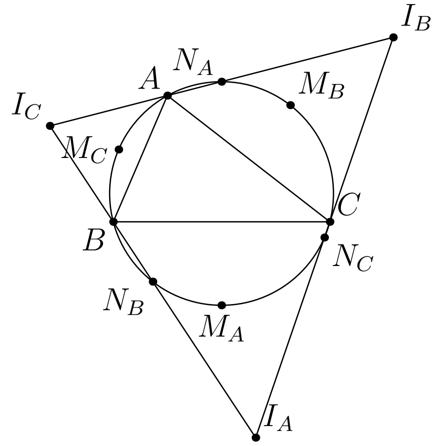

Figure 5: A Magical Nine Point Circle

doesn't seem that we have a point in ABC that really corresponds to the centroid (It would be the concurrency point of I_AN_A, I_BN_B and I_CN_C). We can, however, find something for the circumcenter.

Recall that if P is a point in triangle ABC, then the perpendiculars from A, B, C to the respective sides of the pedal triangle of P are concurrent at the isogonal conjugate of P. Since ABC is the pedal triangle of I in $I_AI_BI_C$, the perpendiculars I_AX_A, I_BY_B and I_CZ_C are concurrent at some point V. This point must be isogonal conjugate of I with respect to $I_AI_BI_C$. As I is the orthocenter, V must be the circumcenter (the isogonal conjugate).

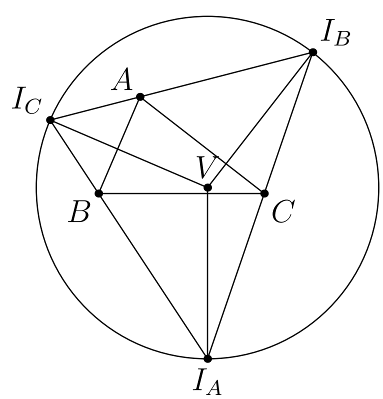

Figure 6: The Bevan Point

V is known as the *Bevan Point* of triangle ABC, and it is the reflection of the incenter about the circumcenter of ABC (because the circumcenter is the reflection of the orthocenter about the nine point center in $I_AI_BI_C$). Another justification of its existence is by using Carnot's theorem[1] and Theorem 1.

[1] *Carnot's theorem* states that if D, E, F are points on the sides BC, CA, AB of triangle ABC respectively, then the perpendiculars at D, E, F to BC, CA, AB respectively are concurrent if and only if,

$$(BD^2 - CD^2) + (CE^2 - AE^2) + (AF^2 - BF^2) = 0$$

2.4 Problems

In the past few pages, we've gone through a large part of the basic theory having to do with excircles. With this knowledge, one can at least tackle most any olympiad excircle problem. Let's have a look at a few problems and their solutions using what we have found so far.

1. (Well known) If r_a, r_b and r_c are the radii of the excircles opposite to A, B and C, prove that

$$\frac{1}{r_a} + \frac{1}{r_b} + \frac{1}{r_c} = \frac{1}{r}$$

Proof. Let F be the tangency point of the incircle with AB. Note that there is a homothety centered at A mapping the incircle to the A-excircle. Using Theorem 1, This homothety has ratio:

$$\frac{AF}{AZ_A} = \frac{r}{r_a} = \frac{s-a}{s}$$

Applying the same procedure with the other two excircles and summing,

$$\frac{r}{r_a} + \frac{r}{r_b} + \frac{r}{r_c} = \frac{s-a}{s} + \frac{s-b}{s} + \frac{s-c}{s} = \frac{3s-2s}{s} = 1$$

The result follows. □

2. (USAMO 1999) Let $ABCD$ be an isosceles trapezoid with $AB \parallel CD$. The inscribed circle ω of triangle BCD meets CD at E. Let F be a point on the (internal) angle bisector of $\angle DAC$ such that $EF \perp CD$. Let the circumscribed circle of triangle ACF meet line CD at C and G. Prove that the triangle AFG is isosceles.

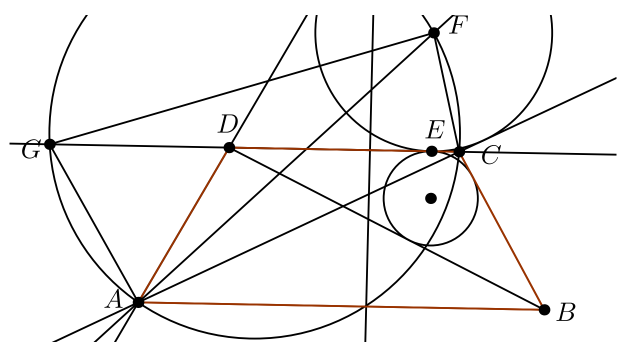

Figure 7: Problem 2

Proof. We mentioned earlier that in triangle ACD, the tangency points of the incircle and the A-excircle with CD are reflections about the midpoint of CD. Since the triangles ACD and BDC are reflections about the common perpendicular bisector of AB and CD, it follows that E is the tangency point of the A-excircle of ACD with CD. Since F lies both on the internal angle bisector of $\angle DAC$ and the perpendicular line to CD at E, it follows that F is the A- excenter of ACD. Thus, noting that CF is the external angle bisector of $\angle ACD$,

$$\angle FAG = \angle FCG = \angle(AC, CF) = \angle FGA$$

This implies that triangle AFG is isosceles. $\square$

3. (IMO 2012) Given triangle ABC the point J is the center of the excircle opposite the vertex A. This excircle is tangent to the side BC at M, and to the lines AB and AC at K and L, respectively. The lines LM and BJ meet at F, and the lines KM and CJ meet at G. Let S be the point of intersection of the lines AF and BC, and let T be the point of intersection of the lines AG and BC. Prove that M is the midpoint of ST.

Proof. First we will do a little bit of angle chasing to prove that $AJKF$ is cyclic:

$$\begin{aligned}\angle MFJ &= 90 - \angle FMK \\ &= 90 - (\angle FMB + \angle BMK) \\ &= 90 - (\angle CML + \angle BMK) \\ &= 90 - (90 - \frac{\gamma}{2} + 90 - \frac{\beta}{2}) \\ &= \frac{\alpha}{2}\end{aligned}$$

Now, since K is the reflection of M about line FJ, $\angle KFJ = \angle MFJ = \frac{\alpha}{2}$. But $\angle KAJ = \frac{\alpha}{2}$. Hence, $\angle KFJ = \angle KAJ$ which implies that $AJKF$ is cyclic.

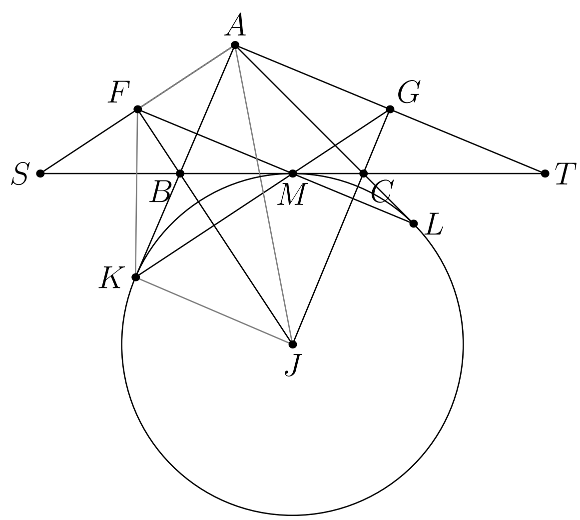

Figure 8: Problem 3

Now, as $\angle AKJ = 90$ we find that $\angle AFJ = 90$, i.e., $JF \perp AS$. However, $JF \perp KM$, so then $AS \parallel KM$. This means that the triangles BAS and BKM are similar. But we know that $BK = BM$, so $BS = BA = c$ similarly, $CT = CA = b$. By Theorem 1, we end up getting that $SM = SB + BM = c + s - c = s$ and $TM = TC + CM = b + s - b = s$. Thus, $SM = TM$ so M is the midpoint of ST.

$\square$

4. (Balkan MO 2013) In a triangle ABC, the excircle ω_a opposite A touches AB at P and AC at Q, and the excircle ω_b opposite B touches BA at M and BC at N. Let K be the projection of C onto MN and let L be the projection of C onto PQ. Show that the quadrilateral $MKLP$ is cyclic.

Proof. In triangle CNK, $\angle CNK = 90 - \frac{\beta}{2}$ (from isosceles triangle BMN), $\angle CKN = 90$ and $CN = s - a$. Hence, $CK = (s-a)\cos\frac{\beta}{2}$. Similarly with triangle CLQ, we get $CL = (s-b)\cos\frac{\alpha}{2}$.

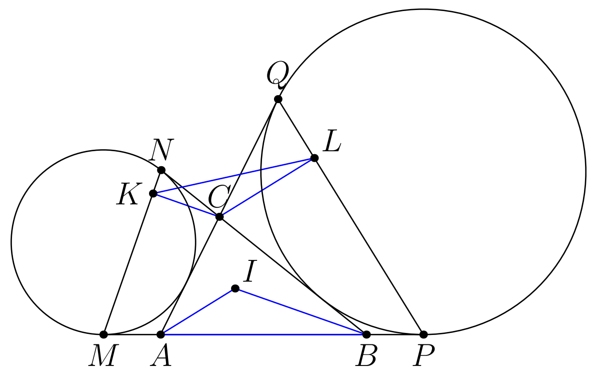

Figure 9: Problem 4

Now let us construct the incenter I of ABC and let us drop the perpendiculars from I to AB and AC at S and T respectively. In triangle IAT, $\angle IAT = \frac{\alpha}{2}$, $\angle ATI = 90$ and $AT = s-a$. Hence, $AI = \frac{s-a}{\cos\frac{\alpha}{2}}$. Similarly, in triangle IBS, $BI = \frac{s-b}{\cos\frac{\beta}{2}}$ (These are fairly well known facts).

So if we compare the triangles CKL and IAB, we observe that:

$$\angle LCK = \angle LCQ + \angle QCN + \angle NCK = \frac{\alpha}{2} + \gamma + \frac{\beta}{2} = 180 - (\frac{\alpha}{2} + \frac{\beta}{2}) = \angle AIB$$

and also,

$$\frac{CK}{CL} = \frac{s-a}{s-b}\frac{\cos\frac{\beta}{2}}{\cos\frac{\alpha}{2}} = \frac{AI}{BI}$$

Hence the triangles CKL and IAB are similar. This means that $\angle CKL = \angle IAB = \frac{\alpha}{2}$. Thus $\angle LKM = 90 + \frac{\alpha}{2} = 180 - (90 - \frac{\alpha}{2}) = 180 - \angle LPM$ and so $MKLP$ is cyclic.

□

5. (IMO 2013) Let the excircle of triangle ABC opposite the vertex A be tangent to the side BC at the point A_1. Define the points B_1 on CA and C_1 on AB analogously, using the excircles opposite B and C, respectively. Suppose that the circumcenter of triangle $A_1B_1C_1$ lies on the circumcircle of triangle ABC. Prove that triangle ABC is right-angled.

Proof. Let the circumcircle of $A_1B_1C_1$ be called Γ_1 and let the excircle opposite to B be called Γ_B.

The largest angle in triangle $A_1B_1C_1$ must be obtuse because the circumcenter, O_1, is outside the triangle. Without loss of generality, assume that $\angle B_1A_1C_1$ is this angle. Then, O_1 lies on the arc BC that contains A. We will prove that $O_1 = N_A$.

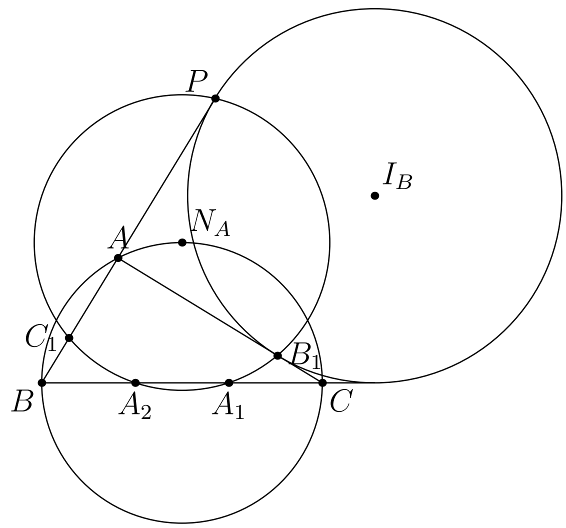

Figure 10: Problem 5

We know that $s - a = BC_1 = CB_1$, $N_AB = N_AC$ and $\angle N_ABC_1 = \angle N_ACB_1$ (since both are subtended by arc AN_A). This means that the triangles N_ABC_1 and N_ACB_1 are congruent. Hence, $N_AB_1 = N_AC_1$. Both O_1 and N_A are points that lie on the arc BC that contains A and that are equidistant from B_1 and C_1, . However, only one point can satisfy both of these properties, so $O_1 = N_A$.

Let the point A_2 be the tangency point of the incircle of ABC with BC. Since $BA_1 = CA_2$ and $BN_A = CN_A$, we find that $N_AA_1 = N_AA_2$. Thus A_2 also lies on Γ_1.

Let P be the tangency point of Γ_B with AC. By Theorem 4, A, N_A and I_B all lie on a line that is the perpendicular bisector of PB_1. Hence, $N_AB_1 = N_AP$ and so P lies on Γ_1.

Finally, using power of a point from B with respect to Γ_1, we get that

$$\begin{gathered}
BC_1 \cdot BP = BA_2 \cdot BA_1 \\
\implies s(s-a) = (s-b)(s-c) \\
\implies (a+b+c)(-a+b+c) = (a-b+c)(a+b-c) \\
\implies (b+c)^2 - a^2 = a^2 - (b-c)^2 \\
\implies a^2 = b^2 + c^2
\end{gathered}$$

Hence $\angle A$ is right.

□

Notice that in all of these problems, we used mostly the basic excircle theory with a few basic geometrical ideas. In even difficult problems, these are often all that are necessary.

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References

[1] H.S.M. Coxeter, S.L. Greitzer, “Geometry Revisited”, 1967

[2] Yuefei Zhao, “Lemmas in Eucliean Geometry”, 2007 http://yufeizhao.com/olympiad/geolemmas.pdf

[3] WolframMathWorld, “Bevan Point” http://mathworld.wolfram.com/BevanPoint.html

[4] Balkan MO 2013 http://www.bmo2013.eu/images/Problems/BMO2013-TheTest-ENGLISH.pdf

[5] IMO offical, Problems http://www.imo-official.org/problems.aspx