Polynomials

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June 10, 2010

1 General facts about polynomials

- Any polynomial of degree n has n complex roots (counted with multiplicity):

$$P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 = a_n \prod_{i=1}^{n}(x - \alpha_i).$$

  If the polynomial has real coefficients, then the roots α_i will consist of the real roots together with pairs of complex conjugate roots. This means that the number of real roots the polynomial has is congruent to its degree modulo 2.

- Although the roots individually cannot in general be expressed in terms of the coefficients of the polynomial, the elementary symmetric functions of them can:

$$e_k(\alpha_1, \ldots, \alpha_n) = (-1)^k \frac{a_{n-k}}{a_n}.$$

- Descartes' Rule of Signs: Let $P(x) = a_n x^n + \cdots + a_0$ be a polynomial with real coefficients, and let C be the number of times the sign changes in the coefficient list $a_n, \ldots, a_0$, where we ignore zeroes. Then the number of *positive real* roots of P (counted with multiplicity) is no larger than C and congruent to C mod 2.

- Rolle's Theorem: Let $P(x)$ be a polynomial with real coefficients. Then between any two zeroes of $P(x)$, there will be (at least one) zero of the derivative $P'(x)$.

- The division algorithm: If $a(x)$ and $b(x)$ are polynomials, then there exist unique polynomials $q(x)$ and $r(x)$ with $\deg(r) < \deg(b)$ such that $a(x) = b(x)q(x) + r(x)$. This can be used to compute GCDs!

- Lagrangian interpolation:

$$f(x) = \sum_{i=0}^{d} f(x_i) \prod_{\substack{0 \le j \le d \\ j \ne i}} \frac{x - x_j}{x_i - x_j}.$$

- Newtonian interpolation:

$$f(n) = \sum_{i=0}^{d} (\Delta^i f)(0) \binom{n}{i}.$$

  Here $\Delta f(x) = f(x+1) - f(x)$ is the finite difference operator.

- If $f(x)$ is a polynomial with leading term cx^d, then $f(x+a) - f(x)$ is a polynomial with leading term $acdx^{d-1}$.

- Finite Taylor expansion:

$$f(x+a) = f(x) + af'(x) + a^2 \frac{f''(x)}{2} + \cdots$$

2 Special types of polynomials

- Chebyshev polynomials: The Chebyshev polynomials $T_0 = 1, T_1 = x, T_2 = 2x^2 - 1, \ldots$ are characterized by the property
$$T_n(\cos\theta) = \cos(n\theta).$$
They have many nice features:
  - A recursion:
$$T_{n+1} = 2xT_n - T_{n-1}.$$
  - A useful identity:
$$T_n\left(\frac{x+x^{-1}}{2}\right) = \frac{x^n + x^{-n}}{2}.$$
  - If $P(x)$ is a degree n polynomial with leading term ax^n, then
$$|P(x)| \geq \frac{a}{2^n} \text{ for some } x \in [-1,1].$$
Equality here is acheived by T_n.

- Cyclotomic polynomials: The nth cyclotomic polynomial is defined by
$$\Phi_n(X) = \prod_{\substack{1\leq j\leq n \\ (j,n)=1}} (X - e^{2\pi ij/n}) = \prod_{d|n} (X^{n/d} - 1)^{\mu(d)} \in \mathbb{Z}[X].$$
They also satisfy the identity
$$X^n - 1 = \prod_{d|n} \Phi_d(X).$$
These polynomials are likely to be relevant whenever you see an expression of the form $X^n - 1$.

- Symmetric and alternating polynomials: A function $f(x_1, \ldots, x_n)$ is called *symmetric* if as a function of any two variables x_i and x_j, it satisfies $f(x_j, x_i) = f(x_i, x_j)$. Similarly, f is called *alternating* if $f(x_j, x_i) = -f(x_i, x_j)$.

  Basic fact about symmetric polynomials: any symmetric polynomial $f(x_1, \ldots, x_n)$ can be written as a (unique) polynomial in the elementary symmetric polynomials $e_i(x_1, \ldots, x_n)$ for $1 \leq i \leq n$.

  Basic fact about alternating polynomials: any alternating polynomial can be written as a symmetric polynomial multiplied by the Vandermonde polynomial
$$\Delta = \prod_{1\leq i<j\leq n} (x_i - x_j) = \sum_{\text{alt}} x_1^{n-1} x_2^{n-2} \cdots x_{n-1},$$

3 Problems

1. (IMO Shortlist, modified) Suppose that P is a polynomial with degree 100 and such that $P(n) = F_n$ for $n = 102, 103, \ldots, 202$. (Here F_n is the nth Fibonacci number, i.e. $F_1 = 1$, $F_2 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 1$.) Find $P(203)$.

2. Determine all polynomials $P(x)$ with real coefficients such that $P(x) + P(1/x) = P(x)P(1/x)$ for all $x \neq 0$.

3. Let a, b, c, d be distinct real numbers with sum 0. Prove that
$$\frac{a^6}{(a-b)(a-c)(a-d)} + \frac{b^6}{(b-a)(b-c)(b-d)} + \frac{c^6}{(c-a)(c-b)(c-d)} + \frac{d^6}{(d-a)(d-b)(d-c)} \geq \frac{1}{3}\left(a^3 + b^3 + c^3 + d^3\right).$$

4. The Lucas numbers are defined by $L_0 = 2, L_1 = 1$, and $L_n = L_{n-1} + L_{n-2}$ for $n \geq 2$. Show that for any positive integer k,

$$L_{2k} = \prod_{j=0}^{k-1} \left(3 + 2\cos\left(\frac{(2i+1)\pi}{2n}\right)\right).$$

5. Find all positive integers n such that every prime divisor of $2^n - 1$ also divides $2^k - 1$ for some $0 < k < n$.

6. (MOP 08) Let P be a polynomial of degree n with real coefficients, and let t be a real number with $t \geq 3$. Show that

$$\max_{0 \leq k \leq n+1} |t^k - P(k)| \geq 1.$$

7 (MOP 01)**.** Let $P(x)$ be a real-valued polynomial with $P(n) = P(0)$. Show that there exist at least n distinct (unordered) pairs of real numbers $\{x, y\}$ such that $x - y$ is a positive integer and $P(x) = P(y)$.

8. Let x, y, z be real numbers satisfying $x + y + z = 5, x^2 + y^2 + z^2 = 9$. Find the minimum and maximum of $P = x^2y + y^2z + z^2x$.

9. Let $M_j = 2^j - 1$ for positive integer j. Prove the identity

$$\sum_{k=0}^{n} \frac{(-1)^k 2^{k(k+1)/2}}{(M_1M_2\cdots M_k)(M_1M_2\cdots M_{n-k})} = (-1)^n$$

for any positive integer n.

10. (FLT for polynomials) Prove that for each $n > 2$ there are no relatively prime non-constant polynomials $a(x), b(x), c(x)$ (with complex coefficients) such that

$$a(x)^n + b(x)^n = c(x)^n.$$

11 (USAMO 88)**.** A certain polynomial product of the form

$$(1-z)^{b_1}(1-z^2)^{b_2}\cdots(1-z^{32})^{b_{32}},$$

where the b_k are positive integers, has the surprising property that if we multiply it out and discard all terms involving z to a power larger than 32, we are left with $1 - 2z$. Find b_{32}.