New Zealand Maths Olympiad Committee
September Problems 2009
Due: October 21

Junior Division

1. *For which values, if any, of the positive integer n is $n(4n+1)$ a perfect square?*

Solution Suppose that $n(4n+1)$ were a perfect square. Since n and $4n+1$ have no common factors, it would be the case that for some integers x and y, $n = x^2$, $4n+1 = y^2$. But then $4n = 4x^2 = (2x)^2$ is also a perfect square, and so we have two perfect squares which differ by 1. However, the minimum difference between positive perfect squares is 3 so this is impossible.

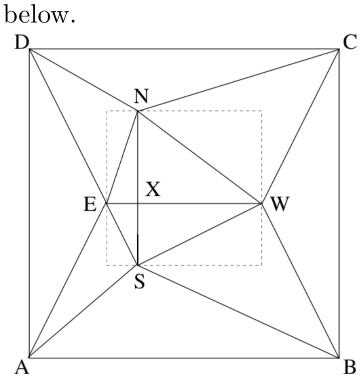
Alternatively, without using the condition about common factors, since $n(4n+1) > (2n)^2$, if it were a perfect square it would have to be the square of a number greater than or equal to $2n+1$. But,

$$(2n+1)^2 = 4n^2 + 4n + 1 > 4n^2 + n = n(4n+1).$$

Hence, $n(4n+1)$ can never be a perfect square.

2. *A square $ABCD$ with sides of length 1 is labeled with A in the lower left corner, and proceeding counter-clockwise. A horizontal line segment EW (E at the left), and a vertical line segment NS (N at the top), both of length $1/2$ lie entirely inside the square, and intersect at a point X. What is the sum of the areas of the triangles EDN, SEA, WSB and NWC?*

Solution Consider the diagram below.



The quadrilateral $NESW$ has area $1/8$ since it divides the indicated square into four triangles, each paired with a congruent triangle to form the entire square. The sum of the areas of triangles DNC and BSA is $1/4$ since each has base of length 1 and their altitudes sum to $1/2$. Similarly, the sum of the areas of triangles DAE and BWC is also $1/4$. The four triangles whose areas we wish to sum form the remainder of the square $ABCD$ after the previously mentioned figures have been removed and the sum of their areas is therefore:

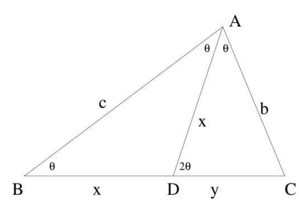
$$1 - 1/8 - 1/4 - 1/4 = 3/8.$$

3. *Let A be a subset of $\{1, 2, 3, \ldots, 2010\}$ having the property that the difference of any two elements of A is not a prime number. What is the largest possible number of elements of A? (Note, 1 is not a prime number).*

   Solution Suppose that $a \in A$. Then none of $a+2$, $a+3$, $a+5$, $a+7$ can belong to A, and among $a+1$, $a+4$ and $a+6$ at most one can belong to A. So, of the eight numbers from a to $a+7$ inclusive, at most 2 belong to A. Therefore, the maximum number of elements possible in A is not more than the integer part of 2010/4 plus 1, namely 503. And, it is easy to find such a set, for instance $\{1, 5, 9, 13, \ldots, 2009\}$.

4. *In $\triangle ABC$, $\angle CAB = 2\angle ABC$. Let the side lengths of BC, CA and AB be a, b and c respectively. Prove that $a^2 = b(b+c)$.*

   Solution Let D be the point of intersection of the angle bisector of $\angle BAC$ with BC. Consider the figure below



   Since $\angle DAB = (1/2)\angle CAB = \angle ABC$, $AD = BD = x$ (say). Let $CD = y$. Since $\angle DAC = \angle ABC$ and $\angle CDA = \angle DBA + \angle BAD = 2\angle ABC = \angle CAB$, the two triangles ABC and DAC are similar. So $c/x = b/y = a/b$. Therefore

$$b(b+c) = b^2 + cb = ay + ax = a^2$$

   as required.

5. *Let x and y be two integers such that $x^2 + 2y$ is a perfect square. Prove that $x^2 + y$ is a sum of two perfect squares.*

   Solution Let $x^2 + 2y = z^2$. Then $y = (z^2 - x^2)/2$, so

$$x^2 + y = \frac{x^2 + z^2}{2}.$$

   Now

$$\frac{x^2 + z^2}{2} = \frac{(x+z)^2 + (x-z)^2}{4} = \left(\frac{x+z}{2}\right)^2 + \left(\frac{x-z}{2}\right)^2.$$

   Since $z^2 - x^2 = 2y$ is even, z and x are either both even or both odd, so this final expression is a sum of two perfect squares.

6. *At a certain math camp there were four times as many boys as girls (sad, but true). One day, all the students sat down around a circular table. One of the adults noticed that among the pairs of students sitting next to each other there were three times as many pairs of the same sex as there were pairs of opposite sexes. What is the smallest possible number of students who were attending the camp?*

Solution Let the number of girls be g, so the number of boys is $4g$. If there are x opposite sex pairs, then there are $4x$ pairs all together. But, the number of pairs is also equal to the total number of students. So $5g = 4x$. Therefore, g must be a multiple of 4 and the least potential number of students is 20. But, if that were indeed possible there would be 5 opposite sex pairs. However, the number of opposite sex pairs must always be even, since one occurs between each block of girls and boys, and reading around the circle there are an even number of blocks all together (since the number of girl blocks is the same as the number of boy blocks). That leaves us with a smallest possibility of 40 students including 8 girls, and needing 10 opposite sex pairs, i.e. 5 blocks of boys and girls respectively (so one possibility would be $(4G)BGBGBGBG(28B)$.)

Senior Division

1. *Let k be a positive integer. Show that the number of powers of 2 that have k digits in decimal notation is at least three and at most four. Given that the largest power of 2 which is less than 10^{2009} is 2^{6673}, for how many k between 1 and 2009 inclusive, are there four powers of 2 that have k digits?*

Solution For the first part, note that the first power of 2 that has k digits lies between 10^{k-1} and $2 \times 10^{k-1}$. Denote this power of 2 by p_k. Then all three of p_k, $2p_k$ and $4p_k$ have k digits. The number $8p_k$ might have k or $k+1$ digits, but the number $16p_k > 10^k$ and so it certainly does not have k digits. So, either three or four powers of two have k digits.

For the second part, from the given information, there are 6674 powers of 2 having 2009 or fewer digits. Since $6674 = 3 \times 2009 + 647$ there must be 647 values of k between 1 and 2009 for which there are four powers of 2 with k digits.

2. *Let a, b, c and d be integers. Show that the equation*

$$x^2 + ax + b = y^2 + cy + d$$

has infinitely many integer solutions (x, y) if and only if $a^2 - 4b = c^2 - 4d$.

Solution If $a^2 - 4b = c^2 - 4d$ then a and c are either both even or both odd. In either case, $(a-c)/2$ is an integer. Given any integer x, we can define $y = x - (a-c)/2$, so $y - c/2 = x - a/2$ and thus:

$$\begin{aligned} x^2 + ax + b &= (x - a/2)^2 - (a^2 - 4b)/4 \\ &= (y - c/2)^2 - (c^2 - 4d)/4 \\ &= y^2 + cy + d. \end{aligned}$$

Conversely, suppose that the equality does not hold and set $k = (a^2 - 4b) - (c^2 - 4d)$. Then, for any solution to the given equation we get (after multiplying by four and completing the squares):

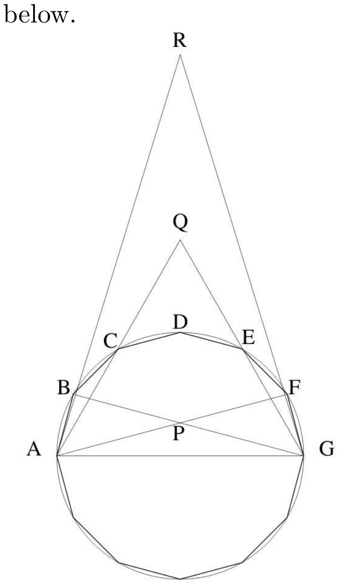
$$(2x + a)^2 - (a^2 - 4b) = (2y + c)^2 - (c^2 - 4d)$$

$$\begin{aligned}(2x+a)^2-(2y+c)^2 &= (a^2-4b)-(c^2-4d)\\ (2x+2y+a+c)(2x-2y+a-c) &= k.\end{aligned}$$

But k has only finitely many different factorizations, and every such factorization determines at most one solution to the equation, so the equation has only finitely many solutions.

3. *Let A, B, C, D, E, F and G be seven consecutive vertices of a regular dodecagon. Segments AF and BG intersect at point P, lines AC and GE meet at point Q, and lines AB and GF meet at point R. Show that P is the orthocentre, and Q is the circumcentre, of triangle ARG. [A triangle's* orthocentre *is the point of intersection of its altitudes, and its* circumcentre *is the centre of the circumscribed circle.]*

Solution Consider the diagram below.



Both $\angle ABG$ and $\angle GFA$ are right, since AG is the diameter of a circle on which both B and F lie. Hence P is the intersection of two of the altitudes of $\triangle ARG$, and so is its orthocentre. To show that Q is its circumcenter we will compute the angles AQG and ARG. Since $\angle AGQ = \angle AGE$ and arc AE subtends one third of the circle, $\angle AGQ = 60°$. By symmetry, or the same argument $\angle GAQ = 60°$, and so $\angle AQG = 60°$ also. Similary, $\angle BAG$ and $\angle RGA$ are each $(5/12)\times 180°$ and hence $\angle ARG = (2/12)\times 180° = 30°$. Clearly Q lies on the perpendicular bisector of AG, and as the angle subtended there by AG is twice the angle subtended at R, it must be the center of the circumcircle of ARG.

4. *Let a number $x \neq 0, 1$, and a positive integer n be given. A sequence of numbers a_0, a_1, a_2, $\ldots$, a_n is defined by: $a_0 = x$, $a_1 = 1 - x$, and $a_k = 1 - a_{k-1}(1 - a_{k-1})$ for $k = 2, 3, \ldots, n$. Prove that:*

$$a_0 a_1 \cdots a_n \left(\frac{1}{a_0} + \frac{1}{a_1} + \cdots + \frac{1}{a_n}\right) = 1.$$

Solution Rewriting the definition of a_k we have

$$\begin{aligned} 1 - a_k &= a_{k-1}(1 - a_{k-1}) \\ &= a_{k-1}a_{k-2}(1 - a_{k-3}) \\ &\cdots \\ &= a_{k-1}a_{k-2}\cdots a_1(1 - a_1) \\ &= a_{k-1}a_{k-2}\cdots a_1 a_0 \end{aligned}$$

That is, $a_k = 1 - a_0 a_1 \cdots a_{k-1}$.

Now we will actually prove that for all $1 \leq k \leq n$,

$$a_0 a_1 \cdots a_k \left(\frac{1}{a_0} + \frac{1}{a_1} + \cdots + \frac{1}{a_k} \right) = 1.$$

This is certainly true for $k = 1$ since

$$x(1 - x) \left(\frac{1}{x} + \frac{1}{1 - x} \right) = 1.$$

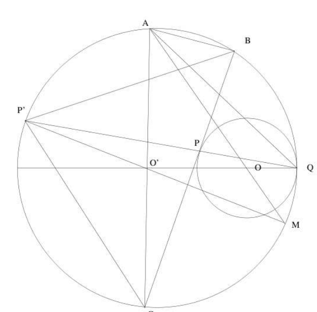
Suppose that it is true for some value j of k. Then, for $k = j + 1$ we have:

$$\begin{aligned} a_0 a_1 \cdots a_{j+1} \left(\frac{1}{a_0} + \frac{1}{a_1} + \cdots + \frac{1}{a_{j+1}} \right) &= a_0 a_1 \cdots a_j \left(\frac{1}{a_0} + \frac{1}{a_1} + \cdots + \frac{1}{a_j} \right) a_{j+1} + a_0 a_1 a_2 \cdots a_j \\ &= a_{j+1} + a_0 a_1 a_2 \cdots a_j \\ &= 1 \end{aligned}$$

as claimed. In particular the result holds for $k = n$ which is what we wanted to prove.

5. *Let ABC be a triangle, and let Γ be its circumcircle. Suppose that a circle with centre O is tangent to the segment BC at a point P, and to the circle Γ at Q which lies in the arc of Γ determined by BC not containing A. Finally suppose that $\angle BAO = \angle CAO$. Prove that $\angle PAO = \angle QAO$.*

Solution We assume that $AB \neq AC$ (because in that case, the two angles we are concerned with are both $0°$). In the diagram below, M is the midpoint of arc BC, O' is the centre of Γ (this is almost on AC as it happens, but it's a tough diagram to get right without some coincidences!), and P' is the other end of the diameter of Γ through M.



Since $\angle CAO = \angle BAO$, the extension of AO must meet the circle Γ at the midpoint of arc BC, i.e. at M. Thus AOM are collinear. Also QOO' are collinear since the two circles share a common tangent at Q. We claim further that $P'PQ$ are collinear. To see this, denote temporarily by R the intersection of of $P'Q$ and the smaller circle (we aim to show that $R = P$). Then

$$\angle MP'R = \angle 0'P'Q = \angle O'QR = \angle OQR = \angle ORQ,$$

where the first and third of these equalities simply arise from taking different points on the corresponding rays, and the second and fourth from the fact that any triangle with two vertices on a circle and the third at its centre is isosceles. But this implies that OR is parallel to MP'. But MP' is perpendicular to BC, and so R must coincide with P the point of tangency of the small circle with BC.

Now, for the same reasons as above we get:

$$\angle QP'M = \angle QP'0' = \angle O'QP' = \angle OQP = \angle QPO$$

and also $\angle QP'M = \angle QAO$ as they both subtend QM on Γ. So $\angle QAO = \angle QPO$. Therefore, $QOAP$ are concyclic, hence $\angle PQO = \angle PAO$. But, $\angle PQO = \angle QPO = \angle QAO$, and we get $\angle PAO = \angle QAO$ as required.

6. *There were 20 players at a chess tournament. Each player played every other player exactly once, and either one of them won, or the game was a draw. It so happened that if a game ended in a draw, then each of the other 18 players beat at least one of the two players involved in the draw. At least two games ended in a draw. Show that it is possible to order the players as P_1, P_2, ..., P_{20} so that P_k beat P_{k+1} for each k from 1 to 19 inclusive.*

Solution We first show that each contestant took part in at most one draw. Suppose otherwise and that A drew with both B and C. Since A drew with B, C must beat either A or B, but drew with A, so beat B. But now B beat neither A nor C who drew with one another, so the conditions of the problem aren't satisfied. Thus, each player can play in at most one draw.

Now consider the longest possible sequence of players P_1, P_2, ..., P_t with P_k losing to P_{k+1} for k from 1 to $t-1$ inclusive. We aim to show that $t = 20$. So, for the sake of contradiction, suppose otherwise, and let A be a player who is not in the sequence.

Then P_1 cannot have beaten A or we could add A at the beginning of the sequence (contradicting the fact that it was "longest possible").

If A beat P_1, then A did not draw with P_2 (since P_1 lost to both). If A beat P_1 and lost to P_2, we could add A between P_1 and P_2 (again a contradiction). So, if A beat P_1, then A beat P_2 as well. But the same argument now applies showing that A must beat P_3, then P_4 etc., and hence that A can be added at the end of the sequence, yet another contradiction.

So, the only remaining possibility is that A drew with P_1. But P_1 drew with at most one player, so we must have $t = 19$ (otherwise we could choose a different player B not belonging to the sequence, who would also have to draw with P_1 for a contradiction.) If A beat P_2 we can repeat the argument above to show that A could be added at the end of the sequence. So A lost to P_2, and indeed must have lost to all of P_2 through P_{19}. But, there were at least two draws – there must be a draw between P_i and P_j for some $i, j \geq 2$. However, A beats neither of these, finally contradicting one of the conditions of the problem.

We conclude that we must have a sequence of players of the type required.