4. Let n be a positive integer and a be a real number. Compute the sum

$$\frac{1}{\cos a - \cos 3a} + \frac{1}{\cos a - \cos 5a} + \cdots + \frac{1}{\cos a - \cos(2n+1)a}$$

5. Let $s_n = \sum_{k=1}^{n} F_k \sin k^\circ$, where F_k is the Fibonacci sequence: $F_0 = 0$, $F_1 = 1$, and $F_{n+1} = F_n + F_{n-1}$. Prove that

$$s_{180} = c(F_{180} \cos 1^\circ + \frac{1}{2}(F_{179} + F_{181}) + 1)$$

where $c = (2\sin 1^\circ + \frac{1}{2}\csc 1^\circ)^{-1}$.

6. Prove the identity

$$\sum_{k=1}^{n} \cot^{-1}(2k^2) = \cot^{-1}(1 + \frac{1}{n}).$$

7. Evaluate the sum

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \tan \frac{a}{2^n},$$

where $a \neq k\pi$, k integer.

8. Prove the identity

$$\sum_{n=1}^{\infty} 3^{n-1} \sin^3 \frac{a}{3^n} = \frac{1}{4}(a - \sin a).$$

9. Prove that for any real number x the following identity holds

$$\prod_{n=1}^{\infty} \cos \frac{x}{2^n} = \frac{\sin x}{x}.$$

10. Prove that for all natural numbers n one has

$$\cos \frac{2\pi}{2^n - 1} \cos \frac{4\pi}{2^n - 1} \cdots \cos \frac{2^n \pi}{2^n - 1} = 2^n.$$

1.10 Trigonometric substitutions

Because of the large number of trigonometric identities, in many problems the choice of a smart trigonometric substitution leads to a very simple solution. This is the case with all the problems presented below. The substitution is usually suggested by the form of an algebraic expression, like in the case of the following problem.

Find all real solutions of the system of equations

$$\begin{cases} x^3 - 3x = y \\ y^3 - 3y = z \\ z^3 - 3z = x \end{cases}$$

Here the presence of $x^3 - 3x$ reminds us of the formula of the cosine of the triple of an angle. Of course the coefficient in front of x^3 is missing, but one can easily take care of this by working with the double of the cosine, instead of the cosine. Let us start by finding the solutions in $[-2,2] \times [-2,2] \times [-2,2]$. We make the notation $x = 2\cos u$, $y = 2\cos v$, $z = 2\cos w$, with $u, v, w \in [0, \pi]$.

The system becomes

$$\begin{cases} 2\cos 3u = 2\cos v \\ 2\cos 3v = 2\cos w \\ 2\cos 3w = 2\cos u \end{cases}$$

This leads to the equation $\cos 27u = \cos u$, which gives $28u = 2k\pi$ of $26u = 2k\pi$, $k \in \mathbf{Z}$. The solutions in the interval $[0, \pi]$ are $u = k\pi/14$, $k = 0, 1, \cdots, 14$ and $u = k\pi/13$, $k = 1, 2, \cdots, 12$.

Consequently, the solutions of the initial equation are
$(2\cos k\pi/14, 2\cos 3k\pi/14, 2\cos 9k\pi/14)$, $k = 0, 1, \cdots, 14$ and
$(2\cos k\pi/13, 2\cos 3k\pi/13, 2\cos 9k\pi/13)$, $k = 1, 2, \cdots, 12$.

Since there are at most $3 \times 3 \times 3 = 27$ solutions and we already found 27, we have found all solutions of the system.

We continue with an example where the tangent is used.

Let $\{x_n\}_{n\geq 0}$ *be a sequence satisfying the reccurence relation*

$$x_{n+1} = \frac{\sqrt{3}x_n - 1}{x_n + \sqrt{3}}.$$

Prove that the sequence is periodic.

The recurrence relation remind us of the formula

$$\tan(x - \frac{\pi}{6}) = \frac{\tan x - \frac{1}{\sqrt{3}}}{1 + \tan x \frac{1}{\sqrt{3}}}.$$

We let $x_0 = \tan t$ for some $t \in \mathbb{R}$. Then $x_1 = \tan(x - \pi/6)$, and inductively $x_n = \tan(x - n\pi/6)$. Since the tangent is periodic of period π, we get that $x_n = x_{n+6}$, which shows that the sequence has period 6.

1. For what values of the real parameter a does there exist a real number x satisfying
$$\sqrt{1 - x^2} \geq a - x$$

2. Find all triples of numbers x, y, $z \in (0, 1)$, satisfying
$$x^2 + y^2 + z^2 + 2xyz = 1.$$

3. Let a, b and c be given positive numbers. Determine all positive real numbers x, y and z such that
$$x + y + z = a + b + c$$
$$4xyz - (a^2x + b^2y + c^2z) = abc.$$

4. Given $-1 \leq a_1 \leq a_2 \leq \cdots \leq a_n \leq 1$, prove that
$$\sum_{i=1}^{n-1} \sqrt{1 - a_i a_{i+1} - \sqrt{(1 - a_i^2)(1 - a_{i+1}^2)}} \leq \frac{\pi\sqrt{2}}{2}$$

5. The sequence $\{x_n\}_n$ satisfies $\sqrt{x_{n+2} + 2} \leq x_n \leq 2$ for all n. Find all possible values of x_{1996}.

6. Find all real solutions of the system of equations
$$\begin{cases} 2x + x^2y = y \\ 2y + y^2z = z \\ 2z + z^2x = x \end{cases}$$

7. Find all real solutions of the system of equations

$$\begin{cases} x_1 - \frac{1}{x_1} = 2x_2 \\ x_2 - \frac{1}{x_2} = 2x_3 \\ x_3 - \frac{1}{x_3} = 2x_4 \\ x_4 - \frac{1}{x_4} = 2x_1 \end{cases}$$

8. Let $x_0 = 0$ and $x_1, x_2, \cdots, x_n > 0$, with $\sum_{k=1}^{n} x_k = 1$. Prove that

$$\sum_{k=1}^{n} \frac{x_k}{\sqrt{1 + x_0 + \cdots + x_{k-1}}\sqrt{x_k + \cdots + x_n}} < \frac{\pi}{2}$$

9. Given four distinct numbers in the interval $(0, 1)$, there exist two of them x and y such that

$$0 < x\sqrt{1 - y^2} - y\sqrt{1 - x^2} < \frac{1}{2}.$$

10. For each real number x, define the sequence $(x_n)_n$ recursively by $x_1 = x$, and

$$s_{n+1} = \frac{1}{1 - x_n} - \frac{1}{1 + x_n}$$

for all n. If $x_n = 1$, then the sequence terminates (for x_{n+1} would be undefined). How many sequences terminate after the eigth term?

11. A sequence of real numbers $a_1, a_2, a_3, \cdots$ has the property that $a_{k+1} = (ka_k + 1)/(k - a_k)$ for any natural number k. Prove that this sequence contains infinitely many positive terms and infinitely many negative terms.