TJUSAMO Contest #3 **Solutions**

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1

Let O be the circumcenter of a convex quadrilateral $ABCD$. Let W, X, Y, Z be the foot of the perpendicular from O to the segments AB, BC, CD, DA, respectively. If brackets denote area, prove that $[ABCD] = [ABZ] + [BCW] + [CDX] + [DAY]$.

1.1 Solution

W, X, Y, Z are just the midpoints of the sides of $ABCD$, so

$$\begin{aligned}
[ABZ] + [BCW] + [CDX] + [DAY] &= \frac{[ABD]}{2} + \frac{[BCA]}{2} + \frac{[CDB]}{2} + \frac{[DAC]}{2} \\
&= \frac{[ABCD]}{2} + \frac{[ABCD]}{2} \\
&= [ABCD]
\end{aligned}$$

2

Prove that for every natural a, there exist infinitely many naturals n such that $10^n a - 1$ is composite.

2.1 Solution

Let p be a prime that divides $10a - 1$. We have $10a \equiv 1 \pmod{p}$, so p must be relatively prime to 10. By Fermat's Little Theorem, $10^{p-1} \equiv 1 \pmod{p}$, so $10^{k(p-1)} 10a \equiv 1 \pmod{p}$ for any natural k. Also, note that $10^{k(p-1)} 10a$ is greater than $10a$, so $10^{k(p-1)} 10a - 1$ is greater than p and also divides p, so it must be composite. Therefore any n of the form $k(p-1) + 1$ makes $10^n a - 1$ composite, and we are done.

3

Find all finite sequences of nonnegative integers $z_0, z_1, \ldots, z_{n-1}$, such that n is a natural, and for any integer i such that $0 \le i < n$, z_i represents the number of integers j such that $0 \le j < n$ and $z_j = i$. One such sequence is $3, 2, 1, 1, 0, 0, 0$.

3.1 Answer

The only sequences that work are $\{1, 2, 1, 0\}, \{2, 0, 2, 0\}, \{2, 1, 2, 0, 0\}$, and for any integer $n > 6$, $z_0 = n - 4, z_1 = 2, z_2 = z_{n-4} = 1$, and $z_i = 0$ for all integral i such that $2 < i < n - 4$ or $n - 4 < i < n$.

3.2 Solution

First of all, it is obvious that $n > 1$ and $z_0 \neq 0$. Also note that $\sum_{i=0}^{n-1} z_i = n \implies \sum_{i=1}^{n-1} z_i = n - z_0$. Additionally, $\sum_{i=1}^{n-1} z_i$ is the number of indices i such that $z_i \neq 0$, but there are $n - z_0 - 1$ such indices between 1 and $n - 1$, inclusive, so we have $n - z_0 - 1$ natural summing up to $n - z_0$. By the Pigeonhole Principle, one of them must equal 2, so the rest must be 1. This means if $i > 0$, $z_i \le 2$, so only one number in the sequence, namely z_0, can be greater than 2. Therefore, if $i > 2$, $z_i \le 1$. So either $z_1 = 2$ or $z_2 = 2$.

Case 1, $z_2 = 2$, $z_1 \neq 2$: There are two twos in the sequence, and the only valid indices for these twos are 0 and 2. Now we have the condition that all numbers in the sequence are less than 2 except z_0 and z_2, so if $i > 2$, $z_i = 0$. The two possibilities remaining, $z_1 = 0$ and $z_1 = 1$, give us the two answers $\{2, 0, 2, 0\}$ and $\{2, 1, 2, 0, 0\}$.

Case 2, $z_1 = 2$, $z_2 \neq 2$: There is at least one two in the sequence, so $z_2 \ge 1$, but since $z_2 \le 2$ and $z_2 \neq 2$, the only possibility is that $z_2 = 1$, so there is only one two in the sequence, so $z_0 \neq 2$. Thus, either $z_0 = 1$ or $z_0 > 2$. If $z_0 = 1$, then if $i > 2$, $z_i = 0$, so we get the answer $\{1, 2, 1, 0\}$. If $z_0 > 2$, we let $n = z_0 + 4$. Now we have $z_{z_0} = 1$ and $z_i = 0$ for all integers i such that $i > 2$ and $i \neq z_0$, so we get our remaining solutions, which then can easily be verified to work.