TJUSAMO Contest #1 **Solutions**

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1

Find all ordered triplets (x, y, z) of integers that satisfy the equation:

$$x^2 + y^2 + z^2 = 2xyz$$

1.1 Answer

(0, 0, 0) is the only solution.

1.2 Proof

We will use proof by contradiction. Assume there exists such a solution other than (0, 0, 0). Divide both sides by the greatest power of 2 that is a divisor of x^2, y^2, and z^2. Then the RHS is an even multiple of xyz, so either one or all of x, y, and z are even. The latter is impossible, as we already divided out by the greatest power of two. In the former case, since one of x, y ,and z is even, the RHS is a multiple of four. All squares are congruent to either 0 or 1 mod 4 depending on if they are even or odd, so we have:

$$0 + 1 + 1 \equiv 0 \pmod 4$$

This is a contradiction, so there can not exist any solutions other than (0, 0, 0).

1.3 Same Proof, Different Style

We will use proof by contradiction. Assume there exists such a solution other than (0, 0, 0). Clearly, either x, y, z are all even, or at least one of them is odd.

Case 1: x, y, z are all even and at least one of them is not zero.

If x, y, z are all even, then let n be the largest power of 2 that divides x, y, z. Since x, y, z are not all zero, n is a finite number. Let $x = 2^n a, y = 2^n b, z = 2^n c$. If a, b, c are all even, 2^{n+1} would divide x, y, z, making our first statement false. At least one of a, b, c must be odd. Now we have $a^2 + b^2 + c + 2 = 2^x abc$, where $x > 2$. A square must be 0 or 1 in mod 4, and the RHS is 0 mod 4, so the LHS must be 0+0+0, and therefore a, b, c are all even. Contradiction. x, y, z are not all even unless they are all zero.

Case 2: At least one of x, y, z is odd.

The RHS is either 0 or 2 mod 4. If it is 0 mod 4, all our solutions are even. Therefore, the RHS must be 2 mod 4. Then the LHS must be $1 + 1 + 0$, so one of x, y, z is even, but then the RHS would be 0 mod 4. Contradiction. None of x, y, z are odd.

There can not exist any solutions other than (0, 0, 0).

1.4 Notes

LHS: Left Hand Side of the equation
RHS: Right Hand Side of the equation
First, we can notice that the LHS has order 2 while the RHS has order 3, so the diophantine equation would be hard to approach directly with algerbra. Thus, we try mods. Seeing squares, the first mod we should try is mod 4. The easiest way to find this solution would be to examine every value of x, y, z in mod 4.
In general, Euler's Totient Theorem ($a^{\phi(n)} \equiv \{0, 1\} \pmod{n}$) can help you decide what mod to take equations in.

2

[Putnam 1993] A deck containing $2n$ cards numbered from 1 to $2n$ is shuffled and n cards are dealt to each of two players, Alice and Bob. Starting with Alice, the two players take turns discarding one of their remaining cards. A player can win at anytime by discarding a card that causes the sum of the numbers of all the discarded cards to be divisible by $2n + 1$. If both Alice and Bob play flawlessly, what is the probability that Alice wins?

2.1 Answer

Alice has 0 probability of winning. In other words, Bob always wins.

2.2 Proof

Whenever it is Bob's turn, he will always have one more card than Alice. Each card Bob plays will allow Alice only one winning card, so at least one of his cards will ensure that Alice does not have a winning move. Since the sum of all the cards is divisible by $2n + 1$, and Bob goes last, Bob can win as long as he keeps Alice from making a winning move.

2.3 Notes

Notice that the random shuffling makes no difference in this problem. To familiarize yourself with game problems, try playing several different starting states of the game. This time, you should realize Bob always wins after playing the game several times. Knowing the answer makes the proof a lot easier to find. Another possible observation you might make is that, while playing as Bob, you don't have to try very hard to win - any move that doesn't cause you to lose immediately seems to be fine. This is one way to motivate the first observation of the problem (that Bob always has one more card than Alice on his turn).

3

[IMO 1985] In a convex cyclic quadrilateral ABCD, a circle has its center on side AB and is tangent to BC, CD, and DA. Prove that AD + BC = AB.

3.1 Proof #1

Let the circle with center at point O touch BC, CD, and DA at P, Q, and R, respectively. Let angle A be α. $OP = OQ = OR$ since they are radii of the same circle. $\angle OPC = \angle OQC = \angle ORD = 90°$. By the Pythagorean Theorem, $\sqrt{OC^2 - OP^2} = \sqrt{OC^2 - OQ^2} = PC = QC$. Similarly, $QD = RD$. ABCD is cyclic, so $\angle A + \angle C = 180°$. Also $\triangle OPC \cong \triangle OQC$, so $\angle OCP = \angle OCQ = \frac{180-\alpha}{2}$.

Let X be a point on line AD on the same side of A as D such that $AO = AX = AR + RX$ (or $-AR + RX$, depending which side of A the point R is on) Since $\triangle AOX$ is isosceles, $\angle AOX = \angle AXO = \frac{180-\alpha}{2}$. Now we have $\triangle ORX \cong \triangle OQC$, and $RX = QC$. Combining multiple equations, we get $AO = AR + RX = AR + QC = AR + PC$. Similarly, $BO = BP + RD$. Adding these last two equations, we get $AB = AD + BC$, which is our desired result. QED.

3.2 Notes

After drawing several large and accurate diagrams, you might notice that $AO = AR + PC$. Working backwords from this point, we can motivate the construction of point X, which is the key to this geometry problem.

3.3 Proof #2

Let AD and BC intersect at E. The circle tangent to BC, CD, and DA must have a center I lying along angle bisectors of D, C, and E and also along AB by the problem condition. Since $ABCD$ is cyclic, $\angle EBA = \angle EDC$ and $\angle EAB = \angle ECD$, so $\triangle CDE \sim \triangle ABE$.

Now reflect AB across EI to $A'B'$. As EC and ED are reflections across EI, A' lies on EC and B' lies on ED. Also, BA' and AB' are reflections of each other, so they are equal and $CB + AD = CA' + DB'$. Thus it suffices to show that $CA' + DB' = A'B'$.

Since $\triangle A'B'E \cong \triangle ABE \sim \triangle CDE$, $A'B' \parallel CD$. Now reflect A' across CI to A''. $\angle A'CI = \angle ICA'' = \angle A'IC = \angle A''IC$, so $CA'IA''$ is a parallelagram. Therefore, $A'I = CA'' = CA'$

(as they are reflections of each other), so $CA'I$ is isoceles. By similar logic, $DB'I$ is isoceles. Then $A'B' = A'I + IB' = A'C + B'D$, so $AD + BC = AB$ and we are done.

3.4 Notes

The reason we draw point E is because we need some way to use the condition of tangency. If we make AD, BC, and CD sides of a triangle, then the tangency condition is equivalent to saying that the incenter or an excenter of CDE lies on AB, which is a much easier condition to work with. You might start with the case where I is the incenter, since this is, again, easier to work with. Whenever angle bisectors appear, a natural thing to try is reflections. Reflecting AB leaves us with a trapezoid $A'B'DC$ where the angle bisectors of CD meet on $A'B'$. This is an even nicer figure to work with. Upon drawing several diagrams, we note that $CA'I$ and $DB'I$ are always isoceles. As we once again have angle bisectors, we reflect again and find a way to show that the resulting figure is a rhombus.