van der Waerden's theorem

Reid Barton

June 21, 2005

Theorem 1 *(van der Waerden) Let n and k be positive integers. Then there exists a positive integer N such that if the numbers 1, 2, …, N are colored in k colors, one color always contains an arithmetic progression of length n.*

We will denote the smallest such integer N by $W(k, n)$.

Problem 1 Prove van der Waerden's theorem as follows:

1. Show that $W(k, 2) = k + 1$ for all k.
2. Find a bound for $W(k, 3)$ in terms of $W(k', 2)$, where k' is some function of k.
3. In general, find a bound for $W(k, n)$ in terms of $W(k', n - 1)$ for some function k' of k and n.

Problem 2 Show that each of the following statements is equivalent to van der Waerden's theorem:

1. If k is a positive integer, and the set of all positive integers is colored in k colors, one color always contains arbitrarily long arithmetic progressions.
2. If $a_0, a_1, \ldots$ is an infinite sequence of integers satisfying $0 < a_{k+1} - a_k < r$ for some fixed r, then the sequence contains arbitrarily long arithmetic progressions.

The best bound currently known for $W(k, n)$ is

$$W(k, n) \leq 2^{2^{k^{2^{2^{n+9}}}}};$$

a proof along the lines of Problem 1 will yield a much worse bound.

van der Waerden's theorem also follows from this harder theorem:

Theorem 2 *(Szemerédi) Let n be an integer and let $\delta > 0$. Then there exists a positive integer N such that if S is a subset of $\{1, 2, \ldots, N\}$ with at least δN elements, S always contains an arithmetic progression of length n.*

Finally, here is a hard olympiad problem whose proof requires van der Waerden's theorem.

Problem 3 (Iran 2004) For every real number x, define $\langle x \rangle = \min(\{x\}, \{1-x\})$, where $\{x\}$ denotes the fractional part of x. Prove that for every irrational number α and every positive real number ϵ there exists a positive integer n such that $\langle n^2\alpha \rangle < \epsilon$.