New Zealand Mathematical Olympiad Committee

2010 Squad Assignment One

Number Theory

Due: Wednesday, 17th February 2010

1. Determine all primes p such that $5^p + 4p^4$ is a square number.

2. For each positive integer a we consider the sequence $\langle a_n \rangle$ with $a_0 = a$ and $a_n = a_{n-1} + 40^{n!}$. Prove that every such sequence contains infinitely many numbers that are divisible by 2009.

3. Find all integers k such that for every integer n, the numbers $4n + 1$ and $kn + 1$ are relatively prime.

4. Let n be a positive integer. Prove that if the sum of all of the positive divisors of n is a perfect power of 2, then the number of those divisors is also a perfect power of 2.

5. (a) Show that there are infinitely many pairs of positive integers (m, n) such that
$$k = \frac{m+1}{n} + \frac{n+1}{m} \tag{1}$$
is a positive integer.
   (b) Find all positive integers k such that (1) has a positive integer solution (m, n).

6. (a) Find all primes p for which $\dfrac{7^{p-1} - 1}{p}$ is a perfect square.
   (b) Find all primes p for which $\dfrac{11^{p-1} - 1}{p}$ is a perfect square.

7. Prove that there exist infinitely many natural numbers n with the following properties: n can be expressed as a sum of two squares, $n = a^2 + b^2$, and as a sum of two cubes, $n = c^3 + d^3$, but can't be expressed as a sum $n = x^6 + y^6$ of two sixth powers, where a, b, c, d, x, y are natural numbers.

8. (a) Let $b, n > 1$ be integers. Suppose that for each $k > 1$ there exists an integer a_k such that $b - a_k^n$ is divisible by k. Prove that $b = A^n$ for some integer A.
   (b) Does the conclusion still hold if we only know that for every *prime* p there is an integer a_p such that $b - a_p^n$ is divisible by p?

9. Show that there are infinitely many pairs of distinct primes (p, q) such that $p \mid (2^{q-1} - 1)$ and $q \mid (2^{p-1} - 1)$.

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