British Mathematical Olympiad

Round 1

2001 to 2004

United Kingdom Mathematics Trust

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Round 1

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Organised by the **United Kingdom Mathematics Trust**

Contents

Introduction

This booklet contains the questions and solutions for the British Mathematical Olympiad Round 1 papers for the academic sessions 2000-01, 2001-02, 2002-03 and 2003-04. BMO1 is a serious test in itself but it also acts as a gateway to BMO2 and further events leading to the selection of the United Kingdom team to compete in the International Mathematical Olympiad – held each summer. Each year, a booklet is produced covering both BMO1 and BMO2 and it is sent to schools who took part. In that booklet, there is a standard comment about the solutions:

> 'These solutions are the result of many hours of work by a large number of people. They have been subject to many drafts and revisions. As such, they do not resemble the first jottings, failed ideas and discarded pages of rough work with which any solution is started. Before looking at the solutions pupils and teachers are encouraged to make a good effort to solve the problems by themselves. Without wrestling with the problem oneself, it is hard to develop a feeling for the question, to understand where the difficulties lie and to appreciate why one method of attack is successful while another may fail. Reading these solutions without first attempting the question is unlikely to be of much benefit.
>
> Many thanks are due to the contestants and to the members of the setting committee who contributed variety, refinement, inventiveness and precision to these written solutions.'

It is hoped that the material in this booklet is tackled in the same spirit.

Further Resources

There is a wide range of books which can help students prepare for BMO 1. The most appropriate is, without doubt,

The Mathematical Olympiad Handbook - An Introduction to Problem Solving based on the First 32 British Mathematical Olympiads 1965-1996

by Tony Gardiner, Oxford University Press, ISBN 0 19 850105 6

Further useful books are

Elementary Number Theory

by David M. Burton, Allyn and Bacon, ISBN 0 205 06978 9

Student Problems from the Mathematical Gazette

The Mathematical Association (www.m-a.org.uk) ISBN 0 906588 49 9

It is hoped that, in due course, a section of the UKMT website will offer further suggestions.

The rubric for all the papers in this booklet is:

British Mathematical Olympiad

Round 1

Time allowed *Three and a half hours.*

Instructions

- *Full written solutions – not just answers – are required, with complete proofs of any assertions you may make. Marks awarded will depend on the clarity of your mathematical presentation. Work in rough first, and then draft your final version carefully before writing up your best attempt.*
  Do not hand in rough work.
- *One* ***complete*** *solution will gain far more credit than several unfinished attempts. It is more important to complete a small number of questions than to try all five problems.*
- *Each question carries 10 marks.*
- *The use of rulers and compasses is allowed, but calculators and protractors are forbidden.*
- *Start each question on a fresh sheet of paper. Write on one side of the paper only. On each sheet of working write the number of the question in the top* ***left*** *hand corner and your name, initials and school in the top* ***right*** *hand corner.*
- *Complete the cover sheet provided and attach it to the front of your script, followed by the questions 1, 2, 3, 4, 5 in order.*
- *Staple all the pages neatly together in the top* ***left*** *hand corner.*

2000-01 Question Paper

Wednesday, 17 January 2001

1. Find all two-digit positive integers N for which the sum of the digits of $10^N - N$ is divisible by 170.

2. Circle S lies inside circle T and touches it at A. From a point P (distinct from A) on T, chords PQ and PR of T are drawn touching S at X and Y respectively. Show that $\angle QAR = 2\angle XAY$.

3. A *tetromino* is a figure made up of four unit squares connected by common edges.
   (i) If we do not distinguish between the possible rotations of a tetromino within its plane, prove that there are seven distinct tetrominoes.
   (ii) Prove or disprove the statement: It is possible to pack all seven distinct tetrominoes into a 4×7 rectangle without overlapping.

4. Define the sequence (a_n) by
$$a_n = n + \{\sqrt{n}\}$$
where n is a positive integer and $\{x\}$ denotes the nearest integer to x, where halves are rounded up if necessary. Determine the smallest integer k for which the terms $a_k, a_{k+1}, \ldots, a_{k+2000}$ form a sequence of 2001 consecutive integers.

5. A triangle has sides of length a, b, c and its circumcircle has radius R. Prove that the triangle is right-angled if and only if $a^2 + b^2 + c^2 = 8R^2$.

2001-02 Question Paper

Wednesday, 5 December 2001

1. Find all positive integers m, n where n is odd, that satisfy
$$\frac{1}{m} + \frac{4}{n} = \frac{1}{12}.$$

2. The quadrilateral $ABCD$ is inscribed in a circle. The diagonals AC, BD meet at Q. The sides DA, extended beyond A, and CB, extended beyond B, meet at P.
Given that $CD = CP = DQ$, prove that $\angle CAD = 60°$.

3. Find all positive real solutions to the equation
$$x + \left\lfloor \frac{x}{6} \right\rfloor = \left\lfloor \frac{x}{2} \right\rfloor + \left\lfloor \frac{2x}{3} \right\rfloor,$$
where $\lfloor t \rfloor$ denotes the largest integer less than or equal to the real number t.

4. Twelve people are seated around a circular table. In how many ways can six pairs of people engage in handshakes so that no arms cross?
(Nobody is allowed to shake hands with more than one person at once.)

5. f is a function from $\mathbb{Z}^+$ to $\mathbb{Z}^+$, where $\mathbb{Z}^+$ is the set of non-negative integers, which has the following properties:–
   a) $f(n + 1) > f(n)$ for each $n \in \mathbb{Z}^+$,
   b) $f(n + f(m)) = f(n) + m + 1$ for all $m, n \in \mathbb{Z}^+$.

   Find all possible values of $f(2001)$.

Wednesday, 11 December 2002

1. Given that

$$34! = 295\ 232\ 799\ cd9\ 604\ 140\ 847\ 618\ 609\ 643\ 5ab\ 000\ 000,$$

determine the digits a, b, c, d.

2. The triangle ABC, where $AB < AC$, has circumcircle S. The perpendicular from A to BC meets S again at P. The point X lies on the line segment AC, and BX meets S again at Q.
Show that $BX = CX$ if and only if PQ is a diameter of S.

3. Let x, y, z be positive real numbers such that $x^2 + y^2 + z^2 = 1$.
Prove that

$$x^2yz + xy^2z + xyz^2 \leqslant \frac{1}{3}.$$

4. Let m and n be integers greater than 1. Consider an $m \times n$ rectangular grid of points in the plane. Some k of these points are coloured red in such a way that no three red points are the vertices of a right-angled triangle two of whose sides are parallel to the sides of the grid.
Determine the greatest possible value of k.

5. Find all solutions in positive integers a, b, c to the equation

$$a!\,b! = a! + b! + c!$$

2003-04 Question Paper

Wednesday, 3 December 2003

1. Solve the simultaneous equations

$$ab + c + d = 3, \; bc + d + a = 5, \; cd + a + b = 2, \; da + b + c = 6,$$

where a, b, c, d are real numbers.

2. $ABCD$ is a rectangle, P is the midpoint of AB, and Q is the point on PD such that CQ is perpendicular to PD.
Prove that the triangle BQC is isosceles.

3. Alice and Barbara play a game with a pack of $2n$ cards, on each of which is written a positive integer. The pack is shuffled and the cards laid out in a row, with the numbers facing upwards. Alice starts, and the girls take turns to remove one card from either end of the row, until Barbara picks up the final card. Each girl's score is the sum of the numbers on her chosen cards at the end of the game.
Prove that Alice can always obtain a score at least as great as Barbara's.

4. A set of positive integers is defined to be **wicked** if it contains no three consecutive integers. We count the empty set, which contains no elements at all, as a wicked set.
Find the number of wicked subsets of the set

$$\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}.$$

5. Let p, q and r be prime numbers. It is given that p divides $qr - 1$, q divides $rp - 1$, and r divides $pq - 1$.
Determine all possible values of pqr.

Solutions to the 2000-01 paper

1. Find all two-digit positive integers N for which the sum of the digits of $10^N - N$ is divisible by 170.

An Arithmetic Solution

Firstly, 10^N has $N + 1$ digits, and we may write

$$10^N = \underbrace{1000\ldots0000}_{N \text{ times}} = \underbrace{999\ldots9900}_{N-2 \text{ times}} + 100.$$

Subtracting the two digits of N from this gives us a number which we can write as

$$\underbrace{999\ldots99}_{N-2 \text{ times}}ab$$

where a and b are single digits such that $10a + b = 100 - N$.

The digit sum of $10^N - N$ is $9(N - 2) + a + b$.

Since a and b are single-digit numbers, $a + b$ cannot exceed $9 + 9 = 18$. Since the two-digit number N is less than 100, the digit sum $9(N - 2) + a + b$ of $10^N - N$ is less than $9 \times 100 + 18 = 918$. It follows that if the digit sum is a multiple of 170, then it must be one of 170, 340, 510, 680, 850.

We now search for values of N such that $9(N - 2)$ is between $170 - 18 = 152$ and 170, between $340 - 18 = 322$ and 340, etc.

It helps our search to note that the nearest multiple of 9 to 170 is $19 \times 9 = 171$.

Range	Multiples of 9	$N - 2$	N	'ab'$= 100 - N$	Digit Sum = $9(N-2)+a+b$
152-170	153	17	19	81	162
	162	18	20	80	170
322-340	324	36	38	62	332
	333	37	39	61	340
492-510	495	55	57	43	502
	504	56	58	42	510
662-680	666	74	76	24	672
	675	75	77	23	680
832-850	837	93	95	05	842
	846	94	96	04	850

The two-digit integers N for which the sum of the digits of $10^N - N$ is divisible by 170 are 20, 39, 58, 77 and 96.

An Algebraic Solution

We can use algebra to eliminate some of the cases considered above. To avoid worrying about 'carried' digits we note that

$$10^N - N = (10^N - 1) - (N - 1).$$

If we let $N - 1 = 10c + d$, where c and d are integers between 0 and 9, then the digit sum of $10^N - N$ is $9(10c + d - 1) + (9 - c) + 9 - d = 89c + 8d + 9$.

We are looking for digits c and d such that $89c + 8d + 9 \equiv 0 \pmod{170}$.

We note first that if c is even then the left-hand side, $89c + 8d + 9$, is odd and cannot satisfy the equation. Hence c is odd.

Let $c = 2e + 1$, where $0 \leqslant e \leqslant 4$.

Then we have (mod 170),

$$\begin{aligned} 89(2e + 1) + 8d + 9 &\equiv 178e + 8d + 98 \\ &\equiv 8e + 8d + 98 \\ &\equiv 2(4e + 4d + 49) \equiv 0. \end{aligned}$$

Since 2 and 85 are coprime factors of 170, it follows that

$$(4e + 4d + 49) \equiv 0 \pmod{85}.$$

Since $0 \leqslant e \leqslant 4$ and $0 \leqslant d \leqslant 9$

$$49 \leqslant 4e + 4d + 49 \leqslant 101$$

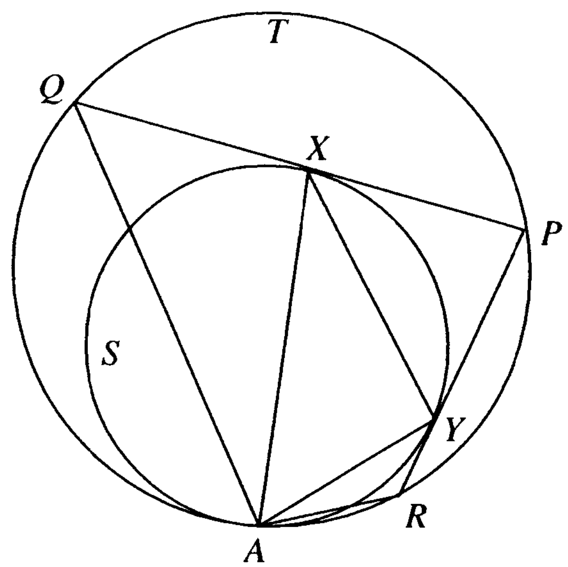
so that $4e + 4d = 85 - 49 = 36$, i.e. $e + d = 9$, offers the only solution.

Running through the possibilities, we get the five answers for N.

e	$c = 2e + 1$	$d = 9 - e$	$N = 10(2e + 1) + d + 1$
0	1	9	20
1	3	8	39
2	5	7	58
3	7	6	77
4	9	5	96

2. Circle S lies inside circle T and touches it at A. From a point P (distinct from A) on T, chords PQ and PR of T are drawn touching S at X and Y respectively. Show that $\angle QAR = 2\angle XAY$.

Solution



The situation described in the question is drawn above with the line XY added.

Since $AQPR$ is a cyclic quadrilateral, $\angle QPR = 180° - \angle QAR$.

Since PX and PY are tangents to circle S, $PX = PY$, so XPY is an isosceles triangle and $\angle PXY = \angle PYX$.

Since the angles in a triangle add up to 180°,

$$\angle PXY = \angle PYX = \frac{180° - (180° - \angle QAR)}{2} = \tfrac{1}{2}\angle QAR.$$

By the alternate segment theorem applied to circle S, $\angle PXY = \angle XAY$. Thus we have $\tfrac{1}{2}\angle QAR = \angle XAY$ and $\angle QAR = 2\angle XAY$ as required.

3. A *tetromino* is a figure made up of four unit squares connected by common edges.
   (i) If we do not distinguish between the possible rotations of a tetromino within its plane, prove that there are seven distinct tetrominoes.
   (ii) Prove or disprove the statement: It is possible to pack all seven distinct tetrominoes into a 4×7 rectangle without overlapping.

Solution

Those familiar with the game *Tetris* will know that the seven distinct shapes are as shown on the right.

We note that all possible tetrominoes can be built up from smaller configurations of connected unit squares (dominoes and triominoes). Since any unit square in a tetromino is connected by an edge to another square, any pair of adjacent squares will be connected to another square etc.

One unit square on its own looks like this. 

We have four sides to which we can connect the next unit square. Whichever we choose, the resulting shape looks like a domino and can be rotated to the shape on the right.

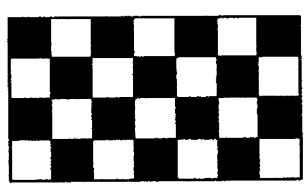
Now we have six positions at which we can add the next unit square. This results in two possible triominoes.

Adding the final unit square to the rectangular triomino in the eight possible positions yields four possible shapes, shown on the right.

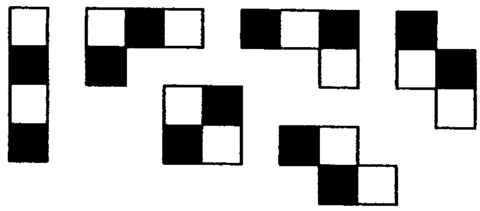
Adding a unit square to the remaining triomino yields three of the shapes above together with three more.

In this way we have systematically exhausted all the possible shapes, resulting in the seven already illustrated.

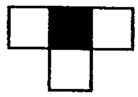
To tackle the second part of the question, we imagine the 4×7 rectangular grid coloured black and white, similar to a chessboard.



If we could fit the seven tetrominoes onto this grid then we could colour the squares of the tetrominoes black and white with no black or white squares adjacent so that fourteen of the squares are white and fourteen are black. However, if we try to colour the seven tetrominoes in this way then six of them are half black and half white (whichever colouring we choose).



The remaining shape has either three white squares and one black square or three white and one black,

i.e. either  or .

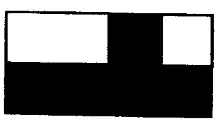
However we choose to colour the tetrominoes to match the chessboard, there will be an odd number of black squares and a different odd number of white squares. It follows that the seven tetrominoes cannot be put together to cover a 4×7 rectangle without overlapping.

A novel approach to the first part

Define a 'nice' polygon as a figure made of unit squares connected by common edges. A tetromino is a 'nice' polygon.

Let the area and the perimeter of a nice polygon be denoted by A unit^2 and P units respectively.

Consider a nice polygon positioned in such a way that its edges either go up and down or right and left. If we start at any vertex and traverse the perimeter of the polygon then we will have travelled units in the left direction the same number of times as we have travelled units in the right direction and similarly for up and down. Consequently the perimeter of a nice polygon is even. We let $P = 2m$ where m is an integer.



Rectangles give the greatest ratios of A relative to P among nice polygons. Suppose, for contradiction, that this is not true. In this case the shape with the maximum value of A relative to P will have dents, such as the one in the picture. By completing the smallest rectangle which fits around the dented shape we get a bigger area. However the perimeter is at most as big as for the dented shape since the perimeter of the dented shape must go as far up and across as the filled-in dents. This contradicts the maximality of the dented shape's area in relation to its perimeter.

Let x represent the length of one side of a rectangle, for example its width. Then, if it has perimeter $2m$, its length is $(m - x)$ and its area $x(m - x)$. By the Arithmetic Mean – Geometric Mean inequality

$$\sqrt{x(m - x)} \leqslant \frac{x + (m - x)}{2}.$$

Therefore its area satisfies the inequality $A \leqslant \left(\frac{m}{2}\right)^2 = \left(\frac{P}{4}\right)^2$. By the maximality argument above, this inequality holds for all nice polygons.

Each unit square of the nice polygon has four edges. P of these edges are on the perimeter. For a nice polygon of area A, there are at least $A - 1$ pairs of edges where one unit square meets another, since every square is connected to another. To confirm this number, imagine the following process. Start at any unit square. At any stage choose as the next unit square one which is connected to one of the squares you have already visited. Stop when there are no more squares to visit. Each time a new square is visited, a new pair of connecting edges is found. Since the area is A, there have to be $A - 1$ unit squares visited in the process, and hence $A - 1$ pairs of connecting edges. By considering the total number of edges of the unit squares, we have shown that $4A \geqslant P + 2(A - 1)$.

It follows that for all nice polygons $\left(\frac{P}{2} - 1\right) \leqslant A \leqslant \left(\frac{P}{4}\right)^2$.

A tetromino is a nice polygon with area 4. The inequality above restricts the tetromino to have perimeter 8 or 10.

For $P = 8$, there are two possibilities for rectangles, a 3×1 and a 2×2. The 2×2 rectangle is a tetromino itself. Because of area restrictions, there are no others.

For $P = 10$, there are also two possibilities for rectangles, a 3×2 and a 4×1. The 4×1 rectangle is a tetromino itself and cannot contain others. The 3×2 rectangle has area 6. Two squares must be removed from the rectangle to leave a tetromino and they must be removed in such a way that the remaining shape is connected.

The problem has now been reduced to checking a very few possibilities and it easily follows that the seven shapes already shown are the only possibilities for tetrominoes.

4. Define the sequence (a_n) by

$$a_n = n + \{\sqrt{n}\}$$

where n is a positive integer and $\{x\}$ denotes the nearest integer to x, where halves are rounded up if necessary. Determine the smallest integer k for which the terms $a_k, a_{k+1}, \ldots, a_{k+2000}$ form a sequence of 2001 consecutive integers.

We start by considering the value of $\{\sqrt{n}\}$ for a positive integer n.

If
$$\left(m - \frac{1}{2}\right)^2 \leqslant n < \left(m + \frac{1}{2}\right)^2,$$
then
$$\left(m - \frac{1}{2}\right) \leqslant \sqrt{n} < \left(m + \frac{1}{2}\right)$$
and
$$\{\sqrt{n}\} = m.$$

Hence we get a consecutive sequence of integers, $a_n = n + m$, from the sequence $a_n = n + \{\sqrt{n}\}$ whilst $m^2 - m + \frac{1}{4} \leqslant n < m^2 + m + \frac{1}{4}$, for some integer m. The sequence will increase by 2 when $n < (m + \frac{1}{2})^2 \leqslant n + 1$ since in this case $a_{n+1} = n + 1 + \{\sqrt{n+1}\} = n + 1 + m + 1 = n + m + 2$ and $a_n = n + m$.

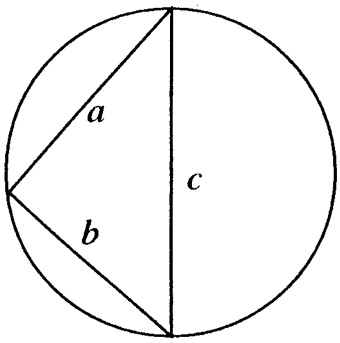
2001 consecutive integers which fall in an interval of the form $m^2 - m + \frac{1}{4} \leqslant n < m^2 + m + \frac{1}{4}$ will yield 2001 consecutive terms in the sequence a_n.

Such an interval contains $2m$ integers from $m^2 - m + 1$ to $m^2 + m$. We simply need $2m \geqslant 2001$.

Since m is an integer, the first such integer is $m = 1001$ and the smallest value of k such that $a_k, a_{k+1}, \ldots, a_{k+2000}$ form a sequence of 2001 consecutive integers is $k = 1001^2 - 1001 + 1 = 1001001$.

5. A triangle has sides of length a, b, c and its circumcircle has radius R. Prove that the triangle is right-angled if and only if $a^2 + b^2 + c^2 = 8R^2$.

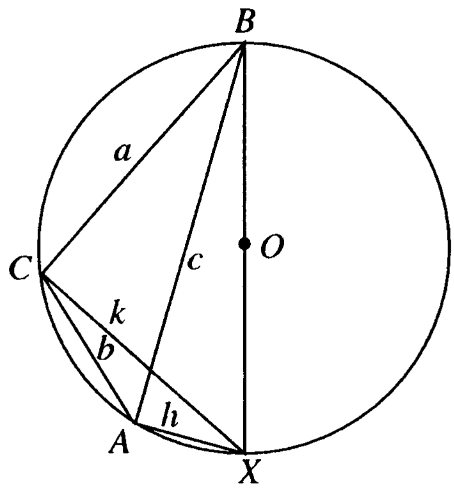
Solution



If the triangle is right-angled then the longest side, c say, is the diameter of the circumcircle, since the angle subtended at the centre is twice the angle at the circumference, i.e. $2 \times 90° = 180°$. In this case, $c = 2R$ and, by Pythagoras' Theorem, $a^2 + b^2 = c^2$.
Hence $a^2 + b^2 + c^2 = 2c^2 = 2 \times (2R)^2 = 8R^2$ as required.

It remains to prove the converse: if $a^2 + b^2 + c^2 = 8R^2$ then the triangle is right-angled.

Geometric Proof (of the converse)

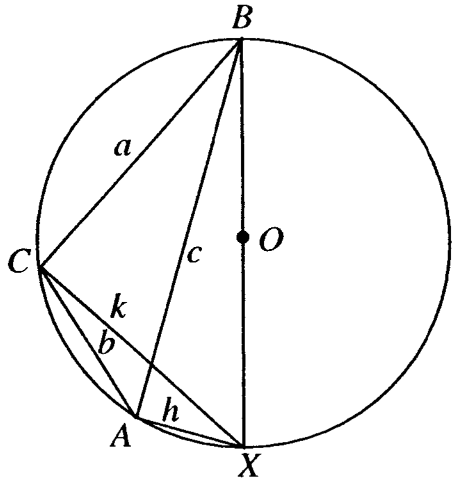


Label the triangle ABC as in the diagram. Draw in the diameter passing from B through the centre of the circle and label its other end X (this is called the *antipodal* point of B). Now add the lines AX and CX. We label their lengths $|AX| = h$ and $|CX| = k$. Note that if $h = 0$ then points A and X are coincident, so that BA is a diameter and $\angle ACB = 90°$. Similarly if $k = 0$ then points C and X are coincident, so that BC is a diameter and $\angle CAB = 90°$. We need only consider the case when $k \neq 0$ and $h \neq 0$.

The triangles BCX and BAX are right-angled since BX is a diameter. Hence, by Pythagoras' Theorem, $a^2 + k^2 = (2R)^2$ and $c^2 + h^2 = (2R)^2$. Adding these two equations together gives $a^2 + c^2 + h^2 + k^2 = 8R^2$.

If the condition of the question is met, i.e. $a^2 + b^2 + c^2 = 8R^2$, then, since $a^2 + c^2 + h^2 + k^2 = 8R^2$, we have $h^2 + k^2 = b^2$. But h, k and b are the sides of the triangle ACX. It follows, by the converse of Pythagoras' Theorem, that ACX is a right-angled triangle with the right angle at X, i.e. the chord CA subtends a right angle at the circumference of the circle. Hence CA is a diameter and $\angle CBA = 90°$ as required.

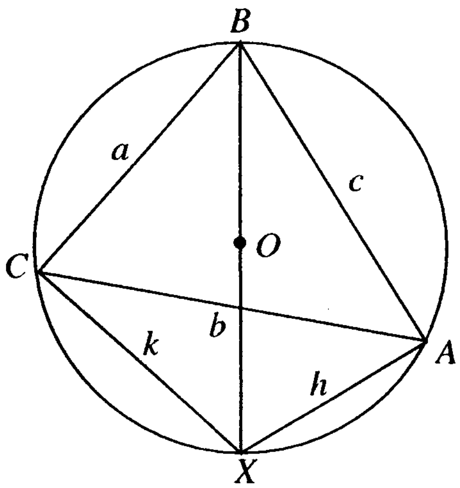
Dr Smith, University of Bath and UK IMO team leader for 2002, has observed that the value of $a^2 + b^2 + c^2$ in comparison with $8R^2$ serves to discriminate between acute, right-angled and obtuse triangles.



If the triangle is obtuse then the circumcentre lies outside the triangle. Label the obtuse angle C and the antipodal point to B, X as before. The angles $\angle CXA$ and $\angle CBA$ are subtended by the same chord, AC, so that they are equal. Since $\angle CBA$ is acute, so is $\angle CXA$. Hence $b^2 < h^2 + k^2$ and

$$a^2 + b^2 + c^2 < a^2 + \left(h^2 + k^2\right) + c^2 = 8R^2$$

where $k = CX$.

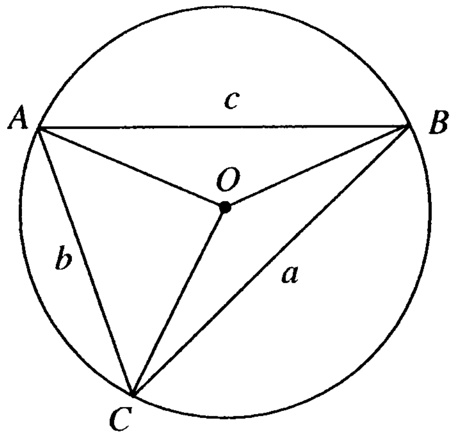


If the triangle is acute then the circumcentre lies inside the triangle and X, the antipodal to point B, lies on the arc of the circle between C and A. The angle $\angle CXA$ is opposite $\angle CBA$ in the cyclic quadrilateral $ABCX$. Since $\angle CBA$ is acute, $\angle CXA$ is obtuse. Hence $b^2 > h^2 + k^2$ and

$$a^2 + b^2 + c^2 > a^2 + \left(h^2 + k^2\right) + c^2 = 8R^2$$

where $b = AC$.

Trigonometric Proof (of the converse)



Draw in the radii, AO, BO and CO, from the vertices of triangle ABC to O, the centre of the circumcircle. We will refer to the angles of the triangle themselves simply as A, B and C. The triangles BOC, COA and AOB are each isosceles with equal sides of length R and with angles $\angle BOC = 2A$, $\angle COA = 2B$ and $\angle AOB = 2C$.

Applying the cosine rule to triangle BOC, we get

$$a^2 = R^2 + R^2 - 2R^2 \cos 2A = 2R^2(1 - \cos 2A).$$

Similarly $b^2 = 2R^2(1 - \cos 2B)$ and $c^2 = 2R^2(1 - \cos 2C)$. Hence, since $a^2 + b^2 + c^2 = 8R^2$, we have

$$2R^2(1 - \cos 2A + 1 - \cos 2B + 1 - \cos 2C) = 8R^2$$

$$\text{i.e. } \cos 2A + \cos 2B + \cos 2C + 1 = 0.$$

Now $A + B + C = 180°$ (angle sum of a triangle) so that

$$\cos 2C = \cos(360° - 2(A + B)) = \cos 2(A + B).$$

Applying the double-angle formula for cosine, we get

$$\cos 2C + 1 = 2\cos^2(A + B).$$

We also make use of the trigonometric addition formula

$$\cos\theta + \cos\phi = 2\cos\left(\frac{\theta + \phi}{2}\right)\cos\left(\frac{\theta - \phi}{2}\right).$$

Then

$$\begin{aligned}
& \cos 2A + \cos 2B + \cos 2C + 1 = 0 \\
\Rightarrow\quad & 2\cos(A + B)\cos(A - B) + 2\cos^2(A + B) = 0 \\
\Rightarrow\quad & 2\cos(A + B)(\cos(A - B) + \cos(A + B)) = 0 \\
\Rightarrow\quad & \cos(A + B)\, 2\cos A \cos B = 0 \\
\Rightarrow\quad & A = 90°,\ B = 90° \text{ or } A + B = 90° \Rightarrow C = 90°.
\end{aligned}$$

Therefore the triangle contains a right angle as required.

Solutions to the 2001-02 paper

1. Find all positive integers m, n where n is odd, that satisfy

$$\frac{1}{m} + \frac{4}{n} = \frac{1}{12}.$$

There are several approaches to this question, all of which eventually boil down to considering cases. It is worth noting straight away that since m and n are positive and $\frac{1}{m} + \frac{4}{n} = \frac{1}{12}$ then $\frac{1}{m} < \frac{1}{12}$ and $\frac{4}{n} < \frac{1}{12}$.
Hence $m > 12$ and $n > 48$.

Solution based on factorisation
Multiply both sides of the given equation by $12mn$, then rearrange in such a way that the expression can be factorised. We have

$$\begin{aligned} \text{i.e.} \quad 12n + 48m &= mn \\ \text{i.e.} \quad mn - 12n - 48m &= 0 \\ \text{i.e.} \quad mn - 12n - 48m + 576 &= 576 \\ \text{i.e.} \quad (m - 12)(n - 48) &= 576. \end{aligned}$$

Hence $m - 12$ and $n - 48$ are integer factors of $576 = 12 \times 48 = 2^6 \times 3^2$.

Since $m > 12$ and $n > 48$, $m - 12$ and $n - 48$ are positive integer factors of 576.

Further, since n is odd, $n - 48$ is an odd positive integer factor of 576.

This limits the possible values of $n - 48$ to 1, 3 and 9, with corresponding values of $m - 12$ being 576, 192 and 64 respectively.

The possible values of (m, n) are (588, 49), (204, 51) and (76, 57).

Checking each of these pairs in turn shows that each is a valid solution.

Solution by making m or n the subject
Rearranging the given equation to make m the subject gives

$$m = \frac{12n}{n - 48}.$$

Since m is an integer, $n - 48$ is a factor of $12n$. It follows that $n - 48$ is a factor of $12n - 12(n - 48) = 576$. The proof now follows as above.

Rearranging the given equation to make n the subject gives

$$n = \frac{48m}{m - 12}.$$

Since n is an integer, $m - 12$ is a factor of $48m$. It follows that $m - 12$ is a factor of $48m - 48(m - 12) = 576$. There are lots of possible values of $m - 12$ amongst the even factors of 576 to consider to complete the proof.

Solution using a different reductive approach

Rearrange the given equation to get $n(m - 12) = 48m$. Since n is odd and the right hand side is divisible by 4, it follows that $m - 12$ is divisible by 4. Hence m is divisible by 4.

If we let $m = 4k$, where k is an integer, then the equation rearranges to

$$\frac{48}{n} = \frac{k - 3}{k}.$$

Thus $\frac{48}{n}$ is equivalent to a fraction of positive integers with its denominator three greater than its numerator.

We know that n is odd and $n > 48$.

We first consider $n = 49$, where the difference between numerator and denominator is 1. There is one fraction which is equivalent to $\frac{48}{49}$ with difference 3, namely $\frac{144}{147}$. This gives $n = 49$, $k = 147$ and $m = 588$.

Next consider $n = 51$. The difference between denominator and numerator in the fraction $\frac{48}{51}$ is 3. This gives $k = 51$ and $m = 204$.

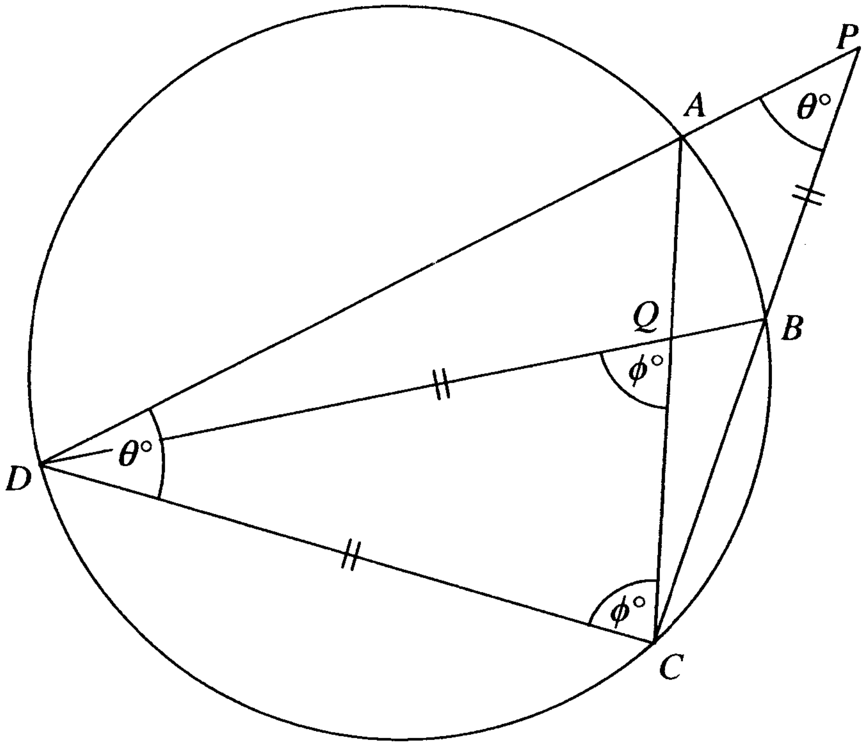
If $n > 51$, then we need to cancel the fraction $\frac{48}{n}$ down to find an equivalent fraction with a reduced difference between numerator and denominator. Since n is odd and $48 = 3 \times 2^4$, there is just one choice of reducing factor, 3. This must correspond to $\frac{48}{n} = \frac{16}{19}$, which yields the third solution, $n = 3 \times 19 = 57$, $m = 4 \times 19 = 76$.

2. The quadrilateral $ABCD$ is inscribed in a circle. The diagonals AC, BD meet at Q. The sides DA, extended beyond A, and CB, extended beyond B, meet at P.

   Given that $CD = CP = DQ$, prove that $\angle CAD = 60°$.

Solution

The situation described in the question is drawn below.



We will express angles in the diagram in terms of two of the angles, $\angle ACD$ and $\angle ADC$. We let $\angle ACD = \phi°$ and $\angle ADC = \theta°$. Then

$\angle DQC = \angle ACD = \phi°$ (isosceles triangle CDQ)

$\angle BDC = 180° - 2\phi°$ (angle sum in triangle CDQ)

$\angle CPD = \angle ADC = \theta°$ (isosceles triangle CPD)

$\angle BCD = 180° - 2\theta°$ (angle sum in triangle CPD)

$\angle CAD = 180° - \phi° - \theta°$ (angle sum in triangle CAD)

$\angle CBD = \angle CAD = 180° - \phi° - \theta°$ (angles subtended by the same chord, CD)

Now consider the angle sum of the triangle BCD:

$$\angle BDC + \angle BCD + \angle CBD = 180°$$
$$180° - 2\phi° + 180° - 2\theta° + 180° - \phi° - \theta° = 180°$$
$$3(\theta° + \phi°) = 360°$$
$$\phi° + \theta° = 120°.$$

Thus
$$\angle CAD = 180° - \theta° - \phi°$$
$$= 180° - 120° = 60°.$$

3. Find all positive real solutions to the equation

$$x + \left\lfloor \frac{x}{6} \right\rfloor = \left\lfloor \frac{x}{2} \right\rfloor + \left\lfloor \frac{2x}{3} \right\rfloor,$$

where $\lfloor t \rfloor$ denotes the largest integer less than or equal to the real number t.

Solution

We note first that, if x is a solution, then $x = \left\lfloor \frac{x}{2} \right\rfloor + \left\lfloor \frac{2x}{3} \right\rfloor - \left\lfloor \frac{x}{6} \right\rfloor$. The right hand side of this equation is a sum of integers and so x is an integer.

We also note that the lowest common denominator of the fractions is 6. Let $x = 6k + t$, where t can be 0, 1, 2, 3, 4, or 5 (t is the remainder when x is divided by 6), and consider the two sides of the equation.

t	x	$\left\lfloor \frac{x}{6} \right\rfloor$	$x + \left\lfloor \frac{x}{6} \right\rfloor$	$\left\lfloor \frac{x}{2} \right\rfloor$	$\left\lfloor \frac{2x}{3} \right\rfloor$	$\left\lfloor \frac{x}{2} \right\rfloor + \left\lfloor \frac{2x}{3} \right\rfloor$
0	$6k$	k	$7k$	$3k$	$4k$	$7k$
1	$6k + 1$	k	$7k + 1$	$3k$	$4k$	$7k$
2	$6k + 2$	k	$7k + 2$	$3k + 1$	$4k + 1$	$7k + 2$
3	$6k + 3$	k	$7k + 3$	$3k + 1$	$4k + 2$	$7k + 3$
4	$6k + 4$	k	$7k + 4$	$3k + 2$	$4k + 2$	$7k + 4$
5	$6k + 5$	k	$7k + 5$	$3k + 2$	$4k + 3$	$7k + 5$

The two sides of the equation are equal in all cases except when $t = 1$. We conclude that the solutions to the equation are all positive integers except those that have remainder 1 when divided by 6.

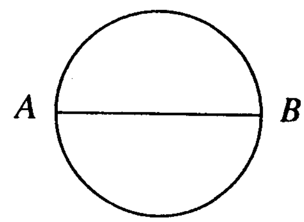
4. Twelve people are seated around a circular table. In how many ways can six pairs of people engage in handshakes so that no arms cross?
(Nobody is allowed to shake hands with more than one person at once.)

Solution

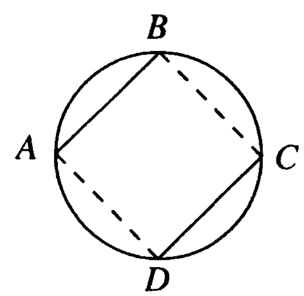
In attempting this question we understand that all handshakes take place between people leaning across the table. There is no pair of people shaking hands behind someone else's back.

We will build up from small cases. Let N_p be the number of ways in which $2p$ people seated around a circular table can shake hands in pairs across the table in such a way that no arms cross, with everyone involved in exactly one handshake. It will be helpful to nominate a person seated at the table as a reference point. Without loss of generality we will refer to this person as King Arthur, and label his position A. Since handshakes occur in pairs, the number of people at the table on any side of King Arthur and his handshaking partner must be zero or even.

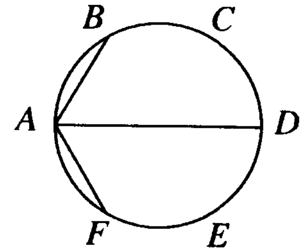
If there are two people, there is only one way in which they can shake hands, so that $N_1 = 1$.



With four people, King Arthur can shake hands with either of his immediate neighbours (B or D in the diagram) but cannot shake hands with his opposite (C), otherwise there would be a crossing pair. We have $N_2 = 2$.



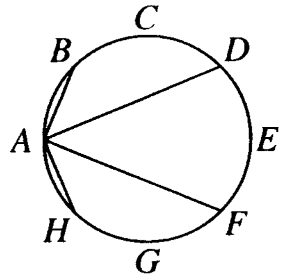
When there are six people, King Arthur can shake hands with either of his neighbours (B and F in the diagram), leaving four people on one side and N_2 ways for them to shake hands, or he can shake hands with D, the person opposite, leaving a pair of people on each side, BC and FE, each with N_1 ways of shaking hands.



This gives $N_3 = 2N_2 + N_1^2 = 5$.

We have established a progressive way to approach the problem which will lead us to N_6.

When there are eight people, King Arthur can shake hands with either of his neighbours (B or H in the diagram), leaving six people on one side who can shake hands in N_3 ways, or he can shake hands with D or F, which would leave two people on one side, with N_1 ways of shaking hands, and four on the other, with N_2 ways.



Hence $N_4 = 2N_3 + 2N_1 \times N_2 = 14$.

Similarly $N_5 = 2N_4 + 2N_3 \times N_1 + N_2^2 = 42$
and $N_6 = 2N_5 + 2N_4 \times N_1 + 2N_3 \times N_2 = 132$.

The number of ways in which six pairs of people seated around a circular table can engage in handshakes so that no arms cross is 132.

The numbers in the sequence 1, 2, 5, 14, 42, 132, ... are known as the Catalan Numbers. If we define $N_0 = 1$ then this sequence is generated by the formula
$$N_k = \sum_{r=0}^{k-1} (N_r \times N_{k-r-1}).$$

5. f is a function from $\mathbb{Z}^+$ to $\mathbb{Z}^+$, where $\mathbb{Z}^+$ is the set of non-negative integers, which has the following properties:–
   a) $f(n+1) > f(n)$ for each $n \in \mathbb{Z}^+$,
   b) $f(n + f(m)) = f(n) + m + 1$ for all $m, n \in \mathbb{Z}^+$.

   Find all possible values of $f(2001)$.

Solution 1

Let $f(0) = k$, where k is a non-negative integer. Then condition (b) gives

$$f(n+k) = f(n) + 1. \tag{1}$$

If $k = 0$ then $f(n) = f(n) + 1$, which is a contradiction. Hence $k \neq 0$.

We note from condition (a) that

$$f(n+k-1) < f(n+k) = f(n) + 1. \tag{2}$$

However, if $k > 1$, then $n + k - 1 \geqslant n + 1$ so that by (a)

$$f(n+k-1) \geqslant f(n+1) \geqslant f(n) + 1. \tag{3}$$

(2) and (3) are contradictory. Hence $k = 1$ is the only possible solution.

When $k = 1$, (1) gives $f(n+1) = f(n) + 1$. This function satisfies conditions (a) and (b) and yields the unique solution $f(2001) = 2002$.

Solution 2

There are alternative ways to establish that $k = 1$. For example, substituting $n = 0$ into (1) gives

$$f(k) = k + 1 \tag{4}$$

and substituting $n = 1$ into (1) gives

$$f(1+k) = f(1) + 1. \tag{5}$$

Substituting $n = 0$ and $m = k$ into condition (b) we get

$$f(0 + f(k)) = f(0) + k + 1$$

$$\text{i.e.} \quad f(k+1) = 2k + 1. \tag{6}$$

From (5) and (6), $f(1) = 2k$.

However, if $k > 1$, then condition (a) states that $f(k) > f(1)$ so that $k + 1 > 2k$, which is a contradiction. Thus $k = 1$ and $f(2001) = 2002$ as above.

The following proof relies on condition (b) alone.

Solution 3

Substituting $n - 1$ for n and k for m into condition (b) we get

$$f\big(n - 1 + f(k)\big) = f(n - 1) + k + 1$$

$$\text{i.e.} \qquad f(n + k) = f(n - 1) + k + 1. \qquad (7)$$

From (1) and (7) $\qquad f(n) + 1 = f(n - 1) + k + 1$

so that $\qquad f(n) = f(n - 1) + k$

for all $n \in \mathbb{Z}^+$. Using this repeatedly gives

$$f(n) = f(n-1) + k = \big[f(n-2) + k\big] + k = f(n-2) + 2k$$

$$= f(n-3) + 3k = \ \ldots \ = f(0) + nk$$

$$\text{i.e.} \qquad f(n) = (n + 1)k. \qquad (8)$$

Substituting into (8) gives $f(k) = (k + 1)k$.

From (4), $f(k) = k + 1$, so that we have

$$k + 1 = k(k + 1)$$

$$\text{i.e.} \qquad k = \pm 1.$$

Since $f(0) = k$ is a non-negative integer, $k = 1$.

Hence $f(2001) = (2001 + 1) \times 1 = 2002$.

We note that condition (a) is satisfied by this value of k.

Solutions to the 2002-03 paper

1. Given that

$$34! = 295\,232\,799\,cd9\,604\,140\,847\,618\,609\,643\,5ab\,000\,000,$$

determine the digits a, b, c, d.

There are many different solutions to this problem, but all use the same essential ideas: to consider which numbers must divide 34!.

Almost all solvers started by considering divisibility by 5 (and 2) in order to establish that $b = 0$. From here one can use reduction mod 8 or mod 10 to obtain the value of a. To obtain c and d, one may either work mod 9 and mod 11, or combine the two approaches by working mod 99. The following two solutions illustrate these different approaches.

Solution 1

$34! = K.11^3.7^4.5^7.3^{15}.2^{32} = K.11^3.7^4.3^{15}.2^{25}.10^7$ for some positive integer K. Hence $b = 0$, $a \neq 0$. Leaving aside the last seven digits which are all zero, the remaining digits must form a number divisible by 2^{25} and hence by 8. The condition for this (since 1000 is divisible by 8) is that the last three digits form a number divisible by 8; and since $352 = 8 \times 44$ while none of 354, 356 or 358 is divisible by 8, it follows that $a = 2$.

To ensure divisibility by 9, we require that the sum of the digits is divisible by 9, whence $141 + c + d \equiv 0 \bmod 9$, giving $c + d \equiv 3 \bmod 9$. To ensure divisibility by 11 we require that the difference between the sum of the odd-placed digits and the sum of the even-placed digits is divisible by 11, whence $80 + d \equiv 61 + c \bmod 11$. From these two congruences it is easy to deduce that $c = 0$ and $d = 3$.

Solution 2

$$5^7 \mid 5 \times 10 \times 15 \times 20 \times 25 \times 30 \mid 34!$$

$$\text{and } 2^7 \mid 2 \times 4 \times 6 \times 8 \mid 34!$$

$$\therefore \text{ as } 2^7 \text{ and } 5^7 \text{ are coprime, } 2^7 \times 5^7 = 10^7 \mid 34!.$$

Hence the decimal representation of 34! ends in 7 zeroes, so the digit b is zero. We can compute $\frac{34!}{10^7}$ by removing these factors of 2^7 and 5^7, leaving

$$\frac{34!}{10^7} = 1 \times 1 \times 3 \times 1 \times 1 \times 3 \times 7 \times 1 \times 9 \times 2 \times (11 \times 12 \times 13 \times 14)$$

$$\times 3 \times (16 \times 17 \times 18 \times 19) \times 4 \times (21 \times 22 \times 23 \times 24) \times 1$$

$$\times (26 \times 27 \times 28 \times 29) \times 6 \times (31 \times 32 \times 33 \times 34).$$

Modulo 10, this is congruent to

$$3 \times 3 \times 7 \times 9 \times 2 \times (1 \times 2 \times 3 \times 4) \times 3 \times (6 \times 7 \times 8 \times 9)$$

$$\times 4 \times (1 \times 2 \times 3 \times 4) \times (6 \times 7 \times 8 \times 9) \times 6 \times (1 \times 2 \times 3 \times 4)$$

$$\equiv 567 \times (1 \times 2 \times 3 \times 4)^3 \times (6 \times 7 \times 8 \times 9)^2 \times 2 \times 3 \times 4 \times 6$$

$$\equiv 7 \times 4^3 \times 4^2 \times 4$$

$$\equiv 7 \times 4096$$

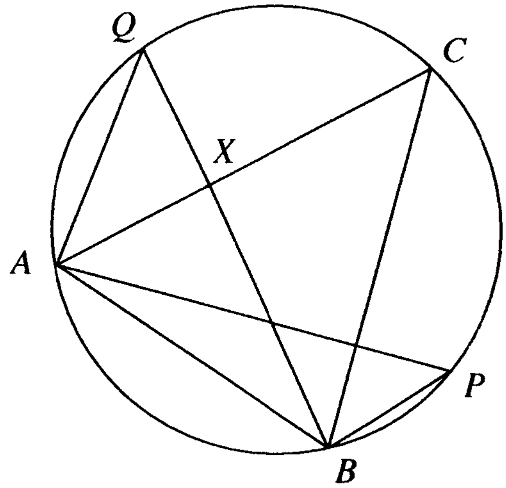
$$\equiv 2.$$

Now as $\dfrac{34!}{10^7} \equiv 2 \bmod 10$, the eighth digit from the right of 34! is 2, i.e. $a = 2$.

$9 \times 11 = 99 \mid 34!$, so adding the digits of the base 100 representation of 34! must give a multiple of 99. As $a = 2$, $b = 0$, this sum is $690 + 10c + d \leqslant 789$ as $10c + d \leqslant 99$. The only multiple of 99 between 690 and 789 is 693, so $c = 0$ and $d = 3$.

2. The triangle ABC, where $AB < AC$, has circumcircle S. The perpendicular from A to BC meets S again at P. The point X lies on the line segment AC, and BX meets S again at Q.

   Show that $BX = CX$ if and only if PQ is a diameter of S.



Solution 1

$$\begin{aligned} PQ \text{ is a diameter} \quad &\Leftrightarrow \quad \angle PAQ = 90^\circ \\ &\Leftrightarrow \quad BC \text{ is parallel to } AQ \\ &\Leftrightarrow \quad \angle AQB = \angle QBC. \end{aligned}$$

Since $\angle AQB = \angle ACB$ (angles in the same segment), if PQ is a diameter, $\angle ACB = \angle QBC$, so ΔBXC is isosceles, with $BX = CX$.

From the reasoning above, if PQ is not a diameter, $\angle AQB \neq \angle QBC$ so $\angle XBC \neq \angle XCB$, hence ΔBXC is not isosceles and $BX \neq CX$.

Solution 2

Join BP.

$$\angle CAP = \angle CBP \text{ (angles in the same segment)}$$

$$\begin{aligned} \text{and} \quad & \angle CAP + \angle BCA = 90^\circ. \\ \therefore \quad & \angle CBP + \angle BCA = 90^\circ. \\ \text{But} \quad & \angle QBC + \angle CBP = \angle QBP. \\ \therefore \text{ subtracting} \quad & \angle BCA - \angle QBC = 90^\circ - \angle QBP. \end{aligned}$$

Hence

$$\begin{aligned} PQ \text{ is a diameter of } S \quad &\Leftrightarrow \quad \angle QBP = 90^\circ \\ &\Leftrightarrow \quad \angle BCA = \angle QBC \\ &\Leftrightarrow \quad BX = CX. \end{aligned}$$

3. Let x, y, z be positive real numbers such that $x^2 + y^2 + z^2 = 1$.
Prove that

$$x^2yz + xy^2z + xyz^2 \leqslant \frac{1}{3}.$$

Solution

We have GM $\leqslant$ AM $\leqslant$ QM, where these are the geometric, arithmetic and quadratic means respectively.

[*The quadratic mean of* $x_1, \ldots, x_n$ *is defined as* $\sqrt{\frac{1}{n}(x_1^2 + \ldots + x_n^2)}$. *By the Cauchy-Schwarz inequality,*

$$(x_1 + \ldots + x_n)^2 \leqslant \left(1^2 + \ldots + 1^2\right)\left(x_1^2 + \ldots + x_n^2\right)$$

which rearranges straightforwardly into AM $\leqslant$ QM.]

In this particular case,

$$\sqrt[3]{xyz} \leqslant \frac{x + y + z}{3} \leqslant \sqrt{\frac{x^2 + y^2 + z^2}{3}} = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \begin{cases} xyz \leqslant \frac{1}{3\sqrt{3}} & (1) \\ x + y + z \leqslant \sqrt{3}. & (2) \end{cases}$$

(1) and (2) together imply

$$xyz(x + y + z) \leqslant \left(\frac{1}{3\sqrt{3}}\right)\left(\sqrt{3}\right)$$

$$\text{or} \qquad x^2yz + xy^2z + xyz^2 \leqslant \frac{1}{3}$$

as required.

4. Let m and n be integers greater than 1. Consider an $m \times n$ rectangular grid of points in the plane. Some k of these points are coloured red in such a way that no three red points are the vertices of a right-angled triangle two of whose sides are parallel to the sides of the grid.

   Determine the greatest possible value of k.

(This problem was found to be quite difficult.)

Solution

Suppose that the grid has m rows and $n \leqslant m$ columns. First note that the greatest possible value is at least $m + n - 2$ since this can be achieved by selecting a particular row and a particular column and colouring red all the dots in this row and column except the one which they have in common.

To show that this is the best possible, suppose that $m + n - 1$ points can be found which satisfy the condition. By the pigeonhole principle they cannot be all in the same row or all in the same column; so there exist two points that are not in the same row and not in the same column.

Now, let us consider adding in the remaining $m + n - 3$ points. Each one must either be in a row that has no other red points, or a column with no other red points, since otherwise we would have a triangle. So each time we add a point, we start on a fresh row or a fresh column.

There are $m - 2$ rows which can be used for additional points, and $n - 2$ columns, and each new point will use up at least one; so the maximum number of points that may be added is $2 + (m - 2) + (n - 2) = m + n - 2$, which is a contradiction.

5. Find all solutions in positive integers a, b, c to the equation

$$a!\,b! = a! + b! + c!$$

Solution

Without loss of generality suppose that $a \geqslant b$. Then the equation can be written

$$a! = \frac{a!}{b!} + 1 + \frac{c!}{b!}.$$

Since three of the terms are integers it follows that the fourth must be as well so $c \geqslant b$. Now since all the terms on the right-hand side are positive integers their sum is at least 3; so $a \geqslant 3$, and $a!$ is even. It now follows that exactly one of $a!/b!$, $c!/b!$ is odd.

Suppose that $\frac{a!}{b!}$ is odd. Then either $a = b$ or $\frac{a!}{b!} = b + 1$, where b is even and $a = b + 1$.

Consider first the case $a = b$. Then $a! = 2 + \frac{c!}{a!}$. If $a = 3$, then $a = b = 3$, $c = 4$ is a solution. If $a > 3$ then $a!$ is divisible by 3 so $a! - 2$ is not, whence c is either $a + 1$ or $a + 2$. Hence $c!/a!$ is either $(a + 1)$ or $(a + 1)(a + 2)$, in which case $a! = a + 3$ or $a! = (a + 1)(a + 2) + 2$. Checking cases we find that $a = 4$ and $a = 5$ do not fit and, for larger a, the left-hand side is much bigger than the right.

Now consider $a = b + 1$ with b even. Then we have to solve

$$(b + 1)! = b + 2 + \frac{c!}{b!}.$$

If $c = b$ we require $(b + 1)! = b + 3$ which has no solutions. Suppose then that $c > b$. The left-hand side is divisible by $b + 1$, as is $c!/b!$. Thus $b + 1$ must divide $b + 2$, a contradiction.

We are left with the case where $a!/b!$ is even and $c!/b!$ is odd. Then $c = b$, or $c = b + 1$ with b even.

If $c = b$ then the equation becomes $a!b! = a! + 2b!$ so $\frac{a!}{b!}(b! - 1) = 2$. Since $\frac{a!}{b!}$ is an even integer we must have $b! - 1 = 1$, so $b = 2$, $a! = 4$ which is impossible.

If $c = b + 1$ then the equation becomes $a!\,b! = a! + (b + 2)b!$, so $a!(b! - 1) = (b + 2)b!$. As $\mathrm{HCF}(b!, b! - 1) = 1$, it follows that $(b! - 1) \mid (b + 2)$; the only possibility for this is $b = 2$, which would give $a! = 8$, a contradiction.

So we have considered all cases, and the only possible solution is (3, 3, 4).

Solutions to the 2003-04 paper

1. Solve the simultaneous equations

$$ab + c + d = 3,\ bc + d + a = 5,\ cd + a + b = 2,\ da + b + c = 6,$$

where a, b, c, d are real numbers.

Most successful solutions relied, sensibly, on exploiting the symmetry in the given system of equations.

One possible solution is:

We number the equations as follows:

$$ab + c + d = 3 \qquad (1)$$
$$bc + d + a = 5 \qquad (2)$$
$$cd + a + b = 2 \qquad (3)$$
$$da + b + c = 6. \qquad (4)$$

We can subtract pairs of equations to derive

$$\begin{aligned} (2) - (3) : &\quad (b - d)(c - 1) = 3 &\quad (5)\\ (4) - (1) : &\quad (d - b)(a - 1) = 3 &\quad (6)\\ (5) - (6) : &\quad (b - d)(c + a - 2) = 0. &\quad (7) \end{aligned}$$

Thus, from (7), either $b = d$ or $c + a = 2$. But the former contradicts (5), so $c + a = 2$. We substitute this into (1) + (2):

$$\begin{aligned} (1) + (2) : \quad (a + c)b + a + c + 2d &= 8\\ \text{i.e.} \qquad 2b + 2d &= 6\\ \text{i.e.} \qquad b + d &= 3. \end{aligned}$$

Thus $a + b + c + d = 5$, so using equations (1) and (2), we get $bc = b + c$, and $ab + 2 = a + b$. Then

$$\begin{aligned} (5) + (6) : &\quad (b - d)(c - a) = 6 &\quad (8)\\ (4) - (3) : &\quad (d - 1)(a - c) = 4 &\quad (9)\\ (2) - (1) : &\quad (b - 1)(c - a) = 2, \text{ whence} &\\ &\quad (2 - 2b)(a - c) = 4. &\quad (10) \end{aligned}$$

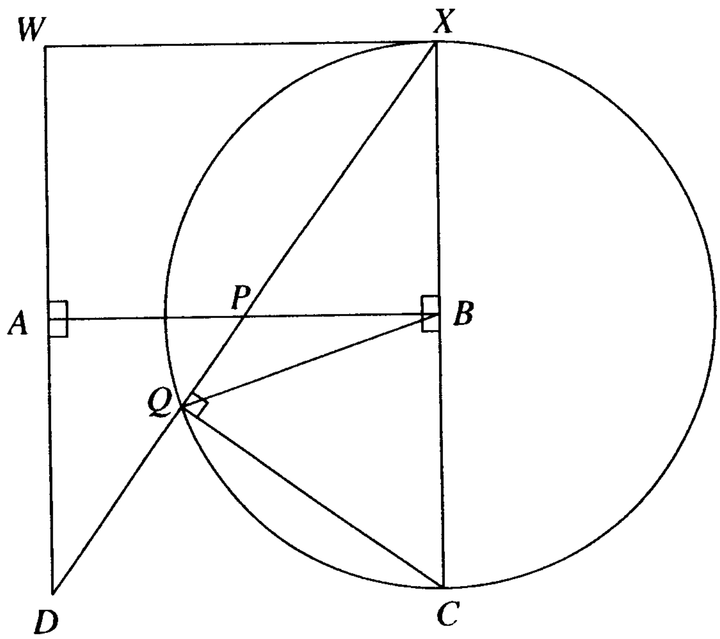
From (9) and (10), $2 - 2b = d - 1$, as $a - c \neq 0$.

Since $d = 3 - b$, $2 - 2b = 3 - b - 1$, so $b = 0$. Hence $d = 3$. Then from (8) we get $-3(c - a) = 6$, so $a - c = 2$. But we showed above that $a + c = 2$, so $a = 2$, $c = 0$. A quick check reveals that $a = 2$, $b = 0$, $c = 0$, $d = 3$ is a solution, and thus the only one.

2. $ABCD$ is a rectangle, P is the midpoint of AB, and Q is the point on PD such that CQ is perpendicular to PD.
   Prove that the triangle BQC is isosceles.

Solution 1

Construct the rectangle $WXBA$ that is the reflection of $DCBA$ in the edge AB, as shown below. Now P is the centre of the rectangle $WXCD$, and so DP extends to X.



Now construct the circle with centre B and radius $BC = BX$, so that CX is a diameter. Since the triangle XQC is right-angled at Q, it follows that Q also lies on this circle. Therefore $BQ = BC$, the radius of the circle, and so ΔBQC is isosceles.

Solution 2

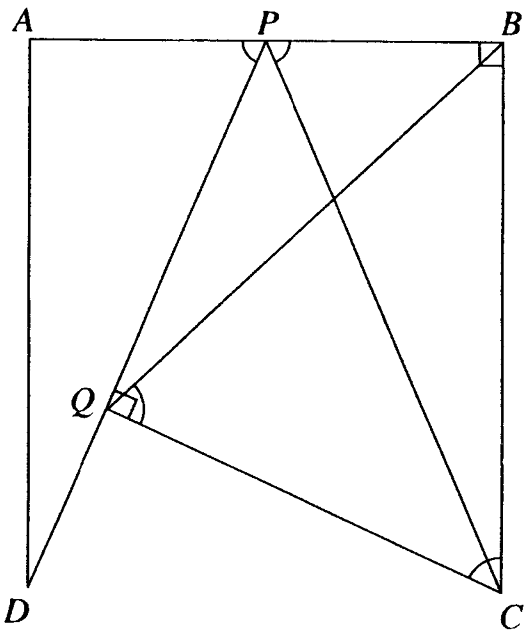
Let R be the mid-point of DC. Now $PB = DR$, both being half the side-length AB of the rectangle, and $PD = BR$, since triangles APD, CRB are congruent, so $PBRD$ is a parallelogram. Also, BR is parallel to PD and CQ is perpendicular to PD, hence also to BR.

Let S be the point where CQ and BR intersect. Triangles RCS and DCQ are both right-angled and share $\angle RCS$, so are similar. Since $RC = \frac{1}{2}DC$, it follows that $SC = \frac{1}{2}QC$; that is, S is the mid-point of CQ.

Finally, we see that triangles BSQ and BSC share side BS, have $SQ = SC$ and both have a right-angle between these sides, so are congruent by the 'SAS' rule. Therefore $BQ = BC$, so ΔBQC is isosceles.

Solution 3

Consider the diagram below, with the line PC drawn in.



As P is the midpoint of AB, $AP = PB$. Also, since ABCD is a rectangle, $AD = BC$ and $\angle DAB = \angle ABC = 90°$. It follows from the 'SAS' rule that triangles APD, BPC are congruent, and so $\angle APD = \angle BPC$.

Since $\angle PBC$ and $\angle PQC$ are both right angles, they add up to 180°, and so $PBCQ$ is a cyclic quadrilateral. Therefore $\angle BQC = \angle BPC$ (angles in the same segment) and $\angle QCB = \angle APD$ (equality of opposite and exterior angles). Since we have seen that $\angle APD = \angle BPC$, it follows that $\angle BQC = \angle QCB$, and so ΔBQC is isosceles.

3. Alice and Barbara play a game with a pack of $2n$ cards, on each of which is written a positive integer. The pack is shuffled and the cards laid out in a row, with the numbers facing upwards. Alice starts, and the girls take turns to remove one card from either end of the row, until Barbara picks up the final card. Each girl's score is the sum of the numbers on her chosen cards at the end of the game.

   Prove that Alice can always obtain a score at least as great as Barbara's.

Solution

Given the cards laid out in a row, we imagine them coloured alternately black and white. Initially the two end cards are of opposite colours. If Alice takes the black end card, then Barbara has a choice of two white cards, and after her move there will again be one white and one black end card, so Alice has a choice again; similarly if Alice initially takes a white end card. In this way Alice can force Barbara to take a card of a particular colour every time.

Let S_0 be the sum of the values on the white cards and S_1 the sum of the values on the black cards. Alice selects the colour corresponding to the larger of these sums (if they are equal she selects a colour arbitrarily), and now chooses the card of that colour available to her at each stage; from the above argument, it follows that Barbara must then pick up all the cards of the other colour. By the choice of colour depending on the above sums, we see that Alice must at least force a draw, and if S_0, S_1 are not equal she will force a win.

4. A set of positive integers is defined to be **wicked** if it contains no three consecutive integers. We count the empty set, which contains no elements at all, as a wicked set.

   Find the number of wicked subsets of the set

$$\{1,\ 2,\ 3,\ 4,\ 5,\ 6,\ 7,\ 8,\ 9,\ 10\}.$$

As with many such problems, the challenge with this one is to divide the problem up in a manageable way. A collection that would be hard to enumerate bare-handed suddenly becomes much more tractable when viewed from a sensible perspective. In this case, several viewpoints permitted one to survey the inherent structure. The approaches below showed particular poise and professionalism.

Solution 1

Call a set *good* if it is not wicked. Let W_n be the number of wicked subsets of $\{1, \ldots, n\}$, and G_n the number of good subsets. Then $W_n = 2^n - G_n$.
Since every good subset contains at least three consecutive integers, we can group the good subsets by the number of the first integer in the first such collection of three or more consecutive integers.
Let $G_{n,i}$ be the number of good subsets of A_n in which the first integer in the first collection of three or more consecutives is i. Then

$$G_n = \sum_{i=1}^{n-2} G_{n,i}.$$

If $i = 1$, the first three integers must be in every subset counted. There are two choices for each of the other $n - 3$ places (either in or out). So $G_{n,1} = 2^{n-3}$.
If $i = 2$, then all subsets counted must contain 2, 3 and 4 but do not contain 1. This leaves two choices for each of the other $n - 4$ places, thus $G_{n,2} = 2^{n-4}$.
If $i \geqslant 3$, then all subsets must contain i, $i + 1$, and $i + 2$ but not $i - 1$. There are 2^{n-i-2} choices for what comes afterwards. There are, however, restrictions on what can come before: it must be wicked, as otherwise we would have counted it with a smaller value of i.
So

$$G_{n,i} = W_{i-2}2^{n-2-i},$$

or, to use the identity above, summing over all i (and introducing the convention that $W_n = 1$ for $n < 1$),

$$W_n = 2^n - \sum_{i=-1}^{n-4} 2^{n-4-i} W_i \text{ for } n \geqslant 1.$$

Using this we can quickly calculate values up to the desired W_{10}:

n		W_n
-1		1
0		1
1		2
2		4
3	$8 - 1 =$	7
4	$16 - 2 - 1 =$	13
5	$2^5 - 4 - 2 - 2 =$	24
6	$2^6 - 8 - 4 - 4 - 4 =$	44

Finally, we can combine these to get

$$W_{10} = 2^{10} - 2^7 W_{-1} - 2^6 W_0 - 2^5 W_1 - 2^4 W_2 - 2^3 W_3 - 2^2 W_4 - 2W_5 - W_6$$

$$= 504.$$

Solution 2

Let W_n be the number of wicked subsets of $\{1, 2, \ldots, n\}$ and let S be one such subset. Then precisely one of the following statements is true:

1. S does not contain n. In which case, S is also a wicked subset of $\{1, 2, \ldots, n - 1\}$, so there are W_{n-1} such subsets.
2. S contains n but not $n - 1$. In this case removing n leaves a wicked subset of $\{1, 2, \ldots, n - 2\}$, so there are W_{n-2} such subsets.
3. S contains n and $n - 1$ but not $n - 2$. In this case removing n and $n - 1$ leaves a wicked subset of $\{1, 2, \ldots, n - 3\}$, so there are W_{n-3} such subsets.

Therefore $W_n = W_{n-1} + W_{n-2} + W_{n-3}$ and since $W_0 = 1$, $W_1 = 2$ and $W_2 = 4$ we can construct the sequence W_i:

$$1, 2, 4, 7, 13, 24, 44, 81, 149, 274, 504, \ldots .$$

Hence $W_{10} = 504$.

5. Let p, q and r be prime numbers. It is given that p divides $qr - 1$, q divides $rp - 1$, and r divides $pq - 1$.
   Determine all possible values of pqr.

Solution

Since $p \mid qr - 1$, $q \mid rp - 1$ and $r \mid pq - 1$, we may multiply these together to infer that $pqr \mid (qr - 1)(rp - 1)(pq - 1)$. Multiplying out this expression, we obtain $pqr \mid p^2q^2r^2 - pqr^2 - pq^2r - p^2qr + pq + qr + rp - 1$. Since pqr always divides the first four terms of the right-hand side, it follows that $pqr \mid pq + qr + rp - 1$, and so the expression

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} - \frac{1}{pqr}$$

must be a positive integer, say k. Since $p, q, r \geqslant 2$, we see that

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} - \frac{1}{pqr} \leqslant \frac{3}{2},$$

and so in fact $k = 1$. Hence $pq + qr + rp - 1 = pqr$.

We now stop to check that p, q, r are distinct; for if $p = q = a$ we would require $a^2 + 2ra - 1 = ra^2$, and hence 1 would be divisible by $a = p \geqslant 2$, a contradiction. (Alternatively, many solvers observed that if, for example, $p = q$, then $q \mid qr - 1$, a contradiction.)

We may therefore assume without loss of generality that $p < q < r$, and deduce from this that for any solution p, q, r, we have $p = 2$ and $q = 3$; for otherwise we would have $p \geqslant 2, q \geqslant 5, r \geqslant 5$, and so

$$\frac{1}{p} + \frac{1}{q} + \frac{1}{r} - \frac{1}{pqr} \leqslant \frac{1}{2} + \frac{1}{5} + \frac{1}{5} = \frac{9}{10} < 1,$$

which is impossible. So we are left with the problem

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{r} - \frac{1}{6r} = 1,$$

which may easily be solved to yield $r = 5$. A quick check shows that, indeed, $2 \mid 3 \times 5 - 1$, $3 \mid 5 \times 2 - 1$ and $5 \mid 3 \times 2 - 1$, so the only possible value of pqr is $2 \times 3 \times 5 = 30$.

This booklet contains the BMO Round 1 question papers for the years 2001 to 2004. It also contains the corresponding answers and solutions.

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