Combinatorial Arguments

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1. (USAMO 1999) Let $p > 2$ be a prime and let a, b, c, d be integers not divisible by p, such that
$$\{ra/p\} + \{rb/p\} + \{rc/p\} + \{rd/p\} = 2$$
for any integer r not divisible by p. Prove that at least two of the numbers $a+b$, $a+c$, $a+d$, $b+c$, $b+d$, $c+d$ are divisible by p. (Note: $\{x\}$ denotes the fractional part of x.)

2. (TST 2000) Let p be a prime number. For integers r, s such that $rs(r^2 - s^2)$ is not divisible by p, let $f(r, s)$ denote the number of integers $n \in \{1, 2, \ldots, p-1\}$ such that $\{rn/p\}$ and $\{sn/p\}$ are either both less than 1/2 or both greater than 1/2. Prove that there exists $N > 0$ such that for $p > N$ and all r, s,
$$\lceil \frac{p-1}{3} \rceil \leq f(r, s) \leq \lfloor \frac{2(p-1)}{3} \rfloor$$
.

3. (TST 2000) Let n be a positive integer. A corner is a finite set S of ordered n-tuples of positive integers such that if $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_n$ are positive integers with $a_k \leq b_k$ for $k = 1, 2, \ldots, n$ and $(a_1, a_2, \ldots, a_n) \in S$, then $(b_1, b_2, \ldots, b_n) \in S$. Prove that among any infinite collection of corners, there exist two corners, one of which is a subset of the other one.

4. (TST 2001) For a set S, let $|S|$ denote the number of elements in S. Let A be a set of positive integers with $|A| = 2001$. Prove that there exists a set B such that 1) $B \subseteq A$; 2) $|B| \geq 668$; 3) for any $u, v \in B$ (not necessarily distinct), $u + v \notin B$.

5. (IMO 2000) Let $n \geq 2$ be a positive integer and λ a positive real number. Initially there are n fleas on a horizontal line, not all at the same point. We define a move of choosing two fleas at some points A and B, with A to the left of B, and letting the flea from A jump over the flea from B to the point C such that $BC/AB = \lambda$. Determine all values of ? such that for any point M on the line and for any initial position of the n fleas, there exists a sequence of moves that will take them all to the position right of M.

6. (IMO 2000) A magician has one hundred cards numbered 1 to 100. He puts them into three boxes, a red one, a white one, and a blue one, so that each box contains at least one card. A member of the audience draws two cards from two different boxes and announces the sum of numbers on those cards. Given this information, the magician locates the box from which no card has been drawn. How many ways are there to put the cards in the three boxes so that the trick works?

7. (IMO Shortlist 2000) Let p and q be relatively prime positive integers. A subset S of non-negative integers is called ideal if $0 \in S$ and for each element $n \in S$, the integers $n + p$ and $n + q$ belong to S. Determine the number of ideal subsets.

8. (IMO 2001) Twenty-one girls and twenty-one boys took part in a mathematical competition. It turned out that each contestant solved at most six problems, and for each pair of a girl and a boy, there was at least one problem that was solved by both the girl and the boy. Show that there is a problem that was solved by at least three girls and at least three boys.

9. (IMO Shortlist 2001) Define a k-clique to be a set of k people such that every pair of them are acquainted with each other. At a certain party, every pair of 3-cliques has at least one person in common, and there are no 5-cliques. Prove that there are two or fewer people at the party whose departure leaves no 3-clique remaining.

10. (IMO Shortlist 2001) For a positive integer n define a sequence of zeros and ones to be balanced if it contains n zeros and n ones. Two balanced sequences a and b are neighbors if you can move one of the $2n$ symbols of a to another position to form b. For instance, when $n = 4$, the balanced sequences 01101001 and 00110101 are neighbors because the third (or fourth) zero in the first sequence can be moved to the first or second position to form the second sequence. Prove that there is a set S of at most $\frac{1}{n+1}\binom{2n}{n}$ balanced sequences such that every balanced sequence is equal to or is a neighbor of at least one sequence in S.

11. (IMO Shortlist 2001) A pile of n pebbles is placed in a vertical column. This configuration is modified according to the following rules. A pebble can be moved if it is at the top of a column that contains at least two more pebbles than the column immediately to its right. (If there are no pebbles to the right, think of this as a column with 0 pebbles.) At each stage, choose a pebble from among those that can be moved (if there are any) and place it at the top of the column to its right. If no pebbles can be moved, the configuration is called a final configuration. For each n, show that no matter what choices are made at each stage, the final configuration is unique. Describe that configuration in terms of n.

12. (IMO 2002) Let $n \geq 3$ be a positive integer. Let $C_1, C_2, C_3, \ldots, C_n$ be unit circles in the plane, with centers $O_1, O_2, O_3, \ldots, O_n$ respectively. If no line meets more than two of the circles, prove that
$$\sum_{1\leq i<j\leq n} \frac{1}{O_iO_j} \leq \frac{(n-1)\pi}{4}$$
.

13. (IMO Shortlist 2002) Among a group of 120 people, some pairs are friends. A weak quartet is a set of four people containing exactly one pair of friends. What is the maximum possible number of weak quartets?

14. (IMO Shortlist 2003) Let $x_1, ..., x_n$ and $y_1, ..., y_n$ be real numbers. Let $A = (a_{ij})_{1\leq i,j\leq n}$ be the matrix with entries $a_{ij} = 1$, if $x_i + y_j \geq 0$; and $a_{ij} = 0$, if $x_i + y_j < 0$. Suppose that B is an n-by-n matrix whose entries are 0, 1 such that the sum of the elements in each row and each column of B is equal to the corresponding sum for the matrix A. Prove that $A = B$.

15. (IMO Shortlist 2004) For an n-by-n matrix A, let X_i be the set of entries in row i and Y_j the set of entries in column j, $1 \leq i, j \leq n$. We say that A is golden if $X_1, ..., X_n, Y_1, \ldots, Y_n$ are distinct sets. Find the least integer n such that there exists a 2004-by-2004 golden matrix with entries in the set $\{1, 2, \ldots, n\}$.

16. (IMO Shortlist 2004) For a finite graph G, let $f(G)$ be the number of triangles and $g(G)$ the number of tetrahedra formed by edges of G. Find the least constant c such that $g(G)^3 \leq c \cdot f(G)^4$ for every graph G.

17. (IMO 2005) In a mathematical competition 6 problems were posed to the contestants. Each pair of problems was solved by more than 2/5 of the contestants. Nobody solved all 6 problems. Show that there were at least 2 contestants who each solved exactly 5 problems.

18. (IMO Shortlist 2005) Let M be a convex n-gon, $n \geq 4$. Some $n-3$ of its diagonals are colored green and some other $n-3$ diagonals are colored red, so that no two diagonals of the same color meet inside M. Find the maximum possible number of intersection points of green and red diagonals inside M.

19. (IMO Shortlist 2006) Consider a convex polyhedron without parallel edges and without an edge parallel to any face other than the two faces adjacent to it. Call a pair of points of the polyheadron antipodal if there exist two parallel planes passing through these points and such that the polyheadron is contained between these planes. Let A be the number of antipodal pairs of vertices, and let B be the number of antipodal pairs of midpoint edges. Determine the difference $A - B$ in terms of the numbers of vertices, edges, and faces.

20. (IMO 2007) Let n be a positive integer. Consider $S = \{(x, y, z) | x, y, z \in \{0, 1, 2, \ldots, n\}, x+y+z > 0\}$ as a set of points in three-dimensional space. Determine the smallest possible number of planes, the union of which contains S but does not include $(0, 0, 0)$.

21. (IMO 2007) In a mathematical competition some competitors are friends. Friendship is always mutual. Call a group of competitors a clique if each two of them are friends. (In particular, any group of fewer than two competitors is a clique.) The number of members of a clique is called its size.

    Given that, in this competition, the largest size of a clique is even, prove that the competitors can be arranged into two rooms such that the largest size of a clique contained in one room is the same as the largest size of a clique contained in the other room.

22. (USAMO 2008) At a certain mathematical conference, every pair of mathematicians are either friends or strangers. At mealtime, every participant eats in one of two large dining rooms. Each mathematician insists upon eating in a room which contains an even number of his or her friends. Prove that the number of ways that the mathematicians may be split between the two rooms is a power of two (i.e., is of the form 2k for some positive integer k).

23. (Russia 2008) On a chess tournament 2n + 3 players take part. Every two play exactly one match. The schedule of the tournament has been made so that the matches are played one after another, and each player, after playing a match, is free in at least n consequent matches. Prove that one of the players who played the opening match on the tournament will also play the closing match.

24. (Russia 2007) Given a (multi-)graph with N vertices. For any set of k vertices there are at most $2k - 2$ edges joining vertices of this set. Prove that the edges may be colored in two colors so that each cycle contains edges of both colors.

25. (Russia 2006) At a tourist camp, each person has at least 50 and at most 100 friends among the other persons at the camp. Show that one can hand out a t-shirt to every person such that the t-shirts have (at most) 1331 different colors, and any person has 20 friends whose t-shirts all have pairwisely different colors.

26. (Russia 2005) One hundred representatives of 25 countries, four from each country, are sitting at a round table. Prove that they can be partitioned into four groups so that every group contains a representative of each country, and no two persons from the same group are sitting next to each other.

27. (Russia 2004) In a country there are several cities; some of these cities are connected by airlines. Each airline belongs to one of k flight companies; two airlines of the same flight company have always a common final point. Show that one can partition all cities in $k + 2$ groups in such a way that two cities from the same group are never connected by an airline with each other.

28. (Russia 2004) A parallelepiped is cut by a plane along a 6-gon. Supposed this 6-gon can be put into a certain rectangle π (which means one can put the rectangle on the parallelepiped's plane such that the 6-gon is completely covered by the rectangle). Show that one also can put one of the parallelepiped's faces into the rectangle π.

29. (Russia 2002?) In a certain city, there are several squares. All streets are one-way and start or terminate only in squares; any two squares are connected by at most one road. It is known that there are exactly two streets that go out of any given square. Show that one can divide the city into 1014 districts so that (i) no street connects two cities in the same district, and (ii) for any two districts, all the streets that connect them have the same direction (either all the streets go from the first district to the second, or vice versa).

30. (Russia 2001) There are 2001 towns in a country. For any town, there exists a road going out of it, and there does not exist a town directly connected by roads with all the rest. A set of towns D is said to be dominating if any town that does not belong to D is directly connected by a road with at least one town from D. It is given that any dominating set consists at least k towns. Prove that the country may be partitioned into $2001 - k$ republics such that no two towns from the same republic will be joined by a road.