

TJUSAMO 2011 – Discrete Calculus

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1 Finite Differences

The type of calculus taught in most high school classes is known as infinite or continuous calculus. It is based on the properties of the derivative operator $\frac{d}{dx}$, defined by

$$\frac{d}{dx}f(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

On the other hand, we can obtain an entirely different type of calculus by restricting ourselves to positive integer values for h . In this case, the closest we can get to 0 is $h = 1$, so we'll define the *difference operator* Δ by

$$\Delta f(x) = f(x+1) - f(x)$$

In calculus classes, one of the first things taught is the derivative of $f(x) = x^n$. In this case, $\frac{d}{dx}f(x) = nx^{n-1}$. Unfortunately, the same result does not hold true for the Δ operator; if we try it, we'll get stuff like $\Delta(x^4) = 4x^3 + 6x^2 + 4x + 1$. However, there is a type of n -th power which has a nice discrete derivative: the falling n -th powers. These are defined by

$$x^{\underline{n}} = x(x-1) \cdots (x-n+1)$$

for integers $n \geq 0$. By definition, $x^{\underline{0}} = 1$. Notice that this is closely related to the factorial function $n!$, as $n! = n^{\underline{n}}$. Looking at the discrete derivative, we see

$$\begin{aligned} \Delta(x^{\underline{n}}) &= (x+1)^{\underline{n}} - x^{\underline{n}} \\ &= (x+1)x(x-1) \cdots (x-n+2) - x(x-1) \cdots (x-n+1) \\ &= nx(x-1) \cdots (x-n+2) \end{aligned}$$

Hence, we obtain a “derivative of a power” rule: $\Delta(x^{\underline{n}}) = nx^{\underline{n-1}}$. We can also obtain linearity rules

$$\Delta(\alpha f(x) + \beta g(x)) = \alpha \Delta f(x) + \beta \Delta g(x)$$

as well as product rules:

$$\begin{aligned} \Delta(f(x)g(x)) &= f(x+1)g(x+1) - f(x)g(x) \\ &= f(x+1)g(x+1) - f(x)g(x+1) + f(x)g(x+1) - f(x)g(x) \\ &= f(x)[g(x+1) - g(x)] + g(x+1)[f(x+1) - f(x)] \\ &= f(x)\Delta g(x) + g(x+1)\Delta f(x) \end{aligned}$$

We can put this into a slightly less messy form by defining $E f(x) := f(x+1)$ so that we get

$$\Delta(fg) = f\Delta g + Eg\Delta f.$$

1.1 Discrete Integration

All of this is nice and good, but we'd like to do more. Infinite calculus has a fundamental theorem relating the derivative to the integral:

$$\int_a^b f(x) dx = F(b) - F(a)$$

where $F'(x) = f(x)$. Thus, we should expect to have a fundamental theorem relating these (discrete) differences to (discrete) sums. Indeed, we'll say

$$\sum_a^b f(x) \delta x = F(b) - F(a)$$

where $\Delta(F) = f$. What is this discrete integral or “anti-difference”? We'll define it so that several important facts hold: $\sum_a^a f(x) \delta x = 0$ and $\sum_a^{a+1} f(x) \delta x = F(a+1) - F(a) = \Delta F = f(a)$. This forces us to take, in general, the definite discrete sum to mean an ordinary sum excluding the value at the upper limit.

$$\sum_a^b f(x) \delta x = \sum_{k=a}^{b-1} f(k)$$

So now what can we do with this discrete integration operator? Well, we can start off by looking at how it behaves on our falling powers. The power rule discussed earlier implies that

$$\sum_a^b x^n \delta x = \frac{x^{n+1}}{n+1} \Big|_a^b \Rightarrow \sum_{k=a}^{b-1} k^n = \frac{b^{n+1} - a^{n+1}}{n+1}.$$

As an example, this gives us an easy way to derive the formula for the sum of n squares (or cubes, fourth powers, etc.). Since $k^2 = k(k-1) + k = k^2 + k^1$, we have that

$$\sum_{k=0}^{n-1} k^2 = \frac{n^3}{3} + \frac{n^2}{2} = \frac{n(n-1)(2n-1)}{6}$$

Setting $n-1 \rightarrow n$ gives us the familiar formula $\frac{n(n+1)(2n+1)}{6}$ for the sum of the first n squares.

2 Finite differences on polynomials

First, we need the following result, which is easily proven by induction on n :

Theorem 2.1. Any polynomial $P \in \mathbb{R}[x]$ of degree n can be written in the form $\sum_{k=0}^n a_k \binom{x}{k}$, where a_0, a_1, \dots, a_n are nonnegative integers and $a_n \neq 0$.

For any polynomial P and nonzero real number a , let $\Delta_a P$ be the polynomial defined by $(\Delta_a P)(x) = P(x+a) - P(x)$. For $a = 1$, we get the “standard” finite difference operator Δ . The main reason we like to apply finite differences to polynomials is that:

Theorem 2.2. For a nonconstant polynomial P , $\deg(\Delta_a P) = \deg(P) - 1$.

Proof. Let $n = \deg P$. First, we define the polynomial $Q(x) = P(ax)$, so that $(\Delta_a P)(ax) = (\Delta Q)(x)$. It is clear that $\deg Q = n$. Then, we write $Q(x) = \sum_{k=0}^n a_k \binom{x}{k}$, where a_0, a_1, \dots, a_n are nonnegative integers and $a_n \neq 0$. We have $\Delta \binom{x}{k} = \binom{x}{k-1}$ for $k > 0$ and $\Delta \binom{x}{0} = 0$, so $\Delta Q(x) = \sum_{k=1}^n a_k \binom{x}{k-1}$, which has degree $n-1$. Therefore, $\deg(\Delta_a P) = \deg(\Delta Q) = n-1 = \deg(P) - 1$. \square

What this means is that induction and finite differences are often seen together. It also means that if P has degree n , then $\Delta^{n+1}P$ is the zero polynomial, where $\Delta^{n+1}P$ represents Δ applied to P $n+1$ times.

It is also worth knowing that for a positive integer k , $(\Delta^k P)(x)$ (Δ applied to P k times) is equal to $\sum_{j=0}^k \binom{k}{j} (-1)^{k-j} P(x+j)$. This, again, is easily proven by induction.

3 Problems

1. Let $H_k = 1 + \frac{1}{2} + \frac{1}{3} + \dots + \frac{1}{k}$. Determine a closed form (H_n allowed) for

$$\sum_{k=0}^n k H_k.$$

2. Let n be a positive integer. Determine the value of $\sum_{k=0}^n \binom{n}{k} (-1)^{n-k} k^n$.
3. Let k be a positive integer. Prove that one can partition the set $\{0, 1, 2, 3, \dots, 2^{k+1} - 1\}$ into two distinct subsets $\{x_1, x_2, \dots, x_{2^k}\}$ and $\{y_1, y_2, \dots, y_{2^k}\}$ such that $\sum_{i=1}^{2^k} x_i^m = \sum_{i=1}^{2^k} y_i^m$ for all $m \in \{1, 2, \dots, k\}$.
4. Prove that every positive integer can be expressed as $\pm 1^2 \pm 2^2 \pm 3^2 \pm \dots \pm n^2$ for some positive integer n and some choice of signs.