Dual Kolmogorov Adversary Method

We define a new formulation of the Kolmogorov Adversary and show that all other methods are tightly bounded by it.
Let

$$DK(f) = \max_{\substack{x,y \\ f(x)\neq f(y)}} \min_{\substack{i \\ x_i \neq y_i}} \frac{\sqrt{2^{-K(x)-K(y|x,i)} \cdot 2^{-K(y)-K(x|y,i)}}}{2^{-K(x,y)}} \tag{1}$$

Lemma 1. $DK(f) = \Omega(SWA(f))$

Proof. Given a weight scheme w, w', we can always construct probability distributions p, p', q in the following manner.
Let $W = \sum_{x,y} w(x,y)$, $wt(x) = \sum_y w(x,y)$ and $v(x,i) = \sum_y w'(x,y,i)$.

$$q(x,y) = \frac{w(x,y)}{W} \tag{2}$$

$$p(x) = \frac{wt(x)}{W} \tag{3}$$

$$p'_{x,i}(y) = \frac{w'(x,y,i)}{v(x,i)} \tag{4}$$

Hence,

$$\sqrt{\frac{wt(x)wt(y)}{v(x,i)v(y,i)}} = \frac{\sqrt{p(x)p'_{x,i}(y)p(y)p'_{y,i}(x)}}{q(x,y)} \tag{5}$$

since $w'(x,y,i) \cdot w'(y,x,i) = w(x,y)^2$
Hence,

$$\max_{w,w'} \min_{\substack{x,y,i \\ f(x)\neq f(y) \\ x_i \neq y_i}} \sqrt{\frac{wt(x)wt(y)}{v(x,i)v(y,i)}} \leq \max_{p,q,p'} \min_{\substack{x,y,i \\ f(x)\neq f(y) \\ x_i \neq y_i}} \frac{\sqrt{p(x)p'_{x,i}(y)p(y)p'_{y,i}(x)}}{q(x,y)} \tag{6}$$

Due to the existence of a universal semicomputable semimeasure μ over S and μ' over S^2 , we have

$$p(x) \leq c \cdot \mu(x) \tag{7}$$

$$p(y) \leq c \cdot \mu(y) \tag{8}$$

$$p'_{x,i}(y) \leq c \cdot \mu_{x,i}(y) \tag{9}$$

$$p'_{y,i}(x) \leq c \cdot \mu_{y,i}(x) \tag{10}$$

Also, for any probability distribution q over S^2, there exists a pair (x, y) s.t

$$q(x,y) \geq \mu'(x,y) \tag{11}$$

Hence, $\forall p, p', q \; \exists x, y$ s.t $\forall i$

$$\frac{\sqrt{p(x)p'_{x,i}(y)p(y)p'_{y,i}(x)}}{q(x,y)} \leq c \cdot \frac{\sqrt{\mu(x)\mu_{x,i}(y)\mu(y)\mu_{y,i}(x)}}{\mu'(x,y)} \tag{12}$$

$\forall p, p', q \; \exists x, y$ s.t.

$$\min_{\substack{i \\ x_i \neq y_i}} \frac{\sqrt{p(x)p'_{x,i}(y)p(y)p'_{y,i}(x)}}{q(x,y)} \leq c \cdot \min_{\substack{i \\ x_i \neq y_i}} \frac{\sqrt{\mu(x)\mu_{x,i}(y)\mu(y)\mu_{y,i}(x)}}{\mu'(x,y)} \tag{13}$$

$\forall p, p', q$

$$\min_{\substack{x,y \\ f(x)\neq f(y)}} \min_{\substack{i \\ x_i \neq y_i}} \frac{\sqrt{p(x)p'_{x,i}(y)p(y)p'_{y,i}(x)}}{q(x,y)} \leq c \cdot \max_{\substack{x,y \\ f(x)\neq f(y)}} \min_{\substack{i \\ x_i \neq y_i}} \frac{\sqrt{\mu(x)\mu_{x,i}(y)\mu(y)\mu_{y,i}(x)}}{\mu'(x,y)} \tag{14}$$

Therefore,

$$\max_{p,q,p'} \min_{\substack{x,y,i \\ f(x)\neq f(y) \\ x_i \neq y_i}} \frac{\sqrt{p(x)p'_{x,i}(y)p(y)p'_{y,i}(x)}}{q(x,y)} \leq c \cdot \max_{\substack{x,y \\ f(x)\neq f(y)}} \min_{\substack{i \\ x_i \neq y_i}} \frac{\sqrt{\mu(x)\mu_{x,i}(y)\mu(y)\mu_{y,i}(x)}}{\mu'(x,y)} \tag{15}$$

$$\max_{p,q,p'} \min_{\substack{x,y,i \\ f(x)\neq f(y) \\ x_i \neq y_i}} \frac{\sqrt{p(x)p'_{x,i}(y)p(y)p'_{y,i}(x)}}{q(x,y)} \leq c \cdot \max_{\substack{x,y \\ f(x)\neq f(y)}} \min_{\substack{i \\ x_i \neq y_i}} \frac{\sqrt{2^{-K(x)-K(y|x,i)} \cdot 2^{-K(y)-K(x|y,i)}}}{2^{-K(x,y)}} \tag{16}$$

Hence, $DK(f) = \Omega(SWA(f))$

Lemma 2. $DK(f) = O(MM(f))$

Proof. We know that for any i with $x_i \neq y_i$

$$K(i|x) \geq K(x,y) - K(x) - K(y|i,x) + K(i|x,y,K(x,y)) - O(1) \tag{17}$$

$$2^{-K(i|x)} \leq c \cdot \frac{2^{-K(x,y)} \cdot 2^{-K(i|x,y,K(x,y))}}{2^{-K(y|i,x)} \cdot 2^{-K(x)}} \tag{18}$$

$$\mu_x(i) \leq c \cdot \frac{\mu'(x,y) \cdot 2^{-K(i|x,y,K(x,y))}}{\mu_{x,i}(y) \cdot \mu(x)} \tag{19}$$

$$\sqrt{\mu_x(i) \cdot \mu_y(i)} \leq c \cdot \frac{\mu'(x,y) \cdot 2^{-K(i|x,y,K(x,y))}}{\sqrt{\mu_{x,i}(y) \cdot \mu(x) \cdot \mu_{y,i}(x) \cdot \mu(y)}} \tag{20}$$

$$\sum_{i:x_i \neq y_i} \sqrt{\mu_x(i) \cdot \mu_y(i)} \leq c \cdot \sum_{i:x_i \neq y_i} \frac{\mu'(x,y) \cdot 2^{-K(i|x,y,K(x,y))}}{\sqrt{\mu_{x,i}(y) \cdot \mu(x) \cdot \mu_{y,i}(x) \cdot \mu(y)}} \tag{21}$$

$$\sum_{i:x_i \neq y_i} \sqrt{\mu_x(i) \cdot \mu_y(i)} \leq c \cdot \max_{i:x_i \neq y_i} \frac{\mu'(x,y)}{\sqrt{\mu_{x,i}(y) \cdot \mu(x) \cdot \mu_{y,i}(x) \cdot \mu(y)}} \cdot \sum_{i:x_i \neq y_i} 2^{-K(i|x,y,K(x,y))} \tag{22}$$

Using Kraft's inequality,

$$\sum_{i:x_i \neq y_i} \sqrt{\mu_x(i) \cdot \mu_y(i)} \leq c \cdot \max_{i:x_i \neq y_i} \frac{\mu'(x,y)}{\sqrt{\mu_{x,i}(y) \cdot \mu(x) \cdot \mu_{y,i}(x) \cdot \mu(y)}} \tag{23}$$

Hence, $\forall x, y$

$$c \cdot \frac{1}{\sum_{i:x_i \neq y_i} \sqrt{\mu_x(i) \cdot \mu_y(i)}} \geq \min_{i:x_i \neq y_i} \frac{\sqrt{\mu_{x,i}(y) \cdot \mu(x) \cdot \mu_{y,i}(x) \cdot \mu(y)}}{\mu'(x,y)} \tag{24}$$

$$c \cdot \max_{\substack{x,y \\ f(x) \neq f(y)}} \frac{1}{\sum_{i:x_i \neq y_i} \sqrt{\mu_x(i) \cdot \mu_y(i)}} \geq \max_{\substack{x,y \\ f(x) \neq f(y)}} \min_{i:x_i \neq y_i} \frac{\sqrt{\mu_{x,i}(y) \cdot \mu(x) \cdot \mu_{y,i}(x) \cdot \mu(y)}}{\mu'(x,y)} \tag{25}$$

$$c \cdot \max_{\substack{x,y \\ f(x) \neq f(y)}} \frac{1}{\sum_{i:x_i \neq y_i} \sqrt{\mu_x(i) \cdot \mu_y(i)}} \geq \max_{\substack{x,y \\ f(x) \neq f(y)}} \min_{\substack{i \\ x_i \neq y_i}} \frac{\sqrt{2^{-K(x)-K(y|x,i)} \cdot 2^{-K(y)-K(x|y,i)}}}{2^{-K(x,y)}} \tag{26}$$

Hence, $DK(f) = O(MM(f))$

Theorem 3. $DK(f) = \Theta(SWA(f)) = \Theta(MM(f))$

Proof. From Spalek & Szegedy's result , we know that

$$MM(f) = SWA(f) \tag{27}$$

Hence, the result holds.