New formulations for $\boldsymbol{Adv(f)}$ and $\boldsymbol{Adv^{\pm}(f)}$

1 Preliminaries

We now state some definitions and theorems from the theory of lower semicomputable semimeasures. For more on this see [1].

Definition 1. *A* discrete semimeasure is a function p from a countable set A to the nonnegative reals that satisfies $\sum_{x\in A} p(x) \leq 1$

Definition 2. *Let $\mathcal{M}$ be a class of discrete semimeasures over a set A. A semimeasure m is universal form $\mathcal{M}$ if $m \in \mathcal{M}$ and for all $p \in \mathcal{M}$, there exists a constant $c_p > 0$ such that for all $x \in A$, we have $m(x) \geq c_p p(x)$.*

Theorem 3. *There is a universal lower semicomputable discrete semimeasure.*

2 $\boldsymbol{KA(f)}$ using Levin's universal semimeasure

Let's recall from[2] the minimax dual formulation of the adversary method:

$$MM(f) = \min_{p} \max_{\substack{x,y:\\ f(x)\neq f(y)}} \frac{1}{\sum_{i:x_i\neq y_i} \sqrt{p_x(i)p_y(i)}} \tag{1}$$

where the p_x are probability distributions over $[n]$.

Fact 1 *The requirement of probability distributions can be relaxed into semimeasures without changing the optimal value.*

Proof. For contradiction, lets assume that we allow semimeasures, the optimal value is attained for p_x and p_y and that at least one of them is a strict semimeasure. Let's say, wlog, that it's p_x. Let $\alpha = 1 - \sum_{i=1}^{n} p_x(i) > 0$. Let $j \in [n]$ be such that $x_j \neq y_j$ (there must be at least one) and define p'_x to be:

$$p'_x(i) = \begin{cases} p_x(i) + \alpha & \text{if } i = j \\ p_x(i) & \text{otherwise} \end{cases}$$

Now we have that:

$$\frac{1}{\sum_{i:x_i\neq y_i} \sqrt{p'_x(i)p_y(i)}} = \frac{1}{(p_x(j)+\alpha)p_y(j) + \sum_{i:x_i\neq y_i\wedge i\neq j} \sqrt{p_x(i)p_y(i)}} \tag{2}$$

$$< \frac{1}{\sum_{i:x_i\neq y_i} \sqrt{p_x(i)p_y(i)}} \tag{3}$$

This contradicts the minimality of the solution attained at p_x and p_y and the contradiction came from assuming that the optimal could be attained at semimeasures. Hence, working with semimeasures instead of probability distributions doesn't change the optimal value.

Now we give a reformulation of the minimax adversary using universal semimeasures.

Definition 4.
$$KA(f) = \max_{\substack{x,y:\\ f(x)\neq f(y)}} \frac{1}{\sum_{i:x_i\neq y_i} m(i)}$$
with m *a universal lower semicomputable semimeasure.*

Proposition 5. $MM(f) = \Theta(KA(f))$

Proof. As noted in Fact 1, the minimization in the definition of $MM(f)$ can be taken over semimeasures. Since m is one particular semimeasure, we have that $MM(f) = O(KA(f))$.

For the lower bound, let p_x and p_y be two semimeasures where the optimal value of $MM(f)$ is attained. By the universality of m we have that there exist two constants c_x and c_y such that $\forall i \in [n]\ m(i) \geq c_x p_x(i) \wedge m(i) \geq c_y p_y(i)$.

So we have that,

$$\frac{1}{\sum_{i:x_i\neq y_i} \sqrt{p_x(i)p_y(i)}} \geq \frac{1}{\sum_{i:x_i\neq y_i} \sqrt{c_x m(i) c_y m(i)}} \tag{4}$$

And hence, $MM(f) = \Omega(KA(f))$

References

1. Ming Li and PMB Vitâanyi. *An introduction to Kolmogorov complexity and its applications*. Springer, 2008.
2. Robert Špalek and Mario Szegedy. All quantum adversary methods are equivalent. In *Automata, Languages and Programming*, pages 1299–1311. Springer, 2005.