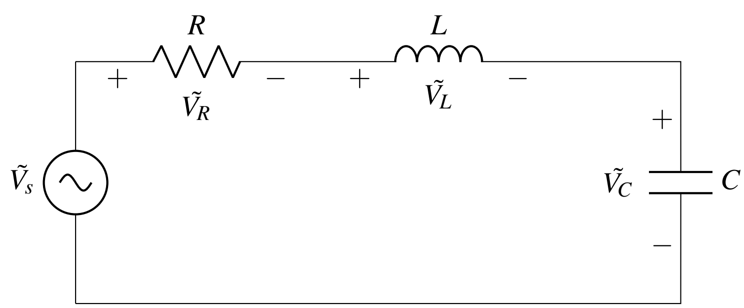
1. RLC Circuit

In this question, we will take a look at an electrical systems described by second-order differential equations and analyze it in the phasor domain. Consider the circuit below where $\tilde{V}_s$ is a sinusoidal signal, $L = 1\,\text{mH}$, and $C = 1\,\text{nF}$:



(a) **Transform the circuit into the phasor domain.**

Solution:

$$Z_R = R$$
$$Z_L = j\omega L$$
$$Z_C = \frac{1}{j\omega C}$$

(b) **Solve for the transfer function $H_C(\omega) = \frac{\tilde{V}_C}{\tilde{V}_s}$ in terms of R, L, and C.**

Solution:

$\widetilde{V}_C$ is a voltage divider where the output voltage is taken across the capacitor.

$$\widetilde{V}_C = \frac{Z_C}{Z_R + Z_L + Z_C}\widetilde{V}_s,$$

$$H_C(\omega) = \frac{Z_C}{Z_R + Z_L + Z_C} = \frac{\frac{1}{j\omega C}}{R + j\omega L + \frac{1}{j\omega C}}.$$

Multiplying the numerator and denominator by $j\omega C$ gives

$$H_C(\omega) = \frac{1}{(j\omega)^2 LC + j\omega RC + 1}$$

(c) **Solve for the transfer function $H_L(\omega) = \frac{\widetilde{V}_L}{\widetilde{V}_s}$ in terms of R, L, and C.**

Solution:

$\widetilde{V}_L$ is a voltage divider where the output voltage is taken across the inductor.

$$\widetilde{V}_L = \frac{Z_L}{Z_R + Z_L + Z_C}\widetilde{V}_s,$$
$$H_L(\omega) = \frac{Z_L}{Z_R + Z_L + Z_C} = \frac{j\omega L}{R + j\omega L + \frac{1}{j\omega C}}.$$

Multiplying the numerator and denominator by $j\omega C$ gives

$$H_L(\omega) = \frac{(j\omega)^2 LC}{(j\omega)^2 LC + j\omega RC + 1}$$

(d) **Solve for the transfer function $H_R(\omega) = \frac{\widetilde{V}_R}{\widetilde{V}_s}$ in terms of R, L, and C.**

Solution:

$\widetilde{V}_R$ is a voltage divider where the output voltage is taken across the resistor.

$$\widetilde{V}_R = \frac{Z_R}{Z_R + Z_L + Z_C}\widetilde{V}_s,$$
$$H_R(\omega) = \frac{Z_R}{Z_R + Z_L + Z_C} = \frac{R}{R + j\omega L + \frac{1}{j\omega C}}.$$

Multiplying the numerator and denominator by $j\omega C$ gives

$$H_R(\omega) = \frac{j\omega RC}{(j\omega)^2 LC + j\omega RC + 1}.$$

2. Bode Plots for Filters

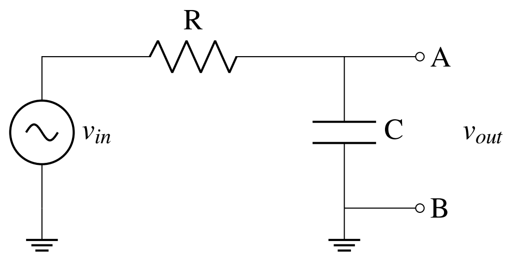


Figure 1: Low Pass Filter

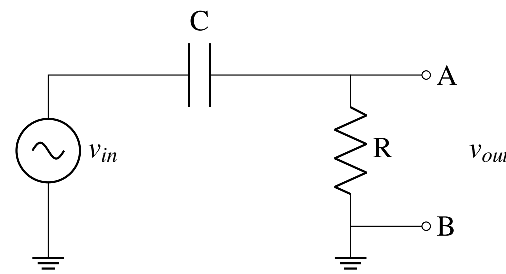


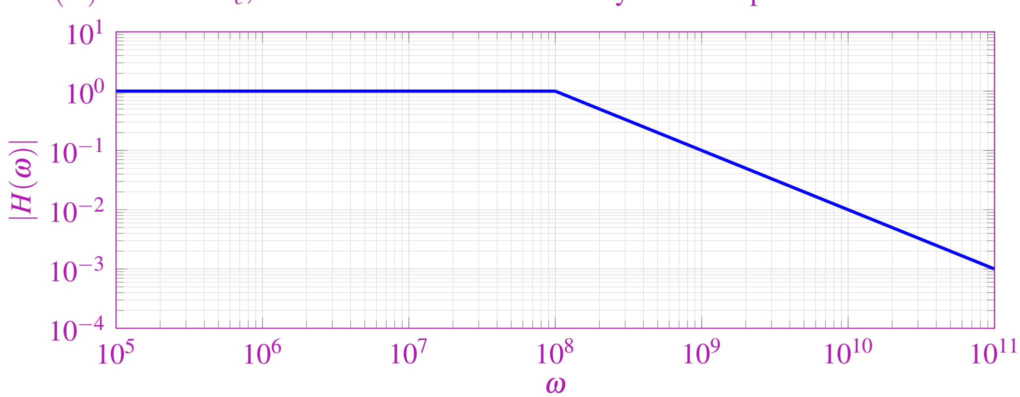
Figure 2: High Pass Filter

(a) First, consider the following transfer function of a Low-Pass Filter:
$H(\omega) = \frac{1}{j\omega C_1 R_1 + 1}$ where $R_1 = 100\Omega$ and $C_1 = 100pF$

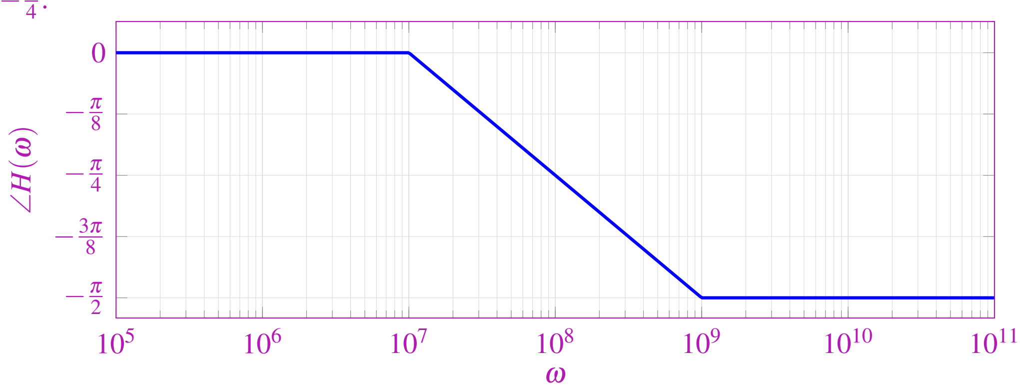
i. What is its cutoff frequency?
Hint: Recall that the cutoff frequency is the frequency at which the magnitude of the transfer function is $\frac{1}{\sqrt{2}}$.

Solution: The cutoff frequency for a lowpass filter is $\frac{1}{RC}$ as prior problems have shown. Thus, $\omega_c = \frac{1}{R_1 C_1} = \frac{1}{10^2 \cdot 10^{-10}} = 10^8$

ii. Sketch its phase and magnitude.
Solution: Magnitude (log-log scale): According to our Bode plot approximation, the magnitude of $H(\omega)$ is 1 until ω_c, after which it decreases linearly with a slope of 1.



Phase (semi-log scale): According to our Bode plot approximation, the phase of $H(\omega)$ is 0 until $\frac{\omega_c}{10}$, after which it decreases linearly until $10\omega_c$, where it stays at $-\frac{\pi}{2}$. At ω_c, the phase is exactly $-\frac{\pi}{4}$.



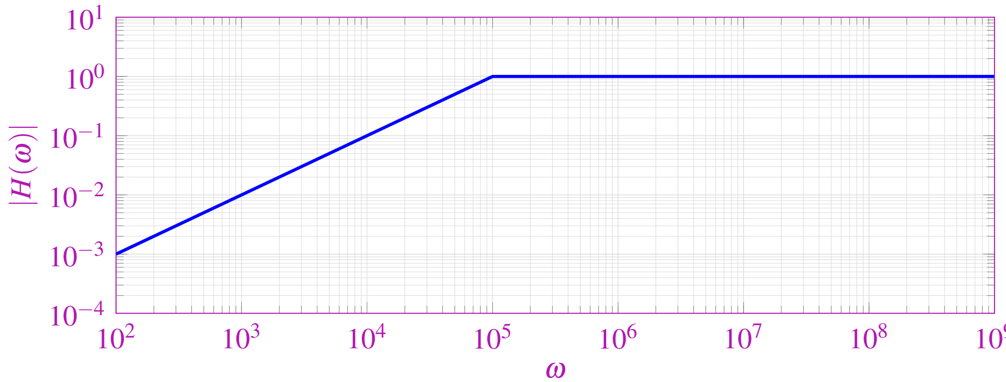
(b) Now consider the transfer function of a High-Pass Filter: $H(\omega) = \frac{j\omega C_2 R_2}{j\omega C_2 R_2 + 1}$ where $R_2 = 1k\Omega$ and $C_1 = 10nF$

i. What is its cutoff frequency?

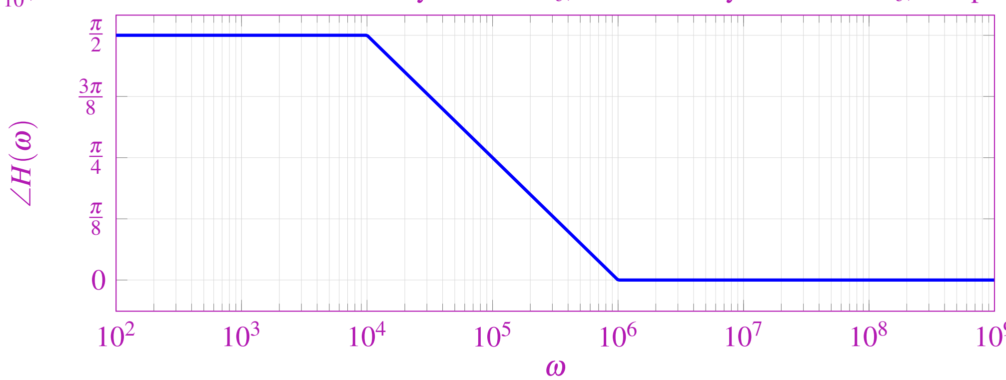
Solution: $\omega_c = \frac{1}{R_2 C_2} = \frac{1}{10^3 \cdot 10^{-8}} = 10^5$

ii. Sketch its phase and magnitude.

Solution: Magnitude (log-log scale): According to our Bode plot approximation, the magnitude of $H(\omega)$ increases linearly with a slope of 1 until ω_c, after which it levels off at 1.



Phase (semi-log scale): According to our Bode plot approximation, the phase of $H(\omega)$ is $\frac{\pi}{2}$ until $\frac{\omega_c}{10}$, after which it decreases linearly until $10\omega_c$, where it stays at 0. At ω_c, the phase is $-\frac{\pi}{4}$.



(c) What happens if we cascade these two filters together with a unity-gain buffer in between them? Consider the resulting transfer function: $H(\omega) = \frac{1}{j\omega C_1 R_1 + 1} \cdot \frac{j\omega C_2 R_2}{j\omega C_2 R_2 + 1}$

i. What are its cutoff frequencies?

Solution: The lower cutoff is the cutoff frequency of the high-pass filter:

$$\omega_{c,l} = \frac{1}{R_2 C_2} = 10^5$$

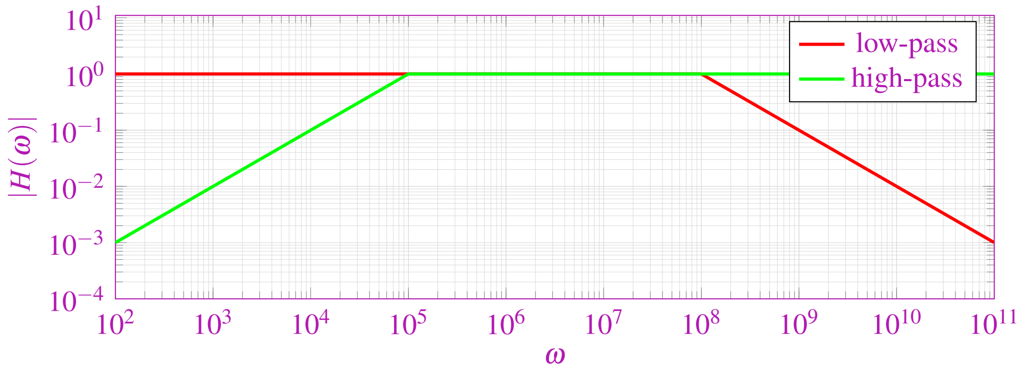
The upper cutoff is the cutoff frequency of the low-pass filter:

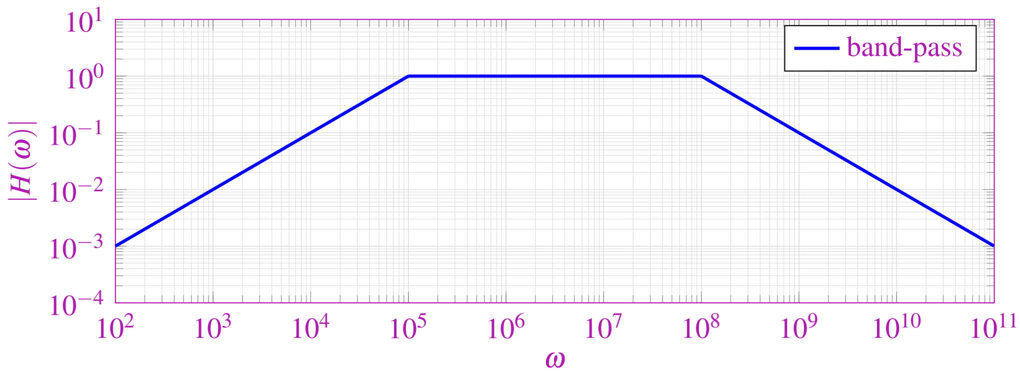
$$\omega_{c,u} = \frac{1}{R_1 C_1} = 10^8$$

ii. Sketch its phase and magnitude. *Hint: How can we combine the plots of the individual filters together?*

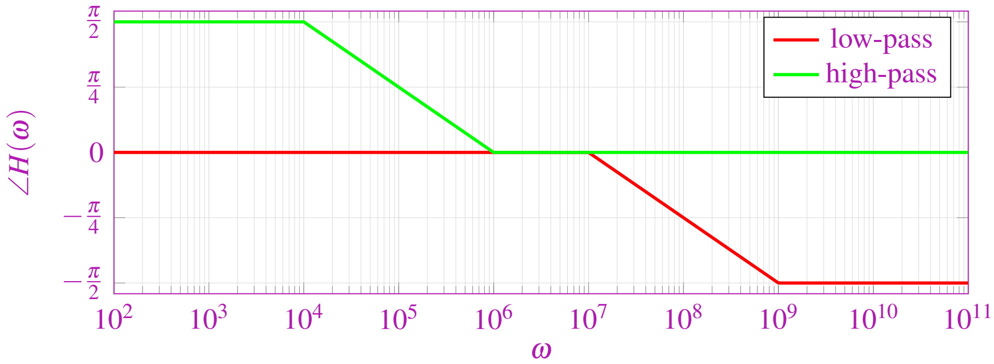
Solution: For both the magnitude and the phase plots, you can "add" the high-pass filter to the low-pass filter.

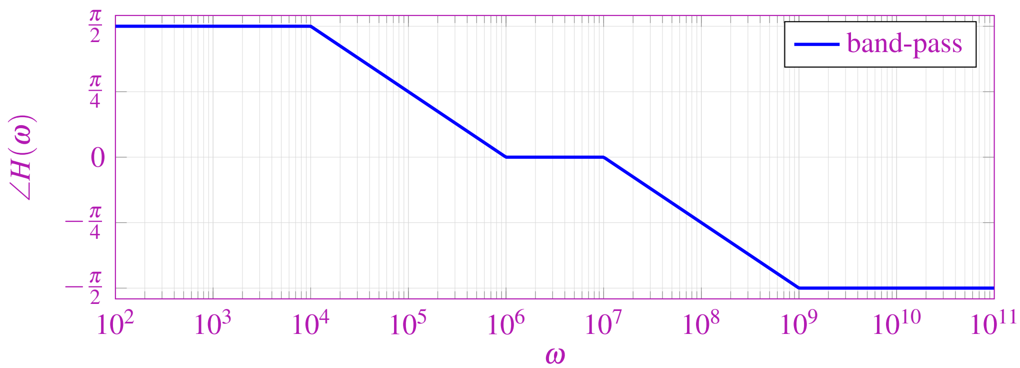
Magnitude plot:





Phase plot:





3. RLC Bandstop Filter

One way to compose a bandpass filter is by combining a low pass and high pass filter via a buffer. Another way to compose a bandpass is to use a RLC circuit of the following form:

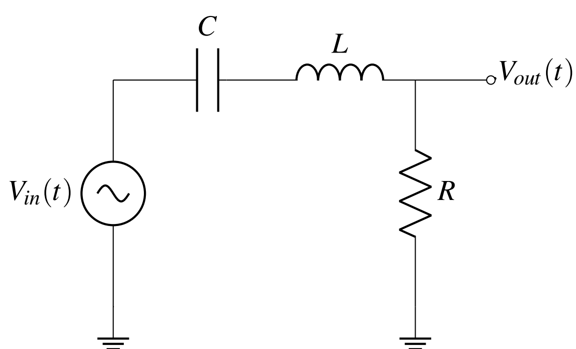


Figure 3: RLC Bandpass Filter

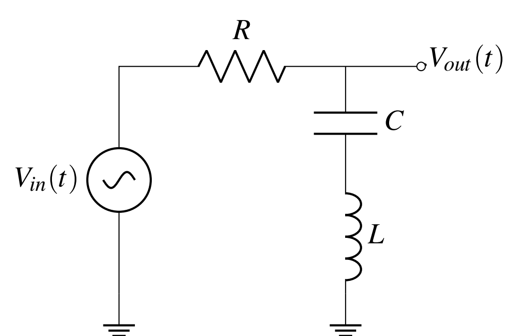


Figure 4: Unknown Behavior

Let's explore what happens when swap the location of the resistor with the location of the capacitor and inductor.

(a) Write out the transfer function of the circuit of unknown behavior.

Solution: Note that $j = -\frac{1}{j}$. This can be proved by multiplying both sides by j. Thus,

$$H_{notch}(j\omega) = \frac{j(L\omega - \frac{1}{C\omega})}{R + j(L\omega - \frac{1}{C\omega})}$$

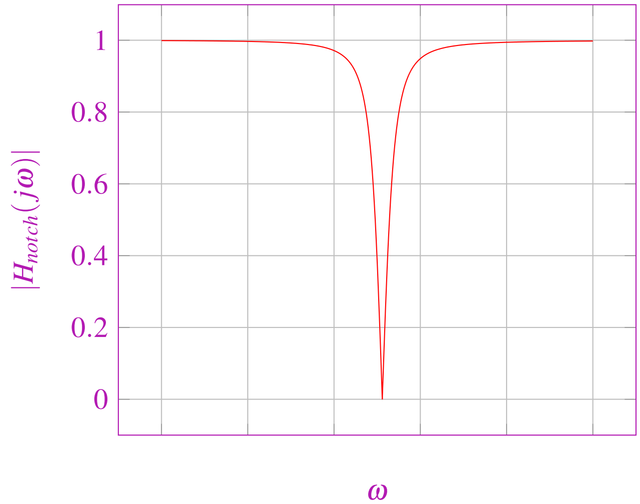
(b) What is the magnitude of the transfer function?

Solution:

$$\left|H_{notch}(j\omega)\right| = \frac{\sqrt{(L\omega - \frac{1}{C\omega})^2}}{\sqrt{R^2 + (L\omega - \frac{1}{Cw})^2}}$$

(c) Sketch the magnitude of this transfer function.

HINT: The name of this question is bandstop filter. What do you think that means?



Solution:

Notice that the magnitude of the transfer function is approximately one outside of some very small region.

(d) When is the magnitude of the transfer function zero? When is it one?

Solution: The magnitude of the transfer function is zero precisely when

$$L\omega = \frac{1}{C\omega}$$

Everywhere else, the magnitude is approximately one. Notice that as $\omega \to \infty$ or $\omega \to 0$, $|H(j\omega)| \approx 1$. The steepness of the curve is controlled by our parameters R, C, ω. Knowing how fast or slow our curve changes isn't in scope for this course.

(e) How would you describe the behavior of this circuit? Why do you think circuits with this type of circuit behavior are classified as notch/bandstop filters? Why is this specific filter called a *resonsant* bandstop filter?

Solution: The circuit's behavior can be thought of as the complement or opposite to the bandpass filter. While the bandpass filter allows a range of frequencies through, the bandstop/notch stops a particular range of frequencies from passing through. This is also why it'd be called bandstop or notch. Bandstop is opposite of bandpass. Notch comes from how the graph of the magnitude looks like there's a notch. This filter is a specific type of bandstop filter in that it's resonant. Resonance occurs when at a specific frequencies, the impedances of circuit elements cancel each other. For this circuit, this occurs when $L\omega = \frac{1}{C\omega}$.

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