CSM
Spring 2022

EECS 16B
Worksheet 3

1. Introduction to Phasor Domain and Impedance

We consider sinusoidal voltages and currents of the general form:

$$v(t) = V_0 \cos(\omega t + \phi_v)$$
$$i(t) = I_0 \cos(\omega t + \phi_i)$$

where:

(a) V_0 is the voltage **magnitude/amplitude** and is the highest value of voltage $v(t)$ will attain at any time. Similarly, I_0 is the current amplitude.

(b) ω is the **frequency** of oscillation, corresponding to the sinusoid's period $T = \frac{2\pi}{\omega}$.

(c) ϕ_v and ϕ_i are the **phase** terms of the voltage and current respectively. These capture a delay, or shift, in time.

We know from Euler's identity that $e^{j\theta} = \cos(\theta) + j\sin(\theta)$. Using this identity, we can obtain an expression for $\cos(\theta)$ in terms of an exponential:

$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2}$$

Extending this to our voltage signal from above:

$$v(t) = V_0 \cos(\omega t + \phi_v) = V_0 \left(\frac{e^{j(\omega t + \phi_v)} + e^{-j(\omega t + \phi_v)}}{2} \right)$$

Now, since we know that the circuit will not change the frequency of the signal (since we saw that the solutions to the systems of differential equations will only produce linear combinations of solutions in the form $e^{j\omega t}$ with the same frequency ω), we can drop the $e^{j\omega t}$ term, as long as we remember that all signals related to the voltage will be sinusoidal with angular frequency ω. We can then notice that the second term in the expression is just the conjugate of the first term:

$$v(t) = V_0 \frac{e^{j\phi}}{2} e^{j\omega t} + V_0 \overline{\frac{e^{j\phi}}{2} e^{j\omega t}}$$

This means that we just need to analyze the component that isn't e^{jwt} and by symmetry don't have to look at the conjugate. The result is called the phasor form of this signal:

$$\boxed{\widetilde{V} = \frac{1}{2} V_0 e^{j\phi_v}}$$

The phasor representation contains the **magnitude** V_0 and **phase** ϕ_v of the signal, but not the time-varying portion. Phasors let us handle sinusoidal signals much more easily, letting us use circuit analysis techniques that we already know to analyze AC circuits. *Note that we can only use this if we know that our signal is a sinusoid.*

Within this standard form, the phasor domain representation is as follows. The general equation that relates cosines to phasors is below, where $\widetilde{V}$ is the phasor.

$$x(t) = A\cos(\omega t + \phi_v) \implies \widetilde{X} = \frac{1}{2}V_0 e^{j\phi_v}$$

In summary, the standard forms for voltage and current phasors are given below:

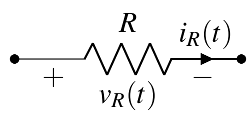
	Time Domain	Phasor Domain
Voltage	$v(t) = V_0\cos(\omega t + \phi_v)$	$\widetilde{V} = \frac{1}{2}V_0 e^{j\phi_v}$
Current	$i(t) = I_0\cos(\omega t + \phi_i)$	$\widetilde{I} = \frac{1}{2}I_0 e^{j\phi_i}$

We define the **impedance** of a circuit component to be: $\boxed{Z = \frac{\widetilde{V}}{\widetilde{I}}}$

$\widetilde{V}$ and $\widetilde{I}$ above represent the phasor representations of voltage across and the current through the component, respectively. Notice how $\widetilde{V} = \widetilde{I}Z$ mirrors Ohm's law for resistors.

In this problem, we will *derive the impedances of resistors, capacitors, and inductors*, which will extend Ohm's law and reveal a common method of phasor-domain analysis for all three circuit elements.

(a) Consider a resistor circuit below, with a sinusoidal current $i_R(t) = I_0\cos(\omega t + \phi)$.



Find the impedance of the resistor, $Z_R = \frac{\widetilde{v_R}}{\widetilde{I_R}}$**.**

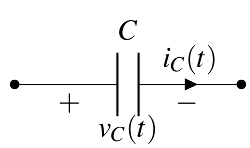
Hint: This part should be straightforward. When you don't know what to do, just write equations and substitute.

Solution: Using the given $i_R(t)$, we can pull out the phasor representation $\widetilde{I_R} = \frac{1}{2}I_0 e^{j\phi}$. To find $\widetilde{V_R}$, we first find $v_R(t)$ in time domain using the Ohm's law relationship. This gives $v_R(t) = Ri_R(t) = RI_0\cos(\omega t + \phi)$. Pulling out the phasor, we have $\widetilde{V_R} = \frac{1}{2}RI_0 e^{j\phi}$.

This leads us to conclude the impedance of a resistor is: $Z_R = \frac{\widetilde{V_R}}{\widetilde{I_R}} = \frac{\frac{1}{2}RI_0 e^{j\phi}}{\frac{1}{2}I_0 e^{j\phi}} = R$.

This means that the $\widetilde{I} - \widetilde{V}$ relationship in phasor domain is purely real.

(b) Consider a capacitor circuit below, with a sinusoidal voltage $v_C(t) = V_0\cos(\omega t + \phi)$.



Find the impedance of the capacitor, $Z_C = \frac{\widetilde{V_C}}{\widetilde{I_C}}$.

Hint: Use the known I-V capacitor relationship starting with the given $v_C(t)$ *to find the coefficient in front of* $\text{Re}(e^{j\omega t})$*, the phasor representation of current.*

Solution: We're given:

$$
\begin{aligned}
v_C(t) &= V_0 \cos(\omega t + \phi) \\
\implies \widetilde{V}_C &= \frac{1}{2} V_0 e^{j\phi}
\end{aligned}
$$

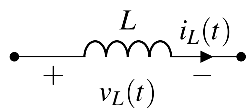
We apply the capacitor relationship to get the current through the capacitor as a function of time:

$$
\begin{aligned}
i_C(t) &= C \frac{d}{dt} v_C(t) \\
&= \frac{d}{dt} C V_0 \cos(\omega t + \phi) \\
&= C V_0 \frac{d}{dt} \cos(\omega t + \phi) \\
&= C V_0 \frac{d}{dt} \left(\frac{1}{2} e^{j(wt+\phi)} + \frac{1}{2} e^{-j(wt+\phi)} \right) \\
&= C V_0 \left((j\omega) \frac{1}{2} e^{j(wt+\phi)} + -(j\omega) \frac{1}{2} e^{-j(wt+\phi)} \right) \\
\implies \widetilde{I}_C &= \frac{1}{2} j\omega C V_0 e^{j\phi}
\end{aligned}
$$

Thus, the impedance of a capacitor is:

$$
\begin{aligned}
Z_C &= \frac{\widetilde{V}_C}{\widetilde{I}_C} \\
&= \frac{\frac{1}{2} V_0 e^{j\phi}}{\frac{1}{2} j\omega C V_0 e^{j\phi}} \\
&= \frac{1}{j\omega C}
\end{aligned}
$$

(c) Consider an inductor circuit below, with a sinusoidal current $i_L(t) = I_0 \cos(\omega t + \phi)$.



Find the impedance of the inductor, $Z_L = \frac{\widetilde{V}_L}{\widetilde{I}_L}$.

Solution: We're given:

$$
\begin{aligned}
i_L(t) &= I_0 \cos(\omega t + \phi) \\
\implies \widetilde{I}_L &= \frac{1}{2} I_0 e^{j\phi}
\end{aligned}
$$

We apply the inductor relationship to get the voltage across the inductor as a function of time:

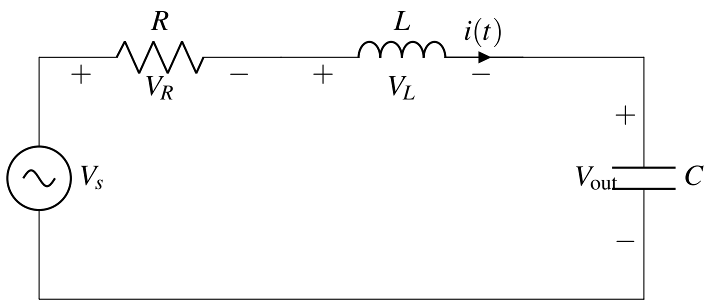
$$\begin{aligned}
v_L(t) &= L\frac{d}{dt}i_L(t) \\
&= \frac{d}{dt}LI_0\cos(\omega t+\phi) \\
&= LI_0\frac{d}{dt}\cos(\omega t+\phi) \\
&= LI_0\frac{d}{dt}\left(\frac{1}{2}e^{j(wt+\phi)}+\frac{1}{2}e^{-j(wt+\phi)}\right) \\
&= LI_0\left((j\omega)\frac{1}{2}e^{j(wt+\phi)}+-(j\omega)\frac{1}{2}e^{-j(wt+\phi)}\right) \\
\implies \widetilde{V}_L &= \frac{1}{2}j\omega LI_0e^{j\phi}
\end{aligned}$$

Thus, the impedance of an inductor is:

$$\begin{aligned}
Z_L &= \frac{\widetilde{V}_L}{\widetilde{I}_L} \\
&= \frac{\frac{1}{2}j\omega LI_0e^{j\phi}}{\frac{1}{2}I_0e^{j\phi}} \\
&= j\omega L
\end{aligned}$$

2. RLC circuit Phasor Analysis

In this question, we will take a look at an electrical systems described by second order differential equations and analyze it using the phasor domain. Consider the circuit below where $R = 3\text{k}\Omega$, $L = 1\text{mH}$, $C = 100\text{nF}$, and $V_s = 5\cos\left(1000t + \frac{\pi}{4}\right)$:



(a) What are the impedances of the resistor, inductor and capacitor, Z_R, Z_L, and Z_C?

Solution:

The impedance of a resistor is the same as its resistance

$$Z_R = 3000\Omega$$

We can find the frequency of the circuit by looking at V_s. The form of a cosine function is $A\cos(\omega t + \phi)$ where A is amplitude, ω is frequency, and ϕ is phase. In this case, the frequency is $1000\frac{\text{rad}}{\text{s}}$

$$Z_L = j\omega L = j1000 * 10^{-3} = j1\Omega$$
$$Z_C = \frac{1}{j\omega C} = \frac{1}{j1000 * 10^{-7}} = -j10^4\Omega$$

(b) Solve for $\widetilde{V}_{\text{out}}$ in phasor form.

Solution:

converting V_s into phasor form, we have

$$\widetilde{V}_s = \frac{1}{2}|A|e^{j\phi} = \frac{5}{2}e^{j\frac{\pi}{4}}$$

The circuit given is a voltage divider. Since impedances act like resistors, we can use the same equation as the resistive voltage divider.

$$\widetilde{V}_{\text{out}} = \widetilde{V}_s \frac{Z_C}{Z_R + Z_L + Z_C} = \widetilde{V}_s * \left|\frac{Z_C}{Z_R + Z_L + Z_C}\right| e^{j*\angle\left(\frac{Z_C}{Z_R+Z_L+Z_C}\right)} \tag{1}$$

We can solve for the magnitude and angle of the divider using

$$\left|\frac{Z_C}{Z_R + Z_L + Z_C}\right| = \frac{|Z_C|}{|Z_R + Z_L + Z_C|} = \frac{10^4}{\sqrt{3000^2 + (1 - 10^4)^2}} = 0.958$$
$$\angle\left(\frac{Z_C}{Z_R + Z_L + Z_C}\right) = \angle(Z_C) - \angle(Z_R + Z_L + Z_C) = \text{atan2}\left(\frac{\mathfrak{Im}(Z_C)}{\mathfrak{Re}(Z_C)}\right) - \text{atan2}\left(\frac{\mathfrak{Im}(Z_R + Z_L + Z_C)}{\mathfrak{Re}(Z_R + Z_L + Z_C)}\right)$$
$$\angle\left(\frac{Z_C}{Z_R + Z_L + Z_C}\right) = -0.2915 \text{ rad}$$

Plugging back into (1)

$$\widetilde{V}_{\text{out}} = \frac{5}{2}e^{j\frac{\pi}{4}} * 0.958e^{-j0.2915} = 2.395e^{j0.494}$$

(c) What is V_{out} in the time domain?

Solution:

$$V_{\text{out}}(t) = 4.79\cos(1000t + 0.494) \text{ V}$$

(d) Solve for the current $i(t)$

Solution:

$$\widetilde{i} = \frac{\widetilde{V}_{\text{s}}}{Z_R + Z_L + Z_C} = \frac{|\widetilde{V}_{\text{s}}|}{|Z_R + Z_L + Z_C|}e^{j\left(\angle\widetilde{V}_{\text{s}} - \angle(Z_R+Z_L+Z_C)\right)} = 2.395 * 10^{-4}e^{j2.0647}$$

Going back to the time domain:

$$i(t) = 4.790 * 10^{-4}\cos(1000t + 2.0647) \text{ A}$$

(e) Solve for the transfer function $H(\omega) = \frac{\widetilde{V}_{\text{out}}}{\widetilde{V}_{\text{s}}}$

Leave your answer in terms of R, L, C, and ω.

Solution:

Looking back at equation (1)

$$\widetilde{V}_{\text{out}} = \widetilde{V}_{\text{s}}\frac{Z_C}{Z_R + Z_L + Z_C}$$

Rearranging we get

$$H(\omega) = \frac{\widetilde{V}_{\text{out}}}{\widetilde{V}_{\text{s}}} = \frac{Z_C}{Z_R + Z_L + Z_C} = \frac{\frac{1}{j\omega C}}{R + j\omega L + \frac{1}{j\omega C}}$$

$$H(\omega) = \frac{1}{LC(j\omega)^2 + jRC\omega + 1}$$

3. **Transfer Functions and Filters**

Earlier in the worksheet we looked at the notion of phasors and how we can use them for circuits with sinusoidal inputs. However, we don't want to perform the earlier computation whenever we have a sinusoidal input. We want to understand the circuit from the perspective of an input/output relationship like we did for 16A (ie. voltage divider, amplifier, etc.).

In 16A, we commonly discussed the input-output relationship of circuits through a term called gain,

$$G = \frac{V_{out}(t)}{V_{in}(t)} \tag{1}$$

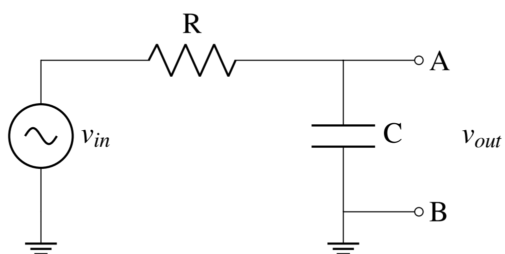
Thus, we will now introduce a similar notion of gain, for frequency/phasor analysis called a transfer function. **The transfer function of a circuit is defined as**

$$H(j\omega) = \frac{\widetilde{V}_{out}}{\widetilde{V}_{in}} \tag{2}$$

Notice that the transfer function depends on the frequency of the input voltage, and will be a complex number for a given frequency.

For the purposes of this question, we will consider various circuit configurations, and study their behavior.

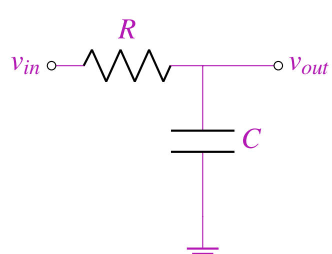
(a) Consider the following RC circuit:



i. **Write out the transfer function $H(j\omega)$.**
ii. **For values of ω approaching 0, find $|H(j\omega)|$.**
iii. **For values of ω approaching ∞, find $|H(j\omega)|$.**
iv. **Calculate the cutoff frequency of this circuit.**

Solution:

i. The circuit can be simplified as:



We recognize the circuit is a voltage divider. Let $\widetilde{V}_{out}$ and $\widetilde{V}_{in}$ be voltage phasors. Using the voltage divider equation with impedance, we have

$$\widetilde{V}_{out} = \frac{Z_C}{Z_R + Z_C}\widetilde{V}_{in} = H(j\omega)\widetilde{V}_{in}.$$

Notice that the above equation uses impedance and voltage phasors instead of resistance and voltage. From the above equation, the circuit has the transfer function

$$H(j\omega) = \frac{Z_C}{Z_R + Z_C} = \frac{\frac{1}{j\omega C}}{R + \frac{1}{j\omega C}} = \frac{1}{1 + j\omega RC},$$

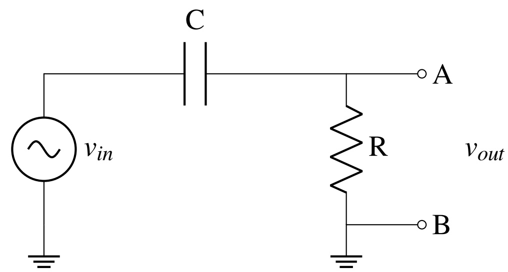
ii. For values of ω close to 0, $\left|H(j\omega)\right|$ approaches 1. In this case, $\widetilde{V}_{out} = \widetilde{V}_{in}$, meaning that low frequencies will pass through this filter.

iii. For values of ω approaching ∞, $\left|H(j\omega)\right|$ approaches 0. In this case, $\widetilde{V}_{out} = 0$, meaning that high frequencies will be blocked by this filter.

Since this filter allows low frequencies to go through and blocks high frequencies, it is called a *low-pass filter*.

iv. The cutoff frequency is the frequency that causes the magnitude of the transfer function to be $\frac{1}{\sqrt{2}}$. Thus the cutoff frequency is $\omega = \frac{1}{RC}$.

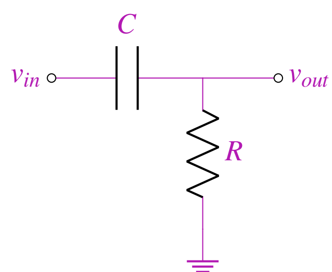
(b) Consider the following CR circuit:



i. **Write out the transfer function $H(j\omega)$.**
ii. **For values of ω approaching 0, find $\left|H(j\omega)\right|$.**
iii. **For values of ω approaching ∞, find $\left|H(j\omega)\right|$.**
iv. **Calculate the cutoff frequency of this circuit.**

Solution:

i. The circuit can be simplified as:



Using the voltage divider equation with impedance, we have

$$\widetilde{V}_{out} = \frac{Z_R}{Z_R + Z_C}\widetilde{V}_{in} = H(j\omega)\widetilde{V}_{in}.$$

Thus, the circuit has the transfer function

$$H(j\omega) = \frac{Z_R}{Z_R + Z_C} = \frac{R}{R + \frac{1}{j\omega C}} = \frac{j\omega RC}{1 + j\omega RC}.$$

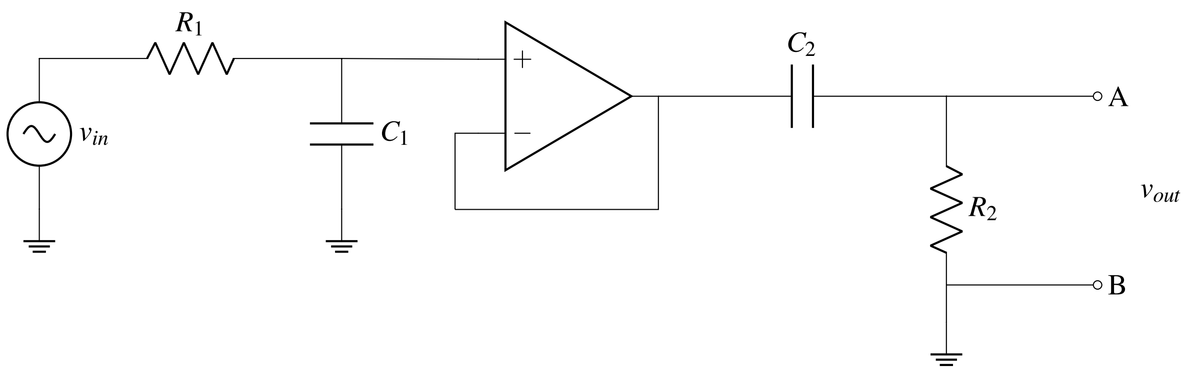
ii. For values of ω close to 0, $\left|H(j\omega)\right|$ approaches 0. In this case, $\widetilde{V}_{out} = 0$, it means that low frequencies will be blocked by this filter.

iii. For values of ω approaching ∞, $\left|H(j\omega)\right|$ approaches 1. In this case, $\widetilde{V}_{out} = \widetilde{V}_{in}$, meaning high frequencies will pass through this filter.

Since this filter allows high frequencies to go through and blocks low frequencies, it is called a *high-pass filter*.

iv. The magnitude of the transfer function is $\frac{1}{\sqrt{2}}$ when $\omega = \frac{1}{RC}$.

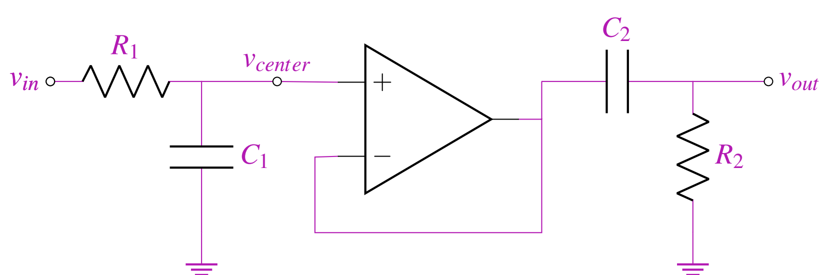
(c) Consider the following cascaded circuit (joined by a buffer):



i. **Write out the transfer function $H(j\omega)$.**
ii. **For values of ω approaching 0, find $\left|H(j\omega)\right|$.**
iii. **For values of ω approaching ∞, find $\left|H(j\omega)\right|$.**

Solution:

i. The circuit can be simplified as:



In this circuit, the op-amp acts as a unity gain buffer which connects the two filters together. On the left of the op-amp is a low-pass filter, and on the right of the op-amp is a high-pass filter. Let $H_1(j\omega)$ and $H_2(j\omega)$ be transfer functions for the low-pass and high-pass filters respectively. We can write $\widetilde{V}_{out}$ in terms of $H_1(j\omega)$, $H_2(j\omega)$, and $\widetilde{V}_{in}$

$$\widetilde{V}_{out} = H_1(j\omega)H_2(j\omega)\widetilde{V}_{in}.$$

Thus, the circuit has the transfer function

$$H(j\omega) = H_1(j\omega)H_2(j\omega) = \frac{1}{1+j\omega R_1C_1}\frac{j\omega R_2C_2}{1+j\omega R_2C_2}.$$

ii. Using the above transfer function, we have

$$\left|H(j\omega)\right| = \left|H_1(j\omega)H_2(j\omega)\right| = \left|H_1(j\omega)\right|\left|H_2(j\omega)\right|.$$

Since $H_1(j\omega)$ is a transfer function of a low-pass filter and $H_2(j\omega)$ is a transfer function of a high-pass filter, for values of ω approaching 0, $\left|H(j\omega)\right|$ approaches 0. Therefore, $\widetilde{V}_{out} = 0$, meaning low frequencies are attenuated by this filter.

iii. For values of ω approaching ∞, $\left|H(j\omega)\right|$ also approaches 0. Therefore, $\widetilde{V}_{out} = 0$; meaning high frequencies are also attenuated by this filter.
Since this filter blocks both low and high frequencies, and it allows signals of a certain frequency range (a band of frequencies) to pass through, it is called a *band-pass filter*.

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