CSM
Spring 2022

EECS 16B
Worksheet 5

1. An Introduction to Systems

Many physical systems such as the motion of a car, can be modeled using a system. Often times, when we are describing a system, we will have a **state variable** $\vec{x}$, that will often be a multivariable function. For a given system, we can often write a differential equation describing its change over time as

$$\frac{d\vec{x}(t)}{dt} = f(\vec{x}(t), \vec{u}(t)) \tag{1}$$

In this problem, we will examine a specific form of systems that can be put in **state-space representation.**

For a continuous-time linear systems (we will define what it means to be linear later) the general state-space representation is shown below:

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + B\vec{u}(t) \tag{2}$$

Similarly, for a discrete-time linear system the general state-space representation is shown below, replacing derivatives with respect to time with recursive difference equations:

$$\vec{x}[n+1] = A\vec{x}[n] + B\vec{u}[n] \tag{3}$$

Where A is the $n \times n$ state matrix, $\vec{x}$ is a state vector in $\mathbb{R}^n$, B is a $n \times d$ input matrix, and $\vec{u}$ is an input vector in $\mathbb{R}^d$. We will usually consider a B as a vector in $\mathbb{R}^n$ and $u(t)$ will be a scalar input. Intuitively, A acts as a linear function that determines how a future state depends on the current state of the world, and B explains how an action or input that we introduce affects our system.

Tying this back to the circuits we've analyzed, an example of a state variable could be the voltage $V_C(t)$ across a capacitor or the current $I_L(t)$ through an inductor, and an example input could be the input voltage of a system $V_{in}(t)$.

Consider the following system:

$$\frac{d}{dt}x_1(t) = 3x_1(t) - 2x_2(t) + 4$$
$$\frac{d}{dt}x_2(t) = -x_1(t) + 5x_2(t) + 2$$

The initial conditions of the state variables are $x_1(0) = 2,\ x_2(0) = 3$.

(a) **What is the state vector $\vec{x}(t)$ for this system?**

Solution: We have to variables x_1 and x_2 therefore we define our state vector as:

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

(b) **What is the initial condition $\vec{x}(0)$ of this system?**

Solution: We have the individual initial conditions for x_1 and x_2 but we must also a define an initial condition for our state vector.

$$\vec{x}(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

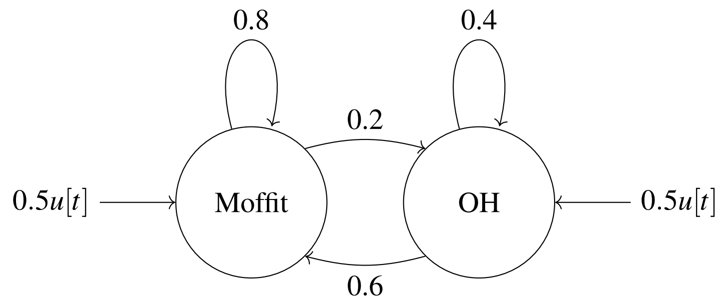
(c) **Write out the system of differential equations the form of a general continuous time state-space model.**

Solution:

$$\frac{d}{dt}\vec{x}(t) = A \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \vec{b} = \begin{bmatrix} 3 & -2 \\ -1 & 5 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

2. Intro to Discrete-Time Systems

Students are studying for the EECS16C exam, and the flow of students from Moffit to Taejin's Office Hours (OH) can be represented as the following:



where $u[t]$ is the number of students that start to study at timestep t. (i.e.: $u[t]$ is the input to the system)

(a) Let our state variables be represented by x_1 and x_2. **Explain in your own words what the state variables x_1 and x_2 could represent in our system** **Solution:** We can represent our state by the number of students in Moffit (x_1) and the number of students in Taejin's office hours (x_2).

(b) Represent the flow of students between the two states as a matrix-vector discrete time system:

$$\vec{x}[t+1] = A\vec{x}[t] + \vec{b}u[t]$$

Find matrix A and vector $\vec{b}$.
Solution:

$$\begin{bmatrix} x_1[t+1] \\ x_2[t+1] \end{bmatrix} = \begin{bmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{bmatrix} \begin{bmatrix} x_1[t] \\ x_2[t] \end{bmatrix} + \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} u[t]$$

(c) Let $\vec{x}[0] = \vec{0}$, and $u[t] = 10$ for all values of t. **What is $\vec{x}[1]$? What is $\vec{x}[2]$? What is $||\vec{x}||$ as $t \to \infty$? Does that make sense in the context of our problem?**
Solution:

$$\vec{x}[1] = A\vec{0} + 10 \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

$$\vec{x}[2] = A \begin{bmatrix} 5 \\ 5 \end{bmatrix} + 10 \begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{bmatrix} \begin{bmatrix} 5 \\ 5 \end{bmatrix} + \begin{bmatrix} 5 \\ 5 \end{bmatrix} = \begin{bmatrix} 12 \\ 8 \end{bmatrix}$$

The input to the system is always positive, and no students ever leave the system, so the magnitude of the state keeps on growing. So,

$$\lim_{t\to\infty} ||\vec{x}[t]|| = \infty$$

This does not make sense in the context of our problem because it is impossible to have an infinite amount of students.

(d) Let $\vec{x}[0] = \begin{bmatrix} 50 \\ 50 \end{bmatrix}$, and $u[t] = -4$ for all values of t. **What is $\vec{x}[1]$? What is $\vec{x}[2]$? What is $||\vec{x}||$ as $t \to \infty$? What sign are the elements of $\vec{x}$? Does that make sense in the context of our problem?**

Solution:

$$\vec{x}[1] = A\begin{bmatrix} 50 \\ 50 \end{bmatrix} - 4\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{bmatrix}\begin{bmatrix} 50 \\ 50 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 68 \\ 28 \end{bmatrix}$$
$$\vec{x}[2] = A\begin{bmatrix} 68 \\ 28 \end{bmatrix} - 4\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{bmatrix}\begin{bmatrix} 68 \\ 28 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 69.2 \\ 22.8 \end{bmatrix}$$

It's ok that the state variables are not whole numbers! Think of them as the average number of students in Moffit and Taejin's OH.

The input to the system is always negative, and no students are ever added to the system, so the magnitude of the state keeps on decreasing linearly. So, **the sign of the elements of $\vec{x}$ is negative** and

$$\lim_{t\to\infty} ||\vec{x}[t]|| = \infty$$

This does not make sense in the context of our problem because there is an infinitely negative amount of students in our system.

(e) Let $\vec{x}[0] = \vec{0}$, $u[0] = 16$, and $u[t > 0] = 0$. **What is $\vec{x}[1]$? What is $\vec{x}[2]$? What is the largest $||\vec{x}||$ can get as $t \to \infty$? Does that make sense in the context of our problem?**

Solution:

$$\vec{x}[1] = A\vec{0} + 16\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 8 \\ 8 \end{bmatrix}$$
$$\vec{x}[2] = A\begin{bmatrix} 8 \\ 8 \end{bmatrix} + 0\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 0.8 & 0.6 \\ 0.2 & 0.4 \end{bmatrix}\begin{bmatrix} 8 \\ 8 \end{bmatrix} = \begin{bmatrix} 11.2 \\ 4.8 \end{bmatrix}$$

The input to the system is 0 after the first time step, and the sum of students in the two states stays the same each timestep, so, by the triangle inequality, the magnitude of $\vec{x}$ will remain less than or equal to the sum of the students in the two states after the first timestep.

$$\lim_{t\to\infty} ||\vec{x}[t]|| \leq 16$$

This *does* make sense in the context of our problem because it is a finite, positive number.

3. BIBO Stability

In this question, we will investigate into the definitions of stability for a scalar system modeled by a first order differential equation of the form with the initial condition $x(0) = x_0$.

$$\frac{d}{dt}x(t) = \lambda x(t) + u(t) \tag{4}$$

When we are discussing the stability of a system, we want to see if this system will produce a bounded output $\vec{y}(t)$for every bounded input $\vec{u}(t)$. Therefore we will say that this system is **BIBO stable** if for every bounded input $u(t)$, the output $y(t)$ is bounded as well.

As a reference, a function $f(t)$ is bounded by a constant B if: $\left|f(t)\right| \leq B < \infty$ for all values of t.

The output $y(t)$ will be a function of $x(t)$ in the form:

$$y(t) = \alpha x(t) + \beta u(t) \tag{5}$$

As $y(t)$ is a linear combination of $x(t)$, and an already assumed to be bounded input $u(t)$, showing that $y(t)$ is bounded is equivalent to showing that $x(t)$ is bounded.

Recall that the particular solution to the differential equation (4), was uniquely determined for $t \geq 0$ as:

$$x_p(t) = x_0 e^{\lambda t} + \int_0^t u(\tau) e^{\lambda(t-\tau)} d\tau \tag{6}$$

Although λ can be complex, we can observe that $|e^{(a+bj)}| = |e^a||e^{bj}| = |e^a||\cos(b) + j\sin(b)| = |e^a|$.

The, $\mathfrak{Im}(\lambda)$ does not affect stability, and will correspond to oscillations that are bounded. Therefore, we will only consider the effects of $\mathfrak{Re}(\lambda)$ which affect the stability of a system, and for the purposes of this question, assume that λ is a real number.

(a) We will start with a bounded input $u(t) = 0$. Check if $x_p(t)$ is bounded for the three following cases:

(i) $\lambda > 0$

(ii) $\lambda = 0$

(iii) $\lambda < 0$

Solution: Remember that in order for a function, $x(t)$, to be bounded, its absolute value must be less than a constant B **for all** values of t.

For this zero input, the solution to the differential equation for $t \geq 0$, will be $x_p(t) = x_0 e^{\lambda t}$.

(i) If $\lambda > 0$, then $x_p(t) = x_0 e^{\lambda t}$, which is a strictly increasing function. As $t \to \infty, x_p(t) \to \infty$ which implies that $x_p(t)$ is unbounded.

(ii) If $\lambda = 0$, then $x_p(t) = x_0 e^{0 \cdot t} = x_0$. Therefore, $x_p(t)$ is bounded by x_0.

(iii) If $\lambda < 0$, then $x_p(t) = x_0 e^{\lambda t}$ which is a strictly decreasing function. As $t \to \infty, x_p(t) \to 0$ so this function is bounded by its initial condition x_0.

(b) True/False: Since $x_p(t)$ is unbounded for $\lambda > 0$, we can say that the system **is not** BIBO stable, for $\lambda > 0$.

Solution: True, remember that a system is BIBO stable if **every** bounded input has a bounded output. However, we gave a bounded input $u(t) = 0$ and the output was unbounded, so this system is NOT BIBO stable.

(c) True/False: Since $x_p(t)$ is bounded for $\lambda \geq 0$, we can say that the system **is** BIBO stable, for $\lambda \geq 0$.

Solution: False, remember that a system is BIBO stable if **every** bounded input has a bounded output. However, we have only shown one example of a bounded input, having a bounded output. Therefore, we cannot say that **every** bounded input has a bounded output. These systems may or may not be BIBO stable, but at the moment, we cannot conclude that they are BIBO stable.

(d) Now let's consider an input $u(t) = e^{\lambda t}$, can you say anything about the BIBO stability for $\lambda = 0$?

Solution: If $u(t) = e^{\lambda t} = 1$ then,

$$x_p(t) = x(0)e^{\lambda t} + \int_0^t u(\tau)e^{\lambda(t-\tau)}d\tau = x(0) + \int_0^t e^{0\cdot(t-\tau)}d\tau = x(0) + \int_0^t 1 d\tau = x(0) + t$$

As $t \to \infty$, $x_p(t) \to \infty$ which implies that for $\lambda = 0$, the system is **not** BIBO stable.

(e) How can we show that when $\lambda < 0$, the system is indeed BIBO stable? ' You should start by assuming you have a bounded input $u(t)$ such that $\left|u(t)\right| \leq B$. *Hint:* $\left|\int x(t)dt\right| \leq \int \left|x(t)\right|$.

Solution: Suppose we have a bounded input $u(t)$ such that $\left|u(t)\right| \leq B$. We know our solution is:

$$x_p(t) = x(0)e^{\lambda t} + \int_0^t u(\tau)e^{\lambda(t-\tau)}d\tau.$$

We can first use the triangle inequality:

$$\left|x_p(t)\right| = \left|x(0)e^{\lambda t} + \int_0^t u(\tau)e^{\lambda(t-\tau)}d\tau\right| \leq \left|x(0)e^{\lambda t}\right| + \left|\int_0^t u(\tau)e^{\lambda(t-\tau)}\right|$$

We already know that $x(0)e^{\lambda t}$ is bounded by $x(0)$ when $\lambda < 0$. Therefore

$$\left|x_p(t)\right| \leq \left|x(0)\right| + \left|\int_0^t u(\tau)e^{\lambda(t-\tau)}\right|$$

It now remains to show that the integral is bounded as well. We start by pulling out the magnitude of $e^{\lambda t}$ out of the absolute value.

$$\left|\int_0^t u(\tau)e^{\lambda(t-\tau)}d\tau\right| = \left|e^{\lambda t}\right|\left|\int_0^t u(\tau)e^{-\lambda(\tau)}d\tau\right|$$

Then we apply the hint,

$$\left|e^{\lambda t}\right|\left|\int_0^t u(\tau)e^{-\lambda(\tau)}d\tau\right| \leq \left|e^{\lambda t}\right|\int_0^t \left|u(\tau)e^{-\lambda\tau}\right|d\tau$$

Then we use the fact that $u(t)$ is bounded by B.

$$\left|e^{\lambda t}\right|\int_0^t \left|u(\tau)e^{-\lambda\tau}\right|d\tau \leq \left|e^{\lambda t}\right|\int_0^t \left|Be^{-\lambda\tau}d\tau\right| = B\left|e^{\lambda t}\right|\int_0^t \left|e^{-\lambda\tau}d\tau\right|$$

Since all of our functions are positive for $t \geq 0$, we can get rid of the absolute values and take the integral from 0 to t:

$$\begin{aligned}
\left|x_p(t)\right| &\leq |x_0| + B\left|e^{\lambda t}\right|\left|\int_0^t\right|e^{-\lambda\tau}d\tau\Big| = |x_0| + Be^{\lambda t}\int_0^t e^{-\lambda\tau}d\tau \\
&= |x_0| + Be^{\lambda t}\left(-\frac{1}{\lambda}e^{-\lambda\tau}\right)\Big|_0^t = |x_0| - \frac{B}{\lambda}e^{\lambda t}(e^{-\lambda t}-1) = |x_0| + \frac{B}{\lambda}(1-e^{\lambda t})
\end{aligned}$$

The quantity $(1-e^{\lambda t})$ is bounded by 1 since $e^{\lambda t}$ is between 0 and 1 for all values of t. Therefore,

$$\left|x_p(t)\right| \leq |x_0| + \frac{B}{\lambda}$$

We have shown that every bounded input $u(t)$ has a bounded output, so the system is BIBO stable for $\lambda < 0$.

4. Aperture Stability

As an intern at Aperture Laboratories, it is your job to make sure the robots being built are stable systems. As a reminder, if the following conditions are met the system will be stable:

- For discrete time systems of the form:

$$\vec{x}[t+1] = A\vec{x}[t] + Bu[t] + \vec{w}[t] \tag{7}$$

All eigenvalues of the matrix A, λ_i, have magnitude $|\lambda_i| < 1$.

- For continuous time systems of the form:

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + Bu(t) + \vec{w}(t) \tag{8}$$

All eigenvalues of the matrix A, λ_i, have real part $\mathfrak{Re}(\lambda_i) < 0$.

(a) According to your boss, the first robot, GLaDOS, can be described with the following discrete time system:

$$\vec{x}[t+1] = \begin{bmatrix} \frac{3}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{3}{8} \end{bmatrix} \vec{x}[t] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u[t]$$

Is she stable?

Solution:

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} \frac{3}{8} - \lambda & \frac{1}{8} \\ \frac{1}{8} & \frac{3}{8} - \lambda \end{bmatrix} \right) = \left(\frac{3}{8} - \lambda\right)\left(\frac{3}{8} - \lambda\right) - \left(\frac{1}{8}\right)\left(\frac{1}{8}\right)$$
$$= \lambda^2 - \frac{3}{4}\lambda + \frac{1}{8} = \left(\lambda - \frac{1}{2}\right)\left(\lambda - \frac{1}{4}\right) = 0$$

Therefore, we see that

$$\lambda = \frac{1}{2}, \frac{1}{4}$$

Since the system is a discrete time system, and both eigenvalues have magnitude smaller than 1, GLaDOS is stable.

(b) Your boss now gives you data on the P-body robot. Is she stable? Her motion is described with the following discrete time system:

$$\vec{x}[t+1] = \begin{bmatrix} -2 & -1 \\ 1 & -2 \end{bmatrix} \vec{x}[t] + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u[t]$$

Solution:

$$\det(A - \lambda I) = \left(\begin{bmatrix} -2 - \lambda & -1 \\ 1 & -2 - \lambda \end{bmatrix} \right) = (-2 - \lambda)(-2 - \lambda) - (-1)(1)$$
$$= \lambda^2 + 4\lambda + 5 = (\lambda - (-2 + j))(\lambda - (-2 - j)) = 0$$

Therefore we can compute the eigenvalues as

$$\lambda = -2 \pm j$$

However, the magnitude of both of these eigenvalues are $|\lambda| = \sqrt{5} \geq 1$. Therefore, Atlas is unstable.

(c) Now your boss gives you data on a more advanced robot, Atlas. Is he stable? His movements can be described with the following continuous time system:

$$\frac{d}{dt}\vec{x}(t) = \begin{bmatrix} -2 & -1 \\ 1 & -2 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$

Solution: This is the exact same system as the previous system but in continuous time. The eigenvalues will be the same:

$$\lambda = -2 \pm j$$

This time, since this is a continuous time system, the conditions for stability have changed. Since $\mathfrak{Re}(\lambda) < 0$ for both eigenvalues, we conclude by saying that this system is stable.

(d) Lastly, your boss gives you data on the Wheatley robot. Is he stable? His motion is described with the following discrete time system:

$$\vec{x}[t+1] = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \vec{x}[t] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u[t]$$

Solution: You can also note that A is the rotation matrix, and make the observation that the eigenvalues are on the unit circle. Without even doing any calculations, we know Wheatley is unstable since $|\lambda| = 1$.

This can also be realized by computing the eigenvalues in the following manner:

$$\det(A - \lambda I) = \det\left(\begin{bmatrix} \frac{\sqrt{3}}{2} - \lambda & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} - \lambda \end{bmatrix} \right) = (\frac{\sqrt{3}}{2} - \lambda)^2 + \frac{1}{4} = \lambda^2 - \sqrt{3}\lambda + 1 = 0$$

Using the quadratic formula, we see that the eigenvalues are:

$$\lambda = \frac{\sqrt{3}}{2} \pm \frac{1}{2}\sqrt{(-\sqrt{3})^2 - 4 \cdot 1} = \frac{\sqrt{3}}{2} \pm \frac{1}{2}j$$

Intuitively, you can see that this system is unstable since it will continue rotating and will never converge to a steady state.

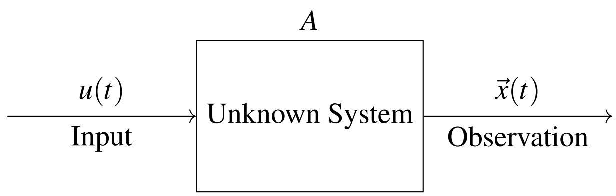
5. System Identification

In this question, we will take a look at how to **identify** a system by taking experimental data taken from a (presumably) linear system to learn a discrete-time linear model for it using the least-squares.

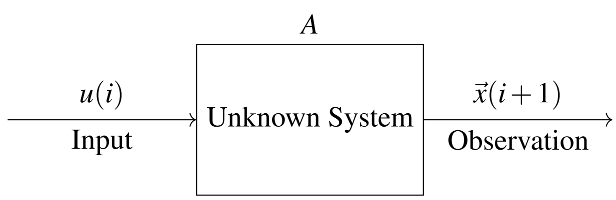
Recall that a **linear, continuous-time,** system can be put in state-space form:

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + \vec{b}u(t) \tag{9}$$

Now let's say we have an **unknown** linear system in which we can give an input $u(t)$ and observe the output $\vec{x}(t)$. We can model the system using the following diagram:



Recall from discussion that if we put a **piecewise constant** input $u(t) = u(i)$ for $t \in [i, i+1)$, then we can observe the output $\vec{x}(t)$ at time $t = i+1$, and form a discretized model of the observation.



If we knew the system, the relationship between $\vec{x}(i+1), \vec{x}(i)$, and $u(i)$ would be:

$$\vec{x}(i+1) = A\vec{x}(i) + \vec{b}u(i) \tag{10}$$

While this relation is useful, we currently do not know what the A matrix or $\vec{b}$ vector are.

Therefore, we will start by creating unknown variables for the A matrix, and $\vec{b}$ vector:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \text{ and } \vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \tag{11}$$

For the purposes of this question, we will be in the space $\mathbb{R}^2$.

(a) Let's say the system initially started at $\vec{x}(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$, and we gave an input at time $t = 0,\ u(0)$. At time $t = 1$, we observe $\vec{x}(1) = \begin{bmatrix} x_1(1) \\ x_2(1) \end{bmatrix}$. **How can you uncouple this matrix/vector equation into a system of linear equations?**

Solution: We start by writing out the matrix/vector equation for our unknown system:

$$\vec{x}(1) = \begin{bmatrix} x_1(1) \\ x_2(1) \end{bmatrix} = A\vec{x}(0) + \vec{b}u(0) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u(0) \tag{12}$$

Uncoupling these equations, we get:

$$x_1(1) = a_{11}x_1(0) + a_{12}x_2(0) + b_1u(0)$$
$$x_2(1) = a_{21}x_1(0) + a_{22}x_2(0) + b_2u(0)$$

(b) Based on the system of linear equations created in the previous part, **how many unknown** variables do we have? Also, if we have a system of linear equations with n unknown variables, at the minimum, **how many equations** would we need to solve our system?

Solution: The unknowns in this system of linear equations are: $a_{11}, a_{12}, a_{21}, a_{22}, b_1, b_2$.

If we have a system of linear equations with n unknown variables, we will need at least n equations to solve the system.

(c) We now give another input at $t = 1,\ u(1)$, and observe $\vec{x}(2)$.

How many more equations do we get from this observation? Also, how many more inputs will we need to observe until we have enough equations?

Solution: We can write out a similar observation as the one made in part(a):

$$\vec{x}(2) = \begin{bmatrix} x_1(2) \\ x_2(2) \end{bmatrix} = A\vec{x}(1) + \vec{b}u(1) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1(1) \\ x_2(1) \end{bmatrix} + \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} u(1) \tag{13}$$

Uncoupling these equations again, we will get:

$$x_1(2) = a_{11}x_1(1) + a_{12}x_2(1) + b_1u(1)$$
$$x_2(2) = a_{21}x_1(1) + a_{22}x_2(1) + b_2u(1)$$

Notice that for every observation we make at a given time step, we will get 2 more equations. This means we will have to look at a total of 3 time steps to get 6 equations. Taking the initial condition $\vec{x}(0)$ into account, we will have to observe a total of 4 inputs.

(d) Assuming we have taken all of the necessary measurements of $x(t)$ at time $t = 0, 1, 2, \ldots$

How can we set up our system of linear equations as a matrix-vector equation?

Solution: We can set up the following system of linear equations:

$$\begin{bmatrix} x_1(0) & x_2(0) & u(0) & 0 & 0 & 0 \\ x_1(1) & x_2(1) & u(1) & 0 & 0 & 0 \\ x_1(2) & x_2(2) & u(2) & 0 & 0 & 0 \\ 0 & 0 & 0 & x_1(0) & x_2(0) & u(0) \\ 0 & 0 & 0 & x_1(1) & x_2(1) & u(1) \\ 0 & 0 & 0 & x_1(2) & x_2(2) & u(2) \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{12} \\ b_1 \\ a_{21} \\ a_{22} \\ b_2 \end{bmatrix} = \begin{bmatrix} x_1(1) \\ x_1(2) \\ x_1(3) \\ x_2(1) \\ x_2(2) \\ x_2(3) \end{bmatrix}$$

This can be written in as a matrix vector equation $D\vec{s} = \vec{y}$ and we can solve for $\vec{s} = D^{-1}\vec{y}$

(e) While we can set up a matrix vector equation and uniquely solve our system, the output of the system can be noisy. Therefore, we update our model by considering a noise term $w(i)$ at time $t = i$.

$$\vec{x}(i+1) = A\vec{x}(i) + \vec{b}u(i) + w(i) \tag{14}$$

How can we set up a system of equations in a similar fashion but with a noise vector $\vec{w}$?

$$\vec{y} = D\vec{s} + \vec{w} \tag{15}$$

Solution:

$$\begin{bmatrix} x_1(1) \\ x_1(2) \\ x_1(3) \\ x_2(1) \\ x_2(2) \\ x_2(3) \end{bmatrix} = \begin{bmatrix} x_1(0) & x_2(0) & u(0) & 0 & 0 & 0 \\ x_1(1) & x_2(1) & u(1) & 0 & 0 & 0 \\ x_1(2) & x_2(2) & u(2) & 0 & 0 & 0 \\ 0 & 0 & 0 & x_1(0) & x_2(0) & u(0) \\ 0 & 0 & 0 & x_1(1) & x_2(1) & u(1) \\ 0 & 0 & 0 & x_1(2) & x_2(2) & u(2) \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{12} \\ b_1 \\ a_{21} \\ a_{22} \\ b_2 \end{bmatrix} + \begin{bmatrix} w(0) \\ w(1) \\ w(2) \\ w(0) \\ w(1) \\ w(2) \end{bmatrix}$$

(f) We can try to solve our system of equations, but we do not know what $\vec{w}$ is.

What we can do however, is to take more measurements, and set up a **least squares** problem as seen in 16A. What would the least squares problem be if we took measurements up to time step $t = 5$?

Solution:

$$\begin{bmatrix} x_1(1) \\ x_1(2) \\ x_1(3) \\ x_1(4) \\ x_1(5) \\ x_2(1) \\ x_2(2) \\ x_2(3) \\ x_2(4) \\ x_2(5) \end{bmatrix} = \begin{bmatrix} x_1(0) & x_2(0) & u(0) & 0 & 0 & 0 \\ x_1(1) & x_2(1) & u(1) & 0 & 0 & 0 \\ x_1(2) & x_2(2) & u(2) & 0 & 0 & 0 \\ x_1(3) & x_2(3) & u(3) & 0 & 0 & 0 \\ x_1(4) & x_2(3) & u(4) & 0 & 0 & 0 \\ 0 & 0 & 0 & x_1(0) & x_2(0) & u(0) \\ 0 & 0 & 0 & x_1(1) & x_2(1) & u(1) \\ 0 & 0 & 0 & x_1(2) & x_2(2) & u(2) \\ 0 & 0 & 0 & x_1(3) & x_2(3) & u(3) \\ 0 & 0 & 0 & x_1(4) & x_2(4) & u(4) \end{bmatrix} \begin{bmatrix} a_{11} \\ a_{12} \\ b_1 \\ a_{21} \\ a_{22} \\ b_2 \end{bmatrix} + \begin{bmatrix} w(0) \\ w(1) \\ w(2) \\ w(0) \\ w(1) \\ w(2) \end{bmatrix}$$

Which can equivalently be written as:

$$\vec{y} = D\vec{s} + \vec{w} \tag{16}$$

For the least squares problem, we will want to minimize $\|\vec{w}\|_2 = \|y - D\vec{s}\|_2$

(g) How would we solve this least squares problem?

Solution: Recall from 16A that if we are given the least squares problem:

$$A\vec{x} = \vec{b} + \vec{e} \tag{17}$$

The solution that minimizes the norm of the residual $\|\vec{e}\|_2$ is:

$$\hat{\vec{x}} = (A^T A)^{-1} A^T \vec{b} \tag{18}$$

Therefore the solution to the least squares problem above will be:

$$\hat{\vec{s}} = (D^T D)^{-1} D^T \vec{y} \tag{19}$$

The $\hat{\vec{s}}$ will give the best possible estimate for the A and $\vec{b}$.

