1. An Introduction to Solving Differential Equations

In this question, we will examine the process behind solving a first order differential equation and provide some motiviation for each step.

Consider the following first order differential equation:

$$a \cdot \frac{dy(t)}{dt} + b \cdot y(t) = c \tag{1}$$

We can divide the equation by a to make the coefficient of $\frac{dy(t)}{dt}$, one.

$$\frac{dy(t)}{dt} + \alpha \cdot y(t) = \beta \tag{2}$$

Our goal is to find a function $y(t)$ such that our differential equation is true for all values of t. To do this, we use a guess and check approach.

(a) Can you think of a function where $\frac{dy(t)}{dt} = y(t)$ for all t?

(b) Now, how can you modify the function above to solve $\frac{dy(t)}{dt} + \alpha y(t) = 0$? This equation is known as the homogenous equation.

You might notice that the solution above is not unique. This is the reason a differential equation will often come with an initial condition such as $y(0) = 2$.

(c) Try using this initial condition to solve for a unique solution to the differential equation above.

(d) Now, let's try solving our original equation:

$$\frac{dy(t)}{dt} + \alpha y(t) = \beta \tag{3}$$

To do this, we will use a change of variables. Let $\widetilde{y}(t) = y(t) - \frac{\beta}{\alpha}$.

i. Try writing the original equation as a differential equation in terms of $\widetilde{y}(t)$.

ii. Does this equation look familiar? How can you solve this equation?

iii. What is the final solution $y(t)$? Assume $y(0)$ is given.

To recap, given a first order differential equation $\frac{dy(t)}{dt} + \alpha y(t) = \beta$, the solution is:

$$y(t) = y(0)e^{-\alpha t} + \frac{\beta}{\alpha}(1 - e^{-\alpha t}) \tag{4}$$

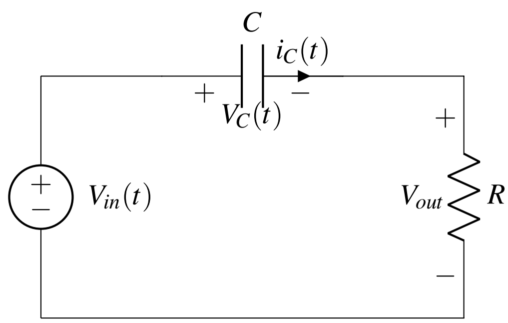
Another form you might find useful is the steady state form:

$$y(t) = y(\infty) + (y(0) - y(\infty))e^{-\alpha t} \tag{5}$$

2. CR Circuit

We're already familiar with the charging RC circuit from the second problem of this worksheet, but what happens if we flip the orientation of the circuit components and switch to discharging?

Consider the CR circuit below:



(a) **Write out the differential equation for the voltage $V_C(t)$ across the capacitor in terms of constants and $V_{in}(t)$.**

(b) Assume that when $t \leq 0$, the capacitor has been fully charged with an input voltage V_{DD}, with the initial condition $V_C(t=0) = V_{DD}$. At $t = 0$, the input voltage switches from high to low, so that $V_{in}(t) = 0$ for $t \geq 0$.

Plug in these conditions to the differential equation from the previous part and solve for $V_C(t)$ for $t \geq 0$.

(c) **What is $V_{out}(t)$?** Sketch a plot of the voltage across the resistor over time, labeling the asymptote it reaches at steady-state.

(d) **What is the steady-state voltage across the capacitor as $t \to \infty$?**

(e) **What is the steady-state current across the capacitor as $t \to \infty$?**

(f) **What circuit element does the capacitor act like at steady-state ($t \to \infty$)?**

3. Change of Coordinates

Many engineering problems can be difficult to solve in its standard xyz coordinates, but may be much easier in a different coordinate system. In this set, we will review the process of **change of basis** between coordinate systems. Remember that a *change of basis* can be represented by a invertible, square matrix.

Let's first start with an example: Consider the vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. When we write a vector in this form, we are implicitly representing it with the **standard basis** for $\mathbb{R}^2$, $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. This means that we can write $\vec{x}$ as a linear combination using standard basis vectors $\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2$. Now, what if we want to represent $\vec{x}$ as a linear combination of another set of basis vectors, say $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$?

This means that we need to find scalars α_1 and α_2 such that $\vec{x} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2$. We can write this equation in matrix form:

$$\begin{bmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Or equivalently:

$$\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Thus we can find α_1 and α_2 by solving a system of linear equations as seen in 16A.
These scalars α_1 and α_2 are called the coordinates of $\vec{x}$ **in the basis** $S = \{\vec{v}_1, \vec{v}_2\}$.

For the following problems, we will look at a vector $\vec{x}$ currently in the standard basis and its representation in a different basis: $S = \{\vec{v}_1, \vec{v}_2\}$.

We will refer to the vector $\vec{x}$ using coordinates from the basis S as $[\vec{x}]_S$. In other words,

if $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and $[\vec{x}]_S = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$, then $\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2$ or $\vec{x} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2$.

(a) Now let's say we have a vector that is originally using coordinates from the basis S. That is $[\vec{x}]_S = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$.

We are told that the basis S is:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

What equation gives the coordinates of $\vec{x}$ in the standard basis?

(b) Let $\vec{x} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$. What equation gives the coordinates of $\vec{x}$ in the basis S? Try to express your answer in matrix-vector form. No need to do the full calculation.

$$\vec{v}_1 = \begin{bmatrix} 4 \\ -2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -3 \\ -3 \end{bmatrix}$$

What equation gives the coordinates of $\vec{v}$ in the standard basis?

Now that we've had some mechanical practice, we'll look at the representation of linear operators through different bases. For a linear transformation, we can represent the input-output relationship with the matrix vector equation: $\vec{y} = A\vec{x}$ where $\vec{x}$ is the input, and $\vec{y}$ is the output vector. In this question we will look at how the linear operator represented by the matrix A looks in a *different basis* S. Remember that the vector $\vec{x}$ is implicitly written in the **standard basis** while the vector $[\vec{x}]_S$ is a vector using coordinates from the **S-basis.**

(c) Let $[\vec{x}]_S$ be a vector using S coordinates, and V be a change of coordinates matrix from the S-basis to the standard basis.
How can we represent $[\vec{x}]_S$ in terms of $\vec{x}$ and V?

(d) Now suppose we have another basis $R = \{w_1, .., w_n\}$ and the change of basis from R to S is represented by the matrix W. This means that if we have a vector $[\vec{x}]_R$ in R-coordinates, to get the coordinate representation in S-coordinates, $[\vec{x}]_S = W[\vec{x}]_R$. What would the change of basis matrix that takes a vector in R coordinates and outputs a vector in standard coordinates look like?

(e) Now let B be a linear operator in β coordinates. This means that it will take in a vector $[\vec{x}]_\beta$ as an input and output $[\vec{y}]_\beta$. Given a vector $\vec{x}$ in standard coordinates, why can't we multiply $B\vec{x}$ to get the output $\vec{y}$ in standard coordinates?

(f) Using our V matrix given above, as the change of coordinates matrix from $S \to \beta$, how can we describe the linear operator B in standard coordinates, that is if $\vec{y} = A\vec{x}$, what is A in the standard basis?

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