1. Eigendecomposition and Change of Basis

Diagonal matrices, matrices where all entries outside of the diagonal are zero, are often desirable since they are easy to analyze. Determining properties such as rank and invertibility, are much simpler on a diagonal matrix as opposed to other non-diagonal matrices. The process of **changing to a basis** in which the linear operator has a diagonal matrix representation is called **eigendecomposition** or **diagonalization.** You can think of eigendecomposition as a change of basis to one entirely made up of eigenvectors.

So what is a **change of basis**? Consider an arbitrary vector in $\mathbb{R}^2$: $\vec{x} = [x_1 \ x_2]^T$. When we write a vector in this form, we are representing it as a linear combination of the *standard basis* vectors for $\mathbb{R}^2$: $\vec{x} = x_1 \left[\begin{smallmatrix}1\\0\end{smallmatrix}\right] + x_2 \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$. Naturally, x_1 and x_2 are the *coordinates* of $\vec{x}$ in the standard basis (as you would refer to them if you graphed $\vec{x}$ on a Cartesian plane).

Now what if we wanted to represent that same vector in a different basis? For example, say you wanted to represent the same vector $\vec{x}$ using the set of basis vectors $\vec{v}_1$ and $\vec{v}_2$. This means that we need to find scalars α_1 and α_2 such that $\vec{x}$ can be written as a linear combination of these new basis vectors: $\vec{x} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2$. To do this, we can just setup and solve a system of linear equations of the form:

$$\begin{bmatrix}\vec{v}_1 & \vec{v}_2\end{bmatrix}\begin{bmatrix}\alpha_1\\ \alpha_2\end{bmatrix} = \begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

In this problem, we'll investigate changing to and from the **eigenbasis** for the following matrix:

$$A = \begin{bmatrix}2 & 2\\ 5 & -1\end{bmatrix}$$

(a) **Find λ_1, λ_2, the eigenvalues of A, ordered from largest to smallest.**

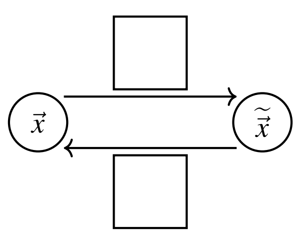
(b) **Find the eigenvectors $\vec{v}_1, \vec{v}_2$ corresponding to the eigenvalues.**

With the eigenvectors we just found, define V to be the matrix:

$$V = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}$$

(c) Let $\widetilde{\vec{x}}$ be the coordinates of $\vec{x}$ in the eigenbasis. This means that for some arbitrary vector represented in the eigenbasis $\widetilde{\vec{x}} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$, the corresponding representation in standard coordinates is a linear combination of the columns of V: $\vec{x} = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2$. **What is $\widetilde{\vec{x}}$ in terms of V and $\vec{x}$?**
(*Hint: Write $\vec{x}$ in terms of V and $\widetilde{\vec{x}}$, then go from there.*)

(d) It is often helpful to visualize the change of basis in a state diagram, where *each arrow represents left-multiplying the variable it's coming out of by the corresponding matrix.* **Fill in the missing matrix operations in the state diagram based on your answer from the previous part.**

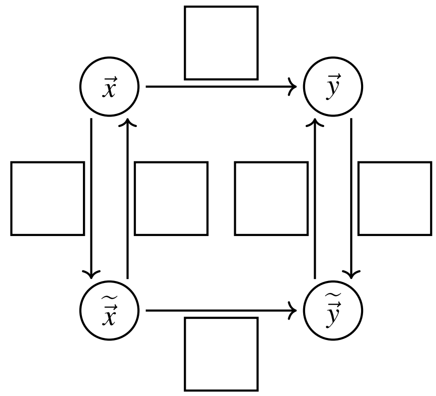


(e) Now that we are able to switch back and forth between the coordinate systems, let's see how the linear transformation brought by A can be viewed as a diagonal scaling transformation in the eigenbasis coordinate system.
Let $\vec{y} = A\vec{x}$, and $\vec{x} = \alpha_1 \vec{v}_1 + \alpha_2 \vec{v}_2$, using the same matrix A and eigenvectors $\vec{v}_1, \vec{v}_2$ from before. Let $\widetilde{\vec{x}}$, $\widetilde{\vec{y}}$ be the coordinates of $\vec{x}$, $\vec{y}$ in the eigenbasis. **Find $\widetilde{\vec{x}}$ and $\widetilde{\vec{y}}$ in terms of $\alpha_1, \alpha_2, \lambda_1, \lambda_2$. What can we say about the relationship between $\widetilde{\vec{x}}$ and $\widetilde{\vec{y}}$?**

(*Hint*: Your answers shouldn't be in terms of the original $\vec{x}$ or $\vec{y}$. Use what you know about the coordinates of a vector in a certain basis; there is no need to invert any matrices or do any major computation.)

(f) **Find the matrix D satisfying $\widetilde{\vec{y}} = D\widetilde{\vec{x}}$ in terms of V and A.**
(*Hint*: Start by writing $\vec{x}, \vec{y}$ in terms of $\widetilde{\vec{x}}$ and $\widetilde{\vec{y}}$. Refer to the state diagram from before.)

(g) Finally, let's visualize this linear transformation A from the perspective of two different coordinate systems in the state diagram below. **Fill in the missing matrix operations in the state diagram. How can you show and explain the diagonalization $A = VDV^{-1}$ (using the state diagram) and the change of basis perspective?**



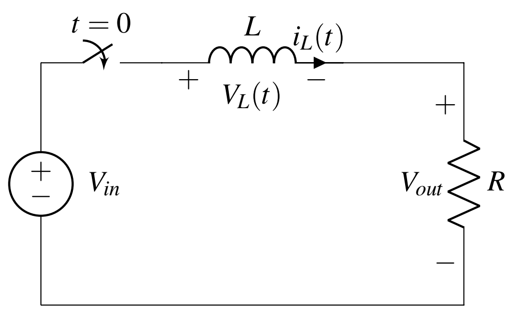
$\vec{x}$
$\vec{y}$
$\widetilde{\vec{x}}$
$\widetilde{\vec{y}}$

2. Introduction to Inductors

Now that we are comfortable solving for the transient behavior of charging and discharging capacitors, we can move to analyzing a new circuit element: the inductor. An inductor has the physical property of *inductance*, represented by a constant L. Inductors are characterized by the following I-V relationship:

$$V_L(t) = L\frac{d}{dt}i_L(t) \tag{1}$$

Let's analyze the following LR circuit, with the initial condition $i_L(0) = 0$ with the switch open before $t = 0$:



(a) **Write out the differential equation for the current $i_L(t)$ of the inductor starting at $t = 0$ when the switch is closed.**

(b) **Solve the differential equation for $i_L(t)$.**

(c) **What is the steady-state current through the inductor as $t \to \infty$?** Sketch a plot of the current through the inductor over time, labeling the asymptote after reaching the steady-state. This should provide you with some intuition as to the physical behavior of an inductor once an inductor is at steady state.

(d) **What is the steady-state voltage drop across the inductor as $t \to \infty$?**

(e) **What circuit element does the inductor act like at steady-state?**

3. Fun with Inductors (Challenge)

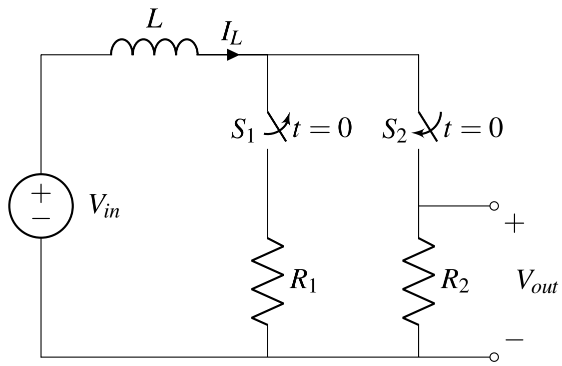


Figure 1: Circuit A

(a) Consider circuit A. Assuming that for $t < 0$, switch S_1 is on and switch S_2 is off (and both switches have been in these states indefinitely), what is $i_L(0)$?

(b) Now let's assume that for $t \geq 0$, S_1 is off and S_2 is on. Solve for $V_{out}(t)$ for $t \geq 0$.

(c) If $V_{in} = 1V$, $L = 1nH$, $R_1 = 1k\Omega$, and $R_2 = 10k\Omega$, what is the maximum value of $V_{out}(t)$ for $t \geq 0$?

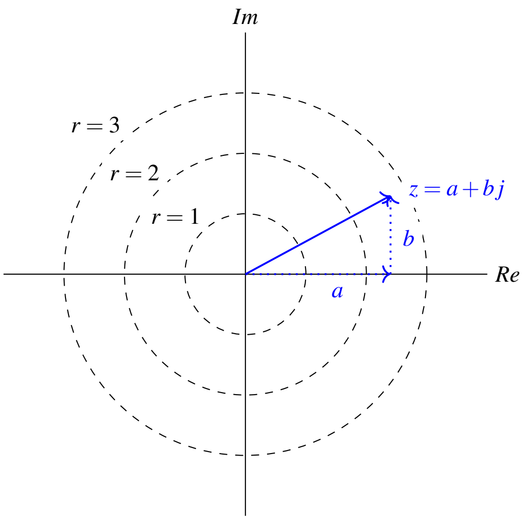
(d) In general, if we want $\max V_{out}(t)$ to be greater than V_{in}, what relationship needs to be maintained between the values of R_1 and R_2?

(e) Now assume that at time $t = t_1$, switch S_2 was turned off, and switch S_1 was turned back on. Solve for $i_L(t)$ for $t > t_1$. If $R_2 > R_1$, how does this $i_L(t)$ for $t > t_1$ compare with the initial condition $i_L(0)$ you found in part (a)?

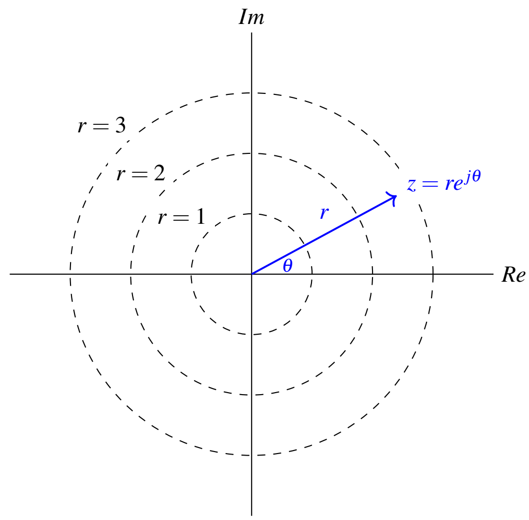
4. Complex Numbers (Optional)

A complex number, z, is composed of a real part and imaginary part. If $z = a + bj$, then $\mathfrak{Re}(z) = a$ (the real portion equals a), and $\mathfrak{Im}(z) = b$ (the imaginary portion equals b). Complex numbers can be expressed in two ways:

Rectangular Form: $z = a + bj$



Polar Form: $z = re^{j\theta}$



In polar form, r represents the magnitude and θ represents the angle of the complex number with respect to the origin of the complex plane. Rectangular form makes adding and subtracting complex numbers easier; whereas, polar form makes multiplying and dividing numbers easier. Some handy equations to switch between forms include:

$$\tan(\theta) = \frac{b}{a} \qquad r = |z| = \sqrt{a^2 + b^2}$$

$$\sin(\theta) = \frac{b}{|z|} \qquad \cos(\theta) = \frac{a}{|z|}$$

(a) **Use the formulas given above to convert between polar and rectangular form.**

i. Convert $10 + 12j$ to polar form.

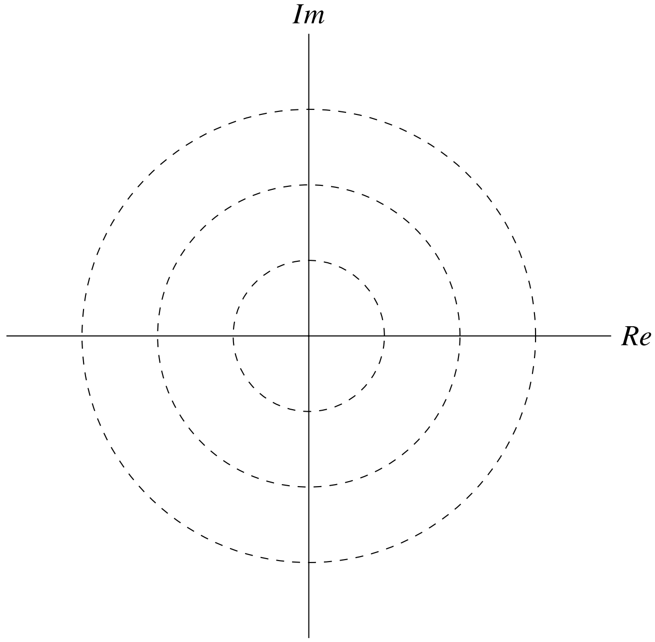
ii. Convert $22e^{23j}$ to rectangular form.

(b) **Plot the following on a polar grid:**

i. 2

ii. 2j

iii. 2 + 2j



(c) **Calculate the magnitude and phase of the following:**

i. 2

ii. $\frac{2}{2j}$

iii. $\frac{3j}{5}$

iv. $\frac{1+2j}{9+7j}$

(d) **Show that $\frac{1}{j} = -j$.**

A complex number, $z = a + bj$ has a **complex conjugate**, $\bar{z} = a - bj$. In polar coordinates, the equivalent expression is $\overline{re^{j\theta}} = re^{-j\theta}$.

Note that the sum of a complex number and its conjugate is always purely real, but the difference between a complex number and its conjugate is always purely imaginary.

(e) **Prove graphically that the sum of any complex number and its conjugate is always real.** *Try plotting an an arbitrary complex number and its conjugate.*

(f) Recall that Euler's Formula states that $e^{j\theta} = \cos(\theta) + j\sin(\theta)$.

Using Euler's identity, show the following identities, which show that sinusoids are sums of complex exponentials:

$$\cos(\theta) = \frac{e^{j\theta} + e^{-j\theta}}{2}$$
$$\sin(\theta) = \frac{e^{j\theta} - e^{-j\theta}}{2j}$$

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