CSM

Spring 2022

EECS 16B

# Worksheet 1

## 1. An Introduction to Solving Differential Equations

In this question, we will examine the process behind solving a first order differential equation and provide some motivation for each step.

Consider the following first order differential equation:

$$a \cdot \frac{dy(t)}{dt} + b \cdot y(t) = c \tag{1}$$

We can divide the equation by $a$ to make the coefficient of $\frac{dy(t)}{dt}$, one.

$$\frac{dy(t)}{dt} + \alpha \cdot y(t) = \beta \tag{2}$$

Our goal is to find a function $y(t)$ such that our differential equation is true for all values of $t$. To do this, we use a guess and check approach.

(a) Can you think of a function where $\frac{dy(t)}{dt} = y(t)$ for all $t$?

**Solution:** It can be seen either through inspection or integration that $y(t) = e^t$.

(b) Now, how can you modify the function above to solve $\frac{dy(t)}{dt} + \alpha y(t) = 0$? This equation is known as the homogenous equation.

**Solution:** We can notice that if $y(t) = e^{rt}$, then $\frac{dy(t)}{dt} = re^{rt}$. Therefore if we subtract $\alpha y(t)$, we get $\frac{dy(t)}{dt} = -\alpha y(t)$. It follows by picking $r = -\alpha$ that $y(t) = e^{-\alpha t}$ satisfies our differential equation.

You might notice that the solution above is not unique. This is the reason a differential equation will often come with an initial condition such as $y(0) = 2$.

(c) Try using this initial condition to solve for a unique solution to the differential equation above.

**Solution:** Notice that our solution isn't unique since $y(t) = Ke^{-\alpha t}$ satisfies our differential equation for any nonzero choice of $K$. Plugging in $t = 0$, we get $y(0) = K = 2$, so our solution is $y(t) = 2e^{-\alpha t}$.

(d) Now, let's try solving our original equation:

$$\frac{dy(t)}{dt} + \alpha y(t) = \beta \tag{3}$$

To do this, we will use a change of variables. Let $\widetilde{y}(t) = y(t) - \frac{\beta}{\alpha}$.

i. Try writing the original equation as a differential equation in terms of $\widetilde{y}(t)$.

**Solution:** Since $y(t) = \widetilde{y}(t) + \beta/\alpha$, $\frac{dy(t)}{dt} = \frac{d\widetilde{y}(t)}{dt}$.

Substituting these values, we see that $\frac{d\widetilde{y}(t)}{dt} + \alpha\widetilde{y}(t) + \beta = \beta$ or $\frac{d\widetilde{y}(t)}{dt} + \alpha\widetilde{y}(t) = 0$.

ii. Does this equation look familiar? How can you solve this equation?

**Solution:** This is the homogenous equation in terms of $\widetilde{y}(t)$!

So by parts (b) and (c), our solution is $\widetilde{y}(t) = Ke^{-\alpha t}$.

iii. What is the final solution $y(t)$? Assume $y(0)$ is given.

**Solution:** Converting back to $y(t)$, our solution is $y(t) = Ke^{-\alpha t} + \beta/\alpha$. Plugging in $t = 0$, we get $y(0) = K + \beta/\alpha$ or $K = y(0) - \beta/\alpha$. Our final solution is $y(t) = y(0)e^{-\alpha t} + \beta/\alpha(1 - e^{-\alpha t})$.

To recap, given a first order differential equation $\frac{dy(t)}{dt} + \alpha y(t) = \beta$, the solution is:

$$y(t) = y(0)e^{-\alpha t} + \frac{\beta}{\alpha}(1 - e^{-\alpha t}) \tag{4}$$

Another form you might find useful is the steady state form:

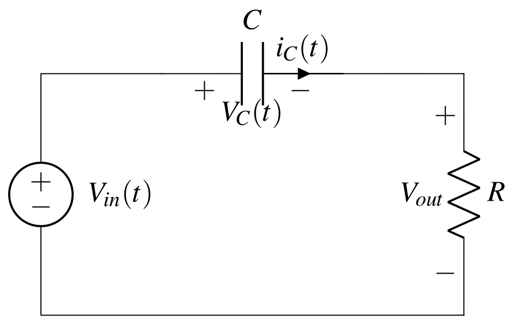
$$y(t) = y(\infty) + (y(0) - y(\infty))e^{-\alpha t} \tag{5}$$

**2. CR Circuit**

We're already familiar with the charging RC circuit from the second problem of this worksheet, but what happens if we flip the orientation of the circuit components and switch to discharging?

Consider the CR circuit below:




(a) **Write out the differential equation for the voltage $V_C(t)$ across the capacitor in terms of constants and $V_{in}(t)$.**

**Solution:** KCL and Ohm's law at the $V_{out}$ node gives us:

$$C\frac{dV_C(t)}{dt} = \frac{V_{in}(t) - V_C(t)}{R}$$
$$\implies \frac{d}{dt}V_C(t) = -\frac{1}{RC}V_C(t) + \frac{1}{RC}V_{in}(t)$$

(b) Assume that when $t \leq 0$, the capacitor has been fully charged with an input voltage $V_{DD}$, with the initial condition $V_C(t=0) = V_{DD}$. At $t = 0$, the input voltage switches from high to low, so that $V_{in}(t) = 0$ for $t \geq 0$.

**Plug in these conditions to the differential equation from the previous part and solve for $V_C(t)$ for $t \geq 0$.**

**Solution:** We can plug in $V_{in}(t) = 0$ as given by the problem:

$$\frac{d}{dt}V_C(t) = -\frac{1}{RC}V_C(t)$$

Now, we see that this is a homogeneous differential equation with the general solution:

$$V_C(t) = Ae^{-\frac{1}{RC}t}$$

We solve for $A$ by plugging in the initial condition $V_C(0) = V_{DD}$:

$$V_C(0) = V_{DD} = Ae^0$$
$$\implies A = V_{DD}$$

Thus, the final solution for the voltage of the discharging capacitor is given by $V_C(t) = V_{DD}e^{-\frac{1}{RC}t}$.

(c) **What is $V_{out}(t)$?** Sketch a plot of the voltage across the resistor over time, labeling the asymptote it reaches at steady-state.

**Solution:** By KVL, $V_{out}(t) = V_{in}(t) - V_C(t) = -V_C(t)$.

(d) **What is the steady-state voltage across the capacitor as $t \to \infty$?**

**Solution:**

$$\begin{aligned}\lim_{t\to\infty} V_C(t) &= \lim_{t\to\infty} V_{DD}(e^{-\frac{1}{RC}t}) \\ &= V_{DD}(0) \\ &= 0\end{aligned}$$

(e) **What is the steady-state current across the capacitor as $t \to \infty$?**

**Solution:**

$$\begin{aligned}\lim_{t\to\infty} i_C(t) &= \lim_{t\to\infty} C\frac{d}{dt}V_C(t) \\ &= \lim_{t\to\infty} C(\frac{d}{dt}V_{DD}(e^{-\frac{1}{RC}t})) \\ &= 0\end{aligned}$$

(f) **What circuit element does the capacitor act like at steady-state ($t \to \infty$)?**

**Solution:** When completely discharged, it acts like a short circuit element, where the voltage across is 0, and the current through it is determined by the circuit components around it, which is 0 in this case.

## 3. Change of Coordinates

Many engineering problems can be difficult to solve in its standard xyz coordinates, but may be much easier in a different coordinate system. In this set, we will review the process of **change of basis** between coordinate systems. Remember that a *change of basis* can be represented by a invertible, square matrix.

Let's first start with an example: Consider the vector $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. When we write a vector in this form, we are implicitly representing it with the **standard basis** for $\mathbb{R}^2$, $\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\vec{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. This means that we can write $\vec{x}$ as a linear combination using standard basis vectors $\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2$. Now, what if we want to represent $\vec{x}$ as a linear combination of another set of basis vectors, say $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}$?

This means that we need to find scalars $\alpha_1$ and $\alpha_2$ such that $\vec{x} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2$. We can write this equation in matrix form:

$$\begin{bmatrix} | & | \\ \vec{v}_1 & \vec{v}_2 \\ | & | \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Or equivalently:

$$\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}.$$

Thus we can find $\alpha_1$ and $\alpha_2$ by solving a system of linear equations as seen in 16A.
These scalars $\alpha_1$ and $\alpha_2$ are called the coordinates of $\vec{x}$ **in the basis** $S = \{\vec{v}_1, \vec{v}_2\}$.

For the following problems, we will look at a vector $\vec{x}$ currently in the standard basis and its representation in a different basis: $S = \{\vec{v}_1, \vec{v}_2\}$.

We will refer to the vector $\vec{x}$ using coordinates from the basis $S$ as $[\vec{x}]_S$. In other words,

if $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$, and $[\vec{x}]_S = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$, then $\vec{x} = x_1\vec{e}_1 + x_2\vec{e}_2$ or $\vec{x} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2$.

(a) Now let's say we have a vector that is originally using coordinates from the basis S. That is $[\vec{x}]_S = \begin{bmatrix} 3 \\ 3 \end{bmatrix}$.

We are told that the basis S is:

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

What equation gives the coordinates of $\vec{x}$ in the standard basis?

**Solution:** Since the vector $\vec{x}$ is currently written in coordinates using the basis, $S = \{\vec{v}_1, \vec{v}_2\}$, we know that $\vec{x} = 3\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 3\begin{bmatrix} 1 \\ -1 \end{bmatrix} = V[\vec{x}]_S$ where $V$ is the matrix $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. Therefore,

$$\vec{x} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 3 \\ 3 \end{bmatrix}$$

(b) Let $\vec{x} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$. What equation gives the coordinates of $\vec{x}$ in the basis S? Try to express your answer in matrix-vector form. No need to do the full calculation.

$$\vec{v}_1 = \begin{bmatrix} 4 \\ -2 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -3 \\ -3 \end{bmatrix}$$

What equation gives the coordinates of $\vec{v}$ in the standard basis?

**Solution:** If we denote the vector in the new basis as $[\vec{x}]_S = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$, then we can write out the following equation: $\vec{x} = \alpha_1\vec{v}_1 + \alpha_2\vec{v}_2 = V[\vec{x}]_S$ where V is the same matrix used in the previous part. Then it follows that $[\vec{x}]_S = V^{-1}\vec{x}$.

Now that we've had some mechanical practice, we'll look at the representation of linear operators through different bases. For a linear transformation, we can represent the input-output relationship with the matrix vector equation: $\vec{y} = A\vec{x}$ where $\vec{x}$ is the input, and $\vec{y}$ is the output vector. In this question we will look at how the linear operator represented by the matrix $A$ looks in a *different basis* $S$. Remember that the vector $\vec{x}$ is implicitly written in the **standard basis** while the vector $[\vec{x}]_S$ is a vector using coordinates from the **S-basis.**

(c) Let $[\vec{x}]_S$ be a vector using $S$ coordinates, and $V$ be a change of coordinates matrix from the $S$-basis to the standard basis.
How can we represent $[\vec{x}]_S$ in terms of $\vec{x}$ and $V$? **Solution:** We are given a matrix $V$ that converts $S$ coordinates to standard coordinates. This means that $V^{-1}$ will be the matrix that converts standard coordinates to the $S$ coordinates. Therefore, in order to get $[\vec{x}]_S$ we multiply $V^{-1}$ with $\vec{x}$, to get $V^{-1}\vec{x} = [\vec{x}]_S$.

(d) Now suppose we have another basis $R = \{w_1, .., w_n\}$ and the change of basis from $R$ to $S$ is represented by the matrix $W$. This means that if we have a vector $[\vec{x}]_R$ in R-coordinates, to get the coordinate representation in S-coordinates, $[\vec{x}]_S = W[\vec{x}]_R$. What would the change of basis matrix that takes a vector in $R$ coordinates and outputs a vector in standard coordinates look like? **Solution:** We want to get represent the vector $\vec{x}$ in standard coordinates. We are looking for a matrix $U$ such that

$$\vec{x} = U[\vec{x}]_R.$$

We currently know that in order to go from S-coordinates to standard, we must multiply by the matrix $V$.

$$\vec{x} = V[\vec{x}]_S.$$

We also know that to go from R-coordinates to S-coordinates, we must multiply by the matrix $W$.

$$[\vec{x}]_S = W[\vec{x}]_R.$$

Therefore, by substituting $[\vec{x}]_S$, we see that

$$\vec{x} = VW[\vec{x}]_R.$$

It follows that the matrix $U = VW$.

(e) Now let $B$ be a linear operator in $\beta$ coordinates. This means that it will take in a vector $[\vec{x}]_\beta$ as an input and output $[\vec{y}]_\beta$. Given a vector $\vec{x}$ in standard coordinates, why can't we multiply $B\vec{x}$ to get the output $\vec{y}$ in standard coordinates?

**Solution:** The transformation $B$ "lives" in a different world. It can only accept vectors in $\beta$ coordinates as inputs. Therefore, in order to solve this, we must convert $\vec{x}$ into $\beta$ coordinates.

(f) Using our $V$ matrix given above, as the change of coordinates matrix from $S \to \beta$, how can we describe the linear operator $B$ in standard coordinates, that is if $\vec{y} = A\vec{x}$, what is $A$ in the standard basis?

**Solution:** There will be two main issues we need to address in this question. First off, we need a $\beta$ coordinate input. Secondly, the output of the $B$ matrix is in $\beta$ coordinates, and we will need to convert that back to standard coordinates.

Therefore, we take the following steps.

1. Let's first make our input into $B$ in $\beta$ coordinates.

$$\text{Let } \vec{v} = V\vec{x}.$$

2. Now if we input $\vec{v}$ we will get some output:

$$\vec{w} = B\vec{v}.$$

3. However, $\vec{w}$ is in $\beta$ coordinates, so we must convert back to standard coordinates using $V^{-1}$.

$$\vec{y} = V^{-1}\vec{w}$$

4. Cascading all of our matrix multiplications, we end up with:

$$\vec{y} = V^{-1}BV\vec{x}.$$

Therefore, we can see that $A = V^{-1}BV$.

The following can also be represented in this state diagram:

Note that when cascading transformations, we apply them to the **left** of the existing transformation.

$$\begin{tikzcd}
\vec{x} \arrow[r, "A"] \arrow[d, shift right, "V^{-1}"'] & \vec{y} \arrow[d, shift left, "V^{-1}"] \\
\vec{v} \arrow[r, "B"'] \arrow[u, shift right, "V"'] & \vec{w} \arrow[u, shift left, "V"]
\end{tikzcd}$$

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