CSM
Spring 2022

EECS 16B
# Worksheet 6

## 1. Controllability Practice

Consider the system

$$\vec{x}[i+1] = \begin{bmatrix} 0 & 1 & -2 \\ 0 & 2 & 0 \\ -1 & 1 & 0 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} u[i]$$

(a) **What is the controllability matrix, $\mathscr{C}$, for this system?** **Solution:** For a 3-dimensional system,

$$\mathscr{C} = \begin{bmatrix} \vec{b} & A\vec{b} & A^2\vec{b} \end{bmatrix}$$

Plugging in this system's $A$ and $\vec{b}$, we have

$$\mathscr{C} = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 0 & 0 \\ 0 & -2 & 0 \end{bmatrix}$$

The column space of $\mathscr{C}$ is the span of its column vectors. The third column is linearily dependent, so $\text{col}(\mathscr{C})$ is equivalent to the span of its first two columns.

$$\text{col}(\mathscr{C}) = \text{span}\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \}$$

(b) **What is the rank of the controllability matrix? Is the system controllable?**

**Solution:** The controllability matrix has rank 2, so this system is not controllable. Note that the initial state $\vec{x}[0]$ has no effect on whether a system is controllable.

(c) **Starting at $\vec{x}[0] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, what possible states can we reach after one timestep? Two timesteps? Three?**

**Solution:** For one timestep,

$$\vec{x}[1] = \begin{bmatrix} 0 & 1 & -2 \\ 0 & 2 & 0 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} u[0]$$

Simplifying, we get

$$\vec{x}[1] = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 2u[0] \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2u[0] \\ 0 \\ -1 \end{bmatrix}$$

Since $u[0]$ is an input we have control over, we can set it arbitrarily, and reach any state of the form

$$\begin{bmatrix} c_1 \\ 0 \\ -1 \end{bmatrix}.$$

For two timesteps,

$$\vec{x}[2] = \begin{bmatrix} 0 & 1 & -2 \\ 0 & 2 & 0 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2u[0] \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} u[1] = \begin{bmatrix} 2+2u[1] \\ 0 \\ -2u[0] \end{bmatrix}$$

Again, since we have control over $u[0]$ and $u[1]$, we can reach any state of the form $\begin{bmatrix} c_1 \\ 0 \\ c_2 \end{bmatrix}$.

For 3 timesteps,

$$\vec{x}[3] = \begin{bmatrix} 0 & 1 & -2 \\ 0 & 2 & 0 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2+2u[1] \\ 0 \\ -2u[0] \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} u[2] = \begin{bmatrix} 4u[0]+2u[2] \\ 0 \\ -2-2u[1] \end{bmatrix}$$

As with 2 timesteps, we can reach any state of the form $\begin{bmatrix} c_1 \\ 0 \\ c_2 \end{bmatrix}$.

(d) **What is the minimum number of timesteps it takes to reach $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$? What about $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$?**

**Solution:** It takes two timesteps to reach $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$.

$\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ is in the form $\begin{bmatrix} c_1 \\ 0 \\ c_2 \end{bmatrix}$, but not in the form $\begin{bmatrix} c_1 \\ 0 \\ -1 \end{bmatrix}$, so it can be reached in at least two timesteps.

It is impossible to reach $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ in any amount of timesteps. The system is not controllable, and we can only reach states of the form $\begin{bmatrix} c_1 \\ 0 \\ c_2 \end{bmatrix}$.

Now, consider the system, with $A$ modified slightly:

$$\vec{x}[i+1] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 2 & -2 \\ -1 & 1 & 0 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} u[i]$$

 

(e) **What is the controllability matrix, $\mathscr{C}$, for this system? What is its column space?** **Solution:**

$$\mathscr{C} = \begin{bmatrix} \vec{b} & A\vec{b} & A^2\vec{b} \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & -2 & 0 \end{bmatrix}$$

The column space of this matrix spans $\mathbb{R}^3$.

(f) **What is the rank of the controllability matrix? Is the system controllable?**

**Solution:** The controllability matrix has rank 3, so the system is controllable.

(g) **Starting at $\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, what possible states can we reach after one timestep? Two timesteps? Three?**

**Solution:** For one timestep,

$$\vec{x}[1] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 2 & -2 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} u[0] = \begin{bmatrix} 2u[0] \\ 0 \\ -1 \end{bmatrix}$$

So, we can reach any state of the form $\begin{bmatrix} c_1 \\ 0 \\ -1 \end{bmatrix}$.

For two timesteps,

$$\vec{x}[2] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 2 & -2 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2u[0] \\ 0 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} u[1] = \begin{bmatrix} 2u[1] \\ 2 \\ -2u[0] \end{bmatrix}$$

We can reach any state of the form $\begin{bmatrix} c_1 \\ 2 \\ c_2 \end{bmatrix}$.

For three timesteps,

$$\vec{x}[3] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 2 & -2 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 2u[1] \\ 2 \\ -2u[0] \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} u[2] = \begin{bmatrix} 2+2u[2] \\ 4+4u[0] \\ 2-2u[1] \end{bmatrix}$$

We can reach any state in $\mathbb{R}^3$ in three timesteps.

(h) **What is the minimum number of timesteps it takes to reach $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$?**

*Hint: Since the system is controllable, we can reach any state in three time steps, however, it may be* *possible to reach a state in fewer than three time steps. Look at your answer to the previous part, and check which possible states we can reach in one, two, and three time steps.*

**Solution:** We unfortunately cannot reach this vector in fewer timesteps because $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ is not in the form $\begin{bmatrix} c_1 \\ 0 \\ -1 \end{bmatrix}$ or $\begin{bmatrix} c_1 \\ 2 \\ c_2 \end{bmatrix}$.

(i) **What is the minimum number of timesteps it takes to reach $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$?** **Solution:** We can reach this vector in two time steps since it is in the form $\begin{bmatrix} c_1 \\ 0 \\ -1 \end{bmatrix}$ or $\begin{bmatrix} c_1 \\ 2 \\ c_2 \end{bmatrix}$.

**2. System Feedback**

Consider the following continuous time system:

$$\frac{d^2}{dt^2}x(t) = -x(t)$$

We convert this system to state space form:

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + B\vec{u}(t) \tag{1}$$

We will pick our state variable $\vec{x} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$, with $x_1(t) = x(t)$ and $x_2(t) = \frac{d}{dt}x(t)$.

(a) What are the values of the $A$ matrix?

**Solution:**

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

(b) Is this continuous time system stable? How would you describe its behavior?

**Solution:** The eigenvalues of the $A$ matrix will be $\pm j$. Since $\mathfrak{Re}(\lambda) = 0$, and is not less than 0, the system is unstable. It can be described as an oscillatory system that never dies out.

We want to change the behavior of the system using a feedback control model:

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = A\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

To do this, we set $u(t) = F\vec{x}(t)$, where $F = \begin{bmatrix} f_1 & f_2 \end{bmatrix}$. Note that $F$ is a $1 \times 2$ matrix.

(c) We will now try to use our knowledge of controllability to stabilize this system. Is this continuous time system controllable?

**Solution:** We can compute our controllability matrix as

$$\mathscr{C} = \begin{bmatrix} \vec{b} & A\vec{b} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

This controllability matrix is full-rank since its columns are linearly independent, so the system is controllable.

(d) After plugging in our input, $u(t) = F\vec{x}(t)$, what will our new system be? How can you write out this system of differential equations in matrix/vector form?

**Solution:** We know that our $B$ matrix is: $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and our input $u(t) = f_1x_1(t) + f_2x_2(t)$.

If we plug this into our system, we will get:

$$\frac{d}{dt}\vec{x}(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ f_1 & f_2 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{2}$$

 

This means our new system will be:

$$\frac{d}{dt}\vec{x}(t) = \begin{bmatrix} 0 & 1 \\ -1+f_1 & f_2 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \tag{3}$$

We can alternatively represent this as a matrix/vector system of differential equations:

$$\frac{d}{dt}\vec{x}(t) = (A+BF)\vec{x}(t) \tag{4}$$

(e) We will define the new matrix derived in the previous part as $A_{cl}$ for the closed loop system. What are the eigenvalues of this new matrix $A_{cl}$?

**Solution:** We first compute the characteristic polynomial of $A_{cl}$.

$$\det(A_{cl} - \lambda I) = \det\left(\begin{bmatrix} -\lambda & 1 \\ -1+f_1 & f_2-\lambda \end{bmatrix}\right) = \lambda(\lambda - f_2) - 1(-1+f_1) = \lambda^2 - f_2\lambda + 1 - f_1$$

The eigenvalues will be the roots of this characteristic polynomial:

$$\lambda = \frac{f_2}{2} \pm \frac{1}{2}\sqrt{f_2^2 - 4(1-f_1)}$$

(f) Suppose $f_1 = 1$ and $f_2 = 2$, what will the eigenvalues of this system be? How about when $f_1 = -1$ and $f_2 = -2$?

**Solution:** If $f_1 = 1$ and $f_2 = 2$, then the eigenvalues will be:

$$1 \pm \frac{1}{2}\sqrt{4-4(1-1)} = 0,\ 2$$

Since both of the eigenvalues have $\mathfrak{Re}(\lambda) = 0, 2 >= 0$, this system will be unstable.

Now suppose $f_1 = -1$ and $f_2 = -2$, then the eigenvalues will be:

$$-1 \pm \frac{1}{2}\sqrt{4-4(1+1)} = -1 \pm j$$

Since $\mathfrak{Re}(\lambda) = -1 < 0$, for both eigenvalues, this system will be stable.

(g) What values of $f_1$ and $f_2$ will remove the oscillatory behavior completely and still stabilize the system?

**Solution:** To remove oscillatory behavior, and stabilize the system, the square root term $\sqrt{f_2^2 - 4(1-f_1)}$ must be positive, and the real part of both eigenvalues must be less than 0.

As example of this would be $f_1 = -1$ and $k_2 = -3$. This gives the system eigenvalues of $-1$ and $-2$ while removing oscillatory behavior and being stable.

 

## 3. Discrete Time Feedback

(a) Consider the scalar system: $x[i+1] = 1.5x[i] + u[i]$. Given the controller $u[i] = fx[i]$, for what value of $f$ can we have the system to behave like:$x[i+1] = \lambda x[i]$ where $\lambda = 0.7$?

**Solution:** To make the system's eigenvalue be 0.7, we can choose $f = -0.8$.

(b) Given the system $\vec{x}[i+1] = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_1[i] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_2[i]$. Let $u_1[i] = f_1 x_1[i]$ and $u_2[i] = f_2 x_2[i]$. What value of $f_1$ and $f_2$ would make the system stable with eigenvalues $\lambda_1 = \lambda_2 = \frac{1}{2}$?

**Solution:** We can choose $f_1 = -2.5$ and $f_2 = -4.5$.

(c) Given the matrix $\begin{bmatrix} 2 & 1 \\ -3+2f_1 & 4+2f_2 \end{bmatrix}$, what should $f_1$ and $f_2$ be for the matrix to have eigenvalues $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = \frac{-1}{3}$?

**Solution:** Coefficient match to find $f_1 = \frac{-1}{4}$ and $f_2 = \frac{-35}{12}$

(d) Given the matrix $\begin{bmatrix} 2+f_1 & 7+f_2 \\ 3 & -1 \end{bmatrix}$, what should $f_1$ and $f_2$ be for the matrix to have eigenvalues $\lambda_1 = 5$ and $\lambda_2 = 2$?

**Solution:** Coefficient match to find $f_1 = 6$ and $f_2 = -13$

(e) Given the system $\vec{x}[i+1] = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u[i]$. Is the system stable?

**Solution:** For a discrete system to be stable, $|\lambda_i| < 1$ Characteristic polynomial:

$$(2-\lambda)^2 + 1 = 0$$
$$\lambda^2 - 4\lambda + 5 = 0$$
$$\lambda_{1,2} = \frac{4 \pm \sqrt{16-20}}{2} = 2 \pm j$$
$$|\lambda_1| = |\lambda_2| = \sqrt{5} > 1$$

The magnitude of the eigenvalues are greater than 1, so the system is unstable.

(f) Given the feedback controller $u[i] = F\vec{x}[i]$ for the previous system, where $K = \begin{bmatrix} f_1 & f_2 \end{bmatrix}$. What should $f_1$ and $f_2$ be for the system to reach $\vec{x}[i] = 0$ from any states in 2 time steps?

**Solution:**

The system can be written as:

$$\vec{x}[i+1] = (A+BF)\vec{x}[i]$$

For this system to converge in 2 steps, we need:

$$\lambda_1 = \lambda_2 = 0$$

 

Where $\lambda_1$ and $\lambda_2$ are the eigenvalues of $(A+BF)$

$$A+BF = \begin{bmatrix} 2 & -1 \\ 1+f_1 & 2+f_2 \end{bmatrix}$$

Characteristic polynomial:

$$(2-\lambda)(2+f_2-\lambda)+1+f_1 = \lambda^2+\lambda(-4-f_2)+5+f_1+2f_2$$

Coefficient match to:

$$\begin{aligned} (\lambda+0)(\lambda+0) &= \lambda^2 \\ -4-f_2 &= 0 \Rightarrow f_2 = -4 \\ 5+f_1+2f_2 = 5+f_1-8 &= 0 \Rightarrow f_1 = 3 \end{aligned}$$

(g) Given the system $\vec{x}[i+1] = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u[i]$. Is the system controllable? What should $f_1$ and $f_2$ be to put the eigenvalues of the system at $\lambda = -1 \pm j$

**Solution:** The controllability matrix

$$\mathscr{C} = \begin{bmatrix} B & AB \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

has full rank. So the system is controllable.

A state feedback controller has the form

$$u[i] = F\vec{x}[i] = \begin{bmatrix} f_1 & f_2 \end{bmatrix} \begin{bmatrix} x_1[i] \\ x_2[i] \end{bmatrix}.$$

With this control the closed loop system is

$$\vec{x}[i+1] = (A+BF)\vec{x}[i] = \begin{bmatrix} 0 & 1 \\ f_1 & f_2 \end{bmatrix} \vec{x}[i].$$

The characteristic equation of the closed loop system is

$$0 = \det(A-\lambda I) = \det \begin{bmatrix} -\lambda & 1 \\ f_1 & f_2-\lambda \end{bmatrix} = \lambda^2 - f_2\lambda - f_1.$$

The desired closed loop characteristic equation is

$$0 = (\lambda-(-1+j))(\lambda-(-1-j)) = \lambda^2+2\lambda+2$$

So we should choose $f_1 = f_2 = -2$.

(h) Given the system $\vec{z}[i+1] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 5 & 11 \end{bmatrix} \vec{z}[i] + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u[i]$. Design state feedback so that the system has eigenvalue $0, 1/2, -1/2$.

 

**Solution:** The closed loop system in $z$ coordinates is given by

$$\widetilde{A}+\widetilde{B}\widetilde{F} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ f_1 & f_2+5 & k_3+11 \end{bmatrix}$$

which has characteristic polynomial $\lambda^3+(-11-k_3)\lambda^2+(-5-f_2)\lambda-f_1$. To place the eigenvalues at 0, 1/2, -1/2, the desired characteristic polynomial is $\lambda(\lambda-\frac{1}{2})(\lambda+\frac{1}{2}) = \lambda^3-\frac{1}{4}\lambda$. So we should choose $f_1=0, f_2=\frac{-19}{4}, k_3=-11$.

 

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