1. **Controllability Practice**

Consider the system

$$\vec{x}[i+1] = \begin{bmatrix} 0 & 1 & -2 \\ 0 & 2 & 0 \\ -1 & 1 & 0 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} u[i]$$

(a) **What is the controllability matrix, $\mathscr{C}$, for this system?**

(b) **What is the rank of the controllability matrix? Is the system controllable?**

(c) **Starting at $\vec{x}[0] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, what possible states can we reach after one timestep? Two timesteps? Three?**

(d) **What is the minimum number of timesteps it takes to reach $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$? What about $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$?**

Now, consider the system, with A modified slightly:

$$\vec{x}[i+1] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 2 & -2 \\ -1 & 1 & 0 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} u[i]$$

(e) **What is the controllability matrix, $\mathscr{C}$, for this system? What is its column space?**

(f) **What is the rank of the controllability matrix? Is the system controllable?**

(g) **Starting at $\vec{x}(0) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, what possible states can we reach after one timestep? Two timesteps? Three?**

(h) **What is the minimum number of timesteps it takes to reach $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$?**

Hint: Since the system is controllable, we can reach any state in three time steps, however, it may be possible to reach a state in fewer than three time steps. Look at your answer to the previous part, and check which possible states we can reach in one, two, and three time steps.

(i) **What is the minimum number of timesteps it takes to reach $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$?**

2. System Feedback

Consider the following continuous time system:

$$\frac{d^2}{dt^2}x(t) = -x(t)$$

We convert this system to state space form:

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + B\vec{u}(t) \tag{1}$$

We will pick our state variable $\vec{x} = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$, with $x_1(t) = x(t)$ and $x_2(t) = \frac{d}{dt}x(t)$.

(a) What are the values of the A matrix?

(b) Is this continuous time system stable? How would you describe its behavior?

We want to change the behavior of the system using a feedback control model:

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = A\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)$$

To do this, we set $u(t) = F\vec{x}(t)$, where $F = \begin{bmatrix} f_1 & f_2 \end{bmatrix}$. Note that F is a 1×2 matrix.

(c) We will now try to use our knowledge of controllability to stabilize this system. Is this continuous time system controllable?

(d) After plugging in our input, $u(t) = F\vec{x}(t)$, what will our new system be? How can you write out this system of differential equations in matrix/vector form?

(e) We will define the new matrix derived in the previous part as A_{cl} for the closed loop system. What are the eigenvalues of this new matrix A_{cl}?

(f) Suppose $f_1 = 1$ and $f_2 = 2$, what will the eigenvalues of this system be? How about when $f_1 = -1$ and $f_2 = -2$?

(g) What values of f_1 and f_2 will remove the oscillatory behavior completely and still stabilize the system?

3. Discrete Time Feedback

(a) Consider the scalar system: $x[i+1] = 1.5x[i] + u[i]$. Given the controller $u[i] = fx[i]$, for what value of f can we have the system to behave like:$x[i+1] = \lambda x[i]$ where $\lambda = 0.7$?

(b) Given the system $\vec{x}[i+1] = \begin{bmatrix} 3 & 0 \\ 0 & 5 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u_1[i] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_2[i]$. Let $u_1[i] = f_1 x_1[i]$ and $u_2[i] = f_2 x_2[i]$. What value of f_1 and f_2 would make the system stable with eigenvalues $\lambda_1 = \lambda_2 = \frac{1}{2}$?

(c) Given the matrix $\begin{bmatrix} 2 & 1 \\ -3+2f_1 & 4+2f_2 \end{bmatrix}$, what should f_1 and f_2 be for the matrix to have eigenvalues $\lambda_1 = \frac{1}{2}$ and $\lambda_2 = \frac{-1}{3}$?

(d) Given the matrix $\begin{bmatrix} 2+f_1 & 7+f_2 \\ 3 & -1 \end{bmatrix}$, what should f_1 and f_2 be for the matrix to have eigenvalues $\lambda_1 = 5$ and $\lambda_2 = 2$?

(e) Given the system $\vec{x}[i+1] = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u[i]$. Is the system stable?

(f) Given the feedback controller $u[i] = F\vec{x}[i]$ for the previous system, where $K = \begin{bmatrix} f_1 & f_2 \end{bmatrix}$. What should f_1 and f_2 be for the system to reach $\vec{x}[i] = 0$ from any states in 2 time steps?

(g) Given the system $\vec{x}[i+1] = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \vec{x}[i] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u[i]$. Is the system controllable? What should f_1 and f_2 be to put the eigenvalues of the system at $\lambda = -1 \pm j$

(h) Given the system $\vec{z}[i+1] = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 5 & 11 \end{bmatrix} \vec{z}[i] + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u[i]$. Design state feedback so that the system has eigenvalue $0, 1/2, -1/2$.

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