CSM
Spring 2022

EECS 16B
# Worksheet 5

## 1. An Introduction to Systems

Many physical systems such as the motion of a car, can be modeled using a system. Often times, when we are describing a system, we will have a **state variable** $\vec{x}$, that will often be a multivariable function. For a given system, we can often write a differential equation describing its change over time as

$$\frac{d\vec{x}(t)}{dt} = f(\vec{x}(t), \vec{u}(t)) \tag{1}$$

In this problem, we will examine a specific form of systems that can be put in **state-space representation.**

For a continuous-time linear systems (we will define what it means to be linear later) the general state-space representation is shown below:

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + B\vec{u}(t) \tag{2}$$

Similarly, for a discrete-time linear system the general state-space representation is shown below, replacing derivatives with respect to time with recursive difference equations:

$$\vec{x}[n+1] = A\vec{x}[n] + B\vec{u}[n] \tag{3}$$

Where $A$ is the $n \times n$ state matrix, $\vec{x}$ is a state vector in $\mathbb{R}^n$, $B$ is a $n \times d$ input matrix, and $\vec{u}$ is an input vector in $\mathbb{R}^d$. We will usually consider a $B$ as a vector in $\mathbb{R}^n$ and $u(t)$ will be a scalar input. Intuitively, $A$ acts as a linear function that determines how a future state depends on the current state of the world, and $B$ explains how an action or input that we introduce affects our system.

Tying this back to the circuits we've analyzed, an example of a state variable could be the voltage $V_C(t)$ across a capacitor or the current $I_L(t)$ through an inductor, and an example input could be the input voltage of a system $V_{in}(t)$.

Consider the following system:

$$\frac{d}{dt}x_1(t) = 3x_1(t) - 2x_2(t) + 4$$
$$\frac{d}{dt}x_2(t) = -x_1(t) + 5x_2(t) + 2$$

The initial conditions of the state variables are $x_1(0) = 2,\ x_2(0) = 3$.

(a) **What is the state vector $\vec{x}(t)$ for this system?**

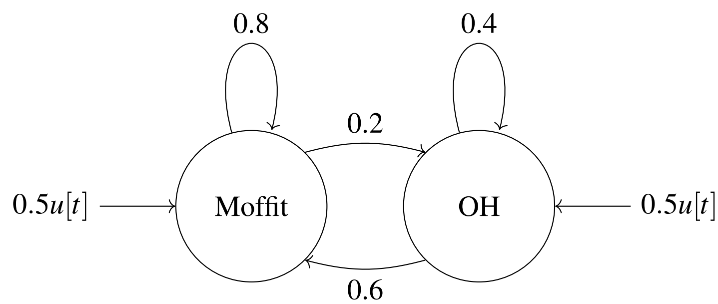
(b) **What is the initial condition $\vec{x}(0)$ of this system?**

(c) **Write out the system of differential equations the form of a general continuous time state-space model.**

**2. Intro to Discrete-Time Systems**

Students are studying for the EECS16C exam, and the flow of students from Moffit to Taejin's Office Hours (OH) can be represented as the following:




where $u[t]$ is the number of students that start to study at timestep $t$. (i.e.: $u[t]$ is the input to the system)

(a) Let our state variables be represented by $x_1$ and $x_2$. **Explain in your own words what the state variables $x_1$ and $x_2$ could represent in our system**

(b) Represent the flow of students between the two states as a matrix-vector discrete time system:

$$\vec{x}[t+1] = A\vec{x}[t] + \vec{b}u[t]$$

**Find matrix $A$ and vector $\vec{b}$.**

(c) Let $\vec{x}[0] = \vec{0}$, and $u[t] = 10$ for all values of $t$. **What is $\vec{x}[1]$? What is $\vec{x}[2]$? What is $||\vec{x}||$ as $t \to \infty$? Does that make sense in the context of our problem?**

(d) Let $\vec{x}[0] = \begin{bmatrix} 50 \\ 50 \end{bmatrix}$, and $u[t] = -4$ for all values of $t$. **What is $\vec{x}[1]$? What is $\vec{x}[2]$? What is $||\vec{x}||$ as $t \to \infty$? What sign are the elements of $\vec{x}$? Does that make sense in the context of our problem?**

(e) Let $\vec{x}[0] = \vec{0}$, $u[0] = 16$, and $u[t > 0] = 0$. **What is $\vec{x}[1]$? What is $\vec{x}[2]$? What is the largest $||\vec{x}||$ can get as $t \to \infty$? Does that make sense in the context of our problem?**

## 3. BIBO Stability

In this question, we will investigate into the definitions of stability for a scalar system modeled by a first order differential equation of the form with the initial condition $x(0) = x_0$.

$$\frac{d}{dt}x(t) = \lambda x(t) + u(t) \tag{4}$$

When we are discussing the stability of a system, we want to see if this system will produce a bounded output $\vec{y}(t)$for every bounded input $\vec{u}(t)$. Therefore we will say that this system is **BIBO stable** if for every bounded input $u(t)$, the output $y(t)$ is bounded as well.

As a reference, a function $f(t)$ is bounded by a constant $B$ if: $\left|f(t)\right| \leq B < \infty$ for all values of $t$.

The output $y(t)$ will be a function of $x(t)$ in the form:

$$y(t) = \alpha x(t) + \beta u(t) \tag{5}$$

As $y(t)$ is a linear combination of $x(t)$, and an already assumed to be bounded input $u(t)$, showing that $y(t)$ is bounded is equivalent to showing that $x(t)$ is bounded.

Recall that the particular solution to the differential equation (**??**), was uniquely determined for $t \geq 0$ as:

$$x_p(t) = x_0 e^{\lambda t} + \int_0^t u(\tau)e^{\lambda(t-\tau)}d\tau \tag{6}$$

Although $\lambda$ can be complex, we can observe that $|e^{(a+bj)}| = |e^a||e^{bj}| = |e^a||\cos(b) + j\sin(b)| = |e^a|$.

The, $\mathfrak{Im}(\lambda)$ does not affect stability, and will correspond to oscillations that are bounded. Therefore, we will only consider the effects of $\mathfrak{Re}(\lambda)$ which affect the stability of a system, and for the purposes of this question, assume that $\lambda$ is a real number.

(a) We will start with a bounded input $u(t) = 0$. Check if $x_p(t)$ is bounded for the three following cases:

  (i) $\lambda > 0$

  (ii) $\lambda = 0$

  (iii) $\lambda < 0$

(b) True/False: Since $x_p(t)$ is unbounded for $\lambda > 0$, we can say that the system **is not** BIBO stable, for $\lambda > 0$.

(c) True/False: Since $x_p(t)$ is bounded for $\lambda \geq 0$, we can say that the system **is** BIBO stable, for $\lambda \geq 0$.

(d) Now let's consider an input $u(t) = e^{\lambda t}$, can you say anything about the BIBO stability for $\lambda = 0$?

(e) How can we show that when $\lambda < 0$, the system is indeed BIBO stable? ' You should start by assuming you have a bounded input $u(t)$ such that $\left|u(t)\right| \leq B$. *Hint:* $\left|\int x(t)dt\right| \leq \int \left|x(t)\right|$.

**4. Aperture Stability**

As an intern at Aperture Laboratories, it is your job to make sure the robots being built are stable systems. As a reminder, if the following conditions are met the system will be stable:

- For discrete time systems of the form:

$$\vec{x}[t+1] = A\vec{x}[t] + Bu[t] + \vec{w}[t] \tag{7}$$

  All eigenvalues of the matrix A, $\lambda_i$, have magnitude $|\lambda_i| < 1$.

- For continuous time systems of the form:

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + Bu(t) + \vec{w}(t) \tag{8}$$

  All eigenvalues of the matrix A, $\lambda_i$, have real part $\mathfrak{Re}(\lambda_i) < 0$.

(a) According to your boss, the first robot, GLaDOS, can be described with the following discrete time system:

$$\vec{x}[t+1] = \begin{bmatrix} \frac{3}{8} & \frac{1}{8} \\ \frac{1}{8} & \frac{3}{8} \end{bmatrix} \vec{x}[t] + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u[t]$$

Is she stable?

(b) Your boss now gives you data on the P-body robot. Is she stable? Her motion is described with the following discrete time system:

$$\vec{x}[t+1] = \begin{bmatrix} -2 & -1 \\ 1 & -2 \end{bmatrix} \vec{x}[t] + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u[t]$$

(c) Now your boss gives you data on a more advanced robot, Atlas. Is he stable? His movements can be described with the following continuous time system:

$$\frac{d}{dt}\vec{x}(t) = \begin{bmatrix} -2 & -1 \\ 1 & -2 \end{bmatrix} \vec{x}(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} u(t)$$

(d) Lastly, your boss gives you data on the Wheatley robot. Is he stable? His motion is described with the following discrete time system:

$$\vec{x}[t+1] = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix} \vec{x}[t] + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u[t]$$

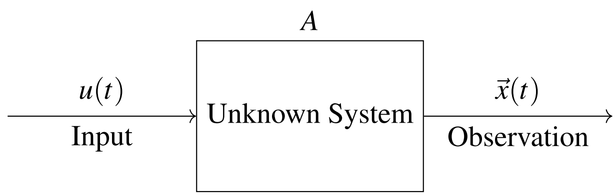
## 5. System Identification

In this question, we will take a look at how to **identify** a system by taking experimental data taken from a (presumably) linear system to learn a discrete-time linear model for it using the least-squares.

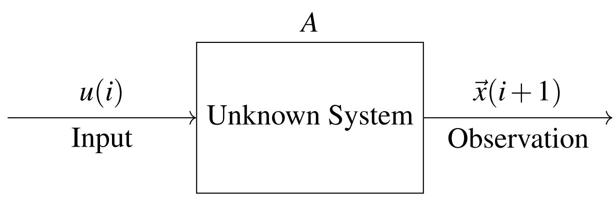
Recall that a **linear, continuous-time,** system can be put in state-space form:

$$\frac{d}{dt}\vec{x}(t) = A\vec{x}(t) + \vec{b}u(t) \tag{9}$$

Now let's say we have an **unknown** linear system in which we can give an input $u(t)$ and observe the output $\vec{x}(t)$. We can model the system using the following diagram:




Recall from discussion that if we put a **piecewise constant** input $u(t) = u(i)$ for $t \in [i, i+1)$, then we can observe the output $\vec{x}(t)$ at time $t = i+1$, and form a discretized model of the observation.




**If** we knew the system, the relationship between $\vec{x}(i+1), \vec{x}(i)$, and $u(i)$ would be:

$$\vec{x}(i+1) = A\vec{x}(i) + \vec{b}u(i) \tag{10}$$

While this relation is useful, we currently do not know what the $A$ matrix or $\vec{b}$ vector are.

Therefore, we will start by creating unknown variables for the $A$ matrix, and $\vec{b}$ vector:

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \text{ and } \vec{b} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} \tag{11}$$

For the purposes of this question, we will be in the space $\mathbb{R}^2$.

(a) Let's say the system initially started at $\vec{x}(0) = \begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix}$, and we gave an input at time $t = 0,\ u(0)$. At time $t = 1$, we observe $\vec{x}(1) = \begin{bmatrix} x_1(1) \\ x_2(1) \end{bmatrix}$. **How can you uncouple this matrix/vector equation into a system of linear equations?**

(b) Based on the system of linear equations created in the previous part, **how many unknown** variables do we have? Also, if we have a system of linear equations with $n$ unknown variables, at the minimum, **how many equations** would we need to solve our system?

(c) We now give another input at $t = 1,\ u(1)$, and observe $\vec{x}(2)$.

**How many more equations do we get from this observation?** Also, how many more inputs will we need to observe until we have enough equations?

(d) Assuming we have taken all of the necessary measurements of $x(t)$ at time $t = 0, 1, 2, ...$

**How can we set up our system of linear equations as a matrix-vector equation?**

(e) While we can set up a matrix vector equation and uniquely solve our system, the output of the system can be noisy. Therefore, we update our model by considering a noise term $w(i)$ at time $t = i$.

$$\vec{x}(i+1) = A\vec{x}(i) + \vec{b}u(i) + w(i) \tag{12}$$

How can we set up a system of equations in a similar fashion but with a noise vector $\vec{w}$?

$$\vec{y} = D\vec{s} + \vec{w} \tag{13}$$

(f) We can try to solve our system of equations, but we do not know what $\vec{w}$ is.

What we can do however, is to take more measurements, and set up a **least squares** problem as seen in 16A. What would the least squares problem be if we took measurements up to time step $t = 5$?

(g) How would we solve this least squares problem?