Final Report

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1 Prerequisites

1.1 Special Theory of Relativity

Consider an inertial frame $\boldsymbol{A}$ with coordinates $(\vec{x}, t)$ and another frame $\boldsymbol{B}$ $(\vec{x}', t')$ attached to a body moving with constant velocity $\vec{v}$, with respect to $\boldsymbol{A}$, along the x_1-axis, i.e., $\vec{v} = v\hat{x_1}$, then, the set of rules for converting from one $\boldsymbol{A}$ to $\boldsymbol{B}$ (i.e., a transformation) is :

$$\begin{aligned} x_1' &= x_1 - vt \\ t' &= t \\ x_2' &= x_2 \\ x_3' &= x_3 \end{aligned}$$

These set of transformations is called a Galilean transformation which is relevant for non-relativistic limits. But in relativity these quantities transform according to Lorentz transformation which is given by :

$$\begin{aligned} x_1' &= \frac{x_1 - vt}{\sqrt{1-(v/c)^2}} \\ t' &= \frac{t - \dfrac{v}{c^2}x_1}{\sqrt{1-(v/c)^2}} \\ x_2' &= x_2 \\ x_3' &= x_3 \end{aligned}$$

Writing out the transformations like this is a bit cumbersome. But one may also notice that Lorentz transformations are linear and hence can be compactly represented in matrix notation. Before we do that we will have to settle for a convention for representing the 4–dimensional vectors of position– and time–co-ordinates. We will denote these vectors as follows

$$x^\mu \equiv (x^0, x^1, x^2, x^3)$$

where $x^0 = ct$ describes the time co-ordinate and $(x^1, x^2, x^3) = \vec{r}$ describes the space coordinates.

Historically, the Lorentz transformations were formulated in a space in which the time component of x^μ was chosen as a purely imaginary number and the space components real. This is because Lorentz transformations leave the quantity

$$s^2 = +(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2 = +(x'^0)^2 - (x'^1)^2 + (x'^2)^2 + (x'^3)^2$$

invariant. One can interpret the quantity s as a distance in a 4–dimensional Euclidean space if one chooses the time component purely imaginary. In such a space Lorentz transformations correspond to 4-dimensional rotations.
Also note that the invariant quantity s^2 can be represented (using the Einstein summation convention) as

$$s^2 = g_{\mu\nu} x^\mu x^\nu$$

where,

$$\boldsymbol{g} = g_{\mu\nu} = \begin{pmatrix} +1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$$

Now, the transformation given in the beginning of the section can be represented as

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu$$

The tensor Λ is such that it leaves s^2, defined earlier, invariant i.e.,

$$g_{\mu\nu} x^\mu x^\nu = \Lambda^\rho{}_\nu g_{\rho\sigma} \Lambda^\sigma{}_\mu x^\mu x^\nu$$

Since this holds for any x^μ it must be true that :

$$g_{\mu\nu} = \Lambda^\rho{}_\nu g_{\rho\sigma} \Lambda^\sigma{}_\mu$$

This is a key property of arbitrary Lorentz transformations and every other useful properties can be derived starting from this.

The most general set of transformations between two inertial frames moving with respect to each other is the Poincare transformation i.e.,$x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu$, where a^μ is a constant 4-vector which represents translation. But in the following sections we will consider only a subset of Lorentz transformation (proper, orthochronous Lorentz transformations).

Scalars : The quantities with the simplest transformation behaviour are so-called scalars which are invariant under Lorentz transformations.

Contravariant and Covariant 4-Vectors : The quantities with the transformation behaviour like that of the position–time vector x^μ are the so-called contravariant 4–vectors a^μ.
Covariant vectors a_μ are defined using their contravariant counterparts as follows : $a_\mu = g_{\mu\nu} a^\nu$

Some important contravariant vectors include the position-time 4-vector(x^μ) and momentum 4-vector :

$$\left(p^\mu = (E, \vec{p}); E = \frac{mc^2}{\sqrt{1-\vec{v}^2/c^2}}; \vec{p} = \frac{m\vec{v}}{\sqrt{1-\vec{v}^2/c^2}}\right)$$

An important example of a covariant 4-vector is the differential operator :

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{1}{c}\frac{\partial}{\partial t}, \nabla\right)$$

NOTE : From this point onwards we will work in natural units i.e., $c = \hbar = 1$. This will allow us to avoid writing a lot of $c\,/\,\hbar$. So the differential operator will now be represented as :

$$\partial_\mu = \frac{\partial}{\partial x^\mu} = \left(\frac{\partial}{\partial t}, \nabla\right)$$

Consequently the d'Alembert operator which is defined as :

$$\partial^\mu \partial_\mu = \partial_t^2 - \nabla^2$$

is a scalar and hence invariant under transformations.

1.2 Formalisms of Classical Field Theories

1.2.1 Lagrangian Field Theory

Most often than not in physics, theories arise from a variational principle. The same is true for field theories as the evolution of the fields is governed by the *Principle of Least Action*. Before doing that, let's understand the notation that we will use extensively in the next sections.

We can think of a real scalar field φ as a function on spacetime i.e., $\varphi : \mathbb{R}^4 \to \mathbb{R}$. Introduce coordinates $x^\mu = (t, x, y, z)$ on $\mathbb{R}^4$. We can now define the Lagrangian density $\mathcal{L}$ as a function of the field φ and its derivatives with respect to each co-ordinate $(\partial_t\varphi, \nabla\varphi)$. This will in turn enable us to define an action functional for the field theory as follows :

$$S[\varphi] = \int \mathcal{L}(\varphi, \partial_\mu\varphi)\, d^4x$$

The principle of least action states that the system evolves on the space-time path for which the action is stationary (no change) to first order. This is also why people sometimes prefer calling it the *principle of stationary action*. So,

$$\delta S = 0$$

$$\delta S = \int \left(\frac{\partial \mathcal{L}}{\partial \varphi}\,\delta\varphi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu\varphi)}\,\delta(\partial_\mu\varphi)\right) d^4x$$

As $\delta(\partial_\mu\varphi) = \partial_\mu(\delta\varphi)$ we can integrate the second term by parts to get :

$$\delta S = \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \Big(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \Big) \right) \delta\varphi + \int d^4x \, \partial_\mu \Big(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \delta\varphi \Big)$$

The last terms, when integrated, involve variations at the surface of region of integration or either at infinity both of which are zero, thus yielding :

$$\delta S = \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \Big(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \Big) \right) \delta\varphi$$

Now as $\delta S = 0$ and it should hold for arbitrary variations of the field configuration, $\delta\varphi$, the integrand must be identically zero. This gives the Euler-Lagrange equation of motion for the field φ :

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \right) - \frac{\partial \mathcal{L}}{\partial \varphi} = 0$$

1.2.2 Hamiltonian Field Theory

Lagrangian formulation of field theory is particularly suited for relativistic dynamics as all expressions are explicitly Lorentz invariant. But the Euler-Lagrange equations are second order in differential equations which is not very useful in quantum mechanics. So now let's briefly consider elements of the Hamiltonian formulation.

So we define the conjugate momentum density $\pi(x)$ as :

$$\pi(x) = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}}$$

And the definition of Hamiltonian density $\mathcal{H}$ and total Hamiltonian H follows :

$$\mathcal{H}\Big(\pi(x), \varphi(x)\Big) = \pi(x)\dot{\varphi}(x) - \mathcal{L}(\varphi, \partial_\mu\varphi)$$
$$H = \int \mathcal{H}\Big(\pi(x), \varphi(x)\Big) \, d^3x$$

Deriving the Hamilton equations of motion for the field is fairly straightforward. Note that :

$$\delta H = \int \Big(\frac{\delta \mathcal{H}}{\delta \pi} \delta\pi + \frac{\delta \mathcal{H}}{\delta \varphi} \delta\varphi \Big) d^3x$$

And also, expanding out $\mathcal{H}$ in the same equation gives :

$$\begin{aligned} \delta H &= \int \Big(\pi \, \delta\dot{\varphi} + \dot{\varphi} \, \delta\pi - \delta\mathcal{L} \Big) d^3x \\ &= \int \Big(\pi \, \delta\dot{\varphi} + \dot{\varphi} \, \delta\pi - \dot{\pi} \, \delta\varphi - \pi \, \delta\dot{\varphi} \Big) d^3x \end{aligned}$$

The last line follows from :

$$
\begin{aligned}
\delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial\varphi}\delta\varphi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\delta(\partial_\mu\varphi) \\
&= \partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\right)\delta\varphi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\delta(\partial_\mu\varphi) \\
&= \partial_0\left(\frac{\partial\mathcal{L}}{\partial(\partial_0\varphi)}\right)\delta\varphi + \frac{\partial\mathcal{L}}{\partial(\partial_0\varphi)}\delta(\partial_0\varphi) \\
&= \dot{\pi}\delta\varphi + \pi\delta\dot{\varphi}
\end{aligned}
$$

We were able to move from equation two to three as we are going to integrate over the entire space in the end, sending the positional derivatives to zero. Comparing the two equations gives us the Hamilton's equations of motion in Classical Field Theory $\left(\frac{\delta\mathcal{H}}{\delta\pi} = \dot{\varphi} \text{ and } -\frac{\delta\mathcal{H}}{\delta\varphi} = \dot{\pi}\right)$.
Also, if you define $L = \int d^3x\,\mathcal{L}$ and $\pi = \frac{\partial L}{\partial\dot{\varphi}}$ $\left(\text{rather than } \frac{\partial\mathcal{L}}{\partial\dot{\varphi}}\right)$ the equations take the form :

$$
\begin{aligned}
\dot{\varphi} &= \frac{\partial H}{\partial\pi} \\
\dot{\pi} &= -\frac{\partial H}{\partial\varphi}
\end{aligned}
$$

1.3 Noether's Theorem

Every continuous symmetry of the Lagrangian gives rise to a conserved 4-vector current j^μ that is $\partial_\mu j^\mu = 0$. Another way of seeing this is by separating the space and time components as :

$$\frac{\partial j^0}{\partial t} + \nabla\cdot\vec{j} = 0$$

A conserved current j^μ gives rise to a global conserved charge Q_{net} defined as :

$$Q_{net} = \int\limits_{R^3} d^3x\, j^0$$

It is easy to verify that :

$$\frac{dQ_{net}}{dt} = \int\limits_{R^3} d^3x\,\frac{\partial j^0}{\partial t} = -\int\limits_{R^3} d^3x\,\nabla\cdot\vec{j} = 0$$

By the divergence theorem the second-last term can be written as integral over a surface at infinity which falls off to zero thus verifying the global or net conservation of Q_{net}. However, the existence of a current is a much stronger statement than the existence of a conserved charge because it implies that charge is conserved locally too.

Proof: We say that a transformation of the field $X(\varphi) = \delta\varphi$ is a symmetry if it changes the Lagrangian by a total derivative $\delta\mathcal{L} = \partial_\mu F^\mu$. Then,

$$\begin{aligned}
\delta\mathcal{L} &= \frac{\partial\mathcal{L}}{\partial\varphi}\,\delta\varphi + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\,\delta(\partial_\mu\varphi) \\
&= \left(\frac{\partial\mathcal{L}}{\partial\varphi} - \partial_\mu\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\Big)\right) + \partial_\mu\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\delta\varphi\Big) \\
&= \partial_\mu\Big(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\delta\varphi\Big)
\end{aligned}$$

Last equation follows from euler-lagrange equations of motion. Now let $X(\varphi) = \delta\varphi$ and $\delta\mathcal{L} = \partial_\mu F^\mu(\varphi)$ to get :

$$\partial_\mu j^\mu = 0 \quad \text{with} \quad j^\mu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}X(\varphi) - F^\mu(\varphi)$$

Let's consider an example :

Energy-Momentum Tensor

Let us consider the transformation caused by translation in space-time coordinates i.e., $x^\nu \to x^\nu - \epsilon^\nu$. The field $\varphi(x)$ and the Lagrangian which is a function of the space-time coordinates for a fixed field, transforms as :

$$\varphi(x) \to \varphi(x) + \epsilon^\nu\partial_\nu\varphi(x) \qquad \mathcal{L}(x) \to \mathcal{L}(x) + \epsilon^\nu\partial_\nu\mathcal{L}(x)$$

As the Lagrangian changes by a total derivative, we have our first symmetry. For every value of ν we have a conserved current $(j^\mu)_\nu$ defined as :

$$(j^\mu)_\nu = \frac{\partial\mathcal{L}}{\partial(\partial_\mu\varphi)}\partial_\nu\varphi - \delta^\mu_\nu\mathcal{L} \equiv T^\mu_\nu$$

as $\partial_\nu\mathcal{L} = \partial_\mu\delta^\mu_\nu\mathcal{L}$. T^μ_ν is called the energy-momentum tensor. The four conserved quantities can be written as :

$$E = \textstyle\int d^3x\,T^{00} \quad \text{and} \quad P^i = \textstyle\int d^3x\,T^{0i}$$

where E is the total energy of the field configuration, while P^i is the total momentum of the field configuration.

These conservation laws can also be seen in the light of Hamiltonian formalism.

Consider an observable Q which is a functional of the field $\varphi(x)$ and the conjugate momentum density $\pi(x)$ and not explicitly dependent on time. Time evolution of this quantity can be can be deduced from the Poisson brackets of that quantity with the Hamiltonian, H,

$$\begin{aligned}
\frac{dQ}{dt} &= \int d^3x \left(\frac{\partial Q}{\partial \varphi} \frac{d\varphi}{dt} + \frac{\partial Q}{\partial \pi} \frac{d\pi}{dt} \right) \\
&= \int d^3x \left(\frac{\partial Q}{\partial \varphi} \frac{\partial H}{\partial \pi} - \frac{\partial Q}{\partial \pi} \frac{\partial H}{\partial \varphi} \right) \\
&= \{Q,\, H\}
\end{aligned}$$

So if the system has conserved charges, Q, since, $\dfrac{dQ}{dt} = 0$, one has for conserved charge, Q,

$$\{Q,\, H\} = 0$$

It is trivial to see that for a Hamiltonian with no explicit time dependence $\dfrac{dH}{dt} = 0$ as $\{H,\, H\} = 0$. Also, its re-assuring to see that that H and E are the same quantity, as :

$$\begin{aligned}
E &= \int d^3x\, T^{00} \\
&= \int d^3x \left(\frac{\partial \mathcal{L}}{\partial(\partial_0 \varphi)} \partial^0 \varphi - \eta^{00} \mathcal{L} \right) \\
&= \int d^3x \left(\pi \dot{\varphi} - \mathcal{L} \right) \\
&= H
\end{aligned}$$

This is why Hamiltonian is interpreted as the total energy of the system or field configuration.

2 First attempts

Let us study the first attempts at integrating Quantum Mechanics with Relativity. We will derive important equations in a **heuristic** fashion.

2.1 Klein-Gordon Equation

Let us consider a very simple system: a spinless, non-relativistic particle with no forces acting on it. In this case, the Hamiltonian is

$$H = \frac{\hat{p}^2}{2m} = -\frac{\nabla^2}{2m}$$

Schrodinger equation for this particle in position basis is :

$$i\frac{\partial}{\partial t}\psi(\vec{x}, t) = -\frac{\nabla^2}{2m}\psi(\vec{x}, t)$$

To make this equation lorentz invariant one can change the Hamiltonian to :

$$\begin{aligned} H &= \sqrt{\hat{p}^2 + m^2} \\ &= \sqrt{-\nabla^2 + m^2} \\ &= m + \frac{\nabla^2}{2m} + ... \end{aligned}$$

With this Hamiltonian, Schrodinger equation becomes :

$$i\frac{\partial}{\partial t}\psi(\vec{x},t) = \sqrt{-\nabla^2 + m^2}\;\psi(\vec{x},t)$$

Unfortunately, this equation presents us with a number of difficulties. One is that it apparently treats space and time on a different footing, something that we would not expect of a relativistic theory. Furthermore, if we expand out the square-root we get an infinite number of spatial derivatives acting on the wavefunction; this implies that theory is not local.

All of these problems can be alleviated if we square both sides of the equation :

$$\begin{aligned} -\frac{\partial^2}{\partial t^2}\psi(\vec{x},t) &= (-\nabla^2 + m^2)\;\psi(\vec{x},t) \\ (-\partial_t^2 + \nabla^2 - m^2)\psi(\vec{x},t) &= 0 \\ (\partial_\mu\partial^\mu + m^2)\psi(\vec{x},t) &= 0 \end{aligned}$$

This is the Klein-Gordon equation which we have established to be lorentz invariant in a previously.

Solving the Klein Gordon equation

We can solve the KG equation over real scalar fields via Fourier's trick. Consider the fourier decomposition of $\psi(\vec{x},t)$:

$$\psi(\vec{x},t) = \left(\frac{1}{2\pi}\right)^{3/2}\int d^3k\;\hat{\varphi}_{\vec{k}}(t)\;e^{i\vec{k}\cdot\vec{x}}$$

Applying the KG equation to this yields the following relation :

$$\ddot{\hat{\varphi}}_{\vec{k}} + (k^2 + m^2)\hat{\varphi}_{\vec{k}} = 0$$

which can easily be solved. As we are considering real fields, coefficients can be appropriately set. Consequently, the most general solution to the KG equation is (for $\omega_{\vec{k}} = \sqrt{k^2 + m^2}$) :

$$\psi(\vec{x},t) = \left(\frac{1}{2\pi}\right)^{3/2}\int d^3k\left(a_{\vec{k}}\,e^{+i\vec{k}\cdot\vec{x}-i\omega_{\vec{k}}t} + a_{\vec{k}}^*\,e^{-i\vec{k}\cdot\vec{x}+i\omega_{\vec{k}}t}\right)$$

Note that :

$$\psi = \psi^+ + \psi^-$$

where,

$$\psi^+ = \left(\frac{1}{2\pi}\right)^{3/2} \int d^3k \left(a_{\vec{k}}\, e^{+i\vec{k}\cdot\vec{x} - i\omega_{\vec{k}} t}\right)$$
$$\psi^- = \left(\frac{1}{2\pi}\right)^{3/2} \int d^3k \left(a_{\vec{k}}^*\, e^{-i\vec{k}\cdot\vec{x} + i\omega_{\vec{k}} t}\right)$$

Each of these is a complex-valued solution to the KG equation. These are called the positive frequency and negative frequency solutions of the KG equation,respectively.

But the positive frequency solution (ψ^+) even satisfies the relativistic Schrodinger equation :

$$i\frac{\partial}{\partial t}\psi^+ = \sqrt{-\nabla^2 + m^2}\ \psi^+$$

The square root operator can be defined from this equation.
But more interesting is the fact that the positive frequency solutions of the KG equation satisfy a Schrodinger equation with Hamiltonian ($H = \sqrt{-\nabla^2 + m^2}$)can be interpreted as the kinetic energy of a relativistic particle. These solutions are sometimes called one-particle wave functions (for particles with zero spin).

Same can't be interpreted for the negative frequency solutions as they have negative norm and kinetic energy.

2.2 Dirac Equation

As the KG equation modeled zero-spin particles, Dirac formulated a similar theory for spin-1/2 particles.
The motivation for Dirac Equation comes from trying to get rid of square root in the relativistic Hamiltonian. We can do that if :

$$-\nabla^2 + m^2 = (-i\vec{\alpha}\cdot\vec{\nabla} + \beta m)^2$$

This imposes these conditions on $\vec{\alpha}$ and β :

$$-\alpha_1^2 = -\alpha_2^2 = -\alpha_3^2 = \beta^2 = 1$$
$$\alpha_i\alpha_j + \alpha_j\alpha_i = 0 \qquad \forall i, j, \quad i \neq j$$
$$\alpha_i\beta + \beta\alpha_i = 0 \qquad \forall i$$

These relations can't be satisfied by scalar numbers. The simplest solution to these are a set of 4x4 matrices.

$$\beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \quad \alpha^i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}$$

Here, σ_i are the famous Pauli matrices.

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Now we are ready to get rid of the square root and write an equation. Note that we will use the standard convention by opening the square root with a positive sign.

$$i\frac{\partial}{\partial t}\psi = (-i\vec{\alpha}\cdot\vec{\nabla} + \beta m)\psi$$
$$\Big[i\Big(\frac{\partial}{\partial t} + \vec{\alpha}\cdot\vec{\nabla}\Big) - \beta m\Big]\psi = 0$$

Multiplying the whole equation by β gives us the Dirac Equation.

$$(i\gamma^{\mu}\partial_{\mu} - m)\psi = 0$$

where,

$$\gamma^0 = \beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \quad \gamma^i = \beta\alpha^i = \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix}$$

3 Canonical Quantisation

Canonical Quantisation is the correct way of going from Hamiltonian formalism of classical field theory to a quantum one. It requires us to promote the field and its conjugate momentum to operators. The Poisson bracket structure of classical mechanics morphs into the structure of commutation relations between operators.
So for two field operators $\phi_a(\vec{x})$ and $\phi_b(\vec{x})$ with corresponding conjugate momentum operators $\pi^a(\vec{x})$ and $\pi^b(\vec{x})$:

$$[\phi_a(\vec{x}), \phi_b(\vec{y})] = [\pi^a(\vec{x}), \pi^b(\vec{y})] = 0$$
$$[\phi_a(\vec{x}), \pi^b(\vec{y})] = \delta^3(\vec{x}-\vec{y})\delta^b_a$$

Note that we are working in the Schrodinger picture where all time dependence sits in the states $|\psi\rangle$ which evolve by the usual Schrodinger equation. The equations, also, don't look lorentz invariant.

3.1 Klein Gordon Field

Let us go back to the Klein-Gordon field. For completeness the Lagrangian density and full Hamiltonian are listed below :

$$\mathcal{L} = \frac{1}{2}(\partial_{\mu}\phi)(\partial^{\mu}\phi) - \frac{1}{2}m^2\phi^2$$
$$H = \int d^3x\,\frac{1}{2}\Big(\pi^2 + (\nabla\phi)^2 + m^2\phi^2\Big)$$

This gives us the Klein-Gordon equation,

$$(\partial_{\mu}\partial^{\mu} + m^2)\phi(\vec{x}, t) = 0$$

with the solution as discussed earlier,

$$\phi(\vec{x},t) = \left(\frac{1}{2\pi}\right)^{3/2} \int d^3p \left(a_{\vec{p}}\, e^{+i\vec{p}\cdot\vec{x} - i\omega_{\vec{p}} t} + a_{\vec{p}}^*\, e^{-i\vec{p}\cdot\vec{x} + i\omega_{\vec{p}} t} \right)$$

We immediately see that the solution is composed of linear superposition of simple harmonic oscillators, each vibrating at a different frequency with a different amplitude at every point in space.
And hence, to construct a quantum theory we need to quantize these infinite number of harmonic oscillators.

So we apply the methods of quantization of the simple harmonic oscillator. We write ϕ and π as a linear sum of an infinite number of creation and annihilation operators $a_{\vec{p}}$ and $a_{\vec{p}}^\dagger$, indexed by the 3-momentum $\vec{p}$,

$$\begin{aligned} \phi(\vec{x}) &= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2\omega_{\vec{p}}}} \left(a_{\vec{p}}\, e^{+i\vec{p}\cdot\vec{x}} + a_{\vec{p}}^\dagger\, e^{-i\vec{p}\cdot\vec{x}} \right) \\ \pi(\vec{x}) &= \int \frac{d^3p}{(2\pi)^3} \sqrt{\frac{\omega_{\vec{p}}}{2}} \left(a_{\vec{p}}\, e^{+i\vec{p}\cdot\vec{x}} - a_{\vec{p}}^\dagger\, e^{-i\vec{p}\cdot\vec{x}} \right) \end{aligned}$$

It is reassuring to know that computing the commutators gives us the right result. We can write similar useful commutation relations for $a_{\vec{p}}$ and $a_{\vec{p}}^\dagger$:

$$\begin{aligned} [a_{\vec{p}}\,,\, a_{\vec{q}}] &= [a_{\vec{p}}^\dagger, a_{\vec{q}}^\dagger] = 0 \\ [a_{\vec{p}}\,,\, a_{\vec{q}}^\dagger] &= (2\pi)^3 \delta^3(\vec{p} - \vec{q}) \end{aligned}$$

Computing the Hamiltonian in terms of $a_{\vec{p}}$ and $a_{\vec{p}}^\dagger$ after much simplification gives :

$$\begin{aligned} H &= \int d^3x\, \frac{1}{2} \left(\pi^2 + (\nabla\phi)^2 + m^2\phi^2 \right) \\ &= \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \omega_{\vec{p}} \left(a_{\vec{p}}\, a_{\vec{p}}^\dagger + a_{\vec{p}}^\dagger\, a_{\vec{p}} \right) \\ &= \int \frac{d^3p}{(2\pi)^3} \omega_{\vec{p}} \left(a_{\vec{p}}^\dagger\, a_{\vec{p}} + \frac{1}{2} [a_{\vec{p}}\,,\, a_{\vec{p}}^\dagger] \right) \\ &= \int \frac{d^3p}{(2\pi)^3} \omega_{\vec{p}} \left(a_{\vec{p}}^\dagger\, a_{\vec{p}} + \frac{1}{2} (2\pi)^3 \delta^3(0) \right) \end{aligned}$$

Here comes our first infinity. There is a term with a factor of delta function evaluated at zero and the integral $\int \frac{d^3p}{(2\pi)^3} \omega_{\vec{p}} = \int \frac{d^3p}{(2\pi)^3} \sqrt{\vec{p}^2 + m^2}$ surely diverges.
The subject of quantum field theory is rife with infinities. Interpreting where these infinities come from tells us a lot about what we're doing wrong. But for brevity's sake I will avoid doing that here.

Usually in physics we are interested in energy differences, so, we just drop the constant infinite term in the Hamiltonian. Now, the Hamiltonian is defined as:

$$H = \int \frac{d^3p}{(2\pi)^3} \omega_{\vec{p}}\, a^\dagger_{\vec{p}}\, a_{\vec{p}}$$

Let's define the vacuum, $|0\rangle$, by insisting that it is annihilated by all $a_{\vec{p}}$ i.e., $a_{\vec{p}}\,|0\rangle = 0 \quad \forall \vec{p}$. Then with the new definition $H\,|0\rangle = 0$.

The new definition can also be seen as a **normal ordered** form of the original. Normal ordering of an operator places all the annihilation operators $a_{\vec{p}}$ to the right of creation operators $a^\dagger_{\vec{p}}$. So :

$$\begin{aligned} : H : &= \frac{1}{2} \int \frac{d^3p}{(2\pi)^3} \omega_{\vec{p}} \Big(a_{\vec{p}}\, a^\dagger_{\vec{p}} + a_{\vec{p}}\, a^\dagger_{\vec{p}} \Big) \\ &= \int \frac{d^3p}{(2\pi)^3} \omega_{\vec{p}}\, a^\dagger_{\vec{p}}\, a_{\vec{p}} \end{aligned}$$

Particles

We can treat particles in the Klein-Gordon field just as excited states in the simple harmonic oscillator.

$$|\vec{p}\rangle = a^\dagger_{\vec{p}}\,|0\rangle$$

This is an eigenstate of the Hamiltonian with energy, $E_{\vec{p}} = \omega_{\vec{p}} = \sqrt{\vec{p}^2 + m^2}$ as

$$\begin{aligned} H\,|\vec{p}\rangle &= \int \frac{d^3q}{(2\pi)^3} \omega_{\vec{q}}\, a^\dagger_{\vec{q}}\, a_{\vec{q}}\, a^\dagger_{\vec{p}}\,|0\rangle \\ &= \int \frac{d^3q}{(2\pi)^3} \omega_{\vec{q}}\, a^\dagger_{\vec{q}}\, [\, a_{\vec{q}}\,, a^\dagger_{\vec{p}}]\,|0\rangle + \int \frac{d^3q}{(2\pi)^3} \omega_{\vec{q}}\, a^\dagger_{\vec{q}}\, a^\dagger_{\vec{p}}\, a_{\vec{q}}\;|0\rangle \\ &= \int \frac{d^3q}{(2\pi)^3} \omega_{\vec{q}}\, a^\dagger_{\vec{q}}\, (2\pi)^3 \delta^3(\vec{p} - \vec{q})\;|0\rangle + 0 \\ &= \omega_{\vec{q}}\;|\vec{p}\rangle \end{aligned}$$

We notice that this energy is same as that of a relativistic particle of mass m and 3-momentum $\vec{p}$ i.e., $E_{\vec{p}} = \omega_{\vec{p}} = \sqrt{\vec{p}^2 + m^2} = \sqrt{\vec{p}^2 c^2 + m^2 c^4}$ in standard units.

So we may interpret the state $|p\rangle$ as a single particle of mass m and momentum $\vec{p}$. We can verify this interpretation by constructing a momentum operator from $P^i = \int d^3x\, T^{0i}$, normal ordering it to get :

$$\vec{P} = \int \frac{d^3p}{(2\pi)^3} \vec{p}\, a^\dagger_{\vec{p}}\, a_{\vec{p}}$$

Acting this on the one-particle state $|\vec{p}\rangle$ we learn that it is indeed also an eigenstate of the momentum operator with eigenvalue $\vec{p}$,

$$\vec{P}\,|\vec{p}\rangle = \vec{p}\,|\vec{p}\rangle$$

We can also study this states angular momentum and doing so will tell us the particle carries no internal angular momentum.

Acting on $|0\rangle$ with multiple creation operators gives us multi-particle states,

$$|\vec{p_1}, \vec{p_2}, ..., \vec{p_n}\rangle = a^\dagger_{\vec{p_1}}\, a^\dagger_{\vec{p_2}}\, ...\, a^\dagger_{\vec{p_n}}\; |0\rangle$$

As $a^\dagger_{\vec{p}}$ and $a^\dagger_{\vec{q}}$ commute

$$|\vec{p}, \vec{q}\rangle = |\vec{q}, \vec{p}\rangle$$

So particles in the Klein-Gordon field are **bosons**.

Taking inspiration from the simple harmonic oscillator we can construct a useful operator N which counts the number of particle,

$$N = \int \frac{d^3q}{(2\pi)^3}\, a^\dagger_{\vec{q}} a_{\vec{q}}$$

such that $N\,|\vec{p_1}, \vec{p_2}, ..., \vec{p_n}\rangle = n\,|\vec{p_1}, \vec{p_2}, ..., \vec{p_n}\rangle$.

One can also verify that $[N, H] = 0$ implying that the total number of particles does not vary with time in this theory.

Relativistic Normalization

The one-particle states in our theory are not normalizable just like the position and momentum eigenstates in quantum mechanics. But we still would like it to be invariant under lorentz transformations.
Before we do that consider the space of one-particle states. The identity operator on in this space should obviously be lorentz invariant.

$$1 = \int \frac{d^3p}{(2\pi)^3}\,|\vec{p}\rangle\,\langle\vec{p}|$$

$$1 = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}\sqrt{2E_{\vec{p}}}\,|\vec{p}\rangle\,\langle\vec{p}|\,\sqrt{2E_{\vec{p}}}$$

Acting by $\langle\vec{p}|\,\sqrt{2E_{\vec{p}}}$ from the left, $\sqrt{2E_{\vec{p}}}\,|\vec{p}\rangle$ from the right on both sides of the equation and dividing by the common term gives us :

$$1 = \int \frac{d^3p}{(2\pi)^3}\frac{1}{2E_{\vec{p}}}\,\langle\vec{p}|\,\sqrt{2E_{\vec{p}}}\sqrt{2E_{\vec{p}}}\,|\vec{p}\rangle$$

The measure $\int (d^3p)/(2E_{\vec{p}})$ is lorentz invariant, so, for the net expression to be lorentz invariant the inner product of $\sqrt{2E_{\vec{p}}}\,|\vec{p}\rangle$ with itself needs to be too.
So finally we learn that the relativistically normalized momentum states are given by :

$$|p\rangle = \sqrt{2E_{\vec{p}}}\,|\vec{p}\rangle$$

Note that we have used $|p\rangle$ for the relativistically state. The identity operator can be represented as :

$$1 = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_{\vec{p}}} |p\rangle \langle p|$$

3.2 Complex Scalar Fields

Let's consider a more interesting field with the Lagrangian :

$$\mathcal{L} = (\partial_\mu \psi^\star)(\partial^\mu \psi) - M^2 \psi^\star \psi$$

We can consider ψ and $\psi^\star$ as independent fields and apply euler-lagrange equations separately to get two equations of motion,

$$(\partial_\mu \partial^\mu + M^2)\psi = 0$$
$$(\partial_\mu \partial^\mu + M^2)\psi^\star = 0$$

Converting ψ and $\psi^\star$ to operators (ψ and $\psi^\dagger$) and expanding it as a sum of plane waves gives :

$$\psi = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(b_{\vec{p}}\, e^{+i\vec{p}\cdot\vec{x}} + c^\dagger_{\vec{p}}\, e^{-i\vec{p}\cdot\vec{x}} \right)$$
$$\psi^\dagger = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \left(b^\dagger_{\vec{p}}\, e^{+i\vec{p}\cdot\vec{x}} + c_{\vec{p}}\, e^{-i\vec{p}\cdot\vec{x}} \right)$$

Note that we get different operators $b_{\vec{p}}$ and $c^\dagger_{\vec{p}}$ unlike the case with Klein-Gordon field as we have no restriction for the field to be real.

So quantising a complex scalar field gives rise to two creation operators, $b^\dagger_{\vec{p}}$ and $c^\dagger_{\vec{p}}$. These have the interpretation of creating two types of particle, both of mass M and both spin zero. To interpret these two types of particles, first consider the following symmetry of the Lagrangian :

$$\psi \to e^{i\alpha}\psi \qquad \text{and} \qquad \psi^\star \to e^{-i\alpha}\psi$$

Working infinitesimally, we see that the change is $\psi \to (1 + i\alpha)\psi$ which gives us $\delta\psi = i\alpha\psi$ and $\delta\psi^\star = -i\alpha\psi^\star$. So the conserved current is :

$$j^\mu = \sum_{\text{all fields}} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} \delta\phi - \text{constant}$$

As the Lagrangian changes by zero the second term can be set to any arbitrary constant. Summing over all fields i.e., ψ and $\psi^\star$, and putting the constant to zero gives us the conserved current $j^\mu = i(\psi\partial^\mu\psi^\star - \psi^\star\partial^\mu\psi)$. The corresponding conserved charge is :

$$Q = i\int d^3x(\dot{\phi}^\star\phi - \dot{\phi}\phi^\star)$$

After normal ordering this becomes the operator :

$$N = \int \frac{d^3q}{(2\pi)^3}\,(c^\dagger_{\vec{q}}c_{\vec{q}} - b^\dagger_{\vec{q}}b_{\vec{q}}) = N_c - N_b$$

We have $[H, N] = 0$, ensuring that N is conserved quantity in the theory. In this theory N_c and N_b are also both conserved. But for a similar complex scalar *interacting* field theory these quantities are individually not conserved but N survives to do so.

This motivates the interpretation that out of the two type of particles in the theory, one is an **anti-particle** of the other. In contrast, for a real scalar field there is only a single type of particle. The particle is its own antiparticle.

4 The Heisenberg Picture

We treat space and time on different footings in the Schrodinger picture making lorentz invariance of our quantized field theory doubtful. To prove that it is not the case we need to build a different way of looking at quantum mechanics.
In the Heisenberg picture of quantum mechanics the states are a constant while the operators evolve in time to get the expectation values that we usually expect from the Schrodinger picture. For example, consider the time evolution of a one-particle state of the Klein-Gordon field in the Schrodinger picture :

$$i\frac{d\left|p(t)\right\rangle}{dt} = H\left|p(t)\right\rangle \qquad \Rightarrow \qquad \left|p(t)\right\rangle = e^{-iHt}\left|p(0)\right\rangle = e^{-iE_{\vec{p}}t}\left|p(0)\right\rangle$$

Let the subscript H and S represent the heisenberg and Schrodinger operators states respectively. Then $\left|p\right\rangle_H = e^{iE_{\vec{p}}t}\left|p(t)\right\rangle_S$ and $\mathcal{O}_H = e^{iE_{\vec{p}}t}\,\mathcal{O}_S\,e^{-iE_{\vec{p}}t}$. By doing this

$$\left|p\right\rangle_H = e^{iE_{\vec{p}}t}\left|p(t)\right\rangle_S = e^{iE_{\vec{p}}t}e^{-iE_{\vec{p}}t}\left|p(0)\right\rangle = \left|p(0)\right\rangle \quad \text{(a constant)}$$

and,

$$\left\langle p\right|_H \mathcal{O}_H \left|p\right\rangle_H = \left\langle p\right|_H e^{iE_{\vec{p}}t}\,\mathcal{O}_S\,e^{-iE_{\vec{p}}t}\left|p\right\rangle_H = \left\langle p\right|_S \mathcal{O}_S \left|p\right\rangle_S$$

Now instead of the states, the operators evolve with the rule :

$$\frac{d\mathcal{O}_H}{dt} = i[H, \mathcal{O}_H]$$

Our new field and canonical momentum now satisfy *equal time* commutation relations :

$$[\phi(\vec{x},t),\phi(\vec{y},t)] = [\pi(\vec{x},t),\pi(\vec{y},t)] = 0$$
$$[\phi(\vec{x},t),\pi(\vec{y},t)] = \delta^3(\vec{x}-\vec{y})$$

To get our final field and canonical momentum operators in the Heisenberg picture we will need to calculate quantities like $e^{iHt}a_{\vec{p}}\,e^{-iHt}$ and $e^{iHt}a^\dagger_{\vec{p}}\,e^{-iHt}$. For that first consider the following commutator :

$$[H,a_{\vec{p}}] = Ha_{\vec{p}} - a_{\vec{p}}H$$

Let this act on an arbitrary multi-particle state $|\psi\rangle$ with energy E_0 then :

$$\begin{aligned}[H,a_{\vec{p}}]\,|\psi\rangle &= Ha_{\vec{p}}\,|\psi\rangle - a_{\vec{p}}H\,|\psi\rangle \\ &= (E-E_{\vec{p}})a_{\vec{p}}\,|\psi\rangle - Ea_{\vec{p}}\,|\psi\rangle \\ &= -E_{\vec{p}}\,a_{\vec{p}}\,|\psi\rangle\end{aligned}$$

So $Ha_{\vec{p}} = a_{\vec{p}}\,(H-E_{\vec{p}})$ and it can easily be seen that $H^n a_{\vec{p}} = a_{\vec{p}}\,(H-E_{\vec{p}})^n$. This also implies that $e^{iHt}a_{\vec{p}} = a_{\vec{p}}\,e^{i(H-E_{\vec{p}})t}$ and $e^{iHt}a^\dagger_{\vec{p}} = a^\dagger_{\vec{p}}\,e^{i(H+E_{\vec{p}})t}$ if a similar analysis was done for $a^\dagger_{\vec{p}}$. So,

$$\begin{aligned}e^{iHt}a_{\vec{p}}\,e^{-iHt} &= a_{\vec{p}}\,e^{i(H-E_{\vec{p}})t}\,e^{-iHt} = a_{\vec{p}}\,e^{-iE_{\vec{p}}t} \\ e^{iHt}a^\dagger_{\vec{p}}\,e^{-iHt} &= a^\dagger_{\vec{p}}\,e^{i(H+E_{\vec{p}})t}\,e^{-iHt} = a^\dagger_{\vec{p}}\,e^{iE_{\vec{p}}t}\end{aligned}$$

This then gives the field operators in the Heisenberg picture,

$$\phi(\vec{x},t) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_{\vec{p}}}}\left(a_{\vec{p}}\,e^{-ip\cdot x} + a^\dagger_{\vec{p}}\,e^{-ip\cdot x}\right)$$

where $p\cdot x = E_{\vec{p}}t - \vec{p}\cdot\vec{x}$. It is easy to verify that this satisfies the Klein-Gordon equation reassuring us of the lorentz invariance of the theory.

5 Interacting Fields

The fields we have studied so far fall under the category of free fields. We found the spectrum of energies in these theories but other than that nothing interesting happens. Interacting fields on the other hand have particles that interact with one another. These interactions may create, destroy or exchange particles. The interaction are brought up by higher order terms in the Lagrangian. For example take the ϕ^4 theory, the Lagrangian describing this theory is :

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi) - \frac{1}{2}m^2\phi^2 + \frac{\lambda}{4!}\phi^4$$

In later sections we will focus on the Scalar Yukawa Theory which couples a complex scalar ψ to a real scalar ϕ (through the λ term). The Lagranian for this theory is :

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) + (\partial_\mu \psi^\star)(\partial^\mu \psi) - \frac{1}{2}m^2\phi^2 - M^2\psi^\star\psi - \lambda\psi^\star\psi\phi$$

5.1 The Interaction Picture

Interaction picture is a useful way to describe situations where we have small perturbations (e.g. interaction terms) to a well-understood Hamiltonian.
In the interaction picture we split up the Hamiltonian as :

$$H = H_0 + H_{\text{int}}$$

Although the split can be arbitrary it is useful when H_0 is solvable (e.g. Hamiltonian of free field theory). The states and operators in the interaction picture will be denoted by a subscript I and are defined by :

$$\begin{aligned} |\psi(t)\rangle_I &= e^{iH_0 t}\,|\psi(t)\rangle_S = e^{iH_{\text{int}}t}\,|\psi(0)\rangle \\ \mathcal{O}_I(t) &= e^{iH_0 t}\mathcal{O}_S e^{-iH_0 t} = e^{iH_0 t}\mathcal{O}(0)e^{-iH_0 t} \end{aligned}$$

The time dependence of operators is governed by H_0, while the time dependence of states is governed by H_{int}. We will denote the interaction Hamiltonian in the interaction picture as H_I such that $H_I = (H_{\text{int}})_I = e^{iH_0 t}H_{\text{int}}e^{-iH_0 t}$. The time evolution of states in the interaction picture can be derived starting from Schrodinger equation :

$$\begin{aligned} i\frac{d\,|\psi\rangle_S}{dt} &= H\,|\psi\rangle_S \\ i\frac{d}{dt}(e^{-iH_0 t}\,|\psi\rangle_I) &= (H_0 + H_{\text{int}})e^{-iH_0 t}\,|\psi\rangle_I \\ ie^{-iH_0 t}\frac{d\,|\psi\rangle_I}{dt} + H_0 e^{-iH_0 t}\,|\psi\rangle_I &= (H_0 + H_{\text{int}})e^{-iH_0 t}\,|\psi\rangle_I \\ i\frac{d\,|\psi\rangle_I}{dt} &= e^{iH_0 t}H_{\text{int}}e^{-iH_0 t}\,|\psi\rangle_I \end{aligned}$$

So we get that,

$$i\,\frac{d\,|\psi\rangle_I}{dt} = H_I(t)\,|\psi\rangle_I$$

Dyson Series

H_I is explicitly time dependent. So the time evolution operator of states $|\psi\rangle_I$ will not be of the usual form e^{-iHt}. To solve for the time evolution operator write the solution to the above equation as :

$$|\psi(t)\rangle_I = U(t,t_0)\,|\psi(0)\rangle_I$$

$U(t,t_0)$ is a unitary operator so that normalized states stay normalized through time evolution. It also has the usual properties that we would expect from such an operator like $U(t_1,t_3) = U(t_1,t_2)U(t_2,t_3)$ and $U(t,t) = 1$. But most importantly it satisfies :

$$i\,\frac{dU}{dt} = H_I(t)U$$

A natural guess could be :

$$U(t,t_0) \overset{?}{=} \exp\bigg(-i\int_{t_0}^{t} H_I(t^{'})dt^{'}\bigg)$$

To see how this operator is not our solution, expand out the exponential

$$\exp\bigg(-i\int_{t_0}^{t} H_I(t^{'})dt^{'}\bigg) = 1-i\int_{t_0}^{t} H_I(t^{'})dt^{'}+\frac{(-i)^2}{2}\int_{t_0}^{t} dt^{''}\int_{t_0}^{t} dt^{'}H_I(t^{'})H_I(t^{''})+\,...$$

Differentiating the quadratic term with respect to time gives us

$$-\frac{1}{2}H_I(t)\bigg(\int_{t_0}^{t} H_I(t^{''})dt^{''}\bigg) - \frac{1}{2}\bigg(\int_{t_0}^{t} H_I(t^{'})dt^{'}\bigg)H_I(t)$$

In the second term of the above expression $H_I(t)$ is on the right side, which is not what we want as $i\dot{U} = H_I(t)U$. So this can't be the solution.

The solution is given by Dyson's Series or Formula.

$$U(t,t_0) = T\exp\bigg(-i\int_{t_0}^{t} H_I(t^{'})dt^{'}\bigg)$$

The T in front of the exponential implies time ordering : operators evaluated at later times must be placed to the left. This implies that :

$$\begin{aligned} T\exp\bigg(-i\int_{t_0}^{t} H_I(t^{'})dt^{'}\bigg) = 1-i\int_{t_0}^{t} H_I(t^{'})dt^{'}+\frac{(-i)^2}{2}\bigg(\int_{t_0}^{t} dt^{''}\int_{t_0}^{t^{''}} dt^{'}H_I(t^{''})H_I(t^{'}) \\ +\int_{t_0}^{t} dt^{''}\int_{t^{''}}^{t} dt^{'}H_I(t^{'})H_I(t^{''})\bigg)+\,... \end{aligned}$$

Actually the last two terms are equal as :

$$\begin{aligned} \int_{t_0}^{t} dt^{''}\int_{t_0}^{t^{''}} dt^{'}H_I(t^{''})H_I(t^{'}) &= \int_{t_0}^{t} dt^{'}\int_{t^{'}}^{t} dt^{''}H_I(t^{''})H_I(t^{'}) \\ &= \int_{t_0}^{t} dt^{''}\int_{t^{''}}^{t} dt^{'}H_I(t^{'})H_I(t^{''}) \end{aligned}$$

We go from the first expression to the second just by changing the order of integration and from second to the third by relabelling $t^{'}$ to $t^{''}$ and $t^{''}$ to $t^{'}$. So finally we have our time evolution operator :

$$
\begin{aligned}
U(t,t_0) &= T\exp\Big(-i\int_{t_0}^{t} H_I(t^{'})dt^{'}\Big) \\
&= 1 - i\int_{t_0}^{t} H_I(t^{'})dt^{'} + (-i)^2 \int_{t_0}^{t} dt^{''}\int_{t_0}^{t^{''}} dt^{'} H_I(t^{''})H_I(t^{'}) + \ ...
\end{aligned}
$$

It is easy to verify that this is the correct solution. Only the left-most integral has t dependence.

5.2 Scattering

Lets apply what we have learned so far in calculating scattering amplitudes in the scalar Yukawa theory. The interaction Hamiltonian is given by :

$$H_{\text{int}} = \lambda \int d^3x\, \psi^{\dagger}\psi\phi$$

To calculate the scattering amplitudes we will consider only initial and final states that are eigenstates of the free theory and assume that they are isolated from each other at $t = \pm\infty$. We take the initial state $|i\rangle$ at $t \to -\infty$ and final state $|f\rangle$ at $t \to +\infty$. As the particles approach each other, they interact briefly, before departing again. So the amplitude to go from $|i\rangle$ to $|f\rangle$ is :

$$\lim_{t_{\pm}\to\pm\infty} \langle f|\, U(t_-, t_+)\, |i\rangle = \langle f|\, S\, |i\rangle$$

Using S in place of $U(-\infty, +\infty)$ is a common convention. It is known as the S-matrix.

Meson Decay

Take the relativistically normalized initial and final states,

$$
\begin{aligned}
|i\rangle &= \sqrt{2E_{\vec{p}}}a^{\dagger}_{\vec{p}}\,|0\rangle \\
|f\rangle &= \sqrt{4E_{\vec{q_1}}E_{\vec{q_2}}}b^{\dagger}_{\vec{q_1}}c^{\dagger}_{\vec{q_2}}\,|0\rangle
\end{aligned}
$$

Lets call the particle of the real scalar field a *meson* and the particles of the complex scalar field *nucleon* and *anti-nucleon*. Then the initial state contains a single meson of momentum $\vec{p}$; the final state contains a nucleon-anti-nucleon pair of momentum $\vec{q_1}$ and $\vec{q_2}$. To first degree in λ, amplitude for the decay of a meson to a nucleon-anti-nucleon pair is :

$$-i\lambda\,\langle f| \int d^4x\, \psi^{\dagger}(x)\psi(x)\phi(x)\,|i\rangle$$

Note that this expression does not contain zeroth order terms as the states $|i\rangle$ and $|f\rangle$ have no overlap. Even if there was an overlap, we really want to calculate $\langle f| S - 1 |i\rangle$ since we're not interested in situations where no scattering occurs.

Expanding out $\phi(x)$ as $a + a^\dagger$ and acting it on $|i\rangle$ gives one part proportional to $|0\rangle$ and an other two meson state. Only the first part can have a non-zero overlap with $\langle f|$. So, to first order :

$$\begin{aligned}
\langle f| S |i\rangle &= -i\lambda \langle f| \int d^4x\, \psi^\dagger(x)\psi(x) \int \frac{d^3k}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{k}}}} a_{\vec{k}} e^{-ik\cdot x} |i\rangle \\
&= -i\lambda \langle f| \int d^4x\, \psi^\dagger(x)\psi(x) \int \frac{d^3k}{(2\pi)^3} \frac{\sqrt{2E_{\vec{p}}}}{\sqrt{2E_{\vec{k}}}} e^{-ik\cdot x} a_{\vec{k}} a^\dagger_{\vec{p}} |0\rangle \\
&= -i\lambda \langle f| \int d^4x\, \psi^\dagger(x)\psi(x) \int \frac{d^3k}{(2\pi)^3} \frac{\sqrt{2E_{\vec{p}}}}{\sqrt{2E_{\vec{k}}}} e^{-ik\cdot x} [a_{\vec{k}}, a^\dagger_{\vec{p}}] |0\rangle \\
&= -i\lambda \langle f| \int d^4x\, \psi^\dagger(x)\psi(x) e^{-ip\cdot x} |0\rangle
\end{aligned}$$

Similarly, now we expand out $\psi = b + c^\dagger$ and $\psi^\dagger = b^\dagger + c$. Terms with b on the right kill the vacuum and $cc^\dagger$ recreates the vacuum which has no overlap with $\langle f|$. So only the $b^\dagger c^\dagger$ contributes.

$$\begin{aligned}
\langle f| S |i\rangle &= -i\lambda \langle 0| \int \frac{d^4x\, d^3k_1\, d^3k_2}{(2\pi)^6} \frac{\sqrt{4E_{\vec{q_1}} E_{\vec{q_2}}}}{\sqrt{4E_{\vec{k_1}} E_{\vec{k_2}}}} c_{\vec{q_2}}\, b_{\vec{q_1}}\, c^\dagger_{\vec{k_1}}\, b^\dagger_{\vec{k_2}}\, e^{i(k_1+k_2-p)\cdot x} |0\rangle \\
&= -i\lambda (2\pi)^4 \delta^{(4)}(q_1 + q_2 - p)
\end{aligned}$$

So we finally have the scattering amplitude (to first order).

5.3 Wick's Theorem

To calculate the scattering amplitude to higher orders we will need to compute quantities like $\langle f| T\{H_I(x_1)H_I(x_2)\ldots H_I(x_n)\} |i\rangle$. To simplify our calculation we would like to move annihilation operators towards the right and creation operators to the left. This process is same as time ordering and Wick's theorem tells us how to go from time ordered products to normal ordered products.

Feynman Propagator

Decompose the real scalar field operator as :

$$\phi(x) = \phi^+(x) + \phi^-(x)$$

where,

$$\phi^+(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \, a_{\vec{p}} \, e^{-ip\cdot x}$$

$$\phi^-(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_{\vec{p}}}} \, a^\dagger_{\vec{p}} \, e^{-ip\cdot x}$$

Then,

$$\begin{aligned}
T\phi(x)\phi(y) &= \phi(x)\phi(y) \qquad (\text{for } x^0 > y^0) \\
&= (\phi^+(x) + \phi^-(x))(\phi^+(y) + \phi^-(y)) \\
&= \phi^+(x)\phi^+(y) + \phi^+(x)\phi^-(y) + \phi^-(x)\phi^+(y) + \phi^-(x)\phi^-(y) \\
&= \phi^+(x)\phi^+(y) + \phi^-(y)\phi^+(x) + \phi^-(x)\phi^+(y) + \phi^-(x)\phi^-(y) + [\phi^+(x), \phi^-(y)] \\
&= \,:\phi(x)\phi(y):\, + [\phi^+(x), \phi^-(y)]
\end{aligned}$$

Similarly for $x^0 < y^0$ we find that $T\phi(x)\phi(y) = \,:\phi(x)\phi(y):\, + [\phi^+(y), \phi^-(x)]$

So putting this together, we have the final expression :

$$T\phi(x)\phi(y) = \,:\phi(x)\phi(y):\, + \Delta_F(x-y)$$

where,

$$\Delta_F(x-y) = \begin{cases} [\phi^+(x), \phi^-(y)] & x^0 > y^0 \\ [\phi^+(y), \phi^-(x)] & x^0 < y^0 \end{cases}$$

$\Delta_F(x-y)$, also known as the Feynman Propagator, has a useful integral representation. The proof involves complex analysis so we won't go there.

$$\Delta_F(x-y) = \int \frac{d^4k}{(2\pi)^4} \frac{ie^{ik\cdot(x-y)}}{k^2 - m^2 + i\epsilon}$$

So the difference between time and normal ordered products, here, is just an operator proportional to identity (a c-number).

Contraction

We define the contraction of a pair of fields in a string of operators $\ldots\phi(x_1)\ldots\phi(x_2)\ldots$ to mean replacing those operators with the Feynman propagator, leaving all other operators untouched. We use the notation,

$$\ldots\overline{\phi(x_n)\ldots\phi(x_m)}\ldots$$

to denote contraction. The operators at the ends of the over-line are replaced with the Feynman propagator.

$$\overline{\phi(x)\phi(y)} = \Delta_F(x-y)$$

A similar discussion for complex scalar field would yield :

$$\overline{\psi(x)\psi^\dagger(y)} = \Delta_F(x-y) \quad \text{and} \quad \overline{\psi(x)\psi(y)} = \overline{\psi^\dagger(x)\psi^\dagger(y)} = 0$$

Wick's Theorem

According to wick's theorem :

$$T(\phi_1\,\phi_2\,...\,\phi_n) = :\phi_1\,\phi_2\,...\,\phi_n: + :\text{all possible contractions}:$$

The proof to this is easier than writing it down. The proof proceeds by induction on n. Suppose it's true for $\phi_2...\phi_n$ and now add ϕ_1. Take $x_1^0 \geq x_k^0 \;\; \forall k$, then,

$$T(\phi_1\,\phi_2\,...\,\phi_n) = (\phi_1^+ + \phi_1^-)(:\phi_2\,...\,\phi_n: + :\text{all contractions without } \phi_1:)$$

The ϕ_1^- term stays where it is since it is already normal ordered. And we pick up a factor of $\overline{\phi_1\phi_k} = \Delta_F(x_1 - x_k)$ from the commutator while moving ϕ_1^+ to the right side.

For example :

$$T(\phi_1\,\phi_2\,\phi_3\,\phi_4) = :\phi_1\,\phi_2\,\phi_3\,\phi_4: + \overline{\phi_1\phi_2}:\phi_3\phi_4: + \overline{\phi_1\phi_3}:\phi_2\phi_4: + \text{4 other terms}$$
$$+ \overline{\phi_1\phi_2}\;\overline{\phi_3\phi_4} + \overline{\phi_1\phi_3}\;\overline{\phi_2\phi_4} + \overline{\phi_1\phi_4}\;\overline{\phi_2\phi_3}$$

5.4 Feynman Diagrams

Nucleon Scattering

Let's look at $\psi\psi \to \psi\psi$ scattering. We have

$$|i\rangle = \sqrt{4E_{\vec{p_1}}E_{\vec{p_2}}}\, b^\dagger_{\vec{p_1}}\, b^\dagger_{\vec{p_2}}\,|0\rangle \equiv |\vec{p_1}, \vec{p_2}\rangle$$
$$|f\rangle = \sqrt{4E_{\vec{p'_1}}E_{\vec{p'_2}}}\, b^\dagger_{\vec{p'_1}}\, b^\dagger_{\vec{p'_2}}\,|0\rangle \equiv |\vec{p'_1}, \vec{p'_2}\rangle$$

Scattering amplitude, $\langle f|\,S-1\,|i\rangle$, to order λ^2 is :

$$\frac{(-i\lambda)^2}{2}\,\langle f|\int d^4x_1\,d^4x_2\,T\Big(\psi^\dagger(x_1)\psi(x_1)\phi(x_1)\psi^\dagger(x_2)\psi(x_2)\phi(x_2)\Big)\,|i\rangle$$

Now, using Wick's theorem we see there is a piece in the string of operators which looks like,

$$:\psi^\dagger(x_1)\psi(x_1)\psi^\dagger(x_2)\psi(x_2):\overline{\phi(x_1)\phi(x_2)}$$

Any other term will give zero contribution. But even after this simplification there is a long way to the final expression, which comes out to be :

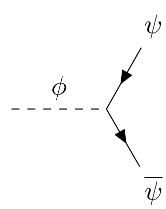
$$i(-i\lambda)^2\left[\frac{1}{(p_1-p'_1)^2 - m^2 + i\epsilon} + \frac{1}{(p_2-p'_2)^2 - m^2 + i\epsilon}\right](2\pi)^4\delta^{(4)}(p_1+p_2-p'_1-p'_2)$$

As the above example demonstrates, to actually compute scattering amplitudes using Wick's theorem is rather tedious. Feynman diagrams present a much easier way to compute these quantities and also provide us with a nice interpretation.

The rules for drawing these diagrams are :

(1) Draw an external line for each particle in the initial state $|i\rangle$ and each particle in the final state $|f\rangle$. We'll choose dotted lines for mesons, and solid lines for nucleons. Assign a directed momentum p to each line. Further, add an arrow to solid lines to denote its charge; we'll choose an incoming (outgoing) arrow in the initial state for nucleon $-\ \psi$ (anti-nucleon $-\ \overline{\psi}$). We choose the reverse convention for the final state, where an outgoing arrow denotes ψ.

(2) Join the external lines together with trivalent vertices like the one drawn below.



Each such diagram you can draw is in 1-1 correspondence with the terms in the expansion of $\langle f|\, S - 1\, |i\rangle$. To calculate these terms from the diagrams lets look at the set of Feynman rules for doing so.

(1) Add a momentum k to each internal line.

(2) To each vertex, write down a factor of

$$(-i\lambda)(2\pi)^4\delta^{(4)}(\sum_i k_i)$$

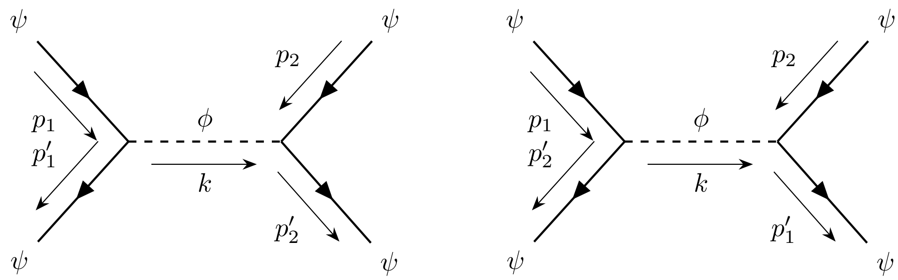
where $\sum_i k_i$ is the sum of all momenta flowing into the vertex.

(3) For each internal line, with momentum k, we write down a factor of

$$\int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2 - m^2 + i\epsilon}$$

For a dotted internal line we take m to be mass of a meson (m) and for a solid internal line we take m to be mass of a nucleon/anti-nucleon (M).

Let's look at how Feynman diagrams work for the $\psi\psi \to \psi\psi$ scattering at order λ^2. We can write down the two simplest diagrams contributing to this process.



Applying the Feynman rules to these diagrams, we get

$$i(-i\lambda)^2\left[\frac{1}{(p_1-p_1')^2-m^2+i\epsilon}+\frac{1}{(p_2-p_2')^2-m^2+i\epsilon}\right](2\pi)^4\delta^{(4)}(p_1+p_2-p_1'-p_2')$$

which agrees with the calculation that we performed earlier.

Loosely speaking, the interpretation for this diagram is that the nucleons exchange a meson during the interaction.
To calculate higher order terms in the scattering amplitude we just have to make elaborate Feynman diagrams with more nodes and internal lines. The scattering amplitudes for other processes like $\psi\overline{\psi} \to \psi\overline{\psi}$, $\psi\phi \to \psi\phi$ etc. can also be just as easily calculated.

6 P.S.

Unfortunately due to lack of time I couldn't write about interactions in ϕ^4 theory or study about quantization of the Dirac Field.

The notes I referred the most and couldn't recommend more are:
`http://www.damtp.cam.ac.uk/user/tong/qft.html`
M. Peskin and D. Schroeder's An *Introduction to Quantum Field Theory* was also very useful at times.