CS 480

Introduction to Artificial Intelligence

March 07, 2024

Announcements / Reminders

- Please follow the Week 09 To Do List instructions (if you haven't already):
- Next week: Spring Break! No office hours.
- Programming Assignment #02: posted
- Written Assignment #04: posted

Plan for Today

- **Inference in Bayes Networks**
- **Decision Networks**

Joint Probability: Marginalization

H: grad	e: female	P(H, e) = P(H ∧ e): P(grad ∧ female)
true	true	0.074
true	false	0.148
false	true	0.086
false	false	0.691
		SUM = 1

Probability P(e)**:**

$$P(e) = P(female = true) = 0.074 + 0.086 \approx 13 / 81$$

Probability P(e)**: "sum of all probabilities where** e **true"**

Joint Probability: Conditionals

H: grad	e: female	P(H, e) = P(H ∧ e): P(grad ∧ female)
true	true	**0.074**
true	false	**0.148**
false	true	**0.086**
false	false	**0.691**
		SUM = 1

From product rule:

$$P(H \wedge e) = P(H \mid e) * P(e)$$

we can derive:

$$P(H \mid e) = \frac{P(H \wedge e)}{P(e)}$$

Joint Probability: Conditionals

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true	true	0.074
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From product rule:

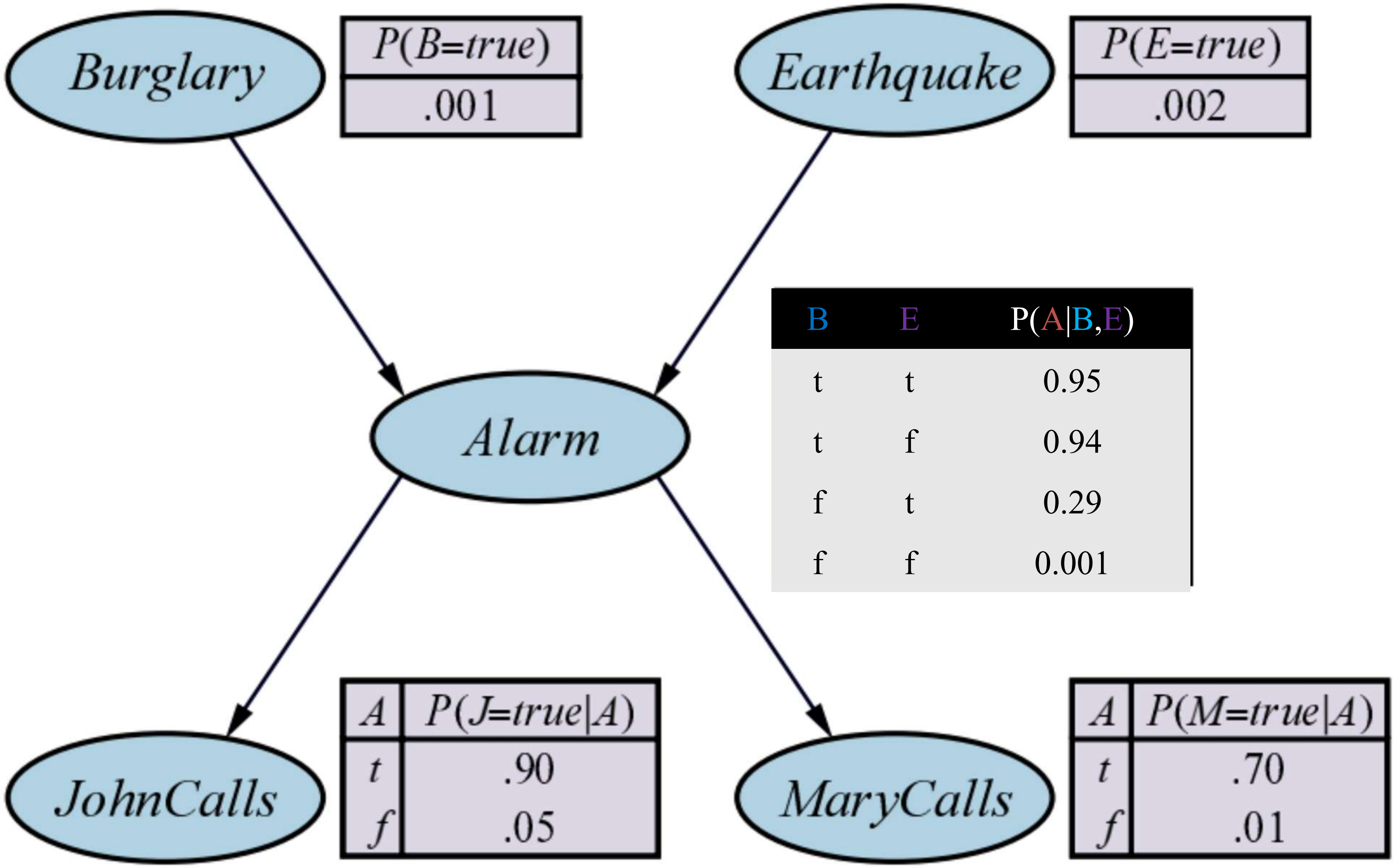
$$P(H \land e) = P(H \mid e) * P(e)$$

we can derive:

$$P(H \mid e) = \frac{P(H \land e)}{P(e)} = \frac{0.074}{0.074 + 0.086} \approx 0.462$$

Inference in Bayes Networks

Inference In Bayes Networks



B	E	P(A|B,E)
t	t	0.95
t	f	0.94
f	t	0.29
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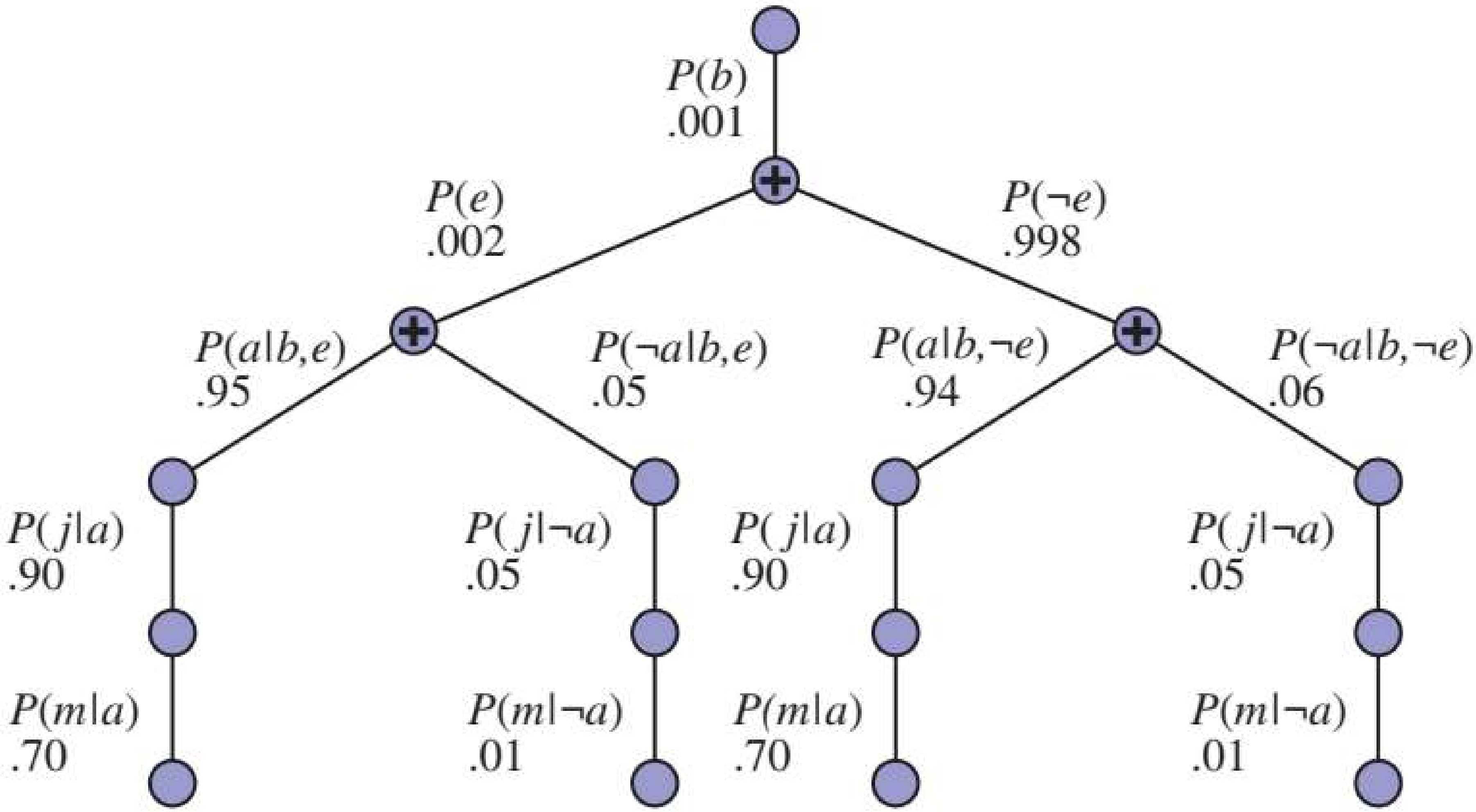
Inference by Enumeration: Example

Query (what is the probability distribution for the following conditional P()?):

$$\boldsymbol{P}(Burglary \mid JohnCalls = true \wedge MaryCalls = true)$$



Joint Probability: Marginalization

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Probability P(H)**:**

$$P(H) = P(grad = true) = 0.074 + 0.148 \approx 18 / 81$$

Probability P(H)**: "sum of all probabilities where** H **true"**

Joint Probability: Marginalization

H: grad	e: female	P(H, e) = P(H ∧ e): P(grad ∧ female)
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Probability P(e)**:**

$$P(e) = P(female = true) = 0.074 + 0.086 \approx 13\,/\,81$$

Probability P(e)**: "sum of all probabilities where** e **true"**

Joint Probability: Conditionals

H: grad	e: female	P(H, e) = P(H ∧ e): P(grad ∧ female)
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From product rule:

$$P(H \wedge e) = P(H \mid e) * P(e)$$

we can derive:

$$P(H \mid e) = \frac{P(H \wedge e)}{P(e)} = \frac{0.074}{0.074 + 0.086} \approx 0.462$$

General Inference Procedure

Given:

- **a query involving a single variable X (in our example: Cavity),**
- **a <u>list</u> of evidence variables $\mathbf{E}$ (in our example: just Toothache),**
- **a <u>list</u> of observed values e for $\mathbf{E}$,**
- **a list of remaining unobserved variables $\mathbf{Y}$ (in our example: just Catch),**

where X, $\mathbf{E}$, and $\mathbf{Y}$ together are a COMPLETE set of variables for the domain, the probability $P(X \mid \mathbf{E})$ can be evaluated as:

$$\boldsymbol{P}(X \mid \boldsymbol{e}) = \alpha * \boldsymbol{P}(X, \boldsymbol{e}) = \alpha * \sum_{\boldsymbol{y}} \boldsymbol{P}(X, \boldsymbol{e}, \boldsymbol{y})$$

where ys are all possible values for $\mathbf{Y}$s, α - normalization constant.

$\mathbf{P}(X, e, y)$ is a subset of probabilities from the joint distribution

Inference

Query:

$$P(Burglary \mid JohnCalls = true \quad MaryCalls = true)$$

Given:

- **a query involving a single variable** X
- **a list of evidence variables** K,
- **a list of observed values** k **for** K,
- **a list of remaining unobserved variables** Y

the probability $P(X \mid K)$ **can be evaluated as:**

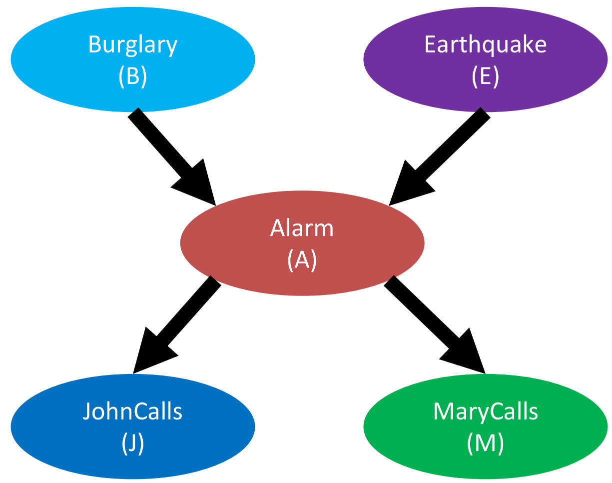
$$P(X \mid k) = \alpha * P(X, k)$$

$$= \alpha * \sum_{y} P(X, k, y)$$

where ys are all possible values for Y**s,** α **- normalization constant.**

P(B)	P(¬B)
0.001	0.999

P(E)	P(¬E)
0.002	0.998



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A	P(J|A)
t	0.90
f	0.05

A	P(M|A)
t	0.70
f	0.01

Inference

Query:

$$P(Burglary \mid JohnCalls = true \wedge MaryCalls = true)$$

Given:

- **a query involving a single variable** X**:** $Burglary$
- **a list of evidence variables** $\mathbf{K}$**:** $JohnCalls$, $MaryCalls$
- **a list of observed values** k **for** $\mathbf{K}$**:** $johnCalls$, $maryCalls$
- **a list of remaining unobserved variables** $\mathbf{Y}$**:** $Earthquake$, $Alarm$

the probability $\mathrm{P}(\mathrm{X} \mid \mathbf{K})$ **can be evaluated as:**

$$\boldsymbol{P}(X \mid \boldsymbol{k}) = \alpha * \sum_{\boldsymbol{y}} \boldsymbol{P}(X, \boldsymbol{k}, \boldsymbol{y})$$

where ys **are all possible values for** Ys**,** α **- normalization constant.**

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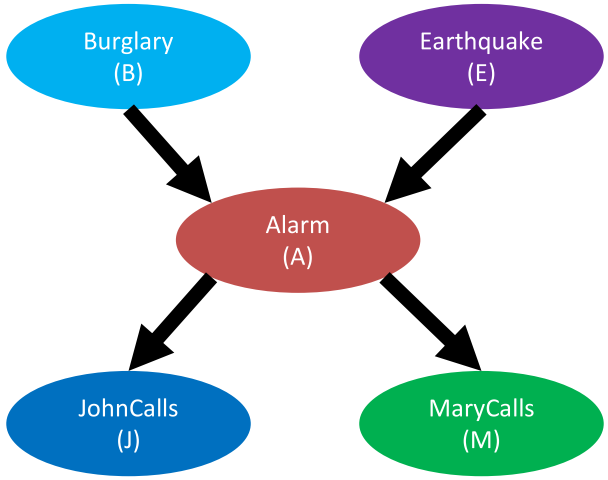

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Inference

Query:

$$P(Burglary \mid JohnCalls = true \wedge MaryCalls = true)$$

Given:

- a query involving a single variable X: B
- a <u>list</u> of **evidence** variables $\mathbf{K}$: J, M
- a <u>list</u> of **observed** values $\mathbf{k}$ for $\mathbf{K}$: j, m
- a list of remaining **unobserved** variables $\mathbf{Y}$: E, A

the probability $\mathrm{P(X \mid \mathbf{K})}$ can be evaluated as:

$$\boldsymbol{P}(X \mid \boldsymbol{k}) = \alpha * \sum_{\boldsymbol{y}} \boldsymbol{P}(X, \boldsymbol{k}, \boldsymbol{y})$$

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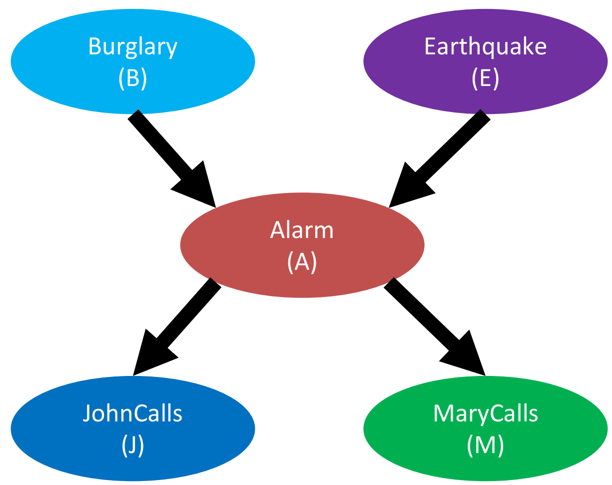

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- **a list of evidence variables** $\mathbf{K}$: J, M
- **a list of observed values** $\mathbf{k}$ **for** $\mathbf{K}$: j, m
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the probability $P(B \mid J, M)$ **can be evaluated as:**

$$P(B \mid j, m) = \alpha * \sum_{e} \sum_{a} P(B, j, m, e, a)$$

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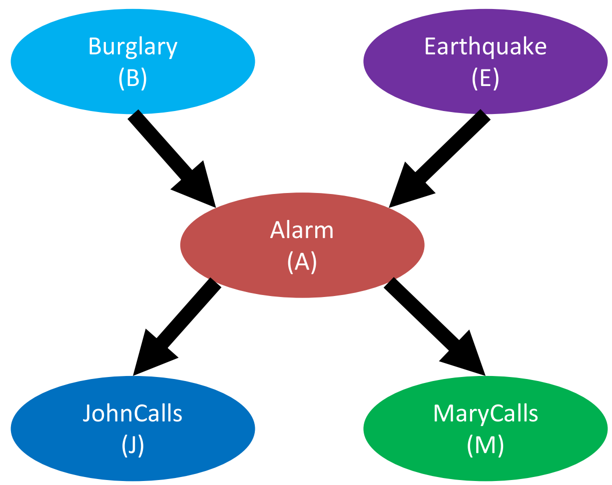

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Inference

Query (let's change it a bit for simplicity):

$P(Burglary = true \mid JohnCalls = true \wedge MaryCalls = true)$

Given:

- a query involving a single variable B
- a <u>list</u> of evidence variables $\mathbf{K}$: J, M
- a <u>list</u> of observed values $\mathbf{k}$ for $\mathbf{K}$: j, m
- a list of remaining unobserved variables $\mathbf{Y}$: E, A

the query can be evaluated as:

$$P(b \mid j, m) = \alpha * \sum_{e} \sum_{a} P(b, j, m, e, a)$$

By Chain rule:

$$P(b, j, m, e\ a)$$
$$= P(b) * P(e) * P(a|b, e) * P(j|a) * P(m|a)$$

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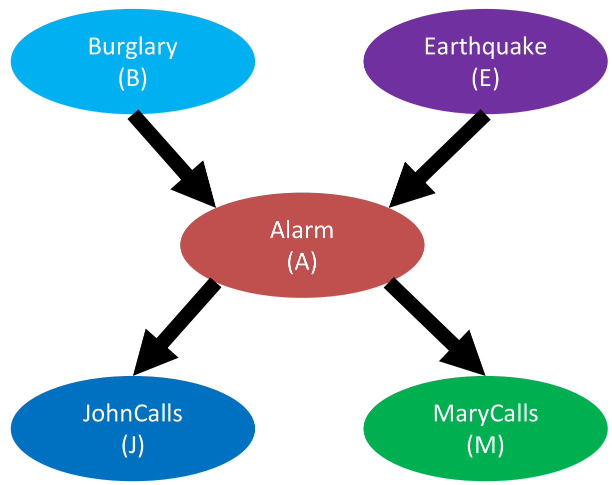

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the query can be evaluated as:

$P(b \mid j, m)$

$$= \alpha * \sum_e \sum_a P(b) * P(e) * P(a|b,e) * P(j|a) * P(m|a)$$

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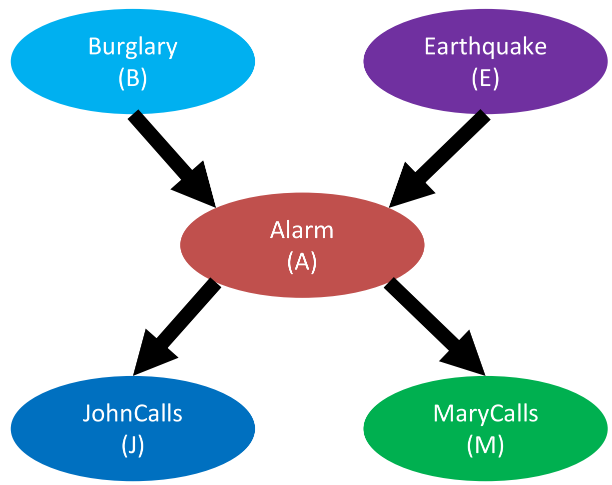

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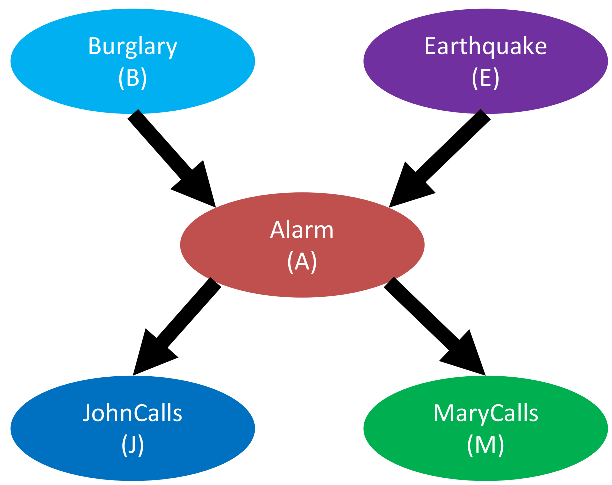

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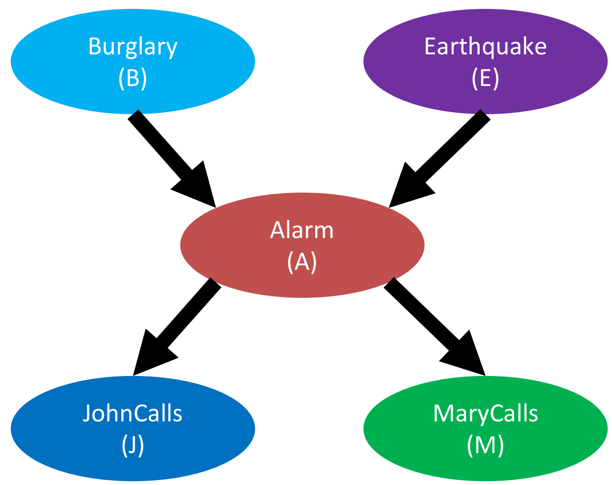

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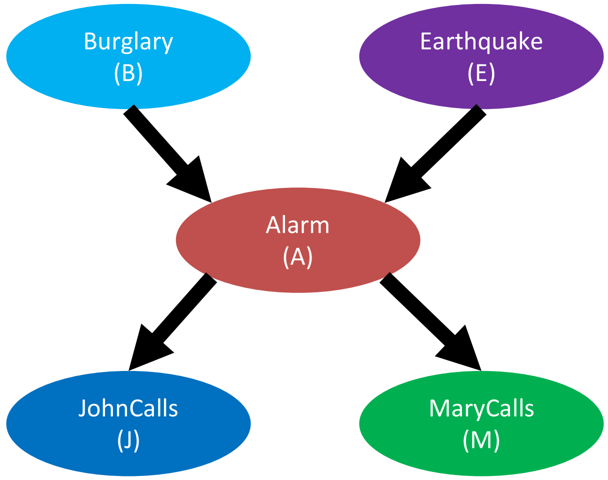

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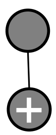
$P(b)$



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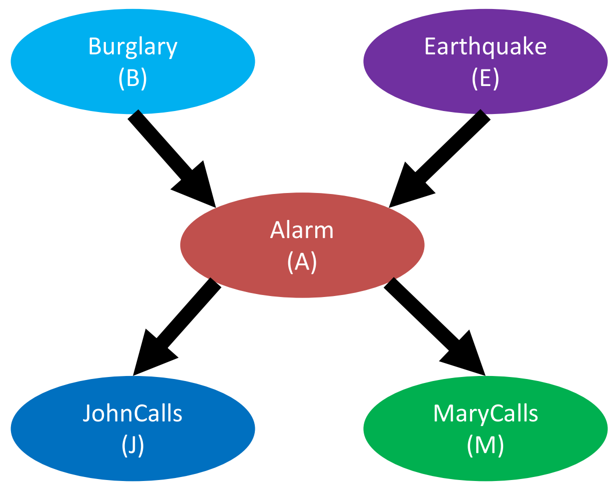

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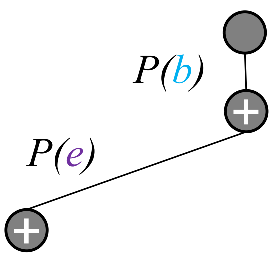

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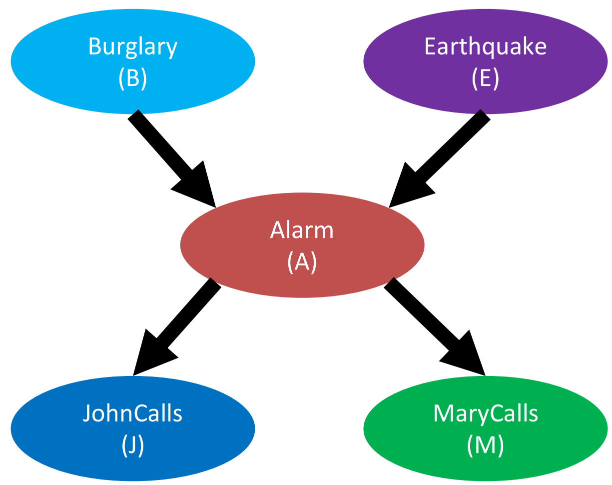

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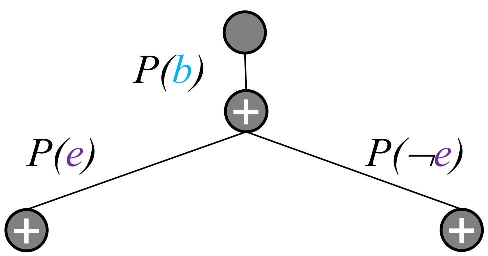

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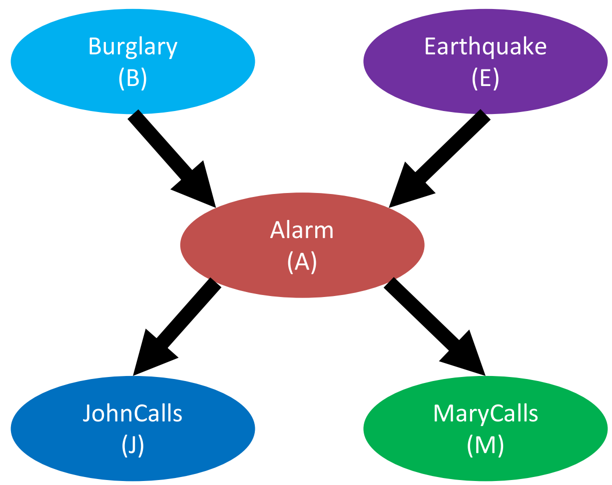

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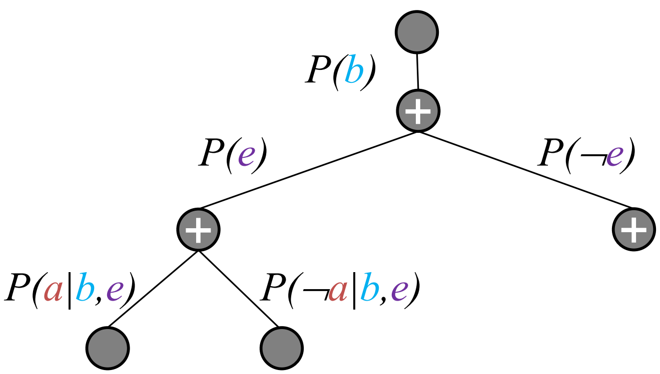

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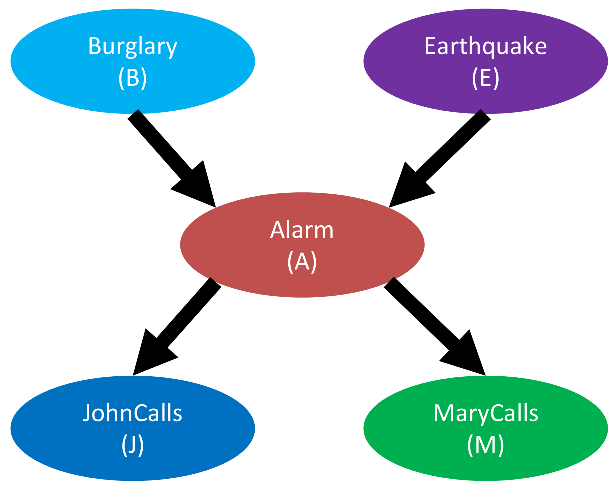

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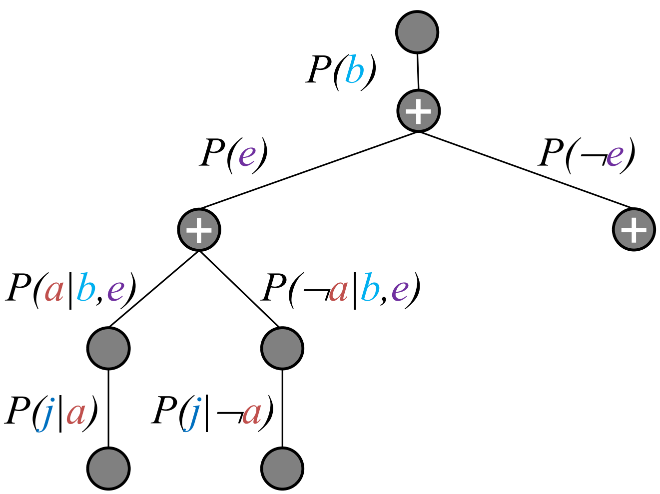

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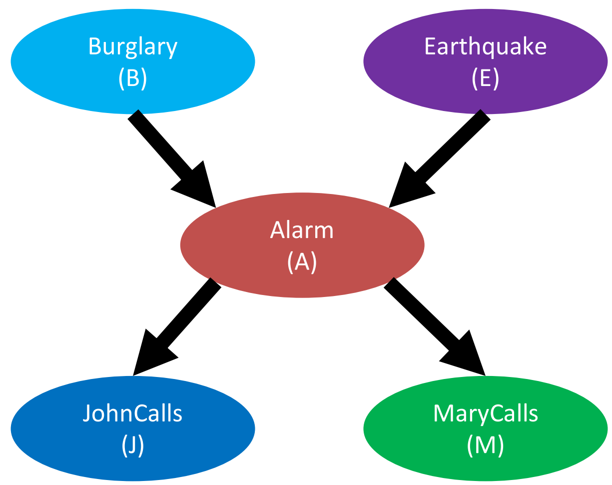

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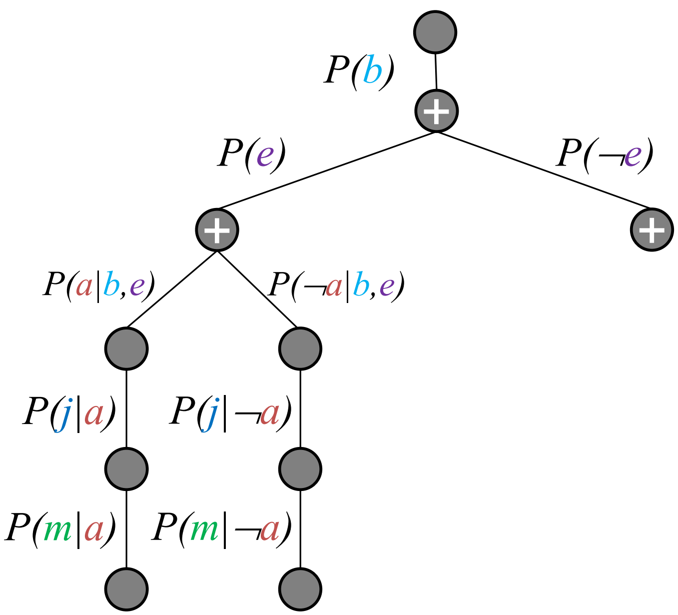
Query rewritten:

$P(b \mid j, m)$

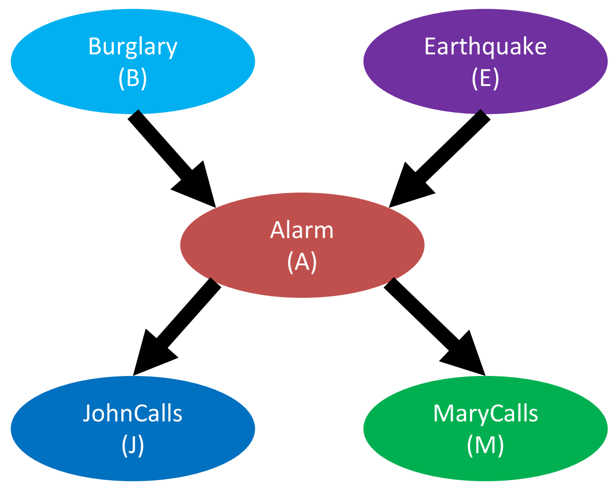
$= \alpha * \sum_{e} \sum_{a} P(b) * P(e) * P(a|b,e) * P(j|a) * P(m|a)$

$= \alpha * P(b) * \sum_{e} P(e) * \boxed{\sum_{a} P(a|b,e) * P(j|a) * \boxed{P(m|a)}}$



P(B)	P(¬B)
0.001	0.999

P(E)	P(¬E)
0.002	0.998



B	E	P(A|B,E)
t	t	0.95
t	f	0.94
f	t	0.29
f	f	0.001

A	P(J|A)
t	0.90
f	0.05

A	P(M|A)
t	0.70
f	0.01

Inference

Query (let's change it a bit for simplicity):

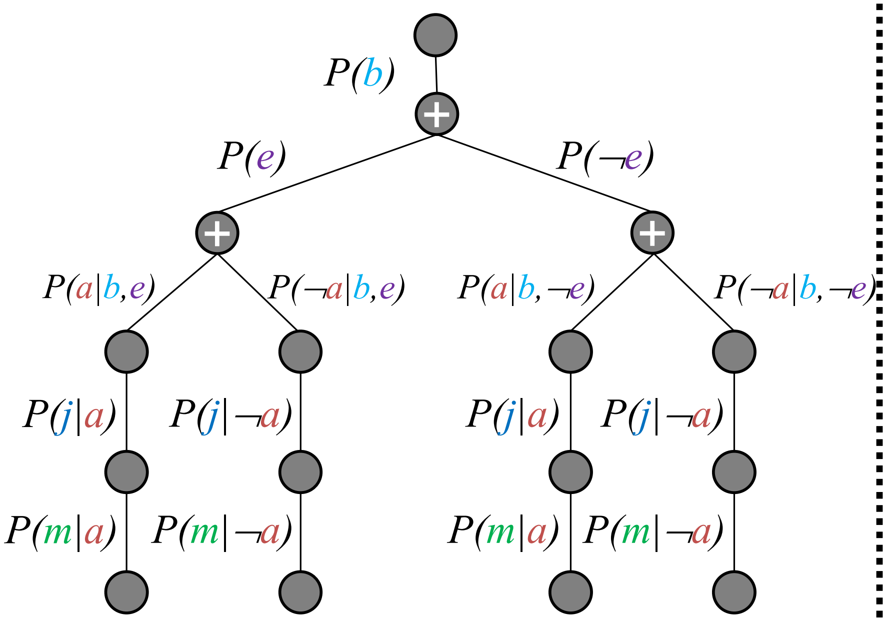
$P(Burglary = true \mid JohnCalls = true \wedge MaryCalls = true)$

Query rewritten:

$$P(b \mid j, m)$$

$$= \alpha * \sum_{e} \sum_{a} P(b) * P(e) * P(a|b,e) * P(j|a) * P(m|a)$$

$$= \alpha * P(b) * \sum_{e} P(e) * \boxed{\sum_{a} P(a|b,e) * P(j|a) * P(m|a)}$$



P(B)	P(¬B)
0.001	0.999

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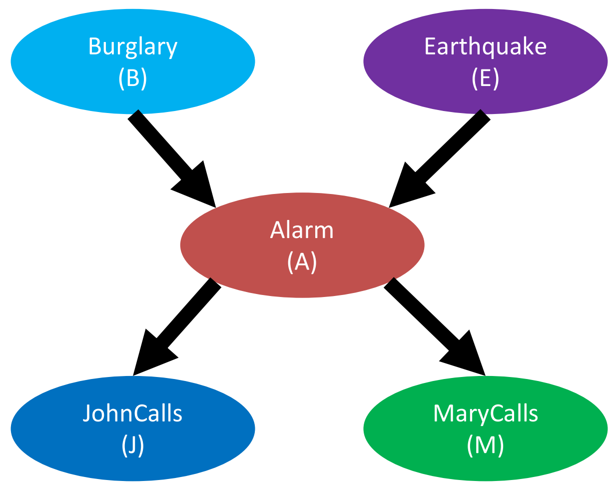
0.002	0.998



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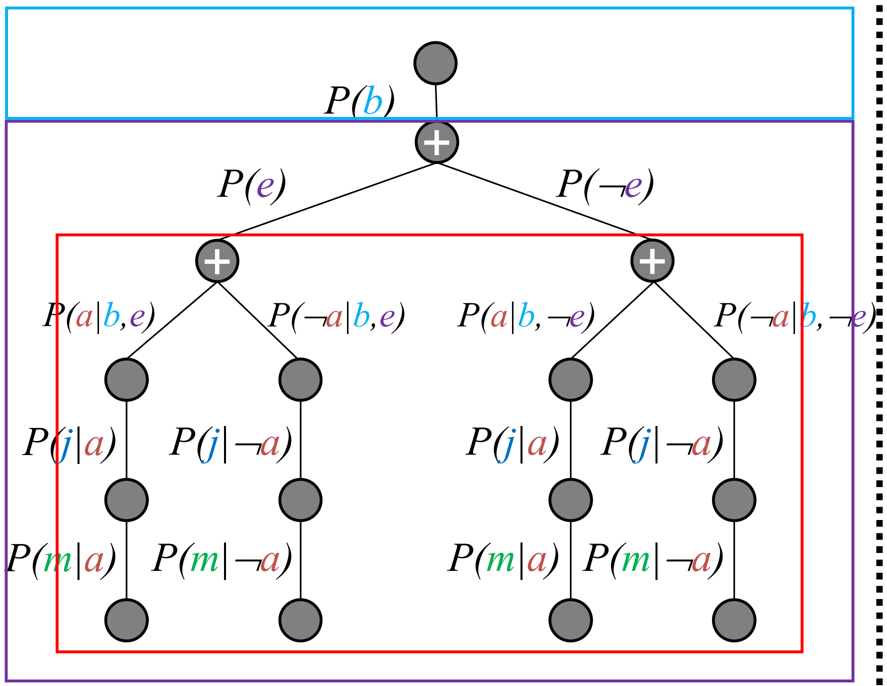
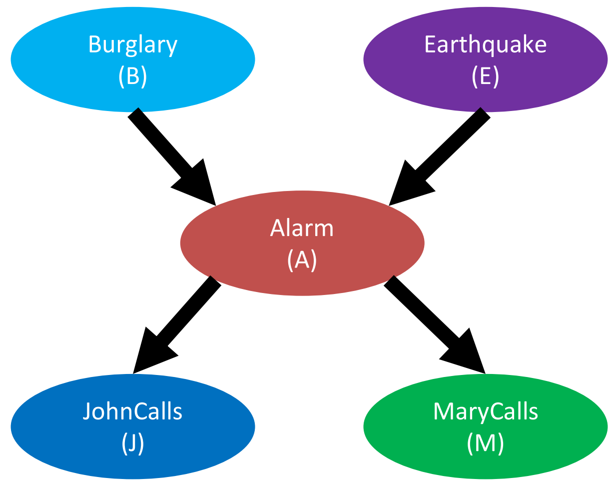
$$= \alpha * \sum_e \sum_a P(b) * P(e) * P(a|b,e) * P(j|a) * P(m|a)$$

$$= \alpha * P(b) * \sum_e P(e) * \sum_a P(a|b,e) * P(j|a) * P(m|a)$$



P(B)	P(¬B)
0.001	0.999

P(E)	P(¬E)
0.002	0.998



B	E	P(A|B,E)
t	t	0.95
t	f	0.94
f	t	0.29
f	f	0.001

A	P(J|A)
t	0.90
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A	P(M|A)
t	0.70
f	0.01

Inference

Query (let's change it a bit for simplicity):

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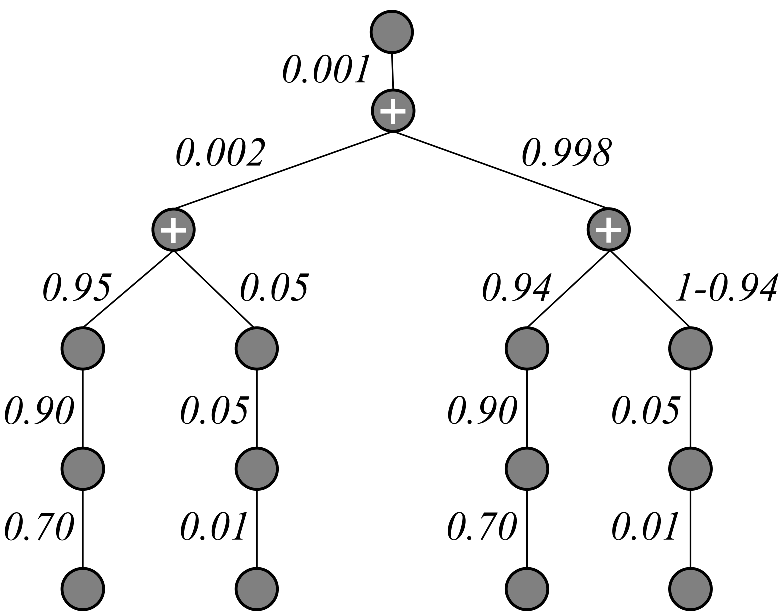
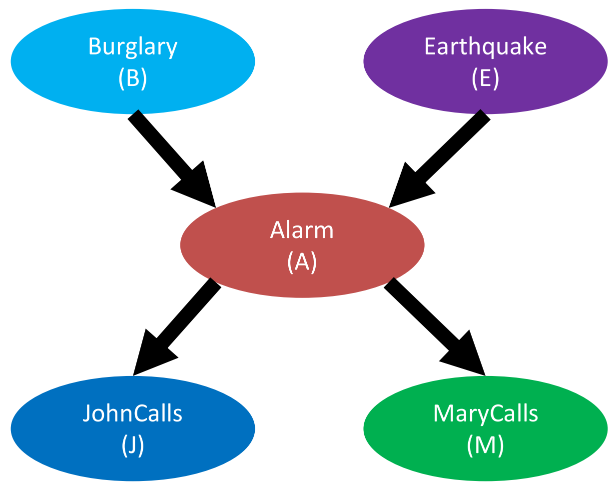


P(B)	P(¬B)
0.001	0.999

P(E)	P(¬E)
0.002	0.998



B	E	P(A|B,E)
t	t	0.95
t	f	0.94
f	t	0.29
f	f	0.001

A	P(J|A)
t	0.90
f	0.05

A	P(M|A)
t	0.70
f	0.01

Inference

Query (let's change it a bit for simplicity):

$P(Burglary = true \mid JohnCalls = true \land MaryCalls = true)$

We can now calculate:

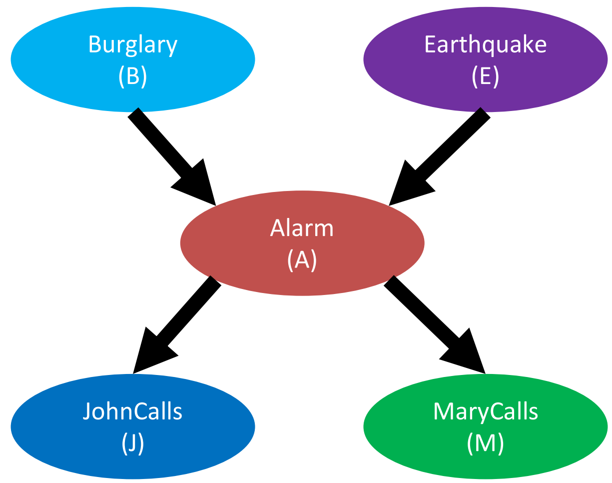
$$P(b \mid j, m) = \alpha * 0.00059224$$

And through similar process (not shown):

$$P(\neg b \mid j, m) = \alpha * 0.0014919$$

P(B)	P(¬B)
0.001	0.999

P(E)	P(¬E)
0.002	0.998



B	E	P(A|B,E)
t	t	0.95
t	f	0.94
f	t	0.29
f	f	0.001

A	P(J|A)
t	0.90
f	0.05

A	P(M|A)
t	0.70
f	0.01

Inference

Query (now we can get joint distribution):

$$\boldsymbol{P}(Burglary \mid JohnCalls = true \wedge MaryCalls = true)$$

We can now calculate:

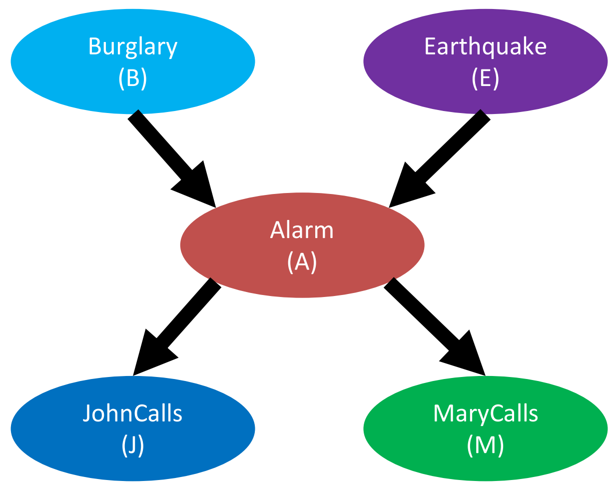
$$\boldsymbol{P}(B \mid j, m) = \alpha * < 0.00059224, 0.0014919 >$$

And after normalization:

$$\boldsymbol{P}(B \mid j, m) \approx < 0.284, 0.716 >$$

P(B)	P(¬B)
0.001	0.999

P(E)	P(¬E)
0.002	0.998

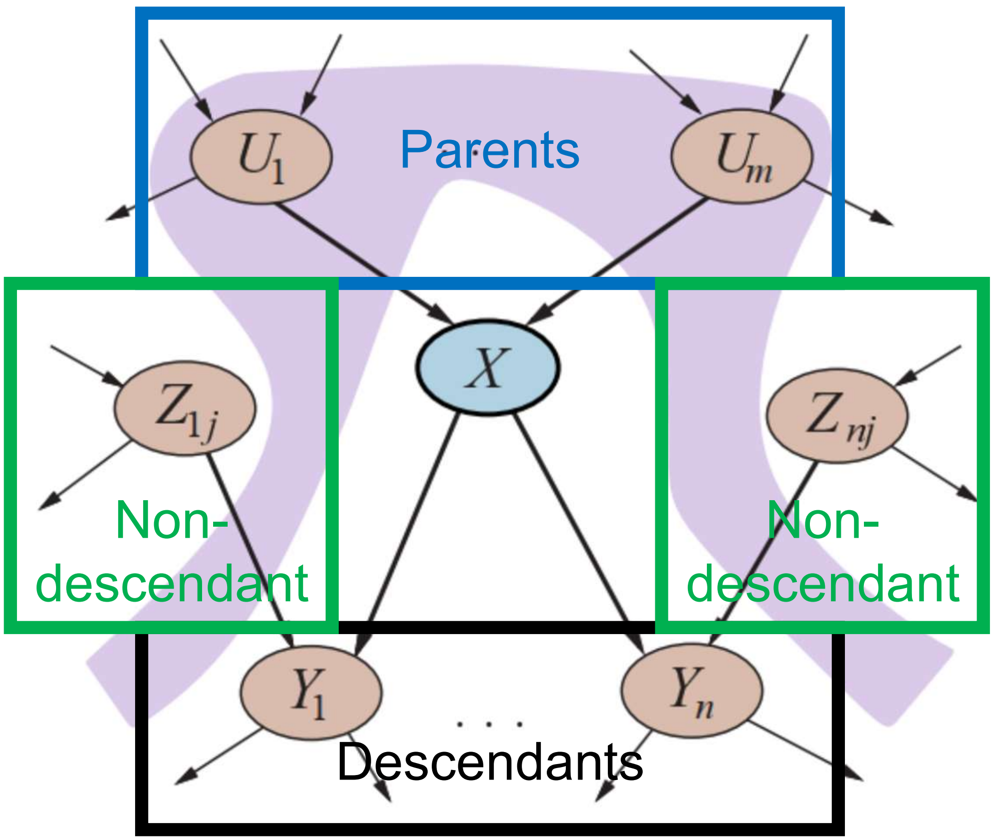


B	E	P(A|B,E)
t	t	0.95
t	f	0.94
f	t	0.29
f	f	0.001

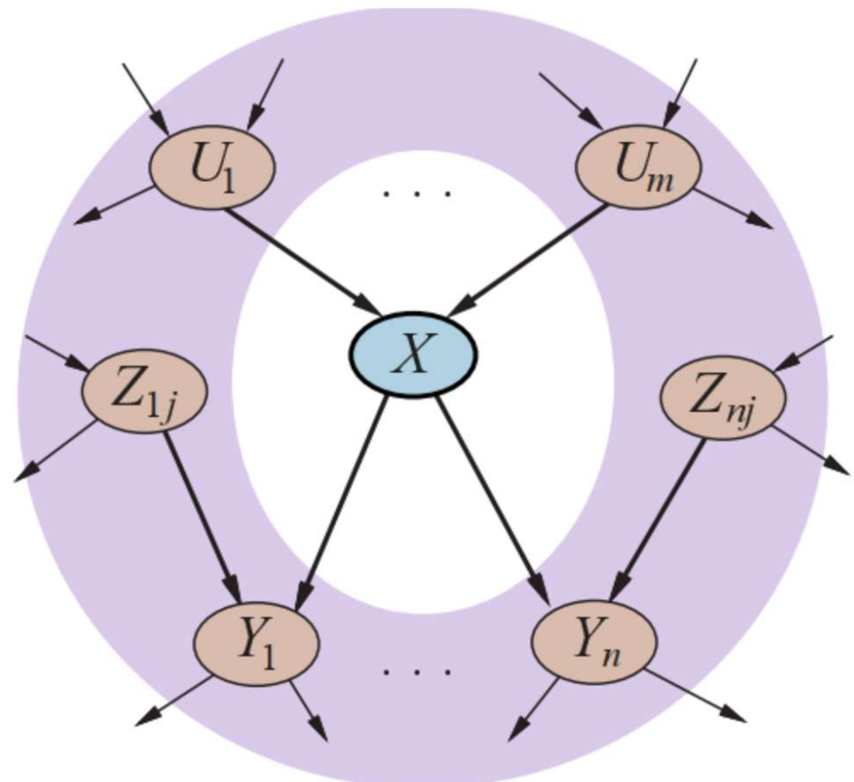
A	P(J|A)
t	0.90
f	0.05

A	P(M|A)
t	0.70
f	0.01

More On Conditional Independence



Node X is conditionally independent of its non-descendants given its parents.



Node X is conditionally independent of ALL other nodes in the network its given its Markov blanket.

Why do we care?

An unconstrained joint probability distribution with N **binary** variables involves 2^N probabilities.
Bayesian network with at most k parents per each node (N) involves $N * 2^k$ probabilities ($k < N$).

Decision Networks

Decision Theory

- Decisions: every plan (actions) leads to an outcome (state)
- Agents have preferences (preferred outcomes)
- Preferences $\rightarrow$ outcome utilities
- Agents have degrees of belief (probabilities) for actions

Decision theory = probability theory + utility theory

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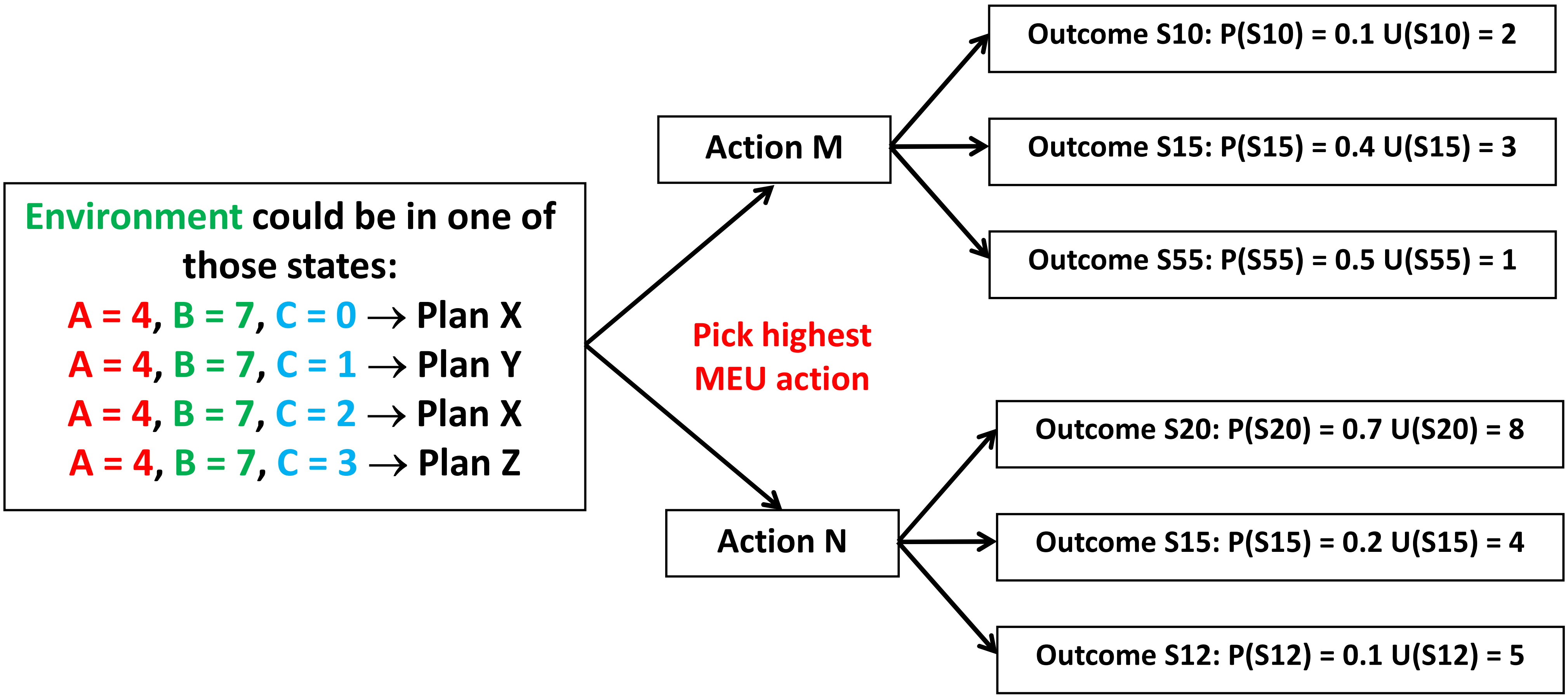
- Agents have degrees of belief (probabilities) for actions

Decision theory = $\underbrace{\text{probability theory}}_{\text{BELIEFS}}$ + $\underbrace{\text{utility theory}}_{\text{DESIRES}}$

Maximum Expected (Average) Utility

$$MEU(M) = P(S10) * U(S10) + P(S15) * U(S15) + P(S55) * U(S55)$$



$$MEU(N) = P(S20) * U(S20) + P(S15) * U(S15) + P(S12) * U(S12)$$

Agents Decisions

Recall that agent ACTIONS change the state:

- **if we are in state** s
- **action** a **is expected to**
- **lead to another state** s' **(outcome)**

Given uncertainty about the current state s **and action outcome** s' **we need to define the following:**

- **probability (belief) of being in state** s**:** $P(s)$
- **probability (belief) of action** a **leading to outcome** s'**:** $P(s' \mid s, a)$

Now:

$$P(s' \mid s, a) = P(\textsc{Result}(a) = s') = \sum_s P(s) * P(s' \mid s, a)$$

Expected Action Utility

The expected utility of an action a given the evidence is the average utility value of all possible outcomes s' of action a, weighted by their probability (belief) of occurence:

$$\mathrm{EU}(\mathrm{a}) = \sum_{s'} \sum_{s} \mathrm{P}(\mathrm{s}) * \mathrm{P}(\mathrm{s}' \mid \mathrm{s}, \mathrm{a}) * \mathrm{U}(\mathrm{s}') = \sum_{s'} \mathrm{P}(Result(a) = s') * \mathrm{U}(\mathrm{s}')$$

Rational agent should choose an action that maximizes the expected utility:

$$\text{chosen action} = \underset{\mathrm{a}}{\operatorname{argmax}} \ \mathrm{EU}(\mathrm{a})$$

State Utility Function

Agent's preferences (desires) are captured by the Utility function $U(s)$.

Utility function assigns a value to each state s to express how desirable this state is to the agent.

Decision Network (Influence Diagram)

Decision networks (also called influence diagrams) are structures / mechanisms for making rational decisions.

Decision networks are based on Bayesian networks, but include additional nodes that represent actions and utilities.

Decision Networks

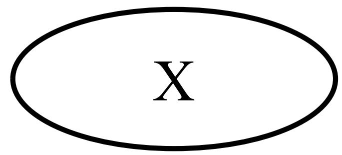
The most basic decision network needs to include:

- **information about current state** s
- **possible actions**
- **resulting state** s' **(after applying chosen action** a**)**
- **utility of the resulting state** $U(s')$

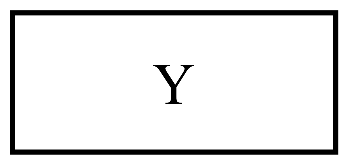
Decision Network Nodes

Decision networks are built using the following nodes:

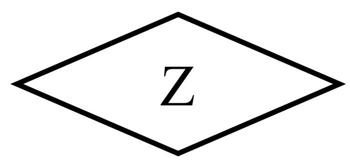
- **chance nodes:**



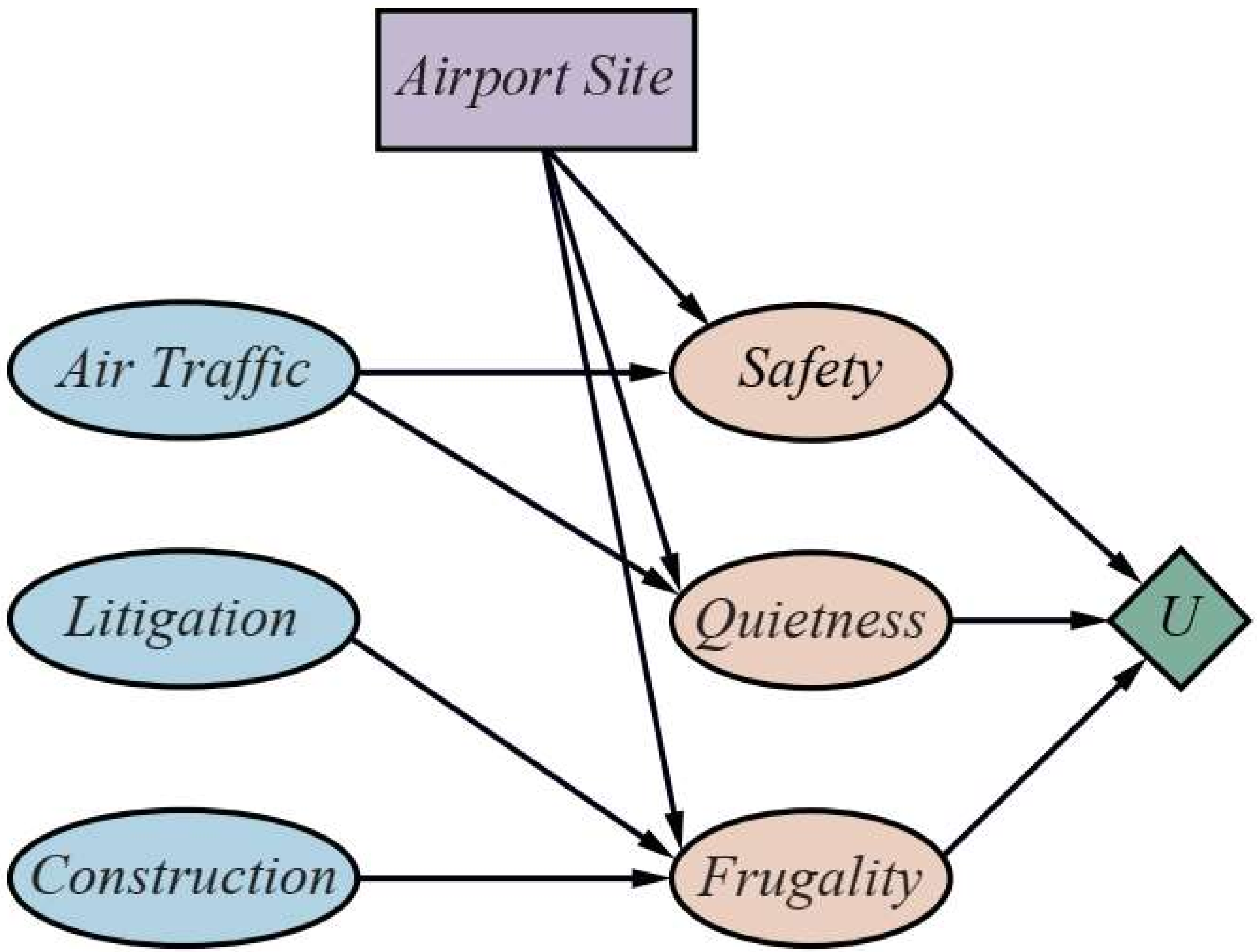
- **decision nodes:**



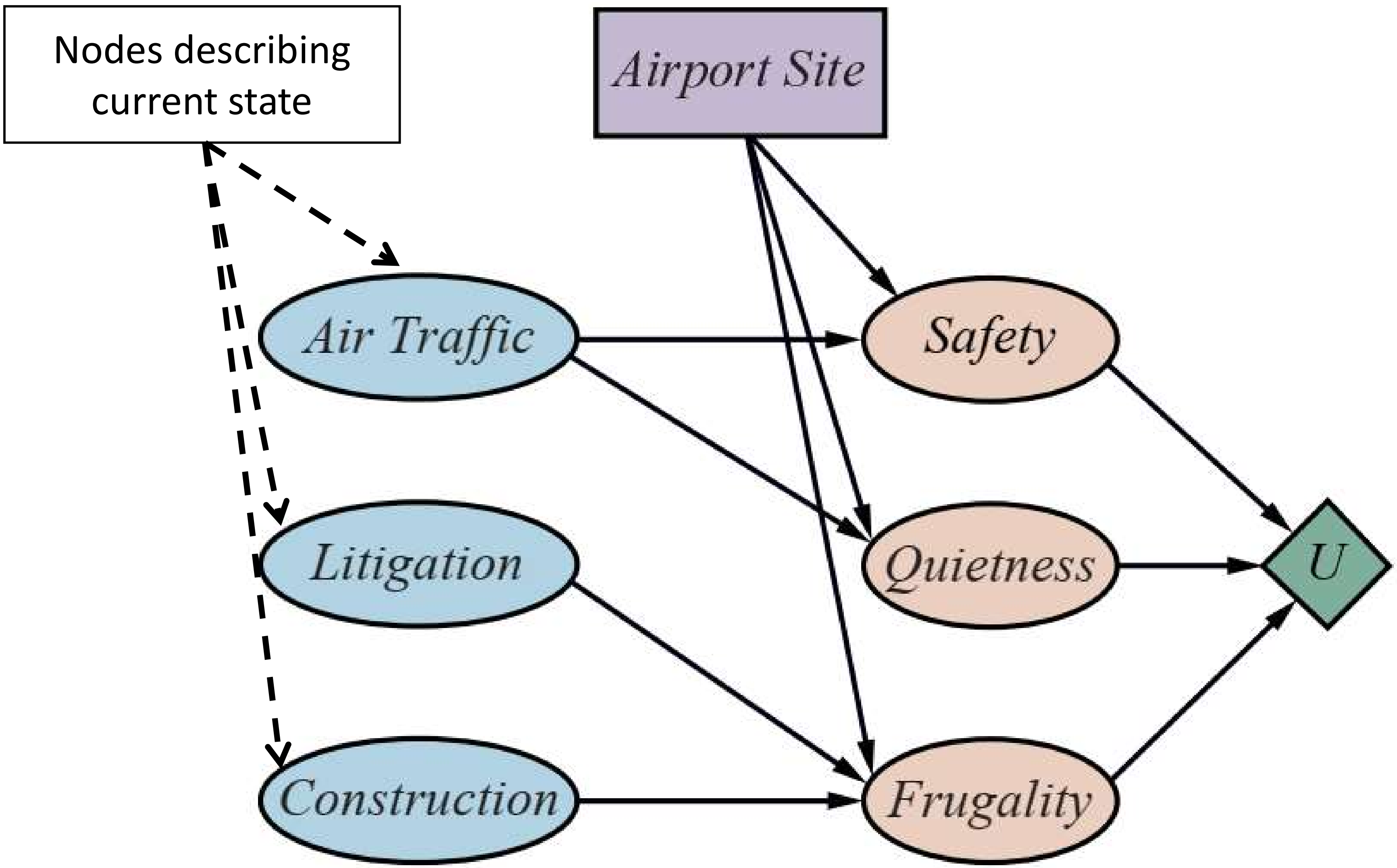
- **utility (or value) nodes**



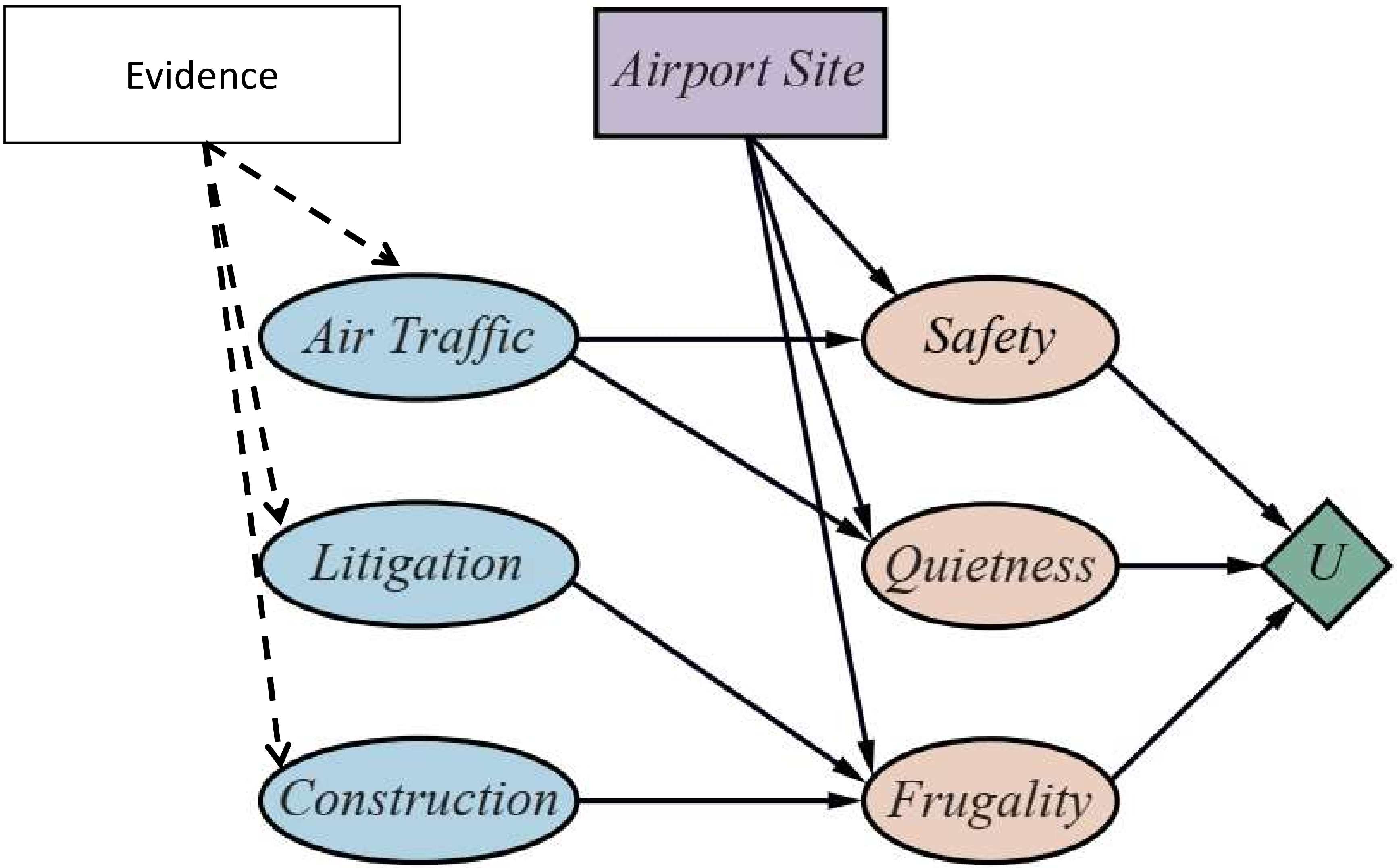
Decision Network: Example



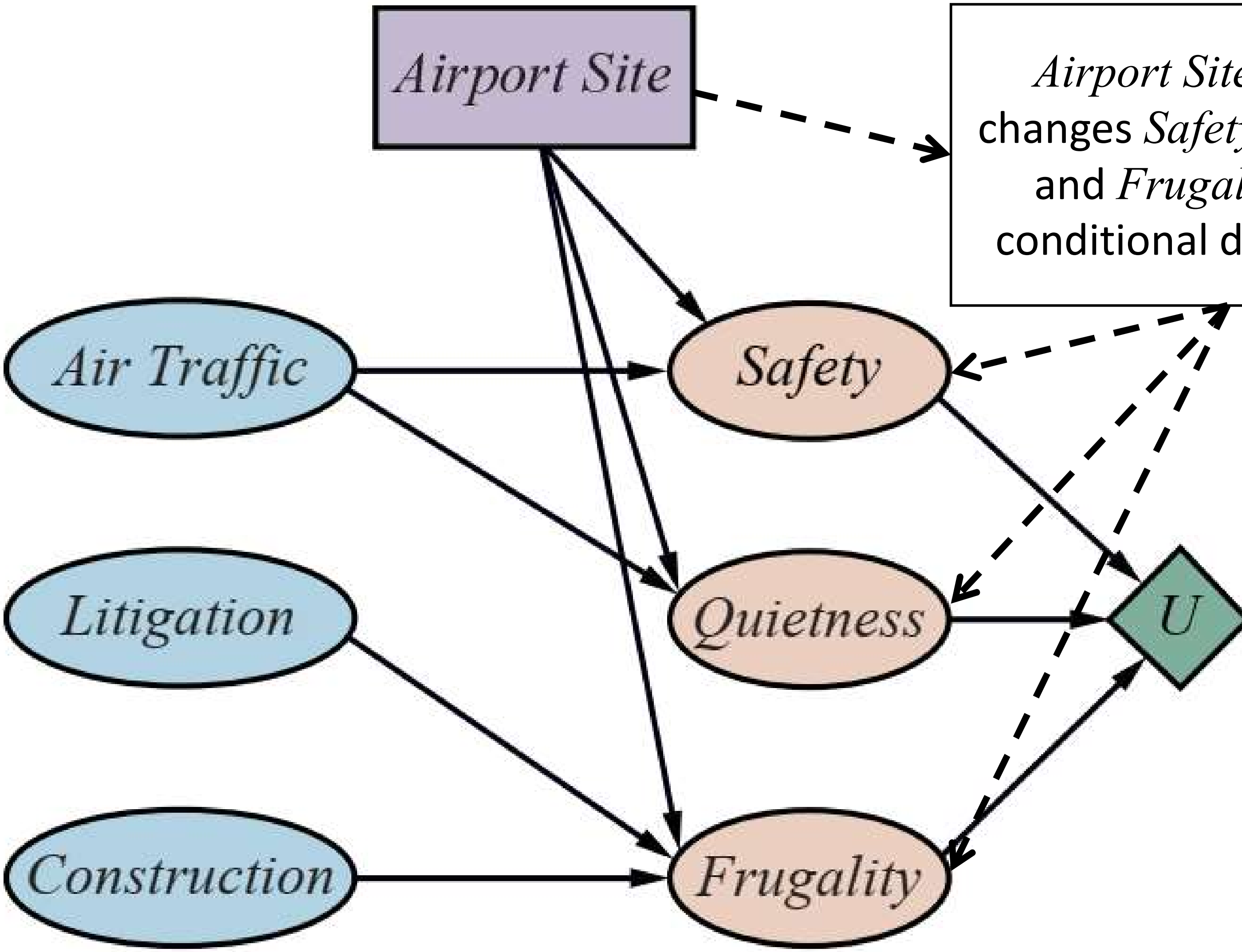
Decision Network: Example



Decision Network: Example

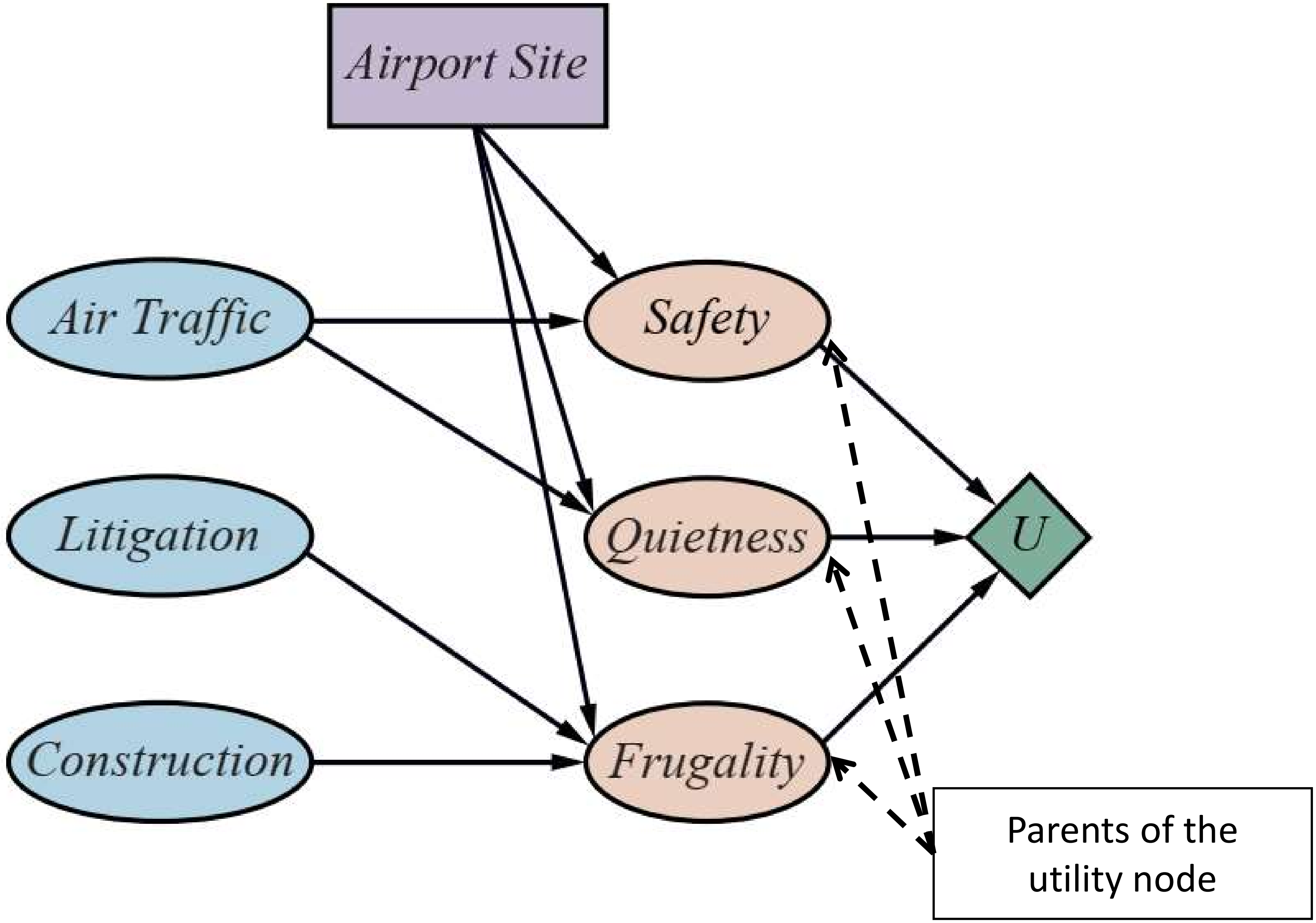


Decision Network: Example

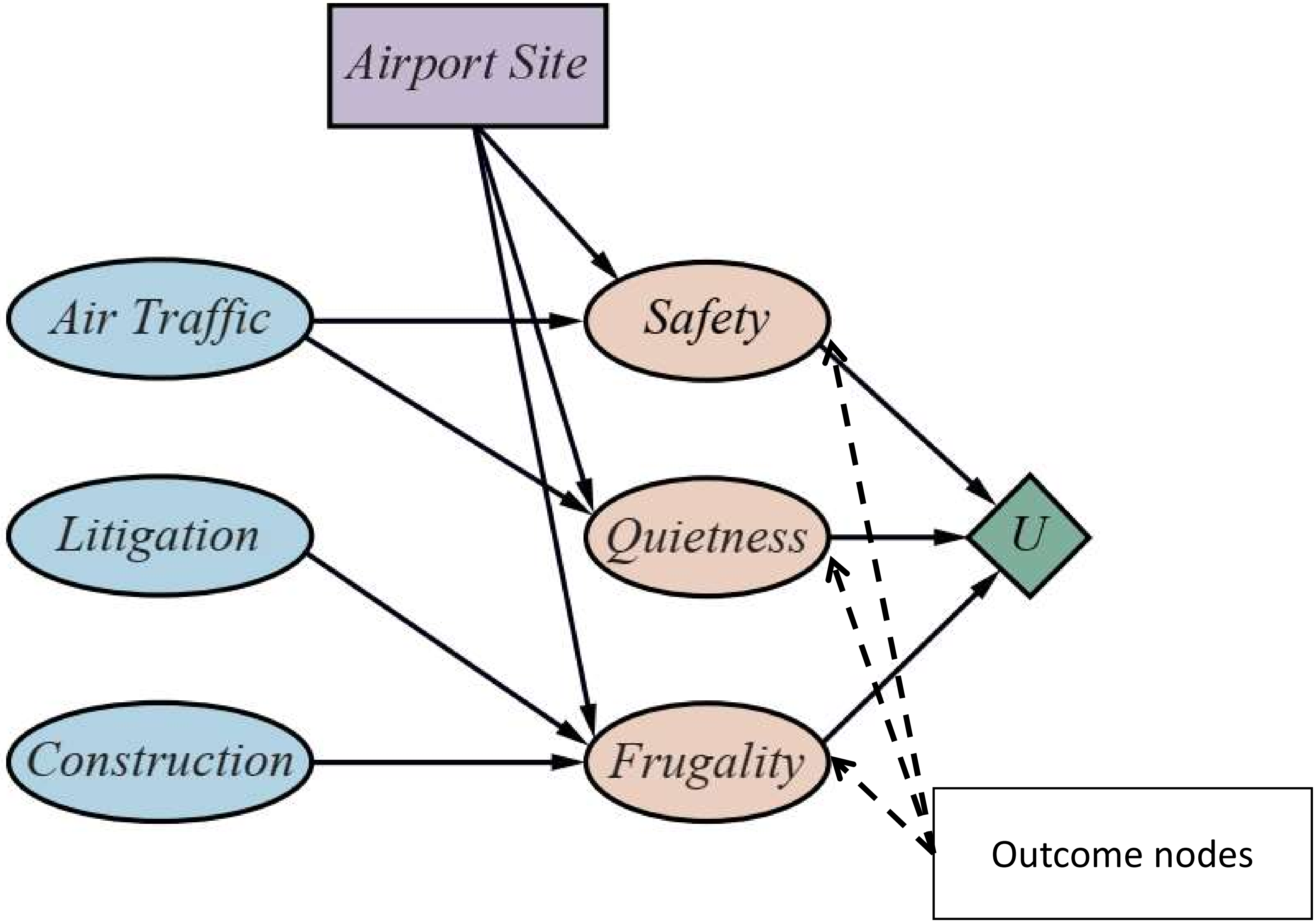


Airport Site decision changes *Safety*, *Quietness*, and *Frugality* nodes conditional distributions

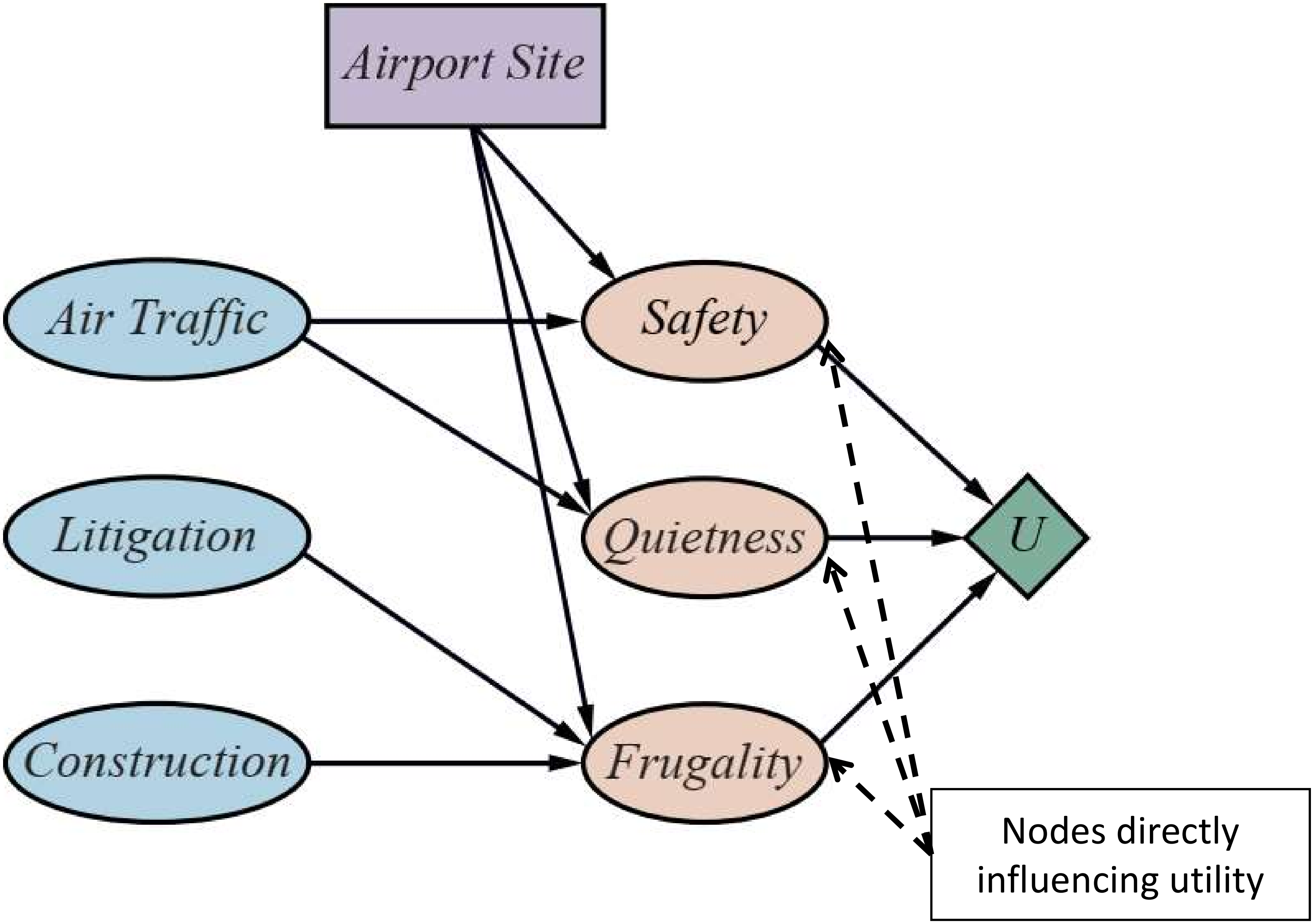
Decision Network: Example



Decision Network: Example



Decision Network: Example

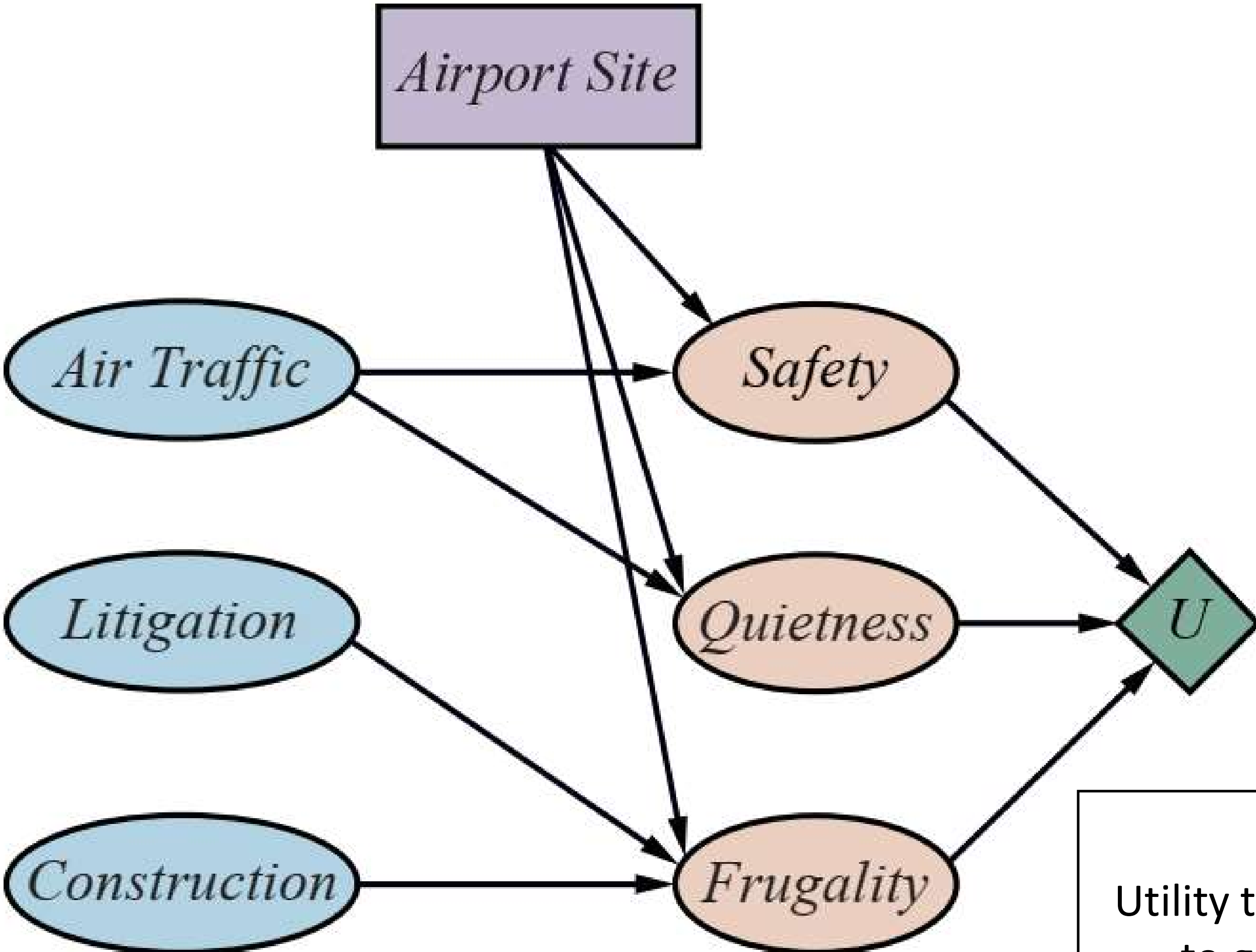


Decision Network: Evaluation

The algorithm for decision network evaluation is as follows:

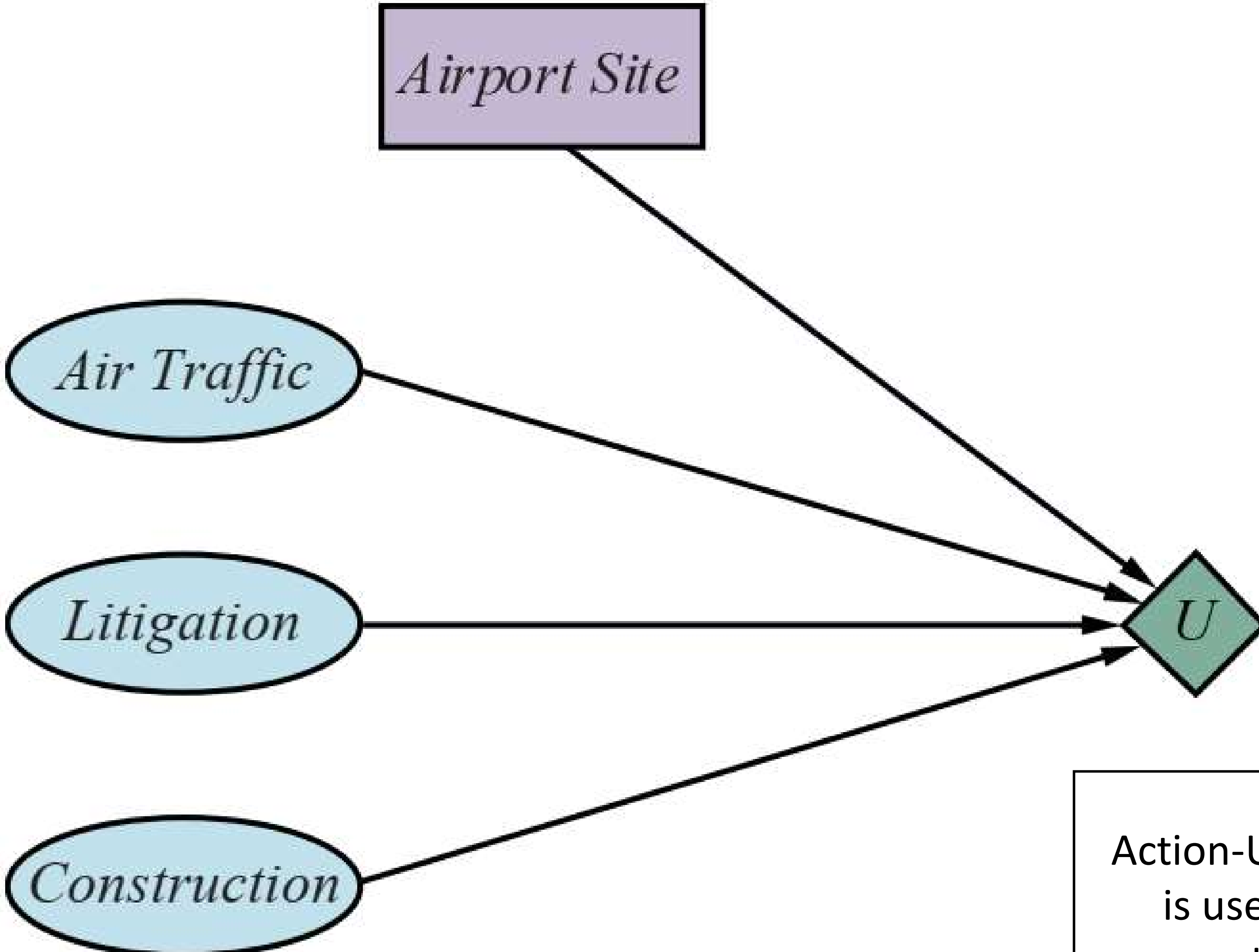
1. Set the evidence variables for the current state
2. For each possible value a of decision node:
    a. Set the decision node to that value
    b. Calculate the posterior probabilities for the parent nodes of the utility node
    c. Calculate the utility for the action / value a
3. Return the action with highest utility

Decision Network: Example



Utility table is used to get utility

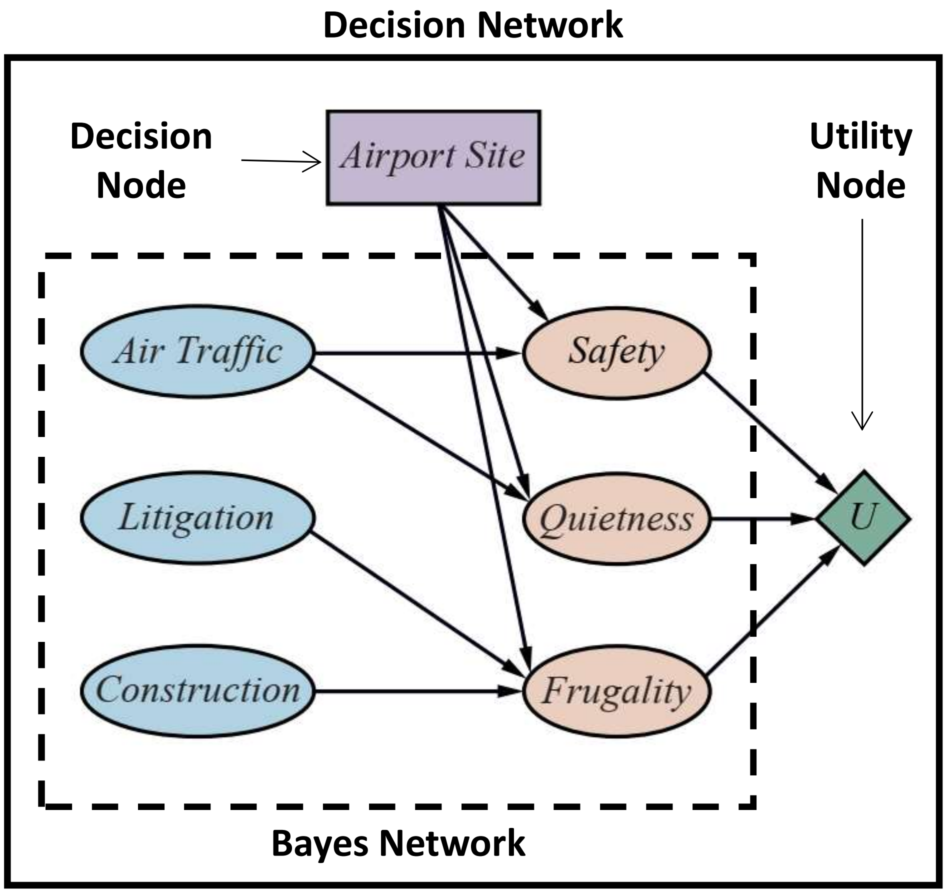
Decision Network: Simplified Form



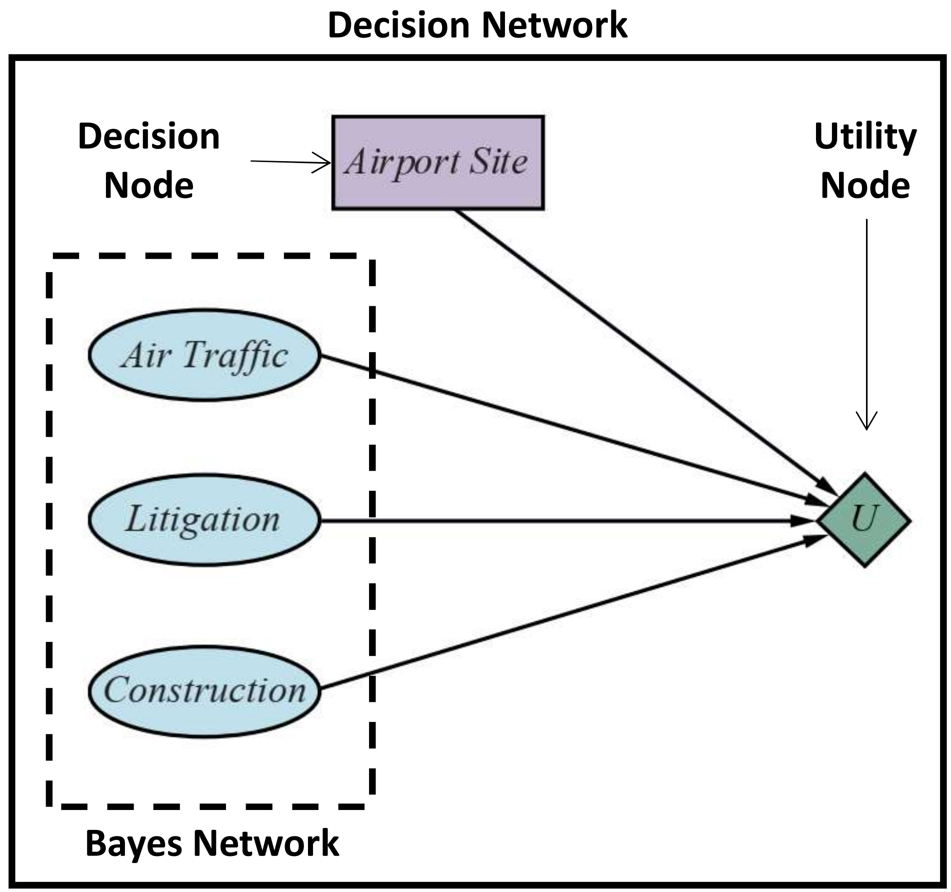
Action-Utility table is used to get expected utility

(Single-Stage) Decision Networks

General Structure

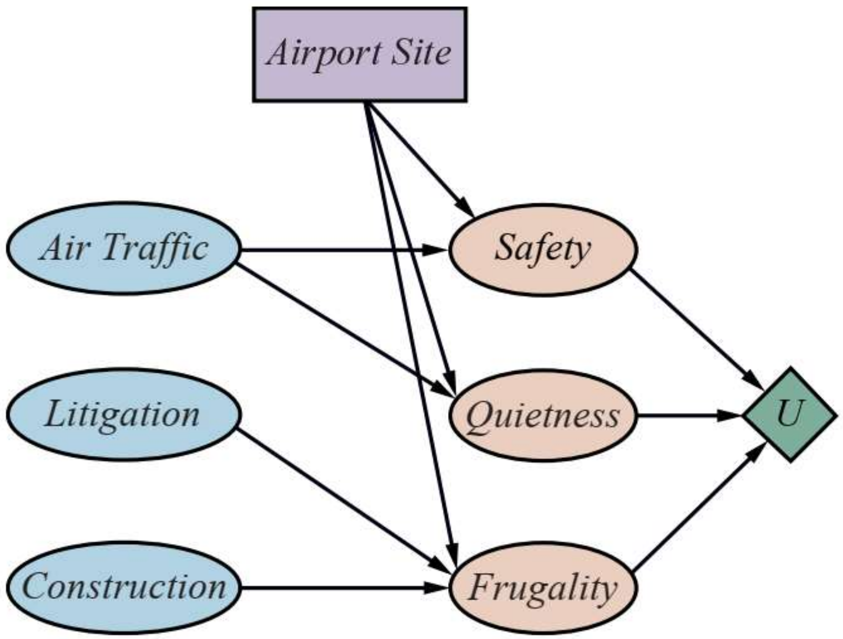


Simplified Structure



(Single-Stage) Decision Networks

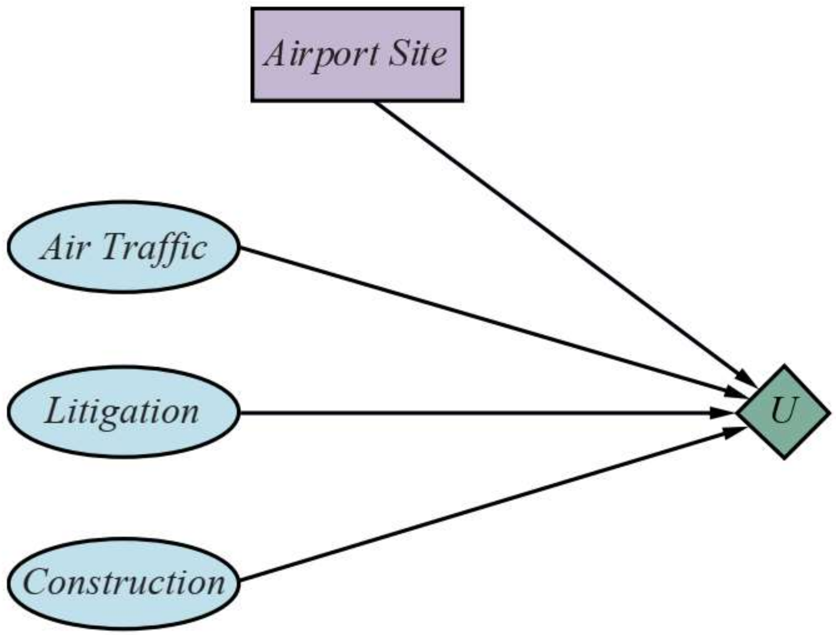
General Structure



Utility Table

S	low	low	low	low	high	high	high	high
Q	low	low	high	high	low	low	high	high
F	low	high	low	high	low	high	low	high
U	10	20	5	50	70	150	100	200

Simplified Structure



Action-Utility Table (not all columns shown)

AT	low	low	low	---	---	high	high	high
L	low	low	high	---	---	low	high	high
C	low	high	low	---	---	high	low	high
AS	A	A	A	---	---	B	B	B
U	10	20	5	---	---	150	100	200

Decision Network: Evaluation

The algorithm for decision network evaluation is as follows:

1. Set the evidence variables for the current state
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Agent's Decisions

Recall that agent ACTIONS change the state:

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Given uncertainty about the current state s **and action outcome** s' **we need to define the following:**

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Now:

$$P(s' \mid s, a) = P(\textsc{Result}(a) = s') = \sum_{s} P(s) * P(s' \mid s, a)$$

Expected Action Utility

The expected utility of an action a **given the evidence is the average utility value of all possible outcomes** s' **of action** a**, weighted by their probability (belief) of occurence:**

$$EU(a) = \sum_{s'} \sum_{s} P(s) * P(s' \mid s, a) * U(s') \;=\; \sum_{s'} P(Result(a) = s') * U(s')$$

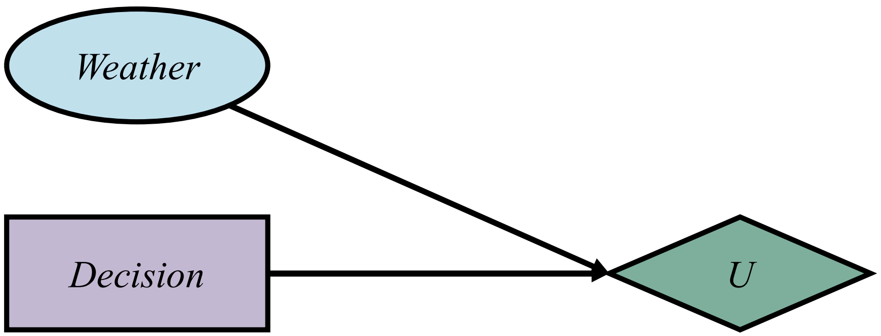
Rational agent should choose an action that maximizes the expected utility:

$$\text{chosen action} \;=\; \underset{a}{\text{argmax}}\; EU(a)$$

Decision Networks: Example

Decision: take umbrella

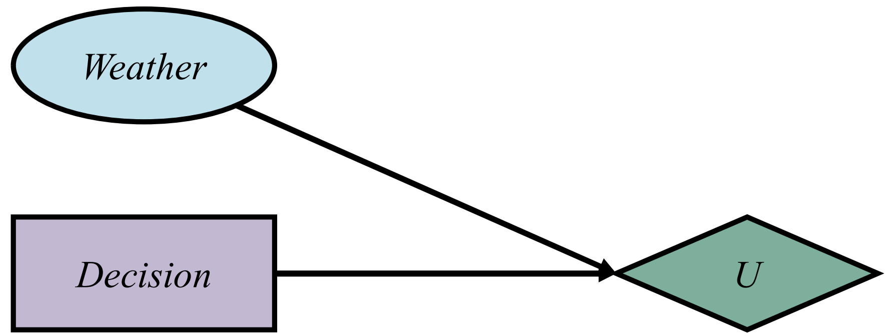
P(W=rain)	P(W=sun)
0.30	0.70



D	W	U
leave	sun	100
leave	rain	0
take	sun	20
take	rain	70

Decision: leave umbrella

P(W=rain)	P(W=sun)
0.30	0.70



D	W	U
leave	sun	100
leave	rain	0
take	sun	20
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Decision Networks: Example

Decision: take umbrella

$$EU(a) = \sum_{s'} P(Result(a) = s') * U(s')$$

P(W=rain)	P(W=sun)
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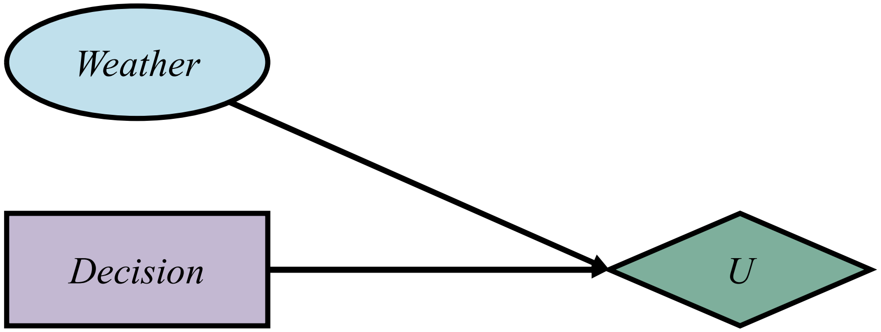


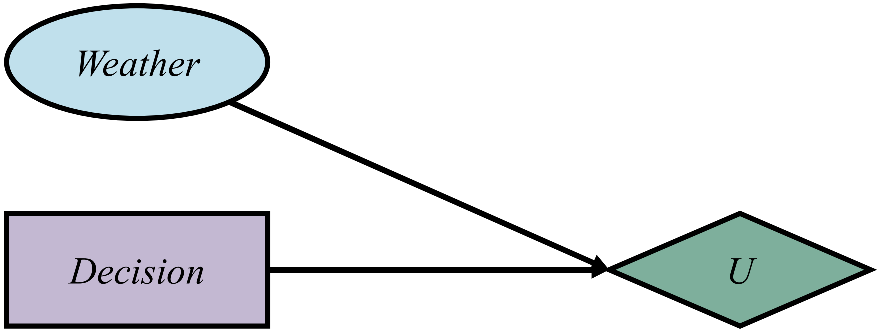
D	W	U
leave	sun	100
leave	rain	0
take	sun	20
take	rain	70

$$EU(take) = ???$$

Decision: leave umbrella

$$EU(a) = \sum_{s'} P(Result(a) = s') * U(s')$$

P(W=rain)	P(W=sun)
0.30	0.70



D	W	U
leave	sun	100
leave	rain	0
take	sun	20
take	rain	70

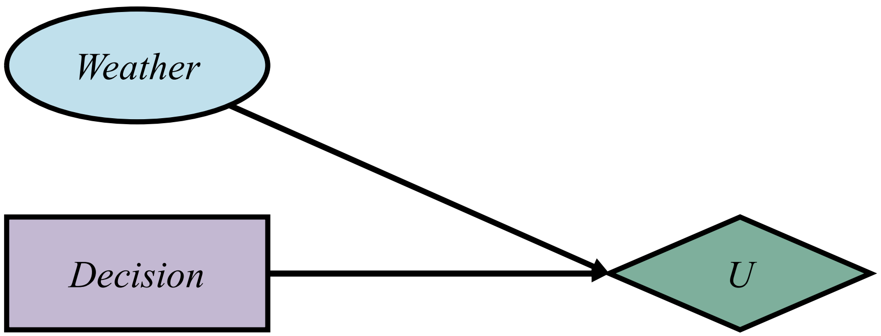
$$EU(leave) = ???$$

Decision Networks: Example

Decision: take umbrella

$$\mathrm{EU}(a) = \sum_{s'} \mathrm{P}(Result(a) = s') * \mathrm{U}(s')$$

P(W=rain)	P(W=sun)
0.30	0.70



S_1': D = take, W = sun
S_2': D = take, W = rain
EU(take) =
P(Result(take) = S_1')*U(S_1') +
P(Result(take) = S_2')*U(S_2') =
0.70 * 20 + 0.30 * 70 = 35

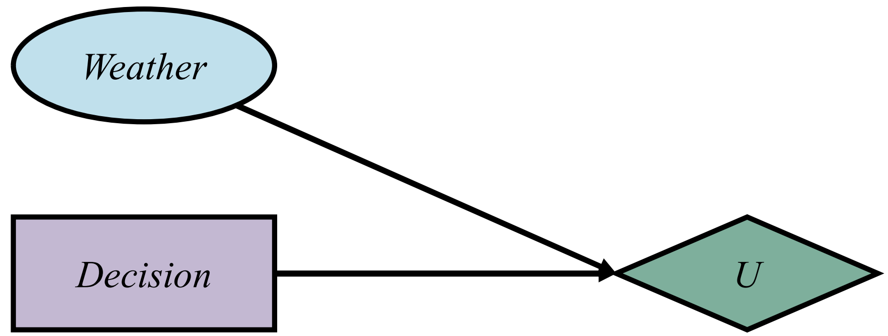
D	W	U
leave	sun	100
leave	rain	0
take	sun	20
take	rain	70

$$\mathrm{EU}(\mathrm{take}) = 35$$

Decision: leave umbrella

$$\mathrm{EU}(a) = \sum_{s'} \mathrm{P}(Result(a) = s') * \mathrm{U}(s')$$

P(W=rain)	P(W=sun)
0.30	0.70



S_3': D = leave, W = sun
S_4': D = leave, W = rain
EU(leave) =
P(Result(leave) = S_3')*U(S_3') +
P(Result(leave) = S_4')*U(S_4') =
0.70 * 100 + 0.30 * 0 = 70

D	W	U
leave	sun	100
leave	rain	0
take	sun	20
take	rain	70

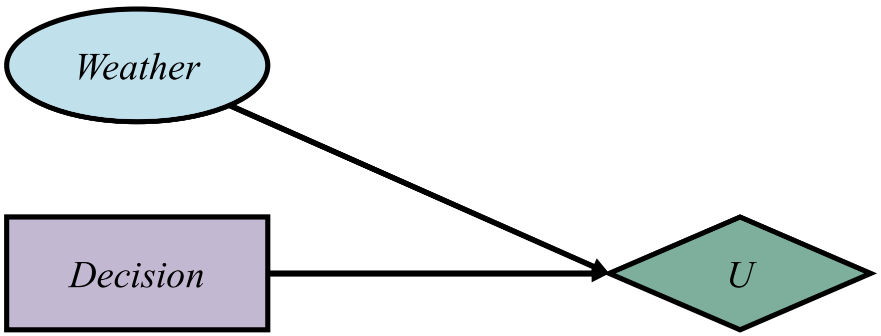
$$\mathrm{EU}(\mathrm{leave}) = 70$$

Decision Networks: Example

Which action to choose: take or leave Umbrella?

$\text{EU}(a) = \sum_{s'} P(Result(a) = s') * \text{U}(s')$

P(W=rain)	P(W=sun)
0.30	0.70

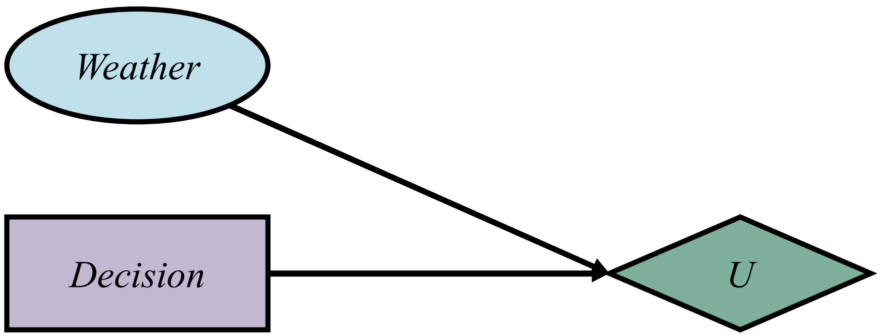


S_1': D = take, W = sun
S_2': D = take, W = rain
EU(take) =
P(Result(take) = S_1')*U(S_1') +
P(Result(take) = S_2')*U(S_2') =
0.70 * 20 + 0.30 * 70 = 35

D	W	U
leave	sun	100
leave	rain	0
take	sun	20
take	rain	70

$\text{EU}(a) = \sum_{s'} P(Result(a) = s') * \text{U}(s')$

P(W=rain)	P(W=sun)
0.30	0.70



S_3': D = leave, W = sun
S_4': D = leave, W = rain
EU(leave) =
P(Result(leave) = S_3')*U(S_3') +
P(Result(leave) = S_4') U(S_4') =
0.70 * 100 + 0.30 * 0 = 70

D	W	U
leave	sun	100
leave	rain	0
take	sun	20
take	rain	70

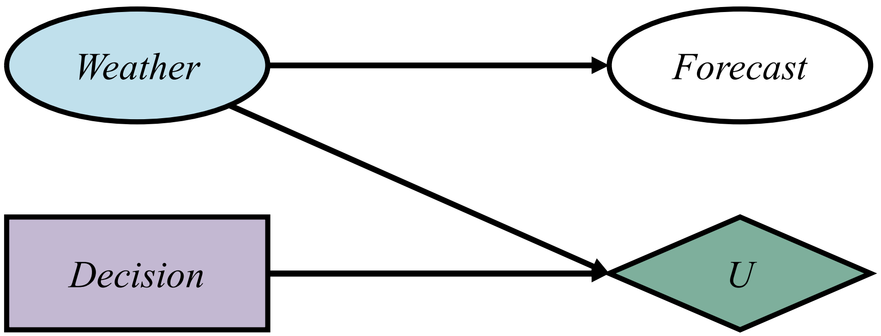
$$\text{action} = \underset{a}{\text{argmax}}\ \text{EU}(a) \mid \max(\text{EU}(take), \underline{\textbf{EU}(leave)}) = \max(35, \mathbf{70}) \rightarrow \text{leave}$$

Decision Networks: Example

Decision: take umbrella

P(W=rain)	P(W=sun)
0.30	0.70

P(F=sun)	P(F=rain)
0.59	0.41

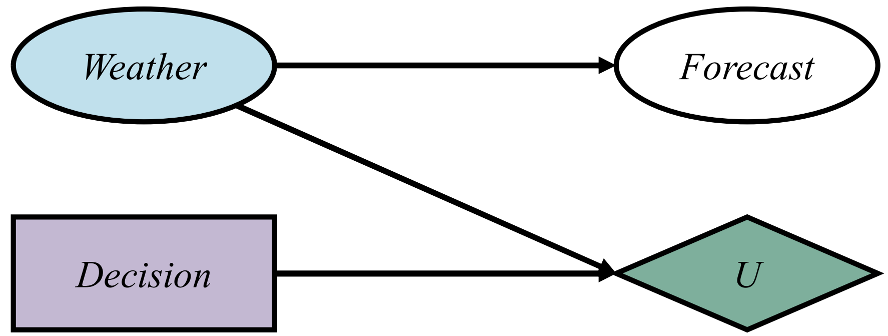


D	W	U
leave	sun	100
leave	rain	0
take	sun	20
take	rain	70

Decision: leave umbrella

P(W=rain)	P(W=sun)
0.30	0.70

P(F=sun)	P(F=rain)
0.59	0.41



D	W	U
leave	sun	100
leave	rain	0
take	sun	20
take	rain	70

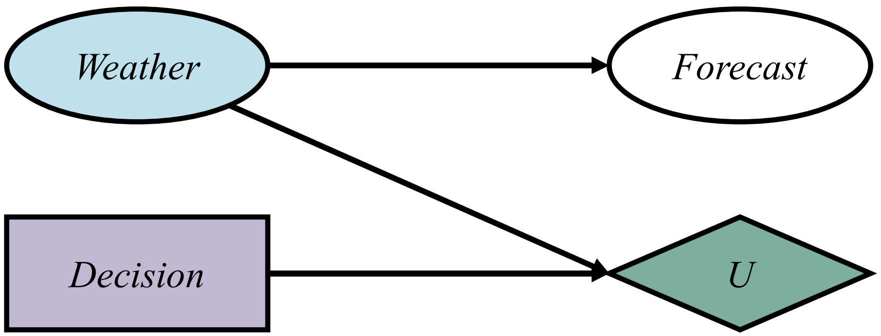
Decision Networks: Example

Decision: take umbrella given e

$$EU(a \mid e) = \sum_{s'} P(Result(a) = s' \mid e) * U(s')$$

P(rain | F)	P(sun | F)
???	???

P(F=sun)	P(F=rain)
0.59	0.41



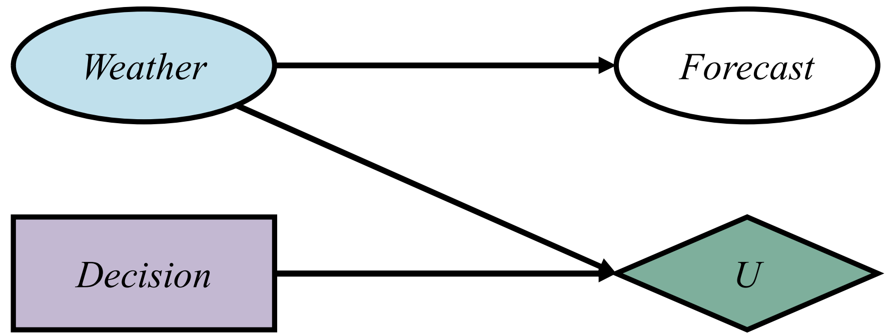
D	W	U
leave	sun	100
leave	rain	0
take	sun	20
take	rain	70

Decision: leave umbrella given e

$$EU(a \mid e) = \sum_{s'} P(Result(a) = s' \mid e) * U(s')$$

P(rain | F)	P(sun | F)
???	???

P(F=sun)	P(F=rain)
0.59	0.41



D	W	U
leave	sun	100
leave	rain	0
take	sun	20
take	rain	70

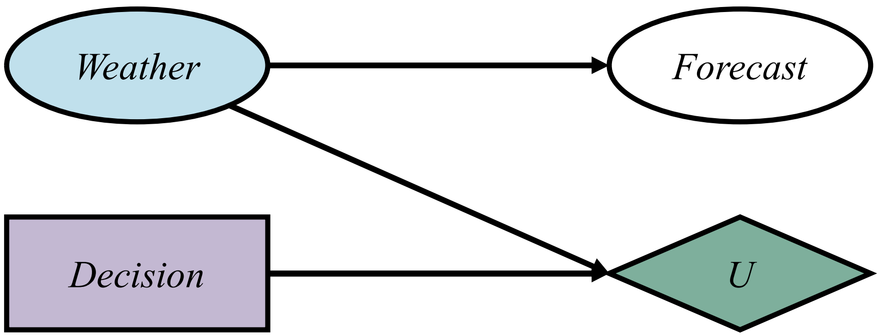
Decision Networks: Example

Decision: take umbrella given e

$$EU(a \mid e) = \sum_{s'} P(Result(a) = s' \mid e) * U(s')$$

P(W=rain)	P(W=sun)
0.30	0.70

P(F=sun)	P(F=rain)
0.59	0.41



D	W	U
leave	sun	100
leave	rain	0
take	sun	20
take	rain	70

Conditional probabilities

Assume that we are given:

F	W	P(F|W)
sun	sun	0.80
rain	sun	0.20
sun	rain	0.10
rain	rain	0.90

By Bayes' Theorem:

$$P(W = sun \mid F = sun) = \frac{P(F = sun \mid W = sun) * P(W = sun)}{P(F = sun)} = \frac{0.80 * 0.70}{0.59} = 0.95$$

$$P(W = sun \mid F = rain) = \frac{P(F = rain \mid W = sun) * P(W = sun)}{P(F = rain)} = \frac{0.20 * 0.70}{0.41} = 0.34$$

$$P(W = rain \mid F = sun) = \frac{P(F = sun \mid W = rain) * P(W = rain)}{P(F = sun)} = \frac{0.10 * 0.30}{0.59} = 0.05$$

$$P(W = rain \mid F = rain) = \frac{P(F = rain \mid W = rain) * P(W = rain)}{P(F = rain)} = \frac{0.90 * 0.30}{0.41} = 0.66$$

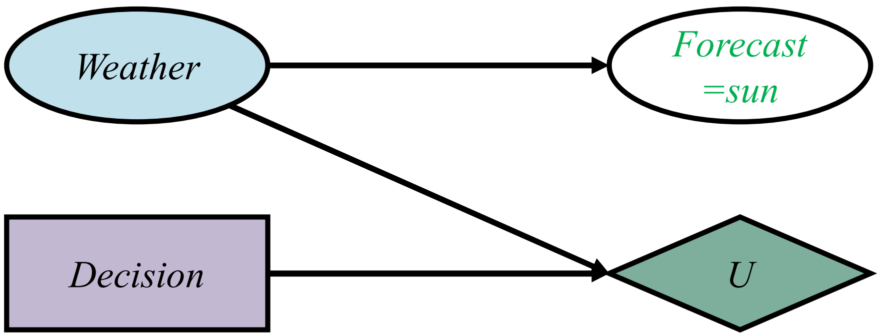
Decision Networks: Example

Decision: take umbrella given sun

$\text{EU}(a \mid e) = \sum_{s'} \text{P}(Result(a) = s' \mid e) * \text{U}(s')$

P(rain | F)	P(sun | F)
0.05	0.95

P(F=sun)	P(F=rain)
0.59	0.41



D	W	U
leave	sun	100
leave	rain	0
take	sun	20
take	rain	70

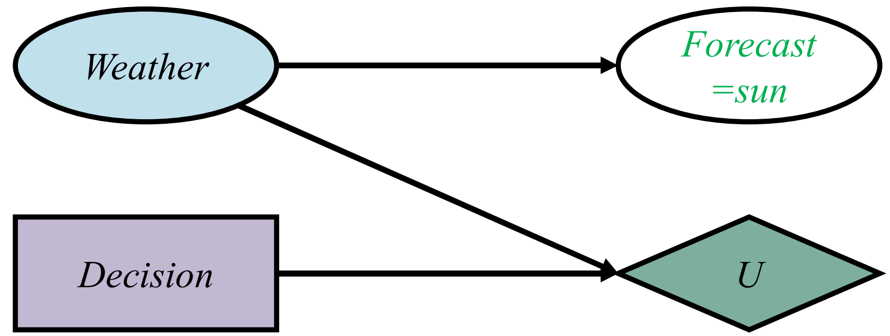
EU(take given sun forecast) = ???

Decision: leave umbrella given sun

$\text{EU}(a \mid e) = \sum_{s'} \text{P}(Result(a) = s' \mid e) * \text{U}(s')$

P(rain | F)	P(sun | F)
0.05	0.95

P(F=sun)	P(F=rain)
0.59	0.41



D	W	U
leave	sun	100
leave	rain	0
take	sun	20
take	rain	70

EU(leave given sun forecast) = ???

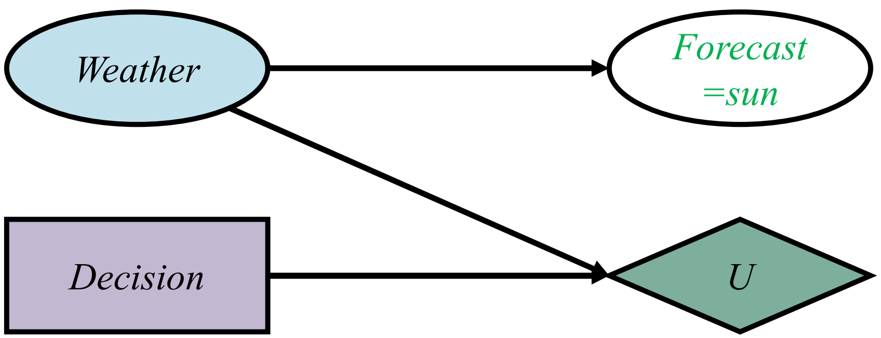
Decision Networks: Example

Decision: take umbrella given sun

$$EU(a \mid e) = \sum_{s'} P(Result(a) = s' \mid e) * U(s')$$

P(rain | F)	P(sun | F)
0.05	0.95

P(F=sun)	P(F=rain)
0.59	0.41



S_1': D = take, W = sun
S_2': D = take, W = rain
EU(take) =
P(Result(take)=S_1'|e)*U(S_1') +
P(Result(take)=S_2'|e)*U(S_2') =
0.95 * 20 + 0.05 * 70 = 22.5

D	W	U
leave	sun	100
leave	rain	0
take	sun	20
take	rain	70

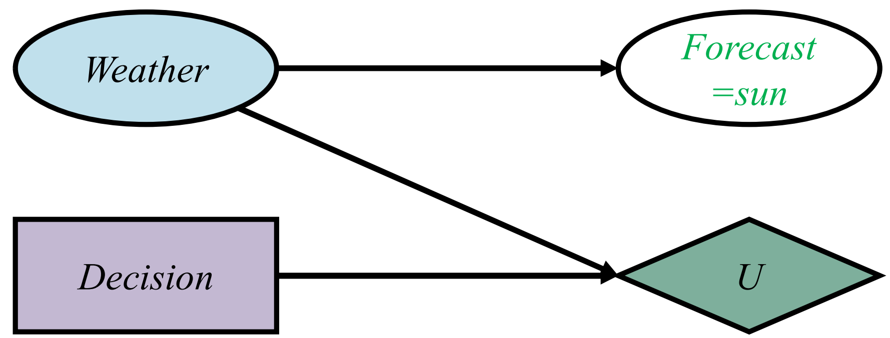
EU(take given sun forecast) = 22.5

Decision: leave umbrella given sun

$$EU(a \mid e) = \sum_{s'} P(Result(a) = s' \mid e) * U(s')$$

P(rain | F)	P(sun | F)
0.05	0.95

P(F=sun)	P(F=rain)
0.59	0.41



S_3': D = leave, W = sun
S_4': D = leave, W = rain
EU(leave) =
P(Result(leave)=S_3'|e)*U(S_3') +
P(Result(leave)=S_4'|e)*U(S_4') =
0.95 * 100 + 0.05 * 0 = 95

D	W	U
leave	sun	100
leave	rain	0
take	sun	20
take	rain	70

EU(leave given sun forecast) = 95

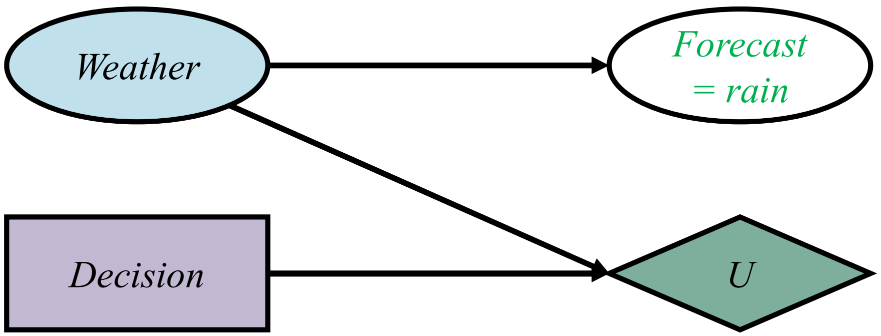
Decision Networks: Example

Decision: take umbrella given rain

$$EU(a \mid e) = \sum_{s'} P(Result(a) = s' \mid e) * U(s')$$

P(rain | F)	P(sun | F)
0.66	0.34

P(F=sun)	P(F=rain)
0.59	**0.41**



D	W	U
leave	sun	100
leave	rain	0
take	sun	20
take	rain	70

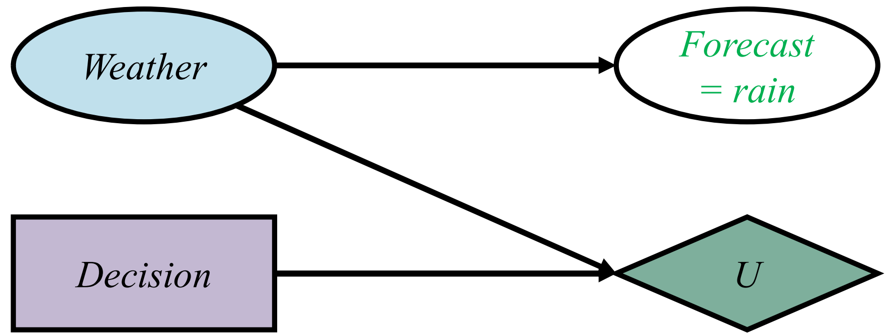
EU(take given rain forecast) = ???

Decision: leave umbrella given rain

$$EU(a \mid e) = \sum_{s'} P(Result(a) = s' \mid e) * U(s')$$

P(rain | F)	P(sun | F)
0.66	0.34

P(F=sun)	P(F=rain)
0.59	**0.41**



D	W	U
leave	sun	100
leave	rain	0
take	sun	20
take	rain	70

EU(leave given rain forecast) = ???

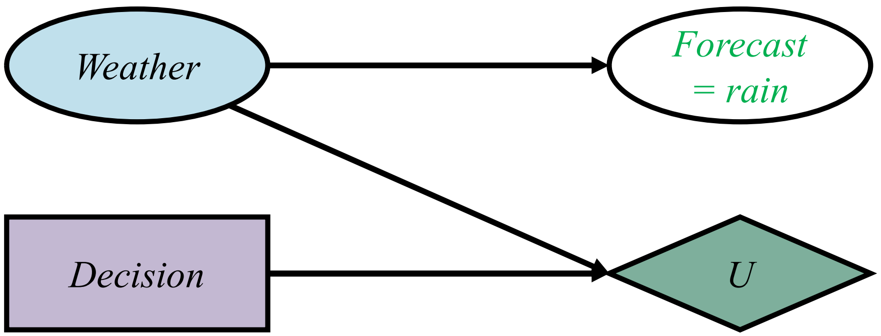
Decision Networks: Example

Decision: take umbrella given rain

$EU(a \mid e) = \sum_{s'} P(Result(a) = s' \mid e) * U(s')$

P(rain | F)	P(sun | F)
0.66	0.34

P(F=sun)	P(F=rain)
0.59	**0.41**



S_1': D = take, W = sun
S_2': D = take, W = rain
EU(take) =
P(Result(take)=S_1'|e)*U(S_1') +
P(Result(take)=S_2'|e)*U(S_2') =
0.34 * 20 + 0.66 * 70 = 53

D	W	U
leave	sun	100
leave	rain	0
take	sun	20
take	rain	70

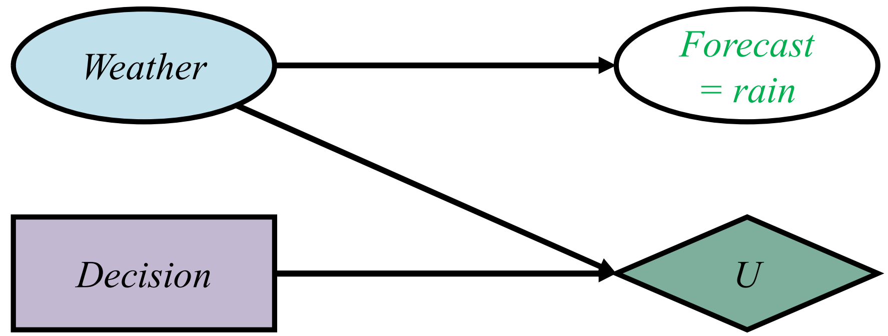
EU(take given rain forecast) = 53

Decision: leave umbrella given rain

$EU(a \mid e) = \sum_{s'} P(Result(a) = s' \mid e) * U(s')$

P(rain | F)	P(sun | F)
0.66	0.34

P(F=sun)	P(F=rain)
0.59	**0.41**



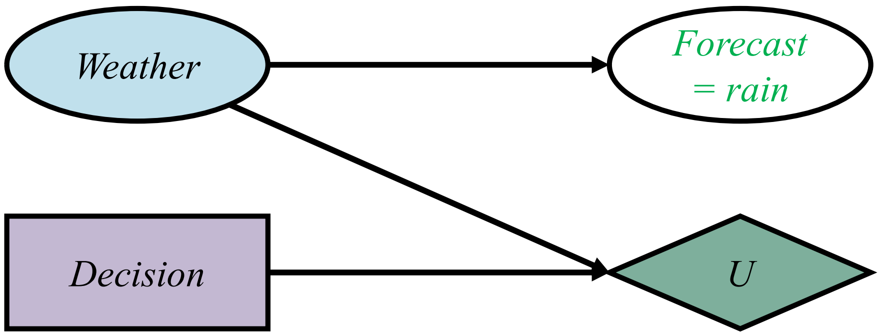
S_3': D = leave, W = sun
S_4': D = leave, W = rain
EU(leave) =
P(Result(leave)=S_3'|e)*U(S_3') +
P(Result(leave)=S_4'|e)*U(S_4') =
0.34 * 100 + 0.66 * 0 = 34

D	W	U
leave	sun	100
leave	rain	0
take	sun	20
take	rain	70

EU(leave given rain forecast) = 34

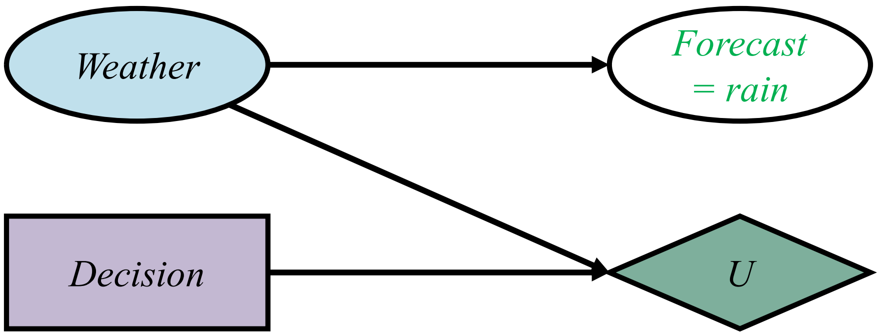
Decision Networks: Example

Decision: take umbrella given rain



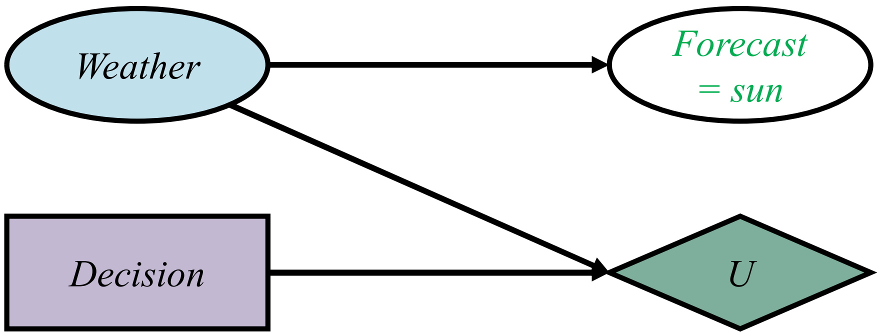
EU(take given rain forecast) = 53

Decision: leave umbrella given rain



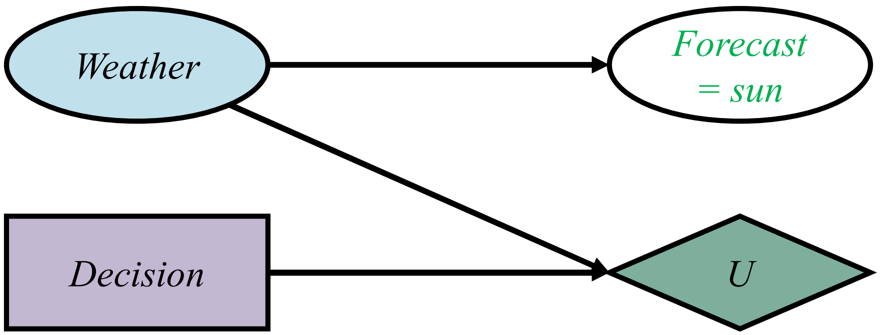
EU(leave given rain forecast) = 34

Decision: take umbrella given sun



EU(take given sun forecast) = 22.5

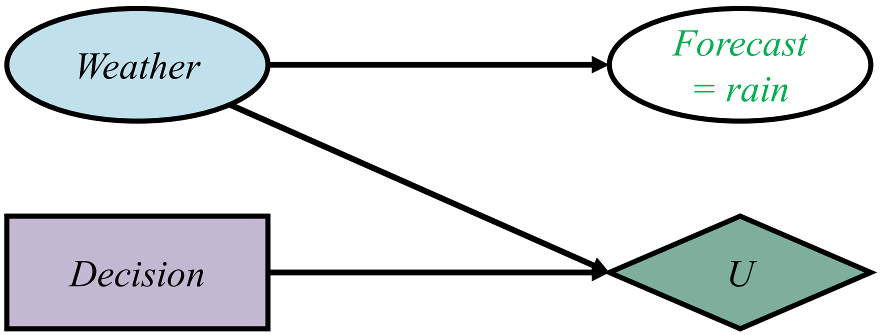
Decision: leave umbrella given sun



EU(leave given sun forecast) = 95

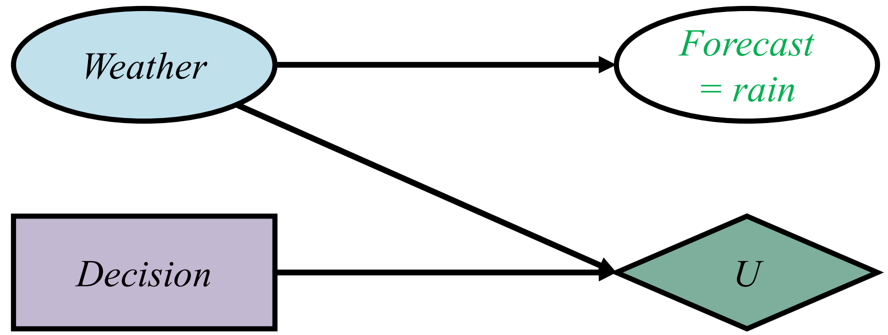
Decision Networks: Example

Decision:take umbrella given rain



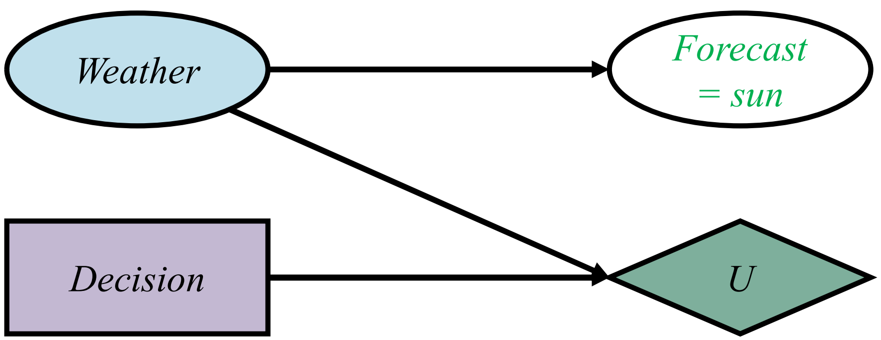
EU(take given rain forecast) = 53

Decision:leave umbrella given rain



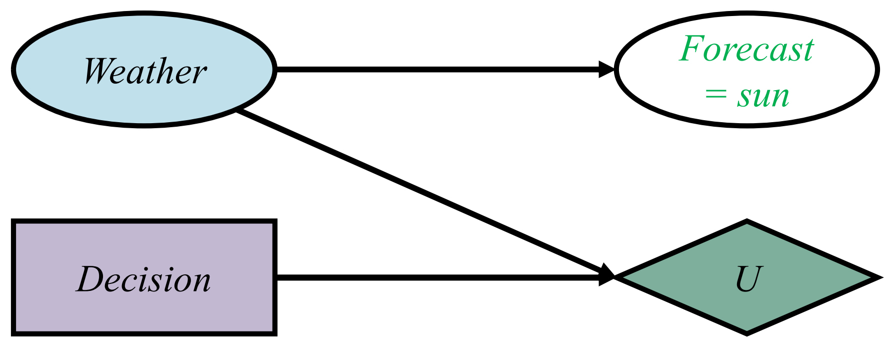
EU(leave given rain forecast) = 34

Decision:take umbrella given sun



EU(take given sun forecast) = 22.5

Decision:leave umbrella given sun



EU(leave given sun forecast) = 95

Value of Perfect Information

The value/utility of best action α without additional evidence (information) is :

$$\mathrm{MEU}(\alpha) = \max_{a} \sum_{s'} \mathrm{P}(Result(a) = s') * \mathrm{U}(s')$$

If we include new evidence/information ($\mathrm{E_j} = \mathrm{e_j}$) given by some variable Ej, value/utility of best action α becomes:

$$\mathrm{MEU}(a_{e_j} \mid e_j) = \max_{a} \sum_{s'} \mathrm{P}(Result(a) = s' \mid e_j) * \mathrm{U}(s')$$

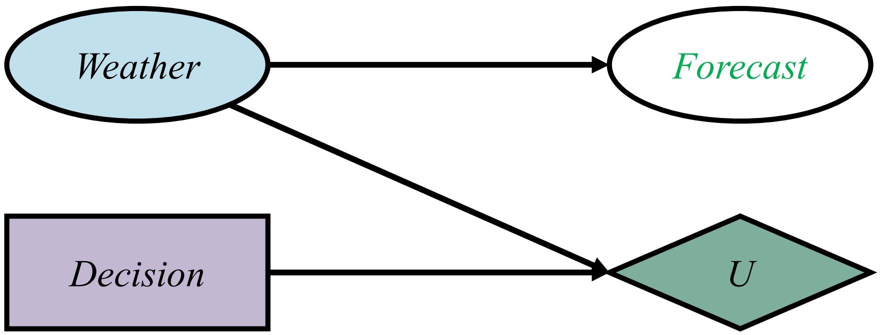
The value of additional evidence/information from Ej is:

$$\mathrm{VPI}(E_j) = \left(\sum_{e_j} \mathrm{P}(E_j = e_j) * \mathrm{MEU}(a_{e_j} \mid E_j = e_j) \right) - MEU(a)$$

using our current beliefs about the world.

Decision Network: Example

Decision network



The value of best action α without additional evidence

$$\text{MEU}(\alpha) = \text{MEU}(\text{leave}) = 70$$

With evidence information ($E_j = e_j$) given by Forecast:

$$\text{MEU}(a_{e_1} \mid e_1) = \text{MEU}(\text{take} \mid F = rain) = 53$$

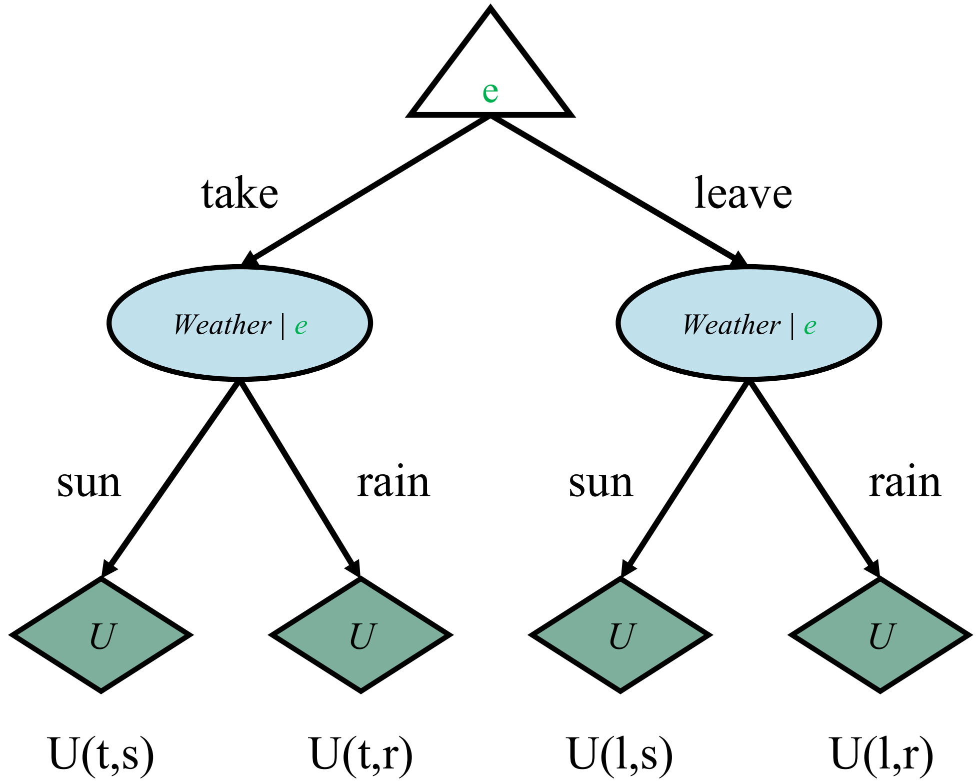
$$\text{MEU}(a_{e_2} \mid e_2) = \text{MEU}(\text{leave} \mid F = sun) = 95$$

The value of additional evidence / information from F is:

$$\text{VPI}(E_j) = \left(\sum_{e_j} \text{P}(E_j = e_j) * \text{MEU}(a_{e_j} \mid E_j = e_j)\right) - MEU(a)$$

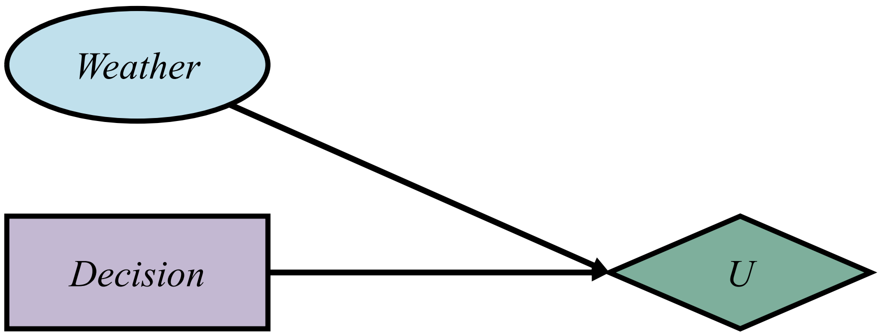
$$\text{VPI}(F) = (\text{P}(F = rain) * \text{MEU}(take \mid F = rain) + \text{P}(F = sun) * \text{MEU}(\text{leave} \mid F = sun)) - MEU(leave) = (0.41 * 53 + 0.59 * 95) - 70 = 7.78$$

Outcome tree



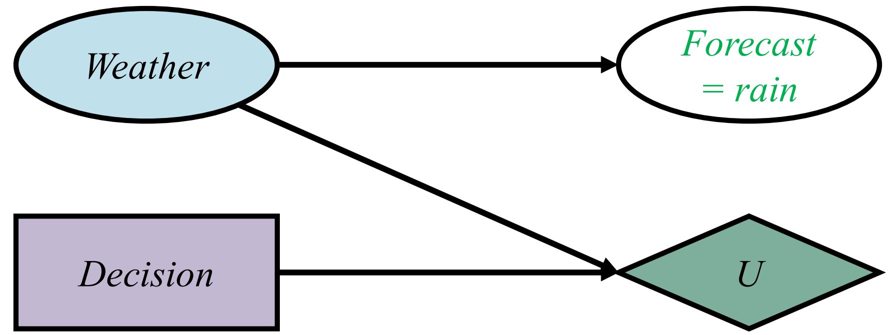
Decision Networks: Example

Decision: leave umbrella



$$EU(\text{leave}) = 70$$

Decision: take umbrella given rain



$$EU(\text{take given rain forecast}) = 53$$

The value of best action α without additional evidence

$$MEU(\alpha) = MEU(\text{leave}) = 70$$

With evidence information ($E_j = e_j$) given by Forecast:

$$MEU(a_{e_1} \mid e_1) = MEU(\text{take} \mid F = rain) = 53$$

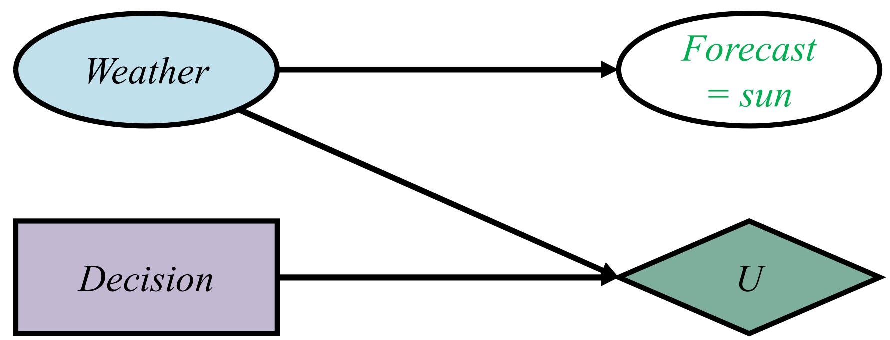
$$MEU(a_{e_2} \mid e_2) = MEU(\text{leave} \mid F = sun) = 95$$

The value of additional evidence / information from F is:

$$VPI(E_j) = \left(\sum_{e_j} P(E_j = e_j) * MEU(a_{e_j} \mid E_j = e_j)\right) - MEU(a)$$

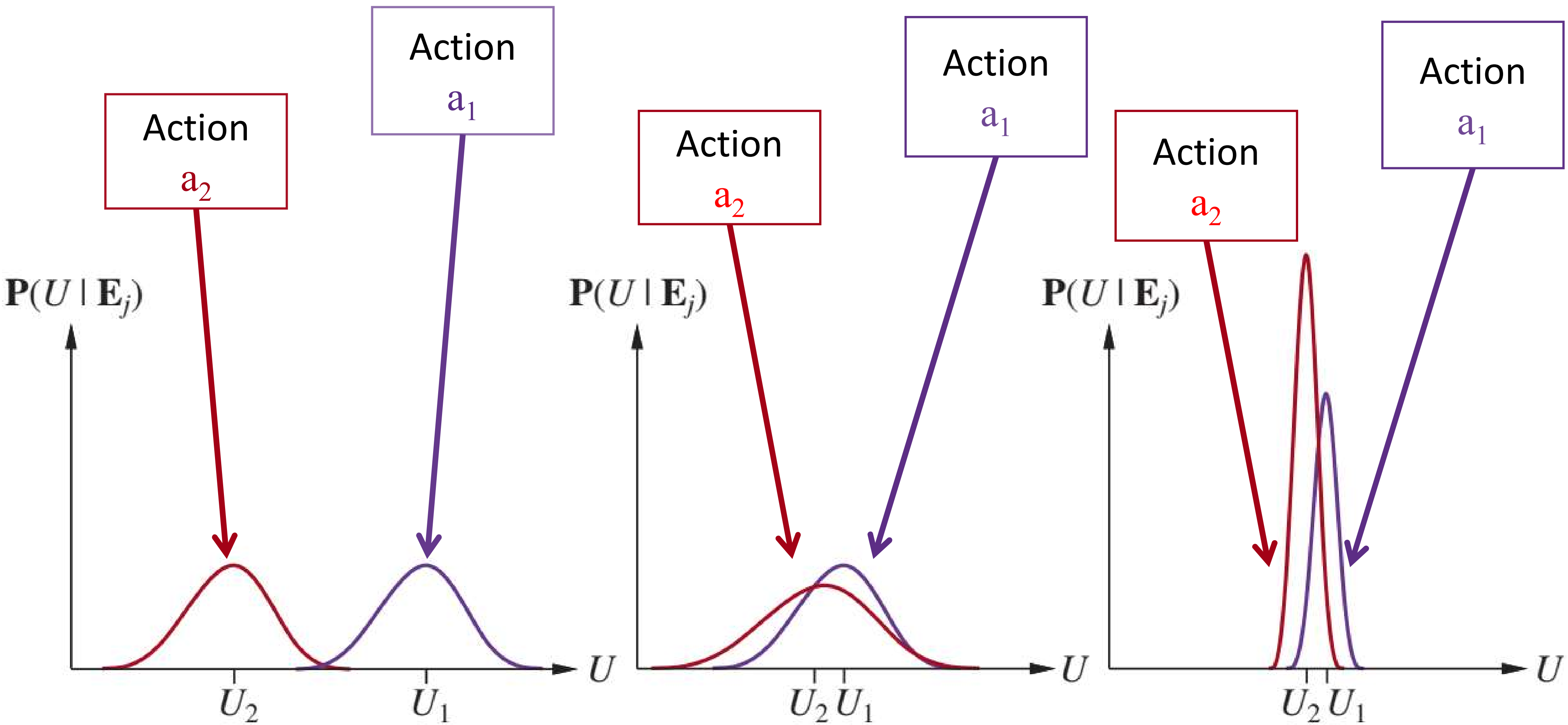
$$VPI(F) = (P(F = rain) * MEU(take \mid F = rain) + P(F = sun) * MEU(\text{leave} \mid F = sun)) - MEU(leave) = (0.41 * 53 + 0.59 * 95) - 70 = 7.78$$

Decision: leave umbrella given sun



$$EU(\text{leave given sun forecast}) = 95$$

Utility & Value of Perfect Information



New information will not help here.

New information may help a lot here.

New information may help a bit here.

VPI Properties

Given a decision network with possible observations $\mathrm{E_j}$ (sources of new information / evidence):

- **The expected value of information is nonnegative:**

$$\forall_{\mathrm{j}}\, \mathrm{VPI}(E_j) \geq 0$$

- **VPI is not additive:**

$$\mathrm{VPI}(E_j, E_k) \neq \mathrm{VPI}(E_j) + \mathrm{VPI}(E_k)$$

- **VPI is order-independent:**

$$\mathrm{VPI}(E_j,\ E_k) = VPI(E_j) + VPI(E_k \mid E_j) = VPI(E_k) + VPI(E_j \mid E_k) = VPI(E_k, E_j)$$

Information Gathering Agent

function INFORMATION-GATHERING-AGENT(*percept*) **returns** an *action*
  persistent: D, a decision network

  integrate *percept* into D
  $j \leftarrow$ the value that maximizes $VPI(E_j) \,/\, C(E_j)$
  if $VPI(E_j) > C(E_j)$
    then return $Request(E_j)$
  else return the best action from D