Name: Aman Kumar
Hawk ID: A20538809
CS 512

NAME :- AMAN KUMAR

HAWK ID :- A20538809

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PART-A (Vector Operations).

$$p = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}, \quad q = \begin{bmatrix} 0 \\ 3 \\ 5 \end{bmatrix}, \quad r = \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix}$$

Find,

i) $3p + 2q$

$$\Rightarrow 3\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 3 \\ 5 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} 6 \\ -3 \\ 12 \end{bmatrix} + \begin{bmatrix} 0 \\ 6 \\ 10 \end{bmatrix} \Rightarrow \begin{bmatrix} 6 \\ 3 \\ 22 \end{bmatrix}$$

2, $\hat{p} = \dfrac{p}{\|p\|}$,

$$\|p\| = \sqrt{2^2+(-1)^2+4^2} = \sqrt{4+1+16} = \sqrt{21}$$

$$\hat{p} = \frac{1}{\sqrt{21}}\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{21} \\ -1/\sqrt{21} \\ 4/\sqrt{21} \end{bmatrix}$$

3, $\|p\| = \sqrt{2^2+(-1)^2+4^2} = \sqrt{4+1+16} = \sqrt{21}$

$$\cos\theta = \frac{p \cdot j}{\|p\|} = \frac{2\times 0 + (-1)\times 1 + (4)\times 0}{\sqrt{21}} = \frac{-1}{\sqrt{21}}$$

$$\theta = \cos^{-1}\left(\frac{-1}{\sqrt{21}}\right).$$

4, $\cos\alpha = \dfrac{x}{\|p\|} = \dfrac{2}{\sqrt{21}}$; $\cos\beta = \dfrac{y}{\|p\|} = \dfrac{-1}{\sqrt{21}}$

$$\cos\gamma = \frac{z}{\|p\|} = \frac{4}{\sqrt{21}}$$

5, $\cos\phi = \dfrac{p \cdot q}{\|p\|\ \|q\|}$

$$\|q\| = \sqrt{0^2+3^2+5^2} = \sqrt{34}$$

$$\cos\phi = \frac{p\cdot q}{\|p\|\,\|q\|} = \frac{(2\cdot 0)+(-1\cdot 3)+(4\cdot 5)}{\sqrt{21}\,\sqrt{34}} = \frac{-3+20}{\sqrt{21}\,\sqrt{34}}$$

$$\cos\phi = \frac{17}{\sqrt{21}\,\sqrt{34}} \quad ; \quad \phi = \cos^{-1}\left(\frac{17}{\sqrt{21}\,\sqrt{34}}\right).$$

5, $$\cancel{p\cdot q = \begin{bmatrix}2\\-1\\4\end{bmatrix}\begin{bmatrix}0\\3\\5\end{bmatrix} = \begin{bmatrix}0\\-3\\20\end{bmatrix} = q\cdot p}.$$

6, $$p\cdot q = (2\cdot 0)+(-1\cdot 3)+(4\cdot 5) = 0-3+20 = 17 = q\cdot p.$$

7, $$p\cdot q = \|p\|\,\|q\|\cdot\cos\phi$$
$$= \sqrt{21}\cdot\sqrt{34}\cdot\frac{17}{\sqrt{21}\,\sqrt{34}} = 17$$

8, $$\text{scalar projection} = \frac{p\cdot q}{\|p\|} = \frac{17}{\sqrt{21}}$$

9, $$\text{perpendicular to } p = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ 2 & -1 & 4\\ 1 & 0 & 0\end{vmatrix}$$
$$= p\times[1,0,0]$$
$$= \hat{i}(0-0) - \hat{j}(0-4) + \hat{k}(0+1)$$
$$= 0\hat{i} + 4\hat{j} + \hat{k}$$

10, $p \times q = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & -1 & 4 \\ 0 & 3 & 5 \end{vmatrix}$

$\hat{i}\,(-1 \cdot 5 - (4 \cdot 3)) - \hat{j}\,(2 \times 5 - 0 \times 4) + \hat{k}\,(2 \times 3 - 0 \times -1)$

$\hat{i}\,(-5 - 12) - \hat{j}\,(10) + \hat{k}\,(6)$

$-17\hat{i} - 10\hat{j} + 6\hat{k}$.

$\Rightarrow q \times p = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 0 & 3 & 5 \\ 2 & -1 & 4 \end{vmatrix}$

$\hat{i}\,(3 \times 4 - (5 \times -1)) - \hat{j}\,(0 \times 4 - 5 \times 2) + \hat{k}\,(0 \times -1 - (3 \times 2))$

$\hat{i}\,(12 + 5) - \hat{j}\,(0 - 10) + \hat{k}\,(0 - 6)$.

$17\hat{i} + 10\hat{j} - 6\hat{k}$.

11, vector perpendicular to both p & q is $p \times q$

$\therefore \quad p \times q = \begin{bmatrix} -17 \\ -10 \\ 6 \end{bmatrix}$

12,

$$c_1 p + c_2 q + c_3 r = 0$$

$$c_1 \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 3 \\ 5 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ -2 \\ 2 \end{bmatrix} = 0$$

$$\cancel{2c_1 + 0c_2 + 1c_3} = 2c_1 + c_3 = 0$$

$$-c_1 + 3c_2 - 2c_3 = 0$$

$$4c_1 + 5c_2 + 2c_3 = 0$$

$$\begin{bmatrix} 2 & 0 & 1 \\ -1 & 3 & -2 \\ 4 & 5 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

determinant of matrix

$$2 \begin{vmatrix} 3 & -2 \\ 5 & 2 \end{vmatrix} - 0 \begin{vmatrix} -1 & -2 \\ 4 & 2 \end{vmatrix} + 1 \begin{vmatrix} -1 & 3 \\ 4 & 5 \end{vmatrix}$$

$$2\left[6 - (-10)\right] - 0 + 1\left[-5 - 12\right]$$

$$2\left[16\right] - 17 = 32 - 17 = 15$$

as,

$\det(\text{matrix}) \neq 0$

so, p, q, r are linearly independent

13, $p^T q$ and pq^T.

$$p^T q = \begin{bmatrix} 2 & -1 & 4 \end{bmatrix}_{1\times3} \begin{bmatrix} 0 \\ 3 \\ 5 \end{bmatrix}_{3\times1} = 2\times0 - 1\times3 + 4\times5$$

$$= \begin{bmatrix} 17 \end{bmatrix}$$

$$pq^T = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}_{3\times1} \begin{bmatrix} 0 & 3 & 5 \end{bmatrix}_{1\times3}$$

$$= \begin{bmatrix} 0 & 6 & 10 \\ 0 & -3 & -5 \\ 0 & 12 & 20 \end{bmatrix}$$

<u>Part-B</u> <u>(Matrix Operations)</u>.

$$X = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 3 & 4 \\ 3 & 2 & -2 \end{bmatrix}, \quad Y = \begin{bmatrix} 4 & -1 & 2 \\ 3 & 0 & -3 \\ 1 & 2 & 1 \end{bmatrix}$$

$$Z = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 4 & 5 \\ 3 & 1 & 2 \end{bmatrix}, \quad S = \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}$$

1, $X + 2Y = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 3 & 4 \\ 3 & 2 & -2 \end{bmatrix} + 2\begin{bmatrix} 4 & -1 & 2 \\ 3 & 0 & -3 \\ 1 & 2 & 1 \end{bmatrix}$

$= \begin{bmatrix} 2 & 1 & 0 \\ -1 & 3 & 4 \\ 3 & 2 & -2 \end{bmatrix} + \begin{bmatrix} 8 & -2 & 4 \\ 6 & 0 & -6 \\ 2 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 10 & -1 & 4 \\ 5 & 3 & -2 \\ 5 & 6 & 0 \end{bmatrix}$

2, $X \cdot Y = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 3 & 4 \\ 3 & 2 & -2 \end{bmatrix} \cdot \begin{bmatrix} 4 & -1 & 2 \\ 3 & 0 & -3 \\ 1 & 2 & 1 \end{bmatrix}$

$= \begin{bmatrix} 8+3+0 & -2+0+0 & 4-3+0 \\ -4+9+4 & 1+0+8 & -2-9+4 \\ 12+6-2 & -3+0-4 & 6-6-2 \end{bmatrix}$

$= \begin{bmatrix} 11 & -2 & 1 \\ 9 & 9 & -7 \\ 16 & -7 & -2 \end{bmatrix}$

$$Y \cdot X = \begin{bmatrix} 4 & -1 & 2 \\ 3 & 0 & -3 \\ 1 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 & 0 \\ -1 & 3 & 4 \\ 3 & 2 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 8+1+6 & 4-3+4 & 0-4-4 \\ 6+0-9 & 3+0-6 & 0+0+6 \\ 2-2+3 & 1+6+2 & 0+8-2 \end{bmatrix}$$

$$= \begin{bmatrix} 15 & 5 & -8 \\ -3 & -3 & 6 \\ 3 & 9 & 6 \end{bmatrix}$$

3, $$(XY)^T = \begin{bmatrix} 11 & -2 & 1 \\ 9 & 9 & -7 \\ 16 & -7 & -2 \end{bmatrix}^T$$

$$= \begin{bmatrix} 11 & 9 & 16 \\ -2 & 9 & -7 \\ 1 & -7 & -2 \end{bmatrix}$$

$$Y^T \cdot X^T = \begin{bmatrix} 4 & 3 & 1 \\ -1 & 0 & 2 \\ 2 & -3 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 3 \\ 1 & 3 & 2 \\ 0 & 4 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} 8+3+0 & -4+9+4 & 12+6-2 \\ -2+0+0 & 1+0+8 & -3+0-4 \\ 4-3+0 & -2-9+4 & 6-6-2 \end{bmatrix}$$

$$= \begin{bmatrix} 11 & 9 & 16 \\ -2 & 9 & -7 \\ 1 & -7 & -2 \end{bmatrix}$$

4

$$|X| = 2 \begin{vmatrix} 3 & 4 \\ 2 & -2 \end{vmatrix} - 1 \begin{vmatrix} -1 & 4 \\ 3 & -2 \end{vmatrix} + 0 \begin{vmatrix} -1 & 3 \\ 3 & 2 \end{vmatrix}$$

$$= 2(-6-8) - 1(2-12) + 0$$

$$= 2(-14) - 1(-10)$$

$$= -28 + 10 = -18$$

$$|Z| = 2 \begin{vmatrix} 4 & 5 \\ 1 & 2 \end{vmatrix} - 0 \begin{vmatrix} 1 & 5 \\ 3 & 2 \end{vmatrix} + (-1) \begin{vmatrix} 1 & 4 \\ 3 & 1 \end{vmatrix}$$

$$= 2(8-5) - 0 + (-1)(1-12)$$

$$= 2 \times 3 + 0 - 1 \times (-11)$$

$$= 6 + 11 = 17.$$

5, for matrix X, $X = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 3 & 4 \\ 3 & 2 & -2 \end{bmatrix}$

row 1 · row 2 $= (2 \cdot -1) + (1 \cdot 3) + (0 \cdot 4) = -2 + 3 = 1$

" 1 · row 3 $= (2 \cdot 3) + (1 \cdot 2) + (0 \cdot -2) = 6 + 2 = 8$

row 2 · row 3 $= (-1 \cdot 3) + (3 \cdot 2) + (4 \cdot -2) = -3 + 6 - 8 = -5$,

dot products are not 0, so, row vectors of X are not orthogonal.

for matrix Y, $Y = \begin{bmatrix} 4 & -1 & 2 \\ 3 & 0 & -3 \\ 1 & 2 & 1 \end{bmatrix}$

row 1 · row 2 $= (4 \cdot 3) + (-1 \cdot 0) + (2 \cdot -3)$

$= 12 - 0 - 6 = 6$

row 1 · row 3 $= (4 \cdot 1) + (-1 \cdot 2) + (2 \cdot 1)$

$= 4 - 2 + 2 = 4$

row 2 · row 3 $= (3 \cdot 1) + (0 \cdot 2) + (-3 \cdot 1)$

$= 3 + 0 - 3 = 0$.

dot products are not all 0 for matrix Y, so, row vectors of Y are not orthogonal.

for matrix Z, $Z = \begin{bmatrix} 2 & 0 & -1 \\ 1 & 4 & 5 \\ 3 & 1 & 2 \end{bmatrix}$

row 1 · row 2 = $(2\cdot1)+(0\cdot4)+(-1\cdot5) = 2-5 = -3$

row 1 · row 3 = $(2\cdot3)+(0\cdot1)+(-1\cdot2) = 6-2 = 4$

row 2 · row 3 = $(1\cdot3)+(4\cdot1)+(5\cdot2) = 3+4+5 = 12$

dot products are not all 0, so matrix Y is not orthogonal.

6, X^{-1} and Y^{-1}

$$X^{-1} = \frac{1}{\det X}[X]^T = \frac{1}{-18}\begin{bmatrix} 2 & -1 & 3 \\ 1 & 3 & 2 \\ 0 & 4 & -2 \end{bmatrix}$$

$$= \begin{bmatrix} -2/18 & 1/18 & -3/18 \\ -1/18 & -3/18 & -2/18 \\ 0 & -4/18 & 2/18 \end{bmatrix}$$

$$Y^{-1} = \frac{1}{\det Y}[Y]^T \Rightarrow \det Y = 4\begin{vmatrix} 0 & -3 \\ 2 & 1 \end{vmatrix} - (-1)\begin{vmatrix} 3 & -3 \\ 1 & 1 \end{vmatrix}$$

$$+ 2\begin{vmatrix} 3 & 0 \\ 1 & 2 \end{vmatrix}$$

$= 4(0+6) + 1(3+3) + 2(6) = 24+6+12 = 42.$

$$= \frac{1}{42}\begin{bmatrix} 4 & 3 & 1 \\ -1 & 0 & 2 \\ 2 & -3 & 1 \end{bmatrix} = \begin{bmatrix} 4/42 & 3/42 & 1/42 \\ -1/42 & 0 & 2/42 \\ 2/42 & -3/42 & 1/42 \end{bmatrix}$$

7, $$Z^{-1} = \frac{1}{\det Z}[Z]^T = \frac{1}{17}\begin{bmatrix} 2 & 1 & 3 \\ 0 & 4 & 1 \\ -1 & 5 & 2 \end{bmatrix}$$

$$= \begin{bmatrix} 2/17 & 1/17 & 3/17 \\ 0 & 4/17 & 1/17 \\ -1/17 & 5/17 & 2/17 \end{bmatrix}$$

8, product $$X_s = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 3 & 4 \\ 3 & 2 & -2 \end{bmatrix}_{3\times 3}\begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}_{3\times 1}$$

$$= \begin{bmatrix} -2+4+0 \\ 1+12+0 \\ -3+8+0 \end{bmatrix} = \begin{bmatrix} 2 \\ 13 \\ 5 \end{bmatrix}$$

9) Scalar projection is $\frac{x \cdot s}{\|s\|}$ or $x \cdot s_{norm}$

where $s_{norm} = \frac{s}{\|s\|}$.

$\|s\| = \sqrt{-1^2 + 4^2 + 0} = \sqrt{17}$.

$$x \cdot s = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 3 & 4 \\ 3 & 2 & -2 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} -2+4+0 \\ 1+12+0 \\ -3+8+0 \end{bmatrix} = \begin{bmatrix} 2 \\ 13 \\ 5 \end{bmatrix}$$

$$\text{Scalar projection} = \frac{1}{\sqrt{17}} \begin{bmatrix} 2 \\ 13 \\ 5 \end{bmatrix} = \begin{bmatrix} 2/\sqrt{17} \\ 13/\sqrt{17} \\ 5/\sqrt{17} \end{bmatrix}$$

10) vector projection = scalar projection × s_{norm}:

$$\cancel{\text{row } 1 = 2/\sqrt{17} \times 1/\sqrt{17} \times (-1) = -2/17}$$

$$\cancel{\text{" } 2 = 13/\sqrt{17} \times 1/\sqrt{17} \times (4) = 52/17}$$

$$\text{vector projection for row } 1 = 2/\sqrt{17} \times 1/\sqrt{17} \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix} = 2/17 \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -2/17 \\ 8/17 \\ 0 \end{bmatrix}$$

vector projection for row 2 $= \frac{13}{\sqrt{17}} \times \frac{1}{\sqrt{17}} \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}$

$$= \frac{13}{17} \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} -13/17 \\ 52/17 \\ 0 \end{bmatrix}$$

vector projection for row 3 $= \frac{5}{\sqrt{17}} \times \frac{1}{\sqrt{17}} \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}$

$$= \frac{5}{17} \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} -5/17 \\ 20/17 \\ 0 \end{bmatrix}$$

!! linear combination $= X \cdot S = \begin{bmatrix} 2 & 1 & 0 \\ -1 & 3 & 4 \\ 3 & 2 & -2 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}$

$$= \begin{bmatrix} -2+4+0 \\ 1+12+0 \\ -3+8+0 \end{bmatrix} = \begin{bmatrix} 2 \\ 13 \\ 5 \end{bmatrix}$$

12) $Yt = s$.

$t = Y^{-1} \cdot s$

$$t = \frac{1}{42}\begin{bmatrix} 4 & 3 & 1 \\ -1 & 0 & 2 \\ 2 & -3 & 1 \end{bmatrix}_{3\times 3} \cdot \begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix}_{3\times 1}$$

$$= \frac{1}{42}\begin{bmatrix} -4+12+0 \\ 1+0+0 \\ -2-12+0 \end{bmatrix} = \frac{1}{42}\begin{bmatrix} 8 \\ 1 \\ -14 \end{bmatrix}$$

$$= \begin{bmatrix} 8/42 \\ 1/42 \\ -14/42 \end{bmatrix}$$

13) $Zt = s$, $t = Z^{-1} s$.

$$t = \frac{1}{17}\begin{bmatrix} 2 & 1 & 3 \\ 0 & 4 & 1 \\ -1 & 5 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 4 \\ 0 \end{bmatrix} = \frac{1}{17}\begin{bmatrix} -2+4+0 \\ 0+16+0 \\ 1+20+0 \end{bmatrix}$$

$$t = \begin{bmatrix} 2/17 \\ 16/17 \\ 21/17 \end{bmatrix}$$

, as, Z is non-singular ~~the~~ ie.. $\det(Z) \neq 0$.
so, it has unique solution.

Part C :- Eigen values & eigen vectors.

$$M = \begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix}, \quad N = \begin{bmatrix} 5 & -3 \\ -3 & 6 \end{bmatrix}, \quad P = \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix}$$

i) eigen values $= \det(M - \lambda I) = \det\left(\begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$

$$= \det\left(\begin{bmatrix} 3-\lambda & 2 \\ -1 & 4-\lambda \end{bmatrix}\right)$$

$$0 = (3-\lambda)(4-\lambda) - (-2)$$

$$(3-\lambda)(4-\lambda) + 2 = 0$$

$$12 - 3\lambda - 4\lambda + \lambda^2 + 2 = 0$$

$$\lambda^2 - 7\lambda + 14 = 0.$$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{7 \pm \sqrt{49 - 4 \times 1 \times 14}}{2 \times 1} = \frac{7 \pm \sqrt{49 - 56}}{2}$$

$$\lambda = \frac{7 \pm \sqrt{7}i}{2} = \frac{7 + \sqrt{7}i}{2}, \quad \frac{7 - \sqrt{7}i}{2},$$

eigen vectors $= \begin{bmatrix} 3-\lambda & 2 \\ -1 & 4-\lambda \end{bmatrix} = \begin{bmatrix} 3 - \left(\frac{7+\sqrt{7}i}{2}\right) & 2 \\ -1 & 4 - \left(\frac{7+\sqrt{7}i}{2}\right) \end{bmatrix}$

$$= \begin{bmatrix} 3 - \left(\frac{7-\sqrt{7}i}{2}\right) & 2 \\ -1 & 4 - \left(\frac{7-\sqrt{7}i}{2}\right) \end{bmatrix}$$

2,

$$\begin{bmatrix} 3-\left(\frac{7+\sqrt{7}i}{2}\right) & 2 \\ -1 & 4-\left(\frac{7+\sqrt{7}i}{2}\right) \end{bmatrix} \begin{bmatrix} 3-\left(\frac{7-\sqrt{7}i}{2}\right) & 2 \\ -1 & 4-\left(\frac{7-\sqrt{7}i}{2}\right) \end{bmatrix}$$

$$= \begin{bmatrix} \left[3-\left(\frac{7+\sqrt{7}i}{2}\right)\right]\left[3-\left(\frac{7-\sqrt{7}i}{2}\right)\right] & 2\times 2 \\ -1\times -1 & \left\{4-\left(\frac{7+\sqrt{7}i}{2}\right)\right\}\left\{4-\left(\frac{7-\sqrt{7}i}{2}\right)\right\} \end{bmatrix}$$

$$= \begin{bmatrix} 9-3\left(\frac{7-\sqrt{7}i}{2}\right)-3\left(\frac{7+\sqrt{7}i}{2}\right)+\left(\frac{49-7i^2}{4}\right) & 4 \\ 1 & 16-4\left(\frac{7-\sqrt{7}i}{2}\right)-4\left(\frac{7+\sqrt{7}i}{2}\right) + \left(\frac{49-7i^2}{4}\right) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{36-6(7-\sqrt{7}i)-6(7+\sqrt{7}i)+(49-7i^2)}{4} & 4 \\ 1 & \frac{64-8(7-\sqrt{7}i)-8(7+\sqrt{7}i)+49-7i^2}{4} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{36-42+\cancel{6\sqrt{7}i}-42-\cancel{6\sqrt{7}i}+49+7}{4} & 4 \\ 1 & \frac{64-56+\cancel{8\sqrt{7}i}-56-\cancel{8\sqrt{7}i}+49+7}{4} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{36-42-42+49+7}{4} & 4 \\ 1 & \frac{64-56-56+56}{4} \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 4 \\ 1 & 2 \end{bmatrix}$$

3, eigen values of $N \Rightarrow \det(N - \lambda I) = 0$

$$= \begin{bmatrix} 5 & -3 \\ -3 & 6 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 5-\lambda & -3 \\ -3 & 6-\lambda \end{bmatrix}$$

$$= (5-\lambda)(6-\lambda) - (-3 \times -3)$$

$$= 30 - 5\lambda - 6\lambda + \lambda^2 - 9$$

$$= \lambda^2 - 11\lambda + 21$$

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{11 \pm \sqrt{121 - 4 \times 21}}{2}$$

$$\lambda = \frac{11 \pm \sqrt{121 - 84}}{2} = \frac{11 \pm \sqrt{37}}{2}$$

$$\lambda_1 = \frac{11 + \sqrt{37}}{2}, \quad \lambda_2 = \frac{11 - \sqrt{37}}{2}$$

$$\text{eigen vector}_1 = \begin{bmatrix} 5 - \left(\frac{11+\sqrt{37}}{2}\right) & -3 \\ -3 & 6 - \frac{(11+\sqrt{37})}{2} \end{bmatrix}$$

$$\text{eigen vector}_2 = \begin{bmatrix} 5 - \left(\frac{11-\sqrt{37}}{2}\right) & -3 \\ -3 & 6 - \left(\frac{11-\sqrt{37}}{2}\right) \end{bmatrix}$$

$$\text{dot product} = \begin{bmatrix} 5 - \left(\frac{11+\sqrt{37}}{2}\right) & -3 \\ -3 & \frac{6-(11+\sqrt{37})}{2} \end{bmatrix} \begin{bmatrix} 5 - \left(\frac{11-\sqrt{37}}{2}\right) & -3 \\ -3 & 6 - \left(\frac{11-\sqrt{37}}{2}\right) \end{bmatrix}$$

$$= \begin{bmatrix} \frac{10-11-\sqrt{37}}{2} & -3 \\ -3 & \frac{12-11-\sqrt{37}}{2} \end{bmatrix} \begin{bmatrix} \frac{10-11+\sqrt{37}}{2} & -3 \\ -3 & \frac{12-11+\sqrt{37}}{2} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{-1-\sqrt{37}}{2} & -3 \\ -3 & \frac{1-\sqrt{37}}{2} \end{bmatrix} \begin{bmatrix} \frac{-1+\sqrt{37}}{2} & -3 \\ -3 & \frac{1+\sqrt{37}}{2} \end{bmatrix}$$

$$= \begin{bmatrix} -\left(\frac{37-1}{2}\right) & 9 \\ 9 & \frac{1-37}{2} \end{bmatrix} = \begin{bmatrix} -18 & 9 \\ 9 & -18 \end{bmatrix}$$

4, orthogonal, because of symmetric matrix.

5, $Pt = 0$

then, $t = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

6, non-trivial solutions for $Pt = 0$.

eigen values of $P = P - \lambda I = \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$= \begin{bmatrix} 2-\lambda & 4 \\ 4 & 8-\lambda \end{bmatrix}$

determinant $= (2-\lambda)(8-\lambda) - 16$

$= \cancel{16} - 2\lambda - 8\lambda + \lambda^2 - \cancel{16}$

$= \lambda^2 - 10\lambda$

$\lambda(\lambda - 10) = 0$

$\lambda = 0, 10.$

$$\begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$2x + 4y = 0$ & $4x + 8y = 0$,

$\therefore \quad x = -2y,$

if, $y = 1$, $x = -2$
& if, $y = 2$, $x = -4$

1st solution $= \begin{bmatrix} -2 \\ 1 \end{bmatrix}$, 2nd solution $= \begin{bmatrix} -4 \\ 2 \end{bmatrix}$

7, $\begin{bmatrix} 3 & 2 \\ -1 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

for, $Mt = 0$.

$3x + 2y = 0$ — 1, & $-x + 4y = 0$,
$x = 4y$ — 2,

putting the value of eq^n 2, in eq^n 1, we get.

$3(4y) + 2y = 0$,
$\therefore 14y = 0$,
$y = 0$

putting the value of y in eq^n 1, we get
$x = 0$

so, the only solution is $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$

reason being, determinant of M is non-zero. which means matrix M has full rank.

Part D :- Neural Network Basics

1, $x_1 = 0.5,\ x_2 = -0.8,\ x_3 = 0.3$

$w_1 = 0.4,\ w_2 = -0.6,\ w_3 = 0.9$

$b = 0.2$

$$\text{output} = \left[\sum_{i=1}^{3} x_i * w_i\right] + b$$

$$= x_1 \times w_1 + x_2 \times w_2 + x_3 \times w_3 + b$$

$$= 0.5 \times 0.4 + (-0.8 \times -0.6) + (0.3 \times 0.9) + 0.2$$

$$= 0.2 + 0.48 + 0.27 + 0.2$$

$$= 1.15$$

2, $$\text{Sigmoid} = \frac{1}{1+e^{-\text{output}}} = \frac{1}{1+e^{-1.15}}$$

$$= \frac{1}{1+0.3166} = \frac{1}{1.3166} = 0.7595$$

3, $$\text{relu} = \max(0, \text{output})$$

$$= \max(0, 1.15)$$

$$= 1.15$$

4

$$w_h = \begin{bmatrix} 0.4 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}_{2\times2}, \quad b_h = \begin{bmatrix} 0.1 \\ -0.1 \end{bmatrix}_{2\times1}, \quad x = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$z = \cancel{w_h \cdot b} \; w_h \cdot x + b_h$$

$$= \begin{bmatrix} 0.4 & 0.3 \\ 0.2 & 0.7 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 0.1 \\ -0.1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.4 + 0.3\times2 \\ 0.2 + 0.7\times2 \end{bmatrix} + \begin{bmatrix} 0.1 \\ -0.1 \end{bmatrix}$$

$$= \begin{bmatrix} 0.4 + 0.6 \\ 0.2 + 1.4 \end{bmatrix} + \begin{bmatrix} 0.1 \\ -0.1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1.6 \end{bmatrix} + \begin{bmatrix} 0.1 \\ -0.1 \end{bmatrix}$$

$$= \begin{bmatrix} 1.1 \\ 1.5 \end{bmatrix}$$

$$\text{relu} = \max(0, z) = \cancel{\begin{bmatrix} 1.1 \\ 1.5 \end{bmatrix}}$$

$$= \begin{bmatrix} \max(0, 1.1) \\ \max(0, 1.5) \end{bmatrix} = \begin{bmatrix} 1.1 \\ 1.5 \end{bmatrix}$$

5, $w_0 = [0.5 \quad -0.3]$, $b_0 = [0.1]$

$$z\text{-output} = w_0 \times z + b_0$$

$$= \underset{1\times 2}{[0.5 \quad -0.3]} \underset{2\times 1}{\begin{bmatrix} 1.1 \\ 1.5 \end{bmatrix}} + [0.1]$$

$$= [1.1 \times 0.5 + (-0.3 \times 1.5)] + 0.1$$

$$= 0.55 - 0.45 + 0.1 = 0.2$$

$$\text{Sigmoid-output} = \frac{1}{1+e^{-(z)}} = \frac{1}{1+e^{-0.2}}$$

$$= \frac{1}{1+0.818} = \frac{1}{1.818} = 0.549$$

6, Gradient loss L w.r.t input components x_1 & x_2.

$$\frac{\partial L}{\partial x_1} = \frac{\partial L}{\partial y} \cdot \frac{\partial y}{\partial z} \cdot \frac{\partial z}{\partial x_1}$$

$$\frac{\partial L}{\partial x_2} = \frac{\partial L}{\partial y} \cdot \frac{\partial y}{\partial z} \cdot \frac{\partial z}{\partial x_2}$$

$\frac{\partial L}{\partial y}$:- gradient of loss w.r.t output.

$\frac{\partial y}{\partial z}$ = derivative of activation f^n used in o/p.

$\frac{\partial z}{\partial x_1}$ & $\frac{\partial z}{\partial x_2}$ = gradient of o/p w.r.t input components

Part-E :- Gradient Calculus :-

1, $f(x) = 2x^2 - 1$, $g(x) = 3x^2 + 4$, $h(x, y) = x^2 + y^2 + xy$.

$$f'(x) = \frac{d}{dx}(2x^2 - 1) = 4x.$$

$$f''(x) = \frac{d^2}{d^2x}(4x) = 4.$$

2, $$\frac{\partial h}{\partial x} = \frac{\partial}{\partial x}(x^2 + y^2 + xy) = 2x + y$$

$$\frac{\partial h}{\partial y} = \frac{\partial}{\partial y}(x^2 + y^2 + xy) = 2y + x.$$

3, $\nabla h(x,y) = \left(\frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}\right)$

$= (2x+y, 2y+x).$

4, $\frac{d}{dx} f(g(x))$

first calculate, $f(g(x))$

$= f(3x^2+4)$

$= 2[3x^2+4]^2 - 1$

$= 2[9x^4 + 16 + 24x^2] - 1$

$= 18x^4 + 32 + 48x^2 - 1$

$= 18x^4 + 48x^2 + 31$

$\frac{d}{dx}[18x^4 + 48x^2 + 31]$

$= 72x^3 + 96x.$