Adam - Bashforth Method $\frac{dy}{dx} = f(x,y)$

- find $y(x_0), y(x_1), y(x_2), y(x_3)$ using say Euler or R-K.
- Predictor: $y(x_4) = y_3 + \frac{h}{24}[55f_3 - 59f_2 + 37f_1 - 9f_0]$

$\boxed{f_k = f(x_k, y_k)}$

- Corrector

$y(x_4) = y_3 + \frac{h}{24}[9f_4 + 19f_3 - 5f_2 + f_1]$

till converges

$\rightarrow y_{i+1} = y_i + h\left[f_i + \frac{1}{2}\nabla f_i + \frac{5}{12}\nabla^2 f_i + \frac{3}{8}\nabla^3 f_i \dots\right]$

EULER'S METHOD, $\frac{dy}{dx} = f(x,y)$

$y(x_0) = y_0$

$y_{i+1} = y_i + h\, f(x_i, y_i) \quad i = 0 \dots \left[\frac{b-a}{h}\right]$

h ↳ steps

Runge - Kutta Method $\frac{dy}{dx} = f(x,y);\ y(x_0) = y_0$

⇒ fourth order

$K_1 = h.f(x_n, y_n)$

$K_2 = h.f(x_n + \frac{h}{2}, y_n + h.\frac{K_1}{2})$

$K_3 = h.f(x_n + \frac{h}{2}, y_n + h.\frac{K_2}{2})$

$K_4 = h.f(x_n + h, y_n + h K_3)$

$y_{n+1} = y_n + \overbrace{\frac{1}{6}(K_1 + 2K_2 + 2K_3 + K_4)}^{K}$

$x_{n+1} = x_n + h$

for $n = 0, 1, 2, 3 \dots$

Simpson's 1/3 rule

⇒ second order

$K_1 = h f(x_n, y_n)$

$K_2 = h.\ f(x_n + h, y_n + K_1)$

$y_{n+1} = y_n + \overbrace{\frac{1}{2}(K_1 + K_2)}^{K}$

$x_{n+1} = x_n + h$

for $n = 0, 1, 2, 3, \dots$

if $f(x,y) = f(x)$

$y(x+h) - y(x) = \int_x^{x+h} f(x)\,dx$

$= \frac{1}{2}(K_1 + K_2) + O(h^3)$

Trapezoidal Rule

Order of convergence (p) ≥ 1

$E_n = \alpha - x_n$

$\lim_{n\to\infty} \left|\frac{E_{n+1}}{E_n^p}\right| = C$

$C > 0$ asymptotic error constant

Solving $f(x) = 0$

Bisection Method

- choose $[a_0, b_0]$ s.t. $f(a_0) f(b_0) < 0$
- $x_0 = a_0$ or b_0 ; $x_{n+1} = \frac{a_n + b_n}{2}$
- $a_{n+1} = a_n,\ b_{n+1} = x_{n+1}$; $f(a_n) f(x_{n+1}) < 0$

$a_{n+1} = x_{n+1},\ b_{n+1} = b_n$; $f(x_{n+1}) f(b_n) < 0$

- till $f(x_{n+1})$ converges.

$|E_n| = |\alpha - x_n| \leq b_n - a_n$

$|E_n| \leq (b_0 - a_0)/2^n$

$\lim_{n\to\infty} \left|\left(\frac{E_{n+1}}{E_n^1}\right)\right| = \frac{1}{2}$

Order of convergence

Secant Method, min. $\left(\frac{1+\sqrt{5}}{2}\right)$ order of convergence

(condition) - choose $[a_0, b_0]$ s.t. $f(a_0) f(b_0) < 0$

x_n, x_{n-1} should be opp. sign for convergence

- $x_{n+1} = \frac{x_{n-1} f(x_n) - f(x_{n-1}) x_n}{f(x_n) - f(x_{n-1})}$

till $f(x_{n+1})$ converges to 0

Newton Raphson Method $x_0 \rightarrow$ initial guess

$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$

check if $f(x_{n+1})$ converges to 0 upto some significant figures

Condition for convergence

$x = \phi(x) = x - \frac{f(x)}{f'(x)}$

$|\phi'(x)| = \left|\frac{f(x) f''(x)}{[f'(x)]^2}\right| < 1$

$x \in [a_0, b_0]$

$E_{n+1} = -\frac{1}{2} E_n^2 \frac{f''(\xi_n)}{f'(x_n)}$

$\lim_{n\to\infty} \left|\frac{E_{n+1}}{E_n^2}\right| = \frac{1}{2} \frac{f''(\alpha)}{f'(\alpha)}$

order of convergence

Fixed point iteration

→ $x_0 = a_0$ or b_0

→ $f(x) = 0 \Rightarrow x = \phi(x)$

s.t. $|\phi'(x)| < 1$ ↳ min.

→ $x_{n+1} = \phi(x_n)$ till (x_i's) converges

Solution of a system of linear eq^n ($Ax = b$)

Direct
(sol^n obtained through finite no. of arithmetical operations)

↓

Cramer's Rule (i^{th} column of $A \leftrightarrow b$)

$$x_i = \frac{\det(A_i)}{\det(A)} \quad i = 1, 2 \ldots n$$

no. of multi. and divis. $= (n+1)!\,(n-1) + n$

Gaussian elimination method

$[A \mid b] \xrightarrow{T} [U \mid b']$

upper Δ matrix using elementary operations → U; solve easily

~~no. of multipli. and divis. $= (n^3 + 3n^2 - n)/3$~~

~~no. of addition and subtraction $= (n^3 + 3n^2 - 4n)/6$~~

Iterative
(a sequence of successive approximations converges to solution)

↓

Gauss Jacobi Method

$$x_i = \frac{1}{a_{ii}}\left[b_i - \sum_{i \neq j} a_{ij} x_j\right] \quad \text{(dominant)}$$

$$x_i^{(k+1)} = \frac{1}{a_{ii}}\left[b_i - \sum_{i \neq j} a_{ij} x_j^{(k)}\right]$$

$x_0^{(0)}$ $(i = 1, 2 \ldots n)$ → initial guess

→ converges if system of eq^n is diagonally dominant

Householder Transform

$A \rightarrow B$ similar symmetric tridiagonal matrix

x_i = ith column of $A \sim B$

$P_i = I - 2 w_i w_i^t$

$$w_i = \frac{(x_i - y_i)^t}{\|x_i - y_i\|_2}$$

$y_i = x_i'$ s.t. $\|x_i\|_2 = \|y_i\|_2$

$B = P_n \ldots P_{i+1} P_i A P_i P_{i+1} \ldots P_n$

s.t. B is tridiagonal

Power Method

Convergence conditions

1. largest (abs. value) eigenvalue is unique
2. $x^{(0)} = \sum c_i v_i$; v_1 eigenvector corresponding to largest eigenvalue $c_1 \neq 0$

↓ if fails → process gives next largest e.value

→ $Z_{k+1} = A y_k$ → normalised vector largest element $= 1$

$$y_{k+1} = \frac{1}{\alpha_{k+1}} Z_{k+1}$$

α_{k+1} → numerically largest element of Z_{k+1}

y_0 → initial $\begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}$

y_k → e.vector corr. to α_k α_k → largest e.value of A as $k \to \infty$

⇒ for smallest e.value, use A^{-1}: and largest e.value $= \dfrac{1}{\text{smallest e.value of } A}$

while eigenvector is same

⇒ for all e.values use $A - x_i P_i^t$ → row of A corresponding to largest element i.e. 1 of x_i

x_i: current largest ~~value e.value~~ corresponding e.vector

largest eigen value is second largest e.value of A and so on

but e.vectors are $x_i - x_{i+k}$ $k = 1, 2 \ldots$

⇒ for e.value closer to β,

use $(A - \beta I_n)^{-1}$ → largest e.value = $\dfrac{1}{\text{smallest e.value of } A - \beta I}$

i.e. $\lambda \rightarrow \frac{1}{\lambda} \rightarrow \beta + \frac{1}{\lambda}$ Ans

$= \dfrac{1}{\text{smallest}} + \beta$