1. Show that the power series $\sum_{n=1}^{\infty} \frac{n! x^n}{n^n}$ converges uniformly on $[-1, 1]$.

2. (a) Find all real x such that $\sum_{n=1}^{\infty} \frac{x^n}{n}$ converges.

   (b) Prove that the nth partial sum of the above series coincides with the Taylor polynomial of degree n around 0 of the function $f(x) = \log \frac{1}{1-x}$.

3. Prove that the series $(1-x)^2 + x(1-x)^2 + x^2(1-x)^2 + \cdots$ is uniformly convergent on $[0, 1]$.

4. Differentiating the series

$$\sum_{n=1}^{\infty} \cos \frac{x}{n} \tag{1}$$

   term by term yields the series $-\sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{x}{n}$ which converges uniformly on $[-r, r]$ for any $r > 0$. But the series in (1) diverges for every $x \in \mathbb{R}$ (why?). Reconcile this with the relevant theorem.

5. Let $f_n : [a, b] \to \mathbb{R}$ be a sequence of continuous non-decreasing functions converging pointwise to the continuous function f. Show that the convergence is uniform.

6. Prove that the series $\sum_{1}^{\infty} \frac{\cos nx}{\{\log(n+1)\}^x}$ is uniformly convergent on any closed interval $[a, b]$ lying within $(0, 2\pi)$.

7. Prove that for all $p > 0$, the series $\sum_{1}^{\infty} \frac{(-1)^{n-1} x^n}{n^p(1+x^n)}$ converges uniformly on $[0, 1]$.

8. Let

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n, \quad |x - x_0| < R,$$

   and there exists a sequence $\{x_n\}$ such that $x_n \neq x_0$, $x_n \to x_0$ and $f(x_n) = 0$ for all n. Show that $f(x) \equiv 0$ on $|x - x_0| < R$.

9. Let $f(x)$ be the sum of the power series $\sum_{n=0}^{\infty} a_n x^n$ on $(-R, R)$ for some $R > 0$. If $f(x) + f(-x) = 0$ for all $x \in (-R, R)$, prove that $a_n = 0$ for all even n.

10. Find the sum of the power series $1 + x + x^2 + \cdots$ on its interval of convergence. Prove that

$$(1-x)^{-3} = 1 + 3x + \frac{3.4}{1.2}x^2 + \frac{3.4.5}{1.2.3}x^3 + \cdots$$

11. Let $f(x) = e^{-1/x^2}$ if $x \neq 0$ and $f(0) = 0$.

    (a) Show that $f^{(n)}(0)$ exists for all $n > 1$.

    (b) Show that the Taylor's series about 0 generated by f converges everywhere on $\mathbb{R}$ but that it represents f only at the origin.

12. If each $a_n > 0$ and if $\sum a_n$ diverges, show that $\sum a_n x^n \to \infty$ as $x \to 1-$. (Assume $\sum a_n x^n$ converges for all $x < 1$.)