Indian Statistical Institute, Kolkata
M.Math I year
Second semester 2024-'25

Analysis II
Assignment 2
Due Date: 10.04.25

Please submit one assignment per group

(1) Prove that there exist functions $C : \mathbb{R} \to \mathbb{R}$ and $S : \mathbb{R} \to \mathbb{R}$ such that
 (i) $C^{''}(x) = -C(x)$ and $S^{''}(x) = -S(x)$ for all $x \in \mathbb{R}$.
 (ii) $C(0) = 1, C^{'}(0) = 0$ and $S(0) = 0, S^{'}(0) = 1$.
 [Hint: Define the functions by $C_1(x) = 1, S_1(x) = x$ and then use the above relations together with integration.] Deduce the power series representation for such functions.

(2) Is the series of functions $\sum_{n=1}^{\infty}(-1)^n(n+x)^{-1}$ convergent for $x \geq 0$? Give reasons for your answer.

(3) (a) Show that $\sum_{k=1}^{n} \sin kx = \frac{\cos x/2 - \cos(n+1)x/2}{2\sin x/2}(x \neq 2l\pi, l \in \mathbb{Z})$.
 (b) Let (c_n) be a decreasing sequence of non negative real numbers such that $\lim_{n\to\infty} c_n = 0$. Show that the trigonometric series $\sum_{n=1}^{\infty} c_n \sin nx$ converges for all $x \in \mathbb{R}$.
 (c) Does there exist a 2π- periodic Riemann integrable function f such that $\sum_{n=1}^{\infty} \frac{\sin nx}{\sqrt{n}}$ is the Fourier series of f? Justify your answer.

(4) Consider the 2π-periodic odd function defined on $[0, \pi]$ by $f(x) = x(\pi - x)$.
 (a) Compute the Fourier coefficients of f.
 (b) Show that $f(x) = \frac{8}{\pi}\sum_{k \text{ odd}, \geq 1} \frac{\sin kx}{k^3}$
 (c) Use Parseval's identity to prove that $\sum_{n=1}^{\infty} \frac{1}{(2n-1)^6} = \frac{\pi^6}{960}$ and $\sum_{n=1}^{\infty} \frac{1}{n^6} = \frac{\pi^4}{945}$.

(5) Show that the trigonometric series

$$\sum_{n\geq 2} \frac{1}{\log n} \sin nx$$

converges for every x, yet it is not the Fourier seies of a Riemann integrable function. Show that the same holds for $\sum_{n\geq 1} \frac{\sin nx}{n^{\alpha}}$ for $0 < \alpha \leq 1/2$.

(6) Show that for α not an integer, the Fourier series of $\frac{\pi}{\sin \pi\alpha} e^{i(\pi - x)\alpha}$ on $[0, 2\pi]$ is given by

$$\sum_{n=1}^{\infty} \frac{e^{inx}}{n + \alpha}.$$

Apply Parseval's formula to show that

$$\frac{1}{(n+\alpha)^2} = \frac{\pi^2}{(\sin \pi\alpha)^2}$$