UNIT-3

Lecture No-18

Boolean Algebra :-

Let B be a set of elements and $\cup$ (union) & $\cap$ (intersection) are two binary operation, then the algebraic structure $(B, \cup, \cap)$ is called Boolean Algebra, if elements of B obey the following Laws:-

(B1) Closure law

for any two elements a & b of B, $a \cup b$ and $a \cap b$ are unique elements of B ie.

(i) $a \cup b \in B \ \forall a, b \in B$ (ii) $a \cap b \in B \ \forall a, b \in B$.

(B2) Commutative law

(i) $a \cup b = b \cup a \ \forall a, b \in B$ (ii) $a \cap b = b \cap a, \forall a, b \in B$.

(B3) Distributive Law

(i) $a \cup (b \cap c) = (a \cup b) \cap (a \cup c) \quad \forall a, b, c \in B$

(ii) $a \cap (b \cup c) = (a \cap b) \cup (a \cap c) \quad \forall a, b, c \in B$.

(B4) Existence of Identity element

There exists two element ϕ (empty set) & I (universal set) in B such that for every $a \in B$ we have

(i) $a \cup \phi = a$ (ii) $a \cap I = a$

ϕ is identity element w.r. to $\cup$

& I is identity element w.r. to $\cap$.

(B5) Complementary Law

(i) $a \cup a' = I$ (ii) $a \cap a' = \phi$

(B6) $\phi \neq I$ ie both identities are not equal.

Definition 2 :

Let B be a set of elements a, b, c ... and + and · are two binary operations, then algebraic structure $(B, +, \cdot)$ is called boolean algebra, if following law holds

(B1) Closure $a+b \in B$, $a \cdot b \in B$ $\forall a \in B$

(B2) Commutative $a+b = b+a$, $a \cdot b = b \cdot a$ $\forall a, b \in B$.

(B3) Distributive $a+(b \cdot c) = (a+b) \cdot (a+c)$ $\forall a, b, c \in B$
$a \cdot (b+c) = a \cdot b + a \cdot c$ $\forall a, b, c \in B$

(B4) Identity $a+0 = 0+a = a$
~~$a \cdot b$~~ $a \cdot 1 = 1 \cdot a = a$

(B5) Complementary Law
$a + a' = a' + a = 1$
$a \cdot a' = a' \cdot a = 0$

(B6) $0 \neq 1$

Theorems of Boolean Algebra

(1) Idempotent law

(i) $a + a = a$

Proof LHS $a + a$
$(a+a) \cdot 1$
$(a+a)(a+\bar{a})$
$a + a\bar{a}$
$a + 0$
a RHS

(ii) $a \cdot a = a$

Proof LHS $a \cdot a$
$a \cdot a + 0$
$a \cdot a + a \cdot \bar{a}$
$a(a+\bar{a})$
$a \cdot 1$
a RHS

$a + 1 = 1$

LHS $a + 1$

$a + a + \bar{a}$

$(a + a) + \bar{a}$

$a + \bar{a}$

1 RHS

(ii) $a \cdot 0 = 0$

LHS $a \cdot 0$

$a \cdot (a \cdot \bar{a})$

$(a \cdot a)\,\bar{a}$

$a \cdot \bar{a}$

0 RHS

3. Absorption Law

(i) $a + (a \cdot b) = a$

LHS $a + (a \cdot b)$

$a \cdot 1 + a \cdot b$

$a(1 + b)$

$a \cdot 1$

a RHS

(ii) $a(a + b) = a$

LHS $a(a + b)$

$(a + 0)(a + b)$

$a + 0 \cdot b$

$a + 0$

a RHS

4. Associative Law :-

(i) $a + (b + c) = (a + b) + c$

(ii) $a(bc) = (ab)c$

5. Complementary law

(i) $\bar{0} = 1$

LHS $\bar{0}$

$= \bar{0} + 0$

$= 1$ RHS

(ii) $\bar{1} = 0$

LHS $\bar{1}$

$= 1 \cdot \bar{1}$

$= 0$ RHS

6. Involution Law $(a')' = a$

Theorem :- The identity element in Boolean algebra are unique.

(i) Identity element 0 is unique for '+'

(ii) Identity element 1 is unique for '·'.

Theorem De Morgan's law

(i) $(a+b)' = a'.b'$ (ii) $(ab)' = (a'+b')$

To prove $(a+b)' = a'.b'$

Here we have to show that the complement of $(a+b)$ is $a'.b'$, for which it is sufficient to prove –

$(a+b) + a'b' = 1$

and $(a+b)(a'b') = 0$

Now $(a+b) + a'b' = [(a+b)+a'][(a+b)+b']$

$= [(a+a')+b][a+(b+b')]$

$= [1+b][a+1]$

$= 1 \cdot 1 = 1$

and $(a+b)(a'b') = (a'b')(a+b)$

$= (a'b')a + (a'b')b$

$= (b'a')a + a(b'b)$

$= b'(a'a) + a(b'b)$

$b' \cdot 0 + a \cdot 0$

$0 + 0 = 0$

Hence $(a+b)' = a'.b'$

(ii) $(ab)' = a'+b'$

Here we need to show that the complement of $a.b$ is $(a'+b')$ for which it is sufficient to prove –

$(a.b) + (a'+b') = 1$

$(a.b)(a'+b') = 0$

Now $(a.b) + (a'+b') = (a'+b') + ab$

$= [(a'+b')+a][a'+b'+b]$

$= [b'+a'+a][a'+b'+b]$

$$= [(b'+a')+a][(a'+b')+b]$$
$$= [b'+(a'+a)][a'+(b'+b)]$$
$$= [b'+1][a'+1]$$
$$= 1 \cdot 1 = 1$$

and. $(ab)(a'+b') = (ab)a' + (ab)b'$
$$= (ba)a' + a(bb')$$
$$= b \cdot 0 + a \cdot 0$$
$$= 0 + 0 = 0$$

Hence $(ab)' = a'+b'$

Qu1 In Boolean algebra B, prove that
$a+b=b \Rightarrow a \cdot b = a$ for $a, b \in B$

Given $a+b=b$

Now $a \cdot b = a(a+b)$
$$= a \cdot a + a \cdot b$$
$$= a + ab$$
$$= a$$

Hence $a+b=b \Rightarrow a \cdot b = a$

Qu2 show that
if $a+x = b+x$ and $a+x' = b+x'$ then $a=b$

Soln since $a = a+0$
$$= a + x \cdot x'$$
$$= (a+x)(a+x')$$
$$= (b+x)(b+x')$$
$$= b + xx'$$
$$= b + 0$$
$$= b \quad \text{Ans}$$

Qu 4 $ab + bc + ca = (a+b)(b+c)(c+a)$

RHS
$$(a+b)(b+c)(c+a)$$
$$(a+b)[(c+b)(c+a)]$$
$$(a+b)[c+ba]$$
$$(a+b)c + (a+b)ba$$
$$ac+bc + a.ab + b.ba$$
$$ac+bc + ab + ba$$
$$ac+bc + (ab + a.b)$$
$$ac+bc+ab \quad \text{LHS}$$

Simplification of Boolean Function

1. Disjunctive Normal Form or (SOP) form :-

Suppose $x_1, x_2 \dots x_n$ are n variables in a Boolean Algebra $(B,+,\cdot)$ and their complements are respectively $x_1', x_2', \dots x_n'$ and suppose $f(x_1, x_2 \dots x_n)$ is an arbitrary Boolean funⁿ of these n variables. The function f is called Disjunctive Normal Form if the function f can be written as the sum of terms of the type $f_1(x_1), f_2(x_2) \dots f_n(x_2)$ where $f_i(x_i) = x_i$ or x_i' $\forall\, i = 1, 2 \dots n$ And no two terms are identical,

In addition to it, 0 & 1 for $n \geq 0$ are called disjunctive normal form in n variables.

Boolean Algebra

Boolean Identitiee

	AND	OR
1. Identity	$x \cdot 1 = x$	$x + 0 = x$
2 Complement	$x \cdot \bar{x} = 0$	$x + \bar{x} = 1$
3. Commutative	$xy = y \cdot x$	$x + y = y + x$
4. Distributive	$x(y+z) = xy + xz$	$x + yz = (x+y)(x+z)$
5. Idempotent	$x \cdot x = x$	$x + x = x$
6. Null	$\cancel{x} \cdot x \cdot 0 = 0$	$x + 1 = 1$
7 Involution	$\bar{\bar{x}} = x$	$\bar{\bar{x}} = x$
8. Absorption	$x(x+y) = x$	$x + xy = x$
9. Associative	$x(yz) = (xy)z$	$x + (y+z) = (x+y) + z$
10. demorgan	$\overline{x \cdot y} = \bar{x} + \bar{y}$	$\overline{x+y} = \bar{x} \cdot \bar{y}$

Gate level Miminization

It concerns with finding optimal gate level implementation of the Boolean functions used to describe a digital circuit.

Various methods to simplify Boolean Expression are :-

1. Algebraic simplification
2. Karnaugh map (K-map) simplification
3. Quine Mc Clanskey Method (Tabulation Method)

• **Max Terms :-**

Any expression can be expressed in Product (AND) of sum (OR) form.

The sum terms are known as Max terms.

eg
$$A \cdot B = (A + B\bar{B})(B + A\bar{A})$$
$$= (A+B)(A+\bar{B})(B+A)(B+\bar{A})$$
$$= (A+B)(A+\bar{B})(\bar{A}+B)$$

Product of sum (POS)

POS are called Max Terms

Binary Values A	B	C	Minterms		Max Terms	
0	0	0	$\bar{A}\bar{B}\bar{C}$	m_0	$A+B+C$	M_0
0	0	1	$\bar{A}\bar{B}C$	m_1	$A+B+\bar{C}$	M_1
0	1	0	$\bar{A}B\bar{C}$	m_2	$A+\bar{B}+C$	M_2
0	1	1	$\bar{A}BC$	m_3	$A+\bar{B}+\bar{C}$	M_3
1	0	0	$A\bar{B}\bar{C}$	m_4	$\bar{A}+B+C$	M_4
1	0	1	$A\bar{B}C$	m_5	$\bar{A}+B+\bar{C}$	M_5
1	1	0	$AB\bar{C}$	m_6	$\bar{A}+\bar{B}+C$	M_6
1	1	1	ABC	m_7	$\bar{A}+\bar{B}+C$	M_7

Ex Simplify the logical Expression

$$\overline{(\overline{A+\bar{B}}) + (\overline{A\cdot\bar{B}}) + (\overline{\bar{A}\cdot B})}$$

Sol^n use De-morgan's law

$$(\overline{\overline{A+\bar{B}}})\cdot(\overline{\overline{A\cdot\bar{B}}})\cdot(\overline{\overline{\bar{A}\cdot B}})$$

$$(A+\bar{B})\,(A\cdot\bar{B})\cdot(\bar{A}\cdot B)$$

$$(A+\bar{B})\,(A\bar{A}\cdot\bar{B}B) = (A+\bar{B})\cdot 0 = 0 \text{ Ans}$$

Simplification of Boolean Function

1. Disjunctive Normal Form or (SOP) form :-

Suppose $x_1, x_2 \cdots x_n$ are n variables in a Boolean Algebra $(B, +, \cdot)$ and their complements are respectively $x_1', x_2', \cdots x_n'$ ~~are n vas~~ and suppose $f(x_1, x_2 \cdots x_n)$ is an arbitrary Boolean funⁿ of these n variables.

The function f is called Disjunctive Normal Form if the function f can be written as the sum of terms of the type $f_1(x_1), f_2(x_2) \cdots f_n(x_2)$

where $f_i(x_i) = x_i$ or x_i' $\forall\, i = 1, 2 \cdots n$

And no two terms are identical,

In addition to it, 0 & 1 for $n \geq 0$ are called disjunctive normal form in n variables.

Complete Disjunctive Normal Form

If the no. of distinct terms in disjunctive normal form of Boolean funⁿ in n variables are 2^n, then it is called complete disjunctive normal form

Ex
$$f(x_1, x_2) = x_1 x_2 + \bar{x}_1 x_2 + x_1 \bar{x}_2 + \bar{x}_1 \bar{x}_2$$

Illustration :-

$f(x_1, x_2, x_3)$ in three variable in DNF may be given by
$$f(x_1, x_2, x_3) = x_1 x_2 \bar{x}_3 + \bar{x}_1 x_2 x_3 + x_1 \bar{x}_2 x_3$$

But in complete DNF, there are $2^3 = 8$ distinct terms

Hence
$$f(x_1 x_2 x_3) = x_1 x_2 x_3 + \bar{x}_1 x_2 x_3 + x_1 \bar{x}_2 x_3 + x_1 x_2 \bar{x}_3 + \bar{x}_1 \bar{x}_2 x_3 + \bar{x}_1 x_2 \bar{x}_3 + \bar{x}_1 \bar{x}_2 \bar{x}_3 + x_1 \bar{x}_2 \bar{x}_3$$

Prove that Complete DNF form in 3 variables has its value as 1.

$$= x_1 x_2 x_3 + x_1 x_2 \bar{x}_3 + x_1 \bar{x}_2 x_3 + \bar{x}_1 x_2 x_3 + x_1 \bar{x}_2 \bar{x}_3 + \bar{x}_1 x_2 \bar{x}_3 + \bar{x}_1 \bar{x}_2 x_3 + \bar{x}_1 \bar{x}_2 \bar{x}_3$$

$$= x_1 x_2 (x_3 + \bar{x}_3) + x_1 \bar{x}_2 (x_3 + \bar{x}_3) + \bar{x}_1 x_2 (x_3 + \bar{x}_3) + \bar{x}_1 \bar{x}_2 (x_3 + \bar{x}_3)$$

$$= x_1 x_2 + x_1 \bar{x}_2 + \bar{x}_1 x_2 + \bar{x}_1 \bar{x}_2$$

$$= x_1 (x_2 + \bar{x}_2) + \bar{x}_1 (x_2 + \bar{x}_2)$$

$$= x_1 + \bar{x}_1$$

$$= 1 \text{ Ans}$$

Conjunctive Normal form :-

A Boolean polynomial is called a Conjunctive Normal Form (or Dual Canonical form) if it is the product of distinct factors where each factor is the sum of variables x_1, x_2, x_3 --- and their complements x_1', x_2', x_3' -- and in each factor variables or their complements do not occur more than once.

The following functions are in CNF :-

(i) $f(x_1, x_2) = (x_1 + \bar{x}_2)(\bar{x}_1 + x_2)(\bar{x}_1 + \bar{x}_2)$

Que Change the funⁿ $f = (x+\bar{y})(\bar{x}+y)(\bar{x}+\bar{y})$ from CN form to DN form or (POS) to (SOP)

Solⁿ $f = (x+\bar{y})(\bar{x}+y)(\bar{x}+\bar{y})$

The complete CN form of two variables x, y is

$$F = (x+\bar{y})(\bar{x}+y)(\bar{x}+\bar{y})(x+y)$$

~~Complete~~ complement of CN form $f' = F - f$

$$f' = (x+y)$$

We know the complement of complement CN form is DN form.

$$f = (f')' = (x+y)'$$

$$= x'y' \text{ Ans}$$

Que obtain the SOP form of Boolean Expression

$x_1 \bar{x}_2 + x_3$

Sol^n $f(x_1, x_2, x_3) = x_1 \bar{x}_2 \cdot 1 + x_3 \cdot 1 \cdot 1$

$= x_1 \bar{x}_2 (x_3 + \bar{x}_3) + x_3 (x_1 + \bar{x}_1)(x_2 + \bar{x}_2)$

$= x_1 \bar{x}_2 x_3 + x_1 \bar{x}_2 \bar{x}_3 + (x_1 x_3 + \bar{x}_1 x_3)(x_2 + \bar{x}_2)$

$= x_1 \bar{x}_2 x_3 + x_1 \bar{x}_2 \bar{x}_3 + x_1 x_2 x_3 + x_1 \bar{x}_2 x_3 + \bar{x}_1 x_2 x_3$
$+ \bar{x}_1 \bar{x}_2 x_3$

Ans

Que Simplify Boolean fun^n

$xy + \bar{x} y + \bar{x} \bar{y}$

$f = xy + \bar{x}(y + \bar{y})$

$xy + \bar{x}$

$= \bar{x} + xy$

$= (\bar{x} + x)(\bar{x} + y)$

$= 1(\bar{x} + y)$

$f = \bar{x} + y$ Ans

Que $f(x_1, x_2, x_3) = (\bar{x}_1 x_2)'(x_1 + x_3)$ in conjunctive Normal form

$f(x_1 x_2 x_3) = (\bar{x}_1 x_2)'(x_1 + x_3)$

$= (\bar{\bar{x}}_1 + \bar{x}_2)(x_1 + x_3)$

$= (x_1 + \bar{x}_2)(x_1 + x_3)$

$= (x_1 + \bar{x}_2 + 0)(x_1 + 0 + x_3)$

$= (x_1 + \bar{x}_2 + x_3 \bar{x}_3)(x_1 + x_2 \bar{x}_2 + x_3)$

$= (x_1 + \bar{x}_2 + x_3)(x_1 + \bar{x}_2 + \bar{x}_3)(x_1 + x_2 + x_3)$
$(x_1 + \bar{x}_2 + x_3)$

Ans

Que Express the following funⁿ in CNF.

$f(x, y, z) = (xy' + xz)' + x'$

$(xy')' \cdot (xz)' + x'$

$(x'+y)(x'+z') + x'$

$(x'+x'+y)(x'+x'+z')$

$(x'+y)(x'+z')$

$(x'+y+zz')(x'+z'+yy')$

$(x'+y+z)(x'+y+z')(x'+z'+y)(x'+z'+y')$ Ans

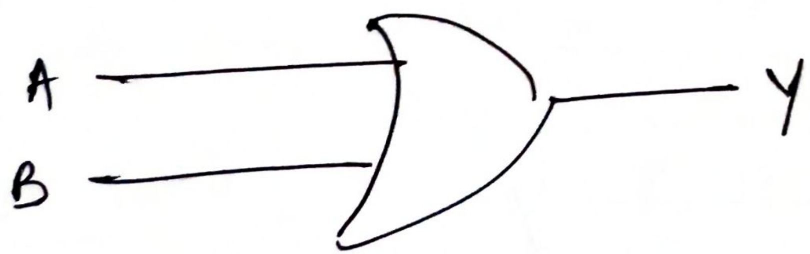
Logic Gates :-

A logic Gate is a digital circuit with one or more inputs but single output.
The most basic logic gates are AND, OR & NOT.
logic Gates are used to design various logical functions.

1. OR Gate :-

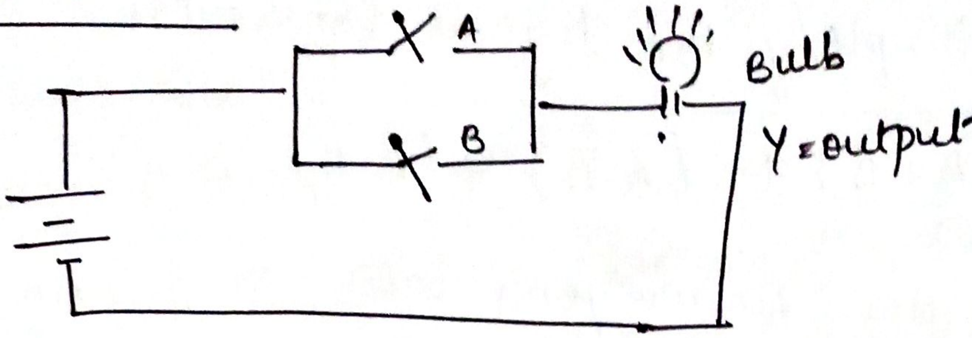
→ It is used to perform logical addition
→ It has two or more input & single output
→ If any of Input is high, output is high.

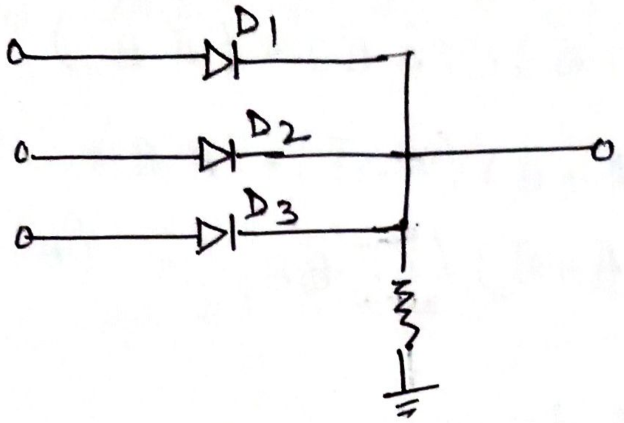
SYMBOL of OR GATE :-



$$\boxed{Y = A + B}$$

electrical circuit





Truth Table

A	B	Y
0	0	0
0	1	1
1	0	1
1	1	1

(2) AND Gate :

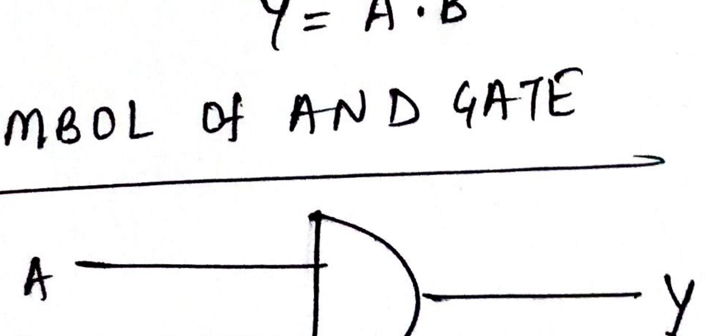
It is used to perform logical multiplication

It has two or more input & single output.

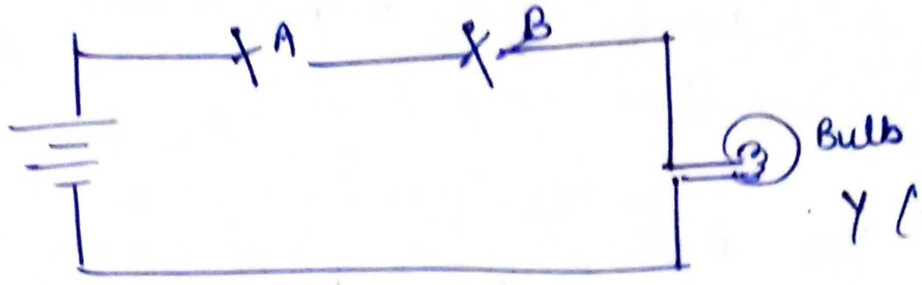
If any of the input is low, output is low

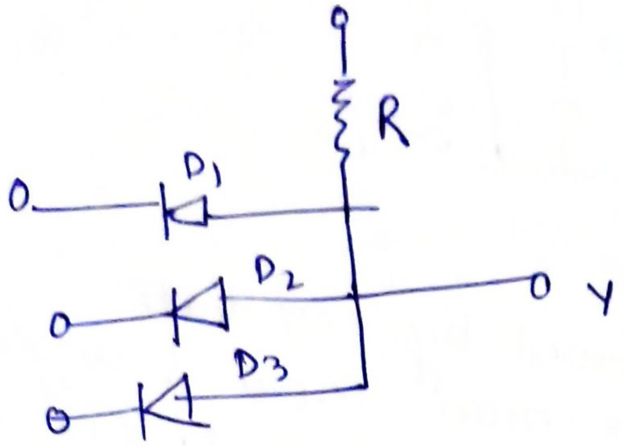
$$Y = A \cdot B$$

SYMBOL OF AND GATE



Electrical Circuit (series connection)





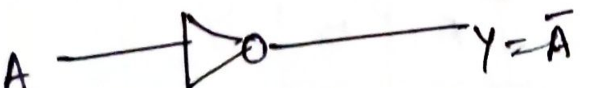
Truth Table

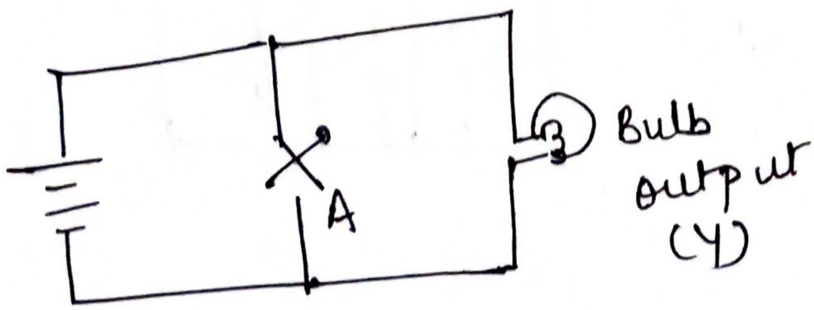
A	B	Y
0	0	0
0	1	0
1	0	0
1	1	1

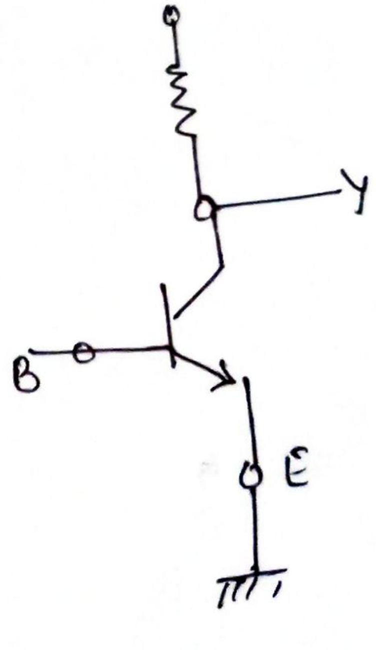
(3) NOT Gate :-

NOT Gate is used to perform complementary operation, it is also called inverter.

$$Y = \bar{A} \text{ or } A'$$







Truth Table

A	$Y = \bar{A}$
0	1
1	0

(4) NOR Gate :-

It is an OR gate followed by NOT Gate.
If any of Input is high, output is low.

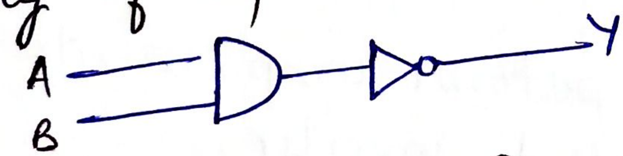
$$Y = \overline{A+B}$$

Truth Table

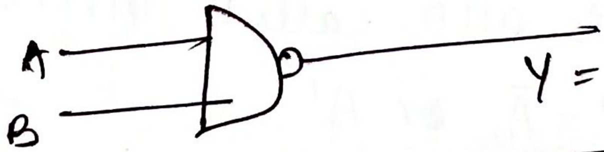
A	B	Y
0	0	1
0	1	0
1	0	0
1	1	0

(5) NAND Gate :-

It is AND Gate followed by NOT Gate
NAND Gate has 2 or more input & single output.
If any of Input is low, output is hight



or



$Y = \overline{A \cdot B}$

$$Y = \overline{A \cdot B}$$

A	B	Y
0	0	1
0	1	1
1	0	1
1	1	0

(6) Exclusive OR (EX-OR) or XOR

→ EX-OR Gate has two or more input & single output

→ If odd no. of inputs are HIGH, its out is HIGH.

→ In Boolean Algebra, it is denoted by ⊕

$$Y = A\bar{B} + \bar{A}B$$

Truth Table

A	B	$\bar{A}$	$\bar{B}$	$\bar{A}B$	$A\bar{B}$	$Y = \bar{A}B + A\bar{B}$
0	0	1	1	0	0	0
0	1	1	0	1	0	1
1	0	0	1	0	1	1
1	1	0	0	0	0	0

(7) Exclusive NOR (EX-NOR) (X-NOR)

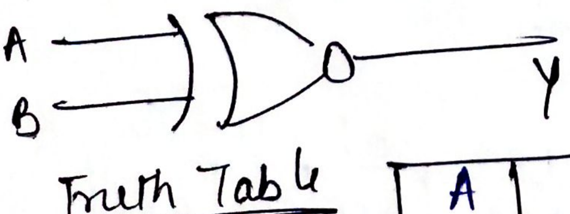
It is EX-OR gate followed by a NOT Gate

If even no. of inputs are high, output is high.

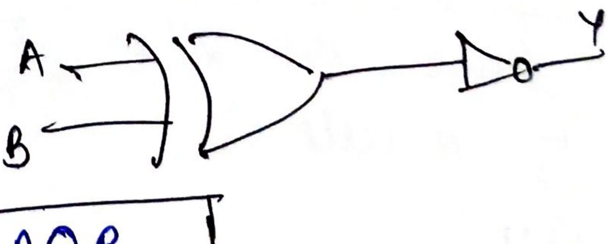
denoted by ⊙

$$\boxed{Y = A \odot B}$$

$$Y = AB + \bar{A}\bar{B}$$



or



Truth Table

A	B	$A \oplus B$	$A \odot B$
0	0	0	1
0	1	1	0
1	0	1	0
1	1	0	1

Max Terms :-

Any expression can be expressed in Product (AND) of sum (OR) form.

The sum terms are known as max terms.

eg

$$A \cdot B = (A + B\bar{B})(B + A\bar{A})$$

$$(A+B)(A+\bar{B})(B+A)(B+\bar{A})$$

$$= \underbrace{(A+B)(A+\bar{B})\cancel{+}(\bar{A}+B)}_{POS}$$

POS (Max terms) The product terms are called Max terms.

Binary Values A	b	C	Minterms	Max terms
0	0	0	$\bar{a}\bar{b}\bar{c} = m_0$	$(a+b+c) = M_0$
0	0	1	$\bar{a}\bar{b}c = m_1$	$(a+b+\bar{c}) = M_1$
0	1	0	$\bar{a}b\bar{c} = m_2$	$(a+\bar{b}+c) = M_2$
0	1	1	$\bar{a}bc = m_3$	$(a+\bar{b}+\bar{c}) = M_3$
1	0	0	$a\bar{b}\bar{c} = m_4$	$(\bar{a}+b+c) = M_4$
1	0	1	$a\bar{b}c = m_5$	$(\bar{a}+b+\bar{c}) = M_5$
1	1	0	$ab\bar{c} = m_6$	$(\bar{a}+\bar{b}+c) = M_6$
1	1	1	$abc = m_7$	$(\bar{a}+\bar{b}+\bar{c}) = M_7$

Canonical Forms (SOP & POS)

1. **SOP**

$$F = (A+BC)(B+\bar{C}A)$$
$$= A(B+\bar{C}A) + BC(B+\bar{C}A)$$
$$= AB + A\bar{C}A + BCB + BC\bar{C}A$$
$$= AB + A\bar{C} + BC + 0$$

(canonical SOP form)

2. **POS**

$$F = (A+BC)(B+\bar{C}A)$$
$$= (A+B)(A+C)(B+\bar{C})(B+A)$$
$$= (A+B)(A+C)(B+\bar{C})$$

(Canonical POS form)

Implementation using only NAND Gate :-

→ Convert expression in SOP form

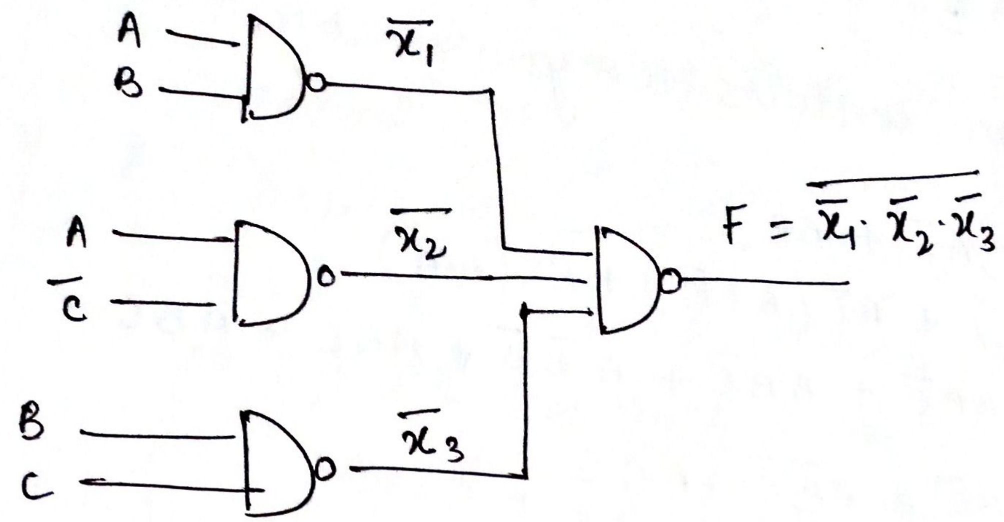
→ $F = \overline{AB + A\bar{C} + BC}$

$$\underset{x_1}{AB} + \underset{x_2}{A\bar{C}} + \underset{x_3}{BC}$$

$$\bar{F} = \overline{x_1 + x_2 + x_3} \Rightarrow \bar{x}_1 \cdot \bar{x}_2 \cdot \bar{x}_3$$

$$\bar{\bar{F}} = \overline{\bar{x}_1 \cdot \bar{x}_2 \cdot \bar{x}_3} \Rightarrow \bar{\bar{x}}_1 + \bar{\bar{x}}_2 + \bar{\bar{x}}_3$$

$$F = x_1 + x_2 + x_3$$



Implementation using only NOR Gate :-

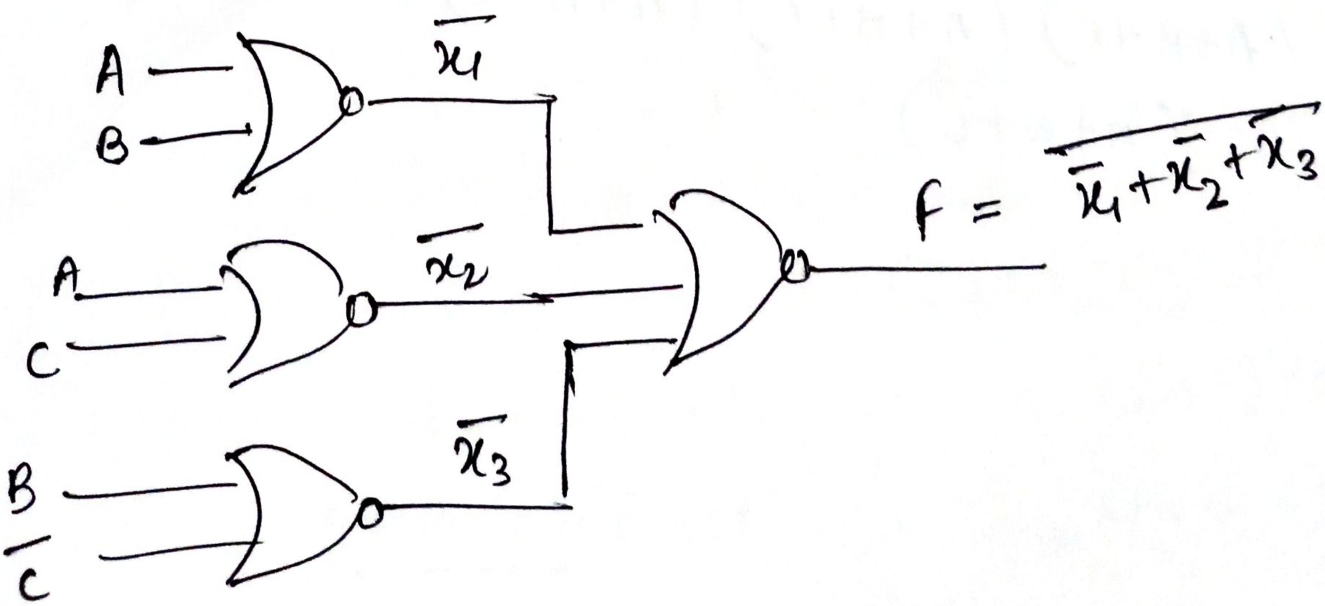
→ Convert expression in POS form

→ $f = (A+B)(A+C)(B+\bar{C})$

$$\bar{F} = \overline{x_1 \cdot x_2 \cdot x_3}$$

$$\bar{F} = \bar{x_1} + \bar{x_2} + \bar{x_3}$$

$$\bar{\bar{F}} = \overline{\bar{x_1} + \bar{x_2} + \bar{x_3}}$$



Standard SOP :

Each min term contains every variable.

Ex

$f = AB + A\bar{C} + BC$

$= AB(C+\bar{C}) + A\bar{C}(B+\bar{B}) + BC(A+\bar{A})$

$= ABC + AB\bar{C} + AB\bar{C} + A\bar{B}\bar{C} + ABC + \bar{A}BC$

$f = ABC + AB\bar{C} + A\bar{B}\bar{C} + \bar{A}BC$

Standard SOP form.

Standard POS :-

Each Max term contains all variables -

$F = (A+B)(A+C)(B+\bar{C})$

$= (A+B+C\bar{C})(A+C+B\bar{B})(B+\bar{C}+A\bar{A})$

$= (A+B+C)(A+B+\bar{C})(A+B+C)(A+C+\bar{B})$

$(B+\bar{C}+A)(B+\bar{C}+\bar{A})$

$f = (A+B+C)(A+B+\bar{C})(A+\bar{B}+C)(A+B+\bar{C})$

$(\bar{A}+B+\bar{C})$

Que1 Write the minterms of ACD + AB

Solution By converting given SOP expression into standard SOP.

$ACD(B+\bar{B}) + AB(C+\bar{C})(D+\bar{D})$

$ABCD + A\bar{B}CD + AB(CD + C\bar{D} + \bar{C}D + \bar{C}\bar{D})$

$ABCD + A\bar{B}CD + ABCD + ABC\bar{D} + AB\bar{C}D + AB\bar{C}\bar{D}$

$ABCD + A\bar{B}CD + ABC\bar{D} + AB\bar{C}D + AB\bar{C}\bar{D}$

$\underbrace{1111}_{15} + \underbrace{1011}_{11} + \underbrace{1110}_{14} + \underbrace{1101}_{13} + \underbrace{1100}_{12}$

$F = \Sigma m(11, 12, 13, 14, 15)$ Ans

Que2 Convert into POS form

$F(A,B,C) = \Sigma m(1,3,7)$

$= m_1 + m_3 + m_7$

$= \bar{A}\bar{B}C + \bar{A}BC + ABC$

$= \bar{A}\bar{B}C + BC(\bar{A}+A) = \bar{A}\bar{B}C + BC$

$= C[\bar{A}\bar{B} + B]$

$= C[\bar{A}+B][\bar{B}+B]$

$= C[\bar{A}+B]$

$= (C + B\bar{B} + A\bar{A})(\bar{A} + B + C\bar{C})$

$= (\bar{A}+B+C)(\bar{A}+B+\bar{C})(A+B\bar{B}+C)(\bar{A}+B\bar{B}+C)$

$= (\bar{A}+B+C)(\bar{A}+B+\bar{C})(A+B+C)(A+\bar{B}+C)(\bar{A}+B+C)(\bar{A}+\bar{B}+C)$

$= M_4 \quad M_5 \quad M_0 \quad M_2 \quad M_6$

Que3 Realise $Y = \overline{A+B+C+D}$ using 2-Input NOR gates only.

$$Y = \overline{A+B+C+D}$$

$$\bar{Y} = \overline{\overline{A+B+C+D}}$$

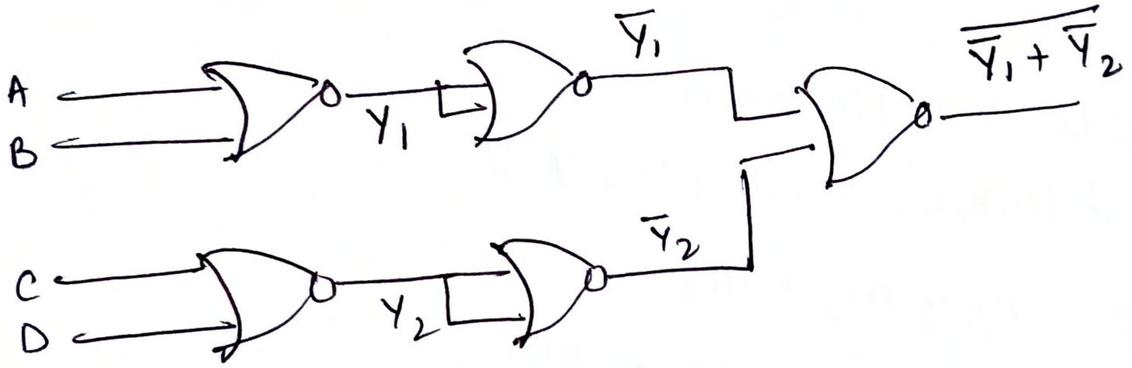
$$\bar{Y} = \overline{\overline{A+B} \cdot \overline{C+D}} = \overline{\overline{A+B}} + \overline{\overline{C+D}}$$

$$Y_1 = \overline{A+B} \qquad Y_2 = \overline{C+D}$$

$$\bar{Y} = \bar{Y}_1 + \bar{Y}_2$$

$$\bar{\bar{Y}} = \overline{\bar{Y}_1 + \bar{Y}_2}$$

$$Y = \overline{\bar{Y}_1 + \bar{Y}_2}$$



Que4 Simplify the expression & implement with two level NAND gate circuit.

$$Y = BD + BC\bar{D} + A\bar{B}\bar{C}\bar{D}$$

$$Y = B[D + C\bar{D}] + A\bar{B}\bar{C}\bar{D}$$

$$= B(D+C)(D+\bar{D}) + A\bar{B}\bar{C}\bar{D}$$

$$B(D+C) + A\bar{B}\bar{C}\bar{D}$$

$$BD + BC + A\bar{B}\bar{C}\bar{D}$$ {simplified Expression}

out is HIG

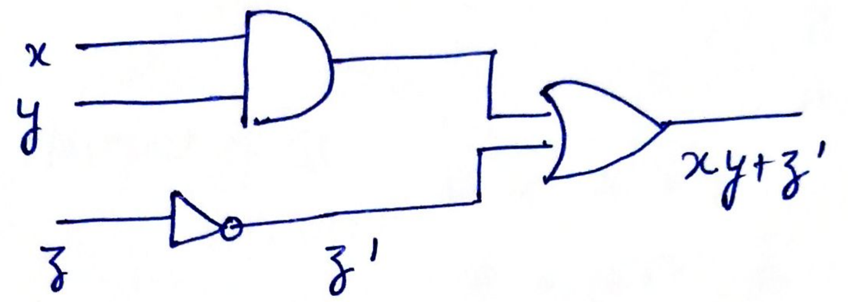
Que Find logic Network corresponding to Boolean Expression -

(i) $f = xy + z'$

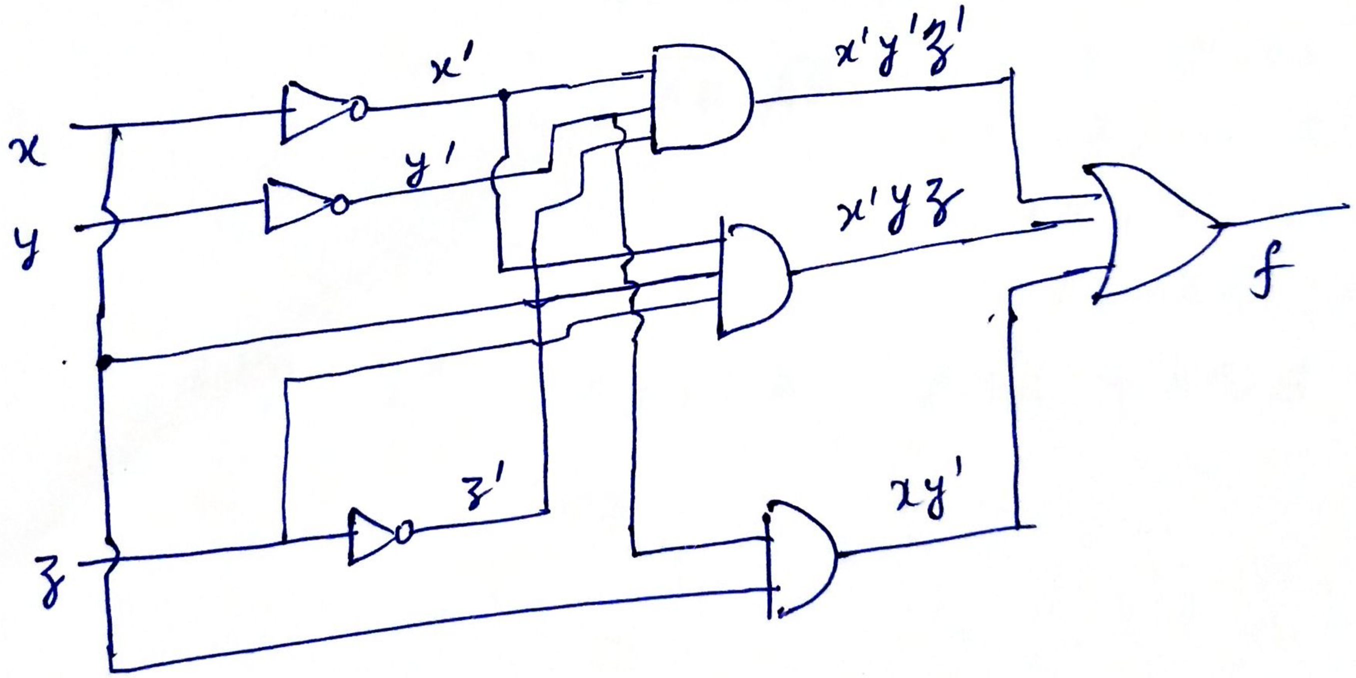
(ii) $f = x'y'z' + x'yz + xy'$

Solⁿ

$f = xy + z'$



(ii) $f = x'y'z' + x'yz + xy'$



Lecture-No-~~07~~ 22

(Ref Pt -3.5)

Gate level Minimization

K-Map Method :-

It is a graphical method for ~~simplifying~~ simplifying boolean functions

→ It is a 2D representation of truth table

→ It has 2^n squares (cells) for n-variable.

2-Variable Map :-

There are $2^n \Rightarrow 2^2 = 4$ minterms ie (map consists of 4 cells)

A \ B	0	1
0	00	01
1	10	11

A \ B	$\bar{B}$	B
$\bar{A}$	m_0 0	m_1 1
A	m_2 2	m_3 3

$m_0 = 00 = \bar{A}\bar{B}$, $m_1 = 01 = \bar{A}B$, $m_2 = 10 = A\bar{B}$, $m_3 = 11 = AB$

3- Variable Map :-

There are $2^n \Rightarrow 2^3 = 8$ minterms, map consists of 8 cells.

A \ BC	00	01	11	10
0	m_0	m_1	m_3	m_2
1	m_4	m_5	m_7	m_6

A \ BC	00	01	11	10
0	$\bar{A}\bar{B}\bar{C}$	$\bar{A}\bar{B}C$	$\bar{A}BC$	$\bar{A}B\bar{C}$
1	$A\bar{B}\bar{C}$	$A\bar{B}C$	ABC	$AB\bar{C}$

Four Variable K-Map

for four variables $n = 4$

$$\text{cell} = 2^n \Rightarrow 2^4 = 16 \text{ cells}.$$

AB \ CD	00 $\bar{C}\bar{D}$	01 $\bar{C}D$	11 CD	10 $C\bar{D}$
00 $\bar{A}\bar{B}$	m_0	m_1	m_3	m_2
01 $\bar{A}B$	m_4	m_5	m_7	m_6
11 AB	m_{12}	m_{13}	m_{15}	m_{14}
10 $A\bar{B}$	m_8	m_9	m_{11}	m_{10}

Grouping of cells for simplification :-

The grouping means combining the adjacent cells to simplify the logic function or truth table on K-map.

→ If 0's are grouped, then result is in POS form

→ If 1's are grouped, then result is in SOP form

3 types of Group can be formed :-

1. **Pair** Group two adjacent cells.

A \ B	$\bar{B}$	B
$\bar{A}$		1
A		1

Pair → $\bar{A}B + AB$

$B(\bar{A} + A) \Rightarrow B$

A \ BC	$\bar{B}\bar{C}$	$\bar{B}C$	BC	$B\bar{C}$
$\bar{A}$	0	0	0	0
A	1	0	0	1

Pair → $A\bar{B}\bar{C} + AB\bar{C}$

$A\bar{C}(\bar{B}+B) = A\bar{C}$

2. Quad :- Group of 4 adjacent cells.

A \ BC	$\bar{B}\bar{C}$	$\bar{B}C$	BC	$B\bar{C}$
$\bar{A}$		1	1	
A		1	1	

Pair ⇒ $A\bar{B}C + \bar{A}\bar{B}C + \bar{A}BC + ABC$

$\bar{B}C(A+\bar{A}) + BC(A+\bar{A})$

$\bar{B}C + BC \Rightarrow C$

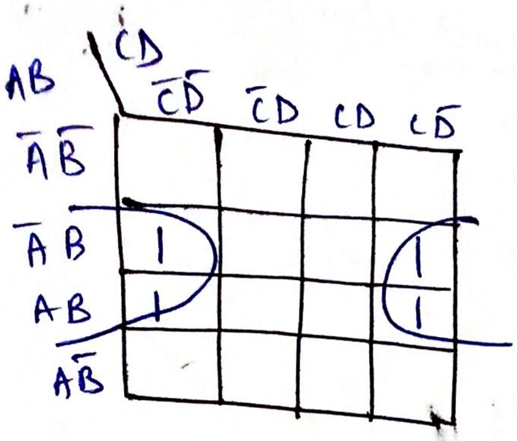
AB \ CD	$\bar{C}\bar{D}$	$\bar{C}D$	CD	$C\bar{D}$
$\bar{A}\bar{B}$				
$\bar{A}B$				
AB	1	1	1	1
$A\bar{B}$				

Quad

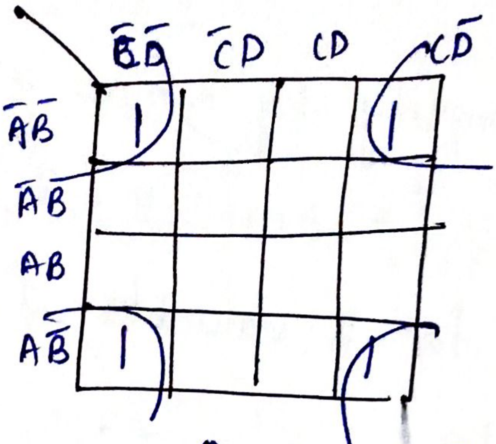
AB

	$\bar{C}\bar{D}$	$\bar{C}D$	CD	$C\bar{D}$
$\bar{A}\bar{B}$		1		
$\bar{A}B$		1		
AB		1		
$A\bar{B}$		1		

Quad ⇒ $\bar{C}D$

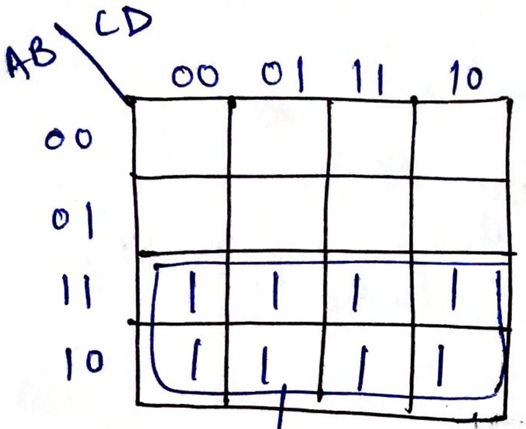


Quad ⇒ $B\bar{D}$

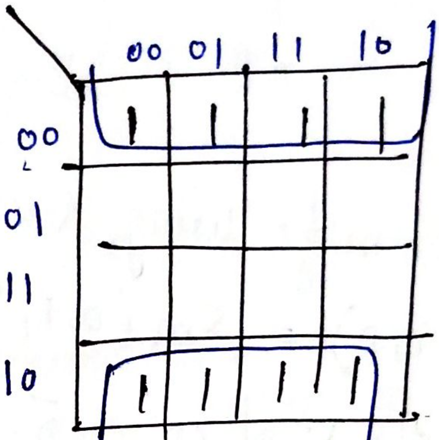


Quad ⇒ $\bar{B}\bar{D}$

3. <u>Octet</u> :- Grouping of 8 octet cells.

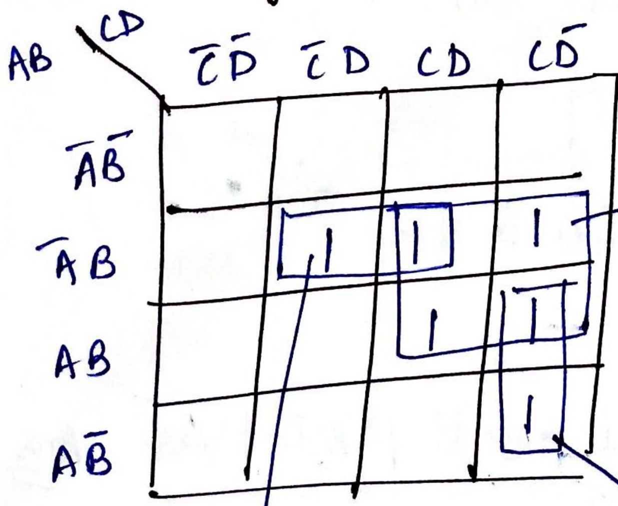


octet ⇒ A



octet ⇒ $\bar{B}$

<u>Overlapping Groups</u> :-



Pair 3 ⇒ BC

Pair 2 ⇒ $AC\bar{D}$

Pair 1 = $A\bar{B}D$

$\boxed{F = A\bar{B}D + AC\bar{D} + BC}$

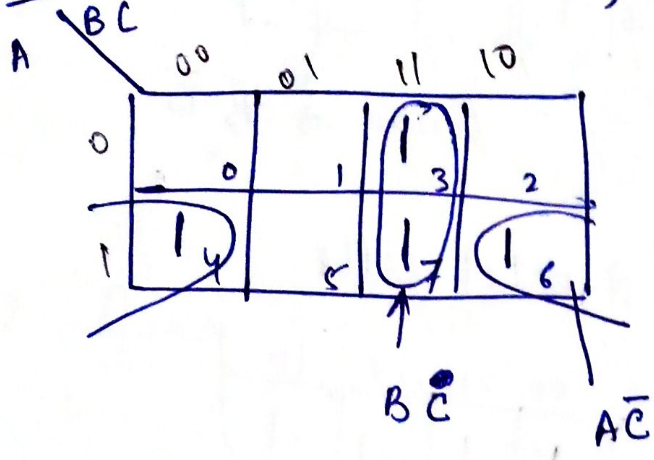
Some Examples (Lecture No 23)

Ref Pt - 3.6

Ques Simplify by K-map using SOP

$$Y = \Sigma m(3, 4, 6, 7)$$

Soln for 3 variables, $2^3 = 8$ cells

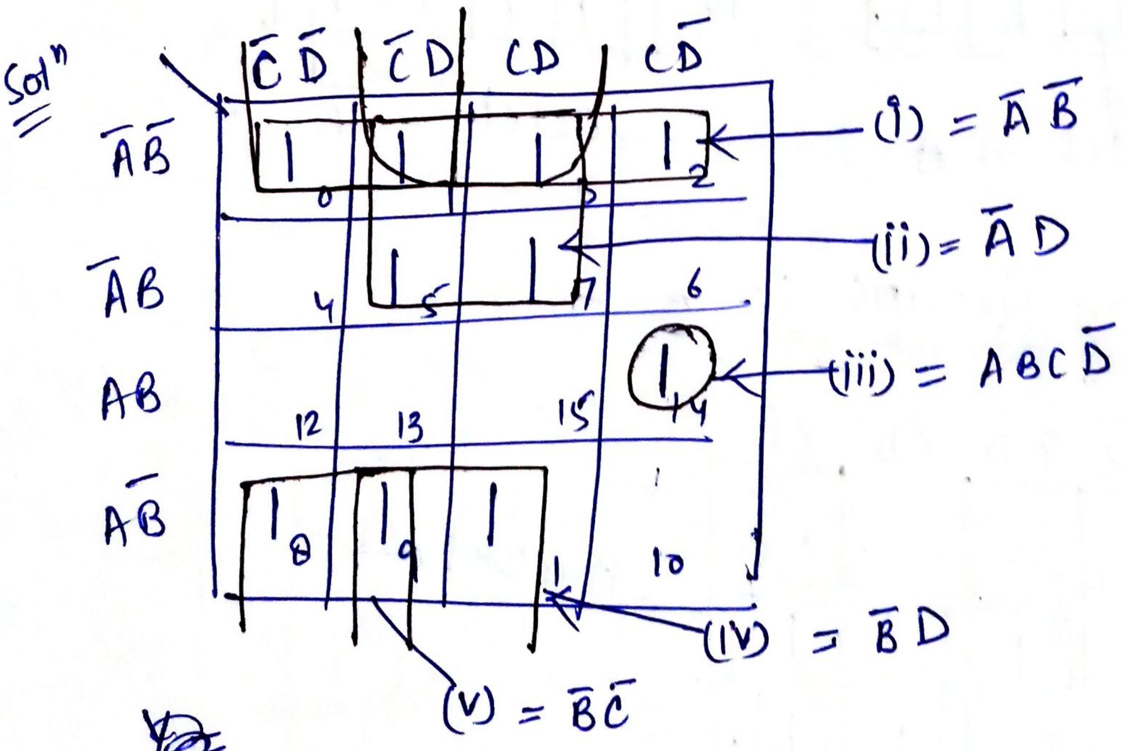


$$\boxed{Y = BC + A\bar{C}}$$ Ans

Ques Minimize using K-map by SOP

$$F(A, B, C, D) = \Sigma m(0, 1, 2, 3, 5, 7, 8, 9, 11, 14)$$

Soln



$$Y = \bar{A}\bar{B} + \bar{A}D + ABC\bar{D} + \bar{B}D + \bar{B}\bar{C}$$ Ans

Ex: Simplify by K-Map using POS.

$$Y = \pi M(0, 1, 2, 5)$$

Soln

A \ BC	00	01	11	10
0	0 (0)	0 (1)	(3)	0 (2)
1	(4)	0 (5)	(7)	(6)

$(B+\bar{C})$ $(A+C)$

$$Y = (B+\bar{C}) \cdot (A+C) \text{ Ans}$$

Que: $A_1 A_0$ and $B_1 B_0$ represent two binary nos. Each of these can have values 00, 01, 10, 11. Draw a logic circuit with inputs $A_1 A_0$ and $B_1 B_0$ such that output is high when binary no. $A_1 A_0 = B_1 B_0$. Use two X-NOR and one AND gate in implementation.

Soln

A_1 A_0	B_1 B_0	F
0 0	0 0	1
0 0	0 1	0
0 0	1 0	0
0 0	1 1	0
0 1	0 0	0
0 1	0 1	1
0 1	1 0	0
0 1	1 1	0
1 0	0 0	0
1 0	0 1	0

A_1 A_0	B_1 B_0	F
1 0	1 0	1
1 0	1 1	0
1 1	0 0	0
1 1	0 1	0
1 1	1 0	0
1 1	1 1	1

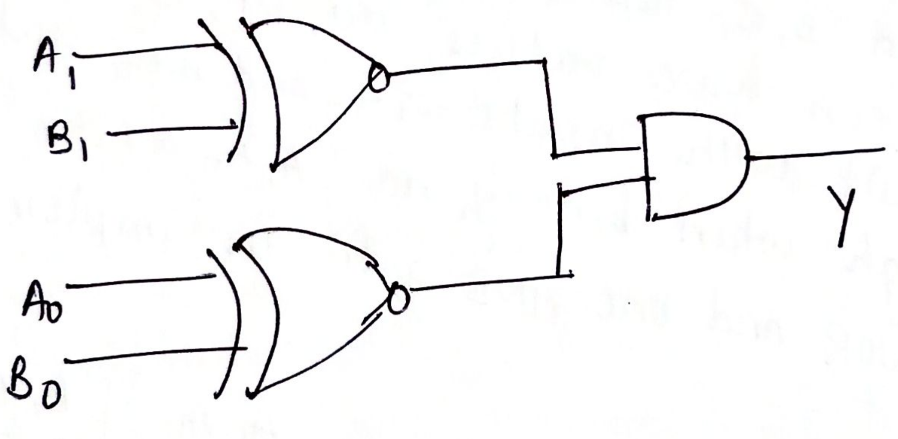
$A_1A_0 \backslash B_1B_0$	00	01	11	10
00	1			
01		1		
10			1	
11				1

$$F = \bar{A}_1 \bar{A}_0 \bar{B}_1 \bar{B}_0 + \bar{A}_1 A_0 \bar{B}_1 B_0 + A_1 A_0 B_1 B_0 + A_1 \bar{A}_0 B_1 \bar{B}_0$$

$$= \bar{A}_1 \bar{B}_1 \left[\bar{A}_0 \bar{B}_0 + A_0 B_0\right] + A_1 B_1 \left[A_0 B_0 + \bar{A}_0 \bar{B}_0 \right)$$

$$\left[A_0 \odot B_0 \right] \left[\bar{A}_1 \bar{B}_1 + A_1 B_1 \right]$$

$$\left[A_0 \odot B_0 \right] \left[A_1 \odot B_1 \right]$$



Que Find using K-Map a minimal form.

$Y = ABC + ABC' + A'BC' + A'B'C'$

A \ BC	$\bar{B}\bar{C}$	$\bar{B}C$	BC	$B\bar{C}$
$\bar{A}$	1			1
A			1	1

$= \bar{A}\bar{C} + AB$ **Ans**

Que $Y = ABC + ABC' + A'BC' + A'B'C + A'B'C'$

A \ BC	$\bar{B}\bar{C}$	$\bar{B}C$	BC	$B\bar{C}$
$\bar{A}$	1	1		1
A			1	1

$= \bar{A}\bar{B} + B\bar{C} + AB$ **Ans**

Lecture No = 24

(Ref Pt 3.7)

K-map upto 5 Variables :-

cell = $2^n \Rightarrow 2^5 = 32$ cells.

BC \ DE	$\bar{D}\bar{E}$	$\bar{D}E$	DE	$D\bar{E}$
$\bar{B}\bar{C}$	m_0	m_1	m_3	m_2
$\bar{B}C$	m_4	m_5	m_7	m_6
BC	m_{12}	m_{13}	m_{15}	m_{14}
$B\bar{C}$	m_8	m_9	m_{11}	m_{10}

$A = 0\ (\bar{A})$

BC \ DE	$\bar{D}\bar{E}$	$\bar{D}E$	DE	$D\bar{E}$
$\bar{B}\bar{C}$	m_{16}	m_{17}	m_{19}	m_{18}
$\bar{B}C$	m_{20}	m_{21}	m_{23}	m_{22}
BC	m_{28}	m_{29}	m_{31}	m_{30}
$B\bar{C}$	m_{24}	m_{25}	m_{27}	m_{26}

$A = 1\ (A)$

Example: Simplify using 5 variable Map

$$F = \bar{A}\bar{B}C\bar{E} + \bar{A}\bar{B}\bar{C}\bar{D} + \bar{B}\bar{D}\bar{E} + \bar{B}C\bar{D} + CD\bar{E} + BD\bar{E}$$

Sol^n Let us find all the min terms

$$\bar{A}\bar{B}C\bar{E} = \bar{A}\bar{B}C\bar{E}(D+\bar{D})$$

$$= \underset{m_6}{\bar{A}\bar{B}CD\bar{E}} + \underset{m_4}{\bar{A}\bar{B}C\bar{D}\bar{E}}$$

$$\bar{A}\bar{B}\bar{C}\bar{D} = \bar{A}\bar{B}\bar{C}\bar{D}(E+\bar{E}) \Rightarrow \bar{A}\bar{B}\bar{C}\bar{D}E + \bar{A}\bar{B}\bar{C}\bar{D}\bar{E}$$

$$\Rightarrow m_1 \qquad m_0$$

$$\bar{B}\bar{D}\bar{E} = (A+\bar{A})\bar{B}\bar{D}\bar{E}(C+\bar{C})$$

$$= (AC + A\bar{C} + \bar{A}C + \bar{A}\bar{C})\bar{B}\bar{D}\bar{E}$$

$$= \underset{m_{20}}{A\bar{B}C\bar{D}\bar{E}} + \underset{m_{16}}{A\bar{B}\bar{C}\bar{D}\bar{E}} + \underset{m_4}{\bar{A}\bar{B}C\bar{D}\bar{E}} + \underset{m_0}{\bar{A}\bar{B}\bar{C}\bar{D}\bar{E}}$$

$$C D \bar{E} \Rightarrow (A+\bar{A})(B+\bar{B}) C D \bar{E}$$

$$\Rightarrow (AB + A\bar{B} + \bar{A}B + \bar{A}\bar{B}) C D \bar{E}$$

$$\Rightarrow ABCD\bar{E} + A\bar{B}CD\bar{E} + \bar{A}BCD\bar{E} + \bar{A}\bar{B}CD\bar{E}$$

$$\Rightarrow m_{30} \quad m_{22} \quad m_{14} \quad m_{6}$$

$$B D \bar{E} = (A+\bar{A}) B (C+\bar{C}) D \bar{E}$$

$$\Rightarrow (AC + A\bar{C} + \bar{A}C + \bar{A}\bar{C}) B D \bar{E}$$

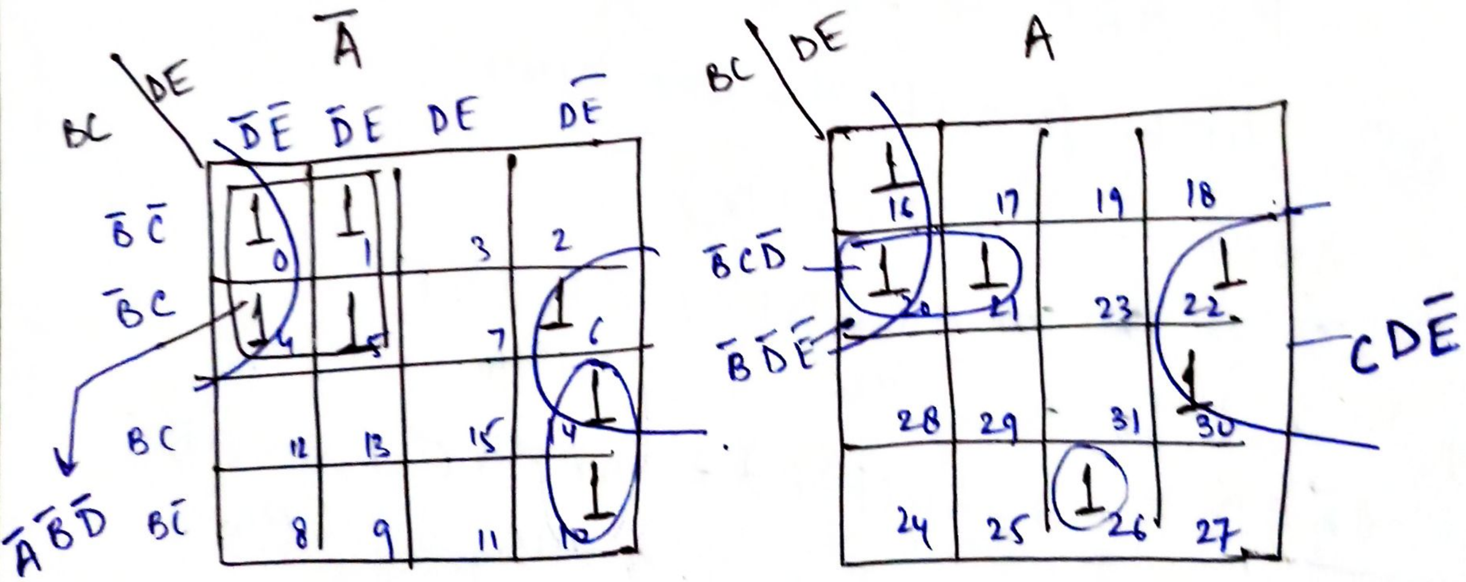
$$\Rightarrow ABCD\bar{E} + AB\bar{C}D\bar{E} + \bar{A}BCD\bar{E} + \bar{A}B\bar{C}D\bar{E}$$

$$\Rightarrow m_{30} \quad m_{26} \quad m_{14} \quad m_{10}$$

In the same way from $\bar{B}C\bar{D}$ we get m_{21}, m_{20}, m_5, m_4

$$F = m_0 + m_1 + m_4 + m_5 + m_6 + m_{10} + m_{14} + m_{16} + m_{20} + m_{21} + m_{22} + m_{26} + m_{30}$$

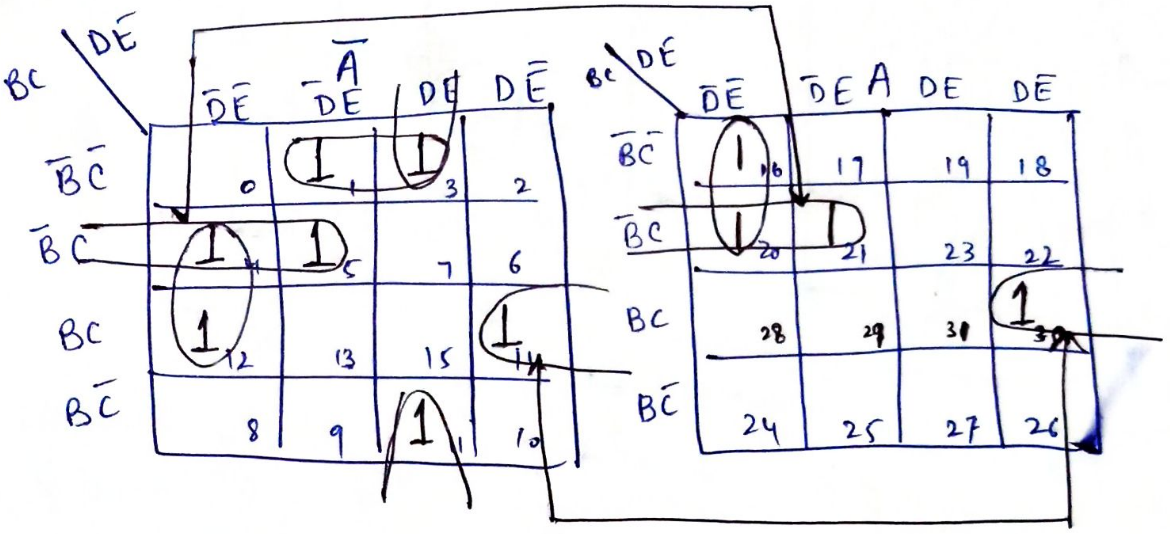
$$F = \Sigma m(0, 1, 4, 5, 6, 10, 14, 16, 20, 21, 22, 26, 30)$$



$$F = \bar{A}\bar{B}\bar{D} + \bar{B}C\bar{D} + \bar{B}\bar{D}E + CD\bar{E} + BD\bar{E} + AB\bar{C}DE$$

Ex $F = \Sigma(1, 3, 4, 5, 11, 12, 14, 16, 20, 21, 30)$

Solution: Since biggest no. is 30, we need to have 5 variables to define this function



$(4, 5, 20, 21)$ - $\bar{B} C \bar{D}$ (since A & E are changing variables, it is eliminated)

$(12, 14) = \bar{A} B C \bar{E}$ (since D is changing variable, so eliminated)

$(14, 30) = B C D \bar{E}$ (since A is changing variable so it is eliminated)

$(3, 11) = \bar{A} \bar{C} D E$

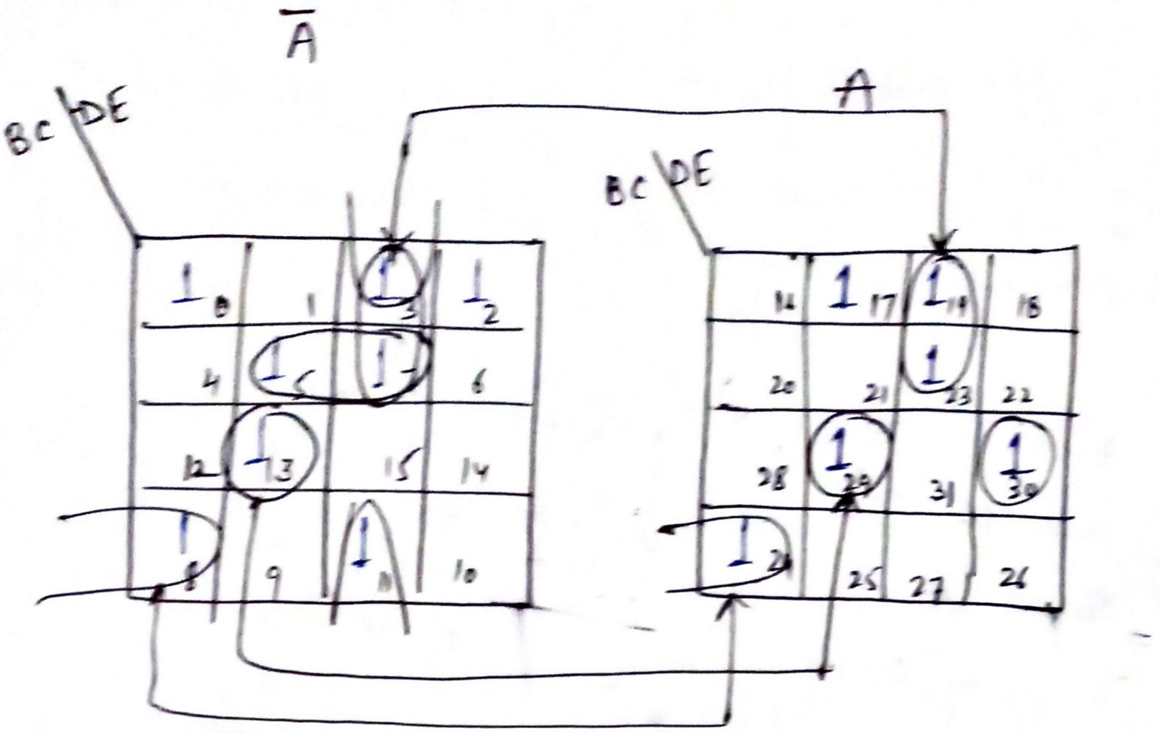
$(16, 20) = A \bar{B} \bar{D} \bar{E}$

$(1, 3) = \bar{A} \bar{B} \bar{C} E$

$$\boxed{f = \bar{B} C \bar{D} + \bar{A} B C \bar{E} + B C D \bar{E} + \bar{A} \bar{C} D E + A \bar{B} \bar{D} \bar{E} + \bar{A} \bar{B} \bar{C} E}$$

Que 2 $F = \Sigma(0, 2, 3, 5, 7, 8, 11, 13, 17, 19, 23, 24, 29, 30)$

Soln: Since biggest no. is 30, so we have to make a 5 variable K-Map.



$(3, 7, 19, 23) = \bar{B}DE$

$(3, 11) = \bar{A}\bar{B}\bar{C}\bar{E}$

$(5, 7) = \bar{A}\bar{B}CE$

$(17, 19) = A\bar{B}\bar{C}E$

$(13, 29) = BC\bar{D}E$

$(8, 24) = B\bar{C}\bar{D}\bar{E}$

$30 = ABCD\bar{E}$

Thus $\boxed{F = \bar{B}DE + \bar{A}\bar{B}\bar{C}\bar{E} + \bar{A}\bar{B}CE + BC\bar{D}E + B\bar{C}\bar{D}\bar{E} + ABCD\bar{E}}$

Ans

Que $F = \Sigma(0,1,2,3,8,9,16,17,20,21,24,25,28,29,30,31)$

Solⁿ

$\bar{A}$

BC \ DE				
	1_0	1_1	1_3	1_2
	4	5	7	6
	12	13	15	14
	1_8	1_9	11	10

A

BC \ DE				
	1_{16}	1_{17}	19	18
	1_{20}	1_{21}	23	22
	1_{28}	1_{29}	1_{31}	1_{30}
	1_{24}	1_{25}	27	26

Group 1 $(0, 1, 3, 2) = \bar{A}\bar{B}\bar{C}$
Group 2 $(0, 1, 16, 17, 8, 9, 24, 25) = \bar{C}\bar{D}$
Group 3 $(28, 29, 30, 31) = ABC$
Group 4 $(16, 17, 20, 21, 24, 25, 28, 29) = A\bar{D}$

$$\boxed{F = A\bar{D} + \bar{C}\bar{D} + ABC + \bar{A}\bar{B}\bar{C}}$$

Que Design a logic circuit having 3 inputs X, Y, Z such that output is 1 when X = 0 or whenever Y = Z = 1.

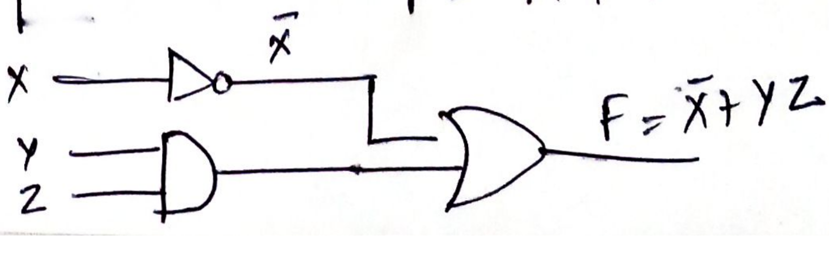
Solⁿ

X	Y	Z	F
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	0
1	0	1	0
1	1	0	0
1	1	1	1

$\Sigma m(0, 1, 2, 3, 7)$

X \ YZ				
	1_0	1_1	1_3	1_2
	4	5	1_7	6

$F = \bar{X} + YZ$



Lecture No-25

(Ref Pt-3 3.8)

K-Map with Don't Care Condition :-

If output value is not certain, it is said to be don't care (X).

$\downarrow$

Either 1 or 0.

SOP simplification (grouping 1's)

$F = \Sigma m(0,1,2,4,5,8,10)$

POS simplification (grouping 0's)

$F = \Pi M(0,2,3,7)$

→ $\boxed{\text{SOP \& POS are complements of each other}}$

$f(A,B,C) = \Sigma m(0,2,3,7) = \Pi M(1,4,5,6)$

Que Logic circuit which operates by a relay has 4 inputs a, b, c, d. The relay is on for abcd = 0000, 0010, 0011, 1000. The states 1001, 1010, 1110, 1111 don't occur. For remaining states the relay is OFF

a) Prepare truth table ~~fo~~ & minimize using K-map in POS.

b) Realize f using NOR gate only.

Inputs a b c d	Output F	
0 0 0 0	1	
0 0 0 1	0	$\rightarrow M_1$
0 0 1 0	1	~~$\rightarrow M$~~
0 0 1 1	1	
0 1 0 0	0	$\rightarrow M_4$
0 1 0 1	0	$\rightarrow M_5$
0 1 1 0	0	$\rightarrow M_6$
0 1 1 1	0	$\rightarrow M_7$
1 0 0 0	1	
1 0 0 1	X	$\rightarrow d_9$
1 0 1 0	X	$\rightarrow d_{10}$
1 0 1 1	0	$\rightarrow M_{11}$
1 1 0 0	0	$\rightarrow M_{12}$
1 1 0 1	0	$\rightarrow M_{13}$
1 1 1 0	X	$\rightarrow d_{14}$
1 1 1 1	X	$\rightarrow d_{15}$

$$F = \pi M(1, 4, 5, 6, 7, 11, 12, 13) + d\pi(9, 10, 14, 15)$$

AB \ CD	00 $(C+D)$	01 $(C+\bar{D})$	11 $(\bar{C}+\bar{D})$	10 $\bar{C}+D$
$(A+B)$ 00		0		
$(A+\bar{B})$ 01	0	0	0	0
$(\bar{A}+\bar{B})$ 11	0	0	X	X
$(\bar{A}+B)$ 10		X	0	X

(i) $C+\bar{D}$

(ii) $\bar{B}$

(iii) $(\bar{A}+\bar{C})$

$$\boxed{F = (C+\bar{D})(\bar{A}+\bar{C})\bar{B}}$$ Ans

Implementation using NOR gate (complementing both side)

$$\bar F = \overline{(C+\bar D)(\bar A+\bar C)\,\bar B}$$

$$\bar F = \overline{(C+\bar D)} + \overline{(\bar A+\bar C)} + \bar{\bar B}$$

$$\bar F = F_1 + F_2 + F_3$$

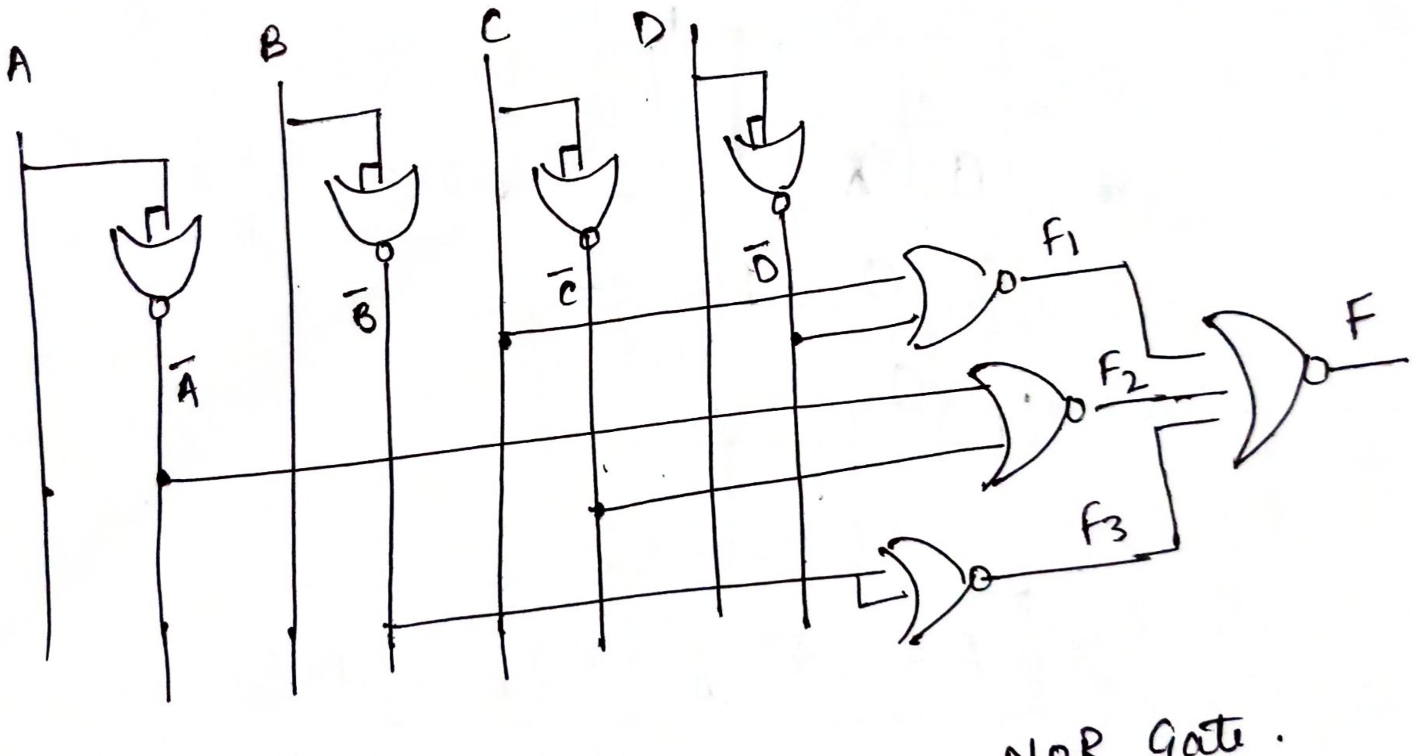
Again complementing both side

$$\bar{\bar F} = \overline{F_1+F_2+F_3}$$

$$F = \overline{F_1+F_2+F_3}$$ where $F_1 = \overline{C+\bar D}$

$F_2 = \overline{\bar A+\bar C}$

$F_3 = \bar{\bar B}$



Implementation using NOR Gate.

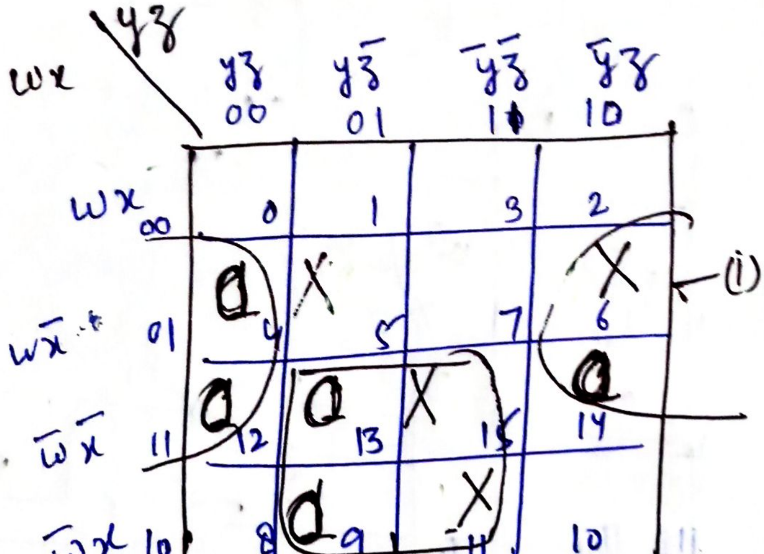
<u>Example</u> Simplify the boolean expression with don't care condition in POS.

$$F(w,x,y,z) = \Sigma(0,1,2,3,7,8,10)$$
$$d(w,x,y,z) = \Sigma(5,6,11,15)$$

<u>Sol</u>

Solⁿ Min terms are given but for product of sum, we need max terms, so initially find maxterms using the complementary nature.

$$f(w,x,y,z) = \Pi M(4,9,12,13,14)$$
$$d(w,x,y,z) = \Pi M(5,6,11,15)$$



(i) = $\bar{x}+z$

(ii) = $\bar{w}+\bar{z}$

$$\boxed{F = (\bar{x}+z)(\bar{w}+\bar{z})}$$ Ans

Applications of logic Gates

logic gates have several applications to the computers. They are used in the following -

(i) Adders, (ii) Encoder (iii) Decoder
(iv) Multiplexer.

1. **Adders :**

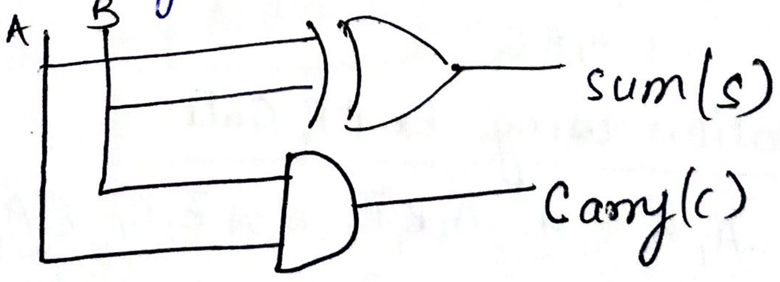
(a) Half Adder :- A logic circuit that perform the addition of two bits is called half adder. The half adder circuit needs two binary input & two binary outputs.

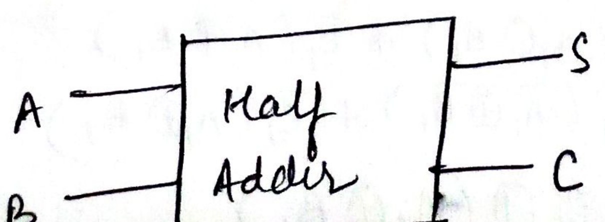
Truth table for adder is -

A	B	Carry (c)	sum (s)
0	0	0	0
0	1	0	1
1	0	0	1
1	1	1	0

$C = AB$ $\qquad$ $S = \bar{A}B + A\bar{B}$

logic circuit is given as-





(ii) Full Adder :

A logic circuit the perform the addition of three bits is a full adder. It consists of 3 inputs & two outputs. The two outputs are sum & carry.

Table for full adder -

A_1	B_1	C_1	Carry(C)	Sum(S)
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

Sum

A_1 \ B_1C_1	00	01	11	10
0		1		1
1	1		1	

$$S = \bar{A}\bar{B}C_1 + \bar{A}_1B_1\bar{C}_1 + A_1\bar{B}_1\bar{C}_1 + \bar{A}_1BC$$

Carry

A_1 \ B_1C_1	00	01	11	10
0			1	
1		1	1	1

$$C = A_1B_1 + B_1C_1 + C_1A_1$$

Implementation using Ex-OR Gate

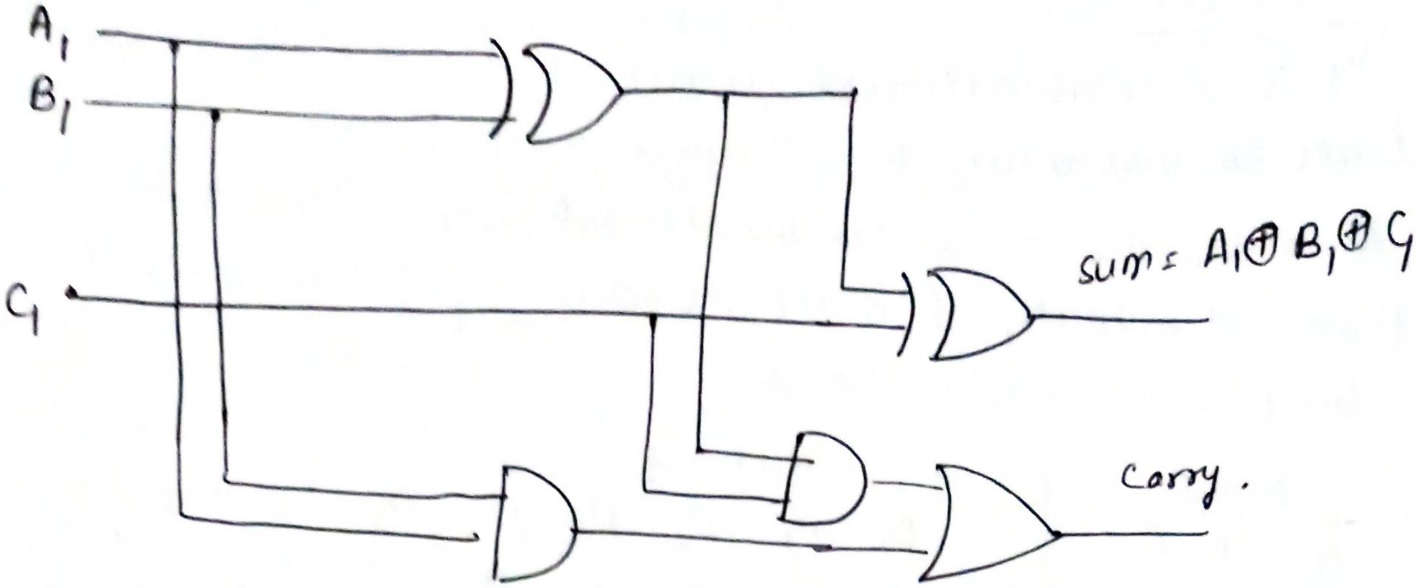
$$S = \bar{A}_1\bar{B}_1C_1 + \bar{A}_1B_1\bar{C}_1 + A\bar{B}_1\bar{C}_1 + A_1B_1C_1$$

$$= C_1(\bar{A}_1\bar{B}_1 + AB_1) + \bar{C}_1(\bar{A}_1B + A\bar{B}_1)$$

$$= C_1(A_1 \odot B_1) + \bar{C}_1(A_1 \oplus B_1)$$

$$= C_1(\overline{A_1 \oplus B_1}) + \bar{C}_1(A_1 \oplus B_1)$$

$$= C_1 \oplus (A_1 \oplus B_1)$$



Carry

$$C = A_1B_1 + B_1C_1 + C_1A_1$$

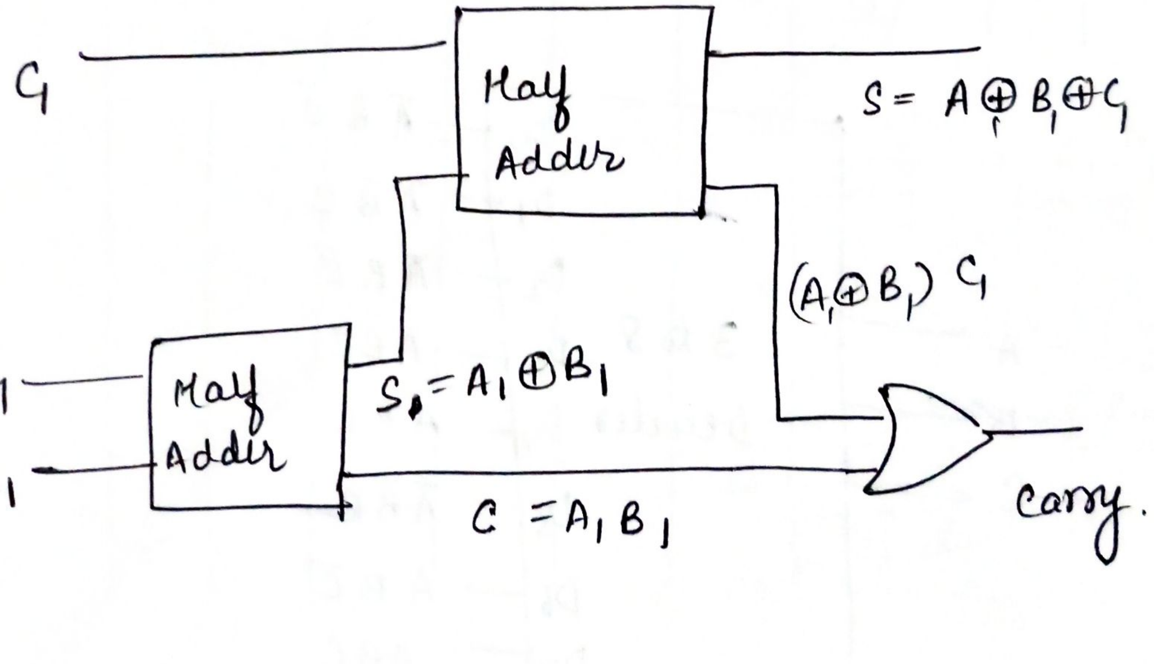
$$= A_1B_1(C_1+\bar{C}_1) + B_1C_1(A_1+\bar{A}_1) + C_1A_1(B_1+\bar{B}_1)$$

$$= A_1B_1C_1 + A_1B_1\bar{C}_1 + A_1B_1C_1 + \bar{A}_1B_1C_1 + A_1B_1C_1 + A_1\bar{B}_1C_1$$

$$= A_1B_1C_1 + A_1B_1\bar{C}_1 + \bar{A}_1B_1C_1 + A_1\bar{B}_1C_1$$

$$= A_1B_1(C_1+\bar{C}_1) + C_1(\bar{A}_1B_1 + A_1\bar{B}_1)$$

$$= A_1B_1 + C_1(A_1 \oplus B_1)$$

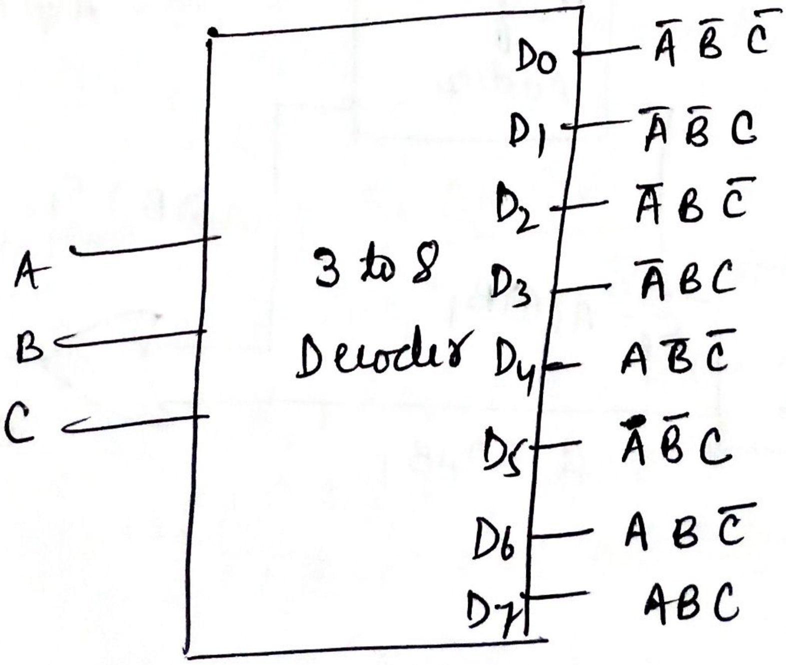


Decoder :

It is a combinational circuit that converts n input lines to maximum of 2^n unique output lines.

A ~~b~~ decoder's ~~is a combinational~~ output will have less than 2^n outputs, if n bit decoded infoⁿ has unused or don't care combinations.

Input			Outputs							
A	B	C	D_0	D_1	D_2	D_3	D_4	D_5	D_6	D_7
0	0	0	1	0	0	0	0	0	0	0
0	0	1	0	1	0	0	0	0	0	0
0	1	0	0	0	1	0	0	0	0	0
0	1	1	0	0	0	1	0	0	0	0
1	0	0	0	0	0	0	1	0	0	0
1	0	1	0	0	0	0	0	1	0	0
1	1	0	0	0	0	0	0	0	1	0
1	1	1	0	0	0	0	0	0	0	1

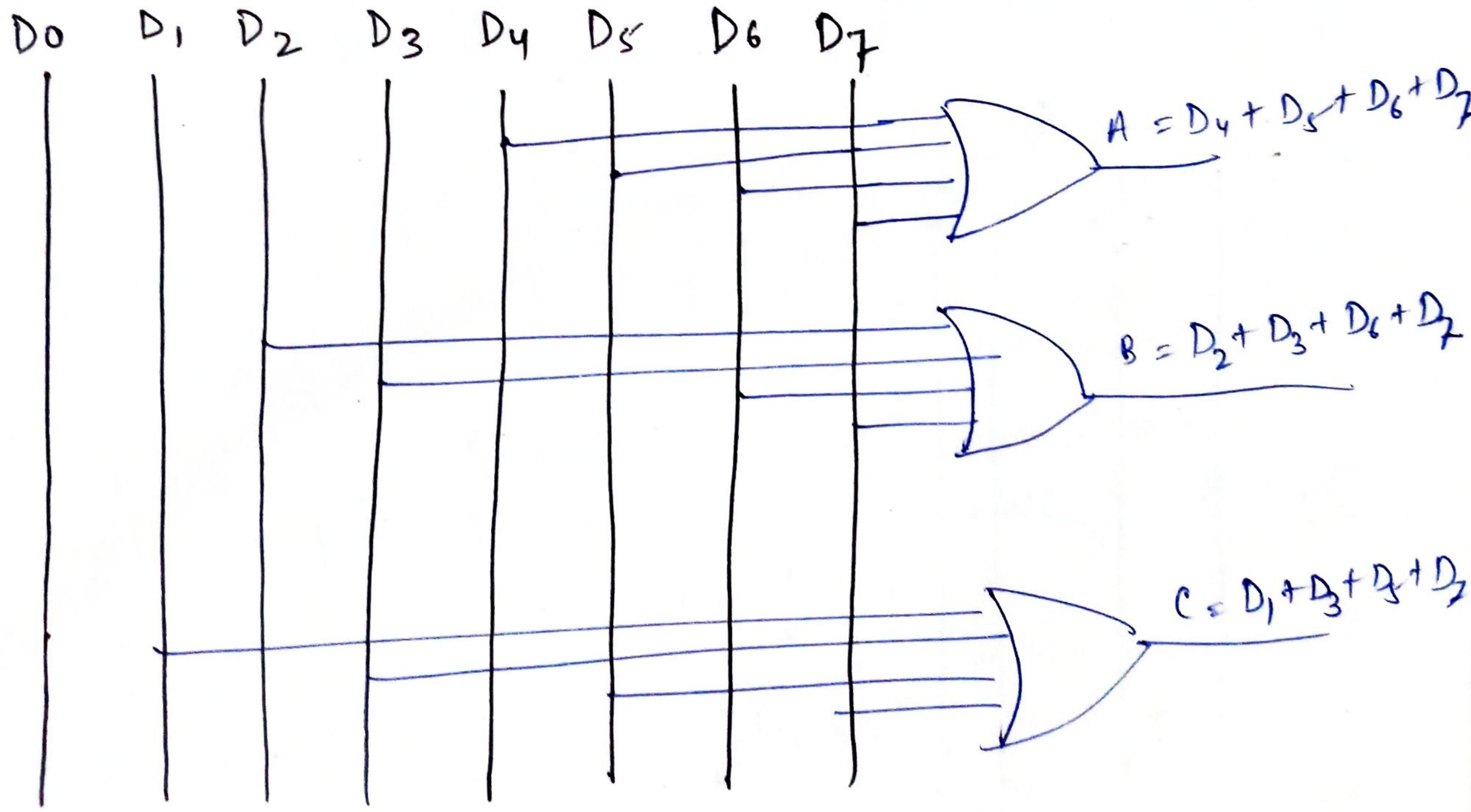


Encoder :

An encoder is a combinational circuit that works just reverse of that of decoder. There are (2^n) or less input lines. There are 2^n input lines and n output lines in an Encoder.

octal to binary Encoder

D_0	D_1	D_2	D_3	D_4	D_5	D_6	D_7	Output A	B	C
1	0	0	0	0	0	0	0	0	0	0
0	1	0	0	0	0	0	0	0	0	1
0	0	1	0	0	0	0	0	0	1	0
0	0	0	1	0	0	0	0	0	1	1
0	0	0	0	1	0	0	0	1	0	0
0	0	0	0	0	1	0	0	1	0	1
0	0	0	0	0	0	1	0	1	1	0
0	0	0	0	0	0	0	1	1	1	1



Multiplexer :

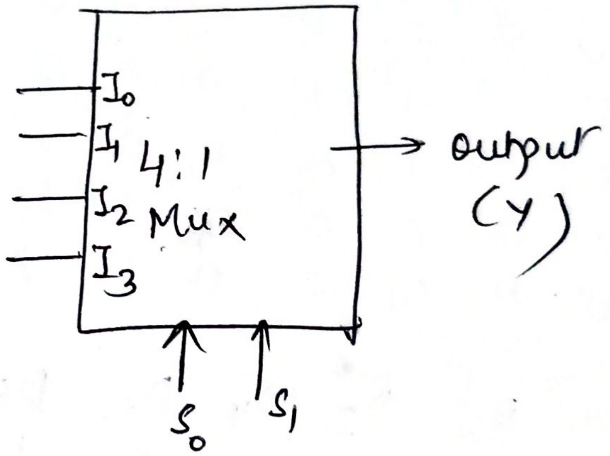
Multiplexing means transmitting a large no. of information units over a small no of channels or lines.

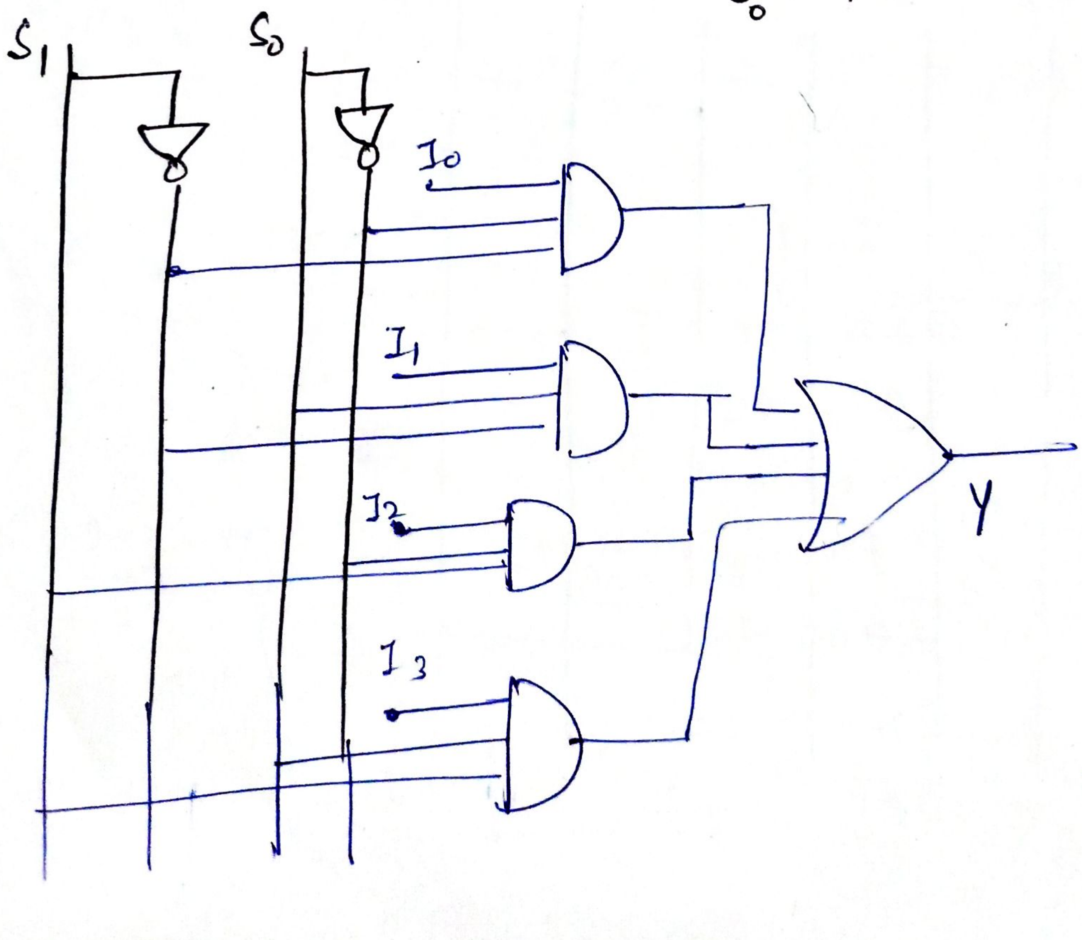
Digital Multiplexer is a combinational circuit that selects binary info^n from one of many input lines & directs it to a single output line.

There are n input lines & m selection lines,

then $\boxed{n = 2^m}$

Selectline S_1	S_0	Input ~~output~~
0	0	I_0
0	1	I_1
1	0	I_2
1	1	I_3





Sequential Circuit

Sequential circuit is obtained by combinational circuit and memory element connected in feedback path

Sequential logic

Output depends not only on current input but also on past values.
Need some type of memory to remember the past input values.

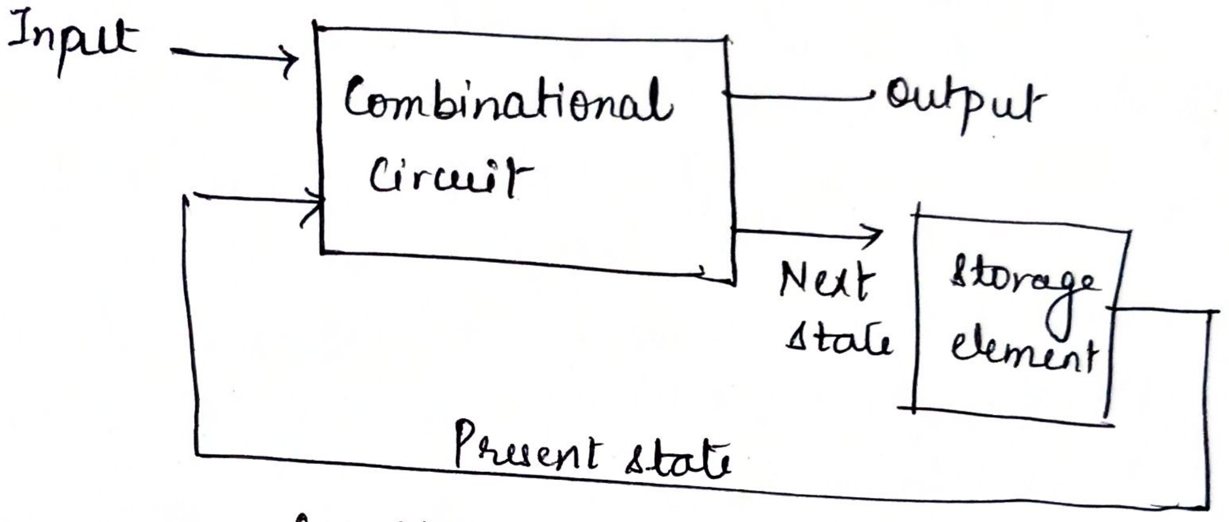


fig: Block Diagram of sequential circuit

Difference between Sequential & Combinational Circuit

Combinational Circuit	Sequential circuit
1. In this output depends only upon present input	In this output depends upon present as well as past input
2. Speed is fast & designe is easy.	Speed is slow & design is tough as compared to combinational circuit.
3.	

3.	There is no feedback b/w Input & output.	There is a feedback path b/w Input & output.
4.	Used for arithmetic as well as boolean opern	Mainly used for storing data.
5.	Elementary building blocks are logic gates.	Elementary building blocks are flip-flops.
6.	These circuit have no memory element.	They have memory element.

UNIT-3

Lecture No-27

Partial Order Set (Poset)

A non empty set A, together with a binary relation R is said to be a partially ordered set or a poset if following conditions are satisfied -

P1 Reflexivity $aRa \ \forall a \in A$

P2 Antisymmetry If $a, b \in A$ then aRb and $bRa \Rightarrow a = b$

P3 Transitivity If $a, b, c \in A$ then $aRb, bRc \Rightarrow aRc$

The relation R on set A is called Partial Order Relation & denoted by (A, R) or $(A, \leq)$. The relation

Example The set S on any collection of sets is partial ordering of S. $\subseteq$ read as "is subset of"

The set S is poset if it is -

P1 Reflexive since $A \subseteq A$ for any subset A of S
$\therefore \subseteq$ is reflexive.

P2 Antisymmetric If $A \subseteq B$ and $B \subseteq A$ then $A = B$ $\forall A, B \in S$ $\therefore \subseteq$ is antisymm

P3 Transitivity If $A \subseteq B$ and $B \subseteq C$ for any sets, $A, B, C \in S$ then $A = C$ $\therefore \subseteq$ is transitive

Hence $(S, \subseteq)$ is a poset.

Que let $A = \{2, 3, 6, 12, 24, 36\}$ and R be the relation in A; which is defined by a divides b then R is partial order in A.

Solⁿ (i) Reflexive $a | a \; \forall a \in A$ {'/' is reflexive}

(ii) Antisymmetric If $a | b$ and $b | a$ $\forall a, b \in A$ then $a = b$ {'/' is antisymmetric}

(iii) Transitivity If $a | b$ and $b | c$ $\forall a, b, c \in A$ then $a | c$

Ex $2 | 6$, $6 | 12 \Rightarrow 2 | 6$ {'/' is transitive}

Hence $(A, '|')$ is a poset.

Comparable :-

Two elements a & b in a poset $(S, \leq)$ is said to be comparable if either $a \leq b$ or $b \leq a$.

Ex In poset $(Z^+, '|')$, the integer 3 & 9 are comparable since $3 | 9$

But 5 & 7 are incomparable since neither $5 | 7$ nor $7 | 5$.

Immediate Predecessor & Immediate Successor

Let $(A, \leq)$ be a poset, & $a, b \in A$.

a is said to be immediate predecessor of b or b is immediate successor of a, written as -

$$a << b$$

If $a < b$ & no elements of A lie b/w a & b ie $\nexists c \in A$ such that $a < c < b$.

Hasse Diagram :

or Representation of Poset

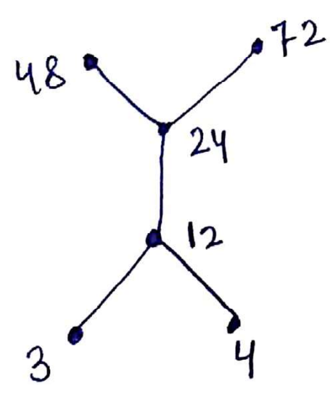
A graphical representation of a partial ordering relation in which all arrow heads are understood to be pointing upward is known as Hasse Diagram

In Hasse Diagram, we draw a rising line from a to b if b covers a.

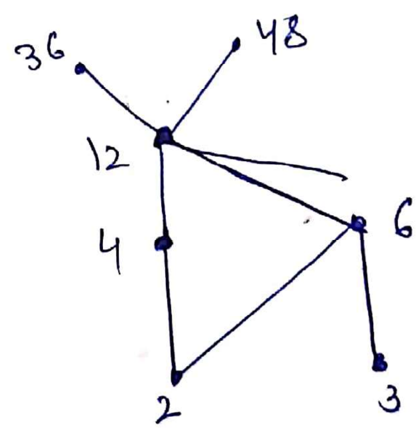
Procedure for drawing Hasse Diagram

1. Draw the diagraph of given relation
2. Delete all cycles of diagraph
3. Eliminate all edges that are implied by transitive relation.
4. Draw the diagram of partial order with edges pointing upward so that arrow may be omitted from edges.
5. Replace the circles representing vertices by dots.

Que Draw Hasse Diagram of $(A, \leq)$ where $A = \{3, 4, 12, 24, 48, 72\}$ & relation $\leq$ be such that $a \leq b$ if a divides b.



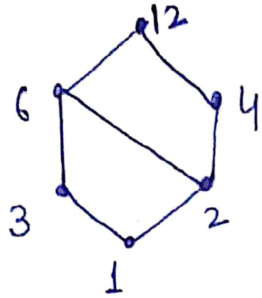
Que2 $B = \{2, 3, 4, 6, 12, 36, 48\}$



Que Draw Hasse diagram for
(i) D_{12} (ii) D_{30} (iii) D_{45}

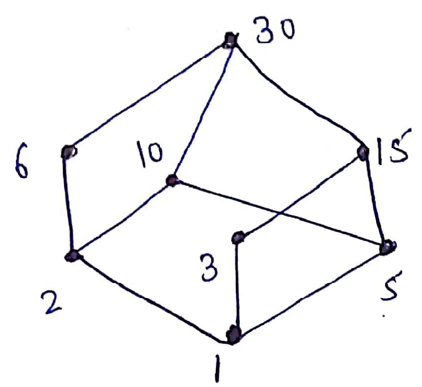
Sol^n for D_{12}

set $A = \{1, 2, 3, 4, 6, 12\}$



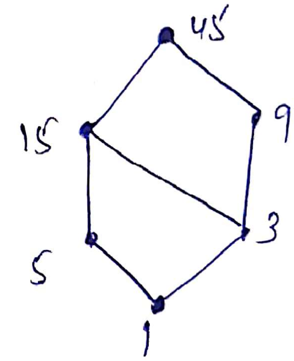
for D_{30}

set $B = \{1, 2, 3, 5, 6, 10, 15\}$

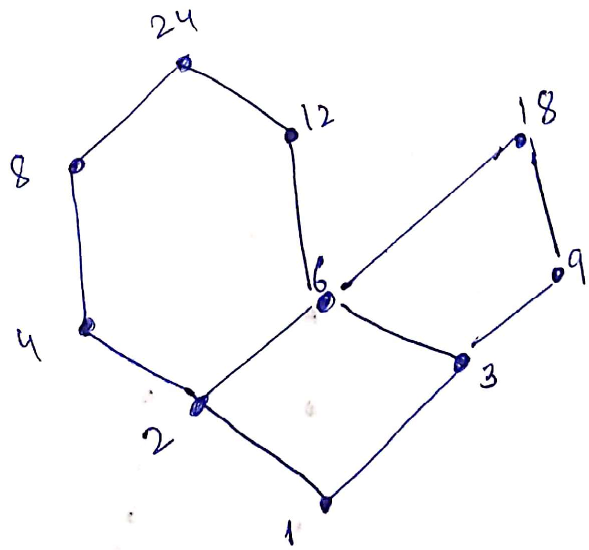


for D_{45}

$\{1, 3, 5, 9, 15, 45\}$



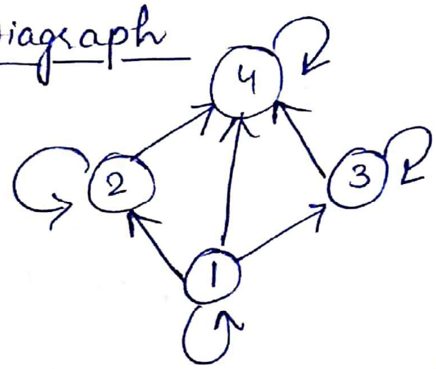
Que $A = \{1, 2, 3, 4, 6, 8, 9, 12, 18, 24\}$ be ordered by relation "a divides b". Draw Hasse Diagram.



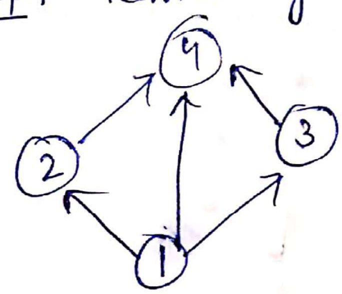
Que. Determine Hassediagram of the Relation R on $A = \{1, 2, 3, 4\}$

$R = \{(1,1)(1,2)(2,2)(2,4)(1,3)(3,3)(3,4)(1,4)(4,4)\}$

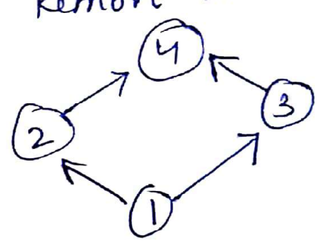
Sol^n Diagraph



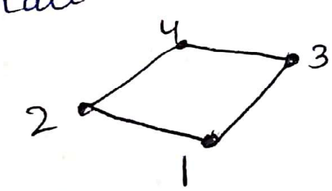
Step 1 Remove cycles



Step 2 Remove transitive edge



Step 3 All edges are pointing upwards, remove arrow from edges, replace circles by dots.



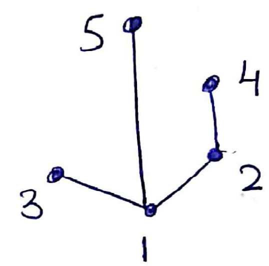
Maximal Element, and Minimal Element :-

An element belonging to a poset $(A, \leq)$ is said to be maximal element of A if there is no element C in A such that $a \leq C$.

An element $b \in A$ is said to be minimal element of A if there is no element c in A such that $c \leq b$

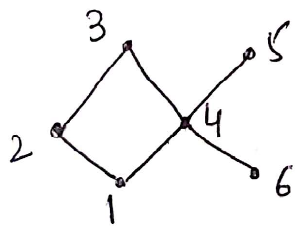
Ex $(P, \leq)$ is a poset. $P = \{1, 2, 3, 4, 5\}$ & $\leq$ is the relation of division. find maximal & minimal element

from Hasse Diagram

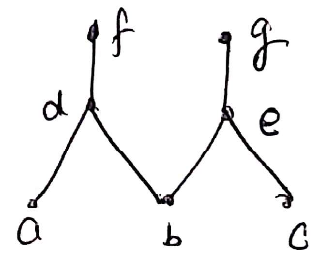


Maximal element 3, 4, 5
Minimal element 1.

Que



Maximal element = 3, 5
Minimal " = 1, 6



Maximal element = ~~a. b~~ f. g.
Minimal " = a, b, c.

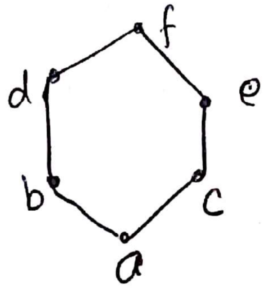
Greatest Element, Least Element

An element $a \in A$ is said to be a Greatest element of A, if $x \leq a \ \forall x \in A$.

An element $a \in A$ is called least element of A if $a \leq x \ \forall \ x \in A$,

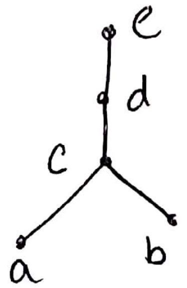
Least element is also called 1st element or zero element of A.

Que find the greatest & least element of following Hasse Diagram.



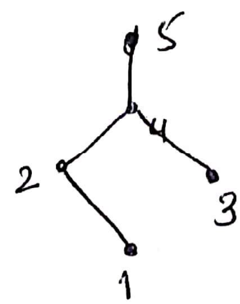
(i)

Greatest = f
least = a



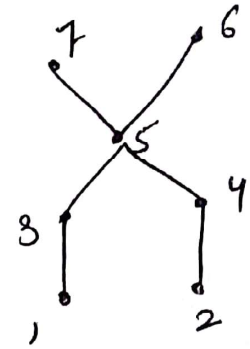
(ii)

greatest = e
least = none



(iii)

greatest = 5
least = None



(iv)

greatest = None
least = None.

Upper Bound & ~~Least Upper~~ Lower Bound

let $(P, \leq)$ be a poset & let A be a subset of P.
An element $x \in P$ is called an upper bound of A if
$a \leq x \ \forall a \in A$.

let $(P, \leq)$ be poset & A be a subset of P.
An element $x \in P$ is called an lower bound of A if
$x \leq a, \forall a \in A$.

Least Upper Bound or (Supremum)

let $(P, \leq)$ be a poset & let $A \leq P$.
An element $x \in P$ is said to be a least upper bound or Supremum of A if x is an upper bound of A
and $x \leq y \ \forall$ upper bounds y of A.

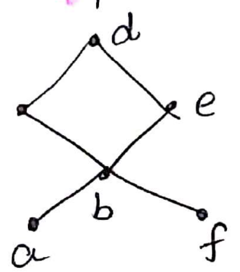
least Upper Bound, if it exists, is unique & will be denoted by "lub" or "sup".

Greatest lower Bound or (Infimum)

Let $(P, \leq)$ be a poset and $A \subseteq P$, An element $x \in P$ is said to be a greatest lower bound or infimum of A, written as $glb(A)$ or $inf(A)$, if x is lower bound and $y \leq x$ $\forall$ lower bound y of A.

Consider the poset whose diagram is given by -

Ex



upper bound for $\{c, e\}$ is d.

$lub\{c, e\} = d$

lower bound for $\{c, e\}$ are b, a & f

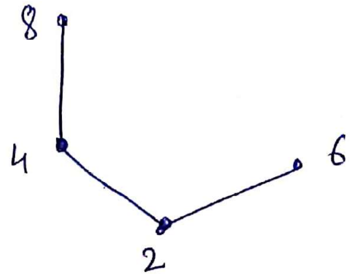
$glb\{c, e\} = b$.

Q

Que Consider the partially order set $A = \{2, 4, 6, 8\}$ where $2/4$ means 2 divide 4, show with reason whether following stmt are true or false.

(i) Every pair of elements in the poset has a glb.
(ii) Every pair of elements in a posed has a lub.
(iii) This poset is lattice.

Solⁿ



lower bound $\{2, 4\} = 2$
" " $\{4, 4\} = 4$
" " $\{4, 6\} = 2$
" " $\{4, 8\} = \{2, 4\}$
" " $\{2, 6\} = \{2\}$
" " $\{2, 8\} = \{2\}$
" " $\{6, 8\} = \{2\}$

table for glb

(i) True

$\wedge$	2	4	6	8
2	2	2	2	2
4	2	4	2	4
6	2	2	6	2
8	2	4	2	8

Here $a \wedge b = HCF$ of (a, b)
Hence every pair of element has glb.

(ii) False, since there exists no lub of 6 & 8.

(iii) False, since 6 & 8 has no lub. This is not lattice. L.

Lecture No. 28
(Ref Pt - 3.11)

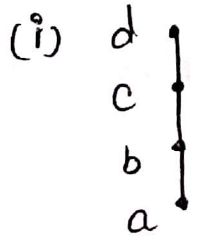
Lattice :-

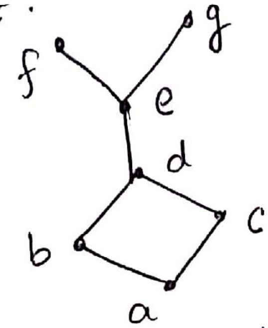
A partially ordered set $(L, \leq)$ is said to be a lattice if every two elements in the set L has unique least upper bound (sup) and a unique greatest lower bound (inf).

The poset $(L, \leq)$ is a lattice if for every $a, b \in L$
$\sup\{a, b\}$ and $\inf\{a, b\}$ exists in L.

$\sup\{a,b\} = a \vee b$ = a join b &
$\text{Inf}\{a,b\} = a \wedge b$ = a meet b.

Que Determine whether following Hasse Diagram represent lattice or not.

(i)


(iii)


(i) Construct the closure table for lub ($\vee$) or supremum or joint and glb ($\wedge$)

$\vee$	a	b	c	d
a	a	b	c	d
b	b	b	c	d
c	c	c	c	d
d	d	d	d	d

$\wedge$	a	b	c	d
a	a	a	a	a
b	a	b	b	b
c	a	b	c	c
d	a	b	c	d

since each subset of two element has least upper bound & a greatest lower bound.
so this is the lattice.

(ii)

V	a	b	c	d	e	f	g
a	a	b	c	d	e	f	g
b	b	b	d	d	e	f	g
c	c	d	c	d	e	f	g
d	d	d	d	d	e	f	g
e	e	e	e	e	e	f	g
f	f	f	f	f	f	f	–
g	g	g	g	g	g	–	g

$\wedge$	a	b	c	d	e	f	g
a	a	a	a	a	a	a	a
b	a	b	a	b	b	b	b
c	a	a	c	c	c	c	c
d	a	b	c	d	d	d	d
e	a	b	c	d	e	e	e
f	a	b	c	d	e	f	e
g	a	b	c	d	e	e	g

since each subset of two elements has not least upper bound but has greatest lower bound, so this is not the lattice.

Dual Lattice

Let $(L, \leq)$ be a lattice, for any $a, b \in L$, the converse of relation $\leq$ denoted by $\geq$ defined as –

$$a \geq b \Leftrightarrow b \leq a$$

Then $(L, \geq)$ is also a lattice called Dual Lattice of $(L, \leq)$

Properties of Lattice :-

1) Idempotent Law

$a \wedge a = a$ $\qquad$ $a \vee a = a$

2) Commutative Law for each $a, b \in L$

$a \wedge b = b \wedge a$ $\qquad$ $a \vee b = b \vee a$

3) Associative Law for any $a, b, c \in L$

(i) $(a \wedge b) \wedge c = a \wedge (b \wedge c)$ (ii) $(a \vee b) \vee c = a \vee (b \vee c)$

4) Absorption law for $a, b \in L$

(i) $a \wedge (a \vee b) = a \wedge (b \wedge c)$

(ii) $a \vee (a \wedge b) = a$

L.

Theorem 1 ; Show that dual of lattice is a lattice.

Solution :- Let $(L, \leq)$ be a lattice and $(L, \geq)$ be its dual, where the relation $\geq$ is defined as -

$$x \geq y \text{ if and only if } y \leq x$$

We now show that $\geq$ is reflexive, antisymmetric & transitive.

(P1) $\geq$ is reflexive : Let $a \in L$. Since $\leq$ is reflexive, we have

$a \leq a \;\forall a \in L \Rightarrow a \geq a \;\forall a \in L$

$\Rightarrow \geq$ is reflexive.

(P2) $\geq$ is anti-symmetric : Let $a, b \in L$ be such that $a \geq b$ and $b \geq a$. Then

$a \geq b$ and $b \geq a \Rightarrow b \leq a$ and $a \leq b$

$\Rightarrow a = b$

Thus $a \geq b$ and $b \geq a \Rightarrow a = b$

Hence, $\geq$ is antisymmetric.

(P3) $\geq$ is transitive.

Let $a, b, c \in L$ be such that $a \geq b$ and $b \geq c$. Then

$a \geq b$ and $b \geq c \Rightarrow b \leq a$ and $a \leq b$

$\Rightarrow c \leq b$ and $b \leq a$

$\Rightarrow c \leq a$

$\Rightarrow a \geq c$

Thus $a \geq b$ and $b \geq c \Rightarrow a \geq c$.

Hence, $\geq$ is transitive.

Therefore, $\geq$ is a partial order relation & L and so $(L, \geq)$ is a poset.

Let $a, b \in L$. Then since $(L, \leq)$ is a lattice,
$\sup\{a,b\}$ exists in $(L, \leq)$

Let $a \vee b = \sup\{a,b\}$ in $(L, \leq)$ Then

Now $a \leq a \vee b$ and $b \leq a \vee b$

$\Rightarrow a \vee b \geq a$ and $a \vee b \geq b$

$\Rightarrow a \vee b$ is a lower bound of $\{a,b\}$ in $(L, \geq)$

We shall show that $a \vee b$ is the greatest lower bound of $\{a,b\}$ in $(L, \geq)$

Let l be any lower bound of $\{a,b\}$ in $(L, \geq)$ Then $l \geq a$ and $l \geq b \Rightarrow a \leq l$ and $b \leq l$

$\Rightarrow l$ is an upper bound of $\{a, b\}$ in $(L, \leq)$

$\Rightarrow \text{lub}\{a,b\} \leq l$ in $(L, \leq)$

$\Rightarrow a \vee b \leq l$ in $(L, \leq)$

$\Rightarrow l \geq a \vee b$

$\Rightarrow a \vee b$ is glb of $\{a,b\}$ in $(L, \geq)$

Similarly, we can show that $a \wedge b$ is the least upper bound of $\{a,b\}$ in $(L, \geq)$. Hence $(L, \geq)$ is a lattice.

* **Distributive lattice :-**

A lattice L is called distributive lattice if for any element a, b & c of L, it satisfies the following properties -

(i) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

(ii) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$

Theorem: Let $a, b, c \in L$ where $(L, \leq)$ is a distributive lattice. Then $a \vee b = a \vee c$ and $a \wedge b = a \wedge c \Rightarrow b = c$.

Proof: We know that

$$
\begin{aligned}
b &= b \vee (b \wedge a) \quad \{\text{Absorption law}\} \\
&= b \vee (a \wedge b) \\
&= b \vee (a \wedge c) \\
&= (b \vee a) \wedge (b \vee c) \\
&= (a \vee b) \wedge (c \vee b) \\
&= (a \vee c) \wedge (c \vee b) \\
&= (c \vee a) \wedge (c \vee b) \Rightarrow c \vee (a \wedge b) = c \vee (a \wedge c) = c \vee (c \wedge a) \\
&= \cancel{c \vee (a \wedge b)}\ c \vee (a \wedge c) \\
&= c \quad \{\text{absorption}\}
\end{aligned}
$$

* **Complete Lattice :**

Let $(L, \leq)$ be lattice. Then L is said to be complete if every subset A of L, $\wedge A$ and $\vee A$ exists in L.
Thus, in every complete lattice $(L, \leq)$, there exists a greatest element g and a least element l.

* **Complement of an Element in a Lattice :-**

Let $(L, \leq)$ be a lattice and let 0 & 1 be its lower & upper bounds. If $a \in L$ is an element then an element b is called complement of a if

$$a \vee b = 1 \text{ and } a \wedge b = 0.$$

* **Complemented Lattice**

* Let $(L, \leq)$ be a lattice with universal bounds 0 & 1. The lattice L is said to be complemented lattice if every element in L has a complement i.e. -

$$\begin{cases} a \vee 1 = 1, & a \wedge 1 = a \\ a \wedge 0 = 0, & a \vee 0 = a \end{cases}$$

The complement of a is denoted by a' or $\bar{a}$. Then

$$a \wedge a' = 0, \quad a \vee a' = 1$$

* ~~Complemented Complete~~ Bounded Lattice :-

Let $(L, \leq)$ be a lattice. Then L is said to be bounded lattice if it has a least element 0 & a greatest element 1.

0 is called the identity of join & 1 is called the identity of meet in a bounded lattice $(L, \vee, \wedge)$

* Modular lattices :-

A lattice L is said to be modular lattice if forall $a, b, c \in L$, $a \leq c \Rightarrow a \vee (b \wedge c) = (a \vee b) \wedge c$.

Theorem : let $L = \{a_1, a_2 .. a_n\}$ be finite lattice then L is bounded.

Proof : Let $L = \{a_1, a_2 ... a_n\}$ be any finite lattice, Then we have to show that L having least & greatest element.

Now
$$b_1 = a_1$$
$$b_2 = a_2 \wedge b_1$$
$$b_3 = a_3 \wedge b_2$$
$$\cdots$$
$$b_n = a_n \wedge b_{n-1}$$

All $b_1, b_2 .. b_n \in L$ and $b_n \leq a_i$ $\forall i = 1, 2, 3 ... n$.

$\Rightarrow b_n$ is the least element of L.

$b_n = a_1 \wedge a_2 \wedge a_3 \dots \wedge a_n$

Similarly, $a_1 \vee a_2 \dots \vee a_n$ is the greatest element of L.
Thus, L is bounded lattice.

Sublattice

A non empty subset M of lattice $(L, \leq)$ is said to be sub-lattice of L if M is closed w.r. to meet $(\wedge)$ and join$(\vee)$ ie -

$$x, y \in M \Rightarrow x \vee y \in M \text{ and } x \wedge y \in M.$$

or

A non-empty subset of M of lattice $(L, \leq)$ is said to be sub-lattice of L if M itself formed lattice w.r. to $\vee$ and $\wedge$ operation.

Example Consider the lattice of all integer 'I' under the opera^n divisibility. The lattice D_n of all divisors of $n > 1$ is a sub-lattice of 'I'. I

Determine all the sub-lattice of D_{30} that Contain at least four elements -

$$D_{30} = \{1, 2, 3, 5, 6, 10, 15, 30\}$$

Sol^n The sublattice of D_{30} that solution at least four elements are as follows -

$\{1, 2, 6, 30\}$ $\{1, 2, 3, 30\}$ $\{1, 5, 15, 30\}$
$\{1, 3, 6, 30\}$ $\{1, 5, 10, 30\}$ $\{1, 3, 15, 30\}$

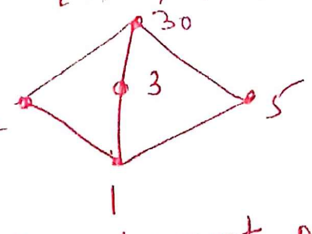
* Complemented Complete lattice

let $(L, \leq)$ be a complete lattice with greatest & lower element g & l, then L is called complemented complete lattice if for each $a \in L$ there exists a' such that

$$a \vee a' = g \text{ and } a \wedge a' = l$$

NOTE: Complemented Distributive lattice is called Boolean Algebra

Que $A = \{1, 2, 3, 5, 30\}$ and $a \leq b$ iff a divides b



find complement of 2.

Sol^n Here ~~lower bound~~ = 1 least element
~~Upper Bound~~ = 30 greatest element.

$2 \wedge 3 = 1, \quad 2 \vee 3 = 30$

$2 \wedge 5 = 1, \quad 2 \vee 5 = 30$

Hence 2, has two complements 3 & 5.

Que Prove that in a distributive lattice, if an element has a complement then this complement is unique.

Proof Let $(L, \leq)$ be a bounded distributive lattice.
Let $a \in L$ having two complement b & c then show $b = c$.
Since b & c be complement of a then

$a \vee b = 1 \quad a \wedge b = 0$

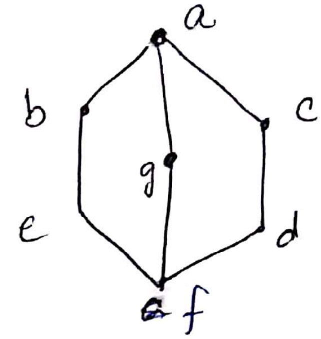
$a \vee c = 1 \quad a \wedge c = 0$

Now $b = b \wedge 1$
$= b \wedge (a \vee c)$
$= (b \wedge a) \vee (b \wedge c)$ {By distributive law}
$= (a \wedge b) \vee (b \wedge c)$
$= 0 \vee (b \wedge c)$
$= (a \wedge c) \vee (b \wedge c)$
$= (a \vee b) \wedge c$
$= 1 \wedge c = c$

Hence complement of a is unique.

Que In the lattice defined by Hasse Diagram find How many complements does element e have?

Sol^n



Since

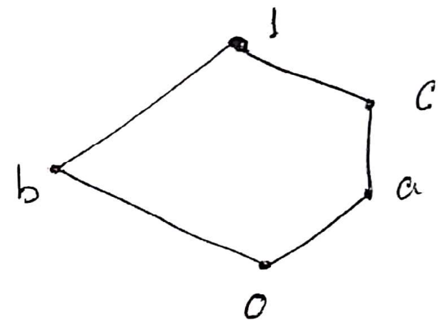
$e \wedge g = f$, $e \vee g = a$

and $e \wedge d = f$, $e \vee d = a$

where a is universal upper bound & f be universal lower bound.

Hence d, g be two complement of a.

Que The pentagonal lattice given below is not modular.



Proof let it be modular.

so for modular lattice

$a \le c \Rightarrow a \vee (b \wedge c) = (a \vee b) \wedge c$

Now let $a \le c$

and $a \vee (b \wedge c)$

$a \vee 0 = a$

$(a \vee b) \wedge c$

$1 \wedge c = c$

so $a \vee (b \wedge c) \neq (a \vee b) \wedge c$

Hence it is not modular.

Theorem :- Let $(L, \leq)$ be a lattice. for any $a, b, c \in L$ the following hold.

$$a \leq c \Leftrightarrow a \vee (b \wedge c) \leq (a \vee b) \wedge c$$

Proof We know that for any $a, b, c \in L$

$$a \leq c \Leftrightarrow a \vee c = c \quad \text{--- (1)}$$

and $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$ —(2)

from (1) & (2)

$$a \leq c \Leftrightarrow a \vee (b \wedge c) \leq (a \vee b) \wedge c$$ Proved

This inequality is known as Modular Inequality.

Theorem :- Let $(L, \leq)$ be a Lattice & $a, b, c, d \in L$. Then following implications hold.

(i) $a \leq b$ and $c \leq d \Rightarrow a \vee c \leq b \vee d$.

(ii) $a \leq b$ and $c \leq d \Rightarrow a \wedge c \leq b \wedge d$.

Proof (i) Suppose $a \leq b$ and $c \leq d$

By definition of join operⁿ $\vee$ in lattice $(L, \leq)$

we have $b \leq b \vee d$ and $d \leq b \vee d$.

Now by transitivity of relation $\leq$, we have

$a \leq b$ and $b \leq b \vee d \Rightarrow a \leq b \vee d$. —(1)

$c \leq d$ and $d \leq b \vee d \Rightarrow c \leq b \vee d$ —(2)

from $a \leq b \vee d$ and $c \leq b \vee d \Rightarrow b \vee d$ is upper bou of a & c.

$\Rightarrow lub\{a, c\} \leq b \vee d$

$\Rightarrow a \vee c \leq b \vee d$

Proved

(ii) Since $a \leq b$ and $c \leq d$. By definition of meet operation $\wedge$ in lattice $(L, \leq)$ we have –
$a \wedge c \leq a$ and $a \wedge c \leq c$

By transitivity $\leq$, we have

$a \wedge c \leq a$ and $a \leq b \Rightarrow a \wedge c \leq b$
$a \wedge c \leq c$ and $c \leq d \Rightarrow a \wedge c \leq d$.

Now $a \wedge c \leq b$ and $a \wedge c \leq d \Rightarrow a \wedge c$ is a lower bound of $\{b, d\}$

$\therefore$ $a \wedge c \leq glb\{b, d\}$
$a \wedge c \leq b \wedge d$ Proved