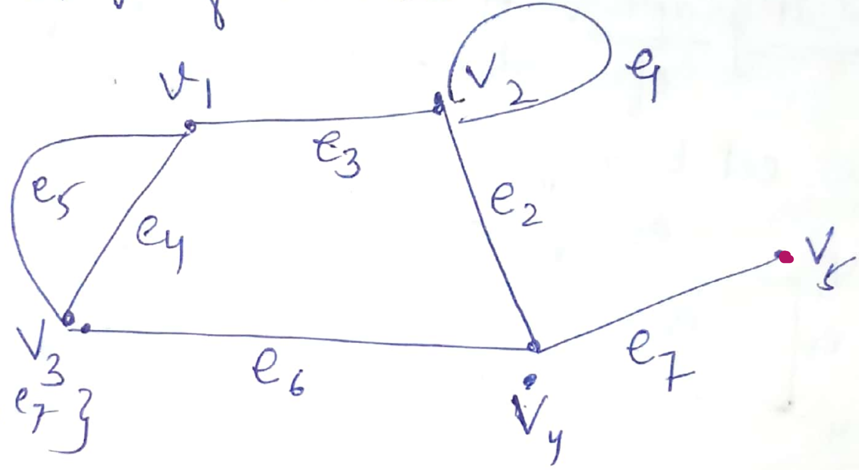
# UNIT-5

**Graph** :- A linear graph (or simply a Graph) $G=(V,E)$ consists of a set of objects $V=(v_1, v_2, \ldots)$ whose elements are called Vertices (or Nodes) and another set $E=\{e_1, e_2, \ldots\}$ whose elements are called edges (or Lines) (or Branches) such that each $e_k$ is identified with an unordered pair $(v_i', v_j)$ of vertices.

$G=(V,E)$

$V=\{v_1, v_2, v_3, v_4, v_5\}$

$E=\{e_1, e_2, e_3, e_4, e_5, e_6, e_7\}$




**Self loop** : An edge whose end vertices are same is called ~~leff~~ self loop.

or an edge connected with a vertex pair $(v_i, v_i)$ is called self loop. eg = $e_1$ is self loop.

**Parallel Edges** : If more than one edge are associated with a given pair of vertices such edge is reffered to as parallel edge.

eg = $e_5$ & $e_4$ are parallel edges.

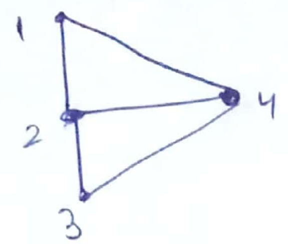
**Degree of a vertex** : No. of edges connected to a vertex is called its degree.

eg. $d(v_1)=3$ $\quad d(v_3)=3$

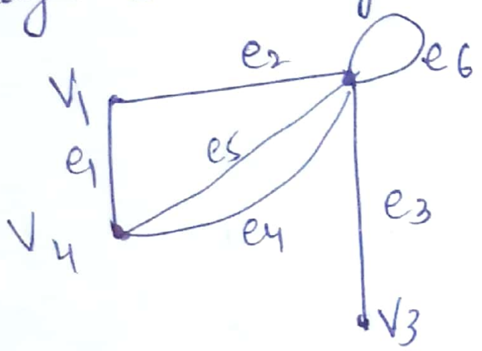
Degree of self loop is counted twice
eg: $d(v_2)=4$

# Types of Graph

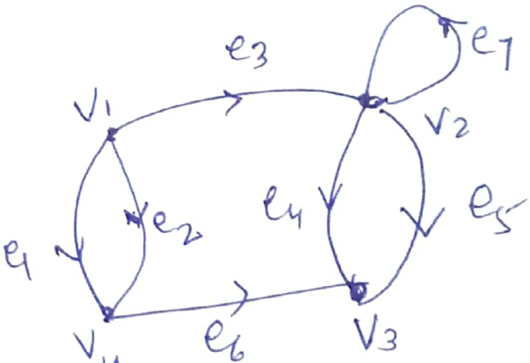
1. **Simple Graph**: A graph with no self loop and no parallel edges is called simple graph.




2. **Multigraph**: A multigraph $G=(V,E)$ consists a set of vertices $V$ and set of edges $E$ such that edge set $E$ may contain multiple edges & self loop.



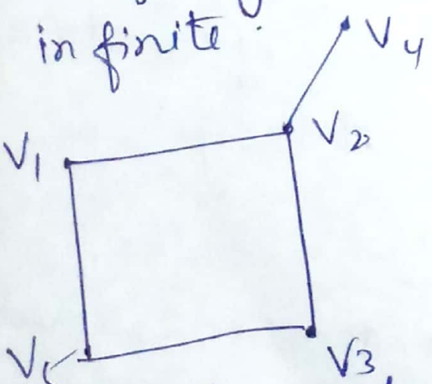

Undirected Multigraph



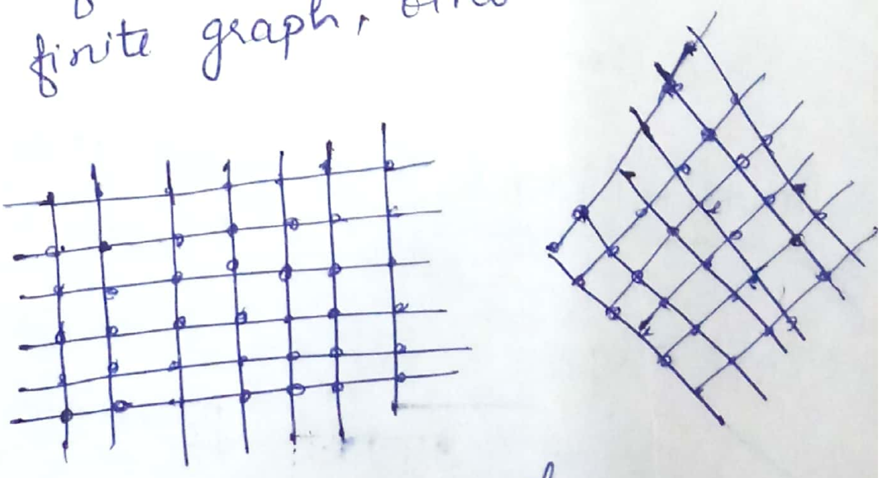

Directed Multigraph.

3. **Finite & Infinite Graph**

A graph with finite no. of vertices as well as finite no. of edges is called finite graph, otherwise it is infinite.



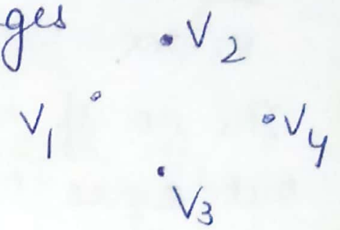

finite graph.



Infinite graph.

4. <u>Null Graph</u>

A graph $G = (V, E)$ is said to be Null if the set of vertices $V$ is non empty but the set of edges is empty.

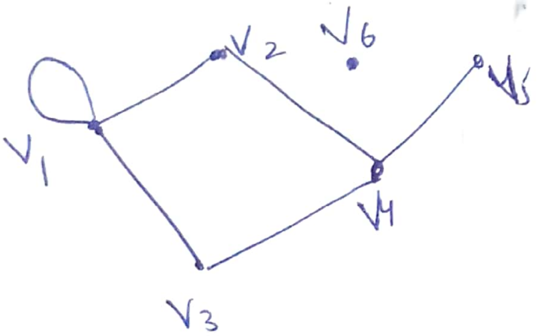



A Null Graph is a graph in which every vertex is isolated vertex.

Isolated Vertex : a vertex on which no edge is incident on it, is called isolated vertex.

Pendant Vertex : A vertex is called pendant if its degree is one ie $d(v) = 1$

Ex




$V_6$ is isolated Vertex

$V_5$ is pendant vertex.

5. <u>Directed Graph (Diagraph)</u>

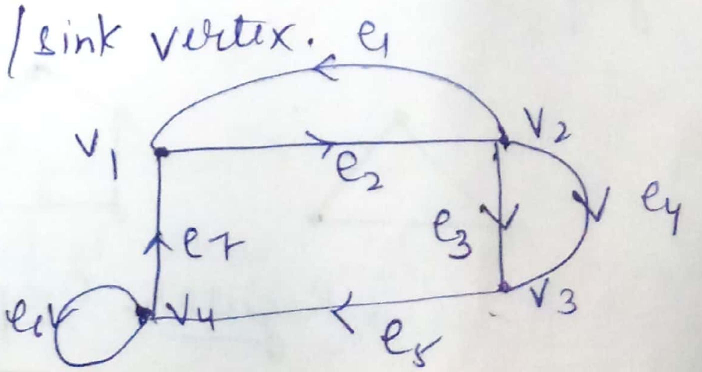
A directed graph is one in which each edge is identified by some ordered pair of vertices $(v_i, v_j)$

A diagraph is also known as oriented graph.

If $e_k = (v_i, v_j)$ is an edge then this edge in diagraph is represented by a line from $v_i$ to $v_j$ with an arrow from $v_i$ to $v_j$.

where $v_i$ is called initial/source vertex.

& $v_j$ is called terminal/sink vertex.



## In-degree & Out-degree of a vertex

In a directed graph, the no. of edges incident into a vertex $v_i$ is called its indegree & denoted by $d^-(v_i)$

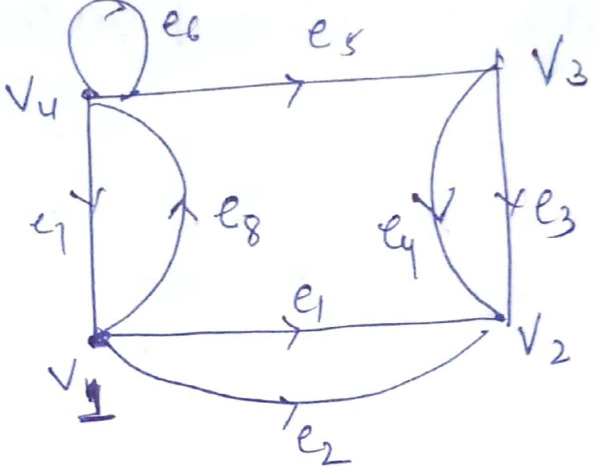
The no. of edges incident out of a vertex $v_i$ is called outdegree of vertex $v_i$ & denoted by $d^+(v_i)$

$d^-(v_1) = 1$, $d^+(v_1) = 3$

$d^-(v_2) = 4$, $d^+(v_2) = 0$

$d^-(v_3) = 1$, $d^+(v_3) = 2$

$d^-(v_4) = 2$, $d^+(v_4) = 3$




It may be seen that

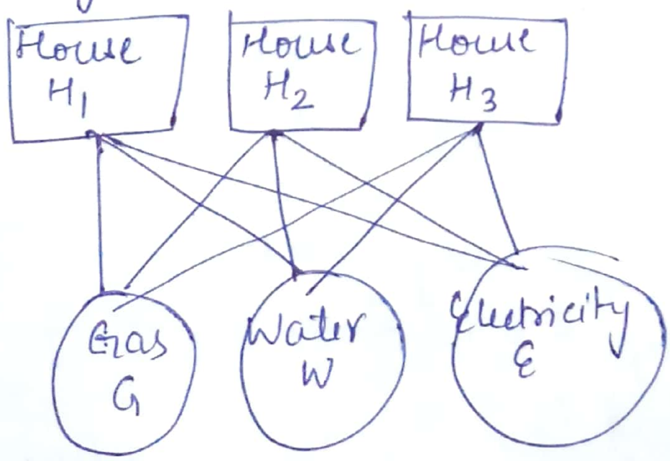
$$d(v) = d^-(v) + d^+(v)$$

## 6. Regular Graph

A graph in which all vertices are of equal degree is called a Regular Graph

Ex

The graph of three utilities is regular graph.




→ Regular Graph

7. Subgraph : A graph $g$ is said to be a subgraph of a graph $G$ if all the vertices & all the edges of $g$ are in $G$, and each edge of $g$ has the same end vertices in $g$ as in $G$.

A graph $H = (V_1, E_1)$ is said to be subgraph of $G = (V, E)$ if

$$V_1 \subseteq V \text{ and } E_1 \subseteq E$$

NOTES

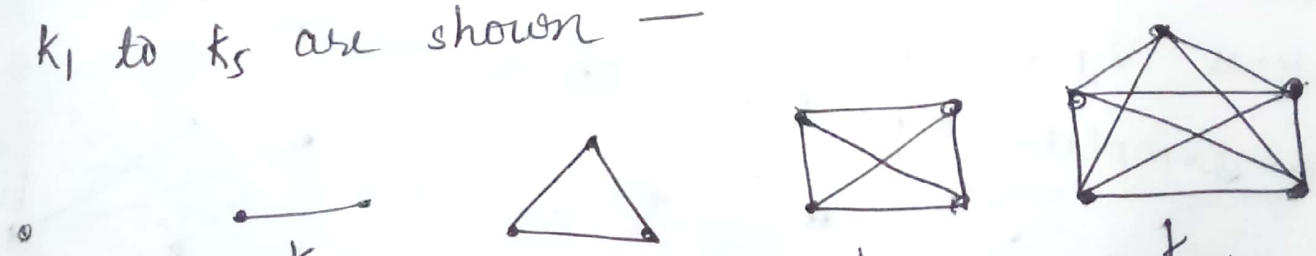
1. Every graph is its own subgraph.
2. A subgraph of a subgraph of $G$ is a subgraph of $G$.
3. A single vertex in a graph $G$ is subgraph of $G$.
4. A single edge in $G$, together with its end vertices is also a subgraph of $G$.

~~Theorems.~~

8. Complete Graph A simple graph, in which there is exactly one edge between each pair of distinct vertices is called a complete graph.

The complete graph of $n$ vertices is denoted by $K_n$

$K_1$ to $K_5$ are shown —



$K_1$ $K_2$ $K_3$ $K_4$ $K_5$

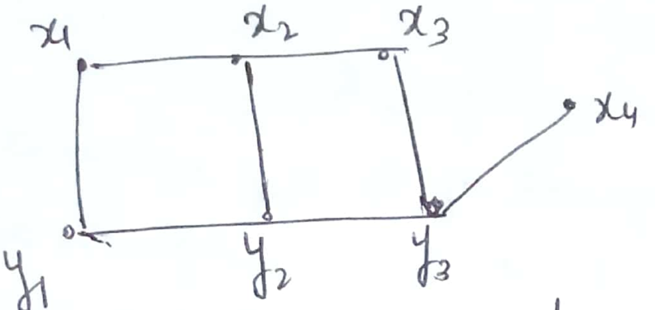
In complete graph $^nC_2$ edges are there.

i.e if there are $n$ vertices then $\frac{n(n-1)}{2}$ edges are there.

9. <u>Bipartite Graph</u>

A graph $G = (V, E)$ is bipartite if the vertex set $V$ can be partitioned into two subsets (disjoint) $V_1$ & $V_2$ such that every edge in $E$ connects a vertex in $V_1$ & a vertex $V_2$ (so that no edge in $G$ connects either two vertices in $V_1$ or in $V_2$).

$(V_1, V_2)$ is called a bipartite graph.




Not a Bipartite graph because no partition of vertices can be made.


Bipartite.

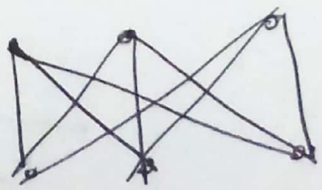
10. <u>Complete Bipartite Graph</u>

The complete Bipartite graph on $m$ & $n$ vertices denoted by $K_{m,n}$ is the graph, whose vertex set is partitioned into sets $V_1$ with $m$ vertices & $V_2$ with $n$ vertices in which there is an edge b/w each pair of vertices $v_{11}$ & $v_{21}$ where $v_{11}$ is in $V_1$ and $v_{21}$ is in $V_2$.

The complete Bipartite graph is shown -

$K_{2,3}$

$K_{2,4}$

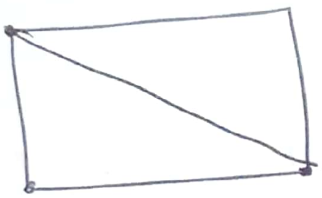


$K_{3,3}$

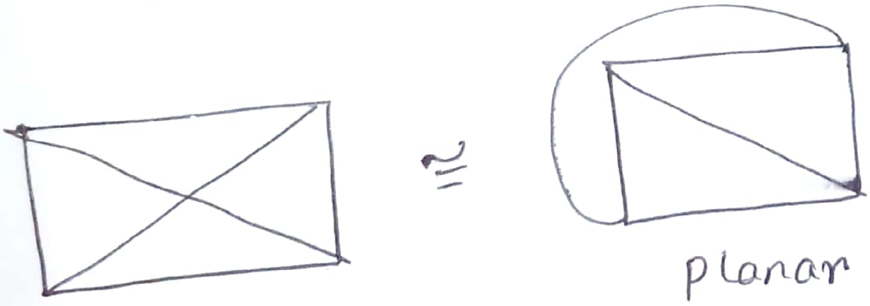
# Planar Graph :

A graph G is said to be planar if there exists some geometric representation of G which can be drawn on a plane such that no two of its edges intersects except only at the common vertex.

or A graph is planar, if it can be drawn on a plane without a cross over b/w its edges crossing.



planar graph



planar graph.

$\boxed{\text{For a graph to be planar } 3v - e \geq 6}$

Que Prove that $K_3$ & $K_4$ are planar graph. $K_5$ is non planar.

Sol$^n$ $K_3$ $v = 3$ $e = \frac{3(3-1)}{2} = 3$

$3v - e \geq 6$

$3 \times 3 - 3 \geq 6 \Rightarrow 6 \geq 6$ $K_3$ is planar.

$K_4$ $v = 4$ $e = \frac{4(4-1)}{2} = 6$

$3 \times 4 - 6 \geq 6 \Rightarrow 6 \geq 6$ $K_4$ is planar

$K_5$ $V = 5$ $e = \frac{5(5-1)}{2} = 10$ $3 \times 5 - 10 \geq 6 \Rightarrow 5 \geq 6$

so $K_5$ is Non-planar.

**Theorem 1** The sum of degree of all vertices in a graph G is equal to twice the no. of edges in G.

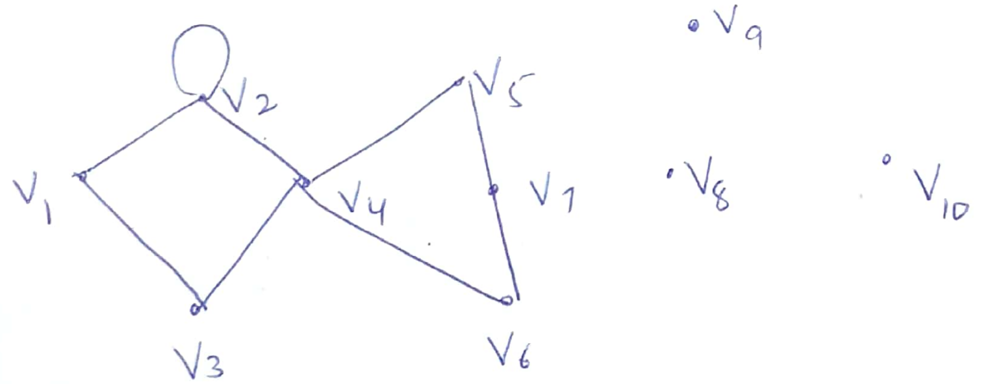
**Proof** :- let $G = (V, E)$ be a graph. Then the no. of edges in G is $|E|$.

Since each edge is incident on two vertices, it contributes to 2 to the sum of degree of graph.

Thus sum of degree of all vertices in G is given by

$$\boxed{\sum_{v \in V} d(v) = 2|E|}$$

This result is also known as "Handshaking lemma".




for this graph -

$d(v_1) = 2$ $\quad d(v_5) = 2$ $\quad d(v_9) = 0$

$d(v_2) = 4$ $\quad d(v_7) = 2$ $\quad d(v_{10}) = 0$

$d(v_3) = 2$ $\quad d(v_6) = 2$

$d(v_4) = 4$ $\quad d(v_8) = 0$

$$\sum_{v \in V} d(v) = 2|E|$$ since edges are = 9

$2+4+2+4+2+2+2 \quad = 2 \times 9$

$18 \quad = 18$

Therefore $\sum_{i=1}^{10} d(v) = 18 \Rightarrow$ twice No of edges.

Theorem-2 The vertices of odd degree in a graph is always even.

Proof :- let $G = (V, E)$ be a graph.

let $V_e$ & $V_o$ denote the set of even & odd vertices in $G$. Then $V_e \subseteq V$ and $V_o \subseteq V$.

such that

$$V = V_e \cup V_o \text{ and } V_e \cap V_o = \phi$$

Hence

$$\sum_{v \in V} d(v) = \sum_{v \in V_e} d(v) + \sum_{v \in V_o} d(v)$$

$$2|E| = 2k + \sum_{v \in V_o} d(v)$$

$$\sum_{v \in V_o} d(v) = 2|E| - 2k$$

$$= 2(|E| - k) = \text{an even no.}$$

## Operations on Graph

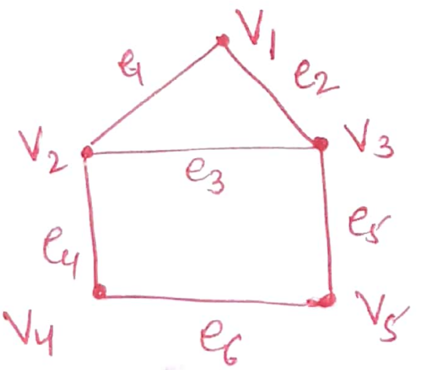
1. Union :- If $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are two graphs then their union $G_1 \cup G_2$ is another graph $G_3 = (V_3, E_3)$ such that $V_3 = V_1 \cup V_2$ and $E_3 = E_1 \cup E_2$.

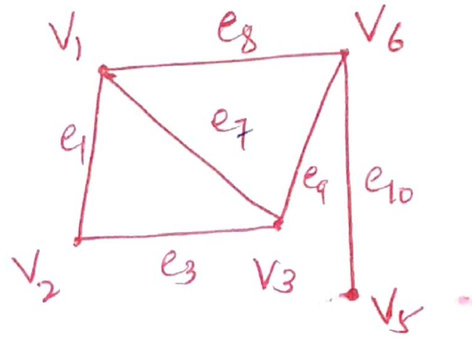
2. Intersection : The intersection of two graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ denote as $G_1 \cap G_2$ is another graph $G_3 = (V_1 \cap V_2, E_1 \cap E_2)$ that is, the vertex set of $G_3$ consists of only those vertices which are common in both $G_1$ & $G_2$, and the edge set of $G_3$ consists of only those edges which are common in both $E_1$ & $E_2$.

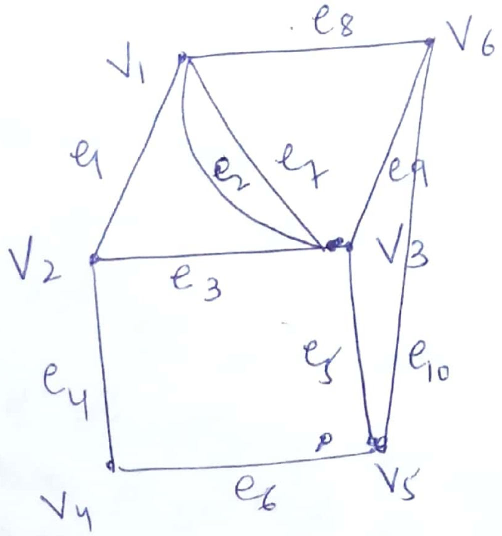
## 3. Ring Sum of Two Graph :-

The ring sum of two graph $G_1 = (V_1, E_1)$ & $G_2 = (V_2, E_2)$ denoted as $G_1 \oplus G_2$ is the graph $G_3 = (V_1 \cup V_2,\ E_1 \cup E_2 - E_1 \cap E_2)$. That is the ringsum of two graph $G_1$ & $G_2$ is a graph consisting of the vertex set $V_1 \cup V_2$ and of edges that are either in $G_1$ or $G_2$ but not in both.

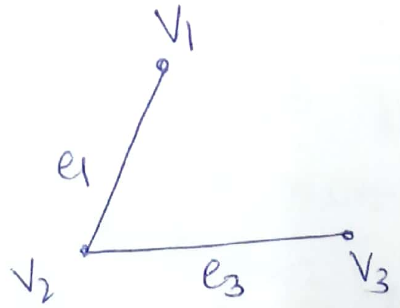
Que



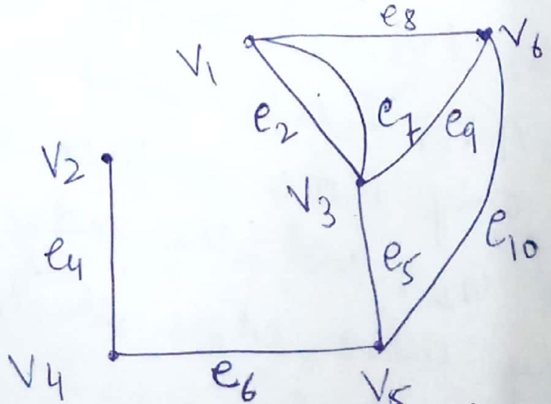




(Union)




(Intersection)




(Ringsum)

# Matrix Representation of Graph

Usually there are two ways to represent graph.

(1) Incidence Matrix (2) Adjacency Matrix

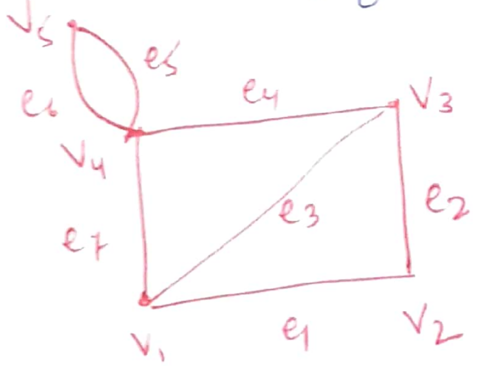
## 1) Incidence Matrix :-

Let $G = (V, E)$ where $V = \{v_1, v_2 \ldots v_n\}$ is set of $n$ vertices & $E = \{e_1\ e_2 \ldots e_m\}$ is set of $m$ edges, be a graph having no self loops.

Then incidence matrix $A = \{a_{ij}\}$ is $n \times m$ matrix defined by

$$a_{ij} = \begin{cases} 1 & \text{if } j^{th} \text{ edge } e_j \text{ is incident on } i^{th} \text{ vertex} \\ 0 & \text{otherwise.} \end{cases}$$

Ex




$$A = \begin{array}{c} \\ v_1 \\ v_2 \\ v_3 \\ v_4 \\ v_5 \end{array} \begin{array}{c} \begin{array}{ccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{array} \\ \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \end{bmatrix} \end{array}$$

**Observations:** Each column has only two one's, since which clearly states that every edge is incident only on two vertex.

2) If a row has all zero's then corresponding vertex is an isolated vertex.

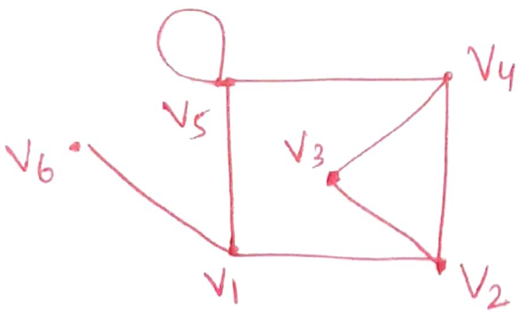
3. In case graph has parallel edges, then corresponding column are identical

4. sum of each row is equal to the degree of corresponding vertex.

## (2) Adjacency Matrix :-

Let $G$ be a graph with n vertices & no parallel edges (self loops are allowed), then the adjacency matrix of $G$ is an $n \times n$ symmetric matrix $A = [a_{ij}]$ defined by -

$$a_{ij} = \begin{cases} 1, & \text{if there is an edge b/w } i^{th} \text{ \& } j^{th} \text{ vertices} \\ 0 & \text{if there is no edge b/w them.} \end{cases}$$




$$\begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \\ V_6 \end{array} \begin{array}{c} \begin{array}{cccccc} V_1 & V_2 & V_3 & V_4 & V_5 & V_6 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

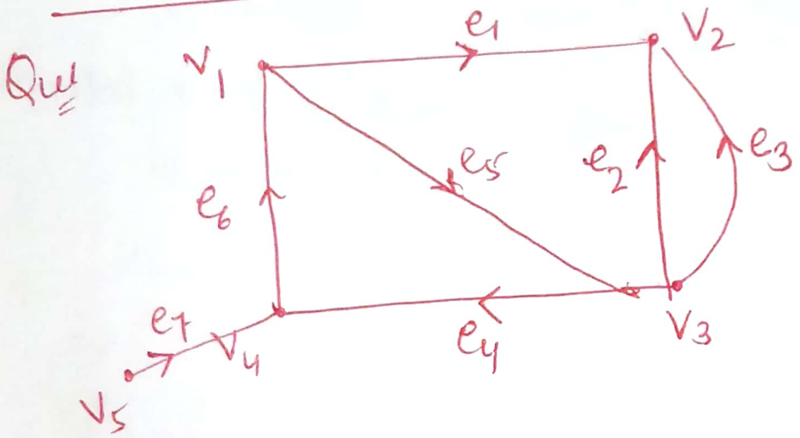
## Observations :-

1. The principal diagonal of A has all 0's if & only if G has no self loop.
2. If G is without self loop, then the sum of any ~~[illegible]~~ row is equal to the degree of corresponding vertex.

## Incidence Matrix for Diagraph

$$a_{ij} = \begin{cases} 1 & \text{if } j^{th} \text{ edge is incident out of } i^{th} \text{ vertex} \\ -1 & \text{if } j^{th} \text{ edge is incident into } i^{th} \text{ vertex} \\ 0 & \text{if } e_j \text{ is not incident on } v_i \end{cases}$$

Que




$$\begin{array}{c} \\ V_1 \\ V_2 \\ V_3 \\ V_4 \\ V_5 \end{array} \begin{array}{c} \begin{array}{ccccccc} e_1 & e_2 & e_3 & e_4 & e_5 & e_6 & e_7 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & -1 & 0 \\ -1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \end{array}$$

Paths

**Walk** :- A walk is a finite or infinte sequence of edges which joins a sequence of vertices.

**Trail** :- is a walk in which all edges are distinct

**Path** :- is a trail in which all vertices are distinct.

or. A path in graph is a finite or infinite sequences of edges which joins a sequence of vertices, ~~that~~ are all distinct (since vertices are distinct, so are the edges)

## Graph Coloring :-

**Coloring** : A coloring of graph G is an assignment of colors to its vertices so that no two adjacent vertices have same color.

**K-Vertex Coloring** :

A K-Vertex coloring of graph G is an assignment of K-colors to the vertices of G such that no two adjacent vertices receive the same color.

Every graph with n vertices is n-colorable.

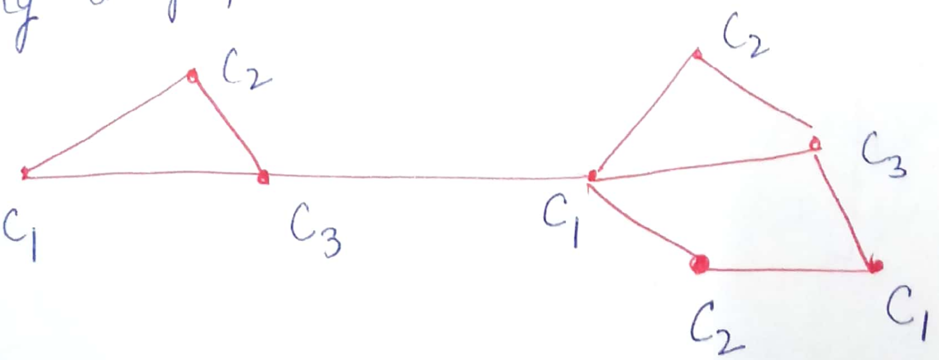
**Color class** :- The set of all points with any one color is independent & is called color class.

**Proper coloring**

If a graph is n-colourable, then vertex set V(G) can be partitioned into n color classes.

The coloring a graph is called Proper Coloring.

Ex




This graph can be colored using 3 colors only.

## Chromatic Number :-

The chromatic no. of a graph G, is the minimum no. of colors required to color the vertices of G, such that no two adjacent vertices receive the same color.

The chromatic no. of a graph G is denoted by $\chi(G)$.

The chromatic no. of a graph properties –

We find the following properties –

(i) The chromatic no. of an isolated vertex is one.

2. The chromatic no. of a graph having atleast one edge is atleast two.

3. The chromatic no. of a path $P_n$ $(n \geq 2)$ is two.

4. The chromatic no. of a wheel graph is ~~6~~ 3, if it has odd no. of vertices & 2 if it has even no. of vertices.

5. The graph G consisting of simply a circuit with $n \geq 3$ is 2 chromatic, if n is even & 3-chromatic if n is odd.

**Theorem** : There exists a k-coloring of a graph G if & only if V(G) can be partitioned into k subset $V_1, V_2 \ldots V_k$ such that no two vertices are adjacent.

**Proof** Let $f: V(G) \rightarrow (1, 2 \ldots k)$ be a coloring of G & let $V_r = \{f(v) = r, 1 \leq r \leq k\}$ Therefore $V_r$ denotes the set of vertices colored r. The $V = V_1 \cup V_2 \ldots \cup V_k$ forms a portion of V. The $V_r = (1 \leq r \leq k)$ are such that no two vertices in are adjacent.

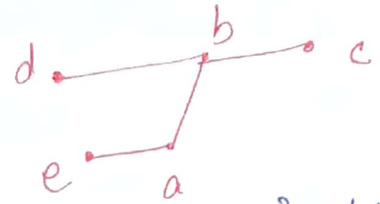
**Conversely** Let $V = V_1 \cup V_2 \ldots \cup V_k$ be a portion of V(G) such that no two vertices in $V_r = (1 \leq r \leq k)$ are adjacent. The funn $f: V(G) \rightarrow (1, 2, \ldots k)$ defined by $f(v) = r$ if $v \in V_r$ is a k-coloring of G.

## Five Color Theorem :-

Every planar graph is 5-colorable.

## Independent Sets

A set of vertices in a graph G is said to be an independent set of vertices if no two vertices in the set are adjacent.




{a, c, d} and {b, e} are independent set.

## Maximally Independent set

Let G be any graph. A subset V' of vertex set V(G) is called a Maximally Independent set of G if -

(i) V' is an independent set of G.

(ii) If V'' is any other independent set of G, then V' is not a proper subset of G.

## Chromatic Polynomial

Let G be a graph on n vertices. If $C_r$ denotes the different ways of properly coloring of graph G, using exactly r distinct colors, then these r colors can be chosen out of $\lambda$ colors.

There are $C_r(\lambda_r)$ distinct ways of properly coloring the graph G using exactly r colors out of $\lambda$ colors.

# Chromatic Polynomial

let $\lambda$ be the no. of available colors for $G$, we want to find a chromatic polynomial $P(G,\lambda)$ that tells us how many ways, we can color a graph with atmost $\lambda$ colors.

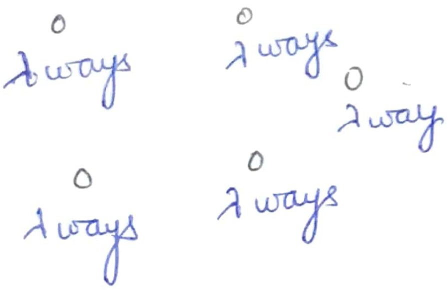
$$\boxed{P_n(G,\lambda) = c_1\,{}^{\lambda}C_1 + c_2\,{}^{\lambda}C_2 + c_3\,{}^{\lambda}C_3 + \cdots c_n\,{}^{\lambda}C_n}$$

where $c_1$ is no. of ways for properly coloring graph with 1 color.

**Rules :**

1. $|V| = n$ & $E = \phi$ then $P(G,\lambda) = \lambda^n$

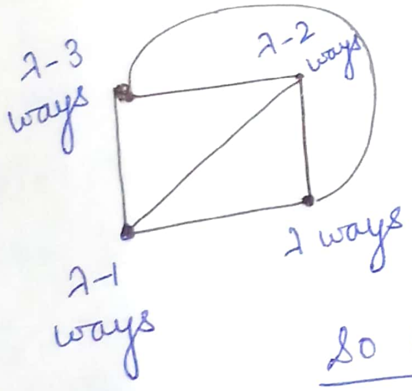
Ex




so total ways $= \lambda^5$

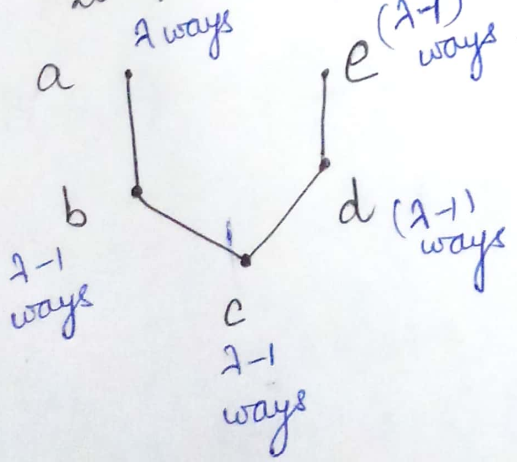
2. $G = K_n$ then $P_n(G,\lambda) = \lambda(\lambda-1)(\lambda-2)\cdots(\lambda-n+1)$

$$= \frac{\lfloor\lambda}{\lfloor\lambda-n}$$




<u>So for $K_4$</u> $\quad P_n(G,\lambda) = \lambda(\lambda-1)(\lambda-2)(\lambda-3)$

3. For path, consider the no. of choices from vertex to vertex




$P_n(G,\lambda) = \lambda(\lambda-1)^4$

$$\boxed{\text{so if we have } n \text{ vertex path then } P_n(G,\lambda) = \lambda(\lambda-1)^{n-1}}$$

4. If G has component, multiply using product rule.

1st component is path of 3 vertex

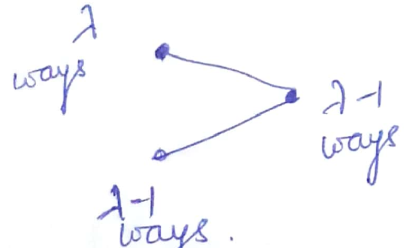
so $P_1(G,\lambda) = \lambda(\lambda-1)^2$

2nd component is $K_3$ so $\frac{|\underline{\lambda}}{|\underline{\lambda-3}}$

$P_2(G,\lambda) = \lambda(\lambda-1)(\lambda-2)$

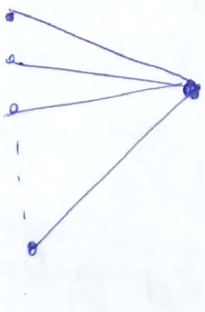
$$P(G,\lambda) = P_1(G,\lambda) \cdot P_2(G,\lambda)$$
$$= (\lambda(\lambda-1))^2 \cdot \lambda(\lambda-1)(\lambda-2)$$

Find $P(G,\lambda)$ for $K_{2,1}$




$\lambda(\lambda-1)^2$

what is $\chi^{(K_{m,1})}$



$\chi^{(K_{m,1})} = 2$

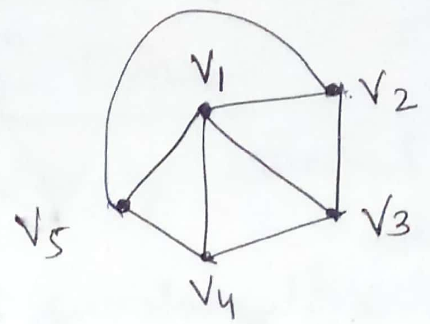
Chromatic No

$P(G,\lambda) = \lambda(\lambda-1)^m$

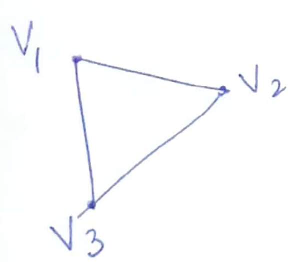
Que find chromatic polynomial for the graph

Sol$^n$

G has triangle so G requires minimum 3 colors for proper coloring




$c_1 = 0 \quad c_2 = 0$




can be properly colored in $\lfloor 3$ ways.

$V_4$ can be assigned the color of $v_2$.

and $V_5$ can be assigned the color of $V_3$

Thus $c_3 = \lfloor 3 = 6$

When we have 4 colors.

We can take any four vertices say $v_1, v_2, v_3$ & $V_4$ can be properly colored in $\lfloor 4$ ways.

$V_5$ can be assigned the color of $V_2$ & $V_3$

Thus $c_4 = 2 \cdot \lfloor 4$

and we have $c_5 = \lfloor 5$

Hence we have chromatic polynomial

$$P(G,\lambda) = \lfloor 3 \frac{\lambda(\lambda-1)(\lambda-2)}{\lfloor 3} + 2 \cdot \lfloor 4 \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{\lfloor 4}$$

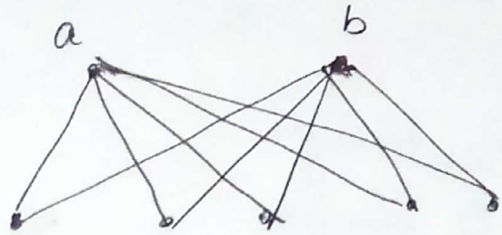
$$+ \lfloor 5 \frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)}{\lfloor 5}$$

$$= \lambda(\lambda-1)(\lambda-2) + 2\lambda(\lambda-1)(\lambda-2)(\lambda-3) + \lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)$$

$$= \lambda(\lambda-1)(\lambda-2)[1 + 2(\lambda-3) + (\lambda-3)(\lambda-4)]$$

$$= \lambda(\lambda-1)(\lambda-2)(\lambda^2-5\lambda+7) \text{ Ans}$$

Que Find chromatic polynomial of complete Bipartite graph $K_{2,5}$




Case I

If vertex a & b have same color (say $\lambda$), then total ways for properly coloring rest vertex is $(\lambda-1)$

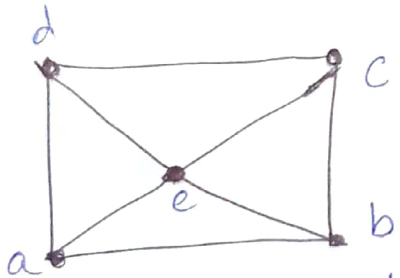
Hence $\lambda(\lambda-1)^5$

Case II

If vertex a & b have different color then $\lambda(\lambda-1)(\lambda-2)^5$

$$P(G,\lambda) = \lambda(\lambda-1)^5 + \lambda(\lambda-1)(\lambda-2)^5 \quad \text{Ans}$$

Que




$c_1 = 0$
$c_2 = 0$

min. colors required are 3

total ways to color from 3 colors is $\lfloor 3$

for 4 colors

Consider four vertex a, e, b, c, total ways $\lfloor 4$ & remaining 'd' can be colored ~~with~~ same as that of 'b'. so only one way.

$\lfloor 4 \times 1 = \lfloor 4$

for 5 colors total ways $= \lfloor 5$

$$P_n(\lambda) = c_1\,{}^{\lambda}C_1 + c_2\,{}^{\lambda}C_2 + c_3\,{}^{\lambda}C_3 + c_4\,{}^{\lambda}C_4 + c_5\,{}^{\lambda}C_5$$
$$= 0 + 0 + \lfloor 3\,{}^{\lambda}C_3 + \lfloor 4\,{}^{\lambda}C_4 + \lfloor 5\,{}^{\lambda}C_5$$
$$= \lambda(\lambda-1)(\lambda-2) + \lambda(\lambda-1)(\lambda-2)(\lambda-3) + \lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)$$

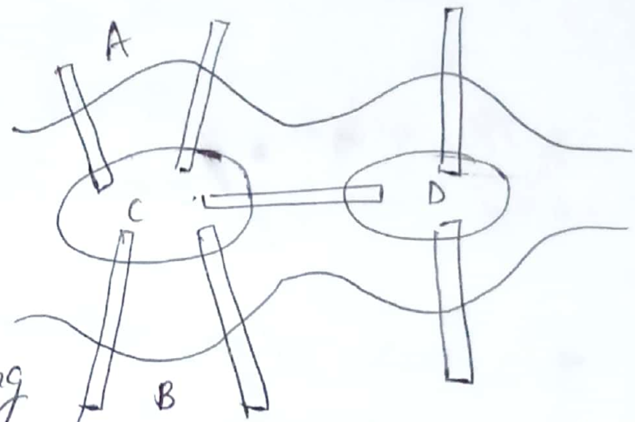
Ans

# Applications of Graph

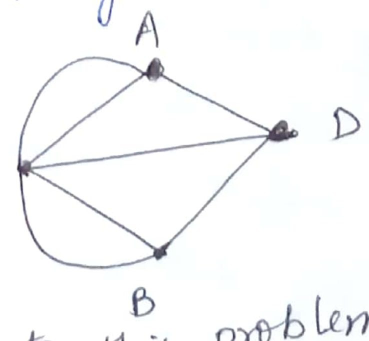
## 1. Konigsberg Bridge Problem :-

This problem was solved by Leonhard Eular in 1736, by means of graph.
Two icelands C & D formed by Pregel River in Konigsberg were connected to each other & to the banks A & B with 7 brigdes (as shown in fig).

The problem was to start at any of the four Land area of the city A, B, C or D, walk over each seven bridges exactly once, & return to the starting point (of course without swimming across the rivers)




Eular represented this situation by means of graph.



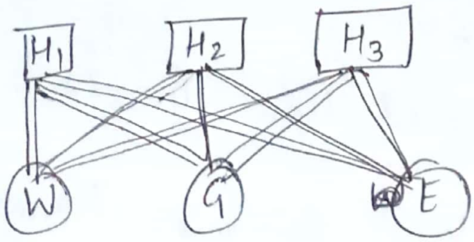

The vertices represent the land areas & the edges represent the bridges.

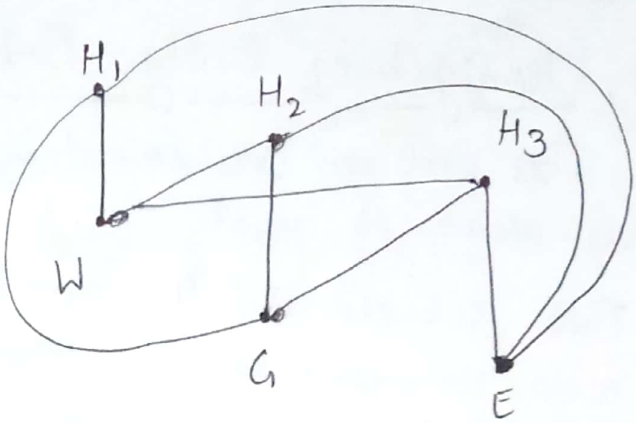
Eular proved that a solution to this problem does not exists.

## 2. Utility Problem :-

There are 3 Houses $H_1$, $H_2$ & $H_3$ each to be connected to each of 3 utilitres - water (W), gas (G) & electricity (E) by means of conduits
The problem is : Is it possible to make such connection

without any cross overs of the conduits?"



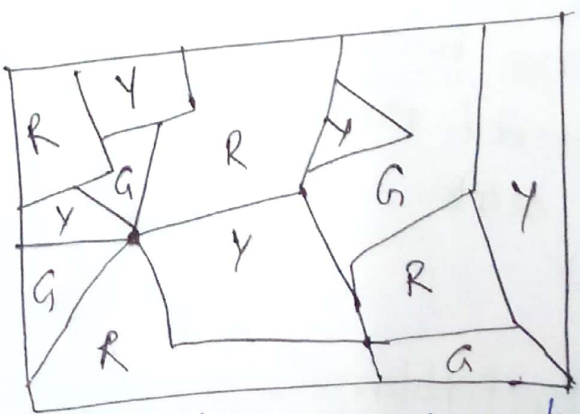

Graph of 3 utility Problem

The graph can't be drawn on plane without edge crossing over. Thus answer is 'No'.

3. <u>Four-color Problem</u>:

In Cartography, a map is normally colored in such a way that no countries with common boundary have the same color. In the map, the four colors R=Red, Y=yellow, G=green B=blue suffice. Cartography soon formulate a question surprisingly difficult to answer:—

Can every map be colored by four colors?




If we replace each country by vertex & join vertices if the corresponding countries have a common boundary, we shall obtain the following equivalent graph-theoretic problem.

## 4. Travelling Sales Man Problem :-

In this problem, a salesperson is required to visit $n$ cities on a trip, The problem is to find a route by which the salesperson can visit each city once & return home, with minimum distance travelled.

The relevant graph to this problem is the complete graph on $n$ vertices, each vertex representing a city. The graph is weighted, each edge being labeled with the distance between two cities represented by its terminals.

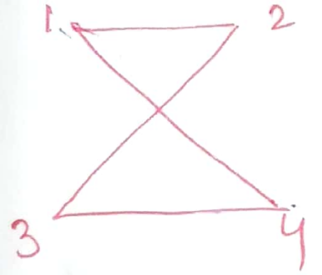
# Euler Graph

**Euler Path :** [vertex can be repeated but edge can't]

It is a path that traverse each edge exactly once & only once.

"A graph that contains an Eular path is called Euler Graph"

Euler graph is always connected, because Eular path contains all the edges of the graph.



$1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 1$ {Eular Circuit}

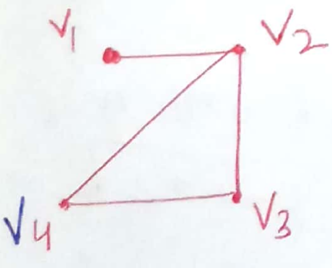
Eular path

{In this each edge is visited only once}

**NOTE :** 1. A connected graph is Euler iff it has atmost two odd degree vertices.

2. In Euler Circuit, each vertex is of even degree.

3. In Euler path, max. two vertices can have odd degree.



$V_1 \rightarrow V_2 \rightarrow V_3 \rightarrow V_4 \rightarrow V_2$

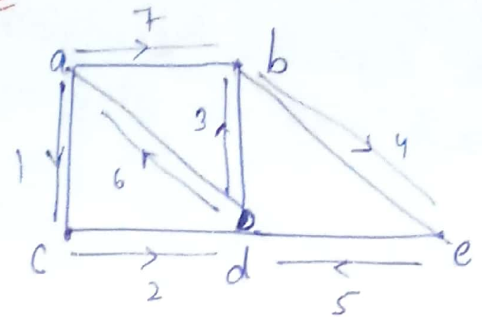
Euler path.

$d(v_1) = 1$ $d(v_2) = 3$ $d(v_3) = 2$ $d(v_4) = 2$

Two vertices of odd degree.

- In Eular path, max two vertices can have odd degree.

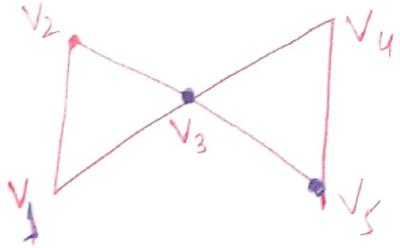
Que Find Euler path for the graph given below




$a \to c \to d \to b \to e \to d \to a \to b$ Euler path

$d(a) = 3$
$d(b) = 3$
$d(c) = 2$
$d(d) = 4$
$d(e) = 2$

(ii)




$V_1 \to V_2 \to V_3 \to V_5 \to V_4 \to V_3 \to V_1$
Eular Circuit.

$d(v_1) = 2$
$d(v_2) = 2$
$d(v_3) = 4$ All are of even degree.
$d(v_4) = 2$
$d(v_5) = 2$

Imp

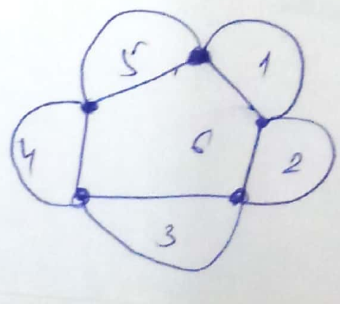
**Eular's formula**: A connected planar graph with $n$ Vertices & $e$ edges has $e-n+2$ regions. $\boxed{r = e-n+2}$

Ex How many edges must a planar graph have, if it has 7 regions & 5 nodes. draw such a graph.

Soln We know $r = e - n + 2$

$7 = e - 5 + 2$

$\boxed{e = 10}$

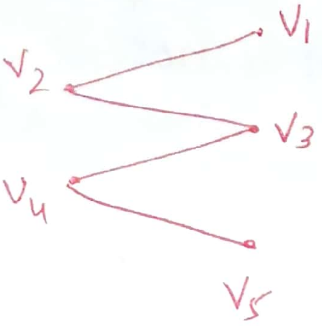



7

# Hamiltonian path & Circuit

**Hamiltonian path** contains each vertex ~~at~~ exactly once.

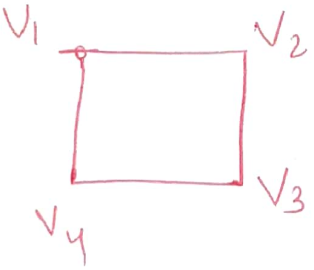
**Hamiltonian circuit**: first and last vertex are same.
{on removing any edge e from Hamiltonian circuit, we are left with Hamiltonian path}.




Hamiltonian path

$V_1 \to V_2 \to V_3 \to V_4 \to V_5$

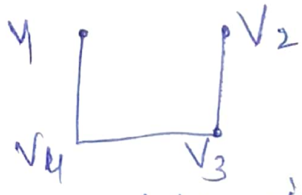
Ex




Hamiltonian circuit

$V_1 \to V_2 \to V_3 \to V_4 \to V_1$

Suppose we remove edge $(V_1, V_2)$



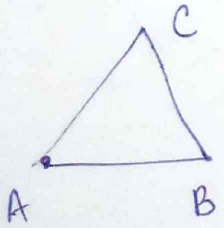

Then ~~we~~ are left with H. Path

$V_1 \to V_4 \to V_3 \to V_2$

- length of Hamiltonian path in a connected graph of n vertices is $(n-1)$ edges

**Theorem**:- Let G be a graph of n vertices than 'G' has a hamiltonian path if for any two vertices u & v of G $\boxed{\deg(u) + \deg(v) \geq n}$

**Proof**

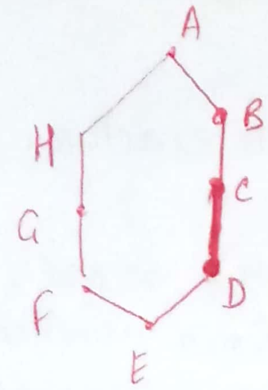



no. of vertices $n = 3$

$\deg(A) = 2 \quad \deg(B) = 2$

$2 + 2 \geq 3$

so there will be a Hamiltonian path.



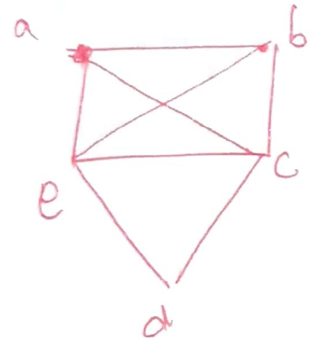

$deg(A) + deg(D)$

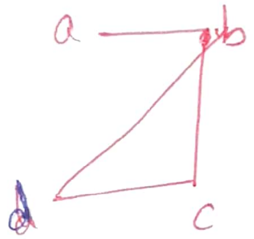
$2 + 2$

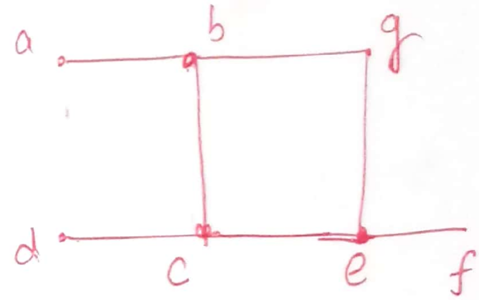
$4 \geq 8$ Not true.

So we can say the condition in theorem is not sufficient or necessary.

Que Which of the following graph has Hamiltonian circuit. If not, a hamiltonian path.




$a \to b \to c \to d \to e \to a$
(Hamiltonian circuit)

$a \to b \to c \to d$
(Hamiltonian path)

No Hamiltonian path
No. Hamiltonian circuit

## Tree

A tree is a connected graph without any circuit.

**Example**

(Tree with 4 vertices)

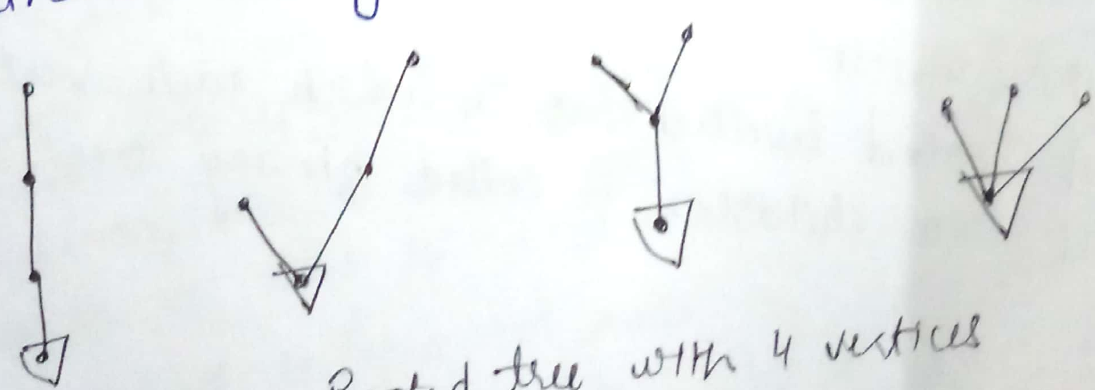
(Tree with 6 vertices)

A tree is obtained by removing branches from a graph, that form circuit. The process of deleting branches from a graph can be done by several methods.

## Rooted Tree:

A rooted tree is a tree with distinguished ~~tree~~ vertex called the Root. Any vertex of a tree may be designed as the root, but we shall customarily draw a rooted tree with the root at the top (or at the bottom) and circled or triangled.

Ex

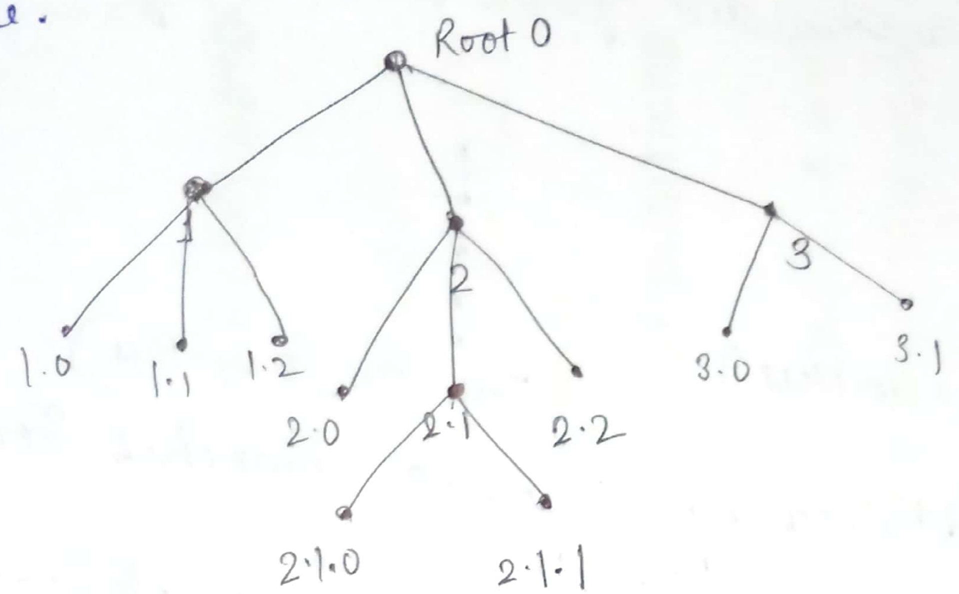


Rooted tree with 4 vertices

## Ordered Rooted Tree :

If edges leaving each vertex of a rooted tree T are labelled, then T is called an ordered rooted tree.

Ex




## Binary Tree :-

"A binary tree is defined as a tree in which there is exactly one vertex of degree 2, and each of the remaining vertices are of degree one or three."

If the level of any vertex $x$ is $i$, then its neighboring vertices on next level $(i+1)$ are called its children or descendants. The vertices on previous level are called father or predecessor.

In other words

" A rooted levelled tree in which each vertex has atmost two children, is called binary tree."

If every vertex of a binary tree has either two children or no child, then such a binary tree is called a full or <u>complete Binary Tree</u>..

Two properties of Binary Tree follow directly from the definition :-

1. The no. of vertices in a binary tree is always odd.

This is because there is exactly one vertex of even degree and remaining $(n-1)$ vertices of odd degree. Since the no. of vertices of odd degree is even. It implies that $(n-1)$ is even. Hence $n$ is odd.

2. Let $p$ be the no. of pendant vertices in Binary Tree T. then $n-p-1$ is the no. of vertices of degree 3.

Therefore no. of edges in T is equal

$$= \frac{1}{2}\left[p + 3(n-p-1) + 2\right] = n-1$$

~~p+3n-3p-3+2=n-1~~

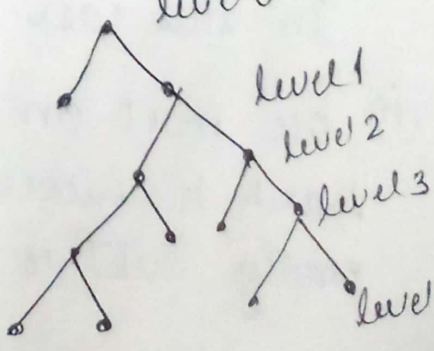
~~2n-2p-1=0~~

$$p = \frac{n+1}{2}$$

The no. of internal vertices in binary tree is one less than the no. of pendant vertices.

<u>Level of Vertices in Binary Tree</u> :-

In a binary tree a vertex $v_i$ is said to be at level $l_i$ if $v_i$ is at a distance of $l_i$ from root node. Thus the root is at level 0.



Binary Search Tree :-

It is a binary tree in which left subtree of node contains values that are less than nodes and right subtree contains values that are larger than the nodes.

Que A binary search tree T and an ITEM of info$^n$ is given, write the algorithm which finds the location of the ITEM or inserts ITEM as a new node in Tree.

(UPTU B.Tech 2008)

Sol$^n$ Algorithm : A binary search tree T and an ITEM of info$^n$ is given.

Step 1 Compare ITEM with root N of the tree.

(i) If ITEM < N, proceed to left child of N.

(ii) If ITEM > N, proceed to right child of N.

Step 2 Repeat step 1, until one of the following obtains -

(i) We meet a node N such the ITEM = N. In this case the search is unsuccessful.

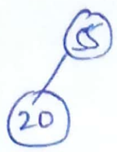
(ii) We meet an empty subtree, which indicates the search is successful. Insert ITEM in place of the empty subtree.

Que Construct Binary Search Tree from the given values.
55 20 63, 10, 28, 60, 93, 5, 11, 40, 68, 25

Soln

Insert 55


Insert 20

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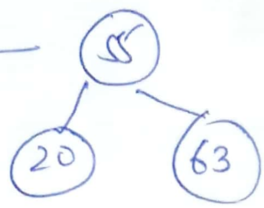


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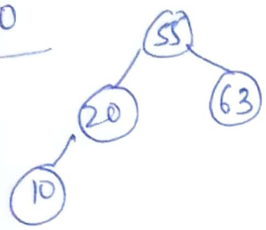


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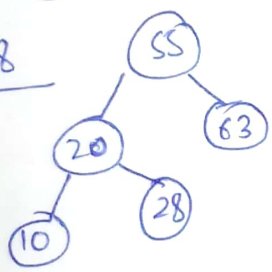


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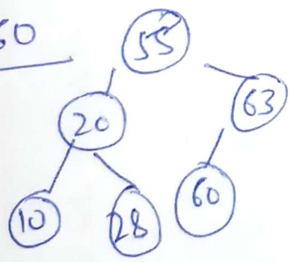


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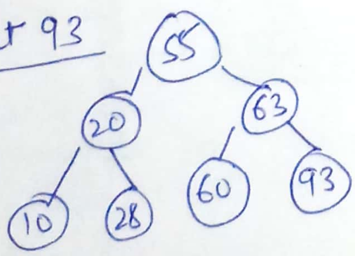


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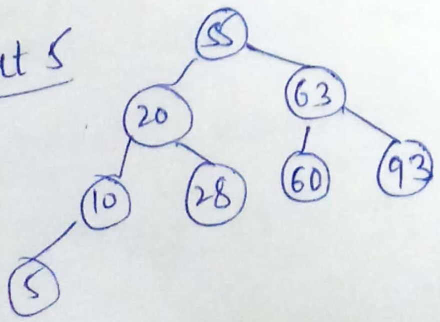


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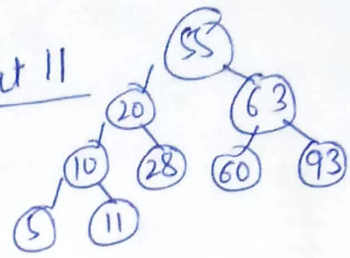


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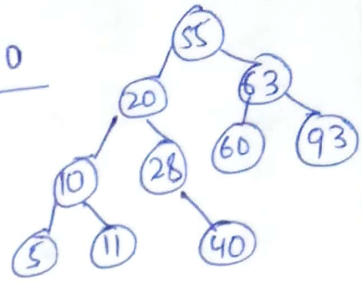


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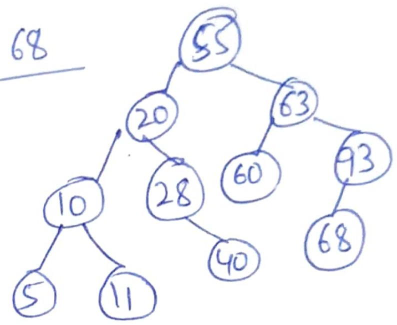


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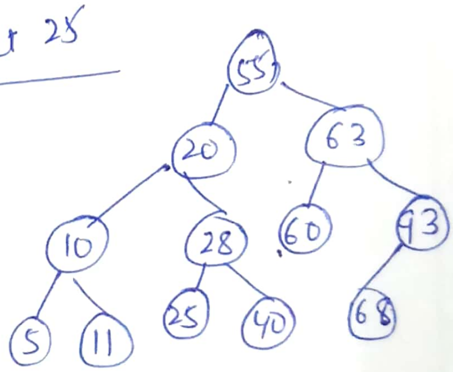


# Binary Tree Traversal

**Tree Traversal** :- A traversal of tree is a process in which each vertex is visited exactly once in a certain manner. For a binary tree, we have three types of traversal.

1. **Preorder Traversal**

Each vertex is visited in following manner -

a) Visit the root (N)

b) Visit the left child (or subtree) of root (L)

c) Visit the right child of root (R).

2 **Post order Traversal**

a) Visit the left child of root (L)

b) Visit the right child of root (R)

c) Visit the root node (N)

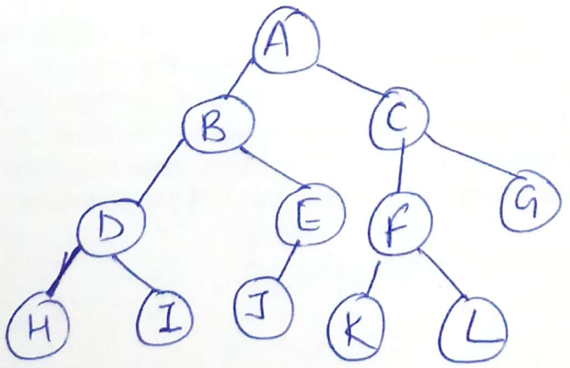
3. **Inorder Traversal**

a) Visit the left child or root (L)

b) Visit the root node (N)

c) Visit the right child (R).

Ex




~~Preorder~~ Postorder Sequence HIDJEBKLFGCA

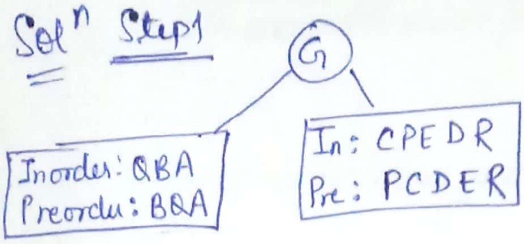
Preorder sequence ABDHIEJCFKLG

Inorder sequence HDIBJEAKFLCG

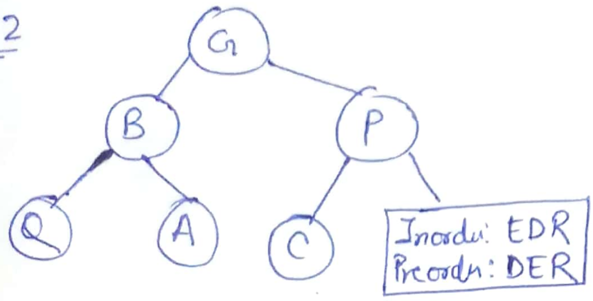
Que Construct a tree whose inorder & preorder traversal given –

Inorder Q B A G C P E D R
Preorder G B Q A P C D E R

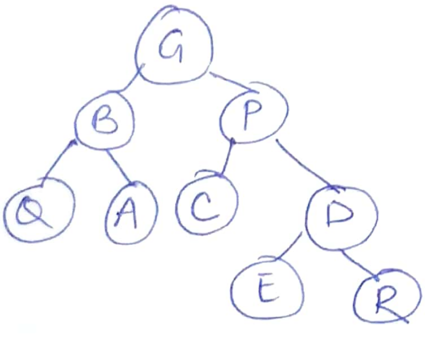
Sol^n Step1




Step 2

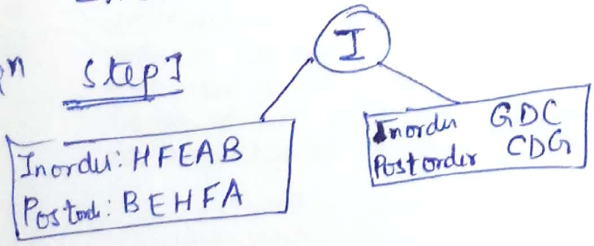



Step3

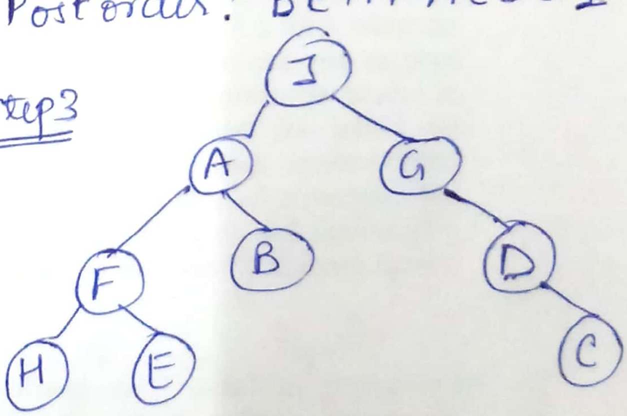



Que Construct Binary Tree, Given :- Postorder: BEHFACDGI
Inorder HFEABIGDC

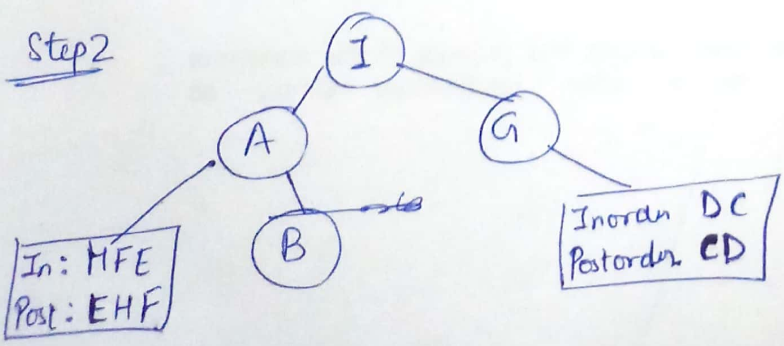
Sol^n Step1




Step3




Step2



## Spanning Tree :-

Let $G = (V, E)$ be any connected graph. A spanning tree in $G$ is a subgraph $T = (V, E')$ of $G$ which is tree.

Note that we have three requirements :-

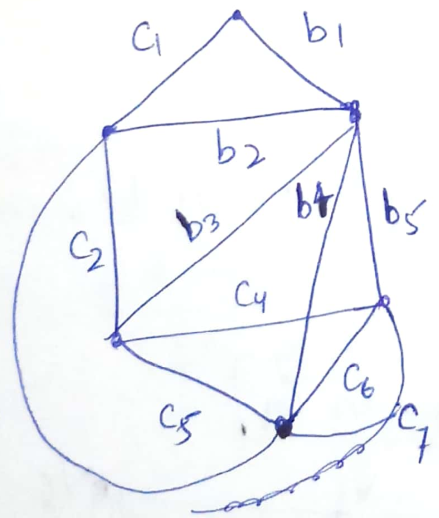
1. T has the same vertex V of vertices as does G.
2. T is a tree
3. T is a subgraph (so $E' \subseteq E$)

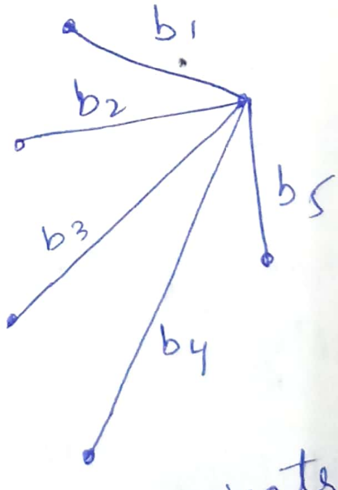
In other words —

- "A subgraph T of a connected graph G is said to be spanning tree of G if the subgraph T is a tree & contains all the vertices of G".

- Edges of spanning tree T are called Branches of T.
- The edge of G that are not in the given ~~tree~~ spanning tree are called chords of T.

Ex




A spanning tree (T)

→ $b_1, b_2, b_3, b_4, b_5$ are branches

→ $c_1, c_2, c_3, c_4, c_5, c_6, c_7$ are chords of this (T) tree.

## Properties of Spanning Tree :-

1. Every connected graph has at least one spanning tree.
2. With respect of any of its spanning tree, a connected graph with n vertices & e edges has $(n-1)$ tree branches and the remaining $(e-n+1)$ chords.
3. There is one & only one path between every pair of vertices in a tree T.
4. A tree with n vertices has $(n-1)$ edges.

Imp

## Rank & Nullity of a Graph :-

Let $G=(V,E)$ be a graph & let $|V|=n$, $|E|=e$
no. of components $=K$.

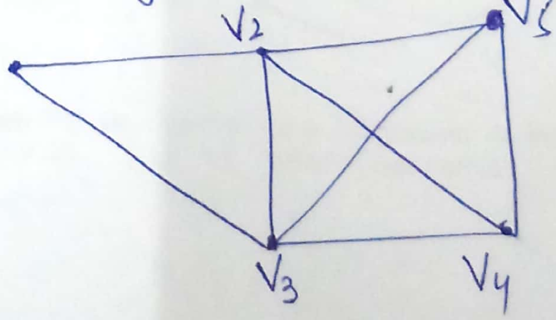
rank $(\gamma) = n-K$

nullity $(\eta) = e-n+K$

Note that rank of connected graph is $(n-1)$,
& nullity $(e-n+1)$

$\boxed{\text{Rank} + \text{Nullity} = \text{No of edges in } G}$

Ex Find rank & nullity of the graph (G)

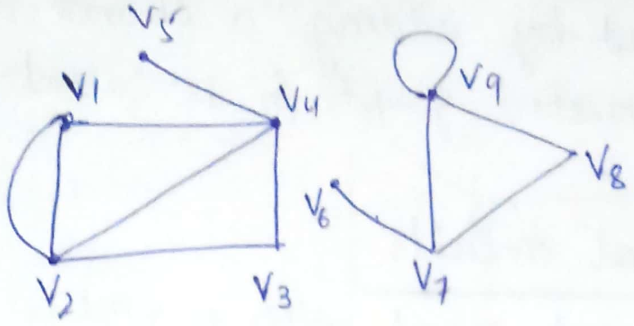



Sol$^n$ $n=5$, $e=8$, $K=1$

Rank $= 5-1=4$

Nullity $= e-n+1 = 8-5+1=4$

Que Find rank & Nullity of following disconnected graph




Soln

$n = 9$ (No of vertices)
$e = 12$ (edges)
$k = 2$ (Two components)

Rank $= n - k \Rightarrow 9 - 2 = 7$

Nullity $= e - n + k$
$= 12 - 9 + 2 = 5$

Imp

In a full Binary tree with n vertices

$\boxed{\text{No of internal vertices } i = \frac{n-1}{2}}$

then no. of leaves $t = n - i \Rightarrow n - \frac{n-1}{2} = \frac{n+1}{2}$

Que What is total no. of nodes in full Binary tree with 20 leaves.

Soln Since we know :-

No. of leaves $t = n - i$ {where $i$ internal vertices given as $i = \frac{n-1}{2}$

$20 = n - \frac{n-1}{2}$

$20 = \frac{n+1}{2}$

$\boxed{n = 39}$ Ans

## Fundamental Circuit :-

A circuit formed by adding a chord to a spanning tree T of a connected graph G is called fundamental circuit of G.

### No. of fundamental circuits

Let G be connected graph with n vertices & e edges.
Let T be a spanning tree of G.
Then G has n-1 tree branches & e-n+1 chords.

→ Since adding a chord to the spanning tree creates one fundamental circuit, it follows that graph has exactly e-n+1 fundamental circuits.

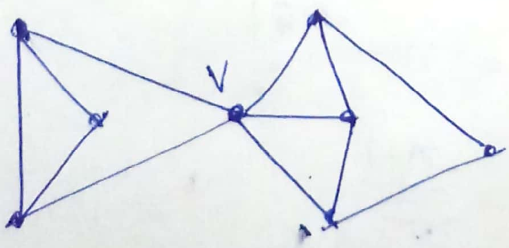
Imp
### Edge-Connectivity

It is defined as the min. no. of edges whose removal reduces the rank of graph by one.

Ex The edge connectivity of tree is one because removal of an edge from tree disconnects the graph.

Imp
**Vertex Connectivity** :- It is defined as the min. no. of vertex whose removal from G leaves the remaining graph disconnected.

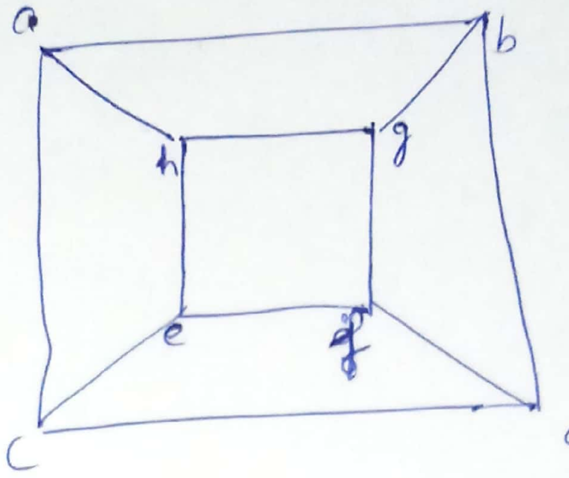
Ex




Find vertex connectivity of the given graph.

Sol$^n$ Vertex connectivity of this graph is one because removal of vertex V disconnects the graph.

Que Find edge connectivity & vertex connectivity of Graph.




Soln If we remove a & f any two vertices doesn't disconnect the graph. However, removal of three vertices a, b & f disconnects the graph, so vertex connectivity is 3.

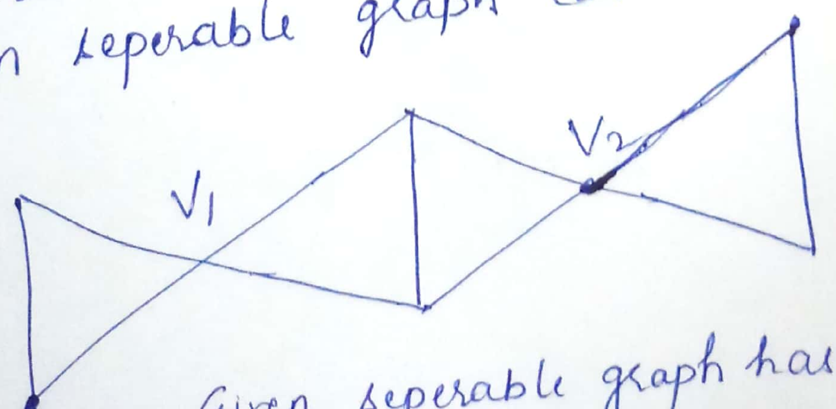
Similarly, if we remove any two edge doesn't disconnects the graph, but the set of any three edges incident on a vertex forms a cut set. Hence edge connectivity is 3.

Seperable Graph :- A connected graph whose vertix connectivity is one is called seperable graph, otherwise it is non-seperable.

Cut-Vertex :- A cut vertex is a vertex in seperable graph whose removal disconnects the graph. Cut-Vertex is also called articulation point.

A given seperable graph can have more than one cut-vertex

Ex




Given seperable graph has two cut vertex.

# Recurrence Relation

A formula which defines any term of a sequence in terms of any no. of its previous terms or which express any term of a sequence as a function of its previous term is called Recursive & the relation is called Recurrence Relation

Ex The $n$th term of the sequence 3, 8, 13, 18 ---
can be written as–

$$a_n = a_{n-1} + 5 \quad \text{for} \quad n \geq 2 \text{ and } a_1 = 3.$$

This formula is called Recursive formula & relation is called recurrence relation, sometimes is called Difference equation.

## Order of Recurrance Relation

Order of recurrance relation or difference equ$^n$ is defined to be the difference b/w highest & lowest subscripts of $f(x)$ or $a_r$ or $y_n$.

Ex equ$^n$ $a_r + 3a_{r-1} = 0$

$r - (r-1)$

$= 1$ first order relation.

$7f(x) + f(x+1) + 5f(x+2) = k(x)$

$x + 2 - x = 2$ 2nd order difference equ$^n$.

**Degree** : Degree of recurrance relation is the highest power of $f(x)$ or $a_r$ or $y_n$.

Ex $y^3_{n+3} + 2y^2_{n+2} + 2y_{n+1} = 0$ degree $= 3$.

A recurrance relation is called Homogenous, if it contains no term that depend only on variable $n$. The recurrance relation which is not homogenous, is called non-homogenous.

Ex
$a_r = a_{r-1} + r^2$ Non Homogenous of order 1, degree = 1

$a_r = r\, a_{r-2} + 2^r$ Non Homogenous of order = 2 degree = 1

$a_r = a_{r-2}^2$ Homogenous of order = 2, degree = 2

## Linear Recurrence Relation with constant coefficients

A recurrance relation is called linear if its degree is 1.

$$C_0 a_r + C_1 a_{r-1} + C_2 a_{r-2} + \cdots a_k a_{r-1} = f(r)$$

$C_0, C_1 \cdots C_k$ are constants.

$f(r)$ is some funⁿ that satisfy the given equⁿ.

## Method to find the solution of Homogenous Relation

let us consider the homogenous diff. equⁿ or linear recurrance relation with constant co-efficient

$$C_0 a_r + C_1 a_{r-1} + \cdots C_k a_{r-k} = 0 \quad (1)$$

The solution of (1) is of the form $A\alpha_1^k$ where $\alpha_1$ is the characteristic root and $A$ is constant determined by boundary conditions.

Now substituting

$$a_r = A\alpha^r, \quad a_{r-1} = A\alpha^{r-1}, \quad a_{r-2} = A\alpha^{r-2} \cdots$$

$$a_{r-k} = A\alpha^{r-k}$$

$$C_0 A\alpha^r + C_1 A\alpha^{r-1} + C_2 A\alpha^{r-2} + \cdots C_k A\alpha^{r-k} = 0$$

$$A\alpha^{r-k}\left(C_0\alpha^k + C_1\alpha^{k-1} + \cdots C_k\right) = 0$$

$$C_0\alpha^k + C_1\alpha^{k-1} + \cdots C_k = 0 \quad \text{——(2)}$$

It is characteristic equ$^n$ of degree $k$.

It has characteristic roots.

| Types of Roots | Solution |
|---|---|
| 1 Distinct Root | $a_r = A_1\alpha_1^r + A_2\alpha_2^r + \cdots A_k\alpha_k^r$ where $A_1, A_2 \cdots A_k$ are constants. |
| 2. Equal Roots | $a_r = (A_m r^{m-1} + A_{m-1} r^{m-2} + \cdots A_2 r + A_1)\alpha_1^r$ If any root is repeated $m$ times. |
| 3. Complex Roots. | $a_r = p^r(A_1 \cos r\theta + A_2 \sin r\theta)$ let $\alpha + i\beta$ and $\alpha - i\beta$ be the roots. $\alpha + i\beta = p(\cos\theta + i\sin\theta)$ $\alpha - i\beta = p(\cos\theta - i\sin\theta)$ |
| 4 Equal Complex Roots | $a_r = p^r[(A_1 + A_2 r)\cos r\theta + (A_3 + A_4 r)\sin r\theta]$ where $p = \sqrt{\alpha^2+\beta^2}$, $\theta = \tan^{-1}(\beta/\alpha)$ |

Que Solve the following recurrance relation
$a_{r+1} - 1.5\, a_r = 0, \quad r > 0$

Put $a_r = A\alpha^r \Rightarrow a_{r+1} = A\alpha^{r+1}$

$A\alpha^{r+1} - 1.5\, A\alpha^r = 0$

$A\alpha^r(\alpha - 1.5) = 0$

But $A\alpha^r \neq 0$

$\alpha - 1.5 = 0$

$\alpha = 1.5$

$a_r = A\alpha^r \Rightarrow A(1.5)^r$ Ans

(2) $a_r = 5a_{r-1} + 6a_{r-2}, \quad r \geq 2, \quad a_0 = a_1 = 3$

$a_r = A\alpha^r$

$A\alpha^r = 5A\alpha^{r-1} + 6A\alpha^{r-2}$

$A\alpha^r\left[1 - \frac{5}{\alpha} + \frac{6}{\alpha^2}\right] = 0$

$A\alpha^r \neq 0 \qquad \alpha^2 - 5\alpha + 6 = 0$

$(\alpha - 3)(\alpha - 2) = 0$

$\alpha = 3, 2$

Distinct roots so solution is given by

$a_r = A_1(\alpha_1)^r + A_2(\alpha_2)^r$

$a_r = A_1 2^r + A_2 3^r$

since $a_0 = a_1 = 3$

for $a_0 = 3, \ r = 0, \qquad a_0 = A_1 + A_2$

$3 = A_1 + A_2 \quad \text{---}(1)$

$a_1 = 3, \ r = 1 \qquad 3 = 2A_1 + 3A_2 \quad \text{---}(2)$

on solving (1) & (2), $A_1 = 6, \quad A_2 = -3$

Required solution $a_r = 6(2)^r - 3(3)^r$ Ans

(3) $a_{r+2} - a_{r+1} - 6a_r = 0$

$A\alpha^{r+2} - A\alpha^{r+1} - 6\alpha^r = 0$

$A\alpha^r(\alpha^2 - \alpha - 6) = 0$

$(\alpha - 3)(\alpha + 2) \Rightarrow \alpha = 3, -2$

solution is $a_r = A_1(3)^r + A_2(-2)^r$ Ans

(4) $a_r = 6a_{r-1} - 11a_{r-2} + 6a_{r-3}$, $r \geq 3$ $a_0 = 2$, $a_1 = 5$

$a_2 = 15$

$A\alpha^r = 6A\alpha^{r-1} - 11A\alpha^{r-2} + 6A\alpha^{r-3}$

$A\alpha^r\left(1 - \frac{6}{\alpha} - \frac{11}{\alpha^2} + \frac{6}{\alpha^3}\right) = 0$

$\alpha^3 - 6\alpha^2 - 11\alpha + 6 = 0$

$(\alpha-1)(\alpha-2)(\alpha-3) = 0$

$\alpha = 1, 2, 3$

$a_r = A_1(1)^r + A_2(2)^r + A_3(3)^r$

$a_0 = 2$

$2 = A_1 + A_2 + A_3$ —— (1)

$a_1 = 5$ $5 = A_1 + 2A_2 + 3A_3$ —— (2)

$a_2 = 15$ $15 = A_1 + 4A_2 + 9A_3$ —— (3)

solving these equ$^n$ we get $A_1 = 1$, $A_2 = -1$, $A_3 = 2$

$a_r = 1 - (2)^r + 2\cdot(3)^r$ Ans

(5) $a_r - 4a_{r-1} + 4a_{r-2} = 0 \qquad a_0 = 1,\ a_1 = 6$

$$A\alpha^r - 4A\alpha^{r-1} + 4A\alpha^{r-2} = 0$$

$$A\alpha^r\left(1 - \frac{4}{\alpha} + \frac{4}{\alpha^2}\right) = 0$$

$$\alpha^2 - 4\alpha + 4 = 0$$

$$(\alpha-2)^2 = 0$$

$$\alpha = 2, 2$$

same roots of characteristic equ$^n$

so solution will be $a_r = (A_1 + A_2 r)\, 2^r$

$a_0 = 1 \qquad 1 = (A_1 + A_2 \cdot 0)\, 2^0$

$$\boxed{A_1 = 1}$$

$a_1 = 6 \qquad 6 = (1 + A_2) \cdot 2$

$$3 = 1 + A_2$$

$$A_2 = 2$$

solution is $a_r = (1 + 2r)\, 2^r$ Ans

(6) $a_r - 4a_{r-1} + 13a_{r-2} = 0$

characteristic equ$^n$ $\alpha^2 - 4\alpha + 13 = 0$

$$\alpha = \frac{4 \pm \sqrt{16-52}}{2} = \frac{4 \pm 6i}{2} = 2 \pm 3i$$

$2 + 3i = p\cos\theta + ip\sin\theta$

$p\cos\theta = 2 \qquad p\sin\theta = 3 \quad \Rightarrow p^2 = 4 + 9 \Rightarrow p = \sqrt{13}$

$\tan\theta = 3/2$

$a_r = p^r(A_1 \cos r\theta + A_2 \sin r\theta) \qquad \theta = \tan^{-1}(3/2)$

$= (13)^{r/2}\left[A_1 \cos r(\tan^{-1} 3/2) + A_2 \sin r(\tan^{-1}(3/2))\right]$ Ans

(7)

$$y_{n+2} - y_{n+1} + y_n = 0 \quad , \quad y_0 = 1 \; , \; y_1 = \frac{1+\sqrt{3}}{2}$$

$$y_n = A\alpha^n$$

$$A\alpha^{n+2} - A\alpha^{n+1} + A\alpha^n = 0$$

$$A\alpha^n(\alpha^2 - \alpha + 1) = 0$$

$$\alpha^2 - \alpha + 1 = 0$$

$$\alpha = \frac{+1 \pm \sqrt{3}\, i}{2} = \frac{1 \pm i\sqrt{3}}{2} \quad \text{——(1)}$$

$$\alpha = p(\cos\theta + i\sin\theta) \quad \text{——(2)}$$

On comparing (1) & (2), we get

$$p\cos\theta = \frac{1}{2} \qquad p\sin\theta = \frac{\sqrt{3}}{2}$$

$$p^2 = \frac{1}{4} + \frac{3}{4} \Rightarrow p = 1$$

$$\theta = \tan^{-1}(\sqrt{3}) = \pi/3$$

$$\tan\theta = \sqrt{3} \Rightarrow$$

$$y_n = (1)^n\left[A_1\cos\left(\frac{\pi}{3}n\right) + A_2\sin\left(\frac{\pi}{3}n\right)\right]$$

The initial condition condition is $y_0 = 1$ &

$$y_1 = \frac{1+\sqrt{3}}{2}$$

$$y_0 = A_1 \Rightarrow A_1 = 1$$

$$y_1 = A_1\cos\frac{\pi}{3} + A_2\sin\frac{\pi}{3} \Rightarrow \frac{1+\sqrt{3}}{2} = \frac{A_1}{2} + \frac{A\sqrt{3}}{2}$$

$$A_2 = 1$$

$$y_n = (1)^n\left(\cos\frac{n\pi}{3} + \sin\frac{n\pi}{3}\right) \quad \underline{\text{Ans}}$$

# Non-Homogeneous Linear Recurrence Relation

$$c_0 a_r + c_1 a_{r-1} + c_2 a_{r-2} \cdots a_k a_{r-k} = f(r)$$

where $c_0, c_1, c_2 \cdots c_k$ are constants.

Total solution of non-homogenous equ$^n$ consists of two parts

(i) Homogenous solution

(ii) particular solution.

## Particular solution

| | Terms in $f(r)$ | Trial solution $= a_r$ |
|---|---|---|
| 1. | $b^r$ | $a_r^{(P)} = A b^r$ |
| 2. | a polynomial of degree k in $r$ | $a_r^{(P)} = A_0 + A_1 r + \cdots\cdots A_K r^K$ |
| 3. | $b^r$ (a polynomial of degree k in $r$) | $a_r^{(P)} = b^r / A_0 + A_1 r + A_2 r^2 + \cdots A_k r^k)$ |
| 4. | $\sin br$ or $\cos br$ | $a_r^{(P)} = A \sin br + B \cos br$ |
| 5. | $b^r \sin br$ | $a_r^{(P)} = b^r (A \cos br + B \sin br)$ |
| 6. | $b^r \cos br$ | $a_r^{(P)} = b^r / A \cos br + B \sin br)$ |
| 7. | $b^r$ (if b is characteristic root with multiplicity m) | $a_r^{(P)} = A b^r r^m$ |
| 8. | Constant term say (c) | $a_r^{(P)} = A$ |

**Que** Determine particular solution for the difference equ$^n$ $a_r - 4a_{r-1} + 4a_{r-2} = 2^r$

**Sol$^n$** Characteristic equ$^n$ is given by

$$\alpha^2 - 4\alpha + 4 = 0$$

$$(\alpha - 2)^2 = 0$$

$$\alpha = 2, 2$$

Here $f(r) = 2^r$, since 2 is characteristic root of multiplicity 2, the P. solution will be

$$a_r^{(P)} = A\, 2^r r^2$$

$a_{r-1} = A\, 2^{r-1}(r-1)^2$ & $a_{r-2} = A\, 2^{r-1}(r-2)^2$

Put these value in question

$$A 2^r r^2 - 4A 2^{r-1}(r-1)^2 + 4A 2^{r-2}(r-2)^2 = 2^r$$

$$\Rightarrow 2A2^r = 2^r$$

$$A = \tfrac{1}{2}$$

Hence particular solution is given

$$a_r = \frac{1}{2} 2^r . r^2 \quad \text{Ans}$$

**Que** $a_r + 6a_{r-1} + 9a_{r-2} = 3$ given that $a_0 = 0$, $a_1 = 1$

$$\alpha^2 + 6\alpha + 9 = 0$$

$$\alpha = -3, -3$$

$$a_r^{(h)} = (A_1 + A_2 r)(-3)^r$$

the particular solution is given as :-

$$a_r^{(P)} = A_3 \quad \{\text{since R.H.S is constant}\}$$

$$A_3 + 6A_3 + 9A_3 = 3 \qquad \boxed{A_3 = 3/16}$$

toral solution = homogenous solution + particular $sol^n$

$$a_r = (A_1 + A_2 r)(-3)^r + \frac{3}{16}$$

Given Boundary conditions $a_0 = 0$, $a_1 = 1$

$$a_0 = A_1 + \frac{3}{16} \quad \Rightarrow \quad A_1 = -\frac{3}{16}$$

$$a_1 = (A_1 + A_2)(-3) + \frac{3}{16}$$

$$A_2 = -\frac{1}{12}$$

Hence required solution

$$a_r = \left(-\frac{3}{16} + -\frac{1}{12} r\right)(-3)^r + \frac{3}{16} \text{ Ans}$$

Que $a_r - 6a_{r-1} + 8a_{r-2} = r \cdot 4^r$ given $a_0 = 8$, $a_1 = 22$

Sol$^n$

$$\alpha^2 - 6\alpha + 8 = 0$$
$$(\alpha-2)(\alpha-4) = 0$$
$$\alpha = 2, 4$$
$$a_r^{(h)} = A_1 \cdot 2^r + A_2 \cdot 4^r \quad \text{——(1)}$$

particular solution.

$$a_r^{(p)} = (A_3 + A_4 r) r 4^r$$

$$(A_3 + A_4 r) r 4^r - 6[A_3 + A_4(r-1)](r-1)4^{r-1} + 8[A_3 + A_4(r-2)](r-2)4^{r-2} = r \cdot 4^r$$

$$r4^r[A_3 + A_4 r] - \frac{\cancel{6}^{3}}{\cancel{4}_{2}}[A_3 + A_4 r - A_4](r-1)4^r + \frac{\cancel{8}}{\cancel{16}_{2}}[A_3 + A_4 r - 2A_4](r-2)4^r = r4^r$$

$$2r4^r[A_3 + A_4 r] - 3\left[\begin{matrix} A_3 r + A_4 r^2 - A_4 r \\ -A_3 - A_4 r + A_4 \end{matrix}\right]4^r + \left[\begin{matrix} A_3 r + A_4 r^2 - 2A_4 r \\ -2A_3 - 2A_4 r + 4A_4 \end{matrix}\right]4^r$$

$$r \cdot 4^r[2A_3 - 3A_3 + 3A_4 + 3A_4 + A_3 - 2A_4 - 2A_4] +$$
$$r^2 4^r[2A_4 - 3A_4 + A_4] + 4^r[+A_3 - 3A_4 + 4A_4 - 2A_3]$$

$$-4A_4 + A_3 - A_3 + 6A_4 = 1 \quad \text{—(1)}$$
$$2A_4 = 1 \Rightarrow \boxed{A_4 = 1/2}$$

$$2\cancel{A_4} - A_3 + A_4 = 0$$

$$A_4 = A_3$$

$$\boxed{A_3 = 1/2}$$

$$a_r^{(p)} = \left(\frac{1}{2} + \frac{1}{2}r\right) r \cdot 4^r$$

$$a_r^{(p)} = (1+r)\frac{r}{2} \cdot 4^r$$

$$a_r = a_r^{(p)} + a_r^{(h)}$$

$$a_r = A_1 \cdot 2^r + A_2 \cdot 4^r + \frac{(1+r) r \cdot 4^r}{2}$$

$\underline{a_0 = 8}$

$$8 = A_1 \cdot \cancel{2} + \cancel{4} A_2 \quad \text{—(1)}$$

$$A_1 + \cancel{2} A_2 = 8$$

$\underline{a_1 = 22}$ $\quad 22 = 2A_1 + 4A_2$

$$A_1 + 2A_2 = 11$$

$$A_1 + A_2 = 8$$

$$\boxed{A_2 = 3} \qquad \boxed{A_1 = 5}$$

$$a_r = 5 \cdot 2^r + 3 \cdot 4^r + \frac{(1+r) \cdot r \cdot 4^r}{2}$$ Ans

## Generating function

If $\{a_1, a_2, a_3 \dots a_n \dots\}$ is a sequence of real or complex numbers, then the power series given by -

$$A(z) = \sum_{r=0}^{\infty} a_r z^r$$

$$= a_0 + a_1 z + a_2 z^2 + \dots + a_n z^n + \dots$$

**Que** Find generating funⁿ for the following series
$1, -1, 1, -1, 1, -1 \dots$

**Solⁿ.** $a_0 = 1, \quad a_1 = -1, \quad a_2 = 1, \quad a_3 = -1$

$$A(z) = \sum_{r=0}^{\infty} a_r z^r = a_0 + a_1 z + a_2 z^2 + \dots$$

$$= 1 - z + z^2 - z^3 + \dots$$

$$= \frac{1}{1+z} \quad \text{Ans}$$

**Que** Find generating funⁿ for $1, 1, 1, 1, 1, 1, 1$

$$A(z) = \sum_{r=0}^{6} a_r z^r$$

$$= a_0 + a_1 z + a_2 z^2 + a_3 z^3 + \dots a_6 z^6$$

$$= 1 + z + z^2 + z^3 + z^4 + z^5 + z^6$$

$$= \frac{1-z^7}{1-z} \quad \text{Ans}$$

**Que** $\{a, a, a, a \dots\}$

$$A(z) = a + az + az^2 + \dots$$

$$a(1 + z + z^2 + \dots)$$

$$\frac{a}{1-z} \quad \text{Ans}$$

**Que** $a_r = b_n = ar^n$

$$A(z) = \sum_{r=0}^{\infty} a_r z^r$$

$$= a + arz + ar^2 z^2 + \dots$$

$$\frac{a}{1-rz} \quad \text{Ans}$$

**Que** $1, 0, 0, 1, 0, 0, 1, 0, 0 \ldots$

$a_0 = 1,\quad a_1 = 0,\quad a_2 = 0,\quad a_3 = 1 \ldots$

$$A(z) = \sum_{r=0}^{\infty} a_r z^r = a_0 + a_1 z + a_2 z^2 + \ldots$$

$$= 1 + 0 + 0 + z^3 + 0 + 0 + z^6 + \ldots$$

$$= 1 + z^3 + z^6 + \ldots \infty$$

$$= \frac{1}{1-z^3} \text{ Ans}$$

**Que** $1, \frac{2}{3}, \frac{3}{9}, \frac{4}{27} \ldots \frac{r+1}{3^r}$

$a_0 = 1,\quad a_1 = \frac{2}{3},\quad a_2 = \frac{3}{9},\quad a_3 = \frac{4}{27}$

$$A(z) = 1 + \frac{2}{3}z + \frac{3}{9}z^2 + \frac{4}{27}z^3 + \ldots$$

$$A(z) = 1 + 2\left(\frac{z}{3}\right) + 3\left(\frac{z}{3}\right)^2 + 4\left(\frac{z}{3}\right)^3 + \ldots$$

Multiplying both sides by $(1 - z/3)$ & then solving.

$$= \frac{1}{(1 - z/3)^2} = \frac{9}{(3-z)^2} \text{ Ans}$$

**Que** $2, 3, 5, 9, 17, 33 \ldots$

$$A(z) = 2 + 3z + 5z^2 + 17z^4 + 33z^5 + \ldots$$

$$= (1 + z + z^2 + z^4 + \ldots) + (1 + 2z + (2z)^2 + (2z)^3 + \ldots)$$

$$= \frac{1}{1-z} + \frac{1}{1-2z} \text{ Ans}$$

$$= \frac{2-3z}{1-3z+2z^2} \text{ Ans}$$

Que Find the generating funⁿ for following numeric funⁿ

(i) $a_r = 7 \cdot 3^r$

$$A(z) = \sum_{r=0}^{\infty} a_r z^r = \sum_{r=0}^{\infty} 7 \cdot 3^r z^r$$

$$= 7\left[1 + 3z + (3z)^2 + (3z)^3 + \cdots\right]$$

$$= 7 \frac{1}{1-3z} = \frac{7}{1-3z} \text{ Ans}$$

(ii) Sol. $a_r = 5^r + (-1)^r 3^r + 8^r + {}^3C_r$

$$A(z) = \sum_{r=0}^{\infty} a_r z^r$$

$$= \sum_{r=0}^{\infty} \{5^r + (-1)^r 3^r + 8^r + {}^3C_r\} z^r$$

$$= (1 + 5z + 5^2 z^2 + \cdots) + (1 - 3z + 3^2 z^2 - 3^3 z^3 + \cdots) + ({}^3C_0 + {}^3C_1 z + {}^3C_2 z^2 + \cdots) + (1 + 8z + 8^2 z^2 + \cdots)$$

$$= \frac{1}{1-5z} + \frac{1}{1+3z} + \frac{1}{1-8z} + (1+z)^3 \text{ Ans}$$

Que Determine numeric funⁿ corresponding to each of following generating funⁿ.

$$A(z) = \frac{1}{5-6z+z^2} = \frac{1}{(5-z)(1-z)} = \frac{1}{4}\left(\frac{1}{1-z} - \frac{1}{5-z}\right)$$

$$= \frac{1}{4} \cdot \frac{1}{1-z} - \frac{1}{20} \cdot \frac{1}{(1-\frac{z}{5})}$$

$$a_r = \frac{1}{4} - \frac{1}{20}\left(\frac{1}{5}\right)^r, \; r \geq 0 \Rightarrow \frac{1}{4}\left[1 - \left(\frac{1}{5}\right)^{r+1}\right]$$

$r \geq 0$

## Solution of Recurrance Relation by Method of Generating funⁿ :-

Que Solve the recurrance relation $a_r - 7a_{r-1} + 10a_{r-2} = 0$ $\forall r \geq 2$, given that $a_0 = 10$, $a_1 = 41$ using generating funⁿ.

Solⁿ $a_r - 7a_{r-1} + 10a_{r-2} = 0$

Multiplying $z^r$ both sides & then take summation $r = 2$ to $\infty$, we find -

$$\sum_{r=2}^{\infty} a_r z^r - 7\sum_{r=2}^{\infty} a_{r-1} z^r + 10\sum_{r=2}^{\infty} a_{r-2} z^r = 0$$

$$(A(z) - a_0 - a_1 z) - 7z(A(z) - a_0) + 10z^2[A(z)] = 0$$

$$A(z) = \frac{a_0 + (a_1 - 7a_0)z}{(1-2z)(1-5z)}$$

$$A(z) = \frac{10 + (-29)z}{(1-2z)(1-5z)} = \frac{c_1}{1-2z} + \frac{c_2}{1-5z}$$

on solving & comparing both sides -

$c_1 = 3, \quad c_2 = 7$

$$A(z) = \frac{3}{(1-2z)} + \frac{7}{(1-5z)}$$

$$\sum_{r=0}^{\infty} a_r z^r = 3\sum_{r=0}^{\infty} 2^r z^r + 7\sum_{r=0}^{\infty} 5^r z^r$$

$$a_r = 3 \cdot 2^r + 7 \cdot 5^r$$

Ans

Que $a_{r+2} - 2a_{r+1} + a_r = 2^r$

solve by method of Generating funⁿ with initial conditions $a_0 = 2$ & $a_1 = 1$.

Solⁿ

$$\sum_{r=0}^{\infty} a_{r+2} z^r - 2\sum_{r=0}^{\infty} a_{r+1} z^r + \sum_{r=0}^{\infty} a_r z^r = \sum_{r=0}^{\infty} 2^r z^r$$

$$\frac{1}{z^2}\sum_{r=0}^{\infty} a_{r+2} z^{r+2} - \frac{2}{z}\sum_{r=0}^{\infty} a_{r+1} z^{r+1} + \sum_{r=0}^{\infty} a_r z^r = \frac{1}{1-2z}$$

$$\frac{A(z) - a_0 - a_1 z}{z^2} - \frac{2}{z}(A(z) - a_0) + A(z) = \frac{1}{1-2z}$$

$$A(z) - 2 - z - 2z(A(z) - 2) + z^2 A(z) = \frac{z^2}{1-2z}$$

$$Az(1 - 2z + z^2) = 2 - 3z + \frac{z^2}{1-2z}$$

$$A(z)(1-z)^2 = 2 - 3z + \frac{z^2}{(1-2z)}$$

$$A(z) = \frac{2}{(1-z)^2} - \frac{3z}{(1-z)^2} + \frac{z^2}{(1-2z)(1-z)^2}$$

By Partial fraction

$$= \frac{1}{(1-z)^2} - \frac{3z}{(1-z)^2} + \frac{1}{(1-2z)}$$

$$\sum_{r=0}^{\infty} a_r z^r = \sum_{r=0}^{\infty} (r+1) z^r - 3\sum_{r=0}^{\infty} r \cdot z^r + \sum_{r=0}^{\infty} 2^r z^r$$

$$a_r = (r+1) - 3r + 2^r \quad \text{Ans}$$

Que Solve $s(n) - 2s(n-1) - 3s(n-2) = 0$, $n \geq 0$ with $s(0) = 3$ & $s(1) = 1$ using generating funⁿ.

Solⁿ

$$A(z) = \sum_{n=0}^{\infty} s(n) z^n$$

$$\sum_{n=2}^{\infty} s(n) z^n - 2 \sum_{n=2}^{\infty} s(n-1) z^n - 3 \sum_{n=2}^{\infty} s(n-2) z^n = 0$$

$$A(z) - s(0) - s(1)z - 2z[A(z) - s(0)] - 3z^2(A(z)) = 0$$

$$A(z) - 3 - z - 2zA(z) + 6z - 3z^2 A(z) = 0$$

$$A(z)[1 - 2z - 3z^2] - 3 + 5z = 0$$

$$A(z) = \frac{3 - 5z}{(1-3z)(1+z)} = \frac{A}{(1-3z)} + \frac{B}{(1+z)}$$

on solving $A = 1$, $B = 2$

$$A(z) = \frac{1}{1-3z} + \frac{2}{1+z}$$

$$\sum_{n=0}^{\infty} s(n) z^n = \sum_{n=0}^{\infty} 3^n z^n + 2 \sum_{n=0}^{\infty} (-1)^n z^n$$

$$s(n) = 3^n + 2(-1)^n$$ Ans

Que Solve By method of Generating funⁿ –

$$a_r - 2a_{r-1} + a_{r-2} = \frac{1}{4}2^r \quad , \quad a_0 = 2 \ \& \ a_1 = 1$$

Solⁿ

$$\sum_{r=2}^{\infty} a_r z^r - 2\sum_{r=2}^{\infty} a_{r-1} z^r + \sum_{r=2}^{\infty} a_{r-2} z^r = \frac{1}{4}\sum_{r=2}^{\infty} 2^r z^r$$

$$\left(A(z) - a_0 - a_1 z\right) - 2z\left(A(z) - a_0\right) + z^2 A(z) = \frac{1}{4}\left[\frac{1}{1-2z} - 2z - 1\right]$$

$$\left(A(z) - 2 - z\right) - 2z\left(A(z) - 2\right) + z^2 A(z) = \frac{1}{4}\left[\frac{1}{1-2z} - 2z - 1\right]$$

$$A(z)\left[1 - 2z + z^2\right] - z - 2 + 4z = \frac{1}{4}\left[\frac{1}{1-2z} - 2z - 1\right]$$

$$A(z)(1-z)^2 = \frac{1}{4}\left[\frac{1}{1-2z} - 14z + 7\right]$$

$$A(z) = \frac{7z^2 - 7z + 2}{(1-2z)(1-z)^2}$$

By partial fraction –

$$\frac{7z^2 - 7z + 2}{(1-2z)(1-z)^2} = \frac{A}{(1-2z)} + \frac{B}{(1-z)} + \frac{Cz}{(1-z)^2}$$

$$A = 1 , \ B = 1 , \ C = -2$$

$$A(z) = \frac{1}{1-2z} + \frac{1}{1-z} - \frac{2z}{(1-z)^2}$$

$$\sum a_r z^r = \sum 2^r z^r + \sum z^r - 2\sum r z^r$$

$$a_r = 2^r + 1 - 2r$$

Ans