UNIT-1

Lecture No-1

Ref Pt - 1.1

Set : The Theory of sets was developed by German Mathematician George Canter (1845-1918 A.D)

A well defined collection of distinct objects is called a set.

(i) The objects in a set are called its members or elements.

(ii) We denote sets by capital letters A, B, C, X, Y, Z etc.

(iii) The elements of a set are represented by small letters a, b, c, x, y, z etc.

If a is an element of a set A, we write $a \in A$ i.e a belongs to A.

If a doesn't belong to A, it is written as $a \notin A$.

Example :-

Collection of all natural nos. denoted by N.

The solution of equ^n $x^2-5x+6=0$ i.e 2 & 3.

Representation of Sets :-

There are two methods of representing a set.

1) Roster or tabular form (2) Rule Method or Set Builder form.

(1) **Roaster or Tabular form**

In this form, all elements of a set are listed within braces { } & are seperated by commas.

Example :- A = set of all factors of 12

A = {1, 2, 3, 4, 6, 12}

B = set of all letters in the word 'MATHEMATICS'

B = {M, A, T, H, E, I, C, S}

2) Rule Method or set Builder form

In this form, we list the property satisfied by all the elements of the set.

It is written as $\{x : x \text{ satisfies the property}\}$

Example :-

Write the set $A = \{1, 2, 3, 4, 5, 6, 7, 8\}$ in set builder form.

$$A = \{x : x \in N \text{ and } x < 9\}$$

Write set $E = \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{6}{7}, \frac{7}{8}, \frac{8}{9}\right\}$ in set builder form

$$E = \left\{x : x = \frac{n}{n+1} \text{ where } n \in N \text{ and } 1 \leq n \leq 8\right\}$$

Types of set :-

1. Empty set

A set containing no element at all is called the empty set or the null set or void set, denoted by ϕ or $\{\ \}$.

Ex
$\{x : x \in N \text{ and } 2 < x < 3\} = \phi$

$\{x : x \in R \text{ and } x^2 = -1\} = \phi$

$\{x : x^2 = 9 \text{ and } x \text{ is an even integer}\} = \phi$

2. Singleton set.

A set containing exactly one element is called singleton set.

Ex
$\{0\}$ is a singleton set whose element is 0.

$\{x : x \in N \text{ and } x^2 - 9 = 0\} = \{3\}$ which is singleton set.

Subsets

A set A is said to be subset of set B. If every element of A is also an element of B and we write, $A \subseteq B$

Super Set

If $A \subseteq B$ then B is called a superset of A & we write $B \supseteq A$.

Proper Subset

If $A \subseteq B$ and $A \neq B$, then A is called a proper subset of B and we write $A \subset B$.

Illustration of subset, superset & proper subset

(a) Let $A = \{2,3,5\}$ and $B = \{2,3,5,7,9\}$ then B but $A \neq B$ every element of A is an element of B

$A \subset B$ ie A is proper subset of B.

(b) since ϕ has no element, we agree to say that ϕ is a subset of every set.

(c) For every set A, we have $A \subseteq A$, since every element of A is an element of A.

Intervals as subsets of R

Let $a, b \in R$ and $a < b$ then, we define -

(i) closed interval $[a,b] = \{x \in R : a \leq x \leq b\}$

(ii) open interval $]a,b[= \{x \in R : a < x < b\}$

(iii) Right half open interval $[a,b[= \{x \in R : a \leq x < b\}$

(iv) Left half open interval $]a,b] = \{x \in R : a < x \leq b\}$

3. Finite and Infinite sets

Set is said to be finite if it consists of only fini[te] no. of elements. A set which is not finite is called an infinite set.

The no. of distinct element in a finite set A is denoted by $n(A)$.

Ex. of finite sets

(a) The set of all persons on the earth is finite set

(b) Let $A = \{2, 4, 6, 8, 10\}$. Then A is finite and $n(A) = 5$

Ex. of infinite sets

(a) N = Set of all natural nos. $= \{1, 2, 3, 4 ----\}$

(b) The set of all points on the arc of a circle is an infinite set.

4. Equal sets :-

Two sets A & B are said to be equal, if they have exactly the same elements & we write $A = B$.

If the sets are not equal, then $A \neq B$.

5. Equivalent sets

Two finite sets A & B are said to be equivalent if $n(A) = n(B)$

Let $A = \{2, 3, 4\}$, $B = \{7, 8, 9\}$

$n(A) = n(B) = 3$

Power Set :

The set of all subsets of a given set A is called power set of A and is denoted by P(A).

If $n(A) = m$ then $n[P(A)] = 2^m$.

Ques Two finite sets have m & n elements. The total no. of subsets of the first set is 56 more than the total no of subsets in the 2nd set. Find values of m & n.

Sol^n In, 1st set total no. of subsets $= 2^m$

In 2nd set total " " " $= 2^n$

$$2^m = 2^n + 56$$

$$2^m - 2^n = 56$$

$$2^n(2^{m-n} - 1) = 56$$

$$2^n(2^{m-n} - 1) = 8 \times 7$$

$$2^n(2^{m-n} - 1) = 2^3 \times 7$$

$\boxed{n = 3}$

$$2^{m-3} = 7 + 1$$

$$2^{m-3} = 2^3$$

$$m - 3 = 3$$

$$\boxed{m = 6}$$

Operations of set :-

1) Union of set

The union of two set A & B, denoted by $A \cup B$, is the set of all those elements which are either in A or in B or in both A & B.

Thus $A \cup B = \{x : x \in A \text{ or } x \in B\}$

$x \in A \cup B \Leftrightarrow x \in A$ or $x \in B$

$x \notin A \cup B \Leftrightarrow x \notin A$ and $x \notin B$.

Algebr

2) Intersection of sets

The intersection of two set A & B denoted by $A \cap B$ is the set of all those elements which are common in both A & B.

Thus, $A \cap B = \{x : x \in A \text{ and } x \in B\}$

$x \in A \cap B \Leftrightarrow x \in A$ and $x \in B$

$x \notin A \cap B \Leftrightarrow x \notin A$ or $x \notin B$.

Disjoint sets : Two sets A & B are said to be disjoint if $A \cap B = \phi$

Intersecting sets : Two sets A & B are said to be intersecting if $A \cap B \neq \phi$

(3) Difference of sets

for any set A & B, their difference A − B is defined as −

$$A - B = \{x : x \in A \text{ and } x \notin B\}$$

$x \in (A - B) \Leftrightarrow x \in A$ and $x \notin B$.

(4) Symmetric Difference of sets

Let A & B are two sets.

Then $(A - B) \cup (B - A)$ is called symmetric difference of A & B and is denoted by $A \Delta B$

$$A \Delta B = (A - B) \cup (B - A)$$
$$= (A - B) \cup (B - A)$$

(5) Complement of a set

Let U be the universal set and let $A \subseteq U$. Then, the complement of A denoted by A′ or $(U - A)$ is defined as −

$$A' = \{x \in U : x \notin A\} \Rightarrow x \in A' \Leftrightarrow x \notin A$$

Algebra of Set Theory :-

There are some very important laws of algebra which the set theory follows.

1) Commutative law Let us consider two sets A & B Then

(i) $A \cup B = B \cup A$ (ii) $A \cap B = B \cap A$

Proof (i) Let $x \in A \cup B \Rightarrow x \in A$ or $x \in B$

$\Rightarrow x \in B$ or $x \in A$

$x \in B$ or A

$x \in B \cup A$

$A \cup B \subseteq B \cup A$ —— (1)

similarly we have $B \cup A \subseteq A \cup B$ —— (2)

from (1) & (2)

$\boxed{A \cup B = B \cup A}$ Proved

Proof (ii) Let $x \in A \cap B \Rightarrow x \in A$ and $x \in B$

$\Rightarrow x \in B$ and $x \in A$

$\Rightarrow x \in (B \cap A)$

$A \cap B \subseteq B \cap A$ —— (1)

similarly $B \cap A \subseteq A \cap B$ —— (2)

from (1) & (2) $\boxed{A \cap B = B \cap A}$ Proved

(2) Associative law :- The set are associative under the proposition of union & intersection ie if A, B & C are three sets.

Then (i) $(A \cup B) \cup C = A \cup (B \cup C)$

(ii) $(A \cap B) \cap C = A \cap (B \cap C)$

Proof (i) Consider an element x, such that

$x \in (A \cup B) \cup C$

$x \in (A \cup B)$ or $x \in C$

$x \in A$ or $x \in B$ or $x \in C$

$x \in A$ or $\{x \in B$ or $x \in C\}$

$x \in A$ or $\{x \in (B \cup C)\}$

$x \in A \cup (B \cup C)$

$(A \cup B) \cup C \subseteq A \cup (B \cup C)$ —— (1)

similarly we can show

$A \cup (B \cup C) \subseteq (A \cup B) \cup C$ —— (2)

from (1) & (2)

$\boxed{A \cup (B \cup C) = (A \cup B) \cup C}$ Proved

Proof (ii) Consider an element x such that

$x \in (A \cap B) \cap C$

$\{x \in A$ and $x \in B\}$ and $x \in C$

$x \in A$ and $\{x \in (B \cap C)\}$

$x \in A \cap (B \cap C)$

$(A \cap B) \cap C \subseteq A \cap (B \cap C)$ —— (1)

similarly $A \cap (B \cap C) \subseteq (A \cap B) \cap C$ —— (2)

from (1) & (2)

$\boxed{A \cap (B \cap C) = (A \cap B) \cap C}$ proved

(3) Idempotent laws If A be any set, then

(i) $A \cup A = A$ (ii) $A \cap A = A$

Proof (i) let any element $x \in A \cup A$, Then

$x \in A \cup A \Rightarrow x \in A$ or $x \in A$

$\Rightarrow x \in A$

$A \cup A \subseteq A$ —— (1)

Again if $x \in A$ then $x \in A$ or $x \in A$

$x \in (A \cup A)$

$\therefore A \subseteq (A \cup A)$ —— (2)

from (1) & (2)

$\boxed{A \cup A = A}$ Pro

) Identity Law The empty set ϕ and Universal set U are the identity elements of union & intersection respectively.

(i) $A \cup \phi = A$ (ii) ~~A~~ $A \cap U = A$

Proof (i) let x be an element such that $x \in A \cup \phi$

$x \in A$ or $x \in \phi$

$x \in A$

$A \cup \phi \subseteq A$ — (1)

Again let $x \in A$ then $x \in A \Rightarrow x \in A$ or $x \in \phi$

$x \in A \cup \phi$

$A \subseteq A \cup \phi$ — (2)

from (1) & (2)

$\boxed{A \cup \phi = A}$ Proved

Proof (ii) In similar way as Proof (i), It can also be proved.

Ordered Pair If $a \in B$ and $b \in B$ then the set $\{\{a\}, \{a,b\}\}$ consisting of ordered pair is the set $\{a\}$. It is represented by pair (a,b) and singleton $\{a\}$. It is represented by (a,b), the element a is called 1st element & b is called 2nd element.

Cartesian Product :- If A & B are two non empty sets then the Cartesian product of A & B is the set of all ordered pairs (a,b) where $a \in A$ and $b \in B$.

Imp **Theorem** :- State and Prove Demorgan's law.

or To prove the following relations.

Theorem

(i) $(A \cup B)' = A' \cap B'$ (ii) $(A \cap B)' = A' \cup B'$

Proof :-

(i) let $x \in (A \cup B)' \Rightarrow x \notin (A \cup B)$

$\Rightarrow x \notin A$ and $x \notin B$

$\Rightarrow \cancel{x \notin (A \cap B)}$

$\Rightarrow x \in A'$ and $x \in B'$

$\Rightarrow x \in (A' \cap B')$

$\therefore (A \cup B)' \subseteq A' \cap B'$ — (1)

Again let $x \in A' \cap B' \Rightarrow x \in A'$ and $x \in B'$

$\Rightarrow x \notin A$ and $x \notin B$

$\Rightarrow x \notin (A \cup B)$

$\Rightarrow x \in (A \cap B)'$

$(A' \cap B') \subseteq (A \cap B)'$ — (2)

from equ^n (1) & (2)

$\boxed{(A \cup B)' = A' \cap B'}$ Proved

Proof (ii) let $x \in (A \cap B)' \Rightarrow x \notin (A \cap B)$

$\Rightarrow x \notin A$ or $x \notin B$

$\Rightarrow x \in A'$ or $x \in B'$

$\Rightarrow x \in A' \cup B'$

$(A \cap B)' \subseteq A' \cup B'$ } — (1)

In the same way $A' \cup B' \subseteq (A \cap B)'$ — (2)

from (1) & (2)

$\boxed{(A \cap B)' = A' \cup B'}$ Proved

Theorem :- If A, B, C be sets, then prove that

$$A-(B\cup C) = (A-B)\cap(A-C)$$

Proof

Let $x \in A-(B\cup C) = x\in A$ and $x \notin (B\cup C)$

$= x \in A$ and $(x\notin B$ and $x\notin C)$

$= x\in A$ and $x\notin B$ and $x \in A$ and $x\notin C$

$= x\in (A-B) \cap x\in (A-C)$

$x \in [(A-B)\cap(A-C)$

$$A-(B\cup C) \subseteq (A-B)\cap(A-C) \quad -(1)$$

Again let $y \in (A-B)\cap(A-C)$

$y\in A$ and $y\notin B$ and $y\in A$ and $y\notin C$

$y \in A$ and $y \notin B$ and $y\notin C$

$y\in A$ and $y\notin (B\cup C)$

$y \in (A-(B\cup C))$

$$(A-B)\cap(A-C) \subseteq A-(B\cup C) \quad -(2)$$

from (1) & (2)

$$A-(B\cup C) = (A-B)\cap(A-C)$$

Lecture No-2

Ref Pt -1.2

Multiset :

A collection of objects that are not necessarily distinct, is called a multiset.

Multiplicity of an Element

The no. of times an element appears in the multiset is called the Multiplicity of the element. In fact, we can characterize a multiset as pair (A, μ) where A is the generic set & μ is the multiplicity fun^n defined as -

$$\mu : A \to \{1, 2, 3 \ldots\}$$

$\mu(a) = k$ where k is the no. of times the element is in the multiset.

Equality of Multiset

If the no. of occurance of each element is the same in both the multiset then the multiset are equal.

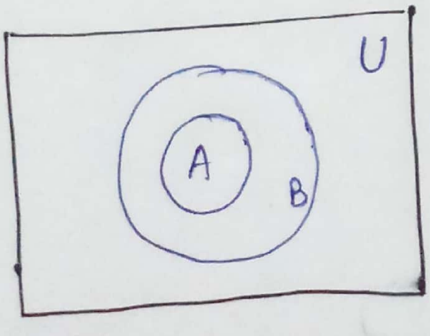
Illustration : $\{a, b, a, a\} = \{a, a, b, a\}$

but $\{a, b, a\} \neq \{a, b\}$

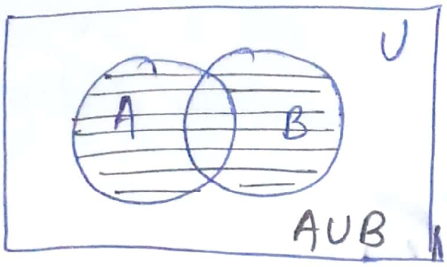
Venn Diagram

In order to express the relationship among set in perspective them pictorially by means of diagram, called Venn Diagram.

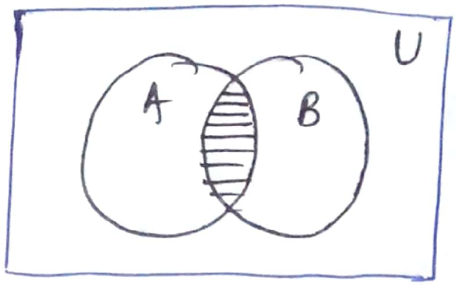
1. Set Inclusion Operation



2. Union of two set

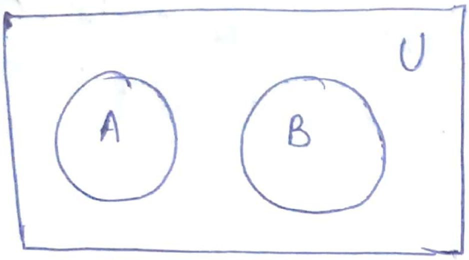


3. Intersection of two se

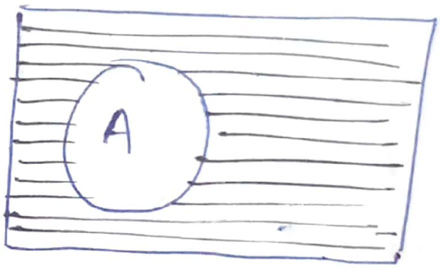


A∩B

4. Disjoint set



5. Complement of a set



1. $n(A \cup B) = n(A) + n(B) - n(A \cap B)$
2. $n(A - B) + n(A \cap B) = n(A)$
3. $n(B - A) + n(A \cap B) = n(B)$
4. $n(A - B) + n(A \cap B) + n(B - A) = n(A \cup B)$
5. $n(A \cup B \cup C) = n(A) + n(B) + n(C) + n(A \cap B \cap C)$
$- [n(A \cap B) + n(B \cap C) + n(A \cap C)]$

Venn Diagram : Application

Qu: In a group of 50 people, 35 speaks Hindi, 25 speaks both English & Hindi and all people speak at least one of the two Languages. How many people speak only English & not Hindi? How many speak English?

Sol^n Let A be the set of people that speak Hindi, B be the set of people that speak English.

$n(A \cup B) = 50$

$n(A) = 35$

$n(A \cap B) = 25$

$n(A \cup B) = n(A) + n(B) - n(A \cap B)$

$50 = 35 + n(B) - 25$

$n(B) = 50 - 35 + 25 = 40$

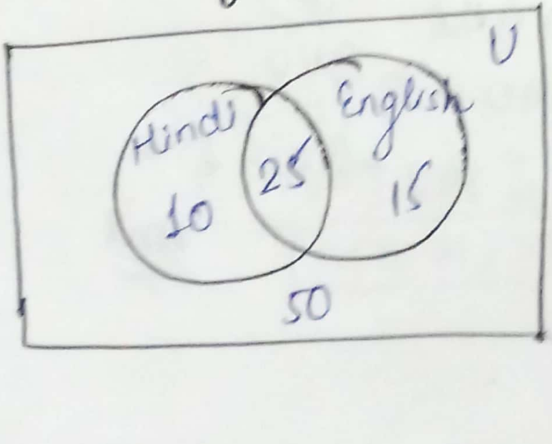
No. of people that speak English only

$n(B) - n(A \cap B)$

$40 - 25 = 15$

No. of people speaking English $= 40$

From Venn Diagram



Que In a group 850 persons, 600 can speak Hindi & [illegible] can speak Tamil.

(i) How many can speak both Hindi & Tamil ?

(ii) How many can speak Hindi only ?

(iii) How many can speak Tamil only.

Sol^n Let A = set of persons who can speak Hindi

B = set of persons who can speak Tamil.

$n(A) = 600, \quad n(B) = 340 \quad n(A \cup B) = 850$

set of persons who can speak both Hindi & Tamil $= A \cap B$

$$n(A \cap B) = n(A) + n(B) - n(A \cup B)$$
$$= 600 + 340 - 850 = 90$$

(ii) set of persons who can speak Hindi only $= n(A-B)$

$$n(A-B) + n(A \cap B) = n(A)$$
$$n(A-B) + 90 = 600$$
$$n(A-B) = 600 - 90 = 510$$

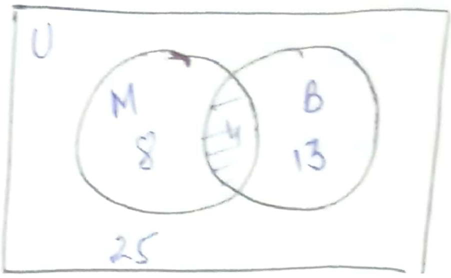
(iii) set of persons who can speak Tamil only $= n(B-A)$

$$n(B-A) + n(A \cap B) = n(B)$$
$$n(B-A) + 90 = 340$$
$$n(B-A) = 340 - 90 = 250$$

Ans

Que In a class of 25 students 12 have taken mathematics
8 have taken mathematics but not biology.
Find no. of students who have taken mathematics & biology and those who took biology but not mathematics.

Soln



$$n(M) = 12$$
$$n(M-B) = 8$$

$$n(M-B) + n(M \cap B) = n(M)$$
$$8 + n(M \cap B) = 12$$

(i) $n(M \cap B) = 4$ Ans

(ii) Those who took biology but not mathematics

$$n(M \cup B) = n(M) + n(B) - n(M \cap B)$$
$$25 = 12 + n(B) - 4$$
$$n(B) = 17$$
$$n(B-M) + n(B \cap M) = n(B)$$
$$n(B-M) + 4 = 17$$
$$\boxed{n(B-M) = 13}$$ Ans

Que In a group of 52 persons, 16 drink tea but not Coffee & 33 drink tea.

(i) How many drink tea & coffee both?

(ii) How many drink coffee but not tea.

Soln

$n(T \cup C) = 52$

$n(T - C) = 16, \quad n(T) = 33$

$n(T-C) + n(T \cap C) = n(T)$

$n(T \cap C) = n(T) - n(T-C)$

$= 33 - 16 = 17$ Ans

(ii) $n(C-T) + n(T \cap C) = n(C)$

$n(C-T) + 17 = n(C) \quad \text{---}(1)$

$n(T \cup C) = n(T) + n(C) - n(T \cap C)$

$52 = 33 + n(C) - 17$

$n(C) = 36$

Putting value of $n(C)$ in eqn (1)

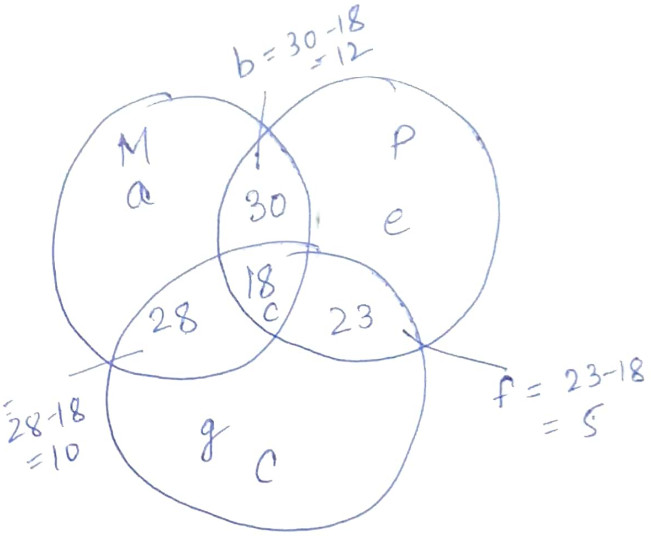
$n(C-T) + 17 = 36$

$n(C-T) = 19$ Ans

A class has 175 students. The following description gives the no. of students studying one or more of subjects in this class.

Maths 100, Phy - 70, ~~Phy~~ Chemistry - 46, Maths & Phy - 30
Maths & chem = 28, Phy & chem - 23, Math, Phy & chem = 18.

Find
(i) How many students are enrolled in Mathematics alone, phy alone & chemistry alone.
(ii) The no. of students who have not offered any of these subjects.



Soln

enrolled in Maths alone -

$= 100 - [30 + 28 - 18)$
$= 60$

enrolled in phy alone

$= 70 - [30 + 23 - 18]$
$= 35$

enrolled in chemistry alone

$= 46 - [28 + 23 - 18]$
$= 13$

(ii) No. of students Not offering any of these subjects

$= 175 - [60 + 12 + 18 + 10 + 35 + 5 + 13)$
$= 175 - 153 = 22$ Ans

Que In a survey, it is found that 21 people like produ 26 people like product B and 29 like product C. If 14 people like products A & B, 15 like B & C, 12 like C & and 8 like all three.

(i) How many people are surveyed in all?

(ii) How many like C only -

Soln

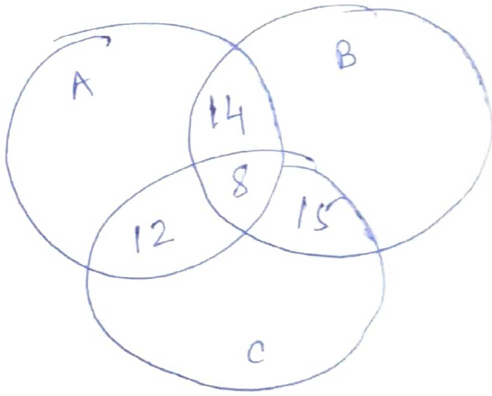
$$n(A\cup B\cup C) = n(A) + n(B) + n(C) + n(A\cap B\cap C) - [n(A\cap B) + n(B\cap C) + n(C\cap A)]$$

$$n(A\cup B\cup C) = 21 + 26 + 29 + 8 - (14 + 15 + 12)$$

$$= 84 - 41$$

$$= 43 \text{ Ans}$$

(ii)



people who like product C only :-

$$= 29 - (\cancel{8+} 15 + 12 - 8)$$

$$= 29 - (27 - 8)$$

$$= 2 + 8 = 10 \text{ Ans}$$

Lecture No-3

Ref pt-1.3

Relations

Binary Relation

Let A & B be non-empty sets, then any subset R of the Cartesian product A×B is called a Relation from A to B, & is denoted by R.

Thus R is a relation from A to B ⇒ $R \subseteq A \times B$.

symbolically, we write

$R = \{(x, y) : x \in A, y \in B \text{ and } xRy\}$.

xRy denotes the x is R related to y.

Example Let $A = \{1, 2, 5\}$ and $B = \{2, 4\}$ be two given sets. Now suppose a relation from set A to B is expressed by statement 'is less than'.

sol^n We have $A \times B = \{(1,2)(1,4)(2,2)(2,4)(5,2)(5,4)\}$ some ordered pair are related & when $x < y$, then some ordered pair are related in some are not.

The subset A×B whose element are related in the relation R is given by -

$R = \{(1,2), (1,4), (2,4)\}$

clearly $R \subset A \times B$.

Total No. of Distinct Binary Relation :

If set A has m elements & set B has n elements then A×B will have mn elements.

Therefor powerset of A×B will have 2^{mn} elements.

Hence, no. of different relations from A to B $= 2^{mn}$

Domain & Range of Relation :-

Let $R = \{(x, y) : x \in A,$ ~~and~~ $y \in B$ and $xRy\}$ be a relation from A to B. Then the set of 1st co-ordinates of every element of R is called Domain R and denoted by Dom(R) or d(R) and the set of 2nd co-ordinates of its every element is called Range of R & denoted by r(R) or Ran(R).

symbolically -

$d(R) =$ domain of $R = \{x : x \in A$ and $(x, y) \in R$ for some $y \in B\}$

$r(R) =$ range of $R = \{y : y \in B$ and $(x, y) \in R$ for some $x \in A\}$

Operations on Relation :-

1. Complement of a Relation

Consider a relation R from set A to B. The complement of relation R denoted by $\bar{R}$ or R' is a relation from A to B such that

$$\bar{R} = \{(a, b) : (a, b) \notin R\}$$

Ex Let R be a relation from X to Y, where $X = \{1, 2, 3\}$ and $Y = \{8, 9\}$ $R = \{(1,8)(2,8)(1,9)(3,9)\}$

Sol^n $X \times Y = \{(1, 8) (1, 9) (2, 8) (2, 9) (3, 8) (3, 9)\}$

Then complement relation $\bar{R}$ w.r. to $X \times Y$

$$\bar{R} = \{(2, 9) (3, 8)\}$$

2. Inverse Relation

let R be a relation from A to B. The inverse of relation R denoted by R^{-1} is a relation from B to A such that $bR^{-1}a$ iff aRb.

symbolically

$$R^{-1} = \{(b,a) : (a,b) \in R\}$$

Ex Find R^{-1} to the relation R on A defined "$x+y$" divisible by 2. For $A = \{1,2,3,4,6\}$

Soln $R = \{(1,1)(2,2)(3,3)(4,4)(6,6)(1,3)(2,4)(2,6)(4,6)$

$\Rightarrow R^{-1} = \{(1,1),(2,2),(3,3),(4,4),(6,6),(3,1),(4,2)(6,2)(6,4)\}$

3. Intersection & Union of Relations

If R & S are two relations then intersection of R & S and union is denoted by $R \cup S$ are denoted by $R \cap S$ that can be formed from R & S. two new relation

Thus $R \cup S = \{(x,y) : xRy \text{ or } xSy\}$

$R \cap S = \{(x,y) : xRy \text{ and } xSy\}$

Illustration let $R_1 = \{(1,1)(2,2)(3,3)(4,4)(3,4)(4,3)\}$

$R_2 = \{(1,1)(2,2)(3,3)(4,4)(1,2)(2,1)\}$

Then $R_1 \cup R_2 = \{(1,1)(2,2)(3,3)(4,4)(3,4)(4,3)(1,2)(2,1)\}$

$R_1 \cap R_2 = \{(1,1)(2,2)(3,3)(4,4)\}$

Lecture No - 4

Ref Pt - 1.4

Properties of Relations :-

A relation R on a set A satisfies certain properties. These properties are defined as -

(i) Reflexive Relation

A relation R on a set A is reflexive if $aRa \ \forall \ a \in A$

ie $(a,a) \in R \ \forall \ a \in R \Rightarrow$ each element a of A is related to itself.

$\boxed{\text{NOTE No of Reflexive Relation from A to A} = 2^{n^2-n}}$

Illustration let $A = \{a,b\}$ & $R = \{(a,a)(a,b)(b,b)\}$
Then R is reflexive as aRa, $bRb \in R$

(ii) Irreflexive Relation

A relation R on set A is irreflexive if, for every $a \in A$, $(a,a) \notin R$

Illustration: let $A = \{1,2\}$ and $R = \{(1,2),(2,1)\}$
Then R is irreflexive, since both $(1,1)$ & $(2,2) \notin R$.

(iii) Non-Reflexive Relation

A relation R on a set A is non-reflexive if R is neither reflexive nor irreflexive.

(iv) Symmetric Relation

If R is a relation in set A, then R is called symmetric relation if a is R related to b then b is also R relation to a.

ie $(a,b) \in R \Rightarrow (b,a) \in R$ or

$aRb \Rightarrow bRa \ \forall \ a,b \in A$

$\boxed{\text{NOTE} \rightarrow \text{No of symmetric relation from A to A} = 2^{\frac{n(n+1)}{2}}}$

s relats that

Illustration

If $A = \{2,4,5,6\}$ and
$R_1 = \{(2,4)(4,2)(4,5)(5,4)(6,6)\}$
$R_2 = \{(2,4)(2,6)(6,2)(5,4)(4,5)\}$

Then relation R_1 is symmetric since
$(2,4) \in R_1 \Rightarrow (4,2) \in R_1$
$(4,5) \in R_1 \Rightarrow (5,4) \in R_1$
$(6,6) \in R_1 \Rightarrow (6,6) \in R_1$

But R_2 is not symmetric
since $(2,4) \in R_2 \Rightarrow (4,2) \notin R_2$.

(v) Antisymmetric Relation :-
A relation R is said to be antisymmetric if aRb and $bRa \Rightarrow a=b$

Illustration :- In the set of natural numbers, the relation a divides b is anti-symmetric, since a divides b and b divides a is possible only when $a=b$.

(vi) Asymmetric Relation :-
A relation R on set A is asymmetric if $(a,b) \in R$ then $(b,a) \notin R$ for $a \neq b$

Illustration let $A = \{1,2,3\}$ and $R = \{(1,2)(1,1)(2,3)(3,1)\}$
The relation R on set A is

(vii) Transitive Relation The relation R on set A is called transitive Relation if aRb and $bRc \Rightarrow aRc$ or $(a,b) \in R$ and $(b,c) \in R \Rightarrow (a,c) \in R \ \forall a,b,c \in A$

Illustration if $A = \{1,3,5\}$ and $R = \{(1,3)(1,5)(3,5)\}$
then $1R3$ and $3R5 \Rightarrow 1R5$

Imp Equality of Relation

or Equivalence Relation

Let A be non-empty set and R be a relation defined on A. Then R is said to be Equivalence Relation if it is

(i) Reflexive ie $aRa \ \forall a \in A$

(ii) Symmetric ie $aRb \Rightarrow bRa \ \forall a, b \in A$

(iii) Transitive ie aRb and $bRc \Rightarrow aRc \ \forall a, b, c \in A$

Composite Relation :-

Let A, B & C be three non-empty sets and R be a relation from A to B and S be a relation from B to C. Then composite Relation of the two relations R & S is a relation from A to C and denoted by $S \circ R$ and defined as :-

$$S \circ R = \{(a, c) : \exists \text{ an element } b \in B \text{ such that } (a, b) \in R \text{ and } (b, c) \in S\} \text{ where } a \in A, c \in C.$$

Hence we can say that

$$(a, b) \in R, \ (b, c) \in S \Rightarrow (a, c) \in S \circ R.$$

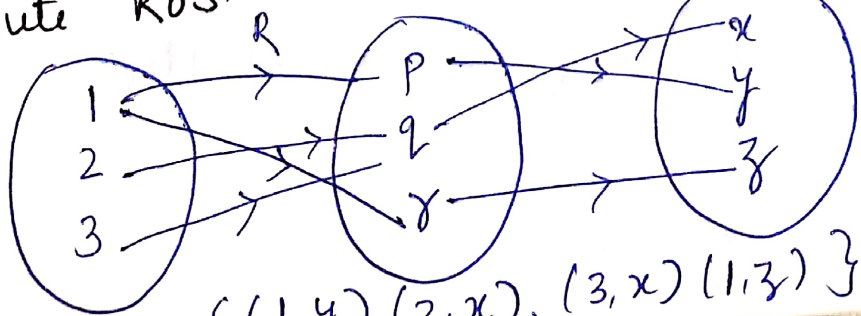
$$R \circ R = R^2$$

similarly $R^3 = R^2 \circ R = R \circ R \circ R$.

Que Let $A = \{1, 2, 3\}$, $B = \{p, q, r\}$ and $C = \{x, y, z\}$

Let $R = \{(1, p), (1, r) (2, q) (3, q)\}$ & $S = \{(p, y) (q, x) (r, z)\}$

Then Compute $R \circ S$.



$\{(1, y) (2, x), (3, x) (1, z)\}$

PRACTICE PROBLEMS

Solved example on equivalence relation on set:

1. A relation R is defined on the set Z by "a R b if a - b is divisible by 5" for a, b ∈ Z. Examine if R is an equivalence relation on Z.

Solution:

(i) Let a ∈ Z. Then a – a is divisible by 5. Therefore aRa holds for all a in Z and R is reflexive.

(ii) Let a, b ∈ Z and aRb hold. Then a – b is divisible by 5 and therefore b – a is divisible by 5.
Thus, aRb ⇒ bRa and therefore R is symmetric.

(iii) Let a, b, c ∈ Z and aRb, bRc both hold. Then a – b and b – c are both divisible by 5.
Therefore a – c = (a – b) + (b – c) is divisible by 5.
Thus, aRb and bRc ⇒ aRc and therefore R is transitive.
Since R is reflexive, symmetric and transitive so, R is an equivalence relation on Z.

2. Three relations R_1, R_2 and R_3 defined on a set $A = \{a, b, c\}$ as follows:

$R_1 = \{(a,a), (a,b), (a,c), (b,b), (b,c), (c,a), (c,b), (c,c)\}$.

$R_2 = \{(a,a)\}$

$R_3 = \{(b,c)\}$

$R_4 = \{(a,b), (b,c), (c,a)\}$.

Find whether or not each of the relations R_1, R_2, R_3, R_4 on A is (i) reflexive (ii) symmetric (iii) transitive.

SOLUTION

We have, $A = \{a, b, c\}$

$R_1 = \{(a,a)(a,b)(a,c)(b,b)(b,c)(c,a)(c,b)(c,c)\}$

R_1 is reflexive *as* $(a,a) \in R_1, (b,b) \in R_1$ & $(c,c) \in R_1$

R_1 is not symmetric *as* $(a,b) \in R_1$ *but* $(b,a) \in R_1$

R_1 is not transitive as $(b,c) \in R_1$ and $(c,a) \in R_1$ but $(b,a) \notin R_1$

$R_2 = \{(a,a)\}$

R_2 is not reflexive as $(b,b) \notin R_2$

R_2 is symmetric and transitive.

$R_3 = \{(b,c)\}$

R_3 is not reflexive as $(b,b) \notin R_3$

R_3 is not symmetric

R_3 is not transitive.

$R_4 = \{(a,b)(b,c)(c,a)\}$

R_4 is not reflexive on set A as $(a,a) \notin R_4$

R_4 is not symmetric as $(a,b) \in R_4$ but $(b,a) \notin R_4$

R_4 is not transitive as $(a,b) \in R_4$ and $(b,c) \in R_4$ but $(a,c) \notin R_4$

Example 3. Let R be the relation on Z × Z such that ((a, b),(c, d)) ∈ R ⇔ a + d = b + c. Show that R is an equivalence relation.

Solution.

R is reflexive:

Suppose (a, b) is an ordered pair in Z × Z.

[We must show that (a, b) R (a, b).]

We have a + b = a + b.

Thus, by definition of R, (a, b) R (a, b).

R is symmetric: Suppose (a, b) and (c, d) are two ordered pairs in Z × Z and (a, b) R (c, d).

[We must show that (c, d) R (a, b).]

Since (a, b) R (c, d), a + d = b + c. But this implies that b + c = a + d, and so, by definition of R, (c, d) R (a, b).

R is transitive: Suppose (a, b),(c, d), and (e, f) are elements of Z × Z, (a, b) R (c, d), and (c, d) R (e, f). **[We must show that (a, b) R (e, f).]**

Since (a, b) R (c, d), a + d = b + c, which means a − b = c − d,

and since (c, d) R (e, f)C, c + f = d + e, which means c − d = e − f.

Thus a − b = e − f, which means a + f = b + e, and so, by definition of R, (a, b) R (e, f).

Lecture No - 5

Ref pt - 1.5

Order of Relation

There are two order of relation

(i) Partial order Relation

The set & A together with partially order relation R on the set A denoted by (A,R) is called partially ordered set or Poset.

(ii) Total order Set

Consider the relation R on the set A. If it is the case, that for all $a, b \in A$, we have either $(a,b) \in R$ or $(b,a) \in R$ or $a = b$, then the relation R is called total order relation on set A.

Closure of Relation

Let R be a relation on a set A. R may or may not have some property P, such as reflexivity, symmetry or transitivity. If there is a relation S with property P containing R such that S is a subset of every relation with P containing R, then S is called the closure of R w.r.to P.

Reflexive closure

A relation $R_1 = R \cup \Delta$ is the reflexive closure of relation R if $R \cup \Delta$ is the smallest relation containing R which is reflexive.

· Illustration $A = \{a, b, c\}$ & R is given by $R = \{(a,a)(a,b)(b,c)\}$

then Reflexive closure $R_1 = R \cup \Delta$

where Δ is set of elements of type (a,a) $\forall a \in A$

$\Delta = \{(a,a)(b,b)(c,c)\}$

$R_1 = R \cup \Delta = \{(a,a)(a,b)(b,b)(b,c)(c,c)\}$

Symmetric Closure

Let R be relation on A which is not symmetric & R^{-1} be Inverse relation of R on A the symmetric closure R^* is defined as $R^* = R \cup R^{-1}$

Example Is $R = \{(1,2)\ (4,3)(2,2)(2,1)(3,1)\}$ be a relation on $S = \{1,2,3,4\}$. find the symmetric closure.

$$R^{-1} = \{(2,1)\ (3,4)(2,2)(1,2)\ (1,3)\}$$

Then $R^* = R \cup R^{-1}$

$$= \{(1,2)(2,1)\ (4,3)(3,4)\ (1,3)(3,1)(2,2)\}$$

Transitive Closure

A relation obtained by adding the least no. of ordered pair to ensure transitivity is called the transitive closure of relation.

$$R^+ = R \cup R^2 \cup R^3 \ldots \cup R^m$$

where R is relation on set A that contains m elements.

Ex Let $A = \{1,2,3,4\}$ & $R = \{(1,2)\ (2,3)(3,4)\}$ be a relation on A. find transitive closure ~~A~~ R^+.

We have $R = \{(1,2)(2,3)(3,4)\}$

$$R^2 = R \circ R = \{(1,2)(2,3)\ (3,4)\} \circ \{(1,2)(2,3)\ (3,4)\}$$
$$= \{(1,3)(2,4)\}$$

$$R^3 = R^2 \circ R = \{(1,3)\ (2,4)\} \circ \{(1,2)(2,3)\ (3,4)\}$$
$$= \{(1,4)\}$$

$$R^4 = R^3 \circ R = \{(1,4)\} \circ \{(1,2)(2,3)(3,4)\} = \phi$$

$$R^+ = R \cup R^2 \cup R^3 \cup R^4$$
$$= \{(1,2)(2,3)\ (3,4)\ (1,3)(2,4)(1,4)\}$$ Ans

Matrix Representation of Relations

Matrix Of Relation

let R be a relation from set A to B

where $\begin{cases} A = \{a_1\ a_2\ a_3 -- a_m\} \\ B = \{b_1\ b_2\ b_3 -- a_n\} \end{cases}$

be finite sets having m & n elements respectively.
Then R can be represented by mn matrix and defined as $M_R = \{m_{ij}\}$

$$m_{ij} = \begin{cases} 1 & \text{if } (a,b) \in R \\ 0 & \text{if } (a,b) \notin R \end{cases}$$

The matrix M_R is called the matrix of relation

$\boxed{\text{NOTE} \quad M_{SOT} = M_T \cdot M_S}$

Example Let R be the relation from the set $A = \{1, 3, 4\}$ on itself, defined by
$R = \{(1,1)(1,3)(3,3)(4,4)\}$ then find relation matrix

$$M_R = \begin{array}{c} \\ 1 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 3 & 4 \end{array} \\ \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \end{array} = \begin{array}{c} \\ 1 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array}$$

Diagraphs:

If A is a finite set and R is a relation on A, we can also represent R pictorially as -

(i) Draw the small circle for each element of A and label the circle with corresponding element. These circles are called vertices

(ii) Draw an arrow, called edge, from vertex a_i to a_j $\Rightarrow a_i R a_j$. The resulting pictorial representation of R is called diagraph of R.

The directed graph representing a relation can be used to determine whether the relation has various properties.

(i) A Relation is reflexive iff there is loop at every vertex of directed graph.
If no vertex has a loop, then relation is irreflexive.

(ii) A relation is symmetric iff for every edge between distinct vertices in its digraph there is an edge in the opposite direction.

(iii) A relation is transitive iff whenever there is a directed edge from a vertex 'a' to a vertex 'b' and from vertex 'b' to 'c', then there is also direct edge from a to c.

Equivalence Classes :

Consider an equivalence relation R on a set A. The equivalence class of an element $a \in A$ is the set of elements of A to which a is related. It is denoted by $[a]$ or $\bar{a}$.

Que Let $A = \{1,2,3,4\}$ and let
$R = \{(1,1)(1,2)(1,3)(2,1)(2,2)(3,1)(2,3)(3,2)(3,3)(4,4)\}$ find rank of R.
Determine the equivalence classes and

Soln The equivalence class of A are

$[1] = R(1) = \{1,2,3\}$

$[2] = R(2) = \{1,2,3\} = [1]$

$[3] = R(3) = \{1,2,3\} = [1]$

$[4] = R(4) = \{4\}$

Hence, there are 2 distinct equivalence classes.
so rank of R is 2.

Lecture No-6

Ref Pt - 1.6

Definitions

1. Domain set of input (also called pre-image) is called domain. It is denoted by D_f.

2. Co-domain set of possible outputs is called co-domain.

3. Range set of actual outputs (also called images) is called Range. It is denoted by R_f.

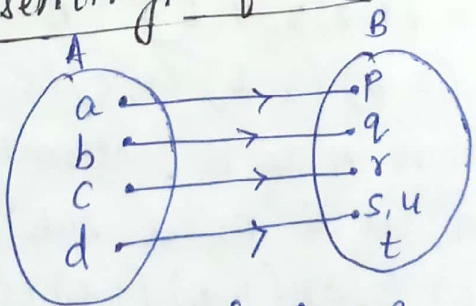
Function (Mathematical Definition)

If A & B are two non-empty sets then a rule f, under which to every element x of the set A, there corresponds one & only one element of set B then the rule f is called the function from A to B, it is denoted by $f: A \to B$.

If a pre-image is denoted by x and an image is denoted by y. then we can write $y = f(x)$

Methods of Representing a function

1) Arrow Diagram



$f: A \to B$

$D_f = \{a, b, c, d\}$

Co-domain $= \{q, p, r, s, t, u\}$

and $f(a) = p$, $f(b) = q$, $f(c) = r$, $f(d) = s$, $f(d) = u$

2) Tabular form

A	x	1	2	3	4	5
B	y	a	b	c	d	e

Let $f: A \to B$ such that
$D_f = \{1, 2, 3, 4, 5\}$ $R_f = \{a, b, c, d, e\}$

Difference between function & a Relation

- If R is a relation from A to B, then domain may be subset of A, but if f is a function or mapping from A to B, then the domain f will be A.
- In relation R, any element of A can be associated to more than one element in B and it is also possible that some elements of A are not associated to any element in B.
But in mapping f, every element of A is associated to one & only one element in B.

Illustration let $A = \{1, 2, 3, 4, 5\}$ $B = \{a, b, c\}$
if $R = \{(1, a), (2, b) (3, b) (4, c) (5, c)\}$
Then R is a fun^n from A to B. clearly B is a fun^n from A to B. Again, let S be a subset of $A \times B$.
where $S = \{(1, a) (2, c) (1, b) (4, c) (5, c)\}$
Here S is a relation from A to b, but S is not a fun^n from A to B because $1 \in A$ is associated with two elements a & b of B.

Classification of Function

1. Real Function

A funⁿ $f: A \to B$ is called real valued if the image of every element of A under f is a real number ie

if $f(x) \in R \ \forall \ x \in A$ or $y = f(x)$

where y is dependent variable & x is independent variable.

2. Algebraic Function

The funⁿ consisting of finite no. of terms involving different power of independent variable (x) & the operation plus (+), minus (−), × & ÷ are called Algebraic funⁿ

eg $2x^2 + x^{1/3} + 4$, $x^2 + 3x + 9$.

3. Polynomial Function

A funⁿ whose domain & co-domain both is the set of real nos. & contain finite no. of terms containing natural number powers of x multiplied by real constant is called polynomial funⁿ.

If $f: R \to R$, such that $f(x) = a_0 + a_1 x + \cdots a_n x^n$.

4. Rational Function

A funⁿ obtained by dividing a polynomial by another polynomial is called a rational funⁿ.

Domain of such funⁿ contains all the real numbers ~~exp~~ except the real no. for which the polynomial in denominator is zero.

$f: A \to R$ $f(x) = \dfrac{P(x)}{Q(x)}$ where $P(x)$ & $Q(x)$ are polynomial funⁿ &

$A = \{x : x \in R$ such that $Q(x) \neq 0\}$

Ex $f(x) = \frac{x^2+5x+7}{x^2-3x+2}$ is a rational funⁿ.

As $x^2 - 3x + 2 \neq 0$, the domain of funⁿ is $R - \{1, 2\}$

5. Irrational Function

The algebraic funⁿ containing one or more terms having non-integral rational power of x are called irrational funⁿ

Ex $g(x) = \frac{2x^{3/2} + 5x}{x^2 - 1}$

6. Modulus Function :-

$f : R \to R$ such that

$$f(x) = |x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

then $f(x)$ is called modulus funⁿ.

7. Signum Function

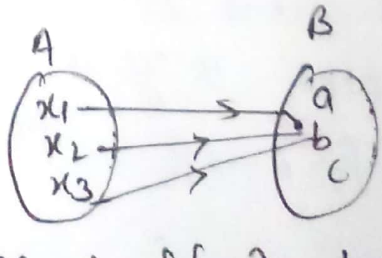
The funⁿ defined by $f(x) = \begin{cases} \frac{|x|}{x} & \text{when } x \neq 0 \\ 0 & \text{when } x = 0 \end{cases}$

Thus we have $f(x) = \begin{cases} 1 & \text{when } x > 0 \\ 0 & \text{when } x = 0 \\ -1 & \text{when } x < 0 \end{cases}$

$D_f = R$, $R_f = \{-1, 0, 1\}$

8. Constant funⁿ Let c be a fixed real number. Then the funⁿ defined by $f(x) = c \ \forall x \in R$ is called Constant funⁿ C.

Clearly $D_f = R$ and $R_f = \{c\}$



$f(x_1) = f(x_2) = f(x_3) = b$

$5/1/21$

Identity function

The funⁿ defined by $f(x) = x \ \forall x \in R$ is called the identity funⁿ.

$D_f = R, \ R_f = R$

10. ## Reciprocal Function

The funⁿ defined by $f(x) = \frac{1}{x}$ is called the reciprocal funⁿ.

The funⁿ $f(x) = \frac{1}{x}$ is not defined if $x = 0$.

11. ## Step Function or Greatest Integer funⁿ

If $x \in R$ then $[x]$ is defined as greatest integer not exceeding x

ie. $[2.01] = 2$, $[2.9] = 2$, $[-1.3] = 2$, $[-1] = -1$

Que Find a set of all real no. x such that $[x] = 2$

Soln $\forall x$ such that $2 \leq x < 3$ we have $f(x) = [x] = 2$

So required set $= \{x \in R : 2 \leq x < 3\} = [2, 3[$

12. ## Exponential funⁿ

$f(x) = e^x$ is defined for all real value of x.

also $y = e^x \Rightarrow x = \log_e y$

13 ## logarithmic funⁿ

$f(x) = \log x$

$\log x$ is not defined when x is zero or negative.

14. ## Trignometric funⁿ $\{\sin x, \cos x, \tan x, \cot x, \sec x, \operatorname{cosec} x\}$

(a) $\sin x$ defined for $x \in R$, Range $[-1, 1]$

(b) $\cos x$ $\forall x \in R$, range $[-1, 1]$

(c) $\tan x$ not defined when $\cos x = 0$

Range $= R$ domain $= R - \{(2n+1)\frac{\pi}{2}, n \in I\}$

(d) $\underline{\cot x}$ not defined when $\sin x = 0$

Domain $= R - \{n\pi : n \in I\}$

and Range $= R$.

(e) $\underline{\sec x}$ Not defined when $\cos x = 0 \Rightarrow \cos x = \cos(2n+1)\frac{\pi}{2}$

$x = (2n+1)\pi/2$. Then domain $= R - \{(2n+1)\frac{\pi}{2}, n \in I$

Range $= R -]-1,1[$

(f) $\underline{\operatorname{cosec} x}$ Not defined when $\sin x = 0 = \sin n\pi$

$x = n\pi$

Domain $= R - \{n\pi : n \in I\}$

Range $= R -]-1,1[$

Que $f(x) = x^2 - \frac{1}{x^2}$ show that $f(x) + f\left(\frac{1}{x}\right) = 0$

Que $y = f(x) = \frac{3x+1}{5x-3}$ prove that $f(y) = x$

Que $f(x) = \log_e\left(\frac{1+x}{1-x}\right)$ show that $f(x) + f(y) = f\left(\frac{x+y}{1+xy}\right)$

Function	Domain	Range
$y = \sin(x)$	All Real Numbers	$\{y \mid -1 \leq y \leq 1\}$
$y = \cos(x)$	All Real Numbers	$\{y \mid -1 \leq y \leq 1\}$
$y = \csc(x)$	$\{x \mid x \neq \ldots\ldots\ldots\ldots -2\pi, -\pi, 0, \pi, 2\pi \ldots\ldots\ldots\ldots\}$	$\{y \mid y \leq -1 \text{ or } y \geq 1\}$
$y = \sec(x)$	$\{x \mid x \neq \ldots\ldots\ldots\ldots -3\pi/2, -\pi/2, \pi/2, 3\pi/2 \ldots\ldots\ldots\ldots\}$	$\{y \mid y \leq -1 \text{ or } y \geq 1\}$
$y = \tan(x)$	$\{x \mid x \neq \ldots\ldots\ldots\ldots -3\pi/2, -\pi/2, \pi/2, 3\pi/2 \ldots\ldots\ldots\ldots\}$	All Real Numbers
$y = \cot(x)$	$\{x \mid x \neq \ldots\ldots\ldots\ldots -2\pi, -\pi, 0, \pi, 2\pi \ldots\ldots\ldots\ldots\}$	All Real Numbers

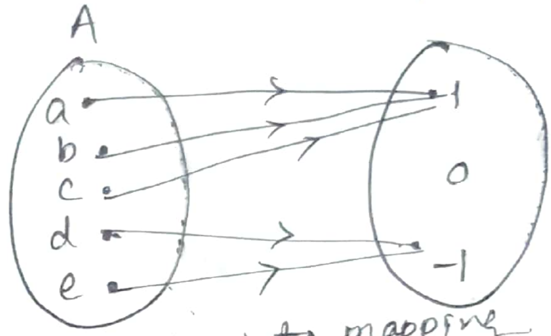
Lecture No-7

Ref. Pt-1.7

Kind of Mappings

1. Into Mapping

If $f: A \to B$ be a mapping such that at least one element of B is not a f-image of any element of the set A, then the mapping f is said to be an into mapping or A into B mapping.



This is into mapping

Symbolltically

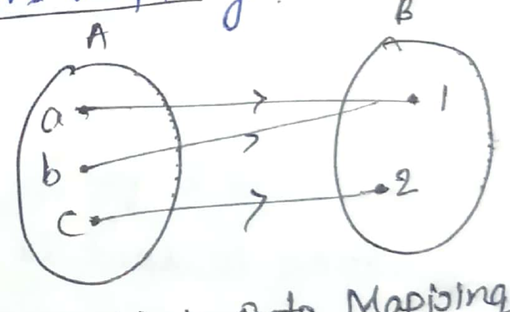
A mapping $f: A \to B$ is said to be into mapping if

$$\{f(x) : x \in A\} \subset B.$$

2. Onto Mapping

If f be a mapping such that each element of B is f-image of at least one element of A, then the mapping f is said to an onto or subjective mapping.

$f: A \to B$ will be an onto mapping if $\{f(x) : x \in A\} = B$



This is Onto Mapping.

3. One-One Mapping

Let $f: X \to Y$ be a mapping. If all distinct element of the domain X has distinct f-images in Y, then the mapping f is said to one-one or injective mapping.

Thus $f: X \to Y$ will be one one or injective mapping.

$$x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$$

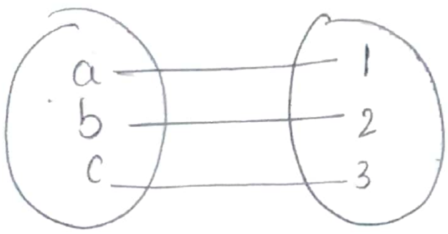
or

$$f(x_1) = f(x_2) \Rightarrow x_1 = x_2$$

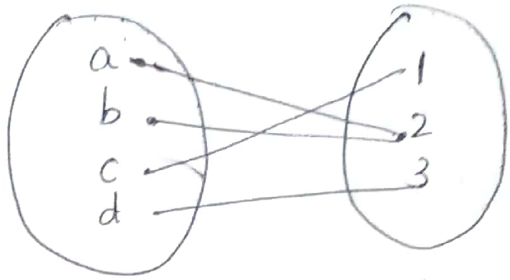
4. Many-One Mapping

Let $f: X \to Y$ be a mapping. If two or more than two elements of the domain X have the same f-image in Y, then the mapping f is said to be many-one mapping.

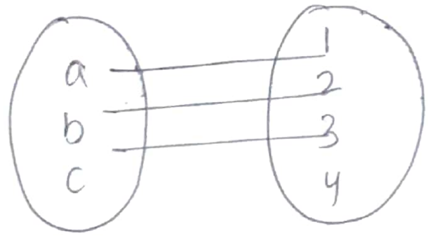
Thus $f: X \to Y$ will be many one mapping, if $x_1 \neq x_2 \Rightarrow$
$f(x_1) = f(x_2)$



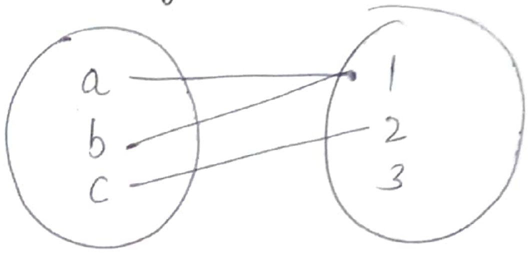
One-one onto mapping
(bijective)



Many-one onto mapping



One-one Into mapping



Many-one into mapping.

Even and Odd funⁿ

A function is said to be

(i) even if $f(-x) = f(x)\ \forall x$ (ii) odd if $f(-x) = -f(x)\ \forall x$.

Que show that a mapping $f: R \to R$ where $f(x) = -\sin x$, $x \in R$ is neither one-one nor onto.

Solⁿ Here $f(x_1) = -\sin x_1$, $f(x_2) = -\sin x_2$, $x_1, x_2 \in R$

$f(x_1) = f(x_2) \Rightarrow -\sin x_1 = -\sin x_2 \Rightarrow \sin x_1 = \sin x_2$

Since $f(x_1) = f(x_2)$ ~~even~~ when $x_1 \neq x_2$

Hence Mapping is not one-one.

Again since numerical value of $\sin x$ can't exceed 1, so mapping is not onto.

Que Discuss the mapping $f: R \to R$ defined by $f(x) = x^2$ where R is set of real numbers.

Soln Here domain is real numbers
& range is set of '+ve' real numbers.
{since square of any real no. is '+ve'}
Thus f-image is proper subset of its domain.
$\{f(x) : x \in R\} \subset R$.
Hence it is Into mapping
Again $f(x_1) = f(-x_1) = x_1^2$ is same element
ie f-image of two distinct element
Thus it is many-one mapping
Thus it is many-one into mapping.
Hence given mapping is many-one into mapping.

Que Show that mapping $f: R \to R$, $f(x) = \frac{1}{x}$, $x \neq 0$ & $x \in R$ is one-one onto, where R is set of non-zero real numbers.

Soln Let $x_1, x_2 \in R$ be any two non-zero real numbers,
then $f(x_1) = f(x_2) \Rightarrow \frac{1}{x_1} = \frac{1}{x_2}$
$\Rightarrow x_1 = x_2$
f is one-one mapping
Let $y \neq 0$ be any real no. such that $y = f(x)$
$y = \frac{1}{x} \Rightarrow x = \frac{1}{y}$
$\because y \neq 0$ $\frac{1}{y}$ is also a non-zero real no.

$f\left(\frac{1}{y}\right) = \frac{1}{1/y} = y$

It shows that $\{f(x) = \frac{1}{x}, x \neq 0, x \in R\} = R$
$\therefore$ f(x) is onto.
Hence given mapping is one-one onto.

Inverse Function :-

Let $f: X \to Y$ be one-one onto mapping. Let $y \in Y$ sin[ce] f is onto, there exists $x \in X$ such that $f(x) = y$.

Again since f is one-one, this is the only element of X such that $f(x) = y$.

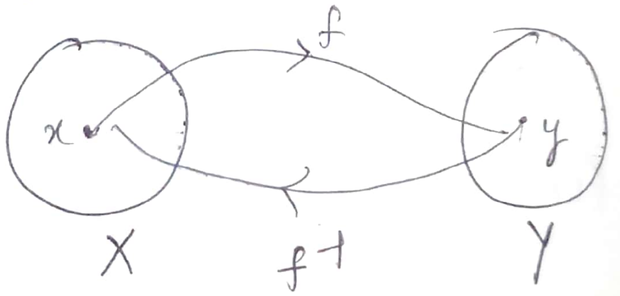
Thus we have seen that for each $y \in Y$, there is a unique $x \in X$ that $f(x) = y$.

This mapping from Y to X is called inverse of f. denoted by f^{-1}.

Thus $f(x) = y \Rightarrow f^{-1}(y) = x$.

Definition If $f: X \to Y$ is a one-one onto mapping, then the mapping $f^{-1}: Y \to X$ which associates each element $y \in Y$ to a unique element $x \in X$ is called inverse mapping of $f: X \to Y$.

$x = f^{-1}(y)$



$y = f(x)$

Theorem: Prove that if $f: A \to B$ is one-one onto mapping then $f^{-1}: B \to A$ will be one-one onto mapping.

(UPTU B Tech 2007)

Sol^n Since f is one one onto mapping.

so we have

$$y = f(x) \Rightarrow f^{-1}(y) = x \quad , \quad x \in A , y \in B$$

let $y_1 = f(x_1)$, $y_2 = f(x_2)$, $x_1, x_2 \in A$ and $y_1, y_2 \in B$

$$x_1 = f^{-1}(y_1) \quad x_2 = f^{-1}(y_2)$$

Now $f^{-1}(y_1) = f^{-1}(y_2) \Rightarrow x_1 = x_2 \Rightarrow f(x_1) = f(x_2)$ $\{\because f \text{ is a fun}^n\}$

$$\Rightarrow y_1 = y_2$$

The mapping f^{-1} is one-one.

Again let x be an arbitrary element of A, then for mapping f there exists an element y in B such that

$y = f(x)$

But $f^{-1}(y) = x \quad \forall \, x \in A$

$$\{f^{-1}(y) : y \in B\} = A$$

$f^{-1}: B \to A$ is onto.

$f^{-1}: B \to A$ is one-one onto.

Hence $f^{-1}: B \to A$ is one-one onto.

Theorem:- If f is one-one onto (bijective) fun^n from set A to set B, then f^{-1} will be a unique function.

Proof:- Suppose there are two inverse g & h of $f: A \to B$.

Now let y be any arbitrary element of B

then $g(y) = x_1$ and $h(y) = x_2$ where $x_1, x_2 \in A$

Hence $g^{-1}(x_1) = y$ and $h^{-1}(x_2) = y$ $\{\because g \,\&\, h \text{ are inverse of } f\}$

~~$g^{-1}(x_1) = g^{-1}(x_2) h^{-1}(x_2)$~~

Hence $f(x_1) = y$ and $f(x_2) = y$

Therefore $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$ {$\because$ g & h are inverse of f} {f is one one onto}

$g(y) = h(y) \;\forall\; y \in B \Rightarrow g = h$

Hence, inverse of f is unique.

<u>Theorem:</u> The inverse of an inversible mapping is unique.

<u>Proof</u>:- Let $f: A \to B$
By invertible mapping, if possible, let $g: B \to A$ and $h: B \to A$ be two different inverse mapping of f. Let $b \in B$ and $g(b) = a_1$, $a_1 \in A$
$h(b) = a_2$, $a_2 \in A$

Now $g(b) = a_1 \Rightarrow b = f(a_1)$
$h(b) = a_2 \Rightarrow b = f(a_2)$
$f(a_1) = f(a_2) \Rightarrow a_1 = a_2$ [$\because$ f is one-one]
This proves that $g(b) = h(b) \;\forall\; b \in B$.
Thus, the inverse of f is unique.

Questions

1. Let $A = \{1, 2, 3, 4, 6\}$ & let R be a relation on A defined "x divides y" indicated as x/y if there exists an integer z, such that $xz = y$.
(i) Write R as set of ordered pairs.
(ii) Find the inverse relation R^{-1} and describe R^{-1} in words.

Soln
(i) $R = \{(1,2)(1,3)(1,4)(1,6)(2,4)(2,6)(3,6)\}$
(ii) $R^{-1} = \{(2,1)(3,1)(4,1)(6,1)(4,2)(6,2)(6,3)\}$
Now Relation R^{-1} can be defined as "x is divisible by y".
or $x R^{-1} y$.

Que 2 Show that there exists one to one mapping from $A \times B$ to $B \times A$. Is it onto also?

Soln let $(x, y) \in A \times B \Rightarrow (y, x) \in B \times A$
Now $f: A \times B \rightarrow B \times A$
let $x_1, y_1, x_2, y_2 \in A \times B$
& f be defined as $f(x, y) = (y, x)$
If $f(x_1, y_1) = f(x_2, y_2)$
$\Rightarrow (y_1, x_1) = (y_2, x_2)$
$\Rightarrow y_1 = y_2$ and $x_1 = x_2$

It is onto also, since every element of co-domain will be f-image of at least one element of a domain.

Que Let $X = \{a, b, c\}$. Define $f : X \to X$ such that
$f = \{(a,b)(b,a)(c,c)\}$ (UPTU)

find f^{-1} (ii) f^2 (iii) f^3 (iv) f^4

Solⁿ $f^{-1} = \{(b,a)(a,b)(c,c)\}$

(ii) $f^2 = fof = fof(a) = f[f(a)] = a$

$fof(b) = f[f(b)] = b$

$fof(c) = f[f(c)] = c$

$f^2 = fof = \{(a,a)(b,b)(c,c)\}$

(iii) $f^3 = f^2 o f \Rightarrow f^2[f(a)] = b$

$f^2[f(b)] = \cancel{b}\, a$

$f^2[f(c)] = c$

$f^3 = \{(a,b)(b,a)(c,c)\}$

(iv) $f^4 = f^3 o f \Rightarrow f^3[f(a)] = a$

$f^3[f(b)] = b$

$f^3[f(c)] = c$

$f^4 = \{(a,a)(b,b)(c,c)\}$

Que Let $X = \{1, 2, 3\}$ & f, g, h, s are funⁿ from X to X given by

$f = \{(1,2)(2,3)(3,1)\}$ $g = \{(1,2)(2,1)(3,3)\}$

$h = \{(1,1)(2,2)(3,1)\}$ $s = \{(1,1)(2,2)(3,3)\}$

find fog, gof, fohog, sog, gos, sos & fos.

Solⁿ $fog = \{(1,3)(2,2)(3,1)\}$ $gof = \{(1,1)(2,3)(3,2)\} \neq fog$

$fohog = \{(1,3)(2,2)(3,2)\}$ $sog = \{(1,2)(2,1)(3,3)\} = g$

$sos = \{(1,1)(2,2)(3,3)\} = s$, $fos = \{(1,2)(2,3)(3,1)\} = f$

Que. If R is an equivalence relation on A, then prove that R^{-1} is also equivalence relation on A.

Sol^n Let $x \in A$, Since R is reflexive relation $(x,x) \in R$

$(x,x) \in R^{-1}$

so R^{-1} is reflexive.

(ii) Let $x, y \in R$ as R is symmetric relation

$(x,y) \in R \Rightarrow (y,x) \in R$

$(y,x) \in R^{-1} \Rightarrow (x,y) \in R^{-1}$

so R^{-1} is symmetric

(iii) Let $x, y, z \in A$, as R is transitive

$(x,y) \in R$ and $(y,z) \in R \Rightarrow (x,z) \in R$

which means $(y,x) \in R^{-1}$ and $(z,y) \in R^{-1} \Rightarrow (z,x) \in R^{-1}$

or $(z,y) \in R^{-1}$ and $(y,x) \in R^{-1} \Rightarrow (z,x) \in R^{-1}$

so R^{-1} is transitive. Hence R^{-1} is equivalence relation.

Que Let $X = \{1, 2, 3 -- 7\}$ and $R = \{(x,y) : (x-y)$ is divisible by $3\}$ show that R is an equivalence relation.

Sol^n Given that $X = \{1, 2, 3, 4, 5, 6, 7\}$

$R = \{(x,y) : (x-y)$ is divisible by $3\}$

Then R is an equivalence relation if —

(1) Reflexive: $\forall x \in X \Rightarrow (x-x)$ is divisible by 3

$(x,x) \in X \quad \forall x \in X$, so R is reflexive.

(ii) Symmetric: Let $x, y \in X$ and $(x,y) \in R$

$\Rightarrow (x-y)$ is divisible by 3

$\Rightarrow (x-y) = 3n_1 \quad \{n_1$ being an integer$\}$

$\Rightarrow (y-x) = -3n_1$

so $y-x$ is divisible by 3, R is symmetric.

(iii) Transitive Let $x, y, z \in X$ and $(x,y) \in R$, $(y,z) \in R$

Then $x-y = 3n_1$; $y-z = 3n_2$

$x-z = 3(n_1+n_2)$ so $x-z$ is also divisible by 3 Hence R is transitive: so R is equivalence relation

Que Let S be the set of all points in a plance. Let R be a relation such that for any two points a & b, $(a, b) \in R$ if b is within 2 cm from a, Determine whethere R is equivalence relation or not?

Soln The relation R will be equivalence relation if –

(i) Reflexive $\forall a \in S \Rightarrow aRa$ ie every element of the plane is related to itself being within the 2cm from itself. so R is Reflexive.

(ii) Symmetric let aRb ie a & b are within 2 cm distance

$\Rightarrow$ b & a will also be within 2 cm distance

$\Rightarrow bRa$

so R is symmetric

(iii) let $aRb \Rightarrow$ a & b are within 2 cm distance ie $|a-b|<2$

and $bRc \Rightarrow$ b & c are within 2cm distance ie $|b-c|<2$

$$\Rightarrow |a-b| + |b-c| < 4$$

$$|a-b+b-c| < 4$$

$$|a-c| < 4 \Rightarrow a \not R c$$

Then R is not transitive

Hence R is not an equivalence relation.

Theorem-1 Let $f: X \to Y$ and $g: Y \to Z$ be one-one & onto mapping. Then the composite funⁿ $gof: X \to Z$ is one-one onto.

$$(gof)^{-1} = f^{-1} o g^{-1}$$

Proof: Since f & g both are one-one onto & therefore f^{-1} & g^{-1} both exists & both are one-one onto.

gof is one-one: Let $x_1, x_2 \in X$ then

$$(gof)_{x_1} = (gof)_{x_2}$$
$$g[f(x_1)] = g[f(x_2)]$$
$$g[y_1] = g[y_2]$$
$$y_1 = y_2$$
$$f(x_1) = f(x_2)$$
$$x_1 = x_2$$

It clearly shows that gof is one-one mapping.

gof is onto: Let $z \in Z$ since g is onto, hence there definitely exists an element $y \in Y$ such that $g(y) = z$. Again f is onto, there exists an element $x \in X$ such that $f(x) = y$.

$$(gof)_x = g[f(x)] = g[y] = z$$

It shows that for each $z \in Z$ there exists an element $x \in X$ such that $f(x) = y$. (gof is onto) Hence gof is one-one onto.

~~$(gof)_x = g[f(x)]$~~

To Prove $(gof)^{-1} = f^{-1} o g^{-1}$

$(gof)^{-1}$ exists

If $f: X \to Y$ be given by $f(x) = y$ where $x \in X$ & $y \in Y$

$g: Y \to Z$ be given by $g(y) = z$ where $y \in Y$ & $z \in Z$.

and $gof: X \to Z$ be given by $(gof)_x = z$

By definition of Inverse mapping

$f^{-1}(y) = x$, $g^{-1}(z) = y$.

$(gof)^{-1}(z) = x$ —(1)

$(f^{-1}og^{-1})_z = f^{-1}[g^{-1}(z)] = f^{-1}(y) = x$ —(2)

$(gof)^{-1}_z = (f^{-1}og^{-1})_z$ {from (1) & (2)}

$(gof)^{-1} = f^{-1}og^{-1}$ Proved

Theorem : Show that the composite of mapping obey associative laws.

Proof $f : X \to Y$, $g : Y \to Z$, $h : Z \to T$

Let x be any arbitrary element of X then

$$
\begin{aligned}
[ho(gof)]_x &= h[(gof)x] \\
&= h[g(f(x))] \\
&= h[g(y)] \\
&= h(z) \quad \text{—(1)}
\end{aligned}
$$

$$
\begin{aligned}
[(hog)of]_x = (hog)f(x) &= (hog)y \\
&= h[g(y)] \\
&= h(z) \quad \text{—(2)}
\end{aligned}
$$

from (1) & (2)

$[ho(gof)] = [(hog)of]$

Proved

Theorem 1

If R is relation from A to B, S is a relation from B to C and T is a relation from C to D.
Then prove that $T\circ(S\circ R) = (T\circ S)\circ R$.

Proof Let M_R, M_S and M_T denote the matrices related to Relation R, S & T respectively, then

$$M_{T\circ(S\circ R)} = M_{S\circ R}\cdot M_T$$
$$= (M_R\cdot M_S)\, M_T$$
$$= M_R\cdot(M_S\cdot M_T)$$ { Multiplication of matrix is associative.
$$= M_R\, M_{T\circ S}$$
$$= M_{(T\circ S)\circ R}$$
$$T\circ(S\circ R) = (T\circ S)\circ R$$

Theorem Let R be a relation from the set A to set B and S be a relation from set B to set C, then

$$(S\circ R)^{-1} = R^{-1}\circ S^{-1}$$

Proof Let $(c,a) \in (S\circ R)^{-1} \Rightarrow (a,c) \in S\circ R \;\forall\, a\in A$ and $c \in C$
There exist an element $b \in B$ with $(a,b) \in R$ and $(b,c) \in S$

$(a,b)\in R$ and $(b,c) \in S \Rightarrow (b,a) \in R^{-1}$ and $(c,b) \in S^{-1}$
$\Rightarrow (c,b) \in S^{-1}$ and $(b,a) \in R^{-1}$
$\Rightarrow (c,a) \in R^{-1}\circ S^{-1}$

$(c,a) \in (S\circ R)^{-1} \Rightarrow (c,a) \in R^{-1}\circ S^{-1}$

$$(S\circ R)^{-1} = R^{-1}\circ S^{-1}.$$

Recursion & Recurrence Relation :-

Let A be given set, the successor of A is the set $A \cup \{A\}$.
It is denoted by A^+.

$$A^+ = A \cup \{A\}$$

Let ϕ be the Null set, then find the successor set of ϕ, these sets are

$$\phi, \phi^+ = \phi \cup \{\phi\}, \quad \phi^{++} = \phi \cup \{\phi\} \cup \{\phi, \{\phi\}\}$$

They can be written as $\phi, \phi^+ = \{\phi\}, \phi^{++} = \{\phi \cup \{\phi\}\}$

Renaming the ϕ as 0 (zero)

$$\phi^+ = 0^+ = \{0\} = 1$$

$$\phi^{++} = 1^+ = \{\phi, \{\phi\}\} = \{0, 1\} = 2$$

We get the set $\{0, 1, 2, 3 \ldots\}$ each element in the above set is a successor set of previous element, except 0.

Now we consider Recursion in terms of successor.

Let s denote the successor, we define.

(i) $x + 0 = x$ (ii) $x + s(y) = s(x+y)$

In the definition (i) is the basis & it defines addition of s. The recursive part defines addition of the successor of y.

Illustration

$$3+2 = 3 + s(1) = s(3+1) = s(s(3+0))$$
$$= s(s+3)$$
$$= s(4) = 5$$

Lecture No-9

Ref Pt - 1.9

Principle of Mathematical Induction

let $P(n)$ be a statement involving the natural no n. To prove that $p(n)$ is true for all natural number $n \geq a$. we proceed as follows -

(i) Verify $P(n)$ for $n = a$
(ii) Assume the result for $n = k > a$
(iii) Using (i) & (ii) prove that $P(k+1)$ is true.

This is known as 1st principle of mathematical induction.
some time the above procedure will not work, then we consider alternative principle called the 2nd principle of mathematical induction, which is given as -

(i) $P(n)$ is true for $n = a$
(ii) Assume that $P(n)$ for $a \leq n \leq K$
(iii) Prove $P(n)$ for $n = K+1$

Example-1 Prove by Induction -

$$1.2.3 + 2.3.4 + \cdots \quad n(n+1)(n+2) = \frac{1}{4} n(n+1)(n+2)(n+3) \quad \forall n \in N$$

Solution for $n = 1$

LHS $1 \cdot 2 \cdot 3 = 6$

RHS $= \frac{1}{4} \cdot 1 \cdot (1+1) \cdot (1+2) \cdot (1+3) = 6$

L.H.S = R.H.S (1) is true for $n = 1$

Assume (1) is true for $n = K$

$$1 \cdot 2 \cdot 3 + 2 \cdot 3 \cdot 4 + \cdots \quad K(K+1)(K+2) = \frac{1}{4} K(K+1)(K+2)(K+3)$$

Now for $n = k+1$

$\underline{L.H.S}$ $\quad \underbrace{1\cdot2\cdot3 + 2\cdot3\cdot4 + \cdots k(k+1)(k+2)} + (k+1)(k+2)(k+3)$

$\frac{1}{4} k(k+1)(k+2)(k+3) + (k+1)(k+2)(k+3)$

$= \frac{1}{4}(k+1)(k+2)(k+3)(k+4)$

R.H.S for $n = k+1$

Hence (1) is true for all $n \in N$.

$\underline{\text{Que}}$ Prove by Induction.

$$1+2+3+4 \cdots + n = \frac{n(n+1)}{2} \quad \forall n \in N \text{ --- } (1)$$

$\underline{\text{Solution}}$ for $n = 1$

$L.H.S = 1$

$R.H.S = \frac{1(1+1)}{2} = 1$

Assume (1) is true for $n = k$

$$1+2+3+ \cdots k = \frac{k(k+1)}{2}$$

Now for $n = k+1$

$$1+2+3+ \cdots k + k+1 = \cancel{k+1}$$

$$\frac{k(k+1)}{2} + (k+1)$$

$$\frac{(k+1)(k+2)}{2}$$

R.H.S for $n = k+1$

Hence (1) is true for all values of $n \in N$.

Que 3 Show that

$$1^2+2^2+ \cdots n^2 = \frac{n(n+1)(2n+1)}{6}, \quad n \geq 1 \quad \text{---(1)}$$

(U.P.T.U)

Solution for $n=1$

LHS of (1) $\Rightarrow 1^2 = 1$

RHS of (1) $\Rightarrow \frac{n(n+1)(2n+1)}{6} = \frac{1(1+1)(2+1)}{6} = 1$

(1) is true for $n=1$.

Assume (1) is true for $n=k$

$$1^2+2^2+3^2+ \cdots k^2 = \frac{k(k+1)(2k+1)}{6}$$

Now for $n=k+1$

L.H.S of (1) $(1^2+2^2+ \cdots k^2) + (k+1)^2$

$$= \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

$$= \frac{(k+1)}{6}\{k(2k+1)+6(k+1)\}$$

$$= \frac{k+1)}{6}\{2k^2+7k+6\}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

RHS for $n=k+1$

Hence (1) is true for all values of $n \in N$.

Que 4 Show that n^3+2n is divisible by 3 for $n \geq 1$ (UPTU) 2007

Solution:: for $n=1$, we have $p(n) = n^3+2n$

$P(1) = 1^3+2(1) = 3$ Hence $P(n)$ is true for $n=1$

3 is divisible by 3.

Assume $p(n)$ is true for $n=k$.

ie $P(k) = k^3+2k$ is divisible by 3

Now for $n=(k+1)$

$$P(k+1) = (k+1)^3 + 2(k+1)$$

$$= k^3+1+3k^2+3k+2k+2$$

$$= k^3+2k+3k+3k^2+3$$

$= (K^3+2K) + 3(K^2+K+1)$

since K^3+2K is divisible by 3 also $3(K^2+K+1)$ being a multiple of 3 is divisible by 3.

$\therefore$ As each term is divisible by 3.

Hence n^3+2n is divisible by 3 is true for all $n \in N$.

Ques $7^{2n} + 2^{3n-3} \cdot 3^{n-1}$ is divisible by 25 $\forall n \in N$.

Solution Let $P(n) = 7^{2n} + 2^{3n-3} \cdot 3^{n-1}$

We have to show that 25 divides $P(n)$

ie. $P(n) = 25 \cdot q$ for some $q \in Z$.

Now $P(n) = 49^n + 8^{n-1} \cdot 3^{n-1}$

$= 49^n + 24^{n-1}$

$P(1) = 49 + 1 = 50 = 25 \times 2$, so 25 divides $P(1)$

Now assume that 25 divides $P(K)$.

ie. $49^k + 24^{k-1} = 25m$ for some $m \in Z$ — (2)

$P(K+1) = 49^{k+1} + 24^k$

$= 49 \cdot 49^k + 24^k$

$= 49(25m - 24^{k-1}) + 24^k$ (from 2)

$= 49 \cdot 25m - 49 \cdot 24^{k-1} + 24^k$

$= \cancel{25[49m} \; 49 \cdot 25m - 24^{k-1}(49-24)$

$= 25[49m - 24^{k-1}]$

$= 25q$ where $q = 49m - 24^{k-1}$

25 divides $P(K) \Rightarrow$ 25 divides $P(K+1)$ — (3)

Hence from (1) (2) & (3) we find that 25 divides $P(n)$ $\forall n \in N$.