LECTURE-12

Course: Mathematics-I **Course Code: KAS-103T**

Module-2: Differential Calculus-I **Teacher: Dr. S. P. Gupta**

Topic: Introduction to limits of function of one Variable and Numerical based on limits

LO: ***Introduce*** and ***recall*** the concept of limit of a function of one variable.

Introduction to limits :- let $f(x)$ be a function of one variable $x \in R$ be fixed.

$\lim_{x \to a}$ is called a limit operator. Here, it is applied to the function f. $\lim_{x \to a} f(x)$ is the real number that $f(x)$ approaches as x approaches a, if such a number exists. If $f(x)$ does, indeed, approach a real number, we denote that number by L (for limit values). we say the limit exists. and we write

$$\lim_{x \to a} f(x) = L \quad \text{or}$$

$$f(x) \to L \text{ as } x \to a$$

if there is no number L with this property, then we say that limit does not exists.

$\lim_{x \to a} f(x)$

exists.	does not exist
The limit is a real number, L	"DNE" $\boxed{\infty}$ $\boxed{-\infty}$

Example (Exists form) :-

① let $f(x) = 3x^2+x-1$, Evaluate $\lim_{x \to 1} f(x)$.

$$\lim_{x \to 1} 3x^2+x-1$$

$$\Rightarrow 3\times(1)^2+(1)-1$$

$$\Rightarrow 3$$

Ans $\rightarrow$ $\lim_{x \to 1} f(x) = 3$ or $f(x) \to 3$ as $x \to 1$

Example (2):- let $f(x) = \frac{2x+1}{x-2}$, Evaluate $\lim_{x \to 3} f(x)$.

$$\lim_{x \to 3} \frac{2x+1}{x-2}$$

$$\Rightarrow \frac{2(3)+1}{(3)-2} \Rightarrow \underline{7}$$

Example (3) find $\lim_{x \to -\pi} 2 = 2$

a constant approaches itself, we can write $2 \to 2$ ('2' approaches '2') as $x \to -\pi$

"DNE" Based problems :-

Indeterminate forms :- There are certain rules to evaluate the limits, But some limits can not be evaluated by using these rules. These limits are known as indeterminate forms. There are seven type of indeterminate forms.

(1) $\frac{0}{0}$, (2) $\frac{\infty}{\infty}$, (3) $0 \times \infty$, (4) $\infty - \infty$, (5) 1^{∞}, (6) 0^{0}

(7) ∞^{0}

These limits can be evaluated by using 'L' Hospital's Rule.

L' Hospital's Rule :- If $f(x)$ and $g(x)$ are two functions of x which can be expanded by Taylor's Series in the nbd of $x = a$ and if $\lim_{x \to a} f(x) = 0 = f(a)$, $\lim_{x \to a} g(x) = g(a) = 0$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

[differentiate Numinator and Denominator seperataly]

Standard Limit forms to be Remember :-

① $\lim_{x\to 0} \frac{\sin x}{x} = 1$, ② $\lim_{x\to 0} \frac{\tan x}{x} = 1$

③ $\lim_{x\to 0} \frac{a^x - 1}{x} = \log a$, ④ $\lim_{x\to 0} \frac{e^x - 1}{x} = 1$

⑤ $\lim_{x\to 0} (1+x)^{1/x} = e$, ⑥ $\lim_{x\to \infty} \left(1+\frac{1}{x}\right)^x = e$

⑦ $\lim_{x\to 0} \frac{\sin^{-1} x}{x} = 1$, ⑧ $\lim_{x\to 0} \frac{\sinh x}{x} = 1$

⑨ $\lim_{x\to 0} \frac{\tan^{-1} x}{x} = 1$

Problem Type ① $\frac{0}{0}$ form :-

Example ① - Evaluate $\lim_{x\to 1} \frac{x - x^x}{1 + \log x - x}$ [form $\frac{0}{0}$]

$$\Rightarrow \lim_{x\to 1} \frac{x - e^{\log x^x}}{1 + \log x - x}$$

$$\Rightarrow \lim_{x\to 1} \frac{x - e^{x\log x}}{1 + \log x - x}$$

$$\Rightarrow \lim_{x\to 1} \frac{1 - e^{x\log x}(1+\log x)}{\frac{1}{x} - 1}$$ [Applying L'Hospital's Rule]

[Again form $\frac{0}{0}$]

$$= \lim_{x\to 1} \frac{-e^{x\log x}(1+\log x)^2 - e^{x\log x}\left(\frac{1}{x}\right)}{-\frac{1}{x^2}}$$ [Again Applying L'Hospital's Rule]

$\Rightarrow$ $\underline{\underline{2}}$ Ans

Example 2 :- Find a, b, c if $\lim_{x \to 0} \frac{ae^{x} - b\cos x + ce^{-x}}{x \sin x} = 2$

Sol^n:

$$2 = \lim_{x \to 0} \frac{ae^{x} - b\cos x + ce^{-x}}{x \sin x}$$

$$= \lim_{x \to 0} \frac{ae^{x} - b\cos x + ce^{-x}}{x \cdot x \left(\frac{\sin x}{x}\right)} \qquad \left[\lim_{x \to 0} \frac{\sin x}{x} = 1\right]$$

$$2 = \lim_{x \to 0} \frac{ae^{x} - b\cos x + ce^{-x}}{x^2}$$

$$2 = \frac{a - b + c}{0} \Rightarrow a - b + c = 0 \quad \text{—— } (1)$$

Now taking

$$2 = \lim_{x \to 0} \frac{ae^{x} - b\cos x + ce^{-x}}{x^2} \qquad \left[\text{form } \frac{0}{0},\ a-b+c=0\right]$$

[Applying L'Hospital's Rule]

$$2 = \lim_{x \to 0} \frac{ae^{x} + b\sin x - ce^{-x}}{2x}$$

$$2 = \frac{a - c}{0} \Rightarrow a - c = 0 \Rightarrow \boxed{a = c} \quad \text{—— } (2)$$

again taking

$$2 = \lim_{x \to 0} \frac{ae^{x} + b\sin x - ce^{-x}}{2x} \qquad \left[\text{Again form } \frac{0}{0},\ a-c=0\right]$$

$$2 = \lim_{x \to 0} \frac{ae^{x} + b\cos x + ce^{-x}}{2} \qquad \left[\text{Again Applying L'Hospital's Rule}\right]$$

$$2 = \frac{a + b + c}{2} \Rightarrow 2a + b = 4 \quad \text{—— } (3)$$

from (1) and (2) $\quad 2a - b = 0 \quad$ —— (4)

On solving eq^n (3) & (4)

$$\boxed{a = 1,\ b = 2,\ c = 1}$$

Evalute the value of the following limits.

① $\lim_{x \to a} \frac{x^2 \log a - a^2 \log x}{x^2 - a^2} = \log a - \frac{1}{2}$

② prove that $\lim_{x \to 0} \frac{x e^x - \log(1+x)}{x^2} = \frac{3}{2}$

③ prove that $\lim_{x \to \frac{\pi}{4}} \frac{1 - \tan x}{1 - \sqrt{2} \sin x} = 2$

④ Find the value of a, b and c so that $\lim_{x \to 0} \frac{x(a + b\cos x) - c\sin x}{x^3} = 1.$

(Ans - $a = 0, b = -3, c = -3$)

⑤ Find the values of a, b and c such that

$\lim_{x \to 0} \frac{ae^x - be^{-x} - cx}{x - \sin x} = 4$ [Ans - $a = 2, b = 2, c = 4$]

Problem Type ② - $\frac{\infty}{\infty}$ form :-

Example - prove that $\lim_{x \to \infty} \frac{x^n}{e^{kx}} = 0$

$\lim_{x \to \infty} \frac{x^n}{e^{kx}}$ [form $\frac{\infty}{\infty}$]

(Applying L'Hospital Rule)

$\Rightarrow \lim_{x \to \infty} \frac{n x^{n-1}}{k e^{kx}}$ (Again $\frac{\infty}{\infty}$)

(again apply L'Hospital Rule)

$\Rightarrow \lim_{x \to \infty} \frac{n(n-1) x^{n-2}}{k^2 e^{kx}}$ (again form $\frac{\infty}{\infty}$)

Applying L'Hospital Rule (n-2) time, we have

$\Rightarrow \lim_{x \to \infty} \frac{n(n-1)(n-2) \cdots 3 \cdot 2 \cdot 1}{k^n e^{kx}} = \lim_{x \to \infty} \frac{n!}{k^n e^{kx}}$

$= 0$ proved

Evaluate the value of the following limits.

1. prove that $\lim_{x \to 0} \frac{\log x}{\cot x} = 0$

2. Prove that $\lim_{x \to 1} \frac{\log(1-x)}{\cot \pi x} = 0$

3. prove that $\lim_{x \to \infty} \frac{\log(1+e^{3x})}{x} = 3$

Problem Type 3 - $0 \times \infty$ form :- $\left[\lim_{x \to a} (f(x) \cdot g(x)) = \lim_{x \to a} f(x) = 0, \ \lim_{x \to a} g(x) = \infty \right.$ Change it to $\frac{0}{0}$ or $\frac{\infty}{\infty}$, then Apply L'Hospital Rule)

Example - prove that $\lim_{x \to 1} \tan^2\left(\frac{\pi x}{2}\right)(1+\sec \pi x) = -2$

$$\lim_{x \to 1} \tan^2\left(\frac{\pi x}{2}\right)(1+\sec \pi x)$$

$\left[\lim_{x \to 1} \tan^2\left(\frac{\pi x}{2}\right) = \infty, \ \lim_{x \to 1} (1+\sec \pi x) = 0 \right]$

[form $\infty \times 0$]

$$\Rightarrow \lim_{x \to 1} \frac{(1+\sec \pi x)}{\cot^2\left(\frac{\pi x}{2}\right)}$$

[change to form $\frac{0}{0}$ or $\frac{\infty}{\infty}$]

Applying L'Hospital Rule,

$$= \lim_{x \to 1} \frac{\pi \sec \pi x \tan \pi x}{2 \cot\left(\frac{\pi x}{2}\right)\left(-\csc^2 \frac{\pi x}{2}\right)\left(\frac{\pi}{2}\right)}$$

$$\Rightarrow -\left(\lim_{x \to 1} \frac{\sec \pi x}{\csc^2 \frac{\pi x}{2}}\right)\left(\lim_{x \to 1} \frac{\tan \pi x}{\cot \frac{\pi x}{2}}\right)$$

$$\Rightarrow -\left(\frac{\sec \pi}{\csc^2 \pi/2}\right)\left(\lim_{x \to 1} \frac{\tan \pi x}{\cot \frac{\pi x}{2}}\right)$$

[form $\frac{0}{0}$]

Again applying L'Hospital Rule

$$\Rightarrow -(-1) \lim_{x \to 1} \frac{\pi \sec^2 \pi x}{-\csc^2 \frac{\pi x}{2}(\pi/2)}$$

$$\Rightarrow -2 \frac{\sec^2 \pi}{\csc^2 \frac{\pi}{2}} = \underline{\underline{-2}}.$$

Evaluate the following limits.

① Prove that $\lim_{x\to 0} x\log x = 0$

② prove that $\lim_{x\to\infty} x^2 e^{-x} = 0$

③ prove that $\lim_{x\to 1} (1+\sec\pi x)\tan\frac{\pi x}{2} = 0.$

Problem Type ④ – form $(\infty-\infty)$ –

Type $\lim_{x\to a}[f(x)-g(x)]$, when $\lim_{x\to a} f(x)=\infty$ and $\lim_{x\to a} g(x)=\infty$

i.e. $(\infty-\infty)$ form. Now the expression is reduced in the form of $\frac{0}{0}$ or $\frac{\infty}{\infty}$ by taking LCM or by rearranging the term and then apply L'Hospital Rule.

Example :- Prove that $\lim_{x\to 0}\left[\frac{\pi}{4x}-\frac{\pi}{2x(e^{\pi x}+1)}\right]=\frac{\pi^2}{8}$

$$\Rightarrow \frac{\pi}{2}\lim_{x\to 0}\left[\frac{1}{2x}-\frac{1}{x(e^{\pi x}+1)}\right] \quad [\text{form}(\infty-\infty)]$$

$$= \frac{\pi}{2}\lim_{x\to 0}\left[\frac{e^{\pi x}+1-2}{2x(e^{\pi x}+1)}\right] \quad \left[\text{form}\left(\frac{0}{0}\right)\right]$$

Applying L'Hospital Rule.

$$= \frac{\pi}{2}\lim_{x\to 0}\left[\frac{\pi e^{\pi x}}{2[e^{\pi x}+1+x\pi e^{\pi x}]}\right]$$

$$= \frac{\pi}{2}\cdot\frac{\pi}{2}\cdot\frac{1}{2} = \frac{\pi^2}{8} \text{ Ans}$$

Evaluate the following limits.

① prove that $\lim_{x\to 0}\left(\cot x-\frac{1}{x}\right)=0$

② Prove that $\lim_{x\to\frac{\pi}{2}}(\sec x-\tan x)=0$

③ Prove that $\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{e^x-1}\right)=\frac{1}{2}$

Problem Type ⑤ ⑥ ⑦ :- form $(1^\infty, \infty^0, 0^0)$

To evaluate the limits of the type $\lim_{x \to a} [f(x)]^{g(x)}$ which takes any one of the above form. for $f(x) > 0$., following method is applied-

let $l = \lim_{x \to a} [f(x)]^{g(x)}$, $f(x) > 0$

$\log l = \lim_{x \to a} [g(x) . \log(f(x))]$, which takes the form $[\infty \times 0]$ and then convert it to $\frac{0}{0}$ form and then apply L'Hospital Rule to Solve.

Example- prove that $\lim_{x \to 0} \left(\frac{1^x + 2^x + 3^x + 4^x}{4} \right)^{1/x} = (24)^{1/4}$

let $l = \lim_{x \to 0} \left(\frac{1^x + 2^x + 3^x + 4^x}{4} \right)^{1/x}$ $[\text{form } 1^\infty]$

taking log on both side.

$$\log l = \lim_{x \to 0} \frac{1}{x} \log \left(\frac{1^x + 2^x + 3^x + 4^x}{4} \right) \quad [\text{form } \infty \times 0]$$

$$\log l = \lim_{x \to 0} \frac{\log \left(\frac{1^x + 2^x + 3^x + 4^x}{4} \right)}{x} \quad \left[\text{form } \frac{0}{0}\right]$$

Applying L'Hospital Rule'

$$= \lim_{x \to 0} \frac{4}{(1^x + 2^x + 3^x + 4^x)} \cdot \left(\frac{1^x \log 1 + 2^x \log 2 + 3^x \log 3 + 4^x \log 4}{4} \right)$$

$$= \frac{1}{4} (\log 2 + \log 3 + \log 4) = \frac{1}{4} \log 24$$

$$\log l = \log (24)^{1/4}$$

Hence $l = (24)^{1/4}$ Ans

Evaluate the following limits.

(1) prove that $\lim_{x\to\infty}\left(\dfrac{1^{\frac{1}{x}}+2^{1/x}+3^{1/x}+4^{1/x}}{4}\right) = 24$

(2) prove that $\lim_{x\to\frac{\pi}{2}} (\sin x)^{\tan^2 x} = \dfrac{1}{\sqrt{e}}$

(3) prove that $\lim_{x\to\infty} \left(1+\dfrac{a}{x}\right)^{x} = e^{a}$

(4) prove that $\lim_{x\to 0} (1+\tan x)^{\cot x} = e.$

Module-02

Differential Calculus I

Limit of a function :- Let f be a real valued function defined on an interval $I \subset \mathbb{R}$ except possibly at $c \in I$.
The function $f(x)$ is said to tend to a limit $l \in \mathbb{R}$ as x tends to c if for each $\epsilon > 0$, ~~that~~ there exists a $\delta = \delta(\epsilon, c) > 0$ such that

$$|f(x) - l| < \epsilon, \text{ wherever } 0 < |x - c| < \delta$$

or $$l - \epsilon < f(x) < l + \epsilon \quad \forall\, x \in (c - \delta, c + \delta).$$

Symbolically, we write

$$\lim_{x \to c} f(x) = l.$$

Left Hand limit :- The function $f(x)$ is said to tend to a limit $l \in \mathbb{R}$ as x tends to c from the left if for each $\epsilon > 0$, there exists a $\delta(\epsilon, c) > 0$, such that

$$|f(x) - l| < \epsilon, \text{ whenever } c - \delta < x < c$$

Symbolically, we write.

$$\lim_{x \to c-0} f(x) = l \text{ or } f(c-0) = l \text{ or } f(c-) = l$$

or $$f(c-0) = \lim_{\substack{h \to 0 \\ h > 0}} f(c-h)$$

Right hand limit :- A function $f(x)$ is said to tend to a limit $l \in \mathbb{R}$ as x tends to c from the right if for each $\epsilon > 0$, there exists a $\delta(\epsilon, c) > 0$, such that

$$|f(x) - l| < \epsilon, \text{ whenever } c < x < c + \delta$$

Symbolically, we write,

$$\lim_{x \to c+0} f(x) = l \text{ or } f(c+0) = l \text{ or } f(c+) = l$$

or $$f(c+0) = \lim_{\substack{h \to 0 \\ h > 0}} f(c+h)$$

Existence of limit :- $\lim_{x \to c-0} f(x) = l = \lim_{x \to c+0} f(x)$

$$\boxed{L \cdot HL = R \cdot HL = l}$$

Working Rule to find R.H.L and L.H.L :-

① RHL :- To find limit on the right, we put $a+h$ for x in $f(x)$ and then take the limits as $h \to 0$. Thus.

$$\lim_{x \to c+0} f(x) = \lim_{h \to 0} f(a+h)$$

② LHL :- To find limit on the left, we put $a-h$ for x in $f(x)$ and then take the limits as $h \to 0$, thus.

$$\lim_{x \to c-0} f(x) = \lim_{h \to 0} f(a-h)$$

Properties of limits :-

① If $\lim_{x \to c} f(x)$ exists then it must be unique.

② If $\lim_{x \to a} f(x) = L$ and $\lim_{x \to a} g(x) = M$, then $\lim_{x \to a} [f(x) + g(x)] = L+M$.

③ If $\lim_{x \to a} f(x) = L$, $\lim_{x \to a} g(x) = M$, then $\lim_{x \to a} f(x) g(x) = LM$

④ If $\lim_{x \to a} f(x) = L$, $\lim_{x \to a} g(x) = M$, then $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{L}{M}$, $M \neq 0$

⑤ If $\lim_{x \to a} f(x) = L$, then $\lim_{x \to a} |f(x)| = |L|$

⑥ If $\lim_{x \to a} f(x) = L$, then for any real number λ, $\lim_{x \to a} \lambda f(x) = \lambda L$.

Q. If $|x-2|<1$, prove that $|x^2-4|<5$

$|x^2-4|<5$, $|x-2|<1$

$|(x+2)(x-2)|<5$, $|x-2|<1$

Now $|(x+2)(x-2)| = |x+2||x-2|$

$|x-2|<1$

$-1<x-2<1$

$1<x<3$

$1+2<x+2<3+2$

$3<x+2<5$

$(x+2) \in (3,5)$

$\epsilon>0$, $\delta=\min(\delta, c) = \min(1, 5)$

$|x-2|<1.$

$\underline{|x^2-4|<5}$

Q. if $|x-3|<\frac{1}{10}$, prove that $|x^2-x-6|<51$

$|x^2-x-6|<51$, $|x-3|<\frac{1}{10}$.

$|x^2-3x+2x-6|<51$

$|x(x-3)+2(x-3)|<51$

$|(x+2)(x-3)|<51$

Now $|(x+2)(x-3)| = |(x+2)||(x-3)|$

$|x-3|<\frac{1}{10}$ $\Rightarrow$

$-\frac{1}{10}<x-3<\frac{1}{10}$

$-\frac{1}{10}+3<x<\frac{1}{10}+3$

$\frac{29}{10}<x<\frac{31}{10}$

$\frac{29}{10}+2<x+2<\frac{31}{10}+2$

$\frac{49}{10}<x+2<\frac{51}{10}$.

$(x+2)\ (\frac{49}{10}, \frac{51}{10})$

$\epsilon>0$;

$\delta=\min(\delta, c)$

$\min(\frac{1}{10}, \frac{51}{10})$

$\Rightarrow |x-3|<\frac{1}{10}$.

$|x^2-x-6|<\frac{51}{10}\times 10$

$\underline{\underline{<51}}$

Example - By using $\epsilon-\delta$ method, prove that

$$\lim_{x \to 3} (x^2+2x) = 15$$

Sol^n: Here $f(x) = x^2+2x$, $L = 15$, $a = 3$.

given $\epsilon > 0$, we must find $\delta > 0$ such that

$$|(x^2+2x)-15| < \epsilon, \quad (0 < |x-3| < \delta) \quad \text{—— (1)}$$

Now $|x^2+5x-3x-15| < \epsilon, \quad (0 < |x-3| < \delta)$

$|x(x+5)-3(x+5)| < \epsilon, \quad (0 < |x-3| < \delta)$

$|(x-3)(x+5)| < \epsilon, \quad (0 < |x-3| < \delta)$

$\Rightarrow |x-3|\,|x+5| < \epsilon, \quad (0 < |x-3| < \delta)$

Now $|(x-3)(x+5)| = |x-3|\,|x+5| \quad \text{—— (1)}$

$|x-3| < 1 = \delta.$

$-1 < x-3 < 1$

$2 < x < 4$

$2+5 < x+5 < 4+5$

$7 < x+5 < 9 \quad \text{—— (11)}$

$(x+5) \in (7, 9)$

$\epsilon > 0, \ \delta = (\delta, c) = \left(1, \frac{\epsilon}{9}\right)$

$|x-3| < \delta$

$|x-3|\,|x+5| < 9 \cdot \frac{\epsilon}{9} = \epsilon.$

Example- By using ε-δ method, prove that $\lim_{x \to 0} x \sin\frac{1}{x} = 0$

Here $f(x) = x \sin\frac{1}{x}$

$l = 0$ and $a = 0$

Here we are to show that $\lim_{x \to 0} f(x) = \lim_{x \to 0} x \sin\frac{1}{x} = 0$

for this we are to show that for any given $\varepsilon > 0$ there exists a number $\delta > 0$ such that

$$|f(x) - 0| < \varepsilon, \quad (0 < |x - 0| < \delta)$$

i.e. $$\left|x \sin\frac{1}{x} - 0\right| < \varepsilon, \quad (0 < |x - 0| < \delta)$$

Now $$\left|x \sin\frac{1}{x} - 0\right| = |x| \left|\sin\frac{1}{x}\right|$$
$$\leq |x| \quad , \text{ since } \left|\sin\frac{1}{x}\right| \leq 1$$
$$\leq |x - 0| \qquad (1)$$

let $\varepsilon > 0$, choose $\delta = \varepsilon$, then for $|x - 0| < \delta$, from (1)

$$\left|x \sin\frac{1}{x} - 0\right| < \varepsilon$$

Hence $\lim_{x \to 0} x \sin\frac{1}{x} = 0$.