Lecture 2 APL 105 4-1-24

Tensors:

tensor of zero order $\longrightarrow$ scalar 3^0 components

of first order $\longrightarrow$ vector 3^1 components

of ~~o~~ second order $\longrightarrow$ 2nd order tensor 3^2 components.

eg. $u_i \rightarrow (u_1, u_2, u_3)$

$v_j \rightarrow (v_1, v_2, v_3)$

$$u_i v_j = c_{ij} = \begin{pmatrix} u_1 v_1 & u_1 v_2 & u_1 v_3 \\ u_2 v_1 & u_2 v_2 & u_2 v_3 \\ u_3 v_1 & u_3 v_2 & u_3 v_3 \end{pmatrix}$$

$$\neq v_i u_j$$

$$\underline{\underline{B}} = b_{ij} \, \underline{e}_i \, \underline{e}_j$$

* 2nd order tensor b_{ij} transforms according to the rule:

$$\boxed{\bar{b}_{kl} = a_{ik}\, a_{jl}\, b_{ij}}$$

eg. $\bar{b}_{12} = a_{i1}\, a_{j2}\, b_{ij}$

$$= a_{11}\, a_{j2}\, b_{1j} + a_{21}\, a_{j2}\, b_{2j} + a_{31}\, a_{j2}\, b_{3j}$$

$\downarrow$

$$a_{11}\, a_{12}\, b_{11}$$
$$+$$
$$a_{11}\, a_{22}\, b_{12}$$
$$+$$
$$a_{11}\, a_{32}\, b_{13}$$

Ten

$$a_{jjm} = a_{11m} + a_{22m} + a_{33m} = b_m$$

Tensor Product

Tensor product of $\underline{A}$ and $\underline{\underline{B}}$ = denoted by $\underline{A} \otimes \underline{\underline{B}}$

$$\underline{A} = a_i \underline{e_i} \qquad \underline{\underline{B}} = b_{jk} \underline{e_j}\, \underline{e_k}$$

$$\underline{A} \otimes \underline{\underline{B}} = a_i b_{jk}\, \underline{e_i}\, \underline{e_j}\, \underline{e_k}$$

$$\neq \underline{\underline{B}} \otimes \underline{A} = b_{ij} a_k\, \underline{e_i}\, \underline{e_j}\, \underline{e_k}$$

Contraction (inner product)

$d_{ijk} \rightarrow 3^{rd}$ order tensor

contract 2 indices say $(i \,\&\, j) \longrightarrow d_{iik}$ or d_{jjk}

$$d_{iik} \downarrow$$

$$d_{11k} + d_{22k} + d_{33k}$$

After contraction a tensor of order N becomes a tensor of order N-2

Inner Product $\underline{\underline{A}} \otimes \underline{\underline{B}}$ (5th order tensor)

$a_{ijk}\, b_{lm}$

contact the last index of 1st tensor & 1st index of 2nd tensor

$a_{ijr}\, b_{rm}$ ← contracted tensor

inner product

~~con~~ inner product $u_i v_j = u_i v_i$ ← dot product

Gradient of a tensor $b_{ij} \rightarrow \nabla(b_{ij})$

gradient of $b_{ij} = \frac{\partial (b_{ij})}{\partial x_k} e_k e_i e_j$

∂x_k

gradient of tensor of order n is a tensor of order $(N+1)$.

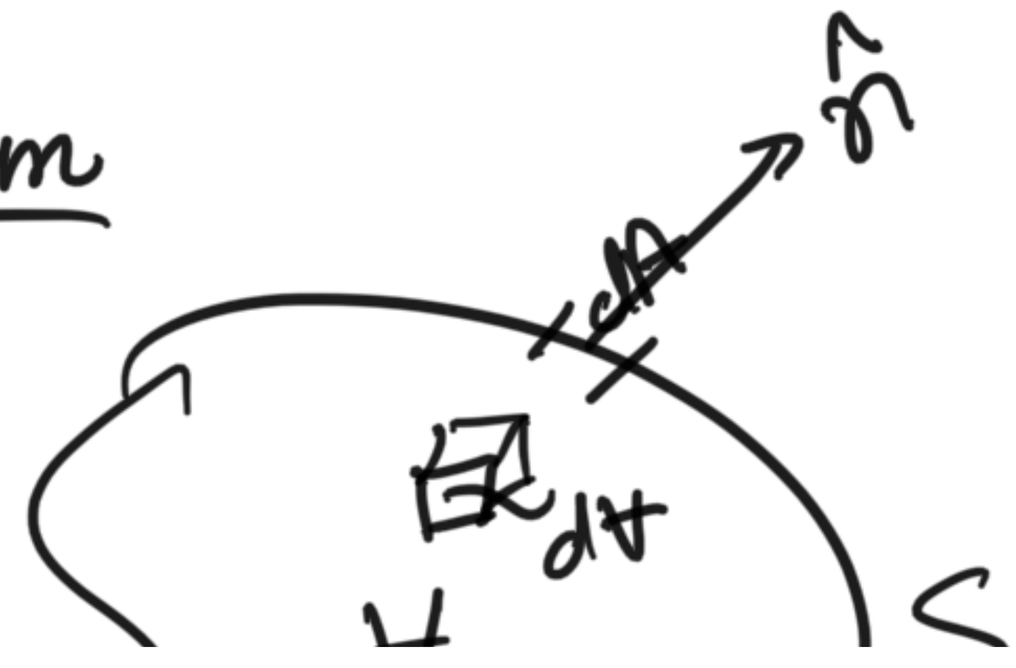
Divergence $\quad \nabla \cdot (\quad)$ $\quad$ inner product of grad and tensor

$\nabla \cdot \underline{\underline{b}} \longrightarrow 1^{st}$ $\quad \dfrac{\partial b_{ij}}{\partial x_k}$ gradient

contract k and i

$\dfrac{\partial b_{ij}}{\partial x_i} \longleftarrow$ divergence of b_{ij}

Divergence Theorem



$$\iiint_{\forall} \{ \quad \} \, d\forall$$

Take $\{\}$

$\downarrow$

add over all volumes

so entire volume is covered

$\downarrow$

$$\iiint_{V} \{ \quad \} \, d\forall$$

Volume enclosed by surface S.

$$\oiint_{S} (\quad) \, dA$$

Divergence Theorem:

$$\iiint_{\forall} \nabla \cdot \{ \vec{V} \} \, d\forall = \oiint_{S} \vec{V} \cdot \hat{n} \, dA$$

$\vec{V}$ is any vector

Generalize to tensors.

$$\oiint_S \{\quad\} \hat{n}\, dA = \iiint_{\forall} \nabla \{\quad\}\, d\forall$$

$\{\quad\}$ is any order tensor

$\varepsilon_{ijk} \rightarrow$ alternating tensor

$\varepsilon_{ijk} = 1$ if i, j, k cyclic 

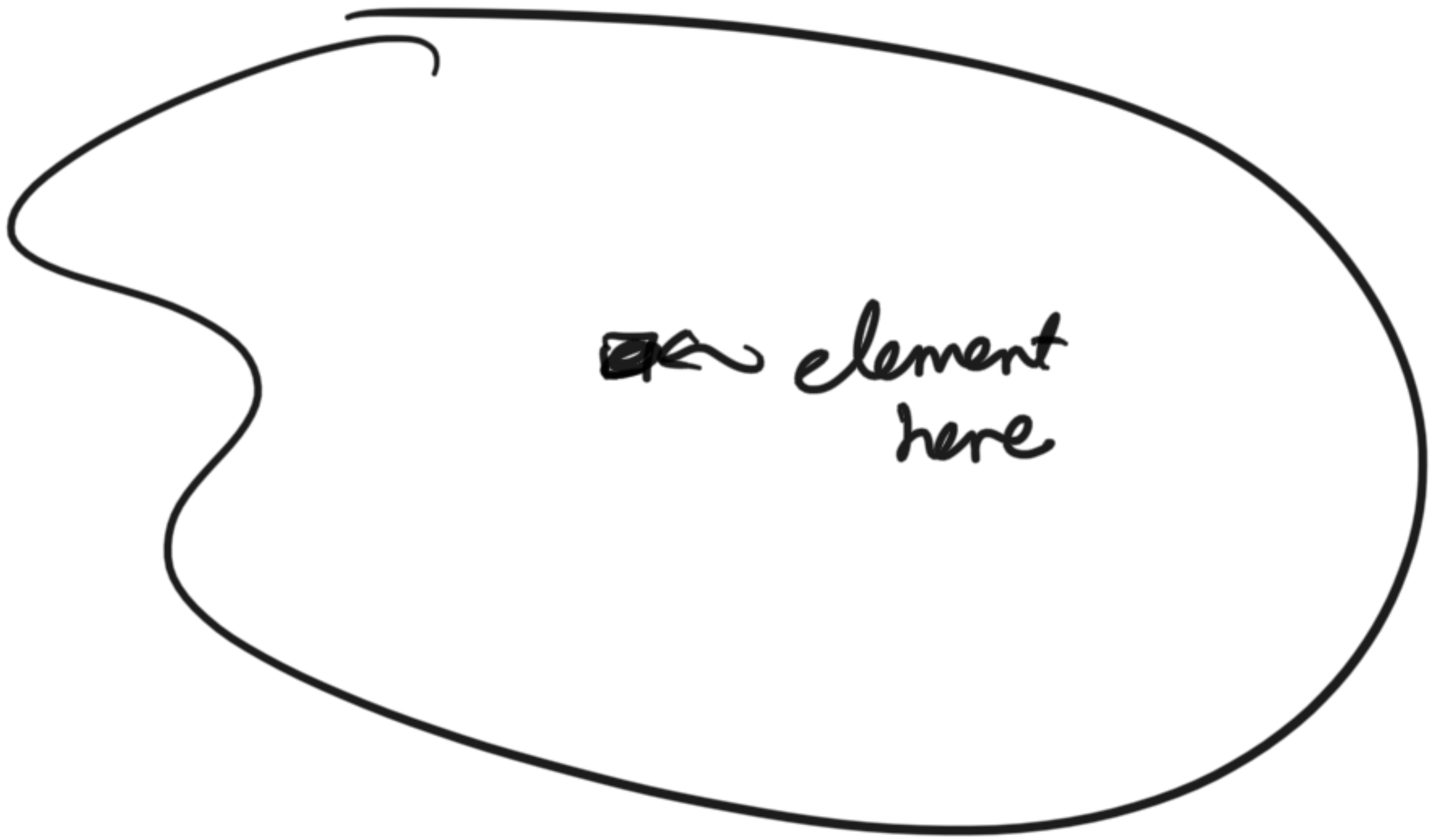
$$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1$$

$\varepsilon_{132} = \varepsilon_{321} = \varepsilon_{213} = -1 \leftarrow$ anticyclic

$= 0$ otherwise

$$\varepsilon_{ijk} u_j u_k = \vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_x & u_y & u_z \\ \cdots \end{vmatrix}$$

$\begin{vmatrix} v_x & v_y & v_z \end{vmatrix}$



Governing equation of motion for this small element

~~D~~ Ex)

two types of forces

i) body forces

(per unit volume)

or per unit mass)

2) surface force (per unit area)

surface force express through traction or stress vector

surface force per unit area, function of normal to area

f_i ← surface force per unit area

$f_i(\hat{n})$

$f_{1i}(-\hat{n})$

Doing a force balance on volume element

$$\left(f_i(\hat{n}) + \cancel{f_{1i}}\, f_{1i}(-n) \right) dA + \rho \vec{g}\, d\forall \quad \text{surface}$$

$$= \rho \vec{a}\, d\forall$$

In lt $\delta\forall \to 0$, $\left.\begin{array}{l} \rho \vec{g}\, d\forall \\ \rho \vec{a}\, d\forall \end{array}\right\} \to 0$

$$f_i(\hat{n}) = -f_{1i}(-\hat{n})$$

$$f_i(\hat{n}) = -f_i(-\hat{n})$$