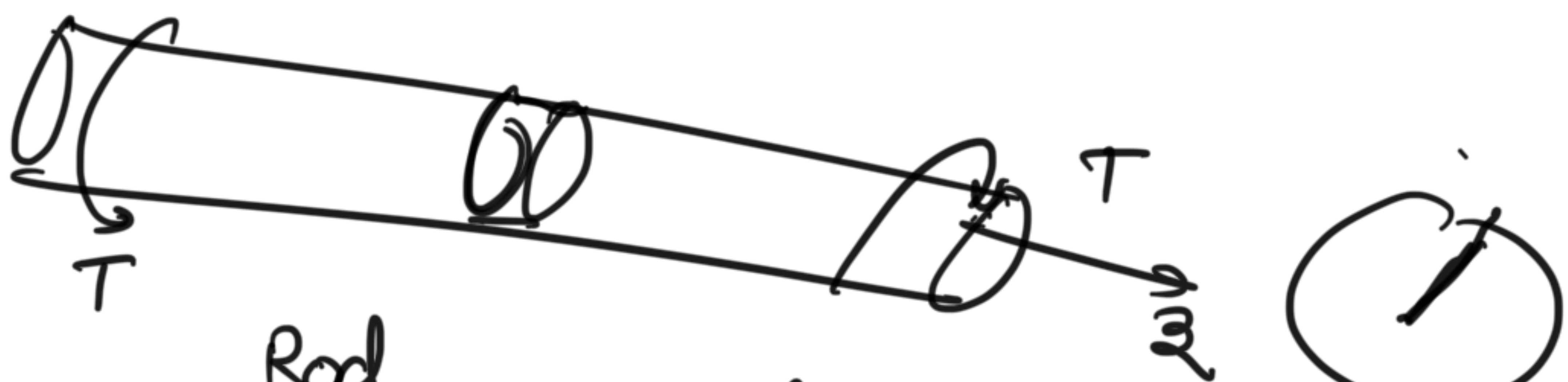
APL 105 Lecture 8

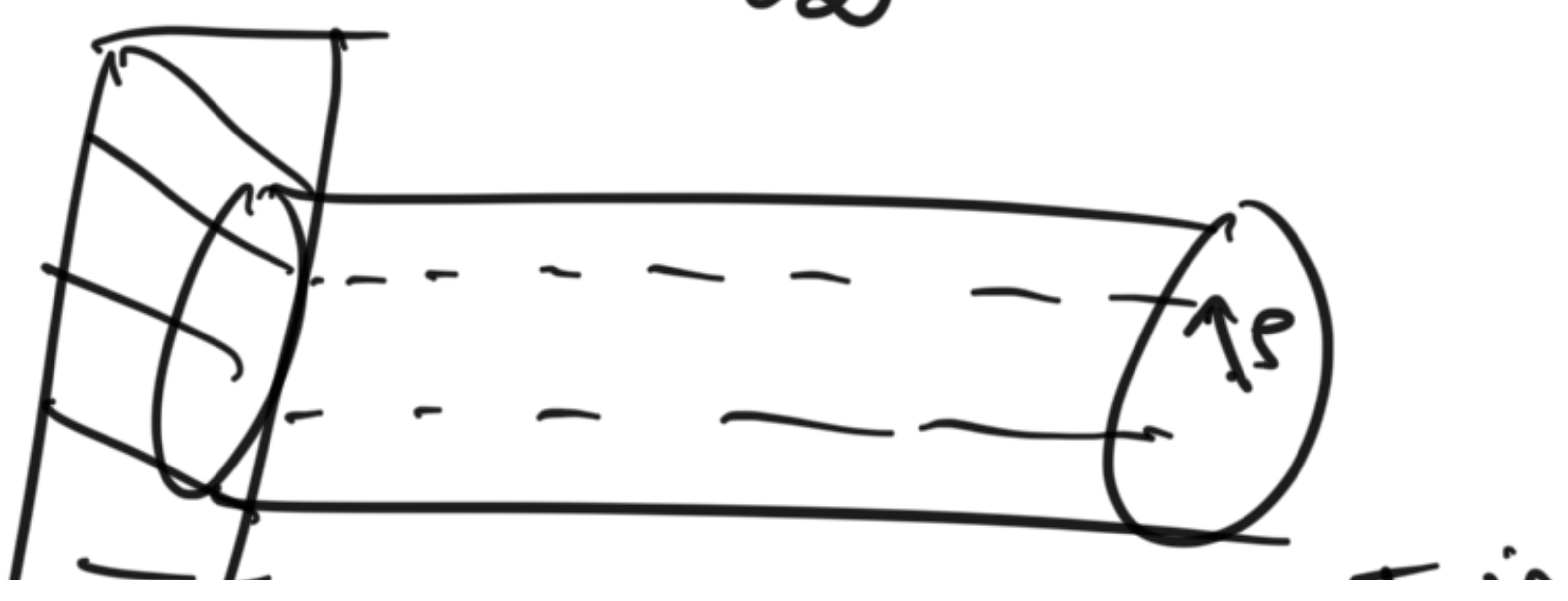
Torsion of bars of circular cross-section



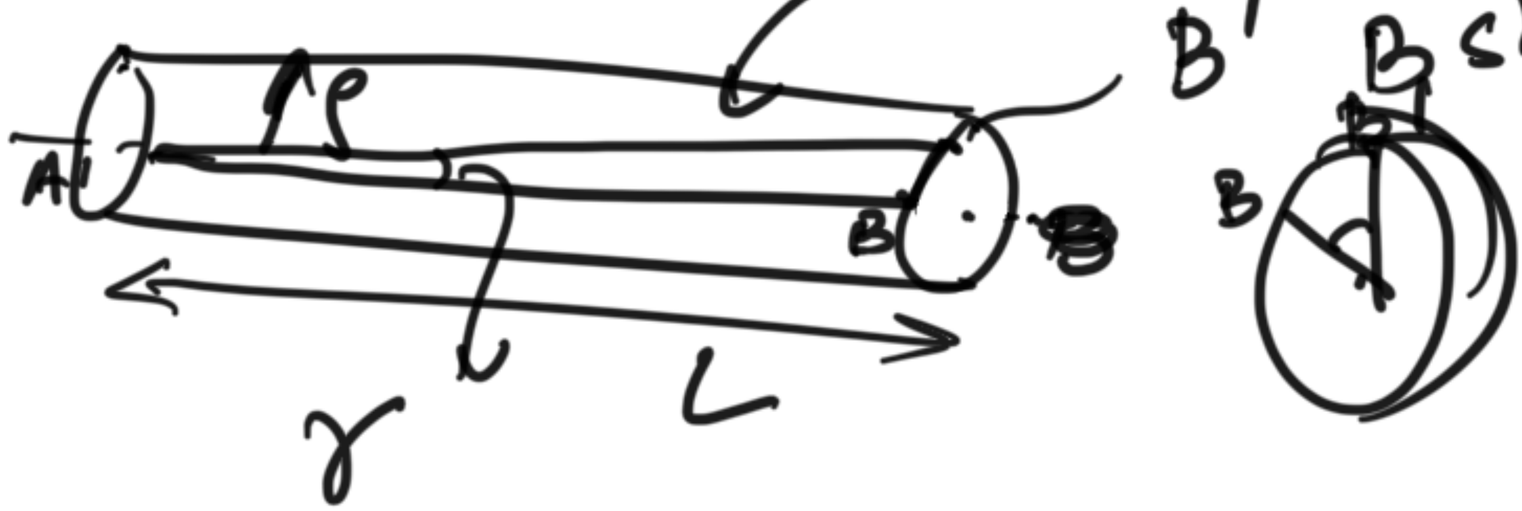
Rod ~~Bar~~ of circular cross-section

Due to axisymmetry, circular section stays circular

$$\tau_{\theta z} = \tau_{z\theta}$$



radius of rod = c
Consider an inner section of radius ρ



AB is straight line without T being applied

when torque T is applied on B, B shifts to B'
(assume end A is fixed)

$\gamma \rightarrow$ shear

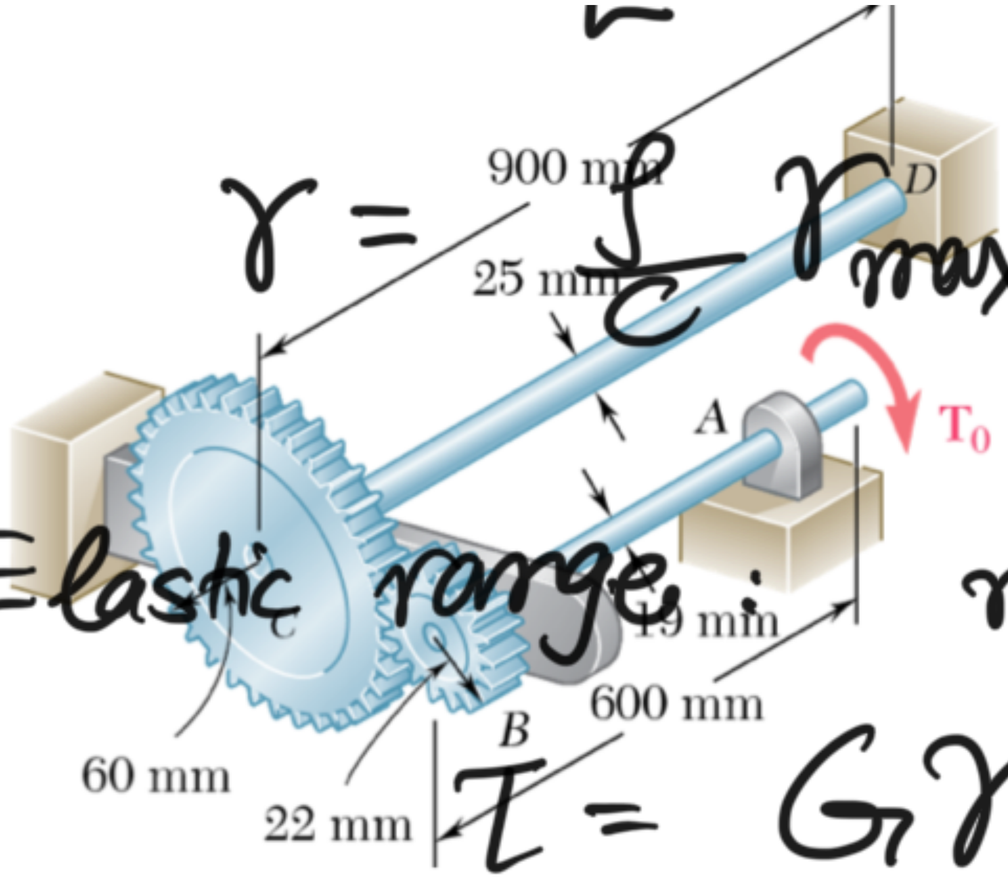
$\phi \rightarrow$ twist

$L\gamma = \rho\phi$ ($\gamma, \phi \rightarrow$ radians)

Sample Problem 3.4

$\gamma = \frac{\rho\phi}{L}$ $\gamma_{max} = \frac{c\phi}{L}$

$$\gamma = \frac{\rho}{c}\gamma_{max}$$



Elastic range : relation between γ and τ

$$\tau = G\gamma$$

shear stress, shear strain

Two solid steel shafts are connected by gears. Knowing that for each shaft G = 77 GPa and that the allowable shearing stress is 55 MPa, determine (*a*) the largest torque T_0 that may be applied to the end of shaft *AB*, (*b*) the corresponding angle through which end *A* of shaft *AB* rotates.

SOLUTION:

- Apply a static equilibrium analysis on the two shafts to find a relationship between T_{CD} and T_0.

- Apply a kinematic analysis to relate the angular rotations of the gears.

- Find the maximum allowable torque on each shaft – choose the smallest.

- Find the corresponding angle of twist for each shaft and the net angular rotation of end *A*.

← 2 Hooke's law (generalized)

Torque T is applied, shear stress τ is produced

if the rod has to stay ~~staf~~ safe,

$$\tau < \text{yield strength } \tau_Y$$

$$G\gamma = \frac{\rho}{c} G\gamma_{max} \quad \text{——— (1)} \times G$$

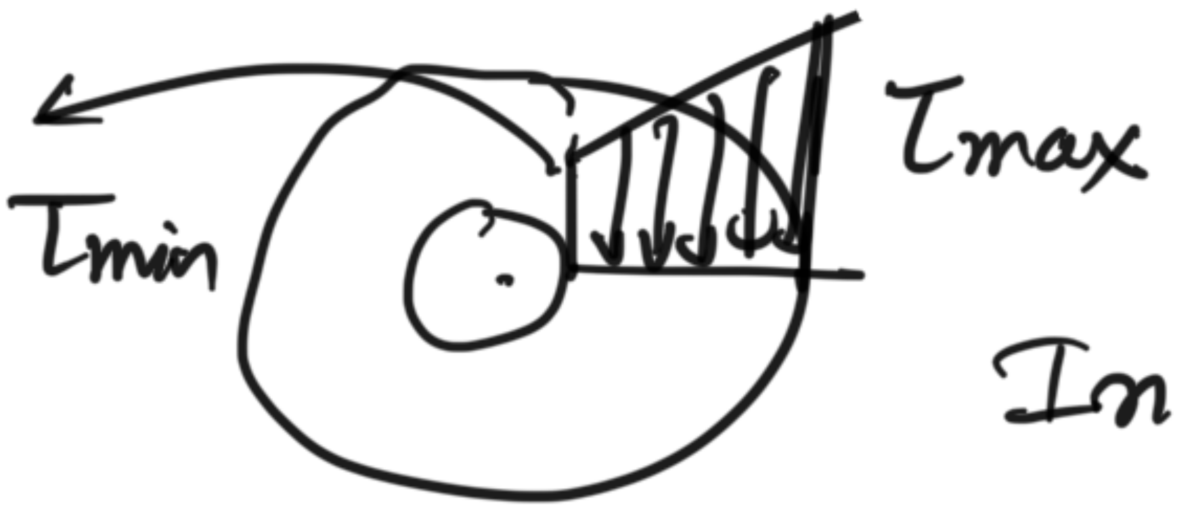
$$\tau = \frac{\rho}{c}\tau_{max}$$

shear stress varies linearly with distance ρ from centre of axis.

same argument holds for hollow circular shafts also.

Hollow tube radius will vary from c_1 to c_2.

inner radius (c_1), outer radius (c_2)



In a hollow shaft

$$\tau_{min} = \frac{c_1}{c_2}\tau_{max}$$

$$T = \int \rho \tau \, dA = \int_0^c \rho\left(\frac{\rho}{c}\tau_{max}\right) 2\pi\rho \, d\rho$$

$$= \frac{\tau_{max}}{c} 2\pi \int_0^c \rho^3 d\rho$$

$= \frac{\tau_{max}}{c} \quad \frac{c}{44}$

$\int \rho^2 dA$ = Polar moment of inertia

$= J$

$J = \frac{\pi c^4}{2}$ (for circular cross section)

$J = \frac{\pi}{2}(c_2^4 - c_1^4)$ (for hollow cross section)

$$\boxed{\begin{aligned} \tau_{max} &= \frac{Tc}{J} \\ \tau &= \frac{T\rho}{J} \end{aligned}}$$

units of $J \rightarrow m^4$

Hollow shaft 1.5 m long $d_i = 40mm$

Hollow shaft 1.5 m long $d_i = 40\,mm$

$d_o = 60\,mm$

Largest Torque which can be applied if
1) max. shear stress = 120 MPa
2) corresponding minimum value of shear stress

$$T = \frac{J\,\tau_{max}}{c}$$

$$J = \frac{\pi}{2}\left(c_2^4 - c_1^4\right) = 1.021\times10^{-6}\,m^4 \qquad c_2 = \frac{d_2}{2},\; c_1 = \frac{d_1}{2}$$

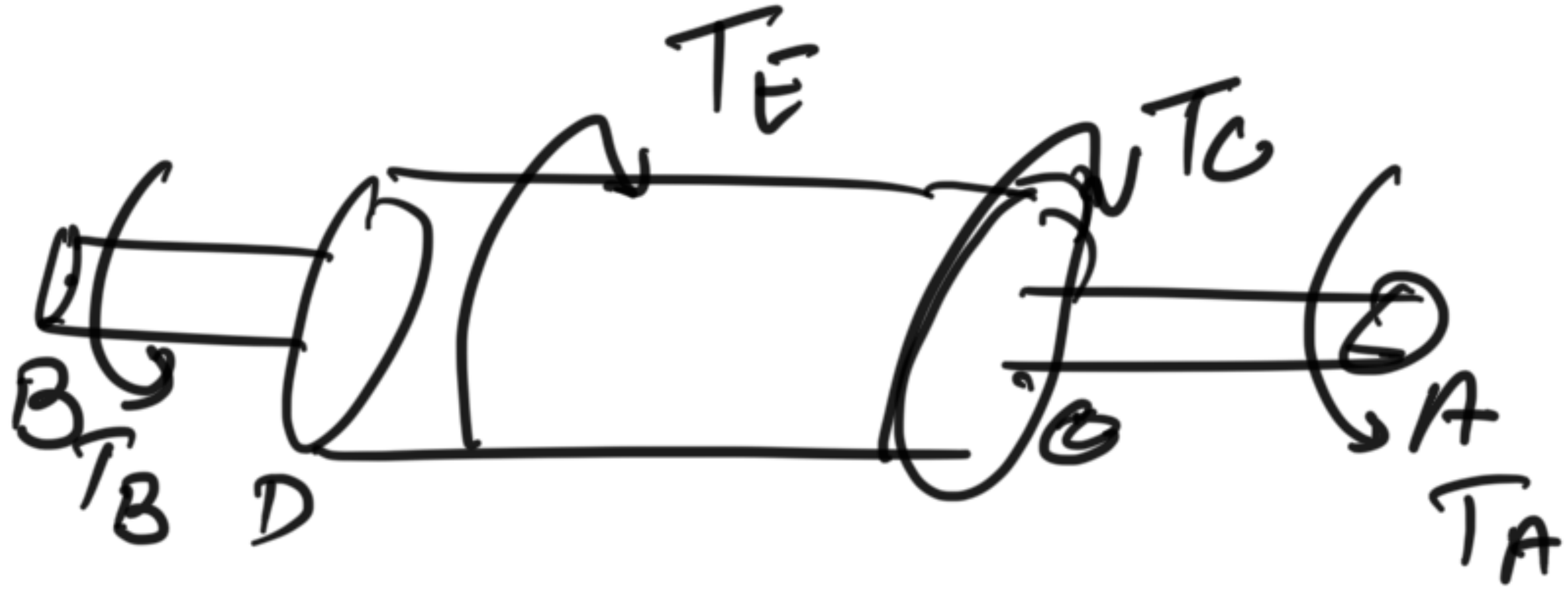
$$T = \frac{(1.021\times10^{-6})\times 120\times10^{6}}{(0.03)} = 4.08\times10^{3}\,N.m$$

$$\tau_{min} = \frac{c_1}{c_2}\,\tau_{max} = 80\ MPa$$

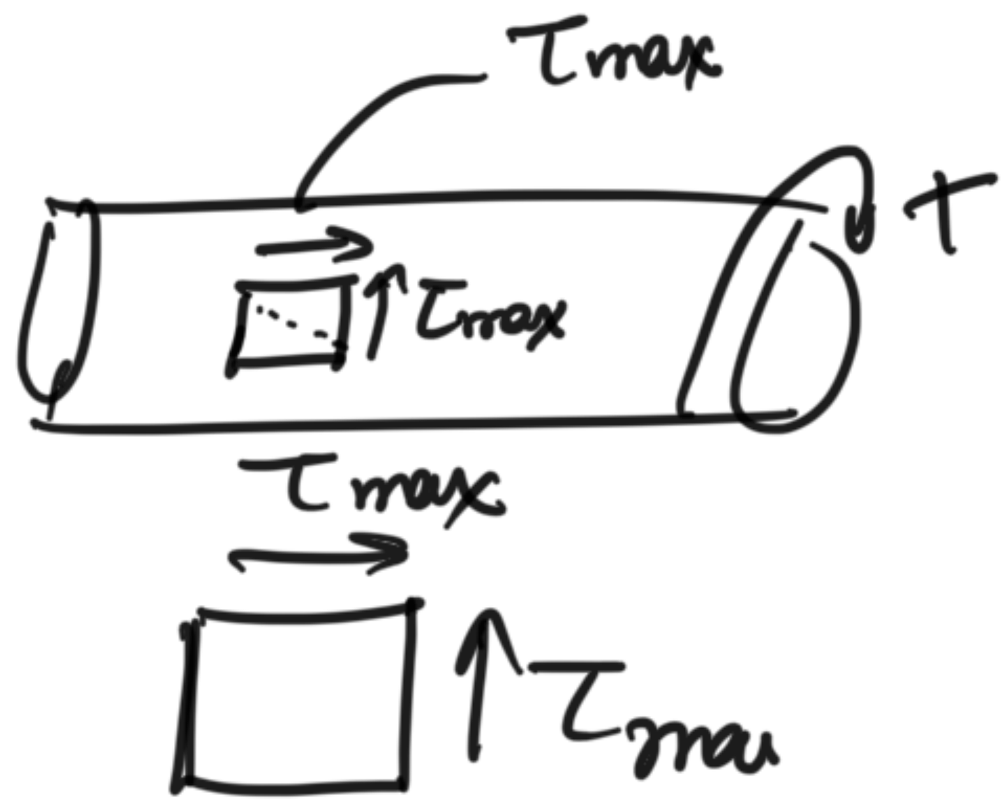
$$\tau_{max} = \frac{Tc}{J}$$ valid for any cross-section

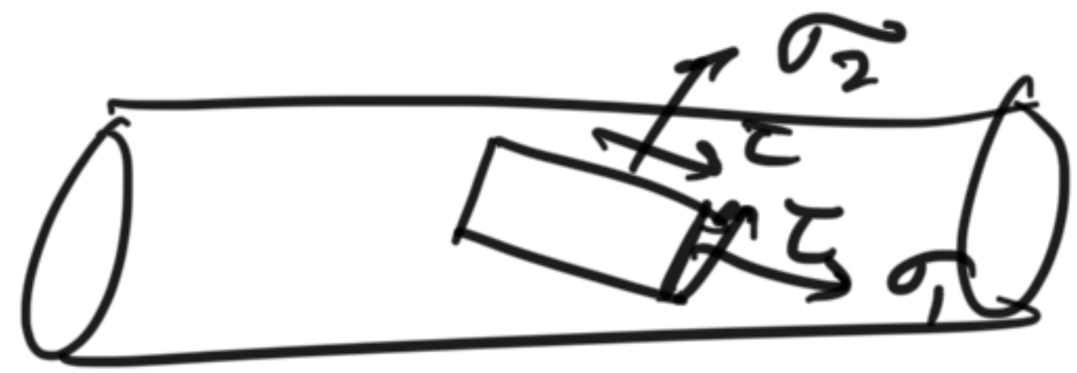
where T = internal torque

J = polar moment of inertia of that section



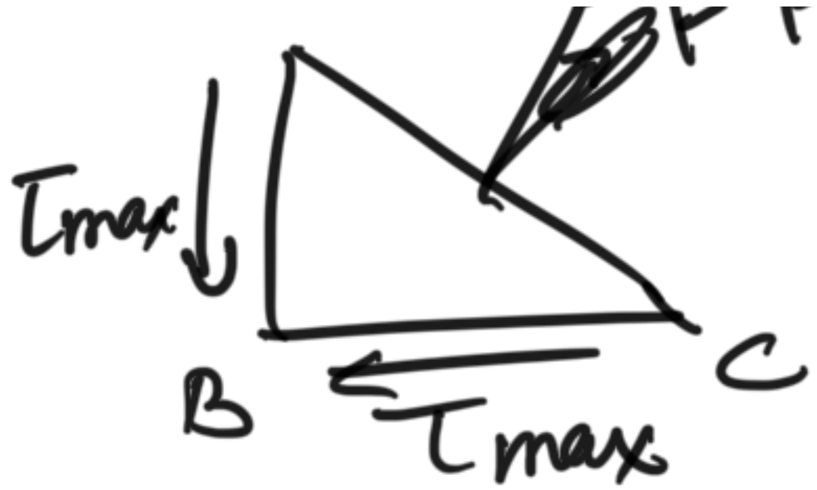
Find Torque at each of the section (AC, CE, ED, BD)





Consider a section at 45°

D



~~Area of CD =~~

Let Area of BC = area of BD = A_0

Area of CD = $\sqrt{2}\, A_0$

A_0 Let

Force Balance

$$-(\tau_{max} A_0)\,\hat{i} - (\tau_{max} A_0)\,\hat{j} + \vec{F} = 0$$

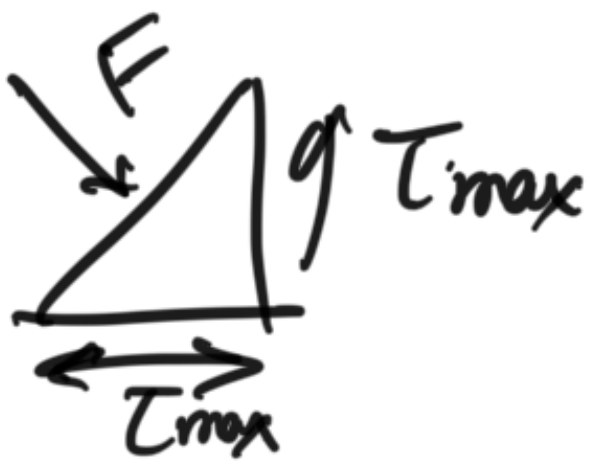
$$\vec{F} = (\tau_{max} A_0)\,(\hat{i} + \hat{j})$$

$$= \sqrt{2}\,(\tau_{max} A_0)\,\frac{(\hat{i} + \hat{j})}{\sqrt{2}}$$

$$= \sqrt{2}\,\tau_{max} A_0\,\hat{n}$$

$$\sigma(\text{on } CD) = \frac{F}{A_{CD}} = \frac{\sqrt{2}\,\tau_{max} A_0}{\sqrt{2}\, A_0}$$

$$= \tau_{max}$$



$$|\sigma| = \tau_{max} \quad \text{(at at angle of } 45^\circ)$$

and shear stress $= 0$

Angle of twist $\longrightarrow \phi$ (in elastic range)

$$\gamma_{max} = \frac{c\phi}{L}$$

$$\gamma_{max} = \frac{\tau_{max}}{G} = \frac{Tc}{JG}$$

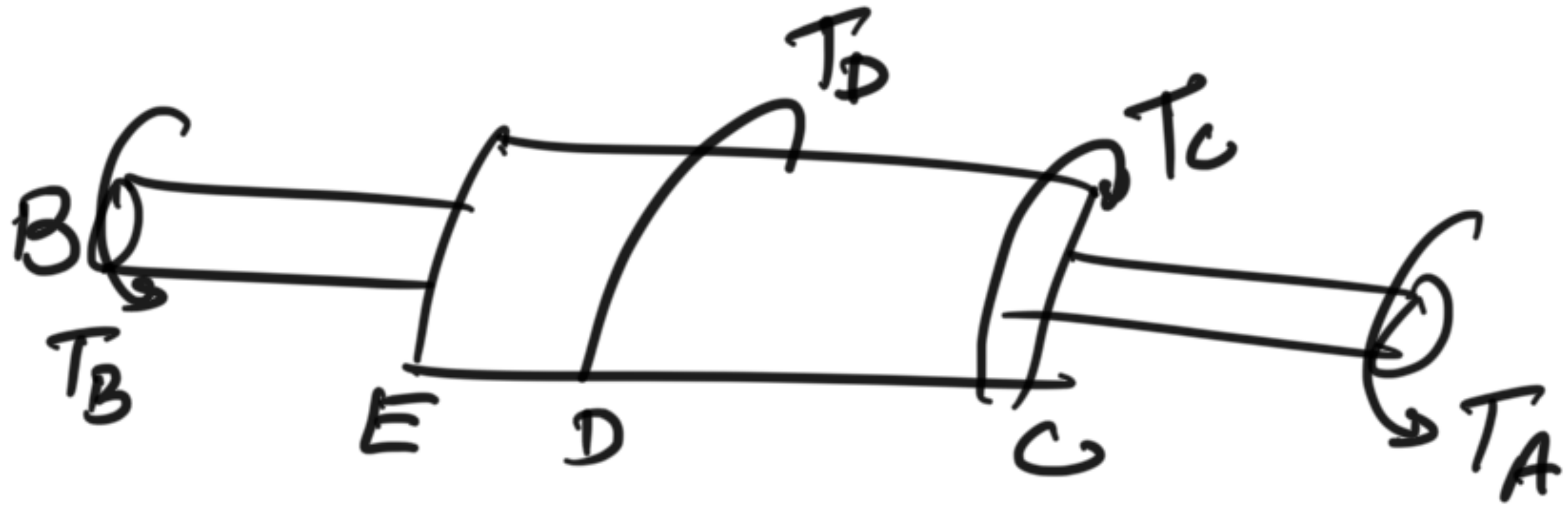
$$\frac{c\phi}{L} = \frac{Tc}{JG} \Rightarrow \boxed{\phi = \frac{TL}{JG}}$$

ϕ in radians.

$$\phi \propto T$$

$\phi \propto L$

One end of shaft was fixed.



If <u>both ends</u> free

Consider AC, CD, DE, EB separately

Total angle of twist which A rotates w.r.t. B is obtained by algebraically adding all angles of twist.

Component $i \rightarrow T_i, L_i, J_i \; G_i$

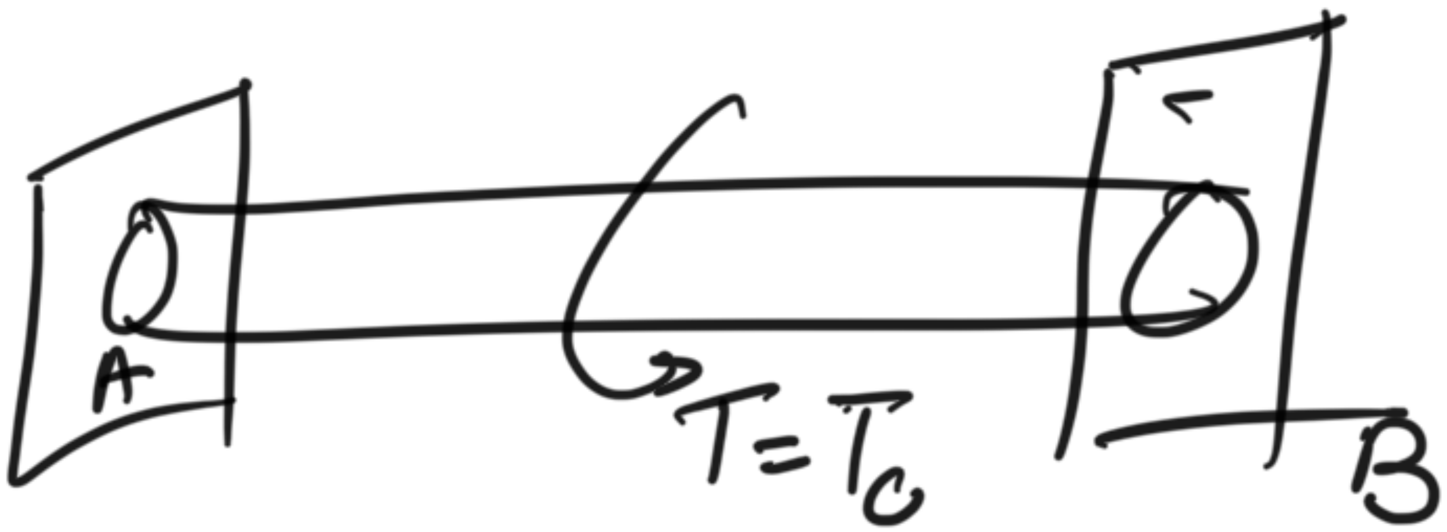
$$\phi = \sum \frac{T_i L_i}{\ldots}$$

$$\phi_{A/B} = \sum_i \frac{\cdots}{J_i G_i}$$

continuously variable cross-section

$$d\phi = \frac{T\,dx}{JG}$$

statically indeterminate shafts



At both ends, $\phi = 0$

Let $\phi_1 = \angle$ of Twist in AC

$\phi_2 = \angle$ of Twist in BC

$$\phi_1 + \phi_2 = 0$$

Transmission shafts

Power to be transmitted through shaft $= P$

rate of rotation $= \omega$

$P = T\omega$ $\quad\omega$ (rad/s)

$\omega = 2\pi f$ $\quad f \rightarrow$ frequency in Hz (s^{-1})

$$T = \frac{P}{2\pi f}$$

$\frac{J}{\ } = \frac{T}{\ }$ $\rightarrow$ Gives minimum

c τ_{max} $\text{allowable } \frac{J}{c}$