Tutorial 1 APL 105 8-1-2024

$\varepsilon_{ijk} \rightarrow$ alternating 'tensor' 3rd order tensor transformation

$$\boxed{b_{ijk} = a_{li}\, a_{mj}\, a_{kn}\, b_{lmn}}$$

$$\begin{tikzcd}
& i \; 1 \arrow[dr] & \\
k \; 3 \arrow[ur] & & j \; 2 \arrow[ll]
\end{tikzcd}$$

If any of ijk repeated $\varepsilon_{ijk} = 0$

$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1$ (cyclic)

$\varepsilon_{132} = \varepsilon_{321} = \varepsilon_{213} = -1$ (anticyclic)

$\cancel{\varepsilon_{ijk} u_j v_k}$ $\quad \underbrace{\varepsilon_{ijk}\, u_j v_k} = w_i$

$\Rightarrow \underline{\vec{w} =} \vec{u} \times \vec{v}$

$$\begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \hat{i}(u_2 v_3 - v_2 u_3) + \hat{j}(v_1 u_3 - u_1 v_3) + \hat{k}(u_1 v_2 - v_1 u_2)$$

$$\varepsilon_{ijk} u_j v_k$$

$$= \varepsilon_{i1k} u_1 v_k + \varepsilon_{i2k} u_2 v_k + \varepsilon_{i3k} u_3 v_k$$

w_i

$$= \varepsilon_{i11} u_1 v_1 + \varepsilon_{i12} u_1 v_2 + \varepsilon_{i13} u_1 v_3$$

$$+ \varepsilon_{i21} u_2 v_1 + \varepsilon_{i22} u_2 v_2 + \varepsilon_{i23} u_2 v_3$$

$$+ \varepsilon_{i31} u_3 v_1 + \varepsilon_{i32} u_3 v_2 + \varepsilon_{i33} u_3 v_3$$

$$w_1 = (u_2 v_3 - u_3 v_2)$$

$$\varepsilon_{ijk}\, \varepsilon_{ikq} = \delta_{ij}\,\delta_{kq} - \delta_{iq}\,\delta_{kj}$$

$$\boxed{\varepsilon_{ijk}\,\varepsilon_{ipq} = \delta_{jp}\,\delta_{kq} - \delta_{jq}\,\delta_{kp}}$$

$$\vec{A} \times (\vec{B} \times \vec{C}) = \vec{D}$$

$$\vec{B} = b_i \qquad \vec{C} = c_i \qquad \vec{A} = a_i$$

$$\underbrace{\vec{B} \times \vec{C}} = \underbrace{\varepsilon_{ijk}\, b_j\, c_k}$$

$$d_p = \vec{A} \times (\vec{B} \times \vec{C}) = \varepsilon_{pqi}\, a_q\, \varepsilon_{ijk}\, b_j\, c_k$$

$$= \varepsilon_{pqi}\,\varepsilon_{ijk}\, a_q\, b_j\, c_k$$

$x_j \delta_{ij} = x_i$

$$= \varepsilon_{ipq}\,\varepsilon_{ijk}\, a_q\, b_j\, c_k$$

$$= (\delta_{pj}\,\delta_{qk} - \delta_{pk}\,\delta_{qj})\, a_q\, b_j\, c_k$$

$$d_p = a_k\, b_p\, c_k - a_q\, b_q\, c_p$$

$$= \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$$

$\cancel{\delta_{pk}\, a_q\, b_j\, c_k}$

$\delta_{pj}\, a_q b_j c_k$

$= (a_q b_p c_k)\, \delta_{qk}$

Q2. $\vec{\omega} = \nabla \times \vec{u} = \varepsilon_{ijk} \dfrac{\partial u_k}{\partial x_j}$

Show $\nabla \cdot \vec{\omega} = 0 \longrightarrow \nabla \cdot \vec{\omega} =$

$\varepsilon_{ijk} S_{jk}$ where S_{jk} is symmetric

$S_{jk} = S_{kj}$

$\varepsilon_{ijk} S_{jk} = \varepsilon_{ijk} S_{kj} = -\varepsilon_{ikj} S_{kj} = -\varepsilon_{ijk} S_{jk}$

interchange j & k

see below

0

symmetry of S_{jk}

cyclic property of ε_{ijk}

$$
\begin{aligned}
-\varepsilon_{ikj} S_{kj} &= -\varepsilon_{ipj} S_{pj} \\
&= -\varepsilon_{ipq} S_{pq} \\
&= -\varepsilon_{ijq} S_{jq} \\
&= -\varepsilon_{ijk} S_{jk}
\end{aligned}
$$

$$
\begin{aligned}
\varepsilon_{ijk}\varepsilon_{jkl} &= \cancel{\varepsilon_{ijk}\varepsilon}\ \varepsilon_{jki}\varepsilon_{jkl} \\
&= \delta_{kk}\delta_{il} - \delta_{kl}\delta_{ik} \\
&= 3\delta_{il} - \delta_{il} = 2\delta_{il}
\end{aligned}
$$

Q1 & Q2 to be submitted as HW1

Show that

$\nabla \cdot \vec{\omega} = 0$ $\qquad$ $\vec{\omega} = \nabla \times \vec{V}$

$$\omega_i = \varepsilon_{ijk} \frac{\partial v_k}{\partial x_j}$$

$$\nabla \cdot \omega_i = \frac{\partial}{\partial x_i} \left(\varepsilon_{ijk} \frac{\partial v_k}{\partial x_j} \right) = \varepsilon_{ijk} \frac{\partial^2 v_k}{\partial x_i \partial x_j}$$

Since $\dfrac{\partial^2 v_k}{\partial x_i \partial x_j}$ is symmetric in i & j

$$\varepsilon_{ijk} \frac{\partial^2 v_k}{\partial x_i \partial x_j} = 0$$