The Math Used To Develop The N^{th} Root Algorithm

Let us put G to represent a Guess and Ng to represent Next-Guess. $\forall N \in \mathbb{R}^+$, $\forall n \in \mathbb{N}^+$; we need to find the n^{th} root of N to a certain level of accuracy. This needs to be done by computing successive guesses as in the *Newton-Raphson* Algorithm for square roots, and in reference to the method outlined in class discussion. The first guess of $(\sqrt[n]{N})$ is likely off by δ and is written as follows: $G = \sqrt[n]{N} + \delta$.
Finding the next guess requires us to find an approximation of δ and subtract it from G. To achieve this goal, we first need to do the long division of $\left(\frac{N}{G^{n-1}}\right)$ in order to get the term $N^{\frac{1}{n}}$.

1 Find an Approximation for δ

We start the process with the following equality: $\frac{N}{G^{n-1}} = \frac{N}{(N^{\frac{1}{n}}+\delta)^{n-1}} (*)$.
Expand the denominator of the right hand side in $(*)$ (Referring to the Binomial Theorem in Mathematics; and since we are looking for an approximation, first few terms and last term will suffice for this purpose):

$$(N^{\frac{1}{n}} + \delta)^{n-1} = \left[N^{\frac{n-1}{n}} + (n-1)N^{\frac{n-2}{n}}\delta + (n-1)N^{\frac{1}{n}}\delta^{n-2} + ... + \delta^{n-1}\right]$$

1.1 Do the polynomial long devision of N over G^{n-1}

$$(1)\left(N^{\frac{n-1}{n}} + (n-1)N^{\frac{n-2}{n}}\delta + (n-1)N^{\frac{1}{n}}\delta^{n-2} + ... + \delta^{n-1}\right)\Big[\, N^{\frac{n}{n}}$$

(2) Divide the 1st term of right hand side by the 1st term of left hand side in (1), the intermediate result is: $N^{\frac{1}{n}}$
(3) Multiply (2) by the whole term in the left hand side in (1): $\left[N + (n-1)N^{\frac{n-1}{n}}\delta + (n-1)N^{\frac{1}{n}}\delta^{n-2} + ... + N^{\frac{1}{n}}\delta^{n-1}\right]$

- ⥁ Subtract (3) from the right hand side in (1) and we get:

$$(4)\begin{cases} N-(3) &= N - \left[N + (n-1)N^{\frac{n-1}{n}}\delta + (n-1)N^{\frac{1}{n}}\delta^{n-2} + ... + N^{\frac{1}{n}}\delta^{n-1}\right] \\ &= \left[-(n-1)N^{\frac{n-1}{n}}\delta - (n-1)N^{\frac{1}{n}}\delta^{n-2} - ... - N^{\frac{1}{n}}\delta^{n-1}\right] \end{cases}$$

- ⥁ Bring down next term in right hand side of (1) & add it to (4):

$$(5)\left\{\left(-(n-1)N^{\frac{n-1}{n}}\delta \underbrace{-(n-1)N^{\frac{1}{n}}\delta^{n-2} - ... - N^{\frac{1}{n}}\delta^{n-1}}_{\textit{Ignore remaining terms in lhs of (4)}}\right) + \overbrace{\mathbf{0}}^{\textit{No terms left in rhs of (1)}}\right.$$

(6) Divide 1st term of (5) by 1st term of (1), and the result is: $[-(n-1)\delta]$
(7) Add (2) and (6) together to get the approximation of the long division as follows:($\cong$): $\left[N^{\frac{1}{n}} - (n-1)\delta\right]$

- ⥁ Ignore the remaining terms as shown in (5) and therefore we get:

$$(8)\begin{cases} (G - \frac{N}{G^{n-1}}) &= \left[N^{\frac{1}{n}} + \delta\right] - \left[N^{\frac{1}{n}} - (n-1)\delta\right] \\ &= n\delta \end{cases}$$

Simplify (8) and we get the approximation: $\boxed{\boldsymbol{\delta \cong \frac{1}{n}\left(G - \frac{N}{G^{n-1}}\right)}}$

2 Find Next Guess' Formula:

$$(9)\begin{cases} Ng &= (G-\delta) \\ &= (G - \frac{1}{n}G + \frac{1}{n}\frac{N}{G^{n-1}}) \\ &= \left(\frac{(n-1)}{n}G + \frac{1}{n}\frac{N}{G^{n-1}}\right) \end{cases}$$

Simplify (9) and we get Ng's Formula: $\boxed{\boldsymbol{Ng = \frac{1}{n}\left((n-1)G + \frac{N}{G^{n-1}}\right)}}$

3 The N^{th} Root Algorithm

Note: The initial guess G can be any positive number. Here it is chosen to be 1.0 when $(N > 1.0)$ and $\left(\dfrac{N}{2.0}\right)$ when $(0 < N < 1.0)$.

Algorithm 1 N^{th} Root Algorithm

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procedure $nth\_root(N, n, limit)$
    //Initializations
  $\boldsymbol{\mathcal{E}} \leftarrow$ *limit;*
  if $\boldsymbol{((N > 0)\&(n > 0))}$ **then**
        //Initialization of First Guess
        if $\boldsymbol{(N > 1.0)}$ **then** $\boldsymbol{G \leftarrow 1.0;}$ **Else** $\boldsymbol{G \leftarrow \frac{N}{2.0};}$
        //Initialization of Next Guess
        $\boldsymbol{Ng \leftarrow \frac{1}{n}\left((n-1)G + \frac{N}{G^{n-1}}\right);}$
        //(First guess) + (Next guess) = 2
        $\boldsymbol{guess\_counter \leftarrow 2;}$
    Loop:
    DO
      $\boldsymbol{G \leftarrow Ng;}$
      $\boldsymbol{Ng \leftarrow \frac{1}{n}((n-1)G + \frac{N}{G^{n-1}});}$
      $\boldsymbol{guess\_counter \leftarrow guess\_counter + 1;}$
      if $\boldsymbol{((G - Ng) \leq \mathcal{E})}$ **then**
          print_results $\boldsymbol{(Ng, guess\_counter);}$
          exit;
    goto *Loop.*
