Zadanie 2 – rozwiązanie równania Ax = e metodami Gaussa-Seidela i gradientów sprzężonych.

$$\mathbf{A} = \begin{bmatrix} 4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & \dots \\ 1 & 4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & \dots \\ 0 & 1 & 4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & \dots \\ 0 & 0 & 1 & 4 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & \dots \\ 1 & 0 & 0 & 1 & 4 & 1 & 0 & 0 & 1 & 0 & 0 & \dots \\ 0 & 1 & 0 & 0 & 1 & 4 & 1 & 0 & 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & 0 & 0 & 1 & 4 & 1 & 0 & 0 & 1 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & 0 & 0 & 1 & 0 & 0 & 1 & 4 & 1 & 0 & 0 & 1 \\ \dots & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 4 & 1 & 0 & 0 \\ \dots & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 4 & 1 & 0 \\ \dots & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 4 & 1 \\ \dots & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 4 \end{bmatrix}$$

A ma wymiary 128x128, natomiast e jest wektorem o wszystkich składowych równych 1.

W metodzie Gaussa-Seidela wartości wyliczam jako:

$$x_i^{(k+1)} = \left(b_i - \sum_{j=1}^{i-1} a_{ij} x_j^{(k+1)} - \sum_{j=i+1}^{N} a_{ij} x_j^{(k)} \right) \Big/ a_{ii}$$

Natomiast metoda gradientów sprzężonych przebiega zgodnie z następującym algorytmem:

$\mathbf{r}_0 := \mathbf{b} - \mathbf{A}\mathbf{x}_0$
if $\mathbf{r}_0$ is sufficiently small, then return $\mathbf{x}_0$ as the result
$\mathbf{p}_0 := \mathbf{r}_0$
$k := 0$
repeat
$\quad \alpha_k := \dfrac{\mathbf{r}_k^\mathsf{T}\mathbf{r}_k}{\mathbf{p}_k^\mathsf{T}\mathbf{A}\mathbf{p}_k}$
$\quad \mathbf{x}_{k+1} := \mathbf{x}_k + \alpha_k \mathbf{p}_k$
$\quad \mathbf{r}_{k+1} := \mathbf{r}_k - \alpha_k \mathbf{A}\mathbf{p}_k$
$\quad$ if $\mathbf{r}_{k+1}$ is sufficiently small, then exit loop
$\quad \beta_k := \dfrac{\mathbf{r}_{k+1}^\mathsf{T}\mathbf{r}_{k+1}}{\mathbf{r}_k^\mathsf{T}\mathbf{r}_k}$
$\quad \mathbf{p}_{k+1} := \mathbf{r}_{k+1} + \beta_k \mathbf{p}_k$
$\quad k := k + 1$
end repeat
return $\mathbf{x}_{k+1}$ as the result

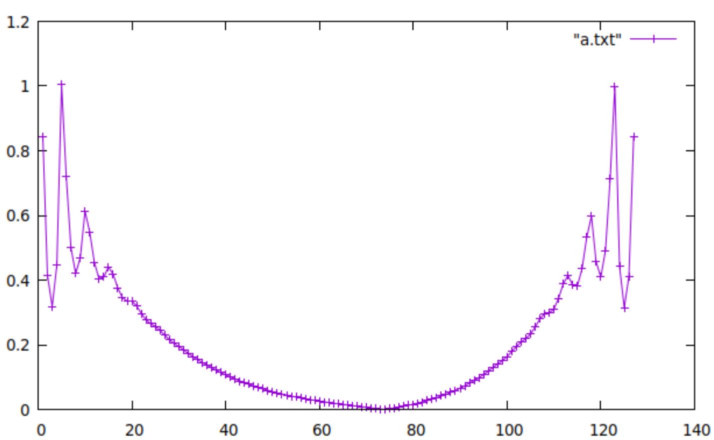
W celu wykonania tego zadania napisałam dwa programy w języku C++.
W lewej kolumnie tabeli przedstawiam wyniki dla metody Gaussa-Seidela, natomiast w prawej kolumnie dla metody gradientów sprzężonych.

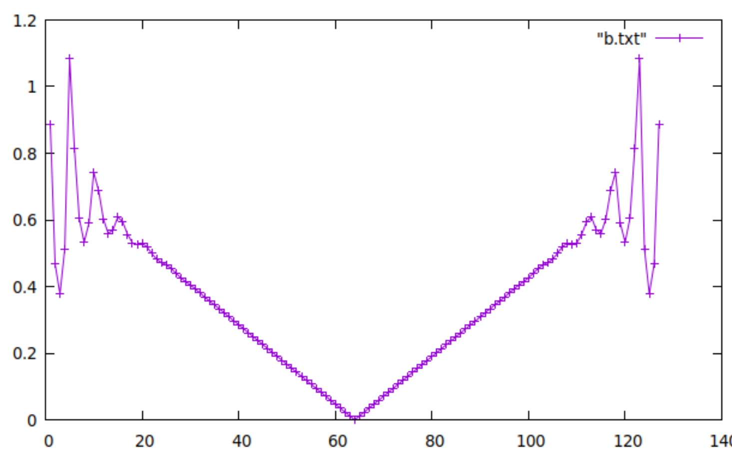
Metoda Gaussa-Seidela	Metoda gradientów sprzężonych
x1 = 0.198291	x1 = 0.19986
x2 = 0.123737	x2 = 0.121552
x3 = 0.160337	x3 = 0.162844
x4 = 0.132176	x4 = 0.129411
x5 = 0.171622	x5 = 0.174738
x6 = 0.0828655	x6 = 0.0790087
x7 = 0.146687	x7 = 0.15109
x8 = 0.102453	x8 = 0.0976596
x9 = 0.139647	x9 = 0.144773
x10 = 0.09812	x10 = 0.0926282
x11 = 0.152323	x11 = 0.158277
x12 = 0.103786	x12 = 0.0974205
x13 = 0.143951	x13 = 0.150655
x14 = 0.108208	x14 = 0.101208
x15 = 0.144405	x15 = 0.151699
x16 = 0.105436	x16 = 0.0978345
x17 = 0.142412	x17 = 0.150295
x18 = 0.109216	x18 = 0.101092
x19 = 0.139707	x19 = 0.14804
x20 = 0.110059	x20 = 0.101533
x21 = 0.139681	x21 = 0.148389
x22 = 0.111337	x22 = 0.102471
x23 = 0.137644	x23 = 0.146641
x24 = 0.113105	x24 = 0.104005
x25 = 0.136558	x25 = 0.145741
x26 = 0.113855	x26 = 0.104607
x27 = 0.135501	x27 = 0.144791
x28 = 0.115177	x28 = 0.105868
x29 = 0.134252	x29 = 0.143557
x30 = 0.116171	x30 = 0.106891
x31 = 0.133414	x31 = 0.142651
x32 = 0.117062	x32 = 0.107889
x33 = 0.132452	x33 = 0.141539
x34 = 0.117989	x34 = 0.109007
x35 = 0.131631	x35 = 0.140489
x36 = 0.118729	x36 = 0.110012
x37 = 0.130904	x37 = 0.139462
x38 = 0.119459	x38 = 0.111078
x39 = 0.130202	x39 = 0.138387
x40 = 0.120107	x40 = 0.112133
x41 = 0.129602	x41 = 0.137351
x42 = 0.120681	x42 = 0.113174
x43 = 0.129046	x43 = 0.136296
x44 = 0.121212	x44 = 0.114233
x45 = 0.128548	x45 = 0.135243
x46 = 0.121677	x46 = 0.115279

x47 = 0.128108 x48 = 0.122097 x49 = 0.127709 x50 = 0.122472 x51 = 0.127358 x52 = 0.122803 x53 = 0.127045 x54 = 0.123099 x55 = 0.126766 x56 = 0.123361 x57 = 0.126518 x58 = 0.123595 x59 = 0.126297 x60 = 0.123805 x61 = 0.126098 x62 = 0.123994 x63 = 0.125917 x64 = 0.124168 x65 = 0.125749 x66 = 0.124332 x67 = 0.125589 x68 = 0.12449 x69 = 0.125432 x70 = 0.124647 x71 = 0.125272 x72 = 0.124811 x73 = 0.125103 x74 = 0.124987 x75 = 0.124919 x76 = 0.125182 x77 = 0.12471 x78 = 0.125405 x79 = 0.124473 x80 = 0.12566 x81 = 0.124194 x82 = 0.125963 x83 = 0.123868 x84 = 0.126312 x85 = 0.123491 x86 = 0.126727 x87 = 0.123036 x88 = 0.127214 x89 = 0.122523 x90 = 0.127762 x91 = 0.121917 x92 = 0.128433 x93 = 0.121206 x94 = 0.129154 x95 = 0.12046 x96 = 0.129995 x97 = 0.119497 x98 = 0.131007 x99 = 0.118543	x47 = 0.134196 x48 = 0.116331 x49 = 0.133142 x50 = 0.117383 x51 = 0.132094 x52 = 0.118431 x53 = 0.131042 x54 = 0.119484 x55 = 0.129991 x56 = 0.120534 x57 = 0.12894 x58 = 0.121585 x59 = 0.127889 x60 = 0.122636 x61 = 0.126839 x62 = 0.123687 x63 = 0.125787 x64 = 0.124738 x65 = 0.124738 x66 = 0.125787 x67 = 0.123687 x68 = 0.126839 x69 = 0.122636 x70 = 0.127889 x71 = 0.121585 x72 = 0.12894 x73 = 0.120534 x74 = 0.129991 x75 = 0.119484 x76 = 0.131042 x77 = 0.118431 x78 = 0.132094 x79 = 0.117383 x80 = 0.133142 x81 = 0.116331 x82 = 0.134196 x83 = 0.115279 x84 = 0.135243 x85 = 0.114233 x86 = 0.136296 x87 = 0.113174 x88 = 0.137351 x89 = 0.112133 x90 = 0.138387 x91 = 0.111078 x92 = 0.139462 x93 = 0.110012 x94 = 0.140489 x95 = 0.109007 x96 = 0.141539 x97 = 0.107889 x98 = 0.142651 x99 = 0.106891

x100 = 0.131923 x101 = 0.117456 x102 = 0.13328 x103 = 0.116011 x104 = 0.134472 x105 = 0.115115 x106 = 0.135717 x107 = 0.113176 x108 = 0.137936 x109 = 0.111707 x110 = 0.138157 x111 = 0.110665 x112 = 0.14107 x113 = 0.106669 x114 = 0.14328 x115 = 0.109232 x116 = 0.143028 x117 = 0.104604 x118 = 0.151612 x119 = 0.0987309 x120 = 0.139118 x121 = 0.102904 x122 = 0.146319 x123 = 0.0831468 x124 = 0.171417 x125 = 0.132339 x126 = 0.160212 x127 = 0.123818 x128 = 0.198259	x100 = 0.143557 x101 = 0.105868 x102 = 0.144791 x103 = 0.104607 x104 = 0.145741 x105 = 0.104005 x106 = 0.146641 x107 = 0.102471 x108 = 0.148389 x109 = 0.101533 x110 = 0.14804 x111 = 0.101092 x112 = 0.150295 x113 = 0.0978345 x114 = 0.151699 x115 = 0.101208 x116 = 0.150655 x117 = 0.0974205 x118 = 0.158277 x119 = 0.0926282 x120 = 0.144773 x121 = 0.0976596 x122 = 0.15109 x123 = 0.0790087 x124 = 0.174738 x125 = 0.129411 x126 = 0.162844 x127 = 0.121552 x128 = 0.19986

Wyliczone normy zapisałam w plikach a.txt i b.txt
Oto wykresy na ich podstawie (odpowiednio dla metody Gaussa-Seidela i gradientów sprzężonych), narysowane w gnuplocie:





Złożoność obliczeniowa rozkładu Cholesky’ego dla tej macierzy wynosi $O(n^3)$
Złożoność metody Gaussa-Seidela wynosi $O(n)$, natomiast gradientów sprzężonych $O(k*n^2)$
gdzie k jest szerokością pasma.
Widać zatem, że te metody są dużo wydajniejsze od metody Cholesky’ego.