Quantum Probability 1982-2017

Hans Maassen, Universities of Nijmegen and Amsterdam

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Villa Mondragone

First conference in quantum probability, 1982

The new scheme

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$p_1 + \ldots + p_n = \mathbb{1}$	$\longleftrightarrow$	exactly one of $p_1, \ldots p_n$ holds .

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* ‘This is nothing new, just operator algebras in quantum mechanics.
* Where is the innovation?’

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  For instance different temperatures require inequivalent Hilbert spaces representations in statistical mechanics.

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Over that bridge ideas could be carried to the other side.

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A coordinate frame now becomes a set of three mutually exclusive propositions!

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These phenomena can be reproduced even without a rabbit.

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This time, we do need the rabbits! They must be placed behind the doors before the questioning begins.

A game with two entangled ρ-mesons

Two rooms, one for A, one for B.
Each room contains a ρ-meson; the pair of mesons is described by

$$\begin{aligned} \mathcal{A} &= M_3 \otimes M_3 \,, \\ \varphi(x) &= \langle \psi, x\psi \rangle, \text{ where } \psi = \frac{1}{\sqrt{3}} \sum_{i=1}^{3} e_i \otimes e_i \,, \end{aligned}$$

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These results are obtained in all frames.

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$$\sum_{i=1}^{3} r_{ji}r_{ki} = \sum_{i=1}^{3} r_{ji}(r_{ik})^* = (rr^*)_{jk} = \delta_{jk} \; .$$

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Now choose $\omega \in \Omega$ sucht that this holds for all 16 frames (α, β, γ) ocurring in the "Sudoku of Asher Peres".

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Contradiction! QED.

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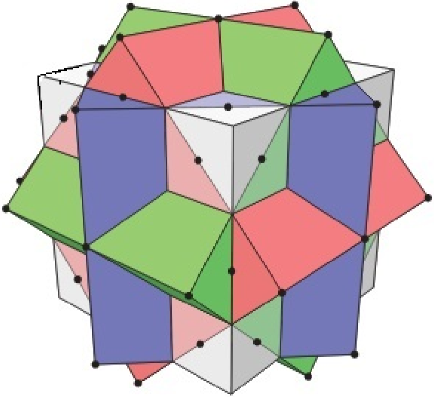

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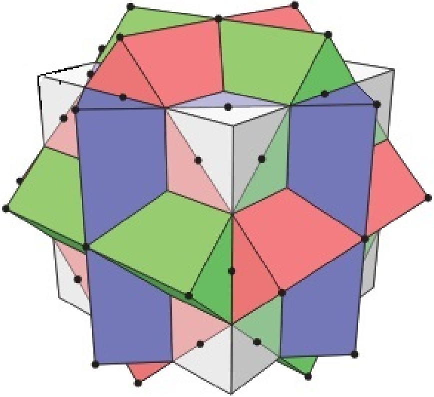

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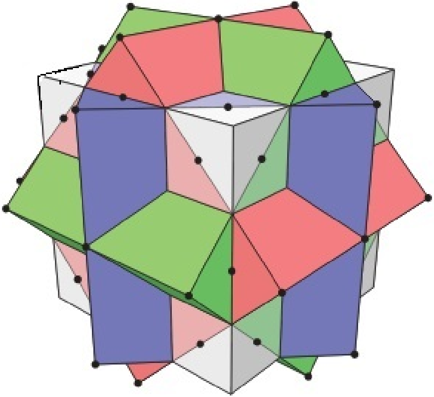

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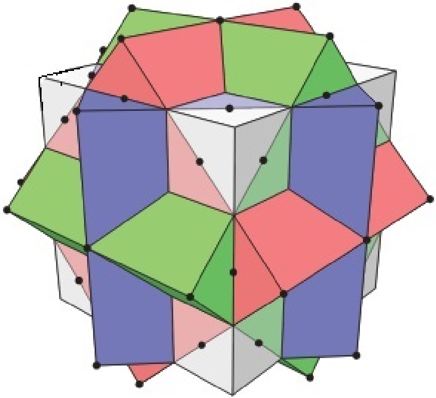

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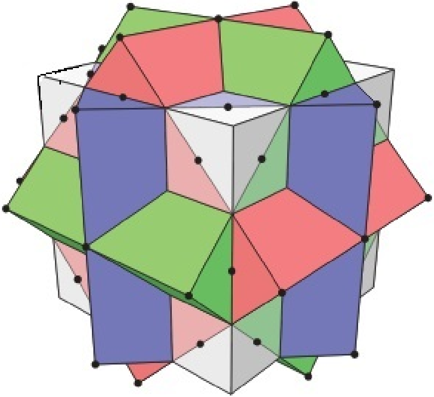

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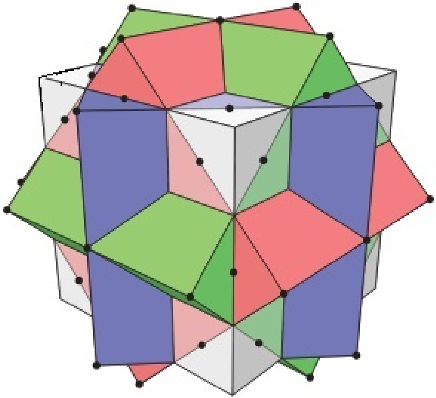

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However, the quantum model is local in the sense that the answer to the question α of A remains the same proposition $p_\alpha \in M_3$, whatever question $\beta \perp \alpha$ his opponent B asks.:

$$\begin{aligned} X_\alpha &= p_\alpha \otimes \mathbb{1} \, ; \\ Y_\beta &= \mathbb{1} \otimes p_\beta \, . \end{aligned}$$

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(Einstein hated this.)

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The rest is easy again.

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