Appendix A

A.1 Block Jacobian matrices for carpet scrapping mechanism

The analytical expressions for the scalar elements of the block Jacobian matrices obtained in (26) from the loop-closure equations are given below:

$$\mathbf{J}_{1,1} \equiv \begin{bmatrix} a_1 s_1 \\ a_1 c_1 \end{bmatrix}, \quad \mathbf{J}_{1,2} \equiv \begin{bmatrix} -a_2 s_2 - a_3 s_3 & -a_3 s_3 \\ -a_2 c_2 - a_3 c_3 & -a_3 c_3 \end{bmatrix}, \quad \mathbf{J}_{2,2} \equiv \begin{bmatrix} -a_2 s_2 + a_{39} s_3 & a_{39} s_3 \\ -a_2 c_2 - a_{39} c_3 & -a_{39} c_3 \end{bmatrix},$$

$$\mathbf{J}_{2,3} \equiv \begin{bmatrix} a_{59} s_5 + (a_{46} - a_{56}) s_4 & a_{59} s_5 & 0 & 0 \\ -a_{59} c_5 - (a_{46} - a_{56}) c_4 & -a_{59} c_5 & 0 & 0 \end{bmatrix},$$

$$\mathbf{J}_{3,3} \equiv \begin{bmatrix} \begin{matrix} -a_{46} s_4 - a_{67} s_6 - \\ a_{97} s_7 + a_{59} s_5 + (a_{46} - a_{56}) s_4 \end{matrix} & a_{59} s_5 & -a_{67} s_6 - a_{97} s_7 & -a_{97} s_7 \\ \begin{matrix} a_{46} c_4 + a_{67} c_6 + \\ a_{97} c_7 - a_{59} c_5 - (a_{46} - a_{56}) c_4 \end{matrix} & -a_{59} c_5 & a_{67} c_6 + a_{97} c_7 & a_{97} c_7 \end{bmatrix}$$

where a_i or a_{ij} represents link length (kinematic parameters), as shown in Fig. 9(a), and s_i and c_i represent sinα_i and cosα_i, respectively. The relations between α_i's and θ_i's are given by $\alpha_1 = \theta_1$, $\alpha_2 = \theta_2$, $\alpha_3 = \pi + \theta_2 + \theta_3$, $\alpha_4 = \theta_4$, $\alpha_5 = \theta_5 + \theta_4 - 2\pi$, $\alpha_6 = \theta_6 + \theta_4 - 2\pi$ and $\alpha_7 = \theta_7 + \theta_6 + \theta_4 - 4\pi$, where α_i represents the angle subtended by a link with the horizontal line and θ_i's are the relative joint angles between the links at the joint locations indicated in the subscripts of θ_i, as shown in Fig. 4.

A.2 Block Jacobian matrices for 3-RRR parallel manipulator

The analytical expressions for the scalar elements of the block Jacobian matrices obtained in (37) from the loop-closure equations are given below

$$\mathbf{J}_{1,1} \equiv \begin{bmatrix} -a_1s_1 - b_1s_4 - d_1s_7 + b_2s_5 & -b_1s_4 - d_1s_7 + b_2s_5 & -d_1s_7 + b_2s_5 & b_2s_5 & 0 \\ a_1c_1 + b_1c_4 - d_1c_7 - b_2c_5 & b_1c_4 - d_1c_7 - b_2c_5 & d_1c_7 - b_2c_5 & -b_2c_5 & 0 \end{bmatrix},$$

$$\mathbf{J}_{2,1} \equiv \begin{bmatrix} -a_1s_1 - b_1s_4 - d_1s_7' + b_3s_6 & -b_1s_4 - d_1s_7' + b_3s_6 & -d_1s_7' + b_3s_6 & 0 & b_3s_6 \\ a_1c_1 + b_1c_4 + d_1c_7' - b_3c_6 & b_1c_4 + d_1c_7' - b_3c_6 & d_1c_7' - b_3c_6 & 0 & -b_3c_6 \end{bmatrix},$$

$$\mathbf{J}_{1,2} \equiv \begin{bmatrix} a_2s_2 \\ -a_2c_2 \end{bmatrix}, \quad \mathbf{J}_{1,3} \equiv \mathbf{J}_{2,2} \equiv \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \quad \mathbf{J}_{2,3} \equiv \begin{bmatrix} a_3s_3 \\ -a_3c_3 \end{bmatrix}$$

where a_i, b_i and d_i represent the kinematic parameters of the 3-RRR parallel manipulator shown in Fig. 9(b). The relations between α_i's and θ_i's are given by $\alpha_1 = \theta_1$, $\alpha_2 = \theta_6$, $\alpha_3 = \theta_7$, $\alpha_4 = \theta_1 + \theta_2$, $\alpha_5 = \theta_1 + \theta_2 + \theta_3 + \theta_4 - \pi$, $\alpha_6 = \theta_1 + \theta_2 + \theta_3 + \theta_5 - (\pi - \pi/3)$, $\alpha_7 = \theta_1 + \theta_2 + \theta_3$ and $\alpha_7' = \alpha_7 + \pi/3$.

A.3 Block Gaussian Elimination of (10)

After carrying out the forward block Gaussian elimination of (10), we arrive at

$$\begin{bmatrix} \mathbf{I} & \mathbf{J}^T \\ \mathbf{O} & \textbf{-}\mathbf{J}\mathbf{I}^{-1}\mathbf{J}^T \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}} \\ -\boldsymbol{\lambda} \end{bmatrix} = \begin{bmatrix} \boldsymbol{\varphi} \\ -\dot{\mathbf{J}}\dot{\mathbf{q}} - \mathbf{J}\mathbf{I}^{-1}\boldsymbol{\varphi} \end{bmatrix} \tag{1}$$

The vector of Lagrange multipliers is thus obtained as

$$\boldsymbol{\lambda} = -(\mathbf{J}\mathbf{I}^{-1}\mathbf{J}^T)^{-1}(\dot{\mathbf{J}}\dot{\mathbf{q}} + \mathbf{J}\mathbf{I}^{-1}\boldsymbol{\varphi}) \tag{2}$$

which is same as (12).

A.4 Block Gaussian Elimination of (23) for the 3-RRR parallel manipulator

Using the expressions of (42), (23) for the 3-RRR parallel manipulator is reproduced here as

$$\begin{bmatrix} \bar{\mathbf{I}}_{1,1} & \bar{\mathbf{I}}_{2,1}^T \\ \bar{\mathbf{I}}_{2,1} & \bar{\mathbf{I}}_{2,2} \end{bmatrix} \begin{bmatrix} \boldsymbol{\lambda}_1 \\ \boldsymbol{\lambda}_2 \end{bmatrix} = \begin{bmatrix} \overline{\boldsymbol{\Psi}}_1 \\ \overline{\boldsymbol{\Psi}}_2 \end{bmatrix} \tag{3}$$

Applying the forward block Gaussian elimination using the block pivot as $\bar{\mathbf{I}}_{2,1}\bar{\mathbf{I}}_{1,1}^{-1}$, on (49) the following block upper triangular matrix on the LHS is obtained

$$\begin{bmatrix} \bar{\mathbf{I}}_{1,1} & \bar{\mathbf{I}}_{2,1}^T \\ \mathbf{O} & \bar{\mathbf{I}}_{2,2} - \bar{\mathbf{I}}_{2,1}\bar{\mathbf{I}}_{1,1}^{-1}\bar{\mathbf{I}}_{2,1}^T \end{bmatrix} \begin{bmatrix} \boldsymbol{\lambda}_1 \\ \boldsymbol{\lambda}_2 \end{bmatrix} = \begin{bmatrix} \overline{\boldsymbol{\Psi}}_1 \\ \overline{\boldsymbol{\Psi}}_2 - \bar{\mathbf{I}}_{2,1}\bar{\mathbf{I}}_{1,1}^{-1}\overline{\boldsymbol{\Psi}}_1 \end{bmatrix} \tag{4}$$

Using the backward substitutions, one can then solve easily for $\boldsymbol{\lambda}_2$ and $\boldsymbol{\lambda}_1$, which are given by

$$\boldsymbol{\lambda}_2 \equiv \tilde{\mathbf{I}}_{2,2}^{-1}\tilde{\boldsymbol{\Psi}}_2 \tag{5}$$

$$\boldsymbol{\lambda}_1 \equiv \tilde{\mathbf{I}}_{1,1}^{-1}(\overline{\boldsymbol{\Psi}}_1 - \bar{\mathbf{I}}_{2,1}^T\boldsymbol{\lambda}_2) \tag{6}$$

where $\tilde{\mathbf{I}}_{2,2} \equiv \bar{\mathbf{I}}_{2,2} - \bar{\mathbf{I}}_{2,1}\bar{\mathbf{I}}_{1,1}^{-1}\bar{\mathbf{I}}_{2,1}^T$ and $\tilde{\boldsymbol{\Psi}}_2 \equiv \overline{\boldsymbol{\Psi}}_2 - \bar{\mathbf{I}}_{2,1}\bar{\mathbf{I}}_{1,1}^{-1}\overline{\boldsymbol{\Psi}}_1$

A.5 Block Gaussian Elimination of the subsystem-level representation of (10) for the 3-RRR parallel manipulator

To illustrate that the results of (51 – 52) can also be obtained from the fundamental formulation given by (10), by using the subsystem-level expressions for the 3-RRR parallel manipulator, i.e., (37 – 40), (10) is re-written as

$$\begin{bmatrix} \mathbf{I}_1 & \mathbf{O} & \mathbf{O} & \mathbf{J}_{1,1}^T & \mathbf{J}_{2,1}^T \\ \mathbf{O} & \mathbf{I}_2 & \mathbf{O} & \mathbf{J}_{1,2}^T & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{I}_3 & \mathbf{O} & \mathbf{J}_{2,3}^T \\ \mathbf{J}_{1,1} & \mathbf{J}_{1,2} & \mathbf{O} & \mathbf{O} & \mathbf{O} \\ \mathbf{J}_{2,1} & \mathbf{O} & \mathbf{J}_{2,3} & \mathbf{O} & \mathbf{O} \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}}_1 \\ \ddot{\mathbf{q}}_2 \\ \ddot{\mathbf{q}}_3 \\ -\boldsymbol{\lambda}_1 \\ -\boldsymbol{\lambda}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{\varphi}_1 \\ \boldsymbol{\varphi}_2 \\ \boldsymbol{\varphi}_3 \\ \boldsymbol{\Psi}_1 \\ \boldsymbol{\Psi}_2 \end{bmatrix} \tag{7}$$

Applying the block Gaussian elimination, the final upper block triangular on the LHS is given by

$$\begin{bmatrix} \mathbf{I}_1 & \mathbf{O} & \mathbf{O} & \mathbf{J}_{1,1}^T & \mathbf{J}_{2,1}^T \\ \mathbf{O} & \mathbf{I}_2 & \mathbf{O} & \mathbf{J}_{1,2}^T & \mathbf{O} \\ \mathbf{O} & \mathbf{O} & \mathbf{I}_3 & \mathbf{O} & \mathbf{J}_{2,3}^T \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & -\bar{\mathbf{I}}_1^{11} - \bar{\mathbf{I}}_2^{11} & -\bar{\mathbf{I}}_1^{12} \\ \mathbf{O} & \mathbf{O} & \mathbf{O} & \mathbf{O} & -\bar{\mathbf{I}}_1^{22} - \bar{\mathbf{I}}_3^{22} + \bar{\mathbf{I}}_1^{21}(\bar{\mathbf{I}}_1^{11} + \bar{\mathbf{I}}_2^{11})^{-1}(\bar{\mathbf{I}}_1^{21})^T \end{bmatrix} \begin{bmatrix} \ddot{\mathbf{q}}_1 \\ \ddot{\mathbf{q}}_2 \\ \ddot{\mathbf{q}}_3 \\ -\boldsymbol{\lambda}_1 \\ -\boldsymbol{\lambda}_2 \end{bmatrix} = \begin{bmatrix} \boldsymbol{\varphi}_1 \\ \boldsymbol{\varphi}_2 \\ \boldsymbol{\varphi}_3 \\ \tilde{\boldsymbol{\Psi}}_1 \\ \tilde{\boldsymbol{\Psi}}_2 \end{bmatrix} \tag{8}$$

where $\tilde{\boldsymbol{\Psi}}_1 \equiv \boldsymbol{\Psi}_1$ - $\mathbf{J}_{1,1}\mathbf{I}_1^{-1}$ $\boldsymbol{\varphi}_1$ - $\mathbf{J}_{1,2}\mathbf{I}_2^{-1}$ $\boldsymbol{\varphi}_2$ and $\tilde{\boldsymbol{\Psi}}_2 \equiv \boldsymbol{\Psi}_2$ - $\mathbf{J}_{2,1}\mathbf{I}_1^{-1}$ $\boldsymbol{\varphi}_1$ - $\mathbf{J}_{2,3}\mathbf{I}_3^{-1}$ $\boldsymbol{\varphi}_3$ - $\mathbf{I}_1^{21}$ ($\mathbf{I}_1^{11}$ + $\mathbf{I}_2^{11})^{-1}\tilde{\boldsymbol{\Psi}}_1$. The Lagrange multipliers are computed next using backward substitutions as

$$\boldsymbol{\lambda}_2 \equiv [\bar{\mathbf{I}}_1^{22} + \bar{\mathbf{I}}_3^{22} - \bar{\mathbf{I}}_1^{12}(\bar{\mathbf{I}}_1^{11} + \bar{\mathbf{I}}_2^{11})^{-1}(\bar{\mathbf{I}}_1^{21})^T]^{-1}\tilde{\boldsymbol{\Psi}}_2 \tag{9}$$

$$\boldsymbol{\lambda}_1 \equiv (\bar{\mathbf{I}}_1^{11} + \bar{\mathbf{I}}_2^{11})^{-1}[\overline{\boldsymbol{\Psi}}_1 - (\bar{\mathbf{I}}_1^{21})^T\boldsymbol{\lambda}_2] \tag{10}$$

Substituting the expressions of (43) in (55 – 56), we get exactly the same expressions for $\boldsymbol{\lambda}_1$ and $\boldsymbol{\lambda}_2$ to that obtained in (51 – 52) and reported in (45 – 46). Note here that one could continue with the back substitutions to solve for the joint accelerations required in the forward dynamics. This is, however, not recommended here, mainly due to the fact that one is not able to exploit the already existing algorithms for the tree-type systems, which may be of recursive in nature, e.g., *ReDySim* of [**?**], with good efficiency and numerical stability.