Toric Code

University of Florida, August 19, 2020

Recap of last time

- Error detection big picture:
  1. Add redundancy $k \to n$ qubits: $|\psi\rangle \to |\psi\rangle_L$
  2. Errors: $|\psi\rangle_L \to E_L |\psi\rangle_L$
  3. Considering only X-type and Z-type errors is enough to account for all errors.
  4. Add ancilla and measure the ancilla
- Stabilizer codes:
  1. $\mathcal{C} = \text{span}\{|\psi\rangle_L \in \mathcal{H} : \; S_i |\psi\rangle_L = |\psi\rangle_L\}$
  2. $S_i^2 = 1$
  3. For error detection, $\{E_L, S_i\} = 0$ for at least one S_i.
  4. Syndrome extraction: $(1 + S_i)E_L |\psi\rangle_L |0\rangle + (1 - S_i)E_L |\psi\rangle_L |1\rangle$

Two approaches to Kitaev toric code (Kitaev, 2003)

As a discrete gauge theory

- It's a $\mathbb{Z}_2$ gauge theory
- It has anyonic excitations
- Long range entanglement and topological order

As a quantum code

- It implements a specific type of quantum code
- Allow error detection and error correction
- Allows for a restricted qubit operations

This talk will focus on the quantum code aspect.

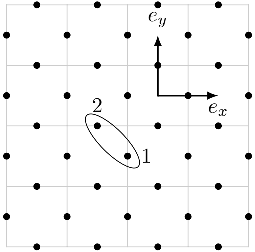
Outline

1. Kitaev toric code
   - The model
   - The code
   - How to perform logical operations
2. Anyonic nature of the excitations

Outline

Kitaev Toric Model

- A lattice model of spin-$1/2$ particles.
- Each unit cell has 2 spin sites, 1 and 2.
- Local operators: $\{\vec{\sigma}_1(\boldsymbol{R}_i), \vec{\sigma}_2(\boldsymbol{R}_i)\}$
- Operators at different lattice sites commute.



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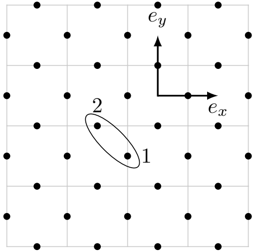
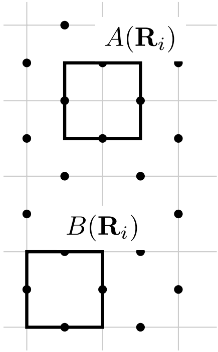


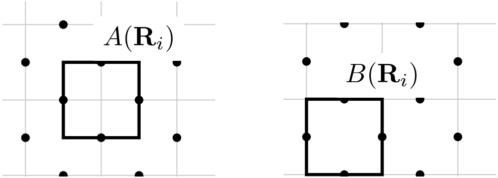
The Hamiltonian:

$$H = -\sum_{\boldsymbol{R}_i} (A(\boldsymbol{R}_i) + B(\boldsymbol{R}_i))$$

$$\begin{aligned} A(\boldsymbol{R}_i) &= \sigma_2^x(\boldsymbol{R}_i)\sigma_1^x(\boldsymbol{R}_i) \\ &\quad \sigma_2^x(\boldsymbol{R}_i + e_x)\sigma_1^x(\boldsymbol{R}_i + e_y), \\ B(\boldsymbol{R}_i) &= \sigma_1^z(\boldsymbol{R}_i)\sigma_2^z(\boldsymbol{R}_i) \\ &\quad \sigma_1^z(\boldsymbol{R}_i - e_x)\sigma_2^z(\boldsymbol{R}_i - e_y) \end{aligned}$$



Ground state of the Kitaev model



Notation:

σ^x: line perpendicular to unit cell edge at the spin site.

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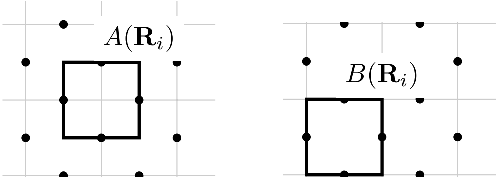

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- $A(\boldsymbol{R}_i)$ and $B(\boldsymbol{R}_i)$ are two different *loops* in the system.
- They only look like loops because of our choice of notation.
- No need for arrows on the loops.
- $A^2(\boldsymbol{R}_i) = 1$ and $B^2(\boldsymbol{R}_i) = 1$. Both have eigenvalues of ± 1.
- $[A(\boldsymbol{R}_i), B(\boldsymbol{R}_i)] = 0$

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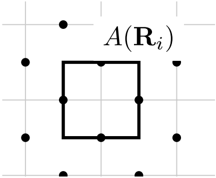

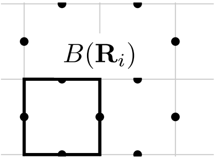

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Ground sates $|\Omega_0\rangle$ of $H = -\sum_{\boldsymbol{R}_i} (A(\boldsymbol{R}_i) + B(\boldsymbol{R}_i))$ is defined by,

$$A(\boldsymbol{R}_i)\,|\Omega_0\rangle = |\Omega_0\rangle\,,\ B(\boldsymbol{R}_i)\,|\Omega_0\rangle = |\Omega_0\rangle$$

The code

We consider a $N \times N$ lattice on a torus.

- The Hilbert space, $\mathcal{H}$, is 2^{2N^2} dimensional.
- Codeword space, $\mathcal{C}$, is defined as

$$\mathcal{C} = \mathsf{span}\{|\Omega_0\rangle \in \mathcal{H} : A(\boldsymbol{R}_i)\,|\Omega_0\rangle = |\Omega_0\rangle\,,\ B(\boldsymbol{R}_i)\,|\Omega_0\rangle = |\Omega_0\rangle\}$$

- This quantum code is called TOR(N), the toric code.
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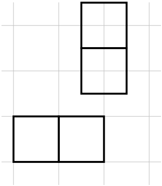
- There are $2N^2 - 2$ independent stabilizers.
  There are N^2 $A(\boldsymbol{R}_i)$, and N^2 $B(\boldsymbol{R}_i)$ operators, but we have the following dependencies,

$$\prod_{\boldsymbol{R}_i} A(\boldsymbol{R}_i) = 1, \prod_{\boldsymbol{R}_i} B(\boldsymbol{R}_i) = 1 \leftarrow \text{No edges left.}$$



- $\mathcal{C}$ is $(2^{2N^2})/(2^{2N^2-2}) = 2^2$ dimensional. It can encode 2 qubits.

What labels the ground states

Since the code stabilizers defines 2^{2N^2-2} 4D subspaces, we must label the 4 states within each subspace by other independent operators that commute with all the $A(\boldsymbol{R}_i)$ and $B(\boldsymbol{R}_i)$.

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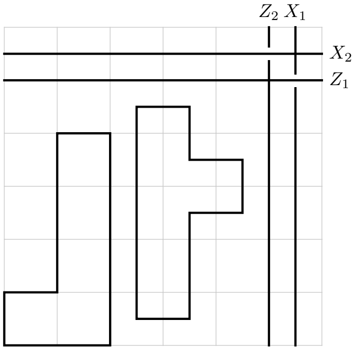
- Every contractible loop can be decomposed into smaller loops of $A(\boldsymbol{R}_i)$ or $B(\boldsymbol{R}_i)$.
- There are 4 different non-contractible loops.
- $\{Z_1, Z_2, X_1, X_2\}$ commute with all contractible loops.
- $\{Z_1, X_1\} = 0$, $\{Z_2, X_2\} = 0$
- The entire 2^{2N^2} Hilbert space can be labeled by the eigenvalues of, $\{Z_1, Z_2, A(\boldsymbol{R}_i), B(\boldsymbol{R}_i)\}$



Errors

A general error can be any linear combination of,

$$E(\{\alpha_i^l, \beta_j^m\}) = \prod_{\substack{\boldsymbol{R}_i, \boldsymbol{R}_i \\ l,m}} (\sigma_l^x(\boldsymbol{R}_i))^{\alpha_i^l} (\sigma_m^z(\boldsymbol{R}_j))^{\beta_j^m},$$

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1. Contain only closed contractible loops. E_1.
2. Contain one or more open strings. E_2.
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Errors of type 3, must at least be N long. And assuming errors act independently on each qubit, these errors would be exponentially suppressed, $e^{-\alpha N}$

Error detection, and correction

Open string operations anticommute with two stabilizer operators, one surrounding each end of the open ended loop.

$$B(\boldsymbol{R}_i)E_2\left|\Omega_0\right\rangle = -E_2\left|\Omega_0\right\rangle,$$

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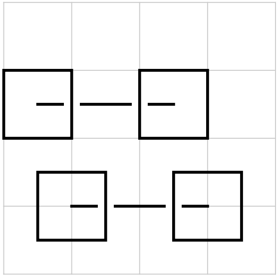

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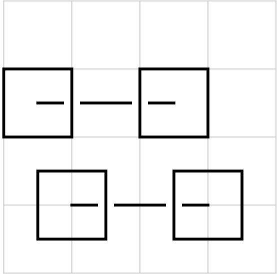

- These errors can be detected using a syndrome measurements.
- Notice that $E_2\left|\Omega_0\right\rangle$ are excited states of the Hamiltonian, with $\Delta E \geqslant 2$.
- Kitaev also suggested fixing these errors by coupling the system to a heat bath and cooling the system down.

Excitations of the toric code

Let's look at low energy excitations of the toric code.

- We cannot have excitation with $\Delta E = 1$. This would violate

$$\prod_{\boldsymbol{R}_i} A(\boldsymbol{R}_i) = 1, \quad \prod_{\boldsymbol{R}_i} B(\boldsymbol{R}_i) = 1.$$

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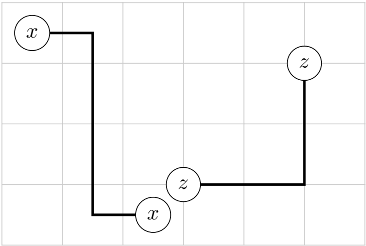
$S^x(t)\,|\Omega_0\rangle\,,\ S^z(t)\,|\Omega_0\rangle$, which depend on

1. The two end points
2. The homotopy of string connecting the two ends, how many non-contractible loops it make. Not the detailed path.



Allowed logic operaion using kitaev model

The non-contractible loops of the toric code behave as Pauli matrices acting on two qubits:

$$[Z_1, Z_2] = 0 \quad [X_1, X_2] = 0 \quad \{Z_1, X_1\} = 0 \quad \{Z_2, X_2\} = 0$$

1. Z operation.
    - Create an a z-type particles pair.
    - Move one around one non-contractible loop. The direction determine which qubit get acted on.
    - Annihilate the two particles.
2. X operation. Using the same steps but with an x-type particle.

These operations do not give us a universal quantum computer.

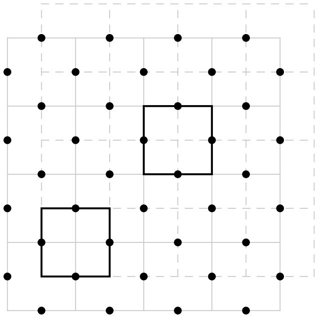
The dual lattice

For the same arrangement of spins there are two ways of defining the lattice. Both of them are equally valid.

- This highlights an important property of the system.

Let $R_y(\theta)$ be the rotation operation around the y-axis then:

$$R_y(90^\circ)A(\boldsymbol{R}_i)R_y^{-1}(90^\circ) = B'(\boldsymbol{R}_i)$$
$$R_y(90^\circ)B(\boldsymbol{R}_i)R_y^{-1}(90^\circ) = A'(\boldsymbol{R}_i)$$



This operation takes an x-type particle to a z-type particle.

Outline

The toric code in different geomtries

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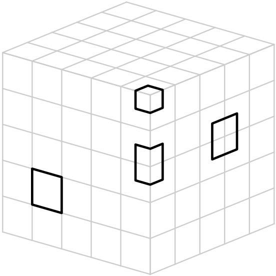
On a *sphere* there are no non-contractible loops. $A(\boldsymbol{R}_i)$ and $B(\boldsymbol{R}_i)$ can label the entire Hilbert space.

- Hilbert space is 2^{12N^2} dimensional.
- $6N^2$ $B(\boldsymbol{R}_i)$ operators.
- $6N^2 + 2$ $A(\boldsymbol{R}_i)$ operators.



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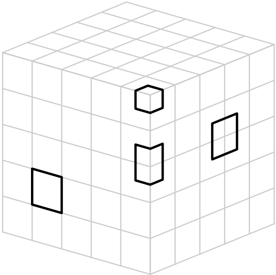
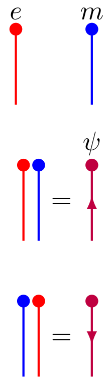


The dependance of the ground state degeneracy on the geometry of the manifold is one of the defining features of topological order.

Particle content of the toric code

- No particles, 1.
- z-type particle, referred to as electric charge, e.
- x-type particle, referred to as a magnetic vortex, m.
- A combinations of an e and an m particle, $\psi = e \times m$
- For convince we drop the lattice from the background, and distinguish different strings by different colors.



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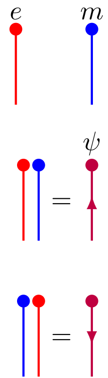
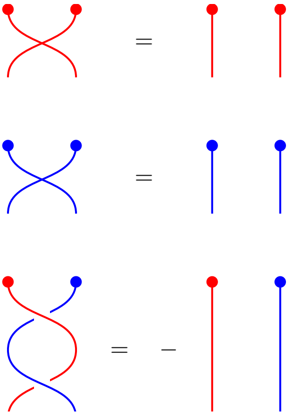


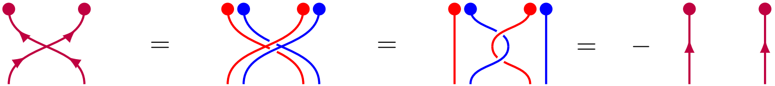
Next we ask what is the statistics of these particles.

- It's natural to consider a braid group because of the strings attached to the particles.
- There are three kinds of strings. The group is then said to be a colored braid group.

Rules and fusion rules



- The electric charge, e is a boson.
- The magnetic vortex, m is a boson.
- e going around m gives a -1.
- ψ is a fermion.
- The braid group is Abelian.



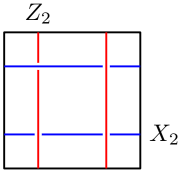
Anyons front and center

Anyons implies the ground state degeneracy.

We can think of the how the anyons braid, as defining the topological order in the system.

- The ground state(s) is a sate with no particles in it. If $|\Omega_0\rangle$ is a ground state, so are $Z_i|\Omega_0\rangle$ and $X_i|\Omega_0\rangle$.
- Braiding rules implies
  1. $Z_1^2 + Z_2^2 + X_1^2 + X_2^2 = 1$
  2. $[Z_1, Z_2] = [X_1, X_2] = 0$
  3. $\{Z_1, X_1\} = \{Z_2, X_2\} = 0.$
- This implies the ground state is fourfold degenerate.

$Z_2X_2Z_2X_2|\Omega_0\rangle \quad =$



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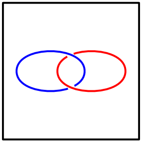

$-|\Omega_0\rangle$

Summary

- Quantum codes encode k qubits into n qubits.
- Quantum codes allow for error correction
- The Kitaev toric code encode $2g$ qubits into a spin lattice.
- The Kitaev code has $e^{-\alpha N}$ probability of missing errors.
- By moving anyons around, one can perform quantum operations on the qubits.
- The anyon content of a theory is enough to define its topological order.

A.Yu. Kitaev. Fault-tolerant quantum computation by anyons. *Annals of Physics*, 303(1):2–30, Jan 2003. ISSN 00034916. doi: 10.1016/S0003-4916(02)00018-0.