Toric Code

University of Florida, August 13, 2020

Two approaches to Kitaev toric code (Kitaev, 2003)

As a discrete gauge theory

- It's a $\mathbb{Z}_2$ gauge theory
- It has anyonic excitations
- Long range entanglement and topological order

As a quantum code

- It implements a specific type of quantum code
- Allow error detection and error correction
- Allows for a restricted qubit operations

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This talk will focus on the quantum code aspect.

Outline

1. Introduction: Noise reduction
   - Classical analog
   - Quantum codes
   - The three-qubit code
2. Kitaev toric code
   - The model
   - The code
   - How to perform logical operations
3. Anyonic nature of the excitations

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   Classical analog
   Quantum codes
   The three-qubit code

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   The model
   The code
   How to perform logical operations

3. Anyonic nature of the excitations

The classical analog of the problem

Communication over a noisy channel,

- We have a string of k bits that we want to *send* over a noisy channel.
- Each bit has a probability, p_f, to flip its value as it go through the channel. Errors can happen when sending data.
- How can we make this communication reliable?

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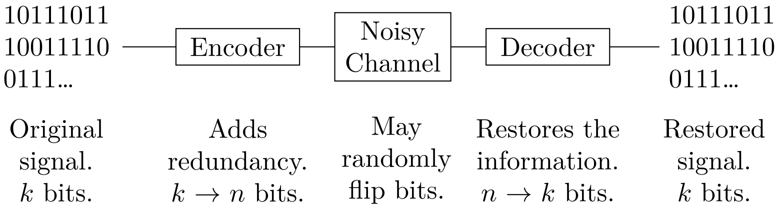
Solutions:

1. Physically build a more reliable channel. (Not always possible)
2. Add redundancy to the sent signal, such that when errors happen we would still have a way of retrieving the original information (the original string of bits).

Classical coding theory



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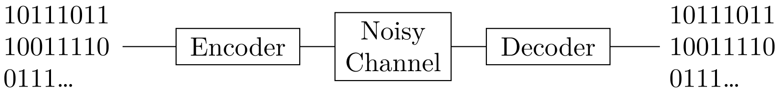

Original signal. k bits.	Adds redundancy. $k \to n$ bits.	May randomly flip bits.	Restores the information. $n \to k$ bits.	Restored signal. k bits.

Example: The repetition code.

Encoder:
$0 \to 000$
$1 \to 111$

Decoder (majority vote):
$\{000, 001, 010, 100\} \to 0$
$\{111, 110, 101, 011\} \to 1$

If we had a probability p_f of making an error (per bit) before, now we have a probability of p_f^2 of making an error.

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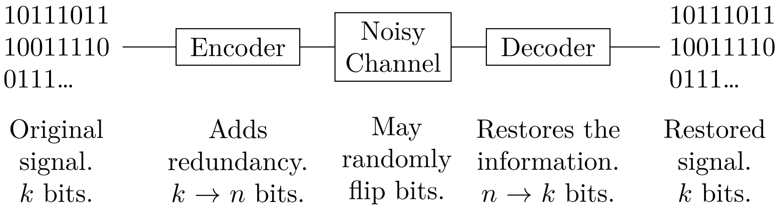

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If we had a probability p_f of making an error (per bit) before, now we have a probability of p_f^2 of making an error.

- The repetition code is not efficient.
- Other problems can be seen as a noisy channel problem.

The (quantum) problem

The same problem is amplified in the quantum world. A qubit is a two level quantum system,

$$|\psi\rangle = c_0 |0\rangle + c_1 |1\rangle$$

Suppose we have prepare a system of k qubits in a specific state,

$$|\Psi\rangle = \bigotimes_k |\psi_k\rangle$$

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Difficulties with the quantum system:

- A qubit cannot duplicated. $|\Psi\rangle \to |\Psi\rangle \otimes |\Psi\rangle$
- A measurement destroys the original state.
- Errors are not just *flips*.

Big picture of error correction

- To correct an error, we first need to detect an error.
- To detect an error, some sort of measurement must be done. (Measuring the error.)
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A breakthrough came in 1995, (Shor, 1995), to first entangle the system to ancillary qubits in such a way that measuring the ancillary qubits would measure the error. After which the appropriate correction can be done. Steps to error correction:

1. Add redundancy, $k \to n$ qubits. A quantum code.
2. Possible errors get introduced to the system.
3. Couple the system ancillary qubits.
4. Measure the errors by measuring the ancillary qubits. Called a syndrome measurement.
5. Correct the error.

Errors

1. Coherent errors. They act as a unitary operator on the state. They don't entangle the qubits to the environment.

$$|\psi\rangle \to E\,|\psi\rangle$$

$$E = \begin{bmatrix} E_{11} & E_{12} \\ E_{21} & E_{22} \end{bmatrix}$$

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2. Incoherent errors. They entangle the qubits to the environment. If $|\Phi\rangle$ represent the state of the environment,

$$|\psi\rangle \otimes |\Phi\rangle \to D\left(|\psi\rangle \otimes |\Phi\rangle\right) = a_{ij}E^{i}\,|\psi\rangle \otimes L^{j}\,|\Phi\rangle$$
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Measuring the ancillary qubits destroys any entanglement between the qubits and the environment. For now we will focus on coherent errors.

Coherent Errors

$$E = \alpha_0 1 + \alpha_x X + \alpha_z Z + \alpha_{xy} XZ$$

There are unaccountably many errors that can occur to one qubit. Part of Shor's breakthrough is that by measuring the error, one force the system into either an X-type error, or a Z-type error.

- X is a bit flip.
- Z is a *phase* flip.

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Being able to measure X, and Z-type errors is enough to account for all errors.

The three-qubit code

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- The codeword space is a 2D subspace of the 8D logic state space,

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$\mathcal{F} = \text{span}\{|100\rangle, |011\rangle, |010\rangle, |101\rangle, |001\rangle, |110\rangle\}$ error subspace

$\mathcal{H} = \mathcal{C} \oplus \mathcal{F}$ full Hilbert space

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- $\{Z_1 Z_2, Z_2 Z_3\}$ are the (generators of) logic qubit stabilizers,

$$\{Z_1 Z_2, Z_2 Z_3\} |\psi\rangle_L = |\psi\rangle_L \quad \text{for } |\psi\rangle_L \in \mathcal{C}$$

Stabilizers can be used to define $\mathcal{C}$.

The three-qubit code

1. Adding redundancy

- $(Z_1Z_2)^2 = 1$ and $(Z_2Z_3)^2 = 1$, both have eigenvalues of ± 1.
- The stabilizers define four 2D orthonormal subspaces, the codeword space is one of them.

	$Z_1Z_2 = 1$	$Z_1Z_2 = -1$
$Z_2Z_3 = 1$	$\{\lvert 000\rangle, \lvert 111\rangle\}$	$\{\lvert 100\rangle, \lvert 011\rangle\}$
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For an error to be detected it needs to take the state *out* of the codeword subspace. For $|\psi\rangle_L \in \mathcal{C}$,

$$\{X_1, X_2, X_3\} |\psi\rangle_L \in \mathcal{F}$$
$$\{Z_1, Z_2, Z_3\} |\psi\rangle_L \in \mathcal{C}$$

The three-qubit code is very limited, it can only detect bit flips errors, and not phase flip errors.

The three-qubit code

2. Errors, and 3. Adding ancillary qubits

Errors:

$$\left|\psi\right\rangle_L \to E_L \left|\psi\right\rangle_L$$

$$E_L = E_1 E_2 E_3$$

$$E_{1,2,3} = \alpha_0 1 + \alpha_x X_{1,2,3} + \alpha_z Z_{1,2,3} + \alpha_{xz} X Z_{1,2,3}$$

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Adding ancillary qubits:

$$
\begin{aligned}
E_L |\psi\rangle_L \to \quad & (1 + Z_1 Z_2)(1 + Z_2 Z_3) E_L |\psi\rangle_L \otimes |00\rangle \\
& + (1 + Z_1 Z_2)(1 - Z_2 Z_3) E_L |\psi\rangle_L \otimes |01\rangle \\
& + (1 - Z_1 Z_2)(1 + Z_2 Z_3) E_L |\psi\rangle_L \otimes |10\rangle \\
& + (1 - Z_1 Z_2)(1 - Z_2 Z_3) E_L |\psi\rangle_L \otimes |11\rangle
\end{aligned}
$$

A syndrome measurement *projects* the logical state into one of the four subspaces defined by the stabilizers, (Shor, 1995).

Measuring the errors

4. Syndrome measurement, and 5. Making corrections

To first order in $|\alpha_x|$, $|\alpha_z|$, $|\alpha_{xz}|$ (which we like to think they are not larger numbers as compared to 1):

Syndrome	Possible errors
00	$1, Z_1, Z_2, Z_3$
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- As mentioned, it cannot detect phase flips.
- For X-errors, we can detect which qubit had its value flipped, hence we can perform a unitary transformation that would undo the error.
- Our communication over the noisy channel has improved assuming that errors act independently on each qubit.
- More elaborate code can detect Z-errors while being also more efficient.

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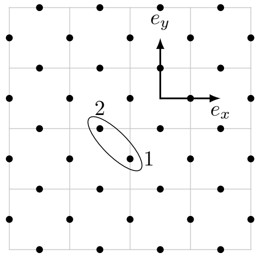
Kitaev Toric Model

- A lattice model of spin-$1/2$ particles.
- Each unit cell has 2 spin sites, 1 and 2.
- Local operators: $\{\vec{\sigma}_1(\boldsymbol{R}_i), \vec{\sigma}_2(\boldsymbol{R}_i)\}$
- Operators at different lattice sites commute.



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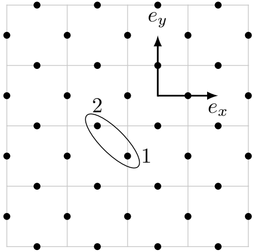

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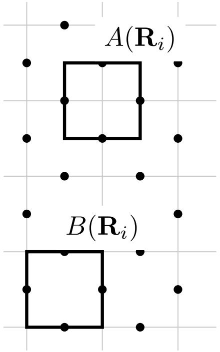
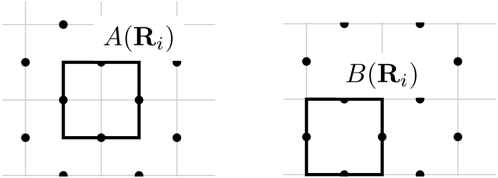
The Hamiltonian:

$$H = -\sum_{\boldsymbol{R}_i} (A(\boldsymbol{R}_i) + B(\boldsymbol{R}_i))$$

$$\begin{aligned} A(\boldsymbol{R}_i) &= \sigma_2^x(\boldsymbol{R}_i)\sigma_1^x(\boldsymbol{R}_i) \\ &\quad \sigma_2^x(\boldsymbol{R}_i + e_x)\sigma_1^x(\boldsymbol{R}_i + e_y), \\ B(\boldsymbol{R}_i) &= \sigma_1^z(\boldsymbol{R}_i)\sigma_2^z(\boldsymbol{R}_i) \\ &\quad \sigma_1^z(\boldsymbol{R}_i - e_x)\sigma_2^z(\boldsymbol{R}_i - e_y) \end{aligned}$$



Ground state of the Kitaev model



Notation:

σ^x: line perpendicular to unit cell edge at the spin site.

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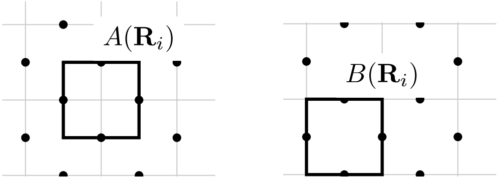

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- $A(\boldsymbol{R}_i)$ and $B(\boldsymbol{R}_i)$ are two different *loops* in the system.
- They only look like loops because of our choice of notation.
- No need for arrows on the loops.
- $A^2(\boldsymbol{R}_i) = 1$ and $B^2(\boldsymbol{R}_i) = 1$. Both have eigenvalues of ± 1.
- $[A(\boldsymbol{R}_i), B(\boldsymbol{R}_i)] = 0$

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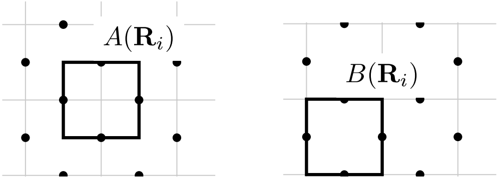

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Ground sates $|\Omega_0\rangle$ of $H = -\sum_{\boldsymbol{R}_i} (A(\boldsymbol{R}_i) + B(\boldsymbol{R}_i))$ is defined by,

$$A(\boldsymbol{R}_i)\,|\Omega_0\rangle = |\Omega_0\rangle\,,\ B(\boldsymbol{R}_i)\,|\Omega_0\rangle = |\Omega_0\rangle$$

The code

We consider a $N \times N$ lattice on a torus.

- The Hilbert space, $\mathcal{H}$, is 2^{2N^2} dimensional.
- Codeword space, $\mathcal{C}$, is defined as

$$\mathcal{C} = \mathsf{span}\{|\Omega_0\rangle \in \mathcal{H} : A(\boldsymbol{R}_i)\,|\Omega_0\rangle = |\Omega_0\rangle\,,\ B(\boldsymbol{R}_i)\,|\Omega_0\rangle = |\Omega_0\rangle\}$$

- This quantum code is called TOR(N), the toric code.
- $A(\boldsymbol{R}_i)$, and $B(\boldsymbol{R}_i)$ are the code stabilizers.

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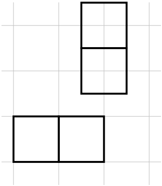
- There are $2N^2 - 2$ independent stabilizers.
  There are N^2 $A(\boldsymbol{R}_i)$, and N^2 $B(\boldsymbol{R}_i)$ operators, but we have the following dependencies,

$$\prod_{\boldsymbol{R}_i} A(\boldsymbol{R}_i) = 1, \prod_{\boldsymbol{R}_i} B(\boldsymbol{R}_i) = 1 \leftarrow \text{No edges left.}$$



- $\mathcal{C}$ is $(2^{2N^2})/(2^{2N^2-2}) = 2^2$ dimensional. It can encode 2 qubits.

What labels the ground states

Since the code stabilizers defines 2^{2N^2-2} 4D subspaces, we must label the 4 states within each subspace by other independent operators that commute with all the $A(\boldsymbol{R}_i)$ and $B(\boldsymbol{R}_i)$.

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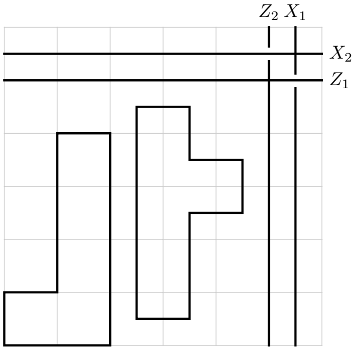
- Every contractible loop can be decomposed into smaller loops of $A(\boldsymbol{R}_i)$ or $B(\boldsymbol{R}_i)$.
- There are 4 different non-contractible loops.
- $\{Z_1, Z_2, X_1, X_2\}$ commute with all contractible loops.
- $\{Z_1, X_1\} = 0$, $\{Z_2, X_2\} = 0$
- The entire 2^{2N^2} Hilbert space can be labeled by the eigenvalues of, $\{Z_1, Z_1, A(\boldsymbol{R}_i), B(\boldsymbol{R}_i)\}$



Errors

A general error can be any linear combination of,

$$E(\{\alpha_i^l, \beta_j^m\}) = \prod_{\substack{\boldsymbol{R}_i, \boldsymbol{R}_i \\ l,m}} (\sigma_l^x(\boldsymbol{R}_i))^{\alpha_i^l} (\sigma_m^z(\boldsymbol{R}_j))^{\beta_j^m},$$

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These can be divided broadly into 3 categories:

1. Contain only closed contractible loops. E_1.
2. Contain one or more open strings. E_2.
3. Contain one or more closed non-contractible loops. E_3.

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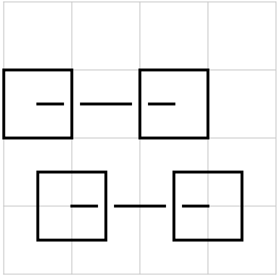
Errors of type 3, must at least be N long. And assuming errors act independently on each qubit, these errors would be exponentially suppressed, $e^{-\alpha N}$

Error detection, and correction

Open string operations anticommute with two stabilizer operators, one surrounding each end of the open ended loop.



$$B(\boldsymbol{R}_i)E_2\left|\Omega_0\right\rangle = -E_2\left|\Omega_0\right\rangle,$$

$$A(\boldsymbol{R}_j)E_2\left|\Omega_0\right\rangle = -E_2\left|\Omega_0\right\rangle.$$

Error detection, and correction

Open string operations anticommute with two stabilizer operators, one surrounding each end of the open ended loop.

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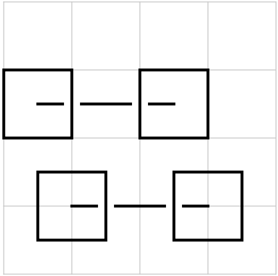

- These errors can be detected using a syndrome measurements.
- Notice that $E_2\left|\Omega_0\right\rangle$ are excited states of the Hamiltonian, with $\Delta E \geqslant 2$.
- Kitaev also suggested fixing these errors by coupling the system to a heat bath and cooling the system down.

Excitations of the toric code

Let's look at low energy excitations of the toric code.

- We cannot have excitation with $\Delta E = 1$. This would violate

$$\prod_{\boldsymbol{R}_i} A(\boldsymbol{R}_i) = 1, \quad \prod_{\boldsymbol{R}_i} B(\boldsymbol{R}_i) = 1.$$

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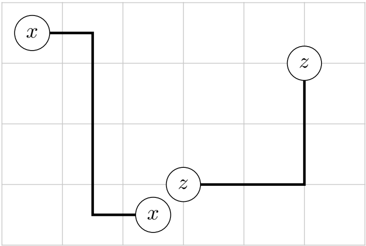
$S^x(t)\left|\Omega_0\right\rangle$, $S^z(t)\left|\Omega_0\right\rangle$, which depend on

1. The two end points
2. The homotopy of string connecting the two ends, how many non-contractible loops it make. Not the detailed path.



Allowed logic operaion using kitaev model

Remember, on a torus geometry, the ground state encode 2 qubits. We can do the following operation in the qubits:

1. Z operation.
   - Create an a z-type particles pair.
   - Move one around one non-contractible loop. The direction determine which qubit get acted on.
   - Annihilate the two particles.
2. X operation. Using the same steps but with an x-type particle.

These operations do not give us a universal quantum computer.

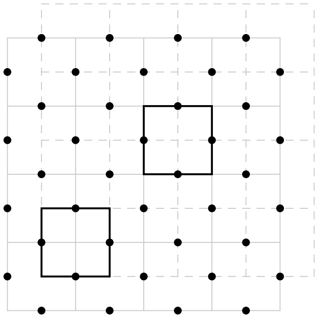
The dual lattice

For the same arrangement of spins there are two ways of defining the lattice. Both of them are equally valid.

- This highlights an important property of the system.

Let $R_y(\theta)$ be the rotation operation around the y-axis then:

$$R_y(90^\circ)A(\boldsymbol{R}_i)R_y^{-1}(90^\circ) = B'(\boldsymbol{R}_i)$$
$$R_y(90^\circ)B(\boldsymbol{R}_i)R_y^{-1}(90^\circ) = A'(\boldsymbol{R}_i)$$



This operation takes an x-type particle to a z-type particle.

Outline

1. Introduction: Noise reduction
   - Classical analog
   - Quantum codes
   - The three-qubit code

2. Kitaev toric code
   - The model
   - The code
   - How to perform logical operations

3. Anyonic nature of the excitations

The toric code in different geomtries

A (surprising) aspect about the toric code is that the ground state degeneracy depends on the genus, g, of the manifold. The toric code is 4^g degenerate.

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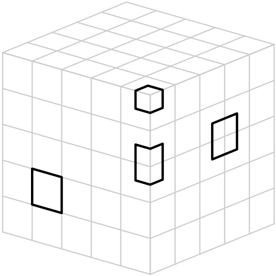
On a *sphere* there are no non-contractible loops. $A(\boldsymbol{R}_i)$ and $B(\boldsymbol{R}_i)$ can label the entire Hilbert space.

- Hilbert space is 2^{12N^2} dimensional.
- $6N^2$ $B(\boldsymbol{R}_i)$ operators.
- $6N^2 + 2$ $A(\boldsymbol{R}_i)$ operators.



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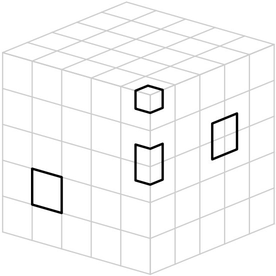

The dependance of the ground state degeneracy on the geometry of the manifold is one of the defining features of topological order.

Particle content of the toric code

- No particles, 1.
- An z-type particle is referred to as electric charge, e.
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- A combinations of an e and an m particle, $\psi = e \times m$

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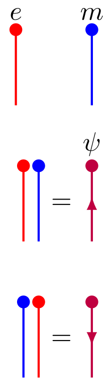
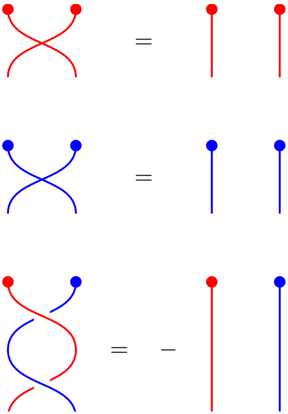
Next we ask what is the statistics of these particles.

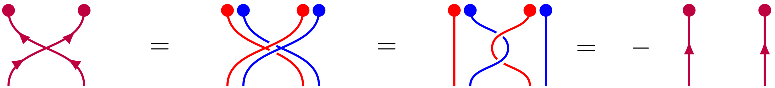


- It's Natural to consider a braid group because of the strings attached to the particles.
- For convince we drop the lattice from the background, and distinguish different strings by different colors.
- There are three kinds of strings. The group is then said to be a colored braid group.

Rules of the braid, and how particles fuse

- The electric charge, e is a boson.
- The magnetic vortex, m is a boson.
- e going around m gives a -1.
- ψ is a fermion.
- The braid group is Abelian.





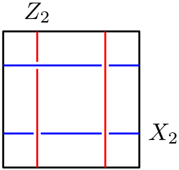
Anyons front and center

Anyons implies the ground state degeneracy.

We want to think of the how the anyons braid, as defining the topological order in the system.

- The ground state(s) is a sate with no particles in it. If $|\Omega_0\rangle$ is a ground state so it $Z_i |\Omega_0\rangle$ and $X_i |\Omega_0\rangle$.

$Z_2 X_2 Z_2 X_2 |\Omega_0\rangle \quad =$



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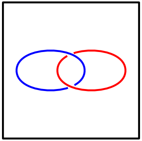

$-|\Omega_0\rangle$

- Braiding rules implies
  1. $Z_i^2 = X_i^2 = 1$
  2. $[Z_1, Z_2] = [X_1, X_2] = [X_1, Z_2] = [Z_1, X_2] = 0$
  3. $\{Z_1, X_1\} = \{Z_2, X_2\} = 0.$
- This implies the ground state is fourfold degenerate.

Summary

- Quantum codes encode k qubits into n qubits.
- Quantum codes allow for error correction
- The Kitaev toric code encode $2g$ qubits into a spin lattice.
- The Kitaev code has $e^{-\alpha N}$ probability of missing errors.
- By moving anyons around, one can perform quantum operations on the qubits.
- The anyon content of a theory is enough to define its topological order.

A.Yu. Kitaev. Fault-tolerant quantum computation by anyons. *Annals of Physics*, 303(1):2–30, Jan 2003. ISSN 00034916. doi: 10.1016/S0003-4916(02)00018-0.

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