Toric Code

University of Florida, August 26, 2020

Recap of last time

- The toric code:

  $$\mathcal{C} = \text{span}\{|\Omega_0\rangle \in \mathcal{H} : A(\boldsymbol{R}_i)\,|\Omega_0\rangle = |\Omega_0\rangle\,, B(\boldsymbol{R}_i)\,|\Omega_0\rangle = |\Omega_0\rangle\}$$

- A *physical* (local) Hamiltonian can have the codeword space as its ground state: $H = -\sum_{\boldsymbol{R}_i}(A(\boldsymbol{R}_i) + B(\boldsymbol{R}_i))$.
- On a torus, the toric code encode 2 qubits.
- The 2 qubits are manipulated using the non-local operators $Z_1, Z_2.X_1, X_2$.
- The toric codes has an $e^{-\alpha N}$ probability of making an error.
- The toric code is said to store the information encoded in the 2 qubits non-locally.
- Excited states: $S^z(t)\,|\Omega_0\rangle$ and $S^x(t)\,|\Omega_0\rangle$. (Open strings.)
- The end of open strings can be thought of as particles. The toric code have an x-type particle and a z-type particle.

Outline

1. Anyonic nature of the excitations

The toric code in different geomtries

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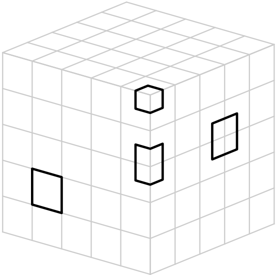
- Hilbert space is 2^{12N^2} dimensional.
- $6N^2\ B(\boldsymbol{R}_i)$ operators.
- $6N^2 + 2\ A(\boldsymbol{R}_i)$ operators.



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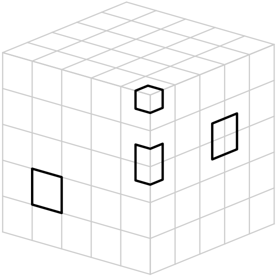
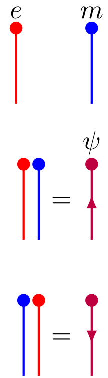


The dependance of the ground state degeneracy on the geometry of the manifold is one of the defining features of topological order.

Particle content of the toric code

- No particles, 1.
- z-type particle, referred to as electric charge, e.
- x-type particle, referred to as a magnetic vortex, m.
- A combinations of an e and an m particle, $\psi = e \times m$
- For convince we drop the lattice from the background, and distinguish different strings by different colors.



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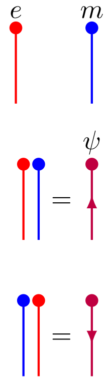

Next we ask what are the statistics of these particles.
That is we ask What happen when we exchange two e, m or ψ particles.

A braid group

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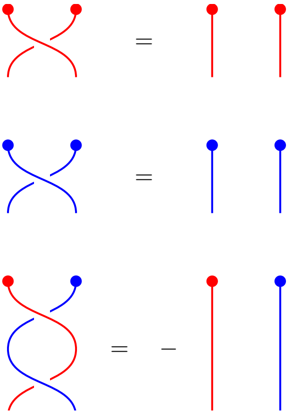
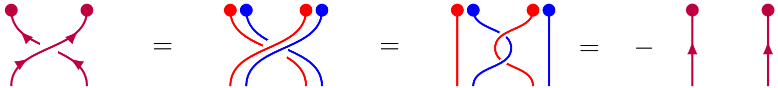
- It's natural to consider a braid group because of the strings attached to the particles.
- There are two kinds of strings. The group is then said to be a colored braid group.
- A state with $2n_e$, and $2n_m$ of e and m particles has n_e, and n_m red and blue strings.
- Strings can go above or under each other depending on which operator act first.

Brading and fusion rules



- The electric charge, e is a boson.
- The magnetic vortex, m is a boson.
- e going around m gives a -1.
- These phases can be explained as Aharonov-Bohm effect
- ψ is a fermion.



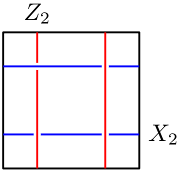
Anyons front and center

Anyons implies the ground state degeneracy.

> We can think of the how the anyons braid, as defining the topological order in the system.

- The ground state(s) is a sate with no particles in it. If $|\Omega_0\rangle$ is a ground state, so are $Z_i\,|\Omega_0\rangle$ and $X_i\,|\Omega_0\rangle$.
- Braiding rules implies
  1. $Z_1^2 = Z_2^2 = X_1^2 = X_2^2 = 1$
  2. $[Z_1, Z_2] = [X_1, X_2] = 0$
  3. $\{Z_1, X_1\} = \{Z_2, X_2\} = 0.$
- This implies the ground state is fourfold degenerate.

$Z_2X_2Z_2X_2\,|\Omega_0\rangle \quad =$



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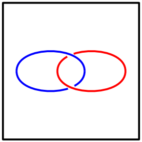

$-\,|\Omega_0\rangle$

Summary

- Quantum codes encode k qubits into n qubits.
- Quantum codes allow for error correction
- The Kitaev toric code encode $2g$ qubits into a spin lattice.
- The Kitaev code has $e^{-\alpha N}$ probability of missing errors.
- Toric code encodes the information non-locally.
- The anyon content of a theory is enough to define its topological order.