PDE. Character 1.

# 第一章. 引言

## §1.1. 基本概念

$\boxed{\text{def1.}}$ 一个 PDE 是一个未知的多元函数及其偏导数有关的方程　e.g. $\Omega \subset R^n$, $u: \Omega \to R$

for $k \in Z^+$, $D^k u$ 表示 u 所有 k 阶偏导数 $\Rightarrow \dfrac{\partial^k u}{\partial x_{i_1} \cdots \partial x_{i_k}}$　($i_1 \cdots i_k$ 从 $1 \sim n$ 中任取)

① $k=1$. $Du = \left(\dfrac{\partial u}{\partial x_1} \cdots \dfrac{\partial u}{\partial x_n}\right) \overset{def}{=} \nabla u$ 梯度　$\boxed{Du = \nabla u = \text{grad}\, u}$

② $k=2$ $D^2 u = \begin{pmatrix} \dfrac{\partial^2 u}{\partial x_1^2} & & \dfrac{\partial^2 u}{\partial x_1 \partial x_n} \\ \vdots & \cdots & \\ \dfrac{\partial^2 u}{\partial x_n \partial x_1} & & \dfrac{\partial^2 u}{\partial x_n^2} \end{pmatrix} \Rightarrow$ 海瑟矩阵　$\Delta u = tr(D^2 u)$ Laplace

③ $\vec{F}: \Omega \to R^n$　$\vec{F} = (F_1 \cdots F_n)$　散度 $\text{div}\vec{F} = \dfrac{\partial F_1}{\partial x_1} + \cdots \dfrac{\partial F_n}{\partial x_n} = \nabla \cdot \vec{F}$

$\therefore \Delta u = \text{div}(\nabla u) = \nabla \cdot (\nabla u)$
$(tr(D^2 u))$

在 $C^2(\Omega)$ 中. $\dfrac{\partial^2 u}{\partial x_i \partial x_j} = \dfrac{\partial^2 u}{\partial x_j \partial x_i}$

$\boxed{\text{def2}}$ $F(D^k u, D^{k-1} u \cdots Du, u, x) = 0$ . $(x \in \Omega$, $F: R^{n^k} \times R^{n^{k-1}} \cdots R \times \Omega \to R)$　(1.1)

称为 k 阶 PDE，$u: \Omega \to R$ 为未知多元函数

$\alpha = (\alpha_1 \cdots \alpha_n)$　$\alpha_1 \cdots \alpha_n \in Z^+ \cup \{0\}$ ,　$|\alpha| = \alpha_1 + \alpha_2 + \cdots \alpha_n = k$

$\Rightarrow D^\alpha u = \dfrac{\partial^{|\alpha|} u}{\partial x_1^{\alpha_1} \cdots \partial x_n^{\alpha_n}}$　$(u \in C^k(\Omega))$ → 古典解 classic. solutions

若 (1.1) 可表示为: ① $\sum\limits_{|\alpha|=0}^{k} a_\alpha(x) D^\alpha u = f(x) \Rightarrow$ $a_\alpha(x)$ 为已知 则称其为线性PDE

② $\sum\limits_{|\alpha|=k} a_\alpha(x) D^\alpha u = f(D^{k-1}u \cdots Du, u, x) \Rightarrow$ 称其为半线性PDE

③ $\sum\limits_{|\alpha|=k} a_\alpha(D^{k-1}u \cdots Du, u, x) D^\alpha u = f(D^{k-1}u \cdots Du, u, x) \Rightarrow$ 拟线性PDE

④ 非线性依赖于 k 阶偏导数 ⇒ 完全非线性PDE

## §1.2. 实例.

A. 线性PDE：　位势方程　$-\Delta u = f(x)$

特征值~：$\Delta u + \lambda u = 0$ ,　$\lambda > 0$

热~ ：$u_t - a^2 \Delta u = f(x,t)$　$u = u(x,t)$. $x \in \Omega$. $t > 0$

Schrodinger ~：$u_t - \Delta u = 0$　…

其余参皆可见PDE教材

较为特殊 Maxwell 方程组：

$$\begin{cases} \vec{E}_t = c \cdot \mathrm{curl}\, \vec{B} \\ \vec{B}_t = -c \cdot \mathrm{curl}\, \vec{E} \\ \mathrm{div}\, \vec{E} = \mathrm{div}\, \vec{B} = 0 \end{cases}$$

$\vec{E} = (E_1, E_2, E_3)$

$\vec{B} = (B_1, B_2, B_3)$

$\mathrm{curl}\, \vec{B} = \nabla \times B$

$\mathrm{curl}\, \vec{E} = \nabla \times E \rightleftharpoons \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ E_1 & E_2 & E_3 \end{vmatrix}$

常用等式

三大类：

(1) 位势方程：$-\Delta u = f(x)$

(2) 热方程：$u_t - a^2 \Delta u = f(x)$

(3) 波动方程：$u_{tt} - a^2 \Delta u = f(x)$

$$\underbrace{\sum_{i,j=1}^{m} a_{ij}(x) u_{x_i x_j}}_{A_{m\times m}} + \sum_{i=1}^{m} b_i(x) u_{x_i} + c(x) + u = f(x)$$

① $m = n$ $\quad a_{ij} = -\delta_{ij} = \begin{cases} -1, & i = j \\ 0, & i \neq j \end{cases} \Rightarrow A = -I$

$b_i = 0$

$c = 0$

② $m = n+1$, $x_{n+1} = t$ $\Rightarrow A = \begin{pmatrix} -1 & & & \\ & -1 & & \\ & & \ddots & \\ & & & -1 \\ & & & & 0 \end{pmatrix}$ $\quad b_{n+1} = 1$, $b_i = 0$, $c = 0$

③ $m = n+1$, $x_{n+1} = t$ $\Rightarrow A = \begin{pmatrix} -a^2 & & & \\ & \ddots & & \\ & & -a^2 & \\ & & & 1 \end{pmatrix}$ $\quad b_i = 0$. $c = 0$

def A特征值

$$\begin{cases} < 0 & \text{椭圆型} \\ (m-1)\text{负} + 0 & \text{抛物型} \\ (m-1)\text{负} + \text{正} & \text{双曲型} \end{cases}$$

$\Leftarrow$ 可化为标准型

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