有限元方法简介

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两点边值问题

求解如下方程

$$
\begin{cases} -u''(x) = f(x), x \in \Omega := (0,1) \\ u(0) = u(1) = 0 \end{cases} \tag{1}
$$

弱导数: 对于函数$v(x)$, 若存在$v_g(\int_a^b |v_g| dx < \infty)$, 满足

$$
\int_a^b v\varphi'(x)dx = -\int_a^b v_g \varphi(x)dx, \forall \varphi(x) \in C_0^\infty(a,b)
$$

则称$v_g(x)$是$v(x)$的弱导数, 也称为广义导数, 记为$v'(x)$.
例:

$$
v(x) = \begin{cases} x, & 0 \leqslant x \leqslant 1; \\ -x, & -1 \leqslant x < 0. \end{cases} \qquad \text{弱导数: } v_g(x) = \begin{cases} 1, & 0 \leqslant x \leqslant 1; \\ -1, & -1 \leqslant x < 0. \end{cases}
$$

$$
H_0^1(\Omega) := \left\{ v : \int_\Omega v^2 dx < \infty, \int_\Omega v_g^2 dx < \infty, v(0) = v(1) = 0 \right\}.
$$

定义

$$\varphi(t) = \begin{cases} 2t, & 0 \leqslant t \leqslant \frac{1}{2}, \\ 1, & t > \frac{1}{2}. \end{cases}$$

对$v(x) \in H_0^1(\Omega)$, 令$w(x) = v(x)\varphi(x)$, 于是, 有

$$\begin{aligned} w(x) &= w(0) + \int_0^x w'(t)dt \\ &= \int_0^x [v'(t)\varphi(t) + v(t)\varphi'(t)]dt \\ &\leqslant \int_0^1 |v'(t)|dt + 2\int_0^1 |v(t)|dt \\ &\leqslant C\|v\|_{H^{1,1}}(\Omega) \end{aligned}$$

因此, $\forall x \geqslant \frac{1}{2}$, 有

$$v(x) = w(x) \leqslant C\|v\|_{H^{1,1}(\Omega)}.$$

同理可证明$\forall 0 \leqslant x < \frac{1}{2}$, 有

$$v(x) \leqslant C\|v\|_{H^{1,1}(\Omega)}$$

对$\forall v(x) \in H_0^1(\Omega), x, y \in (0,1)$, 有

$$|v(x) - v(y)| = |\int_x^y v'(t)dt| \leqslant |x-y|^{1/2}(\int_a^b |v'(t)|^2 dt)^{1/2}$$

因此$v(x)$ Holder连续.

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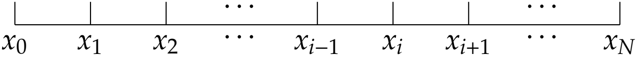
在方程(1)的两端同乘以$v(x) \in H_0^1(\Omega)$, 有

$$-\int_0^1 u''(x)v(x)dx = \int_0^1 f(x)v(x)dx$$

$$\int_0^1 u'(x)v'(x)dx = \int_0^1 f(x)v(x)dx \tag{2}$$

问题(2)是问题(1)的变分问题. 将区间$[0,1]$进行剖分:

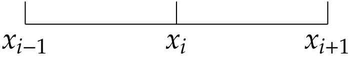
Figure



$x_i, i = 0, 1, 2, \cdots, N$称为剖分的节点, $[x_i, x_{i+1}]$称为单元.
在内节点x_i上, 令$h_i = x_i - x_{i-1}$, 定义

$$\varphi_i(x) = \begin{cases} \frac{x - x_{i-1}}{h_i}, & x \in [x_{i-1}, x_i] \\ \frac{x_{i+1} - x}{h_{i+1}}, & x \in [x_i, x_{i+1}] \\ 0, & \text{其他} \end{cases}$$

Figure



定义: $V_h=\left\{v_h : v_h=\sum_{i=1}^{N-1} v_i\varphi_i(x), v_i\in R\right\}$

有限元问题: 求$u_h=\sum_{j=1}^{N-1} u_j\varphi_j(x)$, 使得

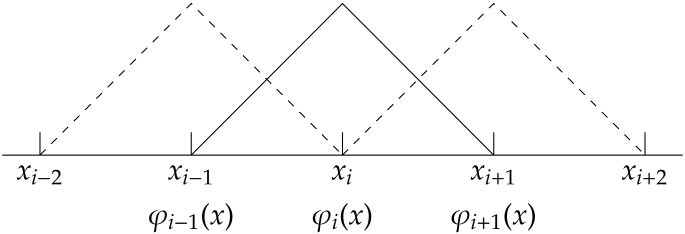
$$\int_0^1 u_h'(x)v_h'(x)dx=\int_0^1 v_h(x)f(x)dx, \forall v_h(x)\in V_h. \qquad (3)$$

等价的问题: $\int_0^1 u_h'(x)\varphi_i'(x)dx=\int_0^1 \varphi_i f(x)dx, i=1,2,\cdots,N-1$, 也即是

$$\sum_{j=1}^{N-1} u_j\int_0^1 \varphi_i'(x)\varphi_j'(x)dx=\int_0^1 f(x)\varphi_i(x)dx$$

只有$j=i-1,i,i+1$时, φ_j与φ_i有共同的支集.

Figure



故有

$$\begin{aligned} u_{i-1}\int_0^1 \varphi'_{i-1}(x)\varphi'_i(x)dx + u_i\int_0^1 \varphi'_i(x)\varphi'_i(x)dx + u_{i+1}\int_0^1 \varphi'_{i+1}(x)\varphi'_i(x)dx \\ = \int_0^1 f\varphi_i(x)dx, \text{令} f_i = \int_0^1 f\varphi'(x)dx \end{aligned}$$

直接计算, 得

$$\begin{aligned}&\int_0^1 \varphi'_{i-1}(x)\varphi'_i(x)dx\\ &= \int_{x_{i-1}}^{x_i} \varphi'_{i-1}(x)\varphi'_i(x)dx = \int_{x_{i-1}}^{x_i}\left(-\frac{1}{h_i}\right)\left(\frac{1}{h_i}\right)dx = -\frac{1}{h_i}\end{aligned}$$

$$\int_0^1 \varphi'_{i+1}(x)\varphi'_i(x)dx = \int_{x_i}^{x_{i+1}}\left(-\frac{1}{h_{i+1}}\right)\left(\frac{1}{h_{i+1}}\right)dx = -\frac{1}{h_{i+1}} \quad .$$

$$\begin{aligned}\int_0^1 \varphi'_i(x)\varphi'_i(x)dx &= \int_{x_{i-1}}^{x_i} \varphi'_i(x)\varphi'_i(x)dx + \int_{x_i}^{x_{i+1}} \varphi'_i(x)\varphi'_i(x)dx\\ &= \frac{1}{h_i} + \frac{1}{h_{i+1}}\end{aligned}$$

即

$$-\frac{u_{i-1}}{h_i} + \left(\frac{1}{h_i} + \frac{1}{h_{i+1}}\right)u_i - \frac{u_{i+1}}{h_{i+1}} = f_i, \quad i = 1, 2, \cdots, N-1. \qquad (4)$$

令$a_i = \frac{1}{h_i} + \frac{1}{h_{i+1}}, i = 1, 2, \cdots N-2$, 并令

$$A_h = \begin{pmatrix} a_1 & -\frac{1}{h_2} & & & & \\ -\frac{1}{h_2} & a_2 & -\frac{1}{h_3} & & & \\ & -\frac{1}{h_3} & a_3 & -\frac{1}{h_4} & & \\ & & \ddots & \ddots & \ddots & \\ & & & -\frac{1}{h_{N-2}} & a_{N-2} & -\frac{1}{h_{N-1}} \\ & & & & -\frac{1}{h_{N-1}} & a_{N-1} \end{pmatrix}$$

与$U_h = (u_1, u_2, \cdots, u_{N-1})^T, \quad F_h = (f_1, f_2, \cdots, f_{N-1})^T$. (4)的矩阵形式为:

$$A_h U_h = F_h.$$

$$f_i = \int_{x_{i-1}}^{x_{i+1}} \phi_i(x) f(x) dx \approx \frac{h_i f(x_{i-1})}{6} + \frac{h_i f(x_i)}{3} + \frac{h_{i+1} f(x_i)}{3} + \frac{h_{i+1} f(x_{i+1})}{6}.$$

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有如下关系式:

$$\int_0^1 u'v'dx = \int_0^1 vfdx$$

$$\int_0^1 u'v_h'(x)dx = \int_0^1 v_h f dx \quad \forall v_n \in V_n$$

$$\int_0^1 u_h'(x)v_h'(x)dx = \int_0^1 v_h(x) f dx \quad \forall v_h \in V_h$$

可得正交性:

$$\int_0^1 (u' - u_h')v_h' dx = 0$$

故$\forall w_h \in V_h$, 有

$$\begin{aligned}
&\int_0^1 (u' - u_h')^2 dx \\
&= \int_0^1 (u' - u_h')(u' - w_h')dx + \int_0^1 (u' - u_h')(w_h' - u_h')dx \\
&\qquad\qquad\qquad\qquad\qquad\qquad (= 0, \text{令} v_h = w_h - u_h \text{即可得}) \\
&= \int_0^1 (u' - u_h')(u' - w_h')dx \\
&\leqslant \left(\int_0^1 (u' - u_h')^2 dx\right)^{1/2} \left(\int_0^1 (u' - w_h')^2 dx\right)^{1/2}
\end{aligned}$$

即有: $\int_0^1 (u' - u_h')^2 dx \leqslant \int_0^1 (u' - w_h')^2 dx$, 任对$w_h \in V_h$成立, 也即是

$$\int_0^1 (u' - u_h')^2 dx \leqslant \inf_{w_h \in V_h} \int_0^1 (u' - w_h')^2 dx$$

定义插值函数$I_h u \in V_h$使得

$$(I_h u)(x_i) = u(x_i), i = 1, 2, \cdots, N$$

变点展开技术: 在单元$[x_i, x_{i+1}]$上, 有:

$$u(x) - I_h u(x) = u(x) - u(x_i)\varphi_i(x) - u(x_{i+1})\varphi_{i+1}(x)$$

展开

$$u(x_{i+1}) = u(x) + u'(x)(x_{i+1} - x) + \int_x^{x_{i+1}} u''(t)(x_{i+1} - t)dt$$

$$u(x_i) = u(x) + u'(x)(x_i - x) + \int_x^{x_i} u''(t)(x_i - t)dt$$

这样

$$
\begin{aligned}
u(x) - I_h u(x) = u(x) &- \left(u(x) + u'(x)(x_i - x)\right) \varphi_i(x) \\
&- \left(u(x) + u'(x)(x_{i+1} - x)\right) \varphi_{i+1}(x) \\
&- \int_x^{x_i} u''(t)(x_i - t)dt \; \varphi_i(x) \\
&- \int_x^{x_{i+1}} u''(t)(x_{i+1} - t)dt \; \varphi_{i+1}(x) \\
= -u'(x) &\left((x_i - x)\varphi_i(x) + (x_{i+1} - x)\, \varphi_{i+1}(x)\right) \\
&- \int_x^{x_i} u''(t)(x_i - t)dt \; \varphi_i(x) \\
&- \int_x^{x_{i+1}} u''(t)(x_{i+1} - t)dt \; \varphi_{i+1}(x) \\
= -&\int_x^{x_i} u''(t)(x_i - t)dt \; \varphi_i(x) \\
&- \int_x^{x_{i+1}} u''(t)(x_{i+1} - t)dt \; \varphi_{i+1}(x)
\end{aligned}
$$

于是:

$$
\begin{aligned}
|u(x) - I_h u(x)| \leqslant & \left(\int_{x_i}^{x_{i+1}} (u''(t))^2 dt\right)^{1/2} \left(\int_{x_i}^{x_{i+1}} (x_i - t)^2 dt\right)^{1/2} \varphi_i(x) \\
& + \left(\int_{x_i}^{x_{i+1}} (u''(t))^2 dt\right)^{1/2} \left(\int_{x_i}^{x_{i+1}} (x_{i+1} - t)^2 dt\right)^{1/2} \varphi_{i+1}(x) \\
& = \frac{1}{\sqrt{3}} \left(\int_{x_i}^{x_{i+1}} (u''(t))^2 dt\right)^{1/2} h_{i+1}^{3/2}
\end{aligned}
$$

于是

$$
\begin{aligned}
\sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1}} (u(x) - I_h u(x))^2 dx & \leqslant \frac{1}{3} \sum_{i=0}^{N-1} \int_{x_i}^{x_{i+1}} (u''(x))^2 dx \, h_{i+1}^4 \\
& = \frac{1}{3} \sum_{i=0}^{N-1} h_{i+1}^4 \int_{x_i}^{x_{i+1}} (u''(x))^2 dx
\end{aligned}
$$

另一方面,

$$
\begin{aligned}
&(u(x) - I_h u(x))' = u'(x) + \frac{u(x_i)}{h_{i+1}} - \frac{u(x_{i+1})}{h_{i+1}} \\
&= u'(x) + \frac{u(x) + u'(x)(x_i - x)}{h_{i+1}} - \frac{u(x) + u'(x)(x_{i+1} - x)}{h_{i+1}} \\
&+ \frac{1}{h_{i+1}} \int_x^{x_i} u''(t)(x_i - t)\,dt - \frac{1}{h_{i+1}} \int_x^{x_{i+1}} u''(t)(x_{i+1} - t)\,dt \\
&= \frac{1}{h_{i+1}} \left(\int_x^{x_i} u''(t)(x_i - t)\,dt - \int_x^{x_{i+1}} u''(t)(x_{i+1} - t)\,dt \right)
\end{aligned}
$$

于是

$$|(u(x) - I_h u(x))'| \leqslant \frac{2}{\sqrt{3} h_{i+1}} \left(\int_{x_i}^{x_{i+1}} (u''(x))^2 dx \right)^{1/2} h_{i+1}^{3/2}$$

这样:

$$\int_0^1 \left((u(x) - I_h u(x))' \right)^2 dx \leq \frac{4}{3} \sum_{i=0}^{N-1} h_{i+1}^2 \int_{x_i}^{x_{i+1}} (u''(x))^2 dx$$

有如下后验误差估计结果：问题(3)的有限元解u_h满足如下估计

$$\|\frac{d(u-u_h)}{dx}\|^2_{L^2(\Omega)} \le C\sum_{i=1}^{n}\eta_i^2(u_h) \tag{5}$$

其中

$$\eta_i(u_h) = h_i\|f + \frac{d^2u_h}{dx^2}\|_{L^2(\Omega_i)}$$

证明：

令$e=u-u_h$为误差，定义插值算子$\Pi: V_0 \to V_{h,0}$为连续分片线性插值, 有

$$
\begin{aligned}
\|\frac{de}{dx}\|^2_{L^2(\Omega)} &= \int_\Omega (\frac{de}{dx})^2\, dx \\
&= \int_\Omega \frac{de}{dx}\frac{d(e-\Pi e)}{dx}\, dx \\
&= \sum_{i=1}^n \int_{x_{i-1}}^{x_i} \frac{de}{dx}\frac{d(e-\Pi e)}{dx}\, dx \\
&= \sum_{i=1}^n \int_{x_{i-1}}^{x_i} (-\frac{d^2e}{dx^2})(e-\Pi e)\, dx + [\frac{de}{dx}(e-\Pi e)]|_{x_{i-1}}^{x_i} \\
&= \sum_{i=1}^n \int_{x_{i-1}}^{x_i} (-\frac{d^2e}{dx^2})(e-\Pi e)\, dx
\end{aligned}
$$

因为$-\frac{d^2e}{dx^2} = -\frac{d^2(u-u_h)}{dx^2} = -\frac{d^2u}{dx^2} + \frac{d^2u_h}{dx^2} = f + \frac{d^2u_h}{dx^2}$

又由Cauchy-Schwarz不等式，以及标准误差估计，有

$$
\begin{aligned}
\|\frac{de}{dx}\|^2_{L^2(\Omega)} &= \sum_{i=1}^{n}\int_{x_{i-1}}^{x_i}(f+\frac{d^2u_h}{dx^2})(e-\Pi e)\,dx \\
&\le \sum_{i=1}^{n}\|f+\frac{d^2u}{dx^2}\|_{L^2(\Omega_i)}\|e-\Pi e\|_{L^2(\Omega_i)} \\
&\le \sum_{i=1}^{n}\|f+\frac{d^2u}{dx^2}\|_{L^2(\Omega_i)}Ch_i\|\frac{de}{dx}\|_{L^2(\Omega_i)} \\
&= C\sum_{i=1}^{n}h_i\|f+\frac{d^2u_h}{dx^2}\|_{L^2(\Omega_i)}\|\frac{de}{dx}\|_{L^2(\Omega_i)} \\
&\le C(\sum_{i=1}^{n}h_i^2\|f+\frac{d^2u_h}{dx^2}\|^2_{L^2(\Omega_i)})^{1/2}(\sum_{i=1}^{n}\|\frac{de}{dx}\|^2_{L^2(\Omega_i)})^{1/2} \\
&= C(\sum_{i=1}^{n}h_i^2\|f+\frac{d^2u_h}{dx^2}\|^2_{L^2(\Omega_i)})^{1/2}\|\frac{de}{dx}\|_{L^2(\Omega)} \quad \text{证毕}
\end{aligned}
$$

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$$
\begin{cases}
-u'' = f & \Omega = (0, L) \\
u'(0) = \kappa_0(u(0) - g_0) \\
u'(L) = -\kappa_L(u(L) - g_L)
\end{cases}
\tag{6}
$$

其中$L = 1$.

在上述问题中取$\kappa_0 = 10^6, \kappa_1 = 0, g_0 = 0$, 取$f(x) = e^{-100(x-0.5)^2}$, 网格情况如下:

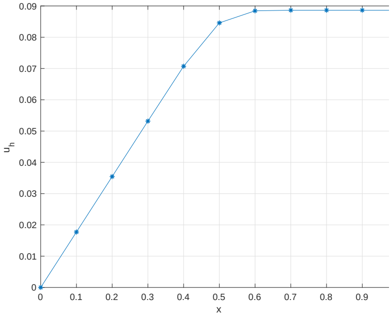

Figure: 均匀剖分10

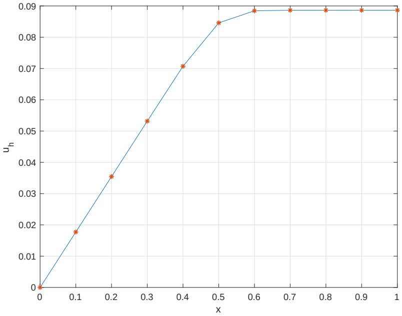

Figure: 自适应初始网格10

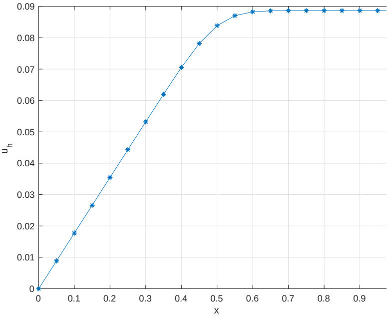

Figure: 均匀剖分20

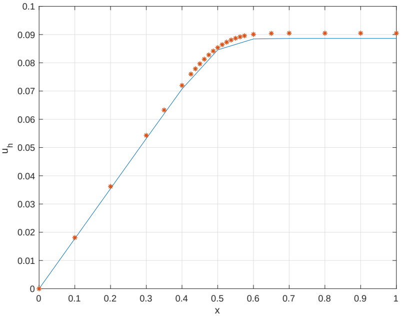

Figure: 自适应网格20

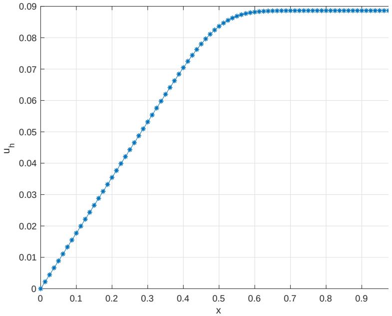

Figure: 均匀剖分80

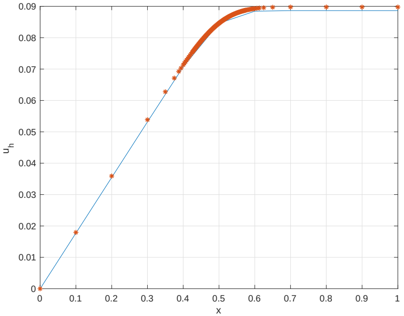

Figure: 自适应网格80

在上述问题中取$\kappa_0 = 10^6, \kappa_1 = 10^5, g_0 = 0, g_L = 0$,
取$f(x) = e^{-100(x-0.5)^2}$, 网格情况如下:

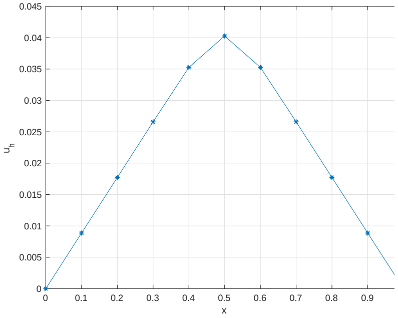

Figure: 均匀剖分10

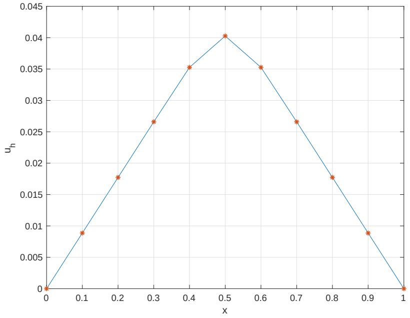

Figure: 自适应初始网格10

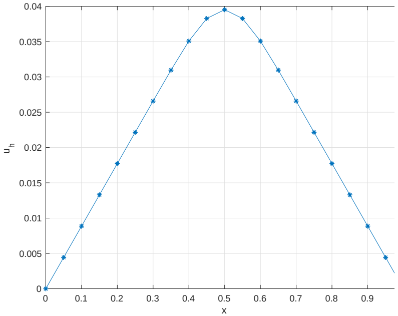

Figure: 均匀剖分20

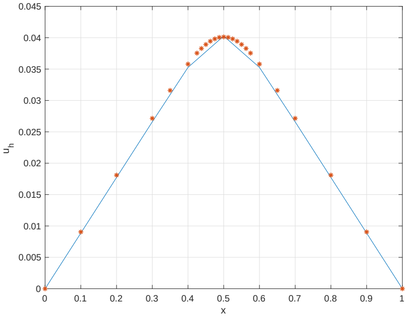

Figure: 自适应网格20

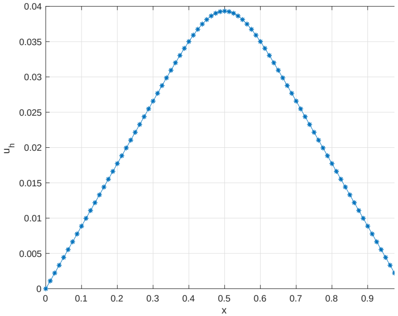

Figure: 均匀剖分80

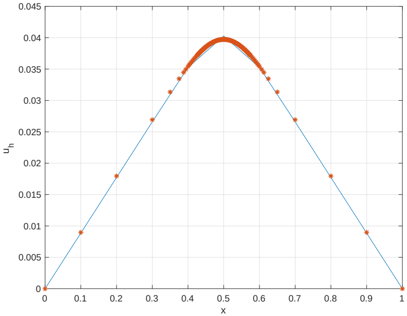

Figure: 自适应网格80

在上述问题中取$\kappa_0 = 10^6, \kappa_1 = 0, g_0 = -1$, 取$f(x) = 0.03(x-6)^4$, 网格情况如下

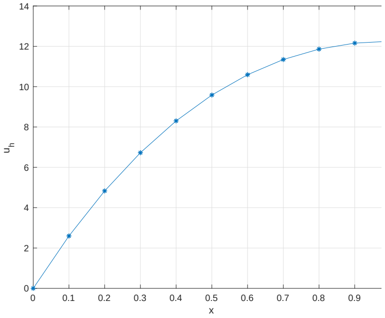

Figure: 均匀剖分10

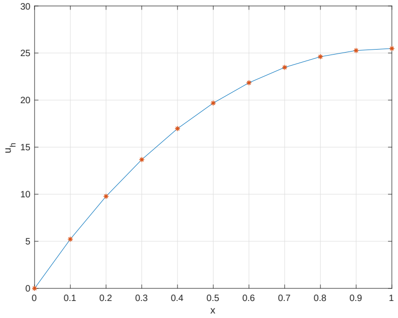

Figure: 自适应初始网格10

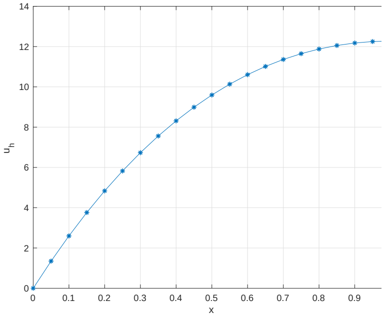

Figure: 均匀剖分20

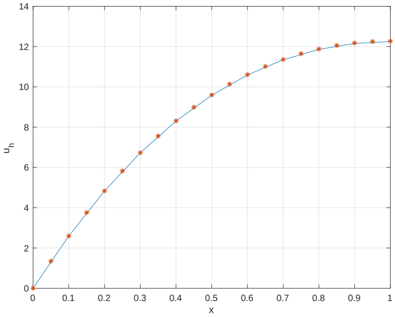

Figure: 自适应网格20

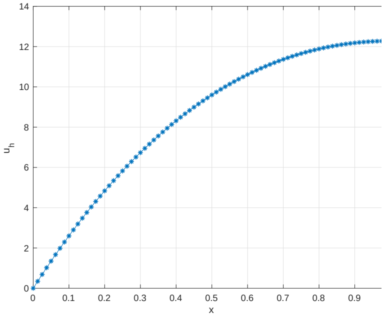

Figure: 均匀剖分80

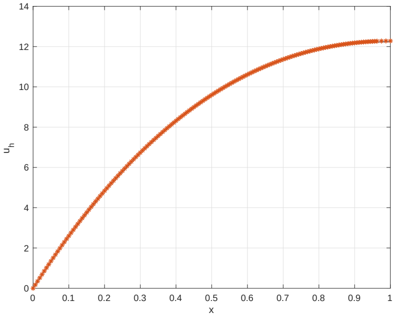

Figure: 自适应网格80

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二维泊松问题

$$\begin{cases} -\Delta u = f(x,y), & x \in \Omega \\ u|_{\partial\Omega} = 0 \end{cases} \tag{7}$$

定义:

$$H_0^1(\Omega) = \left\{ v : \int_\Omega v^2 dxdy + \int_\Omega |\nabla v|^2 dxdy < \infty \right\}$$

在(7)两边同时乘以$v(x,y)$, 分步积分有:

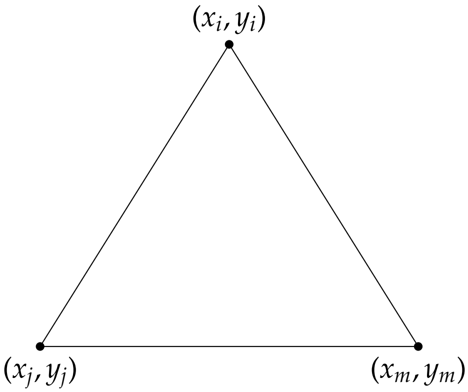
$$\int_\Omega -\Delta u v dxdy = \int_\Omega \nabla u \nabla v dxdy$$

变分问题: 求$u(x,y) \in H_0^1(\Omega)$, 使得

$$\int_\Omega \nabla u \nabla v dxdy = \int_\Omega f(x,y) v(x,y) dxdy, \forall v \in H_0^1(\Omega)$$

重心坐标

Figure



求线性函数$u = ax + by + c$, 使得

$$\begin{cases} ax_i + by_i + c = u_i \\ ax_j + by_j + c = u_j \\ ax_m + by_m + c = u_m \end{cases}$$

这样有:

$$a = \frac{\begin{vmatrix} u_i & y_i & 1 \\ u_j & y_j & 1 \\ u_m & y_m & 1 \end{vmatrix}}{\begin{vmatrix} x_i & y_i & 1 \\ x_j & y_j & 1 \\ x_m & y_m & 1 \end{vmatrix}}, \quad b = \frac{\begin{vmatrix} x_i & u_i & 1 \\ x_j & u_j & 1 \\ x_m & u_m & 1 \end{vmatrix}}{\begin{vmatrix} x_i & y_i & 1 \\ x_j & y_j & 1 \\ x_m & y_m & 1 \end{vmatrix}}, \quad c = \frac{\begin{vmatrix} x_i & y_i & u_i \\ x_j & y_j & u_j \\ x_m & y_m & u_m \end{vmatrix}}{\begin{vmatrix} x_i & y_i & 1 \\ x_j & y_j & 1 \\ x_m & y_m & 1 \end{vmatrix}}$$

也即是

$$
\begin{aligned}
a &= \frac{1}{2\Delta_e}\left[\begin{vmatrix} y_j & 1 \\ y_m & 1 \end{vmatrix} u_i + \begin{vmatrix} y_m & 1 \\ y_i & 1 \end{vmatrix} u_j + \begin{vmatrix} y_i & 1 \\ y_j & 1 \end{vmatrix} u_m\right] \\
b &= \frac{1}{2\Delta_e}\left[-\begin{vmatrix} x_j & 1 \\ x_m & 1 \end{vmatrix} u_i - \begin{vmatrix} x_m & 1 \\ x_i & 1 \end{vmatrix} u_j - \begin{vmatrix} x_i & 1 \\ x_j & 1 \end{vmatrix} u_m\right] \\
c &= \frac{1}{2\Delta_e}\left[\begin{vmatrix} x_j & y_j \\ x_m & y_m \end{vmatrix} u_i + \begin{vmatrix} x_m & y_m \\ x_i & y_i \end{vmatrix} u_j + \begin{vmatrix} x_i & y_i \\ x_j & y_j \end{vmatrix} u_m\right]
\end{aligned}
$$

其中: $2\Delta_e = \begin{vmatrix} x_i & y_i & 1 \\ x_j & y_j & 1 \\ x_m & y_m & 1 \end{vmatrix}.$

定义:

$$\lambda_i = \frac{1}{2\Delta_e}\left[\begin{vmatrix} y_j & 1 \\ y_m & 1 \end{vmatrix} x - \begin{vmatrix} x_j & 1 \\ x_m & 1 \end{vmatrix} y + \begin{vmatrix} x_j & y_j \\ x_m & y_m \end{vmatrix}\right]$$

$$\lambda_j = \frac{1}{2\Delta_e}\left[\begin{vmatrix} y_m & 1 \\ y_i & 1 \end{vmatrix} x - \begin{vmatrix} x_m & 1 \\ x_i & 1 \end{vmatrix} y + \begin{vmatrix} x_m & y_m \\ x_i & y_i \end{vmatrix}\right]$$

$$\lambda_m = \frac{1}{2\Delta_e}\left[\begin{vmatrix} y_i & 1 \\ y_j & 1 \end{vmatrix} x - \begin{vmatrix} \Delta x_i & 1 \\ x_j & 1 \end{vmatrix} y + \begin{vmatrix} x_i & y_i \\ x_j & y_j \end{vmatrix}\right]$$

则有: $u = u_i\lambda_i + u_j\lambda_j + u_m\lambda_m$. 同时也即是

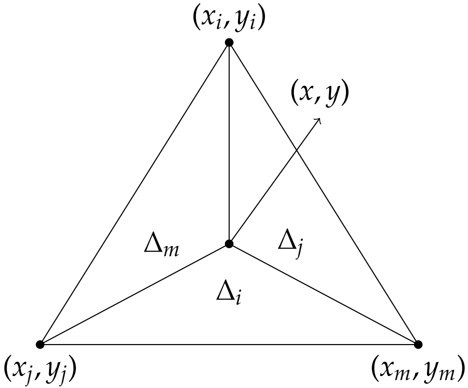
$$\lambda_i = \frac{2\Delta_i}{2\Delta_e}, \lambda_j = \frac{2\Delta_j}{2\Delta_e}, \lambda_m = \frac{2\Delta_m}{2\Delta_e}$$

其中

$$2\Delta_i = \begin{vmatrix} x & y & 1 \\ x_j & y_j & 1 \\ x_m & y_m & 1 \end{vmatrix}, \quad 2\Delta_j = \begin{vmatrix} x & y & 1 \\ x_m & y_m & 1 \\ x_i & y_i & 1 \end{vmatrix}, \quad 2\Delta_m = \begin{vmatrix} x & y & 1 \\ x_i & y_i & 1 \\ x_j & y_j & 1 \end{vmatrix}$$

为下图所示区域面积

Figure



满足$\lambda_i((x_k, y_k)) = \delta_{ik} = \begin{cases} 1, i = k \\ 0, i \neq k \end{cases}$.

同时还有

$$\frac{\partial\lambda_i}{\partial x}=\frac{1}{2\Delta_e}\left(y_j-y_m\right),\quad \frac{\partial\lambda_i}{\partial y}=\frac{x_m-x_j}{2\Delta_e}$$
$$\frac{\partial\lambda_j}{\partial x}=\frac{1}{2\Delta e}\left(y_m-y_i\right),\quad \frac{\partial\lambda_j}{\partial y}=\frac{x_i-x_m}{2\Delta e}$$
$$\frac{\partial\lambda_m}{\partial x}=\frac{y_i-y_j}{2\Delta e},\quad \frac{\partial\lambda_m}{\partial y}=\frac{x_j-x_i}{2\Delta_i}$$

与

$$\begin{aligned}1&=\lambda_i+\lambda_j+\lambda_m\\ x&=x_i\lambda_i+x_j\lambda_j+x_m\lambda_m\\ y&=y_i\lambda_i+y_j\lambda_j+y_m\lambda m\end{aligned}$$

这样就有

$$\begin{aligned}x&=\left(x_i-x_m\right)\lambda_i+\left(x_j-x_m\right)\lambda_j+x_m\\ y&=\left(y_i-x_m\right)\lambda_i+\left(y_j-y_m\right)\lambda_j+y_m\end{aligned}$$

而且

$$\left|\frac{\partial(x,y)}{\partial\left(\lambda_i,\lambda_j\right)}\right|=\begin{vmatrix}x_i-x_m & x_j-x_m\\ y_i-y_m & y_j-y_m\end{vmatrix}=2\Delta_e$$

有限元空间与有限元问题

令

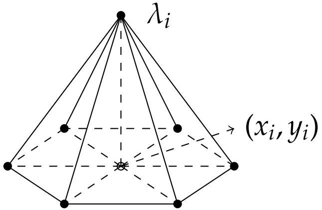
$$V_h := \left\{ v \in C^0(\Omega) : v|_T \in P_1(T), \forall T, v|_{\partial\Omega} = 0 \right\}$$

有限元问题: 求$u_h \in V_h$, 使得

$$\int_\Omega \nabla u_h \cdot \nabla v_h dxdy = \int_\Omega f v_h dxdy, \forall v_h \in V_h$$

有限元基函数

Figure



$u_h = \sum_{j\in\mathcal{N}_0} u_j\lambda_j$, 其中$\mathcal{N}_0$表示内节点的集合.

$$\int_\Omega \nabla u_h \cdot \nabla v_h dxdy = \sum_e \int_e \nabla u_h \cdot \nabla v_h dxdy$$

$$u_h|_e = u_i\lambda_i + u_j\lambda_j + u_m\lambda_m = \begin{bmatrix}\lambda_i \ \lambda_j \ \lambda_m\end{bmatrix}\begin{bmatrix}u_i\\ u_j\\ u_m\end{bmatrix}$$

$$v_h|_e = v_i\lambda_i + u_j\lambda_j + u_m\lambda_m = \begin{bmatrix}v_i \ v_j \ v_m\end{bmatrix}\begin{bmatrix}\lambda_i\\ \lambda_j\\ \lambda_m\end{bmatrix}$$

于是

$$\int_e \nabla u_h \nabla v_h dxdy = [v_i\ v_j\ v_m] \int_e \begin{bmatrix} \nabla\lambda_i \\ \nabla\lambda_j \\ \nabla\lambda_m \end{bmatrix} \cdot [\nabla\lambda_i\ \nabla\lambda_j\ \nabla\lambda_m] dxdy \begin{bmatrix} u_i \\ u_j \\ u_m \end{bmatrix}$$

$$= [v_i\ v_j\ v_m] \int_e \begin{pmatrix} \nabla\lambda_i\cdot\nabla\lambda_i & \nabla\lambda_i\cdot\nabla\lambda_j & \nabla\lambda_i\cdot\nabla\lambda_m \\ \nabla\lambda_j\cdot\nabla\lambda_i & \nabla\lambda_j\cdot\nabla\lambda_j & \nabla\lambda_j\cdot\nabla\lambda_m \\ \nabla\lambda_m\cdot\nabla\lambda_i & \nabla\lambda_m\cdot\nabla\lambda_j & \nabla\lambda_m\cdot\nabla\lambda_m \end{pmatrix} dxdy \begin{bmatrix} u_i \\ u_j \\ u_m \end{bmatrix}$$

令

$$\mathrm{K_e} := \int_e \begin{pmatrix} \nabla\lambda_i\cdot\nabla\lambda_i & \nabla\lambda_i\cdot\nabla\lambda_j & \nabla\lambda_i\cdot\nabla\lambda_m \\ \nabla\lambda_j\cdot\nabla\lambda_i & \nabla\lambda_j\cdot\nabla\lambda_j & \nabla\lambda_j\cdot\nabla\lambda_m \\ \nabla\lambda_m\cdot\nabla\lambda_i & \nabla\lambda_m\cdot\nabla\lambda_j & \nabla\lambda_m\cdot\nabla\lambda_m \end{pmatrix} dxdy$$

这样有

$$\int_\Omega \nabla u_h \cdot \nabla v_h dxdy = \sum_e \left[v_i\ v_j\ v_m\right] \mathrm{K_e} \begin{bmatrix} u_i \\ u_j \\ u_m \end{bmatrix}$$

其中$\mathrm{K_e}$称为单元刚度矩阵.(注: $u_i = v_i = 0$, 若i为边界节点).

怎样组装总刚度矩阵?

$$\mathrm{K_e} \Rightarrow \begin{pmatrix} k_{ii} & \cdots & k_{ij} & \cdots & k_{im} \\ \vdots & & \vdots & & \vdots \\ k_{ji} & \cdots & k_{jj} & \cdots & k_{jm} \\ \vdots & & \vdots & & \vdots \\ k_{mi} & \cdots & k_{mj} & \cdots & k_{mm} \end{pmatrix}, \qquad \mathrm{K}_{N\times N}$$

右端项

$$\int_e f v_h dxdy = [v_i \; v_j \; v_m] \int_e \begin{pmatrix} f\lambda_i \\ f\lambda_j \\ f\lambda_m \end{pmatrix} dxdy$$

令

$$F_e = \int_e \begin{pmatrix} f\lambda_i \\ f\lambda_j \\ f\lambda_m \end{pmatrix} dxdy$$

$$F_e \Rightarrow \begin{pmatrix} f_i \\ \vdots \\ f_j \\ \vdots \\ f_m \end{pmatrix} \quad F_N$$

线性代数方程组

$$A_{N\times N}U_N = F_N$$

目录

类似一维问题, 有

$$\int_\Omega (\nabla u - \nabla u_h)\cdot(\nabla u - \nabla u_h)\, dxdy = \inf_{v_h\in V_h}\int_\Omega |\nabla u - \nabla u_h|^2\, dxdy$$

定义插值函数$I_h u$, 使得

$$I_h u = \sum_{i\in\mathcal{N}} u\left(x_i, y_i\right)\lambda_i$$

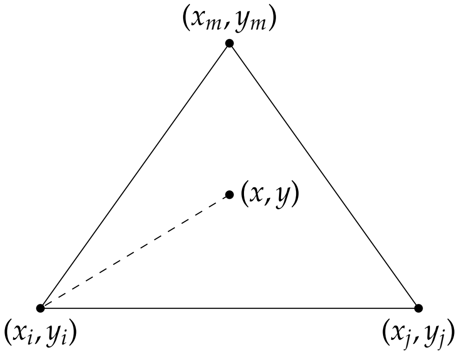
在Δ_e上(顶点为$(x_i, y_i), (x_j, y_j), (x_m, y_m)$), 有

$$\begin{aligned} u(x,y) - I_h u(x,y) = u(x,y) &- u(x_i,y_i)\lambda_i - u(x_j,y_j)\lambda_j \\ &- u(x_m,y_m)\lambda_m \end{aligned}$$

令$A_i=(x_i,y_i), A=(x,y)$, 作变点展开

$$
\begin{aligned}
u\left(x_i,y_i\right)=&\, u(x,y)+\nabla u(x,y)\cdot\left(x_i-x,y_i-y\right)^T\\
&+\frac{\left(x_i-x\right)^2}{2}\int_0^1 s\partial_{xx}u\left(A_i+s\left(A-A_i\right)\right)ds\\
&+\frac{\left(y_i-y\right)^2}{2}\int_0^1 s\partial_{yy}u\left(A_i+s\left(A-A_i\right)\right)ds\\
&+\left(x_i-x\right)\left(y_i-y\right)\int_0^1 s\partial_{xy}u\left(A_i+s\left(A-A_i\right)\right)ds
\end{aligned}
$$

Figure



作极坐标变化, 得

$$
\begin{aligned}
u\left(x_i, y_i\right) = {} & u(x, y) + \nabla u(x, y) \cdot \left(x_i - x, y_i - y\right)^T \\
& + \frac{(\cos\theta_i)^2}{2} \int_0^{l_{i(x,y)}} \partial_{xx} u\left(\left(x_i, y_i\right) - t(\cos\theta_i, \sin\theta_i)\right) t dt \\
& + \frac{(\sin\theta_i)^2}{2} \int_0^{l_{i(x,y)}} \partial_{yy} u\left(\left(x_i, y_i\right) - t(\cos\theta_i, \sin\theta_i)\right) t dt \\
& + \cos\theta_i \sin\theta_i \int_0^{l_{i(x,y)}} \partial_{xy} u\left(\left(x_i, y_i\right) - t(\cos\theta_i, \sin\theta_i)\right) t dt.
\end{aligned}
$$

这样

$$
\begin{aligned}
\nabla u - \nabla I_h u &= \nabla u - \sum_{i=1}^{3} u(x_i, y_i) \nabla \lambda_i \\
&= -\sum_{i=1}^{3} \nabla \lambda_i \Big(\frac{(\cos\theta_i)^2}{2} \int_0^{l_{i(x,y)}} \partial_{xx} u\left((x_i, y_i) - t(\cos\theta_i, \sin\theta_i)\right) t dt \\
&\quad + \frac{(\sin\theta_i)^2}{2} \int_0^{l_{i(x,y)}} \partial_{yy} u\left((x_i, y_i) - t(\cos\theta_i, \sin\theta_i)\right) t dt \\
&\quad + \cos\theta_i \sin\theta_i \int_0^{l_{i(x,y)}} \partial_{xy} u\left((x_i, y_i) - t(\cos\theta_i, \sin\theta_i)\right) t dt \Big).
\end{aligned}
$$

这里用到

$$\sum_{i=1}^{3} u(x,y)\nabla\lambda_i = 0, \quad \sum_{i=1}^{3} \nabla u \cdot (x,y)^T \nabla\lambda_i = 0,$$

$$\begin{aligned}
\sum_{i=1}^{3} \nabla u \cdot (x_i,y_i)^T \nabla\lambda_i &= \frac{\partial u}{\partial x}\sum x_i \nabla\lambda_i + \frac{\partial u}{\partial y}\sum y_i \nabla\lambda_i \\
&= \frac{\partial u}{\partial x}\nabla(\sum x_i\lambda_i) + \frac{\partial u}{\partial y}\nabla(\sum y_i\lambda_i) \\
&= \frac{\partial u}{\partial x}(1,0)^T + \frac{\partial u}{\partial y}(0,1)^T = \nabla u.
\end{aligned}$$

于是

$$
\begin{aligned}
&\int_e |\nabla u - \nabla I_h u|^2 dxdy \\
&= 3\sum_{i=1}^{3} |\nabla\lambda_i|^2 \\
&\quad \times \int_{\alpha_i}^{\beta_i}\int_0^{l_{\theta_i}} \Big(\frac{(\cos\theta_i)^2}{2}\int_0^{l_{i(x,y)}} \partial_{xx}u\left((x_i,y_i)-t(\cos\theta_i,\sin\theta_i)\right)tdt \\
&\quad + \frac{(\sin\theta_i)^2}{2}\int_0^{l_{i(x,y)}} \partial_{yy}u\left((x_i,y_i)-t(\cos\theta_i,\sin\theta_i)\right)tdt \\
&\quad + \cos\theta_i\sin\theta_i\int_0^{l_{i(x,y)}} \partial_{xy}u\left((x_i,y_i)-t(\cos\theta_i,\sin\theta_i)\right)tdt\Big)^2 l_i dl_i d\theta_i \\
&\qquad\qquad (l_i = l_{i(x,y)}).
\end{aligned}
$$

下面估计上式中的每一项

$$
\begin{aligned}
&\int_0^{l_{i(x,y)}} \partial_{xx}u\left((x_i,y_i)-t(\cos\theta_i,\sin\theta_i)\right)tdt\\
\le&\Big(\int_0^{l_{i(x,y)}}(\partial_{xx}u\left((x_i,y_i)-t(\cos\theta_i,\sin\theta_i)\right))^2tdt\Big)^{1/2}\Big(\int_0^{l_{i(x,y)}}tdt\Big)^{1/2}\\
\le&\Big(\int_0^{l_{\theta_i}}(\partial_{xx}u\left((x_i,y_i)-t(\cos\theta_i,\sin\theta_i)\right))^2tdt\Big)^{1/2}\Big(\int_0^{l_{i(x,y)}}tdt\Big)^{1/2}.
\end{aligned}
$$

因此

$$\int_{\alpha_i}^{\beta_i}\int_0^{l_{\theta_i}}\left(\frac{(\cos\theta_i)^2}{2}\int_0^{l_{i(x,y)}}\partial_{xx}u\left((x_i,y_i)-t(\cos\theta_i,\sin\theta_i)\right)tdt\right)^2 l_i dl_i d\theta_i$$

$$\leq \int_{\alpha_i}^{\beta_i}\frac{(\cos\theta_i)^4}{4}\int_0^{l_{\theta_i}}(\partial_{xx}u\left((x_i,y_i)-t(\cos\theta_i,\sin\theta_i)\right))^2 tdt$$

$$\times\left(\int_0^{l_{\theta_i}}\int_0^{l_{i(x,y)}} tdtl_i dl_i\right)d\theta_i$$

$$=\frac{1}{32}\int_{\alpha_i}^{\beta_i} l_{\theta_i}^4(\cos\theta_i)^4\int_0^{l_{\theta_i}}(\partial_{xx}u\left((x_i,y_i)-t(\cos\theta_i,\sin\theta_i)\right))^2 tdtd\theta_i$$

$$\leq \max_{\alpha_i\leq\theta_i\beta_i}\frac{l_{\theta_i}^4(\cos\theta_i)^4}{32}\|\partial_{xx}u\|_{0,e}^2.$$

其它项可以类似处理, 这样

$$\int_e|\nabla u-\nabla I_h u|^2 dxdy\leq C\|\nabla^2 u\|_{0,e}^2\sum_{i=1}^{3}|\nabla\lambda_i|^2 h^4.$$

谢谢！