第十讲：偏微分方程数值解法

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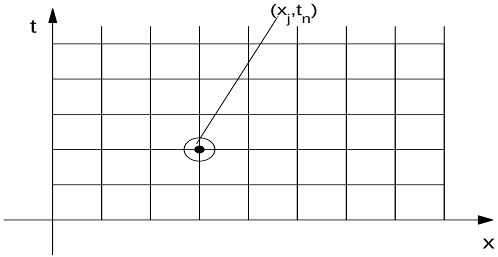
抛物型方程显式格式

$$\begin{cases} \frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u}{\partial x^2} & 0 < x < l, t > 0 \\ u(0,t) = u(l,t) = 0 & t > 0 \\ u(x,0) = f(x) & 0 \le x \le l \end{cases}$$

时间步长k, 空间步长$h = l/J$.

$$x = x_j,\ x_j = jh,\ j = 0, 1, \cdots$$
$$t = t_n,\ t_n = nk,\ n = 0, 1, \cdots$$

网格线, 网格点如下图所示.



抛物型方程显式格式

任取一个内部网格点(x_j, t_n), 即

$$\frac{\partial u}{\partial t}(x_j, t_n) - a^2 \frac{\partial^2 u}{\partial x^2}(x_j, t_n) = 0.$$

$$\frac{\partial u}{\partial t}(x_j, t_n) = \frac{u(x_j, t_{n+1}) - u(x_j, t_n)}{k} - \frac{k}{2}\frac{\partial^2 u}{\partial t^2}(x_j, \mu_n)$$

$$\frac{\partial^2 u}{\partial x^2}(x_j, t_n) = \frac{u(x_{j+1}, t_n) - 2u(x_j, t_n) + u(x_{j-1}, t_n)}{h^2} - \frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}(\xi_j, t_n)$$

有

$$\frac{u(x_j, t_{n+1}) - u(x_j, t_n)}{k} - a^2 \frac{u(x_{j+1}, t_n) - 2u(x_j, t_n) + u(x_{j-1}, t_n)}{h^2} = R_j^n$$

其中

$$R_j^n = \frac{k}{2}\frac{\partial^2 u}{\partial t^2}(x_j, \mu_n) - a^2 \frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}(\xi_j, t_n)$$

抛物型方程显式格式

如果忽略右端的误差项R_j^n, 就得到一个差分方程(又称为差分格式)

$$\frac{U_j^{n+1} - U_j^n}{k} - a^2 \frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{h^2} = 0 \tag{1}$$

其中U_j^n是解$u(x_j, t_n)$的近似解, 称之为数值解. R_j^n称为局部截断误差.

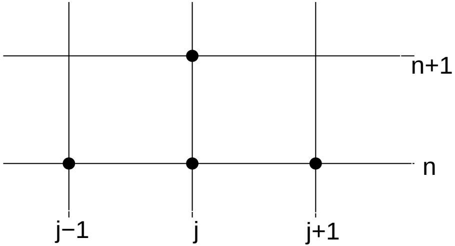
差分方程(1)的精度为$O(k + h^2)$, 即它为一个时间精度为一阶, 空间精度为二阶的格式.

$$U_0^n = 0, \; U_J^n = 0, \; n = 1, 2, \cdots$$
$$U_j^0 = f(x_j), \; j = 0, 1, 2, \cdots, j$$

抛物型方程显式格式

$$\begin{cases} U_j^{n+1} = \lambda U_{j-1}^n + (1-2\lambda)U_j^n + \lambda U_{j+1}^n, \ 0 < j < J \\ U_0^n = 0, \ U_J^n = 0, \ n = 1, 2, \cdots \\ U_j^0 = f_j, \ j = 0, 1, 2, \cdots, J \end{cases}$$

$$\begin{cases} \lambda = a^2 k / h^2 \text{：网格比} \\ f_j = f(x_j) \end{cases}$$



四点显式差分格式

$$U^n = (U_1^n, U_2^n, \cdots, U_{J-1}^n)^T, n = 0, 1, \cdots$$

定义$(J-1)$阶三对角矩阵

$$A = \begin{bmatrix} (1-2\lambda) & \lambda & & & \\ \lambda & \ddots & \ddots & & \\ & \ddots & \ddots & \lambda & \\ & & \lambda & (1-2\lambda) \end{bmatrix}, \text{于是} U^{n+1} = AU^n, n = 0, 1, 2, \cdots$$

抛物型方程显式格式

Example 1.1

$$\begin{cases} \frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} & 0 < x < 1, 0 < t \\ u(0,t) = u(l,t) = 0 & 0 \le t \\ u(x,0) = \sin(\pi x) & 0 \le x \le 1 \end{cases}$$

这个定解问题的分析解是

$$u(x,t) = e^{-\pi^2 t} \sin(\pi x)$$

两种网格

(1) $h = 0.1,\ k = 0.0005$

(2) $h = 0.1,\ k = 0.01$

分别计算$t = 0.5$时刻的数值解, 结果见下表

x_j	$u(x_j, 0.5)$	U_j^{1000}	$\lvert u(x_j, 0.5) - U_j^{1000}\rvert$
0.0	0	0	
0.1	0.00222241	0.00228652	6.411×10^{-5}
0.2	0.00422728	0.00434922	1.219×10^{-4}
0.3	0.00581836	0.00598619	1.678×10^{-4}
0.4	0.00683989	0.00703719	1.973×10^{-4}
0.5	0.00719188	0.00739934	2.075×10^{-4}
0.6	0.00683989	0.00703719	1.973×10^{-4}
0.7	0.00581836	0.00598619	1.678×10^{-4}
0.8	0.00422728	0.00434922	1.219×10^{-4}
0.9	0.00222241	0.00228652	6.511×10^{-5}
1.0	0	0	

Table: 四点显式差分格式在网格(1)上的计算结果

x_j	$u(x_j, 0.5)$	U_j^{50}	$\lvert u(x_j, 0.5) - U_j^{50}\rvert$
0.0	0	0	
0.1	0.00222241	8.19876×10^7	8.199×10^7
0.2	0.00422728	-1.55719×10^8	1.557×10^8
0.3	0.00581836	2.13833×10^8	2.138×10^8
0.4	0.00683989	-2.50642×10^8	2.506×10^8
0.5	0.00719188	2.62685×10^8	2.627×10^8
0.6	0.00683989	-2.49015×10^8	2.490×10^8
0.7	0.00581836	2.11200×10^8	2.112×10^8
0.8	0.00422728	-1.53086×10^8	1.531×10^8
0.9	0.00222241	8.03604×10^7	8.036×10^7
1.0	0	0	

Table: 四点显式差分格式在网格(2)上的计算结果

抛物型方程显式格式

假设初值U^0的舍入误差为e, 即

$$U^0 = f + e$$

其中$f = (f(x_1), f(x_2), \cdots, f(x_{J-1}))^T$, $e = (e_1, e_2, \cdots, e_{J-1})^T$, 则

$$\begin{aligned}
U^1 &= A(f+e) = Af + Ae \\
U^2 &= A^2(f+e) = A^2 f + A^2 e \\
&\cdots \\
U^n &= A^n(f+e) = A^n f + A^n e \\
&\cdots
\end{aligned}$$

如果在向上推进时初始误差没有扩大, 即

$$\|A^n e\| \le \|e\|$$

对所有$n \geq 1$成立, 就称差分格式为数值稳定的, 否则称为数值不稳定的. 显然当且仅当$\|A^n\| \leq 1$, 上述不等式才成立.

$$\rho(A)^n = \rho(A^n) \leq \|A^n\|$$

因此, $\rho(A) \leq 1$是差分格式(1)数值稳定的必要条件. 换句话说, 若$\rho(A) > 1$, 则差分格式(1) 必然数值不稳定的.

上面所定义的矩阵A的特征值为

$$\mu_j = 1 - 4\lambda\Big(\sin(\frac{j\pi}{2J})\Big)^2, \ j = 1, 2, \cdots, J-1 \tag{2}$$

所以(1)数值稳定的条件为

$$\rho(A) = \max_{1 \leq j \leq J-1} |1 - 4\lambda\Big(\sin(\frac{j\pi}{2J})\Big)^2| \leq 1$$

容易看出, 这等价于

$$0 \le \lambda\Big(\sin(\frac{j\pi}{2J})\Big)^2 \le \frac{1}{2}$$

对$j = 1, 2, \cdots, J-1$, 只要求$0 \le \lambda \le 1/2$.
由$\lambda = a^2 k/h^2 \Rightarrow a^2\frac{k}{h^2} \le \frac{1}{2}$.
对网格(1), 有

$$a^2\frac{k}{h^2} = \frac{0.0005}{0.01} = \frac{5}{100} < \frac{1}{2}$$

对网格(2), 有

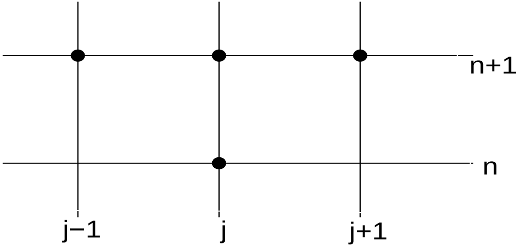
$$a^2\frac{k}{h^2} = \frac{0.01}{0.01} = 1 > \frac{1}{2}$$

抛物型方程隐式格式

四点隐式差分格式

$$\frac{\partial u}{\partial t}(x_j, t_{n+1}) = \frac{u(x_j, t_{n+1}) - u(x_j, t_n)}{k} + \frac{k}{2}\frac{\partial^2 u}{\partial t^2}(x_j, \mu_n)$$

$$\frac{\partial^2 u}{\partial x^2}(x_j, t_{n+1}) = \frac{u(x_{j-1}, t_{n+1}) - 2u(x_j, t_{n+1}) + u(x_{j-1}, t_{n+1})}{h^2} - \frac{h^2}{12}\frac{\partial^4 u}{\partial x^4}(\xi_j, t_{n+1})$$



抛物型方程隐式格式

$$\frac{U_j^{n+1} - U_j^n}{k} = a^2 \frac{U_{j+1}^{n+1} - 2U_j^{n+1} + U_{j-1}^{n+1}}{h^2} \tag{3}$$

$$-\lambda U_{j-1}^{n+1} + (1+2\lambda)U_j^{n+1} - \lambda U_{j+1}^{n+1} = U_j^n$$

$$AU^{n+1} = U^n$$

$$A = \begin{bmatrix} (1+2\lambda) & -\lambda & & \\ -\lambda & \ddots & \ddots & \\ & \ddots & \ddots & -\lambda \\ & & -\lambda & (1+2\lambda) \end{bmatrix}$$

x_j	$u(x_j, 0.5)$	U_j^{50}	$\|u(x_j, 0.5) - U_j^{50}\|$
0.0	0	0	
0.1	0.00222241	0.00289802	6.756×10^{-4}
0.2	0.00422728	0.00551236	1.285×10^{-3}
0.3	0.00581836	0.00758711	1.769×10^{-3}
0.4	0.00683989	0.00891918	2.079×10^{-3}
0.5	0.00719188	0.00937818	2.186×10^{-3}
0.6	0.00683989	0.00891918	2.079×10^{-3}
0.7	0.00581836	0.00758711	1.769×10^{-3}
0.8	0.00422728	0.00551236	1.285×10^{-3}
0.9	0.00222241	0.00289802	6.756×10^{-4}
1.0	0	0	

Table: 四点隐式差分格式在网格(2)上的计算结果

Crank-Nicolson格式:

$$\frac{U_j^{n+1}-U_j^n}{k}-\frac{a^2}{2}\left[\frac{U_{j+1}^{n+1}-2U_j^{n+1}+U_{j-1}^{n+1}}{h^2}+\frac{U_{j+1}^{n}-2U_j^{n}+U_{j-1}^{n}}{h^2}\right]=0$$

$$AU^{n+1}=BU^n$$

其中

$$A=\begin{bmatrix}(1+\lambda) & -\lambda/2 & & \\ -\lambda/2 & \ddots & \ddots & \\ & \ddots & \ddots & -\lambda/2 \\ & & -\lambda/2 & (1+\lambda)\end{bmatrix} B=\begin{bmatrix}(1-\lambda) & \lambda/2 & & \\ \lambda/2 & \ddots & \ddots & \\ & \ddots & \ddots & \lambda/2 \\ & & \lambda/2 & (1-\lambda)\end{bmatrix}$$

Crank-Nicolson格式

x_j	$u(x_j, 0.5)$	U_j^{50}	$\lvert u(x_j, 0.5) - U_j^{50}\rvert$
0.0	0	0	
0.1	0.00222241	0.00230512	8.271×10^{-5}
0.2	0.00422728	0.00438461	1.573×10^{-4}
0.3	0.00581836	0.00603489	2.165×10^{-4}
0.4	0.00683989	0.00709444	2.546×10^{-4}
0.5	0.00719188	0.00745954	2.677×10^{-4}
0.6	0.00683989	0.00709444	2.546×10^{-4}
0.7	0.00581836	0.00603489	2.165×10^{-4}
0.8	0.00422728	0.00438461	1.573×10^{-4}
0.9	0.00222241	0.00230512	8.271×10^{-5}
1.0	0	0	

Table: Crank-Nicolson格式在网格(2)上的计算结果

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$$\begin{cases} \frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 & 0 < t \le T,\ x \in \mathbb{R} \\ u(x,0) = u_0(x) & x \in \mathbb{R} \end{cases}$$

沿直线$\frac{dx}{dt} = a$, 有

$$\frac{du}{dt} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}\frac{dx}{dt} = \frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0$$

即沿$x - at = c$, $u(x,t)$都为常数. 平行直线族$x - at = c$称为双曲型方程的特征线.

$$u(x,t) = u_0(x - at)$$

$(x - at, 0)$称为u在(x,t)点的依赖区域.

局部截断误差$O(k+h)$：

$$\begin{cases} \frac{U_j^{n+1}-U_j^n}{k} + a\frac{U_j^n-U_{j-1}^n}{h} = 0, \ U_j^0 = u_0(x_j) \\ \frac{U_j^{n+1}-U_j^n}{k} + a\frac{U_{j+1}^n-U_j^n}{h} = 0, \ U_j^0 = u_0(x_j) \end{cases}$$

$$U_j^{n+1} = U_j^n - a\lambda(U_j^n - U_{j-1}^n), \, U_j^0 = u_0(x_j),$$

$$\lambda = \frac{k}{h} \text{ 为网格比}$$

$$\tilde{U}_j^{n+1} = \tilde{U}_j^n - a\lambda(\tilde{U}_j^n - \tilde{U}_{j-1}^n), \, \tilde{U}_j^0 = \tilde{u}_0(x_j)$$

$$\epsilon_j^{n+1} = \epsilon_j^n - a\lambda(\epsilon_j^n - \epsilon_{j-1}^n), \, \epsilon_j^0 = \tilde{u}_0(x_j) - u_0(x_j).$$

Fourier分析方法：

设差分格式的解e_j^n为振幅是$v^n(\xi)$的谐波，即

$$e_j^n = v^n(\xi)e^{i\xi x_j}$$

其中$i = \sqrt{-1}$, ξ为任意实数.

$$v^{n+1}(\xi)e^{i\xi x_j} = v^n(\xi)[1 - a\lambda(1 - e^{-i\xi h})]e^{i\xi x_j}$$

消去公因子$e^{i\xi x_j}$, 有

$$v^{n+1}(\xi) = [1 - a\lambda(1 - e^{-i\xi h})]v^n(\xi)$$

记$\theta = \xi h$, $G(\theta) = 1 - a\lambda(1 - e^{-i\theta})$. 称$G(\theta)$为格式的**增长因子**.

Von Neumann条件

Von Neumann条件: 存在$h_0 > 0$, 当$h < h_0$时

$$|G(\theta)| \leq 1.$$

即

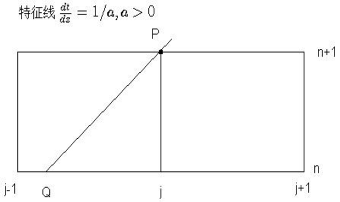
$$|1 - a\lambda(1 - e^{-i\theta})| \leq 1 \Rightarrow \begin{cases} 0 \leq a\frac{k}{h} \leq 1 \\ \text{另一个} -1 \leq a\frac{k}{h} \leq 0 \end{cases}$$

$$\left.\begin{aligned} a > 0 \Rightarrow \text{左偏心格式} \\ a < 0 \Rightarrow \text{右偏心格式} \end{aligned}\right\} \quad \frac{k}{h} \leq \frac{1}{|a|}$$

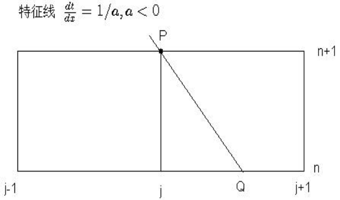
用特征线可以解释上述条件

$$\frac{dx}{dt} = a \Longrightarrow \frac{du}{dt} = 0$$

左右偏心格式



(a) 左偏心格式



(b) 右偏心格式

沿着特征线

$$u(x, t_{n+1}) = u_P = u_Q$$

因此, 只要利用U_{j-1}^n, U_j^n作线性插值得到u_Q的近似值U_j^{n+1}, 即

$$\begin{aligned} U_j^{n+1} &= \left(\frac{x - x_j}{x_{j-1} - x_j} U_{j-1}^n + \frac{x - x_{j-1}}{x_j - x_{j-1}} U_j^n\right)\Big|_Q = \frac{ak}{h} U_{j-1}^n + \frac{h - ak}{h} U_j^n \\ &= U_j^n - a\lambda(U_j^n - U_{j-1}^n) \end{aligned}$$

这就是左偏心格式.

稳定性条件$\frac{k}{h} \le \frac{1}{|a|}$说明, 差分格式的依赖域必须包含微分方程的依赖区域. 这就是**Courant条件**(又称为CFL 条件).

变系数方程:

$$\frac{\partial u}{\partial t} + a(x,t)\frac{\partial u}{\partial x} = 0, \quad 0 < t \le T, x \in \mathbb{R}$$

$$\begin{cases} \frac{U_j^{n+1}-U_j^n}{k} + a_j^n \frac{U_j^n - U_{j-1}^n}{h} = 0 & a_j^n \ge 0 \\ \frac{U_j^{n+1}-U_j^n}{k} + a_j^n \frac{U_{j+1}^n - U_j^n}{h} = 0 & a_j^n < 0 \end{cases} \tag{4}$$

CFL条件

Courant条件

$$\max_{x,t} |a(x,t)|\frac{k}{h} \leq 1$$

$$\left.\begin{array}{l} a > 0,\ u\text{向右传播, 用左偏心格式} \\ a < 0,\ u\text{向左传播, 用右偏心格式} \end{array}\right\} \quad \text{迎风格式}$$

(1) Lax-Friedrichs格式

$$\frac{U_j^{n+1} - \frac{1}{2}(U_{j-1}^n + U_{j+1}^n)}{k} + a\frac{U_{j+1}^n - U_{j-1}^n}{2h} = 0$$

(2) Lax-Wendroff格式

人工粘性

$$\frac{U_j^{n+1} - U_j^n}{k} + a\frac{U_{j+1}^n - U_{j-1}^n}{2h} = \frac{a^2k}{2}\frac{U_{j+1}^n - 2U_j^n + U_{j-1}^n}{h^2}$$

这两个格式$|a|\lambda \leq 1$, CFL条件.

CFL条件

对于抛物型方程

$$\begin{cases} u_t = u_{xx} & 0 < x < 1, t > 0 \\ u(x,0) = u_0(x) & 0 \le x \le 1 \\ u(0,t) = u(l,t) = 0 & t > 0 \end{cases}$$

(用分离变量法)有一族相互独立的非平凡特解

$$u_k(x,t) = e^{-k^2\pi^2 t} \sin(k\pi x),\ k = 1, 2 \cdots,$$

若

$$u_0(x) = \sum_{k=1}^{\infty} a_k \sin(k\pi x)$$

其中

$$a_k = 2\int_0^1 u_0(x)\sin(k\pi x)\,dx,\ k = 1, 2, \cdots,$$

由解的唯一性和线性叠加原理

$$u(x,t) = \sum_{k=0}^{\infty} a_k e^{-k^2\pi^2 t} \sin(k\pi x)$$

驻波:(1)没有波形的传播, 即各点振动相位与位置无关, 按同一方式随时间振动; (2)各点振幅随点x而异.

$$\omega(x,t) = c(t) \cdot \sin(\lambda x) \text{ 单音振动}$$

$$\begin{cases} \frac{U_j^{n+1}-U_j^n}{k} = \frac{U_{j+1}^n-2U_j^n+U_{j-1}^n}{h^2}, 1 \le j \le J-1, n \ge 0 \\ U_j^0 = u_j^0 \\ U_0^n = U_J^n = 0 \end{cases}$$

$$U_j^{n+1} = (1-2\lambda)U_j^n + \lambda(U_{j-1}^n + U_{j+1}^n)$$

CFL条件

Fourier波型

$$U_j^m = g_\ell^m e^{i\ell\pi j/J}, \quad g_\ell\text{为增长因子}$$

$$g_\ell^{n+1} e^{i\ell\pi j/J} = g_\ell^n e^{i\ell\pi j/J}[1 + \lambda(e^{i\ell\pi/J} + e^{-i\ell\pi/J} - 2)]$$

$$g_\ell = 1 - 4\lambda \sin^2 \frac{\ell\pi}{2J}, \quad \ell = J\text{时最大}$$

要求($nk \le t_{\max}$, $1 \le l \le J$, $\tilde{n} = [t_{\max}/k]$)

$$|g_\ell^n| \le \tilde{C} \Longrightarrow |g_\ell| \le \tilde{C}^{1/\tilde{n}} \le 1 + (\tilde{C} - 1)/\tilde{n}$$
$$\Longrightarrow |g_\ell| \le 1 + Ck,\ 1 \le l \le J$$

取$\ell = J$, 即得

$$\lambda \le \frac{1}{2}$$

CFL条件

一阶双曲型方程

$$\begin{cases} \frac{\partial u}{\partial t} + a\frac{\partial u}{\partial x} = 0 \\ u(x,0) = u_0(x) \end{cases}$$

不妨考虑$a > 0$

$$\frac{U_j^{n+1} - U_j^n}{k} + a\frac{U_j^n - U_{j-1}^n}{h} = 0 \quad \lambda = \frac{ak}{h}$$

$$\Longrightarrow U_j^{n+1} = (1-\lambda)U_j^n + \lambda U_{j-1}^n$$

将Fourier波型$U_j^n = g_\ell^n e^{i\ell jh}$代入上式

$$\begin{aligned} g_\ell^{n+1} e^{i\ell jh} &= (1-\lambda) g_\ell^n e^{i\ell jh} + \lambda g_\ell^n e^{i\ell(j-1)h} \\ &= g_\ell^n e^{i\ell jh}((1-\lambda) + \lambda e^{-i\ell h}),\ g_\ell \text{ 增长因子} \end{aligned}$$

$$g_\ell = (1-\lambda) + \lambda e^{-i\ell h} = (1-\lambda) + \lambda\cos(\ell h) - i\lambda\sin(\ell h)$$

$$|g_\ell| = |1 - \lambda(1 - e^{-i\ell h})| \le 1$$

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作业

1. 证明(2).
2. 证明数值格式(3)是无条件稳定的.
3. 证明格式(4)是数值稳定的（在满足CFL条件下）

谢谢！