Cedille2: A proof theoretic redesign of the Calculus of Dependent Lambda Eliminations

by

Andrew Marmaduke

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Thesis Committee: Aaron Stump, Thesis Supervisor
Cesare Tinelli
J. Garrett Morris
Sriram Pemmaraju
William J. Bowman

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PUBLIC ABSTRACT

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INTRODUCTION

Type theory is a tool for reasoning about assertions of some domain of discourse. When applied to programming languages, that domain is the expressible programs and their properties. Of course, a type theory may be rich enough to express detailed properties about a program, such that it halts or returns an even number. Therein lies a tension between what properties a type theory can faithfully (i.e. consistently) encode and the complexity of the type theory itself. If the theory is too complex then it may be untenable to prove that the type theory is well-behaved. Indeed, the design space of type theories is vast, likely infinite. When incorporating features the designer must balance complexity against capability.

Modern type theory arguably began with Martin-Löf in the 1970s and 1980s when he introduced a dependent type theory with the philosophical aspirations of being an alternative foundation of mathematics [38, 39]. Soon after in 1985, the Calculus of Constructions (CC) was introduced by Coquand [11, 12]. Inductive data (e.g. natural numbers, lists, trees) was shown by Guevers to be impossible to derive in CC [23]. Nevertheless, inductive data was added as an extension by Pfenning [46] and the Calculus of Inductive Constructions (CIC) became the basis for the proof assistant Rocq [43].

In the early 1990s Barendregt introduced a generalization to Pure Type Systems (PTS) and studied CC under his now famous λ-cube [5, 4]. The λ-cube demonstrated how CC could be deconstructed into four essential sorts of functions. At its base was the Simply Typed Lambda Calculus (STLC) a type theory introduced in the 1940s by Church to correct logical consistency issues in his (untyped) λ-calculus [8]. The STLC has only basic functions found in all programming languages. System F, a type theory introduced by Girard [25, 26] and independently by Reynolds [50], is obtained from STLC by adding quantification over types (i.e. polymorphic functions). Adding a copy of STLC at the type-layer, functions from types to types, yields System F^ω. Finally, the addition of quantification over terms or functions from terms to types, completes CC. While this is not the only path through the λ-cube to arrive at CC it is the most well-known and the most immediately relevant.

Perhaps surprisingly, all the systems of the λ-cube correspond to a logic. In the 1970s Curry circulated his observations about the STLC corresponding to intuitionistic propositional logic [27]. Reynolds and Girard's combined work demonstrated that System F corresponds to second-order intuitionistic propositional logic [25, 50, 51]. Indeed, Barendregt extended the correspondence to all systems in his λ-cube noting System F^ω as corresponding to higher-order intuitionistic propositional logic and CC as corresponding to higher-order intuitionistic predicate logic [4]. Fundamentally, the Curry-Howard correspondence associates programs of a type theory with proofs of a logic, and types with formula. However, the correspondence is not an isomorphism because the logical view does

$$
\begin{aligned}
t &::= x \mid \mathfrak{b}(\kappa_1, x : t_1, t_2) \mid \mathfrak{c}(\kappa_2, t_1, \ldots, t_{\mathfrak{a}(\kappa_2)}) \\
\kappa_1 &::= \lambda \mid \Pi \\
\kappa_2 &::= \star \mid \square \mid \text{app}
\end{aligned}
$$

$$
\begin{aligned}
\mathfrak{a}(\star) = \mathfrak{a}(\square) &= 0 \\
\mathfrak{a}(\text{app}) &= 2
\end{aligned}
\qquad
\begin{aligned}
\lambda\, x : t_1.\, t_2 &:= \mathfrak{b}(\lambda, x : t_1, t_2) \\
(x : t_1) \to t_2 &:= \mathfrak{b}(\Pi, x : t_1, t_2) \\
t_1\ t_2 &:= \mathfrak{c}(\text{app}, t_1, t_2)
\end{aligned}
\qquad
\begin{aligned}
\star &:= \mathfrak{c}(\star) \\
\square &:= \mathfrak{c}(\square)
\end{aligned}
$$

Figure 1.1: Syntax for System F^ω.

not possess a unique assignment of proofs. The type theory contains potentially *more* information than the proof derivation.

Cedille is a programming language with a core type theory based on CC [55, 57]. However, Cedille took an alternative road to obtaining inductive data than what was done in the 1980s. Instead, CC was modified to add the implicit products of Miquel [40], the dependent intersections of Kopylov [32], and an equality type over untyped terms. The initial goal of Cedille was to find an efficient way to encode inductive data. This was achieved in 2018 with Mendler-style lambda encodings [15]. However, the design of Cedille sacrificed certain properties such as the decidability of type checking. Decidability of type checking was stressed by Kreisel to Scott as necessary to reduce proof checking to type checking because a proof does not, under Kreisel's philosophy, diverge [52]. This puts into contention if Cedille corresponds to a logic at all. What remains is to describe the redesign of Cedille such that it does have decidability of type checking and to argue why this state of affairs is preferable. However, completing this journey requires a deeper introduction into the type theories of the λ-cube.

1.1 System F^ω

The following description of System F^ω differs from the standard presentation in a few important ways:

1. the syntax introduced is of a generic form which makes certain definitions more economical,
2. a bidirectional PTS style is used but weakening is replaced with a well-formed context relation.

These changes do not affect the set of proofs or formula that are derivable internally in the system.

Syntax consists of three forms: variables ($x, y, z, \ldots$), binders ($\mathfrak{b}$), and constructors ($\mathfrak{c}$). Every binder and constructor has an associated discriminate or tag to determine the specific syntactic form. Constructor tags have an associated arity ($\mathfrak{a}$) which determines the number of arguments, or subterms, the specific constructor contains. A particular syntactic expression will be interchangeably called a syntactic form, a term, or a subterm if it exists inside another term in context. See Figure 1.1 for the complete syntax of F^ω. Note that the grammar for the syntax is defined using a BNF-style [17] where $t ::= f(t_1, t_2, \ldots)$ represents a recursive definition defining a category of

$$
\begin{aligned}
FV(x) &= \{x\} \\
FV(\mathfrak{b}(\kappa_1, x : t_1, t_2)) &= FV(t_1) \cup (FV(t_2) - \{x\}) \\
FV(\mathfrak{c}(\kappa_2, t_1, \ldots, t_{\mathfrak{a}(\kappa_2)})) &= FV(t_1) \cup \cdots \cup FV(t_{\mathfrak{a}(\kappa_2)})
\end{aligned}
$$

$$
\begin{aligned}
[y := t]x &= x \\
[y := t]y &= t \\
[y := t]\mathfrak{b}(\kappa_1, x : t_1, t_2) &= \mathfrak{b}(\kappa_1, x : [y := t]t_1, [y := t]t_2) \\
[y := t]\mathfrak{c}(\kappa_2, t_1, \ldots, t_{\mathfrak{a}(\kappa_2)}) &= \mathfrak{c}(\kappa_2, [y := t]t_1, \ldots, [y := t]t_{\mathfrak{a}(\kappa_2)})
\end{aligned}
$$

Figure 1.2: Operations on syntax for System F^ω, including computing free variables and susbtitution.

syntax, t, by its allowed subterms. For convenience a shorthand form is defined for each tag to maintain a more familiar appearance with standard syntactic definitions. Thus, instead of writing $\mathfrak{b}(\lambda, (x : A), t)$ the more common form is used: $\lambda\, x : A.\, t$. Whenever the tag for a particular syntactic form is known the shorthand will always be used instead.

Free variables of syntax is defined by a straightforward recursion that collects variables that are not bound in a set. Likewise, substitution is recursively defined by searching through subterms and replacing the associated free variable with the desired term. See Figure 1.2 for the definitions of substitution and computing free variables. However, there are issues with variable renaming that must be solved. A syntactic form is renamed by consistently replacing bound and free variables such that there is no variable capture. For example, the syntax $\lambda\, x : A.\, y\ x$ cannot be renamed to $\lambda\, y : A.\, y\ y$ because it captures the free variable y with the binder λ. More critically, variable capture changes the meaning of a term. There are several rigorous ways to solve variable renaming including (non-exhaustively): De Bruijn indices (or levels) [13], locally-nameless representations [7], nominal sets [48], locally-nameless sets [49], etc. All techniques incorporate some method of representing syntax uniquely with respect to renaming. For this work the variable bureaucracy will be dispensed with. It will be assumed that renaming is implicitly applied whenever necessary to maintain the meaning of a term. For example, $\lambda\, x : A.\, y\ x = \lambda\, z : A.\, y\ z$ and the substitution $[x := t]\lambda\, x : A.\, y\ x$ unfolds to $\lambda\, x : [x := t]A.\, [z := t](y\ x)$.

The syntax of F^ω has a well understood notion of reduction (or dynamics, or computation) defined in Figure 1.3. This is an *inductive* definition of a two-argument relation on terms. A given rule of the definition is represented by a collection of premises $(P_1, \ldots, P_n)$ written above the horizontal line and a conclusion (C) written below the line. An optional name for the rule (Example) appears to the right of the horizontal line. An inductive definition induces a structural induction principle allowing reasoning by cases on the rules and applying the induction hypothesis on the premises. During inductive proofs it is convenient to name the derivation of a premise $(\mathcal{D}_1, \ldots, \mathcal{D}_n)$. Moreover, to minimize clutter during proofs the name of the rule is removed.

$$\frac{t_1 \rightsquigarrow t_1'}{\mathfrak{b}(\kappa, x : t_1, t_2) \rightsquigarrow \mathfrak{b}(\kappa, x : t_1', t_2)} \qquad \frac{t_2 \rightsquigarrow t_2'}{\mathfrak{b}(\kappa, x : t_1, t_2) \rightsquigarrow \mathfrak{b}(\kappa, x : t_1, t_2')}$$

$$\frac{t_i \rightsquigarrow t_i' \qquad i \in 1, \ldots, \mathfrak{a}(\kappa)}{\mathfrak{c}(\kappa, t_1, \ldots t_i, \ldots t_{\mathfrak{a}(\kappa)}) \rightsquigarrow \mathfrak{c}(\kappa, t_1, \ldots t_i', \ldots t_{\mathfrak{a}(\kappa)})}$$

$$(\lambda\, x{:}A.\, b)\ t \rightsquigarrow [x := t]b$$

Figure 1.3: Reduction rules for System F^ω.

$$\frac{}{t\ R^*\ t}\ \textsc{Reflexive} \qquad \frac{t\ R\ t' \qquad t'\ R^*\ t''}{t\ R^*\ t''}\ \textsc{Transitive}$$

Figure 1.4: Reflexive-transitive closure of a relation R.

$$\frac{P_1 \qquad \ldots \qquad P_n}{C}\ \textsc{Example} \qquad \frac{\overset{\mathcal{D}_1}{P_1} \qquad \ldots \qquad \overset{\mathcal{D}_n}{P_n}}{C}$$

Inductive definitions build a finite tree of rule applications concluding with axioms (or leafs). Axioms are written without premises and optionally include the horizontal line. The reduction relation for F^ω consists of three rules and one axiom. Relations defined in this manner are always the *least* relation that satisfies the definition. In other words, any related terms must have a corresponding inductive tree witnessing the relation.

The reduction relation (or step relation) models function application anywhere in a term via its axiom, called the β-rule. This relation is antisymmetric. There is a *source* term s and a *target* term t, $s \rightsquigarrow t$, where t is the result of one function evaluation in s. Alternatively, $s \rightsquigarrow t$ is read as s *steps* to t. Note that if there is no λ-term applied to an argument (i.e. no function ready to be evaluated) for a given term t then that term cannot be the source term in the reduction relation. A term that cannot be a source is called a *value*. If there exists some sequence of terms related by reduction that end with a value, then all source terms in the sequence are *normalizing*. If *all* possible sequences of related terms end with a value for a particular source term s, then s is *strongly normalizing*. Restricting the set of terms to a normalizing subset is critical to achieve decidability of the reduction relation.

For any relation $-R-$, the reflexive-transitive closure ($-R^*-$) is inductively defined with two rules as shown in Figure 1.4. In the case of the step relation the reflexive-transitive closure, $s \rightsquigarrow^* t$, is called the *multistep relation*. Additionally, when $s \rightsquigarrow^* t$ then s *multisteps* to t. It is easy to show that any reflexive-transitive closure is itself transitive.

Lemma 1.1. *Let R be a relation on a set A and let $a, b, c \in A$. If $a\ R^*\ b$ and $b\ R^*\ c$ then $a\ R^*\ c$*

Proof. By induction on $a\ R^*\ b$.

Case: $\dfrac{}{t\ R^*\ t}$

It must be the case the $a = b$.

Case: $\dfrac{\overset{\mathcal{D}_1}{t\ R\ t'} \qquad \overset{\mathcal{D}_2}{t'\ R^*\ t''}}{t\ R^*\ t''}$

Let $z = t'$, then we have $a\ R\ z$ and $z\ R^*\ b$. By the inductive hypothesis (IH) we have $z\ R^*\ c$ and by the transitive rule we have $a\ R^*\ c$ as desired.

□

Two terms are *convertible*, written $t_1 \equiv t_2$, if $\exists\ t'$ such that $t_1 \rightsquigarrow^* t'$ and $t_2 \rightsquigarrow^* t'$. Note that this is not the only way to define convertibility in a type theory, but it is the standard method for a PTS. Convertibility is used in the typing rules to allow syntax forms to have continued valid types as terms reduce. It may be tempting to view conversion as the reflexive-symmetric-transitive closure of the step relation, but transitivity is not an obvious property. In fact, proving transitivity of conversion is often a significant effort, beginning with the confluence lemma.

Lemma 1.2 (Confluence)**.** *If $s \rightsquigarrow^* t_1$ and $s \rightsquigarrow^* t_2$ then $\exists\ t'$ such that $t_1 \rightsquigarrow^* t'$ and $t_2 \rightsquigarrow^* t'$*

Proof. See Appendix **??** for a proof of confluence involving a larger reduction relation. Note that F^ω's step relation is a subset of this relation and thus is confluent. □

Theorem 1.3 (Transitivity of Conversion)**.** *If $a \equiv b$ and $b \equiv c$ then $a \equiv c$*

Proof. By premises we know $\exists u, v$ such that $a \rightsquigarrow^* u$, $b \rightsquigarrow^* u$, $b \rightsquigarrow^* v$, and $c \rightsquigarrow^* v$. By confluence, $\exists z$ such that $u \rightsquigarrow^* z$ and $v \rightsquigarrow^* z$. By transitivity of multistep reduction, $a \rightsquigarrow^* z$ and $c \rightsquigarrow^* z$. Therefore, $a \equiv c$. □

Figure 1.5 defines the typing relation on terms for F^ω. As previously mentioned this formulation is different from standard presentations. Four relations are defined mutually:

1. $\Gamma \vdash t \rhd T$, to be read as T is the inferred type of the term t in the context Γ or, t infers T in Γ;

2. $\Gamma \vdash t \blacktriangleright T$, to be read as T is the inferred type, possibly after some reduction, of the term t in the context Γ or, t reduction-infers T in Γ;

3. $\Gamma \vdash t \lhd T$, to be read as T is checked against the inferred type of the term t in the context Γ or, t checks against T in Γ;

4. $\vdash \Gamma$, to be read as the context Γ is well-formed, and thus consists only of types that themselves have a type

$$\frac{\Gamma \vdash t \rhd A \qquad A \rightsquigarrow^* B}{\Gamma \vdash t \blacktriangleright B}\ \textsc{RedInf} \qquad \frac{B = \square \vee \Gamma \vdash B \blacktriangleright K \qquad \Gamma \vdash t \rhd A \qquad A \equiv B}{\Gamma \vdash t \lhd B}\ \textsc{Chk}$$

$$\frac{}{\vdash \varepsilon}\ \textsc{CtxEm} \qquad \frac{x \notin FV(\Gamma) \qquad \vdash \Gamma \qquad \Gamma \vdash A \blacktriangleright K}{\vdash \Gamma, x : A}\ \textsc{CtxApp}$$

$$\frac{\vdash \Gamma}{\Gamma \vdash \star \rhd \square}\ \textsc{Axiom} \qquad \frac{\vdash \Gamma \qquad (x : A) \in \Gamma}{\Gamma \vdash x \rhd A}\ \textsc{Var}$$

$$\frac{\Gamma \vdash A \blacktriangleright \square \qquad \Gamma, x : A \vdash B \blacktriangleright \square}{\Gamma \vdash (x : A) \to B \rhd \square}\ \textsc{Pi1} \qquad \frac{\Gamma \vdash A \blacktriangleright K \qquad \Gamma, x : A \vdash B \blacktriangleright \star}{\Gamma \vdash (x : A) \to B \rhd \star}\ \textsc{Pi2}$$

$$\frac{\Gamma \vdash (x : A) \to B \blacktriangleright K \qquad \Gamma, x : A \vdash t \rhd B}{\Gamma \vdash \lambda\, x{:}A.\, t \rhd (x : A) \to B}\ \textsc{Lam} \qquad \frac{\Gamma \vdash f \blacktriangleright (x : A) \to B \qquad \Gamma \vdash a \lhd A}{\Gamma \vdash f\ a \rhd [x := a]B}\ \textsc{App}$$

Figure 1.5: Typing rules for System F^ω. The variable K is a metavariable representing either $\star$ or $\square$.

Note that there are two Pi rules that restrict the domain and codomain pairs of function types to three possibilities: $(\square, \square)$, $(\star, \star)$, and $(\square, \star)$. This is exactly what is required by the λ-cube for this definition to be F^ω. For the unfamiliar reading these rules is arcane, thus exposition explaining a small selected set is provided.

$$\frac{\vdash \Gamma}{\Gamma \vdash \star \rhd \square}\ \textsc{Axiom}$$

The axiom rule has one premise, requiring that the context is well-formed. It concludes that the constant term $\star$ has type $\square$. Intuitively, the term $\star$ should be viewed as a universe of types, or a type of types, often referred to as a *kind*. Likewise, the term $\square$ should be viewed as a universe of kinds, or a kind of kinds. An alternative idea would be to change the conclusion to $\Gamma \vdash \star \rhd \star$. This is called the *type-in-type* rule, and it causes the type theory to be inconsistent [25, 28]. Note that there is no way to determine a type for $\square$. It plays the role of a type only.

$$\frac{\vdash \Gamma \qquad (x : A) \in \Gamma}{\Gamma \vdash x \rhd A}\ \textsc{Var}$$

The variable rule is a context lookup. It scans the context to determine if the variable is anywhere in context and then the associated type is what that variable infers. This rule is what requires the typing relation to mention a context. Whenever a type is inferred or checked it is always desired that the context is well-formed. That is why the variable rule also requires the context to be well-formed as a premise, because it is a leaf relative to the inference relation. Without this additional premise there could be typed terms in ill-formed contexts.

$$\frac{\Gamma \vdash f \blacktriangleright (x : A) \to B \qquad \Gamma \vdash a \lhd A}{\Gamma \vdash f\ a \rhd [x := a]B}\ \textsc{App}$$

The application rule infers the type of the term f and reduces that type until it looks like a function-type. Once a function type is required it is clear that the type of the term a must match the function-type's argument-type. Thus, a is checked against the type A. Finally, the inferred result of the application is the codomain of the function-type B with

the term a substituted for any free occurrences of x in B. This substitution is necessary because this application could be a type application to a type function. For example, let $f = \lambda\, X : \star.\, \mathrm{id}\ X$ where id is the identity term. The inferred type of f is then $(X : \star) \to X \to X$. Let $a = \mathbb{N}$ (any type constant), then $f\ \mathbb{N} \rhd [X := \mathbb{N}](X \to X)$ or $f\ \mathbb{N} \rhd \mathbb{N} \to \mathbb{N}$.

While this presentation of F^ω is not standard Lennon-Bertrand demonstrated that it is equivalent to the standard formulation [33]. In fact, Lennon-Bertrand showed that a similar formulation is logically equivalent for the stronger CIC. Thus, standard metatheoretical results such as preservation and strong normalization still hold.

Lemma 1.4 (Preservation of F^ω)**.** *If* $\Gamma \vdash s \lhd T$ *and* $s \rightsquigarrow^* t$ *then* $\Gamma \vdash t \lhd T$

Proof. See Appendix **??** for a proof of preservation of a conservative extension of F^ω, and thus a proof of preservation for F^ω itself. □

Theorem 1.5 (Strong Normalization of F^ω)**.** *If* $\Gamma \vdash t \rhd T$ *then* t *and* T *are strongly normalizing*

Proof. System F^ω is a subsystem of CC which has several proofs of strong normalization. See (non-exhaustively) proofs using saturated sets [22], model theory [58], realizability [42], etc. □

With strong normalization the convertibility relation is decidable, and moreover, type checking is decidable. Let *red* be a function that reduces its input until it is either $\star$, $\Box$, a binder, or in normal form. Note that this function is defined easily by applying the outermost reduction and matching on the resulting term. Let *conv* test the convertibility of two terms. Note that this function may be defined by reducing both terms to normal forms and comparing them for syntactic identity. Both functions are well-defined because F^ω is strongly normalizing. Then the functions *infer*, *check*, and *wf* can be mutually defined by following the typing rules. Thus, type inference and type checking is decidable for F^ω.

While it is true that F^ω only has function types as primitives several other data types are internally derivable using function types. For example, the type of natural numbers is defined:

$$\mathbb{N} = (X : \star) \to X \to (X \to X) \to X$$

Likewise, pairs and sum types are defined:

$$A \times B = (X : \star) \to (A \to B \to X) \to X$$

$$A + B = (X : \star) \to ((A \to X) \times (B \to X)) \to X$$

The logical constants true and false are defined:

$$\top = (X : \star) \to X \to X$$

$$\bot = (X : \star) \to X$$

Negation is defined as implying false:

$$\neg A = A \to \bot$$

These definitions are called *Church encodings* and originate from Church's initial encodings of data in the λ-calculus [9, 10]. Note that if there existed a term such that $\vdash t \lhd \bot$ then trivially for *any* type T we have $\vdash t\ T \lhd T$. Thus, $\bot$ is both the constant false and the proposition representing the principle of explosion from logic. Moreover, this allows a concise statement of the consistency of F^ω.

Theorem 1.6 (Consistency of System F^ω)**.** *There is no term t such that $\vdash t \lhd \bot$*

Proof. Suppose $\vdash t \lhd \bot$. Let n be the value of t after it is normalized. By preservation $\vdash n \lhd \bot$. Deconstructing the checking judgment we know that $\vdash n \rhd T$ and $T \equiv \bot$, but $\bot$ is a value and values like n infer types that are also values. Thus, $T = \bot$ and we know that $\vdash n \rhd \bot$. By inversion on the typing rules $n = \lambda\, X : \star.\, b$, and we have $X : \star \vdash b \rhd X$. The term b can only be $\star$, $\Box$, or X, but none of these options infer type X. Therefore, there does not exist a term b, nor a term n, nor a term t. □

Recall that induction principles cannot be derived internally for any encoding of data [23]. This is not only cumbersome but unsatisfactory as the natural numbers are in their essence the least set satisfying induction. Ultimately, the issue is that these encodings are too general. They admit theoretical elements that F^ω is not flexible enough to express nor strong enough to exclude.

1.2 Calculus of Constructions and Cedille

As previously mentioned, CC is one extension away from F^ω on the λ-cube. Indeed, the two rules Pi1 and Pi2 can be merged to form CC:

$$\frac{\Gamma \vdash A \blacktriangleright K_1 \qquad \Gamma, x : A \vdash B \blacktriangleright K_2}{\Gamma \vdash (x : A) \to B \rhd K_2}\ \textsc{Pi}$$

where now both K_1 and K_2 are metavariables representing either $\star$ or $\Box$. Note that no other rules, syntax, or reductions need to be changed. Replacing Pi1 and Pi2 with this new Pi rule is enough to obtain a complete and faithful definition of CC.

With this merger types are allowed to depend on terms. From a logical point of view, this is a quantification over terms in formula. Hence, why CC is a predicate logic instead of a propositional one according to the Curry-Howard correspondence. Yet, there is a question about what exactly quantification over terms means. Surely it does not mean quantification over syntactic forms.

It means, at minimum, quantification over well-typed terms, but from a logical perspective these terms correspond to proofs. In first order predicate logic the domain of quantification ranges over a set of *individuals*. The set of individuals represents any potential set of interest with specific individuals identified through predicates expressing their properties. With proofs the situation is different. A proof has meaning relative to its formula, but this meaning may not be relevant as an individual in predicate logic. For example, the proof 2 for a Church encoded natural number is

intuitively data, but a proof that 2 is even is intuitively not. In CC, both are merely proofs that can be quantified over.

Cedille alters the domain of quantification from proofs to (untyped) λ-caluclus terms. Thus, for Cedille, the proof 2 becomes the encoding of 2 and the proof that 2 is even can *also* be the encoding of 2. This is achieved through a notion of *erasure* which removes type information and auxiliary syntactic forms from a term. Additionally, convertibility is modified to be convertibility of λ-calculus terms. However, erasure as it is defined in Cedille enables diverging terms in inconsistent contexts. The result by Abel and Coquand, which applies to a wide range of type theories including Cedille, is one way to construct a diverging term [1].

If terms are able to diverge, in what sense are they a proof? What a proof is or is not is difficult to say. As early as Aristotle there are documented forms of argument, Aristotle's syllogisms [3]. More than a millennium later Euclid's *Elements* is the most well-known example of a mathematical text containing what a modern audience would call proofs. Moreover, visual renditions of *Elements*, initiated by Byrne, challenge the notion of a proof being an algebraic object [6]. However, the study of proof as a mathematical object dates first to Frege [18] followed soon after by Peano's formalism of arithmetic [44] and Whitehead and Russell's *Principia Mathematica* [61]. For the kinds of logics discussed by the Curry-Howard correspondence, structural proof theories, the originator is Gentzen [20, 21]. Gentzen's natural deduction describes proofs as finite trees labelled by rules. Note that this is, of course, a very brief history of mathematical proof.

All of these formulations may be justified as acceptable notions of proof, but the purpose of proof from an epistemological perspective is to provide justification. It is unsatisfactory to have a claimed proof and be unable to check that it is constructed only by the rules of the proof theory. This is the situation with Cedille, although rare, there are terms where reduction diverges making it impossible to check a type. However, it is unfair to levy this criticism against Cedille alone, as well-known type theories also lack decidability of type checking. For example, Nuprl with its equality reflection rule [2], and the proof assistant Lean with its notion of casts [41]. Moreover, Lean has been incredibly successful in formalizing research mathematics including the Liquid Tensor Experiment [34] and Tao's formalization of The Polynomial Freiman-Ruzsa Conjecture [59]. Indeed, not having decidability of type checking does to necessarily prevent a tool from producing convincing arguments.

Ultimately, the definition of proof is a philosophical one with no absolute answer, but this work will follow Gentzen and Kreisel in requiring that a proof is a finite tree, labelled by rules, supporting decidable proof checking. The reader need only asks themselves which proof they would prefer if the option was available: one that potentially diverges, or one that definitely does not. If it is the latter, then striving for decidable type theories that are capable enough to reproduce the results obtained by proof assistants like Lean is a worthy goal.

1.3 Thesis

Cedille is a powerful type theory capable of deriving inductive data with relatively modest extension and modification to CC. However, this capability comes at the cost of decidability of type checking and thus, in the opinion of Kreisel, the cost of a Curry-Howard correspondence to a proof theory. A redesign of Cedille that focuses on maintaining a proof-theoretic view recovers decidability of type checking while still solving the original goals of Cedille. Although this redesign does prevent some constructions from being possible, the new balance struck between capability and complexity is desirable because of a well-behaved metatheory.

1.4 Contributions

Chapter 2 defines the Cedille2 Core (CC2) theory, including its syntax, and typing rules. Erasure from Cedille is rephrased as a projection from proofs to objects. Basic metatheoretical results are proven including: confluence, preservation, and classification.

Chapter 3 models CC2 in F^{ω} obtaining a strong normalization result for proof normalization. This model is a straightforward extension of a similar model for CC. Critically, proof normalization is not powerful enough to show consistency nor object normalization. Additionally, CC2 is shown to be a conservative extension of F^{ω}.

Chapter 4 models CC2 in CDLE obtaining consistency for CC2. Although CDLE is not strongly normalizing it still possess a realizability model which justifies its logical consistency. CC2 is closely related to CDLE which makes this models straightforward to accomplish. Moreover, a selection of axioms added to CC2 is shown to recover much of CDLEs features.

Chapter 5 proves object normalization from proof normalization and consistency. The φ, or cast, rule is the only difficulty after proof normalization and consistency. However, any proof can be translated into a new proof that contains no cast rules. Applying this observation yields an argument to obtain full object normalization.

Chapter 6 with normalization for both proofs and objects a well-founded type checker is defined. This implementation leverages normalization-by-evaluation and other basic techniques like pattern-based unification. The tool it benchmarked to demonstrate reasonable performance.

Chapter 7 contains derivations of generic inductive data, quotient types, large eliminations, constructor subtyping, and inductive-inductive data. All of these constructions are possible in Cedille but require modest modifications to derive in Cedille2.

Chapter 8 concludes with a collection of open conjectures and questions. Cedille2 at the conclusion of this work is still in its infancy.

Chapter 2

THEORY DESCRIPTION AND BASIC METATHEORY

This chapter describes the syntax, reduction, and inference judgment of the core system for Cedille2. Near the conclusion, this chapter also proves basic metatheoretic properties such as a weakening lemma, substitution lemma, classification, and preservation. The presentation is a classical PTS-style with a single inference judgment. As it stands it is not obvious how this judgment admits an inference algorithm, but this situation will be remedied in Chapter 6 with an explicit algorithm.

2.1 Syntax and Reduction

Syntax for the system is defined generically as before. See Figure 2.1 for a complete description. The intended meaning of the syntax is as follows:

1. tags λ_m, Π_m and $\bullet_m$ (application) represent the function fragment of syntax parameterized by three separates *modes*, ω (free), 0 (erased), and τ (type-level);
2. tags $\cap$, pair, proj_1, and proj_2 represent dependent intersections (i.e. dependent pairs);
3. tags eq, refl, ψ (substitution), ϑ (promotion), δ (separation), and φ (cast) represent equality.

At the moment raw syntax has no essential meaning beyond its intended one. Nevertheless, a basic fact about substitution on syntax is provable.

Lemma 2.1. *If $x \neq y$ and $y \notin FV(a)$ then*

$$[x := a][y := b]t = [y := [x := a]b][x := a]t$$

Proof. By induction on t. If t is a binder or a constructor, then substitution unfolds and the IH applied to subterms concludes those cases. Suppose t is a variable, z. If $z = x$, then $z \neq y$ and $t = a$ on both sides because $y \notin FV(a)$. If $z = y$, then $z \neq x$ and $t = [x := a]b$ on both sides. If $z \neq x$ and $z \neq y$, then $t = z$ on both sides. □

Computational meaning is added via reduction rules described in Figure 2.2. The new reductions model projection of pairs (e.g. $[t_1, t_2, t_3].1 \rightsquigarrow t_1$), promotion of equalities (e.g. $\vartheta(\text{refl}(z; Z), a, b; A) \rightsquigarrow \text{refl}(a; A)$) and an elimination form for equality. Note that conversion is different from a traditional PTS. Convertibility with respect to reduction is written: $t \rightleftharpoons s$. A detailed discussion of conversion is delayed until Section 2.3.

Before more important facts about reduction can be discussed it is important to observe the interaction between reduction and substitution. First, note that multistep reduction (i.e. the reflexive-transitive closure of the reduction relation) is congruent with respect to syntax. Second, substitution is shown to commute with multistep reduction through a series of lemmas.

$$
\begin{aligned}
t &::= x_K \mid \mathfrak{b}(\kappa_1, x : t_1, t_2) \mid \mathfrak{c}(\kappa_2, t_1, \ldots, t_{\mathfrak{a}(\kappa_2)}) \\
\kappa_1 &::= \lambda_m \mid \Pi_m \mid \cap \\
\kappa_2 &::= \diamond \mid \star \mid \square \mid \bullet_m \mid \mathrm{pair} \mid \mathrm{proj}_1 \mid \mathrm{proj}_2 \mid \mathrm{eq} \mid \mathrm{refl} \mid \psi \mid \vartheta \mid \delta \mid \varphi \\
m &::= \omega \mid 0 \mid \tau
\end{aligned}
$$

$$
\begin{aligned}
&\mathfrak{a}(\diamond) = \mathfrak{a}(\star) = \mathfrak{a}(\square) = 0 && \mathfrak{a}(\mathrm{pair}) = \mathfrak{a}(\mathrm{eq}) = 3 \\
&\mathfrak{a}(\mathrm{proj}_1) = \mathfrak{a}(\mathrm{proj}_2) = \mathfrak{a}(\delta) = 1 && \mathfrak{a}(\vartheta) = 4 \\
&\mathfrak{a}(\bullet_m) = \mathfrak{a}(\mathrm{refl}) = 2 && \mathfrak{a}(\psi) = \mathfrak{a}(\varphi) = 5
\end{aligned}
$$

$$
\begin{aligned}
\diamond &:= \mathfrak{c}(\diamond) & [t_1, t_2; A] &:= \mathfrak{c}(\mathrm{pair}, t_1, t_2, A) \\
\star &:= \mathfrak{c}(\star) & t.1 &:= \mathfrak{c}(\mathrm{proj}_1, t) \\
\square &:= \mathfrak{c}(\square) & t.2 &:= \mathfrak{c}(\mathrm{proj}_2, t) \\
\lambda_m\, x{:}A.\, t &:= \mathfrak{b}(\lambda_m, x : A, t) & a =_A b &:= \mathfrak{c}(\mathrm{eq}, a, A, b) \\
(x : A) \to_m B &:= \mathfrak{b}(\Pi_m, x : A, B) & \mathrm{refl}(t; A) &:= \mathfrak{c}(\mathrm{refl}, t, A) \\
(x : A) \cap B &:= \mathfrak{b}(\cap, x : A, B) & \vartheta(e, a, b; T) &:= \mathfrak{c}(\vartheta, e, a, b, T) \\
f \bullet_m a &:= \mathfrak{c}(\bullet_m, f, a) & \varphi(a, b, e; A, T) &:= \mathfrak{c}(\varphi, a, b, e, A, T) \\
\psi(e, a, b; A, P) &:= \mathfrak{c}(\psi, e, a, b, A, P) & \delta(e) &:= \mathfrak{c}(\delta, e)
\end{aligned}
$$

Figure 2.1: Generic syntax, there are three constructors, variables, a generic binder, and a generic non-binder. Each is parameterized with a constant tag to specialize to a particular syntactic consruct. The non-binder constructor has a vector of subterms determined by an arity function computed on tags. Standard syntactic constructors are defined in terms of the generic forms.

$$
\frac{t_1 \rightsquigarrow t_1'}{\mathfrak{b}(\kappa, x : t_1, t_2) \rightsquigarrow \mathfrak{b}(\kappa, x : t_1', t_2)} \qquad \frac{t_2 \rightsquigarrow t_2'}{\mathfrak{b}(\kappa, x : t_1, t_2) \rightsquigarrow \mathfrak{b}(\kappa, x : t_1, t_2')}
$$

$$
\frac{t_i \rightsquigarrow t_i' \qquad i \in 1, \ldots, \mathfrak{a}(\kappa)}{\mathfrak{c}(\kappa, t_1, \ldots t_i, \ldots t_{\mathfrak{a}(\kappa)}) \rightsquigarrow \mathfrak{c}(\kappa, t_1, \ldots t_i', \ldots t_{\mathfrak{a}(\kappa)})}
$$

$$
\begin{aligned}
(\lambda_m\, x{:}A.\, b) \bullet_m t &\rightsquigarrow [x := t]b \\
[t_1, t_2; A].1 &\rightsquigarrow t_1 \\
[t_1, t_2; A].2 &\rightsquigarrow t_2 \\
\psi(\mathrm{refl}(z; Z), a, b; A, P) \bullet_\omega t &\rightsquigarrow t \\
\vartheta(\mathrm{refl}(z; Z), a, b; T) &\rightsquigarrow \mathrm{refl}(a; T)
\end{aligned}
$$

$$
s_1 \rightleftharpoons s_2 \text{ iff } \exists\, t.\ s_1 \rightsquigarrow^* t \text{ and } s_2 \rightsquigarrow^* t
$$

Figure 2.2: Reduction and conversion for arbitrary syntax.

Lemma 2.2. *If $t_i \rightsquigarrow^* t_i'$ for any i then,*

1. $\mathfrak{b}(\kappa, (x : t_1), t_2) \rightsquigarrow^* \mathfrak{b}(\kappa, (x : t_1'), t_2')$
2. $\mathfrak{c}(\kappa, t_1, \ldots, t_{\mathfrak{a}(\kappa)}) \rightsquigarrow^* \mathfrak{c}(\kappa, t_1', \ldots, t_{\mathfrak{a}(\kappa)}')$

Proof. Pick any i and apply the reductions to the associate subterm. A straightforward induction on $t_i \rightsquigarrow^* t_i'$ demonstrates that the reductions apply only to the associated subterm. Repeat until all i reductions are applied. □

Lemma 2.3. *If $a \rightsquigarrow b$ then $[x := t]a \rightsquigarrow [x := t]b$*

Proof. By induction on $a \rightsquigarrow b$. Second projection is the same as first projection case and omitted.

Case: $(\lambda_m\, x : A.\, b) \bullet_m t \rightsquigarrow [x := t]b$

$[x := s]((\lambda_m\, y : A.\, b) \bullet_m t) = (\lambda_m\, x : [x := s]A.\, [x := s]b) \bullet_m [x := s]t \rightsquigarrow [y := [x := s]t][x := s]b = [x := s][y := t]b$
Note that the final equality holds by Lemma 2.1.

Case: $[t_1, t_2; A].1 \rightsquigarrow t_1$

$[x := t][t_1, t_2, A].1 = [[x := t]t_1, [x := t]t_2, [x := t]A].1 \rightsquigarrow [x := t]t_1$

Case: $\psi(\mathrm{refl}(z; Z), u, v; A, P) \bullet_\omega b \rightsquigarrow b$

$[x := t]\psi(\mathrm{refl}(z; Z), u, v; A, P) \bullet_\omega b = \psi(\mathrm{refl}([x := t]z; [x := t]Z), [x := t]u, [x := t]v; [x := t]A, [x := t]P) \bullet_\omega [x := t]b \rightsquigarrow [x := t]b$

Case: $\vartheta(\mathrm{refl}(z; Z), u, v; A) \rightsquigarrow \mathrm{refl}(u; A)$

$[x := t]\vartheta(\mathrm{refl}(z; Z), u, v; A) = \vartheta(\mathrm{refl}([x := t]z; [x := t]Z), [x := t]u, [x := t]v; [x := t]A) \rightsquigarrow \mathrm{refl}([x := t]u; [x := t]A) = [x := t]\mathrm{refl}(u; A)$

Case: $$\frac{\overset{\mathcal{D}_1}{t_i \rightsquigarrow t_i'} \qquad i \in 1, \ldots, \mathfrak{a}(\kappa)}{\mathfrak{c}(\kappa, t_1, \ldots t_i, \ldots t_{\mathfrak{a}(\kappa)}) \rightsquigarrow \mathfrak{c}(\kappa, t_1, \ldots t_i', \ldots t_{\mathfrak{a}(\kappa)})}$$

By the IH, $[x := t]t_i \rightsquigarrow [x := t]t_i'$. Note that

$$[x := t]\mathfrak{c}(\kappa, t_1, \ldots, t_{\mathfrak{a}}(\kappa)) = \mathfrak{c}(\kappa, [x := t]t_1, \ldots, [x := t]t_{\mathfrak{a}}(\kappa))$$

Applying the constructor reduction rule and reversing the previous equality concludes the case.

Case: $$\frac{\overset{\mathcal{D}_1}{t_1 \rightsquigarrow t_1'}}{\mathfrak{b}(\kappa, x : t_1, t_2) \rightsquigarrow \mathfrak{b}(\kappa, x : t_1', t_2)}$$

By the IH, $[x := t]t_1 \rightsquigarrow [x := t]t_1'$. Note that

$$[x := t]\mathfrak{b}(\kappa, (y : t_1), t_2) = \mathfrak{b}(\kappa, (y : [x := t]t_1), [x := t]t_2)$$

Applying the first binder reduction rule and reversing the previous equality concludes the case.

□

Lemma 2.4. *If $a \rightsquigarrow^* b$ then $[x := t]a \rightsquigarrow^* [x := t]b$*

Proof. By induction on $a \rightsquigarrow^* b$. The reflexivity case is trivial.

Case: $\dfrac{t \overset{\mathcal{D}_1}{R} t' \qquad t' \overset{\mathcal{D}_2}{R^*} t''}{t \; R^* \; t''}$

Let $z = t'$. By the IH applied to $\mathcal{D}_2$: $[x := t]z \rightsquigarrow^* [x := t]b$. By Lemma 2.3 applied to $\mathcal{D}_1$: $[x := t]a \rightsquigarrow [x := t]z$. Applying the transitivity rule yields $[x := t]a \rightsquigarrow^* [x := t]b$.

□

Lemma 2.5. *If $s \rightsquigarrow t$ then $[x := s]a \rightsquigarrow^* [x := t]a$*

Proof. By induction on a. The $\mathfrak{c}$ case is omitted because it is similar to the $\mathfrak{b}$ case.

Case: x

Rename y. Suppose $x = y$, then $[x := s]y = s \rightsquigarrow t = [x := t]y$. Thus, $[x := s]y \rightsquigarrow^* [x := t]y$. Suppose $x \neq y$, then $[x := s]y = y \rightsquigarrow^* y = [x := t]y$.

Case: $\mathfrak{b}(\kappa_1, x : t_1, t_2)$

By the IH $[x := s]t_1 \rightsquigarrow^* [x := t]t_1$ and $[x := s]t_2 \rightsquigarrow^* [x := t]t_2$. Lemma 2.2 concludes the case.

□

Lemma 2.6. *If $s \rightsquigarrow^* t$ and $a \rightsquigarrow^* b$ then $[x := s]a \rightsquigarrow^* [x := t]b$*

Proof. By induction on $s \rightsquigarrow^* t$. The reflexivity case is Lemma 2.4.

Case: $\dfrac{t \overset{\mathcal{D}_1}{R} t' \qquad t' \overset{\mathcal{D}_2}{R^*} t''}{t \; R^* \; t''}$

Let $z = t'$. By the IH applied to $\mathcal{D}_2$: $[x := z]a \rightsquigarrow^* [x := t]b$. Lemma 2.5 yields $[x := s]a \rightsquigarrow^* [x := z]a$. Transitivity concludes with $[x := s]a \rightsquigarrow^* [x := t]b$.

□

$$\frac{}{x_K \Rrightarrow x_K}\ \textsc{ParVar}$$

$$\frac{t_i \Rrightarrow t_i' \quad \forall\, i \in \{1, \ldots, \mathfrak{a}(\kappa)\}}{\mathfrak{c}(\kappa, t_1, \ldots, t_i, \ldots, t_{\mathfrak{a}(\kappa)}) \Rrightarrow \mathfrak{c}(\kappa, t_1', \ldots, t_i', \ldots, t_{\mathfrak{a}(\kappa)}')}\ \textsc{ParCtor}$$

$$\frac{t_1 \Rrightarrow t_1' \qquad t_2 \Rrightarrow t_2'}{\mathfrak{b}(\kappa, x : t_1, t_2) \Rrightarrow \mathfrak{b}(\kappa, x : t_1', t_2')}\ \textsc{ParBind}$$

$$\frac{t_1 \Rrightarrow t_1' \qquad t_2 \Rrightarrow t_2' \qquad t_3 \Rrightarrow t_3'}{(\lambda_m\, x{:}t_1.\, t_2) \bullet_m t_3 \Rrightarrow [x := t_3']t_2'}\ \textsc{ParBeta}$$

$$\frac{t_1 \Rrightarrow t_1' \qquad t_2 \Rrightarrow t_2' \qquad t_3 \Rrightarrow t_3' \qquad t_4 \Rrightarrow t_4' \qquad t_5 \Rrightarrow t_5' \qquad t_6 \Rrightarrow t_6' \qquad t_7 \Rrightarrow t_7'}{\psi(\mathrm{refl}(t_1; t_2), t_3, t_4; t_5, t_6) \bullet_\omega t_7 \Rrightarrow t_7'}\ \textsc{ParSubst}$$

$$\frac{t_1 \Rrightarrow t_1' \qquad t_2 \Rrightarrow t_2' \qquad t_3 \Rrightarrow t_3'}{[t_1, t_2; t_3].1 \Rrightarrow t_1'}\ \textsc{ParFst}$$

$$\frac{t_1 \Rrightarrow t_1' \qquad t_2 \Rrightarrow t_2' \qquad t_3 \Rrightarrow t_3'}{[t_1, t_2; t_3].2 \Rrightarrow t_2'}\ \textsc{ParSnd}$$

$$\frac{t_1 \Rrightarrow t_1' \qquad t_2 \Rrightarrow t_2' \qquad t_3 \Rrightarrow t_3' \qquad t_4 \Rrightarrow t_4' \qquad t_5 \Rrightarrow t_5'}{\vartheta(\mathrm{refl}(t_1; t_2), t_3, t_4; t_5) \Rrightarrow \mathrm{refl}(t_3'; t_5')}\ \textsc{ParPrm}$$

Figure 2.3: Parallel reduction rules for arbitrary syntax.

Lemma 2.6 is the only fact about the interaction of substitution and reduction that is needed moving forward. A straightforward consequence is a similar lemma about substitution commuting with convertibility w.r.t. reduction.

Lemma 2.7. *If $s \rightleftharpoons t$ and $a \rightleftharpoons b$ then $[x := s]a \rightleftharpoons [x := t]b$*

Proof. By definition $\exists\, z_1, z_2$ such that $t \rightsquigarrow^* z_1$, $s \rightsquigarrow^* z_1$, $a \rightsquigarrow^* z_2$, and $b \rightsquigarrow^* z_2$. Applying Lemma 2.6 twice yields $[x := s]a \rightsquigarrow^* [x := z_1]z_2$ and $[x := t]b \rightsquigarrow^* [x := z_1]z_2$. □

Transitivity, as before, is a consequence of confluence. Confluence is not an obvious property to obtain and can also be an involved property to prove. For example, a natural variant for the ϑ reduction rule is $\vartheta(\mathrm{refl}(t.1)) \rightsquigarrow \mathrm{refl}(t)$, but this breaks confluence. To see why, consider $\vartheta(\mathrm{refl}([x, y; T].1))$. One choice leads to $\vartheta(\mathrm{refl}(x))$, and the other leads to $\mathrm{refl}(x)$. However, these terms are not joinable, hence confluence fails.

2.2 Confluence

The proof of confluence follows the PLFA book [60]. This strategy involves the common technique of defining a parallel reduction variant of the one-step reduction described in Figure 2.2. Parallel

$$
\begin{aligned}
(\!|(\lambda_m\, x{:}t_1.\, t_2) \bullet_m t_3|\!) &= [x := (\!|t_3|\!)](\!|t_2|\!) \\
(\!|\psi(\mathrm{refl}(t_1; t_2), t_3, t_4; t_5, t_6) \bullet_\omega t_7|\!) &= (\!|t_7|\!) \\
(\!|[t_1, t_2; t_3].1|\!) &= (\!|t_1|\!) \\
(\!|[t_1, t_2; t_3].2|\!) &= (\!|t_2|\!) \\
(\!|\vartheta(\mathrm{refl}(t_1; t_2), t_3, t_4; t_5)|\!) &= \mathrm{refl}((\!|t_3|\!); (\!|t_5|\!)) \\
(\!|\mathfrak{c}(\kappa, t_1, \ldots, t_{\mathfrak{a}(\kappa)})|\!) &= \mathfrak{c}(\kappa, (\!|t_1|\!), \ldots, (\!|t_{\mathfrak{a}(\kappa)}|\!)) \\
(\!|\mathfrak{b}(\kappa, (x : t_1), t_2)|\!) &= \mathfrak{b}(\kappa, (x : (\!|t_1|\!)), (\!|t_2|\!)) \\
(\!|x_K|\!) &= x_K
\end{aligned}
$$

Figure 2.4: Definition of a reduction completion function $(\!|-|\!)$ for parallel reduction. Note that this function is defined by pattern matching, applying cases from top to bottom. Thus, the cases at the very bottom are catch-all for when the prior cases are not applicable.

reduction allows reduction steps to occur in any subexpression, but reductions that generate new redexes cannot be reduced in a single step. Figure 2.3 presents the inductive definition of parallel reduction. In fact, it is possible to compute the resulting syntax after all possible redexes are contracted by a single parallel reduction step. This is the *reduction completion* (written $(\!|t|\!)$) of some syntax t. The definition of reduction completion is shown in Figure 2.4. Reduction completion enables the derivation of a triangle property for parallel reduction of which confluence for parallel reduction is a consequence. With confluence for parallel reduction and logical equivalence then confluence of one-step reduction is immediate.

Lemma 2.8. *For any t, $t \Rrightarrow t$*

Proof. Straightforward by induction on t. □

Lemma 2.9. *If $s \rightsquigarrow t$ then $s \Rrightarrow t$*

Proof. By induction on $s \rightsquigarrow t$. The projection and promotion cases are similar to the substitution and beta case and thus omitted. The second structural binder reduction case is omitted.

Case: $(\lambda_m\, x{:}A.\, b) \bullet_m t \rightsquigarrow [x := t]b$

By Lemma 2.8: $t \Rrightarrow t$ and $b \Rrightarrow b$. Applying the ParBeta rule concludes the case.

Case: $\psi(\mathrm{refl}(z; Z), u, v; A, P) \bullet_\omega b \rightsquigarrow b$

Using Lemma 2.8: $z \Rrightarrow z$, $Z \Rrightarrow Z$, $u \Rrightarrow u$, $v \Rrightarrow v$, $A \Rrightarrow A$, $P \Rrightarrow P$, and $b \Rrightarrow b$. Applying the ParSubst rule concludes the case.

Case: $$\frac{t_i \overset{\mathcal{D}_1}{\rightsquigarrow} t_i' \qquad i \in 1, \ldots, \mathfrak{a}(\kappa)}{\mathfrak{c}(\kappa, t_1, \ldots t_i, \ldots t_{\mathfrak{a}(\kappa)}) \rightsquigarrow \mathfrak{c}(\kappa, t_1, \ldots t_i', \ldots t_{\mathfrak{a}(\kappa)})}$$

By the IH applied to $\mathcal{D}_1$: $t_i \Rrightarrow t_i'$. Note that there is only one subderivation. For all $j \neq i$ $t_j \Rrightarrow t_j$ by Lemma 2.8. Using the ParCtor rule concludes the case.

Case: $$\frac{\overset{\mathcal{D}_1}{t_1 \rightsquigarrow t_1'}}{\mathfrak{b}(\kappa, x : t_1, t_2) \rightsquigarrow \mathfrak{b}(\kappa, x : t_1', t_2)}$$

Applying the IH to $\mathcal{D}_1$ yields $t_1 \Rrightarrow t_1'$. By Lemma 2.8: $t_2 \Rrightarrow t_2$. Using the ParBind rule concludes the case.

□

Lemma 2.10. *If $s \rightsquigarrow^* t$ then $s \Rrightarrow^* t$*

Proof. By induction on $s \rightsquigarrow^* t$ applying Lemma 2.9 in the inductive case. □

Lemma 2.11. *If $s \Rrightarrow t$ then $s \rightsquigarrow^* t$*

Proof. By induction on $s \Rrightarrow t$. The projection, promotion, and substitution cases are similar to the beta case with the only difference being applying the associated rule.

Case: $$\frac{}{x_K \Rrightarrow x_K}$$

By reflexivity of reduction.

Case: $$\frac{\overset{\mathcal{D}_i}{t_i \Rrightarrow t_i' \quad \forall\, i \in \{1, \ldots, \mathfrak{a}(\kappa)\}}}{\mathfrak{c}(\kappa, t_1, \ldots, t_i, \ldots, t_{\mathfrak{a}(\kappa)}) \Rrightarrow \mathfrak{c}(\kappa, t_1', \ldots, t_i', \ldots, t'_{\mathfrak{a}(\kappa)})}$$

By the IH applied to each $\mathcal{D}_i$: $t_i \rightsquigarrow^* t_i'$ for all i. Applying Lemma 2.2 concludes the case.

Case: $$\frac{\overset{\mathcal{D}_1}{t_1 \Rrightarrow t_1'} \qquad \overset{\mathcal{D}_2}{t_2 \Rrightarrow t_2'}}{\mathfrak{b}(\kappa, x : t_1, t_2) \Rrightarrow \mathfrak{b}(\kappa, x : t_1', t_2')}$$

As the previous case, the IH yields $t_1 \rightsquigarrow^* t_1$ and $t_2 \rightsquigarrow^* t_2'$. Again using Lemma 2.2 concludes the case.

Case: $$\frac{\overset{\mathcal{D}_1}{t_1 \Rrightarrow t_1'} \qquad \overset{\mathcal{D}_2}{t_2 \Rrightarrow t_2'} \qquad \overset{\mathcal{D}_3}{t_3 \Rrightarrow t_3'}}{(\lambda_m\, x{:}t_1.\, t_2) \bullet_m t_3 \Rrightarrow [x := t_3']t_2'}$$

Applying the IH to all available derivations and using Lemma 2.2 gives $(\lambda_m\, x{:}t_1.\, t_2) \bullet_m t_3 \rightsquigarrow^* (\lambda_m\, x{:}t_1'.\, t_2') \bullet_m t_3'$. Applying the beta rule of reduction with transitivity concludes the case.

□

Lemma 2.12. *If $s \Rightarrow^* t$ then $s \rightsquigarrow^* t$*

Proof. By induction on $s \Rightarrow^* t$ applying Lemma 2.11 in the inductive case. □

Lemma 2.13. *If $s \Rightarrow s'$ and $t \Rightarrow t'$ then $[x := s]t \Rightarrow [x := s']t'$*

Proof. By induction on $t \Rightarrow t'$. The second projection case is omitted because it is the same as the first projection case.

Case: $\dfrac{}{x_K \Rightarrow x_K}$

Rename to y. If $x = y$ then $s \Rightarrow s'$ which is a premise. If $x \neq y$ then no substitution is performed and $y_K \Rightarrow y_K$.

Case: $\dfrac{t_i \overset{\mathcal{D}_i}{\Rightarrow} t'_i \quad \forall\, i \in \{1, \ldots, \mathfrak{a}(\kappa)\}}{\mathfrak{c}(\kappa, t_1, \ldots, t_i, \ldots, t_{\mathfrak{a}(\kappa)}) \Rightarrow \mathfrak{c}(\kappa, t'_1, \ldots, t'_i, \ldots, t'_{\mathfrak{a}(\kappa)})}$

Applying the IH to $\mathcal{D}_i$ yields $[x := s]t_i \Rightarrow [x := s']t'_i$ for all i. Unfolding substitution for $\mathfrak{c}$ and applying the ParCtor rule concludes the case.

Case: $\dfrac{t_1 \overset{\mathcal{D}_1}{\Rightarrow} t'_1 \qquad t_2 \overset{\mathcal{D}_2}{\Rightarrow} t'_2}{\mathfrak{b}(\kappa, x : t_1, t_2) \Rightarrow \mathfrak{b}(\kappa, x : t'_1, t'_2)}$

As above the IH gives $[x := s]t_i \Rightarrow [x := s']t'_i$ for $i = 1$ and $i = 2$. Unfolding substitution for $\mathfrak{b}$ and applying the ParBind rule concludes.

Case: $\dfrac{t_1 \overset{\mathcal{D}_1}{\Rightarrow} t'_1 \qquad t_2 \overset{\mathcal{D}_2}{\Rightarrow} t'_2 \qquad t_3 \overset{\mathcal{D}_3}{\Rightarrow} t'_3}{(\lambda_m\, x{:}t_1.\, t_2) \bullet_m t_3 \Rightarrow [x := t'_3]t'_2}$

By the IH: $[x := s]t_i \Rightarrow [x := s']t'_i$ for $i = 1, 2, 3$. The ParBeta rule gives the following: $[x := s](\lambda_m\, y{:}t_1.\, t_2) \bullet_m t_3 = (\lambda_m\, y{:}[x := s]t_1.\, [x := s]t_2) \bullet_m [x := s]t_3 \Rightarrow [y := t'_3][x := s']t'_2$. Note that y is bound and thus not a free variable in s' and, moreover, by implicit renaming $x \neq y$. Thus, by Lemma 2.1 $[y := t'_3][x := s']t'_2 = [x := s'][y := t'_3]t'_2$.

Case: $\dfrac{t_1 \overset{\mathcal{D}_1}{\Rightarrow} t'_1 \qquad t_2 \overset{\mathcal{D}_2}{\Rightarrow} t'_2 \qquad t_3 \overset{\mathcal{D}_2}{\Rightarrow} t'_3 \qquad t_4 \overset{\mathcal{D}_2}{\Rightarrow} t'_4 \qquad t_5 \overset{\mathcal{D}_2}{\Rightarrow} t'_5 \qquad t_6 \overset{\mathcal{D}_2}{\Rightarrow} t'_6 \qquad t_7 \overset{\mathcal{D}_2}{\Rightarrow} t'_7}{\psi(\mathrm{refl}(t_1; t_2), t_3, t_4; t_5, t_6) \bullet_\omega t_7 \Rightarrow t'_7}$

By the IH: $[x := s]t_i \Rightarrow [x := s']t'_i$ for $i = 1, 2$. The ParSubst rule gives: $[x := s](\psi(\mathrm{refl}(t_1; t_2), t_3, t_4; t_5, t_6) \bullet_\omega t_7) = \psi(\mathrm{refl}([x := s]t_1; [x := s]t_2), [x := s]t_3, [x := s]t_4; [x := s]t_5, [x := s]t_6) \bullet_\omega [x := s]t_7 \Rightarrow [x := s']t'_7$.

Case: $\dfrac{t_1 \overset{\mathcal{D}_1}{\Rightarrow} t'_1 \qquad t_2 \overset{\mathcal{D}_2}{\Rightarrow} t'_2 \qquad t_3 \overset{\mathcal{D}_3}{\Rightarrow} t'_3}{[t_1, t_2; t_3].1 \Rightarrow t'_1}$

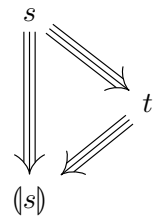
By the IH: $[x := s]t_i \Rrightarrow [x := s']t'_i$ for $i = 1, 2, 3$. The ParFst rule gives: $[x := s][t_1, t_2; t_3].1 = [[x := s]t_1, [x := s]t_2; [x := s]t_3].1 \Rrightarrow [x := s']t'_1$.

Case: $$\frac{t_1 \overset{\mathcal{D}_1}{\Rrightarrow} t'_1 \qquad t_2 \overset{\mathcal{D}_2}{\Rrightarrow} t'_2 \qquad t_3 \overset{\mathcal{D}_3}{\Rrightarrow} t'_3 \qquad t_4 \overset{\mathcal{D}_3}{\Rrightarrow} t'_4 \qquad t_5 \overset{\mathcal{D}_3}{\Rrightarrow} t'_5}{\vartheta(\mathrm{refl}(t_1; t_2), t_3, t_4; t_5) \Rrightarrow \mathrm{refl}(t'_3; t'_5)}$$

By the IH: $[x := s]t_i \Rrightarrow [x := s']t'_i$ for $i = 1, 2, 3$. The ParFst rule gives: $[x := s]\vartheta(\mathrm{refl}(t_1; t_2), t_3, t_4; t_5) = \vartheta(\mathrm{refl}([x := s]t_1; [x := s]t_2), [x := s]t_3, [x := s]t_4; [x := s]t_5) \Rrightarrow \mathrm{refl}([x := s']t'_3; [x := s']t'_5) = [x := s']\mathrm{refl}(t'_3; t'_5)$.

□

The triangle property of parallel reduction is used to complete the set of possible contractible redexes. Thus, if syntax $s \Rrightarrow t$ where t is only partially reduced then both s and t may be completed to $(\!|s|\!)$. To the right the situation is visually depicted. Note that the triangle property is "half" of the diamond property. Indeed, if $s \Rrightarrow t'$ then $t' \Rrightarrow (\!|s|\!)$. Thus, as a consequence of the triangle property, parallel reduction trivially has the diamond property.



Lemma 2.14 (Parallel Triangle)**.** *If $s \Rrightarrow t$ then $t \Rrightarrow (\!|s|\!)$*

Proof. By induction on $s \Rrightarrow t$. The second projection case is omitted.

Case: $$\frac{}{x_K \Rrightarrow x_K}$$

Have $(\!|x_K|\!) = x_K$. Thus, this case is trivial.

Case: $$\frac{t_i \overset{\mathcal{D}_i}{\Rrightarrow} t'_i \quad \forall\, i \in \{1, \ldots, \mathfrak{a}(\kappa)\}}{\mathfrak{c}(\kappa, t_1, \ldots, t_i, \ldots, t_{\mathfrak{a}(\kappa)}) \Rrightarrow \mathfrak{c}(\kappa, t'_1, \ldots, t'_i, \ldots, t'_{\mathfrak{a}(\kappa)})}$$

By the IH applied to $\mathcal{D}_i$: $t'_i \Rrightarrow (\!|t_i|\!)$ for all i. Proceed by cases of $(\!|\mathfrak{c}(\kappa, t_1, \ldots t_{\mathfrak{a}(\kappa)})|\!)$. The second projection case is omitted because it is the same as the first projection case.

Case: $(\!|(\lambda_m\, x\!:\!t_1.\, t_2) \bullet_m t_3|\!) = [x := (\!|t_3|\!)](\!|t_2|\!)$

Note that $\mathfrak{c}(\kappa, t'_1, \ldots t'_{\mathfrak{a}(\kappa)}) = (\lambda_m\, x\!:\!t'_1.\, t'_2) \bullet_m t'_3$. Using the ParBeta rule yields $(\lambda_m\, x\!:\!t'_1.\, t'_2) \bullet_m t'_3 \Rrightarrow [x := (\!|t_3|\!)](\!|t_2|\!)$.

Case: $(\!|\psi(\mathrm{refl}(t_1; t_2), t_3, t_4; t_5, t_6) \bullet_\omega t_7|\!) = (\!|t_7|\!)$

Note that $\mathfrak{c}(\kappa, t'_1, \ldots t'_{\mathfrak{a}(\kappa)}) = \psi(\mathrm{refl}(t'_1; t'_2), t'_3, t'_4; t'_5, t'_6) \bullet_\omega t'_7$. Using the ParSubst rule yields $\psi(\mathrm{refl}(t'_1; t'_2), t'_3, t'_4; t'_5, t'_6) \bullet_\omega t'_7 \Rrightarrow (\!|t_7|\!)$.

Case: $(\!|[t_1, t_2; t_3].1|\!) = (\!|t_1|\!)$

Note that $\mathfrak{c}(\kappa, t'_1, \ldots t'_{\mathfrak{a}(\kappa)}) = [t'_1, t'_2; t'_3].1$. Using the ParFst rule yields $[t'_1, t'_2; t'_3].1 \Rrightarrow (\!|t_1|\!)$.

Case: $(\!|\vartheta(\mathrm{refl}(t_1; t_2), t_3, t_4; t_5)|\!) = \mathrm{refl}((\!|t_3|\!); (\!|t_5|\!))$

Note that $\mathfrak{c}(\kappa, t'_1, \ldots t'_{\mathfrak{a}(\kappa)}) = \vartheta(\mathrm{refl}(t'_1; t'_2), t'_3, t'_4; t'_5)$. Using the ParPrmFst rule yields $\vartheta(\mathrm{refl}(t'_1; t'_2), t'_3, t'_4; t'_5) \Rrightarrow \mathrm{refl}((\!|t_3|\!); (\!|t_5|\!))$.

Case: $(\!|\mathfrak{c}(\kappa, t_1, \ldots t_{\mathfrak{a}(\kappa)})|\!) = \mathfrak{c}(\kappa, (\!|t_1|\!), \ldots (\!|t_{\mathfrak{a}(\kappa)}|\!))$

Using the ParCtor rule concludes the case.

Case: $\dfrac{\overset{\mathcal{D}_1}{t_1 \Rrightarrow t'_1} \qquad \overset{\mathcal{D}_2}{t_2 \Rrightarrow t'_2}}{\mathfrak{b}(\kappa, x : t_1, t_2) \Rrightarrow \mathfrak{b}(\kappa, x : t'_1, t'_2)}$

Note that $(\!|\mathfrak{b}(\kappa, (x : t_1), t_2)|\!) = \mathfrak{b}(\kappa, (x : (\!|t_1|\!)), (\!|t_2|\!))$. By the IH applied to $\mathcal{D}_i$: $t'_i \Rrightarrow (\!|t_i|\!)$ for $i = 1, 2$. Thus, by the ParBind rule $\mathfrak{b}(\kappa, (x : t'_1), t'_2) \Rrightarrow \mathfrak{b}(\kappa, (x : (\!|t_1|\!)), (\!|t_2|\!))$.

Case: $\dfrac{\overset{\mathcal{D}_1}{t_1 \Rrightarrow t'_1} \qquad \overset{\mathcal{D}_2}{t_2 \Rrightarrow t'_2} \qquad \overset{\mathcal{D}_3}{t_3 \Rrightarrow t'_3}}{(\lambda_m\, x{:}t_1.\, t_2) \bullet_m t_3 \Rrightarrow [x := t'_3]t'_2}$

Note that $(\!|(\lambda_m\, x{:}t_1.\, t_2) \bullet_m t_3|\!) = [x := (\!|t_3|\!)](\!|t_2|\!)$. By the IH applied to $\mathcal{D}_i$: $t'_i \Rrightarrow (\!|t_i|\!)$ for $i = 1, 2, 3$. Thus, by Lemma 2.13 $[x := t'_3]t'_2 \Rrightarrow [x := (\!|t_3|\!)](\!|t_2|\!)$.

Case: $\dfrac{\overset{\mathcal{D}_1}{t_1 \Rrightarrow t'_1} \quad \overset{\mathcal{D}_2}{t_2 \Rrightarrow t'_2} \quad \overset{\mathcal{D}_2}{t_3 \Rrightarrow t'_3} \quad \overset{\mathcal{D}_2}{t_4 \Rrightarrow t'_4} \quad \overset{\mathcal{D}_2}{t_5 \Rrightarrow t'_5} \quad \overset{\mathcal{D}_2}{t_6 \Rrightarrow t'_6} \quad \overset{\mathcal{D}_2}{t_7 \Rrightarrow t'_7}}{\psi(\mathrm{refl}(t_1; t_2), t_3, t_4; t_5, t_6) \bullet_\omega t_7 \Rrightarrow t'_7}$

Note that $(\!|\psi(\mathrm{refl}(t_1; t_2), t_3, t_4; t_5, t_6) \bullet_\omega t_7|\!) = (\!|t_7|\!)$. By the IH applied to $\mathcal{D}_i$: $t'_i \Rrightarrow (\!|t_i|\!)$ for $i = 1$ through $i = 7$. Applying the ParBind rule yields $t'_7 \Rrightarrow (\!|t_7|\!)$.

Case: $\dfrac{\overset{\mathcal{D}_1}{t_1 \Rrightarrow t'_1} \qquad \overset{\mathcal{D}_2}{t_2 \Rrightarrow t'_2} \qquad \overset{\mathcal{D}_3}{t_3 \Rrightarrow t'_3}}{[t_1, t_2; t_3].1 \Rrightarrow t'_1}$

Note that $(\!|[t_1, t_2; t_3].1|\!) = (\!|t_1|\!)$. By the IH applied to $\mathcal{D}_i$: $t'_i \Rrightarrow (\!|t_i|\!)$ for $i = 1, 2, 3$. Thus, $t'_1 \Rrightarrow (\!|t_1|\!)$.

Case: $\dfrac{\overset{\mathcal{D}_1}{t_1 \Rrightarrow t'_1} \qquad \overset{\mathcal{D}_2}{t_2 \Rrightarrow t'_2} \qquad \overset{\mathcal{D}_3}{t_3 \Rrightarrow t'_3} \qquad \overset{\mathcal{D}_3}{t_4 \Rrightarrow t'_4} \qquad \overset{\mathcal{D}_3}{t_5 \Rrightarrow t'_5}}{\vartheta(\mathrm{refl}(t_1; t_2), t_3, t_4; t_5) \Rrightarrow \mathrm{refl}(t'_3; t'_5)}$

Note that $(\!|\vartheta_1(\mathrm{refl}(t_1; t_2), t_3, t_4; t_5)|\!) \Rrightarrow \mathrm{refl}((\!|t_3|\!); (\!|t_5|\!))$. By the IH applied to $\mathcal{D}_i$: $t'_i \Rrightarrow (\!|t_i|\!)$ for $i = 1$ through $i = 5$. Thus, $\mathrm{refl}(t'_3; t'_5) \Rrightarrow \mathrm{refl}((\!|t_3|\!); (\!|t_5|\!))$ by the ParCtor rule and Lemma 2.8.

□

Lemma 2.15 (Parallel Strip)**.** *If $s \Rrightarrow t_1$ and $s \Rrightarrow^* t_2$ then $\exists\ t$ such that $t_1 \Rrightarrow^* t$ and $t_2 \Rrightarrow t$*

Proof. By induction on $s \Rrightarrow^* t_2$, pick $t = t_1$ for the reflexivity case. Consider the transitivity case, $\exists\ z_1$ such that $s \Rrightarrow z_1$ and $z_1 \Rrightarrow^* t_2$. Applying Lemma 2.14 to $s \Rrightarrow z_1$ yields $z_1 \Rrightarrow (\!|s|\!)$. By the IH with $z_1 \Rrightarrow (\!|s|\!)$: $\exists\ z_2$ such that $(\!|s|\!) \Rrightarrow^* z_2$ and $t_2 \Rrightarrow z_2$. Using Lemma 2.14 again on $s \Rrightarrow t_1$ yields $t_1 \Rrightarrow (\!|s|\!)$. Now by transitivity $t_1 \Rrightarrow^* z_2$. □

Lemma 2.16 (Parallel Confluence)**.** *If $s \Rrightarrow^* t_1$ and $s \Rrightarrow^* t_2$ then $\exists\ t$ such that $t_1 \Rrightarrow^* t$ and $t_2 \Rrightarrow^* t$*

Proof. By induction on $s \Rrightarrow^* t_1$, pick $t = t_2$ for the reflexivity case. Consider the transitivity case, $\exists\ z_1$ such that $s \Rrightarrow z_1$ and $z_1 \Rrightarrow^* t_1$. By Lemma 2.15 applied with $s \Rrightarrow z_1$ and $s \Rrightarrow^* t_2$ yields $\exists\ z_2$ such that $z_1 \Rrightarrow^* z_2$ and $t_2 \Rrightarrow z_2$. Using the IH with $z_1 \Rrightarrow z_2$ gives $\exists\ z_3$ such that $t_1 \Rrightarrow^* z_3$ and $z_2 \Rrightarrow^* z_3$. By transitivity $t_2 \Rrightarrow^* z_3$. □

Lemma 2.17 (Confluence)**.** *If $s \rightsquigarrow^* t_1$ and $s \rightsquigarrow^* t_2$ then $\exists\ t$ such that $t_1 \rightsquigarrow^* t$ and $t_2 \rightsquigarrow^* t$*

Proof. By Lemma 2.10 applied twice: $s \Rrightarrow^* t_1$ and $s \Rrightarrow^* t_2$. Now by parallel confluence (Lemma 2.16) $\exists\ t$ such that $t_1 \Rrightarrow^* t$ and $t_2 \Rrightarrow^* t$. Finally, two applications of Lemma 2.12 conclude the proof. □

As with F^ω the important consequence of confluence is that convertibility of reduction is an equivalence relation. However, this is *not* the conversion relation that will be used in the inference judgment. Thus, while important, it is still only a stepping stone to showing judgmental conversion is transitive.

Theorem 2.18. *For any s and t the relation $s \rightleftharpoons t$ is an equivalence.*

Proof. Reflexivity is immediate because $s \rightsquigarrow^* s$. Symmetry is also immediate because if $s \rightleftharpoons t$ then $\exists\ z$ such that $s \rightsquigarrow^* z$ and $t \rightsquigarrow^* z$, but logical conjunction is commutative. Transitivity is a consequence of confluence, see Theorem 1.3. □

Additionally, there is a final useful fact about convertibility of reduction that is occasionally used throughout the rest of this work. That is, like reduction, conversion of subexpressions yields conversion of the entire term.

Lemma 2.19. *If $t_i \rightleftharpoons t'_i$ for any i then,*

1. $\mathfrak{b}(\kappa, (x : t_1), t_2) \rightleftharpoons \mathfrak{b}(\kappa, (x : t'_1), t'_2)$
2. $\mathfrak{c}(\kappa, t_1, \ldots, t_{\mathfrak{a}(\kappa)}) \rightleftharpoons \mathfrak{c}(\kappa, t'_1, \ldots, t'_{\mathfrak{a}(\kappa)})$

Proof. By Lemma 2.2 applied on both sides. □

$$
\begin{aligned}
|x_K| &= x_K \\
|\star| &= \star \\
|\Box| &= \Box \\
|\lambda_0\, x{:}A.\, t| &= |t| \\
|\lambda_\omega\, x{:}A.\, t| &= \lambda_\omega\, x{:}\diamond.\, |t| \\
|\lambda_\tau\, x{:}A.\, t| &= \lambda_\tau\, x{:}|A|.\, |t| \\
|(x : A) \to_m B| &= (x : |A|) \to_m |B| \\
|(x : A) \cap B| &= (x : |A|) \cap |B| \\
|f \bullet_0 a| &= |f| \\
|f \bullet_\omega a| &= |f| \bullet_\omega |a| \\
|f \bullet_\tau a| &= |f| \bullet_\tau |a|
\end{aligned}
\qquad
\begin{aligned}
|\diamond| &= \diamond \\
|[t_1, t_2; A]| &= |t_1| \\
|t.1| &= |t| \\
|t.2| &= |t| \\
|x =_A y| &= |x| =_{|A|} |y| \\
|\mathrm{refl}(t; A)| &= \lambda_\omega\, x{:}\diamond.\, x_\star \\
|\psi(e, a, b; A, P)| &= |e| \\
|\vartheta(e, a, b; T)| &= |e| \\
|\delta(e)| &= |e| \\
|\varphi(a, b, e; A, T)| &= |a|
\end{aligned}
$$

Figure 2.5: Erasure of syntax, for type-like and kind-like syntax erasure is homomorphic, for term-like syntax erasure reduces to the untyped lambda calculus.

2.3 Erasure and Pseudo-objects

Cedille has a notion of erasure of syntax that transforms terms into the untyped λ-calculus. This concept is generalized in the core theory of Cedille2 to operate on general syntax. It still called erasure mostly as a holdover, but erasure no longer actually erases all type information of type annotations. Instead, erasure should be thought of as computing the raw syntactic forms of objects. In Section 2.4 the notion of proof will be defined. An object is the erasure of a proof. Erasure is defined in Figure 2.5. With erasure the desired conversion relation is also definable. This definition will enable equating objects in a dependent quantification instead of proofs.

Definition 2.20. $s_1 \equiv s_2$ *iff* $\exists\ t_1, t_2.\ s_1 \rightsquigarrow^* t_1, s_2 \rightsquigarrow^* t_2,$ *and* $|t_1| \rightleftharpoons |t_2|$

Note that the only purpose of the syntactic constructor $\diamond$ is to be a placeholder for erased type annotations of λ_ω syntactic forms. However, for λ_τ variants, the annotation is *not* erased. This is partly why calling this transformation *erasure* is a slight lie, because it does not always erase. Regardless, it is faithful to the interpretation from Cedille when focused on non-type-like syntax. Indeed, any form that is not type-like does reduce to the untyped λ-calculus. For type-like syntax, erasure is instead locally homomorphic. Erasure of raw syntax does not possess much structure, but it is idempotent and commutes with substitution. Additionally, as a consequence an extension of Lemma 2.7 is possible.

Lemma 2.21. $||t|| = |t|$

Proof. By induction on t. □

Lemma 2.22. $|[x := t]b| = [x := |t|]|b|$

Proof. By induction on the size of b.

Case: $\mathfrak{b}(\kappa, (x : t_1), t_2)$

If $b = \lambda_0\, y{:}A.\, b'$, then $|b| = |b'|$ which is a smaller term. Then, by the IH $|[x := t]b'| = [x := |t|]|b'|$. Thus,

$$\begin{aligned}|[x := t]\lambda_0\, y{:}A.\, b'| &= |\lambda_0\, y{:}[x := t]A.\, [x := t]b'| \\ &= |[x := t]b'| = [x := |t|]|b'| = [x := |t|]|\lambda_0\, y{:}A.\, b'|\end{aligned}$$

For the remaining tags, assume w.l.o.g. $\kappa = \cap$. Then $b = (y : A) \cap B$, and by the IH $|[x := t]A| = [x := |t|]|A|$ and $|[x := t]B| = [x := |t|]|B|$. Thus,

$$\begin{aligned}|[x := t]((y : A) \cap B)| &= |(y : [x := t]A) \cap [x := t]B| \\ &= (y : |[x := t]A|) \cap |[x := t]B| = (y : [x := |t|]|A|) \cap [x := |t|]|B|\end{aligned}$$

And, $[x := |t|]|(y : A) \cap B| = (y : [x := |t|]|A|) \cap [x := |t|]|B|$. Thus, both sides are equal.

Case: $\mathfrak{c}(\kappa, t_1, \ldots, t_{\mathfrak{a}(\kappa)})$

If $\kappa \in \{\diamond, \star, \Box\}$ then the equality is trivial.

If $\kappa \in \{\bullet_0, \mathrm{pair}, \mathrm{proj}_1, \mathrm{proj}_2, \psi, \vartheta, \delta, \varphi\}$ then $|\mathfrak{c}(\kappa, t_1, \ldots)| = |t_1|$. Moreover, substitution commutes and both sides of the equality are equal.

If $\kappa \in \{\mathrm{refl}\}$ then the equality is trivial.

If $\kappa \in \{\bullet_\omega, \bullet_\tau, \mathrm{eq}\}$ then w.l.o.g. assume $\kappa = \mathrm{eq}$. Now $|[x := t](a =_A b)| = |[x := t]a| =_{|[x:=t]A|} |[x := t]b|$. By the IH this becomes $[x := |t|]|a| =_{[x:=|t|]|A|} [x := |t|]|b|$. On the right-hand side, $[x := |t|]|a =_A b| = [x := |t|]|a| =_{[x:=|t|]|A|} [x := |t|]|b|$. Thus, both sides are equal.

Case: b variable

Suppose $b = x$, then $|[x := t]x| = |t|$ and $[x := |t|]|x| = |t|$. Suppose $b = y$, then $|[x := t]y| = y$ and $[x := |t|]|y| = y$. Thus, both sides are equal.

□

Lemma 2.23. *If* $|s| \rightleftharpoons |t|$ *and* $|a| \rightleftharpoons |b|$ *then* $|[x := s]a| \rightleftharpoons |[x := t]b|$

Proof. By definition $\exists\, z_1, z_2$ such that $|s| \rightsquigarrow^* z_1$, $|t| \rightsquigarrow^* z_1$, $|a| \rightsquigarrow^* z_2$ and $|b| \rightsquigarrow^* z_2$. By Lemma 2.6 applied twice $[x := |s|]|a| \rightsquigarrow^* [x := |z_1|]z_2$ and $[x := |t|]|b| \rightsquigarrow^* [x := |z_1|]z_2$. Finally, by Lemma 2.22 $[x := |s|]|a| = |[x := s]a|$ and $[x := |t|]|b| = |[x := t]b|$. □

$$\frac{t_1 \text{ pseobj} \qquad t_2 \text{ pseobj} \qquad \kappa \neq \lambda_0}{\mathfrak{b}(\kappa, x : t_1, t_2) \text{ pseobj}} \qquad \frac{\forall\, i \in 1, \ldots, \mathfrak{a}(\kappa).\ t_i \text{ pseobj} \qquad \kappa \neq \text{pair}}{\mathfrak{c}(\kappa, t_1, \ldots, t_{\mathfrak{a}(\kappa)}) \text{ pseobj}}$$

$$\frac{A \text{ pseobj} \qquad t \text{ pseobj} \qquad x \notin FV(|t|)}{\lambda_0\, x\!:\!A.\, t \text{ pseobj}} \qquad \frac{\begin{array}{ccc} & t_1 \text{ pseobj} & \\ t_2 \text{ pseobj} & A \text{ pseobj} & |t_1| \rightleftharpoons |t_2| \end{array}}{[t_1, t_2; A] \text{ pseobj}}$$

$$x_K \text{ pseobj}$$

Figure 2.6: Definition of Pseudo Objects.

Beyond these lemmas more structure needs to be imposed on raw syntax to obtain better behavior with erasure. In particular, the pair case and the λ_0 case are problematic. Indeed, for pairs there is an assumption that the first and second component are convertible. This restriction is what transforms these pairs into something more, an element of an intersection. Likewise, the λ_0 binder is meant to signify that the bound variable does not appear free in the erasure of the body. Imposing these restrictions on syntax retains the spirit of what it means to be an object. However, because syntax is still not a proof, this restriction on syntax instead forms a set of *pseudo-objects*. The inductive definition of pseudo-objects is presented in Figure 2.6.

Note that the restriction for pairs is $|t_1| \rightleftharpoons |t_2|$ as opposed to $t_1 \equiv t_2$. The distinction here is subtle, but it enables proving one of the important properties for the structure of pseudo-objects, that $|t_1| \rightleftharpoons |t_2|$ if and only if $t_1 \equiv t_2$. To reach that goal requires a series of technical lemmas about pseudo-objects and the concepts introduced so far.

Lemma 2.24. *If* s pseobj *and* $s \rightsquigarrow t$ *then* $|s| \rightleftharpoons |t|$

Proof. By induction on s pseobj.

Case: $\dfrac{\overset{\mathcal{D}_1}{t_1 \text{ pseobj}} \qquad \overset{\mathcal{D}_2}{t_2 \text{ pseobj}} \qquad \overset{\mathcal{D}_3}{\kappa \neq \lambda_0}}{\mathfrak{b}(\kappa, x : t_1, t_2) \text{ pseobj}}$

By cases on $s \rightsquigarrow t$, applying the IH and Lemma 2.19.

Case: $\dfrac{\overset{\mathcal{D}_1}{A \text{ pseobj}} \qquad \overset{\mathcal{D}_2}{t \text{ pseobj}} \qquad \overset{\mathcal{D}_3}{x \notin FV(|t|)}}{\lambda_0\, x\!:\!A.\, t \text{ pseobj}}$

By cases on $s \rightsquigarrow t$, applying the IH and Lemma 2.19.

Case: $\dfrac{\overset{\mathcal{D}_1}{\forall\, i \in 1, \ldots, \mathfrak{a}(\kappa).\ t_i \text{ pseobj}} \qquad \overset{\mathcal{D}_2}{\kappa \neq \text{pair}}}{\mathfrak{c}(\kappa, t_1, \ldots, t_{\mathfrak{a}(\kappa)}) \text{ pseobj}}$

By cases on $s \rightsquigarrow t$.

Case: $(\lambda_m\, x\!:\!A.\, b) \bullet_m t \rightsquigarrow [x := t]b$

Note that $\lambda_m\, x{:}A.\, b$ pseobj. If $m = 0$ then $x \notin FV(b)$ and $|[x := t]b| = |b|$. Thus, $|(\lambda_0\, x{:}A.\, b)\bullet_0 t| = |\lambda_0\, x{:}A.\, b| = |b|$. If $m = \omega$, then $|(\lambda_\omega\, x{:}A.\, b)\bullet_\omega t| = (\lambda_\omega\, x.\, b)\bullet_\omega |t|$. By definition of reduction $(\lambda_\omega\, x.\, b)\bullet_\omega |t| \rightleftharpoons [x := |t|]|b|$. Finally, by Lemma 2.22 the goal is obtained. The case of $m = \tau$ is almost exactly the same.

Case: $[t_1, t_2; A].1 \rightsquigarrow t_1$

$|[t_1, t_2; A].1| = |[t_1, t_2; A]| = |t_1|$

Case: $[t_1, t_2; A].2 \rightsquigarrow t_2$

Observe that $|[t_1, t_2; A].2| = |t_1|$ and $[t_1, t_2; A]$ pseobj.
Thus, $|s| = |t_1| \rightleftharpoons |t_2|$.

Case: $\psi(\mathrm{refl}(z; Z), a, b; A, P)\bullet_\omega t \rightsquigarrow t$

$|\psi(\mathrm{refl}(z; Z), a, b; A, P)\bullet_\omega t| = |\mathrm{refl}(z; Z)|\bullet_\omega |t| \rightleftharpoons |t|$

Case: $\vartheta(\mathrm{refl}(z; Z), a, b; A) \rightsquigarrow \mathrm{refl}(a; A)$

$|\vartheta(\mathrm{refl}(z; Z), a, b; A)| = |\mathrm{refl}(z; Z)| = \lambda_\omega\, x{:}\diamond.\, x = |\mathrm{refl}(a; A)|$

Case: $$\frac{\overset{\mathcal{D}_1}{t_i \rightsquigarrow t_i'} \qquad i \in 1, \ldots, \mathfrak{a}(\kappa)}{\mathfrak{c}(\kappa, t_1, \ldots t_i, \ldots t_{\mathfrak{a}(\kappa)}) \rightsquigarrow \mathfrak{c}(\kappa, t_1, \ldots t_i', \ldots t_{\mathfrak{a}(\kappa)})}$$

By the IH, $|t_i| \rightleftharpoons |t_i'|$. The goal is achieved by Lemma 2.19

Case: $$\frac{\overset{\mathcal{D}_1}{t_1 \text{ pseobj}} \qquad \overset{\mathcal{D}_2}{t_2 \text{ pseobj}} \qquad \overset{\mathcal{D}_3}{A \text{ pseobj}} \qquad \overset{\mathcal{D}_4}{|t_1| \rightleftharpoons |t_2|}}{[t_1, t_2; A] \text{ pseobj}}$$

By cases on $s \rightsquigarrow t$, applying the IH and Lemma 2.19.

Case: s variable

By cases on $s \rightsquigarrow t$, t must be a variable. Thus, $|s| = |t|$.

□

Lemma 2.25. *If* s pseobj, $|s| \rightleftharpoons |b|$, *and* $s \rightsquigarrow t$ *then* $|t| \rightleftharpoons |b|$

Proof. By Lemma 2.24 $|s| \rightleftharpoons |t|$ and by Theorem 2.18 $|t| \rightleftharpoons |b|$. □

Lemma 2.26. *If* b pseobj *and* t pseobj *then* $[x := t]b$ pseobj

Proof. By induction on b pseobj. The λ_0 and pair cases are no different from the respective $\mathfrak{b}$ and $\mathfrak{c}$ cases.

Case: $\dfrac{t_1 \overset{\mathcal{D}_1}{\text{ pseobj}} \qquad t_2 \overset{\mathcal{D}_2}{\text{ pseobj}} \qquad \kappa \overset{\mathcal{D}_3}{\neq} \lambda_0}{\mathfrak{b}(\kappa, x : t_1, t_2) \text{ pseobj}}$

By the IH $[x := t]t_1$ pseobj and $[x := t]t_2$ pseobj. Thus, $\mathfrak{b}(\kappa, (y : [x := t]t_1), [x := t]t_2)$ pseobj.

Case: $\dfrac{\forall\, i \in 1, \ldots, \overset{\mathcal{D}_1}{\mathfrak{a}(\kappa)}.\ t_i \text{ pseobj} \qquad \kappa \overset{\mathcal{D}_2}{\neq} \text{pair}}{\mathfrak{c}(\kappa, t_1, \ldots, t_{\mathfrak{a}(\kappa)}) \text{ pseobj}}$

By the IH $[x := t]t_i$ pseobj.
Thus, $\mathfrak{c}(\kappa, [x := t]t_1, \ldots [x := t]t_{\mathfrak{a}(\kappa)})$ pseobj.

Case: s variable

If $s = x$ then $[x := t]x = t$, and t pseobj. Otherwise, $s = y$ with y a variable and y pseobj.

□

Lemma 2.27. *If* s pseobj *and* $s \rightsquigarrow t$ *then* t pseobj

Proof. By induction on s pseobj.

Case: $\dfrac{t_1 \overset{\mathcal{D}_1}{\text{ pseobj}} \qquad t_2 \overset{\mathcal{D}_2}{\text{ pseobj}} \qquad \kappa \overset{\mathcal{D}_3}{\neq} \lambda_0}{\mathfrak{b}(\kappa, x : t_1, t_2) \text{ pseobj}}$

By cases on $s \rightsquigarrow t$. Suppose w.l.o.g. that $t_2 \rightsquigarrow t_2'$. Observe that t_2 pseobj because it is a subterm of s. Then by the IH t_2' pseobj. Thus, $\mathfrak{b}(\kappa, x : t_1, t_2')$ pseobj.

Case: $\dfrac{A \overset{\mathcal{D}_1}{\text{ pseobj}} \qquad t \overset{\mathcal{D}_2}{\text{ pseobj}} \qquad x \overset{\mathcal{D}_3}{\notin} FV(|t|)}{\lambda_0\, x : A.\, t \text{ pseobj}}$

By cases on $s \rightsquigarrow t$. Suppose w.l.o.g that $t \rightsquigarrow t'$. Note that if $x \notin FV(|t|)$ then $x \notin FV(|t'|)$, reduction only reduces the amount of free variables. Observe that t pseobj. Then by the IH t' pseobj. Thus, $\lambda_0\, x : A.\, t'$ pseobj.

Case: $\dfrac{\forall\, i \in 1, \ldots, \overset{\mathcal{D}_1}{\mathfrak{a}(\kappa)}.\ t_i \text{ pseobj} \qquad \kappa \overset{\mathcal{D}_2}{\neq} \text{pair}}{\mathfrak{c}(\kappa, t_1, \ldots, t_{\mathfrak{a}(\kappa)}) \text{ pseobj}}$

By cases on $s \rightsquigarrow t$. The first and second projection cases are very similar to the substitution case.

Case: $(\lambda_m\, x : A.\, b) \bullet_m t \rightsquigarrow [x := t]b$

Observe that b pseobj and t pseobj because both are subterms of s. By Lemma 2.26 $[x := t]b$ pseobj.

Case: $\psi(\mathrm{refl}(z; Z), a, b; A, P) \bullet_\omega t \rightsquigarrow t$

Immediate by the IH: t pseobj.

Case: $\vartheta_1(\mathrm{refl}(z; Z), a, b; A) \rightsquigarrow \mathrm{refl}(a; A)$

Observe that a pseobj and A pseobj. By application of constructor rule $\mathrm{refl}(a; A)$ pseobj.

Case: $$\frac{\overset{\mathcal{D}_1}{t_i \rightsquigarrow t_i'} \qquad i \in 1, \ldots, \mathfrak{a}(\kappa)}{\mathfrak{c}(\kappa, t_1, \ldots t_i, \ldots t_{\mathfrak{a}(\kappa)}) \rightsquigarrow \mathfrak{c}(\kappa, t_1, \ldots t_i', \ldots t_{\mathfrak{a}(\kappa)})}$$

By the IH t_i' pseobj. By application of the constructor rule the goal is obtained.

Case: $$\frac{\overset{\mathcal{D}_1}{t_1 \text{ pseobj}} \qquad \overset{\mathcal{D}_2}{t_2 \text{ pseobj}} \qquad \overset{\mathcal{D}_3}{A \text{ pseobj}} \qquad \overset{\mathcal{D}_4}{|t_1| \rightleftharpoons |t_2|}}{[t_1, t_2; A] \text{ pseobj}}$$

By cases on $s \rightsquigarrow t$. Suppose w.l.o.g. $t_1 \rightsquigarrow t_1'$. Note that t_1 pseobj because it is a subterm of s. By the IH t_1' pseobj. By Lemma 2.25 $|t_1'| \rightleftharpoons |t_2|$. Thus, $[t_1', t_2; A]$ pseobj.

Case: s variable

By cases on $s \rightsquigarrow t$, t must be a variable. Thus, t pseobj.

□

Lemma 2.28. *If s pseobj, $|s| \rightleftharpoons |b|$, and $s \rightsquigarrow^* t$ then $|t| \rightleftharpoons |b|$*

Proof. By induction on $s \rightsquigarrow^* t$. The reflexivity case is trivial. The transitivity case is obtained from Lemma 2.25, Lemma 2.27, and applying the IH. □

Lemma 2.29. *If s pseobj and $s \rightsquigarrow^* t$ then t pseobj*

Proof. By induction on $s \rightsquigarrow^* t$. The reflexivity case is trivial. The transitivity case is obtained from Lemma 2.27 and applying the IH. □

Lemma 2.30. *If s pseobj, $|t| \rightleftharpoons |b|$, and $s \rightsquigarrow^* t$ then $|s| \rightleftharpoons |b|$*

Proof. By induction on $s \rightsquigarrow^* t$. Consequence of Lemma 2.24 and Lemma 2.29. □

Lemma 2.31. *If s pseobj, $s \equiv b$, and $s \rightsquigarrow^* t$ then $t \equiv b$*

Proof. Note that $\exists\ z_1, z_2$ such that $s \rightsquigarrow^* z_1$, $b \rightsquigarrow^* z_2$, and $|z_1| \rightleftharpoons |z_2|$. By confluence $\exists\ z_1'$ such that $z_1 \rightsquigarrow^* z_1'$ and $t \rightsquigarrow^* z_1'$. Then, by Lemma 2.29 z_1 pseobj. Finally, by Lemma 2.28 $|z_1'| \rightleftharpoons |z_2|$. Therefore, $t \equiv b$. □

Unlike with convertibility of reduction, obtaining transitivity of conversion requires the additional assumption that the inner syntax is a pseudo-object. Indeed, the incorporation of erasure into the definition requires this extra structure, because otherwise reductions on pairs would not agree. For example, pick $a = [x, y; T].1$, $b = [x, y; T]$, and $c = [y, x; T].2$. Notice that $|a| = |b|$ but $|b| \neq |c|$, however, $c \rightsquigarrow^* x$, thus $b \equiv c$ and $\neg(b \equiv c)$. There is an inconsistency in the definition because b is not a pseudo-object, it is not the case that $|x| \rightleftharpoons |y|$. Really this is more fundamental than just transitivity as it shows that reduction is not consistent with erasure unless the syntax is a pseudo-object.

Lemma 2.32. *If* b pseobj, $a \equiv b$, *and* $b \equiv c$ *then* $a \equiv c$

Proof. Note that $\exists\ u_1, u_2$ such that $a \rightsquigarrow^* u_1$, $b \rightsquigarrow^* u_2$, and $|u_1| \rightleftharpoons |u_2|$. Additionally, $\exists\ v_1, v_2$ such that $b \rightsquigarrow^* v_1$, $c \rightsquigarrow^* v_2$, and $|v_1| \rightleftharpoons |v_2|$. By confluence, $\exists\ z$ such that $u_2 \rightsquigarrow^* z$ and $v_1 \rightsquigarrow^* z$. Then, by Lemma 2.29 u_2 pseobj and v_1 pseobj. Next, by Lemma 2.28 $|u_1| \rightleftharpoons |z|$ and $|z| \rightleftharpoons |v_2|$. Thus, $|u_1| \rightleftharpoons |v_2|$ by Lemma 2.18 and $a \equiv c$. □

Knowing that $|s| \rightleftharpoons |t|$ if and only if $s \equiv t$ is critical for maintaining the spirit of Cedille. While the core theory of Cedille2 is its own system the purpose is to refine the design of Cedille without losing its essential features. A critical feature of Cedille is that convertibility is done with the untyped λ-calculus (i.e. erased terms) not with annotated terms themselves. Having Theorem 2.33 means that whenever conversion is checked between terms it is safe to instead check convertibility of reduction of objects. Not only does this maintain the spirit of Cedille, but it also enables optimizations in type checking. Indeed, arbitrarily expensive sequences of reductions could potentially be erased when checking $|s| \rightleftharpoons |t|$ instead of $s \equiv t$.

Theorem 2.33. *Suppose* s pseobj *and* t pseobj, *then* $|s| \rightleftharpoons |t|$ *iff* $s \equiv t$

Proof. Case ($\Rightarrow$): Suppose $|s| \rightleftharpoons |t|$. By definition $s \rightsquigarrow^* s$ and $t \rightsquigarrow^* t$. Thus, $s \equiv t$. Case ($\Leftarrow$): Suppose $s \equiv t$, then $\exists\ z_1, z_2$ such that $s \rightsquigarrow^* z_1$, $t \rightsquigarrow^* z_2$, and $|z_1| \rightleftharpoons |z_2|$. By two applications of Lemma 2.30 $|s| \rightleftharpoons |t|$. □

Corollary 2.34. *For* s pseobj *and* t pseobj *the relation* $s \equiv t$ *is an equivalence.*

Finally, a useful lemma about substitution's interaction with conversion is obtained from the effort of pseudo-objects. This lemma is necessary to prove metatheoretic results about the system.

Lemma 2.35. *If* s, t, a, b pseobj, $s \equiv t$, *and* $a \equiv b$ *then* $[x := s]a \equiv [x := t]b$

Proof. By Lemma 2.33 $|s| \rightleftharpoons |t|$ and $|a| \rightleftharpoons |b|$. Then, by Lemma 2.23 $|[x := s]a| \rightleftharpoons |[x := t]b|$. Finally, by Lemma 2.33 again, $[x := s]a \equiv [x := t]b$. □

$$\begin{aligned} \mathrm{dom}_\Pi(\omega, K) &= \star & \mathrm{codom}_\Pi(\omega) &= \star \\ \mathrm{dom}_\Pi(\tau, K) &= K & \mathrm{codom}_\Pi(\tau) &= \Box \\ \mathrm{dom}_\Pi(0, K) &= K & \mathrm{codom}_\Pi(0) &= \star \end{aligned}$$

Figure 2.7: Domain and codomains for function types. The variable K is either $\star$ or $\Box$.

$$\frac{}{\Gamma \vdash \star : \Box}\ \textsc{Axiom} \qquad \frac{x \notin FV(\Gamma_1; \Gamma_2) \qquad \Gamma_1 \vdash A : K}{\Gamma_1; x_m : A; \Gamma_2 \vdash x_K : A}\ \textsc{Var}$$

$$\frac{\Gamma \vdash A : K \qquad \Gamma \vdash t : B \qquad A \equiv B}{\Gamma \vdash t : A}\ \textsc{Conv}$$

$$\frac{\Gamma \vdash A : \mathrm{dom}_\Pi(m, K) \qquad \Gamma; x_m : A \vdash B : \mathrm{codom}_\Pi(m)}{\Gamma \vdash (x : A) \to_m B : \mathrm{codom}_\Pi(m)}\ \textsc{Pi}$$

$$\frac{\Gamma \vdash (x : A) \to_m B : \mathrm{codom}_\Pi(m) \qquad \Gamma; x_m : A \vdash t : B \qquad x \notin FV(|t|) \text{ if } m = 0}{\Gamma \vdash \lambda_m\, x\!:\!A.\, t : (x : A) \to_m B}\ \textsc{Lam}$$

$$\frac{\Gamma \vdash f : (x : A) \to_m B \qquad \Gamma \vdash a : A}{\Gamma \vdash f \bullet_m a : [x := a]B}\ \textsc{App}$$

Figure 2.8: Inference rules for function types, including erased functions. The variable K is either $\star$ or $\Box$.

2.4 Inference Judgment

The inference judgment, presented in Figure 2.8; Figure 2.9; and Figure 2.10, delineate what syntax are *proofs*. As stated previously, the erasure of a proof is an *object*. Thus, for $\Gamma \vdash t : A$, t is a proof and $|t|$ its object. The judgment follows a standard PTS style, but the rules are carefully chosen so that an inference algorithm is possible. Judgments of the form $\Gamma \vdash t : A$ should be read t infers A in Γ.

$\frac{}{\Gamma \vdash \star : \Box}\ \textsc{Axiom}$ The axiom rule is the same as with F^ω. The constant $\star$ should be interpreted as a universe of types, and the constant $\Box$ as a universe of kinds. Thus, the axiom rule states that the universe of types *is* a kind in any context.

$\frac{x \notin FV(\Gamma_1; \Gamma_2) \qquad \Gamma_1 \vdash A : K}{\Gamma_1; x_m : A; \Gamma_2 \vdash x_K : A}\ \textsc{Var}$ The variable rule requires that a variable at a certain type is inside the context. Note that variables are annotated with a mode. Modes take three forms: free (ω); erased (0); or type (τ). The type mode is used for proofs that exist inside the type universe; the free mode for proofs that belong to some type; and the erased mode for proofs that belong to some type but whose bound variable is not computationally relevant in the associated object. Variables are annotated with modes primarily to enable reconstruction of the appropriate binders.

$$\frac{\Gamma \vdash A : \mathrm{dom}_\Pi(m, K) \qquad \Gamma; x_m : A \vdash B : \mathrm{codom}_\Pi(m)}{\Gamma \vdash (x : A) \to_m B : \mathrm{codom}_\Pi(m)}\ \textsc{Pi}$$

The function type formation rule is similar to the rule for CC, but the domain and codomain are restricted. Instead of being part of either a type or kind universe, the respective universes are restricted by the associated mode. If the mode is τ then the domain can be either a type or a kind, but the codomain must be a kind. If the mode is ω then the domain and codomain both must be types. Otherwise, the mode is 0 and the domain may be either a type or kind, but the codomain must be a type. Note that this means polymorphic functions of data are not allowed to use their type argument computationally in the object of a proof.

$$\frac{\Gamma \vdash (x : A) \to_m B : \mathrm{codom}_\Pi(m) \qquad \Gamma; x_m : A \vdash t : B \qquad x \notin FV(|t|) \text{ if } m = 0}{\Gamma \vdash \lambda_m\, x{:}A.\, t : (x : A) \to_m B}\ \textsc{Lam}$$

The function formation rule is again similar to the rule for CC. Unlike the standard PTS CC rule, the codomain of the inferred function type is again restricted to $\mathrm{codom}_\Pi(m)$. Additionally, if the mode is erased then it must be explicitly shown that the bound variable does not appear in the associated object. Note that this is exactly the requirement imposed by pseudo-objects.

$$\frac{\Gamma \vdash f : (x : A) \to_m B \qquad \Gamma \vdash a : A}{\Gamma \vdash f \bullet_m a : [x := a]B}\ \textsc{App}$$

The application rule is not surprising, the only notable feature is that the mode of the function type and the application must match.

$$\frac{\Gamma \vdash A : \star \qquad \Gamma; x_\tau : A \vdash B : \star}{\Gamma \vdash (x : A) \cap B : \star}\ \textsc{Int}$$

The intersection type formation rule is similar to the function type formation rule, but the terms are all restricted to be types. Thus, there are no intersections of kinds in the core Cedille2 system.

$$\frac{\Gamma \vdash (x : A) \cap B : \star \qquad \Gamma \vdash t : A \qquad \Gamma \vdash s : [x := t]B \qquad t \equiv s}{\Gamma \vdash [t, s; (x : A) \cap B] : (x : A) \cap B}\ \textsc{Pair}$$

The pair formation rule is standard for formation of dependent pairs. A third type annotation argument is required in order to make the formula inferable from the proof. Otherwise, the annotation is required to be itself a type, the first component to match the first type, and the second component to match the second type with its free variable substituted with the first component. Additionally, the first and second component must be convertible. This restriction is what makes this a proof of an intersection, as opposed to merely a pair. Note that by Theorem 2.33 this condition is equivalent to $|t| \rightleftharpoons |s|$ which is the restriction imposed by pseudo-objects.

$$\frac{\Gamma \vdash t : (x : A) \cap B}{\Gamma \vdash t.2 : [x := t.1]B}\ \textsc{Snd}$$

The first and second projection rules are unsurprising. Both rules model projection from a pair as expected.

$$\frac{\Gamma \vdash A : \star \qquad \Gamma \vdash a : A \qquad \Gamma \vdash b : A}{\Gamma \vdash a =_A b : \star}\ \textsc{Eq}$$

The equality type formation rule requires that the type annotation is a type and that the left and right-hand sides infer that type. Note that a typed equality like this is standard from the perspective of modern type theory but significantly different from the *untyped* equality of Cedille. Indeed, the equality rules are the area of significant deviation

$$\frac{\Gamma \vdash A : \star \qquad \Gamma; x_\tau : A \vdash B : \star}{\Gamma \vdash (x : A) \cap B : \star}\ \textsc{Int}$$

$$\frac{\Gamma \vdash (x : A) \cap B : \star \qquad \Gamma \vdash t : A \qquad \Gamma \vdash s : [x := t]B \qquad t \equiv s}{\Gamma \vdash [t, s; (x : A) \cap B] : (x : A) \cap B}\ \textsc{Pair}$$

$$\frac{\Gamma \vdash t : (x : A) \cap B}{\Gamma \vdash t.1 : A}\ \textsc{Fst} \qquad\qquad \frac{\Gamma \vdash t : (x : A) \cap B}{\Gamma \vdash t.2 : [x := t.1]B}\ \textsc{Snd}$$

Figure 2.9: Inference rules for intersection types.

from the original Cedille design.

$$\frac{\Gamma \vdash A : \star \qquad \Gamma \vdash t : A}{\Gamma \vdash \mathrm{refl}(t; A) : t =_A t}\ \textsc{Refl}$$

The reflexivity rule is the only value for equality types. It is the standard inductive formulation of the equality type.

$$\frac{\Gamma \vdash A : \star \qquad \Gamma \vdash a : A \qquad \Gamma \vdash b : A \qquad \Gamma \vdash e : a =_A b \qquad \Gamma \vdash P : (y : A) \to_\tau (p : a =_A y_\star) \to_\tau \star}{\Gamma \vdash \psi(e, a, b; A, P) : P \bullet_\tau a \bullet_\tau \mathrm{refl}(a; A) \to_\omega P \bullet_\tau b \bullet_\tau e}\ \textsc{Subst}$$

The substitution rule is a dependent variation of the Leibniz's Law. It is also a variation of Martin-Löf's J rule introduced by Pfenning and Paulin-Mohring [47]. Notice that the only critical difference between this rule and a standard variation of Leibniz's Law is that the predicate may depend on the equality proof as well.

$$\frac{\Gamma \vdash (x : A) \cap B : \star \qquad \Gamma \vdash a : (x : A) \cap B \qquad \Gamma \vdash b : (x : A) \cap B \qquad \Gamma \vdash e : a.1 =_A b.1}{\Gamma \vdash \vartheta(e, a, b; (x : A) \cap B) : a =_{(x:A)\cap B} b}\ \textsc{Prm}$$

The promotion rule enables equational reasoning about intersections. Indeed, because intersections are not inductive it is difficult to reason about them without some auxiliary rule. It states that two elements of an intersection are equal if their first projections are equal. Note that this rule is about dependent intersections. A non-dependent version involving the second projection is internally derivable in the system.

$$\frac{\Gamma \vdash (x : A) \cap B : \star \qquad \Gamma \vdash a : A \qquad \Gamma \vdash b : (x : A) \cap B \qquad \Gamma \vdash e : a =_A b.1}{\Gamma \vdash \varphi(a, b, e; A, (x : A) \cap B) : (x : A) \cap B}\ \textsc{Cast}$$

The cast rule is a typed version of the original cast rule of Cedille. Note that this means this rule enables non-termination. In Chapter 5 it is shown that this rule is the only source of non-termination and a precise condition for when it may be used in a terminating way is devised. The cast rule is critical to the spirit of Cedille. Thus, simply dropping it to obtain a strongly normalizing system is not really a sufficient choice as it throws too much away in its loss. An observant reader may notice that there are some redundant type annotations in the rule. These annotations are present merely to simplify models developed in Chapter 3. A more concise version of the system with fewer redundancies is developed in Chapter 6.

$$\frac{\Gamma \vdash A : \star \qquad \Gamma \vdash a : A \qquad \Gamma \vdash b : A}{\Gamma \vdash a =_A b : \star}\ \textsc{Eq} \qquad \frac{\Gamma \vdash A : \star \qquad \Gamma \vdash t : A}{\Gamma \vdash \mathrm{refl}(t; A) : t =_A t}\ \textsc{Refl}$$

$$\frac{\begin{array}{c}\Gamma \vdash A : \star \\ \Gamma \vdash a : A \qquad \Gamma \vdash b : A \qquad \Gamma \vdash e : a =_A b \qquad \Gamma \vdash P : (y : A) \to_\tau (p : a =_A y_\star) \to_\tau \star\end{array}}{\Gamma \vdash \psi(e, a, b; A, P) : P \bullet_\tau a \bullet_\tau \mathrm{refl}(a; A) \to_\omega P \bullet_\tau b \bullet_\tau e}\ \textsc{Subst}$$

$$\frac{\Gamma \vdash (x : A) \cap B : \star \qquad \Gamma \vdash a : (x : A) \cap B \qquad \Gamma \vdash b : (x : A) \cap B \qquad \Gamma \vdash e : a.1 =_A b.1}{\Gamma \vdash \vartheta(e, a, b; (x : A) \cap B) : a =_{(x:A)\cap B} b}\ \textsc{Prm}$$

$$\frac{\Gamma \vdash (x : A) \cap B : \star \qquad \Gamma \vdash a : A \qquad \Gamma \vdash b : (x : A) \cap B \qquad \Gamma \vdash e : a =_A b.1}{\Gamma \vdash \varphi(a, b, e; A, (x : A) \cap B) : (x : A) \cap B}\ \textsc{Cast}$$

$$\frac{\Gamma \vdash e : \mathrm{ctt} =_{\mathrm{cBool}} \mathrm{cff}}{\Gamma \vdash \delta(e) : (X : \star) \to_0 X_\square}\ \textsc{Sep}$$

Figure 2.10: Inference rules for equality types where $\mathrm{cBool} := (X : \star) \to_0 (x : X_\square) \to_\omega (y : X_\square) \to_\omega X_\square$; $\mathrm{ctt} := \lambda_0 X{:}\star.\, \lambda_\omega x{:}X_\square.\, \lambda_\omega y{:}X_\square.\, x_\star$; and $\mathrm{cff} := \lambda_0 X{:}\star.\, \lambda_\omega x{:}X_\square.\, \lambda_\omega y{:}X_\square.\, y_\star$.

$$\frac{\Gamma \vdash e : \mathrm{ctt} =_{\mathrm{cBool}} \mathrm{cff}}{\Gamma \vdash \delta(e) : (X : \star) \to_0 X_\square}\ \textsc{Sep}$$

The separation rule states only that the equational theory is not degenerate, i.e. that there are at least two distinct objects.

The first critical observation to be made is that the syntax participating in an inference judgment are pseudo-objects. Thus, proofs and their types enjoy transitivity of conversion. Next three standard lemmas are proved about the type system: weakening, substitution, and a sort-hierarchy classification.

Lemma 2.36. *If* $\Gamma \vdash t : A$ *then* t pseobj

Proof. Straightforward by induction. The only interesting case is the pair case, but it is discharged by Theorem 2.33. □

Lemma 2.37. *If* $\Gamma \vdash t : A$ *then* A pseobj

Proof. By induction. The Ax, Pi, Int and Eq rules are trivial. Rules Lam, Pair, Cast, and Conv rules are immediate by applying Lemma 2.36 to a sub-derivation. The Fst and App rules are omitted because it is similar to the Snd rule. Likewise, the Refl rule is omitted because it is similar to the Prm rule.

Case: $\dfrac{x \notin FV(\Gamma_1; \Gamma_2) \qquad \Gamma_1 \vdash \overset{\mathcal{D}_2}{A} : K}{\Gamma_1; x_m : A; \Gamma_2 \vdash x_K : A}$ $\overset{\mathcal{D}_1}{}$

By Lemma 2.36 applied to $\mathcal{D}_2$: A pseobj.

Case: $\dfrac{\Gamma \vdash t : \overset{\mathcal{D}_1}{(x : A) \cap B}}{\Gamma \vdash t.2 : [x := t.1]B}$

By the IH applied to $\mathcal{D}_1$: B pseobj. Using Lemma 2.36 gives t pseobj and thus $t.1$ pseobj. Now by Lemma 2.26: $[x := t.1]B$ pseobj.

Case: $\dfrac{\Gamma \vdash \overset{\mathcal{D}_1}{A} : \star \qquad \Gamma \vdash \overset{\mathcal{D}_2}{a} : A \qquad \Gamma \vdash \overset{\mathcal{D}_3}{b} : A \qquad \Gamma \vdash \overset{\mathcal{D}_4}{e} : a =_A b \qquad \Gamma \vdash P : (y : A) \to_\tau \overset{\mathcal{D}_5}{(p : a =_A y_\star)} \to_\tau \star}{\Gamma \vdash \psi(e, a, b; A, P) : P \bullet_\tau a \bullet_\tau \mathrm{refl}(a; A) \to_\omega P \bullet_\tau b \bullet_\tau e}$

By Lemma 2.36: P, e pseobj. Applying the IH to $\mathcal{D}_1$ gives A, a, b pseobj. Now building up the subexpressions using pseudo-object rules concludes the proof.

Case: $\dfrac{\Gamma \vdash \overset{\mathcal{D}_1}{(x : A)} \cap B : \star \qquad \Gamma \vdash \overset{\mathcal{D}_2}{a} : (x : A) \cap B \qquad \Gamma \vdash \overset{\mathcal{D}_3}{b} : (x : A) \cap B \qquad \Gamma \vdash \overset{\mathcal{D}_4}{e} : a.1 =_A b.1}{\Gamma \vdash \vartheta(e, a, b; (x : A) \cap B) : a =_{(x:A)\cap B} b}$

Applying the IH to $\mathcal{D}_1$ gives that $(x : A) \cap B$ pseobj. Now, by Lemma 2.36: a, b pseobj. Using the pseudo-object rule for equality concludes the case.

Case: $\dfrac{\Gamma \vdash e : \mathrm{ctt} =_{\mathrm{cBool}} \overset{\mathcal{D}_1}{\mathrm{cff}}}{\Gamma \vdash \delta(e) : (X : \star) \to_0 X_\square}$

Immediate by a short sequence of pseudo-object rules.

□

Lemma 2.38 (Weakening)**.** *If* $\Gamma; \Delta \vdash t : A$ *and* $\Gamma \vdash B : K$ *then* $\Gamma; y_m : B; \Delta \vdash t : A$ *for* y *fresh*

Proof. By induction. Most cases are a direct consequence of applying the IH to sub-derivations and applying the associated rule.

Case: $\dfrac{}{\Gamma \vdash \star : \square}$

Trivial by axiom rule.

Case: $\dfrac{x \notin FV(\Gamma_1; \Gamma_2) \qquad \Gamma_1 \vdash \overset{\mathcal{D}_2}{A} : K}{\Gamma_1; x_m : A; \Gamma_2 \vdash x_K : A}$

Note that y is fresh thus $x \neq y$. If y is placed after x then the case is trivial because Γ_2 is only constrained to carry fresh variables. Thus, suppose y is placed before x. Let $\Gamma_1 = \Delta_1; \Delta_2$. Applying the IH to $\mathcal{D}_2$ gives $\Delta_1; y_m : B; \Delta_2 \vdash A : K$. The VAR rule concludes.

Case: $$\frac{\Gamma \vdash A : \mathrm{dom}_\Pi(m, K) \overset{\mathcal{D}_1}{} \qquad \Gamma; x_m : A \vdash B : \mathrm{codom}_\Pi(m) \overset{\mathcal{D}_2}{}}{\Gamma \vdash (x : A) \to_m B : \mathrm{codom}_\Pi(m)}$$

The IH applied to $\mathcal{D}_1$ and $\mathcal{D}_2$ and the pi-rule concludes the case.

□

Lemma 2.39 (Substitution)**.** *Suppose* $\Gamma \vdash b : B$*. If* $\Gamma, y : B, \Delta \vdash t : A$ *then* $\Gamma, [y := b]\Delta \vdash [y := b]t : [y := b]A$

Proof. By induction on $\Gamma, y : B, \Delta \vdash t : A$. The Ax rule is trivial and omitted. The rules Lam and Int are very similar to the Pi rule. The rules Fst, Eq, Refl, Subst, Prm, Cast and Sep rules are proven by applying *1.* to sub-derivations and using the associated rule. Rule Snd is very similar to App and thus omitted. Likewise, Conv is very similar to Pair and thus omitted.

Case: $$\frac{x \notin FV(\Gamma_1; \Gamma_2) \overset{\mathcal{D}_1}{} \qquad \Gamma_1 \vdash A : K \overset{\mathcal{D}_2}{}}{\Gamma_1; x_m : A; \Gamma_2 \vdash x_K : A}$$

Suppose wlog that $y \in \Gamma_1$. Let $\Gamma_1 = \Delta_1; y : B; \Delta_2$. Applying the IH to $\mathcal{D}_1$ gives $\Delta_1; [y := b]\Delta_2 \vdash [y := b]A : K$. Note that $x \notin FV(\Delta_1; [y := b]\Delta_2; [y := b]\Gamma_2)$. Thus by the Var rule: $\Delta_1; [y := b]\Delta_2; x_m : [y := b]A; [y := b]\Gamma_2 \vdash x_K : [y := b]A$.

Case: $$\frac{\Gamma \vdash A : \mathrm{dom}_\Pi(m, K) \overset{\mathcal{D}_1}{} \qquad \Gamma; x_m : A \vdash B : \mathrm{codom}_\Pi(m) \overset{\mathcal{D}_2}{}}{\Gamma \vdash (x : A) \to_m B : \mathrm{codom}_\Pi(m)}$$

Applying *1.* to the sub-derivations gives:

$\mathcal{D}_1$. $\Gamma, [y := b]\Delta \vdash [y := b]A : \mathrm{dom}_\Pi(m, K)$

$\mathcal{D}_2$. $\Gamma, [y := b]\Delta, x_m : [y := b]A \vdash [y := b]B : \mathrm{codom}_\Pi(m)$

Thus, $\Gamma, [y := b]\Delta \vdash (x : [y := b]A) \to_m [y := b]B : \mathrm{codom}_\Pi(m)$.

Case: $$\frac{\Gamma \vdash f : (x : A) \to_m B \overset{\mathcal{D}_1}{} \qquad \Gamma \vdash a : A \overset{\mathcal{D}_2}{}}{\Gamma \vdash f \bullet_m a : [x := a]B}$$

Applying *1.* to $\mathcal{D}_1$ and $\mathcal{D}_2$ gives:

$\mathcal{D}_1$. $\Gamma, [y := b]\Delta \vdash [y := b]f : (x : [y := b]A) \to_m [y := b]B$

$\mathcal{D}_2$. $\Gamma, [y := b]\Delta, x_m : [y := b]A \vdash [y := b]a : [y := b]A$

By the App rule $\Gamma, [y := b]\Delta \vdash [y := b]f \bullet_m [y := b]a : [x := a][y := b]B$. Note that x is fresh to Γ, thus $x \notin FV(b)$. By Lemma 2.1 $[x := a][y := b]B = [y := b][x := a]B$.

Case: $$\frac{\Gamma \vdash (x : A) \cap B : \star \overset{\mathcal{D}_1}{} \qquad \Gamma \vdash t : A \overset{\mathcal{D}_2}{} \qquad \Gamma \vdash s : [x := t]B \overset{\mathcal{D}_3}{} \qquad t \equiv s \overset{\mathcal{D}_4}{}}{\Gamma \vdash [t, s; (x : A) \cap B] : (x : A) \cap B}$$

Applying *1.* to the sub-derivations gives:

$\mathcal{D}_1$. $\Gamma, [y := b]\Delta \vdash (x : [y := b]A) \cap [y := b]B : \star$

$\mathcal{D}_2$. $\Gamma, [y := b]\Delta \vdash [y := b]t : [y := b]A$

$\mathcal{D}_3$. $\Gamma, [y := b]\Delta \vdash [y := b]s : [y := b][x := t]B$

Note that x is locally-bound and thus $x \notin FV(\Gamma)$, thus by Lemma 2.1

$$[y := b][x := t]B = [x := [y := b]t][y := b]B$$

Now by Lemma 2.35: $[y := b]t \equiv [y := b]s$. Thus, by the Pair rule $\Gamma, [y := b]\Delta \vdash [[y := b]t, [y := b]s] : (x : [y := b]A) \cap [y := b]B$.

□

Lemma 2.40. *If* $\Gamma \vdash t : A$ *then* $A = \Box$ *or* $\Gamma \vdash A : K$

Proof. By induction. The Ax, Pi, Lam, Int, Pair, Eq, Cast, and Conv rules are trivial. The Fst rule is omitted because it is similar to Snd rule. Likewise, the Refl rule is omitted because it is similar to the Prm rule.

Case: $\dfrac{x \notin FV(\Gamma_1; \Gamma_2) \qquad \Gamma_1 \vdash \overset{\mathcal{D}_2}{A} : K}{\Gamma_1; x_m : A; \Gamma_2 \vdash x_K : A}$

Immediate by $\mathcal{D}_2$ and weakening.

Case: $\dfrac{\Gamma \vdash f : \overset{\mathcal{D}_1}{(x : A)} \to_m B \qquad \Gamma \vdash \overset{\mathcal{D}_2}{a} : A}{\Gamma \vdash f \bullet_m a : [x := a]B}$

Applying the IH to $\mathcal{D}_1$ gives $\Gamma \vdash (x : A) \to_m B : K$. Now $\Gamma, x : A \vdash B : K$. Using the substitution lemma gives $\Gamma \vdash [x := a]B : K$.

Case: $\dfrac{\Gamma \vdash t : \overset{\mathcal{D}_1}{(x : A)} \cap B}{\Gamma \vdash t.2 : [x := t.1]B}$

By the IH applied to $\mathcal{D}_1$ gives $\Gamma \vdash (x : A) \cap B : K$. Thus, $\Gamma, x : A \vdash B : K$. Applying the substitution lemma gives $\Gamma \vdash [x := t.1]B : K$.

Case: $\dfrac{\Gamma \vdash \overset{\mathcal{D}_1}{A} : \star \qquad \Gamma \vdash \overset{\mathcal{D}_2}{a} : A \qquad \Gamma \vdash \overset{\mathcal{D}_3}{b} : A \qquad \Gamma \vdash \overset{\mathcal{D}_4}{e} : a =_A b \qquad \Gamma \vdash P : \overset{\mathcal{D}_5}{(y : A)} \to_\tau (p : a =_A y_\star) \to_\tau \star}{\Gamma \vdash \psi(e, a, b; A, P) : P \bullet_\tau a \bullet_\tau \mathrm{refl}(a; A) \to_\omega P \bullet_\tau b \bullet_\tau e}$

By the Refl rule: $\Gamma \vdash \mathrm{refl}(a; A) : a =_A a$. Now by the App rule $\Gamma \vdash P \bullet_\tau a \bullet_\tau \mathrm{refl}(a; A) : \star$ and $\Gamma \vdash P \bullet_\tau b \bullet_\tau e : \star$. Using weakening gives $\Gamma, x : P \bullet_\tau a \bullet_\tau \mathrm{refl}(a; A) \vdash P \bullet_\tau b \bullet_\tau e : \star$. Now the Pi rule concludes the case.

Case: $$\frac{\Gamma \vdash (x : \overset{\mathcal{D}_1}{A}) \cap B : \star \qquad \Gamma \vdash a : \overset{\mathcal{D}_2}{(x : A)} \cap B \qquad \Gamma \vdash b : \overset{\mathcal{D}_3}{(x : A)} \cap B \qquad \Gamma \vdash e : \overset{\mathcal{D}_4}{a.1} =_A b.1}{\Gamma \vdash \vartheta(e, a, b; (x : A) \cap B) : a =_{(x:A)\cap B} b}$$

Immediate by applying the Eq rule.

Case: $$\frac{\Gamma \vdash e : \overset{\mathcal{D}_1}{\mathrm{ctt}} =_{\mathrm{cBool}} \mathrm{cff}}{\Gamma \vdash \delta(e) : (X : \star) \to_0 X_\square}$$

Have $\Gamma \vdash (X : \star) \to_\omega X : \star$ via short sequence of rules.

□

The context of a judgment is, for the moment, unrestrained. Indeed, a variable may bind a type represented by arbitrary syntax and as long as that variable is never used in the body of the term there is no issue. To remove these considerations contexts should instead be well-formed:

Definition 2.41. *A context Γ is **well-formed** (written $\vdash \Gamma$) iff for every possible splitting $\Gamma = \Gamma_1, x : A, \Gamma_2$ the variable $x \notin FV(\Gamma_1; \Gamma_2)$ and $\Gamma_1 \vdash A : K$ for some K*

It is not difficult to see that an inference judgment with a well-formed context is obtained from any general inference judgment. Moving forward it will be assumed that the context is well-formed because an equivalent proof is always obtainable under this assumption and the non-well-founded proofs will not add any value.

Lemma 2.42. *If $\Gamma \vdash t : A$ then $\exists\ \Delta$ such that $\Delta \vdash t : A$ and $\vdash \Delta$*

Proof. By Lemma 2.40: $\Gamma \vdash A : K$. Now, the set of free variable $S = FV(t) \cup FV(A)$ determines Δ. Moreover, every occurrence of $x \in S$ in either t or A must be via a Var rule, hence the associated type is a proof. Delete any variables not found in S from Γ to form Δ. □

2.5 Preservation

Preservation states that the status of a term (i.e. both its classification and status as a well-founded proof) do not change with respect to reduction. Note that Cedille only enjoys a semantic version of preservation and not a syntactic version presented below. While this may not matter from the perspective of the semantics it does indicate that syntax is better behaved. The proof follows by induction on the typing derivation and case analysis on the associated reduction.

Definition 2.43. *$\Gamma \rightsquigarrow \Gamma'$ iff there exists a unique $(x_m : A) \in \Gamma$ such that $A \rightsquigarrow A'$*

Lemma 2.44.

1. *If $\Gamma \vdash t : A$ and $t \rightsquigarrow t'$ then $\Gamma \vdash t' : A$*
2. *If $\Gamma \vdash t : A$ and $\Gamma \rightsquigarrow \Gamma'$ then $\Gamma' \vdash t : A$*

3. If $\vdash \Gamma$ *and* $\Gamma \rightsquigarrow \Gamma'$ *then* $\vdash \Gamma'$

Proof. By mutual recursion.

1. Pattern-matching on $\Gamma \vdash t : A$. The Ax and Var cases are impossible by inversion on $t \rightsquigarrow t'$. Int is very similar to Pi, Fst is very similar to Snd. The Refl, Sep, and Conv rules are trivial.

Case: $$\frac{\Gamma \vdash A : \overset{\mathcal{D}_1}{\mathrm{dom}_\Pi}(m, K) \qquad \Gamma; x_m : A \vdash \overset{\mathcal{D}_2}{B} : \mathrm{codom}_\Pi(m)}{\Gamma \vdash (x : A) \to_m B : \mathrm{codom}_\Pi(m)}$$

Suppose $A \rightsquigarrow A'$. Applying *1* to $\mathcal{D}_1$ gives $\Gamma \vdash A' : \mathrm{dom}_\Pi(m, K)$. Note that $\Gamma, x_m : A \rightsquigarrow \Gamma, x_m : A'$. Thus, using *2* with $\mathcal{D}_2$ gives $\Gamma, x_m : A' \vdash B : \mathrm{codom}_\Pi(m)$. Using the Pi rule concludes the case.

Suppose $B \rightsquigarrow B'$. Applying *1* to $\mathcal{D}_2$ gives $\Gamma, x_m : A \vdash B' : \mathrm{codom}_\Pi(m)$. The Pi rule concludes the case.

Case: $$\frac{\Gamma \vdash (x : A) \to_m \overset{\mathcal{D}_1}{B} : \mathrm{codom}_\Pi(m) \qquad \Gamma; x_m : \overset{\mathcal{D}_2}{A} \vdash t : B \qquad \overset{\mathcal{D}_3}{x \notin FV(|t|)} \text{ if } m = 0}{\Gamma \vdash \lambda_m\, x{:}A.\, t : (x : A) \to_m B}$$

Suppose $A \rightsquigarrow A'$. Then $(x : A) \to_m B \rightsquigarrow (x : A') \to_m B$. Now, using *1* with $\mathcal{D}_1$ gives $\Gamma \vdash (x : A') \to_m B : \mathrm{codom}_\Pi(m)$. Note that $\Gamma, x_m : A \rightsquigarrow \Gamma, x_m : A'$. Using *2* with $\mathcal{D}_2$ yields $\Gamma, x_m : A' \vdash t : B$. Applying the Lam rule concludes the case.

Suppose $t \rightsquigarrow t'$. Using *1* with $\mathcal{D}_2$ gives $\Gamma, x_m : A \vdash t' : B$. Note that reduction does not introduce free variables, thus $x \notin FV(|t'|)$ if $m = 0$. The Lam rule concludes.

Case: $$\frac{\Gamma \vdash f : \overset{\mathcal{D}_1}{(x : A)} \to_m B \qquad \Gamma \vdash \overset{\mathcal{D}_2}{a} : A}{\Gamma \vdash f \bullet_m a : [x := a]B}$$

Suppose $f \rightsquigarrow f'$. Applying *1* with $\mathcal{D}_1$ gives $\Gamma \vdash f' : (x : A) \to_m B$. The App rule concludes.

Suppose $a \rightsquigarrow a'$. Using *1* with $\mathcal{D}_2$ gives $\Gamma \vdash a' : A$. Again, the App rule concludes the case.

Suppose $f = \lambda_m\, x : C.\, t$ and $f \bullet_m a \rightsquigarrow [x := a]t$. There must exist C and D such that $\Gamma \vdash C : \mathrm{dom}_\Pi(m, K)$ and $\Gamma, x_m : C \vdash t : D$ with $A \equiv C$ and $B \equiv D$. By classification (Lemma 2.40) and the Conv rule, $\Gamma \vdash a : C$. Now using the substitution lemma (Lemma 2.39) $\Gamma \vdash [x := a]t : [x := a]D$. Using Lemma 2.35 gives $[x := a]B \equiv [x := a]D$. Classification and Conv again yields $\Gamma \vdash [x := a]t : [x := a]B$.

Suppose $f = \psi(\mathrm{refl}(z; Z), u, v; U, P)$ with $m = \omega$ and $f \bullet_\omega a \rightsquigarrow a$. By inversion on

$\mathcal{D}_1$: $A = P \bullet_\tau u \bullet_\tau \mathrm{refl}(u;U)$ and $[x := a]B = P \bullet_\tau v \bullet_\tau \mathrm{refl}(z;Z)$. However, inversion also yields $\Gamma \vdash \mathrm{refl}(z;Z) : u =_U v$ thus $z \equiv u$, $z \equiv v$, and $Z \equiv U$. Thus, $P \bullet_\tau u \bullet_\tau \mathrm{refl}(u;U) \equiv P \bullet_\tau v \bullet_\tau \mathrm{refl}(z;Z)$. The Conv rule concludes the case.

Case: $$\frac{\Gamma \vdash (x : \overset{\mathcal{D}_1}{A}) \cap B : \star \qquad \Gamma \vdash \overset{\mathcal{D}_2}{t} : A \qquad \Gamma \vdash s : \overset{\mathcal{D}_3}{[x := t]B} \qquad t \overset{\mathcal{D}_4}{\equiv} s}{\Gamma \vdash [t, s; (x : A) \cap B] : (x : A) \cap B}$$

Suppose $t \rightsquigarrow t'$. Applying *1* to $\mathcal{D}_2$ gives $\Gamma \vdash t' : A$. Note that $[x := t]B \equiv [x := t']B$ by Lemma 2.35. Moreover, deconstructing $\mathcal{D}_1$ yields $\Gamma, x_\tau : A \vdash B : \star$. By the substitution lemma $\Gamma \vdash [x := t']B : \star$. Thus, by the Conv rule $\Gamma \vdash s : [x := t']B$. Finally, Lemma 2.31 gives $t' \equiv s$ from $\mathcal{D}_4$. The Pair rule concludes the case.

Suppose $s \rightsquigarrow s'$. By *1* applied to $\mathcal{D}_3$: $\Gamma \vdash s' : [x := t]B$. Using Lemma 2.35 with $\mathcal{D}_4$ yields $t \equiv s'$. The Pair rule concludes.

Suppose $A \rightsquigarrow A'$. Then $(x : A) \cap B \rightsquigarrow (x : A') \cap B$. Applying this reduction to *1* with $\mathcal{D}_1$ gives $\Gamma \vdash (x : A') \cap B : \star$. Deconstructing this yields $\Gamma \vdash A' : \star$. Now by the Conv rule $\Gamma \vdash t : A'$. Using the Pair rule concludes.

Suppose $B \rightsquigarrow B'$. Then $(x : A) \cap B \rightsquigarrow (x : A') \cap B$. Applying this reduction to *1* with $\mathcal{D}_1$ gives $\Gamma \vdash (x : A) \cap B' : \star$. Deconstructing this yields $\Gamma, x_m : A' \vdash B' : \star$. Note that $B \rightsquigarrow B'$ implies that $B \equiv B'$. Moreover, using Lemma 2.35 gives $[x := t]B \equiv [x := t]B'$. The substitution lemma gives $\Gamma \vdash [x := t]B' : \star$. Now the Conv rule yields $\Gamma \vdash s[x := t]B'$. The Pair rule concludes the case.

Case: $$\frac{\Gamma \vdash t : \overset{\mathcal{D}_1}{(x : A) \cap B}}{\Gamma \vdash t.2 : [x := t.1]B}$$

Suppose $t \rightsquigarrow t'$. Then applying *1* to $\mathcal{D}_1$ gives $\Gamma \vdash t' : (x : A) \cap B$. Applying the Snd rule concludes the case.

Suppose $t = [t_1, t_2, t_3]$ and $t.2 \rightsquigarrow t_2$. Then we have $\Gamma \vdash [t_1, t_2, t_3] : (x : A) \cap B$. Deconstructing this rule yields $\Gamma \vdash t_1 : A$, $\Gamma, x_\tau : A \vdash B : \star$, and $\Gamma \vdash t_2 : [x := t_1]B$. By the substitution lemma $\Gamma \vdash [x := t.1]B : \star$. Note that $t.1 \rightsquigarrow t_1$ thus $t.1 \equiv t_1$. Now using Lemma 2.35 gives $[x := t.1]B \equiv [x := t_1]B$. Thus, by the Conv rule $\Gamma \vdash t_2 : [x := t.1]B$.

Case: $$\frac{\Gamma \vdash \overset{\mathcal{D}_1}{A} : \star \qquad \Gamma \vdash \overset{\mathcal{D}_2}{a} : A \qquad \Gamma \vdash \overset{\mathcal{D}_2}{b} : A}{\Gamma \vdash a =_A b : \star}$$

Suppose $a \rightsquigarrow a'$. Applying *1* to $\mathcal{D}_2$ gives $\Gamma \vdash a' : A$. The Eq rule concludes.

Suppose $b \rightsquigarrow b'$. Applying *1* to $\mathcal{D}_3$ gives $\Gamma \vdash b' : A$. The Eq rule concludes.

Suppose $A \rightsquigarrow A'$. Applying *1* to $\mathcal{D}_1$ gives $\Gamma \vdash A' : \star$. Note that $A \equiv A'$. Thus, by the Conv rule applied twice: $\Gamma \vdash a : A'$ and $\Gamma \vdash b : A'$. Using the Eq rule concludes the case.

Case: $$\frac{\Gamma \vdash \overset{\mathcal{D}_1}{A} : \star \qquad \Gamma \vdash \overset{\mathcal{D}_2}{a} : A \qquad \Gamma \vdash \overset{\mathcal{D}_3}{b} : A \qquad \Gamma \vdash \overset{\mathcal{D}_4}{e} : a =_A b \qquad \Gamma \vdash P : \overset{\mathcal{D}_5}{(y : A) \to_\tau (p : a =_A y_\star) \to_\tau \star}}{\Gamma \vdash \psi(e, a, b; A, P) : P \bullet_\tau a \bullet_\tau \mathrm{refl}(a; A) \to_\omega P \bullet_\tau b \bullet_\tau e}$$

Suppose $A \rightsquigarrow A'$. Then $a =_A b \equiv a ='_A b$ and $(y : A) \to_t au(p : a =_A y_\star) \to_\tau \star \equiv (y : A) \to_t au(p : a =_A y_\star) \to_\tau$. Thus, by the Conv rule: $\Gamma \vdash a : A'$, $\Gamma \vdash b : A'$, $\Gamma \vdash e : a =_{A'} b$, and $\Gamma \vdash P : (y : A') \to_t au(p : a =_{A'} y_\star) \to_\tau$. Applying *1* to $\mathcal{D}_1$ gives: $\Gamma \vdash A' : \star$. The Subst rule concludes the case.

Suppose $a \rightsquigarrow a'$. Then $a =_A b \equiv a'_A b$ and $(y : A) \to_t au(p : a =_A y_\star) \to_\tau \star \equiv (y : A) \to_t au(p : a' =_A y_\star) \to_\tau$. Thus, by the Conv rule: $\Gamma \vdash e : a' =_A b$ and $\Gamma \vdash P : (y : A) \to_t au(p : a' =_A y_\star) \to_\tau$. Applying *1* to $\mathcal{D}_2$ gives: $\Gamma \vdash a' : A$. The Subst rule concludes the case.

Suppose $b \rightsquigarrow b'$. Then $a =_A b \equiv a =_A b'$ and by the Conv rule $\Gamma \vdash b' : A$. Applying *1* to $\mathcal{D}_3$ gives: $\Gamma \vdash b' : B$. The Subst rule concludes the case.

Suppose $e \rightsquigarrow e'$. Then by *1* applied to $\mathcal{D}_1$: $\Gamma \vdash e' : a =_A b$. The Subst rule concludes the case.

Suppose $P \rightsquigarrow P'$. By *1* applied to $\mathcal{D}_2$: $\Gamma \vdash P : (y : A) \to_\tau (p : a =_A y) \to \tau\star$. The Subst rule concludes the case.

Case: $$\frac{\Gamma \vdash \overset{\mathcal{D}_1}{(x : A) \cap B} : \star \qquad \Gamma \vdash \overset{\mathcal{D}_2}{a} : (x : A) \cap B \qquad \Gamma \vdash \overset{\mathcal{D}_3}{b} : (x : A) \cap B \qquad \Gamma \vdash e : \overset{\mathcal{D}_4}{a.1} =_A b.1}{\Gamma \vdash \vartheta(e, a, b; (x : A) \cap B) : a =_{(x:A)\cap B} b}$$

Suppose $e \rightsquigarrow e'$. Then by *1* applied to $\mathcal{D}_4$: $\Gamma \vdash e'a.1 =_A b.1$ and the Prm rule concludes.

Suppose $a \rightsquigarrow a'$. Then $a.1 =_A b.1 \equiv a'.1 =_A b.1$ and the Conv rule yields $\Gamma \vdash e : a'.1 =_A b.1$. Applying *1* to $\mathcal{D}_2$ gives $\Gamma \vdash a' : (x : A) \cap B$. The Prm rule concludes.

Suppose $b \rightsquigarrow b'$. Then $a.1 =_A b.1 \equiv a.1 =_A b'.1$ and the Conv rule yields $\Gamma \vdash e : a.1 =_A b'.1$. Applying *1* to $\mathcal{D}_3$ gives $\Gamma \vdash b' : (x : A) \cap B$. The Prm rule concludes.

Suppose wlog that $B \rightsquigarrow B'$, the case when $A \rightsquigarrow A'$ is similar. Then $(x : A) \cap B \equiv (x : A) \cap B'$ and the Conv rule yields $\Gamma \vdash a : (x : A) \cap B'$ and $\Gamma \vdash b : (x : A) \cap B'$. Applying *1* to $\mathcal{D}_1$ yields $\Gamma \vdash (x : A) \cap B' : \star$. The Prm rule concludes.

Suppose $e = \mathrm{refl}(z; Z)$ and $\vartheta(e, a, b; (x : A) \cap B) \rightsquigarrow \mathrm{refl}(a; (x : A) \cap B)$. By inversion $\Gamma \vdash \mathrm{refl}(z; Z) : a.1 =_A b.1$, hence $z \equiv a.1$, $z \equiv b.1$. Thus, $a \equiv b$ and $\Gamma \vdash \mathrm{refl}(a; (x : A) \cap B) : a =_{(x:A)\cap B} b$.

Case: $$\frac{\Gamma \vdash (x : \overset{\mathcal{D}_1}{A}) \cap B : \star \qquad \Gamma \vdash \overset{\mathcal{D}_2}{a} : A \qquad \Gamma \vdash b : (\overset{\mathcal{D}_3}{x} : A) \cap B \qquad \Gamma \vdash e : \overset{\mathcal{D}_4}{a} =_A b.1}{\Gamma \vdash \varphi(a, b, e; A, (x : A) \cap B) : (x : A) \cap B}$$

Suppose $a \rightsquigarrow a'$. Then $a =_A b.1 \equiv a' =_A b.1$ and by Conv rule $\Gamma \vdash e : a' =_A b.1$. Applying *1* to $\mathcal{D}_2$ yields $\Gamma \vdash a' : A$. The Cast rule concludes.

Suppose $b \rightsquigarrow b'$, or $A \rightsquigarrow A'$, or $B \rightsquigarrow B'$. These cases are all similar to the previous case.

Suppose $e \rightsquigarrow e'$. Applying *1* to $\mathcal{D}_4$ yields $\Gamma \vdash e' : a =_A b.1$ The Cast rule concludes.

2. Pattern-matching on $\Gamma \vdash t : A$. Note that except Ax and Var all the other cases are immediate by applying *2* to all sub-derivations and using the associated rule.

Case: $$\frac{}{\Gamma \vdash \star : \square}$$

Immediate by the Ax rule, the context does not matter.

Case: $$\frac{x \notin FV(\overset{\mathcal{D}_1}{\Gamma_1}; \Gamma_2) \qquad \Gamma_1 \vdash \overset{\mathcal{D}_2}{A} : K}{\Gamma_1; x_m : A; \Gamma_2 \vdash x_K : A}$$

Suppose $\Gamma_1 \rightsquigarrow \Gamma_1'$. Reduction does not produce free variables, thus $x \notin FV(\Gamma_1'; \Gamma_2)$. Applying *1* to $\mathcal{D}_2$ yields $\Gamma_1' \vdash A : K$. The Var rule concludes.

Suppose $\Gamma_2 \rightsquigarrow \Gamma_2'$. As before $x \notin FV(\Gamma_1; \Gamma_2')$. The Var rule concludes.

Suppose $A \rightsquigarrow A'$. Applying *1* to $\mathcal{D}_2$ gives $\Gamma_1 \vdash A' : K$. The Var rule concludes.

3. Pattern-matching on Γ. If Γ is empty then $\varepsilon \rightsquigarrow \Gamma'$ forces $\Gamma' = \varepsilon$ and $\vdash \varepsilon$. Thus, let $\Gamma = \Delta; x_m : A$.

Suppose $\Delta; x_m : A \rightsquigarrow \Delta'; x_m : A$. Then by *3* applied to Δ: $\vdash \Delta'$. Now, because $\vdash \Delta; x_m : A$ it is the case that $\Delta \vdash A : K$. Using *2* gives $\Delta' \vdash A : K$. Therefore, $\vdash \Delta'; x_m : A$.

Suppose $\Delta; x_m : A \rightsquigarrow \Delta; x_m : A'$. Again $\vdash \Delta; x_m : A$ gives $\Delta \vdash A : K$. Using *1* gives $\Delta \vdash A' : K$.

Thus, $\vdash \Delta; x_m : A'$. □

Lemma 2.45.

1. *If* $\Gamma \vdash t : A$ *and* $t \rightsquigarrow^* t'$ *then* $\Gamma \vdash t' : A$
2. *If* $\Gamma \vdash t : A$ *and* $\Gamma \rightsquigarrow^* \Gamma'$ *then* $\Gamma' \vdash t : A$
3. *If* $\vdash \Gamma$ *and* $\Gamma \rightsquigarrow^* \Gamma'$ *then* $\vdash \Gamma'$

Proof. For each property the proof proceeds by induction on multistep reduction using Lemma 2.44 and the IH in the inductive case. □

Lemma 2.46. *If* $\Gamma \vdash t : A$ *and* $A \rightsquigarrow^* A'$ *then* $\Gamma \vdash t : A'$

Proof. By classification: $\Gamma \vdash A : K$. Using Lemma 2.45 gives $\Gamma \vdash A' : K$. Note that $A \equiv A'$. Thus, by the Conv rule $\Gamma \vdash t : A'$. □

Theorem 2.47 (Preservation)**.** *If* $\Gamma \vdash t : A$, $\Gamma \rightsquigarrow^* \Gamma'$, $t \rightsquigarrow^* t'$, *and* $A \rightsquigarrow^* A'$ *then* $\Gamma' \vdash t' : A'$

Proof. Consequence of Lemma 2.45 and Lemma 2.46. □

2.6 Classification

Classification is a critical property of a system like CC with unified syntax. It allows for the syntax to instead be stratified into levels which would enable an intrinsic presentation of the system. For the core theory of Cedille2 there are only two universes like the original CC, thus the stratification places terms into three separate levels. A term is either a *kind* (thus $A = \Box$), a *type* (thus $\Gamma \vdash A : \Box$), or a *term* (thus $\Gamma \vdash A : \star$). Determining the appropriate level for syntax is also decidable with a classification function defined in Figure 2.11. This function is crafted to maintain preservation of classification after both reduction and erasure. Note that because the function is defined on syntax it is possible that there is no valid level because the syntax is not a proof, in these cases the syntax is given the classification *undefined*.

Definition 2.48.

1. t term *iff* $\mathcal{C}(t) = \text{term}$
2. t type *iff* $\mathcal{C}(t) = \text{type}$
3. t kind *iff* $\mathcal{C}(t) = \text{kind}$

Note that the condition $[x := \lfloor \mathcal{C}(a) \rfloor]t$ type and others like it are necessary. Take for example $\lambda_\tau\, x : \star.\, x_\star$. This is not well-typed and hence not a proof, but it also should not be a kind, type, or term because it will prevent preservation of classification during reduction. If a term then the application will correctly produce a term, but if a type then an application will reduce to a type.

$$\lfloor \text{term} \rfloor = x_\star \qquad \lfloor \text{kind} \rfloor = \star$$
$$\lfloor \text{type} \rfloor = x_\Box \qquad \lfloor \text{undefined} \rfloor = \delta(\star)$$

$\mathcal{C}(x_\Box) = \text{type}$	$\mathcal{C}(\star) = \text{kind}$
$\mathcal{C}(x_\star) = \text{term}$	$\mathcal{C}(\diamond) = \text{type}$
$\mathcal{C}(\lambda_\tau\, x\!:\!A.\, t) = \text{type}$	if (A kind or A type) and t type
$\mathcal{C}(\lambda_0\, x\!:\!A.\, t) = \text{term}$	if (A kind or A type) and t term
$\mathcal{C}(\lambda_\omega\, x\!:\!A.\, t) = \text{term}$	if A type and t term
$\mathcal{C}((x : A) \to_\tau B) = \text{kind}$	if (A kind or A type) and B kind
$\mathcal{C}((x : A) \to_0 B) = \text{type}$	if (A kind or A type) and B type
$\mathcal{C}((x : A) \to_\omega B) = \text{type}$	if A type and B type
$\mathcal{C}((\lambda_\tau\, x\!:\!A.\, t) \bullet_\tau a) = \text{type}$	if (A kind and a type) or (A type and a term) and t type and $[x := \lfloor \mathcal{C}(a) \rfloor]t$ type
$\mathcal{C}(f \bullet_\tau a) = \text{type}$	if (a type or a term) and f type
$\mathcal{C}((\lambda_0\, x\!:\!A.\, t) \bullet_0 a) = \text{term}$	if (A kind and a type) or (A type and a term) and t term and $[x := \lfloor \mathcal{C}(a) \rfloor]t$ term
$\mathcal{C}(f \bullet_0 a) = \text{term}$	if (a type or a term) and f term
$\mathcal{C}((\lambda_\omega\, x\!:\!A.\, t) \bullet_\omega a) = \text{term}$	if A type and a, t term and $[x := \lfloor \mathcal{C}(a) \rfloor]t$ term
$\mathcal{C}(f \bullet_\omega a) = \text{term}$	if a term and f term
$\mathcal{C}((x : A) \cap B) = \text{type}$	if A type and B type
$\mathcal{C}([t_1, t_2; A]) = \text{term}$	if t_1, t_2 term and A type
$\mathcal{C}(t.1) = \text{term}$	if t term
$\mathcal{C}(t.2) = \text{term}$	if t term
$\mathcal{C}(a =_A b) = \text{type}$	if a, b term and A type
$\mathcal{C}(\text{refl}(t; A)) = \text{term}$	if t term and A type
$\mathcal{C}(\vartheta(e, a, b; T)) = \text{term}$	if e, a, b term and T type
$\mathcal{C}(\psi(e, a, b; A, P)) = \text{term}$	if e, a, b term and A, P type
$\mathcal{C}(\varphi(a, b, e; A, T)) = \text{term}$	if a, b, e term and A, T type
$\mathcal{C}(\delta(e)) = \text{term}$	if e term
$\mathcal{C}(t) = \text{undefined}$	otherwise

Figure 2.11: Classification function for sorting raw syntax into three distinct levels: types, kinds, and terms. If the syntactic form does not adhere to the basic structure needed to be correctly sorted then it is assigned undefined and cannot be a proof.

Lemma 2.49. *The definition of $\mathcal{C}(-)$ is terminating*

Proof. The definition is structural except application cases. In particular, application cases require evaluating $\mathcal{C}([x := \lfloor \mathcal{C}(a) \rfloor]t)$ for some subexpressions a and t. Note that computing $\mathcal{C}(-)$ on subxpressions is of course terminating, but moreover $\lfloor - \rfloor$ is a constant function returning a constant syntactic form. Thus, a measure of size can be constructed such that the size of $\lfloor \mathcal{C}(a) \rfloor$ is always zero for any a. Substitution of syntactic forms of zero size do not change the size of the resulting term, therefore $\mathcal{C}([x := \lfloor \mathcal{C}(a) \rfloor]t)$ is a terminating invocation. □

Lemma 2.50. *If $\mathcal{C}(t)$ is defined then $\mathcal{C}(t) = \mathcal{C}(|t|)$*

Proof. By induction on t. Type-like syntax is homomorphic and thus the equation holds by the IH. Term-like syntax eliminates most of the extra structure leaving behind only another term-like syntax. A few cases are presented to illuminate both situations.

Case: $t = a =_A b$

Have $|a =_A b| = |a| =_{|A|} |b|$, and because $\mathcal{C}(a =_A b)$ is defined it must be the case that a, b term and A type. Applying the IH gives $\mathcal{C}(a) = \mathcal{C}(|a|)$, $\mathcal{C}(b) = \mathcal{C}(|b|)$, and $\mathcal{C}(A) = \mathcal{C}(|A|)$. Thus, $|a| =_{|A|} |b|$ type.

Case: $t = (\lambda_0\, x : A.\, t) \bullet_0 a$

Have $|(\lambda_0\, x : A.\, t) \bullet_0 a| = |t|$ and t term. Thus, by the IH $|t|$ term.

Case: $t = \mathrm{refl}(t; A)$

Have $|\mathrm{refl}(t; A)| = \lambda\, x : \diamond.\, x_\star$, and by computation $\lambda_\omega\, x : \diamond.\, x_\star$ term.

□

Lemma 2.51. *If $\mathcal{C}(t)$ and $\mathcal{C}(b)$ are defined then*

$$\mathcal{C}([x := t]b) = \mathcal{C}([x := \lfloor \mathcal{C}(t) \rfloor]b)$$

Proof. If $\mathcal{C}(t)$ is defined then clearly $\mathcal{C}(t) = \mathcal{C}(\lfloor \mathcal{C}(t) \rfloor)$ by definition. The lemma is then shown by induction on b. □

Lemma 2.52. *If $\mathcal{C}(s)$ is defined and $s \rightsquigarrow t$ then $\mathcal{C}(s) = \mathcal{C}(t)$*

Proof. By induction on $s \rightsquigarrow t$, note that $\mathcal{C}(-)$ is structural making the inductive cases trivial. The first projection case is similar to the second projection case and thus omitted.

Case: $(\lambda_m\, x : A.\, b) \bullet_m t \rightsquigarrow [x := t]b$

Suppose wlog that $m = \tau$, then $((\lambda_\tau\, x : A.\, b) \bullet_\tau t)$ type. Note that t type or t term by unraveling the previous definition. Now $[x := \lfloor \mathcal{C}(t) \rfloor]b$ type. By Lemma 2.51 and the above observation: $[x := t]b$ type.

Case: $[t_1, t_2; A].2 \rightsquigarrow t_2$

Have $[t_1, t_2; A]$ term and by deconstructing the definition t_2 term.

Case: $\psi(\text{refl}(z; Z), a, b; A, P) \bullet_\omega t \rightsquigarrow t$

Have $(\psi(\text{refl}(z; Z), a, b; A, P) \bullet_\omega t)$ term and by deconstruction the definition t term.

Case: $\vartheta(\text{refl}(z; Z), a, b; T) \rightsquigarrow \text{refl}(a; T)$

Have $\vartheta(\text{refl}(z; Z), a, b; T)$ term and by deconstruction the definition a term and T type. Thus, $\text{refl}(a; T)$ term.

□

Lemma 2.53. *If $\mathcal{C}(s)$ is defined and $s \rightsquigarrow^* t$ then $\mathcal{C}(s) = \mathcal{C}(t)$*

Proof. By induction on $s \rightsquigarrow^* t$ and Lemma 2.52. □

Theorem 2.54 (Soundness of $\mathcal{C}(-)$)**.**

1. *If $\Gamma \vdash t : A$ and $A = \square$ then t* kind
2. *If $\Gamma \vdash t : A$ and $\Gamma \vdash A : \square$ then t* type
3. *If $\Gamma \vdash t : A$ and $\Gamma \vdash A : \star$ then t* term

Proof. By induction on $\Gamma \vdash t : A$. The Fst amd PrmFst rules are omitted.

Case: $\dfrac{}{\Gamma \vdash \star : \square}$

Have $\star$ kind and $A = \square$, hence trivial.

Case: $\dfrac{x \notin FV(\Gamma_1; \Gamma_2) \qquad \Gamma_1 \vdash \overset{\mathcal{D}_2}{A} : K}{\Gamma_1; x_m : A; \Gamma_2 \vdash x_K : A}$

If $K = \square$ then $x_\square$ type and $\Gamma \vdash A : \square$. Otherwise, $K = \star$ and then $x_\star$ term with $\Gamma \vdash A : \star$.

Case: $\dfrac{\Gamma \vdash A : \overset{\mathcal{D}_1}{\text{dom}_\Pi}(m, K) \qquad \Gamma; x_m : A \vdash \overset{\mathcal{D}_2}{B} : \text{codom}_\Pi(m)}{\Gamma \vdash (x : A) \to_m B : \text{codom}_\Pi(m)}$

Suppose wlog that $m = \tau$, now by the IH applied to $\mathcal{D}_1$: A kind or A type. Applying the IH to $\mathcal{D}_2$ gives B kind. Thus, $(x : A) \to_\tau B$ kind.

Case: $\dfrac{\Gamma \vdash (x : A) \to_m \overset{\mathcal{D}_1}{B} : \text{codom}_\Pi(m) \qquad \Gamma; x_m : \overset{\mathcal{D}_2}{A} \vdash t : B \qquad x \notin \overset{\mathcal{D}_3}{FV}(|t|) \text{ if } m = 0}{\Gamma \vdash \lambda_m\, x{:}A.\, t : (x : A) \to_m B}$

Suppose wlog that $m = \tau$. Applying the IH to $\mathcal{D}_1$ gives A kind or A type. Note by $\mathcal{D}_2$ that $\Gamma, x_\tau : A \vdash B : \square$. Thus, applying the IH to $\mathcal{D}_2$ yields t type. Hence, $\lambda_\tau\, x\!:\!A.\, t$ type.

Case: $$\frac{\Gamma \vdash f : (x : A) \to_m B \overset{\mathcal{D}_1}{} \qquad \Gamma \vdash a : A \overset{\mathcal{D}_2}{}}{\Gamma \vdash f \bullet_m a : [x := a]B}$$

Suppose wlog that $m = \tau$. By classification and inversion with $\mathcal{D}_1$: $\Gamma \vdash (x : A) \to_\tau B : \square$. Deconstructing this judgment yields $\Gamma \vdash A : K$. Applying the IH to $\mathcal{D}_2$ gives a type or a term. Applying the IH to $\mathcal{D}_1$ yields f type. If f is not an abstraction then the proof is done, thus suppose $f = \lambda\, x : A.\, t$. Have A kind or A type, but note that $\Gamma \vdash A : K$ thus the classification of a and A must agree. Moreover, t term. Suppose wlog that a type then $\lfloor \mathcal{C}(a) \rfloor = x_\square$. However, this means that $\Gamma \vdash A : \square$ and that $\Gamma, x_\tau : A \vdash t : B$. Thus, the occurrences of x in t must be annotated as $x_\square$ otherwise the VAR rule for x would fail. Hence, $[x := x_\square]t = t$.

Case: $$\frac{\Gamma \vdash A : \star \overset{\mathcal{D}_1}{} \qquad \Gamma; x_\tau : A \vdash B : \star \overset{\mathcal{D}_2}{}}{\Gamma \vdash (x : A) \cap B : \star}$$

Applying the IH to $\mathcal{D}_1$ and $\mathcal{D}_2$ gives A, B type. Hence, $(x : A) \cap B$ type.

Case: $$\frac{\Gamma \vdash (x : A) \cap B : \star \overset{\mathcal{D}_1}{} \qquad \Gamma \vdash t : A \overset{\mathcal{D}_2}{} \qquad \Gamma \vdash s : [x := t]B \overset{\mathcal{D}_3}{} \qquad t \equiv s \overset{\mathcal{D}_4}{}}{\Gamma \vdash [t, s; (x : A) \cap B] : (x : A) \cap B}$$

Deconstructing $\mathcal{D}_1$ gives $\Gamma \vdash A : \star$ and $\Gamma, x : A \vdash B : \star$. Lemma 2.39 gives $\Gamma \vdash [x := t]B : \star$. Using the IH on $\mathcal{D}_1$, $\mathcal{D}_2$, and $\mathcal{D}_3$ yields $(x : A) \cap B$ type and t, s term. Thus, $[t, s; (x : A) \cap B]$ term.

Case: $$\frac{\Gamma \vdash t : (x : A) \cap B \overset{\mathcal{D}_1}{}}{\Gamma \vdash t.2 : [x := t.1]B}$$

By classification and inversion on $\mathcal{D}_1$: $\Gamma \vdash (x : A) \cap B : \star$. Using the IH on $\mathcal{D}_1$ gives t term. Hence, $t.2$ term.

Case: $$\frac{\Gamma \vdash A : \star \overset{\mathcal{D}_1}{} \qquad \Gamma \vdash a : A \overset{\mathcal{D}_2}{} \qquad \Gamma \vdash b : A \overset{\mathcal{D}_2}{}}{\Gamma \vdash a =_A b : \star}$$

Applying the IH to $\mathcal{D}_1$, $\mathcal{D}_2$, and $\mathcal{D}_3$ yields A type and a, b term. Hence, $a =_A b$ type.

Case: $$\frac{\Gamma \vdash A : \star \overset{\mathcal{D}_1}{} \qquad \Gamma \vdash t : A \overset{\mathcal{D}_2}{}}{\Gamma \vdash \mathrm{refl}(t; A) : t =_A t}$$

Applying the IH to $\mathcal{D}_1$ and $\mathcal{D}_2$ gives A type and t term. Hence, $\mathrm{refl}(t; A)$ term.

Case: $$\frac{\Gamma \vdash \overset{\mathcal{D}_1}{A} : \star \qquad \Gamma \vdash \overset{\mathcal{D}_2}{a} : A \qquad \Gamma \vdash \overset{\mathcal{D}_3}{b} : A \qquad \Gamma \vdash \overset{\mathcal{D}_4}{e} : a =_A b \qquad \Gamma \vdash \overset{\mathcal{D}_5}{P} : (y : A) \to_\tau (p : a =_A y_\star) \to_\tau \star}{\Gamma \vdash \psi(e, a, b; A, P) : P \bullet_\tau a \bullet_\tau \mathrm{refl}(a; A) \to_\omega P \bullet_\tau b \bullet_\tau e}$$

Classification and inversion on $\mathcal{D}_4$ gives $\Gamma \vdash a =_A b : \star$. Likewise, $\Gamma \vdash (y : A) \to_\tau (p : a =_A y_\star) \to_\tau \star : \square$. Applying the IH to all subderivations yields A, P type and a, b, e term. Hence, $\psi(e, a, b; A, P)$ term.

Case: $$\frac{\Gamma \vdash \overset{\mathcal{D}_1}{(x : A)} \cap B : \star \qquad \Gamma \vdash \overset{\mathcal{D}_2}{a} : (x : A) \cap B \qquad \Gamma \vdash \overset{\mathcal{D}_3}{b} : (x : A) \cap B \qquad \Gamma \vdash \overset{\mathcal{D}_4}{e} : a.1 =_A b.1}{\Gamma \vdash \vartheta(e, a, b; (x : A) \cap B) : a =_{(x:A)\cap B} b}$$

By classification, inversion and the IH used with $\mathcal{D}_4$: e term. The IH applied to $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_3$ yields a, b term and $(x : A) \cap B$ type.

Case: $$\frac{\Gamma \vdash \overset{\mathcal{D}_1}{(x : A)} \cap B : \star \qquad \Gamma \vdash \overset{\mathcal{D}_2}{a} : A \qquad \Gamma \vdash \overset{\mathcal{D}_3}{b} : (x : A) \cap B \qquad \Gamma \vdash \overset{\mathcal{D}_4}{e} : a =_A b.1}{\Gamma \vdash \varphi(a, b, e; A, (x : A) \cap B) : (x : A) \cap B}$$

By the IH applied to $\mathcal{D}_1$: $(x : A) \cap B$ type. Thus, by the IH and inversion applied to the remaining derivations: a, b, e term.

Case: $$\frac{\Gamma \vdash \overset{\mathcal{D}_1}{e} : \mathrm{ctt} =_{\mathrm{cBool}} \mathrm{cff}}{\Gamma \vdash \delta(e) : (X : \star) \to_0 X_\square}$$

Classification, inversion, and the IH applied to $\mathcal{D}_1$ gives e term. Hence, $\delta(e)$ term.

Case: $$\frac{\Gamma \vdash \overset{\mathcal{D}_1}{A} : K \qquad \Gamma \vdash \overset{\mathcal{D}_2}{t} : B \qquad A \overset{\mathcal{D}_3}{\equiv} B}{\Gamma \vdash t : A}$$

Classification, inversion, $\mathcal{D}_1$ and $\mathcal{D}_3$ yield $\Gamma \vdash B : K$. Suppose wlog that $K = \star$. Applying the IH to $\mathcal{D}_2$ gives t term.

□

Chapter 3

PROOF NORMALIZATION

There are several techniques for showing strong normalization of a PTS, including saturated sets [22], model theory [58], realizability [42], etc. Geuvers and Nederhof describe yet another technique that models CC inside F^ω where term dependencies are all erased at the type level [24]. In this chapter the technique of Geuvers and Nederhof will be adapted to show strong normalization of proof reduction. Note, this will not entail that objects are strongly normalizing. Moreover, proof normalization ends up being a rather weak property, as it will not entail consistency either. Nevertheless, it is an important stepping stone to strong normalization for objects.

3.1 Model Description

Figure 3.1 describes the syntax of System F^ω augmented with pairs. The reduction relation for this system is presented in Figure 3.2 and the inference judgment in Figure 3.3. System F^ω augmented with pairs is only slightly different from the original PTS description of F^ω. Moreover, it is a subsystem of the Calculus of Inductive Constructions and thus enjoys various metatheoretic properties such as substitution and weakening lemmas, preservation, strong normalization, and consistency.

The model follows all the same principles for the CC fragment of Cedille2. For example, consider the Lam rule.

$$\frac{\Gamma \vdash (x : A) \to_m B : \mathrm{codom}_\Pi(m) \qquad \Gamma; x_m : A \vdash t : B \qquad x \notin FV(|t|) \text{ if } m = 0}{\Gamma \vdash \lambda_m\, x{:}A.\, t : (x : A) \to_m B}\ \textsc{Lam}$$

$$\begin{aligned}
t &::= x \mid \mathfrak{b}(\kappa_1, x : t_1, t_2) \mid \mathfrak{c}(\kappa_2, t_1, \ldots, t_{\mathfrak{a}(\kappa_2)}) \\
\kappa_1 &::= \lambda \mid \Pi \\
\kappa_2 &::= \star \mid \Box \mid \mathrm{app} \mid \mathrm{prod} \mid \mathrm{pair} \mid \mathrm{fst} \mid \mathrm{snd} \\
\mathfrak{a}(\star) &= \mathfrak{a}(\Box) = 0 \\
\mathfrak{a}(\mathrm{fst}) &= \mathfrak{a}(\mathrm{snd}) = 1 \\
\mathfrak{a}(\mathrm{app}) &= \mathfrak{a}(\mathrm{prod}) = \mathfrak{a}(\mathrm{pair}) = 2
\end{aligned}$$

$$\begin{aligned}
\star &:= \mathfrak{c}(\star) \\
\Box &:= \mathfrak{c}(\Box) \\
\lambda\, x{:}t_1.\, t_2 &:= \mathfrak{b}(\lambda, x : t_1, t_2) \\
(x : t_1) \to t_2 &:= \mathfrak{b}(\Pi, x : t_1, t_2) \\
t_1\ t_2 &:= \mathfrak{c}(\mathrm{app}, t_1, t_2)
\end{aligned}
\qquad
\begin{aligned}
t_1 \times t_2 &= \mathfrak{c}(\mathrm{prod}, t_1, t_2) \\
(t_1, t_2) &= \mathfrak{c}(\mathrm{pair}, t_1, t_2) \\
t.1 &= \mathfrak{c}(\mathrm{fst}, t) \\
t.2 &= \mathfrak{c}(\mathrm{snd}, t)
\end{aligned}$$

Figure 3.1: Syntax for System F^ω with pairs.

$$\frac{t_1 \rightsquigarrow t_1'}{\mathfrak{b}(\kappa, x : t_1, t_2) \rightsquigarrow \mathfrak{b}(\kappa, x : t_1', t_2)} \qquad \frac{t_2 \rightsquigarrow t_2'}{\mathfrak{b}(\kappa, x : t_1, t_2) \rightsquigarrow \mathfrak{b}(\kappa, x : t_1, t_2')}$$

$$\frac{t_i \rightsquigarrow t_i' \qquad i \in 1, \ldots, \mathfrak{a}(\kappa)}{\mathfrak{c}(\kappa, t_1, \ldots t_i, \ldots t_{\mathfrak{a}(\kappa)}) \rightsquigarrow \mathfrak{c}(\kappa, t_1, \ldots t_i', \ldots t_{\mathfrak{a}(\kappa)})}$$

$$(\lambda\, x{:}A.\, b)\ t \rightsquigarrow [x := t]b$$
$$[t_1, t_2].1 \rightsquigarrow t_1$$
$$[t_1, t_2].2 \rightsquigarrow t_2$$

Figure 3.2: Reduction rules for System F^ω with pairs.

$$\frac{}{\Gamma \vdash \star : \Box}\ \textsc{Axiom}$$

$$\frac{(x : A) \in \Gamma}{\Gamma \vdash x : A}\ \textsc{Var}$$

$$\frac{\Gamma \vdash A : \Box \qquad \Gamma, x : A \vdash B : \Box}{\Gamma \vdash (x : A) \to B : \Box}\ \textsc{Pi1}$$

$$\frac{\Gamma \vdash (x : A) \to B : K \qquad \Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda\, x{:}A.\, t : (x : A) \to B}\ \textsc{Lam}$$

$$\frac{\Gamma \vdash t : A \times B}{\Gamma \vdash t.1 : A}\ \textsc{Fst}$$

$$\frac{\Gamma \vdash t : A \times B}{\Gamma \vdash t.2 : B}\ \textsc{Snd}$$

$$\frac{\Gamma \vdash A : K \qquad \Gamma \vdash t : B \qquad A \rightleftharpoons B}{\Gamma \vdash t : A}\ \textsc{Conv}$$

$$\frac{\Gamma \vdash A : K \qquad \Gamma, x : A \vdash B : \star}{\Gamma \vdash (x : A) \to B : \star}\ \textsc{Pi2}$$

$$\frac{\Gamma \vdash f : (x : A) \to B \qquad \Gamma \vdash a : A}{\Gamma \vdash f\ a : [x := a]B}\ \textsc{App}$$

$$\frac{\Gamma \vdash A : \star \qquad \Gamma \vdash B : \star}{\Gamma \vdash A \times B : \star}\ \textsc{Int}$$

$$\frac{\Gamma \vdash A \times B : \star \qquad \Gamma \vdash t : A \qquad \Gamma \vdash s : B}{\Gamma \vdash (t, s) : A \times B}\ \textsc{Pair}$$

Figure 3.3: Typing rules for System F^ω with pairs. The variable K is a metavariable representing either $\star$ or $\Box$.

The goal is to find three semantic functions: one for kinds ($V(-)$); one for types ($[\![-]\!]$); and one for terms ($[-]$), such that:

1. $[\![\Gamma]\!] \vdash_\omega [\![(x : A) \to_m B]\!] : V(\mathrm{codom}_\Pi(m))$
2. $[\![\Gamma; x_m : A]\!] \vdash_\omega [t] : [\![B]\!]$
3. $[\![\Gamma]\!] \vdash [\lambda_m\, x{:}A.\, t] : [\![(x : A) \to_m B]\!]$

In order for this to work, term dependencies must all be dropped in function types. Moreover, kinds are squished, such that $V(\Box) = V(\star) = \star$. Thus, the judgment $[\![\Gamma]\!] \vdash_\omega [\![(x : A) \to_m B]\!] : V(\mathrm{codom}_\Pi(m))$ must form an F^ω type. The kind and type semantics is allowed to throw away terms and reductions because it only serves the purpose to maintain a well-typed output. Instead,

$$
\begin{aligned}
V(\Box) &= \star \\
V(\star) &= \star \\
V((x : A) \to_m B) &= V(A) \to V(B) && \text{if } A \text{ kind} \\
V((x : A) \to_m B) &= V(B) && \text{otherwise}
\end{aligned}
$$

$$
\begin{aligned}
[\![\Box]\!] &= 0 \\
[\![\star]\!] &= 0 \\
[\![x_\Box]\!] &= x \\
[\![(x : A) \to_m B]\!] &= (x : V(A)) \to [\![A]\!] \to [\![B]\!] && \text{if } A \text{ kind} \\
[\![(x : A) \to_m B]\!] &= (x : [\![A]\!]) \to [\![B]\!] && \text{if } A \text{ type} \\
[\![\lambda_\tau\, x{:}A.\, t]\!] &= \lambda\, x{:}V(A).\, [\![t]\!] && \text{if } A \text{ kind} \\
[\![\lambda_\tau\, x{:}A.\, t]\!] &= [\![t]\!] && \text{if } A \text{ type} \\
[\![f \bullet_\tau a]\!] &= [\![f]\!]\ [\![a]\!] && \text{if } a \text{ type} \\
[\![f \bullet_\tau a]\!] &= [\![f]\!] && \text{if } a \text{ term} \\
[\![(x : A) \cap B]\!] &= [\![A]\!] \times [\![B]\!] \\
[\![a =_A b]\!] &= \mathrm{Id}
\end{aligned}
$$

$$
\begin{aligned}
[\![x_m : A]\!] &= x : V(A), w_x : [\![A]\!] && \text{if } A \text{ kind} \\
[\![x_m : A]\!] &= x : [\![A]\!] && \text{if } A \text{ type} \\
[\![\varepsilon]\!] &= 0 : \star, \bot : (X : \star) \to X \\
[\![\Gamma, x_m : A]\!] &= [\![\Gamma]\!], [\![x_m : A]\!]
\end{aligned}
$$

Figure 3.4: Model for kinds and types, not that type dependencies are dropped. Define $\mathrm{Id} := (X : \star) \to X \to X$.

it is the term semantics that must take care to preserve all possible reductions such that strong normalization is a consequence of the model.

For dependent intersections, the type semantics is the obvious one: $[\![(x : A) \cap B]\!] = [\![A]\!] \times [\![B]\!]$. Note that because A must be a type, it must be the case that $x \notin FV([\![B]\!])$ otherwise the resulting type is not well-formed in F^ω. This is true already for function types, thus this extension needs no special treatment. For equality the situation is special, the approach taken is to interpret all equalities as the type of the identity function: $[\![a =_A b]\!] = \mathrm{Id}$. There does not appear to be a more sensible choice, as the dependencies a and b must be dropped.

The model interpretation for contexts always introduces two fresh variables, $0 : \star$ which is a canonical type, and $\bot : (X : \star) \to X$ which is used to construct canonical inhabitants for any type or kind. Note that including $\bot$ prevents this model from entailing consistency for the source system. Regardless, F^ω is strongly normalizing in all contexts, thus the addition of $\bot$ does not prevent the model from serving its current purpose. Before exploring more in-depth examples of

$$
\begin{aligned}
c^{B} &= \bot\ B && \text{if } B \text{ type}\\
c^{\star} &= 0\\
c^{(x:A)\to B} &= \lambda\, x\!:\!A.\, c^{B}
\end{aligned}
$$

$$
\begin{aligned}
[*] &= c^{0}\\
[x_{\Box}] &= w_x\\
[x_{\star}] &= x\\
[(x:A)\to_m B] &= c^{0\to 0\to 0}\ [A]\ ([x := c^{V(A)}][w_x := c^{[\![A]\!]}][B]) && \text{if } A \text{ kind}\\
[(x:A)\to_m B] &= c^{0\to 0\to 0}\ [A]\ ([x := c^{[\![A]\!]}][B]) && \text{if } A \text{ type}\\
[\lambda_m\, x\!:\!A.\, t] &= (\lambda\, y\!:\!0.\, \lambda\, x\!:\!V(A).\, \lambda\, w_x\!:\![\![A]\!].\, [t])\ [A] && \text{if } A \text{ kind}\\
[\lambda_m\, x\!:\!A.\, t] &= (\lambda\, y\!:\!0.\, \lambda\, x\!:\![\![A]\!].\, [t])\ [A] && \text{if } A \text{ type}\\
[f \bullet_m a] &= [f]\ [\![a]\!]\ [a] && \text{if } a \text{ type}\\
[f \bullet_m a] &= [f]\ [a] && \text{if } a \text{ term}
\end{aligned}
$$

$$
\begin{aligned}
[(x:A)\cap B] &= c^{0\to 0\to 0}\ [A]\ ([x := c^{[\![A]\!]}][B])\\
[[t_1, t_2; A]] &= (\lambda\, y\!:\!0.\, ([t_1], [t_2]))\ [A]\\
[t.1] &= [t].1\\
[t.2] &= [t].2\\
[a =_A b] &= c^{0\to[\![A]\!]\to[\![A]\!]\to 0}\ [A]\ [a]\ [b]\\
[\mathrm{refl}(t; A)] &= (\lambda\, y_1\!:\!0.\, \lambda\, y_2\!:\![\![A]\!].\, \mathrm{id})\ [A]\ [t]\\
[\psi(e, a, b; A, P)] &= (\lambda\, y_1\!:\!0.\, \lambda\, y_2\ y_3\!:\![\![A]\!].\, \lambda\, y_2\!:\![\![A]\!]\to \mathrm{Id}\to 0.\, [e]\ [\![P]\!])\ [A]\ [a]\ [b]\ [P]\\
[\vartheta(e, a, b; T)] &= (\lambda\, y_1\!:\![\![T]\!].\, \lambda\, y_2\!:\!0.\, \lambda\, y_3\!:\![\![T]\!].\, [e])\ [b]\ [T]\ [a]\\
[\varphi(a, b, e; A, T)] &= (\lambda\, y_1\ y_2\!:\!0.\, \lambda\, y_3\!:\![\![A]\!].\, \lambda\, y_4\!:\![\![T]\!].\, \lambda\, y_5\!:\!\mathrm{Id}.\, c^{[\![T]\!]})\ [A]\ [T]\ [a]\ [b]\ [e]\\
[\delta(e)] &= (\lambda\, y\!:\!\mathrm{Id}.\, \bot)\ [e]
\end{aligned}
$$

Figure 3.5: Model for terms, note that critically every subexpression is represented in the model to make sure no reductions are potentially lost. The definition of c is used to construct a canonical element for any kind or type. Define $\mathrm{id} := \lambda\, X\!:\!\star.\, \lambda\, x\!:\!X.\, x$.

the model the reader is invited to skim to the semantic functions in Figure 3.4 and Figure 3.5.

Consider the following examples to garner intuition for the semantic model:

1. Given $\varepsilon \vdash_{\varsigma_2} \lambda_0\, X{:}\star.\, \lambda_\omega\, x{:}X_\square.\, x_\star : (X:\star) \to_0 X_\square \to_\omega X_\square$ then

$$[\![\varepsilon]\!] = 0 : \star;\ \bot : (X:\star) \to X$$

$$[\lambda_0\, X{:}\star.\, \lambda_\omega\, x{:}X_\square.\, x_\star] = (\lambda\, y{:}0.\, \lambda\, X{:}\star.\, \lambda\, w_X{:}0.\, (\lambda\, y{:}0.\, \lambda\, x{:}X.\, x)\ w_X)\ c^0$$

$$[\![(X:\star) \to_0 X_\square \to_\omega X_\square]\!] = (X:\star) \to 0 \to X \to X$$

2. Given $\Gamma \vdash_{\varsigma_2} t : T$ where $\Gamma = A:\star;\ B:\star;\ a:A_\square;\ f:A_\square \to_\omega (x:A_\square) \cap B_\square$, $t = [(f_\star \bullet_\omega a_\star).1, (f_\star \bullet_\omega a_\star).2; (x:A_\square) \cap B_\square]$, and $T = (x:A_\square) \cap B_\square$ then

$$\begin{aligned}[\![A:\star; B:A \to_\tau \star; a:A; f:A \to_\omega (x:A) \cap B]\!] = \\ 0:\star;\ \bot:(X:\star)\to X;\ A:\star;\ w_A:0;\ B:\star;\ w_B:0; \\ a:A;\ f:A \to A \times B\end{aligned}$$

$$[[(f \bullet_\omega a).1, (f \bullet_\omega a).2]] = (\lambda\, y{:}0.\, ((f\ a).1, (f\ a).2))\ (c^{0\to 0\to 0}\ w_A\ w_B)$$

$$[\![(x:A) \cap B]\!] = A \times B$$

Notice that from the perspective of the type semantics ($[\![-]\!]$) that term dependencies in predicates must be dropped, but that they are preserved in the term semantics ($[-]$). Thus, extra layers of abstraction are added when interpreting function arguments that are kinds to capture the two different usages of that variable in the separate semantic functions.

3.2 Model Soundness

With the model defined the next step is to prove it is sound. The process begins by showing the interpretation of kinds ($V(-)$) is sound. This is not particularly difficult as the kind interpretation is quite simple. After proving soundness lemmas about substitution and conversion are also shown and follow without much difficulty.

Theorem 3.1 (Soundness of V)**.** *If* $\Gamma \vdash_{\varsigma_2} t : \square$ *then* $\Delta \vdash_\omega V(t) : \square$ *for any* Δ

Proof. By induction on $\Gamma \vdash_{\varsigma_2} t : \square$. The cases: Lam, App, Int, Pair, Fst, Snd, Eq, Refl, Subst, Prm, Cast, Sep, and Conv are impossible by inversion.

Case: $\dfrac{}{\Gamma \vdash \star : \square}$

Trivial by the Ax rule.

Case: $\dfrac{x \notin FV(\Gamma_1;\Gamma_2) \qquad \Gamma_1 \vdash \overset{\mathcal{D}_2}{A} : K}{\Gamma_1; x_m : A; \Gamma_2 \vdash x_K : A}$ $\mathcal{D}_1$

By $\mathcal{D}_2$: $\Gamma \vdash \Box : K$ which is impossible.

Case: $\dfrac{\Gamma \vdash A : \overset{\mathcal{D}_1}{\text{dom}_\Pi}(m, K) \qquad \Gamma; x_m : A \vdash \overset{\mathcal{D}_2}{B} : \text{codom}_\Pi(m)}{\Gamma \vdash (x : A) \to_m B : \text{codom}_\Pi(m)}$

Suppose A is a kind, then $\text{dom}_\Pi(m, K) = \Box$ and $V((x : A) \to_m B) = V(A) \to V(B)$. Applying the IH to $\mathcal{D}_1$ and $\mathcal{D}_2$ gives $\Delta_1 \vdash_\omega V(A) : \Box$ and $\Delta_2 \vdash_\omega V(B) : \Box$. However, note that there are no variables in any well-defined $V(t)$ which $V(A)$ and $V(B)$ are. Thus, $\Delta \vdash_\omega V(A) : \Box$ and $\Delta, x : V(A) \vdash_\omega V(B) : \Box$ by properties of F^ω. Now by the Pi1 rule $\Delta \vdash_\omega V(A) \to V(B) : \Box$ as required.

Suppose A is a type, then $\text{dom}_\Pi(m, K) = \star$ and $V((x : A) \to_m B) = V(B)$. By the IH applied to $\mathcal{D}_2$: $\Delta \vdash_\omega V(B) : \Box$.

□

Lemma 3.2. *If* $\Gamma_1 \vdash A : \Box$, $\Gamma_2 \vdash B : \Box$, *and* $A \equiv B$ *then* $V(A) = V(B)$

Proof. By induction on $\Gamma \vdash A : \Box$. Note that A is either $\star$ or $(x : C) \to_\tau D$. Suppose $A = \star$, then because $\star \equiv B$ it must be that $B = \star$. Thus, $V(A) = \star = V(B)$.

Suppose $A = (x : C_1) \to_\tau D_1$, but this forces $B = (x : C_2) \to_\tau D_2$ where $C_1 \equiv C_2$ and $D_1 \equiv D_2$. Note that $\Gamma \vdash C_1 : K$ and $\Gamma, x : C_1 \vdash D_1 : \Box$. Now by the IH: $V(D_1) = V(D_2)$ (note that the contexts need not agree). If C_1 is a kind, then $V((x : C_1) \to_\tau D_1) = V(C_1) \to V(D_1)$ and by the IH $V(C_1) = V(C_2)$. Instead, if C_1 is a type then $V((x : C_1) \to_\tau D_1) = V(D_1)$, but $V(D_1) = V(D_2)$. Thus, $V(A) = V((x : C_1) \to_\tau D_1) = V((x : C_2) \to_\tau D_2) = V(B)$. □

Lemma 3.3. *If* $\Gamma \vdash V(t) : \Box$ *then* $[x := b]V(t) = V(t) = V([x := b]t)$

Proof. By induction on t and inversion on $\Gamma \vdash V(t) : \Box$. Note that there are only two possibilities:

Case: $t = \star$

Have $[x := b]V(\star) = [x := b]\star = \star = V(\star) = V([x := b]\star)$.

Case: $t = (x : A) \to_m B$

Note that A must be a kind or a type because $\Gamma \vdash V(t) : \Box$. Suppose A is a kind, then $V((x : A) \to_m B) = V(A) \to V(B)$. Destructing the judgment gives $\Gamma \vdash V(A) : \Box$ and $\Gamma, x : V(A) \vdash V(B) : \Box$. Thus, by the IH: $[x := b]V(A) = V(A) = V([x := b]A)$ and $[x := b]V(B) = V(B) = V([x := b]B)$. By computation, $V([x := b](x : A) \to_m B) = V((x : [x := b]A) \to_m [x := b]B) = V([x := b]A) \to V([x := b]B) = V(A) \to V(B) = V((x : A) \to_m B)$. Also, by computation $[x := b]V((x : A) \to_m B) = [x := b](V(A) \to V(B)) = [x := b]V(A) \to [x := b]V(B) = V(A) \to V(B) = V((x : A) \to_m B)$.

Suppose A is a type, then $V((x : A) \to_m B) = V(B)$. By the IH: $[x := b]V(B) = V(B) = V([x := b]B)$.

□

Next is demonstrating soundness of the type semantics. Note again that type variables cannot appear free in the result of a well-defined interpretation of types. This is codified in the next lemma, and soundness follows from it and soundness of the model for kinds. A standard substitution lemma is proven after.

Lemma 3.4. *Suppose* $\Gamma \vdash t : A$, $x_m : B \in \Gamma$, *and* B *type, then* $x \notin FV([\![t]\!])$ *where* $A = \Box$ *or* $\Gamma \vdash A : \Box$

Proof. Note that the restrictions on A makes sure that $[\![-]\!]$ is well-defined. The definition of $[\![-]\!]$ intentionally throws away any dependence on terms. Thus, if x is a term, because B is a type, the only places where x may appear in t have all been thrown away. Therefore, $x \notin FV([\![t]\!])$. □

Theorem 3.5 (Soundness of $[\![-]\!]$)**.** *If* $\Gamma \vdash_{\varsigma_2} t : A$ *then* $[\![\Gamma]\!] \vdash_\omega [\![t]\!] : V(A)$ *where* $A = \Box$ *or* $\Gamma \vdash A : \Box$

Proof. By induction on $\Gamma \vdash_{\varsigma_2} t : A$. The cases: Pair, Fst, Snd, Refl, Subst, Prm, Cast, and Sep are impossible by inversion on $A = \Box$ or $\Gamma \vdash A : \Box$.

Case: $\dfrac{}{\Gamma \vdash \star : \Box}$

By computation $[\![\star]\!] = 0$ and $V(\Box) = \star$. Note that $0 : \star \in [\![\Gamma]\!]$ thus this case is concluded by the Var rule.

Case: $\dfrac{x \notin FV(\Gamma_1; \Gamma_2) \qquad \Gamma_1 \vdash \overset{\mathcal{D}_2}{A} : K}{\Gamma_1; x_m : A; \Gamma_2 \vdash x_K : A}$

Note that $A \neq \Box$ by $\mathcal{D}_2$, thus $K = \Box$. By computation $[\![x_\Box]\!] = x$. Moreover, A kind thus $x : V(A) \in [\![\Gamma]\!]$. Thus, $[\![\Gamma]\!] \vdash_\omega x : V(A)$

Case: $\dfrac{\Gamma \vdash \overset{\mathcal{D}_1}{A} : \mathrm{dom}_\Pi(m, K) \qquad \Gamma; x_m : A \vdash \overset{\mathcal{D}_2}{B} : \mathrm{codom}_\Pi(m)}{\Gamma \vdash (x : A) \to_m B : \mathrm{codom}_\Pi(m)}$

By computation $V(\mathrm{codom}_\Pi(m)) = V(\mathrm{dom}_\Pi(m, K)) = \star$. Applying the IH gives:

$\mathcal{D}_1$. $[\![\Gamma]\!] \vdash_\omega [\![A]\!] : \star$

$\mathcal{D}_2$. $[\![\Gamma, x_m : A]\!] \vdash_\omega [\![B]\!] : \star$

Suppose that A is a kind. Then $[\![(x : A) \to_m B]\!] = (x : V(A)) \to [\![A]\!] \to [\![B]\!]$ and $[\![\Gamma, x_m : A]\!] = [\![\Gamma]\!], x : V(A), w_x : [\![A]\!]$. The Pi2 rule applied with the results of the IH gives

$$[\![\Gamma]\!], x : V(A) \vdash_\omega [\![A]\!] \to [\![B]\!] : \star$$

Now by Lemma 3.1 applied to $\mathcal{D}_1$: $[\![\Gamma]\!] \vdash_\omega V(A) : \Box$. Using the Pi1 rule gives $[\![\Gamma]\!] \vdash_\omega V(A) \to [\![A]\!] \to [\![B]\!] : \star$.

Suppose that A is a type. Then $[\![(x : A) \to_m B]\!] = (x : [\![A]\!]) \to [\![B]\!]$ and $[\![\Gamma, x_m : A]\!] = [\![\Gamma]\!], x : [\![A]\!]$. Thus, by the Pi2 rule $[\![\Gamma]\!] \vdash [\![A]\!] \to [\![B]\!] : \star$.

Case: $$\frac{\Gamma \vdash (x : A) \to_m \overset{\mathcal{D}_1}{B} : \mathrm{codom}_\Pi(m) \qquad \Gamma; x_m : \overset{\mathcal{D}_2}{A} \vdash t : B \qquad x \notin FV(\overset{\mathcal{D}_3}{|t|}) \text{ if } m = 0}{\Gamma \vdash \lambda_m\, x{:}A.\, t : (x : A) \to_m B}$$

It must be the case that $\Gamma \vdash (x : A) \to_m B : \Box$. Thus, $m = \tau$. Applying the IH gives:

$\mathcal{D}_1$. $[\![\Gamma]\!] \vdash_\omega [\![(x : A) \to_\tau B]\!] : \star$

$\mathcal{D}_2$. $[\![\Gamma, x_\tau : A]\!] \vdash_\omega [\![t]\!] : V(B)$

Suppose A is a kind. Then $[\![(x : A) \to_\tau B]\!] = (x : V(A)) \to [\![A]\!] \to [\![B]\!]$, $[\![\Gamma, x_m : A]\!] = [\![\Gamma]\!], x : V(A), w_x : [\![A]\!]$, and $[\![\lambda_\tau\, x{:}A.\, t]\!] = \lambda\, x : V(A).\, [\![t]\!]$. Note that $[\![\Gamma]\!] \vdash c^{[\![A]\!]} : [\![A]\!]$. Thus, by substitution lemma for F$^\omega$: $[\![\Gamma]\!], x : V(A) \vdash_\omega [w_x := c^{[\![A]\!]}][\![t]\!] : [w_x := c^{[\![A]\!]}]V(B)$. However, because A is kind and by Lemma 3.4: $[w_x := c^{[\![A]\!]}][\![t]\!] = [\![t]\!]$. Note also that $FV(V(B))$ is empty, thus $[w_x := c^{[\![A]\!]}]V(B) = V(B)$. Thus, $[\![\Gamma]\!], x : V(A) \vdash_\omega [\![t]\!] : V(B)$. Moreover, by Theorem 3.1 it is the case that $[\![\Gamma]\!] \vdash V(A) : \Box$. Using the Lam rule gives $[\![\Gamma]\!] \vdash_\omega \lambda\, x{:}V(A).\, [\![t]\!] : V(A) \to V(B)$.

Suppose A is a type. Then $[\![\Gamma, x_m : A]\!] = [\![\Gamma]\!], x : [\![A]\!]$ and $[\![\lambda_\tau\, x{:}A.\, t]\!] = [\![t]\!]$. Note additionally that $V((x : A) \to_m B) = V(B)$. Note that $[\![\Gamma]\!] \vdash c^{[\![A]\!]} : [\![A]\!]$. By substitution lemma, Lemma 3.4, and as above: $[\![\Gamma]\!] \vdash_\omega [\![t]\!] : V(B)$.

Case: $$\frac{\Gamma \vdash f : \overset{\mathcal{D}_1}{(x : A)} \to_m B \qquad \Gamma \vdash \overset{\mathcal{D}_2}{a} : A}{\Gamma \vdash f \bullet_m a : [x := a]B}$$

Note that it cannot be the case that $[x := a]B = \Box$ by inversion on $\mathcal{D}_1$, thus $\Gamma \vdash [x := a]B : \Box$ which force $m = \tau$. Furthermore, by $\mathcal{D}_1$: $\Gamma \vdash (x : A) \to_\tau B : \Box$. Applying the IH to $\mathcal{D}_1$ thus gives $[\![\Gamma]\!] \vdash_\omega [\![f]\!] : V((x : A) \to_\tau B)$.

Suppose A is a kind, then a is a type. Thus, $V((x : A) \to_\tau B) = V(A) \to V(B)$ and $[\![f \bullet_\tau a]\!] = [\![f]\!]\ [\![a]\!]$. Applying the IH to $\mathcal{D}_2$ gives $[\![\Gamma]\!] \vdash_\omega [\![a]\!] : V(A)$. By the App rule: $[\![\Gamma]\!] \vdash [\![f]\!]\ [\![a]\!] : V(B)$. Now by Lemma 3.3: $V(B) = V([x := a]B)$.

Suppose A is a type, then a is a term. Thus, $V((x : A) \to_\tau B) = V(B)$ and $[\![f \bullet_\tau a]\!] = [\![f]\!]$. But, $[\![\Gamma]\!] \vdash_\omega [\![f]\!] : V(B)$ already. Now by Lemma 3.3: $V(B) = V([x := a]B)$.

Case: $\dfrac{\Gamma \vdash \overset{\mathcal{D}_1}{A} : \star \qquad \Gamma; x_\tau : \overset{\mathcal{D}_2}{A} \vdash B : \star}{\Gamma \vdash (x : A) \cap B : \star}$

Applying the IH gives:

$\mathcal{D}_1$. $[\![\Gamma]\!] \vdash_\omega [\![A]\!] : \star$

$\mathcal{D}_2$. $[\![\Gamma, x_\tau : A]\!] \vdash_\omega [\![B]\!] : \star$

Note that A is a type thus $[\![\Gamma, x_\tau : A]\!] = [\![\Gamma]\!], x : [\![A]\!]$. Applying the LAM rule twice reduces the goal to $[\![\Gamma]\!], [\![A]\!] : \star, [\![B]\!] : \star \vdash_\omega [\![A]\!] \times [\![B]\!] : \star$. However, the pair case is an otherwise simple F^ω type, thus a short sequence of rules concludes the case.

Case: $\dfrac{\Gamma \vdash \overset{\mathcal{D}_1}{A} : \star \qquad \Gamma \vdash \overset{\mathcal{D}_2}{a} : A \qquad \Gamma \vdash \overset{\mathcal{D}_2}{b} : A}{\Gamma \vdash a =_A b : \star}$

By computation $[\![a =_A b]\!] = \mathrm{Id}$ and $V(\star) = \star$. A short sequence of rules in F^ω yields $[\![\Gamma]\!] \vdash \mathrm{Id} : \star$.

Case: $\dfrac{\Gamma \vdash \overset{\mathcal{D}_1}{A} : K \qquad \Gamma \vdash \overset{\mathcal{D}_2}{t} : B \qquad A \overset{\mathcal{D}_3}{\equiv} B}{\Gamma \vdash t : A}$

Note that $A \neq \Box$ by $\mathcal{D}_1$, and furthermore that $K = \Box$. Now by classification and $\mathcal{D}_3$: $\Gamma \vdash B : \Box$. Applying the IH to $\mathcal{D}_2$ gives $[\![\Gamma]\!] \vdash_\omega [\![t]\!] : V(B)$. Using Lemma 3.2 with $\mathcal{D}_3$ gives $V(A) = V(B)$. Thus, the CONV rule concludes the case.

□

Lemma 3.6. *Suppose* $\Gamma \vdash_\omega [\![t]\!] : T$ *then* $[\![[x := b]t]\!] = [x := [\![b]\!]][\![t]\!]$

Proof. By induction on t and inversion on $\Gamma \vdash_\omega [\![t]\!] : T$. Thus, only the cases where $[\![t]\!]$ is well-defined need to be considered.

Case: $t = \star$ or $t = \Box$

The situation is the same because $[\![\star]\!] = [\![\Box]\!]$. By computation $[\![[x := b]\star]\!] = [\![\star]\!] = 0$ and $[x := [\![b]\!]][\![\star]\!] = [x := [\![b]\!]]0 = 0$.

Case: $t = y_\Box$

Suppose $x \neq y$, then by computation $[\![[x := b]y_\Box]\!] = [\![y_\Box]\!] = y$ and $[x := [\![b]\!]][\![y_\Box]\!] = [x := [\![b]\!]]y = y$. Suppose $x = y$, then $[\![[x := b]y_\Box]\!] = [\![b]\!]$ and $[x := [\![b]\!]][\![y_\Box]\!] = [x := [\![b]\!]]y = [\![b]\!]$.

Case: $t = (y : C) \to_m D$

Suppose A is a kind. Then $[\![[x := b](y : C) \to_m D]\!] = [\![(y : [x := b]C) \to_m ([x := b]D)]\!] = (y : V([x := b]A)) \to [\![[x := b]C]\!] \to [\![[x := b]D]\!]$. By Lemma 3.3 and applying the IH:

$$\begin{aligned}
&(y : V([x := b]A)) \to [\![[x := b]C]\!] \to [\![[x := b]D]\!] \\
&\quad = (y : [x := [\![b]\!]]V(A)) \to [x := [\![b]\!]][\![C]\!] \to [x := [\![b]\!]][\![D]\!] \\
&\quad = [x := [\![b]\!]]((y : V(A)) \to [\![C]\!] \to [\![D]\!]) \\
&\quad = [x := [\![b]\!]][\![(y : C) \to_m D]\!]
\end{aligned}$$

Suppose A is a type. Then $[\![[x := b](y : C) \to_m D]\!] = [\![(y : [x := b]C) \to_m ([x := b]D)]\!] = (y : [\![[x := b]C]\!]) \to [\![[x := b]D]\!]$. Applying the IH and chasing similar computations as above concludes the case.

Case: $t = \lambda_\tau\, C : c$.

Suppose C is a kind. Then $[\![[x := b](\lambda_\tau\, x : C.\, c)]\!] = [\![\lambda_\tau\, x : [x := b]C.\, [x := b]c]\!] = \lambda\, x : V([x := b]C).\, [\![[x := b]c]\!]$. By Lemma 3.3 and the IH:

$$\begin{aligned}
&\lambda\, x : V([x := b]C).\, [\![[x := b]c]\!] \\
&\qquad = \lambda\, x : [x := [\![b]\!]]V(C).\, [x := [\![b]\!]][\![c]\!] \\
&\qquad = [x := [\![b]\!]](\lambda\, x : V(C).\, [\![c]\!]) \\
&\qquad = [x := [\![b]\!]][\![\lambda\, x : C.\, c]\!]
\end{aligned}$$

Suppose C is a type. Then $[\![[x := b](\lambda_\tau\, x : C.\, c)]\!] = [\![\lambda_\tau\, x : [x := b]C.\, [x := b]c]\!] = [\![[x := b]c]\!]$. By the IH: $[\![[x := b]c]\!] = [x := [\![b]\!]][\![c]\!] = [x := [\![b]\!]][\![\lambda_\tau\, x : C.\, c]\!]$.

Case: $t = f \bullet_\tau a$

Suppose a is a type. Then $[\![[x := b](f \bullet_\tau a)]\!] = [\![([x := b]f \bullet_\tau [x := b]a)]\!] = [\![[x := b]f]\!]\ [\![[x := b]a]\!]$. Using the IH gives $[\![[x := b]f]\!]\ [\![[x := b]a]\!] = ([x := [\![b]\!]][\![f]\!])\ ([x := [\![b]\!]][\![a]\!]) = [x := [\![b]\!]]([\![f]\!]\ [\![a]\!]) = [x := [\![b]\!]][\![f \bullet_\tau a]\!]$.

Supppose a is a term. Then $[\![[x := b](f \bullet_\tau a)]\!] = [\![([x := b]f \bullet_\tau [x := b]a)]\!] = [\![[x := b]f]\!]$. Using the IH gives $[\![[x := b]f]\!] = [x := [\![b]\!]][\![f]\!] = [x := [\![b]\!]][\![f \bullet_\tau a]\!]$.

Case: $t = (y : C) \cap D$

By computation $[\![[x := b]((y : C) \cap D)]\!] = [\![(y : [x := b]C) \cap [x := b]D]\!] = [\![[x := b]C]\!] \times [\![[x := b]D]\!]$. Using the IH gives $[\![[x := b]C]\!] \times [\![[x := b]D]\!] = ([x := [\![b]\!]][\![C]\!]) \times ([x := [\![b]\!]][\![D]\!]) = [x := [\![b]\!]]([\![C]\!] \times [\![D]\!]) = [x := [\![b]\!]][\![(x : C) \cap D]\!]$.

Case: $t = c =_C d$

By computation $[\![[x := b](c =_C d)]\!] = [\![([x := b]c) =_{[x:=b]C} ([x := b]d)]\!] = \mathrm{Id}$. Again, by computation $[x := [\![b]\!]][\![c =_C d]\!] = [x := [\![b]\!]]\mathrm{Id} = \mathrm{Id}$.

□

Finally, soundness of the term semantics must be shown. This is not as simple as the original argument for CC modelled in F^ω because conversion happens relative to erasure. Luckily, erasure is homomorphic on type-like structure, and because the type semantics drops any term dependencies it will be the case that erasure has no impact on the semantics of types.

Lemma 3.7. *If* $\Gamma \vdash_\omega V(t) : \Box$ *then* $V(t) = V(|t|)$

Proof. By induction on t and inversion on $\Gamma \vdash V(t) : \Box$.

Case: $t = \star$ or $t = \Box$

By computation $V(|\Box|) = V(\Box) = V(\star) = V(|\star|)$.

Case: $t = (x : A) \to_m B$

Suppose A is a kind. By Lemma 2.50: $|A|$ kind. Then $V((x : A) \to_m B) = V(A) \to V(B)$. Note that the subexpressions are well-typed, thus by the IH $V(|A|) = V(A)$ and $V(|B|) = V(B)$. Now by computation $V(|(x : A) \to_m B|) = V((x : |A|) \to_m |B|) = V(|A|) \to V(|B|) = V(A) \to V(B)$.

Suppose A is not a kind. Then $V((x : A) \to_m B) = V(B)$. By the IH $V(|B|) = V(B)$. Thus, by computation $V(|(x : A) \to_m B|) = V((x : |A|) \to_m |B|) = V(|B|) = V(B)$.

□

Lemma 3.8. *If* $\Gamma \vdash_\omega [\![t]\!] : T$ *then* $[\![t]\!] = [\![|t|]\!]$

Proof. By induction on t and inversion on $\Gamma \vdash [\![t]\!] : T$. Erasure is again homomorphic on all remaining syntactic forms after inversion, thus only two cases are presented.

Case: $t = \star$ or $t = \Box$ or $t = x_\Box$

In each case $|t| = t$ thus trivial.

Case: $t = (x : A) \to_m B$

Have $|(x : A) \to_m B| = (x : |A|) \to_m |B|$. Suppose wlog that A is a kind. Then $[\![(x : |A|) \to_m |B|]\!] = (x : V(|A|)) \to [\![|A|]\!] \to [\![|B|]\!]$. By Lemma 3.7 and the IH $(x : V(|A|)) \to [\![|A|]\!] \to [\![|B|]\!] = (x : V(A)) \to [\![A]\!] \to [\![B]\!]$. Likewise, $[\![(x : A) \to_m B]\!] = (x : V(A)) \to [\![A]\!] \to [\![B]\!]$.

□

Now conversion of the kind and type models must be handled relative to erasure. The above lemmas demonstrate that if reduction happens in the erased term it should somehow be mirrored in reduction for the well-typed terms. For kinds this turns out to be simple equality, as any possible dependence involving reduction are always dropped the structure of $V(t)$ for any t is rigid. The type semantics is slightly more complicated, but the same intuition holds: if a reduction where to occur in a term dependency then the resulting type models are equal, otherwise the reduction is exactly mirrored in the model.

Lemma 3.9. *If* $\Gamma \vdash_\omega V(s) : \Box$ *and* $|s| \rightsquigarrow t$ *then* $V(s) = V(t)$

Proof. By induction on $|s| \rightsquigarrow t$. Note that only binder reduction is possible by inversion on $\Gamma \vdash V(s) : \Box$.

Case: $$\frac{\overset{\mathcal{D}_1}{t_1 \rightsquigarrow t_1'}}{\mathfrak{b}(\kappa, x : t_1, t_2) \rightsquigarrow \mathfrak{b}(\kappa, x : t_1', t_2)}$$

Inversion on $\Gamma \vdash V(s) : \Box$ forces $s = (x : A) \to_m B$. Note that $|A| \rightsquigarrow A'$. Suppose A kind, then $V((x : A) \to_m B) = V(A) \to V(B)$. Now by the IH $V(A) = V(A')$ and $V((x : A') \to_m |B|) = V(A') \to V(B)$ by Lemma 3.7. Suppose A is not a kind, then $V((x : A) \to_m B) = V(B) = V((x : A') \to_m |B|)$.

Case: $$\frac{\overset{\mathcal{D}_1}{t_2 \rightsquigarrow t_2'}}{\mathfrak{b}(\kappa, x : t_1, t_2) \rightsquigarrow \mathfrak{b}(\kappa, x : t_1, t_2')}$$

Inversion on $\Gamma \vdash V(s) : \Box$ forces $s = (x : A) \to_m B$. Note that $|B| \rightsquigarrow B'$. Suppose A kind, then $V((x : A) \to_m B) = V(A) \to V(B)$. Now by the IH $V(B) = V(B')$ and $V((x : |A|) \to_m B') = V(A) \to V(B')$ by Lemma 3.7. Suppose A is not a kind, then $V((x : A) \to_m B) = V(B) = V(B') = V((x : |A|) \to_m B')$.

□

Lemma 3.10. *If* $\Gamma \vdash_\omega [\![s]\!] : T$ *and* $|s| \rightsquigarrow t$ *then* $[\![s]\!] \rightsquigarrow [\![t]\!]$ *or* $[\![s]\!] = [\![t]\!]$

Proof. By induction on $|s| \rightsquigarrow t$. Note that only β-reduction is possible, as all other possible reduction steps are erased.

Case: $(\lambda_m\, x : A.\, b) \bullet_m t \rightsquigarrow [x := t]b$

By inversion on $\Gamma \vdash [\![s]\!] : T$ it must be the case that $m = \tau$. Thus, $|s| = (\lambda_\tau\, x : |A|.\, |b|) \bullet_\tau |t|$ and $|s| \rightsquigarrow [x := |t|]|b|$. By Lemma 2.22: $[x := |t|]|b| = |[x := t]b|$. Now, Lemma 3.8 yields $[\![|[x := t]b|]\!] = [\![[x := t]b]\!]$ and $[\![|s|]\!] = [\![s]\!]$. Using Lemma 3.6 gives $[\![[x := t]b]\!] = [x := [\![t]\!]][\![b]\!]$. Suppose A is a kind, and thus t is a type. Then $[\![(\lambda_\tau\, x : A.\, b) \bullet_\tau t]\!] = (\lambda\, x : V(A).\, [\![b]\!])\ [\![t]\!] \rightsquigarrow [x := [\![t]\!]][\![b]\!]$. Suppose A is a type, and thus t is a term. Then $[\![(\lambda_\tau\, x : A.\, b) \bullet_\tau t]\!] = [\![b]\!]$, however this also means that $\Gamma \vdash [\![b]\!] : T$. The internally bound variable x is thrown away, so it cannot be the case that $[\![b]\!]$ is

well-typed in F^ω while $x \in FV(b)$ (Note that x can be renamed to be disjoint from Γ), hence $x \notin FV(b)$. Thus, $[x := [\![t]\!]][\![b]\!] = [\![b]\!]$ and the case is concluded.

Case: $$\frac{\overset{\mathcal{D}_1}{t_i \rightsquigarrow t_i'} \qquad i \in 1, \ldots, \mathfrak{a}(\kappa)}{\mathfrak{c}(\kappa, t_1, \ldots t_i, \ldots t_{\mathfrak{a}(\kappa)}) \rightsquigarrow \mathfrak{c}(\kappa, t_1, \ldots t_i', \ldots t_{\mathfrak{a}(\kappa)})}$$

By inversion on $\Gamma \vdash [\![s]\!] : T$ it must be the case that κ is $*$, $\Box$, $\bullet_\tau$, or eq. However, the cases $*$ and $\Box$ are impossible because they do not reduce. Suppose $|s| = |f| \bullet_\tau |a|$ and assume wlog that $|a| \rightsquigarrow a'$. If a is a term then $[\![|f| \bullet_\tau |a|]\!] = [\![|f|]\!] = [\![|f| \bullet_\tau |a'|]\!]$ and $[\![|f|]\!] = [\![f]\!]$ by Lemma 3.8. Suppose a is a type. Then, by the IH $[\![a]\!] \rightsquigarrow [\![a']\!]$ or $[\![a]\!] = [\![a']\!]$. Now $[\![|f| \bullet_\tau |a|]\!] = [\![|f|]\!]\ [\![|a|]\!]$, but by Lemma 3.8: $[\![|f|]\!]\ [\![|a|]\!] = [\![f]\!]\ [\![a]\!]$. Thus, $[\![f]\!]\ [\![a]\!] \rightsquigarrow [\![f]\!]\ [\![a']\!]$ or $[\![f]\!]\ [\![a]\!] = [\![f]\!]\ [\![a']\!]$.

Suppose $|s| = |a| =_{|A|} |b|$. Note that $[\![u =_U v]\!] = \mathrm{Id}$ for any u, v, U. Thus, $[\![s]\!] = [\![|s|]\!] = [\![t]\!]$.

Case: $$\frac{\overset{\mathcal{D}_1}{t_1 \rightsquigarrow t_1'}}{\mathfrak{b}(\kappa, x : t_1, t_2) \rightsquigarrow \mathfrak{b}(\kappa, x : t_1', t_2)}$$

By inversion on $\Gamma \vdash [\![s]\!] : T$ it must be the case that κ is Π_m, λ_τ, or $\cap$. The $\cap$ and λ_τ cases are similar to the Π_m case and thus omitted. Have $|s| = (x : |A|) \to_m |B|$ and note that $|A| \rightsquigarrow A'$. Suppose wlog that A kind. Now $[\![(x : |A|) \to_m |B|]\!] = (x : V(|A|)) \to [\![|A|]\!] \to [\![|B|]\!]$. By the IH: $[\![A]\!] \rightsquigarrow [\![A']\!]$ or $[\![A]\!] = [\![A']\!]$. Suppose wlog that $[\![A]\!] \rightsquigarrow [\![A']\!]$, then $(x : V(|A|)) \to [\![|A|]\!] \to [\![|B|]\!] \rightsquigarrow (x : V(A')) \to [\![A']\!] \to [\![|B|]\!]$ by Lemma 3.9. Now $[\![(x : A') \to_m |B|]\!] = (x : V(A')) \to [\![A']\!] \to [\![|B|]\!]$.

Case: $$\frac{\overset{\mathcal{D}_1}{t_2 \rightsquigarrow t_2'}}{\mathfrak{b}(\kappa, x : t_1, t_2) \rightsquigarrow \mathfrak{b}(\kappa, x : t_1, t_2')}$$

By inversion on $\Gamma \vdash [\![s]\!] : T$ it must be the case that κ is Π_m, λ_τ, or $\cap$. The $\cap$ and λ_τ cases are similar to the Π_m case and thus omitted. Have $|s| = (x : |A|) \to_m |B|$ and note that $|B| \rightsquigarrow B'$. Suppose wlog that A kind. Now $[\![(x : |A|) \to_m |B|]\!] = (x : V(|A|)) \to [\![|A|]\!] \to [\![|B|]\!]$. By the IH: $[\![B]\!] \rightsquigarrow [\![B']\!]$ or $[\![B]\!] = [\![B']\!]$. Suppose wlog that $[\![B]\!] \rightsquigarrow [\![B']\!]$, then $(x : V(|A|)) \to [\![|A|]\!] \to [\![|B|]\!] \rightsquigarrow (x : V(|A|)) \to [\![|A|]\!] \to [\![B']\!]$. Now $[\![(x : |A|) \to_m B']\!] = (x : V(|A|)) \to [\![|A|]\!] \to [\![B']\!]$.

□

Lemma 3.11. *If* $\Gamma \vdash_\omega [\![s]\!] : T$ *and* $|s| \rightsquigarrow^* t$ *then* $[\![s]\!] \rightsquigarrow^* [\![t]\!]$

Proof. By induction on $|s| \rightsquigarrow^* t$. The reflexivity case is trivial by Lemma 3.8. Suppose $|s| \rightsquigarrow z$ and $z \rightsquigarrow^* t$. By Lemma 3.10 either $[\![s]\!] \rightsquigarrow [\![z]\!]$ or $[\![s]\!] = [\![z]\!]$. If $[\![s]\!] \rightsquigarrow [\![z]\!]$ then by preservation

$\Gamma \vdash [\![z]\!] : T$. Note that $|z| = z$ by Lemma 2.21 and because reduction does not introduce new syntactic forms. Applying the IH to $|z| \rightsquigarrow^* t$ gives $[\![z]\!] \rightsquigarrow^* [\![t]\!]$, thus $[\![s]\!] \rightsquigarrow^* [\![t]\!]$. If $[\![s]\!] = [\![z]\!]$ then obviously $\Gamma \vdash [\![z]\!] : T$ and the same argument as above works. □

With the reduction lemmas handled the required lemma about conversion is straightforward. Finally, soundness of the term semantics is proven by a straightforward induction on the inference judgment of ς_2.

Lemma 3.12. *If* $\Gamma \vdash_\omega [\![A]\!] : T$, $\Gamma \vdash_\omega [\![B]\!] : T$, A, B pseobj, *and* $A \equiv B$ *then* $[\![A]\!] \rightleftharpoons [\![B]\!]$

Proof. By Lemma 2.33 $|A| \rightleftharpoons |B|$. Deconstructing this gives $|A| \rightsquigarrow^* z$ and $|B| \rightsquigarrow^* z$. By Lemma 3.11: $[\![A]\!] \rightsquigarrow^* [\![z]\!]$ and $[\![B]\!] \rightsquigarrow^* [\![z]\!]$. Thus, $[\![A]\!] \rightleftharpoons [\![B]\!]$. □

Lemma 3.13. *If* $\Gamma \vdash_\omega t : T$ *and* $\Gamma \vdash_\omega a : A$ *then* $\Gamma \vdash (\lambda\, x{:}A.\, t)\ a : T$

Proof. Have $\Gamma \vdash_\omega \lambda\, x : A.\, t : A \to T$ because x does not appear free in t. Thus, by the App rule $\Gamma \vdash (\lambda\, x{:}A.\, t)\ a : T$. □

Lemma 3.14. *If* $\Gamma \vdash_\omega A : T$ *and* $(\bot : (X : \star) \to X) \in \Gamma$ *then* $\Gamma \vdash_\omega c^A : A$

Proof. If A type then the proof is trivial. If A kind then the proof follows by induction on the depth of the function type. □

Theorem 3.15 (Soundness of $[-]$)**.** *If* $\Gamma \vdash_{\varsigma_2} t : A$ *then* $[\![\Gamma]\!] \vdash_\omega [t] : [\![A]\!]$

Proof. By induction on $\Gamma \vdash_{\varsigma_2} t : A$. The Fst case is omitted because it is very similar to Snd. The cases Ax, Var, Pi, Lam, and App are the same as the translation from CC to F^ω.

Case: $$\frac{\Gamma \vdash \overset{\mathcal{D}_1}{A} : \star \qquad \Gamma; x_\tau : \overset{\mathcal{D}_2}{A} \vdash B : \star}{\Gamma \vdash (x : A) \cap B : \star}$$

Applying the IH to subderivations:

$\mathcal{D}_1$. $[\![\Gamma]\!] \vdash_\omega [A] : 0$

$\mathcal{D}_2$. $[\![\Gamma, x_\tau : A]\!] \vdash_\omega [B] : 0$

Note that $[\![\Gamma]\!] \vdash_\omega 0 \to 0 \to 0 : \star$. Thus, $[\![\Gamma]\!] \vdash_\omega c^{0\to 0\to 0} : 0 \to 0 \to 0$. By $\mathcal{D}_1$ it is the case that A type, thus $[\![\Gamma, x_\tau : A]\!] = [\![\Gamma]\!], x : [\![A]\!]$. Using Lemma 3.5 on $\mathcal{D}_1$ gives $[\![\Gamma]\!] \vdash_\omega [\![A]\!] : \star$. The substitution lemma yields $[\![\Gamma]\!] \vdash_\omega [x := c^{[\![A]\!]}][B] : 0$. Now applying the App rule two times concludes the case.

Case: $$\frac{\Gamma \vdash \overset{\mathcal{D}_1}{(x : A) \cap B} : \star \qquad \Gamma \vdash \overset{\mathcal{D}_2}{t} : A \qquad \Gamma \vdash s : \overset{\mathcal{D}_3}{[x := t]B} \qquad t \overset{\mathcal{D}_4}{\equiv} s}{\Gamma \vdash [t, s; (x : A) \cap B] : (x : A) \cap B}$$

Applying the IH to subderivations:

$\mathcal{D}_1$. $[\![\Gamma]\!] \vdash_\omega [(x : A) \cap B] : 0$

$\mathcal{D}_2$. $[\![\Gamma]\!] \vdash_\omega [t] : [\![A]\!]$

$\mathcal{D}_3$. $[\![\Gamma]\!] \vdash_\omega [s] : [\![[x := t]B]\!]$

By Lemma 3.6: $[\![[x := t]B]\!] = [x := [\![t]\!]][\![B]\!]$. However, A is a type by $\mathcal{D}_1$ and thus $x \notin FV([\![B]\!])$, hence $[x := [\![t]\!]][\![B]\!] = [\![B]\!]$. Now $[\![\Gamma]\!] \vdash_\omega ([t_1], [t_2]) : [\![A]\!] \times [\![B]\!]$ by the Pair rule. Applying 3.13 concludes the case.

Case: $\dfrac{\Gamma \vdash t : \overset{\mathcal{D}_1}{(x : A) \cap B}}{\Gamma \vdash t.2 : [x := t.1]B}$

Note by $\mathcal{D}_1$ that A is a type, thus $x \notin FV([\![B]\!])$. By Lemma 3.6: $[\![[x := t.1]B]\!] = [x := [\![t.1]\!]][\![B]\!] = [\![B]\!]$. Applying the IH to $\mathcal{D}_1$ gives $[\![\Gamma]\!] \vdash_\omega [t] : [\![A]\!] \times [\![B]\!]$. The Snd rule concludes the case.

Case: $\dfrac{\Gamma \vdash \overset{\mathcal{D}_1}{A} : \star \qquad \Gamma \vdash \overset{\mathcal{D}_2}{a} : A \qquad \Gamma \vdash \overset{\mathcal{D}_2}{b} : A}{\Gamma \vdash a =_A b : \star}$

Applying the IH to subderivations:

$\mathcal{D}_1$. $[\![\Gamma]\!] \vdash_\omega [A] : 0$

$\mathcal{D}_2$. $[\![\Gamma]\!] \vdash_\omega [a] : [\![A]\!]$

$\mathcal{D}_3$. $[\![\Gamma]\!] \vdash_\omega [b] : [\![A]\!]$

Note that $[\![\Gamma]\!] \vdash_\omega 0 \to [\![A]\!] \to [\![A]\!] \to 0 : \star$. Thus, $[\![\Gamma]\!] \vdash_\omega c^{0 \to [\![A]\!] \to [\![A]\!] \to 0} : 0 \to [\![A]\!] \to [\![A]\!] \to 0$. Now applying the App rule three times concludes the case.

Case: $\dfrac{\Gamma \vdash \overset{\mathcal{D}_1}{A} : \star \qquad \Gamma \vdash \overset{\mathcal{D}_2}{t} : A}{\Gamma \vdash \mathrm{refl}(t; A) : t =_A t}$

Applying the IH to subderivations:

$\mathcal{D}_1$. $[\![\Gamma]\!] \vdash_\omega [A] : 0$

$\mathcal{D}_2$. $[\![\Gamma]\!] \vdash_\omega [t] : [\![A]\!]$

Of course, $[\![\Gamma]\!] \vdash_\omega \mathrm{id} : \mathrm{Id}$. Thus, applying Lemma 3.13 twice concludes the case.

Case: $\dfrac{\Gamma \vdash \overset{\mathcal{D}_1}{A} : \star \qquad \Gamma \vdash \overset{\mathcal{D}_2}{a} : A \qquad \Gamma \vdash \overset{\mathcal{D}_3}{b} : A \qquad \Gamma \vdash \overset{\mathcal{D}_4}{e} : a =_A b \qquad \Gamma \vdash P : \overset{\mathcal{D}_5}{(y : A) \to_\tau (p : a =_A y_\star) \to_\tau \star}}{\Gamma \vdash \psi(e, a, b; A, P) : P \bullet_\tau a \bullet_\tau \mathrm{refl}(a; A) \to_\omega P \bullet_\tau b \bullet_\tau e}$

Note that by classification and $\mathcal{D}_1$ it is that case that A type. Applying the IH to subderivations:

$\mathcal{D}_1$. $[\![\Gamma]\!] \vdash_\omega [A] : 0$

$\mathcal{D}_2$. $[\![\Gamma]\!] \vdash_\omega [a] : [\![A]\!]$

$\mathcal{D}_3$. $[\![\Gamma]\!] \vdash_\omega [b] : [\![A]\!]$

$\mathcal{D}_4$. $[\![\Gamma]\!] \vdash_\omega [e] : \mathrm{Id}$

$\mathcal{D}_5$. $[\![\Gamma]\!] \vdash_\omega [P] : [\![A]\!] \to \mathrm{Id} \to 0$

Now $[\![\Gamma]\!] \vdash_\omega [e]\ [\![P]\!] : [\![P]\!] \to [\![P]\!]$. Note also that $[\![P \bullet_\tau a \bullet_\tau \mathrm{refl}(a; A) \to_\omega P \bullet_\tau b \bullet_\tau e]\!] = [\![P]\!] \to [\![P]\!]$ because $P \bullet_\tau a \bullet_\tau \mathrm{refl}(a; A)$ is a type by $\mathcal{D}_3$ and $a, b, e, \mathrm{refl}(a; A)$ are all terms. Applying Lemma 3.13 four times concludes the case.

Case: $$\frac{\Gamma \vdash (x : \overset{\mathcal{D}_1}{A}) \cap B : \star \qquad \Gamma \vdash a : \overset{\mathcal{D}_2}{(x : A)} \cap B \qquad \Gamma \vdash b : \overset{\mathcal{D}_3}{(x : A)} \cap B \qquad \Gamma \vdash e : \overset{\mathcal{D}_4}{a.1} =_A b.1}{\Gamma \vdash \vartheta(e, a, b; (x : A) \cap B) : a =_{(x:A)\cap B} b}$$

Applying the IH to subderivations:

$\mathcal{D}_1$. $[\![\Gamma]\!] \vdash_\omega [(x : A) \cap B] : 0$

$\mathcal{D}_2$. $[\![\Gamma]\!] \vdash_\omega [a] : [\![A]\!] \times [\![B]\!]$

$\mathcal{D}_3$. $[\![\Gamma]\!] \vdash_\omega [b] : [\![A]\!] \times [\![B]\!]$

$\mathcal{D}_4$. $[\![\Gamma]\!] \vdash_\omega [e] : \mathrm{Id}$

Applying Lemma 3.13 three times concludes the case.

Case: $$\frac{\Gamma \vdash (x : \overset{\mathcal{D}_1}{A}) \cap B : \star \qquad \Gamma \vdash \overset{\mathcal{D}_2}{a} : A \qquad \Gamma \vdash b : \overset{\mathcal{D}_3}{(x : A)} \cap B \qquad \Gamma \vdash e : \overset{\mathcal{D}_4}{a} =_A b.1}{\Gamma \vdash \varphi(a, b, e; A, (x : A) \cap B) : (x : A) \cap B}$$

Note by $\mathcal{D}_2$ it is clear that A is a type. Applying the IH to subderivations:

$\mathcal{D}_2$. $[\![\Gamma]\!] \vdash_\omega [(x : A) \cap B] : 0$

$\mathcal{D}_3$. $[\![\Gamma]\!] \vdash_\omega [a] : [\![A]\!]$

$\mathcal{D}_4$. $[\![\Gamma]\!] \vdash_\omega [b] : [\![(x : A) \cap B]\!]$

$\mathcal{D}_4$. $[\![\Gamma]\!] \vdash_\omega [e] : \mathrm{Id}$

Deconstructing $[\![\Gamma]\!] \vdash_\omega [(x : A) \cap B] : 0$ gives $[\![\Gamma]\!] \vdash_\omega [A] : 0$. By Lemma 3.14: $[\![\Gamma]\!] \vdash_\omega c^{[\![T]\!]} : [\![T]\!]$. Applying Lemma 3.13 five times concludes the case.

Case: $$\frac{\Gamma \vdash e : \overset{\mathcal{D}_1}{\mathrm{ctt}} =_{\mathrm{cBool}} \mathrm{cff}}{\Gamma \vdash \delta(e) : (X : \star) \to_0 X_\square}$$

By computation $[\delta(e)] = (\lambda\, x : \mathcal{I}([e]).\, \bot)\ [e]$ and $[\![(X : \star) \to_0 X]\!] = (X : \star) \to X$. Note that $[\![\Gamma]\!] \vdash_\omega \bot : (X : \star) \to X$ and by definition $[\![\Gamma]\!] \vdash_\omega [e] : \mathcal{I}([e])$. Thus, by Lemma 3.13: $[\![\Gamma]\!] \vdash [\delta(e)] : [\![(X : \star) \to_0 X]\!]$.

Case: $$\frac{\Gamma \vdash \overset{\mathcal{D}_1}{A} : K \qquad \Gamma \vdash \overset{\mathcal{D}_2}{t} : B \qquad \overset{\mathcal{D}_3}{A \equiv B}}{\Gamma \vdash t : A}$$

By classification, $\mathcal{D}_1$ and $\mathcal{D}_3$: $\Gamma \vdash B : K$. Now using Theorem 3.5 gives $[\![\Gamma]\!] \vdash_\omega [\![A]\!] : \star$ and $[\![\Gamma]\!] \vdash_\omega [\![B]\!] : \star$. Note that A, B pseobj by Lemma 2.37 and $|A| \rightleftharpoons |B|$ by Lemma 2.33. By Lemma 3.12: $[\![A]\!] \rightleftharpoons [\![B]\!]$. Applying the IH to $\mathcal{D}_2$ gives $[\![\Gamma]\!] \vdash_\omega [t] : [\![B]\!]$. The Conv rule concludes the case.

□

3.3 Normalization

With soundness of the model shown the normalization argument follows in the same way as for CC modelled in F^ω. That is, proof reduction in $ç_2$ is bounded by reduction in F^ω, and thus because F^ω is strongly normalizing it provides a maximum number of reduction steps for which any proof must normalize in $ç_2$. Note that some reduction steps are technical, especially ϑ_i, but they are not conceptually difficult.

Lemma 3.16. $[x := b]c^A = c^{[x:=b]A}$

Proof. Straightforward by unraveling the definition of canonical elements (c) and applying substitution computation rules. □

Lemma 3.17. *If* $\Gamma \vdash t : A$ *and* $(x : B) \in \Gamma$ *then*

1. $[[x := b]a] = [x := [\![b]\!]][w_x := [b]][a]$ *if* B *kind*
2. $[[x := b]a] = [x := [b]][a]$ *if* B *type*

Proof. By induction on $\Gamma \vdash t : A$. Substitution is structural and with Lemma 3.6, Lemma 3.3, and Lemma 3.16 many cases are straightforward by induction. Thus, only the variable cases and the Int case are presented.

Case: $\dfrac{x \notin FV(\Gamma_1; \Gamma_2) \qquad \Gamma_1 \vdash \overset{\mathcal{D}_2}{A} : K}{\Gamma_1; x_m : A; \Gamma_2 \vdash x_K : A}$

Rename to y. Suppose $x \neq y$, then $[[x := b]y_\star] = y$, $[x := [\![b]\!][w_x := [b]][y_\star] = y$, and $[x := [b]][y_\star] = y$. When $y_\square$ the situation is the same. Suppose $x = y$ and that B kind. If B is kind, then it must be the case that $y_\square$. Now $[[x := b]y_\square] = [b]$ and $[x := [\![b]\!][w_x := [b]][y_\square] = [x := [\![b]\!][w_x := [b]]w_y = [b]$. Suppose instead that B type, then $[[x := b]y_\star] = [b]$ and $[x := [b]][y_\star] = [x := [b]]y = [b]$.

Case: $\dfrac{\Gamma \vdash \overset{\mathcal{D}_1}{A} : \star \qquad \Gamma; x_\tau : A \vdash \overset{\mathcal{D}_2}{B} : \star}{\Gamma \vdash (x : A) \cap B : \star}$

Suppose wlog that B is a kind. Then $[[x := b](y : A) \cap B] = [(y : [x := b]A) \cap [x := b]B] = c^{0\to0\to0}[[x := b]A]([y := c^{[\![[x:=b]A]\!]}][[x := b]B])$. Now by the IH, Lemma 3.6, and the fact that $w_x \notin FV([\![A]\!])$ the right-hand side is equal to $c^{0\to0\to0}[x := [\![b]\!][w_x := [b]][A]([y :=$

$c^{[x:=[\![b]\!]][w_x:=[b]][\![A]\!]}[x := [\![b]\!]][w_x := [b]][B])$. Consider $[x := [\![b]\!]][w_x := [b]][(y : A) \cap B] = [x := [\![b]\!]][w_x := [b]]c^{0\to 0\to 0}[A]([y := c^{[\![A]\!]}][B])$. Note that $x, w_x \notin FV(0 \to 0 \to 0)$, thus by Lemma 2.1, Lemma 3.16, and computation rules of substitution this matches the previous right-hand side.

□

Lemma 3.18. *If* $\Gamma \vdash s : T$ *and* $s \rightsquigarrow t$ *then* $[s] \rightsquigarrow^*_{\neq 0} [t]$

Proof. By induction on $s \rightsquigarrow t$. The first projection case is very similar to the second projection case. Note by a simple observation that $[-]$ replicates every subexpression on the left-hand side with a matching invocation of $[-]$ on the right-hand side. Thus, if there is a reduction inside a subexpression it will always be tracked in the corresponding $[-]$ invocation via the inductive hypothesis. For this reason the structural reduction cases are omitted.

Case: $(\lambda_m\, x : A.\, b) \bullet_m t \rightsquigarrow [x := t]b$

Note by $\Gamma \vdash s : T$ that A is either a kind or a type. Suppose A is a kind and note that makes t a type. Then $[(\lambda_m\, x : A.\, b) \bullet_m t] = (\lambda\, y : 0.\, \lambda\, x : V(A).\, \lambda\, w_x : [\![A]\!].\, [b])\ [A]\ [\![t]\!]\ [t]$. The variable y is fresh thus after one β-reduction $(\lambda\, x : V(A).\, \lambda\, w_x : [\![A]\!].\, [b])\ [\![t]\!]\ [t]$. Now applying two more β-reductions yields $[x := [\![t]\!]][w_x := [t]][b]$. Note that $[[x := t]b] = [x := [\![t]\!]][w_x := [t]][b]$ by Lemma 3.17. Thus, $[s] \rightsquigarrow^*_{=3} [t]$, i.e. $[s]$ reduces to $[t]$ in three steps.

Suppose A is a type and note that makes t a term. Then $[(\lambda_m\, x : A.\, b) \bullet_m t] = (\lambda\, y : 0.\, \lambda\, w_x : [\![A]\!].\, [b])\ [A]\ [t]$. The variable y is fresh thus after one β-reduction $(\lambda\, w_x : [\![A]\!].\, [b])\ [t]$. Applying one more β-reduction yields $[x := [t]][b]$. Note that $[[x := t]b] = [x := [t]][b]$ by Lemma 3.17. Thus, $[s] \rightsquigarrow^*_{=2} [t]$.

Case: $[t_1, t_2; A].2 \rightsquigarrow t_2$

Have $[[t_1, t_2; A].2] = ((\lambda\, y : 0.\, ([t_1], [t_2]))\ [A]).2$. Note that the variable y is fresh and thus not in $FV([t_1])$ or $FV([t_2])$. A second projection and one β-reduction yields $[t_2]$. Thus, $[s] \rightsquigarrow^*_{=2} [t_2]$.

Case: $\psi(\mathrm{refl}(t; A_1), a, b; A_2, P) \bullet_\omega t \rightsquigarrow t$

Note that t is a term by inversion on $\Gamma \vdash s : T$. Have $[\psi(\mathrm{refl}(t; A_1); A_2, P) \bullet_\omega t] = (\lambda\, y_1 : 0.\, \lambda\, y_2\ y_3 : [\![A]\!].\, \lambda\, y_4 : [\![A_2]\!] \to \mathrm{Id} \to 0.\, [\mathrm{refl}(t; A_1)]\ [\![P]\!])\ [A_2]\ [a]\ [b]\ [P]\ [t]$. Applying four β-reductions yields $[\mathrm{refl}(t; A_1)]\ [\![P]\!]\ [t]$. Now $[\mathrm{refl}(t; A_1)] = (\lambda\, y_1 : 0.\, \lambda\, y_2 : [\![A_1]\!].\, \mathrm{id})\ [A_1]\ [t]$. Applying two more β-reductions gives $\mathrm{id}\ [\![P]\!]\ [t]$. Finally, applying two remaining β-reductions yields $[t]$. Thus, $[s] \rightsquigarrow^*_{=8} [t]$.

Case: $\vartheta(\mathrm{refl}(t; A), a, b; T) \rightsquigarrow \mathrm{refl}(a; T)$

Have $[\vartheta(\text{refl}(t; A), a, b; T)] = (\lambda\, y_1 : [\![T]\!].\, \lambda\, y_2 : 0.\, \lambda\, y_3 : [\![T]\!].\, ((\lambda\, y_1 : 0.\, \lambda\, y_2 : [\![A]\!].\, \text{id})\ [A]\ [t]))\ [b]\ [T]\ [a]$. Note that all y_i are fresh and thus not in the free variables of any subexpressions. Performing two β-reductions on the interior (the result of $[\text{refl}(t_1; A)]$) and the outermost β-reduction yields: $(\lambda\, y_2 : 0.\, \lambda\, y_3 : [\![T]\!].\, \text{id})\ [T]\ [a]$. Now $[\text{refl}(a; T)] = (\lambda\, y_2 : 0.\, \lambda\, y_3 : [\![T]\!].\, \text{id})\ [T]\ [a]$. Thus, $[s] \rightsquigarrow^*_{=3} [t]$.

□

Theorem 3.19 (Proof Normalization)**.** *If $\Gamma \vdash t : A$ then t is strongly normalizing and there exists a unique value t_n such that $t \rightsquigarrow^* t_n$*

Proof. Using Lemma 3.5 gives $[\![\Gamma]\!] \vdash_\omega [t] : [\![A]\!]$. Note that F^ω with pairs is strongly normalizing with a unique normal form (because it is also confluent). Thus, *all* possible reduction paths to the normal form are terminating. Let $\partial([t])$ be the *maximum* number of reduction steps $[t]$ could take to reach a normal form. Note that this value is computable by brute force search. Pick any sequence of reductions in t bounded by $\partial([t])$. If this sequence concludes in a value then t is strongly normalizing, because the sequence is arbitrary. If t is not a value then $t \rightsquigarrow^*_{>\partial[t]} t'$, but this is impossible by Lemma 3.18. Now by confluence of reduction, all values reached from any arbitrary reduction path must be joinable at a single value. Thus, $t \rightsquigarrow^* t_n$ where t_n is a unique value. □

Chapter 4

CONSISTENCY AND RELATIONSHIP TO CDLE

The Calculus of Dependent Lambda Eliminations (CDLE) was first introduced in 2017 [55] as the core system for the in progress Cedille tool. At that time, CDLE included complicated machinery for lifting lambda terms to the type-level enabling some large eliminations. Over the years, the core system for the Cedille tool was still referred to as CDLE as it evolved culminating in the current core system used in Cedille version 1.1.2 [57]. The ideas leading to CDLE, of course, grew over time with work on efficient lambda encodings in total theories [56]; self-types for encodings [19]; and experiments involving irrelevance [54, 53]. Ultimately, the modern version of CDLE, as presented in this chapter, is the culmination of these efforts.

CDLE is an affirmative answer to the question: is lambda-encoded data enough for a proof assistant? While there may be other philosophical objections the system Mendler-style encodings have been shown to be efficient and enable course-of-values induction [15, 16]. Moreover, the edition of the φ construct, an idea borrowed from the direct computation rule of Nuprl [2], yields efficient data reuse via casts [14]. However, the success of CDLE does not stop there. The successes of CDLE include: quotient subtypes [36]; coinductive data [31]; zero-cost constructor subtypes [37]; monotonic recursive types [30]; simulated large eliminations [29]; and inductive-inductive data [35].

CDLE commits to impredicative (i.e. parametric in sense of F^ω) quantification. With that in mind the well-studied reader may not be surprised at the power and versatility of CDLE. However, taming impredicative quantification without losing logical consistency is a difficult task. Indeed, this is precisely why several proof assistants have discarded impredicative quantification or relegated it into a universe of propositions. A core philosophy behind both Cedille and Cedille2 is to walk a different road and embrace impredicative quantification. To achieve that goal a realizability model was developed for CDLE to demonstrate logical consistency [57]. This chapter will describe a model of ς_2 in CDLE to prove consistency.

4.1 Calculus of Dependent Lambda Eliminations

CDLE is described using an intrinsic style where syntax is presented directly with the typing derivation. However, erasure it still a crucial part of CDLE which gives it an extrinsic philosophy. Whether a system is intrinsic or extrinsic is perhaps not a terribly interesting distinction. Technically, ς_2 is described extrinsically because syntax is defined independently of the typing relation, but there is no essential reason for this choice. Moreover, any intrinsic system necessarily admits a projection of its raw syntax, which would enable an extrinsic presentation. It is better to think about these details via their philosophical import. An intrinsic system wishes to say that raw syntax has no meaning, or at the very least no meaning that anyone should care about. Alternatively,

$$\frac{}{\Gamma \vdash \star} \qquad \frac{\Gamma \vdash A \rhd \star \qquad \Gamma; x : A \vdash \kappa}{\Gamma \vdash \Pi\, x{:}A.\, \kappa} \qquad \frac{\Gamma \vdash \kappa' \qquad \Gamma; x : \kappa' \vdash \kappa}{\Gamma \vdash \Pi\, x{:}\kappa'.\, \kappa}$$

Figure 4.1: Judgment for formation of kinds in CDLE.

$$\frac{(x : \kappa) \in \Gamma}{\Gamma \vdash x \rhd \kappa} \qquad \frac{\Gamma \vdash \kappa \qquad \Gamma; x : \kappa \vdash B \rhd \star}{\Gamma \vdash \forall\, x{:}\kappa.\, B \rhd \star}$$

$$\frac{\Gamma \vdash A \rhd \star \qquad \Gamma; x : A \vdash B \rhd \star}{\Gamma \vdash \forall\, x{:}A.\, B \rhd \star} \qquad \frac{\Gamma \vdash A \rhd \star \qquad \Gamma; x : A \vdash B \rhd \star}{\Gamma \vdash \Pi\, x{:}A.\, B \rhd \star}$$

$$\frac{\Gamma \vdash A \rhd \star \qquad \Gamma; x : A \vdash t \rhd \kappa}{\Gamma \vdash \lambda\, x{:}A.\, t \rhd \Pi\, x{:}A.\, \kappa} \qquad \frac{\Gamma \vdash \kappa' \qquad \Gamma; x : \kappa' \vdash t \rhd \kappa}{\Gamma \vdash \lambda\, x{:}\kappa'.\, t \rhd \Pi\, x{:}\kappa'.\, \kappa}$$

$$\frac{\Gamma \vdash f \rhd \Pi\, x{:}A.\, \kappa \qquad \Gamma \vdash a \lhd A}{\Gamma \vdash f\ a \rhd [x := \chi\ A - a]\kappa} \qquad \frac{\Gamma \vdash f \rhd \Pi\, x{:}\kappa_1.\, \kappa_2 \qquad \Gamma \vdash a \rhd \kappa_1' \qquad \kappa_1 \cong \kappa_1'}{\Gamma \vdash f \cdot a \rhd [x := a]\kappa_2}$$

$$\frac{\Gamma \vdash A \rhd \star \qquad \Gamma; x : A \vdash B \rhd \star}{\Gamma \vdash \iota\, x{:}A.\, B \rhd \star} \qquad \frac{FV(a\ b) \subseteq dom(\Gamma)}{\Gamma \vdash \{a \simeq b\} \rhd \star}$$

Figure 4.2: Inference judgment defining well-formed types and their inferred kind in CDLE.

an extrinsic system wishes to say that types are in some sense only annotations, and it is the raw syntax that is primary.

As one might guess these philosophical positions are not entirely black and white. For example, Pfenning demonstrates how both methods can be combined [45]. Cedille has been historically described as an extrinsic system. With Cedille2 it is more correct to be called a *combined* system, both intrinsic and extrinsic. That is, a *proof* has no meaning as just syntax, but an *object* discards the extra information as mere annotations.

The kind formation rules as presented in Figure 4.1, type formation rules in Figure 4.2, and term annotation rules in Figure 4.3. Lowercase letters are used to refer to metavariables of terms, uppercase letters for metavariables of types, and variations of κ for metavariables of kinds. Call-by-name reduction of the λ-calculus fragment is used in the rules for types and is written $A \rightsquigarrow_n B$. The purpose of this relation is only to reveal a constructor for a type, thus weak-head normal form is sufficient. Conversion for types is presented in Figure 4.4 and kind conversion in Figure 4.5. Note that these conversion relations correspond to β-conversion for types and kinds. Finally, erasure of terms (and only terms) is presented in Figure 4.6. Erasure is only meaningful with terms in CDLE unlike in ς_2 where it is defined for all raw syntax.

The presentation in this work deviates from other descriptions of CDLE by adding a symmetry rule for equality (ς). This rule is admissible using the rewrite rule (ρ), but it is convenient to have available for the model. Otherwise, the presentation is identical to the one by Stump and Jenkins [57].

$$\Gamma \vdash t \blacktriangleright A \text{ iff } \exists\, T.\ (\Gamma \vdash t \triangleright T) \wedge (T \rightsquigarrow_n^* A)$$

$$\frac{(x : A) \in \Gamma}{\Gamma \vdash x \triangleright A}$$

$$\frac{T \rightsquigarrow_n^* \Pi\, x{:}A.\, B \qquad \Gamma; x : A \vdash t \triangleleft B}{\Gamma \vdash \lambda\, x.\, t \triangleleft T}$$

$$\frac{T \rightsquigarrow_n^* \forall\, x{:}\kappa.\, B \qquad \Gamma; x : \kappa \vdash t \triangleleft B}{\Gamma \vdash \Lambda\, x.\, t \triangleleft T}$$

$$\frac{\begin{array}{c} T \rightsquigarrow_n^* \forall\, x{:}A.\, B \\ \Gamma; x : A \vdash t \triangleleft B \qquad x \notin FV(|t|) \end{array}}{\Gamma \vdash \Lambda\, x.\, t \triangleleft T}$$

$$\frac{\begin{array}{c} T \rightsquigarrow_n^* \iota\, x{:}A.\, B \qquad \Gamma \vdash t_1 \triangleleft A \\ \Gamma \vdash t_2 \triangleleft [x := \chi\ A - t_1]B \qquad |t_1| \rightleftharpoons_\eta |t_2| \end{array}}{\Gamma \vdash [t_1, t_2] \triangleleft T}$$

$$\frac{\Gamma \vdash t \blacktriangleright \iota\, x{:}A.\, B}{\Gamma \vdash t.2 \triangleright [x := t.1]B}$$

$$\frac{\Gamma \vdash t \triangleright A \qquad A \cong \{\lambda\, x\ y.\, x \simeq \lambda\, x\ y.\, y\}}{\Gamma \vdash \delta - t \triangleleft T}$$

$$\frac{\Gamma \vdash A \triangleright \star \qquad \Gamma \vdash t \triangleleft A}{\Gamma \vdash \chi\ A - t \triangleright A}$$

$$\frac{\Gamma \vdash e \blacktriangleright \{a \simeq b\}}{\Gamma \vdash \text{ç}\ e \triangleright \{b \simeq a\}}$$

$$\frac{\Gamma \vdash t \triangleright A \qquad A \cong B}{\Gamma \vdash t \triangleleft B}$$

$$\frac{\Gamma \vdash f \blacktriangleright \Pi\, x{:}A.\, B \qquad \Gamma \vdash a \triangleleft A}{\Gamma \vdash f\ a \triangleleft [x := \chi\ A - a]B}$$

$$\frac{\Gamma \vdash f \blacktriangleright \Pi\, x{:}\kappa_1.\, B \qquad \Gamma \vdash a \triangleright \kappa_2 \qquad \kappa_1 \cong \kappa_2}{\Gamma \vdash f \cdot a \triangleleft [x := a]B}$$

$$\frac{\Gamma \vdash f \blacktriangleright \forall\, x{:}A.\, B \qquad \Gamma \vdash a \triangleleft A}{\Gamma \vdash f\ \text{-}a \triangleleft [x := \chi\ A - a]B}$$

$$\frac{\Gamma \vdash f \blacktriangleright \iota\, x{:}A.\, B}{\Gamma \vdash t.1 \triangleright A}$$

$$\frac{\begin{array}{c} T \rightsquigarrow_n^* \{a \simeq b\} \\ FV(t) \subseteq dom(\Gamma) \qquad |a| \rightleftharpoons_\eta |b| \end{array}}{\Gamma \vdash \beta\{t\} \triangleleft T}$$

$$\frac{\begin{array}{c} \Gamma \vdash e \blacktriangleright \{a \simeq b'\} \qquad FV(b') \subseteq dom(\Gamma) \\ |b'| \rightleftharpoons_\eta |b| \qquad \Gamma \vdash [x := b]A \triangleright \star \\ \Gamma \vdash t \triangleleft [x := b]A \qquad [x := a]A \cong T \end{array}}{\Gamma \vdash \rho\ e\ @\ x\ \langle b \rangle.\ A - t \triangleleft T}$$

$$\frac{\begin{array}{c} e \triangleleft \{a \simeq b\} \\ \Gamma \vdash a \triangleright A \qquad FV(b) \subseteq dom(\Gamma) \end{array}}{\Gamma \vdash \varphi\ e - a\ \{b\} \triangleright A}$$

$$\frac{\begin{array}{c} e \triangleleft \{a \simeq b\} \\ \Gamma \vdash a \triangleleft A \qquad FV(b) \subseteq dom(\Gamma) \end{array}}{\Gamma \vdash \varphi\ e - a\ \{b\} \triangleleft A}$$

Figure 4.3: Bidirectional annotation judgment for terms defining when an annotated term infers of checks against a type in CDLE.

$$\frac{A \rightsquigarrow_n^* A' \not\rightsquigarrow_n \qquad B \rightsquigarrow_n^* B' \not\rightsquigarrow_n \qquad A' \cong^t B'}{A \cong B}$$

$$\frac{}{x \cong^t x} \qquad \frac{\kappa_1 \cong \kappa_2 \qquad B_1 \cong B_2}{\forall\, x{:}\kappa_1.\, B_1 \cong^t \forall\, x{:}\kappa_2.\, B_2}$$

$$\frac{A_1 \cong A_2 \qquad B_1 \cong B_2}{\forall\, x{:}A_1.\, B_1 \cong^t \forall\, x{:}A_2.\, B_2} \qquad \frac{A_1 \cong A_2 \qquad B_1 \cong B_2}{\Pi\, x{:}A_1.\, B_1 \cong^t \Pi\, x{:}A_2.\, B_2}$$

$$\frac{A_1 \cong A_2 \qquad B_1 \cong B_2}{\lambda\, x{:}A_1.\, B_1 \cong^t \lambda\, x{:}A_2.\, B_2} \qquad \frac{\kappa_1 \cong \kappa_2 \qquad B_1 \cong B_2}{\lambda\, x{:}\kappa_1.\, B_1 \cong^t \lambda\, x{:}\kappa_2.\, B_2}$$

$$\frac{A_1 \cong A_2 \qquad B_1 \cong B_2}{\iota\, x{:}A_1.\, B_1 \cong^t \iota\, x{:}A_2.\, B_2} \qquad \frac{A_1 \cong^t A_2 \qquad |b_1| \rightleftharpoons_\eta |b_2|}{A_1\ b_1 \cong^t A_2\ b_2}$$

$$\frac{A_1 \cong^t A_2 \qquad B_1 \cong B_2}{A_1 \cdot B_1 \cong^t A_2 \cdot B_2} \qquad \frac{|a_1| \rightleftharpoons_\eta |a_2| \qquad |b_1| \rightleftharpoons_\eta |b_2|}{\{a_1 \simeq b_1\} \cong^t \{a_2 \simeq b_2\}}$$

Figure 4.4: Definition of conversion for types in CDLE.

$$\frac{}{\star \cong \star}$$

$$\frac{A_1 \cong A_2 \qquad \kappa_1 \cong \kappa_2}{\Pi\, x{:}A_1.\, \kappa_1 \cong \Pi\, x{:}A_2.\, \kappa_2} \qquad \frac{\kappa_1' \cong \kappa_2' \qquad \kappa_1 \cong \kappa_2}{\Pi\, x{:}\kappa_1'.\, \kappa_1 \cong \Pi\, x{:}\kappa_2'.\, \kappa_2}$$

Figure 4.5: Defintion of conversion for kinds in CDLE.

$$\begin{aligned}
|x| &= x & |\lambda\, x.\, t| &= \lambda\, x.\, |t| \\
|f\ a| &= |f|\ |a| & |f \cdot a| &= |f| \\
|\Lambda\, x.\, t| &= |t| & |f\ {-a}| &= |f| \\
|[t_1, t_2]| &= |t_1| & |t.1| &= |t| \\
|t.2| &= |t| & |\beta\{t\}| &= |t| \\
|\delta - t| &= \lambda\, x.\, x & |\rho\ e\ @\ x\ \langle a\rangle.\ A - t| &= |t| \\
|\varphi\ e - a\ \{b\}| &= |b| & |\chi\ A - t| &= |t|
\end{aligned}$$

Figure 4.6: Erasure of terms in CDLE, note that erasure is not defined for types or kinds.

A few useful facts about CDLE are needed before defining the model. First, some helpful terms are defined below. Note that an annotation rule (χ) is added to some terms in order to guarantee that each definition always infers a type, as opposed to checks against a type. The Bool definition is a standard Church encoding boolean type, with its two associated values (tt and ff). An identity type, Id, is defined as a desired output of the model for the equality of ς_2. Indeed, CDLEs equality is very flexible in comparison to ς_2. Not only is it untyped, but it allows for any well-scoped term to serve as the erasure (or object) of a reflexivity proof. The irrel definition will not be used in the model and is instead used after to demonstrate that an irrelevance axiom may be added to ς_2 while still preserving consistency.

$$
\begin{aligned}
\text{Bool} &:= \forall\, X{:}\star.\, X \to X \to X \\
\text{tt} &:= \chi\ \text{Bool}\ -\ \Lambda\, X.\, \lambda\, x\ y.\, x \\
\text{ff} &:= \chi\ \text{Bool}\ -\ \Lambda\, X.\, \lambda\, x\ y.\, y \\
\text{Id} &:= \lambda\, A{:}\star.\, \lambda\, a\ b{:}A.\, \iota\, e{:}\{a \simeq b\}.\, \iota\, y{:}\{(\lambda\, x.\, x) \simeq e\}.\, \forall\, X{:}\star.\, X \to X \\
\text{refl} &:= \chi\ \forall\, A{:}\star.\, \forall\, a{:}A.\, \text{Id} \cdot A\ a\ a\ - \\
&\quad \Lambda\, A\ a.\, [\beta\{\lambda\, x.\, x\}, [\beta\{\lambda\, x.\, x\}, \Lambda\, X.\, \lambda\, x.\, x]] \\
\text{delta} &:= \chi\ \text{Id} \cdot \text{Bool tt ff} \to \forall\, X{:}\star.\, X\ - \\
&\quad \lambda\, e.\, (\delta - e.1) \cdot (\text{Id} \cdot \text{Bool tt ff} \to \forall\, X{:}\star.\, X)\ e \\
\text{irrel} &:= \chi\ \forall\, A{:}\star.\, \forall\, a\ b{:}A.\, \text{Id} \cdot A\ a\ b \Rightarrow \text{Id} \cdot A\ a\ b\ - \\
&\quad \Lambda\, A\ a\ b\ e. \\
&\quad [\rho\ e.1\ @\ x\ \langle b\rangle.\ \{x \simeq b\}\ -\ \beta\{\lambda\, x.\, x\}), \\
&\quad [\beta\{\lambda\, x.\, x\}, \Lambda\, X.\, \lambda\, x.\, x]]
\end{aligned}
$$

Aside from the previous terms it is also useful to have terms representing the target output of the substitution and promotion rules of ς_2. All of these terms are constructed to obtain specific erasures.

$$
\begin{aligned}
\text{theta}_1 &:= \chi\ \forall\, A{:}\star.\, \forall\, B{:}A \to \star.\, \forall\, a\ b{:}(\iota\, x{:}A.\, B\ x). \\
&\quad \text{Id} \cdot A\ a.1\ b.1 \to \text{Id} \cdot (\iota\, x{:}A.\, B\ x)\ a\ b\ - \\
&\quad \Lambda\, A\ B\ a\ b.\, \lambda\, e. \\
&\quad \varphi\ (\rho\ e.2.1\ @\ x\ \langle e\rangle.\ \{x \cong e\}\ -\ \beta\{\lambda\, x.\, x\})\ - \\
&\quad (\rho\ e.1\ @\ x\ \langle b\rangle.\ \text{Id} \cdot (\iota\, x{:}A.\, B\ x))\ x\ b\ -\ \text{refl} \cdot (\iota\, x{:}A.\, B\ x)\ \text{-}b) \\
&\quad \{e\}
\end{aligned}
$$

$$
\begin{aligned}
&\text{theta}_2 := \chi\ \forall\, A\ B{:}\star.\, \forall\, a\ b{:}(\iota\, x{:}A.\, B).\\
&\qquad \mathrm{Id}\cdot A\ a.2\ b.2 \to \mathrm{Id}\cdot(\iota\, x{:}A.\, B)\ a\ b\ -\\
&\qquad \Lambda\, A\ B\ a\ b.\, \lambda\, e.\\
&\qquad \varphi\ (\rho\ e.2.1\ @\ x\ \langle e\rangle.\ \{x \cong e\}\ -\ \beta\{\lambda\, x.\, x\})\ -\\
&\qquad (\rho\ e.1\ @\ x\ \langle b\rangle.\ \mathrm{Id}\cdot(\iota\, x{:}A.\, B))\ x\ b\ -\ \mathrm{refl}\cdot(\iota\, x{:}A.\, B)\ \text{-}b)\\
&\qquad \{e\}
\end{aligned}
$$

$$
\begin{aligned}
&\text{subst} := \chi\ \forall\, A{:}\star.\, \forall\, a\ b{:}A.\, \forall\, P{:}(\Pi\, y{:}A.\, \mathrm{Id}\cdot A\ a\ y \to \star).\\
&\qquad \Pi\, e{:}\mathrm{Id}\cdot A\ a\ b.\, P\ a\ (\mathrm{refl}\cdot A\ \text{-}a) \to P\ b\ e\ -\\
&\qquad \Lambda\, A\ a\ b\ P.\, \lambda\, e.\\
&\qquad \rho\ e.2.1\ @\ x\ \langle e\rangle.\ P\ a\ x \to P\ b\ e\ -\\
&\qquad \rho\ e.1\ @\ x\ \langle b\rangle.\ P\ x\ e \to P\ b\ e\ -\\
&\qquad e.2.2\cdot(P\ b\ e)
\end{aligned}
$$

The erasure of each term is designed to match with the erasure of the associated construct in ς_2. While this might not be strictly necessary to obtain a model of ς_2 inside CDLE it makes the process easier. Moreover, carefully crafting terms with specific erasures is a trivial matter in CDLE because of the φ rule.

$$
\begin{aligned}
|\mathrm{tt}| &= \lambda\, x\ y.\, x\\
|\mathrm{ff}| &= \lambda\, x\ y.\, y\\
|\mathrm{refl}\cdot A\ \text{-}a| &= \lambda\, x.\, x\\
|\mathrm{delta}\ e| &= |e|\\
|\mathrm{irrel}\cdot A\ \text{-}a\ \text{-}b\ \text{-}e| &= \lambda\, x.\, x\\
|\mathrm{theta}_1\cdot A\cdot B\ \text{-}a\ \text{-}b\ e| &= |e|\\
|\mathrm{theta}_2\cdot A\cdot B\ \text{-}a\ \text{-}b\ e| &= |e|\\
|\mathrm{subst}\cdot A\ \text{-}a\ \text{-}b\cdot P\ e| &= |e|
\end{aligned}
$$

Finally, each of these terms is shown to infer the desired type. Note that for syntax that is type-like, such as Id and Bool, there is no type-checking rule, only an inference judgment. Moreover, the χ rule only works with term-like syntax. Thus, for these definitions more care is needed to infer the correct kind, but because the definitions are simple there is no real difficulty.

Lemma 4.1.

1. $\vdash_{\varsigma_1} \mathrm{Bool} \rhd \star$
2. $\vdash_{\varsigma_1} \mathrm{tt} \rhd \mathrm{Bool}$

3. $\vdash_{\varsigma_1} \mathrm{ff} \rhd \mathrm{Bool}$

4. $\vdash_{\varsigma_1} \mathrm{Id} \rhd \Pi\, A{:}\star.\, A \to A \to \star$

5. $\vdash_{\varsigma_1} \mathrm{refl} \rhd \forall\, A{:}\star.\, \forall\, a{:}A.\, \mathrm{Id} \cdot A\ a\ a$

6. $\vdash_{\varsigma_1} \mathrm{delta} \rhd \mathrm{Id} \cdot \mathrm{Bool}\ \mathrm{tt}\ \mathrm{ff} \to \forall\, X{:}\star.\, X$

7. $\vdash_{\varsigma_1} \mathrm{irrel} \rhd \forall\, A{:}\star.\, \forall\, a\ b{:}A.\, \mathrm{Id} \cdot A\ a\ b \Rightarrow \mathrm{Id} \cdot A\ a\ b$

8. $\vdash_{\varsigma_1} \mathrm{theta}_1 \rhd \begin{array}{l} \forall\, A{:}\star.\, \forall\, B{:}A \to \star.\, \forall\, a\ b{:}(\iota\, x{:}A.\, B\ x). \\ \mathrm{Id} \cdot A\ a.1\ b.1 \to \mathrm{Id} \cdot (\iota\, x{:}A.\, B\ x)\ a\ b \end{array}$

9. $\vdash_{\varsigma_1} \mathrm{theta}_2 \rhd \begin{array}{l} \forall\, A\ B{:}\star.\, \forall\, a\ b{:}(\iota\, x{:}A.\, B). \\ \mathrm{Id} \cdot A\ a.2\ b.2 \to \mathrm{Id} \cdot (\iota\, x{:}A.\, B)\ a\ b \end{array}$

10. $\vdash_{\varsigma_1} \mathrm{subst} \rhd \begin{array}{l} \forall\, A{:}\star.\, \forall\, a\ b{:}A.\, \forall\, P{:}(\Pi\, y{:}A.\, \mathrm{Id} \cdot A\ a\ y \to \star). \\ \Pi\, e{:}\mathrm{Id} \cdot A\ a\ b.\, P\ a\ (\mathrm{refl} \cdot A\ \text{-}a) \to P\ b\ e \end{array}$

Proof. Straightforward by applying a short sequence of ς_1 rules in each case. These inferences are trivially formalized in the Cedille tool. □

A small collection of additional lemmas about CDLE is needed to prove soundness of the model and presented now. These lemmas are standard, weakening, symmetry of conversion, and transitivity of conversion. The only real difficulty is the bidirectional presentation which requires stating the desired lemma for each variation of judgment.

Lemma 4.2. *Suppose* $\Gamma \vdash_{\varsigma_1} T \rhd K$ *and* x *fresh*

1. *If* t *is a kind and* $\Gamma, \Delta \vdash_{\varsigma_1} t$ *then* $\Gamma, x : T, \Delta \vdash_{\varsigma_1} t$

2. *If* t *is a type and* $\Gamma, \Delta \vdash_{\varsigma_1} t \rhd K$ *then* $\Gamma, x : T, \Delta \vdash_{\varsigma_1} t \rhd K$

3. *If* t *is a term and* $\Gamma, \Delta \vdash_{\varsigma_1} t \rhd A$ *then* $\Gamma, x : T, \Delta \vdash_{\varsigma_1} t \rhd A$

4. *If* t *is a term and* $\Gamma, \Delta \vdash_{\varsigma_1} t \lhd A$ *then* $\Gamma, x : T, \Delta \vdash_{\varsigma_1} t \lhd A$

Proof. Straightforward by mutual recursion on the associated judgments. □

Lemma 4.3.

1. *If* a, b *are terms and* $|a| \rightleftharpoons_\eta |b|$ *then* $|b| \rightleftharpoons_\eta |a|$

2. *If* A, B *are types and values and* $A \cong^t B$ *then* $B \cong^t A$

3. *If* A, B *are types and* $A \cong B$ *then* $B \cong A$

4. *If* A, B *are kinds and* $A \cong B$ *then* $B \cong A$

Proof. Note that *1.* holds because $|a|$ and $|b|$ are untyped λ-calculus terms. For *2.* through *4.* mutual recursion and pattern match on A is sufficient. □

Lemma 4.4.

1. *If a, b, c are terms, $|a| \rightleftharpoons_\eta |b|$, and $|b| \rightleftharpoons_\eta |c|$ then $|a| \rightleftharpoons_\eta |c|$*
2. *If A, B, C are types and values, $A \cong^t B$, and $B \cong^t C$ then $A \cong^t C$*
3. *If A, B, C are types, $A \cong B$, and $B \cong C$ then $A \cong C$*
4. *If A, B, C are kinds, $A \cong B$, and $B \cong C$ then $A \cong C$*

Proof. Note that *1.* holds because $|a|$ and $|b|$ are untyped λ-calculus terms and reduction is confluent. The remainder are proved by mutual recursion. Note that in *3.* the types A, B, and C are reduced using call-name to a weak-head normal form. In particular, this reduction strategy is deterministic, thus $B \rightsquigarrow_n^* B'$ for a unique B'. This combined with using *2.* is sufficient for the *3.* case. The other two cases follow by pattern matching on B, inversion on the respective conversions, and applying the IH. □

4.2 Model

Figure 4.7 describes the model of ς_2 in CDLE. Note that this model is straightforward: abstractions to abstractions, applications to applications, pairs to pairs, etc. The complicated part is the equality type and its constructs, however all the necessary work to find suitable terms for these constructs was already completed above. There is one hiccup involving the ϑ_i constructs. In order to have a fully applied theta_1 it must be the case that the annotation for ϑ_1 is an intersection type. For proofs this will always be the case, but for arbitrary syntax it is not necessarily true. To work around this a catch-all case is also provided where the model only interprets the equality proof e. This choice is largely arbitrary, but it is picked to make sure that one critical property is preserved: erasure.

Lemma 4.5. *If t term then $[\![|t|]\!] = |[\![t]\!]|$*

Proof. By induction on t and inversion on t term. The case of first projection and first equality promotion cases are omitted.

Case: $t = x_\star$

Have $[\![|x_\star|]\!] = [\![x_\star]\!] = x$ and $|[\![x_\star]\!]| = |x| = x$, hence trivial.

Case: $t = \lambda_0\, x : A.\, b$

Have $[\![|\lambda_0\, x : A.\, b|]\!] = [\![|b|]\!]$ and $|[\![\lambda_0\, x : A.\, b]\!]| = |\Lambda\, x.\, [\![b]\!]| = |[\![b]\!]|$. Note that b term, hence by the IH $[\![|b|]\!] = |[\![b]\!]|$.

$$
\begin{aligned}
[\![(x : A) \to_\tau B]\!] &= \Pi\, x : [\![A]\!].\, [\![B]\!] \\
[\![(x : A) \to_\omega B]\!] &= \Pi\, x : [\![A]\!].\, [\![B]\!] \\
[\![(x : A) \to_0 B]\!] &= \forall\, x : [\![A]\!].\, [\![B]\!] \\
[\![\lambda_\tau\, x : A.\, t]\!] &= \lambda\, x : [\![A]\!].\, [\![t]\!] \\
[\![\lambda_\omega\, x : A.\, t]\!] &= \lambda\, x.\, [\![t]\!] \\
[\![\lambda_0\, x : A.\, t]\!] &= \Lambda\, x.\, [\![t]\!]
\end{aligned}
\qquad
\begin{aligned}
[\![\star]\!] &= \star \\
[\![x_K]\!] &= x \\
[\![f \bullet_\tau a]\!] &= [\![f]\!]\ [\![a]\!] && \text{if } a \text{ term} \\
[\![f \bullet_\tau a]\!] &= [\![f]\!] \cdot [\![a]\!] && \text{if } a \text{ type} \\
[\![f \bullet_\omega a]\!] &= [\![f]\!]\ [\![a]\!] \\
[\![f \bullet_0 a]\!] &= [\![f]\!]\ \text{-}[\![a]\!] && \text{if } a \text{ term} \\
[\![f \bullet_0 a]\!] &= [\![f]\!] \cdot [\![a]\!] && \text{if } a \text{ type}
\end{aligned}
$$

$$
\begin{aligned}
[\![(x : A) \cap B]\!] &= \iota\, x : [\![A]\!].\, [\![B]\!] \\
[\![[t_1, t_2, A]]\!] &= [[\![t_1]\!], [\![t_2]\!]]
\end{aligned}
\qquad
\begin{aligned}
[\![t.1]\!] &= [\![t]\!].1 \\
[\![t.2]\!] &= [\![t]\!].2
\end{aligned}
$$

$$
\begin{aligned}
[\![a =_A b]\!] &= \mathrm{Id} \cdot [\![A]\!]\ [\![a]\!]\ [\![b]\!] \\
[\![\mathrm{refl}(t; A)]\!] &= \mathrm{refl} \cdot [\![A]\!]\ \text{-}[\![t]\!] \\
[\![\vartheta_1(e, a, b; (x : A) \cap B)]\!] &= \mathrm{theta}_1 \cdot [\![A]\!] \cdot [\![B]\!]\ \text{-}[\![a]\!]\ \text{-}[\![b]\!]\ [\![e]\!] \\
[\![\vartheta_1(e, a, b; T)]\!] &= [\![e]\!] \\
[\![\vartheta_2(e, a, b; (x : A) \cap B)]\!] &= \mathrm{theta}_2 \cdot [\![A]\!] \cdot [\![B]\!]\ \text{-}[\![a]\!]\ \text{-}[\![b]\!]\ [\![e]\!] \\
[\![\vartheta_2(e, a, b; T)]\!] &= [\![e]\!] \\
[\![\psi(e, a, b; A, P)]\!] &= \mathrm{subst} \cdot [\![A]\!]\ \text{-}[\![a]\!]\ \text{-}[\![b]\!] \cdot [\![P]\!]\ [\![e]\!] \\
[\![\varphi(f, e; A, T)]\!] &= \lambda\, x.\, \varphi\ \text{ç}\ ([\![e]\!]\ x).1 - ([\![f]\!]\ x)\ \{x\} \\
[\![\delta(e)]\!] &= \mathrm{delta}\ [\![e]\!]
\end{aligned}
$$

$$
\begin{aligned}
[\![\varepsilon]\!] &= \varepsilon \\
[\![\Gamma, x : A]\!] &= [\![\Gamma]\!], x : [\![A]\!]
\end{aligned}
$$

Figure 4.7: Model definition interpreting ç_2 in CDLE.

Case: $t = \lambda_\omega\, x : A.\, b$

Have $[\![|\lambda_\omega\, x : A.\, b|]\!] = \lambda\, x.\, [\![|b|]\!]$ and $|[\![\lambda_\omega\, x : A.\, b]\!]| = |\lambda\, x.\, [\![b]\!]| = \lambda\, x.\, |[\![b]\!]|$. Note that b term, hence by the IH $[\![|b|]\!] = |[\![b]\!]|$.

Case: $t = f \bullet_0 a$

Have $[\![|f \bullet_0 a|]\!] = [\![|f|]\!]$ and $|[\![f \bullet_0 a]\!]| = |[\![f]\!]\ \text{-}[\![a]\!]| = |[\![f]\!]|$. Given $f \bullet_0 a$ term it is always the case that f term. Thus, by the IH $[\![|f|]\!] = |[\![f]\!]|$.

Case: $t = f \bullet_\omega a$

Have $[\![|f \bullet_\omega a|]\!] = [\![|f|]\!]\ [\![|a|]\!]$ and $|[\![f \bullet_\omega a]\!]| = |[\![f]\!]|\ |[\![a]\!]|$. Note that f, a term because the mode is ω there is no possibility of a type. Hence, by the IH $[\![|f|]\!] = |[\![f]\!]|$ and $[\![|a|]\!] = |[\![a]\!]|$.

Case: $t = [t_1, t_2; A]$

Have $[\![|[t_1, t_2; A]|]\!] = [\![|t_1|]\!]$ and $|[\![[t_1, t_2; A]]\!]| = |[[\![t_1]\!], [\![t_2]\!]]| = |[\![t_1]\!]|$. By the IH applied to t_1 term: $[\![|t_1|]\!] = |[\![t_1]\!]|$.

Case: $t = t.2$

Have $[\![|t.2|]\!] = [\![|t|]\!]$ and $|[\![t.2]\!]| = |[\![t]\!].2| = |[\![t]\!]|$. By the IH applied to t term: $[\![|t|]\!] = |[\![t]\!]|$.

Case: $t = \text{refl}(a; A)$

Have $[\![|\text{refl}(a; A)|]\!] = [\![\lambda\, x{:}\diamond.\, x_\star]\!] = \lambda\, x.\, x$ and $|[\![\text{refl}(a; A)]\!]| = |\text{refl} \cdot [\![A]\!]\ \text{-}[\![a]\!]| = \lambda\, x.\, x$.

Case: $t = \vartheta_2(e, a, b; T)$

Have $[\![|\vartheta_2(e, a, b; T)|]\!] = [\![|e|]\!]$. Suppose $T = (x : A) \cap B$ then $|[\![\vartheta_2(e, a, b; (x : A) \cap B)]\!]| = |\text{theta}_1 \cdot [\![A]\!] \cdot [\![B]\!]\ \text{-}[\![a]\!]\ \text{-}[\![b]\!]\ [\![e]\!]| = |[\![e]\!]|$. Otherwise, $|[\![\vartheta_2(e, a, b; T)]\!]| = |[\![e]\!]|$. By the IH applied to e term: $[\![|e|]\!] = |[\![e]\!]|$.

Case: $t = \psi(e, a, b; A, P)$

Have $[\![|\psi(e, a, b; A, P)|]\!] = [\![|e|]\!]$ and $|[\![\psi(e, a, b; A, P)]\!]| = |\text{subst} \cdot [\![A]\!]\ \text{-}[\![a]\!]\ \text{-}[\![b]\!] \cdot [\![P]\!]\ [\![e]\!]| = |[\![e]\!]|$. By the IH applied to e term: $[\![|e|]\!] = |[\![e]\!]|$.

Case: $t = \varphi(f, e; A, T)$

Have $[\![|\varphi(f, e; A, T)|]\!] = [\![\lambda\, x : \diamond.\, x_\star]\!] = \lambda\, x.\, x$ and $|[\![\varphi(f, e; A, T)]\!]| = |\lambda\, \varphi([\![e]\!]\ x).1 - ([\![f]\!]\ x)\{x\}.\,| = \lambda\, x.\, x$.

Case: $t = \delta(e)$

Have $[\![|\delta(e)|]\!] = [\![|e|]\!]$ and $|[\![\delta(e)]\!]| = |\text{delta}\ [\![e]\!]| = |[\![e]\!]|$. By the IH applied to e term: $[\![|e|]\!] = |[\![e]\!]|$.

□

To obtain soundness we first need to know that conversion is preserved for the terms, types, and kinds. Luckily, because ç_2 terms are closely matched with CDLE terms lemmas involving reduction can be precise.

Lemma 4.6. $[\![[x := b]t]\!] = [x := [\![b]\!]][\![t]\!]$

Proof. Straightforward by induction on t, substitution is structural with the only exception being variables, but $[\![x_K]\!] = x$. □

Lemma 4.7. *If* t term *and* $|t| \rightsquigarrow t'$ *then* $|[\![t]\!]| \rightsquigarrow [\![t']\!]$

Proof. By induction on t and inversion on t term. The cases: erased lambda, pair, first projection, second projection, promotion (ϑ_i), substitution (ψ), and separation (δ) all erase to a subexpression that is a term. Hence, these cases are very similar to the erased application case and omitted. The erasure of the variable, reflexivity, and cast cases are values and thus do not reduce.

Case: $t = \lambda_\omega\, x : A.\, b$

Have $|\lambda_\omega\, x : A.\, b| = \lambda_\omega\, x : \diamond.\, |b|$ which means $\lambda_\omega\, x : \diamond.\, |b| \rightsquigarrow \lambda_\omega\, x : \diamond.\, b'$. Now b term and $|b| \rightsquigarrow b'$, applying the IH gives $|[\![b]\!]| \rightsquigarrow [\![b']\!]$. Note that $|[\![\lambda_\omega\, x : A.\, b]\!]| = \lambda\, x.\, |[\![b]\!]| \rightsquigarrow \lambda\, x.\, |[\![b']\!]|$. By Lemma 4.6: $|[\![b']\!]| = [\![|b'|]\!]$. However, b' is the result of a contracted redex in an already erased term, hence $|b'| = b'$. Thus, $|[\![\lambda_\omega\, x : A.\, b]\!]| \rightsquigarrow [\![\lambda_\omega\, x : \diamond.\, b']\!]$.

Case: $t = f \bullet_0 a$

Have $|f \bullet_0 a| = |f|$, thus $|f| \rightsquigarrow t'$. Applying the IH gives $|[\![f]\!]| \rightsquigarrow [\![t']\!]$. Note that $|[\![f \bullet_0 a]\!]| = |[\![f]\!]\ \text{-}[\![a]\!]| = |[\![f]\!]|$. Thus, $|[\![f \bullet_0 a]\!]| \rightsquigarrow [\![t']\!]$.

Case: $t = f \bullet_\omega a$

Have $|f \bullet_\omega a| = |f| \bullet_\omega |a|$. Suppose $|f| = \lambda_\omega\, x : \diamond.\, b$ and $|f| \bullet_\omega |a| \rightsquigarrow [x := |a|]b$. Now $|[\![f \bullet_\omega a]\!]| = |[\![f]\!]|\ |[\![a]\!]|$. By Lemma 4.5: $|[\![f]\!]| = [\![|f|]\!] = \lambda\, x.\, [\![b]\!]$. Thus, $(\lambda\, x.\, [\![b]\!])\ |[\![a]\!]| \rightsquigarrow [x := |[\![a]\!]|][\![b]\!]$. Using Lemma 4.5 and Lemma 4.6 gives $[x := |[\![a]\!]|][\![b]\!] = [\![[x := |a|]b]\!]$.

Suppose wlog that $|f| \rightsquigarrow f'$ (the case of $|a| \rightsquigarrow a'$ is very similar). Note that f term, applying the IH gives $|[\![f]\!]| \rightsquigarrow [\![f']\!]$. Now $|[\![f \bullet_\omega a]\!]| = |[\![f]\!]|\ |[\![a]\!]| \rightsquigarrow [\![f']\!]\ |[\![a]\!]| = [\![f' \bullet_\omega |a|]\!]$. The final equality uses Lemma 4.5.

□

Lemma 4.8. *If* t term *and* $|t| \rightsquigarrow^* t'$ *then* $|[\![t]\!]| \rightsquigarrow^* [\![t']\!]$

Proof. By induction on $|t| \rightsquigarrow^* t'$ using Lemma 4.7, Lemma 2.53, and Lemma 2.50. □

Lemma 4.9. *If* a, b term *and* $|a| \rightleftharpoons |b|$ *then* $|[\![a]\!]| \rightleftharpoons |[\![b]\!]|$

Proof. Deconstructing $|a| \rightleftharpoons |b|$ gives $|a| \rightsquigarrow^* z$ and $|b| \rightsquigarrow^* z$. Applying Lemma 4.8 gives $|[\![a]\!]| \rightsquigarrow^* [\![z]\!]$ and $|[\![b]\!]| \rightsquigarrow^* [\![z]\!]$. Thus, $|[\![a]\!]| \rightleftharpoons |[\![b]\!]|$. □

Lemma 4.10. *If* s type *and* $s \rightsquigarrow_n t$ *then* $[\![s]\!] \rightsquigarrow_n [\![t]\!]$

Proof. By induction on s and inversion on s type. Note that only the case where s is a redex is important as all other cases are in weak-head normal form. Thus, suppose $s = f \bullet_\tau a$, $f = \lambda_\tau\, x : A.\, b$, and $f \bullet_\tau a \rightsquigarrow_n [x := a]b$. Suppose wlog that a term. Now $[\![f \bullet_\tau a]\!] = [\![f]\!]\ [\![a]\!] = (\lambda\, x : [\![A]\!].\, [\![b]\!])\ [\![a]\!] \rightsquigarrow [x := [\![a]\!]][\![b]\!]$. Using Lemma 4.6 gives $[x := [\![a]\!]][\![b]\!] = [\![[x := a]b]\!]$. □

Lemma 4.11. *If* s type *and* $s \rightsquigarrow_n^* t$ *then* $[\![s]\!] \rightsquigarrow_n^* [\![t]\!]$

Proof. By induction on $s \rightsquigarrow_n^* t$ using Lemma 4.10 and Lemma 2.53. □

Lemma 4.12.

1. *If* A, B type, A B *are values, and* $A \equiv B$ *then* $[\![A]\!] \cong^t [\![B]\!]$
2. *If* A, B type *and* $A \equiv B$ *then* $[\![A]\!] \cong [\![B]\!]$
3. *If* A, B kind *and* $A \equiv B$ *then* $[\![A]\!] \cong [\![B]\!]$

Proof. By mutual recursion.
1. By induction on A and inversion on A being a value and $A \equiv B$ (hence B must match A). Conversion in ς_1 is structural over weak-head normal forms and in this case A and B must be weak-head normal. Thus, a combination of *1.*, *2.*, *3.*, and Lemma 4.9 on subexpressions in each case is sufficient.
2. By Theorem 3.19, $\exists\ A', B'$ such that $A \rightsquigarrow^* A'$, $B \rightsquigarrow^* B'$ and A', B' are values. Lemma 2.53 gives that A', B' type. Lemma 2.31 gives that $A' \equiv B'$. Thus, applying *1.* concludes.
3. By induction on A and inversion on $A \equiv B$. Again, conversion of kinds is structural in ς_1. Thus, a combination of *2.* and *3.* on subexpressions in each case is sufficient. □

Theorem 4.13 (Soundness of $[\![-]\!]$)**.** *Suppose* $\Gamma \vdash_{\varsigma_2} t : A$

1. *if* $A = \Box$ *then* $[\![\Gamma]\!] \vdash_{\varsigma_1} [\![t]\!]$
2. *if* $\Gamma \vdash_{\varsigma_2} A : \Box$ *then* $[\![\Gamma]\!] \vdash_{\varsigma_1} [\![t]\!] \rhd T$ *and* $T \cong [\![A]\!]$
3. *if* $\Gamma \vdash_{\varsigma_2} A : \star$ *then* $[\![\Gamma]\!] \vdash_{\varsigma_1} [\![t]\!] \lhd [\![A]\!]$

Proof. By induction on $\Gamma \vdash_{\varsigma_2} t : A$. Note that each case is mutually exclusive by classification.

Case: $\dfrac{}{\Gamma \vdash \star : \Box}$

Have $A = \Box$ and $\Gamma \vdash_{\varsigma_1} \star$, hence trivial.

Case: $\dfrac{x \notin FV(\Gamma_1; \Gamma_2) \qquad \Gamma_1 \vdash \overset{\mathcal{D}_2}{A} : K}{\Gamma_1; x_m : A; \Gamma_2 \vdash x_K : A}$ $\mathcal{D}_1$

Let $\Gamma = \Gamma_1; x : A; \Gamma_2$. Have $(x : [\![A]\!]) \in [\![\Gamma]\!]$. Now $[\![\Gamma_1]\!] \vdash_{\varsigma_1} x \rhd [\![A]\!]$ by the IH and $[\![\Gamma]\!] \vdash_{\varsigma_1} x \rhd [\![A]\!]$ by Lemma 4.2. Suppose $K = \Box$ then $[\![A]\!] \cong [\![A]\!]$ and $[\![\Gamma]\!] \vdash_{\varsigma_1} x \rhd [\![A]\!]$. Suppose $K = \star$ then $[\![\Gamma]\!] \vdash_{\varsigma_1} x \lhd [\![A]\!]$.

Case: $\dfrac{\Gamma \vdash \overset{\mathcal{D}_1}{A} : \mathrm{dom}_\Pi(m, K) \qquad \Gamma; x_m : A \vdash \overset{\mathcal{D}_2}{B} : \mathrm{codom}_\Pi(m)}{\Gamma \vdash (x : A) \to_m B : \mathrm{codom}_\Pi(m)}$

Suppose $m = \tau$, then $\mathrm{dom}_\Pi(m, K) = K$ and $\mathrm{codom}_\Pi(m) = \Box$. Applying the IH gives:

$\mathcal{D}_1.\ [\![\Gamma]\!] \vdash_{\varsigma_1} [\![A]\!]$ if $K = \Box$

$\mathcal{D}_1.\ [\![\Gamma]\!] \vdash_{\varsigma_1} [\![A]\!] \rhd \star$ if $K = \star$

$\mathcal{D}_2.\ [\![\Gamma]\!], x : [\![A]\!] \vdash_{\varsigma_1} [\![B]\!]$

The corresponding Π rule for the two possibilities of K concludes the case.

Suppose $m = 0$, then $\mathrm{dom}_\Pi(m, K) = K$ and $\mathrm{codom}_\Pi(m) = \star$. Applying the IH gives:

$\mathcal{D}_1.\ [\![\Gamma]\!] \vdash_{\varsigma_1} [\![A]\!]$ if $K = \Box$

$\mathcal{D}_1.\ [\![\Gamma]\!] \vdash_{\varsigma_1} [\![A]\!] \rhd \star$ if $K = \star$

$\mathcal{D}_2.\ [\![\Gamma]\!], x : [\![A]\!] \vdash_{\varsigma_1} [\![B]\!] \rhd \star$

The corresponding $\forall$ rule for the two possibilities of K concludes the case.

Suppose $m = \omega$, then $\mathrm{dom}_\Pi(m, K) = \star$ and $\mathrm{codom}_\Pi(m) = \star$. Applying the IH gives:

$\mathcal{D}_1.\ [\![\Gamma]\!] \vdash_{\varsigma_1} [\![A]\!] \rhd \star$

$\mathcal{D}_2.\ [\![\Gamma]\!], x : [\![A]\!] \vdash_{\varsigma_1} [\![B]\!] \rhd \star$

The corresponding Π rule concludes the case.

Case: $$\frac{\Gamma \vdash (x : A) \to_m \overset{\mathcal{D}_1}{B} : \mathrm{codom}_\Pi(m) \qquad \Gamma; x_m : \overset{\mathcal{D}_2}{A} \vdash t : B \qquad x \notin FV(|t|) \overset{\mathcal{D}_3}{\text{ if } m = 0}}{\Gamma \vdash \lambda_m\, x{:}A.\, t : (x : A) \to_m B}$$

Suppose $m = \tau$, then $\mathrm{codom}_\Pi(m) = \Box$. Note that this means that t type. Applying the IH gives:

$\mathcal{D}_1.\ [\![\Gamma]\!] \vdash_{\varsigma_1} \Pi\, x{:}[\![A]\!].\, [\![B]\!]$

$\mathcal{D}_2.\ [\![\Gamma]\!], x : [\![A]\!] \vdash_{\varsigma_1} [\![t]\!] \rhd T$ and $T \cong [\![B]\!]$

Suppose $[\![\Gamma]\!] \vdash_{\varsigma_1} [\![A]\!]$, then $[\![\Gamma]\!] \vdash_{\varsigma_1} \lambda\, x{:}[\![A]\!].\, [\![t]\!] \rhd \Pi\, x{:}[\![A]\!].\, T$. By rules of conversion for kinds yields $\Pi\, x{:}[\![A]\!].\, T \cong \Pi\, x{:}[\![A]\!].\, [\![B]\!]$. The case where $[\![A]\!]$ is a type instead of a kind is similar.

Suppose $m = 0$, then $\mathrm{codom}_\Pi(m) = \star$. Note that this means t term. Applying the IH gives:

$\mathcal{D}_1.\ [\![\Gamma]\!] \vdash_{\varsigma_1} \Pi\, x{:}[\![A]\!].\, [\![B]\!] \rhd \star$

$\mathcal{D}_2.\ [\![\Gamma]\!], x : [\![A]\!] \vdash_{\varsigma_1} [\![t]\!] \lhd [\![B]\!]$

Note that $FV(|[\![t]\!]|) \subseteq FV(|t|)$, thus $x \notin FV(|[\![t]\!]|)$. Using the corresponding Λ rule based on the classification of $[\![A]\!]$ concludes the case.

Suppose $m = \omega$, then $\mathrm{codom}_\Pi(m) = \star$. This case is omitted because the previous case is a more general version of it.

Case: $\dfrac{\Gamma \vdash f : (x : \overset{\mathcal{D}_1}{A}) \to_m B \qquad \Gamma \vdash \overset{\mathcal{D}_2}{a} : A}{\Gamma \vdash f \bullet_m a : [x := a]B}$

Suppose $m = \tau$. Classification forces f type, but a is either a term or a type. Applying the IH gives:

$\mathcal{D}_1$. $[\![\Gamma]\!] \vdash_{\varsigma_1} [\![f]\!] \rhd T$ with $T \cong \Pi\, x{:}[\![A]\!].\, [\![B]\!]$

$\mathcal{D}_2$. $[\![\Gamma]\!] \vdash_{\varsigma_1} [\![a]\!] \rhd T_2$ with $T_2 \cong [\![A]\!]$ if a type

$\mathcal{D}_2$. $[\![\Gamma]\!] \vdash_{\varsigma_1} [\![a]\!] \lhd [\![A]\!]$ if a term

Note that because kinds cannot reduce, it must be the case that $\exists\, C, D$ such that $T = \Pi\, x : C.\, D$. Moreover, $C \cong [\![A]\!]$ and $D \cong [\![B]\!]$ by the conversion rules. Suppose a type then using the associated rule yields $[\![\Gamma]\!] \vdash_{\varsigma_1} [\![f]\!] \cdot [\![a]\!] \rhd [x := [\![a]\!]]D$. Now, $[x := [\![a]\!]]D \cong [x := [\![a]\!]][\![B]\!]$ and the case is concluded. Suppose a term then using the associated rule yields $[\![\Gamma]\!] \vdash_{\varsigma_1} [\![f]\!]\ [\![a]\!] \rhd [x := \chi\ C - [\![a]\!]]D$. Again, $[x := \chi\ C - [\![a]\!]]D \cong [x := \chi\ C - [\![a]\!]][\![B]\!]$ and the case is concluded. Note that $[x := \chi\ C - [\![a]\!]][\![B]\!] \cong [x := [\![a]\!]][\![B]\!]$ because the χ is only well-typed in term positions, where it is promptly erased during conversion checking.

Suppose $m = 0$. Classification forces f term, but a is either a term or a type. Applying the IH gives:

$\mathcal{D}_1$. $[\![\Gamma]\!] \vdash_{\varsigma_1} [\![f]\!] \lhd \forall\, x{:}[\![A]\!].\, [\![B]\!]$

$\mathcal{D}_2$. $[\![\Gamma]\!] \vdash_{\varsigma_1} [\![a]\!] \rhd T_2$ with $T_2 \cong [\![A]\!]$ if a type

$\mathcal{D}_2$. $[\![\Gamma]\!] \vdash_{\varsigma_1} [\![a]\!] \lhd [\![A]\!]$ if a term

Deconstructing the checking judgment for $[\![f]\!]$ yields $\exists\, C, D$ such that $[\![\Gamma]\!] \vdash_{\varsigma_1} [\![f]\!] \blacktriangleright \forall\, x : C.\, D$ and $C \cong [\![A]\!]$ and $D \cong [\![B]\!]$. Suppose a type then the associated judgment gives $[\![\Gamma]\!] \vdash_{\varsigma_1} [\![f]\!] \cdot [\![a]\!] \rhd [x := [\![a]\!]]D$. Now, $[x := [\![a]\!]]D \cong [x := [\![a]\!]][\![B]\!]$ and the case is concluded. Suppose a term then the associated judgment gives $[\![\Gamma]\!] \vdash_{\varsigma_1} [\![f]\!]$ -$[\![a]\!] \rhd [x := \chi\ C - [\![a]\!]]D$. Again, $[x := \chi\ C - [\![a]\!]]D \cong [x := \chi\ C - [\![a]\!]][\![B]\!]$ and the case is concluded.

Suppose $m = \omega$ Classification forces f, a term. Applying the IH gives:

$\mathcal{D}_1$. $[\![\Gamma]\!] \vdash_{\varsigma_1} [\![f]\!] \lhd \Pi\, x{:}[\![A]\!].\, [\![B]\!]$

$\mathcal{D}_2$. $[\![\Gamma]\!] \vdash_{\varsigma_1} [\![a]\!] \lhd [\![A]\!]$ if a term

As with the previous case, $\exists\, C, D$ such that $[\![\Gamma]\!] \vdash_{\varsigma_1} [\![f]\!] \blacktriangleright \Pi\, x{:}C.\, D$ and $C \cong [\![A]\!]$ and $D \cong [\![B]\!]$. Applying the associated rule yields $[\![\Gamma]\!] \vdash_{\varsigma_1} [\![f]\!]\ [\![a]\!] \rhd [x := \chi\ C - [\![a]\!]]D$. Now, $[x := \chi\ C - [\![a]\!]]D \cong [x := \chi\ C - [\![a]\!]][\![B]\!]$ and the case is concluded.

Case: $\dfrac{\Gamma \vdash \overset{\mathcal{D}_1}{A} : \star \qquad \Gamma; x_\tau : \overset{\mathcal{D}_2}{A} \vdash B : \star}{\Gamma \vdash (x : A) \cap B : \star}$

Applying the IH gives:

$\mathcal{D}_1.\ [\![\Gamma]\!] \vdash_{\varsigma_1} [\![A]\!] \rhd \star$

$\mathcal{D}_2.\ [\![\Gamma]\!], x : [\![A]\!] \vdash_{\varsigma_1} [\![B]\!] \rhd \star$

Thus, $[\![\Gamma]\!] \vdash_{\varsigma_1} \iota\, x{:}[\![A]\!].\, [\![B]\!] \rhd \star$ as required.

Case: $\dfrac{\Gamma \vdash (x : \overset{\mathcal{D}_1}{A}) \cap B : \star \qquad \Gamma \vdash \overset{\mathcal{D}_2}{t} : A \qquad \Gamma \vdash s : \overset{\mathcal{D}_3}{[x := t]B} \qquad t \overset{\mathcal{D}_4}{\equiv} s}{\Gamma \vdash [t, s; (x : A) \cap B] : (x : A) \cap B}$

Note by classification and $\mathcal{D}_1$: $\Gamma \vdash A : \star$ and $\Gamma, x : A \vdash B : \star$. Applying the IH gives:

$\mathcal{D}_1.\ [\![\Gamma]\!] \vdash_{\varsigma_1} \iota\, x{:}[\![A]\!].\, [\![B]\!] \rhd \star$

$\mathcal{D}_2.\ [\![\Gamma]\!] \vdash_{\varsigma_1} [\![t]\!] \lhd [\![A]\!]$

$\mathcal{D}_3.\ [\![\Gamma]\!] \vdash_{\varsigma_1} [\![s]\!] \lhd [x := [\![t]\!]][\![B]\!]$

Note that $[\![\Gamma]\!] \vdash_{\varsigma_1} [\![t]\!] \lhd [\![A]\!]$ so clearly $[\![\Gamma]\!] \vdash_{\varsigma_1} [\![s]\!] \lhd [x := \chi\ [\![A]\!]\ -\ [\![t]\!]][\![B]\!]$ as the χ merely adds extra typing information. Lemma 4.9 applied to $\mathcal{D}_4$ and using the fact that t, s term gives $|[\![t]\!]| \rightleftharpoons |[\![s]\!]|$. Combining this information yields $[\![\Gamma]\!] \vdash [\![[t, s; A]]\!] \lhd [\![(x : A) \cap B]\!]$.

Case: $\dfrac{\Gamma \vdash t : (\overset{\mathcal{D}_1}{x} : A) \cap B}{\Gamma \vdash t.1 : A}$

By classification t term. Applying the IH to $\mathcal{D}_1$ gives $[\![\Gamma]\!] \vdash_{\varsigma_1} [\![t]\!] \lhd \iota\, x : [\![A]\!].\, [\![B]\!]$. Deconstruct this checking rule and notice that either the inferred type is already an intersection or it must reduce to an intersection. Thus, $\exists\ C\ D$ such that $[\![\Gamma]\!] \vdash_{\varsigma_1} [\![t]\!] \blacktriangleright \iota\, x : [\![C]\!].\, [\![D]\!]$ and $\iota\, x : [\![C]\!].\, [\![D]\!] \cong \iota\, x : [\![A]\!].\, [\![B]\!]$. Deconstructing the congruence yields $[\![C]\!] \cong [\![A]\!]$. Thus, $[\![\Gamma]\!] \vdash_{\varsigma_1} [\![t]\!].1 \lhd [\![A]\!]$

Case: $\dfrac{\Gamma \vdash t : (\overset{\mathcal{D}_1}{x} : A) \cap B}{\Gamma \vdash t.2 : [x := t.1]B}$

By classification t term. Applying the IH to $\mathcal{D}_1$ gives $[\![\Gamma]\!] \vdash_{\varsigma_1} [\![t]\!] \lhd \iota\, x : [\![A]\!].\, [\![B]\!]$. Deconstruct this checking rule and notice that either the inferred type is already an intersection or it must reduce to an intersection. Thus, $\exists\ C\ D$ such that $[\![\Gamma]\!] \vdash_{\varsigma_1} [\![t]\!] \blacktriangleright \iota\, x : [\![C]\!].\, [\![D]\!]$ and $\iota\, x : [\![C]\!].\, [\![D]\!] \cong \iota\, x : [\![A]\!].\, [\![B]\!]$. Deconstructing the congruence yields $[\![D]\!] \cong [\![B]\!]$ and thus $[x := [\![t]\!].1][\![D]\!] \cong [x := [\![t]\!].1][\![B]\!]$. Now $[\![\Gamma]\!] \vdash_{\varsigma_1} [\![t]\!].2 \rhd [x := [\![t]\!].1][\![D]\!]$. Thus, $[\![\Gamma]\!] \vdash_{\varsigma_1} [\![t]\!].2 \lhd [x := [\![t]\!].1][\![B]\!]$.

Case: $\dfrac{\Gamma \vdash \overset{\mathcal{D}_1}{A} : \star \qquad \Gamma \vdash \overset{\mathcal{D}_2}{a} : A \qquad \Gamma \vdash \overset{\mathcal{D}_2}{b} : A}{\Gamma \vdash a =_A b : \star}$

Note that a, b term by $\mathcal{D}_1$. Applying the IH gives:

$\mathcal{D}_1$. $[\![\Gamma]\!] \vdash_{\varsigma_1} [\![A]\!] \rhd \star$

$\mathcal{D}_2$. $[\![\Gamma]\!] \vdash_{\varsigma_1} [\![a]\!] \lhd [\![A]\!]$

$\mathcal{D}_3$. $[\![\Gamma]\!] \vdash_{\varsigma_1} [\![b]\!] \lhd [\![A]\!]$

By Lemma 4.1, Lemma 4.2, and the application rule for ς_1: $[\![\Gamma]\!] \vdash_{\varsigma_1} \mathrm{Id} \cdot [\![A]\!]\ [\![a]\!]\ [\![b]\!] \rhd \star$.

Case: $\dfrac{\Gamma \vdash \overset{\mathcal{D}_1}{A} : \star \qquad \Gamma \vdash \overset{\mathcal{D}_2}{t} : A}{\Gamma \vdash \mathrm{refl}(t; A) : t =_A t}$

Note that t term by $\mathcal{D}_1$. Applying the IH gives:

$\mathcal{D}_1$. $[\![\Gamma]\!] \vdash_{\varsigma_1} [\![A]\!] \rhd \star$

$\mathcal{D}_2$. $[\![\Gamma]\!] \vdash_{\varsigma_1} [\![t]\!] \lhd [\![A]\!]$

By Lemma 4.1, Lemma 4.2, and the application rule for ς_1: $[\![\Gamma]\!] \vdash_{\varsigma_1} \mathrm{refl} \cdot [\![A]\!]\ \text{-}[\![t]\!] \rhd \mathrm{Id} \cdot [\![A]\!]\ [\![t]\!]\ [\![t]\!]$.

Case: $\dfrac{\Gamma \vdash \overset{\mathcal{D}_1}{A} : \star \qquad \Gamma \vdash \overset{\mathcal{D}_2}{a} : A \qquad \Gamma \vdash \overset{\mathcal{D}_3}{b} : A \qquad \Gamma \vdash \overset{\mathcal{D}_4}{e} : a =_A b \qquad \Gamma \vdash \overset{\mathcal{D}_5}{P} : (y : A) \to_\tau (p : a =_A y_\star) \to_\tau \star}{\Gamma \vdash \psi(e, a, b; A, P) : P \bullet_\tau a \bullet_\tau \mathrm{refl}(a; A) \to_\omega P \bullet_\tau b \bullet_\tau e}$

Note by $\mathcal{D}_1$ that a, b term and by classification e term with A, P type. Applying the IH gives:

$\mathcal{D}_1$. $[\![\Gamma]\!] \vdash_{\varsigma_1} [\![A]\!] \rhd \star$

$\mathcal{D}_2$. $[\![\Gamma]\!] \vdash_{\varsigma_1} [\![a]\!] \lhd [\![A]\!]$

$\mathcal{D}_3$. $[\![\Gamma]\!] \vdash_{\varsigma_1} [\![b]\!] \lhd [\![A]\!]$

$\mathcal{D}_4$. $[\![\Gamma]\!] \vdash_{\varsigma_1} [\![e]\!] \lhd \mathrm{Id} \cdot [\![A]\!]\ [\![a]\!]\ [\![b]\!]$

$\mathcal{D}_5$. $[\![\Gamma]\!] \vdash_{\varsigma_1} [\![P]\!] \rhd T$ and $T \cong \forall\, y : [\![A]\!].\, \mathrm{Id} \cdot [\![A]\!]\ [\![a]\!]\ [\![y]\!] \to \star$

By Lemma 4.1, Lemma 4.2, and the application rule for ς_1: $[\![\Gamma]\!] \vdash_{\varsigma_1} \mathrm{subst} \cdot [\![A]\!]\ \text{-}[\![a]\!]\ \text{-}[\![b]\!] \cdot [\![P]\!]\ [\![e]\!] \rhd [\![P]\!]\ [\![a]\!]\ (\mathrm{refl} \cdot [\![A]\!]\ \text{-}[\![a]\!]) \to [\![P]\!]\ [\![b]\!]\ [\![e]\!]$.

Case: $\dfrac{\Gamma \vdash \overset{\mathcal{D}_1}{(x : A)} \cap B : \star \qquad \Gamma \vdash a : \overset{\mathcal{D}_2}{(x : A)} \cap B \qquad \Gamma \vdash b : \overset{\mathcal{D}_3}{(x : A)} \cap B \qquad \Gamma \vdash e : \overset{\mathcal{D}_4}{a.1} =_A b.1}{\Gamma \vdash \vartheta(e, a, b; (x : A) \cap B) : a =_{(x:A)\cap B} b}$

Note by $\mathcal{D}_1$ that a, b term and by classification e term with $(x : A) \cap B$ type. Applying the IH gives:

$\mathcal{D}_1$. $[\![\Gamma]\!] \vdash_{\varsigma_1} \iota\, x : [\![A]\!].\, [\![B]\!] \rhd \star$

$\mathcal{D}_2$. $[\![\Gamma]\!] \vdash_{\varsigma_1} [\![a]\!] \lhd \iota\, x : [\![A]\!].\, [\![B]\!]$

$\mathcal{D}_3$. $[\![\Gamma]\!] \vdash_{\varsigma_1} [\![b]\!] \lhd \iota\, x : [\![A]\!].\, [\![B]\!]$

$\mathcal{D}_4$. $[\![\Gamma]\!] \vdash_{\varsigma_1} [\![e]\!] \lhd \mathrm{Id} \cdot [\![A]\!]\ [\![a]\!].1\ [\![b]\!].1$

Note that $[\![\Gamma]\!], x : [\![A]\!] \vdash_{\varsigma_1} [\![B]\!] \rhd \star$ which means $[\![\Gamma]\!] \vdash_{\varsigma_1} [\![B]\!] \rhd A \to \star$. By Lemma 4.1, Lemma 4.2, and the application rule for ς_1: $[\![\Gamma]\!] \vdash_{\varsigma_1} \mathrm{theta}_1 \cdot [\![A]\!] \cdot [\![B]\!]$ -$[\![a]\!]$ -$[\![b]\!]$ $[\![e]\!] \rhd \mathrm{Id} \cdot (\iota\, x{:}[\![A]\!].\, [\![B]\!])\ [\![a]\!]\ [\![b]\!]$.

Case: $$\frac{\Gamma \vdash (x : \overset{\mathcal{D}_1}{A}) \cap B : \star \qquad \Gamma \vdash \overset{\mathcal{D}_2}{a} : A \qquad \Gamma \vdash b : \overset{\mathcal{D}_3}{(x : A) \cap B} \qquad \Gamma \vdash e : \overset{\mathcal{D}_4}{a} =_A b.1}{\Gamma \vdash \varphi(a, b, e; A, (x : A) \cap B) : (x : A) \cap B}$$

Note by $\mathcal{D}_1$ that f term and by classification e term with T type Applying the IH gives:

$\mathcal{D}_1$. $[\![T]\!] = [\![A]\!] \to \iota\, x{:}[\![A]\!].\, [\![B]\!]$

$\mathcal{D}_2$. $[\![\Gamma]\!] \vdash_{\varsigma_1} [\![T]\!] \rhd \star$

$\mathcal{D}_3$. $[\![\Gamma]\!] \vdash_{\varsigma_1} [\![f]\!] \lhd [\![T]\!]$

$\mathcal{D}_4$. $[\![\Gamma]\!] \vdash_{\varsigma_1} [\![e]\!] \lhd \Pi\, a{:}[\![A]\!].\, \mathrm{Id} \cdot [\![A]\!]\ a\ ([\![f]\!]\ [\![a]\!]).1$

Note that $[\![\varphi(f, e; A, T)]\!] = \lambda\, x.\ \cdots$ and thus by the lambda rule of ς_1 the goal is to show $[\![\Gamma]\!], x : [\![A]\!] \vdash_{\varsigma_1} \varphi\ \varsigma\ ([\![e]\!]\ x).1 - ([\![f]\!]\ x)\ \{x\} \lhd \iota\, x{:}[\![A]\!].\, [\![B]\!]$. By the application and first projection rule and some maneuvering of type conversion: $[\![\Gamma]\!], x : [\![A]\!] \vdash_{\varsigma_1} ([\![e]\!]\ x).1 \lhd \{a \cong (f\ a).1\}$. Using the same manipulations, except now for f, gives $[\![\Gamma]\!], x : [\![A]\!] \vdash_{\varsigma_1} ([\![f]\!]\ x).1 \lhd \iota\, x{:}[\![A]\!].\, [\![B]\!]$. Note that obviously $FV(x) \subseteq dom(\Gamma, x : [\![A]\!])$. Thus, the goal is obtained by the $\simeq$ symmetry and φ rule of ς_1.

Case: $$\frac{\Gamma \vdash e : \mathrm{ctt} \overset{\mathcal{D}_1}{=}_{\mathrm{cBool}} \mathrm{cff}}{\Gamma \vdash \delta(e) : (X : \star) \to_0 X_\square}$$

Applying the IH to $\mathcal{D}_1$ yields $[\![\Gamma]\!] \vdash_{\varsigma_1} \mathrm{IdBoolttff}$. By Lemma 4.1, Lemma 4.2, and the application rule for ς_1: $[\![\Gamma]\!] \vdash_{\varsigma_1} \mathrm{delta}\ [\![e]\!] \rhd \forall\, X{:}\star.\, X$.

Case: $$\frac{\Gamma \vdash \overset{\mathcal{D}_1}{A} : K \qquad \Gamma \vdash \overset{\mathcal{D}_2}{t} : B \qquad \overset{\mathcal{D}_3}{A \equiv B}}{\Gamma \vdash t : A}$$

Suppose $K = \square$. Then by classification and $\mathcal{D}_3$: $\Gamma \vdash B : \square$. Applying the IH to $\mathcal{D}_2$ gives $[\![\Gamma]\!] \vdash [\![t]\!] \rhd T$ with $T \cong [\![B]\!]$. By Lemma 4.12: $[\![A]\!] \cong [\![B]\!]$. Now by Lemma 4.4 and Lemma 4.3: $T \cong [\![B]\!]$.

Suppose $K = \star$. Then by classification and $\mathcal{D}_3$: $\Gamma \vdash B : \star$. Applying the IH to $\mathcal{D}_2$ gives $[\![\Gamma]\!] \vdash [\![t]\!] \rhd [\![B]\!]$. By Lemma 4.12 and Lemma 4.3: $[\![B]\!] \cong [\![A]\!]$. Applying the checking rule of ς_1 yields $[\![\Gamma]\!] \vdash [\![t]\!] \lhd [\![A]\!]$.

□

Theorem 4.14 (Logical Consistency)**.** $\neg(\vdash_{c_2} t : (X : \star) \to_0 X_\square)$

Proof. Proceed using proof by negation. Suppose $\vdash_{c_2} t : (X : \star) \to_0 X_\square$. By Theorem 4.13: $\vdash_{\varsigma_1} [\![t]\!] \lhd \forall\, X{:}\star.\, X$. However, this is impossible by consistency of ς_1. □

Corollary 4.15 (Equational Consistency)**.** $\neg(\vdash_{c_2} t : \mathrm{ctt} =_{\mathrm{cBool}} \mathrm{cff})$

4.3 Irrelevance Axiom

The irrel definition communicates that the axiom $i : (A : \star) \to_0 (a\ b : A) \to_0 a =_A b \to_0 a =_A b$ with the erasure $|i| = \lambda\, x.\, x$ may be added to ς_2 without breaking consistency. Indeed, this axiom would not introduce any proof reduction steps and thus proofs would still normalize. The model could then be trivially extended to interpret i as $[\![i]\!] = \mathrm{irrel}$. This irrelevance axiom enables an unrestricted φ rule:

$$\begin{aligned}
&\mathrm{cast} : (A : \star) \to_0 (B : A_\square \to_\tau \star) \\
&\quad \to_0 (f : A_\square \to_\omega (x : A_\square) \cap B_\square \bullet_\tau x_\star) \\
&\quad \to_0 (e : (a : A_\square) \to_\omega a_\star =_{A_\square} (f_\star \bullet_\omega a_\star).1) \\
&\quad \to_0 A_\square \to_\omega (x : A_\square) \cap B_\square \bullet_\tau x_\star
\end{aligned}$$

$$\begin{aligned}
&\mathrm{cast}\ A\ B\ f\ e = \varphi(f, \\
&\quad \lambda\, a{:}A. \\
&\quad i \bullet_0 A \bullet_0 a \bullet_0 (f \bullet_\omega a_\star).1 \bullet_0 (e \bullet_\omega a_\star); \\
&\quad A, A \to_\omega (x : A) \cap B \bullet_\tau x_\star)
\end{aligned}$$

Note that cast is well-typed because the erasure of i discards all the used subexpressions: A, B, f, and e. Thus, $FV(|i \bullet_0 A \bullet_0 a \bullet_0 (f \bullet_\omega a_\star).1 \bullet_0 (e \bullet_\omega a_\star)|)$ is empty. However, because this axiom side-steps the free variable condition it can be used to produce proofs whose objects are not normalizing. Indeed, as Abel and Coquand remark: normalization is lost when irrelevance of equality evidence is combined with impredicative quantification [1].

Chapter 5

OBJECT NORMALIZATION

Consistency guarantees that the logic and equational theory of ς_2 is non-trivial. Proof normalization guarantees that, at least, inference for kinds and types is decidable. Neither of these properties are strong enough on their own to guarantee decidability of type checking. To obtain decidability of type checking it must be the case that objects are normalizing. However, the technique described below depends on both proof normalization and consistency. While this is not the only possible way it is suggestive how difficult a property object normalization is to prove.

The technique for proving object normalization proceeds in two parts. First, a notion of strict syntax is defined where φ forms are eliminated. Proofs will yield strict proofs and utilizing both proof normalization and consistency one can show that proof reduction bounds strict object reduction. Hence, strict objects will be strongly normalizing. Second, a notion of well-typed observational equivalence where the only observation of interest is divergence of objects is defined. Then it is demonstrated that proofs are observational equivalent to their strict version. Thus, because strict objects are normalizing, it must be the case that objects are normalizing.

5.1 Strict Syntax

An important observation to make about φ forms is that they do not expand the logical theory. Meaning, it is not possible to prove *new* propositions or inhabit *new* types. Instead, the φ form is meant primarily as a means of obtaining efficient objects for an already proven property. To that end a strict variation of syntax is easily defined on arbitrary syntax by replacing any φ form with its internal proof witness. Figure 5.1 describes strictification of syntax. Notice that $\mathfrak{s}(t)$ is structural on all syntax forms *except* the φ form where everything but the internal proof witness is discarded.

An immediate consequence of this definition is that strict proofs remain proofs of the original type. Note, however, that the type itself is not made strict. Strictification of the type would not hold because φ constructs inside the type may be compared via the conversion relation and strict

$$\begin{aligned}
\mathfrak{s}(x_K) &= x_K \\
\mathfrak{s}(\mathfrak{b}(\kappa_1, x : t_1, t_2)) &= \mathfrak{b}(\kappa_1, x : \mathfrak{s}(t_1), \mathfrak{s}(t_2)) \\
\mathfrak{s}(\mathfrak{c}(\kappa_2, t_1, \ldots, t_{\mathfrak{a}(\kappa_2)})) &= \mathfrak{c}(\kappa_2, \mathfrak{s}(t_1), \ldots, \mathfrak{s}(t_{\mathfrak{a}(\kappa_2)})) \text{ if } \kappa_2 \neq \varphi \\
\mathfrak{s}(\varphi(f, e; A, T)) &= \mathfrak{s}(f)
\end{aligned}$$

Figure 5.1: Strictification of raw syntax to remove any φ subexpressions.

objects are not equal to the corresponding object.

Lemma 5.1. *If* $\Gamma \vdash t : A$ *then* $\Gamma \vdash \mathfrak{s}(t) : A$

Proof. By induction on $\Gamma \vdash t : A$, because $\mathfrak{s}$ is structural the φ rule is the only one of interest:

Case: $$\frac{\Gamma \vdash (x : \overset{\mathcal{D}_1}{A}) \cap B : \star \qquad \Gamma \vdash \overset{\mathcal{D}_2}{a} : A \qquad \Gamma \vdash b : (\overset{\mathcal{D}_3}{x} : A) \cap B \qquad \Gamma \vdash e : \overset{\mathcal{D}_4}{a} =_A b.1}{\Gamma \vdash \varphi(a, b, e; A, (x : A) \cap B) : (x : A) \cap B}$$

Immediate by IH applied to $\mathcal{D}_3$.

□

Of course, if strict syntax reduces, then the resulting syntax must be strict. This is an obvious consequence of the fact that reduction never introduces φ forms.

Lemma 5.2. *If* $\mathfrak{s}(s) \rightsquigarrow t$ *then* $t = \mathfrak{s}(t)$

Proof. Without φ constructs $\mathfrak{s}(s) = s$ by a trivial induction. Reduction cannot produce φ constructs, thus t necessarily does not have φ as a subexpression. Hence, $\mathfrak{s}(t) = t$. □

The core observation is that proof reduction in strict proofs upper-bounds reduction in strict objects. Thus, if a strict object steps, and note that this must be a β-step, then there is some strict proof such that the original strict proof reduces to it and the erasures match. There could be many more reductions in the strict proof because syntax forms for equality and intersections are all mostly erased. However, none of these forms will block a β-redex because the proof is well-typed. Note that this property hinges on both proof normalization and equational consistency. Proof normalization is used to eliminate any extraneous redexes that would otherwise be erased. Consistency is used to eliminate the δ case as it could theoretically generate a β-redex after erasure if the theory was not consistent. Of course, φ could also generate a β-redex after erasure, but this is impossible because the syntax under consideration is strict.

Lemma 5.3. *If* $\Gamma \vdash s : A$ *and* $|\mathfrak{s}(s)| \rightsquigarrow t$ *then* $\exists\ t'$ *such that* $\mathfrak{s}(s) \rightsquigarrow^*_{\neq 0} t'$ *and* $|t'| = t$

Proof. By induction on $\Gamma \vdash s : A$. The erasure of the Ax, Var, Refl, and Cast cases are values and thus do not reduce. First projection and first promotion are very similar to second projection and second promotion (respectively). The Int and Eq cases are structural in both erasure and strictification and are thus very similar to the Pi case.

Case: $$\frac{\Gamma \vdash A : \overset{\mathcal{D}_1}{\mathrm{dom}_\Pi}(m, K) \qquad \Gamma; x_m : A \vdash \overset{\mathcal{D}_2}{B} : \mathrm{codom}_\Pi(m)}{\Gamma \vdash (x : A) \to_m B : \mathrm{codom}_\Pi(m)}$$

Have $|\mathfrak{s}((x : A) \to_m B)| = (x : |\mathfrak{s}(A)|) \to_m |\mathfrak{s}(B)|$. Suppose that $|\mathfrak{s}(A)| \rightsquigarrow t$. By the IH applied to $\mathcal{D}_1$: $\exists\ t'$ such that $\mathfrak{s}(A) \rightsquigarrow^*_{\neq 0} t'$ and $|t'| = t$. Thus, $(x : \mathfrak{s}(A)) \to_m \mathfrak{s}(B) \rightsquigarrow^*_{\neq 0} (x : t') \to_m \mathfrak{s}(B)$ and $|(x : t') \to_m \mathfrak{s}(B)| = (x : t) \to_m |\mathfrak{s}(B)|$. The case where a reduction happens in $|\mathfrak{s}(B)|$ is similar.

Case: $\dfrac{\Gamma \vdash (x : A) \to_m \overset{\mathcal{D}_1}{B} : \mathrm{codom}_\Pi(m) \qquad \Gamma; x_m : \overset{\mathcal{D}_2}{A} \vdash t : B \qquad x \notin FV(\overset{\mathcal{D}_3}{|t|}) \text{ if } m = 0}{\Gamma \vdash \lambda_m\, x{:}A.\, t : (x : A) \to_m B}$

Suppose $m = 0$. Have $|\mathfrak{s}(\lambda_0\, x : A.\, b|)| = |\mathfrak{s}(b)|$ with $|\mathfrak{s}(b)| \rightsquigarrow t$. Applying the IH to $\mathcal{D}_2$ concludes the case.

Suppose that $m = \omega$, note that $m = \tau$ is very similar and thus omitted. Have $|\mathfrak{s}(\lambda_\omega\, x : A.\, b)| = \lambda_\omega\, x : \diamond.\, |\mathfrak{s}(b)|$ and $|\mathfrak{s}(b)| \rightsquigarrow t$. Applying the IH to $\mathcal{D}_2$ yields $\exists\ t'$ such that $\mathfrak{s}(b) \rightsquigarrow^*_{\neq 0} t'$ and $|t'| = t$. Now $\lambda_\omega\, x : \mathfrak{s}(A).\, \mathfrak{s}(b) \rightsquigarrow^*_{\neq 0} \lambda_\omega\, x : \mathfrak{s}(A).\, t'$ and $|\lambda_\omega\, x{:}\mathfrak{s}(A).\, t'| = \lambda_\omega\, x{:}\diamond.\, t$.

Case: $\dfrac{\Gamma \vdash f : \overset{\mathcal{D}_1}{(x : A)} \to_m B \qquad \Gamma \vdash \overset{\mathcal{D}_2}{a} : A}{\Gamma \vdash f \bullet_m a : [x := a]B}$

If $m = 0$ then the proof follows by a straightforward application of the IH to $\mathcal{D}_1$.

Suppose that $m = \omega$. Let $|\mathfrak{s}(f)| = \lambda_\omega\, x : \diamond.\, v$ and $|\mathfrak{s}(f)| \bullet_\omega |\mathfrak{s}(a)| \rightsquigarrow [x := |\mathfrak{s}(a)|]v$. By Theorem 3.19 $\mathfrak{s}(f)$ is strongly normalizing in proof reduction. If f contains a projection redex, promotion redex, or erased application redex then produce f_i by contracting that redex. Continue contracting these redexes until none remain, assume k such redexes are contracted, thus $\mathfrak{s}(f) \rightsquigarrow^* f_k$. Note that none of these redexes affect the erasure of f, thus $|\mathfrak{s}(f)| = |f_k|$. Now f_k has only four possibilities: $f_k = \lambda_\omega\, x{:}A.\, b$, or $f_k = \psi(\mathrm{refl}(z; Z), a, b; A, P)$, or $f_k = \delta(\mathrm{refl}(t; A))$, or $f_k = \varphi(f, e; A, T)$. By strictification the φ case is impossible and by Theorem 4.15 the δ case is impossible.

- Suppose $f_k = \lambda_\omega\, x{:}A.\, b$. Now $f_k \bullet_\omega \mathfrak{s}(a) \rightsquigarrow [x := \mathfrak{s}(a)]b$ and $|[x := \mathfrak{s}(a)]b| = [x := |\mathfrak{s}(a)|]v$.
- Suppose $f_k = \psi(\mathrm{refl}(z; Z), a, b; A, P)$. Now $\psi(\mathrm{refl}(z; Z), a, b; A, P) \bullet_\omega \mathfrak{s}(a) \rightsquigarrow \mathfrak{s}(a)$. Note that $|f_k| = |\mathfrak{s}(f)|$, but $|\psi(\mathrm{refl}(z; Z), a, b; A, P)| = \lambda_\omega\, x{:}\diamond.\, x$ and $|\mathfrak{s}(f)| = \lambda_\omega\, x{:}\diamond.\, v$. Thus, $v = x$ and $|\mathfrak{s}(a)| = [x := |\mathfrak{s}(a)|]v$.

Suppose $m = \omega$ and $|\mathfrak{s}(f)| \rightsquigarrow t$. Note that the case where $|\mathfrak{s}(a)| \rightsquigarrow t$ is very similar and thus omitted. Applying the IH to $\mathcal{D}_1$ gives $\exists\ t'$ such that $\mathfrak{s}(f) \rightsquigarrow^*_{\neq 0} t'$ and $|t'| = t$. Now $\mathfrak{s}(f) \bullet_\omega \mathfrak{s}(a) \rightsquigarrow^*_{\neq 0} t' \bullet_\omega \mathfrak{s}(a)$ and $|t' \bullet_\omega \mathfrak{s}(a)| = t \bullet_\omega |\mathfrak{s}(a)|$.

Suppose $m = \tau$ then strictification and erasure are structural. Thus, a β-redex is tracked exactly and any structural redexes are very similar to the $m = \omega$ case.

Case: $\dfrac{\Gamma \vdash \overset{\mathcal{D}_1}{(x : A)} \cap B : \star \qquad \Gamma \vdash \overset{\mathcal{D}_2}{t} : A \qquad \Gamma \vdash s : \overset{\mathcal{D}_3}{[x := t]B} \qquad t \overset{\mathcal{D}_4}{\equiv} s}{\Gamma \vdash [t, s; (x : A) \cap B] : (x : A) \cap B}$

Have $|\mathfrak{s}([t_1, t_2; A])| = |\mathfrak{s}(t_1)|$ and $|\mathfrak{s}(t_1)| \rightsquigarrow t$. Applying the IH to $\mathcal{D}_1$ yields $\exists\ t'$ such that $\mathfrak{s}(t_1) \rightsquigarrow^*_{\neq 0} t'$ and $|t'| = t$. Now $[\mathfrak{s}(t_1), \mathfrak{s}(t_2); \mathfrak{s}(A)] \rightsquigarrow^*_{\neq 0} [t', \mathfrak{s}(t_2); \mathfrak{s}(A)]$ and $|[t', \mathfrak{s}(t_2); \mathfrak{s}(A)]| = t$.

Case: $\dfrac{\Gamma \vdash t : \overset{\mathcal{D}_1}{(x : A) \cap B}}{\Gamma \vdash t.2 : [x := t.1]B}$

Have $|\mathfrak{s}(b.2)| = |\mathfrak{s}(b)|$ and $|\mathfrak{s}(b)| \rightsquigarrow t$. Applying the IH to $\mathcal{D}_1$ gives $\exists\ t'$ such that $\mathfrak{s}(b) \rightsquigarrow^*_{\neq 0} t'$ and $|t'| = t$. Now $(\mathfrak{s}(b)).2 \rightsquigarrow^*_{\neq 0} t'.2$ and $|t'.2| = t$.

Case: $\dfrac{\Gamma \vdash \overset{\mathcal{D}_1}{A} : \star \qquad \Gamma \vdash \overset{\mathcal{D}_2}{a} : A \qquad \Gamma \vdash \overset{\mathcal{D}_3}{b} : A \qquad \Gamma \vdash \overset{\mathcal{D}_4}{e} : a =_A b \qquad \Gamma \vdash P : \overset{\mathcal{D}_5}{(y : A) \to_\tau (p : a =_A y_\star) \to_\tau \star}}{\Gamma \vdash \psi(e, a, b; A, P) : P \bullet_\tau a \bullet_\tau \mathrm{refl}(a; A) \to_\omega P \bullet_\tau b \bullet_\tau e}$

Have $|\mathfrak{s}(\psi(e, a, b; A, T))| = |\mathfrak{s}(e)|$ and $|\mathfrak{s}(e)| \rightsquigarrow t$. Applying the IH to $\mathcal{D}_4$ yields $\exists\ t'$ such that $\mathfrak{s}(e) \rightsquigarrow^*_{\neq 0} t'$ and $|t'| = t$. Now $\psi(\mathfrak{s}(e), \mathfrak{s}(a), \mathfrak{s}(b); \mathfrak{s}(A), \mathfrak{s}(T)) \rightsquigarrow^*_{\neq 0} \psi(t', \mathfrak{s}(a), \mathfrak{s}(b); \mathfrak{s}(A), \mathfrak{s}(T))$ and $|\psi(t', \mathfrak{s}(a), \mathfrak{s}(b); \mathfrak{s}(A), \mathfrak{s}(T))| = t$.

Case: $\dfrac{\Gamma \vdash \overset{\mathcal{D}_1}{(x : A) \cap B} : \star \qquad \Gamma \vdash a : \overset{\mathcal{D}_2}{(x : A) \cap B} \qquad \Gamma \vdash b : \overset{\mathcal{D}_3}{(x : A) \cap B} \qquad \Gamma \vdash e : \overset{\mathcal{D}_4}{a.1} =_A b.1}{\Gamma \vdash \vartheta(e, a, b; (x : A) \cap B) : a =_{(x:A)\cap B} b}$

Have $|\mathfrak{s}(\vartheta_2(e, a, b; (x : A) \cap B))| = |\mathfrak{s}(e)|$ and $|\mathfrak{s}(e)| \rightsquigarrow t$. Applying the IH to $\mathcal{D}_5$ gives $\exists\ t'$ where $\mathfrak{s}(e) \rightsquigarrow^*_{\neq 0} t'$ and $|t'| = t$. Now $\vartheta_2(\mathfrak{s}(e), \mathfrak{s}(a), \mathfrak{s}(b); \mathfrak{s}((x : A) \cap B)) \rightsquigarrow^*_{\neq 0} \vartheta_2(t', \mathfrak{s}(a), \mathfrak{s}(b); \mathfrak{s}((x : A) \cap B))$ and $|\vartheta_2(t', \mathfrak{s}(a), \mathfrak{s}(b); \mathfrak{s}((x : A) \cap B))| = t$.

Case: $\dfrac{\Gamma \vdash e : \overset{\mathcal{D}_1}{\mathrm{ctt}} =_{\mathrm{cBool}} \mathrm{cff}}{\Gamma \vdash \delta(e) : (X : \star) \to_0 X_\square}$

Have $|\mathfrak{s}(\delta(e))| = |\mathfrak{s}(e)|$ and $|\mathfrak{s}(e)| \rightsquigarrow t$. Applying the IH to $\mathcal{D}_1$ gives $\exists\, t'$ where $\mathfrak{s}(e) \rightsquigarrow^*_{\neq 0} t'$ and $|t'| = t$. Now $\delta(\mathfrak{s}(e)) \rightsquigarrow^*_{\neq 0} \delta(t')$ and $|\delta(t')| = t$.

Case: $\dfrac{\Gamma \vdash \overset{\mathcal{D}_1}{A} : K \qquad \Gamma \vdash \overset{\mathcal{D}_2}{t} : B \qquad \overset{\mathcal{D}_3}{A \equiv B}}{\Gamma \vdash t : A}$

Immediate by the IH applied to $\mathcal{D}_2$.

□

Theorem 5.4 (Strict Object Normalization)**.** *If* $\Gamma \vdash t : A$ *then* $|\mathfrak{s}(t)|$ *is strongly normalizing*

Proof. Lemma 5.1 gives $\Gamma \vdash \mathfrak{s}(t) : A$ and thus by Theorem 3.19: $\mathfrak{s}(t)$ is strongly normalizing wrt proof reduction. Let ∂ be the maximum length reduction sequence $\mathfrak{s}(t)$ could take to reach the unique value. Suppose wlog that $|\mathfrak{s}(t)|$ contains a redex. Contract this redex giving $|\mathfrak{s}(t)| \rightsquigarrow e_1$.

By Lemma 5.3: $\exists\ t_1$ such that $\mathfrak{s}(t) \rightsquigarrow^*_{\neq 0} t_1$ and $|t_1| = e_1$. Using preservation of proof reduction: $\Gamma \vdash t_1 : A$ and moreover by Lemma 5.2 $\mathfrak{s}(t_1) = t_1$. Let the number of contracted redexes by the reduction $\mathfrak{s}(t) \rightsquigarrow^*_{\neq 0} t_1$ be k, then there is a maximum of $\partial - k$ redexes in t_1. If redexes remain in e_1 than the process can be repeated because t_1 is a strict proof whose erasure is e_1. However, eventually the number of steps taken must run out, because ∂ is a finite value. Thus, the procedure may be repeated as many times as desired, but e_i, the value after i iterations of this process, must eventually run out of redexes by Lemma 5.3. Therefore, $|\mathfrak{s}(t)|$ is strongly normalizing. □

Strong normalization of strict objects leads to an interesting observation. Recall the definition of conversion: $a \equiv b$ if and only if $\exists\ u, v$ such that $a \rightsquigarrow^* u$, $b \rightsquigarrow^* v$ and $|u| \rightleftharpoons |v|$. An observant reader may wonder why reduction is allowed after two candidates objects, $|u|$ and $|v|$ are obtained. In other words, why not merely compare for equality: $|u| = |v|$. The answer is because φ may generate β-redexes after erasure, and it turns out that it is the only syntax form for which this is possible. Thus, if φ was removed from the system then conversion *could* be defined using equality of objects instead of reduction convertibility of objects. The φ form is unique amongst all the other syntax. It is both not responsible for any new properties, but also responsible for the only interesting reduction in objects.

Another question that the reader may have is why not represent the reduction of φ in the proof system. The answer is that there is no obvious way to make the reduction well-typed, thus preservation would be lost. Indeed, the proof witness of a φ form, $f : (a : A) \to (x : A) \cap B$, is allowed to be as complicated as required to produce a subtype, $(x : A) \cap B$, from the corresponding supertype, A. However, the object for the φ is merely the identity λ-term $(\lambda\, x : \diamond.\, x)$. To make it possible to type this term in the proof system some notion of subtyping would have to be added directly into the rules. It is not immediately clear how to make this move without producing a radically different system. Yet, it does hint that the φ rule is, in some sense, expanding a semantic subtyping relation that is later realized internally via a notion of casts. Indeed, it may be fruitful to view the proof-object distinction as being fundamentally related to subtyping.

5.2 Observational Equivalence of Objects

Unfortunately, the structural relationship between strict objects and regular objects is not easy to exploit, if it can used at all. Observational equivalence gives a method of relating regular objects to strict objects. Objects being the concept of interest means that contexts need to be well-typed because an object is only the erasure of a proof. To make contexts the inductive structure of syntax is reused with a unique fresh free variable, labelled h, that represents a hole. The variable is unique meaning it occurs only once in the given syntax. Context structure could be defined inductively, but this methodology allows reuse of erasure and substitution.

Definition 5.5. *A **context** $\gamma : (\Gamma, A) \to (\Delta, B)$ is a syntactic form with a unique free variable h representing a hole such that if $\Gamma \vdash t : A$ then $\Delta \vdash [h := t]\gamma : B$.*

Observational equivalence is then defined to be equivalence of divergence of the associated objects substituted for h in the given context. There are several possible ways to defined observational equivalence including the choice of what counts as an observation. For the purposes of this chapter divergence is the only observation of interest.

Definition 5.6. *The syntax a and b are **observationally equivalent** at A in Γ (written: $\Gamma \vdash a \approx_A b$) iff for any context $\gamma : (\Gamma, A) \to (\varepsilon, \mathrm{cUnit})$ with unique fresh variable h: $|[h := a]\gamma|$ normalizes iff $|[h := b]\gamma|$ normalizes*

It is easy to see that observational equivalence forms an equivalence relation relative to the parameters Γ and A.

Lemma 5.7. $\Gamma \vdash a \approx_A a$

Proof. Immediate by definition. □

Lemma 5.8. *If $\Gamma \vdash a \approx_A b$ then $\Gamma \vdash b \approx_A a$*

Proof. By definition the stated condition holds via an if-and-only-if. Hence, observational equivalence is symmetric. □

Lemma 5.9. *If $\Gamma \vdash a \approx_A b$ and $\Gamma \vdash b \approx_A c$ then $\Gamma \vdash a \approx_A c$*

Proof. Let $\gamma : (\Gamma, A) \to (\varepsilon, \mathrm{cUnit})$ be an arbitrary context with unique fresh variable h. Suppose $|[h := b]\gamma|$ diverges, then by $\Gamma \vdash b \approx_A c$ it must be the case that $|[h := c]\gamma|$ diverges. By Lemma 5.8: $\Gamma \vdash b \approx_A a$ and thus as above $|[h := a]\gamma|$ diverges. Suppose $|[h := b]\gamma|$ normalizes, then by $\Gamma \vdash b \approx_A c$: $|[h := c]\gamma|$ normalizes. Likewise, using symmetry and the same reasoning: $|[h := a]\gamma|$ normalizes. Hence, $|[h := a]\gamma|$ normalizes if and only if $|[h := c]\gamma|$ normalizes. □

Moreover, because observational equivalence only tracks divergence it must be the case that if an object is observationally equivalent to a strongly normalizing object, then it is also strongly normalizing.

Lemma 5.10. *If $\Gamma \vdash a : A$, $\Gamma \vdash b : A$, $|a|$ is strongly normalizing, and $\Gamma \vdash a \approx_A b$ then $|b|$ is strongly normalizing*

Proof. Immediate by definition of observational equivalence, $|b|$ must be normalizing in any possible context $\gamma : (\Gamma, A) \to (\varepsilon, \mathrm{cUnit})$. Thus, $|b|$ must be strongly normalizing. □

Knowing that φ objects behave like the identity function is critical. Note that the condition $FV(|e|)$ being empty is necessary to ensure the below lemma holds. If this condition where dropped than $e \bullet_\omega a$ for when a is a value would say nothing about the reduction behavior of f. This is because e could be a free variable in Γ and thus the equational proof it is evidence for would not be "actionable" in some sense. It is not clear if this condition is too strict, as one could imagine a more refined or semantic condition that allows rewriting in contexts that are in some sense ordered.

To that point, the free variable condition is communicating that the computational nature of $|e|$ cannot depend on any hypothetical evidence. Thus, the computational nature of $|e|$ is a sort of ground truth. Regardless, the condition is one that works and that is not too onerous as to prevent constructions of interest.

Lemma 5.11. *Let* $T = (a : A) \to_\omega (x : A) \cap B$. *Suppose:*

1. $\Gamma \vdash T : \star$
2. $\Gamma \vdash f : T$
3. $\Gamma \vdash e : (a : A) \to_\omega a_\star =_A (f \bullet_\omega a_\star).1$
4. $FV(|e|)$ *is empty*
5. $\Gamma \vdash a : A$

Then $\Gamma \vdash a \approx_A (f \bullet_\omega a).1$

Proof. Suppose $a \rightsquigarrow^* a'$ where $FV(|a'|)$ is empty then $e \bullet_\omega a$ must reduce to a value because $FV(|e|)$ is empty. Note that theoretically the definition of e could use φ, but because e is erased and Lemma 5.1 shows that φ does not enable any new proofs it can be safely assumed that $e = \mathfrak{s}(e)$. In other words, any usage of φ that contains a φ in e can always be transformed into a usage that does not contain φ in the definition of e without affecting either the typing relation or object reduction. Hence, $e \bullet_\omega a \rightsquigarrow^* \mathrm{refl}(z; Z)$ where $|z|$ is in normal form, $z \equiv a'$, and $z \equiv (f \bullet_\omega a').1$. Thus, $|f| \bullet_\omega |a'| \rightsquigarrow^* |z|$ and $|a'| \rightsquigarrow^* |z|$. Meaning both are normalizing and hence observationally equivalent.

Suppose a is neutral, i.e. $FV(|a|)$ is not empty. In this case $e \bullet_\omega a$ does not necessarily reduce to a value because a is neutral. Let $\gamma : (\Gamma, A) \to (\varepsilon, \mathrm{cUnit})$ be a context with hole variable h. It is possible that a reduces to a value after substitution into h. However, this falls into the purview of the above case, thus suppose wlog that a remains neutral after substitution. If $|[h := a]\gamma|$ diverges then clearly $|[h := f \bullet_\omega a.1]\gamma|$ diverges as well. Thus, suppose $|[h := a]\gamma|$ normalizes. There are two possibilities to consider:

1. Suppose A has no inhabitants and is equivalent to false. In this scenario, $|f|$ must reduce to a term that looks like $\lambda_\omega\, x : \diamond.\, y \bullet_\omega \cdots$ for $y = x$ or $y \in \Gamma$. One alternative possibility is for f to contain a φ rule, but this requires equational evidence that f behaves like an identity and critically that equational evidence cannot depend on $x : A$ or any other variable in Γ. Hence, any φ rule used in f must be via closed terms, but because f produces a $(x : A) \cap B$ the body must contain a variable representing false to inhabit A. Therefore, regardless of the situation, $|f|$ must be neutral and thus $|f| \bullet_\omega |a|$ must be neutral.
2. Suppose A has inhabitants and is not equivalent to false. Let $\vdash v : A$ represent an arbitrary closed value of A. Using the same above argument, $e \bullet_\omega v$ would reduce to a value meaning $|f| \bullet_\omega$

$|v| \rightsquigarrow^* |v|$. Thus, f must reduce to some intermediary f' such that $FV(|f'|)$ is empty. Note that technically f may contain free variables, but they cannot be used in a computationally relevant way. Now $|f'|$ must reduce to a term that looks like $\lambda_\omega\, x : \diamond.\, x \bullet_\omega \cdots$, but when a neutral is applied to this function it produces a neutral. Hence, $|f| \bullet_\omega |a|$ is neutral.

□

The key theorem is that regular proofs are observationally equivalent to their strictification. The only way this theorem could fail is because of φ subexpressions. However, Lemma 5.11 conveys that φ proofs are observationally equivalent to an identity function. Indeed, the previous lemma does all the heavy lifting in making this theorem go through.

Theorem 5.12. *If* $\Gamma \vdash t : A$ *then* $\Gamma \vdash t \approx_A \mathfrak{s}(t)$

Proof. By induction on $\Gamma \vdash t : A$. The Ax and Var cases are trivial because $\mathfrak{s}(\star) = \star$ and $\mathfrak{s}(x_K) = x_K$. Cases that cannot participate in well-typed reductions: Int and Eq are omitted in favor of presenting only the Pi case. The Fst case is very similiar to the Snd case and omitted. Likewise, the PrmSnd case is very similar to the PrmFst case and omitted.

Case: $$\frac{\Gamma \vdash A : \overset{\mathcal{D}_1}{\mathrm{dom}_\Pi(m, K)} \qquad \Gamma; x_m : A \vdash \overset{\mathcal{D}_2}{B} : \mathrm{codom}_\Pi(m)}{\Gamma \vdash (x : A) \to_m B : \mathrm{codom}_\Pi(m)}$$

Have $\mathfrak{s}((x : A) \to_m B) = (x : \mathfrak{s}(A)) \to_m \mathfrak{s}(B)$. Applying the IH to $\mathcal{D}_1$ and $\mathcal{D}_2$ yields $\Gamma \vdash A \approx_{\mathrm{dom}_\Pi(m,K)} \mathfrak{s}(A)$ and $\Gamma; x_m : A \vdash B \approx_{\mathrm{codom}_\Pi(m)} \mathfrak{s}(B)$. Thus, because no reduction in A or B can alter the initial structural shape of this syntax it must be the case that $\Gamma \vdash (x : A) \to_m B \approx_{\mathrm{codom}_\Pi(m)} \mathfrak{s}((x : A) \to_m B)$.

Case: $$\frac{\Gamma \vdash (x : A) \to_m \overset{\mathcal{D}_1}{B} : \mathrm{codom}_\Pi(m) \qquad \Gamma; x_m : \overset{\mathcal{D}_2}{A} \vdash t : B \qquad \overset{\mathcal{D}_3}{x \notin FV(|t|)} \text{ if } m = 0}{\Gamma \vdash \lambda_m\, x : A.\, t : (x : A) \to_m B}$$

Have $\mathfrak{s}(\lambda_m\, x : A.\, t) = \lambda_m\, x : \mathfrak{s}(A).\, \mathfrak{s}(t)$. If $m = \omega$ or $m = \tau$ then the case is similar to the Pi case. Suppose $m = 0$. Let $\gamma : (\Gamma, (x : A) \to_m B) \to (\varepsilon, \mathrm{cUnit})$ be an arbitrary context with hole variable h. Now $|[h := \lambda_m\, x : A.\, t]\gamma| = [h := |t|]|\gamma|$ and $|[h := \lambda_m\, x : \mathfrak{s}(A).\, \mathfrak{s}(t)]\gamma = [h := |\mathfrak{s}(t)|]|\gamma|$. By the IH applied to $\mathcal{D}_2$: $\Gamma; x_m : A \vdash t \approx_B \mathfrak{s}(t)$ which means that $[h := |t|]|\gamma|$ normalizes iff $[h := |\mathfrak{s}(t)|]|\gamma|$. Note that while it is true the type of the contexts differ, the consequence remains the same because observational equivalence is phrased relative to erasure.

Case: $$\frac{\Gamma \vdash f : \overset{\mathcal{D}_1}{(x : A)} \to_m B \qquad \Gamma \vdash \overset{\mathcal{D}_2}{a} : A}{\Gamma \vdash f \bullet_m a : [x := a]B}$$

Suppose $m = \omega$. The only way that $f \bullet_\omega a$ could be non-terminating is if $f = \varphi(f', e; A, T)$. Thus, suppose this is the case wlog. Destructing $\mathcal{D}_1$ and applying

Lemma 5.11 gives $\Gamma \vdash a \approx_A (f' \bullet_\omega a).1$. Applying the IH to $\mathcal{D}_2$ gives $\Gamma \vdash a \approx_A \mathfrak{s}(a)$. Applying the IH to a subderivation of $\mathcal{D}_1$ yields $\Gamma \vdash f' \approx_T \mathfrak{s}(f')$. However, if f' behaves like the identity for any well-typed input it must be the case that $\mathfrak{s}(f')$ does as well. Hence, $\Gamma \vdash f \bullet_\omega a \approx_B \mathfrak{s}(f) \bullet_\omega \mathfrak{s}(a)$

Suppose $m = \tau$. There are no possible β-reductions involving φ, thus by proof normalization and the IH the case is concluded.

Suppose $m = 0$. Let γ be a context with a hole variable h. Now $|[h := f \bullet_0 a]\gamma| = [h := |f|]|\gamma|$ and $|[h := \mathfrak{s}(f \bullet_0 a)]\gamma| = [h := |\mathfrak{s}(f)|]|\gamma|$. Applying the IH to $\mathcal{D}_1$ concludes the case.

Case: $$\frac{\Gamma \vdash (x : \overset{\mathcal{D}_1}{A}) \cap B : \star \qquad \Gamma \vdash \overset{\mathcal{D}_2}{t} : A \qquad \Gamma \vdash s : \overset{\mathcal{D}_3}{[x := t]B} \qquad t \overset{\mathcal{D}_4}{\equiv} s}{\Gamma \vdash [t, s; (x : A) \cap B] : (x : A) \cap B}$$

Let γ be a context with hole variable h. Now $|[h := [t, s; T]]\gamma| = [h := |t|]|\gamma|$ and $|[h := \mathfrak{s}([t, s; T])]\gamma| = [h := |\mathfrak{s}(t)|]|\gamma|$. These objects are observationally equivalent by applying the IH to $\mathcal{D}_2$.

Case: $$\frac{\Gamma \vdash t : \overset{\mathcal{D}_1}{(x : A) \cap B}}{\Gamma \vdash t.2 : [x := t.1]B}$$

Let γ be a context with hole variable h. Now $|[h := t.2]\gamma| = [h := |t|]|\gamma|$ and $|[h := \mathfrak{s}(t.2)]\gamma| = [h := |\mathfrak{s}(t)|]|\gamma$. Applying the IH to $\mathcal{D}_1$ concludes the case.

Case: $$\frac{\Gamma \vdash \overset{\mathcal{D}_1}{A} : \star \qquad \Gamma \vdash \overset{\mathcal{D}_2}{t} : A}{\Gamma \vdash \mathrm{refl}(t; A) : t =_A t}$$

Note that $|\mathrm{refl}(t; A)| = \lambda\, x : \diamond.\, x$ and this erasure does not depend on t or A. Thus, $|\mathfrak{s}(\mathrm{refl}(t; A)) = \lambda\, x : \diamond.\, x$, hence the case is trivial.

Case: $$\frac{\Gamma \vdash \overset{\mathcal{D}_1}{A} : \star \qquad \Gamma \vdash \overset{\mathcal{D}_2}{a} : A \qquad \Gamma \vdash \overset{\mathcal{D}_3}{b} : A \qquad \Gamma \vdash \overset{\mathcal{D}_4}{e} : a =_A b \qquad \Gamma \vdash P : (y : A) \to_\tau \overset{\mathcal{D}_5}{(p : a =_A y_\star)} \to_\tau \star}{\Gamma \vdash \psi(e, a, b; A, P) : P \bullet_\tau a \bullet_\tau \mathrm{refl}(a; A) \to_\omega P \bullet_\tau b \bullet_\tau e}$$

Let γ be a context with a hole variable h. Now $|[h := \psi(e, a, b; A, P)]\gamma| = [h := |e|]|\gamma|$ and $|[h := \mathfrak{s}(\psi(e, a, b; A, P))]\gamma| = [h := |\mathfrak{s}(e)|]|\gamma|$. Applying the IH to $\mathcal{D}_4$ concludes the case.

Case: $$\frac{\Gamma \vdash (x : \overset{\mathcal{D}_1}{A}) \cap B : \star \qquad \Gamma \vdash a : \overset{\mathcal{D}_2}{(x : A) \cap B} \qquad \Gamma \vdash b : \overset{\mathcal{D}_3}{(x : A) \cap B} \qquad \Gamma \vdash e : \overset{\mathcal{D}_4}{a.1} =_A b.1}{\Gamma \vdash \vartheta(e, a, b; (x : A) \cap B) : a =_{(x:A)\cap B} b}$$

Let γ be a context with a hole variable h. Now $|[h := \vartheta_1(e, a, b; T)]\gamma| = [h := |e|]|\gamma|$ and $|[h := \mathfrak{s}(\vartheta_1(e, a, b; T))]\gamma| = [h := |\mathfrak{s}(e)|]|\gamma|$. Applying the IH to $\mathcal{D}_4$ concludes the case.

Case: $$\frac{\Gamma \vdash (x : \overset{\mathcal{D}_1}{A}) \cap B : \star \qquad \Gamma \vdash \overset{\mathcal{D}_2}{a} : A \qquad \Gamma \vdash b : \overset{\mathcal{D}_3}{(x : A) \cap B} \qquad \Gamma \vdash e : \overset{\mathcal{D}_4}{a} =_A b.1}{\Gamma \vdash \varphi(a, b, e; A, (x : A) \cap B) : (x : A) \cap B}$$

Let γ be a context with a hole variable h. Now $|[h := \varphi(f, e; A, T)]\gamma| = [h := \lambda_\omega\, x : \diamond.\, x]|\gamma|$ and $|[h := \mathfrak{s}(\varphi(f, e; A, T))]\gamma| = [h := |\mathfrak{s}(f)|]|\gamma|$. Applying the IH to $\mathcal{D}_3$ gives that $\Gamma \vdash f \approx_T \mathfrak{s}(f)$. Let $\Gamma \vdash a : A$ be an arbitrary proof of a, then by Lemma 5.11: $\Gamma \vdash a \approx_A (f \bullet_\omega a).1$. Hence, f behaves like the identity for any well-typed a proof and thus so does $\mathfrak{s}(f)$ because it is observationally equivalent to f.

Case: $$\frac{\Gamma \vdash e : \mathrm{ctt} =_{\mathrm{cBool}} \mathrm{cff}}{\Gamma \vdash \delta(e) : (X : \star) \to_0 X_\square}$$

Let γ be a context with a hole variable h. Now $|[h := \delta(e)]\gamma| = [h := |e|]|\gamma|$ and $|[h := \mathfrak{s}(\delta(e))]\gamma| = [h := |\mathfrak{s}(e)|]|\gamma|$. Applying the IH to $\mathcal{D}_1$ concludes the case.

Case: $$\frac{\Gamma \vdash \overset{\mathcal{D}_1}{A} : K \qquad \Gamma \vdash \overset{\mathcal{D}_2}{t} : B \qquad \overset{\mathcal{D}_3}{A \equiv B}}{\Gamma \vdash t : A}$$

Applying the IH to $\mathcal{D}_1$ gives $\Gamma \vdash t \approx_B \mathfrak{s}(t)$. Note that $\Gamma \vdash t : A$ by assumption and Lemma 5.1 gives $\Gamma \vdash \mathfrak{s}(t) : A$. With $\mathcal{D}_1$ and $\mathcal{D}_2$ the following derivation is obtained: $\Gamma \vdash \mathfrak{s}(t) : B$. Thus, a context $\gamma : (\Gamma, A) \to (\varepsilon, \mathrm{cUnit})$ is constructed by working through the intermediary context at type B.

□

Theorem 5.13 (Object Normalization)**.** *If $\Gamma \vdash t : A$ then $|t|$ is strongly normalizing*

Proof. By Lemma 5.1: $\Gamma \vdash \mathfrak{s}(t) : A$. Now Theorem 5.12 yields $\Gamma \vdash t \approx_A \mathfrak{s}(t)$. Lemma 5.4 gives $|\mathfrak{s}(t)|$ is strongly normalizing. Therefore, Lemma 5.10 concludes. □

Chapter 6

CEDILLE2: SYSTEM IMPLEMENTATION

6.1 Normalization by Evaluation

6.2 Syntax-directed Bidirectional Type System

6.3 Design Choices

Chapter 7

CEDILLE2: INTERNALLY DERIVABLE CONCEPTS

7.1 Generic Indexed Inductive Types

7.2 Quotient Inductive Subtypes

7.3 Constructor Subtypes

7.4 Example Simulated Large Eliminations

7.5 Example Inductive-Inductive Type

Chapter 8

CONCLUSION

8.1 test

8.2 test

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