# The Calculus of Set Constructions

by

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A thesis submitted in partial fulfillment
of the requirements for the Doctor of Philosophy degree
in Computer Science
in the Graduate College
of The University of Iowa

May 2024

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– Some wise dude

## ACKNOWLEDGMENTS

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## ABSTRACT

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## PUBLIC ABSTRACT


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# CALCULUS OF SET CONSTRUCTIONS AND BASIC METATHEORY

$$
\begin{aligned}
\mathrm{dom}_\Pi(\omega, K) &= \star \\
\mathrm{dom}_\Pi(\tau, K) &= K \\
\mathrm{dom}_\Pi(0, K) &= K
\end{aligned}
$$

$$
\begin{aligned}
\mathrm{codom}_\Pi(\omega) &= \star \\
\mathrm{codom}_\Pi(\tau) &= \Box \\
\mathrm{codom}_\Pi(0) &= \star
\end{aligned}
$$

$$
\begin{aligned}
(\lambda_m\, x\!:\!t_1.\, t_2) \bullet_m t3 &\rightsquigarrow_\beta [x := t_3]t_2 \\
[t_1, t_2, t_3].1 &\rightsquigarrow_\beta t_1 \\
[t_1, t_2, t_3].2 &\rightsquigarrow_\beta t_2 \\
J\ t_1\ t_2\ t_3\ t_4\ (\mathrm{refl}\ t_5)\ t_6 &\rightsquigarrow_\beta t_6 \bullet_0 t_5 \\
\vartheta\ (\mathrm{refl}\ t.1) &\rightsquigarrow_\beta \mathrm{refl}\ t \\
\varphi\ t_1\ t_2\ (\mathrm{refl}\ t_3) &\rightsquigarrow_\beta t_2
\end{aligned}
$$

$$
\begin{aligned}
|x| &= x \\
|\star| &= \star \\
|\square| &= \square \\
|\lambda_0\, x\!:\!A.\, t| &= |t| \\
|\lambda_\omega\, x\!:\!A.\, t| &= \lambda_\omega\, x.\, |t| \\
|\lambda_\tau\, x\!:\!A.\, t| &= \lambda_\tau\, x\!:\!|A|.\, |t| \\
|(x : A) \to_m B| &= (x : |A|) \to_m |B| \\
|(x : A) \cap B| &= (x : |A|) \cap |B| \\
|f \bullet_0 a| &= |f| \\
|f \bullet_\omega a| &= |f| \bullet_\omega |a| \\
|f \bullet_\tau a| &= |f| \bullet_\tau |a| \\
|[s, t, T]| &= |t| \\
|t.1| &= |t| \\
|t.2| &= |t| \\
|x =_A y| &= |x| =_{|A|} |y| \\
|\mathrm{refl}\ t| &= \lambda_\omega\, x.\, x \\
|J\ A\ P\ x\ y\ r\ w| &= |r|\ |w| \\
|\vartheta\ e| &= |e| \\
|\delta\ e| &= |e| \\
|\varphi\ a\ b\ e| &= |a|
\end{aligned}
$$

$$\frac{\vdash \Gamma}{\Gamma \vdash \star \rhd \Box}\textsc{Axiom}$$

$$\frac{\vdash \Gamma \qquad (x : A) \in \Gamma}{\Gamma \vdash x \rhd A}\textsc{Var}$$

$$\frac{\Gamma \vdash A \mathrel{|\!\!\rhd} \mathrm{dom}_\Pi(m, K) \qquad \Gamma, x : A \vdash B \mathrel{|\!\!\rhd} \mathrm{codom}_\Pi(m)}{\Gamma \vdash (x : A) \to_m B \rhd \mathrm{codom}_\Pi(m)}\textsc{Pi}$$

$$\frac{\Gamma \vdash A \mathrel{|\!\!\rhd} \mathrm{dom}_\Pi(m, K) \qquad \Gamma, x : A \vdash t \rhd B \qquad x \notin \mathrm{FV}(|t|) \text{ if } m = 0}{\Gamma \vdash \lambda_m\, x{:}A.\, t : (x : A) \to_m B}\textsc{Lambda}$$

$$\frac{\Gamma \vdash f \mathrel{|\!\!\rhd} (x : A) \to_m B \qquad \Gamma \vdash a \lhd A}{\Gamma \vdash f \bullet_m a \rhd [x := a]B}\textsc{App}$$

$$\frac{\Gamma \vdash A \mathrel{|\!\!\rhd} \star \qquad \Gamma, x : A \vdash B \mathrel{|\!\!\rhd} \star}{\Gamma \vdash (x : A) \cap B \rhd \star}\textsc{Intersection}$$

$$\frac{\Gamma \vdash T \mathrel{|\!\!\rhd} (x : A) \to_\tau B \qquad \Gamma \vdash t \lhd A \qquad \Gamma \vdash s \lhd [x := t]B \qquad t \equiv s}{\Gamma \vdash [t, s; T] \rhd (x : A) \cup B}\textsc{Pair}$$

$$\frac{\Gamma \vdash t \mathrel{|\!\!\rhd} (x : A) \cap B}{\Gamma \vdash t.1 \rhd A}\textsc{First}$$

$$\frac{\Gamma \vdash t \mathrel{|\!\!\rhd} (x : A) \cap B}{\Gamma \vdash t.2 \rhd [x := t.1]B}\textsc{Second}$$

$$\frac{\Gamma \vdash A \mathrel{|\!\!\rhd} \star \qquad \Gamma \vdash a \lhd A \qquad \Gamma \vdash b \lhd A}{\Gamma \vdash a =_A b \rhd \star}\textsc{Equality}$$

$$\frac{\Gamma \vdash t \rhd A}{\Gamma \vdash \mathrm{refl}\ t \rhd t =_A t}\textsc{Refl}$$

$$\frac{\begin{array}{c}\Gamma \vdash A \mathrel{\Vert\!\triangleright} \star \qquad \Gamma \vdash P \triangleleft (x\ y : A) \to_\tau (e : x =_A y) \to_\tau \star \qquad \Gamma \vdash x \triangleleft A \\ \Gamma \vdash y \triangleleft A \qquad \Gamma \vdash r \triangleleft x =_A y \qquad \Gamma \vdash w \triangleleft (a : A) \to_0 P \bullet_\tau a \bullet_\tau a \bullet_\tau \text{refl } a\end{array}}{\Gamma \vdash J\ A\ P\ x\ y\ r\ w \triangleright P \bullet_\tau x \bullet_\tau y \bullet_\tau r}\textsc{EqInduct}$$

$$\frac{\begin{array}{c}\Gamma \vdash e \mathrel{\Vert\!\triangleright} a.1 =_T b.1 \\ \Gamma \vdash a \mathrel{\Vert\!\triangleright} (x : A) \cap B \qquad \Gamma \vdash b \triangleleft (x : A) \cap B \qquad T \equiv A\end{array}}{\Gamma \vdash \vartheta\ e \triangleright a =_{(x:A)\cap B} b}\textsc{Promote}$$

$$\frac{\begin{array}{c}\Gamma \vdash b \mathrel{\Vert\!\triangleright} (x : A) \cap B \\ \Gamma \vdash a \triangleleft A \qquad \Gamma \vdash e \triangleleft a =_A b.1 \qquad \mathrm{FV}(|e|) \subseteq \mathrm{FV}(|a|)\end{array}}{\Gamma \vdash \varphi\ a\ b\ e \triangleright (x : A) \cap B}\textsc{Cast}$$

$$\frac{\begin{array}{c}\text{cBool} := (X : \star) \to_0 (x : X) \to_\omega (y : X) \to_\omega X \\ \text{ctt} := \lambda_0\, X{:}\star.\ \lambda_\omega\, x{:}X.\ \lambda_\omega\, y{:}X.\ x \\ \text{cff} := \lambda_0\, X{:}\star.\ \lambda_\omega\, x{:}X.\ \lambda_\omega\, y{:}X.\ y \qquad \Gamma \vdash e \triangleleft \text{ctt} =_{\text{cBool}} \text{cff}\end{array}}{\Gamma \vdash \delta\ e \triangleright (X : \star) \to_0 X}\textsc{Separation}$$

$$\frac{\Gamma \vdash t \triangleright A \qquad A \rightsquigarrow_\beta^* B}{\Gamma \vdash t \mathrel{\Vert\!\triangleright} B}\textsc{HeadInference}$$

$$\frac{\Gamma \vdash t \triangleright A \qquad \Gamma \vdash B \mathrel{\Vert\!\triangleright} K \qquad A \equiv B}{\Gamma \vdash t \triangleleft B}\textsc{Checking}$$

$$\frac{}{\vdash \cdot}\textsc{ContextEmpty}$$

$$\frac{x \notin \mathrm{FV}(\Gamma) \qquad \vdash \Gamma \qquad \Gamma \vdash A \mathrel{\Vert\!\triangleright} K}{\vdash \Gamma, x : A}\textsc{ContextAppend}$$

For the below theorem we assume we know that $t$ is strongly normalizing and that the theory is consistent.

**Theorem 1.** *$\Gamma \vdash t \rhd A$ implies $|t|$ is strongly normalizing*

*Proof.* By induction on the derivation:

Case: $$\dfrac{\overset{\mathcal{D}_1}{\vdash \Gamma}}{\Gamma \vdash \star \rhd \Box}\textsc{Axiom}$$

Obvious.

Case: $$\dfrac{\overset{\mathcal{D}_1}{\vdash \Gamma} \qquad \overset{\mathcal{D}_2}{(x : A) \in \Gamma}}{\Gamma \vdash x \rhd A}\textsc{Var}$$

Obvious.

Case: $$\dfrac{\Gamma \vdash A \overset{\mathcal{D}_1}{\Rrightarrow} \mathrm{dom}_\Pi(m, K) \qquad \Gamma, x : A \vdash B \overset{\mathcal{D}_2}{\Rrightarrow} \mathrm{codom}_\Pi(m)}{\Gamma \vdash (x : A) \to_m B \rhd \mathrm{codom}_\Pi(m)}\textsc{Pi}$$

By the inductive hypothesis we know that $|A|$ and $|B|$ are strongly normalizing. Thus, $|(x : A) \to_m B| = (x : |A|) \to_m |B|$ is strongly normalizing.

Case: $$\dfrac{\Gamma \vdash A \overset{\mathcal{D}_1}{\Rrightarrow} \mathrm{dom}_\Pi(m, K) \qquad \Gamma, x : A \overset{\mathcal{D}_2}{\vdash} t \rhd B \qquad x \notin \mathrm{FV}(\overset{\mathcal{D}_3}{|t|}) \text{ if } m = 0}{\Gamma \vdash \lambda_m\, x{:}A.\, t : (x : A) \to_m B}\textsc{Lambda}$$

By the inductive hypothesis we know that $|A|$ and $|t|$ are strongly normalizing. Regardless of $m$ the erasure reduces to some combination of these two, thus the erased term is strongly normalizing.

Case: $$\dfrac{\Gamma \vdash f \Rrightarrow \overset{\mathcal{D}_1}{(x : A)} \to_m B \qquad \Gamma \vdash \overset{\mathcal{D}_2}{a} \lhd A}{\Gamma \vdash f \bullet_m a \rhd [x := a]B}\textsc{App}$$

By the inductive hypothesis we have that $|f|$ and $|a|$ are strongly normalizing. Proceed by cases on $f$:

- $f = x$, then no reduction can be performed and $x\ |a|$ is normal.

- $f = \lambda_m\, x{:}A.\, t$, then we have $[x := a]t$, but this infers a type and thus $|[x := a]t|$ is strongly normalizing.
- $f = u \bullet_{m_2} v$
- $f = [t, s; T]$, impossible by inversion on $\mathcal{D}_1$
- $f = t.1$
- $f = t.2$
- $f = \text{refl}\ x$, impossible by inversion on $\mathcal{D}_1$
- $f = J\ A\ P\ x\ y\ r\ w$
- $f = \vartheta\ e$, impossible by inversion on $\mathcal{D}_1$
- $f = \varphi\ a\ b\ e$, impossible by inversion on $\mathcal{D}_1$
- $f = \delta\ e$

Case: $$\dfrac{\Gamma \vdash \overset{\mathcal{D}_1}{A} \mathrel{\|\!\!\rhd} \star \qquad \Gamma, x : \overset{\mathcal{D}_1}{A \vdash} B \mathrel{\|\!\!\rhd} \star}{\Gamma \vdash (x : A) \cap B \rhd \star}\textsc{Intersection}$$

Same as function type case.

Case: $$\dfrac{\Gamma \vdash T \mathrel{\|\!\!\rhd} \overset{\mathcal{D}_1}{(x} : A) \to_\tau B \qquad \Gamma \vdash \overset{\mathcal{D}_2}{t} \lhd A \qquad \Gamma \vdash s \lhd \overset{\mathcal{D}_3}{[x} := t]B \qquad t \overset{\mathcal{D}_4}{\equiv} s}{\Gamma \vdash [t, s; T] \rhd (x : A) \cup B}\textsc{Pair}$$

By the inductive hypothesis we know that $|t|$ is strongly normalizing. Thus, $|t, s; T| = |t|$ is strongly normalizing.

Case: $$\dfrac{\Gamma \vdash t \mathrel{\|\!\!\rhd} \overset{\mathcal{D}_1}{(x} : A) \cap B}{\Gamma \vdash t.1 \rhd A}\textsc{First}$$

Same idea as pair case.

Case: $$\dfrac{\Gamma \vdash t \mathrel{\|\!\!\rhd} \overset{\mathcal{D}_1}{(x} : A) \cap B}{\Gamma \vdash t.2 \rhd [x := t.1]B}\textsc{Second}$$

Same idea as pair case.

Case: $$\frac{\Gamma \vdash \overset{\mathcal{D}_1}{A} \mathrel{|\!\rhd} \star \qquad \Gamma \vdash \overset{\mathcal{D}_2}{a} \lhd A \qquad \Gamma \vdash \overset{\mathcal{D}_2}{b} \lhd A}{\Gamma \vdash a =_A b \rhd \star}\textsc{Equality}$$

Same idea as function type case.

Case: $$\frac{\Gamma \vdash \overset{\mathcal{D}_1}{t} \rhd A}{\Gamma \vdash \text{refl } t \rhd t =_A t}\textsc{Refl}$$

$|\text{refl } x|$ is a value thus strongly normalizing.

Case: $$\frac{\begin{array}{c}\Gamma \vdash \overset{\mathcal{D}_1}{A} \mathrel{|\!\rhd} \star \qquad \Gamma \vdash P \lhd (x\ y : A) \overset{\mathcal{D}_2}{\to_\tau} (e : x =_A y) \to_\tau \star \qquad \Gamma \vdash \overset{\mathcal{D}_3}{x} \lhd A \\ \Gamma \vdash \overset{\mathcal{D}_4}{y} \lhd A \qquad \Gamma \vdash r \overset{\mathcal{D}_5}{\lhd} x =_A y \qquad \Gamma \vdash w \lhd (a : A) \to_0 \overset{\mathcal{D}_6}{P} \bullet_\tau a \bullet_\tau a \bullet_\tau \text{refl } a\end{array}}{\Gamma \vdash J\ A\ P\ x\ y\ r\ w \rhd P \bullet_\tau x \bullet_\tau y \bullet_\tau r}\textsc{EqInduct}$$

All of $|A|, |P|, |x|, |y|, |r|, |w|$ are strongly normalizing by the inductive hypothesis. Suppose that $r = \text{refl } z$. Then by one reduction we have $|r|\ |w|$, but $|r|$ is the identity function, thus this reduces to just $|w|$ which is strongly normalizing. Otherwise, no reduction is possible and $|r|\ |w|$ is normal.

Case: $$\frac{\begin{array}{c}\Gamma \vdash e \mathrel{|\!\rhd} \overset{\mathcal{D}_1}{a}.1 =_T b.1 \\ \Gamma \vdash a \mathrel{|\!\rhd} \overset{\mathcal{D}_2}{(x} : A) \cap B \qquad \Gamma \vdash b \lhd \overset{\mathcal{D}_3}{(x} : A) \cap B \qquad T \overset{\mathcal{D}_4}{\equiv} A\end{array}}{\Gamma \vdash \vartheta\ e \rhd a =_{(x:A)\cap B} b}\textsc{Promote}$$

By the inductive hypothesis $|e|$ is strongly normalizing, thus $|\vartheta\ e| = |e|$ is strongly normalizing.

Case: $$\frac{\begin{array}{c}\Gamma \vdash b \mathrel{|\!\rhd} \overset{\mathcal{D}_1}{(x} : A) \cap B \\ \Gamma \vdash \overset{\mathcal{D}_2}{a} \lhd A \qquad \Gamma \vdash e \lhd \overset{\mathcal{D}_3}{a} =_A b.1 \qquad \text{FV}(|e|) \overset{\mathcal{D}_4}{\subseteq} \text{FV}(|a|)\end{array}}{\Gamma \vdash \varphi\ a\ b\ e \rhd (x : A) \cap B}\textsc{Cast}$$

By the inductive hypothesis $|a|$ is strongly normalizing, thus $|\varphi\ a\ b\ e| = |a|$ is strongly normalizing.

Case:
$$\dfrac{\begin{array}{c}\text{cBool} := (X : \star) \to_0 (x : X) \to_\omega (y : X) \to_\omega X \\ \text{ctt} := \lambda_0\, X \!:\! \star.\, \lambda_\omega\, x \!:\! X.\, \lambda_\omega\, y \!:\! X.\, x \\ \text{cff} := \lambda_0\, X \!:\! \star.\, \lambda_\omega\, x \!:\! X.\, \lambda_\omega\, y \!:\! X.\, y \qquad \Gamma \vdash e \lhd \text{ctt} =_{\text{cBool}} \text{cff}\end{array}}{\Gamma \vdash \delta\ e \rhd (X : \star) \to_0 X}\ \textsc{Separation}$$

By the inductive hypothesis $|e|$ is strongly normalizing, thus $|\delta\ e| = |e|$ is strongly normalizing.

□

**Lemma 1.** *For $\Gamma \vdash t : A$, where $t_w$ is the whnf of $t$ then $t \rightsquigarrow_\beta^* t_w$ iff $|t| \rightsquigarrow_\beta^* |t_w|$*

Chapter 3

# PROPERTIES OF COSC

Chapter 4

# CEDILLE 2.0: IMPLEMENTATION OF SYSTEM COSC

Chapter 5

# DERIVABLE CONSTRUCTS IN SYSTEM COSC

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