# Multivariable Calculus

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# 1. Scalar-valued Functions of Two Variables

A scalar valued function of two variables is a function $f: D \to \mathbb{R}$ whose domain $D$ is a subset of $\mathbb{R}^2$ and the co-domain is $\mathbb{R}$. Examples of such functions:

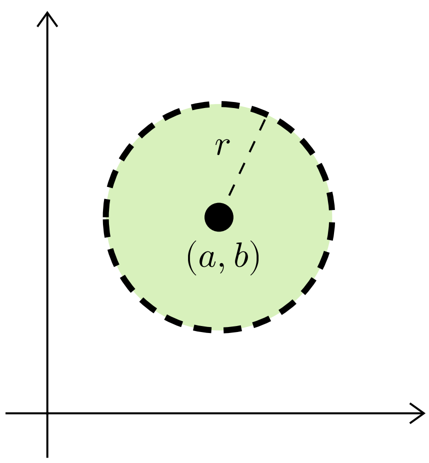
- $f(x,y) = x + y$
- $f(x,y) = \sin(x+y)$
- $f(x,y) = xy\big(e^x + y\big)$

**Open ball**

A concept that we will use in the rest of the document is that of an open ball. An open ball centered at a point $(a,b)$ with radius $r$ is the following set:

$$\left\{(x,y) : (x-a)^2 + (y-b)^2 < r^2\right\}$$

In simple terms, it is the set of all points within the circle centered at $(a,b)$ with radius $r$. Note that the boundary is not a part of an open ball. Visually:




All points in the shaded region belong to the open ball. Note that the boundary is given by a dotted line and is not a part of the open ball.

# 2. Limits and Continuity

## 2.1. Limits of Sequences

Consider $a_n = {}^1\!/_n$. This sequence converges to $0$ as $n \to \infty$. That is, as $n$ increases, $a_n$ gets closer and closer to $0$. We denote this as $a_n \to 0$. A sequence $(a_n, b_n)$ in $\mathbb{R}^2$ converges to $(a,b)$ if $a_n \to a$ and $b_n \to b$. For instance $\left(\frac{1}{n}, n\sin\left(\frac{1}{n}\right)\right) \to (0,1)$. We will use this fact to define limits of functions of two variables.

## 2.2. Limits of functions

Definition

Consider a function $f\colon D \to \mathbb{R}$ where $D \subset \mathbb{R}^2$. Let $(a, b) \in D$. The limit of $f$ at $(a, b)$ exists if there exists a real number $L$ such that:

$$(a_n, b_n) \to (a, b) \implies f(a_n, b_n) \to L$$

for all possible sequences $(a_n, b_n)$ that converge to $(a, b)$.

For simple functions, we can just substitute the value $(a, b)$ into the function to get the limit. For instance:

$$\lim_{(x,y) \to (a,b)} e^x \sin(x + y) = e^a \sin(a + b)$$

This method of substituting the value is not foolproof. There is a useful theorem that can be used to show the non-existence of limits:

Theorem

The limit of a function $f$ at $(a, b)$ exists and is equal to $L$ if for every curve $C$ in the domain $D$ of $f$ passing through $(a, b)$, the limit of $f$ along $C$ is exists and is equal to $L$.

An example of this theorem in action:

$$\lim_{(x,y) \to (0,0)} \frac{x^2 - y^2}{x^2 + y^2}$$

The limit along the $x$-axis is $1$ and the limit along the $y$-axis is $-1$. Since we have two different limits along the two curves, the limit doesn't exist.

## 2.3. Properties of limits

### 2.3.1. Arithmetic

$f$ and $g$ denote two functions on a domain $D$. If $\lim\limits_{(x,y)\to(a,b)} f(x,y) = F$ and $\lim\limits_{(x,y)\to(a,b)} g(x,y) = G$, then:

- $\lim\limits_{(x,y)\to(a,b)} f(x,y) + g(x,y) = F + G$
- $\lim\limits_{(x,y)\to(a,b)} (fg)(x,y) = FG$
- $\lim\limits_{(x,y)\to(a,b)} \dfrac{f(x,y)}{g(x,y)} = \dfrac{F}{G}$, provided $G \neq 0$ and $g$ is defined in a small interval around $(a,b)$.

### 2.3.2. Composition

Let $f$ be a multivariable function and $g$ be a single variable function such that $g \circ f$ is defined. If $\lim\limits_{(x,y)\to(a,b)} f(x,y) = F$ and $\lim\limits_{x\to F} g(x) = L$, then:

$$\lim_{(x,y)\to(a,b)} (g \circ f)(x,y) = L$$

### 2.3.3. Sandwich principle

If $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$ and $\lim\limits_{(x,y)\to(a,b)} g(x,y) = L$, and $f(x) \leqslant h(x) \leqslant g(x)$, we have:

$$\lim_{(x,y)\to(a,b)} h(x,y) = L$$

## 2.4. Continuity

A function $f$ is continuous at the point $(a,b)$ in its domain if the limit at the point exists and is equal to the value of the function at that point. That is:

$$\lim_{(x,y)\to(a,b)} f(x,y) = f(a,b)$$

If a function is continuous at all points in its domain, it is said to be a continuous function.

# 3. Derivatives

## 3.1. Partial Derivatives

The partial derivative of a function $f$ with respect to $x$ and $y$ at the point $(a, b)$ is denoted by $f_x$ and $f_y$ respectively and is given as:

$$f_x(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}$$

$$f_y(a,b) = \lim_{h \to 0} \frac{f(a,b+h) - f(a,b)}{h}$$

For example, $f_x(1,2)$ for $f(x,y) = x^2 + y^2$ can be computed as:

$$\begin{aligned} f_x(1,2) &= \lim_{h \to 0} \frac{(1+h)^2 + 2^2 - \left(1^2 + 2^2\right)}{h} \\ &= \lim_{h \to 0} \frac{(1+h)^2 - 1}{h} \\ &= \lim_{h \to 0} \frac{h(2+h)}{h} \\ &= 2 \end{aligned}$$

$f_x(a,b)$ can be viewed as the instantaneous rate of change of $f$ at $(a,b)$ along the x-axis. A similar idea holds for $f_y(a,b)$. The partial derivatives are themselves functions of $(x,y)$. We denote this as $f_x(x,y)$ and $f_y(x,y)$. To compute the partial derivative with respect to $x$, we can view $f$ as function of $x$ while $y$ remains constant, and differentiate with respect to $x$. For example, if $f(x,y) = x^2 + y^2$, $f_x(x,y) = 2x$ and $f_y(x,y) = 2y$.

## 3.2. Gradient

The partial derivatives can be clubbed together into a vector called the gradient:

$$\nabla f(x,y) = \begin{bmatrix} f_x(x,y) \\ f_y(x,y) \end{bmatrix}$$

For example, if $f(x,y) = x^2 + y^2$, $\nabla f(x,y) = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$.

## 3.3. Directional Derivatives

Directional derivatives extend the notion of partial derivatives to any direction. The directional derivative of $f$ in the direction given by the unit vector $\mathbf{u} = (u_1, u_2)$ at $(a,b)$ is given by $f_{\mathbf{u}}(a,b)$:

$$f_{\mathbf{u}}(a,b) = \lim_{h \to 0} \frac{f(a + hu_1, b + hu_2) - f(a,b)}{h}$$

$f_{\mathbf{u}}(x,y)$ is again a function of $x, y$. As an example, if $f(x,y) = xy$, the directional derivative along $\mathbf{u} = (u_1, u_2)$ can be computed as:

$$\begin{aligned} f_{\mathbf{u}}(x,y) &= \lim_{h \to 0} \frac{f(x + hu_1, y + hu_2) - f(x,y)}{h} \\ &= \lim_{h \to 0} \frac{(x + hu_1)(y + hu_2) - xy}{h} \\ &= \lim_{h \to 0} \frac{xy + xhu_2 + yhu_1 + h^2u_1u_2 - xy}{h} \\ &= yu_1 + xu_2 \end{aligned}$$

Theorem

If the partial derivatives are continuous in an open ball centered at $(a, b)$, the directional derivative at $(a, b)$ is defined for all directions. For any direction given by the unit vector $\mathbf{u} = (u_1, u_2)$, the directional derivative is given by:

$$f_{\mathbf{u}}(a, b) = \nabla f(a, b)^T \mathbf{u}$$

Recall that $u^T v$ is just the dot product of $u$ and $v$.

To see this theorem in action, for $f(x, y) = xy$, we have:

$$\begin{aligned} f_u(x, y) &= \nabla f(x, y)^T \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ &= \begin{bmatrix} y & x \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \\ &= yu_1 + xu_2 \end{aligned}$$

## 3.4. Steepest Ascent and Descent

Since $||\mathbf{u}|| = 1$, we see that:

$$\begin{aligned} f_{\mathbf{u}}(a, b) &= \nabla f(a, b)^T \mathbf{u} \\ &= ||\nabla f(a, b)|| \cdot ||\mathbf{u}|| \cdot \cos\theta \\ &= ||\nabla f(a, b)|| \cdot \cos\theta \end{aligned}$$

The maximum and minimum values of $f_{\mathbf{u}}(a, b)$ occur when $\theta = 0$ and $\theta = 180^\circ$ respectively. If $\theta = 90^\circ$, then the directional derivative is zero. Therefore:

- The direction of steepest ascent of the function at $(a, b)$ is $\dfrac{\nabla f(a, b)}{||\nabla f(a, b)||}$.

- The direction of steepest descent of the function at $(a, b)$ is $\dfrac{-\nabla f(a,b)}{||\nabla f(a,b)||}$.

- The direction in which the function's value doesn't change at $(a, b)$ is orthogonal to $\nabla f(a, b)$.

Note that all these are local phenomenon. When we talk about the direction of steepest ascent at a point $(a, b)$, we are only concerned about the function's behavior in the immediate vicinity of $(a, b)$.

## 3.5. Second Order Partial Derivatives

The second order partial derivatives are given below:

$$f_{xx} = (f_x)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2}$$

$$f_{yy} = (f_y)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2}$$

$$f_{xy} = (f_x)_y = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x}$$

$$f_{yx} = (f_y)_x = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y}$$

$f_{xy}$ and $f_{yx}$ are called mixed partial derivatives.

<u>Theorem</u>

Let $f$ be a function defined on a domain $D$ with a point $(a, b)$ and an open ball around it contained in $D$. If the second order partial derivatives are continuous in an open ball around $(a, b)$, then $f_{xy}(a, b) = f_{yx}(a, b)$.

All the second order partial derivatives can be collected in a matrix termed the Hessian matrix:

$$H = \begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$$

If the theorem above holds, $H$ is a symmetric matrix. For a function of three variables, the Hessian will be a square matrix of order three:

$$H = \begin{bmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix}$$

**Question**

Find all partial derivatives up to order 2 at $(0,0)$.

$$f(x,y) = \begin{cases} \dfrac{xy\left(x^2-y^2\right)}{x^2+y^2}, & (x,y) \neq (0,0) \\ 0, & (x,y) = (0,0) \end{cases}$$

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For $(x,y) = (0,0)$:

$$f_x(0,0) = \lim_{h \to 0} \frac{f(h,0) - f(0,0)}{h} = 0$$

$$f_y(0,0) = \lim_{h \to 0} \frac{f(0,h) - f(0,0)}{h} = 0$$

For $(x,y) \neq (0,0)$:

$$f_x(x,y) = \frac{\left(x^2+y^2\right)\left[3x^2y - y^3\right] - 2x^2y\left(x^2-y^2\right)}{\left(x^2+y^2\right)^2}$$

$$= \frac{x^4y + 4x^2y^3 - y^5}{\left(x^2+y^2\right)^2}$$

$$(x^2+y^2)$$

Therefore:

$$f_x(x,y) = \begin{cases} \dfrac{y\left[x^4+4x^2y^2-y^4\right]}{\left(x^2+y^2\right)^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$$

$$f_{xx}(0,0) = \lim_{h\to 0}\frac{f_x(h,0)-f_x(0,0)}{h} = 0$$

$$f_{xy}(0,0) = \lim_{h\to 0}\frac{f_x(0,h)-f_x(0,0)}{h} = \lim_{h\to 0}\frac{-h^5}{h^5} = -1$$

Similarly, we compute $f_y(x,y)$:

$$f_y(x,y) = \begin{cases} \dfrac{x\left(x^4-4x^2y^2-y^4\right)}{\left(x^2+y^2\right)^2}, & (x,y)\neq(0,0) \\ 0, & (x,y)=(0,0) \end{cases}$$

This gives us $f_{yy}(0,0)=0$ and $f_{yx}(0,0)=1$.

**NOTE: IGNORE FOR END-TERM, CAN BE CONFUSING**

During the live session, there was a mistake in the expression for $f_y(x,y)$. This is what was being attempted. One way to quickly compute $f_y(x,y)$ is to note the following relationship:

$$\begin{aligned} f(x,y) &= -f(y,x) \\ \implies f_y(x,y) &= -f_y(y,x) \end{aligned}$$

$f_y(y,x)$ is the partial derivative of $f$ with respect to the first variable. Therefore, this is the same as partial derivative of $f(x,y)$ with respect to $x$, and then swapping the variables $x$ and $y$. Therefore, we have $f_y(y,x)=f_x(y,x)$. This gives us:

$$f_y(x,y) = -f_x(y,x)$$

# 4. Tangents

For all subsections, consider a function $f$ defined on a domain $D \subset \mathbb{R}^2$ that has *continuous partial derivatives* in an open ball centered at the point $(a, b) \in D$. The existence of continuous partial derivatives implies that the directional derivative exists in all directions and is given by:

$$f_{\mathbf{u}}(a, b) = \nabla f(a, b)^T \mathbf{u}$$

## 4.1. Tangent Lines

The tangent line to a function $f$ at the point $(a, b)$ in the direction of the unit vector $\mathbf{u} = (u_1, u_2)$ is given by the following equivalent forms:

**Form-1**

$$(a, b, f(a, b)) + t(u_1, u_2, f_{\mathbf{u}}(a, b))$$

$$(a, b, f(a, b)) + \mathsf{span}\{(u_1, u_2, f_{\mathbf{u}}(a, b))\}$$

**Form-2**

$$\begin{aligned} x &= a + tu_1 \\ y &= b + tu_2 \\ z &= f(a, b) + tf_{\mathbf{u}}(a, b) \end{aligned}$$

**Form-3**

$$\frac{x - a}{u_1} = \frac{y - b}{u_2} = \frac{z - f(a, b)}{f_{\mathbf{u}}(a, b)}$$

The tangent line is an affine subspace of $\mathbb{R}^3$.

**Question**

Find the tangent line to the function $f(x, y) = xy$ at the point $(1, 3)$ in the direction $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

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$$(1, 3, f(1, 3)) + t\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, f_{\mathbf{u}}(1, 3)\right)$$

$$f_{\mathbf{u}}(1, 3) = \begin{bmatrix} 3 & 1 \end{bmatrix} \begin{bmatrix} 1 / \sqrt{2} \\ 1 / \sqrt{2} \end{bmatrix} = \frac{4}{\sqrt{2}}$$

The tangent line at the point $(1, 3)$ to $f$ in the direction $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ is given by:

$$(1, 3, 3) + t\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, \frac{4}{\sqrt{2}}\right)$$

Alternatively:

$$x = 1 + \frac{t}{\sqrt{2}}$$
$$y = 3 + \frac{t}{\sqrt{2}}$$
$$z = 3 + \frac{4t}{\sqrt{2}}$$

Alternatively:

$$\frac{x - 1}{1 / \sqrt{2}} = \frac{y - 3}{1 / \sqrt{2}} = \frac{z - 3}{4 / \sqrt{2}}$$

## 4.2. Tangent Hyperplanes

The tangent plane to the function at $(a, b)$ is given by:

**Form-1**

$$z = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

**Form-2**

$$(a, b, f(a, b)) + \{(x, y, z) : f_x(a, b)x + f_y(a, b)y - z = 0\}$$

**Form-3**

$$z = f(a, b) + \nabla f(a, b)^T \begin{bmatrix} x - a \\ y - b \end{bmatrix}$$

The vector $(f_x(a, b), f_y(a, b) - 1)$ is orthogonal to the tangent plane.

**Question**

Find the tangent plane to the function $f(x, y) = \sqrt{9 - x^2 - y^2}$ at the point $(1, 2)$.

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$$z - f(1, 2) = f_x(1, 2)(x - 1) + f_y(1, 2)(y - 2)$$

$$f_x = \frac{(-2x)}{2\sqrt{9 - x^2 - y^2}} = \frac{-x}{\sqrt{9 - x^2 - y^2}} = \frac{-1}{2} = \frac{-1}{2}$$

$$f_y = \frac{(-2y)}{2\sqrt{9 - x^2 - y^2}} = \frac{-y}{\sqrt{9 - x^2 - y^2}} = \frac{-2}{2} = -1$$

$$z - 2 = \frac{-1}{2}(x - 1) - (y - 2)$$

The equation of the tangent plane at $(1, 2)$ to $f$ is:

$$x + 2y + 2z = 9$$

## 4.3. Best Linear Approximation

The best linear approximation to the function at $(a, b)$ is given by $L_f(x, y)|_{(a,b)}$:

$$L_f(x, y)|_{(a,b)} = f(a, b) + \nabla f(a, b)^T \begin{bmatrix} x - a \\ y - b \end{bmatrix}$$

Expanding it, we get:

$$L_f(x, y)|_{(a,b)} = f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b)$$

**Question**

Find the best linear approximation to $f$ at $(0,0)$ where, $f(x,y) = e^x \sin y + xe^x$.

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$$L_f(x,y)|_{(0,0)} = f(a,b) + \ \nabla f(a,b)^T \begin{bmatrix} x-a \\ y-b \end{bmatrix}$$

$$f(0,0) = 0$$

$$\nabla f(x,y) = \begin{bmatrix} e^x(\sin y + x + 1) \\ e^x \cos y \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$L_f(x,y)|_{(0,0)} = x + y$$

## 5. Differentiability

Consider a function $f$ defined on a domain $D \subset \mathbb{R}^2$. Let $f_x$ and $f_y$ be defined at $(a,b) \in D$. Then, $f$ is differentiable at $(a,b) \in D$ if:

$$\lim_{(h_1,h_2) \to (0,0)} \quad \frac{f(a+h_1, b+h_2) - f(a,b) - f_x(a,b)h_1 - f_y(a,b)h_2}{\sqrt{h_1^2 + h_2^2}} = 0$$

This is equivalent to:

- The tangent plane to $f$ exists at $(a,b)$.
- The best linear approximation to $f$ exists at $(a,b)$.

Intuitively, this means that very close to the point $(a,b)$ $f$ looks like a plane (behaves like a linear function).

Theorem
If the partial derivatives exist in an open ball at $(a, b)$ and are continuous, then $f$ is differentiable at $(a, b)$.

# 6. Optimization

Consider a function defined over $D \subset \mathbb{R}^2$ and a point $(a, b) \in D$ with an open ball centered at $(a, b)$ that is contained in $D$.

## 6.1. Maxima and Minima

- A point $(a, b)$ is called a local minimum of $f$ if $f(x, y) \geqslant f(a, b)$ for all $(x, y)$ in an open ball centered at $(a, b)$.
- A point $(a, b)$ is called a local maximum of $f$ if $f(x, y) \leqslant f(a, b)$ for all $(x, y)$ in an open ball centered at $(a, b)$.
- A point $(a, b)$ is called a absolute minimum of $f$ if $f(x, y) \geqslant f(a, b)$ for all $(x, y)$ in the domain of $f$.
- A point $(a, b)$ is called a absolute maximum of $f$ if $f(x, y) \leqslant f(a, b)$ for all $(x, y)$ in the domain of $f$.

A point that is a maximum or a minimum is called an extremum.

## 6.2. Critical Points

Consider a function $f$ with domain $D \subset \mathbb{R}^2$. A point $(a, b)$ is a critical point of $f$ if one of two things happen:

- The gradient of $f$ is not defined at $(a, b)$.
- The gradient of $f$ is zero.

For example, consider $f(x, y) = x^2 + 6xy + 4y^2 + 2x - 4y$.

$$\begin{aligned} f_x(x, y) &= 2x + 6y + 2 = 0 \\ f_y(x, y) &= 8y + 6x - 4 = 0 \end{aligned} \implies (x, y) = (2, -1)$$

$(2,-1)$ is a critical point of $f$.

<u>Theorem</u>
If a point $(a, b)$ is a local extremum, then $\nabla f(a, b) = \mathbf{0}$.

To see why this is true, consider what happens if $\nabla f(a, b) \neq \mathbf{0}$. The function's value would increase along the direction $\nabla f(a, b)$ and decrease along $-\nabla f(a, b)$ suggesting that $(a, b)$ is not an extremum. This is a loose argument. For a more concrete argument, consider the slice of the function along $y = b$ and note that for the corresponding single variable function in terms of $x$, $x = a$ is an extremum, which implies that $f_x(a, b) = 0$.

Every local extremum is a critical point. All critical points are not local extrema. A point which is a critical point but not a local extremum is called a **saddle point**. An example is the function $f(x, y) = xy$. This function has a critical point at $(0, 0)$, but this is neither a local maximum nor a local minimum. The function has a minimum along $y = x$ and a maximum along $y = -x$.

## 6.3. Continuous functions on Closed, Bounded Domain

A set $D \subset \mathbb{R}^2$ is

- closed if it contains all its boundary points
- bounded if it is contained within a ball around $\mathbf{0}$ with finite radius.

An example of a closed, bounded domain is a square that contains its boundary:

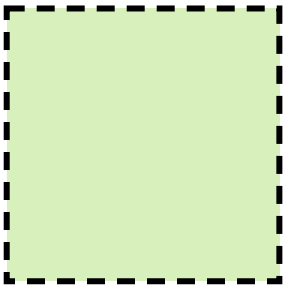
$$\{(x, y) : 0 \leqslant x \leqslant 2, 0 \leqslant y \leqslant 2\}$$

An example of a domain that is bounded but not closed is a square that doesn't contain its boundary:

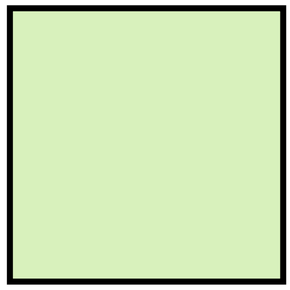
$$\{(x, y) : 0 < x < 2, 0 < y < 2\}$$

Visually:

Open, Bounded



Closed, Bounded



Theorem
Consider a continuous function $f$ on a domain $D$ that is closed and bounded. Then $f$ has a global maximum and global minimum on $D$.

The maximum and minimum could be one of these points:

- A point in the interior of $D$.
- A point on the boundary of $D$.

**Question**
Find the global maximum and minimum of the function $f(x, y) = x^2 + y^2$ on the closed triangular plate bounded by the lines $x = 0, y = 0, y = 2x + 2$ in the second quadrant.

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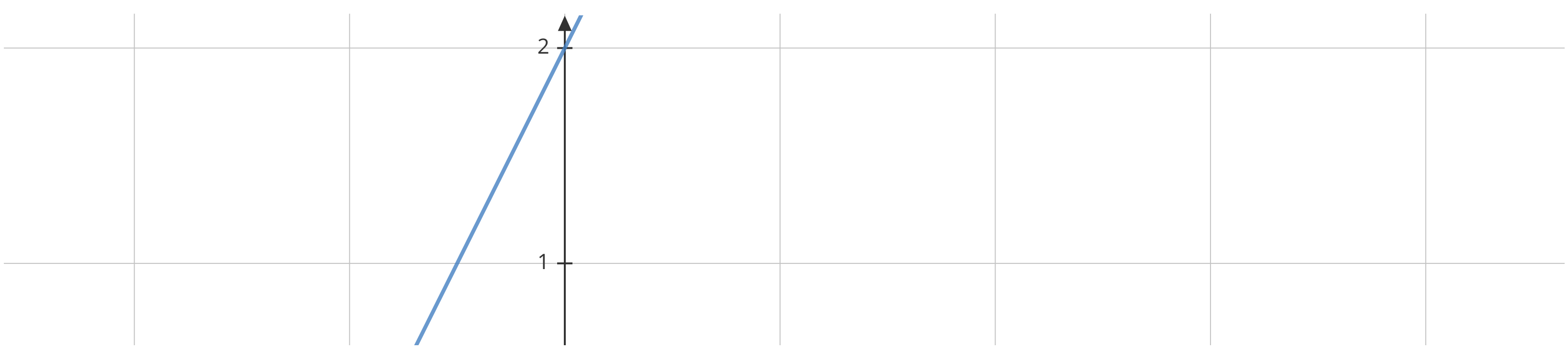
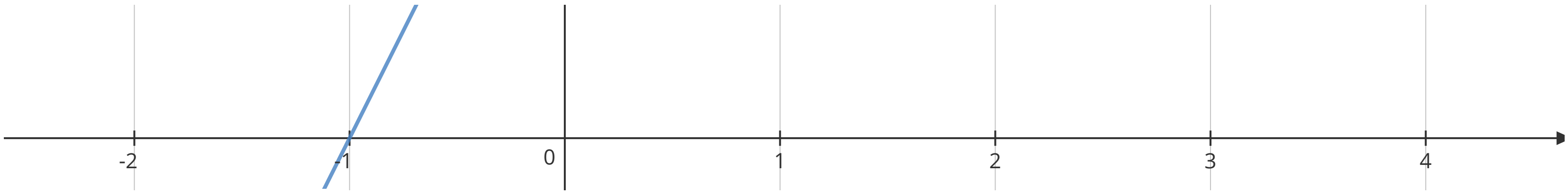
**Step-1: Critical points**

$$\nabla f(x, y) = (2x, 2y)$$

$(0, 0)$ is the only critical point and is one of the candidates for global maximum/minimum.

**Step-2: Graph the region**






Step-3: Check the boundaries

**Boundary-1**: $x = 0$

$$h(y) = f(0, y) = y^2$$

Check for $(0, 0)$ and $(0, 2)$.

**Boundary-2**: $y = 0$

$$h(x) = f(x, 0) = x^2$$

Check for $(0, 0)$ and $(-1, 0)$.

**Boundary-3**: $y = 2x + 2$

$$h(x) = f(x, 2x + 2) = x^2 + (2x + 2)^2 = 5x^2 + 8x + 4$$

Note that $h$ is defined in the interval $[-1, 0]$. In this interval, the global maximum of $h$ can be a critical point or one of the boundary points.

$$h'(x) = 0 \implies 10x + 8 = 0 \implies x = \frac{-4}{5}$$

Check for $(-4\,/\,5, 2\,/\,5)$, $(-1, 0)$ and $(0, 2)$.

| $(a, b)$ | $f(a, b)$ |
|---|---|
| $(0, 0)$ | $0$ |
| $(0, 2)$ | $4$ |

| | |
|---|---|
| $(-1,0)$ | $1$ |
| $(-4\,/\,5,2\,/\,5)$ | $\frac{4}{5}$ |

Therefore, the global maximum of $f$ is $4$ at $(0,2)$ and the global minimum is $0$ at $(0,0)$.

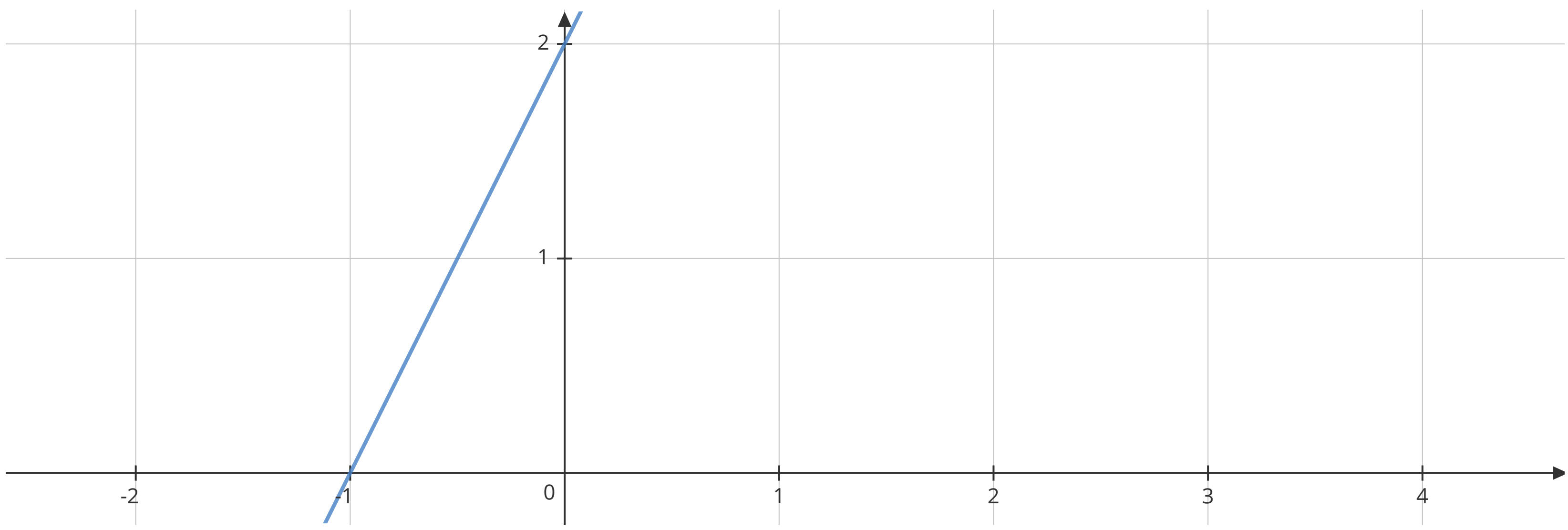
**Aside**

$D$ is the triangular region mentioned above:

$$f: D \to \mathbb{R}$$

$$f(x,y) = x^2 + y^2$$

What is $f_x(0,0)$?




$$f_x(0,0) = \lim_{h\to 0^-} \frac{f(0+h,0) - f(0,0)}{h} = \lim_{h\to 0^-} \frac{h^2 - 0}{h} = 0$$

$$f_y(0,0) = \lim_{h\to 0^+} \frac{f(0,0+h) - f(0,0)}{h} = \lim_{h\to 0^+} \frac{h^2 - 0}{h} = 0$$

## 6.4. Hessian Test

The Hessian test can be used to determine the nature of the critical point. The test may fail to be conclusive in certain situations.

### 6.4.1. Two variables

Consider a function $f$ defined on $D \subset \mathbb{R}^2$ which has a critical point at $(a, b) \in D$ with continuous first and second order partial derivatives in an open ball centered at $(a, b)$. Let $H$ be the Hessian at $(a, b)$ and let $f_{xx}, f_{yy}, f_{xy}, f_{yx}$ denote the second order partial derivatives at $(a, b)$:

- If $|H| > 0$ and $f_{xx} > 0$, $(a, b)$ is a local minimum.
- If $|H| > 0$ and $f_{xx} < 0$, $(a, b)$ is a local maximum.
- If $|H| < 0$, $(a, b)$ is a saddle point.
- If $|H| = 0$, the test is inconclusive.

Here $|H|$ denotes the determinant of $H$.

### 6.4.2. Three variables

Consider a function $f$ defined on $D \subset \mathbb{R}^3$ which has a critical point at $(a, b, c) \in D$ with continuous first and second order partial derivatives in an open ball centered at $(a, b, c)$. Let $H$ be the Hessian at $(a, b, c)$ and let $f_{xx}, f_{yy}, f_{zz}, f_{xy}, f_{yz}, f_{zx}$ denote the second order partial derivatives at $(a, b, c)$:

- If $|H| > 0$, $f_{xx} > 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$, then $(a, b, c)$ is a local minimum.
- If $|H| < 0$, $f_{xx} < 0$ and $f_{xx}f_{yy} - f_{xy}^2 > 0$, then $(a, b, c)$ is a local maximum.
- If $|H| \neq 0$ and the above two cases do not occur, then $(a, b, c)$ is a saddle point.
- If $|H| = 0$, the test is inconclusive.

Here $|H|$ denotes the determinant of $H$. Visually, this test involves the determinants of three matrices:

$$\begin{bmatrix} \color{red}{f_{xx}} & f_{xy} & f_{xz} \\ f_{yx} & f_{yy} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix}, \begin{bmatrix} \color{red}{f_{xx}} & \color{red}{f_{xy}} & f_{xz} \\ \color{red}{f_{yx}} & \color{red}{f_{yy}} & f_{yz} \\ f_{zx} & f_{zy} & f_{zz} \end{bmatrix}, \begin{bmatrix} \color{red}{f_{xx}} & \color{red}{f_{xy}} & \color{red}{f_{xz}} \\ \color{red}{f_{yx}} & \color{red}{f_{yy}} & \color{red}{f_{yz}} \\ \color{red}{f_{zx}} & \color{red}{f_{zy}} & \color{red}{f_{zz}} \end{bmatrix}$$

| $f_{xx}$ | $f_{xx}f_{yy} - f_{xy}^2$ | $\|H\|$ | Outcome |
|---|---|---|---|
| $+$ | $+$ | $+$ | Local minimum |
| $-$ | $+$ | $-$ | Local maximum |

For additional reading, lookup Sylvester's theorem in Wikipedia.

**Question**

Find all the points of local extrema for the function $f(x, y) = x^3 - 3x + y^3 - 3y^2$.

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$$\begin{aligned} f_x(x,y) &= 3x^2 - 3 = 0 \\ f_y(x,y) &= 3y^2 - 6y = 0 \end{aligned} \implies x = \pm 1, y = 0, 2$$

This function has $4$ critical points, namely:

$$(1,0), (1,2), (-1,0), (-1,2)$$

$$\begin{aligned} f_{xx} &= 6x & f_{xy} &= 0 \\ f_{yx} &= 0 & f_{yy} &= 6y - 6 \end{aligned}$$

$$H = \begin{bmatrix} 6x & 0 \\ 0 & 6y - 6 \end{bmatrix}$$

| $(a,b)$ | $f_{xx} = 6x$ | $\lvert H\rvert = 36x(y-1)$ | Outcome |
|---|---|---|---|
| $(1,0)$ | $6$ | $-36$ | Saddle point |
| $(1,2)$ | $6$ | $36$ | Local minimum |
| $(-1,0)$ | $-6$ | $36$ | Local maximum |
| $(-1,2)$ | $-6$ | $-36$ | Saddle point |