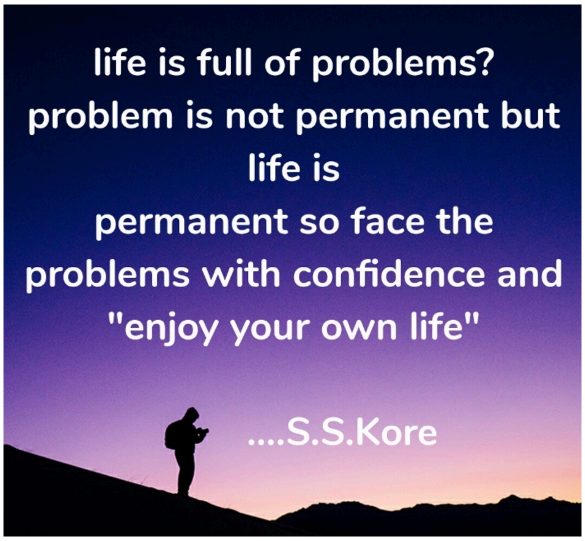
→ Contests.
[O.A].

→ Revision.

Recursion.
↳ Revise



Today's content

→ Addition & multiplication rule

→ Permutation basics

→ Combination basics & properties

→ $^NC_r \, \% \, P$

→ Permutation with repeatition

Given 3 T/F questions, every question have to be answered. In how many ways can we answer all the questions?

F	F	F
F	F	T
F	T	F
T	F	F
F	T	T
T	F	T
T	T	F
T	T	T

$\Rightarrow$ 8 ways

$$\underline{2} * \underline{2} * \underline{2} = 2^3 = \underline{8}.$$

(T F) (T F) (T F)

Eg2 → Given 10 Girls & 7 Boys. How many different pairs?

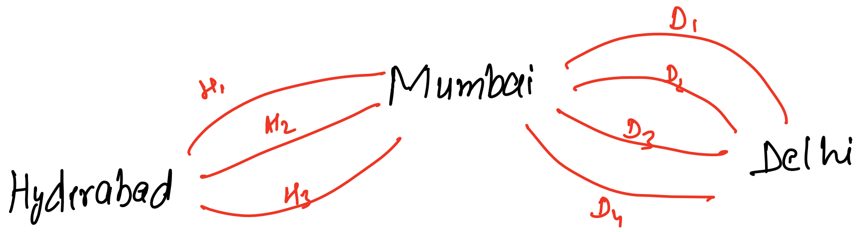
[pair → 1 Girl + 1 Boy]

Boys:	Girls.
B_1	G_1
B_2	G_2
B_4	G_3
B_4	G_4
B_5	G_5
B_6	G_6
B_7	G_7
	G_8
	G_9
	G_{10}

1 Boy & 1 Girls.

$7 * 10 = \underline{70 \text{ ways}}.$

Eg 3.

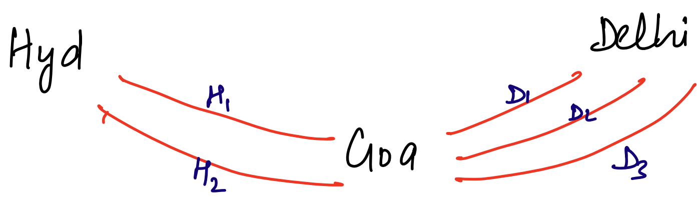


No. of ways to reach Delhi from Hyderabad via Mumbai?

ways Hyd → Mumbai & Mumbai → Delhi

3 * 4 = 12 ways.

Eg 4:

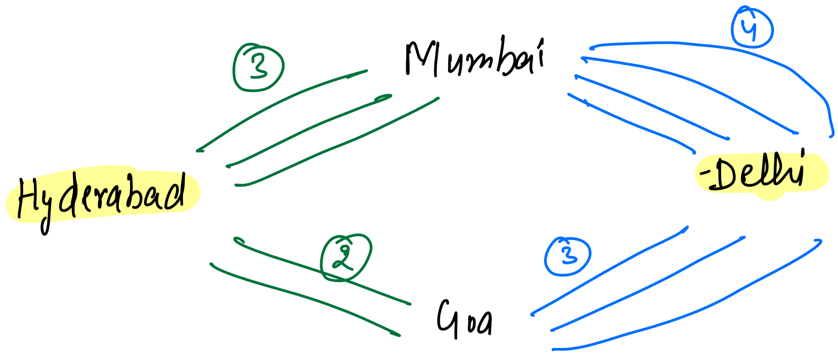


No. of ways to reach Delhi from Hyderabad via Goa?

Hyd → Goa & Goa → Delhi

2 * 3 = 6 ways.

Eg:



No. of ways to reach Delhi from Hyderabad.

Hyd → Delhi via Mumbai or Hyd → Delhi via Goa

$(3*4) + (2*3) = 12 + 6 = 18$ ways.

// Say we need to buy a valentine's gift?

(pen & book) or (flower & chocolate) or (ring)

pen → 3, book → 5, flower → 7, chocolate → 3, ring → 3

No. of ways in which you can select a gift?

$(3 * 5) + (7 * 3) + 3 = 15 + 21 + 3 = 39$ ways.

{AND → *
OR → +}

Permutations : arrangement of objects.

In general $(i, j) \neq (j, i)$: order matters

Q1 Given 3 distinct characters. In how many ways, we can arrange them?

s = "a c d"

$$\left.\begin{array}{l} \to a\ c\ d \\ \to a\ d\ c \\ \to c\ a\ d \\ \to c\ d\ a \\ \to d\ a\ c \\ \to d\ c\ a \end{array}\right\} \to 6 \text{ arrangements}$$

$\underline{3} \times \underline{2} \times \underline{1} = 3! = 6.$

a → c → d
a → d → c

c → a → d
c → d → a

d → a → c
d → c → a

Q2 In how many ways, you can arrange 4 distinct characters?

$$\underset{4}{\underline{\ \ }} \times \underset{3}{\underline{\ \ }} \times \underset{2}{\underline{\ \ }} \ \underset{1}{\underline{\ \ }} = 4!$$

Q: In how many ways n distinct characters can be arranged?

$$\underline{N} * \underline{N-1} * \underline{N-2} * \underline{N-3} \; \_ \; \_ \; \_ \; \_ * \underline{1} = \underline{N!}$$

Q.) Given 5 distinct characters, in how many ways can we arrange 2 characters?

{a b c d e} $\_ \; \_$

a {b c d e}
b {a c d e}
c {a b d e}
d {a b c e}
e {a b c d}

$\Rightarrow$ 20 ways.

Q.) Given N distinct characters. In how many ways can we arrange 3.

$$\underline{N} * \underline{N-1} * \underline{N-2} = N(N-1)(N-2)$$

arrange 4?

$$N * (N-1) * (N-2) * (N-3)$$

Q1 Given N distinct characters. In how many ways can we arrange r characters?

$\underline{N} * \underline{(N-1)} * \underline{(N-2)} * \underline{(N-3)} \quad \_\_ \quad \_\_ \quad \underline{N-(r-1)} = N-r+1$

$$\# \text{ways} = \frac{N(N-1)(N-2) \text{---} (N-r+1) * (N-r) * (N-r-1) * (N-r-2) \text{--} 2*1}{(N-r) * (N-r-1) * (N-r-2) * \text{---} * 2 * 1}$$

$$\Rightarrow \left\{ \frac{N!}{(N-r)!} = {}^{N}P_{R} \right\}$$

Combinations - {selection} → no. of ways to select.

$P_1 \to (i, j)$
$P_2 \to (j, i)$

Q.) Given 4 players, count no. of ways of selecting 3 players.

$[P_1 \quad P_2 \quad P_3 \quad P_4]$

$$\begin{bmatrix} P_1 & P_2 & P_3 \\ P_1 & P_2 & P_4 \\ P_1 & P_3 & P_4 \\ P_2 & P_3 & P_4 \end{bmatrix}$$

{ans = 4}

Given 4 players.

$${}^4P_3 = \frac{4!}{(4-3)!} = \frac{4 \times 3 \times 2 \times 1}{1} = 24.$$

Q) No. of ways to arrange the players in 3 slots.

P_1	P_2	P_3	P_1	P_2	P_4	P_1	P_3	P_4	P_2	P_3	P_4
P_1	P_3	P_2	P_1	P_4	P_2	P_1	P_4	P_3	P_2	P_4	P_3
P_2	P_1	P_3	P_2	P_1	P_4	P_3	P_1	P_4	P_3	P_2	P_4
P_2	P_3	P_1	P_2	P_4	P_1	P_3	P_4	P_1	P_3	P_4	P_2
P_3	P_1	P_2	P_4	P_1	P_2	P_4	P_1	P_3	P_4	P_2	P_3
P_3	P_2	P_1	P_4	P_2	P_1	P_4	P_3	P_1	P_4	P_3	P_2

sel: $\{P_1\ P_2\ P_3\}$ sel: $\{P_1\ P_2\ P_4\}$ sel: $\{P_1\ P_3\ P_4\}$ sel: $\{P_2\ P_3\ P_4\}$

$x * 6 = 24 \Rightarrow x = \frac{24}{6} = 4 \Rightarrow$ no. of selection.

Q: Given N distinct elements, in how many ways can we select r items

// Given N distinct elements, arrange r items ⇒ $^{N}P_r = \frac{N!}{(N-r)!}$

// Arrange r items → $r!$

arrange.	no. of selections.
$r!$	1
$\frac{N!}{(N-r)!}$	x

$x * r! = 1 * \frac{N!}{(N-r)!}$

$$\left\{ \boxed{x = \frac{N!}{(N-r)!\, r!}} \right\}$$

⇑

$^{N}C_r$.

// Given N distinct elements, select r elements →

$$\left\{ \frac{N!}{(N-r)!\; r!} = \frac{NP_r}{r!} = {}^{N}C_r \right\}$$

// $^{N}C_0 = \frac{\cancel{N!}}{\cancel{(N)!}\; 0!} = \frac{1}{1.1} = 1.$

{Don't pick any element}

⇑

[In how many ways, you can select '0' elements from N elements = 1]

Subsets.

$\{a, b, c\}$

$$
\left.
\begin{aligned}
{}^3C_0 &= \{\ \} \\
{}^3C_1 &= \{a\}, \{b\}, \{c\} \\
{}^3C_2 &= \{a, b\}, \{a, c\}, \{b, c\} \\
{}^3C_3 &= \{a, b, c\}
\end{aligned}
\right\}
\quad {}^3C_0 + {}^3C_1 + {}^3C_2 + {}^3C_3 \Rightarrow 2^3 = 8.
$$

$$
\begin{array}{ccccc} a & & b & & c \\ \{ 2 & \times & 2 & \times & 2 \} \Rightarrow 8 \end{array}
$$

$$\boxed{{}^NC_0 + {}^NC_1 + {}^NC_2 + \text{———} \; {}^NC_N = 2^N}$$

// 4 elements

$$
\left.
\begin{aligned}
&{}^4C_0 \rightarrow 1 \\
&+ \\
&{}^4C_1 \rightarrow 4 \\
&+ \\
&{}^4C_2 \rightarrow 6 \\
&+ \\
&{}^4C_3 \rightarrow 4 \\
&+ \\
&{}^4C_4 \rightarrow 1
\end{aligned}
\right\}
\quad \text{ans} = 16
$$

Properties.

N distinct elements, select r items $\rightarrow$ ${}^{N}C_r$

_ _ _ _ _ _ _ _ _ _ N^{th}

select $\rightarrow$ ${}^{N-1}C_{r-1}$

or $\qquad +$

Don't select $\rightarrow$ ${}^{N-1}C_r$

$$\Rightarrow \boxed{{}^{N}C_r = {}^{N-1}C_{r-1} + {}^{N-1}C_r} \rightarrow \text{property 1.}$$

$$= \frac{(N-1)!}{(N-1-(r-1))! * (r-1)!} + \frac{(N-1)!}{(N-1-r)!\, r!}$$

$$= \frac{(N-1)!}{(N-r)! * (r-1)!} + \frac{(N-1)!}{(N-r-1)!\, r!}$$

$$= \frac{(N-1)! * r}{(N-r)\,(N-r-1)! * r * (r-1)!} + \frac{(N-1)!}{(N-r-1)!\, r!}$$

$$\Rightarrow \frac{(N-1)! * r}{(N-r) * (N-r-1)! * r!} + \frac{(N-1)!}{(N-r-1)!\, r!}$$

$$= \frac{(N-1)!}{(N-r-1)!\, r!}\left[\frac{r}{N-r} + 1\right]$$

$$= \frac{(N-1)!}{(N-r-1)!\, r!} \left[\frac{\cancel{r} + N - \cancel{r}}{(N-r)} \right]$$

$$= \frac{N!}{(N-r)!\, r!} = {}^{N}C_{R} .$$

$\therefore$ R.H.S = L.H.S , Hence Proved !!

$\{$ Break. $\rightarrow$ 10:44 $\rightarrow$ 10:50 PM $\}$

// 5 Boys → B_1 B_2 B_3 B_4 B_5

B_1 B_2 : B_3 B_4 B_5

B_1 B_3 : B_2 B_4 B_5

B_1 B_4 : B_2 B_3 B_5

B_1 B_5 : - - -

B_2 B_3 : - - -

B_2 B_4 : - - -

B_2 B_5 : - - -

B_3 B_4 : - - -

B_3 B_5 : - - -

B_4 B_5 : - - -

$$\boxed{{}^{N}C_{R} = {}^{N}C_{N-R}}$$ ← 2nd property

$${}^{5}C_{2} = \frac{5!}{(5-2)!\,2!} = \frac{5!}{3!\,2!}$$

$${}^{5}C_{3} = \frac{5!}{(5-3)!\,3!} = \frac{5!}{2!\,3!}$$

Q) Given N, R and P. Calculate $({}^N C_R) \% P$.

Note → p is prime, $N, R \le P$.

Constraints
$$\left\{ \begin{array}{l} 1 \le (N, R) \le 10^5 \\ R < N < P \\ p \text{ is prime} \end{array} \right\}$$

$${}^N C_R = \left(\frac{N!}{(N-R)!\ R!} \right) \% p$$

$${}^N C_R = \left(N! * ((N-R)!)^{-1} * (R!)^{-1} \right) \% p$$

$$= \underline{N! \% p} * \left\{ \underset{?}{\underline{((N-R)!)^{-1} \% p}} * \underset{?}{\underline{(R!)^{-1} \% p}} \right\} \% p$$

✓

$a^{-1} \% p$ $a^{-1} \% p$

$$\left[\begin{array}{l} T.C \to N + (N-R) + \log P + R + \log P \\ \{ T.C \to O(N + \log P) \} \quad \{ S.C \to O(\log P) \} \end{array} \right]$$

Idea. $a^{-1} \% p \xrightarrow[p \text{ is prime}]{\gcd(a, p) = 1} a^{p-2} \% p$] Fermat's theorem

$\left(\underbrace{(N-R)!}_{a}\right)^{-1} \% \, p$

$\left[\begin{array}{l} \gcd((N-R)!\,,\, p) = 1 \\ p \text{ is prime no.} \end{array}\right]$

$//\ N < p.$ (given)

$(N-R) < p$

$\left[p \text{ is a prime no.} > (N-R) \right]$

$(N-R)! = 1 * \underset{\downarrow}{2}_{\,p} * \underset{\downarrow}{3}_{\,p} * \underset{\downarrow}{4}_{\,p} * \text{ --- ---- } \underset{\downarrow}{(N-R-1)}_{\,p} * \underset{\downarrow}{(N-R)}_{\,p}$

$\gcd((N-R)!\,,\, p) = 1 \Rightarrow$ Apply inverse modulo

$((N-R)!)^{-1} \% \, p = ((N-R)!)^{p-2} \% \, p$

$= \left(\underbrace{(N-R)! \% \, p}_{x}\right)^{p-2} \% \, p$

$= (x)^{p-2} \% \, p \Rightarrow$ Use fast exponentation to calculate $x^{p-2} \% \, p$

$((k!)^{-1}) \% p$

$\gcd(k!, p) = 1$ ✓

p is prime ✓

$$((k!)^{-1}) \% p = ((k!)^{p-2}) \% p$$

$$= ((\underbrace{k! \% p}_{y})^{p-2}) \% p$$

$$= (y^{p-2}) \% p$$

{ use fast exponatation }

→ try to code this { #todo }.

Q: Given N, R, P. Calculate ${}^{N}C_R \% p$. p is not prime

$\Downarrow$

inverse modulo ✗

$$\left\{ {}^{N}C_R = {}^{N-1}C_{R-1} + {}^{N-1}C_R \right\}$$

$${}^{N}C_R \% p = \left({}^{N-1}C_{R-1} + {}^{N-1}C_R\right) \% p$$

```
fun ( N, R, P) { //it will return NCR % P
    if (N < R ) { return 0 }
    if ( R = = 0) { return 1 }

    int a = fun( N-1, R-1, P)
    int b = fun( N-1, R, P)

    return (a+b) % P
}
```

p can be anything.

T.C → $O({}^{N}C_R)$

${}^{N}C_R$

$N=4,\ R=0.$

$${}^{4}C_0 = {}^{3}C_{-1} + {}^{3}C_0$$

$N=5,\ R=5$

$${}^{5}C_5 = {}^{4}C_4 + {}^{4}C_5$$

$N=4$, $R=3$. ${}^4C_3 \to {}^3C_2 \to {}^2C_1 \to {}^1C_0$

R

i \ j	0	1	2	3
0	1	0	0	0
1	1	1	0	0
2	1	2	1	0
3	1	3	3	1
4	1	4	6	4

// fill → top → bottom.

$${}^NC_R = {}^{N-1}C_{R-1} + {}^{N-1}C_R$$

mat[i][j] = mat[i-1][j-1] + mat[i-1][j]

$${}^1C_1 = {}^0C_0 + {}^0C_1 = 1$$

$${}^1C_2 = {}^0C_1 + {}^0C_2 = 0$$

$${}^1C_3 = {}^0C_2 + {}^0C_3 = 0$$

$${}^2C_1 = {}^1C_0 + {}^1C_1 = 2$$

$${}^2C_2 = {}^1C_1 + {}^1C_2 -$$

//pseudo-code.

```
mat [N+1][R+1]
for( j = 1 ; j <= R ; j++) { mat [0][j] = 0 }
for ( i = 0 ; i <= N ; i++) { mat [i][0] = 1 }
  for( i = 1 ; i <= N ; i++) {
        for( j = 1 ; j <= R ; j++) {
              mat[i][j] = (mat[i-1][j-1] + mat[i-1][j]) % P
        }
  }
return mat[N][R]
```

→ D.P.

$\boxed{\begin{array}{l} T.C \to O(N*R) \\ S.C \to O(N*R) \end{array}}$

{→ till Combinatorics}