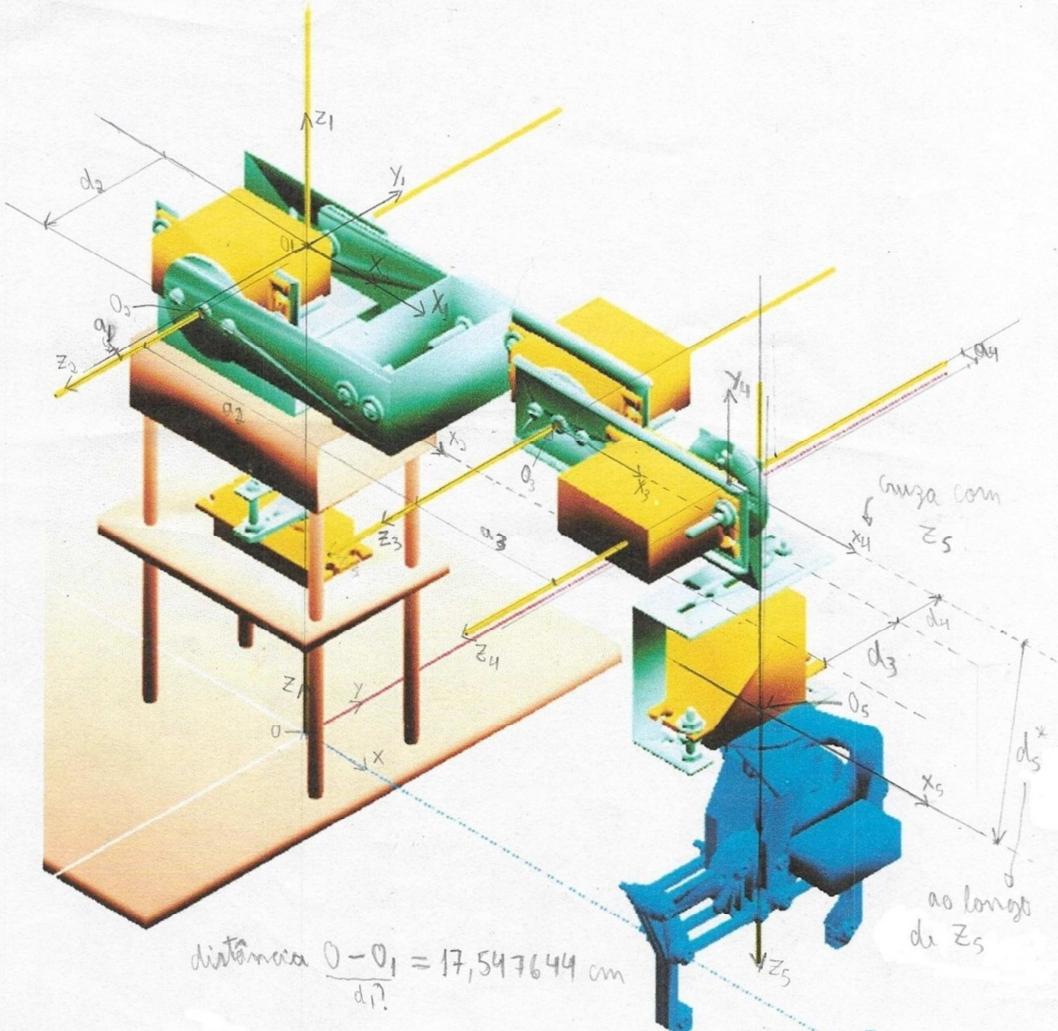


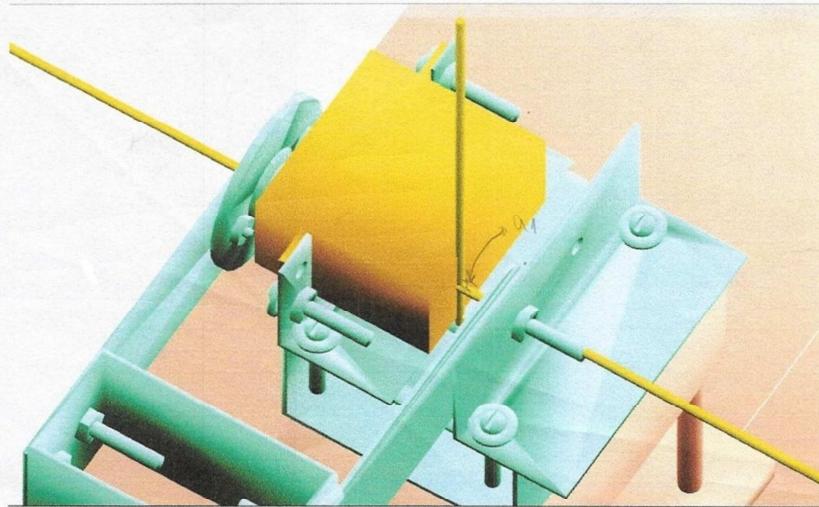
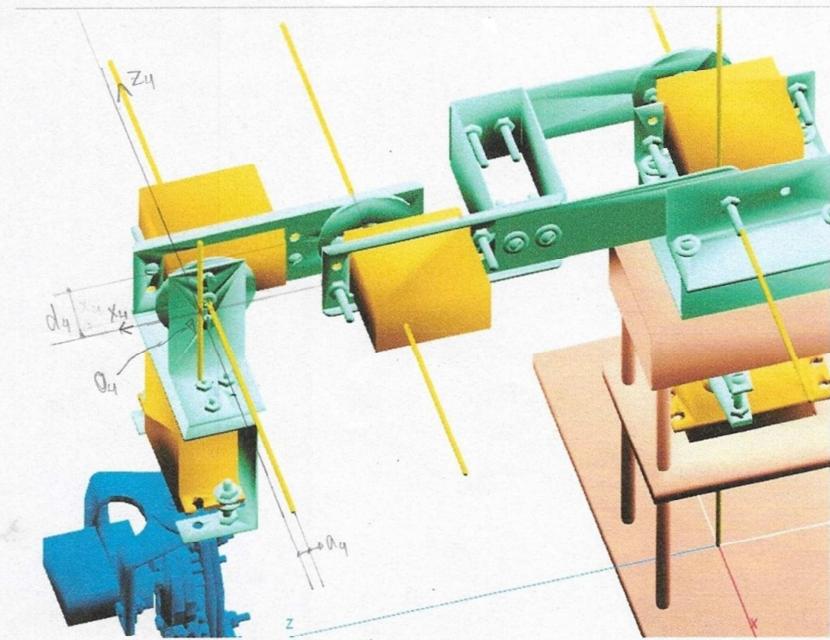
Parâmetros de Denavit-Hartenberg (DH)



i	α_{i-1}	a_{i-1}	d_i	θ_i
1	0	0	0,1	θ_1
2	90°	0,025 cm	4,293516 cm	θ_2
3	0	11,65 cm	-3,438032 cm	θ_3
4	0	5,825 cm	-2,174584 cm	θ_4
5	90°	0,45 cm	8,633297 cm	θ_5

d_3 e d_4 negativos?
(visto invertido no $z_2 + z_3$)

Apesar de termos dado valores não nulos para d_2 , d_3 e d_4 , a_2 e a_4 , na cinemática implementada no software de controle consideramos todos eles como sendo zero, para que os pontos da garra e do pulso fiquem alinhados com a origem, de forma que seja possível definir um plano que corte o braço robô, que passe pela origem da base, pelo pulso e pelo ponto da garra.



Cálculos da Cinemática Direta

$$\begin{aligned}
{}^0_1 T &= \begin{bmatrix} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} & {}^1_2 T &= \begin{bmatrix} c_2 & -s_2 & 0 & a_1 \\ 0 & 0 & -1 & -d_2 \\ s_2 & c_2 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
{}^0_2 T &= \begin{bmatrix} c_1 c_2 & -c_1 s_2 & s_1 & c_1 a_1 + s_1 d_2 \\ s_1 c_2 & -s_1 s_2 & -c_1 & s_1 a_1 - c_1 d_2 \\ s_2 & c_2 & 0 & d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} & {}^2_3 T &= \begin{bmatrix} c_3 & -s_3 & 0 & a_2 \\ s_3 & c_3 & 0 & 0 \\ 0 & 0 & 1 & d_3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
{}^0_3 T &= \begin{bmatrix} c_1 c_2 c_3 - c_1 s_2 s_3 & -c_1 c_2 s_3 - c_1 s_2 c_3 & s_1 & c_1 c_2 a_2 + s_1 d_3 + (c_1 a_1 + s_1 d_2) \\ s_1 c_2 c_3 - s_1 s_2 s_3 & -s_1 c_2 s_3 - s_1 s_2 c_3 & -c_1 & s_1 c_2 a_2 - c_1 d_3 + (s_1 a_1 - c_1 d_2) \\ s_2 c_3 + c_2 s_3 & -s_2 s_3 + c_2 c_3 & 0 & s_2 a_2 + d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
{}^0_4 T &= \begin{bmatrix} c_1 c_{23} & -c_1 s_{23} & s_1 & s_1(d_2 + d_3) + c_1(a_1 + c_2 a_2) \\ s_1 c_{23} & -s_1 s_{23} & -c_1 & s_1(a_1 + c_2 a_2) - c_1(d_2 + d_3) \\ s_{23} & c_{23} & 0 & s_2 a_2 + d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} & {}^3_4 T &= \begin{bmatrix} c_4 & -s_4 & 0 & a_3 \\ s_4 & c_4 & 0 & 0 \\ 0 & 0 & 1 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
{}^0_4 T &= \begin{bmatrix} c_1 c_{23} c_4 - c_1 s_{23} s_4 & -c_1 c_{23} s_4 - c_1 s_{23} c_4 & s_1 & c_1 c_{23} a_3 + s_1 d_4 + s_1(d_2 + d_3) + c_1(a_1 + c_2 a_2) \\ s_1 c_{23} c_4 - s_1 s_{23} s_4 & -s_1 c_{23} s_4 - s_1 s_{23} c_4 & -c_1 & s_1 c_{23} a_3 - c_1 d_4 + s_1(a_1 + c_2 a_2) - c_1(d_2 + d_3) \\ s_{23} c_4 + c_{23} s_4 & -s_{23} s_4 + c_{23} c_4 & 0 & s_{23} a_3 + s_2 a_2 + d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
{}^0_5 T &= \begin{bmatrix} c_1 c_{234} & -c_1 s_{234} & s_1 & s_1(d_2 + d_3 + d_4) + c_1(a_1 + c_2 a_2 + c_{23} a_3) \\ s_1 c_{234} & -s_1 s_{234} & -c_1 & s_1(a_1 + c_2 a_2 + c_{23} a_3) - c_1(d_2 + d_3 + d_4) \\ s_{234} & c_{234} & 0 & s_{23} + s_2 a_2 + d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} & {}^4_5 T &= \begin{bmatrix} c_5 & -s_5 & 0 & a_4 \\ 0 & 0 & -1 & -d_5 \\ s_5 & c_5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
{}^0_5 T &= \begin{bmatrix} c_1 c_{234} c_5 + s_1 s_5 & -c_1 c_{234} s_5 + s_1 s_5 & c_1 s_{234} & c_1 c_{234} a_4 + c_1 s_{234} d_5 + s_1(d_2 + d_3 + d_4) + c_1(a_1 + c_2 a_2 + c_{23} a_3) \\ s_1 c_{234} c_5 - c_1 s_5 & -s_1 c_{234} s_5 - c_1 c_5 & s_1 s_{234} & s_1 c_{234} a_4 + s_1 s_{234} d_5 + s_1(a_1 + c_2 a_2 + c_{23} a_3) - c_1(d_2 + d_3 + d_4) \\ s_{234} c_5 & -s_{234} s_5 & -c_{234} & s_{234} a_4 - c_{234} d_5 + s_{23} a_3 + s_2 a_2 + d_1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
{}^0_5 T &= \boxed{\begin{bmatrix} c_1 c_{234} c_5 + s_1 s_5 & -c_1 c_{234} s_5 + s_1 c_5 & c_1 s_{234} & s_1(d_2 + d_3 + d_4) + c_1(a_1 + c_2 a_2 + c_{23} a_3 + c_{234} a_4 + s_{234} d_5) \\ s_1 c_{234} c_5 - c_1 s_5 & -s_1 c_{234} s_5 - c_1 c_5 & s_1 s_{234} & s_1(a_1 + c_2 a_2 + c_{23} a_3 + c_{234} a_4 + s_{234} d_5) - c_1(d_2 + d_3 + d_4) \\ s_{234} c_5 & -s_{234} s_5 & -c_{234} & d_1 + s_2 a_2 + s_{23} a_3 + s_{234} a_4 - c_{234} d_5 \\ 0 & 0 & 0 & 1 \end{bmatrix}}
\end{aligned}$$

Cálculos da Cinemática Direta – Resultado e Otimizações de Fórmulas

$${}^0_5T = \begin{bmatrix} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{array}{lll} c_i = \cos \theta_i & s_{23} = s_2 c_3 + c_2 s_3 \\ s_i = \sin \theta_i & c_{23} = c_2 s_3 - s_2 c_3 \\ i = 1, \dots, 5 & c_{234} = c_{23} c_4 - s_{23} s_4 \\ & s_{234} = s_{23} c_4 + c_{23} s_4 \end{array}$$

$$r_{11} = c_1 c_{234} c_5 + s_1 s_5$$

$$r_{21} = s_1 c_{234} c_5 - c_1 s_5$$

$$r_{31} = s_{234} c_5$$

$$r_{12} = s_1 c_5 - c_1 c_{234} s_5$$

$$r_{22} = -c_1 c_5 - s_1 c_{234} s_5$$

$$r_{32} = -s_{234} s_5$$

$$r_{13} = c_1 s_{234}$$

$$r_{23} = s_1 s_{234}$$

$$r_{33} = -c_{234}$$

$$d_{234} = d_2 + d_3 + d_4$$

$$f = a_1 + a_2 c_2 + a_3 c_{23} + a_4 c_{234} + d_5 s_{234}$$

$$p_x = d_{234} s_1 + f c_1$$

$$p_y = f s_1 - d_{234} c_1$$

$$p_z = d_1 + a_2 s_2 + a_3 s_{23} + a_4 s_{234} - d_5 c_{234}$$

Utilizamos rotações em ângulos fixos em torno de X, Y e Z, respectivamente, cujos ângulos são γ , β e α , respectivamente. Para encontrar os ângulos γ , β e α , faremos:

$$\begin{aligned} \beta &= \text{atan2}\left(-r_{31}, \sqrt{r_{11}^2 + r_{21}^2}\right) \\ \alpha &= \text{atan2}\left(\frac{r_{21}}{\cos \beta}, \frac{r_{11}}{\cos \beta}\right) \\ \gamma &= \text{atan2}\left(\frac{r_{32}}{\cos \beta}, \frac{r_{33}}{\cos \beta}\right) \end{aligned}$$

As fórmulas para α e γ são válidas para $\beta \neq 0$. Se $\beta = 90^\circ$:

$$\begin{aligned} \alpha &= 0 \\ \gamma &= \text{atan2}(r_{12}, r_{22}) \end{aligned}$$

Se $\beta = -90^\circ$:

$$\begin{aligned} \alpha &= 0 \\ \gamma &= -\text{atan2}(r_{12}, r_{22}) \end{aligned}$$

Para achar as coordenadas x, y e z da garra, basta fazer o seguinte cálculo:

$$\begin{aligned} x &= p_x + (L_3 + L_g) \cdot r_{13} \\ y &= p_y + (L_3 + L_g) \cdot r_{23} \\ z &= p_z + (L_3 + L_g) \cdot r_{33} \end{aligned}$$

Cinemática Inversa – Encontrando a matriz de transformação 0_5T

No software de controle, são fornecidas as coordenadas x, y, z, Rx, Ry e Rz, sendo que estas 3 últimas correspondem aos ângulos γ , β e α . A matriz 0_5T a ser encontrada tem a seguinte forma:

$${}^0_5T = \left[\begin{array}{ccc|c} & {}^5_0R & & {}^0P_{5ORG} \\ \hline 0 & 0 & 0 & 1 \end{array} \right]$$

Para achar 0_0R , faz-se:

$${}^5_0R = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ r_{21} & r_{22} & r_{23} \\ r_{31} & r_{32} & r_{33} \end{bmatrix}$$

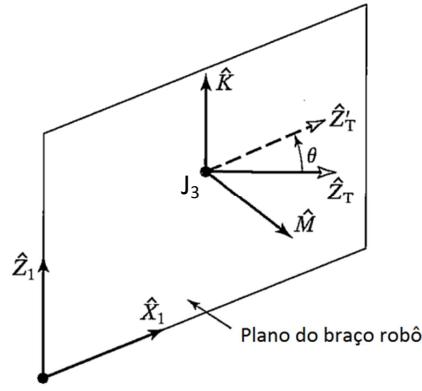
$$\begin{aligned} r_{11} &= cac\beta \\ r_{21} &= sac\beta \\ r_{31} &= -s\beta \\ r_{12} &= cas\beta sy - sacy \\ r_{22} &= sas\beta sy + cac\gamma \\ r_{32} &= c\beta sy \\ r_{13} &= cas\beta c\gamma + sas\gamma \\ r_{23} &= sas\beta c\gamma - cas\gamma \\ r_{33} &= c\beta c\gamma \end{aligned} \quad \text{onde} \quad \begin{cases} s\alpha = \sin \alpha \\ s\beta = \sin \beta \\ s\gamma = \sin \gamma \\ c\alpha = \cos \alpha \\ c\beta = \cos \beta \\ c\gamma = \cos \gamma \end{cases}$$

Para achar ${}^0P_{5ORG}$, faz-se:

$${}^0P_{5ORG} = \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix}$$

$$\begin{aligned} p_x &= x - (L_3 + L_g) \cdot r_{13} \\ p_y &= y - (L_3 + L_g) \cdot r_{23} \\ p_z &= z - (L_3 + L_g) \cdot r_{33} \end{aligned}$$

Cinemática Inversa – Projetando o ponto x, y, z alvo no plano que corta o braço robô



Mesmo que o ponto ${}^0P_{ORG}$ sempre esteja no plano do braço robô, o ponto XYZ alvo da garra nem sempre estará, por conta dos ângulos γ , β e α , que podem, por exemplo, fazer a garra apontar para a direção do vetor \hat{Z}_T mostrado na figura. Para fazer a projeção do XYZ alvo da garra, seguem os cálculos:

$$\hat{M} = \frac{1}{\sqrt{p_x^2 + p_y^2}} \begin{bmatrix} -p_y \\ p_x \\ 0 \end{bmatrix} \quad \hat{Z}_T = \begin{bmatrix} r_{13} \\ r_{23} \\ r_{33} \end{bmatrix} \quad \hat{Y}_T = \begin{bmatrix} r_{12} \\ r_{22} \\ r_{32} \end{bmatrix}$$

$$\hat{K} = \hat{M} \times \hat{Z}_T$$

$$\hat{Z}'_T = \hat{K} \times \hat{M}$$

$$\cos \theta = \hat{Z}_T \cdot \hat{Z}'_T$$

$$\sin \theta = (\hat{Z}_T \times \hat{Z}'_T) \cdot \hat{K}$$

Fórmula de Rodrigues:

$$\hat{Y}'_T = \cos \theta \cdot \hat{Y}_T + \sin \theta (\hat{K} \times \hat{Y}_T) + (1 - \cos \theta) (\hat{K} \cdot \hat{Y}_T) \hat{K}$$

$$\hat{X}'_T = \hat{Y}'_T \times \hat{Z}'_T$$

A nova matriz de rotação passa a ser (apenas para a cinemática inversa):

$${}^0R' = [\hat{X}'_T \quad \hat{Y}'_T \quad \hat{Z}'_T]$$

De forma que o ponto XYZ alvo da garra passa a ser:

$$\begin{aligned} x &= p_x + (L_3 + L_g) \cdot r'_{13} \\ y &= p_y + (L_3 + L_g) \cdot r'_{23} \\ z &= p_z + (L_3 + L_g) \cdot r'_{33} \end{aligned}$$

E as rotações passam a ser

$$\begin{aligned} \beta &= \text{atan2}\left(-r'_{31}, \sqrt{{r'_{11}}^2 + {r'_{21}}^2}\right) \\ \alpha &= \text{atan2}\left(\frac{r'_{21}}{\cos \beta}, \frac{r'_{11}}{\cos \beta}\right) \\ \gamma &= \text{atan2}\left(\frac{r'_{32}}{\cos \beta}, \frac{r'_{33}}{\cos \beta}\right) \end{aligned}$$

As fórmulas para α e γ são válidas para $\beta \neq 0$. Se $\beta = 90^\circ$:

$$\begin{aligned}\alpha &= 0 \\ \gamma &= \text{atan}2(r'_{12}, r'_{22})\end{aligned}$$

Se $\beta = -90^\circ$:

$$\begin{aligned}\alpha &= 0 \\ \gamma &= -\text{atan}2(r'_{12}, r'_{22})\end{aligned}$$

Cálculos da Cinemática Inversa – θ_1 e θ_5

$$\begin{aligned} {}^1T = {}^0T^{-1} &= \left[\begin{array}{ccc|c} {}^0R^T & & & -{}^0R^T \cdot {}^0P_{1ORG} \\ \hline 0 & 0 & 0 & 1 \end{array} \right] & {}^0T = \left[\begin{array}{ccc|c} c_1 & -s_1 & 0 & 0 \\ s_1 & c_1 & 0 & 0 \\ 0 & 0 & 0 & d_1 \\ \hline 0 & 0 & 0 & 1 \end{array} \right] &= \left[\begin{array}{ccc|c} {}^1R & & & {}^0P_{1ORG} \\ \hline 0 & 0 & 0 & 1 \end{array} \right] \\ {}^0T^{-1} &= \left[\begin{array}{ccc|c} c_1 & s_1 & 0 & \left[\begin{array}{ccc|c} -c_1 & -s_1 & 0 & 0 \\ s_1 & -c_1 & 0 & 0 \\ 0 & 0 & -1 & d_1 \end{array} \right] \\ \hline 0 & 0 & 1 & 1 \end{array} \right] = \left[\begin{array}{cccc} c_1 & s_1 & 0 & 0 \\ -s_1 & c_1 & 0 & 0 \\ 0 & 0 & 1 & -d_1 \\ 0 & 0 & 0 & 1 \end{array} \right] & {}^5T = \left[\begin{array}{cccc} r_{11} & r_{12} & r_{13} & p_x \\ r_{21} & r_{22} & r_{23} & p_y \\ r_{31} & r_{32} & r_{33} & p_z \\ 0 & 0 & 0 & 1 \end{array} \right] \end{aligned}$$

$${}^1T^{-1} {}^0T = {}^0T^{-1}({}^0T {}^1T {}^2T {}^3T {}^4T {}^5T) = {}^1T {}^2T {}^3T {}^4T {}^5T$$

$${}^1T^{-1} {}^0T = \left[\begin{array}{cccc} c_1r_{11} + s_1r_{21} & c_1r_{12} + s_1r_{22} & c_1r_{13} + s_1r_{23} & c_1p_x + s_1p_y \\ -s_1r_{11} + c_1r_{21} & -s_1r_{12} + c_1r_{22} & -s_1r_{13} + c_1r_{23} & -s_1p_x + c_1p_y \\ r_{31} & r_{32} & r_{33} & p_z - d_1 \\ 0 & 0 & 0 & 1 \end{array} \right]$$

$${}^1T {}^2T {}^3T {}^4T {}^5T = \left[\begin{array}{cccc} t_{11} & t_{12} & t_{31} & v_x \\ t_{21} & t_{22} & t_{32} & v_y \\ t_{31} & t_{32} & t_{33} & v_z \\ 0 & 0 & 0 & 1 \end{array} \right]$$

$$t_{11} = c_1^2 c_{234} c_5 + c_1 s_1 s_5 + s_1^2 c_{234} c_5 - s_1 c_1 s_5 = c_{234} c_5 (c_1^2 + s_1^2) = c_{234} c_5$$

$$t_{21} = -s_1 c_1 c_{234} c_5 - s_1^2 s_5 + c_1 s_1 c_{234} c_5 - c_1^2 s_5 = -s_5$$

$$t_{31} = s_{234} c_5$$

$$t_{12} = -c_1^2 c_{234} s_5 + c_1 s_1 c_5 - s_1^2 c_{234} s_5 - s_1 c_1 c_5 = -c_{234} s_5$$

$$t_{22} = s_1 c_1 c_{234} s_5 - s_1^2 c_5 - c_1 s_1 c_{234} s_5 - c_1^2 c_5 = -c_5$$

$$t_{32} = -s_{234} s_5$$

$$t_{13} = c_1^2 s_{234} + s_1^2 s_{234} = s_{234}$$

$$t_{23} = -s_1 c_1 s_{234} + c_1 s_1 s_{234} = 0$$

$$t_{33} = -c_{234}$$

$$v_x = c_1 s_1 d_{234} - s_1 c_1 f + c_1 s_1 f - s_1 c_1 d_{234} = f$$

$$v_y = -s_1^2 d_{234} - s_1 c_1 f + c_1 s_1 f - c_1^2 d_{234} = -d_{234}$$

$$v_z = s_2 a_2 + s_{23} a_3 + s_{234} a_4 - c_{234} d_5$$

Para o cálculo de θ_1 (todas as soluções), temos:

$$-s_1 p_x + c_1 p_y = -d_{234}$$

$$\boxed{\theta_1 = \text{atan2}(-p_x, p_y) \pm \text{atan2}\left(\sqrt{p_y^2 + p_x^2 - d_{234}^2}, -d_{234}\right)} \quad (\text{soluções 1 e 2})$$

$$-s_1 r_{13} + c_1 r_{23} = 0$$

$$\boxed{\theta_1 = \text{atan2}(r_{23}, r_{13})} \quad (\text{solução 3})$$

$$\boxed{\theta_1 = \text{atan2}(-r_{23}, -r_{13})} \quad (\text{solução 4})$$

Conhecido θ_1 , temos, para θ_5 :

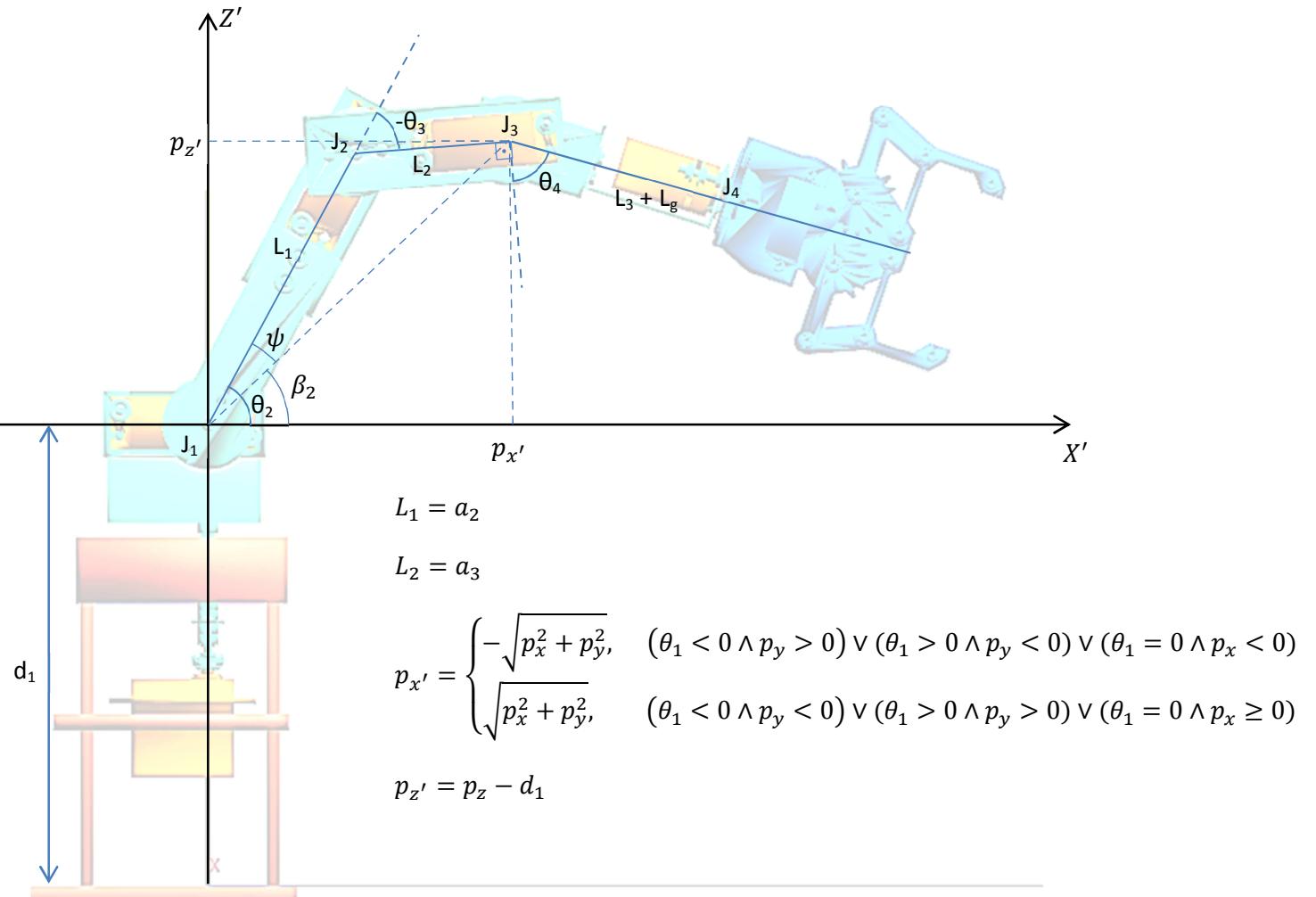
$$\begin{cases} -s_1 r_{11} + c_1 r_{21} = -s_5 \\ -s_1 r_{12} + c_1 r_{22} = -c_5 \end{cases}$$

$$\begin{cases} s_5 = s_1 r_{11} - c_1 r_{21} \\ c_5 = s_1 r_{12} - c_1 r_{22} \\ \theta_5 = \text{atan2}(s_5, c_5) \end{cases}$$

Conhecido θ_1 , temos, para θ_{234} :

$$\begin{cases} s_{234} = c_1 r_{13} + s_1 r_{23} \\ c_{234} = -r_{33} \\ \theta_{234} = \text{atan2}(s_{234}, c_{234}) \end{cases}$$

Cinemática Inversa – Abordagem geométrica para θ_2, θ_3 e θ_4



Para esta abordagem, trataremos o pulso da garra (J_4) como se estivesse junto com J_3 , de forma que p_x , p_y e p_z se referem aos valores de coordenada de J_3 . Isso torna possível a abordagem geométrica como se o braço robô fosse um manipulador planar.

Para θ_3 , temos, a partir do triângulo formado por J_1 , J_2 e J_3 :

$$p_{x'}^2 + p_{z'}^2 = L_1^2 + L_2^2 - 2 \cdot L_1 \cdot L_2 \cdot \cos(180^\circ + \theta_3)$$

$$\cos \theta_3 = \frac{p_{x'}^2 + p_{z'}^2 - a_2^2 - a_3^2}{2 \cdot a_2 \cdot a_3}$$

Como θ_3 será sempre menor ou igual a zero, então:

$$\sin \theta_3 = -\sqrt{1 - \cos^2 \theta_3}$$

Logo:

$$\theta_3 = \text{atan2}(\sin \theta_3, \cos \theta_3)$$

Para θ_2 , calculamos os ângulos β_2 e ψ :

$$\beta_2 = \text{atan2}(p_{z'}, p_{x'})$$

$$\cos \psi = \frac{p_{x'}^2 + p_{z'}^2 + a_2^2 - a_3^2}{2 \cdot a_2 \cdot \sqrt{p_{x'}^2 + p_{z'}^2}}$$

Como ψ será sempre maior ou igual à zero, então:

$$\sin \psi = \sqrt{1 - \cos^2 \psi}$$

$$\psi = \text{atan2}(\sin \psi, \cos \psi)$$

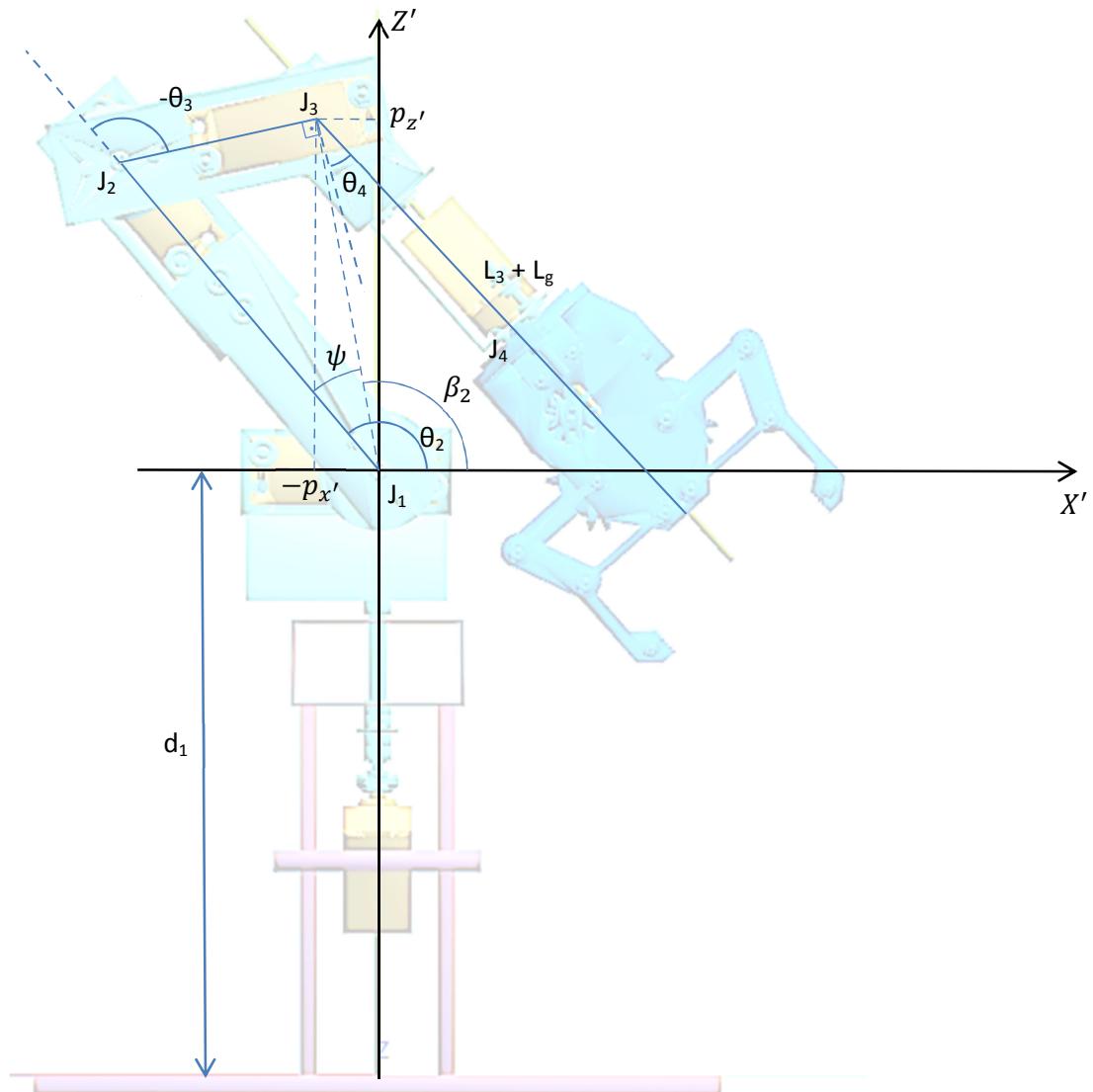
Logo:

$$\theta_2 = \beta_2 + \psi$$

Logo, para θ_4 , temos:

$$\theta_4 = \theta_{234} - \theta_2 - \theta_3$$

Braço robô na posição de repouso



Observe que, nesta posição, o $p_{x'}$ deve assumir valor negativo, pois está no lado negativo do eixo X' , o que justifica a fórmula do $p_{x'}$ ser:

$$p_{x'} = \begin{cases} -\sqrt{p_x^2 + p_y^2}, & (\theta_1 < 0 \wedge p_y > 0) \vee (\theta_1 > 0 \wedge p_y < 0) \vee (\theta_1 = 0 \wedge p_x < 0) \\ \sqrt{p_x^2 + p_y^2}, & (\theta_1 < 0 \wedge p_y < 0) \vee (\theta_1 > 0 \wedge p_y > 0) \vee (\theta_1 = 0 \wedge p_x \geq 0) \end{cases}$$

de forma que o ângulo β_2 seja calculado no quadrante correto.